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EXISTENCE AND SOLUTION SETS FOR SYSTEMS OF IMPULSIVE DIFFERENTIAL INCLUSIONS

Abstract. In this paper, we consider the existence of solutions and some properties of the set of solutions, as well as the solution operator for a system of differential inclusions with impulse effects. For the Cauchy problem, under various assumptions on the nonlinear term, we present several existence results. We appeal to some fixed point theorems in vector metric spaces. Finally, we prove some characterizing geometric properties about the structure of the solution set such as AR, R_δ, contractibility and acyclicity, with these properties corresponding to Aronszajn–Browder–Gupta type results.

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Key words and phrases. System of differential inclusions, impulsive, fixed point, existence, vector metric space, R_δ-set, acyclic, matrix.

რეზიუმე. ნაშრომში შესწავლილია იმპულსური ეფექტის მქონე დიფერენციალური ჩანართების სისტემის ამონახსნის არსებობა და ამონახსნთა სიმრავლის ზოგიერთი თვისება, აგრეთვე განხილულია ამოხსნის ოპერატორი. კოშის ამოცანისთვის, სხვადასხვა დაშვებებით არაწრფივ წევრზე, მოყვანილია ამონახსნის არსებობის რამდენიმე შედეგი. კვლევისას გამოყენებულია ზოგიერთი თეორემა ვექტორულ მეტრიკულ სივრცეებში უძრავი წერტილის შესახებ. დაბოლოს, დამტკიცებულია ამონახსნთა სიმრავლის სტრუქტურის ზოგიერთი დამახასიათებელი გეომეტრიული თვისება, როგორიცაა AR, R_δ, კუმშვადობა და აციკლურობა. ეს თვისებები შეესაბამება აირონშაინ-ბროუდერ-გუფთას ტიპის შედეგებს.

1 Introduction

Differential equations with impulses were considered for the first time by Milman and Myshkis [41] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [31]. The dynamics of many processes in physics, population dynamics, biology, medicine, and so on, may be subject to abrupt changes such as shocks or perturbations (see, e.g., [1, 39, 40] and the references therein). These perturbations may be seen as impulses. For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. Their models are described by impulsive differential equations and inclusions. Important contributions to the study of the mathematical aspects of such equations have been undertaken in [25, 37, 50] among others.

In this work, we consider the following problem:

$$\begin{cases} x'(t) \in F_1(t, x(t), y(t)), & \text{a.e. } t \in [0,1], \\ y'(t) \in F_2(t, x(t), y(t)), & \text{a.e. } t \in [0,1], \\ x(t_k^+) = x(t_k^-) + I_{1,k}(x(t_k), y(t_k)), & k = 1, \dots, m, \\ y(t_k^+) = y(t_k^-) + I_{2,k}(x(t_k), y(t_k)), & k = 1, \dots, m, \\ x(0) = x_0, \quad y(0) = y_0, \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < \cdots < t_m < 1$, $F_i : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$, $i = 1, 2$, is a multifunction and $I_{1,k}, I_{2,k} \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. The notations $x(t_k^+) = \lim\limits_{h \to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim\limits_{h \to 0^+} x(t_k - h)$ stand for the right and the left limits of the function y at $t = t_k$, respectively.

For single valued framework, the above system was used to analyze initial value and boundary value problems for nonlinear competitive or cooperative differential systems from mathematical biology [42] and mathematical economics [34]; this can be set in the operator form (1.1).

Recently, Precup [48] proved the role of matrix convergence and vector metric in the study of semilinear operator systems. In recent years, many authors studied the existence of solutions for systems of differential equations and impulsive differential equations by using vector version of fixed point theorems (see [11, 12, 26, 32, 35, 44–46, 49] and in the references therein).

In general, for the ordinary Cauchy problems, the uniqueness property does not hold. Kneser [36] proved in 1923 that the solution set is a continuum, i.e., closed and connected. For differential inclusions, Aronszajn [7] proved in 1942 that the solution set is, in fact, compact and acyclic, and he even specified this continuum to be an R_δ-set.

An analogous result was obtained for differential inclusions with upper semi-continuous (*u.s.c.*) convex valued nonlinearities by several authors (we cite [2–4, 6, 24, 30, 33]).

The topological and geometric structure of solution sets for impulsive differential inclusions on compact intervals, which were investigated in [18, 27–29, 53], are a contractibility, AR, acyclicity and R_δ-sets. Also, the topological structure of solution sets for some Cauchy problems without impulses posed on non-compact intervals were studied by various techniques in [4, 10, 16, 17].

The goal of this paper is to study the existence of solutions and solution sets for systems of impulsive differential inclusions with initial conditions. The paper is organized as follows. In Section 2, we recall some definitions and facts which will be needed in our analysis. In Section 3, we prove some existence results based on a nonlinear alternative of Leray–Schauder type theorem in generalized Banach spaces in the convex case, and a multivalued version of Perov's fixed point theorem (Theorem 2.3) for the nonconvex case. Finally, we present some topological and geometric structures for solution sets of (1.1).

2 Preliminaries

In this section, we introduce notations and definitions which are used throughout this paper.

Denote by

$$\begin{aligned}
\mathcal{P}(X) &= \big\{Y \subset X : \;\; Y \neq \varnothing\big\};\\
\mathcal{P}_{cl}(X) &= \big\{Y \in \mathcal{P}(X) : \;\; Y \text{ closed}\big\};\\
\mathcal{P}_{b}(X) &= \big\{Y \in \mathcal{P}(X) : \;\; Y \text{ bounded}\big\};\\
\mathcal{P}_{cv}(X) &= \big\{Y \in \mathcal{P}(X) : \;\; Y \text{ convex}\big\};\\
\mathcal{P}_{cp}(X) &= \big\{Y \in \mathcal{P}(X) : \;\; Y \text{ compact}\big\};\\
\mathcal{M}_{n\times n}(\mathbb{R}_+) &: \text{Designate the set of real nonnegative } \; n\times n \; \text{ matrices.}
\end{aligned}$$

Definition 2.1. Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:

(i) $d(u, v) \geq 0$ for all $u, v \in X$, if $d(u, v) = 0$ if and only if $u = v$;

(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

We call the pair (X, d) a generalized metric space. For $r = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, we denote by

$$B(x_0, r) = \big\{x \in X : \;\; d(x_0, x) < r\big\}$$

the open ball of radius r centered at x_0 and by

$$\overline{B(x_0, r)} = \big\{x \in X : \;\; d(x_0, x) \leq r\big\}$$

the closed ball of radius r centered at x_0.

We mention that for a generalized metric space, the notation of an open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces. If, $x, y \in \mathbb{R}^n, x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \dots, n$. Also, $|x| = (|x_1|, \dots, |x_n|)$ and $\max(x, y) = (\max(x_1, y_1), \dots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \dots, n$.

Definition 2.2. A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc (i.e., $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $\det(M - \lambda I) = 0$, where I denotes the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$).

Theorem 2.1 ([51])**.** *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) *M is convergent towards zero;*

(ii) *$M^k \to 0$ as $k \to \infty$;*

(iii) *the matrix $(I - M)$ is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots ;$$

(iv) *the matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

Definition 2.3. We say that a non-singular matrix $A = (a_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ has the absolute value property if

$$A^{-1}|A| \leq I,$$

where

$$|A| = \big(|a_{ij}|\big)_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}).$$

Definition 2.4. Let (X,d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix M such that

$$d(N(x),N(y)) \le Md(x,y), \quad \forall\, x,y \in X.$$

Theorem 2.2 ([23, 47])**.** *Let (X,d) be a complete generalized metric space and $N : X \to X$ be a contractive operator with Lipschitz matrix M. Then N has a unique fixed point x_* and for each $x_0 \in X$ we have*

$$d(N^k(x_0),x_*) \le M^k(I-M)^{-1}d(x_0,n(x_0)), \quad \forall\, k \in \mathbb{N}.$$

Let (X,d) be a metric space. We denote by H_{d_*} the Pompeiu–Hausdorff pseudo-metric distance on $\mathcal{P}(X)$ defined as

$$H_{d_*} : \mathcal{P}(X)\times\mathcal{P}(X) \longrightarrow \mathbb{R}_+\cup\{\infty\}, \quad H_{d_*}(A,B) = \max\Big\{\sup_{a\in A} d_*(a,B),\ \sup_{b\in B} d_*(A,b)\Big\},$$

where $d_*(A,b) = \inf\limits_{a\in A} d_*(a,b)$ and $d_*(a,B) = \inf\limits_{b\in B} d_*(a,b)$. Then $(\mathcal{P}_{b,cl}(X),H_{d_*})$ is a metric space and $(\mathcal{P}_{cl}(X),H_{d_*})$ is a generalized metric space. In particular, H_{d_*} satisfies the triangle inequality.

Let (X,d) be a generalized metric space with

$$d(x,y) := \begin{pmatrix} d_1(x,y) \\ \vdots \\ d_n(x,y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if d_i, $i=1,\dots,n$, are metrics on X. Consider the generalized Hausdorff pseudo-metric distance

$$H_d : \mathcal{P}(X)\times\mathcal{P}(X) \longrightarrow \mathbb{R}^n_+\cup\{\infty\}$$

defined by

$$H_d(A,B) := \begin{pmatrix} H_{d_1}(A,B) \\ \vdots \\ H_{d_n}(A,B) \end{pmatrix}.$$

Definition 2.5. Let (X,d) be a generalized metric space. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is said to be contractive if there exists a metrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that

$$M^k \to 0 \ \text{ as } \ k\to\infty$$

and

$$H_d(N(u),N(v)) \le Md(u,v), \quad \forall\, u,v \in X.$$

Theorem 2.3 ([23])**.** *Let (X,d) be a generalized complete metric space, and let $N : X \to \mathcal{P}_{cl}(X)$ be a multivalued map. Assume that there exist $A,B,C \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that*

$$H_d(N(x),N(y)) \le Ad(x,y) + Bd(y,N(x)) + Cd(x,N(x)), \tag{2.1}$$

where $A+C$ converges to zero. Then there exists $x \in X$ such that $x \in N(x)$.

Definition 2.6. Let E be a vector space on $K=\mathbb{R}$ or $\mathbb{C}$. By a vector-valued norm on E we mean a map $\|\cdot\| : E \to \mathbb{R}^n$ with the following properties:

(i) $\|x\| \ge 0$ for all $x\in E$; if $\|x\| = 0$, then $x = (0,\dots,0)$;

(ii) $\|\lambda x\| = |\lambda|\,\|x\|$ for all $x\in E$ and $\lambda \in K$;

(iii) $\|x+y\| \le \|x\| + \|y\|$ for all $x,y \in E$.

The pair $(E, \|\cdot\|)$ is called a generalized normed space. If the generalized metric generated by $\|\cdot\|$ (i.e., $d(x,y) = \|x-y\|$) is complete, then the space $(E, \|\cdot\|)$ is called a generalized Banach space.

Lemma 2.1 ([43, Theorem 19.7]). *Let Y be a separable metric space and $F : [a,b] \to \mathcal{P}(Y)$ be a measurable multi-valued map with nonempty closed values. Then F has a measurable selection.*

Lemma 2.2 ([38]). *Let X be a Banach space. Let $F : [a,b] \times X \to \mathcal{P}_{cp,cv}(X)$ be an L^1-Carathéodory multifunction with $S_{F,y} \neq \varnothing$, and let Γ be a continuous linear operator from $L^1([a,b],X)$ to $C([a,b],X)$. Then the operator*

$$\begin{aligned} \Gamma \circ S_F : C([0,b],X) &\longrightarrow \mathcal{P}_{cp,cv}(C([a,b],X)), \\ y &\longrightarrow (\Gamma \circ S_F)(y) := \Gamma(S_{F,y}) \end{aligned}$$

has a closed graph in $C([a,b],X) \times C([a,b],X)$, where

$$S_{F,y} = \big\{ v \in L^1([0,b],X) : \ \ v(t) \in F(t,y(t)); \ \ t \in [a,b] \big\}.$$

Lemma 2.3 ([23, 47]). *Let X be a generalized Banach space and $F : X \to \mathcal{P}_{cp,cv}(X)$ be an u.s.c. compact multifunction. Moreover, assume that the set*

$$\mathcal{A} = \big\{ x \in X : \ \ x \in \lambda N(x) \ \ \textit{for some} \ \ \lambda \in (0,1) \big\}$$

is bounded. Then N has at least one fixed point.

Theorem 2.4 ([23]). *Let X be a generalized Banach space and $N : X \to X$ be a continuous compact mapping. Moreover, assume that the set*

$$\mathcal{K} = \big\{ x \in X : \ \ x = \lambda N(x) \ \ \textit{for some} \ \ \lambda \in (0,1) \big\}$$

is bounded. Then N has a fixed point.

Definition 2.7. Let X be a Banach space. A is called $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $I \times D$, where I is Lebesgue measurable in $[a,b]$ and D is Borel measurable in X.

Definition 2.8. A subset $B \subset L^1([a,b],X)$ is decomposable if for all $u, v \in A$ and for every Lebesgue measurable set $I \subset [a,b]$, we have

$$u\chi_{_I} + v\chi_{_{[a,b]\setminus I}} \in B,$$

where $\chi_{_I}$ stands for the characteristic function of the set I.

Let $F : J \times X \to \mathcal{P}_{cl}(X)$ be multi-valued. Assign to F the multi-valued operator $\mathcal{F} : C(J,X) \to \mathcal{P}(L^1([a,b],X))$ defined by $\mathcal{F}(y) = S_{F,y}$. The operator $\mathcal{F}$ is called the Nemyts'kiĭ operator associated to F.

Definition 2.9. Let $F : J \times X \to \mathcal{P}_{cp}(X)$ be multi-valued. We say that F is of lower semi-continuous type (*l.s.c.* type) if its associated Nemyts'kiĭ operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Lemma 2.4 ([19]). *Let $F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ be an integrable bounded multi-valued map such that*

(a) $(t,x,y) \to F(t,x,y)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable;*

(b) $(x,y) \to F(t,x,y)$ *is l.s.c. a.e.* $t \in [a,b]$.

Then F is lower semi-continuous.

Next, we state a classical selection theorem due to Bressan and Colombo.

Theorem 2.5 ([13, 20]) (Theorem of "Bressan–Colombo" selection). *Let X be a metric separable space, and let E be a Banach space. Then each l.s.c. operator $N : X \to \mathcal{P}_{cl}(L^1([a,b],X))$ which has a decomposable closed value, also has a continuous selection.*

2.1 σ-selectionable multi-valued maps

The following four definitions and the theorem can be found in [22,30] (see also [8, p. 86]). Let (X,d) and (Y,d') be two metric spaces.

Definition 2.10. We say that a map $F : X \to \mathcal{P}(Y)$ is σ-Ca-selectionable if there exists a decreasing sequence of compact-valued *u.s.c.* maps $F_n : X \to Y$ satisfying:

(a) F_n has a Carathédory selection for all $n \geq 0$ (F_n are called Ca-selectionable);

(b) $F(x) = \bigcap_{n\geq 0} F_n(x)$ for all $x \in X$.

Definition 2.11. A single-valued map $f : [0,a]\times X \to Y$ is said to be measurable-locally-Lipschitz (mLL) if $f(\,\cdot\,,x)$ is measurable for every $x \in X$, and for every $x \in X$ there exist a neighborhood $V_x \subset X$ of x and an integrable function $L_x : [0,a] \to [0,\infty)$ such that

$$d'(f(t,x_1), f(t,x_2)) \leq L_x(t)d(x_1,x_2) \ \text{ for every } \ t \in [0,a], \ \ x_1,x_2 \in V_x.$$

Definition 2.12. A multi-valued mapping $F : [0,a]\times X \to \mathcal{P}(Y)$ is mLL-selectionable if it has an mLL-selection.

Definition 2.13. We say that a multi-valued map $\phi : [0,a]\times E \to \mathcal{P}(E)$ with closed values is upper-Scorza–Dragoni if, given $\delta > 0$, there exists a closed subset $A_\delta \subset [0,a]$ such that the measure $\mu([0,a]\setminus A_\delta) \leq \delta$ and the restriction ϕ_δ of ϕ to $A_\delta \times E$ is *u.s.c.*

Theorem 2.6 (see [22, Theorem 19.19])**.** *Let E, E_1 be two separable Banach spaces and let $F : [a,b]\times E \to \mathcal{P}_{cp,cv}(E_1)$ be an upper-Scorza–Dragoni map. Then F is σ-*Ca*-selectionable, the maps $F_n : [a,b]\times E \to \mathcal{P}(E_1)$, $n\in\mathbb{N}$, are almost upper semicontinuous, and we have*

$$F_n(t,e) \subset \overline{co}\Big(\bigcup_{x\in E} F(t,x)\Big).$$

*Moreover, if F is integrably bounded, then F is σ-*mLL*-selectionable.*

Lemma 2.5 ([9])**.** *For an u.s.c. multifunction $F : X \to \mathcal{P}_{cp}(Y)$, we have*

$$\forall\, x_0 \in X, \ \ \lim_{x\to x_0}\sup F(x) \subseteq F(x_0).$$

Lemma 2.6 ([9])**.** *Let $(K_n)_n \subset K$ such that K is a compact subset of X, and X is a separable Banach space. Then*

$$\overline{co}\Big(\lim_{n\to\infty}\sup K_n\Big) = \bigcap_{N>0}\overline{co}\Big(\bigcup_{n\geq N} K_n\Big),$$

where co is the convex envelope.

Lemma 2.7 ([21])**.** *Let X be a metric compact space. If X is R_δ-set, then X is an acyclic space.*

Theorem 2.7 ([22])**.** *Let E be a normed space, X be a metric space, and let $f : X \to E$ be a continuous map. Then $\forall\,\varepsilon > 0$ there is a locally Lipschitz function $f_\varepsilon : X \to E$ such that*

$$\|f(x) - f_\varepsilon(x)\| \leq \varepsilon, \ \ \forall\, x \in X. \tag{2.2}$$

Theorem 2.8 (Theorem of Browder and Gupta, [14])**.** *Let $(E,\|\cdot\|)$ be a Banach space, $f : X \to E$ be a proper map, and suppose that for every $\varepsilon > 0$, we have a proper map $f_\varepsilon : X \to E$ satisfying:*

(i) *$\|f_\varepsilon(x) - f(x)\| < \varepsilon$ for all $x \in X$;*

(ii) *for all $u \in E$ such that $\|u\| \leq \varepsilon$, the equation $f_\varepsilon(x) = u$ has a unique solution.*

Then the set $S = f^{-1}(0)$ is R_δ.

3 Existence results

Let $J := [0, 1]$. In order to define a solution for problem (1.1), consider the space $PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$, where

$$PC(J,\mathbb{R}) := \Big\{ y : \ J \to \mathbb{R},\ \ y \in C(J \setminus \{t_k\}, \mathbb{R}) : \ \ k = 1, \ldots, m, \\ y(t_k^-) \text{ and } y(t_k^+) \text{ exist and satisfy } y(t_k^-) = y(t_k) \Big\}.$$

Endowed with the norm

$$\|y\|_{PC} = \sup\big\{\|y(t)\| : \ \ t \in J\big\},$$

PC is a Banach space.

3.1 Convex case

Theorem 3.1. *Assume there exist a continuous nondecreasing map $\psi : [0, +\infty) \to (0, +\infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that*

$$\|F_i(t, u, v)\| \leq p(t)\psi\big(\|u\| + \|v\|\big) \ \ a.e.\ \ t \in J, \ \ i \in \{1, 2\}, \ \ (u, v) \in \mathbb{R}^2.$$

Assume also that $F_1, F_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ are Carathéodory. Then problem (1.1) has at least one solution.

Proof. Consider the operator $N : PC \times PC \to \mathcal{P}(PC \times PC)$ defined by

$$N(x,y) = \left\{ (h_1, h_2) \in PC \times PC : \ \ \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} x_0 + \int\limits_0^t f_1(s)\,ds + \sum\limits_{0<t_k<t} I_1(x(t_k), y(t_k)), \ \ t \in J \\ y_0 + \int\limits_0^t f_2(s)\,ds + \sum\limits_{0<t_k<t} I_2(x(t_k), y(t_k)), \ \ t \in J \end{pmatrix} \right\},$$

where $f_i \in S_{F_i} = \{f \in L^1(J, \mathbb{R}) : f(t) \in F_i(t, x(t), y(t)), \text{ a.e. } t \in J\}$. Fixed points of the operator N are the solutions of problem (1.1).

We are going to prove that N is *u.s.c.* compact and that N has convex compact values. The proof is given by the following steps.

Step 1. $N(x, y)$ is convex for all $(x, y) \in PC \times PC$.

Let $(h_1, h_2), (h_3, h_4) \in N(x, y)$. So, there exist $f_1, f_3 \in S_{F_1(\,\cdot\,, x(\,\cdot\,), y(\,\cdot\,))}$ and $f_2, f_4 \in S_{F_2(\,\cdot\,, x(\,\cdot\,), y(\,\cdot\,))}$ such that for all $t \in J$, we have

$$h_1(t) = x_0 + \int\limits_0^t f_1(s)\,ds + \sum_{0<t_k<t} I_1(x(t_k), y(t_k)),$$

$$h_2(t) = y_0 + \int\limits_0^t f_2(s)\,ds + \sum_{0<t_k<t} I_2(x(t_k), y(t_k))$$

and

$$h_3(t) = x_0 + \int\limits_0^t f_3(s)\,ds + \sum_{0<t_k<t} I_1(x(t_k), y(t_k)),$$

$$h_4(t) = y_0 + \int\limits_0^t f_4(s)\,ds + \sum_{0<t_k<t} I_2(x(t_k), y(t_k)).$$

Let $l \in [0,1]$. For each $t \in J$, we have

$$\left(l \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + (1-l) \begin{pmatrix} h_3 \\ h_4 \end{pmatrix} \right)(t) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \int\limits_0^t (lf_1 + (1-l)f_3)(s)\,ds \\ \int\limits_0^t (lf_2 + (1-l)f_4)(s)\,ds \end{pmatrix} + \begin{pmatrix} \sum\limits_{0<t_k<t} I_1(x(t_k), y(t_k)) \\ \sum\limits_{0<t_k<t} I_2(x(t_k), y(t_k)) \end{pmatrix}.$$

As S_{F_1} and S_{F_2} are convex (since F_1 and F_2 have convex values),

$$l \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + (1-l) \begin{pmatrix} h_3 \\ h_4 \end{pmatrix} \in N(x,y).$$

Step 2. N transforms every bounded set to a bounded set in $PC \times PC$.

It suffices to show that

$$\exists\, \ell := \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} > 0 \ \text{ such that}$$
$$\forall\, (x,y) \in \mathcal{B}_q := \left\{ (x,y) \in PC \times PC : \ \|(x,y)\|_{PC\times PC} \leq q, \ \ q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} > 0 \right\},$$
$$\text{if } \ (h,g) \in N(x,y), \ \text{ then we have } \ \|(h,g)\|_{PC\times PC} \leq \ell.$$

Let $(h,g) \in N(x,y)$, then there exist $f_1 \in S_{F_1(\,\cdot\,,x(\,\cdot\,),y(\,\cdot\,))}$ and $f_2 \in S_{F_2(\,\cdot\,,x(\,\cdot\,),y(\,\cdot\,))}$ such that for all $t \in J$,

$$h(t) = x_0 + \int\limits_0^t f_1(s)\,ds + \sum_{0<t_k<t} I_1(x(t_k), y(t_k)),$$

$$g(t) = y_0 + \int\limits_0^t f_2(s)\,ds + \sum_{0<t_k<t} I_2(x(t_k), y(t_k)),$$

$$\|(h,g)\|_{PC\times PC} = \begin{pmatrix} \|h\|_{PC} \\ \|g\|_{PC} \end{pmatrix}.$$

For all $t \in J$, we have

$$\begin{aligned} \|h(t)\| &\leq \|x_0\| + \int\limits_0^t \|f_1(s)\|\,ds + \sum_{0<t_k<t} \big\|I_1(x(t_k), y(t_k))\big\| \\ &\leq \|x_0\| + \int\limits_0^1 \big\|F_1(s, x(s), y(s))\big\|\,ds + \sum_{k=1}^m \sup_{(x,y)\in\mathcal{B}_q} \|I_1(x,y)\| \\ &\leq \|x_0\| + \psi(q_1+q_2)\|p\|_{L^1} + \sum_{k=1}^m \sup_{(x,y)\in\mathcal{B}_q} \|I_1(x,y)\| := \widetilde{\ell} \end{aligned}$$

and

$$\begin{aligned} \|g(t)\| &\leq \|y_0\| + \int\limits_0^t \|f_2(s)\|\,ds + \sum_{0<t_k<t} \big\|I_2(x(t_k), y(t_k))\big\| \\ &\leq \|y_0\| + \int\limits_0^b \|F_2(s, x(s), y(s))\|\,ds + \sum_{k=1}^m \sup_{(x,y)\in\mathcal{B}_q} \|I_2(x,y)\| \end{aligned}$$

$$\leq \|y_0\| + \psi(q_1+q_2)\|p\|_{L^1} + \sum_{k=1}^{m} \sup_{(x,y)\in\mathcal{B}_q} \|I_2(x,y)\| := \widetilde{\widetilde{\ell}}.$$

Then

$$\begin{pmatrix} \|h\|_{PC} \\ \|g\|_{PC} \end{pmatrix} \leq \begin{pmatrix} \widetilde{\ell} \\ \widetilde{\widetilde{\ell}} \end{pmatrix} := \ell.$$

Step 3. N transforms every bounded set to an equicontinuous set in $PC \times PC$.

Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, and let $\mathcal{B}_q$ be as above in Step 2. For each $(x,y) \in \mathcal{B}_q$ and $(h,g) \in N(x,y)$, there exist $f_1 \in S_{F_1(\,\cdot\,,x(\,\cdot\,),y(\,\cdot\,))}$ and $f_2 \in S_{F_2(\,\cdot\,,x(\,\cdot\,),y(\,\cdot\,))}$ such that for all $t \in J$, we have

$$h(t) = x_0 + \int_0^t f_1(s)\,ds + \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k)),$$

$$g(t) = y_0 + \int_0^t f_2(s)\,ds + \sum_{0<t_k<t} I_{2,k}(x(t_k), y(t_k)).$$

Then

$$\begin{aligned} \|h(\tau_2) - h(\tau_1)\| &\leq \int_{\tau_1}^{\tau_2} \|f_1(s)\|\,ds + \sum_{\tau_1 \leq t_k < \tau_2} \big\|I_{1,k}(x(t_k), y(t_k))\big\| \\ &\leq \psi(q_1+q_2)\int_{\tau_1}^{\tau_2} p(s)\,ds + \sum_{\tau_1 \leq t_k < \tau_2} \sup_{(x,y)\in\mathcal{B}_q} \|I_{1,k}(x,y)\| \longrightarrow 0 \ \text{ as } \ \tau_2 \to \tau_1 \end{aligned}$$

and

$$\begin{aligned} \|g(\tau_2) - g(\tau_1)\| &\leq \int_{\tau_1}^{\tau_2} \|f_2(s)\|\,ds + \sum_{\tau_1 \leq t_k < \tau_2} \big\|I_{2,k}(x(t_k), y(t_k))\big\| \\ &\leq \psi(q_1+q_2)\int_{\tau_1}^{\tau_2} p(s)\,ds + \sum_{\tau_1 \leq t_k < \tau_2} \sup_{(x,y)\in\mathcal{B}_q} \|I_{2,k}(x,y)\| \longrightarrow 0 \ \text{ as } \ \tau_2 \to \tau_1. \end{aligned}$$

So, by Step 2 and Step 3, N is compact.

Step 4. The graph of N is closed.

Let $(x_n, y_n) \to (x_*, y_*)$, $(h_n, g_n) \in N(x_n, y_n)$, and $h_n \to h_*$ and $g_n \to g_*$. It suffices to show that there exist $f_1 \in S_{F_1(\,\cdot\,,x_*(\,\cdot\,),y_*(\,\cdot\,))}$ and $f_2 \in S_{F_2(\,\cdot\,,x_*(\,\cdot\,),y_*(\,\cdot\,))}$ such that for all $t \in J$, we have

$$h_*(t) = x_0 + \int_0^t f_1(s)\,ds + \sum_{0<t_k<t} I_{1,k}(x_*(t_k), y_*(t_k)),$$

$$g_*(t) = y_0 + \int_0^t f_2(s)\,ds + \sum_{0<t_k<t} I_{2,k}(x_*(t_k), y_*(t_k)).$$

With $(h_n, g_n) \in N(x_n, y_n)$, there exist $f_{1,n} \in S_{F_1(\,\cdot\,,x_n(\,\cdot\,),y_n(\,\cdot\,))}$ and $f_{2,n} \in S_{F_2(\,\cdot\,,x_n(\,\cdot\,),y_n(\,\cdot\,))}$ such that for all $t \in J$,

$$h_n(t) = x_0 + \int_0^t f_{1,n}(s)\,ds + \sum_{0<t_k<t} I_{1,k}(x_n(t_k), y_n(t_k)),$$

$$g_n(t) = y_0 + \int_0^t f_{2,n}(s)\,ds + \sum_{0<t_k<t} I_{2,k}(x_n(t_k), y_n(t_k)).$$

Since $I_{i,k}$, $k = 1, \dots, m$, $i = 1, 2$, are continuous,

$$\Big\| \Big(h_n(t) - x_0 - \sum_{0<t_k<t} I_{1,k}(x_n(t_k), y_n(t_k))\Big) - \Big(h_*(t) - x_0 - \sum_{0<t_k<t} I_{1,k}(x_*(t_k), y_*(t_k))\Big) \Big\|_{PC} \longrightarrow 0$$

and

$$\Big\| \Big(g_n(t) - y_0 - \sum_{0<t_k<t} I_{2,k}(x_n(t_k), y_n(t_k))\Big) - \Big(g_*(t) - y_0 - \sum_{0<t_k<t} I_{2,k}(x_*(t_k), y_*(t_k))\Big) \Big\|_{PC} \longrightarrow 0$$

as $n \to \infty$.

Let Γ be a continuous linear operator defined as

$$\begin{aligned} \Gamma : L^1(J, \mathbb{R}) &\longrightarrow PC(J, \mathbb{R}), \\ r &\longrightarrow \Gamma(r) \end{aligned}$$

such that

$$\Gamma(r)(t) = \int_0^t r(s)\,ds, \quad \forall\, t \in J.$$

By Lemma 2.2, the operator $\Gamma \circ S_F$ has a closed graph and, moreover, we have

$$\Big(h_n(t) - x_0 - \sum_{0<t_k<t} I_{1,k}(x_n(t_k), y_n(t_k))\Big) \in \Gamma(S_{F_1(\,\cdot\,, x_n(\,\cdot\,), y_n(\,\cdot\,))})$$

and

$$\Big(g_n(t) - y_0 - \sum_{0<t_k<t} I_{2,k}(x_n(t_k), y_n(t_k))\Big) \in \Gamma(S_{F_2(\,\cdot\,, x_n(\,\cdot\,), y_n(\,\cdot\,))}).$$

So,

$$\begin{aligned} \Big(h_*(t) - x_0 - \sum_{0<t_k<t} I_{1,k}(x_*(t_k), y_*(t_k))\Big) &= \int_0^t f_1(s)\,ds, \\ \Big(g_*(t) - y_0 - \sum_{0<t_k<t} I_{2,k}(x_*(t_k), y_*(t_k))\Big) &= \int_0^t f_2(s)\,ds, \end{aligned}$$

and then $f_1 \in S_{F_1(\,\cdot\,, x_*(\,\cdot\,), y_*(\,\cdot\,))}$ and $f_2 \in S_{F_2(\,\cdot\,, x_*(\,\cdot\,), y_*(\,\cdot\,))}$.

Step 5. A priori estimation.

Let $(x, y) \in PC(J, \mathbb{R})$ such that $(x, y) \in \lambda N(x, y)$, and $0 < \lambda < 1$. So, $\exists\, f_1 \in S_{F_1(\,\cdot\,, x(\,\cdot\,), y(\,\cdot\,))}$ and $\exists\, f_2 \in S_{F_2(\,\cdot\,, x(\,\cdot\,), y(\,\cdot\,))}$ such that for all $t \in [0, t_1]$,

$$\begin{aligned} x(t) &= \lambda x_0 + \lambda \int_0^t f_1(s, x(s), y(s))\,ds, \\ y(t) &= \lambda y_0 + \lambda \int_0^t f_2(s, x(s), y(s))\,ds. \end{aligned}$$

Then

$$\|x(t)\| \le \|x_0\| + \int_0^t p(s)\psi\big(\|x(s)\| + \|y(s))\|\big)\,ds, \quad t \in [0, t_1],$$

$$\|y(t)\| \leq \|y_0\| + \int_0^t p(s)\psi\big(\|(x(s)\| + \|y(s))\|\big)\,ds, \ \ t \in [0, t_1].$$

Consider the functions ϑ_1, $\mathcal{W}_1$ defined by

$$\vartheta_1(t) = \|x_0\| + \int_0^t p(s)\psi\big(\|(x(s)\| + \|y(s))\|\big)\,ds, \ \ t \in [0, t_1],$$

$$\mathcal{W}_1(t) = \|y_0\| + \int_0^t p(s)\psi\big(\|(x(s)\| + \|y(s))\|\big)\,ds, \ \ t \in [0, t_1].$$

So,

$$(\vartheta_1(0), \mathcal{W}_1(0)) = \big(\|x_0\|, \|y_0\|\big), \quad \|x(t)\| \leq \vartheta_1(t), \quad \|y(t)\| \leq \mathcal{W}_1(t), \ \ t \in [0, t_1],$$

and

$$\dot{\mathcal{W}}_1(t) = \dot{\vartheta}_1(t) = p(t)\psi\big(\|(x(t)\| + \|y(t))\|\big), \ \ t \in [0, t_1].$$

As ψ is a nondecreasing map, we have

$$\dot{\vartheta}_1(t) \leq p(t)\psi(\vartheta_1(t)), \quad \dot{\mathcal{W}}_1(t) \leq p(t)\psi(\mathcal{W}_1(t)), \ \ t \in [0, t_1].$$

This implies that for every $t \in [0, t_1]$,

$$\int_{\vartheta_1(0)}^{\vartheta_1(t)} \frac{du}{\psi(u)} \leq \int_0^{t_1} p(s)\,ds, \quad \int_{\mathcal{W}_1(0)}^{\mathcal{W}_1(t)} \frac{du}{\psi(u)} \leq \int_0^{t_1} p(s)\,ds.$$

The maps $\Gamma_0^1(z) = \int_{\vartheta_1(0)}^{z} \frac{du}{\psi(u)}$ and $\Gamma_0^2(z) = \int_{\mathcal{W}_1(0)}^{z} \frac{du}{\psi(u)}$ are continuous and increasing. Then $(\Gamma_0^1)^{-1}$ and $(\Gamma_0^2)^{-1}$ exist and are increasing, and we get

$$\vartheta_1(t) \leq (\Gamma_0^1)^{-1}\bigg(\int_0^{t_1} p(s)\,ds\bigg) := M_0, \quad \mathcal{W}_1(t) \leq (\Gamma_0^2)^{-1}\bigg(\int_0^{t_1} p(s)\,ds\bigg) := \ell_0.$$

As for every $t \in [0, t_1]$, $\|x(t)\| \leq \vartheta_1(t)$ and $\|y(t)\| \leq \mathcal{W}_1(t)$, so,

$$\sup_{t\in[0,t_1]} \|y(t)\| \leq \ell_0, \quad \sup_{t\in[0,t_1]} \|x(t)\| \leq M_0.$$

Now, for $t \in (t_1, t_2]$, we have

$$\|x(t_1^+)\| \leq \big\|I_{1,1}(x(t_1), y(t_1))\big\| + \|x(t_1)\| \leq \sup_{(\alpha,\beta)\in\overline{B}(0,M_0)\times\overline{B}(0,\ell_0)} \|I_{1,1}(\alpha,\beta)\| + M_0 := N_1,$$

$$\|y(t_1^+)\| \leq \big\|I_{2,1}(x(t_1), y(t_1))\big\| + \|y(t_1)\| \leq \sup_{(\alpha,\beta)\in\overline{B}(0,M_0)\times\overline{B}(0,\ell_0)} \|I_{2,1}(\alpha,\beta)\| + \ell_0 := D_1.$$

Also,

$$x(t) = \lambda\big(x(t_1) + I_{1,1}(x(t_1), y(t_1))\big) + \lambda\int_{t_1}^{t} f_1(s, x(s), y(s))\,ds,$$

$$y(t) = \lambda\big(y(t_1) + I_{2,1}(x(t_1), y(t_1))\big) + \lambda\int_{t_1}^{t} f_2(s, x(s), y(s))\,ds,$$

and so,

$$\|x(t)\| \leq N_1 + \int_{t_1}^{t} p(s)\psi\big(\|(x(s)\| + \|y(s))\|\big)\, ds, \ \ t \in [t_1, t_2],$$

$$\|y(t)\| \leq D_1 + \int_{t_1}^{t} p(s)\psi\big(\|(x(s)\| + \|y(s))\|\big)\, ds, \ \ t \in [t_1, t_2].$$

Let us consider the maps ϑ_2 and $\mathcal{W}_2$ defined by

$$\vartheta_2(t) = N_1 + \int_{t_1}^{t} p(s)\psi\big(\|(x(s)\| + \|y(s))\|\big)\, ds, \quad \mathcal{W}_2(t) = D_1 + \int_{t_1}^{t} p(s)\psi\big(\|(x(s)\| + \|y(s))\|\big)\, ds, \ \ t \in [t_1, t_2].$$

Then

$$\vartheta_2(t_1^+) = N_1, \quad \|x(t)\| \leq \vartheta_2(t), \ \ t \in [t_1, t_2],$$
$$\mathcal{W}_2(t_1^+) = D_1, \quad \|y(t)\| \leq \mathcal{W}_2(t), \ \ t \in [t_1, t_2],$$

and

$$\dot{\vartheta}_2(t) = p(t)\psi\big(\|(x(t)\| + \|y(t))\|\big), \quad \dot{\mathcal{W}}_2(t) = p(t)\psi\big(\|(x(t)\| + \|y(t))\|\big), \ \ t \in [t_1, t_2].$$

As ψ is nondecreasing,

$$\dot{\vartheta}_2(t) \leq p(t)\psi(\vartheta_2(t)), \quad \dot{\mathcal{W}}_2(t) \leq p(t)\psi(\mathcal{W}_2(t)), \ \ t \in [t_1, t_2].$$

This implies that for every $t \in [t_1, t_2]$,

$$\int_{\vartheta_2(t_1^+)}^{\vartheta_2(t)} \frac{du}{\psi(u)} \leq \int_{t_1}^{t_2} p(s)\, ds, \qquad \int_{\mathcal{W}_2(t_1^+)}^{\mathcal{W}_2(t)} \frac{du}{\psi(u)} \leq \int_{t_1}^{t_2} p(s)\, ds.$$

If we consider the maps $\Gamma_1^1(z) = \int_{\vartheta_2(t_1^+)}^{z} \frac{du}{\psi(u)}$ and $\Gamma_1^2(z) = \int_{\mathcal{W}_2(t_1^+)}^{z} \frac{du}{\psi(u)}$, we get

$$\vartheta_2(t) \leq (\Gamma_1^1)^{-1}\Bigg(\int_{t_1}^{t_2} p(s)\, ds\Bigg) := M_1,$$

$$\mathcal{W}_2(t) \leq (\Gamma_1^2)^{-1}\Bigg(\int_{t_1}^{t_2} p(s)\, ds\Bigg) := \ell_1.$$

For all $t \in [t_1, t_2]$, $\|x(t)\| \leq \vartheta_2(t)$ and $\|y(t)\| \leq \mathcal{W}_2(t)$, and then

$$\sup_{t \in [t_1, t_2]} \|x(t)\| \leq M_1, \qquad \sup_{t \in [t_1, t_2]} \|y(t)\| \leq \ell_1.$$

We continue the process to the interval $(t_m, 1]$. We get the existence of M_m and ℓ_m such that

$$\sup_{t \in [t_m, 1]} \|x(t)\| \leq (\Gamma_m^1)^{-1}\Bigg(\int_{t_m}^{1} p(s)\, ds\Bigg) := M_m, \qquad \sup_{t \in [t_m, 1]} \|y(t)\| \leq (\Gamma_m^2)^{-1}\Bigg(\int_{t_m}^{1} p(s)\, ds\Bigg) := \ell_m.$$

As we chose y arbitrarily, then for all solutions of problem (1.1), we get

$$\|(x, y)\|_{PC \times PC} \leq \max\left\{ \begin{pmatrix} M_k \\ \ell_k \end{pmatrix} : \ \ k = 0, 1, \ldots, m \right\} := b^*.$$

Then the set
$$\mathcal{A} = \Big\{(x,y) \in PC \times PC : \ \ (x,y) \in \lambda N(x,y), \ \ \lambda \in (0,1)\Big\}$$
is bounded. So, $N : PC \times PC \to \mathcal{P}_{cv}(PC \times PC)$ is compact and *u.s.c.* Then, by Lemma 2.3, we obtain that problem (1.1) has at least one solution. □

3.2 Nonconvex case

Assume that the following conditions hold:

($\mathcal{H}_1$) $F_i : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$, $t \to F_i(t,u,v)$ are measurable for each $u, v \in \mathbb{R}$, $i = 1,2$.

($\mathcal{H}_2$) There exist the functions $l_i \in L^1(J, \mathbb{R}^+)$, $i = 1, \ldots, 4$, such that
$$H_d\big(F_1(t,u,v), F_1(t,\overline{u},\overline{v})\big) \le l_1(t)\|u-\overline{u}\| + l_2(t)\|v-\overline{v}\|, \ \ t \in J, \ \ \forall\, u, \overline{u}, v, \overline{v} \in \mathbb{R},$$
$$H_d\big(F_2(t,u,v), F_2(t,\overline{u},\overline{v})\big) \le l_3(t)\|u-\overline{u}\| + l_4(t)\|v-\overline{v}\|, \ \ t \in J, \ \ \forall\, u, \overline{u}, v, \overline{v} \in \mathbb{R}$$
and
$$H_d(0, F_1(t,0,0)) \le l_1(t) \ \text{ for a.e. } \ t \in J, \quad H_d(0, F_2(t,0,0)) \le l_3(t) \ \text{ for a.e. } \ t \in J.$$

($\mathcal{H}_3$) There exist the constants $a_i, b_i \ge 0$, $i = 1,2$, such that
$$\big\|I_1(u,v) - I_1(\overline{u} - \overline{v}\big\| \le a_1\|u-\overline{u}\| + a_2\|v-\overline{v}\|, \ \ \forall\, u, \overline{u}, v, \overline{v} \in \mathbb{R}$$
and
$$\big\|I_2(u,v) - I_2(\overline{u} - \overline{v}\big\| \le b_1\|u-\overline{u}\| + b_2\|v-\overline{v}\|, \ \ \forall\, u, \overline{u}, v, \overline{v} \in \mathbb{R}.$$

Theorem 3.2. *Assume that* ($\mathcal{H}_1$)–($\mathcal{H}_3$) *are satisfied and the matrix*
$$M = \begin{pmatrix} \|l_1\|_{L^1} + a_1 & \|l_2\|_{L^1} + a_2 \\ \|l_3\|_{L^1} + b_1 & \|l_4\|_{L^1} + b_2 \end{pmatrix}$$
converges to zero. Then problem (1.1) *has at least one solution.*

Proof. Consider the operator $N : PC \times PC \to \mathcal{P}(PC \times PC)$ defined by
$$N(x,y) = \left\{ (h_1,h_2) \in PC \times PC : \ \ \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} x_0 + \int\limits_0^t f_1(s)\,ds + \sum\limits_{0<t_k<t} I_1(x(t_k), y(t_k)), \ \ t \in J \\ y_0 + \int\limits_0^t f_2(s)\,ds + \sum\limits_{0<t_k<t} I_2(x(t_k), y(t_k)), \ \ t \in J \end{pmatrix} \right\},$$
where
$$f_i \in S_{F_i} = \Big\{ f \in L^1(J,\mathbb{R}) : \ \ f(t) \in F_i(t, x(t), y(t)), \ \ \text{a.e. } \ t \in J \Big\}.$$
Fixed points of the operator N are the solutions of problem (1.1).

Let, for $i = 1,2$,
$$N_i(x,y) = \left\{ h \in PC : \ \ h(t) = x_i(t) + \int\limits_0^t f_i(s)\,ds + \sum_{0<t_k<t} I_i(x(t_k), y(t_k)), \ \ t \in J \right\},$$
where $x_1 = x_0$ and $x_2 = y_0$. We show that N satisfies the assumptions of Theorem 2.3.

Let $(x,y),(\overline{x},\overline{y}) \in PC\times PC$ and $(h_1,h_2)\in N(x,y)$. Then there exist $f_i \in S_{F_i}, i=1,2,$, such that

$$\begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} x_0 + \int\limits_0^t f_1(s)\,ds + \sum\limits_{0<t_k<t} I_1(x(t_k),y(t_k)), & t\in J \\ y_0 + \int\limits_0^t f_2(s)\,ds + \sum\limits_{0<t_k<t} I_2(x(t_k),y(t_k)), & t\in J \end{pmatrix}.$$

$(\mathcal{H}_2)$ implies that

$$H_{d_1}\big(F_1(t,x(t),y(t)),F_1(t,\overline{x}(t),\overline{y}(t))\big) \le l_1(t)|x(t)-\overline{x}(t)| + l_2(t)|y(t)-\overline{y}(t)|, \;\; t\in J,$$

and

$$H_{d_2}\big(F_2(t,x(t),y(t)),F_2(t,\overline{x}(t),\overline{y}(t))\big) \le l_3(t)|x(t)-\overline{x}(t)| + l_4(t)|y(t)-\overline{y}(t)|, \;\; t\in J.$$

Hence, there is some $(\omega,\overline{\omega})\in F_1(t,\overline{x}(t),\overline{y}(t))\times F_2(t,\overline{x}(t),\overline{y}(t))$ such that

$$|f_1(t)-\omega| \le l_1(t)|x(t)-\overline{x}(t)| + l_2(t)|y(t)-\overline{y}(t)|, \;\; t\in J,$$

and

$$|f_2(t)-\overline{\omega}| \le l_3(t)|x(t)-\overline{x}(t)| + l_4(t)|y(t)-\overline{y}(t)|, \;\; t\in J.$$

Consider the multi-valued maps $U_i : J\to\mathcal{P}(\mathbb{R})$, $i=1,2$, defined by

$$U_1(t) = \Big\{\omega\in F_1(t,\overline{x}(t),\overline{y}(t)) : \;\; |f_1(t)-\omega| \le l_1(t)|x(t)-\overline{x}(t)| + l_2(t)|y(t)-\overline{y}(t)|, \;\; \text{a.e. } t\in J\Big\}$$

and

$$U_2(t) = \Big\{\omega\in F_2(t,\overline{x}(t),\overline{y}(t)) : \;\; |f_1(t)-\omega| \le l_1(t)|x(t)-\overline{x}(t)| + l_2(t)|y(t)-\overline{y}(t)|, \;\; \text{a.e. } t\in J\Big\}.$$

Then each $U_i(t)$ is a nonempty set and Theorem III.4.1 in [15] implies that U_i is measurable. Moreover, the multi-valued intersection operator $V_i(\cdot) := U_i(\cdot)\cap F_i(\cdot,\overline{x}(\cdot),\overline{y}(\cdot))$ is measurable. Therefore, for each $i=1,2$, by Lemma 2.1, there exists a function $t\to\overline{f}_i(t)$, which is a measurable selection for V_i, that is, $\overline{f}_i(t)\in F_i(t,\overline{x}(t),\overline{y}(t))$ and

$$|f_1(t)-\overline{f}_1(t)| \le l_1(t)|x(t)-\overline{x}(t)| + l_2(t)|y(t)-\overline{y}(t)|, \;\; \text{a.e. } t\in J,$$

and

$$|f_2(t)-\overline{f}_2(t)| \le l_3(t)|x(t)-\overline{x}(t)| + l_4(t)|y(t)-\overline{y}(t)|, \;\; \text{a.e. } t\in J.$$

Define $\overline{h}_1$ and $\overline{h}_2$ by

$$\overline{h}_1(t) = x_0 + \int\limits_0^t \overline{f}_1(s)\,ds + \sum_{0<t_k<t} I_1(\overline{x}(t_k),\overline{y}(t_k)), \;\; t\in J,$$

and

$$\overline{h}_2(t) = y_0 + \int\limits_0^t \overline{f}_2(s)\,ds + \sum_{0<t_k<t} I_2(\overline{x}(t_k),\overline{y}(t_k)), \;\; t\in J.$$

Then for $t\in J$,

$$|h_1(t)-\overline{h}_1(t)| \le \big(\|l_1\|_{L^1}+a_1\big)|x-\overline{x}|_{PC} + \big(\|l_2\|_{L^1}+a_2\big)\|y-\overline{y}\|_{PC}.$$

Thus

$$\|h_1-\overline{h}_1\|_{PC} \le \big(\|l_1\|_{L^1}+a_1\big)|x-\overline{x}|_{PC} + \big(\|l_2\|_{L^1}+a_2\big)\|y-\overline{y}\|_{PC}.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, we finally arrive at the estimate

$$H_{d_1}\big(N_1(x,y),N_1(\overline{x},\overline{y})\big) \le \big(\|l_1\|_{L^1}+a_1\big)\|x-\overline{x}\|_{PC}+\big(\|l_2\|_{L^1}+a_2\big)\|y-\overline{y}\|_{PC}.$$

Similarly, we get

$$H_{d_2}\big(N_2(x,y),N_2(\overline{x},\overline{y})\big) \le \big(\|l_3\|_{L^1}+b_1\big)\|x-\overline{x}\|_{PC}+\big(\|l_4\|_{L^1}+b_2\big)\|y-\overline{y}\|_{PC}.$$

Therefore,

$$H_d\big(N(x,y),N(\overline{x},\overline{y})\big) \le M\big(\|x-\overline{x}\|_{PC},\|y-\overline{y}\|_{PC}\big),\ \ \forall\,(x,y),(\overline{x},\overline{y})\in PC\times PC.$$

Hence, by Theorem 2.3, the operator N has at least one fixed point which is a solution of (1.1). □

Theorem 3.3. *Assume, for each $i=1,2$, that there exist a continuous nondecreasing map $\psi_i : [0,+\infty[\to(0,+\infty)$ and $p_i\in L^1(J,\mathbb{R}_+)$ such that*

$$\|F_i(t,u,v)\| \le p_i(t)\psi_i\big(\|u\|+\|v\|\big)\ \ a.e.\ \ t\in J,\ \ (u,v)\in\mathbb{R}^2.$$

Assume also that $F_1,F_2 : J\times\mathbb{R}\times\mathbb{R}\to\mathcal{P}_{cp,cv}(\mathbb{R})$ are Carathéodory, and

(a) $(t,x,y)\to F_i(t,x,y)$ *is $\mathcal{L}\otimes\mathcal{B}$ measurable for $i=1,2$.*

(b) $(x,y)\to F_i(t,x,y)$ *is l.s.c. a.e. $t\in J$.*

Then problem (1.1) *has at least one solution.*

Proof. For each $i=1,2$, since F_i is *l.s.c.*, by Theorem 2.5, there exists a continuous function $f_i : PC\to L^1(J,\mathbb{R})$ such that $f_i(x,y)\in S_{F_i(\,\cdot\,,x,y)}$ for all $(x,y)\in PC(J,\mathbb{R})\times PC(J,\mathbb{R})$. Consider the impulsive system

$$\begin{cases} x'(t)=f_1(t,x,y), & \text{a.e. } t\in J,\\ y'(t)=f_2(t,x,y), & \text{a.e. } t\in J,\\ x(t_k^+)-x(t_k^-)=I_1(x(t_k),y(t_k)), & k=1,2,\dots,m,\\ y(t_k^+)-y(t_k^-)=I_2(x(t_k),y(t_k)), & k=1,2,\dots,m,\\ x(0)=x_0,\quad y(0)=y_0. \end{cases}\tag{3.1}$$

It is clear that if (x,y) is a solution of problem (3.1), then (x,y) is also a solution of problem (1.1). When the proof of Theorem 3.1 is applied to the operator $N_*: PC\times PC\to\mathcal{P}(PC\times PC)$ defined by

$$N_*(x,y)=\left\{(h_1,h_2)\in PC\times PC:\ \ \begin{pmatrix}h_1(t)\\ h_2(t)\end{pmatrix}=\begin{pmatrix} x_0+\displaystyle\int_0^t f_1(s)\,ds+\sum_{0<t_k<t} I_1(x(t_k),y(t_k)), & t\in J\\ y_0+\displaystyle\int_0^t f_2(s)\,ds+\sum_{0<t_k<t} I_2(x(t_k),y(t_k)), & t\in J\end{pmatrix}\right\},$$

there is a solution of problem (1.1). □

4 Structure of solutions sets

Consider the first-order impulsive single-valued problem

$$\begin{cases} x'(t)=f_1(t,x(t),y(t)), & \text{a.e. } t\in[0,1],\\ y'(t)=f_2(t,x(t),y(t)), & \text{a.e. } t\in[0,1],\\ x(t_k^+)-x(t_k^-)=I_1(x(t_k),y(t_k)), & k=1,\dots,m,\\ y(t_k^+)-y(t_k^-)=I_2(x(t_k),y(t_k)), & k=1,\dots,m,\\ x(0)=x_0,\quad y(0)=y_0, \end{cases}\tag{4.1}$$

where $f_1, f_2 \in L^1(J \times \mathbb{R}^2, \mathbb{R})$ are te given functions and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$. Then (x, y) is a solution of (4.1) if and only if (x, y) is a solution of the impulsive integral system

$$\begin{cases} x(t) = x_0 + \int\limits_0^t f_1(s, x(s), y(s))\,ds + \sum\limits_{0<t_k<t} I_1(x(t_k), y(t_k)), \;\text{ a.e. } t \in J, \\ y(t) = y_0 + \int\limits_0^t f_2(s, x(s), y(s))\,ds + \sum\limits_{0<t_k<t} I_2(x(t_k), y(t_k)), \;\text{ a.e. } t \in J. \end{cases} \tag{4.2}$$

Denote by $S(f_{1,2}, (x_0, y_0))$ the set of all solutions of problem (4.1).

Theorem 4.1. *Suppose that there are the functions $\ell_i \in L^1(J, \mathbb{R}_+)$, $i = 1, 2$, such that*

$$\big|f_i(t, x_1, y_1) - f_i(t, x_2, y_2)\big| < \ell_i(t)\big(|x_1 - x_2| + |y_1 - y_2|\big), \;\; \forall\, (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2.$$

Then problem (4.1) *has a unique solution.*

Proof.
1. The existence:

- We consider problem (4.1) on $[0, t_1]$,

$$\begin{gathered} x'(t) = f_1(t, x(t), y(t)), \quad y'(t) = f_2(t, x(t), y(t)), \;\text{ a.e. } t \in [0, t_1], \\ x(0) = x_0, \quad y(0) = y_0. \end{gathered} \tag{4.3}$$

We consider the operator N_1 defined by

$$\begin{gathered} N_1 : C([0, t_1], \mathbb{R}) \times C([0, t_1], \mathbb{R}) \longrightarrow C([0, t_1], \mathbb{R}) \times C([0, t_1], \mathbb{R}), \\ (x, y) \longrightarrow N_1(x, y), \end{gathered}$$

$$N_1(x, y)(t) = \left(x_0 + \int\limits_0^t f_1(s, x(s), y(s))\,ds; y_0 + \int\limits_0^t f_2(s, x(s), y(s))\,ds\right), \;\; t \in [0, t_1].$$

Let $(x_1, y_1), (x_2, y_2) \in C([0, t_1], \mathbb{R}) \times C([0, t_1], \mathbb{R})$, $t \in [0, t_1]$, and

$$\big\|N_1(x_1, y_1)(t) - N_1(x_2, y_2)(t)\big\| = \big\|(\alpha, \beta)\big\| = \begin{pmatrix} \|\alpha\| \\ \|\beta\| \end{pmatrix},$$

where

$$\alpha = \int\limits_0^t \big(f_1(s, x_1(s), y_1(s)) - f_1(s, x_2(s), y_2(s))\big)\,ds$$

and

$$\beta = \int\limits_0^t \big(f_2(s, x_1(s), y_1(s)) - f_2(s, x_2(s), y_2(s))\big)\,ds.$$

Then

$$\begin{aligned} \|\alpha\| &\le \int\limits_0^t \ell_1(s)\big\|(x_1(s), y_1(s)) - (x_2(s), y_2(s))\big\|\,ds \\ &\le \frac{1}{\tau} \int\limits_0^t \tau\ell(s) e^{\tau L(s)}\,ds \left\|\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}\right\| |_{BC} \le \frac{1}{\tau}\, e^{\tau L(t)} \left\|\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}\right\|_{BC} = \frac{1}{\tau}\, e^{\tau L(t)} \left\|\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}\right\|_{BC} \\ &= \frac{1}{\tau}\, e^{\tau L(t)}\big(\|x_1 - x_2\| + \|y_1 - y_2\|\big) = e^{\tau L(t)}\Big(\frac{1}{\tau}\, \|x_1 - x_2\| + \frac{1}{\tau}\, \|y_1 - y_2\|\Big), \end{aligned}$$

where

$$L(t) = \int_0^t \ell(s)\,ds, \quad \text{and} \quad \tau > 2.$$

Similarly,

$$\|\beta\| \le e^{\tau L(t)}\Big(\frac{1}{\tau}\,\|x_1 - x_2\| + \frac{1}{\tau}\,\|y_1 - y_2\|\Big).$$

Thus

$$e^{-\tau L(t)}\big\|N_1(x_1,y_1)(t) - N_1(x_2,y_2)(t)\big\| \le \begin{pmatrix} \frac{1}{\tau} & \frac{1}{\tau} \\ \frac{1}{\tau} & \frac{1}{\tau} \end{pmatrix} \begin{pmatrix} \|x_1 - x_2\| \\ \|y_1 - y_2\| \end{pmatrix}, \quad t \in [0, t_1].$$

Then

$$\big\|N_1(x_1,y_1) - N_1(x_2,y_2)\big\|_{BC} \le \frac{1}{\tau}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \|x_1 - x_2\| \\ \|y_1 - y_2\| \end{pmatrix},$$

where

$$\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_{BC} = \sup_{t\in[0,t_1]} e^{-\tau L(t)} \left\|\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\right\|.$$

Let

$$B = \frac{1}{\tau}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then we have

$$\det(B - \lambda I) = \Big(\frac{1}{\tau} - \lambda\Big)^2 - \frac{1}{\tau^2},$$

hence $\rho(B) = \frac{2}{\tau}$. For $\tau \in (2, +\infty)$, N_1 is contractive, so there exists a unique

$$(x^0, y^0) \in C([0,t_1], \mathbb{R}) \times C([0,t_1], \mathbb{R}) \ \text{ such that } \ N_1(x^0, y^0) = (x^0, y^0).$$

Then (x^0, y^0) is the solution of (4.3).

- We consider problem (4.1) on $(t_1, t_2]$,

$$\begin{aligned} &x'(t) = f_1(t, x(t), y(t)), \quad y'(t) = f_2(t, x(t), y(t)), \ \text{ a.e. } t \in J_1 = (t_1, t_2], \\ &x(t_1^+) = x^0(t_1) + I_1(x^0(t_1), y^0(t_1)), \quad y(t_1^+) = y^0(t_1) + I_1(x^0(t_1), y^0(t_1)). \end{aligned} \tag{4.4}$$

  Consider the space $C_* = \{(x,y) \in C(J_1, \mathbb{R}) \times C(J_1, \mathbb{R}) : \ (x(t_1^+), y(t_1^+)) \text{ exist}\}$, $(C_*, \|\cdot\|_{J_1})$ is a Banach space.

Let

$$\begin{aligned} N_2 : C_* &\longrightarrow C_*, \\ (x,y) &\longrightarrow N_2(x,y), \end{aligned}$$

$$\begin{aligned} N_2(x,y)(t) = \Bigg(&x^0(t_1) + I_1(x^0(t_1), y^0(t_1)) + \int_{t_1}^t f_1(s, x(s), y(s))\,ds, \\ &\qquad y^0(t_1) + I_2(x^0(t_1), y^0(t_1)) + \int_{t_1}^t f_2(s, x(s), y(s))\,ds\Bigg), \quad t \in (t_1, t_2]. \end{aligned}$$

Let $(x_1,y_1), (x_2,y_2) \in C_* \times C_*$, and $t \in (t_1, t_2]$,

$$\big\|N_2(x_1,y_1)(t) - N_2(x_2,y_2)(t)\big\| = \|(\alpha, \beta)\| = \begin{pmatrix} \|\alpha\| \\ \|\beta\| \end{pmatrix},$$

where

$$\|\alpha\| \leq \int_{t_1}^{t} \ell(s)\big\|(x_1(s), y_1(s)) - (x_2(s), y_2(s))\big\|\, ds \leq \frac{1}{\tau}\int_{t_1}^{t} \tau\ell(s)e^{\tau L(s)}\, ds \left\|\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}\right\|_{BC}$$
$$\leq \frac{1}{\tau}\, e^{\tau L(t)} \left\|\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}\right\|_{BC} = e^{\tau L(t)}\Big(\frac{1}{\tau}\, \|x_1 - x_2\| + \frac{1}{\tau}\, \|y_1 - y_2\|\Big)$$

and

$$L(t) = \int_{t_1}^{t} \ell(s)\, ds.$$

Similarly,

$$\|\beta\| \leq e^{\tau L(t)}\Big(\frac{1}{\tau}\, \|x_1 - x_2\| + \frac{1}{\tau}\, \|y_1 - y_2\|\Big).$$

So,

$$e^{-\tau L(t)}\big\|N_2(x_1, y_1)(t) - N_2(x_2, y_2)(t)\big\| \leq \frac{1}{\tau}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \|x_1 - x_2\| \\ \|y_1 - y_2\| \end{pmatrix}, \quad t \in (t_1, t_2].$$

Then

$$\big\|N_2(x_1, y_1) - N_2(x_2, y_2)\big\|_{BC} \leq \frac{1}{\tau}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \|x_1 - x_2\| \\ \|y_1 - y_2\| \end{pmatrix}.$$

Then for $\tau \in (2, +\infty)$, N_2 is a contraction and, so, there exists a unique $(x^1, y^1) \in C((t_1, t_2], \mathbb{R})$ such that

$$N_2(x^1, y^1) = (x^1, y^1).$$

We have

$$(x^1, y^1)(t_1^+) = N_2(x^1, y^1)(t_1^+) = \bigg(x^0(t_1) + I_1(x^0(t_1), y^0(t_1)) + \lim_{t\to t_1}\int_{t_1}^{t} f_1(s, x(s), y(s))\, ds,$$
$$y^0(t_1) + I_1(x^0(t_1), y^0(t_1)) + \lim_{t\to t_1}\int_{t_1}^{t} f_2(s, x(s), y(s))\, ds\bigg).$$

Then (x^1, y^1) is the solution of problem (4.4). As a consequence, arguing inductively, the solution of problem (4.1) is given by

$$(x^*, y^*)(t) := \begin{cases} (x^0, y^0)(t), & t \in [0, t_1], \\ (x^1, y^1)(t), & t \in (t_1, t_2], \\ \qquad\vdots & \\ (x^m, y^m)(t), & t \in (t_m, 1]. \end{cases}$$

2. The uniqueness:

Let $(x^*, y^*), (x^{**}, y^{**})$ be two solutions of problem (4.1). We are going to show that

$$(x^*, y^*)(t) = (x^{**}, y^{**})(t), \ \ \forall\, t \in J = [0, 1].$$

Again, the process is inductive.

If $t \in J_0 = [0, t_1]$, then $(x^*, y^*)(t) = (x^{**}, y^{**})(t)$, $\forall\, t \in [0, t_1]$.

Now, suppose that if $t \in J_i = (t_i, t_{i+1}]$, then $(x^*, y^*)(t) = (x^{**}, y^{**})(t)$, $\forall\, t \in (t_i, t_{i+1}]$. It is enough to show that $(x^*, y^*)(t_k^+) = (x^{**}, y^{**})(t_k^+)$, $k \in \{1, 2, \ldots, m\}$. To that end, we have

$$(x^*, y^*)(t_i^+) - (x^*, y^*)(t_i^-) = \big(I_{1i}(x^*(t_i), y^*(t_i)), I_{2i}(x^*(t_i), y^*(t_i))\big),$$

which implies that

$$(x^*,y^*)(t_i^+) = (x^*,y^*)(t_i^-) + I_{1i}(x^*(t_i),y^*(t_i))$$

and

$$I_{2i}(x^*(t_i),y^*(t_i)) = (x^{**},y^{**})(t_i) + \big(I_{1i}(x^{**}(t_i),y^{**}(t_i)), I_{2i}(x^{**}(t_i),y^{**}(t_i))\big) = (x^{**},y^{**})(t_i^+). \quad \square$$

Theorem 4.2. *Suppose there exist a continuous function $\psi : [0,\infty) \to (0,\infty)$ which is nondecreasing, and a function $p \in L^1(J,\mathbb{R}_+)$ such that*

$$\|f^i(t,x,y)\| \le p(t)\psi\big(\|x\|+\|y\|\big), \ \ \forall\, t \in J, \ \ \forall\, x,y \in \mathbb{R},$$

with

$$\int_0^1 p(s)\,ds < \int_{\|x_0\|}^{\infty} \frac{du}{\psi(u)}\,.$$

Then problem (4.1) *has at least one solution.*

Proof. For the proof we use "the nonlinear alternative of Leray–Schauder". Consider the operator

$$N : PC(J,\mathbb{R}) \times PC(J,\mathbb{R}) \longrightarrow PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$$

defined by

$$N(x,y)(t) = \Bigg(x_0 + \int_0^t f_1(s,x(s),y(s))\,ds + \sum_{0<t_k<t} I_{1,k}(x(t_k),y(t_k)),$$
$$y_0 + \int_0^t f_2(s,x(s),y(s))\,ds + \sum_{0<t_k<t} I_{2,k}(x(t_k),y(t_k))\Bigg).$$

The fixed points of N are the solutions of problem (4.1). It is enough to prove that N is completely continuous. This is established in the following steps.

Step 1. N is continuous.

Let $(x_n,y_n)_n$ be a sequence in $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ such that $(x_n,y_n) \to (x,y)$. It is enough to prove that $N(x_n,y_n) \to N(x,y)$. For all $t \in J$, we have

$$N(x_n,y_n)(t) = \Bigg(x_0 + \int_0^t f_1(s,x_n(s),y_n(s))\,ds + \sum_{0<t_k<t} I_{1,k}(x_n(t_k),y_n(t_k)),$$
$$y_0 + \int_0^t f_2(s,x_n(s),y_n(s))\,ds + \sum_{0<t_k<t} I_{2,k}(x_n(t_k),y_n(t_k))\Bigg).$$

Then

$$\big\|N(x_n,y_n)(t) - N(x,y)(t)\big\| = \big\|(\alpha,\beta)\big\| = \begin{pmatrix}\|\alpha\| \\ \|\beta\|\end{pmatrix},$$

where

$$\|\alpha\| = \Bigg\| \int_0^t \big(f_1(s,x_n(s),y_n(s)) - f_1(s,x(s),y(s))\big)\,ds + \sum_{0<t_k<t} \big(I_{1,k}(x_n(t_k),y_n(t_k)) - I_{1,k}(x(t_k),y(t_k))\big)\Bigg\|$$
$$\le \int_0^t \big\|f_1(s,x_n(s),y_n(s)) - f_1(s,x(s),y(s))\big\|\,ds + \sum_{0<t_k<t} \big\|I_{1,k}(x_n(t_k),y_n(t_k)) - I_{1,k}(x(t_k),y(t_k))\big\|.$$

As I_k, $k = 1, \dots, m$, are continuous functions, and f^1 and f^2 are L^1-Carathéodory functions, by the Lebesgue dominated convergence theorem, we have

$$\|\alpha\| \le \int_0^b \big\|f_1(s, x_n(s), y_n(s)) - f_1(s, x(s), y(s))\big\|\, ds + \sum_{k=1}^m \big\|I_{1,k}(x_n(t_k), y_n(t_k)) - I_{1,k}(x(t_k), y(t_k))\big\| \longrightarrow 0 \ \text{ as } \ n \to \infty.$$

Similarly,

$$\|\beta\| \le \int_0^b \big\|f_2(s, x_n(s), y_n(s)) - f_2(s, x(s), y(s))\big\|\, ds + \sum_{k=1}^m \big\|I_{2,k}(x_n(t_k), y_n(t_k)) - I_{2,k}(x(t_k), y(t_k))\big\| \longrightarrow 0 \ \text{ as } \ n \to \infty.$$

So,

$$\big\|N(x_n, y_n) - N(x, y)\big\| \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \text{ as } \ n \to \infty.$$

Then N is continuous.

Step 2. N transforms every bounded set into a bounded set in $PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$.

It suffices to show that

$$\forall\, q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} > 0, \ \ \exists\, \ell = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} > 0 \ \text{ such that } \ \forall\, (x, y) \in \mathcal{B}_q = \big\{(x, y) \in PC \times PC : \ \ \|(x, y)\| \le q\big\}, \ \text{ we have } \ \|N(x, y)\| \le \ell.$$

Let $(x, y) \in \mathcal{B}_q$. We have

$$\|N(x, y)\| \le \Bigg(\|x_0\| + \int_0^b \|f_1(s, x(s), y(s))\|\, ds + \sum_{k=1}^m \|I_{1,k}(x(t_k), y(t_k))\|, \ \|y_0\| + \int_0^b \|f_2(s, x(s), y(s))\|\, ds + \sum_{k=1}^m \|I_{2,k}(x(t_k), y(t_k))\|\Bigg) = (\alpha, \beta),$$

where

$$\begin{aligned} \|\alpha\| &\le \|x_0\| + \int_0^b p(t)\psi\big(\|x\|_{PC} + \|y\|_{PC}\big)\, dt + \sum_{k=1}^m \|I_{1,k}(x(t_k), y(t_k))\| \\ &\le \|x_0\| + \int_0^b p(t)\psi\big(\|x\|_{PC} + \|y\|_{PC}\big)\, dt + \sum_{k=1}^m \sup_{(x,y) \in \overline{B_q}} \|I_{1,k}(x, y)\| := \ell_1. \end{aligned}$$

Similarly,

$$\|\beta\| \le \|y_0\| + \int_0^b p(t)\psi\big(\|x\|_{PC} + \|y\|_{PC}\big)\, dt + \sum_{k=1}^m \sup_{(x,y) \in \overline{B_q}} \|I_{2,k}(x, y)\| := \ell_2.$$

Step 3. N transforms every bounded set into an equicontinuous set to $PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$.

Let τ_1, $\tau_2 \in J$, $\tau_1 < \tau_2$ and let $\mathcal{B}_q$ be as in Step 2.

Let $(x, y) \in \mathcal{B}_q$. Then:

1. If $\tau_1 \neq t_k$ (or $\tau_2 \neq t_k$), $\forall\, k \in \{1, 2, \dots, m\}$, we have

$$\begin{aligned}\big\|N(x,y)(\tau_2) - N(x,y)(\tau_1)\big\| \leq \Bigg(&\int_{\tau_1}^{\tau_2} p(s)\psi(q_1+q_2)\,ds + \sum_{\tau_1 \leq t_k < \tau_2} \sup_{(x,y)\in\overline{B_q}} \|I_{1,k}(x,y)\|,\\ &\int_{\tau_1}^{\tau_2} p(s)\psi(q_1+q_2)\,ds + \sum_{\tau_1 \leq t_k < \tau_2} \sup_{(x,y)\in\overline{B_q}} \|I_{2,k}(x,y)\|\Bigg) \longrightarrow \begin{pmatrix}0\\0\end{pmatrix} \text{ as } \tau_1 \to \tau_2.\end{aligned}$$

2. If $\tau_1 = t_i^-$, we consider $\delta_1 > 0$ such that $\{t_k, k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \varnothing$, so, for $0 < h < \delta_1$, we have

$$\begin{aligned}&\big\|N(x,y)(t_i) - N(x,y)(t_i - h)\big\|\\ &\qquad \leq \Bigg(\int_{t_i-h}^{t_i} p(s)\psi(q_1+q_2)\,ds, \int_{t_i-h}^{t_i} p(s)\psi(q_1+q_2)\,ds\Bigg) \longrightarrow \begin{pmatrix}0\\0\end{pmatrix} \text{ as } h \to 0.\end{aligned}$$

3. If $\tau_2 = t_i^+$, we consider $\delta_2 > 0$ such that $\{t_k, k \neq i\} \cap [t_i - \delta_2, t_i + \delta_2] = \varnothing$, so, for $0 < h < \delta_2$, we have

$$\begin{aligned}&\big\|N(x,y)(t_i + h) - N(x,y)(t_i)\big\|\\ &\qquad \leq \Bigg(\int_{t_i}^{t_i+h} p(s)\psi(q_1+q_2)\,ds, \int_{t_i}^{t_i+h} p(s)\psi(q_1+q_2)\,ds\Bigg) \longrightarrow \begin{pmatrix}0\\0\end{pmatrix} \text{ as } h \to 0.\end{aligned}$$

So by Steps 1, 2 and 3, and by Arzelà-Ascoli's theorem, N is completely continuous.

Step 4. A Priori Estimates.

Let $(x,y) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ such that $(x,y) = \lambda N(x,y)$, and $0 < \lambda < 1$. Then for all $t \in [0, t_1]$, we have

$$x(t) = \lambda x_0 + \lambda \int_0^t f_1(s, x(s), y(s))\,ds,$$

$$y(t) = \lambda y_0 + \lambda \int_0^t f_2(s, x(s), y(s))\,ds,$$

and so,

$$\|(x,y)(t)\| \leq \Bigg(\|x_0\| + \int_0^t p(s)\psi\big(\|x(s)\| + \|y(s)\|\big)\,ds, \|y_0\| + \int_0^t p(s)\psi\big(\|x(s)\| + \|y(s)\|\big)\,ds\Bigg), \quad t \in [0, t_1].$$

Consider the map $\vartheta = (\vartheta_1, \vartheta_2)$ such that

$$\vartheta_1(t) = \|x_0\| + \int_0^t p(s)\psi\big(\|x(s)\| + \|y(s)\|\big)\,ds, \quad t \in [0, t_1],$$

$$\vartheta_2(t) = \|y_0\| + \int_0^t p(s)\psi\big(\|x(s)\| + \|y(s)\|\big)\,ds, \quad t \in [0, t_1].$$

Then we have

$$\vartheta(0) = \big(\|x_0\|, \|y_0\|\big), \quad \|(x,y)(t)\| \le \vartheta(t), \;\; t \in [0,t_1],$$

and

$$\dot{\vartheta}_i(t) = p(t)\psi\big(\|x(s)\| + \|y(t)\|\big), \;\; \forall\, i = 1,2, \;\; t \in [0,t_1].$$

As ψ is a nondecreasing map, we have

$$\dot{\vartheta}_i(t) \le p(t)\psi(\vartheta_i(t)), \;\; \forall\, i = 1,2, \;\; t \in [0,t_1],$$

which implies that for every $t \in [0,t_1]$,

$$\int\limits_{\vartheta_i(0)}^{\vartheta_i(t)} \frac{du}{\psi(u)} \le \int\limits_0^{t_1} p(s)\,ds, \;\; \forall\, i = 1,2.$$

The map $\Gamma_{i,0}(z) = \int\limits_{\vartheta_i(0)}^{z} \frac{du}{\psi(u)}$, $i = 1,2$, is continuous and increasing. Then $\Gamma_{i,0}^{-1}$ exists and is increasing, and we get

$$\vartheta_i(t) \le \Gamma_{i,0}^{-1}\Bigg(\int\limits_0^{t_1} p(s)\,ds\Bigg) := M_{i,0}, \;\; i = 1,2.$$

As for all $t \in [0,t_1]$, $\|(x,y)(t)\| \le \vartheta(t)$, and so,

$$\sup_{t\in[0,t_1]} \|(x,y)(t)\| \le \begin{pmatrix} M_{1,0} \\ M_{2,0} \end{pmatrix}.$$

Now, for $t \in (t_1,t_2]$, we have

$$\|x(t_1^+)\| \le \|I_{1,1}(x(t_1),y(t_1))\| + \|x(t_1)\| \le \sup_{(x,y)\in\overline{B}_q} \|I_{1,1}(x,y)\| + M_{1,0} := N_1,$$
$$\|y(t_1^+)\| \le \|I_{2,1}(x(t_1),y(t_1))\| + \|y(t_1)\| \le \sup_{(x,y)\in\overline{B}_q} \|I_{2,1}(x,y)\| + M_{2,0} := N_2,$$

where

$$q = \begin{pmatrix} M_{1,0} \\ M_{2,0} \end{pmatrix},$$

$$y(t) = \lambda\big(x(t_1) + I_{1,1}(x(t_1),y(t_1))\big) + \lambda\int\limits_{t_1}^{t} f_1(s,x(s),y(s))\,ds,$$

$$y(t) = \lambda\big(y(t_1) + I_{2,1}(x(t_1),y(t_1))\big) + \lambda\int\limits_{t_1}^{t} f_2(s,x(s),y(s))\,ds.$$

Then

$$\|x(t)\| \le N_1 + \int\limits_{t_1}^{t} p(s)\psi\big(\|x(s)\| + \|y(s)\|\big)\,ds, \;\; t \in [t_1,t_2],$$

$$\|y(t)\| \le N_2 + \int\limits_{t_1}^{t} p(s)\psi\big(\|x(s)\| + \|y(s)\|\big)\,ds, \;\; t \in [t_1,t_2].$$

Consider the map $W = (W_1, W_2)$ such that

$$W_1(t) = N_1 + \int_{t_1}^{t} p(s)\psi\big(\|x(s)\| + \|y(s)\|\big)\, ds, \quad t \in [t_1, t_2],$$

$$W_2(t) = N_2 + \int_{t_1}^{t} p(s)\psi\big(\|x(s)\| + \|y(s)\|\big)\, ds, \quad t \in [t_1, t_2].$$

So,

$$W(t_1^+) = (N_1, N_2), \quad \|(x, y)(t)\| \le W(t), \quad t \in [t_1, t_2],$$

and

$$\dot{W}_i(t) = p(t)\psi\big(\|x(s)\| + \|y(t)\|\big), \quad \forall\, i = 1, 2, \quad t \in [t_1, t_2].$$

Since ψ is nondecreasing, we get

$$\dot{W}_i(t) \le p(t)\psi(W_i(t)), \quad \forall\, i = 1, 2, \quad t \in [t_1, t_2],$$

what implies that for every $t \in [t_1, t_2]$, we have

$$\int_{W_i(t_1^+)}^{W_i(t)} \frac{du}{\psi(u)} \le \int_{t_1}^{t_2} p(s)\, ds, \quad i = 1, 2.$$

If we consider the map $\Gamma_{i,1}(z) = \int\limits_{W_i(t_1^+)}^{z} \frac{du}{\psi(u)}$, $i = 1, 2$, we get

$$W_i(t) \le \Gamma_{i,1}^{-1}\left(\int_{t_1}^{t_2} p(s)\, ds\right) := M_{i,1}, \quad i = 1, 2.$$

For all $t \in [t_1, t_2]$,

$$\|(x, y)(t)\| = \begin{pmatrix} \|x(t)\| \\ \|y(t)\| \end{pmatrix} \le \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix},$$

so,

$$\sup_{t \in [t_1, t_2]} \|(x, y)(t)\| \le \begin{pmatrix} M_{1,1} \\ M_{2,1} \end{pmatrix}).$$

We continue this process to the interval $(t_m, 1]$, and $(x, y)|_{(t_m,1]}$ is the solution of the problem $(x, y) = \lambda N(x, y)$ for $0 < \lambda < 1$. There exists $M_{i,m}$, $i = 1, 2$, such that

$$\sup_{t \in [t_m, b]} \|(x, y)(t)\| \le \Gamma_{i,m}^{-1}\left(\int_{t_m}^{b} p(s)\, ds\right) := M_{i,m}.$$

As we choose (x, y) arbitrarily, for all solution of problem (4.1) we have

$$\|(x, y)\| \le \begin{pmatrix} \max\limits_{k=0,1,\ldots,m} (M_{1,k}) \\ \max\limits_{k=0,1,\ldots,m} (M_{2,k}) \end{pmatrix} := \begin{pmatrix} b_1^* \\ b_2^* \end{pmatrix}.$$

Thus, the set

$$\mathcal{K} = \Big\{(x, y) \in PC \times PC : \quad (x, y) = \lambda N(x, y), \quad \lambda \in (0, 1)\Big\}.$$

Since $N : PC \times PC \to PC \times PC$ is completely continuous and the set $\mathcal{K}$ is bounded, from Theorem 2.4, N has a fixed point $(x, y) \in PC \times PC$ which is the solution of problem (4.1). $\square$

Theorem 4.3. *Suppose that the conditions of Theorem 4.2 hold. Then the set of all solutions of problem* (4.1) *is nonempty, compact,* R_δ*, and acyclic. Moreover, the solution operator* S *is u.s.c., where*

$$S:\mathbb{R}\times\mathbb{R}\longrightarrow\mathcal{P}_{cp}(PC\times PC),$$
$$(x_0,y_0)\longrightarrow S(x_0,y_0),$$
$$S(x_0,y_0)=\Big\{(x,y)\in PC\times PC:\ (x,y)\ \textit{is a solution of problem}\ (4.1)\ \textit{with}\ (x(0),y(0))=(x_0,y_0)\Big\}.$$

Proof.

- The solution set is compact.

Let $(a,b)\in\mathbb{R}\times\mathbb{R}$,

$$S(a,b)=\Big\{(x,y)\in PC\times PC:\ (x,y)\text{ is a solution of problem (4.1) with }(x(0),y(0))=(a,b)\Big\}.$$

1. $S(a,b)$ is a closed set.
Let $(x_q,y_q)_q$ be a sequence in $S(a,b)$ such that

$$\lim_{q\to\infty}(x_q,y_q)=(x,y).$$

Let

$$Z_1(t)=a+\int_0^t f_1(s,x(s),y(s))\,ds+\sum_{0<t_k<t}I_{1,k}(x(t_k),y(t_k)),\quad t\in[0,1],$$
$$Z_2(t)=b+\int_0^t f_2(s,x(s),y(s))\,ds+\sum_{0<t_k<t}I_{2,k}(x(t_k),y(t_k)),\quad t\in[0,1].$$

For $t\in[0,1]$, we have

$$\begin{aligned}&\|x_q(t)-Z_1(t)\|\\&\le\int_0^t\big\|f_1(s,x_q(s),y_q(s))-f_1(s,x(s),y(s))\big\|\,ds+\sum_{0<t_k<t}\big\|I_{1,k}(x_q(t_k),y_q(t))-I_{1,k}(x(t_k),y(t_k))\big\|\\&\quad\le\int_0^1\big\|f_1(s,x_q(s),y_q(s))-f_1(s,x(s),y(s))\big\|\,ds+\sum_{k=1}^m\big\|I_{1,k}(x_q(t_k),y_q(t))-I_{1,k}(x(t_k),y(t_k))\big\|.\end{aligned}$$

By the Lebesgue dominated convergence theorem, we have

$$\|x_q(t)-Z_1(t)\|\longrightarrow 0\ \text{ as }\ q\to\infty.$$

Similarly,

$$\|y_q(t)-Z_2(t)\|\longrightarrow 0\ \text{ as }\ q\to\infty.$$

So, $\lim_{q\to\infty}(x_q,y_q)=(x,y)=(Z_1,Z_2)\in S(a,b)$.

2. $S(a,b)$ is bounded uniformly.
Let $(x,y)\in S(a,b)$; then (x,y) is a solution of problem (4.1) and hence, $\exists\, b^*>0$ such that

$$\|(x,y)\|\le(b^*,b^*).$$

3. $S(a,b)$ is equicontinuous.

Let $r_1, r_2 \in [0,1]$, $r_1 < r_2$ and $(x,y) \in S(a,b)$. Then

$$\|(x,y)(r_1)-(x,y)(r_2)\| \leq \Big(\int_{r_1}^{r_2} \|f_1(s,x(s),y(s))\|\,ds + \sum_{r_1<t_k<r_2} \|I_{1,k}(x(t),y(t))\|,$$
$$\int_{r_1}^{r_2} \|f_2(s,x(s),y(s))\|\,ds + \sum_{r_1<t_k<r_2} \|I_{2,k}(x(t),y(t))\|\Big)$$

and

$$\int_{r_1}^{r_2} \|f_1(s,x(s),y(s))\|\,ds + \sum_{r_1<t_k<r_2} \|I_{1,k}(x(t),y(t))\|$$
$$\leq \int_{r_1}^{r_2} p(s)\psi\big(\|x(s)\|+\|y(s)\|\big)\,ds + \sum_{r_1<t_k<r_2} \sup_{(x,y)\in \overline{B}_{b^*}} \|I_{1,k}(x,y)\|$$
$$\leq \int_{r_1}^{r_2} p(s)\psi(b_1^*+b_2^*)\,ds + \sum_{r_1<t_k<r_2} \sup_{(x,y)\in \overline{B}_{b^*}} \|I_{1,k}(x,y)\| \longrightarrow 0 \ \text{ as } \ r_1 \to r_2.$$

Then $S(a,b)$ is compact.

- The solution set $S(a,b)$ is R_δ.

Let $N : PC \times PC \longrightarrow PC \times PC$ be defined by

$$N(x,y)(t) = \Big(a + \int_0^t f_1(s,x(s),y(s))\,ds + \sum_{0<t_k<t} I_{1,k}(x(t_k),y(t_k)),$$
$$b + \int_0^t f_2(s,x(s),y(s))\,ds + \sum_{0<t_k<t} I_{2,k}(x(t_k),y(t_k))\Big), \quad t \in J.$$

Then $\operatorname{Fix} N = S(a,b)$, and by Step 4 of the proof of Theorem 4.2, $\exists\, b^* > 0$ such that

$$\|(x,y)\| \leq (b^*,b^*), \quad \forall\, (x,y) \in S(a,b).$$

For $i = 1,2$, we define

$$\widetilde{f_i}(t,y(t)) = \begin{cases} f_i(t,x(t),y(t)), & \text{if } \|(x,y)(t)\| \leq (b^*,b^*), \\ f_i\Big(t, \dfrac{b^* x(t)}{\|x(t)\|}, \dfrac{b^* y(t)}{\|y(t)\|}\Big), & \text{if } \|(x,y)(t)\|_{PC\times PC} \geq (b^*,b^*) \end{cases}$$

and

$$\widetilde{I}_{i,k}(x(t),y(t)) = \begin{cases} I_{i,k}(x(t),y(t)) & \text{if } \|(x,y)(t)\| \leq (b^*,b^*), \\ I_{i,k}\Big(\dfrac{b_1^* x(t)}{\|x(t)\|}, \dfrac{b_2^* y(t)}{\|y(t)\|}\Big) & \text{if } \|(x,y)(t)\| \geq (b^*,b^*). \end{cases}$$

Since the functions $f_i,\ i=1,2$, are L^1-Carathéodory, $\widetilde{f}^i$ are also L^1-Carathéodory, and $\exists\, h \in L^1(J,\mathbb{R}_+)$ such that

$$\|\widetilde{f}_i(t,x,y)\| \leq h(t), \quad \forall i=1,2, \quad \text{a.e. } t \in J, \quad \text{and} \quad (x,y) \in \mathbb{R}\times\mathbb{R}. \tag{4.5}$$

Consider the problem

$$\begin{cases} \dot{x}(t) = \widetilde{f}_1(t, x(t), y(t)), & t \in [0,1], \\ \dot{y}(t) = \widetilde{f}_2(t, x(t), y(t)), & t \in [0,1], \\ x(t_k^+) - x(t_k^-) = \widetilde{I}_{1,k}(x(t_k), y(t_k^-)), & k = 1, 2, \dots, m, \\ y(t_k^+) - y(t_k^-) = \widetilde{I}_{2,k}(x(t_k), y(t_k^-)), & k = 1, 2, \dots, m, \\ x(0) = a, \quad y(0) = b. \end{cases}$$

We can easily prove that $\operatorname{Fix} N = \operatorname{Fix} \widetilde{N}$, where $\widetilde{N} : PC \times PC \to PC \times PC$ is defined by

$$\widetilde{N}(x,y)(t) = \Big(a + \int_0^t \widetilde{f}_i(s, x(s), y(s))\, ds + \sum_{0<t_k<t} \widetilde{I}_{1,k}(x(t_k), y(t_k)),$$
$$b + \int_0^t \widetilde{f}_2(s, x(s), y(s))\, ds + \sum_{0<t_k<t} \widetilde{I}_{2,k}(x(t_k), y(t_k))), \ \ t \in J.$$

By inequalities (4.5) and the continuity of $I_{i,k}$, $i = 1, 2$, we get

$$\|\widetilde{N}(x,y)\| \le \Big(\|a\| + \|h\|_{L^1} + \sum_{k=1}^{m} \sup_{(x,y)\in \overline{B}_b^*} \|I_{1,k}(x,y)\|,$$
$$\|b\| + \|h\|_{L^1} + \sum_{k=1}^{m} \sup_{(x,y)\in \overline{B}_b^*} \|I_{2,k}(x,y)\| \Big) := (r_1, r_2) = r.$$

Then $\widetilde{N}$ is bounded uniformly.

We can easily prove that the function $\mathcal{M}$ defined by $\mathcal{M}(x,y) = (x,y) - \widetilde{N}(x,y)$ is well defined, and since $\widetilde{N}$ is compact, by the Lasota–Yorke theorem (Theorem 2.7), it is easy to prove that the conditions of Theorem 2.8 are satisfied. Then the set $\mathcal{M}^{-1}(0) = \operatorname{Fix} \widetilde{N} = S(a,b)$ is the R_δ-set and, by Lemma 2.7, it is also acyclic.

- The solution operator is *u.s.c.*

1. S has a closed graph.

To see this, first we note that the graph of S is the set

$$G_S = \Big\{ ((a,b),(x,y)) \in (\mathbb{R} \times \mathbb{R}) \times (PC \times PC) : \ \ (x,y) \in S(a,b) \Big\}.$$

Let $((a_q,b_q),(x_q,y_q))_q$ be a sequence in G_S, and let $((a_q,b_q),(x_q,y_q))_q \to ((a,b),(x,y))$ as $q \to \infty$. Since $(x_q,y_q) \in S(a_q,b_q)$, we have

$$x_q(t) = a_q + \int_0^t f_1(s, x_q(s), y_q(s))\, ds + \sum_{0<t_k<t} I_{1,k}(x_q(s), y_q(t_k)), \ \ t \in J,$$
$$y_q(t) = b_q + \int_0^t f_2(s, x_q(s), y_q(s))\, ds + \sum_{0<t_k<t} I_{2,k}(x_q(s), y_q(t_k)), \ \ t \in J.$$

Let

$$Z(t) = (Z_1(t), Z_2(t)) = \Big(a + \int_0^t f_1(s, x(s), y(s))\, ds + \sum_{0<t_k<t} I_{1,k}(x(s), y(t_k)),$$
$$b + \int_0^t f_2(s, x(s), y(s))\, ds + \sum_{0<t_k<t} I_{2,k}(x(s), y(t_k)) \Big), \ \ t \in J.$$

Let $t \in J$, then

$$\|(x_q, y_q)(t) - Z(t)\|$$
$$\leq \Bigg(\|a_q - a\| + \int_0^b \big\| f_1(s, x_q(s), y_q(s)) - f_1(s, x(s), y(s)) \big\| \, ds + \sum_{k=1}^m \big\| I_{1,k}(x_q(t), y_q(t)) - I_{1,k}(x(t), y(t)) \big\|,$$
$$\|b_q - b\| + \int_0^b \big\| f_2(s, x_q(s), y_q(s)) - f_2(s, x(s), y(s)) \big\| \, ds + \sum_{k=1}^m \big\| I_{2,k}(x_q(t), y_q(t)) - I_{2,k}(x(t), y(t)) \big\| \Bigg)$$

and, by the Lebesgue dominated convergence theorem, we have

$$\|(x_q, y_q)(t) - Z(t)\| \longrightarrow 0 \text{ as } q \to \infty.$$

Then

$$(x, y)(t) = Z(t),$$

which implies that $(x, y) \in S(a, b)$.

2. S transforms every bounded set into a relatively compact set.
Let $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} > 0$ and $\overline{B}_r := \{(x, y) \in PC \times PC : \ \|(x, y)\| \leq r\}$.

(a) $S(\overline{B}_r)$ is bounded uniformly.

Let $(x, y) \in S(\overline{B}_r)$, then there exists $(a, b) \in \overline{B}_r$ such that

$$x(t) = a + \int_0^t f_1(s, x(s), y(s)) \, ds + \sum_{0 < t_k < t} I_{1,k}(x(t_k), y(t_k)), \quad t \in J,$$
$$y(t) = b + \int_0^t f_2(s, x(s), y(s)) \, ds + \sum_{0 < t_k < t} I_{2,k}(x(t_k), y(t_k)), \quad t \in J.$$

By the same method detailed in Step 4 of the proof of Theorem 4.2, we find that there exists $b^* > 0$ such that

$$\|(x, y)\|_{PC \times PC} \leq (b^*, b^*).$$

(b) $S(\overline{B}_r)$is an equicontinuous set.

Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, and $(x, y) \in S(\overline{B}_r)$. Then

$$\|(x, y)(\tau_2) - (x, y)(\tau_1)\|$$
$$\leq \Bigg(\int_{\tau_1}^{\tau_2} \|f_1(s, x(s), y(s))\| \, ds + \sum_{\tau_1 < t_k < \tau_2} \|I_{1,k}(x(t_k), y(t_k))\|,$$
$$\int_{\tau_1}^{\tau_2} \|f_2(s, x(s), y(s))\| \, ds + \sum_{\tau_1 < t_k < \tau_2} \|I_{2,k}(x(t_k), y(t_k))\| \Bigg)$$
$$\leq \Bigg(\int_{\tau_1}^{\tau_2} p(s)\psi(\|x(s)\| + \|y(s)\|) \, ds + \sum_{\tau_1 < t_k < \tau_2} \|I_{1,k}(x(t_k), y(t_k))\|,$$
$$\int_{\tau_1}^{\tau_2} p(s)\psi(\|x(s)\| + \|y(s)\|) \, ds + \sum_{\tau_1 < t_k < \tau_2} \|I_{2,k}(x(t_k), y(t_k))\| \Bigg)$$

$$\leq \Big(\int_{\tau_1}^{\tau_2} p(s)\psi(b_1^*+b_2^*)\,ds + \sum_{\tau_1<t_k<\tau_2}\sup_{(x,y)\in\overline{B_{b^*}}}\|I_{1,k}(x,y)\|,$$
$$\int_{\tau_1}^{\tau_2} p(s)\psi(b_1^*+b_2^*)\,ds + \sum_{\tau_1<t_k<\tau_2}\sup_{(x,y)\in\overline{B_{b^*}}}\|I_{2,k}(x,y)\|\Big) \longrightarrow 0 \;\text{ as }\; \tau_1\to\tau_2.$$

Thus the set $\overline{S(\overline{B}_r)}$ is compact.

The operator S is locally compact and has a closed graph, so, S is *u.s.c.* □

Theorem 4.4. *Assume that the conditions of Theorem 3.1 hold, where $F_1, F_2 : J\times\mathbb{R}\times\mathbb{R}\to\mathcal{P}_{cp,cv}(\mathbb{R})$ are Carathédory, u.s.c. and* mLL*-sectionnable. Then the set of all solutions of problem* (1.1) *is contractible.*

Proof. Let $f^i \in S_{F_i}$ be a locally Lipschitzian measurable selection of F_i, $i=1,2$. Let us consider the problem

$$\begin{cases} x'(t) = f_1(t,x(t),y(t)), & \text{a.e. } t\in J,\\ y'(t) = f_2(t,x(t),y(t)), & \text{a.e. } t\in J,\\ x(t_k^+)-x(t_k^-) = I_{1k}(x(t_k),y(t_k)), & k=1,\dots,m,\\ y(t_k^+)-y(t_k^-) = I_{2k}(x(t_k),y(t_k)), & k=1,\dots,m,\\ x(0)=x_0, \quad y(0)=y_0. \end{cases} \tag{4.6}$$

By Theorem 4.1, problem (4.6) has a unique solution.

Consider a homotopy function $h : S(x_0,y_0)\times[0,1]\to S(x_0,y_0)$ defined by

$$h((x,y),\alpha)(t) = \begin{cases} (x,y)(t) & \text{if } 0\le t\le\alpha,\\ (x^*,y^*)(t) & \text{if } \alpha<t\le 1,\end{cases}$$

where (x^*,y^*) is the solution of problem (4.6), and $S(x_0,y_0)$ is the set of all solutions of problem (1.1). In particular

$$h((x,y),\alpha) = \begin{cases} (x,y), & \text{if } \alpha=1,\\ (x^*,y^*), & \text{if } \alpha=0.\end{cases}$$

Thus to prove that $S(x_0,y_0)$ is contractible, it is enough to show that the homotopy h is continuous. Let $((x_n,y_n),\alpha_n)\in S(x_0,y_0)\times[0,1]$ be such that $((x_n,y_n),\alpha_n)\to((x,y),\alpha)$ as $n\to\infty$. We have

$$h((x_n,y_n),\alpha_n)(t) = \begin{cases} (x_n,y_n)(t) & \text{if } 0\le t\le\alpha_n,\\ (x^*,y^*)(t) & \text{if } \alpha_n<t\le 1.\end{cases}$$

(a) If $\lim\limits_{n\to\infty}\alpha_n = 0$, then

$$h((x,y),0)(t) = (x^*,y^*)(t) \;\text{ for all }\; t\in J.$$

Thus

$$\big\|h((x_n,y_n),\alpha_n) - h((x,y),\alpha)\big\|_\infty \le \|(x_n,y_n)-(x^*,y^*)\|_{[0,\alpha_n]} \longrightarrow 0 \;\text{ as }\; n\to\infty.$$

(b) If $\lim\limits_{n\to\infty}\alpha_n = 1$, then

$$h((x,y),1)(t) = (x,y)(t) \;\text{ for all }\; t\in J.$$

Thus

$$\big\|h((x_n,y_n),\alpha_n) - h((x,y),\alpha)\big\|_\infty \le \|(x_n,y_n)-(x,y)\|_{[0,\alpha_n]} \longrightarrow 0 \;\text{ as }\; n\to\infty.$$

(c) If $0<\lim_{n\to\infty}\alpha_n=\alpha<1$, then we distinguish the following two cases.

(1) If $t \in [0,\alpha]$, we have $(x_n, y_n) \in S(x_0, y_0)$, thus there exists $(v_{1n}, v_{2n}) \in S_{F_1} \times S_{F_2}$ such that for all $t \in [0, \alpha_n]$,

$$x_n(t) = x_0 + \int_0^t v_{1n}(s)\,ds + \sum_{0<t_k<t} I_{1,k}(x_n(t_k), y_n(t_k)),$$

$$y_n(t) = y_0 + \int_0^t v_{2n}(s)\,ds + \sum_{0<t_k<t} I_{2,k}(x_n(t_k), y_n(t_k)).$$

By Step 5 of the proof of Theorem 3.1, we have

$$\|(x_n, y_n)\|_{PC\times PC} \le b^* = \begin{pmatrix} b_1^* \\ b_2^* \end{pmatrix},$$

and, by hypothesis, we get

$$\|(v_{1n}, v_{2n})(t)\| \le p(t)\psi(b_1^* + b_2^*)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \ \text{ for all } \ n \in \mathbb{N} \Longrightarrow (v_{1n}, v_{2n})(t) \in p(t)\psi(b_1^* + b_2^*)\overline{B}(0,1).$$

The sequences $\{v_{1n}(\,\cdot\,), v_{2n}(\,\cdot\,)\}_{n\in\mathbb{N}}$ are integrably bounded. By the Dunford–Pettis theorem [52], there are subsequences, still denoted by $(v_{1n})_{n\in\mathbb{N}}$, $(v_{2n})_{n\in\mathbb{N}}$ which converge weakly to elements $v_1(\,\cdot\,) \in L^1$ and $v_2(\,\cdot\,) \in L^1$, respectively. Mazur's Lemma implies the existence of $\alpha_i^n \ge 0$, $i = n, \dots, k(n)$, such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinations $g_n^i(\,\cdot\,) = \sum_{j=1}^{k(n)} \alpha_j^n v_{ij}(\,\cdot\,)$, $i = 1, 2$, converges strongly to v_i in L^1. Since F_1 and F_2 take convex values, using Lemma 2.6, we obtain

$$\begin{aligned} v_i(t) &\in \bigcap_{n\ge 1} \overline{\{g_n^i(t)\}}, \quad \text{a.e. } \ t \in J, \\ &\subset \bigcap_{n\ge 1} \overline{co}\big\{v_{ik}(t), \ \ k \ge n\big\} \subset \bigcap_{n\ge 1} \overline{co}\Big\{\bigcup_{k\ge n} F_i(t, x_k(t), y_k(t))\Big\} \\ &= \overline{co}\Big(\limsup_{k\to\infty} F_i(t, x_k(t), y_k(t))\Big). \end{aligned} \tag{4.7}$$

Since F is $u.s.c.$ with compact values, by Lemma 2.5, we have

$$\limsup_{n\to\infty} F_i(t, x_n(t), y_n(t)) \subseteq F_i(t, x(t), y(t)) \ \text{ for a.e. } \ t \in [0,\alpha].$$

This, together with (4.7), imply that

$$v_i(t) \in \overline{co}\, F_i(t, x(t), y(t)), \quad i = 1, 2.$$

Hence, for every $t \in [0,\alpha]$,

$$x(t) = x_0 + \int_0^t v_1(s)\,ds + \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k))$$

and

$$y(t) = y_0 + \int_0^t v_2(s)\,ds + \sum_{0<t_k<t} I_{2,k}(x(t_k), y(t_k)).$$

(2) If $t \in \,]\alpha_n, 1]$, then

$$h((x_n, y_n), \alpha_n)(t) = h((x, y), \alpha)(t) = (x^*, y^*)(t).$$

Thus

$$\big\|h((x_n, y_n), \alpha_n) - h((x, y), \alpha)\big\| \to 0 \ \text{ as } \ n \to \infty.$$

Hence, h is continuous, so, the set $S(x_0, y_0)$ is contractible. □

Theorem 4.5. *Suppose the conditions of Theorem 3.1 hold, and $F_1, F_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R} \times \mathbb{R})$ are Carathéodory, u.s.c. and σ-*Ca*-selectionnable. Then the set of all solutions of problem* (1.1) *is R_δ-contractible and acyclic.*

Proof. Let $f^i \in S_{F_i}$ be a Carathéodory selection of F_i, $i = 1, 2$. Consider the homotopy multifunction $\Pi : S(x_0, y_0) \times [0,1] \to \mathcal{P}(S(x_0, y_0))$ defined by

$$\Pi((x,y),\alpha) = \begin{cases} S(x_0,y_0)(t) & \text{if } 0 \le t \le \alpha, \\ S(f,\alpha,(x,y)) & \text{if } \alpha < t \le 1, \end{cases}$$

where

- $S(x_0, y_0)$ is the set of all solutions of problem (1.1);
- $S(f, \alpha, (x, y))$ is the set of all solutions of the problem

$$\begin{cases} z_1'(t) = f_1(t, z_1(t), z_2(t)), & \text{a.e. } t \in [\alpha, 1], \\ z_2'(t) = f_2(t, z_1(t), z_2(t)), & \text{a.e. } t \in [\alpha, 1], \\ z_1(t_k^+) - z_1(t_k^-) = I_{1,k}(z_1(t_k), z_2(t_k)), & k = 1, \dots, m, \\ z_2(t_k^+) - z_2(t_k^-) = I_{2,k}(z_1(t_k), z_2(t_k)), & k = 1, \dots, m, \\ z_1(\alpha) = x(\alpha), \quad z_2(\alpha) = y(\alpha). \end{cases} \tag{4.8}$$

By the definition of Π, for all $(x,y) \in S(x_0,y_0)$, $(x,y) \in \Pi((x,y),1)$ and $\Pi((x,y),0) = S(f,0,(x,y))$, which is an R_δ-set by Theorem 4.3.

It remains to show that Π is *u.s.c.* and $\Pi((x,y),\alpha)$ is an R_δ-set for all $((x,y),\alpha) \in S(x_0,y_0) \times [0,1]$. The proof is given by the following steps.

Step 1. Π is locally compact.

(a) The multifunction $\widetilde{S} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(PC(J,\mathbb{R}) \times PC(J,\mathbb{R}))$ defined by

$$\widetilde{S}\big(\widetilde{t}, (\widetilde{x}, \widetilde{y})\big) = S\big(f, \widetilde{t}, (\widetilde{x}, \widetilde{y})\big)$$

is *u.s.c.* where $S(f, \widetilde{t}, (\widetilde{x}, \widetilde{y}))$ is the set of all solutions of the problem

$$\begin{cases} z_1'(t) = f_1(t, z_1(t), z_2(t)), & \text{a.e. } t \in [\widetilde{t}, 1], \\ z_2'(t) = f_2(t, z_1(t), z_2(t)), & \text{a.e. } t \in [\widetilde{t}, 1], \\ z_1(t_k^+) - z_1(t_k^-) = I_{1,k}(z_1(t_k), z_2(t_k)), & k = 1, \dots, m, \\ z_2(t_k^+) - z_2(t_k^-) = I_{2,k}(z_1(t_k), z_2(t_k)), & k = 1, \dots, m, \\ z_1(\widetilde{t}) = \widetilde{x}, \quad z_2(\widetilde{t}) = \widetilde{y}. \end{cases} \tag{4.9}$$

Assume the opposite, i.e., $\widetilde{S}$ is not *u.s.c.* Then for some point $(\widetilde{t}, (\widetilde{x}, \widetilde{y}))$, there is an open neighborhood U of $\widetilde{S}(\widetilde{t}, (\widetilde{x}, \widetilde{y}))$ in $PC([0,1],\mathbb{R}) \times PC([0,1],\mathbb{R})$ such that for any open neighborhood V of $(\widetilde{t}, (\widetilde{x}, \widetilde{y}))$ in $[0,1] \times \mathbb{R} \times \mathbb{R}$, there exists $(\widetilde{t_1}, (\widetilde{x_1}, \widetilde{y_1})) \in V$ such that $\widetilde{S}(\widetilde{t_1}, (\widetilde{x_1}, \widetilde{y_1})) \not\subset U$.

Let

$$V_n = \left\{ (t,(x,y)) \in [0,1] \times \mathbb{R} \times \mathbb{R} : \ d\big((t,(x,y)), (\widetilde{t}, (\widetilde{x}, \widetilde{y}))\big) < \begin{pmatrix} \frac{1}{n} \\ \frac{1}{n} \\ \frac{1}{n} \end{pmatrix} \right\}, \quad n \in \mathbb{N},$$

where d is the generalized metric of the space $[0,1] \times (\mathbb{R} \times \mathbb{R})$. Then for each $n \in \mathbb{N}$ we take $(t_n, (x_n, y_n)) \in V_n$ and $(x_n, y_n) \in \widetilde{S}(t_n, (x_n, y_n))$ such that $(x_n, y_n) \notin U$. We define the functions

$$G_{\widetilde{t},(\widetilde{x},\widetilde{y})}, F_{\widetilde{t},(\widetilde{x},\widetilde{y})} : PC([0,1],\mathbb{R}) \times PC([0,1],\mathbb{R}) \longrightarrow PC([0,1],\mathbb{R}) \times PC([0,1],\mathbb{R})$$

by

$$F_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x,y)(t)=\Big(\widetilde{x}+\int_{\widetilde{t}}^{t} f_1(s,(x(s),y(s)))\,ds+\sum_{\widetilde{t}<t_k<t} I_{1k}(x(t_k),y(t_k)),$$

$$\widetilde{y}+\int_{\widetilde{t}}^{t} f_2(s,(x(s),y(s)))\,ds+\sum_{\widetilde{t}<t_k<t} I_{2k}(x(t_k),y(t_k))\Big),\quad t\in[\widetilde{t},1],$$

$$G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x,y)=(x,y)-F_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x,y)\ \text{ for }\ t\in[0,1],\quad (x,y)\in PC(J,\mathbb{R})\times PC(J,\mathbb{R}).$$

Then for $(x,y)\in PC(J,\mathbb{R})\times PC(J,\mathbb{R}), t,\widetilde{t}\in[0,1]$, and $(\widetilde{x},\widetilde{y})\in\mathbb{R}\times\mathbb{R}$, we have

$$F_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x,y)(t)=(\widetilde{x},\widetilde{y})-F_{0,(\widetilde{x},\widetilde{y})}(x,y)(\widetilde{t})+F_{0,(\widetilde{x},\widetilde{y})}(x,y)(t).$$

Consequently,

$$G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x,y)(t)=-(\widetilde{x},\widetilde{y})+F_{0,(\widetilde{x},\widetilde{y})}(x,y)(t)+G_{0,(\widetilde{x},\widetilde{y})}(x,y)(t).$$

Then, we obtain

$$\widetilde{S}(\widetilde{t},(\widetilde{x},\widetilde{y}))=G^{-1}_{\widetilde{t},(\widetilde{x},\widetilde{y})}(0)\ \text{ for all }\ (\widetilde{t},(\widetilde{x},\widetilde{y}))\in[0,1]\times\mathbb{R}\times\mathbb{R}.$$

Since $F_{\widetilde{t},(\widetilde{x},\widetilde{y})}$ is compact (see the proof of Theorem 4.3), $G_{\widetilde{t},(\widetilde{x},\widetilde{y})}$ is proper. And as $(x_n,y_n)\in\widetilde{S}(t_n,(x_n,y_n))$, we have

$$x_n(t)=x_n(t_n)+\int_{t_n}^{t} f_1(s,x_n(s),y_n(s))\,ds+\sum_{t_n<t_k<t} I_{1,k}(x_n(t_k),y_n(t_k)),\quad t\in[t_n,1],$$

$$y_n(t)=y_n(t_n)+\int_{t_n}^{t} f_2(s,x_n(s),y_n(s))\,ds+\sum_{t_n<t_k<t} I_{2,k}(x_n(t_k),y_n(t_k)),\quad t\in[t_n,1],$$

which in turn gives

$$0=G_{t_n,(x_n,y_n)}(x_n,y_n)(t)=-(x_n,y_n)(t_n)+F_{0,(x_n,y_n)}(x_n,y_n)(t_n)+G_{0,(x_n,y_n)}(x_n,y_n)(t)$$

and

$$G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x_n,y_n)(t)=-(\widetilde{x},\widetilde{y})+F_{0,(\widetilde{x},\widetilde{y})}(x_n,y_n)(\widetilde{t})+G_{0,(\widetilde{x},\widetilde{y})}(x_n,y_n)(t).$$

Then

$$\begin{aligned}&\big\|G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x_n,y_n)(t)-G_{t_n,(x_n,y_n)}(x_n,y_n)(t)\big\|=\big\|G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x_n,y_n)(t)\big\|\\ &\quad=\big\|-(\widetilde{x},\widetilde{y})+(x_n,y_n)(t_n)+F_{0,(\widetilde{x},\widetilde{y})}(x_n,y_n)(\widetilde{t})-F_{0,(x_n,y_n)}(x_n,y_n)(t_n)\big\|=\left\|\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right\|=\begin{pmatrix}\|\alpha\|\\ \|\beta\|\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}\alpha=-\widetilde{x}+x_n(t_n)+\Big(\widetilde{x}+\int_0^{\widetilde{t}} f_1(s,x_n(s),y_n(s))\,ds+\sum_{0<t_k<\widetilde{t}} I_{1,k}(x_n(t_k),y_n(t_k))\Big)\\ -\Big(x_n(t_n)+\int_0^{t_n} f_1(s,x_n(s),y_n(s))\,ds+\sum_{0<t_k<t_n} I_{1,k}(x_n(t_k),y_n(t_k))\Big).\end{aligned}$$

Therefore,

$$\|\alpha\| \leq \int_{t_n}^{\widetilde{t}} \|f_1(s, x_n(s), y_n(s))\|\, ds + \sum_{t_n < t_k < \widetilde{t}} \|I_{1,k}(x_n(t_k), y_n(t_k))\|$$
$$\leq \int_{t_n}^{\widetilde{t}} p(s)\psi(b_1^* + b_2^*)\, ds + \sum_{t_n < t_k < \widetilde{t}} \|I_{1,k}(x_n(t_k), y_n(t_k))\|.$$

Similarly,

$$\beta = -\widetilde{y} + y_n(t_n) + \left(\widetilde{y} + \int_0^{\widetilde{t}} f_2(s, x_n(s), y_n(s))\, ds + \sum_{0 < t_k < \widetilde{t}} I_{2,k}(x_n(t_k), y_n(t_k))\right)$$
$$- \left(y_n(t_n) + \int_0^{t_n} f_2(s, x_n(s), y_n(s))\, ds + \sum_{0 < t_k < t_n} I_{2,k}(x_n(t_k), y_n(t_k))\right),$$

$$\|\beta\| \leq \int_{t_n}^{\widetilde{t}} p(s)\psi(b_1^* + b_2^*)\, ds + \sum_{t_n < t_k < \widetilde{t}} \|I_{2,k}(x_n(t_k), y_n(t_k))\|.$$

Now,

$$\lim_{n\to\infty} (x_n, y_n) = (\widetilde{x}, \widetilde{y}) \text{ and } \lim_{n\to\infty} t_n = \widetilde{t}$$

imply that

$$\lim_{n\to\infty} G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x_n, y_n) = 0.$$

Then the set $A = \overline{\{G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x_n, y_n)\}}$ is compact, thus $G^{-1}_{\widetilde{t},(\widetilde{x},\widetilde{y})}(A)$ is also compact. It is clear that $\{(x_n, y_n)\} \subset A$. As $\lim_{n\to\infty} (x_n, y_n) = (\widetilde{x}, \widetilde{y})$, it follows $(\widetilde{x}, \widetilde{y}) \in \widetilde{S}(\widetilde{t}, (\widetilde{x}, \widetilde{y})) \subset U$, so we have a contradiction to the hypothesis $(x_n, y_n) \notin U$ for every n.

(b) Π is locally compact.

For $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} > 0$, consider the set

$$B \times I = \Big\{((x, y), \alpha) \in S(x_0, y_0) \times [0, 1] : \ \|(x, y)\| \leq r\Big\},$$

and let $\{u_n\} \in \Pi(B \times I)$. Then there exists $((x_n, y_n), \alpha_n) \in B \times I$ such that

$$u_n(t) = \begin{cases} (x_n, y_n) & \text{if } 0 \leq t \leq \alpha_n, \\ v_n(t) & \text{if } \alpha_n < t \leq 1, \ \ v_n \in S(f, \alpha_n, (x_n, y_n)). \end{cases}$$

Since $S(x_0, y_0)$ is compact, there exists a subsequence of $(x_n, \alpha_n)_n$ which converges to $((x, y), \alpha)$. $\widetilde{S}$ is *u.s.c.* implies that for all $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that $v_n(t) \in \widetilde{S}(t, (x, y)) = S(f, \alpha, (x, y))$ for all $n \geq n_0(\varepsilon)$, and by the compactness of $S(f, \alpha, (x, y))$, it is concluded that there is a subsequence of $\{v_n\}$ which converges towards $v \in S(f, \alpha, (x, y))$. Hence Π is locally compact.

Step 2. Π has a closed graph.

Let $((x_n, y_n), \alpha_n) \to ((x_*, y_*), \alpha), h_n \in \Pi(x_n, y_n, \alpha_n)$ and $h_n \to h_*$ as $n \to +\infty$. We are going to prove that $h_* \in \Pi((x_*, y_*), \alpha)$. Now, $h_n \in \Pi((x_n, y_n), \alpha_n)$ implies that there exists $z_n \in S(f^i, \alpha_n, (x_n, y_n))$ such that for all $t \in J$,

$$h_n(t) = \begin{cases} (x_n, y_n) & \text{if } 0 \leq t \leq \alpha_n, \\ z_n(t) & \text{if } \alpha_n < t \leq 1. \end{cases}$$

Therefore, it is enough to prove that there exists $z_* \in S(f^i, \alpha, (x_*, y_*))$ such that for all $t \in J$,

$$h_*(t) = \begin{cases} (x_*, y_*) & \text{if } 0 \le t \le \alpha, \\ z_*(t) & \text{if } \alpha < t \le 1. \end{cases}$$

It is clear that $(\alpha_n, (x_n, y_n)) \to (\alpha, (x_*, y_*))$ as $n \to \infty$, and it can easily be proved that there exists a subsequence of $\{z_n\}$ which converges to z_*. So, we can handle the cases $\alpha = 0$ and $\alpha = 1$ as we did in the proof of Theorem 4.4, and we obtain finally that $z_* \in S(f, \alpha, (x_*, y_*))$.

Step 3. $\Pi((x,y),\alpha)$ is an R_δ-set for all $((x,y),\alpha) \in S(x_0, y_0) \times [0,1]$.

Since F is σ-Ca-selectionnable, there is a decreasing sequence of multifunctions $F_k : [0,b]\times\mathbb{R}\times\mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R}\times\mathbb{R})$, $k \in \mathbb{N}$, which admit Carathéodory selections and

$$F_{k+1}(t,u) \subset F_k(t,u) \ \text{ for all } \ t \in [0,1], u \in \mathbb{R}\times\mathbb{R},$$

and

$$F(t,u) = \bigcap_{k=0}^{\infty} F_k(t,u), \quad u \in \mathbb{R}\times\mathbb{R}.$$

Then

$$\Pi((x,y),\alpha) = \bigcap_{k=0}^{\infty} S(F_k, (x,y)).$$

By Theorem 4.3, the sets $\Pi((x,y),\alpha)$ and $S(F_k,(x,y))$ are compact. Furthermore, by Theorem 4.4, the set $S(F_k,(x,y))$ is contractible. Thus, $\Pi((x,y),\alpha)$ is an R_δ-set. □

Lemma 4.1. *Suppose that the multifunction $F : J \times \mathbb{R}\times\mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is Carathéodory and u.s.c. of the type of Scorza–Dragoni. Then the set of all solutions of problem* (1.1) *is R_δ-contractible.*

Proof. By Theorem 2.6, we have that F is σ-Ca-selectionnable. Thus we have the same conditions of the last theorem. □

5 Summary/Conclusion

In this paper, we investigate the existence of a solution for the system of differential inclusions under various assumptions on the multi-valued right-hand side nonlinearity. Also, we have studied some properties of solution sets of those results, such as topological properties (compactness), acyclicity properties, geometric topological properties, R_δ, etc. Theorem 4.3 is a major result entailing some of the topological properties, while Section 4 is devoted to geometric topological properties.

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SOLVABILITY AND NUMERICAL APPROXIMATION OF THE SHELL EQUATION DERIVED BY THE Γ-CONVERGENCE

Abstract. A mixed boundary value problem for the Lamé equation in a thin layer $\Omega^h = \mathcal{C} \times [-h, h]$ around a surface $\mathcal{C}$ with the Lipshitz boundary is investigated. The main goal is to find out what happens when the thickness of the layer tends to zero, $h \to 0$. To this end, we reformulate BVP into an equivalent variational problem and prove that the energy functional has the Γ-limit of the energy functional on the mid-surface $\mathcal{C}$. The corresponding BVP on $\mathcal{C}$, considered as the Γ-limit of the initial BVP, is written in terms of Günter's tangential derivatives on $\mathcal{C}$ and represents a new form of the shell equation. It is shown that the Neumann boundary condition from the initial BVP on the upper and lower surfaces transforms into the right-hand side of the basic equation of the limit BVP. The finite element method is established for the obtained BVP.

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Key words and phrases. Hypersurface, Günter's derivatives, Lamé equation, Γ-convergence, shell equation.

რეზიუმე. გამოკვლეულია ლიფშიცის საზღვრიანი $\mathcal{C}$ ზედაპირის მომცველ $\Omega^h = \mathcal{C} \times [-h, h]$ თხელ ფენაში განსაზღვრული ლამეს განტოლებისთვის დასმული შერეული სასაზღვრო ამოცანა. გამოკვლევის მთავარი მიზანია დადგინდეს რა ხდება, როდესაც ფენის h სისქე ნულისკენ მიისწრაფვის, $h \to 0$. ამისთვის, ჩვენ, დასმული სასაზღვრო ამოცანა შევცვალეთ ექვივალენტური ვარიაციული ამოცანით და ვაჩვენეთ, რომ ენერგიის ფუნქციონალი, Γ-კრებადობის აზრით, მიისწრაფვის $\mathcal{C}$ შუა ზედაპირზე განსაზღვრული ფუნქციონალისკენ. ამ ფუნქციონალის შესაბამისი სასაზღვრო ამოცანა განიხილება როგორც თავდაპირველი სასაზღვრო ამოცანის Γ-ზღვარი. ის ჩაწერილია $\mathcal{C}$ ზედაპირზე განსაზღვრულ გიუნტერის მხებ წარმოებულებში და წარმოადგენს გარსის განტოლების ახალ ფორმას. ნაჩვენებია, რომ თავდაპირველი ამოცანის ნეიმანის სასაზღვრო პირობები, ზედა და ქვედა საზღვარზე, გარდაიქმნება ზღვრული განტოლების მარჯვენა მხარედ. მიღებული სასაზღვრო ამოცანისთვის დაფუძნებულია სასრულ ელემენტთა მეთოდი.

1 Introduction

In the present paper, we study a mixed boundary value problem for the Lamé equation in a thin layer $\Omega^h := \mathcal{C} \times [-h, h]$ of thickness $2h$ around a smooth mid-hypersurface $\mathcal{C} \subset \mathbb{R}^3$ written in terms of Günter's derivatives and the energy functional associated to it. We show that when thickness of the layer tends to zero, $h \to 0$, the corresponding energy functional, scaled properly, converges in the Γ-limit sense to some functional defined on mid-surface $\mathcal{C}$ of the layer, which corresponds to the two-dimensional boundary value problem for associated Euler–Lagrange equation in terms of Günter's derivatives. The obtained equations together with boundary conditions can be considered as a boundary value problem defined on a shell model. We employ Galerkin's method to establish numerical approximation for solutions of the obtained BVP.

The equations of three-dimensional linearized elasticity have been studied mostly in Cartesian coordinates. The linear shell theory justified in the present paper is based on the natural curvilinear coordinates, defined on the mid-surface $\mathcal{C}$ extended by the normal vector field of this surface, which "follow the geometry" of the shell in a most natural way. Accordingly, the purpose of the present preliminary section is to provide a thorough derivation and a mathematical treatment of the equations of linearized three-dimensional elasticity in terms of special curvilinear coordinates.

Let $\mathcal{C} \subset \mathbb{R}^3$ be an open surface with the boundary $\Gamma = \partial\mathcal{C}$ in the Euclidean space $\mathbb{R}^3$, represented by a single coordinate function $\theta : \omega \to \mathcal{C}$ (the case of multiple coordinate function is similar and we skip this case for the simplicity). Let $\boldsymbol{\nu}(\mathcal{X}) = (\nu_1(\mathcal{X}), \nu_2(\mathcal{X}), \nu_3(\mathcal{X}))^\top$, $\mathcal{X} \in \mathcal{C}$, be the normal vector field on $\mathcal{C}$ and $\nu(x) = (\mathcal{N}_1(x), \mathcal{N}_2(x), \mathcal{N}_3(x))^\top$ be its extension in the neighbourhood $U_{\mathcal{C}}$ of the surface $\mathcal{C}$. It is known that such extension is unique under the assumption that the extension, as the field on the surface itself, is a gradient vector field $\partial_j\mathcal{N}_k = \partial_k\mathcal{N}_j$ for all $j, k = 1, 2, 3$ and is called the proper extension (see [6] for details).

The 3-tuple of tangential vector fields to the surface $\mathbf{g}_1 := \partial_1\Theta, \mathbf{g}_2 := \partial_2\Theta$ (the covariant basis) together with the proper extension $\mathbf{g}_3 := \mathcal{N}$ of normal vector field $\boldsymbol{\nu}$ from the surface $\mathcal{C}$ into the neighborhood Ω^h depend only on the variable $x' \in \mathcal{C}$ and constitute a basis in Ω^h. That means that an arbitrary vector field $\mathbf{U} = \sum\limits_{j=1}^{3} U_j\mathbf{e}^j$ can also be represented with this basis in "curvilinear coordinates". Along with the covariant basis, the use is made of the contravariant basis $\mathbf{g}^1$, $\mathbf{g}^2$ which is the bi-orthogonal system to the covariant basis $\langle \mathbf{g}_j, \mathbf{g}^k \rangle = \delta_{jk}$, where δ_{jk} denotes Kroneker's symbol, $j, k = 1, 2$ (see, e.g., [3, 4]). In the classical geometry, the covariant $\{\langle \mathbf{g}_i, \mathbf{g}_k \rangle\}_{j,k=1,2}$ and contravariant $\{\langle \mathbf{g}^i, \mathbf{g}^k \rangle\}_{j,k=1,2}$ metric tensors together with the Christofell symbols $\Gamma^i_{jk} := \langle \mathbf{g}^i, \partial_j\mathbf{g}_k \rangle$ are the main tools of the calculus. For example, the covariant derivatives on the surface $\mathcal{C}$ are defined by $v_{i\|j} := \partial_j v_i - \sum\limits_{k=1}^{2} \Gamma^k_{ij} v_k$.

Our calculus on the surface $\mathcal{C}$ is based on a different curvilinear system of coordinates than the covariant and contravariant vector fields used usually by mathematicians and mechanists to derive the shell equations (see, e.g., P. Ciarlet [3,4]). Moreover, the system of curvilinear coordinates introduced below is linearly dependent but, surprisingly, many partial differential equations are written in this system in a simple form, including Laplace–Beltramy and shell equations on a hypersurface (see [5].

From now on, if not stated otherwise, we stick to the following notation: the terms with repeated indices are implicitly summed from 1 to 3 if indices are Greek $(\alpha, \beta, \gamma, \dots)$ and are summed from 1 to 4 if indices are Latin $(j, k, l, \dots)$, as shown in the following examples:

$$a_\alpha b_\alpha := \sum_{\alpha=1}^{3} a_\alpha b_\alpha, \quad b^2_\alpha := \sum_{\alpha=1}^{3} b^2_\alpha, \quad c_j d_j := \sum_{j=1}^{4} c_j d_j, \quad c^2_j := \sum_{j=1}^{4} c^2_j.$$

We consider a deformation of an isotropic layer domain $\Omega^h := \mathcal{C} \times (-h, h)$ of thickness $2h$ around the mid-surface $\mathcal{C}$ which has the nonempty Lipschitz boundary $\partial\mathcal{C}$. The deformation is governed by the Lamé equation with the classical mixed boundary conditions, Dirichlet conditions on the lateral surface $\Gamma^h_L := \partial\mathcal{C}\times(-h,h)$ and Neumann conditions on the upper and lower surfaces $\Gamma^\pm := \mathcal{C}\times\{\pm h\}$:

$$\begin{aligned} &\mathcal{L}_{\Omega^h}\mathbf{U}(x)=\mathbf{F}(x),\ \ x\in\Omega^h:=\mathcal{C}\times(-h,h),\\ &\mathbf{U}^+(t)=\mathbf{G}(t),\ \ t\in\Gamma^h_L:=\partial\mathcal{C}\times(-h,h),\\ &(\mathfrak{T}(\mathcal{x},\nabla)\mathbf{U})^+(\mathcal{x})=\mathbf{H}(\mathcal{x},\pm h),\ \ (\mathcal{x},t)\in\Gamma^\pm=\mathcal{C}\times\{\pm h\}. \end{aligned}\tag{1.1}$$

Here $\mathbf{U}(x)=(U_1(x),U_2(x),U_3(x))^\top$ is the displacement vector, $\mathcal{L}_{\Omega^h}$ is the Lamé differential operator and $\mathfrak{T}(\mathcal{x},\nabla)$ is the traction operator

$$\begin{aligned} &\mathcal{L}_{\Omega^h}\mathbf{U}=-\mu\,\boldsymbol{\Delta}\mathbf{U}-(\lambda+\mu)\,\nabla\operatorname{div}\mathbf{U},\\ &[\mathfrak{T}(\mathcal{x},\nabla)\mathbf{U}]_\beta=\lambda\,\nu_\beta\partial_\gamma\mathbf{U}_\gamma+\mu\,\nu_\gamma\partial_\beta\mathbf{U}_\gamma+\mu\,\partial_{\boldsymbol{\nu}}\mathbf{U}_\beta,\ \ \beta=1,2,3. \end{aligned}\tag{1.2}$$

The BVP (1.1) we consider in the following weak classical setting:

$$\mathbf{U}\in\mathbb{H}^1(\Omega^h),\ \ \mathbf{F}\in\widetilde{\mathbb{H}}^{-1}(\Omega^h),\ \ \mathbf{G}\in\mathbb{H}^{1/2}(\Gamma^h_L),\ \ \mathbf{H}(\,\cdot\,,\pm h)\in\mathbb{H}^{-1/2}(\mathcal{C}).\tag{1.3}$$

For definitions of Bessel potential spaces $\mathbb{H}^s$, $\widetilde{\mathbb{H}}^s$ see, e.g., [8].

Let us consider the following subspace of $\mathbb{H}^1(\Omega^h)$:

$$\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L):=\Big\{\mathbf{V}\in\mathbb{H}^1(\Omega^h):\ \ \mathbf{V}^+(t)=0\ \text{ for all }\ t\in\Gamma^h_L\Big\}.\tag{1.4}$$

Theorem 1.1. *The BVP* (1.1) *in the weak classical setting* (1.3) *has a unique solution.*

Proof. The Lamé operator $\mathcal{L}_{\Omega^h}$ is strictly positive on the subspace $\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$,

$$\langle\mathcal{L}_{\Omega^h}\mathbf{V},\mathbf{V}\rangle\geqslant M\|\mathbf{V}\|^2\quad\forall\,\mathbf{V}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L),$$

and the proof follows easily from the Lax-Milgram Lemma (a similar proof see, e.g., in [7]). □

To find what happens with the BVP (1.1), (1.3) as $h\to0$, we first reformulate this BVP into the equivalent variational problem: Find the vector $\mathbf{U}$ which minimizes the energy functional $\mathcal{E}_{\Omega^h}(\mathbf{U})$ (see (3.4)) under the same constraints (1.3). It is proved that if the weak limits

$$\lim_{h\to0}\mathbf{F}(\mathcal{x},h\tau)=\mathbf{F}(\mathcal{x}),\quad \lim_{h\to0}\frac{1}{2h}\big[\mathbf{H}(\mathcal{x},+h)-\mathbf{H}(\mathcal{x},-h)\big]=\mathbf{H}^{(1)}(\mathcal{x}),\ \ \mathbf{F},\mathbf{H}^{(1)}\in\mathbb{L}_2(\mathcal{C}),$$

exist in $\mathbb{L}_2(\Omega^h)$ and $\mathbb{L}_2(\mathcal{C})$, respectively, then there exists the Γ-limit of the energy functional $\lim\limits_{h\to0}\mathcal{E}_{\Omega^h}(\mathbf{U})=\mathcal{E}^3_{\mathcal{C}}(\overline{U})$ (cf. (4.2)), and the equivalent BVP on the surface $\mathcal{C}$, using Einstein's convention, is written as follows:

$$\begin{cases}\mu\Big[\Delta_{\mathcal{C}}\overline{U}_\alpha+\mathcal{D}_\beta\mathcal{D}_\alpha\overline{U}_\beta-2\mathcal{H}_{\mathcal{C}}\nu_\beta\mathcal{D}_\alpha\overline{U}_\beta-\mathcal{D}_\gamma(\nu_\alpha\nu_\beta\mathcal{D}_\gamma\overline{U}_\beta)\Big]\\ \qquad\qquad+\dfrac{4\lambda\mu}{\lambda+2\mu}\Big[\mathcal{D}_\alpha\mathcal{D}_\beta\overline{U}_\beta-2\mathcal{H}_{\mathcal{C}}\nu_\alpha\mathcal{D}_\beta\overline{U}_\beta\Big]=\dfrac12\,F_\alpha+H^{(1)}_\alpha\ \text{ on }\ \mathcal{C}, \qquad \alpha=1,2,3.\\ \overline{U}_\alpha(t)=0\ \text{ on }\ \Gamma=\partial\mathcal{C},\end{cases}\tag{1.5}$$

In (1.5), $\boldsymbol{\nu}:=(\nu_1,\nu_2,\nu_3)^\top$ is the unit normal vector filed on $\mathcal{C}$, $\mathcal{H}_{\mathcal{C}}$ is the mean curvature of $\mathcal{C}$, $\mathcal{D}_\alpha:=\partial_\alpha-\nu_\alpha\partial_{\boldsymbol{\nu}}$, $\alpha=1,2,3$, are Günter's tangential derivatives on $\mathcal{C}$ (see Section 2) and $\overline{\mathbf{U}}:=(U_1(\mathcal{x},0),U_2(\mathcal{x},0),U_3(\mathcal{x},0))^\top$, $\mathcal{x}\in\mathcal{C}$, is the trace of the displacement vector field

$$\mathbf{U}(\mathcal{x},t):=\big(U_1(\mathcal{x},t),U_2(\mathcal{x},t),U_3(\mathcal{x},t)\big)^\top,\ \ (\mathcal{x},t)\in\Omega^h:=\mathcal{C}\times(-h,h),$$

on the mid-surface $\mathcal{C}$ (see Theorem 4.3).

The BVP (1.5) represents a new 2D shell equation in terms of Günter's tangential derivatives on the mid-surface $\mathcal{C}$.

2 Auxiliaries

We commence with the definition of a new system of coordinates: the system of 4-vectors

$$\mathbf{d}^j := \mathbf{e}^j - \mathcal{N}_j\mathcal{N}, \quad j = 1,2,3, \quad \text{and} \quad \mathbf{d}^4 := \mathcal{N}, \tag{2.1}$$

where $\mathbf{e}^1 = (1,0,0)^\top$, $\mathbf{e}^2 = (0,1,0)^\top$, $\mathbf{e}^3 = (0,0,1)^\top$ is the Cartesian basis in $\mathbb{R}^3$; the first 3 vectors $\mathbf{d}^1$, $\mathbf{d}^2$, $\mathbf{d}^3$ are projections of the Cartesian vectors and are tangential to the surface $\mathcal{C}$, while the last one $\mathbf{d}^4 = \mathcal{N}$ is orthogonal to it and, thus, to $\mathbf{d}^1$, $\mathbf{d}^2$, $\mathbf{d}^3$. The system is linearly dependent, but full, and any vector field $\mathbf{U} = U_\alpha \mathbf{e}^\alpha$ in Ω_h can be written in the following form:

$$\begin{aligned} \mathbf{U} = U_\alpha \mathbf{e}^\alpha = U_j^0 \mathbf{d}^j = \mathbf{U}^0 = \mathbf{U}_0 + U_4^0 \mathcal{N}, \\ \mathbf{U}_0 := \mathbf{U} - \langle \mathcal{N}, \mathbf{U} \rangle \mathcal{N}, \quad U_4^0 := \langle \mathcal{N}, \mathbf{U} \rangle = \mathcal{N}_\alpha U_\alpha, \end{aligned} \tag{2.2}$$

and the vector $\mathbf{U}_0 := (U_1^0, U_2^0, U_3^0)^\top$ is chosen to be tangential to the surface $\langle \mathcal{N}, \mathbf{U}_0 \rangle = 0$.

Since the proper extension depends only on the surface variable $\mathcal{N}(x,t) = \mathcal{N}(x)$ (see [6]), the same is true for the entire basis $\mathbf{d}^j(x,t) = \mathbf{d}^j(x)$, $j = 1,2,3,4$.

Note that

$$\mathcal{N}_4 := \langle \mathcal{N}, \mathcal{N} \rangle = 1.$$

Although the system $\{\mathbf{d}^j\}_{j=1}^4$ is linearly dependent, the following holds.

In [2, Lemma 1], it is proved that representation (2.2) is unique, that is,

$$\text{if} \ \ \mathbf{U}^0 = U_j^0 \mathbf{d}^j = 0, \ \ \text{then} \ \ U_1^0 = U_2^0 = U_3^0 = U_4^0 = 0.$$

Moreover, the scalar product and, consequently, the distance between two vectors in the Cartesian and new coordinate systems coincide:

$$\langle \mathbf{U}^0, \mathbf{V}^0 \rangle = U_j^0 V_j^0 = U_\alpha V_\alpha = \langle \mathbf{U}, \mathbf{V} \rangle, \ \ \|\mathbf{U}^0 - \mathbf{V}^0\| = \|\mathbf{U} - \mathbf{V}\|$$

for arbitrary vectors $\mathbf{U} = (U_1, U_2, U_3)^\top$, $\mathbf{V} = (V_1, V_2, V_3)^\top \in \mathbb{R}^3$.

Günter's derivatives

$$\mathcal{D}_\alpha \varphi := \partial_\alpha \varphi - \nu_\alpha \partial_\nu \varphi, \quad \alpha = 1,2,3, \tag{2.3}$$

represent tangential differential operators on the surface $\mathcal{C}$ (orthogonal projections of the coordinate derivatives ∂_1, ∂_2, ∂_3) and have the extensions

$$\mathcal{D}_\alpha \varphi := \partial_\alpha \varphi - \mathcal{N}_\alpha \partial_\mathcal{N} \varphi$$

in the neighbourhood of the surface $\mathcal{C}$. The system $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ is, obviously, linearly dependent, but full: any tangential linear differential operator on the surface $\mathbf{A}(D)$ is written in the following form:

$$\mathbf{A}(D) = a_\alpha(x)\partial_\alpha = a_\alpha(x)\mathcal{D}_\alpha, \ \ \text{provided} \ \ a_\alpha(x)\nu_\alpha(x) \equiv 0, \ \ x \in \mathcal{C}.$$

In particular,

$$\partial_\mathbf{U} = U_\alpha \partial_\alpha = U_j^0 \mathcal{D}_j.$$

The adjoint operator to $\mathcal{D}_j$, $j = 1,2,3$, is

$$\mathcal{D}_j^* \varphi = -\mathcal{D}_j \varphi + 2\nu_j \mathcal{H}_\mathcal{C} \varphi, \ \ \varphi \in \mathbb{C}^1(\mathcal{C}),$$

where

$$\mathcal{H}_\mathcal{C}(x) := \frac{1}{2}\mathcal{D}_\alpha \nu_\alpha(x) = \frac{1}{2}\mathcal{D}_\alpha \mathcal{N}_\alpha(x), \ \ x \in \mathcal{C}, \tag{2.4}$$

is the mean curvature of the surface $\mathcal{C}$.

Definition 2.1. For a function $\varphi \in \mathbb{W}^1(\Omega^h)$, we define the extended gradient

$$\nabla_{\Omega^h} \varphi = \big\{\mathcal{D}_1\varphi, \mathcal{D}_2\varphi, \mathcal{D}_3\varphi, \mathcal{D}_4\varphi\big\}^\top, \quad \mathcal{D}_4\varphi := \partial_{\mathcal{N}}\varphi, \tag{2.5}$$

and, for a vector field $\mathbf{U} = U_\alpha \mathbf{e}^\alpha = U^0_j \mathbf{d}^j \in \mathbb{W}^1(\Omega^h)$, we define the extended divergence

$$\operatorname{div}_{\Omega^h} \mathbf{U} := \mathcal{D}_j U^0_j + 2\mathcal{H}_{\mathcal{C}} U^0_4 = -\nabla^*_{\Omega^h}\mathbf{U}, \tag{2.6}$$

where $\nabla^*_{\Omega^h}$ denotes the formally adjoint operator to the gradient ∇_{Ω^h}, $\mathcal{H}_{\mathcal{C}}$ is the mean curvature (cf. (2.4)) and

$$\mathcal{D}_4 U^0_4 := \partial_{\mathcal{N}} U^0_4 = \langle \mathcal{N}, \partial_{\mathcal{N}}\mathbf{U}\rangle = (\mathcal{D}_4\mathbf{U})^0_4.$$

Caution: While defining the extended divergence in (2.6), we have to use only the representation $\mathbf{U} = U^0_j\mathbf{d}^j$ (cf. (2.2)), because any other representation differs from the indicated one by the vector $c\mathcal{N}$, where $c(\mathcal{x})$ is an arbitrary function. Then the extended divergences will differ by the summand $\operatorname{div}_{\Omega^h}(c(\mathcal{x})\mathcal{N}(\mathcal{x})) = \partial_{\mathcal{N}} c(\mathcal{x}) + 2c(\mathcal{x})\mathcal{H}_{\mathcal{C}}(\mathcal{x})$.

Lemma 2.2. *The classical gradient $\nabla\varphi := \{\partial_1\varphi, \partial_2\varphi, \partial_3\varphi\}^\top$, written in the full system of vectors $\{\mathbf{d}^j\}^4_{j=1}$ in* (2.1)*, coincides with the extended gradient $\nabla\varphi = \nabla_{\Omega^h}\varphi$ in* (2.5)*.*

The classical divergence $\operatorname{div}\mathbf{U} := \partial_\alpha U_\alpha$ *of a vector field $\mathbf{U} := U_\alpha\mathbf{e}^\alpha$, written in the full system* (2.1)*, coincides with the extended divergence* $\operatorname{div}\mathbf{U} = \operatorname{div}_{\Omega^h}\mathbf{U}^0$ *in* (2.6)*.*

The gradient and the negative divergence are the adjoint operators, $\nabla^*_{\Omega^h} = -\operatorname{div}_{\Omega^h}$ with respect to the scalar product induced from the ambient Euclidean space $\mathbb{R}^n$.

In the domain Ω_h, the classical Laplace operator

$$\Delta_{\Omega^h}\varphi(x) := (\operatorname{div}_{\Omega^h}\nabla_{\Omega^h}\varphi)(x) = -\big(\nabla^*_{\Omega^h}(\nabla_{\Omega^h}\varphi)\big)(x), \quad x \in \Omega^h,$$

written in the full system (2.1)*, acquires the following form:*

$$\Delta_{\Omega^h}\varphi = \mathcal{D}^2_j\varphi + 2\mathcal{H}_{\mathcal{C}}\mathcal{D}_4\varphi, \quad \varphi \in \mathbb{W}^2(\Omega^h).$$

Proof see in [2, Lemma 2].

The Lamé operator

$$\begin{aligned}\mathcal{L}\mathbf{U} = -\mu\,\boldsymbol{\Delta}\mathbf{U} - (\lambda+\mu)\,\nabla\operatorname{div}\mathbf{U} &= -\big[\mu\delta_{\alpha\beta}\partial^2_k + (\lambda+\mu)\partial_\alpha\partial_\beta\big]_{3\times 3}\mathbf{U}\\ &= -\big[c_{\alpha\gamma\beta\omega}\partial_\gamma\partial_\omega\big]_{3\times3}\mathbf{U}, \quad c_{\alpha\gamma\beta\omega} = \lambda\delta_{\alpha\gamma}\delta_{\beta\omega} + \mu(\delta_{\alpha\beta}\delta_{\gamma\omega} + \delta_{\alpha\omega}\delta_{\beta\gamma})\end{aligned}$$

is formally the self-adjoint differential operator of the second order and, written in the full system (2.1), acquires the form

$$\mathcal{L}_{\Omega^h}\mathbf{U}^0 = -\mu\,\boldsymbol{\Delta}_{\Omega^h}\mathbf{U}^0 - (\lambda+\mu)\,\nabla_{\Omega^h}\operatorname{div}_{\Omega^h}\mathbf{U}^0.$$

To reformulate the BVP (1.1) in curvilinear coordinates we introduce the traction operator (cf. (1.2))

$$\mathfrak{T}(x,\partial)\mathbf{U} = (\mathfrak{T}_{\alpha\beta}(x,\partial)\mathbf{U}_\beta)e^\alpha = \Big(\big\{\lambda\nu_\alpha\partial_\beta + \mu\nu_\beta\partial_\alpha + \delta_{\alpha\beta}\mu\,\partial_{\boldsymbol{\nu}}\big\}U_\beta\Big)\mathbf{e}^\alpha, \quad \mathbf{U} = (U_1,U_2,U_3)^\top = U_\alpha\mathbf{e}^\alpha$$

and Gunter's derivatives (see [2, (25)])

$$\begin{aligned}\mathfrak{T}(\mathcal{x},\mathcal{D}) &= \mathbf{e}^\alpha\otimes\mathbf{e}^\beta\big\{\lambda\nu_\alpha\partial_\beta + \mu\nu_\beta\partial_\alpha + \delta_{\alpha\beta}\mu\,\partial_{\boldsymbol{\nu}}\big\}\\ &= \lambda\mathbf{d}^4\otimes(\mathbf{d}^\beta + \nu_\beta\mathbf{d}^4)(\mathcal{D}_\beta + \nu_\beta\mathcal{D}_4)\\ &\quad + \mu(\mathbf{d}^\beta + \nu_\alpha\mathbf{d}^4)\otimes(\mathbf{d}^\beta + \nu_\beta\mathbf{d}^4)\mathcal{D}_4 + \mu(\mathbf{d}^\beta + \nu_\beta\mathbf{d}^4)\otimes\mathbf{d}^4(\mathcal{D}_\beta + \nu_\beta\mathcal{D}_4)\\ &= \begin{bmatrix}\mu\mathcal{D}_4 & 0 & 0 & \mu\mathcal{D}_1\\ 0 & \mu\mathcal{D}_4 & 0 & \mu\mathcal{D}_2\\ 0 & 0 & \mu\mathcal{D}_4 & \mu\mathcal{D}_3\\ \lambda\mathcal{D}_1 & \lambda\mathcal{D}_2 & \lambda\mathcal{D}_3 & (\lambda+2\mu)\mathcal{D}_4\end{bmatrix}.\end{aligned}$$

Let us recall some results related to the uniqueness of solutions to an arbitrary elliptic equation.

Definition 2.3. Let Ω be an open subset with the Lipschitz boundary $\partial\Omega \neq \varnothing$ either on a Lipschitz hypersurface $\mathcal{C} \subset \mathbb{R}^n$, or in the Euclidean space $\mathbb{R}^{n-1}$.

We say that a class of functions $\mathcal{U}(\Omega)$, defined in a domain Ω in $\mathbb{R}^n$, has the **strong unique continuation property** if every $u \in \mathcal{U}(\Omega)$ in this class which vanishes to infinite order at one point must vanish identically.

If a surface $\mathcal{C}$ is C^∞-smooth, any elliptic operator on $\mathcal{C}$ has the strong unique continuation property due to Holmgren's theorem. But we can have more.

Lemma 2.4. *Let $\mathcal{C}$ be a C^2-smooth hypersurface in $\mathbb{R}^n$. The class of solutions to a second order elliptic equation $\mathbb{A}(\mathcal{x}, \mathcal{D})u = 0$ with the Lipschitz continuous top order coefficients on a surface $\mathcal{C}$ has the strong unique continuation property.*

In particular, if the solution $u(\mathcal{x}) = 0$ vanishes in any open subset of $\mathcal{C}$, it vanishes identically on entire $\mathcal{C}$.

Proof see in [1, Lemma 1.7.2].

Lemma 2.5. *Let $\mathcal{C}$ be a C^2-smooth hypersurface in $\mathbb{R}^n$ with the Lipschitz boundary $\Gamma := \partial\mathcal{C}$ and $\gamma \subset \Gamma$ be an open part of the boundary Γ. Let $\mathbb{A}(\mathcal{x}, \mathcal{D})$ be a second order elliptic system with the Lipschitz continuous top order matrix coefficients on a surface $\mathcal{C}$.*

The Cauchy problem

$$\begin{cases} \mathbb{A}(\mathcal{x}, \mathcal{D})u = 0 & \text{on } \mathcal{C}, \quad u \in \mathbb{H}^1(\Omega), \\ u(\mathfrak{s}) = 0 & \text{for all } \mathfrak{s} \in \gamma, \\ (\partial_{\mathbf{V}} u)(\mathfrak{s}) = 0 & \text{for all } \mathfrak{s} \in \gamma, \end{cases}$$

where $\mathbf{V}$ is a non-tangential vector to Γ, but tangent to $\mathcal{C}$, has only a trivial solution $u(\mathcal{x}) = 0$ on entire $\mathcal{C}$.

Proof see in [1, Lemma 1.7.3].

3 Variational reformulation of the problem

To apply the method of Γ-convergence, we have to reformulate the BVP (1.1) into an equivalent variational problem for the energy functional. To this end, we have to consider the BVP with the vanishing Dirichlet condition on the lateral surface:

$$\begin{gathered} \mathcal{L}_{\Omega^h} \mathbf{U}_0(x) = \mathbf{F}_0(x), \quad x \in \Omega^h := \mathcal{C} \times (-h, h), \\ \mathbf{U}_0^+(t) = 0, \quad t \in \Gamma_L^h := \partial\mathcal{C} \times (-h, h), \\ (\mathfrak{T}(\mathcal{x}, \nabla)\mathbf{U}_0)^+(\mathcal{x}, \pm h) = \mathbf{H}_0(\mathcal{x}, \pm h), \quad \mathcal{x} \in \mathcal{C}. \end{gathered}$$

It is possible to rewrite the BVP (1.1) in the equivalent BVP (3.2). Indeed, consider the BVP

$$\begin{gathered} \mathcal{L}_{\Omega^h} \mathbf{V}(x) = 0, \quad x \in \Omega^h := \mathcal{C} \times (-h, h), \\ \mathbf{V}^+(t) = \mathbf{G}(t), \quad t \in \Gamma_L^h, \\ (\mathfrak{T}((\mathcal{x}, \nabla)\mathbf{V})^+(\mathcal{x}, \pm h) = 0, \quad (\mathcal{x}, \pm h) \in \Gamma^\pm = \mathcal{C} \times \{\pm h\}, \end{gathered} \tag{3.1}$$

which has a unique solution $\mathbf{V} \in \mathbb{W}^1(\Omega^h)$ (see Theorem 1.1) and note that the difference $\mathbf{U}_0 := \mathbf{U} - \mathbf{V}$ of solutions to BVPs (1.1) and (3.1) is a solution to the BVP (3.2), where $\mathbf{F}_0(\mathcal{x}) = \mathbf{F}(\mathcal{x}) - \mathcal{L}_{\Omega^h}\mathbf{V}(\mathcal{x})$, $= \mathbf{H}_0(\mathcal{x}, \pm h) == \mathbf{H}(\mathcal{x}, \pm h) - (\mathfrak{T}((\mathcal{x}, \nabla)\mathbf{V})^+(\mathcal{x}, \pm h)$. Vice versa, a solution to the BVP (1.1) is recovered as the sum of solutions $\mathbf{U} = \mathbf{U}_0 + \mathbf{V}$ of the BVPs (3.2) and (3.1).

Thus, in the BVP (1.1) we can assume, without restricting generality, that $\mathbf{G} = 0$ and consider the BVP

$$\begin{gathered} \mathcal{L}_{\Omega^h} \mathbf{U}(x) = \mathbf{F}(x), \quad x \in \Omega^h := \mathcal{C} \times (-h, h), \\ \mathbf{U}^+(t) = 0, \quad t \in \Gamma_L^h := \partial\mathcal{C} \times (-h, h), \\ (\mathfrak{T}(\mathcal{x}, \nabla)\mathbf{U})^+(\mathcal{x}, \pm h) = \mathbf{H}(\mathcal{x}, \pm h), \quad \mathcal{x} \in \mathcal{C}. \end{gathered} \tag{3.2}$$

Theorem 3.1. *Problem* (3.2) *with the constraints*

$$\mathbf{U}\in\mathbb{H}^1(\Omega^h,\Gamma^h_L),\quad \mathbf{F}\in\widetilde{\mathbb{H}}^{-1}(\Omega^h),\quad \mathbf{H}(\,\cdot\,,\pm h)\in\mathbb{H}^{-1/2}(\mathcal{C}) \tag{3.3}$$

is reformulated into the following equivalent variational problem: Under the same constraints (3.3), *look for a displacement vector-function* $\mathbf{U}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$, *which is a stationary point of the following functional:*

$$\begin{aligned}\mathcal{E}_{\Omega^h}(\mathbf{U}) :=& \frac{1}{2}\int\limits_{\Omega^h}\Big[\mu\partial_\beta\mathbf{U}_\alpha\cdot\partial_\beta\mathbf{U}_\alpha+\mu\partial_\beta\mathbf{U}_\alpha\cdot\partial_\alpha\mathbf{U}_\beta+\lambda\partial_\alpha\mathbf{U}_\alpha\cdot\partial_\gamma\mathbf{U}_\gamma+2\mathbf{F}_\beta\cdot\mathbf{U}_\beta\Big]\,dx\\ &+\int\limits_{\mathcal{C}}\Big[\big\langle\mathbf{H}(\mathcal{x},+h),\mathbf{U}^+(\mathcal{x},+h)\big\rangle-\big\langle\mathbf{H}(\mathcal{x},-h),\mathbf{U}^+(\mathcal{x},-h)\big\rangle\Big]\,d\sigma\\ =&\frac{1}{2}\int\limits_{-h}^{h}\int\limits_{\mathcal{C}}\Big[\mu\partial_\beta\mathbf{U}_\alpha\cdot\partial_\beta\mathbf{U}_\alpha+\mu\partial_\beta\mathbf{U}_\alpha\cdot\partial_\alpha\mathbf{U}_\beta+\lambda\partial_\alpha\mathbf{U}_\alpha\cdot\partial_\gamma\mathbf{U}_\gamma+2\mathbf{F}_\beta\cdot\mathbf{U}_\beta\\ &+\frac{1}{h}\Big[\big\langle\mathbf{H}(\mathcal{x},+h),\mathbf{U}^+(\mathcal{x},+h)\big\rangle-\big\langle\mathbf{H}(\mathcal{x},-h),\mathbf{U}^+(\mathcal{x},-h)\big\rangle\Big]\Big]\,d\sigma\,dt,\end{aligned} \tag{3.4}$$

Proof see in [2, Theorem 2].

Remark 3.2. The integral on $\mathcal{C}$ in (3.4) is understood in the sense of duality between the spaces $\widetilde{\mathbb{H}}^{1/2}(\mathcal{C})$ and $\mathbb{H}^{-1/2}(\mathcal{C})$ because $\mathbf{H}(\,\cdot\,,\pm h)\in\mathbb{H}^{-1/2}(\mathcal{C}_N)$ and the condition $\mathbf{U}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$ implies the inclusion $\mathbf{U}^+(\,\cdot\,,\pm h)\in\widetilde{\mathbb{H}}^{1/2}(\mathcal{C}_N)$.

Let us prove the following auxiliary lemma.

Lemma 3.3. *Let* $\mu>0$ *and* $\mu+\lambda>0$. *Then the quantity* $n(\mathbf{E}):=2\mu\,|\mathbf{E}|^2+\lambda(\operatorname{Trace}\mathbf{E})^2$ *is non-negative,* $n(\mathbf{E})\geqslant 0$ *for an arbitrary matrix* $\mathbf{E}=[E_{\alpha\beta}]_{3\times 3}$.

Proof. We proceed as follows:

$$\begin{aligned}n(\mathbf{E})&=2\mu\sum_{\alpha\neq\beta}E^2_{\alpha\beta}+2\mu\sum_{\alpha}E^2_{\alpha\alpha}+(\mu+\lambda)\sum_{\alpha,\beta}E_{\alpha\alpha}E_{\beta\beta}-\mu\sum_{\alpha,\beta}E_{\alpha\alpha}E_{\beta\beta}\\ &=2\mu\sum_{\alpha\neq\beta}E^2_{\alpha\beta}+(\mu+\lambda)\Big(\sum_{\alpha}E_{\alpha\alpha}\Big)^2+\mu\Big[2\sum_{\alpha}E^2_{\alpha\alpha}-\sum_{\alpha\neq\beta}E_{\alpha\alpha}E_{\beta\beta}\Big]\\ &=2\mu\sum_{\alpha\neq\beta}E^2_{\alpha\beta}+(\mu+\lambda)\Big(\sum_{\alpha}E_{\alpha\alpha}\Big)^2+\mu\sum_{\alpha\neq\beta}(E_{\alpha\alpha}-E_{\beta\beta})^2\geq 0,\end{aligned}$$

since $\mu>0$, $\mu+\lambda>\frac{2\mu+3\lambda}{3}>0$ (see (1.2)). □

4 Shell operator is non-negative

The main theorem of the present paper, Theorem 4.3, will be proved later. Here we recall the main results about Γ-limit of the energy functional $\mathcal{E}_{\Omega^h}(\mathbf{U})$ in (3.4).

Next, we perform the scaling of the variable $t=h\tau$, $-1<\tau<1$, in the modified kernel $Q_4(\nabla\mathbf{U})$ of the quadratic part of energy functional (3.4) and divide by h.

Lemma 4.1. *The scaled and divided by* h *energy functional*

$$\mathcal{E}^0_{\Omega^h}(\widetilde{\mathbf{U}}^h)=\frac{1}{h}\,\mathcal{E}_{\Omega^h}(\widetilde{\mathbf{U}}^h)=\frac{1}{2}\,\mathcal{Q}^0_4(\widetilde{\mathbf{U}}^h)-\mathcal{F}^0(\widetilde{\mathbf{U}}^0_h) \tag{4.1}$$

with the quadratic and linear parts

$$\mathcal{Q}_4^0(\widetilde{\mathbf{U}}^h)=\int\limits_{-1}^{1}\int\limits_{\mathcal{C}} Q_4^0(\nabla_{\Omega^h}\widetilde{\mathbf{U}}^h(x,\tau))\,d\sigma\,d\tau,$$

$$\mathcal{F}^0(\widetilde{\mathbf{U}}_h^0)=-\int\limits_{-h}^{h}\int\limits_{\mathcal{C}}\Big[\big\langle\widetilde{\mathbf{F}}_h^0,\mathbf{U}_h^0\big\rangle+\frac{1}{h}\Big[\big\langle\widetilde{\mathbf{H}}(x,+h),\widetilde{\mathbf{U}}^{0,+}(x,+h)\big\rangle-\big\langle\widetilde{\mathbf{H}}^0(x,-h),\widetilde{\mathbf{U}}^{0,+}(x,-h)\big\rangle\Big]\Big],$$

$$\mathbf{F}_h^0(x,\tau):=\big(F_1^0(x,h\tau),F_2^0(x,h\tau),F_3^0(x,h\tau),F_4^0(x,h\tau)\big)^\top,\quad F_4^0=\mathcal{N}_\alpha F_\alpha,$$
$$\widetilde{\mathbf{H}}_h^0(x,\tau):=\big(H_1^0(x,h\tau),H_2^0(x,h\tau),H_3^0(x,h\tau),H_4^0(x,h\tau)\big)^\top,\quad H_4^0=\mathcal{N}_\alpha H_\alpha,$$

is correctly defined on the space $\widetilde{\mathbb{H}}^1(\Omega^1,\Gamma_L^1)$ *(see* (1.4)*) and is convex:*

$$\mathcal{E}_{\Omega^h}^0(\theta\widetilde{\mathbf{U}}^h+(1-\theta)\widetilde{\mathbf{V}}^h)\leqslant\mathcal{E}_{\Omega^h}^0(\widetilde{\mathbf{U}}^h)+(1-\theta)\mathcal{E}_{\Omega^h}^0(\widetilde{\mathbf{V}}^h),\quad 0<\theta<1,$$

for arbitrary vector $\widetilde{\mathbf{V}}^h(x,\tau):=(V_1(x,h\tau),V_2(x,h\tau),V_3^0(x,h\tau),V_4(x,h\tau))^\top$, $\widetilde{\mathbf{V}}^h\in\widetilde{\mathbb{H}}^1(\Omega^1,\Gamma_L^1)$.

Moreover, if $\widetilde{\mathbf{F}}_h^0(x,\tau):=\mathbf{F}^0(x,h\tau)$ *are uniformly bounded in* $\mathbb{L}_2(\Omega^1)$*, i.e.,*

$$\sup_{h<h_0}\|\widetilde{\mathbf{F}}_h^0|\mathbb{L}_2(\Omega^1)\|<\infty$$

for some $h_0>0$*, the energy functional has the following quadratic estimate: there exist positive constants* C_1*,* C_2 *and* C_3 *independent of the parameter* h *such that*

$$C_1\int\limits_{\Omega^1}\Big[\big(\mathcal{D}_\alpha U_j^0(x,h\tau)\big)^2+\Big(\frac{1}{h}\,\frac{\partial U_j^0(x,h\tau)}{\partial\tau}\Big)^2\Big]\,dx-C_2\leqslant\mathcal{E}_{\Omega^h}^0(\widetilde{\mathbf{U}}^h)$$
$$\leqslant C_3\Big\{1+\int\limits_{\Omega^1}\Big[\big(\mathcal{D}_\alpha U_j^0(x,h\tau)\big)^2+\Big(\frac{1}{h}\,\frac{\partial U_j^0(x,h\tau)}{\partial\tau}\Big)^2\Big]\,dx\Big\}$$

for all $\widetilde{\mathbf{U}}^h\in\widetilde{\mathbb{H}}^1(\Omega^1,\Gamma_L^1)$.

Proof see in [2, Lemma 5].

Theorem 4.2. *Let the weak limits*

$$\lim_{h\to0}\mathbf{F}(x,h\tau)=\mathbf{F}(x),\quad\lim_{h\to0}\frac{1}{2h}\big[\mathbf{H}(x,+h)-\mathbf{H}(x,-h)\big]=\mathbf{H}^{(1)}(x),\quad\mathbf{F},\mathbf{H}^{(1)}\in\mathbb{L}_2(\mathcal{C}),$$

in $\mathbb{L}_2(\Omega^h)$ *and* $\mathbb{L}_2(\mathcal{C})$*, respectively, exist. Then the* Γ*-limit of the energy functional* $\mathcal{E}_{\Omega^h}^0(\widetilde{\mathbf{U}}^h)$ *exists:*

$$\Gamma-\lim_{h\to0}\mathcal{E}_{\Omega^h}^0(\widetilde{\mathbf{U}}^h)=\mathcal{E}_{\mathcal{C}}^3(\overline{\mathbf{U}}):=\int\limits_{\mathcal{C}}Q_3(\overline{\mathbf{U}}(x))\,d\sigma,\tag{4.2}$$

where

$$Q_3(\overline{\mathbf{U}})=\frac{\mu}{2}\Big[\big[\mathcal{D}_\alpha\overline{U}_\beta+\mathcal{D}_\beta\overline{U}_\alpha\big]^2-2\nu_\beta\nu_\gamma\mathcal{D}_\alpha\overline{U}_\beta\mathcal{D}_\alpha\overline{U}_\gamma\Big]$$
$$+\frac{2\lambda\mu}{\lambda+2\mu}\,(\mathcal{D}_\alpha\overline{U}_\alpha)^2+\big\langle\mathbf{F}(x)+2\mathbf{H}^{(1)}(x),\overline{U}(x)\big\rangle\tag{4.3}$$

and

$$\overline{\mathbf{U}}(x):=\big(\overline{U}_1(x),\overline{U}_2(x),\overline{U}_3(x)\big)^\top,\quad\overline{U}_\alpha(x):=U_\alpha(x,0),\quad\alpha=1,2,3.$$

Proof see in [2, Theorem 3].

Theorem 4.3. *Let $\mathbf{F},\mathbf{H}^{(1)}\in\mathbb{L}_2(\mathcal{C})$. The vector-function $\overline{\mathbf{U}}\in\widetilde{\mathbb{H}}^1(\mathcal{C})$ which minimizes the energy functional $\mathcal{E}^3_{\mathcal{C}}(\overline{\mathbf{U}})$ in (4.2), (4.3) is a solution to the following boundary value problem:*

$$\begin{cases}(\mathcal{L}_{\mathcal{C}}\overline{\mathbf{U}})_\alpha:=\mu\Big[\Delta_{\mathcal{C}}\overline{U}_\alpha+\mathcal{D}_\beta\mathcal{D}_\alpha\overline{U}_\beta-2\mathcal{H}_{\mathcal{C}}\nu_\beta\mathcal{D}_\alpha\overline{U}_\beta-\mathcal{D}_\gamma(\nu_\alpha\nu_\beta\mathcal{D}_\gamma\overline{U}_\beta)\Big]\\ \qquad\qquad+\dfrac{4\lambda\mu}{\lambda+2\mu}\left[\mathcal{D}_\alpha\mathcal{D}_\beta\overline{U}_\beta-2\mathcal{H}_{\mathcal{C}}\nu_\alpha\mathcal{D}_\beta\overline{U}_\beta\right]=\dfrac{1}{2}\,F_\alpha+H^{(1)}_\alpha\ \ on\ \mathcal{C}, \qquad \alpha=1,2,3.\\ \overline{U}_\alpha(t)=0\quad on\quad \Gamma=\partial\mathcal{C},\end{cases} \tag{4.4}$$

Vice versa: on the solution $\overline{\mathbf{U}}\in\widetilde{\mathbb{H}}^1(\mathcal{C})$ to the boundary value problem (4.4) under the condition $\mathbf{F},\mathbf{H}^{(1)}\in\mathbb{L}_2(\mathcal{C})$, the energy functional $\mathcal{E}^3_{\mathcal{C}}(\overline{\mathbf{U}})$ in (4.2), (4.3) attains the minimum.

Moreover, the operator $\mathcal{L}_{\mathcal{C}}$ in the left-hand side of the shell equation (4.4) is elliptic, positive definite and has finite dimensional kernel consisting of the solutions to the following system of equations:

$$\mathcal{D}_\alpha\overline{U}_\beta+\mathcal{D}_\beta\overline{U}_\alpha-\sum_\gamma\left[\nu_\alpha\nu_\gamma(\mathcal{D}_\beta\overline{U}_\gamma)+\nu_\beta\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma)\right]\equiv 0,\ \ \alpha,\beta=1,2,3. \tag{4.5}$$

The boundary value problem (4.4) has a unique solution in the classical setting:

$$\overline{\mathbf{U}}:=(\overline{U}_1,\overline{U}_2,\overline{U}_3)^\top\in\mathbb{H}^1(\mathcal{C}),\ \ \frac{1}{2}\,\mathbf{F}+\mathbf{H}^{(1)}\in\mathbb{L}_2(\mathcal{C}).$$

Proof. The first part of the theorem, that BVP (4.4) is the Γ-limit of the BVP (3.2) (i.e., the solution to the BVP (4.4) $\overline{\mathbf{U}}\in\widetilde{\mathbb{H}}^1(\mathcal{C})$ minimizes the energy functional $\mathcal{E}^3_{\mathcal{C}}(\overline{\mathbf{U}})$ in (4.2), (4.3)) is proved in [2, Theorem 4].

Ellipticity of the operator $\mathcal{L}_{\mathcal{C}}$ in the left-hand side of the shell equation (4.4) is checked directly and from the Lax–Milgram Lemma, it follows that it is the Fredholm operator in the setting $\mathcal{L}_{\mathcal{C}}:\mathbb{H}^{-1}(\mathcal{C})\to\mathbb{H}^1(\mathcal{C})$ (see [7, Theorem 14]) for a similar proof). Therefore, $\mathcal{L}_{\mathcal{C}}$ has the finite dimensional kernel.

Let us start with the energy functional and recall the quadratic part of the energy functional (see (4.1) and formulae [2, (33)]):

$$\begin{gathered}\mathcal{Q}^0_4(\mathbf{U})=\int\limits_{-h}^{h}\int\limits_{\mathcal{C}}Q^0_4(\nabla\mathbf{U}(\mathcal{x},t))\,d\sigma\,dt,\\ Q^0_4(\mathbf{F})=2\mu|\mathbf{E}|^2+\lambda(\operatorname{Trace}\mathbf{E})^2,\ \ \mathbf{E}=\frac{1}{2}\,(\mathbf{F}+\mathbf{F}^\top),\end{gathered} \tag{4.6}$$

where $\mathbf{F}=[F_{\alpha\beta}]_{3\times3}$ and $\mathbf{E}=[E_{\alpha\beta}]_{3\times3}$ are the 3×3 matrices and $|\mathbf{E}|^2=\operatorname{Trace}(\mathbf{E}^\top\mathbf{E})=\sum\limits_{\alpha,\beta}E^2_{\alpha\beta}$. From Lemma 3.3 it follows that the kernel $Q^0_4(\mathbf{F})$ is non-negative:

$$Q^0_4(\mathbf{F})=2\mu\sum_{\alpha\neq\beta}E^2_{\alpha\beta}+(\mu+\lambda)\Big(\sum_\alpha E_{\alpha\alpha}\Big)^2+\mu\sum_{\alpha\neq\beta}(E_{\alpha\alpha}-E_{\beta\beta})^2\geq 0. \tag{4.7}$$

Let us rewrite the kernel $Q^0_4(\nabla\mathbf{U})$ of the quadratic part $\mathcal{Q}^0_4(\mathbf{U})$ of the energy functional in (4.1), (4.6), (4.7) by using the equalities

$$\mathbf{F}=\nabla\mathbf{U}=[\partial_\alpha U_\beta]_{3\times3},\quad (\operatorname{Def}\mathbf{U}):=\frac{1}{2}\left((\nabla\mathbf{U})+(\nabla\mathbf{U})^\top\right)=\Big[\frac{1}{2}\,(\partial_\alpha U_\beta+\partial_\beta U_\alpha)\Big]_{3\times3}$$

and (2.3) as follows:

$$\begin{aligned}Q_4(\nabla\mathbf{U})&=2\mu\sum_{\alpha\neq\beta}(\operatorname{Def}\mathbf{U})^2_{\alpha\beta}+(\mu+\lambda)\Big(\sum_\alpha\partial_\alpha U_\alpha\Big)^2+\mu\sum_{\alpha\neq\beta}\left[\partial_\alpha U_\alpha-\partial_\beta U_\beta\right]^2\\ &=2\mu\sum_{\alpha\neq\beta}\Big[(\operatorname{Def}\mathbf{U})_{\alpha\beta}+\frac{\nu_\alpha\mathcal{D}_4U_\alpha+\nu_\beta\mathcal{D}_4U_\beta}{2}\Big]^2+(\mu+\lambda)\Big(\sum_\alpha\mathcal{D}_\alpha U_\alpha+\mathcal{D}_4U_4\Big)^2\end{aligned}$$

$$
\begin{aligned}
&+\mu\sum_{\alpha\neq\beta}\big[\mathcal{D}_\alpha U_\alpha-\mathcal{D}_\beta U_\beta+\nu_\alpha\mathcal{D}_4U_\alpha-\nu_\beta\mathcal{D}_4U_\beta\big]^2\\
=&\,2\mu\sum_{\alpha\neq\beta}\Big[(\operatorname{Def}\mathbf{U})_{\alpha\beta}+\frac{\nu_\alpha\mathcal{D}_4U_\alpha+\nu_\beta\mathcal{D}_4U_\beta}{2}\Big]^2+(\mu+\lambda)\Big(\sum_\alpha\mathcal{D}_\alpha U_\alpha+\mathcal{D}_4U_4\Big)^2\\
&+\mu\sum_{\alpha,\beta}\big[\mathcal{D}_\alpha U_\alpha-\mathcal{D}_\beta U_\beta+\nu_\alpha\mathcal{D}_4U_\alpha-\nu_\beta\mathcal{D}_4U_\beta\big]^2,
\end{aligned}\tag{4.8}
$$

where

$$
\operatorname{Def}\mathbf{U})_{\alpha\beta}:=\frac{\mathcal{D}_\alpha U_\beta+\mathcal{D}_\beta U_\alpha}{2},\quad \alpha,\beta=1,2,3.
$$

Next, we perform the scaling of the variable $t=h\tau$, $-1<\tau<1$, in the modified kernel $Q_4(\nabla\mathbf{U})$ of the quadratic part of energy functional (4.8), divide by h and study the following kernel in the scaled domain $\Omega^1=\mathcal{C}\times(1,1)$:

$$
\begin{aligned}
&Q_4^0(\nabla_{\Omega^h}\widetilde{\mathbf{U}}^h(x,\tau))=\frac{1}{h}\,Q_4(\nabla\mathbf{U}(x,h\tau))\\
&=\frac{\mu}{2}\sum_{\alpha\neq\beta}\Big[\mathcal{D}_\alpha U_\beta(x,h\tau)+\mathcal{D}_\beta U_\alpha(x,h\tau)+\frac{\nu_\alpha}{h}\,\frac{\partial U_\beta(x,h\tau)}{\partial\tau}+\frac{\nu_\beta}{h}\,\frac{\partial U_\alpha(x,h\tau)}{\partial\tau}\Big]^2\\
&\qquad+(\mu+\lambda)\Big(\sum_\alpha\mathcal{D}_\alpha U_\alpha(x,h\tau)+\frac{1}{h}\,\frac{\partial U_4(x,h\tau)}{\partial\tau}\Big)^2\\
&+\mu\sum_{\alpha,\beta}\Big[\mathcal{D}_\alpha U_\alpha(x,h\tau)-\mathcal{D}_\beta U_\beta(x,h\tau)+\frac{\nu_\alpha}{h}\,\frac{\partial U_\alpha(x,h\tau)}{\partial\tau}-\frac{\nu_\beta}{h}\,\frac{\partial U_\beta(x,h\tau)}{\partial\tau}\Big]^2,
\end{aligned}\tag{4.9}
$$

where

$$
\widetilde{\mathbf{U}}^h(x,\tau):=\big(U_1^0(x,h\tau),U_2^0(x,h\tau),U_3^0(x,h\tau),U_4^0(x,h\tau)\big)^\top,\ \ U_4^0=\mathcal{N}_\alpha U_\alpha.
$$

For this, let us rewrite Q_4^0 in (4.9) in the form

$$
\begin{aligned}
&Q_4^0(\nabla_{\Omega^h}\widetilde{\mathbf{U}}^h(x,\tau))=\frac{\mu}{2}\sum_{\alpha\neq\beta}\Big[\mathcal{D}_\alpha U_\beta(x,h\tau)+\mathcal{D}_\beta U_\alpha(x,h\tau)+\mathcal{N}_\alpha\xi_\beta+\mathcal{N}_\beta\xi_\alpha\Big]^2\\
&+(\mu+\lambda)\Big(\sum_\alpha\mathcal{D}_\alpha U_\alpha(x,h\tau)+\xi_4\Big)^2+\mu\sum_{\alpha,\beta}\Big[\mathcal{D}_\alpha U_\alpha(x,h\tau)-\mathcal{D}_\beta U_\beta(x,h\tau)+\mathcal{N}_\alpha\xi_\alpha-\mathcal{N}_\beta\xi_\beta\Big]^2\\
&\qquad=\frac{\mu}{2}\sum_{\alpha\neq\beta}\Big[\mathcal{D}_\alpha U_\beta(x,h\tau)+\mathcal{D}_\beta U_\alpha(x,h\tau)+\mathcal{N}_\alpha\xi_\beta+\mathcal{N}_\beta\xi_\alpha\Big]^2\\
&+(\mu+\lambda)(\mathcal{D}iv\,\mathbf{U}(x,h\tau)+\xi_4)^2+\mu\sum_{\alpha,\beta}\Big[\mathcal{D}_\alpha U_\alpha(x,h\tau)-\mathcal{D}_\beta U_\beta(x,h\tau)+\mathcal{N}_\alpha\xi_\alpha-\mathcal{N}_\beta\xi_\beta\Big]^2,
\end{aligned}\tag{4.10}
$$

where the variables

$$
\xi_\alpha=\xi_\alpha(x,h\tau):=\frac{1}{h}\,\frac{\partial U_\alpha(x,h\tau)}{\partial\tau},\quad \alpha=1,2,3,\quad \xi_4=\mathcal{N}_\alpha\xi_\alpha
$$

depend on h and we find minimum of the kernel $Q_4^0(\nabla_{\Omega^h}\widetilde{\mathbf{U}}(x,\tau))$ with respect to the variables ξ_1,ξ_2,ξ_3. It was shown in [2] that by $Q_4^0(\nabla_{\Omega^h}\widetilde{\mathbf{U}}^h(x,\tau))$ the Γ-limit is attained on the following values of the variables:

$$
\xi_4=-\frac{\lambda}{\lambda+2\mu}\,\mathcal{D}_\beta U_\beta=-\frac{\lambda}{\lambda+2\mu}\,\mathcal{D}iv\,\mathbf{U},\tag{4.11}
$$

$$
\xi_\alpha=-\mathcal{N}_\gamma(\mathcal{D}_\alpha U_\gamma)-\frac{\lambda}{\lambda+2\mu}\,\mathcal{N}_\alpha\mathcal{D}iv\,\mathbf{U},\quad \alpha=1,2,3,\tag{4.12}
$$

where we remind that $\mathcal{D}iv\,\mathbf{U}=\mathcal{D}_\alpha U_\alpha$. From (4.11), (4.12) and (4.10) we find the Γ-limit $Q_3^0(\overline{\mathbf{U}})$ (the same as in [2], but written in a different form):

$$\begin{aligned}
Q_3^0(\overline{\mathbf{U}}) &= \min_{\xi_1,\xi_2,\xi_3} Q_4^0(\nabla_{\Omega^h}\widetilde{\mathbf{U}}^h)\\
&= \frac{\mu}{2}\sum_{\alpha\neq\beta}\Big[\mathcal{D}_\alpha\overline{U}_\beta+\mathcal{D}_\beta\overline{U}_\alpha-\sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathcal{D}_\beta\overline{U}_\gamma)+\nu_\beta\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma)\big]-\frac{2\lambda}{\lambda+2\mu}\,\nu_\alpha\nu_\beta\mathcal{D}iv\,\overline{\mathbf{U}}\Big]^2\\
&\qquad\qquad+(\mu+\lambda)\Big(\mathcal{D}iv\,\overline{\mathbf{U}}-\frac{\lambda}{\lambda+2\mu}\,\mathcal{D}iv\,\overline{\mathbf{U}}\Big)^2\\
&\qquad+\mu\sum_{\alpha,\beta}\Big[\mathcal{D}_\alpha\overline{U}_\alpha-\mathcal{D}_\beta\overline{U}_\beta-\sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma)-\nu_\beta\nu_\gamma(\mathcal{D}_\beta\overline{U}_\gamma)\big]\\
&\qquad\qquad-\frac{\lambda}{\lambda+2\mu}\,\nu_\alpha^2\mathcal{D}iv\,\overline{\mathbf{U}}+\frac{\lambda}{\lambda+2\mu}\,\nu_\beta^2\mathcal{D}iv\,\overline{\mathbf{U}}\Big]^2\\
&= \frac{\mu}{2}\sum_{\alpha\neq\beta}\Big[\mathcal{D}_\alpha\overline{U}_\beta+\mathcal{D}_\beta\overline{U}_\alpha-\sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathcal{D}_\beta\overline{U}_\gamma)+\nu_\beta\nu_\gamma\mathcal{D}_\alpha\overline{U}_\gamma\big]-\frac{2\lambda}{\lambda+2\mu}\,\nu_\alpha\nu_\beta\mathcal{D}iv\,\overline{\mathbf{U}}\Big]^2\\
&\quad+\frac{4\mu^2(\mu+\lambda)}{(\lambda+2\mu)^2}\,[\mathcal{D}iv\,\overline{\mathbf{U}}]^2+\mu\sum_{\alpha,\beta}\Big[\mathcal{D}_\alpha\overline{U}_\alpha-\mathcal{D}_\beta\overline{U}_\beta-\sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma)-\nu_\beta\nu_\gamma(\mathcal{D}_\beta\overline{U}_\gamma)\big]\Big]^2. \qquad (4.13)
\end{aligned}$$

From (4.13) it follows that $Q_3^0(\overline{\mathbf{U}})$ is a nonnegative quadratic form $Q_3^0(\overline{\mathbf{U}})\geqslant 0$ for all $\mathbf{U}\in\mathbb{H}^1(\mathcal{C},\Gamma)$, $\Gamma:=\partial\mathcal{C}$. □

5 Shell operator is positive definite

If $Q_3^0(\overline{\mathbf{U}})\equiv 0$, from (4.13) we get

$$\begin{gathered}
\mathcal{D}iv\,\overline{\mathbf{U}}\equiv 0,\\
\mathcal{D}_\alpha\overline{U}_\alpha-\mathcal{D}_\beta\overline{U}_\beta-\sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma)-\nu_\beta\nu_\gamma(\mathcal{D}_\beta\overline{U}_\gamma)\big]\equiv 0,\quad \alpha\neq\beta=1,2,3,\\
\mathcal{D}_\alpha\overline{U}_\beta+\mathcal{D}_\beta\overline{U}_\alpha-\sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathcal{D}_\beta\overline{U}_\gamma)+\nu_\beta\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma)\big]\equiv 0,\quad \alpha\neq\beta=1,2,3.
\end{gathered}\qquad (5.1)$$

By taking the sum with respect to β in the second equality in (5.1), we get

$$\mathcal{D}_\alpha\overline{U}_\alpha=\sum_\gamma\nu_\alpha\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma),\quad \alpha=1,2,3.$$

Note that the obtained equality implies both, the first and the second equalities from (5.1). Moreover, it coincides with the third equality in (5.1) if we allow there $\alpha=\beta=1,2,3$. Thus, equation (4.5) implies all three equalities in (5.1) and describes the kernel $\operatorname{Ker}\mathcal{L}_\mathcal{C}$ of the shell equation $\mathcal{L}_\mathcal{C}$ in (4.4).

Now we rewrite the obtained equation in the following form:

$$\begin{aligned}
\mathcal{D}_\alpha\overline{U}_\alpha&=\sum_\gamma\nu_\alpha\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma)=\nu_\alpha\mathcal{D}_\alpha\Big(\sum_\gamma\nu_\gamma\overline{U}_\gamma\Big)-\sum_\gamma\nu_\alpha(\mathcal{D}_\alpha\nu_\gamma)\overline{U}_\gamma\\
&=\nu_\alpha(\mathcal{D}_\alpha\overline{U}_4)-\sum_\gamma\nu_\alpha(\mathcal{D}_\alpha\nu_\gamma)\overline{U}_\gamma,\quad \overline{U}_4=\sum_\gamma\nu_\gamma\overline{U}_\gamma,\quad \alpha=1,2,3. \qquad (5.2)
\end{aligned}$$

Similarly to (5.2), from equality (4.5) (see the third equality in (5.1) we derive

$$\mathcal{D}_\alpha\overline{U}_\beta+\mathcal{D}_\beta\overline{U}_\alpha=\nu_\alpha\mathcal{D}_\beta\overline{U}_4+\nu_\beta\mathcal{D}_\alpha\overline{U}_4-\sum_\gamma\big[\nu_\alpha(\mathcal{D}_\beta\nu_\gamma)+\nu_\beta(\mathcal{D}_\alpha\nu_\gamma)\big]\overline{U}_\gamma,\quad \alpha,\beta=1,2,3. \qquad (5.3)$$

Besides the equalities (4.5), (5.2), (5.3) we have the following equality

$$\begin{aligned}\sum_{\alpha,\beta}\Big[\big[\mathcal{D}_\alpha\overline{U}_\beta+\mathcal{D}_\beta\overline{U}_\alpha\big]^2-2\sum_\gamma\nu_\beta\nu_\gamma\mathcal{D}_\alpha\overline{U}_\beta\mathcal{D}_\alpha\overline{U}_\gamma\Big]\\
=\sum_{\alpha,\beta}\Big[\big[\mathcal{D}_\alpha\overline{U}_\beta+\mathcal{D}_\beta\overline{U}_\alpha\big]^2\Big]-2\sum_\alpha(\mathcal{D}_\alpha\overline{U}_4)^2-2\sum_{\alpha,\beta,\gamma}(\mathcal{D}_\alpha\nu_\beta)(\mathcal{D}_\alpha\nu_\gamma)\overline{U}_\beta\overline{U}_\gamma\\
+2\sum_{\alpha,\beta}(\mathcal{D}_\alpha\nu_\beta)(\mathcal{D}_\alpha\overline{U}_4)\overline{U}_\beta-2\sum_{\alpha,\gamma}(\mathcal{D}_\alpha\nu_\gamma)(\mathcal{D}_\alpha\overline{U}_4)\overline{U}_\gamma\equiv 0,\end{aligned}\tag{5.4}$$

which follows from (4.3) if we apply the first equality from (5.1) and recall that $Q_3^0(\overline{\mathbf{U}})=0$.

If $\overline{U}_\alpha(\mathfrak{s})=0$, $\alpha=1,2,3$, equalities (5.2)–(5.4) simplify:

$$\begin{gathered}\mathcal{D}_\alpha(\mathfrak{s})\overline{U}_\alpha(\mathfrak{s})=\nu_\alpha(\mathfrak{s})\mathcal{D}_\alpha\overline{U}_4(\mathfrak{s}),\\
\mathcal{D}_\alpha\overline{U}_\beta(\mathfrak{s})+\mathcal{D}_\beta\overline{U}_\alpha(\mathfrak{s})=\nu_\alpha(\mathfrak{s})\mathcal{D}_\beta\overline{U}_4(\mathfrak{s})+\nu_\beta(\mathfrak{s})\mathcal{D}_\alpha\overline{U}_4(\mathfrak{s}),\quad\alpha,\beta=1,2,3,\\
\sum_{\alpha,\beta}\Big[\big[\mathcal{D}_\alpha\overline{U}_\beta(\mathfrak{s})+\mathcal{D}_\beta\overline{U}_\alpha(\mathfrak{s})\big]^2\Big]=2\sum_\alpha(\mathcal{D}_\alpha\overline{U}_4(\mathfrak{s}))^2,\quad\mathfrak{s}\in\partial\mathcal{C}.\end{gathered}\tag{5.5}$$

We can see that not only the first equality in (5.5) is the consequence of the second one (by taking $\alpha=\beta$), but also the third equality follows from the second one if we take into account that $\sum\limits_\alpha\nu_\alpha^2=1$ and $\sum\limits_\alpha\nu_\alpha\mathcal{D}_\alpha=0$.

By inserting the first equality from (5.5) into the second one we get

$$\mathcal{D}_\alpha\overline{U}_\beta(\mathfrak{s})+\mathcal{D}_\beta\overline{U}_\alpha(\mathfrak{s})=\frac{\nu_\alpha(\mathfrak{s})}{\nu_\beta(\mathfrak{s})}\,\mathcal{D}_\beta\overline{U}_\beta(\mathfrak{s})+\frac{\nu_\beta(\mathfrak{s})}{\nu_\alpha(\mathfrak{s})}\,\mathcal{D}_\alpha\overline{U}_\alpha(\mathfrak{s}),\quad\alpha,\beta=1,2,3.$$

If we succeed in proving that

$$\mathcal{D}_\alpha\overline{U}_4(\mathfrak{s})\equiv 0,\quad \mathfrak{s}\in\partial\mathcal{C},\quad\alpha=1,2,3,\tag{5.6}$$

then from (5.5) and (5.6) will follow

$$\mathcal{D}_\alpha\overline{U}_\beta(\mathfrak{s})+\mathcal{D}_\beta\overline{U}_\alpha(\mathfrak{s})\equiv 0,\quad \mathfrak{s}\in\partial\mathcal{C},\quad\alpha,\beta=1,2,3.\tag{5.7}$$

The latter implies that

$$\mathcal{D}_\alpha\overline{U}_\beta(\mathfrak{s})\equiv 0\qquad\forall\,\alpha,\beta=1,2,3,\quad\forall\,\mathfrak{s}\in\partial\mathcal{C}.\tag{5.8}$$

Indeed (cf. [1, Lemma 1.7.4]), among directing tangential vector fields $\{\mathbf{d}^k(\mathfrak{s})\}_{k=1}^3$ generating Günter's derivatives $\mathcal{D}_k=\partial_{\mathbf{d}^k}$, $k=1,2,3$, only 2 are linearly independent (one of these vectors might even collapse at a point $\mathbf{d}^k(\mathfrak{s})=0$ if the corresponding basis vector $\mathbf{e}^k$ is orthogonal to the surface at $\mathfrak{s}\in\mathcal{C}$). One of these vectors might be tangential to the boundary curve $\partial\mathcal{C}$ and, at least one, say $\mathbf{d}^3(\mathfrak{s})$, is non-tangential to $\partial\mathcal{C}$. The vector $\mathbf{d}^\alpha$ for $\alpha=1,2,3$, is a linear combination $\mathbf{d}^\alpha(\mathfrak{s})=c_1(\mathfrak{s})\mathbf{d}^3(\mathfrak{s})+c_2(\mathfrak{s})\boldsymbol{\tau}^\alpha(\mathfrak{s})$ of the non-tangential vector $\mathbf{d}^3(\mathfrak{s})$ and of the projection $\boldsymbol{\tau}^\alpha(\mathfrak{s}):=\pi_{\partial\mathcal{C}}\mathbf{d}^\alpha(\mathfrak{s})$ of the vector $\mathbf{d}^\alpha(\mathfrak{s})$ to the boundary curve $\partial\mathcal{C}$ at the point $\mathfrak{s}\in\partial\mathcal{C}$. Then

$$(\mathcal{D}_\alpha U_3)(\mathfrak{s})=c_1(\mathfrak{s})(\partial_{\mathbf{d}^3}U_3)(\mathfrak{s})+c_2(\mathfrak{s})(\partial_{\boldsymbol{\tau}^\alpha}U_3)(\mathfrak{s})=c_1(\mathfrak{s})(\mathcal{D}_3U_3)(\mathfrak{s})\tag{5.9}$$

for all $\mathfrak{s}\in\gamma$ and all $\alpha=1,2,3$, since $(\mathcal{D}_{\mathbf{d}^3}U_3)(\mathfrak{s})=(\mathcal{D}_3U_3)(\mathfrak{s})U_3$, U_3 vanishes identically on $\partial\mathcal{C}$ and the derivative $(\partial_{\boldsymbol{\tau}^j}U_3^0)(\mathfrak{s})=0$ vanishes, as well.

On the other hand, from (5.7) for $\beta=\alpha=3$ follows $2\mathcal{D}_3U_3(\mathfrak{s})=0$ and, together with (5.9), gives $(\mathcal{D}_\alpha U_3)(\mathfrak{s})=0$ for all $\mathfrak{s}\in\gamma$, $\beta=1,2,3$. Then, due to (5.7), $(\mathcal{D}_3U_\alpha)(\mathfrak{s})=(\mathcal{D}_\alpha U_3)(\mathfrak{s})=0$ and, due to (5.7), $(\mathcal{D}_\alpha U_\alpha)(\mathfrak{s})=0$ for all $\mathfrak{s}\in\gamma$, $\alpha=1,2,3$. Applying again the above arguments, exposed for U_3, we prove equalities (5.8).

6 Numerical approximation of the shell equation

Consider the boundary value problem (4.4)

$$\begin{cases}(\mathcal{L}_{\mathcal{C}}\overline{\mathbf{U}})_\alpha := \mu\Big[\Delta_{\mathcal{C}}\overline{U}_\alpha + \mathcal{D}_\beta\mathcal{D}_\alpha\overline{U}_\beta - 2\mathcal{H}_{\mathcal{C}}\nu_\beta\mathcal{D}_\alpha\overline{U}_\beta - \mathcal{D}_\gamma(\nu_\alpha\nu_\beta\mathcal{D}_\gamma\overline{U}_\beta)\Big] \\ \qquad\qquad + \dfrac{4\lambda\mu}{\lambda+2\mu}\left[\mathcal{D}_\alpha\mathcal{D}_\beta\overline{U}_\beta - 2\mathcal{H}_{\mathcal{C}}\nu_\alpha\mathcal{D}_\beta\overline{U}_\beta\right] = \dfrac{1}{2}\,G_\alpha \ \text{ on } \ \mathcal{C}, \\ \overline{U}_\alpha(t) = 0 \ \text{ on } \ \Gamma = \partial\mathcal{C}, \ \ \alpha = 1,2,3,\end{cases}$$

where $G_\alpha = F_\alpha + 2H^{(1)}_\alpha \in [\mathbb{L}_2(\mathcal{C})]$, $\alpha = 1,2,3$.

In [2, Theorem 4], it is proved that if $\overline{\mathbf{U}} \in [\widetilde{\mathbb{H}}^1(\mathcal{C})]^3$ is a solution of BVP (4.4) and $\overline{\mathbf{V}} \in [\widetilde{\mathbb{H}}^1(\mathcal{C})]^3$, then

$$\int\limits_{\mathcal{C}}\left\{2\mu\Big[\mathcal{D}_\beta\overline{U}_\alpha\mathcal{D}_\beta\overline{V}_\alpha + \mathcal{D}_\alpha\overline{U}_\beta\mathcal{D}_\beta\overline{V}_\alpha - \nu_\alpha\nu_\beta\mathcal{D}_\gamma\overline{U}_\beta\mathcal{D}_\gamma\overline{V}_\alpha\Big] + \frac{4\lambda\mu}{\lambda+2\mu}\,\mathcal{D}_\beta\overline{U}_\beta\mathcal{D}_\alpha\overline{V}_\alpha\right\}d\sigma$$
$$= \int\limits_{\mathcal{C}}\langle\overline{G}_\alpha,\overline{V}_\alpha\rangle\,d\sigma. \tag{6.1}$$

Therefore, the BVP (4.4) can be reformulated in the following way.

Find a vector $\overline{\mathbf{U}} \in [\widetilde{\mathbb{H}}^1(\mathcal{C})]^3$ satisfying equation (6.1) for any $\overline{\mathbf{V}} \in [\widetilde{\mathbb{H}}^1(\mathcal{C})]^3$:

$$(c_{\alpha\beta\gamma\zeta}(x)\mathcal{D}_\beta U_\alpha, \mathcal{D}_\zeta V_\gamma) = (G_\alpha, V_\alpha) \ \ \forall\,\mathbf{V} \in [\widetilde{\mathbb{H}}^1(\mathcal{C})]^3, \tag{6.2}$$

where

$$c_{\alpha\beta\gamma\zeta}(x) = \frac{4\lambda\mu}{\lambda+2\mu}\,\delta_{\alpha\beta} + 2\mu(\delta_{\alpha\gamma}\delta_{\beta\zeta} + \delta_{\alpha\zeta}\delta_{\beta\gamma} - \nu_\alpha\nu_\gamma\delta_{\beta\zeta})$$

and $(\,\cdot\,,\,\cdot\,)$ denotes an inner product

$$(f,g) = \int\limits_{\mathcal{C}}\langle f,g\rangle\,d\sigma.$$

Due to (4.13), the sesquilinear form

$$a(U,V) := (c_{\alpha\beta\gamma\zeta}\mathcal{D}_\beta U_\alpha, \mathcal{D}_\zeta V_\gamma)$$

is bounded and coercive in $\mathbb{H}^1_0(\mathcal{C})$,

$$M_1\big\|U \mid \mathbb{H}^1(\mathcal{C})\big\|^2 \geq a(U,U) \geq M\big\|U \mid \mathbb{H}^1(\mathcal{C})\big\|^2 \ \ \forall\,U \in [\mathbb{H}^1_0(\mathcal{C})]^3$$

for some $M > 0$, $M_1 > 0$. Therefore, by the Lax–Milgram Theorem problem (6.2) possesses a unique solution.

Now, let us consider the discrete counterpart of the problem.

Let X_h be a family of finite-dimensional subspaces approximating $[\mathbb{H}^1(\mathcal{C})]^3$, i.e., such that $\bigcup\limits_h X_h$ is dense in $[\mathbb{H}^1(\mathcal{C})]^3$.

Consider equation (6.2) in the finite-dimensional space X_h,

$$a(U_h,V_h) = \widetilde{g}(V_h) \ \ \forall\,V \in X_h, \tag{6.3}$$

where $\widetilde{g}(V_h) = -(G,V_h)_{\mathcal{C}}$.

Theorem 6.1. *Equation* (6.3) *has the unique solution* $U_h \in X_h$ *for all* $h > 0$*. This solution converges in* $[\mathbb{H}^1(\mathcal{C})]^3$ *to the solution* U *of* (6.2) *as* $h \to 0$.

Proof. Immediately follows from the coercivity of sesquilinear form a:

$$c_1 \left\| U_h \mid [\mathbb{H}^1(\mathcal{C})]^3 \right\|^2 \leq a(U_h, U_h) = |\widetilde{f}(U_h)| \leq c_2 \left\| U_h \mid [\mathbb{H}^1(\mathcal{C})]^3 \right\| \text{ for all } h. \tag{6.4}$$

Let U_h be the unique solution of the homogeneous equation

$$a(U_h, \psi_h) = 0 \text{ for all } \psi_h \in X_h.$$

Then (6.4) implies $\|U_h \mid [\mathbb{H}^1(\mathcal{C})]^3\| = 0$ and, consequently, $U_h = 0$. Therefore, equation (6.3) has a unique solution. From (6.4) it also follows that

$$\left\| U_h \mid [\mathbb{H}^1(\mathcal{C})]^3 \right\|^2 \leq \frac{c_2}{c_1} \left\| U_h \mid [\mathbb{H}^1(\mathcal{C})]^3 \right\|.$$

Hence, the sequence $\{\|U_h \mid [\mathbb{H}^1(\mathcal{C})]^3\|\}$ is bounded and we can extract a subsequence $\{U_{h_k}\}$ which converges weakly to some $U \in \mathbb{H}^1(\mathcal{C})$.

Let us take an arbitrary $V \in [\mathbb{H}^1(\mathcal{C})]^3$ and for each $h > 0$ choose $V_h \in X_h$ such that $V_h \to V$ in $[\mathbb{H}^1(\mathcal{C})]^3$. Then from (6.3) we have

$$a(U, V) = \widetilde{g}(V) \ \ \forall V \in [\mathbb{H}^1(\mathcal{C})]^3.$$

Hence, U solves (6.2). Note that since (6.2) is uniquely solvable, each subsequence $\{U_{h_k}\}$ converges weakly to the same solution U and, consequently, the whole sequence $\{U_h\}$ also converges weakly to U.

Now, let us prove that it converges in the space $[\mathbb{H}^1(\mathcal{C})]^3$.

Indeed, due to (6.4), we have

$$\begin{aligned} c_1\|U_h - U\|^2 &\leq |a(U_h - U, U_h - U)| \leq |a(U_h, U_h - U) - a(U, U_h - U)| \\ &= c_1|\widetilde{g}(U_h) - a(U_h, U) - \widetilde{g}(U_h - U)| \longrightarrow c_1|\widetilde{g}(U) - a(U, U)| = 0, \end{aligned}$$

which completes the proof. □

We can choose spaces X_h in different ways.

In particular, consider a case where $\omega = U_\alpha$ in the above parametrization is a square part of $\mathbb{R}^2$:

$$\omega = \big\{(x_1, x_2) : \ 0 < x_1 < 1, \ 0 < x_2 < 1\big\}, \ \ \zeta(\omega) = \mathcal{C}.$$

Allocate N^2 nodes $P_{ij} = (i/(N+1), j/(N+1))$, $i, j = 1, \ldots, N$, on ω.

Let α_k, $k = 1, \ldots, N$, be piecewise linear functions defined on the segment $[0, 1]$ as follows:

$$\alpha_k(x) = \begin{cases} 0, & 0 \leq x \leq \dfrac{k-1}{N+1}, \\ (N+1)(x - \dfrac{k-1}{N+1}), & \dfrac{k-1}{N+1} < x \leq \dfrac{k}{N+1}, \\ (N+1)(\dfrac{k+1}{N+1} - x), & \dfrac{k}{N+1} < x \leq \dfrac{k+1}{N+1}, \\ 0, & \dfrac{k+1}{N+1} < x \leq 1, \\ j = k, \ldots, N. \end{cases}$$

Denote by φ_{ij}, $i, j = 1, \ldots, N$, the functions

$$\varphi_{ij}(x_1, x_2) = \alpha_i(x_1)\alpha_j(x_2), \ \ i, j = 1, \ldots N, \ \ (x_1, x_2) \in \omega.$$

Evidently, φ_{ij} are continuous functions, which take their maximal value $\varphi_{ij}(P_{ij}) = 1$ at the point P_{ij} and vanish outside the set

$$\omega_{ij} = \omega \cap \left\{(x_1, x_2) : \ 0 \leq \left|x_1 - \frac{i}{N+1}\right| \leq 1, \ 0 \leq \left|x_2 - \frac{j}{N+1}\right| \leq 1\right\}.$$

Consequently, they belong to $\mathbb{H}^1(\omega)$ and are linearly independent.

Denote by X_N the linear span of the functions $\widehat{\varphi}_{ij} = \varphi_{ij} \circ \vartheta$, $i,j = 1,\dots,N$. The space X_N is N^2-dimensional space contained in $\mathbb{H}^1(\mathcal{C})$.

Let $\widetilde{\varphi}^{(k)}_{ij} = (\delta_{1k}, \delta_{2k}, \delta_{3k})\widehat{\varphi}_{ij} \in [X_N]^3$, $k = 1,2,3$, $i,j = 1,\dots,N$.

Consider equation (6.3) in the space $[X_N]^3$

$$a(U,V) = \widetilde{g}(V) \ \ \forall\, V \in [X_N]^3. \tag{6.5}$$

We sought for the solution $U \in [X_N]^3$ of equation (6.5) in the form

$$U = \sum_{m=1}^{3} \sum_{i,j=1}^{N} C^{(m)}_{ij} \widetilde{\varphi}^{(m)}_{ij},$$

where $C^{(m)}_{ij}$ are unknown coefficients. Substituting U into (6.5) and replacing V successively by $\widetilde{\varphi}^{(m)}_{ij}$, $m = 1,2,3$, $i,j = 1,\dots,N$, we get the equivalent system of $3N^2$ linear algebraic equations

$$\sum_{m=1}^{3} \sum_{i,j=1}^{N} A^{(m,n)}_{ijkl} C^{(m)}_{ij} = g^{(n)}_{kl}, \quad n = 1,2,3, \ \ k,l = 1,\dots,N, \tag{6.6}$$

where

$$A^{(m,n)}_{ijkl} = a\big(\widetilde{\varphi}^{(m)}_{ij}, \widetilde{\varphi}^{(n)}_{kl}\big), \quad g^{(n)}_{kl} = \widetilde{g}\big(\widetilde{\varphi}^{(n)}_{kl}\big).$$

The matrix $A = A^{(m,n)}_{(ijkl)}$ is Gram's matrix defined by the positive semidefinite bilinear form a attached to basis vectors $\widetilde{\varphi}^{(m)}_{ij}$, $m = 1,2,3$, $i,j = 1,\dots,N$, of $[X_N]^3$. Therefore, it is a nonsingular matrix and equation (6.6) has a unique solution

$$U = \sum_{i,j,k,l=1}^{N} (A^{-1})^{(m,n)}_{ijkl} \widetilde{\varphi}^{(m)}_{ij} g^{(n)}_{kl}.$$

To calculate explicitly $A^{(m,n)}_{ijkl}$ and $g^{(n)}_{kl}$ we note that

$$\begin{aligned}
\mathcal{D}_r \widetilde{\varphi}^{(m)}_{ij}(y) &= \partial_{y_r} \widetilde{\varphi}^{(m)}_{ij}(y) + \nu_r \partial_\nu \widetilde{\varphi}^{(m)}_{ij}(y)\\
&= \sum_{p=1}^{2} \partial_p \varphi_{ij}(\vartheta(y))\big(\partial_r \vartheta_p(y) + \nu_r \nu_l \partial_l \vartheta_p(y)\big)(\delta_{m1}, \delta_{m2}, \delta_{m3})\\
&= \sum_{p=1}^{2} \partial_p \varphi_{ij}(\vartheta(y)) \mathcal{D}_r \vartheta_p(y)(\delta_{m1}, \delta_{m2}, \delta_{m3}),\\
A^{(m,n)}_{ijkl} &= a\big(\widetilde{\varphi}^{(m)}_{ij}, \widetilde{\varphi}^{(n)}_{kl}\big) = (c_{qrst}\delta_{rm}\delta_{tn}\mathcal{D}_q\varphi_{ij}, \mathcal{D}_s\varphi_{kl})\\
&= \sum_{\alpha,\beta=1}^{2} \int\limits_{\omega_{ij}\cap\omega_{kl}} c_{qmsn}(\vartheta(y))\big(\partial_\alpha \varphi_{ij}(\vartheta(y))\big)\big(\partial_\beta \varphi_{kl}(\vartheta(y))\big)\mathcal{D}_q\vartheta_\alpha(y)\mathcal{D}_s\vartheta_\beta(y)|\sigma'(y)|\,dy,\\
g^{(n)}_{kl} &= -(g,\, \widetilde{\varphi}^{(n)}_{kl})_{\mathcal{C}} = -\int\limits_{\omega_{ij}\cap\omega_{kl}} g(\vartheta(y))\, \varphi^{(n)}_{kl}(\vartheta(y))|\sigma'(y)|\,dy,
\end{aligned}$$

where $|\sigma'(y)|$ is a surface element of $\mathcal{C}$

$$|\sigma'(y)| = |\partial_1\vartheta(y) \times \partial_2\vartheta(y)|.$$

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ON THE EXISTENCE AND STABILITY OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL SYSTEMS DRIVEN BY THE G-BROWNIAN MOTION

Abstract. In this paper, we study the Carathéodory approximate solution for a class of stochastic differential systems driven by G-Brownian motion. Based on the Carathéodory approximation scheme, we prove under some suitable conditions that our system has a unique solution and show that the Carathéodory approximate solutions converge to the solution of the system. Moreover, we prove a stability theorem for our system.

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Key words and phrases. G-expectation, G-brownian motion, G-stochastic differential equations, Carathéodory approximation scheme.

რეზიუმე. სტატიაში შესწავლილია კარათეოდორის მიახლოებითი ამონახსნი სტოქასტურ დიფერენციალურ სისტემათა კლასისთვის, რომელიც განპირობებულია G-ბრაუნის მოძრაობით. კარათეოდორის მიახლოების სქემაზე დაყრდნობით დამტკიცებულია, რომ შესაფერის პირობებში სისტემას აქვს ერთადერთი ამონახსნი და ნაჩვენებია, რომ კარათეოდორის მიახლოებითი ამონახსნები კრებადია სისტემის ამ ამონახსნისკენ. მოცემული სისტემისთვის აგრეთვე დამტკიცებულია მდგრადობის თეორემა.

1 Introduction

This paper is intended to study stochastic differential equations (SDE, for short) which have been the object of sustained attention in recent years because of their interesting structure and usefulness in various applied fields. The motivation for studying SDEs comes originally from the stochastic optimal control theory, that is, the adjoint equation in the Pontryagin type maximum principle. After this, extensive study of SDEs was initiated, and potential for its application was found in applied and theoretical areas such as stochastic control, mathematical finance, differential geometry, et al. It is worth pointing out that the SDEs have also been successfully applied to model and to resolve some interesting problems in mathematical finance, such as problems involving term structure of interest rates and hedging contingent claims for large investors, etc. See, e.g., [1, 3, 11, 13, 15, 16, 18, 20, 21] and [24–27, 29].

Recently, the theory of G-Brownian motion was introduced by S. Peng. The existence and uniqueness of solutions for some stochastic differential equations under G-Brownian motion (G-SDEs) with Lipschitz continuous coefficients were developed by Peng and Gao. In 2006, Peng in [24] (for more details see [10] and [19, 24–29]) introduced the theory of nonlinear expectation, the G-Brownian motion and defined the related stochastic calculus, especially, stochastic integrals of Itô's type with respect to the G-Brownian motion, and derived the related Itô's formula. In addition, the notion of G-normal distribution plays the same important role in the theory of nonlinear expectation as that of the normal distribution with the classical probability. In 2009, Gao in [10] studied pathwise properties and homeomorphic property with respect to the initial values for stochastic differential equations driven by the G-Brownian motion. Later, Faizullah et *al.* extended this theory (see, e.g., [4–9]).

In general, one cannot obtain the explicit solutions of SDEs. The fact that these systems model phenomena of the real world, the important mathematical questions that concern them are: the existence and uniqueness of a solution, stability, asymptotic behavior of a solution, etc.

There are many theoretical, analytical and numerical methods and techniques for processing and studying SDEs. We find this in the references mentioned and others. In this work, we will focus on the Carathéodory approximation scheme that has been used by many mathematicians to prove the existence theorem of solutions of ordinary differential equations under weak regularity conditions (see, e.g., [2, 5, 14, 18, 22, 23]).

Furthermore, in [5], Faizullah introduced the Carathéodory approximation scheme for vector-valued stochastic differential equations under the G-Brownian motion. It is shown that the Carathéodory approximate solutions converge to the unique solution of the equation. The existence and uniqueness theorem for G-SDEs is established by using the stated Lipschitz method and the linear growth conditions

$$X(t) = X(0) + \int_0^t f(s, X(s))\, ds + \int_0^t g(s, X(s))\, d\langle B\rangle(s) + \int_0^t h(s, X(s))\, dB(s), \quad t \in [0, T]. \tag{1.1}$$

The existence and the uniqueness of the solution $X(t)$ for G-SDEs (1.1) under different conditions were proved in [1, 4–10, 15, 17] and [19, 24–29].

In this paper, we study the existence, uniqueness and stability of the solution for the following stochastic differential system driven by the G-Brownian motion (SG-DEs):

$$\begin{cases} X_1(t) = X_1(0) + \int\limits_0^t f_{1,1}(s, X_1(s), \dots, X_n(s))\, ds \\ \qquad\qquad + \int\limits_0^t f_{2,1}(s, X_1(s), \dots, X_n(s))\, d\langle B\rangle(s) + \int\limits_0^t f_{3,1}(s, X_1(s), \dots, X_n(s))\, dB(s), \\ \quad \vdots \\ X_n(t) = X_n(0) + \int\limits_0^t f_{1,n}(s, X_1(s), \dots, X_n(s))\, ds \\ \qquad\qquad + \int\limits_0^t f_{2,n}(s, X_1(s), \dots, X_n(s))\, d\langle B\rangle(s) + \int\limits_0^t f_{3,n}(s, X_1(s), \dots, X_n(s))\, dB(s), \end{cases} \tag{1.2}$$

where $(X_1(0), \dots, X_n(0))$ is the given initial condition, $(\langle B(t)\rangle)_{t\geq 0}$ is the quadratic variation process of the G-Brownian motion $(B(t))_{t\geq 0}$, and all the coefficients $f_{i,j}(t, x_1, \dots, x_n)$ for $1 \leq i \leq 3$ and $1 \leq j \leq n$ satisfy the Lipschitz and the linear growth conditions with respect to $(x_1, \dots, x_n)$. These results are obtained by using the technics adopted by F. Faizullah [5] in the case where the Lipschitz and the linear growth constants are time dependant.

The article is organized as follows. In Section 2, we provide some results and definitions necessary to understand the content of this work. Section 3 is devoted to the existence and uniqueness of the solution of system (1.2) using the Carathéodory approximation scheme. In the last Section 4 we give a result of the stability.

2 Preliminaries

In this section, we recall some basic notions, definitions and theorems necessary to understand the content of this work. For more details concerning this section see, e.g., [5, 10–12, 15, 26–28] and [24].

Let Ω be a given non-empty set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω such that any arbitrary constant $c \in \mathcal{H}$ and if $X \in \mathcal{H}$, then $|X| \in \mathcal{H}$. We consider that $\mathcal{H}$ is the space of random variables.

Definition 2.1. A functional $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ is called sublinear expectation, if for all X, Y in $\mathcal{H}$, c in $\mathbb{R}$ and $\lambda \geq 0$, the following properties are satisfied:

(i) (Monotonicity): if $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;

(ii) (Constant preserving): $\mathbb{E}[c] = c$;

(iii) (Sub-additivity): $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$;

(iv) (Positive homogeneity): $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space.

We assume that if $X_1, X_2, \dots, X_n \in \mathcal{H}$, then $\varphi(X_1, X_2, \dots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{\ell,\mathrm{Lip}}(\mathbb{R}^n)$, the set of functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ satisfying the condition:

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y| \ \text{ for all } \ x, y \in \mathbb{R}^n,$$

where C is a positive constant and $m \in \mathbb{N}^*$ depending only on φ.

Definition 2.2. Let X, Y be two n-dimensional random vectors defined on nonlinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$, respectively. They are called identically distributed, denoted by $X \overset{d}{=} Y$, if

$$\mathbb{E}_2[\varphi(Y)] = \mathbb{E}_1[\varphi(X)] \ \text{ for each } \ \varphi \in C_{\ell,\mathrm{Lip}}(\mathbb{R}^n).$$

Definition 2.3. In a sublinear expectation space $(\Omega,\mathcal{H},\mathbb{E})$, a random vector $Y\in\mathcal{H}^n$ is said to be independent of another random vector $X\in\mathcal{H}^m$ if

$$\mathbb{E}[\varphi(X,Y)]=\mathbb{E}\big[\mathbb{E}[\varphi(x,Y)]_{x=X}\big]\ \ \forall\,\varphi\in C_{\ell,\mathrm{Lip}}(\mathbb{R}^m\times\mathbb{R}^n).$$

$\widetilde{X}$ is called an independent copy of X if $\widetilde{X}\overset{d}{=}X$ and $\widetilde{X}$ is independent of X.

Let Γ be a closed bounded and convex subset of $\mathbb{S}_+(d)$, the set of positive and symmetric d-dimensional matrices. Let

$$\Sigma=\big\{\gamma\gamma^{\mathrm{Tr}}:\ \gamma\in\Gamma\big\}$$

and let $G:\mathbb{S}_+(d)\to\mathbb{R}$ is defined by

$$G(A)=\frac{1}{2}\sup_{\gamma\in\Gamma}\mathrm{Tr}(\gamma\gamma^{\mathrm{Tr}}A)$$

Definition 2.4. In a sublinear expectation space $(\Omega,\mathcal{H},E)$, a d-dimensional vector of random variables $X\in\mathcal{H}^d$ is G-normal distributed if for each $\varphi\in C_{\ell,\mathrm{Lip}}(\mathbb{R}^d)$, the function $u(t,x)=\mathbb{E}(\varphi(x+\sqrt{t}X))$ is the unique viscosity solution of the following parabolic equation called the G-heat equation:

$$\begin{cases}\dfrac{\partial u}{\partial t}=G(D^2u), & \\ u(0,x)=\varphi(x), & \end{cases}\qquad (t,x)\in\mathbb{R}_+\times\mathbb{R}^d,$$

where $D^2u=(\partial^2_{x_ix_j}u)^d_{i,j}$ is the Hessian matrix of u.

Remark 2.5. In fact, if $d=1$, we have $G(\alpha)=\frac{1}{2}(\overline{\sigma}^2\alpha^+-\underline{\sigma}^2\alpha^-)$, where $\overline{\sigma}^2=\mathbb{E}[X^2]$, $\underline{\sigma}^2=-\mathbb{E}[-X^2]$, $\alpha^+=\max(\alpha,0)$ and $\alpha^-=\max\{-\alpha,0\}$ (for more details see [24]). We write $X\sim\mathcal{N}(0;[\underline{\sigma}^2,\overline{\sigma}^2])$.

Definition 2.6. A process $(B(t))_{t\geq 0}$ in a sublinear expectation space (Ω,H,E) is called a G-Brownian motion if the following properties are satisfied:

(i) $B(0)=0$;

(ii) for each t, $s\geq 0$, the increment $B(t+s)-B(t)$ is $N(0;[\underline{\sigma}^2s,\overline{\sigma}^2s]$-distributed and is independent of $(B(t_1),\dots,B(t_n))$ for each $n\in\mathbb{N}$ and $0\leq t_1\leq\cdots\leq t_n\leq t$.

We denote by $\Omega=C_0(\mathbb{R})$ the space of all $\mathbb{R}$-valued continuous functions ω defined on $\mathbb{R}_+$ such that $\omega(0)=0$, equipped with the distance

$$\rho(\omega_1,\omega_2)=\sum_{i=1}^{\infty}2^{-i}\max_{t\in[0,i]}\big[\big|(\omega_1(t)-\omega_2(t))\wedge 1\big|\big].$$

For each fixed $T>0$, let

$$\Omega_T=\big\{\omega(\,.\,_{\wedge T}),\ \ \omega\in\Omega\big\},$$

$$\mathrm{Lip}(\Omega_T)=\Big\{\varphi(B(t_1),\dots,B(t_m)),\ \ m\geq 1,\ \ t_1,\dots,t_m\in[0,T],\ \ \varphi\in C_{\ell,\mathrm{Lip}}(\mathbb{R}^m)\Big\},$$

where

$$\mathrm{Lip}(\Omega)=\bigcup_{n=1}^{\infty}\mathrm{Lip}(\Omega_n).$$

In [24], Peng constructs a sublinear expectation $\mathbb{E}$ on $(\Omega,\mathrm{Lip}(\Omega))$ under which the canonical process $(B(t))_{t\geq 0}$ (i.e., $B(t,\omega)=\omega(t)$) is a G-Brownian motion. In what follows, we consider this G-Brownian motion.

We denote by $L^p_G(\Omega_T)$, $p\geq 1$, the completion of $\mathrm{Lip}(\Omega_T)$ under the norm $||X||_p=(\mathbb{E}[|X|^p])^{\frac{1}{p}}$. Similarly, we denote by $L^p_G(\Omega)$ the completion space of $\mathrm{Lip}(\Omega)$. It was shown in [28] and [24] that there exists a family of probability measures $\mathcal{P}$ on Ω such that

$$\mathbb{E}[X]=\sup_{P\in\mathcal{P}}E^P[X]\ \ \text{for}\ \ X\in L^1_G(\Omega),$$

where E^P stands for the linear expectation under the probability P. We say that a property holds quasi surely (q.s.) if it holds for each $P \in \mathcal{P}$.

For a finite partition of $[0,T]$, $\pi_T = \{t_0, t_1, \dots, t_N\}$, we set

$$\mu(\pi_T) = \max\big\{|t_{i+1} - t_i|, \ \ 0 \le i \le N-1\big\}.$$

Consider the collection $M_G^{p,0}(0,T)$ of simple processes defined by

$$\eta(t,\omega) = \sum_{i=0}^{N-1} \xi_i(\omega) I_{[t_i;t_{i+1}[}(t),$$

where

$$\xi_i \in L_G^p(\Omega_{t_i}), \ \ 0 \le i \le N-1 \ \text{ and } \ p \ge 1.$$

The completion of $M_G^{p,0}(0,T)$ under the norm

$$\|\eta\| = \left\{\frac{1}{T}\int_0^T \mathbb{E}\big[|\eta(t)|^p\big]\,dt\right\}^{\frac{1}{p}}$$

is denoted by $M_G^p(0,T)$. Note that

$$M_G^q(0,T) \subset M_G^p(0,T) \ \text{ for } \ 1 \le p \le q.$$

Definition 2.7. For each $\eta \in M_G^{2,0}(0,T)$, the G-Itô integral is defined by

$$I(\eta) = \int_0^T \eta(v)\,dB(v) = \sum_{i=0}^{N-1} \xi_i(B(t_{i+1}) - B(t_i)).$$

The mapping $\eta \longmapsto I(\eta)$ can be extended continuously to $M_G^2(0,T)$.

Definition 2.8. The increasing continuous process $(\langle B\rangle(t))_{t\ge0}$ with $\langle B\rangle(0) = 0$ defined by

$$\langle B\rangle(t) = B^2(t) - 2\int_0^t B(v)\,dB(v)$$

is called the quadratic variation process of $(B(t))_{t\ge0}$. Note that $\langle B\rangle(t)$ can be regarded as the limit in $L_G^2(\Omega_t)$ of $\sum_{j=1}^{N}(B(t_{i+1}^N) - B(t_i^N))^2$, where $\pi_T^N = \{t_0^N, t_1^N, \dots, t_k^N\}$ is a sequence of partitions of $[0,T]$ such that $\mu(\pi_T^N)$ tends to 0 when N goes to infinity.

The following Burkholder–Davis–Gundy inequalities play an important role in the study of our system (see [10] and [29]).

Lemma 2.9. *Let $p \ge 1$, $\eta \in M_G^p(0,T)$ and $0 \le s \le t \le T$. Then*

$$\mathbb{E}\bigg[\sup_{s\le u\le t}\bigg|\int_s^u \eta(r)\,d\langle B\rangle(r)\bigg|^p\bigg] \le C_1(t-s)^{p-1}\int_s^t \mathbb{E}\big[|\eta(u)|^p\big]\,du,$$

where $C_1 > 0$ is a constant independent of η.

Lemma 2.10. *Let $p \ge 2$, $\eta \in M_G^p(0,T)$ and $0 \le s \le t \le T$. Then*

$$\mathbb{E}\bigg[\sup_{s\le u\le t}\bigg|\int_s^u \eta(r)\,dB(r)\bigg|^p\bigg] \le C_2|t-s|^{\frac{p}{2}-1}\int_s^t \mathbb{E}\big[|\eta(u)|^p\big]\,du,$$

where $C_2 > 0$ is a constant independent of η.

3 Existence and uniqueness results

In this section, we are interested in the study of the existence and uniqueness of the solution to the SG-SDE (1.2), where the initial condition $(X_1(0),\dots,X_n(0)) \in (\mathbb{R}^d)^n$ is a given constant and $f_{i,j}(t,x_1,\dots,x_n) \in M_G^2(0,T;(\mathbb{R}^d)^n)$ for $0 \le i \le 3$ and $1 \le j \le n$.

For system (1.2), the Carathéodory approximation scheme is given as follows. For any integer $k \ge 1$, we define

$$(X_1^k(t),\dots,X_n^k(t)) = (X_1(0),\dots,X_n(0)), \text{ if } t \in]-1,0],$$

and for $t \in]0,T]$, we have

$$\begin{cases} X_1^k(t) = X_1(0) + \int\limits_0^t f_{1,1}\Big(s, X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,ds \\ \qquad\qquad + \int\limits_0^t f_{2,1}\Big(s, X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,d\langle B\rangle(s) \\ \qquad\qquad + \int\limits_0^t f_{3,1}\Big(s, X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,dB(s), \\ \qquad \vdots \\ X_n^k(t) = X_n(0) + \int\limits_0^t f_{1,n}\Big(s, X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,ds \\ \qquad\qquad + \int\limits_0^t f_{2,n}\Big(s, X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,d\langle B\rangle(s) \\ \qquad\qquad + \int\limits_0^t f_{3,n}\Big(s, X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,dB(s). \end{cases} \tag{3.1}$$

We assume the following assumptions (A1) and (A2) for $f_{i,j}$, $0 \le i \le 3$ and $1 \le j \le n$:

(A1)

$$\big|f_{i,j}(t,x_1,x_2,\dots,x_n)\big|^2 \le g(t)\Big(1+\sum_{j=1}^n |x_j|^2\Big)$$

for each $x_1,\dots,x_n \in \mathbb{R}^d$ and $t \in [0,T]$, where g is a positive and continuous function on $[0,T]$.

(A2)

$$\big|f_{i,j}(t,x_1,\dots,x_n) - f_{i,j}(t,y_1,\dots,y_n)\big|^2 \le h(t)\Big(\sum_{j=1}^n |y_j - x_j|^2\Big)$$

for each $x_1,y_1,\dots,x_n,y_n \in \mathbb{R}^d$ and $t \in [0,T]$, where h is a positive and continuous function on $[0,T]$.

In the sequel, the space of processes in $(M_G^2(0,T;\mathbb{R}^d))^n$ will be equipped with the norm

$$\|(X_1,\dots,X_n)\| = \mathbb{E}^{\frac{1}{2}}\Big[\sup_{0\le t\le T}\Big(\sum_{j=1}^n |X_{j(t)}|^2\Big)\Big].$$

We note that this is a Banach space.

Now, we give first main result of this work.

Theorem 3.1. *Under the assumptions* (A1) *and* (A2)*, system* (1.2) *has a unique solution q.s.,*

$$(X_1(t),\dots,X_n(t)) \in (M_G^2(0,T;\mathbb{R}^d))^n.$$

In order to prove this theorem, we need some important lemmas.

Lemma 3.2. *For all integers n, $k \ge 1$ and $0 \le s < t \le T$, we have*

$$\sup_{0\le t\le T} \mathbb{E}\Big[\sum_{j=1}^{n} |X_j^k(t)|^2\Big] \le K_n \exp\Big(C_n \int_0^T g(t)\,dt\Big),$$

where

$$K_n = 1 + 4\sum_{j=1}^{n} \mathbb{E}\big[|X_j(0)|^2\big], \quad C_n = 4n(T + C_1T + C_2).$$

Proof. By using (3.1) and the fact that $\big(\sum_{j=1}^{n} a_j\big)^2 \le n\sum_{j=1}^{n} a_j^2$ for each positive constants a_j, $1 \le j \le n$, for all $t \in [0,T]$, we have

$$|X_j^k(t)|^2 \le 4|X_j(0)|^2 + 4\Big|\int_0^t f_{1,j}\Big(s, X_1^k\Big(s-\frac{1}{k}\Big), \dots, X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,ds\Big|^2$$

$$+4\Big|\int_0^t f_{2,j}\Big(s, X_1^k\Big(s-\frac{1}{k}\Big), \dots, X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,d\langle B\rangle(s)\Big|^2 + 4\Big|\int_0^t f_{3,j}\Big(s, X_1^k\Big(s-\frac{1}{k}\Big), \dots, X_n^k\Big(s-\frac{1}{k}\Big)\Big)\,dB(s)\Big|^2,$$

which, due to Lemmas 2.9 and 2.10, the G-Hölder inequality and the assumption (A1), implies that

$$\begin{aligned}\sup_{0\le v\le t} \mathbb{E}\big[|X_j^k(v)|^2\big] &\le 4\mathbb{E}\big[|X_j(0)|^2\big] + 4(T + C_1T + C_2)\int_0^t g(s)\Big(1 + \mathbb{E}\Big[\sum_{j=1}^{n}\Big|X_j^k\Big(s-\frac{1}{k}\Big)\Big|^2\Big]\Big)\,ds \\ &\le 4\mathbb{E}\big[|X_j(0)|^2\big] + 4(T + C_1T + C_2)\int_0^t g(s)\Big(1 + \sup_{0\le v\le s}\mathbb{E}\Big[\sum_{j=1}^{n}|X_j^k(v)|^2\Big]\Big)\,ds.\end{aligned}$$

Thus

$$1 + \sup_{0\le v\le t}\mathbb{E}\Big[\sum_{j=1}^{n}|X_j^k(t|^2\Big] \le 1 + 4\sum_{j=1}^{n}\mathbb{E}\big[|X_j(0)|^2\big] + C_n\int_0^t g(s)\Big(1 + \sup_{0\le v\le s}\mathbb{E}\Big[\sum_{j=1}^{n}|X_j^k(v)|^2\Big]\Big)\,ds,$$

where $C_n = 4n(T + C_1T + C_2)$. Applying Gronwall's lemma, we conclude that

$$1 + \sup_{0\le v\le t}\mathbb{E}\Big[\sum_{j=1}^{n}|X_j^k(v)|^2\Big] \le K_n \exp\Big(C_n\int_0^t g(s)\,ds\Big)$$

and, consequently,

$$\sup_{0\le t\le T}\mathbb{E}\Big[\sum_{j=1}^{n}|X_j^k(t)|^2\Big] \le K_n \exp\Big(C_n\int_0^T g(t)\,dt\Big). \qquad \square$$

Lemma 3.3. *For all integers n, $k \ge 1$ and $0 \le s < t \le T$, we have*

$$\mathbb{E}\Big[\sum_{j=1}^{n}|X_j^k(t) - X_j^k(s)|^2\Big] \le L_n[G(t) - G(s)],$$

where

$$G(t) = \int_0^t g(s)\,ds \ \text{ and } \ L_n = \frac{3}{4}C_n\Big[1 + K_n\exp\Big(C_n\int_0^T g(t)\,dt\Big)\Big].$$

Proof. We have

$$X_j^k(t) - X_j^k(s) = \int\limits_s^t f_{1,j}\Big(w, X_1^k\Big(w-\frac{1}{k}\Big), \dots, X_n^k\Big(w-\frac{1}{k}\Big)\Big)\,dw$$

$$+\int\limits_s^t f_{2,j}\Big(w, X_1^k\Big(w-\frac{1}{k}\Big), \dots, X_n^k\Big(w-\frac{1}{k}\Big)\Big)\,d\langle B\rangle(w)+\int\limits_s^t f_{3,j}\Big(w, X_1^k\Big(w-\frac{1}{k}\Big), \dots, X_n^k\Big(w-\frac{1}{k}\Big)\Big)\,dB(w)$$

and so, for each $0 \le s \le v \le u \le t \le T$, we have

$$\mathbb{E}\Big[\sup_{s\le v\le u\le t} |X_j^k(u) - X_j^k(v)|^2\Big] \le 3\mathbb{E}\Big[\sup_{s\le v\le u\le t}\Big|\int\limits_v^u f_{1,j}\Big(w, X_1^k\Big(w-\frac{1}{k}\Big), \dots, X_n^k\Big(w-\frac{1}{k}\Big)\Big)\,dw\Big|^2\Big]$$

$$+ 3\mathbb{E}\Big[\sup_{s\le v\le u\le t}\Big|\int\limits_v^u f_{2,j}\Big(w, X_1^k\Big(w-\frac{1}{k}\Big), \dots, X_n^k\Big(w-\frac{1}{k}\Big)\Big)\,d\langle B\rangle(w)\Big|^2\Big]$$

$$+ 3\mathbb{E}\Big[\sup_{s\le v\le u\le t}\Big|\int\limits_v^u f_{3,j}\Big(w, X_1^k\Big(w-\frac{1}{k}\Big), \dots, X_n^k\Big(w-\frac{1}{k}\Big)\Big)\,dB(w)\Big|^2\Big].$$

Owing to Lemmas 2.9, 2.10 and the assumption (A1), we obtain

$$\mathbb{E}\Big[\sup_{s\le v\le u\le t} |X_j^k(u) - X_j^k(v)|^2\Big] \le 3T\int\limits_s^t \mathbb{E}\Big[\Big|f_{1,j}(w, X_j^k\Big(w-\frac{1}{k}\Big), \dots, X_n^k\Big(w-\frac{1}{k}\Big)\Big)\Big|^2\Big]\,dw$$

$$+ 3C_1T\int\limits_s^t \mathbb{E}\Big[\Big|f_{2,j}(w, X_j^k\Big(w-\frac{1}{k}\Big), \dots, X_n^k\Big(w-\frac{1}{k}\Big)\Big)\Big|^2\Big]\,dw$$

$$+ 3C_2\int\limits_s^t \mathbb{E}\Big[\Big|f_{3,j}(w, X_j^k\Big(w-\frac{1}{k}\Big), \dots, X_n^k\Big(w-\frac{1}{k}\Big)\Big)\Big|^2\Big]\,dw$$

$$\le 3(T + C_1T + C_2)\int\limits_s^t g(w)\Big(1 + \mathbb{E}\Big[\sum_{j=1}^n\Big|X_j^k\Big(w-\frac{1}{k}\Big)\Big|^2\Big]\Big)\,dw$$

$$\le 3(T + C_1T + C_2)[G(t) - G(s)] + 3(T + C_1T + C_2)\int\limits_s^t g(w)\mathbb{E}\Big[\sum_{j=1}^n\Big|X_j^k\Big(w-\frac{1}{k}\Big)\Big|^2\Big]\,dw.$$

Using Lemma 3.2, we get

$$\mathbb{E}\Big[\sup_{s\le v\le u\le t} |X_j^k(u) - X_j^k(v)|^2\Big] \le 3(T + C_1T + C_2)\Big[1 + K_n\exp\Big(C_n\int\limits_0^T g(t)\,dt\Big)\Big]\,[G(t) - G(s)].$$

Thus

$$\sum_{j=1}^n\mathbb{E}\Big[\sup_{s\le v\le u\le t} |X_j^k(u) - X_j^k(v)|^2\Big] \le 3n(T + C_1T + C_2)\Big[1 + K_n\exp\Big(C_n\int\limits_0^T g(t)\,dt\Big)\Big]\,[G(t) - G(s)].$$

Then

$$\sum_{j=1}^n\mathbb{E}\big[|X_j^k(t) - X_j^k(s)|^2\big] \le L_n[G(t) - G(s)],$$

where

$$L_n = \frac{3}{4}\, C_n \left[1 + K_n \exp\left(C_n \int_0^T g(t)\, dt\right)\right],$$

which proves the desired result. □

Proof of Theorem 3.1. We will prove the theorem in three steps.

Step 1: Suppose that $(X_1(t), \dots, X_n(t))$ and $(Y_1(t), \dots, Y_n(t))$ are two solutions of system (1.2) with the initial conditions $(X_1(0), \dots, X_n(0))$ and $(Y_1(0), \dots, Y_n(0))$, respectively. Then we for $1 \le j \le n$, we have

$$\begin{aligned} |Y_j(t) - X_j(t)|^2 \le{}& 4|X_j(0) - Y_j(0)|^2 \\ &+ 4\Big|\int_0^t f_{1,j}(s, X_1(s), \dots, X_n(s)) - f_{1,j}(s, Y_1(s), \dots, Y_n(s))\, ds\Big|^2 \\ &+ 4\Big|\int_0^t f_{2,j}(s, X_1(s), \dots, X_n(s)) - f_{2,j}(s, Y_1(s), \dots, Y_n(s))\, d\langle B\rangle(s)\Big|^2 \\ &+ 4\Big|\int_0^t f_{3,j}(s, X_1(s), \dots, X_n(s)) - f_{3,j}(s, Y_1(s), \dots, Y_n(s))\, dB(s)\Big|^2. \end{aligned}$$

Now, by using Lemmas 2.9, 2.10 and the assumption (A2), for $0 \le r \le t \le T$, we have

$$\begin{aligned} &\mathbb{E}\left[\Big|\int_0^r \big(f_{1,j}(s, X_1(s), \dots, X_n(s)) - f_{1,j}(s, Y_1(s), \dots, Y_n(s))\big)\, ds\Big|^2\right] \\ &\qquad \le T\int_0^t \mathbb{E}\Big[\big|f_{1,j}(s, X_1(s), \dots, X_n(s)) - f_{1,j}(s, Y_1(s), \dots, Y_n(s))\big|^2\Big]\, ds \\ &\qquad \le T\int_0^t h(s)\mathbb{E}\Big[\Big(\sum_{j=1}^n |Y_j(s) - X_j(s)|^2\Big)\Big]\, ds, \end{aligned}$$

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\le r\le t}\Big|\int_0^r f_{2,j}(s, X_1(s), \dots, X_n(s)) - f_{2,j}(s, Y_1(s), \dots, Y_n(s))\, d\langle B\rangle(s)\Big|^2\right] \\ &\qquad \le C_1 T\int_0^t \mathbb{E}\Big[\big|f_{2,j}(s, X_1(s), \dots, X_n(s)) - f_{2,j}(s, Y_1(s), \dots, Y_n(s))\big|^2\Big]\, ds \\ &\qquad \le C_1 T\int_0^t h(s)\mathbb{E}\Big[\Big(\sum_{j=1}^n |Y_j(s) - X_j(s)|^2\Big)\Big]\, ds, \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\le r\le t}\Big|\int_0^r f_{3,j}(s, X_1(s), \dots, X_n(s)) - f_{3,j}(s, Y_1(s), \dots, Y_n(s))\, dB(s)\Big|^2\right] \\ &\qquad \le C_2\int_0^t \mathbb{E}\Big[\big|f_{3,j}(s, X_1(s), \dots, X_n(s)) - f_{3,j}(s, Y_1(s), \dots, Y_n(s))\big|^2\Big]\, ds \end{aligned}$$

$$\leq C_2 \int_0^t h(s)\mathbb{E}\Big[\Big(\sum_{j=1}^n |Y_j(s) - X_j(s)|^2\Big)\Big]\, ds.$$

Therefore,

$$\mathbb{E}\Big[\sup_{0\leq r\leq t} |Y_j(r) - X_j(r)|^2\Big]$$
$$\leq 4|Y_j(0) - X_j(0)|^2 + 4(T + C_1T + C_2)\int_0^t h(s)\mathbb{E}\Big[\sum_{j=1}^n |Y_j(s) - X_j(s)|^2\Big]\, ds.$$

We obtain

$$\mathbb{E}\Big[\sup_{0\leq r\leq t}\Big(\sum_{j=1}^n |Y_j(r) - X_j(r)|^2\Big)\Big] \leq 4\sum_{j=1}^n |Y_j(0) - X_j(0)|^2 + C_n\int_0^t h(s)\mathbb{E}\Big[\sum_{j=1}^n |Y_j(s) - X_j(s)|^2\Big]\, ds.$$

Using Gronwall's lemma, we get

$$\mathbb{E}[\sup_{0\leq r\leq t}\Big(\sum_{j=1}^n |Y_j(r) - X_j(r)|^2\Big)\Big] \leq 4\sum_{j=1}^n |Y_j(0) - X_j(0)|^2 \exp\Big(C_n\int_0^t h(s)\, ds\Big).$$

Now, taking

$$(X_1(0), \dots, X_n(0)) = (Y_1(0), \dots, Y_n(0)),$$

we can see that for $t = T$,

$$\mathbb{E}\Big[\sup_{0\leq r\leq T}\Big(\sum_{j=1}^n |Y_j(r) - X_j(r)|^2\Big)\Big] = 0,$$

which implies

$$(X_1(t), \dots, X_n(t)) = (Y_1(t), \dots, Y_n(t)) \text{ q.s. for each } t \in [0, T].$$

Step 2: We now prove that $(X_1^k(t), \dots, X_n^k(t))_{k\geq 1}$ in $(M_G^2(0, T; \mathbb{R}^d))^n$ is a Cauchy sequence for each $t \in [0, T]$. By the same arguments as those used in the previous step, for each $\ell > k$, we have

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big(\sum_{j=1}^n |X_j^\ell(t) - X_j^k(t)|^2\Big)\Big] \leq \frac{3}{4}\, C_n\int_0^T h(s)\mathbb{E}\Big[\Big(\sum_{j=1}^n \Big|X_j^\ell\Big(s - \frac{1}{\ell}\Big) - X_j^k\Big(s - \frac{1}{k}\Big)\Big|^2\Big)\Big]\, ds.$$

Since

$$\mathbb{E}\Big[\sum_{j=1}^n \Big|X_j^\ell\Big(s - \frac{1}{\ell}\Big) - X_j^k\Big(s - \frac{1}{k}\Big)\Big|^2\Big]$$
$$\leq 2\mathbb{E}\Big[\sum_{j=1}^n \Big|X_j^\ell\Big(s - \frac{1}{\ell}\Big) - X_j^k\Big(s - \frac{1}{\ell}\Big)\Big|^2\Big] + 2\mathbb{E}\Big[\sum_{j=1}^n \Big|X_j^k\Big(s - \frac{1}{\ell}\Big) - X_j^k\Big(s - \frac{1}{k}\Big)\Big|^2\Big]$$
$$\leq 2\mathbb{E}\Big[\sup_{0\leq u\leq s}\Big(\sum_{j=1}^n |X_j^\ell(u) - X_j^k(u)|^2\Big)\Big] + 2\mathbb{E}\Big[\Big(\sum_{j=1}^n \Big|X_j^k\Big(s - \frac{1}{\ell}\Big) - X_j^k\Big(s - \frac{1}{k}\Big)\Big|^2\Big)\Big],$$

using Lemma 3.3 we get

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big(\sum_{j=1}^n |X_j^\ell(t) - X_j^k(t)|^2\Big)\Big]$$
$$\leq \frac{3}{2}\, C_n\int_0^T h(r)\mathbb{E}\Big[\sup_{0\leq u\leq r}\Big(\sum_{j=1}^n |X_j^\ell(u) - X_j^k(u)|^2\Big)\Big]\, dr + \frac{3}{2}\, C_nL_n\Big[G\Big(s - \frac{1}{\ell}\Big) - G\Big(s - \frac{1}{k}\Big)\Big]\int_0^T h(r)\, dr.$$

Thus, by Gronwall's lemma,

$$\mathbb{E}\Big[\sup_{0\le t\le T}\Big(\sum_{j=1}^{n}|X_j^{\ell}(t)-X_j^{k}(t)|^2\Big)\Big]\le M_n\Big(\frac{1}{k}-\frac{1}{\ell}\Big)\exp\Big(\frac{3}{2}\,C_n\int_0^T h(s)\,ds\Big),$$

where

$$M_n=\frac{3}{2}\,TC_nL_n\sup_{0\le t\le T}[g(t)]\sup_{0\le t\le T}[h(t)],$$

which means that $(X_1^k(t),\dots,X_n^k(t))_{k\ge1}$ is a Cauchy sequence.

Step 3: Here we prove that the limit $(X_1(t),\dots,X_n(t))$ in $(M_G^2(0,T;\mathbb{R}^d))^n$ of $(X_1^k(t),\dots,X_n^k(t))$ is the solution of system (1.2). For the existence, let the initial condition $(X_1(0),\dots,X_n(0))\in(\mathbb{R}^d)^n$ be a given constant.

This results in

$$\begin{aligned}|X_j(u)-X_j^k(u)|^2\le 3\Big|\int_0^u f_{1,j}\Big(s,X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)-f_{1,j}(s,X_1(s),\dots,X_n(s))\,ds\Big|^2\\
+3\Big|\int_0^u f_{2,j}\Big(s,X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)-f_{2,j}(s,X_1(s),\dots,X_n(s))\,d\langle B\rangle(s)\Big|^2\\
+3\Big|\int_0^u f_{3,j}\Big(s,X_1^k\Big(s-\frac{1}{k}\Big),\dots,X_n^k\Big(s-\frac{1}{k}\Big)\Big)-f_{3,j}(s,X_1(s),\dots,X_n(s))\,dB(s)\Big|^2.\end{aligned}$$

Using Lemmas 2.9, 2.10 and the assumption (A2), we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le u\le T}\big(|X_j^k(u)-X_j(u)|^2\big)\Big]&\le 3(T+C_1T+C_2)\int_0^T h(s)\mathbb{E}\Big[\sum_{j=1}^n\Big|X_j^k\Big(s-\frac{1}{k}\Big)-X_j(s)\Big|^2\Big]\,ds\\
&\le 6(T+C_1T+C_2)\int_0^T h(s)\mathbb{E}\Big[\sum_{j=1}^n\Big|X_j^k\Big(s-\frac{1}{k}\Big)-X_j^k(s)\Big|^2\Big]\,ds\\
&\quad+6(T+C_1T+C_2)\int_0^T h(s)\mathbb{E}\Big[\sum_{j=1}^n|X_j^k(s)-X_j(s)|^2\Big]\,ds.\end{aligned}$$

Thus, using Lemma 3.3,

$$\mathbb{E}\Big[\sup_{0\le u\le T}\big(|X_j^k(u)-X_j(u)|^2\big)\Big]\le\frac{M_n}{nk}+\frac{3C_n}{2n}\int_0^T\mathbb{E}\Big[\sup_{0\le u\le s}\Big(\sum_{j=1}^n|X_j^k(u)-X_j(u)|^2\Big)\Big]\,ds,$$

which implies that

$$\mathbb{E}\Big[\sup_{0\le u\le T}\Big(\sum_{j=1}^n\big(|X_j^k(u)-X_j(u)|^2\big)\Big)\Big]\le\frac{M_n}{k}+\frac{3}{2}\,C_n\int_0^T h(s)\mathbb{E}\Big[\sup_{0\le u\le s}\sum_{j=1}^n\big(|X_j^k(u)-X_j(u)|^2\big)\Big]\,ds.$$

Applying Gronwall's lemma again, we get directly

$$\mathbb{E}\Big[\sup_{0\le u\le T}\Big(\sum_{j=1}^n|X_j^k(u)-X_j(u)|^2\Big)\Big]\le\frac{M_n}{k}\exp\Big(\frac{3}{2}\,C_n\int_0^T h(s)\,ds\Big),$$

which shows our result. □

4 Stability result

In this section, we prove another important result on the stability of the following G-SDEs depending on a parameter ε ($\varepsilon \geq 0$) (for more information, see, e.g., [13, 15, 27, 28, 30]):

$$\begin{cases} X_1^\varepsilon(t) = X_1^\varepsilon(0) + \int_0^t f_{1,1}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s))\, ds \\ \qquad\qquad + \int_0^t f_{2,1}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s))\, d\langle B\rangle(s) + \int_0^t f_{3,1}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s))\, dB(s), \\ \qquad \vdots \\ X_n^\varepsilon(t) = X_n^\varepsilon(0) + \int_0^t f_{1,n}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s))\, ds \\ \qquad\qquad + \int_0^t f_{2,n}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s))\, d\langle B\rangle(s) + \int_0^t f_{3,n}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s))\, dB(s). \end{cases}$$

We assume the following assumptions (B1), (B2) and (B3) for $f_{i,j}^\varepsilon$, $0 \leq i \leq 3$ and $1 \leq j \leq n$:

(B1)
$$|f_{i,j}^\varepsilon(t, x_1, \dots, x_n)|^2 \leq g(t)\Big(1 + \sum_{j=1}^n |x_j|^2\Big)$$
for each $x_1,\ x_2, \dots, x_n \in \mathbb{R}^d$ and $t \in [0,T]$, where g is a positive and continuous function on $[0,T]$.

(B2)
$$\big|f_{i,j}^\varepsilon(t, x_1, \dots, x_n) - f_{i,j}^\varepsilon(t, y_1, \dots, y_n)\big|^2 \leq h(t)\Big(\sum_{j=1}^n |y_j - x_j|^2\Big)$$
for each $x_1, y_1, \dots, x_n, y_n \in \mathbb{R}^d$ and $t \in [0,T]$, where h is a positive and continuous function on $[0,T]$.

(B3) (i) $\forall\, t \in [0,T]$,
$$\lim_{\varepsilon \to 0} \int_0^t \mathbb{E}\Big[\big|f_{i,j}^\varepsilon(s, X_1^0(s), \dots, X_n^0(s)) - f_{i,j}^0(s, X_1^0(s), \dots, X_n^0(s))\big|\Big]\, ds = 0;$$

(ii)
$$\lim_{\varepsilon \to 0} (X_1^\varepsilon(0), \dots, X_n^\varepsilon(0)) = (X_1^0(0), \dots, X_n^0(0)).$$

Remark 4.1. The assumptions (B1) and (B2) guarantee, for any $\varepsilon \geq 0$, the existence of a unique solution
$$(X_1^\varepsilon(t), \dots, X_n^\varepsilon(t)) \in (M_G^2(0,T;\mathbb{R}^d))^n$$
of our system, while the assumption (B3) allows us to deduce the stability theorem for the system.

The following lemmas are very important, they will be used in the upcoming result. For the proofs see [15].

Lemma 4.2. *For every $p \geq 1$ and for any $T > 0$ and $\eta \in M_G^p(0,T)$, we have*

$$\mathbb{E}\Big[\Big|\int_0^T \eta(t)\,dt\Big|^p\Big] \leq T^{p-1}\int_0^T \mathbb{E}\big[|\eta(t)|^p\big]\,dt,$$

$$\mathbb{E}\Big[\Big|\int_0^T \eta(t)\,d\langle B\rangle(t)\Big|^p\Big] \leq T^{p-1}\int_0^T \mathbb{E}\big[|\eta(t)|^p\big]\,dt.$$

Lemma 4.3. *For every $p \geq 2$, there exists a positive constant C_p such that, for any $T > 0$ and $\eta \in M_G^p(0,T)$,*

$$\mathbb{E}\Big[\Big|\int_0^T \eta(t)\,dB(t)\Big|^p\Big] \leq C_p T^{\frac{p}{2}-1}\int_0^T \mathbb{E}\big[|\eta(t)|^p\big]\,dt.$$

Now, we present our second main result of this work.

Theorem 4.4. *Under the assumptions* (B1), (B2) *and* (B3), *we have*

$$\forall\, t \in [0,T],\quad \lim_{\varepsilon\to 0}\mathbb{E}\Big[\sum_{j=1}^n |X_j^\varepsilon(t) - X_j^0(t)|^2\Big] = 0.$$

Proof. For all $1 \leq j \leq n$, we have

$$\begin{aligned}
X_j^\varepsilon(t) &= X_j^\varepsilon(0) + \int_0^t f_{1,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s))\,ds \\
&\quad + \int_0^t f_{2,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s))\,d\langle B\rangle_s + \int_0^t f_{3,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s))\,dB(s), \\
X_j^0(t) &= X_j^0(0) + \int_0^t f_{1,j}^0(s, X_1^0(s), \dots, X_n^0(s))\,ds \\
&\quad + \int_0^t f_{2,j}^0(s, X_1^0(s), \dots, X_n^0(s))\,d\langle B\rangle(s) + \int_0^t f_{3,j}^0(s, X_1^0(s), \dots, X_n^0(s))\,dB(s).
\end{aligned}$$

Then

$$\begin{aligned}
X_j^\varepsilon(t) - X_j^0(t) &= X_j^\varepsilon(0) - X_j^0(0) \\
&\quad + \int_0^t \Big[f_{1,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) - f_{1,j}^0(s, X_1^0(s), \dots, X_n^0(s))\Big]\,ds \\
&\quad + \int_0^t \Big[f_{2,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) - f_{2,j}^0(s, X_1^0(s), \dots, X_n^0(s))\Big]\,d\langle B\rangle(s) \\
&\quad + \int_0^t \Big[f_{3,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) - f_{3,j}^0(s, X_1^0(s), \dots, X_n^0(s))\Big]\,dB(s)
\end{aligned}$$

and

$$\begin{aligned} X_j^\varepsilon(t) - X_j^0(t) = X_j^\varepsilon(0) - X_j^0(0) + &\\ + \int\limits_0^t \Big(& f_{1,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) - f_{1,j}^0(s, X_1^0(s), \dots, X_n^0(s)) \\ &+ f_{1,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) - f_{1,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) \Big)\, ds \\ + \int\limits_0^t \Big(& f_{2,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) - f_{2,j}^0(s, X_1^0(s), \dots, X_n^0(s)) \\ &+ f_{2,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) - f_{2,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) \Big)\, d\langle B\rangle(s) \\ + \int\limits_0^t \Big(& f_{3,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) - f_{3,j}^\varepsilon(s, X_1^0(s), \dots, X_n^0(s)) \\ &+ f_{3,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) - f_{3,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) \Big)\, dB(s). \end{aligned}$$

We have

$$\begin{aligned} |X_j^\varepsilon(t) - X_j^0(t)|^2 \le 7|X_j^\varepsilon(0) - X_j^0(0)|^2 & \\ + 7\Big| \int\limits_0^t \Big[& f_{1,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) - f_{1,j}^\varepsilon(s, X_1^0(s), \dots, X_n^0(s)) \Big]\, ds \Big|^2 \\ + 7\Big| \int\limits_0^t \Big[& f_{1,j}^\varepsilon(s, X_1^0(s), \dots, X_n^0(s)) - f_{1,j}^0(s, X_1^0(s), \dots, X_n^0(s)) \Big]\, ds \Big|^2 \\ + 7\Big| \int\limits_0^t \Big[& f_{2,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) - f_{2,j}^\varepsilon(s, X_s^0, y_s^0) \Big]\, d\langle B\rangle(s) \Big|^2 \\ + 7\Big| \int\limits_0^t \Big[& f_{2,j}^\varepsilon(s, X_1^0(s), \dots, X_n^0(s)) - f_{2,j}^0(s, X_1^0(s), \dots, X_n^0(s)) \Big]\, d\langle B\rangle(s) \Big|^2 \\ + 7\Big| \int\limits_0^t \Big[& f_{3,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) - f_{3,j}^\varepsilon(s, X_1^0(s), \dots, X_n^0(s)) \Big]\, dB(s) \Big|^2 \\ + 7\Big| \int\limits_0^t \Big[& f_{3,j}^\varepsilon(s, X_1^0(s), \dots, X_n^0(s)) - f_{3,j}^0(s, X_1^0(s), \dots, X_n^0(s)) \Big]\, dB(s) \Big|^2. \end{aligned}$$

Taking the G-expectation on both sides of the above relation, from Lemmas 4.2 and 4.3 we get

$$\begin{aligned} \mathbb{E}|X_j^\varepsilon(t) - X_j^0(0)|^2 \le 7\mathbb{E}\big[|X_j^\varepsilon(0) - X_j^0(0)|^2\big] & \\ + 7T \int\limits_0^t \mathbb{E}\Big[& \Big| f_{1,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) - f_{1,j}^\varepsilon(s, X_1^0(s), \dots, X_n^0(s)) \Big|^2 \Big]\, ds \\ + 7T\Big| \int\limits_0^t \mathbb{E}\Big[& \Big| f_{1,j}^\varepsilon(s, X_1^0(s), \dots, X_n^0(s)) - f_{1,j}^0(s, X_1^0(s), \dots, X_n^0(s)) \Big|^2 \Big]\, ds \Big| \\ + 7T \int\limits_0^t \mathbb{E}\Big[& \Big| f_{2,j}^\varepsilon(s, X_1^\varepsilon(s), \dots, X_n^\varepsilon(s)) - f_{2,j}^\varepsilon(s, X_1^0(s), \dots, X_n^0(s)) \Big|^2 \Big]\, ds \end{aligned}$$

$$+ 7T \int_0^t \mathbb{E}\Big[\Big|f^{\varepsilon}_{2,j}(s, X^0_1(s), \ldots, X^0_n(s)) - f^0_{1,j}(s, X^0_1(s), \ldots, X^0_n(s))\Big|^2\Big]\, ds$$
$$+ 7C \int_0^t \mathbb{E}\Big[\Big|f^{\varepsilon}_{3,j}(s, X^{\varepsilon}_1(s), \ldots, X^{\varepsilon}_n(s)) - f^{\varepsilon}_{3,j}(s, X^0_1(s), \ldots, X^0_n(s))\Big|^2\Big]\, ds$$
$$+ 7C \int_0^t \mathbb{E}\Big[\Big|f^{\varepsilon}_{3,j}(s, X^0_1(s), \ldots, X^0_n(s)) - f^0_{3,j}(s, X^0_1(s), \ldots, X^0_n(s))\Big|^2\Big]\, ds.$$

By the assumptions (B1)–(B3), we obtain

$$\mathbb{E}\big[|X^{\varepsilon}_j(t) - X^0_j(t)|^2\big] \le C^{\varepsilon}(T) + 7(2T + C) \int_0^t \mathbb{E}\Big(h(s) \sum_{j=1}^n |X^{\varepsilon}_j(s) - X^0_j(s)|^2\Big)\, ds,$$

where

$$\begin{aligned} C^{\varepsilon}(t) = 7\mathbb{E}\big[|X^{\varepsilon}_j(0) - X^0_j(0)|^2\big] \\ + 7T \int_0^t \mathbb{E}\Big[\Big|f^{\varepsilon}_{1,j}(s, X^0_1(s), \ldots, X^0_n(s)) - f^0_{1,j}(s, X^0_1(s), \ldots, X^0_n(s))\Big|^2\Big]\, ds \\ + 7T \int_0^t \mathbb{E}\Big[\Big|f^{\varepsilon}_{2,j}(s, X^0_1(s), \ldots, X^0_n(s)) - f^0_{2,j}(s, X^0_1(s), \ldots, X^0_n(s))\Big|^2\Big]\, ds \\ + 7C \int_0^t \mathbb{E}\Big[\Big|f^{\varepsilon}_{3,j}(s, X^0_1(s), \ldots, X^0_n(s)) - f^0_{3,j}(s, X^0_1(s), \ldots, X^0_n(s))\Big|^2\Big]\, ds. \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}\Big[\sum_{j=1}^n |X^{\varepsilon}_j(t) - X^0_j(t)|^2\Big] &\le \sum_{j=1}^n \mathbb{E}|X^{\varepsilon}_j(t) - X^0_j(t)|^2 \\ &\le C^{\varepsilon}_n(T) + C_n(T) \int_0^t h(s) \sum_{j=1}^n \mathbb{E}|X^{\varepsilon}_j(s) - X^0_j(s)|^2\, ds, \end{aligned}$$

where

$$C^{\varepsilon}_n(T) = nC^{\varepsilon}(T) \ \text{ and } \ C_n(T) = 7n(2T + C).$$

Hence, by Gronwall's inequality, we have

$$\mathbb{E}\Big[\sum_{j=1}^n |X^{\varepsilon}_j(t) - X^0_j(t)|^2\Big] \le C^{\varepsilon}_n(T) \exp\Big(C_n(T) \int_0^T h(t)\, dt\Big).$$

Since $C^{\varepsilon}_n(T) \to 0$ as $\varepsilon \to 0$, we finally get

$$\forall\, t \in [0, T], \ \ \lim_{\varepsilon \to 0} \mathbb{E}\Big[\sum_{j=1}^n |X^{\varepsilon}_j(t) - X^0_j(t)|^2\Big] = 0,$$

hence the desired result follows. □

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EXPONENTIAL STABILITY FOR COUPLED FLEXIBLE STRUCTURE SYSTEM WITH DISTRIBUTED DELAY

Abstract. In this paper, we consider a coupled flexible structure system with distributed delay in two equations. We first give the well-posedness of the system by using a semigroup method. Then, by using the perturbed energy method and constructing some Lyapunov functionals, we obtain the exponential decay result.

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რეზიუმე. ნაშრომში განხილულია ორი განტოლებით წარმოდგენილი შეწყვილებული მოქნილი სტრუქტურის სისტემა განაწილებული დაგვიანებით. ჯერ ნახევარჯგუფების მეთოდით დადგენილია სისტემის კორექტულობა, ხოლო შემდეგ, ენერგიის შეშფოთების მეთოდის გამოყენებით და გარკვეული ლიაპუნოვის ფუნქციონალის აგების საშუალებით, დადგენილია ამონახსნის ექსპონენციალური ქრობა.

1 Introduction

In this article, we study the well-posedness and exponential stability for coupled flexible structure system with distributed delay in two equations

$$\begin{cases} m_1(x)u_{tt} - (p_1(x)u_x + 2\delta_1(x)u_{xt})_x + \mu_0 u_t + \int\limits_{\tau_1}^{\tau_2} \mu_1(s)u_t(x,t-s)\,ds = 0, \\ m_2(x)v_{tt} - (p_2(x)v_x + 2\delta_2(x)v_{xt})_x + \mu_0' v_t + \int\limits_{\tau_1}^{\tau_2} \mu_2(s)v_t(x,t-s)\,ds = 0, \end{cases} \tag{1.1}$$

where $(x,t) \in (0,L) \times (0,+\infty)$, with the following initial and boundary conditions:

$$\begin{gathered} u(\,\cdot\,,0) = u_0(x), \;\; u_t(\,\cdot\,,0) = u_1(x), \;\; \forall\, x \in [0,L], \\ u(0,t) = u(L,t) = 0, \;\; \forall\, t \geq 0, \\ v(\,\cdot\,,0) = v_0(x), \;\; v_t(\,\cdot\,,0) = v_1(x), \;\; \forall\, x \in [0,L], \\ v(0,t) = v(L,t) = 0, \;\; \forall\, t \geq 0, \\ u_t(x,-t) = f_0(x,t), \;\; 0 < t \leq \tau_2, \\ v_t(x,-t) = g_0(x,t), \;\; 0 < t \leq \tau_2, \end{gathered} \tag{1.2}$$

where $u(x,t), v(x,t)$ are the displacements of a particle at position $x \in (0,L)$ and time $t > 0$. u_0, v_0 are initial data, and f_0, g_0 are the history function. The parameters $m_i(x)$, $\delta_i(x)$ and $p_i(x)$ (for $i = 1,2$) are responsible for the non-uniform structure of the body, where $m_i(x)$ denotes mass per unit length of the structure, $\delta_i(x)$ is a coefficient of internal material damping and $p_i(x)$ is a positive function related to the stress acting on the body at a point x. We recall the assumptions of the functions $m_i(x), \delta_i(x)$ and $p_i(x)$ in [1] such that

$$m_i, \delta_i, p_i \in W^{1,\infty}(0,L), \;\; m_i(x), \delta_i(x), p_i(x) > 0, \;\; \forall\, x \in [0,L] \;\text{ for }\; i = 1,2.$$

The coefficients μ_0, μ_0' are positive constants, and $\mu_1, \mu_2 : [\tau_1; \tau_2] \to \mathbb{R}$ are the bounded functions, where τ_1 and τ_2 are two real numbers satisfying $0 \leq \tau_1 < \tau_2$. Here, we prove the well-posedness and stability results for the problem on the under the assumption

$$\begin{cases} \mu_0 > \int\limits_{\tau_1}^{\tau_2} |\mu_1(s)|\,ds, \\ \mu_0' > \int\limits_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds. \end{cases} \tag{1.3}$$

During the last few decades, the theory of stabilisation of flexible structural system has been a topic of interest in view of vibration control of various structural elements. In [6], Gorain established the uniform exponential stability of the problem

$$m(x)u_{tt} - \big(p(x)u_x + 2\delta(x)u_{xt}\big)_x = f(x) \;\text{ on }\; (0,L) \times \mathbb{R}^+,$$

which describes the vibrations of an inhomogeneous flexible structure with an exterior disturbing force f. Indeed, it is physically relevant to take into account thermal effects in flexible structures: in 2014, M. Siddhartha et al. [9] showed the exponential stability of the vibrations of a inhomogeneous flexible structure with thermal effect governed by the Fourier law,

$$\begin{cases} m(x)u_{tt} - \big(p(x)u_x + 2\delta(x)u_{xt}\big)_x + \kappa\theta_x = f, \\ \theta_t - \theta_{xx} + \kappa u_{tx} = 0. \end{cases}$$

It is known that the dynamic systems with delay terms have become a major research subject in the differential equation since the 1970s of the last century (see, e.g., [2–4, 7, 8, 11–15, 18]). It may not only destabilize a system which is asymptotically stable in the absence of delay, but may also lead to the well-posedness (see [5, 17] and the references therein). Therefore, the stability issue of systems with delay is of great theoretical and practical importance. In [8], the authors consider a non-uniform flexible structure system with time delay under Cattaneo's law of heat condition

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \eta\theta_x + \mu u_t(x, t-\tau_0) = 0, & x \in (0, L), \quad t > 0, \\ \theta_t + \kappa q_x + \eta u_{tx} = 0, & x \in (0, L), \quad t > 0, \\ \tau q_t + \beta q + \kappa\theta_x = 0, & x \in (0, L), \quad t > 0, \end{cases} \tag{1.4}$$

with the boundary condition

$$u(0, t) = u(L, t) = 0, \quad \theta(0, t) = \theta(L, t) = 0, \quad t \geq 0, \tag{1.5}$$

and the initial condition

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x), \quad q(x, 0) = q_0(x), \quad x \in [0, L]. \tag{1.6}$$

They proved that system (1.4)–(1.6) is well-posed, and the system is an exponential decay under a small condition on time delay. M. S. Alves et al. (see [1]) considered system (1.4)–(1.6) without delay term, and obtained an exponential stability result for one set of boundary conditions and at least a polynomial for another set of boundary conditions.

In [14], Nicaise and Pignotti considered the wave equation with linear frictional damping and internal distributed delay

$$u_{tt} - \triangle u + \mu_1 u_t + a(x) \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(t-s)\, ds = 0$$

in $\Omega \times (0, \infty)$, with initial and mixed Dirichlet–Neumann boundary conditions and a as a function, chosen in an appropriate space. They established exponential stability of the solution under the assumption

$$\|a\|_\infty \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds < \mu_1.$$

The authors also obtained the same result when the distributed delay acted on a part of the boundary.

Motivated by the above results, in the present work we consider system (1.1), (1.2), prove the well-posedness and establish exponential stability results.

We now briefly sketch the outline of the paper. In Section 2, we state and prove the well-posedness of system (1.1), (1.2) by using the semigroup method. In Section 3, we establish an exponential stability by using the perturbed energy method and construct some Lyapunov functionals.

2 The well-posedness

In this section, we give a brief idea about the existence and uniqueness of solutions for (1.1), (1.2) using the semigroup theory [16]. As in [14], we introduce the new variables

$$\begin{aligned} z_1(x, \rho, t, s) &= u_t(x, t - \rho s), \quad x \in (0, L), \quad \rho \in (0, 1), \quad s \in (\tau_1, \tau_2), \quad t > 0, \\ z_2(x, \rho, t, s) &= v_t(x, t - \rho s), \quad x \in (0, L), \quad \rho \in (0, 1), \quad s \in (\tau_1, \tau_2), \quad t > 0. \end{aligned}$$

Then we have

$$s z_{it}(x, \rho, t, s) + z_{i\rho}(x, \rho, t, s) = 0 \ \text{ in } \ (0, L) \times (0, 1) \times (0, \infty) \times (\tau_1, \tau_2) \ \text{ for } \ i = 1, 2.$$

Therefore, problem (1.1) takes the form

$$\begin{cases} m_1(x)u_{tt} - (p_1(x)u_x + 2\delta_1(x)u_{xt})_x + \mu_0 u_t + \int\limits_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s)\,ds = 0, \\ s z_{1t}(x,\rho,t,s) + z_{1\rho}(x,\rho,t,s) = 0, \\ m_2(x)v_{tt} - (p_2(x)v_x + 2\delta_2(x)v_{xt})_x + \mu_0' v_t + \int\limits_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s)\,ds = 0, \\ s z_{2t}(x,\rho,t,s) + z_{2\rho}(x,\rho,t,s) = 0, \end{cases} \tag{2.1}$$

with the following initial and boundary conditions:

$$\begin{cases} u(\,\cdot\,,0) = u_0(x), \;\; u_t(\,\cdot\,,0) = u_1(x), \;\; \forall\, x \in [0,L], \\ u(0,t) = u(L,t) = 0, \;\; \forall\, t \geq 0, \\ v(\,\cdot\,,0) = v_0(x), \;\; v_t(\,\cdot\,,0) = v_1(x), \;\; \forall\, x \in [0,L], \\ v(0,t) = v(L,t) = 0, \;\; \forall\, t \geq 0, \\ z_1(x,0,t,s) = u_t(x,t) \;\text{ on }\; (0,L)\times(0,\infty)\times(\tau_1,\tau_2), \\ z_2(x,0,t,s) = v_t(x,t) \;\text{ on }\; (0,L)\times(0,\infty)\times(\tau_1,\tau_2), \\ z_1(x,\rho,0,s) = f_0(x,\rho s) \;\text{ on }\; (0,L)\times(0,1)\times(\tau_1,\tau_2), \\ z_2(x,\rho,0,s) = g_0(x,\rho s) \;\text{ on }\; (0,L)\times(0,1)\times(\tau_1,\tau_2). \end{cases} \tag{2.2}$$

Introducing the vector function $U = (u, \varphi, z_1, v, \psi, z_2)^T$, where $\varphi = u_t$ and $\psi = v_t$, system (2.1), (2.2) can be written as

$$\begin{cases} U'(t) + \mathcal{A}U(t) = 0, \quad t > 0, \\ U(0) = U_0 = (u_0, u_1, f_0, v_0, v_1, g_0)^T, \end{cases} \tag{2.3}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} -\varphi \\ -\dfrac{1}{m_1(x)}\,(p_1(x)u_x + 2\delta_1(x)\varphi_x)_x + \dfrac{\mu_0}{m_1(x)}\,\varphi + \dfrac{1}{m_1(x)} \int\limits_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s)\,ds \\ s^{-1} z_{1\rho} \\ -\psi \\ -\dfrac{1}{m_2(x)}\,(p_2(x)v_x + 2\delta_2(x)\psi_x)_x + \dfrac{\mu_0'}{m_2(x)}\,\psi + \dfrac{1}{m_2(x)} \int\limits_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s)\,ds \\ s^{-1} z_{2\rho} \end{pmatrix}.$$

Next, we define the energy space as

$$\begin{aligned} \mathcal{H} = H_0^1(0,L) \times L^2(0,L) \times L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)) \\ \times H_0^1(0,L) \times L^2(0,L) \times L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)), \end{aligned}$$

equipped with the inner product

$$\begin{aligned} \langle U, \widetilde{U} \rangle_{\mathcal{H}} = \int\limits_0^L p_1(x) u_x \widetilde{u}_x\,dx + \int\limits_0^L m_1(x)\varphi\widetilde{\varphi}\,dx + \int\limits_0^L \int\limits_0^1 \int\limits_{\tau_1}^{\tau_2} s|\mu_1(s)| z_1(x,\rho,s)\widetilde{z}_1(x,\rho,s)\;ds\,d\rho\,dx \\ + \int\limits_0^L p_2(x) v_x \widetilde{v}_x\,dx + \int\limits_0^L m_2(x)\psi\widetilde{\psi}\,dx + \int\limits_0^L \int\limits_0^1 \int\limits_{\tau_1}^{\tau_2} s|\mu_2(s)| z_2(x,\rho,s)\widetilde{z}_2(x,\rho,s)\;ds\,d\rho\,dx. \end{aligned}$$

Then the domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{c} U \in \mathcal{H} \mid \ u, v \in H^2(0,L) \cap H_0^1(0,L), \quad \varphi, \psi \in H_0^1(0,L), \\ z_1, z_{1\rho}, z_2, z_{2\rho} \in L^2((0,L) \times (0,1) \times (\tau_1, \tau_2)), \\ z_1(x,0,s) = \varphi(x), \quad z_2(x,0,s) = \psi(x) \end{array} \right\}.$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

The well-posedness of problem (2.3) is ensured by

Theorem 2.1. *Assume that $U_0 \in \mathcal{H}$ and (1.3) holds, then problem (2.3) has a unique solution $U \in C(\mathbb{R}^+; \mathcal{H})$. Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C(\mathbb{R}^+; D(\mathcal{A})) \cap C^1(\mathbb{R}^+; \mathcal{H}).$$

Proof. The result follows from the Lumer–Phillips theorem provided we prove that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a maximal monotone operator. First, we prove that $\mathcal{A}$ is monotone. For any $U = (u, \varphi, z_1, v, \psi, z_2)^T \in D(\mathcal{A})$, by using the inner product and integrating by parts, we obtain

$$\begin{aligned} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = 2 \int_0^L \delta_1(x) \varphi_x^2 \, dx + \int_0^L \varphi \int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s) \, ds \, dx + \mu_0 \int_0^L \varphi^2 \, dx \\ + \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(s)| z_1(x,\rho,s) z_{1\rho}(x,\rho,s) \, ds \, d\rho \, dx + 2 \int_0^L \delta_2(x) \psi_x^2 \, dx \\ + \int_0^L \psi \int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s) \, ds \, dx + \mu_0' \int_0^L \psi^2 \, dx + \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z_2(x,\rho,s) z_{2\rho}(x,\rho,s) \, ds \, d\rho \, dx. \end{aligned}$$

Integrating by parts in ρ, we have

$$\begin{aligned} \int_0^L \int_{\tau_1}^{\tau_2} \int_0^1 |\mu_i(s)| z_i(x,\rho,s) z_{i\rho}(x,\rho,s) d\rho \, ds \, dx \\ = \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} |\mu_i(s)| \left[z_i^2(x,1,s) - z_i^2(x,0,s) \right] ds \, dx \quad \text{for} \quad i = 1, 2. \end{aligned}$$

Using the fact that $z_1(x,0,s,t) = \varphi$ and $z_2(x,0,s,t) = \psi$, we obtain

$$\begin{aligned} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = 2 \int_0^L \delta_1(x) \varphi_x^2 \, dx + \int_0^L \varphi \int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s) \, ds \, dx + \left(\mu_0 - \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_1(s)| \, ds \right) \int_0^L \varphi^2 \, dx \\ + \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} |\mu_1(s)| z_1^2(x,1,s) \, ds \, dx + 2 \int_0^L \delta_2(x) \psi_x^2 \, dx + \int_0^L \psi \int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s) \, ds \, dx \\ + \left(\mu_0' - \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, ds \right) \int_0^L \psi^2 \, dx + \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| z_2^2(x,1,s) \, ds \, dx. \end{aligned} \tag{2.4}$$

Now, using Young's inequality, we can estimate

$$\int_0^L \varphi \int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s) \, ds \, dx \geq -\frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_1(s)| \, ds \int_0^1 \varphi^2 \, dx - \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(s)| z_1^2(x,1,s) \, ds \, dx \tag{2.5}$$

and

$$\int_0^L \psi \int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s)\, ds\, dx \geq -\frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_0^1 \psi^2\, dx - \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z_2^2(x,1,s)\, ds\, dx. \tag{2.6}$$

Substituting (2.5) and (2.6) in (2.4), and using (1.3), we obtain

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \geq 2\int_0^L \delta_1(x)\varphi_x^2\, dx + \left(\mu_0 - \int_{\tau_1}^{\tau_2} |\mu_1(s)|\, ds\right)\int_0^L \varphi^2\, dx$$
$$+ 2\int_0^L \delta_2(x)\psi_x^2\, dx + \left(\mu_0' - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)\int_0^L \psi^2\, dx \geq 0.$$

Hence, $\mathcal{A}$ is monotone. Next, we prove that the operator $I + \mathcal{A}$ is surjective, i.e., for any $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$, there exists $U = (u, \varphi, z_1, v, \psi, z_2)^T \in D(\mathcal{A})$ satisfying

$$(I + \mathcal{A})U = F, \tag{2.7}$$

which is equivalent to

$$\begin{cases} u - \varphi = f_1, \\ (m_1(x) + \mu_0)\varphi - (p_1(x)u_x + 2\delta_1(x)\varphi_x)_x + \int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s)\, ds = m_1(x) f_2, \\ s z_1 + z_{1\rho} = s f_3, \\ v - \psi = f_4, \\ (m_2(x) + \mu_0')\psi - (p_2(x)v_x + 2\delta_2(x)\psi_x)_x + \int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s)\, ds = m_2(x) f_5, \\ s z_2 + z_{2\rho} = s f_6. \end{cases} \tag{2.8}$$

Suppose that we have found u and v. Then equations $(2.8)_1$ and $(2.8)_4$ yield

$$\begin{cases} \varphi = u - f_1, \\ \psi = v - f_4. \end{cases} \tag{2.9}$$

It is clear that $\varphi \in H_0^1(0,L)$ and $\psi \in H_0^1(0,L)$. Equations $(2.8)_3$ and $(2.8)_6$ with (2.9), recalling $z_1(x,0,t,s) = \varphi$, $z_2(x,0,t,s) = \psi$, yield

$$z_1(x,\rho,s) = u(x)e^{-\rho s} - f_1(x)e^{-\rho s} + s e^{-\rho s}\int_0^\rho f_3(x,\tau,s) e^{\tau s}\, d\tau \tag{2.10}$$

and

$$z_2(x,\rho,s) = v(x)e^{-\rho s} - f_4(x)e^{-\rho s} + s e^{-\rho s}\int_0^\rho f_6(x,\tau,s) e^{\tau s}\, d\tau. \tag{2.11}$$

Clearly, $z_1, z_{1\rho}, z_2, z_{2\rho} \in L^2((0,L)\times(0,1)\times(\tau_1,\tau_2))$.

Inserting $(2.9)_1$ and (2.10) into $(2.8)_2$, and inserting $(2.9)_2$ and (2.11) into $(2.8)_5$, we get

$$\begin{cases} \eta_1 u - (p_1(x)u_x + 2\delta_1(x)\varphi_x)_x = g_1, \\ \eta_2 v - (p_2(x)v_x + 2\delta_2(x)\psi_x)_x = g_2, \\ u_x - \varphi_x = g_3, \\ v_x - \psi_x = g_4, \end{cases} \tag{2.12}$$

where

$$\eta_1 = m_1(x) + \mu_0 + \int_{\tau_1}^{\tau_2} \mu_1(s)e^{-s}\,ds, \quad \eta_2 = m_2(x) + \mu_0' + \int_{\tau_1}^{\tau_2} \mu_2(s)e^{-s}\,ds,$$

$$g_1 = \eta_1 f_1 + m_1(x) f_2 - \int_{\tau_1}^{\tau_2} s\mu_1(s)e^{-s} \int_0^1 f_3(x,\tau,s)e^{\tau s}\,d\tau\,ds,$$

$$g_2 = \eta_2 f_4 + m_2(x) f_5 - \int_{\tau_1}^{\tau_2} s\mu_2(s)e^{-s} \int_0^1 f_6(x,\tau,s)e^{\tau s}\,d\tau\,ds,$$

$$g_3 = f_{1x}, \quad g_4 = f_{4x}.$$

The variational formulation corresponding to equation (2.12) takes the form

$$B((u,v)^T, (\widetilde{u},\widetilde{v})^T) = G(\widetilde{u},\widetilde{v})^T, \tag{2.13}$$

where

$$B : \left[H_0^1(0,L) \times H_0^1(0,L)\right]^2 \longrightarrow \mathbb{R}$$

is the bilinear form given by

$$B\big((u,v)^T, (\widetilde{u},\widetilde{v})^T\big) = \eta_1 \int_0^L u\widetilde{u}\,dx + \int_0^L (p_1(x) + 2\delta_1(x))u_x\widetilde{u}_x\,dx + \eta_2 \int_0^L v\widetilde{v}\,dx + \int_0^L (p_2(x) + 2\delta_2(x))v_x\widetilde{v}_x\,dx,$$

and

$$G : \left[H_0^1(0,L) \times H_0^1(0,L)\right] \longrightarrow \mathbb{R}$$

is the linear form defined by

$$G(\widetilde{u},\widetilde{v})^T = \int_0^L g_1\widetilde{u}\,dx + \int_0^L g_2\widetilde{v}\,dx + \int_0^L 2\delta_1(x)g_3\widetilde{u}_x\,dx + \int_0^L 2\delta_2(x)g_4\widetilde{v}_x\,dx.$$

Now, we introduce the Hilbert space $V = H_0^1(0,L) \times H_0^1(0,L)$ equipped with the norm

$$\|(u,v)\|_V^2 = \|u\|_2^2 + \|u_x\|_2^2 + \|v\|_2^2 + \|v_x\|_2^2.$$

It is clear that $B(\,\cdot\,,\,\cdot\,)$ and $G(\,\cdot\,)$ are bounded. Furthermore, we can find that there exists a positive constant α such that

$$B((u,v)^T, (u,v)^T) = \eta_1 \int_0^L u^2\,dx + \int_0^L (p_1(x) + 2\delta_1(x))u_x^2\,dx + \eta_2 \int_0^L v^2\,dx + \int_0^L (p_2(x) + 2\delta_2(x))v_x^2\,dx \geq \alpha\|(u,v)\|_V^2,$$

which implies that $B(\,\cdot\,,\,\cdot\,)$ is coercive.

Consequently, applying the Lax–Milgram lemma, we obtain that (2.13) has a unique solution $(u,v)^T \in V$.

Then, by substituting u, v into (2.9), we get

$$\varphi, \psi \in H_0^1(0, L).$$

Next, it remains to show that

$$u, v \in H^2(0, L) \cap H_0^1(0, L).$$

Furthermore, if $\widetilde{v} \equiv 0 \in H_0^1(0, L)$, then (2.13) reduces to

$$-\int_0^L \big[(p_1(x) + 2\delta_1(x))u_x\big]_x \widetilde{u}\, dx = \int_0^L g_1 \widetilde{u}\, dx - \int_0^L 2(\delta_1(x) g_3)_x \widetilde{u}\, dx - \eta_1 \int_0^L u\widetilde{u}\, dx$$

for all $\widetilde{u}$ in $H_0^1(0, L)$, which implies

$$\big[(p_1(x) + 2\delta_1(x))u_x\big]_x = \eta_1 u - g_1 + 2(\delta_1(x) g_3)_x \in L^2(0, L).$$

Thus, by the L^2 theory for the linear elliptic equations, we obtain

$$u \in H^2(0, L) \cap H_0^1(0, L).$$

In a similar way, we have

$$v \in H^2(0, L) \cap H_0^1(0, L).$$

Finally, the application of the classical regularity theory for the linear elliptic equations guarantees the existence of unique solution $U \in D(\mathcal{A})$ which satisfies (2.7). Therefore, the operator $\mathcal{A}$ is maximal.

Hence, the result of Theorem 2.1 follows. □

3 Exponential stability

In this section, we prove the exponential decay for problem (2.1), (2.2). This will be achieved by using the perturbed energy method. We define the energy functional $E(t)$ as

$$\begin{aligned}
E(t) &= E_1(t) + E_2(t), \\
E_1(t) &= \frac{1}{2}\int_0^L \big[m_1(x)u_t^2 + p_1(x)u_x^2\big]\, dx + \frac{1}{2}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_1(s)| z_1^2(x, \rho, z, t)\, ds\, d\rho\, dx, \\
E_2(t) &= \frac{1}{2}\int_0^L \big[m_2(x)u_t^2 + p_2(x)u_x^2\big]\, dx + \frac{1}{2}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z_2^2(x, \rho, z, t)\, ds\, d\rho\, dx.
\end{aligned} \tag{3.1}$$

We have the following exponentially stable result.

Theorem 3.1. *Let $(u, u_t, z_1, v, v_t, z_2)$ be a solution of* (2.1), (2.2) *and assume that* (1.3) *holds. Then there exists positive constants λ_0, λ_1 such that the energy $E(t)$ associated with problem* (2.1), (2.2) *satisfies*

$$E(t) \le \lambda_0 e^{-\lambda_1 t}, \quad t \ge 0. \tag{3.2}$$

To prove this result, we will state and prove some useful lemmas in advance.

Lemma 3.2 (**Poincaré-type Scheeffer's inequality**, [10])**.** *Let $h \in H_0^1(0, L)$. Then*

$$\int_0^L |h|^2\, dx \le \frac{L^2}{\pi^2} \int_0^L |h_x|^2\, dx. \tag{3.3}$$

Lemma 3.3 (**Mean value theorem**, [1]). *Let (u, u_t, v, v_t) be a solution to system $(1.1), (1.2)$ with an initial datum in $D(\mathcal{A})$. Then, for any $t > 0$, there exists a sequence of real numbers (depending on t), denoted by $\zeta_i, \xi_i \in [0, L] (i = 1, \ldots, 6)$, such that*

$$\begin{aligned}
&\int_0^L p_1(x) u_x^2\,dx = p_1(\zeta_1) \int_0^L u_x^2\,dx, \quad \int_0^L m_1(x) u_t^2\,dx = m_1(\zeta_2) \int_0^L u_t^2\,dx,\\
&\int_0^L m_1(x) u^2\,dx = m_1(\zeta_3) \int_0^L u^2\,dx, \quad \int_0^L \delta_1(x) u^2\,dx = \delta_1(\zeta_4) \int_0^L u^2\,dx,\\
&\int_0^L \delta_1(x) u_x^2\,dx = \delta_1(\zeta_5) \int_0^L u_x^2\,dx, \quad \int_0^L \delta_1(x) u_{xt}^2\,dx = \delta_1(\zeta_6) \int_0^L u_{xt}^2\,dx,\\
&\int_0^L p_2(x) v_x^2\,dx = p_2(\xi_1) \int_0^L v_x^2\,dx, \quad \int_0^L m_2(x) v_t^2\,dx = m_2(\xi_2) \int_0^L v_t^2\,dx,\\
&\int_0^L m_2(x) v^2\,dx = m_2(\xi_3) \int_0^L v^2\,dx, \quad \int_0^L \delta_2(x) v^2\,dx = \delta_2(\xi_4) \int_0^L v^2\,dx,\\
&\int_0^L \delta_2(x) v_x^2\,dx = \delta_2(\xi_5) \int_0^L v_x^2\,dx, \quad \int_0^L \delta_2(x) v_{xt}^2\,dx = \delta_2(\xi_6) \int_0^L v_{xt}^2\,dx.
\end{aligned}$$

Lemma 3.4. *Let $(u, u_t, z_1, v, v_t, z_2)$ be a solution of $(2.1), (2.2)$. Then the energy functional satisfies*

$$\begin{aligned}
E'(t) &= E_1'(t) + E_2'(t) \le 0, \quad \forall\, t \ge 0,\\
E_1'(t) &\le -2 \int_0^L \delta_1(x) u_{xt}^2\,dx + \left(\int_{\tau_1}^{\tau_2} |\mu_1(s)|\,ds - \mu_0 \right) \int_0^L u_t^2\,dx \le 0,\\
E_2'(t) &\le -2 \int_0^L \delta_2(x) v_{xt}^2\,dx + \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds - \mu_0' \right) \int_0^L v_t^2\,dx \le 0.
\end{aligned}$$

Proof. Multiplying $(2.1)_1$ and $(2.1)_3$ by u_t and v_t, respectively, and integrating over $(0, L)$, using integration by parts and the boundary conditions in (2.2), we get

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_0^L \left[m_1(x) u_t^2 + p_1(x) u_x^2 \right] dx\\
&\qquad = -2 \int_0^L \delta_1(x) u_{xt}^2\,dx - \mu_0 \int_0^L u_t^2\,dx - \int_0^L u_t \int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x, 1, t, s)\,ds\,dx, \qquad (3.4)\\
&\frac{1}{2} \frac{d}{dt} \int_0^L \left[m_2(x) v_t^2 + p_2(x) v_x^2 \right] dx\\
&\qquad = -2 \int_0^L \delta_2(x) v_{xt}^2\,dx - \mu_0' \int_0^L v_t^2\,dx - \int_0^L v_t \int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x, 1, t, s)\,ds\,dx. \qquad (3.5)
\end{aligned}$$

On the other hand, multiplying $(2.1)_2$ and $(2.1)_4$ by $|\mu_1(s)| z_1$ and $|\mu_2(s)| z_2$, respectively, and integrating over $(0, L) \times (0, 1) \times (\tau_1, \tau_2)$, and recalling $z_1(x, 0, t, s) = u_t$ and $z_2(x, 0, t, s) = v_t$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_1(s)|z_1^2(x,\rho,s,t)\,ds\,d\rho\,dx$$
$$= -\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_1(s)|z_1^2(x,1,s,t)\,ds\,dx + \frac{1}{2}\int_0^L u_t^2\int_{\tau_1}^{\tau_2}|\mu_1(s)|\,ds\,dx, \tag{3.6}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z_2^2(x,\rho,s,t)\,ds\,d\rho\,dx$$
$$= -\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|z_2^2(x,1,s,t)\,ds\,dx + \frac{1}{2}\int_0^L v_t^2\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\,dx. \tag{3.7}$$

A combination of (3.4) and (3.6) gives

$$E_1'(t) = -2\int_0^L \delta_1(x)u_{xt}^2\,dx - \mu_0\int_0^L u_t^2\,dx - \int_0^L u_t\int_{\tau_1}^{\tau_2}\mu_1(s)z_1(x,1,t,s)\,ds\,dx$$
$$-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_1(s)|z_1^2(x,1,s,t)\,ds\,dx + \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_1(s)|\,ds\int_0^L u_t^2\,dx. \tag{3.8}$$

Also, (3.5) and (3.7) give

$$E_2'(t) = -2\int_0^L \delta_2(x)v_{xt}^2\,dx - \mu_0'\int_0^L v_t^2\,dx - \int_0^L v_t\int_{\tau_1}^{\tau_2}\mu_2(s)z_2(x,1,t,s)\,ds\,dx$$
$$-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|z_2^2(x,1,s,t)\,ds\,dx + \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_0^L v_t^2\,dx. \tag{3.9}$$

Now, using Young's inequality, we obtain

$$-\int_0^L u_t\int_{\tau_1}^{\tau_2}\mu_1(s)z_1(x,1,t,s)\,ds\,dx \le \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_1(s)|\,ds\int_0^1 u_t^2\,dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_1(s)|z_1^2(x,1,s)\,ds\,dx, \tag{3.10}$$
$$-\int_0^L v_t\int_{\tau_1}^{\tau_2}\mu_2(s)z_2(x,1,t,s)\,ds\,dx \le \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_0^1 v_t^2\,dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z_2^2(x,1,s)\,ds\,dx. \tag{3.11}$$

Substituting (3.10) into (3.8), (3.11) into (3.9), and using (1.3), we obtain (3.4), which completes the proof. □

Next, in order to construct a Lyapunov functional equivalent to the energy, we prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

Lemma 3.5. *Let* $(u, u_t, z_1, v, v_t, z_2)$ *be a solution of* (2.1), (2.2). *Then the functions*

$$I_1(t) := \int_0^L \delta_1(x)u_x^2\,dx + \int_0^L m_1(x)u_t u\,dx,$$

$$F_1(t) := \int_0^L \delta_2(x)v_x^2\,dx + \int_0^L m_2(x)v_t v\,dx$$

satisfy, for all $\varepsilon_1, \varepsilon_2 > 0$ and $\varepsilon_1', \varepsilon_2' > 0$, the estimates

$$I_1'(t) \leq -\left(p_1(\zeta_1) - \frac{L^2\mu_0^2}{\pi^2}\,\varepsilon_1 - \frac{L^2\varepsilon_2}{\pi^2}\right)\int_0^L u_x^2\,dx + \left(m_1(\zeta_2) + \frac{1}{4\varepsilon_1}\right)\int_0^L u_t^2\,dx$$
$$+ \frac{\mu_0}{4\varepsilon_2}\int_0^L\int_{\tau_1}^{\tau_2} |\mu_1(s)| z_1^2(x,1,s,t)\,ds\,dx, \tag{3.12}$$

$$F_1'(t) \leq -\left(p_2(\xi_1) - \frac{L^2\mu_0'^2}{\pi^2}\,\varepsilon_1' - \frac{L^2\varepsilon_2'}{\pi^2}\right)\int_0^L v_x^2\,dx + \left(m_2(\xi_2) + \frac{1}{4\varepsilon_1'}\right)\int_0^L v_t^2\,dx$$
$$+ \frac{\mu_0'}{4\varepsilon_2'}\int_0^L\int_{\tau_1}^{\tau_2} |\mu_2(s)| z_2^2(x,1,s,t)\,ds\,dx. \tag{3.13}$$

Proof. By differentiating $I_1(t)$ with respect to t, using $(2.1)_1$ and integrating by parts, we obtain

$$I_1'(t) = -\int_0^L p_1(x)u_x^2\,dx - \mu_0\int_0^L u_t u\,dx - \int_0^L u\int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,s,t)\,ds\,dx + \int_0^L m_1(x)u_t^2\,dx.$$

By using Young's inequality, Lemma 3.2 and $(1.3)_1$, for $\varepsilon_1, \varepsilon_2 > 0$ we get

$$-\mu_0\int_0^L u_t u\,dx \leq \frac{L^2\mu_0^2}{\pi^2}\varepsilon_1\int_0^L u_x^2\,dx + \frac{1}{4\varepsilon_1}\int_0^L u_t^2\,dx, \tag{3.14}$$

$$-\int_0^L u\int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,s,t)\,ds\,dx \leq \frac{L^2\varepsilon_2}{\pi^2}\int_0^L u_x^2\,dx + \frac{\mu_0}{4\varepsilon_2}\int_0^L\int_{\tau_1}^{\tau_2} |\mu_1(s)| z_1^2(x,1,s,t)\,ds\,dx. \tag{3.15}$$

Consequently, using Lemma 3.3, (3.14) and (3.15), we establish (3.12).
Similarly, we prove (3.13). □

Lemma 3.6. *Let $(u, u_t, z_1, v, v_t, z_2)$ be a solution of (2.1), (2.2). Then the functions*

$$I_2(t) := \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_1(s)| z_1^2(x,\rho,s,t)\,ds\,d\rho\,dx,$$
$$F_2(t) := \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)| z_2^2(x,\rho,s,t)\,ds\,d\rho\,dx,$$

satisfy, for some positive constants n_1 and n_2, the estimates

$$I_2'(t) \leq -n_1\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_1(s)| z_1^2(x,\rho,s,t)\,ds\,d\rho\,dx$$
$$-n_1\int_0^L\int_{\tau_1}^{\tau_2} |\mu_1(s)| z_1^2(x,1,s,t)\,ds\,dx + \mu_0\int_0^L u_t^2\,dx, \tag{3.16}$$

$$F_2'(t) \leq -n_2\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z_2^2(x,\rho,s,t)\,ds\,d\rho\,dx$$

$$- n_2 \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| z_2^2(x,1,s,t)\, ds\, dx + \mu_0' \int_0^L v_t^2\, dx. \tag{3.17}$$

Proof. By differentiating $I_2(t)$ with respect to t and using equation $(2.1)_2$, we obtain

$$\begin{aligned}
I_2'(t) &= -2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_1(s)| z_1(x,\rho,s,t) z_{1\rho}(x,\rho,s,t)\, ds\, d\rho\, dx \\
&= -\frac{d}{d\rho} \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_1(s)| z_1^2(x,\rho,s,t)\, ds\, d\rho\, dx - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_1(s)| z_1^2(x,\rho,s,t)\, ds\, d\rho\, dx \\
&= - \int_0^L \int_{\tau_1}^{\tau_2} |\mu_1(s)| [e^{-s} z_1^2(x,1,s,t) - z_1^2(x,0,s,t)]\, ds\, dx - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_1(s)| z_1^2(x,\rho,s,t)\, ds\, d\rho\, dx.
\end{aligned}$$

Using the fact that $z_1(x,0,s,t) = u_t$ and $e^{-s} \le e^{-s\rho} \le 1$, for all $0 < \rho < 1$, we obtain

$$\begin{aligned}
I_2'(t) \le &- \int_0^L \int_{\tau_1}^{\tau_2} e^{-s} |\mu_1(s)| z_1^2(x,1,s,t)\, ds\, dx \\
&+ \int_{\tau_1}^{\tau_2} |\mu_1(s)|\, ds \int_0^L u_t^2\, dx - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_1(s)| z_1^2(x,\rho,s,t)\, ds\, d\rho\, dx.
\end{aligned}$$

Since $-e^{-s}$ is an increasing function, we have $-e^{-s} \le -e^{-\tau_2}$ for all $s \in [\tau_1, \tau_2]$.

Finally, setting $n_1 = e^{-\tau_2}$ and recalling $(1.3)_1$, we obtain (3.16).

Similarly, we prove (3.17). □

Next, we define a Lyapunov functional L and show that it is equivalent to the energy functional E.

Lemma 3.7. *Let $N, N_1, N_2 > 0$ and a functional be defined by*

$$L(t) := NE(t) + I_1(t) + N_1 I_2(t) + F_1(t) + N_2 F_2(t). \tag{3.18}$$

For two positive constants c_1 and c_2, we have

$$c_1 E(t) \le L(t) \le c_2 E(t), \quad \forall\, t \ge 0. \tag{3.19}$$

Proof. Let

$$\mathcal{L}(t) := I_1(t) + N_1 I_2(t) + F_1(t) + N_2 F_2(t).$$

Then

$$\begin{aligned}
|\mathcal{L}(t)| \le &\int_0^L \delta_1(x) u_x^2\, dx + \frac{1}{2} \int_0^L m_1(x) u_t^2\, dx + \frac{1}{2} \int_0^L m_1(x) u^2\, dx \\
&+ N_1 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_1(s)| z_1^2(x,\rho,s,t)\, ds\, d\rho\, dx + \int_0^L \delta_2(x) v_x^2\, dx + \frac{1}{2} \int_0^L m_2(x) v_t^2\, dx \\
&+ \frac{1}{2} \int_0^L m_2(x) v^2\, dx + N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z_2^2(x,\rho,s,t)\, ds\, d\rho\, dx \le c' E_1(t) + c'' E_2(t) \le c_0 E(t),
\end{aligned}$$

where $c_0 = \max\{c', c''\}$, with

$$c' = 1 + \frac{L^2 m_1(\zeta_3)}{\pi^2 p_1(\zeta_1)} + \frac{2\delta_1(\zeta_5)}{p_1(\zeta_1)} + 2N_1, \quad c'' = 1 + \frac{L^2 m_2(\xi_3)}{\pi^2 p_2(\xi_1)} + \frac{2\delta_2(\xi_5)}{p_2(\xi_1)} + 2N_2.$$

Consequently, $|L(t) - NE(t)| \leq c_0 E(t)$, which yields

$$(N - c_0)E(t) \leq L(t) \leq (N + c_0)E(t).$$

Choosing N large enough, we obtain estimate (3.19). □

Now, we prove the main result of this section.

Proof of Theorem 3.1. Differentiating (3.18) and recalling (3.4), (3.12), (3.13), (3.16) and (3.17), we obtain

$$\begin{aligned}
L'(t) \leq \Big[\Big(\int_{\tau_1}^{\tau_2} |\mu_1(s)|\,ds - \mu_0\Big)N + \Big(m_1(\zeta_2) + \frac{1}{4\varepsilon_1}\Big) + N_1\mu_0\Big] \int_0^L u_t^2\,dx \\
- \Big[p_1(\zeta_1) - \frac{L^2\mu_0^2}{\pi^2}\,\varepsilon_1 - \frac{L^2}{\pi^2}\,\varepsilon_2\Big] \int_0^L u_x^2\,dx - 2N\int_0^L \delta_1(x)u_{xt}^2\,dx \\
- n_1N_1 \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_1(s)|z_1^2(x,\rho,s,t)\,ds\,d\rho\,dx - \Big[n_1N_1 - \frac{\mu_0}{4\varepsilon_2}\Big]\int_0^L\int_{\tau_1}^{\tau_2} |\mu_1(s)|z_1^2(x,1,s,t)\,ds\,dx \\
+ \Big[\Big(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds - \mu_0'\Big)N + \Big(m_2(\xi_2) + \frac{1}{4\varepsilon_1'}\Big) + N_2\mu_0'\Big] \int_0^L v_t^2\,dx \\
- \Big[p_2(\xi_1) - \frac{L^2\mu_0'^2}{\pi^2}\varepsilon_1' - \frac{L^2}{\pi^2}\varepsilon_2'\Big] \int_0^L v_x^2\,dx - 2N\int_0^L \delta_2(x)v_{xt}^2\,dx \\
- n_2N_2 \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z_2^2(x,\rho,s,t)\,ds\,d\rho\,dx - \Big[n_2N_2 - \frac{\mu_0'}{4\varepsilon_2'}\Big]\int_0^L\int_{\tau_1}^{\tau_2} |\mu_2(s)|z_2^2(x,1,s,t)\,ds\,dx.
\end{aligned}$$

Using Lemma 3.2 and Lemma 3.3, we get

$$\begin{aligned}
L'(t) \leq -\Big[\gamma_1 N - \frac{L^2}{\pi^2}\Big(m_1(\zeta_2) + \frac{1}{4\varepsilon_1}\Big) - \frac{L^2\mu_0}{}N_1\Big]\int_0^L u_{tx}^2\,dx - \Big[p_1(\zeta_1) - \frac{L^2\mu_0^2}{\pi^2}\,\varepsilon_1 - \frac{L^2}{\pi^2}\varepsilon_2\Big]\int_0^L u_x^2\,dx \\
- n_1N_1 \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_1(s)|z_1^2(x,\rho,s,t)\,ds\,d\rho\,dx - \Big[n_1N_1 - \frac{\mu_0}{4\varepsilon_2}\Big]\int_0^L\int_{\tau_1}^{\tau_2} |\mu_1(s)|z_1^2(x,1,s,t)\,ds\,dx \\
- \Big[\gamma_2 N - \frac{L^2}{\pi^2}\Big(m_2(\xi_2) + \frac{1}{4\varepsilon_1'}\Big) - \frac{L^2\mu_0'}{\pi^2}\,N_2\Big]\int_0^L v_{tx}^2\,dx - \Big[p_2(\xi_1) - \frac{L^2\mu_0'^2}{\pi^2}\,\varepsilon_1' - \frac{L^2}{\pi^2}\,\varepsilon_2'\Big]\int_0^L v_x^2\,dx \\
- n_2N_2 \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z_2^2(x,\rho,s,t)\,ds\,d\rho\,dx - \Big[n_2N_2 - \frac{\mu_0'}{4\varepsilon_2'}\Big]\int_0^L\int_{\tau_1}^{\tau_2} |\mu_2(s)|z_2^2(x,1,s,t)\,ds\,dx, \quad (3.20)
\end{aligned}$$

where

$$\gamma_1 = 2\delta_1(\zeta_6) - \frac{L^2}{\pi^2}\Big(\int_{\tau_1}^{\tau_2} |\mu_1(s)|\,ds - \mu_0\Big) > 0,$$

$$\gamma_2 = 2\delta_2(\xi_6) - \frac{L^2}{\pi^2}\left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds - \mu_0'\right) > 0.$$

At this point, we need to choose our constants very carefully.

First, we choose $\varepsilon_2 < \frac{\pi^2}{2L^2} p_1(\zeta_1)$ and $\varepsilon_2' < \frac{\pi^2}{2L^2} p_2(\xi_1)$ so that

$$p_1(\zeta_1) - \frac{L^2}{\pi^2}\varepsilon_2 > \frac{p_1(\zeta_1)}{2}, \quad p_2(\xi_1) - \frac{L^2}{\pi^2}\varepsilon_2' > \frac{p_2(\xi_1)}{2}.$$

Next, we choose N_1 and N_2 large enough so that

$$n_1 N_1 - \frac{\mu_0}{4\varepsilon_2} > 0, \quad n_2 N_2 - \frac{\mu_0'}{4\varepsilon_2'} > 0.$$

Then, we choose ε_1 and ε_1' small enough satisfying

$$\frac{p_1(\zeta_1)}{2} - \frac{L^2\mu_0^2}{\pi^2}\varepsilon_1 > 0, \quad \frac{p_2(\xi_1)}{2} - \frac{L^2\mu_0'^2}{\pi^2}\varepsilon_1' > 0.$$

Finally, we choose N large enough so that

$$\gamma_1 N - \frac{L^2}{\pi^2}\left(m_1(\zeta_2) + \frac{1}{4\varepsilon_1}\right) - \frac{L^2\mu_0}{\pi^2} N_1 > 0,$$
$$\gamma_2 N - \frac{L^2}{\pi^2}\left(m_2(\xi_2) + \frac{1}{4\varepsilon_1'}\right) - \frac{L^2\mu_0'}{\pi^2} N_2 > 0.$$

By (3.1), we deduce that there exists a positive constant c_3 such that (3.20) becomes

$$L'(t) \le -c_3 E(t), \ \ \forall\, t \ge 0. \tag{3.21}$$

The combination of (3.19) and (3.21) gives

$$L'(t) \le -\lambda_1 L(t), \ \ \forall\, t \ge 0, \tag{3.22}$$

where $\lambda_1 = \frac{c_3}{c_2}$. Then a simple integration of (3.22) over $(0,t)$ yields

$$c_1 E(t) \le L(t) \le L(0)e^{-\lambda_1 t}, \ \ \forall\, t \ge 0. \tag{3.23}$$

Finally, combining (3.19) and (3.23), we obtain (3.2) with $\lambda_0 = \frac{c_2 E(0)}{c_1}$, which completes the proof. □

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STUDY OF STABILITY IN NONLINEAR NEUTRAL DYNAMIC EQUATIONS USING KRASNOSELSKII–BURTON'S FIXED POINT THEOREM

Abstract. Let $\mathbb{T}$ be an unbounded above and below time scale such that $0 \in \mathbb{T}$. Let $id - \tau : [0, \infty) \cap \mathbb{T}$ be such that $(id - \tau)([0, \infty) \cap \mathbb{T})$ is a time scale. We use Krasnoselskii–Burton's fixed point theorem to obtain stability results about the zero solution for the following nonlinear neutral dynamic equation with a variable delay:

$$x^{\Delta}(t) = -a(t)h(x^{\sigma}(t)) + Q(t, x(t - \tau(t)))^{\Delta} + G\big(t, x(t), x(t - \tau(t))\big).$$

The stability of the zero solution of this equation is provided by $h(0) = Q(t, 0) = G(t, 0, 0) = 0$. The Carathéodory condition is used for the functions Q and G. The results obtained here extend the work of Mesmouli, Ardjouni and Djoudi [21].

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Key words and phrases. Krasnoselskii–Burton's theorem, large contraction, neutral dynamic equation, integral equation, stability, time scales.

რეზიუმე. ვთქვათ, $\mathbb{T}$ ზემოდან და ქვემოდან შემოუსაზღვრელი ისეთი დროის სკალაა, რომ $0 \in \mathbb{T}$, ხოლო $id - \tau : [0, \infty) \cap \mathbb{T}$ ისეთია, რომ $(id - \tau)([0, \infty) \cap \mathbb{T})$ წარმოადგენს დროის სკალას. განხილულია შემდეგი არაწრფივი ნეიტრალური დინამიკური განტოლება ცვლადი დაგვიანებით

$$x^{\Delta}(t) = -a(t)h(x^{\sigma}(t)) + Q(t, x(t - \tau(t)))^{\Delta} + G\big(t, x(t), x(t - \tau(t))\big)$$

ამ განტოლების ნულოვანი ამონახსნის მდგრადობის დასადგენად გამოყენებულია კრასნოსელ-სკი-ბარტონის უძრავი წერტილის თეორემა. ნულოვანი ამოხსნის მდგრადობას უზრუნველყოფს პირობა $h(0) = Q(t, 0) = G(t, 0, 0) = 0$. Q და G ფუნქციებისთვის გამოიყენება კარათეოდორის პირობა. მიღებული შედეგები ანზოგადებს მესმულის, არჯაუნისა და ჯაუდის [21] ნაშრომში მიღებულ შედეგებს.

1 Introduction

The concept of time scales analysis is a fairly new idea. In 1988, it was introduced by the German mathematician Stefan Hilger in his Ph.D. thesis [17]. It combines the traditional areas of continuous and discrete analysis into one theory. After the publication of two textbooks in this area by Bohner and Peterson [9] and [10], more and more researchers were getting involved in this fast-growing field of mathematics. The study of dynamic equations brings together the traditional research areas of differential and difference equations. It allows one to handle these two research areas simultaneously, hence shedding light on the reasons for their seeming discrepancies. In fact, many new results for the continuous and discrete cases have been obtained by studying the more general time scales case (see [1, 4–6, 18] and the references therein).

There is no doubt that the Lyapunov method have been used successfully to investigate stability properties of wide variety of ordinary, functional and partial equations. Nevertheless, the application of this method to the problem of stability in differential equations with a delay has encountered serious difficulties if the delay is unbounded or if the equation has an unbounded term. It has been noticed that some of theses difficulties vanish by using the fixed point technic. Other advantages of fixed point theory over Lyapunov's method is that the conditions of the former are average, while those of the latter are pointwise (see [2–4, 6–8, 12–15, 18–22] and the references therein).

In this paper, we consider the nonlinear neutral dynamic equations with a variable delay given by

$$x^{\Delta}(t) = -a(t)h(x^{\sigma}(t)) + (Q(t, x(t-\tau(t))))^{\Delta} + G\big(t, x(t), x(t-\tau(t))\big), \tag{1.1}$$

with an assumed initial function

$$x(t) = \psi(t), \quad t \in [m_0, 0] \cap \mathbb{T},$$

where $\mathbb{T}$ is an unbounded above and below time scale such that $0 \in \mathbb{T}$.

Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due to Burton (see [12, Theorem 3]) to show the asymptotic stability and the stability of the zero solution for equation (1.1). Clearly, the present problem is totally nonlinear so that the variation of parameters cannot be applied directly. Then we resort to the idea of adding and subtracting a linear term. As is noted by T. A. Burton in [12], the added term destroys a contraction already present in part of the equation but it replaces it with the so-called large contraction mapping which is suitable for the fixed point theory. During the process we have to transform (1.1) into an integral equation written as a sum of two mappings; one is a large contraction and the other is compact. After that, we use a variant of Krasnoselskii's fixed point theorem to show the asymptotic stability and the stability of the zero solution for equation (1.1). In the special case $\mathbb{T} = \mathbb{R}$, Mesmouli, Ardjouni and Djoudi [21] show that the zero solution of (1.1) is asymptotically stable by using Krasnoselskii–Burton's fixed point theorem. Then the results presented in this paper extend the main results obtained in [21].

The paper is organized as follows. In Section 2, we present some preliminary material that we will need through the remainder of the paper. We will state some facts about the exponential function on a time scale. In Section 3, we present the inversion of (1.1) and state the modification of Krasnoselskii's fixed point theorem established by Burton (see [10, Theorem 3] and [14]). For details on Krasnoselskii's theorem, we refer the reader to [23]. We present our main results on the stability in Section 4.

In this paper, we give the assumptions below that will be used in the main results.

(H1) $\tau : [0,\infty) \cap \mathbb{T} \to \mathbb{T}$ is a positive right dense continuous (rd-continuous) function, $id - \tau : [0,\infty) \cap \mathbb{T} \to \mathbb{T}$ is an increasing mapping such that $(id-\tau)([0,\infty)\cap\mathbb{T})$ is closed, where id is the identity function. Moreover, there exists a constant $l_2 > 0$ such that for $0 \le t_1 < t_2$

$$|\tau(t_2) - \tau(t_1)| \le l_2 |t_2 - t_1|.$$

(H2) $\psi : [m_0, 0] \cap T \to \mathbb{R}$ is a rd-continuous function with $m_0 = -\tau(0)$.

(H3) $a : [0,\infty) \cap \mathbb{T} \to (0,\infty)$ is a bounded rd-continuous function and there exists a constant $l_3 > 0$ such that for $0 \le t_1 < t_2$,

$$\left| \int_{t_1}^{t_2} a(u)\,\Delta u \right| \le l_3 |t_2 - t_1|.$$

(H4) $Q : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function and $Q(t,0) = 0$, that is, for $t_1, t_2 \geq 0$ and $x, y \in [-R, R]$, where $R \in (0,1]$, there exist the constants $l_0, E_Q > 0$ such that

$$|Q(t_1, x) - Q(t_2, y)| \leq l_0|t_1 - t_2| + E_Q|x - y|.$$

Also, Q is a bounded function satisfying the Carathéodory condition with respect to $L^1_\Delta([0,\infty) \cap \mathbb{T})$ such that

$$\left|Q(t, \varphi(t - \tau(t)))\right| \leq q_R(t) \leq \frac{\alpha_1}{2} R,$$

where α_1 is a positive constant.

(H5) The function $G : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition with respect to $L^1_\Delta([0,\infty) \cap \mathbb{T})$, G/a is a bounded function and $G(t,0,0) = 0$ such that for $t \geq 0$,

$$\left|G(t, \varphi(t), \varphi(t - \tau(t)))\right| \leq g_{\sqrt{2}\,R}(t) \leq \alpha_2 a(t) R,$$

where α_2 is a positive constant.

(H6) There exists a constant $J > 3$ such that

$$J(\alpha_1 + \alpha_2) \leq 1$$

and

$$(E_Q + E_Q l_2) l_1 + l_0 + 3R\Big(\frac{\alpha_1}{2} + \alpha_2 + \frac{2}{J}\Big) l_3 < l_1,$$

where l_1 is a positive constant.

(H7) $h : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing on $[-R, R]$, $h(0) = 0$, h is differentiable on $(-R, R)$ with $h'(x) \leq 1$ for $x \in (-R, R)$.

(H8) For $\gamma > 0$ small enough,

$$[1 + E_Q]\gamma + (E_Q + E_Q l_2) l_1 + l_0 + 3R\Big(\frac{\alpha_1}{2} + \alpha_2 + \frac{2}{J}\Big) l_3 \leq l_1$$

and

$$[1 + E_Q]\gamma e_{\ominus a}(t, 0) + \frac{3R}{J} \leq R.$$

Also,

$$\max\big\{|H(-R)|, |H(R)|\big\} \leq \frac{2R}{J},$$

where $H(x) = x^\sigma - h(x^\sigma)$.

(H9) $t - \tau(t) \to \infty$, $e_{\ominus a}(t, 0) \to 0$, $q_R(t) \to 0$ and $\frac{g_{\sqrt{2}R}(t)}{a(t)} \to 0$ as $t \to \infty$.

2 Preliminaries

In this section, we consider some advanced topics in the theory of dynamic equations on a time scales. Again, we remind that for a review of this topic we direct the reader to the monographs of Bohner and Peterson [9] and [10].

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t \in \mathbb{T}$, the forward jump operator σ and the backward jump operator ρ, respectively, are defined as $\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}$. These operators allow the elements in the time scale to be classified as follows. We say t is right scattered if $\sigma(t) > t$ and right dense if $\sigma(t) = t$. We say t is left scattered if $\rho(t) < t$ and left dense if $\rho(t) = t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ and gives the distance between an element and its successor. We set $\inf \varnothing = \sup \mathbb{T}$ and $\sup \varnothing = \inf \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$. Otherwise, we define $\mathbb{T}_k = \mathbb{T}$.

Let $t \in \mathbb{T}^k$ and let $f : \mathbb{T} \to \mathbb{R}$. The delta derivative of $f(t)$, denoted by $f^\Delta(t)$, is defined to be the number (if any) with the property that for each $\varepsilon > 0$, there is a neighborhood U of t such that

$$\big|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]\big| \le \varepsilon|\sigma(t) - s|$$

for all $s \in U$. If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

A function f is rd-continuous, $f \in C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at every right dense point $t \in \mathbb{T}$ and its left-hand limits exist at each left dense point $t \in \mathbb{T}$. The function $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}^k$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$.

We are now ready to state some properties of the delta-derivative of f. Note that $f^\sigma(t) = f(\sigma(t))$.

Theorem 2.1 ([9, Theorem 1.20])**.** *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and let α be a scalar.*

(i) $(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$.

(ii) $(\alpha f)^\Delta(t) = \alpha f^\Delta(t)$.

(iii) *The product rules*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t),$$
$$(fg)^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t).$$

(iv) *If $g(t)g^\sigma(t) \neq 0$, then*

$$\Big(\frac{f}{g}\Big)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)}.$$

The next theorem is the chain rule on time scales (see [9, Theorem 1.93]).

Theorem 2.2 (Chain Rule)**.** *Assume $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. Let $w : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $v^\Delta(t)$ and $w^{\widetilde{\Delta}}(v(t))$ exist for $t \in \mathbb{T}^k$, then*

$$(w \circ v)^\Delta = (w^{\widetilde{\Delta}} \circ v)v^\Delta.$$

In the sequel, we will need to differentiate and integrate functions of the form $f(t-\tau(t)) = f(v(t))$, where $v(t) := t - \tau(t)$. Our next theorem is the substitution rule (see [9, Theorem 1.98]).

Theorem 2.3 (Substitution)**.** *Assume $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and v is differentiable with an rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)v^\Delta(t)\,\Delta t = \int_{v(a)}^{v(b)} (f \circ v^{-1})(s)\,\widetilde{\Delta} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$. The set of all positively regressive functions $\mathcal{R}^+$ is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t,s) = \exp\bigg(\int_s^t \frac{1}{\mu(z)} \log(1 + \mu(z)p(z))\,\Delta z\bigg).$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given by the following lemma.

Lemma 2.1 ([9, Theorem 2.36]). *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$,

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$,

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$,

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$,

(vi) $e_p^{\Delta}(\,\cdot\,, s) = pe_p(\,\cdot\,, s)$ *and* $(\frac{1}{e_p(\,\cdot\,,s)})^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\,\cdot\,,s)}$.

Lemma 2.2 ([1]). *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t,s) \le \exp\Big(\int_s^t p(u)\,\Delta u\Big), \quad \forall\, t \in \mathbb{T}.$$

Corollary 2.1 ([1]). *If $p \in \mathcal{R}^+$ and $p(t) < 0$ for all $t \in \mathbb{T}$, then for all $s \in \mathbb{T}$ with $s \le t$ we have*

$$0 < e_p(t,s) \le \exp\Big(\int_s^t p(u)\,\Delta u\Big) < 1.$$

3 The inversion and the fixed point theorem

We begin this section with the following

Lemma 3.1. *x is a solution of equation* (1.1) *if and only if*

$$\begin{aligned} x(t) = \big[\psi(0) - Q(0, \psi(-\tau(0)))\big]e_{\ominus a}(t,0) + \int_0^t a(s)e_{\ominus a}(t,s)H(x(s))\,\Delta s + Q(t, x(t-\tau(t))) \\ + \int_0^t e_{\ominus a}(t,s)\Big[-a(s)Q^{\sigma}(s, x(s-\tau(s))) + G\big(s, x(s), x(s-\tau(s))\big)\Big]\,\Delta s, \end{aligned} \quad (3.1)$$

where

$$H(x) = x^{\sigma} - h(x^{\sigma}). \quad (3.2)$$

Proof. Let x be a solution of (1.1). Rewrite equation (1.1) as

$$\begin{aligned} \big(x(t) - Q(t, x(t-\tau(t)))\big)^{\Delta} + a(t)\big[x^{\sigma}(t) - Q^{\sigma}(t, x(t-\tau(t)))\big] \\ = a(t)\big[x^{\sigma}(t) - h(x^{\sigma}(t))\big] - a(t)Q^{\sigma}(t, x(t-\tau(t))) + G\big(t, x(t), x(t-\tau(t))\big). \end{aligned}$$

Multiplying both sides of the above equation by $e_a(t,0)$ and then integrating from 0 to t, we obtain

$$\begin{aligned} \int_0^t \big((x(s) - Q(s, x(s-\tau(s))))e_a(s,0)\big)^{\Delta}\,\Delta s = \int_0^t a(s)\big[x^{\sigma}(s) - h(x^{\sigma}(s))\big]e_a(s,0)\,\Delta s \\ + \int_0^t \Big[-a(s)Q^{\sigma}(s, x(s-\tau(s))) + G\big(s, x(s), x(s-\tau(s))\big)\Big]e_a(s,0)\,\Delta s. \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}\big[x(t) - Q(t, x(t-\tau(t)))\big]e_a(t,0) - \psi(0) + Q(0, \psi(-\tau(0)))\\ = \int_0^t a(s)[x^\sigma(s) - h(x^\sigma(s))]e_a(s,0)\,\Delta s\\ + \int_0^t \Big[-a(s)Q^\sigma(s, x(s-\tau(s))) + G\big(s, x(s), x(s-\tau(s))\big)\Big]e_a(s,0)\,,\Delta s.\end{aligned}$$

By dividing both sides of the above equation by $e_a(t,0)$, we obtain

$$\begin{aligned}x(t) - Q(t, x(t-\tau(t))) - \big[\psi(0) - Q(0, \psi(-\tau(0)))\big]e_{\ominus a}(t,0)\\ = \int_0^t a(s)[x^\sigma(s) - h(x^\sigma(s))]e_{\ominus a}(t,s)\,\Delta s\\ + \int_0^t \Big[-a(s)Q^\sigma(s, x(s-\tau(s))) + G\big(s, x(s), x(s-\tau(s))\big)\Big]e_{\ominus a}(t,s)\,\Delta s.\end{aligned} \tag{3.3}$$

The converse implication is easily obtained and the proof is complete. □

Now, we give some definitions which will be used in this paper.

Definition 3.1. The map $f : [0,\infty) \cap \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$ is an L^1_Δ-Carathéodory function if it satisfies the following conditions:

(i) for each $z \in \mathbb{R}^n$, the mapping $t \mapsto f(t,z)$ is Δ-measurable,

(ii) for almost all $t \in [0,\infty) \cap \mathbb{T}$, the mapping $z \mapsto f(t,z)$ is continuous on $\mathbb{R}^n$,

(iii) for each $r > 0$, there exists $\alpha_r \in L^1_\Delta([0,\infty) \cap \mathbb{T}, \mathbb{R}^+)$ such that for almost all $t \in [0,\infty) \cap \mathbb{T}$ and for all z such that $|z| < r$, we have $|f(t,z)| \le \alpha_r(t)$.

T. A. Burton studied the theorem of Krasnoselskii (see [14] and [23]) and observed (see [11]) that Krasnoselskii's result may be more interesting in applications with certain changes, and formulated Theorem 3.1 below (see [11] for its proof).

Definition 3.2. Let $(\mathcal{M}, d)$ be a metric space and assume that $B : \mathcal{M} \to \mathcal{M}$. B is said to be a large contraction if for $\varphi, \phi \in \mathcal{M}$, with $\varphi \neq \phi$, we have $d(B\varphi, B\phi) < d(\varphi, \phi)$, and if $\forall\, \varepsilon > 0$, $\exists\, \delta < 1$ such that

$$\big[\varphi, \phi \in \mathcal{M},\ d(\varphi,\phi) \ge \varepsilon\big] \Longrightarrow d(B\varphi, B\phi) < \delta d(\varphi,\phi).$$

It is proved in [11] that a large contraction defined on a closed bounded and complete metric space has a unique fixed point.

Theorem 3.1 (Krasnoselskii–Burton)**.** *Let $\mathcal{M}$ be a closed bounded convex nonempty subset of a Banach space $(\chi, \|\cdot\|)$. Suppose that A and B map $\mathcal{M}$ into $\mathcal{M}$ such that*

(i) *A is continuous and $A\mathcal{M}$ is contained in a compact subset of $\mathcal{M}$,*

(ii) *B is large contraction,*

(iii) *$x, y \in \mathcal{M}$, implies $Ax + By \in \mathcal{M}$.*

Then there exists $z \in \mathcal{M}$ with $z = Az + Bz$.

Here we manipulate the function spaces defined on infinite t-intervals. So, for the compactness, we need an extension of Arzela–Ascoli's theorem. This extension is taken from [14, Theorem 1.2.2, p. 20] and is presented as follows.

Theorem 3.2. *Let $q : [0,\infty) \cap \mathbb{T} \to \mathbb{R}^+$ be an rd-continuous function such that $q(t) \to 0$ as $t \to \infty$. If $\{\varphi_n(t)\}$ is an equicontinuous sequence of $\mathbb{R}^m$-valued functions on $[0,\infty) \cap \mathbb{T}$ with $|\varphi_n(t)| \le q(t)$ for $t \in [0,\infty) \cap \mathbb{T}$, then there is a subsequence that converges uniformly on $[0,\infty) \cap \mathbb{T}$ to an rd-continuous function $\varphi(t)$ with $|\varphi(t)| \le q(t)$ for $[0,\infty) \cap \mathbb{T}$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^m$.*

4 The stability by Krasnoselskii–Burton's theorem

From the existence theory, which can be found in [14] or [16], we conclude that for each rd-continuous initial function $\psi \in C_{rd}([m_0,0] \cap \mathbb{T}, \mathbb{R})$, there exists an rd-continuous solution $x(t,0,\psi)$ which satisfies (1.1) on an interval $[0,\sigma) \cap \mathbb{T}$ for some $\sigma > 0$ and $x(t,0,\psi) = \psi(t)$, $t \in [m_0,0] \cap \mathbb{T}$. We refer the reader to [14] for the stability definitions.

Definition 4.1. The zero solution of (1.1) is said to be stable at $t = 0$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\psi : [m_0,0] \cap \mathbb{T} \to (-\delta,\delta)$ implies that $|x(t)| < \varepsilon$ for $t \ge m_0$.

Definition 4.2. The zero solution of (1.1) is said to be asymptotically stable if it is stable at $t = 0$ and there exists $\delta > 0$ such that for any rd-continuous function $\psi : [m_0,0] \cap \mathbb{T} \to (-\delta,\delta)$, the solution x with $x(t) = \psi(t)$ on $[m_0,0] \cap \mathbb{T}$ tends to zero as $t \to \infty$.

To apply Theorem 3.1, we need to define a Banach space χ, a closed bounded convex subset $\mathcal{M}$ of χ and construct two mappings; one large contraction and the other a compact operator. So, let $\omega : [m_0,\infty) \cap \mathbb{T} \to [1,\infty)$ be any strictly increasing and rd-continuous function with $\omega(m_0) = 1$, $\omega(t) \to \infty$ as $t \to \infty$. Let $(S, |\cdot|_\omega)$ be the Banach space of rd-continuous $\varphi : [m_0,\infty) \cap \mathbb{T} \to \mathbb{R}$ for which

$$|\varphi|_\omega = \sup_{t \ge m_0} \left| \frac{\varphi(t)}{\omega(t)} \right| < \infty.$$

Let $R \in (0,1]$ and define the set

$$\begin{aligned} \mathcal{M} := \Big\{ \varphi \in S :\ & \varphi \text{ is } l_1\text{-Lipschitzian}, \\ & |\varphi(t)| \le R,\ t \in [m_0,\infty) \cap \mathbb{T} \text{ and } \varphi(t) = \psi(t) \text{ if } t \in [m_0,0] \cap \mathbb{T} \Big\}. \end{aligned}$$

Clearly, if $\{\varphi_n\}$ is a sequence of l_1-Lipschitzian functions converging to some function φ, then

$$\begin{aligned} |\varphi(t) - \varphi(s)| &= \big|\varphi(t) - \varphi_n(t) + \varphi_n(t) - \varphi_n(s) + \varphi_n(s) - \varphi(s)\big| \\ &\le |\varphi(t) - \varphi_n(t)| + |\varphi_n(t) - \varphi_n(s)| + |\varphi_n(s) - \varphi(s)| \\ &\le l_1 |t - s| \end{aligned}$$

as $n \to \infty$, which implies that φ is l_1-Lipschitzian. It is clear that $\mathcal{M}$ is closed convex and bounded. For $\varphi \in \mathcal{M}$ and $t \ge 0$, we define by (3.1) the mapping $P : \mathcal{M} \to S$ as follows:

$$\begin{aligned} (P\varphi)(t) = \big[\psi(0) - Q(0,\psi(-\tau(0)))\big] e_{\ominus a}(t,0) + \int_0^t a(s) e_{\ominus a}(t,s) H(\varphi(s))\,\Delta s + Q(t,\varphi(t-\tau(t))) \\ + \int_0^t e_{\ominus a}(t,s) \Big[-a(s) Q^\sigma(s,\varphi(s-\tau(s))) + G\big(s,\varphi(s),\varphi(s-\tau(s))\big) \Big]\,\Delta s. \quad (4.1) \end{aligned}$$

Therefore, we express mapping (4.1) as

$$P\varphi = A\varphi + B\varphi,$$

where $A, B : \mathcal{M} \to S$ are given by

$$(A\varphi)(t) = Q(t, \varphi(t-\tau(t))) + \int_0^t e_{\ominus a}(t,s)\Big[-a(s)Q^\sigma(s,\varphi(s-\tau(s))) + G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\Big]\,\Delta s, \tag{4.2}$$

$$(B\varphi)(t) = \big[\psi(0) - Q(0,\psi(-\tau(0)))\big]e_{\ominus a}(t,0) + \int_0^t a(s)e_{\ominus a}(t,s)H(\varphi(s))\,\Delta s. \tag{4.3}$$

By applying Theorem 3.1, we need to prove that P has a fixed point φ on the set $\mathcal{M}$, where $x(t,0,\psi) = \varphi(t)$ for $t \geq 0$ and $x(t,0,\psi) = \psi(t)$ on $[m_0,0]\cap\mathbb{T}$, $x(t,0,\psi)$ satisfies (1.1) and $|x(t,0,\psi)| \leq R$ with $R \in (0,1]$.

By a series of steps we will prove the fulfillment of (i), (ii) and (iii) of Theorem 3.1.

Lemma 4.1. *For A defined in* (4.2)*, suppose that* (H1)–(H6) *hold. Then $A : \mathcal{M} \to \mathcal{M}$ and A is continuous and $A\mathcal{M}$ is contained in a compact subset of $\mathcal{M}$.*

Proof. Let A be defined by (4.2). Then for any $\varphi \in \mathcal{M}$, we have

$$\begin{aligned} |(A\varphi)(t)| &\leq |Q(t,\varphi(t-\tau(t)))| \\ &\quad + \int_0^t e_{\ominus a}(t,s)\Big[a(s)|Q^\sigma(s,\varphi(s-\tau(s)))| + \big|G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\big|\Big]\,\Delta s \\ &\leq q_R(t) + R\int_0^t e_{\ominus a}(t,s)\Big(a(s)\,\frac{q_R(s)}{R} + \frac{g_{\sqrt{2}R}(s)}{R}\Big)\,\Delta s \leq \frac{\alpha_1}{2}\,R + \frac{\alpha_1}{2}\,R + \alpha_2 R \leq \frac{R}{J} < R. \end{aligned}$$

That is, $|(A\varphi)(t)| < R$. Second, we show that for any $\varphi \in \mathcal{M}$, the function $A\varphi$ is l_1-Lipschitzian. Let $\varphi \in \mathcal{M}$, and let $0 \leq t_1 < t_2$, then

$$\begin{aligned} \big|(A\varphi)(t_2) - (A\varphi)(t_1)\big| &\leq \Big|Q(t_2,\varphi(t_2-\tau(t_2))) - Q(t_1,\varphi(t_1-\tau(t_1)))\Big| \\ &\quad + \Bigg|\int_0^{t_2} e_{\ominus a}(t_2,s)\Big[-a(s)Q^\sigma(s,\varphi(s-\tau(s))) + G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\Big]\,\Delta s \\ &\qquad - \int_0^{t_1} e_{\ominus a}(t_1,s)\Big[-a(s)Q^\sigma(s,\varphi(s-\tau(s))) + G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\Big]\,\Delta s\Bigg|. \end{aligned} \tag{4.4}$$

By hypotheses (H1), (H3) and (H4), we have

$$\begin{aligned} &\Big|Q(t_2,\varphi(t_2-\tau(t_2))) - Q(t_1,\varphi(t_1-\tau(t_1)))\Big| \\ &\qquad \leq l_0|t_2-t_1| + E_Q l_1\big|(t_2-t_1) - (\tau(t_2)-\tau(t_1))\big| \leq (l_0 + E_Q l_1 + E_Q l_1 l_2)|t_2-t_1|, \end{aligned} \tag{4.5}$$

where l_1 is the Lipschitz constant of φ. In the same way, by (H3)–(H5), we have

$$\begin{aligned} &\Bigg|\int_0^{t_2} e_{\ominus a}(t_2,s)\Big[-a(s)Q^\sigma(s,\varphi(s-\tau(s))) + G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\Big]\,\Delta s \\ &\qquad - \int_0^{t_1} e_{\ominus a}(t_1,s)\Big[-a(s)Q^\sigma(s,\varphi(s-\tau(s))) + G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\Big]\,\Delta s\Bigg| \end{aligned}$$

$$\le \Big|\int_0^{t_1}\Big[-a(s)Q^\sigma(s,\varphi(s-\tau(s)))+G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\Big]\,\Delta s\cdot e_{\ominus a}(t_1,s)(e_{\ominus a}(t_2,t_1)-1)\,\Delta s\Big|$$

$$+\Big|\int_{t_1}^{t_2} e_{\ominus a}(t_2,s)\Big[-a(s)Q^\sigma(s,\varphi(s-\tau(s)))+G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\Big]\,\Delta s\Big|$$

$$\le \Big(\frac{\alpha_1}{2}+\alpha_2\Big)R|e_{\ominus a}(t_2,t_1)-1|\int_0^{t_1} a(s)e_{\ominus a}(t_1,s)\,\Delta s$$

$$+\int_{t_1}^{t_2} e_{\ominus a}(t_2,s)\big(a(s)q_R(s)+g_{\sqrt{2}R}(s)\big)\,\Delta s$$

$$\le \Big(\frac{\alpha_1}{2}+\alpha_2\Big)R\int_{t_1}^{t_2} a(s)\,\Delta s+\int_{t_1}^{t_2} a(s)e_{\ominus a}(t_2,s)\Big(\int_{t_1}^{s}\big(a(r)q_R(r)+g_{\sqrt{2}R}(r)\big)\,\Delta r\Big)^{\Delta}\Delta s$$

$$\le \Big(\frac{\alpha_1}{2}+\alpha_2\Big)R\int_{t_1}^{t_2} a(s)\,\Delta s+\Big[e_{\ominus a}(t_2,s)\int_{t_1}^{s}\big(a(r)q_R(r)+g_{\sqrt{2}R}(r)\big)\,\Delta r\Big]_{t_1}^{t_2}$$

$$+\int_{t_1}^{t_2} a(s)e_{\ominus a}(t_2,s)\int_{t_1}^{s}\big(a(r)q_R(r)+g_{\sqrt{2}R}(r)\big)\,\Delta r\,\Delta s$$

$$\le \Big(\frac{\alpha_1}{2}+\alpha_2\Big)R\int_{t_1}^{t_2} a(s)\,\Delta s+\int_{t_1}^{t_2}\big(a(s)q_R(s)+g_{\sqrt{2}R}(s)\big)\,\Delta s\Big(1+\int_{t_1}^{t_2} a(s)e_{\ominus a}(t_2,s)\,\Delta s\Big)$$

$$\le \Big(\frac{\alpha_1}{2}+\alpha_2\Big)R\int_{t_1}^{t_2} a(s)\,\Delta s+2\int_{t_1}^{t_2}\big(a(s)q_R(s)+g_{\sqrt{2}R}(s)\big)\,\Delta s$$

$$\le \Big(\frac{\alpha_1}{2}+\alpha_2\Big)R\int_{t_1}^{t_2} a(s)\,\Delta s+2\Big(\frac{\alpha_1}{2}+\alpha_2\Big)R\int_{t_1}^{t_2} a(s)\,\Delta s\le 3R\Big(\frac{\alpha_1}{2}+\alpha_2\Big)l_3|t_2-t_1|. \tag{4.6}$$

Thus, by substituting (4.5) and (4.6) into (4.4), we obtain

$$\big|(A\varphi)(t_2)-(A\varphi)(t_1)\big|\le (l_0+E_Ql_1+E_Ql_1l_2)|t_2-t_1|+3R\Big(\frac{\alpha_1}{2}+\alpha_2\Big)l_3|t_2-t_1|\le l_1|t_2-t_1|.$$

This shows that $A\varphi$ is l_1-Lipschitzian if φ is. This completes the proof that $A:\mathcal{M}\to\mathcal{M}$.

Since $A\varphi$ is l_1-Lipschitzian, we have that $A\mathcal{M}$ is equicontinuous, which implies that the set $A\mathcal{M}$ resides in a compact set in the space $(S,|\cdot|_\omega)$.

Now, we show that A is continuous in the weighted norm letting $\varphi_n\in\mathcal{M}$, where n is a positive integer such that $\varphi_n\to\varphi$ as $n\to\infty$. Then

$$\Big|\frac{(A\varphi_n)(t)-(A\varphi)(t)}{\omega(t)}\Big|\le \Big|Q(t,\varphi_n(t-\tau(t)))-Q(t,\varphi(t-\tau(t)))\Big|_\omega$$

$$+\int_0^t a(s)e_{\ominus a}(t,s)\Big|Q^\sigma(s,\varphi_n(s-\tau(s)))-Q^\sigma(s,\varphi(s-\tau(s)))\Big|_\omega\,\Delta s$$

$$+\int_0^t e_{\ominus a}(t,s)\Big|G\big(s,\varphi_n(s),\varphi_n(s-\tau(s))\big)-G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\Big|_\omega\,\Delta s.$$

By the dominated convergence theorem, $\lim_{n\to\infty} |(A\varphi_n)(t) - (A\varphi)(t)|_\omega = 0$. Then A is continuous. This completes the proof that $A : \mathcal{M} \to \mathcal{M}$ is continuous and $A\mathcal{M}$ is contained in a compact subset of $\mathcal{M}$. □

Now, we state an important result implying that the mapping H given by (3.2) is a large contraction on the set $\mathcal{M}$. This result was already obtained in [1] and for convenience we present below its proof.

Theorem 4.1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying* (H7). *Then the mapping H in* (3.2) *is a large contraction on the set $\mathcal{M}$.*

Proof. Let $\varphi^\sigma, \phi^\sigma \in \mathcal{M}$ with $\varphi^\sigma \neq \phi^\sigma$. Then $\varphi^\sigma(t) \neq \phi^\sigma(t)$ for some $t \in \mathbb{T}$. Let us denote the set of all such t by $D(\varphi, \phi)$, i.e.,

$$D(\varphi, \phi) = \big\{t \in \mathbb{T} :\ \varphi^\sigma(t) \neq \phi^\sigma(t)\big\}.$$

For all $t \in D(\varphi, \phi)$, we have

$$\begin{aligned}|(H\varphi)(t) - (H\phi)(t)| &\le \big|\varphi^\sigma(t) - \phi^\sigma(t) - h(\varphi^\sigma(t)) + h(\phi^\sigma(t))\big| \\ &\le |\varphi^\sigma(t) - \phi^\sigma(t)| \left|1 - \frac{h(\varphi^\sigma(t)) - h(\phi^\sigma(t))}{\varphi^\sigma(t) - \phi^\sigma(t)}\right|. \end{aligned} \tag{4.7}$$

Since h is a strictly increasing function, we have

$$\frac{h(\varphi^\sigma(t)) - h(\phi^\sigma(t))}{\varphi^\sigma(t) - \phi^\sigma(t)} > 0 \ \text{ for all } \ t \in D(\varphi, \phi). \tag{4.8}$$

For each fixed $t \in D(\varphi, \phi)$, we define the interval $I_t \subset [-R, R]$ by

$$I_t = \begin{cases} (\varphi^\sigma(t), \phi^\sigma(t)) & \text{if } \varphi^\sigma(t) < \phi^\sigma(t), \\ (\phi^\sigma(t), \varphi^\sigma(t)) & \text{if } \phi^\sigma(t) < \varphi^\sigma(t). \end{cases}$$

The Mean Value Theorem implies that for each fixed $t \in D(\varphi, \phi)$ there exists a real number $c_t \in I_t$ such that

$$\frac{h(\varphi^\sigma(t)) - h(\phi^\sigma(t))}{\varphi^\sigma(t) - \phi^\sigma(t)} = h'(c_t).$$

By (H7), we have

$$0 \le \inf_{s\in(-R,R)} h'(s) \le \inf_{s\in I_t} h'(s) \le h'(c_t) \le \sup_{s\in I_t} h'(s) \le \sup_{s\in(-R,R)} h'(s) \le 1. \tag{4.9}$$

Hence, by (4.7)–(4.9), we obtain

$$|(H\varphi)(t) - (H\phi)(t)| \le |\varphi^\sigma(t) - \phi^\sigma(t)| \left|1 - \inf_{s\in(-R,R)} h'(s)\right| \tag{4.10}$$

for all $t \in D(\varphi, \phi)$. This implies a large contraction in the supremum norm. To see this, choose a fixed $\varepsilon \in (0, 1)$ and assume that φ and ϕ are two functions in $\mathcal{M}$ satisfying

$$\varepsilon \le \sup_{t\in(-R,R)} |\varphi(t) - \phi(t)| = \|\varphi - \phi\|.$$

If $|\varphi^\sigma(t) - \phi^\sigma(t)| \le \frac{\varepsilon}{2}$ for some $t \in D(\varphi, \phi)$, then we get by (4.9) and (4.10) that

$$\big|(H\varphi)(t) - (H\phi)(t)\big| \le \frac{1}{2}\, |\varphi^\sigma(t) - \phi^\sigma(t)| \le \frac{1}{2}\, \|\varphi - \phi\|. \tag{4.11}$$

Since h is continuous and strictly increasing, the function $h(s + \frac{\varepsilon}{2}) - h(s)$ attains its minimum on the closed and bounded interval $[-R, R]$. Thus, if $\frac{\varepsilon}{2} \le |\varphi^\sigma(t) - \phi^\sigma(t)|$ for some $t \in D(\varphi, \phi)$, then by (H7) we conclude that

$$1 \ge \frac{h(\varphi^\sigma(t)) - h(\phi^\sigma(t))}{\varphi^\sigma(t) - \phi^\sigma(t)} > \lambda,$$

where

$$\lambda := \frac{1}{2R}\min\Big\{h\Big(s+\frac{\varepsilon}{2}\Big)-h(s):\ s\in[-R,R]\Big\}>0.$$

Hence, (4.7) implies

$$\big|(H\varphi)(t)-(H\phi)(t)\big|\le(1-\lambda)\|\varphi-\phi\|. \tag{4.12}$$

Consequently, combining (4.11) and (4.12) we obtain

$$\big|(H\varphi)(t)-(H\phi)(t)\big|\le\delta\|\varphi-\phi\|,$$

where

$$\delta=\max\Big\{\frac{1}{2},1-\lambda\Big\}.$$

□

The relations of (H8) will be used below in Lemma 4.2 and Theorem 4.2 to show that if $\varepsilon=R$ and $\|\psi\|<\gamma$, then the solution satisfies $|x(t,0,\psi)|<\varepsilon$.

Lemma 4.2. *Let B be defined by* (4.3). *Suppose that* (H1)–(H3), (H7) *and* (H8) *hold. Then $B:\mathcal{M}\to\mathcal{M}$ and B is a large contraction.*

Proof. Let B be defined by (4.3). Obviously, B is continuous with the weighted norm. Let $\varphi\in\mathcal{M}$,

$$\begin{aligned}|(B\varphi)(t)|&\le\big|\psi(0)-Q(0,\psi(-\tau(0)))\big|e_{\ominus a}(t,0)+\int_0^t a(s)e_{\ominus a}(t,s)|H(\varphi(s))|\,\Delta s\\&\le[1+E_Q]\gamma e_{\ominus a}(t,0)+\int_0^t a(s)e_{\ominus a}(t,s)\max\big\{|H(-R)|,|H(R)|\big\}\,\Delta s\le R,\end{aligned}$$

and we use a method like in Lemma 4.1 and deduce that for any $\varphi\in\mathcal{M}$, the function $B\varphi$ is l_1-Lipschitzian, which implies $B:\mathcal{M}\to\mathcal{M}$.

By Theorem 4.1, H is a large contraction on $\mathcal{M}$, then for any $\varphi,\phi\in\mathcal{M}$ with $\varphi\ne\phi$ and for any $\varepsilon>0$, from the proof of that theorem, we have found that $\delta<1$ such that

$$\Big|\frac{B\varphi(t)-B\phi(t)}{\omega(t)}\Big|\le\int_0^t a(s)e_{\ominus a}(t,s)\big|H(\varphi(s))-H(\phi(s))\big|_\omega\,\Delta s\le\delta|\varphi-\phi|_\omega.$$

□

Theorem 4.2. *Assume that* (H1)–(H8) *hold. Then the zero solution of* (1.1) *is stable.*

Proof. By Lemmas 4.1 and 4.3, $A:\mathcal{M}\to\mathcal{M}$ is continuous and $A\mathcal{M}$ is contained in a compact set. Also, from Lemma 4.2, the mapping $B:\mathcal{M}\to\mathcal{M}$ is a large contraction. First, we show that if $\varphi,\phi\in\mathcal{M}$, we have $\|A\varphi+B\phi\|\le R$. Let $\varphi,\phi\in\mathcal{M}$ with $\|\varphi\|,\|\phi\|\le R$, then

$$\|A\varphi+B\phi\|\le(1+E_Q)\gamma e_{\ominus a}(t,0)+(\alpha_1+\alpha_2)R+\frac{2R}{J}\le(1+E_Q)\gamma e_{\ominus a}(t,0)+\frac{R}{J}+\frac{2R}{J}\le R.$$

Next, we prove that for any $\varphi,\phi\in\mathcal{M}$, the function $A\varphi+B\phi$ is l_1-Lipschitzian. Let $\varphi,\phi\in\mathcal{M}$, and let $0\le t_1<t_2$, then

$$\begin{aligned}&\big|(A\varphi+B\phi)(t_2)-(A\varphi+B\phi)(t_1)\big|\\&\quad\le\Big([1+E_Q]\gamma+(E_Q+E_Ql_2)l_1+l_0+3R\Big(\frac{\alpha_1}{2}+\alpha_2+\frac{2}{J}\Big)l_3\Big)|t_2-t_1|\le l_1|t_2-t_1|.\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii–Burton theorem are satisfied. Thus there exists a fixed point $z\in\mathcal{M}$ such that $z=Az+Bz$. By Lemma 3.1, this fixed point is a solution of (1.1). Hence, the zero solution of (1.1) is stable. □

Remark 1. When $\mathbb{T}=\mathbb{R}$, Theorem 4.2 reduces to Theorem 4 of [21]. Therefore, Theorem 4.2 is a generalization of Theorem 4 of [21].

Now, for the asymptotic stability, define $\mathcal{M}_0$ by

$$\mathcal{M}_0 := \Big\{\varphi \in S:\ \varphi \text{ is } l_1\text{-Lipschitzian},\ |\varphi(t)| \le R,\ t \in [m_0,\infty)\cap\mathbb{T},$$
$$\varphi(t)=\psi(t) \text{ if } t\in[m_0,0]\cap\mathbb{T} \text{ and } |\varphi(t)|\to 0 \text{ as } t\to\infty\Big\}.$$

All calculations in the proof of Theorem 4.2 hold with $\omega(t)=1$, when $|\cdot|_\omega$ is replaced by the supremum norm $\|\cdot\|$.

Lemma 4.3. *Let* (H1)–(H6) *and* (H9) *hold. Then the operator A maps $\mathcal{M}$ into a compact subset of $\mathcal{M}$.*

Proof. First, we deduce by Lemma 4.1 that $A\mathcal{M}$ is equicontinuous. Next, we notice that for an arbitrary $\varphi\in\mathcal{M}$, we have

$$|(A\varphi)(t)| \le q_R(t) + \int_0^t e_{\ominus a}(t,s)a(s)\Big(q_R(s)+\frac{g_{\sqrt{2}R}(s)}{a(s)}\Big)\,\Delta s := q(t).$$

We see that $q(t)\to 0$ as $t\to\infty$ which implies that the set $A\mathcal{M}$ resides in a compact set in the space $(S,\|\cdot\|)$ by Theorem 3.2. □

Theorem 4.3. *Assume that* (H1)–(H9) *hold. Then the zero solution of* (1.1) *is asymptotically stable.*

Proof. Note that all of the steps in the proof of Theorem 4.2 hold with $\omega(t)=1$ when $|\cdot|_\omega$ is replaced by the supremum norm $\|\cdot\|$. It suffices to show that for $\varphi\in\mathcal{M}_0$ we have $A\varphi\to 0$ and $B\varphi\to 0$. Let $\varphi\in\mathcal{M}_0$ be fixed, we will prove that $|(A\varphi)(t)|\to 0$ as $t\to\infty$. As above, we get

$$\begin{aligned}|(A\varphi)(t)| \le{}& \big|Q(t,\varphi(t-\tau(t)))\big|\\ &+\int_0^t e_{\ominus a}(t,s)\Big[a(s)\big|Q^\sigma(s,\varphi(s-\tau(s)))\big| + \big|G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\big|\Big]\,\Delta s.\end{aligned}$$

First of all, we have

$$\big|Q(t,\varphi(t-\tau(t)))\big| \le q_R(t)\to 0 \ \text{ as } \ t\to\infty.$$

Second, let $\varepsilon>0$ be given. Find T such that $|\varphi(t-\tau(t))|, |\varphi(t)|<\varepsilon$ for $t\ge T$. Then we have

$$\begin{aligned}&\int_0^t e_{\ominus a}(t,s)\Big[a(s)\big|Q^\sigma(s,\varphi(s-\tau(s)))\big| + \big|G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\big|\Big]\,\Delta s\\ &\quad= e_{\ominus a}(t,T)\int_0^T e_{\ominus a}(T,s)\Big[a(s)\big|Q^\sigma(s,\varphi(s-\tau(s)))\big| + \big|G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\big|\Big]\,\Delta s\\ &\qquad+\int_T^t e_{\ominus a}(t,s)\Big[a(s)\big|Q^\sigma(s,\varphi(s-\tau(s)))\big| + \big|G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\big|\Big]\,\Delta s\\ &\qquad\le e_{\ominus a}(t,T)\Big(\frac{\alpha_1}{2}+\alpha_2\Big)R+\Big(\frac{\alpha_1}{2}+\alpha_2\Big)\varepsilon.\end{aligned}$$

By (H9), the term $e_{\ominus a}(t,T)(\frac{\alpha_1}{2}+\alpha_2)R$ is arbitrarily small as $t\to\infty$. In the same way, we obtain $B\varphi\to 0$. Then, by the Krasnoselskii–Burton theorem, there exists a fixed point $z\in\mathcal{M}_0$ such that $z=Az+Bz$. By Lemma 3.1, this fixed point is a solution of (1.1). Hence, the zero solution of (1.1) is asymptotically stable. □

Remark 2.

1) When $\mathbb{T} = \mathbb{R}$, Theorem 4.3 reduces to Theorem 5 of [21]. Therefore, Theorem 4.3 is a generalization of Theorem 5 of [21].
2) The sufficient conditions (H1)–(H9) of Theorem 4.3 are essential for applying Theorems 3.1 and 3.2.

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STABILIZATION OF BILINEAR TIME-VARYING SYSTEMS WITH NORM-BOUNDED CONTROLS

Abstract. In this paper, we consider a class of bilinear time-varying systems. We study the stabilization problem for these systems with norm-bounded controls by using Lyapunov techniques and the solutions of Riccati differential equations. A numerical example is given to illustrate the efficiency of the obtained result.

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რეზიუმე. ნაშრომში განხილულია ორადწრფივი, დროის მიმართ ცვლადი სისტემების კლასი. ლიაპუნოვის ტექნიკისა და რიკატის დიფერენციალური განტოლებების ამონახსნების გამოყენებით შესწავლილია მდგრადობის ამოცანა ასეთი სისტემებისთვის ნორმით შეზღუდული მართვით. მიღებული შედეგის ეფექტურობის საილუსტრაციოდ მოყვანილია რიცხვითი მაგალითი.

1 Introduction

The problem of controllability and stabilizability for linear control systems has received a considerable amount of interest in the last few years [5,9,10]. This problem is an extension of the classical Kalman result [3] on the controllability and stability of linear control systems. Linear nonautonomous control systems are usually represented in the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in \mathbb{R}^+, \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. We assume that $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ are the matrices, continuously depending on t. The global null-controllability (GNC) problem of the linear system (1.1) concerns the question of finding an admissible control $u(t)$ which leads an arbitrary state x_0 to the origin. The stabilization problem is aimed by means of a linear control to find a control $u(t) = K(t)x(t)$ such that the zero solution of the closed-loop system

$$\dot{x}(t) = \big[A(t) + B(t)K(t)\big]x(t), \quad t \geq 0,$$

is asymptotically stable in the Lyapunov sense. In this case one says that the system is stabilizable with the stabilizing feedback control $u(t) = K(t)x(t)$. For linear time-varying (LTV) systems, the first result on the relationship between GNC problem and Riccati differential equation (RDE) was given in [3] where it was proven that if the LTV control system (1.1) is GNC, then the RDE

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) - P(t)B(t)B^T(t)P(t) + Q(t) = 0, \tag{1.2}$$

where $Q(t) \geq 0$, has a positive semi-definite solution $P(t)$. However, the existence of the positive definite solution $P(t)$ of the above RDE is not sufficient for the GNC. In [2], the authors prove that the system is completely stabilizable if it is uniformly globally null-controllable. In [6], the authors have developed the relationship between the exact controllability and complete stabilizability for linear time-varying control systems in Hilbert spaces. In [7], the authors study the stabilization of linear nonautonomous systems with norm-bounded controls (1.1), where the control $u(t)$ satisfies the following condition:

$$\|u(t)\| \leq r, \quad t \in \mathbb{R}^+.$$

For autonomous systems, where the constant matrix A satisfies some appropriate spectral properties, Slemrod [8] proposed a nonsmooth feedback control of the form

$$u(t) = \begin{cases} \dfrac{-rB^Tx(t)}{\|B^Tx(t)\|} & \text{if } \|B^Tx(t)\| \geq r, \\ -B^Tx(t) & \text{if } \|B^Tx(t)\| \leq r. \end{cases}$$

In this paper, we consider the following bilinear time-varying (BTV) control system:

$$\dot{x}(t) = A(t)x(t) + u(t)B(t)x(t), \quad t \in \mathbb{R}^+, \tag{1.3}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times n}$.

The purpose of this paper is to discuss the problem of global uniform stabilization of the BTV control system (1.3) with norm-bounded controls by using the Lyapunov techniques.

2 Preliminary results

We start by recalling some classical notation and definitions that will be useful throughout the paper.

- $\mathbb{R}^+$ denotes the set of all real nonnegative numbers.
- $\mathbb{R}^n$ denotes the n-dimensional space.
- $\langle x, y\rangle$ or x^Ty denote the scalar inner product of two vectors $x, y \in \mathbb{R}^n$.

- $\|x\|$ denotes the Euclidean vector norm of x.
- $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ matrices.
- I_n denotes the identity matrix.

Let $A \in \mathbb{R}^{n\times n}$:

- A^T denotes the transpose matrix of A; A is symmetric if and only if $A^T = A$.
- $\lambda(A)$ denotes the set of all eigenvalues of A.
- $\lambda_{\max}(A) = \max\{\mathrm{Re}(\lambda) : \ \lambda \in \lambda(A)\}$, $\lambda_{\min}(A) = \min\{\mathrm{Re}(\lambda) : \ \lambda \in \lambda(A)\}$.
- $\mu(A)$ denotes the matrix measure of the matrix A defined by
$$\mu(A) = \frac{1}{2}\,\lambda_{\max}(A + A^T).$$
- $\mathbf{L}_2([t,s],\mathbb{R})$ denotes the set of all square integrable $\mathbb{R}$-valued functions on $[t,s]$.
- The matrix A is bounded on $\mathbb{R}^+$ if there exists $M > 0$ such that $\sup\limits_{t\geq 0}\|A(t)\| \leq M$.
- The matrix $A \in \mathbb{R}^{n\times n}$ is positive semi-definite ($A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$.
- $\mathbf{M}([0,\infty),\mathbb{R}^n_+)$ is the set of all symmetric positive semi-definite matrix functions, continuous and bounded on $[0,\infty)$.
- The matrix function $A(t)$ is positive definite ($A(t) > 0$) if there exists a constant $c > 0$ such that $\langle A(t)x, x\rangle \geq c\|x\|^2$ for all $x \in \mathbb{R}^n$, $t \geq 0$.

Now, we recall some classical definitions and results.
Let the system is described by the equation
$$\dot{x} = f(t,x), \tag{2.1}$$
where the map $f : \mathbb{R} \times U \to \mathbb{R}^n$ is continuous locally Lipschitz with respect to x, $f(t,0) = 0\ \forall\, t \geq 0$, and U is an open set of $\mathbb{R}^n$ $(0 \in U)$. Denote by $x(t,t_0)$ the solution of (2.1) starting at x_0 at time t_0.

Definition 2.1. The equilibrium point $x = 0$ of system (2.1) is said to be

(i) stable if $\forall\, \varepsilon > 0$, $\forall\, t_0 \geq 0$, $\exists\, \delta = \delta(t_0,\varepsilon) > 0$ such that $\forall\, x_0 \in \mathbb{R}^n$ one has
$$\|x_0\| < \delta \Longrightarrow \|x(t,t_0)\| < \varepsilon, \ \ \forall\, t \geq t_0;$$

(ii) uniformly stable if (i) holds where $\delta = \delta(\varepsilon)$ is independent of t_0;

(iii) attractive if there exists a neighborhood $\mathcal{V}$ of 0 such that for any initial condition x_0 belonging to $\mathcal{V}$, the corresponding solution $x(t,t_0)$ is defined for all $t \geq 0$ and $\lim\limits_{t\to+\infty} x(t,t_0) = 0$. If $\mathcal{V} = \mathbb{R}^n$, then $x = 0$ is globally attractive;

(iv) asymptotically stable if it is stable and attractive;

(v) uniformly asymptotically stable if it is uniformly stable and, in addition, there exists $c > 0$ such that for all $\varepsilon > 0$, there exists $\tau > 0$ such that for all $x_0 \in \mathbb{R}^n$
$$\|x_0\| < c \Longrightarrow \|x(t,t_0)\| < \varepsilon, \ \ \forall\, t \geq \tau + t_0;$$

(vi) globally uniformly asymptotically stable if it is uniformly stable, $\delta(\varepsilon)$ can be chosen to satisfy $\lim\limits_{\varepsilon\to+\infty} \delta(\varepsilon) = +\infty$, and for all $c > 0$ and for all $\varepsilon > 0$, there exists $\tau > 0$ such that for all $x_0 \in \mathbb{R}^n$,
$$\|x_0\| < c \Longrightarrow \|x(t,t_0)\| < \varepsilon, \ \ \forall\, t \geq \tau + t_0.$$

Definition 2.2. The pair $(A(t), B(t))$ is said to be GNC if the associated linear control system (1.1) is GNC in the following sense:

for every $x_0 \in \mathbb{R}^n$, there exist a number $\tau > 0$ and an admissible control $u(t)$ such that $x(\tau) = 0$.

We recall the following controllability criterion that will be used later.

Proposition 2.1 ([1,3])**.** *The pair $(A(t), B(t))$ is GNC if and only if one of the following conditions holds:*

(i) *there exist $t > 0$ and $c > 0$ such that*

$$\int_0^t \left\|B^T(s)U^T(t,s)\right\| ds \geq c\,\|U^T(t,0)\|^2, \quad \forall\, x \in \mathbb{R}^n;$$

(ii) *$A(t)$, $B(t)$ are analytic on $\mathbb{R}_+$ and the rank $M(t_0) = n$ for some $t_0 > 0$, where*

$$M(t) = \big[M_0(t), M_1(t), \dots, M_{n-1}(t)\big];$$
$$M_0 := B(t), \quad M_{i+1}(t) = -A(t)M_i(t) + \frac{d}{dt}M_i(t), \quad i = 0, 1, \dots, n-2.$$

Definition 2.3. A scalar continuous function $\alpha(r)$ defined for $r \in [0, a[$ belongs to the class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It belongs to the class $\mathcal{K}_\infty$ if it is defined for all $r \geq 0$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Theorem 2.1 ([4])**.** *Let $r > 0$ and denote $\mathcal{B}_r = \{x \in \mathbb{R}^n, \ \ \|x\| < r\}$. Let $V : \mathbb{R}^+ \times \mathcal{B}_r \to \mathbb{R}$ be a smooth function. If there exists functions α_1, α_2 and α_3 of the class $\mathcal{K}$ defined on $[0, a[$ and satisfying: $\forall\, t \geq t_0$ and $\forall\, x \in \mathcal{B}_r$,*

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|), \tag{2.2}$$
$$\dot{V}(t,x) \leq -\alpha_3(\|x\|), \tag{2.3}$$

then the origin $x = 0$ is uniformly asymptotically stable (UAS). If $\mathcal{B}_r = \mathbb{R}^n$ and α_1 and α_2 are two functions of the class $\mathcal{K}_\infty$, then the origin $x = 0$ is globally uniformly asymptotically stable (GUAS).

To solve the stabilization problem of the bilinear system (1.3) the RDE (1.2) is useful.

Theorem 2.2 ([6])**.** *The following statements are equivalent:*

(i) *the pair $(A(t), B(t))$ is GNC;*

(ii) *for $Q \in \mathbf{M}([0,\infty), \mathbb{R}^n_+)$, the RDE (1.2) has a solution $P \in \mathbf{M}([0,\infty), \mathbb{R}^n_+)$.*

3 The main results

Let us consider the BTV control system (1.3)

$$\dot{x}(t) = A(t)x(t) + u(t)B(t)x(t), \quad t \in \mathbb{R}^+,$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $A(t)$, $B(t)$ are matrix functions, continuous and bounded on $[0,\infty)$. Suppose that the pair $(A(t), B(t))$ is GNC. Then for $Q \in \mathbf{M}([0,\infty), \mathbb{R}^n_+)$, the RDE (1.2) has a solution $P \in \mathbf{M}([0,\infty), \mathbb{R}^n_+)$. Denote

$$b = \sup_{t \geq 0} \|B(t)\|, \quad p = \sup_{t \geq 0} \|P(t)\|.$$

In what follows, we need the following assumptions:

(H_1) The BTV control system (1.3) is GNC.

(H_2) $\eta = \inf\limits_{t\geq 0} \|Q(t)\|$ satisfies $\eta > p^2b^2$.

Proposition 3.1. *Let $B(t)$ and $P(t)$ be bounded matrix functions. Then for $r>0$, the function*

$$g(t,x) = -r\Big(\frac{\|B(t)\|\,\|P(t)\|\,\|x\|}{1+\|B(t)\|\,\|P(t)\|\,\|x\|}\Big)B(t)x$$

is globally Lipschitz with respect to $x\in\mathbb{R}^n$.

Proof. Let $x_1, x_2 \in \mathbb{R}^n$, $t\geq 0$. We have

$$\begin{aligned}
\|g(t,x_1)-g(t,x_2)\| &= r\Big\|\frac{\|B(t)\|\,\|P(t)\|\,\|x_2\|}{1+\|B(t)\|\,\|P(t)\|\,\|x_2\|}B(t)x_2 - \frac{\|B(t)\|\,\|P(t)\|\,\|x_1\|}{1+\|B(t)\|\,\|P(t)\|\,\|x_1\|}B(t)x_1\Big\| \\
&\leq r\|B(t)\|^2\|P(t)\|\,\Big\|\frac{\|x_2\|x_2}{1+\|B(t)\|\,\|P(t)\|\,\|x_2\|} - \frac{\|x_1\|x_1}{1+\|B(t)\|\,\|P(t)\,\|\|x_1\|}\Big\| \\
&\leq r\|B(t)\|^2\|P(t)\|\,\Big\|\frac{\|x_2\|x_2 - \|x_1\|x_1 + \|B(t)\|\,\|P(t)\|\,\|x_1\|\,\|x_2\|(x_2-x_1)}{(1+\|B(t)\|\,\|P(t)\|\,\|x_1\|)(1+\|B(t)\|\,\|P(t)\|\,\|x_2\|)}\Big\|.
\end{aligned}$$

Since

$$\begin{aligned}
\Big\|\|x_2\|x_2 - \|x_1\|x_1\Big\| &= \Big\|\|x_2\|x_2 - \|x_1\|x_2 + \|x_1\|x_2 - \|x_1\|x_1\Big\| \\
&\leq \|x_2\|\,\|x_2-x_1\| + \|x_1\|\,\|x_2-x_1\| \\
&\leq \|x_2-x_1\|\,\big(\|x_2\|+\|x_1\|\big),
\end{aligned}$$

we get

$$\begin{aligned}
&\|g(t,x_1)-g(t,x_2)\| \\
&\quad\leq r\|B(t)\|^2\|P(t)\|\,\|x_1-x_2\|\,\Big[\frac{\|x_2\|+\|x_1\|+\|B(t)\|\,\|P(t)\|\,\|x_1\|\,\|x_2\|}{(1+\|B(t)\|\,\|P(t)\|\,\|x_1\|)(1+\|B(t)\|\,\|P(t)\|\,\|x_2\|)}\Big] \\
&\quad\leq r\|B(t)\|^2\|P(t)\|\,\|x_1-x_2\|\,\Big[\frac{\|x_1\|+\|x_2\|(1+\|B(t)\|\,\|P(t)\|\,\|x_1\|)}{(1+\|B(t)\|\,\|P(t)\|\,\|x_1\|)(1+\|B(t)\|\,\|P(t)\|\,\|x_2\|)}\Big] \\
&\quad\leq r\|B(t)\|\,\|x_1-x_2\|\,\Big[\frac{\|B(t)\|\,\|P(t)\|\,\|x_1\|}{(1+\|B(t)\|\,\|P(t)\|\,\|x_1\|)(1+\|B(t)\|\,\|P(t)\|\,\|x_2\|)} \\
&\qquad\qquad + \frac{\|B(t)\|\,\|P(t)\|\,\|x_2\|}{1+\|B(t)\|\,\|P(t)\|\,\|x_2\|}\Big] \\
&\quad\leq 2rb\|x_1-x_2\|.
\end{aligned}$$

Therefore the function $g(t,x)$ is a globally Lipschitz function with respect to x. □

Theorem 3.1. *Suppose that the conditions (H_1) and (H_2) are fulfilled. Then if we choose $0<r<\frac{\eta-p^2b^2}{2pb}$, the feedback function*

$$u(t,x) = -r\Big(\frac{\|B(t)\|\,\|P(t)\|\,\|x\|}{1+\|B(t)\|\,\|P(t)\|\,\|x\|}\Big),\ \ t\in\mathbb{R}^+,\ \ x\in\mathbb{R}^n, \tag{3.1}$$

is bounded and makes system (1.3) GUAS.

Proof. Let us consider the Lyapunov function

$$V(t,x) = \langle P(t)x,x\rangle,\ \ t\in\mathbb{R}^+,\ \ x\in\mathbb{R}^n.$$

Since P is a positive definite symmetric matrix, we can reduce condition (2.2) of Theorem 2.1 by choosing $\alpha_1(\|x\|) = c\|x\|^2$ and $\alpha_2(\|x\|) = p\|x\|^2$. Furthermore, the derivative of $V(t,x)$ along the solutions of the closed-loop system (1.1) by the feedback (3.1) is

$$\begin{aligned}
\dot{V}(t,x) &= \langle \dot{P(t)}x,x\rangle + 2\langle P(t)\dot{x},x\rangle \\
&\leq -\eta\|x\|^2 + \big\langle P(t)B(t)B(t)^TP(t)x,x\big\rangle - 2r\,\frac{\|B(t)\|\,\|P(t)\|\,\|x\|}{1+\|B(t)\|\,\|P(t)\|\,\|x\|}\,\langle P(t)B(t)x,x\rangle.
\end{aligned}$$

Since

$$\big|\langle P(t)B(t)x, x\rangle\big| \leq \|P(t)\|\,\|B(t)\|\,\|x\|^2,$$

we get

$$\begin{aligned}\dot V(t,x) &\leq -\eta\|x\|^2 + \|P(t)\|^2\|B(t)\|^2\|x\|^2 + 2r\big|\langle P(t)B(t)x, x\rangle\big| \\ &\leq -\eta\|x\|^2 + \|P(t)\|^2\|B(t)\|^2\|x\|^2 + 2r\|P(t)\|\,\|B(t)\|\,\|x\|^2 \\ &\leq (p^2b^2 - \eta + 2rpb)\|x\|^2.\end{aligned}$$

By choosing $\alpha_3(\|x\|) = (\eta - p^2b^2 - 2rpb)\|x\|^2$, condition (2.3) of Theorem 2.1 is well checked. So, the closed loop system (1.3) is globally uniformly asymptotically stable. Moreover, $|u(t,x)| \leq r$, $\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. □

Now, let us consider the dynamical control system

$$\dot x(t) = A(t)x(t) + u(t)B(t)x(t) + F(t,x), \quad t \in \mathbb{R}^+, \tag{3.2}$$

where $x(t) \in \mathbb{R}^n$, $F : [0,+\infty[\, \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear continuous function which is locally Lipschitz with respect to x.

Theorem 3.2. *If $F(t,x)$ satisfies the condition*

$$\|F(t,x)\| \leq \gamma\|x\,\|, \quad \forall\, t \geq 0, \quad \forall\, x \in \mathbb{R}^n,$$

where γ is a positive number satisfying

$$0 < \gamma < \frac{\eta - p^2b^2 - 2rpb}{2p}\,, \tag{3.3}$$

then the closed loop system (3.2) *by the feedback function* (3.1) *is GUAS.*

Proof. Let us consider the Lyapunov function

$$V(t,x) = \langle P(t)x, x\rangle, \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}^n,$$

and let the feedback control be of form (3.1). The derivative of V along the solutions of the system (3.2) by using the chosen feedback control (3.1) and the RDE (1.2), results in

$$\begin{aligned}\dot V(t,x) &= \big\langle \dot P(t)x, x\rangle + 2\langle P(t)\dot x, x\big\rangle \\ &\leq (p^2b^2 - \eta + 2rpb)\|x\|^2 + 2\big\langle P(t)F(t,x), x(t)\big\rangle \\ &\leq (p^2b^2 - \eta + 2rpb)\|x\|^2 + 2\|P(t)\|\,\|F(t,x)\|\,\|x(t)\| \\ &\leq (p^2b^2 - \eta + 2rpb)\|x\|^2 + 2\gamma\|P(t)\|\,\|x(t)\|\|x(t)\| \\ &\leq (p^2b^2 - \eta + 2rpb + 2p\gamma)\|x\|^2.\end{aligned}$$

The proof of the theorem is completed by using condition (3.3) and Theorem 2.1. □

Theorem 3.3. *If $F(t,x)$ satisfies*

$$\|F(t,x)\| \leq \gamma\|x\|^d, \quad \forall\, t \geq 0, \quad \forall\, x \in \mathbb{R}^n,$$

where $d > 1$, then the closed loop system (3.2) *by the feedback function* (3.1) *is locally uniformly asymptotically stable.*

Proof. Let us consider the Lyapunov function

$$V(t,x) = \langle P(t)x, x\rangle, \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^n,$$

and let the feedback control be of form (3.1).

The derivative of V along the solutions of system (3.2) by using the chosen feedback control (3.1) and the RDE (1.2) gives

$$\begin{aligned}\dot{V}(t,x) &= \langle \dot{P}(t)x, x\rangle + 2\langle P(t)\dot{x}, x\rangle \\ &\le (p^2b^2 - \eta + 2rpb)\|x\|^2 + 2\big\langle P(t)F(t,x), x(t)\big\rangle \\ &\le (p^2b^2 - \eta + 2rpb)\|x\|^2 + 2\|P(t)\|\,\|F(t,x)\|\,\|x(t)\| \\ &\le (p^2b^2 - \eta + 2rpb)\|x\|^2 + 2\gamma\|x\|^d\|P(t)\|\,\|x(t)\| \\ &\le \big(p^2b^2 - \eta + 2rpb + 2p\gamma\|x\|^{d-1}\big)\|x\|^2.\end{aligned}$$

So, for x in a small neighborhood of the origin, $p^2b^2 - \eta + 2rpb + 2p\gamma\|x\|^{d-1} < -\rho < 0$. Then $\dot{V}(t,x) \le -\rho\|x\|^2$, which implies that the origin is locally uniformly asymptotically stable. $\square$

4 Example

Let us consider the bilinear time-varying control system

$$\dot{x}(t) = A(t)x(t) + u(t)B(t)x(t), \tag{4.1}$$

where $x(t) \in \mathbb{R}^2$,

$$A(t) = \begin{pmatrix} -e^{-t} & 1 \\ -1 & e^{-t}\end{pmatrix} \quad \text{and} \quad B(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t}\end{pmatrix}.$$

To verify the global null-controllablity of system (4.1), we apply Proposition (2.1)(ii). Denote

$$M(t) = \begin{bmatrix} e^{-t} & 0 & e^{-2t} - e^{-t} & -e^{-t} \\ 0 & e^{-t} & e^{-t} & -e^{-2t} - e^{-t}\end{bmatrix}.$$

It is easy to verify that $rank(M(t)) = 2$ for all $t \ge 0$. By taking $Q = 100I_2 \in \mathbf{M}([0,\infty), \mathbb{R}^2_+)$, the RDE (1.2) has a solution $P(t) \in \mathbf{M}([0,\infty), \mathbb{R}^2_+)$.

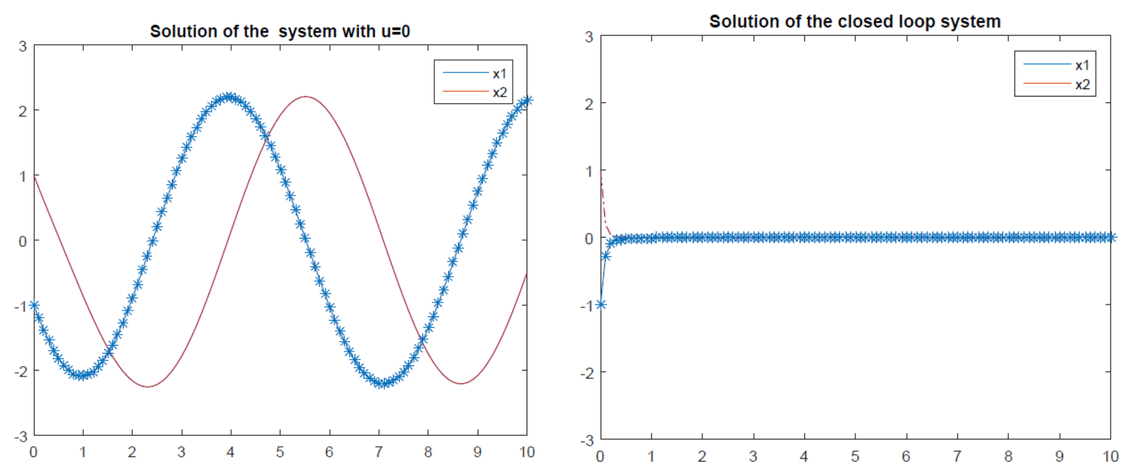

Figure 1. Dynamics of the closed BTV system $\dot{x}(t) = A(t)x(t) + u(t,x)B(t)x(t)$.

Using the Lyapunov function

$$V(t,x) = \langle P(t)x, x\rangle$$

and the feedback function

$$u(t,x) = -20\,\frac{\|B(t)\|\,\|P(t)\|\,\|x\|}{1 + \|B(t)\|\,\|P(t)\|\,\|x\|},$$

we verify that there exists $\alpha > 0$ such that

$$\dot{V}(t,x) \le -\alpha\|x\|^2, \quad \forall t \in \mathbb{R}_+, \quad \forall x \in \mathbb{R}^n.$$

So, according to Theorem 3.1, system (4.1) is GUAS (see Figure 1).

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EXISTENCE OF SOLUTION FOR A THIRD-ORDER DIFFERENTIAL INCLUSION WITH THREE-POINT BOUNDARY VALUE PROBLEM INVOLVING CONVEX MULTIVALUED MAPS

Abstract. In this paper, we discuss the existence of solutions for a third-order differential inclusions with three-point boundary conditions involving convex multivalued maps. The obtained results are based on a nonlinear alternative of the Leray–Schauder type. Finally, some examples are given to illustrate our results.

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Key words and phrases. Third-order differential inclusion, three point boundary value problem, fixed point theorem, selection theorem.

რეზიუმე. ნაშრომში შესწავლილია ამონახსნის არსებობა მესამე რიგის დიფერენციალური ჩანართებისთვის დასმული სასაზღვრო ამოცანისთვის სამწერტილიანი სასაზღვრო პირობებით, რომელიც მოიცავს ამოზნექილ მრავალსახა ასახვებს. მიღებული შედეგები ემყარება ლერეი-შაუდერის ტიპის არაწრფივ ალტერნატივას. ნაშრომის ბოლოს, შედეგების საილუსტრაციოდ, მოყვანილია რამდენიმე მაგალითი.

1 Introduction

Differential inclusions arising in the mathematical modeling of certain problems in economics, optimal control, stochastic analysis, and so forth, are widely studied by many authors (see [3–5, 14, 15, 18, 19] and the references therein). This work is concerned with the existence of solutions for boundary value problems (BVP, for short). In Section 3, we study the three-point boundary value problems of the third order differential inclusion, when the right-hand side is convex

$$-u'''(t) \in F(t, u(t)), \quad t \in (0, 1), \tag{1.1}$$

with the boundary conditions

$$u(0) = \alpha u'(0), \quad u(1) = \beta u'(\eta), \quad u'(1) = \gamma u'(\eta), \tag{1.2}$$

where $\eta \in (0, 1)$, $\alpha, \beta, \gamma \in \mathbb{R}$, with $(1+\alpha)\gamma \le \beta \le \frac{\gamma}{2}$, and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map; and with

$$u'(0) = u''(0) = \beta u(\eta), \quad u(1) = \alpha u(\eta), \tag{1.3}$$

where $\eta \in (0, 1)$, $\alpha, \beta \in \mathbb{R}$, and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map.

The present paper is motivated by the recent papers [15], by S. A. Guezane-Lakoud, N. Hamidane, and [10], by R. Khaldi and D. Liu and Z. Ouyang, where problems (1.1), (1.2) and (1.1), (1.3) with single valued $F(\,\cdot\,,\,\cdot\,)$, respectively, are considered, and several existence results are obtained by using fixed point techniques.

The aim of our paper is to extend the study in [10] and [15] to the set-valued framework and to present some existence results for problems (1.1), (1.2) and (1.1), (1.3). Our results are based on the nonlinear alternative of Leray–Schauder type [9]. The method used is standard, however, its exposition in the framework of problems (1.1), (1.2) and (1.1), (1.3) are new. In Section 4, we complete our work by giving some examples to illustrate the obtained results.

2 Preliminaries

We begin this section by introducing some notation. Let $C([0,1];\mathbb{R})$ denote the Banach space of all continuous functions $u : [0,1] \to \mathbb{R}$ with the norm

$$\|u\| = \sup\big\{|u(t)| \ \text{ for all } t \in [0,1]\big\},$$

let $L^1([0,1];\mathbb{R})$ be the Banach space of measurable functions $u : [0,1] \to \mathbb{R}$ which are Lebesgue integrable, normed by

$$\|u\|_{L^1} = \int_0^1 |u(t)|\,dt,$$

and $AC^i([0,1];\mathbb{R})$ be the space of i-times differentiable functions $u : [0,1] \to \mathbb{R}$, whose ith derivative $u^{(i)}$ is absolutely continuous. Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Denote

$$\begin{aligned}
\mathcal{P}_0(X) &= \big\{A \in \mathcal{P}(X) : \ A \neq \varnothing\big\},\\
\mathcal{P}_{cl}(X) &= \big\{A \in \mathcal{P}_0(X) : \ A \text{ is closed}\big\},\\
\mathcal{P}_b(X) &= \big\{A \in \mathcal{P}_0(X) : \ A \text{ is bounded}\big\},\\
\mathcal{P}_c(X) &= \big\{A \in \mathcal{P}_0(X) : \ A \text{ is convex}\big\},\\
\mathcal{P}_{comp}(X) &= \big\{A \in \mathcal{P}_0(X) : \ A \text{ is compact}\big\}.
\end{aligned}$$

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\Big\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\Big\},$$

where
$$d(a,B)=\inf_{b\in B} d(a,b) \text{ and } d(b,A)=\inf_{a\in A} d(a,b).$$
Then $(\mathcal{P}_{b,cl}(X),H_d)$ is a metric space and $(\mathcal{P}_{cl}(X),H_d)$ is a generalized metric space (see [12]).

Let E be a separable Banach space, Y be a nonempty closed subset of E and $G:Y\to\mathcal{P}_{cl}(E)$ be a multivalued operator. G has a fixed point if there is $x\in Y$ such that $x\in G(x)$. G is said to be completely continuous if $G\ (\Omega)$ is relatively compact for every $\Omega\in\mathcal{P}_b(Y)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is upper semicontinuous $(u.s.c)$ if and only if G has a closed graph, that is, $x_n\to x_*, y_n\to y_*, y_n\in G\ (x_n)$ imply that $y_*\in G\ (x_*)$. For more details on the multi-valued maps, see the books by Aubin and Cellina [1], by Aubin and Frankowska [2], by Deimling [7], by Gorniewicz [8] and by Hu and Papageorgiou [11].

Definition 2.1. A multivalued map $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(1) $t\to F(t,u)$ is measurable for each $u\in\mathbb{R}$,

(2) $u\to F(t,u)$ is upper semicontinuous for almost all $t\in(0,1)$,

and, further, a Carathéodory function F is called L^1-Carathéodory if

(3) for each $r>0$, there exists $\Phi_r\in L^1((0,1);\mathbb{R}^+)$ such that
$$\|F(t,u)\|=\sup\big\{|v|:\ v\in F(t,u)\big\}\leqslant\Phi_r(t)$$
for all $\|u\|\le r$ and for a.e. $t\in(0,1)$.

For each $u\in C((0,1);\mathbb{R})$, define the set of selections of F by
$$S_{F,u}=\Big\{v\in L^1((0,1);\mathbb{R}):\ v(t)\in F(t,u(t)) \text{ for a.e. } t\in(0,1)\Big\}.$$

Lemma 2.1 ([13]). *Let E be a Banach space, let $F:[0,T]\times\to\mathcal{P}_{comp,c}(E)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1],E)$ to $C([0,1],E)$. Then the operator*
$$\Theta\circ S_F: C([0,1],E)\to\mathcal{P}_{comp,c}(C([0,1],E)),\ \ u\to(\Theta\circ S_F)(u)=\Theta(S_{F,u})$$
is a closed graph operator in $C([0,1],E)\times C([0,1],E)$.

Lemma 2.2. *Assume*
$$\xi=2\big(\eta(\alpha(\gamma+1)-\beta)+(\beta-\alpha)\big)-\gamma-1\neq0,$$
then for $y\in C([0,1];\mathbb{R})$, the problem
$$\begin{gathered}u'''(t)+y(t)=0,\ \ t\in(0,1),\\ u(0)=\alpha u'(0),\quad u(1)=\beta u'(\eta),\quad u'(1)=\gamma u'(\eta)\end{gathered}\tag{2.1}$$
has a unique solution
$$\begin{aligned}u(t)=&-\frac12\int_0^t(t-s)^2y(s)\,ds-\frac1\xi\left[t^2(\beta-\gamma-\alpha\gamma)+(t+\alpha)(\gamma-2\beta)\right]\int_0^\eta(\eta-s)y(s)\,ds\\&+\frac1\xi\int_0^1(1-s)\left[\frac{t^2}{2}\left(s-2\alpha+2\beta-\gamma-s\gamma-1\right)+(t+\alpha)(\gamma\eta-2\beta\eta-s+s\gamma\eta)\right]y(s)\,ds.\end{aligned}$$

Lemma 2.3. *Assume*
$$\xi=1-\beta\Big(\frac{\eta^2}{2}+\eta-\frac32\Big)-\alpha\neq0,$$

then for $y \in C([0,1];\mathbb{R})$*, the problem*

$$u'''(t) + y(t) = 0, \quad t \in (0,1),$$
$$u'(0) = u''(0) = \beta u(\eta), \quad u(1) = \alpha u(\eta) \tag{2.2}$$

has a unique solution

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + \frac{1}{2\xi}\Big(-\frac{\beta}{2}t^2 - \beta t + \Big(\frac{3}{2}\beta - \alpha\Big)\Big)\int_0^\eta (\eta - s)^2 y(s)\,ds$$
$$+ \frac{1}{2\xi}\Big(\frac{\beta}{2}\,t^2 + \beta t + 1 - \frac{1}{2}\,\beta\eta^2 - \beta\eta\Big)\int_0^1 (1-s)^2 y(s)\,ds.$$

The proofs of Lemmas 2.2 and 2.3 are given by integrating three times $u'''(t) + y(t) = 0$ over the interval $[0,t]$. We obtain

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + A_1 t^2 + A_2 t + A_3, \quad \text{where } A_1, A_2, A_3 \in \mathbb{R}.$$

The constants A_1, A_2 and A_3 in Lemmas 2.2 and 2.3 are given by the three-point boundary conditions (2.1) and (2.2), respectively.

3 Main results

Before presenting the existence result for problem (1.1), (1.2), let us introduce the following hypotheses which are assumed hereafter:

(H_1) $F : [0,1] \times \mathbb{R} \to \mathcal{P}_c(\mathbb{R})$ is Carathéodory;

(H_2) there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,1];\mathbb{R}^+)$ such that

$$\|F(t,u)\|_{\mathcal{P}} = \sup\big\{|w| : \; w \in F(t,u)\big\} \le p(t)\psi(\|u\|) \text{ for each } (t,u) \in [0,1] \times \mathbb{R}.$$

Definition 3.1. A function $u \in AC^2((0,1);\mathbb{R})$ is called a solution to the BVP (1.1), (1.2) if u satisfies the differential inclusion (1.1) a.e. on $(0,1)$ and conditions (1.2).

Theorem 3.1. *Assume that* (H_1), (H_2) *hold and let the function* ψ *be bounded satisfying the condition: there exists a number* $M > 0$ *such that*

$$\left(\frac{1}{2} + \frac{\frac{1}{2} + \gamma + |\alpha - \beta| - \eta(\beta + \alpha\gamma)}{|\xi_1|}\right)\psi(\|u\|)\|p\|_{L^1} < M.$$

Then the BVP (1.1), (1.2) *has at least one solution on* $[0,1]$.

Proof. Define the operator $T : C([0,1];\mathbb{R}) \to \mathcal{P}(C[0,1];\mathbb{R})$ by

$$T(u) = \Big\{h \in C([0,1];\mathbb{R}) : \quad h(t) = -\frac{1}{2}\int_0^t (t-s)^2 f(u)\,ds$$
$$+ \frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1 - s + 2(\alpha-\beta) + \gamma(1+s)\big) + (t+\alpha)\big(\eta(\gamma - 2\beta) - s(1+\gamma\eta)\big)\Big) f(u)\,ds$$
$$- \frac{1}{\xi_1}\big(t^2(\beta - (1+\alpha)\gamma) + (t+\alpha)(\gamma - 2\beta)\big)\int_0^\eta (\eta - s) f(u)\,ds\Big\}$$

for $f \in \mathcal{S}_{F,u}$. It is not difficult to show that T has a fixed point which is a solution of problem (1.1), (1.2). We show that T satisfies the assumptions of the nonlinear alternative of Leray–Schauder type. The proof consists of several steps.

Step 1. First, we show that T is convex for each $u \in C([0,1];\mathbb{R})$.

Let $h_1, h_2 \in Tu$. Then there exist $w_1, w_2 \in \mathcal{S}_{F,u}$ such that for each $t \in [0,1]$, we have

$$\begin{aligned} h_i(t) &= -\frac{1}{2}\int_0^t (t-s)^2 w_i(s)\,ds \\ &+ \frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t+\alpha)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big)w_i(s)\,ds \\ &- \frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)w_i(s)\,ds, \quad i=1,2. \end{aligned}$$

Let $0 \le \mu \le 1$. So, for each $t \in [0,1]$, we have

$$\begin{aligned} \mu h_1(t)+(1-\mu)h_2(t) &= \frac{1}{2}\int_0^t (t-s)^2\big(\mu w_1(s)+(1-\mu)w_2(s)\big)\,ds \\ &+ \frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t+\alpha)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big) \\ &\qquad \times \big(\mu w_1(s)+(1-\mu)w_2(s)\big)\,ds \\ &- \frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)(\mu w_1(s+(1-\mu)w_2(s))\,ds. \end{aligned}$$

Since $\mathcal{S}_{F,u}$ is convex, it follows that $\mu h_1+(1-\mu)h_2 \in Tu$.

Step 2. Here we show that T maps bounded sets into bounded sets in $C([0,1];\mathbb{R})$.

For a positive number r, let $B_r = \{u \in C([0,1];\mathbb{R}) : \|u\| \le r\}$ be a bounded ball in $C([0,1];\mathbb{R})$. So, for each $h \in Tu$, $u \in B_r$, there exists $w \in \mathcal{S}_{F,u}$ such that

$$\begin{aligned} h(t) &= -\frac{1}{2}\int_0^t (t-s)^2 w(s)\,ds \\ &+ \frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t+\alpha)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big)w(s)\,ds \\ &- \frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)w(s)\,ds. \end{aligned}$$

If $(1+\alpha)\gamma \le \beta \le \frac{\gamma}{2}$, we obtain

$$\begin{aligned} |h(t)| \le \frac{\psi(\|u\|)}{2}\int_0^1 p(s)\,ds &+ \frac{\psi(\|u\|)}{|\xi_1|}\Big[\frac{1}{2}+|\alpha-\beta|+|\gamma|-\beta\eta\Big]\int_0^1 p(s)\,ds \\ &- \frac{\psi(\|u\|)}{|\xi_1|}\,\gamma\eta\Big(\frac{1}{2}+\alpha\Big)\int_0^\eta p(s)\,ds, \end{aligned}$$

Thus

$$\|h\| \le \frac{\psi(\|u\|)}{2}\int_0^1 p(s)\,ds + \frac{\psi(\|u\|)}{|\xi_1|}\Big[1+|\alpha-\beta|+2\gamma-\beta\eta\Big]\int_0^1 p(s)\,ds$$
$$-\frac{\psi(\|u\|)}{|\xi_1|}\gamma\eta\Big(\frac{1}{2}+\alpha\Big)\int_0^\eta p(s)\,ds.$$

Step 3. Now we show that T maps the bounded sets into equicontinuous sets of $C([0,1];\mathbb{R})$.

Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and let B_r be a bounded set of $C([0,1];\mathbb{R})$. Then, for each $h\in Tu$, we obtain that the bounded sets of $C([0,1];\mathbb{R})$ are mapped into the equicontinuous sets,

$$|h(t_2)-h(t_1)| \le \frac{1}{2}\int_{t_1}^{t_2}(t_2-s)^2|w(s)|\,ds + \frac{1}{2}\int_0^{t_1}\big((t_1-s)^2-(t_2-s)^2\big)|w(s)|\,ds$$
$$+\frac{1}{|\xi_1|}\int_0^1(1-s)\Big(\frac{-(t_2^2-t_1^2)}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t_2-t_1)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big)|w(s)|\,ds$$
$$+\frac{1}{|\xi_1|}\big((t_2^2-t_1^2)(\beta-(1+\alpha)\gamma)+(t_2-t_1)(\gamma-2\beta)\big)\int_0^\eta(\eta-s)|w(s)|\,ds,$$
$$\le \frac{\psi(\|u\|)}{2}\int_{t_1}^{t_2}(t_2-s)^2p(s)\,ds + \frac{\psi(\|u\|)}{2}\int_0^{t_1}\big((t_1-s)^2-(t_2-s^2\big)p(s)\,ds$$
$$+\frac{\psi(\|u\|)}{|\xi_1|}\int_0^1(1-s)\Big(\frac{-(t_2^2-t_1^2)}{2}(1-s+2(\alpha-\beta)+\gamma(1+s))+(t_2-t_1)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big)p(s)\,ds$$
$$+\frac{\eta\psi(\|u\|)}{|\xi_1|}\big((t_2^2-t_1^2)(\beta-(1+\alpha)\gamma)+(t_2-t_1)(\gamma-2\beta)\big)\int_0^\eta p(s)\,ds.$$

Obviously, the right-hand side of the above inequality tends to zero independently of $u\in B_r$ as $t_2-t_1\to 0$. Since T satisfies the above three assumptions, it follows by the Ascoli–Arzelèa theorem that $T: C([0,1];\mathbb{R})\to P(C[0,1];\mathbb{R})$ is completely continuous.

Step 4. We show that T has a closed graph.

Let $u_n\to u_*$, $h_n\in T(u_n)$ and $h_n\to h_*$. Then we need to show that $h_*\in Tu_*$.

Associated with $h_n\in T(u_n)$, there exists $w_n\in\mathcal{S}_{F,u_n}$ such that for each $t\in[0,1]$, we have

$$h(t) = -\frac{1}{2}\int_0^t(t-s)^2w_n(s)\,ds$$
$$+\frac{1}{\xi_1}\int_0^1(1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t+\alpha)(\eta(\gamma-2\beta)-s(1+\gamma\eta))\Big)w_n(s)\,ds$$
$$-\frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta(\eta-s)w_n(s)\,ds.$$

Thus we have to show that there exists $w_*\in\mathcal{S}_{F,u_*}$ such that for each $t\in[0,1]$,

$$h_*(t) = -\frac{1}{2}\int_0^t (t-s)^2 w_*(s)\,ds$$
$$+\frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t+\alpha)(\eta(\gamma-2\beta)-s(1+\gamma\eta))\Big)w_*(s)\,ds$$
$$-\frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)w_*(s)\,ds.$$

Let us consider the continuous linear operator $\Theta : L^1([0,1];\mathbb{R}) \to C([0,1];\mathbb{R})$ given by

$$w \longrightarrow \Theta w(t) = -\frac{1}{2}\int_0^t (t-s)^2 w(s)\,ds$$
$$+\frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t+\alpha)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big)w(s)\,ds$$
$$-\frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)w(s)\,ds.$$

Observe that

$$\|h_n(t)-h_*(t)\| = \Bigg\| -\frac{1}{2}\int_0^t (t-s)^2 (w_n(s)-w_*(s))\,ds$$
$$+\frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t+\alpha)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big)(w_n(s)-w_*(s))\,ds$$
$$-\frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)(w_n(s)-w_*(s))\,ds\Bigg\|,$$

then $\|h_n(t)-h_*(t)\| \to 0$ as $n\to\infty$.

Thus, it follows by Lemma 2.1 that $\Theta\circ\mathcal{F}$ is a closed graph operator.

Further, we have $h_n(t)\in\Theta(S_{F,u_n})$. Since $u_n\to u_*$, we get

$$h_*(t) = -\frac{1}{2}\int_0^t (t-s)^2 w_*(s)\,ds$$
$$+\frac{1}{\xi_1}\int_0^1 (1-s)\Big[\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t+\alpha)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big]w_*(s)\,ds$$
$$-\frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)w_*(s)\,ds$$

for some $w_*\in S_{F,u_*}$.

Step 5. We discuss a priori bounds on solutions.

Let u be a solution of (1.1), (1.2). So, there exists $w\in L^1([0,1];\mathbb{R})$ with $w\in S_{F,u}$ such that for $t\in[0,1]$, we have

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 w(s)\,ds$$

$$+\frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t+\alpha)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big)w(s)\,ds$$

$$-\frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)w(s)\,ds.$$

In view of (H_2), for each $t \in [0,1]$, and $(1+\alpha)\gamma \le \beta \le \frac{\gamma}{2}$, we obtain

$$|u(t)| \le \frac{\psi(\|u\|)}{2}\int_0^1 p(s)\,ds + \frac{\psi x(\|u\|)}{|\xi_1|}\int_0^1 \Big(\frac{1}{2}+|\alpha-\beta|+\gamma+\frac{1}{2}\eta(\gamma-2\beta)\Big)\int_0^1 p(s)\,ds$$

$$-\frac{1}{2}\gamma\eta\,\frac{\psi(\|u\|)}{|\xi_1|}((2\alpha+1))\int_0^1 p(s)\,ds.$$

Consequently,

$$\frac{\|u\|}{\big(\frac{1}{2}+\frac{\frac{1}{2}+\gamma+|\alpha-\beta|-\eta(\beta+\alpha\gamma)}{|\xi_1|}\big)\psi(\|u\|)\|p\|_{L^1}} \le 1.$$

So, there exists M such that $\|u\| \ne M$. Let us set $U = \{u \in C([0,1];\mathbb{R}) : \ \|u\| < M+1\}$. Note that the operator $T : \overline{U} \to \mathcal{P}C([0,1];\mathbb{R})$ is upper semicontinuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u \in \lambda Tx$ for some $\lambda \in (0,1)$.

Consequently, by the nonlinear alternative of Leray–Schauder type [19], we deduce that T has a fixed point $u \in \overline{U}$ which is a solution of problem (1.1), (1.2). This completes the proof. □

The next result concerns the four-point BVP (1.1), (1.3). Before stating and proving this result, we give the definition of a solution of the four-point BVP (1.1), (1.3).

Definition 3.2. A function $u \in AC^2((0,1);\mathbb{R})$ is called a solution to the BVP (1.1), (1.3) if u satisfies the differential inclusion (1.1) a.e. on $(0,1)$ and conditions (1.3).

Theorem 3.2. *Assume that (H_1), (H_2) hold and let the function ψ be bounded satisfying the condition: there exists a number $M > 0$ such that*

$$\Big(\frac{1}{2}+\frac{1}{2|\xi|}\Big(\eta^2|\alpha|+\Big(\frac{7}{2}\eta^2+\eta+\frac{3}{2}\Big)|\beta|+1\Big)\Big)\psi(\|u\|)\|p\|_{L^1} < M.$$

Then the BVP (1.1), (1.3) has at least one solution on $[0,1]$.

Proof. Define the operator $T : C([0,1];\mathbb{R}) \to \mathcal{P}(C[0,1];\mathbb{R})$ by

$$T(u) = \Big\{h \in C([0,1];\mathbb{R}) : \ h(t) = -\frac{1}{2}\int_0^t (t-s)^2 f(u)\,ds$$

$$+\frac{1}{2\xi}\Big(-\frac{\beta}{2}t^2-\beta t+\Big(\frac{3}{2}\beta-\alpha\Big)\Big)\int_0^\eta (\eta-s)^2 f(u)\,ds$$

$$+\frac{1}{2\xi}\Big(\frac{\beta}{2}t^2+\beta t+1-\frac{1}{2}\beta\eta^2-\beta\eta\Big)\int_0^1 (1-s)^2 f(u)\,ds\Big\}$$

for $f \in \mathcal{S}_{F,u}$. We can easily show that T has a fixed point which is a solution of problem (1.1), (1.3), following the steps of Theorem 3.1. We omit the details. □

4 Examples

Example 4.1. Consider the boundary value problem

$$\begin{gathered} -u'''(t) \in F(t,u(t)), \ \ t \in (0,1), \\ u(0) = -u'(0), \quad u(1) = \frac{1}{3}\,u'\Big(\frac{1}{3}\Big), \quad u'(1) = u'\Big(\frac{1}{3}\Big), \end{gathered} \tag{4.1}$$

where $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$F(t,u) = \left[\frac{\exp(u)}{3+\exp(u)}, -2\log(t+1) + t^3 + t + 1\right].$$

For $f \in F$, we have

$$|f| \leqslant \max\left(\frac{\exp(u)}{3+\exp(u)}\,, -2\log(t+1) + t^3 + t + 1\right) \leqslant 2, \ \ u \in \mathbb{R}.$$

Thus

$$\|F(t,u)\|_{\mathcal{P}} = \sup\big\{|w| : \ w \in F(t,u)\big\} \leq 2 = p(t)\psi(\|u\|), \ \ u \in \mathbb{R},$$

with $p(t) = \frac{1}{2}$, $\psi(\|u\|) = 4$. Further, using the condition

$$\left(\frac{1}{2} + \frac{\frac{1}{2} + \gamma + |\alpha - \beta| - \eta(\beta + \alpha\gamma)}{|\xi|}\right)\psi(\|u\|)\|p\|_{L^1} < M.$$

we find that $M > \frac{63}{8}$. By Theorem 3.1, the boundary value problem (4.1), has at least one solution on $[0,1]$.

Example 4.2. Consider the boundary value problem

$$\begin{gathered} -u'''(t) \in F(t,u(t)), \ \ t \in (0,1), \\ u'(0) = u''(0) = -u\Big(\frac{1}{7}\Big), \ \ u(1) = -2u\Big(\frac{1}{7}\Big), \end{gathered} \tag{4.2}$$

where $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$F(t,u) = \Big[\sin(u), \frac{u}{\exp u} + t\Big].$$

For $f \in F$, we have

$$|f| \leqslant \max\Big(\sin(u), \frac{u}{\exp u} + t\Big) \leqslant 1 + t, \ \ u \in \mathbb{R}.$$

Thus

$$\|F(t,u)\|_{\mathcal{P}} = \sup\big\{|w| : \ w \in F(t,u)\big\} \leq 1 + t = p(t)\psi(\|u\|), \ \ u \in \mathbb{R},$$

with $p(t) = 1 + t$, $\psi(\|u\|) = 1$. Further, we use the condition

$$\left(\frac{1}{2} + \frac{1}{2|\xi|}\,(\eta^2|\alpha| + \Big(\frac{7}{2}\,\eta^2 + \eta + \frac{3}{2}\Big)|\beta| + 1)\right)\psi(\|u\|)\|p\|_{L^1} < M$$

with $M > 2$. By Theorem 3.2, the boundary value problem (4.2) has at least one solution on $[0,1]$.

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