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Mouffak Benchohra, Sara Litimein, Atika Matallah, Yong Zhou

GLOBAL EXISTENCE AND CONTROLLABILITY FOR SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY IN FRÉCHET SPACES

Abstract. The sufficient conditions are given ensuring the existence and the controllability of mild solutions for a semi-linear fractional differential equation with state-dependent delay in Fréchet space. We use in the study a generalization of Darboux's fixed point theorem combined with measures of non-compactness.

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რეზიუმე. მოცემულია საკმარისი პირობები, რომლებიც უზრუნველყოფს სუსტი ამონახსნების არსებობას და მართვას ფრეშეს სივრცეში ნახევრადწრფივი ფრაქციული დიფერენციალური განტოლებისთვის შინაგან მდგომარეობაზე დამოკიდებული დაგვიანებით. კვლევაში გამოყენებულია დარბუს უძრავი წერტილის თეორემის განზოგადების კომბინაცია არაკომპაქტურობის ზომებთან.

1 Introduction

This paper deals with the existence and controllability of mild solutions for a semi-linear fractional differential equation with state-dependent delay in Fréchet spaces. In Section 3, we examine semilinear fractional differential equations with state-dependent delay given by

$$^{c}D^{\alpha}y(t) = Ay(t) + f(t, y(t - \rho(y(t)))), \text{ a.e. } t \in J = [0, +\infty), \ 0 < \alpha < 1, \tag{1.1}$$
$$y(t) = \phi(t), \ t \in [-r, 0], \tag{1.2}$$

and, in Section 4, we investigate the controllability of semi-linear fractional differential equation with state-dependent delay

$$^{c}D^{\alpha}y(t) = Ay(t) + f(t, y(t - \rho(y(t)))) + Bu(t), \text{ a.e. } t \in J = [0, +\infty), \ 0 < \alpha < 1, \tag{1.3}$$
$$y(t) = \phi(t), \ t \in [-r, 0], \tag{1.4}$$

where $^{c}D^{\alpha}$ is the standard Caputo fractional derivative, $f : J \times E \to E$ is a given function, $A : D(A) \subset E \to E$ is an almost sectorial operator, that is, $A \in \Theta_{\omega}^{\gamma}(E)$ $(-1 < \gamma < 0, 0 < \omega < \frac{\Pi}{2})$, $\Theta_{\omega}^{\gamma}(E)$ is a space of almost sectorial operator to be specified later, the control function u is given in $L^2(J, U)$, a Banach space of admissible control functions, B is a bounded linear operator from U into E, $\phi : [-r, 0] \to E$ is a given continuous function and $(E, \| \cdot \|)$ is a Banach space, ρ is a positive bounded continuous function on $C([-r, 0], E)$, r is the maximal delay defined by

$$r = \sup_{y \in C} |\rho(y)| < \infty.$$

Recently, fractional calculus takes a great interest, in cause, in part to both the intensive development of the theory of fractional calculus itself and the applications of such constructions to different sciences such as physics, mechanics, chemistry, engineering, etc. (for details, see the monographs [17, 21, 23] and the references therein). Newly, several works have been published on the existence and uniqueness of mild solutions for various types of fractional differential equations using different approaches and techniques such as fixed point theorems, probability density functions, lower and upper solutions method, coincidence degree theory, etc. (see, e.g., [2, 3, 12, 15, 28]).

Moreover, the existence of solutions on the half-line of the integer order differential equations has been investigated in [1, 5, 6, 8, 16, 22]. Quite recently, in [25], Su considered the existence of solutions to the boundary value problems of fractional differential equations on unbounded domains by using the Darboux fixed point theorem. The attractiveness of fractional evolution equations with almost sectorial operators has been proved by Zhou [29].

The problem of controllability for linear and nonlinear systems shown by ODEs in a finite-dimensional space has been extensively examined. Certain authors have enlarged the controllability concept to the infinite-dimensional systems in Banach space with unbounded operators (for more details see [11, 20]). N. Carmichael and M. D. Quinn [24] proved that the controllability problem can be translated into a fixed point problem. Interesting controllability results of various classes of fractional differential equations defined on a bounded and unbounded intervals are given in many papers (see e.g., [4, 7, 10, 19]).

Our investigations are considered in the Fréchet spaces by using a generalization of the classical Darboux fixed point theorem with the concept of a family of measures of noncompactness.

The paper is organized as follows. In Section 2, we recall briefly some basic definitions and preliminary facts that will be used throughout the paper. In Section 3, we discuss the existence of mild solutions for problem (1.1), (1.2). In Section 4, we testify the controllability of mild solutions for problem (1.3), (1.4). The investigation on semilinear fractional differential equations with almost sectorial operators have not been shown yet in the Fréchet spaces, so the present results make a valuable contribution to this study.

2 Preliminaries

Let $J = [0,b]$, $b > 0$, be a compact interval in $\mathbb{R}$, $C(J,E)$ be the Banach space of all continuous functions from J to E with the norm

$$\|y\|_\infty = \sup_{t\in J} \|y(t)\|.$$

Let $B(E)$ denote the Banach space of bounded linear operators from E into E.

A measurable function $y : J \to E$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable.

Let $L^1(J,E)$ denote the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b \|y(t)\|\, dt.$$

Definition 2.1. A function $f : J \times E \to E$ is said to be Carathéodory if

(i) for each $t \in J$ the function $f(t, \cdot\,) : E \to E$ is continuous;

(ii) for each $y \in E$ the function $f(\,\cdot\,, y) : J \to E$ is measurable.

Definition 2.2 ([17])**.** The fractional primitive of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \to E$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_0^\alpha h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds.$$

Definition 2.3 ([17])**.** The Riemann–Liouville derivative of order $\alpha > 0$ with the lower limit t_0 for a function $f : \mathbb{R}^+ \to E$ is given by

$$D^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^t (t-s)^{n-\alpha-1} f(s)\, ds, \quad t > t_0, \quad n-1 < \alpha < n.$$

Definition 2.4 ([17])**.** The Caputo fractional derivative of order $\alpha > 0$ with the lower limit t_0 for a function $f : \mathbb{R}^+ \to E$ is given by

$${}^cD^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds.$$

We denote by $D(A)$ the domain of A, by $\sigma(A)$ its spectrum, while $\rho(A) = \mathbf{C} \setminus \sigma(A)$ is the resolvent set of A, and denote by $R(z,A) = (zI - A)^{-1}$, $z \in \rho(A)$, the family of bounded linear operators which are the resolvents of A.

Definition 2.5. Let $-1 < \gamma < 0$ and $0 < \omega < \frac{\Pi}{2}$. By $\Theta_\omega^\gamma(E)$ we denote the family of all linear closed operators $A : D(A) \subset E \to E$ which satisfy the following conditions:

(a) $\sigma(A) \subset S_\omega = \{z \in \mathbb{C} \setminus \{0\}; |\arg z| \le \omega\} \cup \{0\}$;

(b) for every $\omega < \mu < \Pi$, there exists a constant C_μ such that

$$\|R(z;A)\| \le C_\mu |z|^\gamma \quad \text{for all} \quad z \in \mathbb{C} \setminus S_\mu.$$

A linear operator A is said to be an almost sectorial operator on E if $A \in \Theta_\omega^\gamma(E)$.

Let A be an operator in the class $\Theta^{\gamma}_{\omega}(E)$ and $-1 < \gamma < 0$, $0 < \omega < \frac{\Pi}{2}$. Define the operator families $\{\mathcal{S}_\alpha(t)\}_{t \in S^0_{\frac{\Pi}{2}-\omega}}$, $\{\mathcal{P}_\alpha(t)\}_{t \in S^0_{\frac{\Pi}{2}-\omega}}$ by

$$\mathcal{S}_\alpha(t) = E_\alpha(-zt^\alpha)(A) = \frac{1}{2\Pi i}\int\limits_{\Gamma_\theta} E_\alpha(-zt^\alpha)R(z,A)\,dz,$$
$$\mathcal{P}_\alpha(t) = e_\alpha(-zt^\alpha)(A) = \frac{1}{2\Pi i}\int\limits_{\Gamma_\theta} e_\alpha(-zt^\alpha)R(z,A)\,dz,$$

where the integral contour $\Gamma_\theta = \{\mathbb{R}_+ e^{i\theta}\} \cup \{\mathbb{R}_+ e^{-i\theta}\}$ is oriented counter-clockwise and $\omega < \theta < \mu < \frac{\Pi}{2} - |\arg t|$. Now, we present the following important results about the operators $\mathcal{S}_\alpha$ and $\mathcal{P}_\alpha$.

Theorem 2.6 ([27])**.** *For each fixed $t \in S^0_{\frac{\Pi}{2}-\omega}$, $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are the bounded linear operators on E. Moreover, there exist the constants $C_s = C(\alpha,\gamma) > 0$, $C_p = C(\alpha,\gamma) > 0$ such that for all $t > 0$,*

$$\|\mathcal{S}_\alpha(t)\| \le C_s t^{-\alpha(1+\gamma)}, \quad \|\mathcal{P}_\alpha(t)\| \le C_p t^{-\alpha(1+\gamma)}.$$

Also,

$$\mathcal{S}_\alpha(t)x = \int\limits_0^\infty \Psi_\alpha(s)T(st^\alpha)x\,ds, \ \ t \in S^0_{\frac{\Pi}{2}-\omega}, \ \ x \in E,$$

and

$$\mathcal{P}_\alpha(t)x = \int\limits_0^\infty \alpha s\Psi_\alpha(s)T(st^\alpha)x\,ds, \ \ t \in S^0_{\frac{\Pi}{2}-\omega}, \ \ x \in E,$$

where $T(\,\cdot\,)$ is a semigroup associated with A.

Theorem 2.7 ([27])**.** *For $t > 0$, $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are continuous in the uniform operator topology.*

Consider the problem

$$^cD^\alpha y(t) - Ay(t) = f(t), \ \ t \in (0,b], \tag{2.1}$$
$$y(0) = y_0, \tag{2.2}$$

where $^cD^\alpha$, $0 < \alpha < 1$, is the Caputo fractional derivative, $f \in L^1(J,E)$ and $y_0 \in E$.

Definition 2.8 ([27])**.** A function $y \in C([0,b],E)$ is called a mild solution of Problem (2.1), (2.2) if

$$y(t) = \mathcal{S}_\alpha(t)y_0 + \int\limits_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s)\,ds, \ \ t \in [0,b].$$

Let $C(\mathbb{R}_+)$ be the Fréchet space of all continuous functions ν from $\mathbb{R}_+$ into E, equipped with the family semi-norms

$$\|\nu\|_n = \sup_{t\in[0,n]} \|\nu(t)\|, \ \ n \in \mathbb{N},$$

and the distance

$$d(u,v) = \sum_{n=1}^\infty 2^{-n}\,\frac{\|u-v\|_n}{1+\|u-v\|_n}\,, \ \ u,v \in C(\mathbb{R}_+).$$

(For more details about measures of noncompactness see [13, 14].)

Definition 2.9. Let $\mathcal{M}_X$ be the family of all nonempty and bounded subsets of a Fréchet space X. A family of functions $\{\mu_n\}_{n\in\mathbb{N}}$, where $\mu_n : \mathcal{M}_X \to [0,\infty)$ is said to be a family of measures of noncompactness in the real Fréchet space X if for all $B, B_1, B_2 \in \mathcal{M}_X$ it satisfies the following conditions:

(a) $\{\mu_n\}_{n\in\mathbb{N}}$ is full, that is, $\mu_n(B)=0$ for $n\in\mathbb{N}$ if and only if B is precompact;

(b) $\mu_n(B_1)<\mu_n(B_2)$ for $B_1\subset B_2$ and $n\in\mathbb{N}$;

(c) $\mu(ConvB)=\mu(B)$ for $n\in\mathbb{N}$;

(d) if $\{B\}$ is a sequence of closed sets from $\mathcal{M}_X$ such that $B_{i+1}\subset B_i$, $i=1,\dots$, and if $\lim\limits_{i\to\infty}\mu_n(B_i)=0$, for each $n\in\mathbb{N}$, then the intersection set $B_\infty=\bigcap\limits_{i=1}^{\infty}B_i$ is nonempty.

Definition 2.10. A nonempty subset $B\subset X$ is said to be bounded if for $n\in\mathbb{N}$, there exists $M_n>0$ such that

$$\|y\|_n\le M_n,\ \text{ for each }\ y\in B.$$

Lemma 2.11 ([9])**.** *If Y is a bounded subset of the Banach space X, then for each $\varepsilon>0$, there is a sequence $\{y_k\}_{k=1}^{\infty}\subset Y$ such that*

$$\mu(Y)\le 2\mu\big(\{y_k\}_{k=1}^{\infty}\big)+\varepsilon.$$

Lemma 2.12 ([18])**.** *If $\{u_k\}_{k=1}^{\infty}\subset L^1(I)$ is uniformly integrable, then $\mu(\{u_k\}_{k=1}^{\infty})$ is measurable for $n\in\mathbb{N}$ and*

$$\mu\Bigg(\bigg\{\int_0^t u_k(s)\,ds\bigg\}_{k=1}^{\infty}\Bigg)\le 2\int_0^t\mu\big(\{u_k(s)\}_{k=1}^{\infty}\big)\,ds$$

for each $t\in[0,n]$.

Definition 2.13. Let Ω be a nonempty subset of a Fréchet space X, and let $A:\Omega\to X$ be a continuous operator which transforms bounded subsets onto the bounded ones. One says that A satisfies the Darboux condition with constants $(k_n)_{n\in\mathbb{N}}$ with respect to a family of measures of noncompactness $(\mu_n)_{n\in\mathbb{N}}$ if

$$\mu_n(A(B))\le k_n\mu_n(B)$$

for each bounded set $B\subset\Omega$ and $n\in\mathbb{N}$. If $k_n<1$, $n\in\mathbb{N}$, then A is called a contraction with respect to $\{\mu_n\}_{n\in\mathbb{N}}$.

In the sequel, we will make use of the following generalization of the classical Darboux fixed point theorem for the Fréchet spaces.

Theorem 2.14 ([13,14])**.** *Let Ω be a nonempty, bounded, closed and convex subset of a Fréchet space F and let $V:\Omega\to\Omega$ be a continuous mapping. Suppose that V is a contraction with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$. Then V has at least one fixed point in the set Ω.*

3 The main result

Influenced by [27] with $\phi(0)\in D(A^{\beta})$, $\beta>1+\gamma$, we define a mild solution of problem (1.1), (1.2) by the following

Definition 3.1. We say that a continuous function $y:\mathbb{R}\to E$ is a mild solution of problem (1.1), (1.2) if $y(t)=\phi(t)$ for all $t\in[-r,0]$ and y satisfies the integral equation

$$y(t)=\mathcal{S}_\alpha(t)\phi(0)+\int_0^t(t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f\big(s,y(s-\rho(y(s)))\big)\,ds\ \text{ for each }\ t\in J.$$

Let us include the hypotheses.

(H1) The function $f:J\times E\to E$ is Carathéodory.

(H2) There exist a function $p \in L^1_{loc}(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : J \to [0, +\infty)$ such that
$$\|f(t,u)\| \leq p(t)\psi(\|u\|) \text{ for a.e. } t \in J \text{ and each } u \in E.$$

(H3) There exists a function $l \in L^1_{loc}(J, \mathbb{R}^+)$ such that for any bounded set $B \subset E$, and for each $t \in J$, we have
$$\alpha((f, B)) \leq l(t)\alpha(B).$$

(H4) There exists $r_n > 0$ such that
$$C_s n^{-\alpha(1+\gamma)}|\phi(0)| + C_p\psi(r_n) \sup_{t\in[0,n]} \left\{ \int_0^t (t-s)^{-(1+\alpha\gamma)} p(s)\, ds \right\} \leq r_n.$$

For $n \in \mathbb{N}$, we define on $C([-r, +\infty), E)$ the family of measures of noncompactness by
$$\mu_n(V) = \omega_0^n(V) + \sup_{t\in[0,n]} e^{-Lt}\mu(V(t)),$$
where $V(t) = \{v(t) \in E : v \in V\}, t \in [0, n]$, and $L > 0$ is a constant chosen so that
$$l_n = 4C_p \sup_{t\in[0,n]} \int_0^t e^{-L(t-s)}(t-s)^{-(1+\alpha\gamma)} l(s)\, ds < 1.$$

Remark 3.2. Notice that if the set V is equicontinuous, then $\omega_0^n(V) = 0$.

Theorem 3.3. *Assume* (H1)–(H4) *are satisfied. Then problem* (1.1), (1.2) *admits at least one mild solution.*

Proof. Consider the operator $N : C([-r, +\infty), E) \to C([-r, +\infty), E)$ given by
$$(Ny)(t) = \begin{cases} \phi(t) & \text{if } t \in [-r, 0]; \\ \mathcal{S}_\alpha(t)\phi(0) + \displaystyle\int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) f\big(s, y(s-\rho(y(s)))\big)\, ds & \text{if } t \in J. \end{cases}$$
We shall check that the operator N satisfies all conditions of Theorem 2.14. The proof is given in several steps.

Let
$$B_{r_n} = \big\{u \in C([-r, +\infty), E) : \|u\|_n \leq r_n\big\},$$
where r_n is the constant given by (H4). It is obvious that the subset B_{r_n} is closed, bounded and convex.

Step 1. $N(B_{r_n}) \subset B_{r_n}$.

For any $n \in \mathbb{N}$ and for each $y \in B_{r_n}$ and $t \in [0, n]$, we have
$$\begin{aligned} \|(Ny)(t)\| &\leq \|\mathcal{S}_\alpha(t)\|\, |\phi(0)| + \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|\, \big\|f\big(s, y(s-\rho(y(s)))\big)\big\|\, ds \\ &\leq C_s t^{-\alpha(1+\gamma)}|\phi(0)| + \int_0^t (t-s)^{-(1+\alpha\gamma)} C_p p(s)\psi(\|y(s)\|)\, ds \\ &\leq C_s n^{-\alpha(1+\gamma)}|\phi(0)| + C_p\psi(r_n) \sup_{t\in[0,n]} \left\{ \int_0^t (t-s)^{-(1+\alpha\gamma)} p(s)\, ds \right\} \\ &\leq r_n. \end{aligned}$$

Thus

$$\|N(y)\|_n \leq r_n.$$

Step 2. N is continuous on B_{r_n}.

Let y_n be a sequence such that $y_n \longrightarrow y$ in B_{r_n}. Then for each $t \in [0, n]$, we have

$$\begin{aligned}
&\big\|(Ny_n)(t) - (Ny)(t)\big\| \\
&\quad \leq \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\| \, \Big\| f\big(s, y_n(s-\rho(y_n(s)))\big) - f\big(s, y(s-\rho(y(s)))\big)\Big\| \, ds \\
&\qquad \leq C_p \int_0^t (t-s)^{-(1+\alpha\gamma)} \Big\| f\big(s, y_n(s-\rho(y_n(s)))\big) - f\big(s, y(s-\rho(y(s)))\big)\Big\| \, ds.
\end{aligned}$$

Since f is a Carathéodory function for $t \in [0, n]$, from the continuity of ρ, the Lebesgue dominated convergence theorem implies that

$$\|N(y_n) - N(y)\|_n \longrightarrow 0 \text{ as } n \to \infty.$$

Step 3. $N(B_{r_n})$ is bounded which is clear.

Step 4. For each bounded equicontinuous subset V of B_{r_n}, $\mu_n(N(V)) \leq k_n \mu_n(V)$.

From Lemmas 2.11 and 2.12, for any $V \subset B_{r_n}$ and any $\epsilon > 0$, there exists a sequence $\{y_k\}_{k=0}^{\infty} \subset V$ such that for all $t \in [0, n]$,

$$\begin{aligned}
\mu((NV)(t)) &= \mu\Bigg(\Bigg\{\mathcal{S}_\alpha(t)\phi(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) f\big(s, y(s-\rho(y(s)))\big)\, ds, v \in V\Bigg\}\Bigg) \\
&\leq 2\mu\Bigg(\Bigg\{\int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) f\big(t, y_k(s-\rho(y_k(s)))\big)\, ds\Bigg\}_{k=1}^{\infty}\Bigg) + \epsilon \\
&\leq 4C_p \int_0^t (t-s)^{-(1+\alpha\gamma)} \mu\Big(\Big\{f\big(t, y_k(s-\rho(y_k(s)))\big)\Big\}_{k=1}^{\infty}\Big)\, ds + \epsilon \\
&\leq 4C_p \int_0^t (t-s)^{-(1+\alpha\gamma)} l(s)\mu\big(\{(y_k(s))\}_{k=1}^{\infty}\big)\, ds + \epsilon \\
&\leq 4C_p \int_0^t e^{Ls}(t-s)^{-(1+\alpha\gamma)} e^{-Ls} l(s)\mu\big(\{(y_k(s))\}_{k=1}^{\infty}\big)\, ds + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have

$$\mu(N(V)) \leq 4C_p \int_0^t e^{-L(t-s)}(t-s)^{-(1+\alpha\gamma)} l(s)\mu_n(V)\, ds.$$

Thus

$$\mu_n(N(V)) \leq l_n \mu_n(V).$$

As a conclusion, N has at least one fixed point in B_{r_n}. □

4 Controllability of semilinear fractional differential equations with state-dependent delay

In this section, we prove a controllability result for system (1.3), (1.4).

Definition 4.1. System (1.3), (1.4) is said to be controllable if for any continuous function $\phi \in [-r, 0]$, any $y_1 \in E$ and for each $n \in \mathbb{N}$ there exists a control $u \in L^2([0,n], E)$ such that the mild solution $y(\cdot)$ of (1.3), (1.4) satisfies $y(n) = y_1$.

Let us introduce the following hypotheses:

(H4′) There exists $r'_n > 0$ such that

$$C_s n^{-\alpha(1+\gamma)}|\phi(0)|\Big[1+\frac{n^{-\alpha\gamma}}{-\alpha\gamma}\Big] + |y_1| C_p M_1 M_2 \frac{n^{-\alpha\gamma}}{-\alpha\gamma} + C_p \psi(r'_n)\int_0^n (t-s)^{-(1+\alpha\gamma)} p(s)\,ds \cdot \Big[1+\frac{n^{-\alpha\gamma}}{-\alpha\gamma} C_p M_1 M_2\Big] \le {r'}_n.$$

(H5) For each $n > 0$, the linear operator $W : L^2([0,n], U) \to E$ is defined by

$$Wu = \int_0^n (t-s)^{\alpha-1} P_\alpha(n-s)(Bu(s))\,ds,$$

and

(i) the operator W has a pseudo-invertible operator W^{-1} which takes values in $L^2([0,n], U)/\operatorname{Ker} W$ and there exist positive constants M_1, M_2 such that

$$\|B\| \le M_1 \quad\text{and}\quad \|W^{-1}\| \le M_2,$$

(ii) there exist $\eta_W(t) \in L^\infty(J, \mathbb{R}^+)$, $C_B \ge 0$, for any bounded sets $V_1 \subset E$, $V_2 \subset U$,

$$\mu((W^{-1}V_1)(t)) \le \eta_W(t)\mu(V_1(t)), \quad \mu((BV_2)) \le C_B \mu_U(V_2).$$

Theorem 4.2. *Suppose that hypotheses* (H1)–(H3) *and* (H4′)–(H5) *hold. Further, assume that the inequality*

$$l_n\Big(1 + 2C_p C_B \|\eta_W\|_{L^\infty} \frac{n^{-\alpha\gamma}}{\alpha\gamma}\Big) < 1$$

holds, then problem (1.3), (1.4) *is controllable.*

Proof. We define in $C((-\infty, r], E)$ the family of measures of noncompactness by

$$\mu_n(V) = \omega_0^n(V) + \sup_{t\in[0,n]} e^{-Lt}\mu(V(t)),$$

where $V(t) = \{v(t) \in E : v \in V\}$.

Consider the operator $N_1 : C((-\infty, r], E) \to C((-\infty, r], E)$ defined by

$$(N_1 y)(t) = \begin{cases} \phi(t) & \text{if } t \in [-r, 0]; \\ \mathcal{S}_\alpha(t)\phi(0) + \displaystyle\int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) f\big(s, y(s-\rho(y(s)))\big)\,ds & \\ \quad + \displaystyle\int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) B u_y(s)\,ds & \text{if } t \in J. \end{cases}$$

Using assumption (H5), for an arbitrary function $y(\cdot)$, we define the control

$$u_y(t) = W^{-1}\left[y_1 - \mathcal{S}_\alpha(t)\phi(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f\big(s, y(s-\rho(y(s)))\big)\,ds\right](t).$$

Noting that

$$\|u_y(t)\| \le \|W^{-1}\|\left[|y_1| + \|\mathcal{S}_\alpha(t)\phi(0)\| + \int_0^n (n-\tau)^{\alpha-1}\mathcal{P}_\alpha(n-\tau)f\big(\tau, y(\tau-\rho(y(\tau)))\big)\,d\tau\right],$$

by (H2) we get

$$\|u_y(t)\| \le M_2\left[|y_1| + C_s t^{-\alpha(1+\gamma)}|\phi(0)| + \int_0^n C_p(n-\tau)^{-(1+\alpha\gamma)}p(\tau)\|y(\tau)\|\,d\tau\right]. \tag{4.1}$$

Next, for any $n \in \mathbb{N}$,

$$B_{r'_n} = B(0, r'_n) = \big\{w \in C([-r,\infty), E) : \ \|w\|_n \le r'_n\big\},$$

where $r'_n > 0$ is the constant defined in (H4′). Obviously, the subset $B_{r'n}$ is closed, bounded and convex.

Step 1. $N_1(B_{r_n}) \subset B_{r_n}$.
For any $n \in \mathbb{N}$, and each $y \in B_{r'n}$, by (4.1) we have

$$\begin{aligned}
\|(N_1y)(t)\| &\le \|\mathcal{S}_\alpha(t)\|\,|\phi(0)| + \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|\,\big\|f\big(s, y(s-\rho(y(s)))\big)\big\|\,ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|\|Bu_y(s)\|\,ds \\
&\le C_s n^{-\alpha(1+\gamma)}|\phi(0)| + C_p\psi(r'_n)\int_0^t (t-s)^{-(1+\alpha\gamma)}p(s)\,ds \\
&\quad + C_pM_1M_2\int_0^t (t-s)^{-(1+\alpha\gamma)}\bigg[|y_1| + C_s n^{-\alpha(1+\gamma)}|\phi(0)| \\
&\qquad + C_p\psi(r'_n)\int_0^n (n-\tau)^{-(1+\alpha\gamma)}p(\tau)\,d\tau\bigg]\,ds \\
&\le C_s n^{-\alpha(1+\gamma)}|\phi(0)|\left[1 + \frac{n^{-\alpha\gamma}}{-\alpha\gamma}\right] + |y_1|C_pM_1M_2\,\frac{n^{-\alpha\gamma}}{-\alpha\gamma} \\
&\quad + C_p\psi(r'_n)\int_0^n (t-s)^{-(1+\alpha\gamma)}p(s)\,ds\cdot\left[1 + \frac{n^{-\alpha\gamma}}{-\alpha\gamma}\,C_pM_1M_2\right] \\
&\le r'_n.
\end{aligned}$$

Step 2. N_1 is continuous on $B_{r'_n}$.

Let y_n be a sequence such that $y_n \longrightarrow y$ in $B_{r'_n}$. Then for each $t \in [0,n]$, and by the Lebesgue dominated convergence theorem, we obtain

$$\|(N_1 y_n)(t) - (N_1 y)(t)\| \leq \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\| \Big\| f\big(s, y_n(s-\rho(y_n(s)))\big) - f\big(s, y(s-\rho(y(s)))\big)\Big\| \, ds + \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\| \, \big\|Bu_{y_n}(s) - Bu_y(s)\big\| \, ds \longrightarrow 0 \text{ as } n\to\infty.$$

Thus N_1 is continuous.

Step 3. Since $N_1(B_{r_n}) \subset B_{r'_n}$ and $B_{r'_n}$ is bounded, we find that $N_1(B_{r'_n})$ is bounded.

Step 4. For each bounded subset V of $B_{r'_n}$, $\mu_n(N_1(V)) \leq k_n \mu_n(V)$. □

From Lemmas 2.11 and 2.12, for any $V \subset B_{r'_n}$ and any $\epsilon > 0$, there exists a sequence $\{y_k\}_{k=0}^\infty \subset V$ such that for all $t \in [0,n]$, we have

$$\begin{aligned}
\mu((N_1 V)(t)) &= \mu\bigg(\bigg\{\mathcal{S}_\alpha(t)\phi(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)\Big[f(s, y(s-\rho(y(s)))) + Bu_y(s)\Big]\, ds, v \in V\bigg\}\bigg) \\
&\leq 2\mu\bigg(\bigg\{\int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)\Big[f\big(s, y_k(s-\rho(y_k(s)))\big) + Bu_{y_k}(s)\Big]\, ds\bigg\}_{k=1}^\infty\bigg) + \epsilon \\
&\leq 4C_p \int_0^t (t-s)^{-(1+\alpha\gamma)}\mu\Big(\Big\{f\big(s, y_k(s-\rho(y_k(s)))\big) + Bu_{y_k}(s)\Big\}_{k=1}^\infty\Big) + \epsilon \\
&\leq 4C_p \int_0^t (t-s)^{-(1+\alpha\gamma)} l(s)\mu\big(\{y_k(s)\}_{k=1}^\infty\big) + \epsilon \\
&\qquad + 4C_p \int_0^t (t-s)^{-(1+\alpha\gamma)} C_B \mu\big(\{u_{y_k}(s)\}_{k=1}^\infty\big)\, ds.
\end{aligned}$$

Now, let us calculate $\mu(\{u_{y_k}(s))\}_{k=1}^\infty)$.

By (H5) we have

$$\begin{aligned}
\mu\big(\{u_{y_k}(t)\}_{k=1}^\infty\big) &\leq 2\eta_W(t) C_p \int_0^t (t-s)^{-(1+\alpha\gamma)} l(s)\mu\big(\{(y_k(s))\}_{k=1}^\infty\big)\, ds \\
&\leq \frac{1}{2}\,\eta_W(t) C_p 4 \int_0^t (t-s)^{-(1+\alpha\gamma)} e^{Ls} e^{-Ls} l(s) \mu\big(v\{(y_k(s))v\}_{k=1}^\infty v\big)\, ds.
\end{aligned}$$

Then

$$\mu_n(u(V)) \leq \frac{1}{2}\, l_n \eta_W(t) \mu_n(V). \tag{4.2}$$

Since $\epsilon > 0$ is arbitrary, by (4.2) we obtain

$$\mu(N_1(V)) \leq l_n \mu_n(V) + 2 l_n C_p C_B \,\frac{t^{-\alpha\gamma}}{\alpha\gamma}\, \|\eta_W\|_{L^\infty} \mu_n(V).$$

Thus

$$\mu_n(N_1(V)) \leq l_n \Big(1 + 2C_p C_B \|\eta_W\|_{L^\infty}\,\frac{n^{-\alpha\gamma}}{\alpha\gamma}\Big)\mu_n(V).$$

As a conclusion, we have achieved that N_1 has at least one fixed point in $B_{r'_n}$.

5 An example

We consider the fractional differential equation with state-dependent delay of the form

$$\begin{cases} {}_0^c\partial_t^\alpha u(t,x) = \partial_x^2 u(t,x) + Q(t)|u(t-\tau(u(t,x)),x)|, & x \in [0,\pi],\ \ t \in [0,\infty), \\ u(t,x) = u_0(t,x), & x \in [0,\pi],\ \ -\tau_{\max} \le t \le 0, \\ u(t,0) = u(t,\pi) = 0, & t \in [0,\infty), \end{cases} \tag{5.1}$$

where $u_0 \in C^2([-\tau_{\max}, 0] \times [0,\pi], \mathbb{R})$ Q is a continuous function from $[0,+\infty)$ to $\mathbb{R}$, the delay function τ is the bounded positive continuous function in $\mathbb{R}^n$, and $\tau_{\max}$ is the maximal delay which is defined by

$$\tau_{\max} = \sup_{x\in\mathbb{R}} \tau(x).$$

Consider the space of Hölder continuous functions $E = C^l([0,\pi],\mathbb{R})$ $(0 < l < 1)$, and let ${}_0^c\partial^\alpha$ be the regularized Caputo fractional partial derivative of order $0 < \alpha < 1$ with respect to t defined by

$$({}_0^c\partial^\alpha u)(t,x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} u(t,x)\, ds - t^{-\alpha} u(0,x) \right).$$

Next, we introduce the operator

$$A = -\partial_x^2, \quad D(A) = \big\{ u \in C^{2+l}([0,\pi]) :\ u(t,0) = u(t,\pi) = 0 \big\}$$

in the space $C^l([0,\pi],\mathbb{R})$. It follows from [26] that ν exists, $\epsilon > 0$ such that $A + \nu \in \Theta^{\frac{l}{2}-1}_{\frac{\pi}{2}-\epsilon}(X)$. Set

$$\begin{aligned} y(t)(x) &= u(t,x),\ \ t \in (-\infty,0],\ \ x \in [0,\pi], \\ \phi(t)(x) &= u_0(t,x),\ \ t \in [-\tau_{\max},0],\ \ x \in [0,\pi], \\ f(t,\varphi)(x) &= Q(t)\big|u(t-\tau(u(t,x)),x)\big|,\ \ \varphi \in E, \quad t \in [0,+\infty),\ \ -\infty < \theta \le 0,\ \ x \in [0,\pi]. \end{aligned}$$

Then system (5.1) can be written in the abstract form as (1.1), (1.2). As a consequence of Theorem 2.14, system (5.1) has a mild solution.

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Authors' addresses:

Mouffak Benchohra

1. Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria.

2. Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia.

E-mail: benchohra@yahoo.com

Sara Litimein

Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria.

E-mail: sara_litimein@yahoo.fr

Atika Matallah

Ecole supérieure de management de Tlemcen, Algérie.

E-mail: atika_matallah@yahoo.fr

Yong Zhou

Faculty of Mathematics and Computational Science, Xiangtan University, Hunan 411105, P.R. China.

E-mail: yzhou@xtu.edu.cn

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Aurelian Cernea

ON SOME FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSIONS WITH ERDÉLYI–KOBER FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

Abstract. We study two classes of fractional integro-differential inclusions with Erdélyi–Kober fractional integral boundary conditions and we obtain existence results in the case of the set-valued map has nonconvex values.

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რეზიუმე. შესწავლილია ფრაქციული ინტეგრო-დიფერენციალური ჩართვების ორი კლასი ერდელ-კობერის ფრაქციული ინტეგრალური სასაზღვრო პირობებით და მიღებულია არსებობის შედეგები იმ შემთხვევაში, როცა მრავალმნიშვნელოვანი ასახვა ღებულობს არაამოზნექილ მნიშვნელობებს.

1 Introduction

In recent years, the systems defined by fractional order derivatives have attracted increasing interest mainly due to their applications in different fields of science and engineering. The main reason is that a lot of phenomena in nature can be better explained using fractional-order systems (see, e.g., [5, 10, 13, 15, 16], etc.).

The present paper is concerned with the following boundary value problems. First, we consider a fractional integro-differential inclusion defined by the Caputo fractional derivative

$$D_c^q x(t) \in F(t, x(t), V(x)(t)) \quad a.e. \ ([0,T]) \tag{1.1}$$

with the boundary conditions of the form

$$\begin{aligned} x(0) &= \alpha \frac{1}{\Gamma(p)} \int_0^\zeta (\zeta - s)^{p-1} x(s)\, ds = \alpha J^p x(\zeta), \\ x(T) &= \beta \frac{\eta \xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^\xi \frac{s^{\eta\gamma+\eta-1}}{(\xi^\eta - s^\eta)^{1-\delta}}\, x(s)\, ds = \beta I_\eta^{\gamma,\delta} x(\xi), \end{aligned} \tag{1.2}$$

where $q \in (1,2]$, D_c^q is the Caputo fractional derivative of order q, $0 < \zeta, \xi < T$, $\alpha, \beta, \gamma \in \mathbb{R}$, $p, \delta, \eta > 0$, J^p is the Riemann–Liouville fractional integral of order p, $I_\eta^{\gamma,\delta}$ is the Erdélyi–Kober fractional integral of order $\delta > 0$ with $\eta > 0$ and $\gamma \in \mathbb{R}$, $F : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map and $V : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t,s,x(s))\, ds$ with $k(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a given function. We note that the fractional derivative introduced by Caputo in [6] and afterwards adopted in the theory of linear visco-elasticity allows to use Cauchy conditions with physical meanings.

Next, we consider the problem

$$D^q x(t) \in F(t, x(t), V(x)(t)) \ \ a.e. \ ([0,T]) \tag{1.3}$$

with the boundary conditions of the form

$$x(0) = 0, \quad \alpha x(T) = \sum_{i=1}^m \beta_i I_{\eta_i}^{\gamma_i,\delta_i} x(\xi_i), \tag{1.4}$$

where D^q is the Riemann–Liouville fractional derivative of order $q \in (1,2]$, $0 < \xi_i < T$, $\alpha, \beta_i, \gamma_i \in \mathbb{R}$, $\delta_i, \eta_i > 0$, $i = 1, 2, \dots, m$, F and V are as above.

Our aim is to obtain the existence of solutions for problems (1.1), (1.2) and (1.3), (1.4) in case where the set-valued map F has nonconvex values, but is assumed to be Lipschitz in the second and third variable. Our results use Filippov's techniques (see [12]); namely, the existence of solutions is obtained by starting from a given "quasi" solution. In addition, the result provides an estimate between the "quasi" solution and the solution obtained.

Note that in the case when F does not depend on the last variable and is single-valued, the existence results for problem (1.1), (1.2) may be found in [2], and in the situation when F does not depend on the last variable, the existence results for problem (1.3), (1.4) are given in [1]. All the results in [1, 2] are proved by using several suitable theorems from fixed point theory.

Our results improve some existence theorems in [1] and, respectively, in [2] in the case where the right-hand side is Lipschitz in the second variable. Moreover, these results may be regarded as generalizations to the case where the right-hand side contains a nonlinear Volterra integral operator. It should be also mentioned that the method used in our approach is known in the theory of differential inclusions; similar results for other classes of fractional differential inclusions have been obtained in our previous papers (see [7–9], etc.). However, the exposition of this method in the framework of problems (1.1), (1.2) and (1.3), (1.4) is new.

The paper is organized as follows. In Section 2, we recall some preliminary results that we need in the sequel and in Section 3, we prove our main results.

2 Preliminaries

Let (X,d) be a metric space. Recall that the Pompeiu–Hausdorff distance of the closed subsets $A,B\subset X$ is defined by

$$d_H(A,B)=\max\left\{d^*(A,B),d^*(B,A)\right\},\quad d^*(A,B)=\sup\left\{d(a,B);\ a\in A\right\},$$

where $d(x,B)=\inf\limits_{y\in B}d(x,y)$.

Let $I=[0,T]$, we denote by $C(I,\mathbb{R})$ the Banach space of all continuous functions from I to $\mathbb{R}$ with the norm $\|x(\cdot)\|_C=\sup_{t\in I}|x(t)|$, and $L^1(I,\mathbb{R})$ is the Banach space of integrable functions $u(\cdot):I\to\mathbb{R}$ endowed with the norm $\|u(\cdot)\|_1=\int\limits_0^T|u(t)|\,dt$.

The fractional integral of order $\alpha>0$ of a Lebesgue integrable function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$J^\alpha f(t)=\int\limits_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds,$$

provided the right-hand side is defined pointwise on $(0,\infty)$, and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha)=\int\limits_0^\infty t^{\alpha-1}e^{-t}\,\mathrm{d}t$.

The Riemann–Liouville fractional derivative of order $\alpha>0$ of a Lebesgue integrable function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$D^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int\limits_0^t(t-s)^{-\alpha+n-1}f(s)\,ds,$$

where $n=[\alpha]+1$, provided the right-hand side is defined pointwise on $(0,\infty)$.

The Caputo fractional derivative of order $\alpha>0$ of a function $f:[0,\infty)\to\mathbb{R}$ is defined by

$$D_c^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int\limits_0^t(t-s)^{-\alpha+n-1}f^{(n)}(s)\,\mathrm{d}s,$$

where $n=[\alpha]+1$. It is assumed implicitly that f is n times differentiable whose n-th derivative is absolutely continuous.

The Erdélyi–Kober fractional integral of order $\delta>0$ *with* $\eta>0$ *and* $\gamma\in\mathbb{R}$ of a continuous function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$I_\eta^{\gamma,\delta}f(t)=\frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\int\limits_0^t\frac{s^{\eta\gamma+\eta-1}}{(t^\eta-s^\eta)^{1-\delta}}f(s)\,ds,$$

provided the right-hand side is defined pointwise on $(0,\infty)$.

We recall that for $\eta=1$,

$$I_1^{\gamma,\delta}f(t)=\frac{t^{-(\delta+\gamma)}}{\Gamma(\delta)}\int\limits_0^t\frac{s^\gamma}{(t-s)^{1-\delta}}f(s)\,ds$$

is the Kober operator introduced by Kober in [14]. If $\gamma=0$, the Kober operator reduces to the Riemann–Liouville fractional integral with a power weight

$$I_1^{0,\delta}f(t)=\frac{t^{-\delta}}{\Gamma(\delta)}\int\limits_0^t\frac{f(s)}{(t-s)^{1-\delta}}\,ds.$$

Lemma 2.1 ([2])**.** *Let $\delta,\eta>0$ and $\gamma,q\in\mathbb{R}$. Then*

$$I_\eta^{\gamma,\delta}(t^q)=\frac{t^q\Gamma(\gamma+\frac{q}{\eta}+1)}{\Gamma(\gamma+\frac{q}{\eta}+\delta+1)}\,.$$

By definition, a function $x(\,\cdot\,)\in C^2(I,\mathbb{R})$ is called a solution of problem (1.1), (1.2) *if there exists $f(\,\cdot\,)\in L^1(I,\mathbb{R})$ such that $f(t)\in F(t,x(t),V(x)(t))$ a.e. (I), $D_c^q x(t)=f(t)$ a.e. (I) and conditions* (1.2) *are satisfied.*

Lemma 2.2 ([2])**.** *For $f(\,\cdot\,)\in AC(I,\mathbb{R})$, $x(\,\cdot\,)\in C^2(I,\mathbb{R})$ is a solution of the problem*

$$D_c^q x(t)=f(t)\ \text{ a.e. } (I),$$

with the boundary conditions (1.2) *if and only if*

$$x(t)=J^qf(t)+\frac{\alpha}{\Lambda}\,(v_4-tv_3)J^{p+q}f(\zeta)+\frac{1}{\Lambda}\,(v_2+tv_1)\big(\beta I_\eta^{\gamma,\delta}J^qf(\xi)-J^qf(T)\big),$$

where

$$\Lambda=v_1v_4+v_2v_3\neq 0,\quad v_1=1-\alpha\,\frac{\zeta^p}{\Gamma(p+1)}\,,\quad v_2=\alpha\,\frac{\zeta^{p+1}}{\Gamma(p+2)}\,,$$
$$v_3=1-\beta\,\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\delta+1)}\,,\quad v_4=T-\beta\zeta\,\frac{\Gamma(\gamma+\frac{1}{\eta}+1)}{\Gamma(\gamma+\frac{1}{\eta}+\delta+1)}\,.$$

Remark 2.3. The solution $x(\,\cdot\,)$ in Lemma 2.2 can be written as

$$\begin{aligned}
x(t)&=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\,f(s)\,ds+\frac{\alpha}{\Lambda}\,\frac{(v_4-tv_3)}{\Gamma(q)}\int_0^\zeta(\zeta-s)^{p+q-1}f(s)\,ds\\
&\quad+\frac{\beta(v_2+tv_1)}{\Lambda}\,\frac{\eta\xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\int_0^\xi\frac{s^{\eta\gamma+\eta-1}}{(\xi^\eta-s^\eta)^{1-\delta}}\Big(\frac{1}{\Gamma(q)}\int_0^s(s-u)^{q-1}f(u)\,du\Big)\,ds\\
&\quad-\frac{1}{\Lambda}\,(v_2+tv_1)\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}f(s)\,ds\\
&=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s)\,ds+\frac{\alpha}{\Lambda}\,\frac{(v_4-tv_3)}{\Gamma(q)}\int_0^\zeta(\zeta-s)^{p+q-1}f(s)\,ds\\
&\quad+\frac{\beta(v_2+tv_1)}{\Lambda\Gamma(q)}\,\frac{\eta\xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\int_0^\xi\Big(\int_u^\xi\frac{s^{\eta\gamma+\eta-1}}{(\xi^\eta-s^\eta)^{1-\delta}}\,(s-u)^{q-1}\,ds\Big)f(u)\,du\\
&\quad-\frac{1}{\Lambda}\,(v_2+tv_1)\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}\,f(s)\,ds\\
&=\int_0^T G_1(t,s)f(s)\,ds,
\end{aligned}$$

where

$$\begin{aligned}
G_1(t,u)&=\frac{(t-u)^{q-1}}{\Gamma(q)}\,\chi_{[0,t]}(u)+\frac{\alpha}{\Lambda}\,\frac{(v_4-tv_3)}{\Gamma(q)}\,(\zeta-u)^{p+q-1}\chi_{[0,\zeta]}(u)\\
&\quad+\frac{\beta(v_2+tv_1)}{\Lambda\Gamma(q)}\,\frac{\eta\xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\int_u^\xi\frac{s^{\eta\gamma+\eta-1}}{(\xi^\eta-s^\eta)^{1-\delta}}\,(s-u)^{q-1}\,ds\chi_{[0,\xi]}(u)-\frac{v_2+tv_1}{\Lambda\Gamma(q)}\,(T-u)^{q-1},
\end{aligned}$$

$\chi_S(\cdot)$ denotes the characteristic function of the set S.

Using the fact that $q>1$ and taking into account Lemma 2.1, one has

$$\frac{\eta\xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\int\limits_u^\xi \frac{s^{\eta\gamma+\eta-1}}{(\xi^\eta-s^\eta)^{1-\delta}}(s-u)^{q-1}\,ds \le \frac{\eta\xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\int\limits_0^\xi \frac{s^{\eta\gamma+\eta-1}}{(\xi^\eta-s^\eta)^{1-\delta}}s^{q-1}\,ds = \frac{\xi^{q-1}\Gamma(\gamma+\frac{q-1}{\eta}+1)}{\Gamma(\gamma+\frac{q-1}{\eta}+\delta+1)}.$$

Therefore, for any $t,u\in I$,

$$|G_1(t,u)| \le \frac{T^{q-1}}{\Gamma(q)} + \frac{|\alpha|(|v_4|+T|v_3|)\zeta^{p+q-1}}{|\Lambda|\Gamma(q)} + \frac{|\beta|(|v_2|+T|v_1|)}{|\Lambda|\Gamma(q)}\frac{\xi^{q-1}\Gamma(\gamma+\frac{q-1}{\eta}+1)}{\Gamma(\gamma+\frac{q-1}{\eta}+\delta+1)} + \frac{(|v_2|+T|v_1|)T^{q-1}}{|\Lambda|\Gamma(q)} =: K_1.$$

By definition, a function $x(\cdot)\in C^2(I,\mathbb{R})$ is called a solution of problem (1.3), (1.4) if there exists $f(\cdot)\in L^1(I,\mathbb{R})$ such that $f(t)\in F(t,x(t),V(x)(t))$ a.e. (I), $D_c^q x(t)=f(t)$ a.e. (I) and conditions (1.4) are satisfied.

Lemma 2.4 ([1]). *For $f(\cdot)\in AC(I,\mathbb{R})$, $x(\cdot)\in C^2(I,\mathbb{R})$ is a solution of the problem*

$$D_c x(t) = f(t) \ \text{ a.e. } (I),$$

with the boundary conditions (1.4) *if and only if*

$$x(t) = J^q f(t) - \frac{t^{q-1}}{\Lambda}\Big(\alpha J^q f(t) - \sum_{i=1}^m \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q f(\xi_i)\Big),$$

where

$$\Lambda = \alpha T^{q-1} - \sum_{i=1}^m \frac{\beta_1\xi_i^{q-1}\Gamma(\gamma_i+\frac{q-1}{\eta_i}+1)}{\Gamma(\gamma_i+\frac{q-1}{\eta_i}+\delta_i+1)} \ne 0.$$

Remark 2.5. The solution $x(\cdot)$ in Lemma 2.4 can be written as $x(t)=\int\limits_0^T G_2(t,s)f(s)\,ds$, where

$$G_2(t,u) = \frac{(t-u)^{q-1}}{\Gamma(q)}\chi_{[0,t]}(u) - \frac{\alpha t^{q-1}}{\Lambda\Gamma(q)}(t-u)^{q-1}\chi_{[0,t]}(u) + \sum_{i=1}^m \frac{\beta_i t^{q-1}}{\Lambda\Gamma(q)}\frac{\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\delta_i)}\int\limits_u^{\xi_i}\frac{s^{\eta_i\gamma_i+\eta_i-1}}{(\xi_i^{\eta_i}-s^{\eta_i})^{1-\delta_i}}(s-u)^{q-1}\,ds\,\chi_{[0,\xi_i]}(u).$$

As in Remark 2.3, for $i=1,2,\dots,m$, one has

$$\frac{\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\delta_i)}\int\limits_u^{\xi_i}\frac{s^{\eta_i\gamma_i+\eta_i-1}}{(\xi_i^{\eta_i}-s^{\eta_i})^{1-\delta_i}}(s-u)^{q-1}\,ds \le \frac{\xi_i^{q-1}\Gamma(\gamma_i+\frac{q-1}{\eta_i}+1)}{\Gamma(\gamma_i+\frac{q-1}{\eta_i}+\delta_i+1)}$$

and thus, for any $t,u\in I$,

$$|G_2(t,u)| \le \frac{T^{q-1}}{\Gamma(q)} + \frac{T^{q-1}}{|\Lambda|\Gamma(q)}\left[|\alpha|T^{q-1} + \sum_{i=1}^m \frac{|\beta_i|\xi_i^{q-1}\Gamma(\gamma_i+\frac{q-1}{\eta_i}+1)}{\Gamma(\gamma_i+\frac{q-1}{\eta_i}+\delta_i+1)}\right] =: K_2.$$

3 The main results

First, we recall a selection result (see [4]) which is a version of the celebrated Kuratowski and Ryll–Nardzewski selection theorem.

Lemma 3.1. *Suppose X is a separable Banach space, B is the closed unit ball in X, $H : I \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g : I \to X$, $L : I \to \mathbb{R}_+$ are measurable functions. If*

$$H(t) \cap (g(t) + L(t)B) \neq \varnothing \ \text{ a.e. } (I),$$

then the set-valued map $t \to H(t) \cap (g(t) + L(t)B)$ has a measurable selection.

In order to prove our results, we need the following hypotheses.

Hypothesis 3.2.

(i) $F(\,\cdot\,,\,\cdot\,) : I \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has nonempty closed values and is* $\mathcal{L}(I) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R})$ *measurable.*

(ii) *There exists $L(\,\cdot\,) \in L^1(I, (0, \infty))$ such that, for almost all $t \in I$, $F(t, \cdot\,, \cdot\,)$ is $L(t)$-Lipschitz in the sense that*

$$d_H\big(F(t, x_1, y_1), F(t, x_2, y_2)\big) \le L(t)\big(|x_1 - x_2| + |y_1 - y_2|\big) \ \ \forall\, x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

(iii) $k(\,\cdot\,,\,\cdot\,,\,\cdot\,) : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a function such that $\forall\, x \in \mathbb{R}$, $(t, s) \to k(t, s, x)$ is measurable.*

(iv) $|k(t, s, x) - k(t, s, y)| \le L(t)|x - y|$ *a.e.* $(t, s) \in I \times I$, $\forall\, x, y \in \mathbb{R}$.

Next, we use the notation

$$M(t) := L(t)(1 + \int\limits_0^t L(u)\, du), \ \ t \in I, \ \ K_0 = \int\limits_0^T M(t)\, dt.$$

Theorem 3.3. *Assume that Hypothesis 3.2 is satisfied and $K_1 K_0 < 1$. Let $y(\,\cdot\,) \in C^2(I, \mathbb{R})$ be such that $y(0) = \alpha J^p y(\zeta)$, $y(T) = \beta I_\eta^{\gamma,\delta} y(\xi)$ and there exist $p(\,\cdot\,) \in L^1(I, \mathbb{R}_+)$ with*

$$d\big(D_c^q y(t), F(t, y(t), V(y)(t))\big) \le p(t) \ \text{ a.e. } (I).$$

Then there exists a solution $x(\,\cdot\,) : I \to \mathbb{R}$ of problem (1.1), (1.2) satisfying for all $t \in I$ the inequality

$$|x(t) - y(t)| \le \frac{K_1}{1 - K_1 K_0}\, \|p(\,\cdot\,)\|_1.$$

Proof. The set-valued map $t \to F(t, y(t), V(y)(t))$ is measurable with closed values and

$$F(t, y(t), V(y)(t)) \cap \big\{D_c^q y(t) + p(t)[-1, 1]\big\} \neq \varnothing \ \text{ a.e. } (I).$$

It follows from Lemma 3.1 that there exists a measurable selection $f_1(t) \in F(t, y(t), V(y)(t))$ a.e. (I) such that

$$|f_1(t) - D_c^q y(t)| \le p(t) \ \text{ a.e. } (I). \tag{3.1}$$

Define $x_1(t) = \int\limits_0^T G_1(t, s) f_1(s)\, ds$. One has

$$|x_1(t) - y(t)| \le M_1 \int\limits_0^T p(t)\, dt.$$

We construct two sequences $x_n(\cdot) \in C(I,\mathbb{R})$, $f_n(\cdot) \in L^1(I,\mathbb{R})$, $n \geq 1$, with the following properties:

$$x_n(t) = \int_0^T G_1(t,s) f_n(s)\,ds, \quad t \in I, \tag{3.2}$$

$$f_n(t) \in F\big(t, x_{n-1}(t), V(x_{n-1})(t)\big) \ \text{ a.e. } (I), \tag{3.3}$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)\left(|x_n(t) - x_{n-1}(t)| + \int_0^t L(s)|x_n(s) - x_{n-1}(s)|\,ds\right) \ \text{ a.e. } (I). \tag{3.4}$$

If this is done, then from (3.1)–(3.4) for almost all $t \in I$ we have

$$|x_{n+1}(t) - x_n(t)| \leq K_1 (K_1 K_0)^n \int_0^T p(t)\,dt \ \ \forall\, n \in \mathbf{N}.$$

Indeed, assume that the last inequality is true for $n-1$ and we prove it for n. One has

$$\begin{aligned}
|x_{n+1}(t) - x_n(t)| &\leq \int_0^T |G_1(t,t_1)|\,|f_{n+1}(t_1) - f_n(t_1)|\,dt_1 \\
&\leq K_1 \int_0^T L(t_1)\left[|x_n(t_1) - x_{n-1}(t_1)| + \int_0^{t_1} L(s)|x_n(s) - x_{n-1}(s)|\,ds\right] dt_1 \\
&\leq K_1 \int_0^T L(t_1)\left(1 + \int_0^{t_1} L(s)\,ds\right) dt_1 \cdot K_1^n K_0^{n-1} \int_0^T p(t)\,dt \\
&= K_1 (K_1 K_0)^n \int_0^T p(t)\,dt.
\end{aligned}$$

Therefore, $\{x_n(\cdot)\}$ is a Cauchy sequence in the Banach space $C(I,\mathbb{R})$ converging uniformly to some $x(\cdot) \in C(I,\mathbb{R})$. Hence, by (3.4), for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy sequence in $\mathbb{R}$. Let $f(\cdot)$ be the pointwise limit of $f_n(\cdot)$.

At the same time, one has

$$\begin{aligned}
|x_n(t) - y(t)| &\leq |x_1(t) - y(t)| + \sum_{i=1}^{n-1} |x_{i+1}(t) - x_i(t)| \\
&\leq M_1 \int_0^T p(t)\,dt + \sum_{i=1}^{n-1} \left(K_1 \int_0^T p(t)\,dt\right)(K_1K_0)^i = \frac{K_1 \int_0^T p(t)\,dt}{1 - K_1 K_0}.
\end{aligned} \tag{3.5}$$

On the other hand, from (3.1), (3.4) and (3.5) for almost all $t \in I$ we obtain

$$|f_n(t) - D_c^q y(t)| \leq \sum_{i=1}^{n-1} |f_{i+1}(t) - f_i(t)| + |f_1(t) - D_c^q y(t)| \leq L(t)\,\frac{K_1 \int_0^T p(t)\,dt}{1 - K_1 K_0} + p(t).$$

Hence the sequence $f_n(\cdot)$ is integrably bounded and therefore $f(\cdot) \in L^1(I,\mathbb{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.2), (3.3), we deduce that $x(\cdot)$ is a solution of (1.1), (1.2). Finally, passing to the limit in (3.5), we obtain the desired estimate on $x(\cdot)$.

It remains to construct the sequences $x_n(\,\cdot\,)$, $f_n(\,\cdot\,)$ with the properties in (3.2)–(3.4). The construction will be done by induction.

Since the first step is already realized, assume that for some $N \geq 1$ we have already constructed $x_n(\,\cdot\,) \in C(I,\mathbb{R})$ and $f_n(\,\cdot\,) \in L^1(I,\mathbb{R})$, $n = 1,2,\dots,N$, satisfying (3.2), (3.4) for $n = 1,2,\dots,N$ and (3.3) for $n = 1,2,\dots,N-1$. The set-valued map $t \to F(t, x_N(t), V(x_N)(t))$ is measurable. Moreover, the map

$$t \longrightarrow L(t)\bigg(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|\,ds\bigg)$$

is measurable. By the lipschitzianity of $F(t,\,\cdot\,)$ for almost all $t \in I$ we have

$$F\big(t, x_N(t), V(x_N)(t)\big) \cap \bigg\{ f_N(t) + L(t)\bigg(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|\,ds\bigg)[-1,1]\bigg\} \neq \varnothing.$$

Lemma 3.1 yields that there exists a measurable selection $f_{N+1}(\,\cdot\,)$ of $F(\,\cdot\,, x_N(\,\cdot\,), V(x_N)(\,\cdot\,))$ such that for almost all $t \in I$,

$$|f_{N+1}(t) - f_N(t)| \leq L(t)\bigg(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|\,ds\bigg).$$

We define $x_{N+1}(\,\cdot\,)$ as in (3.2) with $n = N+1$. Thus $f_{N+1}(\,\cdot\,)$ satisfies (3.3) and (3.4) and the proof is complete. □

The assumption in Theorem 3.3 is satisfied, in particular, for $y(\,\cdot\,) = 0$ and therefore with $p(\,\cdot\,) = L(\,\cdot\,)$. We obtain the following consequence of Theorem 3.3.

Corollary 3.4. *Assume that Hypothesis* 3.2 *is satisfied,* $d(0, F(t,0,0) \leq L(t)$ *a.e.* (I) *and* $K_1K_0 < 1$. *Then there exists a solution* $x(\,\cdot\,)$ *of problem* (1.1), (1.2) *satisfying for all* $t \in I$, *the inequality*

$$|x(t)| \leq \frac{K_1}{1 - K_1K_0}\,\|L(\,\cdot\,)\|_1.$$

Example 3.5. Consider

$$q = \frac{3}{2}\,,\ \ T = 1,\ \ \alpha = \frac{6}{13}\,,\ \ p = \frac{1}{2}\,,\ \ \zeta = \frac{1}{4}\,,$$
$$\beta = \frac{\sqrt{7}}{9}\,,\ \ \gamma = \frac{3}{4}\,,\ \ \delta = \frac{\sqrt{7}}{5}\,,\ \ \eta = \frac{1}{6}\,,\ \ \xi = \frac{3}{4}\,.$$

Denote by K_1^0 the corresponding estimate of $G_1(\,\cdot\,,\,\cdot\,)$ in Remark 2.3 and take $a \in \big(0, -1 + \sqrt{1 + \frac{2}{K_1^0}}\big)$.

Define $F(\,\cdot\,,\,\cdot\,) : I \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by

$$F(t,x,y) = \Big[-a\,\frac{|x|}{1+|x|}\,, 0\Big] \cup \Big[0, a\,\frac{|y|}{1+|y|}\Big]$$

and $k(\,\cdot\,,\,\cdot\,,\,\cdot\,) : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $k(t,s,x) = ax$.

Since

$$\sup\big\{|u| :\ u \in F(t,x,y)\big\} \leq a\ \ \forall\, t \in [0,1],\ \ x,y \in \mathbb{R},$$
$$\mathrm{d}_H\big(F(t,x_1,y_1), F(t,x_2,y_2)\big) \leq a|x_1 - x_2| + a|y_1 - y_2|\ \ \forall\, x_1,x_2,y_1,y_2 \in \mathbb{R},$$

in this case $p(t) \equiv L(t) \equiv a$, $M(t) = a(1 + at)$ and $K_0 = a + \frac{a^2}{2}$.

According to the choice of a, we are able to apply Corollary 3.4 in order to deduce the existence of a solution of the problem

$$D_c^{\frac{3}{2}} x(t) \in \Big[-a\, \frac{|x(t)|}{1+|x(t)|}\,, 0\Big] \cup \Big[0, a^2\, \frac{\big|\int_0^t x(s)\, ds\big|}{1+a\big|\int_0^t x(s)\, ds\big|}\Big],$$
$$x(0) = \frac{6}{13}\, J^{\frac{1}{2}} x\Big(\frac{1}{4}\Big), \quad x(1) = \frac{\sqrt{7}}{9}\, I_{\frac{1}{6}}^{\frac{3}{4}, \frac{\sqrt{7}}{5}} x\Big(\frac{3}{4}\Big)$$

that satisfies

$$|x(t)| \leq \frac{K_1^0 a}{1-(a+\frac{a^2}{2})K_1^0} \quad \forall\, t \in [0,1].$$

If F does not depend on the last variable, Hypothesis 3.2 becames

Hypothesis 3.6.

(i) $F(\,\cdot\,,\,\cdot\,) : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has nonempty closed values and is* $\mathcal{L}(I) \otimes \mathcal{B}(\mathbb{R})$ *measurable.*

(ii) *There exists* $L(\,\cdot\,) \in L^1(I,(0,\infty))$ *such that for almost all* $t \in I$, $F(t,\,\cdot\,)$ *is* $L(t)$*-Lipschitz in the sense that*
$$d_H\big(F(t,x_1), F(t,x_2)\big) \leq L(t)|x_1 - x_2| \;\; \forall\, x_1, x_2 \in \mathbb{R}.$$

Denote $L_0 = \int\limits_0^T L(t)\, dt.$

Corollary 3.7. *Assume that Hypothesis* 3.6 *is satisfied,* $d(0, F(t,0) \leq L(t)$ *a.e.* (I) *and* $K_1 L_0 < 1$. *Then there exists a solution* $x(\,\cdot\,)$ *of the fractional differential inclusion*

$$D_c^q x(t) \in F(t, x(t)) \;\; \text{a.e. } (I),$$

with the boundary conditions (1.2) *satisfying for all* $t \in I$

$$|x(t)| \leq \frac{K_1 L_0}{1-K_1 L_0}\,. \tag{3.6}$$

Remark 3.8. If $F(\,\cdot\,,\,\cdot\,)$ is a single-valued map, the fractional differential inclusion reduces to the fractional differential equation
$$D_c^q x(t) = f(t, x(t)) \;\; \text{a.e. } (I).$$
In this case, a similar result to the one in Corollary 3.7 may be found in [2], namely, Theorem 3.1. It is assumed that the Lipschitz constant of $f(t,\,\cdot\,)$ does not depend on t and its proof is done by using the Banach fixed point theorem. Therefore, our Corollary 3.7 extends Theorem 3.1 in [2] to the situation when the Lipschitz constant of $f(t,\,\cdot\,)$ depends on t and to the set-valued framework. Moreover, Corollary 3.7 provides a priori bounds for the solution, as in (3.6).

The proof of the next theorem is similar to that of Theorem 3.3.

Theorem 3.9. *Assume that Hypothesis* 3.2 *is satisfied and* $K_2 K_0 < 1$. *Let* $y(\,\cdot\,) \in C^2(I,\mathbb{R})$ *be such that* $y(0) = 0$, $\alpha y(T) = \sum\limits_{i=1}^m \beta_i I_{\eta_i}^{\gamma_i, \delta_i} y(\xi_i)$ *and let there exist* $p(\,\cdot\,) \in L^1(I,\mathbb{R})$ *with*

$$d\big(D^q y(t), F(t, y(t, V(y)(t)))\big) \leq p(t) \;\; \text{a.e. } (I).$$

Then there exists a solution $x(\,\cdot\,) : I \to \mathbb{R}$ *of problem* $(1.3), (1.4)$ *satisfying for all* $t \in I$

$$|x(t) - y(t)| \leq \frac{K_2}{1-K_2 K_0}\, \|p(\,\cdot\,)\|_1.$$

Example 3.10. Consider

$$q=\frac{3}{2}\,,\ \ T=5,\ \ m=3,\ \ \alpha=\frac{2}{3}\,,\ \ \beta_1=\frac{e}{2}\,,\ \ \beta_2=\frac{\pi}{3}\,,\ \ \beta_3=\frac{\sqrt{\pi}}{6}\,,$$
$$\eta_1=\frac{\sqrt{3}}{5}\,,\ \ \eta_2=\frac{\sqrt{2}}{5}\,,\ \ \eta_3=\frac{e}{3}\,,\ \ \gamma_1=\frac{5}{3}\,,\ \ \gamma_2=\frac{2}{9}\,,\ \ \gamma_3=\frac{\sqrt{e}}{2}\,,$$
$$\delta_1=\frac{3}{7}\,,\ \ \delta_2=\frac{\sqrt{3}}{8}\,,\ \ \delta_3=\frac{e^2}{4}\,,\ \ \xi_1=\frac{4}{3}\,,\ \ \xi_2=\frac{3}{2}\,,\ \ \xi_3=\frac{2}{7}\,.$$

Denote by K_2^0 the corresponding estimate of $G_2(\,\cdot\,,\,\cdot\,)$ in Remark 2.5 and take $a\in\big(0,\frac{1}{5}\,(-1+\sqrt{1+\frac{2}{K_2^0}})\big)$.

Define $F(\,\cdot\,,\,\cdot\,):I\times\mathbb{R}\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ by

$$F(t,x,y)=\Big[-a\,\frac{|x|}{1+|x|},0\Big]\cup\Big[0,a\,\frac{|y|}{1+|y|}\Big]$$

and $k(\,\cdot\,,\,\cdot\,,\,\cdot\,):I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by $k(t,s,x)=ax$.

As above,

$$\sup\big\{|u|:\ u\in F(t,x,y)\big\}\le a\ \ \forall\,t\in[0,1],\ \ x,y\in\mathbb{R},$$
$$\mathrm{d}_H\big(F(t,x_1,y_1),F(t,x_2,y_2)\big)\le a|x_1-x_2|+a|y_1-y_2|\ \ \forall\,x_1,x_2,y_1,y_2\in\mathbb{R},$$

and, therefore, $p(t)\equiv L(t)\equiv a$, $M(t)=a(1+at)$ and $K_0=5a+\frac{25a^2}{2}$.

Taking into account the choice of a, we can apply Theorem 3.9 with $y(\,\cdot\,)=0$ and deduce the existence of a solution of the problem

$$D^{\frac{3}{2}}x(t)\in\Big[-a\,\frac{|x(t)|}{1+|x(t)|}\,,0\Big]\cup\Big[0,a^2\,\frac{\big|\int_0^t x(s)\,ds\big|}{1+a\big|\int_0^t x(s)\,ds\big|}\Big],$$
$$x(0)=0,\quad \frac{2}{3}\,x(5)=\frac{e}{2}\,I_{\frac{\sqrt{3}}{5}}^{\frac{5}{3},\frac{3}{7}}x\Big(\frac{4}{3}\Big)+\frac{\pi}{3}\,I_{\frac{\sqrt{2}}{5}}^{\frac{2}{9},\frac{\sqrt{3}}{8}}x\Big(\frac{3}{2}\Big)+\frac{\sqrt{\pi}}{6}\,I_{\frac{e}{3}}^{\frac{\sqrt{e}}{2},\frac{e^2}{4}}x\Big(\frac{2}{7}\Big)$$

that satisfies

$$|x(t)|\le\frac{5K_2^0a}{1-(5a+\frac{25a^2}{2})K_2^0}\ \ \forall\,t\in[0,5].$$

Remark 3.11. If $F(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ does not depend on the last variable and $y(\,\cdot\,)=0$, similar results to the one in Theorem 3.9 can be found in [1], namely, Theorem 3.1 and Theorem 4.2. Even if our hypothesis concerning the set-valued map is weaker than in [1] (in Theorem 3.1 of [1] it is assumed that F has the approximate end point property and in Theorem 4.2 of [1] it is assumed that F is a generalized contraction), our approach does not require for the values of F to be compact as in [1] and also provides a priori bounds for solutions.

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Author's addresses:

1. Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, 010014 Bucharest, Romania.

2. Academy of Romanian Scientists, Splaiul Independenţei 54, 050094 Bucharest, Romania.

E-mail: acernea@fmi.unibuc.ro

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George Chkadua

INTERACTION PROBLEMS OF ACOUSTIC WAVES AND ELECTRO-MAGNETO-ELASTIC STRUCTURES

Abstract. In the paper, is consider a three-dimensional model of fluid-solid acoustic interaction when an electro-magneto-elastic body occupying a bounded region Ω^+ is embedded in an unbounded fluid domain $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. In this case in the domain Ω^+ is a five-dimensional electro-magneto-elastic field (the displacement vector with three components, electric potential and magnetic potential), while in the unbounded domain Ω^- is a scalar acoustic pressure field. The physical kinematic and dynamic relations mathematically are described by appropriate boundary and transmission conditions. In the paper, less restrictions are considered on matrix differential operator of electro-magneto-elasticity and asymptotic classes are introduced. In particular, corresponding characteristic polynomial of the matrix differential operator can have multiple real zeros. With the help of the potential method and theory of pseudodifferential equations, for above mentioned fluid-solid acoustic interaction mathematical problems the uniqueness and existence theorems are proved in Sobolev–Slobodetskii spaces.

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Key words and phrases. Boundary-transmission problems, fluid-solid interaction, potential method, pseudodifferential equations, Helmholtz equation, steady state oscillations, Jones modes, Jones eigenfrequencies.

რეზიუმე. ნაშრომში განხილულია სითხისა და სხეულის აკუსტიკური ურთიერთქმედების სამგანზომილებიანი მოდელი, როდესაც ელექტრო-მაგნეტო-დრეკადი სხეულს უკავია Ω^+ შემოსაზღვრული არე, რომელიც ჩადგმულია $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$ შემოუსაზღვრელ არეში. ამ შემთხვევაში შემოსაზღვრულ Ω^+ არეში არის ხუთგანზომილებიანი ელექტრო-მაგნეტო-დრეკადი ველი (გადაადგილების ვექტორის სამი კომპონენტი, ელექტრული პოტენციალი და მაგნიტური პოტენციალი), ხოლო Ω^- შემოუსაზღვრელ არეში - აკუსტიკური წნევის სკალარული ველი. ფიზიკური კინემატიკური და დინამიკური ურთიერთქმედებები მათემატიკურად აღწერილია შესაბამისი სასაზღვრო და ტრანსმისიის პირობებით. ნაშრომში მოთხოვნილია ნაკლები შეზღუდვები ელექტრო-მაგნეტო-დრეკადობის დიფერენციალურ ოპერატორზე და შემოღებულია შესაბამისი ასიმპტოტური კლასები. კერძოდ, მატრიცული დიფერენციალური ოპერატორის შესაბამის მახასიათებელ პოლინომს შეიძლება გააჩნდეს ჯერადი ნამდვილი ნულები. პოტენციალთა მეთოდისა და ფსევდოდიფერენციალურ განტოლებათა თეორიის გამოყენებით დამტკიცებულია ზემოთ აღნიშნული სითხისა და სხეულის აკუსტიკური ურთიერთქმედების მათემატიკური ამოცანების ამონახსნების ერთადერთობისა და არსებობის თეორემები სობოლევ-სლობოდეცკის სივრცეებში.

1 Formulation of the problems

1.1 Introduction

Interaction problems of different dimensional fields of this type appear in mathematical models of electro-magneto transducers. Further examples of similar models are related to phased array microphones, ultrasound equipment, inkjet droplet actuators, sonar transducers, bioimaging, immunochemistry, and acousto-biotherapeutics (see [38, 39]).

Due to the rapidly increasing use of composite materials in modern industrial and technological processes on the one hand, and in biology and medicine on the other hand, mathematical modeling related to complex composite structures and their mathematical analysis became very important from the theoretical and practical points of view in recent years.

The Dirichlet, Neumann and mixed type interaction problems of acoustic waves and piezoelectric structures are studied in [9, 11, 12].

Similar interaction problems for the classical model of elasticity has been investigated by a number of authors. An exhaustive information concerning theoretical and numerical results, for the case when the both interacting media are isotropic, can be found in [1–4, 15, 17–19, 26, 27, 31]. The cases when the elastic body is homogeneous and anisotropic, and the fluid is isotropic, has been considered in [25, 35, 36]. In this case, one has a three-dimensional elastic field, the displacement vector with three components in the bounded domain Ω^+, and a scalar pressure field in the unbounded domain Ω^-.

In our case, in the domain Ω^+ we have an additional electric and magnetic fields which essentially complicate the investigation of the transmission problems in question. In contrast to the classical elasticity, the differential operator of electro-magneto-elasticity is not self-adjoint and is not positive-definite.

We consider less restrictions on the matrix differential operator of electro-magneto-elasticity by introducing asymptotic classes $M_{m_1,m_2,m_3}(\mathbf{P})$, where $\mathbf{P}$ is determinant of the electro-magneto-elasticity matrix operator, in particular, we allow for the corresponding characteristic polynomial of the matrix differential operator to have multiple real zeros. This class is generalization of the Sommerfeld-Kupradze class.

We investigate the above problems with the use of the boundary integral equations method and the theory of pseudodifferential equations on manifolds and prove the existence and uniqueness theorems in Sobolev–Slobodetskii spaces.

1.2 Piezoelectric field

Let Ω^+ be a bounded three-dimensional domain in $\mathbb{R}^3$ with a compact C^∞-smooth boundary $S = \partial\Omega^+$ and let $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. Assume that the domain Ω^+ is filled with an anisotropic homogeneous piezoelectro-magnetic material.

The basic equations of steady state oscillations of piezoelectro-magneticity for anisotropic homogeneous media are written as follows:

$$
\begin{aligned}
c_{ijkl}\partial_i\partial_l u_k + \rho_1\omega^2\delta_{jk}u_k + e_{lij}\partial_l\partial_i\varphi + q_{lij}\partial_i\partial_l\psi + F_j = 0, \quad j = 1,2,3,\\
-e_{ikl}\partial_i\partial_l u_k + \varepsilon_{il}\partial_i\partial_l\varphi + a_{il}\partial_i\partial_l\psi + F_4 = 0,\\
-q_{ikl}\partial_i\partial_l u_k + a_{il}\partial_i\partial_l\varphi + \mu_{il}\partial_i\partial_l\psi + F_5 = 0,
\end{aligned}
$$

or in the matrix form

$$A(\partial,\omega)U + F = 0 \ \text{ in } \Omega^+,$$

where $U = (u,\varphi,\psi)^\top$, $u = (u_1,u_2,u_3)^\top$ is the displacement vector, $\varphi = u_4$ is the electric potential, $\psi = u_5$ is the magnetic potential and $F = (F_1,F_2,F_3,F_4,F_5)^\top$ is a given vector-function. The three-dimensional vector (F_1,F_2,F_3) is the mass force density, while $-F_4$ is the electric charge density, $-F_5$

is the electric current density, and $A(\partial,\omega)$ is the matrix differential operator,

$$A(\partial,\omega)=[A_{jk}(\partial,\omega)]_{5\times5}, \tag{1.1}$$
$$A_{jk}(\partial,\omega)=c_{ijkl}\partial_i\partial_l+\rho_1\omega^2\delta_{jk},\quad A_{j4}(\partial,\omega)=e_{lij}\partial_l\partial_i,\quad A_{j5}(\partial,\omega)=q_{lij}\partial_l\partial_i,$$
$$A_{4k}(\partial,\omega)=-e_{ikl}\partial_i\partial_l,\quad A_{44}(\partial,\omega)=\varepsilon_{il}\partial_i\partial_l,\quad A_{45}(\partial,\omega)=a_{il}\partial_i\partial_l,$$
$$A_{5k}(\partial,\omega)=-q_{ikl}\partial_i\partial_l,\quad A_{54}(\partial,\omega)=a_{il}\partial_i\partial_l,\quad A_{55}(\partial,\omega)=\mu_{il}\partial_i\partial_l,$$

$j,k=1,2,3$, where $\omega\in\mathbb{R}$ is a frequency parameter, ρ_1 is the density of the piezoelectro-magnetic material, c_{ijlk}, e_{ikl}, q_{ikl}, ε_{il}, μ_{il}, a_{il} are elastic, piezoelectric, piezomagnetic, dielectric, magnetic permeability and electromagnetic coupling constants, respectively, δ_{jk} is the Kronecker symbol and summation over repeated indices is meant from 1 to 3, if not stated otherwise. These constants satisfy the standard symmetry conditions

$$c_{ijkl}=c_{jikl}=c_{klij},\ \ e_{ijk}=e_{ikj},\ \ q_{ijk}=q_{ikj},\ \ \varepsilon_{ij}=\varepsilon_{ji},\ \ \mu_{jk}=\mu_{kj},\ \ a_{jk}=a_{kj},\ \ i,j,k,l=1,2,3.$$

Moreover, from physical considerations related to positiveness of the internal energy, it follows that the quadratic forms $c_{ijkl}\xi_{ij}\xi_{kl}$ and $\varepsilon_{ij}\eta_i\eta_j$ are positive definite:

$$c_{ijkl}\xi_{ij}\xi_{kl}\geq c_0\xi_{ij}\xi_{ij}\ \ \forall\,\xi_{ij}=\xi_{ji}\in\mathbb{R}, \tag{1.2}$$
$$\varepsilon_{ij}\eta_i\eta_j\geq c_2|\eta|^2,\ \ q_{ij}\eta_i\eta_j\geq c_3|\eta|^2,\ \ \mu_{ij}\eta_i\eta_j\geq c_1|\eta|^2\ \ \forall\,\eta=(\eta_1,\eta_2,\eta_3)\in\mathbb{R}^3, \tag{1.3}$$

where c_0, c_1, c_2 and c_3 are positive constants.

More careful analysis related to the positive definiteness of the potential energy insures that the matrix

$$\Lambda:=\begin{pmatrix}[\varepsilon_{kj}]_{3\times3} & [a_{kj}]_{3\times3}\\ [a_{kj}]_{3\times3} & [\mu_{kj}]_{3\times3}\end{pmatrix}_{6\times6}$$

is positive definite, i.e.,

$$\varepsilon_{kj}\zeta'_k\overline{\zeta'_j}+a_{kj}\big(\zeta'_k\overline{\zeta''_j}+\overline{\zeta'_k}\zeta''_j\big)+\mu_{kj}\zeta''_k\overline{\zeta''_j}\geq c_4\big(|\zeta'|^2+|\zeta''|^2\big)\ \ \forall\,\zeta',\zeta''\in\mathbb{C}^3, \tag{1.4}$$

where c_4 some positive constant.

The principal homogeneous symbol matrix of the operator $A(\partial,\omega)$ has the following form:

$$A^{(0)}(\xi)=\begin{pmatrix}[-c_{ijlk}\xi_i\xi_l]_{3\times3} & [-e_{lij}\xi_l\xi_i]_{3\times1} & [-q_{lij}\xi_l\xi_i]_{3\times1}\\ [e_{ikl}\xi_i\xi_l]_{1\times3} & -\varepsilon_{il}\xi_i\xi_l & -a_{il}\xi_i\xi_l\\ [q_{ikl}\xi_i\xi_l]_{1\times3} & -a_{il}\xi_i\xi_l & -\mu_{il}\xi_i\xi_l\end{pmatrix}_{5\times5}.$$

With the help of inequalities (1.2) and (1.3) it can be easily shown that

$$-\operatorname{Re}A^{(0)}(\xi)\zeta\cdot\zeta\geq c|\zeta|^2|\xi|^2\ \ \forall\,\zeta\in\mathbb{C}^4,\ \ \forall\,\xi\in\mathbb{R}^3,\ \ c=const>0,$$

implying that $A(\partial,\omega)$ is a strongly elliptic, formally nonselfadjoint differential operator.

Here and in the sequel, $a\cdot b$ denotes the scalar product of two vectors $a,b\in\mathbb{C}^N$, $a\cdot b:=\sum\limits_{k=1}^{N}a_k\overline{b}_k$.

In the theory of electro-magneto-elasticity, the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n=(n_1,n_2,n_3)$ have the form

$$\sigma_{ij}n_i:=c_{ijlk}n_i\partial_lu_k+e_{lij}n_i\partial_l\varphi+q_{lij}n_i\partial_l\psi,\ \ j=1,2,3,$$

while the normal component of the electric displacement vector $D=(D_1,D_2,D_3)^\top$ and the normal component of the magnetic induction vector $B=(B_1,B_2,B_3)^\top$ read as

$$-D_in_i=-e_{ikl}n_i\partial_lu_k+\varepsilon_{il}n_i\partial_l\varphi+a_{il}n_i\partial_l\psi,$$
$$-B_in_i=-q_{ikl}n_i\partial_lu_k+a_{il}n_i\partial_l\varphi+\mu_{il}n_i\partial_l\psi.$$

Let us introduce the boundary matrix differential operator

$$
\begin{gathered}
T(\partial,n)=[T_{jk}(\partial,n)]_{5\times 5},\\
T_{jk}(\partial,n)=c_{ijlk}n_i\partial_l,\quad T_{j4}(\partial,n)=e_{lij}n_i\partial_l,\quad T_{j5}(\partial,n)=q_{lij}n_i\partial_l,\\
T_{4k}(\partial,n)=-e_{ikl}n_i\partial_l,\quad T_{44}(\partial,n)=\varepsilon_{il}n_i\partial_l,\quad T_{45}(\partial,n)=a_{il}n_i\partial_l,\\
T_{5k}(\partial,n)=-q_{ikl}n_i\partial_l,\quad T_{54}(\partial,n)=a_{il}n_i\partial_l,\quad T_{55}(\partial,n)=\mu_{il}n_i\partial_l,
\end{gathered}
$$

$j,k=1,2,3$. For a vector $U=(u,\varphi,\psi)^\top$, we have

$$
T(\partial,n)U=(\sigma_{1j}n_j,\sigma_{2j}n_j,\sigma_{3j}n_j,-D_in_i,\,-B_in_i)^\top. \tag{1.5}
$$

The components of the vector TU given by (1.5) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of electro-magneto-elasticity, while the fourth one is the normal component of the electric displacement vector and the fifth one is the normal component of the magnetic induction vector.

In Green's formulae, one also has the following boundary operator associated with the adjoint differential operator $A^*(\partial,\omega)=A^\top(-\partial,\omega)=A^\top(\partial,\omega)$,

$$
\widetilde{T}(\partial,n)=[\widetilde{T}_{jk}(\partial,n)]_{5\times 5},
$$

where

$$
\begin{gathered}
\widetilde{T}_{jk}(\partial,n)=T_{jk}(\partial,n),\quad \widetilde{T}_{j4}(\partial,n)=-T_{j4}(\partial,n),\quad \widetilde{T}_{j5}(\partial,n)=-T_{j5}(\partial,n),\\
\widetilde{T}_{4k}(\partial,n)=-T_{4k}(\partial,n),\quad \widetilde{T}_{44}(\partial,n)=T_{44}(\partial,n),\quad \widetilde{T}_{45}(\partial,n)=T_{45}(\partial,n),\\
\widetilde{T}_{5k}(\partial,n)=-T_{5k}(\partial,n),\quad \widetilde{T}_{54}(\partial,n)=T_{54}(\partial,n),\quad \widetilde{T}_{55}(\partial,n)=T_{55}(\partial,n),
\end{gathered}
$$

$j,k=1,2,3$.

1.3 Green's formulae for electro-magneto-elastic vector fields

For arbitrary vector-functions $U=(u_1,u_2,u_3,u_4,u_5)^\top\in[C^2(\overline{\Omega^+})]^5$ and $V=(v_1,v_2,v_3,v_4,v_5)^\top\in[C^2(\overline{\Omega^+})]^5$, we have the following Green's formulae (see [6]):

$$
\begin{gathered}
\int\limits_{\Omega^+}\big[A(\partial,\omega)U\cdot V+E(U,\overline{V})\big]\,dx=\int\limits_{S}\{TU\}^+\cdot\{V\}^+\,dS,\\
\int\limits_{\Omega^+}\big[A(\partial,\omega)U\cdot V-U\cdot A^*(\partial,\omega)V\big]\,dx=\int\limits_{S}\big[\{TU\}^+\cdot\{V\}^+-\{U\}^+\cdot\{\widetilde{T}V\}^+\big]\,dS,
\end{gathered}
$$

where

$$
\begin{aligned}
E(U,\overline{V})=c_{ijlk}\partial_iu_j\partial_l\overline{v}_k-\rho_1\omega^2u\cdot v+e_{lij}(\partial_lu_4\partial_i\overline{v}_j-\partial_iu_j\partial_l\overline{v}_4)&\\
+q_{lij}(\partial_lu_5\partial_i\overline{v}_j-\partial_iu_j\partial_l\overline{v}_5)+\varepsilon_{jl}\partial_ju_4\partial_l\overline{v}_4+a_{jl}(\partial_lu_4\partial_j\overline{v}_5-\partial_ju_5\partial_l\overline{v}_4)+\mu_{jl}\partial_ju_5\partial_l\overline{v}_5&
\end{aligned}
$$

with $u=(u_1,u_2,u_3)^\top$ and $v=(v_1,v_2,v_3)^\top$. The symbol $\{\,\cdot\,\}^+$ denotes the one-sided limits (the trace operator) on S from Ω^+. Note that by the standard limiting procedure, the above Green's formulae can be generalized to the vector-functions $U\in[H^1(\Omega^+)]^5$ and $V\in[H^1(\Omega^+)]^5$ with $A(\partial,\omega)U\in[L_2(\Omega^+)]^5$ and $A^*(\partial,\omega)V\in[L_2(\Omega^+)]^5$.

With the help of these Green's formulae, we can define a generalized trace vector $\{T(\partial,n)U\}^+\in[H^{-1/2}(S)]^5$ for a function $U\in[H^1(\Omega^+)]^5$ with $A(\partial,\omega)U\in[L_2(\Omega^+)]^5$:

$$
\big\langle\{T(\partial,n)U\}^+,\{V\}^+\big\rangle_S:=\int\limits_{\Omega^+}\big[A(\partial,\omega)U\cdot V+E(U,\overline{V})\big]\,dx,
$$

where $V\in[H^1(\Omega^+)]^5$ is an arbitrary vector-function.

Here and in what follows, the symbol $\langle\,\cdot\,,\,\cdot\,\rangle_S$ denotes the duality between the mutually adjoint function spaces $[H^{-1/2}(S)]^N$ and $[H^{1/2}(S)]^N$, which extends the usual L_2 scalar product

$$\langle f,g\rangle_S=\int\limits_S\sum_{j=1}^N f_j\overline{g}_j\,dS \ \text{ for } \ f,g\in[L_2(S)]^N.$$

1.4 Scalar acoustic pressure field and Green's formulae

We assume that the exterior domain Ω^- is filled with a homogeneous isotropic inviscid fluid medium with the constant density ρ_2. Further, let the propagation of acoustic wave in Ω^- be described by a complex-valued scalar function (scalar field) w, being a solution of the homogeneous Helmholtz equation

$$\Delta \mathrm{w}+\rho_2\omega^2\mathrm{w}=0 \ \text{ in } \Omega^-, \tag{1.6}$$

where $\Delta=\sum\limits_{j=1}^3\frac{\partial^2}{\partial x_j^2}$ is the Laplace operator and $\omega>0$. The function $\mathrm{w}(x)=P^{sc}(x)$ is the pressure of a scattered acoustic wave.

We say that a solution w to the Helmholtz equation (1.6) belongs to the class $Som_p(\Omega^-)$, $p=1,2$, if w satisfies the classical Sommerfeld radiation condition

$$\frac{\partial \mathrm{w}(x)}{\partial |x|}+i(-1)^p\sqrt{\rho_2}\,\omega \mathrm{w}(x)=O(|x|^{-2}) \ \text{ as } \ |x|\to\infty. \tag{1.7}$$

Note that if a solution w of the Helmholtz equation (1.6) in Ω^- satisfies the Sommerfeld radiation condition (1.7), then (see [43])

$$\mathrm{w}(x)=O(|x|^{-1}) \ \text{ as } \ |x|\to\infty.$$

Let Ω be a domain in $\mathbb{R}^3$ with a compact simply connected boundary $\partial\Omega\in C^\infty$.

We denote by $H^s(\Omega)$ $(H^s_{loc}(\Omega))$ and $H^s(\partial\Omega)$ $s\in\mathbb{R}$, the L_2 based Sobolev–Slobodetskii (Bessel potential) spaces in Ω and on the closed manifold $\partial\Omega$.

Respectively, we denote by $H^s_{comp}(\Omega)$ the subspace of $H^s(\Omega)$ $(H^s_{loc}(\Omega))$ consisting of functions with compact supports.

If M is a smooth proper submanifold of a manifold $\partial\Omega$, then we denote by $\widetilde{H}^s(M)$ the following subspace of $H^s(\partial\Omega)$:

$$\widetilde{H}^s(M):=\big\{g:\ g\in H^s(\partial\Omega),\ \operatorname{supp} g\subset\overline{M}\big\},$$

while $H^s(M)$ denotes the space of restrictions to M of functions from $H^s(\partial\Omega)$,

$$H^s(M):=\big\{r_M f:\ f\in H^s(\partial\Omega)\big\},$$

where r_M is the restriction operator to M.

Let $\mathrm{w}_1\in H^1_{loc}(\Omega^-)\cap Som_p(\Omega^-)$, $p=1,2,$, $\Delta\mathrm{w}_1\in L_{2,loc}(\Omega^-)$, $\mathrm{w}_2\in H^1_{comp}(\overline{\Omega^-})$, then the following Green's first formula holds:

$$\int\limits_{\Omega^-}(\Delta+k^2)\mathrm{w}_1\overline{\mathrm{w}}_2\,dx+\int\limits_{\Omega^-}\nabla\mathrm{w}_1\nabla\overline{\mathrm{w}}_2\,dx-k^2\int\limits_{\Omega^-}\mathrm{w}_1\overline{\mathrm{w}}_2\,dx=-\big\langle\{\partial_n\mathrm{w}_1\}^-,\{\mathrm{w}_2\}^-\big\rangle_S, \tag{1.8}$$

where $n=(n_1,n_2,n_3)$ is the exterior unit normal vector to S directed outward with respect to the domain Ω^+, and $\partial_n=\frac{\partial}{\partial n}$ denotes the normal derivative.

1.5 Formulation of the Dirichlet and Neumann type interaction problems for steady state oscillation equations

Now we formulate the fluid-solid interaction problems. We assume that $S=\partial\Omega^+=\partial\Omega^-\in C^\infty$.

Dirichlet type problem (D_ω)**:** Find a vector-function $U = (u, u_4, u_5)^\top = (u, \varphi, \psi)^\top \in [H^1(\Omega^+)]^5$ and a scalar function $\mathrm{w} \in H^1_{loc}(\Omega^-) \cap Som_1(\Omega^-)$ satisfying the differential equations

$$A(\partial, \omega)U = 0 \;\text{ in } \Omega^+, \tag{1.9}$$

$$\Delta \mathrm{w} + \rho_2\omega^2 \mathrm{w} = 0 \text{in } \Omega^-, \tag{1.10}$$

the transmission conditions

$$\{u \cdot n\}^+ = b_1\{\partial_n \mathrm{w}\}^- + f_0 \;\text{ on } S, \tag{1.11}$$

$$\big\{[T(\partial, n)U]_j\big\}^+ = b_2\{\mathrm{w}\}^- n_j + f_j \;\text{ on } S, \;\; j = 1, 2, 3, \tag{1.12}$$

and the Dirichlet boundary conditions

$$\{\varphi\}^+ = f_1^{(D)} \;\text{ on } S, \tag{1.13}$$

$$\{\psi\}^+ = f_2^{(D)} \;\text{ on } S, \tag{1.14}$$

where b_1 and b_2 are the given complex constants satisfying the conditions

$$b_1 b_2 \neq 0 \;\text{ and }\; \mathrm{Im}[\overline{b}_1 b_2] = 0, \tag{1.15}$$

and $f_0 \in H^{-1/2}(S)$, $f_j \in H^{-1/2}(S)$, $j = 1, 2, 3$, $f_1^{(D)} \in H^{1/2}(S)$, $f_2^{(D)} \in H^{1/2}(S)$.

Neumann type problem (N_ω)**:** Find a vector-function $U = (u, u_4, u_5) = (u, \varphi, \psi)^\top \in [H^1(\Omega^+)]^5$ and a scalar function $\mathrm{w} \in H^1_{loc}(\Omega^-) \cap Som_1(\Omega^-)$ satisfying the differential equations (1.9), (1.10), the transmission conditions (1.11), (1.12) and the Neumann boundary conditions

$$\big\{[T(\partial, n)U]_4\big\}^+ = f_1^{(N)} \;\text{ on } S, \tag{1.16}$$

$$\big\{[T(\partial, n)U]_5\big\}^+ = f_2^{(N)} \;\text{ on } S, \tag{1.17}$$

where b_1 and b_2 are the given complex constants satisfying conditions (1.15), and $f_0 \in H^{-1/2}(S)$, $f_j \in H^{-1/2}(S)$, $j = 1, 2, 3$, $f_1^{(N)} \in H^{-1/2}(S)$, $f_2^{(N)} \in H^{-1/2}(S)$.

The transmission conditions (1.11), (1.12) are called the *kinematic and dynamic* conditions. For an interaction problem of fluid and electro-magneto-elastic body

$$\begin{gathered} b_1 = [\rho_2\omega^2]^{-1}, \quad b_2 = -1, \quad f_0(x) \equiv f_0^{inc}(x) = [\rho_2\omega^2]^{-1}\partial_n P^{inc}(x), \\ f_j = -P^{inc}(x)\, n_j(x), \;\; j = 1, 2, 3, \end{gathered} \tag{1.18}$$

where P^{inc} is an incident plane wave,

$$P^{inc}(x) = e^{id\cdot x}, \quad d = \omega\sqrt{\rho_2}\,\eta, \;\; \eta \in \mathbb{R}^3, \;\; |\eta| = 1.$$

2 The uniqueness of solutions of the problems (D_ω) and (N_ω)

2.1 Jones modes and Jones eigenfrequencies

We denote by $J_D(\Omega^+)$ the set of values of the frequency parameter $\omega > 0$ for which the following boundary value problem

$$A(\partial, \omega)U = 0 \;\text{ in } \Omega^+, \tag{2.1}$$

$$\{u \cdot n\}^+ = 0 \;\text{ on } S, \tag{2.2}$$

$$\big\{[T(\partial, n)U]_j\big\}^+ = 0 \;\text{ on } S, \;\; j = 1, 2, 3, \tag{2.3}$$

$$\{\varphi\}^+ = 0 \;\text{ on } S, \tag{2.4}$$

$$\{\psi\}^+ = 0 \;\text{ on } S, \tag{2.5}$$

has a nontrivial solution $U = (u, \varphi, \psi)^\top \in [H^1(\Omega^+)]^5$ (cf. [25]).

We denote by $J_N(\Omega^+)$ the set of values of the frequency parameter $\omega > 0$ for which the following boundary value problem

$$A(\partial, \omega)U = 0 \ \text{ in } \Omega^+, \tag{2.6}$$

$$\{u \cdot n\}^+ = 0 \ \text{ on } S, \tag{2.7}$$

$$\big\{[T(\partial, n)U]\big\}^+ = 0 \ \text{ on } S, \tag{2.8}$$

has a nontrivial solution $U = (u, \varphi, \psi)^\top \in [H^1(\Omega^+)]^5$ (cf. [25]).

Nontrivial solutions of problems (2.1)–(2.5) and (2.6)–(2.8) will be referred as *Jones modes*, while the corresponding values of ω are called *Jones eigenfrequencies*, as they were first discussed by D. S. Jones [25] in a related context (a thin layer of ideal fluid between an elastic body and a surrounding elastic exterior). For example, Jones eigenfrequencies exist for any axisymmetric body, such bodies can sustain torsional oscillations in which only the azimuthal component of displacement is nonzero. However, we do not expect Jones eigenfrequencies to exist for an arbitrary body. The spaces of Jones modes corresponding to ω we denote by $X_{D,\omega}(\Omega^+)$ and $X_{N,\omega}(\Omega^+)$, respectively.

Let $J^*_D(\Omega^+)$ be the set of values of the frequency parameter $\omega > 0$ for which the following boundary value problem

$$A^*(\partial, \omega)V = 0 \ \text{ in } \Omega^+, \tag{2.9}$$

$$\{v \cdot n\}^+ = 0 \ \text{ on } S, \tag{2.10}$$

$$\big\{[\widetilde{T}(\partial, n)V]_j\big\}^+ = 0 \ \text{ on } S, \ \ j = 1, 2, 3, \tag{2.11}$$

$$\{v_4\}^+ = 0 \ \text{ on } S, \tag{2.12}$$

$$\{v_5\}^+ = 0 \ \text{ on } S \tag{2.13}$$

has a nontrivial solution $V = (v, v_4, v_5)^\top \in [H^1(\Omega^+)]^5$.

Let $J^*_N(\Omega^+)$ be the set of values of the frequency parameter $\omega > 0$ for which the following boundary value problem

$$A^*(\partial, \omega)V = 0 \ \text{ in } \Omega^+, \tag{2.14}$$

$$\{v \cdot n\}^+ = 0 \ \text{ on } S, \tag{2.15}$$

$$\big\{[\widetilde{T}(\partial, n)V]\big\}^+ = 0 \ \text{ on } S \tag{2.16}$$

has a nontrivial solution $V = (v, v_4, v_5)^\top \in [H^1(\Omega^+)]^5$.

The spaces of Jones modes corresponding to ω for the differential operator $A^*(\partial, \omega)$ we denote by $X^*_{D,\omega}(\Omega^+)$, and $X^*_{N,\omega}(\Omega^+)$, respectively.

It can be shown that $J_D(\Omega^+)$ is at most countable, while $J_N(\Omega^+) \equiv \mathbb{R}$, since for an arbitrary nonzero constants c_1 and c_2, the vector $(0, 0, 0, c_1, c_2)^\top$ is a Jones eigenvector: $(0, 0, 0, c_1, c_2)^\top \in X_{N,\omega}(\Omega^+)$ for arbitrary ω. The same is true for $J^*_D(\Omega^+)$ and $J^*_N(\Omega^+)$. Note that for each ω the corresponding spaces of Jones modes $X_{D,\omega}(\Omega^+)$, $X_{N,\omega}(\Omega^+)$, $X^*_{D,\omega}(\Omega^+)$ and $X^*_{N,\omega}(\Omega^+)$ are of a finite dimension.

2.2 The uniqueness theorems for the problems (D_ω) and (N_ω)

Theorem 2.1. *Let a pair (U, w) be a solution of the homogeneous problem (D_ω) and $\omega > 0$. Then $\mathrm{w} = 0$ in Ω^- and either $U = 0$ in Ω^+ if $\omega \notin J_D(\Omega^+)$ or $U \in X_{D,\omega}(\Omega^+)$ if $\omega \in J_D(\Omega^+)$.*

Proof. Let us write Green's formula for the Helmholtz equation in the domain $\Omega_R := \Omega^- \cap B(0, R)$, where $\overline{\Omega^+} \subset B(0, R)$ with $B(0, R)$ being the ball of radius R and centered at the origin,

$$\int\limits_{\Omega_R} \big[(\Delta + \rho_2\omega^2)\mathrm{w}\overline{\mathrm{w}} - \mathrm{w}(\Delta + \rho_2\omega^2)\overline{\mathrm{w}}\big]\, dx = \int\limits_{S(0,R)} \partial_n \mathrm{w}\overline{\mathrm{w}}\, dS - \int\limits_{S(0,R)} \partial_n \overline{\mathrm{w}}\, \mathrm{w}\, dS - \big\langle \{\partial_n \mathrm{w}\}^-, \{\mathrm{w}\}^- \big\rangle_S + \big\langle \{\partial_n \overline{\mathrm{w}}\}^-, \{\overline{\mathrm{w}}\}^- \big\rangle_S, \tag{2.17}$$

where $S(0,R)=\partial B(0,R)$ is the boundary of the ball $B(0,R)$.

We have also the following Green's formula for the operator $A(\partial,\omega)$ in the domain Ω^+:

$$\int\limits_{\Omega^+}\Big[[A(\partial,\omega)U]_j\overline{u}_j+[\overline{A(\partial,\omega)U}]_4u_4+[\overline{A(\partial,\omega)U}]_5u_5+\mathcal{E}(U,\overline{U})\Big]\,dx$$
$$=\big\langle\{TU\}_j^+,\{u_j\}^+\big\rangle_S+\big\langle\{\overline{TU}\}_4^+,\{\overline{u}_4\}^+\big\rangle_S+\big\langle\{\overline{TU}\}_5^+,\{\overline{u}_5\}^+\big\rangle_S,\quad(2.18)$$

where $\mathcal{E}(U,\overline{U})=c_{ijlk}\partial_iu_j\partial_l\overline{u}_k-\rho_1\omega^2|u|^2+\varepsilon_{il}\partial_iu_4\partial_l\overline{u}_4+\mu_{jl}\partial_ju_5\partial_l\overline{u}_5$. Clearly, $\operatorname{Im}\mathcal{E}(U,\overline{U})=0$ for an arbitrary vector-function U.

With the help of (1.9), (1.10), (1.13), and (1.14), we obtain from (2.17) and (2.18) the following equalities:

$$\int\limits_{S(0,R)}\partial_n\mathrm{w}\overline{\mathrm{w}}\,dS-\int\limits_{S(0,R)}\partial_n\overline{\mathrm{w}}\mathrm{w}\,dS-\big\langle\{\partial_n\mathrm{w}\}^-,\{\mathrm{w}\}^-\big\rangle_S+\big\langle\{\partial_n\overline{\mathrm{w}}\}^-,\{\overline{\mathrm{w}}\}^-\big\rangle_S=0,\quad(2.19)$$

$$\operatorname{Im}\big\langle\{[TU]_j\}^+,\{u_j\}^+\big\rangle_S=0.\quad(2.20)$$

The homogeneous transmission conditions yield

$$\big\langle\{[TU]_j\}^+,\{u_j\}^+\big\rangle_S=\big\langle b_2\{\mathrm{w}\}^-\,n_j,\{u_j\}^+\big\rangle_S=b_2\overline{b}_1\big\{\partial_n\overline{\mathrm{w}}\}^-,\{\overline{\mathrm{w}}\}^-\big\rangle_S.\quad(2.21)$$

Since $\operatorname{Im}[\overline{b}_1b_2]=0$, from (2.20) and (2.21) it follows that

$$\operatorname{Im}\big\langle\{\partial_n\overline{\mathrm{w}}\}^-,\{\overline{\mathrm{w}}\}^-\big\rangle_S=0,$$

and from (2.19) we derive that

$$\operatorname{Im}\int\limits_{S(0,R)}\partial_n\overline{\mathrm{w}}\mathrm{w}\,dS=0.\quad(2.22)$$

Taking into account the Sommerfeld radiation condition, from (2.22) we conclude that

$$\lim_{R\to\infty}\int\limits_{S(0,R)}|\mathrm{w}|^2\,dS=0.$$

Using the Rellich-Vekua lemma, we find that $\mathrm{w}=0$ in the domain Ω^- (see [13, 43]). Then from the homogeneous boundary conditions it follows that the vector-function $U=(u,\varphi,\psi)^\top$ solves problem (2.1)–(2.4), i.e., either $U=0$ in Ω^+ if $\omega\notin J_D(\Omega^+)$ or $U\in X_{D,\omega}(\Omega^+)$ if $\omega\in J_D(\Omega^+)$, which completes the proof. □

The following assertions can be proved quite analogously.

Theorem 2.2. *Let a pair (U,w) be a solution of the homogeneous problem (N_ω). Then $U\in X_{N,\omega}(\Omega^+)$ and $\mathrm{w}=0$ in Ω^-.*

Remark 2.3. Let a pair $(V,\mathrm{w})\in[H^1(\Omega^+)]^5\times[H^1_{loc}(\Omega^-)\cap Som_2(\Omega^-)]$ be a solution of the homogeneous problem

$$A^*(\partial,\omega)V=0\ \text{ in }\Omega^+,$$
$$(\Delta+\rho_2\omega^2)\mathrm{w}=0\ \text{ in }\Omega^-,$$
$$\{v\cdot n\}^++\overline{b}_2^{\,-1}\{\partial_n\mathrm{w}\}^-=0\ \text{ on }S,$$
$$\big\{[\widetilde{T}(\partial,n)V]_j\big\}^++\overline{b}_1^{\,-1}\{\mathrm{w}\}^-n_j=0\ \text{ on }S,\quad j=1,2,3,$$
$$\{v_4\}^+=0\ \text{ on }S,$$
$$\{v_5\}^+=0\ \text{ on }S,$$

where b_1 and b_2 are the given complex constants satisfying the conditions (1.15).

Then $\mathrm{w}=0$ in Ω^- and either $V=0$ in Ω^+ if $\omega\notin J^*_D(\Omega^+)$ or $V\in X^*_{D,\omega}(\Omega^+)$ if $\omega\in J^*_D(\Omega^+)$.

Remark 2.4. Let a pair $(V,\mathrm{w}) \in [H^1(\Omega^+)]^5 \times [H^1_{loc}(\Omega^-) \cap Som_2(\Omega^-)]$ be a solution of the homogeneous problem

$$\begin{gathered}
A^*(\partial,\omega)V = 0 \ \text{ in } \Omega^+, \\
(\Delta + \rho_2\omega^2)\mathrm{w} = 0 \ \text{ in } \Omega^-, \\
\{v\cdot n\}^+ + \overline{b}_2^{\,-1}\{\partial_n \mathrm{w}\}^- = 0 \ \text{ on } S, \\
\big\{[\widetilde{T}(\partial,n)V]_j\big\}^+ + \overline{b}_1^{\,-1}\{\mathrm{w}\}^- n_j = 0 \ \text{ on } S, \quad j=1,2,3, \\
\big\{[\widetilde{T}(\partial,n)V]_4\big\}^+ = 0 \ \text{ on } S, \\
\big\{[\widetilde{T}(\partial,n)V]_5\big\}^+ = 0 \ \text{ on } S,
\end{gathered}$$

where b_1 and b_2 are the given complex constants satisfying conditions (1.15).

Then $V \in X^*_{N,\omega}(\Omega^+)$ and $\mathrm{w} = 0$ in Ω^-.

3 Layer potentials

3.1 Potentials associated with the Helmholtz equation

Let us introduce the single and double layer potentials,

$$\begin{aligned}
V_\omega(g)(x) &:= \int_S \gamma(x-y,\omega) g(y)\, d_yS, \quad x \notin S, \\
W_\omega(f)(x) &:= \int_S \partial_{n(y)}\gamma(x-y,\omega) f(y)\, d_yS, \quad x \notin S,
\end{aligned}$$

where

$$\gamma(x,\omega) := -\frac{\exp(i\sqrt{\rho_2}\,\omega|x|)}{4\pi|x|}$$

is the fundamental solution of the Helmholtz equation (1.6). These potentials satisfy the Sommerfeld radiation condition, i.e., belong to the class $Som_1(\Omega^-)$.

For these potentials the following theorems are valid (see [13, 37]).

Theorem 3.1. *Let $g \in H^{-1/2}(S)$, $f \in H^{1/2}(S)$. Then on the manifold S the following jump relations hold:*

$$\begin{gathered}
\{V_\omega(g)\}^\pm = \mathcal{H}_\omega(g), \quad \{W_\omega(f)\}^\pm = \pm 2^{-1} f + \mathcal{K}^*_\omega(f), \\
\{\partial_n V_\omega(g)\}^\pm = \mp 2^{-1} g + \mathcal{K}_\omega(g), \quad \{\partial_n W_\omega(f)\}^+ = \{\partial_n W_\omega(f)\}^- =: \mathcal{L}_\omega(f),
\end{gathered}$$

where $\mathcal{H}_\omega$, $\mathcal{K}^*_\omega$ and $\mathcal{K}_\omega$ are integral operators with the weakly singular kernels,

$$\begin{aligned}
\mathcal{H}_\omega(g)(z) &:= \int_S \gamma(z-y,\omega) g(y)\, d_yS, \quad z \in S, \\
\mathcal{K}^*_\omega(f)(z) &:= \int_S \partial_{n(y)}\gamma(z-y,\omega) f(y)\, d_yS, \quad z \in S, \\
\mathcal{K}_\omega(g)(z) &:= \int_S \partial_{n(z)}\gamma(z-y,\omega) g(y)\, d_yS, \quad z \in S,
\end{aligned}$$

while $\mathcal{L}_\omega$ is a singular integro-differential operator (pseudodifferential operator) of order 1.

Theorem 3.2. *The operators*

$$\mathcal{N} := -2^{-1} I_1 + \mathcal{K}^*_\omega + \mu\mathcal{H}_\omega : H^{1/2}(S) \to H^{1/2}(S), \tag{3.1}$$

$$\mathcal{M} := \mathcal{L}_\omega + \mu\big(2^{-1} I_1 + \mathcal{K}_\omega\big) : H^{1/2}(S) \to H^{-1/2}(S), \tag{3.2}$$

are invertible provided that $\operatorname{Im}\mu \neq 0$. Here I_1 is the scalar identity operator.

The mapping properties of the above potentials and the boundary integral operators are described in Appendix.

3.2 Fundamental solution and potentials of the steady state oscillation equations of electro-magneto-elasticity

Let us consider the equation

$$\Phi_A(\xi,\omega) := \det A(i\xi,\omega) = \det \begin{pmatrix} [c_{ijlk}\xi_i\xi_l - \rho_1\omega^2\delta_{jk}]_{3\times 3} & [e_{lij}\xi_l\xi_i]_{3\times 1} & [q_{lij}\xi_l\xi_i]_{3\times 1} \\ [-e_{ikl}\xi_i\xi_l]_{1\times 3} & \varepsilon_{il}\xi_i\xi_l & a_{il}\xi_i\xi_l \\ [-q_{ikl}\xi_i\xi_l]_{1\times 3} & a_{il}\xi_i\xi_l & \mu_{il}\xi_i\xi_l \end{pmatrix}_{5\times 5} = 0, \quad (3.3)$$

$$\xi \in \mathbb{R}^3 \setminus \{0\}, \quad \omega \in \mathbb{R}, \quad i,j,k,l = 1,2,3,$$

where $\Phi_A(\xi,\omega)$ is the characteristic polynomial of the operator $A(\partial,\omega)$. The origin is an isolated zero of (3.3).

We are interested in the real zeros of the function $\Phi_A(\xi,\omega)$, $\xi \in \mathbb{R}^3 \setminus \{0\}$.

Denote

$$\lambda := \frac{\rho_1\omega^2}{|\xi|^2}, \quad \widehat{\xi} := \frac{\xi}{|\xi|} \ \text{ for } \ |\xi| \neq 0,$$

$$B(\lambda,\widehat{\xi}) := \begin{pmatrix} [c_{ijkl}\widehat{\xi}_i\widehat{\xi}_l - \lambda\delta_{jk}]_{3\times 3} & [A_{j4}(\widehat{\xi})]_{3\times 1} & [A_{j5}(\widehat{\xi})]_{3\times 1} \\ [-A_{j4}(\widehat{\xi})]_{1\times 3} & \varepsilon_{il}\widehat{\xi}_i\widehat{\xi}_l & a_{il}\widehat{\xi}_i\widehat{\xi}_l \\ [-A_{j5}(\widehat{\xi})]_{1\times 3} & a_{il}\widehat{\xi}_i\widehat{\xi}_l & \mu_{il}\widehat{\xi}_i\widehat{\xi}_l \end{pmatrix}_{5\times 5}.$$

Then (3.3) can be rewritten as

$$\Psi(\lambda,\widehat{\xi}) := \det B(\lambda,\widehat{\xi}) = 0. \quad (3.4)$$

This is a cubic equation in λ with real coefficients.

Theorem 3.3. *Equation* (3.4) *possesses three real positive roots* $\lambda_1(\widehat{\xi})$*,* $\lambda_2(\widehat{\xi})$*,* $\lambda_3(\widehat{\xi})$*.*

Proof. Let $\widehat{\xi} \in \Sigma_1 := \{x \in \mathbb{R}^3 : |x| = 1\}$ and $\Psi(\lambda,\widehat{\xi}) = 0$. Then there is a non-trivial vector $\eta \in \mathbb{C}^5 \setminus \{0\}$ such that $B(\lambda,\widehat{\xi})\,\eta = 0$, i.e.,

$$(c_{ijkl}\widehat{\xi}_i\widehat{\xi}_l - \lambda\delta_{jk})\eta_k + e_{lij}\widehat{\xi}_l\widehat{\xi}_i\eta_4 + q_{lij}\widehat{\xi}_l\widehat{\xi}_i\eta_5 = 0, \ \ j = 1,2,3, \quad (3.5)$$

$$-e_{ikl}\widehat{\xi}_i\widehat{\xi}_l\eta_k + \varepsilon_{il}\widehat{\xi}_i\widehat{\xi}_l\eta_4 + a_{il}\widehat{\xi}_i\widehat{\xi}_l\eta_5 = 0, \quad (3.6)$$

$$-q_{ikl}\widehat{\xi}_i\widehat{\xi}_l\eta_k + a_{il}\widehat{\xi}_i\widehat{\xi}_l\eta_4 + \mu_{il}\widehat{\xi}_i\widehat{\xi}_l\eta_5 = 0, \quad (3.7)$$

Multiply the first three equations by $\overline{\eta}_j$, the complex conjugate of the fourth equation by η_4, the complex conjugate of the fifth equation by η_5 and sum them to obtain

$$c_{ijkl}\widehat{\xi}_i\widehat{\xi}_l\eta_k\overline{\eta}_j - \lambda|\eta'|^2 + e_{lij}\widehat{\xi}_l\widehat{\xi}_i\eta_4\overline{\eta}_j + q_{lij}\widehat{\xi}_l\widehat{\xi}_i\eta_5\overline{\eta}_j$$
$$- e_{ijl}\widehat{\xi}_i\widehat{\xi}_l\overline{\eta}_j\eta_4 + \varepsilon_{il}\widehat{\xi}_i\widehat{\xi}_l|\eta_4|^2 + a_{il}\widehat{\xi}_i\widehat{\xi}_l\overline{\eta}_5\eta_4 - q_{ijl}\widehat{\xi}_i\widehat{\xi}_l\overline{\eta}_j\eta_5 + a_{il}\widehat{\xi}_i\widehat{\xi}_l\overline{\eta}_4\eta_5 + \mu_{il}\widehat{\xi}_i\widehat{\xi}_l|\eta_5|^2 = 0, \quad (3.8)$$

where $\eta' = (\eta_1,\eta_2,\eta_3)$.

Due to the symmetry property of the coefficients e_{lij} and q_{lij},

$$e_{lij}\widehat{\xi}_l\widehat{\xi}_i\eta_4\overline{\eta}_j = e_{ijl}\widehat{\xi}_i\widehat{\xi}_l\overline{\eta}_j\eta_4, \quad q_{lij}\widehat{\xi}_l\widehat{\xi}_i\eta_5\overline{\eta}_j = q_{ijl}\widehat{\xi}_i\widehat{\xi}_l\overline{\eta}_j\eta_5.$$

Therefore, we derive from (3.8) that

$$c_{ijkl}\widehat{\xi}_i\widehat{\xi}_l\eta_k\overline{\eta}_j - \lambda|\eta'|^2 + \varepsilon_{il}\widehat{\xi}_i\widehat{\xi}_l|\eta_4|^2 + \mu_{il}\widehat{\xi}_i\widehat{\xi}_l|\eta_5|^2 + 2\,\mathrm{Re}\,a_{il}\widehat{\xi}_i\widehat{\xi}_l\overline{\eta}_5\eta_4 = 0. \quad (3.9)$$

Next, we note that $c_{ijkl}\widehat{\xi}_i\widehat{\xi}_l\eta_k\overline{\eta}_j = c_{ijkl}\overline{\varkappa}_{ij}\varkappa_{kl} \geq \delta_0\varkappa_{kl}\overline{\varkappa}_{kl} \geq 0$ with $\varkappa_{kl} = 2^{-1}(\widehat{\xi}_l\eta_k + \widehat{\xi}_k\eta_l)$.

Moreover, due to the strict inequalities $\varepsilon_{il}\widehat{\xi}_i\widehat{\xi}_l \geq \delta_1 > 0$, $\mu_{il}\widehat{\xi}_i\widehat{\xi}_l \geq \delta_2 > 0$, and (1.4), it follows that $|\eta'| \neq 0$, since otherwise from (3.9) we get $\eta_4 = 0$, which contradicts the inclusion $\eta = (\eta', \eta_4, \eta_5) \in \mathbb{C}^5 \setminus \{0\}$. Therefore, from (3.9) we finally conclude that $\lambda > 0$. □

Denote the roots of equation (3.4) by $\lambda_1,\ \lambda_2,\ \lambda_3$. Clearly, the equation of the surface $S_{\omega,j}$, $j=1,2,3$, in the spherical coordinates reads as

$$r=r_j(\theta,\varphi)=\frac{\sqrt{\rho_1}\,\omega}{\sqrt{\lambda_j(\widehat{\xi})}}\,,$$

where $\xi_1=r\cos\varphi\sin\theta$, $\xi_2=r\sin\varphi\sin\theta$, $\xi_3=r\cos\theta$ with $0\le\varphi\le 2\pi$, $0\le\theta\le\pi$, $r=|\xi|$.

We also have the following identity:

$$\Phi_A(\xi,\omega)=\det A(i\xi,\omega)=\Phi_A(\widehat{\xi},0)\,r^4\prod_{j=1}^{3}\big(r^2-r_j^2(\widehat{\xi})\big)=\Phi_A(\widehat{\xi},0)\,r^4\prod_{j=1}^{3}P_j(\xi).$$

It can easily be shown that the vector

$$n(\xi)=(-1)^j|\nabla\Phi_A(\xi,\omega)|^{-1}\nabla\Phi_A(\xi,\omega),\ \ \xi\in S_{\omega,j},$$

is an external unit normal vector to $S_{\omega,j}$ at the point ξ.

Further, we assume that the following conditions are fulfilled (cf. [10, 33, 41, 42]):

(i) if $\Phi_A(\xi,\omega)=\Phi_A(\widehat{\xi},0)\,r^4P_1(\xi)P_2(\xi)P_3(\xi)$, then $\nabla_\xi(P_1(\xi)P_2(\xi)P_3(\xi))\neq 0$ at real zeros $\xi\in\mathbb{R}^3\setminus\{0\}$ of the polynomial (3.3), or

if $\Phi_A(\xi,\omega)=\Phi_A(\widehat{\xi},0)\,r^4P_1^2(\xi)P_2(\xi)$, then $\nabla_\xi(P_1(\xi)P_2(\xi))\neq 0$ at real zeros $\xi\in\mathbb{R}^3\setminus\{0\}$ of the polynomial (3.3), or

if $\Phi_A(\xi,\omega)=\Phi_A(\widehat{\xi},0)\,r^4P_1^3(\xi)$, then $\nabla_\xi P_1(\xi)\neq 0$ at real zeros $\xi\in\mathbb{R}^3\setminus\{0\}$ of the polynomial (3.3);

(ii) the Gaussian curvature of the surface, defined by the real zeros of the polynomial $\Phi_A(\xi,\omega)$, $\xi\in\mathbb{R}^3\setminus\{0\}$, does not vanish anywhere.

It follows from the above conditions (i) and (ii) that the real zeros $\xi\in\mathbb{R}^3\setminus\{0\}$ of the polynomial $\Phi_A(\xi,\omega)$ form non-self-intersecting, closed, convex two-dimensional surfaces $S_{\omega,1},S_{\omega,2},S_{\omega,3}$, enclosing the origin. For an arbitrary unit vector $\eta=x/|x|$ with $x\in\mathbb{R}^3\setminus\{0\}$, there exists only one point on each $S_{\omega,j}$, namely, $\xi^j=(\xi_1^j,\xi_2^j,\xi_3^j)\in S_{\omega,j}$ such that the outward unit normal vector $n(\xi^j)$ to $S_{\omega,j}$ at the point ξ^j has the same direction as η, i.e., $n(\xi^j)=\eta$. In this case, we say that the points ξ^j, $j=1,2,3$, correspond to the vector η.

From (i), we see that the surfaces $S_{\omega,j}$ $j=1,2,3$, might have multiplicities.

We say that a vector-function $U=(u_1,u_2,u_3,u_4,u_5)^\top$ belongs to the class $M_{m_1,m_2,m_3}(\mathbf{P})$ if $U\in[C^\infty(\Omega^-)]^5$ and the relation

$$U(x)=\sum_{p=1}^{5}u^p(x)$$

holds, where u^p has the following uniform asymptotic expansion as $r=|x|\to\infty$:

$$u^p\sim\sum_{j=1}^{3}e^{-ir\xi^j}\Big\{d^p_{0,m_j}(\eta)r^{m_j-2}+\sum_{q=1}^{\infty}d^p_{q,m_j}(\eta)r^{m_j-2-q}\Big\},\ \ p=1,2,3,\tag{3.10}$$

$$u^4(x)=O(r^{-1}),\quad \partial_ku^4(x)=O(r^{-2}),\quad u^5(x)=O(r^{-1}),\quad \partial_ku^5(x)=O(r^{-2}),\ \ k=1,2,3,$$

here $\mathbf{P}=\det A(i\partial_x,\omega)$ and $d^p_{q,m_j}\in C^\infty$, $j=1,2,3$ (see [10]).

These conditions are generalization of Sommerfeld–Kupradze type radiation conditions in the anisotropic elasticity (cf. [28, 33]).

From condition (i) it follows that our class $M_{m_1,m_2,m_3}(\mathbf{P})$ is $M_{1,1,1}(\mathbf{P})$ (when there is no multiplicity, i.e., surfaces do not coincide) or $M_{2,1}(\mathbf{P})$ (when two surfaces coincide) or $M_3(\mathbf{P})$ (when all three surfaces coincide).

The class $M_{1,1,1}(\mathbf{P})$ is a subset of the generalized Sommerfeld–Kupradze class.

We can show the following uniqueness theorems.

Theorem 3.4. *The homogeneous exterior Dirichlet boundary value problem*

$$A(\partial,\omega)U = 0 \ \text{ in } \Omega^-, \quad \{U\}^- = 0 \ \text{ on } S,$$

has only the trivial solution in the class $[H^1_{loc}(\Omega^-)]^5 \cap M_{m_1,m_2,m_3}(\mathbf{P})$.

Theorem 3.5. *The homogeneous exterior Dirichlet boundary value problem*

$$A^*(\partial,\omega)V = 0 \ \text{ in } \Omega^-, \quad \{V\}^- = 0 \ \text{ on } S,$$

has only the trivial solution in the class $[H^1_{loc}(\Omega^-)]^5 \cap M_{m_1,m_2,m_3}(\mathbf{P}^*)$, *where* $\mathbf{P}^* = \det A^*(\partial,\omega)$.

If surfaces $S_{\omega,j}$ $j = 1, 2, 3$, have no multiplicity, Theorems 3.4 and 3.5 are valid in generalized the Sommerfeld–Kupradze class (cf. [28]).

Denote by $\Gamma(x,\omega)$ the fundamental matrix of the operator $A(\partial,\omega)$. By means of the Fourier transform method and the limiting absorption principle, we can construct this matrix explicitly (see Ch. 1, Section 1, also see [42])

$$\Gamma(x,\omega) = \lim_{\varepsilon\to 0+} F^{-1}_{\xi\to x}\big[A^{-1}(i\xi,\omega+i\varepsilon)\big], \tag{3.11}$$

where F^{-1} is the inverse Fourier transform. The columns of the matrix $\Gamma(x,\omega)$ are infinitely differentiable in $\mathbb{R}^3 \setminus \{0\}$ and belong to the class $M_{m_1,m_2,m_3}(\mathbf{P})$.

Further, we introduce the single and double layer potentials associated with the differential operator $A(\partial,\omega)$,

$$\mathbf{V}_\omega(g)(x) = \int_S \Gamma(x-y,\omega)g(y)\,d_yS, \quad x \in \Omega^\pm,$$

$$\mathbf{W}_\omega(f)(x) = \int_S \big[\widetilde{T}(\partial_y,n(y))\Gamma^\top(x-y,\omega)\big]^\top f(y)\,d_yS, \quad x \in \Omega^\pm,$$

where $g = (g_1,\dots,g_4)^\top$ and $f = (f_1,\dots,f_4)^\top$ are density vector-functions.

For a solution $U \in [H^1(\Omega^+)]^5$ to the homogeneous equation (1.9) in Ω^+ we have the integral representation

$$U = \mathbf{W}_\omega(\{U\}^+) - \mathbf{V}_\omega(\{TU\}^+) \ \text{ in } \Omega^+.$$

For these potentials the following theorem holds (see [6, 7]).

Theorem 3.6. *Let* $g \in [H^{-1+s}(S)]^4$ *and* $f \in [H^s(S)]^4$, $s > 0$. *Then*

$$\{\mathbf{V}_\omega(g)(z)\}^\pm = \mathbf{H}_\omega(g)(z), \quad z \in S,$$

$$\{\mathbf{W}_\omega(f)(z)\}^\pm = \pm 2^{-1}f(z) + \widetilde{\mathbf{K}}_\omega(f)(z), \quad z \in S,$$

$$\big\{T(\partial_y,n(y))\mathbf{V}_\omega(g)(z)\big\}^\pm = \mp 2^{-1}g(z) + \mathbf{K}_\omega(g)(z), \quad z \in S,$$

$$\big\{T(\partial_z,n(z))\mathbf{W}_\omega(f)(z)\big\}^+ = \big\{T(\partial_z,n(z))\mathbf{W}_\omega(f)(z)\big\}^- := \boldsymbol{L}_\omega(f)(z), \quad z \in S,$$

where $\mathbf{H}_\omega$ *is a weakly singular integral operator,* $\widetilde{\mathbf{K}}_\omega$ *and* $\mathbf{K}_\omega$ *are singular integral operators, while* $\boldsymbol{L}_\omega$ *is a pseudodifferential operator of order* 1,

$$\mathbf{H}_\omega(g)(z) := \int_S \Gamma(z-y,\omega)g(y)\,d_yS, \quad z \in S,$$

$$\widetilde{\mathbf{K}}_\omega(f)(z) := \int_S \big[\widetilde{T}(\partial_y,n(y))\Gamma^\top(z-y,\omega)\big]^\top f(y)\,d_yS, \quad z \in S,$$

$$\mathbf{K}_\omega(g)(z) := \int_S T(\partial_z,n(z))\Gamma(z-y,\omega)g(y)\,d_yS, \quad z \in S.$$

The mapping properties of these potentials and boundary integral operators are described in Appendix.

4 The Dirichlet and Neumann type interaction problems for pseudo-oscillation equations

In this section, we consider the Dirichlet and Neumann type interaction problems for the so-called pseudo-oscillation equations. These problems are intermediate auxiliary problems for investigation of interaction problems for the steady state oscillation equations.

4.1 Formulation of the problems

The matrix differential operator corresponding to the basic pseudo-oscillation equations of the electro-magneto-elasticity for anisotropic homogeneous media is written as follows:

$$\begin{gathered}A(\partial,\tau)=[A_{jk}(\partial,\tau)]_{5\times 5},\\ A_{jk}(\partial,\tau)=c_{ijkl}\partial_i\partial_l+\rho_1\tau^2\delta_{jk},\quad A_{j4}(\partial,\tau)=e_{lij}\partial_l\partial_i,\quad A_{j5}(\partial,\tau)=q_{lij}\partial_l\partial_i,\\ A_{4k}(\partial,\tau)=-e_{ikl}\partial_i\partial_l,\quad A_{44}(\partial,\tau)=\varepsilon_{il}\partial_i\partial_l,\quad A_{45}(\partial,\tau)=a_{il}\partial_i\partial_l,\\ A_{5k}(\partial,\tau)=-q_{ikl}\partial_i\partial_l,\quad A_{54}(\partial,\tau)=a_{il}\partial_i\partial_l,\quad A_{55}(\partial,\tau)=\mu_{il}\partial_i\partial_l,\end{gathered}$$

$j,k=1,2,3$, where τ is a purely imaginary complex parameter: $\tau=i\sigma$, $\sigma\neq 0$, $\sigma\in\mathbb{R}$.

Dirichlet type problem (D_τ)**:** Find a vector-function $U=(u,u_4,u_5)^\top\in[H^1(\Omega^+)]^5$ and a scalar function $\mathrm{w}\in H^1_{loc}(\Omega^-)\cap Som_1(\Omega^-)$ satisfying the differential equations

$$A(\partial,\tau)U=0\ \text{ in } \Omega^+, \tag{4.1}$$

$$\Delta\mathrm{w}+\rho_2\omega^2\mathrm{w}=0\ \text{ in } \Omega^-, \tag{4.2}$$

the transmission conditions

$$\{u\cdot n\}^+=b_1\{\partial_n\mathrm{w}\}^-+f_0\ \text{ on } S, \tag{4.3}$$

$$\{[TU]_j\}^+=b_2\{\mathrm{w}\}^-n_j+f_j\ \text{ on } S,\ \ j=1,2,3, \tag{4.4}$$

and the Dirichlet boundary conditions

$$\{u_4\}^+=f_1^{(D)}\ \text{ on } S, \tag{4.5}$$

$$\{u_5\}^+=f_2^{(D)}\ \text{ on } S, \tag{4.6}$$

where b_1 and b_2 are the given complex constants satisfying conditions (1.15), $f_0\in H^{-1/2}(S)$, $f_j\in H^{-1/2}(S)$, $j=1,2,3$, $f_1^{(D)}\in H^{1/2}(S)$, $f_2^{(D)}\in H^{1/2}(S)$.

Neumann type problem (N_τ)**:** Find a vector-function $U=(u,u_4,u_5)^\top\in[H^1(\Omega^+)]^5$ and a scalar function $\mathrm{w}\in H^1_{loc}(\Omega^-)\cap Som_1(\Omega^-)$ satisfying the differential equations (4.1) and (4.2), respectively, transmission conditions (4.3), (4.4), and the Neumann boundary conditions

$$\{[TU]_4\}^+=f_1^{(N)}\ \text{ on } S \text{ with } f_1^{(N)}\in H^{-1/2}(S), \tag{4.7}$$

$$\{[TU]_5\}^+=f_2^{(N)}\ \text{ on } S \text{ with } f_2^{(N)}\in H^{-1/2}(S). \tag{4.8}$$

4.2 Uniqueness theorems for problems (D_τ) and (N_τ)

Theorem 4.1. *Let $\tau=i\sigma$, $\sigma\neq 0$, $\sigma\in\mathbb{R}$. The homogeneous problem (D_τ) has only the trivial solution, while the general solution of the homogeneous problem (N_τ) is the vector $(0,0,0,c_1,c_2)$, where c_1 and c_2 are an arbitrary complex scalar constants.*

Proof. Let (U,w) be a solution of the homogeneous problem (D_τ).

Let us write Green's formula for the Helmholtz equation (4.2) in the domain $\Omega_R:=\Omega^-\cap B(0,R)$, where $\overline{\Omega^+}\subset B(0,R)$,

$$\int\limits_{\Omega_R} \big[(\Delta+\rho_2\omega^2)\mathrm{w}\overline{\mathrm{w}} - \mathrm{w}(\Delta+\rho_2\omega^2)\overline{\mathrm{w}}\big]\,dx = \int\limits_{S(0,R)} \partial_n \mathrm{w}\overline{\mathrm{w}}\,dS - \int\limits_{S(0,R)} \partial_n \overline{\mathrm{w}}\mathrm{w}\,dS - \big\langle\{\partial_n\mathrm{w}\}^-,\{\mathrm{w}\}^-\big\rangle_S + \big\langle\{\partial_n\overline{\mathrm{w}}\}^-,\{\overline{\mathrm{w}}\}^-\big\rangle_S. \quad (4.9)$$

Now write Green's formula for the operator $A(\partial,\tau)$ in the domain Ω^+,

$$\int\limits_{\Omega^+} \Big[[A(\partial,\tau)U]_j\overline{u}_j + [\overline{A(\partial,\tau)U}]_4 u_4 + [\overline{A(\partial,\tau)U}]_5 u_5 + \mathcal{E}(U,\overline{U})\Big]\,dx = \big\langle\{TU\}_j^+,\{u_j\}^+\big\rangle_S + \big\langle\{\overline{TU}\}_4^+,\{\overline{u}_4\}^+\big\rangle_S + \big\langle\{\overline{TU}\}_5^+,\{\overline{u}_5\}^+\big\rangle_S, \quad (4.10)$$

where $\mathcal{E}(U,\overline{U}) = c_{ijlk}\partial_i u_j\partial_l\overline{u}_k + \rho_1\sigma^2|u|^2 + \varepsilon_{il}\partial_i u_4\partial_l\overline{u}_4 + \mu_{jl}\partial_j u_5\partial_l\overline{u}_5$. Using (4.1), (4.2), and (4.5), from (4.9) and (4.10) we obtain the following equalities:

$$\int\limits_{S(0,R)} \partial_n \mathrm{w}\overline{\mathrm{w}}\,dS - \int\limits_{S(0,R)} \partial_n \overline{\mathrm{w}}\mathrm{w}\,dS - \big\langle\{\partial_n\mathrm{w}\}^-,\{\mathrm{w}\}^-\big\rangle_S + \big\langle\{\partial_n\overline{\mathrm{w}}\}^-,\{\overline{\mathrm{w}}\}^-\big\rangle_S = 0, \quad (4.11)$$

$$\operatorname{Im}\big\langle\{[TU]_j\}^+,\{u_j\}^+\big\rangle_S = 0, \quad j=1,2,3. \quad (4.12)$$

In view of the homogeneous transmission conditions, we get

$$\big\langle\{[TU]_j\}^+,\{u_j\}^+\big\rangle_S = \big\langle b_2\{\mathrm{w}\}^- n_j,\{u_j\}^+\big\rangle_S = b_2\overline{b}_1\big\langle\{\partial_n\overline{\mathrm{w}}\}^-,\{\overline{\mathrm{w}}\}^-\big\rangle_S. \quad (4.13)$$

Since $\operatorname{Im}[\overline{b}_1 b_2]=0$, from (4.12) and (4.13) we get

$$\operatorname{Im}\big\langle\{\partial_n\overline{\mathrm{w}}\}^-,\{\overline{\mathrm{w}}\}^-\big\rangle_S = 0,$$

and from (4.11) we derive that

$$\operatorname{Im}\int\limits_{S(0,R)} \partial_n\overline{\mathrm{w}}\mathrm{w}\,dS = 0. \quad (4.14)$$

By the Sommerfeld radiation condition, from (4.14) we conclude that

$$\lim_{R\to\infty}\int\limits_{S(0,R)} |\mathrm{w}|^2\,dS = 0.$$

Using the Rellich–Vekua lemma, we find that $\mathrm{w}=0$ in the domain Ω^-.

Then from Green's formula (4.10) it follows that

$$\int\limits_{\Omega^+} \mathcal{E}(U,\overline{U})\,dx = 0. \quad (4.15)$$

Using (1.2) and (1.3), it is easy to see that for a complex vector $u=(u_1,u_2,u_3)^\top$ and a complex functions u_4, u_5,

$$c_{ijlk}\partial_i u_j\partial_l\overline{u}_k \ge 0, \quad \varepsilon_{jl}\partial_l u_4\partial_j\overline{u}_4 \ge 0, \quad \mu_{jl}\partial_l u_5\partial_j\overline{u}_5 \ge 0. \quad (4.16)$$

Taking into account (4.16), from (4.15) we obtain

$$\int\limits_{\Omega^+} \Big[c_{ijlk}\partial_i u_j\partial_l\overline{u}_k + \rho_1\sigma^2|u|^2 + \varepsilon_{jl}\partial_l u_4\partial_j\overline{u}_4 + \mu_{jl}\partial_l u_5\partial_j\overline{u}_5\Big]\,dx = 0, \quad (4.17)$$

implying that $u=0$ in Ω^+ and $u_4=c_1$, $u_5=c_2$ in Ω^+, where c_1, c_2 are arbitrary constants. Since $\{u_4\}^+=\{u_5\}^+=0$ on S, we deduce that $u_4=u_5=0$ in the domain Ω^+.

Applying the same arguments, we can show that the general solution of the homogeneous problem (N_τ) is a vector $(0,0,0,c_1,c_2)^\top$, where c_1 and c_2 are arbitrary complex scalar constants. □

4.3 Fundamental solution and potentials for the pseudo-oscillation equations of piezoelectro-magneto-elasticity

The full symbol of the pseudo-oscillation operator $A(\partial,\tau)$ is elliptic provided $\tau=i\sigma$, $\sigma\neq 0$, $\sigma\in\mathbb{R}$, i.e.,

$$\det A(-i\xi,\tau)\neq 0 \ \ \forall\xi\in\mathbb{R}^3\setminus\{0\}.$$

Moreover, the entries of the inverse matrix $A^{-1}(-i\xi,\tau)$ are locally integrable functions decaying at infinity as $O(|\xi|^{-2})$. Therefore, we can construct the fundamental matrix $\Gamma(x,\tau)=[\Gamma_{kj}(x,\tau)]_{5\times5}$ of the operator $A(\partial,\tau)$ by the Fourier transform technique,

$$\Gamma(x,\tau)=F^{-1}_{\xi\to x}[A^{-1}(-i\xi,\tau)]. \tag{4.18}$$

Note that in a neighbourhood of the origin the following estimates hold ($0<|x|<1$):

$$\big|\Gamma_{jk}(x,\tau)-\Gamma_{jk}(x,\omega)\big|\leq c(\tau,\omega), \tag{4.19}$$
$$\big|\partial_l\big[\Gamma_{jk}(x,\tau)-\Gamma_{jk}(x,\omega)\big]\big|\leq c(\tau,\omega)\ln|x|^{-1}, \tag{4.20}$$
$$\big|\partial^\alpha\big[\Gamma_{jk}(x,\tau)-\Gamma_{jk}(x,\omega)\big]\big|\leq c(\tau,\omega)|x|^{1-|\alpha|}, \ \ j,k=\overline{1,5}, \tag{4.21}$$

where $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ is a multi-index with $|\alpha|=\alpha_1+\alpha_2+\alpha_3\geq 2$, while $c(\tau,\omega)$ is a positive constant depending on $\tau=i\sigma$ and ω with $\sigma,\omega\in\mathbb{R}\setminus\{0\}$ (cf. [33]).

Let us introduce the single and double layer pseudo-oscillation potentials

$$\mathbf{V}_\tau(h)=\int\limits_S\Gamma(x-y,\tau)h(y)\,d_yS,$$
$$\mathbf{W}_\tau(h)=\int\limits_S\big[\widetilde{T}(\partial_y,n(y))\Gamma^\top(x-y,\tau)\big]^\top h(y)\,d_yS,$$

where $h=(h_1,h_2,h_3,h_4,h_5)^\top$ is a density vector-function.

These pseudo-oscillation potentials have the following jump properties (see [6]).

Theorem 4.2. *Let $h^{(1)}\in[H^{-1+s}(S)]^5$, $h^{(2)}\in[H^s(S)]^5$, $s>0$. Then the following jump relations hold on S:*

$$\big\{\mathbf{V}_\tau(h^{(1)})(z)\big\}^\pm=\int\limits_S\Gamma(z-y,\tau)h^{(1)}(y)\,d_yS,$$
$$\big\{\mathbf{W}_\tau(h^{(2)})(z)\big\}^\pm=\pm2^{-1}h^{(2)}(z)+\int\limits_S\big[\widetilde{T}(\partial_y,n(y))\Gamma^\top(z-y,\tau)\big]^\top h^{(2)}(y)\,d_yS,$$
$$\big\{T\mathbf{V}_\tau(h^{(1)})(z)\big\}^\pm=\mp2^{-1}h^{(1)}(z)+\int\limits_S T(\partial_z,n(z))\Gamma(z-y,\tau)h^{(1)}(y)\,d_yS,$$
$$\big\{T\mathbf{W}_\tau(h^{(2)})(z)\big\}^+=\big\{T\mathbf{W}_\tau(h^{(2)})(z)\big\}^-.$$

Further, we introduce the boundary operators

$$\mathbf{H}_\tau(h)(z)=\int\limits_S\Gamma(z-y,\tau)h(y)\,d_yS,$$
$$\mathbf{K}_\tau(h)(z)=\int\limits_S T(\partial_z,n(z))\Gamma(z-y,\tau)h(y)\,d_yS,$$
$$\widetilde{\mathbf{K}}_\tau(h)(z)=\int\limits_S\big[\widetilde{T}(\partial_y,n(y))\Gamma^\top(z-y,\tau)\big]^\top h(y)\,d_yS,$$
$$\mathbf{L}_\tau(h)(z)=\big\{T\mathbf{W}_\tau(h)(z)\big\}^+=\big\{T\mathbf{W}_\tau(h)(z)\big\}^-.$$

Note that $\mathbf{H}_\tau$ is a weakly singular integral operator (pseudodifferential operator of order -1), $\mathbf{K}_\tau$ and $\widetilde{\mathbf{K}}_\tau$ are singular integral operators (pseudodifferential operator of order 0), and $\mathbf{L}_\tau$ is a pseudodifferential operator of order 1.

The mapping properties of these potentials are described in Appendix.

4.4 Existence of solutions of problem (D_τ)

By Theorem 6.4 (see Appendix) the operator $\mathbf{H}_\tau : [H^s(S)]^5 \to [H^{s+1}(S)]^5$ is invertible for all $s \in \mathbb{R}$ and we can look for a solution of problem (D_τ) in the following form

$$U = \mathbf{V}_\tau \mathbf{H}_\tau^{-1} g \ \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h \ \text{ in } \Omega^-, \ \ \mu \in \mathbb{C}, \ \ \mathrm{Im}\,\mu \neq 0,$$

where $g = (\widetilde{g}, g_4, g_5)^\top \in [H^{1/2}(S)]^5$, $\widetilde{g} = (g_1, g_2, g_3)^\top$, $h \in H^{1/2}(S)$ are unknown densities. From Theorems 6.1, 6.3 and 6.4 (see Appendix) it follows that $U \in [H^1(\Omega^+)]^5$ and $\mathrm{w} \in H^1_{loc}(\Omega^-)$.

Transmission conditions (4.3), (4.4) and the Dirichlet type conditions (4.5), (4.6) lead to the following system of pseudodifferential equations with respect to the unknowns $\widetilde{g}$, g_4, g_5 and h:

$$\widetilde{g} \cdot n - b_1 \mathcal{M}(h) = f_0 \ \text{ on } S, \tag{4.22}$$

$$\big[(-2^{-1} I_5 + \mathbf{K}_\tau)\mathbf{H}_\tau^{-1} g\big]_j - b_2 n_j \mathcal{N}(h) = f_j \ \text{ on } S, \ \ j = 1,2,3, \tag{4.23}$$

$$g_4 = f_1^{(D)} \ \text{ on } S, \tag{4.24}$$

$$g_5 = f_2^{(D)} \ \text{ on } S, \tag{4.25}$$

where $\mathcal{N} = -2^{-1} I_1 + \mathcal{K}^*_\omega + \mu \mathcal{H}_\omega$, $\mathcal{M} = \mathcal{L}_\omega + \mu(2^{-1} I_1 + \mathcal{K}_\omega)$.

Here and in what follows, I_m stands for the $m \times m$ unit matrix.

The matrix operator generated by the left-hand side expressions in system (4.22)–(4.25) reads as

$$\mathcal{P}_{\tau,D} := \begin{pmatrix} [n]_{1\times 3} & 0 & 0 & -b_1\mathcal{M} \\ [\mathcal{A}_\tau^{jk}]_{3\times 3} & [\mathcal{A}_\tau^{j4}]_{3\times 1} & [\mathcal{A}_\tau^{j5}]_{3\times 1} & [-b_2 n_j \mathcal{N}]_{3\times 1} \\ [0]_{1\times 3} & I_1 & 0 & 0 \\ [0]_{1\times 3} & 0 & I_1 & 0 \end{pmatrix}_{6\times 6}, \quad j,k = 1,2,3,$$

where

$$\mathcal{A}_\tau := \big(-2^{-1} I_5 + \mathbf{K}_\tau\big)\mathbf{H}_\tau^{-1} = [\mathcal{A}_\tau^{jk}]_{5\times 5}, \ \ j,k = \overline{1,5}, \tag{4.26}$$

is the Steklov–Poincaré type operator on S. This operator is a strongly elliptic pseudodifferential operator of order 1 (see [6] for details).

By Theorems 6.2 and 6.4 (see Appendix), the operator $\mathcal{P}_{\tau,D}$ possesses the following mapping property:

$$\mathcal{P}_{\tau,D} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^5 \times H^{1/2}(S). \tag{4.27}$$

In view of (4.24) and (4.25), equations (4.22) and (4.23) can be rewritten in the following equivalent form as a system with respect to $\widetilde{g}$ and h:

$$\widetilde{g} \cdot n - b_1 \mathcal{M}(h) = f_0 \ \text{ on } S, \tag{4.28}$$

$$[\mathcal{A}_\tau(\widetilde{g}, 0, 0)^\top]_j - b_2 n_j \mathcal{N}(h) = F_j \ \text{ on } S, \ \ j = 1,2,3, \tag{4.29}$$

where $F_j := f_j - \mathcal{A}_\tau^{j4} f_1^{(D)} - \mathcal{A}_\tau^{j5} f_2^{(D)}$, $j = 1,2,3$.

Denote by $\mathcal{R}_{\tau,D}$ the operator corresponding to system (4.28), (4.29)

$$\mathcal{R}_{\tau,D} := \begin{pmatrix} [n]_{1\times 3} & -b_1\mathcal{M} \\ \widetilde{\mathcal{A}}_\tau & [-b_2 n_k \mathcal{N}]_{3\times 1} \end{pmatrix}_{4\times 4},$$

where $\widetilde{\mathcal{A}}_\tau := [\mathcal{A}_\tau^{jk}]_{3\times 3}$, $j,k = 1,2,3$.

Clearly, the operator

$$\mathcal{R}_{\tau,D} : [H^{1/2}(S)]^4 \to [H^{-1/2}(S)]^4 \tag{4.30}$$

is bounded.

Let us represent the operator $\mathcal{R}_{\tau,D}$ as the sum of two operators

$$\mathcal{R}_{\tau,D} = \mathcal{R}^{(1)}_{\tau,D} + \mathcal{R}^{(2)}_{\tau,D},$$

where

$$\mathcal{R}^{(1)}_{\tau,D} = \begin{pmatrix} [0]_{1\times 3} & -b_1\mathcal{M} \\ \widetilde{\mathcal{A}}_\tau & [0]_{3\times 1} \end{pmatrix}_{4\times 4}, \quad \mathcal{R}^{(2)}_{\tau,D} = \begin{pmatrix} [n]_{1\times 3} & 0 \\ [0]_{3\times 3} & [-b_2 n_k \mathcal{N}]_{3\times 1} \end{pmatrix}_{4\times 4}.$$

It is easy to see that the operator $\mathcal{N} : H^{1/2}(S) \to H^{-1/2}(S)$ is compact due to Theorem 3.2 and Rellich compact embedding theorem. Therefore, the operator $\mathcal{R}^{(2)}_{\tau,D} : [H^{1/2}(S)]^4 \to [H^{-1/2}(S)]^4$ is compact. Further, we show that the operator $\widetilde{\mathcal{A}}_\tau$ is Fredholm. Indeed,

$$\mathcal{A}_\tau : [H^{1/2}(S)]^5 \to [H^{-1/2}(S)]^5$$

is strongly elliptic pseudodifferential operator of order 1 (see [6]), i.e.,

$$\operatorname{Re}\mathfrak{S}(\mathcal{A}_\tau; x, \xi)\zeta \cdot \zeta \geq c|\xi|\,|\zeta|^2,$$

where c is a positive constant and $\mathfrak{S}(\mathcal{A}_\tau; x, \xi)$ with $x \in S$, $\xi \in \mathbb{R}^2 \setminus \{0\}$, is the principal homogeneous symbol of the operator $\mathcal{A}_\tau$ in some local coordinate system. Therefore, $\forall\,\xi \in \mathbb{R}^2 \setminus \{0\}$, $\forall\,\zeta' \in \mathbb{C}^3$ the following estimate holds:

$$\operatorname{Re}\mathfrak{S}(\widetilde{\mathcal{A}}_\tau; x, \xi)\zeta' \cdot \zeta' = \operatorname{Re}\mathfrak{S}(\mathcal{A}_\tau; x, \xi)(\zeta', 0)^\top \cdot (\zeta', 0)^\top \geq c|\xi||\zeta'|^2.$$

Thus $\widetilde{\mathcal{A}}_\tau$ is a strongly elliptic pseudodifferential operator of order 1. Therefore, by virtue of the general theory of elliptic pseudodifferential operators on a compact manifold without boundary (see [16, Ch. 19], [14, Ch. 5]), we conclude that

$$\widetilde{\mathcal{A}}_\tau : [H^{1/2}(S)]^3 \to [H^{-1/2}(S)]^3$$

is a Fredholm operator. From the strong ellipticity property it also follows that the index of the operator $\widetilde{\mathcal{A}}_\tau$ is zero (see [16, Ch. 6], [14, Ch. 2]). Taking into account Theorem 3.2, we find that the operator $\mathcal{R}^{(1)}_{\tau,D}$ is Fredholm with index zero. Therefore, operators (4.30) and, consequently, (4.27) are Fredholm with index zero.

Now we show that the operator $\mathcal{R}_{\tau,D}$ is injective. Let $(\widetilde{g}, h)^\top$ with $\widetilde{g} \in [H^{1/2}(S)]^3$ and $h \in H^{1/2}(S)$ be some solution of the homogeneous system

$$\mathcal{R}_{\tau,D}(\widetilde{g}, h)^\top = 0,$$

and set

$$\widetilde{U} = (\widetilde{u}, \widetilde{u}_4, \widetilde{u}_5)^\top = \mathbf{V}_\tau \mathbf{H}_\tau^{-1}(\widetilde{g}, 0, 0), \quad \widetilde{\mathrm{w}} = (W_\omega + \mu V_\omega)h, \quad \operatorname{Im}\mu \neq 0.$$

Evidently, $\widetilde{U}$ and $\widetilde{\mathrm{w}}$ solve the homogeneous problem (D_τ).

It follows from the uniqueness result for problem (D_τ) (see Theorem 4.1) that $\widetilde{U} = 0$ in Ω^+ and $\widetilde{\mathrm{w}} = 0$ in Ω^-. Then $\{\widetilde{U}\}^+ = (\widetilde{g}, 0, 0)^\top = 0$ on S. Since $\{\widetilde{\mathrm{w}}\}^- = \mathcal{N}(h) = 0$ and $\mathcal{N}$ is invertible operator, we obtain $h = 0$ on S. Consequently, the operators

$$\mathcal{R}_{\tau,D} : [H^{1/2}(S)]^4 \to [H^{-1/2}(S)]^4,$$
$$\mathcal{P}_{\tau,D} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^5 \times H^{1/2}(S)$$

are invertible.

Therefore, system (4.22)–(4.25) is uniquely solvable. Thus the following assertion holds.

Theorem 4.3. *Let $\tau = i\sigma$, $\sigma \neq 0$, $\sigma \in \mathbb{R}$, and let $f_0 \in H^{-1/2}(S)$, $f_j \in H^{-1/2}(S)$, $j = 1,2,3$, and $f^{(D)} \in H^{1/2}(S)$. Then problem (D_τ) has a unique solution (U, w), $U \in [H^1(\Omega^+)]^5$, $\mathrm{w} \in H^1_{loc}(\Omega^-) \cap Som_1(\Omega^-)$, which can be represented as*

$$U = \mathbf{V}_\tau \mathbf{H}_\tau^{-1} g \ \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h \ \text{ in } \Omega^-,$$

where the densities $g \in [H^{1/2}(S)]^5$ and $h \in H^{1/2}(S)$ are defined from the uniquely solvable system (4.22)–(4.25).

4.5 Existence of solutions of problem (N_τ)

As in the previous subsection, we can look for a solution of problem (N_τ) in the following form:

$$U = \mathbf{V}_\tau \mathbf{H}_\tau^{-1} g \ \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h \ \text{ in } \Omega^-, \ \ \mu \in \mathbb{C}, \ \ \mathrm{Im}\, \mu \neq 0,$$

where $g = (\widetilde{g}, g_4, g_5)^\top \in [H^{1/2}(S)]^5$ and $h \in H^{1/2}(S)$ are unknown densities. From Theorems 6.1, 6.3 and 6.4 of Appendix it follows that $U \in [H^1(\Omega^+)]^5$ and $\mathrm{w} \in H^1_{loc}(\Omega^-)$.

Transmission conditions (4.3), (4.4), and the Neumann type condition (4.7) lead to the following system of pseudodifferential equations with respect to the unknowns g and h:

$$\widetilde{g} \cdot n - b_1 \mathcal{M}(h) = f_0 \ \text{ on } S, \tag{4.31}$$
$$[\mathcal{A}_\tau g]_j - b_2 n_j \mathcal{N}(h) = f_j \ \text{ on } S, \ \ j = 1,2,3, \tag{4.32}$$
$$[\mathcal{A}_\tau g]_4 = f_1^{(N)} \ \text{ on } S, \tag{4.33}$$
$$[\mathcal{A}_\tau g]_5 = f_2^{(N)} \ \text{ on } S, \tag{4.34}$$

where $\mathcal{N}$ and $\mathcal{M}$ are defined in (3.1) and (3.2), while $\mathcal{A}_\tau$ is defined in (4.26).

The operator generated by the left-hand side of the system (4.31)–(4.33) reads as

$$\mathcal{P}_{\tau,N} := \begin{pmatrix} [(n,0,0)]_{1\times 5} & -b_1\mathcal{M} \\ [\mathcal{A}_\tau^{jk}]_{3\times 5} & [-b_2 n_j \mathcal{N}]_{3\times 1} \\ [\mathcal{A}_\tau^{4j}]_{1\times 4} & [0]_{1\times 2} \\ [\mathcal{A}_\tau^{5j}]_{1\times 4} & [0]_{1\times 2} \end{pmatrix}_{6\times 6}, \quad j = 1,2,3, \ \ k = \overline{1,5}.$$

The operator $\mathcal{P}_{\tau,N}$ possesses the following mapping property:

$$\mathcal{P}_{\tau,N} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^6.$$

From equation (4.31), we define h,

$$h = b_1^{-1}\mathcal{M}^{-1}(\widetilde{g} \cdot n) - b_1^{-1}\mathcal{M}^{-1} f_0,$$

and substitute this into equation (4.32). We obtain the system

$$[\mathcal{A}_\tau g]_j - b_2 b_1^{-1} n_j \mathcal{N}\mathcal{M}^{-1}(\widetilde{g} \cdot n) = F_j \ \text{ on } S, \ \ j = 1,2,3, \tag{4.35}$$
$$[\mathcal{A}_\tau g]_4 = f_1^{(N)} \ \text{ on } S, \tag{4.36}$$
$$[\mathcal{A}_\tau g]_5 = f_2^{(N)} \ \text{ on } S, \tag{4.37}$$

where $F_j = f_j - b_1^{-1} b_2 n_j \mathcal{N}\mathcal{M}^{-1} f_0$.

Denote by $\mathcal{R}_{\tau,N}$ the operator generated by the left-hand side of system (4.35)–(4.37),

$$\mathcal{R}_{\tau,N} = \begin{pmatrix} [C_\tau]_{3\times 3} & [\mathcal{A}_\tau^{j4}]_{3\times 1} & [\mathcal{A}_\tau^{j5}]_{3\times 1} \\ [\mathcal{A}_\tau^{4j}]_{1\times 3} & \mathcal{A}_\tau^{44} & \mathcal{A}_\tau^{45} \\ [\mathcal{A}_\tau^{5j}]_{1\times 3} & \mathcal{A}_\tau^{54} & \mathcal{A}_\tau^{55} \end{pmatrix}_{5\times 5},$$

where

$$[C_\tau]_{3\times3} = [\mathcal{A}_\tau^{jk}]_{3\times3} - b_2 b_1^{-1}[n_j\mathcal{N}]_{3\times1}[\mathcal{M}^{-1}n_k]_{1\times3}, \quad j,k = 1,2,3.$$

Note that the difference $\mathcal{A}_\tau - \mathcal{R}_{\tau,N} : [H^{1/2}(S)]^5 \to [H^{-1/2}(S)]^5$ is a compact operator.

Since the Steklov–Poincaré type operator $\mathcal{A}_\tau$ is strongly elliptic pseudodifferential operator of order 1, it follows that the operator $\mathcal{A}_\tau : [H^{1/2}(S)]^5 \to [H^{-1/2}(S)]^5$ is Fredholm with index zero. Hence the operators

$$\mathcal{R}_{\tau,N} : [H^{1/2}(S)]^5 \to [H^{-1/2}(S)]^5, \quad \mathcal{P}_{\tau,N} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^6$$

are Fredholm with index zero.

Now let us investigate the null space of the operator $\mathcal{P}_{\tau,N}$. Let $g \in [H^{1/2}(S)]^5$ and $h \in H^{1/2}(S)$ be solutions of the homogeneous system (4.31)–(4.33)

$$\mathcal{P}_{\tau,N}(g,h)^\top = 0,$$

and put

$$\widetilde{U} = (\widetilde{u}, \widetilde{u}_4, \widetilde{u}_5)^\top = \mathbf{V}_\tau \mathbf{H}_\tau^{-1} g, \quad \widetilde{\mathrm{w}} = (W_\omega + \mu V_\omega)h.$$

Evidently, $\widetilde{U}$ and $\widetilde{\mathrm{w}}$ solve the homogeneous problem (N_τ).

From the structure of a solution to the homogeneous problem (N_τ) (see Theorem 4.1) we have

$$\widetilde{U} = (0,0,0,c_1,c_2)^\top \ \text{ in } \Omega^+, \quad \widetilde{\mathrm{w}} = 0 \ \text{ in } \Omega^-,$$

where c_1 and c_2 are arbitrary constants. Then $\{\widetilde{U}\}^+ = (0,0,0,c_1,c_2)^\top = g$ on S, i.e. $g_1 = g_2 = g_3 = 0$, $g_4 = c_1$ and $g_5 = c_2$. Since $\{\mathrm{w}\}^- = \mathcal{N}h = 0$ on S, the invertibility of the operator $\mathcal{N}$ yields that $h = 0$ on S. Whence we obtain that if $\mathcal{P}_{\tau,N}(g,h)^\top = 0$, then $g = (0,0,0,c_1,c_2)^\top$ and $h = 0$.

Therefore, the dimension of the null space of the operator $\mathcal{P}_{\tau,N}$ equals to 2, $\dim\operatorname{Ker}\mathcal{P}_{\tau,N} = 2$. Thus $\dim\operatorname{Ker}\mathcal{P}^*_{\tau,N} = 2$, where $\mathcal{P}^*_{\tau,N} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^6$ is the operator adjoint to $\mathcal{P}_{\tau,N} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^6$.

Now we can formulate the following existence theorem.

Theorem 4.4. *Let $\tau = i\sigma$, $\sigma \neq 0$, $\sigma \in \mathbb{R}$, and let $f_0 \in H^{-1/2}(S)$, $f_j \in H^{-1/2}(S)$, $j = 1,2,3$, and $f_1^{(N)} \in H^{-1/2}(S)$, $f_2^{(N)} \in H^{-1/2}(S)$. Then problem (N_τ) is solvable if and only if the condition*

$$\langle f_0, \phi_1\rangle_S + \sum_{j=1}^{3}\langle f_j, \phi_{j+1}\rangle_S + \langle f_1^{(N)}, \phi_5\rangle_S + \langle f_2^{(N)}, \phi_6\rangle_S = 0 \tag{4.38}$$

is fulfilled, where $\phi = (\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6)^\top$ is a nontrivial solution of the homogeneous equation $\mathcal{P}^*_{\tau,N}\phi = 0$. If condition (4.38) holds, then solutions of problem (N_τ) are represented by the potentials

$$U = \mathbf{V}_\tau \mathbf{H}_\tau^{-1} g \ \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h \ \text{ in } \Omega^-,$$

where the densities $g \in [H^{1/2}(S)]^5$ and $h \in H^{1/2}(S)$ are defined from system (4.31)–(4.35), and they are defined modulo the addend vector $(0,0,0,c_1,c_2)^\top$ with arbitrary complex constants c_1 and c_2.

5 Existence results for the steady state oscillation problems (D_ω) and (N_ω)

5.1 Existence of solution of the Dirichlet type problem (D_ω)

We look for a solution of problem (D_ω) in the form

$$U = \mathbf{V}_\omega g \ \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h \ \text{ in } \Omega^-, \ \ \mu \in \mathbb{C}, \ \operatorname{Im}\mu \neq 0,$$

where $g \in [H^{-1/2}(S)]^5$ and $h \in H^{1/2}(S)$ are unknown densities, and $\omega \in \mathbb{R}\setminus\{0\}$. From Theorems 6.1 and 6.3 of Appendix it follows that $U \in [H^1(\Omega^+)]^5$ and $\mathrm{w} \in H^1_{loc}(\Omega^-)$.

Transmission conditions (1.11), (1.12) and the Dirichlet boundary conditions (1.13), (1.14) lead to the following system of pseudodifferential equations with respect to the unknowns g and h:

$$[\mathbf{H}_\omega g]_l n_l - b_1 \mathcal{M}(h) = f_0 \ \text{ on } S, \tag{5.1}$$

$$\big[(-2^{-1} I_4 + \mathbf{K}_\omega) g\big]_j - b_2 n_j \mathcal{N}(h) = f_j \ \text{ on } S, \ \ j = 1,2,3, \tag{5.2}$$

$$[\mathbf{H}_\omega g]_4 = f_1^{(D)} \ \text{ on } S, \tag{5.3}$$

$$[\mathbf{H}_\omega g]_5 = f_2^{(D)} \ \text{ on } S. \tag{5.4}$$

The operator generated by the left-hand side of system (5.1)–(5.4) reads as

$$Q_{\omega,D} = \begin{pmatrix} [n_l \mathbf{H}_\omega^{lk}]_{1\times 5} & -b_1 \mathcal{M} \\ \big[(-2^{-1} I_5 + \mathbf{K}_\omega)^{jk}\big]_{3\times 5} & [-b_2 n_j \mathcal{N}]_{3\times 1} \\ [\mathbf{H}_\omega^{4k}]_{1\times 5} & 0 \\ [\mathbf{H}_\omega^{5k}]_{1\times 5} & 0 \end{pmatrix}_{6\times 6}, \quad j = \overline{1,3}, \ \ k = \overline{1,5}.$$

By Theorem 6.5, the operator

$$Q_{\omega,D} : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{-1/2}(S)]^4 \times [H^{1/2}(S)]^2$$

is bounded.

In view of estimates (4.19)-(4.21) it follows that the main parts of the operators $\mathbf{H}_\omega$ and $\mathbf{H}_\tau$ (as well as the main parts of the operators $\mathbf{K}_\omega$ and $\mathbf{K}_\tau$) are the same, implying that the operators

$$\mathbf{H}_\omega - \mathbf{H}_\tau : [H^{-1/2}(S)]^5 \to [H^{1/2}(S)]^5, \tag{5.5}$$

$$\mathbf{K}_\omega - \mathbf{K}_\tau : [H^{-1/2}(S)]^5 \to [H^{-1/2}(S)]^5 \tag{5.6}$$

are compact. Hence the operator

$$Q_{\omega,D} - Q_{\tau,D} : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{-1/2}(S)]^4 \times [H^{1/2}(S)]^2$$

is compact, where $Q_{\tau,D} := \mathcal{P}_{\tau,D} \mathcal{T}_\tau$ with

$$\mathcal{T}_\tau := \begin{pmatrix} \mathbf{H}_\tau & [0]_{4\times 1} \\ [0]_{1\times 4} & I_1 \end{pmatrix}_{5\times 5}. \tag{5.7}$$

Therefore, from the invertibility of the operators $\mathcal{P}_{\tau,D} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^5 \times H^{1/2}(S)$ and $\mathcal{T}_\tau : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{1/2}(S)]^6$ (see Section 4) the invertibility of the operator $Q_{\tau,D} : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{-1/2}(S)]^5 \times H^{1/2}(S)$ follows. In turn, this implies that the operator

$$Q_{\omega,D} : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{-1/2}(S)]^4 \times [H^{1/2}(S)]^2 \tag{5.8}$$

is Fredholm with index zero.

Let us show that for $\omega \notin J_D(\Omega^+)$ the operator $Q_{\omega,D}$ is injective. Indeed, let $g \in [H^{-1/2}(S)]^5$ and $h \in H^{1/2}(S)$ be solutions of the homogeneous system

$$Q_{\omega,D}(g,h)^\top = 0 \ \text{ on } S.$$

Construct a vector-function $U = \mathbf{V}_\omega g$ and a scalar function $\mathrm{w} = (W_\omega + \mu V_\omega)h$ with $\mu \in \mathbb{C}$, $\operatorname{Im}\mu \neq 0$; Clearly, the pair (U, w) solves the homogeneous problem (D_ω). Since $\omega \notin J_D(\Omega^+)$, from Theorem 2.1 we have that

$$U = \mathbf{V}_\omega g = 0 \ \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h = 0 \ \text{ in } \Omega^-.$$

In view of the equation $\{\mathrm{w}\}^- = \mathcal{N}(h) = 0$ on S and the invertibility of the operator $\mathcal{N}$ we deduce that $h = 0$ on S. From continuity of a single layer potential we have $\{U\}^+ = \{U\}^- = 0$ on S.

Thus $U = \mathbf{V}_\omega g$ solves the exterior homogeneous Dirichlet problem

$$A(\partial,\omega)U = 0 \ \text{ on } \Omega^-, \quad \{U\}^- = 0 \ \text{ on } S. \tag{5.9}$$

$U = \mathbf{V}_\omega g \in M_{m_1,m_2,m_3}(\mathbf{P})$ and, by Theorem 3.4, $U = \mathbf{V}_\omega g \equiv 0$ in Ω^-. Using the jump formula $\{TU\}^- - \{TU\}^+ = g$ on S, we get $g = 0$ on S. Thus the null space of the Fredholm operator (5.8) is trivial and since the index equals to zero we conclude that (5.8) is invertible.

These results imply the following assertion.

Theorem 5.1. *If $\omega \notin J_D(\Omega^+)$, then problem (D_ω) is uniquely solvable.*

Now let us consider the case where ω is Jones's frequency, $\omega \in J_D(\Omega^+)$.

The operator adjoint to $Q_{\omega,D}$ has the following form:

$$Q^*_{\omega,D} = \begin{pmatrix} [\mathbf{H^*}^{kl}_\omega n_l]_{5\times1} & \big[(-2^{-1}I_4 + \mathbf{K}^*_\omega)^{kj}\big]_{5\times3} & [\mathbf{H^*}^{k4}_\omega]_{5\times1} & [\mathbf{H^*}^{k5}_\omega]_{5\times1} \\ -\overline{b}_1\mathcal{M}^* & [-\overline{b}_2\mathcal{N}^* n_j]_{1\times3} & 0 & 0 \end{pmatrix}_{6\times6}, \quad j=\overline{1,3}, \quad k=\overline{1,5},$$

where

$$\mathbf{H}^*_\omega(g)(z) = \int\limits_S [\overline{\Gamma(y-z,\omega)}]^\top g(y)\,d_yS, \quad z\in S,$$
$$\mathbf{K}^*_\omega(g)(z) = \int\limits_S \big[T(\partial_y,n(y)\overline{\Gamma(y-z,\omega)})\big]^\top g(y)d_yS, \quad z\in S,$$
$$\mathcal{N}^*(h)(z) = \big(-2^{-1}I_1 + \overline{\mathcal{K}_\omega}\big)(h)(z) + \overline{\mu}\mathcal{H}^*_\omega(h)(z), \quad z\in S,$$
$$\mathcal{M}^*(h)(z) = \mathcal{L}^*_\omega(h)(z) + \overline{\mu}\big(2^{-1}I_1 + \overline{\mathcal{K}}^*_\omega\big)(h)(z), \quad z\in S,$$

while

$$\overline{\mathcal{K}}_\omega(h)(z) = \int\limits_S \partial_{n(z)}\overline{\gamma(z-y,\omega)}h(y)\,d_yS, \quad z\in S,$$
$$\overline{\mathcal{K}}^*_\omega(h)(z) = \int\limits_S \partial_{n(y)}\overline{\gamma(z-y,\omega)}h(y)\,d_yS, \quad z\in S,$$
$$\mathcal{H}^*_\omega(h)(z) = \int\limits_S \overline{\gamma(z-y,\omega)}h(y)\,d_yS, \quad z\in S,$$
$$\mathcal{L}^*_\omega(h)(z) = \big\{\partial_{n(z)}\widetilde{W}_\omega(h)(z)\big\}^\pm, \quad z\in S,$$
$$\widetilde{W}_\omega(h)(x) = \int\limits_S \partial_{n(y)}\overline{\gamma(x-y,\omega)}h(y)\,d_yS, \quad x\notin S,$$
$$\widetilde{V}_\omega(h)(x) = \int\limits_S \overline{\gamma(x-y,\omega)}h(y)\,d_yS, \quad x\notin S.$$

The adjoint operator possesses the following mapping property:

$$Q^*_{\omega,D} : [H^{1/2}(S)]^4 \times [H^{-1/2}(S)]^2 \to [H^{1/2}(S)]^5 \times H^{-1/2}(S).$$

Let $\Psi := (\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6)^\top \in [H^{1/2}(S)]^4 \times [H^{-1/2}(S)]^2$ be a solution of the homogeneous adjoint system

$$Q^*_{\omega,D}\Psi = 0. \tag{5.10}$$

Construct the potentials

$$\widetilde{U} = \widetilde{\mathbf{V}}_\omega\Psi^{(1)} + \widetilde{\mathbf{W}}_\omega\Psi^{(2)} + \widetilde{\mathbf{V}}_\omega\Psi^{(3)} \ \text{ in } \Omega^-, \tag{5.11}$$
$$\widetilde{\mathrm{w}} = -\overline{b}_1\widetilde{W}_\omega\psi_1 - \overline{b}_2\widetilde{V}_\omega[\Psi'\cdot n] \ \text{ in } \Omega^+, \tag{5.12}$$

where

$$\Psi^{(1)} := (n\psi_1, 0)^\top, \quad \Psi^{(2)} := (\Psi', 0)^\top, \quad \Psi^{(3)} := (0,0,0,\psi_5,\psi_6)^\top, \quad \Psi' = (\psi_2,\psi_3,\psi_4)^\top,$$

$$\widetilde{\mathbf{V}}_\omega(g)(x) := \int\limits_S [\overline{\Gamma(y-x,\omega)}]^\top g(y)\, d_yS, \ \ x \in \Omega^+,$$

$$\widetilde{\mathbf{W}}_\omega(g)(x) := \int\limits_S \big[T(\partial_y, n(y))\overline{\Gamma(y-x,\omega)}\,\big]^\top g(y)\, d_yS, \ \ x \in \Omega^+.$$

The vectors $\widetilde{\mathbf{V}}_\omega(g)$ and $\widetilde{\mathbf{W}}_\omega(g)$ are the single and double layer potentials associated with the operator $A^*(\partial,\omega)$.

From (5.10) it follows that

$$\{\widetilde{U}\}^- = 0 \ \text{ and } \ \{\partial_n \widetilde{\mathrm{w}} + \overline{\mu}\widetilde{\mathrm{w}}\}^+ = 0 \ \text{ on } S,$$

where $\mu = \mu_1 + i\mu_2$, $\mu_2 \neq 0$.

Since the vector $\widetilde{U} \in [H^1_{loc}(\Omega^-)]^5 \cap M_{m_1,m_2,m_3}(\mathbf{P}^*)$ and solves the homogeneous Dirichlet problem

$$A^*(\partial,\omega)\widetilde{U} = 0 \ \text{ in } \Omega^-, \quad \{\widetilde{U}\}^- = 0 \ \text{ on } S,$$

the uniqueness Theorem 3.5 implies that $\widetilde{U} = 0$ in Ω^-.

On the other hand, the function $\widetilde{\mathrm{w}} \in H^1(\Omega^+)$ solves the homogeneous Robin type problem

$$(\Delta + \rho_2\omega^2)\widetilde{\mathrm{w}} = 0 \ \text{ in } \Omega^+, \tag{5.13}$$

$$\{\partial_n \widetilde{\mathrm{w}} + \overline{\mu}\widetilde{\mathrm{w}}\}^+ = 0 \ \text{ on } S. \tag{5.14}$$

This problem possesses only the trivial solution. Indeed, the following Green's first formula holds:

$$\int\limits_{\Omega^+} (\Delta + \rho_2\omega^2)\widetilde{\mathrm{w}}\overline{\widetilde{\mathrm{w}}}\, dx + \int\limits_{\Omega^+} |\nabla \widetilde{\mathrm{w}}|\, dx - \rho_2\omega^2 \int\limits_{\Omega^+} |\widetilde{\mathrm{w}}|\, dx = \big\langle \{\partial_n \widetilde{\mathrm{w}}\}^+, \{\widetilde{\mathrm{w}}\}^+ \big\rangle_S, \tag{5.15}$$

Taking into account equation (5.13) and the boundary condition (5.14), from (5.15) we get

$$\int\limits_{\Omega^+} |\nabla \widetilde{\mathrm{w}}|\, dx - \rho_2\omega^2 \int\limits_{\Omega^+} |\widetilde{\mathrm{w}}|\, dx = -\mu_1 \int\limits_S \big|\{\widetilde{\mathrm{w}}\}^+\big|^2 dS + i\mu_2 \int\limits_S \big|\{\widetilde{\mathrm{w}}\}^+\big|^2 dS.$$

Therefore, $\{\widetilde{\mathrm{w}}\}^+ = 0$. For a solution $\widetilde{\mathrm{w}} \in H^1(\Omega^+)$ to the homogeneous equation (5.13) we have the following integral representation:

$$\widetilde{\mathrm{w}} = W_\omega\big(\{\widetilde{\mathrm{w}}\}^+\big) - V_\omega\big(\{\partial_n \widetilde{\mathrm{w}}\}^+\big) \ \text{ in } \Omega^+. \tag{5.16}$$

Since $\{\widetilde{\mathrm{w}}\}^+ = 0$ and $\{\partial_n \widetilde{\mathrm{w}}\}^+ = 0$, from the representation formula (5.16) we find that $\widetilde{\mathrm{w}} = 0$ in Ω^+.

Using the jump formulae for potentials (5.11) and (5.12), we derive that on the surface S the following relations hold:

$$\begin{aligned} \{\widetilde{\mathrm{w}}\}^- &= \overline{b}_1\psi_1, \\ \{\partial_n \widetilde{\mathrm{w}}\}^- &= -\overline{b}_2\, \Psi' \cdot n, \\ \{[\widetilde{T}\widetilde{U}]_j\}^+ &= -n_j\psi_1, \ \ j = 1,2,3, \\ \{[\widetilde{T}\widetilde{U}]_4\}^+ &= -\psi_5, \\ \{[\widetilde{T}\widetilde{U}]_5\}^+ &= -\psi_6, \\ \{\widetilde{U}\}^+ &= (\Psi', 0)^\top, \\ \{\widetilde{U}_4\}^+ &= 0, \\ \{\widetilde{U}_5\}^+ &= 0. \end{aligned}$$

Hence we deduce that $\widetilde{U}=(\widetilde{U}_1,\widetilde{U}_2,\widetilde{U}_3,\widetilde{U}_4,\widetilde{U}_5)^\top=(\widetilde{U}',\widetilde{U}_4,\widetilde{U}_5)^\top$ with $\widetilde{U}'=(\widetilde{U}_1,\widetilde{U}_2,\widetilde{U}_3,)^\top$ and $\widetilde{\mathrm{w}}$ solve the following homogeneous transmission problem:

$$\begin{aligned}
A^*(\partial,\omega)\widetilde{U}&=0 \ \text{ in } \Omega^+,\\
(\Delta+\rho_2\omega^2)\widetilde{\mathrm{w}}&=0 \ \text{ in } \Omega^-,\\
\{\widetilde{U}'\cdot n\}^+ + \overline{b}_2^{\,-1}\{\partial_n\widetilde{\mathrm{w}}\}^-&=0 \ \text{ on } S,\\
\big\{[\widetilde{T}(\partial,n)\widetilde{U}]_j\big\}^+ + \overline{b}_1^{\,-1}\{\widetilde{\mathrm{w}}\}^- n_j&=0 \ \text{ on } S, \quad j=1,2,3,\\
\{\widetilde{U}_4\}^+&=0 \ \text{ on } S,\\
\{\widetilde{U}_5\}^+&=0 \ \text{ on } S,
\end{aligned}$$

From the uniqueness result (see Remark 2.3) it follows that $\widetilde{\mathrm{w}}=0$ in Ω^- and $\widetilde{U}\in X^*_{D,\omega}(\Omega^+)$, i.e., $\widetilde{U}$ belongs to the space of Jones modes $X^*_{D,\omega}(\Omega^+)$. Then we obtain

$$\psi_1=0,\quad \psi_{j+1}=\{\widetilde{U}_j\}^+ \ \ j=1,2,3,\quad \psi_5=-\big\{[\widetilde{T}\widetilde{U}]_4\big\}^+,\quad \psi_6=-\big\{[\widetilde{T}\widetilde{U}]_5\big\}^+.$$

Vice versa, if $\widetilde{U}\in X^*_{D,\omega}(\Omega^+)$, then from the representation formula

$$\widetilde{U}=\widetilde{\mathbf{W}}_\omega\{\widetilde{U}\}^+ - \widetilde{\mathbf{V}}_\omega\{\widetilde{T}\widetilde{U}\}^+ \ \text{ in } \Omega^+ \tag{5.17}$$

it is easy to show that the vector-function $\widetilde{\Psi}:=(0,\{\widetilde{U}_1\}^+,\{\widetilde{U}_2\}^+,\{\widetilde{U}_3\}^+,-\{[\widetilde{T}\widetilde{U}]_4\}^+,-\{[\widetilde{T}\widetilde{U}]_5\}^+)^\top$ is a solution of the adjoint homogeneous system (5.10). Indeed, let us substitute $\widetilde{\Psi}$ in system (5.10). Therefore, we obtain the equalities

$$\big[(-2^{-1}I_4+\mathbf{K}^*_\omega)^{kj}\big]_{5\times 3}\{\widetilde{U}'\}^+ - [\mathbf{H}^{*k4}_{\ \omega}]_{5\times 1}\big\{[\widetilde{T}\widetilde{U}]_4\big\}^+ - [\mathbf{H}^{*k5}_{\ \omega}]_{5\times 1}\big\{[\widetilde{T}\widetilde{U}]_5\big\}^+=0, \quad j=\overline{1,3}, \ \ k=\overline{1,5}, \tag{5.18}$$

$$-\overline{b}_2\mathcal{N}^*\big(\{\widetilde{U}'\}^+\cdot n\big)=0, \tag{5.19}$$

where $\widetilde{U}'=(\widetilde{U}_1,\widetilde{U}_2,\widetilde{U}_3)^\top$.

By taking a trace of the representation formula (5.17), we get

$$\{\widetilde{U}\}^+=2^{-1}\{\widetilde{U}\}^+ + \mathbf{K}^*_\omega\{\widetilde{U}\}^+ - \mathbf{H}^*_\omega\{\widetilde{T}\widetilde{U}\}^+ \ \text{ on } S,$$

i.e., we have

$$(-2^{-1}I+\mathbf{K}^*_\omega)\{\widetilde{U}\}^+ - \mathbf{H}^*_\omega\{\widetilde{T}\widetilde{U}\}^+=0 \ \text{ on } S. \tag{5.20}$$

Since $\widetilde{U}\in X^*_{D,\omega}(\Omega^+)$, we have

$$\{\widetilde{U}_4\}^+=0,\quad \{\widetilde{U}_5\}^+=0,\quad \big\{[\widetilde{T}\widetilde{U}]_j\big\}^+=0, \ \ j=1,2,3, \tag{5.21}$$

$$\{\widetilde{U}'\}^+\cdot n=0. \tag{5.22}$$

Therefore, taking into account (5.21) in equality (5.20), we find that (5.18) is true, and it follows from (5.22) that (5.19) is true.

Therefore,

$$\dim\ker Q_{\omega,D}=\dim\ker Q^*_{\omega,D}=\dim X^*_{D,\omega}(\Omega^+).$$

Thus the orthogonality condition

$$\sum_{j=1}^3\big\langle f_j,\{\widetilde{U}_j\}^+\big\rangle_S - \Big\langle\big\{[\overline{\widetilde{T}\widetilde{U}}]_4\big\}^+,\overline{f}_1^{(D)}\Big\rangle_S - \Big\langle\big\{[\overline{\widetilde{T}\widetilde{U}}]_5\big\}^+,\overline{f}_2^{(D)}\Big\rangle_S=0 \ \ \forall\,\widetilde{U}\in X^*_{D,\omega}(\Omega^+), \tag{5.23}$$

is necessary and sufficient for the system of pseudodifferential equations (5.1)–(5.4) to be solvable.

We can now formulate the following existence theorem.

Theorem 5.2. *If $\omega \in J_D(\Omega^+)$, then the Dirichlet type problem (D_ω) is solvable if and only if the orthogonality condition* (5.23) *holds, and a solution is defined modulo Jones modes $X_{D,\omega}(\Omega^+)$.*

Remark 5.3. Let $(f_1, f_2, f_3) = n\psi$, where ψ is a scalar function and n is the unit normal vector to S (see (1.18)). Then the necessary and sufficient condition (5.23) reads as

$$\Big\langle \big\{[\widetilde{T}\widetilde{U}]_4\big\}^+, f_1^{(D)} \Big\rangle_S + \Big\langle \big\{[\widetilde{T}\widetilde{U}]_5\big\}^+, f_2^{(D)} \Big\rangle_S = 0 \ \ \forall \widetilde{U} \in X^*_{D,\omega}(\Omega^+).$$

Clearly, if the Dirichlet datum for the electric potential and magnetic potential are constant, or $\omega \notin J^*_D(\Omega^+)$, then problem (D_ω) is always solvable.

5.2 Existence of solution to the Neumann type problem (N_ω)

We look for a solution of the Neumann type problem (N_ω) in the form of the following potentials:

$$U = \mathbf{V}_\omega g \ \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h \ \text{ in } \Omega^-,$$

where $g \in [H^{-1/2}(S)]^5$ and $h \in H^{1/2}(S)$ are unknown densities. From Theorems 6.1 and 6.3 of Appendix it follows that $U \in [H^1(\Omega^+)]^5$ and $\mathrm{w} \in H^1_{loc}(\Omega^-)$.

Transmission conditions (1.11), (1.12) and the Neumann boundary conditions (1.16), (1.17) lead to the following system of pseudodifferential equations with respect to the unknowns g and h:

$$[\mathbf{H}_\omega g]_l n_l - b_1 \mathcal{M}(h) = f_0 \ \text{ on } S, \tag{5.24}$$

$$\big[(-2^{-1}I_5 + \mathbf{K}_\omega)g\big]_j - b_2 n_j \mathcal{N}(h) = f_j \ \text{ on } S, \ \ j = 1,2,3, \tag{5.25}$$

$$\big[(-2^{-1}I_5 + \mathbf{K}_\omega)g\big]_4 = f_1^{(N)} \ \text{ on } S, \tag{5.26}$$

$$\big[\big(-2^{-1}I_5 + \mathbf{K}_\omega\big)g\big]_5 = f_2^{(N)} \ \text{ on } S. \tag{5.27}$$

The operator generated by the left-hand side of system (5.24)–(5.27) reads as

$$Q_{\omega,N} = \begin{pmatrix} [n_l \mathbf{H}^{lk}_\omega]_{1\times 5} & -b_1\mathcal{M} \\ \big[(-2^{-1}I_5 + \mathbf{K}_\omega)^{jk}\big]_{3\times 5} & [-b_2 n_j \mathcal{N}]_{3\times 1} \\ \big[(-2^{-1}I_5 + \mathbf{K}_\omega)^{4k}\big]_{1\times 5} & 0 \\ \big[(-2^{-1}I_5 + \mathbf{K}_\omega)^{5k}\big]_{1\times 5} & 0 \end{pmatrix}_{6\times 6}, \quad j = \overline{1,3}, \ \ k = \overline{1,5}.$$

Due to Theorem 6.5 (see Appendix), it is evident that the operator

$$Q_{\omega,N} : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{-1/2}(S)]^6$$

is bounded.

It follows from (5.5) and (5.6) that the operator

$$Q_{\omega,N} - Q_{\tau,N} : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{-1/2}(S)]^6$$

is compact, where $Q_{\tau,N} := \mathcal{P}_{\tau,N}\mathcal{T}_\tau$ with the operator $\mathcal{T}_\tau$ defined in (5.7). Since the operator $Q_{\tau,N}$ is Fredholm with index zero (see Section 4), we have that the operator

$$Q_{\omega,N} : [H^{-1/2}(S)]^5 \times H^{1/2}(S) \to [H^{-1/2}(S)]^6$$

is Fredholm with index zero.

Recall that $J_N(\Omega^+) = \mathbb{R}$, due to Theorem 2.2 (see the end of Subsection 2.1).

The operator adjoint to $Q_{\omega,N}$ has the form

$$Q^*_{\omega,N} = \begin{pmatrix} [\mathbf{H}^{*kl}_{\ \omega} n_l]_{5\times 1} & \big[(-2^{-1}I_5 + \mathbf{K}^*_\omega)^{kj}\big]_{5\times 3} & [(-2^{-1}I_5 + \mathbf{K}^*_\omega)^{k4}]_{5\times 1} & \big[(-2^{-1}I_5 + \mathbf{K}^*_\omega)^{k5}\big]_{5\times 1} \\ -\bar{b}_1\mathcal{M}^* & \big[-\bar{b}_2\mathcal{N}^* n_j\big]_{1\times 3} & 0 & 0 \end{pmatrix}_{6\times 6},$$

$$j = \overline{1,3}, \ \ k = \overline{1,5},$$

and

$$Q^*_{\omega,N} : [H^{1/2}(S)]^6 \to [H^{1/2}(S)]^5 \times H^{-1/2}(S)$$

is bounded.

Let $\Phi := (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6)^\top \in [H^{1/2}(S)]^6$ be a solution of the homogeneous adjoint system

$$Q^*_{\omega,N}\Phi = 0. \tag{5.28}$$

Construct the potentials

$$\widetilde{U} = \widetilde{\mathbf{V}}_\omega \Phi^{(1)} + \widetilde{\mathbf{W}}_\omega \Phi^{(2)} \ \text{ in } \Omega^-, \tag{5.29}$$

$$\widetilde{\mathrm{w}} = -\overline{b}_1 \widetilde{W}_\omega \varphi_1 - \overline{b}_2 \widetilde{V}_\omega [\Phi' \cdot n] \ \text{ in } \Omega^+, \tag{5.30}$$

where $\Phi^{(1)} := (n\varphi_1, 0)^\top$, $\Phi^{(2)} := (\Phi', \varphi_5, \varphi_6)^\top$, $\Phi' := (\varphi_2, \varphi_3, \varphi_4)^\top$.

From (5.28) we have

$$\{\widetilde{U}\}^- = 0 \ \text{ on } S,$$
$$\big\{\partial_n \widetilde{\mathrm{w}} + \overline{\mu}\widetilde{\mathrm{w}}\big\}^+ = 0 \ \text{ on } S,$$

where $\widetilde{U} \in [H^1_{loc}(\Omega^-)]^5 \cap M_{m_1,m_2,m_3}(\mathbf{P}^*)$ and $\widetilde{\mathrm{w}} \in H^1(\Omega^+)$.

Therefore, from the uniqueness results for the exterior Dirichlet problem (see Theorem 3.5) and interior Robin type problem, we conclude that $\widetilde{U} = 0$ in Ω^- and $\widetilde{\mathrm{w}} = 0$ in Ω^+.

From jump formulae for potentials (5.29) and (5.30) we find that on the surface S the following relations hold:

$$\{\widetilde{\mathrm{w}}\}^- = \overline{b}_1 \varphi_1, \tag{5.31}$$

$$\{\partial_n \widetilde{\mathrm{w}}\}^- = -\overline{b}_2 \Phi' \cdot n, \tag{5.32}$$

$$\{\widetilde{U}\}^+ = (\Phi', \varphi_5, \varphi_6)^\top, \tag{5.33}$$

$$\big\{[\widetilde{T}\widetilde{U}]_j\big\}^+ = -n_j \varphi_1, \ \ j = 1,2,3, \tag{5.34}$$

$$\big\{[\widetilde{T}\widetilde{U}]_4\big\}^+ = 0, \tag{5.35}$$

$$\big\{[\widetilde{T}\widetilde{U}]_5\big\}^+ = 0. \tag{5.36}$$

Hence we obtain that $\widetilde{U} = (\widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3, \widetilde{U}_4, \widetilde{U}_5)^\top = (\widetilde{U}', \widetilde{U}_4, \widetilde{U}_5)^\top$ with $\widetilde{U}' = (\widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3)^\top$ and $\widetilde{\mathrm{w}}$ solve the following homogeneous problem:

$$A^*(\partial,\omega)\widetilde{U} = 0 \ \text{ in } \Omega^+,$$
$$(\Delta + \rho_2\omega^2)\widetilde{\mathrm{w}} = 0 \ \text{ in } \Omega^-,$$
$$\{\widetilde{U}' \cdot n\}^+ + \overline{b}_2^{\,-1}\{\partial_n \widetilde{\mathrm{w}}\}^- = 0 \ \text{ on } S,$$
$$\big\{[\widetilde{T}(\partial,n)\widetilde{U}]_j\big\}^+ + \overline{b}_1^{\,-1}\{\widetilde{\mathrm{w}}\}^- n_j = 0 \ \text{ on } S, \ \ j = 1,2,3,$$
$$\big\{[\widetilde{T}\widetilde{U}]_4\big\}^+ = 0 \ \text{ on } S,$$
$$\big\{[\widetilde{T}\widetilde{U}]_5\big\}^+ = 0 \ \text{ on } S.$$

From uniqueness result (see Remark 2.4) we have $\widetilde{\mathrm{w}} = 0$ in Ω^- and $\widetilde{U} \in X^*_{N,\omega}(\Omega^+)$, i.e., $\widetilde{U}$ belongs to the space of Jones modes $X^*_{N,\omega}(\Omega^+)$.

From (5.31) and (5.33) we get

$$\varphi_1 = 0, \ \ \varphi_{j+1} = \{\widetilde{U}_j\}^+, \ \ j = \overline{1,5}.$$

On the other hand, if $\widetilde{U} \in X^*_{N,\omega}(\Omega^+)$, then using the representation formula (5.17) it is easy to show that the vector-function $\widetilde{\Phi} := (0, \{\widetilde{U}_1\}^+, \{\widetilde{U}_2\}^+, \{\widetilde{U}_3\}^+, \{\widetilde{U}_4\}^+, \{\widetilde{U}_5\}^+)^\top$ is a solution of the homogeneous adjoint system (5.28). Indeed, let us substitute $\widetilde{\Phi}$ in system (5.28). Therefore, we obtain the equalities

$$\big[(-2^{-1}I_5 + \mathbf{K}^*_\omega)\big]\{\widetilde{U}\}^+ = 0, \tag{5.37}$$

$$-\bar{b}_2\mathcal{N}^*\big(\{\widetilde{U}'\}^+ \cdot n\big) = 0. \tag{5.38}$$

Taking the trace of the representation formula (5.17), we get

$$(-2^{-1}I + \mathbf{K}^*_\omega)\{\widetilde{U}\}^+ - \mathbf{H}^*_\omega\{\widetilde{T}\widetilde{U}\}^+ = 0 \text{ on } S. \tag{5.39}$$

Since $\widetilde{U} \in X^*_{N,\omega}(\Omega^+)$, we have

$$\{\widetilde{T}\widetilde{U}\}^+ = 0, \tag{5.40}$$

$$\{\widetilde{U}'\}^+ \cdot n = 0. \tag{5.41}$$

Therefore, taking into account (5.40) in equality (5.39), we obtain that (5.37) is true, and it follows from (5.41) that (5.38) is true.

Therefore,

$$\dim\ker Q_{\omega,N} = \dim\ker Q^*_{\omega,N} = \dim X^*_{N,\omega}(\Omega^+).$$

Thus the orthogonality condition

$$\sum_{j=1}^{3}\big\langle f_j, \{\widetilde{U}_j\}^+\big\rangle_S + \big\langle f_1^{(N)}, \{\widetilde{U}_4\}^+\big\rangle_S + \big\langle f_2^{(N)}, \{\widetilde{U}_5\}^+\big\rangle_S = 0 \ \ \forall \widetilde{U} \in X^*_{N,\omega}(\Omega^+) \tag{5.42}$$

is necessary and sufficient for the system of pseudodifferential equations (5.24)-(5.27) to be solvable.

The following existence theorem follows directly.

Theorem 5.4. *The Neumann type problem (N_ω) is solvable if and only if the orthogonality condition* (5.42) *holds, and a solution is defined modulo Jones modes $X_{N,\omega}(\Omega^+)$.*

Remark 5.5. If $(f_1, f_2, f_3) = n\psi$, where ψ is a scalar function and n is the unit normal vector to S (see (1.18)), then the necessary and sufficient condition (5.42) can be written in the form

$$\big\langle f_1^{(N)}, \{\widetilde{U}_4\}^+\big\rangle_S + \big\langle f_2^{(N)}, \{\widetilde{U}_5\}^+\big\rangle_S = 0 \ \ \forall \widetilde{U} \in X^*_{N,\omega}(\Omega^+).$$

Clearly, if $f_1^{(N)} = f_2^{(N)} = 0$, then problem (N_ω) is always solvable.

6 Appendix

For the readers convenience, we collect here some results describing properties of the layer potentials. Here, we preserve the notation from the main text of the paper. For the potentials associated with the Helmholtz equation, the following theorems hold (see [13, 20, 32, 37]).

Theorem 6.1. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Then the single and double layer scalar potentials can be extended to the following continuous operators:*

$$V_\omega : H^s(S) \to H^{s+3/2}(\Omega^+), \quad V_\omega : Hs(S) \to H^{s+3/2}_{loc}(\Omega^-),$$
$$W_\omega : H^s(S) \to H^{s+1/2}(\Omega^+), \quad W_\omega : H^s(S) \to H^{s+1/2}_{loc}(\Omega^-).$$

Theorem 6.2. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Then the operators*

$$\mathcal{H}_\omega : H^s(S) \to H^{s+1}(S),$$
$$\mathcal{K}_\omega, \mathcal{K}^*_\omega : H^s(S) \to H^{s+1}(S),$$
$$\mathcal{L}_\omega : H^s(S) \to H^{s-1}(S)$$

are continuous.

For the potentials of steady state oscillation and pseudo-oscillation equations, the following theorems hold (see [5–8, 12]).

Theorem 6.3. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Then the vector potentials $\mathbf{V}_\omega$, $\mathbf{W}_\omega$, $\mathbf{V}_\tau$ and $\mathbf{W}_\tau$ are continuous in the following spaces:*

$$\mathbf{V}_\omega, \mathbf{V}_\tau : [H^s(S)]^5 \to [H^{s+3/2}(\Omega^+)]^5 \quad \Big([H^s(S)]^5 \to [H^{s+3/2}_{loc}(\Omega^-)]^5 \Big),$$
$$\mathbf{W}_\omega, \mathbf{W}_\tau : [H^s(S)]^5 \to [H^{s+1/2}_p(\Omega^+)]^5 \quad \Big([H^s(S)]^5 \to [H^{s+1/2}_{loc}(\Omega^-)]^5 \Big).$$

Theorem 6.4. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Then the operators*

$$\mathbf{H}_\tau : [H^s(S)]^5 \to [H^{s+1}(S)]^5,$$
$$\mathbf{K}_\tau, \widetilde{\mathbf{K}}_\tau : [H^s(S)]^5 \to [H^s(S)]^5,$$
$$\mathbf{L}_\tau : [H^s(S)]^5 \to [H^{s-1}(S)]^5$$

are bounded.

The operators $\mathbf{H}_\tau$ and $\boldsymbol{L}_\tau$ are strongly elliptic pseudodifferential operators of order -1, and 1 respectively, while the operators $\pm 2^{-1} I_5 + \mathbf{K}_\tau$ and $\pm 2^{-1} I_5 + \widetilde{\mathbf{K}}_\tau$ are elliptic pseudodifferential operators of order 0.

Moreover, the operators $\mathbf{H}_\tau$, $2^{-1} I_5 + \widetilde{\mathbf{K}}_\tau$ and $2^{-1} I_5 + \mathbf{K}_\tau$ are invertible, whereas the operators $\mathbf{L}_\tau$, $-2^{-1} I_5 + \widetilde{\mathbf{K}}_\tau$ and $-2^{-1} I_5 + \mathbf{K}_\tau$ are Fredholm operators with index zero.

Theorem 6.5. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Then the operators*

$$\mathbf{H}_\omega : [H^s(S)]^5 \to [H^{s+1}(S)]^5,$$
$$\pm 2^{-1} I_5 + \mathbf{K}_\omega : [H^s(S)]^5 \to [H^s(S)]^5,$$
$$\pm 2^{-1} I_5 + \widetilde{\mathbf{K}}_\omega : [H^s(S)]^5 \to [H^s(S)]^5,$$
$$\mathbf{L}_\omega : [H^s(S)]^5 \to [H^{s-1}(S)]^5$$

are bounded Fredholm operators with index zero.

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Author's address:

Department of Mathematical Physics, Andrea Razmadze Mathematical Institute of Ivane Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia

E-mail: g.chkadua@gmail.com

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Yuqiang Feng, Yuanyuan Wang, Deyi Li

COMPARISON THEOREM AND SOLVABILITY OF THE BOUNDARY VALUE PROBLEM OF A FRACTIONAL DIFFERENTIAL EQUATION

Abstract. When the nonlinearities satisfy the growth conditions on a finite interval, some existence results of solutions to the boundary value problems of fractional differential equations are established via comparison theorem, upper and lower solutions method and fixed point theorems. An example is presented to illustrate the applications of the obtained results.

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Key words and phrases. Comparison theorem, fractional differential equation, upper and lower solutions method, the Banach contraction principle, Shauder's fixed-point theorem.

რეზიუმე. შედარების თეორემის, ზედა და ქვედა ამონახსნების მეთოდისა და უძრავი წერტილის თეორემების საშუალებით დადგენილია რამდენიმე შედეგი ფრაქციული დიფერენციალური განტოლებისთვის სასაზღვრო ამოცანების ამონახსნის არსებობის შესახებ, როდესაც არაწრფივობა აკმაყოფილებს ზრდის პირობებს სასრულ ინტერვალზე. მიღებული შედეგების გამოყენების საილუსტრაციოდ მოყვანილია მაგალითი.

1 Introduction

Fractional calculus has played a significant role in engineering, science, economy, and other fields. Most of papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations in terms of special functions. Recently, there appeared some papers dealing with the existence of solutions (or positive solutions) of nonlinear initial value problems of fractional differential equation using the techniques of nonlinear analysis (see [2, 9] and the references therein).

In the literature, ${}^cD^{\alpha}_{0+}u(t) + f(t, u(t)) = 0$ is known as a single-term equation. This kind of fractional differential equation has many applications and has been studied widely. Equations containing more than one fractional differential terms are called multi-term fractional differential equations; they have some concrete applications in many fields. Due to the complexity of such a kind of equations, it seems that there has been no result for a general multi-term fractional differential equation. Only some special cases have been investigated. A classical example is the so-called Bagley–Torvik equation (B-T equation for short) [12],

$$Au''(t) + B{}^cD^{\frac{3}{2}}_{0+}u(t) + Cu(t) = f(t),$$

where A, B and C are certain constants, ${}^cD^{\alpha}_{0+}$ is the Caputo fractional derivative and f is a given function. This equation arises from the mathematical model of the motion of a thin plate in a Newtonian fluid. The B-T equation, as well as various generalizations, have wide applications in fluid dynamics and hence attracted much attention. The analytic solution and the numerical solution for the B-T equation were studied in [4] and [5], respectively.

J. Cermak et al. [3] investigated the two-term fractional differential equation

$$u''(t) + B{}^cD^{\beta}_{0+}u(t) + bu(t) = 0$$

with coefficients $a, b \in R$ and positive real orders $0 < \beta < 2$. It contains the important case such as the B-T equation for $\beta = \frac{3}{2}$. Qualitative properties of the true and numerical solutions were described and numerical stability regions for the classical and fractional models were compared.

In [14], S. Zhang discussed the following boundary-value problems for two-point nonlinear fractional differential equation:

$$\begin{cases} D^{\alpha}_{0+}u(t) + q(t)f\big(u(t), u'(t), u''(t), \dots, u^{(n-2)}(t)\big) = 0, \ \ t \in (0, 1), \\ u(0) = u'(1) = u''(0) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0, \end{cases}$$

where α is a positive number, D^{α}_{0+} is the Riemann–Liouville's fractional derivative, q may be singular at $t = 0$ and $f(x_0, x_1, \dots, x_{n-2})$ may be singular at $x_0 = 0, x_1 = 0, x_2 = 0, \dots, x_{(n-2)} = 0$. The existence of positive solutions to the problem is obtained by the fixed point theorem for the mixed monotone operator.

In [7], the authors have investigated the existence of solutions for two-point boundary value problems

$$\begin{cases} D^{\alpha}_{0+}u(t) + f\big(t, u(t), D^{\alpha-2}_{0+}u(t)\big) = 0, \ \ t \in (0, 1), \\ u^{(k)}(0) = 0, \ \ k = 0, 1, \dots, n-3, \ \ n = [\alpha] + 1, \\ D^{\alpha-2}_{0+}u(1) = D^{\alpha-1}_{0+}u(0) = 0, \end{cases}$$

for fractional differential equations of arbitrary order $\alpha > 2$, by applying upper and lower solutions method together with Schauder's fixed point theorem. First, they transformed the posed problem to an ordinary first order initial value problem that they modified to prove the existence of solutions for the problem. Moreover, they gave the explicit expression of the upper and lower solutions of the problem.

Recently, in [13], the authors considered the existence of solutions of the boundary-value problem for two-term three-point nonlinear fractional differential equation:

$$\begin{cases} \lambda D^{\alpha}_{0+}u(t) + D^{\beta}_{0+}u(t) = f(t, u(t)), \ \ t \in [0, T], \\ u(0) = 0, \ \ \mu D^{\gamma_1}_{0+}u(T) + D^{\gamma_2}_{0+}u(\eta) = \gamma_3, \end{cases}$$

where $1 < \alpha \leq 2$, $1 \leq \beta < \alpha$, $0 < \lambda \leq 1$, $0 \leq \mu \leq 1$, $0 \leq \gamma_1 \leq \alpha - \beta$, $\gamma_2 \geq 0$, $0 < \eta < T$ are the constants, D_{0+}^{α}, D_{0+}^{β} are the Riemann–Liouville fractional derivative, and $f : [0,T] \times R \to R$ is continuous. By means of the fixed point theorems and Gronwall type inequality, some results on the existence of solutions and the Hyers–Ulam stability are obtained. (For more results see [1,6,10,11] and the references therein.)

Motivated by the above results, in this paper we deal with the boundary value problem of the two-term fractional differential equation:

$$\begin{cases} D_{0+}^{2+\alpha} u(t) + f\big(t, u(t), D_{0+}^{\alpha} u(t)\big) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad D_{0+}^{\alpha} u(t)\big|_{t=0} = D_{0+}^{\alpha} u(t)\big|_{t=1} = 0, \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$ is a real number and D_{0+}^{α} is the standard Riemann–Liouville fractional derivative, $f : [0,1] \times R^2 \to R$ is continuous. We prove a new comparison theorem, and then establish the existence of solutions for the above-given problem using the comparison theorem, fixed point theory and the method of upper and lower solutions. By these methods, we can obtain the iterative scheme for this problem, which implies that the solutions are computable.

The paper is organized as follows. In Section 2, a new comparison theorem is proved. The existence results for problem (1.1) are established in Section 3. In the same section, we give the proof of the main result. An example is presented in the last section to illustrate the application of our results.

2 Preliminaries and comparison theorem

In this section, we first recall some standard definitions and notation.

Let $\alpha > 0$ be a constant.

Definition 2.1 ([8])**.** The Riemann–Liouville fractional integral $I_{a+}^{\alpha} f$ of order α is defined by

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(x)}{(t-x)^{1-\alpha}}\, dx, \quad t > a,$$

provided that the right-hand side is defined point-wisely, where Γ is the Gamma function.

Definition 2.2 ([8])**.** The Riemann–Liouville fractional derivatives $D_{a+}^{\alpha} f$ of order α are defined by

$$D_{a+}^{\alpha} f(t) = \Big(\frac{d}{dx}\Big)^n (I_{a+}^{n-\alpha} f)(t) = \Big(\frac{d}{dx}\Big)^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\, dx, \quad n = [\alpha] + 1, \quad t > a,$$

provided that the right-hand side is defined point-wisely, where $[\alpha]$ denotes the integer part of α.

Lemma 2.3 ([8])**.** *Let $m \in N_+$ and $D = d/dt$. If the fractional derivatives $(D_{a+}^{\alpha} f)(t)$ and $(D_{a+}^{\alpha+m} f)(t)$ exist, then*

$$(D^m D_{a+}^{\alpha} f)(t) = (D_{a+}^{\alpha+m} f)(t).$$

Remark 2.4.

(1) The Riemann–Liouville fractional integral satisfies the equality

$$I_{0+}^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}\, t^{\beta+\alpha}, \quad \alpha > 0, \quad \beta > -1, \quad t > 0.$$

(2) The equality $D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t)$ holds for $u \in L(0,1)$.

(3) If $\alpha \in (0,1]$, then for $u \in L(0,1)$, $D_{a+}^{\alpha} u \in L(0,1)$ and arbitrary $c \in R$, the equality

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c t^{\alpha-1}$$

holds.

The following comparison theorem is crucial in this paper.

Lemma 2.5. *Let λ_1, λ_2 be two nonnegative numbers, $r > 0$ be a constant. If $m(t) \in C^2[0,1]$ satisfies*

$$m''(t) \geq \frac{\lambda_1}{\Gamma(r)} \int_0^t (t-s)^{r-1} m(s)\, ds + \lambda_2 m(t), \quad 0 < t < 1, \quad m(0) \leq 0, \quad m(1) \leq 0,$$

then $m(t) \leq 0$, $\forall\, t \in [0,1]$, provided that $0 \leq \lambda_1 + \lambda_2 \Gamma(r+1) \leq 2\Gamma(r+1)$.

Proof. We will verify the assertion in the following cases.

Case 1. If $\lambda_1 = \lambda_2 = 0$, then we have $m''(t) \geq 0$, which implies that $m(t)$ is a convex function on $[0,1]$. Hence, we have $m(t) \leq \min\{m(0), m(1)\} \leq 0$, $t \in [0,1]$.

Case 2. Let $\lambda_1 = 0, 0 < \lambda_2 < 2$.

Conversely, suppose there exists $t_0 \in (0,1)$ such that $m_0 = m(t_0) = \max m(t) > 0$, then $m'(t_0) = 0$, $m''(t_0) \leq 0$. But $m''(t_0) \geq \lambda_2 m(t_0)$ implies $m''(t_0) > 0$, which is a contradiction.

Case 3. Let $\lambda_1 > 0$, $\lambda_2 \geq 0$ and $0 < \lambda_1 + \lambda_2 \Gamma(r+1) \leq 2\Gamma(r+1)$.

Assume that there exists $t_0 \in (0,1)$ such that $m_0 = m(t_0) = \max\limits_{0 \leq t \leq 1} m(t) > 0$, then $m'(t_0) = 0$, $m''(t_0) \leq 0$. Hence, by

$$0 \geq m''(t_0) \geq \frac{\lambda_1}{\Gamma(r)} \int_0^{t_0} (t_0 - s)^{r-1} m(s)\, ds + \lambda_2 m(t_0),$$

we have $\int_0^{t_0} (t_0 - s)^{r-1} m(s)\, ds < 0$.

This implies that there is $t_1 \in [0, t_0)$ such that $m_1 = m(t_1) = \min\limits_{t \in [0,t_0]} m(t) < 0$. According to Taylor's formula, there is $\lambda \in (t_1, t_0)$ such that

$$m_1 = m(t_1) = m(t_0) + m'(t_0)(t_1 - t_0) + \frac{m''(\lambda)}{2}(t_1 - t_0)^2.$$

Since $m_1 < 0$, we have

$$m''(\lambda) = \frac{2(m_1 - m_0)}{(t_1 - t_0)^2} < \frac{2m_1}{(t_1 - t_0)^2}\,.$$

Hence

$$\begin{aligned} 2m_1 > m''(\lambda) \geq \frac{\lambda_1}{\Gamma(r)} \int_0^{\lambda} (\lambda - s)^{r-1} m(s)\, ds + \lambda_2 m(\lambda) &\geq \frac{\lambda_1}{\Gamma(r)} \int_0^{\lambda} (\lambda - s)^{r-1} m_1\, ds + \lambda_2 m_1 \\ &= \frac{\lambda_1}{\Gamma(r+1)}\, \lambda^r m_1 + \lambda_2 m_1 > \frac{\lambda_1}{\Gamma(r+1)}\, m_1 + \lambda_2 m_1. \end{aligned}$$

This implies that $\lambda_1 + \lambda_2 \Gamma(r+1) > 2\Gamma(r+1)$, which contradicts the assumption that $0 \leq \lambda_1 + \lambda_2 \Gamma(r+1) \leq 2\Gamma(r+1)$.

This ends the proof. □

Corollary 2.6. *Let λ_1, λ_2 be two nonnegative numbers, $0 < \alpha \leq 1$ be a constant. If $h(t) \in C^3[0,1]$ satisfies*

$$\begin{cases} D_{0+}^{2+\alpha} h(t) \geq \lambda_1 h(t) + \lambda_2 D_{0+}^{\alpha} h(t), \quad 0 < t < 1, \\ h(0) = 0, \quad D_{0+}^{\alpha} h(t)\big|_{t=0} \leq 0, \quad D_{0+}^{\alpha} h(t)\big|_{t=1} \leq 0, \end{cases}$$

then $h(t) \leq 0$, $\forall\, t \in [0,1]$ provided that $0 \leq \lambda_1 + \lambda_2 \Gamma(\alpha+1) \leq 2\Gamma(\alpha+1)$.

Proof. Let $m(t) = D_{0+}^{\alpha}h(t)$. Since $h(0) = 0$, we have

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m(s)\,ds, \quad m''(t) = D_{0+}^{2+\alpha}h(t)$$

and

$$m''(t) \geq \frac{\lambda_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m(s)\,ds + \lambda_2 m(t), \quad 0 < t < 1, \quad m(0) \leq 0, \quad m(1) \leq 0.$$

Due to Lemma 2.5, we have $m(t) \leq 0$, $\forall\, t \in [0,1]$. Hence

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m(s)\,ds \leq 0, \quad \forall\, t \in [0,1].$$

This ends the proof. □

3 The existence criteria

Throughout this section, we assume that $f : [0,1] \times R^2 \to R$ is continuous and there exist non-negative numbers λ_1, λ_2 such that

(H$_1$) for $t \in [0,1]$, $z \in R$, $x_1 \geq x_2$, $y_1 \geq y_2$

$$f(t, x_1, y_1) - f(t, x_2, y_2) \geq -\lambda_1(x_1 - x_2) - \lambda_2(y_1 - y_2).$$

(H$_2$) $0 \leq \lambda_1 + \lambda_2\Gamma(\alpha+1) \leq 2\Gamma(\alpha+1)$.

Definition 3.1. A function $u \in C[0,1]$ is called a solution of problem (1.1) if $D_{0+}^{\alpha}u \in C[0,1]$, and u satisfies the equation in (1.1) for $t \in [0,1]$ and the boundary condition in (1.1).

Lemma 3.2. *If $u \in C[0,1]$ is a solution of the following boundary value problem*

$$\begin{cases} (D_{0+}^{\alpha}u(t))'' + f\big(t, u(t), D_{0+}^{\alpha}u(t)\big) = 0, & t \in (0,1), \\ u(0) = 0, \quad D_{0+}^{\alpha}u(t)\big|_{t=0} = D_{0+}^{\alpha}u(t)\big|_{t=1} = 0, \end{cases} \tag{3.1}$$

then u is a solution of (1.1).

Proof. According to Lemma 2.3, we have

$$(D^2 D_{a+}^{\alpha}u)(t) = (D_{a+}^{\alpha+2}u)(t),$$

i.e.,

$$(D_{0+}^{\alpha}u)''(t) = (D_{0+}^{\alpha+2}u)(t).$$

So, if $u \in C[0,1]$ is a solution of (3.1), it is a solution of (1.1). □

The main result reads as follows.

Theorem 3.3. *If $\min\limits_{0 \leq t \leq 1} f(t,0,0) \geq 0$ and there exists $c > 0$ such that*

$$\max\left\{ f(t,x,y) \mid \ (t,x,y) \in [0,1] \times \left[0, \frac{c}{\Gamma(3+\alpha)} (\frac{1+\alpha}{2})^{1+\alpha}\right] \times \left[0, \frac{c}{4}\right] \right\} \leq 2c,$$

then (1.1) *has a solution u^* satisfying*

$$0 \leq u^*(t) \leq c\Big(\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}\Big).$$

Proof. Let $X = C[0,1]$, the norm on X be $\|\cdot\| : \|x\| = \max\limits_{0\le t\le 1} |x(t)|$ for $x \in X$. Let $K = \{x \in X \mid \; x(t) \ge 0,\; 0 \le t \le 1\}$ and the partial order "$\le$" on X be induced by K: for $x, y \in X$, $y \le x \Longleftrightarrow x - y \in K$, then (X, K) is an ordered Banach space.

Having in mind (3.1) (with $D^{\alpha}_{0+}u$ replaced by h), we discuss the problem

$$\begin{cases} -h''(t) = f\big(t, I^{\alpha}_{0+}h(t), h(t)\big), \\ h(0) = h(1) = 0, \end{cases} \tag{3.2}$$

Let $D = \{h \in X \mid \; h'' \in X,\; h(0) = h(1) = 0\}$. Define $L : D \subset X \to X$ and $N : X \to X$ as follows:

$$Lh = -h''(t) + \lambda_1 I^{\alpha}_{0+}h(t) + \lambda_2 h(t),$$
$$Nh = f\big(t, I^{\alpha}_{0+}h(t), h(t)\big) + \lambda_1 I^{\alpha}_{0+}h(t) + \lambda_2 h(t).$$

By the definition of L and N, (3.2) can be rewritten as

$$Lh = Nh. \tag{3.3}$$

Step 1. *$L : D \subset X \to X$ is a reversible mapping.*

Given $\eta \in X$, we consider the following boundary value problem:

$$\begin{cases} -h''(t) + \lambda_1 I^{\alpha}_{0+}h(t) + \lambda_2 h(t) = \eta(t), \\ h(0) = h(1) = 0. \end{cases}$$

It is known that h is the solution of the above problem if and only if h is the fixed point of the operator $A_\eta : X \to X$, where

$$A_\eta h(t) = \int_0^1 G(t,s)\big[\eta(s) - \lambda_1 I^{\alpha}_{0+}h(s) - \lambda_2 h(s)\big]\, ds$$

and

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Since $\max\limits_{t\in[0,1]} \int_0^1 G(t,s)\, ds = \frac{1}{8}$, we have

$$|A_\eta x(t) - A_\eta y(t)| = \int_0^1 G(t,s)\Big[\lambda_1 I^{\alpha}_{0+}(y(s) - x(s)) + \lambda_2 (y(s) - x(s))\Big]\, ds$$
$$\le \int_0^1 G(t,s)\Big[\lambda_1 I^{\alpha}_{0+}\|x - y\| + \lambda_2 \|x - y\|\Big]\, ds \le \frac{1}{8}\Big[\frac{\lambda_1}{\Gamma(\alpha+1)} + \lambda_2\Big]\|x - y\| \le \frac{1}{4}\,\|x - y\|$$

for all $t \in [0,1]$, $x, y \in X$, which implies that $A_\eta : X \to X$ is contractive.

By the completeness of X and an application of the Banach contraction principle, there exists a unique $h \in X$ such that $A_\eta h = h$, i.e., $Lh = \eta$. In fact, $h \in D$. Hence $L : D \subset X \to X$ is reversible.

Step 2. *$L^{-1} : X \to D$ is continuous.*

Let $\eta \in X$, $\{\eta_n\} \subset X$, $\eta_n \to \eta$, $L^{-1}\eta = x$, $L^{-1}\eta_n = x_n$, then

$$x_n(t) = \int_0^1 G(t,s)\big[\eta_n(s) - \lambda_1 I^{\alpha}_{0+}x_n(s) - \lambda_2 x_n(s)\big]\, ds,$$

$$x(t) = \int_0^1 G(t,s)\big[\eta(s) - \lambda_1 I^{\alpha}_{0+}x(s) - \lambda_2 x(s)\big]\, ds.$$

As a result,

$$\begin{aligned}
|x_n(t) - x(t)| &= \Big| \int_0^1 G(t,s)\Big[\eta_n(s) - \eta(s) + \lambda_1 I_{0+}^{\alpha}(x - x_n)(s) + \lambda_2(x(s) - x_n(s))\Big]\, ds \Big| \\
&\le \int_0^1 G(t,s)\Big[|\eta_n(s) - \eta(s)| + \lambda_1 I_{0+}^{\alpha}|x - x_n|(s) + \lambda_2 |x(s) - x_n(s)|\Big]\, ds \\
&\le \frac{1}{8}\left[\|\eta_n - \eta\| + \Big(\frac{\lambda_1}{\Gamma(\alpha+1)} + \lambda_2\Big)\|x - x_n\|\right] \\
&\le \frac{1}{8}\,\|\eta_n - \eta\| + \frac{1}{4}\,\|x - x_n\|.
\end{aligned}$$

We have

$$\|x_n - x\| \le \frac{1}{6}\,\|\eta_n - \eta\|.$$

Consequently, $x_n \to x$, when $\eta_n \to \eta$. Therefore, $L^{-1} : X \to D$ is continuous.

Step 3. *$L^{-1} : X \to D$ is compact.*

Let $S \subset X$ be a bounded subset, i.e., there exists a constant $M > 0$ such that $\|\eta\| \le M$ for any $\eta \in S$.

Let $\eta \in S, L^{-1}\eta = x$, then

$$x(t) = \int_0^1 G(t,s)\big[\eta(s) - \lambda_1 I_{0+}^{\alpha} x(s) - \lambda_2 x(s)\big]\, ds.$$

As a result,

$$\|x\| \le \frac{1}{8}\,\|\eta\| + \frac{1}{8}\Big(\frac{\lambda_1}{\Gamma(\alpha+1)} + \lambda_2\Big)\|x\| \le \frac{1}{8}\,\|\eta\| + \frac{1}{4}\,\|x\|,$$

hence

$$\|x\| \le \frac{1}{6}\,\|\eta\| \le \frac{1}{6}\,M,$$

which implies that $L^{-1}(S)$ is bounded.

Furthermore, let $t_1, t_2 \in [0,1]$, $t_1 < t_2$, then for any $x \in L^{-1}(S)$, there exists $\eta \in D$ such that $L^{-1}\eta = x$ and

$$\begin{aligned}
|x(t_1) - x(t_2)| &= |A_\eta x(t_1) - A_\eta x(t_2)| \\
&= \Big| \int_0^1 (G(t_1,s) - G(t_2,s))\big[\eta(s) - \lambda_1 I_{0+}^{\alpha} x(s) - \lambda_2 x(s)\big]\, ds \Big| \\
&\le \int_0^1 |G(t_1,s) - G(t_2,s)|\, |\eta(s) - \lambda_1 I_{0+}^{\alpha} x(s) - \lambda_2 x(s)|\, ds \\
&\le \int_0^1 |G(t_1,s) - G(t_2,s)|\, ds \left[\|\eta\| + \Big(\frac{\lambda_1}{\Gamma(\alpha+1)} + \lambda_2\Big)\|x\|\right] \\
&\le \frac{4M}{3} \int_0^1 |G(t_1,s) - G(t_2,s)|\, ds.
\end{aligned}$$

Due to the uniform continuity of $G(t,s)$ on $[0,1] \times [0,1]$, for $\forall\, \varepsilon > 0$, there exists $\sigma > 0$ such that $|t_2 - t_1| < \sigma$ implies

$$|G(t_1,s) - G(t_2,s)| < \frac{3}{4M}\,\varepsilon.$$

At the same time, we have

$$|x(t_1) - x(t_2)| \le \frac{4M}{3} \int_0^1 |G(t_1, s) - G(t_2, s)|\, ds < \frac{4M}{3} \frac{3}{4M} \varepsilon = \varepsilon.$$

Hence $L^{-1}(S)$ is equi-continuous.

Since $L^{-1}(S)$ is bounded and equi-continuous, $L^{-1} : X \to D$ is compact.

Step 4. *$L^{-1}N : X \to D$ is continuous and increasing.*

Since f is continuous, by the definition of N and Step 3, $N : X \to X$ and $L^{-1}N : X \to D$ are continuous.

Moreover, for arbitrary $\eta_1, \eta_2 \in X$, $\eta_1 \le \eta_2$, (H_1) implies $N\eta_1 \le N\eta_2$. Let $v_1 = L^{-1}N\eta_1$, $v_2 = L^{-1}N\eta_2$, then $Lv_1 = N\eta_1 \le N\eta_2 = Lv_2$. Hence we have $L(v_1 - v_2) \le 0$, i.e.,

$$-(v_1 - v_2)''(t) + \frac{\lambda_1}{\Gamma(r)} \int_0^t (t-s)^{r-1}(v_1(s) - v_2(s))\, ds + \lambda_2(v_1(t) - v_2(t)), \ \ 0 < t < 1,$$
$$(v_1 - v_2)(0) = (v_1 - v_2)(1) = 0.$$

By Lemma 2.5, we obtain $(v_1 - v_2)(t) \le 0$ for $t \in [0, 1]$, i.e., $v_1 \le v_2$. Hence $L^{-1}N : X \to D$ is increasing.

Step 5. *There exist $x, y \in D$, $x \le y$ such that $Lx \le Nx$ and $Ly \ge Ny$.*

Let $v(t) = 0$. Since

$$\min_{0 \le t \le 1} f(t, 0, 0) \ge 0,$$

we have

$$\begin{cases} D_{0+}^{2+\alpha} v(t) + f(t, v(t), D_{0+}^{\alpha} v(t)) \ge 0, \ \ t \in (0,1) \\ v(0) = 0, \ \ D_{0+}^{\alpha} v(t)\big|_{t=0} \le 0, \ \ D_{0+}^{\alpha} v(t)\big|_{t=1} \le 0. \end{cases}$$

Let

$$w(t) = c\Big(\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}\Big).$$

Noting that for $t \in [0, 1]$,

$$D_{0+}^{2+\alpha} w(t) = 2c, \ \ w(t) \in \Big[0, \frac{c}{\Gamma(3+\alpha)} \Big(\frac{1+\alpha}{2}\Big)^{1+\alpha}\Big], \ \ D_{0+}^{\alpha} w(t) \in \Big[0, \frac{c}{4}\Big]$$

and

$$\max\Big\{ f(t, x, y) \mid \ (t, x, y) \in [0, 1] \times \Big[0, \frac{c}{\Gamma(3+\alpha)} \Big(\frac{1+\alpha}{2}\Big)^{1+\alpha}\Big] \times \Big[0, \frac{c}{4}\Big] \Big\} \le 2c,$$

we get

$$\begin{cases} D_{0+}^{2+\alpha} w(t) + f\big(t, w(t), D_{0+}^{\alpha} w(t)\big) \le 0, \ \ t \in (0,1), \\ w(0) = 0, \ \ D_{0+}^{\alpha} w(t)\big|_{t=0} \ge 0, \ \ D_{0+}^{\alpha} w(t)\big|_{t=1} \ge 0. \end{cases}$$

By Step 1, there exist $x, y \in D$ such that

$$Lx = N(D_{0+}^{\alpha} v(t)), \quad Ly = N(D_{0+}^{\alpha} w(t)).$$

Next, we assert that

(1) $x \le y$;

(2) $D_{0+}^{\alpha} v(t) \le x$ and $Lx \le Nx$;

(3) $y \le D_{0+}^{\alpha} w(t)$ and $Ly \ge Ny$.

Since N is nondecreasing, we have $N(D_{0+}^{\alpha}v(t)) \leq N(D_{0+}^{\alpha}w(t))$, hence $Lx \leq Ly$. Lemma 2.5 implies $x \leq y$. Assertion (1) is verified.

Next, we verify assertion (2).

In fact, by the definition of x, we have

$$\begin{cases} -x''(t) + \lambda_1 I_{0+}^{\alpha}x(t) + \lambda_2 x(t) = f\big(t, v(t), D_{0+}^{\alpha}v(t)\big) + \lambda_1 v(t) + \lambda_2 D_{0+}^{\alpha}v(t), \\ x(0) = x(1) = 0. \end{cases} \tag{3.4}$$

Let $\phi(t) = D_{0+}^{\alpha}v(t)$. Then

$$\begin{cases} -\phi''(t) + \lambda_1 I_{0+}^{\alpha}\phi(t) + \lambda_2 \phi(t) \leq f\big(t, v(t), D_{0+}^{\alpha}v(t)\big) + \lambda_1 v(t) + \lambda_2 D_{0+}^{\alpha}v(t), \\ \phi(0) \leq 0, \quad \phi(1) \leq 0. \end{cases} \tag{3.5}$$

(3.4), (3.5) together with the assumption (H_2) lead to

$$\begin{cases} -(x(t) - \phi(t))'' + \lambda_1 I_{0+}^{\alpha}(x - \phi)(t) + \lambda_2(x(t) - \phi(t)) \geq 0, \\ (x(0) - \phi(0)) \geq 0, \quad (x(1) - \phi(1)) \geq 0. \end{cases}$$

By virtue of Lemma 2.5, we have $x(t) - \phi(t) \geq 0$ i.e., $x(t) \geq \phi(t)$. The nondecreasing of N gives $Nx \geq N\phi$, hence $Lx = N\phi \leq Nx$.

$y \leq D_{0+}^{\alpha}w(t)$, $Ny \leq Ly$ can be verified similarly.

Step 6. *Problem* (1.1) *has a solution* $u^*(t)$ *satisfying* $v(t) \leq u^*(t) \leq w(t)$.

Step 4 and Step 5 implies that the operator $L^{-1}N$ maps $[x, y] \cap D$ into $[x, y] \cap D$. Since $[x, y] \cap D$ is convex, closed and bounded and $L^{-1}N$ is completely continuous, an application of Schauder's fixed point theorem implies that $Lh = Nh$ has a solution h^* in $[x, y]$. Let

$$u^*(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h^*(s)\, ds,$$

then $u^*(t)$ is a solution of problem (1.1) satisfying $v(t) \leq u^*(t) \leq w(t)$. □

Theorem 3.4. *If* $\max_{0 \leq t \leq 1} f(t, 0, 0) \leq 0$ *and there exists* $c > 0$ *such that*

$$\min\left\{ f(t, x, y) \mid (t, u, v) \in [0, 1] \times \Big[-\frac{c}{\Gamma(3 + \alpha)} \Big(\frac{1 + \alpha}{2}\Big)^{1 + \alpha}, 0 \Big] \times \Big[-\frac{c}{4}, 0 \Big] \right\} \geq -2c,$$

then (1.1) *has a solution* u^* *satisfying*

$$0 \geq u^*(t) \geq -c\Big(\frac{t^{\alpha}}{\Gamma(1 + \alpha)} - \frac{2t^{2 + \alpha}}{\Gamma(3 + \alpha)} \Big).$$

Proof. In Step 5 of the proof of Theorem 3.3, let

$$v(t) = -c\Big(\frac{t^{\alpha}}{\Gamma(1 + \alpha)} - \frac{2t^{2 + \alpha}}{\Gamma(3 + \alpha)} \Big), \quad w(t) \equiv 0.$$

Then the conclusion of Theorem 3.4 can be verified in a similar way. □

Theorem 3.5. *If there exists* $c > 0$ *such that*

$$\max\left\{ f(t, x, y) \mid (t, x, y) \in [0, 1] \times \Big[0, \frac{c}{\Gamma(3 + \alpha)} \Big(\frac{1 + \alpha}{2}\Big)^{1 + \alpha} \Big] \times \Big[0, \frac{c}{4} \Big] \right\} \leq 2c,$$
$$\min\left\{ f(t, x, y) \mid (t, u, v) \in [0, 1] \times \Big[-\frac{c}{\Gamma(3 + \alpha)} \Big(\frac{1 + \alpha}{2}\Big)^{1 + \alpha}, 0 \Big] \times \Big[-\frac{c}{4}, 0 \Big] \right\} \geq -2c,$$

then (1.1) *has a solution* u^* *satisfying*

$$-c\Big(\frac{t^{\alpha}}{\Gamma(1 + \alpha)} - \frac{2t^{2 + \alpha}}{\Gamma(3 + \alpha)} \Big) \leq u^*(t) \leq c\Big(\frac{t^{\alpha}}{\Gamma(1 + \alpha)} - \frac{2t^{2 + \alpha}}{\Gamma(3 + \alpha)} \Big).$$

Proof. In Step 5 of the proof of Theorem 3.3, let

$$v(t) = -c\Big(\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}\Big), \quad w(t) = c\Big(\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}\Big).$$

Then the conclusion of Theorem 3.5 can be verified in a similar way. □

4 Example and remark

Example 4.1. Consider the following boundary value problem for the fractional differential equation:

$$\begin{cases} D_{0+}^{\frac{5}{2}} u(t) + \cos u(t) + \arctan(D_{0+}^{\frac{1}{2}} u(t)) = 0, \\ u(0) = 0, \ \ D_{0+}^{\frac{1}{2}} u(t)\big|_{t=0} = D_{0+}^{\frac{1}{2}} u(t)\big|_{t=1} = 0. \end{cases}$$

Let

$$f(t,x,y) = \cos x + \arctan y.$$

Then $f(t,0,0) > 0$ and f satisfies $(H_1 - H_2)$ with $\lambda_1 = 1$, $\lambda_2 = 0$, $\alpha = \frac{1}{2}$.

Furthermore, let $c = 4$, we have

$$\max\Big\{f(x,y) \mid \ (x,y) \in \Big[0, \frac{c}{\Gamma(3+\alpha)}\Big(\frac{1+\alpha}{2}\Big)^{1+\alpha}\Big] \times \Big[0, \frac{c}{4}\Big]\Big\} = 1 + \frac{\pi}{4} \le 2c.$$

Then Theorem 3.3 assures the above problem has a solution between 0 and

$$\frac{8t^{\frac{1}{2}}}{\sqrt{\pi}}\Big(1 - \frac{8t^2}{15}\Big).$$

Remark 4.2. By the proof of Theorem 3.3, we know that the solution of problem (3.3) can be obtained by iterative sequence $\{x_n\}$ or $\{y_n\}$, where

$$\begin{aligned} Lx_{n+1} &= N(x_n), \ \ x_0 = x, \ \ n = 0,1,2,\dots; \\ Ly_{n+1} &= N(y_n), \ \ y_0 = y, \ \ n = 0,1,2,\dots. \end{aligned}$$

This implies that the solution of problem (1.1) is computable.

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Authors' addresses:

Yuqiang Feng

1. School of Science, Wuhan University of Science and Technology, Wuhan 430065, Hubei, P.R. China.

2. Hubei Province Key Laboratory of Systems Science in Metallurgical Process, Wuhan 430065, Hubei, P.R. China.

E-mail: yqfeng6@126.com

Yuanyuan Wang, Deyi Li

School of Science, Wuhan University of Science and Technology, Wuhan 430065, Hubei, P.R. China.

E-mails: wangyuanyuan@wust.edu.cn; lixueyuan@wust.edu.cn

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Roland Gachechiladze

DYNAMICAL CONTACT PROBLEMS WITH REGARD TO FRICTION OF COUPLE-STRESS VISCOELASTICITY FOR INHOMOGENEOUS ANISOTROPIC BODIES

Abstract. The paper deals with the three-dimensional boundary-contact problems of couple-stress viscoelasticity for inhomogeneous anisotropic bodies with friction. The uniqueness theorem is proved by using the corresponding Green's formulas and positive definiteness of the potential energy. To analyze the existence of solutions, the problem under consideration is reduced equivalently to a spatial variational inequality. A special parameter-dependent regularization of this variational inequality is considered, which is equivalent to the relevant regularized variational equation depending on a real parameter, and its solvability is studied by the Faedo–Galerkin method. Some a priori estimates for solutions of the regularized variational equation are established and with the help of an appropriate limiting procedure the existence theorem for the original contact problem with friction is proved.

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Key words and phrases. Couple-stress elasticity theory, viscoelasticity, contact problem with friction, variational inequality, variational equation, Faedo–Galerkin method.

რეზიუმე. ნაშრომში განხილულია ბლანტი დრეკადობის მომენტური თეორიის დინამიკის სამ-განზომლებიანი სასაზღვრო-საკონტაქტო ამოცანა არაერთგვაროვანი, ანიზოტროპული სხეულისთვის ხახუნის ეფექტის გათვალისწინებით. შესწავლილია ამოცანის სუსტი ამონახსნის არსებობისა და ერთადერთობის საკითხი. ამონახსნის ერთადერთობის დადგენა ეფუძნება გრინის ფორმულებს და პოტენციალური ენერგიის დადებითად განსაზღვრულობას. ამონახსნის არსებობის შესწავლისთვის სასაზღვრო-საკონტაქტო ამოცანა ეკვივალენტურად დაიყვანება სივრცით ვარიაციულ უტოლობაზე, რომელიც, თავის მხრივ, ეკვივალენტურია მცირე პარამეტრ-ზე დამოკიდებული რეგულარიზებული განტოლების. ამ განტოლების ამონახსნის არსებობა შესწავლილია ფაედო-გალიორკინის მეთოდის მეშვეობით და მიღებულია ამონახსნის გარკვე-ული აპრიორული შეფასებები. ეს შეფასებები იძლევა ზღვარზე გადასვლის საშუალებას ჯერ განზომილების, ხოლო შემდეგ მცირე პარამეტრის მიმართ. და ბოლოს, ნაჩვენებია, რომ ზღვარითი ფუნქცია წარმოადგენს დასმული სასაზღვრო-საკონტაქტო ამოცანის ამონახსნს.

1 Introduction

The general and widespread use of the linear theory of viscoelasticity has been observed since the early seventies of the past century. Activity in this area is associated with a wide application of polymeric materials with properties that can obviously be described neither by elastic nor by viscous models, but combine the features of both models. Mathematical strictly grounded theory of linear viscoelasticity with numerous practical applications is contained in the monographs of D. R. Bland and R. M. Christensen (see [1,2] and the references therein).

Viscoelastic materials are those supplied with the "memory" in the sense that the state at time t depends on all the deformations that the material undergoes. A particularly important class of "viscoelastic equations of state" is associated with materials for which there is a linear relationship between the time derivatives of the stress and strain tensors. We will consider viscoelastic materials with short-term memory, i.e., when the stress of the moment at time t depends only on the deformations, the moment at time t and the nearest previous moments of time. In the considered model of the theory of elasticity, as distinct from the classical theory, every elementary medium particle undergoes both displacement and rotation. In this case, all mechanical values are expressed in terms of the displacement and rotation vectors. In their work [4], E. Cosserat and F. Cosserat created and presented the model of a solid medium in which every material point has six degrees of freedom, three of which are defined by the displacement components and the other three by the components of rotation (for the history of the model of elasticity see [6, 24, 27, 31] and the references therein). The main equations of that model are interrelated and generate a matrix second order differential operator of dimension 6×6. The basic boundary value problems and also the transmission problems of the hemitropic theory of elasticity for smooth and non-smooth Lipschitz domains were studied in [28]. The one-sided contact problems of statics of the hemitropic theory of elasticity, free from friction, were investigated in [11, 12, 16, 18, 21], and the contact problems of statics and dynamics with a friction were considered in [9, 10, 13–15, 17, 19, 20]. Analogous, one-sided problems of classical linear theory of elasticity have been considered in many works and monographs (see [5, 7, 8, 22, 23] and the references therein). Particular problems of the viscoelasticity theory are considered in [1, 2]. As for the dynamical and quasistatical boundary-contact problems of viscoelasticity with friction, we have considered them in [5].

The paper is organized as follows. First, we present general field equations of the linear theory of couple-stress viscoelasticity and formulate the boundary-contact problem of dynamics with regard to the friction. We prove the uniqueness theorem by using Green's formulas and positive definiteness of the potential energy. Afterwards, the contact problem is equivalently reduced to a spacial variational inequality. The latter is in its turn replaced by the relevant regularized equation depending on a real positive parameter ε, and its solvability is studied by the Faedo–Galerkin method in appropriate approximate function spaces of dimension m. Furthermore, some a priori estimates are established, which allow us to pass to the limit with respect to dimension m as $m \to \infty$ and to parameter ε as $\varepsilon \to 0$. As a result, we prove that the limiting function is a solution of the variational inequality and, consequently, the limiting function solves the original contact problem.

2 Field equations and Green's formulas

2.1 Basic equations

Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected domain with C^∞ smooth boundary $S := \partial\Omega$, $\overline{\Omega} = \Omega \cup S$. Throughout the paper, $n(x) = (n_1(x), n_2(x), n_3(x))$ denotes the outward unit normal vector at a point $x \in S$.

The basic equilibrium equations of dynamics of couple-stress viscoelasticity for inhomogeneous anisotropic bodies read as

$$\begin{aligned} \partial_i \sigma_{ij}(x,t) + \varrho F_j(x,t) &= \varrho \, \frac{\partial^2 u_j(x,t)}{\partial t^2}\,, \\ \partial_i \mu_{ij}(x,t) + \varepsilon_{ikj}\sigma_{ik}(x,t) + \varrho G_j(x,t) &= \mathcal{J} \, \frac{\partial^2 \omega_j(x,t)}{\partial t^2}\,, \end{aligned} \tag{2.1}$$

where t is the time variable, $\partial = (\partial_1, \partial_2, \partial_3)$ with $\partial_i = \frac{\partial}{\partial x_i}$, ϱ is the mass density of the elastic material, $\mathcal{J}$ is the moment of inertia per unit volume, $F = (F_1, F_2, F_3)^\top$ and $G = (G_1, G_2, G_3)^\top$ are, respectively, the body force and body couple vectors per unit mass, $u = (u_1, u_2, u_3)^\top$ is the *displacement vector*, $\omega = (\omega_1, \omega_2, \omega_3)^\top$ is the *micro-rotation vector*, ε_{ikj} is the permutation (Levi–Civita) symbol;

Here and in what follows, the symbol $(\,\cdot\,)^\top$ denotes transposition and the repetition of the index means summation over this index from 1 to 3. For the *force stress tensor* $\{\sigma_{ij}\}$ and the *couple-stress tensor* $\{\mu_{ij}\}$, we have

$$\begin{aligned}\sigma_{ij}(x,t) &:= \sigma_{ij}(U(t))\\ &= a^{(0)}_{ijlk}(x)\zeta_{lk}(U(t)) + b^{(0)}_{ijlk}(x)\eta_{lk}(U(t)) + a^{(1)}_{ijlk}(x)\partial_t\zeta_{lk}(U(t)) + b^{(1)}_{ijlk}(x)\partial_t\eta_{lk}(U(t)),\\ \mu_{ij}(x,t) &:= \mu_{ij}(U(t))\\ &= b^{(0)}_{ijlk}(x)\zeta_{lk}(U(t)) + c^{(0)}_{ijlk}(x)\eta_{lk}(U(t)) + b^{(1)}_{ijlk}(x)\partial_t\zeta_{lk}(U(t)) + c^{(1)}_{ijlk}(x)\partial_t\eta_{lk}(U(t)),\end{aligned}$$

where $U(t) := U(x,t) = (u(x,t), \omega(x,t))^\top$, $\zeta_{lk}(U(t)) = \partial_l u_k(x,t) - \varepsilon_{lkm}\omega_m(x,t)$ and $\eta_{lk}(U(t)) = \partial_l \omega_k(x,t)$ are the so-called strain and torsion (curvature) tensors; the real-valued functions $a^{(0)}_{ijlk}$, $b^{(0)}_{ijlk}$, $c^{(0)}_{ijlk}$ (respectively, $a^{(1)}_{ijlk}$, $b^{(1)}_{ijlk}$, $c^{(1)}_{ijlk}$), called the elastic constants (respectively, viscosity constants), satisfy certain smoothness and symmetry conditions

(i) $a^{(q)}_{ijlk}, b^{(q)}_{ijlk}, c^{(q)}_{ijlk} \in C^1(\overline{\Omega})$,

(ii) $a^{(q)}_{ijlk} = a^{(q)}_{lkij}$, $c^{(q)}_{ijlk} = c^{(q)}_{lkij}$,

(iii) there exists $\alpha_0 > 0$ such that $\forall\, x \in \overline{\Omega}$ and $\forall\, \xi_{ij}, \eta_{ij} \in R$:

$$a^{(q)}_{ijlk}(x)\xi_{ij}\xi_{lk} + 2b^{(q)}_{ijlk}(x)\xi_{ij}\eta_{lk} + c^{(q)}_{ijlk}(x)\eta_{ij}\eta_{lk} \geq \alpha_0(\xi_{ij}\xi_{ij} + \eta_{ij}\eta_{ij}) \ \ (q = 0, 1).$$

We introduce a matrix differential operator corresponding to the left-hand side of system (2.1):

$$\mathcal{M}(x,\partial) = \begin{bmatrix} \mathcal{M}^{(1)}(x,\partial) & \mathcal{M}^{(2)}(x,\partial)\\ \mathcal{M}^{(3)}(x,\partial) & \mathcal{M}^{(4)}(x,\partial)\end{bmatrix}_{6\times 6}, \quad \mathcal{M}^{(p)}(x,\partial) = \Big[\mathcal{M}^{(p)}_{jk}(x,\partial)\Big]_{3\times 3}, \ \ p = \overline{1,4},$$

where

$$\begin{aligned}\mathcal{M}^{(1)}_{jk}(x,\partial) &= \partial_i\big(\big[a^{(0)}_{ijlk}(x) + a^{(1)}_{ijlk}(x)\partial_t\big]\partial_l\big),\\ \mathcal{M}^{(2)}_{jk}(x,\partial) &= \partial_i\big(\big[b^{(0)}_{ijlk}(x) + b^{(1)}_{ijlk}(x)\partial_t\big]\partial_l\big) - \varepsilon_{lrk}\partial_i\big[a^{(0)}_{ijlr}(x) + a^{(1)}_{ijlr}(x)\partial_t\big];\\ \mathcal{M}^{(3)}_{jk}(x,\partial) &= \partial_i\big(\big[b^{(0)}_{lkij}(x) + b^{(1)}_{lkij}(x)\partial_t\big]\partial_l\big) + \varepsilon_{irj}\big[a^{(0)}_{irlk}(x) + a^{(1)}_{irlk}(x)\partial_t\big]\partial_l;\\ \mathcal{M}^{(4)}_{jk}(x,\partial) &= \partial_i\big(\big[c^{(0)}_{ijlk}(x) + c^{(1)}_{ijlk}(x)\partial_t\big]\partial_l\big) - \varepsilon_{lrk}\partial_i\big[b^{(0)}_{lrij}(x) + b^{(1)}_{lrij}(x)\partial_t\big]\\ &\qquad + \varepsilon_{irj}\big[b^{(0)}_{irlk}(x) + b^{(1)}_{irlk}(x)\partial_t\big]\partial_l - \varepsilon_{ipj}\varepsilon_{lrk}\big[a^{(0)}_{iplr}(x) + a^{(1)}_{iplr}(x)\partial_t\big].\end{aligned}$$

Denote by $\mathcal{N}(\partial, n)$ the generalized 6×6 matrix differential stress operator

$$\mathcal{N}(\partial,n) = \begin{bmatrix} \mathcal{N}^{(1)}(\partial,n) & \mathcal{N}^{(2)}(\partial,n)\\ \mathcal{N}^{(3)}(\partial,n) & \mathcal{N}^{(4)}(\partial,n)\end{bmatrix}_{6\times 6}, \quad \mathcal{N}^{(p)}(\partial,n) = \Big[\mathcal{N}^{(p)}_{jk}(\partial,n)\Big]_{3\times 3}, \ \ p = \overline{1,4},$$

where

$$\begin{aligned}\mathcal{N}^{(1)}_{jk}(\partial,n) &= \big[a^{(0)}_{ijlk} + a^{(1)}_{ijlk}\partial_t\big]n_i\partial_l;\\ \mathcal{N}^{(2)}_{jk}(\partial,n) &= \big[b^{(0)}_{ijlk} + b^{(1)}_{ijlk}\partial_t\big]n_i\partial_l - \varepsilon_{lrk}\big[a^{(0)}_{ijlr} + a^{(1)}_{ijlr}\partial_t\big]n_i;\\ \mathcal{N}^{(3)}_{jk}(\partial,n) &= \big[b^{(0)}_{lkij} + b^{(1)}_{lkij}\partial_t\big]n_i\partial_l;\\ \mathcal{N}^{(4)}_{jk}(\partial,n) &= \big[c^{(0)}_{ijlk} + c^{(1)}_{ijlk}\partial_t\big]n_i\partial_l - \varepsilon_{lrk}\big[b^{(0)}_{lrij} + b^{(1)}_{lrij}\partial_t\big]n_i.\end{aligned} \tag{2.2}$$

Here $\partial_n = \partial/\partial n$ denotes the directional derivative along the vector n (normal derivative). In the sequel, for the force stress and couple-stress vectors we use the following notation:

$$\mathcal{T}U = \mathcal{N}^{(1)}u + \mathcal{N}^{(2)}\omega, \quad MU = \mathcal{N}^{(3)}u + \mathcal{N}^{(4)}\omega,$$

where $\mathcal{N}^{(p)}$, $p = 1, 2, 3, 4$, is defined by formula (2.2).

The system of equations (2.1) can be rewritten in the matrix form

$$\mathcal{M}(x,\partial)U(x,t) + \mathcal{G}(x,t) = P\,\frac{\partial^2 U(x,t)}{\partial t^2}\,, \;\; x\in\Omega, \;\; 0<t<T, \tag{2.3}$$

where T is an arbitrary positive number, $U = (u,\omega)^\top$, $\mathcal{G} = (\varrho F, \varrho G)^\top$, $P = [p_{ij}]_{6\times 6}$, $p_{ii} = \varrho$, when $i = 1, 2, 3$, $p_{ii} = \mathcal{J}$, when $i = 4, 5, 6$, and $p_{ij} = 0$, when $i \neq j$.

Throughout the paper, $L_p(\Omega)$ $(1 \le p \le \infty)$, $L_2(\Omega) = H^0(\Omega)$ and $H^s(\Omega) = H^s_2(\Omega)$, $s \in \mathbb{R}$, denote the Lebesgue and Bessel potential spaces (see, e.g., [25, 32]). We denote the corresponding norms by the symbols $\|\cdot\|_{L_p(\Omega)}$ and $\|\cdot\|_{H^s(\Omega)}$, respectively. Denote by $D(\Omega)$ the class of $C^\infty(\Omega)$ functions with a support in the domain Ω. If M is an open proper part of the manifold $\partial\Omega$, i.e., $M \subset \partial\Omega$, $M \neq \partial\Omega$: then we denote by $H^s(M)$ the restriction of the space $H^s(\partial\Omega)$ on M,

$$H^s(M) := \big\{ r_{_M}\varphi : \; \varphi \in H^s(\partial\Omega) \big\},$$

where $r_{_M}$ stands for the restriction operator on the set M. Further, let

$$\widetilde{H}^s(M) := \big\{ \varphi \in H^s(\partial\Omega) : \; \operatorname{supp}\varphi \subset \overline{M} \big\}.$$

The total strain energy of the respective media has the form

$$\mathcal{B}^{(q)}(U,V) = \int\limits_{\Omega} \Big\{ a^{(q)}_{ijlk}(x)\zeta_{ij}(U)\zeta_{lk}(V) + b^{(q)}_{ijlk}(x)\zeta_{ij}(U)\eta_{lk}(V) + b^{(q)}_{ijlk}(x)\zeta_{ij}(V)\eta_{lk}(U) + c^{(q)}_{ijlk}(x)\eta_{ij}(U)\eta_{lk}(V) \Big\}\, dx,$$

where $q = 1, 2$, $U = (u,\omega)^\top$, $V = (v,w)^\top$ and $\zeta_{ij}(U) = \partial_i u_j - \varepsilon_{ijr}\omega_r$, $\eta_{ij}(U) = \partial_i \omega_j$.

From properties (ii) and (iii), it is clear that $\mathcal{B}^{(q)}(U,V) = \mathcal{B}^{(q)}(V,U)$ and $\mathcal{B}^{(q)}(U,U) \ge 0$. Moreover, there exist positive constants C_1 and C_2, depending only on the material parameters, such that Korn's type inequality (cf., [8, Part I, § 12], [3, § 6.3])

$$\mathcal{B}^{(q)}(U,U) \ge C_1\|U\|^2_{[H^1(\Omega)]^6} - C_2\|U\|^2_{[L_2(\Omega)]^6}, \;\; q = 1, 2, \tag{2.4}$$

holds for an arbitrary real-valued vector function $U \in [H^1(\Omega)]^6$.

Remark 2.1. If $U \in [H^1(\Omega)]^6$ and on some open part $S^* \subset \partial\Omega$ the trace $\{U\}^+$ vanishes, i.e., $r_{_{S^*}}\{U\}^+ = 0$, then we have the strict Korn's inequality

$$\mathcal{B}^{(q)}(U,U) \ge c\|U\|^2_{[H^1(\Omega)]^6}$$

with some positive constant $c > 0$ which does not depend on the vector U. This follows from (2.4) and the fact that in this case $\mathcal{B}^{(q)}(U,U) > 0$ for $U \neq 0$ (see [29], [26, Ch. 2, Exercise 2.17]).

2.2 Green's formulas

For the real-valued vector functions $U(t) = (u(t),\omega(t))^\top$ and $\widetilde{U}(t) = (\widetilde{u}(t),\widetilde{\omega}(t))^\top$ of the class $[C^2(\overline{\Omega})]^6$ and for an arbitrary $t \in [0;T]$, the following Green's formula (see [13])

$$\int\limits_{\Omega} \mathcal{M}(x,\partial)U(t)\cdot\widetilde{U}(t)\,dx = \int\limits_{S} \big\{\mathcal{N}(\partial,n)U(t)\big\}^+ \cdot \{\widetilde{U}(t)\}^+\,dS - \Big\{\mathcal{B}^{(0)}(U(t),\widetilde{U}(t)) + \partial_t\mathcal{B}^{(1)}(U(t),\widetilde{U}(t))\Big\} \tag{2.5}$$

holds, where $\{\,\cdot\,\}^+$ denotes the trace operator on S from Ω.

By the standard limiting arguments, Green's formula (2.5) can be extended to the Lipschitz domains and to vector functions $U, \widetilde{U} \in [H^1(\Omega)]^6$ with $\mathcal{M}(x,\partial)U(t) \in [L_2(\Omega)]^6$ (see [25, 29]),

$$\int_\Omega \mathcal{M}(x,\partial)U(t)\cdot \widetilde{U}(t)\,dx = \Big\langle \big\{\mathcal{N}(\partial,n)U(t)\big\}^+ \cdot \{\widetilde{U}(t)\}^+\Big\rangle_s\, dS \\ - \Big\{\mathcal{B}^{(0)}(U(t),\widetilde{U}(t)) + \partial_t \mathcal{B}^{(1)}(U(t),\widetilde{U}(t))\Big\},\ \ t\in(0;T), \quad (2.6)$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_S$ denotes the duality between the spaces $[H^{-1/2}(S)]^6$ and $[H^{1/2}(S)]^6$, which generalizes the usual inner product in the space $[L_2(\partial\Omega)]^6$. By this relation, the generalized trace of the stress operator $\{\mathcal{N}(\partial,n)U\}^+ \in [H^{-1/2}(S)]^6$ is well defined.

The following assertion describes the null space of the energy quadratic form $\mathcal{B}^{(q)}(U(t),U(t))$ (see [13]).

Lemma 2.2. *Let for an arbitrary $t\in(0;T)$, $U(t) = (u(t),\omega(t))^\top \in [C^1(\overline{\Omega})]^6$ and $\mathcal{B}^{(q)}(U(t),U(t)) = 0$ in Ω. Then*

$$u(t) = [a^{(q)} \times x] + b^{(q)},\ \ \omega(t) = a^{(q)},\ \ x\in\Omega,$$

where $a^{(q)}$ and $b^{(q)}$ are arbitrary three-dimensional constant vectors and the symbol $[\,\cdot\times\cdot\,]$ denotes the cross product of two vectors.

The vectors of type $([a^{(q)}\times x]+b^{(q)}, a^{(q)})$ are called *generalized rigid displacement vectors*. Observe that a generalized rigid displacement vector vanishes, i.e., $a^{(q)} = b^{(q)} = 0$, if it is zero at a single point.

3 Contact problems with friction

3.1 Coulomb's law

Let the boundary S of the domain Ω be divided into two open, connected and non-overlapping parts S_1 and S_2 of positive measure, $S = \overline{S_1}\cup\overline{S_2}$, $S_1\cap S_2 = \varnothing$. Assume that the viscoelastic body occupying the domain Ω is in a contact with another rigid body along the subsurface S_2. Denote by $F(x,t)$ the force stress vector by which the hemitropic body acts upon the rigid body at the point $x\in S_2$. Throughout the paper, F_n and F_s stand for the normal and tangential components of the vector F, respectively: $F_n = F\cdot n$ and $F_s = F - (F\cdot n)\,n$. Further, let $\mathcal{F}(x)$ be the *friction coefficient* at the point $x\in S_2$. It is a nonnegative scalar function which depends on the geometry of the contacting surfaces and also on the physical properties of the interacting materials.

Coulomb's law describing the contact interaction of materials with friction reads as follows (for details see [5]):

If the contact of two bodies is described by the force vector F, then

$$|F_s(x,t)| \le \mathcal{F}(x)|F_n(x,t)|\,.$$

Moreover, if

$$|F_s(x,t)| < \mathcal{F}(x)|F_n(x,t)|,$$

then

$$\frac{\partial u_s(x,t)}{\partial t} = 0,$$

and if

$$|F_s(x,t)| = \mathcal{F}(x)|F_n(x,t)|,$$

then there exist nonnegative functions λ_1 and λ_2 not vanish simultaneously such that

$$\lambda_1(x,t)\,\frac{\partial u_s(x,t)}{\partial t} = -\lambda_2(x,t)F_s(x,t).$$

3.2 Pointwise and variational formulation of the contact problem

Let X be a Banach space with the norm $\|\cdot\|_X$. We denote by $L_p(0,T;X)$ $(1\le p\le\infty)$ the space of measurable functions $t\mapsto f(t)$ defined on the interval $(0;T)$ with values in the space X such that

$$\|f\|_{L_p(0,T;X)} := \bigg\{\int\limits_0^T \|f(t)\|_X^p\,dt\bigg\}^{1/p} < \infty \ \text{ for } \ 1\le p<\infty$$

and

$$\|f\|_{L_\infty(0,T;X)} := \operatorname*{ess\,sup}_{t\in(0;T)} \big\{\|f(t)\|_X\big\} < \infty \ \text{ for } \ p=\infty.$$

Definition 3.1. The vector-function $U:(0;T)\to[H^1(\Omega)]^6$ is said to be a weak solution of equation (2.3) for $\mathcal{G}:(0;T)\to[L_2(\Omega)]^6$ if

$$U(t),U'(t)\in L_\infty(0,T;[H^1(\Omega)]^6),\ \ U''(t)\in L_\infty(0,T;[L_2(\Omega)]^6),$$

and for every $\Phi\in[\mathcal{D}(\Omega)]^6$,

$$(PU''(t),\Phi)+\mathcal{B}^{(0)}(U(t),\Phi)+\mathcal{B}^{(1)}(U'(t),\Phi)=(\mathcal{G}(t),\Phi).$$

Here and in what follows, the symbol $(\,\cdot\,,\,\cdot\,)$ denotes the scalar product in the space $L_2(\Omega)$.
Further, let

$$\mathcal{G}:(0,T)\to[L_2(\Omega)]^6,\quad \varphi:(0;T)\to[H^{-1/2}(S_2)]^3,\quad f:(0;T)\to L_\infty(S_2),$$

and set

$$g:=\mathcal{F}|f|\ge 0. \tag{3.1}$$

Consider the following contact problem of dynamics with friction.

Problem (A_0)**.** Find a weak solution $U:(0;T)\to[H^1(\Omega)]^6$ of the equation

$$\mathcal{M}(x,\partial)U(x,t)+\mathcal{G}(x,t)=P\,\frac{\partial^2U(x,t)}{\partial t^2}\,,\ \ x\in\Omega,\ \ t\in(0;T), \tag{3.2}$$

satisfying the inclusion $r_{S_2}\{(\mathcal{T}U)_s\}^+\in[L_\infty(S_2\times(0;T))]^3$, the initial conditions

$$U(x,0)=0,\ \ x\in\Omega, \tag{3.3}$$

$$U'(x,0)=0,\ \ x\in\Omega, \tag{3.4}$$

and the boundary contact conditions

$$r_{S_1}\{U\}^+=0\ \text{ on }\ S_1\times(0;T), \tag{3.5}$$

$$r_{S_2}\{(\mathcal{T}U)_n\}^+=f\ \text{ on }\ S_2\times(0;T), \tag{3.6}$$

$$r_{S_2}\{MU\}^+=\varphi\ \text{ on }\ S_2\times(0;T), \tag{3.7}$$

$$r_{S_2}\Big\{\frac{\partial u_s}{\partial t}\Big\}^+=0\ \text{ if }\ \big|r_{S_2}\{(\mathcal{T}U)_s\}^+\big|<g\ \text{ on }\ S_2\times(0;T), \tag{3.8}$$

and if $|r_{S_2}\{(\mathcal{T}U)_s\}^+|=g$, then there exist nonnegative functions λ_1 and λ_2 do not vanishing simultaneously, such that

$$\lambda_1(x,t)r_{S_2}\Big\{\frac{\partial u_s}{\partial t}\Big\}^+=-\lambda_2(x,t)\,r_{S_2}\{(\mathcal{T}U)_s\}^+\ \text{ on }\ S_2\times(0;T). \tag{3.9}$$

This problem can be reformulated in terms of a variational inequality. To this end, on the space $[H^1(\Omega)]^6$ we introduce the continuous convex functional

$$j(V)=\int\limits_{S_2} g|\{v_s\}^+|\,dS,\quad V=(v,w)^\top:(0;T)\to[H^1(\Omega)]^6 \tag{3.10}$$

and the closed convex sets $\mathcal{K}$ and $\mathcal{K}_0$:

$$\mathcal{K} := \Big\{ V \mid \ V(t), V'(t) \in L_\infty(0,T;[H^1(\Omega)]^6), \\ V''(t) \in L_\infty(0,T;[L_2(\Omega)]^6), \ r_{_{S_1}}\{V\}^+ = 0, \ V(0) = V'(0) = 0 \Big\};$$

$$\mathcal{K}_0 := \Big\{ V \mid \ V \in [H^1(\Omega)]^6, \ r_{_{S_1}}\{V\}^+ = 0 \Big\}.$$

Consider the following variational inequality: Find a $(u,\omega)^\top \in \mathcal{K}$ such that the variational inequality

$$(PU''(t), V - U'(t)) + \mathcal{B}^{(0)}(U(t), V - U'(t)) + \mathcal{B}^{(1)}(U'(t), V - U'(t)) + j(V) - j(U'(t)) \\ \geq (\mathcal{G}(t), V - U'(t)) + \int\limits_{S_2} f(t)\{v_n - u_n'(t)\}^+ \, dS + \big\langle \varphi(t), r_{_{S_2}}\{w - \omega'(t)\}^+ \big\rangle_{S_2} \tag{3.11}$$

holds for all $V = (v,w)^\top \in \mathcal{K}_0$.

Here and in what follows, the symbol $\langle \cdot \, , \, \cdot \rangle$ denotes the duality relation between the corresponding dual pairs $X^*(M)$ and $X(M)$. In particular, $\langle \cdot \, , \, \cdot \rangle_{S_2}$ in (3.11) denotes the duality relation between the spaces $[H^{-1/2}(S_2)]^3$ and $[\widetilde{H}^{1/2}(S_2)]^3$.

4 Equivalence theorem

Here we prove the following equivalence result.

Theorem 4.1. *If $U : (0;T) \to [H^1(\Omega)]^6$ is a solution of problem (A_0), then U is a solution of the variational inequality* (3.11)*, and vice versa.*

Proof. Let $U = (u,\omega)^\top : (0;T) \to [H^1(\Omega)]^6$ be a solution of problem (A_0), and $V = (v,w)^\top \in \mathcal{K}_0$. By virtue of the interior regularity theorems (see [8]), we have $U(t) \in [H^2(\Omega')]^6$ for every domain $\overline{\Omega'} \subset \Omega$. Hence the equation

$$\mathcal{M}(x,\partial)U(x,t) + \mathcal{G}(x,t) = P\,\frac{\partial^2 U(x,t)}{\partial t^2}\,, \ \ x \in \Omega, \ \ t \in (0;T)$$

holds almost everywhere in the domain Ω. By virtue of Green's formula (2.6), we get

$$(PU''(t), V - U'(t)) - \big\langle \{\mathcal{T}U\}^+, \{v - u'(t)\}^+ \big\rangle_S - \big\langle \{MU\}^+, \{w - \omega'(t)\}^+ \big\rangle_S \\ + \mathcal{B}^{(0)}(U(t), V - U'(t)) + \mathcal{B}^{(1)}(U'(t), V - U'(t)) = (\mathcal{G}(t), V - U'(t)). \tag{4.1}$$

Taking into account the boundary conditions (3.5), (3.6), (3.7) and the form of the functional (3.10), we deduce that for all $V = (v,w)^\top \in \mathcal{K}_0$ from (4.1), we have

$$(PU''(t), V - U'(t)) + \mathcal{B}^{(0)}(U(t), V - U'(t)) + \mathcal{B}^{(1)}(U'(t), V - U'(t)) + j(V) - j(U'(t)) \\ = (\mathcal{G}(t), V - U'(t)) + \int\limits_{S_2} f(t)\{v_n - u_n'(t)\}^+ \, dS + \big\langle \varphi(t), r_{_{S_2}}\{w - \omega'(t)\}^+ \big\rangle_{S_2} \\ + \int\limits_{S_2} \Big[\{(\mathcal{T}U)_s\}^+ \cdot \{v_s - u_s'(t)\}^+ + g\big(|\{v_s\}^+| - |\{u_s'(t)\}^+|\big) \Big] \, dS.$$

It is easy to see that if conditions (3.8) and (3.9) hold, then

$$r_{_{S_2}}\{(\mathcal{T}U)_s\}^+ \cdot r_{_{S_2}}\{v_s - u_s'(t)\}^+ + g\big(|r_{_{S_2}}\{v_s\}^+| - |r_{_{S_2}}\{u_s'(t)\}^+|\big) \geq 0.$$

Hence we have

$$(PU''(t), V - U'(t)) + \mathcal{B}^{(0)}(U(t), V - U'(t)) + \mathcal{B}^{(1)}(U'(t), V - U'(t)) + j(V) - j(U'(t))$$
$$\geq (\mathcal{G}(t), V - U'(t)) + \int\limits_{S_2} f(t)\{v_n - u_n'(t)\}^+ \, dS + \big\langle \varphi(t), r_{S_2}\{w - \omega'(t)\}^+ \big\rangle_{S_2}$$

for all $V = (v, w)^\top \in \mathcal{K}_0$. Thus $U = (u, \omega)^\top : (0;T) \to [H^1(\Omega)]^6$ is a solution of the variational inequality (3.11).

Let now $U = (u, \omega)^\top \in \mathcal{K}$ be a solution of the variational inequality (3.11). Substituting $U'(t) \pm \Phi$ instead of V in (3.11) with an arbitrary $\Phi \in [\mathcal{D}(\Omega)]^6$, we obtain

$$(PU''(t), \Phi) + \mathcal{B}^{(0)}(U(t), \Phi) + \mathcal{B}^{(1)}(U'(t), \Phi) = (\mathcal{G}(t), \Phi) \quad \forall\, \Phi \in [\mathcal{D}(\Omega)]^6,$$

which implies that U is a weak solution of equation (3.2). Again, by virtue of the interior regularity theorem (see [8]), equation (3.2) is satisfied almost everywhere in the domain Ω. Thus, taking into account the fact that $r_{S_1}\{V - U'(t)\}^+ = 0$ for all $V = (v, w)^\top \in \mathcal{K}_0$, Green's formula (2.6) yields

$$(PU''(t), V - U'(t)) + \mathcal{B}^{(0)}(U(t), V - U'(t)) + \mathcal{B}^{(1)}(U'(t), V - U'(t))$$
$$= (\mathcal{G}(t), V - U'(t)) + \Big\langle r_{S_2}\{(\mathcal{T}U)_n\}^+, r_{S_2}\{v_n - u_n'(t)\}^+ \Big\rangle_{S_2}$$
$$+ \Big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{v_s - u_s'(t)\}^+ \Big\rangle_{S_2} + \Big\langle r_{S_2}\{MU\}^+, r_{S_2}\{w - \omega'(t)\}^+ \Big\rangle_{S_2} \quad \forall\, V \in \mathcal{K}_0.$$

Subtracting the above equality from (3.11), we obtain

$$\Big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{v_s - u_s'(t)\}^+ \Big\rangle_{S_2} + \int\limits_{S_2} g\big(|\{v_s\}^+| - |\{u_s'(t)\}^+|\big)\, dS$$
$$+ \Big\langle r_{S_2}\{(\mathcal{T}U)_n\}^+ - f(t), r_{S_2}\{v_n - u_n'(t)\}^+ \Big\rangle_{S_2} + \Big\langle r_{S_2}\{MU\}^+ - \varphi(t), r_{S_2}\{w - \omega'(t)\}^+ \Big\rangle_{S_2} \geq 0 \tag{4.2}$$

for all $V = (v, w)^\top \in \mathcal{K}_0$. For an arbitrary t from the interval $(0;T)$, we choose $V = (v, w)^\top \in \mathcal{K}_0$ such that $r_{S_2}\{w\}^+ = r_{S_2}\{\omega'(t)\}^+$, $r_{S_2}\{v_s\}^+ = r_{S_2}\{u_s'(t)\}^+$, and $r_{S_2}\{v_n\}^+ = r_{S_2}[\{u_n'(t)\}^+ \pm \psi]$, where $\psi \in \widetilde{H}^{1/2}(S_2)$ is an arbitrary scalar function. Then from (4.2) we infer

$$r_{S_2}\{(\mathcal{T}U)_n\}^+ = f(t), \tag{4.3}$$

i.e., condition (3.6) is fulfilled. Taking into account (4.3), from (4.2) we find that

$$\Big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{v_s - u_s'(t)\}^+ \Big\rangle_{S_2} + \int\limits_{S_2} g\big(|\{v_s\}^+| - |\{u_s'(t)\}^+|\big)\, dS$$
$$+ \Big\langle r_{S_2}\{MU\}^+ - \varphi(t), r_{S_2}\{w - \omega'(t)\}^+ \Big\rangle_{S_2} \geq 0 \quad \forall\, V = (v, w)^\top \in \mathcal{K}_0. \tag{4.4}$$

Let now the vector-function $V = (v, w)^\top \in \mathcal{K}_0$ be such that $r_{S_2}\{v_s\}^+ = r_{S_2}\{u_s'(t)\}^+$ and $r_{S_2}\{w\}^+ = r_{S_2}[\{\omega'(t)\}^+ \pm \psi]$, where $\psi \in [\widetilde{H}^{1/2}(S_2)]^3$ is an arbitrary vector-function. Then (4.4) yields

$$r_{S_2}\{MU\}^+ = \varphi(t)\,. \tag{4.5}$$

Consequently, condition (3.7) is satisfied. Note that conditions (3.5), (3.3) and (3.4) are automatically fulfilled, since $U = (u, \omega)^\top \in \mathcal{K}$. Taking into account condition (4.5), from (4.4) we obtain

$$\Big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{v_s - u_s'(t)\}^+ \Big\rangle_{S_2} + \int\limits_{S_2} g\big(|\{v_s\}^+| - |\{u_s'(t)\}^+|\big)\, dS \geq 0 \quad \forall\, V = (v, w)^\top \in \mathcal{K}_0, \tag{4.6}$$

whence

$$\Big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{v_s-u_s'(t)\}^+\Big\rangle_{S_2} + \int\limits_{S_2} g\big|\{v_s\}^+-\{u_s'(t)\}^+\big|\,dS \geq 0 \quad \forall\, V=(v,w)^\top\in\mathcal{K}_0. \tag{4.7}$$

Further, let us choose the vector-function $V=(v,w)^\top\in\mathcal{K}_0$ such that $r_{S_2}\{w\}^+=r_{S_2}\{\omega'(t)\}^+$, $r_{S_2}\{v_n\}^+=r_{S_2}\{u_n'(t)\}^+$, and $r_{S_2}\{v_s\}^+=r_{S_2}\{u_s'(t)\}^+\pm r_{S_2}\psi_s$, where $\psi\in[\widetilde{H}^{1/2}(S_2)]^3$ is an arbitrary vector-function. Then from (4.7) we obtain

$$\pm\big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi_s\big\rangle_{S_2} + \int\limits_{S_2} g|\psi_s|\,dS \geq 0. \tag{4.8}$$

For an arbitrary $\psi\in[\widetilde{H}^{1/2}(S_2)]^3$, we have $|r_{S_2}\psi_s|\leq|r_{S_2}\psi|$ and

$$\big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi_s\big\rangle_{S_2} = \big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi\big\rangle_{S_2}.$$

Therefore, from (4.8) we derive

$$\Big|\big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi\big\rangle_{S_2}\Big| \leq \int\limits_{S_2} g|\psi|\,dS \quad \forall\,\psi\in[\widetilde{H}^{1/2}(S_2)]^3. \tag{4.9}$$

Let $t\in(0;T)$ and consider in the space $[\widetilde{H}^{1/2}(S_2)]^3$ the linear functional

$$\Phi_t(\psi)=\big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi\big\rangle_{S_2}, \;\; \psi\in[\widetilde{H}^{1/2}(S_2)]^3.$$

Due to inequality (4.9), this functional is continuous on the space $[\widetilde{H}^{1/2}(S_2)]^3$ with respect to the topology induced by the space $[L_1(S_2)]^3$. Since the space $[\widetilde{H}^{1/2}(S_2)]^3$ is dense in $[L_1(S_2)]^3$, the functional Φ_t can be continuously extended to the whole space $[L_1(S_2)]^3$ preserving the norm. Since the dual of $[L_1(S_2)]^3$ is isomorphic to $[L_\infty(S_2)]^3$, there exists a function $\Phi_t^*\in[L_\infty(S_2)]^3$ such that

$$\Phi_t(\psi)=\int\limits_{S_2}\Phi_t^*\cdot\psi\,dS \quad \forall\,\psi\in[L_1(S_2)]^3.$$

Hence

$$r_{S_2}\{(\mathcal{T}U)_s\}^+=\Phi_t^*\in[L_\infty(S_2)]^3.$$

Using again inequality (4.9) we derive

$$\int\limits_{S_2}\big[\pm\{(\mathcal{T}U)_s\}^+\cdot\psi-g|\psi|\big]\,dS\leq 0 \quad \forall\,\psi\in[\widetilde{H}^{1/2}(S_2)]^3, \tag{4.10}$$

whence the inequality

$$\big|r_{S_2}\{(\mathcal{T}U)_s\}^+\big|\leq g \;\text{ almost everywhere on }\; S_2\times(0;T)$$

follows. Indeed, it is well known that for an arbitrary essentially bounded function $\widetilde{\psi}\in L_\infty(S_2)$ there is a sequence $\widetilde{\varphi}_l\in C^\infty(S_2)$ with supports in S_2 for which (see [30, Lemma 1.4.2])

$$\lim_{l\to\infty}\widetilde{\varphi}_l(x)=\widetilde{\psi}(x) \;\text{ for almost all }\; x\in S_2 \;\text{ and }\; |\widetilde{\varphi}_l(x)|\leq\operatorname*{ess\,sup}_{y\in S_2}|\widetilde{\psi}(y)|$$

for almost all $x\in S_2$. Therefore, from inequality (4.10), by the Lebesgue dominated convergence theorem, it follows that

$$\int\limits_{S_2}\big[\pm\{(\mathcal{T}U)_s\}^+\cdot\psi-g|\psi|\big]\,dS\leq 0 \quad \forall\,\psi\in[L_\infty(S_2)]^3,$$

whence we get

$$\pm r_{S_2}\{(\mathcal{T}U)_s\}^+\cdot\psi - g|\psi| \le 0$$

on S_2 for every $\psi\in[L_\infty(S_2)]^3$. Substituting $\psi = r_{S_2}\{(\mathcal{T}U)_s\}^+$ in the above inequality, we finally get the inequality

$$\big|r_{S_2}\{(\mathcal{T}U)_s\}^+\big| \le g. \tag{4.11}$$

Now let us set

$$\vartheta_s := r_{S_2}\{v_s\}^+, \quad \vartheta_{0s} := r_{S_2}\{u_s'(t)\}^+. \tag{4.12}$$

Clearly, $\vartheta_s, \vartheta_{0s}\in[H^{1/2}(S_2)]^3$. Due to the inclusion

$$r_{S_2}\{(\mathcal{T}U)_s\}^+ \in \big[L_2(S_2\times(0;T))\big]^3,$$

from (4.6) we get

$$\big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+,\vartheta_s\big\rangle_{S_2} + \int\limits_{S_2} g|\vartheta_s|\,dS - \big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+,\vartheta_{0s}\big\rangle_{S_2} - \int\limits_{S_2} g|\vartheta_{0s}|\,dS \ge 0. \tag{4.13}$$

Let $\psi\in[H^{1/2}(S_2)]^3$ be an arbitrary vector-function. Substitute in (4.13) $\vartheta_s = q\psi$ for a nonnegative number $q\ge 0$, and take into consideration that $|\psi_s|\le|\psi|$ and $r_{S_2}\{(\mathcal{T}U)_s\}^+\cdot\psi_s = r_{S_2}\{(\mathcal{T}U)_s\}^+\cdot\psi$ to obtain

$$q\int\limits_{S_2}\big[\{(\mathcal{T}U)_s\}^+\cdot\psi + g|\psi|\,\big]\,dS - \int\limits_{S_2}\big[\{(\mathcal{T}U)_s\}^+\cdot\vartheta_{0s} + g|\vartheta_{0s}|\,\big]\,dS \ge 0.$$

Sending q to 0, we arrive at the inequality

$$\int\limits_{S_2}\big[\{(\mathcal{T}U)_s\}^+\cdot\vartheta_{0s} + g|\vartheta_{0s}|\,\big]\,dS \le 0,$$

whence by (4.11) and (4.12) we arrive at the equation

$$r_{S_2}\{(\mathcal{T}U)_s\}^+\cdot r_{S_2}\{u_s'(t)\}^+ + g\big|r_{S_2}\{u_s'(t)\}^+\big| = 0. \tag{4.14}$$

Clearly, if $|r_{S_2}\{(\mathcal{T}U)_s\}^+|<g$, then it follows from (4.14) that $r_{S_2}\{u_s'(t)\}^+=0$. But if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| = g$, then (4.14) can be rewritten in the form

$$g\big|r_{S_2}\{u_s'(t)\}^+\big|(\cos\alpha+1) = 0 \ \text{ on } \ S_2\times(0;T),$$

where α is the angle lying between the vectors $r_{S_2}\{u_s'(t)\}^+$ and $r_{S_2}\{(\mathcal{T}U)_s\}^+$ at the point $x\in S_2$. Consequently, there exist the functions λ_1 and λ_2 such that $\lambda_1(x,t)+\lambda_2(x,t)>0$ and

$$\lambda_1(x,t)\,r_{S_2}\{u_s'(t)\}^+ = -\lambda_2(x,t)r_{S_2}\{(\mathcal{T}U)_s\}^+ \ \text{ on } \ S_2\times(0;T).$$

Moreover, we may assume that λ_1 belongs to the same class as $\{(\mathcal{T}U)_s\}^+$, while λ_2 belongs to the same class as $\{u_s'(t)\}^+$. This completes the proof. □

5 The uniqueness theorem

We start the investigation of the variational inequality (3.11) with the following uniqueness result.

Theorem 5.1. *The variational inequality* (3.11) *and hence Problem* (A_0) *have at most one weak solution.*

Proof. Let $U=(u,\omega)^\top\in\mathcal{K}$ and $\widetilde{U}=(\widetilde{u},\widetilde{\omega})^\top\in\mathcal{K}$ be two solutions of inequality (3.11). Substituting in (3.11) $\widetilde{U}'(t)$ instead of V, we obtain

$$\begin{aligned}\big(PU''(t),\widetilde{U}'(t)-U'(t)\big)+\mathcal{B}^{(0)}\big(U(t),\widetilde{U}'(t)-U'(t)\big)+\mathcal{B}^{(1)}\big(U'(t),\widetilde{U}'(t)-U'(t)\big)+j(\widetilde{U}'(t))-j(U'(t))\\ \geq(\mathcal{G}(t),\widetilde{U}'(t)-U'(t))+\int\limits_{S_2}f(t)\big\{\widetilde{u}_n'(t)-u_n'(t)\big\}^+\,dS+\big\langle\varphi(t),r_{S_2}\{\widetilde{\omega}'(t)-\omega'(t)\}^+\big\rangle_{S_2}.\end{aligned}\tag{5.1}$$

Analogously, substituting $U(t)=\widetilde{U}(t)$ and $V=U'(t)$ in (3.11), we get

$$\begin{aligned}\big(P\widetilde{U}''(t),U'(t)-\widetilde{U}'(t)\big)+\mathcal{B}^{(0)}\big(\widetilde{U}(t),U'(t)-\widetilde{U}'(t)\big)+\mathcal{B}^{(1)}\big(\widetilde{U}'(t),U'(t)-\widetilde{U}'(t)\big)+j(U'(t))-j(\widetilde{U}'(t))\\ \geq(\mathcal{G}(t),U'(t)-\widetilde{U}'(t))+\int\limits_{S_2}f(t)\big\{u_n'(t)-\widetilde{u}_n'(t)\big\}^+\,dS+\big\langle\varphi(t),r_{S_2}\{\omega'(t)-\widetilde{\omega}'(t)\}^+\big\rangle_{S_2}.\end{aligned}\tag{5.2}$$

Combining (5.1) and (5.2) and denoting the difference $U(t)-\widetilde{U}(t)$ by $W(t)$, we obtain

$$-(PW''(t),W'(t))-\mathcal{B}^{(0)}(W(t),W'(t))-\mathcal{B}^{(1)}(W'(t),W'(t))\geq 0,\tag{5.3}$$

Note that

$$(PW''(t),W'(t))=\frac{1}{2}\,\frac{d}{dt}\Big(\sqrt{P}\,W'(t),\sqrt{P}\,W'(t)\Big)=\frac{1}{2}\,\frac{d}{dt}\Big[\big\|\sqrt{P}\,W'(t)\big\|^2_{[L_2(\Omega)]^6}\Big]$$

and

$$\mathcal{B}^{(0)}(W(t),W'(t))=\frac{1}{2}\,\frac{d}{dt}\,\mathcal{B}^{(0)}(W(t),W(t)),$$

where $\sqrt{P}=[\sqrt{p_{ij}}]_{6\times 6}$ with $\sqrt{p_{ii}}=\sqrt{\varrho}$ for $i=1,2,3$, $\sqrt{p_{ii}}=\sqrt{\mathcal{J}}$ for $i=4,5,6$, and $p_{ij}=0$ if $i\neq j$.

Then, from (5.3) we get

$$\frac{1}{2}\,\frac{d}{dt}\Big\{\big\|\sqrt{P}\,W'(t)\big\|^2_{[L_2(\Omega)]^6}+\mathcal{B}^{(0)}(W(t),W(t))\Big\}+\mathcal{B}^{(1)}(W'(t),W'(t))\leq 0.\tag{5.4}$$

Since $\mathcal{B}^{(1)}(W'(t),W'(t))$ is nonnegative, (5.4) can be rewritten as

$$\frac{1}{2}\,\frac{d}{dt}\Big\{\big\|\sqrt{P}\,W'(t)\big\|^2_{[L_2(\Omega)]^6}+\mathcal{B}^{(0)}(W(t),W(t))\Big\}\leq 0.\tag{5.5}$$

On the basis of (5.5), we can conclude that the scalar function

$$\big\|\sqrt{P}\,W'(t)\big\|^2_{[L_2(\Omega)]^6}+\mathcal{B}^{(0)}(W(t),W(t))$$

decreases on the interval $(0;T)$. Since $\mathcal{B}^{(0)}(W(t),W(t))\geq 0$ $\forall\,t\in(0;T)$ and $W(0)=W'(0)=0$, we see that $\mathcal{B}^{(0)}(W(t),W(t))=0$. Hence, by virtue of Lemma 2.2, we conclude that $W(t)=0$, which completes the proof. □

6 The existence results

The existence of a solution to the variational inequality (3.11) is obtained by the following scheme. First, we reduce the variational inequality (3.11) to an equivalent regularized variational equation depending on a small parameter ε whose solvability is studied by the Faedo–Galerkin approximation method. Then we establish some a priori estimates which allow us to pass to the limit with respect to the dimension m of the approximation space of test functions as $m\to+\infty$ and with respect to the parameter as $\varepsilon\to 0$. We will show that the limiting function solves the variational inequality (3.11) and, consequently, by virtue of Theorem 4.1, it will be a solution of problem (A_0), as well. The assumptions which are to be satisfied by the data of problem (A_0) will be given below in the course of discussions and, finally, we will formulate the basic existence theorem.

6.1 Reduction to regularized variational equation

To reduce the variational inequality (3.11) to the regularized variational equation, we consider on the space $\mathcal{K}_0$ the convex differentiable functional

$$j_\varepsilon(V) = \int\limits_{S_2} g(x)\varphi_\varepsilon(|\{v_s\}^+|)\,dS, \quad V = (v,w)^\top \in \mathcal{K}_0, \tag{6.1}$$

where ε is an arbitrary positive number, $\varphi_\varepsilon : \mathbb{R} \to (0;\infty)$ is defined by

$$\varphi_\varepsilon(\lambda) = \sqrt{\lambda^2 + \varepsilon^2}\,,$$

g is defined by (3.1) and, in what follows, we assume that it does not depend on the time variable t. Denote by $\mathcal{K}_0'$ the dual space to $\mathcal{K}_0$ and by j_ε' the Gâteaux derivative of the functional (6.1). It is easy to show that for almost all t from the interval $(0;T)$,

$$j_\varepsilon' : \mathcal{K}_0 \to \mathcal{K}_0'$$

is given by

$$\big\langle j_\varepsilon'(V), U\big\rangle_{S_2} = \int\limits_{S_2} g(x)\,\frac{\{v_s\}^+\cdot\{u_s\}^+}{\sqrt{|\{v_s\}^+|^2+\varepsilon^2}}\,dS \quad \forall\, V = (v,w)^\top \in \mathcal{K}_0, \ \ \forall\, U = (u,\omega)^\top \in \mathcal{K}_0. \tag{6.2}$$

Consider the following regularized variational equation: Find $U_\varepsilon \in \mathcal{K}$ satisfying for almost all t from the interval $(0;T)$, the equation

$$(PU_\varepsilon''(t),V) + \mathcal{B}^{(0)}(U_\varepsilon(t),V) + \mathcal{B}^{(1)}(U_\varepsilon'(t),V) + \big\langle j_\varepsilon'(U_\varepsilon'(t)),V\big\rangle_{S_2} = \langle \Psi(t),V\rangle_{\mathcal{K}_0}, \tag{6.3}$$

where $V = (v,w)^\top \in \mathcal{K}_0$ and the linear functional $\Psi(t)$ is defined as

$$\langle \Psi(t),V\rangle_{\mathcal{K}_0} := (\mathcal{G}(t),V) + \int\limits_{S_2} f(t)\{v_n\}^+\,dS + \big\langle \varphi(t), r_{S_2}\{w\}^+\big\rangle_{S_2} \tag{6.4}$$

with $\mathcal{G}$, f, and φ involved in the formulation of Problem (A_0).

It can be easily shown that the variational inequality (3.11), in which U and j are replaced, respectively, by U_ε and j_ε, is equivalent to the regularized variational equation (6.3). Therefore, we investigate the regularized variational equation (6.3).

Since the space $\mathcal{K}_0$ is separable, there exists a countable basis $W_1, W_2, \dots, W_m, \dots$ in the sense that for every m the system of vectors $W_1, W_2, \dots, W_m$ is linearly independent and the space of all finite linear combinations is dense in $\mathcal{K}_0$. We denote by $\mathbf{W}_m := [W_1, W_2, \dots, W_m]$ the linear span of elements $W_1, W_2, \dots, W_m$.

Consider the auxiliary problem: Find a vector-function $U_{\varepsilon m} : (0;T) \to \mathbf{W}_m$ such that $U_{\varepsilon m}, U_{\varepsilon m}'$, $U_{\varepsilon m}'' \in L_\infty(0,T;\mathbf{W}_m)$ and the variational equation

$$(PU_{\varepsilon m}''(t),V) + \mathcal{B}^{(0)}(U_{\varepsilon m}(t),V) + \mathcal{B}^{(1)}(U_{\varepsilon m}'(t),V) + \big\langle j_\varepsilon'(U_{\varepsilon m}'(t)),V\big\rangle_{S_2} = \langle \Psi(t),V\rangle_{\mathcal{K}_0} \tag{6.5}$$

and the initial conditions

$$U_{\varepsilon m}(0) = 0, \tag{6.6}$$

$$U_{\varepsilon m}'(0) = 0 \tag{6.7}$$

are satisfied for almost all t from the interval $(0;T)$ and $\forall\, V \in \mathbf{W}_m$.

Let us look for a solution of the above problem in the form of a linear combination with unknown coefficients $C_{j\varepsilon m}(t)$:

$$U_{\varepsilon m}(t) = \sum_{j=1}^m C_{j\varepsilon m}(t)W_j. \tag{6.8}$$

Replace in (6.5) the test vector-function V by W_k and instead of $U_{\varepsilon m}$ substitute the above linear combination to obtain

$$\sum_{j=1}^{m}(PW_j,W_k)C''_{j\varepsilon m}(t)+\sum_{j=1}^{m}\mathcal{B}^{(0)}(W_j,W_k)C_{j\varepsilon m}(t)+\sum_{j=1}^{m}\mathcal{B}^{(1)}(W_j,W_k)C'_{j\varepsilon m}(t)$$
$$+\Big\langle j'_\varepsilon\Big(\sum_{j=1}^{m}C'_{j\varepsilon m}(t)W_j\Big),W_k\Big\rangle_{S_2}=\langle\Psi(t),W_k\rangle_{\mathcal{K}_0},\ \ k=1,2,\dots,m. \tag{6.9}$$

Introduce the notation:

$$\Phi_k(C'_{1\varepsilon m},\dots,C'_{m\varepsilon m}):=\Big\langle j'_\varepsilon\Big(\sum_{j=1}^{m}C'_{j\varepsilon m}(t)W_j\Big),W_k\Big\rangle_{S_2},\quad \Phi:=(\Phi_1,\dots,\Phi_m)^\top,$$
$$\mathcal{P}_k(t):=\big\langle\Psi(t),W_k\big\rangle_{\mathcal{K}_0},\ \ k=\overline{1,m},\quad \mathcal{P}:=(\mathcal{P}_1,\mathcal{P}_2,\dots,\mathcal{P}_m)^\top,$$
$$\mathcal{B}:=\big[(PW_j,W_k)\big]_{m\times m},\quad D^{(0)}:=\big[\mathcal{B}^{(0)}(W_j,W_k)\big]_{m\times m},$$
$$D^{(1)}:=\big[\mathcal{B}^{(1)}(W_j,W_k)\big]_{m\times m},\quad C_{\varepsilon m}(t):=\big(C_{1\varepsilon m}(t),C_{2\varepsilon m}(t),\dots,C_{m\varepsilon m}(t)\big)^\top.$$

System (6.9) can be then rewritten as

$$\mathcal{B}C''_{\varepsilon m}(t)+D^{(1)}\,C'_{\varepsilon m}(t)+D^{(0)}\,C_{\varepsilon m}(t)+\Phi(C'_{\varepsilon m}(t))=\mathcal{P}(t). \tag{6.10}$$

The initial conditions (6.6) and (6.7) result in

$$C_{\varepsilon m}(0)=C'_{\varepsilon m}(0)=0. \tag{6.11}$$

Note that $\det\mathcal{B}\neq 0$, since the system of vectors $W_1,W_2,\dots,W_m$ is linearly independent, and hence from (6.10) we get

$$C''_{\varepsilon m}(t)+\mathcal{B}^{-1}D^{(1)}C'_{\varepsilon m}(t)+\mathcal{B}^{-1}D^{(0)}\,C_{\varepsilon m}(t)+\mathcal{B}^{-1}\Phi(C'_{\varepsilon m}(t))=\mathcal{B}^{-1}\mathcal{P}(t). \tag{6.12}$$

To reduce system (6.12) to the normal type, we introduce the notation

$$S_{\varepsilon m}(t):=C'_{\varepsilon m}(t),\quad Y_{\varepsilon m}(t):=\big(S_{\varepsilon m}(t),C_{\varepsilon m}(t)\big)^\top$$

and

$$\mathcal{L}(t,Y_{\varepsilon m}):=\begin{bmatrix}\mathcal{B}^{-1}\mathcal{P}(t)-\mathcal{B}^{-1}\Phi(S_{\varepsilon m})-\mathcal{B}^{-1}D^{(1)}C'_{\varepsilon m}-\mathcal{B}^{-1}D^{(0)}C_{\varepsilon m}\\ S_{\varepsilon m}\end{bmatrix}_{2m\times 1}.$$

Then equation (6.12) and the initial conditions (6.11) take the form

$$Y'_{\varepsilon m}(t)=\mathcal{L}(t,Y_{\varepsilon m}),\quad Y_{\varepsilon m}(0)=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}_{2m\times 1}. \tag{6.13}$$

Let us show that the matrix function $\mathcal{L}$ is continuous with respect to the first argument t. To this end, we estimate the difference

$$\big|\mathcal{P}_k(t+\Delta t)-\mathcal{P}_k(t)\big|=\big|\big\langle\Psi(t+\Delta t)-\Psi(t),W_k\big\rangle_{\mathcal{K}_0}\big|$$
$$=\bigg|\big(\mathcal{G}(t+\Delta t)-\mathcal{G}(t),W_k\big)+\int\limits_{S_2}\big(f(t+\Delta t)-f(t)\big)\{(\xi_k)_n\}^+\,dS+\big\langle\varphi(t+\Delta t)-\varphi(t),r_{S_2}\{\eta_k\}^+\big\rangle_{S_2}\bigg|$$
$$\leq\Big(\big\|\mathcal{G}(t+\Delta t)-\mathcal{G}(t)\big\|_{[L_2(\Omega)]^6}+\big\|f(t+\Delta t)-f(t)\big\|_{L_2(S_2)}$$
$$+\big\|\varphi(t+\Delta t)-\varphi(t)\big\|_{[H^{-1/2}(S_2)]^3}\Big)\|W_k\|_{[H^1(\Omega)]^6},$$

where $W_k = (\xi_k, \eta_k)^\top \in \mathcal{K}_0$.

In what follows, we assume that

$$\mathcal{G}, \mathcal{G}', \mathcal{G}'' \in L_2(0,T;[L_2(\Omega)]^6), \quad f \in L_\infty(S_2), \quad \varphi, \varphi', \varphi'' \in L_2(0,T;[H^{-1/2}(S_2)]^3). \tag{6.14}$$

Note that the further analysis of the problem shows that g cannot be dependent on t, and hence f also cannot be dependent on t. Assumptions $\mathcal{G}$, f, and φ are continuously differentiable with respect to t almost everywhere in the interval $(0;T)$, and hence $|\mathcal{P}_k(t+\Delta t) - \mathcal{P}_k(t)| \to 0$ as $\Delta t \to 0$, implying that the function $\mathcal{L}$ is continuous with respect to the first argument.

To prove the continuity of the function $\mathcal{L}$ with respect to $Y_{\varepsilon m}$, it suffices to consider only the term $\Phi(S_{\varepsilon m})$. By formula (6.2), we have

$$\Phi_k(S_{\varepsilon m}) = \Big\langle j_\varepsilon'\Big(\sum_{j=1}^m S_{j\varepsilon m} W_j\Big), W_k\Big\rangle_{S_2} = \int_{S_2} g(x)\, \frac{\Big(\sum\limits_{j=1}^m S_{j\varepsilon m}\{(\xi_j)_s\}^+\Big)\cdot\{(\xi_k)_s\}^+}{\sqrt{\Big|\sum\limits_{j=1}^m S_{j\varepsilon m}\{(\xi_j)_s\}^+\Big|^2 + \varepsilon^2}}\, dS.$$

It is easily seen that Φ_k is continuous and continuously differentiable with respect to the variables $S_{j\varepsilon m}$. Moreover, Φ_k and its derivatives with respect to $S_{j\varepsilon m}$ are bounded by an absolute constant depending on ε. Therefore, the function $\mathcal{L}$ satisfies the Lipschitz condition in the second argument. Consequently, system (6.13) possesses at most one solution.

Any vector function $Y_{\varepsilon m}$ that is a solution to problem (6.13) possesses second order continuous derivatives with respect to t. The same is valid for $U_{\varepsilon m}(t)$ defined by formula (6.8) with $C_{j\varepsilon m}(t)$, being a solution of problem (6.13). It can be shown that $U_{\varepsilon m}(t)$ possesses actually continuous third order derivatives with respect to t and solves problem (6.5)–(6.7).

In the next subsections we derive some a priori estimates which we need to perform the limiting procedure with respect to the dimension m.

6.2 A priori estimates I

Insert the solution of system (6.13) in (6.8) and then substitute $U'_{\varepsilon m}(t)$ instead of V into (6.5) to obtain

$$\begin{aligned}(PU''_{\varepsilon m}(t), U'_{\varepsilon m}(t)) &+ \mathcal{B}^{(0)}(U_{\varepsilon m}(t), U'_{\varepsilon m}(t)) \\ &+ \mathcal{B}^{(1)}(U'_{\varepsilon m}(t), U'_{\varepsilon m}(t)) + \big\langle j'_\varepsilon(U'_{\varepsilon m}(t)), U'_{\varepsilon m}(t)\big\rangle_{S_2} = \big\langle \Psi(t), U'_{\varepsilon m}(t)\big\rangle_{\mathcal{K}_0}.\end{aligned}$$

Since

$$\big\langle j'_\varepsilon(U'_{\varepsilon m}(t)), U'_{\varepsilon m}(t)\big\rangle_{S_2} = \int_{S_2} g(x)\, \frac{|\{(u'_{\varepsilon m}(t))_s\}^+|^2}{\sqrt{|\{(u'_{\varepsilon m}(t))_s\}^+|^2 + \varepsilon^2}}\, dS \ge 0$$

and $\mathcal{B}^{(1)}(U'_{\varepsilon m}(t), U'_{\varepsilon m}(t)) \ge 0$, from the preceding equality we have

$$\frac{d}{dt}\Big\{\big\|\sqrt{P}\, U'_{\varepsilon m}(t)\big\|^2_{[L_2(\Omega)]^6} + \mathcal{B}^{(0)}(U_{\varepsilon m}(t), U_{\varepsilon m}(t))\Big\} \le 2\big\langle \Psi(t), U'_{\varepsilon m}(t)\big\rangle_{\mathcal{K}_0}.$$

Consequently, due to the homogeneous initial conditions, we arrive at the inequality

$$\big\|\sqrt{P}\, U'_{\varepsilon m}(t)\big\|^2_{[L_2(\Omega)]^6} + \mathcal{B}^{(0)}(U_{\varepsilon m}(t), U_{\varepsilon m}(t)) \le 2\int_0^t \big\langle \Psi(\sigma), U'_{\varepsilon m}(\sigma)\big\rangle_{\mathcal{K}_0}\, d\sigma.$$

By virtue of (2.4), we get

$$\big\|\sqrt{P}\, U'_{\varepsilon m}(t)\big\|^2_{[L_2(\Omega)]^6} + C_1\|U_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6} \le C_2\|U_{\varepsilon m}(t)\|^2_{[L_2(\Omega)]^6} + 2\int_0^t \big\langle \Psi(\sigma), U'_{\varepsilon m}(\sigma)\big\rangle_{\mathcal{K}_0}\, d\sigma \tag{6.15}$$

with C_1 and C_2 from (2.4). Since $U_{\varepsilon m}(0)=0$, we can write

$$U_{\varepsilon m}(t)=\int\limits_0^t U'_{\varepsilon m}(\sigma)\,d\sigma,$$

whence

$$\|U_{\varepsilon m}(t)\|^2_{[L_2(\Omega)]^6}\leq\int\limits_0^t\|U'_{\varepsilon m}(\sigma)\|^2_{[L_2(\Omega)]^6}\,d\sigma. \tag{6.16}$$

For the last term in (6.15) we have

$$\begin{aligned}2\int\limits_0^t\big\langle\Psi(\sigma),U'_{\varepsilon m}(\sigma)\big\rangle_{\mathcal{K}_0}\,d\sigma&=2\big\langle\Psi(t),U_{\varepsilon m}(t)\big\rangle_{\mathcal{K}_0}-2\int\limits_0^t\big\langle\Psi'(\sigma),U_{\varepsilon m}(\sigma)\big\rangle_{\mathcal{K}_0}\,d\sigma\\&\leq\frac{1}{\delta}\,\|\Psi(t)\|^2_{\mathcal{K}'_0}+\delta\|U_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6}+\int\limits_0^t\big(\|\Psi'(\sigma)\|^2_{\mathcal{K}'_0}+\|U_{\varepsilon m}(\sigma)\|^2_{[H^1(\Omega)]^6}\big)\,d\sigma\\&\leq C_3+\delta\|U_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6}+\int\limits_0^t\|U_{\varepsilon m}(\sigma)\|^2_{[H^1(\Omega)]^6}\,d\sigma.\end{aligned} \tag{6.17}$$

Taking into account estimates (6.16) and (6.17) and choosing δ in inequality (6.17) smaller than C_1 from (6.15), we finally get

$$\|U'_{\varepsilon m}(t)\|^2_{[L_2(\Omega)]^6}+\|U_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6}\leq C_4\int\limits_0^t\big(\|U'_{\varepsilon m}(\sigma)\|^2_{[L_2(\Omega)]^6}+\|U_{\varepsilon m}(\sigma)\|^2_{[H^1(\Omega)]^6}\big)\,d\sigma+C_5$$

with some constants C_4 and C_5 independent of m and ε. Now, by using Gronwall's lemma, we obtain

$$\|U'_{\varepsilon m}(t)\|^2_{[L_2(\Omega)]^6}+\|U_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6}\leq C \tag{6.18}$$

with the constant C independent of m and ε.

6.3 A priori estimates II

Differentiating (6.5) with respect to t and replacing V with the vector-function $U''_{\varepsilon m}(t)$, we obtain

$$\begin{aligned}&(PU'''_{\varepsilon m}(t),U''_{\varepsilon m}(t))+\mathcal{B}^{(0)}(U'_{\varepsilon m}(t),U''_{\varepsilon m}(t))\\&\qquad+\mathcal{B}^{(1)}(U''_{\varepsilon m}(t),U''_{\varepsilon m}(t))+\Big\langle\frac{d}{dt}\,j'_\varepsilon(U'_{\varepsilon m}(t)),U''_{\varepsilon m}(t)\Big\rangle_{S_2}=\big\langle\Psi'(t),U''_{\varepsilon m}(t)\big\rangle_{\mathcal{K}_0}.\end{aligned} \tag{6.19}$$

Due to formula (6.2), for every $W=(\xi,\eta)^\top\in\mathcal{K}_0$ and $V=(v,w)^\top\in\mathcal{K}_0$, we have

$$\langle j'_\varepsilon(W(t)),V\rangle_{S_2}=\int\limits_{S_2}g(x)Q_\varepsilon(\xi_s(t))\cdot\{v_s\}^+\,dS, \tag{6.20}$$

where

$$Q_\varepsilon\big(\xi_s(t)\big):=\frac{r_{S_2}\{\xi_s(t)\}^+}{\sqrt{|r_{S_2}\{\xi_s(t)\}^+|^2+\varepsilon^2}}\,.$$

Equality (6.20) yields

$$\Big\langle\frac{d}{dt}\,j'_\varepsilon(W(t)),V\Big\rangle_{S_2}=\int\limits_{S_2}g(x)\lim_{h\to 0}\frac{1}{h}\,\big[Q_\varepsilon(\xi_s(t+h))-Q_\varepsilon(\xi_s(t))\big]\cdot\{v_s\}^+\,dS.$$

Replace here V by the vector-function $W'(t)$, then

$$\Big\langle \frac{d}{dt}\, j'_\varepsilon(W(t)), W'(t)\Big\rangle_{S_2} = \int\limits_{S_2} g(x) \lim_{h\to 0} \frac{1}{h}\,\big[Q_\varepsilon(\xi_s(t+h)) - Q_\varepsilon(\xi_s(t))\big]\cdot \frac{1}{h}\,\big\{\xi_s(t+h)-\xi_s(t)\big\}^+\, dS.$$

Since j_ε is a convex differentiable functional on $\mathcal{K}_0$, the operator $j'_\varepsilon : \mathcal{K}_0 \to \mathcal{K}'_0$ is monotone and we have

$$\begin{aligned} 0 &\le \Big\langle j'_\varepsilon(W(t+h)) - j'_\varepsilon(W(t)), W(t+h)-W(t)\Big\rangle_{S_2} \\ &= \int\limits_{S_2} g(x)Q_\varepsilon(\xi_s(t+h))\cdot\{\xi_s(t+h)-\xi_s(t)\}^+\, dS + \int\limits_{S_2} g(x)Q_\varepsilon(\xi_s(t))\cdot\big\{\xi_s(t)-\xi_s(t+h)\big\}^+\, dS \\ &= \int\limits_{S_2} g(x)\big[Q_\varepsilon(\xi_s(t+h)) - Q_\varepsilon(\xi_s(t))\big]\cdot\big\{\xi_s(t+h)-\xi_s(t)\big\}^+\, dS. \end{aligned}$$

Thus we obtain

$$\Big\langle \frac{d}{dt}\, j'_\varepsilon(W(t)), W'(t)\Big\rangle_{S_2} \ge 0. \tag{6.21}$$

Taking into account (6.21), it follows from (6.19) that

$$(PU'''_{\varepsilon m}(t), U''_{\varepsilon m}(t)) + \mathcal{B}^{(0)}(U'_{\varepsilon m}(t), U''_{\varepsilon m}(t)) + \mathcal{B}^{(1)}(U''_{\varepsilon m}(t), U''_{\varepsilon m}(t)) \le \big\langle \Psi'(t), U''_{\varepsilon m}(t)\big\rangle_{\mathcal{K}_0},$$

whence, since $\mathcal{B}^{(1)}(U''_{\varepsilon m}(t), U''_{\varepsilon m}(t))$ is nonnegative, we have

$$\frac{1}{2}\,\frac{d}{dt}\,\Big\{\big\|\sqrt{P}\,U''_{\varepsilon m}(t)\big\|^2_{[L_2(\Omega)]^6} + B^{(0)}(U'_{\varepsilon m}(t), U'_{\varepsilon m}(t))\Big\} \le \big\langle \Psi'(t), U''_{\varepsilon m}(t)\big\rangle_{\mathcal{K}_0}.$$

Using (2.4) and the homogeneous initial condition (6.7), by the integration of the foregoing formula we get

$$\begin{aligned} &\big\|\sqrt{P}\,U''_{\varepsilon m}(t)\big\|^2_{[L_2(\Omega)]^6} + C_1\|U'_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6} \\ &\qquad \le C_2\|U'_{\varepsilon m}(t)\|^2_{[L_2(\Omega)]^6} + \big\|\sqrt{P}\,U''_{\varepsilon m}(0)\big\|^2_{[L_2(\Omega)]^6} + 2\int\limits_0^t \big\langle \Psi'(\sigma), U''_{\varepsilon m}(\sigma)\big\rangle_{\mathcal{K}_0}\, d\sigma \end{aligned} \tag{6.22}$$

with C_1 and C_2 from (2.4). Since

$$\int\limits_0^t \big\langle \Psi'(\sigma), U''_{\varepsilon m}(\sigma)\big\rangle_{\mathcal{K}_0}\, d\sigma = \big\langle \Psi'(t), U'_{\varepsilon m}(t)\big\rangle_{\mathcal{K}_0} - \int\limits_0^t \big\langle \Psi''(\sigma), U'_{\varepsilon m}(\sigma)\big\rangle_{\mathcal{K}_0}\, d\sigma, \tag{6.23}$$

using the inclusions (6.14), we infer that $\Psi'' \in L_2(0,T;\mathcal{K}'_0)$, and hence for an arbitrary positive δ it follows from (6.23) that

$$\begin{aligned} \int\limits_0^t \langle \Psi'(\sigma), U''_{\varepsilon m}(\sigma)\rangle_{\mathcal{K}_0}\, d\sigma &\le \frac{1}{2\delta}\,\|\Psi'(t)\|^2_{\mathcal{K}'_0} + \frac{\delta}{2}\,\|U'_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6} \\ &\quad + C_3\int\limits_0^t \|\Psi''(\sigma)\|^2_{\mathcal{K}'_0}\, d\sigma + C_4\int\limits_0^t \|U'_{\varepsilon m}(\sigma)\|^2_{[H^1(\Omega)]^6}\, d\sigma. \end{aligned} \tag{6.24}$$

Taking now into account the inequality

$$\|\Psi'(t)\|^2_{\mathcal{K}'_0} \le 2\int\limits_0^t \|\Psi''(\sigma)\|^2_{\mathcal{K}'_0}\, d\sigma + 2\|\Psi'(0)\|^2_{\mathcal{K}'_0} \le C_5,$$

from (6.24) we get

$$\int_0^t \big\langle \Psi'(\sigma), U''_{\varepsilon m}(\sigma)\big\rangle_{\mathcal{K}_0}\, d\sigma \le C_6 + \frac{\delta}{2}\|U'_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6} + C_4 \int_0^t \|U'_{\varepsilon m}(\sigma)\|^2_{[H^1(\Omega)]^6}\, d\sigma. \tag{6.25}$$

Choosing δ sufficiently small and taking into account estimates (6.25) and

$$\|U'_{\varepsilon m}(t)\|^2_{[L_2(\Omega)]^6} \le \int_0^t \|U''_{\varepsilon m}(\sigma)\|^2_{[L_2(\Omega)]^6}\, d\sigma,$$

from (6.22) we derive

$$\begin{aligned} &\big\|\sqrt{P}\,U''_{\varepsilon m}(t)\big\|^2_{[L_2(\Omega)]^6} + \|U'_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6} \\ &\quad \le C_7\big\|\sqrt{P}\,U''_{\varepsilon m}(0)\big\|^2_{[L_2(\Omega)]^6} + C_8 \int_0^t \Big[\big\|\sqrt{P}\,U''_{\varepsilon m}(\sigma)\big\|^2_{[L_2(\Omega)]^6} + \|U'_{\varepsilon m}(\sigma)\|^2_{[H^1(\Omega)]^6}\Big]\, d\sigma + C_9. \end{aligned} \tag{6.26}$$

Let us now estimate $\|\sqrt{P}U''_{\varepsilon m}(0)\|_{[L_2(\Omega)]^6}$. Substituting $t = 0$ in (6.5), we obtain

$$(PU''_{\varepsilon m}(0), V) = \big\langle \Psi(0), V\big\rangle_{\mathcal{K}_0} \quad \forall\, V \in \mathbf{W}_m, \tag{6.27}$$

where, in view of (6.4),

$$\big\langle \Psi(0), V\big\rangle_{\mathcal{K}_0} = (\mathcal{G}(0), V) + \int_{S_2} f(0)\{v_n\}^+\, dS + \big\langle \varphi(0), r_{S_2}\{w\}^+\big\rangle_{S_2}.$$

Here we formulate one more restriction on the data of the problem: we assume that there exists a vector-function $U_0 \in [L_2(\Omega)]^6$ such that

$$\big\langle \Psi(0), V\big\rangle_{\mathcal{K}_0} = (U_0, V) \quad \forall\, V \in \mathcal{K}_0. \tag{6.28}$$

Note that if $\varphi \in L_2(0,T;[L_2(S_2)]^3)$, then (6.28) holds.

Since $U''_{\varepsilon m}(0) \in \mathbf{W}_m$, we can take $U''_{\varepsilon m}(0)$ instead of V in (6.27) and, using (6.28), we arrive at the inequality

$$\big\|\sqrt{P}\,U''_{\varepsilon m}(0)\big\|^2_{[L_2(\Omega)]^6} = (U_0, U''_{\varepsilon m}(0)) \le \|U_0\|_{[L_2(\Omega)]^6}\|U''_{\varepsilon m}(0)\|_{[L_2(\Omega)]^6},$$

whence

$$\big\|\sqrt{P}\,U''_{\varepsilon m}(0)\big\|^2_{[L_2(\Omega)]^6} \le C_{10}$$

with C_{10} independent of ε and m. Therefore (6.26) takes the form

$$\begin{aligned} &\big\|\sqrt{P}\,U''_{\varepsilon m}(t)\big\|^2_{[L_2(\Omega)]^6} + \|U'_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6} \\ &\qquad \le C_{11} + C_{12} \int_0^t \Big[\big\|\sqrt{P}\,U''_{\varepsilon m}(\sigma)\big\|^2_{[L_2(\Omega)]^6} + \|U'_{\varepsilon m}(\sigma)\|^2_{[H^1(\Omega)]^6}\Big]\, d\sigma. \end{aligned}$$

Using again Gronwall's lemma, we find

$$\|U''_{\varepsilon m}(t)\|^2_{[L_2(\Omega)]^6} + \|U'_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6} \le C, \tag{6.29}$$

where C does not depend on ε and m.

6.4 The basic existence theorem

First, we pass to the limit with respect to the dimension m. The estimates (6.18) and (6.29) show that $U_{\varepsilon m}$ and $U'_{\varepsilon m}$ (respectively, $U''_{\varepsilon m}$) are bounded by the constants independent of ε and m in the space $L_\infty(0,T;\mathcal{K}_0)$ (respectively, in the space $L_\infty(0,T;[L_2(\Omega)]^6)$. Thus we can choose from the sequence $U_{\varepsilon m}$ a subsequence, which we again denote by $U_{\varepsilon m}$, such that

$$\begin{aligned} &U_{\varepsilon m} \to U_\varepsilon \ \ *\text{-weakly in } L_\infty(0,T;\mathcal{K}_0) \text{ as } m\to\infty,\\ &U'_{\varepsilon m} \to U'_\varepsilon \ \ *\text{-weakly in } L_\infty(0,T;\mathcal{K}_0) \text{ as } m\to\infty,\\ &U''_{\varepsilon m} \to U''_\varepsilon \ \ *\text{-weakly in } L_\infty(0,T;[L_2(\Omega)]^6) \text{ as } m\to\infty. \end{aligned} \tag{6.30}$$

Let us show that the limiting function U_ε satisfies the regularized variational equation (6.3) with the homogeneous initial conditions for $t=0$. We proceed as follows. Let $\vartheta_j\in C^1([0;T])$, $\vartheta_j(T)=0$, $j=\overline{1,\infty}$, be smooth scalar functions and consider the vector-function $\Phi(t)=\sum\limits_{j=1}^{m_0}\vartheta_j(t)W_j$ with a natural number m_0. It is easy to see that $\Phi\in\mathbf{W}_m$ for every $m\geq m_0$ and $\forall\, t\in[0;T]$ and, consequently, from (6.5) we have

$$\begin{aligned} &(PU''_{\varepsilon m}(t),\Phi(t))+\mathcal{B}^{(0)}(U_{\varepsilon m}(t),\Phi(t))\\ &\qquad\qquad +\mathcal{B}^{(1)}(U'_{\varepsilon m}(t),\Phi(t))+\big\langle j'_\varepsilon(U'_{\varepsilon m}(t)),\Phi(t)\big\rangle_{S_2}=\big\langle\Psi(t),\Phi(t)\big\rangle_{\mathcal{K}_0}. \end{aligned} \tag{6.31}$$

Integrate (6.31) with respect to t from 0 to T,

$$\begin{aligned} &\int\limits_0^T\Big[(PU''_{\varepsilon m}(t),\Phi(t))+\mathcal{B}^{(0)}(U_{\varepsilon m}(t),\Phi(t))\\ &\qquad\qquad +\mathcal{B}^{(1)}(U'_{\varepsilon m}(t),\Phi(t))+\big\langle j'_\varepsilon(U'_{\varepsilon m}(t)),\Phi(t)\big\rangle_{S_2}\Big]\,dt=\int\limits_0^T\big\langle\Psi(t),\Phi(t)\big\rangle_{\mathcal{K}_0}\,dt. \end{aligned}$$

Taking now into account (6.30) and passing to the limit in the last equality as $m\to\infty$, we get

$$\begin{aligned} &\int\limits_0^T\Big[(PU''_\varepsilon(t),\Phi(t))+\mathcal{B}^{(0)}(U_\varepsilon(t),\Phi(t))\\ &\qquad\qquad +\mathcal{B}^{(1)}(U'_\varepsilon(t),\Phi(t))+\big\langle j'_\varepsilon(U'_\varepsilon(t)),\Phi(t)\big\rangle_{S_2}\Big]\,dt=\int\limits_0^T\big\langle\Psi(t),\Phi(t)\big\rangle_{\mathcal{K}_0}\,dt. \end{aligned} \tag{6.32}$$

Since the finite linear combinations $\sum\limits_j\vartheta_j(t)W_j$ are dense in $\mathcal{K}_0$ for every $t\in[0;T]$, equality (6.32) allows us to conclude that

$$\begin{aligned} &\int\limits_0^T\Big[(PU''_\varepsilon(t),V)+\mathcal{B}^{(0)}(U_\varepsilon(t),V)\\ &\qquad +\mathcal{B}^{(1)}(U'_\varepsilon(t),V)+\big\langle j'_\varepsilon(U'_\varepsilon(t)),V\big\rangle_{S_2}-\big\langle\Psi(t),\Phi(t)\big\rangle_{\mathcal{K}_0}\Big]\,dt=0\quad\forall\,V\in\mathcal{K}_0. \end{aligned} \tag{6.33}$$

To obtain equality (6.3), it remains to derive a pointwise equation from the integral equality (6.33). To this end, we take an arbitrary fixed number $\tau\in(0;T)$ and an arbitrary vector-function $W\in\mathcal{K}_0$. Consider the family of neighborhoods of the point τ,

$$\Gamma_k=\Big(\tau-\frac{1}{k}\,,\ \tau+\frac{1}{k}\Big),$$

and define the function $V(t)$ as follows:

$$V(t)=\begin{cases}0, & \text{if } t\notin\Gamma_k,\\ W, & \text{if } t\in\Gamma_k.\end{cases}$$

Denoting the measure of Γ_k by $|\Gamma_k|$, from (6.33) we find that

$$\Big(\frac{1}{|\Gamma_k|}\int\limits_{\Gamma_k} PU''_\varepsilon(t)\,dt, W\Big)+\mathcal{B}^{(0)}\Big(\frac{1}{|\Gamma_k|}\int\limits_{\Gamma_k} U_\varepsilon(t)\,dt, W\Big)+\mathcal{B}^{(1)}\Big(\frac{1}{|\Gamma_k|}\int\limits_{\Gamma_k} U'_\varepsilon(t)\,dt, W\Big)$$
$$+\Big\langle j'_\varepsilon\Big(\frac{1}{|\Gamma_k|}\int\limits_{\Gamma_k} U'_\varepsilon(t)\,dt\Big), W\Big\rangle_{S_2}-\frac{1}{|\Gamma_k|}\int\limits_{\Gamma_k}\big\langle\Psi(t),W\big\rangle_{\mathcal{K}_0}\,dt=0. \tag{6.34}$$

According to the Lebesgue theorem, since

$$\frac{1}{|\Gamma_k|}\int\limits_{\Gamma_k}\psi(t)\,dt\longrightarrow\psi(\tau)\ \text{ as }\ k\to\infty$$

for almost all τ, it follows from (6.34) that

$$(PU''_\varepsilon(\tau),W)+\mathcal{B}^{(0)}(U_\varepsilon(\tau),W)+\mathcal{B}^{(1)}(U'_\varepsilon(\tau),W)+\big\langle j'_\varepsilon(U'_\varepsilon(\tau)),W\big\rangle_{S_2}=\big\langle\Psi(\tau),W\big\rangle_{\mathcal{K}_0}\quad\forall\,W\in\mathcal{K}_0,$$

that is, the limiting function U_ε satisfies the regularized variational equation (6.3). As for the initial conditions for $t=0$, we notice that the conditions (6.30) allow us to conclude that $U_\varepsilon(t)$ and $U'_\varepsilon(t)$ are the continuous mappings of the interval $[0;T]$ onto $\mathcal{K}_0$. Thus $U_\varepsilon(0)$ and $U'_\varepsilon(0)$ are well defined and, in view of (6.30), we see that $U_{\varepsilon m}(0)$ and $U'_{\varepsilon m}(0)$ converge weakly in $\mathcal{K}_0$ to $U_\varepsilon(0)$ and $U'_\varepsilon(0)$, respectively. Since $U_{\varepsilon m}(0)=0$ and $U'_{\varepsilon m}(0)=0$, we can show that $U_\varepsilon(0)=0$ and $U'_\varepsilon(0)=0$, i.e., the initial conditions are fulfilled.

It remains to pass to the limit in equality (6.3) with respect to the parameter ε. Repeating the arguments applied above, we can derive the estimate

$$\|U_\varepsilon(t)\|_{[H^1(\Omega)]^6}+\|U'_\varepsilon(t)\|_{[H^1(\Omega)]^6}+\|U''_\varepsilon(t)\|_{[L_2(\Omega)]^6}\le C$$

with the constant C independent of ε. Thus from the sequence $\{U_\varepsilon(t)\}$ we can choose a subsequence, which we denote again by $\{U_\varepsilon\}$, such that

$$\begin{aligned}&U_\varepsilon\to U\ \ *\text{-weakly in } L_\infty(0,T;\mathcal{K}_0)\ \text{ as }\ \varepsilon\to0,\\ &U'_\varepsilon\to U'\ \ *\text{-weakly in } L_\infty(0,T;\mathcal{K}_0)\ \text{ as }\ \varepsilon\to0,\\ &U''_\varepsilon\to U''\ \ *\text{-weakly in } L_\infty(0,T;[L_2(\Omega)]^6)\ \text{ as }\ \varepsilon\to0.\end{aligned}$$

Let us show that the limiting function U satisfies the variational inequality (3.11). Replacing in (6.3) V by the vector-function $W-U'_\varepsilon(t)$, where $W\in\mathcal{K}_0$ is arbitrary, we have

$$\begin{aligned}(PU''_\varepsilon(t),W-U'_\varepsilon(t))&+\mathcal{B}^{(0)}(U_\varepsilon(t),W-U'_\varepsilon(t))\\ &+\mathcal{B}^{(1)}(U'_\varepsilon(t),W-U'_\varepsilon(t))+j_\varepsilon(W)-j_\varepsilon(U'_\varepsilon(t))-\big\langle\Psi(t),W-U'_\varepsilon(t)\big\rangle_{\mathcal{K}_0}\\ &=j_\varepsilon(W)-j_\varepsilon(U'_\varepsilon(t))-\big\langle j'_\varepsilon(U'_\varepsilon(t)),W-U'_\varepsilon(t)\big\rangle_{S_2}\quad\forall\,W\in\mathcal{K}_0.\end{aligned}\tag{6.35}$$

The right-hand side of the above inequality is non-negative. Indeed, since the functional j_ε is convex, we find that

$$\begin{aligned}&j_\varepsilon(W)-j_\varepsilon(U'_\varepsilon(t))-\big\langle j'_\varepsilon(U'_\varepsilon(t)),W-U'_\varepsilon(t)\big\rangle_{S_2}\\ &\qquad=j_\varepsilon(W)-j_\varepsilon(U'_\varepsilon(t))-\lim_{h\to0}\frac{1}{h}\big[j_\varepsilon(hW+(1-h)U'_\varepsilon(t))-j_\varepsilon(U'_\varepsilon(t))\big]\ge0.\end{aligned}$$

Taking into account the last inequality, from (6.35) we have

$$\int_0^T \Big[(PU_\varepsilon''(t),W)+\mathcal{B}^{(0)}(U_\varepsilon(t),W)+\mathcal{B}^{(1)}(U_\varepsilon'(t),W)+j_\varepsilon(W)-\big\langle\Psi(t),W-U_\varepsilon'(t)\big\rangle_{\mathcal{K}_0}\Big]\,dt$$
$$\geq\int_0^T \Big[(PU_\varepsilon''(t),U_\varepsilon'(t))+\mathcal{B}^{(0)}(U_\varepsilon(t),U_\varepsilon'(t))+\mathcal{B}^{(1)}(U_\varepsilon'(t),U_\varepsilon'(t))+j_\varepsilon(U_\varepsilon'(t))\Big]\,dt.$$

On the other hand, the equality

$$\int_0^T \Big[(PU_\varepsilon''(t),U_\varepsilon'(t))+\mathcal{B}^{(0)}(U_\varepsilon(t),U_\varepsilon'(t))+\mathcal{B}^{(1)}(U_\varepsilon'(t),U_\varepsilon'(t))+j_\varepsilon(U_\varepsilon'(t))\Big]\,dt$$
$$=\frac{1}{2}\Big[\big\|\sqrt{P}\,U_\varepsilon'(T)\big\|^2_{[L_2(\Omega)]^6}+\mathcal{B}^{(0)}(U_\varepsilon(T),U_\varepsilon(T))\Big]+\int_0^T\big[\mathcal{B}^{(1)}(U_\varepsilon'(t),U_\varepsilon'(t))+j_\varepsilon(U_\varepsilon'(t))\big]\,dt$$

with the help of the inequality

$$\liminf_{\varepsilon\to 0}\mathcal{B}^{(0)}(U_\varepsilon(T),U_\varepsilon(T))\geq\mathcal{B}^{(0)}(U(T),U(T))$$

leads to the inequality

$$\int_0^T \Big[(PU''(t),W-U'(t))+\mathcal{B}^{(0)}(U(t),W-U'(t))+\mathcal{B}^{(1)}(U'(t),W-U'(t))$$
$$+j(W)-j(U'(t))-\big\langle\Psi(t),W-U'(t)\big\rangle_{\mathcal{K}_0}\Big]\,dt\geq 0\quad\forall\,W\in\mathcal{K}_0. \tag{6.36}$$

From the integral relation (6.36) we can derive as above the pointwise inequality

$$(PU''(t),W-U'(t))+\mathcal{B}^{(0)}(U(t),W-U'(t))$$
$$+\mathcal{B}^{(1)}(U'(t),W-U'(t))+j(W)-j(U'(t))-\big\langle\Psi(t),W-U'(t)\big\rangle_{\mathcal{K}_0}\geq 0\quad\forall\,W\in\mathcal{K}_0,$$

and by an analogous reasoning we conclude that the homogeneous initial conditions are fulfilled. Thus we have proved the following existence theorem.

Theorem 6.1. *Let conditions* (6.14) *be fulfilled, g be independent of t, and let there exist a vector-function $U_0\in[L_2(\Omega)]^6$ such that*

$$(U_0,V)=\big(\mathcal{G}(0),V\big)+\int_{S_2}f(0)\,\{v_n\}^+\,dS+\big\langle\varphi(0),r_{S_2}\{w\}^+\big\rangle_{S_2}\quad\forall\,V=(v,w)^\top\in\mathcal{K}_0.$$

Then there exists one and only one function $U\in\mathcal{K}$ which is a solution of the variational inequality (3.11) *and, according to Theorem* 4.1*, it is a solution of problem* (A_0)*, as well.*

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Author's addresses:

1. Andrea Razmadze Mathematical Institute of Ivane Javakishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia.

2. Department of Mathematics, Georgian Technical University, 77 M. Kostava Str., Tbilisi 0171, Georgia.

E-mail: r.gachechiladze@yahoo.com

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Ridha Selmi, Mounia Zaabi

MATHEMATICAL STUDY TO A REGULARIZED $3D$-BOUSSINESQ SYSTEM

Abstract. We prove existence of weak solution to a regularized Boussinesq system in Sobolev spaces under the minimal regularity to the initial data. Continuous dependence on initial data (and then uniqueness) is proved provided that the initial fluid velocity is mean free. If the temperature is also mean free, we prove that the solution decays exponentially fast, as time goes to infinity. Moreover, we show that the unique solution converges to a Leray–Hopf solution of the three-dimensional Boussinesq system, as the regularizing parameter alpha vanishes. The mean free technical condition appears because the nonlinear part of the fluid equation is subject to regularization. The main tools are the energy methods, the compactness method, the Poincaré inequality and some Grönwall type inequalities. To handle the long time behaviour, a time dependent change of function is used.

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რეზიუმე. დამტკიცებულია რეგულარიზებული ბუსინესკის სისტემის სუსტი ამონახსნის არსებობა სობოლევის სივრცეებში საწყისი მონაცემების მინიმალური რეგულარობის პირობებში. დამტკიცებულია ამონახსნის უწყვეტი დამოკიდებულება საწყის მონაცემებზე (ხოლო შემდეგ ერთადერთობა), თუ სითხის საწყისი სიჩქარე საშუალოდ თავისუფალია. თუ ტემპერატურაც საშუალოდ თავისუფალია, მაშინ ჩვენ ვამტკიცებთ, რომ ამონახსნი ექსპონენციალურად სწრაფად ქრება, როცა დრო უსასრულობისკენ მიისწრაფის. გარდა ამისა, დამტკიცებულია, რომ ერთადერთი ამონახსნი კრებადია სამგანზოლებიანი ბუსინესკის სისტემის ლერეი-ჰოფის ამონახსნისკენ, როცა მარეგულირებელი ალფა პარამეტრი ნულისკენ მიისწრაფის. საშუალო თავისუფლების ტექნიკური პირობა გამოჩნდება იმიტომ, რომ ხდება სითხის განტოლების არაწრფივი ნაწილის რეგულარიზაცია. კვლევის მთავარი ინსტრუმენტებია ენერგეტიკული მეთოდები, კომპაქტურობის მეთოდი, პუანკარეს უტოლობა და გრონველის ტიპის უტოლობები. იმისათვის, რომ შევისწავლოთ ყოფაქცევა ხანგრძლივი დროის განმავლობაში, გამოყენებულია ფუნქციის ცვალებადობის დროზე დამოკიდებულება.

1 Introduction

We consider the following system denoted by (Bq_α):

$$\begin{aligned} &\partial_t\theta - \Delta\theta + (u\cdot\nabla)\theta = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3,\\ &\partial_t v - \Delta v + (v\cdot\nabla)u = -\nabla p + \theta e_3, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3,\\ &v = u - \alpha^2\Delta u, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3,\\ &\operatorname{div} u = \operatorname{div} v = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3,\\ &(u,\theta)\big|_{t=0} = (u^0,\theta^0), \quad x\in\mathbb{T}^3, \end{aligned}$$

where the unknown vector field u, the scalars p and θ denote, respectively, the velocity, the pressure and the temperature of the fluid at the point $(t,x)\in\mathbb{R}_+\times\mathbb{T}^3$. Here, $\mathbb{T}^3$ is the three-dimensional torus and $\alpha>0$ is a real parameter that has to go to zero. The data θ^0 and u^0 are initial temperature and initial divergence free velocity. In [7], the author explained motivations behind considering regularized systems such as (Bq_α), and he gave a wide review of related literature. Here, we just recall that alpha-regularization consists in replacing the velocity u in some of its occurrences by the most regular field $v=u-\alpha^2\Delta u$. So, contrarily to the non-regularized fluid mechanic equation, we have the existence of a unique three-dimensional solution that depends continuously on initial data. Moreover, as explained in [2], these models can be implemented in a relatively simple way in numerical computation of the three-dimensional fluid equations. Thus, they are to be known as regularization stimulated by numerical motivations. In the framework of computational fluid dynamics, for zero valued temperature, it was proved in [4] that the model we are actually considering, provides a computationally sound analytical subgrid scale model for large eddy simulation of turbulence. More important is that when the regularizing parameter α tends to zero, the solution of (Bq_α) coincides with the solution of Boussinesq system $(Bq_{\alpha=0})$. Furthermore, as time tends to infinity, the system $(Bq_{\alpha>0})$ behaves like $(Bq_{\alpha=0})$.

In this paper, we will investigate the weak solution to the modified Leray-alpha model for the Boussinesq system. More than the linear part, the nonlinear part of the fluid equation is to be regularized as well. This is one of the main differences between systems we considered in [7] and [3], where we regularized only the linear part and studied, respectively, the weak and the strong solutions.

Our first result is the existence of the weak solution to the system (Bq_α) in the context of the minimal regularity to the initial data.

Theorem 1.1. *Let $\theta^0\in L^2(\mathbb{T}^3)$ and let $u^0\in H^1(\mathbb{T}^3)$ be a divergence-free vector field. Then there exists a unique weak solution (u_α,θ_α) of system (Bq_α) such that u_α belongs to $C(\mathbb{R}_+,H^1(\mathbb{T}^3))\cap L^2(\mathbb{R}_+,H^2(\mathbb{T}^3))$ and θ_α belongs to $C(\mathbb{R}_+,L^2(\mathbb{T}^3))\cap L^2(\mathbb{R}_+,H^1(\mathbb{T}^3))$. Moreover, this solution satisfies the energy estimate*

$$\begin{aligned} &\|\theta_\alpha\|_{L^2}^2 + \|u_\alpha\|_{L^2}^2 + \alpha^2\|\nabla u_\alpha\|_{L^2}^2 + 2\int_0^t \|\nabla\theta_\alpha\|_{L^2(\mathbb{T}^3)}^2\,d\tau\\ &\qquad + 2\int_0^t \big(\|\nabla u_\alpha\|_{L^2}^2 + \alpha^2\|\Delta u_\alpha\|_{L^2}^2\big)\,d\tau \le \|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 + \sigma_\alpha(t), \quad (1.1) \end{aligned}$$

where

$$\sigma_\alpha(t) = (e^{2t}-1)\big(\|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2\big).$$

If the initial velocity is mean free, the solution is continuously dependent on the initial data on any bounded interval $[0,T]$. In particular, it is unique.

The proof is done in the frequency space and uses the compactness method. To close the energy estimates, the buoyancy force presents some difficulties that we have overcome by Grönwall' s lemma, without useless sharpness. More than the uniqueness, we have continuous dependence of the weak solution on the initial data. This is the main advantage provided by alpha regularization, since such dependence plays an important role in numerical schemes.

To prove continuous dependence with respect to the initial data, we consider the system satisfied by the difference of two solutions and apply energy methods. The Young product inequalities and suitable Sobolev products allow to estimate the nonlinear terms. Grönwall's type differential inequality finishes the proof. In particular, we infer the uniqueness of solution. Compared to [7] and [3], the mean free condition is compulsory, since we are regularizing the nonlinear term and thus the Poincaré inequality turns to be a necessary tool to run the argument of the continuous dependence to initial data.

Our next result asserts that for long time, the regularized temperature and the regularized velocity fields vanish exponentially fast as time tends to infinity. This convergence is uniform with respect to α. One recovers, for $\alpha > 0$, a similar property of the long time behavior to the Leray–Hopf solution of the non-regularized system.

Theorem 1.2. *Let $a \in (0,1)$. Let θ_α and u_α be the family of solutions from Theorem 1.1. If θ^0 and u^0 are both mean free and satisfy the inequality*

$$\|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 \leq 1 - a,$$

then θ_α and u_α decay exponentially fast to zero as time tends to infinity as soon as the initial data (hence the solution) are mean free:

$$\|\theta_\alpha(t)\|_{L^2} + \|u_\alpha(t)\|_{H^1} \leq (1-a)e^{-at} \quad \forall\, t \geq 0.$$

To prove this result, we use a change of the function that depends explicitly on time. This leads to an energy estimate that is sharper than the one of the existence result. For zero-mean valued temperature and velocity, this estimation allows to derive the vanishing limit and the rate of convergence, as time tends to infinity.

Our last result describes the weak and strong convergence, as $\alpha \to 0$, of the unique weak solution of the regularized system (Bq_α) to the Leray–Hopf solution of the system (Bq_0). This convergence asserts that as smaller is alpha, as better we describe reality.

Theorem 1.3. *Let $T > 0$, $(u_\alpha, \theta_\alpha)$ be the unique solution of system (Bq_α). Then there exist the subsequences u_{α_k}, v_{α_k} and θ_{α_k}, a scalar function θ, and a divergence-free vector field u, both belonging to $L^\infty([0,T], L^2(\mathbb{T}^3)) \cap L^2([0,T], H^1(\mathbb{T}^3))$, such that as $\alpha_k \to 0^+$, we have:*

1. *The sequence u_{α_k} converges to u and θ_{α_k} converges to θ weakly in $L^2([0,T], H^1(\mathbb{T}^3))$ and strongly in $L^2([0,T], L^2(\mathbb{T}^3))$.*
2. *The sequence v_{α_k} converges to u weakly in $L^2([0,T], L^2(\mathbb{T}^3))$ and strongly in $L^2([0,T], H^{-1}(\mathbb{T}^3))$.*
3. *The sequence u_{α_k} converges to u and θ_{α_k} converges to θ weakly in $L^2(\mathbb{T}^3)$ and uniformly over $[0,T]$. Furthermore, (u,θ) is the weak solution of the Boussinesq system (Bq_0) on $[0,T]$ associated with the initial data (u^0, θ^0) satisfying for all $t \in [0,T]$ the energy inequality*

$$\|\theta\|_{L^2}^2 + \|u\|_{L^2}^2 + \int_0^t \|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\, d\tau \leq \|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \sigma_0(t). \tag{1.2}$$

Here, (Bq_0) and σ_0 denote, respectively, (Bq_α) and σ_α for $\alpha = 0$.

The purpose of the proof is to extract subsequences that converge to the solution of (Bq) as $\alpha \to 0^+$. First, we derive a uniform bound independent of the parameter α. This gives the weak convergence. Then, following the lines of the existence proof, we establish strong convergence of such subsequences in suitable spaces. This strong convergence allows to take the limit in the quadratic terms, and hence a weak convergence of the unique weak solution of (Bq) to a weak solution of (Bq) is proved and the associated energy estimate is derived.

The remainder of the paper is organized as follows. We start with recalling some useful background. Section 3 is devoted to the proof of the existence result and the continuous dependence of the weak solution on the initial data, in particular, uniqueness. In Section 4, we investigate the long time behaviour of the regularized temperature and the regularized velocity. Section 5 is devoted to proving several convergence results, as the regularizing parameter α vanishes.

2 Preliminary results

For $n \in N$, let P_n denote the projection into the Fourier modes of order up to n, that is,

$$P_n\Big(\sum_{k\in Z^3} \widehat{u}_k e^{ik\cdot x}\Big) = \sum_{|k|\leq n} \widehat{u}_k e^{ik\cdot x}.$$

We define for $s \geq 0$ the operator Λ^s acting on $H^s(\mathbb{T}^3)$ by

$$\Lambda^s u(x) = \sum_{k\in Z^3} |k|^s \widehat{u}_k e^{ik\cdot x} \in L^2(\mathbb{T}^3).$$

Moreover, we denote by $\|\cdot\|_{\dot{H}^s}$ the seminorm $\|\cdot\|_{L^2}$. This is, of course, compatible with the definition of the Sobolev norm that $\|\cdot\|_{H^s}$ is equivalent to $\|\cdot\|_{L^2} + \|\cdot\|_{\dot{H}^s}$. We will also make use of the fact that $\|u\|_{\dot{H}^s} \leq \|u\|_{\dot{H}^t}$ if $0 < s \leq t$ and $\Lambda^2 = -\Delta$. Moreover, if $\operatorname{div} u = 0$, we have $(v\cdot\nabla u, u)_{L^2(\mathbb{T}^3)} = 0$ and $(u\cdot\nabla\theta, \theta)_{L^2(\mathbb{T}^3)} = 0$. Finally, we recall the version of the Aubin–Lions Theorem that will be used.

Lemma 2.1. *Let X_0, X and X_1 be three Banach spaces with $X_0 \subset X \subset X_1$. Suppose that X_0 is compactly embedded in X and X is continuously embedded in X_1. For $1 \leq p,\ q \leq \infty$, let*

$$\mathcal{W} = \Big\{u \in L^p([0,T], X_0) : \ \frac{du}{dt} \in L^q([0,T], X_1)\Big\}.$$

- *If $p < +\infty$, then the embedding of $\mathcal{W}$ into $L^p([0,T]; X)$ is compact.*
- *If $p = +\infty$ and $q > 1$, then the embedding of $\mathcal{W}$ into $C([0,T]; X)$ is compact.*

Also, we need the following inequalities:

$$\|\vartheta\|_{L^3} \leq \|\vartheta\|_{L^2}^{1/2}\|\nabla\vartheta\|_{L^2}^{1/2}, \tag{2.1}$$

$$\|\vartheta\|_{L^\infty} \leq \|\vartheta\|_{\dot{H}^1}^{1/2}\|\vartheta\|_{\dot{H}^2}^{1/2}, \tag{2.2}$$

$$\|\vartheta\|_{L^6} \leq \|\nabla\vartheta\|_{L^2}. \tag{2.3}$$

3 Existence and uniqueness results

Let $u_n = P_n u$. One approximates the continuous problem (Bq_α) by the following problem denoted by $(Bq_\alpha)_n$:

$$\partial_t\theta_n - \Delta\theta_n + P_n \operatorname{div}(\theta_n u_n) = 0, \tag{3.1}$$

$$\partial_t v_n - \Delta v_n + P_n \operatorname{div}(v_n u_n) - \theta_n e_3 = P_n \nabla\Delta^{-1}\Big(\sum_{i,j=1}^{3} \partial_i\partial_j(v_n^i u_n^j) - \partial_3\theta_n\Big), \tag{3.2}$$

$$v_n = u_n - \alpha^2\Delta u_n, \tag{3.3}$$

$$\operatorname{div} u_n = \operatorname{div} v_n = 0, \tag{3.4}$$

$$(u_n, \theta_n)_{t=0} = (u_n^0, \theta_n^0) = (P_n u^0, P_n\theta^0). \tag{3.5}$$

The ordinary differential equation theory implies that there exists some maximal $T_n^* > 0$ and a unique local solution $u_n \in C^\infty([0,T_n^*) \times \mathbb{T}^3)$ to $(Bq_\alpha)_n$. Taking the inner product of (3.1) by θ_n and (3.2) by u_n, applying the Cauchy–Schwarz inequality to the forcing term $< \theta_n e_3, u_n >_{L^2}$ and dropping the viscous term, we obtain

$$\frac{d}{dt}\left(\|\theta_n\|_{L^2}^2 + \|u_n\|_{L^2}^2 + \alpha^2\|\nabla u_n\|_{L^2}^2\right) \leq 2\big(\|\theta_n\|_{L^2}^2 + \|u_n\|_{L^2}^2 + \alpha^2\|\nabla u_n\|_{L^2}^2\big).$$

Let

$$\phi(t) = \|\theta_n\|_{L^2}^2 + \|u_n\|_{L^2}^2 + \alpha^2\|\nabla u_n\|_{L^2}^2,$$

then the above equation reads $\phi'(t) \le 2\phi(t)$. Applying Grönwall' s inequality and integrating over $[0,t]$, we obtain $\phi(t) \le \phi(0)e^{2t}$. Thus,

$$\|\theta_n(t)\|_{L^2}^2 + \|u_n(t)\|_{L^2}^2 + \alpha^2\|\nabla u_n(t)\|_{L^2}^2 \le (\|\theta_n^0\|_{L^2}^2 + \|u_n^0\|_{L^2}^2 + \alpha^2\|\nabla u_n^0\|_{L^2}^2)e^{2t}.$$

This implies that

$$\|\theta_n(t)\|_{L^2}^2 + \|u_n(t)\|_{L^2}^2 + \alpha^2\|\nabla u_n(t)\|_{L^2}^2 + 2\int_0^t \|\nabla\theta_n(\tau)\|_{L^2(\mathbb{T}^3)}^2\, d\tau$$
$$+ 2\int_0^t (\|\nabla u_n(\tau)\|_{L^2}^2 + \alpha^2\|\Delta u_n(\tau)\|_{L^2}^2)\, d\tau \le \|\theta_n^0\|_{L^2}^2 + \|u_n^0\|_{L^2}^2 + \alpha^2\|\nabla u_n^0\|_{L^2}^2 + \sigma_\alpha(t),$$

where

$$\sigma_\alpha(t) = (e^{2t}-1)\big(\|\theta_n^0\|_{L^2}^2 + \|u_n^0\|_{L^2}^2 + \alpha^2\|\nabla u_n^0\|_{L^2}^2\big).$$

So, the maximal solution to problem (3.1)–(3.5) is global and $T_n^* = +\infty$.

Using the product laws and interpolation inequality, we obtain

$$\|\operatorname{div}(v_n \otimes u_n)\|_{\dot H^{-2}} \le \|v_n\|_{L^2}\|u_n\|_{L^2}^{1/2}\|u_n\|_{\dot H^1}^{1/2}.$$

Hence, $\frac{d}{dt}v_n \in L^2([0,T],\dot H^{-2})$. We denote by $\mathcal{W}$ the set of functions defined by

$$\mathcal{W} = \Big\{u_n :\ u_n \in L^2([0,T],\ \dot H^2(\mathbb{T}^3)),\ \frac{du_n}{dt} \in L^2([0,T],L^2(\mathbb{T}^3))\Big\}.$$

By the Aubin–Lions Theorem, we conclude that there is a subsequence $u_{n'}$ such that $u_{n'} \rightharpoonup u_\alpha$ weakly in $L^2([0,T],\dot H^2(\mathbb{T}^3))$, and $u_{n'} \to u_\alpha$ strongly in $L^2([0,T],\dot H^1(\mathbb{T}^3))$, moreover, $u_{n'} \to u_\alpha$ in $C([0,T],L^2(\mathbb{T}^3))$. Likewise, if we denote

$$\mathcal{W}' = \Big\{\theta_n :\ \theta_n \in L^2([0,T],\ \dot H^1(\mathbb{T}^3)),\ \frac{d\theta_n}{dt} \in L^2([0,T],\dot H^{-1}(\mathbb{T}^3))\Big\},$$

then there exists θ_α such that $\theta_{n'} \rightharpoonup \theta_\alpha$ weakly in $L^2([0,T],\dot H^1(\mathbb{T}^3))$, and $\theta_{n'} \to \theta_\alpha$ strongly in $L^2([0,T],L^2(\mathbb{T}^3))$, moreover, $\theta_{n'} \to \theta_\alpha$ in $C([0,T],\dot H^{-1}(\mathbb{T}^3))$. Further, we relabel $u_{n'}$, $v_{n'}$ and $\theta_{n'}$ by u_n, v_n and θ_n and note that the strong convergence is compulsory when taking the limit in the nonlinear term. Let us begin with proving that

$$\lim_{n\to+\infty} P_n[(u_n\nabla)\theta_n] = [(u_\alpha\nabla)\theta_\alpha]$$

in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. Let $\Psi \in \dot H^2$ be a vector divergence-free test function, $\Phi \in \dot H^1$ be a scalar test function, and $\forall t \in \mathbb{R}^+$,

$$I_n^1 = \int_0^t \big\langle P_n\big[(u_n - u_\alpha)\nabla\theta_n\big], \Phi\big\rangle_{L^2}\, d\tau,$$
$$I_n^2 = \int_0^t \big\langle P_n\big[(u_\alpha)\nabla(\theta_n - \theta_\alpha)\big], \Phi\big\rangle_{L^2}\, d\tau,$$
$$I_n^3 = \int_0^t \big\langle (P_n - I)(u_\alpha\nabla)\theta_\alpha, \Phi\big\rangle_{L^2}\, d\tau.$$

Using, respectively, the Cauchy–Schwarz inequality and Sobolev product laws, we obtain

$$|I_n^1| \le \|u_n - u_\alpha\|_{L^2([0,T],\dot H^1)}\|\theta_n\|_{L^2([0,T],\dot H^1)}\|\Phi\|_{\dot H^1},$$
$$|I_n^2| \le \|u_\alpha\|_{L^2([0,T],\dot H^2)}\|\theta_n - \theta_\alpha\|_{L^2([0,T],L^2)}\|\Phi\|_{\dot H^1}.$$

As for I_n^3, first, we estimate the term

$$\big\langle (P_n - I)(u_\alpha \nabla)\theta_\alpha, \Phi \big\rangle_{L^2} = \int_{\mathbb{T}^3} \sum_{|k|>n} \widehat{(u_{\alpha,k}\nabla)\theta}_{\alpha,k} e^{ik\cdot x} \Phi \, dx$$
$$\leq \int_{\mathbb{T}^3} \sum_{|k|>n} \frac{|k|}{n} \widehat{(u_{\alpha,k}\nabla)\theta}_{\alpha,k} e^{ik\cdot x} \Phi \, dx \leq \frac{1}{n} \int_{\mathbb{T}^3} \Lambda(\mathrm{div}(u_\alpha \theta_\alpha)) \Phi \, dx.$$

Then, by inequality (2.2) and Hölder's inequality, we obtain

$$|I_n^3| \leq \frac{1}{n} \int_0^t \big\| \Lambda(\mathrm{div}(u_\alpha \theta_\alpha)) \big\|_{\dot{H}^{-1}} \|\Phi\|_{\dot{H}^1} \, d\tau \leq \frac{1}{n} \|u_\alpha\|_{L^2([0,T],\dot{H}^2)} \|\theta_\alpha\|_{L^2([0,T],\dot{H}^1)} \|\Phi\|_{\dot{H}^1}.$$

Now, let us prove that

$$\lim_{n\to+\infty} P_n(v_n \cdot \nabla) u_n = (v_\alpha \cdot \nabla) u_\alpha$$

in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. Let

$$J_n^1 = \int_0^t \big\langle P_n(v_n - v_\alpha) \cdot \nabla u_n, \Psi \big\rangle_{L^2} \, d\tau,$$
$$J_n^2 = \int_0^t \big\langle P_n v_\alpha \cdot \nabla (u_n - u_\alpha), \Psi \big\rangle_{L^2} \, d\tau,$$
$$J_n^3 = \int_0^t \big\langle (P_n - I)(v_\alpha \cdot \nabla) u_\alpha, \Psi \big\rangle_{L^2} \, d\tau.$$

As for J_n^1, we have

$$|J_n^1| \leq \int_0^t \big\| (v_n - v_\alpha) \cdot \nabla u_n \big\|_{\dot{H}^{-2}} \|\Psi\|_{\dot{H}^2} \, d\tau$$
$$\leq c \int_0^t \|v_n - v_\alpha\|_{\dot{H}^{-1}} \|\nabla u_n\|_{\dot{H}^{1/2}} \|\Psi\|_{\dot{H}^2} \, d\tau \leq c \|v_n - v_\alpha\|_{L^2([0,T],\dot{H}^{-1})} \|u_n\|_{L^2([0,T],\dot{H}^2)} \|\Psi\|_{\dot{H}^2}.$$

Since u_n is bounded in $L^2([0,T],\dot{H}^2)$ and $v_n \to v_\alpha$ in $L^2([0,T],\dot{H}^{-1})$, we get $\lim\limits_{n\to+\infty} J_n^1 = 0$. Applying the Cauchy–Schwarz inequality and Sobolev product laws, we have

$$|J_n^2| \leq \int_0^t \big\| v_\alpha \cdot \nabla (u_n - u_\alpha) \big\|_{\dot{H}^{-2}} \|\Psi\|_{\dot{H}^2} \, d\tau$$
$$\leq \int_0^t \|v_\alpha\|_{\dot{H}^{-1/2}} \big\| \nabla (u_n - u_\alpha) \big\|_{L^2} \|\Psi\|_{\dot{H}^2} \, d\tau \leq \|v_\alpha\|_{L^2([0,T],L^2)} \|u_n - u_\alpha\|_{L^2([0,T],\dot{H}^1)} \|\Psi\|_{\dot{H}^2}.$$

Since v_α is bounded in $L^2([0,T],L^2)$ and $u_n \to u_\alpha$ strongly in $L^2([0,T],\dot{H}^1)$, we get $\lim\limits_{n\to+\infty} J_n^2 = 0$. As for J_n^3, at a first step, we estimate the term

$$\big\langle (P_n - I)(v_\alpha \cdot \nabla) u_\alpha, \Psi \big\rangle_{L^2} = \int_{\mathbb{T}^3} (P_n - I)(v_\alpha \cdot \nabla) u_\alpha \Psi \, dx \leq \frac{1}{n} \int_{\mathbb{T}^3} \Lambda(\mathrm{div}(v_\alpha \otimes u_\alpha)) \Psi \, dx,$$

where we have used the divergence-free condition and a standard calculation. Then, by the Cauchy–Schwarz inequality and Sobolev product laws, we get

$$|J_n^3| \leq \frac{1}{n}\int_0^t \big\langle \Lambda(\operatorname{div}(v_\alpha \otimes u_\alpha)), \Psi\big\rangle_{L^2}\, d\tau$$

$$\leq \frac{1}{n}\int_0^t \big\|\Lambda(\operatorname{div}(v_\alpha \otimes u_\alpha))\big\|_{\dot H^{-2}} \|\Psi\|_{\dot H^2}\, d\tau \leq \frac{1}{n}\, \|v_\alpha\|_{L^2([0,T],L^2)} \|u_\alpha\|_{L^2([0,T],\dot H^2)} \|\Psi\|_{\dot H^2}.$$

To prove the continuity of the solution, it suffices to prove at a first step that for all $t_0 \in \mathbb{R}_+$,

$$\|\theta_\alpha(t) - \theta_\alpha(t_0)\|_{L^2(\mathbb{T}^3)} \to 0 \;\text{ as }\; t \to t_0.$$

Towards this end, we have to prove that the function $t \longmapsto \|\theta_\alpha(t)\|_{L^2}$ is continuous and the function $t \longmapsto \theta_\alpha(t)$ is weakly continuous with value in $L^2(\mathbb{T}^3)$. We have $\theta_\alpha \in L^\infty(\mathbb{R}_+, L^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+, \dot H^1(\mathbb{T}^3))$, so, $\frac{d}{dt}\|\theta_\alpha(t)\|_{L^2}^2$ belongs to $L^1([0,T])$. Hence, $\|\theta_\alpha(t)\|_{L^2}^2$ belongs to $C([0,T])$. Since $\theta_\alpha \in L^2(\mathbb{R}_+, \dot H^1(\mathbb{T}^3))$ and $\Phi \in \dot H^1$, we find that as t tends to t_0, the inequality

$$\Big|\int_{t_0}^t \big\langle \nabla\theta_\alpha, \nabla\Phi\big\rangle_{L^2}\, d\tau\Big| \leq \Big(\int_{t_0}^t \|\nabla\theta_\alpha(\tau)\|_{L^2}^2\, d\tau\Big)^{1/2} \Big(\int_{t_0}^t \|\nabla\Phi(\tau)\|_{L^2}^2\, d\tau\Big)^{1/2}$$

tends to zero. Using inequality (2.2) and the Cauchy–Schwarz and Hölder inequalities, we find that

$$\Big|\int_{t_0}^t \big\langle \operatorname{div}(\theta_\alpha u_\alpha), \Phi >_{L^2}\; d\tau\Big| \leq \Big(\int_{t_0}^t \|\theta_\alpha\|_{L^2}^2\, d\tau\Big)^{1/2} \Big(\int_{t_0}^t \|u_\alpha\|_{\dot H^2}^2\, d\tau\Big)^{1/2} \|\Phi\|_{\dot H^1}$$

tends to zero as t tends to t_0. Therefore $langle\theta_\alpha(t), \Phi\rangle_{L^2} \to \langle\theta(t_0), \Phi\rangle_{L^2}$ as $t \to t_0$ for every $\Phi \in \dot H^1$. In particular, $\theta_\alpha(t) \in L^2$ and $\Phi \in \dot H^1 \subset L^2$. Since the Sobolev space $\dot H^1$ is dense in L^2, we have for $t \in [0,T]$, $\langle\theta_\alpha(t), \Phi\rangle_{L^2} \to \langle\theta(t_0), \Phi\rangle_{L^2}$ as $t \to t_0$ for every $\Phi \in L^2$. Hence, $\theta_\alpha \in C([0,T), L^2)$. Similarly, we obtain $\|\nabla u_\alpha(t) - \nabla u_\alpha(t_0)\|_{L^2}^2 \to 0$ as $t \to t_0$.

To prove continuous dependence of solutions on initial data, we assumer that (u,θ) and $(\overline{u}, \overline{\theta})$ are any two solutions of the system (Bq_α) on the interval $[0,T]$, with initial values (u^0, θ^0) and $(\overline{u}^0, \overline{\theta}^0)$, respectively. Let us denote $v = u - \alpha^2\Delta u$, $\overline{v} = \overline{u} - \alpha^2\overline{\Delta u}$, $\delta u = u - \overline{u}$, $\delta v = v - \overline{v}$, $\delta\theta = \theta - \overline{\theta}$, and by $\delta p = p - \overline{p}$. Then

$$\begin{gathered}
\partial_t\delta\theta - \Delta\delta\theta + (\delta u\cdot\nabla)\theta + (\overline{u}\cdot\nabla)\delta\theta = 0,\\
\partial_t\delta v - \Delta\delta v + (\delta v\cdot\nabla)u + (\overline{v}\cdot\nabla)\delta u = -\nabla\delta p + \delta\theta e_3,\\
\delta v = \delta u - \alpha^2\Delta\delta u,\\
\operatorname{div}\delta u = \operatorname{div}\delta v = 0,\\
(\delta u, \delta\theta)_{t=0} = (u^0 - \overline{u}^0, \theta^0 - \overline{\theta}^0).
\end{gathered}$$

We have $\frac{d}{dt}\,\delta\theta \in L^2([0,T], \dot H^{-1})$ and $\delta\theta \in L^2([0,T], \dot H^1)$. Moreover, $\frac{d}{dt}\,\delta v$ belongs to $L^2([0,T], \dot H^{-2})$ and $\delta u \in L^2([0,T], \dot H^2)$. By appropriate duality action, for almost every time t in $[0,T]$ we have

$$\Big\langle \frac{d}{dt}\,\delta\theta, \delta\theta\Big\rangle_{\dot H^{-1}} + \|\nabla\delta\theta\|_{L^2}^2 + \big\langle \delta u\cdot\nabla\theta, \delta\theta\big\rangle_{\dot H^{-1}} = 0,$$

$$\Big\langle \frac{d}{dt}\,\delta v, \delta u\Big\rangle_{\dot H^{-2}} + \big(\|\nabla\delta u\|_{L^2}^2 + \alpha^2\|\Delta\delta u\|_{L^2}^2\big) + \langle \delta v\cdot\nabla u, \delta u\rangle_{\dot H^{-2}} = \langle\delta\theta, \delta u\rangle_{\dot H^{-1}}.$$

Using the fact that (see, e.g., [8, Chapter 3, p. 169])

$$\Big\langle \frac{d}{dt}\,\delta\theta, \delta\theta\Big\rangle_{\dot H^{-1}(\mathbb{T}^3)} = \frac{1}{2}\,\frac{d}{dt}\,\|\delta\theta\|_{L^2(\mathbb{T}^3)}^2,$$

$$\Big\langle \frac{d}{dt}\,\delta v, \delta u\Big\rangle_{\dot H^{-2}(\mathbb{T}^3)} = \frac{1}{2}\,\frac{d}{dt}\,\big(\|\delta u\|_{L^2(\mathbb{T}^3)}^2 + \alpha^2\|\nabla\delta u\|_{L^2(\mathbb{T}^3)}^2\big),$$

and summing up, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big(\|\delta u\|^2_{L^2(\mathbb{T}^3)}+\alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)}+\|\delta\theta\|^2_{L^2(\mathbb{T}^3)}\big)\\ +\big(\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)}+\alpha^2\|\Delta\delta u\|^2_{L^2(\mathbb{T}^3)}\big)+\|\nabla\delta\theta\|^2_{L^2(\mathbb{T}^3)}\\ =\langle\delta\theta,\delta u\rangle_{\dot H^{-1}(\mathbb{T}^3)}\underbrace{-\langle\delta v\cdot\nabla u,\delta u\rangle_{\dot H^{-2}(\mathbb{T}^3)}}_{I_2}\underbrace{-\langle\delta u\cdot\nabla\theta,\delta\theta\rangle_{\dot H^{-1}(\mathbb{T}^3)}}_{I_3}.\end{aligned}$$

Using, respectively, the Cauchy–Schwarz and Young's inequalities, we obtain

$$\big|\langle\delta\theta,\delta u\rangle_{\dot H^{-1}(\mathbb{T}^3)}\big|\le\frac{1}{2}\big(\|\delta u\|^2_{L^2}+\|\delta\theta\|^2_{L^2}\big). \tag{3.6}$$

For I_2, we note that

$$\big|\langle\delta v\cdot\nabla u,\delta u\rangle_{\dot H^{-2}(\mathbb{T}^3)}\big|=\big|\langle\delta v\cdot\nabla u,\delta u\rangle_{L^2(\mathbb{T}^3)}\big|\le\|\delta u\|_{L^\infty(T^3)}\|\nabla u\|_{L^2(T^3)}\|\delta v\|_{L^2(T^3)}.$$

Using inequality (2.2), we obtain

$$|I_2|\le C\|\delta v\|_{L^2(T^3)}\|\nabla u\|_{L^2(T^3)}\|\delta u\|^{1/2}_{\dot H^1(T^3)}\|\delta u\|^{1/2}_{\dot H^2(T^3)}.$$

The velocity has zero average for positive times, thus we have

$$\|\delta v\|_{L^2(T^3)}\le(c+\alpha^2)\|\Delta\delta u\|_{L^2(T^3)}, \tag{3.7}$$

using (3.7) and Young's inequality, we obtain

$$\begin{aligned}|I_2|&\le C(c+\alpha^2)\|\nabla u\|_{L^2(T^3)}\|\delta u\|^{1/2}_{\dot H^1(T^3)}\|\delta u\|^{3/2}_{\dot H^2(T^3)}\\ &\le\frac{C}{\alpha^6}(c+\alpha^2)^4\|\nabla u\|^4_{L^2(T^3)}\|\nabla\delta u\|^2_{L^2(T^3)}+\frac{\alpha^2}{2}\|\Delta\delta u\|^2_{L^2(T^3)}.\end{aligned} \tag{3.8}$$

To estimate I_3, we use the Cauchy–Schwarz inequality twice to obtain

$$\big|\langle\delta u\cdot\nabla\theta,\delta\theta\rangle_{\dot H^{-1}(\mathbb{T}^3)}\big|\le\|\delta u\|_{L^3}\|\nabla\theta\|_{L^2}\|\delta\theta\|_{L^6}.$$

Next, inequalities (2.1), (2.3) and Sobolev's norm definition imply that

$$\big|\langle\delta u\cdot\nabla\theta,\delta\theta\rangle_{\dot H^{-1}(\mathbb{T}^3)}\big|\le\|\delta u\|^{1/2}_{L^2}\|\delta u\|^{1/2}_{\dot H^1}\|\nabla\theta\|_{L^2}\|\delta\theta\|_{\dot H^1}\le\|\delta u\|^{1/2}_{L^2}\|\nabla\delta u\|^{1/2}_{L^2}\|\nabla\theta\|_{L^2}\|\nabla\delta\theta\|_{L^2}.$$

Using twice the Young product inequality, we obtain

$$|I_3|\le\frac{1}{4\alpha}\big(\|\delta u\|^2_{L^2}+\alpha^2\|\nabla\delta u\|^2_{L^2}\big)\|\nabla\theta\|^2_{L^2}+\frac{1}{2}\|\nabla\delta\theta\|^2_{L^2}. \tag{3.9}$$

Summing up estimates (3.6), (3.8) and (3.9), we infer that

$$\begin{aligned}\frac{d}{dt}\big(\|\delta u\|^2_{L^2}+\alpha^2\|\nabla\delta u\|^2_{L^2}+\|\delta\theta\|^2_{L^2}\big)+\big(\|\nabla\delta u\|^2_{L^2}+\alpha^2\|\Delta\delta u\|^2_{L^2}\big)+\|\nabla\delta\theta\|^2_{L^2}\\ \le g(t)\big(\|\delta u\|^2_{L^2(\mathbb{T}^3)}+\alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)}+\|\delta\theta\|^2_{L^2(\mathbb{T}^3)}\big),\end{aligned}$$

where

$$g(t)=\Big(1+C\Big(\frac{1}{\alpha^8}+1\Big)\|\nabla u\|^4_{L^2}+\frac{1}{2\alpha}\|\nabla\theta\|^2_{L^2}\Big).$$

Dropping the dissipative positive term from the left-hand side, we obtain

$$\frac{d}{dt}\big(\|\delta u\|^2_{L^2(\mathbb{T}^3)}+\alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)}+\|\delta\theta\|^2_{L^2(\mathbb{T}^3)}\big)\le g(t)\big(\|\delta u\|^2_{L^2(\mathbb{T}^3)}+\alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)}+\|\delta\theta\|^2_{L^2(\mathbb{T}^3)}\big).$$

Since $\theta\in L^2([0,T],\dot H^1)$ and $u\in L^\infty([0,T],\dot H^1)$, Grönwall's lemma (cf. [5, Appendix A, p. 377]) leads to

$$\big(\|\delta u\|^2_{L^2(\mathbb{T}^3)}+\alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)}+\|\delta\theta\|^2_{L^2(\mathbb{T}^3)}\big)\le\big(\|\delta u^0\|^2_{L^2(\mathbb{T}^3)}+\alpha^2\|\nabla\delta u^0\|^2_{L^2(\mathbb{T}^3)}+\|\delta\theta^0\|^2_{L^2(\mathbb{T}^3)}\big)e^{\int_0^t g(s)\,ds}.$$

This implies the continuous dependence of the weak solution on the initial data in any bounded interval of time $[0,T]$. In particular, the solution is unique.

4 Decay results

Following [1], we introduce the change of functions $\varphi_n := \mathcal{F}^{-1}(e^{at|k|}\widehat{\theta}_n)$ and $w_n := \mathcal{F}^{-1}(e^{at|k|}\widehat{u}_n)$. Applying Fourier transform to (3.1) and to (3.2), we obtain

$$\partial_t \widehat{\varphi}_n + |k|(|k|-a)\widehat{\varphi}_n + e^{at|k|}\mathcal{F}(P_n(u_n \cdot \nabla\theta_n)) = 0, \tag{4.1}$$

$$(1+\alpha^2|k|^2)\big(\partial_t \widehat{w}_n + |k|(|k|-a)\widehat{w}_n\big) - \widehat{\varphi}_n e_3 + e^{at|k|}\mathcal{F}(P_n(v_n \cdot \nabla\theta_n)) = 0. \tag{4.2}$$

We note that under the divergence free condition, the pressure term vanishes. The Plancherel identity implies that the trilinear expressions vanish as $(v\cdot\nabla u, u)_{L^2} = 0$ and $(u\cdot\nabla\theta,\theta)_{L^2} = 0$. Taking the combinations $\overline{(4.1)}\widehat{\varphi}_n + (4.1)\overline{\widehat{\varphi}_n}$ and $\overline{(4.2)}\widehat{w}_n + (4.2)\overline{\widehat{w}_n}$, using the Cauchy–Schwarz inequality and the fact that

$$(1-a)|k|^2 \le |k|(|k|-a) \quad \forall\, k\in Z^3,$$

one obtains

$$\partial_t|\widehat{\varphi}_n|^2 + 2(1-a)|k|^2|\widehat{\varphi}_n|^2 = 0 \tag{4.3}$$

and

$$(1+\alpha^2|k|^2)\partial_t|\widehat{w}_n|^2 + 2(1-a)|k|^2(1+\alpha^2|k|^2)|\widehat{w}_n|^2 \le |\widehat{\varphi}_n|\,|\widehat{w}_n|. \tag{4.4}$$

Integrating (4.3) with respect to time and summing up over $k\in Z^3$, we obtain

$$\|\varphi(t,\cdot)\|_{L^2}^2 + (1-a)\int_0^t \|\nabla\varphi(\tau)\|_{L^2}^2\,d\tau \le \|\theta^0\|_{L^2}^2. \tag{4.5}$$

Integrating (4.4) with respect to time and summing up over $k\in Z^3$, we obtain

$$\|w(t)\|_{L^2}^2 + \alpha^2\|\nabla w(t)\|_{L^2}^2 + (1-a)\int_0^t \|\nabla w(s)\|_{L^2}^2 + \alpha^2\|\Delta w(s)\|_{L^2}^2\,ds$$
$$\le \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 + \|\theta^0\|_{L^2}\int_0^t \|w(\tau)\|_{L^2}\,d\tau.$$

Since $\partial_t|\widehat{w}_n|^2 \le |\widehat{\varphi}_n||\widehat{w}_n|$, we can deduce that

$$\|w(t)\|_{L^2}^2 + \alpha^2\|\nabla w(t)\|_{L^2}^2 + (1-a)\int_0^t \|\nabla w(s)\|_{L^2}^2 + \alpha^2\|\Delta w(s)\|_{L^2}^2\,ds$$
$$\le \big(\|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 + t\|\theta^0\|_{L^2}\big)^2. \tag{4.6}$$

Summing up estimates (4.5) and (4.6), one obtains

$$\|\varphi(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \alpha^2\|\nabla w(t)\|_{L^2}^2 + (1-a)\int_0^t \|\nabla\varphi(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \alpha^2\|\Delta w(t)\|_{L^2}^2$$
$$\le \big(\|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 + t\|\theta^0\|_{L^2}\big)^2.$$

As for the existence result, this energy estimate allows to run a standard compactness argument and to obtain the existence of (φ, w) such that $\varphi\in C(\mathbb{R}^+, L^2)\cap L^2(\mathbb{R}^+, H^1)$ and $w\in C(\mathbb{R}^+, H^1)\cap L^2(\mathbb{R}^+, H^2)$. In particular,

$$\sum_{k\in Z^3} e^{2at|k|}\big(|\theta(t,k)|^2 + (1+\alpha^2|k|^2)|u(t,k)|^2\big) \le \big(\|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 + t\|\theta^0\|_{L^2}\big)^2. \tag{4.7}$$

For zero-mean valued (θ, u), multiplying by $\exp(-2at)$, we deduce that θ and u vanish, respectively, in the L^2 and H^1 norm as time tends to infinity. Note that estimation (4.7) does not allow to deduce the decay result, so a sharper estimation is needed.

5 Convergence results

As α is destined to vanish, we can suppose that there exists a fixed α_0 such that $0 < \alpha \leq \alpha_0$. It follows that

$$\|\theta_\alpha\|_{L^2}^2 + \|u_\alpha\|_{L^2}^2 + \alpha^2\|\nabla u_\alpha\|_{L^2}^2 + 2\int_0^t \|\nabla\theta_\alpha\|_{L^2(\mathbb{T}^3)}^2\, d\tau$$
$$+ 2\int_0^t \left(\|\nabla u_\alpha\|_{L^2}^2 + \alpha^2\|\Delta u_\alpha\|_{L^2}^2\right) d\tau \leq \|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \alpha_0^2\|\nabla u^0\|_{L^2}^2 + \sigma_{\alpha_0}(t). \quad (5.1)$$

This implies that θ_α and u_α are uniformly bounded in $L^2([0,T],\dot H^1(\mathbb{T}^3))$ and v_α is uniformly bounded in $L^2([0,T],L^2(\mathbb{T}^3))$, then the Banach–Alaoglu theorem [6] allows to extract subsequences (u_α), (v_α), and (θ_α) such that $(\theta_\alpha,u_\alpha) \rightharpoonup (\theta,u)$ weakly in $L^2([0,T],\dot H^1(\mathbb{T}^3))$ and $v_\alpha \rightharpoonup u$ weakly in $L^2([0,T],L^2(\mathbb{T}^3))$ as $\alpha \to 0$. Using the energy estimate, we infer that (u_α,θ_α) converges to (u,θ) weakly in $L^2(\mathbb{T}^3)$ and uniformly over $[0,T]$. At this step, we have proved the two first results of statements 1 and 2 and the third statement of Theorem 1.3.

About time derivatives, since θ_α is uniformly bounded independently on α in the space $L^2([0,T],\dot H^1(\mathbb{T}^3))$, we find that $\Delta\theta_\alpha$ belongs to $L^2([0,T],\dot H^{-1}(\mathbb{T}^3))$. Furthermore, the energy estimate (5.1) implies that

$$\int_0^T \|\operatorname{div}\theta_\alpha u_\alpha\|_{\dot H^{-3/2}}^2 \leq \|\theta_\alpha\|_{L^\infty([0,T],L^2)}^2\|u_\alpha\|_{L^2([0,T],\dot H^1)}^2$$
$$\leq \frac{1}{2}\left(\|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2 + \alpha_0^2\|\nabla u^0\|_{L^2}^2 + \sigma_{\alpha_0}(t)\right)^2.$$

Then we obtain

$$\Big\|\frac{d}{dt}\,\theta_\alpha\Big\|_{L^2([0,T],\dot H^{-3/2})} \leq K_1,$$

where K_1 is a real positive constant. To handle the velocity derivatives, we apply the operator $(I-\alpha^2\Delta)^{-1}$ to the equation (3.2) and obtain

$$\frac{d}{dt}\,u_\alpha = \Delta u_\alpha - (I-\alpha^2\Delta)^{-1}(v_\alpha\cdot\nabla)u_\alpha + (I-\alpha^2\Delta)^{-1}\nabla p_\alpha + (I-\alpha^2\Delta)^{-1}\theta_\alpha e_3. \quad (5.2)$$

We have that u_α is uniformly bounded independently of α in $L^2([0,T],\dot H^1(\mathbb{T}^3))$, and it follows that Δu_α belongs to $L^2([0,T],\dot H^{-1}(\mathbb{T}^3))$. First, we note that

$$\big\|\,|(I-\alpha^2\Delta)^{-1}|\,\big\| \leq 1.$$

Then we use the Sobolev norms definition and product laws to get

$$\int_0^T \big\|(I-\alpha^2\Delta)^{-1}\operatorname{div}(v_\alpha\otimes u_\alpha)\big\|_{\dot H^{-5/2}}^2 \leq \int_0^T \big\|\operatorname{div}(v_\alpha\otimes u_\alpha)\big\|_{\dot H^{-5/2}}^2$$
$$\leq \int_0^T \|v_\alpha\|_{L^2}^2\|u_\alpha\|_{L^2}^2 \leq \|u_\alpha\|_{L^\infty([0,T],L^2)}^2\|v_\alpha\|_{L^2([0,T],L^2)}^2.$$

Thus, estimate (5.1) allows to bound the convective term. The linear terms are not problematic. Equation (5.2) implies that $\|\frac{d}{dt}\,u_{\alpha_k}\|_{L^2([0,T],\dot H^{-5/2}(\mathbb{T}^3))} \leq K$, where K is a real positive constant, and so on for $\frac{d}{dt}v_{\alpha_k}$ in the space $L^2([0,T],\dot H^{-9/2}(\mathbb{T}^3))$.

At this step, using Aubin's compactness theorem, we can extract subsequences of θ_α, u_α that converge strongly in $L^2([0,T],L^2(\mathbb{T}^3))$ and subsequence of v_α converging strongly in $L^2([0,T],\dot H^{-1}(\mathbb{T}^3))$.

Thus, as in the existence section, using Aubin's compactness theorem, we can take the weak limit in the variational formulation associated to the system (Bq_α). For $t \in [0;T]$ one obtains

$$(\theta(t),\Phi) - (\theta(0),\Phi) - \int_0^t (\theta,\Delta\Phi)\,d\tau + \int_0^t ((u\nabla)\theta,\Phi)\,d\tau = 0,$$

$$(u(t),\Psi) - (u(0),\Psi) - \int_0^t (u,\Delta\Psi)\,d\tau + \int_0^t ((u\nabla)u,\Psi)\,d\tau - \int_0^t (\theta e_3,\Psi)\,d\tau = 0$$

for all Φ and Ψ belonging to the space of infinitely differentiable functions with a compact support $\mathcal{D}(\mathbb{T}^3 \times [0,T))$.

On the other hand, θ_α converges weakly to θ and u_α converges weakly to u in $L^2([0,T],L^2(\mathbb{T}^3)) \cap L^2([0,T],\dot{H}^1(\mathbb{T}^3))$, which are Hilbert spaces. So, for all non-negative time t, we have

$$\|\theta\|_{L^2}^2 + \|u\|_{L^2}^2 \le \liminf_{\alpha\to 0} \left(\|\theta_\alpha\|_{L^2}^2 + \|u_\alpha\|_{L^2}^2 + \alpha^2\|\nabla u_\alpha\|_{L^2}^2\right),$$

and

$$2\int_0^t \|\nabla\theta\|_{L^2(\mathbb{T}^3)}^2\,d\tau + 2\int_0^t \|\nabla u\|_{L^2}^2\,d\tau \le \liminf_{\alpha\to 0} 2\int_0^t \|\nabla\theta_\alpha\|_{L^2(\mathbb{T}^3)}^2\,d\tau + 2\int_0^t \left(\|\nabla u_\alpha\|_{L^2}^2 + \alpha^2\|\Delta u_\alpha\|_{L^2}^2\right)d\tau.$$

Taking the lower limit as α tends to zero in the energy inequality (1.1), we obtain (1.2).

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Authors' addresses:

Ridha Selmi

1. Department of Mathematics, College of Sciences, Northern Border University, Arar, 91431, Kingdom of Saudi Arabia.

2. Department of Mathematics, Faculty of Sciences, University of Gabès, Gabès, 6000, Tunisia.

3. Department of Mathematics, Faculty of Sciences of Tunis, PDEs Lab, (LR03ES04), University of Tunis El Manar, 2092, Tunisia.

E-mails: Ridha.selmi@nbu.edu.sa; Ridha.selmi@isi.rnu.tn; ridhaselmiridhaselmi@gmail.com

Mounia Zaabi

Department of Mathematics, Faculty of Sciences of Tunis, PDEs Lab, (LR03ES04), University of Tunis El Manar, 2092, Tunisia.

E-mail: mounia.zaabi@fst.utm.tn

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Zurab Vashakidze

AN APPLICATION OF THE LEGENDRE POLYNOMIALS FOR THE NUMERICAL SOLUTION OF THE NONLINEAR DYNAMICAL KIRCHHOFF STRING EQUATION

Abstract. In the present work, the classical nonlinear Kirchhoff string equation is considered. A three-layer symmetrical semi-discrete scheme with respect to the temporal variable is applied for finding an approximate solution to the initial-boundary value problem for this equation, in which the value of the gradient of a non-linear term is taken at the middle point. This approach is essential because the inversion of the linear operator is sufficient for computations of approximate solutions for each temporal step. The variation method is applied to the spatial variable. Differences of the Legendre polynomials are used as coordinate functions. This choice of Legendre polynomials is also important for numerical realization. This way makes it possible to get a system whose structure does not essentially differ from the corresponding system of difference equations allowing us to use the methods developed for solving a system of difference equations. An application of the suggested variational-difference scheme for the numerical treatment of the stated nonlinear problem gives us an opportunity to solve the system of linear equations instead of a nonlinear one. It is proved that a matrix of the system of Galerkin's linear equations is positively defined and the stability of the factorization method is established.

The program of the numerical implementation with the corresponding interface is created based on the suggested algorithm, and numerical computations are carried out for the model problems.

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Key words and phrases. Non-linear Kirchhoff string equation, Cauchy problem, three-layer semi-discrete scheme, Galerkin method, Cholesky decomposition.

რეზიუმე. ნაშრომში განხილულია კირხჰოფის არაწრფივი კლასიკური სიმის განტოლება. ამ განტოლებისათვის დასმული საწყის-სასაზღვრო ამოცანის მიახლოებითი ამოხსნისათვის გამოყენებულია სამშრიანი, სიმეტრიული, ნახევრადდისკრეტული სქემა დროითი ცვლადის მიხედვით, სადაც გრადიენტის მნიშვნელობა არაწრფივ წევრში აღებულია შუა წერტილში. ეს მიდგომა მნიშვნელოვანია, რადგან მიახლოებითი ამონახსნის ყოველ დროით ბიჯზე გამოთვლისათვის საკმარისია წრფივი ოპერატორის შებრუნება. სივრცითი ცვლადების მიხედვით გამოყენებულია ვარიაციული მეთოდი. საკოორდინატო ფუნქციებად აღებულია ლეჟანდრის პოლინომების სხვაობა. საბაზისო ფუნქციებად ლეჟანდრის პოლინომების სხვაობის აღება მნიშვნელოვანია რიცხვითი რეალიზაციის თვალსაზრისით. ამ გზით მიიღება ისეთი სისტემა, რომლის სტრუქტურა არსებითად არ განსხვავდება შესაბამის სხვაობიან განტოლებათა სისტემისგან, რაც გვაძლევს საშუალებას გამოყენებულ იქნას სხვაობიანი სისტემის ამოხსნისთვის დამუშავებული მეთოდები. დასმული არაწრფივი ამოცანის რიცხვითი ამონახსნის საპოვნელად შემოთავაზებული ვარიაციულ-სხვაობიანი სქემის გამოყენება საშუალებას გვაძლევს ამოიხსნას წრფივი განტოლებათა სისტემა ნაცვლად არაწრფივისა. დამტკიცებულია გალიორკინის წრფივ განტოლებათა სისტემის მატრიცის დადებითად განსაზღვრულობა და დადგენილია ფაქტორიზაციის მეთოდის მდგრადობა.

შემოთავაზებული ალგორითმის საფუძველზე შეიქმნა რიცხვითი რეალიზაციის პროგრამა შესაბამისი ინტერფეისით, ჩატარდა რიცხვითი გამოთვლები მოდელური ამოცანებისათვის.

1 Introduction

For the first time, G. Kirchhoff generalized D'Alembert's classical linear model with the addition of a nonlinear term (see [14]). The issues on the existence and uniqueness of local and global solutions of initial-boundary value problems for the Kirchhoff string equation were first studied by S. Bernstein in 1940 (see [4]). The issues of the solvability of the classical and generalized Kirchhoff equations were later considered by many authors: Arosio, Panizzi [1], Arosio and Spagnolo [2], Berselli, Manfrin [5], D'Ancona, Spagnolo [7,8], Manfrin [17], Medeiros [19], Liu, Rincon [15], Matos [18] and Nishihara [20]. To the approximate solutions of initial-boundary value problems for classical equations the following works are devoted: Christie, Sanz-Serna [6], Peradze [3, 21, 22] and Temimi et al. [28]. Construction of algorithms of finding approximate solutions and their investigations for initial-boundary value problems of some classes integro-differential equations are considered in the monograph of Jangveladze, Kiguradze and Neta [13]. As far as we know, issues on the approximate solution in terms of a part of numerical realization to the Kirchhoff string equation are less studied.

We consider the nonlinear dynamical Kirchhoff string equation and look for an approximate solution to a Cauchy problem for this equation using the symmetric three-layer semi-discrete scheme with respect to the temporal variable. The value of the gradient in the nonlinear term of the equation is taken at the middle point. This type of semi-discrete schemes for a generalized Kirchhoff equation have been studied by Rogava and Tsiklauri [24–26]. Inversion of the liner operator makes it possible to find an approximate solution at each temporal step. The variation method is applied to a spatial variable. The differences of the Legendre polynomials are used as coordinate functions. An application of the Legendre polynomials to boundary value problems of equations of the theory of elasticity are considered in the monograph of Vashakmadze [30]. The Gauss-Legendre quadrature (see [16, 27]) is applied for numerical integration, where $[-1, 1]$ is the domain.

The results of the numerical computations of test problems are presented at the end of the paragraph. According to the numerical experiments, the order of convergence of the scheme is practically stated and it is shown that the constructed scheme describes well the behavior of an oscillating solution.

2 Statement of the problem and discretization for a temporal variable

Let us consider the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \left(\alpha + \beta \int_{-1}^{1} \left[\frac{\partial u(x,t)}{\partial x}\right]^2 dx\right) \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad (x,t) \in]-1,1[\times]0,T], \tag{2.1}$$

where $\alpha > 0$ and $\beta > 0$; $f(x,t)$ is a continuous function; $u(x,t)$ is an unknown function.

For equation (2.1), the following initial-boundary conditions

$$u(x,0) = \psi_0(x), \quad u'_t(x,0) = \psi_1(x), \tag{2.2}$$

$$u(-1,t) = 0, \quad u(1,t) = 0 \tag{2.3}$$

hold, where $\psi_0(x)$ and $\psi_1(x)$ are continuous functions, and, in addition, the compatibility condition $\psi_0(-1) = 0$, $\psi_0(1) = 0$ is fulfilled.

The segment $[0, 1]$ is divided into equal parts with uniform meshes τ, i.e.,

$$0 = t_0 < t_1 < \cdots < t_M = T,$$

where

$$t_k = k\tau \ \ (k = 0, 1, \dots, M), \ \ \tau = \frac{T}{M}\,.$$

We would like to find an approximate solution of problem (2.1)–(2.3) by using the following semi-discrete scheme:

$$\frac{u_{k+1}(x)-2u_k(x)+u_{k-1}(x)}{\tau^2}-\frac{1}{2}\,q_k\Big(\frac{d^2u_{k+1}(x)}{dx^2}+\frac{d^2u_{k-1}(x)}{dx^2}\Big)=f_k(x),\quad k=1,2,\dots,M-1,\tag{2.4}$$

where $f_k(x)=f(x,t_k)$,

$$q_k=\alpha+\beta\int_{-1}^{1}\Big(\frac{du_k(x)}{dx}\Big)^2\,dx.$$

As an approximate solution of $u(x,t)$ of problem (2.1)–(2.3) at the point $t_k=k\tau$, we declare $u_k(x)$, $u(x,t_k)\approx u_k(x)$.

From equation (2.4) we obtain

$$\Big(2I-\tau^2q_k\,\frac{d^2}{dx^2}\Big)u_{k+1}(x)=g_k(x),\tag{2.5}$$

where

$$g_k(x)=2\tau^2f_k(x)+4u_k(x)+\tau^2q_k\,\frac{d^2u_{k-1}(x)}{dx^2}-2u_{k-1}(x).$$

The values of the unknown functions on the zeroth and first layers are described by the initial conditions (2.2) and equation (2.1),

$$u_0(x)=\psi_0(x),\tag{2.6}$$

$$u_1(x)=\psi_0(x)+\tau\psi_1(x)+\frac{1}{2}\,\tau^2\Big(q_0\,\frac{d^2\psi_0(x)}{dx^2}+f_0(x)\Big).\tag{2.7}$$

Let us rewrite the boundary conditions (2.3) in the following form:

$$u_k(-1)=0,\quad u_k(1)=0.\tag{2.8}$$

3 A solution of the system of equations with the Galerkin method using the Legendre polynomials as coordinate functions

To find approximate solutions of problem (2.1)–(2.3) per temporal step we apply the following linear combination:

$$\widetilde{u}_k(x)=\sum_{m=1}^{N}c_m^k\varphi_m(x),\tag{3.1}$$

where the coordinate functions $\varphi_m(x)$ represent differences of the Legendre polynomials, i.e.,

$$\varphi_m(x)=\sqrt{\frac{2m+1}{2}}\int_{-1}^{x}P_m(s)\,ds=A_m\big(P_{m+1}(x)-P_{m-1}(x)\big),\quad A_m=\frac{1}{\sqrt{2(2m+1)}}\,.\tag{3.2}$$

For any $(k+1)$-th layers, the coefficients c_m^{k+1} ($k=1,2,\dots,M-1$) can be found from the following equation:

$$\bigg(\Big(2I-\tau^2q_k\,\frac{d^2}{dx^2}\Big)u_{k+1}(x)-g_k(x),\varphi_m(x)\bigg)=0.\tag{3.3}$$

Putting (3.1) into equation (3.3), we finally get

$$\bigg(\sum_{i=1}^{N}c_i^{k+1}\Big(2I-\tau^2q_k\,\frac{d^2}{dx^2}\Big)\varphi_i(x),\varphi_m(x)\bigg)=\big(g_k(x),\varphi_m(x)\big).\tag{3.4}$$

The key property of the Legendre polynomials is given (see [9, 12]) in the form

$$\int\limits_{-1}^{1} P_i(x)P_n(x)\,dx = \frac{2}{\sqrt{(2i+1)(2n+1)}}\,\delta_{in}, \tag{3.5}$$

where δ_{in} is the Kronecker symbol.

We introduce the notation

$$\widetilde{P}_i(x) = \sqrt{\frac{2i+1}{2}}\,P_i(x).$$

It is easy to see that

$$\varphi'_m(x) = \widetilde{P}_m(x). \tag{3.6}$$

If we apply the integration by parts with the boundary conditions (2.8), we get

$$\int\limits_{-1}^{1} \Big(\frac{du_k(x)}{dx}\Big)^2\,dx = -\int\limits_{-1}^{1} \frac{d^2u_k(x)}{dx^2}\,u_k(x)\,dx. \tag{3.7}$$

The usage of the integration by parts, due to (3.5) and (3.6), yields

$$\int\limits_{-1}^{1} \frac{d^2\varphi_i(x)}{dx^2}\,\varphi_m(x)\,dx = -\delta_{im}. \tag{3.8}$$

Now, let us rewrite equality (3.5) in terms of A_i and A_m:

$$\int\limits_{-1}^{1} P_i(x)P_m(x)\,dx = 4A_iA_m\delta_{im}. \tag{3.9}$$

According to (3.9), we get

$$\int\limits_{-1}^{1} \varphi_i(x)\varphi_m(x)\,dx = 4A_iA_m\big(A_{i+1}A_{m+1}\delta_{i+1,m+1} - A_{i+1}A_{m-1}\delta_{i+1,m-1} - A_{i-1}A_{m+1}\delta_{i-1,m+1} + A_{i-1}A_{m-1}\delta_{i-1,m-1}\big). \tag{3.10}$$

If we take equalities (3.7) and (3.8) into account, we obtain

$$q_k = \alpha + \beta\sum_{m=1}^{N}\big(c_m^k\big)^2. \tag{3.11}$$

From (3.10) we get

$$\begin{aligned}\big(u_{k+1}(x),\varphi_m(x)\big) &= \sum_{i=1}^{N} c_i^{k+1}\int\limits_{-1}^{1}\varphi_i(x)\varphi_m(x)\,dx\\ &= 4\big(-A_{m-2}A_{m-1}^2A_mc_{m-2}^{k+1} + A_m^2(A_{m-1}^2+A_{m+1}^2)c_m^{k+1} - A_mA_{m+1}^2A_{m+2}c_{m+2}^{k+1}\big),\end{aligned}$$

Let us introduce the following notation:

$$B_m = 4A_{m-1}A_m^2A_{m+1}, \qquad B_m = \frac{1}{(2m+1)\sqrt{(2m-1)(2m+3)}}, \tag{3.12}$$

$$C_m = 4A_m^2(A_{m-1}^2+A_{m+1}^2) = 8A_{m-1}^2A_{m+1}^2, \qquad C_m = \frac{2}{(2m-1)(2m+3)}. \tag{3.13}$$

According to (3.12) and (3.13), the inner product of $(u_{k+1}(x),\varphi_m(x))$ can be rewritten in the following form:

$$\big(u_{k+1}(x),\varphi_m(x)\big)=-B_{m-1}c_{m-2}^{k+1}+C_mc_m^{k+1}-B_{m+1}c_{m+2}^{k+1}. \tag{3.14}$$

From (3.8) we conclude that

$$\Big(\frac{d^2u_{k+1}(x)}{dx^2},\varphi_m(x)\Big)=-c_m^{k+1}. \tag{3.15}$$

Finally, if we use (3.14) and (3.15), for the calculation of inner product of the left-hand side of equation (3.4), we get the equality

$$\Big(\sum_{i=1}^{N}c_i^{k+1}(2I-\tau^2q_k\frac{d^2}{dx^2})\varphi_i(x),\varphi_m(x)\Big)=-2B_{m-1}c_{m-2}^{k+1}+(2C_m+\tau^2q_k)c_m^{k+1}-2B_{m+1}c_{m+2}^{k+1}. \tag{3.16}$$

For the right-hand side of equation (3.4), we have

$$\begin{aligned}\big(g_k(x),\varphi_m(x)\big)&=-2B_{m-1}(2c_{m-2}^{k}-c_{m-2}^{k-1})\\&\quad+2C_m(2c_m^k-c_m^{k-1})-\tau^2(q_kc_m^{k-1}-2I_m^k)-2B_{m+1}(2c_{m+2}^k-c_{m+2}^{k-1}).\end{aligned} \tag{3.17}$$

For every $k=1,2,\dots,M-1$, we obtain the following system of linear equations:

$$\begin{aligned}&-2B_{m-1}c_{m-2}^{k+1}+(2C_m+\tau^2q_k)c_m^{k+1}-2B_{m+1}c_{m+2}^{k+1}\\&\qquad=-2B_{m-1}(2c_{m-2}^{k}-c_{m-2}^{k-1})+2C_m(2c_m^k-c_m^{k-1})\\&\qquad\quad-\tau^2(q_kc_m^{k-1}-2I_m^k)-2B_{m+1}(2c_{m+2}^k-c_{m+2}^{k-1}).\end{aligned} \tag{3.18}$$

To find coefficients c_m^{k+1} $(k=1,2,\dots,M-1)$, we have first to find c_m^0 and c_m^1. To this end, we calculate the inner products $(u_0(x),\varphi_m(x))$ and $(u_1(x),\varphi_m(x))$:

$$-B_{m-1}c_{m-2}^0+C_mc_m^0-B_{m+1}c_{m+2}^0=\widetilde{I}_m^0, \tag{3.19}$$

$$-B_{m-1}c_{m-2}^1+C_mc_m^1-B_{m+1}c_{m+2}^1=\widetilde{I}_m^0+\tau\widetilde{I}_m^1-\frac{1}{2}\,\tau^2(q_0c_m^0-I_m^0). \tag{3.20}$$

The values of summands with negative indices in (3.18), (3.19) and (3.20) we set equal to zeros.

The notation of I_m^k, $\widetilde{I}_m^0$ and $\widetilde{I}_m^1$ denote the inner products $(f_k(x),\varphi_m(x))$, $(u_0(x),\varphi_m(x))$ and $(u_1(x),\varphi_m(x))$, respectively. We calculate approximately the already-mentioned inner products using the Gauss–Legendre quadrature rule (see [16,27]), which is exact for polynomials of degree $2N-1$ or less.

We rewrite the system of linear equations (3.18) in a matrix form. Let us introduce the following notation:

$$\begin{aligned}D_m^k&=2C_m+\tau^2q_k,\\F_m^k&=-2B_{m-1}(2c_{m-2}^k-c_{m-2}^{k-1})+2C_m(2c_m^k-c_m^{k-1})\\&\quad-\tau^2(q_kc_m^{k-1}-2I_m^k)-2B_{m+1}(2c_{m+2}^k-c_{m+2}^{k-1}).\end{aligned}$$

According to the above-mentioned notation, the system of linear equations has the form

$$\begin{pmatrix}D_1^k&0&-2B_2&0&\cdots&0\\0&D_2^k&0&-2B_3&\ddots&\vdots\\-2B_2&0&D_3^k&0&\ddots&0\\0&-2B_3&0&\ddots&\ddots&-2B_{m-1}\\\vdots&\ddots&\ddots&\ddots&D_{m-1}^k&0\\0&\cdots&0&-2B_{m-1}&0&D_m^k\end{pmatrix}\begin{pmatrix}c_1^{k+1}\\c_2^{k+1}\\c_3^{k+1}\\c_4^{k+1}\\\vdots\\c_m^{k+1}\end{pmatrix}=\begin{pmatrix}F_1^k\\F_2^k\\F_3^k\\F_4^k\\\vdots\\F_m^k\end{pmatrix}. \tag{3.21}$$

The following statement takes place.

Theorem 3.1. *The matrix of the system of Galerkin's linear equations* (3.21) *is positively defined.*

This theorem is a result of the following

Lemma 3.1. *Let us consider a general operator equation in a Hilbert space H,*

$$Au = f, \quad f \in H,$$

where the operator A is symmetric and satisfies the condition

$$(Au, u) \geq \alpha(Bu, u) + \nu\|u\|^2, \quad \forall\, u \in D(A) \subset D(B), \tag{3.22}$$

B is also a symmetric operator, besides $D(A) \subset D(B)$; α and ν are the positive constants.

The matrix of the system of linear equations (3.21) *is positively defined when the basis functions $\{\varphi_k\}_{k=1}^{\infty}$ are B-orthogonal, which means that*

$$(B\varphi_k, \varphi_i) = \delta_{ki}. \tag{3.23}$$

Proof. We denote the Galerkin system of equations by S_N. Let us introduce the vector

$$v_N = (c_1, c_2, \dots, c_N)^{\top}.$$

We can straightforwardly show that

$$S_N v_N = \big((Au_N, \varphi_1), (Au_N, \varphi_2), \dots, (Au_N, \varphi_N)\big)^T,$$

where

$$u_N = \sum_{k=1}^{N} c_k \varphi_k. \tag{3.24}$$

Indeed,

$$(Au_N, \varphi_i) = \Big(\sum_{k=1}^{N} c_k A\varphi_k, \varphi_i\Big) = \sum_{k=1}^{N} (A\varphi_k, \varphi_i) c_k \;\; (i = 1, 2, \dots, N). \tag{3.25}$$

Due to (3.25), we have

$$\begin{aligned}(S_N v_N, v_N) &= c_1(Au_N, \varphi_1) + c_2(Au_N, \varphi_2) + \dots + c_N(Au_N, \varphi_N) \\ &= (Au_N, c_1\varphi_1) + (Au_N, c_2\varphi_2) + \dots + (Au_N, c_N\varphi_N) = \Big(Au_N, \sum_{k=1}^{N} c_k\varphi_k\Big) = (Au_N, u_N),\end{aligned}$$

and obtain

$$(S_N v_N, v_N) = (Au_N, u_N). \tag{3.26}$$

From (3.22) and (3.26) it follows that

$$(S_N v_N, v_N) \geq \alpha(Bu_N, u_N) + \nu\|u_N\|^2. \tag{3.27}$$

Inserting (3.24) into inequality (3.27) and also taking into account the B-orthogonality (3.23), we get

$$\begin{aligned}(S_N v_N, v_N) &\geq \alpha\Big(\sum_{k=1}^{N} c_k B\varphi_k, \sum_{i=1}^{N} c_i B\varphi_i\Big) + \nu\|u_N\|^2 \\ &\geq \alpha \sum_{k=1}^{N}\sum_{i=1}^{N} c_k c_i (B\varphi_k, \varphi_i) = \alpha \sum_{k=1}^{N} c_k^2 = \alpha\|v_N\|^2.\end{aligned}$$

□

Remark 3.1. Obviously, for equation (2.5) we have

$$(Au, u) = 2\|u\|^2 + \tau^2 q_k(Bu, u),$$

where $A = 2I + \tau^2 q_k B$ and $B = -\frac{d^2}{dx^2}$, $D(A) = D(B) = \{u(x) \in C^2([-1,1]) \mid u(-1) = u(1) = 0\}$. It is well-known that the operator B is positive (see [23]).

Remark 3.2. The matrix of system (3.21) is diagonally dominant of order $\mathcal{O}(\frac{1}{m^3})$ and the following inequality holds:

$$C_m + \frac{m+4}{(2m-1)(2m+3)(m-1)(m+1)} > B_{m-1} + B_{m+1} \ \ (m = 3, 4, \dots, N-2).$$

Proof. We note that for the coefficient B_m $(m = 2, 3, \dots, N-1)$ in (3.12) the following double inequality holds:

$$(2m)^2 < (2m-1)(2m+3) < (2m+1)^2 \tag{3.28}$$

Due to (3.28), for B_{m-1} and B_{m+1}, the inequalities

$$4(m-1)^2 < (2m-3)(2m+1) < (2m-1)^2 \tag{3.29}$$

and

$$4(m+1)^2 < (2m+1)(2m+5) < (2m+3)^2 \tag{3.30}$$

are fulfilled, respectively.

Let us evaluate the expression $B_{m-1} + B_{m+1} - C_m$ $(m = 3, 4, \dots, N-2)$. Taking into account (3.29) and (3.30) we get

$$\frac{16}{(2m-1)^2(2m+3)^2} < B_{m-1} + B_{m+1} - C_m < \frac{m+4}{(2m-1)(2m+3)(m-1)(m+1)}.$$

For the first two and the last two rows of the matrix of system (3.21), we have the following estimations:

$$\begin{gathered}
\frac{7}{20} < C_1 - B_2 < \frac{9}{25}\,, \\
\frac{1}{14} < C_2 - B_3 < \frac{11}{147}\,, \\
\frac{2N-9}{2(2N-3)(2N+1)(N-2)} < C_{N-1} - B_{N-2} < \frac{2N-7}{(2N-3)^2(2N+1)}\,, \\
\frac{2N-7}{2(2N-1)(2N+3)(N-1)} < C_N - B_{N-1} < \frac{2N-5}{(2N-1)^2(2N+3)}\,.
\end{gathered}$$

□

For the solution of system (3.21) we consider the so-called Cholesky decomposition (see [10, 11, 27, 29])

$$A = LDL^\top \tag{3.31}$$

of a symmetric, positively defined matrix $A = (a_{i,j})_{N\times N}$, where L is a lower triangular matrix having identities of the main diagonal, $L^\top$ is the transposed matrix of L and D is a diagonal matrix. Applying the decomposition similar to (3.31), the system of linear equations

$$Ax = b$$

can be split into the following sub-systems:

$$\begin{cases} Lz = b, \\ Dy = z, \\ L^\top x = y. \end{cases}$$

For the system of equations on the layers $k = 0$ and $k = 1$, we get

$$Ac^{(n)} = b^{(n)}, \ \ n = 0, 1, \tag{3.32}$$

a solution of system (3.32) has the following form ($n = 0, 1$):

$$\begin{cases} z_m^{(n)} = b_m^{(n)}, & m \in \{1, 2\}; \\ z_m^{(n)} = b_m^{(n)} + \dfrac{B_{m-1}}{d_{m-2}}\, z_{m-2}^{(n)}, & m \in \{3, 4, \dots, N\}; \\ y_m^{(n)} = \dfrac{z_m^{(n)}}{d_m}, & m \in \{1, 2, \dots, N\}; \\ c_m^{(n)} = y_m^{(n)}, & m \in \{N, N-1\}; \\ c_m^{(n)} = y_m^{(n)} + \dfrac{B_{m+1}}{d_m}\, c_{m+2}^{(n)}, & m \in \{N-2, N-3, \dots, 1\}, \end{cases}$$

where

$$\begin{cases} d_m = C_m, & m \in \{1, 2\}; \\ d_m = C_m - \dfrac{B_{m-1}^2}{d_{m-2}}, & m \in \{3, 4, \dots, N\}. \end{cases}$$

Any $(k+1)$-th layers, a solution of linear algebraic system of equations $A^{(k)} c^{(k+1)} = F^{(k)}$, where $k = 1, 2, \dots, M-1$, has the following form:

$$\begin{cases} z_m^{(k+1)} = F_m^{(k)}, & m \in \{1, 2\}; \\ z_m^{(k+1)} = F_m^{(k)} + \dfrac{2B_{m-1}}{d_{m-2}^{(k)}}\, z_{m-2}^{(k+1)}, & m \in \{3, 4, \dots, N\}; \\ y_m^{(k+1)} = \dfrac{z_m^{(k+1)}}{d_m^{(k)}}, & m \in \{1, 2, \dots, N\}; \\ c_m^{(k+1)} = y_m^{(k+1)}, & m \in \{N, N-1\}; \\ c_m^{(k+1)} = y_m^{(k+1)} + \dfrac{2B_{m+1}}{d_m^{(k)}}\, c_{m+2}^{(k+1)}, & m \in \{N-2, N-3, \dots, 1\}, \end{cases}$$

where

$$\begin{cases} d_m^{(k)} = 2C_m + \tau^2 q_k, & m \in \{1, 2\}; \\ d_m^{(k)} = (2C_m + \tau^2 q_k) - \dfrac{4B_{m-1}^2}{d_{m-2}^{(k)}}, & m \in \{3, 4, \dots, N\}. \end{cases}$$

4 Analysis of the numerical results

Let us consider the initial-boundary value problem (2.1)–(2.3) with the constants $\alpha = \beta = 1$ and $t \in [0, 1]$. For this problem we take two cases of tests, which are also considered in [25].

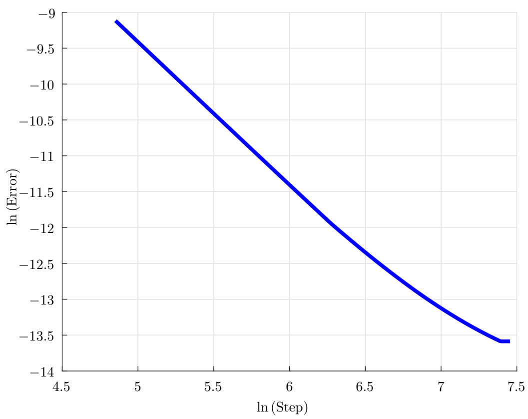
Test 1:

$$\psi_0(x) = 0, \quad \psi_1(x) = m\pi \sin(\pi x),$$
$$f(x,t) = \pi^2\big(-m^2 + (\alpha + \beta\pi^2 \sin^2(m\pi t))\big) \sin(m\pi t) \sin(\pi x).$$

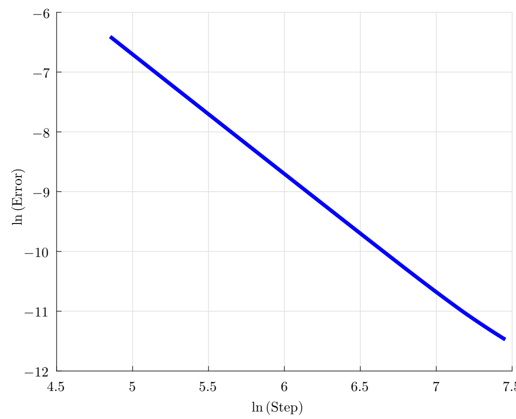
Test 2:

$$\psi_0(x) = \sin(m\pi x), \quad \psi_1(x) = \pi \sin(m\pi x),$$
$$f(x,t) = \pi^2\big(1 + m^2(\alpha + \beta m^2 \pi^2 \mathrm{e}^{2\pi t})\big) \mathrm{e}^{\pi t} \sin(m\pi x).$$

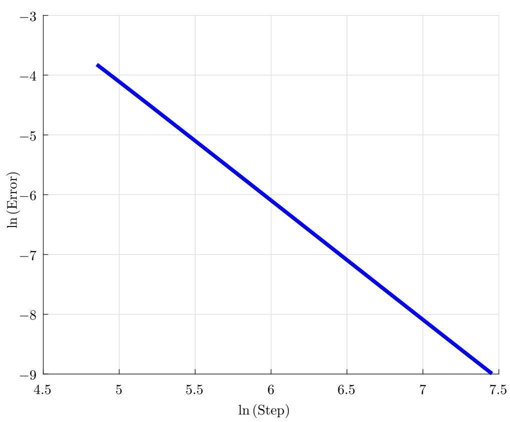
The solutions of Test 1 and Test 2 are $u(x,t) = \sin(m\pi t)\sin(\pi x)$ and $u(x,t) = \mathrm{e}^{\pi t}\sin(m\pi x)$, respectively.



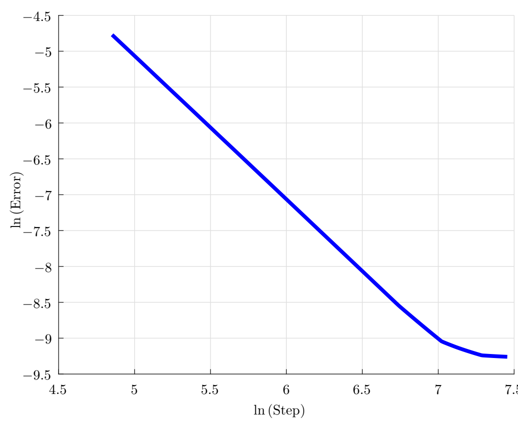
(a) $m = 1$.



(b) $m = 3$.



(c) $m = 5$.



(d) $m = 7$.

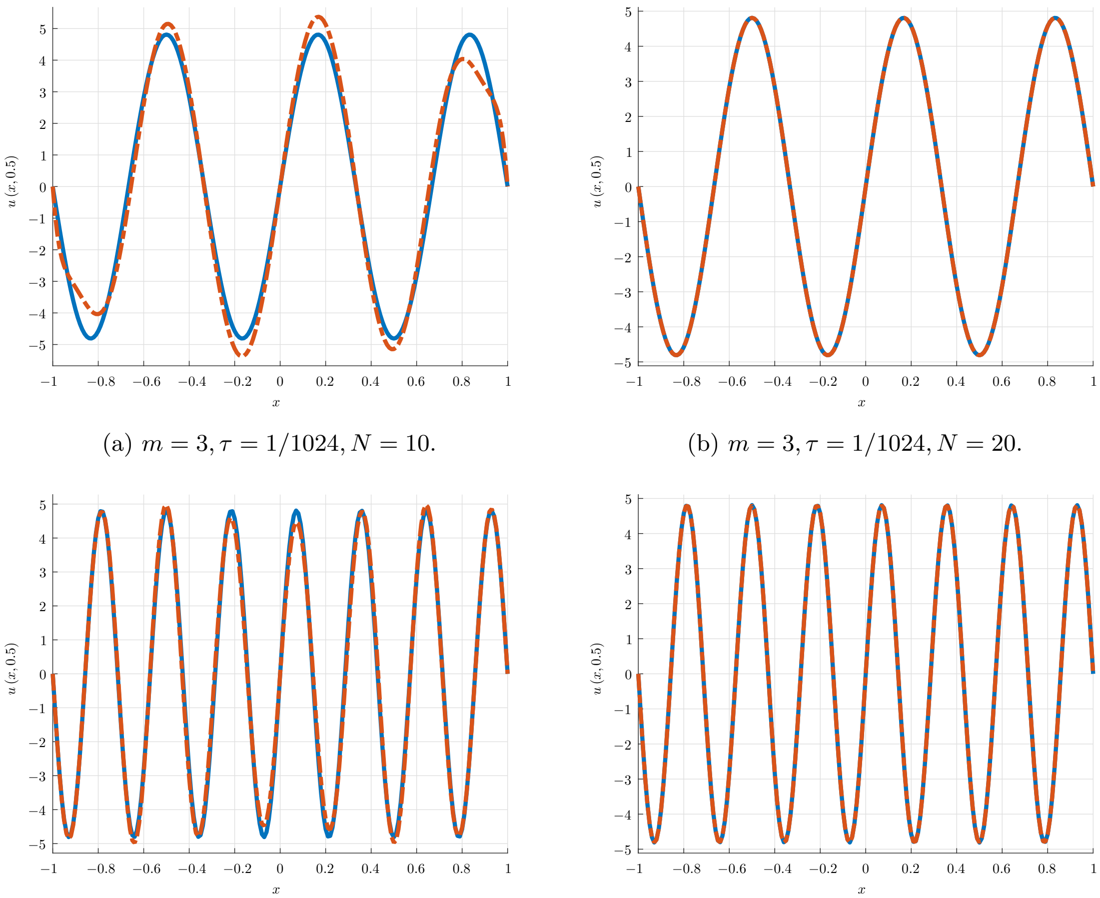
Figure 1: Dependence of logarithm of relative error on logarithm of the temporal step.

In Figure 1, there is a dependence of the logarithm of relative error of the approximated solution of Test 1 on the logarithm of the temporal step. On the horizontal axis there is the logarithm of temporal step, and on the vertical axis there is the logarithm of a relative error of the approximated solution. In all the four pictures, starting from the certain time step, the curve approaches the line, whose angular coefficient is -2, which confirms that the approximate solution obtained by the considered scheme is of the second order accuracy. For this case, eleven ($N = 11$) coordinate functions are taken and the errors of each temporal step are calculated with a maximum norm.

In Figure 2, there are approximate and exact solutions of Test 2 at the point $t = 0.5$. The approximate and exact solutions are shown as dashed and continuous curves, respectively. The errors between the exact and approximate solutions are calculated by a maximum norm and in each cases they represent the following values:

$$\begin{aligned}
\|u(x,0.5) - \widetilde{u}(x,0.5)\|_\infty &\approx 1.00 \times 10^{0}, \\
\|u(x,0.5) - \widetilde{u}(x,0.5)\|_\infty &\approx 4.44 \times 10^{-5}, \\
\|u(x,0.5) - \widetilde{u}(x,0.5)\|_\infty &\approx 3.43 \times 10^{-1}, \\
\|u(x,0.5) - \widetilde{u}(x,0.5)\|_\infty &\approx 3.31 \times 10^{-5}
\end{aligned}$$

with respect to the cases (a), (b), (c) and (d). In Figure 2, (a) and (b) represent the case $m = 3$, and (c) and (d) represent the case $m = 7$. In figures (a) and (b), the value of τ is the same, but the amount of the coordinate functions is different. Analogously, figures (c) and (d) have the same



(a) $m = 3, \tau = 1/1024, N = 10$.

(b) $m = 3, \tau = 1/1024, N = 20$.

(c) $m = 7, \tau = 1/4096, N = 30$.

(d) $m = 7, \tau = 1/4096, N = 35$.

Figure 2: Exact and approximate solutions at the point of 0.5 with respect to the temporal variable, which are represented by solid and dashed lines, respectively.

mesh length, however, the number of the coordinate functions is not equal to each others. As the tests show, increasing of only temporal layers is not enough to reach high order accuracy, we need to rise the amount of the coordinate functions. Nevertheless, there exists some relationship between numbers of layers and the coordinate functions.

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Author's addresses:

1. Institute of Mathematics, School of Science and Technology, The University of Georgia, 77a M. Kostava St., Tbilisi 0171, Georgia.

2. Ilia Vekua Institute of Applied Mathematics of Ivane Javakhishvili Tbilisi State University, 2 University St., Tbilisi 0186, Georgia.

E-mails: zurab.vashakidze@gmail.com, z.vashakidze@ug.edu.ge

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