E ISSN 1512-3391

ANDREA RAZMADZE MATHEMATICAL INSTITUTE

of Ivane Javakhishvili Tbilisi State University

REPORTS OF QUALITDE

Volume 2

Tbilisi, Georgia, 2023

International Workshop on the Qualitative Theory of Differential Equations – QUALITDE-2023
December 9-11, 2023

Andrea Razmadze Mathematical Institute of Ivane Javakhishvili Tbilisi State University
Tbilisi, Georgia

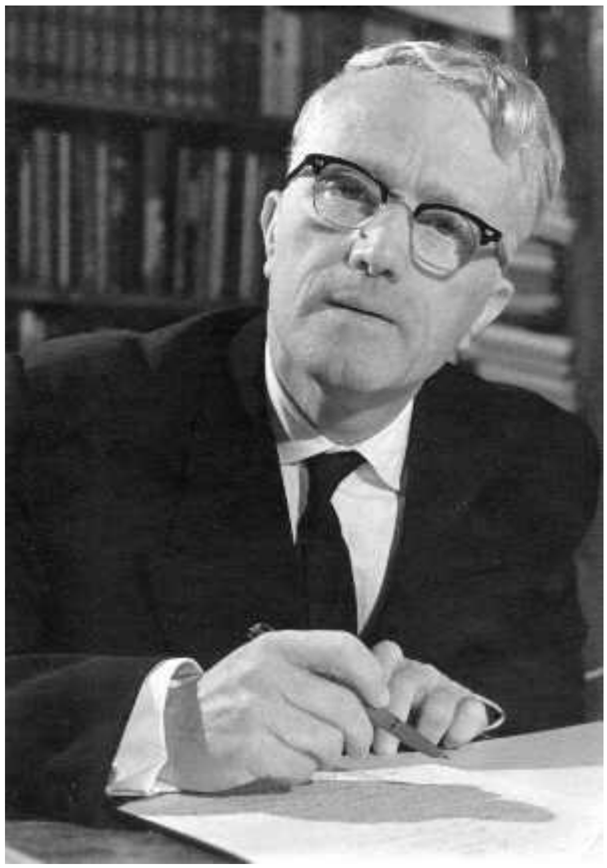
REPORTS OF QUALITDE

Volume 2

Editor: Nino Partsvania

Publisher: Tbilisi State University Press

Dedicated to the 120th birthday anniversary
of Professor V. Kupradze



02.11.1903 - 25.04.1985

On the Criterion of Well-Posedness of the Modified Cauchy Problem for Singular Systems of Linear Ordinary Differential Equations

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Let $[a,b]\subset\mathbb{R}$ be a finite and closed interval non-degenerated in the point.

Consider the modified initial problem for a linear system of generalized ordinary differential equations with singularities

$$dx=dA(t)\cdot x+df(t)\ \text{ for }\ t\in[a,b[\,, \tag{1}$$

$$\lim_{t\to b-}\big(\Phi^{-1}(t)\,x(t)\big)=0, \tag{2}$$

where $A=(a_{ik})_{i,k=1}^n$ is an $n\times n$-matrix valued function and $f=(f_k)_{k=1}^n$ is an n-vector valued function, both of them have a locally bounded variation on $[a,b[$; $\Phi=\operatorname{diag}(\varphi_1,\dots,\varphi_n)$ is a diagonal $n\times n$-matrix valued function, defined on $[a,b[$ and having an inverse $\Phi^{-1}(t)$ for each $t\in[a,b[$.

Along with system (1) consider the perturbed singular systems

$$dx=dA_m(t)\cdot x+df_m(t)\ \text{ for }\ t\in[a,b[\tag{3}$$

$(m=1,2,\dots)$ under conditions (2), where A_m is an $n\times n$-matrix valued function and f_m is an n-vector valued function, both of them have a locally bounded variation on $[a,b[$.

We are interested to established the necessary and sufficient conditions whether the unique solvability of problem $(1),(2)$ guarantees the unique solvability of problem $(3),(2)$ and nearness of its solution in the definite sense if matrix-functions A_m and A and vector-functions f_m and f are nearly among themselves.

We assume $A(a)=A_m(a)=O_{n\times n}$ and $f(a)=f_m(a)=0_n$ $(m=1,2,\dots)$ without loss of generality.

The same and related problems for ordinary differential systems with singularities $\frac{dx}{dt}=P(t)\,x+q(t)$, where $P\in L_{loc}([a,b[,\mathbb{R}^{n\times n})$, $q\in L_{loc}([a,b[,\mathbb{R}^n)$, have been investigated in [7,9] (see, also, the references therein).

The singularity of system (1) consists in the fact that both A and f need not to have bounded variations on any interval containing the point t_0.

The solvability question of the generalized differential problem $(1),(2)$ has been investigated in [6]. The well-posedness of problem $(1),(2)$ with singularity has been considered in [4]. To our knowledge, the necessary and sufficient conditions for well-posedness of problem $(1),(2)$ with singularity has not been investigated up to now.

Some singular boundary problems for the generalized differential system (1) are investigated in [1,2] (see, also, the references therein).

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see [1–6,8,10,11] and the references therein).

In the paper, we give necessary and sufficient conditions for the so called strongly Φ-well-posedness of problem (1), (2).

Throughout the paper we use the following notation and definitions.

$\mathbb{R} =]-\infty,+\infty[$. $\mathbb{R}_+ =]0,+\infty[$. $\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices with the standard norm.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all column n-vectors $x = (x_i)_{i=1}^n$.

If $X = (x_{ik})_{i,k=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = (|x_{ik}|)_{i,k=1}^{n,m}$, $[X]_{\mp} = \frac{1}{2}\,(|X| \mp X)$.

$O_{n\times m}$ (or O) is the zero $n\times m$-matrix, 0_n (or 0) is the zero n-vector.

I_n is identity $n\times n$-matrix.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X.

The inequalities between the matrices are understood componentwisely.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

If $X : \mathbb{R} \to \mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_a^b(X)$ is the sum of total variations on $[a,b]$ of its components; $\bigvee\limits_a^{b-}(X) = \lim\limits_{t\to b-} \bigvee\limits_a^t(X)$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$ at the point t.

$\mathrm{BV}([c,d],\mathbb{R}^{n\times m})$ is the set of bounded variation matrix-functions on $[c,d]$.

$\mathrm{BV}_{loc}([a,b[;\mathbb{R}^{n\times m})$ is the set of all locally bounded matrix-functions.

If $X \in \mathrm{BV}_{loc}([a,b[;\mathbb{R}^{n\times m})$, then

$$[X(t)]^v_- \equiv \frac{1}{2}\,(V(X)(t) - X(t)), \quad [X(t)]^v_+ \equiv \frac{1}{2}\,(V(X)(t) + X(t)).$$

$s_1, s_2, s_c : \mathrm{BV}_{loc}([a,b[;\mathbb{R}) \to \mathrm{BV}_{loc}([a,b[;\mathbb{R})$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0, \quad s_c(x)(a) = x(a),$$
$$s_1(x)(t) = s_1(x)(a) + \sum_{a<\tau\le t} d_1x(\tau), \quad s_2(x)(t) = s_2(x)(a) + \sum_{a\le\tau<t} d_2x(\tau),$$
$$s_c(x)(t) = s_c(x)(a) + x(t) - x(a) - \sum_{j=1}^{2} s_j(x)(t) \ \text{ for } \ a<t<b.$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function and $x : [a,b] \to \mathbb{R}$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,ds_c(g)(\tau) + \sum_{s<\tau\le t} x(\tau)\,d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)\,d_2g(\tau)$$
$$\text{for } \ s<t;\ s,t \in [a,b],$$

where $\int\limits_{]s,t[} x(\tau)\,ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure corresponding to the function $s_c(g)$. So $\int\limits_s^t x(\tau)\,dg(\tau)$ is the Kurzweil integral ([10,11]).

We put $\int\limits_s^{t-} x(\tau)\,dg(\tau) = \lim_{\delta\to 0+} \int\limits_s^{t-\delta} x(\tau)\,dg(\tau)$.

If $G=(g_{ik})_{i,k=1}^{l,n} : [a,b]\to\mathbb{R}^{l\times n}$ and $X=(x_{kj})_{k,j=1}^{n,m} : [a,b]\to\mathbb{R}^{n\times m}$, then

$$\int\limits_a^t dG(\tau)\cdot X(\tau) \equiv \bigg(\sum_{k=1}^n \int\limits_a^t x_{kj}(\tau)dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}.$$

We introduce the operators $\mathcal{A}(X,Y)$, $\mathcal{B}(X,Y)$ and $\mathcal{I}(X,Y)$ in the following way:

(a) if $X\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $\det(I_n+(-1)^j d_jX(t))\neq 0$ for $t\in I$ $(j=1,2)$, and $Y\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m})$, then $\mathcal{A}(X,Y)(a)=O_{n\times m}$,

$$\begin{aligned}\mathcal{A}(X,Y)(t)\equiv Y(t)-Y(a)+\sum_{a<\tau\le t} d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)\\ -\sum_{a\le\tau<t} d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau);\end{aligned}$$

(b) if $X\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ and $Y: I\to\mathbb{R}^{n\times m}$, then $\mathcal{B}(X,Y)(a)=O_{n\times m}$,

$$\mathcal{B}(X,Y)(t)\equiv X(t)Y(t)-X(a)Y(a)-\int\limits_a^t dX(\tau)\cdot Y(\tau);$$

(c) if $X\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $\det(X(t))\neq 0$, and $Y: I\to\mathbb{R}^{n\times n}$, then

$$\mathcal{I}(X,Y)(a)=O_{n\times m},\ \ \mathcal{I}(X,Y)(t)\equiv\int\limits_a^t d\big(X(\tau)+\mathcal{B}(X,Y)(\tau)\big)\cdot X^{-1}(\tau).$$

In addition, let $\mathcal{V}_j(\Phi,A_*,\cdot\,):\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times l})\to\mathbb{R}$ $(j=1,2)$ be operators defined, respectively, by

$$\mathcal{V}_1(\Phi,A_*,F)(t,\tau)=\int\limits_t^\tau \Phi^{-1}(s)\,d\,\mathrm{V}(\mathcal{A}(A_*,F))(s)\cdot\Phi(s)\ \text{ and}$$

$$\mathcal{V}_2(\Phi,A_*,F)(t,\tau)=\int\limits_t^\tau \Phi^{-1}(s)\,d\,\mathrm{V}(\mathcal{A}(A_*,A_*))(s)\cdot|F(s)|\ \text{ for }\ a\le t<\tau<b.$$

A vector-function $x: I\to\mathbb{R}^n$ is said to be a solution of system (1) if $x\in\mathrm{BV}_{loc}(I,\mathbb{R}^n)$ and

$$x(t)=x(a)+\int\limits_a^t dA(\tau)\cdot x(\tau)+f(t)-f(a)\ \text{ for }\ t\in I.$$

We assume that $\det(I_n+(-1)^j d_j A(t))\neq 0$ for $t\in I$ $(j=1,2)$.

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems, i.e., for the case when $A\in \mathrm{BV}([a,c];\mathbb{R}^{n\times n})$ and $f\in \mathrm{BV}([a,c];\mathbb{R}^n)$ for every $c\in I$.

Let a matrix-function $A_*=(a_{*ik})_{i,k=1}^n\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ be such that $\det(I_n+(-1)^j d_j A_*(t))\neq 0$ for $t\in I$ $(j=1,2)$.

Then a matrix-function $C_*: I\times I\to\mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the homogeneous system $dx=dA_*(t)\cdot x$, if, for each interval $J\subset I$ and $\tau\in J$, the restriction of the matrix-function $C_*(\,\cdot\,,\tau):I\to\mathbb{R}^{n\times n}$ on J is the fundamental matrix of the system, satisfying the condition $C_*(\tau,\tau)=I_n$. Therefore, C_* is the Cauchy matrix of the system if and only if the restriction of C_* on $J\times J$ is the Cauchy matrix of the system in the regular case. Let $X_*(\tau)\equiv C_*(\,\cdot\,,\tau)$.

Definition 1. Problem $(1),(2)$ is said to be weakly Φ-well-posed with respect to the matrix-function A_* if it has the unique solution x_0 and for every sequences of A_m and f_m $(m=1,2,\dots)$ such that

$$\det\big(I_n+(-1)^j d_j A_m(t)\big)\neq 0 \ \text{ for } \ t\in I \ \ (j=1,2), \tag{4}$$

for each sufficiently large m, and the conditions

$$\lim_{m\to+\infty}\big\|\mathcal{V}_1(\Phi,A_*,A_m-A)(t,b-)\big\|=0, \tag{5}$$

$$\lim_{m\to+\infty}\big\|\mathcal{V}_2(\Phi,A_*,f_m-f)(t,b-)\big\|=0, \tag{6}$$

$$\lim_{m\to+\infty}\big\|\Phi^{-1}(t)(f_m(t)-f(t))-\Phi^{-1}(b-)(f_m(b-)-f(b-))\big\|=0 \tag{7}$$

hold uniformly on I, problem $(3),(2)$ has the unique solution x_m for each sufficiently large m and

$$\lim_{m\to+\infty}\big\|\Phi^{-1}(t)\,(x_m(t)-x_0(t))\big\|=0 \ \text{ uniformly on } I. \tag{8}$$

Definition 2. Problem $(1),(2)$ is said to be strongly Φ-well-posed with respect to the matrix-function A_* if it has the unique solution x_0 and for every sequences of matrix-and vector-functions A_m and f_m $(m=1,2,\dots)$ such that condition (4) holds for every sufficiently large m and the conditions (6) and

$$\lim_{m\to+\infty}\big\|\mathcal{V}_1(\Phi,A_*,f_m-f)(t,b-)\big\|=0$$

hold uniformly on I, problem $(3),(2)$ has the unique solution x_m for each sufficiently large m and condition (8) holds.

Remark 1. If problem $(1),(2)$ is strongly well-posed, then it is weakly well-posed, as well, because

$$\begin{aligned}\big\|\mathcal{V}_1(\Phi,A_*,f_m-f)(t,\tau)\big\|\leq \big\|\Phi^{-1}(t)(f_m(t)-f(t))-\Phi^{-1}(\tau)(f_m(\tau)-f(\tau))\big\|&\\ +\big\|\mathcal{V}_2(\Phi,A_*,f_m-f)(t,\tau)\big\| \ \text{ for } \ a\leq t<\tau<b.&\end{aligned}$$

Definition 3. We say that the sequence (A_m,f_m) $(m=1,2,\dots)$ belongs to the set $\mathcal{S}_{A_*}(A,f;\Phi,b)$, i.e.,

$$\big((A_m,f_m)\big)_{m=1}^{+\infty}\in\mathcal{S}_{A_*}(A,f;\Phi), \tag{9}$$

if problem $(3),(2)$ has the unique solution x_m for each sufficiently large m and condition (8) holds.

Let $I(\delta)=[b-\delta,b[$ for every $\delta>0$.

Theorem 1. *Let there exist nonnegative constant $n\times n$ matrices B_0 and B such that*

$$r(B)<1, \tag{10}$$

the estimates $|C_*(t,\tau)|\le\Phi(t)B_0\Phi^{-1}(\tau)$ for $b-\delta\le t\le\tau<b$ and

$$\bigg|\int\limits_t^{b-}|C_*(t,s)|\,d\,\mathrm{V}(\mathcal{A}(A_*,A-A_*))(s)\cdot\Phi(s)\bigg|\le H(t)B \ \text{ for } \ t\in I(\delta)$$

fulfilled for some $\delta>0$. Let, moreover,

$$\lim_{t\to b-}\bigg\|\int\limits_t^{b-}\Phi^{-1}(t)\,C_*(t,\tau)\,d\mathcal{A}(A_*,f)(\tau)\bigg\|=0.$$

Then problem (1), (2) is weakly Φ-well-posed with respect to A_*.

Theorem 2. *Let there exist a constant matrix $B=(b_{ik})_{i,k=1}^n\in\mathbb{R}_+^{n\times n}$ such that conditions (10) and*

$$\big[(-1)^jd_ja_{ii}(t)\big]_+>-1 \ \text{ for } \ t\in I \ \ (j=1,2;\ i=1,\dots,n)$$

hold, and the estimates

$$c_i(t,\tau)\le b_0\,\frac{h_i(t)}{h_i(\tau)} \ \text{ for } \ b-\delta\le t\le\tau<b \ \ (i=1,\dots,n);$$

$$\bigg|\int\limits_t^{b-}c_i(t,\tau)h_i(\tau)\,d[a_{ii}(\tau)]_-^v\bigg|\le b_{ii}\,h_i(t) \ \text{ for } \ t\in I(\delta) \ \ (i=1,\dots,n),$$

$$\bigg|\int\limits_t^{b-}c_i(t,\tau)h_k(\tau)\,d\,\mathrm{V}(\mathcal{A}(a_{*ii},a_{ik}))(\tau)\bigg|\le b_{ik}h_i(t) \ \text{ for } \ t\in I(\delta) \ \ (i\ne k;\ i,k=1,\dots,n)$$

fulfilled for some $b_0>0$ and $\delta>0$. Let, moreover,

$$\lim_{t\to b-}\int\limits_t^{b-}\frac{c_i(t,\tau)}{h_i(t)}\,d\,\mathrm{V}(\mathcal{A}(a_{*ii},f_i))(\tau)=0 \ \ (i=1,\dots,n),$$

where $a_{*ii}(t)\equiv[a_{ii}(t)]_+^v$ $(i=1,\dots,n)$, and c_i is the Cauchy function of the equation $dx=x\,da_{*ii}(t)$. Then problem (1), (2) is weakly Φ-well-posed with respect to the matrix-function $A_*(t)\equiv\mathrm{diag}(a_{*11}(t),\dots,a_{*nn}(t))$.

Theorem 3. *Let conditions of Theorem 1 be fulfilled and let there exist a sequence of non-degenerated matrix-functions $H_m\in\mathrm{BV}_{loc}([a,b[;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ such that*

$$\lim_{m\to+\infty}\big\|\Phi^{-1}(t)H_m^{-1}(t)\Phi(t)-I_n\big\|=0, \tag{11}$$

$$\lim_{m\to+\infty}\big\|\mathcal{V}_1(\Phi,A_*,A_m^*-A)(t,b-)\big\|=0, \tag{12}$$

$$\lim_{m\to+\infty}\big\|\mathcal{V}_2(\Phi,A_*,f_m^*-f)(t,b-)\big\|=0, \tag{13}$$

$$\lim_{m\to+\infty}\big\|\Phi^{-1}(t)(f_m^*(t)-f(t))-\Phi^{-1}(b-)(f_m^*(b-)-f(b-))\big\|=0 \tag{14}$$

hold uniformly on I, where $A_m^*(t)\equiv\mathcal{I}(H_m,A_m)(t)$ and $f_m^*(t)\equiv\mathcal{B}(H_m,f_m)(t)$. Then inclusion $((A_m^*,f_m^*))_{m=1}^{+\infty}\in\mathcal{S}_{A_*}(A,f;\Phi)$ holds.

Theorem 3 has the following form for $H_m(t) \equiv I_n$ $(m = 1, 2, \dots)$.

Corollary 1. *Let conditions of Theorem 1 be fulfilled and conditions* (5)–(7) *hold uniformly on I. Then inclusion* (9) *holds.*

Theorem 4. *Let conditions of Theorem 1 be fulfilled and let, moreover,*

$$\|B_0\|\,\|(I_n - B)^{-1}\| < 1 \tag{15}$$

and

$$\limsup_{t\to b-} \bigg\|\Phi^{-1}(t)\int\limits_t^{b-} dV(A)(s)\cdot\Phi(s)\bigg\| < +\infty.$$

Then inclusion (9) *holds if and only if there exist the sequence of matrix functions $H_m \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(m = 1, 2, \dots)$ such that*

$$\limsup_{t\to b-} \bigg\|\int\limits_t^{b-} \Phi^{-1}(s)\,d\,\mathrm{V}(\mathcal{A}(A_*, A_*))(s)\cdot\Phi(s)\bigg\| < +\infty \ \ \text{for} \ \ a \le t < \tau < b,$$

$$\limsup_{t\to b-} \bigg(\|\Phi^{-1}(t)(f^*_m(t) - f(t))\| + \bigg\|\Phi^{-1}(t)\int\limits_t^{b-} d\,\mathrm{V}(A)(s)\cdot|f^*_m(s) - f(s)|\bigg\|\bigg) = 0 \tag{16}$$

and conditions (11)–(14) *hold uniformly on I, where the matrix- and vector functions A^*_m and f^*_m $(m = 1, 2, \dots)$ are defined as in Theorem 3.*

Theorem 4′. *Let conditions of Theorem 4 be fulfilled. Then inclusion* (9) *holds if and only if conditions* (13), (14) *and*

$$\lim_{m\to+\infty} \big\|\Phi^{-1}(t)(X_m(t) - X_0(t))\big\| = 0$$

hold uniformly on I, where X_0, X_m are the fundamental matrices of systems (1), (3)*, respectively, and $f^*_m(t) \equiv \mathcal{B}(X_0X_m^{-1}, f_m)(t)$ $(m = 1, 2, \dots)$.*

Remark 2. In Theorem 4, condition (15) is essential and it cannot be neglected, i.e., if the condition is violated, then the conclusion of the theorem is not true, in general. Below we present an example.

Let $I = [0, 1]$, $n = 1$, $b = 1$, $B = 0$, $B_0 = 1$, $\Phi(t) \equiv 1-t$; $A(t) = A_m(t) = A_*(t) \equiv \ln(1-t)$ $(m = 1, 2, \dots)$;

$$f(t) \equiv 0, \ \ f_m(t) \equiv -\frac{1}{m}\int\limits_0^t \cos\frac{\ln(1-t)}{m} \ \ (m = 1, 2, \dots).$$

Then $C_*(t, \tau) \equiv 1 - t(1-\tau)^{-1}$, $x_0(t) \equiv 0$, $x_m(t) \equiv (1-t)\sin\frac{\ln(1-t)}{m}$ $(m = 1, 2, \dots)$. So, all conditions of Theorem 4 are fulfilled, except of (15), but condition (8) is not fulfilled uniformly on I.

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General Decreasing Solutions to the Equation Arises in Cryochemistry

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1 Introduction

We investigate properties of a solution to the ordinary differential equation arises in mathematical models describing the physico-chemical processes occurring during a cryochemical modification of drug substances (see [6, 7]).

Under these assumptions, the thermal conductivity equation with mass transfer for the one–dimensional case can be used to calculate the temperature field created by the carrier gas stream:

$$\frac{\partial T}{\partial t} = V\,\frac{\partial T}{\partial x} - \frac{\mu}{\rho C_V}\cdot\frac{\partial}{\partial x}\Big(\lambda\,\frac{\partial T}{\partial x}\Big). \tag{1.1}$$

Here ρ, μ, λ are the density ($\mathrm{kg/m^3}$), molecular weight (kg/mol), thermal conductivity ($\mathrm{W/(m\cdot K)}$) of the carrier gas, respectively, C_V is the molar heat capacity of the carrier gas at constant volume ($\mathrm{J/(mol\cdot K)}$), V is the linear velocity of the carrier-gas flow front (m/s).

In stationary mode we have $\partial T/\partial t = 0$ and equation (1.1) reduces to the ordinary differential equation

$$\frac{dT}{dx} - \frac{\mu}{\rho V C_V}\cdot\frac{d}{dx}\Big(\lambda\,\frac{dT}{dx}\Big) = 0. \tag{1.2}$$

The flow rate of the carrier gas is controlled during the experiment with the help of an external device (an industrial gas pipeline with accuracy, according to its passport data, not worse than 5%). The regulated gas stream of the carrier, passing through a heated copper screen (a mixed molecular flow shaper) of cylindrical shape, heats up to a certain temperature, captures the vapors of the initial substance and takes them out into the vacuum space. Let the nozzle area of the mixed molecular flow shaper be $S(\mathrm{m}^2)$. Then the molar flow rate of the carrier gas is dN/dt(mol/s) and can be written as

$$\dot N = \frac{dN}{dt} = \frac{\rho V S}{\mu}\,.$$

In this case, the ratio of the molar flow rate of the carrier gas dN/dt (mol/s) to the nozzle area of the mixed molecular flow shaper, that is, the density of the carrier gas flow dn/dt ($\mathrm{mol/(m^2\cdot s)}$) can be represented as

$$\dot n = \frac{dn}{dt} = \frac{\dot N}{S} = \frac{\rho V}{\mu}\,.$$

Therefore, equation (1.2) can be written as

$$\frac{dT}{dx} - \frac{d}{dx}\Big(\frac{\lambda}{C_V\dot{n}}\cdot\frac{dT}{dx}\Big) = 0.$$

It can be solved analytically, taking into account the dependence of the thermal conductivity of the carrier gas on the temperature. An interesting fact is that the heat capacity of gases in a wide range of pressures practically does not depend on the pressure. This circumstance received its explanation from the molecular kinetic theory. A large number of gases, such as nitrogen, helium, argon, carbon dioxide, etc., have the square-root dependence of the thermal conductivity on the temperature expressed by the approximate formula

$$\lambda = \frac{ik}{3\pi^{3/2}d^2}\sqrt{\frac{RT}{\mu}}\,, \tag{1.3}$$

where

i is the sum of translational and rotational degrees of freedom of molecules (5 for diatomic gases, 3 for monatomic ones),

k is the Boltzmann constant,

μ is the molar mass,

T is the absolute temperature,

d is the effective diameter of molecules,

R is the universal gas constant.

Representing λ in (1.3) as $\alpha\sqrt{T}$ with the appropriate coefficient α, we obtain

$$\frac{\lambda}{C_V\dot{n}} = \frac{\alpha\sqrt{T}}{C_V\dot{n}} = b\sqrt{T} \ \text{ with } \ b = \frac{\alpha}{C_V\dot{n}}\,.$$

Now the thermal conductivity equation with mass transfer of these process for the one–dimensional case can be transformed to the ordinary differential equation [5]:

$$\frac{d}{dx}\Big(T - b\sqrt{T}\,\frac{dT}{dx}\Big) = 0, \ \ b > 0. \tag{1.4}$$

We study the dependence of the temperature on the distance under three types of boundary conditions, namely the Dirichlet, Neumann, and Robin ones.

The Dirichlet condition specifies the temperature value at the boundary.

The Neumann condition specifies the boundary value for the derivative of the temperature.

In the Robin condition, we specify a linear combination of the temperature value and the derivative of the temperature at the boundary.

The coefficient of the temperature value in the Robin condition is the Biot number (the ratio of the conductive thermal resistance inside the object to the convective resistance at the surface of the object).

The mathematical model was discussed with colleagues from the Department of Chemistry of M. V. Lomonosov Moscow State University T. A. Shabatina, and Yu. Morozov.

2 General decreasing solutions

Theorem 2.1. *Each positive solution T to equation* (1.4) *is either constant or strictly monotonic. Each strictly decreasing solution has the form*

$$T(x) = c^2\, \Theta\Big(\frac{x - x^*}{bc}\Big)^2, \tag{2.1}$$

where x^* and $c > 0$ are arbitrary constants, while Θ is a decreasing function $(-\infty; 0) \to (0; 1)$ implicitly defined by

$$x = 2\Theta(x) + \ln \frac{1 - \Theta(x)}{1 + \Theta(x)}\,. \tag{2.2}$$

The left-hand side of (1.4) *contains an expression in parentheses which must be constant and, for the solution defined by* (2.1)*, equals c^2.*

If maximally extended, such T is defined on the interval $(-\infty; x^*)$ and satisfies

$$T(x) \to c^2 \ \ \textit{and}\ \ T'(x) \to 0 \ \ \textit{as}\ \ x \to -\infty, \tag{2.3}$$

$$T(x) \to 0 \ \ \textit{and}\ \ T'(x) \to -\infty \ \ \textit{as}\ \ x \to x^*. \tag{2.4}$$

Proof. First, by the substitution $T = Z^2$ with $Z > 0$ we convert equation (1.4) into the form

$$(Z^2 - 2bZ^2 Z')' = 0,$$

which immediately yields

$$Z^2 - 2bZ^2\, Z' = C = const$$

with further transformations depending on $\operatorname{sgn} C$.

If $C = 0$, then either $Z \equiv 0$ or $1 = 2b\, Z'$, which entails that $Z' > 0$ and Z is strictly increasing.

If $C = -c^2 < 0$, then we obtain $Z^2 + c^2 = 2bZ^2 Z'$. This shows again that $Z' > 0$.

Finally, if $C = c^2 > 0$ with $c > 0$, then we obtain

$$Z^2 - c^2 = 2bZ^2 Z'. \tag{2.5}$$

Now, if $Z(x) = c$ at some point x, then, by the uniqueness theorem, Z must coincide with the constant solution $Z \equiv c$. If not, then either $Z > c$ on the whole domain or $Z < c$. We reject the first case (with $Z' > 0$ due to (2.5)) as well as the previous constant one.

In the second case we put

$$Z(x) = c\, z\Big(\frac{x}{bc}\Big), \ \ 0 < z < 1,$$

which converts (2.5) into

$$z^2 - 1 = 2z^2 z'. \tag{2.6}$$

This can be written as

$$1 = \frac{2z^2 z'}{z^2 - 1} = \Big(2 + \frac{2}{z^2 - 1}\Big) z',$$

whence, for $0 < z < 1$,

$$x - a = \int\limits_0^{z(x)} \Big(2 + \frac{2}{\zeta^2 - 1}\Big)\, d\zeta = 2z(x) + \ln \frac{1 - z(x)}{1 + z(x)}$$

with some a. We have a general family of implicitly defined strictly decreasing solutions to (2.6) satisfying $0 < z < 1$. One of them, with $a = 0$, is just Θ defined by (2.2). All others can be obtained from Θ by a horizontal shift. Thus, we have (2.1).

It follows from (2.2) that

$$\Theta(x) \to 0 \ \text{ as } \ x \to 0,$$
$$\Theta(x) \to 1 \ \text{ as } \ x \to -\infty.$$

Then, using (2.6), we obtain

$$\Theta'(x) \to -\infty \ \text{ as } \ x \to 0,$$
$$\Theta'(x) \to 0 \ \text{ as } \ x \to -\infty.$$

These limits, together with (2.1), produce the first three limits in (2.3) and (2.4). For the fourth one, we use (2.5) to obtain

$$T' = 2ZZ' = \frac{Z^2 - c^2}{2bZ} = \frac{T - c^2}{2b\sqrt{T}} \to -\infty \ \text{ as } \ T \to 0. \quad \square$$

3 On existence and uniqueness of solutions

Theorem 3.1. *For any constants $x_0 < x_1$ and $T_1 > T_0 > 0$, equation* (1.4) *has a unique solution T defined on $[x_0; x_1]$ and satisfying the conditions*

$$T(x_0) = T_0, \quad T(x_1) = T_1. \tag{3.1}$$

Proof. The boundary conditions show that, according to Theorem 2.1, the solution T must strictly decrease and therefore have the form given by (2.1) and (2.2). So, the boundary conditions become

$$\frac{\sqrt{T_j}}{c} = \Theta\Big(\frac{x_j - x^*}{bc}\Big), \ \ j \in \{0, 1\},$$

or, by using (2.2),

$$\frac{x_j - x^*}{bc} = 2\frac{\sqrt{T_j}}{c} + \ln \frac{1 - \frac{\sqrt{T_j}}{c}}{1 + \frac{\sqrt{T_j}}{c}}, \ \ j \in \{0, 1\}. \tag{3.2}$$

Thus, we have to prove the existence and uniqueness of a pair (x^*, c) satisfying (3.2). Putting

$$q := \sqrt{\frac{T_1}{T_0}} \in (0;1) \ \text{ and } \ k := \frac{\sqrt{T_0}}{c} \in (0;1), \tag{3.3}$$

we write the difference of the two equations (3.2) as

$$\frac{k(x_1 - x_0)}{b\sqrt{T_0}} = 2k(q-1) + \ln \frac{(1-qk)(1+k)}{(1+qk)(1-k)}$$

or

$$\frac{x_1 - x_0}{2b\sqrt{T_0}} = F_q(k) \tag{3.4}$$

with

$$F_q(k) := f(k) - qf(qk), \tag{3.5}$$
$$f(k) := \frac{1}{2k}\,\ln \frac{1+k}{1-k} - 1. \tag{3.6}$$

Lemma 3.1. *For each $A > 0$ and $q \in (0;1)$, there exists a unique $k \in (0;1)$ such that $F_q(k) = A$ with F_q defined by* (3.5) *and* (3.6)*. The mapping $(A,q) \mapsto k$ is a C^1 function $(0;+\infty)\times(0;1) \to (0;1)$ strictly increasing with respect to both A and q.*

Proof. Note that

$$f(k) = \frac{\ln(1+k)}{2k} - \frac{\ln(1-k)}{2k} - 1,$$

whence $f(k) \to 0$ as $k \to 0$ (by L'Hôpital's rule) and $f(k) \to +\infty$ as $k \to 1$.

Now we study the derivative of f by using its Taylor series uniformly converging on any subsegment of the interval $(0,1)$.

$$\begin{aligned} f'(k) &= \frac{1}{2k(1+k)} - \frac{\ln(1+k)}{2k^2} + \frac{1}{2k(1-k)} + \frac{\ln(1-k)}{2k^2} = \frac{1}{k(1-k^2)} - \frac{\ln(1+k)}{2k^2} + \frac{\ln(1-k)}{2k^2} \\ &= \frac{1}{k}\sum_{n=0}^{\infty} k^{2n} + \frac{1}{2k^2}\sum_{n=1}^{\infty}\frac{((-1)^n-1)k^n}{n} = \frac{1}{k}\sum_{n=0}^{\infty} k^{2n} - \frac{1}{k^2}\sum_{m=0}^{\infty}\frac{k^{2m+1}}{2m+1} \\ &= \frac{1}{k}\sum_{n=0}^{\infty}\Big(1 - \frac{1}{2n+1}\Big)k^{2n} = \frac{1}{k}\sum_{n=1}^{\infty}\frac{2n}{2n+1}k^{2n} = \sum_{n=1}^{\infty}\frac{2n}{2n+1}k^{2n-1} > 0, \end{aligned}$$

whence $f(k) > 0$ as well.

Further,

$$f''(k) = \sum_{n=1}^{\infty}\frac{2n(2n-1)}{2n+1}\,k^{2n-2} > 0,$$

whence f' is strictly increasing and

$$\frac{dF_q}{dk}(k) = f'(k) - q^2 f'(qk) > 0.$$

So, F_q is strictly increasing in k, $F_q(k) \to 0$ as $k \to 0$, and

$$F_q(k) = (1-q)f(k) + q(f(k) - f(qk)) > (1-q)f(k) \to +\infty \ \text{ as } \ k \to 1.$$

Therefore, F_q must attain, exactly once, each $A > 0$, which proves the first part of Lemma 3.1.

The second part follows immediately from the implicit function theorem and the evident inequalities

$$\begin{aligned} \frac{\partial(F_q(k) - A)}{\partial A} &= -1 < 0, \\ \frac{\partial(F_q(k) - A)}{\partial q} &= -f(qk) - qkf'(qk) < 0. \end{aligned}$$ □

We return to proving Theorem 3.1. Having the unique value of k satisfying (3.4), we obtain, from (3.2) and (3.3), the unique values

$$c = \frac{\sqrt{T_0}}{k} > \sqrt{T_0} \ \text{ and } \ x^* = x_1 - 2b\sqrt{T_1} - bc\ln\frac{c-\sqrt{T_1}}{c+\sqrt{T_1}}$$

to satisfy (3.2). This completes the proof of Theorem 3.1. □

Now we will to prove two theorems concerning other boundary conditions for equation (1.4).

Theorem 3.2. *For any real constants $x_0 < x_1$, $T_0 > 0$, and $U_1 < 0$, equation* (1.4) *has a unique solution T defined on $[x_0; x_1]$ and satisfying the conditions*

$$T(x_0) = T_0, \quad T'(x_1) = U_1.$$

Theorem 3.3. *For any real constants $x_0 < x_1$, $T_0 > 0$, and $U_1 < 0$, equation* (1.4) *has a unique solution T defined on $[x_0; x_1]$ and satisfying the conditions*

$$T(x_0) = T_0, \quad T'(x_1) = U_1 T(x_1).$$

Proof. We try to prove the existence and uniqueness of a constant $T_1 \in (0; T_0)$ such that the unique solution T existing according to Theorem 3.1 satisfies the boundary conditions of the related theorem.

According to Theorem 2.1, $T - b\sqrt{T}\,T' = c^2$, whence, using notation (3.3),

$$\begin{aligned} T'(x_1) &= \frac{T(x_1) - c^2}{b\sqrt{T(x_1)}} = \frac{q^2 T_0 - T_0/k^2}{bq\sqrt{T_0}} = \frac{k^2q^2 - 1}{k^2 q} \cdot \frac{\sqrt{T_0}}{b}\,, \\ \frac{T'(x_1)}{T(x_1)} &= \frac{k^2q^2 - 1}{k^2q^3} \cdot \frac{1}{b\sqrt{T_0}}\,, \end{aligned}$$

where $k \in (0;1)$ is chosen, depending on $q \in (0;1)$, to provide the boundary conditions (3.1) for the solution T defined by (2.1).

It follows from Lemma 3.1 that $k \in (0;1)$ strictly increases with respect to $q \in (0;1)$. So, in both right-hand sides of the last equations, the numerator $k^2q^2 - 1$ is negative and strictly increases in q, while its absolute value decreases. The denominators are positive and also strictly increase. Thus, the fractions are negative with strictly decreasing absolute values.

Now consider their limits at 0 and 1.

Both fractions tend to $-\infty$ as $q \to 0$. As for $q \to 1$, there must exist $k_1 = \lim\limits_{q\to 1} k \in (0;1]$.

If $k_1 < 1$, then it follows from (3.4)–(3.6) that

$$0 < \frac{x_1 - x_0}{2b\sqrt{T_0}} = F_1(k_1) = f(k_1) - 1 \cdot f(1 \cdot k_1) = 0.$$

This contradiction shows that $k_1 = 1$. (For this k_1, no contradiction arises because $f(k) \to +\infty$ as $k \to 1$.) Hence

$$T'(x_1) \to 0 \ \text{ and } \ \frac{T'(x_1)}{T(x_1)} \to 0 \ \text{ as } \ q \to 1.$$

So, both expressions strictly increase from $-\infty$ to 0 as q increases from 0 to 1 (i.e. as T_1 increases from 0 to T_0). Therefore, they both must attain, exactly once, each negative value, and this proves Theorems 3.2 and 3.3. $\square$

Remark 3.1. The authors' results connected with mathematical modeling in other physical processes can be found in [1–4].

Acknowledgements

The reported study was partially supported by RSF, Project # 20-11-20272 and MSU Program of Development, Project # 23-SCH05-26.

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Dimensions of Subspaces Defined by the Lyapunov Exponents of Regular Linear Differential Systems with Parametric Perturbations Vanishing at Infinity as Functions of the Parameter

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For a given $n \in \mathbb{N}$ let $\mathcal{M}_n$ denote the class of linear differential systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{1}$$

with continuous bounded coefficients defined on the half-axis $\mathbb{R}_+$. In what follows, we identify system (1) with its defining function $A(\,\cdot\,) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and therefore write $A \in \mathcal{M}_n$ and the like. The vector space of solutions to system (1) will be denoted by $\mathcal{S}(A)$. Recall that *the characteristic exponent* (or *the Lyapunov exponent*) of a non-zero solution $x(\,\cdot\,)$ to system (1) is the quantity [7, p. 552], [1, p. 25]

$$\lambda[x] = \overline{\lim_{t\to+\infty}} \, \frac{1}{t} \, \ln \|x(t)\|;$$

for the zero solution let it equal $-\infty$. As is well-known [7, p. 561], [1, p. 38], system (1) has exactly n Lyapunov exponents, counting multiplicity, which we denote by $\lambda_1(A) \leqslant \lambda_2(A) \leqslant \cdots \leqslant \lambda_n(A)$.

For each $\alpha \in \mathbb{R}$, let

$$L_\alpha(A) = \big\{x \in \mathcal{S}(A) : \ \lambda[x] < \alpha\big\} \ \text{ and } \ N_\alpha(A) = \big\{x \in \mathcal{S}(A) : \ \lambda[x] \leqslant \alpha\big\}.$$

Clearly [6, p. 2], for every $\alpha \in \mathbb{R}$, the sets $L_\alpha(A)$ and $N_\alpha(A)$ are vector subspaces of the space $\mathcal{S}(A)$. Let us denote by $d_\alpha(A)$ and $D_\alpha(A)$ respectively their dimensions. In particular, the number $d_0(A)$ is called the *exponential stability index* and, as follows from its definition, coincides with the dimension of the subspace of solutions to system (1) that decay exponentially at infinity.

O. Perron constructed [8], see also [6, p. 13], an example of a two-dimensional diagonal system $A \in \mathcal{M}_2$ and its perturbation $Q \in \mathcal{M}_2$ decaying exponentially at infinity such that the following relations hold:

$$d_a(A) = 1, \quad D_a(A) = 2, \quad d_a(A+Q) = 0, \quad D_a(A+Q) = 1, \tag{2}$$

where a is a positive number. Moreover, it is fairly easy to see that in equalities (2) the number a can be taken arbitrary. This assertion follows from an obvious fact that by adding the matrix γI_n (I_n being the $n \times n$ identity matrix and $\gamma \in \mathbb{R}$) to the coefficient matrix A of system (1), we change the Lyapunov exponents of all its solutions by γ.

Thus, by virtue of the Perron example, the quantities d_α and D_α are not invariant under vanishing at infinity perturbations of system coefficients and hence [9, Lemma 7.3], they are not semicontinuous in the topology of uniform convergence over the half-axis $\mathbb{R}_+$ on the space $\mathcal{M}_n$.

System (1) is said [7, p. 563], [1, p. 61] to be *regular*, if the following two conditions are met:

1) the limit
$$T(A) = \lim_{t\to+\infty} t^{-1} \int\limits_0^t \operatorname{tr} A(\tau)\, d\tau$$
exists, where $\operatorname{tr}(\,\cdot\,)$ stands for the trace of a matrix;

2) the equality
$$\lambda_1(A) + \lambda_2(A) + \cdots + \lambda_n(A) = T(A)$$
holds.

The class of regular n-dimensional systems will be denoted by $\mathcal{R}_n$.

A. M. Lyapunov demonstrated [7, pp. 576–578] that if a nonlinear system (under natrural assumptions on the right-hand side) has a regular first approximation system with the exponential stability index equal to $k \in \{1, \ldots, n\}$, then the nonlinear system possesses exactly k-dimensional exponentially stable manifold passing through the origin (i.e. any solution to the nonlinear system starting on this manifold decays exponentially; furthermore, such a solution $x(\,\cdot\,)$ admits the estimate
$$\|x(t)\| \leqslant C_\varepsilon \exp\bigl\{(\lambda_k(A) + \varepsilon)t\bigr\}\|x(0)\| \ \text{ for every } \ \varepsilon > 0,$$
where $\lambda_k(A)$ is the k-th Lyapunov exponent of the first approximation system). Taking into account this fundamental result, one may conjecture that the exponential stability index of a regular system (and along with it the quantities d_α and D_α for all $\alpha \in \mathbb{R}$) is invariant under vanishing at infinity perturbations of its coefficients. Let us note that for exponentially decaying perturbations of a regular system the mentioned invariance does indeed take place [3, 4].

The conjecture stated above had been around for quite some time, until R. È. Vinograd in the paper [10] gave an example of systems $A, B \in \mathcal{R}_2$ for which the relations
$$d_0(A) = 0, \quad D_0(A) = 2, \quad D_0(B) = d_0(B) = 1, \quad \lim_{t\to+\infty} \|A(t) - B(t)\| = 0$$
are valid.

Let M be a metric space. Consider a family of systems
$$\dot{x} = \mathcal{A}(t, \mu)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+, \tag{3}$$
such that for each fixed $\mu \in M$ the matrix-valued function $\mathcal{A}(\,\cdot\,, \mu)$ has continuous and bounded coefficients, i.e. $\mathcal{A}(\,\cdot\,, \mu) \in \mathcal{M}_n$.

Here and subsequently, $\mathcal{Z}_n$ stands for the set $\{0, 1, \ldots, n\}$. For each $\alpha \in \mathbb{R}$, define the functions $d_\alpha(\,\cdot\,; \mathcal{A}), D_\alpha(\,\cdot\,; \mathcal{A}) : M \to \mathcal{Z}_n$ by
$$d_\alpha(\mu; \mathcal{A}) = d_\alpha(\mathcal{A}(\,\cdot\,, \mu)) \ \text{ and } \ D_\alpha(\mu; A) = D_\alpha(\mathcal{A}(\,\cdot\,, \mu)), \ \ \mu \in M.$$

Let $\mathcal{R}^n(M)$ denote the class of families of systems (1) with coefficient matrices of the form $\mathcal{A}(t, \mu) = B(t) + Q(t, \mu)$, where a matrix-valued function $B : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is continuous and bounded, the system $\dot{x} = B(t)x$ is regular, and $Q : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ is continuous and satisfies the condition
$$\sup_{\mu\in M} \|Q(t, \mu)\| \to 0 \ \text{ as } \ t \to +\infty.$$

Given $\alpha,\beta\in\mathbb{R}$, let

$$R^n_{\alpha,\beta}(M)\equiv\big\{(d_\alpha(\,\cdot\,,\mathcal{A}),D_\beta(\,\cdot\,,\mathcal{A})):\ \mathcal{A}\in\mathcal{R}^n(M)\big\}.$$

The problem is to obtain a complete function-theoretic description of the classes $R^n_{\alpha,\beta}(M)$ for any metric space M and numbers $n\geq 2$ and $\alpha,\beta\in\mathbb{R}$. This problem can be viewed as a generalization of Vinograd's example [10] of instability of the Lyapunov exponents of a regular system under vanishing at infinity perturbations of its coefficient matrix.

Following [5, p. 264], for a number $r\in\mathbb{R}$ and function $f:M\to\mathbb{R}$, we write $[f\geqslant r]$ for the *Lebesgue* set $\{\mu\in M:\ f(\mu)\geq r\}$.

Before stating the main result of the report, let us recall [5, p. 156] that a subset of a metric space is said to be an F_σ-set, if it can be represented as a countable union of closed subsets, and an $F_{\sigma\delta}$-set, if it can be represented as a countable intersection of F_σ-sets.

The following statement solves the problem posed above.

Theorem. *For any metric space M, real numbers α, β and integer $n\geq 2$, a vector function $(g,h):M\to\mathcal{Z}_n\times\mathcal{Z}_n$ belongs to the class $R^n_{\alpha,\beta}(M)$, if and only if for every $r\in\mathbb{R}$, the set $[g\geq r]$ is an F_σ-set and $[h\geq r]$ is an $F_{\sigma\delta}$-set, and for all $\mu\in M$, we have either $h(\mu)\geq g(\mu)$ or $h(\mu)\leq g(\mu)$, depending on whether $\beta\geq\alpha$ or $\beta<\alpha$.*

Remark 1. A complete description of an analogous class of vector functions corresponding to families (3) with coefficients of the form $\mathcal{A}(t,\mu)=B(t)+Q(t,\mu)$, where $B:\mathbb{R}_+\to\mathbb{R}^{n\times n}$ is continuous and bounded, and $Q:\mathbb{R}_+\times M\to\mathbb{R}^{n\times n}$ is continuous and decays exponentially (uniformly in μ) as $t\to+\infty$, is obtained in the paper [2] and coincides with the one stated above.

Remark 2. The class $R^1_{\alpha,\beta}(M)$ consists of pairs of constant functions $M\to\{0,1\}$, namely:

$$R^1_{\alpha,\beta}(M)=\begin{cases}\{(0,0),(1,1),(0,1)\}, & \text{if }\ \beta\geq\alpha,\\ \{(0,0),(1,1),(1,0)\}, & \text{if }\ \beta<\alpha.\end{cases}$$

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On Solvability Conditions for the Cauchy Problem for Second Order Linear Non-Volterra Functional Differential Equations

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Consider the Cauchy problem for the most general case of linear second order non-Volterra functional differential equations, which can be written in the operator form:

$$\begin{cases} \ddot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), \ \ t \in [0,1], \\ x(0) = c_0, \quad \dot{x}(0) = c_1, \end{cases} \tag{1}$$

where T^+ and T^- are linear positive operators acting from the space of real continuous functions $\mathbf{C}[0,1]$ into the space of real integrable functions $\mathbf{L}[0,1]$ (positive operators map non-negative functions into non-negative ones), c_0, $c_1 \in \mathbb{R}$, $f \in \mathbf{L}[0,1]$ is integrable.

Let p^+ and p^- be two given non-negative integrable functions. Suppose that positive operators T^+ and T^- satisfy the equalities

$$(T^+\mathbf{1}\,)(t) = p^+(t), \quad (T^-\mathbf{1}\,)(t) = p^-(t), \ \ t \in [0,1], \tag{2}$$

where $\mathbf{1}$ is the unit function, $\mathbf{1}(t) = 1$ for all $t \in [0,1]$. By imposing various restrictions on the functions p^+ and p^-, we can obtain various conditions for the solvability of problem (1) for all operators T^+, T^- satisfying equalities (2) and additional restrictions.

All known solvability conditions of this kind for many boundary value problems were obtained under the same types of restrictions on the operators T^+, T^-, that is only under pointwise restrictions or only under integral ones [2, 4–11]. We can obtain solvability conditions under mixed restrictions, when pointwise restrictions are imposed on the action of one of the operators T^+, T^-, and integral restrictions are imposed on the other operator.

Let us present several obtained statements.

First of all, using ideas of [1, 3, 5, 6], we formulate necessary and sufficient solvability conditions for pointwise restrictions.

Put

$$k(t) \equiv 1 - \int_0^t (t-s)(p^+(s) - p^-(s))\,ds.$$

Theorem 1. *Let non-negative functions p^+, $p^- \in \mathbf{L}[0,1]$ be given.*

The Cauchy problem (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that $T^+\mathbf{1} = p^+$, $T^-\mathbf{1} = p^-$ if and only if*

$$\int_0^1 (1-s)p^+(s)\,ds < 1$$

and

$$\Bigg(1-\int\limits_0^{t_3}(t_1-s)p^+(s)\,ds+\int\limits_{t_3}^{t_1}(t_1-s)p^-(s)\,ds\Bigg)k(1)$$
$$+\Bigg(\int\limits_0^{t_3}(1-s)p^+(s)\,ds-\int\limits_{t_3}^{1}(1-s)p^-(s)\,ds\Bigg)k(t_1)>0$$

for all $0\le t_3\le t_1\le 1$.

Corollary 1. *Let a non-negative function* $p^-\in\mathbf{L}[0,1]$ *be given.*

The Cauchy problem

$$\begin{cases}\ddot{x}(t)=-(T^-x)(t)+f(t), & t\in[0,1],\\ x(0)=c_0, \quad \dot{x}(0)=c_1,\end{cases}$$

is uniquely solvable for all linear positive operators $T^-:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ *such that* $T^-\mathbf{1}\ =p^-$ *if and only if the inequality*

$$\Delta_-\equiv\Bigg(1+\int\limits_{t_3}^{t_1}(t_1-s)p^-(s)\,ds\Bigg)\Bigg(1+\int\limits_0^{1}(1-s)p^-(s)\,ds\Bigg)$$
$$-\int\limits_{t_3}^{1}(1-s)p^-(s)\,ds\Bigg(1+\int\limits_0^{t_1}(t_1-s)p^-(s))\,ds\Bigg)>0$$

holds for all $0\le t_3\le t_1\le 1$.

Corollary 2. *If*

$$p^-(t)\le 16,\quad p^-(t)\not\equiv 16\ \ or$$
$$p^-(t)\le 487t^2(1-t)^2\ \ or\ \ p^-(t)\le 39t\ \ or\ \ p^-(t)\le 24.7e^{-t},$$
$$p^-(t)\le 9.8e^{t}\ \ or\ \ p^-(t)\le\frac{10.4}{\sqrt{1-t}}\ \ or\ \ p^-(t)\le 32\sin(10\pi t),$$

then the Cauchy problem

$$\begin{cases}\ddot{x}(t)=-(T^-x)(t)+f(t), & t\in[0,1],\\ x(0)=c_0, \quad \dot{x}(0)=c_1\end{cases}$$

is uniquely solvable for all linear positive operators $T^-:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ *such that* $T^-\mathbf{1}\ =p^-$.

With the help of Theorem 1 we can obtain necessary and sufficient solvability conditions for mixed restrictions.

Theorem 2. *Let a non-negative function* $p^-\in\mathbf{L}[0,1]$ *and a number* $\mathcal{P}^+\ge 0$ *be given.*

The Cauchy problem (1) *is uniquely solvable for all linear positive operators* $T^+,T^-:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ *such that*

$$T^-\mathbf{1}\ =p^-,\qquad\int\limits_0^1(1-s)(T^+\mathbf{1}\,)(s)\,ds=\mathcal{P}^+$$

if and only if

$$\mathcal{P}^+ < 1,$$

$$\Delta_-(t_3, t_1, p^-) > \mathcal{P}^+\Big(1 + \int\limits_{t_3}^{t_1} (t_1 - s)p^-(s)\,ds\Big), \ \ 0 \le t_3 \le t_1 \le 1,$$

$$\Delta_-(t_3, t_1, p^-) \ge \mathcal{P}^+\Big(t_1 + (1 - t_1)\int\limits_0^{t_3} sp^-(s)\,ds\Big), \ \ 0 \le t_3 \le t_1 \le 1.$$

Corollary 3. *Let two non-negative numbers $\mathcal{P}^+$, $\mathcal{P}^-$ be given.*

The Cauchy problem (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$\int\limits_0^1 (1 - s)(T^+\mathbf{1})(s)\,ds \le \mathcal{P}^+ \ \ \textit{and} \ \ (T^-\mathbf{1})(t) \le \mathcal{P}^-, \ \ t \in [0,1],$$

if and only if

$$\mathcal{P}^+ < 1 \ \ \textit{and} \ \ \mathcal{P}^- < 8\Big(1 + \sqrt{1 - \mathcal{P}^+}\Big).$$

Theorem 3. *Let constants $\mathcal{P}^+ \ge 0$, $\mathcal{P}^- \ge 0$ be given.*

The Cauchy problem (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$(T^-\mathbf{1})(t) \le \mathcal{P}^-, \ \ t \in [0,1], \quad \int\limits_0^1 (1 - s)(T^+\mathbf{1})(s)\,ds \le \mathcal{P}^+,$$

if and only if

$$\mathcal{P}^+ < 1, \quad \mathcal{P}^- < 8\Big(1 + \sqrt{1 - \mathcal{P}^+}\Big).$$

Theorem 4. *Let $\alpha \ge -1$. Let a non-negative function $p^+\mathbf{L}[0,1]$ and a number $\mathcal{P}^- \ge 0$ be given.*

The Cauchy problem (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$T^+\mathbf{1} = p^+, \quad \int\limits_0^1 (1 + \alpha s)(T^-\mathbf{1})(s)\,ds = \mathcal{P}^-$$

if and only if

$$\mathcal{P}^+ \equiv \int\limits_0^1 (1 - s)p^+(s)\,ds < 1,$$

$$\mathcal{P}^- \le \beta(t_1)(1 + t_1\mathcal{T}_3^+ - \mathcal{T}_2^+) + \frac{\mathcal{T}_1^+ - \mathcal{T}_2^+}{t_1} + 2\sqrt{\beta(t_1)(1 - \mathcal{T}_2^+)\Big(\mathcal{T}_3^+ + 1 - \mathcal{T}^+ + \frac{\mathcal{T}_1^+ - \mathcal{T}_2^+}{t_1}\Big)}, \ \ 0 < t_3 \le t_1 < 1,$$

where

$$\beta(t_1) = \frac{1+\alpha t_1}{t_1(1-t_1)}\,,$$

$$\mathcal{T}_1^+ = \int_0^{t_1} (t_1 - s)p^+(s)\,ds,\quad \mathcal{T}_2^+ = \int_0^{t_3} (t_1 - s)p^+(s)\,ds,\quad \mathcal{T}_3^+ = \int_0^{t_3} (1 - s)p^+(s)\,ds.$$

Corollary 4. *Let $\alpha \geq -1$. Let a non-negative function $p^+ \in \mathbf{L}[0,1]$ and a number $\mathcal{P}^- \geq 0$ be given, and $p^+(t) = 0$ for $t \in [0, \frac{1}{1+\sqrt{1+\alpha}}]$.*

The Cauchy problem (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$T^+\mathbf{1} = p^+, \qquad \int_0^1 (1+\alpha s)(T^-\mathbf{1})(s)\,ds = \mathcal{P}^-$$

if and only if

$$\mathcal{P}^+ < 1, \quad \mathcal{P}^- + 1 - \mathcal{P}^+ \leq \left(1 + \sqrt{1+\alpha} + \sqrt{1-\mathcal{P}^-}\right)^2.$$

Corollary 5. *The Cauchy problem* (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$(T^+\mathbf{1})(t) \leq 2, \quad (T^+\mathbf{1})(t) \not\equiv 2, \;\; t \in [0,1],$$
$$\int_0^1 (T^-\mathbf{1})(s)\,ds \leq \min_{t\in(0,1)} \left(\frac{1}{t(1-t)} + t + \sqrt{1+t}\right) \approx 6.9.$$

Corollary 6. *The Cauchy problem* (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$(T^+\mathbf{1})(t) \leq 1, \quad t \in [0,1],$$
$$\int_0^1 (T^-\mathbf{1})(s)\,ds \leq \min_{t\in(0,1)} \left(\frac{1}{t(1-t)} + \frac{t}{2} + \sqrt{\frac{(2-t^2)(1+t)}{t(1-t)}}\right) \approx 7.4.$$

The constants of the solvability conditions from Corollaries 5 and 6 are exact and cannot be increased.

Finally we obtain solvability conditions under integral restrictions on both operators T^+, T^-.

Theorem 5. *Let $\alpha \geq -1$. Let constants $\mathcal{P}^+ \geq 0$, $\mathcal{P}^- \geq 0$ be given.*

The Cauchy Problem (1) *is uniquely solvable for all linear positive operators $T^+, T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$\int_0^1 (1+\alpha s)(T^-\mathbf{1})(s)\,ds \leq \mathcal{P}^-, \quad \int_0^1 (1-s)(T^+\mathbf{1})(s)\,ds \leq \mathcal{P}^+,$$

if and only if

$$\mathcal{P}^+ < 1, \quad \mathcal{P}^- - \mathcal{P}^+ + 1 \leq \left(1 + \sqrt{1+\alpha} + \sqrt{1-\mathcal{P}^+}\right)^2.$$

Acknowledgements

This work was supported by the Russian Scientific Foundation (Project # 22-21-00517).

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On the Massera's Theorem of Existence of Periodic Solutions of Linear Differential Systems

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Many areas of modern physics and technology are substantially based on various oscillatory processes or use them. Oscillatory processes also play an important, and sometimes determining role in a significant part of natural phenomena. These circumstances determine the relevance of research in the oscillation theory and the necessity for its development. Although an effective apparatus for studying oscillations in nonlinear systems is developed in modern oscillation theory, the "linear" part of the theory remains an important and demanded part of it both in theoretical and practical aspects. At the same time, the center of gravity of practical research methods has been largely shifted to systems of linear differential equations with periodic coefficients (see, for example, [1, 3, 12] and many other works).

Let us dwell in more detail on some of the studies of the Uruguayan mathematician J. L. Massera on the problem of the existence of periodic solutions of ordinary differential periodic systems, which are directly related to this paper. For quite a long time, up to the middle of the 20th century, it was believed in the theory of oscillations that the period of a periodic differential system and the period of its periodic solution are commensurable. And only in 1950 J. L. Massera showed the fallacy of this assumption. Moreover, he obtained (also for linear systems) the conditions for the existence of solutions whose period is incommensurable with the period of the system itself [7]. Subsequently, such solutions, because of their unusual nature, were called *strongly irregular* [2, p. 17].

In the same 1950, J. L. Massera published another paper [8] on the existence of a periodic solution of a periodic differential system of the same period as the system. In particular, he established the following remarkable result: in the linear case the existence of a bounded solution of a periodic system entails the existence of a periodic solution of the same period as the system. In other words, the necessary and sufficient condition for a periodic linear system to have a periodic solution of the same period as the system is the existence of a bounded solution of the system. Consequently, this Massera's theorem reduces the problem of the existence of periodic solution of a periodic linear differential system with the same period as the system to the problem of the existence of a bounded solution. The latter problem is simplier than the original one, since the class of bounded continuously differentiable vector functions is much broader than its subclass consisiting of periodic vector functions.

Thus we have the following rather unexpected property: if a linear periodic system has a solution from a wide class (bounded solutions), it also has a solution from narrow class (periodic solutions of the same period as the system), – a rather rare situation in mathematics in general, if we take into account the fact that only the existence of some object would imply the existence of an object with additional properties. This result of J. L. Massera was transferred or generalized to other types of systems and their solutions in [4–6, 9–11, 13, 14] and others.

As a cosequence of Massera's theorem a natural question arises: is it possible to replace in its formulation the class of bounded solutions by some broader class so that modified theorem remains true. The present paper is devoted to the solution of this problem.

Recall that a set in a topological space is called *nowhere dense* if the interior of its closure is empty, and *a set of the first category according to Baer*, if it can be represented as a countable union of nowhere dense in this space sets. A set that is not a set of the first category is called *a set of the second category according to Baer*.

If M is a topological space and $M_0 \subset M$, then we will say that the space M is *an essential extension* of a subspace M_0 if M_0 has the first category in the space M.

Let M be some set of vector functions defined on the entire numerical axis $\mathbb{R}$. A metric in M given by the equality

$$\operatorname{dist}_{\mathrm{u}}(f,g) = \min\Big\{1, \sup_{t\in\mathbb{R}} \|f(t)-g(t)\|\Big\} \ \text{ for all } \ f,g \in M,$$

is called the metric of *uniform convergence on the axis*, and a metric given by the equality

$$\operatorname{dist}_{\mathrm{c}}(f,g) = \sup_{t\in\mathbb{R}} \min\big\{\|f(t)-g(t)\|, |t|^{-1}\big\} \ \text{ for all } \ f,g \in M,$$

– the metric of *uniform convergence on segments*. It is easy to see that convergence of the sequence $(f_n)_{n\in\mathbb{N}} \subset M$ in the metric $\operatorname{dist}_{\mathrm{u}}$ is equivalent to uniform convergence on the axis, and convergence in the metric $\operatorname{dist}_{\mathrm{c}}$ is equivalent to uniform convergence on each segment.

Next we denote by $\mathcal{B}$ the set of bounded continuously differentiable vector functions $\mathbb{R} \to \mathbb{R}^n$, and by $\mathcal{P}_\omega$ its subset consisting of ω-periodic vector functions. Let us introduce the metric $\operatorname{dist}_{\mathrm{u}}$ of uniform convergence on the axis on the set $\mathcal{B}$ and denote the obtained metric space by $\mathcal{B}_{\mathrm{u}}$.

Consider a linear differential system

$$\dot{x} = A(t)x + f(t), \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}, \tag{1}$$

where $n \in \mathbb{N}$ is fixed, with continuous ω-periodic $n \times n$ coefficient matrix $A(t)$ and free term $f(t)$. Its solutions are continuously differentiable vector functions $x(\,\cdot\,) : \mathbb{R} \to \mathbb{R}^n$. As stated above, according to Massera's theorem, if the system (1) has a bounded solution, then it also has an ω-periodic solution. Let us emphasize that we do not assert the ω-periodicity of this bounded solution, but only the fact that the system (1) has an ω-periodic solution. In general, a bounded solution of the ω-periodic system (1) may be neither ω-periodic nor periodic.

The problem stated above has the following formal formulation.

Problem. Is it possible to extend the class $\mathcal{B}$ of bounded vector-functions to some class so that the fact that the ω-periodic system (1) has a solution in this wider class implies that it also has an ω-periodic solution?

Further, while comparing a class of functions and some subclass of it, we will use the language of Baire's categories to understand the relation between them. Thus, Massera's theorem, which reduces the question of the existence of a solution from the set $\mathcal{P}_\omega$ to the question of the existence of a solution from the set $\mathcal{B}$, means that the latter question is much simplier, since, as the following statement shows, the space $\mathcal{B}_{\mathrm{u}}$ is an essential extension of its subspace $\mathcal{P}_\omega$.

Indeed,there is

Proposition. *The set $\mathcal{P}_\omega$ is closed and nowhere dense in the space $\mathcal{B}_{\mathrm{u}}$; in particular, it has the first Baire category in $\mathcal{B}_{\mathrm{u}}$.*

Thus, almost all in the sense of Baire's categories functions of the space $\mathcal{B}_{\mathrm{u}}$ are not ω-periodic. Nevertheless, according to Massera's theorem, only the fact of existing of a solution belonging to the "wide" class (class $\mathcal{B}$) implies the existing of a solution belonging to the "narrow" class (class $\mathcal{P}_\omega$).

Let us give the following

Definition. We will say that a vector function $x(\,\cdot\,) : \mathbb{R} \to \mathbb{R}^n$ *grows slower than a linear function*, if at least one of the following relations holds

$$\lim_{t\to-\infty} \frac{\|x(t)\|}{t} = 0 \quad \text{or} \quad \lim_{t\to+\infty} \frac{\|x(t)\|}{t} = 0. \tag{2}$$

We denote by $\mathcal{L}$ the class of continuously differentiable vector functions $\mathbb{R} \to \mathbb{R}^n$, which grow slower than a linear function. Clearly, $\mathcal{B} \subset \mathcal{L}$ and this is a proper inclusion. Indeed, for example, unbounded on $\mathbb{R}$ vector function $(\ln(t^2+1), 1, \dots, 1)^\top$ satisfies the condition (2), i.e. grows slower than a linear function. Therefore, the following statement strengthens Massera's theorem.

Theorem. *An ω-periodic system* (1) *has an ω-periodic solution if and only if it has a solution that grows slower than a linear function.*

The proof of necessity follows obviously from the chain of inclusions $\mathcal{P}_\omega \subset \mathcal{B} \subset \mathcal{L}$. The proof of sufficiency of the statement of the theorem is equivalent to proving that if the system (1) has no ω-periodic solutions, then it also has no solutions that grow slower than a linear function.

The question naturally arises how significant is extension $\mathcal{L}$ of the set $\mathcal{B}$. If we consider in $\mathcal{L}$ the metric $\mathrm{dist}_{\mathrm{u}}$ of uniform convergence on the axis, then from the point of view of categories there is no difference between $\mathcal{L}$ and $\mathcal{B}$, since, as it is easy to see, in this metric $\mathcal{L}$ is the union of two open sets: the set $\mathcal{B}$ of interest and its complement $\mathcal{L} \setminus \mathcal{B}$.

Consider in $\mathcal{L}$ the metric $\mathrm{dist}_{\mathrm{c}}$ of uniform convergence on segments. We denote the obtained metric space by $\mathcal{L}_{\mathrm{c}}$.

The set $\mathcal{B}$ has the first Baire's category in the space $\mathcal{L}_{\mathrm{c}}$. Thus almost all functions in the metric space $\mathcal{L}_{\mathrm{c}}$ are not bounded on the axis in the sense of categories, i.e. do not belong to the set $\mathcal{B}$. Consequently, the space $\mathcal{L}_{\mathrm{c}}$ is an essential extension of the subspace $\mathcal{B}$.

Acknowledgements

The authors are grateful to E. A. Barabanov for helpful discussions of the results.

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Asymptotic Proximity Between Equations with Mean Curvature Operator and Linear Equation

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1 Introduction

Consider the nonlinear equations

$$\big(a(t)\Phi_E(x')\big)' + b(t)F(x) = 0, \quad t \in I = [1,\infty), \tag{1.1}$$

and

$$\big(a(t)\Phi_R(x')\big)' + b(t)F(x) = 0, \quad t \in I = [1,\infty), \tag{1.2}$$

where the functions a and b are continuous and positive on $[1,\infty)$, the function F is continuous on $\mathbb{R}$ with $F(u)u > 0$ for $u \neq 0$, and the functions $\Phi_E : \mathbb{R} \to (-1,1)$ and $\Phi_R : (-1,1) \to \mathbb{R}$ are defined as

$$\Phi_E(u) = \frac{u}{\sqrt{1+u^2}}\,, \quad \Phi_R(u) = \frac{u}{\sqrt{1-u^2}}\,.$$

The operator Φ_E is called the *Euclidean mean curvature operator*. It arises in the search for radial solutions to partial differential equations which model fluid mechanics problems, in particular capillarity-type phenomena for compressible and incompressible fluids. The operator Φ_R is called the *Minkowski mean curvature operator* or, sometimes, the relativity operator. It originates from studying certain extrinsic properties of the mean curvature of hypersurfaces in the relativity theory, see e.g., [1,2] and the references therein.

The operators Φ_E and Φ_R are strictly related: the inverse of Φ_E is Φ_R and vice-versa. This fact plays an important role in the study of equations (1.1), (1.2), as we show below.

Here we consider the problem associated with (1.1) and (1.2) to find necessary and sufficient conditions for the existence of solutions such that

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} a(t)x'(t) = 0. \tag{1.3}$$

Observe that sometimes such solutions are called *intermediate solutions*, see, e.g., [3]. Other boundary value problems concerning Kneser-type boundary value problems for (1.1), or (1.2), are in [7]. More details on Kneser boundary value problems can be found in [11, Sections 13.1, 13.2 and 16.1].

Denote by J_a, J_b, J_1 the following integrals

$$J_a = \int\limits_1^\infty \frac{1}{a(t)}\,dt, \quad J_b = \int\limits_1^\infty b(t)\,dt, \quad J_{ab} = \int\limits_1^\infty b(t)\bigg(\int\limits_1^t \frac{1}{a(s)}\,ds\bigg)\,dt.$$

If the nonlinearity F is odd and satisfies the conditions

$$\liminf_{u\to\infty}\frac{F(u)}{u}>0, \quad \limsup_{u\to\infty}\frac{F(u)}{u}<\infty, \tag{1.4}$$

that, is, roughly speaking, F has a linear growth near infinity, we show that equations (1.1) and (1.2) are closely related with the linear equation

$$\big(a(t)x'\big)'+b(t)x=0. \tag{1.5}$$

Indeed, the well-known Leighton criterion states that (1.5) is oscillatory if $J_a=J_b=\infty$, see, e.g., [6] or [12, Theorem 2.24]. This oscillation result is valid also for equations with the curvature operator, see below. Further, the qualitative similarity between equations with the curvature operator and the linear case continues to hold also when (1.5) is nonoscillatory. More precisely, concerning the intermediate solutions in the linear case, the following holds, see, e.g., [5, Theorems 1 and 2].

Theorem 1.1. *Assume that $J_a=\infty, J_b<\infty$. If the linear equation* (1.5) *is nonoscillatory, then* (1.5) *has eventually positive solutions x satisfying* (1.3) *if and only if $J_{ab}=\infty$.*

In the following we illustrate how Theorem 1.1 continues to hold for equations (1.1) and (1.2).

2 Main results

We start by considering equation (1.1). The following oscillation result can be viewed as an extension of the quoted Leighton criterion.

Theorem 2.1. *Let $J_a=\infty$, $J_b=\infty$ and $\liminf_{u\to\infty}F(u)>0$. Then any continuable solution at infinity of equation* (1.1) *is oscillatory.*

Theorem 2.1 is proved in [3, Theorem 2.1 (ii)], see also [8, Theorem 4.1], by using a different argument to the one in [6] or [12, Theorem 2.24] for linear equation.

The next result concerns the asymptotic proximity between the intermediate solutions to equations (1.1) and (1.5). The following holds.

Theorem 2.2. *Let $J_a=\infty$, $\liminf_{t\to\infty}a(t)>0$, conditions* (1.4) *hold and $F_M=\sup_{u\geq1}F(u)/u$.*

If the linear equation

$$\Big(\frac{\sqrt{3}}{2}\,a(t)w'\Big)'+F_M\,b(t)w=0 \tag{2.1}$$

is nonoscillatory, then equation (1.1) *has infinitely many solutions x satisfying* (1.3) *if and only if*

$$J_b<\infty, \ \ J_{ab}=\infty. \tag{2.2}$$

Theorem 2.2 follows from [8, Theorem 3.1, Theorem 4.2]. Observe that Theorem 2.2 requires the existence of a suitable nonoscillatory linear equation (2.1) which, roughly speaking, can be viewed with respect to (1.1), as a dominant equation. This assumption can be verified by comparing (2.1) with known linear auxiliary equations such as, for instance, the Euler equation or the Riemann–Weber equation. More precisely, consider the Euler equation $w''+4^{-1}t^{-2}w=0$. Using the substitution $z(t)=t^{-\lambda}w$ we get that the linear equation

$$(c(t)z')'+d(t)z=0, \tag{2.3}$$

where $c(t)=t^{2\lambda}$, $d(t)=(\lambda-2^{-1})^2t^{2(\lambda-1)}$, is nonoscillatory. If $\lambda<2^{-1}$, then $J_c=\infty$, $J_d<\infty$ and $J_{cd}=\infty$. Hence, from Theorem 2.2 we have the following, see [8, Corollary 5.1].

Corollary 2.1. *Let* (1.4) *be verified. Assume that there exists* $\lambda \in (0, 2^{-1})$ *such that for large* t

$$a(t) \geq \frac{2}{\sqrt{3}}\, t^{2\lambda}, \quad b(t) \leq \frac{(\lambda - 2^{-1})^2}{F_M}\, t^{2(\lambda-1)},$$

where F_M *is given in Theorem 2.2. Then equation* (1.1) *has a solution* x *satisfying* (1.3).

Clearly, any other nonoscillatory linear equation of type (2.3) satisfying $J_c = \infty$, $J_d < \infty$ and $J_{cd} = \infty$ can be used as majorant equation.

Now, we study the qualitative similarity between (1.2) and (1.5). The oscillation for (1.2) is a more subtle problem, see, e.g., [3]. The following holds.

Theorem 2.3. *Let* $J_b = \infty$, $\liminf_{u\to\infty} F(u) > 0$ *and for any* $\lambda > 0$

$$\int_1^\infty \Phi_E\Big(\frac{\lambda}{a(t)}\Big)\, dt = \infty.$$

Then any continuable solution at infinity of equation (1.2) *is oscillatory.*

Theorem 2.3 follows, with minor changes, from a more general result stated in [3, Theorem 2.1]. Concerning the existence of intermediate solutions to (1.2), the following holds.

Theorem 2.4. *Let* $J_a = \infty$, $J_b < \infty$, $J_{ab} = \infty$, $\liminf\limits_{t\to\infty} a(t) > 0$, *conditions* (1.4) *hold and* $F_M = \sup\limits_{u\geq 1} F(u)/u$. *If* (2.2) *holds and the linear equation*

$$\big(a(t)w'\big)' + F_M\, b(t) w = 0 \tag{2.4}$$

is nonoscillatory, then equation (1.2) *has infinitely many solutions* x *satisfying* (1.3).

Theorem 2.4 is proved in [8, Theorem 5.1]. Moreover, in [8, Section 5] some necessary conditions for existence of intermediate solutions to (1.2) are given too.

3 Concluding remarks

We start by presenting the idea of the proof of Theorem 2.2. It is based on an important feature on the operator Φ_E and its inverse Φ_R. Setting

$$w = x, \quad z = a(t)\Phi_E(x'),$$

an easy calculation shows that the problem (1.1), (1.3) is equivalent to the problem

$$\begin{cases} w' = \Phi_R\Big(\dfrac{z}{a(t)}\Big) = \dfrac{z}{\sqrt{a^2(t) - z^2}}\,, \quad z' = -b(t)F(w), \quad t \in I, \\[2ex] \lim\limits_{t\to\infty} w(t) = \infty, \quad \lim\limits_{t\to\infty} \Phi_R\Big(\dfrac{z(t)}{a(t)}\Big) = 0. \end{cases} \tag{3.1}$$

For solving (3.1), we use a fixed point result, which originates from [4, Theorem 1.4], jointly with some asymptotic properties of the principal solution of a linear equation, see, e.g., [10, Chapter 11, Section 6]. We briefly describe our approach.

Let Ω be a nonempty, closed, convex and bounded subset of $C([1,\infty),\mathbb{R}^2)$ and for any $(u,v)\in\Omega$ consider the linear boundary value problem

$$\begin{cases} \xi' = \dfrac{\eta}{\sqrt{a^2(t)-v^2(t)}}\,, \quad \eta' = -b(t)\,\dfrac{F(u)}{u(t)}\,\xi, \quad t\in I, \\ \lim\limits_{t\to\infty}\xi(t)=\infty, \quad \lim\limits_{t\to\infty}\eta(t)=0. \end{cases} \tag{3.2}$$

For any $(u,v)\in\Omega$ denote by (ξ_{uv},η_{uv}) the principal solution of the linear system in (3.2) such that $\eta_{uv}(1)=k_a$, where k_a is a suitable positive fixed constant. Let T be the operator which maps (u,v) into (ξ_{uv},η_{uv}). Defining in an appropriate way the set Ω and using some comparison results on the behavior of the principal solution, it is easy to show that T has a fixed point $(\widehat{\xi},\widehat{\eta})$, which clearly is a solution of (3.1).

Observe that the linear system in (3.2) is equivalent to the second order linear equation

$$\Big(\sqrt{a^2(t)-v^2(t)}\,y'\Big)' + b(t)\,\frac{F(u)}{u(t)}y = 0 \tag{3.3}$$

and so the principal solution of the linear system in (3.2) coincides with the principal solution y_0 of (3.3). Thus, roughly speaking, this approach reduces the solvability of (3.1) to the solvability of a boundary value problem for a suitable associated second order linear equation. Clearly, a similar approach, with minor changes, is valid for proving the existence of intermediate solutions to (1.2).

Using the disconjugacy theory and some comparison results for principal solutions of linear equations, we can extend Theorems 2.2 and 2.4 by obtaining the so-called global positiveness of intermediate solutions, that is their positiveness on the whole interval I. Observe that, in general, this fact does not occur, because nonoscillatory solutions can have an arbitrary finite number of zeros, also in the linear case. This result is a consequence of a more general criterion in the forthcoming paper [9] and reads as follows.

Theorem 3.1. *Let the assumptions of Theorem 2.2 [Theorem 2.4] be valid. In addition, if the linear equation* (2.1) [(2.4)] *has the principal solution which is positive on I, then* (1.1) [(1.2)] *has infinitely many solutions x which are positive nondecreasing on I and satisfy* (1.3).

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On the Optimization Problem of One Market Relation Containing the Delay Functional Differential Equation

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In the paper, for a market relation theoretical model is constructed in the form of the controlled delay functional differential equation. Moreover, for the corresponding optimization problem the necessary conditions of optimality are formulated.

1 Mathematical model

Let us for the production of goods i_1 and i_2 require substitutable raw materials with concentration $x_1(t)$ and $x_2(t)$, respectively, at the moment t. Let the dynamic of these concentrations is described by the system of differential equations

$$\begin{cases} \dot{x}_1(t) = ax_1(t) + bx_2(t), \\ \dot{x}_2(t) = cx_1(t) + dx_2(t), \end{cases}$$

where a, b, c, d are given numbers.

Let market relation demand and supply for the good i_1 are described by functions $D_1(t,\omega)$ and $S_1(t,x_1,x_2,u)$ and for the good i_2 are described by functions $D_2(t,\vartheta)$ and $S_2(t,x_1,x_2,v)$. Let cost of the goods i_1 and i_2 at the moment t be $u(t)$ and $v(t)$, respectively. Suppose that at time t consumer demand will be satisfied on the good i_1 which has been ordered at time $t-\rho$, where $\rho>0$ is a fixed delay parameter and on the good i_2 which has been ordered at time $t-\theta$, where $\theta>0$, in general, is non fixed delay. The function

$$E_1(t) = D_1(t-\rho, u(t-\rho)) - S_1\big(t, x_1(t-\tau), x_2(t-\tau), u(t)\big), \quad t \in I$$

we call the disbalance index for the good i_1. We assume that for the production of the good i_1 requires the amount of raw materials $x_1(t-\tau)$ and $x_2(t-\tau)$ allocated at moments $t-\tau$, where $\tau>0$, in general, is non fixed delay. Here, it is taken into account that the production of i_1 good is carried out after some time from the allocation of raw materials.

Similarly, the function

$$E_2(t)=D_2(t-\theta,v(t-\theta))-S_2\big(t,x_1(t-\tau),x_2(t-\tau),v(t)\big),\ \ t\in I$$

is called the disbalance index for the good i_2.

If $E_1(t)=0$, then at the moment t we do not have disbalance between demand and supply with respect to good i_1, and the customer will buy exactly the quantity of good i_1 he needs. At time t, if $E_1(t)>0$, then demand exaggerates supply, if $E_1(t)<0$, then supply exaggerates demand. Analogously we can consider above described cases for $E_2(t)$.

In order to characterize the dynamics of the disbalance in time, we introduce the integral indices of the disbalance for the moment t

$$x_3(t)=x_{30}+\int\limits_{t_0}^{t}\Big[D_1(\xi-\rho,u(\xi-\rho))-S_1\big(\xi,x_1(\xi-\tau),x_2(\xi-\tau),u(\xi)\big)\Big]\,d\xi,$$

$$x_4(t)=x_{40}+\int\limits_{t_0}^{t}\Big[D_2(\xi-\theta,v(\xi-\theta))-S_2\big(\xi,x_1(\xi-\tau),x^2(\xi-\tau),v(\xi)\big)\Big]\,d\xi,$$

where x_{i0}, $i=3,4$ are given numbers. Thus, in the framework of the above mentioned conditions, we can describe the market relationship with the following system of controlled functional differential equation containing delays in phase coordinates and controls

$$\begin{cases}\dot x_1(t)=ax_1(t)+bx_2(t),\\ \dot x_2(t)=cx_1(t)+dx_2(t),\\ \dot x_3(t)=D_1(t-\rho,u(t-\rho))-S_1(t,x_1(t-\tau),x_2(t-\tau),u(t)),\\ \dot x_4(t)=D_2(t-\theta,v(t-\theta))-S_2(t,x_1(t-\tau),x_2(t-\tau),v(t)).\end{cases}\tag{1.1}$$

Finally, we note that one dimensional models for the market relation and corresponding optimization problems when right-hand side of the differential equation depends only on control both without delay and with delay were discussed in [1–4].

2 Statement of the problem and necessary conditions of optimality

Let $I=[t_0,t_1]$ be a given interval and $\tau_2>\tau_1>0$, $\rho>0$ and $\theta_2>\theta_1>0$ be given numbers with $t_1-t_0>\max\{\tau_2,\rho,\theta_2\}$. Suppose that the functions $\varphi_i(t)\in R_+=(0,\infty)$, $i=1,2$ are continuously differentiable on the interval $[\widehat\tau,t_0]$, where $\widehat\tau=t_0-\tau_2$. Further, denote by Ω and V the set of piecewise continuous control functions $u(t)\in[0,\widehat u]$, $t\in[t_0-\rho,t_1]$ and continuously differentiable control functions $v(t)\in[0,\widehat v]$, $t\in[t_0-\theta_2,t_1]$, respectively, where $\widehat u>0$, $\widehat v>0$ are given numbers.

To each element $w=(\tau,\theta,u(t),v(t))\in W=(\tau_1,\tau_2)\times(\theta_1,\theta_2)\times\Omega\times V$ we assign the delay functional differential equation (1.1) with the initial condition

$$x_i(t)=\varphi_i(t),t\in[\widehat\tau,t_0),x_i(t_0)=x_{i0},i=1,2;x_i(t_0)=x_{i0},i=3,4,\tag{2.1}$$

where $x_{i0} \in R_+$, $i = 1,2$ and, in general, $\varphi_i(t_0) \neq x_{i0}$, $i = 1,2$ (so called discontinuous part of the condition (2.1)). In the equation (1.1) it is assumed that the function $D_1(t,\omega), (t,\omega) \in [t_0-\rho, t_1] \times [0,\widehat{u}]$ is continuous and continuously differentiable with respect to ω; the function $S_1(t,x_1,x_2,u)$, $(t,x_1,x_2,u) \in I \times R_+^2 \times [0,\widehat{u}]$ is continuous and continuously differentiable with respect to x_1, x_2, u; the function $D_2(t,\vartheta)$, $(t,\vartheta) \in [t_0-\theta_2, t_1] \times [0,\widehat{v}]$ is continuous and continuously differentiable with respect to ϑ; the function $S_2(t,x_1,x_2,v)$, $(t,x_1,x_2,v) \in I \times R_+^2 \times [0,\widehat{v}]$ is continuous and continuously differentiable with respect to x_1, x_2, v.

Definition 1. Let $w = (\tau,\theta,u(t),v(t)) \in W$. The collection of functions

$$\Big\{x_i(t) = x_i(t;w) \in R_+,\ t \in [\widehat{\tau}, t_1],\ i = 1,2;\ \ x_i(t) = x_i(t;w),\ t \in I,\ i = 3,4\Big\}$$

is called a solution of the equation (1.1) with the initial condition (2.1) or a solution corresponding to the element w, if it satisfies the condition (2.1) and the functions $x_i(t)$, $i = 1,2,3,4$ are absolutely continuous on the interval I and satisfy the equation (1.1) almost everywhere on I.

Denote by W_0 the set of $w \in W$ for which there exists a solution. We assume that $W_0 \neq \varnothing$.

Definition 2. An element $w_0 = (\tau_0,\theta_0,u_0(t),v_0(t)) \in W_0$ is said to be optimal if for an arbitrary element $w \in W_0$ the inequality

$$J(w_0) \leq J(w)$$

holds, where

$$J(w) = \int\limits_{t_0}^{t_1} \big[x_3^2(t) + x_4^2(t)\big]\,dt$$

and $x_i(t) = x_i(t;w)$, $i = 3,4$.

Theorem 1. *Let w_0 be an optimal element and*

$$\Big\{x_{i0}(t) = x_i(t;w_0) \in R_+,\ t \in [\widehat{\tau}, t_1],\ i = 1,2;\ \ x_{i0}(t) = x_i(t;w_0),\ t \in I,\ i = 3,4\Big\}$$

be the corresponding solution. Let the function $u_0(t)$ be continuous at the point $t_0+\tau_0$. Then there exists a solution $\{\psi_i(t),\ t \in [t_0, t_1+\tau_0],\ i = 1,2,3,4\}$ of the equation

$$\begin{cases}\dot{\psi}_1(t) = -a\psi_1(t) - c\psi_2(t) + S_{1x_1}[t+\tau_0]\psi_3(t+\tau_0) + S_{2x_1}[t+\tau_0]\psi_4(t+\tau_0),\\ \dot{\psi}_2(t) = -b\psi_1(t) - d\psi_2(t) + S_{1x_2}[t+\tau_0]\psi_3(t+\tau_0) + S_{2x_2}[t+\tau_0]\psi_4(t+\tau_0),\\ \dot{\psi}_3(t) = 2x_{30}(t),\\ \dot{\psi}_4(t) = 2x_{40}(t),\\ t \in I\end{cases}$$

with the initial condition

$$\psi_i(t) = 0,\ \ t \in [t_1, t_1+\tau_0],\ \ i = 1,2,3,4,$$

where

$$S_{1x_1}[t] = \frac{\partial}{\partial x_1}\, S_1\big(t, x_{10}(t-\tau_0), x_{20}(t-\tau_0), u_0(t)\big),$$
$$S_{2x_1}[t] = \frac{\partial}{\partial x_1}\, S_2\big(t, x_{10}(t-\tau_0), x_{20}(t-\tau_0), v_0(t)\big)$$

such that the following conditions hold:

1) *the condition for the delay τ_0*

$$\widehat{S}_1\psi_3(t_0+\tau_0)+\widehat{S}_2\psi_4(t_0+\tau_0)$$
$$=\int\limits_{t_0}^{t_1}\Big\{\big[S_{1x_1}[t]\psi_3(t)+S_{2x_1}[t]\psi_4(t)\big]\dot{x}_{10}(t-\tau_0)+\big[S_{1x_2}[t]\psi_3(t)+S_{2x_2}[t]\psi_4(t)\big]\dot{x}_{20}(t-\tau_0)\Big\}\,dt,$$

where

$$\widehat{S}_1=S_1\big(t_0+\tau_0,\varphi_1(t_0),\varphi_2(t_0),u_0(t_0+\tau_0)\big)-S_1\big(t_0+\tau_0,x_{10},x_{20},u_0(t_0+\tau_0)\big),$$
$$\widehat{S}_2=S_2\big(t_0+\tau_0,\varphi_1(t_0),\varphi_2(t_0),v_0(t_0+\tau_0)\big)-S_2\big(t_0+\tau_0,x_{10},x_{20},v_0(t_0+\tau_0)\big);$$

2) *the condition for the delay θ_0*

$$\int\limits_{t_0}^{t_1}D_{2\vartheta}(t-\theta_0,v_0(t-\theta_0))\psi_4(t)\dot{v}_0(t-\theta_0)\,dt=0;$$

3) *the condition for the control $u_0(t)$*

$$\int\limits_{t_0}^{t_1}\Big\{\psi_3(t)\big[-S_{1u}[t]u_0(t)+D_{1\omega}(t-\rho,u_0(t-\rho))u_0(t-\rho)\big]\Big\}\,dt$$
$$=\max_{u(t)\in\Omega}\int\limits_{t_0}^{t_1}\Big\{\psi_3(t)\big[-S_{1u}[t]u(t)+D_{1\omega}(t-\rho,u_0(t-\rho))u(t-\rho)\big]\Big\}\,dt;$$

4) *the condition for the control $v_0(t)$*

$$\int\limits_{t_0}^{t_1}\Big\{\psi_4(t)\big[-S_{2v}[t]v_0(t)+D_{2w}(t-\theta_0,v_0(t-\theta_0))v_0(t-\theta_0)\big]\Big\}\,dt$$
$$=\max_{v(t)\in V}\int\limits_{t_0}^{t_1}\Big\{\psi_4(t)\big[-S_{2v}[t]v(t)+D_{2w}(t-\theta_0,v_0(t-\theta_0))v(t-\theta_0)\big]\Big\}\,dt.$$

It is clear that if $\varphi_i(t_0)=x_{i0}$, $i=1,2$, then $\widehat{S}_i=0$, $i=1,2$. Theorem 1 is proved by the scheme given in [5].

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Asymptotic Behaviour of Solutions of One Class of Nonlinear Differential Equations of Fourth Order

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We consider a two-membered non-autonomous fourth-order differential equation of the form

$$y^{(4)} = \alpha_0 p_0(t)[1+r(t)]e^{\sigma y} \quad (\sigma \neq 0), \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p_0 : [a,\omega[\,\to\,]0,+\infty[$ is a continuous or continuously differentiable function, $-\infty < a < \omega \le +\infty$, $r : [a,\omega[\,\to\,]-1,+\infty[$ is a continuous function such that

$$\lim_{t\uparrow\omega} r(t) = 0.$$

It is easy to see that in this equation the function $e^{\sigma y}$ $(\sigma \neq 0)$ is a fast-variable function when $y \to Y_0 = \pm\infty$ (by Karamata). We can choose the intervals Δ_{Y_0} of the points $Y_0 = \pm\infty$ as the neighbourhood of Δ_{Y_0}

$$\Delta_{Y_0} = \left[\begin{array}{ll}]0,+\infty[\,, & \text{if } Y_0 = +\infty, \\ ]-\infty,0[\,, & \text{if } Y_0 = -\infty. \end{array}\right.$$

Definition 1. A solution y of the differential equation (1) is called a $P_\omega(Y_0,\lambda_0)$-solution where $-\infty \le \lambda_0 \le +\infty$, if it is defined on the interval $[t_0,\omega[\,\subset [a,\omega[$ and satisfies the following conditions

$$y(t) \in \Delta_{Y_0} \text{ or } t \in [t_0,\omega[\,, \quad \lim_{t\uparrow\omega} y(t) = Y_0 = \pm\infty,$$

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \left[\begin{array}{l} \text{or } 0, \\ \text{or } \pm\infty, \end{array}\right. \quad (k=1,2,3), \qquad \lim_{t\uparrow\omega} \frac{[y^{(3)}(t)]^2}{y^{(2)}(t)y^{(4)}(t)} = \lambda_0.$$

From this definition, in particular, it follows that the number of

$$\nu_0 = \begin{cases} 1, & \text{or } Y_0 = +\infty, \\ -1, & \text{or } Y_0 = -\infty \end{cases}$$

determines the signs of any $P_\omega(Y_0,\lambda_0)$-solution and its first derivative in any left neighbourhood of ω. In [1] for $P_\omega(Y_0,\lambda_0)$-solutions at $\lambda_0 \in \mathbb{R} \setminus \{0,\frac{1}{2},\frac{2}{3},1\}$ (non special case) the following two theorems were obtained, but to formulate them we need to introduce additional auxiliary notations

$$K(\lambda_0) = \frac{(\lambda_0-1)^3}{\lambda_0(2\lambda_0-1)}\,, \quad \pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty, \\ t-\omega, & \text{if } \omega < +\infty, \end{cases}$$

$$J_0(t) = \int_{A_0}^{t} \pi_\omega^3(\tau)p_0(\tau)\,d\tau, \quad J_1(t) = \int_{A_1}^{t} \frac{p_0(\tau)}{J_0(\tau)}\,d\tau, \quad J_i(t) = \int_{A_i}^{t} J_{i-1}(\tau)\,d\tau \; (i=2,3),$$

$$Y(t) = -\frac{1}{\sigma}\ln\Big(\alpha_0\Big(-\frac{1}{\sigma}\Big)K(\lambda_0)J_0(t)\Big), \quad q(t) = \frac{Y'(t)}{\alpha_0 J_3(t)}\,,$$

where the integration boundary A_i is chosen to be equal to either ω or constant a and is defined in such a way that at this value of A_i the integral tends either to 0 or to $\pm\infty$. The following two theorems were established for equation (1) in [1].

Theorem 1. *Let $\lambda_0 \in \mathbb{R}\setminus\{0, \frac{1}{2}, \frac{2}{3}, 1\}$. For the differential equation* (1) *to have $P_\omega(Y_0, \lambda_0)$-solutions, the following inequalities*

$$\alpha_0\nu_0\lambda_0(2\lambda_0-1)(3\lambda_0-2)>0, \quad \alpha_0\nu_1 K(\lambda_0)\pi_\omega(t)>0 \ \ at \ \ t\in\,]a,\omega[\,, \tag{2}$$

and the following conditions

$$\begin{gathered}\alpha_0\sigma K(\lambda_0)J_0(t)<0 \ \ at \ \ t\in\,]a,\omega[\,,\\ \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_0'(t)}{J_0(t)}=\pm\infty, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_1'(t)}{J_1(t)}=\frac{1}{\lambda_0-1}\,, \quad \lim_{t\uparrow\omega}q(t)=1\end{gathered} \tag{3}$$

must be satisfied and each such solution admits at $t\uparrow\omega$ the following asymptotic mappings

$$y(t)=-\frac{1}{\sigma}\,\ln\Big(\alpha_0\Big(-\frac{1}{\sigma}\Big)K(\lambda_0)J_0(t)\Big)+o(1), \quad y^{(k)}(t)=\alpha_0J_{4-k}(t)[1+o(1)] \ \ (k=1,2,3).$$

Theorem 2. *Let $\lambda_0\in\mathbb{R}\setminus\{0,\frac{1}{2},\frac{2}{3},1\}$, the function p_0 be continuous and conditions* (2), (3) *be satisfied. Let, in addition*

$$\lim_{t\uparrow\omega}(1-q(t))|Y(t)|^{\frac{3}{4}}=0 \ \ and \ \ \alpha_0\sigma>0. \tag{4}$$

Then the differential equation (1) *has a two-parameter family $P_\omega(Y_0,\lambda_0)$ of solutions which satisfy at $t\uparrow\omega$ the asymptotic mappings*

$$\begin{gathered}y(t)=Y(t)+o(1), \quad y'(t)=\alpha_0J_3(t)\Big[1+\frac{o(1)}{|Y(t)|^{\frac{3}{4}}}\Big], \quad y''(t)=\alpha_0J_2(t)\Big[1+\frac{o(1)}{|Y(t)|^{\frac{1}{2}}}\Big],\\ y'''(t)=\alpha_0J_1(t)\Big[1+\frac{o(1)}{|Y(t)|^{\frac{1}{4}}}\Big].\end{gathered}$$

In Theorem 2, the first of conditions (4) is rather rigid. In the present paper an attempt is made to eliminate it.

Theorem 3. *Let $\lambda_0\in\mathbb{R}\setminus\{0,\frac{1}{2},\frac{2}{3},1\}$, the function p_0 be continuously differentiable and conditions* (2), (3) *be satisfied. Suppose, in addition, that the second condition in* (4) *is satisfied and there exists a finite or equal to $\pm\infty$ limit*

$$\lim_{t\uparrow\omega}\pi_\omega(t)q'(t).$$

Then the differential equation (1) *has a two-parameter family $P_\omega(Y_0,\lambda_0)$ of solutions which satisfy at $t\uparrow\omega$ the asymptotic mappings*

$$\begin{gathered}y(t)=Y(t)+o(1), \quad y'(t)=\alpha_0J_3(t)\Big[q(t)+\frac{o(1)}{|Y(t)|^{\frac{3}{4}}}\Big], \quad y''(t)=\alpha_0J_2(t)\Big[1+\frac{o(1)}{|Y(t)|^{\frac{1}{2}}}\Big],\\ y'''(t)=\alpha_0J_1(t)\Big[1+\frac{o(1)}{|Y(t)|^{\frac{1}{4}}}\Big].\end{gathered} \tag{5}$$

Sketch of the proof

First, it is easy to prove that
$$\lim_{t\uparrow\omega} \pi_\omega(t)q'(t) = 0.$$
In the same way as in the proof of Theorem 2 of [1], equation (1) by the transformation
$$y(t) = Y(t) + y_1(t), \quad y^{(k)}(t) = \alpha_0 J_{4-k}(t)[1 + y_{k+1}(t)] \ \ (k = 1, 2, 3) \tag{6}$$
is reduced to a system of differential equations of the form
$$\begin{aligned}
y_1' &= \alpha_0 J_3(t)\big[1 - q(t) + y_2\big],\\
y_2' &= \frac{J_3'(t)}{J_3(t)}\,(y_3 - y_2),\\
y_3\prime &= \frac{J_2'(t)}{J_2(t)}\,(y_4 - y_3),\\
y_4' &= \frac{J_1'(t)}{J_1(t)}\,\big[r(t) + (1 + r(t))y_1 - y_4 + R(t, y_1)\big].
\end{aligned}$$
We will consider this system on the set
$$\Omega = [t_1, \omega[\times D, \ \text{ where } \ D = \Big\{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4_{\frac{1}{2}} : \ \ |y_i| \le \frac{1}{2}\,, \ \ (i = 1, \dots, 4)\Big\},$$
$$\text{where } \ |R(t, y_1| \leqslant y_1^2 \ \text{ at } \ |y_1| \leqslant \delta \ \text{ for some } \ 0 < \delta < \frac{1}{2}\,.$$
Further we will use the obtained system on the set $\Omega_0 = [t_1, \omega[\,\times \mathbb{R}^4_\delta$.

In contrast to Theorem 2, let us make an additional transformation
$$y_1(t) = z_1(t), \quad y_2(t) = z_2(t) + q(t) - 1, \quad y_3(t) = z_3(t), \quad y_4(t) = z_4(t), \tag{7}$$
the sense of which is to exclude the summand $(1 - q(t))$ from the first equation of the system and as a result we obtain a system of differential equations of the form
$$\begin{aligned}
z_1' &= \frac{Y(t)}{\pi_\omega(t)}\,\big\{\xi_1(t)z_2\big\},\\
z_2' &= \frac{1}{\pi_\omega(t)}\,\big\{\xi_2(t)(z_3 - z_2) - \pi_\omega(t)q'(t)\big\},\\
z_3' &= \frac{1}{\pi_\omega(t)}\,\big\{\xi_3(t)(z_4 - z_3)\big\},\\
z_4' &= \frac{1}{\pi_\omega(t)}\,\Big\{\xi_4(t)\big[r(t) + (1 + r(t))z_1 - z_4 + R(t, z_1)\big]\Big\},
\end{aligned} \tag{8}$$
where
$$\lim_{t\uparrow\omega} \xi_1(t) = \frac{3\lambda_0 - 2}{\lambda_0 - 1}\,, \quad \lim_{t\uparrow\omega} \xi_2(t) = \frac{2\lambda_0 - 1}{\lambda_0 - 1}\,, \quad \lim_{t\uparrow\omega} \xi_3(t) = \frac{\lambda_0}{\lambda_0 - 1}\,, \quad \lim_{t\uparrow\omega} \xi_4(t) = \frac{1}{\lambda_0 - 1}\,.$$
To asymptotically equalise the multipliers at $t \uparrow \omega$ in the right-hand side of the equations of the system (8), we apply the following transformation to it:
$$z_1(t) = \upsilon_1(t), \quad z_2(t) = |Y(t)|^{-\frac{3}{4}}\upsilon_2(t), \quad z_3(t) = |Y(t)|^{-\frac{1}{2}}\upsilon_3(t), \quad z_4(t) = |Y(t)|^{-\frac{1}{4}}\upsilon_4(t). \tag{9}$$

As a result, we obtain a system of quasilinear differential equations for which all the conditions of Theorem 2.2 of [2] are fulfilled. The limit matrix of coefficients at υ_1, υ_2, υ_3, υ_4 of the obtained quasilinear system has the form

$$C = \begin{pmatrix} 0 & \dfrac{3\lambda_0 - 2}{\lambda_0 - 1}\Big(\dfrac{\nu_0}{\operatorname{sign}\sigma}\Big) & 0 & 0 \\ 0 & 0 & \dfrac{2\lambda_0 - 1}{\lambda_0 - 1} & 0 \\ 0 & 0 & 0 & \dfrac{\lambda_0}{\lambda_0 - 1} \\ \dfrac{1}{\lambda_0 - 1} & 0 & 0 & 0 \end{pmatrix},$$

and has, taking into account the sign conditions (2), (3), a characteristic equation of the form

$$\lambda^4 + \frac{\alpha_0}{\sigma}\,\frac{|3\lambda_0 - 2|\,|2\lambda_0 - 1|\,|\lambda_0|}{(\lambda_0 - 1)^4} = 0.$$

The characteristic equation has two pairs of complex-conjugate roots with real parts different from zero. Then the system of differential equations has a two-parameter family of solutions $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4 : [t_2, \omega[\to \mathbb{R}^4_\delta$ ($t_2 \in [t_0, \omega[$), which tend to 0 at $t \uparrow \omega$. To each such solution, taking into account substitutions (6), (7), (9), corresponds a solution $y : [t_2, \omega[\to \mathbb{R}$ of the differential equation (1) for which the asymptotic representations (5) take place at $t \uparrow \omega$. It is also easy to check, taking into account these asymptotic representations and the form of equation (1), that the solutions we have constructed are $P_\omega(Y_0, \lambda_0)$-solutions.

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Qualitative Behavior of the Trajectories Impulsive Semigroup for the Hyperbolic Equation

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1 Introduction

The study is devoted to an important class of evolutionary systems characterized by the presence of impulsive disturbances when the system trajectory reaches a fixed subset in the phase space. The systematic study of such systems began relatively recently and was mostly focused on the finite-dimensional case [1, 2, 4, 10–12]. The results regarding the limit behavior of infinite-dimensional impulsive dynamic systems are contained in works [3, 6, 8], however, in both the parabolic and hyperbolic cases, the impulsive parameters are "finite-dimensional" in nature, i.e., the situation was considered when only a finite number of coordinates of the phase vector were subjected to an impulsive disturbance. The novelty of this study is that we consider the case when the entire infinite-dimensional phase vector undergoes an impulsive disturbance when the energy functional reaches a certain threshold value.

2 Setting of the problem and the main results

Let a triple of Hilbert spaces $V \subset H \subset V^*$ with compact dense embeddings be given, $\|\cdot\|$ be the norm and $(\cdot\,,\cdot)$ be the scalar product in H, $A : V \to V^*$ be a linear, continuous, self-adjoint, coercive operator, $\|u\|_V := \langle A^{\frac{1}{2}}u, u\rangle$ be the norm in V, $\langle\cdot\,,\cdot\rangle$ be the scalar product in V.

Let us consider an evolution problem

$$\begin{cases} \dfrac{d^2y}{dt^2} + 2\beta\,\dfrac{dy}{dt} + Ay = 0, \\ y\big|_{t=0} = y_0 \in V, \\ y_t\big|_{t=0} = y_1 \in V. \end{cases} \tag{2.1}$$

Problem (2.1) in phase space $X = V \times H$ generates a continuous semigroup $G : \mathbb{R}_+ \times X \to X$ [13], where for $z_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \in X$

$$G(t, z_0) = z(t) = \begin{pmatrix} y(t) \\ y_t(t) \end{pmatrix} =$$

$$= e^{-\beta t}\sum_{j=1}^{\infty}\begin{pmatrix} (y_0,\varphi_j)\cos\omega_j t + \big(\beta(y_0,\varphi_j)+(y_1,\varphi_j)\big)\dfrac{1}{\omega_j}\sin\omega_j t \\ (y_1,\varphi_j)\cos\omega_j t - \big(\lambda_j^2(y_0,\varphi_j)+\beta(y_1,\varphi_j)\big)\dfrac{1}{\omega_j}\sin\omega_j t \end{pmatrix}, \quad (2.2)$$

where $\omega_j = \sqrt{\lambda_j^2-\beta^2}$, $\{\lambda_j\}_{j=1}^{\infty}$, $\{\varphi_j\}_{j=1}^{\infty}$ are solutions of the spectral problem

$$A\varphi_j = \lambda_j\varphi_j, \;\; j\geq 1,$$

$\{\varphi_j\}_{j=1}^{\infty}$ is the orthonormal basis in H, $0<\lambda_1\leq\lambda_2\leq\cdots$, $\lambda_j\to+\infty$, $j\to\infty$, and without limitation of generality we will assume that $\lambda_1>\beta$.

Consider the functional $\Psi : X\to\mathbb{R}_+$, that for $z=\begin{pmatrix}u\\ v\end{pmatrix}\in X$ is determined by the rule

$$\Psi(z) = \|z\|_X^2 = \|u\|_V^2 + \|v\|^2. \quad (2.3)$$

The impulsive problem is formulated as follows: if at some point in time $t>0$ at the solution $z=\begin{pmatrix}y\\ y_t\end{pmatrix}$ the functional (2.3) reaches the value Ψ_0, then the system instantly moves to a new position

$$z^+ = \varphi(z)+\alpha, \quad (2.4)$$

where $\alpha\in X$, $\varphi : X\to X$ are given.

In [9] we prove that, under certain conditions on the parameters, the problem (2.1), (2.3), (2.4) generates in X an impulsive dynamical system $\widetilde{G} : \mathbb{R}_+\times X\to X$ (see Definition 3.1 below), for which, for each $z_0\in X$, the ω-boundary set is nonempty, compact, and the limit relation is true

$$\operatorname{dist}_X\big(\widetilde{G}(t,z_0),\widetilde{\omega}(z_0)\big)\to 0, \;\; t\to\infty.$$

3 ω-Boundary set for impulsive dynamical systems

Following the work [7], we will describe the general construction of the impulsive dynamical system. Suppose that a continuous semigroup $G : \mathbb{R}_+\times X\to X$ is given on the phase space X, the trajectories of semigroup, when they reach a fixed subset $M\subset X$ (impulsive set), are moved by the mapping I (impulsive mapping) to a new position

$$z^+ := Iz.$$

For the correctness of such construction, the following conditions must be met

$$\begin{gathered} G:\mathbb{R}_+\times X\to X \text{ is continuous semigroup,}\\ \text{i.e. for all } z\in X \text{ and } t,s\geq 0\text{: } G(0,z)=z,\ G(t+s,z)=G(t,G(s,z)), \\ \text{map } (t,z)\mapsto G(t,z) \text{ is continuous on } \mathbb{R}_+\times X; \end{gathered} \quad (3.1)$$

$$M \text{ is closed set}, \;\; M\cap IM=\varnothing; \quad (3.2)$$

$$\forall\, z\in M \;\; \exists\,\tau=\tau(z)>0 \;\; \forall\, t\in(0,\tau): \;\; G(t,z)\notin M. \quad (3.3)$$

Under the conditions (3.1)–(3.3) it is known [6] that if for $z\in X$

$$M^+(z) := \Big(\bigcup_{t>0} G(t,z)\Big)\cap M\neq\varnothing,$$

then there exists $\widetilde{s} := \widetilde{s}(z) > 0$ such that

$$\forall\, t \in (0, \widetilde{s}): \;\; G(t, z) \notin M, \;\; G(\widetilde{s}, z) \in M.$$

Using the introduced notations z^+, $M^+(z)$, $\widetilde{s}$, the impulsive trajectory $\widetilde{G}(\,\cdot\,, z_0)$ starting from $z_0 \in X$ is constructed as follows:

- if $M^+(z_0) = \varnothing$, then $\widetilde{G}(t, z_0) = G(t, z_0)$, $t \geq 0$;
- if $M^+(z_0) \neq \varnothing$, then for $s_0 := \widetilde{s}(z_0)$ let's mark $z_1 := G(s_0, z_0)$, so

$$\widetilde{G}(t, z_0) = \begin{cases} G(t, z_0), & t \in [0, s_0), \\ z_1^+, & t = s_0; \end{cases}.$$

- if $M^+(z_1^+) = \varnothing$, then $\widetilde{G}(t, z_0) = G(t - s_0, z_1^+)$, $t \geq s_0$;
- if $M^+(z_1^+) \neq \varnothing$, then for $s_1 := \widetilde{s}(z_1^+)$ let's mark $z_1 := G(s_1, z_1^+)$, so

$$\widetilde{G}(t, z_0) = \begin{cases} G(t - s_0, z_1^+), & t \in [s_0, s_0 + s_1), \\ z_2^+, & t = s_0 + s_1; \end{cases}$$

and so on. Continuing this process, we will obtain a finite or infinite number of impulsive points

$$z_{n+1}^+ = IG(s_n, z_n^+), \;\; z_0^+ := z_0, \;\; n \geq 0,$$

and corresponding sequence of time moments

$$T_{n+1} := \sum_{k=0}^{n} s_k, \;\; T_0 := 0, \;\; n \geq 0.$$

At the same time, $\widetilde{G}$ is given by the formula

$$\widetilde{G}(t, z_0) = \begin{cases} G(t - T_n, z_n^+), & t \in [T_n, T_{n+1}), \\ z_{n+1}^+, & t = T_{n+1}. \end{cases} \tag{3.4}$$

It should be noted that in such a system there may be "beating effects" or "Zeno"-modes, when moments of impulsive occur so often that the trajectory (3.4) is destroyed in a finite time [5].

Since we are interested in the behavior of (3.4) when $t \to \infty$, then we will make the following assumption:

$$\begin{cases} \text{for each } \; z_0 \in X \; \text{ there are either no impulsive points,} \\ \text{or their number is finite, or } \; T_n \to \infty, \;\; n \to \infty. \end{cases} \tag{3.5}$$

The condition (3.5) guarantees that for an arbitrary $z_0 \in X$ the function $t \mapsto \widetilde{G}(t, z_0)$ is defined on $[0, +\infty)$.

Definition 3.1. The mapping $\widetilde{G} : \mathbb{R}_+ \times \mathbb{X} \to X$ constructed above is called an **impulsive dynamic system**. We will say that $\{V, M, I\}$ generate an impulsive dynamic system, if the conditions (3.1)–(3.3), (3.5) are met.

It is known that under the conditions (3.1) – (3.3), (3.5) the mapping $\widetilde{G} : \mathbb{R}_+ \times \mathbb{X} \to X$ is a semigroup whose trajectories are continuous from the right.

In addition, by construction for arbitrary $z_0 \in X$ and $t > 0$:

$$\widetilde{G}(t, z_0) \cap M = \varnothing.$$

The main object of study in this paper is the ω-boundary set:

$$\widetilde{\omega}(z_0) = \Big\{ \xi \in X : \ \ \exists \{t_n\}_{n=1}^{\infty} : t_n \nearrow \infty, \ \ \xi = \lim_{n\to\infty} \widetilde{G}(t_n, z_0) \Big\}.$$

Lemma 3.1. *Let $\{V, M, I\}$ generate an impulsive dynamic system $\widetilde{G}$ and for $z_0 \in X$ the following conditions be fulfilled:*

(1) *set $\widetilde{\gamma} := \bigcup\limits_{t\geq 0} \widetilde{G}(t, z_0)$ is bounded;*

(2) *for each $z \in \widetilde{\gamma} : G(t, z) = G_1(t, z) + G_2(t, z)$, where $\{G_1(t, z), t \geq 0, z \in \widetilde{\gamma}\}$ is precompact, $\sup\limits_{z\in\widetilde{\gamma}} G_2(t, z) \to 0$, $t \to \infty$.*

(3) *if $\widetilde{\gamma}$ has an infinite number of impulsive points $\{z_n^+\}_{n\geq 0}\}$, then $\{z_n^+\}_{n\geq 0}$ is precompact.*

Then the set $\widetilde{\omega}(z_0) \neq \varnothing$ is compact and $\operatorname{dist}_X(\widetilde{G}(t, z_0), \widetilde{\omega}(z_0)) \to 0$, $t \to \infty$.

Remark 3.1. Fulfillment of the condition (1) can be guaranteed under the following conditions

$$\begin{gathered}
\exists\, C_1, C_2 \geq 0 \ \ \exists\, \delta > 0 \ \ \forall\, z \in \widetilde{\gamma} \ \ \forall\, t \geq 0 \\
\|G(t, z)\|_X \leq \|z\|_X e^{-\delta t} + C_1, \\
\|Iz\|_x \leq \|z\|_X + C_2,
\end{gathered}$$

and if C$\{s_k\}_{k\geq 0}$ are the distances between impulses along $\widetilde{\gamma}$, then

$$\overline{s} := \inf_{k\geq 0} s_k > 0.$$

Remark 3.2. The condition (3) can be replaced by the following:

$$\text{if } \{z_n\} \text{ is bounded, then } \{I z_n\} \text{ is precompact.}$$

We cannot expect that $\widetilde{\omega}(z_0)$ to be stable in any sense, since this is not true even in the non-impulsive case. The stability property can be guaranteed for more massive objects – uniform attractors [6]. However, we can ensure the invariance of the non-impulsive part of $\widetilde{\omega}(z_0)$. For this, it is necessary to impose conditions on trajectories starting from initial data close to $\widetilde{\omega}(z_0)$.

Lemma 3.2. *Let $\{V, M, I\}$ generate impulsive dynamical system $\widetilde{G}$, the conditions of Lemma 3.1 be fulfilled for $z_0 \in X$, and, in addition*

$$I : M \to X \ \ \textit{be continuous};$$

if $\xi \in \widetilde{\omega}(z_0) \setminus M$, then for $\xi_n \to \xi$

$$\begin{cases} \widetilde{s}(\xi) = \infty, & \textit{if } \ \widetilde{s}(\xi_n) = \infty \ \textit{ for infinitely many } n, \\ \widetilde{s}(\xi_n) \to \widetilde{s}(\xi), & \textit{otherwise.} \end{cases}$$

Then for each $t \geq 0$

$$\widetilde{G}\big(t, \widetilde{\omega}(z_0) \setminus M\big) \subset \widetilde{\omega}(z_0) \setminus M.$$

If in addition for $\xi \in \widetilde{\omega}(z_0) \cap M$ *and for* $\xi_m \to \xi$, $\xi_m \notin M$,

$$\widetilde{s}(\xi_n) = \infty \ \textit{ for infinitely many } n \textit{ or } \widetilde{s}(\xi_n) \to 0,$$

then for arbitrary $t \geq 0$

$$\widetilde{G}\big(t, \widetilde{\omega}(z_0)\big) \supset \widetilde{\omega}(z_0) \setminus M.$$

Remark 3.3. If we add the following condition to the conditions of Lemma 3.2:

$$\text{for } t_n \nearrow \infty \text{ by subsequence } G(t_n, z_0) \to y \notin M, \tag{3.6}$$

then for an arbitrary $t \geq 0$:

$$\widetilde{G}\big(t, \widetilde{\omega}(z_0) \setminus M\big) = \widetilde{\omega}(z_0) \setminus M.$$

The condition (3.6) means that the ω-boundary set of the non-impulsive half-flow G does not intersect with M.

4 Limit modes of the impulsive problem (2.1), (2.3), (2.4)

For the problem (2.1), (2.3), (2.4), the phase space is the Hilbert space $X = V \times H$, on which the solutions of the evolutionary problem (2.1) generate a continuous semigroup $G : \mathbb{R}_+ \times X \to X$ according to the formula (2.2).

The set M is given by (2.3) according to the formula

$$M = \left\{ z = \begin{pmatrix} u \\ v \end{pmatrix} \in X : \ \ \Psi(z) = \Psi_0 \right\}, \ \ \Psi_0 > 0.$$

We will consider that the following conditions are fulfilled

$$\|\varphi(z)\|_X \leq \|z\|_X, \ \ \Psi_0 < \frac{1}{4} \|\alpha\|_X^2. \tag{4.1}$$

In [9] we have checked the fulfillment of the conditions (3.1)–(3.3) and (3.5). Thus, it is proved that the problem (2.1), (2.3), (2.4) generates an impulsive dynamic system, and each impulsive trajectory has an infinite number of impulsive points.

Theorem. *Suppose that for the problem* (2.1), (2.3), (2.4) *the conditions* (4.1) *and the following are fulfilled*

$$\frac{1}{\sqrt{\lambda_1}} < \frac{1}{8\beta} \ln\Big(\frac{\|\alpha\|_X^2}{2\Psi_0} - 1 \Big), \tag{4.2}$$

$$\varphi : M \to X \ \ \textit{is a compact mapping.}$$

Then, for the corresponding impulsive dynamical system $\widetilde{G}$*, we have that for an arbitrary* $z_0 \in X$ ω*-limit set* $\widetilde{\omega}(z_0) \neq \varnothing$*, it is compact and*

$$\operatorname{dist}_X \big(\widetilde{G}(t, z_0), \widetilde{\omega}(z_0)\big) \to 0, \ \ t \to \infty.$$

Remark 4.1. The condition (4.2) can be removed by requiring the limit $\lim\limits_{k\to\infty} s_k$ to exist instead.

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On the Representation Formula of a Solution for a Class of Perturbed Controlled Neutral Functional Differential Equation

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The neutral functional-differential equation is a mathematical model of such a system whose behavior at a given moment depends on the velocity of the system in the past. In the paper an analytic relation between solutions of the original Cauchy problem and a corresponding perturbed problem is established for the controlled neutral functional-differential equation with the discontinuous initial condition, whose right-hand side is linear with respect to the prehistory of the phase velocity. In the representation formula of a solution the effects of perturbations of the delay parameter containing in the phase coordinates, of the initial and control functions are revealed. Such analytic relation plays an important role in proving the necessary conditions of optimality [1, 6]. Besides, such relation allows one to get an approximate solution of the perturbed equation and to carry out a sensitive analysis of mathematical models.

Let $I=[t_0,t_1]$ be a given interval. Let $\mathbb{R}^n$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$ and let $O\subset\mathbb{R}^n$, $U\subset\mathbb{R}^r$ be convex open sets; let $\sigma>0$ and $\tau_2>\tau_1>0$ be given numbers, with $t_0+\max\{\sigma,\tau_2\}<t_1$. Suppose that the $n\times n$ -dimensional matrix function $A(t,x,y)$ is continuous on the set $I\times O^2$ and continuously differentiable with respect to x^i, $i=1,2,\dots,n$ and y^j, $j=1,2,\dots,n$; moreover, there exists $M_1>0$ such that

$$|A(t,x,y)|+\sum_{i=1}^{n}|A_{x^i}(\,\cdot\,)|+\sum_{j=1}^{n}|A_{y^j}(\,\cdot\,)|\le M_1\quad\forall\,(t,x,y)\in I\times O\times O.$$

Let the n-dimensional function $f(t,x,y,u)$ be continuous on the set $I\times O^2\times U$ and continuously differentiable with respect to x, y, u; moreover, there exists $M_2>0$ such that

$$|f(t,x,y,u)|+|f_x(\,\cdot\,)|+|f_y(\,\cdot\,)|+|f_u(\,\cdot\,)|\le M_2\quad\forall\,(t,x,y,u)\in I\times O^2\times U.$$

Further, denote by Φ and Ω the sets of continuous differentiable functions $\varphi(t)\in O$, $t\in[\widehat{\tau},t_0]$, where $\widehat{\tau}=t_0-\max\{\sigma,\tau_2\}$ and measurable functions $u(t)\in U$, $t\in I$, respectively, with the set $\operatorname{cl}u(I)$ is compact and $\operatorname{cl}u(I)\subset U$.

To each element
$$\mu=(\tau,x_0,\varphi(t),u(t))\in\Lambda=(\tau_1,\tau_2)\times O\times\Phi\times\Omega$$
we assign the quasi-linear neutral functional-differential equation
$$\dot x(t)=A(t,x(t),x(t-\tau))\dot x(t-\sigma)+f\big(t,x(t),x(t-\tau),u(t)\big),\ \ t\in I \tag{1}$$
with the initial condition
$$x(t)=\varphi(t),\ \ t\in[\widehat\tau,t_0),\quad x(t_0)=x_0. \tag{2}$$
Condition (2) is called the discontinuous initial condition because in general $x(t_0)\neq\varphi(t_0)$. Discontinuity at the initial moment t_0 may be related to the instant change in a dynamical process (for example, change of an investment, environment and so on).

Definition. Let $\mu\in\Lambda$. A function $x(t)=x(t;\mu)\in O$, $t\in I_1=[\widehat\tau,t_1]$ is called a solution of equation (1) with condition (2) or a solution corresponding to the element μ and defined on the interval I_1 if it satisfies condition (2) and is absolutely continuous on the interval I and satisfies equation (1) almost everywhere on I.

Let us introduce the notations:
$$|\mu|=|\tau|+|x_0|+\|\varphi\|_1+\|u\|,\quad \Lambda_\varepsilon(\mu_0)=\big\{\mu\in\Lambda:\ |\mu-\mu_0|\le\varepsilon\big\},$$
where
$$\|\varphi\|_1=\sup\big\{|\varphi(t)|+|\dot\varphi(t)|:\ t\in I_1\big\},\quad \|u\|=\sup\big\{|u(t)|:\ t\in I\big\},$$
$\varepsilon>0$ is a fixed number and $\mu_0=(\tau_0,x_{00},\varphi_0(t),u_0(t))\in\Lambda$ is a fixed element; furthermore,
$$\delta\tau=\tau-\tau_0,\quad \delta x_0=x_0-x_{00},\quad \delta\varphi(t)=\varphi(t)-\varphi_0(t),\quad \delta u(t)=u(t)-u_0(t),$$
$$\delta\mu=\mu-\mu_0=(\delta\tau,\delta x_0,\delta\varphi(t),\delta u(t)).$$

Let $x(t;\mu_0)$ be a solution corresponding to the element $\mu_0\in\Lambda$ and defined on the interval I_1. Then there exists a number $\varepsilon_1>0$ such that to each element $\mu=\mu_0+\delta\mu\in\Lambda_{\varepsilon_1}(\mu_0)$ corresponds a solution $x(t;\mu)$, $t\in I_1$, [2, 6], i.e. Cauchy's perturbed problem has a solution, defined on the interval I_1.

Theorem 1. *Let $x_0(t):=x(t;\mu_0)$ be a solution corresponding to the element $\mu_0=(\tau_0,x_{00},\varphi_0,u_0)\in\Lambda$ and defined on the interval I_1, with $t_0+\tau_0\notin\{t_1-\sigma,t_1-2\sigma,\dots\}$. Moreover, let the function $u_0(t)$ be continuous at the point $t_0+\tau_0$. Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta>0$ such that for arbitrary $\mu\in\Lambda_{\varepsilon_2}(\mu_0)$ on the interval $[t_1-\delta,t_1]\subset[t_0+\tau_0,t_1]$ the following representations hold:*
$$x(t;\mu)=x_0(t)+\delta x(t;\delta\mu)+o(t;\delta\mu), \tag{3}$$
$$\delta x(t;\delta\mu)=\Psi(t_0;t)\delta x_0+\int_{t_0-\sigma}^{t_0}Y(\xi+\sigma;t)A[\xi+\sigma]\dot{\delta\varphi}(\xi)\,d\xi$$
$$+\int_{t_0-\tau_0}^{t_0}Y(\xi+\tau_0;t)\Big\{\frac{\partial}{\partial y}\big[A[\xi+\tau_0]\dot x_0(\xi+\tau_0-\sigma)\big]+f_y[\xi+\tau_0]\Big\}\delta\varphi(\xi)\,d\xi+\int_{t_0}^{t}Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi$$
$$-\Big\{Y(t_0+\tau_0;t)\big(\widehat A\dot x_0(t_0+\tau_0-\sigma)+\widehat f\big)+\int_{t_0}^{t}Y(\xi;t)\Big(\frac{\partial}{\partial y}\big[A[\xi]\dot x_0(\xi-\sigma)\big]+f_y[\xi]\Big)\dot x_0(\xi-\tau_0)\,d\xi\Big\}\delta\tau. \tag{4}$$

Here,

$$A[\xi]=A(\xi,x_0(\xi),x_0(\xi-\tau_0)),\quad f_y[\xi]=f_y\big(\xi,x_0(\xi),x_0(\xi-\tau_0),u_0(\xi)\big),$$
$$\frac{\partial}{\partial y}\big[A[\xi]\dot x_0(\xi-\sigma)\big]=\frac{\partial}{\partial y}\big[A(t,x,y)\dot y_0(\xi-\sigma)\big]_{x=x_0(\xi),\,y=x_0(\xi-\tau_0)},$$
$$\widehat{A}=A\big(t_0+\tau_0,x_0(t_0+\tau_0),x_{00}\big)-A\big(t_0+\tau_0,x_0(t_0+\tau_0),\varphi_0(t_0)\big),$$
$$\widehat{f}=f\big(t_0+\tau_0,x_0(t_0+\tau_0),x_{00},u_0(t_0+\tau_0)\big)-f\big(t_0+\tau_0,x_0(t_0+\tau_0),\varphi_0(t_0),u_0(t_0+\tau_0)\big);$$

$\Psi(\xi;t)$ and $Y(\xi;t)$ are $n\times n$ matrix functions satisfying the system

$$\begin{cases}\Psi_\xi(\xi;t)=-Y(\xi;t)\Big\{\dfrac{\partial}{\partial x}\big[A[\xi]\dot x_0(\xi-\sigma)\big]+f_x[\xi]\Big\}\\ \qquad\qquad -Y(\xi+\tau_0;t)\Big(\dfrac{\partial}{\partial y}\big[A[\xi+\tau_0]\dot x_0(\xi+\tau_0-\sigma)\big]+f_y[\xi+\tau_0]\Big),\\ Y(\xi;t)=\Psi(\xi;t)+Y(\xi+\sigma;t)A[\xi+\sigma],\\ \qquad\qquad \xi\in(t_0,t),\ \ t\in(t_0,t_1]\end{cases}\tag{5}$$

and the condition

$$\Psi(\xi;t)=Y(\xi;t)=\begin{cases}E, & \xi=t,\\ \Theta, & \xi>t,\end{cases}\tag{6}$$

where E is the identity matrix and Θ is the zero matrix.

Some Comments

The function $\delta x(t;\delta\mu)$ in (3) is called the first variation of the solution $x_0(t)$. The expression (4) is called the local variation formula of the solution. The term “variation formula of the solution” has been introduced by R. V. Gamkrelidze and proved for ordinary differential equation in [6].

The addend $\Psi(t_0;t)\delta x_0$ in formula (4) is the effect of perturbation of the initial vector x_{00}.

The expression

$$\int\limits_{t_0-\sigma}^{t_0}Y(\xi+\sigma;t)A[\xi+\sigma]\dot\delta\varphi(\xi)\,d\xi$$
$$+\int\limits_{t_0-\tau_0}^{t_0}Y(\xi+\tau_0;t)\Big\{\frac{\partial}{\partial y}\big[A[\xi+\tau_0]\dot x_0(\xi+\tau_0-\sigma)\big]+f_y[\xi+\tau_0]\Big\}\delta\varphi(\xi)\,d\xi$$

in formula (4) is the effect of perturbation of the initial function $\varphi_0(t)$.

The addend
$$\int\limits_{t_0}^{t}Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi$$
in formula (4) is the effect of perturbation of the control function $u_0(t)$.

The expression

$$\Big\{Y(t_0+\tau_0;t)\big(\widehat{A}\dot x_0(t_0+\tau_0-\sigma)+\widehat{f}\big)+\int\limits_{t_0}^{t}Y(\xi;t)\Big(\frac{\partial}{\partial y}\big[A[\xi]\dot x_0(\xi-\sigma)\big]+f_y[\xi]\Big)\dot x_0(\xi-\tau_0)\,d\xi\Big\}\delta\tau$$

in formula (4) is the effect of perturbation of the delay τ_0, where $Y(t_0+\tau_0;t)(\widehat{A}\dot{x}_0(t_0+\tau_0-\sigma)+\widehat{f})$ is the effect of the discontinuous initial condition (2). If $x_0(t_0)=\varphi_0(t_0)$, then $\widehat{A}=0$ and $\widehat{f}=0$.

Formula (3) allows us to obtain an approximate solution of the perturbed equation in the analytical form on the interval $[t_1-\delta,t_1]$. In fact, for a small $|\delta\mu|$ from (3) it follows

$$x(t;\mu)\approx x_0(t)+\delta x(t;\delta\mu),$$

where $\delta x(t;\delta\mu)$ has the form (4). We note that to construct $\delta x(t;\delta\mu)$ it is sufficient to find a solution to the linear problem (5), (6).

Theorem 1 is proved by the scheme given in [2,6]. The case when $A(t,x,y)=A(t)$ is considered in [2, 3, 6] and the case when $A(t,x,y)=0$ is considered in [4, 5].

Acknowledgments

This work was supported by Shota Rustaveli National Science Foundation of Georgia (SRNSFG), Grant # YS-21-554.

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Positive Periodic Solutions for Functional Differential Equations with Super-Linear Growth

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Consider a functional differential equation

$$u'(t)=\ell(u)(t)+\lambda F(u)(t) \quad \text{for a. e. } t\in\mathbb{R}, \tag{1}$$

where $\ell : C_\omega(\mathbb{R}) \to L_\omega(\mathbb{R})$ is a linear bounded operator, $F : C_\omega(\mathbb{R}) \to L_\omega(\mathbb{R})$ is a continuous operator satisfying the Carathéodory conditions, and $\lambda \in \mathbb{R}$ is a parameter. By an ω-periodic solution to the equation (1) we understand a locally absolutely continuous ω-periodic function $u : \mathbb{R} \to \mathbb{R}$ that satisfies the equation (1) almost everywhere in $\mathbb{R}$. We say that an ω-periodic solution u to (1) is positive if $u(t) > 0$ for $t \in \mathbb{R}$.

Notation 1.

$\mathbb{Z}$ is the set of integers, $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0,+\infty[\,$.

$C_\omega(\mathbb{R})$ is the Banach space of ω-periodic continuous functions $v : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|v\|_{C_\omega}=\max\big\{|v(t)|:\ t\in[0,\omega]\big\}.$$

$C_\omega(\mathbb{R}_+)=\{v\in C_\omega(\mathbb{R}):\ v(t)\in\mathbb{R}_+ \text{ for } t\in\mathbb{R}\}$.

$L_\omega(\mathbb{R})$ is the Banach space of ω-periodic locally Lebesgue integrable functions $p : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|p\|_{L_\omega}=\int_0^\omega |p(s)|\,ds.$$

$L_\omega(\mathbb{R}_+)=\big\{p\in L_\omega(\mathbb{R}):\ p(t)\in\mathbb{R}_+ \text{ for a. e. } t\in\mathbb{R}\big\}$.

If $A : C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ is a linear bounded operator, by $\|A\|$ we denote the norm of A.

Notation 2. Let $c \in (0,1)$. Then $\mathcal{K}_c \subset C_\omega(\mathbb{R}_+)$ is a set of functions such that $cu(s) \le u(t)$ for $s,t \in \mathbb{R}$.

Definition 1. We say that an operator ℓ belongs to the set $\mathcal{U}_c^+$ if every function $u \in C_\omega(\mathbb{R})$ that is locally absolutely continuous and satisfies

$$u'(t)\ge \ell(u)(t) \quad \text{for a. e. } t\in\mathbb{R},$$

belongs to $\mathcal{K}_c$.

It can be easily seen that if $\ell \in \mathcal{U}_c^+$, then the only ω-periodic solution to the homogeneous equation

$$u'(t) = \ell(u)(t) \quad \text{for a. e. } t \in \mathbb{R} \tag{2}$$

is the trivial solution.

Now we formulate the assumptions laid on the nonlinear operator F.

(H.1) F transforms $C_\omega(\mathbb{R}_+)$ into $L_\omega(\mathbb{R}_+)$ and it is not the zero operator, i.e., there exists $x_0 \in C_\omega(\mathbb{R}_+)$ such that

$$\int_0^\omega F(x_0)(s)\,ds > 0.$$

(H.2) F is super-linear with respect to $\mathcal{K}_c$, i.e., there exists a Carathéodory function $\eta : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F(v)(t) \geq \eta(t, \|v\|_{C_\omega}) \quad \text{for a. e. } t \in \mathbb{R}, \ \ v \in \mathcal{K}_c$$

and

$$\lim_{x\to+\infty} \frac{1}{x} \int_0^\omega \eta(s,x)\,ds = +\infty.$$

(H.3) For every $0 \neq x \in \mathcal{K}_c$ there exists $\delta_x > 0$ such that for every $\delta \in (0, \delta_x]$ we have

$$\delta F(x)(t) \geq F(v)(t) \quad \text{for a. e. } t \in \mathbb{R} \ \text{ whenever } \ v \in \mathcal{K}_c, \quad v(t) \leq \delta x(t) \quad \text{for } t \in \mathbb{R}$$

and

$$\delta_0 \int_0^\omega F(x)(s)\,ds > \int_0^\omega F(\delta_0 x)(s)\,ds \quad \text{for some } \ \delta_0 \in (0, \delta_x).$$

Note that the assumptions (H.1) and (H.3) imply $F(0)(t) = 0$ for a. e. $t \in \mathbb{R}$.

Notation 3. Let $\lambda \in \mathbb{R}$. Then by $\mathcal{S}(\lambda)$ we denote the set of all positive ω-periodic solutions to (1) for corresponding λ.

Theorem 1. *Let $c \in (0,1)$ be such that $\ell \in \mathcal{U}_c^+$ and F satisfies* (H.1)–(H.3). *Then there exists a critical value $\lambda_c \in (0, +\infty]$ such that*

(i) *Eq.* (1) *has at least one positive ω-periodic solution provided $\lambda \in (0, \lambda_c)$,*

(ii) *Eq.* (1) *has no positive ω-periodic solution provided $\lambda \notin (0, \lambda_c)$.*

Moreover,

$$\lim_{\lambda\to\lambda_c^-} \sup\big\{\|u\|_{C_\omega} : \ u \in \mathcal{S}(\lambda)\big\} = 0,$$
$$\lim_{\lambda\to 0^+} \inf\big\{\|u\|_{C_\omega} : u \in \mathcal{S}(\lambda)\big\} = +\infty.$$

Because the ω-periodic solutions to (1) belong to $\mathcal{K}_c$, the latter means that the solutions uniformly tends to $+\infty$ as λ tends to zero.

Suppose that the operator F includes a linear part, i.e.,

$$F(v)(t) = \widetilde{F}(v,v)(t) \quad \text{for a. e. } t \in \mathbb{R}, \ \ v \in C_\omega(\mathbb{R}), \tag{3}$$

where $\widetilde{F} : C_\omega(\mathbb{R}) \times C_\omega(\mathbb{R}) \to L_\omega(\mathbb{R})$ is a continuous operator satisfying the Carathéodory conditions and it is linear and nondecreasing in the first variable. Therefore, instead of (1) we consider the equation

$$u'(t) = \ell(u)(t) + \lambda \widetilde{F}(u,u)(t) \quad \text{for a. e. } t \in \mathbb{R},$$

where ℓ and λ are the same as in (1) and $\widetilde{F}$ is described above.

Theorem 2. *Let $c \in (0,1)$ be such that $\ell \in \mathcal{U}_c^+$ and F given by (3) satisfies (H.1)–(H.3). Let, moreover, $\widetilde{F}(\cdot,0) : C_\omega(\mathbb{R}) \to L_\omega(\mathbb{R})$ be a non-zero operator. Then, $\lambda_c < +\infty$ and the equation*

$$u'(t) = \ell(u)(t) + \lambda_c \widetilde{F}(u,0)(t) \quad \textit{for a. e. } t \in \mathbb{R} \tag{4}$$

has a positive solution u_c, the set of solutions to (4) is one-dimensional (generated by u_c), and

$$T_\lambda \in \mathcal{U}_c^+ \quad \textit{for } \lambda \in]0, \lambda_c[\,,$$

where

$$T_\lambda(v)(t) \overset{def}{=} \ell(v)(t) + \lambda \widetilde{F}(v,0)(t) \quad \textit{for a. e. } t \in \mathbb{R}, \ \ v \in C_\omega(\mathbb{R}).$$

If $\widetilde{F}(\cdot,0) : C_\omega(\mathbb{R}) \to L_\omega(\mathbb{R})$ is a zero operator, then $\lambda_c = +\infty$.

Theorem 2 gives us a method how to calculate the precise value of λ_c in the cases where F includes a linear part. Indeed, define an operator $A : C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ by

$$A(x)(t) \overset{def}{=} \int\limits_{t-\omega}^{t} G(t,s)\widetilde{F}(x,0)(s)\,ds \quad \text{for } t \in \mathbb{R}, \ \ x \in C_\omega(\mathbb{R}),$$

where G is Green's function to the ω-periodic problem for (2). Then

$$u_c(t) = \lambda_c A(u_c)(t) \quad \text{for } t \in \mathbb{R},$$

i.e., $1/\lambda_c$ is the first eigenvalue to A corresponding to the positive eigenfunction u_c. Therefore, according to Krasnoselski's theory and Gelfand's formula,

$$\lambda_c = \lim_{n \to +\infty} \frac{1}{\sqrt[n]{\|A^n\|}}\,.$$

Corollaries

Consider a differential equation with deviating arguments

$$u'(t) = -g(t)u(\sigma(t)) + \lambda p(t)\, \frac{u(\tau(t))[a_0 + a_1 u(\mu_1(t)) + a_2 u(\mu_2(t)) u(\mu_3(t))]}{b_0 + b_1 u(\nu(t))} \quad \text{for a. e. } t \in \mathbb{R}, \tag{5}$$

where

- $p, g \in L_\omega(\mathbb{R}_+)$, $p \neq 0$, $g \neq 0$,
- σ, τ, μ_i, ν are measurable ω-periodic functions,
- $a_i > 0$, $b_i > 0$ are constants.

Corollary 1. *Let*

$$\int\limits_{\widetilde{\sigma}(t)}^{t} g(s)\,ds \le \frac{1}{e} \quad \textit{for a. e. } t \in [0,\omega], \tag{6}$$

where $\widetilde{\sigma}(t) = \sigma(t) - z\omega$ *if* $\sigma(t) \in [t+(z-1)\omega, t+z\omega)$ $(z \in \mathbb{Z})$, *and let*

$$\frac{a_1}{b_1}\,\exp\bigg(-e\int\limits_0^{\omega} g(s)\,ds\bigg) > \frac{a_0}{b_0}\,. \tag{7}$$

Then there exists a critical value $\lambda_c \in (0,+\infty)$ *such that* (5) *has a positive* ω*-periodic solution iff* $\lambda \in (0,\lambda_c)$. *Moreover,*

$$\lim_{\lambda\to\lambda_c^-} \sup\big\{\|u\|_{C_\omega} :\ u \in \mathcal{S}(\lambda)\big\} = 0, \quad \lim_{\lambda\to 0^+} \inf\big\{\|u\|_{C_\omega} :\ u \in \mathcal{S}(\lambda)\big\} = +\infty.$$

The condition (6) guarantees that the operator

$$\ell(v)(t) \overset{def}{=} -g(t)v(\sigma(t)) \quad \text{for a. e. } t \in \mathbb{R},\ \ v \in C_\omega(\mathbb{R})$$

belongs to the set $\mathcal{U}_c^+$ with

$$c = \exp\bigg(-e\int\limits_0^{\omega} g(s)\,ds\bigg),$$

and the condition (7) guarantees that the assumption (H.3) is fulfilled with F defined by

$$F(v)(t) \overset{def}{=} p(t)\,\frac{v(\tau(t))[a_0 + a_1 v(\mu_1(t)) + a_2 v(\mu_2(t))v(\mu_3(t))]}{b_0 + b_1 v(\nu(t))} \quad \text{for a. e. } t \in \mathbb{R},\ \ v \in C_\omega(\mathbb{R}).$$

Now consider a differential equation with deviating arguments

$$u'(t) = -g(t)u(\sigma(t)) + \lambda p(t)\,\frac{u^{1+n}(\tau(t))(a+u(\mu(t)))^m}{(b+u^k(\nu(t)))}\,, \tag{8}$$

where

- $p, g \in L_\omega(\mathbb{R}_+)$, $p \not\equiv 0$, $g \not\equiv 0$,
- σ, τ, μ, ν are measurable ω-periodic functions,
- $a > 0$, $b > 0$, $n > 0$, $m > 0$, $k > 0$.

Corollary 2. *Let* $n + m > k$, *and let*

$$\int\limits_{\widetilde{\sigma}(t)}^{t} g(s)\,ds \le \frac{1}{e} \quad \textit{for a. e. } t \in [0,\omega],$$

where $\widetilde{\sigma}(t) = \sigma(t) - z\omega$ *if* $\sigma(t) \in [t+(z-1)\omega, t+z\omega)$ $(z \in \mathbb{Z})$. *Then* (8) *has a positive* ω*-periodic solution for every* $\lambda > 0$. *Moreover,*

$$\lim_{\lambda\to+\infty} \sup\big\{\|u\|_{C} :\ u \in \mathcal{S}(\lambda)\big\} = 0, \quad \lim_{\lambda\to 0^+} \inf\big\{\|u\|_{C} :\ u \in \mathcal{S}(\lambda)\big\} = +\infty.$$

Lyapunov Stability of Time-Fractional Stochastic Volterra Equations

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Time-fractional stochastic differential models became popular in applications, and its analysis is presented in multiple highly cited monographs and articles, for example, [1–3, 5, 6].

The target of this report is a stochastic fractional-in-time Volterra equation defined with multiple deterministic and stochastic time scales:

$$dx(t) = \sum_{j=1}^{m} \Big[f_j(t, (H_{1j}x)(t))\,(dt)^{\alpha_j} + g_j(t, (H_{2j}x)(t))\,dB_j(t) \Big] \ \ (t \geq 0). \tag{1}$$

Here $f_j(\omega, t, v)$ and $g_j(\omega, t, v)$ are random functions, H_{1j} and H_{2j} are linear delay operators, $0 < \alpha_j \leq 1$, $dB_j(t)$ are Itô differentials generated by the standard scalar Wiener processes (Brownian motions) B_j, m is the number of the deterministic/stochastic time-scales and $x(t)$ is an unknown stochastic process on $\Re$ satisfying, in addition to (1), the initial condition

$$x(s) = \varphi(s) \ \ (s \leq 0), \tag{2}$$

where $\varphi(\omega, s)$ is some random function (not necessarily continuous). Throughout the paper we tacitly assume that

$$f_j(\,\cdot\,,\,\cdot\,,0) = 0 \ \text{ and } \ g_j(\,\cdot\,,\,\cdot\,,0) = 0 \ \ (P \otimes \mu)\text{-almost everywhere}$$

(μ is the Lebesgue measure on $\Re$), which simply means that $x \equiv 0$ satisfies Eq. (1) and the initial condition (2) with $\varphi \equiv 0$. A solution of the initial value problem (1), (2) is a progressively measurable stochastic process x almost surely satisfying (2) for μ-almost all $s \in \Re_-$ and the integral equation

$$x(t) - \varphi(0) = \sum_{j=1}^{m} \Big[\int_0^t \alpha_j (t-s)^{\alpha_j - 1} f_j(s, (H_{1j}x)(s))\,ds + \int_0^t g_j(s, (H_{2j}x)(s))\,dB_j(s) \Big]$$

for all $t \in \Re_+$. It is assumed that the initial value problem (1), (2) has a unique solution $x(t, \varphi)$ for all admissible φ (see Definition 1).

Below we keep fixed the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in\Re}, P)$ satisfying the standard conditions [1] assuming, in addition, that $\mathcal{F}_t = \mathcal{F}_0$ for all $t \leq 0$. All stochastic processes in this paper are supposed to be progressively measurable w.r.t. this stochastic basis or parts of it.

Basic notation used below:

- $\Re = (-\infty, \infty)$, $\Re_+ = [0, \infty)$, $\Re_- = (-\infty, 0)$.
- μ is the Lebesgue measure defined on $\Re$ or its subintervals.
- E is the expectation.
- $|\cdot|$ is the fixed norm in $\Re^n$ and $\|\cdot\|$ is the associated matrix norm $\|\cdot\|$.
- $B_j(t)$ $(t \in \Re_+,\ j = 1, \dots, m)$ are the standard scalar Brownian motions (Wiener processes).

The constants used below:

- $n \in N$ is the dimension of the phase space, i.e. the size of the solution vector.
- $m \in N$ is the number of the deterministic/stochastic time-scales.
- The indices i, j satisfy $1 \leq i \leq 2$, $1 \leq j \leq m$.
- $0 < \alpha_j \leq 1$ define the time scales.
- p is a fixed real constant appearing in the p-stability we assume that $p \geq 2$ and $p > \alpha_j^{-1}$.

Let $J \subset \Re_+$. The following spaces of random variables and stochastic processes are used below as well:

- The space k_p^n consists of all n-dimensional, $\mathcal{F}_0$-measurable random variables $\{\xi :\ E|\xi|^p < \infty\}$.
- $\mathcal{L}_p(J, \Re^l)$ contains all progressively measurable l-dimensional stochastic processes $x(t)$ $(t \in J)$ such that
$$\int\limits_J E|x(t)|^p \, dt < \infty.$$
- For a given positive continuous function $\gamma(t)$, $t \in J$, the space $\mathcal{M}_p^\gamma(J, \Re^l)$ consists of all progressively measurable l-dimensional stochastic processes $x(t)$ $(t \in J)$ such that
$$\sup_{t\in J} E|\gamma(t)x(t)|^p < \infty.$$
- For $l = n$ and $J = \Re_+$ we define $\mathcal{M}_p^\gamma \equiv \mathcal{M}_p^\gamma(\Re_+, \Re^n)$, and if, in addition, $\gamma = 1$, then we put $\mathcal{M}_p \equiv \mathcal{M}_p^1(\Re_+, \Re^n)$.
- The Banach space $\mathcal{U}$ is the direct product of $2m$ copies of the space $\mathcal{M}_p(\Re_+, \Re^l)$ equipped with the natural norms.

In the well-known definition of the stochastic Lyapunov stability below we assume that $\varphi \in \mathcal{M}_p(\Re_- \cup \{0\}, \Re^n)$.

Definition 1. Eq. (1) is called globally

- p-stable if there exists $c > 0$ such that
$$E|x(t, \varphi)|^p \leq c \sup_{s\leq 0} E|\varphi(s)|^p \ \text{ for all } \ t \in \Re_+;$$

- asymptotically p-stable if it is p-stable and, in addition,

$$\lim_{t\to\infty} E|x(t,\varphi)|^p = 0;$$

- exponentially p-stable if there exist $c > 0$ and $\beta > 0$ such that the inequality

$$E|x(t,\varphi)|^p \le c\exp\{-\beta t\}\sup_{s\le 0} E|\varphi(s)|^p \quad \text{for all} \quad t \in \Re_+$$

holds.

To study Lyapunov stability of the solutions of Eq. (1), it is convenient to rewrite it as a multi-time scale stochastic Volterra equation with predefined controls:

$$dy(t) = \sum_{j=1}^{m} \Big[(F_j(y,u_{1j}))(t)\,(dt)^{\alpha_j} + (G_j(y,u_{2j}))(t)\,dB_j(t)\Big] \quad (t \ge 0), \tag{3}$$

where $u_{ij} = u_{ij}(t,\omega)$ $(t \in \Re_+)$ belong to the space $\mathcal{M}_p(\Re_+,\Re^l)$, F_j and G_j are some nonlinear Volterra mappings. The way to construct u_{ij}, F_j and G_j is described in the paper [7]. Note that Eq. (3) only requires the initial condition for $t = 0$

$$y(0) = y_0 \in k_p^n. \tag{4}$$

Given $u_{ij} \in \mathcal{M}_p(\Re_+,\Re^l)$, by a solution of the control problem (3), (4) we understand a progressively measurable stochastic process $y(t)$ almost surely satisfying the initial condition (4) and the integral equation

$$y(t) - y_0 = \sum_{j=1}^{m}\Big[\int_0^t \alpha_j(t-s)^{\alpha_j-1}F_j(y,u_{1j})(s)\,ds + \int_0^t G_j(y,u_{2j})(s)\,dB_j(s)\Big]$$

for all $t \in \Re_+$. Two integrals here are understood in the sense of Lebesgue and Itô, respectively. In the sequel, we will assume that the restrictions on the operators F_j and G_j ensure the existence of these integrals and existence and uniqueness of the solution $y(t,y_0,u)$ of the control problem (3), (4) for all $u_{ij} \in \mathcal{M}_p(\Re_+,\Re^l)$ and $y_0 \in k_p^n$.

The Lyapunov stability of the solutions of Eq. (1) will be, then, replaced by a particular version of the input-to-state stability, which is well-known in the control theory. Below, we call this version $\mathcal{M}_p^\gamma$-stability.

Definition 2. We say that Eq. (3) is $\mathcal{M}_p^\gamma$-stable if for all $y_0 \in k_p^n$ and $u_{ij} \in \mathcal{M}_p(\Re_+,\Re^l)$

- $y(\,\cdot\,,y_0,u) \in \mathcal{M}_p^\gamma$;
- there exists $K > 0$ such that

$$\|y(\,\cdot\,,y_0,u)\|_{\mathcal{M}_p^\gamma} \le K\big(\|y_0\|_{k_p^n} + \|u\|_{\mathcal{U}}\big).$$

Under some very natural conditions on γ (see [7] for the details) the $\mathcal{M}_p^\gamma$-stability of solutions of Eq. (3) implies p-stability, asymptotic p-stability and exponential p-stability of solutions of Eq. (1). This result is exploited in this report.

To study the property of $\mathcal{M}_p^\gamma$-stability for Eq. (3) it is convenient to start with choosing some simpler linear equation, which already has this property:

$$dy(t)=\sum_{j=1}^{m}\Big[\big((Q_jy)(t)+z_{1j}(t)\big)\,(dt)^{\alpha_j}+z_{2j}(t)\,dB_j(t)\Big]\ \ (t\in\Re_+). \tag{5}$$

Here $Q_j:\mathcal{M}_p\to\mathcal{L}_{p_j}(\Re_+,\Re^n)$ $(p_j>\frac{1}{\alpha_j})$ are k_p^1-linear operators, $z_{1j}\in\mathcal{L}_{p_j}(\Re_+,\Re^n)$ and $z_{2j}\in\mathcal{L}_2(\Re_+,\Re^n)$. Assuming the existence and uniqueness property for Eq. (5) for any initial condition (4) and using the linearity of Q_j, we obtain the following representation of its solutions:

$$y(t)=U(t)\chi(0)+(Wz)(t),$$

where $U(t)$ is the fundamental matrix of the associated homogeneous equation, which is an $n\times n$-matrix whose columns satisfy this homogeneous equation and $U(0)=I_n$ and

$$W:\prod_{j=1}^{m}\big(\mathcal{L}_{p_j}(\Re_+,\Re^n)\times\mathcal{L}_2(\Re_+,\Re^n)\big)\to\mathcal{M}_p$$

is Green's operator for (5), $(Wz)(0)=0$ and Wz is a solution of Eq. (5) for any z from the domain of W. Using the solutions representation of the auxiliary equation we can regularize Eq. (3) by rewriting it as

$$y(t)=U(t)y_0+\sum_{j=1}^{m}\Big[\big(W_{1j}(-Q_jy+F_j(y,u_{1j}))\big)(t)+\sum_{j=1}^{m}(W_{2j}G_j(y,u_{2j}))(t)\Big]\ \ (t>0).$$

Given a continuous function $\gamma:\Re_+\to(0,\infty)$, an initial value $y_0=[y_{01},\to,y_{0n}]^T\in k_p^n$, a control $u=(u_{ij}:\ i=1,2,\ j=1,\dots,m)$, $u_{ij}\in\mathcal{M}_p(\Re_+,\Re^l)$, which produce the solution of Eq. (3)

$$y(t,y_0,u)=\big[y_1(t,y_0,u),\dots,y_n(t,y_0,u)\big]^T$$

and a nonnegative stopping time η, we define

- $\overline{y}_0=[\overline{y}_{01},\dots,\overline{y}_{0n}]^T$, where
$$\overline{y}_{0\nu}=(E|y_{0\nu}|^p)^{1/p}\equiv\|y_{0\nu}\|_{k_p^1};$$
- $\overline{y}^{\,\eta}=[\overline{y}_1^{\,\eta},\dots,\overline{y}_n^{\,\eta}]^T$, where
$$\overline{y}_\nu^{\,\eta}=\sup_{0\le t\le\eta}\big(E|\gamma(t)y_\nu(t,y_0,u)|^p\big)^{1/p},$$

so that $\overline{y}_\nu^{\,\eta}=\overline{y}_\nu^{\,\eta}(\gamma,p)$, $\overline{y}^{\,\eta}=\overline{y}^{\,\eta}(\gamma,p)$ and $\overline{y}_\nu^{\,\eta}=\overline{y}_\nu^{\,\eta}(\gamma,p)$ for $\nu=1,\dots,n$. These notations allow us to formulate and prove the main result of this report.

Theorem 1. *Suppose there exist a real $n\times n$-matrix C and two constants $K_1>0$ and $K_2>0$ such that I_n-C is inverse-positive and for any stopping time $0\le\eta<\infty$ the vector $\overline{y}^{\,\eta}=\overline{y}^{\,\eta}(\gamma,p)$ satisfies the matrix inequality*

$$\overline{y}^{\,\eta}\le C\overline{y}^{\,\eta}+K_1\overline{y}_0+K_2\|u\|_{\mathcal{U}}e_n\ \ (e_n=[1,\dots,1]^T\in\Re^n).$$

Then Eq. (3) is $\mathcal{M}_p^\gamma$-stable.

The proof of the theorem can be found in [7].

Using this theorem, one can conveniently study different kinds of Lyapunov stability of the solutions of Eq. (1), choosing an appropriate weight γ and an auxiliary equation (5).

The illustrative example below demonstrates applications of Theorem 1. The universal constant c_p used in the example comes from the following estimate:

$$E\bigg|\int\limits_0^t f(s)\,dB(s)\bigg|^{2p} \le c_p^{2p} E\bigg(\int\limits_0^t |f(s)|^2\,ds\bigg)^p \ \ (t \in \Re_+, \ \ p \ge 1), \tag{6}$$

where $B(t)$ $(t \in \Re_+)$ is the standard scalar Brownian motion and $f(s)$ ia an arbitrary scalar, progressive measurable stochastic process on $\Re_+$; some explicit formulae for c_p can be found in the literature, for instance, in [4], where $c_p = 2\sqrt{12}p$, which, however, is not best possible, as evidently, $c_1 = 1$,

Example. Let $1 \le p < \infty$. Consider the following system of linear equations

$$dx(t) = -\sum_{j=1}^{m}\Big[A^{(j)}x(h_j(t))\,(dt)^{\alpha_j} + \sum_{\tau=1}^{m_j} A^{(j,\tau)}x(h_{j\tau}(t))\,d\mathcal{B}_j(t)\Big] \ \ (t \ge 0), \tag{7}$$

where $A^{(j)} = (a_{sl}^{(j)})_{s,l=1}^n$, $j = 1, \dots, m$, $A^{(j,\tau)} = (a_{sl}^{(j,\tau)})_{s,l=1}^n$, $j = 1, \dots, m$, $\tau = 1, \dots, m_i$ are real $n \times n$-matrices and h_j, $h_{j\tau}$, $j = 1, \dots, m$, $\tau = 1, \dots, m_j$ are continuous functions such that $h_j(t) \le t$, $h_{j\tau} \le t$, $t \ge 0$, $j = 1, \dots, m$, $\tau = 1, \dots, m_j$, $0 < \alpha_j \le 1$, $j = 1, \dots, m$, $A^{(1)}$ is a diagonal matrix with the positive diagonal entries $a_\nu^{(1)}$ and $\alpha_1 = 1$.

Let C be the $n \times n$-matrix with the entries

$$c_{\nu\kappa} = \sum_{j=2}^{m}\Bigg[|a_{\nu\kappa}^{(j)}|\Bigg(\exp\{-\alpha_j\}\Big(\frac{\alpha_j}{a_{\nu\nu}^{(1)}}\Big)^{\alpha_j} + \Gamma\Big(\frac{\alpha_j+1}{a_{\nu\nu}^{(1)}}\Big)^{\alpha_j}\Bigg)\Bigg] + \sum_{j=1}^{m}\sum_{\tau=1}^{m_j} c_p\Bigg[\frac{|a_{\nu\kappa}^{(j,\tau)}|}{\sqrt{2a_{\nu\nu}}}\Bigg] \ \ (\nu, \kappa = 1, \dots, n). \tag{8}$$

Then the system (7) will be globally $2p$-stable if the matrix $I_n - C$ defined by (8) is inverse-positive. Here c_p is the universal constant from the estimate (6).

In this case one uses the constant weigth function $\gamma(t) = 1$ and an ordinary scalar equation (5).

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A Version of Anti-Perron Effect of Changing Positive Exponents of Linear Approximation to Negative Ones Under Perturbations of Higher Order of Smallness

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We consider the linear differential systems

$$\dot{x} = A(t)x, \quad x \in R^n, \quad t \geq t_0, \tag{1}$$

with bounded infinitely differentiable coefficients and characteristic exponents $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$. Along with them, we consider the nonlinear systems

$$\dot{y} = A(t)y + f(t,y), \quad y \in R^n, \quad t \geq t_0, \tag{2}$$

with m-perturbations $f(t,y)$ also with infinitely differentiable coefficients of order $m > 1$ smallness in the neighbourhood of the origin $y = 0$ and admissible growth outside it:

$$\|f(t,y)\| \leq C_f \|y\|^m, \quad m > 1, \quad C_f = const, \quad y \in R^n, \quad t \geq t_0. \tag{3}$$

Perron's effect [6], [5, pp. 50–51] in a two-dimensional case establishes the existence of system (1) with negative exponents and 2 – perturbation (3) such that all nontrivial solutions of the two-dimensional system (2) are infinitely extendable to the right, and a part of them have coinciding positive exponents, and the remaining, nonempty part, has a negative exponent. This effect of changing negative exponents of system (1) to positive for solutions of system (2) is investigated by us (including the joint work with S. K. Korovin) in a cycle of works [1,2] which are completed by a full description of the sets of positive and negative (and in their absence) exponents of all nontrivial solutions of system (2).

Of greater interest for its possible applications is the anti-Perron effect [3,4], i.e., the effect of changing all positive exponents of linear approximation (1) to negative ones for the solutions of perturbed systems with small perturbations (with linear exponentially decreasing and tending to zero at infinity; nonlinear of higher order of smallness). Moreover, in [3], the change of exponents

$$\lambda_1(A) > 0 \mapsto \lambda_{n-1}(A+Q) < 0 < \lambda_n(A+Q)$$

is realized by exponentially decreasing linear perturbations $f(t,y) = Q(t)y$ (the case $\lambda_n(A+Q) < 0$ remains open), while in [4] – a complete change of exponents $\lambda_1(A) > 0 \mapsto \lambda_n(A+Q) < 0$ is realized by perturbations $Q(t) \to 0$ for $t \to +\infty$.

In this report, we have realized the following version of the anti-Perron effect of changing the positive exponents of the two-dimensional linear approximation (1) to a negative one for a nontrivial solution of the nonlinear system (2) with m-perturbation (3).

The following theorem is valid.

Theorem. *For any parameters $m > 1$, $\theta > 1$ and $\lambda > 0$ there exist:*

1) *two-dimensional linear system* (1) *with a bounded infinitely differentiable matrix of coefficients $A(t)$ and characteristic exponents $\lambda_1(A) = \lambda_2(A) = \lambda > 0$;*

2) *also infinitely differentiable with respect to its arguments m-perturbation*

$$f(t, y) : [t_0, +\infty) \times R^2 \to R^2,$$

such that the perturbed nonlinear system (2) *has a solution $y(t)$ with the Lyapunov exponent*

$$\lambda[y] = -\lambda \, \frac{\theta + 1}{m\theta - 1} < 0.$$

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On Decomposition Method for Bitsadze–Samarskii Nonlocal Boundary Value Problem for Nonlinear Two-Dimensional Second Order Elliptic Equations

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The present note is devoted to the Bitsadze–Samarskii nonlocal boundary value problem for nonlinear two-dimensional second-order elliptic equations. The sequential and parallel domain decomposition algorithms are considered.

The different kinds of problems with nonlocal boundary conditions arise very often. Nonlocal boundary value problems are quite an interesting generalization of classical problems and at the same time, they are naturally obtained when constructing mathematical models of real processes and phenomena in physics, engineering, sociology, ecology, etc. (see, for example, [1, 3, 5, 10] and the references therein).

The nonlocal problems for ordinary differential equations, elliptic and other models are studied in many works (see, for example, [1–3, 5–7, 18] and the references therein). One of the main publications in this direction is work [3] by A. Bitsadze and A. Samarskii, in which by means of the method of integral equations the theorems of existence and uniqueness of a solution for the second-order multi-dimensional elliptic equations are proved. There are given some classes of problems for which the proposed method also works.

Numerous scientific papers deal with the investigation and numerical solution of problems considered in [3] and their modifications and generalizations. Many scientific papers are devoted to the construction and investigation of discrete analogs of the above-mentioned models. One of the first among them was the work [6] where the iterative method of proving the existence of a solution for the Laplace equation was proposed. By the approach proposed in the work [6], the nonlocal problem reduced to the classical Dirichlet problems, which yields the possibility to apply the elaborated effective methods for the numerical resolution of these problems. After this work, many scientists have been investigating nonlocal problems by using the same or different methods for elliptic equations and, among them, nonlinear models as well (see, for example, [1,2,5,7–14,16] and the references therein). Nevertheless, there are still many open questions in this direction.

It is well known that, in order to find the approximate solutions, it is important to construct useful cost-effective algorithms. For constructing such algorithms, the method of domain decomposition has great importance (see, for example, [19] and the references therein). There are several reasons why the domain decomposition techniques might be attractive. Applying this method, the whole problem can be reduced to relative subproblems on the domains which are comparatively less in size than the one considered at the beginning. At the same time, it's worth noting that, in addition to the sequential count algorithm on each of these domains, it is often possible to apply a parallel count algorithm as well. In the works [10–14, 16] domain decomposition method based on the Schwarz alternative method [4] is given for the study of nonlocal problems for Laplace [11–14,16] and nonlinear elliptic equations [10].

It is well known how a great role takes place variational formulation of boundary problems in modern mathematics. This question for nonlocal elliptic problems is at the beginning of study so far (see, for example, [13, 14] and the references therein).

The results of this paper are partially published in the work [10].

The outline of this note is as follows. The Bitsadze–Samarskii nonlocal boundary value problem for the nonlinear-second order two-dimensional elliptic equation in a rectangle is considered. The convergence of the Schwarz-type iterative sequential algorithm as well as the same question for the parallel algorithm is studied.

In the plane Oxy, let us consider the rectangle $G=\{(x,y)|-a<x<0,\ 0<y<b\}$, where a and b are the given positive constants. We denote the boundary of the rectangle G by ∂G and the intersection of the line $x=t$ with the set $\overline{G}=G\cup\partial G$ by Γ_t correspondingly.

Consider the following nonlocal Bitsadze–Samarskii boundary value problem:

$$\begin{gathered}
F\Big(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial x\partial y},\frac{\partial^2 u}{\partial y^2}\Big)=0,\\
u(x,y)\big|_{\Gamma}=0,\\
u(x,y)\big|_{\Gamma_{-\xi}}=u(x,y)\big|_{\Gamma_0},
\end{gathered}\tag{1}$$

where $\Gamma=\partial G\setminus\Gamma_0$, $\xi\in(0,a)$; $u(x,y)\in C(\overline{G})\cap C^2(G)$ is an unknown function, F is the analytic function of its arguments: u, $u_x=p$, $u_y=q$, $u_{xx}=r$, $u_{xy}=s$, $u_{yy}=t$ and

$$4F_rF_t-F_s^2\geq const>0,\quad F_u\leq 0.$$

For the problem (1) let's consider the following sequential iterative procedure:

$$\begin{gathered}
F\Big(x,y,u_1^k,\frac{\partial u_1^k}{\partial x},\frac{\partial u_1^k}{\partial y},\frac{\partial^2 u_1^k}{\partial x^2},\frac{\partial^2 u_1^k}{\partial x\partial y},\frac{\partial^2 u_1^k}{\partial y^2}\Big)=0,\ \ (x,y)\in G_1,\\
u_1^k(x,y)\big|_{\Gamma^1}=0,\quad u_1^k(x,y)\big|_{\Gamma_{-\xi_1}}=u_2^{k-1}(x,y)\big|_{\Gamma_{-\xi_1}},
\end{gathered}\tag{2}$$

$$\begin{gathered}
F\Big(x,y,u_2^k,\frac{\partial u_2^k}{\partial x},\frac{\partial u_2^k}{\partial y},\frac{\partial^2 u_2^k}{\partial x^2},\frac{\partial^2 u_2^k}{\partial x\partial y},\frac{\partial^2 u_2^k}{\partial y^2}\Big)=0,\ \ (x,y)\in G_2,\\
u_2^k(x,y)\big|_{\Gamma^2}=0,\quad u_2^k(x,y)\big|_{\Gamma_{-\xi}}=u_2^k(x,y)\big|_{\Gamma_0}=u_1^k(x,y)\big|_{\Gamma_{-\xi}},\\
k=1,2,\dots\,.
\end{gathered}\tag{3}$$

Here we utilize the following notations:

$$G_1=\{(x,y)\mid\ -a<x<-\xi_1,\ \ 0<y<b\},\quad G_2=\{(x,y)\ -\xi<x<0,\ \ 0<y<b\},$$

where $-\xi_1$ is a fixed point of the interval $(-\xi,0)$, $\Gamma^1=\partial G_1\setminus\Gamma_{-\xi_1}$, $\Gamma^2=\partial G_2\setminus(\Gamma_{-\xi}\cup\Gamma_0)$ and $u_2^0(-\xi_1,y)\equiv 0$.

The iterative procedure (2), (3) reduces the nonlocal nonclassical problem (1) to the sequence of classical Dirichlet boundary value problems on every step of the iteration.

As we have already noted, algorithm (2), (3) for the solution of the problem (1) has a sequential form. Now, let us consider one more approach to the solution of the problem (1). In this case, the search for approximate solutions on domains G_1 and G_2 will be carried out not by means of a sequential algorithm, but in a parallel way.

Consider the following parallel iterative process:

$$\begin{gathered}F\Big(x,y,u_1^k,\frac{\partial u_1^k}{\partial x},\frac{\partial u_1^k}{\partial y},\frac{\partial^2 u_1^k}{\partial x^2},\frac{\partial^2 u_1^k}{\partial x\partial y},\frac{\partial^2 u_1^k}{\partial y^2}\Big)=0,\ \ (x,y)\in G_1,\\ u_1^k(x,y)\big|_{\Gamma^1}=0,\quad u_1^k(x,y)\big|_{\Gamma_{-\xi_1}}=u_2^{k-1}(x,y)\big|_{\Gamma_{-\xi_1}},\end{gathered}\tag{4}$$

$$\begin{gathered}F\Big(x,y,u_2^k,\frac{\partial u_2^k}{\partial x},\frac{\partial u_2^k}{\partial y},\frac{\partial^2 u_2^k}{\partial x^2},\frac{\partial^2 u_2^k}{\partial x\partial y},\frac{\partial^2 u_2^k}{\partial y^2}\Big)=0,\ \ (x,y)\in G_2,\\ u_2^k(x,y)\big|_{\Gamma^2}=0,\quad u_2^k(x,y)\big|_{\Gamma_{-\xi}}=u_2^k(x,y)\big|_{\Gamma_0}=u_1^{k-1}(x,y)\big|_{\Gamma_{-\xi}},\end{gathered}\tag{5}$$

$$k=1,2,\dots,$$

where $u_1^0(-\xi,0)\equiv u_2^0(-\xi_1,0)\equiv 0$.

The following statements are true.

Theorem 1. *The sequential iterative process* (2), (3) *converges to a solution of the problem* (1) *uniformly in the domain* $\overline{G}$.

Theorem 2. *The parallel iterative process* (4), (5) *converges to a solution of the problem* (1) *uniformly in the domain* $\overline{G}$.

Remark 1. In the case of the Poisson equation, in Theorem 1 the following estimations are valid too for the sequential iterative process (2), (3):

$$|u(x,y)-u_1^k(x,y)|\le Cq^{k-1},\ \ (x,y)\in\overline{G}_1,$$
$$|u(x,y)-u_2^k(x,y)|\le Cq^{k-1},\ \ (x,y)\in\overline{G}_2,$$

and

$$|u(x,y)-u_1^k(x,y)|\le Cq^{\frac{k}{2}-1},\ \ (x,y)\in\overline{G}_1,$$
$$|u(x,y)-u_2^k(x,y)|\le Cq^{\frac{k}{2}-1},\ \ (x,y)\in\overline{G}_2,$$

for the parallel iterative process (4), (5).

Here $q\in(0,1)$ and C are constants independent of functions: $u(x,y), u_1^k(x,y), u_2^k(x,y)$.

Remark 2. The Bitsadze–Samarskii nonlocal boundary value problem for the above-mentioned nonlinear equation by using iterative process analogical to (2), (3) at first was studied in [14].

Remark 3. Theorems analogous to the above Theorems 1 and 2 are valid for the sequential as well as parallel algorithms for multi-grid domain decomposition case too.

Acknowledgments

This work is supported by the Shota Rustaveli National Science Foundation (SRNSFG), # FR-21-2101.

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Nonoscillation Theory of Nonlinear Differential Equations of Emden–Fowler Type with Variable Exponents

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1 Introduction

Since the publication of the pioneering papers by Koplatadze [2, 3], Koplatadze and Kvinikadze [4] and Yoshida [5, 6] there has been an increasing interest in the qualitative study of nonlinear differential equations with variable exponents. See also Došlá and Fujimoto [1].

In this lecture we take up second order Emden–Fowler type differential equations of the form

$$\big(p(t)\varphi_{\alpha(t)}(x')\big)' + q(t)\varphi_{\beta(t)}(x) = 0, \tag{A}$$

for which it is assumed that

(a) the coefficients $p(t)$ and $q(t)$ are positive continuous functions on $I = [a, \infty)$, $a \geq 0$;

(b) the exponents $\alpha(t)$ and $\beta(t)$ are positive continuous functions on I which tend to the non-zero limits $\alpha(\infty)$ and $\beta(\infty)$, respectively, as $t \to \infty$ in the extended real number system;

(c) the symbol $\varphi_\gamma(t)$ with a positive continuous function $\gamma(t)$ on I denotes the operator on $C(I)$ defined by
$$\varphi_{\gamma(t)}(u(t)) = |u(t)|^{\gamma(t)} \operatorname{sgn} u(t), \ \ u \in C(I).$$

We are concerned exclusively with nonoscillatory solutions of equation (A), that is, those solutions $x(t)$ of (A) which are defined on an interval of the form $J = [T, \infty)$, $T \geq a$, and eventually positive or negative there. For any solution $x(t)$ of (A) we define

$$D_\alpha x(t) = p(t)\varphi_{\alpha(t)}(x'(t)),$$

and call it the *quasi-derivative* of $x(t)$. It is easy to see that if $x(t)$ is a nonoscillatory solution of (A) on J, then (A) implies that its quasi-derivative $D_\alpha x(t)$ is eventually monotone on J so that $x'(t)$ is eventually of constant sign, and this means that $x(t)$ is eventually monotone on J. Thus it turns out that both $D_\alpha x(t)$ and $x(t)$ have the limits as $t \to \infty$ in the extended real number system. The pair of these limits $(x(\infty), D_\alpha x(\infty))$ is referred to as the *terminal state* of the solution $x(t)$. The terminal state of $x(t)$ can be a crucial indicator of the asymptotic behavior of $x(t)$ as $t \to \infty$.

Given an equation of the form (A), consider the set of all terminal states of its nonoscillatory solutions. This set is divided into a finite number of subsets, each of which claims its own pattern (or type) of asymptotic behavior shared by all members of that subset. It is expected that all these patterns specific to (A), once precisely analyzed, will provide us with a deeper insight into the overall asymptotic behavior at infinity of solutions of (A). As the object of our investigation we choose two classes of equations of the form (A), equations of category I and category II, which are defined in terms of the integrals

$$I_p = \int_a^\infty p(t)^{-\frac{1}{\alpha(t)}}\, dt \ \text{ and } \ I_q = \int_a^\infty q(t)\, dt,$$

as follows. Equation (A) is said to be of category I or of category II according to whether $I_p = \infty$ and $I_q < \infty$ or $I_p < \infty$ and $I_q = \infty$, respectively. In this work we focus our attention on equations of these two categories, leaving equations of the remaining categories for later studies. Equation (A) of category I is studied in Section 2, where it turns out that there are three different patterns of terminal states of solutions of (A). This means that the entirety of solutions of (A) can be divided into three groups whose members exhibit different astmptotic behaviors as $t \to \infty$. Our most important task is to answer the question about the existence of solutions of (A) having these three patterns of asymptotic behavior. As it turns out, the question is too difficult to gain a complete answer. Section 3 is devoted to the study of equation (A) of category II. This equation of new category can also be handled by way of the standard analysis as developed in Section 1. However, we present here a surprisingly convenient means named *Duality Principle* which makes it possible to derive the desired results for equations of category II *almost automatically* from the results already known for equations of category I.

2 Nonoscillatory solutions of equation (A) of category I

We start with equation (A) of category I. Use is made of the following functions:

$$P_\alpha(t) = \int_a^t p(s)^{-\frac{1}{\alpha(s)}}\, ds, \quad \rho(t) = \int_t^\infty q(s)\, ds, \quad t \geq a.$$

It is clear that $P_\alpha(t) \to \infty$ and $\rho(t) \to 0$ as $t \to \infty$.

Let $x(t)$ be any nonoscillatory solution of (A) on $J = [T, \infty)$, $T \geq a$. We may assume without loss of generality that $x(t) > 0$ on J. Then, it can be shown that $D_\alpha x(t) > 0$ on J, so that the terminal states of $x(t)$ is divided into the following three patterns:

I-(i) $\{x(\infty) = \infty,\ < D_\alpha x(\infty) < \infty\}$,

I-(ii) $\{x(\infty) = \infty,\ D_\alpha x(\infty) = 0\}$,

I-(iii) $\{0 < x(\infty) < \infty,\ D_\alpha x(\infty) = 0\}$.

A solution of (A) having the asymptotic pattern I-(i), I-(ii) or I-(iii) is named a *maximal solution*, an *intermediate solution* or a *minimal solution* of (A). Note that maximal and intermediate solutions are unbounded on J. It is important to recognize that the order of growth of a maximal solution $x(t)$ of (A) as $t \to \infty$ is precisely determined by the value $D_\alpha x(\infty)$ as follows:

$$D_\alpha x(\infty) = d \in (0, \infty) \implies \lim_{t \to \infty} \frac{x(t)}{P_\alpha(t)} = d^{\frac{1}{\alpha(\infty)}}.$$

On the other hand, as for an intermediate solution $x(t)$ of (A) nothing precise can be said about its growth at infinity except that it satisfies $\lim_{t\to\infty} x(t)/P_\alpha(t) = 0$ because of $D_\alpha x(\infty) = 0$.

Our primary goal in this section is to characterize the existence of solutions with three different asymptotic patterns. More specifically, we want to find necessary and sufficient conditions for (A) to have maximal, intermediate and minimal solutions. This, however, is a difficult task in general. The first result concerns necessary conditions for the existence of maximal solutions of (A).

Theorem 2.1. *Let* (A) *be of category* I. *Suppose that* (A) *has a maximal solution* $x(t)$ *such that*

$$\lim_{t\to\infty} D_\alpha x(t) = d \ \text{ or equivalently } \ \lim_{t\to\infty} \frac{x(t)}{P_\alpha(t)} = d^{\frac{1}{\alpha(\infty)}} \ \text{ for some constant } \ d > 0. \tag{2.1}$$

(i) *If* $d > 1$, *then it holds that*

$$\int_a^\infty q(t) P_\alpha(t)^{\beta(t)}\, dt < \infty. \tag{2.2}$$

(ii) *Let the condition* $\beta(\infty) < \infty$ *be added to (A). If* $0 < d \le 1$, *then* (2.2) *is satisfied.*

The second result gives sufficient conditions for (A) to have maximal solutions.

Theorem 2.2. *Let* (A) *be of category* I. *Suppose that* (2.2) *is satisfied.*

(i) *Equation* (A) *has a maximal solution* $x(t)$ *satisfying* (2.1) *for any given* $d < 1$.

(ii) *Equation* (A) *with* $\beta(\infty) < \infty$ *has a maximal solution* $x(t)$ *satisfying* (2.1) *for any given* $d \ge 1$.

From the above two theorems combined we have the following result characterizing the existence of maximal solutions for (A).

Theorem 2.3. *Let* (A) *be of category* I. *Assume that* $\beta(\infty) < \infty$. *Then,* (A) *has a maximal solution* $x(t)$ *satisfying* (2.1) *for any positive constant* d *if and only if* (2.2) *is satisfied.*

Let us turn our attention to minimal solutions $x(t)$ of equation (A) having the asymptotic pattern

$$\lim_{t\to\infty} D_\alpha x(t) = 0, \quad \lim_{t\to\infty} x(t) = \omega \ \text{ for some constants } \ \omega \ne 0. \tag{2.3}$$

Such solutions can also be handled in essentially the same way as maximal solutions, and we are led to the following results.

Theorem 2.4. *Let* (A) *be of category* I. *Suppose that* (A) *has a minimal solution* $x(t)$ *satisfying* (2.3) *for some non-zero constant* ω.

(i) *If* $|\omega| > 1$, *then it holds that*

$$\int_a^\infty \big(p(t)^{-1}\rho(t)\big)^{\frac{1}{\alpha(t)}}\, dt < \infty. \tag{2.4}$$

(ii) *Let the condition* $\beta(\infty) < \infty$ *be added to* (A). *If* $0 < |\omega| \le 1$, *then* (2.4) *is satisfied.*

Theorem 2.5. *Let* (A) *be category* I. *Suppose that* (2.4) *is satisfied.*

(i) *Equation* (A) *has a minimal solution* $x(t)$ *satisfying* (2.3) *for any given* ω *with* $0 < |\omega| \le 1$.

(ii) *Equation* (A) *with* $\beta(\infty) < \infty$ *has a minimal solution* $x(t)$ *satisfying* (2.3) *for any given* ω *with* $|\omega| > 1$.

Theorem 2.6. *Let* (A) *be of category* I. *Assume that* $\beta(\infty) < \infty$. *Then,* (A) *has a minimal solution* $x(t)$ *satisfying* (2.3) *for any positive constant* ω *if and only if* (2.4) *is satisfied.*

The analysis of intermediate solutions seems to be extremely difficult. What we have been able to achieve so far is to prove the existence of such a solution $x(t)$ satisfying

$$\lim_{t\to\infty} x(t) = \infty \ \text{ and } \ \lim_{t\to\infty} \frac{x(t)}{P_\alpha(t)} = 0,$$

only for the sublinear case of equation (A). We call equation (A) *sublinear* if $\alpha(t)$ decreases to $\alpha(\infty) > 0$, $\beta(t)$ increases to $\beta(\infty) > 0$ and $\alpha(\infty) > \beta(\infty)$. Our result reads:

Theorem 2.7. *Let* (A) *be of category* I *and sublinear. There exists an intermediate solution of* (A) *if*

$$\int_a^\infty \left(p(t)^{-1}\rho(t)\right)^{\frac{1}{\alpha(t)}} dt = \infty \ \text{ and } \ \int_a^\infty q(t)P_\alpha(t)^{\beta(t)}\, dt < \infty.$$

3 Nonoscillatory solutions of equation (A) of category II

This section is concerned with equation (A) of category II. Naturally this equation of new category can also be analyzed by the method similar to that employed in Section 2 for equations of category I. Here we avoid the routine approach, but instead we introduce a surprisingly convenient means called *Duality Principle* that makes it possible to derive all the desired asymptotic results for category II equations almost automatically from the known results for category I equations.

Let there be given equation (A). Putting $y(t) = -p(t)\varphi_{\alpha(t)}(x'(t))$ split (A) into the cyclic differential system

$$x'(t) = -p(t)^{-\frac{1}{\alpha(t)}}\, \varphi_{\frac{1}{\alpha(t)}}(y(t)), \quad y'(t) = q(t)\varphi_{\beta(t)}(x(t)), \tag{3.1}$$

and eliminate $x(t)$ and $x'(t)$ from (3.1). We then obtain the following differential equation for $y(t)$:

$$\left(q(t)^{-\frac{1}{\beta(t)}}\varphi_{\frac{1}{\beta(t)}}(y')\right)' + p(t)^{-\frac{1}{\alpha(t)}}\varphi_{\frac{1}{\alpha(t)}}(y) = 0. \tag{B),}$$

Equation (B) is called the *reciprocal equation* of (A). Equation (B) is of the same type as (A) with different exponents $\alpha^*(t) = 1/\beta(t)$ and $\beta^*(t) = 1/\alpha(t)$. It is clear that the assumption (b) required for equation (A) is also satisfied for equation (B). As is easily seen, (A) is the reciprocal equation of (B).

A simple but noteworthy relationship between (A) and (B) called the duality principle will play a vital role in the whole development of this section.

Duality Principle

If equation (A) is of category I (resp. category II), then equation (B) is of category II (resp. category I), and vice versa.

We start with equation (A) for $x(t)$ of category II. To gain information about the asymptotic properties of its solutions, proceed as follows. First, form the reciprocal equation (B) for $y(t)$. Since (B) is of category I, all the results obtained in Section 1 can be applied to (B) so that we have a list of the main theorems describing the asymptotic properties of solutions $y(t)$ of (B). All the theorems in the list need to be rewritten as the statements regarding solutions $x(t)$ of equation (A). The new theorems thus obtained provide the asymptotic results we want to establish for equations of category II.

In the process of rewriting it is imperative to make correct use of the precise relationship between the data on (A) and those on (B) which are generated by (3.1) or equivalently by $y(t) = -D_\alpha x(t)$ and $x(t) = D_{\frac{1}{\beta}} y(t)$. Some of the main results obtained for (A) of category II via the duality principle are mentioned below.

Classification of solutions

Let (A) be of category II. If $x(t)$ is a positive solution on J of (A), then $D_\alpha x(t) < 0$ on J and its terminal state is one of the three patterns:

II-(i): $0 < x(\infty) < \infty$, $D_\alpha x(\infty) = -\infty$;

II-(ii): $x(\infty) = 0$, $D_\alpha x(\infty) = -\infty$;

II-(iii): $x(\infty) = 0$, $-\infty < D_\alpha x(\infty) < 0$.

A solution satisfying II-(i), II-(ii) or II-(iii) are called, respectively, a maximal solution, an intermediate solution, or a minimal solution of equation (A) of category II.

Using the functions

$$\pi_\alpha(t) = \int_t^\infty p(s)^{-\frac{1}{\alpha(s)}}\,ds, \quad Q(t) = \int_a^t q(s)\,ds,$$

which satisfy $\pi_\alpha(t) \to 0$ and $Q(t) \to \infty$ as $t \to \infty$, it is easy to show that the asymptotic behavior of a maximal solution or a minimal solution can be expressed as

$$\lim_{t\to\infty} x(t) = c, \quad \lim_{t\to\infty} \frac{D_\alpha x(t)}{Q(t)} = -c^{\frac{1}{\alpha(\infty)}}, \ \text{ for some constant } \ c > 0, \tag{3.2}$$

or

$$\lim_{t\to\infty} D_\alpha x(t) = -d, \quad \lim_{t\to\infty} \frac{x(t)}{\pi_\alpha(t)} = d^{\frac{1}{\alpha(\infty)}} \ \text{ for some constant } \ d > 0. \tag{3.3}$$

Existence of maximal and minimal solutions

Only the rewritten versions of Theorems 2.3 and 2.6 applied to (B) are presented here. The condition $\beta(\infty) < \infty$ needed in Theorems 2.3 and 2.6 is dispensed with for an obvious reason.

Theorem 3.1. *Equation* (A) *of category* II *has a maximal solution satisfying* (3.2) *for any* $c > 0$ *if and only if*

$$\int_a^\infty \Big(p(t)^{-1} Q(t)\Big)^{\frac{1}{\alpha(t)}}\,dt < \infty.$$

Theorem 3.2. *Equation* (A) *of category* II *has a minimal solution satisfying* (3.3) *for any* $d > 0$ *if and only if*

$$\int_a^\infty q(t)\pi_\alpha(t)^{\beta(t)}\,dt < \infty.$$

Existence of an intermediate solution

Let (A) be of category II. If in addition (A) is sublinear, then so is its reciprocal equation (B) of category I. First, apply Theorem 2.5 on equation (A) of category I to (B) and formulate a proposition on the existence of intermediate solutions $y(t)$ of (B). Then, using Duality Principle, translate the result into a theorem on intermediate solutions $x(t)$ of (A). It should read as follows:

Theorem 3.3. *Sublinear equation* (A) *of category II has an intermediate solution if*

$$\int_a^\infty q(t)\pi_\alpha(t)^{\beta(t)}\,dt = \infty \ \ \textit{and} \ \ \int_a^\infty \Big(p(t)^{-1}Q(t)\Big)^{\frac{1}{\alpha(t)}}\,dt < \infty.$$

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On the Solvability of a Periodic Problem in an Infinite Stripe for Second Order Hyperbolic Equations

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In the infinite stripe $D_T := \{(x,t) \in \mathbb{R}^2,\ x \in \mathbb{R},\ 0 < t < T\}$ of the plane of independent variables x, t we consider the problem of finding a regular solution $u = u(x,t)$ of the hyperbolic equation

$$au_{tt} + 2bu_{tx} + cu_{xx} = f(x,t), \quad (x,t) \in D_T, \ \ a,b,c := const, \ \ a \neq 0, \tag{1}$$

satisfying the periodic boundary conditions with respect to the variable t

$$u(x,0) = u(x,T), \quad u_t(x,0) = u_t(x,T), \ \ x \in \mathbb{R} := (-\infty, +\infty). \tag{2}$$

For hyperbolic equations and systems time periodic problems have been the subject of research by many authors (see, for example, works [2–6] and the references therein), in which questions of existence, absence, uniqueness and representation of solutions are studied.

Assuming that

$$b^2 - ac > 0, \ \ f \in C^1(\overline{D}_T), \tag{3}$$

the regular solution $u \in C^2(\overline{D}_T)$ of the equation (1) can be represented in the form

$$u(x,t) = \frac{\lambda_2 \varphi(x - \lambda_1 t) - \lambda_1 \varphi(x - \lambda_2 t)}{\lambda_2 - \lambda_1} + \frac{1}{\lambda_2 - \lambda_1} \int\limits_{x-\lambda_2 t}^{x-\lambda_1 t} \psi(\tau)\,d\tau + \frac{1}{a(\lambda_2 - \lambda_1)} \int\limits_{D_{x,t}} f(\xi,\tau)\,d\xi\,d\tau, \tag{4}$$

where λ_i, $i = 1,2$ by virtue (3) are the different real roots of the quadratic equation $a\lambda^2 - 2b\lambda + c = 0$ and $D_{x,t}$ is the triangular domain bounded by an axis Ox and characteristic lines of the equation (1) coming from the point $(x,t) \in D_T$ and

$$\varphi(x) := u(x,0), \quad \psi(x) := u_t(x,0), \ \ x \in \mathbb{R}.$$

By applying the representation (4), the problem (1), (2) is equivalently reduced to a system of functional equations

$$\begin{cases} \psi(x) + \dfrac{1}{\lambda_2 - \lambda_1} \Big[\lambda_1 \psi(x - \lambda_1 T) - \lambda_2 \psi(x - \lambda_2 T) + \lambda_1 \lambda_2 \varphi'(x - \lambda_1 T) - \lambda_1 \lambda_2 \varphi'(x - \lambda_2 T)\Big] \\ \qquad = \dfrac{1}{a(\lambda_2 - \lambda_1)} \displaystyle\int\limits_0^T \Big\{ -\lambda_1 f\big[x - \lambda_1(T - \tau), \tau\big] + \lambda_2 f\big[x - \lambda_2(T - \tau), \tau\big]\Big\}\,d\tau, \\ \varphi'(x) + \dfrac{1}{\lambda_2 - \lambda_1} \Big[-\lambda_2 \varphi'(x - \lambda_1 T) + \lambda_1 \varphi'(x - \lambda_2 T) - \psi(x - \lambda_1 T) + \psi(x - \lambda_2 T)\Big] \\ \qquad = \dfrac{1}{a(\lambda_2 - \lambda_1)} \displaystyle\int\limits_0^T \Big\{ f\big[x - \lambda_1(T - \tau), \tau\big] - f\big[x - \lambda_2(T - \tau), \tau\big]\Big\}\,d\tau. \end{cases} \tag{5}$$

In the notation $v := \big(\psi, \varphi'\big)$ we write the system of equations (5) in the form

$$v(x) + \sum_{i=1}^{2} A_i v(x - \lambda_i T) = F(x), \quad x \in \mathbb{R}, \tag{6}$$

where

$$A_1 := \frac{1}{\lambda_2 - \lambda_1} \begin{Vmatrix} \lambda_1 & \lambda_1\lambda_2 \\ -1 & -\lambda_2 \end{Vmatrix}, \quad A_2 := \frac{1}{\lambda_2 - \lambda_1} \begin{Vmatrix} -\lambda_2 & -\lambda_1\lambda_2 \\ 1 & \lambda_1 \end{Vmatrix}, \tag{7}$$

and

$$F(x) := \frac{1}{a(\lambda_2 - \lambda_1)} \begin{Vmatrix} \int\limits_0^T \Big\{ -\lambda_1 f\big[x - \lambda_1(T-\tau), \tau\big] + \lambda_2 f\big[x - \lambda_2(T-\tau), \tau\big] \Big\}\, d\tau \\ \int\limits_0^T \Big\{ f\big[x - \lambda_1(T-\tau), \tau\big] - f\big[x - \lambda_2(T-\tau), \tau\big] \Big\}\, d\tau \end{Vmatrix}.$$

If we introduce the notations

$$\omega_i := A_i v, \quad i = 1, 2, \tag{8}$$

by virtue of (7) and taking into account the facts that: $A_1 A_2 = A_2 A_1 = O$ and $A_i^2 := -A_i$, $i = 1, 2$, from the equation (6) with respect to the unknown functions ω_i, $i = 1, 2$, we get the following independent from each other equations

$$\omega_i(x) - A_i \omega_i(x - \lambda_i T) = A_i F(x), \quad x \in \mathbb{R}, \quad i = 1, 2. \tag{9}$$

For arbitrary $\alpha, \beta \in \mathbb{R}$ let's introduce the following spaces:

$$C_{\alpha,\beta}(\mathbb{R}) := \Big\{ v \in C(\mathbb{R}) : \sup_{x \in (-\infty, 0)} e^{-\alpha x} |v(x)| + \sup_{x \in (0, +\infty)} e^{-\beta x} |v(x)| < +\infty \Big\},$$

$$C^2_{\alpha,\beta}(\overline{D}_T) := \Big\{ u \in C^2(\overline{D}_T) : \sup_{(x,t) \in (-\infty,0) \times [0,T]} e^{-\alpha x} \big(|u(x,t)| + |u_t(x,t)| \big) + \sup_{(x,t) \in (0,+\infty) \times [0,T]} e^{-\beta x} \big(|u(x,t)| + |u_t(x,t)| \big) < +\infty \Big\},$$

$$C^1_{\alpha,\beta}(\overline{D}_T) := \Big\{ f \in C^1(\overline{D}_T), \sup_{x \in (-\infty,0) \times [0,T]} e^{-\alpha x} |f(x,t)| + \sup_{x \in (0,+\infty) \times [0,T]} e^{-\beta x} |f(x,t)| < +\infty \Big\},$$

and the notation

$$I_{\alpha,\beta} := \big[\min(\alpha, \beta), \max(\alpha, \beta) \big].$$

Remark 1. It is easy to check that from the equalities (8) the vector function v is uniquely determined if and only if

$$\lambda_1 \lambda_2 = \frac{c}{a} \neq 0. \tag{10}$$

Throughout Theorems 1–4 formulated below we will assume that the condition (10) is satisfied. Based on Bochner's results [1] regarding to the functional equation (9) in the space $C_{\alpha,\beta}(\mathbb{R})$ there are proved the following:

Theorem 1. *If $\alpha\beta > 0$, then for any right-hand side $f \in C^1_{\alpha,\beta}(\overline{D}_T)$ the problem (1), (2) has a unique solution in the space $C^2_{\alpha,\beta}(\overline{D}_T)$.*

Theorem 2. *If $\alpha < 0$ and $\beta > 0$, then for any right-hand side $f \in C^1_{\alpha,\beta}(\overline{D}_T)$ there exists a solution of the problem (1), (2) in the space $C^2_{\alpha,\beta}(\overline{D}_T)$, besides the corresponding homogeneous problem has an infinite number of linearly independent solutions in the same space.*

Theorem 3. *If $\alpha > 0$ and $\beta < 0$, then the problem* $(1),(2)$ *in the space* $C^2_{\alpha,\beta}(\overline{D}_T)$ *cannot have more than one solution and for its solvability it is necessary and sufficient that the function* $f \in C^1_{\alpha,\beta}(\overline{D}_T)$ *satisfy the following condition*

$$\Lambda_\gamma(f) = 0, \quad \gamma \in \mathbb{R},$$

where Λ_γ *- is a well-defined linear functional on the space* $C^1_{\alpha,\beta}(\overline{D}_T)$*, depending on a real parameter* γ*.*

Theorem 4. *If $\alpha\beta = 0$, then the problem* $(1),(2)$ *is not solvable even in the Hausdorff's sense in the space* $C^2_{\alpha,\beta}(\overline{D}_T)$*, when* $f \in C^1_{\alpha,\beta}(\overline{D}_T)$*, i.e. the set of functions* f *from the space* $C^1_{\alpha,\beta}(\overline{D}_T)$ *for which the problem* $(1),(2)$ *is solvable in the space* $C^2_{\alpha,\beta}(\overline{D}_T)$ *is not closed in the space* $C^1_{\alpha,\beta}(\overline{D}_T)$*.*

Remark 2. Note also that in the case, when the condition (10) is violated, i.e. when $c = 0$, to the necessary conditions for the solvability of the problem $(1),(2)$ in space $C^2_{\alpha,\beta}(\overline{D}_T)$ there will be added the following condition

$$\int\limits_0^T f(x,\tau)\,d\tau = 0 \quad \forall\, x \in \mathbb{R},$$

imposed on the function $f \in C^1_{\alpha,\beta}(\overline{D}_T)$.

Acknowledgements

This work was supported by the Shota Rustaveli National Science Foundation, Grant # FR-21-7307.

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Stability of Global Attractors for the Chafee–Infante Equation w.r.t. Boundary Disturbances

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We consider the following initial boundary-value problem:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + f(u(t,x)), \ \ t>0, \ \ x\in(0,l), \\ u\big|_{x=0} = d_1(t), \quad u\big|_{x=l} = d_2(t), \\ u\big|_{t=0} = u_0(x). \end{cases} \tag{1}$$

Here $u = u(t,x)$ is an unknown function, $f \in \mathbb{C}^1(\mathbb{R})$ is a given nonlinear function satisfying conditions

$$\begin{gathered} \exists\, C_1 > 0 \quad \forall\, s \in \mathbb{R} \quad |f(s)| \le C_1(1+|s|^3), \\ \exists\, C_2 > 0, \ \ \alpha > 0 \quad \forall\, s \in \mathbb{R} \quad f(s)\cdot s \ge -\alpha s^4 - C_2, \\ \exists\, C_3 > 0 \quad \forall\, s \in \mathbb{R} \quad |f'(s)| \le C_3(1+|s|^2). \end{gathered} \tag{2}$$

We consider bounded $d = d_1, d_2$ as a boundary disturbances.

It is well-known [6] that the corresponding undisturbed problem ($d \equiv 0$)

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + f(u(t,x)), \\ u\big|_{x=0} = u\big|_{x=l} = 0, \\ u\big|_{t=0} = u_0(x), \end{cases} \tag{3}$$

for every $u_0 \in X = L^2(0,l)$ has a unique weak solution defined on $[0,+\infty)$.

Such solutions generate semigroup $\{S(t): \ X \mapsto X\}_{t\ge 0}$ which has a global attractor $\Theta \subset X$ [6].

Definition. A compact set $\Theta \subset X$ is called a global attractor of a semigroup $\{S(t): X \mapsto X\}_{t\ge 0}$ if

- $\forall\, t \ge 0 \quad \Theta = S(t)\Theta$ (invariance);
- $\forall\, r > 0 \quad \sup\limits_{\|u_0\|\le r} \operatorname{dist}(S(t)u_0, \Theta) \to 0$ as $t \to \infty$ (attraction).

The structure of the global attractor of problem (2) can be rather complicated, but it is well understood and can be investigated by analytical and numerical methods [3,5,6].

In particular, the set Θ is bounded in $L^\infty(0,l)$ and in $H^2(0,l)$, and

$$\Theta = W^u(N),$$

where N is a set of stationary solutions of (3), and $W^u(N)$ is an unstable set emanating from N, i.e., Θ consist of points lying on complete trajectories $u(\,\cdot\,)$ of (3) such that

$$\operatorname{dist}(u(t), N) \to 0 \ \text{ as } \ t \to \infty.$$

Moreover, the global attractor is stable in the Lyapunov sense, i.e.,

$$\forall\,\varepsilon>0 \quad \exists\,\delta>0 \quad \forall\,\xi \quad \text{such that } \ \|\xi\|_\Theta := \operatorname{dist}(\xi,\Theta)<\delta$$

we have that

$$\forall\,t\geq 0 \quad \|S(t)\xi\|_\Theta<\varepsilon.$$

So, for the undisturbed problem (3), we have that all trajectories eventually get to any neighborhood of the stable invariant set Θ.

The natural question arises: does this limit behaviour remain true under the presence of disturbances? The problem is that the disturbed problem is non-autonomous, and we have no guarantee in general, that it's solutions converge to Θ as $t\to\infty$. But we can expect that such attractivity property are affected only slightly by disturbances of small magnitude [2]. In [1] it was given a positive answer for this question in the case of external disturbances, i.e. when bounded functions $d=d(t,x)$ appears in the right-hand part of equation (3).

This property, named robust stability with respect to (w.r.t.) disturbances, can be effectively described in the Input-to-State Stability (ISS) framework [4]. In this work we apply this approach to the case of boundary disturbances.

Let us introduce the following classes of functions:

$$\begin{aligned} K&=\big\{\gamma:[0,+\infty)\mapsto[0,+\infty) \mid \ \ \gamma \text{ is continuous strictly increasing, } \gamma(0)=0\big\};\\ KL&=\big\{\beta:[0,+\infty)\times[0,+\infty)\mapsto[0,+\infty) \mid \ \ \beta \text{ is continuous}\big\},\\ &\qquad \forall\,t>0 \quad \beta(\,\cdot\,,t)\in K, \quad \forall\,s>0 \quad \beta(s,\,\cdot\,) \ \text{ is strictly decreasing to zero.}\end{aligned}$$

We prove that for every $d=\{d_1,d_2\}\in L^\infty[0,+\infty)$ and for every $u_0\in X=L^2(0,l)$ problem (1) has a unique weak solution $u(t)=S_d(t,u_0)$ defined on $[0,+\infty)$.

We also prove that for a shift-invariant subset $U\subset L^\infty[0,+\infty)$ the family $\{S_d\}_{d\in U}$ generates the semiprocess family, i.e.,

$$S_d(t+h,u_0)=S_{d(\,\cdot\,+h)}(t,S_d(h,u_0)).$$

Our main results are the following:

Theorem 1. *The semiprocess family $\{S_d\}_{d\in U}$, generated by* (1)*, is locally ISS w.r.t. Θ, i.e., there exists $r>0$, $\beta\in KL$, and $\gamma\in K$ such that for any $\|u_0\|_\Theta\leq r$ and $\|d\|_\infty\leq r$ it holds that*

$$\forall\,t\geq 0 \quad \|S_d(t,u_0)\|_\Theta\leq\beta\big(\|u_0\|_\Theta,t\big)+\gamma(\|d\|_\infty). \tag{4}$$

Theorem 2. *The semiprocess family $\{S_d\}_{d\in U}$, generated by* (1)*, satisfies the asymptotic gain* (*AG*) *property w.r.t. Θ, i.e. there exists $\gamma\in K$ such that $\forall\,u_0\in X$ $\forall\,d\in U$ it holds that*

$$\limsup_{t\to\infty}\|S_d(t,u_0)\|_\Theta\leq\gamma(\|d\|_\infty). \tag{5}$$

It should be noted that the methods of proving (4) and (5) are different. To prove (4), we use Lyapunov's function technique. To prove (5), we use results on upper semicontinuity of global attractors with respect to parameters.

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Solutions of Some Type n-th Order Differential Equations

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1 Statement of the problem

We consider the following nth-order differential equation

$$\big(r(t)u^{(m)}\big)^{(n-m)} = \sum_{k=0}^{m} p_k u^{(k)}, \quad n \geq 2, \tag{1.1}$$

where $p_k \in C_{loc}([a;+\infty[)$ $(k = 0, \dots, m)$,

$$\lim_{t\to+\infty} \frac{p_0(t)}{q(t)} = \sigma, \quad \sigma = \operatorname{sign}(p_0(a)), \tag{1.2}$$

$r(t)$ and $q(t)$ are positive twice differentiable on the set $[a;+\infty[$ functions, $C_{loc}([a;+\infty[)$ is the space of locally continuous functions on the interval $[a;+\infty[$, $L([a;+\infty[)$ is the Banach space of Lebesgue integrable functions.

In case $m = 0$ we have the equation

$$\big(r(t)u^{(m)}\big)^{(n-m)} \pm qy = 0, \quad n \geq 2,$$

that was considered, and for which the asymptotic images of the solutions were obtained, in Hinton's work [2].

In case $s \equiv 1$ and $m = n - 1$ the equation

$$u^{(n)} = \sum_{k=0}^{n-1} p_k(t) u^{(k)}$$

was considered in the work by I. T. Kiguradze [3], the corresponding asymptotic images of the solutions were obtained when various conditions were imposed on the coefficients.

The purpose of this work is to establish the asymptotic images of the solutions of equation (1.1) as $t \to +\infty$.

2 Main results

The following theorem has been obtained.

Theorem. *Let for equation* (1.1) *condition* (1.2) *and the following conditions be satisfied*

$$\Big(\frac{q}{r}\Big)^{\frac{1}{n}} \notin L([a;+\infty[), \tag{2.1}$$

$$\bigg(\frac{r'}{r} \cdot \Big(\frac{q}{r}\Big)^{-\frac{1}{n}}\bigg)' \in L([a;+\infty[), \quad \bigg(\frac{q'}{q} \cdot \Big(\frac{q}{r}\Big)^{-\frac{1}{n}}\bigg)' \in L([a;+\infty[), \tag{2.2}$$

$$\left(\frac{r'}{r}\right)^2 \cdot \left(\frac{q}{r}\right)^{-\frac{1}{n}} \in L([a;+\infty[), \quad \left(\frac{q'}{q}\right)^2 \cdot \left(\frac{q}{r}\right)^{-\frac{1}{n}} \in L([a;+\infty[), \tag{2.3}$$
$$\frac{p_{k-1}(t)}{q(t)} \cdot \left(\frac{q}{r}\right)^{\frac{k-1}{n}} \in L([a;+\infty[) \;\; (k=2,\dots,m), \quad \frac{p_m(t)}{r(t)q(t)} \cdot \left(\frac{q}{r}\right)^{\frac{m}{n}} \in L([a;+\infty[).$$

Then equation (1.1) *has a fundamental system of solutions* u_j $(j = 1,\dots,n)$*, which admit the asymptotic images*

$$u_j^{k-1} = q(t)^{-\alpha_k} \cdot r(t)^{-\beta_k} \cdot \exp\left[\lambda_j \cdot \int_a^t \left(\frac{q}{r}\right)^{\frac{1}{n}}\right] \cdot \left[\lambda_j^{k-1} + o(1)\right] \;\; (k,j=1,\dots,n) \tag{2.4}$$

in which λ_j^0 *are the roots of the equation*

$$\lambda^n = \sigma. \tag{2.5}$$

To prove this theorem, the following transformations were applied to equation (1.1):

$$\begin{cases} u^{(i)}(t) = z_{i+1}(t), \;\; 0 \le i \le m-1, \\ u^{(m)}(t) = \dfrac{z_{m+1}(t)}{r(t)}, \\ (r(t)u^{(m)})^{(i-m)} = z_{i+1}(t), \;\; m+1 \le i \le n-1, \;\; m \ne n-1. \end{cases}$$

The following system of quasi-linear equations equivalent to equation (1.1) is obtained

$$\begin{cases} z'_{(i)}(t) = z_{i+1}(t), \;\; 1 \le i \le n-1, \;\; i \ne m, \\ z'_{(m)}(t) = \dfrac{z_{m+1}(t)}{r(t)}, \\ z'_n = p_0(t) z_1 + \displaystyle\sum_{i=1}^{m-1} p_i(t) \cdot z_{i+1} + \dfrac{p_m(t)}{r(t)} \cdot z_{m+1}. \end{cases} \tag{2.6}$$

Let's write system (2.6) in matrix form:

$$Z' = P \cdot Z, \tag{2.7}$$

where

$$P = (p_{ij})_1^n, \quad p_{ij} = \begin{cases} 1, & 1 \le i \le n-1, \;\; i \ne m, \;\; j = i+1, \\ \dfrac{1}{r(t)}, & i = m, \;\; j = i+1, \\ p_{i-1}, & i = n, \;\; 1 \le j \le m, \\ \dfrac{p_m}{r}, & i = n, j = m+1, \\ 0, & \text{otherwise}. \end{cases}$$

Further, the following transformation can be applied to system (2.7):

$$Z(t) = Q(t) \cdot W(t), \tag{2.8}$$

in which

$$Q(t) = \operatorname{diag}\left[q^{\alpha_1} r^{\beta_1} \cdots q^{\alpha_n} r^{\beta_n}\right].$$

As a result of transformation (2.8), we have a system

$$W' = \big[Q^{-1}PQ - Q^{-1}Q'\big] \cdot W. \tag{2.9}$$

Also we note that the following statements are true

$$Q^{-1} = \operatorname{diag}\Big[\frac{1}{q^{\alpha_1} r^{\beta_1}} \cdots \frac{1}{q^{\alpha_n} r^{\beta_n}}\Big],$$

$$P \cdot Q = (a_{ij})_1^n, \quad a_{ij} = \begin{cases} q^{\alpha_{i+1}} \cdot r^{\beta_{i+1}}, & 1 \le i \le n-1, \;\; i \ne m, \;\; j = i+1, \\ q^{\alpha_{m+1}} \cdot r^{\beta_{m+1}-1}, & i = m, \;\; j = i+1, \\ p_{i-1} \cdot q^{\alpha_i} \cdot r^{\beta_i}, & i = n, \;\; 1 \le j \le m, \\ p_m \cdot q^{\alpha_{m+1}} \cdot r^{\beta_{m+1}-1}, & i = n, \;\; j = m+1, \\ 0, & \text{otherwise}. \end{cases}$$

$$Q^{-1}PQ = (b_{ij})_1^n, \quad b_{ij} = \begin{cases} q^{\alpha_{i+1}-\alpha_i} \cdot r^{\beta_{i+1}-\beta_i}, & 1 \le i \le n-1, \;\; i \ne m, \;\; j = i+1, \\ q^{\alpha_{m+1}-\alpha_m} \cdot r^{\beta_{m+1}-\beta_m-1}, & i = m, \;\; j = i+1, \\ p_{i-1} \cdot q^{\alpha_i-\alpha_n} \cdot r^{\beta_i-\beta_n}, & i = n, \;\; 1 \le j \le m, \\ p_m \cdot q^{\alpha_{m+1}-\alpha_n} \cdot r^{\beta_{m+1}-\beta_n-1}, & i = n, \;\; j = m+1, \\ 0, & \text{otherwise}, \end{cases}$$

$$Q^{-1}Q = \frac{q'}{q} \cdot D_1 + \frac{r'}{r} \cdot D_2, \quad D_1 = \operatorname{diag}[\alpha_1, \dots, \alpha_n], \quad D_2 = \operatorname{diag}[\beta_1, \dots, \beta_n].$$

We choose α_i and β_i in such a way that

$$\begin{gathered} \alpha_2 - \alpha_1 = \alpha_3 - \alpha_2 = \cdots = \alpha_{m+1} - \alpha_m = \alpha_n - \alpha_{n-1} = 1 + \alpha_1 - \alpha_n = \tau_\alpha, \\ \beta_2 - \beta_1 = \beta_3 - \beta_2 = \cdots = \beta_{m+1} - \beta_m - 1 = \beta_n - \beta_{n-1} = \beta_1 - \beta_n = \tau_\beta. \end{gathered}$$

It follows from the last equalities that $\tau_\alpha = \frac{1}{n}$, $\tau_\beta = -\frac{1}{n}$.
Then we have the following equalities:

$$\begin{cases} \alpha_1 - \alpha_n = \dfrac{1}{n} - 1, \\ \alpha_2 - \alpha_n = \dfrac{1}{n} - \dfrac{n-1}{n}, \\ \;\dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha_{m+1} - \alpha_n = \dfrac{1}{n} - \dfrac{n-m}{n}. \end{cases}$$

Then let's $Q^{-1}PQ = (\frac{q}{r})^{\frac{1}{n}} \cdot [K+V]$, where $K = (k_{ij})_1^n$, $V = (v_{ij})_1^n$,

$$k_{ij} = \begin{cases} 1, & 1 \le i \le n-1, \;\; j = i+1, \\ \sigma, & i = n, \;\; j = 1, \\ 0, & \text{otherwise}, \end{cases} \qquad v_{ij} = \begin{cases} \dfrac{p_0}{q} - \sigma, & i = n, \;\; j = 1, \\ \dfrac{p_{i-1}}{q} \cdot \Big(\dfrac{q}{r}\Big)^{\frac{i-1}{n}}, & i = n, \;\; 2 \le j \le m, \\ \dfrac{p_m}{q} \cdot \Big(\dfrac{q}{r}\Big)^{\frac{m}{n}}, & i = n, \;\; j = m+1, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, system (2.9) turns into the following system

$$W' = \Big[\Big(\frac{q}{r}\Big)^{\frac{1}{n}} \cdot [K+V] - \frac{q'}{q} \cdot D_1 + \frac{r'}{r} \cdot D_2\Big] \cdot W. \tag{2.10}$$

We apply the following transformation to system (2.10) once again:

$$h(t)=\int\limits_a^t\Big(\frac{q(\varsigma)}{r(\varsigma)}\Big)^{\frac{1}{n}}\,d\varsigma. \tag{2.11}$$

Let, moreover, g be a function inverse of the function h and for all $t>a$, $g(h(t))=t$.

Since conditions (2.1)–(2.3) of the theorem are fulfilled, then $h(t)\to\infty$ as $t\to\infty$. We also have that $W(s)=Z(g(s))$. As a result of transformation (2.11), we obtain a system

$$W'=\big[K+V-\alpha(s)\cdot D_1+\beta(s)\cdot D_2\big]\cdot W,$$

in which

$$\alpha(s)=\Big(\frac{q(t)}{r(t)}\Big)^{-\frac{1}{n}}\,\frac{q'}{q}\,,\quad \beta(s)=\Big(\frac{q(t)}{r(t)}\Big)^{-\frac{1}{n}}\,\frac{r'}{r}.$$

It also follows from conditions (2.1)–(2.3) that

$$\int\limits_0^\infty|\alpha'(s)|\,ds=\int\limits_a^\infty\bigg|\Big(\Big(\frac{q(t)}{r(t)}\Big)^{-\frac{1}{n}}\frac{q'}{q}\Big)'\bigg|\,ds<+\infty$$

and

$$\int\limits_0^\infty\alpha^2(s)\,ds=\int\limits_a^\infty\Big(\Big(\frac{q(t)}{r(t)}\Big)^{-\frac{1}{n}}\Big(\frac{q'}{q}\Big)^2\Big)\,ds<+\infty.$$

Similar results are valid for $\beta(s)$.

From all that has been shown and taking into account the conditions of the theorem, we proved that the conditions of the well-known Levinson result are satisfied [1, Theorem 8.1], the equation (1.1) is in some sense asymptotically equivalent to the corresponding binomial differential equation of the nth order. Therefore, equation (1.1) has a fundamental system of solutions u_j $(j=1,\dots,n)$, which admit the asymptotic images (2.4).

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Antiperiodic Problem for One Class of Nonlinear Partial Differential Equations

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In the plane of variables x and t consider a nonlinear partial differential equation of the form

$$L_f u := \frac{\partial^2 u}{\partial t^2} - \frac{\partial^4 u}{\partial x^4} + \lambda \frac{\partial^2 u}{\partial x^2} + f(u) = F, \tag{1}$$

where f, F are given, while u unknown function, $\lambda = const$.

For the equation (1) we consider the following antiperiodic problem: find in the domain D_T : $0 < x < l, 0 < t < T$ a solution $u = u(x,t)$ of the equation (1) according to the boundary conditions

$$u(x,0) = -u(x,T), \quad u_t(x,0) = -u_t(x,T), \quad 0 \le x \le l, \tag{2}$$

$$\frac{\partial^i u}{\partial x^i}\,(0,t) = -\frac{\partial^i u}{\partial x^i}\,(l,t), \quad 0 \le t \le T, \quad i = 0,1,2,3. \tag{3}$$

Note that to the study of antiperiodic problems for nonlinear partial differential equations, having a structure different from (1), is devoted numerous literature (see, for example, [1–7] and the references therein).

Denote by $C^{1,2}(\overline{D}_T)$ the space of functions continuous in $\overline{D}_T$, having in $\overline{D}_T$ continuous partial derivatives $\frac{\partial^i u}{\partial t^i}$, $i = 1,2$, $\frac{\partial^j u}{\partial x^j}$, $j = 1,2,3,4$. Let

$$C^{1,2}_{-}(\overline{D}_T) := \Big\{ u \in C^{1,2}(\overline{D}_T) : \ \frac{\partial^i u}{\partial t^i}\,(x,0) = -\frac{\partial^i u}{\partial t^i}\,(x,T), \ \ 0 \le x \le l, \ \ i = 0,1, \\ \frac{\partial^j u}{\partial x^j}\,(0,t) = -\frac{\partial^j u}{\partial x^j}\,(l,t), \ \ 0 \le t \le T, \ \ j = 0,1,2,3 \Big\}.$$

Consider the Hilbert space $W^{1,2}_{-}(D_T)$ as a completion of the classical space $C^{1,2}_{-}(\overline{D}_T)$ with respect to the norm

$$\|u\|^2_{W^{1,2}_{-}(D_T)} = \int\limits_{D_T} \Big[u^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 + \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial^2 u}{\partial x^2}\Big)^2 \Big] \, dx \, dt. \tag{4}$$

Remark 1. It follows from (4) that if $u \in W^{1,2}_{-}(D_T)$ then $u \in W^1_2(D_T)$ and $\frac{\partial^2 u}{\partial x^2} \in L_2(D_T)$. Here $W^1_2(D_T)$ is the well-known Sobolev space consisting of the elements $L_2(D_T)$, having up to the first order generalized derivatives from $L_2(D_T)$.

Below, for function f in the equation (1) we require that

$$f \in C(\mathbb{R}), \ \ |f(u)| \le M_1 + M_2 |u|^\alpha, \ \ \alpha = const > 1, \ \ u \in \mathbb{R}, \tag{5}$$

where $M_i = const \ge 0$, $i = 1,2$.

Remark 2. As it is known, since the dimension of the domain $D_T \subset \mathbb{R}^2$ equals two, then the embedding operator
$$I : W_2^1(D_T) \to L_q(D_T)$$
is linear and compact operator for any fixed $q = const > 1$. At the same time the Nemitskii operator $K : L_q(D_T) \to L_2(D_T)$, acting by formula $Ku = f(u)$. where $u \in L_q(D_T)$, and function f satisfies the condition (5) is bounded and continuous, when $q \geq 2\alpha$. Therefore, if we take $q = 2\alpha$ then the operator
$$K_0 = KI : W_2^1(D_T) \to L_2(D_T)$$
will be continuous and compact. Whence, in particular, we have that if $u \in W_2^1(D_T)$, then $f(u) \in L_2(D_T)$ and from $u_n \to u$ in the space $W_2^1(D_T)$ it follows $f(u_n) \to f(u)$ in the space $L_2(D_T)$.

Remark 3. Let $u \in C_-^{1,2}(\overline{D}_T)$ be a classical solution of the problem (1)–(3). Multiplying the both sides of the equation (1) by an arbitrary function $\varphi \in C_-^{1,2}(\overline{D}_T)$ and integrating obtained equality over the domain D_T with taking into account that the functions from the space $C_-^{1,2}(\overline{D}_T)$ satisfy the boundary conditions (2) and (3), we get

$$\int\limits_{D_T} \Big[\frac{\partial u}{\partial t}\,\frac{\partial \varphi}{\partial t} + \frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2 \varphi}{\partial x^2} + \lambda\,\frac{\partial u}{\partial x}\,\frac{\partial \varphi}{\partial x}\Big]\,dx\,dt - \int\limits_{D_T} f(u)\varphi\,dx\,dt = -\int\limits_{D_T} F\varphi\,dx\,dt \quad \forall\,\varphi \in C_-^{1,2}(\overline{D}_T). \tag{6}$$

We take the equality (6) as a basis of definition of a weak generalized solution of the problem (1)–(3).

Definition 1. Let a function f satisfy the condition (5). A function $u \in W_-^{1,2}(D_T)$ is named a weak generalized solution of the problem (1)–(3), if the integral equality (6) holds for any function $\varphi \in W_-^{1,2}(\overline{D}_T)$, i.e.

$$\int\limits_{D_T} \Big[\frac{\partial u}{\partial t}\,\frac{\partial \varphi}{\partial t} + \frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2 \varphi}{\partial x^2} + \lambda\,\frac{\partial u}{\partial x}\,\frac{\partial \varphi}{\partial x}\Big]\,dx\,dt - \int\limits_{D_T} f(u)\varphi\,dx\,dt = -\int\limits_{D_T} F\varphi\,dx\,dt \quad \forall\,\varphi \in W_-^{1,2}(D_T). \tag{7}$$

Note that due to Remark 2 the integral $\int\limits_{D_T} f(u)\varphi\,dx\,dt$ in the left-hand side of the equality (7) is defined correctly since from $u \in W_-^{1,2}(D_T)$ it follows that $f(u) \in L_2(D_T)$, and since $\varphi \in L_2(D_T)$, then $f(u)\varphi \in L_1(D_T)$.

It is easy to see that if a weak generalized solution u of the problem (1)–(3) in the sense of Definition 1 belongs to the class $C_-^{1,2}(\overline{D}_T)$, then it is a classical solution to this problem.

Under fulfillment of the condition
$$\lambda \geq 0 \tag{8}$$
in the space $C_-^{1,2}(\overline{D}_T)$ together with the scalar product

$$(u,v)_0 = \int\limits_{D_T} \Big[uv + \frac{\partial u}{\partial t}\,\frac{\partial v}{\partial t} + \frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x} + \frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2 v}{\partial x^2}\Big]\,dx\,dt \tag{9}$$

with norm $\|\cdot\|_0 = \|\cdot\|_{W_-^{1,2}(D_T)}$, defined by the right-hand side of the equality (4), let us consider the following scalar product

$$(u,v)_1 = \int\limits_{D_T} \Big[\frac{\partial u}{\partial t}\,\frac{\partial v}{\partial t} + \frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2 v}{\partial x^2} + \lambda\,\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x}\Big]\,dx\,dt \tag{10}$$

with the norm

$$\|u\|_1^2 = \int\limits_{D_T} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \lambda \left(\frac{\partial u}{\partial x}\right)^2 \right] dx, dt, \tag{11}$$

where $u, v \in C_-^{1,2}(\overline{D}_T)$.

The following inequalities

$$c_1\|u\|_0 \le \|u\|_1 \le c_2\|u\|_0 \quad \forall\, u \in C_-^{1,2}(\overline{D}_T)$$

with positive constants c_1 and c_2, independent of u, are valid. Whence due to (8)–(11) it follows that if we complete the space $C_-^{1,2}(\overline{D}_T)$ with respect to the norm (11), then we obtain the same Hilbert space $W_-^{1,2}(D_T)$ with the equivalent scalar products (9) and (10). Using this circumstance, one can prove the unique solvability of the linear problem corresponding to (1)–(3), when $f = 0$, i.e. for any $F \in L_2(D_T)$ there exists a unique solution $u = L_0^{-1}F \in W_-^{1,2}(D_T)$ to this problem, where the linear operator

$$L_0^{-1} : L_2(D_T) \to W_-^{1,2}(D_T)$$

is continuous.

Remark 4. From the above reasoning, it follows that the nonlinear problem (1)–(3) is reduced equivalently to the functional equation

$$u = L_0^{-1}[f(u) - F] \tag{12}$$

in the Hilbert space $W_-^{1,2}(D_T)$.

Supposing that

$$\lim_{|u|\to\infty} \sup \frac{f(u)}{u} \le 0, \tag{13}$$

it can be proved a priori estimate for the solution of the functional equation (12) in the space $W_-^{1,2}(D_T)$, whence, due to Remarks 2 and 4, it follows the existence of the solution of the equation (12), and, therefore, of the problem (1)–(3) in the specified space. Thus, the following theorem is valid.

Theorem 1. *Let the conditions* (5)*,* (8) *and* (13) *be fulfilled. Then for any $F \in L_2(D_T)$ the problem* (1)*–*(3) *has at least one weak generalized solution u in the space $W_-^{1,2}(D_T)$.*

Note that the monotonicity of the function f can provide the uniqueness of the solution of the problem (1)–(3).

Theorem 2. *If the conditions* (5)*,* (8) *are fulfilled and f is a non-strictly decreasing function, i.e.*

$$(f(s_2) - f(s_1))(s_2 - s_1) \le 0 \quad \forall\, s_1, s_2 \in \mathbb{R}, \tag{14}$$

then for any $F \in L_2(D_T)$ the problem (1)*–*(3) *can not have more than one weak generalized solution in the space $W_-^{1,2}(\overline{D}_T)$.*

These theorems imply the following theorem.

Theorem 3. *Let the conditions* (5)*,* (8) *and* (13)*,* (14) *be fulfilled. Then for any $F \in L_2(D_T)$ the problem* (1)*–*(3) *has a unique weak generalized solution u in the space $W_-^{1,2}(D_T)$.*

Note that if the condition (13) is violated, then the problem (1)–(3) may be unsolvable. Indeed, there is valid the following theorem.

Theorem 4. *Let the function f satisfy the conditions* (5), (8) *and*

$$f(u) \leq -|u|^{\alpha} \quad \forall\, u \in \mathbb{R}, \;\; \alpha = const > 1, \tag{15}$$

and the function $F = \mu F_0$, where $F_0 \in L_2(D_T)$, $F_0 > 0$ in the domain D_T, $\mu = const > 0$. Then there exists a number $\mu_0 = \mu_0(F_0, \alpha) > 0$ such that for $\mu > \mu_0$ the problem (1)–(3) *can not have a weak generalized solution in the space $W^{1,2}_{-}(D_T)$.*

It is easy to see that when the condition (15) is fulfilled, then the condition (13) is violated.

Acknowledgements

The work was supported by the Shota Rustavely National Science Foundation, Grant # FR-21-7307.

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Blow-Up Solutions of the Cauchy Problem for Nonlinear Delay Ordinary Differential Equations

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Problems on the existence and asymptotic estimates of blow-up solutions occupy an important place in the qualitative theory of ordinary differential equations and have been studied in sufficient detail for a wide class of nonlinear nonautonomous ordinary differential equations (see [1–9] and the references therein). However, for delay differential equations this problem remained practically unstudied. Most probably, [10] is the first work done in this direction. Here theorems on the existence of blow-up solutions are proved for the equation that does not contain intermediate derivatives. In the present paper, similar results are given for the equation of general type.

On a finite interval $[0,b[$ we investigate the delay differential equation

$$u^{(n)}(t)=f\big(t,u(\tau(t)),\dots,u^{(n-1)}(\tau(t))\big) \tag{1}$$

with the initial conditions

$$u^{(i-1)}(t)=c_i(t) \ \text{ for } \ a\leq t\leq 0 \ \ (i=1,\dots,n). \tag{2}$$

Here n is an arbitrary natural number, $f:[0,b]\times\mathbb{R}_+^n\to\mathbb{R}_+$ is a continuous function, $\mathbb{R}_+=[0,+\infty[$, $\tau:[0,b]\to\mathbb{R}$ is a continuous function, satisfying the conditions

$$\begin{gathered}\tau(t)<t \ \text{ for } \ 0\leq t<b, \quad \tau(b)=b,\\ a=\min\big\{\tau(t): \ 0\leq t\leq b\big\},\end{gathered} \tag{3}$$

and $c_i:[a,0]\to\mathbb{R}_+$ $(i=1,\dots,n)$ are also continuous functions.

Definition 1. Let $t_0\in[0,b[$ and

$$t_*=\min\big\{\tau(t): \ t_0\leq t\leq b\big\}.$$

An n-times continuously differentiable function $u:[t_0,b[\,\to\mathbb{R}_+$ is said to be **a solution of equation** (1) **in the interval** $[t_0,b[$ if

$$u^{(i-1)}(t)\geq 0 \ \text{ for } \ t_0\leq t<b \ \ (i=1,\dots,n),$$

and there exist continuous functions $u_{0i}:[t_*,t_0]\to\mathbb{R}_+$ $(i=1,\dots,n)$ such that in that interval equality (1) is satisfied, where

$$u^{(i-1)}(t)=u_{0i}(t) \ \text{ for } \ t_*\leq t\leq t_0 \ \ (i=1,\dots,n).$$

A solution u of equation (1), defined in the interval $[0,b[$ and satisfying the initial conditions (2), is said to be **a solution of problem** (1), (2).

Definition 2. A solution u of equation (1) defined in some interval $[t_0,b[$ is said to be **blow-up** if

$$\lim_{t\to b} u^{(n-1)}(t)=+\infty.$$

A blow-up solution $u:[t_0,b[\to\mathbb{R}_+$ is said to be **strongly blow-up (weakly blow-up)** if

$$\lim_{t\to b} u(t)=+\infty \quad \big(\lim_{t\to b} u(t)<+\infty\big).$$

Definition 3. A solution $u:[t_0,b[\to\mathbb{R}_+$ of equation (1), having the finite limits $\lim\limits_{t\to b} u^{(i-1)}(t)$ $(i=1,\dots,n)$, is said to be **regular**.

According to condition (3), there exists an increasing sequence of numbers $t_i\in]0,b[$ $(i=1,2,\dots)$ such that

$$\tau(t)<0 \ \text{ for } \ 0\le t<t_1, \quad \tau(t_1)=0,$$
$$\tau(t)<t_i \ \text{ for } \ t_i\le t<t_{i+1}, \quad \tau(t_{i+1})=t_i \ \ (i=1,2,\dots),$$
$$\lim_{i\to+\infty} t_i=b.$$

From this fact it immediately follows

Lemma 1. *For arbitrarily fixed continuous functions $c_i:[a,0]\to\mathbb{R}_+$ $(i=1,2,\dots,n)$, problem $(1),(2)$ in the interval $[0,b[$ has a unique solution u and for any natural number k the equality*

$$u(t)=u_k(t) \ \text{ for } \ 0\le t\le t_k$$

is valid, where

$$u_1^{(i-1)}(t)=c_i(t) \ \text{ for } \ a\le t\le 0 \ \ (i=1,\dots,n), \quad u_1(t)=\sum_{i=1}^n \frac{c_i(0)}{(i-1)!}\,t^{i-1}$$
$$+\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f\big(s,c_1(\tau(s)),\dots,c_n(\tau(s))\big)\,ds \ \text{ for } \ 0\le t\le t_1,$$

$$u_{k+1}^{(i-1)}(t)=c_i(t) \ \text{ for } \ a\le t\le 0 \ \ (i=1,\dots,n), \quad u_{k+1}(t)=\sum_{i=1}^n \frac{c_i(0)}{(i-1)!}\,t^{i-1}$$
$$+\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f\big(s,u_k(\tau(s)),\dots,u_k^{(n-1)}(\tau(s))\big)\,ds \ \text{ for } \ 0\le t\le t_{k+1} \ \ (k=1,2,\dots).$$

Theorem 1. *Let along with* (3) *the condition*

$$f(t,x_1,\dots,x_n)\ge f_0(t,x_1,\dots,x_n) \ \text{ for } \ t_0\le t\le b, \ \ (x_1,\dots,x_n)\in\mathbb{R}_+^n$$

be satisfied, where $t_0\in]0,b[$, and $f_0:[0,b]\times\mathbb{R}_+^n\to\mathbb{R}_+$ is a nondecreasing in the phase variables continuous function such that the differential equation

$$v^{(n)}(t)=f_0\big(t,v(\tau(t)),\dots,v^{(n-1)}(\tau(t))\big)$$

in the interval $[t_0,b[$ has a blow-up solution v. Then there exist numbers $r>0$ and $t^*\in]t_0,b]$ such that if

$$c_n(0)>r, \tag{4}$$

then the solution u of problem $(1),(2)$ is blow-up as well and admits the estimates

$$u^{(i-1)}(t)\ge v^{(i-1)}(t) \ \text{ for } \ t^*\le t<b \ \ (i=1,\dots,n).$$

Based on this comparison theorem, effective criteria for the existence of blow-up solutions of problem (1), (2) are obtained. In particular, the following statement is true.

Corollary 1. *Let the functions f and τ satisfy the inequalities*

$$f(t,x_1,\dots,x_n)\geq \ell(b-t)^{\mu}x_k^{\lambda} \ \text{ for } \ t_0\leq t\leq b, \ \ (x_1,\dots,x_n)\in\mathbb{R}_+^n,$$
$$\alpha(t-b)+b\leq \tau(t)<t \ \text{ for } \ 0\leq t<b,$$

where $k\in\{1,\dots,n\}$, $t_0\in\,]0,b[\,$, $\ell>0$, $\mu\geq 0$, $\lambda>1$, $\alpha>1$. *Then for an arbitrary* $\gamma>0$ *there exists a positive number* $r=r(\gamma)$ *such that if inequality* (4) *holds, then the solution* u *of problem* (1), (2) *is strongly blow-up and admits the estimate*

$$\inf\Big\{(b-t)^{\gamma}u(t): \ t_0\leq t<b\Big\}>0. \tag{5}$$

An important particular case of (1) is the differential equation

$$u^{(n)}(t)=\sum_{i=1}^{n-1}p_i(t)\big(u^{(i-1)}(\alpha(t-b)+b)\big)^{\lambda_i}, \tag{6}$$

where $p_i:\mathbb{R}_+\to\mathbb{R}_+$ $(i=1,\dots,n)$ are continuous functions, $\lambda>1$, $\alpha>1$.

For this equation we consider the Cauchy problem with the initial conditions (2), where $a=-(\alpha-1)b$, and $c_i:[a,0]\to\mathbb{R}_+$ $(i=1,\dots,n)$ are continuous functions.

Corollary 2. *There exists* $\varepsilon>0$ *such that if*

$$\sum_{i=1}^{n}c_i(t)<\varepsilon \ \text{ for } \ a\leq t\leq 0,$$

then the solution of problem (6), (2) *is regular. And if*

$$p_k(t)\geq \ell(b-t)^{\mu} \ \text{ for } \ 0\leq t\leq b,$$

where $k\in\{1,\dots,n\}$, $\ell>0$, $\mu\geq 0$, *then for an arbitrary* $\gamma>0$ *there exists a positive number* $r=r(\gamma)$ *such that in the case where inequality* (4) *holds, the solution* u *of problem* (6), (2) *is strongly blow-up and admits estimate* (5).

The first part of the corollary can be easily obtained from Lemma 1, while the second part follows from Corollary 1.

Example 1. Let $n>2$, $\alpha>1$, $\lambda>2$, $\ell_0=((\lambda-1)\alpha^{\frac{\lambda}{1-\alpha}})^{\frac{1}{1-\lambda}}$, $b>0$, $a=-(\alpha-1)b$. We choose positive numbers ρ_i $(i=1,\dots,n)$ so that the function, defined by the equality

$$u(t)=\sum_{i=1}^{n-1}\frac{(t-a)^{i-1}}{(i-1)!}\,\rho_i+(-1)^{n-1}\ell_0\prod_{i=1}^{n-1}\Big(n-i-\frac{1}{\lambda-1}\Big)^{-1}(b-t)^{n-1-\frac{1}{\lambda-1}} \ \text{ for } \ a\leq t<b,$$

satisfies the conditions

$$u^{(i-1)}(t)\geq 0 \ \text{ for } \ a\leq t\leq 0 \ \ (i=1,\dots,n-1).$$

Then the restriction of the function u to $[0,b[$ is a solution of the differential equation

$$u^{(n)}(t)=\big(u^{(n-1)}(\alpha(t-b)+b)\big)^{\lambda} \tag{7}$$

with the initial functions

$$c_i(t) = u^{(i-1)}(t) \;\text{ for }\; a \le t \le 0 \;\; (i = 1, \dots, n).$$

Moreover, it is clear that $u^{(i-1)}$ $(i = 1, \dots, n-1)$ have finite limits

$$u^{(i-1)}(b-0) \;\; (i = 1, \dots, n-1),$$

and

$$\lim_{t \to b} u^{(n-1)}(t) = +\infty.$$

Consequently, u is a weakly blow-up solution of equation (6).

On the other hand, by virtue of Corollary 2 equation (7) has infinite sets of strongly blow-up and regular solutions.

The example constructed above shows that if the functions f and τ satisfy the conditions of either Theorem 1 or one of its corollaries, then equation (7) can simultaneously have strongly blow-up, weakly blow-up and regular solutions.

Example 2. Theorem 1 and its corollaries are specific for delay equations and they have no analogs for equations without delay. To make sure of this, in the interval $[0, b[$ we consider the differential equation

$$u^{(n)}(t) = (u^{(n-1)}(t))^{\lambda}, \tag{8}$$

where $n \ge 2$, $\lambda > 2$. We choose positive numbers ρ_i $(i = 1, \dots, n)$ so that the function, defined by the equality

$$u_0(t) = \sum_{i=1}^{n-1} \frac{\rho_i}{(i-1)!}\, t^{i-1} + (-1)^{n-1} (\lambda - 1)^{\frac{1}{1-\lambda}} \prod_{i=1}^{n-1} \Big(n - i - \frac{1}{\lambda - 1} \Big)^{-1} (b-t)^{n-1-\frac{1}{1-\lambda}} \;\text{ for }\; 0 \le t < b,$$

satisfies the conditions

$$u_0^{(i-1)}(0) \ge 0 \;\; (i = 1, \dots, n-1).$$

Then every solution of equation (8), defined in the interval $[0, b[$ and blowing up at the point b, has the form $u(t) \equiv u_0(t)$, and, consequently, it is weakly blow-up. On the other hand, no matter how the number

$$r > u_0^{(n-1)}(0)$$

is, equation (8) does not have a solution $u : [0, b[\to \mathbb{R}_+$, satisfying the inequalities

$$u^{(i-1)}(0) \ge 0 \;\; (i = 1, \dots, n-1), \quad u^{(n-1)}(0) \ge r.$$

Theorem 2. *Let $n \ge 2$,*

$$\tau(t) = \frac{b(t - t_0)}{b - t_0} \;\textit{ for }\; 0 \le t \le b,$$

and let the function f satisfy the inequality

$$f(t, x_1, \dots, x_n) \ge \ell (b-t)^{\mu} \omega(x_k) \;\textit{ for }\; t_0 \le t \le b, \;\; (x_1, \dots, x_n) \in \mathbb{R}_+^n,$$

where $t_0 \in]0, b[$, $k \in \{1, \dots, n\}$, $\ell > 0$, $\mu \ge 0$, and $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. Let, moreover, there exist a number $\lambda > 1$ such that

$$\int_0^x \omega(y)\, dy > x^{\lambda} - 1 \;\textit{ for }\; x \ge 0. \tag{9}$$

Then for an arbitrary $\gamma > 0$ there exists a positive number $r = r(\gamma)$ such that if inequality (4) *holds, then the solution u of problem* (1), (2) *is strongly blow-up and admits estimate* (5).

Example 3. Consider the differential equation

$$u^{(n)}(t) = (b-y)^{\mu}\omega\Big(u\Big(\frac{b(t-t_0)}{b-t_0}\Big)\Big), \tag{10}$$

where $\mu \geq 0$, $t_0 \in]a,b[$, and $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function which along with (9) satisfies the condition

$$\omega(x_m) = 0 \ \ (m = 1, 2, \dots). \tag{11}$$

Here $\lambda > 1$, and $x_m \in \mathbb{R}_+$ $(m = 1, 2, \dots)$ is an increasing sequence of numbers converging to $+\infty$. The example of such a function is constructed in [10, p. 44].

In view of (11), Theorem 1 and their corollaries leave open the question on the existence of blow-up solutions of equation (10). On the other hand, by Theorem 2 this equation has an infinite set of blow-up solutions.

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Nonlocal Boundary Value Problems for Second Order Linear Hyperbolic Systems

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In the rectangle $\Omega = [0, \omega_1] \times [0, \omega_2]$ consider the problem

$$u_{xy} = P_0(x,y)u + P_1(x,y)u_x + P_2(x,y)u_y + q(x,y), \tag{1}$$

$$\ell(u(\,\cdot\,,y)) = \varphi(y), \quad h(u_x(x,\,\cdot\,)) = \psi(x), \tag{2}$$

where $P_j \in C(\Omega; \mathbb{R}^{n\times n})$ $(j = 0,1,2)$, $q \in C(\Omega;\mathbb{R}^n)$, $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi \in C([0,\omega_1];\mathbb{R}^n)$, and $\ell : C([0,\omega_1];\mathbb{R}^n) \to \mathbb{R}^n$ and $h : C([0,\omega_2];\mathbb{R}^n) \to \mathbb{R}^n$ are bounded linear operators that are *commutative*, i.e., the operators ℓ and h satisfy the equality

$$\ell \circ h(z) = h \circ \ell(z) \quad \text{for} \quad z \in C(\Omega;\mathbb{R}^n). \tag{3}$$

One may think that the boundary conditions

$$\ell(u(\,\cdot\,,y)) = \varphi(y), \quad h(u(x,\,\cdot\,)) = \Psi(x) \tag{$\widetilde{2}$}$$

are more natural than conditions (2). All the more so, conditions $(\widetilde{2})$ obviously imply conditions (2). The main reason for studying problem $(1),(2)$ instead of problem $(1),(\widetilde{2})$ is that problem $(1),(\widetilde{2})$ is ill-posed, since functions φ and ψ should satisfy certain compatibility conditions. Indeed, if $u \in C(\Omega)$ is an arbitrary function satisfying conditions $(\widetilde{2})$, then, in view of (3), we have

$$\ell(\psi) = \ell \circ h(u) = h \circ \ell(u) = h(\varphi).$$

By a solution of problem $(1),(2)$ we understand a *classical* solution, i.e., a function $u \in C^{1,1}(\Omega)$ satisfying equation (1) and boundary conditions (2) everywhere in Ω.

Along with problem $(1),(2)$ consider its corresponding homogeneous problem

$$u_{xy} = P_0(x,y)u + P_1(x,y)u_x + P_2(x,y)u_y, \tag{1_0}$$

$$\ell(u(\,\cdot\,,y)) = 0, \quad h(u_x(x,\,\cdot\,)) = 0, \tag{2_0}$$

as well as the problems

$$v' = P_2(x,y^*)v, \tag{1_1}$$

$$\ell(v) = 0 \tag{2_1}$$

and

$$v' = P_2(x^*,y)v, \tag{1_2}$$

$$h(v) = 0. \tag{2_2}$$

Problems $(1_1),(2_1)$ are $(1_2),(2_2)$ called ***associated problems*** of problem $(1),(2)$. Notice that problem $(1_1),(2_1)$ (problem $(1_2),(2_2)$) is a boundary value problem for a linear ordinary differential equation depending on a parameter y^* (a parameter x^*).

The concept of $\boldsymbol{\sigma}$-associated problems for n-dimensional periodic problems was introduced in [4], and for two-dimensional Dirichlet type problems in [3].

Definition 1. Problem (1), (2) is called *well-posed*, if it is uniquely solvable for arbitrary $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi \in C([0,\omega_1];\mathbb{R}^n)$ and $q \in C(\Omega)$, and its solution u admits the estimate

$$\|u\|_{C^{1,1}(\Omega)} \leq M\Big(\|\varphi\|_{C^1([0,\omega_2])} + \|\psi\|_{C([0,\omega_1])} + \|q\|_{C(\Omega)}\Big),$$

where M is a positive constant independent of φ, ψ and q.

Theorem 1. *Let problem* (1), (2) *be solvable for arbitrary* $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$ *and* $\psi \in C([0,\omega_1];\mathbb{R}^n)$. *Then the problem*

$$z' = 0, \quad \ell(z) = 0 \tag{4}$$

has only the trivial solution.

Remark 1. If problem (4) has only the trivial solution, then problem $(1_0), (2_0)$ is equivalent to the homogeneous problem

$$u_{xy} = P_0(x,y)\,u + P_1(x,y)\,u_x + P_2(x,y)\,u_y, \tag{1_0}$$

$$\ell(u(\,\cdot\,,y)) = 0, \quad h(u(x,\,\cdot\,)) = 0. \tag{$\widetilde{2_0}$}$$

Theorem 2. *Let P_j $(j=1,2)$ be constant matrices, let problem $(1_1), (2_1)$ have a nontrivial solution, and let the following conditions hold:*

$$h(P_0(x,\,\cdot\,)z(\,\cdot\,)) = h(P_0(x,\,\cdot\,))\,h(z(\,\cdot\,)) \ \textit{ for every } \ z \in C([0,\omega_2];\mathbb{R}^n),$$
$$h(P_1 z(\,\cdot\,)) = P_1\,h(z(\,\cdot\,)) \ \textit{ for every } \ z \in C([0,\omega_2];\mathbb{R}^n),$$
$$h(P_2 z(\,\cdot\,)) = P_2\,h(z(\,\cdot\,)) \ \textit{ for every } \ z \in C([0,\omega_2];\mathbb{R}^n).$$

Then for solvability of problem (1), (2) *it is necessary that the problem*

$$v' = P_2 v + (P_0 + P_2 P_1)\Psi(x) + h(q(x,\,\cdot\,)),$$
$$\ell(v) = h(\varphi') - \ell(P_1\,\Psi)$$

is solvable, where Ψ is a solution of the problem

$$z' = \psi(x), \quad \ell(z) = h(\varphi).$$

Theorem 3. *Let P_j $(j=1,2)$ be constant matrices, let problem $(1_2), (2_2)$ have a nontrivial solution, and let along with* (4) *the following conditions hold:*

$$\ell(P_0(\,\cdot\,,y)z(\,\cdot\,)) = \ell(P_0(\,\cdot\,,y))\,\ell(z(\,\cdot\,)) \ \textit{ for every } \ z \in C([0,\omega_1];\mathbb{R}^n),$$
$$\ell(P_1 z(\,\cdot\,)) = P_1\,\ell(z(\,\cdot\,)) \ \textit{ for every } \ z \in C([0,\omega_1];\mathbb{R}^n),$$
$$\ell(P_2 z(\,\cdot\,)) = P_2\,\ell(z(\,\cdot\,)) \ \textit{ for every } \ z \in C([0,\omega_1];\mathbb{R}^n).$$

Then for solvability of problem (1), (2) *it is necessary that the problem*

$$v' = P_1 v + (P_0 + P_1 P_2)\varphi(y) + h(q(\,\cdot\,,y)), \tag{5}$$

$$h(v) = \ell(\psi) - h(P_2\,\varphi) \tag{6}$$

is solvable.

Remark 2. Solvability of the ill-posed nonhomogenous problem (5), (6) means additional compatibility conditions between the boundary values φ and ψ, matrices P_0, P_1 and P_2, and the vector function q. Indeed, consider the problem

$$u_{xy} = P_0(x,y)u + q(x,y), \tag{7}$$

$$u(0,y) = \varphi(y), \quad u_x(x,0) = u_x(x,\omega_2). \tag{8}$$

Let u be a solution of problem (7), (8). Set $v(y) = u_x(0,y)$. Then v is a solution of the problem

$$v' = P_0(0,y)\,\varphi(y) + q(0,y), \tag{9}$$

$$v(0) = v(\omega_2). \tag{10}$$

In other words the solvability of (9), (10) is necessary for the solvability of problem (7), (8). Problem (9), (10) itself is ill-posed. It is solvable if and only if the following equality holds

$$\int\limits_0^{\omega_2} \big(P_0(0,t)\,\varphi(t) + q(0,t)\big)\,dt = 0.$$

Remark 3. Solvability of the ill-posed nonhomogenous problem (5), (6) is necessary for solvability of problem (1), (2), but by no means sufficient. Indeed, consider the two-dimensional problem

$$\begin{aligned} v_{1xy} &= w - q_1(x),\\ w_{2xy} &= -v + q_2(x), \end{aligned} \tag{11}$$

$$\begin{gathered} v(0,y) = 0, \quad v(\omega_1,y) = 0,\\ v_x(x,0) = v_x(x,\omega_2), \quad w_x(x,0) = w_x(x,\omega_2). \end{gathered} \tag{12}$$

Let us show that the corresponding homogeneous problem has only the trivial solution. Let

$$\begin{pmatrix} v(x,y)\\ w(x,y) \end{pmatrix}$$

be an arbitrary solution of the homogeneous system

$$v_{1xy} = w, \tag{13}$$

$$w_{2xy} = -v, \tag{14}$$

satisfying conditions (12). Multiply (13) by w, integrate over Ω. After integrating by parts and taking into account conditions (12), we arrive at the equality

$$-\int\limits_0^{\omega_1}\int\limits_0^{\omega_2} v_x(x,y)w_y(x,y)\,dy\,dx = \int\limits_0^{\omega_1}\int\limits_0^{\omega_2} w^2(x,y)\,dy\,dx. \tag{15}$$

Similarly, after multiplying (14) by v and integrating over Ω, we get

$$-\int\limits_0^{\omega_1}\int\limits_0^{\omega_2} w_y(x,y)v_x(x,y)\,dy\,dx = -\int\limits_0^{\omega_1}\int\limits_0^{\omega_2} v^2(x,y)\,dy\,dx. \tag{16}$$

After subtracting (16) from (15) we arrive at the equality

$$\int\limits_0^{\omega_1}\int\limits_0^{\omega_2} (v^2(x,y) + w^2(x,y))\,dy\,dx = 0.$$

Consequently the homogeneous problem (13), (14), (12) has only the trivial solution. Therefore, problem (11), (12) has at most one solution. Hence, the only possible (*ω_2-periodic with respec to the second variable*) solution of problem (11), (12) should be independent of y. Consequently,

$$\begin{pmatrix} v(x,y) \\ w(x,y) \end{pmatrix} = \begin{pmatrix} q_1(x) \\ q_2(x) \end{pmatrix}$$

is the only possible solution of problem (11), (12). It is clear that u is a weak solution but not a classical one, if q_1 and q_2 are *nowhere differentiable* continuous functions.

Theorem 4. *Let the following conditions hold:*

(A_0) *problem* (3) *has only the trivial solution;*

(A_1) *problem* $(1_1), (2_1)$ *has only the trivial solution for every* $y^* \in [0, \omega_2]$*;*

(A_2) *problem* $(1_2), (2_2)$ *have only the trivial solution for every* $x^* \in [0, \omega_1]$.

Then problem (1), (2) *has the Fredholm property, i.e. the following assertions hold:*

(i) *problem* $(1_0), (2_0)$ *has a finite dimensional space of solutions;*

(ii) *if problem* $(1_0), (2_0)$ *has only the trivial solution, then problem* (1), (2) *is uniquely solvable, and its solution* u *admits estimate*

$$\|u\|_{C^{1,1}(\Omega)} \leq M\Big(\|q\|_{C(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])} + \|\psi\|_{C([0,\omega_1])}\Big), \tag{17}$$

where M *is a positive constant independent of* φ, ψ *and* q.

Definition 2. Problem (1), (2) is called *well-posed*, if it is uniquely solvable for arbitrary $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi \in C([0,\omega_1];\mathbb{R}^n)$ and $q \in C(\Omega;\mathbb{R}^n)$, and its solution u admits the estimate (17), where M is a positive constant independent of φ, ψ and q.

Theorem 5. *Let problem* (1), (2) *be well-posed. Then conditions* (A_1) *and* (A_2) *of Theorem 4 hold.*

Remark 4. Consider the problem

$$u_{xy} = p(x)\,u_x + p(x)\,u_y - p^2(x)\,u + q(x,y), \tag{18}$$

$$u(0,x) = 2u(\omega_1,y), \quad u_x(x,0) = u_x(x,0), \tag{19}$$

where $p \in C^\infty([0,\omega_1])$ is a *nonnegative* function and $q \in C^\infty(\Omega)$. Let

$$q(x,y) = p(x)\,\widetilde{q}(x,y).$$

Set: $I_p = \{x \in [0,\omega_1] : \ p(x) = 0\}$. Then:

(i) problem (18), (19) is well-posed if and only if $I_p = \varnothing$. Moreover, if $I_p = \varnothing$, then a unique solution of problem (18), (19) belongs to $C^\infty(\Omega)$;

(ii) if $\widetilde{q} \in L^\infty([0,\omega_1])$, then problem (18), (19) has a unique weak solution if and only if $\operatorname{mes} I_p = 0$, and has *infinite dimensional set* of *nonclassical* weak solutions otherwise. If $\widetilde{q} \in C([0,\omega_2])$ and $\operatorname{mes} I_p = 0$, then that unique weak solution is a classical solution;

(iii) If $\widetilde{q} \in C([0,\omega_2])$, then problem (18), (19) has a *unique classical* solution if and only if I_p is nowhere dense in $[0,\omega_1]$, and has *infinite dimensional set* of classical solutions otherwise;

(iv) problem (18), (19) has a *unique classical* solution and *infinite dimensional set* of *weak* solutions if I_p is a nowhere dense set of a positive measure;

(v) if $q(x,y) = 1$ and $I_p \neq \varnothing$, then problem (18), (19) has *no classical solution* despite the fact that the coefficients of equation (18) belong to $C^\infty(\Omega)$.

Theorem 6. *Let conditions* (A_0), (A_1), (A_2) *of Theorem 4 hold, and let* $P_2 \in C^{0,1}(\Omega)$ *be such that*

$$h(v) = 0 \implies h(P_2(\,\cdot\,,y)v(\,\cdot\,)) = 0 \ \textit{for} \ y \in [0,\omega_2]$$

for every function $v \in C([0,\omega_2])$*. Then there exists* $\varepsilon > 0$ *such that if*

$$\big\|P(x,y) + P_1(x,y)\,P_2(x,y) - P_{2\,y}(x,y)\big\| \le \varepsilon \ \textit{for} \ (x,y) \in \Omega,$$

then problem (1), (2) *is well-posed. In particular, if*

$$P(x,y) + P_1(x,y)\,P_2(x,y) - P_{2\,y}(x,y) = \mathrm{O},$$

then the solution of problem (1), (2_0) *admits the representation*

$$u(x_1,x_2) = \int\limits_0^{\omega_1}\int\limits_0^{\omega_2} G_1(x,s,y)\,G_2(y,t,s)\,q(s,t)\,dt\,ds,$$

where G_j *is Green's matrix of problem* (1_j), (2_j) $(j = 1,2)$.

Let $n = 2m$, $u = (v,w)$, and $v,w \in \mathbb{R}^m$. For the system

$$\begin{aligned} v_{xy} &= A_1(y)w_x + B_1(x)w_y + Q_1(x,y)w + q_1(x,y),\\ w_{xy} &= A_2(y)v_x + B_2(x)v_y + Q_2(x,y)v + q_2(x,y,) \end{aligned} \tag{20}$$

consider the following boundary conditions of Nicoletti type

$$w(0,y) = 0, \quad v(\omega_1,y) = 0, \quad w_x(x,0) = 0, \quad v_x(x,\omega_2) = 0, \tag{21}$$

and the periodic boundary conditions

$$\begin{aligned} v(0,y) &= v(\omega_1,y), \quad & w(0,y) &= w(\omega_1,y),\\ v_x(x,0) &= v_x(x,\omega_2), \quad & w_x(x,0) &= w_x(x,\omega_2). \end{aligned} \tag{22}$$

Corollary 1. *Let* $A_1 \in C([0,\omega_2];\mathbb{R}^{m\times m})$, $A_2 \in C([0,\omega_2];\mathbb{R}^{m\times m})$, $B_1 \in C([0,\omega_1];\mathbb{R}^{m\times m})$ *and* $B_2 \in C([0,\omega_1];\mathbb{R}^{m\times m})$ *be positive semi-definite symmetric matrix functions, and let there exist* $\delta > 0$ *such that the following conditions hold:*

$$Q_1(x,y)\,w\cdot w \ge \delta\|w\|^2 \ \textit{for} \ (x,y,w) \in \Omega\times\mathbb{R}^m, \tag{23}$$

$$Q_2(x,y)\,v\cdot v \le -\delta\|v\|^2 \ \textit{for} \ (x,y,w) \in \Omega\times\mathbb{R}^m. \tag{24}$$

Then problem (20), (21) *is well-posed.*

Corollary 2. *Let* $A_1 \in C([0,\omega_2];\mathbb{R}^{m\times m})$, $A_2 \in C([0,\omega_2];\mathbb{R}^{m\times m})$, $B_1 \in C([0,\omega_1];\mathbb{R}^{m\times m})$ *and* $B_2 \in C([0,\omega_1];\mathbb{R}^{m\times m})$ *be positive definite symmetric matrix functions, and let there exist* $\delta > 0$ *such that conditions* (23) *and* (24) *hold. Then problem* (20), (22) *is well-posed.*

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In the rectangle Ω consider the boundary value problem

$$u_{xy} = f(x, y, u_x, u_y, u), \tag{1}$$

$$\ell\big(u(\,\cdot\,,y)\big) = \varphi(y), \quad h\big(u_x(x,\,\cdot\,)\big) = \psi(x), \tag{2}$$

where $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi \in C([0,\omega_1];\mathbb{R}^n)$, $\ell : C([0,\omega_1];\mathbb{R}^n) \to \mathbb{R}^n$ and $h : C([0,\omega_2];\mathbb{R}^n) \to \mathbb{R}^n$ are bounded linear operators that are *commutative*, i.e., the operators ℓ and h satisfy the equality

$$\ell \circ h(z) = h \circ \ell(z) \quad \text{for} \quad z \in C(\Omega;\mathbb{R}^n).$$

By $\mathbf{B}^1(z;r)$ denote the closed ball of radius r centered at z in space $C^1(\Omega;\mathbb{R}^n)$, i.e.,

$$\mathbf{B}^1(z;r) = \big\{\zeta \in C^1(\Omega) : \ \|\zeta - z\|_{C^1(\Omega)} \le r\big\}.$$

If $f(x,y,v,w,z)$ is differentiable with respect to the phase variables, set:

$$F_1(x,y,v,w,z) = \frac{\partial f(x,y,v,w,z)}{\partial v}\,, \quad F_2(x,y,v,w,z) = \frac{\partial f(x,y,v,w,z)}{\partial w}\,,$$

$$F_0(x,y,v,w,z) = \frac{\partial f(x,y,v,w,z)}{\partial z}\,,$$

$$P_j[u](x,y) = F_j\big(x,y,u_x(x,y),u_y(x,y),u(x,y)\big) \ \ (j=0,1,2).$$

A vector function $(\widetilde{f},\widetilde{\varphi},\widetilde{\psi})$ s called an *admissible perturbation* if $\widetilde{f} \in C(\Omega \times \mathbb{R}^{3n};\mathbb{R}^n)$ is locally Lipschitz continuous with respect to the first $2n$ phase variables, $\widetilde{\varphi} \in C^1([0,\omega_2];\mathbb{R}^n)$, $\widetilde{\psi} \in C([0,\omega_1];\mathbb{R}^n)$. Set: $\widetilde{F}_1(x,y,v,w,z) = \widetilde{f}_v(x,y,v,w,z)$ and $\widetilde{F}_2(x,y,v,w,z) = \widetilde{f}_w(x,y,v,w,z)$.

Definition 1. Let u_0 be a solution of problem (1), (2), and $r > 0$. Problem (1), (2) is said to be (u_0,r)-*well-posed* if:

(i) $u_0(x,y)$ is the unique solution of the problem in the ball $\mathbf{B}^1(u_0;r)$;

(ii) There exist a positive constant δ_0 and an increasing continuous function $\varepsilon : [0,\delta_0] \to [0,+\infty)$ such that $\varepsilon(0) = 0$ and for any $\delta \in (0,\delta_0]$ and an arbitrary admissible perturbation $(\widetilde{f},\widetilde{\varphi},\widetilde{\psi})$ satisfying the following conditions

$$\|\widetilde{F}_1(x,y,v,w,z)\| + \|\widetilde{F}_2(x,y,v,w,z)\| \le \delta_0 \ \text{ for } \ (x,y,v,w,z) \in \Omega \times \mathbb{R}^{3n}, \tag{3}$$

$$\begin{gathered} \|\widetilde{f}(x,y,v,w,z)\| < \delta \ \text{ for } \ (x,y,v,w,z) \in \Omega \times \mathbb{R}^{3n}, \\ \|\widetilde{\varphi}\|_{C^1([0,\omega_2])} + \|\widetilde{\psi}\|_{C([0,\omega_1])} \le \delta, \end{gathered} \tag{4}$$

the problem

$$u_{xy} = f(x, y, u_x, u_y, u) + \widetilde{f}(x, y, u_x, u_y, u), \tag{$\widetilde{1}$}$$

$$\ell\big(u(\,\cdot\,, y)\big) = \varphi(y) + \widetilde{\varphi}(y), \quad h\big(u_x(x, \,\cdot\,)\big) = \psi(x) + \widetilde{\psi}(x) \tag{$\widetilde{2}$}$$

has at least one solution in the ball $\mathbf{B}^1(u_0; r)$, and each such solution belongs to the ball $\mathbf{B}^1(u_0; \varepsilon(\delta))$.

Definition 2. Let u_0 be a solution of problem (1), (2), and $r > 0$. Problem (1), (2) is said to be *strongly* (u_0, r)*-well-posed* if:

(i) Problem (1), (2) is (u_0, r)-well-posed;

(ii) There exist positive numbers M_0 and δ_0 such that for arbitrary $\delta \in (0, \delta_0)$ an arbitrary admissible perturbation $(\widetilde{f}, \widetilde{\varphi}, \widetilde{\psi})$ satisfying inequalities (3), (4), problem $(\widetilde{1})$, $(\widetilde{2})$ has at least one solution in the ball $\mathbf{B}^1(u_0; r)$, and each such solution belongs to the ball $\mathbf{B}^1(u_0; M_0\, \delta)$.

Definition 3. Problem (1), (2) is called well-posed if it is (u_0, r)-well-posed for every $r > 0$.

Definition 4. A solution u_0 of problem (1), (2) is called *strongly isolated*, if problem (1), (2) is strongly (u_0, r)-well-posed for some $r > 0$.

The concepts of strong well-posedness and a strongly isolated solution of a boundary value problem for a nonlinear ordinary differential system were introduced in [1]. Definitions 2 and 4 are adaptations of the idea of Definitions 3.1 and 3.2 from [1] to problem (1), (2).

The linear case of system (1), i.e. the system

$$u_{xy} = P_1(x, y)u_x + P_2(x, y)u_y + P_0(x, y)u + q(x, y) \tag{5}$$

was studied in [2].

Along with problem (5), (2) consider its corresponding homogeneous problem

$$u_{xy} = P_1(x, y)u_x + P_2(x, y)u_y + P_0(x, y)u, \tag{5_0}$$

$$\ell(u(\,\cdot\,, y)) = 0, \quad h(u_x(x, \,\cdot\,)) = 0. \tag{2_0}$$

Definition 5. Problem (5), (2) is called *well-posed*, if it is uniquely solvable for arbitrary $\varphi \in C^1([0, \omega_2]; \mathbb{R}^n)$, $\psi \in C([0, \omega_1]; \mathbb{R}^n)$ and $q \in C(\Omega)$, and its solution u admits the estimate

$$\|u\|_{C^{1,1}(\Omega)} \le M\Big(\|\varphi\|_{C^1([0,\omega_2])} + \|\psi\|_{C([0,\omega_1])} + \|q\|_{C(\Omega)}\Big),$$

where M is a positive constant independent of φ, ψ and q.

Remark 1. Notice that for the linear problem (5), (2) (u_0, r)-well-posedness is equivalent to the strong well-posedness. Furthermore, for problem (5), (2), Definitions 1, 2 and 3 are equivalent to Definition 5.

Theorem 1. *Let f be a continuously differentiable function with respect to the phase variables v, w and z, and let problem (1), (2) be strongly (u_0, r)-well-posed for some $r > 0$. Then problem (5_0), (2_0) is well-posed, where $P_j(x, y) = P_j[u_0](x, y)$ $(j = 0, 1, 2)$.*

Theorem 2. *Let f be a continuously differentiable function with respect to the phase variables v, w and z, and let there exist matrix functions $P_{ij} \in C(\Omega; \mathbb{R}^{n\times n})$ $(i = 1, 2;\ j = 0, 1, 2)$ such that*

(A_0)

$$P_{1j}(x,y) \le F_j(x,y,v,w,z) \le P_{2j}(x,y) \ \text{ for } \ (x,y,v,w,z) \in \Omega \times \mathbb{R}^{3n} \ \ (j=0,1,2,);$$

(A_1) *for every $x^* \in [0,\omega_1]$ and arbitrary measurable matrix function $P_1 : [0,\omega_2] \to \mathbb{R}^{n\times n}$ satisfying the inequalities*

$$P_{11}(x^*,y) \le P_1(y) \le P_{21}(x^*,y) \ \text{ for } \ y \in [0,\omega_2],$$

the homogeneous problem

$$v' = P_1(y)v, \quad h(v) = 0$$

has only the trivial solution;

(A_2) *for every $y^* \in [0,\omega_2]$ and arbitrary measurable matrix function $P_2 : [0,\omega_1] \to \mathbb{R}^{n\times n}$ satisfying the inequalities*

$$P_{12}(x,y^*) \le P_2(x) \le P_{22}(x,y^*) \ \text{ for } \ x \in [0,\omega_1],$$

the homogeneous problem

$$v' = P_2(x)v, \quad \ell(v) = 0$$

has only the trivial solution;

(A_3) *for arbitrary measurable matrix function $P_j : \Omega \to \mathbb{R}^{n\times n}$ $(j=0,1,2)$ satisfying the inequalities*

$$P_{1j}(x,y) \le P_j(x,y) \le P_{2j}(x,y) \ \text{ for } \ (x,y) \in \Omega \ \ (j=0,1,2),$$

problem $(5_0),(2_0)$ has only the trivial solution.

*Then problem $(1),(2)$ is **strongly** well-posed.*

Remark 2. Conditions (A_1) and (A_2) of Theorem 2 are key and cannot be weakened. Violation of either of conditions (A_1) and (A_2) may lead to additional compatibility conditions between the boundary values (2) and the right-hand side of system (1).

Indeed, consider the problem

$$u_{xy} = P_2\, u_y + q(x,y,u), \tag{6}$$

$$u(0,y) = \varphi(y), \quad u_x(x,0) - u_x(x,\omega_2) = 0, \tag{7}$$

where $P_2 \in \mathbb{R}^{n\times n}$ is an arbitrary matrix, and $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$ and $q \in C(\Omega \times \mathbb{R};\mathbb{R}^n)$ satisfy the equalities

$$\varphi(0) = \varphi(\omega_2), \quad q(x,0,z) = q(x,\omega_2,z).$$

Let u be a solution of problem $(6),(7)$. Set $v(y) = u_x(0,y) - P_2\, u(0,y)$. Then v is a solution of the problem

$$v' = q(0,y,\varphi(y)), \tag{8}$$

$$v(0) - v(\omega_2) = 0. \tag{9}$$

In other words the solvability of $(8),(9)$ is necessary for the solvability of problem $(6),(7)$. Problem $(8),(9)$ itself is ill-posed. It is solvable if and only if the following equality holds

$$\int\limits_0^{\omega_2} q\big(0,t,\varphi(t)\big)\,dt = 0.$$

Remark 3. The fulfillment of additional compatibility conditions is necessary for solvability of problem (1), (2), but by no means sufficient. Indeed, consider the two-dimensional problem

$$\begin{aligned} u_{1xy} &= u_2^3 - \cos x, \\ u_{2xy} &= -u_1^5 + \sin x, \end{aligned} \tag{10}$$

$$\begin{gathered} u_1(0,y)=0, \quad u_1(\omega_1,y)=0, \\ u_{1x}(x,0)=u_{1x}(x,\omega_2), \quad u_{2x}(x,0)=u_{2x}(x,\omega_2). \end{gathered} \tag{11}$$

Let us show that problem (10), (11) has at most one solution. Indeed, let

$$u(x,y)=\begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix} \quad \text{and} \quad \widetilde{u}(x,y)=\begin{pmatrix} \widetilde{u}_1(x,y) \\ \widetilde{u}_2(x,y) \end{pmatrix}$$

be arbitrary solutions of problem (10), (11). Then, in view of (10), we have

$$\big(u_1(x,y)-\widetilde{u}_1(x,y)\big)_{xy} = u_2^3(x,y)-\widetilde{u}_2^3(x,y), \tag{12}$$

$$\big(u_2(x,y)-\widetilde{u}_2(x,y)\big)_{xy} = -\big(u_1^5(x,y)-\widetilde{u}_1^5(x,y)\big). \tag{13}$$

Multiply (12) by $u_2-\widetilde{u}_2$, integrate over Ω. After integrating by parts and taking into account conditions (11), we arrive at the equality

$$-\int_0^{\omega_1}\int_0^{\omega_2} \big(u_1(x,y)-\widetilde{u}_1(x,y)\big)_x \big(u_2(x,y)-\widetilde{u}_2(x,y)\big)_y \, dy\, dx \\ = \int_0^{\omega_1}\int_0^{\omega_2} \big(u_2^3(x,y)-\widetilde{u}_2^3(x,y)\big)\big(u_2(x,y)-\widetilde{u}_2(x,y)\big)\, dy\, dx. \tag{14}$$

Similarly, after multiplying (13) by $u_1-\widetilde{u}_1$ and integrating over Ω, we get

$$-\int_0^{\omega_1}\int_0^{\omega_2} \big(u_2(x,y)-\widetilde{u}_2(x,y)\big)_y \big(u_1(x,y)-\widetilde{u}_1(x,y)\big)_x \, dy\, dx \\ = -\int_0^{\omega_1}\int_0^{\omega_2} \big(u_1^5(x,y)-\widetilde{u}_1^5(x,y)\big)\big(u_1(x,y)-\widetilde{u}_1(x,y)\big)\, dy\, dx. \tag{15}$$

After subtracting (15) from (14) we arrive at the equality

$$\int_0^{\omega_1}\int_0^{\omega_2} \big(u_2^3(x,y)-\widetilde{u}_2^3(x,y)\big)\big(u_2(x,y)-\widetilde{u}_2(x,y)\big)\, dy\, dx \\ + \int_0^{\omega_1}\int_0^{\omega_2} \big(u_1^5(x,y)-\widetilde{u}_1^5(x,y)\big)\big(u_1(x,y)-\widetilde{u}_1(x,y)\big)\, dy\, dx = 0.$$

The latter equality implies $u_k(x,y)\equiv \widetilde{u}_k(x,y)$ $(k=1,2)$, i.e., $u=\widetilde{u}$. In other words, problem (10), (11) has at most one solution. Therefore, due to uniqueness, the only possible solution of problem (10), (11) should be independent of y. Consequently,

$$u(x)=\begin{pmatrix} \cos^{\frac{1}{2}} x \\ \sin^{\frac{1}{5}} x \end{pmatrix}$$

is the only possible solution of problem (10), (11). It is clear that u is a weak solution but not a classical one since u is not differentiable at points $x = \frac{\pi}{2}\, m$ $(m = 0, 1, 2, 3, 4)$. Thus problem (10), (11) has no (classical) solution despite the fact that the right-hand side of system (10) and the boundary values are analytic functions.

Consider the system

$$u_{xy} = f(x, y, u_x, u_y, u) + q(x, y, u). \tag{16}$$

Theorem 3. *Let f satisfy all of the conditions of Theorem 2, and $q(x, y, z)$ be an arbitrary continuous function such that*

$$\lim_{\|z\| \to +\infty} \frac{\|q(x, y, z)\|}{\|z\|} = 0 \tag{17}$$

uniformly on Ω. Then problem (16), (2) has at least one solution.

For the quasi-linear system

$$u_{xy} = P_1(x, y)u_x + P_2(x, y)u_y + P_0(x, y)u + q(x, y, u) \tag{18}$$

Theorem 2 immediately implies

Corollary 1. *Let problem $(5_0), (2_0)$ be well-posed, and let $q(x, y, z)$ be an arbitrary continuous function satisfying condition (17) uniformly on Ω. Then problem (18), (2) has at least one solution.*

Let $n = 2m$, $u = (v, w)$, and $v, w \in \mathbb{R}^m$. For the system

$$\begin{aligned} v_{xy} &= A_1(y)w_x + B_1(x)w_y + f_1(x, y, w) + q_1\big(x, y, v, w\big), \\ w_{xy} &= A_2(y)v_x + B_2(x)v_y + f_2(x, y, v) + q_2\big(x, y, v, w\big) \end{aligned} \tag{19}$$

consider the boundary conditions of Nicoletti type

$$w(0, y) = 0, \quad v(\omega_1, y) = 0, \quad w_x(x, 0) = 0, \quad v_x(x, \omega_2) = 0, \tag{20}$$

and the periodic boundary conditions

$$v(0, y) = v(\omega_1, y), \quad w(0, y) = w(\omega_1, y), \quad v_x(x, 0) = v_x(x, \omega_2), \quad w_x(x, 0) = w_x(x, \omega_2). \tag{21}$$

Here $f_i = \big(f_{ik}\big)_{k=1}^m \in C(\Omega \times \mathbb{R}^m; \mathbb{R}^m)$ $(i = 1, 2)$, $q_i \in C(\Omega \times \mathbb{R}^{2m}; \mathbb{R}^m)$ $(i = 1, 2)$, and $A_i \in C([0, \omega_2]; \mathbb{R}^{m\times m})$ and $B_i \in C([0, \omega_1]; \mathbb{R}^{m\times m})$ are symmetric matrix functions.

Corollary 2. *Let $A_1 \in C([0, \omega_2]; \mathbb{R}^{m\times m})$, $A_2 \in C([0, \omega_2]; \mathbb{R}^{m\times m})$, $B_1 \in C([0, \omega_1]; \mathbb{R}^{m\times m})$ and $B_2 \in C([0, \omega_1]; \mathbb{R}^{m\times m})$ be positive semi-definite symmetric matrix functions, and let there exist $\delta > 0$ such that the following conditions hold:*

$$f_{1k}(x, y, w_1, \dots, w_m)\, w_k \geq \delta\, w_k^2 f_1(x, y, w) \cdot w \geq \delta \|w\|^2 \ \text{ for } \ (x, y, w_1, \dots, w_m) \in \Omega \times \mathbb{R}^m, \tag{22}$$

$$f_{2k}(x, y, v_1, \dots, v_m)\, v_k \leq -\delta\, v_k^2 f_2(x, y, v) \cdot v \leq -\delta \|v\|^2 \ \text{ for } \ (x, y, v_1, \dots, v_m) \in \Omega \times \mathbb{R}^m, \tag{23}$$

$$\lim_{\|v\|,\, \|w\| \to +\infty} \frac{\|q_1(x, y, v, w)\| + \|q_2(x, y, v, w)\|}{\|v\| + \|w\|} = 0 \quad \text{uniformly on } \Omega. \tag{24}$$

Then problem (19), (20) has at least one solution.

Corollary 3. *Let $A_1 \in C([0, \omega_2]; \mathbb{R}^{m\times m})$, $A_2 \in C([0, \omega_2]; \mathbb{R}^{m\times m})$, $B_1 \in C([0, \omega_1]; \mathbb{R}^{m\times m})$ and $B_2 \in C([0, \omega_1]; \mathbb{R}^{m\times m})$ be positive definite symmetric matrix functions, and let there exist $\delta > 0$ such that conditions (22)–(24) hold. Then problem (19), (21) has at least one solution.*

Remark 4. In Theorem 2 it is assumed that the function $f(x,y,v,w,z)$ has at most linear growth with respect to the phase variables v, w and z. Corollaries 2 and 3 cover the case where the right-hand side of system (19) has an arbitrary growth order in some phase variables. As an example, consider the systems

$$\begin{aligned} v_{xy} &= y^2 w_x + (1+x^2) w_y + w + \sinh(w) + \sin(x^2\,y^3)\,w^{\frac{4}{5}}, \\ w_{xy} &= \sin^2 x\,v_y - 2v - \sinh(v^3) + \ln(1 + x^2y^2 + v^6 + w^8) \end{aligned} \tag{25}$$

and

$$\begin{aligned} v_{xy} &= (1+y^2) w_x + (1+x^4)\,x_y w + \sinh(w) + \sin(x^2\,y^3)\,w^{\frac{4}{5}}, \\ w_{xy} &= e^y\,v_x + (1+\sin^2 x)\,v_y - 2v - \sinh(v^3) + \ln(1 + x^2y^2 + v^6 + w^8). \end{aligned} \tag{26}$$

System (25) satisfies all of the conditions of Corollary 2, and system (26) satisfies all of the conditions of Corollary 3. Therefore, by Corollaries 2 and 3, problems (25), (20) and (26), (21) are solvable.

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Numerical Solution for One Nonlinear Integro-Differential Equation Applying Deep Neural Network

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Studying the propagation of an electromagnetic field into a substance, along with its mathematical modeling, investigation, and numerical solution, stands as one of the important tasks in applied mathematics. Typically, this phenomenon is associated with the generation of thermal energy, leading to alterations in the permeability of the medium and influencing the diffusion process. The mathematical representation of this process, like numerous other applied problems, results in the nonlinear partial differential and integro-differential equations and systems thereof. In a quasistationary scenario, the corresponding system of Maxwell equations takes the form outlined in [9]:

$$\frac{\partial H}{\partial t} = -\nabla \times (\nu_m \nabla \times H), \tag{1}$$

$$c_\nu \frac{\partial \theta}{\partial t} = \nu_m (\nabla \times H)^2, \tag{2}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, c_ν and ν_m characterize the heat capacity and electrical conductivity of the medium which are functions of θ. As demonstrated in [3], the system represented by equations (1) and (2) can be rewritten into the following nonlinear parabolic-type integro-differential equation

$$\frac{\partial H}{\partial t} = -\nabla \times \left[a\bigg(\int_0^t |\nabla \times H|^2 \, d\tau \bigg) \nabla \times H \right], \tag{3}$$

where the function $a = a(S)$ is defined for $S \in [0, \infty)$.

Assuming that the magnetic field has the form $H = (0, 0, U)$, and $U = U(x, t)$, we get the following nonlinear integro-differential equation:

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x} \left[a\bigg(\int_0^t \Big(\frac{\partial U}{\partial x} \Big)^2 d\tau \bigg) \frac{\partial U}{\partial x} \right] = 0. \tag{4}$$

The aim of the current note is to extend the investigation initiated in [8] and employ a Deep Neural Network (DNN) for the nonlinear equation (4) featuring the diffusion coefficient $a(S) = (1 + S)^p$, $0 < p \le 1$.

Thus, our goal is to apply DNN for the approximate solution of the following nonlinear initial-boundary value problem

$$\begin{gathered}
\frac{\partial U(x,t)}{\partial t} - \frac{\partial}{\partial x} \left[\bigg(1 + \int_0^t \Big(\frac{\partial U(x,t)}{\partial x} \Big)^2 d\tau \bigg)^p \frac{\partial U(x,t)}{\partial x} \right] = f(x,t), \quad (x,t) \in \Omega, \\
U(0,t) = U(1,t) = 0, \quad t \in [0, T], \\
U(x,0) = U_0(x), \quad x \in [0, 1],
\end{gathered} \tag{5}$$

where $\Omega = (0,\,1) \times (0,\,T)$, $T = const > 0$, f and U_0 are the given functions.

Qualitative and quantitative properties, as well as the numerical solution for the problem (5) and its even more intricate nonlinear counterparts, have been extensively explored in the literature (refer to, for instance, [1, 3–12, 14] and the references therein). As previously stated, our objective is to investigate an alternative approach to solving partial differential equations (PDEs) through Machine Learning methods. Specifically, we aim to train the DNN to serve as a surrogate model capable of predicting the solution of the PDE at any given point $(x,\,t) \in \Omega$. DNNs can consist of multiple layers, including input and output layers, and may feature any number of inner layers referred to as hidden layers (see, for example, Fig. 1). The deep of the network is determined by the number of hidden layers (columns of yellow circles – neurons).

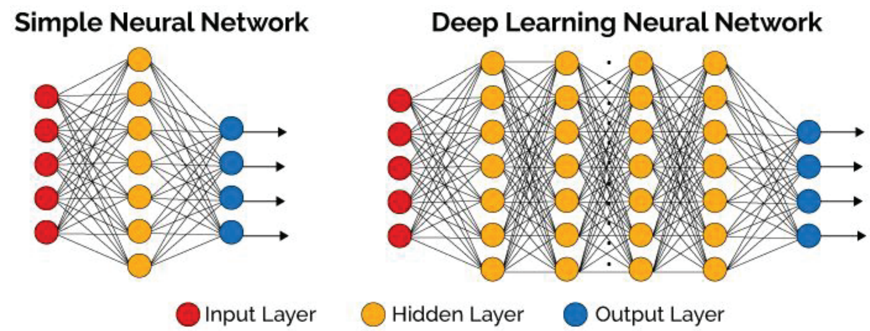

Figure 1. Example of the Neural Network architectures.

The DNN constructs approximation for the solution of problem (5) $u(x,t,\rho) \approx U(x,t)$, where $u(x,t,\rho)$ represents the function obtained from the DNN, and ρ is the variable encompassing all DNN parameters that need for optimization during the training process. As highlighted in [8],the training of the DNN necessitates a substantial amount of training data, serving as the DNN's input. Nevertheless, utilizing the DNN for approximating solutions to PDEs offers an advantage by incorporating physics, thus reducing the size of the required training data (see, for example, [2, 13]).

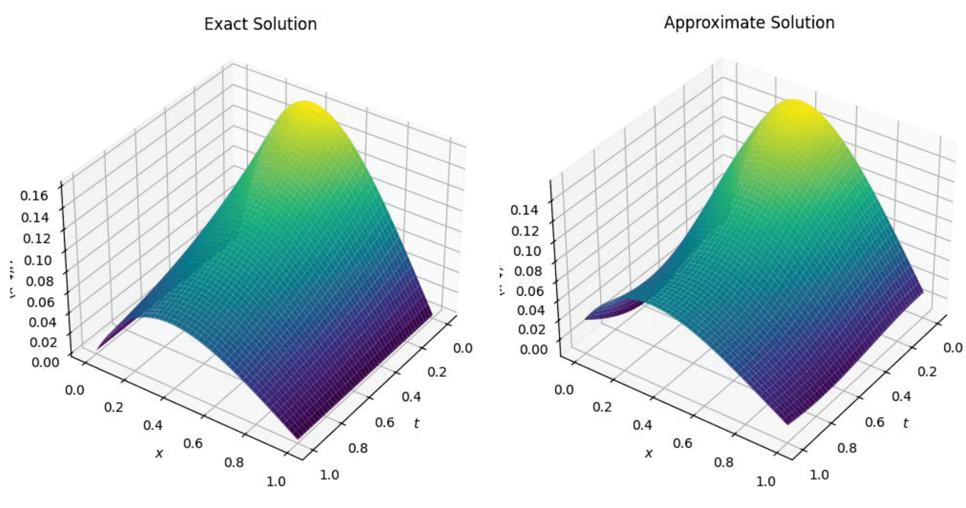

Figure 2. Exact and numerical solutions ($p = 0$).

Following the methodology outlined in [2, 8, 13], we can construct the residual of the nonlinear

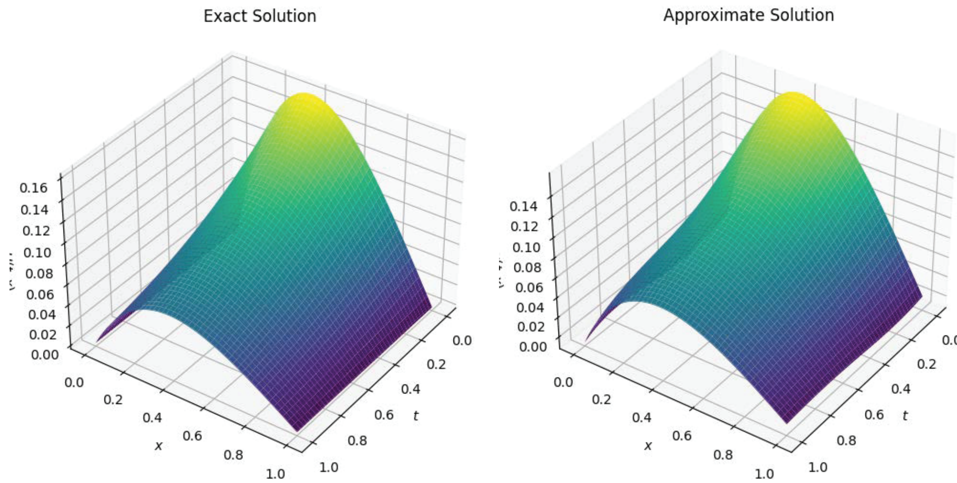

Figure 3. Exact and numerical solutions ($p = 0.5$).

problem (5) to be assessed at a designated set of training points

$$R(x,t,\rho) = \frac{\partial u(x,t,\rho)}{\partial t} - \frac{\partial}{\partial x}\left[\left(1 + \int\limits_0^t \left(\frac{\partial u(x,t,\rho)}{\partial x}\right)^2 d\tau\right)^p \frac{\partial u(x,t,\rho)}{\partial x}\right] - f(x,t). \tag{6}$$

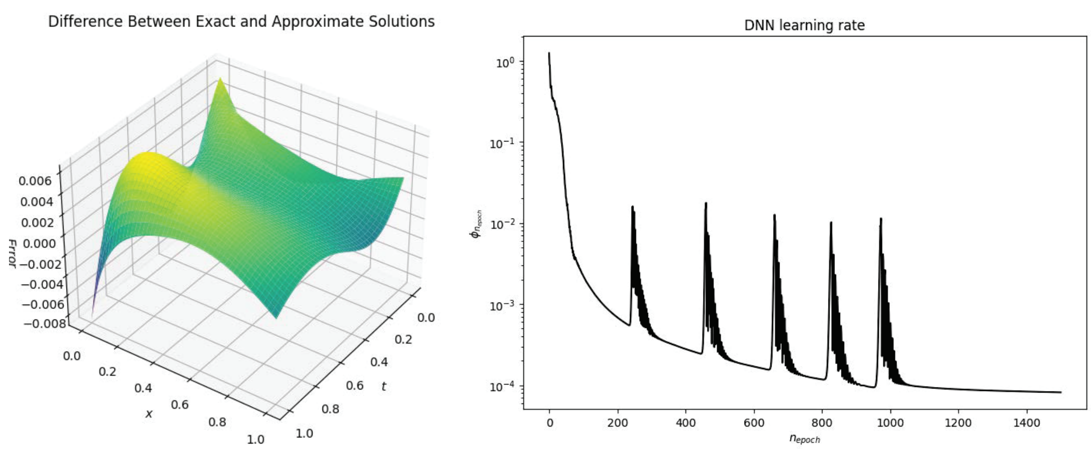

Figure 4. Difference between exact and numerical solutions and learning rate ($p = 0.5$).

Additionally, a cost function $\mathcal{F}(x,t,\rho)$ encompassing the residual (6), along with initial and boundary conditions can be built and minimized by a Deep Neural Network during the training.

In the test experiments, we adopted the same example and parameters as provided in [8]. The right-hand side $f(x,t)$ of the problem (5) was selected to yield an exact solution as follows $U(x,t) = x(1-x)\exp(-x-t)$, accompanied by the corresponding initial solution $U_0(x) = x(1-x)\exp(-x)$. The training of the neural network was conducted using the NumPy library for scientific computing and the TensorFlow library for machine learning.

In Fig. 2 we replicated results of the numerical experiment given in [8] that is for the case $p = 0$ in the problem (5).

The results of the numerical experiment for $p = 0.5$ is given on Fig. 3. The difference between exact and numerical solutions is given in Fig. 4 (left). In the same figure, the DNN learning rate for 1500 epochs is given on the right.

Acknowledgments

This work is supported by the Shota Rustaveli National Science Foundation (SRNSFG), # FR-21-2101.

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On the Instability of Millionshchikov Linear Systems with Smooth Dependence on a Parameter

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Consider a one-parameter family of two-dimensional linear differential systems

$$\dot{x} = A_\mu(t)x, \quad x \in \mathbb{R}^2, \quad t \geq 0 \tag{1_μ}$$

with the matrices

$$A_\mu(t) := \begin{cases} d_k(\mu)\operatorname{diag}[1,-1], & 2k-2 \leq t < 2k-1, \\ (\mu + \gamma(\mu) + b_k)J, & 2k-1 \leq t < 2k, \end{cases}$$

where $k \in \mathbb{N}$ and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and a real parameter μ; the conditions on the numbers $b_k \in \mathbb{R}$ and the functions $d_k(\,\cdot\,), \gamma(\,\cdot\,) : \mathbb{R} \to \mathbb{R}$ will be indicated below.

It was proved in [2] that the upper Lyapunov exponent of system (1_μ) considered as a function of the parameter μ is positive on a set of positive Lebesgue measure for the case in which the functions $d_k(\,\cdot\,)$ are independent of μ, positive, and separated from zero uniformly in $k \in \mathbb{N}$ (i.e., $d_k(\mu) \equiv d_k \geqslant d > 0$, $k \in \mathbb{N}$). Complex matrices of a special kind are substantially used in the proof of this result. Another method for proving the theorem in [1] based on an application of the Parseval equality for trigonometric sums can be found in [3].

Let $\alpha_n \in \mathbb{R}$, $n \in \mathbb{N}$ be arbitrary numbers. Set

$$d_k(\mu) \equiv d(\mu) > 0, \quad b_{2^{n-1}} = \alpha_n, \quad k \in \mathbb{N}, \quad \mu \in \mathbb{R}. \tag{2}$$

Denote the Cauchy matrix of system (1_μ) by $X_{A_\mu}(t,s)$, $t,s \geq 0$. For each $\varphi \in \mathbb{R}$, the matrix of clockwise rotation by the angle φ will be denoted by

$$U(\varphi) \equiv \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

One can readily verify that if the matrix $A_\mu(\,\cdot\,)$ is determined by conditions (2), then

$$X_{A_\mu}(2^{k+1}, 0) = U(\alpha_{k+1} - \alpha_k) X^2_{A_\mu}(2^k, 0) \ \text{ for any } \ k \in \mathbb{N}.$$

Systems with coefficients chosen according to (2) have a number of properties that permit one to construct one-parameter families with various asymptotic characteristics. In particular, if the sequence $\{\alpha_n\}_{n=1}^\infty$ converges, then the matrix $A_\mu(\,\cdot\,)$ is the limit of a sequence of periodic matrices uniformly with respect to $t \geq 0$. V. M. Millionshchikov used such systems in [5–7] (see also [1]) to prove the existence of Lyapunov improper linear differential systems with limit-periodic and quasiperiodic coefficients.

In the paper [4], it was proved under conditions (2) in which $\gamma(\,\cdot\,) \equiv 0$ and in the case of a continuous function $d(\,\cdot\,)$ that there exists a parameter value $\mu \in \mathbb{R}$ such that system (1_μ) is unstable. In the present talk we show that the upper Lyapunov exponent of system (1_μ) considered as a function of the parameter μ is positive on a set of positive Lebesgue measure for the case in which the functions $d_k(\,\cdot\,)$ and $\gamma(\,\cdot\,)$ are differentiable and under the conditions

$$\widetilde{C} := \inf_{\mu\in\mathbb{R}} (1+\gamma'(\mu)) > 2|d'(\mu)|e^{4d(\mu)}, \ \ \mu\in\mathbb{R}, \tag{3}$$

$$\int\limits_0^\pi d(\mu)\,d\mu > 2^{10}(1+\widetilde{C}^{-1}). \tag{4}$$

For any $k\in\mathbb{N}$ and $\mu\in\mathbb{R}$ we recursively define real numbers $\eta_k=\eta_k(\mu)\geq 1$ and $\psi_k=\psi_k(\mu)$ as follows. Set

$$\eta_1(\mu)=e^{d(\mu)}, \quad \psi_1(\mu):\equiv 0,$$
$$\xi_k=\xi_k(\mu):=2\psi_k(\mu)+\alpha_k+\mu+\gamma(\mu), \quad q_k(\mu):=2\pi\big[2^{-1}\pi^{-1}\xi_k(\mu)\big]$$

(here $[\,\cdot\,]$ denotes the integer part of number). Since $\eta_k\geq 1$ and hence $\operatorname{sh}(2\ln\eta_k)\geq 0$, it follows that there exist unique $1\leq\eta_{k+1}\in\mathbb{R}$ and $\varphi_k=\varphi_k(\mu)\in[q_k(\mu)-2^{-1}\pi, q_k(\mu)+2^{-1}\pi)$ such that

$$\operatorname{sh}\ln\eta_{k+1}=(\operatorname{sh}(2\ln\eta_k))|\cos\xi_k|,$$
$$\operatorname{ctg}\varphi_k=(\operatorname{ch}(2\ln\eta_k))\operatorname{ctg}\xi_k \ \text{ if } \ \sin\xi_k\neq,$$
$$\varphi_k=\xi_k \ \text{ if } \ \sin\xi_k=0.$$

Finally, we set

$$\psi_{k+1}(\mu):=\psi_k(\mu)+2^{-1}\varphi_k(\mu)+\frac{\pi}{2}\,\beta(\mu),$$

where

$$\beta(\mu)=0 \ \text{ if } \ \xi_k(\mu)\in\bigcup_{n\in\mathbb{Z}}[2\pi n-2^{-1}\pi, 2\pi n+2^{-1}\pi),$$

$\beta(\mu)=1$ for all others $\mu\in\mathbb{R}$.

In what follows, we will assume that conditions (2) and (3) hold.

Lemma 1. *For any $n\in\mathbb{N}$ and $\mu\in\mathbb{R}$ the functions η_k and ψ_k are differentiable on μ and one has the representation*

$$X_{A_\mu}(2^n-1,0)=U(\psi_n)\begin{pmatrix}\eta_n & 0\\ 0 & \eta_n^{-1}\end{pmatrix}U(\psi_n).$$

Lemma 2. *For any $k\in\mathbb{N}$ an equality holds*

$$\psi_k(\pi)-\psi_k(0)=(2^{k-1}-2^{-1})\pi.$$

Besides of that for all $\mu\in\mathbb{R}$ we have the estimation

$$\psi_k'(\mu)>0.$$

Lemma 3. *For any $k\in\mathbb{N}$ the inequality is true*

$$\int\limits_0^\pi \ln|\cos\xi_k(\mu)|\,d\mu\geq -2^5k-2\pi\ln(1+\widetilde{C}^{-1}).$$

Theorem. *If conditions (2)–(4) are satisfied, then there exists a set $J\subset\mathbb{R}$ of positive Lebesgue measure such that the upper Lyapunov exponent $\lambda_2(A_\mu)$ of system (1_μ) considered as a function of the parameter μ is positive for all $\mu\in J$.*

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Periodic Solutions of a Pendulum Like Planar System

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We consider the periodic problem

$$u' = f(t,u,v), \;\; v' = p(t)\sin u + q(t); \quad u(0) = u(\omega), \;\; v(0) = v(\omega). \tag{1}$$

Here we assume that $p, q \in L([0,\omega])$, $p \not\equiv 0$, and $f \in Car([0,\omega] \times \mathbb{R}^2)$ satisfies the conditions

$$f(t,x,y)\operatorname{sgn} y \geq 0, \quad |f(t,x,y)| \leq h(t,|y|) \quad \text{for } \; t \in [0,\omega], \;\; x, y \in \mathbb{R},$$

where $h \in Car([0,\omega] \times \mathbb{R}_+)$ is non-decreasing in the second argument.

The theory of BVPs for non-autonomous and non-resonant systems is quite well developed (see, [3]). However, (1) is a resonant type problem. Some particular cases of (1) are studied in the literature, but usually under the assumption that $\int\limits_0^\omega q(s)\,\mathrm{d}s = 0$ (see, e.g., [2, 4]). As for the case $\int\limits_0^\omega q(s)\,\mathrm{d}s \neq 0$, there are only a few results available in the existing literature (see, [1, 5]). Below we present new results concerning the existence, multiplicity, and localization of solutions of (1).

We use the following notation:

$$[x]_\pm = \frac{1}{2}\,(|x| \pm x),$$

$$q_0(t) = \max\big\{\|[q]_+\|_L, \|[q]_-\|_L\big\}, \quad H(y) = \int\limits_0^\omega h(s,|y|)\,\mathrm{d}s,$$

$$a(\ell) = \frac{\pi}{2} + \frac{1}{4}\,H(\ell), \quad b(\ell) = \frac{\pi}{2} - \frac{1}{4}\,H(\ell),$$

$$I_{ak}(\ell) = \,]-a(\ell) + 2k\pi, a(\ell) + 2k\pi[\,, \;\; I_{bk}(\ell) = \,]-b(\ell) + 2k\pi, b(\ell) + 2k\pi[\,,$$

$$J_{ak}(\ell) = \,]-a(\ell) + (2k+1)\pi, a(\ell) + (2k+1)\pi[\,, \;\; I_{bk}(\ell) = \,]-b(\ell) + (2k+1)\pi, b(\ell) + (2k+1)\pi[\,,$$

$$B(\ell) = \Big\{v \in C([0,\omega]) : \;\; \|v\|_C \leq \ell, \;\; v(t_0) = 0 \;\text{ for some }\; t_0 \in [0,\omega]\Big\}.$$

Theorem 1. *Let $\sigma \in \{-1,1\}$, $\ell \overset{\text{def}}{=} \|[\sigma p]_-\|_L + q_0$, and the conditions*

$$H(\ell) < 2\pi, \tag{2}$$

$$\|[\sigma p]_-\|_L + \bigg|\int\limits_0^\omega q(s)\,\mathrm{d}s\bigg| \leq \|[\sigma p]_+\|_L \cos\frac{H(\ell)}{4} \tag{3}$$

hold. Then, for any $k \in \mathbb{Z}$, problem (1) *possesses a solution (u_k, v_k) such that $v_k \in B(\ell)$, and*

$$\operatorname{Range} u_k \subseteq \overline{I_{ak}(\ell)}, \quad \overline{I_{bk}(\ell)} \cap \operatorname{Range} u_k \neq \varnothing \quad \text{if } \; \sigma = 1,$$

and

$$\operatorname{Range} u_k \subseteq \overline{J_{ak}(\ell)}, \quad \overline{J_{bk}(\ell)} \cap \operatorname{Range} u_k \neq \varnothing \quad \text{if } \; \sigma = -1.$$

Remark 1. Inequality (3) is optimal for the solvablity of (1) and cannot be replaced by

$$\|[\sigma p]_-\|_L + \left|\int\limits_0^\omega q(s)\,\mathrm{d}s\right| \le (1+\varepsilon)\|[\sigma p]_+\|_L \cos\frac{H(\ell)}{4}\,,$$

no matter how small $\varepsilon > 0$ is.

Theorem 1 guarantees that problem (1) possesses infinitely many solutions (u_k, v_k). However, it may happen that $u_{k+1} \equiv u_k + 2\pi$ (for example, if $f(t, x+2\pi, y) \equiv f(t,x,y)$). Introduce the following definition.

Definition 1. Solutions (u_1, v_1) and (u_2, v_2) of (1) are said to be geometrically distinct (g.d.) if $u_1 - u_2 \not\equiv 2\pi n$ for $n \in \mathbb{Z}$.

Theorem 2. *Let $\sigma \in \{-1, 1\}$, $\ell \overset{\text{def}}{=} \frac{1}{2}\left(\|p]\|_L + \|q]\|_L\right)$, inequality* (2) *hold, and*

$$\|[\sigma p]_-\|_L + \left|\int\limits_0^\omega q(s)\,\mathrm{d}s\right| < \|[\sigma p]_+\|_L \cos\frac{H(\ell)}{4}\,.$$

Then, for any $k \in \mathbb{Z}$, problem (1) *possesses a pair of g.d. solutions (u_{1k}, v_{1k}) and (u_{2k}, v_{2k}) such that $v_{ik} \in B(\ell)$ for $i = 1, 2$, and*

$$\operatorname{Range} u_{1k} \subset I_{ak}(\ell),\ \ I_{bk}(\ell) \cap \operatorname{Range} u_{1k} \neq \varnothing,\ \ \textit{and}\ \ \operatorname{Range} u_{2k} \subset J_{ak}(\ell),\ \ J_{bk}(\ell) \cap \operatorname{Range} u_{2k} \neq \varnothing.$$

Definition 2. Solutions (u_1, v_1) and (u_2, v_2) of (1) are said to be consecutive if $u_1(t) \le u_2(t)$ for $t \in [0, \omega]$, $u_1 \not\equiv u_2$, and problem (1) has no solution (u, v) satisfying $u_1(t) \le u(t) \le u_2(t)$ for $t \in [0, \omega]$, $u \not\equiv u_1$, and $u \not\equiv u_2$.

It is worth mentioning that a pair of consecutive solutions may not be geometrically distinct and vice versa.

In order to formulate the next theorem, we need to introduce the following hypothesis:

$$\left.\begin{array}{c}\text{the function } f(t,x,\cdot): \mathbb{R} \to \mathbb{R} \text{ is non-decreasing for a. e. } t \in [0,\omega] \text{ and all } x \in \mathbb{R},\\ \operatorname{mes}\big\{t \in [0,\omega]:\ f(t,x,y) \neq 0\big\} > 0 \ \text{ for } x, y \in \mathbb{R},\ \ y \neq 0,\\ \text{for every } \varepsilon > 0 \text{ and } r > 0 \text{ there exists } f_{\varepsilon r} \in L([0,\omega]) \text{ such that}\\ |f(t,x_2,y) - f(t,x_1,y)| \le f_{\varepsilon r}(t) \ \text{ for } t \in [0,\omega],\ \ |x_2 - x_1| \le \varepsilon,\ \ |y| \le r.\end{array}\right\} \tag{A}$$

Theorem 3. *Let $\sigma \in \{-1, 1\}$, $\ell \overset{\text{def}}{=} \|[\sigma p]_-\|_L + q_0$, $\ell^* \overset{\text{def}}{=} \frac{1}{2}\left(\|p]\|_L + \|q]\|_L\right)$, and hypothesis* (A) *hold. Let, moreover, $H(\ell^*) < \pi$ and*

$$\|[\sigma p]_-\|_L - \|[\sigma p]_+\|_L \cos\frac{H(\ell^*)}{2} < \left|\int\limits_0^\omega q(s)\,\mathrm{d}s\right| \le \|[\sigma p]_+\|_L \cos\frac{H(\ell)}{4} - \|[\sigma p]_-\|_L\,.$$

Then, for any $k \in \mathbb{Z}$, problem (1) *possesses a pair of consecutive solutions (u_{1k}, v_{1k}) and (u_{2k}, v_{2k}) such that either (u_{1k}, v_{1k}) or (u_{2k}, v_{2k}) is Lyapunov unstable, $v_{ik} \in B(\ell)$ for $i = 1, 2$, and*

$$\operatorname{Range}(u_{1k} - 2k\pi) \subseteq \Big[-a(\ell), \frac{\pi}{2}\Big[,\quad \operatorname{Range}(u_{2k} - 2k\pi) \subset \Big]\frac{\pi}{2}, \frac{5\pi}{2}\Big[\quad \textit{if } \sigma\int\limits_0^\omega q(s)\,\mathrm{d}s \ge 0,$$

and

$$\operatorname{Range}(u_{1k}-2k\pi)\subset \Big]-\frac{5\pi}{2}\,,-\frac{\pi}{2}\Big[,\quad \operatorname{Range}(u_{2k}-2k\pi)\subseteq \Big]-\frac{\pi}{2}\,,a(\ell)\Big] \quad \text{if } \sigma\int\limits_0^\omega q(s)\,\mathrm{d}s\leq 0.$$

Moreover, if, for $i\in\{1,2\}$, the inequality $(-1)^i\sigma\int\limits_0^\omega q(s)\,\mathrm{d}s\geq 0$ holds, then, for every solution (u,v) of problem (1), *the condition*

$$\Big\{(-1)^i\,\frac{\pi}{2}+2\pi n:\ n\in\mathbb{Z}\Big\}\cap \operatorname{Range} u\neq\varnothing$$

is satisfied.

Now, we consider a particular case of (1), namely, the problem

$$u'=h(t)\varphi(v),\ \ v'=p(t)\sin u+q(t);\quad u(0)=u(\omega),\ \ v(0)=v(\omega), \tag{4}$$

where $\varphi:\mathbb{R}\to\mathbb{R}$ is an increasing continuous function satisfying the conditions $\varphi(-y)=-\varphi(y)$ and $\varphi(y)>0$ for $y>0$, and $h\in L([0,\omega])$ is a non-trivial non-negative function. Moreover, we assume that

$$\varphi^*(x,y)\overset{\text{def}}{=}\frac{\varphi(x)-\varphi(y)}{x-y}\ \text{ is continuous}$$

and we put

$$\varphi^*_r\overset{\text{def}}{=}\max\big\{\varphi^*(x,y):x,y\in[-r,r]\big\}.$$

Definition 3. A pair of solutions (u_1,v_1) and (u_2,v_2) of (4) is called a fundamental system of solutions if, for any solution (u,v) of (4), there exists $k\in\mathbb{Z}$ such that either $u\equiv u_1+2k\pi$ or $u\equiv u_2+2k\pi$.

Theorem 4. *Let $\sigma\in\{-1,1\}$, $\ell^*\overset{\text{def}}{=}\frac{1}{2}\,(\|p]\|_L+\|q]\|_L)$, and*

$$\sigma p(t)\geq 0\quad \text{for }\ t\in[0,\omega]. \tag{5}$$

Let, moreover,

$$\|h\|_L\varphi(\ell^*)<\pi,\quad \varphi^*_{\ell^*}\|h\|_L\|p\|_L\leq 16,\quad \text{and}\quad \bigg|\int\limits_0^\omega q(s)\,\mathrm{d}s\bigg|<\|p\|_L\cos\frac{\|h\|_L\varphi(\ell^*)}{2}\,.$$

Then, problem (4) *possesses a fundamental system of solutions (u_1,v_1) and (u_2,v_2) such that $v_1,v_2\in B(\ell^*)$, and*

$$\operatorname{Range}u_1\subset\Big]-\frac{\pi}{2}\,,\frac{\pi}{2}\Big[\quad \text{and}\quad \operatorname{Range}u_2\subset\Big]\frac{\pi}{2}\,,\frac{3\pi}{2}\Big[.$$

Moreover, for $\sigma=1$, (u_1,v_1) is unstable, while for $\sigma=-1$, (u_2,v_2) is unstable.

As an example, we consider the so-called relativistic problem

$$u'=h(t)\,\frac{v}{\sqrt{1+v^2}}\,,\ \ v'=p(t)\sin u+q(t);\quad u(0)=u(\omega),\ \ v(0)=v(\omega). \tag{6}$$

It is clear that $H(y)=\|h\|_L\,\frac{|y|}{\sqrt{1+y^2}}$ in this case. Therefore, taking into account that $H(y)<\|h\|_L$ and the monotonicity of the cosine function, we get from Theorem 3 the following corollary.

Corollary 1. *Let $\sigma \in \{-1,1\}$, $\|h\|_L \leq 2\pi$, and*

$$\bigg|\int\limits_0^\omega q(s)\,\mathrm{d}s\bigg| \leq \|[\sigma p]_+\|_L \cos\frac{\|h\|_L}{4} - \|[\sigma p]_-\|_L.$$

Then, problem (6) *possesses a pair of g.d. solutions* (u_1,v_1) *and* (u_2,v_2). *If, moreover,* $\|h\|_L < \pi$ *and*

$$\bigg|\int\limits_0^\omega q(s)\,\mathrm{d}s\bigg| \geq \|[\sigma p]_-\|_L - \|[\sigma p]_+\|_L \cos\frac{\|h\|_L}{4}\,,$$

then (u_1,v_1) *and* (u_2,v_2) *are consecutive solutions and at least one of them is unstable.*

Theorem 4 implies the following corollary.

Corollary 2. *Let $\sigma \in \{-1,1\}$ and* (5) *be fulfilled. Let, moreover,*

$$\|h\|_L \leq \pi, \quad \|h\|_L\|p\|_L \leq 16, \quad \text{and} \quad \bigg|\int\limits_0^\omega q(s)\,\mathrm{d}s\bigg| \leq \|p\|_L \cos\frac{\|h\|_L}{2}\,.$$

Then, the conclusions of Theorem 4 hold for problem (6).

At last we mention that the above theorems also guarantee a localization of the second component of solutions (see, the conditions like $v \in B(\ell)$). Therefore, our results can be applied to some singular problems as well. For example, let us consider the so-called mean curvature problem

$$u' = f(t,u)\,\frac{v}{\sqrt{1-v^2}}\,, \quad v' = p(t)\sin u + q(t); \quad u(0)=u(\omega), \;\; v(0)=v(\omega),$$

where $f \in \mathit{Car}([0,\omega]\times\mathbb{R})$ and $0 \leq f(t,x) \leq h(t)$ for $t \in [0,\omega]$, $x \in \mathbb{R}$. Theorem 1 yields the following corollary.

Corollary 3. *Let $\sigma \in \{-1,1\}$, $\ell \overset{\text{def}}{=} \|[\sigma p]_-\|_L + q_0$, $\ell < 1$, and inequalities* (2) *and* (3) *be satisfied with $H(\ell) \overset{\text{def}}{=} \frac{\|h\|_L\,\ell}{\sqrt{1-\ell^2}}$. Then, problem* (6) *possesses infinitely many solutions.*

Acknowledgements

The research is supported by the internal grant # FSI-S-23-8161 of FME BUT.

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Branches of Positive Solutions of a Superlinear Indefinite Prescribed Curvature Problem

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1 The result

This contribution is based on our recent paper [4] where we analyzed the set of positive regular solutions of the quasilinear Neumann problem

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1+(u')^2}}\Big)' = \lambda a(x) f(u), \quad 0<x<1, \\ u'(0)=u'(1)=0. \end{cases} \tag{1.1}$$

Here, $\lambda \in \mathbb{R}$ is a parameter and the functions a and f satisfy:

(a_1) $a \in L^\infty(0,1)$, $\int\limits_0^1 a(x)\,dx < 0$, *and there is* $z \in (0,1)$ *such that* $a(x) > 0$ *almost everywhere in* $(0,z)$ *and* $a(x) < 0$ *almost everywhere in* $(z,1)$*;*

(f_1) $f \in \mathcal{C}^0[0,+\infty)$, $f(s) > 0$ *if* $s > 0$*, and, for some constant* $p > 1$, $\lim\limits_{s\to 0^+} \frac{f(s)}{s^p} = 1$.

As a is sign indefinite and f is superlinear at zero, (1.1) is a *superlinear indefinite* elliptic problem. These problems have attracted a huge amount of attention during the last few decades.

The problem (1.1) can be regarded as a simple prototype of its more sophisticated multidimensional counterpart, which plays a central role in the mathematical analysis of a number of important geometrical and physical issues, ranging from prescribed mean curvature problems for cartesian surfaces in the Euclidean space, to the study of capillarity phenomena for compressible or incompressible fluids, as well as to the analysis of reaction-diffusion processes where the flux features saturation at high regimes.

Although the study of (1.1) is often settled in the space of *bounded variation* functions (see, e.g., [5–8]), here we will be instead concerned with the *regular* solutions of (1.1), that is, functions $u \in W^{2,1}(0,1)$ which fulfill the differential equation almost everywhere in $(0,1)$, as well as the boundary conditions.

A function $u \in \mathcal{C}^0[0,1]$ is said to be *positive* if $\min\limits_{[0,1]} u \geq 0$ and $\max\limits_{[0,1]} u > 0$, whereas it is said *strictly positive* if $\min\limits_{[0,1]} u > 0$. Here, the positive solutions of (1.1) are regarded as couples (λ, u). Naturally, for any given $\lambda \geq 0$, a couple (λ, u) is said to be a positive, or strictly positive, solution of (1.1) if u is a positive, or strictly positive, solution of (1.1), respectively. Note that, under conditions (a_1) and (f_1), the strong maximum principle (see, e.g., [5, Theorem 2.1]) yields the strict positivity of any positive regular solution of (1.1).

Subsequently, we denote by $\mathscr{S}^+$ the set of all couples $(\lambda, u) \in [0, \infty) \times C^1[0,1]$ such that (λ, u) is a positive, and hence strictly positive, regular solution of (1.1).

The following result establishes the existence of an unbounded closed connected subset $\mathscr{C}^+$ of $\mathscr{S}^+$, bifurcating from $u = 0$ as $\lambda \to +\infty$, and provides simultaneously some sharp information on its localization. The existence of unstable solutions, however not necessarily belonging to $\mathscr{C}^+$, is also detected.

Theorem 1.1. *Assume (a_1) and (f_1). Then, there exists an unbounded closed connected subset $\mathscr{C}^+$ of $\mathscr{S}^+$ for which the following properties hold:*

(i) *there is $\lambda^* > 0$ such that $[\lambda^*, \infty) \subseteq \mathrm{proj}_{\mathbb{R}}(\mathscr{C}^+)$;*

(ii) *there are functions α and β, explicitly defined by* (2.6) *and* (2.7) *respectively, such that, for every $(\lambda, u_\lambda) \in \mathscr{C}^+$, one has*

$$u_\lambda(x_\lambda) < \lambda^{\frac{1}{1-p}} \alpha(x_\lambda), \ \ \textit{for some} \ \ x_\lambda \in [0, z),$$

and

$$u_\lambda(y_\lambda) > \lambda^{\frac{1}{1-p}} \beta(y_\lambda), \ \ \textit{for some} \ \ y_\lambda \in [0, 1];$$

(iii) *there is $C > 0$ such that, for every $(\lambda, u\lambda) \in \mathscr{C}^+$,*

$$\|u_\lambda'\|_{L^\infty(0,1)} < C\lambda^{\frac{1}{1-p}}.$$

Moreover, for every $\lambda \in [\lambda_*, \infty)$, there exists at least one Lyapunov unstable solution $u \in \mathscr{S}^+$ of (1.1) *satisfying the conditions expressed by properties* (ii) *and* (iii).

Theorem 1.1 is a substantial sharpening of some previous results obtained in [6–8]. Unlike in these papers, here the proof exploits an alternative method based on viewing (1.1) as a perturbation of a semilinear problem, on constructing some non-ordered lower and upper solutions, and on using the Leray–Schauder degree. This approach, which appears of interest in its own, yields, in addition, the localization and the instability information established by Theorem 1.1, which is a novel result in the context of the problem (1.1).

2 The proof

2.1 Reformulation of (1.1) as a perturbation of a semilinear problem

Since $f(0) = 0$ and we are focusing attention on the positive solutions of (1.1), without loss of generality we can extend f to the whole of $\mathbb{R}$ as an even function. By performing the change of variable

$$u = \varepsilon v, \ \ \varepsilon = \lambda^{\frac{1}{1-p}}, \tag{2.1}$$

and setting

$$h(s) = \begin{cases} \dfrac{f(s)}{|s|^p} & \text{if } \ s \neq 0, \\ 1 & \text{if } \ s = 0, \end{cases} \tag{2.2}$$

the problem (1.1) can be equivalently written in the form

$$\begin{cases} -v'' = a(x)|v|^p\, h(\varepsilon v)\,(1+(\varepsilon v')^2)^{\frac{3}{2}}, & 0<x<1, \\ v'(0)=v'(1)=0. \end{cases} \tag{2.3}$$

Throughout the rest of this proof, for every $r>0$, we consider the auxiliary function

$$\ell_r(x,s) = \begin{cases} |s|^p & \text{if } \ s\le 0, \\ a(x)\, s^p & \text{if } \ 0<s\le r, \\ a(x)\, s^p\,(r+1-s) & \text{if } \ r<s\le r+1, \\ -s+r+1 & \text{if } \ s>r+1, \end{cases}$$

as well as the associated problem

$$\begin{cases} -v'' = \ell_r(x,v)\, h(\varepsilon v)\,(1+(\varepsilon v')^2)^{\frac{3}{2}}, & 0<x<1, \\ v'(0)=v'(1)=0. \end{cases} \tag{2.4}$$

It is obvious that any solution v of (2.4), with $0\le v\le r$ in $[0,1]$, solves (2.3). Moreover, due to (2.2), the problem (2.4) perturbs, as $\varepsilon>0$ separates away from 0, from the semilinear x problem

$$\begin{cases} -v'' = \ell_r(x,v), & 0<x<1, \\ v'(0)=v'(1)=0. \end{cases} \tag{2.5}$$

2.2 Existence of non-ordered strict lower and upper solutions of (2.5)

Construction of a lower solution α

Let $\mu_1>0$ be the principal eigenvalue of the linear weighted eigenvalue problem

$$\begin{cases} ll - \varphi'' = \mu\, a(x)\,\varphi, & 0<x<\dfrac{z}{2}\,, \\ \varphi'(0)=0, \ \ \varphi\Big(\dfrac{z}{2}\Big)=0. \end{cases}$$

Denote by φ_1 any positive eigenfunction associated to μ_1 and let $\overline{x}\in(0,\frac{z}{2})$ be such that

$$\varphi_1(\overline{x})+\varphi_1'(\overline{x})(x-\overline{x})=0.$$

Then, we define, for $c>0$,

$$\alpha(x) = \begin{cases} c\varphi_1(x) & \text{if } \ 0\le x<\overline{x}, \\ c\varphi_1(\overline{x})+c\varphi_1'(\overline{x})(x-\overline{x}) & \text{if } \ \overline{x}\le x<z, \\ 0 & \text{if } \ z\le x\le 1. \end{cases} \tag{2.6}$$

Construction of an upper solution β

For every $\kappa>0$, let us denote by z_κ the unique solution of the linear problem

$$\begin{cases} -z'' = \left(a(x) - \displaystyle\int_0^1 a(t)\,dt\right)\kappa^p, & 0<x<1, \\ z'(0)=z'(1)=0, \ \ \displaystyle\int_0^1 z(t)\,dt = 0. \end{cases}$$

Then, we define

$$\beta = z_\kappa + \kappa. \tag{2.7}$$

By making a suitable choice of c and the following conclusions about α and β can be inferred.

Proposition 2.1. *There exists a constant $r_0 > 0$ such that, for all $r \geq r_0$, the problem* (2.5) *admits a lower solution α and an upper solution β, respectively defined by* (2.6) *and* (2.7)*, such that:*

(i) *$\beta - \alpha$ changes sign in $[0,1]$;*

(ii) *any solution v of* (2.5) *such that $\alpha \leq v$ in $[0,1]$, satisfies $\alpha(x) < v(x)$ for all $x \in [0,1]$;*

(iii) *any solution v of* (2.5) *such that $v \leq \beta$ in $[0,1]$, satisfies $v(x) < \beta(x)$ for all $x \in [0,1]$.*

2.3 Positivity and a priori bounds for the solutions of (2.5)

Proposition 2.2. *Fix any $r > 0$. Then, the following assertions hold:*

(i) *every solution of* (2.5) *is non-negative;*

(ii) *every positive solution of* (2.5) *is strictly positive.*

Proposition 2.3. *The following assertions hold:*

(i) *for every $r > 0$, any solution v of* (2.5) *satisfies*

$$0 \leq v(x) \leq r + 1, \ \ \textit{for all} \ \ x \in [0,1],$$

and

$$\|v'\|_{L^\infty(0,1)} < C = \|a\|_{L^1(0,1)}(r+1)^{p+1}; \tag{2.8}$$

(ii) *for every $r \geq r_0$, any solution v of* (2.5)*, with $v(x_0) \leq \alpha(x_0)$ for some $x_0 \in [0,1]$, satisfies*

$$\max_{[0,1]} v < R = \|\alpha\|_{L^\infty(0,1)} + \|\alpha'\|_{L^\infty(0,1)}. \tag{2.9}$$

2.4 Existence of ordered strict lower and upper solutions of (2.5)

Proposition 2.4. *Fix any $r \geq r_0$. The constants $\alpha_1 = -1$ and $\beta_1 = r + 2$ are, respectively, a lower solution and an upper solution of* (2.5) *satisfying*

$$\alpha_1 < 0 \leq \alpha(x), \ \ \beta(x) \leq r_0 < \beta_1, \ \ \textit{for all} \ \ x \in [0,1]. \tag{2.10}$$

Moreover, every solution v of (2.5) *is such that $\alpha_1 < v(x) < \beta_1$, for all $x \in [0,1]$.*

2.5 Degree computations

Fix any $r \geq \max\{r_0, R\}$, where R is the constant defined in (2.9). Then, C being the constant introduced in (2.8), define the following open bounded subsets of $C^1[0,1]$:

$$\begin{aligned}
\Omega_1 &= \Big\{v \in \mathcal{C}^1[0,1] : \ \ \alpha_1 < v(x) < \beta_1 \ \text{ for all } \ x \in [0,1], \ \ \|v'\|_\infty < C\Big\}, \\
\Omega_2 &= \Big\{v \in \mathcal{C}^1[0,1] : \ \ \alpha_1 < v(x) < \beta(x) \ \text{ for all } \ x \in [0,1], \ \ \|v'\|_\infty < C\Big\}, \\
\Omega_3 &= \Big\{v \in \mathcal{C}^1[0,1] : \ \ \alpha(x) < v(x) < \beta_1 \ \text{ for all } \ x \in [0,1], \ \ \|v'\|_\infty < C\Big\}, \\
\Omega = \Omega_1 \setminus \overline{\Omega_2 \cup \Omega_3} &= \Big\{v \in \Omega_1 : \ \ v(x_0) < \alpha(x_0) \ \text{ and } \ \beta(y_0) < v(y_0) \ \text{ for some } \ x_0, y_0 \in [0,1]\Big\}.
\end{aligned}$$

From (2.10), it follows that $\Omega_2 \cup \Omega_3 \subset \Omega_1$. Moreover, we have that $\Omega_2 \cap \Omega_3 = \varnothing$ by Proposition 2.1.

Let us denote by $\mathcal{T} : [0,\infty)\times\mathcal{C}^1[0,1] \to \mathcal{C}^1[0,1]$ the operator sending each $(\varepsilon, v) \in [0,\infty)\times\mathcal{C}^1[0,1]$ to the unique solution $w \in W^{2,\infty}(0,1)$ of the linear problem

$$\begin{cases} -w'' + w = \ell_r(x,v)\, h(\varepsilon v)\, (1+(\varepsilon v')^2)^{\frac{3}{2}} + v, \quad 0 < x < 1, \\ w'(0) = w'(1) = 0. \end{cases}$$

It is clear that $\mathcal{T}$ is completely continuous and that its fixed points are the solutions of the problem (2.4). Moreover, by Propositions 2.1 and 2.3 and our choice of C, the operator $T(0,\,\cdot\,)$ cannot have fixed points on $\partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3$. Thus, by the additivity property of the degree, we infer that

$$\begin{aligned} &\deg_{LS}\big(\mathcal{I} - \mathcal{T}(0,\,\cdot\,), \mathcal{O}\big) \\ &\qquad = \deg_{LS}\big(\mathcal{I} - \mathcal{T}(0,\,\cdot\,), \Omega_1\big) - \deg_{LS}\big(\mathcal{I} - \mathcal{T}(0,\,\cdot\,), \Omega_2\big) - \deg_{LS}\big(\mathcal{I} - \mathcal{T}(0,\,\cdot\,), \Omega_3\big). \end{aligned}$$

As, from, e.g., [1, Chapter III], we know that $\deg_{LS}(\mathcal{I} - \mathcal{T}(0,\,\cdot\,), \Omega_i) = 1$, for $i = 1,2,3$, we can conclude that $\deg_{LS}(\mathcal{I} - \mathcal{T}(0,\,\cdot\,), \Omega) = -1$. Therefore, by the existence property of the degree, the problem (2.5) possesses a solution $v \in \Omega$, where necessarily $x_0 \in [0,z)$, because $\alpha(x_0) > v(x_0) > 0$ and $\alpha = 0$ on $[z,1]$. In addition, having chosen $r > R$, Proposition 2.3 guarantees that $v(x) < r$ for all $x \in [0,1]$ and therefore v is a solution of the problem (2.3) for $\varepsilon = 0$. Hence, if we define

$$\mathcal{O} = \Big\{ v \in \Omega : \ \min_{[0,1]} v > 0, \ \ \max_{[0,1]} v < r \Big\},$$

then every solution $v \in \Omega$ must belong to $\mathcal{O}$. Thus, the excision property of the degree yields

$$\deg_{LS}\big(\mathcal{I} - \mathcal{T}(0,\,\cdot\,), \mathcal{O}\big) = -1.$$

2.6 Existence of a continuum and conclusion of the proof

The boundedness of $\partial\mathcal{O}$ in $\mathcal{C}^1[0,1]$ and the complete continuity of the operator $\mathcal{T}$ guarantee the existence of some $\varepsilon^* > 0$ such that $\mathcal{T}(\varepsilon,\,\cdot\,)$ has no fixed points on $\partial\mathcal{O}$ for all $\varepsilon \in [0,\varepsilon^*]$. Consequently, the homotopy property of the degree implies that $\deg_{LS}(\mathcal{I} - \mathcal{T}(0,\varepsilon), \mathcal{O}) = -1$ for all $\varepsilon \in [0,\varepsilon^*]$, and hence the existence of at least one solution $v = v_\varepsilon \in \mathcal{O}$ of the problem (2.3) for all $\varepsilon \in [0,\varepsilon^*]$. Actually, the Leray–Schauder continuation theorem [3, p. 63] provides us with a continuum $\mathscr{K}^+$ of solutions $(\varepsilon, v_\varepsilon)$ of (2.3) with $\varepsilon \in [0,\varepsilon^*]$ and $v_\varepsilon \in \mathcal{O}$.

The change of variables (2.1) then implies the existence of a closed connected set $\mathscr{C}^+$ of solutions (λ, u_λ) of (1.1), where $\lambda = \varepsilon^{1-p} \in [\lambda_*, \infty)$, with $\lambda_* = (\varepsilon^*)^{1-p}$, and

$$u_\lambda = \varepsilon v_\varepsilon = \lambda^{\frac{1}{1-p}} v_\varepsilon.$$

It is apparent that every $(\lambda, u_\lambda) \in \mathscr{C}^+$ is strictly positive and satisfies conditions (ii) and (iii).

Finally, adapting the results in [2], we can prove the existence, for each $\varepsilon \in [0,\varepsilon^*]$, of a Lyapunov unstable solution $v \in \mathcal{O}$ of (2.4). Consequently, for every $\lambda \in [\lambda_*, \infty)$ there is at least one unstable solution u_λ of (1.1) which is strictly positive and satisfies (ii) and (iii). This concludes the proof of Theorem 1.1.

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Some Tests for Regularity and Almost Reducibility for Limit Periodic Linear Systems

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Consider a linear differential system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^2, \quad t \in \mathbb{R}, \tag{1}$$

with piecewise continuous and bounded coefficient matrix A of the form

$$A(t) = \sum_{k=0}^{+\infty} A_k(t), \tag{2}$$

where A_k, $k = 0, \dots, +\infty$, are periodic matrices with the periods T_k. If each matrix A_k is everywhere continuous and series (2) converges uniformly on the entire time axis $\mathbb{R}$, then the matrix A is limit-periodic [1, p. 32] and, therefore, almost periodic. The problem on Lyapunov regularity of linear systems with almost periodic coefficients was posed by N. P. Erugin at a mathematical seminar at the Institute of Physics and Mathematics of Byelorussian Academy of Sciences in 1956. The formulation of this problem was published in [3, pp. 121, 137], see also [4].

In [6], using some results of [5] V. M. Millionshchikov has proved the existence of some Lyapunov-irregular linear system with limit periodic coefficients. To this end V. M. Millionshchikov has introduced some special class of linear systems. A comprehensive study of systems from Millionshchikov class was made by A. V. Lipntskii in [7–14]. In particular, an explicit example of Lyapunov-irregular system from the Millionshchikov class is given in [7], see also [17].

On the other hand, it is well known [5,15,16], that the set of Lyapunov-regular (and even almost reducible, for the definition of almost reducibility see [2]) systems with almost periodic coefficients is large in some natural sense. However no effective tools to recognize these properties are known.

Our aim here is to give some sufficient conditions for linear systems from Millionshchikov class to be Lyapunov regular or almost reducible. The conditions of regularity and almost reducibility provided by Theorem 1 below are not coefficient, but may be useful in constructing systems from Millionshchikov class with prescribed asymptotic properties.

In what follows we suppose that $T_0 = 2$, $T_k \in \mathbb{N}$, and $T_{k+1}/T_k = m_i \in \mathbb{N}$ for all $k = 0, \dots, +\infty$. We also suppose that $m_k > 1$, $k = 0, \dots, +\infty$. Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Take some continuous function $\omega : [0,1] \to \mathbb{R}$ such that $\omega(0) = \omega(1) = 0$ and $\int\limits_0^1 \omega(t)\,dt = 1$. Take also a sequence $\varphi : \mathbb{N} \to [0, \pi/2[$. As usually, the values of the sequence φ we denote by φ_k, $k \in \mathbb{N}$.

Now let us define the matrices A_k by the following equalities:

$$A_0(t) = \begin{cases} \omega(t)D, & \text{for } t \in [0,1[\,, \\ 0, & \text{for } t \in [1,2[\,, \end{cases} \tag{3}$$

for $k=0$ and

$$A_k(t)=\begin{cases}-\varphi_k\omega(t)J, & \text{for } t\in[0,1[\,,\\ 0, & \text{for } t\in[1,T_i[\,,\end{cases}\tag{4}$$

for all $k=1,\dots,+\infty$.

Lemma 1. *If* $\sum\limits_{k=1}^{\infty}\varphi_k<+\infty$, *then system* (1) *with the coefficient matrix A defined by* (3) *and* (4) *is limit periodic.*

Let $S_m(t)=\sum\limits_{k=0}^{m}A_k(t)$, $m=1,\dots,+\infty$, where A_k are defined by (3) and (4). It can be easily seen that each matrix S_m is T_m-periodic. Now for arbitrary $m\in\mathbb{N}$ consider a periodic linear system

$$\dot z=S_m(t)z,\ \ z\in\mathbb{R}^2,\ \ t\in\mathbb{R}.\tag{5}$$

Denote the Cauchy matrix of system (5) by Z_m. Then the monodromy matrix of system (5) can be written as $Z_m(T_m,0)$. Hence the eigenvalues of $Z_m(T_m,0)$ are the Floquet multipliers of system (5).

Definition. We say that system (1) with the coefficient matrix A defined by (3) and (4) is a real-type system if all Floquet multipliers of each corresponding system (5) with $m\in\mathbb{N}$ are real.

Remark. Note that the condition $\varphi_k\in[0,\pi/2[$ guarantees that the Floquet multipliers of system (5) are positive.

Lemma 2. *If system* (1) *with the coefficient matrix A defined by* (3) *and* (4) *is a real-type system, then all eigenvectors of matrices $Z_m(T_m,0)$, $m\in\mathbb{N}$ lie in the first quadrant, i.e. have positive coordinates.*

Suppose that system (1) with the coefficient matrix A defined by (3) and (4) is a real-type system. Let ζ_1^m and ζ_2^m be some eigenvectors of $Z_m(T_m,0)$, where each vector ζ_2^m corresponds to greater eigenvalue of $Z_m(T_m,0)$. Denote the angle between ζ_1^m and ζ_2^m by β_m.

Theorem 1. *The following statements are valid:*

(i) *If the angle β_k is separated from zero, then system* (1) *is almost reducible.*

(ii) *If* $\varliminf\limits_{k\to\infty}T_k^{-1}\ln\beta_k=0$, *then system* (1) *is Lyapunov regular.*

To prove the first statement we use the fact that system (1) lies in the closure of the set of reducible systems. The second statement is based on the following lemma.

Lemma 3. *Let x_{mj} be the solution of system* (1) *satisfying the condition $x_{mj}(jT_m)=\zeta_2^m$ for some $j\in\mathbb{N}$. Then the vectors $x_{mj}(t)$ lie between ζ_1^m and ζ_2^m for all $t=(j+l)T_m$, $l\in\mathbb{N}$.*

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Uncertainty Estimates for Target Functionals Values to a Class of Continuous-Discrete Systems with Incomplete Information

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1 Introduction

A linear control system with continuous and discrete times and discrete memory is considered. The model includes an uncertainty in the description of operators implementing control actions. This uncertainty is a consequence of random disturbances under the assumption of their uniform distribution over known intervals.

With each implementation a corresponding trajectory arises from random perturbations, and in the aggregate – an ensemble of trajectories. Thus, there arises a set of values of target functionals in the control problem. For each functional, the probabilistic description is given in the form of corresponding probability density functions. To construct these functions, the previously obtained representation of the Cauchy operator of the system under consideration is used. The proposed probabilistic description allows one to find their standard characteristics, including expectation and variance, as well as the entire possible range of values. The results are constructive in nature and allow for effective computer implementation.

2 Description of the problem

Fix a finite segment $[0,T] \subset \mathbb{R}$. Denote by $L^n = L^n[0,T]$ the space of summable functions $v : [0,T] \to \mathbb{R}^n$ with the norm $\|v\|_{L^n} = \int\limits_0^T |v(s)|_n \, ds$, where $|\cdot|_n$ stands for a norm in $\mathbb{R}^n$; $L_2^r = L_2^r[0,T]$ is the space of square summable functions $v : [0,T] \to \mathbb{R}^r$ with the inner product $\langle u, v\rangle = \int\limits_0^T u'(s) \cdot v(s)\, ds$ ($(\,\cdot\,)'$ stands for transposition); $AC^n = AC^n[0,T]$ is the space of absolutely continuous functions $x : [0,T] \to \mathbb{R}^n$ with the norm $\|x\|_{AC^n} = |x(0)|_n + \|\dot{x}\|_{L^n}$. Next we fix the set $J = \{t_0, t_1, \dots, t_\mu\}$, $0 = t_0 < t_1 < \dots < t_\mu = T$ and denote by $FD^\nu(\mu) = FD^\nu\{t_0, t_1, \dots, t_\mu\}$ the space of functions of discrete argument $z : J \to \mathbb{R}^\nu$ with the norm

$$\|z\|_{FD^\nu(\mu)} = \sum_{i=0}^{\mu} |z(t_i)|_\nu .$$

We consider the continuous-discrete system with discrete memory

$$\dot{x}(t) = \sum_{j:\, t_j < t} A_j(t) x(t_j) + \sum_{j:\, t_j < t} B_j(t) z(t_j) + (Fu)(t), \quad t \in [0,T], \tag{2.1}$$

$$z(t_i) = \sum_{j<i} D_{ij} x(t_j) + \sum_{j<i} H_{ij} z(t_j) + (Gu)(t_i), \quad i = 1, \dots, \mu. \tag{2.2}$$

Here the columns of $(n \times n)$-matrix A_j and $(n \times \nu)$-matrix B_j belong to L^n; D_{ij} and H_{ij} are constant matrices of dimension $(\nu \times n)$ and $(\nu \times \nu)$, respectively; $F : L_2^r \to L^n$, $G : L_2^r \to FD^\nu(\mu)$ are linear bounded Volterra [1] operators. Discreteness of the memory to all operators acting onto the state variable in (2.1), (2.2) is defined by their construction.

For the system under control (2.1), (2.2) with a given initial state

$$x(0) = \alpha, \quad z(0) = \delta \tag{2.3}$$

we consider the control problem with the aim of control given by the equality

$$\ell(x, z) = \beta \in \mathbb{R}^{\mathcal{N}}, \tag{2.4}$$

where $\ell : AC^n \times FD^\nu(\mu) \to \mathbb{R}^N$ is a linear bounded vector-functional.

Conditions of the solvability to the problem (2.1), (2.2) within the class of programmed control are obtained for the case of unconstrained control and for the case of point-wise polyhedral constraints [2,4,7,8]. Here we study the question on the impact of random disturbances of operators F and G onto the values of the target vector-functional $\ell(x, z)$ when the control is known. Without loss of generality we suppose the initial position of the system (2.1), (2.2) to be zero: $\alpha = 0$, $\delta = 0$.

Define the form of disturbances in the action of the operators F and G:

$$(Fu)(t) = (F_0 u)(t) + \Delta F \cdot u(t), \;\; t \in [0, T],$$

$$(Gu)(t_j) = (G_0 u)(t_j) + \Delta G_j \cdot \int_0^{t_j} u(s)\, ds, \;\; j = 1, \ldots, \mu.$$

Here ΔF and ΔG_j are matrices of dimension $n \times r$ and $\nu \times r$, respectively, with the elements ΔF^{ik} and ΔG_j^{ik} being random values distributed uniformly on the segments $[a^{ik}, b^{ik}]$ and $[a_j^{ik}, b_j^{ik}]$, respectively (we write for short $\Delta F^{ik} \sim U^{ik}$ and $\Delta G_j^{ik} \sim U_j^{ik}$). The operators F_0 and G_0 are assumed to be acting with no disturbances.

In [9], a component-by-component probabilistic description is obtained for the components of $x(t)$ and $z(t_j)$. This description is given in the form of a set of probability density functions parametrized by the current time. To construct these functions, the previously obtained representation of the Cauchy operator of the system under consideration is used.

The system (2.1), (2.2) is a particular case of the general continuous-discrete system considered in [5]. Theorem 1 [5] gives the presentation to solution of (2.1), (2.2) with zero initial values:

$$\begin{pmatrix} x \\ z \end{pmatrix} = \mathcal{C} \begin{pmatrix} Fu \\ Gu \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} \\ \mathcal{C}_{21} & \mathcal{C}_{22} \end{pmatrix} \begin{pmatrix} Fu \\ Gu \end{pmatrix}, \tag{2.5}$$

where $z = \operatorname{col}(z(t_1), \ldots, z(t_\mu))$, $\mathcal{C}$ is the Cauchy operator with block components $\mathcal{C}_{ij}$, $i, j = 1, 2$.

As applied to the case under consideration, the explicit representation of $\mathcal{C}_{ij}$ in the terms of matrix parameters of (2.1), (2.2) is obtained in [6]. In the sequel, we use the components

$$(\mathcal{C}_{11} f)(t) = \int_0^t C_{11}(t, s) f(s)\, ds, \quad (\mathcal{C}_{12} g)(t) = \int_0^t \sum_{j:\, t_j < t} C_{12}(t_j, s) g(t_j)\, ds, \;\; t \in [0, T],$$

$$\mathcal{C}_{21}^i f = \int_0^{t_i} \sum_{j<i} C_{21}^i(t_j, s) f(s)\, ds, \quad \mathcal{C}_{22}^i g = \sum_{j \le i} C_{22}^i(j) g(t_j), \;\; i = 1, \ldots, \mu.$$

Here the upper index in notations $\mathcal{C}_{22}^i$ and $\mathcal{C}_{22}^i$ stands for the number of a ν-column in a column from $\mathbb{R}^{\nu\mu}$.

Each component of the solution (x, z) includes the determined term $x_i^0(t)$, $z_i^0(t_j)$ correspondingly to operators F_0 and G_0 and the random term $\xi_i(t)$, $\eta_i(t_j)$ that corresponds to matrices ΔF and ΔG_j:

$$x_i(t) = x_i^0(t) + \xi_i(t), \quad z_i(t_j) = z_i^0(t_j) + \eta_i(t_j).$$

Thus for $\ell x = \operatorname{col}(\ell_1(x,z), \dots, \ell_N(x,z))$ we have

$$\ell_i(x,z) = \ell_i(x^0, z^0) + \ell_i(\xi, \eta),$$

and we are aimed at the description of the distribution to random values $\lambda_i \equiv \ell_i(\xi, \eta)$.

Let us recall the general form of $\ell : AC^n \times FD^\nu(\mu) \to \mathbb{R}^N$:

$$\ell(x,z) = \Psi x(0) + \int\limits_0^T \Phi(s)\dot{x}(s)\,ds + \sum_{j=0}^{\mu} \Gamma_j z(t_j),$$

covering various special cases of target vector-functionals such as multipoint, integral and many others.

Due to (2.5), we have

$$\xi_i(t) = \sum_{\ell=1}^{n}\sum_{k=1}^{r} {}^{11}\theta^i_{\ell k}(t)\,\Delta F^{\ell k} + \sum_{j:\, t_j<t}\ \sum_{\ell=1}^{\nu}\sum_{k=1}^{r} {}^{12}\theta^i_{j\ell k}(t)\,\Delta G_j^{\ell k} \tag{2.6}$$

and

$$\eta_i(t_j) = \sum_{m=1}^{n}\sum_{k=1}^{r} {}^{21}\theta^{ij}_{mk}\Delta F^{mk} + \sum_{m=1}^{\nu}\sum_{k=1}^{r}\sum_{\ell=1}^{\mu} {}^{22}\theta^{ij}_{\ell mk}\Delta G_\ell^{mk}, \tag{2.7}$$

where matrices ${}^{11}\theta^{ij}$, ${}^{12}\theta^{ij}$, ${}^{21}\theta^{ij}$, ${}^{22}\theta^{ij}$ are defined in [9].

In the way described in [9] we rewrite (2.6), (2.7) in the form

$$\xi_i(t) = \sum_{q=1}^{N} \varphi^i_q(t)\cdot(b_q - a_q)\cdot c_q + \sigma_i(t), \quad \sigma_i(t) = \sum_{q=1}^{N} \varphi^i_q(t)\cdot a_q$$

and

$$\eta_i(t_j) = \sum_{q=1}^{N} \psi^i_q(t_j)\cdot(b_q - a_q)\cdot c_q + \omega_i(t_j), \quad \omega_i(t_j) = \sum_{q=1}^{N} \psi^i_q(t_j)\cdot a_q,$$

where $N = n\cdot r + \nu\cdot\mu\cdot r$.

In the sequel, we will use the component-wise representation of the target vector-functional ℓ:

$$\ell_i(x,z) = {}^1\ell_i(x) + {}^2\ell_i(z) = \sum_{j=1}^{n} {}^1\ell_i^j(x_j) + \sum_{j=1}^{\nu}\sum_{k=0}^{\mu} {}^2\ell_i^j(z_j(t_k)).$$

Hence it follows that

$$\lambda_i = \ell_i(\xi,\eta) = \sum_{q=1}^{N} \varkappa_i^q\cdot(b_q - a_q)\cdot c_q + \gamma_i, \tag{2.8}$$

where

$$\varkappa_i^q = \sum_{j=1}^{n} {}^1\ell_i^j(\varphi_q^j) + \sum_{j=1}^{\nu}\sum_{k=0}^{\mu} {}^2\ell_i^{jk}(\psi_q^j(t_k)), \quad \gamma_i = \sum_{j=1}^{n} {}^1\ell_i^j(\sigma_j) + \sum_{j=1}^{\nu}\sum_{k=0}^{\mu} {}^2\ell_i^{jk}(\omega_j^i(t_k)). \tag{2.9}$$

3 Main result

For any $y_1 \in \mathbb{R}$, we define in $\mathbb{R}^{N-1}$ the polyhedral set $\mathcal{M}_i(y_1)$:

$$\mathcal{M}_i(y_1) = \Big\{(y_2, \dots, y_N)' \in \mathbb{R}^{N-1}: \ \ 0 \le y_q \le 1, \ \ q = 2, \dots, N;$$
$$\frac{1}{\varkappa_i^1 \cdot (b_1 - a_1)} \cdot y_1 - 1 \le \sum_{q=2}^{N} \frac{\varkappa_i^q \cdot (b_q - a_q)}{\varkappa_i^1 \cdot (b_1 - a_1)} \cdot y_q \le \frac{1}{\varkappa_i^1 \cdot (b_1 - a_1)} \cdot y_1 \Big\}.$$

Theorem. *Let $\varkappa_i^q$, $i = 1, \dots, n$, $q = 1, \dots, N$, and γ_i, $i = 1, \dots, n$, be defined by equalities* (2.9)*, and $\varkappa_i^1 \ne 0$. Then the probability density function $f_{\lambda_i}(y_1)$ of the random variable* (2.8) *is defined by the equality*

$$f_{\lambda_i}(y_1) = \frac{\mathbf{V}^{N-1}[\mathcal{M}_i(y_1 - \gamma_1)]}{|\varkappa_i^1| \cdot (b_1 - a_1)} \,,$$

where $\mathbf{V}^{N-1}[\mathcal{M}]$ is the Lebesgue measure of a set $\mathcal{M} \subset \mathbb{R}^{N-1}$.

Emphasize in conclusion that this result allows to find a segment I_i of all possible values for each component of the target vector-functional and calculate the probability $P(\lambda_i \in J_i)$ for any subset $J_i \subset I_i$. This can be useful when studying control problems with a given target set (see, for instance, [3] and the references therein).

Acknowledgements

This work is supported by the Russian Science Foundation, Project # 22–21–00517.

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On the Stability of Toroidal Manifold for One Class of Dynamical System

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We study the stability of the invariant toroidal manifold of one class, the linear extension of a dynamical system on a torus. The result is used to study the question of the existence of an invariant manifold of a nonlinear system of differential equations.

In the direct product of an m-dimensional torus T_m and Euclidean space R^n we consider a system of differential equations

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = P(\varphi)x, \tag{1}$$

where $\varphi = col(\varphi_1, \dots, \varphi_m)$, $x = col(x_1, \dots, x_n)$, $a(\varphi), P(\varphi)$ are, respectively, vector and matrix functions continuous and 2π-periodic in each component φ_j $(j = 1, \dots, m)$ and defined on the m-dimensional torus T_m. Assume that the function $a(\varphi)$ satisfies the Lipschitz condition with respect to φ, a constant L, i.e. for any two points $\varphi', \varphi'' \in T_m$ we have

$$\|a(\varphi') - a(\varphi'')\| \le L\|\varphi' - \varphi''\|. \tag{2}$$

We establish sufficient conditions for the asymptotic stability of the trivial torus of system (1) and use these results for the investigation of nonlinear system of differential equations more complicated than system (1) and defined in the direct product $T_m \times R^n$.

In what follows, we need a generalization of the Wazewski inequality [2]. By $\varphi_t(\varphi)$ we denote the solution of the first equation in system (1) and consider a system of equation

$$\frac{dx}{dt} = P(\varphi_t(\varphi))x \tag{2}$$

for x. According to the Wazewski theorem [2], any solution $x_t(t_0, \varphi, x_0)$, $x_{t_0}(t_0, \varphi, x_0) = x_0$ of this system admits

$$\|x_0\| e^{\int\limits_{t_0}^{t} \lambda(\varphi_s(\varphi))\, ds} \le \|x_t(t_0, \varphi, x_0)\| \le \|x_0\| e^{\int\limits_{t_0}^{t} \Lambda(\varphi_s(\varphi))\, ds}, \tag{3}$$

where $\lambda(\varphi)$ and $\Lambda(\varphi)$ are, respectively, the maximum and minimum eigenvalues of the symmetric matrix

$$\widehat{P}(\varphi) = \frac{1}{2}\big(P(\varphi) + P^T(\varphi)\big),$$

$P^T(\varphi)$ is the matrix transposed with respect to the matrix $P(\varphi)$.

Inequality (3) yields the estimate

$$\|x_0\|e^{\lambda(t-t_0)} \leq \|x_t(t_0,\varphi,x_0)\| \leq \|x_0\|e^{\Lambda(t-t_0)}, \ \ t \geq t_0,$$

where

$$\lambda = \min_{\varphi\in T_m} \lambda(\varphi), \quad \Lambda = \max_{\varphi\in T_m} \Lambda(\varphi).$$

On the basic of this estimate, we can make the following conclusion: If the matrix $P(\varphi)$ in system (1) is such that $\Lambda < 0$, than the nontrivial torus of this system is exponentially stable because the matricant $\Omega^t_{t_0}(\varphi)$ of system (2) admits the estimate

$$\|\Omega^t_{t_0}(\varphi)\| \leq Ke^{-\gamma(t-t_0)}, \ \ t \geq t_0, \tag{4}$$

$\varphi \in T_m$, $K \geq 1$, $\gamma > 0$.

We now show that a similar conclusion concerning the exponential stability of the trivial torus of the system of equations (1) can be made under weaker conditions imposed on the matrix $P(\varphi)$.

Recall [5] that a point $\varphi \in T_m$ of the dynamical system on a torus

$$\frac{d\varphi}{dt} = a(\varphi) \tag{5}$$

is called wandering if there exist its neighborhood $U(\varphi)$ and a positive number T such that

$$U(\varphi) \cap \varphi_t(U(\varphi)) = \varnothing \ \text{ for } \ t \geq T.$$

By W we denote the set of wandering points and by $\Omega = T_m \setminus W$ we denote the set of nonwandering points. The set W of wandering points is invariant and open because, together with φ, all points of the neighborhood $U(\varphi)$ are wandering.

In view of the compactness of the torus T_m, the set of nonwandering point Ω is a nonempty closed invariant set.

It is clear that Ω is also a compact set as a closed set on the torus.

As shown in [5], any solution of system (5) eventually approaches the set of nonwandering points. More precisely, for any $\varepsilon > 0$, every phase point $\varphi_t(\varphi)$ lies outside the ε-neihghborhood $U_\varepsilon(\Omega)$ of the set Ω only for a finite time interval not larger than $T(\varepsilon)$.

To prove the theorem presented below, we use the property of nonwandering points.

Theorem 1. *If the matrix $P(\varphi)$ in the system of equations* (1) *such that the maximum eigenvalue $\Lambda(\varphi)$ of the symetric matrix $\widehat{P}(\varphi)$ is negative on the set Ω of nonwandering points of the dynamical system* (5)*, then the trivial torus of system* (1) *is exponentially stable.*

Proof. We fix sufficiently small ε-neighborhood $U_\varepsilon(\Omega)$ of the set Ω. Since $\Lambda(\varphi) < 0$ for all $\varphi \in \Omega$ and Ω is closed compact set, one can find a sufficiently small positive number ε_0 such that $\Lambda(\varphi) < -\gamma(\varepsilon)$ for any $0 \leq \varepsilon \leq \varepsilon_0$, $\Lambda(\varphi) < -\gamma(\varepsilon)$ and all $\varphi \in U_\varepsilon(\Omega)$, where $\gamma(\varepsilon)$ is a positive monotonically nonincreasing function of the parameter ε such that $\gamma(\varepsilon) \to \gamma(0)$ as $\varepsilon \to 0$, where

$$-\gamma(0) = \max_{\varphi\in\Omega} \Lambda(\varphi).$$

If $\varphi_t(\varphi)$ is nonwandering trajectory, then, for any solution from inequality (3), we get the following estimate:

$$\|x_t(t_0,\varphi,x_0)\| \leq \|x_0\|e^{-\gamma(0)(t-t_0)}, \ \ t \geq t_0, \ \ \varphi \in \Omega.$$

If $\varphi_t(\varphi)$ is a wandering trajectory, then one can find a positive number $T(\varepsilon)$ such that the time of stay of this trajectory outside the set $U_\varepsilon(\Omega)$ is not greater than $T(\varepsilon)$. By using inequality (3), we obtain

$$\|x_t(t_0,\varphi,x_0)\| \le \|x_0\| e^{\Lambda T(\varepsilon)} \cdot e^{-\gamma(\varepsilon)(t-t_0)}, \ \ t \ge t_0, \ \ \varphi \in W.$$

Hence, under the conditions of the theorem, any solution $x_t(t_0,\varphi,x_0)$ of system (2) exponentially approaches zero as $t \to \infty$ for any $\varphi \in T_m$. Therefore, the matricant $\Omega^t_{t_0}(\varphi)$ of the system admits an estimate of the form (4), which completes the proof of the theorem. □

We now present one more class of system (1) for which the trivial torus is asymptotically stable.

Theorem 2. *If the matrix function $P(\varphi)$ in the system of equations* (1) *satisfies the condition*

$$\langle P(\varphi)x, x\rangle \le \gamma(\varphi)\langle x, x\rangle$$

for all $\varphi \in T_m$ and $x \in R^n$, where $\gamma(\varphi)$ is a function continuous and 2π-periodic in each component φ_j $(j = 1,\dots,n)$ and negative on the set Ω of nonwandering points of the dynamical system (5), *then the trivial torus of the original system* (1) *is asymptotically stable.*

Proof. For any solution $x_t(t_0,\varphi,x_0)$ of system (2), we obtain:

$$\begin{aligned}\frac{d}{dt}\,\|x_t(t_0,\varphi,x_0)\|^2 &= \frac{d}{dt}\,\big\langle x_t(t_0,\varphi,x_0), x_t(t_0,\varphi,x_0)\big\rangle \\ &= 2\big\langle P(\varphi_t(\varphi))x_t(t_0,\varphi,x_0), x_t(t_0,\varphi,x_0)\big\rangle \le 2\gamma(\varphi_t(\varphi))\|x_t(t_0,\varphi,x_0)\|^2.\end{aligned}$$

Integrating the last inequality, we find

$$\|x_t(t_0,\varphi,x_0)\| \le e^{\int\limits_{t_0}^{t}\gamma(\varphi_S(\varphi))\,dS}\|x_0\|, \ \ t \ge t_0, \ \ \varphi \in T_m.$$

Reasoning as in the proof of the previous theorem, we conclude that the exponential stability of the trivial torus of the original system follows from the last inequality. □

To prove that the invariant torus is stable (unstable), we can use the direct Lyapunov method. We now present a theorem that partially supplements the result of classical investigations in this field presented in the monographs [3, 7].

Theorem 3. *Suppose that, for the system of equations* (1), *there exists a positive-definite quadratic form*

$$V(\varphi,x) = \langle S(\varphi)x, x\rangle$$

with symmetric matrix $S(\varphi)$ such that its total derivative composed with the use of the original system (1), *i.e., the quadratic form*

$$\frac{d}{dt}V(\varphi,x) = \langle \widehat{S}(\varphi)x, x\rangle,$$

where

$$\widehat{S}(\varphi) = \frac{\partial S(\varphi)}{\partial \varphi}\cdot a(\varphi) + S(\varphi)P(\varphi) + P^T(\varphi)S(\varphi),$$

is negative-definite of the set Ω of nonwandering points of system (5). *Then the trivial torus of the system of equations* (1) *is exponentially stable.*

It is natural to study the problem of existence of the quadratic form $V(\varphi, x)$, atisfying the conditions of Theorem 3.

We now present an example in which this form exists and makes it possible to state that the trivial torus of system (1) is exponentially stable.

Theorem 4. *Suppose that* $P(\varphi)$ *in system* (1) *is a constant matrix* $P(\varphi) = P_0$ *on the set* Ω. *If the real parts of the eigenvalues* $\operatorname{Re}\lambda_j(P_0)$ *of the matrix* P_0 *are negative, then there exists a positive-definite quadratic form* $v(\varphi, x) = \langle S(\varphi)x, x\rangle$ *with symmetric matrix* $S(\varphi)$ *such that its derivative, according to system* (1)*, is a negative-definite quadratic form on the set* Ω*, and, hence, the trivial torus of system* (1) *is asymptotically stable.*

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On the Existence of Bifurcation Points for Periodic Problems to Distributional Differential Equations

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1 Introduction

The contribution is a continuation of the research started in [7] and [2]. We deal with periodic problems for nonlinear distributional (measure) differential equations with a parameter. In particular, we are interested in the existence of the bifurcation points for such problems.

The concept of distributional differential equations arose more or less together with the concept of systems with impulses. In general, they can describe some physical or biological problems, such as heartbeat, blood flow, pulse/frequency modulated systems, biological neural networks and/or models arising in control theory in which measures can be suitable controls, cf. e.g. [10]. Of course, differential equations with measures appear also in non-smooth mechanics. In these models, derivatives are understood in the sense of distributions and the solutions are generally discontinuous, but not too bad from another point of view, i.e. they are usually regulated or have bounded variation. For some early results, see e.g. [1] and references therein.

In this article we consider distributional differential systems of the form

$$Dx = f(\lambda, x, t) + g(x, t) \cdot Dh, \tag{1.1}$$

where D stands for the distributional derivatives and λ is a parameter. To this end, a handful tool are generalized ordinary differential equations (we write simply GODEs) introduced by Kurzweil in [3, 4] in the middle of 1950's. Since then, many authors have dealt with the potentialities of this theory (see e.g. [5, 9, 11] and references therein). In particular, measure differential equations of the form (1.1) as well as equations with impulses acting in fixed times are their special cases.

Throughout $G[0,T]$ is the Banach space of regulated functions (functions having all onesided limits) with values in $\mathbb{R}^n$ and equipped with the supremal norm and $BV[0,T] \subset G[0,T]$ is the space of functions with bounded variation on $[0,T]$. As usual, we denote $\Delta^+ x(s) = x(s+) - x(s)$ and $\Delta^- x(t) = x(t) - x(t-)$ for $x \in G[0,T]$. Our basic assumptions are the following:

Assumptions 1.1. $T \in (0,\infty)$, $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}$ are open sets, $f : \Lambda \times \Omega \times [0,T] \to \mathbb{R}^n$, $g : \Omega \times [0,T] \to \mathbb{R}^n$, $h : \mathbb{R} \to \mathbb{R}$ has a bounded variation on $[0,T]$ and is left-continuous on $[0,T]$, while $h(0-) = h(0)$ and $h(T+) = h(T)$.

2 Distributional differential equations

By distributions we understand linear continuous n-vector functionals on the topological vector space $\mathcal{D}^n$ of functions $\varphi : \mathbb{R} \to \mathbb{R}^n$ possessing for any $j \in N \cup \{0\}$ a derivative $\varphi^{(j)}$ of the order j which is continuous on $\mathbb{R}$ and such that $\varphi^{(j)}(t) = 0$ if $t \notin (0,T)$. The space $\mathcal{D}^n$ is endowed by the topology in which the sequence $\varphi_k \in \mathcal{D}$ tends to $\varphi_0 \in \mathcal{D}$ in $\mathcal{D}$ if and only if $\lim_k \|\varphi_k^{(j)} - \varphi_0^{(j)}\|_\infty = 0$ for all non negative integers j.

The space of n-vector distributions on $[0,T]$ (dual of $\mathcal{D}^n$) is denoted by $\mathcal{D}^{n*}$. Instead of $\mathcal{D}^{1*}$ we write $\mathcal{D}^*$. Given a distribution $f \in \mathcal{D}^{n*}$ and a test function $\varphi \in \mathcal{D}^n$, $\langle f, \varphi \rangle$ is the value of the functional f on φ. Of course, reasonable real valued point functions are naturally included into distributions. The zero distribution $0 \in \mathcal{D}^{n*}$ on $[0,T]$ can be identified with an arbitrary measurable function vanishing a.e. on $[0,T]$. Obviously, if $f \in G[0,T]$ is left-continuous on $(0,T]$, then $f = 0 \in \mathcal{D}^{*n}$ if and only if $f(t) \equiv 0$.

For $h \in \mathcal{D}^*$, the symbol Dh stands for its distributional derivative, i.e.

$$Dh : \varphi \in \mathcal{D} \to \langle Dh, \varphi \rangle = -\langle h, \varphi' \rangle \text{ for all } \varphi \in \mathcal{D}.$$

If $f \in AC[0,T]$, then $Df = f'$, of course.

The term $g(t,x) \cdot Dh$ on the right hand side of (1.1) is a symbol for the distributional product of the function $\widetilde{g}_x : t \in [0,T] \to g(x(t),t)$ and the derivative Dh of h. As in the Schwartz setting no general rule how to define a product of an arbitrary couple of distributions is available, some more explanation is desirable. In text-books one can find the trivial examples. However, the product occurring in (1.1) is not covered by these cases. Fortunately, it turned out that, for this aim, a good tool is provided by the Kurzweil–Stieltjes integral. The following definition has been introduced in [12] and was used in [9, Section 8.4], as well.

Definition 2.1. If $g : [0,T] \to \mathbb{R}^n$ and $h : [0,T] \to \mathbb{R}$ are such that the Kurzweil–Stieltjes integral $\int_0^T g\,dh$ exists, then the product $g \cdot Dh$ is the distributional derivative of the indefinite integral $H(t) = \int_0^t g\,dh$, i.e. $g \cdot Dh = DH$.

The multiplication operation given by Definition 2.1 has all the usual properties excepting that (cf. [12, Remark 4.1] and [9, Theorem 6.4.2]) the expected formula $D(f \cdot g) = Df \cdot g + f \cdot Dg$ does not hold, in general. More precisely, if f and g are regulated and at least one of them has a bounded variation, then

$$D(f \cdot g) = Df \cdot g + f \cdot Dg + Df \cdot \Delta^+\widetilde{g} - \Delta^-\widetilde{f} \cdot Dg,$$

where

$$\Delta^+\widetilde{g}(t) = \begin{cases} \Delta^+ g(t) & \text{if } t < T, \\ 0 & \text{if } t = T \end{cases} \quad \text{and} \quad \Delta^-\widetilde{f}(t) = \begin{cases} 0 & \text{if } t = 0, \\ \Delta^- f(t) & \text{if } t > 0. \end{cases}$$

Now, we can define solutions of (1.1) as follows:

Definition 2.2. A couple $(x,\lambda) \in G[0,T] \times \Lambda$ is a solution of (1.1) if x is left-continuous on $(0,T]$, $x(t) \in \Omega$ for all $t \in [0,T]$, the distributional product $\widetilde{g}_x \cdot Dh$ of the function $\widetilde{g}_x : t \in [0,T] \to g(x(t),t) \in \mathbb{R}^n$ with Dh has a sense and the equality (1.1) is satisfied in the distributional sense, i.e. $\langle Dx, \varphi \rangle = \langle \widetilde{f}_{\lambda,x}, \varphi \rangle + \langle \widetilde{g}_x \cdot Dh, \varphi \rangle$ for all $\varphi \in \mathcal{D}^n$, where $\widetilde{f}_{\lambda,x} : t \in [0,T] \to f(\lambda, x(t), t) \in \mathbb{R}^n$.

Together with (1.1) let us consider two related equations

$$x(t)=x(0)+\int_0^t f(\lambda,x(s),s)\,ds+\int_0^t g(x(s),s)\,dh(s)\ \text{ for }\ t\in[0,T] \tag{2.1}$$

and (GODE)

$$x(t)=x(0)+\int_{t_0}^t DF(x(\tau),t)\ \Big(\text{i.e. } \frac{dx}{d\tau}=DF(x,t)\Big). \tag{2.2}$$

We have

Theorem 2.1. *Let Assumptions 1.1 and*

$$\begin{cases} f(\lambda,\,\cdot\,,\,\cdot\,)\ \textit{ is Carathéodory on }\ \Omega\times[0,T]\ \textit{ for any }\ \lambda\in\Lambda,\\ g(\,\cdot\,,t)\ \textit{ is continuous on } \Omega \textit{ for } t\in[0,T] \textit{ and there is } m_h \textit{ such that:}\\ \qquad\qquad \displaystyle\int_0^T m_h(s)\,d[\operatorname{var}_0^s h]<\infty\ \textit{ and }\ \|g(x,t)\|\le m_h(t)\ \textit{ for }\ (\lambda,x,t)\in\Lambda\times\Omega\times[0,T]. \end{cases}$$

hold and let

$$F(\lambda,x,t)=\int_0^t f(\lambda,x,s)\,ds+\int_0^t g(x,s)\,dh(s)\ \textit{ for }\ (\lambda,x,t)\in\Lambda\times\Omega\times[0,T].$$

Then the equations (1.1), (2.1) *and* (2.2) *are equivalent.*

3 Bifurcations

In the rest we assume that the assumptions of Theorem 2.1 are satisfied. Let us consider the equivalent periodic problems

$$Dx=f(\lambda,x,t)+g(x,t)\,\cdot\,Dh,\quad x(0)=x(T) \tag{3.1}$$

and

$$x(t)=x(T)+\int_0^t f(\lambda,x(s),s)\,ds+\int_0^t g(x(s),s)\,dh(s).$$

Put

$$\Phi(\lambda,x)(t)=x(T)+\int_0^t f(\lambda,x(s),s)\,ds+\int_0^t g(x(s),s)\,dh(s)\ \text{ for }\ \lambda\in\lambda,\ \ x\in B(x_0,\rho),\ \ t\in[0,T].$$

Then $\Phi(\lambda,\,\cdot\,)$ maps $B(x_0,\rho)$ into $G[0,T]$ for any $\lambda\in\Lambda$ and (3.1) is equivalent to finding couples (x,λ) such that $x=\Phi(\lambda,x)$.

Definition 3.1. Let x_0 be a solution of (3.1) for all $\lambda\in\Lambda$ and let $\rho>0$ be such that $x(t)\in\Omega$ for all $t\in[0,T]$ whenever $\|x-x_0\|<\rho$. Then (λ_0,x_0) a bifurcation point of (3.1) if every its neighborhood in $\Lambda\times G[0,T]$ contains a solution (λ,x) such that $x\neq x_0$.

Next result is taken from [2].

Theorem 3.1. *In addition to the assumptions of Theorem 2.1, let x_0 and ρ be as in Definition 3.1 and*

$$\begin{cases} \text{there is a } \gamma:[0,T]\to\mathbb{R} \text{ nondecreasing and such that for any } \varepsilon>0 \text{ there is a } \delta>0 \\ \text{such that } \left\| \int_s^t \big[f(\lambda_2,x,r)-f(\lambda_1,x,r)\big]\,dr \right\| < \varepsilon|\gamma(t)-\gamma(s)| \\ \qquad \text{for } x\in\Omega,\ \ t,s\in[0,T] \text{ and } \lambda_1,\lambda_2\in\Lambda \text{ such that } |\lambda_1-\lambda_2|<\delta \end{cases}$$

and let $[\lambda_1^*,\lambda_2^*]\subset\Lambda$ be such that x_0 is an isolated fixed point of both $\Phi(\lambda_1^*,\,\cdot\,)$ and $\Phi(\lambda_2^*,\,\cdot\,)$ and

$$\deg_{LS}\big(Id-\Phi(\lambda_1^*,\,\cdot\,),B(x_0,\rho),0\big)\neq\deg_{LS}\big(Id-\Phi(\lambda_2^*,\,\cdot\,),B(x_0,\rho),0\big).$$

Then there is $\lambda_0\in[\lambda_1^*,\lambda_2^*]$ such that (x_0,λ_0) is a bifurcation point of (3.1).

The conditions necessary for the pair $(\lambda_0,x_9)\in\Lambda\times\Omega$ to be a bifurcation point of (3.1) are presented in our upcoming paper [8]. One of the equivalent formulations of the main result reads as follows:

Theorem 3.2. *Besides the assumptions of Theorem 3.1, let us assume also*

- *f has a total differential $f_x'(\lambda,x,t)$ for $(\lambda,x,t)\in\Lambda\times\Omega\times[0,T]$ fulfilling Carathéodory conditions withe respect to (x,t);*
- *g has a total differential $g_x'(x,t)$ for $(x,t)\in\Omega\times[0,T]$ which is bounded on $\Omega\times[0,T]$ and continuous with respect to $x\in\Omega$ for each $t\in[0,T]$ and there is $\Theta_h:[0,T]\to\mathbb{R}$ such that*

$$\int_0^T \Theta_h(s)\,d\,[\mathrm{var}_0^s\,h]<\infty \ \ \text{and}\ \ \|g_x'(x,t)\|\le\Theta_h(t);$$

- *there is a nondecreasing function $\gamma:[0,T]\to\mathbb{R}$ such that for any $\varepsilon>0$ there is a $\delta>0$ such that*

$$\left\|\int_s^t \big[f_x'(\lambda_1,x,r)-f_x'(\lambda_2,y,r)\big]\,dr+\int_s^t\big[g_x'(x,r)-g_x'(y,r)\big]\,dh(r)\right\|<\varepsilon|\gamma(t)-\gamma(s)|,$$

whenever $|\lambda_1-\lambda_2|+\|x-y\|<\delta$.

Then the couple $(\lambda_0,x_9)\in\Lambda\times\Omega$ is not a bifurcation point for (3.1) *whenever the homogeneous system*

$$z(r)=z(T)-\int_0^t f_x'(\lambda_0,x_0,\tau)z(\tau)\,d\tau-\int_0^t g_x'(x_0,\tau)z(\tau)\,dh(\tau),\ \ r\in[0,T]$$

have only trivial solutions.

Remark. It is worth noting that in the proofs of theorems 3.1 and 3.2, reformulating the given problem to GODEs proved useful.

Example 3.1. Consider the periodic impulse problem $x' = \lambda\, b(t)x + c(t)x^2$, $\Delta^+ x(\frac{1}{2}) = x^2(\frac{1}{2})$, $x(0) = x(1)$ with $b, c \in L^1[0,1]$ and $\int\limits_0^1 b\, ds \neq 0$. One can verify that, by Theorem 3.1, the couple $(0,0)$ is its bifurcation point, while by Theorem 3.2 the couple $(\lambda, 0)$ can not be a bifurcation point whenever $\lambda \neq 0$.

Example 3.2. One can verify that $u_0(t) = (2+\cos t)^3$ solves for all $\lambda \in \mathbb{R}$ the impulsive problem related to the Liebau valveless pumping phenomena

$$u'' = \lambda\big((2+\cos t)u' + 3(\sin t)u\big) + (6.6 - 5.7\cos t - 9\cos^2 t)u^{1/3} - 0.3u^{2/3},$$
$$\Delta^+ u'\Big(\frac{\pi}{2}\Big) = \Big(64 - u^2\Big(\frac{\pi}{2}\Big)\Big), \quad u(0) = u(2\pi), \;\; u'(0) = u'(2\pi).$$

By Theorem 3.2 and using the result by A. Lomtatidze (cf. [6, Theorem 11.1 and Remark 0.5]) and with some help of the software system Mathematica we can conclude that the couple $(x_0, 0)$ can not be its bifurcation point.

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Sturm–Liouville Operators with Strongly Singular Coefficients: Semi-Boundedness and Self-Adjointness

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The problem of symmetric operators being self-adjoint is one of the main problems in theory of differential operators and serves as the basis for the analysis of their spectral properties and scattering problems. Investigation of this problem for Sturm–Liouville and Schrödinger operators in the space $L^2(\mathbb{R})$ is inspired by the problems of mathematical physics and has numerous applications. The results of that obtained for this problem in the case of regular coefficients are rather complete.

We introduce and investigate symmetric operators L_0 associated in the complex Hilbert space $L^2(\mathbb{R})$ with a formal differential expression

$$l[u] := -(pu')' + qu + i((ru)' + ru') \tag{1}$$

under minimal conditions on the regularity of the coefficients. They are assumed to satisfy conditions

$$q = Q' + s; \quad \frac{1}{\sqrt{|p|}}, \frac{Q}{\sqrt{|p|}}, \frac{r}{\sqrt{|p|}} \in L^2_{loc}(\mathbb{R}), \ \ s \in L^1_{loc}(\mathbb{R}), \tag{2}$$

where the derivative of the function Q is understood in the sense of distributions, and all functions p, Q, r, s are real-valued. In particular, the coefficients q and r' may be Radon measures on $\mathbb{R}$, while function p may be discontinuous. Our main results are two sufficient conditions on coefficients p which provide that the operator L_0 being semi-bounded implies it being self-adjoint.

If these coefficients of (1) are regular enough, then the mapping

$$\mathrm{L}_{00} : u \mapsto l[u], \ \ u \in C_0^\infty(\mathbb{R})$$

defines a densely defined in the complex Hilbert space $L^2(\mathbb{R})$ preminimal symmetric operator L_{00}. Here naturally arises question whether the closure of this operator $\mathrm{L}_0 := (\mathrm{L}_{00})^\sim$ is self-adjoint. A large number of papers are devoted to this problem (see, e.g. the references in [12]). For instance, Hartman [5] and Rellich [10] established that if operator L_{00} is bounded from below and

$$r \equiv 0, \quad 0 < p \in C^2(\mathbb{R}), \ \ q \text{ is piecewise continuous on } \mathbb{R},$$

and function p satisfies the condition

$$\int\limits_0^\infty p^{-1/2}(t)\,dt = \int\limits_{-\infty}^0 p^{-1/2}(t)\,dt = \infty, \tag{3}$$

then the minimal operator L_0 corresponding to l is self-adjoint. In the paper [11], the conditions on the regularity of the coefficients of l were weakened:

$$r \equiv 0, \quad 0 < p \ \text{ is locally Lipschitz}, \ \ q \in L^2_{loc}(\mathbb{R}).$$

Another sufficient condition for the operator L_0 to be self-adjoint was obtained in [1]. It may be written in the form

$$\|p\|_{L^\infty(-\rho,-\rho/2)}, \ \ \|p\|_{L^\infty(\rho/2,\rho)} = O(\rho^2), \ \ \rho \to \infty. \tag{4}$$

Here the coefficients of (1) satisfy the conditions

$$r \equiv 0, \quad 0 < p \in W^1_{2,loc}(\mathbb{R}), \ \ q \in L^1_{loc}(\mathbb{R}).$$

Examples show that conditions (3) and (4) are independent (see [1]).

We propose to consider the operators generated by the formal differential expression (1) as quasi-differential operators, which are defined applying compositions of differential operators with locally summable coefficients. These operators are defined using the Shin–Zettl matrix function specifically chosen to correspond to the coefficients of l (see [2–4, 13]).

In our case it has the form

$$A(x) = \begin{pmatrix} \dfrac{Q+ir}{p} & \dfrac{1}{p} \\ -\dfrac{Q^2+r^2}{p} + s & -\dfrac{Q-ir}{p} \end{pmatrix}$$

and, due to our assumptions, belongs to the class $L^1_{loc}(\mathbb{R}, \mathbb{C}^2)$.

It can be used to define corresponding quasi-derivatives as follows:

$$\begin{aligned} u^{[0]} &:= u, \\ u^{[1]} &:= pu' - (Q+ir)u, \\ u^{[2]} &:= (u^{[1]})' + \frac{Q-ir}{p}\, u^{[1]} + \Big(\frac{Q^2+r^2}{p} - s\Big)u. \end{aligned}$$

A formal differential expression (1) may now be defined as quasi-differential:

$$l[u] := -u^{[2]}, \quad \mathrm{Dom}(l) := \big\{u : \mathbb{R} \to \mathbb{C} \mid \ u, u^{[1]} \in \mathrm{AC}_{loc}(\mathbb{R})\big\}.$$

This definition is motivated by the fact that

$$\langle -u^{[2]}, \varphi\rangle = \big\langle -(pu')' + qu + i((ru)' + ru'), \varphi\big\rangle \quad \forall\, \varphi \in \mathrm{C}_0^\infty(\mathbb{R})$$

in the sense of distributions.

We define for the quasi-differential expression l the operators L and L_{00} as:

$$\begin{gathered} \mathrm{L}u := l[u], \quad \mathrm{Dom}(\mathrm{L}) := \big\{u \in L^2(\mathbb{R}) \mid \ u, u^{[1]} \in \mathrm{AC}_{loc}(\mathbb{R}), \ l[u] \in L^2(\mathbb{R})\big\}, \\ \mathrm{L}_{00}u := \mathrm{L}u, \quad \mathrm{Dom}(\mathrm{L}_{00}) := \big\{u \in \mathrm{Dom}(\mathrm{L}) \mid \ \operatorname{supp} u \Subset \mathbb{R}\big\}. \end{gathered}$$

The operators L and L_{00} are maximal and preminimal operators for expression l, respectively. Their definitions coincide with the classical ones if the coefficients l are sufficiently smooth. It can be shown that the operator L_{00} is densely defined in $L^2(\mathbb{R})$ and is symmetric.

Let us formulate the main results of the paper in the form of two theorems. The first of them is a natural generalization of the above-mentioned result of Hartman and Rellich.

Theorem 1. *Let the coefficients of the formal differential expression* (1) *satisfy the assumptions* (2) *and also*

(i) $p \in W^1_{2,loc}(\mathbb{R})$, $p > 0$,

(ii) $\int\limits_{-\infty}^{0} p^{-1/2}(t)\,dt = \int\limits_{0}^{\infty} p^{-1/2}(t)\,dt = \infty.$

Then, if operator L_{00} *is bounded from below, then it is essentially self-adjoint and* $\mathrm{L}_{00}^* = \mathrm{L} = \mathrm{L}^*$.

For the case $p \equiv 1$, $r \equiv 0$, Theorem 1 was previously established in [6].

In the second theorem, additional conditions on the coefficient p are imposed not on the entire axis, but only on a sequence of finite intervals. However, outside of these intervals the function p may vanish and be discontinuous.

Theorem 2. *Suppose the assumptions* (2) *are satisfied and the operator* L_{00} *is bounded from below. Suppose the sequence of intervals* $\Delta_n := [a_n, b_n]$ *exists such that*

$$-\infty < a_n < b_n < \infty, \quad b_n \to -\infty, \ \ n \to -\infty, \quad a_n \to \infty, \ \ n \to \infty,$$

where the coefficients p *satisfy the additional conditions:*

(i) $p_n := p|_{\Delta_n} \in W_2^1(\Delta_n)$, $p_n > 0$;

(ii) $\exists\, C > 0 : \ p_n(x) \le C|\Delta_n|^2$, $n \in \mathbb{Z}$, *where* $|\Delta_n|$ *is the length of interval* Δ_n.

Then operator L_{00} *is essentially self-adjoint and* $\mathrm{L}_{00}^* = \mathrm{L} = \mathrm{L}^*$.

For the case $p \equiv 1$, $r \equiv 0$, necessary and sufficient conditions for semi-boundedness of operator L_{00} were obtained in [8].

The proofs of Theorem 1 and Theorem 2 can be found in [9].

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Pointwise Conditions of Solvability of a Periodic Problem for Higher Order Functional Differential Equations

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Abstract

On the interval $I := [a, b]$, we study the higher order linear functional-differential equation

$$u^{(n)}(t) = \ell(u)(t) + q(t), \tag{1}$$

where $q \in L(I; R)$, $\ell : C(I; R) \to L^\infty(I; R)$ is a linear bounded operator, under the periodic type boundary conditions

$$u^{(i)}(\omega) - u^{(i)}(0) = c_i \ \ (i = 0, \dots, n-1). \tag{2}$$

The obtained pointwise efficient sufficient conditions of unique solvability of our problem are non-improvable and moreover, for such important classes of functional-differential equations as differential or integro-differential equations with deviated argument are these conditions take into account the effect of argument deviation and generalize some previously known results (see, for example, [1–3]). Also on the basis of the mentioned results for the linear problem, there are proved non-improvable efficient sufficient conditions of solvability of the periodic type problem for the nonlinear functional differential equation

$$u^{(n)}(t) = F(u)(t) + f_0(t) \ \text{ for } \ t \in [0, \omega], \tag{3}$$

where $F : C(I; R) \to L(I; R)$ is a Carathéodory's local class operator and $f_0 \in L(I, R)$.

Main results

Let $m \in N$, $\sigma \in \{-1, 1\}$, and consider the numbers defined by the following equations

$$\gamma_{n,\sigma} = \begin{cases} 1 & \text{for } \ n = 2m, \ \ \sigma = (-1)^m, \\ 0 & \text{for } \ n = 2m, \ \ \sigma = (-1)^{m+1}, \\ 0 & \text{for } \ n = 2m+1, \ \ \sigma \in \{-1, 1\}. \end{cases}$$

Let also for an arbitrary $x \in I = [0, \omega]$ and a monotone linear operator ℓ, the nonnegative functions $\Delta_x \in C(I; R_0^+)$, and $\rho_\ell \in L^\infty(I; R_0^+)$ be defined by the equalities

$$\Delta_x(t) = |t - x|, \quad \rho_\ell(t) = \frac{2\pi}{\omega} \Big(\ell(1)(t) \int\limits_0^\omega \ell(\Delta_s)(s)\, ds \Big)^{1/2}.$$

Then the following theorem is true.

Theorem 1. *Let $\sigma \in \{-1, 1\}$, and the monotone linear operator $\ell : C(I;R) \to L^\infty(I;R)$ satisfy the conditions*

$$\sigma \int\limits_0^\omega \ell(1)(s)\,ds > 0,$$

and

$$\gamma_{n,\sigma}|\ell(1)(t)| + \rho_\ell(t) < \Big(\frac{2\pi}{\omega}\Big)^n \ \text{ for } \ t \in I.$$

Then problem (1), (2) is uniquely solvable.

Due to the definition of the constant $\gamma_{n,\sigma}$ from our theorem it immediately follows

Corollary 1. *Let $m \in N$, $\ell : C(I;R) \to L^\infty(I;R)$ be the monotone linear operator, and*

$$n = 2m+1 \ \text{ and } \int\limits_0^\omega \ell(1)(s)\,ds \neq 0,$$

or

$$n = 2m \ \text{ and } \ (-1)^{m+1} \int\limits_0^\omega \ell(1)(s)\,ds > 0.$$

Then the condition

$$\ell(1)(t) \int\limits_0^\omega \ell(\Delta_s)(s)\,ds < \Big(\frac{2\pi}{\omega}\Big)^{2(n-1)} \ \text{ for } \ t \in I$$

guarantees the unique solvability of problem (1), (2).

Now assume that $\ell(u)(t) = p(t)u(t)$, where $p \in L^\infty(I;R)$, i.e. we assume that (1) is the ordinary differential equation

$$u^{(n)}(t) = p(t)u(t) + q(t) \ \text{ for } \ t \in I. \tag{4}$$

Then it is clear that $\ell(\Delta_t)(t) = p(t)|t-t| \equiv 0$, and therefore from our theorem it follows:

Corollary 2. *Let $\sigma \in \{-1, 1\}$, and a constant sign function $p \in L^\infty(I;R)$ satisfy the conditions*

$$\sigma \int\limits_0^\omega p(s)\,ds > 0 \ \text{ and } \ \gamma_{n,\sigma}|p(t)| < \Big(\frac{2\pi}{\omega}\Big)^n \ \text{ for } \ t \in I.$$

Then problem (4), (2) is uniquely solvable.

But this proposition is I. Kiguradze and T. Kusano's theorem from [1], and there was shown that $(\frac{2\pi}{\omega})^n$ is optimal.

Now we consider the nonlinear problem (3), (2). To formulate the main theorem we need the following definition.

Definition. Let $\sigma \in \{-1, 1\}$. We will say that the operator $h : C(I;R) \to L^\infty(I;R)$ belongs to the class $\mathcal{K}_\omega^{\sigma,n}$ if h is a nonnegative linear operator,

$$h(1)(t) \not\equiv 0,$$

and for an arbitrary $\alpha \in L^\infty(I;R)$ such that

$$\alpha \not\equiv 0, \ \ 0 \le \alpha(t) \le 1 \ \text{ for } \ t \in I,$$

the homogeneous problem

$$v^{(n)}(t) = \sigma\alpha(t)h(v)(t) \ \text{ for } \ t \in I,$$
$$v^{(i)}(\omega) - v^{(i)}(0) = 0 \ \ (i = 0, \dots, n-1)$$

has no nontrivial solution.

Note that in the given theorem the function $\eta : I \times R_0^+ \to R_0^+$ is summable in the first argument, nondecreasing in the second one, and satisfies the condition

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_0^\omega \eta(s, \rho)\, ds = 0.$$

Theorem 2. *Let the linear nonnegative operator $h : C(I;R) \to L^\infty(I;R)$, the function $g_0 \in L(I;R)$, and numbers $\sigma \in \{-1, 1\}$, $r_0 > 0$ be such that the condition*

$$g_0(t) \le \sigma F(x)(t) \operatorname{sign} h(x)(t) \le |h(x)(t)| + \eta(t, \, ||x||_{C^{n-1}}) \ \ \textit{if} \ \ ||x||_{C^{n-1}} \ge r_0,$$

on I, and the inclusion

$$h \in \mathcal{K}_\omega^{\sigma,n}$$

hold. Moreover, let $g \in L(I;R)$ be such that on I the condition

$$g(t) \le \sigma F(x)(t) \operatorname{sign} h(x)(t) \ \ \textit{if} \ \ \min_{t \in I} |x(t)| \ge r_0$$

is fulfilled, and

$$\int_0^\omega g(s)\, ds - \left| \int_0^\omega f_0(s)\, ds \right| \ge |c_{n-1}|.$$

Then problem $(3), (2)$ *has at least one solution.*

Now we give a corollary of our theorem for the following ordinary differential equation

$$u''(t) = f(t, x(\tau(t))) + f_0(t) \ \text{ for } \ t \in I. \tag{5}$$

Corollary 3. *Let numbers $\sigma \in \{-1, 1\}$, $r_0 > 0$, functions $h \in L^\infty(I;R)$, $g_0 \in L(I;R)$, and a measurable function $\tau : I \to I$ be such that conditions*

$$\gamma_{\sigma,n} h(t) + \rho_h(t) < \left(\frac{2\pi}{\omega}\right)^n \ \textit{for} \ \ t \in I,$$
$$g_0(t) \le \sigma f(t, x) \operatorname{sign} x \le h(t)|x| + \eta(t, |x|) \ \ \textit{for} \ \ |x| \ge r_0, \ \ t \in I,$$

and

$$\int_0^\omega g_0(s)\, ds - \left| \int_0^\omega f_0(s)\, ds \right| \ge |c_{n-1}|$$

hold. Then problem $(5), (2)$ *has at least one solution.*

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On Admissible Perturbations of 3D Autonomous Polynomial ODE Systems

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1 Introduction

In paper [3] V. I. Mironenko introduced the concept of a reflecting function to study the qualitative behavior of solutions of ODE systems. This function is now known as the *Mironenko reflecting function* (MRF) and has been successfully used to solve many problems in qualitative theory of ODE [1, 4–7, 13, 14].

ODE systems with the same MRF have the same translation operator (see [2]) on any interval $(-\beta,\, \beta)$, and 2ω-periodic ODE systems with the same MRF have the same mapping on the period $[-\omega, \omega]$ (Poincare mapping). Therefore, some qualitative properties (such as the existence of periodic solutions and their stability) of solutions of ODE systems that have the same MRF are common. Thus, the study of the qualitative properties of solutions of a whole class of systems with the same MRF can be reduced to the corresponding study of a simple (well-studied) system. In such cases, non-autonomous systems can be studied on the basis of corresponding autonomous systems. In other words, some (well-studied) autonomous system can be transformed into a non-autonomous one with the help of special perturbations that preserve the MRF, which are called *admissible perturbations.* This provides new chances for researchers when modeling real-world processes and exploring novel (unstudied) ODE systems.

To search for admissible perturbations, we can use Theorem 1 from [5], which we formulate here in the form of the following lemma.

Lemma 1.1. *If the vector functions $\Delta_i(t,x)$ ($i=\overline{1,m}$, where $m\in\mathbb{N}$ or $m=\infty$) satisfy the identity*

$$\frac{\partial \Delta_i(t,x)}{\partial t}+\frac{\partial \Delta_i(t,x)}{\partial x}X(t,x)-\frac{\partial X(t,x)}{\partial x}\,\Delta_i(t,x)\equiv 0, \tag{1.1}$$

then the systems $\dot{x}=X(t,x)$ and $\dot{x}=X(t,x)+\sum\limits_{i=1}^{m}\alpha_i(t)\Delta_i(t,x)$ have identical MRF, where $t\in\mathbb{R}$, $x=(x_1,x_2,\ldots,x_n)\in D\subset\mathbb{R}^n$, $\alpha_i(t)$ – arbitrary continuous scalar odd functions.

As initial systems, we consider well-known autonomous polynomial ODE systems (i.e. systems whose right-hand side $X(t,x)\equiv X(x)$, as well as the components of $X(x)$ are polynomials). The search for admissible perturbations is carried out by the method of undetermined coefficients, using identity (1.1) for vector functions $\Delta_i(t,x)\equiv\Delta_i(x)$ whose components are polynomials. That is, in this case, identity (1.1) is transformed to the form

$$\frac{\partial \Delta_i(x)}{\partial x}\,X(x)\equiv\frac{\partial X(x)}{\partial x}\,\Delta_i(x).$$

2 Examples of admissibly perturbed systems and their studies

Using Lemma 1.1 and the approach outlined above, for the Hindmarsh–Rose neuron model

$$\begin{aligned}\dot{x} &= y - ax^3 + bx^2 - z + I,\\ \dot{y} &= c - dx^2 - y, \qquad\qquad x, y, z, a, b, c, d, I, r, s, \alpha \in \mathbb{R}\\ \dot{z} &= r(s(x-\alpha) - z),\end{aligned}$$

admissible perturbations are obtained in [11]. Numerical examples show that admissibly perturbed systems have similar bifurcation diagrams, periodic attractors and strange attractor as the original Hindmarsh–Rose system.

In [10] admissible perturbations are obtained for the Lorentz-84 system, which models the general circulation of the atmosphere in mid-latitudes:

$$\begin{aligned}\dot{x} &= -ax - y^2 - z^2 + aF,\\ \dot{y} &= -y + xy - bxz + G, \qquad a, b, F, G, x, y, z \in \mathbb{R}.\\ \dot{z} &= -z + bxy + xz,\end{aligned} \tag{2.1}$$

In particular, it has been proven that the MRF of system (2.1) and the system

$$\begin{aligned}\dot{x} &= (-ax - y^2 - z^2 + aF)(1 + \alpha_1(t)),\\ \dot{y} &= (-y + xy - bxz)(1 + \alpha_1(t)) - z\alpha_2(t),\\ \dot{z} &= (-z + bxy + xz)(1 + \alpha_1(t)) + y\alpha_2(t)\end{aligned} \tag{2.2}$$

coincide if $G = 0$ and $\alpha_i(t)$ are arbitrary continuous scalar odd functions ($i = \overline{1,2}$). The results of the analysis of the qualitative behavior of solutions of the original system (2.1) are extended to the perturbed system (2.2) and the following theorem is proved.

Theorem 2.1. *Suppose that $\alpha_i = \alpha_i(t)$ $(i = \overline{1,2})$ are continuous functions (not necessarily odd). Then the following statements hold:*

(1) *if $a > 0$, $F < 1$ and $\alpha_1(t) \geqslant c > -1$ $\forall\, t \geqslant 0$ (c is a constant), then the equilibrium solution $x = F$, $y = z = 0$ of system (2.2) is globally exponentially stable (exponentially stable in the large);*

(2) *if $a \geqslant 0$, $F \leqslant 1$ and $\alpha_1(t) \geqslant -1$ $\forall\, t \geqslant 0$, then the equilibrium solution $x = F$, $y = z = 0$ of system (2.2) is globally uniformly Lipschitz stable;*

(3) *if $a > 0$, $F > 1$ and $\alpha_1(t) \geqslant c > -1$ $\forall\, t \geqslant 0$ (c is a constant), then the equilibrium solution $x = F$, $y = z = 0$ of system (2.2) is Lyapunov unstable.*

For the Langford system, which models turbulence in a liquid, presented in the form (more often found in Russian-language literature):

$$\begin{aligned}\dot{x} &= (2a-1)x - y + xz,\\ \dot{y} &= x + (2a-1)y + yz, \qquad a, x, y, z \in \mathbb{R},\\ \dot{z} &= -az - (x^2 + y^2 + z^2),\end{aligned}$$

admissible perturbations are obtained in [8]. And for the Langford system, presented in the form:

$$\begin{aligned}\dot{x} &= (a-1)x - y + xz,\\ \dot{y} &= x + (a-1)y + yz, \qquad a, x, y, z \in \mathbb{R},\\ \dot{z} &= az - (x^2 + y^2 + z^2),\end{aligned}$$

admissible perturbations are obtained in [9].

In [12], admissible perturbations are obtained for the generalized Langford system

$$\begin{aligned} \dot{x} &= ax + by + xz, \\ \dot{y} &= cx + dy + yz, \qquad\qquad a,b,c,d,e,x,y,z \in \mathbb{R}. \\ \dot{z} &= ez - (x^2 + y^2 + z^2), \end{aligned} \tag{2.3}$$

In particular, it has been proven that the MRF of system (2.3) and the system

$$\begin{aligned} \dot{x} &= (ax + by + xz)(1 + \alpha_1(t)) + x(a + z)\alpha_2(t) + y\alpha_3(t) \\ &\qquad - y(x^2 + y^2)(4az + x^2 + y^2 + 2z^2)\alpha_4(t), \\ \dot{y} &= (-bx + ay + yz)(1 + \alpha_1(t)) + y(a + z)\alpha_2(t) - x\alpha_3(t) \\ &\qquad + x(x^2 + y^2)(4az + x^2 + y^2 + 2z^2)\alpha_4(t), \\ \dot{z} &= -(2az + x^2 + y^2 + z^2)(1 + \alpha_1(t) + \alpha_2(t)) \end{aligned} \tag{2.4}$$

coincide if $c = -b$, $d = a$, $e = -2a$ and $\alpha_i(t)$ are arbitrary continuous scalar odd functions ($i = \overline{1,4}$). The obtained result allows us to extend the results of the analysis of the qualitative behavior of solutions of the original system (2.3) to solutions of the perturbed system (2.4). In particular, the following statements are proven in [12].

Theorem 2.2. *Let $\alpha_i(t)$ $(i = \overline{1,4})$ be scalar continuous functions (not necessarily odd).*

(1) *If $a = 0$ and $\alpha_1(t) + \alpha_2(t) \geqslant l > -1\ \forall\, t \geqslant 0$ ($l = const$), then the solution $x = y = z = 0$ of system (2.4) is Lyapunov unstable.*

(2) *If $b = 0$ and the function $\alpha_3(t) + a^4\alpha_4(t)$ is ω-periodic and $\exists\, k \in \mathbb{Z}$ such that $\int_0^\omega (\alpha_3(s) + a^4\alpha_4(s))\,\mathrm{d}s = 2\pi k$, then the solution*

$$\begin{aligned} x(t) &= a\sin\left(bt + \int_0^t (b\alpha_1(s) + \alpha_3(s) + a^4\alpha_4(s))\,\mathrm{d}s\right), \\ y(t) &= a\cos\left(bt + \int_0^t (b\alpha_1(s) + \alpha_3(s) + a^4\alpha_4(s))\,\mathrm{d}s\right), \\ z(t) &= -a \end{aligned} \tag{2.5}$$

of system (2.4) is ω-periodic (the period is not necessarily minimal).

(3) *If $b \neq 0$ and the function $b\alpha_1(t) + \alpha_3(t) + a^4\alpha_4(t)$ is $2\pi/\,|b|$-periodic and $\int_0^{2\pi/b} (b\alpha_1(s) + \alpha_3(s) + a^4\alpha_4(s))\,\mathrm{d}s = 0$, then solution (2.5) of system (2.4) is $2\pi/|b|$-periodic (the period is not necessarily minimal).*

Theorem 2.3. *Let $\alpha_i(t)$ $(i = \overline{1,4})$ be scalar twice continuously differentiable odd functions, $b \neq 0$ and the right side of system (2.4) be $2\pi/|b|$-periodic in t. If $\exists\, k \in \mathbb{Z}$ such that $\int_0^{-2\pi/|b|} (b\alpha_1(s) + \alpha_3(s) + a^4\alpha_4(s))\,\mathrm{d}s = 2\pi k$, then solution (2.5) of system (2.4) is $2\pi/|b|$-periodic.*

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Rapidly Growing Solutions to Two-Dimensional Nonlinear Differential Systems

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Let $a > 0$, and let $f_i : [a, +\infty[\times]0, +\infty[\to]0, +\infty[$ $(i = 1, 2)$ be continuous functions satisfying the local Lipschitz condition in the second argument.

We consider the differential system

$$u_1' = f_1(t, u_2), \ \ u_2' = f_2(t, u_1). \tag{1}$$

A solution to that system in an arbitrary interval $I \subset [a, +\infty[$ is sought on the set of two-dimensional continuously differentiable vector functions with positive components.

A solution (u_1, u_2) to system (1) defined on some infinite interval $[t_0, +\infty[\subset [a, +\infty[$ is said to be **proper**. Obviously, the components of an arbitrary proper solution (u_1, u_2) to system (1) are increasing functions and satisfy one of the following two conditions:

$$\lim_{t \to +\infty} u_i(t) = +\infty \ \ (i = 1, 2);$$

$$\lim_{t \to +\infty} u_k(t) < +\infty \ \text{ for some } \ k \in \{1, 2\}.$$

In the first case the above mentioned solution is said to be **rapidly growing**, while in the second case it is said to be **slowly growing**.

A solution (u_1, u_2) to system (1) defined on some finite interval $[t_0, t_1[\subset [a, +\infty[$ is said to be **blow-up** if

$$\lim_{t \to t_1} \big(u_1(t) + u_2(t)\big) = +\infty.$$

By a solution to the system under consideration we mean a solution that is maximally extended to the right. Thus every solution to that system is either proper or blow-up.

A particular case of system (1) is the second order differential equation

$$u'' = f(t, u) \tag{2}$$

with a continuous right-hand side $f : [a, +\infty[\times]0, +\infty[\to]0, +\infty[$.

A solution to that equation in an arbitrary interval $I \subset [a, +\infty[$ is sought on the set of twice continuously differentiable functions, satisfying the inequalities

$$u(t) > 0, \ \ u'(t) \geq 0,$$

and by a solution it is meant a maximally extended to the right solution.

According to the above definitions, a solution to equation (2) defined on some infinite interval $[t_0, +\infty[\subset [a, +\infty[$ is said to be **proper**. A proper solution u to equation (2) is said to be **rapidly growing** if

$$\lim_{t \to +\infty} u'(t) = +\infty,$$

and it is said to be **slowly growing** otherwise. As for the solution to equation (2) defined on some finite interval $[t_0, t_1[\subset [a, +\infty[$, it is said to be **blow-up** if

$$\lim_{t\to t_1} u(t) = +\infty.$$

R. Emden and R. H. Fowler have investigated in detail asymptotic properties of proper monotone solutions to the frequently occurring in applications differential equation

$$u'' = t^\sigma u^\lambda.$$

The results obtained by them are reflected in the monograph by R. Bellman ([2], Ch. VII). The theory of monotone solutions to the Emden–Fowler type differential equation with general coefficient

$$u'' = p(t)u^\lambda$$

was constructed by I. T. Kiguradze [8] (see, also [13], Ch. V). The asymptotic theory of nonoscillatory and oscillatory solutions to two-dimensional differential systems was constructed by J. D. Mirzov [15].

The foundations of the asymptotic theory of monotone solutions to an arbitrary order differential equations were laid back in the late sixties of the last century and it still remains relevant (see [1, 3–7, 9–14] and the references therein).

The results on the existence of rapidly growing solutions and on their asymptotic estimates given in the present work are obtained based on the method proposed by I. T. Kiguradze and G. G. Kvinikadze [14].

We investigate the case, where

$$f_1(t,x) \geq f_1(s,y) \;\text{ for }\; t \geq s, \;\; x \geq y, \quad f_2(t,x) \geq f_2(s,y) \;\text{ for }\; t \leq s, \;\; x \geq y. \tag{3}$$

Consequently, the function f_1 is assumed to be nondecreasing in both arguments, while the function f_2 is assumed to be nonincreasing in first argument and nondecreasing in the second argument.

Everywhere below we use the following notation.

$$f_{0i}(t,x) = \int\limits_0^x f_i(t,y)\,dy \;\text{ for }\; t \geq a, \;\; x > y;$$

φ_0 is a function defined from the equality

$$f_{01}(t, \varphi_0(t,x)) = x \;\text{ for }\; t \geq a, \;\; x > 0;$$
$$\varphi(t,x) = f_1\big(t, \varphi_0(t, f_{02}(t,x))\big) \;\; t \geq a, \;\; x > 0.$$

Theorem 1. *Let conditions* (3) *be fulfilled and let the differential equation*

$$v' = \varphi(t,v) \tag{4}$$

have no proper solution. Then any solution to the differential system (1) *is blow-up.*

Theorem 2. *Let conditions* (3) *be fulfilled and let the differential equation* (4) *have a unique solution, satisfying the limit condition*

$$\lim_{t\to+\infty} v(t) = +\infty. \tag{5}$$

Then for any $t_0 \in [a, +\infty[$ *there exists a positive number* γ *such that if*

$$c_1 \geq 0, \quad c_2 \geq \gamma, \tag{6}$$

then the solution (u_1, u_2) *to the differential system* (1)*, satisfying the initial conditions*

$$u_1(t_0) = c_1, \quad u_2(t_0) = c_2, \tag{7}$$

is blow-up.

Theorem 3. *Let along with* (3) *the condition*

$$\int\limits_a^{+\infty} f_2\Big(t, x + \int\limits_a^t f_1(s,x)\,ds\Big)\,dt < +\infty \quad \textit{for} \quad x > 0$$

hold. If, moreover, problem (4), (5) *has a unique solution* v*, then the differential system* (1) *along with two-parametric set of slowly growing solutions has a one-parametric set of rapidly growing solutions whose first component for large* t_0 *admits the estimate*

$$u_1(t) \leq v(t) \quad \textit{for} \quad t \geq t_0.$$

As an example, we consider the Emden–Fowler type differential system

$$u_1' = p_1(t)u_2^{\lambda_1}, \quad u_2' = p_2(t)u_1^{\lambda_2}, \tag{8}$$

where λ_1 and λ_2 are positive numbers such that

$$\lambda_1\lambda_2 > 1,$$

$p_1 : [a, +\infty[\to]0, +\infty[$ is a nondecreasing continuous function, and $p_2 : [a, +\infty[\to]0, +\infty[$ is a nonincreasing continuous function.

System (8) can be obtained from system (1) in the case, where

$$f_1(t,x) = p_1(t)x^{\lambda_1}, \quad f_2(t,x) = p_2(t)x^{\lambda_2}.$$

In that case the above defined functions f_{0i} $(i = 1, 2)$, φ_0, φ have the form

$$\begin{aligned}
f_{0i}(t,x) &= \frac{1}{1+\lambda_1} p_i(t) x^{1+\lambda_i} \quad (i = 1, 2),\\
\varphi_0(t,x) &= \Big(\frac{1+\lambda_1}{p_1(t)}\Big)^{\frac{1}{1+\lambda_1}} x^{\frac{1}{1+\lambda_1}},\\
\varphi(t,x) &= \Big(\frac{1+\lambda_1}{1+\lambda_2}\Big)^{\frac{\lambda_1}{1+\lambda_2}} \big(p_1(t)p_2^{\lambda_1}(t)\big)^{\frac{1}{1+\lambda_1}} x^{\frac{\lambda_1+\lambda_1\lambda_2}{1+\lambda_1}}.
\end{aligned}$$

Thus Theorems 1–3 yield the following statements.

Corollary 1. *If*

$$\int\limits_a^{+\infty} \big(p_1(t)p_2^{\lambda_1}(t)\big)^{\frac{1}{1+\lambda_1}}\,dt = +\infty,$$

then any solution to system (8) *is blow-up.*

Corollary 2. *If*

$$\int\limits_a^{+\infty} \big(p_1(t)p_2^{\lambda_1}(t)\big)^{\frac{1}{1+\lambda_1}}\,dt < +\infty, \tag{9}$$

then for any $t_0 \in [a, +\infty[$ *there exists a positive number* γ *such that if inequalities* (6) *are satisfied, then the solution to problem* (8), (7) *is blow-up.*

Corollary 3. *Let along with* (9) *the condition*

$$\int\limits_a^{+\infty} p_2(t)\bigg(\int\limits_a^t p_1(s)\,ds\bigg)^{\lambda_2}\,dt < +\infty$$

hold. Then the differential system (8) *along with two-parametric set of slowly growing solutions has a one-parametric set of rapidly growing solutions whose first component for large* t_0 *admits the estimate*

$$u_1(t) \le \ell\bigg(\int\limits_t^{+\infty} \big(p_1(s)p_2^{\lambda_1}(s)\big)^{\frac{1}{1+\lambda_1}}\,ds\bigg)^{-\frac{1}{\lambda}} \ \ \textit{for} \ \ t \ge t_0,$$

where

$$\lambda = \frac{\lambda_1\lambda_2 - 1}{1+\lambda_1}, \ \ \ell = (1+\lambda_1)^{-\frac{1}{1+\lambda_1}}(1+\lambda_2)^{-\frac{\lambda_1}{1+\lambda_1}}(\lambda_1\lambda_2-1)^{-1}.$$

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Investigation of the Behavior of Solutions of Stochastic Ito Differential Equations

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1 Introduction

In this paper, we investigate the conditions of the existence and the general appearance of locally invariant curves of a perturbed differential equation by a random Wiener process of the "white noise" type in the form of Ito. Random perturbations occur along the phase velocity vector of the corresponding undisturbed differential (deterministic) equation. In [3], the conditions for the existence and uniqueness of solutions of stochastic differential equations are presented. Construction and study of the phase portrait of stochastic Ito differential equations with a degenerate diffusion matrix was carried out in [4]. For nonlinear stochastic Ito differential equations with Markov switching, some sufficient conditions for invariance, stochastic stability, stochastic asymptotic stability, and instability of invariant sets of equations are obtained in [5]. There is the significant literature devoted to the invariant sets of ordinary differential equations, functional differential equations, and stochastic differential equations, and we here mention [2, 5, 7]. The conditions for the existence of bounded solutions of linear and nonlinear pulsed systems were obtained in [1, 6].

In this paper, the conditions under which the locally phase trajectories of the corresponding deterministic differential equation can be locally invariant curves of the perturbed equation are established. A model example describing a certain class of problems related to the study of random harmonic oscillators is given. The conducted researches in an example illustrate application of the received results for construction and the analysis of stochastic differential equations of Ito.The obtained conditions make it possible to build classes of stochastic differential equations for which the given set is invariant.

2 Setting of the problem and the main results

Consider a system of stochastic differential equations

$$d\xi(t) = a(\xi(t)\,dt + b(\xi(t)\,dw(t), \;\; \xi(0) = x^0, \tag{2.1}$$

where $a(x) = (a_1(x), a_2(x))$, $b(x) = (b_1(x), b_2(x))$ – continuous-differential functions in a certain open domain $D \subset R_2$. Denote by $w(t)$ the one-dimensional Wiener process defined in probabilistic space (Ω, F, P), $x = (x_1, x_2)$ – point in D, $x^0 \in D$. It is known [3] that under the given conditions for coefficients of the equation, there is a continuous with probability 1 unique strong solution $\xi(t)$ for all $t \geq 0$ of this equation.

Denote by $\Gamma_D(G)$ the set of the form $\Gamma = \{x : G(x) = C\} \subset D$, where C is a definite constant, $G(x)$ – a twice continuous-differential function in D and has no special points for all $x \in \Gamma$.

If for all $x^0 \in \Gamma_D(G)$

$$P\Big\{\sup_{0\leq t\leq \tau_{D(x^0)}} \big|G(\xi(t)) - G(x^0)\big| = 0\Big\} = 1,$$

where $\tau_{D(x^0)}$ is the moment of the first exit of the solution from the domain D, then the curve $\Gamma_{D(G)}$ is a locally invariant curve of the corresponding equation (2.1).

Consider the problem of investigating the conditions under which the locally phase trajectories of a deterministic differential equation can be locally invariant curves of the corresponding perturbed equation by a random Wiener process of the "white noise" type in the Ito form.

According to [4], the locally invariant curve $\Gamma_{D(G)}$ of equation (2.1) coincides with the locally phase trajectory of equation

$$\frac{dx(t}{dt} = b(x(t)), \quad x(0) = x^0. \tag{2.2}$$

That is $G(x(t)) = G(x^0)$, for all for $t \geq 0$ whom $x(t) \in D$. Since, $(\nabla G(x), b(x)) = 0$, then the phase velocity vector $b(x)$ of equation (2.2) is directed along the tangent to the phase trajectory $G(x) = G(x^0)$ at point x. Thus we obtained the following theorem.

Theorem 2.1. *Locally phase trajectory $\Gamma_{D(G)}$ of equation* (2.2)*, in which $|b(x)| > 0$ for all $x \in \Gamma_{D(G)}$, there can be a local phase curve of equation* (2.1)*, only when the random perturbation of equation* (2.2) *by Ito-shaped "white noise" processes occurs along the phase velocity vector of equation* (2.2)*.*

We obtain the following result for the case $(\nabla G(x), a(x)) = 0$ for all $x \in \Gamma_{D(G)}$.

Since we have a given function $G(x)$, it follows from the necessary condition [4] that

$$b(x) = \big(-G'_{x_2}(x)g(x), G'_{x_1}(x)g(x)\big)$$

for each $x \in \Gamma_D(G)$, where $g(x)$ is an arbitrary continuous-differential function.

Therefore, from the necessary condition of local invariance [4], we have equality $Q(x)g^2(x) = 0$ for all $x \in \Gamma_{D(G)}$, where

$$Q(x) = G''_{x_1x_1}(x)(G'_{x_2})^2(x) + G''_{x_2x_2}(x)(G'_{x_1})^2(x) - 2G''_{x_1x_2}(x)G'_{x_1}(x)G'_{x_2}(x).$$

Theorem 2.2. *The locally phase trajectory $\Gamma_D(G)$ of equation* (2.2) *can be a locally invariant curve of equation* (2.1) *in which $(\nabla G(x), a(x)) = 0$ for all $x \in \Gamma_D(G)$, only when the curve consists only of equilibrium points of equation* (2.2) *($|b(x)| = 0$), and points where the curvature of the curve $\Gamma_D(G)$ is zero.*

Theorem 2.3. *Let the curves $\Gamma_D(G)$ be the set of locally phase trajectories of equation* (2.2) *for all C. If the curvature of the curve $\Gamma_D(G)$ is not equal to zero at the point $x^0 \in D$, $|b(x^0)| > 0$ and $(\nabla G(x), a(x)) = 0$ for all $x \in D$, then the solution of equation* (2.2) *instantly deviates from $\Gamma_D(G)$ the direction of convexity of the curve at the point x^0.*

In order for the solution of equation (2.2) to remain on the phase trajectory $\Gamma_D(G)$ in case of random perturbations along the phase velocity vector $b(x)$ by the Ito-shaped "white noise" process, it is necessary to additionally introduce the corresponding control vector $a(x)$ in equation (2.2).

3 Application to the perturbed limit cycle

For qualitative analysis of stochastic differential equations, it is convenient to use the polar coordinate system $x_1 = r\cos\phi$, $x_2 = r\cos\phi$.

Therefore, we present an auxiliary statement about the connection of the stochastic differential equation (2.1) with the corresponding stochastic differential equation in polar coordinates. We consider a system of stochastic differential equations in the domain $D = \{r > 0, -\infty < \phi < +\infty\}$:

$$\begin{cases} dr(t) = a_1(r,\phi)\,dt + b_1(r,\phi)\,dw(t) \\ d\phi(t) = a_2(r,\phi)\,dt + b_2(r,\phi)\,dw(t), \end{cases} \tag{3.1}$$

where the flow σ-algebra F_t and the one-dimensional process $w(t)$ are the same as in equation (2.1). The coefficients of system are such that there is one strong solution of the system until the moment of the first exit τ_D from the domain D.

Process $\xi(t) = (r(t)\cos\phi(t), r(t)\sin\phi(t))$ for $t < \tau_D$ is the solution of the stochastic equation (2.1).

If for $t < \tau_D$ there is a unique solution of equation (2.1), then for $t < \tau_D$, $r(t)$ is the radial characteristic of the process $\xi(t)$ and $\phi(t)$ is the angular characteristic of the process $\xi(t)$.

Consider equation (2.1) with the corresponding coefficients:

$$a_1(x) = x_2 q(x) + \alpha x_1(1 - |x|^2), \quad a_2(x) = -x_1 q(x) + \alpha x_2(1 - |x|^2),$$
$$b_1 = x_2 g, \quad b_2 = -x_1 g,$$

where α, g are constants, $q(x)$ is arbitrary continuous-differential function in R^2 and $|x| = \sqrt{x_1^2 + x_2^2}$.

The given system describes a certain class of harmonic oscillators that depend on the parameters α, g.

In this case, the phase trajectories of the corresponding deterministic equation (2.2) are circles $x_1^2 + x_2^2 = C$, where $C > 0$ and equilibrium point $(0;0)$.

To study the phase "picture" of this equation (2.1), consider the process$\eta(t) = G(\xi(t))$, where $G(x) = x_1^2 + x_2^2$.

According to the formula Ito we obtain the equation:

$$d\eta(t) = \eta(t)\big[2\alpha(1 - \eta(t)) + g^2\big]\,dt, \quad \eta(0) = |x^0|^2. \tag{3.2}$$

The invariant set of equation (2.1) is the circle $|x|^2 = 1 + g^2(2\alpha)^{-1}$ at $\alpha > 0$ and at $2\alpha < -g^2$. If $2\alpha = -g^2$ or $\alpha = 0$, then the invariant set will be a point $(0;0)$.

If $-g^2 < 2\alpha < 0$, then there are no invariant curves for this equation (2.1).

Suppose $\alpha = 0$, then from equation (3.2) we have

$$\eta(t) = |x^0|^2 e^{gt^2}$$

for all $t \geq 0$ and therefore $\eta(t) \to \infty$ for $t \to \infty$.

If $\alpha \neq 0$ and $|x^0| > 0$, then with probability 1 for all $t \geq 0$ it holds

$$\eta(t) = \frac{1 + g^2(2\alpha)^{-1}}{1 + C_0\exp\{-(2\alpha + g^2)t\}}\,, \tag{3.3}$$

where

$$C_0 = |x^0|^{-2}\big[1 + g^2(2\alpha)^{-1} - |x^0|^2\big].$$

From the analysis of solution (3.3), we have the following:

(a) If $1 + g^2(2\alpha)^{-1} > 0$, then $|x^0|^2 = 1 + g^2(2\alpha)^{-1}$ is an invariant circle and $|\xi(t)|^2 = 1 + g^2(2\alpha)^{-1}$ with a probability of 1 for all $t \geq 0$.

If in this case $\alpha > 0$ and $|x^0|^2 \neq 1 + g^2(2\alpha)^{-1}$, then $|\xi(t)|^2 \to 1 + g^2(2\alpha)^{-1}$ with a probability of 1 at $t \to \infty$ (stability with probability 1).

If $\alpha < 0$ and $|x^0|^2 < 1 + g^2(2\alpha)^{-1}$, then $|\xi(t)|^2 \to 0$ with a probability of 1 at $t \to \infty$.

If $\alpha < 0$ and $|x^0|^2 > 1 + g^2(2\alpha)^{-1}$, then $|\xi(t)|^2 \to \infty$ with a probability of 1 at $t \to t_0$, where

$$t_0 = \frac{-\ln(-1/C_0)}{2 + g^2}\,.$$

(b) If $1 + g^2(2\alpha)^{-1} < 0$ and $\alpha < 0$, then there are no invariant curves.

The equation for the process argument $\xi(t) = (r(t)\cos\phi(t), r(t)\sin\phi(t))$ in this case takes the form

$$d\phi(t) = -q_1(\eta(t), \phi(t))\,dt - g\,dw(t), \tag{3.4}$$

where

$$q_1(\eta(t), \phi(t)) = q\Big(\sqrt{\eta(t)\cos\phi(t)}, \sqrt{\eta(t)\sin\phi(t)}\Big).$$

The systems of equations (3.2), (3.4) provide opportunities for a more detailed study of the behavior of the solution $\xi(t)$.

In particular, if $q(x) = q_0$, where q_0 - constant, then $\frac{w(t)}{t} \to 0$ with probability of 1 at $t \to \infty$ and $\frac{\phi(t)}{t} \to -q_0$ with probability of 1 at $t \to \infty$.

In the case of $q(x) = 0$, process $\phi(t)$ has a normal distribution$N(\phi(0), g^2 t)$ for all $t > 0$.

Note that when $|x^0| = 1 + g^2(2\alpha)^{-1}$ we obtain $\eta(t) = (2\alpha)^{-1}g^2$ with probability of 1 for all $t > 0$.

Equation (3.4) will turn into an equation with one variable $\phi(t)$, which greatly simplifies its study.

By changing the values of the parameters of this example, we can obtain various models of stochastic oscillators.

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The Method of Local Linear Approximation in the Theory of Nonlinear Impulse Systems

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Abstract

For nonlinear differential equations with impulsive perturbations, a general assertion about the existence of bounded solutions is given. With the help of this assertion necessary and sufficient conditions for the existence and uniqueness of bounded solutions of analogous linear equations are obtained. The equations are studied using the method of local linear approximation of nonlinear equations.

1 Problem statement

A method of studying nonlinear differential equations with impulse disturbances is proposed, which uses the approximation of these equations by linear systems on spheres with radii dependent on these systems. In the case of linear momentum equations, this method provides not only sufficient, but also necessary conditions for the existence and unity of bounded solutions of the corresponding equations.

2 Basic notation, spaces and problem

Let $\mathbb{R}$ and $\mathbb{Z}$ – the set of all real and integer numbers, respectively, $\mathbb{T} = \{t_n : \ n \in \mathbb{Z}\}$ – the set of real numbers for which $t_n < t_{n+1}$ for all $n \in \mathbb{Z}$, $\lim\limits_{n\to-\infty} t_n = -\infty$ and $\lim\limits_{n\to+\infty} t_n = +\infty$, E – a finite-dimensional Banach space over the field of real or complex numbers with norm $\|\cdot\|_E$ and $L(X,Y)$ – Banach space of linear continuous operators $A : X \to Y$ with the norm

$$\|A\|_{L(X,Y)} = \sup_{\|x\|_X=1} \|Ax\|_Y,$$

where X and Y – Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ in accordance.

Denote through $C^0(\mathbb{R},\mathbb{T},E)$ the Banach space of defined, continuous and bounded on $\mathbb{R}\setminus\mathbb{T}$ functions $x = x(t)$ with values in E, for each of which there are finite boundaries $\lim\limits_{t\to t_n-0} x(t) = x(t_n - 0)$ and $\lim\limits_{t\to t_n+0} x(t) = x(t_n + 0)$ to all $n \in \mathbb{Z}$, with the norm

$$\|x\|_{C^0(\mathbb{R},\mathbb{T},E)} = \sup_{t\in\mathbb{R}\setminus\mathbb{T}} \|x(t)\|_E,$$

through $C^1(\mathbb{R},\mathbb{T},E)$ denote the Banach space of continuously differentiable by $\mathbb{R}\setminus\mathbb{T}$ functions $x\in C^0(\mathbb{R},\mathbb{T},E)$, for each of which $dx/dt\in C^0(\mathbb{R},\mathbb{T},E)$, with the norm

$$\|x\|_{C^1(\mathbb{R},\mathbb{T},E)}=\max\Big\{\sup_{t\in\mathbb{R}\setminus\mathbb{T}}\|x(t)\|_E,\ \sup_{t\in\mathbb{R}\setminus\mathbb{T}}\Big\|\frac{dx(t)}{dt}\Big\|_E\Big\},$$

and through $\mathfrak{M}(\mathbb{Z},E)$ – Banach space of two-way sequences $\mathfrak{g}=g_n$ elements g_n, $n\in\mathbb{Z}$, space E with the norm

$$\|\mathfrak{g}\|_{\mathfrak{M}(\mathbb{Z},E)}=\sup_{n\in\mathbb{Z}}\|g_n\|_E.$$

Let us also consider the Banach space $C^i(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E)$, where $i\in\{0,1\}$, pairs $(x,\mathfrak{g})$ of elements $x=x(t)\in C^i(\mathbb{R},\mathbb{T},E)$ and $\mathfrak{g}=g_n\in\mathfrak{M}(\mathbb{Z},E)$ with the norm

$$\|(x,\mathfrak{g})\|_{C^i(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E)}=\max\big\{\|x\|_{C^i(\mathbb{R},\mathbb{T},E)},\|\mathfrak{g}\|_{\mathfrak{M}(\mathbb{Z},E)}\big\}.$$

For function jumps $x\in C^0(\mathbb{R},\mathbb{T},E)$ in the points of the set $\mathbb{T}$ similarly, as in [2,3], we will use the notation

$$\Delta x|_{t=t_n}=x(t_n+0)-x(t_n-0),\ \ n\in\mathbb{Z}.$$

Consider a continuous display $F:(\mathbb{R}\setminus\mathbb{T})\times E\to E$, for which for every bounded set $\mathcal{M}\subset E$ a function $F(t,x)$ is bounded on the set $(\mathbb{R}\setminus\mathbb{T})\times\mathcal{M}$ and this function is uniformly continuous on every bounded subset $\mathcal{N}$ plural $(\mathbb{R}\setminus\mathbb{T})\times E$. Also consider continuous mappings $G_n:E\to E$, $n\in\mathbb{Z}$, for which $\sup\limits_{n\in\mathbb{Z},\ x\in\mathcal{M}}\|G_n(x)\|_E<+\infty$ for every bounded set $\mathcal{M}\subset E$.

From the conditions that satisfy F, it follows that for each $x\in C^0(\mathbb{R},\mathbb{T},E)$ the function $y=F(t,x(t))$ is an element of the space $C^0(\mathbb{R},\mathbb{T},E)$.

We will be interested in the conditions under which the system of differential equations with an impulse disturbance is fulfilled

$$\begin{cases}\dfrac{dx(t)}{dt}+F(t,x(t))=f(t), & t\in\mathbb{R}\setminus\mathbb{T},\\ \Delta x|_{t=t_n}+G_n(x(t_n-0))=g_n, & n\in\mathbb{Z}\end{cases}\tag{2.1}$$

for each function $f=f(t)\in C^0(\mathbb{R},\mathbb{T},E)$ and sequences $\mathfrak{g}=g_n\in\mathfrak{M}(\mathbb{Z},E)$ will have at least one solution $x=x(t)\in C^1(\mathbb{R},\mathbb{T},E)$.

The left part of the system of equations (2.1) operator is generated $\mathscr{I}$, that works with $C^1(\mathbb{R},\mathbb{T},E)$ in $C^0(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E)$. If you use operators $\mathcal{L}:C^1(\mathbb{R},\mathbb{T},E)\to C^0(\mathbb{R},\mathbb{T},E)$ and $\mathcal{D}:C^0(\mathbb{R},\mathbb{T},E)\to\mathfrak{M}(\mathbb{Z},E)$, which are defined by equalities

$$(\mathcal{L}x)(t)=\frac{dx(t)}{dt}+F(t,x(t)),\ \ t\in\mathbb{R}\setminus\mathbb{T},$$

and

$$(\mathcal{D}x)_n=\Delta x\big|_{t=t_n}+G_n(x(t_n-0)),\ \ n\in\mathbb{Z},$$

then according to (2.1) operator $\mathscr{I}:C^1(\mathbb{R},\mathbb{T},E)\to C^0(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E)$ is given by the ratio

$$\mathscr{I}x=(\mathcal{L}x,\mathcal{D}x),\ \ x\in C^1(\mathbb{R},\mathbb{T},E).$$

Let $R(\mathscr{I})$ – set of operator values $\mathscr{I}$, i.e. $\{\mathscr{I}x:x\in C^1(\mathbb{R},\mathbb{T},E)\}$.

System of equations (2.1) and the corresponding operator $\mathscr{I}$ in the general case are nonlinear and clarification for system (2.1) conditions for the existence of bounded solutions for each function $f=f(t)\in C^0(\mathbb{R},\mathbb{T},E)$ and sequences$\mathfrak{g}=g_n\in\mathfrak{M}(\mathbb{Z},E)$ or similarly, finding out the conditions of execution for the operator$\mathscr{I}$ equality

$$R(\mathscr{I})=C^0(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E)$$

are not trivial tasks.

3 The main result

When finding out the conditions for the existence of limited solutions of system (2.1) we will use the auxiliary linear systems with impulse disturbance of appearance

$$\begin{cases} \dfrac{dx(t)}{dt} + A(t)x(t) = f(t), & t \in \mathbb{R} \setminus \mathbb{T}, \\ \Delta x\big|_{t=t_n} + B_n x(t_n - 0) = g_n, & n \in \mathbb{Z}, \end{cases} \tag{3.1}$$

coefficients $A(t)$ and B_n of which in a certain sense (see the formulation of Theorem 3.1 and the relation (3.5)) differ little on closed spheres of space E from $F(t,\cdot)$ and $G_n(\cdot)$ in accordance.

Let's use a set of pairs $(A,\mathfrak{B})$ defined and continuous on $\mathbb{R} \setminus \mathbb{T}$ functions $A = A(t)$ with values in $L(E,E)$ and bilateral sequences $\mathfrak{B} = B_n \in L(E,E)$, $n \in \mathbb{Z}$, which are elements of spaces $C^0(\mathbb{R},\mathbb{T},L(E,E))$ and $\mathfrak{M}(\mathbb{Z},L(E,E))$ in accordance.

For a pair of $(A,\mathfrak{B})$ let's match the linear continuous operator

$$\mathfrak{L}_{(A,\mathfrak{B})} : C^1(\mathbb{R},\mathbb{T},E) \to C^0(\mathbb{R},\mathbb{T},E) \times \mathfrak{M}(\mathbb{Z},E),$$

which is given by the ratio

$$\mathfrak{L}_{(A,\mathfrak{B})} x = (\mathscr{L}x, \mathscr{D}x), \ \ x \in C^1(\mathbb{R},\mathbb{T},E), \tag{3.2}$$

where

$$(\mathscr{L}x)(t) = \frac{dx(t)}{dt} + A(t)x(t), \ \ t \in \mathbb{R} \setminus \mathbb{T}, \tag{3.3}$$

and

$$(\mathscr{D}x)_n = \Delta x|_{t=t_n} + B_n x(t_n - 0), \ \ n \in \mathbb{Z}. \tag{3.4}$$

Set of linear operators $\mathfrak{L}_{(A,\mathfrak{B})} : C^1(\mathbb{R},\mathbb{T},E) \to C^0(\mathbb{R},\mathbb{T},E) \times \mathfrak{M}(\mathbb{Z},E)$, dependent on $(A,\mathfrak{B})$, each of which is determined by the left part of system (3.1), i.e. ratios (3.2)–(3.4), and has an inverse continuous operator $\mathfrak{L}^{-1}_{(A,\mathfrak{B})} : C^0(\mathbb{R},\mathbb{T},E) \times \mathfrak{M}(\mathbb{Z},E) \to C^1(\mathbb{R},\mathbb{T},E)$, denote by $\mathcal{O}$.

Theorem 3.1 ([1]). *Suppose for each number $H > 0$ there are such number $r > 0$ and $\mathfrak{L}_{(A,\mathfrak{B})} \in \mathcal{O}$ that*

$$\begin{aligned} \sup_{x \in \mathcal{B}^0[0,r]} \max \Big\{ \sup_{t \in \mathbb{R} \setminus \mathbb{T}} \|F(t,x(t)) - A(t)x(t)\|_E, \sup_{n \in \mathbb{Z}} \|G_n(x(t_n - 0)) - B_n x(t_n - 0)\|_E \Big\} \\ \leq r \|\mathfrak{L}^{-1}_{(A,\mathfrak{B})}\|^{-1}_{L(C^0(\mathbb{R},\mathbb{T},E) \times \mathfrak{M}(\mathbb{Z},E), C^1(\mathbb{R},\mathbb{T},E))} - H. \end{aligned} \tag{3.5}$$

Then for each $f \in C^0(\mathbb{R},\mathbb{T},E)$ and $\mathfrak{g} \in \mathfrak{M}(\mathbb{Z},E)$ the system of equations (2.1) *has at least one solution $x \in C^1(\mathbb{R},\mathbb{T},E)$.*

Remark 3.1. In system (2.1) the reflection $F(t,\cdot)$, $t \in \mathbb{R} \setminus \mathbb{T}$, and $G_n(\cdot)$, $n \in \mathbb{Z}$, may be non-Lipschitz.

4 The case of linear impulse systems

Let's fix an arbitrary function $Q = Q(t) \in C^0(\mathbb{R},\mathbb{T},L(E,E))$ and a sequence

$$\mathfrak{R} = R_n \in \mathfrak{M}(\mathbb{Z},L(E,E)).$$

Consider the corresponding system of linear differential equations with an impulse disturbance

$$\begin{cases} \dfrac{dx(t)}{dt} + Q(t)x(t) = f(t), & t \in \mathbb{R} \setminus \mathbb{T}, \\ \Delta x\big|_{t=t_n} + R_n x(t_n - 0) = g_n, & n \in \mathbb{Z}, \end{cases}$$

where $f = f(t) \in C^0(\mathbb{R}, \mathbb{T}, E)$ and $\mathfrak{g} = g_n \in \mathfrak{M}(\mathbb{Z}, E)$, and the linear differential operator $\mathfrak{L}_{(Q,\mathfrak{R})} : C^1(\mathbb{R}, \mathbb{T}, E) \to C^0(\mathbb{R}, \mathbb{T}, E) \times \mathfrak{M}(\mathbb{Z}, E)$, which is given by the ratio

$$\mathfrak{L}_{(Q,\mathfrak{R})} x = (\mathscr{L}_1 x, \mathscr{D}_1 x), \;\; x \in C^1(\mathbb{R}, \mathbb{T}, E),$$

where

$$(\mathscr{L}_1 x)(t) = \frac{dx(t)}{dt} + Q(t)x(t), \;\; t \in \mathbb{R} \setminus \mathbb{T},$$

and

$$(\mathscr{D}_1 x)_n = \Delta x\big|_{t=t_n} + R_n x(t_n - 0), \;\; n \in \mathbb{Z}.$$

Let's use Theorem 3.1 and operators $\mathfrak{L}_{(A,\mathfrak{B})} \in \mathcal{O}$, which are determined by ratios (3.2)–(3.4). The following two statements are true.

Theorem 4.1 ([1])**.** *For each number $H > 0$ there are such number $r > 0$ and the operator $\mathfrak{L}_{(A,\mathfrak{B})} \in \mathcal{O}$, for which*

$$\sup_{x \in \mathcal{B}^0[0,r]} \max\Big\{ \sup_{t \in \mathbb{R}\setminus\mathbb{T}} \|Q(t)x(t)) - A(t)x(t)\|_E, \sup_{n \in \mathbb{Z}} \|R_n x(t_n - 0) - B_n x(t_n - 0)\|_E \Big\} \\ < r \|\mathfrak{L}^{-1}_{(A,\mathfrak{B})}\|^{-1}_{L(C^0(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E), C^1(\mathbb{R},\mathbb{T},E))} - H,$$

if and only if the linear operator $\mathfrak{L}_{(Q,\mathfrak{R})} : C^1(\mathbb{R}, \mathbb{T}, E) \to C^0(\mathbb{R}, \mathbb{T}, E) \times \mathfrak{M}(\mathbb{Z}, E)$ has an inverse continuous operator.

Theorem 4.2 ([1])**.** *Operator $\mathfrak{L}_{(Q,\mathfrak{R})} : C^1(\mathbb{R}, \mathbb{T}, E) \to C^0(\mathbb{R}, \mathbb{T}, E) \times \mathfrak{M}(\mathbb{Z}, E)$ has an inverse continuous operator if and only if the operator exists $\mathfrak{L}_{(A,\mathfrak{B})} \in \mathcal{O}$, for which*

$$\sup_{x \in \mathcal{B}^0[0,1]} \max\Big\{ \sup_{t \in \mathbb{R}\setminus\mathbb{T}} \|Q(t)x(t)) - A(t)x(t)\|_E, \sup_{n \in \mathbb{Z}} \|R_n x(t_n - 0) - B_n x(t_n - 0)\|_E \Big\} \\ < \|\mathfrak{L}^{-1}_{(A,\mathfrak{B})}\|^{-1}_{L(C^0(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E), C^1(\mathbb{R},\mathbb{T},E))}.$$

5 Perturbations of linear impulse systems are small at infinity

Consider a system of differential equations with an impulse disturbance

$$\begin{cases} \dfrac{dx(t)}{dt} + A(t)x(t) = F(t, x(t)) + f(t), & t \in \mathbb{R} \setminus \mathbb{T}, \\ \Delta x\big|_{t=t_n} + B_n(x(t_n - 0)) = G_n(x(t_n - 0)) + g_n, & n \in \mathbb{Z}, \end{cases} \tag{5.1}$$

in which function $A = A(t)$, $f = f(t)$ and sequences $\mathfrak{B} = B_n$, $\mathfrak{g} = g_n$, $n \in \mathbb{Z}$ are such as in system (3.1), and non-linear mappings $F(t, \cdot) : E \to E$, $t \in \mathbb{R} \setminus \mathbb{T}$, and $G_n(\cdot) : E \to E$, $n \in \mathbb{Z}$ are such as in system (2.1).

We assume that the operator $\mathfrak{L}_{(A,\mathfrak{B})} : C^1(\mathbb{R},\mathbb{T},E) \to C^0(\mathbb{R},\mathbb{T},E) \times \mathfrak{M}(\mathbb{Z},E)$, which is known as the left part of system (3.1), has an inverse continuous operator $\mathfrak{L}^{-1}_{(A,\mathfrak{B})}$ and

$$\varliminf_{r\to+\infty} r^{-1} \sup_{x\in\mathcal{B}^0[0,r]} \max\Big\{ \sup_{t\in\mathbb{R}\setminus\mathbb{T}} \|F(t,x(t))\|_E, \sup_{n\in\mathbb{Z}} \|G_n(x(t_n-0)\|_E\Big\} \\ < \|\mathfrak{L}^{-1}_{(A,\mathfrak{B})}\|^{-1}_{L(C^0(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E),C^1(\mathbb{R},\mathbb{T},E))}. \quad (5.2)$$

A special case of Theorem 4.2 is

Theorem 5.1 ([1])**.** *System of equations* (5.1) *for each* $(f,\mathfrak{g}) \in C^0(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E)$ *has at least one solution* $x \in C^1(\mathbb{R},\mathbb{T},E)$*.*

Remark 5.1. Ratio (5.2) is performed if

$$\sup_{(t,x)\in\mathbb{R}\times E} \|F(t,x)\|_E + \sup_{(n,x)\in\mathbb{Z}\times E} \|G_n(x)\|_E < +\infty.$$

Remark 5.2. Reflection $F(t,\cdot\,) : E \to E$, $t \in \mathbb{R}\setminus\mathbb{T}$, and $G_n(\,\cdot\,) : E \to E$, $n \in \mathbb{Z}$, in system (5.1) can be such that the relation (5.2) holds and

$$\varlimsup_{r\to+\infty} r^{-1} \sup_{x\in\mathcal{B}^0[0,r]} \max\Big\{ \sup_{t\in\mathbb{R}\setminus\mathbb{T}} \|F(t,x(t))\|_E, \sup_{n\in\mathbb{Z}} \|G_n(x(t_n-0))\|_E\Big\} = +\infty.$$

Remark 5.3. The method of local linear approximation in the theory of nonlinear differential, difference, and differential functional equations is considered in [4].

Theorems 3.1, 4.1, 4.2, 5.1 are substantiated using the theory of c-continuous operators, the elements of which are laid out in [1, 4].

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Existence an Uniqueness of Weak Solutions of Stochastic Functional-Differential Neutral Equations in Hilbert Space

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We consider the following stochastic functional-differential neutral equation on Hilbert space:

$$d(u(t)-g(u_t)) = (f(u_t)+Au)\,dt + \sigma(u_t)\,dW(t), \quad t \geq 0, \tag{0.1}$$

$$u(t) = \phi(t), \quad t \in [-h,0], \tag{0.2}$$

where

- $u_t = u(t+\theta)$, $\theta \in [-h,0]$;
- A – linear operator on separable Hilbert space H;
- $W(t)$ – Q-Wiener process on separable Hilbert space K;
- $u(t)$ – state process;
- f – functional from $C([-h,0],H)$ into H;
- σ – mapping from same space to special space of Hilbert–Smidt operators;
- $\phi : [-h,0] \to H$ – initial condition,

while existence and uniqueness of a mild solution of the given equation (0.1), (0.2) is known, weak solutions is relatively undiscovered field.

Thus, we consider existence of weak solutions of equation (0.1), (0.2).

1 Preliminaries

Let's assume that K and H are Hilbert spaces, and V, V' is such Banach spaces that

$$V \subset H = H' \subset V'$$

is a Gelfand triple.

Let (Ω, F, P) be a complete probability space equipped with a normal filtration $\{F_t; t \geq 0\}$ generated by the Q-Wiener process W on (Ω, F, P) with the linear bounded covariance operator such that $\operatorname{tr} Q < \infty$.

We assume that there exist a complete ortonormal system e_k in K and a sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \ldots$, and

$$\sum_{k=1}^{\infty} \lambda_k < \infty.$$

The Wiener process admits the expansion $W(t)=\sum\limits_{k=1}^{\infty}\lambda_k\beta_k(t)e_k$, where $\beta_k(t)$ are real valued Brownian motions mutually independent on (Ω,F,P).

Let $U_0=Q^{\frac{1}{2}}(U)$ and $L_0^2=L_2(U_0,H)$ be the space of all Hilbert–Schmidt operators from U_0 to H with the inner product $(\Phi,\Psi)L_0^2=\operatorname{tr}[\Phi Q\Psi^*]$ and the norm $\|\Phi\|_{L_2^0}$, respectively.

$C:=C([-h,0];H)$ is the space of continuous mappings from $[-h,0]$ to H equipped with the norm $\|u\|_C=\sup\limits_{\theta\in[-h,0]}\|u(\theta)\|$, and $L_2^V:=L_2((-h,0);V)$ is the space of V-valued mappings with the norm

$$\|u\|_V^2:=\int\limits_{-h}^{0}\|u(t)\|_V^2\,dt.$$

2 Conditions on functions

To ensure existence and uniqueness of a solution, we have to impose additional conditions on functions A, f, σ, g.

Conditions on A:

(A1) Domain of A - $D(A)$ is dense in H such that $A:V\to V'$;

(A2) For any $u,v\in V$ there exist $\alpha>0$:

$$|\langle Au,v\rangle|\le\alpha\|u\|_V\|v\|_V;$$

(A3) A satisfies the coercitivity condition: $\exists\,\beta>0$, γ:

$$\langle Av,v\rangle\le-\beta\|v\|_V^2+\gamma\|v\|_V^2,\ \ \forall\,v\in V.$$

Conditions on g:

(G1) g are mapping from $C\cap L_V^2$ to H;

(G2) (Growth condition) $\exists\,K>0$:

$$\|g(\phi)\|_V^2\le K(1+\|\phi\|_{L_2^V}^2),\ \ \forall\,\phi\in L_2^V;$$

(G3) (Lipshitz condition) $\exists 1/2>L>0$:

$$\|g(\phi)-g(\psi)\|_V\le L\|\phi(t)-\psi(t)\|_V,\ \ \forall\,t\in V.$$

Composite conditions:

(C1) f is a mapping from $C\cap L_V^2$ to H, σ is a mapping from $C\cap L_V^2$ to L_2^0;

(C2) (Growth condition) There $\exists\,K>0$, $\theta\ge1$:

$$\|f(\phi)\|_V\le K\bigg(1+\Big(\int\limits_{-h}^{0}\|\phi(t)\|_V\,dt\Big)^{\theta}+\|\phi\|_V^{\theta}\bigg)$$

and

$$\|\sigma(\phi)\|_{L_2^0}^2\le K\big(1+\|\phi\|_C^2\big)$$

$\forall\,\phi\in C\cap L_V^2$.

(C3) (Coercitivity condition) There $\exists\,\beta > 0$, λ, C_1: $\forall\,\phi \in C \cap L^2_V$:

$$\langle A\phi(0), \phi(0)\rangle + \langle f(\phi), \phi(0)\rangle + \|\sigma(\phi)\|^2_{L_2^V} \le -\beta\|\phi(0)\|^2_V + \lambda\|\phi\|^2_C + C_1.$$

(C4) (Monotonicity condition) There $\exists\,\delta > 0$: $\forall\,\phi, \phi_1 \in C \cap L^2_V$:

$$2\big\langle A(\phi(0) - \phi_1(0)), \phi(0) - \phi_1(0)\big\rangle \\ + 2\big\langle f(\phi) - f(\phi_1), \phi(0) - \phi_1(0)\big\rangle + \|\sigma(\phi) - \sigma(\phi_1\|^2_{L_2^0} \le \delta\|\phi - \phi_1\|^2_C\,.$$

3 Main results

Definition. We call an F_t adapted random process $(u(t)) \in V$ weak solution for equation (0.1), (0.2) if:

(1) $u(t) = \phi(t)$, $t \in [-h, 0]$;

(2) $u \in L_2(\Omega \times [0, T], V)$;

(3) $\forall\, v \in V$, $t \in [0, T]$:

$$(u(t) - g(u_t), v) = (\phi - g(\phi), v) + \int_0^t (f(u_s) + Au, v)\, ds + \int_0^t (\sigma(u_s), v)\, dW(s).$$

Theorem (Existence and uniqueness)**.** *Suppose that conditions* (A1)–(A3)*,* (G1)–(G3) *and* (C1)–(C4) *hold, then* $\forall\phi \in C \cap L^2_V$ *equation* (0.1)*,* (0.2) *has a unique weak solution on* $[0, T]$ *such that*

$$u \in C([0, T] \times \Omega; H) \cap L_2([0, T] \times \Omega; V).$$

Moreover, the energy equation holds:

$$\|u - g(u_t)\|^2 = \|\phi - g(\phi)\|^2 \\ + \int_0^t \big\langle Au(s) + f(u_s), u(s)\big\rangle\, ds + \int_0^t \|\sigma(u_s)\|^2_{L_2^0}\, ds + \int_0^t \langle\sigma(u_s), u(s)\rangle\, dW(s).$$

Sketch of the proof:

Step 1: We consider projections of equation (0.1), (0.2) into sequence of finite-dimensional subspaces which looks as follows:

$$d\big(u^n(t) - g^n(u^n_t)\big) = \big(f^n(u^n_t) + Au^n(t)\big)\, dt + \sigma^n(u^n_t)\, dW^n(t), \tag{3.1}$$

$$u^n(t) = \phi^n(t), \quad t \in [-h, 0], \tag{3.2}$$

assuming that P_n is generated by $\{e_k; k = 1, \dots, n\}$ of H and P'_n its restriction on V – projectors of H and V correspondingly:

(1) $A^n = P'_n A$;

(2) $u^n(t) = P'_n u(t)$;

(3) $\phi^n(t) = P'_n \phi(t)$;

(4) $f^n = P_n f$;

(5) $g^n = P_n g$;

(6) $\sigma^n = P_n \sigma$.

Then prove that each of $(3.1), (3.2)$ has exactly one solution.

Step 2: Then we create a priory estimate on solutions of projected equations, which looks as follows:

$$E \sup_{t\in[0,t_1]} \big(\|u^n(t)\|_V^2 + \|g^n u_t^n\|_V^2\big) + E\bigg(\int\limits_0^{t_1} \|u^n(t)\|_V^2 \, dt\bigg) \le A$$

for some $A > 0$.

Those estimates are uniform (not dependant on dimension) and t_1 depends only on predefined coercitivity constants from (A3) and (C3), which implies that sequence of solutions are weak compact, hence holds weakly converging subsequence and can be iteratively continued on further intervals.

Step 3: After that we prove that we can make $n \to \infty$ in projected equations.

There we use the monotonicity condition and the growth conditions (G2) and (C2).

Additionally we prove that the energy equation holds, which implies existence and continuous dependence on initial data.

Corollary (Continuous dependence on the initial data)**.** *Let the conditions of the theorem above hold. Let ϕ and ϕ_1 be initial data for the solutions $u(t,\phi)$ and $u(t,\phi_1)$ of equation* (0.1)*,* (0.2)*. Then there exist a constant $C(T)$ such that*

$$E \sup_{t\in[0,T]} \big(\|u_t(\phi) - u_t(\phi_1)\|_C^2\big) \le C(T)\|\phi - \phi_1\|_C^2.$$

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The Optimal Control Problem of a System of Integro-Differential Equations on Infinite Horizon

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Abstract

We consider the problem of optimal control for a system of integro-differential equations on the half-axis. Sufficient optimality conditions are derived in terms of the right-hand side of the system and the functions involved in the cost function. This task is distinctive in that it is analyzed up to the moment when the solution reaches the boundary of the region, which depends on the control. The proof of existence is based on the compactness approach with the identification of a minimizing sequence, followed by a limit transition in the equation and the cost function.

1 Problem statement

We consider the optimal control problem for a system of integro-differential equations:

$$\begin{cases} \dot{x} = f_1(t,x) + f_2(t,x)u(t) + \int\limits_0^t f_3(t,s,x)u(s)\,ds, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

with a cost function on the infinite interval:

$$J(u) = \int\limits_0^\tau e^{-\gamma t} L(t, x(t), u(t))\,dt \to \inf, \tag{1.2}$$

where $x_0 \in D$ is a fixed vector, $t \in [0,\infty)$, $x \in D$ is the phase vector, D is a bounded region in $\mathbb{R}^d$, ∂D is the boundary of D, τ is the first moment when the solution $x(t)$ reaches to ∂D, $u \in U \subset \mathbb{R}^m$ is the control vector, U is a convex, closed set in $\mathbb{R}^m$, and $0 \in U$.

Let the following conditions be satisfied:

(A) Vector function $f_1(t,x) : [0,\infty) \times D \to \mathbb{R}^d$, matrix $f_2(t,x) : [0,\infty) \times D \to \mathbb{R}^d \times \mathbb{R}^m$, and matrix $f_3(t,s,x) : [0,\infty) \times [0,\infty) \times D \to \mathbb{R}^d \times \mathbb{R}^m$ are continuous with respect to all variables.

(B) Functions $f_1(t,x)$, $f_2(t,x)$, $f_3(t,s,x)$ satisfy the Lipschitz condition, i.e. there exists a constant $H > 0$ such that for any $x_1, x_2 \in D$, $t \geq 0$, and $u \in U$, the following inequalities hold:
$$\begin{gathered}
|f_1(t,x_1) - f_1(t,x_2)| \leq H|x_1 - x_2|, \\
\|f_2(t,x_1) - f_2(t,x_2)\| \leq H|x_1 - x_2|, \\
\|f_3(t,s,x_1) - f_3(t,s,x_2)\| \leq H|x_1 - x_2|,
\end{gathered}$$

here we have $|\cdot|$ – vector norm, $\|\cdot\|$ – matrix norm.

The functions $L(t,x,u)$, $L_x(t,x,u)$, and $L_u(t,x,u)$ are continuous with respect to all variables and the following conditions hold:

(1) $L(t,x,u) \geq 0$ for $t \in [0,\infty)$, $x \in D$ and $u \in U$;

(2) there exist constants $C > 0$ and $p \geq 2$ such that for any $t \in [0,\infty)$, $x \in D$, $u \in U$, the following inequality holds:
$$L(t,x,u) \leq C\big(1 + |u|^p\big);$$

(3) there exists a constant $K > 0$ such that for any $t \in [0,\infty)$, $x \in D$, $u \in U$, the next inequality holds:
$$|L_x(t,x,u)| + |L_u(t,x,u)| \leq K\big(1 + |u|^{p-1}\big);$$

(4) $L(t,x,u)$ is convex with respect to u for any fixed $t \in [0,\infty)$, $x \in D$.

Control $u(t)$ is considered admissible if:

(a1) $u(t) \in L_p([0,\infty])$;

(a2) $u(t) \in U$, for $t \in [0,\infty]$;

(a3) there exists a constant $C_1 > 0$ that is independent of $u(t)$ and the next condition holds:
$$\int\limits_0^\infty |u(t)|^p \, dt \leq C_1;$$

(a4) $|J(u)| < \infty$.

The set of admissible controls is marked as "V" for problem (1.1), (1.2).

For systems of ordinary differential equations, similar problems were studied in works [3], for stochastic in [4], for functional-differential systems in [1], and for impulsive systems in [2].

2 Main result

The main result of this paper concerns the existence of a solution of problem (1.1), (1.2). We obtained the following theorem.

Theorem 2.1. *Let system* (1.1) *with the quality criterion* (1.2) *satisfy conditions* (A), (B), *and* (1)–(3). *Then problem* (1.1), (1.2) *has a solution in the class of admissible controls* V, *i.e. there exists an optimal control* $u^*(t)$ *which minimizes the cost function* (1.2).

Proof. Because the cost function is a non-negative quantity, there exists a non-negative lower bound m for the values of $J(u)$. Therefore, there exists a sequence of admissible controls $\{u_n(t), n \geq 1\}$ such that $J(u_n) \to m$ monotonically as $n \to \infty$.

Since U is a convex and closed set, then using Mazur's lemma we obtain $u^*(t)$ almost everywhere for t.

For the solutions $x_n(t)$, we have the integral representation:

$$x_n(t) = x_0 + \int_0^t \left[f_1(t, x_n(t)) + f_2(t, x_n(t))u_n(t) + \int_0^s f_3(s, \sigma, x_n(s))u(\sigma)\, d\sigma \right] dt.$$

Using the functions x_n, we build functions $y_n(t)$ defined on $[0, \infty)$ as follows:

$$y_n(t) = \begin{cases} x_n(t), & t \in [0, \tau_n), \\ x_n(\tau_n), & t \geq 0. \end{cases}$$

It is easy to see the set of function y_n is compact in the space of continuous functions defined on $[0, T]$ for arbitrary $t > 0$. So, there exist a subsequence $\{y_{n_k}(t), n \geq 1\}$ of a sequence $\{y_n(t), n \geq 1\}$ such that $\{y_{n_k}(t), n \geq 1\}$ uniformly on the interval $[0, T]$.

Using the diagonal method, we can show that some subsequence of the sequence $\{y_{n_n}(t), n \geq 1\}$ converges pointwise to a continuous function $y^*(t)$ for any $t \in [0, \infty)$.

For convenience, we denote this subsequence again as $\{y_n(t), n \geq 1\}$ and the corresponding control sequence as $\{u_n(t), n \geq 1\}$.

Let τ^* denote the moment of the first exit of $y^*(t)$ to the boundary ∂D, so

$$\tau^* = \begin{cases} \inf \big\{t \geq 0 : \ y^*(t) \in \partial D\big\}, \\ \infty, \ \text{if } \ y^*(t) \in D, \ \ \forall\, t \geq 0, \end{cases}$$

$$\tau_n = \begin{cases} \inf \big\{t \geq 0 : \ y_n(t) \in \partial D\big\}, \\ \infty, \ \text{if } \ y_n(t) \in D, \ \ \forall\, t \geq 0. \end{cases}$$

We will show that $\tau^* \leq \lim\limits_{n\to\infty} \inf \tau_n$.

Really assume that this is not true. Then $\tau^* > \lim\limits_{n\to\infty} \inf \tau_n = \tau$. Let's consider two cases:

(1) Suppose $\tau^* < \infty$. Choose any $T_1 \in [0, \infty)$ such that $T_1 \geq \tau^*$. On the interval $[0, T_1]$, $y_n(t) \to y^*(t)$, $n \to \infty$.

By the characterization theorem of the lower limit, for any $\delta > 0$ the set $\{n \in \mathbb{N} : \ \tau_n < \tau + \delta\}$ is infinite. Choose δ such that $\tau + \delta < \tau^*$. Then, there exists a subsequence $\{\tau_{n_k}, n_k \geq 1\}$ of $\{\tau_n, n \geq 1\}$ such that $\tau_{n_k} < \tau + \delta$. Choose a moment t_0 such that $t_0 \in (\tau + \delta, \tau^*)$. Then $y_{n_k}(t_0) = x_{n_k}(\tau_{n_k}) \in \partial D$.

From the uniform convergence of $y_n(t)$ to $y^*(t)$ on $[0, T_1]$, we have for any $\varepsilon > 0$ that there exists $N \in \mathbb{N}$ such that for any $n_k \geq N$, the following inequality holds:

$$|y^*(t) - y_{n_k}(t)| < \varepsilon.$$

However, by choosing ε such that $0 < \varepsilon < \inf\limits_{\upsilon \in \partial D} |y^*(t_0) - \upsilon|$, then for a fixed $t_0 \in (\tau + \delta, \tau^*)$, we obtain

$$|y^*(t_0) - y_{n_k}(t)| = \big|y^*(t_0) - x_{n_k}(\tau_{n_k})\big| > \varepsilon.$$

So, we get a contradiction.

(2) Suppose $\tau^* = \infty$ and $\lim\limits_{n\to\infty} \inf \tau_n < \infty$. Similarly to the previous case, we have

$$\tau^* \leq \lim_{n\to\infty} \inf \tau_n.$$

Set $x^*(t) = y^*(t)$ for $t \in [0, \tau^*]$, in the case of finite τ^* and $x^*(t) = y^*(t)$ for $t \in [0, \infty)$ in the case of $\tau^* = \infty$.

Now, let's show that $x^*(t)$ is a solution to system (1.1), for all t until it reaches the boundary, corresponding to the control $u^*(t)$.

Take any $t \in [0, \tau^*]$, for $\tau^* < \infty$, and $t \in [0, \infty)$ for $\tau^* = \infty$. Choose a sufficiently large $T \geq 0$ so that for any such t, $y_n(t) = x_n(t)$ for sufficiently large n. Since $y_n(t) \to y^*t)$ as $n \to \infty$ uniformly on $[0, T]$, $x_n(t) \to x^*(t)$ uniformly on $[0, \tau_1^*]$, where

$$\tau_1^* = \begin{cases} \inf \big\{ t \in [0, T] : \text{ if } x^*(t) \in \partial D \big\}, \\ T, \ \text{ if } \ x^*(t) \in D \setminus \partial D, \ \ \forall\, t \geq 0. \end{cases}$$

Since $x_n(t)$ is a solution to system (1.1), we have

$$\begin{aligned} x_n(t) = x_0 + \int\limits_0^t \bigg(f_1(s, x_n(s)) + f_2(s, x_n(s)) u^*(s) + \int\limits_0^s f_3(s, \sigma, x_n(s)) u^*(\sigma)\, d\sigma \bigg)\, ds \\ + \int\limits_0^t \big(f_2(s, x_n(s)) - f_2(s, x^*(s)) \big) (u_n(s) - u^*(s))\, ds \\ + \int\limits_0^t \int\limits_0^s \big(f_3(s, \sigma, x_n(s)) - f_3(s, \sigma, x^*(s)) \big) (u_n(\sigma) - u^*(\sigma))\, d\sigma\, ds \\ + \int\limits_0^t f_2(s, x^*(s)) (u_n(s) - u^*(s))\, ds + \int\limits_0^t \int\limits_0^s f_3(s, \sigma, x^*(s)) (u_n(\sigma) - u^*(\sigma))\, d\sigma\, ds. \end{aligned}$$

The convergence of each integral can be easily proven by Lebesgue's dominated convergence theorem, and the definition of weak convergence.

Then, by taking the limit as $n \to \infty$, we obtain

$$x^*(t) = x_0 + \int\limits_0^t \bigg(f_1(s, x^*(s)) + f_2(s, x^*(s)) u^*(s) + \int\limits_0^s f_3(s, \sigma, x^*(s)) u^*(\sigma)\, d\sigma \bigg)\, ds$$

for any $t \in [0, \tau_1^*]$.

So, we conclude that $x^*(t)$ is a solution to system (1.1), corresponding to the control $u^*(t)$ for $t \in [0, \tau_1^*]$.

Since the time moment T is chosen arbitrarily, we have that $x^*(t)$ is a solution to system (1.1) corresponding to the control $u^*(t)$ for $t \geq 0$ until the solution reaches the boundary of the region.

As $x_n(t)$ coincides with $y_n(t)$ up to this moment, the sequences $\{x_n(t), n \geq 1\}$ converge pointwise to $x^*(t)$ for any $t \in [0, \tau_1^*]$.

Now, we prove that the control $u^*(t)$ is optimal. Consider two cases:

(1) Suppose $x^*(\tau^*) \in \partial D$. Since $L(t, x, \cdot\,)$ is convex, the following inequality holds:

$$e^{-\gamma t} \big(L(t, x^*(t), \upsilon(t)) \big) \geq e^{-\gamma t} \big(L(t, x^*(t), u^*(t)) \big) + (\upsilon(t) - u^*(t)) e^{-\gamma t} \big(L_\upsilon(t, x^*(t), u^*(t)) \big),$$

$\upsilon \in U$, $t \in [0, \tau^*]$.

Set $\upsilon = u_n(t)$. Then we easily get the following inequality

$$\lim_{n\to\infty} \int_0^{\tau^*} e^{-\gamma t} L(t, x^*(t), u_n(t))\, dt \geq \int_0^{\tau^*} e^{-\gamma t} L(t, x^*(t), u^*(t))\, dt.$$

Also, we have:

$$J(u^*) \leq \lim_{n\to\infty} \inf \int_0^{\tau^*} e^{-\gamma t} L(t, x_n(t), u_n(t))\, dt \leq \lim_{n\to\infty} J(u_n(t)).$$

Since

$$\inf_{u\in U} J(u) \leq J(u^*) \leq \lim_{n\to\infty} \inf J(u_n) = m,$$

we conclude that

$$J(u^*) = m.$$

Therefore, $u^*(t)$ is the optimal control.

(2) Now let $\tau^* = \infty$ and $x^*(t) \in D \setminus \partial D$, $t \geq 0$.

It is easy to show that the function $L(t, x^*(t), u_n(t))$ is integrable on $[0, \infty)$. Since $L(t, x, \cdot)$ is convex, the following inequality holds:

$$e^{-\gamma t} L(t, x^*(t), \upsilon(t)) \geq e^{-\gamma t} L(t, x^*(t), u^*(t)) + (\upsilon(t) - u^*(t)) e^{-\gamma t} L_\upsilon(t, x^*(t), u^*(t)),$$

$\upsilon(t) \in V$, $t \in [0, \infty)$.

Furthermore, due to weak convergence, we obtain the satisfaction of the following inequality:

$$\lim_{n\to\infty} \int_0^{\infty *} e^{-\gamma t} L(t, x^*(t), u_{n_k}(t))\, dt \geq \int_0^{\infty} e^{-\gamma t} L(t, x^*(t), u^*(t))\, dt.$$

Let's also define $J(u_n)$ as follows:

$$J(u_n) = \int_0^{\infty} e^{-\gamma t} L(t, x_n(t), u_n(t))\, dt = \int_0^{\infty} e^{-\gamma t} \big[L(t, x_n(t), u_n(t)) - L(t, x^*(t), u_n(t)) \big]\, dt$$
$$+ \int_0^{\infty} e^{-\gamma t} \big[L(t, x^*(t), u_n(t)) - L(t, x^*(t), u^*(t)) \big]\, dt + \int_0^{\infty} e^{-\gamma t} L(t, x^*(t), u^*(t))\, dt.$$

We obtain

$$\lim_{n\to\infty} \inf J(u_n(t)) \geq J(u^*).$$

Since

$$\inf_{u\in U} J(u) \leq J(u^*) \leq \lim_{n\to\infty} \inf J(u_n(t)) = m,$$

then

$$J(u^*) = m.$$

Thus, $u^*(t)$ is the optimal control. $\square$

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On Investigation of Differential-Algebraic Systems with Two-Point Non-Linear Boundary Conditions

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We study the differential-algebraic two-point non-linear boundary value problem on a compact interval

$$x'(t) = f(t, x(t), y(t)), \quad t \in [a, b], \tag{1}$$

$$y(t) = g(t, x(t), y(t)), \quad t \in [a, b]; \tag{2}$$

$$B(x(a), y(b)) = d, \tag{3}$$

where $f : [a, b] \times \Omega_{\varrho_0} \times \Omega_{\varrho_1} \to \mathbb{R}^p$, $g : [a, b] \times \Omega_{\varrho_0} \times \Omega_{\varrho_1} \to \mathbb{R}^q$ and $B : \Omega_{\varrho_0} \times \Omega_{\varrho_1} \to \mathbb{R}^p$ are continuous functions defined on certain bounded sets $\Omega_{\varrho_0} \subset \mathbb{R}^p$, $\Omega_{\varrho_1} \subset \mathbb{R}^q$ specified below (see (7) and (9)), $d \in \mathbb{R}^p$. We assume that the functions f, g, B satisfy the Lipschitz conditions

$$\big|f(t, u_1, v_1) - f(t, u_2, v_2)\big| \le K_1|u_1 - u_2| + K_2|v_1 - v_2|, \tag{4}$$

$$\big|g(t, u_1, v_1) - g(t, u_2, v_2)\big| \le K_3|u_1 - u_2| + K_4|v_1 - v_2|, \tag{5}$$

for $t \in [a, b]$, $\{u_1, u_2\} \subset \Omega_{\varrho_0}$, $\{v_1, v_2\} \subset \Omega_{\varrho_1}$, where K_1, K_2, K_3, K_4 are non-negative matrices of dimensions $p \times p$, $p \times q$, $q \times p$, $q \times q$ such that the spectral radii of Q and K_4 satisfy the inequalities

$$r(Q) < 1, \quad r(K_4) < 1, \tag{6}$$

where

$$Q = \frac{3}{10}\,(b - a)K$$

and K is the $(p+q)\times(p+q)$ block matrix $K=\begin{pmatrix} K_1 & K_2\\ K_3 & K_4\end{pmatrix}$. By a solution of problem (1)–(3) we understand a pair of a continuously differentiable $x:[a,b]\to\Omega_{\varrho 0}$ and continuous $y:[a,b]\to\Omega_{\varrho 1}$ functions, satisfying (1)–(3).

We show that techniques similar to those of [1, 2] can be effectively applied to the study of problem (1), (2). Note that, although condition (6) ensures that equation (2) can theoretically be solved with respect to y, it may be difficult or impossible to do this explicitly, and this is not required in our approach.

In the sequel, 1_l is the unit matrix of dimension l. For any $x=\operatorname{col}(x_1,\dots,x_l)$, $y=\operatorname{col}(y_1,\dots,y_l)$, we write $|x|=\operatorname{col}(|x_1|,\dots,|x_l|)$ and understand $x\le y$ as $x_i\le y_i$ for all $i=1,2,\dots,l$. The operations max and min for vector functions are also understood componentwise. Given any non-negative vector $\varrho\in\mathbb{R}^l$, we put

$$O_\varrho(z)=\big\{\xi\in\mathbb{R}^l:\ \ |\xi-z|\le\varrho\big\}$$

for $z\in\mathbb{R}^l$ and

$$O_\varrho(U)=\bigcup_{z\in U}O_\varrho(z)$$

for a set $U\subset\mathbb{R}^l$. The set $O_\varrho(U)$ may be called the componentwise ϱ-neighbourhood of U.

Fix certain compact convex sets $D_0\subset\mathbb{R}^p$, $D_1\subset\mathbb{R}^p$ and put

$$\Omega_{\varrho_0}=O_{\varrho_0}(D_{a,b}), \tag{7}$$

where ϱ_0 is a non-negative vector and

$$D_{a,b}=\big\{(1-\theta)z+\theta\eta:\ \ z\in D_0,\ \ \eta\in D_1,\ \ \theta\in[0,1]\big\}.$$

Choose some $\widetilde{y}\in\mathbb{R}^q$ and put

$$Y=\max_{(t,z,\eta)\in[a,b]\times D_0\times D_1}\big|g(t,x_0(t,z,\eta),\widetilde{y})-\widetilde{y}\big|,$$

where

$$x_0(t,z,\eta)=\Big(1-\frac{t-a}{b-a}\Big)z+\frac{t-a}{b-a}\,\eta,\ \ t\in[a,b], \tag{8}$$

for any $z\in D_0$, $\eta\in D_1$. Take a non-negative vector ϱ_1 and put

$$\Omega_{\varrho_1}=O_{\varrho_1}(\overline{y}_1), \tag{9}$$

where

$$\overline{y}_1=\max_{(t,z,\eta)\in[a,b]\times D_0\times D_1}|g(t,x_0(t,z,\eta),\widetilde{y})|.$$

We assume in what follows that the non-negative vectors ϱ_0, ϱ_1 can be chosen so that

$$\varrho_0\ge\frac{1}{2}\,(b-a)\delta_{\varrho_0,\varrho_1}(f), \tag{10}$$

$$\varrho_1\ge(1_q-K_4)^{-1}\Big(\frac{1}{2}\,(b-a)K_3\delta_{\varrho_0,\varrho_1}(f)+K_4Y\Big)+Y, \tag{11}$$

where

$$\delta_{\varrho_0,\varrho_1}(f)=\frac{1}{2}\,\Big(\max_{(t,x,y)\in[a,b]\times\Omega_{\varrho_0}\times\Omega_{\varrho_1}}f(t,x,y)-\min_{(t,x,y)\in[a,b]\times\Omega_{\varrho_0}\times\Omega_{\varrho_1}}f(t,x,y)\Big).$$

Together with inequalities (6), conditions (10), (11) may be regarded as smallness conditions on the functions f and g in a neighbourhood of sets D_0 and D_1. When some of these conditions are violated, one can apply a technique from [3] in order to construct convergent iterations.

Instead of the original boundary value problem (1)–(3), consider the family of auxiliary two-point boundary value problems

$$\begin{aligned} x'(t) &= f(t, x(t), y(t)), \quad t \in [a,b], \\ y(t) &= g(t, x(t), y(t)), \quad t \in [a,b]; \end{aligned} \tag{12}$$

$$x(a) = z, \quad x(b) = \eta, \tag{13}$$

where $z \in D_0$ and $\eta \in D_1$ are free parameters. We focus on continuously differentiable $x : [a,b] \to \Omega_{\varrho_0}$ and continuous $y : [a,b] \to \Omega_{\varrho_1}$ solutions of problem (12), (13) with values $x(a) \in D_0$ and $x(b) \in D_1$. As it will be indicated below, one can then go back to the original problem by choosing the values of z and η appropriately.

In relation to the two-point boundary value problem (12), (13), introduce the sequences of functions

$$\begin{aligned} x_{m+1}(t,z,\eta) = z + \int_a^t f\big(s, x_m(s,z,\eta), y_m(s,z,\eta)\big)\, ds \\ - \frac{t-a}{b-a} \int_a^b f\big(s, x_m(s,z,\eta), y_m(s,z,\eta)\big)\, ds + \frac{t-a}{b-a}\,(\eta - z), \end{aligned} \tag{14}$$

$$y_{m+1}(t,z,\eta) = g(t, x_m(t,z,\eta), y_m(t,z,\eta)), \ \ t \in [a,b], \ \ m = 1, 2, \dots, \tag{15}$$

where

$$y_0(t,z,\eta) = \widetilde{y}, \ \ t \in [a,b],$$

with a fixed value of $\widetilde{y}$ and the function x_0 given by (8).

For $1 \le i_1 < i_2 \le n$, let J_{i_1,i_2} be the $(i_2 - i_1 + 1) \times n$ block matrix with the unit matrix of dimension $i_2 - i_1 + 1$ placed starting from the i_1th column, that is

$$J_{i_1,i_2} = \begin{pmatrix} 0 & 1_{i_2-i_1+1} & 0 \end{pmatrix}, \tag{16}$$

where the symbols 0 stand for the zero blocks of appropriate dimensions.

Theorem 1. *Let conditions* (4)–(6) *and* (10), (11) *be fulfilled. Then, for all fixed* $z \in D_0$ *and* $\eta \in D_1$*:*

1) *The functions of sequence* (14) *have range in* Ω_{ϱ_0}*, satisfy the two-point conditions* (13)*, and the limit*

$$x_\infty(t,z,\eta) = \lim_{m\to\infty} x_m(t,z,\eta)$$

exists uniformly on $[a,b] \times D_0 \times D_1$*. The function* $x_\infty(\,\cdot\,,z,\eta)$ *satisfies conditions* (13) *and is continuously differentiable.*

2) *The functions of sequence* (15) *have range in* Ω_{ϱ_1} *and converge to a continuous limit function*

$$y_\infty(t,z,\eta) = \lim_{m\to\infty} y_m(t,z,\eta)$$

uniformly on $[a,b] \times D_0 \times D_1$*.*

3) *The functions $x = x_\infty(\,\cdot\,,z,\eta)$, $y = y_\infty(\,\cdot\,,z,\eta)$ satisfy equation (2) and the Cauchy problem with a constant forcing term:*

$$x'(t) = f(t,x(t),y(t)) + \frac{1}{b-a}\,\Delta(z,\eta), \ \ t\in[a,b]; \ \ x(a)=z, \tag{17}$$

where $\Delta : D_0\times D_1 \to \mathbb{R}^p$ is the mapping given by the formula

$$\Delta(z,\eta) = \eta - z - \int\limits_a^b f\big(s, x_\infty(s,z,\eta), y_\infty(s,z,\eta)\big)\,ds. \tag{18}$$

Other couples of functions (x,y) having range in the set $\Omega_{\varrho_0}\times\Omega_{\varrho_1}$ and satisfying (2), (17) do not exist.

4) *The estimates*

$$\big|x_\infty(t,z,\eta) - x_m(t,z,\eta)\big| \le \frac{10}{9}\,\alpha_1(t) J_{1,p} K^\sigma Q^{m-\sigma} (1_{p+q}-Q)^{-1} \begin{pmatrix} \delta_{\varrho_0,\varrho_1}(f) \\ Y \end{pmatrix},$$
$$\big|y_\infty(t,z,\eta) - y_m(t,z,\eta)\big| \le \frac{10}{9}\,\alpha_1(t) J_{p+1,p+q} K^{\sigma+1} Q^{m-\sigma-1} (1_{p+q}-Q)^{-1} \begin{pmatrix} \delta_{\varrho_0,\varrho_1}(f) \\ Y \end{pmatrix}$$

hold for all $t\in[a,b]$ and m sufficiently large, where

$$\sigma = \begin{cases} 0 & \text{if } b-a > \dfrac{10}{3}\,, \\[2mm] 1 & \text{if } b-a \le \dfrac{10}{3}\,. \end{cases}$$

By (16), the left multiplication of a vector column by $J_{1,p}$ (resp., $J_{p+1,p+q}$) means the selection of the components $1,2,\dots,p$ (resp., $p+1,\dots,p+q$).

For $z\in\Omega_{\varrho_0}$, $\eta\in\Omega_{\varrho_1}$, let us put

$$\Lambda(z,\eta) = B\big(x_\infty(a,z,\eta), y_\infty(b,z,\eta)\big) - d.$$

The functions Δ and Λ determine the relation of the functions $x_\infty(\,\cdot\,,z,\eta)$ and $y_\infty(\,\cdot\,,z,\eta)$ to solutions of the original problem (1)–(3).

Theorem 2. *Under the assumptions of Theorem 1, the couple of functions $(x_\infty(\,\cdot\,,z,\eta), y_\infty(\,\cdot\,,z,\eta))$ is a solution of the boundary value problem (1)–(3) if and only if the parameters z, η satisfy the system of $2p$ algebraic or transcendental equations*

$$\Delta(z,\eta)=0, \quad \Lambda(z,\eta)=0. \tag{19}$$

The next statement proves that the system of determining equations (19) determines all possible solutions of the original non-linear boundary value problem (1)–(3) in the regions Ω_{ϱ_0}, Ω_{ϱ_1}.

Theorem 3. *Under the assumptions of Theorem 1, the following assertions hold:*

1) *If there exist a pair of vectors $(z_*,\eta_*)\in D_0\times D_1$ satisfying the system of determining equations (19), then the boundary value problem (1)–(3) has a solution (x_*,y_*) such that $x_*([a,b])\subset\Omega_{\varrho_0}$, $y_*([a,b])\subset\Omega_{\varrho_1}$ and*

$$x_*(a)=z_*, \quad x_*(b)=\eta_*.$$

Moreover, this solution has the form

$$(x_*,y_*) = \big(x_\infty(\,\cdot\,,z_*,\eta_*), y_\infty(\,\cdot\,,z_*,\eta_*)\big).$$

2) *If problem* (1)–(3) *has a solution* (x_*, y_*) *with* $(x_*(a), x_*(b)) \in D_0 \times D_1$ *and range in* $\Omega_{\varrho_0} \times \Omega_{\varrho_1}$, *then the system of determining equations* (19) *is satisfied with* $z = x_*(a)$, $\eta = x_*(b)$.

The practical investigation of problem (1)–(3) is carried out by studying the approximate determining equations

$$\Delta_m(z, \eta) = 0, \quad \Lambda_m(z, \eta) = 0, \tag{20}$$

where

$$\Lambda_m(z, \eta) = B\big(x_m(a, z, \eta), y_m(b, z, \eta)\big) - d$$

and

$$\Delta_m(z, \eta) = \eta - z - \int_a^b f\big(s, x_m(s, z, \eta), y_m(s, z, \eta)\big)\, ds,$$

for some fixed m. Assuming that the Lipschitz condition holds for B,

$$|B(u_1, v_1) - B(u_2, v_2)| \le K_5 |u_1 - u_2| + K_6 |v_1 - v_2|,$$

under additional assumptions, we prove the existence of a solution of the original problem (1)–(3) by showing that the solvability of the approximate determining equations (20) in the respective region implies that of (19). A practical computation using Maple confirms the constructiveness of the proposed approach.

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Definition and Some Properties of Measures of Stability and Instability of a Differential System

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For a given $n\in\mathbb{N}$ and zero neighborhood $G\subset\mathbb{R}^n$, we consider the differential system

$$\dot{x}=f(t,x),\quad f(t,0)\equiv 0,\ \ t\in\mathbb{R}_+\equiv[0,+\infty),\ \ x\in G, \tag{1}$$

where $f,f'_x\in C(\mathbb{R}_+,G)$. Let's put

$$B_\delta\equiv\big\{x_0\in\mathbb{R}^n\ |\ \ 0<|x_0|<\delta\big\},\quad \Delta\equiv\sup\big\{\delta\ |\ \ B_\delta\subset G\big\},$$

and denote by $x(\,\cdot\,,x_0)$ a non-extendable solution of system (1) with the initial value $x(0,x_0)=x_0$.

The differential system (1) is completely deterministic, however, it is possible to give a natural stochastic meaning to its measures of stability $\mu_\varkappa(f)$ or instability $\nu_\varkappa(f)$ [1,2]. They allow us to estimate from below the possibility or impossibility of randomly selecting the initial value x_0 of perturbed solution $x(\,\cdot\,,x_0)$, arbitrarily close to zero, so that its graph falls into a given tube of the zero solution in any of the following senses [3,4]:

(a) immediately on the entire time semi-axis (the Lyapunov stability for $\varkappa=\lambda$);

(b) at least episodically, but at arbitrarily late points in time (the Perron stability for $\varkappa=\pi$);

(c) at least from some moment, but then forever (the upper-limit stability for $\varkappa=\sigma$).

The forerunners of the described measures were the recent concepts of almost stability and almost complete instability [5], which provide the corresponding properties of solutions with a full measure.

Definition 1. We will say that system (1) has the following property of the *Lyapunov, Perron* or *upper-limit* type:

(a) *stability* (*almost stability*) if for any $\varepsilon>0$ there exists $\delta\in(0,\Delta)$ such that any (respectively, almost any in the sense of the Lebesgue measure) initial value $x_0\in B_\delta$ satisfies the corresponding requirement

$$\sup_{t\in\mathbb{R}_+}|x(t,x_0)|<\varepsilon,\quad \varliminf_{t\to+\infty}|x(t,x_0)|<\varepsilon,\quad \varlimsup_{t\to+\infty}|x(t,x_0)|<\varepsilon; \tag{2}$$

(b) *complete instability* (*almost complete instability*) if there exist $\varepsilon>0$ and $\delta\in(0,\Delta)$ such that any (respectively, almost any) initial value $x_0\in B_\delta$ does not satisfy the corresponding requirement (2) (which is considered to be unfulfilled by definition, in particular, when the solution $x(\,\cdot\,,x_0)$ is not defined on the entire ray $\mathbb{R}_+$).

Definition 2. For system (1), the number

$$\mu_\varkappa(f)\in[0,1],\quad \varkappa=\lambda,\pi,\sigma,$$

is called, respectively, *the Lyapunov, Perron* and *upper-limit measure of stability*, if system (1):

(a) for each $\mu < \mu_\varkappa(f)$ is *μ-stable*, i.e. for any $\varepsilon > 0$ there exists $\delta_\varepsilon \in (0, \Delta)$ such that for every $\delta \in (0, \delta_\varepsilon)$ all values $x_0 \in B_\delta$, satisfying the corresponding requirement (2), form a subset, whose relative measure (in the Lebesgue sense) in B_δ is

$$\mathrm{M}_\varkappa(f, \varepsilon, \delta) \geq \mu;$$

(b) for each $\mu > \mu_\varkappa(f)$ is not μ-stable.

Definition 3. For system (1), the number

$$\nu_\varkappa(f) \in [0, 1], \quad \varkappa = \lambda, \pi, \sigma,$$

is called, respectively, *the Lyapunov, Perron* and *upper-limit measure of instability*, if system (1):

(a) for each $\nu < \nu_\varkappa(f)$ is *ν-unstable*, i.e. for any $\varepsilon > 0$ there exists $\delta_\varepsilon \in (0, \Delta)$ such that for every $\delta \in (0, \delta_\varepsilon)$ all values $x_0 \in B_\delta$, unsatisfying the corresponding requirement (2), form a subset, whose relative measure (in the Lebesgue sense) in B_δ is

$$\mathrm{N}_\varkappa(f, \varepsilon, \delta) \geq \nu;$$

(b) for each $\nu > \nu_\varkappa(f)$ is not ν-unstable.

The correctness of Definitions 2 and 3 is justified by the following theorems.

Theorem 1. *For any system* (1), *any* $\varepsilon > 0$ *and each of the requirements* (2), *the sets of all points* $x_0 \in G$, *both satisfying this requirement and not satisfying it, are measurable.*

Theorem 2. *For any system* (1) *the set of all values* $\mu \in [0, 1]$ *for which it is Lyapunov, Perron or upper-limit μ-stable, as well as all values* $\nu \in [0, 1]$, *for which it is ν-unstable, obviously contains the point 0 and represents an interval, possibly degenerate to this point.*

The following two theorems offer specific formulas for measures of stability and instability and define a set of basic relations linking various measures.

Theorem 3. *For each system* (1), *the entire six of its Lyapunov, Perron and upper-limit measures of stability or instability are uniquely defined, which are respectively given by the formulas*

$$\mu_\varkappa(f) = \lim_{\varepsilon \to +0} \varliminf_{\delta \to +0} \mathrm{M}_\varkappa(f, \varepsilon, \delta), \quad \nu_\varkappa(f) = \lim_{\varepsilon \to +0} \varliminf_{\delta \to +0} \mathrm{N}_\varkappa(f, \varepsilon, \delta), \tag{3}$$

where the limits at $\varepsilon \to +0$ *can be replaced by the lower or, respectively, upper exact bound on* $\varepsilon > 0$.

Theorem 4. *For any system* (1) *the inequalities are satisfied*

$$0 \leqslant \mu_\lambda(f) \leqslant \mu_\sigma(f) \leqslant \mu_\pi(f) \leqslant 1, \quad 0 \leqslant \nu_\pi(f) \leqslant \nu_\sigma(f) \leqslant \nu_\lambda(f) \leqslant 1, \tag{4}$$

$$0 \leqslant \mu_\varkappa(f) + \nu_\varkappa(f) \leqslant 1. \tag{5}$$

Almost stability and almost complete instability are naturally associated with single values of the corresponding measures, but this logical connection turns out to be only one-way.

Theorem 5. *System* (1) *has almost stability or almost complete instability (of some type) if and only if it is 1-stable or, accordingly,* 1*-unstable (of that type), and then its measures of stability and instability (of the same type) are equal to* 1 *and* 0 *or, respectively, vice versa.*

Theorem 6. *For $n = 2$, there are two autonomous systems of the form* (1)*, which have neither almost stability nor almost complete instability of any of the three types: one of them has measures of stability and instability of all three types equal to* 1 *and* 0*, respectively, and the other is the opposite.*

In the case of a linear system, the Lyapunov and upper-limit measures can only take their extreme values, which are obviously also realized on the Perron measures – this is what the following two theorems establish.

Theorem 7. *For any linear system* (1)*, only the following two situations are possible, and in formulas* (3) *for all measures of stability and instability mentioned in them, the lower limits for $\delta \to +0$ are exact:*

(a) *either the relations are satisfied*

$$\mu_\lambda(f) = \mu_\sigma(f) = \mu_\pi(f) = 1 > 0 = \nu_\pi(f) = \nu_\sigma(f) = \nu_\lambda(f)$$

and system (1) *has stability of all three types;*

(b) *either the relations are satisfied*

$$\mu_\lambda(f) = \mu_\sigma(f) = 0 < 1 = \nu_\sigma(f) = \nu_\lambda(f)$$

and system (1) *has the Lyapunov and upper-limit almost complete (possibly even complete) instability.*

In addition, in the linear case, the upper-limit complete instability follows from the Lyapunov one, but the Perron instability does not follow, and not to any extent.

Theorem 8. *For any $n \in \mathbb{N}$, each of the situations listed in Theorem 7 is realized on some limited scalar linear system of the form* (1)*, and the second situation is realized on at least two systems: one of them is autonomous and has the Perron complete instability, i.e.*

$$\mu_\pi(f) = 0 < 1 = \nu_\pi(f),$$

and the other – the Perron stability, i.e.

$$\mu_\pi(f) = 1 > 0 = \nu_\pi(f).$$

The set of all possible sets of different measures of stability and instability of one-dimensional systems is finite.

Theorem 9. *For $n = 1$, the measures of stability and instability of any system* (1) *satisfy the relations*

$$\mu_\lambda(f) = \mu_\sigma(f) \leqslant \mu_\pi(f), \quad \nu_\pi(f) \leqslant \nu_\sigma(f) = \nu_\lambda(f), \tag{6}$$

$$\mu_\varkappa(f), \nu_\varkappa(f) \in \{0,\ 1/2,\ 1\}, \quad \mu_\varkappa(f) + \nu_\varkappa(f) = 1, \quad \varkappa = \lambda, \pi, \sigma. \tag{7}$$

Theorem 10. *For $n = 1$, both inequalities in chains* (6) *for some limited linear system* (1) *are strict, and the cases of all equalities in these chains for each pair of measures of stability and instability specified by conditions* (7) *are implemented on some autonomous systems* (1)*.*

Theorem 6 simultaneously confirms the realizability of both zero and one values by all measures of stability or instability for two-dimensional autonomous systems. Moreover, for such systems the set of implementable sets of all measures turns out to be quite rich.

Theorem 11. *For $n = 2$, for each individual non-strict inequality in chains* (4) *and* (5) *there are two autonomous systems of the form* (1)*: for one of them it turns into an equality, and for the other into a strict inequality.*

Theorem 12. *For $n = 2$, for any $r > 0$ there exists an autonomous system* (1)*, in which the measures of stability of all three types take the same positive value, as well as all measures of instability, and the ratio of these two values equals r, and the right inequality in chain* (5) *turns into equality.*

The following two theorems implement the most contrasting situations in the autonomous arbitrarily non-one-dimensional case.

Theorem 13. *For every integer $n > 1$, some autonomous system* (1) *satisfies the relations*

$$\mu_\lambda(f) = \mu_\sigma(f) = 0 < 1 = \mu_\pi(f), \quad \nu_\pi(f) = \nu_\sigma(f) = 1 > 0 = \nu_\lambda(f).$$

Theorem 14. *For every integer $n > 1$, some autonomous system* (1) *satisfies the relations*

$$\mu_\lambda(f) = 0 < 1 = \mu_\sigma(f) = \mu_\pi(f), \quad \nu_\pi(f) = 1 > 0 = \nu_\sigma(f) = \nu_\lambda(f).$$

In the one-dimensional autonomous case, two contrasting situations described in Theorems 13 and 14 are impossible.

Theorem 15. *For $n = 1$, for any autonomous system* (1) *the equalities are satisfied*

$$\mu_\lambda(f) = \mu_\sigma(f) = \mu_\pi(f), \quad \nu_\pi(f) = \nu_\sigma(f) = \nu_\lambda(f).$$

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A System of Singularly Perturbed Differential Equations with an Unstable Turning Point of the First Kind

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Let us consider the system of differential equations with turning point:

$$\varepsilon Y'(x,\varepsilon) - A(x,\varepsilon)Y(x,\varepsilon) = H(x), \tag{0.1}$$

where

$$A(x,\varepsilon) = A_0(x) + \varepsilon A_1(x)$$

is a known matrix, where

$$\mathbf{A_0}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -b(x) & -a(x) & 0 \end{pmatrix}, \quad \mathbf{A_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

when $\varepsilon \to 0$, $x \in [-l, 0]$, $Y(x,\varepsilon) \equiv Y_k(x,\varepsilon) = colomn(y_1(x,\varepsilon), y_2(x,\varepsilon), y_3(x,\varepsilon))$ is an unknown vector function, $H(x) = colomn(0, 0, h(x))$ is a given vector function.

Needs of modern physics, mathematics, biology and their applied fields require us solving problems of a more complex nature, i.e. research behavior of the function in asymptotic models, which are reduced to problems (0.1).

Let us investigate the problem of constructing uniform asymptotics of solutions of a singularly perturbed system (0.1) for which the conditions are fulfilled:

S1. $A_0(x)$, $H(x) \in C^{\infty}[-l, 0]$.

S2. $a(x) = x\widetilde{a}(x)$, $\widetilde{a}(x) < 0$, $b(x) \neq 0$.

This case has the following feature: the turning point is unstable [1] and the construction of asymptotics requires a separate technique, since the results of previous studies cannot be simply extended to this case.

Conducted research in [1] showed that for construction of uniform asymptotic under conditions **S2**, i.e. when $\widetilde{a}(x) < 0$, $b(x) > 0$ when $x \in [-l, 0]$, the second form must be used the Airy equation, the solutions of which are the so-called Airy–Langer functions: Ai(t) and Bi(t).

$$U''(t) - tU(t) = 0.$$

That is, in this case, the model operator for a homogeneous system is the Airy model operator. And to construct the asymptotics of the solution of a heterogeneous system, we will use an essentially special function $\nu(t)$

$$U''(t) - tU(t) = \pi^{-1}.$$

Some aspects of problem (0.1) we studied in [3]. In [4], a developed algorithm for constructing uniform asymptotics of solutions to systems of singularly perturbed differential equations is proposed. In [2] constructive conditions for the existence of the asymptotics of the solution of the system of singularly perturbed differential equations of the fourth order with a differential turning point are established, and the algorithm for constructing the corresponding solution is proposed.

The characteristic equation that corresponds to the SP system (0.1) is as follows:

$$|A_0(x)-\lambda E| = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ -b(x) & -a(x) & -\lambda \end{vmatrix} = -\lambda^3 - x\widetilde{a}(x)\lambda = 0.$$

The roots of this equation are

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm\sqrt{x\widetilde{a}(x)}\,.$$

The purpose of this work is to construct a uniform asymptotic solution with an unstable turning point of the first kind.

1 Regularization of singularly perturbed systems of differential equations

In order to save all essential singular functions, that appear in the solution of system (0.1) due to the special point $\varepsilon = 0$, a regularizing variable is introduced $t = \varepsilon^{-p}\cdot\varphi(x)$, where exponent p and regularizing function $\varphi(x)$ are to be determined.

Instead of $Y_k(x,\varepsilon)$ function, $\widetilde{Y}_k(x,t,\varepsilon)$ transformation function will be studied, also the transformation will be performed in such a way that the following identity is true

$$\widetilde{Y}(x,t,\varepsilon)\big|_{t=\varepsilon^{-p}\varphi(x)} \equiv Y(x,\varepsilon),$$

which is the necessary condition for suggested method.

The vector equation (0.1) can be written as

$$\widetilde{L}_\varepsilon \widetilde{Y}_k(x,t,\varepsilon) \equiv \mu\varphi' \frac{\partial\widetilde{y}(x,t,\varepsilon)}{\partial t} + \mu^3 \frac{\partial\widetilde{y}(x,t,\varepsilon)}{\partial x} - A(x,\varepsilon)\widetilde{Y}_k(x,t,\varepsilon) = H(x). \tag{1.1}$$

Asymptotic forms of solutions for equation (1.1) are constructed in the form of the series

$$\widetilde{Y}_k(x,t,\varepsilon) = \sum_{i=1}^{2} D_i(x,t,\varepsilon) + f(x,\varepsilon)\nu(t) + \varepsilon^\gamma g(x,\varepsilon)\nu'(t) + \omega(x,\varepsilon),$$

$$\sum_{i=1}^{2} D_i(x,t,\varepsilon) = \begin{pmatrix} \varepsilon^{s1}\alpha_{k1}(x,\varepsilon) \\ \varepsilon^{s2}\alpha_{k2}(x,\varepsilon) \\ \varepsilon^{s3}\alpha_{k3}(x,\varepsilon) \end{pmatrix} U_i(t) + \varepsilon^\gamma \begin{pmatrix} \varepsilon^{k1}\beta_{k1}(x,\varepsilon) \\ \varepsilon^{k2}\beta_{k2}(x,\varepsilon) \\ \varepsilon^{k3}\beta_{k3}(x,\varepsilon) \end{pmatrix} U_i{}'(t),$$

where $U_1(t)$, $U_2(t)$ are the Airy–Langer functions [1] and $\alpha_{ik}(x,\varepsilon)$, $\beta_{ik}(x,\varepsilon)$, $f_k(x,\varepsilon)$, $g_k(x,\varepsilon)$, $\omega_k(x,\varepsilon)$, $k = 1,2,3$ are analytic functions with reference to a small parameter and are infinitely differentiable functions of variable $x \in [-l;0]$ which are still to be determined.

For convenience, we introduce the notation $U_1(t) \equiv \mathrm{Ai}(t)$, $U_2(t) \equiv \mathrm{Bi}(t)$.

First of all, the analysis how transformation operator $\widetilde{L}_\varepsilon$ operates on vector function $D_k(x,t,\varepsilon)$ will be performed, and then the obtained result will be utilized in the homogeneous transformation equation (0.1). Then, after equating corresponding coefficients of essential singular functions $U_k(t)$, $k = 1,2$ and their derivatives two following vector equations are obtained:

$$U_i'(t) : \varepsilon^{1-p}\alpha_{ik}(x,\varepsilon)\varphi'(x) - \varepsilon^\gamma[A_0(x)+\varepsilon A_1]\beta_{ik}(x,\varepsilon) = -\varepsilon^{1+\gamma}\beta_{ik}'(x,\varepsilon), \tag{1.2}$$

$$U_i(t) : -\varepsilon^{1+\gamma-2p}\beta_{ik}(x,\varepsilon)\varphi(x)\varphi'(x) - [A_0(x)+\varepsilon A_1]\alpha_{ik}(x,\varepsilon) = -\varepsilon\alpha_{ik}'(x,\varepsilon). \tag{1.3}$$

2 Construction of formal solutions of a homogeneous transformation system

The unknown coefficients of the vector equations (1.2) and (1.3) are sought as following vector function series ($i=1,2$):

$$\alpha_{ik}(x,\varepsilon)=\sum_{r=0}^{+\infty}\mu^r\alpha_{ikr}(x),\quad \beta_{ik}(x,\varepsilon)=\sum_{r=0}^{+\infty}\mu^r\beta_{ikr}(x).$$

To determine vector function components

$$\alpha_{ikr}=colomn\big(\alpha_{i1r}(x),\alpha_{i2r}(x),\alpha_{i3r}(x)\big)$$

and

$$\beta_{ikr}(x)=colomn\big(\beta_{i1r}(x),\beta_{i2r}(x),\beta_{i3r}(x)\big),$$

the following recurrent systems of equations are obtained:

$$\begin{aligned}&\Phi(x)Z_{k0}(x)=0,\ \ r=0,1,2,\\&\Phi(x)Z_{kr}(x)=FZ_{k(r-3)}(x),\ \ r\geq 3.\end{aligned}$$

At the moment, the regularizing function has not yet been defined; therefore, it will be defined as a solution of the problem

$$\varphi(x)\varphi'^2(x)=-a(x)\equiv -x\widetilde{a},\quad \varphi(0)=0,$$

which is the following function

$$\varphi(x)=\bigg(\frac{3}{2}\int\limits_0^x\sqrt{-x\widetilde{a}(x)}\,dx\bigg)^{\frac{2}{3}}.$$

The regularizing function of such kind has been considered in [1,5].

Due to such a choice of the regularizing variable $\varphi(x)$ there is a nontrivial solution of the homogeneous system $\Phi(x)Z_{kr}(x)=0$, $r=0,1,2$, that is

$$Z_{k0}(x)=colomn\Big(0,\frac{1}{\varphi'(x)}\,\beta_{i30}(x),-\varphi(x)\varphi'(x)\beta_{i20}(x),0,\beta_{i20}(x),\beta_{i30}(x)\Big),$$

where $\beta_{ikr}(x)$ ($i=1,2$; $k=2,3$) are arbitrary up to some point and sufficiently smooth function at $x\in[-l;0]$.

Solving systems of recurrent equations at the third step, i.e., when $r=3$, and taking into account that the functions are arbitrary, $\beta_{is0}(x)=\beta^0_{is0}\cdot\hat{\beta}_{is0}(x)$ ($i=1,2$; $s=2,3$), where $\beta^0_{is0}(x)$ are arbitrary constants, $\hat{\beta}_{is0}(x)$ is a partial and sufficiently smooth for all $x\in[-l;0]$ solutions of homogeneous equations. This definition of vector functions $Z_{ik0}(x)$ implies that there are following solutions of inhomogeneous systems of the algebraic equations (1.2) and (1.3):

$$\begin{gathered}Z_{k3}(x)=colomn(z_{i13},z_{i23},z_{i33},z_{i43},z_{i53},z_{i63}),\\ z_{i13}=\frac{1}{\varphi'(x)}\,\beta_{i20}(x),\ \ z_{i23}=\frac{-\beta'_{i20}(x)+\beta_{i33}(x)}{\varphi'(x)}\,,\\ z_{i33}=\frac{-\beta'_{i30}(x)-a(x)\beta_{i23}(x)-b(x)(\varphi(x))^{-1}(\varphi'(x))^{-2}\beta_{i30}}{\varphi'(x)}\,,\\ z_{i43}=(\varphi(x))^{-1}(\varphi'(x))^{-2}\beta_{i20}(x),\ \ z_{i53}=\beta_{i21}(x),\ \ z_{i63}=\beta_{i31}(x),\end{gathered}$$

where $\beta_{i21}(x)$ and $\beta_{i31}(x)$ are arbitrary up to some point and sufficiently smooth functions for all $x \in [-l; 0]$.

Thus, gradual solving of systems of equations (1.2) and (1.3) gives two formal solutions of the transformation vector equation (0.1)

$$D_{ik}(x, \varepsilon^{-\frac{2}{3}}\varphi(x), \varepsilon) = \sum_{r=0}^{\infty} \varepsilon^r \Big[\alpha_{ikr}(x) U_i(\varepsilon^{-\frac{2}{3}}\varphi(x)) + \varepsilon^{\frac{1}{3}} \beta_{ikr}(x, \varepsilon) U_i'(\varepsilon^{-\frac{2}{3}}\varphi(x))\Big]. \tag{2.1}$$

The third formal solution of the homogeneous vector equation (0.1) is then constructed as a series

$$\omega(x, \varepsilon) \equiv \sum_{r=0}^{\infty} \varepsilon^r \omega_r(x) \equiv colomn\Big(\sum_{r=0}^{\infty} \varepsilon^r \omega_{1r}(x), \sum_{r=0}^{\infty} \varepsilon^r \omega_{2r}(x), \sum_{r=0}^{\infty} \varepsilon^r \omega_{3r}(x)\Big).$$

3 Construction of formal partial solutions

Similarly to the previous steps, in order to construct asymptotic forms of partial solutions of the inhomogeneous transformation vector equation (0.1), let us analyze how transformation operator operates on an element from the space of non-resonant solutions:

$$\begin{aligned}
&\widetilde{L}_\varepsilon\big(f_k(x,\varepsilon)\nu(t) + \mu g_k(x,\varepsilon)\nu'(t) + \omega_k(x,\varepsilon)\big) \\
&\quad = \mu f_k(x,\varepsilon)\varphi'(x)\nu(t) + g_k(x,\varepsilon)\varphi'(x)\varphi(x)\nu(t) - A(x,\varepsilon) f_k(x,\varepsilon)\nu(t) - \mu A(x,\varepsilon) g_k(x,\varepsilon)\nu'(t) \\
&\qquad + \mu^3 f_k'(x)\nu(t) + \mu^4 g_k'(x)\nu'(t) + \mu^2 \varphi'(x) g_k(x)\pi^{-1} + \mu^3 \omega'(x) - A(x,\varepsilon)\omega_k(x) = H(x).
\end{aligned}$$

In order to have smooth solutions of the systems, the asymptotic forms of the solutions are constructed as series

$$f_k(x,\varepsilon) = \sum_{r=-2}^{+\infty} \mu^r f_r(x), \quad g_k(x,\varepsilon) = \sum_{r=-2}^{+\infty} \mu^r g_r(x), \quad \overline{\omega}(x,\varepsilon) = \sum_{r=0}^{+\infty} \mu^r \overline{\omega}_r(x).$$

Therefore, the partial solution of the transformation vector equation (0.1) is then defined as the series

$$\widetilde{Y}_k^{part.}(x,t,\varepsilon) = \sum_{r=-2}^{\infty} \mu^r \big[f_{kr}(x)\nu(t) + \mu g_{kr}(x)\nu'(t)\big] + \sum_{r=0}^{\infty} \mu^r \overline{\omega}_{kr}(x). \tag{3.1}$$

4 Conclusions

Therefore, we constructed a uniform asymptotic solution for a system of singularly perturbed differential equations with an unstable turning point (0.1) in the form (2.1) and (3.1).

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Parameter-Dependent Periodic Problems for Non-Autonomous Duffing Equations with a Sign-Changing Forcing Term

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The extended abstract concerns the parameter-dependent periodic problem

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u + \mu f(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega), \tag{1}$$

where $p, h, f \in L([0,\omega])$, $h \geq 0$ a.e. on $[0,\omega]$, $\lambda > 1$, and $\mu \in \mathbb{R}$ is a parameter. By a solution to problem (1), as usual, we understand a function $u : [0,\omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies the given equation almost everywhere, and meets the periodic conditions. The text is based on the paper [4].

We first show where problem (1) may appear from. Consider a forced oscillator consisting of two fixed charged bodies of charges $q > 0$ and a charged mass body of weight m and charge $Q > 0$ (see Fig. 1).

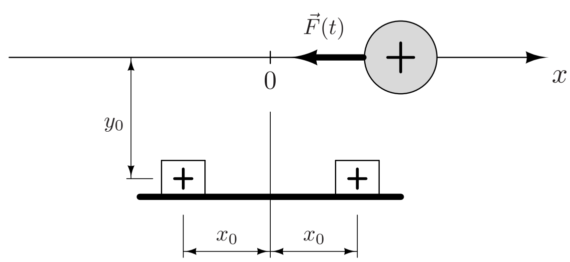

Figure 1. Nonlinear undamped forced oscillator.

Assume that the mass body moves horizontally without any friction and the charges q of the fixed bodies change ω-periodically, i.e., $q : \mathbb{R} \to]0, +\infty[$ is an ω-periodic function. This is a system with one degree of freedom described by the coordinate x, whose equation of motion is of the form

$$mx'' - \frac{Qq(t)}{4\pi\varepsilon_r\varepsilon_0}\left(\frac{x + x_0}{[(x + x_0)^2 + y_0^2]^{3/2}} + \frac{x - x_0}{[(x - x_0)^2 + y_0^2]^{3/2}}\right) = F(t), \tag{2}$$

where ε_r is the relative permittivity and ε_0 is the vacuum permittivity.

Numeric simulations show that if $y_0^2 < 2x_0^2$, then equation (2) with $q(t) \equiv Const.$ and $F(t) \equiv 0$ has exactly three equilibria $x_1 := 0$, $x_2 > 0$, and $x_3 = -x_2$. Approximating the non-linearity in (2) by the third degree Taylor polynomial centred at 0, we obtain the equation

$$x'' = -\frac{Qq(t)(2x_0^2 - y_0^2)}{2\pi\varepsilon_r\varepsilon_0 m(x_0^2 + y_0^2)^{5/2}}\, x + \frac{3Qq(t)(24x_0^2y_0^2 - 3y_0^4 - 8x_0^4)}{\pi\varepsilon_r\varepsilon_0 m(x_0^2 + y_0^2)^{9/2}}\, x^3 + \frac{F(t)}{m}\,,$$

which is a particular case of the differential equation in (1) with $\mu = 1$, where

$$p(t) := -\frac{Qq(t)(2x_0^2 - y_0^2)}{2\pi\varepsilon_r\varepsilon_0 m(x_0^2 + y_0^2)^{5/2}}, \quad h(t) := \frac{3Qq(t)(24x_0^2y_0^2 - 3y_0^4 - 8x_0^4)}{\pi\varepsilon_r\varepsilon_0 m(x_0^2 + y_0^2)^{9/2}},$$

$f(t) := \frac{F(t)}{m}$, and $\lambda := 3$. Assuming that $(4 - 2\sqrt{10/3}\,)x_0^2 < y_0^2 < 2x_0^2$ and $F(t) \not\equiv 0$, it is easy to show that the functions p and h are negative and positive, respectively.

To formulate our results, we need the following definition.

Definition ([2])**.** We say that a function p belongs to the set $\mathcal{V}^-(\omega)$ (resp. $\mathcal{V}^+(\omega)$) if, for any function $u \in AC^1([0,\omega])$ satisfying

$$u''(t) \geq p(t)u(t) \quad \text{for a. e. } t \in [0,\omega], \quad u(0) = u(\omega), \ \ u'(0) \geq u'(\omega),$$

the inequality $u(t) \leq 0$ (resp. $u(t) \geq 0$) holds for $t \in [0,\omega]$. By $\mathcal{U}(\omega)$, we denote the set of pairs (p,f) such that the problem

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega) \tag{3}$$

has a unique solution which is positive. The set $\mathcal{V}_0(\omega)$ consists of all the functions p such that problem (3) with $f(t) \equiv 0$ possesses a positive solution.

Remark 1. The effective conditions guaranteeing the inclusions $p \in \mathcal{V}^-(\omega)$, $p \in \mathcal{V}^+(\omega)$, $p \in \mathcal{V}_0(\omega)$, as well as $(p,f) \in \mathcal{U}(\omega)$ are provided in [2] (see also [1,5]).

Below we discuss the existence/non-existence as well as the exact multiplicity of positive solutions to problem (1) depending on the choice of the parameter μ provided that $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. Let us show, as a motivation, what happens in the autonomous case of (1). Hence, we consider the equation

$$x'' = -ax + b|x|^\lambda \operatorname{sgn} x + \mu. \tag{4}$$

In view of our hypotheses $h \geq 0$ a. e. on $[0,\omega]$, $h(t) \not\equiv 0$ and since $-a \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ if only if $a > 0$, we assume that $a, b > 0$. By direct calculation, the phase portraits of equation (4) can be elaborated depending on the choice of the parameter $\mu \in \mathbb{R}$ (see, Fig. 2) and, thus, one can prove the following proposition concerning the periodic solutions to equation (4).

Proposition 1. *Let $\lambda > 1$ and $a, b > 0$. Then, the following conclusions hold:*

(i) *If $\mu > \frac{(\lambda-1)a}{\lambda}\,(\frac{a}{\lambda b})^{\frac{1}{\lambda-1}}$, then equation* (4) *has a unique negative equilibrium (saddle) and no other periodic solutions occur.*

(ii) *If $\mu = \frac{(\lambda-1)a}{\lambda}\,(\frac{a}{\lambda b})^{\frac{1}{\lambda-1}}$, then equation* (4) *has a unique positive equilibrium (cusp), a unique negative equilibrium (saddle), and no other periodic solutions occur.*

(iii) *If $0 < \mu < \frac{(\lambda-1)a}{\lambda}\,(\frac{a}{\lambda b})^{\frac{1}{\lambda-1}}$, then equation* (4) *possesses exactly two positive equilibria $x_1 > x_2$ (x_1 is a saddle and x_2 is a center), a unique negative equilibrium x_3 (saddle), and non-constant (positive and possibly sign-changing) periodic solutions with different periods. Moreover, all the non-constant periodic solutions oscillate around x_2 between x_3 and x_1.*

(iv) *If $\mu = 0$, then equation* (4) *possesses a unique positive equilibrium x_0 (saddle), a trivial equilibrium (center), a unique negative equilibrium $-x_0$, and non-constant sign-changing periodic solutions with different periods. Moreover, all the non-constant periodic solutions oscillate around* 0 *between $-x_0$ and x_0.*

(v) *For $\mu < 0$, the conclusions are "symmetric" as compared with the items* (i)–(iii)*, see Fig. 2.*

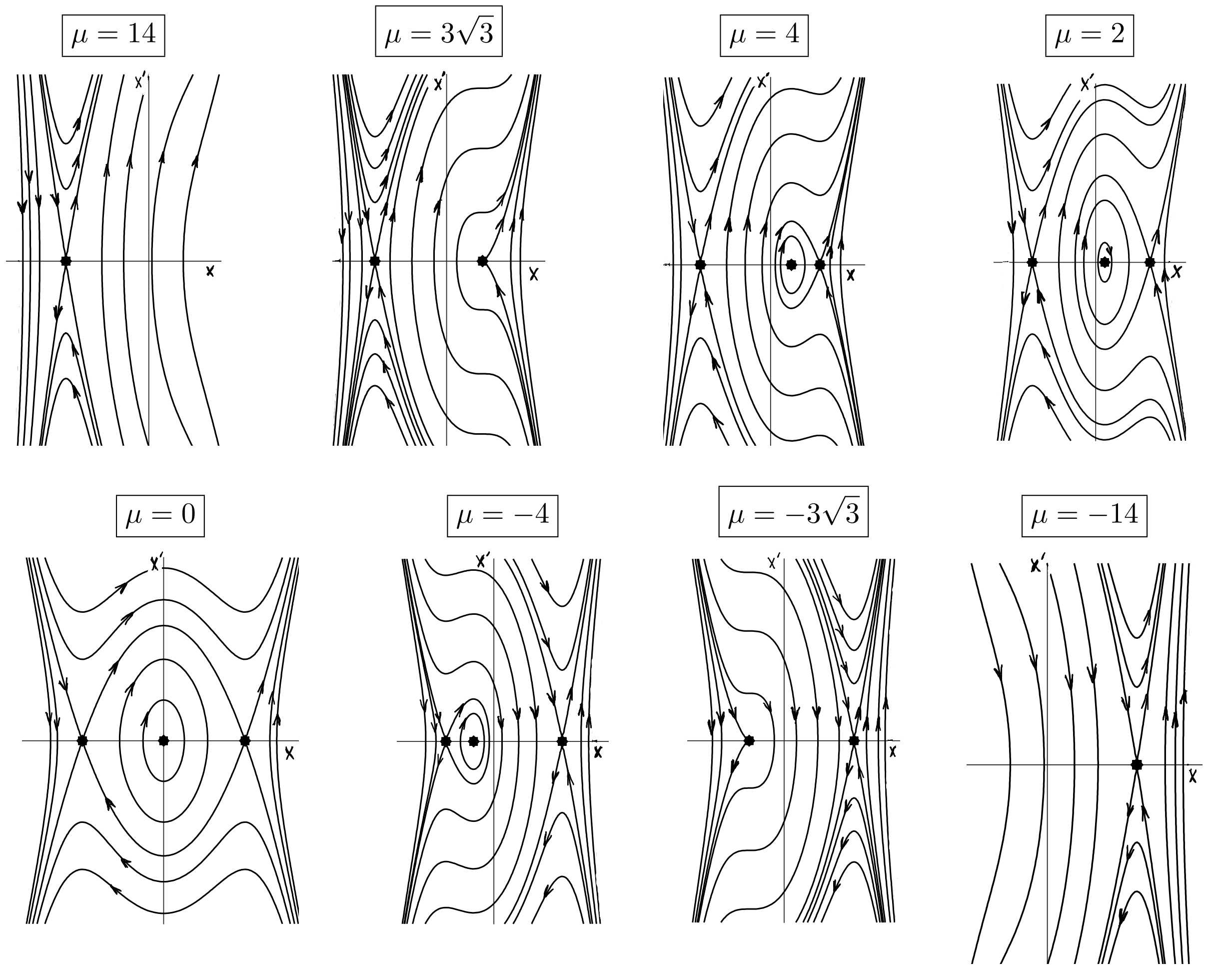

Figure 2. Phase portraits of equation (4) with $a = 9$, $b = 4$, and $\lambda = 3$.

We start with the most general statement of the text, which provides the existence/non-existence results in the case of $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. This condition is satisfied, e.g., if $\int_0^\omega p(s)\mathrm{d}s \le 0$, $p(t) \not\equiv 0$.

Theorem 1. *Let $\lambda > 1$, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $f(t) \not\equiv 0$, and*

$$h(t) > 0 \quad \text{for a. e. } t \in [0, \omega]. \tag{5}$$

Then, there exist $-\infty \le \mu_* < 0$ and $0 < \mu^* \le +\infty$ such that the following conclusions hold:

(1) *For any $\mu \in]\mu_*, \mu^*[$, problem (1) has a positive solution u^* such that every solution u to problem (1) satisfies*

$$\text{either } u(t) < u^*(t) \quad \text{for } t \in [0, \omega], \quad \text{or } u(t) \equiv u^*(t). \tag{6}$$

Moreover, any couple of distinct positive solutions u_1, u_2 to (1) different from u^* satisfies

$$\min\big\{u_1(t) - u_2(t) : t \in [0, \omega]\big\} < 0, \quad \max\big\{u_1(t) - u_2(t) : t \in [0, \omega]\big\} > 0.$$

(2) *If $\mu^* < +\infty$ (e.g. provided that $\int_0^\omega f(s)\mathrm{d}s > 0$), then*

(a) *for* $\mu > \mu^*$*, problem* (1) *has no positive solution,*

(b) *for* $\mu = \mu^*$*, problem* (1) *has a unique non-negative solution* u^* *and every solution* u *to* (1) *satisfies* (6)*.*

(3) *If* $\mu_* > -\infty$ *(e.g. provided that* $\int\limits_0^\omega f(s)\mathrm{d}s < 0$*), then*

(a) *for* $\mu < \mu_*$*, problem* (1) *has no positive solution,*

(b) *for* $\mu = \mu_*$*, problem* (1) *has a unique non-negative solution* u^* *and every solution* u *to* (1) *satisfies* (6)*.*

It is clear that u is a solution to problem (1) if and only if $-u$ is a solution to the problem

$$z'' = p(t)z + h(t)|z|^\lambda \operatorname{sgn} z - \mu f(t); \quad z(0) = z(\omega), \;\; z'(0) = z'(\omega).$$

Therefore, we get the following corollary from Theorem 1.

Corollary. *Let* $\lambda > 1$*,* $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$*,* $f(t) \not\equiv 0$*, and condition* (5) *hold. Then, there exists* $0 < \mu_0 < +\infty$ *such that, for any* $\mu \in \,]-\mu_0, \mu_0[\,$*, problem* (1) *has a negative solution* u_* *and a positive solution* u^* *such that every solution* u *to problem* (1) *different from* u_**,* u^* *satisfies*

$$u_*(t) < u(t) < u^*(t) \quad \text{for } t \in [0, \omega].$$

We showed in [3, Example 2.8] that assuming $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, hypothesis (5) in Theorem 1 (i.e. the positivity of h a. e. on $[0, \omega]$) is essential for the existence of a positive solution to problem (1) with $\mu = 0$ and cannot be weakened to the non-negativity of h. However, under a stronger assumption on the coefficient p (namely, $p \in \mathcal{V}^+(\omega)$), hypothesis (5) of Theorem 1 can be relaxed to

$$h(t) \geq 0 \quad \text{for a. e. } t \in [0, \omega], \;\; h(t) \not\equiv 0. \tag{7}$$

Theorem 2. *Let* $\lambda > 1$*,* $p \in \mathcal{V}^+(\omega)$*,* h *satisfy* (7)*, and*

$$(p, f) \in \mathcal{U}(\omega), \qquad \int_0^\omega f(s)\mathrm{d}s > 0. \tag{8}$$

Then, there exist $-\infty \leq \mu_* < 0$ *and* $0 < \mu^* < +\infty$ *such that the following conclusions hold:*

(1) *For any* $\mu > \mu^*$*, problem* (1) *has no positive solution.*

(2) *For* $\mu = \mu^*$*, problem* (1) *has a unique positive solution* u^* *and, moreover, every solution* u *to problem* (1) *satisfies* (6)*.*

(3) *For* $\mu \in \,]0, \mu^*[\,$*, problem* (1) *has exactly two positive solutions* u_1*,* u_2 *and these solutions satisfy*

$$u_1(t) > u_2(t) > 0 \quad \text{for } t \in [0, \omega].$$

Moreover, every solution u *to problem* (1) *different from* u_1 *is such that*

$$u(t) < u_1(t) \quad \text{for } t \in [0, \omega].$$

(4) *For* $\mu = 0$*, problem* (1) *has exactly three solutions: a positive solution* u_0*, the trivial solution, a negative solution* $-u_0$*.*

(5) *For* $\mu \in \,]\mu_*, 0[\,$*, problem* (1) *has either one or two positive solutions. Moreover,* (1) *has a positive solution* u^* *such that every solution* u *to problem* (1) *satisfies* (6)*.*

(6) *If $\mu_* > -\infty$, then, for any $\mu < \mu_*$, problem* (1) *has no positive solution.*

Open questions. The following two questions remain open in Theorem 2:

(a) Does the inequality $\mu_* > -\infty$ hold without any additional assumption?
(b) What happens in the case of $\mu = \mu_*$, if $\mu_* > -\infty$ and $h(t) = 0$ on a set of positive measure?

Remark 2. Assuming $f(t) \geq 0$ for a. e. $t \in [0, \omega]$, $f(t) \not\equiv 0$, the conclusions of Theorems 1 and 2 can be substantially refined (see [4, Theorems 3.6 and 3.14]).

Theorem 2 guarantees the existence of certain "critical" values μ_*, μ^* of the parameter μ such that crossing these values, a bifurcation of positive solutions to problem (1) occurs. From an application point of view, the estimates of these numbers are also needed.

Proposition 2. *Let $\lambda > 1$, $p \in \operatorname{Int} \mathcal{V}^+(\omega)$, h satisfy* (7), *and*

$$\int_0^\omega [f(s)]_+ \mathrm{d}s > \Gamma(p) \int_0^\omega [f(s)]_- \mathrm{d}s > 0,$$

where the number $\Gamma(p)$, depending only on p, is defined in [2, Section 6]. Then, the numbers μ_*, μ^* appearing in the conclusion of Theorem 2 satisfy

$$\mu_* \leq -\frac{(\lambda-1)[\Delta(p)]^{-\frac{\lambda}{\lambda-1}}}{\lambda\big[\lambda \int_0^\omega h(s)\mathrm{d}s\big]^{\frac{1}{\lambda-1}} \int_0^\omega [f(s)]_- \mathrm{d}s},$$

where $\Delta(p)$ denotes a norm of Green's operator of problem (8) *(see [4, Remark 2.5]), and*

$$\frac{(\lambda-1)[\Delta(p)]^{-\frac{\lambda}{\lambda-1}}}{\lambda\big[\lambda \int_0^\omega h(s)\mathrm{d}s\big]^{\frac{1}{\lambda-1}} \int_0^\omega [f(s)]_+ \mathrm{d}s} \leq \mu^* < \frac{(\lambda-1)\big[\Gamma(p) \int_0^\omega [p(s)]_- \mathrm{d}s - \int_0^\omega [p(s)]_+ \mathrm{d}s\big]^{\frac{\lambda}{\lambda-1}}}{\lambda\big[\lambda \int_0^\omega h(s)\mathrm{d}s\big]^{\frac{1}{\lambda-1}} \big[\int_0^\omega [f(s)]_+ \mathrm{d}s - \Gamma(p) \int_0^\omega [f(s)]_- \mathrm{d}s\big]}.$$

Acknowledgements

The research is supported by the internal grant # FSI-S-23-8161 of FME BUT.

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Solvability of BVPs for Sequential Fractional Differential Equations at Resonance

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1 Introduction

Let $T > 0$ be given, $J = [0, T]$ and $\|x\| = \max\{|x(t)| : t \in J\}$ be the norm in $C(J)$.

We discuss the fractional boundary value problem

$$^{c}\!D^{\alpha}x(t) - p(t, x(t))\,^{c}\!D^{\alpha-1}x(t) = f(t, x(t)), \tag{1.1}$$

$$x(0) = x(T), \quad x'(0) = 0, \tag{1.2}$$

where $\alpha \in (1, 2]$, $p, f \in C(J \times \mathbb{R})$ and $^{c}\!D$ denotes the Caputo fractional derivative.

Definition 1.1. We say that $x : J \to \mathbb{R}$ is *a solution of equation* (1.1) if $x', {}^{c}\!D^{\alpha}x \in C(J)$ and (1.1) holds for $t \in J$. A solution x of (1.1) satisfying the boundary condition (1.2) is called *a solution of problem* (1.1), (1.2).

The special case of (1.1) is the differential equation $x'' - p(t, x)x' = f(t, x)$. Problem (1.1), (1.2) is at resonance, because each constant function x on J is a solution of problem $^{c}\!D^{\alpha}x - p(t, x)\,^{c}\!D^{\alpha-1}x = 0$, (1.2).

The aim of this paper is to give conditions guaranteeing the existence and uniqueness of solutions to problem (1.1), (1.2). It is shown that this problem is reduced to the existence of a fixed point of an integral operator $\mathcal{S}$ in the set $C(J) \times \mathbb{R}$. The Schaefer fixed point theorem [1] is applied for solving $\mathcal{S}(x, c) = (x, c)$.

2 Preliminaries

We recall the definitions of the Riemann–Liouville fractional integral and the Caputo fractional derivative [2, 3].

The Riemann–Liouville fractional integral $I^{\gamma}x$ of order $\gamma > 0$ of a function $x : J \to \mathbb{R}$ is defined as

$$I^{\gamma}x(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\, x(s)\,\mathrm{d}s,$$

where Γ is the Euler gamma function. I^0 is the identical operator.

The Caputo fractional derivative $^{c}\!D^{\gamma}x$ of order $\gamma > 0$, $\gamma \notin \mathbb{N}$, of a function $x : J \to \mathbb{R}$ is given as

$$^{c}\!D^{\gamma}x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)} \Big(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} s^k \Big)\,\mathrm{d}s,$$

where $n=[\gamma]+1$, $[\gamma]$ means the integral part of the fractional number γ. If $\gamma\in\mathbb{N}$, then ${}^cD^\gamma x=x^{(\gamma)}$. In particular,

$$ {}^cD^\gamma x(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\,(x(s)-x(0))\,\mathrm{d}s=\frac{\mathrm{d}}{\mathrm{d}t}\,I^{1-\gamma}(x(t)-x(0))\ \text{ if }\ \gamma\in(0,1). $$

Let $\mathcal{P},\mathcal{F}:C(J)\to C(J)$ be the Nemytskii operators associated to p, f,

$$ \mathcal{P}x(t)=p(t,x(t)),\quad \mathcal{F}x(t)=f(t,x(t)). $$

Equation (1.1) can be written as

$$ {}^cD^\alpha x(t)-\mathcal{P}x(t){}^cD^{\alpha-1}x(t)=\mathcal{F}x(t). $$

Let an operator $\mathcal{Q}$ acting on $C(J)$ be defined by the formula

$$ \mathcal{Q}x(t)=\int_0^t\mathcal{F}x(s)\exp\left(\int_s^t\mathcal{P}x(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s,\quad t\in J. $$

Then $\mathcal{Q}x(t)|_{t=0}=0$, $\mathcal{Q}:C(J)\to C^1(J)$ and for $x\in C(J)$, $t\in J$,

$$ (\mathcal{Q}x(t))'=\mathcal{P}x(t)\mathcal{Q}x(t)+\mathcal{F}x(t), $$

$$ I^{\alpha-1}\mathcal{Q}x(t)=I^\alpha(\mathcal{Q}x(t))'=I^\alpha(\mathcal{P}x(t)\mathcal{Q}x(t)+\mathcal{F}x(t)). \tag{2.1} $$

The following result deals with solutions x of equation (1.1) satisfying the initial condition

$$ x(0)=c,\quad x'(0)=0, \tag{2.2} $$

where $c\in\mathbb{R}$.

Lemma 2.1. *If x is a solution of the initial value problem* (1.1), (2.2), *then*

$$ x(t)=c+I^{\alpha-1}\mathcal{Q}x(t),\quad t\in J. \tag{2.3} $$

Also vice versa if $x\in C(J)$ satisfies (2.3), *then x is a solution of problem* (1.1), (2.2).

Let $\mathcal{S}:C(J)\times\mathbb{R}\to C(J)\times\mathbb{R}$ be an operator defined by

$$ \mathcal{S}(x,c)=\Big(c+I^{\alpha-1}\mathcal{Q}x(t),c-I^{\alpha-1}\mathcal{Q}x(t)\big|_{t=T}\Big). $$

The relation between fixed points of $\mathcal{S}$ and solutions of problem (1.1), (1.2) is given in the following result.

Lemma 2.2. *If $(x,c)\in C(J)\times\mathbb{R}$ is a fixed point of $\mathcal{S}$, then x is a solution of problem* (1.1), (1.2) *and $c=x(0)$. If x is a solution of problem* (1.1), (1.2), *then $(x,x(0))\in C(J)\times\mathbb{R}$ is a fixed point of $\mathcal{S}$.*

Proof. Let $(x,c)\in C(J)\times\mathbb{R}$ be a fixed point of $\mathcal{S}$. Then

$$ x(t)=c+I^{\alpha-1}\mathcal{Q}x(t),\quad t\in J, \tag{2.4} $$

$$ I^{\alpha-1}\mathcal{Q}x(t)\big|_{t=T}=0. \tag{2.5} $$

Now we conclude from Lemma 2.1 and equality (2.4) that x is a solution of (1.1) and $x(0)=c$, $x'(0)=0$. The equality $x(T)=c$ follows from (2.4) and (2.5). Hence x is a solution of problem (1.1), (1.2).

Let x be a solution of problem (1.1), (1.2) and let $x(0)=c$. Then (see (1.2)) $x(T)=c$. By Lemma 2.1, x satisfies equality (2.3) which together with $x(T)=c$ gives $I^{\alpha-1}\mathcal{Q}x(t)|_{t=T}=0$. Consequently, (x,c) is a fixed point of $\mathcal{S}$. □

Lemma 2.3. *Operator $\mathcal{S}$ is completely continuous.*

3 Existence results

Theorem 3.1. *Let*

(H_1) *$p(t,x)$ be bounded and nonnegative on $J \times \mathbb{R}$,*

(H_2) *there exist $D > 0$ such that*

$$xf(t,x) > 0 \ \ \textit{for} \ \ t \in J, \ \ |x| \geq D,$$

(H_3) *there exist $A, B \in [0, \infty)$ such that*

$$|f(t,x)| \leq A + B|x| \ \ \textit{for} \ \ t \in J, \ \ x \in \mathbb{R}.$$

Then problem (1.1), (1.2) *has at least one solution. In addition, $|x(0)| < D$ for each solution x of this problem.*

Proof. Keeping in mind Lemma 2.2, we need to prove that the operator $\mathcal{S}$ admits a fixed point in $C(J) \times \mathbb{R}$. Since $\mathcal{S}$ is a completely continuous operator by Lemma 2.3, the Schaefer fixed point theorem guarantees the existence of a fixed point of $\mathcal{S}$ if the set

$$\mathcal{M} = \Big\{ (x,c) \in C(J) \times \mathbb{R} : \ \ (x,c) = \lambda \mathcal{S}(x,c) \ \text{ for some } \ \lambda \in (0,1) \Big\}$$

is bounded in $C(J) \times \mathbb{R}$.

In order to prove the boundedness of $\mathcal{M}$, let $(x,c) = \lambda \mathcal{S}(x,c)$ for some $(x,c) \in C(J) \times \mathbb{R}$ and $\lambda \in (0,1)$. Then

$$x(t) = \lambda \big(c + I^{\alpha-1} \mathcal{Q}x(t) \big), \ \ t \in J, \tag{3.1}$$

$$c(\lambda - 1) = \lambda I^{\alpha-1} \mathcal{Q}x(t) \big|_{t=T}. \tag{3.2}$$

It follows from (2.1) and (3.1) that

$$x'(t) = \lambda \frac{\mathrm{d}}{\mathrm{d}t} I^{\alpha-1} \mathcal{Q}x(t) = \lambda I^{\alpha-1} \big(\mathcal{P}x(t) \mathcal{Q}x(t) + \mathcal{F}x(t) \big), \ \ t \in J, \tag{3.3}$$

and $x'(0) = 0$. We claim that

$$|x(0)| < D, \tag{3.4}$$

where D is from (H_2). Suppose $x(0) \geq D$. Then $\mathcal{F}x(t)|_{t=0} = f(0, x(0)) > 0$ by (H_2), and therefore $\mathcal{F}x > 0$ on $[0, \rho]$ for some $\rho \in (0, T]$. Since $\mathcal{P}x(t) = p(t, x(t)) \geq 0$ on J by the assumption, we have $\mathcal{Q}x > 0$ on $(0, \rho]$ and then (see (3.3)) $x' > 0$ on this interval. Thus x is increasing on $[0, \rho]$ and so $x > D$ on $(0, \rho]$. Analysis similar to the above interval $[0, \rho]$ shows that $x \geq D$ on J. Hence $\mathcal{F}x > 0$ on J and therefore $\lambda I^{\alpha-1} \mathcal{Q}x(t)|_{t=T} > 0$ contrary to (3.2) since $c(\lambda - 1) < 0$. We have proved $x(0) < D$. Similarly we can prove $x(0) > -D$. Consequently, estimate (3.4) is valid.

Since (see (3.1)) $x(0) = \lambda c$, we have

$$x(t) = x(0) + \lambda I^{\alpha-1} \mathcal{Q}x(t), \ \ t \in J.$$

Now by applying (3.4), (H_1) and (H_3), some calculations give

$$|x(t)| \leq L_1 + L_2 \int_0^t |x(s)| \, \mathrm{d}s, \ \ t \in J,$$

where L_1, L_2 are positive constants independent of λ. By the Gronwall–Bellman lemma, $\|x\| \leq L_1 e^{L_2 T}$.

In order to give the bound for c, two cases if $\lambda \in (0, 1/2]$ or $\lambda \in (1/2, 1)$ are discussed. $\square$

Example 3.1. Let $k > 0$, $\rho \in (0,1)$, $q \in C(J)$ and $w, r \in C(J \times \mathbb{R})$ be bounded, $|r(t,x)| \le P$ for $(t,x) \in J \times \mathbb{R}$. Then the function

$$f(t,x) = r(t,x) + q(t)|x|^\rho + kx$$

satisfies condition (H_3) for $A = P + \|q\|$, $B = k + \|q\|$. Since

$$\lim_{x \to \pm\infty} \frac{P + \|q\||x|^\rho}{x} = 0,$$

there exists $D > 0$ such that

$$\frac{P + \|q\||x|^\rho}{x} > -k \ \text{ for } \ x \le -D, \quad \frac{P + \|q\|x^\rho}{x} < k \ \text{ for } \ x \ge D.$$

Hence f satisfies condition (H_2). By Theorem 3.1, there exists a solution of problem

$$\begin{gathered} {}^{c}\!D^\alpha x - |w(t,x)|\,{}^{c}\!D^{\alpha-1} x = r(t,x) + q(t)|x|^\rho + kx, \\ x(0) = x(T), \quad x'(0) = 0. \end{gathered}$$

4 Uniqueness results

In this section we assume that the function $p(t,x)$ in equation (1.1) is independent of the variable x, that is, $p(t,x) = p(t)$. Hence we discuss the fractional differential equation

$${}^{c}\!D^\alpha x(t) - p(t)\,{}^{c}\!D^{\alpha-1} x(t) = f(t, x(t)), \tag{4.1}$$

where $p \in C(J)$. According to Lemma 2.2, x is a solution of problem (4.1), (1.2) if and only if $x \in C(J)$,

$$x(t) = x(0) + I^{\alpha-1} \mathcal{Q}x(t) \ \text{ for } \ t \in J \ \text{ and } \ x(0) = x(T),$$

where

$$\mathcal{Q}x(t) = \int_0^t \mathcal{F}x(s) \exp\left(\int_s^t p(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s.$$

Let $\mathcal{A}$ be the set of all solutions to problem (4.1), (1.2). Under conditions (H_2), (H_3) and $p \ge 0$ on J, $\mathcal{A} \ne \varnothing$ and $|x(0)| < D$ for $x \in \mathcal{A}$ by Theorem 3.1. We are interested in the structure of the set $\mathcal{A}$, especially when $\mathcal{A}$ is a singleton set, that is, when problem (4.1), (1.2) has a unique solution.

Lemma 4.1. *Let $p \ge 0$ on J and let (H_2), (H_3),*

(H_4) *for each $t \in J$, $f(t,x)$ is increasing in the variable x on $\mathbb{R}$*

hold. Then $u(0) = v(0)$ for $u, v \in \mathcal{A}$.

The following theorem says that if $u, v \in \mathcal{A}$ and $u \ne v$, then the function $u - v$ vanishes at points t_n of a sequence $\{t_n\} \subset (0,T)$.

Theorem 4.1. *Let (H_2)–(H_4) hold and let $p \ge 0$ on J. Let $u, v \in \mathcal{A}$ and $u \ne v$. Then there exists a decreasing sequence $\{t_n\} \subset (0,T)$, $\lim_{n\to\infty} t_n = 0$, such that*

$$u(t_n) - v(t_n) = 0 \ \text{ for } \ n \in \mathbb{N}.$$

We are now in the position to give the conditions for the existence of a unique solution to problem (4.1), (1.2).

Theorem 4.2. *Let $p \geq 0$ on J and let (H_2)–(H_4),*

(H_5) *f satisfies the local Lipschitz condition on $J \times \mathbb{R}$, that is, for each $S > 0$ there is $L = L(S) > 0$ such that*

$$|f(t,x_1) - f(t,x_2)| \leq L|x_1 - x_2| \ \text{ for } \ t \in J, \ \ x_1, x_2 \in [-S, S]$$

hold. Then problem $(4.1), (1.2)$ *has a unique solution.*

Example 4.1. Let $k > 0$, $\rho \in (0,1)$, $q, r \in C(J)$ and $f(t,x) = r(t) + |x|^\rho \arctan x + kx$. Then f satisfies conditions (H_2) and (H_3) for $D = \|r\|/k$ and $A = \|r\| + \pi/2$, $B = k + \pi/2$. Since the function $\phi(x) = |x|^\rho \arctan x + kx$ has continuous derivative on $\mathbb{R}$, $\frac{\partial f}{\partial x} \in C(J \times \mathbb{R})$, and therefore f satisfies condition (H_5). Clearly, f satisfies condition (H_4). Consequently, by Theorem 4.2, there exists a unique solution of problem

$$\begin{gathered}
{}^cD^\alpha x - |q(t)|{}^cD^{\alpha-1} x = r(t) + |x|^\rho \arctan x + kx, \\
x(0) = x(T), \quad x'(0) = 0.
\end{gathered}$$

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Periodic Solutions in Dynamic Equations on Time Scales and Their Relationship with Differential Equations

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1 Basic concepts of the theory of time scales

A time scale, denoted by $\mathbb{T}$, is defined as an arbitrary nonempty closed subset of the real axis. To refer to a subset of the time scale, we use the notation $A_{\mathbb{T}}$, where $A_{\mathbb{T}}$ represents the intersection of set A of the real axis with the time scale $\mathbb{T}$.

For every time scale there are defined two operators, the forward jump operator σ and the backward jump operator ρ, which are integral to this theory. The forward jump operator is defined as $\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\}$, while the backward jump operator is defined as $\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}$. It's important to note that in this context, we assume that $\inf\varnothing := \sup\mathbb{T}$ and $\sup\varnothing := \inf\mathbb{T}$.

A key component of time scale theory is the graininess function, denoted as μ, which maps elements of the time scale to the interval $[0,\infty]$. It is defined as $\mu(t) = \sigma(t) - t$.

Additionally, a point $t \in \mathbb{T}$ is characterized as left-dense (LD), left-scattered (LS), right-dense (RD), or right-scattered (RS) based on conditions involving the operators ρ and σ. If $\mathbb{T}$ contains a left-scattered maximum M, we define $\mathbb{T}^k = \mathbb{T} \setminus M$; otherwise, $\mathbb{T}^k = \mathbb{T}$.

Moreover, a function $f : \mathbb{T} \to \mathbb{R}^n$ is considered Δ-differentiable at $t \in \mathbb{T}^k$ if the limit

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}$$

exists in $\mathbb{R}^n$.

Let us recall the following classical results (see [1,2]):

(a) If $t \in \mathbb{T}^k$ is a right-dense point of $\mathbb{T}$, then f is Δ-differentiable at t if and only if the limit

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}$$

exists in $\mathbb{R}^n$.

(b) If $t \in \mathbb{T}^k$ is a right-scattered point of $\mathbb{T}$, and if f is continuous at t, then f is Δ-differentiable at t, and

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}\,.$$

To delve into the properties of periodicity in dynamic equations on time scales, it is essential to establish a clear understanding of periodicity on these time scales.

We say that a time scale $\mathbb{T}$ is called a periodic time scale if

$$\Pi := \big\{\tau \in \mathbb{R}: \ \ t \pm \tau \in \mathbb{T}, \ \ \forall\, t \in \mathbb{T}\big\} \neq \{0\}.$$

The smallest positive $\tau \in \Pi$ is called the period of the time scale.

Definition 1.1 ([5])**.** Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period τ. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic if there exists a natural number n such that $P = n\tau$, $f(t+P) = f(t)$ for all $t \in \mathbb{T}$. The smallest positive number P is called the period of function f if $f(t+P) = f(t)$ for all $t \in \mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $P > 0$, if P is the smallest positive number such that $f(t+P) = f(t)$ for all $t \in \mathbb{T}$.

It's worth noting that if $\mathbb{T}$ is a periodic time scale with period τ, then the forward jump operator σ exhibits periodic behavior, where $\sigma(t+n\tau) = \sigma(t)+n\tau$. This periodicity extends to the graininess function, as

$$\mu(t+n\tau) = \sigma(t+n\tau) - (t+n\tau) = \sigma(t) - t = \mu(t).$$

In our subsequent study, we consider a set of periodic time scales denoted as $\mathbb{T}_\lambda$, where $\lambda \in \Lambda \subset \mathbb{R}$ and $\lambda = 0$ serves as a limit point of the set Λ. It is assumed that for any $\lambda \in \Lambda$, $\inf \mathbb{T}_\lambda = -\infty$, $\sup \mathbb{T}_\lambda = \infty$, and the point $t = 0$ is a part of $\mathbb{T}_\lambda$ for all $\lambda \in \Lambda$.

Let $\mathbb{T}_\lambda$ be a periodic time scale with period $\tau_\lambda = \frac{\omega}{n(\lambda)}$, where $n(\lambda)$ is a natural number. We set $\mu_\lambda := \sup\limits_{t\in\mathbb{T}_\lambda} \mu_\lambda(t)$, where $\mu_\lambda(t) : \mathbb{T}_\lambda \to [0,\infty)$ is the graininess function. If $\mu_\lambda \to 0$ as $\lambda \to 0$, then $\mathbb{T}_\lambda$ converges to the continuous time scale $\mathbb{T}_0 = \mathbb{R}$, and the dynamic equation system on the time scale transforms into the corresponding system of differential equations. Due to the periodicity of the graininess function $\mu\lambda(t)$, on each subset of the time scale $[t;t+\tau]_\lambda \subset \mathbb{T}_\lambda$, the following equality holds:

$$\sup_{t\in[t;t+\tau]_\lambda} \mu_\lambda(t) = \mu_\lambda.$$

Hence, it is naturaly to expect that, under certain conditions, the existence of a periodic solution in a differential equation implies the existence of a corresponding solution in the dynamic equation on the periodic time scale $\mathbb{T}_\lambda$, and vice versa.

2 Problem statement and auxiliary results

We consider the system of differential equations

$$\frac{dx}{dt} = X(t,x), \tag{2.1}$$

where $x \in D$, $D \subset \mathbb{R}^n$ is a domain in the space $\mathbb{R}^d$, and the corresponding system of equations defined on $\mathbb{T}_\lambda$

$$x_\lambda^\Delta = X(t,x_\lambda), \tag{2.2}$$

where $t \in \mathbb{T}_\lambda$, $\lambda \in \Lambda \subset \mathbb{R}$, $\lambda = 0$ is a limit point of the set Λ, $x_\lambda : \mathbb{T}_\lambda \to \mathbb{R}^n$, and $x_\lambda^\Delta(t)$ is the Δ-derivative of $x_\lambda(t)$ in $\mathbb{T}_\lambda$.

Assume that $X(t,x)$ is continuously differentiable and bounded with its partial derivatives, i.e. there exists $C > 0$ such that

$$|X(t,x)| + \Big|\frac{\partial X(t,x)}{\partial t}\Big| + \Big\|\frac{\partial X(t,x)}{\partial x}\Big\| \leq C$$

for $t \in \mathbb{R}$ and $x \in D$. Here $\frac{\partial X}{\partial x}$ is the corresponding Jacobian matrix, $|\cdot|$ is the Euclidian norm of a vector, and $\|\cdot\|$ is the norm of a matrix.

In addition, we also assume that the function $X(t,x)$ is periodic in t with a period ω, i.e.

$$X(t+\omega,x) = X(t,x), \;\; t \in \mathbb{R}, \;\; x \in D.$$

We need a lemma to address the evaluation of the discrepancy between the solutions of a Cauchy problem for a system of differential equations and the corresponding solutions of dynamic equations on time scales, given that they share the same initial conditions.

Lemma 2.1 ([4])**.** *Let $t_0 \in \mathbb{T}_\lambda$, $t_0 + T \in \mathbb{T}_\lambda$, x_λ and $x(t)$ are the solutions of* (2.2) *and* (2.1) *on* $[t_0, t_0+T]$ *and* $[t_0, t_0+T]_{\mathbb{T}_\lambda}$*, respectively. Then if the initial conditions $x(t_0) = x_\lambda(t_0) = x_0$, $x_0 \in D$ are satisfied, the following inequality holds*

$$|x(t) - x_\lambda(t)| \leq \mu(\lambda)K(T),$$

where

$$\mu(\lambda) = \sup_{t \in [t_0, t_0+T]_{\mathbb{T}_\lambda}} \mu_\lambda(t) \;\; \textit{for} \;\; t \in [t_0, t_0+T]_{\mathbb{T}_\lambda},$$

$$K(T) = e^{C(T+1)}\Big(C + \frac{C^2T}{4}\Big) + 3C \;\; \textit{is constant.}$$

Let us define the notion of asymptotic stability for solutions of dynamic equations on time scales, drawing parallels with the definition of asymptotic stability in the context of differential equations as outlined in [3].

Definition 2.1. A solution $x_\lambda(t)$ of system (2.2), defined on a family of time scales $\mathbb{T}_\lambda$, is called uniformly in t_0 and λ asymptotically stable if for any $\varepsilon > 0$ there exist $\delta > 0$ and $T > 0$, which do not depend on t_0 and λ, such that if $y_\lambda(t)$ is a solution of system (2.2) and

$$|x_\lambda(t_0) - y_\lambda(t_0)| < \delta,$$

then

$$|x_\lambda(t) - y_\lambda(t)| < \varepsilon, \;\; \text{if} \;\; t \geq t_0,$$
$$|x_\lambda(t) - y_\lambda(t)| \leq \frac{\delta}{2}, \;\; \text{if} \;\; t \in [t_0 + T, \infty)_{\mathbb{T}_\lambda}.$$

3 Main results

We were considering the sequence of periodic time scales $\mathbb{T}_\lambda$ with the smallest period $\tau(\lambda)$ such that $\tau(\lambda)$ approaches 0 as $\lambda \to 0$, and $\frac{\tau(\lambda)}{\omega}$ is rational. And we delineated prerequisites for the existence of a periodic solution to the system described by equation (2.1), provided that the system (2.2) already possesses a periodic solution.

Theorem 3.1. *Suppose there exists a positive value λ_0 such that for all λ less than λ_0, the system of differential equations* (2.2) *has a uniformly along $t_0 \in \mathbb{T}_\lambda$ and λ asymptotically stable periodic solution $x_\lambda(t)$, which belongs to the domain D along with ρ-neighborhood. Then the dynamic system* (2.1) *also has a periodic solution with period $p = r\omega$, where r is an integer.*

Proof. Since $x_\lambda(t)$ is asymptotically stable, then for any $\varepsilon > 0$ ($\varepsilon < \frac{\rho}{2}$) there exist $\delta > 0$ ($\delta < \varepsilon$) and $\widetilde{T} > 0$, which are independent of t_0 and λ, such that if

$$|x_\lambda(t_0) - y_\lambda(t_0)| \le \delta,$$

then

$$|x_\lambda(t) - y_\lambda(t)| < \varepsilon, \ \text{ if } \ t \ge 0, \tag{3.1}$$

$$|x_\lambda(t) - y_\lambda(t)| \le \frac{\delta}{2}\,, \ \text{ if } \ t \in [\widetilde{T}, \infty)_\lambda. \tag{3.2}$$

Without loss of generality, it can be assumed that $t_0(\lambda) = 0$. Let T be the smallest point right from $\widetilde{T}$ such that $T = r_0\omega$, r_0 is an integer.

Let us choose λ_0 such that for any $\lambda < \lambda_0$ and for the defined $\delta > 0$ and T the following conditions hold:

(1) the corresponding time scale $\mathbb{T}_\lambda$ with the graininess function μ_λ has the period $\tau_\lambda = \frac{\omega}{m_0}$, m_0 is an integer;

(2) if $y_\lambda(t)$ is a solution of the dynamic system (2.2) on time scale $\mathbb{T}_\lambda$ and $\varphi(t)$ is a solution of the differential system (2.1) such that $\varphi(t_k) = y_\lambda(t_k)$, $t_k \in \mathbb{T}_\lambda$, then the following inequality holds:
$$|\varphi(t) - y_\lambda(t)| < \frac{\delta}{2}\,, \ \ t \in [t_k, t_{k+1}]_\lambda, \tag{3.3}$$
where t_{k+1} is the smallest point in the interval $[t_k + T, t_k + T + 1]_{\mathbb{T}_\lambda}$ such that $t_{k+1} = i_{k+1}\tau_\lambda$, with $i_{k+1} \in \mathbb{N}$. As $\lambda \to 0$ both μ_λ and τ_λ tend to zero, which ensures the existence of such a point for sufficiently small graininess function.

Since we can choose λ_0 such that for any $\lambda < \lambda_0$ it holds $\mu_\lambda K(T+1) \le \delta/2$, then, by Lemma 2.1, the inequality (3.3) holds.

For the corresponding μ_λ, according to the conditions of Theorem 3.1 and Definition 1.1, the system (2.2) has a periodic asymptotically stable solution $x_\lambda(t)$ with a period $P_\lambda = n_0\tau_\lambda$.

We consider the δ-neighborhood of the point $x_\lambda(0)$. Let y_0 be any point in this neighborhood. Then
$$|x_\lambda(0) - y_0| \le \delta.$$

Let $\varphi(t, y_0)$ be a solution of the system (2.1), and let $y_\lambda(t)$ be a solution of the system (2.2), both satisfying the initial condition $\varphi(0, y_0) = y_\lambda(0) = y_0$ at the point $t_0(\lambda) = 0$.

Let's consider the interval $[0, T]_\lambda$. Since $T = r_0\omega$, $\omega = m_0\tau_\lambda$ and $i_1 := r_0m_0$, then $T = r_0m_0\tau_\lambda = 0 + i_1\tau_\lambda = t_1 \in \mathbb{T}_\lambda$. So, from the inequalities (3.1) and (3.2) it follows that

$$|y_\lambda(T) - x_\lambda(T)| \le \frac{\delta}{2}\,.$$

Consequently, considering (3.2), (3.3), we obtain

$$|x_\lambda(T) - \varphi(T, y_0)| \le |x_\lambda(T) - y_\lambda(T)| + |y_\lambda(T) - \varphi(T, y_0)| < \delta.$$

Thus, the solution $\varphi(t)$ of the system (2.1), which starts in the δ-neighborhood of $x_\lambda(0)$, does not leave the 2ε-neighborhood of the solution $x_\lambda(t)$ of the system (2.2) on the interval $[0, T]_\lambda$ of time scale $\mathbb{T}_\lambda$, returns to the δ-neighborhood of $x_\lambda(t)$ at time T, provided that the solution $\varphi(t)$ is defined on the interval $[0, T]$.

Let $\widehat{y}_\lambda(t)$ be a solution of the system (2.2) such that its initial conditions coincide with the initial conditions of the solution $\varphi(t)$ at time T:

$$\varphi(T) = \widehat{y}_\lambda(T).$$

Let's consider the interval $[T, 2T]_{\mathbb{T}_\lambda}$. If $i_2 := 2r_0 m_0$, then we get

$$2T = 2r_0 m_0 \tau_\lambda = 0 + i_2 \tau_\lambda = t_2 \in \mathbb{T}_\lambda.$$

So,

$$|\widehat{y}_\lambda(2T) - x_\lambda(2T)| \leq \frac{\delta}{2}$$

and we have

$$|x_\lambda(2T) - \varphi(2T)| < \delta.$$

Continuing this process, we obtain on each interval $[(k-1)T, kT]$

$$|x_\lambda(kT) - \varphi(kT)| < \delta.$$

Recall that the time scale T_λ has a period $\tau_\lambda = \frac{\omega}{m_0}$, and, according to the definition 1.1, the solution x_λ has a period $P_\lambda = n_0 \tau_\lambda$.

Then, at the point $t_{k_M} = M\tau_\lambda := r\omega$ from the set of points $\{t_k = kT\}$, r is divisible by n_0, we have:

$$|x_\lambda(t_{k_M}) - \varphi(r\omega)| < \delta,$$

where M is a common multiple of m_0, r_0 and n_0.

Because $x_\lambda(t_{k_M}) = x_\lambda(0)$, $\pi : y_0 \to \varphi(r\omega, y_0)$ maps the ball of radius δ onto itself. Thus, there exists a fixed point y_1 of the mapping π such that

$$\varphi(r\omega, y_1) = y_1.$$

This implies that the solution of the system (2.1) with the initial condition $\varphi(0) = y_1$ is periodic with a period $r\omega$, which completes the proof. $\square$

The next theorem establishes the existence of a periodic solution of the system (2.2) on $\mathbb{T}_\lambda$, if the system (2.1) has the corresponding periodic solution.

Theorem 3.2. *Suppose the system of dynamic equations* (2.1) *has an asymptotically stable periodic solution $x(t)$ with a period ω, which belongs to the domain D with ρ-neighborhood. Then there exist $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$ the differential system* (2.2) *has at least one periodic solution with a period $r\omega$ on $\mathbb{T}_\lambda$, where r is an integer.*

Acknowledgement

The work of Oleksandr Stanzhytskyi was partially supported by the National Research Foundation of Ukraine # F81/41743 and by the Ukrainian Government Scientific Research Grant # 210BF38-01.

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Existence and Asymptotic Behavior of Nonoscillatory Solutions of Quasilinear Differential Equations with Variable Exponents

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1 Introduction

Recently, there has been an increasing interest in studying differential equation with variable exponents, that is, differential equations with $\alpha(t)$-Laplacian (Generally referred to as $p(t)$-Laplacian and see below for the details) of the form

$$\big(p(t)\varphi_{\alpha(t)}(x')\big)' + q(t)\varphi_{\beta(t)}(x) = 0, \ \ t \geqq a, \tag{A}$$

where $\alpha(t)$, $\beta(t)$, $p(t)$ and $q(t)$ are positive continuous functions on $[a,\infty)$, $a \geqq 1$ and use is made of the notation

$$\varphi_{\gamma(t)}(\xi) = |\xi|^{\gamma(t)-1}\xi = |\xi|^{\gamma(t)} \operatorname{sgn}\xi, \ \ \xi \in \mathbb{R}, \ \ \gamma \in C[1,\infty).$$

By a solution of (A) we mean a function $x \in C^1[T_x,\infty)$, $T_x \geqq a$, which has the property $p(t)\varphi_{\alpha(t)}(x') \in C^1[T_x,\infty)$ and satisfies the equation at all points $t \geqq T_x$. A nontrivial solution $x(t)$ of (A) is said to be nonoscillatory if $x(t) \neq 0$ for all large t and oscillatory otherwise. In this talk we restrict our attention to its eventually positive solutions.

The first interest in $\alpha(\,\cdot\,)$-type Laplacian (i.e., $p(\,\cdot\,)$-type Laplacian) was in function spaces called variable exponent spaces. Variable exponent spaces appeared in the literature for the first time in a 1931 paper by Orlicz ([9]). In 2000 and 2011, Růžička and Diening et al. studied equation with non-standard $p(x)$-growth in the modeling of the so-called electrorheological fluids ([10]) and the Lebesgue and Sobolev spaces with variable exponetns([2]), respectively. The mathematically and physically importance of $p(\,\cdot\,)$-type Laplacian was recognized by the above-mentioned Růžička's monograph (see [8]).

In recent years, there has been well analyzed the oscillatory and nonoscillatory behavior of the equation with $p(t)$-Laplacian

$$\big(a(t)\varphi_{p(t)-1}(x')\big)' \pm b(t)\varphi_{q(t)-1}(x) = 0, \ \ t \geqq a, \ \ (p(t) = q(t) \ \text{ or } \ p(t) \neq q(t)),$$

which is of the same type as (A) but written in a different representation of $p(t) = a(t)$, $q(t) = b(t)$, $\alpha(t) = p(t) - 1$ $(p(t) > 1)$ and $\beta(t) = q(t) - 1$ $(q(t) > 1)$ in equation (A) (see [1, 3–7]).

To the best of the author's knowledge, detail is unknown about nonoscillatory behavior of (A), and so in this talk we make an attempt to investigate in detail the existence and asymptotic behavior of eventually positive solutions of (A).

2 Existence of positive solutions

In this talk we make the following assumptions without further mentioning:

$$\int\limits_a^\infty \Big[\frac{k}{p(t)}\Big]^{\frac{1}{\alpha(t)}}\,dt = \infty \tag{2.1}$$

for every constant $k > 0$, and employ the notation

$$P_{\alpha(t),k}(t) = \int\limits_T^t \Big[\frac{k}{p(s)}\Big]^{\frac{1}{\alpha(s)}}\,ds, \ \ t \geqq T \geqq a. \tag{2.2}$$

From (2.1) and (2.2) it is obvious that

$$P_{\alpha(T),k}(T) = 0, \ \ \lim_{t\to\infty} P_{\alpha(t),k}(t) = \infty \ \text{ for every } \ k > 0,$$
$$P_{\alpha(t),k}(t) > P_{\alpha(t),l}(t), \ \ t > T \ \text{ for } \ k > l > 0 \ \text{ and } \ \lim_{k\to 0} P_{\alpha(t),k}(t) = 0 \ \text{ for each } \ t \geqq T.$$

First of all, we begin by classifying all possible positive solutions of equation (A) according to their asymptotic behavior as $t \to \infty$.

Lemma 2.1. *One and only one of the following cases occurs for each positive solution $x(t)$ of* (A)*:*

I. $\lim\limits_{t\to\infty} p(t)\varphi_{\alpha(t)}(x'(t)) = const. > 0$, $\lim\limits_{t\to\infty} x(t) = \infty$;

II. $\lim\limits_{t\to\infty} p(t)\varphi_{\alpha(t)}(x'(t)) = 0$, $\lim\limits_{t\to\infty} x(t) = \infty$;

III. $\lim\limits_{t\to\infty} p(t)\varphi_{\alpha(t)}(x'(t)) = 0$, $\lim\limits_{t\to\infty} x(t) = const > 0$.

We want to obtain criteria for the existence of positive solutions of (A) of type I, II and III.

Theorem 2.1. *Suppose that for each fixed $k > 0$ and $T \geqq a$,*

$$\lim_{l\to 0} \frac{P_{\alpha(t),l}(t)}{P_{\alpha(t),k}(t)} = 0 \tag{2.3}$$

uniformly on any interval of the form $[T_1, \infty)$, $T_1 > T$. Then equation (A) *possesses a positive solution of type* I *if and only if*

$$\int\limits_a^\infty q(t)(P_{\alpha(t),k}(t))^{\beta(t)}\,dt < \infty \ \text{ for some constant } \ k > 0. \tag{2.4}$$

Theorem 2.2. *Equation* (A) *possesses a positive solution of type* III *if and only if*

$$\int\limits_a^\infty \bigg[\frac{1}{p(t)} \int\limits_t^\infty q(s)c^{\beta(s)}\,ds\bigg]^{\frac{1}{\alpha(t)}} dt < \infty \ \text{ for some constant } \ c > 0. \tag{2.5}$$

Unlike the solution of the types I and III it is difficult to characterize the type II solution of (A), and so we content ourselves with sufficient conditions for the existence of such solutions of (A).

Theorem 2.3. *Suppose that* (2.3) *holds. Equation* (A) *possesses a positive solution of type* II *if* (2.4) *holds for some constant $k > 0$ and*

$$\int\limits_a^\infty \bigg[\frac{1}{p(t)} \int\limits_t^\infty q(s)d^{\beta(s)}\,ds\bigg]^{\frac{1}{\alpha(t)}} dt = \infty \tag{2.6}$$

for every constant $d > 0$.

3 Examples

We now present some examples illustrating Theorem 2.1 obtained in Section 2.

Example 3.1. Consider the equations with variable exponents of nonlinearity

$$\big(e^{-(t^2-1)}\varphi_t(x')\big)' + q_1(t)\varphi_t(x) = 0, \ \ t \geqq e, \tag{E$_1$}$$
$$\big(e^{-(t^2-1)}\varphi_t(x')\big)' + q_2(t)\varphi_{\frac{1}{t}}(x) = 0, \ \ t \geqq e, \tag{E$_2$}$$

and

$$\big(e^{-(1-\frac{1}{t^2})}\varphi_{\frac{1}{t}}(x')\big)' + q_3(t)\varphi_{\frac{1}{t}}(x) = 0, \ \ t \geqq e, \tag{E$_3$}$$

where the functions $q_i(t)$, $i = 1, 2, 3$, are

$$q_1(t) = e^{-(t^2-1)}\Big(1+\frac{1}{t^2}\Big)^t\Big\{\frac{2}{t^2+1} - \log\Big(1+\frac{1}{t^2}\Big)\Big\},$$
$$q_2(t) = e^{-(1-\frac{1}{t^2})}\Big(1+\frac{1}{t^2}\Big)^t\Big\{\frac{2}{t^2+1} - \log\Big(1+\frac{1}{t^2}\Big)\Big\},$$

and

$$q_3(t) = e^{-(1-\frac{1}{t^2})}\frac{1}{t^2}\Big(1+\frac{1}{t^2}\Big)^{\frac{1}{t}}\Big\{\frac{2}{t^2+1} + \log\Big(1+\frac{1}{t^2}\Big)\Big\},$$

respectively. They are special cases of (A) with $\alpha(t) = t$ in (E$_i$), $i = 1, 2$, $\alpha(t) = 1/t$ in (E$_3$), $\beta(t) = t$ in (E$_1$), $\beta(t) = 1/t$ in (E$_i$), $i = 2, 3$, $p(t) = e^{-(t^2-1)}$ in (E$_i$), $i = 1, 2$, $p(t) = e^{-(1-\frac{1}{t^2})}$ in (E$_3$) and $q(t) = q_i(t)$, $i = 1, 2, 3$ in the above. The functions $p(t) = e^{-(t^2-1)}$ and $p(t) = e^{-(1-\frac{1}{t^2})}$ satisfy (2.1) with $k = 1$ and, in addition, the function $P_{\alpha(t),1}(t)$ associated with (E$_i$), $i = 1, 2, 3$ is

$$P_{\alpha(t),1}(t) = \int_e^t \Big[\frac{1}{p(s)}\Big]^{\frac{1}{\alpha(s)}} ds = \int_e^t e^{s-\frac{1}{s}}\, ds \ \sim \ e^{t-\frac{1}{t}} \ \text{ as } \ t \to \infty$$

by (2.2), where the symbol $\sim$ is used to denote the asymptotic equivalence

$$f(t) \sim g(t) \ \text{ as } \ t \to \infty \iff \lim_{t\to\infty}\frac{f(t)}{g(t)} = 1.$$

Since

$$\begin{aligned}
\int_e^\infty q_1(t)(P_{t,1}(t))^t\, dt &= \int_e^\infty e^{-(t^2-1)}\Big(1+\frac{1}{t^2}\Big)^t\Big\{\frac{2}{t^2+1} - \log\Big(1+\frac{1}{t^2}\Big)\Big\}(e^{t-\frac{1}{t}})^t\, dt \\
&= \int_e^\infty \Big(1+\frac{1}{t^2}\Big)^t\Big\{\frac{2}{t^2+1} - \log\Big(1+\frac{1}{t^2}\Big)\Big\}\, dt < \infty, \\
\int_e^\infty q_2(t)(P_{t,1}(t))^{\frac{1}{t}}\, dt &= \int_e^\infty \Big(1+\frac{1}{t^2}\Big)^t\Big\{\frac{2}{t^2+1} - \log\Big(1+\frac{1}{t^2}\Big)\Big\}\, dt < \infty,
\end{aligned}$$

and

$$\int_e^\infty q_3(t)(P_{\frac{1}{t},1}(t))^{\frac{1}{t}}\, dt = \int_e^\infty \frac{1}{t^2}\Big(1+\frac{1}{t^2}\Big)^{\frac{1}{t}}\Big\{\frac{2}{t^2+1} + \log\Big(1+\frac{1}{t^2}\Big)\Big\}\, dt < \infty,$$

we can apply Theorem 2.1 to conclude that there exists a positive solution of type I such that $x(t) = e^{t-\frac{1}{t}}$, which satisfies

$$\lim_{t\to\infty} p(t)\varphi_t(x'(t)) = \lim_{t\to\infty} \Big(1 + \frac{1}{t^2}\Big)^t = 1, \quad \lim_{t\to\infty} x(t) = \infty$$

for (E_i), $i = 1, 2$ and that for (E_3)

$$\lim_{t\to\infty} p(t)\varphi_{\frac{1}{t}}(x'(t)) = \lim_{t\to\infty} \Big(1 + \frac{1}{t^2}\Big)^{\frac{1}{t}} = 1, \quad \lim_{t\to\infty} x(t) = \infty.$$

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Some Properties of Topological Entropy of Families of Dynamical Systems on the Cantor Set

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Following [1], we give the definition of topological entropy that will be necessary hereafter. Let X be a compact metric space with a metric d and $f : X \to X$ a continuous mapping. Along with the original metric d, we define an additional system of metrics on X:

$$d_n^f(x, y) = \max_{0 \le i \le n-1} d\big(f^i(x), f^i(y)\big), \quad x, y \in X, \quad n \in \mathbb{N},$$

where f^i, $i \in \mathbb{N}$, is the i-th iteration of f, $f^0 \equiv \mathrm{id}_X$. For any $n \in \mathbb{N}$ and $\varepsilon > 0$, denote by $N_d(f, \varepsilon, n)$ the maximum number of points in X, pairwise d_n^f-distances between which are greater than ε. Then *the topological entropy* of the mapping f is defined by the formula

$$h_d(f, x) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \ln N_d(f, \varepsilon, n).$$

Let $C(X, X)$ denote the set of continuous mappings from X to X with the metric

$$\rho(f, g) = \max_{x \in X} d(f(x), g(x)).$$

Consider the function

$$f \longmapsto h_{\mathrm{top}}(f). \tag{1}$$

It was proved in [2] that function (1) belongs to the second Baire class on the space $C(X, X)$, and the set of points in the space $C(X, X)$ at which function (1) is lower semicontinuous contains an everywhere dense G_δ set. It was established in [3] that the set of points of lower semicontinuity itself is an everywhere dense G_δ set in $C(X, X)$.

If X coincides with the Cantor set $\mathcal{K}$ on the interval $[0, 1]$ with the metric induced by the natural metric of the real line, then function (1) is everywhere discontinuous and is lower semicontinuous only at the points where the topological entropy is equal to zero [3]. It was demonstrated in [4] that function (1) does not belong to the first Baire class even on the subspace of homeomorphisms satisfying the Lipschitz condition.

Let us denote by $E_h(f)$ the set of *limiting realizable* values of topological entropy, i.e. those that are obtained for arbitrarily small uniform perturbations of the mapping f:

$$E_h(f) = \bigcap_{n \in \mathbb{N}} \big\{h_{\mathrm{top}}(g) : \ \rho(f, g) < n^{-1}\big\}.$$

Theorem 1 ([5])**.** *For each continuous mapping $f : \mathcal{K} \to \mathcal{K}$, the equality $E_h(f) = [0; +\infty]$ holds.*

Given a metric space $\mathcal{M}$ and a continuous mapping $f : \mathcal{M} \to C(X, X)$ let us construct a function

$$\mu \longmapsto h_{\mathrm{top}}(f(\mu, \cdot\,). \tag{2}$$

From [2] and [3] it follows that the set of points in the space $\mathcal{M}$ at which function (2) is lower semicontinuous is an everywhere dense G_δ set. In the case $\mathcal{M} = X = \mathcal{K}$ for any everywhere dense G_δ set $A \subset \mathcal{M}$, there is a continuous mapping $f : \mathcal{M} \to C(X, X)$ such that $h_{\mathrm{top}}(f(A, \cdot\,) = 0$ and $h_{\mathrm{top}}(f(\mathcal{M} \setminus A, \cdot\,) = +\infty$ [5]. In particular, the set of points of lower semicontinuity of function (2) coincides with the set A. It turns out that using the method of [5] one can prove the following

Theorem 2. *If $\mathcal{M} = X = \mathcal{K}$, then for any number $h > 0$ and an everywhere dense G_δ set $A \subset \mathcal{M}$, there is a continuous mapping $f : \mathcal{M} \to C(X, X)$ such that the equalities $h_{\mathrm{top}}(f(A, \cdot\,)) = 0$ and $h_{\mathrm{top}}(f(\mathcal{M} \setminus A, \cdot\,)) = h$ are satisfied.*

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General Theory of the Higher-Order Linear Quaternion q-Difference Equations

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Abstract

The general theory of quaternion q-difference equations is essentially different from the traditional q-difference equations in the complex space for the non-commutative algebraic structure of the quaternionic algebra. In this talk, we will present some basic results such as the Wronskian, Liouville formula and the general solution structure theorems of the higher-order linear quaternion q-difference equations (short for QQDCEs) with constant and variable coefficients by applying the quaternion characteristic polynomial and the quaternion determinant algorithm.

1 Introduction

In 1843, Hamilton initiated the quaternion space $\mathbb{H}$ to extend and develop the complex field $\mathbb{C}$ and applied it to mechanics in three-dimensional space. With the large development of the quaternion algebra, it demonstrates a great superiority over the real-valued vectors and has been widely applied to depicting some complex phenomena in physics, space geometric analysis, especially in the aspects of flight dynamics, molecular dynamics and three-dimensional rotations, etc. (see [1, 2]).

The unified form called dynamic equations on time scales were introduced to combine these both continuous and discrete forms and the common features of the continuous and discrete dynamic equations have been extensively studied (see [5]). Recently, the quaternion differential and difference equations have been widely studied in both theoretical aspects and application area, the quaternionic dynamics described by these equations perfectly present the dynamical behavior of the status of the objects comparing with the complex equations since the various shift transforms such as the rotations in the quaternion space can be easily expressed and accurately calculated. In 2020–2021, the authors established some basic results of quaternion dynamic equations on time scales, and some real applications were provided (see [3, 4]).

In [6], Wang, Chen and Li established the general theory of the higher-order quaternion linear difference equations via the complex adjoint matrix and the quaternion characteristic polynomial and it is largely different from the general theory of the traditional difference equations since the non-commutativity under the quaternion multiplication (i.e., $ab \neq ba$ for $a, b \in \mathbb{H}$). In [7], the general theory of the higher-order linear quaternion q-difference equations was established.

2 Preliminaries

We assume that $0 < q < 1$, $\overline{q^{\mathbb{N}}} := \{q^n : \ n \in \mathbb{N}\} \cup \{0\}$. For convenience, we introduce some notations. The symbol $\mathbb{N}$ denotes the set of non-negative integers, $\mathbb{C}$ the complex numbers, $M_m(\mathbb{H})$ the $m \times m$-order quaternionic matrices and $M_m(\mathbb{C})$ the $m \times m$-order complex matrices.

Next, some basic knowledge of the quaternion algebra is necessary. Let $\widetilde{q}, \widetilde{q}\,' \in \mathbb{H}$, if there exists $w \in \mathbb{H} \setminus \{0\}$ such that $\widetilde{q}\,' = w\widetilde{q}w^{-1}$, then we say that $\widetilde{q}$ is equivalent to $\widetilde{q}\,'$, for convenience, we denote it by $\widetilde{q}\,' \sim \widetilde{q}$. Let also $\widetilde{q} = \widetilde{q}_0 + \widetilde{q}_1\,\mathbf{i} + \widetilde{q}_2\,\mathbf{j} + \widetilde{q}_3\,\mathbf{k} \in \mathbb{H}$, we define $\chi : \mathbb{H} \to \mathbb{R}$ by $\chi(\widetilde{q}) = \widetilde{q}_0$ and define the set $[\widetilde{q}] = \{\widetilde{q}\,' \in \mathbb{H} | \widetilde{q}\,' \sim \widetilde{q}\}$, then the following results hold:

(i) $|\widetilde{q}\,'| = |w\widetilde{q}w^{-1}| = |\widetilde{q}|$;

(ii) if $\widetilde{q}\,' \sim \widetilde{q}$, then $\chi(\widetilde{q}\,') = \chi(\widetilde{q})$;

(iii) $[\widetilde{q}] \subset \big\{\widetilde{q}\,' \in \mathbb{H} : \ \chi(\widetilde{q}\,') = \chi(\widetilde{q}), |\widetilde{q}\,'| = |\widetilde{q}|\big\}$;

(iv) if $\widetilde{q} \sim \widetilde{q}_0 + \mathbf{i}\sqrt{\widetilde{q}_1^{\,2} + \widetilde{q}_2^{\,2} + \widetilde{q}_3^{\,2}}$, then

$$[\widetilde{q}\,] = \Big\{\widetilde{q}_0 + x_1\,\mathbf{i} + x_2\,\mathbf{j} + x_3\,\mathbf{k} : \ x_1^2 + x_2^2 + x_3^2 = \widetilde{q}_1^{\,2} + \widetilde{q}_2^{\,2} + \widetilde{q}_3^{\,2}\Big\}.$$

For any $a \in \mathbb{C}$, we introduce the q-shifted factorial by

$$\begin{cases} (a,q)_n = 1, & n = 0; \\ (a,q)_n = \displaystyle\prod_{l=0}^{n-1} (1 - aq^l), & n \in \mathbb{N}; \\ (a,q)_\infty = \displaystyle\prod_{l=0}^{\infty} (1 - aq^l). \end{cases}$$

Moreover, we obtain

$$(a,q)_\infty = \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n-1)}{2}} \, \frac{a^n}{(q;q)_n}$$

if $\lim\limits_{n\to\infty} (a;q)_n$ exists.

Now, we will introduce the definition of the q-difference operator.

Definition 2.1 (see [7])**.** Let $f : \overline{q^{\mathbb{N}}} \to \mathbb{H}$, the q-difference operator is defined by

$$D_q f(t) = \begin{cases} \dfrac{f(qt) - f(t)}{qt - t}\,, & t \neq 0, \\ \lim\limits_{n\to\infty} \dfrac{f(q^n) - f(0)}{q^n}\,, & t = 0. \end{cases}$$

The concept of integrable quaternion-valued functions on $\overline{q^{\mathbb{N}}}$ can be introduced naturally as follows.

Definition 2.2 (see [7])**.** Let $f : \overline{q^{\mathbb{N}}} \to \mathbb{H}$, $t, s \in \overline{q^{\mathbb{N}}}$, $f(t) = f^{(0)}(t) + f^{(1)}(t)\,\mathbf{i} + f^{(2)}(t)\,\mathbf{j} + f^{(3)}(t)\,\mathbf{k}$. If $f^{(l)}(t)$ is integrable for $l = 0, 1, 2, 3$, then we say that $f(t)$ is integrable, i.e.,

$$\int_s^t f(t)\, d_q t = \int_s^t f^{(0)}(t)\, d_q t + \int_s^t f^{(1)}(t)\, d_q t\, \mathbf{i} + \int_s^t f^{(2)}(t)\, d_q t\, \mathbf{j} + \int_s^t f^{(3)}(t)\, d_q t\, \mathbf{k}.$$

Now, let $b \in \mathbb{H}$, we define the quaternion q-exponential by

$$e_q^b := \sum_{l=0}^{\infty} \frac{b^l}{(q;q)_l}\,, \quad l \in \mathbb{N}.$$

Consider the following initial value problem of the quaternion q-difference equation:

$$D_q x(t) = bx(t), \quad x(t_0) = x_0, \tag{2.1}$$

where $b, x_0 \in \mathbb{H}$.

Lemma 2.1 (see [7])**.** *The solution of* (2.1) *with the q-exponential form can be given as*

$$x(t) = e_q^{bt(1-q)} x_0 = \sum_{l=0}^{\infty} \frac{(bt(1-q))^l}{(q;q)_l}\, x_0.$$

Based on Lemma 2.1, one can obtain the following functions immediately.

Definition 2.3 (see [7])**.** Let $b \in \mathbb{H}$, $t \in \overline{q^{\mathbb{N}}}$, we define the sine and cosine functions by

$$\begin{cases} \sin_q(bt) := \dfrac{1}{2\,\mathbf{i}}\left(e_q^{\mathbf{i}bt(1-q)} - e_q^{-\mathbf{i}bt(1-q)}\right) = \dfrac{1}{\mathbf{i}}\displaystyle\sum_{l=0}^{\infty}\frac{(\mathbf{i}bt(1-q))^{2l+1}}{(q;q)_{2l+1}}\,, \\ \cos_q(bt) := \dfrac{1}{2}\left(e_q^{\mathbf{i}bt(1-q)} + e_q^{-\mathbf{i}bt(1-q)}\right) = \displaystyle\sum_{l=0}^{\infty}\frac{(\mathbf{i}bt(1-q))^{2l}}{(q;q)_{2l}}\,. \end{cases}$$

3 Existence and uniqueness of the solution for the higher-order linear quaternion q-difference equations

Consider the higher-order linear quaternion q-difference equations as follows

$$\begin{cases} a_m(t)D_q^m x(t) + a_{m-1}(t)D_q^{m-1}x(t) + \cdots + a_1(t)D_q x(t) + a_0(t)x(t) = B(t), \quad t \in \overline{q^{\mathbb{N}}}, \\ D_q^l x(t_0) = v_l, \end{cases} \tag{3.1}$$

where $a_m, \ldots, a_1, a_0, B : \overline{q^{\mathbb{N}}} \to \mathbb{H}$, $v_l \in \mathbb{H}$, $0 \le l \le m-1$. Let $b_l(t) = a_m^{-1}(t)a_l(t)$, then (3.1) is equivalent to the following q-difference equation

$$\begin{cases} D_q^m x(t) + b_{m-1}(t)D_q^{m-1}x(t) + \cdots + b_1(t)D_q x(t) + b_0(t)x(t) = \widetilde{B}(t), \quad t \in \overline{q^{\mathbb{N}}}, \\ D_q^l x(t_0) = v_l, \end{cases} \tag{3.2}$$

where

$$\widetilde{B}(t) = a_m^{-1}(t)B(t).$$

Below, though applying the transforms $x_0(t) = x(t)$, $x_1(t) = D_q x(t), \ldots, x_{m-1}(t) = D_q^{m-1}x(t)$, one has that (3.2) is equivalent to

$$\begin{aligned} D_q x_l(t) &= f_l\big(t, x_0(t), x_1(t), \ldots, x_{m-1}(t)\big) \\ &= \begin{cases} x_{l+1}(t), & l = 0, 1, \ldots, m-2, \\ f(t, x_0(t), x_1(t), \ldots, x_{m-1}(t)), & l = m-1, \end{cases} \end{aligned} \tag{3.3}$$

where

$$f\big(t, x_0(t), x_1(t), \ldots, x_{m-1}(t)\big) = \sum_{l=0}^{m-1}\big(-b_l(t)D_q^l x(t)\big) + \widetilde{B}(t), \quad x_l(t_0) = v_l, \;\; 0 \le l \le m-1.$$

Definition 3.1 (see [7])**.** Let $t \in \overline{q^{\mathbb{N}}}$, we define the set $D_l(\mathbb{H})$ as

$$D_l(\mathbb{H}) := \big\{x_l(\,\cdot\,) \in \mathbb{H} : \ |x_l(t) - v_l| < \alpha\big\},$$

where $\{x_l(t)\}_{l=0}^{m-1}$ is continuous at $t = t_0$ and bounded at $t \neq t_0$, i.e., $\lim\limits_{t \to t_0} x_l(t) = x_l(t_0) = v_l \in \mathbb{H}$ and there exists $M > 0$ such that $|x(t)| \leq M$ for all $t \in \overline{q^{\mathbb{N}}}$, $\alpha > 0$, $l = 0, 1, \dots, m-1$, $|p| = \sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2}$ for $p = p_0 + p_1\,\mathbf{i} + p_2\,\mathbf{j} + p_3\,\mathbf{k} \in \mathbb{H}$.

Next, we will establish the solutions of (3.3) with the initial value $x_l(t_0) = v_l$.

Theorem 3.1 (see [7])**.** *Let $t \in \overline{q^{\mathbb{N}}}$, $f_l : \overline{q^{\mathbb{N}}} \times D_0(\mathbb{H}) \times D_1(\mathbb{H}) \times \cdots \times D_{m-1}(\mathbb{H}) \to \mathbb{H}$, $0 \leq l \leq m-1$. If $f_l(t, x_0(t), x_1(t), \dots, x_{m-1}(t))$ satisfies the following conditions:*

(i) *$f_l(t, x_0(t), x_1(t), \dots, x_{m-1}(t))$ is continuous at $t = t_0$ and boundedat $t \neq t_0$.*

(ii) *$f_l(t, x_0(t), x_1(t), \dots, x_{m-1}(t))$ satisfies the Lipschitz condition, i.e., there exists a constant $K > 0$ such that*

$$\Big|f_l\big(t, x_0(t), x_1(t), \dots, x_{m-1}(t)\big) - f_l\big(t, \widetilde{x}_0(t), \widetilde{x}_1(t), \dots, \widetilde{x}_{m-1}(t)\big)\Big| \leq K \sum_{l=0}^{m-1} \big(|x_l(t) - \widetilde{x}_l(t)|\big),$$

where $x_l(\,\cdot\,), \widetilde{x}_l(\,\cdot\,) \in D_l(\mathbb{H})$.

Then (3.3) with the initial value $x_l(t_0) = v_l$ has an unique solution on $[t_0 - h, t_0 + h] \cap \overline{q^{\mathbb{N}}}$, where $h = \min\{\frac{1}{Km(1-q)}, \frac{\alpha}{B}\}$,

$$B := \max_{0 \leq l \leq m-1} \sup_{|x_l(t) - v_l| < \alpha} \Big|f_l\big(t, x_0(t), x_1(t), \dots, x_{m-1}(t)\big)\Big|, \quad \alpha > 0.$$

Theorem 3.2 (see [7])**.** *Let $t_0 \in \overline{q^{\mathbb{N}}}$, $h > 1 - q$, where $h = \min\{\frac{1}{Km(1-q)}, \frac{\alpha}{B}\}$. If (3.3) has an initial value $x_l(t_0) = v_l$, then (3.3) has an unique solution on $\overline{q^{\mathbb{N}}}$.*

4 Solving higher-order linear quaternion q-difference equations

In this section, we shall consider the following initial value problem:

$$D_q W(t) = A(t) W(t), \quad W(0) = W_0, \tag{4.1}$$

where $W : \overline{q^{\mathbb{N}}} \to \mathbb{H}^d$, $W_0 \in \mathbb{H}^d$, $d \in \mathbb{N}$, $\mathbb{H}^d$ is the d-dimensional quaternion space, $A(t) = (a_{uv}(t))_{u,v=1}^d \in \mathbb{H}^d$, $a_{uv}(t)$ is continuous at $t = 0$ and bounded at $t \neq 0$.

Theorem 4.1 (see [7])**.** *If $I - (1-q) q^l t A(q^l t)$ is invertible for $l \in \mathbb{N}$. Then the solution of (4.1) can be represented by*

$$W(t) = \prod_{l=0}^{\infty} \big[I - (1-q) q^l t A(q^l t)\big]^{-1} W_0,$$

where $t \in \overline{q^{\mathbb{N}}}$, $A(t)$ is a $d \times d$ quaternion matrix and I is an identity matrix.

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Stability and Exponential Stability Indices of a Linear System Depending on a Parameter

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1 Introduction

For a given integer $n \geq 2$ let $\mathcal{M}_n$ denote the class of linear differential systems

$$\dot{x} = A(t)x, \;\; x \in \mathbb{R}^n, \;\; t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{1.1}$$

with continuous bounded coefficients defined on $\mathbb{R}_+$.

Let us identify the system (1.1) with the matrix-valued function $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and use the following notation: $A \in \mathcal{M}_n$.

Recall that the *characteristic exponent* (or the *Lyapunov exponent*) of a function $x(\,\cdot\,)$ is [6, p. 552], [1, p. 25]

$$\chi(x) = \varlimsup_{t\to+\infty} \ln \|x(t)\|^{1/t}.$$

We denote by $\mathrm{s}(A)$ the stability index of a system $A \in \mathcal{M}_n$, i.e. the dimension of the linear subspace of bounded solutions to this system, and by $\mathrm{es}(A)$ its exponential stability index, that is the dimension of the linear subspace of solutions to this system having negative characteristic exponents. One can see that the following inequality holds:

$$\mathrm{s}(A) \geq \mathrm{es}(A).$$

O. Perron in his paper [7] constructed an example of a system $A \in \mathcal{M}_2$ and its continuous exponentially decaying (2×2)-dimensioned perturbation $Q(\,\cdot\,)$ such that the initial system has its exponential stability index equal to 2, and the perturbed system

$$\dot{x} = (A(t) + Q(t))x, \;\; x \in \mathbb{R}^2, \;\; t \in \mathbb{R}_+,$$

has both its stability indices equal to 1.

There is another interesting example presented by O. Perron [8]. He describes a diagonal system $A \in \mathcal{M}_2$ with the exponential stability index equal to 2 (and with the same stability index) and its continuous higher-order perturbation $f(\,\cdot\,,\,\cdot\,)$ such that the stability index of the perturbed system $\dot{x} = A(t)x + f(t,x)$ equals 0 (and so does its exponential stability index).

Thus, both examples demonstrate the effect of loss of stability. These examples had initiated a lot of research aimed to learn how perturbations of different types can affect stability of systems in $\mathcal{M}_n$. The results obtained in this direction form a considerable part of the contemporary Lyapunov exponent theory. When a perturbation is in a certain sense “small”, the effect of loss of stability is called the Perron effect [5, Chapter 4]. Starting with the paper [4], this term is used only when perturbations do not decrease the Lyapunov exponents of the initial system, and we adhere to this terminology.

2 Statement of the problem and the main result

In this paper we present a kind of generalization of the Perron effect. To this end, for a system $A \in \mathcal{M}_n$ and a metric space M we consider the class $\mathcal{E}_n[A](M)$ consisting of jointly continuous matrix-valued functions $Q : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ satisfying the following two conditions.

The first one is the estimate

$$\|Q(t,\mu)\| \le C_Q \exp(-\sigma_Q t) \ \text{ for all } \ (t,\mu) \in \mathbb{R}_+ \times M,$$

where C_Q and σ_Q are positive constants (generally, different for each function Q).

The second condition is that the stability and exponential stability indices of the perturbed system $A+Q$, being functions of $\mu \in M$ and denoted by $\mathrm{s}(\,\cdot\,; A+Q)$ and $\mathrm{es}(\,\cdot\,; A+Q)$, do not exceed the corresponding stability indices of the system A, i.e.

$$\mathrm{s}(\mu; A+Q) \le \mathrm{s}(A) \ \text{ and } \ \mathrm{es}(\mu; A+Q) \le \mathrm{es}(A) \ \text{ for all } \ \mu \in M.$$

We state the problem in the following way. Our task is for any integer $n \ge 2$ and metric space M to give a complete functional description of the class of pairs $((\mathrm{s}(A), \mathrm{es}(A)), (\mathrm{s}(\,\cdot\,; A+Q), \mathrm{es}(\,\cdot\,; A+Q)))$ composed of the stability indices of the initial system A and those of the perturbed system $A+Q$. The system A here ranges over $\mathcal{M}_n$, and for every A the matrix-valued function Q ranges over the set $\mathcal{E}_n[A](M)$. Thus, our problem is to present a complete functional description of the following class:

$$\Sigma\,\mathcal{E}_n(M) \equiv \Big\{\big((\mathrm{s}(A), \mathrm{es}(A)), (\mathrm{s}(\,\cdot\,; A+Q), \mathrm{es}(\,\cdot\,; A+Q))\big) \mid \ A \in \mathcal{M}_n, \ \ Q \in \mathcal{E}_n[A](M)\Big\}.$$

Before we could formulate the main result, let us remind the reader that a function $f : M \to \mathbb{R}$ is called [3, pp. 266–267] a function of the class $(F_\sigma, {}^*)$ if for any $r \in \mathbb{R}$ the preimage of the half-line $(r, +\infty)$ is an F_σ-set in the space M, i.e. it can be represented as a countable union of closed subsets of M. In particular, the class $(F_\sigma, {}^*)$ is a subclass of Baire class 2 [3, p. 294]. Let us also denote the set $\{0, 1, \ldots, n\}$ by $\mathcal{Z}_n$.

The solution to the problem is stated by the following

Theorem 1. *Let M be a metric space and $n \geqslant 2$ an integer. A pair $((\alpha_0, \beta_0), (\alpha(\,\cdot\,), \beta(\,\cdot\,))$ with $\alpha_0, \beta_0 \in \mathcal{Z}_n$ and $\alpha(\,\cdot\,), \beta(\,\cdot\,) : M \to \mathcal{Z}_n$ belongs to the class $\Sigma\,\mathcal{E}_n(M)$ if and only if the following conditions are met:*

1) $\alpha_0 \ge \beta_0$;

2) $\alpha(\mu) \ge \beta(\mu)$ *for all* $\mu \in M$;

3) $\alpha(\mu) \le \alpha_0$,$\beta(\mu) \le \beta_0$ *for all* $\mu \in M$;

4) *the functions* $\alpha(\,\cdot\,)$ *and* $\beta(\,\cdot\,)$ *are of the class* $(F_\sigma, {}^*)$.

3 Corollaries and remarks

Let M be a metric space. For an integer $n \ge 2$ we consider a family of linear systems depending on a parameter $\mu \in M$ of the form

$$\dot{x} = \mathcal{A}(t,\mu)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+, \tag{3.1}$$

such that for each fixed $\mu \in M$ the matrix-valued function $\mathcal{A}(\,\cdot\,,\mu) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is continuous and bounded (generally, the bounding constant is different for each μ).

Usually, a family of mappings $\mathcal{A}(\,\cdot\,,\mu)$, $\mu \in M$, is considered under one of the two following natural assumptions: the family is continuous in **a)** the compact-open topology or **b)** the uniform topology. The case **a)** is equivalent to the condition that if a sequence $(\mu_k)_{k\in\mathbb{N}}$ of points from M converges to a point $\mu_0 \in M$, then the sequence $(\mathcal{A}(t,\mu_k))_{k\in\mathbb{N}}$ of matrices converges uniformly over each interval $[0,T] \subset \mathbb{R}_+$ to the matrix $\mathcal{A}(t,\mu_0)$ as $k \to +\infty$. The case **b)** differs from **a)** in that the convergence is uniform over the whole half-line $\mathbb{R}_+$. We denote the class of families (3.1) that are continuous in the compact-open topology by $\mathcal{C}^n(M)$ and those that are continuous in the uniform topology by $\mathcal{U}^n(M)$. Clearly, $\mathcal{U}^n(M) \subset \mathcal{C}^n(M)$.

Further, we identify the family (3.1) with the matrix-valued function $\mathcal{A}(\,\cdot\,,\,\cdot\,)$ specifying it and use the following notation: $\mathcal{A} \in \mathcal{C}^n(M)$ or $\mathcal{A} \in \mathcal{U}^n(M)$.

Corollary. *Let M be a metric space. For any integer $n \geq 2$, the classes of pairs of functions*

$$\Sigma\,\mathcal{C}_n(M) \equiv \big\{(\mathrm{s}(\,\cdot\,;\mathcal{A}), \mathrm{es}(\,\cdot\,;\mathcal{A}))\mid\ \mathcal{A} \in \mathcal{C}^n(M)\big\}$$

and

$$\Sigma\,\mathcal{U}_n(M) \equiv \big\{(\mathrm{s}(\,\cdot\,;\mathcal{A}), \mathrm{es}(\,\cdot\,;\mathcal{A}))\mid\ \mathcal{A} \in \mathcal{U}^n(M)\big\}$$

coincide with one another and consist of the pairs $(\alpha(\,\cdot\,),\beta(\,\cdot\,))$ of functions $M \to \mathcal{Z}_n$ of the class $(F_\sigma,\,^*)$ that satisfy the inequality $\alpha(\mu) \geq \beta(\mu)$ for all $\mu \in M$.

Remark. The description of the classes composed of the second elements of the pairs from $\Sigma\,\mathcal{C}_n(M)$ and $\Sigma\,\mathcal{U}_n(M)$ was obtained in the paper [2]. Those classes coincide with one another and consist of functions $M \to \mathcal{Z}_n$ from the class $(F_\sigma,\,^*)$.

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C o n t e n t s