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NONLINEAR ATANGANA–BALEANU FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING THE MITTAG–LEFFLER INTEGRAL OPERATOR

Abstract. This paper intends to investigate the existence and uniqueness of solutions for some nonlinear Atangana–Baleanu fractional differential equations involving the Mittag–Leffler integral operator. By means of Schauder's fixed point theorem and Banach's fixed point theorem, the existence and uniqueness results are obtained. A generalized fractional order free electron laser equation is given as an application.

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Key words and phrases. Atangana–Baleanu fractional derivative, Schauder's fixed point theorem, Mittag–Leffler operator.

რეზიუმე. ნაშრომი ეძღვნება ამოხსნების არსებობისა და ერთადერთობის გამოკვლევას ზოგიერთი არაწრფივი ატანგანა-ბალეანუს წილადურ დიფერენციალური განტოლებებისთვის, რომლებიც შეიცავს მიტაგ-ლეფლერის ინტეგრალურ ოპერატორს. შაუდერის უძრავი წერტილის თეორემისა და ბანახის უძრავი წერტილის თეორემის საშუალებით მიღებულია არსებობისა და ერთადერთობის შედეგები. გამოყენების სახით მოცემულია თავისუფალ ელექტრონებზე მომუშავე ლაზერის წილადური რიგის განზოგადებული განტოლება.

1 Introduction

In the last decades, several significant results related to the qualitative properties of fractional differential equations have been recorded because of their ability to model real-world problems in many fields such as science, technology and engineering [11, 12, 19, 21–23, 26, 29].

Recently, the interest of many researchers interested in fractional calculus has gone to a new type of fractional derivative with non-singular kernel introduced by Caputo and Fabrizio [10], this derivative is based on the exponential kernel. Later, Atangana and Baleanu [7] developed another version which used the generalized Mittag–Leffler function as non-local and non-singular kernel which appears naturally in several physical problems and the field of science and engineering [3–6, 8, 14, 25, 30, 31].

On the other hand, the Mittag–Leffler function and its generalizations play a fundamental role in fractional calculus and its applications such as modelling groundwater fractal flow, viscoelasticity and probability theory [1, 13].

In [24], Prabhakar studied a singular integral equation with a general Mittag–Leffler function in the kernel, namely,

$$\int_a^t (t-s)^{\delta-1}\mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t-s)^{\sigma})\phi(s)\,ds = g(t), \;\; t\in[a,b],$$

where

$$\mathbb{E}^{\lambda}_{\sigma,\delta}(z) = \sum_{k=0}^{\infty}\frac{(\lambda)_k}{\Gamma(\sigma k+\delta)}\frac{z^k}{k!} \quad (\sigma,\delta,\lambda\in\mathbb{C}, \;\; \mathrm{Re}(\sigma)>0).$$

The function $\mathbb{E}^{\lambda}_{\sigma,\delta}(z)$ is the three-parameter Mittag–Leffler function and $(\lambda)_k$ is the Pochhammer symbol defined as

$$(\lambda)_k = \begin{cases}(\lambda)(\lambda+1)\cdots(\lambda+k-1), & k\in\mathbb{N},\\ 1, & k=0, \;\; \lambda\neq 0.\end{cases}$$

When $\lambda=1$, $\mathbb{E}^{1}_{\sigma,\delta}(z)$ coincides with the classical two-parameter Mittag–Leffler function

$$\mathbb{E}_{\sigma,\delta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\sigma k+\delta)}\,.$$

It is useful to mention that the three-parameter Mittag–Leffler function is closely connected with the phenomenon of Havriliak–Negami relaxation [15].

In [17], Kilbas et al. investigated an integro-differential equation of the form

$$D^{\alpha}_{a^+}y(t) = \gamma\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;a^+}y(t)+f(t), \;\; a<t\leq b, \tag{1.1}$$

where $\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;a^+}$ is the Mittag–Leffler integral operator defined by

$$\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;a^+}y(t) = \int_a^t (t-s)^{\delta-1}\mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t-s)^{\sigma})y(s)\,ds, \tag{1.2}$$

where $\sigma,\delta,\nu,\lambda\in\mathbb{C}$, $\mathrm{Re}(\sigma)>0$, $\mathrm{Re}(\delta)>0$.

Obviously, $\mathbb{E}^{0}_{\sigma,\delta,\nu;a^+}$ is the Riemann–Liouville fractional integral operator of order δ. Therefore, operator (1.2) and its inverse can be considered as generalization of fractional integral and derivative operators involving $\mathbb{E}^{\lambda}_{\sigma,\delta}(z)$ in their kernels.

In this paper, we consider the following nonlinear Atangana–Baleanu fractional differential equation involving the Mittag–Leffler integral operator

$$\begin{cases} {}^{ABC}D^{\alpha}_{0^+}x(t) = \mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}f(t,x(t)), \;\; \alpha\in(0,1], \;\; t\in[0,1],\\ \qquad x(0)=x_0\in\mathbb{R},\end{cases} \tag{1.3}$$

where ${}^{ABC}D^{\alpha}_{0^+}$ denotes the Atangana–Baleanu fractional derivative of order α in Caputo sense, $\sigma, \delta, \nu, \lambda \in \mathbb{R}$, $\sigma, \delta > 0$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

The importance of studying such equations like (1.1) and (1.3) is that they describe the unsaturated behavior of the free electron laser [9, 27, 28], which is a kind of laser whose lasing medium consists of very-high-speed electrons moving freely through a magnetic structure.

2 Preliminaries

In [7], Atangana and Baleanu improved the Caputo–Fabrizio fractional derivative with non-singular kernel to another one with non-local and non-singular kernel. We present the basic definitions of the new fractional order derivatives.

Definition 2.1 (see [7])**.** Let $h \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$, then the Atangana–Baleanu fractional derivative in Caputo sense is given by

$$ {}^{ABC}D^{\alpha}_{a^+} h(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t \mathbb{E}_{\alpha}\Big[-\alpha \frac{(t-s)^{\alpha}}{1-\alpha} \Big] h'(s)\, ds, \tag{2.1} $$

where $B(\alpha)$ denotes a normalization function such that $B(0) = B(1) = 1$ and $\mathbb{E}_{\alpha}$ denotes the Mittag–Leffler function defined by

$$ \mathbb{E}_{\alpha}(-t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(-t)^{\alpha k}}{\Gamma(\alpha k + 1)} . $$

However, when $\alpha = 0$, they did not recover the original function, except when at the origin the function vanishes. To avoid this issue, they proposed the following definition.

Definition 2.2 (see [7])**.** Let $h \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$, and it is not necessary differentiable, then the Atangana–Baleanu fractional derivative in Riemann–Liouville sense is given by

$$ {}^{ABR}D^{\alpha}_{a^+} h(t) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t \mathbb{E}_{\alpha}\Big[-\alpha \frac{(t-s)^{\alpha}}{1-\alpha} \Big] h(s)\, ds. \tag{2.2} $$

Equations (2.1) and (2.2) have a non-local kernel. Also in equation (2.1), when the function is constant, we get zero. For more details and properties, see [7, 10].

Definition 2.3 (see [7])**.** Let $h \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$, then the Atangana–Baleanu fractional integral, associate to the new fractional derivative with non-local kernel is given by

$$ {}^{AB}I^{\alpha}_{a^+} h(t) = \frac{1-\alpha}{B(\alpha)}\, h(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\, ds, $$

where $\Gamma(\cdot)$ denotes the well-known gamma function. The initial function is recovered when the fractional order turns to zero. Also, when the order turns to 1, we have the classical integral.

To end this section, we collect some useful lemmas.

Lemma 2.4 (see [2])**.**

$$ I^{\alpha}_{0^+} \mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(\phi) = \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}(\phi), \quad \mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} \mathbb{E}^{\eta}_{\sigma,\mu,\nu;0^+}(\phi) = \mathbb{E}^{\lambda+\eta}_{\sigma,\delta+\mu,\nu;0^+}(\phi), $$
$$ \|\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(\phi)\|_C \le \mathbb{E}^{\lambda}_{\sigma,\delta+1}(|\nu|)\|(\phi)\|_C . $$

Lemma 2.5 (see [2])**.** *Suppose $z \ge 0$ is fixed, $\sigma, \delta, \lambda > 0$.*

(i) *If $0 \le \lambda \le 1$, then $\mathbb{E}^{\lambda}_{\sigma,\delta}(z) \le \mathbb{E}_{\sigma,\delta}(z)$.*

(ii) *If* $\lambda \geq 1$, *then* $\mathbb{E}^{\lambda}_{\sigma,\delta}(z) \geq \mathbb{E}_{\sigma,\delta}(z)$.

Lemma 2.6 (see [18])**.** *Assume that* $\sigma, \delta, \nu, \lambda \in \mathbb{R}$, $(\sigma, \delta > 0)$, *then for a continuous function* $\phi \in C([0,1])$ *and positive integer* n, *where* $\delta > n$,

$$\frac{d^n}{dt^n}\,\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(\phi) = \mathbb{E}^{\lambda}_{\sigma,\delta-n,\nu;0^+}(\phi).$$

Lemma 2.7 (see [20])**.** *Suppose* $\sigma, \delta, \nu, \lambda \in \mathbb{R}$, $(\sigma, \delta > 0,\ \delta > \alpha \geq 0)$, *then for a continuous function* $\phi \in C([0,1])$,

$$D^{\alpha}_{0^+}\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(\phi) = \mathbb{E}^{\lambda}_{\sigma,\delta-\alpha,\nu;0^+}(\phi).$$

Lemma 2.8 (Ascoli–Arzelà theorem)**.** *Let* $S = \{s(t)\}$ *be a function family of continuous mappings on a closed and bounded interval* $[a,b]$, $s : [a,b] \to \mathbb{X}$.

If S *is uniformly bounded and equicontinuous, and for any* $t^* \in [a,b]$, *the set* $\{s(t^*)\}$ *is relatively compact, then there exists a uniformly convergent function sequence* $\{s_n(t)\}$ $(n = 1,2,\dots,\ t \in [a,b])$ *in* S.

Lemma 2.9 (Schauder's fixed point theorem)**.** *If* U *is a closed, bounded and convex subset of a Banach space* $\mathbb{X}$ *and* $\mathcal{T} : U \to U$ *is completely continuous, then* $\mathcal{T}$ *has a fixed point in* U.

3 The Existence and Uniqueness Results

Let $C([0,1])$ be the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\|_C = \max\{|x(t)| :\ t \in [0,1]\}$.

Definition 3.1 ([16, Theorem 3.1])**.** A function $x \in C([0,1])$ is said to be a solution of equation (1.3) with $x(0) = x_0$ if $x(t)$ satisfies the integral equation

$$x(t) = x_0 +^{AB} I^{\alpha}_{0^+}\big(\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t, x(t))\big). \tag{3.1}$$

In view of Definition 2.3, together with Lemma 2.4, equation (3.1) can be reformulated as follows:

$$\begin{aligned} x(t) &= x_0 +^{AB} I^{\alpha}_{0^+}\big(\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t, x(t))\big) \\ &= x_0 + \frac{1-\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t,x(t)) + \frac{\alpha}{B(\alpha)}\, I^{\alpha}_{0^+}\big(\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t,x(t))\big) \\ &= x_0 + \frac{1-\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t,x(t)) + \frac{\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+} f(t,x(t)). \end{aligned} \tag{3.2}$$

We introduce the following assumptions:

$(A1)$ The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

$(A2)$ There exists a constant $L_f > 0$ such that

$$|f(t,x) - f(t,y)| \leq L_f|x-y| \ \text{ for each }\ t \in [0,1], \ \text{ and all }\ x, y \in \mathbb{R}.$$

3.1 Existence result via Schauder's fixed point theorem

Theorem 3.2. *Assume that* $(A1)$ *and* $(A2)$ *are satisfied. Then the Atangana–Baleanu fractional differential equation* (1.3) *has at least one solution on* $[0,1]$.

Proof. We define the operator $\mathcal{T} : C([0,1]) \to C([0,1])$ by

$$(\mathcal{T}x)(t) = x_0 + \frac{1-\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t,x(t)) + \frac{\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+} f(t,x(t)), \quad t \in [0,1]. \tag{3.3}$$

Note that the operator $\mathcal{T}$ is well-defined on $C([0,1])$ due to $(A1)$.

Consider the set $B_r = \{x \in C([0,1]) : \ \|x\|_C \le r\}$. Clearly, the set B_r is closed, bounded and convex. The proof is divided into several steps.

Step 1. $\mathcal{T}$ is continuous.

Let x_n be a sequence such that $x_n \to x$ in B_r . Then for each $t \in [0,1]$, we have

$$\begin{aligned}
\big|(\mathcal{T}x_n)(t) - (\mathcal{T}x)(t)\big| &= \bigg|\frac{1-\alpha}{B(\alpha)}\Big(\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t,x_n(t)) - \mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t,x(t))\Big) \\
&\quad + \frac{\alpha}{B(\alpha)}\Big(\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+} f(t,x_n(t)) - \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+} f(t,x(t))\Big)\bigg| \\
&\le \frac{1-\alpha}{B(\alpha)}\Big|\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(f(t,x_n(t)) - f(t,x(t)))\Big| + \frac{\alpha}{B(\alpha)}\Big|\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}(f(t,x_n(t)) - f(t,x(t)))\Big| \\
&\le \Big(\frac{1-\alpha}{B(\alpha)}\,\|\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(1)\| + \frac{\alpha}{B(\alpha)}\,\|\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}(1)\|\Big)\big\|f(\,\cdot\,,x_n(\,\cdot\,)) - f(\,\cdot\,,x(\,\cdot\,))\big\|_C \\
&\le \Big(\frac{1-\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+1}(|\nu|) + \frac{\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha+1}(|\nu|)\Big)\big\|f(\,\cdot\,,x_n(\,\cdot\,)) - f(\,\cdot\,,x(\,\cdot\,))\big\|_C\,,
\end{aligned}$$

which implies that

$$\|\mathcal{T}x_n - \mathcal{T}x\|_C \le \Big(\frac{1-\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+1}(|\nu|) + \frac{\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha+1}(|\nu|)\Big)\big\|f(\,\cdot\,,x_n(\,\cdot\,)) - f(\,\cdot\,,x(\,\cdot\,))\big\|_C\,.$$

By $(A1)$, the continuity of the function f implies that $\mathcal{T}$ is continuous.

Step 2. $\mathcal{T}$ maps bounded sets into bounded sets in B_r.

Indeed, it is enough to show that for any $r > 0$, there exists a positive constant ℓ such that for each $x \in B_r$, one has $\|\mathcal{T}x\|_C \le \ell$. For $t \in [0,1]$, $x \in B_r$ and in view of $(A1)$, we define $M_f = \sup_{(t,x)\in[0,1]\times B_r}\|f(t,x)\|$ and, consequently, we have

$$\begin{aligned}
|(\mathcal{T}x)(t)| &= \Big|x_0 + \frac{1-\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t,x(t)) + \frac{\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+} f(t,x(t))\Big| \\
&\le |x_0| + \frac{(1-\alpha)M_f}{B(\alpha)}\,\|\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(1)\| + \frac{\alpha M_f}{B(\alpha)}\|\,\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}(1)\| \\
&\le |x_0| + \frac{(1-\alpha)M_f}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+1}(|\nu|) + \frac{\alpha M_f}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha+1}(|\nu|) := \ell.
\end{aligned}$$

Hence, $\|\mathcal{T}x\|_C \le \ell$. This implies that $\mathcal{T}(B_r) \subset B_r$.

Step 3. $\mathcal{T}$ maps bounded sets into equicontinuous sets of B_r.

Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and for any $x \in B_r$, we have

$$\begin{aligned}
\big|(\mathcal{T}x)(t_2) - (\mathcal{T}x)(t_1)\big| &\le \bigg|\frac{1-\alpha}{B(\alpha)}\Big(\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t_2,x(t_2) - \mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+} f(t_1,x(t_1))\Big)\bigg| \\
&\quad + \bigg|\frac{\alpha}{B(\alpha)}\Big(\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+} f(t_2,x(t_2) - \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+} f(t_1,x(t_1))\Big)\bigg| \\
&\le \frac{1-\alpha}{B(\alpha)}\bigg|\int_0^{t_2}(t_2-s)^{\delta-1}\mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_2-s)^{\sigma})f(s,x(s)\,ds \\
&\qquad - \int_0^{t_1}(t_1-s)^{\delta-1}\mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1-s)^{\sigma})f(s,x(s)\,ds\bigg| \\
&\quad + \frac{\alpha}{B(\alpha)}\bigg|\int_0^{t_2}(t_2-s)^{\delta+\alpha-1}\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_2-s)^{\sigma})f(s,x(s)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\qquad - \int_0^{t_1} (t_1 - s)^{\delta+\alpha-1} \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_1 - s)^{\sigma}) f(s, x(s)\, ds \Big| \\
&= \frac{1-\alpha}{B(\alpha)}\, I_1 + \frac{\alpha}{B(\alpha)}\, I_2,
\end{aligned}
$$

where

$$
I_1 = \Big| \int_0^{t_2} (t_2 - s)^{\delta-1} \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_2 - s)^{\sigma}) f(s, x(s)\, ds - \int_0^{t_1} (t_1 - s)^{\delta-1} \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) f(s, x(s)\, ds \Big|
$$

and

$$
\begin{aligned}
I_2 = \Big| &\int_0^{t_2} (t_2 - s)^{\delta+\alpha-1} \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_2 - s)^{\sigma}) f(s, x(s)\, ds \\
&\qquad - \int_0^{t_1} (t_1 - s)^{\delta+\alpha-1} \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_1 - s)^{\sigma}) f(s, x(s)\, ds \Big|.
\end{aligned}
$$

For I_1, we have

$$
\begin{aligned}
I_1 \le \Big[&\int_0^{t_2} (t_2 - s)^{\delta-1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_2 - s)^{\sigma}) - \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \big| \, \| f(s, x(s)) \| \, ds \\
&+ \int_0^{t_1} \big| (t_2 - s)^{\delta-1} - (t_1 - s)^{\delta-1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \| f(s, x(s)) \| \, ds \\
&+ \int_{t_1}^{t_2} (t_2 - s)^{\delta-1} \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \| f(s, x(s)) \| \, ds \Big] \\
\le M_f \Big[&\int_0^{1} (t_2 - s)^{\delta-1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_2 - s)^{\sigma}) - \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \big| \, ds \\
&+ \int_0^{1} \big| (t_2 - s)^{\delta-1} - (t_1 - s)^{\delta-1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \, ds \\
&+ \int_0^{1} \big| (t_2 - s)^{\delta-1} - (t_1 - s)^{\delta-1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \, ds \Big] \\
\le M_f \Big[&\Big(\int_0^{1} \big| (t_2 - s)^{\delta-1} \big|^2 \, ds \Big)^{1/2} \Big(\int_0^{1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_2 - s)^{\sigma}) - \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \big|^2 \, ds \Big)^{1/2} \\
&+ 2 \Big(\int_0^{1} \big| (t_2 - s)^{\delta-1} - (t_1 - s)^{\delta-1} \big|^2 \, ds \Big)^{1/2} \Big(\int_0^{1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1 - s)^{\sigma}) \big|^2 \, ds \Big)^{1/2} \Big].
\end{aligned}
$$

Similarly, I_2 can be estimated as

$$
I_2 \le M_f \Big[\Big(\int_0^{1} \big| (t_2 - s)^{\delta+\alpha-1} \big|^2 \, ds \Big)^{1/2} \Big(\int_0^{1} \big| \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_2 - s)^{\sigma}) - \mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_1 - s)^{\sigma}) \big|^2 \, ds \Big)^{1/2}
$$

$$+2\Big(\int_0^1 \big|(t_2-s)^{\delta+\alpha-1}-(t_1-s)^{\delta+\alpha-1}\big|^2\,ds\Big)^{1/2}\Big(\int_0^1 \big|\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_1-s)^{\sigma})\big|^2\,ds\Big)^{1/2}\Big].$$

Hence, we get

$$\begin{aligned}\big|(\mathcal{T}x)(t_2)-(\mathcal{T}x)(t_1)\big| &\le \frac{(1-\alpha)M_f}{B(\alpha)}\Big[\Big(\int_0^1 \big|(t_2-s)^{\delta-1}\big|^2\,ds\Big)^{1/2}\\ &\quad\times\Big(\int_0^1 \big|\mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_2-s)^{\sigma})-\mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1-s)^{\sigma})\big|^2\,ds\Big)^{1/2}\\ &\quad+2\Big(\int_0^1 \big|(t_2-s)^{\delta-1}-(t_1-s)^{\delta-1}\big|^2\,ds\Big)^{1/2}\Big(\int_0^1 \big|\mathbb{E}^{\lambda}_{\sigma,\delta}(\nu(t_1-s)^{\sigma})\big|^2\,ds\Big)^{1/2}\Big]\\ &\quad+\frac{\alpha M_f}{B(\alpha)}\Big[\Big(\int_0^1 \big|(t_2-s)^{\delta+\alpha-1}\big|^2\,ds\Big)^{1/2}\\ &\quad\times\Big(\int_0^1 \big|\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_2-s)^{\sigma})-\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_1-s)^{\sigma})\big|^2\,ds\Big)^{1/2}\\ &\quad+2\Big(\int_0^1 \big|(t_2-s)^{\delta+\alpha-1}-(t_1-s)^{\delta+\alpha-1}\big|^2\,ds\Big)^{1/2}\Big(\int_0^1 \big|\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha}(\nu(t_1-s)^{\sigma})\big|^2\,ds\Big)^{1/2}\Big].\end{aligned}$$

As a result, we immediately find that the right-hand side of the above inequality tends to zero as $t_2 \to t_1$. Therefore, $\mathcal{T}(B_r)$ is an equicontinuous set. It is also uniformly bounded.

Consequently, from Steps $1-3$ together with the Ascoli–Arzelà theorem (Lemma 2.8), we show that the operator $\mathcal{T}$ is completely continuous. Hence, by Schauder's fixed point theorem (Lemma 2.9), we conclude that the operator $\mathcal{T}$ has at least one fixed point which is a solution of the Atangana–Baleanu fractional differential equation (1.3) on $[0,1]$. The proof is completed. □

3.2 Uniqueness result via the Banach fixed point theorem

Theorem 3.3. *If the assumptions* $(A1)$ *and* $(A2)$ *hold, then the Atangana–Baleanu fractional differential equation* (1.3) *has a unique solution on* $[0,1]$*, provided that*

$$\Lambda := \Big(\frac{1-\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+1}(|\nu|)+\frac{\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha+1}(|\nu|)\Big)L_f<1. \tag{3.4}$$

Proof. Consider the operator $\mathcal{T}$ defined in (3.3). In what follows, we show that the operator $\mathcal{T}$ is a contraction. Repeating the same procedure as in Step 2 of the proof of Theorem 3.2, we obtain $\mathcal{T}(B_r)\subset B_r$.

Now, for $x,y\in C([0,1])$ and for each $t\in[0,1]$, by using $(A2)$, we have

$$\begin{aligned}\big|(\mathcal{T}x)(t)-(\mathcal{T}y)(t)\big| &= \Big|\frac{1-\alpha}{B(\alpha)}\Big(\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}f(t,x(t))-\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}f(t,y(t))\Big)\\ &\quad+\frac{\alpha}{B(\alpha)}\Big(\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}f(t,x(t))-\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}f(t,y(t))\Big)\Big|\\ &\le \frac{1-\alpha}{B(\alpha)}\big|\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(f(t,x(t))-f(t,y(t)))\big|+\frac{\alpha}{B(\alpha)}\big|\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}(f(t,x(t))-f(t,y(t)))\big|\\ &\le \Big(\frac{1-\alpha}{B(\alpha)}\,\|\mathbb{E}^{\lambda}_{\sigma,\delta,\nu;0^+}(1)\|+\frac{\alpha}{B(\alpha)}\,\|\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha,\nu;0^+}(1)\|\Big)L_f\|x-y\|_C\end{aligned}$$

$$\leq \Big(\frac{1-\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+1}(|\nu|) + \frac{\alpha}{B(\alpha)}\,\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha+1}(|\nu|)\Big) L_f \|x-y\|_C\,.$$

Hence,

$$\|\mathcal{T}x - \mathcal{T}y\|_C \leq \Lambda \|x-y\|_C\,.$$

If condition (3.4) is satisfied, then, as a consequence of the Banach fixed point theorem, we conclude that the operator $\mathcal{T}$ has a unique fixed point. Thus, the Atangana–Baleanu fractional differential equation (1.3) has a unique solution. The proof is completed. □

4 An application

In this section, we consider the following generalized fractional order free electron laser equation as an application of the Atangana–Baleanu fractional differential equation (1.3).

Example 4.1.

$$\begin{cases} {}^{ABC}D^{\frac{1}{2}}_{0^+} x(t) = \mathbb{E}^{\frac{2}{5}}_{1,\frac{1}{2},2;0^+} \dfrac{|x(t)|}{50(1+e^t)(1+|x(t)|)}\,, & t \in [0,1], \\ x(0) = 0. \end{cases} \tag{4.1}$$

Here, t is a dimensionless time ranging from 0 to 1 and $x(t)$ is a complex-field amplitude which is assumed dimensionless and satisfies the initial condition $x(0) = 0$.

Set $\alpha = \frac{1}{2}$, $\sigma = 1$, $\delta = \frac{1}{2}$, $\nu = 2$, $\lambda = \frac{2}{5}$ and $f(t,x) = \frac{x}{50(1+e^t)(1+x)}$. Since

$$\begin{aligned} |f(t,x) - f(t,y)| &= \Big|\frac{x}{50(1+e^t)(1+x)} - \frac{y}{50(1+e^t)(1+y)}\Big| \\ &\leq \frac{|x-y|}{50(1+e^t)(1+x)(1+y)} \leq \frac{1}{50(1+e^t)}|x-y| \leq \frac{1}{100}\,\|x-y\|_C\,, \end{aligned}$$

we get the assumption $(A2)$ with $L_f = \frac{1}{100}$.

Moreover, using Lemma 2.5 and the fact that $\Gamma(k+2) \leq \Gamma(k+\frac{5}{2})$, the condition (3.4) gives

$$\begin{aligned} \Lambda &= \Big(\frac{1-\alpha}{B(\alpha)}\mathbb{E}^{\lambda}_{\sigma,\delta+1}(|\nu|) + \frac{\alpha}{B(\alpha)}\mathbb{E}^{\lambda}_{\sigma,\delta+\alpha+1}(|\nu|)\Big) L_f \\ &= \frac{1}{100}\Big(\frac{1-\frac{1}{2}}{B(\frac{1}{2})}\,\mathbb{E}^{\frac{2}{5}}_{1,\frac{1}{2}+1}(|2|) + \frac{\frac{1}{2}}{B(\frac{1}{2})}\,\mathbb{E}^{\frac{2}{5}}_{1,\frac{1}{2}+\frac{1}{2}+1}(|2|)\Big) = \frac{1}{100}\Big(\frac{1}{2}\,\mathbb{E}^{\frac{2}{5}}_{1,\frac{5}{2}}(|2|) + \frac{1}{2}\,\mathbb{E}^{\frac{2}{5}}_{1,2}(|2|)\Big) \\ &\leq \frac{1}{100}\Big(\frac{1}{2}\,\mathbb{E}_{1,\frac{5}{2}}(|2|) + \frac{1}{2}\,\mathbb{E}_{1,2}(|2|)\Big) = \frac{1}{100}\Big(\frac{1}{2}\sum_{k=0}^{\infty}\frac{2^k}{\Gamma(k+\frac{5}{2})} + \frac{1}{2}\sum_{k=0}^{\infty}\frac{2^k}{\Gamma(k+2)}\Big) \\ &\leq \frac{1}{100}\Big(\frac{1}{2}\sum_{k=0}^{\infty}\frac{2^k}{\Gamma(k+2)} + \frac{1}{2}\sum_{k=0}^{\infty}\frac{2^k}{\Gamma(k+2)}\Big) = \frac{1}{100}\Big(\frac{1}{2}\sum_{k=0}^{\infty}\frac{2^k}{(k+1)!} + \frac{1}{2}\sum_{k=0}^{\infty}\frac{2^k}{(k+1)!}\Big) \\ &= \frac{1}{100}\Big(\frac{1}{2}\,\frac{e^2-1}{2} + \frac{1}{2}\,\frac{e^2-1}{2}\Big) = \frac{e^2-1}{200} = 0.03194528049 < 1. \end{aligned}$$

Therefore, all the assumptions of Theorem 3.3 are satisfied. Hence, the Atangana–Baleanu fractional differential equation (4.1) has a unique solution on $[0,1]$.

Finally, according to formula (3.2), we can obtain a unique solution $x(t)$, which is the complex-field amplitude of the generalized fractional order free electron laser equation (4.1), from the following Volterra integral equation:

$$x(t) = \frac{1}{100(1+e^t)}\Bigg[\int_0^t (t-s)^{-\frac{1}{2}}\mathbb{E}^{\frac{2}{5}}_{1,\frac{1}{2}}(2(t-s))\,\frac{x(s)}{1+x(s)}\,ds + \int_0^t \mathbb{E}^{\frac{2}{5}}_{1,1}(2(t-s))\,\frac{x(s)}{1+x(s)}\,ds\Bigg],$$

where

$$\mathbb{E}^{\frac{2}{5}}_{1,\frac{1}{2}}(2(t-s)) = \sum_{k=0}^{\infty} \frac{2^k(\frac{2}{5})_k}{\Gamma(k+\frac{1}{2})} \frac{(t-s)^k}{k!}$$

and

$$\mathbb{E}^{\frac{2}{5}}_{1,1}(2(t-s)) = \sum_{k=0}^{\infty} \frac{2^k(\frac{2}{5})_k}{\Gamma(k+1)} \frac{(t-s)^k}{k!}\,.$$

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TWO EXPLICIT NON-ALGEBRAIC CROSSING LIMIT CYCLES FOR A FAMILY OF PIECEWISE LINEAR SYSTEMS

Abstract. For a given family of planar piecewise linear differential systems, it is a very difficult problem to determine an upper bound for the number of its limit cycles and its explicit expressions. In this paper, we give a family of planar discontinuous piecewise linear differential systems formed by two regions separated by a straight line and having only one focus whose limit cycles can be explicitly described by using the first integrals. We show that these systems may have at most two explicit non-algebraic limit cycles.

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1 Introduction

The study of piecewise linear differential systems goes back to Andronov, Vitt and Khaikin [1] and still continues to receive attention by researchers. Piecewise linear systems often appear in the descriptions of many real processes such as dry friction in mechanical systems or switches in electronic circuits (see, e.g., [5, 15, 18, 19]). This kind of systems is generally modeled by ordinary differential equations with discontinuous right-hand sides which can exhibit very complicated dynamics and rich bifurcation phenomena.

A limit cycle is a periodic orbit of a differential system in $\mathbb{R}^2$ isolated in the set of all periodic orbits of that system. There are two types of limit cycles in the planar discontinuous piecewise linear differential systems, the crossing and sliding ones. The "sliding limit cycles" contain some arc of the lines of discontinuity that separate the different linear differential systems (more precise definition can be found in [17]). The "crossing limit cycles" contain only isolated points of the lines of discontinuity. In this paper, we consider only the crossing limit cycles of some planar discontinuous piecewise linear differential systems separated by one straight line.

Limit cycles of discontinuous piecewise linear differential systems separated by a straight line have been studied by many authors (see, e.g., [2, 7, 8, 10, 11, 13] and the references therein). There are examples of such systems exhibiting three limit cycles (see [3, 4, 9, 12, 14]), but at present moment we do not know whether discontinuous piecewise linear differential systems separated by a straight line may have more than three limit cycles.

On the other hand, it seems intuitively clear that "most" limit cycles of discontinuous piecewise linear differential systems have to be non-algebraic. Nevertheless, in all these papers devoted to the study of the crossing limit cycles of piecewise linear differential systems, explicit non-algebraic limit cycles do not appear, their existence is proved by using different methods as the first integrals, the averaging theory, the Poincaré map, the Newton–Kantorovich Theorem, the Melnikov function.

The goal of this paper is to give a discontinuous piecewise linear differential systems separated by a straight line for which we can get two explicit limit cycles which are not algebraic. As far as we know, there are no examples of this situation in the literature.

We consider planar piecewise linear systems with two linearity regions separated by a straight line $\Sigma = \{(x, y) \in \mathbb{R}^2 : \ x = 0\}$, where we assume that the two linearity regions in the phase plane are the left and right half-planes

$$\Sigma_- = \big\{(x, y) \in \mathbb{R}^2 : \ x < 0\big\}, \quad \Sigma_+ = \big\{(x, y) \in \mathbb{R}^2 : \ x > 0\big\}.$$

We suppose that one of the two linear differential systems has no equilibria, neither real nor virtual, and the other one has a focus at the origin. We prove that these two systems are integrable. Moreover, we determine sufficient conditions for a discontinuous piecewise linear differential systems to possess two or one explicit non-algebraic limit cycles. Concrete examples exhibiting the applicability of our result are introduced.

2 Preliminaries

The following normal form for the linear differential system in $\mathbb{R}^2$ and its first integral will help us to prove our main result.

Lemma 2.1. *A linear differential system having a focus at the origin can be written as*

$$\dot{x} = (2\lambda - \delta)x + \beta y, \quad \dot{y} = -\frac{1}{\beta}\big((\lambda - \delta)^2 + \omega^2\big)x + \delta y \tag{2.1}$$

with $\omega > 0$. Moreover, this system has the first integral

$$H_1(x, y) = \Big(\big((\lambda - \delta)^2 + \omega^2\big)x^2 + 2\beta(\lambda - \delta)xy + \beta^2 y^2\Big)e^{-\frac{2\lambda}{\omega}\arctan\left(\frac{\omega x}{(\lambda-\delta)x+\beta y}\right)}.$$

Proof. Consider a general linear differential system

$$\dot{x} = \alpha x + \beta y, \quad \dot{y} = \eta x + \delta y. \tag{2.2}$$

The eigenvalues of this system are

$$\lambda_{1,2} = \frac{1}{2}\Big(\alpha + \delta \pm \sqrt{(\alpha-\delta)^2 + 4\beta\eta}\Big).$$

We know that system (2.2) has a real focus if $\frac{1}{2}(\alpha+\delta) = \lambda$, and $(\alpha-\delta)^2 + 4\beta\eta = -4\omega^2$, for some $\omega > 0$, $\beta\eta < 0$ and $\lambda \in \mathbb{R}$, then

$$\alpha = 2\lambda - \delta, \quad \eta = -\frac{1}{\beta}\big((\lambda-\delta)^2 + \omega^2\big).$$

Therefore, we obtain system (2.1).

Since the unique equilibrium is located at the origin $O(0,0)$ and is of focus type, any orbit of system (2.1) crosses the straight line $x = 0$ at least at one point, namely, $(0, C)$, $C \in \mathbb{R}$, thus the general solution of (2.1) is given by

$$x(t) = \frac{\beta}{\omega} Ce^{t\lambda} \sin t\omega, \quad y(t) = \frac{1}{\omega} Ce^{t\lambda} v\big(\omega \cos t\omega + (\delta - \lambda)\sin t\omega\big), \tag{2.3}$$

where $C \in \mathbb{R}$. So, from the first equation of (2.3), we obtain

$$e^{t\lambda} \sin \omega t = \frac{\omega}{\beta C}\, x.$$

Substituting this last expression into the second equation, we get

$$e^{t\lambda} \cos \omega t = \frac{1}{C\beta}\big((\lambda-\delta)x + \beta y\big).$$

Therefore,

$$\tan \omega t = \frac{\omega x}{(\lambda-\delta)x + \beta y}\,.$$

From the last equation, we obtain

$$t = \frac{1}{\omega} \arctan\Big(\frac{\omega x}{(\lambda-\delta)x + \beta y}\Big).$$

Substituting the previous expressions in the first equation of (2.3) and simplifying, we obtain

$$\Big(\big((\lambda-\delta)^2 + \omega^2\big)x^2 + 2\beta(\lambda-\delta)xy + \beta^2 y^2\Big)e^{-\frac{2\lambda}{\omega}\arctan\left(\frac{\omega x}{(\lambda-\delta)x + y\beta}\right)} = h,$$

where $h = (\beta C)^2 \in \mathbb{R}$. □

It is known that if the vector field has no equilibrium points, it can be written as

$$\dot{x} = ax + by + c, \quad \dot{y} = \mu a x + \mu b y + d, \tag{2.4}$$

where a, b, c, μ and d are real constants such that $d \neq \mu c$ and $\mu \neq 0$.

The following Lemma provides a first integral for an arbitrary linear differential system without equilibrium points.

Lemma 2.2. *For system* (2.4)*, the following statements hold.*

(i) *If $a + b\mu = 0$, then system* (2.4) *is Hamiltonian and all its solutions are algebraic and given by parabolas. Moreover, this system has the first integral*

$$H_2(x,y) = b\mu^2 x^2 - 2b\mu xy - 2\,dx + by^2 + 2cy.$$

(ii) *If $a + b\mu \neq 0$, the only algebraic invariant curve of* (2.4) *is an invariant line. Moreover, this system has the first integral*

$$H_3(x,y) = \big((a+b\mu)(ax+by) + ac + bd\big)e^{\frac{a+b\mu}{d-c\mu}(\mu x - y)}.$$

Proof.

(i) Via the change of variables $x = v$, $u = \frac{1}{d-c\mu}(y - \mu x)$, where $d - c\mu \neq 0$, system (2.4) is transformed into

$$\dot{v} = (a + b\mu)v + b(d - c\mu)u + c, \quad \dot{u} = 1. \tag{2.5}$$

If $a + b\mu = 0$, the last system is Hamiltonian and it has the first integral

$$H_2(v, u) = v - \frac{b(d - c\mu)}{2}\, u^2 - cu,$$

and statement (i) follows.

(ii) If $a + b\mu \neq 0$, the general solution of (2.5) is

$$\begin{aligned} v(t) &= \frac{1}{(a + b\mu)^2}\Big((a + b\mu)^2(C_2 + e^{at+bt\mu}C_1) - ac - bd + b(c\mu - d)(a + b\mu)t\Big), \\ u(t) &= -\frac{1}{b(d - c\mu)}\big((a + b\mu)C_2 + b(c\mu - d)t\big), \end{aligned} \tag{2.6}$$

where C_1 and C_2 are real constants. So, from the second equation of (2.6), we obtain

$$t = \frac{(a + b\mu)C_2 + bu(d - c\mu)}{b(d - c\mu)}.$$

Substituting the expression of t into the first equation of (2.6), we get

$$\Big(b(d - c\mu)(a + b\mu)u + (a + b\mu)^2 v + ac + bd\Big)e^{-(a+b\mu)u} = C_1(a + b\mu)^2 e^{\frac{C_2(a+b\mu)^2}{bd-bc\mu}}.$$

Going back through the changes of variables, we obtain

$$\big((a + b\mu)(ax + by) + ac + bd\big)e^{\frac{(a+b\mu)}{d-c\mu}(\mu x - y)} = h, \tag{2.7}$$

where $h = C_1(a + b\mu)^2 e^{\frac{C_2(a+b\mu)^2}{bd-bc\mu}} \in \mathbb{R}$. From (2.7), we define a first integral of (2.4) as follows:

$$H_3(x, y) = \big((a + b\mu)(ax + by) + ac + bd\big)e^{\frac{a+b\mu}{d-c\mu}(\mu x - y)},$$

statement (ii) holds. $\square$

Suppose that we have a discontinuous piecewise linear differential system separated by Σ. We assume, without loss of generality, that the left half-system has no equilibria, neither real nor virtual, and the right half-system is of focus type at the origin. By Lemma 2.1, and using the normal form (2.4), we can write such a discontinuous piecewise linear differential system as

$$\begin{aligned} &\dot{x} = (2\lambda - \delta)x + \beta y, \qquad \dot{y} = -\frac{1}{\beta}\big((\lambda - \delta)^2 + \omega^2\big)x + \delta y \ \text{ in } \Sigma_+, \\ &\dot{x} = ax + by + c, \qquad\quad\ \dot{y} = \mu a x + \mu b y + d \ \text{ in } \Sigma_-. \end{aligned} \tag{2.8}$$

In order to state precisely our results, we introduce first some notations and definitions. Consider the piecewise differential system (2.8) defined in $\Sigma_\pm$. We use the techniques and approaches presented by Filippov in [6] and by di Bernardo et al. in [5] to establish these notations. An equilibrium point is called a real (resp. virtual) singular point of the right system of (2.8) if this point locates in the region Σ_+ (resp. Σ_-). A similar definition can be done for the left system of (2.8). Otherwise it is called a virtual equilibrium point. In order to extend the definition of a trajectory to $\Sigma = \{(x, y) \in \mathbb{R}^2 : \ x = 0\}$, we split Σ into three parts depending on whether or not the vector field points towards it:

1. Crossing region:
$$\Sigma_c = \big\{(0, y) \in \Sigma : \ \ \beta(by + c)y > 0\big\},$$

2. Attractive sliding region:

$$\Sigma_{as} = \Big\{(0,y) \in \Sigma : \quad \beta y < 0, \quad by + c > 0\Big\},$$

3. Repulsive sliding region:

$$\Sigma_{rs} = \Big\{(0,y) \in \Sigma : \quad \beta y > 0, \quad by + c < 0\Big\}.$$

These three regions are relatively open in Σ and may have several connected components. Therefore, their definitions exclude the so-called tangency points, that is, points where one of the two vector fields is tangent to Σ, which can be characterized by

$$\Big\{(0,y) \in \Sigma : \quad y = 0 \ \text{ or } \ by + c = 0\Big\}.$$

These points are on the boundary of the regions Σ_c, Σ_{as} and Σ_{rs}.

Periodic orbits that have neither sliding part nor tangent points are called crossing periodic orbits, otherwise, they are called sliding periodic orbits. We say that an isolated periodic orbit Γ is an algebraic limit cycle if all its points are contained in the level sets of polynomials. Otherwise, they are called non-algebraic limit cycles.

3 Main result

Our main result is contained in the following

Theorem 3.1. *The discontinuous piecewise linear differential system* (2.8) *may have at most two non-algebraic crossing limit cycles. Moreover, there are the systems in this class having one or two non-algebraic crossing limit cycles.*

Theorem 3.1 is proved in Section 4.

The next Propositions show that there are discontinuous piecewise linear differential systems of the form (2.8) (in case the left half-linear system of (2.8) is non-Hamiltonian) with two, or one (respectively) non-algebraic crossing limit cycles.

Proposition 3.1. *For $a = \mu+1$, $c = -1$, $d = -\mu-3$, $b = -1$, $\mu \neq 0$ and $\lambda = -\frac{1}{2}\,\omega$, the discontinuous piecewise linear differential system* (2.8) *defined by*

$$\begin{aligned} \dot{x} &= -(\omega+\delta)x + \beta y, \quad \dot{y} = -\frac{1}{\beta}\left(\delta^2 + \delta\omega + \frac{5}{4}\,\omega^2\right)x + \delta y \ \text{ in } \Sigma_+, \\ \dot{x} &= (\mu+1)x - y - 1, \quad \dot{y} = \mu(\mu+1)x - \mu y - (\mu+3) \ \text{ in } \Sigma_-, \end{aligned} \tag{3.1}$$

when $\omega > 1.7525$, $\mu \neq 0$ and $\beta < 0$, has exactly two nested crossing limit cycles. Moreover, these limit cycles are hyperbolic, non-algebraic and given by

$$\Gamma_1 = \left\{(x,y) \in \Sigma_+ : \quad \left(\left(\delta^2 + \delta\omega + \frac{5}{4}\,\omega^2\right)x^2 - \beta(2\delta+\omega)xy + \beta^2 y^2\right)e^{-\arctan\left(\frac{2\omega x}{(2\delta+\omega)x - 2\beta y}\right)} = 50.971\beta^2\right\}$$
$$\cup \left\{(x,y) \in \Sigma_- : \quad ((1+\mu)x - y + 2)e^{\frac{1}{3}\,y - \frac{\mu}{3}\,x} = 0.846\,03\right\},$$
$$\Gamma_2 = \left\{(x,y) \in \Sigma_+ : \quad \left(\left(\delta^2 + \delta\omega + \frac{5}{4}\,\omega^2\right)x^2 - \beta(2\delta+\omega)xy + \beta^2 y^2\right)e^{-\arctan\left(\frac{2\omega x}{(2\delta+\omega)x - 2\beta y}\right)} = 19.825\beta^2\right\}$$
$$\cup \left\{(x,y) \in \Sigma_- : \quad ((1+\mu)x - y + 2)e^{\frac{1}{3}\,y - \frac{\mu}{3}\,x} = 1.462\,7\right\}.$$

This proposition will be proved in Section 5.

Proposition 3.2. *For $a = \mu - 1$, $c = -3$, $d = -(3\mu + 10)$, $b = -1$, $\lambda = -\omega$ and $\mu \neq 0$, the discontinuous piecewise linear differential system* (2.8) *defined by*

$$\begin{aligned} \dot{x} &= -(2\omega + \delta)x + \beta y, \quad \dot{y} = -\frac{1}{\beta}\left((\omega+\delta)^2 + \omega^2\right)x + \delta y \ \text{ in } \Sigma_+, \\ \dot{x} &= (\mu - 1)x - y - 3, \qquad \dot{y} = \mu(\mu-1)x - \mu y - (3\mu + 10) \ \text{ in } \Sigma_-, \end{aligned} \tag{3.2}$$

when $\omega > 5.315$, $\mu \neq 0$ and $\beta < 0$, has exactly one explicit hyperbolic non-algebraic crossing limit cycle given by

$$\Gamma = \Big\{(x,y) \in \Sigma_+ : \ \Big(\big((\omega+\delta)^2 + \omega^2\big)x^2 - 2\beta(\omega+\delta)xy + \beta^2 y^2\Big)e^{-2\arctan\left(\frac{\omega x}{(\delta+\omega)x - \beta y}\right)} = 32.1\beta^2\Big\}$$
$$\cup \Big\{(x,y) \in \Sigma_- : \ (x + y - \mu x + 13)e^{\frac{\mu}{10}x - \frac{1}{10}y} = 12.925\Big\}.$$

This proposition will be proved in Section 6.

The next proposition shows that there are discontinuous piecewise linear differential systems of the form (2.8) (in case the left half-linear system of (2.8) is Hamiltonian) with one crossing non-algebraic limit cycle.

Proposition 3.3. *For $a = \mu$, $\lambda = -\frac{\omega}{2}$, $c = -3$, $b = -1$, $d = -(1 + 3\mu)$ and $\mu \neq 0$, the discontinuous piecewise linear differential system defined by*

$$\begin{aligned} \dot{x} &= -(\delta + \omega)x + \beta y, \quad \dot{y} = -\frac{1}{\beta}\Big(\delta^2 + \delta\omega + \frac{5}{4}\omega^2\Big)x + \delta y \ \text{ in } \Sigma_+, \\ \dot{x} &= \mu x - y - 3, \quad \dot{y} = \mu^2 x - \mu y - (1 + 3\mu) \ \text{ in } \Sigma_-, \end{aligned} \tag{3.3}$$

when $\omega > 0.34337$, $\mu \neq 0$ and $\beta < 0$, has exactly one explicit hyperbolic non-algebraic crossing limit cycle given by

$$\Gamma = \Big\{(x,y) \in \Sigma_+ : \ \Big(\Big(\delta^2 + \delta\omega + \frac{5}{4}\omega^2\Big)x^2 - \beta(2\delta + \omega)xy + \beta^2 y^2\Big)e^{\arctan\left(\frac{-2x\omega}{(2\delta+\omega)x - 2\beta y}\right)} = 57.375\beta^2\Big\}$$
$$\cup \Big\{(x,y) \in \Sigma_- : \ -\mu^2 x^2 + 2\mu xy + 2(1 + 3\mu)x - y^2 - 6y = -11.927\Big\}.$$

This proposition will be proved in Section 7.

Remark 3.1. The assumption $\beta < 0$ in Propositions 3.1, 3.2 and 3.3 is a necessary condition for the existence of crossing limit cycles of system (3.1) (resp. (3.2) and (3.3)). Effectively, if the crossing region of (3.1) (resp. (3.2) and (3.3)) exists with $\beta > 0$, then the inequality $y(-y-1) > 0$ (resp. $y(-y-3) > 0$) implies that the crossing region is an open interval $(-1, 0)$ (resp. $(-3, 0)$) of the line Σ. Since the right half-system is of focus type at the origin, any orbit starting at the point $(0, y_0)$ with $y_0 < 0$ goes into the left zone Σ_- under the flow of the left linear differential systems. If these orbits can reach Σ again at some point $(0, y_1)$ after some time $t > 0$, must be $y_1 > 0$ and so, the condition $\beta > 0$ precludes the existence of crossing limit cycles.

4 Proof of Theorem 3.1

Suppose that we have a discontinuous piecewise linear differential system (2.8). In order to investigate the crossing limit cycles of this system, we use the first integrals for the right and the left side systems of (2.8). Due to Lemmas 2.1 and 2.2, these first integrals are

$$H_1(x,y) = \Big(\big((\lambda-\delta)^2 + \omega^2\big)x^2 + 2\beta(\lambda-\delta)xy + \beta^2 y^2\Big)e^{-\frac{2\lambda}{\omega}\arctan\left(\frac{\omega x}{(\lambda-\delta)x + \beta y}\right)},$$
$$H_2(x,y) = \begin{cases} \big((a + b\mu)(ax + by) + ac + bd\big)e^{\frac{a+b\mu}{d-c\mu}(\mu x - y)} & \text{if } a + b\mu \neq 0, \\ b\mu^2 x^2 - 2b\mu xy - 2\,dx + by^2 + 2cy & \text{if } a + b\mu = 0 \end{cases}$$

in Σ_+ and Σ_-, respectively. Suppose that this discontinuous piecewise differential system has some limit cycles intersecting Σ at two points, namely, $(0, y_0)$ with $y_0 < 0$, and $(0, y_1)$ with $y_1 > 0$. Then the first integrals H_1 and H_2 must satisfy the following two equations:

$$\begin{aligned} H_1(0, y_0) - H_1(0, y_1) &= 0, \\ H_2(0, y_0) - H_2(0, y_1) &= 0, \end{aligned} \tag{4.1}$$

it is easy to see that the implicit form of the orbit arc of (2.8) in Σ_+ which starting at the point $(0, y_0)$, where $y_0 < 0$ when $t = 0$, is given by $H_1(x, y) - \beta^2 y_0^2 = 0$, this last orbit can be given also by the analytic curves $(x_+(t), y_+(t))$, where

$$\begin{aligned} x_+(t) &= \frac{\beta}{\omega} y_0 e^{\lambda t} \sin \omega t, \\ y_+(t) &= \frac{1}{\omega} y_0 e^{\lambda t} (\omega \cos \omega t + (\delta - \lambda) \sin \omega t). \end{aligned}$$

Denote by t_+ the minimum positive time such that $x(t_+) = x(0) = 0$, then $t_+ = \frac{\pi}{\omega}$. Since the orbits starting at the point $(0, y_0)$ go into the left zone Σ_- under the flow of the left linear differential systems and since these orbits can reach Σ again at some point $(0, y_1)$ after the time $t_+ = \frac{\pi}{\omega}$, we have

$$y_1 = y(t_+) = -y_0 e^{\frac{\lambda\pi}{\omega}},$$

which is proves that $H_1(0, y_0) - H_1(0, y_1) = 0$. Now, it is easy to see that the existence of crossing periodic solutions of discontinuous piecewise linear differential system (2.8) is equivalent to the existence of negative values of y_0 satisfying

$$H_2(0, y_0) = H_2(0, -y_0 e^{\frac{\lambda\pi}{\omega}}). \tag{4.2}$$

Here, we have to separate the proof of Theorem 3.1 in two cases.

Case 1. $a + \mu b = 0$.

In this case (4.2) becomes

$$y_0 \big(b(1 - e^{\frac{2\lambda\pi}{\omega}}) y_0 + 2c(1 + e^{\frac{\lambda\pi}{\omega}})\big) = 0. \tag{4.3}$$

It is easy to see that when $b = 0$ or $c = 0$, the unique solution of (4.3) is $y_0 = 0$. So, in this case, the discontinuous piecewise linear differential system (2.8) has no limit cycles.

When $b \neq 0$ and $c \neq 0$, equation (4.3) has two roots: $y_{01} = 0$, which cannot contribute a limit cycle and $y_0 = \frac{2c(e^{\frac{\lambda\pi}{\omega}}+1)}{b(e^{\frac{2\lambda\pi}{\omega}}-1)} \neq 0$. Moreover, we can choose the appropriate parameters b, c, λ and ω in such a way that (4.3) has exactly one real negative root $y_0 = \frac{2c(e^{\frac{\lambda\pi}{\omega}}+1)}{b(e^{\frac{2\lambda\pi}{\omega}}-1)}$, thus obtaining at most one limit cycle for the discontinuous piecewise linear differential system (2.8). Using the first integrals of both linear differential systems and knowing that the non-algebraic crossing periodic orbit passes through the point $(0, y_0)$ when $t = 0$ and through the point $(0, -y_0 e^{\frac{\lambda\pi}{\omega}})$ when $t = \frac{\pi}{\omega}$, where $y_0 = \frac{2c(e^{\frac{\lambda\pi}{\omega}}+1)}{b(e^{\frac{2\lambda\pi}{\omega}}-1)} < 0$, we get the expression

$$\begin{aligned} \Gamma = \Big\{(x, y) \in \Sigma_+ :\ \Big(\big((\lambda - \delta)^2 + \omega^2\big) x^2 + 2\beta(\lambda - \delta) xy + \beta^2 y^2\Big) e^{-\frac{2\lambda}{\omega} \arctan\left(\frac{\omega x}{(\lambda-\delta)x + \beta y}\right)} = \beta^2 y_0^2\Big\} \\ \cup \Big\{(x, y) \in \Sigma_- :\ b\mu^2 x^2 - 2b\mu xy - 2\,dx + by^2 + 2cy = (2c + by_0) y_0\Big\} \end{aligned}$$

So, Theorem 3.1 is proved in Case 1.

Case 2. $a + \mu b \neq 0$.

In this case (4.2) becomes

$$\big(ac + bd - b(a + b\mu) e^{\pi \frac{\lambda}{\omega}} y_0\big) e^{\frac{a+b\mu}{d-c\mu} y_0 e^{\frac{\lambda\pi}{\omega}}} = (b(a + b\mu) y_0 + ac + bd) e^{-\frac{a+b\mu}{d-c\mu} y_0}. \tag{4.4}$$

Then the existence of crossing periodic solutions of discontinuous piecewise linear differential system (2.8) is equivalent to the existence of zeros for equation (4.4) with respect to the variable y_0. On the other hand, this equation can be rewritten as

$$\big(ac+bd-b(a+b\mu)e^{\frac{\lambda\pi}{\omega}}y_0\big)e^{\frac{(a+b\mu)(e^{\frac{\lambda\pi}{\omega}}+1)}{d-c\mu}y_0}-b(a+b\mu)y_0-ac-bd=0.$$

For convenience, we use the notation

$$f(y)=\big(ac+bd-b(a+b\mu)e^{\frac{\lambda\pi}{\omega}}y\big)e^{\frac{(a+b\mu)(e^{\frac{\lambda\pi}{\omega}}+1)}{d-c\mu}y}-b(a+b\mu)y-ac-bd. \tag{4.5}$$

Now, solving (4.4) is equivalent to finding the solutions y_{0j} of the equation $f(y)=0$. In order to investigate a number of solutions of $f(y)=0$, and since f is a differentiable function in $\mathbb{R}$, we use the first two derivatives of the function f. Simple calculations yield

$$\begin{aligned}
f'(y)=&\ \frac{a+b\mu}{c\mu-d}\Big(be^{\frac{\lambda\pi}{\omega}}(1+e^{\frac{\lambda\pi}{\omega}})(a+b\mu)y-ac-bd-c(a+b\mu)e^{\frac{\lambda\pi}{\omega}}\Big)e^{\frac{(e^{\frac{\lambda\pi}{\omega}}+1)(a+b\mu)}{d-c\mu}y}\\
&-\frac{b(a+b\mu)(d-c\mu)}{d-c\mu},\\
f''(y)=&-\Big(be^{\pi\frac{\lambda}{\omega}}(1+e^{\pi\frac{\lambda}{\omega}})(a+b\mu)y-e^{\pi\frac{\lambda}{\omega}}(ac-bd+2bc\mu)-bd-ac\Big)(e^{\pi\frac{\lambda}{\omega}}+1)\\
&\times\frac{(a+b\mu)^2}{(d-c\mu)^2}\,e^{\frac{(e^{\frac{\lambda}{\omega}\pi}+1)(a+b\mu)}{d-c\mu}y}.
\end{aligned}$$

It is easy to see that f' and f'' are continuous functions in $\mathbb{R}$.

It is obvious that $f^{''}(y)=0$ has at most one root y_0, thus the equation $f'(y)=0$ has at most two zeros y_{0j}, $j=1,2$, and the equation $f(y)=0$ has at most three roots y_{0i}, $i=1,2,3$.

Note that the equation $f(y)=0$ has the solution $y_0=0$, which cannot contribute a limit cycle. So, in this case, the equation $f(y)=0$ may have eventually two real solutions, $y_{0j}\neq 0$ for $j=1,2$ that can provide at most 2 limit cycles for the discontinuous piecewise linear differential system (2.8). Moreover, we can choose the appropriate parameters a, b, c, d, λ, δ, μ and ω in such a way that $f(y)=0$ has exactly 2 real negative roots y_{0i}, $i=1,2$, that can provide 2 limit cycles for the discontinuous piecewise linear differential system (2.8).

Using the first integrals of both linear differential systems and knowing that the non-algebraic crossing periodic orbits pass through the points $(0,y_{0i})$ when $t=0$, and through the point $(0,-y_{0i}e^{\frac{\lambda\pi}{\omega}})$ when $t=\frac{\pi}{\omega}$, where y_{0i}, $i=1,2$, are the zeros of $f(y)=0$. Thus the expressions for these orbits are:

$$\begin{aligned}
\Gamma_i=&\Big\{(x,y)\in\Sigma_+:\ \ \Big(\big((\lambda-\delta)^2+\omega^2\big)x^2+2\beta(\lambda-\delta)xy+\beta^2y^2\Big)e^{-\frac{2\lambda}{\omega}\arctan(\frac{\omega x}{(\lambda-\delta)x+\beta y})}=\beta^2y_{0i}^2\Big\}\\
&\cup\Big\{(x,y)\in\Sigma_-:\ \big((a+b\mu)(ax+by)+ac+bd\big)e^{\frac{a+b\mu}{d-c\mu}(\mu x-y)}=\big(b(a+b\mu)y_{0i}+ac+bd\big)e^{\frac{a+b\mu}{c\mu-d}y_{0i}}\Big\}.
\end{aligned}$$

This completes the proof of Theorem 3.1 in Case 2.

Remark 4.1. The orbit arc passing through the crossing point $(0,-y_0e^{\frac{\lambda\pi}{\omega}})$ is $H_1(x,y)-\beta^2(y_0e^{\frac{\lambda\pi}{\omega}})^2=0$, this orbit, when $(\lambda-\delta)x+\beta y\neq 0$ and $((\lambda-\delta)^2+\omega^2)x^2+2\beta(\lambda-\delta)xy+\beta^2y^2\neq 0$, can be rewritten as

$$\tan\Big(\frac{-\omega}{2\lambda}\ln\frac{\beta^2y_0^2}{\big(((\lambda-\delta)^2+\omega^2)x^2+2\beta(\lambda-\delta)xy+\beta^2y^2\big)}-\pi\Big)=\frac{\omega x}{(\lambda-\delta)x+\beta y},$$

thus

$$\tan\Big(\frac{-\omega}{2\lambda}\ln\frac{\beta^2y_0^2}{\big(((\lambda-\delta)^2+\omega^2)x^2+2\beta(\lambda-\delta)xy+\beta^2y^2\big)}\Big)=\frac{\omega x}{(\lambda-\delta)x+\beta y},$$

this last equation is equivalent to

$$H_1(x,y)-\beta^2y_0^2=0,$$

and shows that $H_1(0,y_0)-H_1(0,-y_0e^{\frac{\lambda\pi}{\omega}})=0$.

5 Proof of Proposition 3.1

We prove that the discontinuous piecewise linear differential system (3.1) has exactly two hyperbolic non-algebraic limit cycles. It is easy to see that the left half-system has no equilibria, neither real nor virtual, and since $-\frac{1}{2} \pm i\omega$, $\omega > 0$ are the eigenvalues of the matrices of the right half-system of (3.1), this system has its equilibria as focus type at the origin.

The two linear differential systems of (3.1) have the following first integrals:

$$\begin{aligned} H_1(x,y) &= \Big(\Big(\delta^2 + \delta\omega + \frac{5}{4}\omega^2\Big)x^2 - \beta(2\delta+\omega)xy + \beta^2 y^2\Big)e^{-\arctan(\frac{2\omega x}{(2\delta+\omega)x - 2\beta y})}, \\ H_2(x,y) &= ((1+\mu)x - y + 2)e^{\frac{1}{3}y - \frac{\mu}{3}x} \end{aligned}$$

in Σ_+ and Σ_-, respectively. The parametric solution of the right half-system of (2.8) starting at the point $(0, y_0)$ with $y_0 < 0$ when $t = 0$, is

$$\begin{aligned} x_+(t) &= \frac{\beta}{\omega} y_0 e^{-\frac{\omega}{2}t} \sin \omega t, \\ y_+(t) &= \frac{1}{\omega} y_0 e^{-\frac{\omega}{2}t}\Big(\omega \cos \omega t + \Big(\delta + \frac{\omega}{2}\Big)\sin \omega t\Big). \end{aligned}$$

Let t_+ denote the minimum positive time such that $x(t_+) = x(0) = 0$, then $t_+ = \frac{\pi}{\omega}$. Since the orbits starting at the point $(0, y_0)$ go into the left zone Σ_- under the flow of the left linear differential systems and since these orbits can reach Σ again at some point $(0, y_1)$ after the time $t_+ = \frac{\pi}{\omega}$, we have

$$y_1 = y(t_+) = -y_0 e^{-\frac{\pi}{2}}.$$

Then, for the discontinuous piecewise linear differential system (3.1), the function (4.5) becomes

$$f(y) = (ye^{\frac{-\pi}{2}} + 2)e^{-\frac{1}{3}(e^{\frac{-\pi}{2}}+1)y} + y - 2.$$

The graphic of this function is given in Figure 5.1.

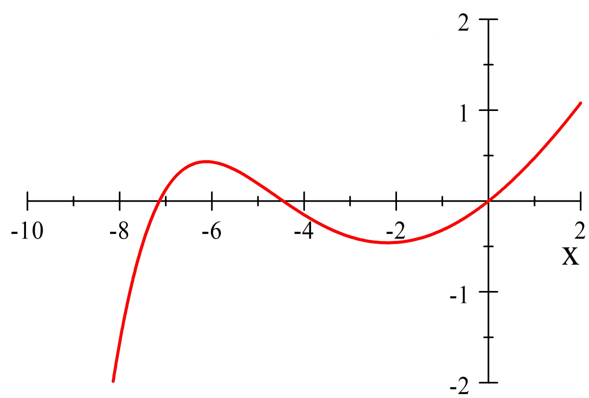

Figure 5.1. The graphic of the function $f(y)$.

The equation $f(y) = 0$ has exactly three zeros $y_{00} = 0$, $y_{01} = -4.4522$ and $y_{02} = -7.1392$. From these values of y_{0i}, $i = 0, 1, 2$, we get the values $y_{10} = 0$, $y_{11} = 0.92558$ and $y_{12} = 1.4841$.

Straightforward computations show that the solution passing through the crossing points $(0, y_{01})$ and $(0, y_{11})$ corresponds to

$$\begin{aligned} \Gamma_1 = &\Big\{(x,y) \in \Sigma_+ : \ \Big(\Big(\delta^2+\delta\omega+\frac{5}{4}\omega^2\Big)x^2 - \beta(2\delta+\omega)xy + \beta^2y^2\Big)e^{-\arctan\frac{2\omega x}{(2\delta+\omega)x-2\beta y}} = 19.825\beta^2\Big\} \\ &\cup \Big\{(x,y) \in \Sigma_- : \ ((1+\mu)x - y + 2)e^{\frac{1}{3}y - \frac{\mu}{3}x} = 1.4627\Big\}, \end{aligned}$$

and the solution passing through the crossing points $(0, y_{02})$ and $(0, y_{12})$ corresponds to

$$\Gamma_2 = \Big\{(x,y) \in \Sigma_+ : \ \Big(\Big(\delta^2 + \delta\omega + \frac{5}{4}\,\omega^2\Big)x^2 - \beta(2\delta+\omega)xy + \beta^2 y^2\Big)e^{-\arctan\frac{2\omega x}{(2\delta+\omega)x - 2\beta y}} = 50.971\beta^2\Big\}$$
$$\cup \Big\{(x,y) \in \Sigma_- : \ ((1+\mu)x - y + 2)e^{\frac{1}{3}\,y - \frac{\mu}{3}\,x} = 0.84603\Big\}.$$

Moreover, Γ_1 and Γ_2 are non-algebraic and travel in a counterclockwise sense around the sliding segment $\Sigma_{rs} = \{(0,y) \in \Sigma : \ -1 \le y \le 0\}$. Clearly, Γ_1 and Γ_2 are nested, and Γ_1 is the inner one and Γ_2 is the outer one. Now we prove that these non-algebraic crossing periodic orbits are the hyperbolic limit cycles.

Let T be the period of the periodic solution

$$\Gamma : \ \big\{(x(t), y(t)), \ t \in [0,T]\big\}.$$

To see that Γ is, in fact, a limit cycle, we recall a classic result characterizing limit cycles among other periodic orbits for a smooth differential system in the plane (see, e.g., Perko [16] for more details), which means that $\Gamma(t)$ is a hyperbolic limit cycle when

$$\int\limits_0^T \operatorname{div}(\Gamma(t)\,dt \neq 0, \tag{5.1}$$

stable if $\int\limits_0^T \operatorname{div}(\Gamma(t))\,dt < 0$, and instable if $\int\limits_0^T \operatorname{div}(\Gamma(t))\,dt > 0$.

Using the form parametric $(x_{-i}(t), y_{-i}(t))$ of the curve $H_2(x,y) = (-y_{1i} + 2)e^{\frac{1}{3}y_{1i}}$ starting at the point $(0, y_{1i})$ in the half-plane Σ_-

$$\begin{aligned} x_{-i}(t) &= y_{1i} - 3t + (2 - y_{1i})e^t - 2, \\ y_{-i}(t) &= y_{1i} - 2\mu - (3\mu+3)t + (2\mu - \mu y_{1i})e^t + \mu y_{1i}, \end{aligned}$$

where $i = 1,2$ and $y_{1i} = -y_{0i}e^{-\frac{\pi}{2}}$, it is easy to check that the periodic orbits Γ_1 and Γ_2 have periods $T_1 = 1.7926$ and $T_2 = 2.8745$, respectively.

Formula (5.1) can be extended to the discontinuous piecewise linear differential systems considered here, then for the discontinuous piecewise linear differential system, we have

$$\begin{aligned} \Gamma_1 &: \Big\{(x_{+1}(t), y_{+1}(t)), \ t \in \Big[0, \frac{\pi}{\omega}\Big]\Big\} \cup \Big\{(x_{-1}(t), y_{-1}(t)), \ t \in \Big[\frac{\pi}{\omega}, T\Big]\Big\}, \\ \Gamma_2 &: \Big\{(x_{+2}(t), y_{+2}(t)), \ t \in \Big[0, \frac{\pi}{\omega}\Big]\Big\} \cup \Big\{(x_{-2}(t), y_{-2}(t)), \ t \in \Big[\frac{\pi}{\omega}, T\Big]\Big\}, \end{aligned}$$

where

$$\begin{aligned} x_{+i}(t) &= \frac{\beta}{\omega}\,y_{0i}e^{-\frac{1}{2}\,\omega t}\sin\omega t, \\ y_{+i}(t) &= \frac{1}{\omega}\,y_{0i}e^{-\frac{1}{2}\,\omega t}\Big(\omega\cos\omega t + \Big(\delta + \frac{1}{2}\,\omega\Big)\sin\omega t\Big). \end{aligned}$$

Thus

$$\begin{aligned} \int\limits_0^{T_1} \operatorname{div}(\Gamma_1(t))\,dt &= \int\limits_0^{\frac{\pi}{\omega}} -\omega\,dt + \int\limits_{\frac{\pi}{\omega}}^{1.7926} dt = 1.7926 - \frac{\pi}{\omega} - \pi, \\ \int\limits_0^{T_2} \operatorname{div}(\Gamma_2(t))\,dt &= \int\limits_0^{\frac{\pi}{\omega}} -\omega\,dt + \int\limits_{\frac{\pi}{\omega}}^{2.8745} dt = 2.8745 - \frac{\pi}{\omega} - \pi. \end{aligned}$$

Since $\omega > 1.7525$, we have $\frac{\pi}{\omega} < 1.7926$, thus $\int\limits_0^{T_1} \operatorname{div}(\Gamma_1(t))\,dt \neq 0$ and $\int\limits_0^{T_2} \operatorname{div}(\Gamma_2(t))\,dt \neq 0$, so we obtain two hyperbolic non-algebraic crossing limit cycles.

Example 5.1. When $\mu = 2$, $\beta = -1$, $\omega = 2$ and $\delta = 1$, system (3.1) reads as

$$\begin{aligned} \dot{x} &= -3x - y, \quad \dot{y} = 8x + y \ \text{ in } \Sigma_+, \\ \dot{x} &= 3x - y - 1, \quad \dot{y} = 6x - 2y - 5 \ \text{ in } \Sigma_-. \end{aligned} \tag{5.2}$$

This system has exactly two explicit hyperbolic and non-algebraic crossing limit cycles Γ_i, $i = 1, 2$. The smallest one Γ_1 intersects the switching line Σ at two points

$$y_{01} = -4.4522, \quad y_{11} = 0.92558$$

and is given by

$$\Gamma_1 = \left\{(x,y) \in \Sigma_+ : \ (8x^2 + 4xy + y^2)e^{-\arctan\frac{2x}{2x+y}} = 19.825\right\} \\ \cup \left\{(x,y) \in \Sigma_- : \ (3x - y + 2)e^{\frac{1}{3}y - \frac{2}{3}x} = 1.4627\right\}.$$

The biggest limit cycle Γ_2 intersects the switching line Σ at two points

$$y_{02} = -7.1392, \quad y_{12} = 1.4841.$$

and the expression of this limit cycle is given by

$$\Gamma_2 = \left\{(x,y) \in \Sigma_+ : \ (8x^2 + 4xy + y^2)e^{-\arctan\frac{2x}{2x+y}} = 50.971\right\} \\ \cup \left\{(x,y) \in \Sigma_- : \ ((1+\mu)x - y + 2)e^{\frac{1}{3}y - \frac{\mu}{3}x} = 0.84603\right\}$$

(see Figure 5.2).

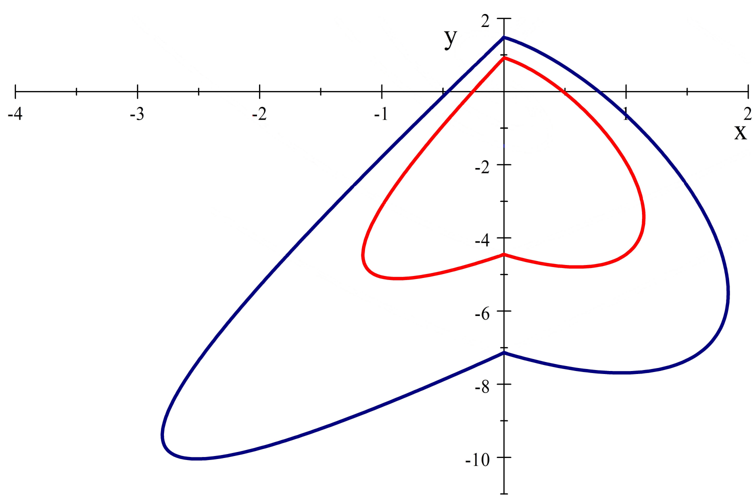

Figure 5.2. The two crossing non-algebraic limit cycles of the discontinuous piecewise linear differential systems (5.2).

6 Proof of Proposition 3.2

We consider the planar piecewise linear system (3.2), for this system it is easy to check that the left linear differential system has neither real nor virtual equilibria and the right linear differential system is a focus with eigenvalues $-1 \pm \omega i$, $\omega > 0$. In order to prove that the discontinuous piecewise linear differential system (3.2) has exactly one hyperbolic non-algebraic limit cycle, we use the first integrals for the right and the left side systems of (3.2).

The first integrals of the two linear differential systems of (3.2) are

$$H_1(x,y) = \Big((\omega^2 + (\delta+\omega)^2)x^2 - 2\beta(\delta+\omega)xy + \beta^2y^2\Big)e^{2\arctan(\frac{\omega x}{(\delta+\omega)x-\beta y})},$$
$$H_2(x,y) = (x + y - x\mu + 13)e^{\frac{\mu}{10}x - \frac{1}{10}y}$$

in Σ_+ and Σ_-, respectively. The solution $(x_+(t), y_+(t))$ of right half-system of (3.2) such that $(x_+(0), y_+(0)) = (0, y_0)$ with $y_0 < 0$ is

$$x_+(t) = \frac{\beta}{\omega}\, y_0 e^{-\omega t}\sin\omega t,$$
$$y_+(t) = \frac{1}{\omega}\, y_0 e^{-\omega t}\big(\omega\cos\omega t + (\delta-\lambda)\sin\omega t\big).$$

The time t_+ that the solution $(x_+(t), y_+(t))$ contained in Σ_+ needs to reach the point $(0, y_1)$ is $t_+ = \frac{\pi}{\omega}$. Therefore,

$$y_1 = y(t_+) = -y_0e^{-\pi}.$$

Then, for the discontinuous piecewise linear differential system (3.2), the function (4.5) becomes

$$f(y) = -(e^{-\pi}y - 13)e^{\frac{1}{10}(e^{-\pi}+1)y} - y - 13.$$

The graphic of this function is given in Figure 6.1.

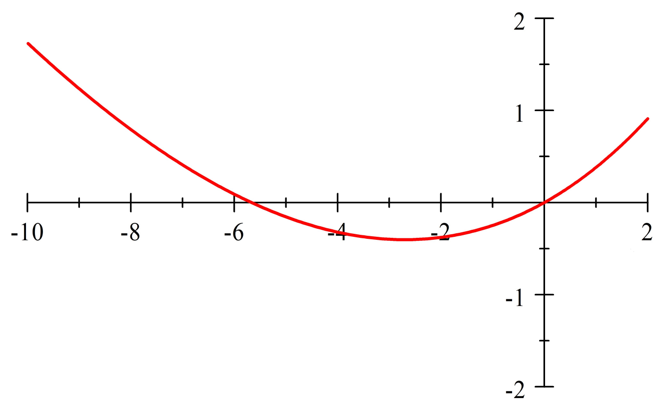

Figure 6.1. The graphic of the function $f(y)$.

The unique solution $y_0 \neq 0$ of the equation $f(y) = 0$ is $y_0 = -5.6657$. From this value of y_0, we get the value of $y_1 = 0.24484$.

Thus, the solution passing through the crossing points $(0, y_0)$ and $(0, y_1)$ corresponds to

$$\Gamma = \Big\{(x,y) \in \Sigma_+ : \ \Big(((\omega+\delta)^2 + \omega^2)x^2 - 2\beta(\omega+\delta)xy + \beta^2y^2\Big)e^{-2\arctan(\frac{\omega x}{(\delta+\omega)x-\beta y})} = 32.1\beta^2\Big\}$$
$$\cup \Big\{(x,y) \in \Sigma_- : \ (x + y - \mu x + 13)e^{\frac{\mu}{10}x - \frac{1}{10}y} = 12.925\Big\}.$$

Moreover, Γ is non-algebraic and travels in a counterclockwise sense around the sliding segment $\Sigma_{rs} = \{(0,y) \in \Sigma : \ -3 \leq y \leq 0\}$.

Using the form parametric $(x_-(t), y_-(t))$ of the curve $H_2(x,y) = (y_1 + 2)e^{\frac{-1}{10}y_1}$ starting at the point $(0, y_1)$ in the half-plane Σ_-

$$x_-(t) = 10t - y_1 + e^{-t}(y_1 + 13) - 13,$$
$$y_-(t) = y_1 - 13\mu + 10(\mu-1)t - \mu y_1 + \mu(13 + y_1)e^{-t},$$

where $y_1 = -y_0 e^{-\pi}$, it is easy to check that the periodic orbit Γ has period $T = 0.59108$.
Then, for the discontinuous piecewise linear differential system (3.2), we have

$$\Gamma: \left\{(x_+(t), y_+(t)),\ t \in \left[0, \frac{\pi}{\omega}\right]\right\} \cup \left\{(x_-(t), y_-(t)),\ t \in \left[\frac{\pi}{\omega}, T\right]\right\},$$

and

$$\int_0^T \operatorname{div}(\Gamma(t))\, dt = \int_0^{\frac{\pi}{\omega}} -2\omega\, dt - \int_{\frac{\pi}{\omega}}^{0.59108} dt = \frac{\pi}{\omega} - 2\pi - 0.59108.$$

Since $\omega > 5.315$, $\frac{\pi}{\omega} < 0.59108$ which leads to $\int_0^T \operatorname{div}(\Gamma(t))\, dt < 0$, hence the non-algebraic crossing periodic orbit Γ is a stable and hyperbolic limit cycle. This completes the proof of Proposition 3.2.

Example 6.1. When $\mu = -2$, $\beta = -1$, $\delta = 1$ and $\omega = 8$, system (3.2) reads as

$$\begin{aligned} \dot{x} &= -17x - y, \quad \dot{y} = 145x + y \ \text{ in } \Sigma_+, \\ \dot{x} &= -3x - y - 3, \quad \dot{y} = 6x + 2y - 4 \ \text{ in } \Sigma_-. \end{aligned} \tag{6.1}$$

Then, this system has exactly one explicit hyperbolic and non-algebraic crossing limit cycle Γ. This limit cycle intersects the switching line Σ at two points

$$y_0 = -5.6657, \quad y_1 = 0.24484$$

and is given by

$$\Gamma = \left\{(x, y) \in \Sigma_+ : \ (145x^2 + 18xy + y^2)e^{-2\arctan(\frac{8x}{9x+y})} = 32.1\right\} \\ \cup \left\{(x, y) \in \Sigma_- : \ (x + y + 2x + 13)e^{\frac{-1}{5}x - \frac{1}{10}y} = 12.925\right\}.$$

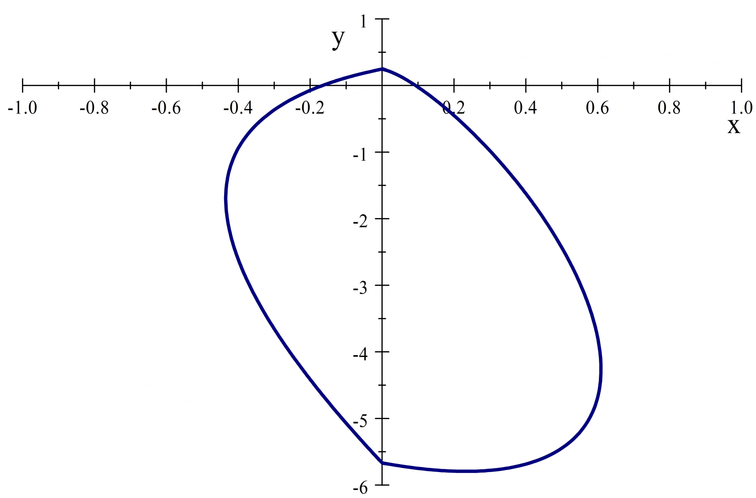

Figure 6.2. The unique crossing non-algebraic limit cycle of system (6.1).

7 Proof of Proposition 3.3

Suppose that we have a discontinuous piecewise linear differential system (3.3). It is easy to see that the left half-system is Hamiltonian without equilibrium points and, since $-\frac{1}{2} \pm i\omega$, $\omega > 0$ are the eigenvalues of the matrices of the right half-system, this system has its equilibria as focus type at the origin. In order for the piecewise linear differential system (3.3) to have exactly one hyperbolic non-algebraic limit cycle, it must intersect the discontinuous curve Σ at two points. Let $(0, y_0)$ with $y_0 < 0$, and $(0, y_1)$ with $y_1 > 0$ be two intersecting points. Then, taking into account that

$$H_1(x,y) = \Big(\Big(\delta^2 + \delta\omega + \frac{5}{4}\,\omega^2\Big)x^2 - \beta(2\delta+\omega)\Big)xy + \beta^2 y^2)e^{\arctan(\frac{-2x\omega}{(2\delta+\omega)x-2\beta y})},$$
$$H_2(x,y) = -\mu^2x^2 + 2\mu xy + 2(1+3\mu)x - y^2 - 6y$$

are first integrals of the two linear differential systems of (3.3) in Σ_+ and Σ_-, respectively, these two points satisfy equations (4.1).

The solution of the right half-system of (3.3) starting at the point $(0, y_0)$, $y_0 < 0$ when $t = 0$, is

$$x_+(t) = \frac{\beta}{\omega}\, y_0 e^{-\frac{\omega}{2}t}(\sin\omega t),$$
$$y_+(t) = \frac{1}{\omega}\, y_0 e^{-\frac{\omega}{2}t}\Big(\omega\cos\omega t + \Big(\delta + \frac{\omega}{2}\Big)\sin\omega t\Big).$$

The time t_+ that the solution $(x_+(t), y_+(t))$ contained in Σ_+ needs to reach the point $(0, y_1)$ is $t_+ = \frac{\pi}{\omega}$. Since the orbits starting at the point $(0, y_0)$ go into the left zone Σ_- under the flow of the left linear differential systems and since these orbits can reach Σ again at some point $(0, y_1)$ after the time $t_+ = \frac{\pi}{\omega}$, we have

$$y_1 = y(t_+) = -y_0 e^{-\frac{\pi}{2}}.$$

This proves that $H_1(0, y_0) - H_1(0, y_1) = 0$. Then, for the discontinuous piecewise linear differential system (3.3), equation (4.2) becomes

$$\big((e^{-\pi} - 1)y_0 - 6(1 + e^{\frac{-\pi}{2}})\big)y_0 = 0.$$

The unique solution $y_0 \neq 0$ of this last equation is

$$y_0 = \frac{6(e^{-\frac{\pi}{2}} + 1)}{e^{-\pi} - 1} = -7.5746.$$

From this value of y_0, we get the value of $y_1 = 1.5746$.
Therefore, the solution passing through the crossing points $(0, y_0)$ and $(0, y_1)$ is written as

$$\Gamma = \Big\{(x,y)\in\Sigma_+ :\ \Big(\Big(\delta^2 + \delta\omega + \frac{5}{4}\,\omega^2\Big)x^2 - \beta(2\delta+\omega)xy + \beta^2y^2\Big)e^{\arctan(\frac{-2x\omega}{(2\delta+\omega)x-2\beta y})} = 57.375\beta^2\Big\}$$
$$\cup\Big\{(x,y)\in\Sigma_- :\ -\mu^2x^2 + 2\mu xy + 2(1+3\mu)x - y^2 - 6y = -11.927\Big\}.$$

Moreover, Γ is non-algebraic and travels in a counterclockwise sense around the sliding segment $\Sigma_{rs} = \{(0,y)\in\Sigma :\ -3 \le y \le 0\}$.

Now, we prove that this non-algebraic crossing periodic orbit is a hyperbolic limit cycle. From the analytical form $(x_-(t), y_-(t))$ of the curve $H_2(x,y) = -(6+y_1)y_1$ starting at the point $(0, y_1)$ in the half-plane Σ_-, we have

$$x_-(t) = -\frac{1}{2}\,t^2 - t(y_1 + 3),$$
$$y_-(t) = \frac{1}{2}\,\mu t^2 - (3\mu+1)t + y_1,$$

where $y_1 = -y_0e^{-\pi}$, it is easy to check that the periodic orbit Γ has period $T = 9.1492$.

Then, for the discontinuous piecewise linear differential system (3.3), we have

$$\Gamma :\ \Big\{(x_+(t), y_+(t)),\ t\in\Big[0, \frac{\pi}{\omega}\Big]\Big\} \cup \Big\{(x_-(t), y_-(t)),\ t\in\Big[\frac{\pi}{\omega}, T\Big]\Big\}$$

and

$$\int_0^T \operatorname{div}(\Gamma(t))\,dt = \int_0^{\frac{\pi}{\omega}} -\omega\,dt = -\pi < 0,$$

hence, the non-algebraic crossing periodic orbit Γ is a stable and hyperbolic limit cycle. This completes the proof of Proposition 3.3.

Example 7.1. When $\beta = -1$, $\mu = -2$, $\delta = 1$ and $\omega = 1$, system (3.3) reads as

$$\begin{aligned} \dot{x} &= -2x - y, \quad \dot{y} = \frac{13}{4}x + y \ \text{ in } \Sigma_+, \\ \dot{x} &= -2x - y - 3, \quad \dot{y} = 4x + 2y + 6 \ \text{ in } \Sigma_-. \end{aligned} \tag{7.1}$$

Then, this system has exactly one explicit hyperbolic, non-algebraic crossing limit cycle Γ. This limit cycle intersects the switching line Σ at two points

$$y_0 = -7.5746, \quad y_1 = 1.5746$$

and is given by

$$\Gamma = \left\{(x,y) \in \Sigma_+ : \ \frac{1}{4}(13x^2 + 12xy + 4y^2)e^{-\arctan\left(\frac{2x}{3x+2y}\right)} = 57.375\right\} \cup \left\{(x,y) \in \Sigma_- : \ -4x^2 - 4xy - y^2 - 10x - 6y = -11.927\right\}$$

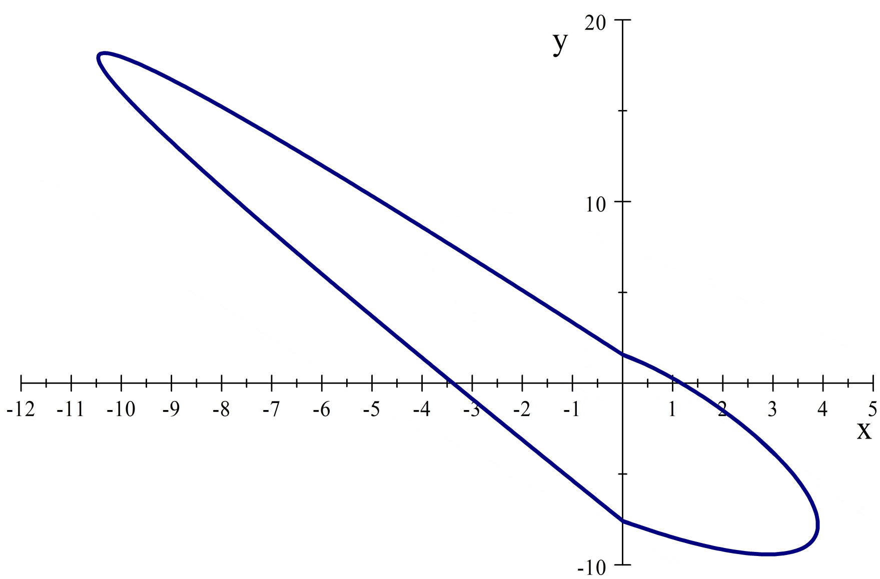

Figure 7.1. The unique crossing non-algebraic limit cycle of system (7.1).

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THE HARTLEY TRANSFORM VIA SUSY QUANTUM MECHANICS

Abstract. We present the connection between Hartley transform (HT) and a one-dimensional realization by difference-differential operator of $N = \frac{1}{2}$-supersymmetric quantum mechanics elaborated by S. Post, L. Vinet and A. Zhedanov. The key feature of our approach is that the Hartley transform commutes with the supercharge and provides the overcomplete bases of the HT eigenvectors.

2010 Mathematics Subject Classification. 37K20, 81Q60.

Key words and phrases. Special functions, supersymmetry.

რეზიუმე. წარმოგიდგენთ ჰარტლის გარდაქმნის (HT) კავშირს $N = \frac{1}{2}$ სუპერსიმეტრიული კვანტური მექანიკის დიფერენციალურ-სხვაობიანი ოპერატორის ერთგანზომილებიან რეალიზაციასთან, რომელიც შემუშავებულია ს. პოსტის, ლ. ვინეტის და ა. ჟედანოვის მიერ. ჰარტლის გარდაქმნა კომუტირებს სუპერმუხტთან და წარმოქმნის HT საკუთრივი ვექტორების ზესრულ ბაზისს - სწორედ ეს არის ჩვენი მიდგომის მთავარი დამახასიათებელი თვისება.

1 Preliminaries

The Fourier transform of a suitable function f is defined by the formula

$$(\mathcal{F}f)(\lambda) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} f(x) e^{i\lambda t}\, dx.$$

Recently, the one-dimensional harmonic oscillator has been approached by the Fourier transform method (see [9, 13, 15, 17]). Let us recall some remarks related to the Fourier transform and harmonic oscillator. In one-dimension coordinates, the representation of the creation and annihilation operators $a^\dagger$, a and the harmonic oscillator H are given by

$$a = \frac{1}{\sqrt{2}}\,(x + ip_x), \quad a^\dagger = \frac{1}{\sqrt{2}}\,(x - ip_x), \quad H = -\frac{1}{2}\,p_x^2 + \frac{1}{2}\,x^2, \quad p_x = -i\frac{d}{dx}\,. \tag{1.1}$$

They satisfy

$$[a, a^\dagger] = 1, \quad [H, a] = -a, \quad [H, a^\dagger] = a^\dagger,$$

where $[A, B] = AB - BA$ denotes the usual commutator of A and B.

The wave functions $\psi_n(x)$ of the linear harmonic oscillator,

$$\int\limits_{-\infty}^{\infty} \psi_n(x)\, \psi_m(x)\, dx = \delta_{n\,m}, \quad n, m = 0, 1, 2, \ldots,$$

are explicitly given as

$$\psi_n(x) = \left(\sqrt{\pi}\, n! 2^n\right)^{-\frac{1}{2}} e^{-x^{\frac{2}{2}}}\, H_n(x),$$

where $H_n(x)$ is the Hermite polynomial of degree n, which is orthogonal over the real line $\mathbb{R}$ with respect to the weight function $w(x) = e^{-x^2}$ [14]. In quantum mechanics, the wave functions emerge as eigenfunctions of the Hamiltonian H,

$$H\psi_n(x) = \left(n + \frac{1}{2}\right)\psi_n(x), \quad n = 0, 1, 2, \ldots. \tag{1.2}$$

The Fourier transform simply changes the basis from the coordinate basis x to the momentum basis p_x and, consequently, commutes with the harmonic oscillator H. Namely, we have

$$\mathcal{F}H = H\mathcal{F}. \tag{1.3}$$

Form (1.3) in the standard algebraic way expresses the fact that the Hamiltonian H and the Fourier transform $\mathcal{F}$ have a common set of eigenfunctions $\psi_n(x)$. More precisely, the wave functions $\psi_n(x)$ are eigenfunctions of the Fourier transform associated with the eigenvalues i^n, that is,

$$\mathcal{F}(\psi_n)(x) = i^n\, \psi_n(x).$$

The one-dimensional harmonic oscillator was also studied by Schrödinger via Laplace transform when discussing the radial eigenfunction of the hydrogen atom [19], and later, Englefield approached the Schrödinger equation with Coulomb, oscillator, exponential, and Yamaguchi potentials [10].

The fundamental purpose of the present work is to extended the integral approach of the harmonic oscillator to the setting of supersymmetric quantum mechanics "SUSY QM". Let us first recall some mathematical aspects of the supersymmetric quantum mechanics. The "SUSY QM", introduced by Witten [23], may be generated by three operators Q_-, Q_+ and H satisfying

$$Q_\pm^2 = 0, \quad [Q_\pm, H] = 0, \quad \{Q_-, Q_+\} = H,$$

with $\{A, B\} = AB + BA$ denoting the anti-commutator of A and B.

For a complete correspondence with the quantum mechanical oscillator problem, the supersymmetric quantum mechanics models need an analogue of the Fourier transformation. In the present work we fill this gap. Indeed, we propose the Hartley transform as an alternative of the Fourier transform approach to the SUSY quantum mechanics.

Recall that the Hartley transform of a suitable function $f(x)$ is defined by

$$(\mathcal{H}f)(\lambda) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x) \operatorname{cas}(\lambda x)\, dx,$$

where the kernel of the integral, known as cas function, is defined as $\operatorname{cas}(x) = \cos(x) - \sin(x)$. The relation between the Hartley transform and the Fourier transform is given by

$$(\mathcal{H}f)(\lambda) = \sqrt{2}\left(\Re((\mathcal{F}f)(\lambda)) - \Im((\mathcal{F}f)(\lambda))\right),$$

where $\Re$ and $\Im$ denote, respectively, the real and imaginary parts of the Fourier transform. Compared to the Fourier transform, the Hartley transform has the advantages of transforming real functions to real functions (as opposed to requiring complex numbers), also this transform has complementary symmetry properties with respect to their real and imaginary axis and of being its own inverse.

The paper is organized as follows. In Section 2, we recall general properties of the supersymmetric quantum mechanics with reflection. In Section 3, we give some details related to the Hartley transform and difference-differential operator. Finally, in Section 4, we develop the connection between HT and SUSY Quantum Mechanics and exploit it to obtain overcomplete bases for Hartley transform eigenvectors.

2 The Hartley transform

Our first observation in this section is the following representation of the function $\operatorname{cas}(x)$ defined in (2.2) by the power series:

$$\operatorname{cas}(x) = \sum_{n=0}^{\infty} \frac{(-1)^{\binom{n+1}{2}}}{n!} x^n, \tag{2.1}$$

where $\binom{n}{2}$ is the binomial coefficient given by

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

Theorem 2.1. *For $\lambda \in \mathbb{C}$, the function $\operatorname{cas}(\lambda x)$ is the unique analytic solution of the problem*

$$\begin{cases} (\partial_x R)u(x) = \lambda u(x), \\ u(0) = 0. \end{cases}$$

Proof. From the well known identity for binomial coefficients

$$\binom{n+1}{2} = \binom{n}{1} + \binom{n}{2} = n + \binom{n}{2},$$

we have

$$\partial_x \operatorname{cas}(\lambda x) = \lambda \sum_{n=1}^{\infty} \frac{(-1)^{\binom{n+1}{2}}}{(n-1)!} (\lambda x)^{n-1} = \lambda \sum_{n=0}^{\infty} \frac{(-1)^{\binom{n+2}{2}}}{n!} (\lambda x)^n = -\lambda \operatorname{cas}(-\lambda x).$$

Hence $(\partial_x R)u(x) = \lambda u(x)$. $\square$

Since

$$(-1)^{\binom{2n}{2}} = (-1)^n, \quad (-1)^{\binom{2n+1}{2}} = (-1)^n,$$

the sum in (2.1) turns to be

$$\operatorname{cas}(x) = \cos(x) - \sin(x). \tag{2.2}$$

The Hartley transform pair for f in a suitable functions class is given by (see [4, 12])

$$\begin{cases} (\mathcal{H}f)(\lambda) = \dfrac{1}{\sqrt{\pi}} \displaystyle\int_{\mathbb{R}} f(x) \operatorname{cas}(\lambda x)\, dx, \\ f(x) = \dfrac{1}{\sqrt{\pi}} \displaystyle\int_{\mathbb{R}} (\mathcal{H}f)(\lambda) \operatorname{cas}(\lambda x)\, d\lambda. \end{cases}$$

Accordingly,

$$\mathcal{H}^2 = I.$$

The function $\operatorname{cas}(x)$ satisfies the product formula

$$\operatorname{cas}(x)\operatorname{cas}(y) = \frac{1}{2}\left((1-R)\operatorname{cas}\right)(x+y) + \frac{1}{2}\left((1+R)\operatorname{cas}\right)(x-y).$$

This allows us to define the generalized translation operator related to the differential-difference operator ∂R by

$$\tau_y f(x) = \frac{1}{2}\left((1-R)f\right)(x+y) + \frac{1}{2}\left((1+R)f\right)(x-y),$$

and the convolution product by

$$f * g(x) = \int_{\mathbb{R}} f(y) \tau_x g(y)\, dy.$$

The Hartley transform has the following properties:

$$\mathcal{H}(\tau_x f)(\lambda) = \operatorname{cas}(\lambda x)\mathcal{H}(f)(\lambda), \quad \mathcal{H}(f*g)(\lambda) = \mathcal{H}(f)(\lambda)\mathcal{H}(g)(\lambda).$$

3 SUSY QM with reflection

Let us first recall some mathematical aspects of the supersymmetric quantum mechanics. The “SUSY QM” introduced by Witten [23] can be generated by three operators Q_-, Q_+ and H satisfying

$$Q_\pm^2 = 0, \quad [Q_\pm, H] = 0, \quad \{Q_-, Q_+\} = H \tag{3.1}$$

(with $\{A, B\} = AB + BA$ denoting the anti-commutator of A and B) to facilitate the comparison with the usual harmonic oscillator. The minimal version of $N = 1$ supersymmetric quantum mechanics is achieved by taking the supercharges Q_+ and (Q_-) as product of the bosonic operator a ($a^\dagger$) defined in (1.1) and the fermionic operator ψ ($\psi^\dagger$). Namely, we have

$$Q = a\psi^\dagger, \quad Q^\dagger = a^\dagger \psi,$$

where the matrix fermionic creation and annihilation operators are defined via

$$\psi = \sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \psi^\dagger = \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Thus, ψ and $\psi^\dagger$ obey the usual algebra of the fermionic creation and annihilation operators, namely,

$$\{\psi^\dagger, \psi\} = 1, \quad \{\psi^\dagger, \psi^\dagger\} = \{\psi, \psi\} = 0.$$

They also satisfy the commutation relation

$$[\psi^\dagger, \psi] = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The SUSY Hamiltonian can be rewritten in the form

$$H = QQ^\dagger + Q^\dagger Q = -\frac{d^2}{dx^2} + \frac{1}{4}\,x^2 - \frac{1}{2}\,[\psi, \psi^\dagger].$$

Note that if the supercharge Q in (3.1) is self-adjoint, i.e., $Q^\dagger = Q$. Then $H = 2Q^2$, and the model is said to be $N = \frac{1}{2}$ supersymmetric.

In [18], the authors developed several realizations of $N = \frac{1}{2}$ supersymmetric quantum mechanics in one-dimension by taking the supercharge as the following Dunkl-type difference-differential operator:

$$Q = \frac{1}{\sqrt{2}}\,(\partial_x R + UR + V),$$

where $U(x)$ is even, $V(x)$ is odd, and the operator R is the reflection operator which acts as $Rf(x) = f(-x)$. In this case, the SUSY Hamiltonian takes the form

$$\widehat{H} = Q^2 = -\frac{1}{2}\,\frac{d^2}{dx^2} + \frac{1}{2}\,(U^2 + V^2) + \frac{1}{2}\,\frac{dU}{dx} - \frac{1}{2}\,\frac{dV}{dx}\,R.$$

The wave functions for such systems have been obtained in [18], where it was shown that they define orthogonal polynomials, expressed in terms of Hermite and Jacobi polynomials.

Consider the supercharge

$$Q = \frac{1}{\sqrt{2}}\,(\partial_x R + x). \tag{3.2}$$

Note that this supercharge corresponds to the case $U(x) = 0$ and $V(x) = x$ in (3). Upon computing Q^2, we readily find that

$$\widehat{H} = Q^2 = -\frac{1}{2}\,\frac{d^2}{dx^2} + \frac{1}{2}\,x^2 - \frac{1}{2}\,R.$$

The spectrum of the supersymmetric Hamiltonian $\widehat{H}$ is easily obtained by observing that

$$\widehat{H} = H - \frac{1}{2}\,R,$$

where

$$H = -\frac{1}{2}\,\frac{d^2}{dx^2} + \frac{1}{2}\,x^2.$$

Since

$$R\psi_n(x) = (-1)^n \psi_n(x),$$

it follows from (1.2) that

$$\widehat{H}\psi_n = E_n \psi_n,$$

where

$$E_n = n + \frac{1 - (-1)^n}{2}, \quad n = 0, 1, \dots .$$

Therefore, the spectrum will only consist of even numbers. Each level is degenerate, except for the ground state, which is unique.

4 Eigenfunctions of the Hartley transform

Now, we are interested in finding all eigenfunctions of the Hartley transform operator explicitly. Since mutually commuting operators have the same set of eigenfunctions, one can solve this problem by defining such a self-adjoint operator with a simple spectrum of distinct eigenvalues, which commutes with the Hartley transform.

In what follows, the following lemma is needed.

Lemma 4.1. *For $\alpha, \beta \in \mathbb{R}$ such that $\alpha \neq -\beta$, the problem*

$$\begin{cases} -u'(-x) + x(u(x) - u(-x)) + \alpha u(-x) = \beta u(x), \\ u(0) = 1. \end{cases} \tag{4.1}$$

has a unique analytic solution given by

$$u(x) = {}_1F_1\left(\begin{matrix} \dfrac{\alpha^2-\beta^2}{4} \\ \dfrac{1}{2} \end{matrix}; x^2\right) + (\alpha-\beta)x\ {}_1F_1\left(\begin{matrix} \dfrac{2+\alpha^2-\beta^2}{4} \\ \dfrac{3}{2} \end{matrix}; x^2\right).$$

where

$${}_1F_1\left(\begin{matrix} a \\ b \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$$

is Kummer's confluent hypergeometric function (*see* [14]).

Proof. Note that one can always write u as the superposition $u = u_e + u_o$ of an even function u_e and of an odd function u_o by the formulae

$$u_e(x) = \frac{u(x) + u(-x)}{2}, \quad u_o(x) = \frac{u(x) - u(-x)}{2}.$$

Further, this decomposition is unique. This allows us to rewrite the eigenvalue equation (4.1) equivalently as a system of two linear differential equations of first order:

$$\begin{cases} u_e' = (\alpha+\beta)u_o, \\ u_o' - 2xu_o = -(\alpha-\beta)u_e. \end{cases} \tag{4.2}$$

We can eliminate the function $u_o(x)$ from system (4.2) and obtain for $u_e(x)$ a second-order differential equation

$$u_e''(x) - 2xu_e'(x) = -(\alpha^2-\beta^2)u_e(x). \tag{4.3}$$

We choose $t = x^2$ as a new variable and reduce equation (4.3) to

$$tv'' + \left(\frac{1}{2} - t\right)v' = -\frac{\alpha^2-\beta^2}{4} w,$$

so that the general solution of (4.3) can be written in the form

$$u_e(x) = A\ {}_1F_1\left(\begin{matrix} \dfrac{\alpha^2-\beta^2}{4} \\ \dfrac{1}{2} \end{matrix}; x^2\right) + Bx\ {}_1F_1\left(\begin{matrix} \dfrac{2+\alpha^2-\beta^2}{4} \\ \dfrac{3}{2} \end{matrix}; x^2\right),$$

where A and B are constants depending on λ, α and β. Since the function $u_e(x)$ is even, we have

$$u_e(x) = A\ {}_1F_1\left(\begin{matrix} \dfrac{\alpha^2-\beta^2}{4} \\ \dfrac{1}{2} \end{matrix}; x^2\right).$$

From (4.2), for the function $u_o(x)$ we obtain

$$u_o(x) = A\,\frac{\alpha-\beta}{2}x\;{}_1F_1\left(\begin{matrix}1+\dfrac{\alpha^2-\beta^2}{4}\\ \dfrac{3}{2}\end{matrix};x^2\right).$$

We have the general solution of (4.5)

$$u(x) = A\;{}_1F_1\left(\begin{matrix}\dfrac{\alpha^2-\beta^2}{4}\\ \dfrac{1}{2}\end{matrix};x^2\right) + A(\alpha-\beta)x\;{}_1F_1\left(\begin{matrix}1-\dfrac{\alpha^2-\beta^2}{4}\\ \dfrac{3}{2}\end{matrix};x^2\right).$$

From the initial condition in (4.1), we get $A = 1$. $\square$

The following theorem states that the Hartley transform commutes with the supercharge Q defined in (3.2).

Theorem 4.2. *We have*

$$\mathcal{H}Q = Q\mathcal{H}.$$

Proof. Using integration by parts, we can show that the Hartley transform satisfies the following intertwining relations:

$$\mathcal{H}R = R\mathcal{H}, \quad \mathcal{H}\partial_x R = x\mathcal{H}, \quad \mathcal{H}x = \partial_x R\mathcal{H}.$$

The two last intertwining relations provide the proof of the theorem. $\square$

The ground state wave function $\psi_0(x)$ is given by $\psi_0(x) = e^{-x^{\frac{2}{2}}}$ and satisfies $Q\psi_0 = 0$. Let us now carry out the gauge transformation of Q with the ground state ψ_0. Let

$$\widetilde{Q} = \psi_0^{-1}Q\psi_0. \tag{4.4}$$

It is not difficult to see that

$$\widetilde{Q} = \frac{1}{\sqrt{2}}\frac{d}{dx}R + \frac{1}{\sqrt{2}}x(1-R).$$

From Theorem 4.2, we see that the eigenfunctions of the Hartley transform can be obtained by finding the eigenvalues of the supercharge Q. So, in this way, one reduces the problem of funding the eigenfunctions of the Hartley transform into one of solving the following difference-differential equation

$$-u'(-x) + x(u(x) - u(-x)) = \sqrt{2}\,\lambda u(x). \tag{4.5}$$

From Lemma (4.1), the general solution of (4.5) is given by

$$u(x) = A\left({}_1F_1\left(\begin{matrix}-\dfrac{\lambda^2}{2}\\ \dfrac{1}{2}\end{matrix};x^2\right) - \sqrt{2}\,\lambda x\;{}_1F_1\left(\begin{matrix}1-\dfrac{\lambda^2}{2}\\ \dfrac{3}{2}\end{matrix};x^2\right)\right). \tag{4.6}$$

It can be is easily seen that polynomial solutions are possible only if $\lambda = \pm\sqrt{2n}$, $n = 0, 1, 2, \dots$. If $\lambda = \pm\sqrt{2n}$, then the first term in (4.6) is a polynomial of degree $2n$ and the second term is a polynomial of degree $2n-1$.

Let us by $\widehat{\psi}_{\pm,n}(x)$ denote the eigenfunction of Q corresponding to the eigenvalue $\lambda_n = \pm\sqrt{2n}$. Then we have the following explicit expressions:

$$\widehat{\psi}_{\pm n}(x) = \kappa_n^{\pm}e^{-x^{\frac{2}{2}}}\left({}_1F_1\left(\begin{matrix}-n\\ \dfrac{1}{2}\end{matrix};x^2\right) \pm 2\sqrt{n}\,x\;{}_1F_1\left(\begin{matrix}1-n\\ \dfrac{3}{2}\end{matrix};x^2\right)\right).$$

The normalized constants $\kappa_{\pm n}$ are also chosen so that

$$\int_{-\infty}^{\infty} |\widehat{\psi}_{\pm n}| \, ds = 1.$$

A simple computation shows that $\kappa_n^{-1} = \kappa_{-n}^{-1} = \pi^{\frac{1}{4}} 2^{n+\frac{1}{2}} (2n!)^{-\frac{1}{2}} n!$, $n = 0, 1, 2, \dots$. We denote by $\widehat{H}_n(x)$, $n \in \mathbb{Z}$, the orthogonal polynomial extracts that form the orthogonal function $\widehat{\psi}_{\pm n}(x)$. That is,

$$\widehat{\psi}_n(x) = \kappa_n e^{-x^{\frac{2}{2}}} \widehat{H}_n(x).$$

Using the well known explicit expressions of Hermite polynomials in terms of the Confluent hypergeometric series

$$H_{2n}(x) = (-1)^n \frac{(2n)!}{n!} \, {}_1F_1 \begin{pmatrix} -n \\ \frac{1}{2} \end{pmatrix} ; x^2 \Bigg) ,$$

$$H_{2n+1}(x) = (-1)^n \frac{(2n+1)!}{n!} 2x \, {}_1F_1 \begin{pmatrix} -n \\ \frac{3}{2} \end{pmatrix} ; x^2 \Bigg) ,$$

we obtain

$$\widehat{H}_{\pm n}(x) = \frac{(-1)^n n!}{(2n)!} \Big(H_{2n}(x) \mp 2\sqrt{n}\, H_{2n-1}(x) \Big), \quad n = 0, 1, 2, \dots .$$

They satisfy the orthogonality relations

$$\int_{\mathbb{R}} \widehat{H}_n(x) \widehat{H}_m(x) e^{-x^2} \, dx = \sqrt{\pi}\, 2^{2|n|+1} \frac{(|n|!)^2}{(2|n|)!} \delta_{nm}, \quad n, m \in \mathbb{Z}.$$

The system $\{\widehat{\psi}_{\pm n}(x)\}_{n \in \mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{R}, dx)$ and it is complete by the same argument which was used to prove that the classical Hermite functions form a complete orthogonal set in $L^2(\mathbb{R}, dx)$. Further, the operator Q with domain $D(Q) = \mathcal{S}(\mathbb{R})$ ($\mathcal{S}(\mathbb{R})$ is the Schwartz space) is essentially self-adjoint; the spectrum of its closure is discrete and, by (4.4), we easily obtain that

$$Q\widehat{\psi}_{\pm n}(x) = \pm\sqrt{2n}\, \widehat{\psi}_{\pm n}(x), \quad n = 0, 1, 2, \dots .$$

Theorem 4.3. *For $n \in \mathbb{Z}$, we have*

$$\int_{-\infty}^{\infty} \operatorname{cas}(xy) \widehat{H}_n(x) e^{-x^{\frac{2}{2}}} \, dx = (-1)^n \widehat{H}_n(x) e^{-x^{\frac{2}{2}}} .$$

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A BICUBIC SPLINES METHOD FOR SOLVING A TWO-DIMENSIONAL OBSTACLE PROBLEM

Abstract. The objective of this paper is to develop a numerical method for solving a bidimensional unilateral obstacle problem. This is based on the bicubic splines collocation method and the generalized Newton method. In this paper, we obtain an approximate expression for solving a bidimensional unilateral obstacle problem. We show that the approximate formula obtained by the bicubic splines collocation method is effective. Next, we prove the convergence of the proposed method. The method is applied to some test examples and the numerical results have been compared with the exact solutions. The obtained results show the computational efficiency of the method. It can be concluded that computational efficiency of the method is effective for the two-dimensional obstacle problem.

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რეზიუმე. ნაშრომის მიზანია რიცხვითი მეთოდის შემუშავება ორგანზომილებიანი ცალმხრივი დაბრკოლების ამოცანის ამოსახსნელად. იგი ეფუძნება ბიკუბური სპლაინების კოლოკაციის მეთოდს და ნიუტონის განზოგადებულ მეთოდს. ნაშრომში მიღებულია მიახლოებითი გამოსახულება ორგანზომილებიანი ცალმხრივი დაბრკოლების პრობლემის გადასაჭრელად. ვაჩვენებთ, რომ მიახლოებითი ფორმულა, რომელიც მიღებულია ბიკუბური სპლაინების კოლოკაციის მეთოდით, ეფექტურია. შემდეგ დავამტკიცებთ შემოთავაზებული მეთოდის კრებადობას. მეთოდი გამოყენებულია ზოგიერთ სატესტო მაგალითზე და რიცხვითი შედეგები შედარებულია ზუსტ ამოხსნებთან. მიღებული შედეგები აჩვენებს მეთოდის გამოთვლით ეფექტურობას. შეიძლება დავასკვნათ, რომ მეთოდის გამოთვლითი ეფექტურობა ძალაშია ორგანზომილებიანი ამოცანებისთვის დაბრკოლებით.

1 Introduction

In this paper, we consider the following unilateral obstacle problem:

$$\text{Find } u \in K \text{ such that } \int_\Omega \nabla u \cdot \nabla(v-u)\,dx + \int_\Omega f(v-u)\,dx \geq 0, \quad \forall\, v \in K, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open domain with $n \geq 2$, with a smooth boundary $\partial\Omega$, f is an element of $L^2(\Omega)$ and $K = \{v \in H_0^1(\Omega) \mid \ v \geq \psi \text{ a.e. in } \Omega\}$. The main point here is that we are considering an irregular obstacle function ψ which is an element of $H^1(\Omega)$ with $\psi \leq 0$ on $\partial\Omega$. It is well known that problem (1.1) admits a unique solution u, and if $\Delta\psi \in L^2(\Omega)$, then u is an element of $H^2(\Omega)$ (see [10,14]), and the solution u of problem (1.1) is an element of $H^2(\Omega)$ that can be characterized as (see [10], for instance)

$$\begin{cases} -\Delta u + f \geq 0 & \text{a.e. on } \Omega, \\ (-\Delta u + f)(u-\psi) = 0 & \text{a.e. on } \Omega, \\ u - \psi \geq 0 & \text{a.e. on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

As a classical subject in the field of partial differential equations, the obstacle problem is aimed to find a solution which is constrained by a given obstacle to some extent. It has numerous applications in various fields including economics, engineering, biology, computer science, etc. There are several numerical solution methods of the obstacle problem (see, e.g., [1,6,9–11,13,17,26]). Numerical solution by penalty methods have been considered, e.g., in [9,24]. In this paper, we develop a numerical method for solving a two-dimensional obstacle problem by using the generalized tension splines collocation method and the generalized Newton method. First, problem (1.1) is approximated by a sequence of nonlinear equation problems by using the penalty method given in [14, 16]. Then we apply the GB-spline collocation method to approximate the solution of a boundary value problem of second order. The discret problem is formulated as to find the generalized tension splines coefficients of a nonsmooth system $\varphi(Y) = Y$, where $\varphi : \mathbb{R}^m \to \mathbb{R}^m$. In order to solve the nonsmooth equation, we apply the generalized Newton method (see, e.g., [4,5,25]). We prove that the generalized tension splines collocation method converges quadratically provided a property, coupling the penalty parameter ε and the discretization parameter h is satisfied.

Numerical methods to approximate the solution of boundary value problems have been considered by several authors. We only mention the papers [3, 15] and the references therein, which use the bicubic spline collocation method for solving the boundary value problems.

The present paper is organized as follows. In Section 2, we present the penalty method to approximate the obstacle problem by a sequence of second order boundary value problems, we also construct a bicubic spline to approximate the solution of the boundary problem, and we present the generalized Newton method. In Section 3, we show the convergence of the generalized tension spline to the solution of the boundary problem and provide an error estimate. Some numerical results are given in Section 4 to validate our methodology. The study ends with conclusions and remarks in Section 5.

2 Bicubic spline collocation method

In this section, we construct a bicubic spline which approximates the solution u_ε of problem (2.1), with Ω being the interval $I \times J = (a,b)^2 \subset \mathbb{R}^2$. We denote by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^{(n+1)(n+1)}$, by $\|\cdot\|_\infty$ the uniform norm, by $\otimes$ Kronecker product (tensor product) and by $\odot$ the biproduct of matrices.

By using the penalty method (see [14, p. 110], [16]), an approximate solution u_ε of problem (1.1) can be characterized as the following boundary value problem (see [14, p. 107], [16]):

$$\begin{cases} -\Delta u_\varepsilon = \max(-\Delta\psi + f, 0)\theta_\varepsilon(u_\varepsilon - \psi) - f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where θ_ε is a sequence of Lipschitz functions which tend to the function θ defined by

$$\theta_\varepsilon(t) = \begin{cases} 1, & t \le 0, \\ 1 - \dfrac{t}{\varepsilon}, & 0 \le t \le \varepsilon, \\ 0, & t \ge \varepsilon. \end{cases} \tag{2.2}$$

If we put

$$J_\varepsilon(x, y, u_\varepsilon(x, y)) = \max(-\Delta\psi(x, y) + f(x, y), 0)\Theta_\varepsilon$$

with

$$\Theta_\varepsilon = \theta_\varepsilon(u_\varepsilon(x, y) - \psi(x, y)) - f(x, y),$$

then problem (2.1) becomes

$$\begin{cases} -\Delta u_\varepsilon = J_\varepsilon(\,\cdot\,, u_\varepsilon) & \text{on } \Omega, \\ u_\varepsilon(a, y) = u_\varepsilon(x, b) = 0, & x, y \in (a, b). \end{cases} \tag{2.3}$$

It is easy to see that J_ε is a nonlinear continuous function on u_ε; and for any two functions u_ε and v_ε, J_ε satisfies the following Lipschitz condition:

$$\big|J_\varepsilon(x, y, u_\varepsilon(x, y)) - J_\varepsilon(x, y, v_\varepsilon(x, y))\big| \le L_\varepsilon |u_\varepsilon(x, y) - v_\varepsilon(x, y)| \ \text{ a.e. on } (x, y) \in \Omega, \tag{2.4}$$

where

$$L_\varepsilon = \frac{1}{\varepsilon}\, \| - \Delta\psi + f\|_\infty = \frac{1}{\varepsilon} \max_{(x,y)\in\Omega} | - \Delta\psi(x, y) + f(x, y)|.$$

Now, let

$$\Pi_x = \big\{a = x_{-3} = \cdots = x_0 < x_1 < \cdots < x_{n+1} = \cdots = x_{n+3} = b\big\},$$
$$\Pi_y = \big\{a = y_{-3} = \cdots = y_0 < y_1 < \cdots < y_{n+1} = \cdots = y_{n+3} = b\big\}$$

be the subdivisions of the intervals I and J, respectively, with $x_i = a + ih$ and $y_j = a + jh$, where $0 \le i, j \le n$ and $h = (b - a)/n$. The partition $\Pi_{xy} = \Pi_x \otimes \Pi_y$ subdivides Ω into smaller rectangles in the plane:

$$T = \Big\{(x, y) : \ x_i \le x \le x_{i+1},\ y_j \le y \le y_{j+1},\ i, j = -3, \ldots, n - 1\Big\}.$$

Denote by

$$S_4^{bicu}(\Omega, \Pi_{xy}) = S_4^{cub}(I, \Pi_x) \otimes S_4^{cub}(J, \Pi_y)$$

a bicubic spline with respect to the partition Π_{xy} with $S_4^{cub}(I, \Pi_x)$ (resp. $S_4^{cub}(J, \Pi_y)$), the space of piecewise polynomials of degree 3 over the subdivision Π_x (resp. Π_y) and of class $\mathcal{C}^2$ everywhere on I (resp. J).

Moreover, let $\{B^x_{-3}, B^x_{-2}, \ldots, B^x_{n-1}\}$ (resp. $\{B^y_{-3}, \ldots, B^y_{n-1}\}$) be a B-spline basis of $S_4^{cub}(I, \Pi_x)$ (resp. $S_4^{cub}(J, \Pi_y)$). By applying the tensor product method (see [19]), we obtain the following bicubic spline interpolation.

Proposition 2.1 (see [19])**.** *Let u_ϵ be a solution of problem* (2.3)*. Then there exists a unique bicubic spline interpolant $S_\epsilon \in S_4^{bicu}(\Omega, \Pi_{xy})$ of u_ϵ which satisfies*

$$S_\epsilon(\tau_i^x, \tau_j^y) = u_\epsilon(\tau_i^x, \tau_j^y), \ \ i, j = 0, \ldots, n + 2,$$

where

$$\tau_0^x = x_0,\ \ \tau_i^x = \frac{x_i + x_{i-1}}{2},\ \ 1 \le i \le n,\ \ \tau_{n+1}^x = x_{n-1},\ \ \tau_{n+2}^x = x_n,$$
$$\tau_0^y = y_0,\ \ \tau_j^y = \frac{y_j + y_{j-1}}{2},\ \ 1 \le j \le n,\ \ \tau_{n+1}^y = y_{n-1},\ \ \tau_{n+2}^y = y_n.$$

If we put

$$S_\varepsilon(x,y) = \sum_{p,q=-3}^{n-1} c_{p,q,\varepsilon} B_p^x(x) B_q^y(y),$$

then by using the boundary conditions of problem (2.3) we obtain

$$c_{-3,q,\varepsilon} = S_\varepsilon(a,y) = u_\varepsilon(a,y) = 0, \quad q = -3, \dots, n-1,$$

and

$$c_{p,n-1,\varepsilon} = S_\varepsilon(x,b) = u_\varepsilon(x,b) = 0, \quad p = -3, \dots, n-1.$$

Hence

$$S_\varepsilon(x,y) = \sum_{p,q=-2}^{n-2} c_{p,q,\varepsilon} B_p^x(x) B_q^y(y).$$

Furthermore, for any $u_\varepsilon \in H^4(\Omega)$, where $H^4(\Omega) = \{u \in L^2(\Omega); \partial^\alpha u \in L^2(\Omega), |\alpha| \leq 4\}$ is the Sobolev space (see [8]), we have

$$-\Delta S_\varepsilon(\tau_i^x, \tau_j^y) = J_\varepsilon(\tau_i^x, \tau_j^y, u_\varepsilon) + O(1), \quad i,j = 1, \dots, n+1. \tag{2.5}$$

The bicubic spline collocation method, presented in this paper, constructs numerically a bicubic spline $\widetilde{S}_\varepsilon = \sum\limits_{p,q=-3}^{n-1} \widetilde{c}_{p,q,\varepsilon} B_p^x B_q^y$ which satisfies equation (2.3) at the points (τ_i^x, τ_j^y), $i,j = 0, \dots, n+2$. It is easy to see that

$$\widetilde{c}_{-3,q,\varepsilon} = \widetilde{c}_{p,n-1,\varepsilon} = 0 \ \text{ for } \ p,q = -3, \dots, n-1$$

and the coefficients $\widetilde{c}_{p,q,\varepsilon}$, $p,q = -2, \dots, n-2$, satisfy the following nonlinear system with $(n+1)^2$ equations:

$$\sum_{p,q=-2}^{n-2} \widetilde{c}_{p,q,\varepsilon} \Delta B_p^x(\tau_i^x) B_q^y(\tau_j^y) = -J_\varepsilon(\tau_i^x, \tau_j^y, \sum_{p,q=-2}^{n-2} \widetilde{c}_{p,q,\varepsilon} B_p^x(\tau_i^x) B_q^y(\tau_j^y)) \ \text{ for } \ i,j = 1, \dots, n+1. \tag{2.6}$$

Since

$$\Delta B_p^x(\tau_i^x) B_q^y(\tau_j^y) = B_p^x(\tau_i^x) \Delta B_q^y(\tau_j^y) + B_q^y(\tau_j^y) \Delta B_p^x(\tau_i^x),$$

relations (2.5) and (2.6) can be written in the matrix form, respectively, as follows:

$$\begin{aligned} 2(A_h \odot B_h) C_\varepsilon &= -F_\varepsilon - \widehat{E}_\varepsilon, \\ 2(A_h \odot B_h) \widetilde{C}_\varepsilon &= -F_{\widetilde{C}_\varepsilon}, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} A_h \odot B_h &= \frac{1}{2} \left(A_h \otimes B_h + B_h \otimes A_h \right), \\ C_\varepsilon &= \left[(c_{-2,q,\varepsilon})_{-2 \leq q \leq n-2}, \dots, (c_{n-2,q,\varepsilon})_{-2 \leq q \leq n-2} \right]^T, \\ \widetilde{C}_\varepsilon &= \left[(\widetilde{c}_{-2,q,\varepsilon})_{-2 \leq q \leq n-2}, \dots, (\widetilde{c}_{n-2,q,\varepsilon})_{-2 \leq q \leq n-2} \right]^T, \end{aligned}$$

for any integer i such that $1 \leq i \leq n+1$,

$$\begin{aligned} F_\varepsilon &= \Big[J_\varepsilon\big(\tau_i^x, \tau_1^y, u_\varepsilon(\tau_i^x, \tau_1^y)\big), \dots, J_\varepsilon\big(\tau_i^x, \tau_{n+1}^y, u_\varepsilon(\tau_i^x, \tau_{n+1}^y)\big) \Big]^T, \\ F_{\widetilde{C}_\varepsilon} &= \Big[J_\varepsilon\big(\tau_i^x, \tau_1^y, \widetilde{S}_\varepsilon(\tau_i^x, \tau_1^y)\big), \dots, J_\varepsilon\big(\tau_i^x, \tau_{n+1}^y, \widetilde{S}_\varepsilon(\tau_i^x, \tau_{n+1}^y)\big) \Big]^T, \end{aligned}$$

and $\widehat{E}_\varepsilon$ is a vector, where each component is of order $O(1)$. It is well known that $A_h = \frac{1}{h^2} A$ and $B_h = B$, where A and B are the matrices independent of h given as follows:

$$A = \begin{bmatrix} \frac{-15}{4} & \frac{1}{4} & \frac{1}{2} & 0 & \dots & & & 0 \\ \frac{3}{4} & \frac{-3}{4} & \frac{-1}{2} & \frac{1}{2} & 0 & \dots & & 0 \\ 0 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & \dots & & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{-3}{4} & \frac{3}{4} \\ 0 & \dots & & & 0 & \frac{1}{2} & \frac{1}{4} & \frac{-15}{4} \\ 0 & \dots & & & 0 & 1 & \frac{-5}{2} & \frac{3}{2} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{57}{96} & \frac{25}{96} & \frac{1}{48} & 0 & \dots & & & 0 \\ \frac{3}{96} & \frac{45}{96} & \frac{23}{48} & \frac{1}{48} & 0 & \dots & & 0 \\ 0 & \frac{1}{48} & \frac{23}{48} & \frac{45}{96} & \frac{3}{96} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{1}{48} & \frac{23}{48} & \frac{45}{96} & \frac{3}{96} & 0 \\ 0 & \dots & & & 0 & \frac{1}{48} & \frac{25}{96} & \frac{57}{96} \\ 0 & \dots & & & 0 & \frac{1}{6} & \frac{7}{12} & \frac{1}{4} \end{bmatrix}.$$

Then relation (2.7) becomes

$$\begin{aligned} (A \odot B)C_\varepsilon &= -\frac{1}{2} h^4 F_\varepsilon - E_\varepsilon, \\ (A \odot B)\widetilde{C}_\varepsilon &= -\frac{1}{2} h^2 F_{\widetilde{C}_\varepsilon} \end{aligned} \tag{2.8}$$

with E_ε being a vector, where each of its components is of order $O(h^2)$.

As the matrices A and B are invertible (see [18]), then $A \odot B$ is invertible (see [12]) and

$$(A \odot B)^{-1} = A^{-1} \odot B^{-1}. \tag{2.9}$$

Proposition 2.2. *Assume that the penalty parameter ε and the discretization parameter h satisfy the following relation:*

$$h^2 \| - \Delta\psi + f\|_\infty \|A^{-1} \odot B^{-1}\|_\infty < 2\varepsilon. \tag{2.10}$$

Then there exists a unique bicubic spline which approximates the exact solution u_ε of problem (2.3).

Proof. From relation (2.8), we have

$$\widetilde{C}_\varepsilon = -\frac{1}{2}\, h^2 A^{-1} \odot B^{-1} F_{\widetilde{C}_\varepsilon}.$$

Let $\varphi : \mathbb{R}^{(n+1)(n+1)} \to \mathbb{R}^{(n+1)(n+1)}$ be a function defined by

$$\varphi(Y) = -\frac{1}{2}\, h^2 A^{-1} \odot B^{-1} F_Y.$$

To prove the existence of bicubic spline collocation, it suffices to prove that φ admits a unique fixed point. Indeed, let Y_1 and Y_2 be two vectors of $\mathbb{R}^{(n+1)(n+1)}$. Then we have

$$\|\varphi(Y_1) - \varphi(Y_2)\| \le \frac{1}{2}\, h^2 \|A^{-1} \odot B^{-1}\|_\infty \|F_{Y_1} - F_{Y_2}\|_\infty. \tag{2.11}$$

Using relation (2.4) and the fact that $\sum\limits_{p,q=-2}^{n-2} B_p^x B_q^y \le 1$, we get

$$\begin{aligned}\Big| J_\varepsilon\big(\tau_i^x, \tau_j^y, S_{Y_1}(\tau_i^x, \tau_j^y)\big) - J_\varepsilon\big(\tau_i^x, \tau_j^y, S_{Y_2}(\tau_i^x, \tau_j^y)\big)\Big| \\ \le L_\varepsilon \big| S_{Y_1}(\tau_i^x, \tau_j^y) - S_{Y_2}(\tau_i^x, \tau_j^y)\big| \le L_\varepsilon \|Y_1 - Y_2\|_\infty,\end{aligned}$$

where $L_\varepsilon = \frac{1}{\varepsilon}\, \| - \Delta\psi + f\|_\infty$. Then we obtain

$$\|F_{Y_1} - F_{Y_2}\|_\infty \le L_\varepsilon \|Y_1 - Y_2\|_\infty.$$

From relation (2.11), we conclude that

$$\|\varphi(Y_1) - \varphi(Y_2)\| \le L_\varepsilon\, \frac{1}{2}\, h^2 \|A^{-1} \odot B^{-1}\|_\infty \|Y_1 - Y_2\|_\infty.$$

Thus we have

$$\|\varphi(Y_1) - \varphi(Y_2)\| \le k \|Y_1 - Y_2\|_\infty,$$

with $k = \frac{1}{2}\, h^2 \|A^{-1} \odot B^{-1}\|_\infty$, by relation (2.10). Hence the function φ admits a unique fixed point. □

In order to calculate the coefficients of the generalized tension spline collocation given by the nonsmooth system

$$\widetilde{C_\varepsilon} = \varphi(\widetilde{C}_\varepsilon),$$

we propose the generalized Newton method defined by

$$\widetilde{C}_\varepsilon^{(k+1)} = \widetilde{C}_\varepsilon^{(k)} - (I_{n+1} - V_k)^{-1}\big(\widetilde{C}_\varepsilon^{(k)} - \varphi(\widetilde{C}_\varepsilon^{(k)})\big),$$

where $I_{(n+1)(n+1)}$ is the unit matrix of order $(n+1)(n+1)$ and V_k is the generalized Jacobian of the function $\widetilde{C}_\varepsilon \mapsto \varphi(\widetilde{C}_\varepsilon)$ (see, e.g., [4, 5, 25]).

3 Convergence of the method

Theorem 3.1. *If we assume that the penalty parameter ε and the discretization parameter h satisfy the relation*

$$h^2 \| - \Delta\psi + f\|_\infty \|A^{-1} \odot B^{-1}\|_\infty < \varepsilon. \tag{3.1}$$

then the bicubic spline $\widetilde{S}_\varepsilon$ converges to the solution u_ε. Moreover, the error estimate $\|u_\varepsilon - \widetilde{S}_\varepsilon\|_\infty$ is of order $O(h^2)$.

Proof. From (2.8) and (2.9), we have

$$C_\varepsilon - \widetilde{C}_\varepsilon = -\frac{1}{2}\,h^4 A^{-1} \odot B^{-1}(F_\varepsilon - F_{\widetilde{C}_\varepsilon}) - A^{-1} \odot B^{-1} E_\varepsilon.$$

Since E_ε is of order $O(h^2)$, there exists a constant K_1 such that $\|E_\varepsilon\|_\infty \leq k_1 h^2$. Hence, we get

$$\|C_\varepsilon - \widetilde{C}_\varepsilon\|_\infty \leq \frac{1}{2}\,h^2 \|A^{-1} \odot B^{-1}\|_\infty \|F_\varepsilon - F_{\widetilde{C}_\varepsilon}\|_\infty + K_1 \|A^{-1} \odot B^{-1}\|_\infty h^2. \tag{3.2}$$

On the other hand, we have

$$\begin{aligned}&\Big|J_\varepsilon\big(\tau_i^x, \tau_j^y, u_\varepsilon(\tau_i^x, \tau_j^y)\big) - J_\varepsilon\big(\tau_i^x, \tau_j^y, \widetilde{S}(\tau_i^x, \tau_j^y)\big)\Big| \\ &\qquad \leq L_\varepsilon \big|u_\varepsilon(\tau_i^x, \tau_j^y) - \widetilde{S}_\varepsilon(\tau_i^x, \tau_j^y)\big| \leq L_\varepsilon \big|u_\varepsilon(\tau_i^x, \tau_j^y) - S_\varepsilon(\tau_i^x, \tau_j^y)\big| + L_\varepsilon \big|S_\varepsilon(\tau_i^x, \tau_j^y) - \widetilde{S}_\varepsilon(\tau_i^x, \tau_j^y)\big|.\end{aligned}$$

Since S_ε is the bicubic spline interpolation of u_ε, there exists a constant K_2 such that

$$\|u_\varepsilon - S_\varepsilon\|_\infty \leq K_2 h^2. \tag{3.3}$$

Using the fact that

$$|S_\varepsilon - \widetilde{S}_\varepsilon| \leq \|C_\varepsilon - \widetilde{C}_\varepsilon\|_\infty \sum_{p,q=-2}^{n-2} B_p^x B_q^y \leq \|C_\varepsilon - \widetilde{C}_\varepsilon\|_\infty, \tag{3.4}$$

we obtain

$$|F_\varepsilon - F_{\widetilde{C}_\varepsilon}| \leq L_\varepsilon \|C_\varepsilon - \widetilde{C}_\varepsilon\|_\infty + L_\varepsilon K_2 h^4.$$

By using relation (3.2) and assumption (3.1), it is easy to see that

$$\begin{aligned}\|C_\varepsilon - \widetilde{C}_\varepsilon\|_\infty &\leq \frac{\frac{1}{2}\,h^2 \|A^{-1} \odot B^{-1}\|_\infty}{1 - L_\varepsilon \frac{1}{2}\,h^2 \|A^{-1} \odot B^{-1}\|_\infty}\,(K_2 L_\varepsilon h^2 + 2K_1) \\ &\leq h^2 \|A^{-1} \odot B^{-1}\|_\infty (K_2 L_\varepsilon h^2 + 2K_1).\end{aligned} \tag{3.5}$$

Thus

$$\|u_\varepsilon - \widetilde{S}_\varepsilon\|_\infty \leq \|u_\varepsilon - S_\varepsilon\|_\infty + \|S_\varepsilon - \widetilde{S}_\varepsilon\|_\infty.$$

Therefore, from relations (3.3), (3.4) and (3.5), we deduce that $\|u_\varepsilon - \widetilde{S}_\varepsilon\|_\infty$ is of order $O(h^2)$. Hence, the proof is complete. □

Remark 3.1. Theorem 3.1 provides a relation coupling the penalty parameter ε and the discretization parameter h, which guarantees the quadratic convergence of the bicubic spline collocation $\widetilde{S}_\varepsilon$ to the solution u_ε of the penalty problem.

We have the interesting properties.

Theorem 3.2 ([14, p. 110], [16])**.** *Let u denote the solution of the variational inequality problem* (1.1) *and u_ε, $\varepsilon > 0$, denote the solution of the penalty problem* (2.1) *with θ_ε defined by relation* (2.2)*. Then $\{u_\varepsilon\}$ is a nondecreasing sequence and*

$$u(x,y) \leq u_\varepsilon(x,y) \leq u(x,y) + \varepsilon, \quad (x,y) \in \Omega, \quad \text{for } \varepsilon > 0.$$

Theorem 3.3. *Suppose that $u(x,y)$ is the solution of* (1.1) *and $u_{bc}(x,y)$ is the approximate solution by our presented method. Then we have*

$$\|u(x,y) - u_{bc}(x,y)\|_\infty \leq \epsilon + kh^2, \quad (x,y) \in \Omega, \quad \text{for } \varepsilon > 0,$$

where k is a finite constant. Therefore, for sufficiently small ϵ and h, the solution of presented scheme (2.8) *converges to the solution of the variational inequality problem* (1.1) *in the discrete L_∞-norm and the rates of convergence are $O(\epsilon + h^2)$.*

4 Numerical examples

In this section, we give the numerical experiments in order to validate the theoretical results presented in this paper. We report numerical results for solving a two-dimensional obstacle problem by using the bicubic spline method to approximate the solution of the penalty problem (2.3), and the generalized Newton method [23] to determine the coefficients of the bicubic spline collocation.

As a numerical experiment, the example by Bartels and Carstensen [2] with $\Omega = (-1.5, 1.5)^2$ is considered, however, with an additional mass term. For the obstacle $\psi = 0$ and volume force $f = 2$, the exact solution is

$$u(x, y) = \begin{cases} -\dfrac{r^2}{2} - \ln(r) - \dfrac{1}{2} & \text{if } \ r = |x|_2 \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

As a stopping criteria for the generalized Newton iterations, we have considered that the absolute value of the difference between the input coefficients and the output coefficients is less than 10^{-5}.

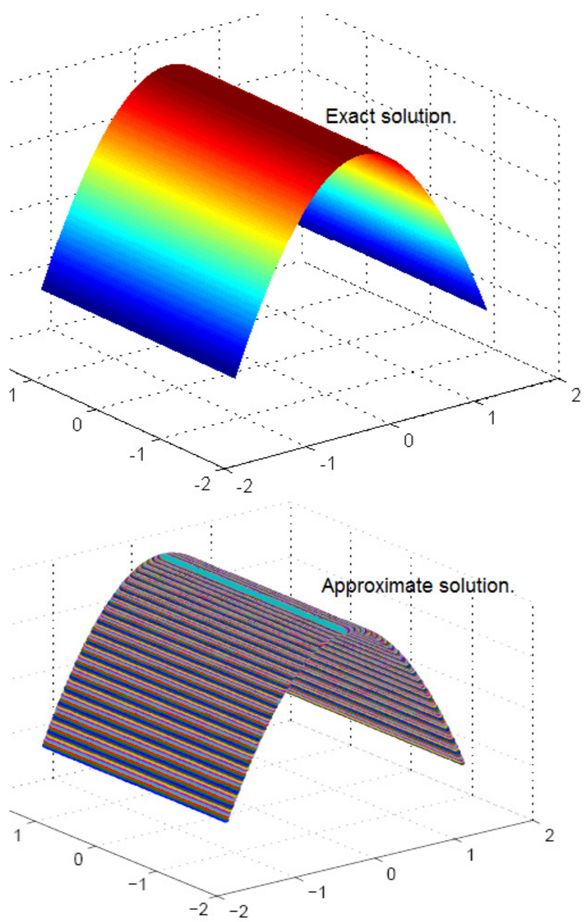

Figure 1. Exact and Approximate solution.

Table 1 shows, for different values of the discretization parameter h, the error between the bicubic spline collocation $\widetilde{S}_\varepsilon$ and the true solution u. We note that the convergence of the solution $\widetilde{S}_\varepsilon$ to the function u depends on the discretization parameter h and the penalty parameter ε. Theorem 3.1 implies that for a fixed h, this convergence is guaranteed only if there exists $\varepsilon_h > 0$ such that $\varepsilon \geq \varepsilon_h$. Some experimental values of ε_h are given in Table 1.

Theorem 3.3 implies that we have the error estimate between the exact solution and the discrete penalty solution given by $\|u(x,y)-u_{bc}(x,y)\|_\infty \leq \epsilon + kh^2$. The obtained results show the convergence of the discrete penalty solution to the solution of the original obstacle problem as the parameters h and ε get smaller provided they satisfy relation (3.1). Moreover, the numerical error estimates behave like $\varepsilon + kh^2$ which confirms what we were expecting.

Table 1. Numerical results

ϵ	10^{-2}	10^{-3}	5×10^{-4}	$2 \times 10^{-4} = \varepsilon_h$
For $h = 0.05$				
$\|u - \widetilde{S}_\varepsilon\|_\infty$	5×10^{-3}	10.61×10^{-4}	10.12×10^{-4}	9.84×10^{-4}
For $h = 0.02$				
$\|u - \widetilde{S}_\varepsilon\|_\infty$	4.7×10^{-3}	7.21×10^{-4}	2.34×10^{-4}	2.03×10^{-4}
For $h = 0.01$				
$\|u - \widetilde{S}_\varepsilon\|_\infty$	4.63×10^{-4}	7.03×10^{-5}	3.15×10^{-6}	1.84×10^{-6}

5 Concluding remarks

In this paper, we have considered an approximation of a bidimensional unilateral obstacle problem by a sequence of penalty problems, which are nonsmooth equation problems, presented in [14, 16]. Then we have developed a numerical method for solving each nonsmooth equation, based on a bicubic collocation spline method and the generalized Newton method. We have shown the convergence of the method provided that the penalty and discret parameters satisfy relation (3.1). Moreover, we have provided an error estimate of order $O(h^2)$ with respect to the norm $\|\cdot\|_\infty$. The obtained numerical results show the convergence of the approximate penalty solutions to the exact one and confirm the error estimates provided in this paper.

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ANALYSIS OF A FRICTIONAL UNILATERAL CONTACT PROBLEM FOR PIEZOELECTRIC MATERIALS WITH LONG-TERM MEMORY AND ADHESION

Abstract. This paper deals with the study of a mathematical model that describes a frictional contact between a piezoelectric body and an obstacle. The material behavior is described with an electro-elastic constitutive law with long memory and the contact is modelled with Signorini conditions associated with the non-local friction law in which the adhesion between the contact surfaces is taken into account. We establish a variational formulation of the model in the form of a system involving the displacement, stress, electric displacement, electric potential and adhesion field. Under the assumption that the coefficient of friction is small enough, we prove the existence of a unique weak solution to the problem. The proof is based on arguments of variational inequalities, nonlinear evolutionary equations with monotone operators, differential equations and the Banach fixed-point theorem.

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რეზიუმე. ნაშრომი ეძღვნება მათემატიკური მოდელის შესწავლას, რომელიც ხახუნის გათვალისწინებით აღწერს პიეზოელექტრულ სხეულსა და დაბრკოლებას შორის კონტაქტს. სხეულის მასალის ყოფაქცევა აღწერილია ელექტრო-დრეკადი ძირითადი თანაფარდობით, რომელიც ხანგრძლივ მეხსიერებას უზრუნველყოფს. კონტაქტის მოდელირება ხდება არალოკალურ ხახუნთან ასოცირებული სინიორინის პირობებით, რომლებშიც გათვალისწინებულია ადგეზია კონტაქტურ ზედაპირებს შორის. ჩამოყალიბებულია მოდელის ვარიაციული ფორმულირება სისტემის სახით, რომელიც შეიცავს გადაადგილებას, ძაბვას, ელექტრულ ძვრას, ელექტრულ პოტენციალს და ადგეზიურ ველს. იმ დაშვებით, რომ ხახუნის კოეფიციენტი საკმარისად მცირეა, ჩვენ ვამტკიცებთ ამოცანის ერთადერთი სუსტი ამონახსნის არსებობას. დამტკიცება ემყარება ვარიაციულ უტოლობებს, არაწრფივ ევოლუციურ განტოლებებს მონოტონური ოპერატორებით, დიფერენციალურ განტოლებებსა და ბანახის უძრავი წერტილის თეორემას.

1 Introduction

Contact problems involving deformable bodies are common in industry and in everyday life and play an important role in structural and mechanical systems, especially, the so-called piezoelectric materials, which consider the interaction of mechanical and electrical properties. Contact processes involve complicated surface phenomena and are modeled with highly nonlinear initial boundary value problems. Taking into account various conditions associated with more and more complex behavior laws lead to introducing new and nonstandard models, expressed by the aid of evolution variational inequalities. An early attempt to study contact problems within the framework of variational inequalities is due to Duvaut and Lions [5], to find the state of mathematical, mechanical, and numerical art (see [22, 26]). Several authors have studied unilateral frictional contact problems involving the Signorini state with or without adhesion (see, e.g., the references in [7, 9, 18, 26, 28]), as well as the models of viscoelastic adhesive materials and piezoelectric effect models (see [6, 12, 13, 15, 20]).

In this paper, we study a mathematical model that describes a problem of frictional and adhesive contact between a supposed long-memory electro-elastic body and a foundation. Recall that a frictionless contact problem with short memory has been studied in [25]. In the present work, we assume that the contact is modeled with a unilateral constraint and the law of non-local friction with adhesion. The bonding field evolution is described by a first-order differential equation. As in [10, 11], we use it as an internal surface variable with values between zero and one to describe the fractional density of active bonds. We refer the reader to the extensive bibliography on the subject in [4, 17, 22, 25].

The present paper aims to extend the results established in the study of a unilateral and frictional contact problem with adhesion. Novelty is the introduction of a non-local friction law in unilateral adhesive contact problem for an elastic body with long memory. We contribute to the solution of this problem by proposing a variational formulation for this model, then, we prove that under the assumption of the smallness of the coefficient of the friction and suitable regularity assumptions on the data, the problem admits a unique weak solution where we specify its regularity. The proof of this result requires proving several technical lemmas by arguments on variational inequalities, monotone operators, differential equations, and Banach's fixed-point theorem.

The paper is organized as follows. In Section 2, we state the mechanical model; we list the assumption on the problem data; we present some notations and give a variational formulation. Finally, in Section 3, under the assumption of the smallness of the coefficient of friction, we state and prove our main existence and uniqueness result.

2 Problem statement and variational formulation

First, we explain some notations used in this paper. We denote by $\mathbf{S}_d$ the space of second order symmetric tensors on $\mathbb{R}^d (d = 2, 3)$, while '$\cdot$' and $\|\cdot\|$ represent the inner product and the Euclidean norm on $\mathbf{S}_d$ and $\mathbb{R}^d$, respectively. Thus, for every $u, v \in \mathbb{R}^d$, $u \cdot v = u_i v_i$, $\|v\| = (v \cdot v)^{\frac{1}{2}}$ and for every σ, $\tau \in \mathbf{S}_d$, $\sigma \cdot \tau = \sigma_{ij}\tau_{ij}$, $\|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}}$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, given by $v_\nu = v \cdot \nu = v_i \nu_i$, $v_\tau = v - v_\nu \nu$, $\sigma_\nu = \sigma\nu \cdot \nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$.

We consider the following physical setting. An electro-elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with the Lipschitz boundary $\partial\Omega = \Gamma$. The boundary Γ is partitioned into three disjoint measurable parts Γ_1, Γ_2, Γ_3 on the one hand, and on two disjoint measurable parts Γ_a and Γ_b on the other hand, such that $\mathrm{meas}(\Gamma_1) > 0$, $\mathrm{meas}(\Gamma_a) > 0$ and $\Gamma_3 \subset \Gamma_b$. Let $T > 0$ and let $[0, T]$ denote the time interval of interest. We assume the body is clamped on Γ_1 and therefore the displacement field vanishes there. A volume forces of density φ_0 act in Ω and surface tractions of density φ_2 act on Γ_2. The body is submitted to electrical constraints for which we assume the electric potential is zero on Γ_a, the body is subjected to an electric charge of density q_0 act on Ω and a surface electric charge of density q_0 act on Γ_b. On Γ_3, the body is in unilateral contact with adhesion following the nonlocal friction law with an insulator obstacle, the so-called foundation.

Thus, the formulation of the mechanical problem is written as follows.

Problem (P). *Find a displacement field* $u : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\sigma : \Omega \times [0,T] \to \mathbf{S}_d$, *an electric potential* $\varphi : \Omega \times [0,T] \to \mathbb{R}$, *an electric displacement field* $D : \Omega \times [0,T] \to \mathbb{R}^d$ *and a bonding field* $\beta : \Gamma_3 \times [0,T] \to \mathbb{R}$ *such that for all* $t \in [0,T]$,

$$\sigma(t) = \mathcal{B}\varepsilon(u(t)) + \int_0^t \mathcal{F}(t-s)\varepsilon(u(s))\,ds - \mathcal{E}^* E(\varphi(t)), \tag{2.1}$$

$$D(t) = \mathcal{E}\varepsilon(u(t)) + \mathcal{C}E(\varphi(t)), \tag{2.2}$$

$$\operatorname{Div}\sigma(t) + \varphi_0(t) = 0 \quad in\ \Omega, \tag{2.3}$$

$$\operatorname{div} D(t) + q_0(t) = 0 \quad in\ \Omega, \tag{2.4}$$

$$u(t) = 0 \quad on\ \Gamma_1, \tag{2.5}$$

$$\sigma\nu(t) = \varphi_2(t) \quad on\ \Gamma_2, \tag{2.6}$$

$$u_\nu(t) \le 0, \quad \sigma_\nu(t) - \gamma_\nu\beta^2 R_\nu(u_\nu(t)) \le 0, \quad u_\nu(t)(\sigma_\nu(t) - \gamma_\nu\beta^2(t)R_\nu(u_\nu(t))) = 0 \quad on\ \Gamma_3, \tag{2.7}$$

$$\dot\beta(t) = -[\beta(t)((\gamma_\nu R_\nu u_\nu(t))^2 + \gamma_\tau\|R_\tau(u_\tau(t))\|^2) - \epsilon_a]_+ \quad on\ \Gamma_3, \tag{2.8}$$

$$\varphi(t) = 0 \quad on\ \Gamma_a, \tag{2.9}$$

$$D\nu(t) = q_2(t) \quad on\ \Gamma_b, \tag{2.10}$$

$$\beta(0) = \beta_0 \quad on\ \Gamma_3, \tag{2.11}$$

$$\begin{cases} \left\|\sigma_\tau(t) + \gamma_\tau\beta^2(t)R_\tau(u_\tau(t))\right\| \leqslant \mu|R\sigma_\nu(u(t))|, \\ \left\|\sigma_\tau(t) + \gamma_\tau\beta^2(t)R_\tau(u_\tau(t))\right\| < \mu|R\sigma_\nu(u(t))| \Longrightarrow u_\tau = 0 \\ \left\|\sigma_\tau(t) + \gamma_\tau\beta^2(t)R_\tau(u_\tau(t))\right\| = \mu|R\sigma_\nu(u(t))| \Longrightarrow \exists\,\lambda \geqslant 0 \ \ such\ that \\ \qquad \sigma_\tau(t) + \gamma_\tau^2\beta^2(t)R_\tau(u_\tau(t)) = -\lambda u_\tau(t) \end{cases} \quad on\ \Gamma_3. \tag{2.12}$$

We now describe the equations and conditions involved in our model above.

First, equations (2.1) and (2.2) present an elastic constitutive law with long memory in which u is the displacement field, $D = (D_1, \ldots, D_d)$ is the electric displacement field, $\sigma = (\sigma_{ij})$ is the stress tensor, $\varepsilon(u)$ denote the linearised deformation tensor defined by $\varepsilon(u) = (\varepsilon_{ij}(u))$, $\varepsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$; $\mathcal{B}$ is an operator of elasticity, $\mathcal{F}$ is the tensor of relaxation, $\mathcal{E} = (e_{ijk})$ is the third order piezoelectric operator, $\mathcal{E}^* = (e^*_{ijk})$ is its transpose. $E(\varphi) = -\nabla\varphi$ is the electric field, where $\nabla\psi = (\partial_i\psi)$ and $\mathcal{C} = (\mathcal{C}_{ij})$ is a positive definite symmetric tensor, called the electric permittivity. More details on the constitutive equations of forms (2.1) and (2.2) can be found in [1] and [2]. Next, (2.3) is the equation of motion describing the evolution of the displacement u where $\operatorname{Div}\sigma = (\partial_j\sigma_{ij})$ and (2.4) is the equation describing the evolution of the electric displacement D. Conditions (2.5) and (2.6) are the displacement and traction boundary conditions, whereas (2.7) are the Signorini contact conditions with adhesion, with zero gap, in which γ_ν denotes an adhesion coefficient which may be dependent on $x \in \Gamma_3$. R_ν and R_τ are the truncation operators defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > L, \end{cases} \qquad R_\tau(s) = \begin{cases} s & \text{if } |s| \le L, \\ L\,\dfrac{s}{|s|} & \text{if } |s| > L, \end{cases}$$

where $L > 0$ is the characteristic length of the bond.

The differential equation (2.8) describes the evolution of the bonding field β. Here, γ_ν, γ_τ and ϵ_a are positive coefficients of adhesion, where $[r]_+ = \max\{0, r\}$. In (2.9), we assume that the potential vanishes on Γ_a, and we express the fact that the electric charge density q_2 is imposed on Γ_b by (2.10). Finally, (2.11) is the initial condition and (2.12) represent Coulomb's law of dry friction with adhesion, where μ denotes the coefficient of friction.

Now, to obtain a variational formulation of Problem (P), we will use the spaces

$$H = L^2(\Omega)^d, \quad Q = \{\tau = (\tau_{ij});\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\},$$
$$H_1 = \{u = (u_i) :\ u_i \in H^1(\Omega),\ \ i = \overline{1,d}\}, \quad Q_1 = \{\sigma \in Q :\ \operatorname{Div}\sigma \in H\}$$

H, Q, H_1, H_d are the real Hilbert spaces endowed with the respective inner products

$$(u,v)_H = \int_\Omega u_i v_i \, dx, \quad \langle \sigma, \tau \rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij}\, dx,$$
$$(u,v)_{H_1} = \langle u,v\rangle_H + (\varepsilon(u),\varepsilon(v))_Q, \quad (\sigma, \tau_{H_d}) = \langle \sigma,\tau\rangle_Q + (\operatorname{Div}\sigma, \operatorname{Div}\tau)_H.$$

We denote respectively the norms associated with $\|\cdot\|_H$, $\|\cdot\|_Q$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{H_d}$.

Recall that the following *Green's* formula holds:

$$\langle \sigma, \varepsilon(v)\rangle_Q + (\operatorname{Div}\sigma, v)_H = \int_\Gamma \sigma\nu\cdot v\, da, \ \ \forall\, v \in H_1 \tag{2.13}$$

where da is the measure surface element.

The displacement fields will be sought in the space $V = \{v \in H_1 : \ \gamma v = 0 \text{ a.e. on } \Gamma_1\}$.

Since $\operatorname{meas}(\Gamma_1) > 0$, the Korn inequality holds, i.e., there exists a constant $C_0 > 0$ such that

$$\|\varepsilon(v)\|_Q \geqslant C_0 \|v\|_{H_1}, \ \ \forall\, v \in V,$$

and V is a Hilbert space with the inner product $(u,v)_V = (\varepsilon(u),\varepsilon(v))_Q$ and the associated norm $\|\cdot\|_V$.

For $v \in H_1$, we use the same symbol v for its trace on Γ. Given the Sobolev trace theorem, there is a constant $C_\Omega > 0$ such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leqslant C_\Omega \|v\|_V, \ \ \forall\, v \in V. \tag{2.14}$$

We use the set of admissible displacements fields given by $U_{ad} = \{v \in V : \ v_\nu \le 0 \text{ a.e. on } \Gamma_3\}$.

For the electric displacement field, we need the following two Hilbert spaces:

$$W = \big\{\psi \in H^1 : \ \gamma\psi = 0 \text{ a.e on } \Gamma_a\big\}, \quad W_a = \big\{D = (D_i) : \ D_i \in L^2(\Omega), \ \operatorname{div} D \in L^2(\Omega)\big\}$$

endowed, respectively, with the inner products

$$(\psi,\phi)_W = (\nabla\psi, \nabla\phi)_H, \quad (D,E)_{Wa} = (D,E)_H + (\operatorname{div} D, \operatorname{div} E)_{L^2(\Omega)},$$

and we denote the norms associated with $\|\cdot\|_W$ and $\|\cdot\|_{W_a}$.

Since $\operatorname{meas}(\Gamma_a) > 0$, the *Friedrichs–Poincaré* inequality holds and we have a constant $C_F > 0$ such that

$$\|\nabla\psi\|_W \ge C_F \|\psi\|_{H^1(\Omega)}, \ \ \forall\, \psi \in W.$$

Moreover, if $D \in W_a$ is sufficiently regular, the following *Green's* formula holds:

$$(D, \nabla\psi)_H + (\operatorname{div} D, \psi)_{L^2(\Omega)} = \int_{\Gamma_b} D\nu \cdot \psi \, da, \ \ \forall\, \psi \in W. \tag{2.15}$$

We will also need the space Q_∞ of fourth order tensors defined by

$$Q_\infty = \Big\{\mathcal{A} = (\mathcal{A}_{ijkh}); \ \mathcal{A}_{ijkh} = \mathcal{A}_{jikh} = \mathcal{A}_{khij} \in L^\infty(\Omega)\Big\}.$$

Q_∞ is a Banach space with the norm defined by

$$\|\mathcal{A}\|_{Q_\infty} = \max_{0 \le i,j,k,h \le d} \|\mathcal{A}_{ijkh}\|_{L^\infty(\Omega)}.$$

Let $T > 0$. For every real Hilbert space X, we use the usual notation for the spaces $L^p(0,T;X)$, $k \in [0,\infty]$ and $W^{1,\infty}(0,T;X)$. Recall that the norm of the space $W^{1,\infty}(0,T;X)$ is defined by $\|u\|_{W^{1,\infty}(0,T;X)} = \|u\|_{L^\infty(0,T;X)} + \|\dot{u}\|_{L^\infty(0,T;X)}$, where $\dot{u}$ denotes the first derivative of u with respect to time. We also use the space of continuous functions $C([0,T];X)$ with the norm $\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X$.

Finally, we introduce the space of bonding field denoted as $\mathbf{B}$ by

$$\mathbf{B} = \Big\{\beta :\ [0,T] \to L^2(\Gamma_3);\ 0 \le \beta(t) \le 1,\ \forall\, t \in [0,T],\ a.e.\ on\ \Gamma_3\Big\}.$$

For the study of Problem (P) we adopt the following assumptions on the data.

The operator $\mathcal{B}$ and the tensors $\mathcal{F}$, $\mathcal{C}$, $\mathcal{E}$ and $\mathcal{E}^*$ satisfy the following hypotheses:

$$\begin{cases} \text{(a)} & \mathcal{B} : \Omega \times S_d \to S_d, \\ \text{(b)} & \mathcal{B} \in Q_\infty \ \text{ and there exists a constant } M_{\mathcal{B}} > 0 \ \text{ such that} \\ & \qquad \|\mathcal{B}(x,\xi_1) - \mathcal{B}(x,\xi_2)\| \le M_{\mathcal{B}}\|\xi_1 - \xi_2\|,\ \ \forall\, \xi_1,\xi_2 \in S_d,\ \text{ a.e. in } \Omega, \\ \text{(c)} & \text{There exists a constant } m_{\mathcal{B}} > 0 \ \text{ such that } \ \mathcal{B}\xi\cdot\xi \geqslant m_{\mathcal{B}}\|\xi\|^2,\ \ \forall\,\xi \in S_d \ \text{ a.e. in } \Omega, \\ \text{(d)} & \text{The function } \ x \to \mathcal{B}(x,\xi) \ \text{ is measurable on } \Omega \text{ a.e } \xi \in S_d; \end{cases} \tag{2.16}$$

$$\mathcal{F} \in C([0,T]; Q_\infty); \tag{2.17}$$

$$\begin{cases} \text{(a)} & \mathcal{C} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \\ \text{(b)} & \mathcal{C}(x,E) = (c_{ij}(x)E_j),\ \ \forall\, E = (E_{ij}) \in \mathbb{R}^d \ \text{ a.e. in } \Omega, \quad c_{ij} = c_{ji} \in L^\infty(\Omega), \\ \text{(c)} & \text{There exists a constant } \ m_{\mathcal{C}} > 0 \ \text{ such that} \\ & \qquad c_{ij}(x)E_iE_j \geqslant m_{\mathcal{C}}\|E\|^2 \ \ \forall\,\xi \in S_d \ \text{ a.e. in } \Omega; \end{cases} \tag{2.18}$$

$$\begin{cases} \text{(a)} & y\mathcal{E} : \Omega \times S_d \to \mathbb{R}^d, \\ \text{(b)} & \mathcal{E}(x,\xi) = (e_{ijk}(x)\xi_{ij}),\ \ \forall\,\xi = (\xi_{ij}) \in \mathbf{S}_d \ \text{ a.e. in } \Omega, \\ \text{(c)} & e_{ijk} = e_{ikj} \in L^\infty(\Omega); \end{cases} \tag{2.19}$$

$$\mathcal{E}\sigma\cdot v = \sigma\cdot\mathcal{E}^*v,\ \ \forall\,\sigma \in \mathbf{S}_d,\ \ \forall\, v \in \mathbb{R}^d. \tag{2.20}$$

where the components of the tensor $\mathcal{E}^*$ are given by $e^*_{ijk} = e_{kij}$.

In addition, we assume that adhesion coefficients satisfy

$$\gamma_\tau, \gamma_\nu, \epsilon_a \in L^\infty(\Gamma_3), \quad \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\tau, \gamma_\nu, \epsilon_a \geqslant 0 \ \text{ a.e. } x \in \Gamma_3, \tag{2.21}$$

and the following regularity on φ_0 and q_0:

$$\varphi_0 \in C([0,T]; H),\ \ \varphi_2 \in C([0,T]; L^2(\Gamma_2)^d), \tag{2.22}$$

$$q_0 \in C([0,T]; H),\ \ q_2 \in C([0,T]; L^2(\Gamma_b)^d). \tag{2.23}$$

To reflect that the foundation is isolated, we assume

$$q_0(t) = 0 \ \text{ on } \Gamma_3,\ \ \forall\, t \in [0,T]. \tag{2.24}$$

The initial data β_0 satisfy

$$\beta_0 \in L^2(\Gamma_3),\ \ 0 \le \beta_0 \le 1 \ \text{ a. e. on } \Gamma_3. \tag{2.25}$$

The friction coefficient μ is such that

$$\mu \in L^\infty(\Gamma_3),\ \ \mu(x) \ge 0 \ \text{ a. e. on } \Gamma_3. \tag{2.26}$$

Finally, R is linear and continuous mapping, where

$$R : H^{-\frac{1}{2}}(\Gamma) \to L^2(\Gamma_3). \tag{2.27}$$

By the representation theorem of *Riesz–Fréchet*, for all $t \in [0,T]$, we define $f(t) \in V$ and $q(t) \in W$ as follows:

$$(f(t), v)_V = \int_\Omega \varphi_0(t)\cdot v\,dx + \int_{\Gamma_2} \varphi_2(t)\cdot v\,da,\ \ \forall\, v \in V,$$

$$(q(t), \psi)_V = \int_\Omega q_0(t)\cdot\psi\,dx + \int_{\Gamma_2} q_2(t)\cdot\psi\,da,\ \ \forall\,\psi \in W,$$

which imply that $f \in C([0,T];H)$ and $q \in C([0,T];W)$. Next, we consider V_0, the subset of regularity defined by $V_0 = \{v \in H_1 : \ \mathrm{Div}\,\sigma(v) \in H\}$. Let us denote by $j_{ad} : L^\infty(\Gamma_3) \times V_0 \times V \to \mathbb{R}$ and $j_{fr} : V_0 \times V \to \mathbb{R}$, respectively, the functionals given by

$$j_{ad}(\beta, u, v) = \int_{\Gamma_3} \big(-\gamma_\nu \beta^2 R_\nu(u_\nu) v_\nu + \gamma_\tau \beta^2 R_\tau(u_\tau)\cdot v_\tau \big)\, da,$$

$$j_{fr}(u, v) = \int_{\Gamma_3} \mu |R\sigma_\nu(u)| \|v_\tau\|\, da, \ \ \forall\,(u,v) \in V_0 \times V.$$

If (v, φ) is a solution of Problem (P_V) stated below, then $\sigma(t) = \sigma(u(t), \varphi(t)) \in Q$ a.e. $t \in [0,T]$ and therefore

$$j_{fr}(u(t), v) = \int_{\Gamma_3} \mu |R\sigma_\nu(u(t))| \|v_\tau\|\, da, \ \ \forall\, v \in V.$$

Using the *Green's* formula (2.13) and (2.15), we prove that if u, σ, φ and D are regular and satisfy equations and conditions (2.1)–(2.12), then

$$(\sigma(t), \varepsilon(u(t)))_Q + j_{ad}(\beta(t), u(t), v) + j_{fr}(u(t), v) - j_{fr}(u(t), u(t)) \geq (f(t), v - u(t))_V, \quad \forall\, v \in V, t \in [0,T], \tag{2.28}$$

$$(D(t), \nabla\psi)_H + (q(t), \psi)_W = 0, \ \ \forall\,\psi \in W. \tag{2.29}$$

Taking $\sigma(t)$ in (2.28) by the expression given by (2.1), and $D(t)$ by the expression given by (2.2) , we derive the following variational formulation of Problem (P).

Problem (P_V)**.** *Find a displacement field $u \in C([0,T];V)$, an electric potential $\varphi \in C([0,T];W)$ and a bonding field $\beta \in W^{1,\infty}([0,T];L^2(\Gamma_3)) \cap \mathbf{B}$ such that $u(t) \in U_{ad} \cap V_0$ for all $t \in [0,T]$ and*

$$\begin{aligned}
&\big(\mathcal{B}\varepsilon(u(t)), \varepsilon(v - u(t))\big)_Q + \Big(\int_0^t \mathcal{F}(t-s)\varepsilon(u(s))\, ds, \varepsilon(v-u(t)) \Big)_Q \\
&\quad + \big(\mathcal{E}^*\nabla\varphi(t), \varepsilon(v - u(t))\big)_Q + j_{ad}(\beta(t), u(t), v - u(t)) \\
&\quad + j_{fr}(u(t), v) - j_{fr}(u(t), u(t)) \geq (f(t), v - u(t))_V, \ \ \forall\, v \in U_{ad}, \ \ t \in [0,T],
\end{aligned} \tag{2.30}$$

$$(\mathcal{C}\nabla\varphi(t), \nabla\psi)_H - (\mathcal{E}\varepsilon(u(t), \nabla\psi)_H = (q(t), \psi)_W, \ \ \forall\,\psi \in W, \ \ t \in [0,T], \tag{2.31}$$

$$\dot\beta(t) = -\Big[\beta(t)((\gamma_\nu R_\nu u_\nu(t))^2 + \gamma_\tau \|R_\tau(u_\tau(t))\|^2) - \epsilon_a\Big]_+, \ \ t \in [0,T], \tag{2.32}$$

$$\beta(0) = \beta_0. \tag{2.33}$$

3 Existence and uniqueness

Our main existence and uniqueness result that we state and prove is the following

Theorem 3.1. *Assume that assumptions* (2.16)–(2.27) *hold. Then there exists a constant $\mu_0 > 0$ such that if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, then Problem (P_V) has a unique solution (u, φ, β).*

We carry out the proof of Theorem 3.1 in several steps. We define intermediate problems and prove their unique solvability, and then we construct a contraction mapping whose unique fixed point is the solution of Problem (P_V). First, we consider the closed subset $Z = \{\theta \in C([0,T];\ L^2(\Gamma_3)) \cap B;\ \theta(0) = \beta_0\}$, where the Banach space $C([0,T];L^2(\Gamma_3))$ is endowed with the norm

$$\|\theta\|_k = \max_{t\in[0,T]} \big[e^{-kt}\|\theta\|_{L^2(\Gamma_3)}\big], \ \ k > 0.$$

For a given $\beta \in Z$, we consider the following auxiliary problem.

Problem (P_V^β)**.** Find a displacement field $u_\beta \in C([0,T];V)$ and an electric potential $\varphi_\beta \in C([0,T];W)$ such that $u_\beta(t) \in U_{ad} \cap V_0$ for all $t \in [0,T]$ and

$$\begin{aligned}\big(\mathcal{B}\varepsilon(u_\beta(t)), \varepsilon(v - u_\beta(t))\big)_Q + \bigg(\int_0^t \mathcal{F}(t-s)\varepsilon(u_\beta(s))\,ds, \varepsilon(v - u_\beta(t))\bigg)_Q \\ +\big(\mathcal{E}^*\nabla\varphi_\beta(t), \varepsilon(v - u_\beta(t))\big)_Q + j_{ad}(\beta(t), u_\beta(t), v - u_\beta(t)) \\ +j_{fr}(u_\beta(t), v) - j_{fr}(u_\beta(t), u_\beta(t)) \geq (f(t), v - u_\beta(t))_V, \ \ \forall\, v \in U_{ad}, \ \ t \in [0,T], \end{aligned} \tag{3.1}$$

$$\big(\mathcal{C}\nabla\varphi_\beta(t), \nabla\psi\big)_H - \big(\mathcal{E}\varepsilon(u_\beta(t), \nabla\psi)\big)_H = (q(t), \psi)_W, \ \ \forall\psi \in W, \ \ t \in [0,T]. \tag{3.2}$$

We have the following result.

Theorem 3.2. *Problem* (P_V^β) *has a unique solution* $(u_\beta, \varphi_\beta) \in C([0,T]; V \times W)$.

We consider the product Hilbert space $X = V \times W$ with the inner product defined by

$$\langle x, y\rangle = \big\langle (u,\varphi), (v,\psi)\big\rangle = \langle u, v\rangle + \langle \varphi, \psi\rangle, \ \ x, y \in X,$$

and the associated norm $\|\cdot\|_X$. In the sequel, let $X_1 = U_{ad} \times W$.

To prove Theorem 3.2 for all $\eta \in C([0,T];Q)$ and $t \in [0,T]$, we consider the following problem.

Problem (P_η^1)**.** Find $x_{\beta\eta} \in C([0,T];X)$ such that $x_{\beta\eta}(t) \in X_1$ for all $t \in [0,T]$ and

$$\begin{aligned}\big(\mathcal{B}\varepsilon(u_{\beta\eta}(t)), \varepsilon(v - u_{\beta\eta}(t))\big)_Q + \big(\mathcal{E}^*\nabla\varphi_{\beta\eta}(t), \varepsilon(v - u_{\beta\eta}(t))\big)_Q + \big(\mathcal{C}\nabla\varphi_{\beta\eta}(t), \nabla\psi\big)_H - \big(\mathcal{E}\varepsilon(u_{\beta\eta}(t), \nabla\psi)\big)_H \\ + (\eta(t), \varepsilon(v - u_{\beta\eta}(t)))_Q + j_{ad}\big(\beta(t), u_{\beta\eta}(t), v - u_{\beta\eta}(t)\big) + j_{fr}(u_{\beta\eta}(t), v) - j_{fr}(u_{\beta\eta}(t), u_{\beta\eta}(t)) \\ \geq (f(t), v - u_{\beta\eta}(t))_V + (q(t), \psi)_W, \ \ \forall v \in U_{ad}, \ \ \forall\,\psi \in W, \ \ t \in [0,T]. \end{aligned} \tag{3.3}$$

Since *Riesz's* representation theorem implies that there exists an element $f_\eta(t) \in X$ defined for all $x = (u, \varphi)$ by

$$\langle f_\eta(t), x\rangle = (f(t), u)_V + (q(t), \varphi)_W - (\eta(t), \varepsilon(v))_Q,$$

we introduce the operator $\Lambda_\beta : [0,T] \times X \to X$ defined as

$$\begin{aligned}\langle \Lambda_\beta(t)x, X\rangle &= (\mathcal{B}\varepsilon(u), \varepsilon(v))_Q + (\mathcal{E}^*\nabla\varphi, \varepsilon(v))_Q \\ &\quad + (\mathcal{C}\nabla\varphi, \nabla\psi)_H - (\mathcal{E}\varepsilon(u), \nabla\psi)_H + j_{ad}(\beta(t), u, v), \ \ \text{for all} \ \ x = (u,\varphi), \ \ y = (v,\psi) \in X\end{aligned}$$

denoted by $\widetilde{X} = X \times X$, we introduce $\widetilde{j}_{fr} : \widetilde{X} \to \mathbb{R}$ defined by

$$\widetilde{j}_{fr}(y, x) = j_{fr}(u, v) \ \ \text{for all} \ \ x = (u,\varphi), \ \ y = (v,\psi) \in X.$$

Then Problem (P_η^1) is equivalent to

Problem (P_η^2)**.** Find $x_{\beta\eta} : [0,T] \to X_1$ such that

$$\begin{aligned}\big\langle \Lambda_\beta(t)x_{\beta\eta}(t), y - x_{\beta\eta}(t)\big\rangle + \widetilde{j}_{fr}(y, x_{\beta\eta}(t)) - \widetilde{j}_{fr}(x_{\beta\eta}(t), x_{\beta\eta}(t)) \\ \geq \langle f_\eta(t), y - x_{\beta\eta}(t)\rangle, \ \ \forall\, y \in X, \ \ t \in [0,T]. \end{aligned} \tag{3.4}$$

Remark. The two precedent Problems (P_η^1) and (P_η^2) are equivalent in the way that if $x_{\beta\eta} = (u_\beta, \varphi_{\beta\eta}) \in C([0,T];X)$ is a solution of one of the problems, it is also a solution of the other problem.

We now have the following

Lemma 3.1. *There exists a constant* $\mu_0 > 0$ *such that if* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$*, Problem* (P_η^2) *has a unique solution* $x_{\beta\eta} \in C([0,T];X)$.

We prove Lemma 3.1 by steps. The functional j_{ad} is linear over the third term and therefore

$$j_{ad}(\beta, u, -v) = -j_{ad}(\beta, u, v). \tag{3.5}$$

Using the properties of truncation operators, we deduce that there exists $c > 0$ such that

$$j_{ad}(\beta_1, u_1, u_2 - u_1) + j_{ad}(\beta_2, u_2, u_1 - u_2) \le c \int_{\Gamma_3} |\beta_1 - \beta_2| \, \|u_1 - u_2\|_V \, ds. \tag{3.6}$$

Taking $\beta = \beta_1 = \beta_2$ in the last inequality, we obtain

$$j_{ad}(\beta, u_1, u_2 - u_1) + j_{ad}(\beta, u_2, u_1 - u_2) \le 0. \tag{3.7}$$

Choosing $u_1 = v$ and $u_2 = 0$ in (3.7) and using (3.5) and the equality $R_\nu(0) = R_\tau(0) = 0$, we obtain

$$j_{ad}(\beta, v, v) \ge 0. \tag{3.8}$$

Similar computations based on the properties of R_ν and R_τ show that there exists a constant $c > 0$ such that

$$|j_{ad}(\beta, u_1, v) - j_{ad}(\beta, u_2, v)| \le c\|u_1 - u_2\|_V \|v\|_V. \tag{3.9}$$

For $t \in [0, T]$ and for all $x_1 = (u_1, \varphi_1)$ and $x_2 = (u_2, \varphi_2)$, using (3.4), we have

$$\begin{aligned}
\big\langle \Lambda_\beta(t)x_1 - \Lambda_\beta(t)x_2, x_1 - x_2 \big\rangle &= \big(\mathcal{B}\varepsilon(u_1) - \mathcal{B}\varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2)\big)_Q \\
&\quad + \big(\mathcal{E}^*\nabla\varphi_1 - \mathcal{E}^*\nabla\varphi_2, \varepsilon(u_1) - \varepsilon(u_2)\big)_Q + \big(\mathcal{C}\nabla\varphi_1 - \mathcal{C}\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2\big)_H \\
&\quad - \big(\mathcal{E}\varepsilon(u_1) - \mathcal{E}\varepsilon(u_2), \nabla\varphi_1 - \nabla\varphi_2\big)_H + j_{ad}(\beta, u_1, u_2) - j_{ad}(\beta, u_2, u_1),
\end{aligned}$$

and, by (2.20), we have

$$\big(\mathcal{E}^*\nabla\varphi_1 - \mathcal{E}^*\nabla\varphi_2, \varepsilon(u_1) - \varepsilon(u_2)\big)_Q = \big(\mathcal{E}\varepsilon(u_1) - \mathcal{E}\varepsilon(u_2), \nabla\varphi_1 - \nabla\varphi_2\big)_H.$$

Then, by (3.8), (2.16)(c) and (2.18)(c) we deduce

$$\begin{aligned}
\big\langle \Lambda_\beta(t)x_1 - \Lambda_\beta(t)x_2, x_1 - x_2 \big\rangle &\ge \big(\mathcal{B}\varepsilon(u_1) - \mathcal{B}\varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2)\big)_Q \\
&\quad + \big(\mathcal{C}\nabla\varphi_1 - \mathcal{C}\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2\big)_H \ge m_{\mathcal{B}}\|u_1 - u_2\|_V^2 + m_{\mathcal{C}}\|\varphi_1 - \varphi_2\|_W^2.
\end{aligned}$$

Then the operator $\Lambda_\beta(t)$ is strongly monotone, and for $C_m = \min(m_{\mathcal{B}}, m_{\mathcal{C}})$ it satisfies

$$\big\langle \Lambda_\beta(t)x_1 - \Lambda_\beta(t)x_2, x_1 - x_2 \big\rangle \ge C_m\|x_1 - x_2\|_X^2, \ \ \forall\, x, y \in X. \tag{3.10}$$

For $y = (v, \psi)$, using (2.14), (2.16)(b), (2.18) and (3.9), we get

$$\big\langle \Lambda_\beta(t)x_1 - \Lambda_\beta(t)x_2, y \big\rangle \le c\Big(\|u_1 - u_2\|_V \big(\|v\|_V + \|\psi\|_W\big) + \|\varphi_1 - \varphi_2\|_W \big(\|v\|_V + \|\psi\|_W\big)\Big),$$

thus, $\Lambda_\beta(t)$ is a Lipschitz continuous operator and there exists a constant $L_0 > 0$ such that

$$\|\Lambda_\beta(t)x_1 - \Lambda_\beta(t)x_2\| \le L_0\|x_1 - x_2\|_X, \ \ \forall\, x, y \in X.$$

Next, let the non-empty subset $L^2_+(\Gamma_3)$ be defined by

$$L^2_+(\Gamma_3) = \big\{g \in L^2(\Gamma_3);\ g \geqslant 0 \text{ a.e. on } \Gamma_3\big\}.$$

For each $g \in L^2_+(\Gamma_3)$, we define the functional $h(g, \cdot\,) : X \to \mathbb{R}$ by

$$h(g, y) = \int_{\Gamma_3} \mu g\|w_\tau\| \, da, \ \ \forall\, y = (w, \varphi) \in X,$$

and introduce an intermediate problem as follows.

Problem (P_1^g)**.** Find $x_{\beta\eta} : [0, T] \to X_1$ such that

$$\big\langle \Lambda_\beta(t)x_{\beta\eta g}(t), y - x_{\beta\eta g}(t) \big\rangle + h(g, y) - h(g, x_{\beta\eta g}(t)) \geqslant (f, y - x_{\beta\eta g}(t))_V, \ \ \forall\, y \in X. \tag{3.11}$$

Lemma 3.2. *Problem* (P_1^g) *has a unique solution.*

Proof. The functional $h(g,\cdot)$ is convex and lower semi-continuous, Λ_β is Lipschitz continuous and strongly monotone, we deduce that Problem (P_1^g) has a unique solution (see [13]). $\square$

Now, to prove Lemma 3.1, for each $t \in [0,T]$ we define on $L^2_+(\Gamma_3)$ the map $\Psi_t : g \longmapsto \Psi_t(g) = |R\sigma_\nu(u_{_{\beta\eta g}}(t))|$. Then we show the following

Lemma 3.3. *There exists a constant* $\mu_1 > 0$ *such that if* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1$*, the mapping* Ψ *has a unique fixed point* g^**, and* $x_{\beta\eta g^*}$ *is a unique solution to Problem* (P_η^2)*.*

Proof. For $i = 1,2$, define the following

Problem $(P^2_{\eta g i})$. *Find* $x_{\beta\eta g i} = (u_{_{\beta\eta g_i}}, \varphi_{_{\beta\eta g_i}}) \in X_1$ *such that*

$$\langle \Lambda_\beta(t) x_{\beta\eta g i}, y\rangle + h(g_i, y) - h(g_i, x_{\beta\eta g i}) \geqslant (f, y - x_{\beta\eta g i})_V, \quad \forall\, y \in V.$$

Take $y = x_{\beta\eta g_2}$ in inequality (3.11) written for $g = g_1$, then take $y = x_{\beta\eta g_1}$ in (3.11) written for $g = g_2$, by adding the resulting inequalities, we get

$$\big\langle \Lambda_\beta(t)(x_{\beta\eta g_1} - x_{\beta\eta g_2}), x_{\beta\eta g_1} - x_{\beta\eta g_2}\big\rangle \leq h(g_1, x_{\beta\eta g_1}) - h(g_1, x_{\beta\eta g_2}) + h(g_2, x_{\beta\eta g_2}) - h(g_2, x_{\beta\eta g_1}).$$

Then using (2.14) and (3.10), we have

$$C_m \|x_{\beta\eta g_1}(t) - x_{\beta\eta g_2}(t)\|_X^2 \leqslant C_\Omega \|g_1 - g_2\|_{L^2(\Gamma_3)} \int\limits_{\Gamma_3} \mu\big(|u_{\beta\eta g 1\tau}(t)| - |u_{\beta\eta g 2\tau}(t)|\big)\, da. \tag{3.12}$$

Using (2.27), it follows that there exists a constant c_0 such that

$$\|\Psi(g_1) - \Psi(g_2)\|_{L^2(\Gamma_3)} \leqslant c_0 \big\|\sigma_\nu(u_{_{\beta\eta g_1}}(t)) - \sigma_\nu(u_{_{\beta\eta g_2}}(t))\big\|_{H^{-\frac{1}{2}}(\Gamma)}. \tag{3.13}$$

Moreover, using (2.16), we prove that there is a constant $c_1 > 0$ such that

$$\big\|\sigma_\nu(u_{_{\beta\eta g_1}}(t)) - \sigma_\nu(u_{_{\beta\eta g_2}}(t))\big\|_{H^{-\frac{1}{2}}(\Gamma)} \leq c_1 \|x_{_{\beta\eta g_1}}(t) - x_{_{\beta\eta g_2}}(t)\|_X. \tag{3.14}$$

Hence, taking into account (2.14) and combining (3.12), (3.13) and (3.14), after some calculus we find

$$\|\Psi(g_1) - \Psi(g_2)\|_{L^2(\Gamma_3)} \leqslant \frac{c_0 c_1 C_\Omega}{C_m} \|\mu\|_{L^\infty(\Gamma_3)} \|g_1 - g_2\|_{L^2(\Gamma_3)}.$$

Let $\mu_1 = \frac{C_m}{c_0 c_1 C_\Omega}$, then we deduce that if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1$, Ψ is a contraction and, so, it admits a unique fixed point denoted by g^*.

Keeping in mind that there is a unique element $x_{\beta\eta g^*}$ satisfying the inequality

$$\big\langle \Lambda_\beta(t) x_{\beta\eta g^*}, y - x_{\beta\eta g^*}\big\rangle + h(\Psi(g^*), y) - h(\Psi(g^*), x_{\beta\eta g^*}) \geqslant (f, y - x_{\beta\eta g^*})_V, \quad \forall\, y \in X,$$

and $h \circ \Psi = j$, we prove that $x_{\beta\eta}(t) = x_{\beta\eta g^*}$ is a unique solution of Problem (P_η^2). We shall now see that $x_{\beta\eta} \in C([0,T];X)$. Indeed, let t_1, $t_2 \in [0,T]$, take $y = x_{\beta\eta}(t_2)$ in (3.3) written for $t = t_1$ and take $y = x_{\beta\eta}(t_1)$ in the same inequality written for $t = t_2$. Using (2.16), (2.27) and the properties of R_ν and R_τ, we prove that there exists a constant $c > 0$ such that

$$\|x_{\beta\eta}(t_1) - x_{\beta\eta}(t_2)\|_X \leq c\big(\|\beta(t_1) - \beta(t_2)\|_{L^2(\Gamma_3)} + \|f(t_1) - f(t_2)\|_H + \|\eta(t_1) - \eta(t_2)\|_Q\big).$$

Then, as $f \in C([0,T];H)$, $\eta \in C([0,T];Q)$ and $\beta \in C([0,T];L^2(\Gamma_3))$, we immediately conclude that $x_{\beta\eta} \in C([0,T];X)$. We also have that $u_{\beta\eta}(t) \in U_{ad} \cap V_0$, $\forall\, t \in [0,T]$. Indeed, for each $t \in [0,T]$, denote $\sigma(u_{_{\beta\eta}}(t)) = \mathcal{B}\varepsilon(u_{_{\beta\eta}}(t)) - \mathcal{E}^* E(\varphi_{_{\beta\eta}}(t)) + \eta(t)$ and using *Green's* formula with the regularity $\varphi_0(t) \in H$, we get $\operatorname{div} \sigma(u_{_{\beta\eta}}(t))) \in H$ and then $u_{\beta\eta}(t) \in V_0$. $\square$

Now, we define the operator $\digamma_\beta : C([0,T];Q) \to C([0,T];Q)$ by

$$\digamma_\beta \eta(t) = \int_0^t \mathcal{F}(t-s)\varepsilon(u_{\beta\eta}(s))\, ds, \quad \forall\, \eta \in C(0,T;Q), \quad t \in [0,T].$$

We have the following

Lemma 3.4. *The operator $\digamma_\beta$ has a unique fixed point η_β.*

Proof. Let $\eta_1, \eta_1 \in C([0,T];Q)$. By a standard computation based on (2.17) and (3.3), we prove that there exists a constant $c_2 > 0$ such that

$$\|\digamma_\beta \eta_1(t) - \digamma_\beta \eta_2(t)\|_Q \le c_2 \int_0^t \|\eta_1(t) - \eta_2(t)\|_Q\, ds, \quad \forall\, t \in [0,T].$$

By iteration, for any positive integer n we deduce the estimate

$$\|\digamma_\beta^n \eta_1 - \digamma_\beta^n \eta_2\|_{C([0,T];Q)} \le \frac{c_2^n T^n}{n!} \|\eta_1 - \eta_2\|_{C([0,T];Q)}.$$

As $\lim\limits_{n\to+\infty} \frac{c_2^n T^n}{n!} = 0$, it follows that for a positive integer n sufficiently large, $\digamma_\beta^n$ is a contraction on the space $C([0,T];Q)$. Then, by using the Banach fixed point theorem, $\digamma_\beta^n$ has a unique fixed point $\eta_\beta \in C([0,T];Q)$ which is also a unique fixed point of $\digamma_\beta$, i.e.,

$$\digamma_\beta \eta_\beta(t) = \eta_\beta(t), \quad \forall\, t \in [0,T].$$ □

Next, we denote $u_\beta = u_{\beta\eta}$ and $\varphi_\beta = \varphi_{\beta\eta}$ and deduce that the couple (u_β, φ_β) is a solution of Problem (P_V^β). The uniqueness follows from the fixed point of the operator $\digamma$, which completes the proof of Theorem 3.2.

In the following step, we use u_β, the solution obtained by Theorem 3.2, to state the following Cauchy problem.

Problem (P_{ad})**.** *Find a bonding field $\theta_\beta : [0,T] \to L^\infty(\Gamma_3)$ such that*

$$\dot\theta_\beta(t) = -\Big[\theta_\beta(t)\big((\gamma_\nu R_\nu u_{\beta\nu}(t))^2 + \gamma_\tau \|R_\tau(u_{\beta^*\tau}(t))\|^2\big) - \epsilon_a\Big]_+ \quad a.e.\ t \in [0,T], \tag{3.15}$$

$$\theta_\beta(0) = \beta_0. \tag{3.16}$$

Lemma 3.5. *Problem (P_{ad}) has a unique solution θ_β which satisfies $\theta_\beta \in W^{1,\infty}([0,T];L^\infty(\Gamma_2)) \cap Z$.*

Proof. Consider the mapping $\mathcal{F} : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$\mathcal{F}_\beta(t,\theta) = -\Big[\theta\big((\gamma_\nu R_\nu u_{\beta\nu}(t))^2 + \gamma_\tau \|R_\tau(u_{\beta\tau}(t))\|^2\big) - \epsilon_a\Big].$$

For all $t \in [0,T]$ and $\theta \in L^2(\Gamma_3)$, it follows from the properties of the truncation operators R_ν and R_τ that $\mathcal{F}_\beta$ is Lipschitz continuous uniformly in time with respect to β. Moreover, for any $\theta \in L^2(\Gamma_3)$, the mapping $t \to \mathcal{F}_\beta(t,\theta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Using now a version of the Cauchy–Lipschitz theorem (see [15]), we obtain a unique function $\theta_\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ satisfying (3.15) and (3.16). We note that the restriction $0 \le \theta_\beta \le 1$ is implicitly included in the variational Problem P_V and, therefore, from the definition of the sets B and Z, we find that $\theta_\beta \in Z$, which concludes the proof of lemma. □

Consider the mapping $\Phi : Z \to Z$ defined by $\Phi\beta = \theta_\beta$.

The third step consists in the following result.

Lemma 3.6. *There exists a unique element $\beta^* \in Z$ such that $\Phi\beta^* = \beta^*$.*

Proof. Indeed, let β_i, $i=1,2$, be two elements of Z. Denote by u_{β_i}, φ_{β_i}, θ_{β_i} the functions obtained in Theorem 3.2 and Lemma 3.5 and denote $\theta_{\beta_i}=\theta_i$. It follows from (3.15) that

$$\theta_i(t)=\beta_0-\int_0^t\Big[\beta_i(s)\big((\gamma_\nu R_\nu u_{\beta_i\nu}(s))^2+\gamma_\tau\|R_\tau(u_{\beta_i\tau}(s))\|^2\big)-\epsilon_a\Big]_+ ds$$

and there exists a constant $c>0$ such that

$$\|\theta_1(t)-\theta_2(t)\|_{L^2(\Gamma_3)}\le c\int_0^t\Big\|\beta_1(s)R_\nu(u_{\beta_{1\nu}}(s))^2-\beta_2(s)R_\nu(u_{\beta_{2\nu}}(s))^2\Big\|_{L^2(\Gamma_3)}ds$$
$$+\int_0^t\Big\|\beta_1(s)\|R_\tau(u_{\beta_{1\tau}}(s))\|^2-\beta_2(s)\|R_\tau(u_{\beta_{2\tau}}(s))\|^2\Big\|_{L^2(\Gamma_3)}ds.$$

Using the properties of the operators R_ν and R_τ, we get

$$\|\theta_1(t)-\theta_2(t)\|_{L^2(\Gamma_3)}\le c_3\Big(\int_0^t\|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}ds+\int_0^t\|u_{\beta_1}(s)-u_{\beta_2}(s)\|_{L^2(\Gamma_3)^d}ds\Big)\tag{3.17}$$

for some constant $c_3>0$. □

Now, to continue the proof, we need to prove the following

Lemma 3.7. *There exists a constant $\mu_2>0$ such that if $\|\mu\|_{L^\infty(\Gamma_3)}<\mu_2$, we have*

$$\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_{L^2(\Gamma_3)^d}\le c\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)},\quad\forall t\in[0,T].$$

Proof. Let $t\in[0;T]$. We take $\psi=\psi-\varphi_\beta(t)$ in (3.2) and by adding with (3.1) we get

$$\begin{aligned}&\big(\mathcal{B}\varepsilon(u_\beta(t)),\varepsilon(v-u_\beta(t))\big)_Q+\Big(\int_0^t\mathcal{F}(t-s)\varepsilon(u(s))\,ds,\varepsilon(v-u(t))\Big)_Q\\&\quad+\big(\mathcal{E}^*\nabla\varphi_\beta(t),\varepsilon(v-u_\beta(t))\big)_Q+j_{ad}(\beta(t),u_\beta(t),v-u_\beta(t))+\big(\mathcal{C}\nabla\varphi_\beta(t),\nabla\psi-\nabla\varphi_\beta(t)\big)_H\\&\qquad-\big(\mathcal{E}\varepsilon(u_\beta(t),\nabla\psi-\nabla\varphi_\beta(t))\big)_H+j_{fr}(u_\beta(t),v)-j_{fr}(u_\beta(t),u_\beta(t))\\&\qquad\ge(f(t),v-u_\beta(t))_V+(q(t),\psi-\varphi_\beta(t))_W,\quad\forall\,v\in U_{ad},\quad\forall\,\psi\in W,\quad t\in[0,T].\end{aligned}\tag{3.18}$$

Taking $v=u_{\beta_2}(t)$ and $\psi=\varphi_{\beta_2}$ in (3.18) satisfied by $(u_{\beta_1}(t),\varphi_{\beta_1})$, and then taking $v=u_{\beta_1}(t)$ and $\psi=\varphi_{\beta_1}$ in the same inequality satisfied by $(u_{\beta_2}(t),\varphi_{\beta_2})$, by adding the resulting inequalities and using (2.20), we obtain

$$\begin{aligned}&\Big(\mathcal{B}\varepsilon(u_{\beta_1}(t))-\mathcal{B}\varepsilon(u_{\beta_2}(t)),\varepsilon(u_{\beta_1}(t))-\varepsilon(u_{\beta_2}(t))\Big)_Q+\Big(\mathcal{C}\nabla\varphi_{\beta_1}(t)-\mathcal{C}\nabla\varphi_{\beta_2}(t),\nabla\varphi_{\beta_1}(t)-\nabla\varphi_{\beta_2}(t)\Big)_H\\&\qquad\le\Big(\int_0^t\mathcal{F}(t-s)\big(\varepsilon(u_{\beta_1}(t))-\varepsilon(u_{\beta_2}(t))\big)\,ds,\varepsilon(u_{\beta_2}(t))-\varepsilon(u_{\beta_1}(t))\Big)_Q\\&\quad+j_{ad}\big(\beta_1(t),u_{\beta_1}(t),u_{\beta_2}(t)-u_{\beta_1}(t)\big)+j_{fr}(u_{\beta_1}(t),u_{\beta_2}(t))+j_{ad}\big(\beta_2(t),u_{\beta_2}(t),u_{\beta_1}(t)-u_{\beta_2}(t)\big)\\&\qquad+j_{fr}(u_{\beta_2}(t),u_{\beta_1}(t))-j_{fr}(u_{\beta_1}(t),u_{\beta_1}(t))-j_{fr}(u_{\beta_2}(t),u_{\beta_2}(t)).\end{aligned}$$

Using (2.16)(c) and (2.18)(c), we deduce

$$\begin{aligned}&m_{\mathcal{B}}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2+m_{\mathcal{C}}\|\varphi_{\beta_1}(t)-\varphi_2(t)\|_W\\&\qquad\le\Big(\int_0^t\mathcal{F}(t-s)\big(\varepsilon(u_{\beta_1}(s))-\varepsilon(u_{\beta_2}(s))\big)\,ds,\varepsilon(u_{\beta_2}(t))-\varepsilon(u_{\beta_1}(t))\Big)_Q\\&\quad+j_{ad}\big(\beta_1(t),u_{\beta_1}(t),u_{\beta_2}(t)-u_{\beta_1}(t)\big)+j_{fr}(u_{\beta_1}(t),u_{\beta_2}(t))+j_{ad}\big(\beta_2(t),u_{\beta_2}(t),u_{\beta_1}(t)-u_{\beta_2}(t)\big)\\&\qquad+j_{fr}(u_{\beta_2}(t),u_{\beta_1}(t))-j_{fr}(u_{\beta_1}(t),u_{\beta_1}(t))-j_{fr}(u_{\beta_2}(t),u_{\beta_2}(t)),\end{aligned}$$

thus

$$
\begin{aligned}
m_{\mathcal{B}}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2 \le \Big(\int_0^t \mathcal{F}(t-s)\big(\varepsilon(u_{\beta_1}(s))-\varepsilon(u_{\beta_2}(s))\big)\,ds, \varepsilon(u_{\beta_2}(t))-\varepsilon(u_{\beta_1}(t)) \Big)_Q \\
+ j_{ad}\big(\beta_1(t),u_{\beta_1}(t),u_{\beta_2}(t)-u_{\beta_1}(t)\big) + j_{ad}\big(\beta_2(t),u_{\beta_2}(t),u_{\beta_1}(t)-u_{\beta_2}(t)\big) \\
+ j_{fr}(u_{\beta_1}(t),u_{\beta_2}(t)) - j_{fr}(u_{\beta_1}(t),u_{\beta_1}(t)) + j_{fr}(u_{\beta_2}(t),u_{\beta_1}(t)) - j_{fr}(u_{\beta_2}(t),u_{\beta_2}(t)). \quad (3.19)
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\Big(\int_0^t \mathcal{F}(t-s)\big(\varepsilon(u_{\beta_1}(s))-\varepsilon(u_{\beta_2}(s))\big)\,ds, \varepsilon(u_{\beta_2}(t))-\varepsilon(u_{\beta_1}(t)) \Big)_Q \\
&\qquad \le \Big(\int_0^t \|\mathcal{F}(t-s)\|_{Q_\infty}\|u_{\beta_1}(s)-u_{\beta_2}(s)\|_V\,ds \Big)\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V \\
&\qquad\qquad \le c_4\Big(\int_0^t \|u_{\beta_1}(s)-u_{\beta_2}(s)\|_V\,ds \Big)\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V
\end{aligned}
$$

for some positive constant c_4. Using Young's inequality, we find that

$$
\begin{aligned}
&\Big(\int_0^t \mathcal{F}(t-s)\big(\varepsilon(u_{\beta_1}(s))-\varepsilon(u_{\beta_2}(s))\big)\,ds, \varepsilon(u_{\beta_2}(t))-\varepsilon(u_{\beta_1}(t)) \Big)_Q \\
&\qquad \le \frac{c_4^2}{m_{\mathcal{B}}}\Big(\int_0^t \|u_{\beta_1}(s)-u_{\beta_2}(s)\|_V^2\,ds \Big) + \frac{m_{\mathcal{B}}}{4}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2. \quad (3.20)
\end{aligned}
$$

Using (3.6) and Young's inequality, we deduce that there exists a positive constant c_5 such that

$$
j_{ad}(\beta_1,u_1,u_2-u_1)+j_{ad}(\beta_2,u_2,u_1-u_2) \le c_5\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}^2 + \frac{m_{\mathcal{B}}}{4}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2. \quad (3.21)
$$

Moreover, we have

$$
\begin{aligned}
&j_{fr}(u_{\beta_1}(t),u_{\beta_2}(t)) - j_{fr}(u_{\beta_1}(t),u_{\beta_1}(t)) + j_{fr}(u_{\beta_2}(t),u_{\beta_1}(t)) - j_{fr}(u_{\beta_2}(t),u_{\beta_2}(t)) \\
&\qquad \le \int_{\Gamma_3} \mu R|\sigma_\nu(u_{\beta_{1\nu}}(t))-\sigma_\nu(u_{\beta_{2\nu}}(t))|\,\|u_{\beta_1}(t)-u_{\beta_2}(t)\|\,da.
\end{aligned}
$$

Keeping in mind (3.14) and using (2.14), we get

$$
\begin{aligned}
&j_{fr}(u_{\beta_1}(t),u_{\beta_2}(t)) - j_{fr}(u_{\beta_1}(t),u_{\beta_1}(t)) + j_{fr}(u_{\beta_2}(t),u_{\beta_1}(t)) - j_{fr}(u_{\beta_2}(t),u_{\beta_2}(t)) \\
&\qquad \le c_1C_\Omega^2\|\mu\|_{L^\infty(\Gamma_3)}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2. \quad (3.22)
\end{aligned}
$$

We now combine inequalities (3.19), (3.20), (3.21) and (3.22) to deduce

$$
\begin{aligned}
&m_{\mathcal{B}}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2 \\
&\qquad \le c_5\|\beta_1-\beta_2\|_{L^2(\Gamma_3)}^2 + \frac{m_{\mathcal{B}}}{4}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2 + c_1C_\Omega^2\|\mu\|_{L^\infty(\Gamma_3)}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2 \\
&\qquad\qquad + \frac{c_4^2}{m_{\mathcal{B}}}\Big(\int_0^t \|u_{\beta_1}(s)-u_{\beta_2}(s)\|_V\,ds \Big)^2 + \frac{m_{\mathcal{B}}}{4}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2.
\end{aligned}
$$

Hence, we have

$$\Big(\frac{m_{\mathcal{B}}}{2} - c_1 C_\Omega^2 \|\mu\|_{L^\infty(\Gamma_3)}\Big)\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2$$
$$\leq c_5\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}^2 + \frac{c_4^2}{m_{\mathcal{B}}}\int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V^2\,ds.$$

Further, if

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_2 = \frac{m_{\mathcal{B}}}{2c_1 C_\Omega^2}\,,$$

we deduce that there exists a constant $c_8 > 0$ such that

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 \leq c_8\bigg(\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}^2 + \int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V^2\,ds\bigg).$$

Hence, using Cornwall's argument, it follows that there exists a constant $c_9 > 0$ such that

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 \leq c_9\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}^2,\ \ \forall\, t \in [0,T]. \tag{3.23}$$

Now, to end the proof of Lemma 3.6 we use (3.17) and (3.23) to obtain

$$\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_{L^2(\Gamma_3)} \leq c_9\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\,ds,\ \ \forall\, t \in [0,T],$$

where $c_7 > 0$. We have

$$e^{-kt}\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_{L^2(\Gamma_3)} \leq c_9 e^{-kt}\int_0^t e^{ks}e^{-ks}\|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\,ds,$$

then

$$\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_k \leq c_9 e^{-kt}\|\beta_1(t) - \beta_2(t)\|_k\int_0^t e^{ks}\,ds,\ \ \forall\, t \in [0,T].$$

So, we deduce that

$$\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_k \leq \frac{c_{10}}{k}\|\beta_1(t) - \beta_2(t)\|_k,\ \ \forall\, t \in [0,T], \tag{3.24}$$

where $c_{10} > 0$. Inequality (3.24) shows that for $k > c_{10}$, Φ is a contraction on Z. Then Φ has a unique fixed point which satisfies (3.15) and (3.16). □

Thus, we have all the ingredients to prove Theorem 3.1.

Existence. Consider β^*, the fixed point of the operator Φ, and $x^* = (u^*, \varphi^*)$, the solution of Problem $(P_V^{\beta^*})$, i.e., $u^* = u_{\beta^*}$ and $\varphi^* = \varphi_{\beta^*}$.

By (3.1), (3.2), (3.15) and (3.16), we conclude that the triple $(u^*, \varphi^*, \beta^*)$ is a solution to Problem (P_V).

Uniqueness. The uniqueness arises from the uniqueness of the fixed point of the operator Φ, which completes the proof of Theorem 3.1.

Indeed, let (u, φ, β) be a solution of Problem (P_V), it follows from (3.1) and (3.2) that u is a solution of Problem (P_V^β) and, by Theorem 3.2, this problem has a unique solution (u_β, φ_β), where $u_\beta = u$ and $\varphi_\beta = \varphi$.

Taking $u = u_\beta$ and $\varphi = \varphi_\beta$ in Problem (P_V), we deduce that β is a solution of Problem (P_{ad}). From the result of Lemma 3.5, Problem (P_{ad}) has a unique solution β^*, so we find $\beta^* = \beta$, and then we conclude that $(u^*, \varphi^*, \beta^*)$ is a unique solution to Problem (P_V).

Let now σ^* and D^* be the functions defined by (2.1) and (2.2), respectively, which correspond to (u^*, φ^*). Then it results from (2.16)–(2.20) that $\sigma^* \in C([0,T];Q)$ and $D^* \in C([0,T];H)$. Using also a standard argument, it follows from (2.30) and (2.31) that

$$\begin{aligned} \operatorname{Div} \sigma^*(t) + \varphi_0(t) = 0 \ \text{ in } \Omega, \\ \operatorname{div} D^*(t) + q_0(t) = 0 \ \text{ in } \Omega. \end{aligned}$$

Therefore, using (2.22) and (2.23), we deduce that $\operatorname{Div} \sigma^*(u^*(t), \varphi^*(t)) \in H$ for each $t \in [0,T]$ and $\operatorname{div} D^* \in C([0,T]; L^2(\Omega))$, which implies that $\sigma^* \in C([0,T];Q_1)$ and $D^* \in C([0,T];W_a)$. The triple $(u^*, \varphi^*, \beta^*)$ which satisfies (2.30)–(2.33) is called a weak solution of Problem (P). We conclude that under stated assumptions, Problem (P) has a unique weak solution $(u^*, \varphi^*, \beta^*, \sigma^*, D^*)$ with the regularity $u^* \in C([0,T];V)$, $\varphi^* \in C([0,T];W)$, $\beta^* \in W^{1,\infty}((0,T;L^2(\Gamma_3))) \cap B$, $\sigma^* \in C([0,T];Q_1)$ and $D^* \in C([0,T];W_a)$.

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A FAMILY OF PLANAR DIFFERENTIAL SYSTEMS WITH EXPLICIT EXPRESSION FOR ALGEBRAIC AND NON-ALGEBRAIC LIMIT CYCLES

Abstract. This paper is devoted to the study of a family of planar polynomial differential systems. First, we prove that the considered family has invariant algebraic curves which are given explicitly. Then, we introduce an explicit expression for their first integral. Moreover, we provide sufficient conditions for the systems to possess two limit cycles explicitly given: one is an algebraic and the other is shown to be non-algebraic. The applicability of our result was illustrated by concrete examples.

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რეზიუმე. ნაშრომი ეძღვნება ბრტყელი პოლინომიალური დიფერენციალური სისტემების ოჯახის შესწავლას. თავდაპირველად დავამტკიცებთ, რომ განხილულ ოჯახს გააჩნია ცხადი სახით მოცემული ინვარიანტული ალგებრული წირები. შემდეგ შემოგვაქვს მათი პირველი ინტეგრალის გამოსახულება ცხადი სახით. ამის გარდა, მოვძებნით საკმარის პირობებს, რათა სისტემებს ჰქონდეს ცხადი სახით მოცემული ორი ზღვრული ციკლი: ერთი - ალგებრული, ხოლო მეორე - არაალგებრული. შედეგის საილუსტრაციოდ მოყვანილია კონკრეტული მაგალითები.

1 Introduction

One of the main problems in the qualitative theory of differential equations is the study of limit cycles of planar differential systems and especially of the planar polynomial differential systems of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = P(x,y), \\ y' = \dfrac{dy}{dt} = Q(x,y), \end{cases} \tag{1.1}$$

where $P(x,y)$ and $Q(x,y)$ are real polynomials in the variables x and y. The degree of the system is the maximum of the degrees of the polynomials P and Q.

Recall that:

- A limit cycle of system (1.1) is an isolated periodic orbit in the set of its periodic orbits and is said to be algebraic if it is contained in the zero set of an invariant algebraic curve of the system.

- An algebraic curve defined by $U(x,y) = 0$ is an invariant curve for (1.1) if there exists a polynomial $K(x,y)$ (called the cofactor) such that

$$P(x,y)\frac{\partial U(x,y)}{\partial x} + Q(x,y)\frac{\partial U(x,y)}{\partial y} = K(x,y)U(x,y).$$

- System (1.1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non-constant analytic function $H : \Omega \to \mathbb{R}$, called a first integral, such that

$$\frac{dH(x,y)}{dt} = P(x,y)\,\frac{\partial H(x,y)}{\partial x} + Q(x,y)\,\frac{\partial H(x,y)}{\partial y} \equiv 0.$$

Among the important and attractive problems in the qualitative theory of differential equations [8, 14] is the study of limit cycles of system (1.1) related to the Hilbert's 16th problem [11]; several works and papers in this field investigate their number, stability and location in the phase plane [1,12].

The notion of integrability of (1.1) is based on the existence of a first integral [5, 16]. There is a strong relationship between the integrability of polynomial systems and the number of invariant algebraic curves they have [7], and questions about the existence of a first integral, determining its expression explicitly, when it exists, are always presents.

The results and examples [2–4, 9, 10] about algebraic and non-algebraic limit cycle are given, but it is not easy work to decide whether a limit cycle is algebraic or not. Thus, the well-known limit cycle of the van der Pol differential system exhibited in 1926 (see [15]), was not proved until 1995 by Odani [13] that it was non-algebraic. An invariant algebraic curve is a principal topic for several authors and researchers because of its importance in understanding the dynamics of a system (we refer to [6] for an exhaustive survey on this topic).

In this paper, we give an explicit expression of invariant algebraic curves, then we prove that these systems are integrable, and we introduce an explicit expression of a first integral of a multi-parameter planar polynomial differential system of thirteenth degree of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = x + (x^2+y^2)^2\big(P_3(x,y) - x(x^2+y^2)^3R_2(x,y)\big), \\ y' = \dfrac{dy}{dt} = y + (x^2+y^2)^2\big(Q_3(x,y) - y(x^2+y^2)^3R_2(x,y)\big), \end{cases} \tag{1.2}$$

where

$$\begin{aligned} P_3(x,y) &= ax^3 + bx^2y + cxy^2 - dy^3, \\ Q_3(x,y) &= ax^2y + dx^3 + (b+2\,d)xy^2 + cy^3, \\ R_2(x,y) &= (a+1)x^2 + (b+d)xy + (c+1)y^2, \end{aligned}$$

in which a, b, c, d are the real constants.

Moreover, we provide sufficient conditions for a polynomial differential system to possess two limit cycles explicitly given: one is algebraic and the other is shown to be non-algebraic. Concrete examples exhibiting the applicability of our result are introduced.

We define the trigonometric functions

$$G(\theta) = \frac{a+c}{2} + \frac{a-c}{2}\cos 2\theta + \frac{b+d}{2}\sin 2\theta,$$
$$A(\theta) = \int_0^\theta \frac{6+6G(t)}{d}\exp\left(\int_0^t \frac{-12-6G(\omega)}{d}\,d\omega\right)dt,$$
$$B(\theta) = \exp\left(\int_0^\theta \frac{-12-6G(\omega)}{d}\,d\omega\right).$$

Our main result is contained in the following theorem.

Theorem 1.1. *For system* (1.2)*, the following statements hold.*

(1) *If $d \neq 0$, then the origin of coordinates $O(0,0)$ is the unique critical point of system* (1.2) *at a finite distance.*

(2) *The curve $U(x,y) = x^6 + 3\,x^4y^2 + 3\,x^2y^4 + y^6 - 1$ is an invariant algebraic curve of system* (1.2) *with a cofactor*

$$K(x,y) = -6(x^2+y^2)^3\Big(1 + (x^2+y^2)^2\big((a+1)x^2 + (b+d)xy + (c+1)y^2\big)\Big).$$

(3) *System* (1.2) *has the first integral*

$$H(x,y) = \frac{(1-(x^2+y^2)^3)A(\arctan\frac{y}{x}) + B(\arctan\frac{y}{x})}{(x^2+y^2)^3 - 1}.$$

(4) *System* (1.2) *has an explicit limit cycle, given in Cartesian coordinates by*

$$(\Gamma_1): x^6 + 3\,x^4y^2 + 3\,x^2y^4 + y^6 - 1 = 0.$$

(5) *If $d < 0$, $-2-(a+c) > |b+d| + |c-a|$ and $4+a+c > |b+d| + |c-a|$, then system* (1.2) *has non-algebraic limit cycle (Γ_2), explicitly given in the polar coordinates (r,θ) by*

$$r(\theta, r_*) = \left(\frac{(B(\theta)+A(\theta))(B(2\pi)-1)+A(2\pi)}{A(\theta)(B(2\,\pi)-1)+A(2\pi)}\right)^{\frac{1}{6}}.$$

Moreover, the algebraic limit cycle (Γ_1) lies inside the non-algebraic limit cycle (Γ_2).

2 Proof of Theorem 1.1

Proof of Statement (1). By definition, $A(x_0,y_0) \in \mathbb{R}^2$ is a critical point of system (1.2) if

$$\begin{cases} x_0 + (x_0^2+y_0^2)\big(P_3(x_0,y_0) - x_0(x_0^2+y_0^2)^3R_2(x_0,y_0)\big) = 0, \\ y_0 + (x_0^2+y_0^2)\big(Q_3(x_0,y_0) - y_0(x_0^2+y_0^2)^3R_2(x_0,y_0)\big) = 0, \end{cases}$$

and we have

$$(x_0^2+y_0^2)^2(y_0P_3(x_0,y_0) - x_0Q_3(x_0,y_0)) = -d({x_0}^2+{y_0}^2)^4.$$

Since $d \neq 0$, we have that $(x_0,y_0) = (0,0)$ is the unique solution of this equation. Thus the origin is the unique critical point at a finite distance.

This completes the proof of Statement (1) of Theorem 1.1.

Proof of Statement (2). A computation shows that

$$U(x,y) = x^6 + 3\,x^4y^2 + 3\,x^2y^4 + y^6 - 1$$

satisfies the linear partial differential equation

$$\frac{\partial U(x,y)}{\partial x}\,P(x,y) + \frac{\partial U(x,y)}{\partial y}\,Q(x,y) = U(x,y)K(x,y),$$

the associated cofactor being

$$K(x,y) = -6(x^2+y^2)^3\Big(1 + (x^2+y^2)^2\big((a+1)x^2 + (b+d)xy + (c+1)y^2\big)\Big).$$

This completes the proof of Statement (2) of Theorem 1.1.

Proof of Statement (3). To prove Statement (3), we need to convert system (1.2) in polar coordinates (r,θ) given by $x = r\cos\theta$ and $y = r\sin\theta$, then system (1.2) takes the form

$$\begin{cases} r' = \dfrac{dr}{dt} = r + G(\theta)r^7 + (-G(\theta)-1)r^{13}, \\ \theta' = \dfrac{d\theta}{dt} = dr^6. \end{cases} \tag{2.1}$$

Taking θ as an independent variable, we obtain the equation

$$\frac{dr}{d\theta} = \frac{1}{d}\,r^{-5} + \frac{G(\theta)}{d}\,r + \frac{-G(\theta)-1}{d}\,r^7. \tag{2.2}$$

Using the change of variables $\rho = r^6$, equation (2.2) is transformed into the Riccati equation

$$\frac{d\rho}{d\theta} = \frac{6}{d} + \frac{6G(\theta)}{d}\,\rho + \frac{-6G(\theta)-6}{d}\,\rho^2. \tag{2.3}$$

This equation is integrable, since it possesses the particular solution $\rho = 1$.

By introducing the standard change of variables $z = \rho - 1$, we obtain the Bernoulli equation

$$\frac{dz}{d\theta} = \frac{-6-6G(\theta)}{d}\,z^2 + \frac{-12-6G(\theta)}{d}\,z. \tag{2.4}$$

We note that $z = 0$ is the solution for (2.4), and by introducing the standard change of variables $y = \frac{1}{z}$, we obtain the linear equation

$$\frac{dy}{d\theta} = -\frac{6+6G(\theta)}{d} - \frac{12+6G(\theta)}{d}\,y. \tag{2.5}$$

The general solution of linear equation (2.5) is

$$y(\theta) = \frac{\alpha + A(\theta)}{B(\theta)}\,,$$

where $\alpha \in \mathbb{R}$. Then the general solution of equation (2.4) is

$$z(\theta) = 0, \quad z(\theta) = \frac{B(\theta)}{\alpha + A(\theta)}, \ \text{ where } \ \alpha \in \mathbb{R}.$$

The general solution of equation (2.3) is

$$\rho(\theta) = 1, \quad \rho(\theta) = \frac{\alpha + A(\theta) + B(\theta)}{\alpha + A(\theta)}, \ \text{ where } \ \alpha \in \mathbb{R}.$$

Consequently, the general solution of (2.2) is

$$r(\theta)=1,\quad r(\theta)=\Big(\frac{\alpha+A(\theta)+B(\theta)}{\alpha+A(\theta)}\Big)^{\frac{1}{6}},\ \text{ where }\ \alpha\in\mathbb{R}.$$

From this solution we obtain a first integral in the variables (x,y) of the form

$$H(x,y)=\frac{(1-(x^2+y^2)^3)A(\arctan\frac{y}{x})+B(\arctan\frac{y}{x})}{(x^2+y^2)^3-1}\,.$$

Hence, Statement (3) of Theorem 1.1 is proved.

Proof of Statement (4). The curves $H=h$ with $h\in\mathbb{R}$, which are formed by trajectories of the differential system (1.2), in Cartesian coordinates are written as

$$x^2+y^2=1,\quad (x^2+y^2)^3=\frac{\alpha+A(\theta)+B(\theta)}{\alpha+A(\theta)}\,,$$

where $\alpha\in\mathbb{R}$.

Notice that system (1.2) has a periodic orbit if and only if equation (2.2) has a strictly positive 2π-periodic solution. This, moreover, is equivalent to the existence of a solution of (2.2) that fulfils $r(0,r_*)=r(2\pi,r_*)$ and $r(\theta,r_*)>0$ for any θ in $[0,2\pi]$.

The solution $r(\theta,r_0)$ of the differential equation (2.2) such that $r(0,r_0)=r_0$ is

$$r(\theta,r_0)=\left(\frac{\frac{1}{r_0^6-1}+A(\theta)+B(\theta)}{\frac{1}{r_0^6-1}+A(\theta)}\right)^{\frac{1}{6}},$$

where $r_0=r(0)$.

We have the particular solution $\rho(\theta)=1$ of the differential equation (2.3); from this solution we obtain $r^6(\theta)=1>0$ for all θ in $[0,2\pi]$, which is a particular solution of the differential equation (2.2).

This is an algebraic limit cycle for the differential systems (1.2), corresponding, of course, to an invariant algebraic curve $U(x,y)=0$.

More precisely, in Cartesian coordinates $r^2=x^2+y^2$ and $\theta=\arctan(\frac{y}{x})$ the curve (Γ_1) defined by this limit cycle is $(\Gamma_1): x^6+3x^4y^2+3x^2y^4+y^6-1=0$.

Hence, Statement (4) of Theorem 1.1 is proved.

Proof of Statement (5). A periodic solution of system (1.2) must satisfy the condition $r(0,r_*)=r(2\pi,r_*)$, which leads to a unique value $r_0=r_*$ given by

$$r_*=\Big(\frac{A(2\pi)+B(2\pi)-1}{A(2\pi)}\Big)^{\frac{1}{6}}.$$

The value r_* is the intersection of the periodic orbit with the OX_+ axis. After the substitution of this value r_* into $r(\theta,r_0)$, we obtain

$$r(\theta,r_*)=\left(\frac{(B(\theta)+A(\theta))(B(2\,\pi)-1)+A(2\,\pi)}{A(\theta)(B(2\,\pi)-1)+A(2\,\pi)}\right)^{\frac{1}{6}}.$$

In what follows, it is proved that $r(\theta,r_*)>0$. Indeed,

$$A(2\pi)-A(\theta)=\int\limits_{\theta}^{2\pi}\frac{6+6G(t)}{d}\,\exp\left(\int\limits_0^t\frac{-12-6G(\omega)}{d}\,\mathrm{d}\omega\right)\mathrm{d}t.$$

According to $d<0$, $-2-(a+c)>|b+d|+|c-a|$ and $4+a+c>|b+d|+|c-a|$, hence $\frac{-2-G(\theta)}{d}$ and $\frac{1+G(\theta)}{d}>0$ for all θ in $[0,2\pi]$, then we have $A(2\pi)-A(\theta)>0$ and $B(2\pi)>1$; therefore, we have

$r_* > 0$ and $r(\theta, r_*) > 0$ for all θ in $[0, 2\pi]$. This is the second limit cycle for the differential system (1.2), we denote it by (Γ_2). This limit cycle is not algebraic, due to the expression

$$B(\theta) = \exp\Big(\int_0^\theta \frac{-12 - 6G(\omega)}{d}\,\mathrm{d}\omega\Big).$$

More precisely, in the Cartesian coordinates $r^2 = x^2 + y^2$ and $\theta = \arctan(\frac{y}{x})$, the curve defined by this limit cycle (Γ_2) is $F(x, y) = 0$, where

$$F(x,y) = (x^2+y^2)^3 - \frac{\big(B(\arctan\frac{y}{x}) + A(\arctan\frac{y}{x})\big)(B(2\pi)-1) + A(2\pi)}{A(\arctan\frac{y}{x})(B(2\pi)-1) + A(2\pi)}.$$

If the limit cycle is algebraic, this curve should be given by a polynomial, but a polynomial $F(x, y)$ in the variables x and y satisfies that there is a positive integer n such that $\frac{\partial^n F(x,y)}{\partial x^n} = 0$, but this is not the case, therefore, the curve $(\Gamma_2) : F(x, y) = 0$ is non-algebraic and the limit cycle will also be non-algebraic.

According to $d < 0$, $-2 - (a + c) > |b + d| + |c - a|$ and $4 + a + c > |b + d| + |c - a|$, we get

$$r_* = (1 + \frac{B(2\pi) - 1}{A(2\pi)})^{\frac{1}{6}} > 1,$$

and

$$r(\theta, r_*) = \left(1 + \frac{B(\theta)}{\frac{1}{r_*^6 - 1} + A(\theta)}\right)^{\frac{1}{6}} > 1.$$

We conclude that system (1.2) has two limit cycles, the algebraic (Γ_1) lies inside the non-algebraic one (Γ_2).

This completes the proof of Statement (5) of Theorem 1.1.

3 Examples

Example 3.1. We take $a = c = -\frac{6}{5}$, $d = -5$ and $b = \frac{51}{10}$, then system (1.2) reads as

$$\begin{cases} x' = x + (x^2+y^2)^2\Big(-\frac{6}{5}x^3 + \frac{51}{10}x^2y - \frac{6}{5}xy^2 + 5y^3\Big) - x(x^2+y^2)^5\Big(-\frac{1}{5}x^2 + \frac{1}{10}xy - \frac{1}{5}y^2\Big), \\ y' = y + (x^2+y^2)^2\Big(-\frac{6}{5}x^2y - 5x^3 - \frac{49}{10}xy^2 - \frac{6}{5}y^3\Big) - y(x^2+y^2)^5\Big(-\frac{1}{5}x^2 + \frac{1}{10}xy - \frac{1}{5}y^2\Big). \end{cases} \tag{3.1}$$

In this case, we get

$$A(\theta) = -\frac{3}{50}\int_0^\theta (\sin(2t) - 4)\exp\Big(\frac{3}{100} + \frac{24}{25}t - \frac{3}{100}\cos(2\theta)\Big)\,dt,$$
$$B(\theta) = \exp\Big(-\frac{3}{100}\cos(2\theta) + \frac{24}{25}\theta + \frac{3}{100}\Big).$$

The intersection of the non-algebraic limit cycle (Γ_2) with the OX_+ axis is the point

$$r_* = \Big(\frac{116.8 + \exp(\frac{48\pi}{25}) - 1}{116.8}\Big)^{\frac{1}{6}} \simeq 1.2876.$$

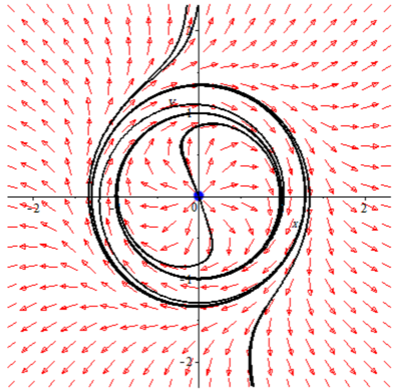

Figure 3.1. Limit cycles of system (3.1).

Example 3.2. We take $a = \frac{-11}{10}$, $c = \frac{-115}{100}$, $d = -7$ and $b = \frac{141}{20}$, then system (1.2) reads as

$$\begin{cases} x' = x + (x^2+y^2)^2\Big(\dfrac{-11}{10}\,x^3 + \dfrac{141}{20}\,x^2y - \dfrac{23}{20}\,xy^2 + 7y^3\Big) \\ \qquad\qquad -x(x^2+y^2)^5\Big(-\dfrac{1}{10}\,x^2 + \dfrac{1}{20}\,xy - \dfrac{3}{20}y^2\Big), \\ y' = y + (x^2+y^2)^2\Big(-\dfrac{11}{10}\,x^2y - 7\,x^3 - \dfrac{139}{20}\,xy^2 - \dfrac{23}{20}\,y^3\Big) \\ \qquad\qquad -y(x^2+y^2)^5\Big(-\dfrac{1}{10}\,x^2 + \dfrac{1}{20}\,xy - \dfrac{3}{20}\,y^2\Big). \end{cases} \tag{3.2}$$

In this case, we get

$$A(\theta) = -\frac{3}{140}\int_0^\theta (\cos(2t)+\sin(2t)-5)\exp\Big(\frac{3}{280} + \frac{3}{280}\,\sin(2t) - \frac{3}{280}\,\cos(2t) + \frac{3}{4}\Big)\,dt,$$

$$B(\theta) = \exp\Big(-\frac{3}{280}\,\sin(2\theta) - \frac{3}{280}\,\cos(2\theta) + \frac{3}{4}\,\theta + \frac{3}{280}\Big).$$

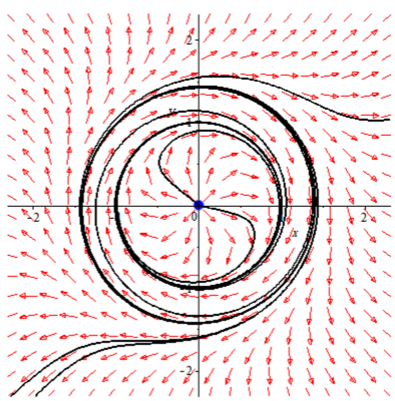

Figure 3.2. Limit cycles of system (3.2).

The intersection of the non-algebraic (Γ_2) limit cycle with the OX_+ axis is the point

$$r_* = \Big(\frac{16.509 + \exp(\frac{2\pi}{3}) - 1}{16.509}\Big)^{\frac{1}{6}} \simeq 1.4047.$$

Example 3.3. We take $a = \frac{-101}{100}$, $c = \frac{-105}{100}$, $d = -1$ and $b = \frac{151}{150}$, then system (1.2) reads as

$$\begin{cases} x' = x + (x^2+y^2)^2\Big(-\dfrac{101}{100}\,x^3 + \dfrac{151}{50}\,x^2y - \dfrac{21}{20}\,xy^2 + y^3\Big) \\ \qquad -x(x^2+y^2)^5\Big(-\dfrac{1}{100}x^2 + \dfrac{1}{50}\,xy - \dfrac{1}{20}\,y^2\Big), \\ y' = y + (x^2+y^2)^2\Big(-\dfrac{101}{100}\,x^2y - x^3 + \dfrac{149}{150}\,xy^2 - \dfrac{21}{20}y^3\Big) \\ \qquad -y(x^2+y^2)^5\Big(-\dfrac{1}{100}\,x^2 + \dfrac{1}{150}\,xy - \dfrac{1}{20}\,y^2\Big). \end{cases} \tag{3.3}$$

In this case, we get

$$A(\theta) = -\frac{1}{50}\int_0^\theta (6\cos(2t) + \sin(2t) - 9)\exp\Big(\frac{1}{100} + \frac{3}{50}\,\sin(2t) - \frac{1}{100}\,\cos(2t) + \frac{291}{50}\,t\Big)\,dt,$$

$$B(\theta) = \exp\Big(\frac{3}{50}\,\sin(2\theta) - \frac{1}{100}\cos(2t) + \frac{291}{50}\,\theta + \frac{1}{100}\Big).$$

The intersection of the non-algebraic limit cycle (Γ_2) with the OX_+ axis is the point

$$r_* = \Big(\frac{1.019\times 10^{14} + \exp(\frac{291\pi}{25}) - 1}{1.019\times 10^{14}}\Big)^{\frac{1}{6}} \simeq 2.0566.$$

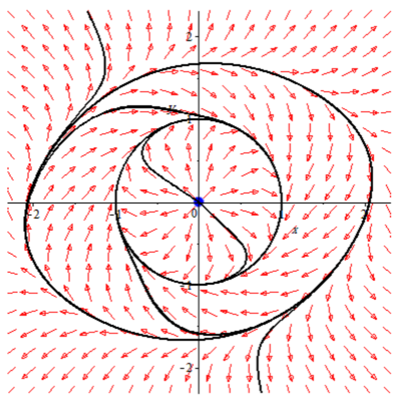

Figure 3.3. Limit cycles of system (3.3).

Example 3.4. We take $a = \frac{-107}{100}$, $c = \frac{-109}{100}$, $d = -5$ and $b = \frac{507}{100}$, then system (1.2) reads as

$$\begin{cases} x' = x + (x^2+y^2)^2\Big(-\dfrac{107}{100}x^3 + \dfrac{507}{100}\,x^2y - \dfrac{109}{100}\,xy^2 + 5y^3\Big) \\ \qquad -x(x^2+y^2)^5\Big(-\dfrac{7}{100}\,x^2 + \dfrac{7}{100}\,xy - \dfrac{9}{100}\,y^2\Big), \\ y' = y + (x^2+y^2)^2\Big(-\dfrac{107}{100}\,x^2y - 5x^3 - \dfrac{493}{100}\,xy^2 - \dfrac{109}{100}\,y^3\Big) \\ \qquad -y(x^2+y^2)^5\Big(-\dfrac{7}{100}\,x^2 + \dfrac{7}{100}\,xy - \dfrac{9}{100}\,y^2\Big). \end{cases} \tag{3.4}$$

In this case, we get

$$A(\theta) = -\frac{3}{500}\int_0^\theta (2\cos(2t) + 7\sin(2t) - 16)\exp\Big(\frac{21}{1000} + \frac{3}{500}\,\sin(2t) - \frac{21}{1000}\,\cos(2t) + \frac{138}{125}\,t\Big)\,dt,$$

$$B(\theta) = \exp\Big(\frac{3}{500}\,\sin(2\theta) - \frac{21}{1000}\,\cos(2t) + \frac{138}{25}\,\theta + \frac{21}{1000}\Big).$$

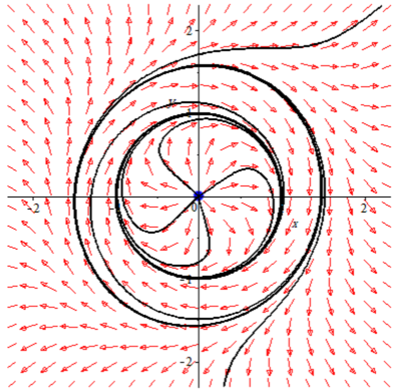

Figure 3.4. Limit cycles of system (3.4).

The intersection of the non-algebraic limit cycle (Γ_2) with the OX_+ axis is the point

$$r_* = \Big(\frac{104.804 + \exp(\frac{276\pi}{125}) - 1}{104.804}\Big)^{\frac{1}{6}} \simeq 1.4870.$$

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THEORETICAL AND NUMERICAL RESULTS ON BIRKHOFF NORMAL FORMS AND RESONANCES IN THE BORN–OPPENHEIMER APPROXIMATION

Abstract. This paper mainly focuses on the Birkhoff normal form theorem for the Born–Oppenheimer Hamiltonians. Normal forms are accessible via those of the effective Hamiltonian obtained by the Grushin reduction method and the pseudodifferential calculus with operator-valued symbols. Resonance situations are discussed; the theoretical computations of Birkhoff normal form in the 1 : 1 resonance are written explicitly. Our approach gives compatible numerical results while using a computer program.

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Key words and phrases. Birkhoff normal form, Born–Oppenheimer approximation, effective Hamiltonian, resonance.

რეზიუმე. ნაშრომი, ძირითადად, ეხება ბირკჰოფის ნორმალური ფორმის თეორემას ბორნ-ოპენჰაიმერის ჰამილტონიანებისთვის. ნორმალური ფორმები ხელმისაწვდომია ეფექტური ჰამილტონიანების საშუალებით, რომლებიც მიღებულია გრუშინის დაყვანის მეთოდით და ოპერატორ-მნიშვნელობბებიანი სიმბოლოების ფსევდოდიფერენციალური აღრიცხვით. განხილულია რეზონანსული სიტუაციები; ბირკჰოფის ნორმალური ფორმის თეორიული გამოთვლები 1 : 1 რეზონანსში ცხადად არის ჩაწერილი. ჩვენი მიდგომა იძლევა თავსებად ციფრულ შედეგებს კომპიუტერული პროგრამის გამოყენების დროს.

1 Introduction

The question of the stability of the multi-body problems dates back to the 18th century. The problem was analyzed by means of series expansions and the canonical approach. The method of normal forms is one of the main tools for studying this stability. The idea of the method is to transform a differential operator into a simpler one by a change of the variables.

The Poincaré theory of normal forms has a counterpart in the Hamiltonian formalism, due to Birkhoff and then extended to the resonant case by Gustavson. Thus, by carefully choosing transformations, one changes a Hamiltonian system into a form with a well understood part, integrable part, under a sufficiently small perturbation, such a transformation will conserve the Hamiltonian structure [2, 8]. Precisely, the well-known Birkhoff theorem states that, in some neighbourhood of the origin, there exists a canonical transformation under which a smooth semiclassical Schrödinger operator $-h^2\Delta_x + V$, for energies close to a non-degenerate minimum of V, can be replaced by a suitable perturbation of a harmonic oscillator.

Some results on Birkhoff normal forms have been proved by Birkhoff [2], Ghomari and Messirdi [5,6] and Ghomari, Messirdi and Vu Ngoc [7] for Schrödinger operators. Nevertheless, no result of the existence, constructions and applications of Birkhoff normal forms was known up to now, for Born–Oppenheimer Hamiltonians. In [9], one can find a description of the question without theoretical details and numerical analysis.

The main objective of this work is the construction of a Birkhoff normal forms method for the Born–Oppenheimer Hamiltonians in the semiclassical limit of type $P = -h^2\Delta_x + Q(x)$, where $Q(x)$ is an operator in the electronic y variables that depends only parametrically on the nuclear x variables, and h^2 stands for the ratio between the electronic and nuclear masses, $h \to 0^+$. $Q(x)$ is referred to as the electronic Hamiltonian, its spectrum is typically discrete in the low energy region and continuous above the threshold energy. Since Q is an operator, it becomes necessary to use the pseudodifferential calculus with operator-valued symbols. We are typically interested in the relationship between the spectrum of the operator P and the classical dynamics of its principal operator-valued symbol.

The main novelty in this work is the introduction of the Birkhoff normal form theorem for Born–Oppenheimer Hamiltonians. The idea is to combine the usual Birkhoff normal forms method with the reduction process to an effective Hamiltonian. If $Q(x)$ and $\lambda_1(x)$, the lowest eigenvalue of $Q(x)$, are smooth and, under suitable assumptions, the Grushin operator associated with P and $\lambda_1(x)$ is invertible as a pseudodifferential operator near the bottom of $\lambda_1(x)$, then, in particular, we get a reduction result, namely, the spectral study of P is close, at least modulo $\mathcal{O}(h^2)$, to one of $P_e = -h^2\Delta_x + \lambda_1(x)$, the effective Hamiltonian in the Born–Oppenheimer approximation. This allows to get asymptotic expansions of the discrete spectrum and the eigenfunctions of P (see, e.g., [6,10–12]) In fact, P_e can explain the complete spectral picture of P modulo errors in h. We first present in Section 2 the general framework of normal forms for semiclassical Schrödinger operators $-h^2\Delta_x + V(x)$, where we give a rigourous proof of the Birkhoff normal form theorem. Furthermore, in Section 3, we explain the core of the mathematical form of the Born–Oppenheimer approximation and describe the construction of the effective Hamiltonian. Namely, the possibility to approximate, for large nuclear masses, the true molecular Hamiltonian, a Schrödinger operator with an operator-valued potential, by some effective Hamiltonian. The effective Hamiltonian is a good approximation to the true molecular Hamiltonian with error-terms of order h^∞ concerning smooth interaction potentials only.

Thanks to the reduction of P to its effective Hamiltonian P_e in x variables, it is now possible to define the Birkhoff normal forms of the full Hamiltonian P by those of P_e. Consequently, in Section 4, we introduce the Birkhoff normal form theorem for P, near an equilibrium point in the Born–Oppenheimer approximation, via the effective Hamiltonian P_e, using the results of Section 2, where the function $\lambda_1(x)$ plays the role of an effective potential function and h tends to zero. Our main ingredient is the use successively two reductions, first the reduction to an effective Hamiltonian and then the classical Birkhoff normal form reduction. We show that one can recover the Birkhoff normal form for the Born–Oppenheimer operator near an equilibrium point and we give a connection the between Birkhoff normal form and resonances that occurs in terms of frequencies of the corresponding harmonic oscillator. As an application, we study the dynamics near a local extremum of the effective Hamiltonian, for which the frequencies are in $1:1$ resonance. Our mathematical results are of physical or chemical relevance, up to some controlled error depending on the semiclassical parameter h. In Section 5, we use a computer program to compute easily the Birkhoff normal form for a given effective Schrödinger Hamiltonian in 1 : 1 resonance. Our numerical results are compatible with the theoretical ones.

2 Generalities on Birkhoff normal forms

The purpose of this section is to apply the fundamental results on the quantum Birkhoff normal forms for semiclassical Schrödinger operators. In the classical setting, the operator to be discussed is of the type $P = -h^2\Delta + V$, where V is the multiplication operator by a smooth potential function. In the molecular case, the corresponding object is Q. Q is neither a multiplication operator, nor smooth if V is a non-smooth potential. The general philosophy consists in finding adequate transformations in which P can be written as a commuting perturbation of the harmonic oscillator. Precisely, there exists a formal real canonical transformation generated by a power series such that P is transformed into a Hamiltonian which is a power series in one-dimensional uncoupled harmonic oscillator Hamiltonians. The procedure for transforming to Birkhoff's normal form is reviewed and enriched here.

Let $V \in C^\infty(\mathbb{R}^N)$, $N \in \mathbb{N}$, $N \geq 1$, and assume that the Hessian matrix $V''(0)$ is diagonal, let $(\nu_1^2, \dots, \nu_N^2)$ be its eigenvalues, with $\nu_j > 0$ and $\nu = (\nu_1, \dots, \nu_N)$. The rescaling $x_j \to \sqrt{\nu_j}x_j$, $x = (x_1, \dots, x_N)$, transforms P into $P = H + W(x)$, where H is the harmonic oscillator $\sum\limits_{j=1}^{N} \frac{\nu_j}{2}(-h^2 \frac{\partial^2}{\partial x_j^2} + x_j^2)$ and $W(x)$ is a smooth function such that $W(x) = \mathcal{O}(|x|^3)$ as $|x| \to 0$. In general, W does not commute with H, on the other hand, we do not have enough information on this perturbation, for that we will use the Birkhoff normal form of P which is a transformation of the previous type, but more adapted and less restrictive.

Let $m, d \in \mathbb{R}$, and $S^{m,d}$ be the space of smooth functions $a(x, \xi; h) : \mathbb{R}_x^N \times \mathbb{R}_\xi^N \times]0, 1] \to \mathbb{C}$ such that for all $\alpha \in \mathbb{N}^{2N}$, $|\partial_{(x,\xi)}^\alpha a(x, \xi; h)| \leq C_\alpha h^d(1 + |x|^2 + |\xi|^2)^{m/2}$ uniformly with respect to x, ξ and h, $C_\alpha > 0$. $S^d(m)$ is called the semiclassical space of symbols of order d and degree m. For $a \in S^{m,d}$ and $u \in C_0^\infty(\mathbb{R}^{2N})$, we set

$$(Op_w(a)u)(x) = (2\pi h)^{-N} \int\limits_{\mathbb{R}^{2n}} e^{ih^{-1}\langle x-x',\xi\rangle} a\Big(\frac{x + x'}{2}, \xi; h\Big) u(x')\, dx'\, d\xi. \tag{2.1}$$

$Op_w(a)$ is an unbounded linear operator on $L^2(\mathbb{R}^N)$ with domain $C_0^\infty(\mathbb{R}^{2N})$, the space of infinitely differentiable functions on $\mathbb{R}^{2N}$ with a compact support. $Op_w(a) : C_0^\infty(\mathbb{R}^{2N}) \to C^\infty(\mathbb{R}^{2N})$ is called a semiclassical pseudodifferential operator (or h-Weyl quantization) with h-Weyl symbol a of order d and degree m. Different classes of symbols can also be defined, but for our purpose this class is enough. For example, the h-Weyl symbol of the harmonic oscillator H is the polynomial $H(x, \xi) = \sum\limits_{j=1}^{N} \frac{\nu_j}{2}(x_j^2 + \xi_j^2)$. Now, we introduce the space $\mathcal{S}$ to be the set of formal series:

$$\mathcal{S} = \Big\{ \sum_{\alpha,\beta\in\mathbb{N}^N,\ \ell\in\mathbb{N}} t_{\alpha,\beta,l}\, x^\alpha \xi^\beta h^\ell : \ \ t_{\alpha,\beta,l} \in \mathbb{C} \Big\},$$

where the degree of $x^\alpha \xi^\beta h^\ell$ is defined by $|\alpha| + |\beta| + 2\ell$, for technical reasons that of h is double-counted. Let $M \in \mathbb{N}$ and $\mathcal{D}_M$ be the finite-dimensional vector space spanned by monomials $x^\alpha \xi^\beta h^\ell$ of degree M and let $\mathcal{O}_M$ be the subspace of $\mathcal{S}$ consisting of formal series, whose coefficients of degree $< M$ vanish,

$$\mathcal{D}_M = \Big\{ \sum_{\alpha,\beta\in\mathbb{N}^N,\ \ell\in\mathbb{N};\ |\alpha|+|\beta|+2\ell=M} t_{\alpha,\beta,l}\, x^\alpha \xi^\beta h^\ell : \ \ t_{\alpha,\beta,l} \in \mathbb{C} \Big\},$$

$$\mathcal{O}_M = \Big\{ \sum_{\alpha,\beta\in\mathbb{N}^N,\ \ell\in\mathbb{N}} t_{\alpha,\beta,l}\, x^\alpha \xi^\beta h^\ell : \ \ t_{\alpha,\beta,l} = 0 \ \text{ if } \ |\alpha| + |\beta| + 2\ell < M \Big\}.$$

Note that $(\mathcal{O}_M)_{M\in\mathbb{N}}$ is a filtration, $\mathcal{S}=\mathcal{O}_0\supset\mathcal{O}_1\supset\cdots$, $\bigcap\limits_{M\in\mathbb{N}}\mathcal{O}_M=\{0\}$.

Let $\langle f,g\rangle_W=\widehat{f}\widehat{g}-\widehat{g}\widehat{f}$ be the Weyl bracket on $\mathcal{S}$, where $\widehat{f}$ and $\widehat{g}$ are the h-Weyl quantizations of symbols f and g, respectively. Precisely,

$$\langle f_T,g_T\rangle_W=\sigma_W(\widehat{f}\widehat{g}-\widehat{g}\widehat{f}),$$

where f_T and g_T are the formal Taylor series at the origin of f and g in $\mathcal{S}$, respectively, and σ_W denotes the h-Weyl symbol. Then $\langle\cdot,\cdot\rangle_W$ is antisymmetric satisfying the Jacobi identity

$$\langle\langle f_T,g\rangle_W,h_T\rangle_W+\big\langle\langle h_T,f_T\rangle_W,g_T\big\rangle_W+\big\langle\langle g_T,h_T\rangle_W,f_T\big\rangle_W=0$$

and the Leibniz identity

$$\langle f_T,g_Th_T\rangle_W=\langle f_T,g_T\rangle_Wh_T+g_T\langle f_T,h_T\rangle_W.$$

Thus, the space $\mathcal{S}$ equipped with the Weyl bracket is a Lie algebra such that if $x=(x_1,\dots,x_N)$ and $\xi=(\xi_1,\dots,\xi_N)\in\mathbb{R}^N$, then

$$\langle h,x_j\rangle_W=\langle h,\xi_j\rangle_W=0\ \text{ and }\ \langle\xi_j,x_j\rangle_W=-ih\ \text{ for every }\ j=1,\dots,N.$$

The filtration of $\mathcal{S}$ has a nice behaviour with respect to the Weyl bracket, if $M_1+M_2\geq 2$, $f\in\mathcal{O}_{M_1}$ and $g\in\mathcal{O}_{M_2}$, then $h^{-1}\langle f,g\rangle_W\in\mathcal{O}_{M_1+M_2-2}$. For any $S\in\mathcal{S}$, we define the map ad_S, called the adjoint action:

$$\begin{aligned}ad_S:\ &\mathcal{S}\longrightarrow\mathcal{S}\\ &S'\longmapsto ad_S(S')=\langle S,S'\rangle_W.\end{aligned}$$

Let us consider the important special case of this concept, which is the adjoint action ad_S for $S\in\mathcal{D}_2$ and, especially, $ad_{H(x,\xi)}$. Let $\mathbb{C}[z,\overline{z},h]$ be the $\mathbb{C}$-linear space of polynomials spanned by $z^\alpha\overline{z}^\beta h^\ell$ of degree $|\alpha|+|\beta|+2\ell$; $\alpha,\beta\in\mathbb{N}^N$, $\ell\in\mathbb{N}$, where $z=(x_1+i\xi_1,\dots,x_N+i\xi_N)\in\mathbb{C}^N$ and $\overline{z}=(x_1-i\xi_1,\dots,x_N-i\xi_N)$ is the complex conjugate of z. Then $\mathcal{B}=\{z^\alpha\overline{z}^\beta:\ z\in\mathbb{C}^N,\ \alpha,\beta\in\mathbb{N}^N\}$ is a natural basis of $\mathbb{C}[z,\overline{z},h]$. We are particularly interested in the adjoint action of elements of the subspace $\mathcal{D}_2$ of $\mathcal{S}$. Such elements are of the form hH_0+H, where $H_0\in\mathbb{C}$ and H is a quadratic form in (x,ξ). Furthermore, when H is positive, it can be written as harmonic oscillators in some canonical coordinates.

The next proposition gives some important properties and results on $ad_{H(x,\xi)}$ denoted by ad_H for short, where $H(x,\xi)=\sum\limits_{j=1}^N\frac{\nu_j}{2}(x_j^2+\xi_j^2)$.

Proposition 2.1 ([5, 6])**.**

(1) $ih^{-1}ad_H(S)=\{H(x,\xi),S\}$, *where* $S\in\mathcal{S}$ *and* $\{H(x,\xi),S\}=\sum\limits_{j=1}^N\frac{\partial H}{\partial\xi_j}\frac{\partial S}{\partial x_j}-\frac{\partial H}{\partial x_j}\frac{\partial S}{\partial\xi_j}$ *is the classical Poisson bracket.*

(2) ad_H *is diagonal on* $\mathcal{B}$*, in the sense that* $ad_H(z^\alpha\overline{z}^\beta)=h\langle\beta-\alpha,\nu\rangle z^\alpha\overline{z}^\beta$, $\alpha,\beta\in\mathbb{N}^N$.

We say that an element G in $\mathcal{D}_2$ is admissible when the algebraic sum $\ker(ad_G)+\mathrm{Im}(ad_G)$ of the kernel of ad_G and the image of ad_G coincides with $\mathcal{D}_M$, $M\in\mathbb{N}$. A typical example is the harmonic oscillator $H(x,\xi)$.

Example. $H(x,\xi)=\sum\limits_{j=1}^N\frac{\nu_j}{2}(x_j^2+\xi_j^2)$ is admissible on $\mathcal{D}_M$ for all $M\in\mathbb{N}$. Indeed, let $S\in\mathcal{D}_M$, then

$$\begin{aligned}S&=\sum_{\alpha,\beta\in\mathbb{N}^N,\ \ell\in\mathbb{N};\ |\alpha|+|\beta|+2\ell=M}t_{\alpha,\beta,l}z^\alpha\overline{z}^\beta h^\ell\\ &=\sum_{|\alpha|+|\beta|+2\ell=M;\ \langle\beta-\alpha,\nu\rangle=0}t_{\alpha,\beta,l}z^\alpha\overline{z}^\beta h^\ell+\sum_{|\alpha|+|\beta|+2\ell=M;\ \langle\beta-\alpha,\nu\rangle\neq0}t_{\alpha,\beta,l}z^\alpha\overline{z}^\beta h^\ell,\end{aligned}$$

where $t_{\alpha,\beta,l} \in \mathbb{C}$ and $\nu = (\nu_1, \dots, \nu_N)$. By using Proposition 2.1, we obtain

$$\langle \beta - \alpha, \nu \rangle = 0 \Longleftrightarrow z^\alpha \overline{z}^\beta \in \ker(ad_H),$$

thus

$$\sum_{|\alpha|+|\beta|+2\ell=M;\ \langle \beta-\alpha,\ \nu\rangle=0} t_{\alpha,\beta,l} z^\alpha \overline{z}^\beta h^\ell \in \ker(ad_H),$$

$$\langle \beta - \alpha, \nu \rangle \neq 0 \Longleftrightarrow z^\alpha \overline{z}^\beta = \frac{h^{-1}}{\langle \beta - \alpha, \nu \rangle} ad_H(z^\alpha \overline{z}^\beta) \Longleftrightarrow z^\alpha \overline{z}^\beta \in \mathrm{Im}(ad_H),$$

and hence

$$\sum_{|\alpha|+|\beta|+2\ell=M;\ \langle \beta-\alpha,\nu\rangle\neq 0} t_{\alpha,\beta,l} z^\alpha \overline{z}^\beta h^\ell \in \mathrm{Im}(ad_H).$$

The Birkhoff normal form theorem can be expressed as follows.

Theorem 2.1. *Let $H \in \mathcal{D}_2$ be the harmonic oscillator and $L \in \mathcal{O}_3$. Then there exist S and K in the subspace $\mathcal{O}_3$ such that*

$$e^{ih^{-1}ad_S}(H + L) = H + K,$$

where $K = K_3 + K_4 + \cdots$ and $K_j \in \mathcal{D}_j$ commutes with H : $\langle H, K \rangle_W = 0$. Notice that the sum $e^{ih^{-1}ad_S}(H + L) = \sum\limits_l \frac{1}{l!} (\frac{i}{h} ad_S)^l (H + L)$ is convergent on $\mathcal{S}$ because $\frac{i}{h} ad_S(\mathcal{O}_M) \subset \mathcal{O}_{M+1}$. Moreover, if L has real coefficients, then S and K can be chosen to have real coefficients, as well.

Proof. We construct S and K by successive approximations. Let $M \geq 1$, we show that there exist $S_M \in \mathcal{O}_3$ and $K \in \mathcal{O}_3$ such that

$$e^{ih^{-1}ad_{S_M}}(H + L) = H + K_3 + \cdots + K_{M+1} + R_{M+2} + \mathcal{O}_{M+3}, \tag{2.2}$$

where $S_M = B_3 + B_4 + \cdots + B_{M+1}$, $B_i \in \mathcal{D}_i$, $K_i \in \mathcal{D}_i$, K_i commutes with H and $R_{M+2} \in \mathcal{D}_{M+2}$. Indeed, if $M = 2$, find $S_2 = B_3 \in \mathcal{D}_3$ and $K_3 \in \mathcal{D}_3$ which commutes with H and $R_4 \in \mathcal{D}_4$ such that

$$e^{ih^{-1}ad_{B_2}}(H + L) = H + K_3 + R_4 + \mathcal{O}_5 = H + K_3 + \mathcal{O}_4, \tag{2.3}$$

$$(2.3) \Longleftrightarrow H + L + ih^{-1}\langle B_3, H + L \rangle_W + \cdots = H + K_3 + \mathcal{O}_4.$$

As $L \in \mathcal{O}_3$, then $L = L_1 + L_2$ with $L_1 \in \mathcal{D}_3$ and $L_2 \in \mathcal{O}_4$. So,

$$(2.3) \Longleftrightarrow H_2 + L_1 + L_2 + ih^{-1}\langle B_3, H \rangle_W + ih^{-1}\langle B_3, L \rangle_W + \cdots = H + K_3 + \mathcal{O}_4.$$

Since H is admissible, it follows that $\mathcal{D}_3 = \ker(ih^{-1}ad_H) \oplus \mathrm{Im}(ih^{-1}ad_H)$ and $L_1 = L_1' + ih^{-1}\langle H, L_1'' \rangle_W$, where $L_1' \in \mathcal{D}_3$ and commutes with H, $L_1'' \in \mathcal{D}_3$. Thus, since $ih^{-1}\langle B_3, L \rangle_W \in \mathcal{O}_4$, we have

$$(2.3) \Longleftrightarrow H_2 + L_1' + ih^{-1}\langle H, L_1'' \rangle_W - ih^{-1}\langle H, B_3 \rangle_W + \mathcal{O}_4 = H + K_3 + \mathcal{O}_4.$$

So, it suffices to take $K_3 = L_1'$ and $S_2 = B_3 = L_1''$.

If $M = 3$, we need to find $B_4 \in \mathcal{D}_4$ and $K_4 \in \mathcal{D}_4$, K_4 commutes with H, such that

$$e^{ih^{-1}ad_{S_3}}(H + L) = H + K_3 + K_4 + \mathcal{O}_5, \tag{2.4}$$

where $S_3 = S_2 + B_4 = B_3 + B_4$. Using again the fact that H is admissible, we find

$$\begin{aligned}
(2.4) &\Longleftrightarrow e^{ih^{-1}ad_{B_4}}(e^{ih^{-1}ad_{B_3}}(H + L)) = H + K_3 + K_4 + \mathcal{O}_5 \\
&\Longleftrightarrow e^{ih^{-1}ad_{B_4}}(H + K_3 + R_4 + \mathcal{O}_5) = H + K_3 + K_4 + \mathcal{O}_5 \\
&\Longleftrightarrow H + K_3 + R_4 + \mathcal{O}_5 + ih^{-1}\langle B_4, H + K_3 + R_4 + \mathcal{O}_5 \rangle_W + \cdots = H + K_3 + K_4 + \mathcal{O}_5 \\
&\Longleftrightarrow R_4' + ih^{-1}\langle H, R_4'' \rangle_W - ih^{-1}\langle H, B_4 \rangle_W + \mathcal{O}_5 = K_4 + \mathcal{O}_5
\end{aligned}$$

with $R_4 = R_4' + ih^{-1}\langle H, R_4''\rangle_W$.

We then take $K_4 = R_4' \in \mathcal{D}_4$ and $B_4 = R_4'' \in \mathcal{D}_4$. Assume that the statement (2.2) holds for some arbitrary natural number $M \geq 1$, and prove that (2.2) holds for $M+1$. Thus, we want to find $B_{M+2} \in \mathcal{D}_{M+2}$, where $S_{M+1} = S_M + B_{M+2}$, and $K_{M+2} \in \mathcal{D}_{M+2}$, K_{M+2} commutes with H, so that

$$e^{ih^{-1}ad_{S_{M+1}}}(H+L) = H + K_3 + \cdots + K_{M+1} + K_{M+2} + \mathcal{O}_{M+3}; \tag{2.5}$$

$$\begin{aligned}
(2.5) \iff & e^{ih^{-1}ad_{B_{M+2}}}(e^{ih^{-1}ad_{S_M}}(H+L)) = H + K_3 + \cdots + K_{M+1} + K_{M+2} + \mathcal{O}_{M+3} \\
\iff & e^{ih^{-1}ad_{B_{M+2}}}(H + K_3 + \cdots + K_{M+1} + R_{M+2} + \mathcal{O}_{M+3}) \\
& \qquad = H + K_3 + \cdots + K_{M+1} + K_{M+2} + \mathcal{O}_{M+3} \\
\iff & H + K_3 + \cdots + K_{M+1} + R_{M+2} - ih^{-1}\langle H, B_{M+2}\rangle_W + \mathcal{O}_{M+3} \\
& \qquad = H + K_3 + \cdots + K_{M+1} + K_{M+2} + \mathcal{O}_{M+3} \\
\iff & R_{M+2} - ih^{-1}\langle H, B_{M+2}\rangle_W + \mathcal{O}_{M+3} = K_{M+2} + \mathcal{O}_{M+3} \\
\iff & R'_{M+2} + ih^{-1}\langle H, R''_{N+2}\rangle_W - ih^{-1}\langle H, B_{M+2}\rangle_W + \mathcal{O}_{M+3} = K_{M+2} + \mathcal{O}_{M+3}.
\end{aligned}$$

We can therefore take $K_{M+2} = R'_{M+2}$ and $B_{M+2} = R''_{M+2}$.

Now, if we assume that L and K_j, $j \leq M+1$, have real coefficients, then R_{M+2} is real, too. $ih^{-1}ad_H$ is a real endomorphism on each $\mathcal{D}_4$, hence (2.5) can be solved with real coefficients. $\square$

Remark 2.1. The Birkhoff normal form theorem remains valid for any element of the subspace $\mathcal{D}_2$ of $\mathcal{S}$ and in a neighborhood of the origin, via similar canonical transformations defined near 0.

3 Born–Oppenheimer approximation

The Born–Oppenheimer approximation is based on the fact that the mass of the nucleus is much greater than that of the electron [3]. This principle is exploited in order to approximate the complete molecular Schrödinger operator by a reduced Hamiltonian, acting on the positions of the nuclei only, and in which the electrons are involved through the effective electric potential they create only. The Born–Oppenheimer approximation shows how the electronic motions can be approximately separated from the nuclear motions. Let us explain the results on the Born Oppenheimer reduction for diatomic molecules with singular Coulomb-type interactions.

Consider a molecule system composed of two atomic nuclei A and B whose positions are defined by the vectors x_A and x_B and one electron of position x_e. The nuclei are assumed to be heavy with a mass of order $M \gg 1$ and the electron is light with a mass one. The Hamiltonian of the system is given by

$$\mathcal{P} = -\frac{1}{2M}\,\partial^2_{x_A} - \frac{1}{2M}\,\partial^2_{x_B} - \frac{1}{2}\,\partial^2_{x_e} + V(x_A - x_e) + V(x_B - x_e) + W(x_A - x_B),$$

where V and W represent the Coulomb interactions $V(x) = -\frac{\alpha}{|x|}$ and $W(x) = \frac{\beta}{|x|}$; α and β are real constants, $\alpha > 0$, $\beta > 0$. $\mathcal{P}$ is the sum of kinetic energy of the atomic nuclei $-\frac{1}{2M}\,\partial^2_{x_A} - \frac{1}{2M}\,\partial^2_{x_B}$, kinetic energy of the electrons $-\frac{1}{2}\,\partial^2_{x_e}$, internuclear repulsion $W(x_A - x_B)$, and electronic-nuclear attraction $V(x_A - x_e) + V(x_B - x_e)$. Removing the center of mass motion of this system and choosing properly the coordinates, one can correctly describe this approximation. Indeed, we consider the center of a mass coordinate system

$$R = \frac{Mx_A + Mx_B + x_e}{2M+1}, \quad x = x_A - x_B, \quad y = x_e - \frac{x_A + x_B}{2}.$$

In these coordinates, the Hamiltonian $\mathcal{P}$ becomes

$$\begin{aligned}
\mathcal{P} &= -\frac{1}{2(2M+1)}\,\partial^2_R + P, \\
P &= -\frac{1}{M}\,\partial^2_x - \frac{1}{2}\Big(1 + \frac{1}{2M}\Big)\partial^2_y + V\Big(\frac{x}{2} - y\Big) + V\Big(\frac{x}{2} + y\Big) + W(x).
\end{aligned}$$

If we remove the center of mass motion, the study of $\mathcal{P}$ is reduced to that of the operator P on $L^2(\mathbb{R}^6)$, where the spectrum of P defines the energy levels of the molecule. The Born–Oppenheimer approximation is a very important method for analyzing this spectrum when M, the mass of nuclei, tends to infinity. In general, molecular systems of $n+p+1$ particles ($n+1$ nuclei and p electrons) in the semiclassical limit, where the mass ratio h^2 of electronic to nuclear mass tends to zero, are described by the many-body Hamiltonians of the type

$$P = -h^2\Delta_x - \Delta_y + V(x,y),$$

where V is the sum of all interactions between the particles, $x \in \mathbb{R}^N$, $N = 3n$, denote the relative positions of the nuclei, and $y \in \mathbb{R}^{N'}$, $N' = 3p$, those of the electrons. P is defined on $L^2(\mathbb{R}^N_x \times \mathbb{R}^{N'}_y)$, we denote by $Q(x)$ the electronic Hamiltonian $-\Delta_y + V(x,y)$ on $L^2(\mathbb{R}^{N'}_y)$. Then, one can define the so-called electronic levels being the discrete eigenvalues $\lambda_1(x) < \lambda_2(x) \leq \cdots$ of the operator $Q(x)$. Born and Oppenheimer [3] realized that the study of P can be approximately reduced, when h is small, to the diagonal matrix $\operatorname{diag}(-h^2\Delta_x + \lambda_j(x))$, $j = 1, 2, \ldots$ on $\bigoplus_j L^2(\mathbb{R}^N_x)$. In particular, when, for example, the first simple eigenvalue $\lambda_1(x)$ admits a non-degenerate point well at some energy level E, the eigenvalues of P near E should admit a complete asymptotic expansion in half-powers of h (WKB expansions). This principle has been widely used by chemists, but the mathematically rigorous justifications of this reduction and WKB expansions for eigenfunctions and eigenvalues of a diatomic molecule are more recent. Such a result was proved for smooth interactions (see, e.g., [4]), it was generalized later by Belmouhoub and Messirdi to singular Coulombic potentials where they introduced some x-dependent changes in the y-variables that will regularize the associated eigenfunctions, localize in a compact region the x-dependent singularities with respect to y in the interactions and construct a kind of semiclassical pseudodiffcrcntial calculus, adapted to these changes [1].

3.1 Pseudodifferential calculus with operator-valued symbols

In the literature, there exist several versions of operator-valued pseudodifferential calculus, each adopted to some particular, more or less general, situation. We recall here the constructions made mainly in [10]. Let Ω be a bounded open subset of $\mathbb{R}^N_x$, and $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ be complex Hilbert spaces. $\mathcal{B}(\mathcal{H},\mathcal{K})$ is the algebra of all continuous linear operators from $\mathcal{H}$ into $\mathcal{K}$. We denote by $C^\infty(\Omega,\Lambda)$ the set of all infinitely differentiable functions from Ω to $\Lambda = \mathcal{H},\mathcal{K},\mathcal{L}$. Given $\psi \in C^\infty(\Omega,\mathbb{R})$ and $\mathcal{V}$ a neighborhood of 0 in $\mathbb{R}^N_x$, we set

$$\Omega^* = \big\{(x,\xi) \in \Omega \times \mathbb{C}^N : \ \ \xi - i\nabla\psi(x) \in \mathcal{V}\big\}.$$

Pseudodifferential operators can be considered in the following more general context. For $m \in \mathbb{R}$, consider the spaces of formal power series

$$S^m(\Omega,\mathcal{H}) = \left\{ \sum_{j=0}^{\infty} h^{-m+j/2} s_j(x) : \ \ s_j \in C^\infty(\Omega,\mathcal{H}) \right\},$$

$$e^{-\psi(x)/h} S^m(\Omega,\mathcal{H}) = \left\{ \sum_{j=0}^{\infty} h^{-m+j/2} e^{-\psi(x)/h} s_j(x) : \ \ s_j \in C^\infty(\Omega,\mathcal{H}) \right\},$$

$$S^0(\Omega^*,\mathcal{B}(\mathcal{H},\mathcal{K})) = \left\{ \sum_{j=0}^{\infty} h^j a_j(x,\xi) : \ \ a_j \in C^\infty(\Omega^*,\mathcal{B}(\mathcal{H},\mathcal{K})) \right\}.$$

The operator-valued functions in $S^0(\Omega^*,\mathcal{B}(\mathcal{H},\mathcal{K}))$ are called symbols. For any symbol $a = a(x,\xi;h)$ in $S^0(\Omega^*,\mathcal{B}(\mathcal{H},\mathcal{K}))$, one can define an operator $Op(a)$ from $e^{-\psi(x)/h}S^m(\Omega,\mathcal{H})$ into $e^{-\psi(x)/h}S^m(\Omega,\mathcal{K})$ by the formula

$$Op(a)(e^{-\psi(x)/h}s(x,h)) = e^{-\psi(x)/h} \sum_{\alpha\in\mathbb{N}^N} \frac{h^{|\alpha|}}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha a(x,i\nabla\psi(x);h) \partial_y^\alpha (e^{\chi(x,y)/h} s(y,h))_{y=x},$$

$\chi(x,y) = \psi(y) - \psi(x) - (y-x).\nabla\psi(x) = \mathcal{O}(|x-y|^2)$, $s \in S^m(\Omega,\mathcal{H})$. $Op(a)$ is called an h-pseudodifferential the operator with operator-valued symbol $a(x,\xi;h) = \sum\limits_{j=0}^{\infty} h^j a_j(x,\xi)$. The function $a_0(x,\xi)$ (coefficient of h^0) is called principal symbol of $Op(a)$. Furthermore, such operators verify $e^{\psi(x)/h}Op(a)(e^{-\psi(x)/h}s(x,h)) \in S^m(\Omega,\mathcal{H})$ and can be composed by using the formula

$$Op(b) \circ Op(a) = Op(b \sharp a), \tag{3.1}$$
$$b \sharp a(x,\xi;h) = \sum_{\alpha\in\mathbb{N}^N} \frac{h^{|\alpha|}}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha b(x,\xi;h)\partial_x^\alpha a(x,\xi;h) \in S^0(\Omega^*,\mathcal{B}(\mathcal{H},\mathcal{K})).$$

where $a \in S^0(\Omega^*,\mathcal{B}(\mathcal{H},\mathcal{K}))$, $b \in S^0(\Omega^*,\mathcal{B}(\mathcal{K},\mathcal{L}))$ and the range of $Op(a)$ is contained in the domain of $Op(b)$. This formula makes it possible to inverse asymptotically operators $Op(a)$, whose principal symbol $a_0(x,\xi)$ is invertible as a linear operator from $\mathcal{H}$ into $\mathcal{K}$.

3.2 Representation of the effective Hamiltonian

Let $\Omega \subset \mathbb{R}_x^N$ be an open neighborhood of 0 and $V \in C^\infty(\Omega,\mathcal{B}(H^2(\mathbb{R}_y^{N'}),L^2(\mathbb{R}_y^{N'})))$ be Δ_y-compact:

$$V(x,y)(-\Delta_y+1)^{-1} \in C^\infty(\Omega,\mathcal{B}(L^2(\mathbb{R}_y^{N'}))). \tag{3.2}$$

Thus, P is self-adjoint on $L^2(\mathbb{R}_x^N \times \mathbb{R}_y^{N'})$ with domain the Sobolev space $H^2(\mathbb{R}_x^N \times \mathbb{R}_y^{N'})$, as well as the operator $Q(x)$ is self-adjoint on $L^2(\mathbb{R}_y^{N'})$ with domain $H^2(\mathbb{R}_y^{N'})$.

For the sake of simplicity, we take into account only the first electronic level $\lambda_1(x) = \inf(\sigma(Q(x))$ and call $u_1(x,y)$ the first eigenfunction of $Q(x)$ associated to $\lambda_1(x)$ and normalized, $\|u_1(x,\cdots)\|_{L^2(\mathbb{R}_y^{N'})} = 1$ in $L^2(\mathbb{R}_y^{N'})$ for any $x \in \mathbb{R}^N$. We also assume that $\lambda_1(x)$ is separated by a constant gap from the rest of the spectrum $\sigma(Q(x))$, i.e.,

$$\inf_{x\in\mathbb{R}^N}\Big(\inf\big(\sigma(Q(x)) \setminus \{\lambda_1(x)\}\big)\Big) > 0, \tag{3.3}$$

and $\lambda_1(x)$ has a unique and non-degenerate minimum at 0:

$$\lambda_1(x) \geq 0, \ \ \lambda_1^{-1}(0) = \{0\}, \ \ \lambda_1'(0) = 0, \ \ \lambda_1''(0) > 0. \tag{3.4}$$

It can be shown that $\lambda_1 \in C^\infty(\Omega,\mathbb{R})$ and $u_1 \in C^\infty(\Omega,H^2(\mathbb{R}_y^{N'}))$ (cf. [10]). In particular, the assumptions (3.2) and (3.3) imply that the orthogonal projection $\Pi(x)$ on the subspace of $L^2(\mathbb{R}_y^{N'})$, spanned by $u_1(x,\cdots)$, $x \in \Omega$, is C^2-regular with respect to x (see [4]). To construct the effective Hamiltonian of P, the idea here is to use the pseudodifferential calculus with operator-valued symbols developed previously.

For $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda < \inf(\sigma(Q(x)) \setminus \{\lambda_1(x)\})$, we consider the Grushin operator

$$P_\lambda = \begin{pmatrix} P-\lambda & u_1 \\ \langle\,\cdot\,,u_1\rangle_y & 0 \end{pmatrix}$$

acting on $L^2(\mathbb{R}_x^N \times \mathbb{R}_y^{N'}) \oplus L^2(\mathbb{R}_y^{N'})$, where $\langle\,\cdot\,,u_1\rangle_y$ is the inner product in $L^2(\mathbb{R}_y^{N'})$. It follows from the assumptions that $P_\lambda = Op(a_\lambda)$ is an h-pseudodifferential operator in x, from $e^{-\psi(x)/h}S^m(\Omega,H^2(\mathbb{R}_y^{N'}))$ into $e^{-\psi(x)/h}S^m(\Omega,L^2(\mathbb{R}_y^{N'}))$, with the operator-valued symbol a_λ,

$$a_\lambda(x,\xi) = \begin{pmatrix} \xi^2 + Q(x) - \lambda & u_1 \\ \langle\,\cdot\,,u_1\rangle_y & 0 \end{pmatrix} \in S^0(\Omega^*,\mathcal{B}(H^2(\mathbb{R}_y^{N'}) \oplus \mathbb{C}, L^2(\mathbb{R}_y^{N'}) \oplus \mathbb{C})),$$

where $\psi(x)$ is the Agmon distance associated to the metric $\lambda_1(x)\,dx^2$. We show that P_λ is invertible and describe a method for finding its inverse. Using the fact that $(\nabla\psi)^2(x) = \lambda_1(x)$ and the gap assumption (3.3), one can easily show that for $|\lambda|$ small enough and ξ close enough to $i\nabla\psi(x)$, $\mathrm{Re}(\widehat{\Pi}(x)Q(x)\widehat{\Pi}(x)-\lambda)>0$ and thus a_λ is invertible with inverse

$$b_0(x,\xi;\lambda)=\begin{pmatrix}\widehat{\Pi}(x)(\xi^2+\widehat{\Pi}(x)Q(x)\widehat{\Pi}(x)-\lambda)^{-1}\widehat{\Pi}(x) & u_1\\ \langle\,\cdot\,,u_1\rangle_y & \lambda-\xi^2-\lambda_1(x)\end{pmatrix},$$

where $\widehat{\Pi}(x)=1-\Pi(x)$ (see, e.g., [1]). In particular, $b_0(x,\xi;\lambda)\in S^0(\Omega^*,\mathcal{B}(L^2(\mathbb{R}_y^{N'})\oplus\mathbb{C},H^2(\mathbb{R}_y^{N'})\oplus\mathbb{C}))$. Then using the composition formula (3.2), it is easy to construct a symbol

$$b_\lambda(x,\xi;h)=b_0(x,\xi;\lambda)+hb_1(x,\xi;\lambda)+h^2b_2(x,\xi;\lambda)+\cdots$$
$$b_\lambda(x,\xi;h)\in S^0(\Omega^*,\mathcal{B}(L^2(\mathbb{R}_y^{N'})\oplus\mathbb{C},H^2(\mathbb{R}_y^{N'})\oplus\mathbb{C})),$$

such that $a_\lambda\,\sharp\,b_\lambda(x,\xi;h)=1$ and $Op(a_\lambda)\circ Op(b_\lambda)=I$, where I is the identity operator on $e^{-\psi(x)/h}S^m(\Omega,L^2(\mathbb{R}_y^{N'})\oplus\mathbb{C})$. Let us pose

$$Op(b_\lambda)=\begin{pmatrix}E(\lambda) & E_+(\lambda)\\ E_-(\lambda) & E_\mp(\lambda)\end{pmatrix}.$$

By Lemma 3.1 in [1], we know that $E_\mp(\lambda)=Op(e_\lambda(x,\xi;\lambda))$ is h-pseudodifferential operator with the symbol $e_\lambda(x,\xi;\lambda)\in S^0(\Omega^*,\mathbb{C})$ and its principal symbol is $e_0(x,\xi;\lambda)=\lambda-\xi^2-\lambda_1(x)$. In particular, $\lambda-E_\mp(\lambda)$ is a scalar h-pseudodifferential operator with the principal symbol $\xi^2+\lambda_1(x)$. Moreover, we have the following fundamental spectral reduction:

$$\lambda\in\sigma(P)\Longleftrightarrow\lambda\in\sigma(\lambda-E_\mp(\lambda)).$$

Hence, the spectral study of the Hamiltonian P on $L^2(\mathbb{R}_x^N\times\mathbb{R}_y^{N'})$ is reduced to that of the h-pseudodifferential operator $\lambda-E_\mp(\lambda)$ on $L^2(\mathbb{R}_x^N)$, the so-called effective Hamiltonian of P. In fact, one can show in many situations that $\lambda-E_\mp(\lambda)=P_e+\mathcal{O}(h^2)$, which makes it easy to compare (using, for example, the maximum principle) the eigenvalues of P and those of $P_e=-h^2\Delta_x+\lambda_1(x)$, and then identify them when h decays to zero fast enough [4]. In the next section, this reduction will justify our definition of the normal Birkhoff forms for P as those of the effective Hamiltonian P_e.

4 The Birkhoff normal forms for the Born–Oppenheimer Hamiltonian and resonances

In the previous section, it has been established that the Born–Oppenheimer Hamiltonian P can be reduced to the effective Hamiltonian $P_e=-h^2\Delta_x+\lambda_1(x)$ on $L^2(\mathbb{R}_x^N)$, modulo $\mathcal{O}(h^2)$. Thus, it is natural to define the Birkhoff normal forms of P as those of P_e modulo $\mathcal{O}(h^2)$.

Definition 4.1. We call normal forms of the Born–Oppenheimer Hamiltonian P the Birkhoff normal forms of the associated effective Hamiltonian P_e when the semiclassical parameter h tends to zero.

Assumption (3.4) implies that $\lambda_1(x)\in\mathcal{O}_3$, and since $H+\lambda_1(x)\in\mathcal{D}_2$, one can obtain the quantum Birkhoff normal forms for P_e as a direct consequence of the Birkhoff normal form theorem (Theorem 2.1), when the potential energy operator $V(x)=\lambda_1(x)$ is regular and the Hessian matrix $\lambda_1''(0)$ is diagonal with the eigenvalues $(\nu_1^2,\dots,\nu_N^2)$, $\nu_j>0$. The complicated behavior of the dynamics and spectrum of a molecular system happens under a resonance. In this case, to decide wether the Hamiltonian has resonance frequencies or not, we need the following definitions.

Definition 4.2. The frequencies vector $\nu=(\nu_1,\dots,\nu_N)$ is non-resonant if $k\cdot\nu=\sum\limits_{j=1}^N k_j\nu_j\neq 0$ for all $k\in\mathbb{Z}^N\setminus\{0\}$. ν is resonant if $\nu_1,\dots,\nu_N$ are dependent over $\mathbb{Z}$, i.e., there exist integers $k_1,\dots,k_N\in\mathbb{Z}$, not all zero, such that $k_1\nu_1+\dots+k_N\nu_N=0$. The number $r=\sum\limits_{j=1}^N|k_j|$ is called the degree of resonance of P_e. In the particular resonant case, where $\nu_j=\nu_ck_j$ for every $j=1,\dots,N$, with $\nu_c>0$ and $k_1,\dots,k_N\in\mathbb{N}$, the frequencies vector ν is said to be completely resonant.

For a theoretical definition of resonances, the interested reader may consult the excellent paper [10].

As an application we study the dynamics near a local extremum of the effective Hamiltonian, for which the frequencies are in 1 : 1 Darling–Dennison resonance (ν_j, ν_j). This is a well-known effect in the overtone spectroscopy of molecules such as water molecule H_2O, acetylene C_2H_2, methylidynephosphane (phosphaethyne) $HCP, \dots$.

In what follows, we explicitly give the computations of Birkhoff normal forms in the 1 : 1 resonance for P, therefore, for the effective Hamiltonian P_e of P, the situation which can be encountered in physical models, like small molecules. So, all the following computations are valid modulo $\mathcal{O}(h^2)$.

Consider the semiclassical harmonic oscillator with the resonant frequencies vector $\nu = (1, 1)$:

$$H = \frac{1}{2}\Big(-h^2\frac{\partial^2}{\partial x_1^2} + x_1^2\Big) + \frac{1}{2}\Big(-h^2\frac{\partial^2}{\partial x_2^2} + x_2^2\Big)$$

and the symbol $H(z_1, z_2) = \frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2$, where $z_j = x_j + i\xi_j$, $j = 1, 2$.

To find a Birkhoff normal form for P, we construct a formal series K_3 in $\mathcal{D}_3$ such that $\langle H_2, K_3\rangle_W = 0$. Thus, $K_3 = \sum\limits_{\alpha,\beta\in\mathbb{N}^2,\ 2\ell+|\alpha|+|\beta|=3} h^\ell z^\alpha \overline{z}^\beta$ and we should verify the resonance relation $\langle \nu, \beta - \alpha\rangle = 0$. Let $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$,

$$\langle \nu, \beta - \alpha\rangle = 0 \Longleftrightarrow \beta_1 - \alpha_1 + \beta_2 - \alpha_2 = 0 \Longleftrightarrow \alpha_1 + \alpha_2 = \beta_1 + \beta_2. \tag{4.1}$$

We then look for all monomials of order 3 of type $z_1^{\alpha_1} z_2^{\alpha_2} \overline{z}_1^{\beta_1} \overline{z}_2^{\beta_2}$ satisfying the resonance relation (4.1). The system

$$\begin{cases} \alpha_1 + \alpha_2 + \beta_1 + \beta_2 = 3, \\ \alpha_1 + \alpha_2 = \beta_1 + \beta_2 \end{cases}$$

does not admit solutions in $\mathbb{N}$. Thus, there is no monomial in $\mathcal{D}_3$ verifying $|\alpha| + |\beta| = 3$ and the resonance relation (4.1), $K_3 = 0$, but one can calculate $K_4 \in \mathcal{D}_4$. The couples $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ and $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$ which verify the system $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = 4$ and $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$, are

$$\alpha = \beta = (1, 1);\quad \alpha = \beta = (2, 0);\quad \alpha = \beta = (0, 2);$$
$$\alpha = (2, 0)\ \text{ and }\ \beta = (0, 2);\quad \alpha(0, 2)\ \text{ and }\ \beta = (2, 0).$$

Therefore, K_4 is generated by the monomials $|z_1|^4$; $|z_2|^4$; $|z_1|^2|z_2|^2$; $z_1^2\overline{z}_2^2$; $\overline{z}_1^2 z_2^2$ and h^2. Since K_4 is real, we have

$$K_4 = a_1|z_1|^4 + a_2|z_2|^4 + a_3|z_1|^2|z_2|^2 + a_4\,\mathrm{Re}(z_1^2\,\overline{z}_2^2) + \mathcal{O}(h^2);\quad a_1, a_2, a_3, a_4 \in \mathbb{R}.$$

We can use Taylor series for $\lambda_1(x)$ to determine the coefficients a_1, a_2, a_3 and a_4. Remember that $P_e = H + \lambda_1^{(3)}(x) + \lambda_1^{(4)}(x) + \cdots$,

$$\lambda_1^{(3)}(x) = \frac{1}{12\sqrt{2}}\frac{\partial^3\lambda_1}{\partial x_1^3}(0)x_1^3 + \frac{1}{4\sqrt{2}}\frac{\partial^3\lambda_1}{\partial x_1^2\partial x_2}(0)x_1^2x_2 + \frac{1}{4\sqrt{2}}\frac{\partial^3\lambda_1}{\partial x_1\partial x_2^2}(0)x_1x_2^2 + \frac{1}{12\sqrt{2}}\frac{\partial^3\lambda_1}{\partial x_2^3}(0)x_2^3.$$

By setting $y_j = \frac{1}{\sqrt{2}}(z_j + \overline{z}_j)$, $j = 1, 2$, and after a long but straightforward calculation, we can determine all monomials that are in K_4,

$$\begin{aligned}
&-\frac{5}{48}\bigg[\Big(\frac{\partial^3\lambda_1}{\partial x_1^3}(0)\Big)^2 + \Big(\frac{\partial^3\lambda_1}{\partial x_1^2\partial x_2}(0)\Big)^2\bigg]|z_1|^4 - \frac{5}{48}\bigg[\Big(\frac{\partial^3\lambda_1}{\partial x_2^3}(0)\Big)^2 + \Big(\frac{\partial^3\lambda_1}{\partial x_1\partial x_2^2}(0)\Big)^2\bigg]|z_2|^4 \\
&+\frac{1}{8}\bigg[\frac{\partial^3\lambda_1}{\partial x_1^3}(0)\,\frac{\partial^3\lambda_1}{\partial x_1^2\partial x_2}(0) + \frac{\partial^3\lambda_1}{\partial x_2^3}(0)\,\frac{\partial^3\lambda_1}{\partial x_2^2\partial x_1}(0) \\
&\qquad\qquad + \frac{\partial^3\lambda_1}{\partial x_1^3}(0)\,\frac{\partial^3\lambda_1}{\partial x_2^2\partial x_1}(0) + \frac{\partial^3\lambda_1}{\partial x_2^3}(0)\,\frac{\partial^3\lambda_1}{\partial x_1^2\partial x_2}(0)\bigg]|z_1|^2|z_2|^2 \\
&+\frac{1}{6}\bigg[\bigg(\Big(\frac{\partial^3\lambda_1}{\partial x_1^2\partial x_2}(0)\Big)^2 + \Big(\frac{\partial^3\lambda_1}{\partial x_2^2\partial x_1}(0)\Big)^2\bigg)\bigg]|z_1|^2|z_2|^2
\end{aligned}$$

$$- \frac{1}{192}\left[\frac{\partial^3\lambda_1}{\partial x_1^3}(0)\,\frac{\partial^3\lambda_1}{\partial x_1^2\partial x_2}(0)+\frac{\partial^3\lambda_1}{\partial x_2^3}(0)\,\frac{\partial^3\lambda_1}{\partial x_1\partial x_2^2}(0)\right.$$
$$\left.+\frac{\partial^3\lambda_1}{\partial x_1^3}(0)\,\frac{\partial^3\lambda_1}{\partial x_1\partial x_2^2}(0)+\frac{\partial^3\lambda_1}{\partial x_2^3}(0)\,\frac{\partial^3\lambda_1}{\partial x_1^2\partial x_2}(0)\right]\mathrm{Re}(z_1^2\overline{z}_2^2).$$

The fourth degree Taylor polynomial for $\lambda_1(x)$ at 0 is given by

$$\lambda_1^{(4)}(x_1,x_2)=\frac{1}{4!}\,\frac{\partial^4\lambda_1}{\partial x_1^4}(0)x_1^4+\frac{1}{4!}\,\frac{\partial^4\lambda_1}{\partial x_2^4}(0)x_2^4+\frac{1}{6}\,\frac{\partial^4\lambda_1}{\partial x_1^3\partial x_2}(0)x_1^3x_2$$
$$+\frac{1}{6}\,\frac{\partial^4\lambda_1}{\partial x_1\partial x_2^3}(0)x_1x_2^3+\frac{1}{4}\,\frac{\partial^4\lambda_1}{\partial x_1^2\partial x_2^2}(0)x_1^2x_2^2.$$

It is easy to see that only $\frac{1}{4!}\frac{\partial^4\lambda_1}{\partial x_1^4}(0)x_1^4$, $\frac{1}{4}\frac{\partial^4\lambda_1}{\partial x_1^2\partial x_2^2}(0)x_1^2x_2^2$ and $\frac{1}{4!}\frac{\partial^4\lambda_1}{\partial x_2^4}(0)x_2^4$ contain the terms of K_4, the remainder terms are absorbed by the rest of the Taylor series

$$y_1^4=\frac{1}{4}(z_1+\overline{z}_1)^4=\frac{1}{4}\Big(z_1^4+4z_1^2|z_1|^2+\underbrace{6|z_1|^4}_{\in K_4}+4\overline{z}_1^2|z_1|^2+\overline{z}_1^4\Big),$$
$$y_2^4=\frac{1}{4}(z_2+\overline{z}_2)^4=\frac{1}{4}\Big(z_2^4+4z_2^2|z_2|^2+\underbrace{6|z_2|^4}_{\in K_4}+4\overline{z}_2^2|z_2|^2+\overline{z}_2^4\Big),$$
$$y_1^2y_2^2=\frac{1}{4}(z_1+\overline{z}_1)^2(z_2+\overline{z}_2)^2=\frac{1}{4}\,z_1^2z_2^2+\frac{1}{4}\,\underbrace{z_1^2\overline{z}_2^2}_{\in K_4}+\frac{1}{2}\,\overline{z}_1^2|z_2|^2$$
$$+\frac{1}{4}\,\underbrace{\overline{z}_1^2z_2^2}_{\in K_4}+\frac{1}{4}\,\overline{z}_1^2\overline{z}_2^2+\frac{1}{2}\,\overline{z}_1^2|z_2|^2+\frac{1}{2}\,z_2^2|z_1|^2+\frac{1}{2}\,|z_1|^2\overline{z}_2^2+\underbrace{|z_1|^2|z_2|^2}_{\in K_4}.$$

Therefore,

$$a_1=\frac{1}{16}\,\frac{\partial^4\lambda_1}{\partial x_1^4}(0)-\frac{5}{48}\left[\Big(\frac{\partial^3\lambda_1}{\partial x_1^3}(0)\Big)^2+\Big(\frac{\partial^3\lambda_1}{\partial x_1^2\partial x_2}(0)\Big)^2\right],\tag{4.2}$$
$$a_2=\frac{1}{16}\,\frac{\partial^4\lambda_1}{\partial x_2^4}(0)-\frac{5}{48}\left[\Big(\frac{\partial^3\lambda_1}{\partial x_2^3}(0)\Big)^2+\Big(\frac{\partial^3\lambda_1}{\partial x_1\partial x_2^2}(0)\Big)^2\right],$$
$$a_3=\frac{1}{4}\,\frac{\partial^4\lambda_1}{\partial x_1^2\partial x_2^2}(0)+\frac{1}{8}\left[\Big(\frac{\partial^3\lambda_1}{\partial x_1^3}(0)\,\frac{\partial^3\lambda_1}{\partial x_1^2\partial x_2}(0)+\frac{\partial^3\lambda_1}{\partial x_2^3}(0)\,\frac{\partial^3\lambda_1}{\partial x_2^2\partial x_1}(0)\Big)\right.$$
$$\left.+\Big(\frac{\partial^3\lambda_1}{\partial x_1^3}(0)\,\frac{\partial^3\lambda_1}{\partial x_2^2\partial x_1}(0)+\frac{\partial^3\lambda_1}{\partial x_2^3}(0)\frac{\partial^3\lambda_1}{\partial x_1^2\partial x_2}(0)\Big)\right]$$
$$+\frac{1}{6}\left[\Big(\Big(\frac{\partial^3\lambda_1}{\partial x_1^2\partial x_2}(0)\Big)^2+\Big(\frac{\partial^3\lambda_1}{\partial x_2^2\partial x_1}(0)\Big)^2\Big)\right],$$
$$a_4=\frac{1}{8}\,\frac{\partial^4\lambda_1}{\partial x_1^2\partial x_2^2}(0)-\frac{1}{192}\left[\frac{\partial^3\lambda_1}{\partial x_1^3}(0)\,\frac{\partial^3\lambda_1}{\partial x_1^2\partial x_2}(0)+\frac{\partial^3\lambda_1}{\partial x_2^3}(0)\,\frac{\partial^3\lambda_1}{\partial x_1\partial x_2^2}(0)\right.$$
$$\left.+\frac{\partial^3\lambda_1}{\partial x_1^3}(0)\,\frac{\partial^3\lambda_1}{\partial x_1\partial x_2^2}(0)+\frac{\partial^3\lambda_1}{\partial x_2^3}(0)\,\frac{\partial^3\lambda_1}{\partial x_1^2\partial x_2}(0)\right].$$

The Weyl quantization $Op_w(K_4)$ of K_4 is given by

$$Op_w(K_4)=a_1Op_w\big(|z_1|^4\big)+a_2Op_w\big(|z_2|^4\big)+a_3Op_w\big(|z_1|^2|z_2|^2\big)+a_4Op_w\big(\mathrm{Re}(z_1^2\overline{z}_2^2)\big)+\mathcal{O}(h^2).$$

Furthermore,

$$|z_1|^4=x_1^4+\xi_1^4+2x_1^2\xi_1^2,$$
$$|z_2|^4=x_2^4+\xi_2^4+2x_2^2\xi_2^2,$$
$$|z_1|^2|z_2|^2=x_1^2x_2^2+x_1^2\xi_2^2+x_2^2\xi_1^2+\xi_1^2\xi_2^2,$$

$$\mathrm{Re}(z_1^2\overline{z}_2^2) = x_1^2x_2^2 - x_1^2\xi_2^2 - x_2^2\xi_1^2 + \xi_1^2\xi_2^2 + 4x_1x_2\xi_1\xi_2,$$

then the Weyl quantization of every monomial gives

$$\begin{aligned}
Op_w\big(|z_1|^4\big) &= x_1^4 + h^4\frac{\partial^4}{\partial x_1^4} - h^2\Big[2x_1^2\frac{\partial^2}{\partial x_1^2} + 1\Big],\\
Op_w\big(|z_2|^4\big) &= x_2^4 + h^4\frac{\partial^4}{\partial x_2^4} - h^2\Big[2x_2^2\frac{\partial^2}{\partial x_2^2} + 1\Big],\\
Op_w\big(|z_1|^2|z_2|^2\big) &= x_1^2x_2^2 - h^2\Big[x_1^2\frac{\partial^2}{\partial x_2^2} + x_2^2\frac{\partial^2}{\partial x_1^2} - h^2\frac{\partial^4}{\partial x_1^2\partial x_2^2}\Big],\\
Op_w(\mathrm{Re}(z_1^2\overline{z}_2^2)) &= x_1^2x_2^2 - h^2\Big[-x_1^2\frac{\partial^2}{\partial x_2^2} - x_2^2\frac{\partial^2}{\partial x_1^2} - h^2\frac{\partial^4}{\partial x_1^2\partial x_2^2} + 2x_1x_2\frac{\partial^2}{\partial x_1\partial x_2} + 2\Big].
\end{aligned}$$

Finally, we obtain the following Birkhoff normal form in the 1 : 1 resonance of the Hamiltonian P with the electronic energy level $\lambda_1(x)$:

$$\begin{aligned}
H + Op_w(K_4) = {} & \frac{1}{2}\Big(-h^2\frac{\partial^2}{\partial x_1^2} + x_1^2\Big) + \frac{1}{2}\Big(-h^2\frac{\partial^2}{\partial x_2^2} + x_2^2\Big)\\
& + a_1\Big[x_1^4 + \hbar^4\frac{\partial^4}{\partial x_1^4} - h^2\Big(x_1^2\frac{\partial^2}{\partial x_1^2} + 2\Big)\Big]\\
& + a_2\Big[x_2^4 + h^4\frac{\partial^4}{\partial x_2^4} - h^2\Big(x_2^2\frac{\partial^2}{\partial x_2^2} + 2\Big)\Big]\\
& + a_3\Big[x_1^2x_2^2 - h^2\Big(x_1^2\frac{\partial^2}{\partial x_2^2} + x_2^2\frac{\partial^2}{\partial x_1^2} - h^2\frac{\partial^4}{\partial x_1^2\partial x_2^2}\Big)\Big]\\
& + a_4\Big[x_1^2x_2^2 + h^2x_1^2\frac{\partial^2}{\partial x_2^2} + h^2x_2^2\frac{\partial^2}{\partial x_1^2} + h^4\frac{\partial^4}{\partial x_1^2\partial x_2^2}\\
& \qquad - 4h^2x_1x_2\frac{\partial^2}{\partial x_1\partial x_2} + 2h^2x_1\frac{\partial}{\partial x_1} + 2h^2x_2\frac{\partial}{\partial x_2}\Big] + \mathcal{O}(h^2).
\end{aligned}$$

Remark 4.1. To study just a small neighborhood of some fixed energy level, it suffices to take the first electronic level $\lambda_1(x)$ of $Q(x)$. However, in order to study a larger range of energy, we shall as well treat the case of several electronic levels $\lambda_1(x), \dots, \lambda_N(x)$ (N arbitrary), and assume that there exists a gap between them and the rest of the spectrum of $Q(x)$. In such a case, the effective Hamiltonian is an $N \times N$ matrix of pseudodifferential operators; does this general situation lead to the same Birkhoff normal form theorem? We hope to investigate this interesting question in a future work.

5 Numerical results for the 1 : 1 resonance

The 1 : 1 symbol $H(x,\xi) = \frac{1}{2}(x_1^2+\xi_1^2) + \frac{1}{2}(x_2^2+\xi_2^2)$, $x=(x_1,x_2)$, $\xi=(\xi_1,\xi_2)$, of the harmonic oscillator is defined by using the Maple notation as follows:

```
| let H = Maple. to_poly "0.5 * x[1]^2+0.5 * xi[1]^2+0.5 * x[2]^2+0.5 * xi[2]^2";;
```

H is converted in the complex coordinates to $H(z_1,z_2) = \frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2$, $z_j = \frac{1}{\sqrt{2}}(x_j + i\xi_j)$, $j=1,2$. In order to deal with harmonic oscillators in real variables (x_j,ξ_j), we need to use the new variables $x_j' = \frac{1}{\sqrt{2}}(x_j + i\xi_j)$, $\xi_j' = \frac{1}{\sqrt{2}}(x_j - i\xi_j)$, $j=1,2$. The harmonic oscillator has now the required form $H = x_1'\xi_1' + x_2'\xi_2'$.

```
| let  Hz = coordz  H;;
| Maple. of_poly  Hz;;
| - : string = "1 * x[1]^1 * xi[1]^1+1 * x[2]^1 * xi[2]^1"
```

We add now a simple perturbation $\lambda_1(x_1, x_2) = x_1^2 x_2^2$, which we convert to complex coordinates:

```
| let λ1 = Maple.to_poly "x[1]^2 * x[2]^2";;
| let λ1z = coordz λ1;;
| Maple.of_poly vz;;
| − : string =
| "1.0606601810596428 * x[2]^1 * xi[2]^2+0.3535533936865476 * x[2]^3+
| 1.0606601810596428 * x[2]^2 * xi[2]^1+0.3535533936865476 * xi[2]^3"
```

Thus, in the complex coordinates (x'_j, ξ'_j) we have

$$\lambda_1 = x_1^2 x_2^2 = 0,25\, x_1'^2 x_2'^2 + 0,5\, x_1'^2 x_2' \xi_2' + 0,25\, x_1'^2 \xi_2'^2 + 0,5\, x_1' x_2'^2 \xi_1' \\ + x_1' x_2' \xi_1' \xi_2' + 0,5\, x_1' \xi_1' \xi_2'^2 + 0,25\, x_2'^2 \xi_1'^2 + 0,5\, x_2' \xi_1'^2 \xi_2' + 0,25\, \xi_1'^2 \xi_2'^2.$$

We consider now the Hamiltonian $P_e = H + \lambda_1$:

```
| let Hz = Weyl . add Hz  vz;;
```

Define the frequency vector [1; 1] and apply Birkhoff procedure at order 4:

```
| let  freq = [ | one; of_int 1 | ]; ;
| let  kz = birkhoff  freq  hz  4;;
```

Then we get the normalized Hamiltonian kz, which we convert in the real coordinates and print the result:

```
| let  k = coordx  kz;;
| Maple. of_poly  k;;
| − : string  =
| "0.5 * x[1]^2+0.5 * xi[1]^2+0.5 * x[2]^2+0,5 * xi[2]^2+1,5 * x[1]^2 * x[2]^2
|  +0,5 * x[1]^2 * xi[2]^2+0,5 * x[2]^2 * xi[1]^2+1,5 * xi[1]^2 * xi[2]^2
|  +2 * x[1] * x[2] * xi[1] * xi[2]
```

We see from formula (4.2) that $a_1 = a_2 = 0$, $a_3 = \frac{1}{4} \frac{\partial^4 \lambda_1}{\partial x_1^2 \partial x_2^2}(0) = 1$ and $a_4 = \frac{1}{8} \frac{\partial^4 \lambda_1}{\partial x_1^2 \partial x_2^2}(0) = \frac{1}{2}$. Hence,

$$\begin{aligned} K_4 &= a_3 |z_1|^2 |z_2|^2 + a_4 \operatorname{Re}(z_1^2 \overline{z_2}^2) + \mathcal{O}(h^2) \\ &= x_1^2 x_2^2 + x_1^2 \xi_2^2 + x_2^2 \xi_1^2 + \xi_1^2 \xi_2^2 + \frac{1}{2} \left(x_1^2 x_2^2 - x_1^2 \xi_2^2 - x_2^2 \xi_1^2 + \xi_1^2 \xi_2^2 + 4 x_1 x_2 \xi_1 \xi_2 \right) + \mathcal{O}(h^2) \\ &= \frac{3}{2}\, x_1^2 x_2^2 + \frac{1}{2}\, x_1^2 \xi_2^2 + \frac{1}{2}\, x_2^2 \xi_1^2 + \frac{3}{2}\, \xi_1^2 \xi_2^2 + 2 x_1 x_2 \xi_1 \xi_2 + \mathcal{O}(h^2) \end{aligned}$$

and

$$H + K_4 = \frac{1}{2}\, x_1^2 + \frac{1}{2}\, \xi_1^2 + \frac{1}{2}\, x_2^2 + \frac{1}{2}\, \xi_2^2 + \frac{3}{2}\, x_1^2 x_2^2 + \frac{1}{2}\, x_1^2 \xi_2^2 + \frac{1}{2}\, x_2^2 \xi_1^2 + \frac{3}{2}\, \xi_1^2 \xi_2^2 + 2 x_1 x_2 \xi_1 \xi_2 + \mathcal{O}(h^2).$$

These results are qualitatively identical to those obtained above over a Maple module, the Birkhoff module and the normal form algorithm.

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QUALITATIVE ANALYSIS OF GENERALIZED PROPORTIONAL FRACTIONAL FUNCTIONAL INTEGRO-DIFFERENTIAL LANGEVIN EQUATION WITH VARIABLE COEFFICIENT AND NONLOCAL INTEGRAL CONDITIONS

Abstract. In this paper, the existence and uniqueness of solutions for a nonlinear generalized proportional fractional functional integro-differential Langevin equation involving variable coefficient via nonlocal multi-point integral conditions are investigated by using Banach's, Schaefer's and Krasnoselskii's fixed point theorems. Different types of Ulam–Hyers stability are also established. Finally, an example is given to demonstrate applicability to the theoretical findings.

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Key words and phrases. Existence and uniqueness, fixed point theorem, fractional Langevin equation, generalized proportional fractional derivative, nonlocal integral condition, Ulam–Hyers stability.

რეზიუმე. ნაშრომში ბანახის, შეფერისა და კრასნოსელსკის უძრავი წერტილის თეორემების გამოყენებით გამოკვლეულია ამონახსნების არსებობა და ერთადერთობა არაწრფივი განზოგადებული პროპორციული წილად-ფუნქციონალური ინტეგრო-დიფერენციალური ლანჟევანის განტოლებისთვის ცვლადი კოეფიციენტის შემცველი არალოკალური მრავალწერტილიანი ინტეგრალური პირობებით, აგრეთვე დადგენილია სხვადასხვა ტიპის ულამ-ჰაიერსის მდგრადობა. ნაშრომის ბოლოს მოცემულია თეორიული დასკვნების გამოყენების მაგალითი.

1 Introduction

Fractional differential equations have used to be an excellent instrument in the mathematical modelling of dynamical systems and real world problems, such as aerodynamics, polymer science, fractals and chaotic, nonlinear control theory, signal and image processing, bioengineering and chemical engineering, etc. However, a number of various definitions of fractional derivative and integral operators of non-integer order can be found in literature. For more details, we refer the reader to the books [20, 24, 29, 32]. Recently, Jarad *et al.* [22] introduced a new type of fractional derivative operator, the so-called generalized proportional fractional (GPF) derivatives extended by local derivatives [9]. The characteristic of the new derivative is that it involves two fractional orders, preserves the semigroup property, possesses nonlocal character and upon limiting cases it converges to the original function and its derivative. The GPF derivative is well behaved and has a various helpful over the classical derivatives in the sense that it generalizes previously defined derivatives in the literature. We list some recent papers which have been refined in frame of GPF derivative and other related works [2, 7, 8, 37].

Several interesting and important areas of investigation fractional differential equations are devoted to the existence theory and stability analysis of the solutions. In recent years, many authors have discussed the questions on existence, uniqueness and different types of Ulam–Hyers (UH) stability of solutions of initial and boundary value problems for fractional differential equations. The UH stability is the essential and special type of stability analysis that researchers studied in the field of mathematical analysis. The concept of Ulam stability of functional equations was firstly initiated by Ulam [40, 41] and Hyers [21] who presented the partial answer to the Ulam question in the case of Banach space. Thereafter, this type of stability is called the UH stability. In 1950, the Hyers stability was generalized by Aoki [10]. Rassias [33,34] provided an interesting generalization of the UH stability of linear and nonlinear mappings. The UH stability was initially applied to a linear differential equation by Obloza [31]. We refer the reader to the recent works [1,5,11,12,14,17,23,28,36,42,43]. It should be noted that the above-said areas of interest (existence and stability) have been fabulously deliberated within the Riemann–Liouville, Caputo, Hilfer or Hadamard derivatives.

In 1908, Paul Langevin [26] introduced a concept of Langevin equation in a sense of ordinary derivative which is an important equation of mathematical physics. It is well known that a Langevin equation have been widely used to describe the dynamical processes of various fluctuating environments such as physics, chemistry and electrical engineering [16, 30, 44]. However, for a system in complex media, the ordinary Langevin equation does not provide the correct representation of dynamical systems. One of the possible ways of the ordinary Langevin equation is to replace the ordinary (integer-order) derivative by the fractional-order derivative. The fractional Langevin equation was studied by various researchers (for some recent works on fractional Langevin equations, see [6,13,15,18,27,38,39,45]). It is to be noted that most exiting in literature results dealt with a fractional Langevin equation, have been reported in the case of a constant coefficient $\mathcal{H}(t)$. However, the paper [4] has first discussed fractional Langevin equation containing variable coefficient and supplemented with nonlocal-terminal fractional boundary conditions. On the other hand, we claim that our approach in this paper is totally different from paper [4] in the sense that different fractional derivative is accommodated, different boundary conditions are associated, different fixed point theorems are used and UH stability is discussed which has not studied in [4].

Motivated by [4, 15, 38, 39], in this paper we study th existence, uniqueness and different types of UH stability for a nonlinear GPF functional integro-differential Langevin equation involving a variable coefficient via nonlocal multi-point integral conditions:

$$\begin{cases} {}^{C}_{a}\mathcal{D}^{\beta,\rho}\big({}^{C}_{a}\mathcal{D}^{\alpha,\rho} + \mathcal{H}(t)\big)x(t) = f(t, x(t), x(\theta(t)), (\mathcal{S}x)(t)), \quad t \in (a, T], \quad a > 0, \\ x(a) = \gamma, \quad x(\eta) = \displaystyle\sum_{i=1}^{m} \delta_i {}_{a}\mathcal{I}^{\mu_i,\rho} x(\xi_i) + \kappa, \end{cases} \tag{1.1}$$

where ${}^{C}_{a}\mathcal{D}^{q,\rho}$ denotes the GPF derivative operator of Caputo type of order $q \in \{\alpha, \beta\}$, $0 < \alpha, \beta \le 1$, $1 < \alpha + \beta \le 2$, $\rho > 0$, ${}_{a}\mathcal{I}^{\mu_i,\rho}$ denotes the GPF integral opertator of order $\mu_i > 0$, $\rho > 0$, $i = 1, \ldots, m$,

$\mathcal{H} \in C([a,T],\mathbb{R})$, $f \in C([a,T]\times\mathbb{R}^3,\mathbb{R})$, $\theta : [a,T] \to [a,T]$,

$$(\mathcal{S}x)(t) = \int_a^t \phi(t,s,x(s))\,ds, \quad t \in [a,T],$$

$\phi : [a,T]^2 \times \mathbb{R} \to [a,\infty)$ is a continuous function. $\gamma, \kappa, \delta_i \in \mathbb{R}$ and $\eta, \xi_i \in (a,T)$, $i = 1,2,\dots,m$.

The manuscript is structured as follows. In Section 2, we give some definitions and lemmas. In Section 3, we establish some appropriate conditions for the existence results of solutions of problem (1.1) by applying a variety of fixed point theorems due to Banach, Schaefer and Krasnoselskii. In Section 4, we set up applicable results for different types of Ulam–Hyers stability to the solution of problem (1.1). An example illustrating our results is given in Section 5.

2 Preliminaries

This section is devoted to definitions and lemmas that will be used throughout the paper. For their justifications and proofs, we refer the reader to [22].

Definition 2.1 ([22])**.** For $0 < \rho \le 1$, $\alpha \in \mathbb{C}$ and $\mathrm{Re}(\alpha) > 0$, the GPF integral of f of order α is

$$({}_a\mathcal{I}^{\alpha,\rho} f)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1} f(s)\,ds = \rho^{-\alpha} e^{\frac{\rho-1}{\rho}t}\, {}_a\mathcal{I}^{\alpha}(e^{\frac{1-\rho}{\rho}s} f)(t),$$

where ${}_a\mathcal{I}^\alpha$ is the Riemann–Liouville fractional integral [24].

Definition 2.2 ([22])**.** For $0 < \rho \le 1$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) \ge 0$, the Caputo type GPF derivative of f of order α is

$$({}^C_a\mathcal{D}^{\alpha,\rho} f)(t) = \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{n-\alpha-1} (\mathcal{D}^{n,\rho} f)(s)\,ds,$$

where $n = [\mathrm{Re}(\alpha)] + 1$ and $[\mathrm{Re}(\alpha)]$ represents the integer part of the real number α.

Lemma 2.1 ([22])**.** *For $0 < \rho \le 1$ and $n = [\mathrm{Re}(\alpha)] + 1$, we have $({}^C_a\mathcal{D}^{\alpha,\rho}\ {}_a\mathcal{I}^{\alpha,\rho} f)(t) = f(t)$, and*

$$({}_a\mathcal{I}^{\alpha,\rho}\ {}^C_a\mathcal{D}^{\alpha,\rho} f)(t) = f(t) - e^{\frac{\rho-1}{\rho}(t-a)} \sum_{k=0}^{n-1} \frac{(\mathcal{D}^{k,\rho} f)(a)}{\rho^k k!}(t-a)^k.$$

Lemma 2.2 ([22])**.** *Let $\alpha, \beta \in \mathbb{C}$ be such that $\mathrm{Re}(\alpha) \ge 0$ and $\mathrm{Re}(\beta) > 0$. Then, for any $0 < \rho \le 1$ and $n = [\mathrm{Re}(\alpha)] + 1$, we have*

(i)
$$\left({}_a\mathcal{I}^{\alpha,\rho} e^{\frac{\rho-1}{\rho}s}(s-a)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\rho^\alpha \Gamma(\beta+\alpha)} e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta+\alpha-1}, \quad \mathrm{Re}(\alpha) > 0.$$

(ii)
$$\left({}^C_a\mathcal{D}^{\alpha,\rho} e^{\frac{\rho-1}{\rho}s}(s-a)^{\beta-1}\right)(t) = \frac{\rho^\alpha \Gamma(\beta)}{\Gamma(\beta-\alpha)} e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-\alpha-1}, \quad \mathrm{Re}(\beta) > n.$$

(iii)
$$\left({}^C_a\mathcal{D}^{\alpha,\rho} e^{\frac{\rho-1}{\rho}s}(s-a)^{k}\right)(t) = 0, \quad \mathrm{Re}(\alpha) > n, \quad k = 0,1,\dots,n-1.$$

Lemma 2.3 (Arzelá–Ascoli theorem [3])**.** *A subset $\mathbb{M}$ in $C([a,b],\mathbb{R})$ with norm*

$$\|f\| = \sup_{t\in[a,b]} |f(t)|$$

is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.

Lemma 2.4 (Banach's fixed point theorem [19]). *Let $\mathbb{M}$ be a non-empty closed subset of a Banach space $\mathbb{E}$. Then any contraction mapping $\mathbb{T}$ from $\mathbb{M}$ into itself has a unique fixed point.*

Lemma 2.5 (Schaefer's fixed point theorem [19]). *Let $\mathbb{M}$ be a Banach space and $\mathbb{T}: \mathbb{M} \to \mathbb{M}$ be a completely continuous operator and let the set $\mathbb{G} = \{x \in \mathbb{M} : \ x = \kappa \mathbb{T} x, \ 0 < \kappa \leq 1\}$ be bounded. Then $\mathbb{T}$ has a fixed point in $\mathbb{M}$.*

Lemma 2.6 (Krasnoselskii's fixed point theorem [25]). *Let $\mathbb{M}$ be a closed, bounded, convex and nonempty subset of a Banach space $\mathbb{X}$. Let $\mathcal{A}$, $\mathcal{B}$ be the operators such that*

(i) *$\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$ whenever $x, y \in \mathbb{M}$;*

(ii) *$\mathcal{A}$ is compact and continuous;*

(iii) *$\mathcal{B}$ is a contraction mapping.*

Then there exists $z \in \mathbb{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$.

For the sake of computational convenience, we make use of the following constants:

$$\Lambda := \frac{(\eta - a)^{\alpha} e^{\frac{\rho-1}{\rho}(\eta - a)}}{\rho^{\alpha}\Gamma(\alpha+1)} - \sum_{i=1}^{m} \frac{\delta_i (\xi_i - a)^{\alpha+\mu_i} e^{\frac{\rho-1}{\rho}(\xi_i - a)}}{\rho^{\alpha+\mu_i}\Gamma(\alpha+\mu_i+1)} \neq 0, \tag{2.1}$$

$$\Omega_1 := \frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)} \left(\sum_{i=1}^{m} \frac{|\delta_i|(\xi_i - a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)} + \frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \right), \tag{2.2}$$

$$\Omega_2 := \frac{(T-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)} + \frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)} \left(\sum_{i=1}^{m} \frac{|\delta_i|(\xi_i - a)^{\alpha+\beta+\mu_i+1}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+2)} + \frac{(\eta-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)} \right), \tag{2.3}$$

$$\Omega_3 := {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(T) + \frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)} \left(\sum_{i=1}^{m} |\delta_i| {}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta) \right), \tag{2.4}$$

$$\Omega_4 := \frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)} \left(\sum_{i=1}^{m} \frac{|\gamma|\,|\delta_i|(\xi_i - a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa| \right) + |\gamma|. \tag{2.5}$$

Let $\mathbb{E} = C([a,T], \mathbb{R})$ be the Banach space of all continuous functions from $[a,T]$ into $\mathbb{R}$ equipped with the norm $\|x\|_{\mathbb{E}} = \sup_{t \in [a,T]} \{|x(t)|\}$. In order to transform the main problem into a fixed point problem, problem (1.1) must be converted to an equivalent Volterra integral equation. Next, we provide the following lemma.

Lemma 2.7. *Let $h : [a,T] \to \mathbb{R}$ be a continuous function, $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, and $\rho, \mu_i > 0$, $i = 1, 2, \ldots, m$. Then the function $x \in \mathbb{E}$ is the solution to the following linear GPF Langevin equation equipped with the nonlocal integral conditions*

$$\begin{cases} {}_a^C\mathcal{D}^{\beta,\rho}\big({}_a^C\mathcal{D}^{\alpha,\rho} + \mathcal{H}(t)\big)x(t) = h(t), \quad t \in (a,T], \\ x(a) = \gamma, \quad x(\eta) = \displaystyle\sum_{i=1}^{m} \delta_i {}_a\mathcal{I}^{\mu_i,\rho} x(\xi_i) + \kappa, \end{cases} \tag{2.6}$$

if and only if x satisfies the following Volterra integral equation:

$$\begin{aligned} x(t) = {}_a\mathcal{I}^{\alpha+\beta,\rho}h(t) - {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(t)x(t) \\ + \frac{(t-a)^{\alpha}e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}h(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho}h(\eta) \\ - \sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(\xi_i)x(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(\eta)x(\eta) \\ + \sum_{i=1}^{m}\frac{\gamma\delta_i(\xi_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)} - \gamma e^{\frac{\rho-1}{\rho}(\eta-a)} + \kappa\Bigg) + \gamma e^{\frac{\rho-1}{\rho}(t-a)}, \end{aligned} \tag{2.7}$$

where Λ is given by (2.1)*.*

Proof. Let x be a solution of problem (2.6). By using Lemma 2.1 with Lemma 2.2(i), the first equation of (2.6) can be written as an equivalent integral equation

$$x(t) = {}_a\mathcal{I}^{\alpha+\beta,\rho}h(t) - {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(t)x(t) + c_1\frac{(t-a)^{\alpha}e^{\frac{\rho-1}{\rho}(t-a)}}{\rho^{\alpha}\Gamma(\alpha+1)} + c_2e^{\frac{\rho-1}{\rho}(t-a)}, \tag{2.8}$$

where $c_1, c_2 \in \mathbb{R}$.

From the first condition, $x(a) = \gamma$, we get $c_2 = \gamma$. Taking the GPF integral operator ${}_a\mathcal{I}^{\mu_i,\rho}$ into both sides of (2.8), we have

$${}_a\mathcal{I}^{\mu_i,\rho}x(t) = {}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}h(t) - {}_a\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(t)x(t) + c_1\frac{(t-a)^{\alpha+\mu_i}e^{\frac{\rho-1}{\rho}(t-a)}}{\rho^{\alpha+\mu_i}\Gamma(\alpha+\mu_i+1)} + \frac{\gamma(t-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(t-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)}.$$

From the second condition, we obtain c_1 as follows:

$$\begin{aligned} c_1 = \frac{1}{\Lambda}\Bigg(\sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}h(\xi_i) - {}_aI^{\alpha+\beta,\rho}h(\eta) - \sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(\xi_i)x(\xi_i) \\ + {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(\eta)x(\eta) + \sum_{i=1}^{m}\frac{\gamma\delta_i(\xi_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)} - \gamma e^{\frac{\rho-1}{\rho}(\eta-a)} + \kappa\Bigg), \end{aligned}$$

where Λ is defined by (2.1). Substituting c_1 and c_2 into (2.8), we get the Volterra integral equation (2.7).

Conversely, it is easily shown by direct calculation that the solution $x(t)$ is given by (2.7) and satisfies problem (2.6) under the given boundary conditions. □

3 Main results

In this section, we establish the existence results of solutions for problem (1.1), which is studied by applying Banach's, Schaefer's and Krasnolselskii's fixed point theorems. Throughout this paper, the expression ${}_a\mathcal{I}^{b,\rho}f(s, x(s), x(\theta(s)), (\mathcal{S}x)(s))(c)$ means that

$${}_a\mathcal{I}^{b,\rho}F_x(s)(c) := \frac{1}{\rho^b\Gamma(b)}\int_a^c e^{\frac{\rho-1}{\rho}(c-s)}(c-s)^{b-1}F_x(s)\,ds, \quad c \in [a, T],$$

where $b \in \{\alpha, \alpha+\mu_i, \alpha+\beta, \alpha+\beta+\mu_i\}$ and $c \in \{t, T, \eta, \xi_i\}$, $i = 1, 2, \ldots, m$. For simplicity, we set

$$F_x(t) = f\big(s, x(s), x(\theta(s)), (\mathcal{S}x)(s)\big)(t).$$

In view of Lemma 2.7, an operator $\mathcal{A}:\mathbb{E}\to\mathbb{E}$ is defined by

$$\begin{aligned}(\mathcal{A}x)(t) &= {}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(t) - {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(t)\\
&\quad+\frac{(t-a)^{\alpha}e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}F_x(s)(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(\eta)\\
&\quad-\sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(s)x(s)(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(\eta)\\
&\quad+\sum_{i=1}^{m}\frac{\gamma\delta_i(\xi_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)} - \gamma e^{\frac{\rho-1}{\rho}(\eta-a)}+\kappa\Bigg)+\gamma e^{\frac{\rho-1}{\rho}(t-a)},\end{aligned}\tag{3.1}$$

where Λ is defined by (2.1).

To proceed further, we introduce the following assumptions:

(H_1) The functions $f:[a,T]\times\mathbb{R}^3\to\mathbb{R}$ and $\mathcal{H}:[a,T]\to\mathbb{R}$ are continuous.

(H_2) There exist the positive constants L_1, L_2 such that

$$\big|f(t,u_1,u_2,u_3)-f(t,v_1,v_2,v_3)\big| \le L_1\big(|u_1-v_1|+|u_2-v_2|\big)+L_2|u_3-v_3|,$$

for each $t\in[a,T]$ and u_i, $v_i\in\mathbb{R}$, $i=1,2,3$.

(H_3) The function $\phi:[a,T]^2\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a constant $\phi_0>0$ such that

$$|\phi(t,s,u)-\phi(t,s,v)|\le\phi_0|u-v|,$$

for each t, $s\in[a,T]$ and u, $v\in\mathbb{R}$.

(H_4) There exist the functions σ, τ, φ, $\omega\in C([a,T],\mathbb{R}^+)$ such that

$$|f(t,u,v,w)|\le\sigma(t)+\tau(t)|u|+\varphi(t)|v|+\omega(t)|w|,\quad u,v,w\in\mathbb{R},\quad t\in[a,T],$$

with

$$\sigma^*=\sup_{t\in[a,T]}\sigma(t),\quad \tau^*=\sup_{t\in[a,T]}\tau(t),\quad \varphi^*=\sup_{t\in[a,T]}\varphi(t),\quad \omega^*=\sup_{t\in[a,T]}\omega(t).$$

(H_5) $|f(t,u,v,w)|\le g(t)$, $\forall\,(t,u,v,w)\in[a,T]\times\mathbb{R}^3$ and $g\in C([a,T],\mathbb{R}^+)$.

3.1 Existence and uniqueness result via Banach's fixed point theorem

The existence and uniqueness result of a solution for problem (1.1) will be proved by using Banach's fixed point theorem (Banach contraction mapping principle).

Theorem 3.1. *Assume that* (H_1)–(H_3) *hold. If* $\mathcal{L}<1$, *where*

$$\mathcal{L}:=2L_1\Omega_1+L_2\phi_0\Omega_2+\Omega_3,\tag{3.2}$$

and Ω_i, $i=1,2,3$, *are given by* (2.2)–(2.4), *respectively, then* (1.1) *has a unique solution in* $\mathbb{E}$.

Proof. Firstly, we transform problem (1.1) into a fixed point problem, $x=\mathcal{A}x$, where $\mathcal{A}$ is defined as in (3.1). Observe that the fixed points of the operator $\mathcal{A}$ are solutions of problem (1.1). Applying Banach's fixed point theorem, we show that $\mathcal{A}$ has a fixed point which is a unique solution of problem (1.1).

Let $\sup_{t\in[a,T]}|f(t,0,0,0)|:=M_1<\infty$. Next, we define a set $B_{r_1}:=\{x\in\mathbb{E}:\ \|x\|_{\mathbb{E}}\le r_1\}$ with

$$r_1\ge\frac{\Omega_1M_1+\Omega_4}{1-[2L_1\Omega_1+L_2\phi_0\Omega_2+\Omega_3]}.$$

Notice that B_{r_1} is a bounded, closed and convex subset of $\mathbb{E}$. The proof is divided into two steps.

Step 1. We show that $\mathcal{A}B_{r_1} \subset B_{r_1}$.

For any $x \in B_{r_1}$, we have

$$
\begin{aligned}
|(\mathcal{A}x)(t)| \le\ & {}_a\mathcal{I}^{\alpha+\beta,\rho}|F_x(s)|(T) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(T) \\
&+ \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}|F_x(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha+\beta,\rho}|F_x(s)|(\eta) \\
&+ \sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|\,|x(s)|(\xi_i) + {}_aI^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(\eta) \\
&+ \sum_{i=1}^{m}\frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa|\Bigg) + |\gamma| \\
\le\ & {}_a\mathcal{I}^{\alpha+\beta,\rho}\big(|F_x(s)-f(s,0,0,0)| + |f(s,0,0,0)|\big)(T) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(T) \\
&+ \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}\big(|F_x(s)-f(s,0,0,0)| + |f(s,0,0,0)|\big)(\xi_i) \\
&+ {}_a\mathcal{I}^{\alpha+\beta,\rho}\big(|F_x(s)-f(s,0,0,0)| + |f(s,0,0,0)|\big)(\eta) + \sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|\,|x(s)|(\xi_i) \\
&+ {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(\eta) + \sum_{i=1}^{m}\frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa|\Bigg) + |\gamma|.
\end{aligned}
$$

By using the property $0 < e^{\frac{\rho-1}{\rho}(u-s)} \le 1$ for $a \le s < u < t \le T$ and (H_2)–(H_3), we obtain

$$
\begin{aligned}
|(\mathcal{A}x)(t)| \le\ & \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^T (T-s)^{\alpha+\beta-1}\big((2L_1 + L_2\phi_0(s-a))r_1 + M_1\big)\,ds + r_{1a}I^{\alpha,\rho}|\mathcal{H}(s)|(T) \\
&+ \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\frac{|\delta_i|}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i)} \\
&\times \int_a^{\xi_i}(\xi_i-s)^{\alpha+\beta+\mu_i-1}\big((2L_1 + L_2\phi_0(s-a))r_1 + M_1\big)\,ds \\
&+ \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^{\eta}(\eta-s)^{\alpha+\beta-1}\big((2L_1 + L_2\phi_0(s-a))r_1 + M_1\big)\,ds + r_{1a}\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta) \\
&+ r_1\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i) + \sum_{i=1}^{m}\frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa|\Bigg) + |\gamma| \\
=\ & (2L_1r_1 + M_1)\Bigg[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)} \\
&\times \Bigg(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)} + \frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Bigg)\Bigg] \\
&+ L_2\phi_0r_1\Bigg[\frac{(T-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)} + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)} \\
&\times \Bigg(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i+1}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+2)} + \frac{(\eta-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)}\Bigg)\Bigg] \\
&+ r_1\Bigg[{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(T) + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta)\Bigg)\Bigg]
\end{aligned}
$$

$$+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)}+|\gamma|+|\kappa|\Bigg)+|\gamma|$$
$$\le (2L_1r_1+M_1)\Omega_1+L_2\phi_0\Omega_2r_1+\Omega_3r_1+\Omega_4\le r_1,$$

then $\|\mathcal{A}x\|_{\mathbb{E}}\le r_1$, which implies that $\mathcal{A}B_{r_1}\subset B_{r_1}$.

Step 2. We show that the operator $\mathcal{A}:\mathbb{E}\to\mathbb{E}$ is a contraction mapping.

Let $x,y\in\mathbb{E}$. Then for $t\in[a,T]$, we have

$$\begin{aligned}
|(\mathcal{A}x)(t)-(\mathcal{A}y)(t)| &\le {}_a\mathcal{I}^{\alpha+\beta,\rho}|F_x(s)-F_y(s)|(T)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)-y(s)|(T)\\
&+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}|F_x(s)-F_y(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha+\beta,\rho}\big(|F_x(s)-F_y(s)|\big)(\eta)\\
&+\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|\,|x(s)-y(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)-y(s)|(\eta)\Bigg)\\
&\le\Bigg\{2L_1\Bigg[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)}+\frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Bigg)\Bigg]\\
&+L_2\phi_0\Bigg[\frac{(T-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)}+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i+1}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+2)}+\frac{(\eta-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)}\Bigg)\Bigg]\\
&+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(T)+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta)\Bigg)\Bigg\}\|x-y\|_{\mathbb{E}}\\
&=\big[2L_1\Omega_1+L_2\phi_0\Omega_2+\Omega_3\big]\|x-y\|_{\mathbb{E}}=\mathcal{L}\|x-y\|_{\mathbb{E}},
\end{aligned}$$

which implies that $\|\mathcal{A}x-\mathcal{A}y\|_{\mathbb{E}}\le\mathcal{L}\|x-y\|_{\mathbb{E}}$. As $\mathcal{L}<1$, hence, by Banach's fixed point theorem (Lemma 2.4), the operator $\mathcal{A}$ is a contraction mapping. Therefore, $\mathcal{A}$ has only one fixed point, which implies that problem (1.1) has a unique solution in $\mathbb{E}$. □

3.2 Existence result via Schaefer's fixed point theorem

Next, the second existence result is based on Schaefer's fixed point theorem.

Theorem 3.2. *Assume that* (H_1) *and* (H_4) *hold. Then problem* (1.1) *has at least one solution on* $[a,T]$.

Proof. To show that $\mathcal{A}$ has at least a fixed point in $\mathbb{E}$, the proof is divided into four steps.

Step 1. We show that the operator $\mathcal{A}$ is continuous.

Let $\{x_n\}$ be a sequence such that $x_n\to x$ in $\mathbb{E}$. Then, for each $t\in[a,T]$, we get

$$\begin{aligned}
|(\mathcal{A}x_n)(t)-(\mathcal{A}x)(t)| &\le {}_a\mathcal{I}^{\alpha+\beta,\rho}|F_{x_n}(s)-F_x(s)|(T)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x_n(s)-x(s)|(T)\\
&+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}|F_{x_n}(s)-F_x(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha+\beta,\rho}|F_{x_n}(s)-F_x(s)|(\eta)\\
&+\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|\,|x_n(s)-x(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)||x_n(s)-x(s)|(\eta)\Bigg)\\
&\le\Bigg[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)}\\
&+\frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Bigg)\Bigg]\|F_{x_n}-F_x\|_{\mathbb{E}}+\Bigg[{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(T)\\
&+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta)\Bigg)\Bigg]\|x_n-x\|_{\mathbb{E}}\\
&=\Omega_1\|F_{x_n}-F_x\|_{\mathbb{E}}+\Omega_3\|x_n-x\|_{\mathbb{E}}.
\end{aligned}$$

Since f and $\mathcal{H}$ are continuous, by the Lebesgue dominated convergent theorem, we have

$$|(\mathcal{A}x_n)(t) - (\mathcal{A}x)(t)| \longrightarrow 0 \ \text{ as } \ n \to \infty.$$

Hence,

$$\|\mathcal{A}x_n - \mathcal{A}x\|_{\mathbb{E}} \longrightarrow 0 \ \text{ as } \ n \to \infty.$$

Therefore, the operator $\mathcal{A}$ is continuous.

Step 2. We show that the operator $\mathcal{A}$ maps a bounded set into the bounded set in $\mathbb{E}$.

Indeed, we show that for any $r_2 > 0$, there exists a constant $M_2 > 0$ such that for each $x \in \overline{B}_{r_2} = \{x \in \mathbb{E} : \|x\|_{\mathbb{E}} \le r_2\}$, we have $\|\mathcal{A}x\|_{\mathbb{E}} \le M_2$.

Then, for any $t \in [a, T]$ and $x \in \overline{B}_{r_2}$, we have

$$\begin{aligned}
|(\mathcal{A}x)(t)| &\le {}_a\mathcal{I}^{\alpha+\beta,\rho}|F_x(s)|(T) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(T) \\
&\quad + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\left(\sum_{i=1}^m |\delta_i|{}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}|F_x(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha+\beta,\rho}|F_x(s)|(\eta)\right. \\
&\quad + \sum_{i=1}^m |\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|\,|x(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(\eta) \\
&\quad \left. + \sum_{i=1}^m \frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa|\right) + |\gamma| \\
&\le {}_a\mathcal{I}^{\alpha+\beta,\rho}\Big(\sigma(s) + \tau(s)|x(s)| + \varphi(s)|x(\theta(s))| + \omega(s)|(\mathcal{S}x)(s)|\Big)(T) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(T) \\
&\quad + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\left(\sum_{i=1}^m |\delta_i|{}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}\Big(\sigma(s) + \tau(s)|x(s)| + \varphi(s)|x(\theta(s))| + \omega(s)|(\mathcal{S}x)(s)|\Big)(\xi_i)\right. \\
&\quad + {}_a\mathcal{I}^{\alpha+\beta,\rho}\Big(\sigma(s) + \tau(s)|x(s)| + \varphi(s)|x(\theta(s))| + \omega(s)|(\mathcal{S}x)(s)|\Big)(\eta) \\
&\quad \left. + \sum_{i=1}^m |\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|\,|x(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(\eta) + \sum_{i=1}^m \frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa|\right) + |\gamma| \\
&\le (\sigma^* + \tau^* r_2 + \varphi^* r_2)\left[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\right. \\
&\quad \left. \times\left(\sum_{i=1}^m \frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)} + \frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\right)\right] \\
&\quad + \omega^* r_2\left[\frac{(T-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)} + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\right. \\
&\quad \left. \times\left(\sum_{i=1}^m \frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i+1}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+2)} + \frac{(\eta-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)}\right)\right] \\
&\quad + r_2\left[{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(T) + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\left(\sum_{i=1}^m |\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta)\right)\right] \\
&\quad + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\left(\sum_{i=1}^m \frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa|\right) + |\gamma| \\
&= (\sigma^* + \tau^* r_2 + \varphi^* r_2)\Omega_1 + (\omega^*\Omega_2 + \Omega_3)r_2 + \Omega_4,
\end{aligned}$$

and we get the estimate

$$\|\mathcal{A}x\|_{\mathbb{E}} \le \big[(\tau^* + \varphi^*)\Omega_1 + \omega^*\Omega_2 + \Omega_3\big]r_2 + \sigma^*\Omega_1 + \Omega_4 := M_2,$$

where Ω_i, $i = 1, 2, 3, 4$, are given by (2.2)–(2.5), respectively.

Step 3. We show that the operator $\mathcal{A}$ is equicontinuous.

Let $\overline{B}_{r_2}$ be a bounded set of $\mathbb{E}$ as defined in Step 2, then, for $x \in \overline{B}_{r_2}$ and $t_1, t_2 \in [a, T]$ with $t_1 < t_2$, we have

$$
\begin{aligned}
&|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)| \\
&\quad \leq \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^{t_1} \left| e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha+\beta-1} - e^{\frac{\rho-1}{\rho}(t_1-s)}(t_1-s)^{\alpha+\beta-1} \right| |F_x(s)|\, ds \\
&\qquad + \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha+\beta-1}|F_x(s)|\, ds \\
&\qquad + \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_a^{t_1} \left| e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha-1} - e^{\frac{\rho-1}{\rho}(t_1-s)}(t_1-s)^{\alpha-1} \right| |\mathcal{H}(s)|\, |x(s)|\, ds \\
&\qquad + \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_{t_1}^{t_2} e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha-1}|\mathcal{H}(s)|\, |x(s)|\, ds + |\gamma| \left| e^{\frac{\rho-1}{\rho}(t_2-a)} - e^{\frac{\rho-1}{\rho}(t_1-a)} \right| \\
&\qquad + \frac{\left|(t_2-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t_2-a)} - (t_1-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t_1-a)}\right|}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)} \left(\sum_{i=1}^{m} \frac{|\delta_i|}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i)} \right. \\
&\qquad \times \int_a^{\xi_i} e^{\frac{\rho-1}{\rho}(\xi_i-s)}(\xi_i-s)^{\alpha+\beta+\mu_i-1} \left| f\big(s, x(s), x(\theta(s)), (\mathcal{S}x)(s)\big) \right| ds \\
&\qquad + \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^{\eta} e^{\frac{\rho-1}{\rho}(\eta-s)}(\eta-s)^{\alpha+\beta-1} \left| f\big(s, x(s), x(\theta(s)), (\mathcal{S}x)(s)\big) \right| ds \\
&\qquad + \sum_{i=1}^{m} \frac{|\delta_i|}{\rho^{\alpha+\mu_i}\Gamma(\alpha+\mu_i)} \int_a^{\xi_i} e^{\frac{\rho-1}{\rho}(\xi_i-s)}(\xi_i-s)^{\alpha+\mu_i-1}|\mathcal{H}(s)|\, |x(s)|\, ds \\
&\qquad \left. + \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_a^{\eta} e^{\frac{\rho-1}{\rho}(\eta-s)}(\eta-s)^{\alpha-1}|\mathcal{H}(s)|\, |x(s)|\, ds + \sum_{i=1}^{m} \frac{|\gamma|\, |\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa| \right) \\
&\quad \leq \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^{t_1} \left| e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha+\beta-1} - e^{\frac{\rho-1}{\rho}(t_1-s)}(t_1-s)^{\alpha+\beta-1} \right| \\
&\qquad \times \big(\sigma^* + \tau^* r_2 + \varphi^* r_2 + \omega^* r_2 (s-a)\big)\, ds + \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha+\beta-1} \\
&\qquad \times \big(\sigma^* + \tau^* r_2 + \varphi^* r_2 + \omega^* r_2 (s-a)\big)\, ds \\
&\qquad + \frac{r_2}{\rho^{\alpha}\Gamma(\alpha)} \int_a^{t_1} \left| e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha-1} - e^{\frac{\rho-1}{\rho}(t_1-s)}(t_1-s)^{\alpha-1} \right| |\mathcal{H}(s)|\, ds \\
&\qquad + \frac{r_2}{\rho^{\alpha}\Gamma(\alpha)} \int_{t_1}^{t_2} e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha-1}|\mathcal{H}(s)|\, ds \\
&\qquad + |\gamma| \left| e^{\frac{\rho-1}{\rho}(t_2-a)} - e^{\frac{\rho-1}{\rho}(t_1-a)} \right| + \frac{\left|(t_2-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t_2-a)} - (t_1-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t_1-a)}\right|}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Bigg(\sum_{i=1}^{m}\frac{|\delta_i|}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i)}\int_a^{\xi_i}(\xi_i-s)^{\alpha+\beta+\mu_i-1}\big(\sigma^*+\tau^*r_2+\varphi^*r_2+\omega^*r_2(s-a)\big)\,ds\\
&\quad+\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^{\eta}(\eta-s)^{\alpha+\beta-1}\big(\sigma^*+\tau^*r_2+\varphi^*r_2+\omega^*r_2(s-a)\big)\,ds\\
&\quad+r_2\sum_{i=1}^{m}\frac{|\delta_i|}{\rho^{\alpha+\mu_i}\Gamma(\alpha+\mu_i)}\int_a^{\xi_i}e^{\frac{\rho-1}{\rho}(\xi_i-s)}(\xi_i-s)^{\alpha+\mu_i-1}|\mathcal{H}(s)|\,ds\\
&\quad+\frac{r_2}{\rho^{\alpha}\Gamma(\alpha)}\int_a^{\eta}e^{\frac{\rho-1}{\rho}(\eta-s)}(\eta-s)^{\alpha-1}|\mathcal{H}(s)|\,ds+\sum_{i=1}^{m}\frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)}+|\gamma|+|\kappa|\Bigg),
\end{aligned}
$$

which implies that

$$
|(\mathcal{A}x)(t_2)-(\mathcal{A}x)(t_1)|\longrightarrow 0,\ \text{ as }\ t_1\to t_2,
$$

As a result of Steps 1–3 together with the Arzelá–Ascoli theorem (Lemma 2.3), we conclude that the operator $\mathcal{A}:\mathbb{E}\to\mathbb{E}$ is completely continuous.

Step 4. We show that the set $\mathbb{D}=\{x\in\mathbb{E}:\ x=\varepsilon\mathcal{A}x,\ 0<\varepsilon<1\}$ is bounded (A priori bounds).

Let $x\in\mathbb{D}$, then $x=\varepsilon\mathcal{A}x$. For any $t\in[a,T]$, one can get the estimate

$$
\begin{aligned}
(\mathcal{A}x)(t)=\varepsilon\Bigg[&{}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(t)-{}_aI^{\alpha,\rho}\mathcal{H}(s)x(s)(t)\\
&+\frac{(t-a)^{\alpha}e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\delta_i{}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}F_x(s)(\xi_i)-{}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(\eta)\\
&\quad-\sum_{i=1}^{m}\delta_i{}_a\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(s)x(s)(\xi_i)+{}_aI^{\alpha,\rho}\mathcal{H}(s)x(s)(\eta)\\
&\quad+\sum_{i=1}^{m}\frac{\gamma\delta_i(\xi_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)}-\gamma e^{\frac{\rho-1}{\rho}(\eta-a)}+\kappa\Bigg)+\gamma e^{\frac{\rho-1}{\rho}(t-a)}\Bigg].
\end{aligned}
$$

It follows from (H_3)-(H_4) and $0<\varepsilon<1$ that for any $t\in[a,T]$,

$$
\begin{aligned}
|x(t)|=|\varepsilon(\mathcal{A}x)(t)|\le{}&(\sigma^*+\tau^*r_2+\varphi^*r_2)\Bigg[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\\
&+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)}+\frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Bigg)\Bigg]\\
&+\omega^*r_2\Bigg[\frac{(T-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)}+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\\
&\quad\times\Bigg(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i+1}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+2)}+\frac{(\eta-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)}\Bigg)\Bigg]\\
&+r_2\Bigg[{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(T)+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta)\Bigg)\Bigg]\\
&+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)}+|\gamma|+|\kappa|\Bigg)+|\gamma|\\
={}&(\sigma^*+\tau^*r_2+\varphi^*r_2)\Omega_1+(\omega^*\Omega_2+\Omega_3)r_2+\Omega_4.
\end{aligned}
$$

Thus,

$$
\|x\|_{\mathbb{E}}\le\big[(\tau^*+\varphi^*)\Omega_1+\omega^*\Omega_2+\Omega_3\big]r_2+\sigma^*\Omega_1+\Omega_4:=N<\infty.
$$

This implies that $\mathbb{D}$ is bounded.

Hence, as a consequence of Schaefer's fiexd point theorem (Lemma 2.5), the operator $\mathcal{A}$ has at least one fixed point which is the solution of problem (1.1). $\square$

3.3 Existence result via Krasnoselskii's fixed point theorem

By using Krasnoselskii's fixed point theorem, we obtain the last existence theorem.

Theorem 3.3. *Assume that* (H_1), (H_3), (H_5) *hold. Then problem* (1.1) *has at least one solution on* $[a,T]$ *if* $\Omega_3<1$, *where* Ω_3 *is defined by* (2.4).

Proof. Let $\sup\limits_{t\in[a,T]}|g(t)|=\|g\|_{\mathbb{E}}$. By choosing a suitable $B_{\overline{r}_3}=\{x\in\mathbb{E}:\ \|x\|_{\mathbb{E}}\le\overline{r}_3\}$, where

$$\overline{r}_3\ge\frac{\Omega_1\|g\|_{\mathbb{E}}+\Omega_4}{1-\Omega_3}$$

with $\|g\|_{\mathbb{E}}=\sup\limits_{t\in[a,T]}|g(t)|$, we define the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ on $B_{\overline{r}_3}$ by

$$\begin{aligned}(\mathcal{A}_1x)(t)&={}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(t)+\frac{(t-a)^\alpha e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^\alpha\Gamma(\alpha+1)}\Big(\sum_{i=1}^m\delta_i{}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}F_x(s)(\xi_i)-{}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(\eta)\Big),\\
(\mathcal{A}_2x)(t)&=\frac{(t-a)^\alpha e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^\alpha\Gamma(\alpha+1)}\Bigg({}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(\eta)-\sum_{i=1}^m\delta_i{}_a\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(s)x(s)(\xi_i)\\
&\quad+\sum_{i=1}^m\frac{\gamma\delta_i(\xi_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)}-\gamma e^{\frac{\rho-1}{\rho}(\eta-a)}+\kappa\Bigg)+\gamma e^{\frac{\rho-1}{\rho}(t-a)}-{}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(t).\end{aligned}$$

To show that $\mathcal{A}_1x+\mathcal{A}_2y\in B_{\overline{r}_3}$, let $x,y\in B_{\overline{r}_3}$. Then we have

$$\begin{aligned}\|\mathcal{A}_1x+\mathcal{A}_2y\|_{\mathbb{E}}&\le\sup_{t\in[a,T]}\Bigg\{{}_a\mathcal{I}^{\alpha+\beta,\rho}|F_x(s)|(t)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|y(s)|(t)\\
&\quad+\frac{(t-a)^\alpha e^{\frac{\rho-1}{\rho}(t-a)}}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^m|\delta_i|{}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}|F_x(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha+\beta,\rho}|F_x(s)|(\eta)\\
&\quad+\sum_{i=1}^m|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|\,|y(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|y(s)|(\eta)\\
&\quad+\sum_{i=1}^m\frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)}+|\gamma|e^{\frac{\rho-1}{\rho}(\eta-a)}+|\kappa|\Bigg)+|\gamma|e^{\frac{\rho-1}{\rho}(t-a)}\Bigg\}\\
&\le\|g\|_{\mathbb{E}}\Bigg[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^m\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)}+\frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Bigg)\Bigg]\\
&\quad+\|x\|_{\mathbb{E}}\Bigg[{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(T)+\frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^m|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta)\Bigg)\Bigg]\\
&\quad+\frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^m\frac{|\gamma||\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)}+|\gamma|+|\kappa|\Bigg)+|\gamma|\le\Omega_1\|g\|_{\mathbb{E}}+\Omega_3\overline{r}_3+\Omega_4\le\overline{r}_3.\end{aligned}$$

This implies that $\mathcal{A}_1x+\mathcal{A}_2y\in B_{\overline{r}_3}$, which satisfies assumption (i) of Lemma 2.6.

Show that assumption (ii) of Lemma 2.6 is satisfied, the continuity of f and $\mathcal{H}$ implies that the operator $\mathcal{A}_1$ is continuous. For $x\in B_{\overline{r}_3}$, we obtain $\|\mathcal{A}_1x\|_{\mathbb{E}}\le\Omega_1\|g\|_{\mathbb{E}}$. This means that the operator $\mathcal{A}_1$ is uniformly bounded on $B_{\overline{r}_3}$. Next, we show that the operator $\mathcal{A}_1$ is equicontinuous. Setting

$$\sup_{(t,z_1,z_2,z_3)\in[a,T]\times B^3_{\overline{r}_3}}|f(t,z_1,z_2,z_3)|=f^*<\infty,$$

for $a \le t_1 < t_2 \le T$, we have

$$\begin{aligned}
&|(\mathcal{A}_1 x)(t_2) - (\mathcal{A}_1 x)(t_1)| \le \left| {}_a\mathcal{I}^{\alpha+\beta,\rho} F_x(s)(t_2) - {}_a\mathcal{I}^{\alpha+\beta,\rho} F_x(s)(t_1) \right| \\
&+ \frac{\left|(t_2-a)^\alpha e^{\frac{\rho-1}{\rho}(t_2-a)} - (t_1-a)^\alpha e^{\frac{\rho-1}{\rho}(t_1-a)}\right|}{|\Lambda|\rho^\alpha \Gamma(\alpha+1)} \left(\sum_{i=1}^m |\delta_i| {}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho} |F_x(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha+\beta,\rho} |F_x(s)|(\eta) \right) \\
&\le f^* \Bigg[\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \left(\left|(t_2-a)^{\alpha+\beta} - (t_1-a)^{\alpha+\beta} - (t_2-t_1)^{\alpha+\beta}\right| + (t_2-t_1)^{\alpha+\beta} \right) \\
&+ \frac{\left|(t_2-a)^\alpha e^{\frac{\rho-1}{\rho}(t_2-a)} - (t_1-a)^\alpha e^{\frac{\rho-1}{\rho}(t_1-a)}\right|}{|\Lambda|\rho^{\alpha+\beta} \Gamma(\alpha+1)} \left(\frac{(\eta-s)^{\alpha+\beta}}{\rho^\alpha \Gamma(\alpha+\beta+1)} + \sum_{i=1}^m \frac{|\delta_i|(\xi_i-s)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)} \right) \Bigg],
\end{aligned}$$

which is independent of x and $|(\mathcal{A}_1 x)(t_2) - (\mathcal{A}_1 x)(t_1)| \to 0$ as $t_1 \to t_2$. Therefore, the operator $\mathcal{A}_1$ is equicontinuous. So, the operator $\mathcal{A}_1$ is relatively compact on $B_{\overline{r}_3}$. Then, by the Arzelá–Ascoli theorem, the operator $\mathcal{A}_1$ is compact on $B_{\overline{r}_3}$, and assumption (ii) of Lemma 2.6 is satisfied. It is easy to see that, using $\Omega_3 < 1$, we come to the conclusion that the operator $\mathcal{A}_2$ is a contraction mapping, and also assumption (iii) of Lemma 2.6 holds. Hence, the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy all assumptions of Krasnoselskii's fixed point theorem (Lemma 2.6). Therefore, problem (1.1) has at least one solution on $[a, T]$. □

4 Ulam–Hyers stability results

In this section, we investigate some necessary and sufficient conditions for Ulam–Hyers (UH) stability, generalized Ulam–Hyers (GUH) stability, Ulam–Hyers–Rassias (UHR) stability, and generalized Ulam–Hyers–Rassias (GUHR) stability of problem (1.1).

Definition 4.1 ([35])**.** Problem (1.1) is UH stable if there exists a real number $\Phi > 0$ such that for $\epsilon > 0$ and solution $z \in \mathbb{E}^1 = C^1([a,T],\mathbb{R})$ of the inequality

$$\left| {}_a^C\mathcal{D}^{\beta,\rho}\left({}_a^C\mathcal{D}^{\alpha,\rho} + \mathcal{H}(t)\right)z(t) - f\big(t, z(t), z(\theta(t)), (\mathcal{S}z)(t)\big) \right| \le \epsilon, \quad t \in [a,T], \tag{4.1}$$

there exists a solution $x \in \mathbb{E}^1$ of problem (1.1) with

$$|z(t) - x(t)| \le \Phi\epsilon, \quad t \in [a,T].$$

Definition 4.2 ([35])**.** Problem (1.1) is GUH stable if there exists $\Phi_f \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\Phi_f(0) = 0$ such that for each solution $z \in \mathbb{E}^1$ of inequality (4.1) there exists a solution $x \in \mathbb{E}^1$ of problem (1.1) such that

$$|z(t) - x(t)| \le \Phi_f\epsilon, \quad t \in [a,T].$$

Definition 4.3 ([35])**.** Problem (1.1) is UHR stable with respect to $\Phi_f \in C([a,T], \mathbb{R}^+)$ if there exists a real number $C_{f,\Phi} > 0$ such that for $\epsilon > 0$ and for each solution $z \in \mathbb{E}^1$ of the inequality

$$\left| {}_a^C\mathcal{D}^{\beta,\rho}\left({}_a^C\mathcal{D}^{\alpha,\rho} + \mathcal{H}(t)\right)z(t) - f\big(t, z(t), z(\theta(t)), (\mathcal{S}z)(t)\big) \right| \le \epsilon\Phi_f(t), \quad t \in [a,T], \tag{4.2}$$

there exists a solution $x \in \mathbb{E}^1$ of problem (1.1) with

$$|z(t) - x(t)| \le C_{f,\Phi}\epsilon\Phi_f(t), \quad t \in [a,T].$$

Definition 4.4 ([35])**.** Problem (1.1) is GUHR stable with respect to $\Phi_f \in C([a,T], \mathbb{R}^+)$ if there exists a real number $C_{f,\Phi} > 0$ such that for each solution $z \in \mathbb{E}^1$ of the inequality

$$\left| {}_a^C\mathcal{D}^{\beta,\rho}\left({}_a^C\mathcal{D}^{\alpha,\rho} + \mathcal{H}(t)\right)z(t) - f\big(t, z(t), z(\theta(t)), (\mathcal{S}z)(t)\big) \right| \le \Phi_f(t), \quad t \in [a,T],$$

there exists a solution $x \in \mathbb{E}^1$ of problem (1.1) such that

$$|z(t) - x(t)| \le C_{f,\Phi}\Phi_f(t), \quad t \in [a,T].$$

Remark 4.1. It is clear that

(i) Definition 4.1 $\Longrightarrow$ Definition 4.2;

(ii) Definition 4.3 $\Longrightarrow$ Definition 4.4;

(iii) Definition 4.3 for $\Phi_f(\,\cdot\,) = 1 \Longrightarrow$ Definition 4.1.

Remark 4.2. A function $z \in \mathbb{E}^1$ is a solution of inequality (4.1) if and only if there exists a function $v \in C([a,T],\mathbb{R})$ (dependent on z) such that

(i) $|v(t)| \le \epsilon$, $\forall\, t \in [a,T]$.

(ii) ${}_a^C\mathcal{D}^{\beta,\rho}({}_a^C\mathcal{D}^{\alpha,\rho} + \mathcal{H}(t))z(t) = f(t, z(t), z(\theta(t)), (\mathcal{S}z)(t)) + v(t)$, $t \in [a,T]$.

By Remark 4.2, the solution of the problem

$$ {}_a^C\mathcal{D}^{\beta,\rho}({}_a^C\mathcal{D}^{\alpha,\rho} + \mathcal{H}(t))z(t) = f\big(t, z(t), z(\theta(t)), (\mathcal{S}z)(t)\big) + v(t), \quad t \in [a,T], $$

can be written by

$$ \begin{aligned} z(t) &= {}_a\mathcal{I}^{\alpha+\beta,\rho}F_z(s)(t) - {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)z(s)(t) \\ &\quad + \frac{(t-a)^{\alpha}e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}F_z(s)(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho}F_z(s)(\eta) \\ &\quad - \sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(s)z(s)(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)z(s)(\eta) \\ &\quad + \sum_{i=1}^{m}\frac{\gamma\delta_i(\xi_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)} - \gamma e^{\frac{\rho-1}{\rho}(\eta-a)} + \kappa\Bigg) + \gamma e^{\frac{\rho-1}{\rho}(t-a)} + {}_a\mathcal{I}^{\alpha+\beta,\rho}v(s)(t) \\ &\quad + \frac{(t-a)^{\alpha}e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)}\Big(\sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}v(s)(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho}v(s)(\eta)\Big). \end{aligned} $$

Firstly, we present an important lemma that will be used in the proofs of the first stability theorem.

Lemma 4.1. *If $z \in \mathbb{E}^1$ satisfies* (4.1)*, then the function z is a solution of the inequality*

$$ |z(t) - (\mathcal{A}z)(t)| \le \Omega_1\epsilon, \quad 0 < \epsilon \le 1, \tag{4.3} $$

where Ω_1 is given by (2.2)*.*

Proof. From Remark 4.2, we obtain the inequality

$$ \begin{aligned} |z(t) - (\mathcal{A}z)(t)| &\le \Bigg|{}_a\mathcal{I}^{\alpha+\beta,\rho}v(s)(t) + \frac{(t-a)^{\alpha}e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)} \\ &\qquad \times \Big(\sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}v(s)(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho}v(s)(\eta)\Big)\Bigg| \\ &\le \Bigg[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)} \\ &\qquad \times \Big(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)} + \frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Big)\Bigg]\epsilon = \Omega_1\epsilon, \end{aligned} $$

where Ω_1 is given by (2.2), from which inequality (4.3) follows. $\square$

Now, we present the UH and GUH results.

Theorem 4.1. *Assume that* (H_1), (H_2), (H_3) *are satisfied with* $\mathcal{L} < 1$, *where* $\mathcal{L}$ *is defined by* (3.2). *Then problem* (1.1) *is both UH stable and GUH stable on* $[a, T]$.

Proof. Let $z \in \mathbb{E}^1$ be a solution of (4.1) and let x be the unique solution of problem (1.1),

$$\begin{cases} {}^C_a\mathcal{D}^{\beta,\rho}({}^C_a\mathcal{D}^{\alpha,\rho} + \mathcal{H}(t))x(t) = f\big(t, x(t), x(\theta(t)), (\mathcal{S}x)(t)\big), \quad t \in (a, T] \\ x(a) = \gamma, \quad x(\eta) = \displaystyle\sum_{i=1}^{m} \delta_{ia} I^{\mu_i,\rho} x(\xi_i) + \kappa. \end{cases}$$

By applying the triangle inequality $|u - v| \leq |u| + |v|$ and Lemma 4.1, we have

$$\begin{aligned} |z(t) - x(t)| = \Bigg| & z(t) - {}_a\mathcal{I}^{\alpha+\beta,\rho} F_x(s)(t) + {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(t) \\ & - \frac{(t-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)} \Bigg(\sum_{i=1}^{m} \delta_{ia}\mathcal{I}^{\alpha+\beta+\mu_i,\rho} F_x(s)(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho} F_x(s)(\eta) \\ & - \sum_{i=1}^{m} \delta_{ia}\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(s)x(s)(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(\eta) \\ & + \sum_{i=1}^{m} \frac{\gamma\delta_i(\xi_i - a)^{\mu_i} e^{\frac{\rho-1}{\rho}(\xi_i - a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)} - \gamma e^{\frac{\rho-1}{\rho}(\eta-a)} + \kappa \Bigg) - \gamma e^{\frac{\rho-1}{\rho}(t-a)} \Bigg| \\ = \big| z(t) - (\mathcal{A}z)(t) + (\mathcal{A}z)(t) - (\mathcal{A}x)(t) \big| & \leq |z(t) - (\mathcal{A}z)(t)| + |(\mathcal{A}z)(t) - (\mathcal{A}x)(t)| \leq \Omega_1\epsilon + \mathcal{L}|z(t) - x(t)|. \end{aligned}$$

This yields

$$|z(t) - x(t)| \leq \frac{\Omega_1\epsilon}{1 - \mathcal{L}}.$$

By setting $\Phi = \frac{\Omega_1}{1-\mathcal{L}}$ and $\mathcal{L} < 1$, we end up with

$$|z(t) - x(t)| \leq \Phi\epsilon.$$

Hence, problem (1.1) is UH stable. Moreover, if we set $\Phi_f(\epsilon) = \Phi\epsilon$, with $\Phi_f(0) = 0$, then problem (1.1) is GUH stable. □

Remark 4.3. A function $z \in \mathbb{E}^1$ is a solution of inequality (4.2) if and only if there exists a function $w \in C([a, T], \mathbb{R})$ (dependent on z) such that

(i) $|\Theta(t)| \leq \epsilon\Psi_{\Theta}(t)$, $\forall\, t \in [a, T]$.

(ii) ${}^C_aD^{\beta,\rho}({}^C_aD^{\alpha,\rho} + \lambda(t))z(t) = f(t, z(t), z(\theta(t)), (\mathcal{S}z)(t)) + \Theta(t)$, $t \in [a, T]$.

By Remark 4.3, the solution of the problem

$${}^C_a\mathcal{D}^{\beta,\rho}({}^C_a\mathcal{H}^{\alpha,\rho} + \mathcal{H}(t))z(t) = f\big(t, z(t), z(\theta(t)), (\mathcal{S}z)(t)\big) + \Theta(t), \quad t \in [a, T],$$

can be written by

$$\begin{aligned} z(t) = {} & {}_a\mathcal{I}^{\alpha+\beta,\rho} F_z(s)(t) - {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)z(s)(t) \\ & + \frac{(t-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)} \Bigg(\sum_{i=1}^{m} \delta_{ia}\mathcal{I}^{\alpha+\beta+\mu_i,\rho} F_z(s)(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho} F_z(s)(\eta) \\ & - \sum_{i=1}^{m} \delta_{ia}\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(s)z(s)(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)z(s)(\eta) \\ & + \sum_{i=1}^{m} \frac{\gamma\delta_i(\xi_i - a)^{\mu_i} e^{\frac{\rho-1}{\rho}(\xi_i - a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)} - \gamma e^{\frac{\rho-1}{\rho}(\eta-a)} + \kappa \Bigg) + \gamma e^{\frac{\rho-1}{\rho}(t-a)} + {}_a\mathcal{I}^{\alpha+\beta,\rho} w(s)(t) \\ & + \frac{(t-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)} \Big(\sum_{i=1}^{m} \delta_{ia}\mathcal{I}^{\alpha+\beta+\mu_i,\rho} w(s)(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho} w(s)(\eta) \Big). \end{aligned}$$

Next, we construct lemma that will be used in the proofs of the second stability theorem.

Lemma 4.2. *Let $z \in \mathbb{E}^1$ be a solution of inequality* (4.2). *Then the function z satisfies the inequality*

$$|z(t) - (\mathcal{A}z)(t)| \leq \Omega_1 \Psi_\Theta(t)\epsilon, \quad 0 < \epsilon \leq 1, \tag{4.4}$$

where Ω_1 is given by (2.2).

Proof. From Remark 4.3, we obtain the inequality

$$\begin{aligned}
|z(t) - (\mathcal{A}z)(t)| \leq & \Bigg| {}_a\mathcal{I}^{\alpha+\beta,\rho}\Theta(s)(t) + \frac{(t-a)^\alpha e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^\alpha\Gamma(\alpha+1)} \\
& \times \left(\sum_{i=1}^m \delta_{ia}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}\Theta(s)(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho}\Theta(s)(\eta) \right) \Bigg| \\
\leq & \Bigg[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)} \\
& \times \left(\sum_{i=1}^m \frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)} + \frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \right) \Bigg] \Psi_\Theta(t)\epsilon \\
= & \, \Omega_1\Psi_\Theta(t)\epsilon,
\end{aligned}$$

where Ω_1 is given by (2.2), which leads to inequality (4.4). □

Next, we are ready to prove UHR and GUHR stability results.

Theorem 4.2. *If assumptions* (H_1), (H_2), (H_3) *are satisfied, $\mathcal{L} < 1$, where $\mathcal{L}$ is defined by* (3.2), *then problem* (1.1) *is both UHR stable and GUHR stable on* $[a, T]$.

Proof. Let $z \in \mathbb{E}^1$ be a solution of inequality (4.2) and let x be the unique solution of problem (1.1). By applying the triangle inequality and Lemma 4.1, we get

$$\begin{aligned}
|z(t) - x(t)| = & \Bigg| z(t) - {}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(t) + {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(t) \\
& - \frac{(t-a)^\alpha e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^\alpha\Gamma(\alpha+1)} \Bigg(\sum_{i=1}^m \delta_{ia}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}F_x(s)(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(\eta) \\
& - \sum_{i=1}^m \delta_{ia}\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(s)x(s)(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(\eta) \\
& + \sum_{i=1}^m \frac{\gamma\delta_i(\xi_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)} - \gamma e^{\frac{\rho-1}{\rho}(\eta-a)} + \kappa \Bigg) - \gamma e^{\frac{\rho-1}{\rho}(t-a)} \Bigg| \\
= & \left| z(t) - (\mathcal{A}z)(t) + (\mathcal{A}z)(t) - (\mathcal{A}x)(t) \right| \\
\leq & |z(t) - (\mathcal{A}z)(t)| + |(\mathcal{A}z)(t) - (\mathcal{A}x)(t)| \\
\leq & \, \Omega_1\Psi_\Theta(t)\epsilon + \mathcal{L}|z(t) - x(t)|,
\end{aligned}$$

where $\mathcal{L}$ is defined by (3.2), which implies that

$$|z(t) - x(t)| \leq \frac{\Omega_1\Psi_\Theta(t)\epsilon}{1-\mathcal{L}}.$$

By setting $C_{f,\Phi} = \frac{\Omega_1}{1-\mathcal{L}}$ with $\mathcal{L} < 1$, we get the inequality

$$|z(t) - x(t)| \leq C_{f,\Phi}\epsilon\Psi_\Theta(t).$$

Hence, problem (1.1) is UHR stable. Moreover, if we set $\Phi_f(t) = \epsilon\Psi_\Theta(t)$, with $\Phi_f(0) = 0$, then problem (1.1) is GUHR stable. □

5 An example

In this section, we present an example which illustrates the validity and applicability of the main results.

Example. Consider the following boundary value problem for the nonlinear GPF integro-differential Langevin equation

$$\begin{cases} {}_0^C\mathcal{D}^{\frac{\sqrt{\pi}}{2},\frac{\sqrt{2}}{2}}\left({}_0^C\mathcal{D}^{\frac{\sqrt{3}}{2},\frac{\sqrt{2}}{2}} + \frac{1}{16}\,(t-a)^2 e^{\frac{\rho-1}{\rho}(t-a)}\right)x(t) = f\big(t, x(t), x(\theta(t)), (\mathcal{S}x)(t)\big), \quad t \in [0,2], \\ x(0) = 0, \quad x(1) = \sqrt{2}\,{}_0\mathcal{I}^{\frac{1}{2},\frac{\sqrt{2}}{2}} x\Big(\frac{1}{2}\Big) - \frac{1}{2}\,{}_0\mathcal{I}^{\frac{3}{2},\frac{\sqrt{2}}{2}} x\Big(\frac{4}{3}\Big) - {}_0\mathcal{I}^{\frac{5}{2},\frac{\sqrt{2}}{2}} x\Big(\frac{3}{2}\Big) + \frac{1}{10}\,. \end{cases} \tag{5.1}$$

Here,

$$\begin{gathered} \alpha = \frac{\sqrt{3}}{2}\,, \quad \beta = \frac{\sqrt{\pi}}{2}\,, \quad \rho = \frac{\sqrt{2}}{2}\,, \\ a = 0, \quad T = 2, \quad m = 3, \quad \gamma = 0, \quad \eta = 1, \\ \kappa = \frac{1}{10}\,, \quad \mu_1 = \frac{1}{2}\,, \quad \mu_2 = \frac{3}{2}\,, \quad \mu_3 = \frac{5}{2}\,, \\ \xi_1 = \frac{1}{2}\,, \quad \xi_2 = \frac{4}{3}\,, \quad \xi_3 = \frac{3}{2}\,, \\ \delta_1 = \sqrt{2}\,, \quad \delta_2 = -\frac{1}{2}\,, \quad \delta_3 = -1, \quad \theta(t) = \frac{t}{2} \end{gathered}$$

and

$$\mathcal{H}(t) = \frac{1}{16}\,(t-a)^2 e^{\frac{\rho-1}{\rho}\,(t-a)}.$$

Obviously, the function $\mathcal{H}$ satisfies the assumption (H_1) for all $t \in [a,T]$. From the all given all data, we obtain that $\Lambda \approx 1.49603 \neq 0$, $\Omega_1 \approx 8.26497$, $\Omega_2 \approx 4.17132$, $\Omega_3 \approx 0.17389$ and $\Omega_4 \approx 0.17303$.

(i) Let $f : [a,T] \times \mathbb{R}^3 \to \mathbb{R}$ be a function defined by

$$f(t, x(t), x(\theta(t)), (\mathcal{S}x)(t)) = \frac{1}{4} + \frac{1}{9}\,t^3 + \frac{2\sin^2(\pi t)}{(t+5)^2}\,\frac{|x|}{1+|x|} - \frac{x(1.5t)}{(t+5)^2} + \frac{(t+1)^3}{e^t+2}\int_a^t \frac{\cos^2(\pi t)}{(e^{s^2}+3)^2}\,x(s)\,ds.$$

For $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$ and $t \in [a,T]$, we have

$$\begin{gathered} \big|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\big| \le \frac{1}{25}\,\big(|x_1-y_1| + |x_2-y_2|\big) + \frac{1}{3}\,|z_1-z_2|, \\ |\phi(t,s,x_1) - \phi(t,s,y_1)| \le \frac{1}{16}\,|x_1-y_1|. \end{gathered}$$

The assumptions (H_1)–(H_3) are satisfied with $L_1 = \frac{1}{25}$, $L_2 = \frac{1}{3}$, and $\phi_0 = \frac{1}{16}$. Hence

$$\mathcal{L} := 2L_1\Omega_1 + L_2\phi_0\Omega_2 + \Omega_3 \approx 0.92199 < 1.$$

This ensures the existence of the unique solution for (5.1) according to Theorem 3.1. Further, we compute

$$\Phi := \frac{\Omega_1}{1-\mathcal{L}} \approx 105.95156 > 0.$$

Thus, by Theorem (4.1), problem (5.1) is UH stable and, consequently, GUH stable.

(ii) Let $f : [a,T] \times \mathbb{R}^3 \to \mathbb{R}$ be a function defined by

$$\begin{aligned} f\big(t, x(t), x(\theta(t)), (\mathcal{S}x)(t)\big) &= \frac{e^{-t}}{(t+8)^2} + \frac{6e^{-2t}}{(t+8)^2}\,\frac{|x|}{2+|x|} \\ &\quad + \frac{5}{4(2+t)^2}\,\frac{|x(0.25t)|}{|x(0.25t)|+9} + \frac{(t+3)^3\cos^2(\pi t)}{(e^t+2)^2}\int_a^t \frac{\sin^2(t-s)}{(e^{t-s}+2)^2}\,x(s)\,ds. \end{aligned}$$

It is easy to see that for all $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$ and $t \in [a, T]$, we get

$$\begin{aligned} \big|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\big| &\le \frac{1}{32}\big(|x_1-y_1|+|x_2-y_2|\big) + \frac{1}{3}\,|z_1-z_2|, \\ |\phi(t,s,x_1) - \phi(t,s,y_1)| &\le \frac{1}{9}|x_1-y_1|. \end{aligned}$$

The assumptions (H_1)–(H_3) are satisfied with $L_1 = \frac{1}{32}$, $L_2 = \frac{1}{3}$, and $\phi_0 = \frac{1}{9}$. Hence

$$\mathcal{L} := 2L_1\Omega_1 + L_2\phi_0\Omega_2 + \Omega_3 \approx 0.84495 < 1.$$

Furthermore, for $x, y, z \in \mathbb{R}$ and $t \in [a, T]$, it follows that

$$|f(t,x,y,z)| \le \frac{e^{-t}}{(t+8)^2} + \frac{2e^{-2t}}{(t+8)^2}\,|x| + \frac{1}{8(2+t)^2}\,|y| + \frac{27}{(e^t+2)^4}\,|z|.$$

The hypothesis (H_4) is also valid with

$$\sigma(t) = \frac{e^{-t}}{(t+8)^2}\,,\quad \tau(t) = \frac{2e^{-2t}}{(t+8)^2}\,,\quad \varphi(t) = \frac{1}{8(2+t)^2}\,,\quad \omega(t) = \frac{27}{(e^t+2)^4}$$

and

$$\sigma^* = \frac{1}{64}\,,\quad \tau^* = \frac{1}{32}\,,\quad \varphi^* = \frac{1}{32}\,,\quad \omega^* = \frac{1}{3}\,.$$

Therefore, all the assumptions of Theorem (3.2) are fulfilled, which allow to conclude that system (5.1) has at least one solution on $[a, T]$. Moreover, we obtain

$$C_{f,\Phi} := \frac{\Omega_1}{1-\mathcal{L}} \approx 53.30408555 > 0.$$

Thus, by Theorem 4.2, system (5.1) is UHR stable and, consequently, GUHR stable.

6 Conclusion

In this paper, we construct the equivalence between problem (1.1) and the Volterra integral equation. We prove the existence results of solutions for the GPF integro-differential Langevin equation via a variable coefficient with nonlocal integral conditions (1.1) using a variety of fixed point theorems due to Banach, Schaefer and Krasnoselskii. Moreover, we discuss the stability analysis of UH, GUH, UHR and GUHR for the proposed problem (1.1). In addition, an example was given to illustrate our main results. We believe that the all results of this paper will provide considerable potential to interested researchers to develop relevant results concerning qualitative properties of nonlinear GPF differential equations. In a forthcoming work, we shall focus on studying the different types of existence results and stability analysis to an impulsive GPF differential equation with nonlocal integral multi-point conditions.

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