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ASYMPTOTIC REPRESENTATIONS OF SOLUTIONS WITH SLOWLY VARYING DERIVATIVES OF ESSENTIALLY NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

Abstract. Differential equations of the second order with nonlinearities of rather general type that are in some sense near to the power ones are considered. For some class of solutions with derivatives of the first order that are slowly varying functions as the argument tends to the critical point, the conditions of the existence and asymptotic representations are found.

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Let us consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y') \exp\big(R(|\ln|yy'||)\big), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ are continuous functions, $R :]0; +\infty[\to]0; +\infty[$ is a continuously differentiable function, that is, regularly varying at infinity of order μ, $0 < \mu < 1$, and has a monotone derivative. Here, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either the interval $[y_i^0; Y_i[$ [1], or the interval $]Y_i; y_i^0]$ $(i = 0, 1)$. Moreover, it is supposed that every function φ_i $(i = 0, 1)$ is regularly varying of order σ_i [4, Chapter 1, § 1.1, p. 9] as the argument tends to Y_i and $\sigma_0 + \sigma_1 \ne 1$.

The solution y of equation (1) defined on the interval $[t_0, \omega[\subset [a, \omega[$ is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution $(-\infty \le \lambda_0 \le +\infty)$ if the conditions

$$\lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0 \tag{2}$$

are satisfied.

Let the function $\varphi : \Delta_Y \to]0, +\infty[$ be regularly varying of order σ as $z \to Y$ $(z \in \Delta_Y, Y \in \{0, \infty\}$, Δ_Y is a one-sided neighborhood of $Y)$. We say that the function φ satisfies the condition S if for any slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $L : \Delta_{Y_i} \to]0; +\infty[$ such that

$$\lim_{\substack{z\to Y\\ z\in\Delta_Y}} \frac{zL'(z)}{L(z)} = 0,$$

the equality

$$\Theta(zL(z)) = \Theta(z)(1 + o(1)) \ \text{ as } \ z \to Y \ \ (z \in \Delta_Y)$$

takes place, where $\Theta(z) = \varphi(z)|z|^{-\sigma}$.

Some classes of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equation (1) were investigated earlier (see, e.g., [3]). The sufficiently important class of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equations like (1) has been considered only for the cases $R(z) \equiv 0$ and $\varphi_0(z)|z|^{-\sigma_0}$ satisfies the condition S. Later, it has turned out to extend the results on more general cases (see, e.g., [1]). But the functions that do not satisfy the condition S, but contain in the left-hand side the derivative of an unknown function as in a general case of equation (1), have not been considered before. Notice that the derivative of every $P_\omega(Y_0, Y_1, \pm\infty)$-solution is a slowly varying function as $t \uparrow \omega$. It makes a lot of difficulties when conducting investigations.

We need the following auxiliary notation

$$\pi_\omega(t) = \begin{cases} t & \text{as } \ \omega = +\infty, \\ t - \omega & \text{as } \ \omega < +\infty, \end{cases} \quad \Theta_i(z) = \varphi_i(z)|z|^{-\sigma_i} \ \ (i = 0, 1)$$

and in case $\lim_{t\uparrow\omega} |\pi_\omega(t)| \operatorname{sign} y_0^0 = Y_0$,

$$N(t) = \alpha_0 p(t)|\pi_\omega(t)|^{\sigma_0+1}\Theta_0\big(|\pi_\omega(t)| \operatorname{sign} y_0^0\big) \ \text{ as } \ t \in [b, \omega[\,,$$

$$I_0(t) = \alpha_0 \int_{A_\omega^0}^{t} p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\Theta_0\big(|\pi_\omega(\tau)| \operatorname{sign} y_0^0\big)\, d\tau,$$

$$A_\omega^0 = \begin{cases} b & \text{as } \displaystyle\int_b^\omega p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\Theta_0\big(|\pi_\omega(\tau)| \operatorname{sign} y_0^0\big)\, d\tau = +\infty, \\ \omega & \text{as } \displaystyle\int_b^\omega p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\Theta_0\big(|\pi_\omega(\tau)| \operatorname{sign} y_0^0\big)\, d\tau < +\infty. \end{cases}$$

Here, we choose $b \in [a, \omega[$ in such a way that $|\pi_\omega(t)| \operatorname{sign} y_0^0 \in \Delta_0$ as $t \in [b, \omega[\,$.

[1] If $Y_i = +\infty$ $(Y_i = -\infty)$, we respectively suppose that $y_i^0 > 0$ $(y_i^0 < 0)$.

Theorem 1. *The conditions*

$$Y_0 = \begin{cases} \pm\infty & \text{as } \omega = +\infty, \\ 0 & \text{as } \omega < +\infty, \end{cases} \qquad \pi_\omega(t) y_0^0 y_1^0 > 0 \;\; \text{as} \;\; t \in [a; \omega[\tag{3}$$

are necessary for the existence of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation (1)*. If the function φ_0 satisfies the condition S and*

$$\lim_{t \uparrow \omega} \frac{R'(|\ln|\pi_\omega(t)||) I_0(t)}{\pi_\omega(t) I_0'(t)} = 0, \tag{4}$$

then the conditions

$$\begin{gathered} y_1^0 I_0(t)(1 - \sigma_0 - \sigma_1) > 0 \;\; \text{as} \;\; t \in [a, \omega[\,, \\ \lim_{t \uparrow \omega} y_1^0 |I_0(t)|^{\frac{1}{1-\sigma_0-\sigma_1}} = Y_1, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) I_0'(t)}{I_0(t)} = 0 \end{gathered} \tag{5}$$

together with conditions (3) *are necessary and sufficient for the existence of the above-mentioned solutions of equation* (1)*. Moreover, for each $P_\omega(Y_0, Y_1, \pm\infty)$-solution of equation* (1) *the asymptotic representations*

$$\begin{gathered} \frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t)) \exp(R(|\ln|y(t)y'(t)||))} = (1 - \sigma_0 - \sigma_1) I_0(t)[1 + o(1)], \\ \frac{y'(t)}{y(t)} = \frac{1}{\pi_\omega(t)} [1 + o(1)] \end{gathered} \tag{6}$$

take place as $t \uparrow \omega$.

If condition (4) is not valid, there takes place the next theorem with another condition (7). Note that if the limit of the left-hand side of equality (4) is equal to infinity, then condition (7) takes place in most cases.

Theorem 2. *Let the function p in equation* (1) *be continuously differentiable in its domain. If the function φ_0 satisfies the condition S and*

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) N'(t)}{R'(|\ln|\pi_\omega(t)||) N(t)} = 0, \tag{7}$$

then the conditions

$$\begin{gathered} \alpha_0 y_1^0 (1 - \sigma_0 - \sigma_1) \ln|\pi_\omega(t)| > 0 \;\; \text{as} \;\; t \in [a, \omega[\,, \\ \lim_{t \uparrow \omega} y_1^0 \exp\Big(\frac{1}{1 - \sigma_0 - \sigma_1} R\big(|\ln|\pi_\omega(t)||\big) \Big) = Y_1 \end{gathered} \tag{8}$$

together with conditions (3) *are necessary and sufficient for the existence of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation* (1)*. Moreover, for every such solution the asymptotic representations*

$$\begin{gathered} \frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t)) \exp(R(|\ln|y(t)y'(t)||))} = \frac{(1 - \sigma_0 - \sigma_1) N(t)}{R'(|\ln|\pi_\omega(t)||)} [1 + o(1)], \\ \frac{y'(t)}{y(t)} = \frac{1}{\pi_\omega(t)} [1 + o(1)] \end{gathered} \tag{9}$$

take place as $t \uparrow \omega$.

Proof of Theorem 1. The necessity. Let the function $y : [t_0, \omega[\to \Delta_{Y_0}$ be a $P_\omega(Y_0, Y_1, \pm\infty)$-solution of equation (1). By virtue of (2), the equality

$$\frac{y''(t) y(t)}{(y'(t))^2} = \Big(\frac{y'(t)}{y(t)} \Big)' \cdot \Big(\frac{y'(t)}{y(t)} \Big)^{-2} + 1$$

implies that

$$\Big(\frac{y'(t)}{y(t)} \Big)' \cdot \Big(\frac{y'(t)}{y(t)} \Big)^{-2} = -1 + o(1) \;\; \text{as} \;\; t \uparrow \omega.$$

From this, in view of (2), we have the following asymptotic representations:

$$y(t) = \pi_\omega(t)y'(t)[1+0(1)], \quad y''(t) = o\Big(\frac{y'(t)}{\pi_\omega(t)}\Big) \text{ as } t\uparrow\omega. \tag{10}$$

From the first formula we get the first one of representations (6) and condition (3). It also follows from (10) that there exists a slowly varying continuously differentiable function $L : \Delta_{Y_0} \to]0,+\infty[$ such that $y(t) = \pi_\omega(t)L(\pi_\omega(t))$. By the condition S, we obtain $\Theta_0(y(t)) = \Theta_0(|\pi_\omega(t)|\operatorname{sign} y_0^0)[1+o(1)]$ as $t\uparrow\omega$.

Moreover, from the first formula of (10), using the properties of logarithmic functions and the function R, we find that the asymptotic representations

$$R\big(|\ln|y(t)y'(t)||\big) = R\big(|\ln|\pi_\omega(t)||\big)[1+o(1)], \quad R'\big(|\ln|y(t)y'(t)||\big) = R'\big(|\ln|\pi_\omega(t)||\big)[1+o(1)] \tag{11}$$

take place as $t\uparrow\omega$.

Let us rewrite (1) in the form

$$\frac{y''(t)}{\varphi_1(y'(t))|y'(t)|^{\sigma_0}} = I_0'(t)\exp\big(R(|\ln|y(t)y'(t)||)\big)[1+o(1)] \text{ as } t\uparrow\omega. \tag{12}$$

Suppose now that condition (4) holds and denote

$$\lim_{t\uparrow\omega} I_0(t) = J_0.$$

Let us show that the function $\exp(R(|\ln|y(I_0^{-1}(z))y'(I_0^{-1}(z))||))$ is slowly varying as $z\to J_0$. Here, I_0^{-1} is the function, inverse to I_0. By conditions (4), (11) and (10), we have

$$\lim_{z\to J_0} \frac{z\big(\exp(R(|\ln|y(I_0^{-1}(z))y'(I_0^{-1}(z))||))\big)'}{\exp\big(R(|\ln|y(I_0^{-1}(z))y'(I_0^{-1}(z))||)\big)}$$
$$= \lim_{z\to J_0} \frac{z\exp(R\big(|\ln|y(I_0^{-1}(z))y'(I_0^{-1}(z))||\big))R'\big(|\ln|y(I_0^{-1}(z))y'(I_0^{-1}(z))||\big)}{I_0'(I_0^{-1}(z))\exp\big(R(|\ln|y(I_0^{-1}(z))y'(I_0^{-1}(z))||)\big)}\Big(\frac{y'(I_0^{-1}(z))}{y(I_0^{-1}(z))} + \frac{y''(I_0^{-1}(z))}{y'(I_0^{-1}(z))}\Big)$$
$$= \lim_{z\to J_0} \frac{zR'\big(|\ln|y(I_0^{-1}(z))y'(I_0^{-1}(z))||\big)}{I_0'(I_0^{-1}(z))}\,\frac{y'(I_0^{-1}(z))}{y(I_0^{-1}(z))}\Big(1+\frac{y''(I_0^{-1}(z))y(I_0^{-1}(z))}{(y'(I_0^{-1}(z)))^2}\Big) = 0.$$

Therefore, using (12), we get

$$\frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)y'(t)||))} = (1-\sigma_0-\sigma_1)I_0(t)[1+o(1)] \text{ as } t\uparrow\omega. \tag{13}$$

Thus representation (6) is valid. Taking into account the sign of the function $y'(t)$, we obtain the first and the second of conditions (5). Using the second of relations (10), by (13) and (12), we have

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_0'(t)\varphi_1(y'(t))}{|y'(t)|^{1-\sigma_0}} = 0.$$

The third of conditions (5) follows from the latter relation, and thus the necessity is proved.

The sufficiency. Suppose that the function φ_1 satisfies the condition S and conditions (3)–(5) of the theorem hold. We denote $g(v_0,v_1) = \exp(R(|\ln|v_0v_1||))L_1(v_1)$, where $L_1 : \Delta_{Y_1} \to]0,+\infty[$ is a slowly varying function as $z\to Y_1$ ($z\in\Delta_{Y_1}$) such that

$$L_1(z) = \Theta_1(z)[1+o(1)] \text{ as } z\to Y_1 \ (z\in\Delta_{Y_1}), \qquad \lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_1}}}\frac{zL_1'(z)}{L_1(z)} = 0. \tag{14}$$

According to the properties of the function R and (14), we get

$$\lim_{\substack{v_i\to Y_i\\ v_i\in\Delta_{Y_i}}}\frac{v_i\frac{\partial g}{\partial v_i}(v_0,v_1)}{g(v_0,v_1)} = 0 \text{ uniformly by } v_j\in\Delta_{Y_j}, \ j\neq i, \ i,j=0,1. \tag{15}$$

So, we can take $\widetilde{\Delta}_{Y_i} \subset \Delta_{Y_i}$ $(i = 0, 1)$ in a form such that

$$\left| \frac{v_i \frac{\partial g}{\partial v_i}(v_0, v_1)}{g(v_0, v_1)} \right| < \zeta \ \ (i = 0, 1) \ \text{ as } \ (v_0, v_1) \in \widetilde{\Delta}_{Y_0} \times \widetilde{\Delta}_{Y_1}. \tag{16}$$

Here, $0 < \zeta < \frac{|1-\sigma_0-\sigma_1|}{8}$, ζ is sufficiently small and

$$\widetilde{\Delta}_{Y_i} = \begin{cases} [\widetilde{y}_i^{\,0}, Y_i[\,, & \text{if } \ \Delta_{Y_i} = [y_i^0, Y_i[\,, \ \ y_i^0 \le \widetilde{y}_i^{\,0} < Y_i, \\ ]Y_i, \widetilde{y}_i^{\,0}], & \text{if } \ \Delta_{Y_i} =]Y_i, y_i^0], \ \ Y_i > \widetilde{y}_i^{\,0} \ge y_i^0, \end{cases} \quad i = 0, 1.$$

Consider now the function

$$F(s_0, s_1) = \begin{pmatrix} \dfrac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0, s_1)} \\ \dfrac{s_1}{s_0} \end{pmatrix}$$

on the set $\widetilde{\Delta}_{Y_0} \times \widetilde{\Delta}_{Y_1}$. Using (15), we have

$$\lim_{\substack{s_1 \to Y_1 \\ s_1 \in \widetilde{\Delta}_{Y_1}}} \frac{s_1 \left(\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0, s_1)} \right)'_{s_1}}{\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0, s_1)}} = 1 - \sigma_0 - \sigma_1 \ \text{ uniformly by } \ s_0 \in \widetilde{\Delta}_{Y_0}, \tag{17}$$

$$\lim_{\substack{s_0 \to Y_0 \\ s_0 \in \widetilde{\Delta}_{Y_0}}} \frac{s_0 \left(\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0, s_1)} \right)'_{s_0}}{\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0, s_1)}} = -R'(|\ln|s_0 s_1||)\,\mathrm{sign}(s_0) = 0 \ \text{ uniformly by } \ s_1 \in \widetilde{\Delta}_{Y_1}.$$

Therefore, we get

$$\lim_{\substack{s_1 \to Y_1 \\ s_1 \in \widetilde{\Delta}_{Y_1}}} \frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0, s_1)} = \Upsilon \ \text{ uniformly by } \ s_0 \in \widetilde{\Delta}_{Y_0},$$

$$\Upsilon = \begin{cases} +\infty, & \text{if } \ Y_1 = \infty, \ \ 1 - \sigma_0 - \sigma_1 > 0, \ \text{ or } \ Y_1 = 0, \ \ 1 - \sigma_0 - \sigma_1 < 0, \\ 0, & \text{if } \ Y_1 = \infty, \ \ 1 - \sigma_0 - \sigma_1 < 0, \ \text{ or } \ Y_0 = 0, \ \ 1 - \sigma_0 - \sigma_1 > 0. \end{cases}$$

Let us show that F establishes the one-to-one correspondence between the set $\widetilde{\Delta}_{Y_0} \times \widetilde{\Delta}_{Y_1}$ and the set

$$F(\widetilde{\Delta}_{Y_0} \times \widetilde{\Delta}_{Y_1}) = \begin{cases} \Big[\dfrac{|\widetilde{y}_0^{\,1}|^{1-\sigma_0-\sigma_1}}{g(\widetilde{y}_0^{\,0}, \widetilde{y}_0^{\,1})}; \Upsilon \Big) \times \Delta_0 & \text{as } \ \dfrac{|\widetilde{y}_0^{\,1}|^{1-\sigma_0-\sigma_1}}{g(\widetilde{y}_0^{\,0}, \widetilde{y}_0^{\,1})} < \Upsilon, \\ \Big(\Upsilon; \dfrac{|\widetilde{y}_0^{\,1}|^{1-\sigma_0-\sigma_1}}{g(\widetilde{y}_0^{\,0}, \widetilde{y}_0^{\,1})} \Big] \times \Delta_0 & \text{as } \ \dfrac{|\widetilde{y}_0^{\,1}|^{1-\sigma_0-\sigma_1}}{g(\widetilde{y}_0^{\,0}, \widetilde{y}_0^{\,1})} > \Upsilon. \end{cases} \tag{18}$$

Here,

$$\Delta_0 = \begin{cases} \Big[\dfrac{\widetilde{y}_0^{\,1}}{\widetilde{y}_0^{\,0}}; Y_0^0 \Big) & \text{as } \ \lambda_0 < 0, \ \ \dfrac{\widetilde{y}_0^{\,1}}{\widetilde{y}_0^{\,0}} < Y_0^0, \\ \Big(Y_0^0; \dfrac{\widetilde{y}_0^{\,1}}{\widetilde{y}_0^{\,0}} \Big] & \text{as } \ \lambda_0 < 0, \ \ \dfrac{\widetilde{y}_0^{\,1}}{\widetilde{y}_0^{\,0}} > Y_0^0, \end{cases} \tag{19}$$

$$Y_0^0 = \begin{cases} 0 & \text{as } \ Y_0 = 0, \\ -\infty & \text{as } \ Y_0 = 0, \ \ \omega < +\infty, \\ +\infty & \text{as } \ Y_0 = 0, \ \ \omega = +\infty. \end{cases}$$

Let us consider the behavior of the function $\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0, s_1)}$ on the straight lines

$$s_0 = k s_1, \ \ k \in \mathbb{R} \setminus \{0\}. \tag{20}$$

On every such a line we have

$$\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_0,s_1)} = \frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(ks_1,s_1)}\,.$$

Moreover, we get

$$\left(\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(ks_1,s_1)}\right)'_{s_1} = \frac{|s_1|^{1-\sigma_0-\sigma_1}}{s_1 g(ks_1,s_1}\left(1-\sigma_0-\sigma_1-\frac{s_1L_1'(s_1)}{L(s_1)}-2ks_1R'(|\ln|ks_1^2||)\operatorname{sign}(\ln|ks_1^2|)\right).$$

Taking into account (16), from the latter equality we obtain

$$\operatorname{sign}\left(\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(ks_1,s_1)}\right)'_{s_1} = \operatorname{sign}(y_1^0(1-\sigma_0-\sigma_1)).$$

Therefore, the function $\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(ks_1,s_1)}$ is strongly monotone on every line of type (20). Suppose that the correspondence F is not of one-to-one type. Then

$$\exists\,(p_0,p_1),(q_0,q_1)\in\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1},\;\; p_0,p_1)\neq(q_0,q_1): \quad F(p_0,p_1)=F(q_0,q_1).$$

Taking into account the definitions of the sets $\widetilde{\Delta}_{Y_0},\widetilde{\Delta}_{Y_1}$, the latter equality implies that

$$\frac{|p_1|^{1-\sigma_0-\sigma_1}}{g(p_0,p_1)} = \frac{|q_1|^{1-\sigma_0-\sigma_1}}{g(q_0,q_1)}\,,\quad \frac{p_0}{p_1}=\frac{q_0}{q_1}=c\in\mathbb{R}\setminus\{0\}. \tag{21}$$

Thus, the points (p_0,p_1) and (q_0,q_1) lie on a line of type (20). But in this case equalities (21) fail to take place, because the function $\frac{|s_1|^{1-\sigma_0-\sigma_1}}{g(s_1,cs_1)}$ is strongly monotone on the line. Therefore there exists the inverse function $F^{-1}: F(\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1})\to\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1}$. Taking into account the character of the function F, we have

$$F^{-1}(w_0,w_1)=\begin{pmatrix}F_1^{-1}(w_0,w_1)\\ F_0^{-1}(w_0,w_1)\end{pmatrix}=\begin{pmatrix}F_1^{-1}(w_0,w_1)\\ \dfrac{1}{w_0}F_1^{-1}(w_0,w_1)\end{pmatrix}.$$

Since by (16) the Jakobian of the function F is different from zero as $(s_0,s_1)\in\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1}$, the function F^{-1} is continuously differentiable on $F(\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1})$.

Taking

$$\begin{cases}\dfrac{|y'(t)|^{1-\sigma_0}}{\varphi_1(y')\exp(R(|\ln|y(t)y'(t)||))}=(1-\sigma_0-\sigma_1)I_0(t)\operatorname{sign}(y')[1+z_1(x)],\\ \dfrac{y'(t)}{y(t)}=\dfrac{1}{\pi_\omega(t)}\,[1+z_2(x)],\end{cases} \tag{22}$$

where

$$x=\beta\ln|\pi_\omega(t)|,\quad \beta=\begin{cases}1 & \text{as } \omega=+\infty,\\ -1 & \text{as } \omega<\infty,\end{cases}$$

we can reduce equation (1) to the system

$$\begin{cases} z_1'=\beta G_0(x)[1+z_1]\Bigg(\left(1-\sigma_0-\sigma_1-\dfrac{\Psi_1(x,z_1,z_2)L_1'(\Psi_1(x,z_1,z_2))}{L_1(\Psi_1(x,z_1,z_2))}\right)\\ \qquad\times\dfrac{K_1(x,z_1,z_2)}{[1+z_1]|1+z_2|^{\sigma_0}}-K_2(x,z_1,z_2)\,\dfrac{R'(|\ln|\pi_\omega(t)||)}{G_0(x)}\left(1+\dfrac{K_1(x,z_1,z_2)G_0(x)}{[1+z_1][1+z_2]^{\sigma_0-1}}\right)-1\Bigg),\\ z_2'=\beta[1+z_2]\left(\dfrac{G_0(x)K_1(x,z_1,z_2)}{(1-\sigma_0-\sigma_1)[1+z_1]\,[1+z_2]^{\sigma_0}}-z_2\right),\end{cases} \tag{23}$$

where

$$\Psi_0(x,z_1,z_2) = F_0^{-1}\bigg((1-\sigma_0-\sigma_1)I_0(t(X))[1+z_1(x)],\, \frac{1}{\pi_\omega(t(x))}\,[1+z_2(x)]\bigg),$$
$$\Psi_1(x,z_1,z_2) = F_1^{-1}\bigg((1-\sigma_0-\sigma_1)I_0(t(x))[1+z_1(x)],\, \frac{1}{\pi_\omega(t(x))}\,[1+z_2(x)]\bigg),$$
$$G_0(x) = \frac{\pi_\omega(t(x))I_0'(t(x))}{I_0(t(x))}\,,$$
$$K_1(x,z_1,z_2) = \frac{\Theta_0(\Psi_0(t(x),z_1,z_2))}{(1-\sigma_0-\sigma_1)\Theta_0(|\pi_\omega(t(x))|\operatorname{sign} y_0^0)}\,,$$
$$K_2(x,z_1,z_2) = \frac{R'\big(|\ln|\Psi_0(t(x),z_1,z_2)\Psi_1(t(x),z_1,z_2)||\big)}{R'(|\ln|\pi_\omega(t(x))||)}\,.$$

By (3), it is clear that

$$\lim_{t\uparrow\omega}\frac{1}{\pi_\omega(t)} = Y_1.$$

Moreover, it follows from the first and the second of conditions (5) that

$$\lim_{t\uparrow\omega}(1-\sigma_0-\sigma_1)I_0(t) = \Upsilon.$$

Therefore, we can choose $t_0\in[a,\omega[$ in a form such that

$$\begin{pmatrix}(1-\sigma_0-\sigma_1)I_0(t)[1+z_1(x)]\\ \dfrac{1}{\pi_\omega(t)}\,[1+z_2(x)]\end{pmatrix}\in F(\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1})\ \text{ as }\ t\in[t_0,\omega[,\quad |z_i|\le\frac{1}{2}\,,\quad i=1,2.$$

Then we consider the system of differential equations (23) on the set

$$\Omega = [x_0,+\infty[\times D,\ \text{ where }\ x_0=\beta\ln|\pi_\omega(t_0)|,$$
$$D = \Big\{(z_1,z_2):\ |z_i|\le\frac{1}{2}\,,\ i=1,2\Big\}.$$

Rewrite the system in the form

$$\begin{cases} z_1' = G_0(x)(A_{11}z_1 + A_{12}z_2 + R_1(x,z_1,z_2) + R_2(z_2)),\\ z_2' = A_{21}z_1 + A_{22}z_2 + R_3(x,z_1,z_2) + R_4(z_2),\end{cases}\tag{24}$$

where

$$A_{11} = A_{22} = -\beta,\quad A_{12} = -\beta\sigma_0,\quad A_{21} = 0,$$
$$R_1(x,z_1,z_2) = -\beta[1+z_1]\bigg(K_2(x,z_1,z_2)\,\frac{R'(|\ln|\pi_\omega(t(x))||)}{G_0(x)}\Big(1+\frac{K_1(x,z_1,z_2)G_0(x)}{(1+z_1)(1+z_2)^{\sigma_0-1}}\Big)$$
$$+\frac{K_1(x,z_1,z_2)}{(1+z_1)|1+z_2|^{\sigma_0}}\,\frac{\Psi_1(x,z_1,z_2)L_1'(\Psi_1(x,z_1,z_2))}{L_1(\Psi_1(x,z_1,z_2))}\bigg)$$
$$+\beta\,\frac{|K_1(x,z_1,z_2)|(1-\sigma_0-\sigma_1)-1}{|1+z_2|^{\sigma_0}}\,,$$
$$R_2(z_2) = \beta\big(|1+z_2|^{-\sigma_0}+\sigma_0 z_2\big),$$
$$R_3(x,z_1,z_2) = \beta\,\frac{[1+z_2]G_0(x)K_1(x,z_1,z_2)}{(1-\sigma_0-\sigma_1)[1+z_1]\,[1+z_2]^{\sigma_0}}\,,\quad R_4(z_2) = -\beta z_2^2.$$

For $(w_0,w_1)\in F(\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1})$, we have the equality

$$\frac{|F_1^{-1}(w_0,w_1)|^{1-\sigma_0-\sigma_1}}{g(F_0^{-1}(w_0,w_1),F_1^{-1}(w_0,w_1))} = w_1.$$

Since (16), (3) and the second of conditions (4) are filfilled, it follows from this equality that

$$\lim_{x\to\infty} \Psi_i(t(x), z_1, z_2) = Y_i \ \text{ uniformly by } \ (z_1, z_2) \in \Big[-\frac{1}{2}\,;\frac{1}{2}\Big] \times \Big[-\frac{1}{2};\frac{1}{2}\Big]$$

as $i = 0, 1$. Therefore, by (14), we have

$$\lim_{x\to\infty} \frac{\Psi_1(t(x), z_1, z_2) L_1'(\Psi_1(t(x), z_1, z_2))}{L_1(\Psi_1(t(x), z_1, z_2))} = 0 \ \text{ uniformly by } \ (z_1, z_2) \in \Big[-\frac{1}{2}\,;\frac{1}{2}\Big] \times \Big[-\frac{1}{2}\,;\frac{1}{2}\Big]. \tag{25}$$

Moreover, it follows from the properties of the function F and conditions (3)–(5) that the function $\Psi_1(t, z_1, z_2)$ is slowly varying as $t \uparrow \omega$ uniformly by $(z_1, z_2) \in [-\frac{1}{2}\,;\frac{1}{2}] \times [-\frac{1}{2}\,;\frac{1}{2}]$. Since

$$\Psi_0(t, z_1, z_2) = \frac{\pi_\omega(t)\Psi_1(t, z_1, z_2)}{1 + z_2}\,,$$

and the function φ_0 together with the logarithmic function satisfy the condition S, we have

$$\lim_{x\to\infty} K_1(x, z_1, z_2) = \frac{1}{1 - \sigma_0 - \sigma_1} \ \text{ uniformly by } \ (z_1, z_2) \in \Big[-\frac{1}{2}\,;\frac{1}{2}\Big] \times \Big[-\frac{1}{2}\,;\frac{1}{2}\Big], \tag{26}$$

$$\lim_{x\to\infty} K_2(x, z_1, z_2) = 1 \ \text{ uniformly by } \ (z_1, z_2) \in \Big[-\frac{1}{2}\,;\frac{1}{2}\Big] \times \Big[-\frac{1}{2}\,;\frac{1}{2}\Big]. \tag{27}$$

Since the function R is regularly varying at infinity of order μ, $0 < \mu < 1$, we obtain

$$\lim_{t\uparrow\omega} R'\big(|\ln|\pi_\omega(t)||\big) = 0. \tag{28}$$

Third of conditions (5) implies

$$\lim_{x\to\infty} G_0(x) = 0. \tag{29}$$

By (4) and (25)–(29), we get the limit relations

$$\lim_{|z_1|+|z_2|\to 0} \frac{R_i(z_2)}{|z_1| + |z_2|} = 0 \ \text{ uniformly by } x: \ x \in \,]x_0, +\infty[$$

as $i = 2, 4$ and

$$\lim_{x\to+\infty} R_i(x, z_1, z_2) = 0 \ \text{ uniformly by } z_1, z_2: \ (z_1, z_2) \in D$$

as $i = 1, 3$.

By the definition of the function G_0 it is clear that $\int\limits_{x_0}^{\infty} G_0(x)\,dx = \infty$.

So, for the system of differential equations (24) all conditions of Theorem 2.8 from [2] are fulfilled. According to this theorem, system (24) has at least one solution $\{z_i\}_{i=1}^2 : [x_1, +\infty[\, \to \mathbb{R}^2$ $(x_1 \geq x_0)$ tending to zero as $x \to +\infty$. By (22) and (23), this solution corresponds to such solution y of equation (1) that admits asymptotic representations (6) as $t \uparrow \omega$. By our representations and (1), it is clear that the obtained solution is indeed the $P_\omega(Y_0, Y_1, \pm\infty)$-solution. □

Proof of Theorem 2. The necessity. Let the function $y : [t_0, \omega[\, \to R$ be a $P_\omega(Y_0, Y_1, \pm\infty)$-solution of equation (1). We obtain (10) and (12) just as in the proof of Theorem 1. The second of representations (9) follows from these relations. Let us rewrite (12) by using the first of asymptotic representations (10) in the form

$$\frac{y''(t)}{\varphi_1(y'(t))|y'(t)|^{\sigma_0}} = \frac{N(t)\exp(R(|\ln|y(t)y'(t)||))y'(t)[1 + o(1)]}{y(t)} \ \text{ as } \ t \uparrow \omega. \tag{30}$$

Suppose that conditions (7) are valid. By the properties of the function R, there exists a twice continuously differentiable function $\widetilde{R} :]0; +\infty[\, \to \,]0; +\infty[$ such that

$$\widetilde{R}(z) = R(z)[1 + o(1)], \quad \widetilde{R}'(z) = R'(z)[1 + o(1)], \quad \lim_{z\to+\infty} \frac{\widetilde{R}''(z)R(z)}{(\widetilde{R}'(z))^2} = \frac{\mu}{\mu - 1}\,. \tag{31}$$

By conditions (2), (11), (31), (7) and the first of asymptotic representations (10), from the equality

$$\left(\frac{N(t)\exp(R(|\ln|y(t)y'(t)||))}{\widetilde{R}'(|\ln|\pi_\omega(t)||)}\right)' = \frac{N(t)\exp(R(|\ln|y(t)y'(t)||))y'(t)}{y(t)}$$
$$\times\left(\frac{y(t)}{\pi_\omega(t)y'(t)}\left(\frac{N'(t)\pi_\omega(t)}{N(t)\widetilde{R}'(|\ln|\pi_\omega(t)||)} - \frac{\widetilde{R}(|\ln|\pi_\omega(t)||)}{(\widetilde{R}'(|\ln|\pi_\omega(t)||))^2}\,\frac{\widetilde{R}''(|\ln|\pi_\omega(t)||)}{\widetilde{R}'(|\ln|\pi_\omega(t)||)}\right) + 1 + \frac{y(t)y''(t)}{(y'(t))^2}\right)$$

we have the following representation

$$\left(\frac{N(t)\exp(R(|\ln|y(t)y'(t)||))}{\widetilde{R}'(|\ln|\pi_\omega(t)||)}\right)' = \frac{N(t)\exp(R(|\ln|y(t)y'(t)||))}{\widetilde{R}'(|\ln|\pi_\omega(t)||)}\,[1+o(1)]$$

as $i = 1, 3$. So, using the properties of the function φ_1 and (30), we get

$$\frac{y'(t)}{\varphi_1(y'(t))|y'(t)|^{\sigma_0}} = \frac{N(t)\exp(R(|\ln|y(t)y'(t)||))}{\widetilde{R}'(|\ln|\pi_\omega(t)||)}\,(1-\sigma_0-\sigma_1)[1+o(1)] \ \text{ as } \ t\uparrow\omega.$$

The first of representations (9) follows from this relations by using (31). Taking into account the sign of the function $y'(t)$, we obtain conditions (8). The necessity is proved.

The sufficiency. Suppose that the function φ_1 satisfies the condition S and there take place conditions (3), (7), (8). Consider the twice continuously differentiable function $\widetilde{R} :]0;+\infty[\to]0;+\infty[$ that satisfies (31), just as in the proof of Theorem 1. We use the same function F with the same properties as in the proof of Theorem 1.

Taking

$$F(y'(t), y(t)) = \begin{pmatrix} \dfrac{|1-\sigma_0-\sigma_1|N(t)}{\widetilde{R}'(|\ln|\pi_\omega(t)||)}\,[1+z_1(x)] \\ \dfrac{1}{\pi_\omega(t)}\,[1+z_2(x)] \end{pmatrix},$$

where

$$x = \beta\ln|\pi_\omega(t)|, \quad \beta = \begin{cases} 1 & \text{as } \ \omega = +\infty, \\ -1 & \text{as } \ \omega = +\infty, \end{cases}$$

we can reduce equation (1) to the system

$$\begin{cases} z_1' = \beta G_0(x)\Bigg[\dfrac{K_1(x,z_1,z_2)|1+z_2|^{\sigma_0}}{(1-\sigma_0-\sigma_1)}\Big(1-\sigma_0-\sigma_1 - \dfrac{\Psi_1(x,z_1,z_2)L_1'(\Psi_1(x,z_1,z_2))}{L_1(\Psi_1(x,z_1,z_2))}\Big) \\ \qquad -G_1(x)[1+z_1] - K_2(x,z_1,z_2)\Big(1+z_1+\dfrac{K_1(x,z_1,z_2)G_0(x)|1+z_2|^{\sigma_0}}{|1-\sigma_0-\sigma_1|}\Big) \\ \qquad\qquad +\dfrac{G_2(x)[1+z_1]}{R(|\ln|\pi_\omega(t)||)}\Bigg], \\ z_2' = \beta[1+z_2]\left[\dfrac{K_1(x,z_1,z_2)G_0(x)|1+z_2|^{\sigma_0}}{|1-\sigma_0-\sigma_1|[1+z_1]} - z_2\right], \end{cases} \tag{32}$$

where

$$G_0(x) = \widetilde{R}'\big(|\ln|\pi_\omega(t(x))||\big), \quad G_1(x) = \frac{\pi_\omega(t(x))N'(t(x))}{\widetilde{R}'(|\ln|\pi_\omega(t(x))||)N(t(x))},$$
$$G_2(x) = \frac{\widetilde{R}''\big(|\ln|\pi_\omega(t(x))||\big)\widetilde{R}(|\ln|\pi_\omega(t(x))||)}{(\widetilde{R}'(|\ln|\pi_\omega(t(x))||))^2},$$
$$\Psi_0(x,z_1,z_2) = F_0^{-1}\left(\frac{(1-\sigma_0-\sigma_1)N(t(x))}{\widetilde{R}'(|\ln|\pi_\omega(t(x))||)}[1+z_1], \frac{1}{\pi_\omega(t(x))}[1+z_2]\right),$$
$$\Psi_1(x,z_1,z_2) = F_1^{-1}\left(\frac{(1-\sigma_0-\sigma_1)N(t(x))}{\widetilde{R}'(|\ln|\pi_\omega(t(x))||)}[1+z_1], \frac{1}{\pi_\omega(t(x))}[1+z_2]\right),$$

$$K_1(x,z_1,z_2) = \frac{\Theta_0(\Psi_0(t(x),z_1,z_2))}{\Theta_0(|\pi_\omega(t(x))|)}\,,$$
$$K_2(x,z_1,z_2) = \frac{\widetilde{R}'(|\ln|\Psi_0(t(x),z_1,z_2)\Psi_1(t(x),z_1,z_2)||)}{\widetilde{R}'(|\ln|\pi_\omega(t(x))||)}\,.$$

We get

$$\lim_{k\to\infty} K_i(x,z_1,z_2) = 1 \ \text{ uniformly by } \ (z_1,z_2) \in \Big[-\frac{1}{2};\frac{1}{2}\Big] \times \Big[-\frac{1}{2};\frac{1}{2}\Big], \tag{33}$$

as in the proof of Theorem 1.

By (3), it is clear that

$$\lim_{t\uparrow\omega} \frac{1}{\pi_\omega(t)} = Y_0^0.$$

Moreover, it follows from (7) and (8) that

$$\lim_{t\uparrow\omega} \frac{|1-\sigma_0-\sigma_1|N(t)}{\widetilde{R}'(|\ln|\pi_\omega(t)||)} = \Upsilon.$$

Therefore, we can choose $t_0 \in [a,\omega[$ such that

$$\begin{pmatrix} \dfrac{(1-\sigma_0-\sigma_1)N(t)}{\widetilde{R}'(|\ln|\pi_\omega(t)||)}[1+z_1(x)] \\ \dfrac{1}{\pi_\omega(t)}[1+z_2(x)] \end{pmatrix} \in F(\widetilde{\Delta}_{Y_0}\times\widetilde{\Delta}_{Y_1}) \ \text{ as } \ t\in[t_0,\omega[\,, \quad |z_i| \le \frac{1}{2} \ \ (i=1,2).$$

Further, we consider system (32) on the set

$$\Omega = [x_0,+\infty[\,\times D, \ \text{ where } \ x_0 = \beta\ln|t_0|,$$
$$D = \Big\{(z_1,z_2):\ |z_i| \le \frac{1}{2}\ (i=1,2)\Big\}$$

and rewrite system (32) in the form

$$\begin{cases} z_1' = G_0(x)\big[A_{11}z_1 + A_{12}z_1 + R_1(x,z_1,z_2) + R_2(z_2)\big], \\ z_2' = A_{21}z_1 + A_{22}z_2 + R_3(x,z_1,z_2) + R_4(z_2), \end{cases}$$

where

$$A_{11} = A_{22} = -\beta, \quad A_{12} = \beta\sigma_0, \quad A_{21} = 0,$$
$$R_1(x,z_1,z_2) = \beta\bigg((K_1(x,z_1,z_2)-1)|1+z_2|^{\sigma_0} - (K_2(x,z_1,z_2)-1) - G_1(x)[1+z_1]$$
$$+ \frac{G_2(x)}{R(|\ln|\pi_\omega(x)||)[1+z_2]} - \frac{K_1(x,z_1,z_2)}{|1-\sigma_0-\sigma_1|}\,\frac{\Psi_1(x,z_1,z_2)L_1'(\Psi_1(x,z_1,z_2))}{L_1(\Psi_1(x,z_1,z_2))}$$
$$- \frac{G_0(x)K_2(x,z_1,z_2)K_1(x,z_1,z_2)}{|1-\sigma_0-\sigma_1|}|1+z_2|^{\sigma_0}\bigg),$$
$$R_2(z_2) = \big(|1+z_2|^{\sigma_0} - \sigma_0 z_2 - 1\big),$$
$$R_3(x,z_1,z_2) = \beta\,\frac{G_0(x)K_1(x,z_1,z_2)}{|1-\sigma_0-\sigma_1|}\,\frac{|1+z_2|^{\sigma_0+1}}{1+z_1}\,,$$
$$R_4(z_2) = -\beta z_2^2.$$

It follows from (3) and (7) that

$$\lim_{x\to\infty} G_i(x) = 0 \ \ (i=0,1), \qquad \lim_{x\to\infty} G_2(x) = \frac{\mu-1}{\mu}\,.$$

By the character of the function G_0, it is clear that

$$\int_{x_0}^{\infty} G_0(x)\,dx = \infty.$$

So, using (33), we have

$$\lim_{|z_1|+|z_2|\to 0} \frac{R_i(z_2)}{|z_1|+|z_2|} = 0 \ \text{ uniformly by } x: \ x \in]x_0, +\infty[$$

as $i = 2, 4$ and

$$\lim_{x\to+\infty} R_i(x, z_1, z_2) = 0 \ \text{ uniformly by } z_1, z_2: \ (z_1, z_2) \in D$$

as $i = 1, 3$.

Thus, for the system of differential equations (32) all conditions of Theorem 2.8 from [2] are fulfilled. According to this theorem, system (32) has at least one solution $\{z_i\}_{i=1}^2 : [x_1, +\infty[\to \mathbb{R}^2$ $(x_1 \geq x_0)$ that tends to zero as $x \to +\infty$. This solution corresponds to such solution y of equation (1) that admits asymptotic representations (9) as $t \uparrow \omega$. By our representations and (1), it is clear that the obtained solution is indeed the $P_\omega(Y_0, Y_1, \pm\infty)$-solution. □

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GREEN'S FUNCTIONS FOR THE FOURTH-ORDER DIFFERENTIAL EQUATIONS

Abstract. The purpose of this work is to establish and study some useful properties of Green's functions of the fourth-order linear differential equation before using them together with the Guo–Krasnosel'skiĭ's fixed point theorem for proving the existence of positive periodic solutions of the fourth-order nonlinear differential equation.

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რეზიუმე. სტატიის მიზანია მეოთხე რიგის წრფივი დიფერენციალური განტოლებისთვის დასმული პერიოდული ამოცანის გრინის ფუნქციის ზოგიერთი მნიშვნელოვანი თვისების შესწავლა. შესწავლილი თვისებები გუო–კრასნოსელსკის უძრავი წერტილის პრინციპთან ერთად საშუალებას გვაძლევს დავადგინოთ დადებითი პერიოდული ამონახსნების არსებობის პირობები მეოთხე რიგის არაწრფივი დიფერენციალური განტოლებისთვის.

1 Introduction

In this work we are essentially interested in studding the existence of positive periodic solutions for certain classes of fourth-order nonlinear differential equations which are ubiquitous in different scientific disciplines and arise especially in the beam theory, viscoelastic and inelastic flows and electric circuits.

There is a vast literature related to this topic, for instance, in the middle of the past century, the existence and uniqueness of solutions for higher-order differential equations have been extensively studied by many researches (see, e.g., [1–7]). During the last two decades, there has been increasing activity in the study of periodic problems of higher-order nonlinear differential equations (see [12] and the references therein).

Some mathematicians used transformation in order to reduce the equation to a more simple one, or to a system of equations, or used synthetic division, others gave the solution in a form of series which converges to the exact solution and some of them dealt with the fourth-order differential equations by using numerical techniques such as the Ritz, finite difference, finite element, cubic spline and multi derivative methods. In this paper, these usual methods may seem inefficient to establish the existence of positive periodic solutions for the fourth-order nonlinear differential equations. For this, inspired by the method presented in [9], we convert the ordinary differential equation to an integral equation in which the kernel is a Green's function, before using the fixed point theorem in cones.

The paper is organized as follows.

The main goal of the next section is to give the Green's functions of the fourth-order constant-coefficient linear differential equation

$$u'''' + au''' + bu'' + cu' + du = h(t), \tag{1.1}$$

where $a, b, c, d \in \mathbb{R}$ and $h \in C(\mathbb{R}, (0, +\infty))$ is a w-periodic function with the period $w > 0$.

The associated homogeneous equation of (1.1) is

$$u'''' + au''' + bu'' + cu' + du = 0, \tag{1.2}$$

where its characteristic equation is

$$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0. \tag{1.3}$$

In this work we assume that $d \neq 0$ and we will study only the situation when the roots λ_1, λ_2, λ_3, λ_4 are real numbers. These roots satisfy one of the following five cases:

(1) $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4$;

(2) $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$;

(3) $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$;

(4) $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$;

(5) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$.

In the third section, some useful properties of the obtained Green's functions are established. Finally, in the last part, by using the fixed point theorem in cones, we establish the existence of positive periodic solutions of the fourth-order nonlinear differential equation

$$u'''' + au''' + bu'' + cu' + du = f(t, u(t)), \tag{1.4}$$

where $f \in C(\mathbb{R}\times[0, +\infty), [0, +\infty))$ and $f(t, u) > 0$, for $u > 0$.

2 Green's functions

Theorem 2.1. *If $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4$, then equation* (1.1) *has a unique w-periodic solution of the form*

$$u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds,$$

where $s \in [t, t+w]$ and

$$\begin{aligned} G_1(t,s) = & \frac{e^{\lambda_1(w+t-s)}}{(1-e^{w\lambda_1})(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{e^{\lambda_2(w+t-s)}}{(1-e^{w\lambda_2})(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} \\ & + \frac{e^{\lambda_3(w+t-s)}}{(1-e^{w\lambda_3})(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} + \frac{e^{\lambda_4(w+t-s)}}{(1-e^{w\lambda_4})(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}. \end{aligned}$$

Proof. For $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4$, it is easy to see that the general solution of the homogeneous equation (1.2) is

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t} + c_4 e^{\lambda_4 t},$$

and that $u(t) \equiv 0$ is its unique solution. Applying the method of variation of parameters, we obtain

$$\begin{aligned} c_1'(t) &= h(t)\,\frac{e^{-t\lambda_1}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}, \quad & c_2'(t) &= -h(t)\,\frac{e^{-t\lambda_2}}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}, \\ c_3'(t) &= h(t)\frac{e^{-t\lambda_3}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)}, \quad & c_4'(t) &= -h(t)\,\frac{e^{-t\lambda_4}}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}, \end{aligned}$$

whence

$$\begin{aligned} c_1(t+w) &= c_1(t) + \int_t^{t+w} h(s)\,\frac{e^{-s\lambda_1}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\,ds, \\ c_2(t+w) &= c_2(t) - \int_t^{t+w} h(s)\,\frac{e^{-s\lambda_2}}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\,ds, \\ c_3(t+w) &= c_3(t) + \int_t^{t+w} h(s)\,\frac{e^{-s\lambda_3}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)}\,ds, \\ c_4(t+w) &= c_4(t) - \int_t^{t+w} h(s)\,\frac{e^{-s\lambda_4}}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}\,ds. \end{aligned}$$

Since we are looking for w-periodic solutions of (1.1), we have

$$\begin{aligned} c_1(t) &= \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}}{(1-e^{w\lambda_1})(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\,ds, \\ c_2(t) &= -\int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_2}}{(1-e^{w\lambda_2})(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\,ds, \\ c_3(t) &= \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_3}}{(1-e^{w\lambda_3})(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)}\,ds, \\ c_4(t) &= -\int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_4}}{(1-e^{w\lambda_4})(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}\,ds. \end{aligned}$$

Therefore,

$$u(t+w) = \int_t^{t+w} G_1(t+w, \theta+w) h(\theta+w)\, d\theta = \int_t^{t+w} G_1(t,s) h(s)\, ds = u(t),$$

which proves the periodicity of u.

Assume that u_1 and u_2 are two w-periodic solutions of (1.1), then $v(t) = u_1(t) - u_2(t)$ is a w-periodic solution of (1.2), i.e., $v(t) = 0$, hence the uniqueness of the w-periodic solution for (1.1) is guaranteed. □

Theorem 2.2. *If $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$, then equation* (1.1) *has a unique w-periodic solution of the form*

$$u(t) = \int_t^{t+w} G_2(t,s) h(s)\, ds,$$

where $s \in [t, t+w]$ and

$$\begin{aligned} G_2(t,s) = {} & \frac{e^{(t+w-s)\lambda_1}\big(w(\lambda_1-\lambda_3)(\lambda_1-\lambda_4) - (e^{w\lambda_1}-1)\big(\lambda_3 - 2\lambda_1 + \lambda_4 - s(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)\big)\big)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2} \\ & + t\, \frac{e^{(t+w-s)\lambda_1}}{(1-e^{w\lambda_1})(\lambda_1^2 - \lambda_1\lambda_3 - \lambda_1\lambda_4 + \lambda_3\lambda_4)} + \frac{e^{(t+w-s)\lambda_3}}{(1-e^{w\lambda_3})(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)} \\ & - \frac{e^{(t+w-s)\lambda_4}}{(1-e^{w\lambda_4})(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)}. \end{aligned}$$

Proof. For $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$, it is easy to see that the general solution of the homogeneous equation (1.2) is

$$u(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t} + c_3 e^{\lambda_3 t} + c_4 e^{\lambda_4 t}.$$

Applying the method of variation of parameters, we obtain

$$c_1'(t) = h(t)\, \frac{e^{-t\lambda_1}}{(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}\big(\lambda_3 - 2\lambda_1 + \lambda_4 - t(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)\big),$$

$$c_2'(t) = \frac{h(t)e^{-t\lambda_1}}{(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}, \quad c_3'(t) = \frac{h(t)e^{-t\lambda_3}}{(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)}, \quad c_4'(t) = -\frac{h(t)e^{-t\lambda_4}}{(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)}.$$

Since $u(t)$, $u'(t)$, $u''(t)$ and $u'''(t)$ are supposed to be continuous functions, we get

$$c_1(t) = \int_t^{t+w} h(s)\, \frac{e^{\lambda_1(w-s)}\big(w(\lambda_1-\lambda_3)(\lambda_1-\lambda_4) - (e^{w\lambda_1}-1)\big(\lambda_3-2\lambda_1+\lambda_4-s(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)\big)\big)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}\, ds,$$

$$c_2(t) = \int_t^{t+w} \frac{h(s)e^{(w-s)\lambda_1}}{(1-e^{w\lambda_1})(\lambda_1^2 - \lambda_1\lambda_3 - \lambda_1\lambda_4 + \lambda_3\lambda_4)}\, ds,$$

$$c_3(t) = \int_t^{t+w} \frac{h(s)e^{(w-s)\lambda_3}}{(1-e^{w\lambda_3})(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)}\, ds,$$

$$c_4(t) = -\int_t^{t+w} h(s)\, \frac{e^{(w-s)\lambda_4}}{(1-e^{w\lambda_4})(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)}\, ds.$$

Therefore,

$$u(t) = c_1(t)e^{\lambda_1 t} + c_2(t) t e^{\lambda_1 t} + c_3(t) e^{\lambda_3 t} + c_4(t) e^{\lambda_4 t} = \int_t^{t+w} G_2(t,s) h(s)\, ds.$$

In the same way as in the proof of Theorem 2.1, we can prove the uniqueness and periodicity of the solution. □

Theorem 2.3. *If $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$, then equation* (1.1) *has a unique w-periodic solution of the form*

$$u(t) = \int_t^{t+w} G_3(t,s)h(s)\,ds,$$

where $s \in [t, t+w]$ and

$$\begin{aligned} G_3(t,s) = &- \frac{e^{(w+t-s)\lambda_1}\big((1-e^{w\lambda_1})(s\lambda_1 - s\lambda_4 + 2) + w\lambda_4 - w\lambda_1\big)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3} + t\,\frac{e^{(w+t-s)\lambda_1}}{(1-e^{w\lambda_1})(\lambda_1-\lambda_4)^2} \\ &- \frac{e^{(w+t-s)\lambda_4}\big((e^{w\lambda_4}-1)(s\lambda_4 - s\lambda_1 + 2) + w\lambda_4 - w\lambda_1\big)}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^3} + t\,\frac{e^{(w+t-s)\lambda_4}}{(1-e^{w\lambda_4})(\lambda_1-\lambda_4)^2}\,. \end{aligned}$$

Proof. For $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$, (1.2) has the general solution

$$u(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t} + c_3 e^{\lambda_4 t} + c_4 t e^{\lambda_4 t}.$$

Applying the method of variation of parameters, we obtain

$$\begin{aligned} c_1'(t) &= -h(t)\,\frac{e^{-t\lambda_1}}{(\lambda_1-\lambda_4)^3}\,(t\lambda_1 - t\lambda_4 + 2), \quad c_2'(t) = h(t)\,\frac{e^{-t\lambda_1}}{(\lambda_1-\lambda_4)^2}\,, \\ c_3'(t) &= h(t)\,\frac{e^{-t\lambda_4}}{(\lambda_1-\lambda_4)^3}\,(t\lambda_4 - t\lambda_1 + 2), \quad c_4'(t) = h(t)\,\frac{e^{-t\lambda_4}}{(\lambda_1-\lambda_4)^2}\,. \end{aligned}$$

Since $u(t)$, $u'(t)$, $u''(t)$ and $u'''(t)$ are continuous, we have

$$\begin{aligned} c_1(t) &= \int_t^{t+w} -h(s)\,\frac{e^{-\lambda_1(s-w)}\big((1-e^{w\lambda_1})(s\lambda_1 - s\lambda_4 + 2) + w\lambda_4 - w\lambda_1\big)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\,ds, \\ c_2(t) &= \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}}{(1-e^{w\lambda_1})(\lambda_1-\lambda_4)^2}\,ds, \\ c_3(t) &= \int_t^{t+w} -h(s)\,\frac{e^{-\lambda_4(s-w)}\big((e^{w\lambda_4}-1)(s\lambda_4 - s\lambda_1 + 2) + w\lambda_4 - w\lambda_1\big)}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^3}\,ds, \\ c_4(t) &= \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_4}}{(1-e^{w\lambda_4})(\lambda_1-\lambda_4)^2}\,ds. \end{aligned}$$

Therefore,

$$u(t) = c_1(t)e^{\lambda_1 t} + c_2(t)te^{\lambda_1 t} + c_3(t)e^{\lambda_4 t} + c_4(t)te^{\lambda_4 t} = \int_t^{t+w} G_3(t,s)h(s)\,ds.$$

The uniqueness and periodicity of the solution can again be shown in the same way as in the proof of Theorem 2.1. □

Theorem 2.4. *If $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$, then equation* (1.1) *has a unique w-periodic solution of the form*

$$u(t) = \int_t^{t+w} G_4(t,s)h(s)\,ds,$$

where $s \in [t, t+w]$ *and*

$$G_4(t,s) = e^{(t+w-s)\lambda_1}\,\frac{(1-e^{w\lambda_1})\big((e^{w\lambda_1}-1)\big(((s-t)(\lambda_1-\lambda_4)+1)^2+1\big)\big)}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3} + \frac{e^{(t+w-s)\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}$$
$$+ \frac{e^{(t+w-s)\lambda_1}\big((1-e^{w\lambda_1})\big(w(\lambda_1-\lambda_4)(2(s-t)(\lambda_1-\lambda_4)+2)\big)\big)}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}$$
$$- \frac{w^2e^{(t+w-s)\lambda_1}(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\,.$$

Proof. For $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$, (1.2) has the general solution

$$u(t) = c_1e^{\lambda_1 t} + c_2te^{\lambda_1 t} + c_3t^2e^{\lambda_1 t} + c_4te^{\lambda_4 t}.$$

The application of the method of variation of parameters gives

$$c_1'(t) = h(t)\,\frac{e^{-t\lambda_1}(t^2\lambda_1^2 - 2t^2\lambda_1\lambda_4 + t^2\lambda_4^2 + 2t\lambda_1 - 2t\lambda_4 + 2)}{2(\lambda_1-\lambda_4)^3}\,,$$

$$c_2'(t) = -h(t)\,\frac{e^{-t\lambda_1}(t\lambda_1 - t\lambda_4 + 1)}{(\lambda_1-\lambda_4)^2}\,, \quad c_3'(t) = h(t)\,\frac{e^{-t\lambda_1}}{2(\lambda_1-\lambda_4)}\,, \quad c_4'(t) = -h(t)\,\frac{e^{-t\lambda_4}}{(\lambda_1-\lambda_4)^3}\,.$$

Since $u(t)$, $u'(t)$, $u''(t)$ and $u'''(t)$ are continuous functions, we have

$$c_1(t) = \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}(s^2\lambda_1^2 - 2s^2\lambda_1\lambda_4 + s^2\lambda_4^2 + 2s\lambda_1 - 2s\lambda_4 + 2)}{2(1-e^{w\lambda_1})(\lambda_1-\lambda_4)^3}\,ds$$
$$+ w\,\frac{1}{(1-e^{w\lambda_1})}\int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}\big((e^{w\lambda_1}-1)(s(\lambda_1-\lambda_4)+1) + w(\lambda_1-\lambda_4)\big)}{(1-e^{w\lambda_1})^2(\lambda_1-\lambda_4)^2}\,ds$$
$$- w^2\,\frac{1}{(1-e^{w\lambda_1})}\int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}}{2(\lambda_1-\lambda_4)(1-e^{w\lambda_1})}\,ds,$$

$$c_2(t) = \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}\big((e^{w\lambda_1}-1)(s(\lambda_1-\lambda_4)+1) + w(\lambda_1-\lambda_4)\big)}{(1-e^{w\lambda_1})^2(\lambda_1-\lambda_4)^2}\,ds,$$

$$c_3(t) = \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}}{2(\lambda_1-\lambda_4)(1-e^{w\lambda_1})}\,ds,$$

$$c_4(t) = \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\,ds.$$

Therefore,

$$u(t) = c_1(t)e^{\lambda_1 t} + c_2(t)te^{\lambda_1 t} + c_3(t)t^2e^{\lambda_1 t} + c_4(t)te^{\lambda_4 t} = \int_t^{t+w} G_4(t,s)h(s)\,ds.$$

In the same way as in the proof of Theorem 2.1 we can prove the uniqueness and periodicity of the solution. □

Theorem 2.5. *If* $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, *then equation* (1.1) *has a unique* w*-periodic solution of the form*

$$u(t) = \int_t^{t+w} G_5(t,s)h(s)\,ds,$$

where $s \in [t, t+w]$ *and*

$$G_5(t,s) = e^{(t+w-s)\lambda_1}\,\frac{(s-t)^3(e^{w\lambda_1}-1)^3 + 3w(s-t)^2(e^{w\lambda_1}-1)^2}{6(e^{w\lambda_1}-1)^4} + e^{(t+w-s)\lambda_1}\,\frac{3w^2(s-t)(e^{2w\lambda_1}-1) + w^3(e^{2(w\lambda_1)}+4e^{w\lambda_1}+1)}{6(e^{w\lambda_1}-1)^4}.$$

Proof. For $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, (1.2) has the general solution

$$u(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t} + c_3 t^2 e^{\lambda_1 t} + c_4 t^3 e^{\lambda_1 t}.$$

By the method of variation of parameters, we arrive at

$$c_1'(t) = -\frac{1}{6}\,ht^3 e^{-t\lambda_1}, \quad c_2'(t) = \frac{1}{2}\,ht^2 e^{-t\lambda_1}, \quad c_3'(t) = -\frac{1}{2}\,hte^{-t\lambda_1}, \quad c_4'(t) = \frac{1}{6}\,he^{-t\lambda_1}.$$

Since $u(t)$, $u'(t)$, $u''(t)$ and $u'''(t)$ are continuous functions, we get

$$\begin{aligned}
c_1(t) = {} & \int_t^{t+w} -h(s)\,\frac{s^3 e^{(w-s)\lambda_1}}{6(1-e^{w\lambda_1})}\,ds \\
& + \int_t^{t+w} -h(s)\,\frac{we^{(w-s)\lambda_1}\left(s^2e^{2(w\lambda_1)} - 2s^2e^{w\lambda_1} + s^2 + 2swe^{w\lambda_1} - 2sw + w^2e^{w\lambda_1} + w^2\right)}{2(1-e^{w\lambda_1})(e^{w\lambda_1}-1)^3}\,ds \\
& - \frac{w^2}{(1-e^{w\lambda_1})}\int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}(w - s + se^{w\lambda_1})}{2(1-e^{w\lambda_1})^2}\,ds \\
& + \frac{w^3}{(1-e^{w\lambda_1})}\int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}}{6(1-e^{w\lambda_1})}\,ds, \\
c_2(t) = {} & \int_t^{t+w} -h(s)\,\frac{e^{(w-s)\lambda_1}\left(s^2e^{2(w\lambda_1)} - 2s^2e^{w\lambda_1} + s^2 + 2swe^{w\lambda_1} - 2sw + w^2e^{w\lambda_1} + w^2\right)}{2(e^{w\lambda_1}-1)^3}\,ds, \\
c_3(t) = {} & \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}(w - s + se^{w\lambda_1})}{2(1-e^{w\lambda_1})^2}\,ds, \\
c_4(t) = {} & \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}}{6(1-e^{w\lambda_1})}\,ds.
\end{aligned}$$

Therefore,

$$u(t) = c_1(t)e^{\lambda_1 t} + c_2(t)te^{\lambda_1 t} + c_3(t)t^2e^{\lambda_1 t} + c_3(t)t^3e^{\lambda_1 t} = \int_t^{t+w} G_5(t,s)h(s)\,ds.$$

In the same way as in the proof of Theorem 2.1, we can prove the uniqueness and the periodicity of the solution. □

3 Properties of the Green's functions

We denote

$$\begin{aligned}
\mathcal{C}_w^+ &= \left\{u \in \mathcal{C}(\mathbb{R}, (0, +\infty)) : \ \ u(t+w) = u(t)\right\}, \\
\mathcal{C}_w^- &= \left\{u \in \mathcal{C}(\mathbb{R}, (-\infty, 0)) : \ \ u(t+w) = u(t)\right\}.
\end{aligned}$$

Case 1. *If* $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4$. *For ease of exposition, we use the following abbreviations:*

$$
\begin{aligned}
g_{1,1}(t,s) &= \frac{e^{(w+t-s)\lambda_1}}{(1-e^{(\lambda_1)w})(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}, \\
g_{1,2}(t,s) &= \frac{e^{(w+t-s)\lambda_2}}{(1-e^{(\lambda_2)w})(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}, \\
g_{1,3}(t,s) &= \frac{e^{(w+t-s)\lambda_3}}{(1-e^{(\lambda_3)w})(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}, \\
g_{1,4}(t,s) &= \frac{e^{(w+t-s)\lambda_4}}{(1-e^{(\lambda_4)w})(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)};
\end{aligned}
$$

$$
\begin{aligned}
A_{1,1} = &-\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} - \frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)} \\
&+ \frac{1}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} + \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}, \\
A_{1,2} = &-\frac{1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{e^{w\lambda_2}}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} \\
&- \frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)} + \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)};
\end{aligned}
$$

$$
\begin{aligned}
B_{1,1} = &-\frac{1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} - \frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)} \\
&+ \frac{e^{w\lambda_2}}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} + \frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}, \\
B_{1,2} = &-\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{1}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} \\
&- \frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)} + \frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)};
\end{aligned}
$$

$$
\begin{aligned}
n_{1,1} = &\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)}, \\
n_{1,2} = &+\frac{1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)} \\
&- \frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}, \\
n_{1,3} = &+\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)} \\
&- \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}, \\
n_{1,4} = &\frac{1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)}, \\
n_{1,5} = &+\frac{1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)};
\end{aligned}
$$

$$
\begin{aligned}
p_{1,1} &= \frac{1}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} + \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}, \\
p_{1,2} &= \frac{e^{w\lambda_2}}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)},
\end{aligned}
$$

$$p_{1,3} = \frac{1}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)},$$
$$p_{1,4} = \frac{e^{w\lambda_2}}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} + \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)},$$
$$p_{1,5} = \frac{e^{w\lambda_2}}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} + \frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}.$$

Theorem 3.1. *For all $t \in [0, w]$ and $s \in [t, t+w]$, we have*

$$\int_t^{t+w} G_1(t,s)\,ds = \frac{1}{\lambda_1\lambda_2\lambda_3\lambda_4}.$$

Proof. We have

$$\int_t^{t+w} g_{1,1}(t,s)\,ds = -\frac{1}{\lambda_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)},$$
$$\int_t^{t+w} g_{1,2}(t,s)\,ds = \frac{1}{\lambda_2(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)},$$
$$\int_t^{t+w} g_{1,3}(t,s)\,ds = -\frac{1}{\lambda_3(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)},$$
$$\int_t^{t+w} g_{1,4}(t,s)\,ds = \frac{1}{\lambda_4(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)},$$

and

$$\int_t^{t+w} G_1(t,s)\,ds = \int_t^{t+w} g_{1,1}(t,s)\,ds + \int_t^{t+w} g_{1,2}(t,s)\,ds + \int_t^{t+w} g_{1,3}(t,s)\,ds + \int_t^{t+w} g_{1,4}(t,s)\,ds = \frac{1}{\lambda_1\lambda_2\lambda_3\lambda_4}.$$ □

We have four different roots satisfying one of the five cases:

- All roots are positive.
- Three roots are positive and one root is negative.
- Three roots are negative and one root is positive.
- Two roots are positive and two roots are negative.
- All roots are negative.

If all roots are positive, we suppose that $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$ (the other situations can be proved by using the same method), and we have

Theorem 3.2. *If $p_{1,1} > n_{1,1}$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$, then*

$$0 < A_{1,1} \le G_1(t,s) \le B_{1,1}.$$

Proof. If $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$, the study of the derivatives of $g_{1,i}$, $i = \overline{1,4}$, with respect to s, gives $\frac{\partial}{\partial s} g_{1,1}(t,s) > 0$, $\frac{\partial}{\partial s} g_{1,2}(t,s) < 0$, $\frac{\partial}{\partial s} g_{1,3}(t,s) > 0$ and $\frac{\partial}{\partial s} g_{1,4}(t,s) < 0$. This implies that

$$\begin{aligned} g_{1,1}(t,t) + g_{1,2}(t,t+w) + g_{1,3}(t,t) + g_{1,4}(t,t+w) \\ \le G_1(t,s) \le g_{1,1}(t,t+w) + g_{1,2}(t,t) + g_{1,3}(t,t+w) + g_{1,4}(t,t). \end{aligned}$$

From the above double inequality and the assumption $p_{1,1} > n_{1,1}$, we obtain $0 < A_{1,1} \le G_1(t,s) \le B_{1,1}$. □

Corollary 3.1. *If $h \in \mathcal{C}_w^+$ and $p_{1,1} > n_{1,1}$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds.$$

Example 3.1. Consider the equation

$$u'''' - 0.56u''' + 0.0311u'' - 5.56 \times 10^{-4}u' + 3 \times 10^{-6}u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda - 0.5)(\lambda - 0.03)(\lambda - 0.02)(\lambda - 0.01) = 0$ has four roots $\lambda_1 = 0.5$, $\lambda_2 = 0.03$, $\lambda_3 = 0.02$, $\lambda_4 = 0.01$.

Since $p_{1,1} = 2.086\,4 \times 10^5 > n_{1,1} = 1.764\,3 \times 10^5$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds$, with $\int_t^{t+w} G_1(t,s)\,ds = 3.3333 \times 10^5$ and $0 < 32210 < G_1(t,s) < 73894$.

If three roots are positive and one root is negative, we suppose that $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and $\lambda_4 < 0$ (the other situations can be proved by using the same method), and we have

Theorem 3.3. *If $p_{1,2} < n_{1,2}$, $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and $\lambda_4 < 0$, then*

$$A_{1,1} \le G_1(t,s) \le B_{1,1} < 0.$$

Proof. If $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and $\lambda_4 < 0$, the study of the derivatives of $g_{1,i}$, $i = \overline{1,4}$, with respect to s gives $\frac{\partial}{\partial s} g_{1,1}(t,s) > 0$, $\frac{\partial}{\partial s} g_{1,2}(t,s) < 0$, $\frac{\partial}{\partial s} g_{1,3}(t,s) > 0$ and $\frac{\partial}{\partial s} g_{1,4}(t,s) < 0$. Similarly, as in the proof of Theorem 3.2, we obtain $A_{1,1} \le G_1(t,s) \le B_{1,1} < 0$. □

Corollary 3.2. *If $h \in \mathcal{C}_w^-$, $p_{1,2} < n_{1,2}$, $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and $\lambda_4 < 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds.$$

Example 3.2. We consider the equation

$$u'''' - 0.59u''' + 0.104u'' - 0.0049u' - 0.00006u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda - 0.3)(\lambda - 0.2)(\lambda - 0.1)(\lambda + 0.01) = 0$ has the roots $\lambda_1 = 0.3$, $\lambda_2 = 0.2$, $\lambda_3 = 0.1$, $\lambda_4 = -0.01$.

Since $p_{1,2} = 665.64 < n_{1,2} = 2702.1$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds$, with $\int_t^{t+w} G_1(t,s)\,ds = -16667$ and $-3268.1 < G_1(t,s) < -2036.5 < 0$.

If three roots are negative and one root is positive, we suppose that $\lambda_1 < \lambda_2 < \lambda_3 < 0$ and $\lambda_4 > 0$ (the other situations can be proved by using the same method), and we have

Theorem 3.4. *If $p_{1,3} < n_{1,3}$, $\lambda_1 < \lambda_2 < \lambda_3 < 0$ and $\lambda_4 > 0$, then*

$$B_{1,1} \le G_1(t,s) \le A_{1,1} < 0.$$

Proof. If $\lambda_1 < \lambda_2 < \lambda_3 < 0$ and $\lambda_4 > 0$, the study of the derivatives of $g_{1,i}$, $i = \overline{1,4}$, with respect to s gives $\frac{\partial}{\partial s} g_{1,1}(t,s) < 0$, $\frac{\partial}{\partial s} g_{1,2}(t,s) > 0$, $\frac{\partial}{\partial s} g_{1,3}(t,s) < 0$ and $\frac{\partial}{\partial s} g_{1,4}(t,s) > 0$. Similarly, as in the proof of Theorem 3.2, we obtain $B_{1,1} \le G_1(t,s) \le A_{1,1} < 0 < 0$. □

Corollary 3.3. *If $h \in \mathcal{C}_w^-$, $p_{1,3} < n_{1,3}$, $\lambda_1 < \lambda_2 < \lambda_3 < 0$ and $\lambda_4 > 0$, then equation (1.1) has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds.$$

Example 3.3. Consider the equation

$$u'''' + 0.59u''' + 0.104u'' + 0.0049u' - 0.00006u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda+0.3)(\lambda+0.2)(\lambda+0.1)(\lambda-0.01) = 0$ has the roots $\lambda_1 = -0.3$, $\lambda_2 = -0.2$, $\lambda_3 = -0.1$, $\lambda_4 = 0.01$.

Since $p_{1,3} = 665.64 < n_{1,3} = 2702.1$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds$ with $\int_t^{t+w} G_1(t,s)\,ds = -16667$ and $-3268.1 < G_1(t,s) < -2036.5 < 0$.

If two roots are negative and two roots are positive, we suppose that $\lambda_1 < \lambda_2 < 0$ and $\lambda_3 > \lambda_4 > 0$ (the other situations can be proved by using the same method) and have

Theorem 3.5. *If $p_{1,4} > n_{1,4}$, $\lambda_1 < \lambda_2 < 0$ and $\lambda_3 > \lambda_4 > 0$, then*

$$0 < A_{1,2} \le G_1(t,s) \le B_{1,2}.$$

Proof. If $\lambda_1 < \lambda_2 < 0$ and $\lambda_3 > \lambda_4 > 0$, the study of the derivatives of $g_{1,i}$, $i = \overline{1,4}$, with respect to s gives $\frac{\partial}{\partial s} g_{1,1}(t,s) < 0$, $\frac{\partial}{\partial s} g_{1,2}(t,s) > 0$, $\frac{\partial}{\partial s} g_{1,3}(t,s) > 0$ and $\frac{\partial}{\partial s} g_{1,4}(t,s) < 0$. Similarly, as in the proof of Theorem 3.2, we obtain $0 < A_{1,2} \le G_1(t,s) \le B_{1,2}$. □

Corollary 3.4. *If $h \in \mathcal{C}_w^+$, $p_{1,4} > n_{1,4}$, $\lambda_1 < \lambda_2 < 0$ and $\lambda_3 > \lambda_4 > 0$, then equation (1.1) has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds.$$

Example 3.4. Consider the equation

$$u'''' - 0.054u'' - 4.9304 \times 10^{-32}u' + 0.0004u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda+0.2)(\lambda+0.1)(\lambda-0.2)(\lambda-0.1) = 0$ has the roots $\lambda_1 = -0.2$, $\lambda_2 = -0.1$, $\lambda_3 = 0.2$, $\lambda_4 = 0.1$.

Since $p_{1,4} = 381.19 > n_{1,4} = 232.97$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds$ with $\int_t^{t+w} G_1(t,s)\,ds = 2500$ and $0 < 148.22 < G_1(t,s) < 648.22$.

If all roots are negative, we suppose that $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 0$ (the other situations can be proved by using the same method), and we have

Theorem 3.6. *If $p_{1,5} > n_{1,5}$ and $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 0$, then*

$$0 < B_{1,1} \le G_1(t,s) \le A_{1,1}.$$

Proof. If $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 0$, the study of the derivatives of $g_{1,i}$, $i = \overline{1,4}$, with respect to s gives $\frac{\partial}{\partial s} g_{1,1}(t,s) < 0$, $\frac{\partial}{\partial s} g_{1,2}(t,s) > 0$, $\frac{\partial}{\partial s} g_{1,3}(t,s) < 0$ and $\frac{\partial}{\partial s} g_{1,4}(t,s) > 0$. Similarly, as in the proof of Theorem 3.2, we obtain $0 < B_{1,1} \le G_1(t,s) \le A_{1,1}$. □

Corollary 3.5. *If $h \in \mathcal{C}_w^+$, $p_{1,5} > n_{1,5}$ and $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds.$$

Example 3.5. Consider the equation

$$u'''' + 0.56u''' + 0.0311u'' + 5.56 \times 10^{-4}u' + 3.0 \times 10^{-6}u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda+0.5)(\lambda+0.03)(\lambda+0.02)(\lambda+0.01) = 0$ has the roots $\lambda_1 = -0.5$, $\lambda_2 = -0.03$, $\lambda_3 = -0.02$, $\lambda_4 = -0.01$. Since $p_{1,5} = 2.0864 \times 10^5 > n_{1,5} = 1.7643 \times 10^5$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds$, with $\int_t^{t+w} G_1(t,s)\,ds = 3.333 \times 10^5$ and $0 < 32210 < G_1(t,s) < 73894$.

Case 2. *If $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$, $\lambda_1, \lambda_3, \lambda_4 \in \mathbb{R}$. We use the following abbreviations:*

$$g_{2,1}(t,s) = \frac{e^{(t+w-s)\lambda_1}\big(w(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)\big(\lambda_3-2\lambda_1+\lambda_4-s(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)\big)\big)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2},$$

$$g_{2,2}(t,s) = t\,\frac{e^{(t+w-s)\lambda_1}}{(1-e^{w\lambda_1})(\lambda_1^2-\lambda_1\lambda_3-\lambda_1\lambda_4+\lambda_3\lambda_4)},$$

$$g_{2,3}(t,s) = \frac{e^{(t+w-s)\lambda_3}}{(1-e^{w\lambda_3})(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)},$$

$$g_{2,4}(t,s) = -\frac{e^{(t+w-s)\lambda_4}}{(1-e^{w\lambda_4})(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)};$$

$$h_{2,1}(s,t) = \frac{\big((\lambda_1^2-\lambda_3\lambda_4)(e^{w\lambda_1}-1)+w\lambda_1(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)\big)e^{\lambda_1(t-s+w)}}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2},$$

$$h_{2,2}(s,t) = \frac{(s\lambda_1+1)e^{\lambda_1(t-s+w)}}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)};$$

$$\begin{aligned}
A_{2,1} = {} & \frac{we^{w\lambda_1}(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_3-2\lambda_1+\lambda_4)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2} \\
& - w\,\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} - \frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)} \\
& + \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)},
\end{aligned}$$

$$\begin{aligned}
A_{2,2} = {} & \frac{1}{\lambda_1}\,\frac{\big((\lambda_1^2-\lambda_3\lambda_4)(e^{w\lambda_1}-1)+w\lambda_1(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)\big)e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2} \\
& + \frac{1}{\lambda_1}\,\frac{2w\lambda_1+1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} - \frac{we^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} \\
& - \frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)} + \frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)};
\end{aligned}$$

$$\begin{aligned}
B_{2,1} = {} & \frac{\big(we^{w\lambda_1}(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_3-2\lambda_1+\lambda_4)\big)e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2} \\
& - \frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)} + \frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)},
\end{aligned}$$

$$B_{2,2} = \frac{(\lambda_1^2-\lambda_3\lambda_4)(e^{w\lambda_1}-1)+w\lambda_1(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}$$

$$+\frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)}+\frac{e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}$$
$$-\frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)};$$

$$n_{2,1}=w\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}+\frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)},$$
$$n_{2,2}=\frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)}-\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)},$$
$$n_{2,3}=+\frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)}-\frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)},$$
$$n_{2,4}=\frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)}-\frac{1}{\lambda_1}\frac{2w\lambda_1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)};$$

$$p_{2,1}=\frac{we^{w\lambda_1}(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_3-2\lambda_1+\lambda_4)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}$$
$$+\frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)},$$
$$p_{2,2}=\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}$$
$$\times\Big(we^{w\lambda_1}(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_3-2\lambda_1+\lambda_4)\Big),$$
$$p_{2,3}=\frac{1}{\lambda_1}\frac{\big((\lambda_1)(\lambda_1)-\lambda_3\lambda_4\big)(e^{w\lambda_1}-1)+w\lambda_1(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}$$
$$+\frac{e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)},$$
$$p_{2,4}=\frac{e^{w\lambda_1}\big((\lambda_1^2-\lambda_3\lambda_4)(e^{w\lambda_1}-1)+w\lambda_1(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)\big)}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}$$
$$+\frac{1}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}$$
$$+\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)}-w\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}.$$

Theorem 3.7. *For all $t\in[0,w]$ and $s\in[t,t+w]$, we have*

$$\int_t^{t+w}G_2(t,s)\,ds=\frac{1}{\lambda_1^2\lambda_3\lambda_4}.$$

Proof. We have

$$\int_t^{t+w}g_{2,1}(t,s)\,ds=\frac{3\lambda_1^2-2\lambda_1\lambda_3-2\lambda_1\lambda_4+\lambda_3\lambda_4+t\lambda_1^3-t\lambda_1^2\lambda_3-t\lambda_1^2\lambda_4+t\lambda_1\lambda_3\lambda_4}{\lambda_1^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2},$$
$$\int_t^{t+w}g_{2,2}(t,s)\,ds=-\frac{t}{\lambda_1(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)},$$
$$\int_t^{t+w}g_{2,3}(t,s)\,ds=-\frac{1}{\lambda_3(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)},$$

$$\int_t^{t+w} g_{2,4}(t,s)\,ds = \frac{1}{\lambda_4(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)}\,,$$

and

$$\int_t^{t+w} G_2(t,s)\,ds = \int_t^{t+w} g_{2,1}(t,s)\,ds + \int_t^{t+w} g_{2,2}(t,s)\,ds + \int_t^{t+w} g_{2,3}(t,s)\,ds + \int_t^{t+w} g_{2,4}(t,s)\,ds = \frac{1}{\lambda_1^2\lambda_3\lambda_4}\,. \qquad \square$$

We have three different roots satisfying one of the following four cases:

- All roots are positive.
- Two roots are positive and one root is negative.
- Two roots are negative and one root is positive.
- All roots are negative.

If all roots are positive, we suppose that $\lambda_1 > \lambda_3 > \lambda_4 > 0$ (the other situations can be proved by using the same method), and we have

Theorem 3.8. *If $p_{2,1} > n_{2,1}$ and $\lambda_1 > \lambda_3 > \lambda_4 > 0$, then*

$$0 < A_{2,1} \le G_2(t,s) \le B_{2,1}.$$

Proof. If $\lambda_1 > \lambda_3 > \lambda_4 > 0$, the study of the derivatives of $g_{1,i}$, $i = \overline{1,4}$, with respect to s gives $\frac{\partial}{\partial s}\, g_{2,1}(s,t) < 0$, $\frac{\partial}{\partial s}\, g_{2,2}(s,t) > 0$, $\frac{\partial}{\partial s}\, g_{2,3}(s,t) > 0$ and $\frac{\partial}{\partial s}\, g_{2,4}(s,t) < 0$. This implies that

$$g_{2,1}(t,t+w) + g_{2,2}(t,t) + g_{2,3}(t,t) + g_{2,4}(t,t+w) \le G_2(t,s) \le g_{2,1}(t,t) + g_{2,2}(t,t+w) + g_{2,3}(t,t+w) + g_{2,4}(t,t).$$

It is easy to check that

$$0 < \frac{we^{w\lambda_1}(\lambda_1-\lambda_3)(\lambda_1-\lambda_4) - (e^{w\lambda_1}-1)(\lambda_3-2\lambda_1+\lambda_4)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2} \le g_{2,1}(t,t+w),$$

$$0 < g_{2,1}(t,t) \le \frac{\big(we^{w\lambda_1}(\lambda_1-\lambda_3)(\lambda_1-\lambda_4) - (e^{w\lambda_1}-1)(\lambda_3-2\lambda_1+\lambda_4)\big)e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}\,,$$

$$-w\,\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} \le g_{2,2}(t,t) \le 0, \quad g_{2,2}(t,t+w) \le 0.$$

By using the last double inequality together with the assumption $p_{2,1} > n_{2,1}$, we arrive at $0 < A_{2,1} \le G_1(t,s) \le B_{2,1}$. $\square$

Corollary 3.6. *If $h \in \mathcal{C}_w^+$, $p_{2,1} > n_{2,1}$ and $\lambda_1 > \lambda_3 > \lambda_4 > 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_2(t,s)h(s)\,ds.$$

Example 3.6. Consider the equation

$$u'''' - 0.51u''' + 0.085u'' - 0.0048u' + 0.00004u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda-0.2)^2(\lambda-0.1)(\lambda-0.01)=0$ has the roots $\lambda_1=0.2$, $\lambda_3=0.1$, $\lambda_4=0.01$. Since $p_{2,1}=5249.8>n_{2,1}=2844$, the equation has a unique 2π-periodic solution $u(t)=\int_t^{t+w} G_2(t,s)h(s)\,ds$ with $\int_t^{t+w} G_2(t,s)\,ds=25000$ and $0<2405.8<G_2(t,s)<5552.5$.

If two roots are positive and one root is negative, we suppose that $\lambda_1>\lambda_3>0$ and $\lambda_4<0$ (the other situations can be proved by using the same method), and we have

Theorem 3.9. *If $p_{2,2}<n_{2,2}$, $\lambda_1>\lambda_3>0$ and $\lambda_4<0$, then*

$$A_{2,1}\le G_2(t,s)\le B_{2,1}<0.$$

Proof. If $\lambda_1>\lambda_3>0$ and $\lambda_4<0$, the study of the derivatives of $g_{1,i}$, $i=\overline{1,4}$, with respect to s gives $\frac{\partial}{\partial s}g_{2,1}(s,t)<0$, $\frac{\partial}{\partial s}g_{2,2}(s,t)>0$, $\frac{\partial}{\partial s}g_{2,3}(s,t)>0$ and $\frac{\partial}{\partial s}g_{2,4}(s,t)<0$. Similarly, as in the proof of Theorem 3.8, we obtain $A_{2,1}\le G_2(t,s)\le B_{2,1}<0$. □

Corollary 3.7. *If $h\in\mathcal{C}_w^-$, $p_{2,2}<n_{2,2}$, $\lambda_1>\lambda_3>0$ and $\lambda_4<0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t)=\int_t^{t+w} G_2(t,s)h(s)\,ds.$$

Example 3.7. Consider the equation

$$u''''-0.49u'''+0.075u''-0.0032u'-0.00004u=h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda-0.2)^2(\lambda-0.1)(\lambda+0.01)=0$ has the roots $\lambda_1=0.2$, $\lambda_3=0.1$, $\lambda_4=-0.01$. Since $p_{2,2}=1567.2<n_{2,2}=4218.5$, the equation has a unique 2π-periodic solution $u(t)=\int_t^{t+w} G_2(t,s)h(s)\,ds$ with $\int_t^{t+w} G_2(t,s)\,ds=-25000$ and $-5305.9<G_2(t,s)<-2651.3<0$.

If two roots are negative and one root is positive, we suppose that $\lambda_1<\lambda_3<0$ and $\lambda_4>0$ (the other situations can be proved by using the same method), and we have

Theorem 3.10. *If $p_{2,3}<n_{2,3}$, $\lambda_1<\lambda_3<0$ and $\lambda_4>0$, then*

$$A_{2,2}\le G_2(t,s)\le B_{2,2}<0.$$

Proof. We have $g_{2,1}(s,t)=h_{2,1}(s,t)+h_{2,2}(s,t)$. If $\lambda_1<\lambda_3<0$ and $\lambda_4>0$, the study of the derivatives with respect to s gives $\frac{\partial}{\partial s}h_{2,1}(s,t)>0$, $\frac{\partial}{\partial s}h_{2,2}(s,t)<0$, $\frac{\partial}{\partial s}g_{2,2}(s,t)>0$, $\frac{\partial}{\partial s}g_{2,3}(s,t)<0$ and $\frac{\partial}{\partial s}g_{2,4}(s,t)>0$. This implies that

$$\begin{aligned}h_{2,1}(t,t)+h_{2,2}(t,t+w)+g_{2,2}(t,t)+g_{2,3}(t,t+w)+g_{2,4}(t,t)&\\ \le G_2(t,s)\le h_{2,1}(t,t+w)+h_{2,2}(t,t)+g_{2,2}(t,t+w)+g_{2,3}(t,t)+g_{2,4}(t,t+w).&\end{aligned}$$

It is easy to check that

$$\frac{e^{w\lambda_1}(w\lambda_1+1)}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\le h_{2,2}(t,t)\le\frac{e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)},$$
$$\frac{2w\lambda_1+1}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\le h_{2,2}(t,t+w)\le\frac{w\lambda_1+1}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}.$$

The above double inequality and the assumption $p_{2,3}>n_{2,3}$ lead to $0<A_{2,2}\le G_2(t,s)\le B_{2,2}$. □

Corollary 3.8. *If $h\in\mathcal{C}_w^-$, $p_{2,3}<n_{2,3}$, $\lambda_1<\lambda_3<0$ and $\lambda_4>0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t)=\int_t^{t+w} G_2(t,s)h(s)\,ds.$$

Example 3.8. Consider the equation

$$u'''' + 0.49u''' + 0.075u'' + 0.0032u' - 0.00004u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda+0.2)^2(\lambda+0.1)(\lambda-0.01) = 0$ has the roots $\lambda_1 = -0.2$, $\lambda_3 = -0.1$, $\lambda_4 = 0.01$. Since $p_{2,3} = 1329.1 < n_{2,3} = 4218.5$, the equation has a unique 2π-periodic solution $u(t) = \int\limits_t^{t+w} G_2(t,s)h(s)\,ds$ with $\int\limits_t^{t+w} G_2(t,s)\,ds = -25000$, $-5367.0 < G_2(t,s) < -2889.4 < 0$.

If all roots are negative, we suppose that $\lambda_1 < \lambda_3 < 0 < \lambda_4 < 0$ (the other situations can be proved by using the same method), and we have

Theorem 3.11. *If $p_{2,4} > n_{2,4}$ and $\lambda_1 < \lambda_3 < \lambda_4 < 0$, then*

$$0 < A_{2,2} \le G_2(t,s) \le B_{2,2}.$$

Proof. The study of the derivatives with respect to s gives $\frac{\partial}{\partial s} h_{2,1}(s,t) > 0$, $\frac{\partial}{\partial s} h_{2,2}(s,t) < 0$, $\frac{\partial}{\partial s} g_{2,2}(s,t) > 0$, $\frac{\partial}{\partial s} g_{2,3}(s,t) < 0$ and $\frac{\partial}{\partial s} g_{2,4}(s,t) > 0$. Similarly, as in the proof of Theorem 3.8, we obtain $0 < A_{2,2} \le G_2(t,s) \le B_{2,2}$. ☐

Corollary 3.9. *If $h \in \mathcal{C}_w^+$, $p_{2,4} > n_{2,4}$ and $\lambda_1 < \lambda_3 < \lambda_4 < 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int\limits_t^{t+w} G_2(t,s)h(s)\,ds.$$

Example 3.9. Consider the equation

$$u'''' + 0.51u''' + 0.085u'' + 0.0048u' + 0.00004u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda+0.2)^2(\lambda+0.1)(\lambda+0.01) = 0$ has the roots $\lambda_1 = -0.2$, $\lambda_3 = -0.1$, $\lambda_4 = -0.01$. Since $p_{2,4} = 5644.5 > n_{2,4} = 3306.3$, the equation has a unique 2π-periodic solution $u(t) = \int\limits_t^{t+w} G_2(t,s)h(s)\,ds$ with $\int\limits_t^{t+w} G_2(t,s)\,ds = 25000$ and $0 < 2338.3 < G_2(t,s) < 5289.4$.

Case 3. *If $\lambda_1 = \lambda_2 \ne \lambda_3 = \lambda_4$. We use the following abbreviations:*

$$g_{3,1}(t,s) = -\frac{\big((1-e^{w\lambda_1})(s\lambda_1 - s\lambda_4 + 2) - w(\lambda_1-\lambda_4)\big)e^{(w+t-s)\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3} + t\,\frac{e^{(w+t-s)\lambda_1}}{(1-e^{w\lambda_1})(\lambda_1-\lambda_4)^2},$$

$$g_{3,2}(t,s) = -\frac{\big((e^{w\lambda_4}-1)(s\lambda_4 - s\lambda_1 + 2) - w(\lambda_1-\lambda_4)\big)e^{(w+t-s)\lambda_4}}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^3} + t\,\frac{e^{(w+t-s)\lambda_4}}{(1-e^{w\lambda_4})(\lambda_1-\lambda_4)^2};$$

$$h_{3,1}(s,t) = \frac{e^{\lambda_4(t-s+w)}\big(\lambda_4(s-t)(e^{w\lambda_4}-1) + e^{w\lambda_4} + w\lambda_4 - 1\big)}{\lambda_4(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^2},$$

$$h_{3,2}(s,t) = -\frac{e^{\lambda_4(t-s+w)}(\lambda_1+\lambda_4)}{\lambda_4(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3},$$

$$h_{3,3}(t,s) = \frac{1}{\lambda_1}\,\frac{e^{\lambda_1(t-s+w)}\big(\lambda_1(s-t)(e^{w\lambda_1}-1)(\lambda_1-\lambda_4) + w\lambda_1(\lambda_1-\lambda_4) - (e^{w\lambda_1}-1)(\lambda_4-2\lambda_1)\big)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3},$$

$$h_{3,4}(t,s) = \frac{1}{\lambda_1}\,\lambda_4\,\frac{e^{\lambda_1(t-s+w)}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^3},$$

$$h_{3,5}(t,s) = -\frac{e^{\lambda_4(t-s+w)}(1-(s-t)(\lambda_1-\lambda_4))}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3},$$

$$\begin{aligned}
h_{3,6}(t,s) &= -\frac{e^{\lambda_4(t-s+w)}}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^3}\,(e^{w\lambda_4}-w\lambda_1+w\lambda_4-1),\\
h_{3,7}(t,s) &= \frac{e^{\lambda_1(t-s+w)}\big(w\lambda_1(\lambda_1-\lambda_4)+(\lambda_1+\lambda_4)(e^{w\lambda_1}-1)\big)}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3},\\
h_{3,8}(t,s) &= \frac{e^{\lambda_1(t-s+w)}(s\lambda_1-t\lambda_1+1)}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^2},\\
h_{3,9}(t,s) &= w\,\frac{e^{\lambda_4(t-s+w)}}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^2},\\
h_{3,10}(t,s) &= -\frac{e^{\lambda_4(t-s+w)}(2-(s-t)(\lambda_1-\lambda_4))}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3};
\end{aligned}$$

$$\begin{aligned}
A_{3,1} &= \frac{2e^{w\lambda_1}+w\lambda_1e^{w\lambda_1}-w\lambda_4e^{w\lambda_1}-2}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}-\frac{e^{w\lambda_4}(\lambda_1+\lambda_4)}{\lambda_4(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\\
&\quad+\frac{(e^{w\lambda_4}+w\lambda_4e^{w\lambda_4}-1)}{\lambda_4(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^2},\\
A_{3,2} &= \frac{\lambda_4-2\lambda_1+2\lambda_1e^{w\lambda_1}-\lambda_4e^{w\lambda_1}+w\lambda_1^2e^{w\lambda_1}-w\lambda_1\lambda_4e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\\
&\quad+\frac{\lambda_4e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^3}-\frac{(w\lambda_4-w\lambda_1+1)}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\\
&\quad-\frac{(e^{w\lambda_4}-w\lambda_1+w\lambda_4-1)e^{w\lambda_4}}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^3},\\
A_{3,3} &= \frac{\big((\lambda_1+\lambda_4)(e^{w\lambda_1}-1)+w\lambda_1(\lambda_1-\lambda_4)\big)e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}+\frac{we^{w\lambda_4}}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^2}\\
&\quad+\frac{1}{\lambda_1}\,\frac{w\lambda_1+1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^2}-\frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\,(w\lambda_4-w\lambda_1+2);\\
B_{3,1} &= \frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\,(2e^{w\lambda_1}+w\lambda_1-w\lambda_4-2)-\frac{1}{\lambda_4}\,\frac{\lambda_1+\lambda_4}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\\
&\quad+\frac{1}{\lambda_4}\,\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^2}\,(e^{w\lambda_4}+w\lambda_4-1),\\
B_{3,2} &= \frac{1}{\lambda_1}\,\frac{e^{w\lambda_1}(\lambda_4-2\lambda_1+w\lambda_1^2+2\lambda_1e^{w\lambda_1}-\lambda_4e^{w\lambda_1}-w\lambda_1\lambda_4)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\\
&\quad+\frac{1}{\lambda_1}\,\frac{\lambda_4}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^3}-\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\\
&\quad-\frac{1}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^3}\,(e^{w\lambda_4}-w\lambda_1+w\lambda_4-1),\\
B_{3,3} &= -\frac{1}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\,(\lambda_1+\lambda_4-w\lambda_1^2-\lambda_1e^{w\lambda_1}-\lambda_4e^{w\lambda_1}+w\lambda_1\lambda_4)\\
&\quad+\frac{1}{\lambda_1}\,\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^2}+\frac{w}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^2}-2\,\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3};
\end{aligned}$$

$$\begin{aligned}
n_{3,1} &= \frac{e^{w\lambda_4}(\lambda_1+\lambda_4)}{\lambda_4(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3},\\
n_{3,2} &= -\frac{\lambda_4e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^3}-\frac{w(\lambda_1-\lambda_4)}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3},\\
n_{3,3} &= \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}(w\lambda_4-w\lambda_1+2)-\frac{w}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^2};
\end{aligned}$$

$$p_{3,1} = \frac{2e^{w\lambda_1} + w\lambda_1 e^{w\lambda_1} - w\lambda_4 e^{w\lambda_1} - 2}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3} + \frac{e^{w\lambda_4} + w\lambda_4 e^{w\lambda_4} - 1}{\lambda_4(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^2},$$

$$p_{3,2} = \frac{\lambda_4 - 2\lambda_1 + 2\lambda_1 e^{w\lambda_1} - \lambda_4 e^{w\lambda_1} + w\lambda_1^2 e^{w\lambda_1} - w\lambda_1\lambda_4 e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3} - \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3} - \frac{(e^{w\lambda_4} - w\lambda_1 + w\lambda_4 - 1)e^{w\lambda_4}}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^3},$$

$$p_{3,3} = \frac{\big((\lambda_1+\lambda_4)(e^{w\lambda_1}-1) + w\lambda_1(\lambda_1-\lambda_4)\big)e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3} + \frac{1}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^2} + w\,\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^2}.$$

Theorem 3.12. *For all $t \in [0, w]$ and $s \in [t, t+w]$, we have*

$$\int_t^{t+w} G_3(t,s)\,ds = \frac{1}{\lambda_1^2\lambda_4^2}.$$

Proof. We have

$$\int_t^{t+w} G_3(t,s)\,ds = \int_t^{t+w} g_{3,1}(t,s)\,ds + \int_t^{t+w} g_{3,2}(t,s)\,ds = -\frac{\lambda_4 - 3\lambda_1}{\lambda_1^2(\lambda_1-\lambda_4)^3} + \frac{\lambda_1 - 3\lambda_4}{\lambda_4^2(\lambda_1-\lambda_4)^3} = \frac{1}{\lambda_1^2\lambda_4^2}. \quad \square$$

We have two different roots satisfying one of the following three cases:

- Two positive roots.
- One positive root and one negative root.
- Two negative roots.

If all roots are positive, we suppose that $\lambda_1 > \lambda_4 > 0$ (the situation when $\lambda_4 > \lambda_1 > 0$ can be proved by using the same method), and we have

Theorem 3.13. *If $p_{3,1} > n_{3,1}$ and $\lambda_1 > \lambda_2 > 0$, then*

$$0 < A_{3,1} \le G_3(t,s) \le B_{3,1}.$$

Proof. We write $g_{3,2}(t,s) = h_{3,1}(t,s) + h_{3,2}(t,s)$. If $\lambda_1 > \lambda_2 > 0$, the study of the derivatives with respect to s gives $\frac{\partial}{\partial s} g_{3,1}(t,s) < 0$, $\frac{\partial}{\partial s} h_{3,1}(t,s) < 0$ and $\frac{\partial}{\partial s} h_{3,2}(t,s) > 0$. This implies that

$$g_{3,1}(t,t+w) + h_{3,1}(t,t+w) + h_{3,2}(t,t) \le G_3(t,s) \le g_{3,1}(t,t) + h_{3,1}(t,t) + h_{3,2}(t,t+w).$$

This double inequality together with the assumption $p_{3,1} > n_{3,1}$ lead to $0 < A_{3,1} \le G_3(t,s) \le B_{3,1}$. $\square$

Corollary 3.10. *If $h \in \mathcal{C}_w^+$, $p_{3,1} > n_{3,1}$ and $\lambda_1 > \lambda_2 > 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_3(t,s)h(s)\,ds.$$

Example 3.10. Consider the equation

$$u'''' - 0.06\,u''' + 0.0013u'' - 1.2\times 10^{-5}u' + 4.0\times 10^{-8}u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation is $(\lambda-0.02)^2(\lambda-0.01)^2 = 0$ has two roots $\lambda_1 = 0.02$ and $\lambda_4 = 0.01$. Since $p_{3,1} = 5.0241\times 10^7 > n_{3,1} = 4.9262\times 10^7$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_3(t,s)h(s)\,ds$ with $\int_t^{t+w} G_3(t,s)\,ds = 2.5\times 10^7$ and $0 < 9.7887\times 10^5 < G_3(t,s) < 6.9789\times 10^6$.

If one root is positive and one root is negative, we suppose that $\lambda_1 > 0$ and $\lambda_4 < 0$ (the situation when $\lambda_1 < 0$ and $\lambda_4 > 0$ can be proved by using the same method), and we have

Theorem 3.14. *If $p_{3,2} > n_{3,2}$, $\lambda_1 > 0$ and $\lambda_4 < 0$, then*

$$0 < A_{3,2} \le G_3(t,s) \le B_{3,2}.$$

Proof. We write $g_{3,1}(s,t) = h_{3,3}(s,t) + h_{3,4}(s,t)$ and $g_{3,2}(s,t) = h_{3,5}(s,t) + h_{3,6}(s,t)$. If $\lambda_1 > 0$ and $\lambda_4 < 0$, the study of the derivatives with respect to s gives $\frac{\partial}{\partial s} h_{3,3}(t,s) < 0$, $\frac{\partial}{\partial s} h_{3,4}(t,s) > 0$, $\frac{\partial}{\partial s} h_{3,5}(s,t) < 0$ and $\frac{\partial}{\partial s} h_{3,6}(s,t) > 0$. Similarly, as in the proof of Theorem 3.13, we obtain $0 < A_{3,2} \le G_3(t,s) \le B_{3,2}$. □

Corollary 3.11. *If $h \in \mathcal{C}_w^+$, $p_{3,2} > n_{3,2}$, $\lambda_1 > 0$ and $\lambda_4 < 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_3(t,s)h(s)\,ds.$$

Example 3.11. Consider the equation

$$u'''' - 0.02u'' + 0.0001u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation

$$(\lambda - 0.1)^2(\lambda + 0.1)^2 = 0$$

has two roots $\lambda_1 = 0.1$ and $\lambda_4 = -0.1$. Since $p_{3,2} = 1609.8 > n_{3,2} = 604.66$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_3(t,s)h(s)\,ds$ with $\int_t^{t+w} G_3(t,s)\,ds = 10000$, $1005.2 < G_3(t,s) < 2178.6$.

If all roots are negative, we suppose that $\lambda_1 < \lambda_4 < 0$ (the situation when $\lambda_4 < \lambda_1 < 0$ can be proved by using the same method), and we have

Theorem 3.15. *If $p_{3,3} > n_{3,3}$ and $\lambda_1 < \lambda_4 < 0$, then*

$$0 < A_{3,3} \le G_3(t,s) \le B_{3,3}.$$

Proof. We write $g_{3,1}(s,t) = h_{3,7}(s,t) + h_{3,8}(s,t)$, $g_{3,2}(s,t) = h_{3,9}(s,t) + h_{3,10}(s,t)$. If $\lambda_1 < \lambda_4 < 0$, the study of the derivatives with respect to s gives $\frac{\partial}{\partial s} h_{3,7}(t,s) < 0$, $\frac{\partial}{\partial s} h_{3,8}(t,s) < 0$, $\frac{\partial}{\partial s} h_{3,9}(t,s) > 0$ and $\frac{\partial}{\partial s} h_{3,10}(t,s) < 0$. Similarly, as in the proof of Theorem 3.13, we obtain $0 < A_{3,3} \le G_3(t,s) \le B_{3,3}$. □

Corollary 3.12. *If $h \in \mathcal{C}_w^+$, $p_{3,3} > n_{3,3}$, $\lambda_1 < \lambda_4 < 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_3(t,s)h(s)\,ds.$$

Example 3.12. Consider the equation

$$u'''' + 0.22u''' + 0.0141u'' + 0.00022u' + 1.0 \times 10^{-6}u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation

$$(\lambda + 0.1)^2(\lambda + 0.01)^2 = 0$$

has two roots $\lambda_1 = -0.1$ and $\lambda_4 = -0.01$. Since $2.027 \times 10^5 > n_{3,3} = 59450$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_3(t,s)h(s)\,ds$ with $\int_t^{t+w} G_3(t,s)\,ds = 1000000$ and $1.4325 \times 10^5 < G_3(t,s) < 1.7506 \times 10^5$.

Case 4. *If $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$. We use the following abbreviations:*

$$
\begin{aligned}
g_{4,1}(t,s) = {} & e^{(t+w-s)\lambda_1}\,\frac{(1-e^{w\lambda_1})\big((e^{w\lambda_1}-1)(((s-t)(\lambda_1-\lambda_4)+1)^2+1)\big)}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\\
& + \frac{e^{(t+w-s)\lambda_1}(1-e^{w\lambda_1})\big(w(\lambda_1-\lambda_4)(2(s-t)(\lambda_1-\lambda_4)+2)\big)}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\\
& - \frac{w^2 e^{(t+w-s)\lambda_1}(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\,,
\end{aligned}
$$

$$
g_{4,2}(t,s) = \frac{e^{(t+w-s)\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\,;
$$

$$
\begin{aligned}
h_{4,1}(t,s) = {} & -\frac{\lambda_1(s-t)(e^{w\lambda_1}-1)e^{\lambda_1(t-s+w)}}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)}\Big(\lambda_1(s-t)(e^{w\lambda_1}-1)+2(e^{w\lambda_1}+w\lambda_1-1)\Big)\\
& - \frac{e^{\lambda_1(t-s+w)}}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)}\Big(w^2\lambda_1^2(e^{w\lambda_1}+1)+2(e^{w\lambda_1}-1)(e^{w\lambda_1}+w\lambda_1-1)\Big),
\end{aligned}
$$

$$
\begin{aligned}
h_{4,2}(t,s) = {} & -\frac{1}{\lambda_1^2}\,\lambda_4\,\frac{e^{\lambda_1(t-s+w)}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\Big(w\lambda_1(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_4-2\lambda_1)\Big)\\
& - \frac{1}{\lambda_1}\,\lambda_4 e^{\lambda_1(t-s+w)}\,\frac{s-t}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^2}\,,
\end{aligned}
$$

$$
\begin{aligned}
h_{4,3}(t,s) = {} & -\frac{\big(2w\lambda_1\lambda_4(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)+w^2\lambda_1^2(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2\big)e^{\lambda_1(t-s+w)}}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\\
& - \frac{e^{\lambda_1(t-s+w)}\big(\lambda_1(s-t)(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^2(s\lambda_1-t\lambda_1+2)\big)}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\\
& - \frac{e^{\lambda_1(t-s+w)}\big(2(e^{w\lambda_1}-1)^2(\lambda_1^2-\lambda_1\lambda_4+\lambda_4^2)\big)}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\,,
\end{aligned}
$$

$$
h_{4,4}(t,s) = \frac{1}{\lambda_1^2}\,\frac{e^{\lambda_1(t-s+w)}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^2}\,(s\lambda_1-t\lambda_1+1)(\lambda_4-w\lambda_1^2-\lambda_4 e^{w\lambda_1}+w\lambda_1\lambda_4);
$$

$$
\begin{aligned}
A_{4,1} = {} & -e^{w\lambda_1}\,\frac{2(e^{w\lambda_1}-1)^2+w^2(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2+2w(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\\
& + \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\,,
\end{aligned}
$$

$$
\begin{aligned}
A_{4,2} = {} & \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}-\frac{e^{w\lambda_1}\big(w^2\lambda_1^2(e^{w\lambda_1}+1)+2(e^{w\lambda_1}-1)^2+2w\lambda_1(e^{w\lambda_1}-1)\big)}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)}\\
& - \frac{\lambda_4}{\lambda_1^2(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\Big(w\lambda_1 e^{w\lambda_1}(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_4-2\lambda_1)\Big),
\end{aligned}
$$

$$
\begin{aligned}
A_{4,3} = {} & -\frac{1}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\Big((\lambda_1-\lambda_4)^2\big(w^2\lambda_1^2(e^{w\lambda_1}+1)+w\lambda_1(w\lambda_1+2)(e^{w\lambda_1}-1)^2\big)\Big)\\
& - \frac{1}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\Big(2w\lambda_1\lambda_4(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)+2(e^{w\lambda_1}-1)^2(\lambda_1^2-\lambda_1\lambda_4+\lambda_4^2)\Big)\\
& + \frac{1}{\lambda_1^2}\,\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^2}\,(\lambda_4-w\lambda_1^2-\lambda_4 e^{w\lambda_1}+w\lambda_1\lambda_4)\\
& + \frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\,;
\end{aligned}
$$

$$B_{4,1} = -\frac{2we^{w\lambda_1}(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)+w^2e^{w\lambda_1}(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2+2(e^{w\lambda_1}-1)^2}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}$$
$$+\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3},$$
$$B_{4,2} = \frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}$$
$$-\frac{w^2\lambda_1^2e^{w\lambda_1}(e^{w\lambda_1}+1)+2(e^{w\lambda_1}-1)^2+2w\lambda_1e^{w\lambda_1}(e^{w\lambda_1}-1)}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)}$$
$$-\frac{1}{\lambda_1^2}\frac{\lambda_4e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\big(w\lambda_1(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_4-2\lambda_1)\big),$$
$$B_{4,3} = -\frac{e^{\lambda_1(w)}}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\Big(2w\lambda_1\lambda_4(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)+w^2\lambda_1^2(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2\Big)$$
$$-\frac{e^{\lambda_1(w)}}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\Big(2(e^{w\lambda_1}-1)^2(\lambda_1^2-\lambda_1\lambda_4+\lambda_4^2)\Big)$$
$$+\frac{(w\lambda_1+1)(\lambda_4-w\lambda_1^2-\lambda_4e^{w\lambda_1}+w\lambda_1\lambda_4)}{\lambda_1^2(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^2}-\frac{2e^{w\lambda_4}-w\lambda_1e^{w\lambda_4}+w\lambda_4e^{w\lambda_4}-2}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^3};$$

$$n_{4,1} = e^{w\lambda_1}\,\frac{2(e^{w\lambda_1}-1)^2+w^2(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2+2w(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3},$$
$$n_{4,2} = -\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}$$
$$+\frac{w^2\lambda_1^2e^{w\lambda_1}(e^{w\lambda_1}+1)+2(e^{w\lambda_1}-1)^2+2w\lambda_1e^{w\lambda_1}(e^{w\lambda_1}-1)}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)},$$
$$n_{4,3} = \frac{1}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\Big(2w\lambda_1\lambda_4(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)+w^2\lambda_1^2(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2\Big)$$
$$+\frac{2(e^{w\lambda_1}-1)^2(\lambda_1^2-\lambda_1\lambda_4+\lambda_4^2)}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}+\frac{w\lambda_1(w\lambda_1)(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^2}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}$$
$$-\frac{1}{\lambda_1^2}\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^2}(\lambda_4-w\lambda_1^2-\lambda_4e^{w\lambda_1}+w\lambda_1\lambda_4);$$

$$p_{4,1} = \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3},$$
$$p_{4,2} = -\frac{\lambda_4e^{w\lambda_1}\big(w\lambda_1(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_4-2\lambda_1)\big)}{\lambda_1^2(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3},$$
$$p_{4,3} = -\frac{1}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\Big(w\lambda_1(2)(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^2\Big)$$
$$+\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}.$$

Theorem 3.16. *For all $t\in[0,w]$ and $s\in[t,t+w]$, we have*

$$\int_t^{t+w} G_4(t,s)\,ds = \frac{1}{\lambda_1^3\lambda_4}.$$

Proof. We have

$$\int_t^{t+w} g_{4,1}(s,t)\,ds + \int_t^{t+w} g_{4,2}(s,t)\,ds = -\frac{3\lambda_1^2 - 3\lambda_1\lambda_4 + \lambda_4^2}{\lambda_1^3(\lambda_1-\lambda_4)^3} + \frac{1}{\lambda_4(\lambda_1-\lambda_4)^3} = \frac{1}{\lambda_1^3\lambda_4}\,. \qquad \square$$

We have two different roots satisfying one of the three cases:

- Two positive roots.
- One positive root and one negative root.
- Two negative roots.

If all roots are positive, we suppose that $\lambda_1 > \lambda_4 > 0$ (the situation when $\lambda_4 > \lambda_1 > 0$ can be proved by using the same method), and we have

Theorem 3.17. *If $p_{4,1} > n_{4,1}$ and $\lambda_1 > \lambda_4 > 0$, then*

$$0 < A_{4,1} \le G_4(t,s) \le B_{4,1}.$$

Proof. If $\lambda_1 > \lambda_4 > 0$, the study of the derivatives with respect to s gives $\frac{\partial}{\partial s} g_{4,1}(s,t) > 0$ and $\frac{\partial}{\partial s} g_{4,2}(s,t) < 0$. So $g_{4,1}(t,t) + g_{4,2}(t,t+w) \le g_{4,1}(t,s) \le g_{4,1}(t,t+w) + g_{4,2}(t,t)$. This double inequality together with the assumption $p_{4,1} > n_{4,1}$ give $0 < A_{4,1} \le G_4(t,s) \le B_{4,1}$. $\square$

Corollary 3.13. *If $h \in \mathcal{C}_w^+$, $p_{4,1} > n_{4,1}$ and $\lambda_1 > \lambda_4 > 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_4(t,s)h(s)\,ds.$$

Example 3.13. Consider the equation

$$u'''' - 0.61u''' + 0.126u'' - 0.0092u' + 0.00008u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda-0.2)^3(\lambda-0.01) = 0$ has the roots $\lambda_1 = 0.2$ and $\lambda_4 = 0.01$. We compute $p_{4,1} = 2248.2 > n_{4,1} = 404.33$, and hence the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_4(t,s)h(s)\,ds$ with $\int_t^{t+w} G_4(t,s)\,ds = 2.5 \times 10^5$ and $0 < 1843.9 < G_4(t,s) < 2135.5\,.$

If one root is positive and one root is negative, we suppose that $\lambda_1 > 0$ and $\lambda_4 < 0$ (the situation when $\lambda_1 < 0$ and $\lambda_4 > 0$ can be proved by using the same method), and we have

Theorem 3.18. *If $p_{4,2} < n_{4,2}$, $\lambda_1 > 0$ and $\lambda_4 < 0$ then*

$$A_{4,2} \le G_4(t,s) \le B_{4,2} < 0.$$

Proof. We have $g_{4,1}(t,s) = h_{4,1}(t,s) + h_{4,2}(t,s)$. If $\lambda_1 > 0$ and $\lambda_4 < 0$, the study of the derivatives with respect to s gives $\frac{\partial}{\partial s} h_{4,1}(s,t) > 0$, $\frac{\partial}{\partial s} h_{4,2}(s,t) < 0$ and $\frac{\partial}{\partial s} g_{4,2}(s,t) < 0$. Similarly, as in the proof of Theorem 3.17, we obtain $A_{4,2} \le G_4(t,s) \le B_{4,2} < 0$. $\square$

Corollary 3.14. *If $h \in \mathcal{C}_w^-$, $p_{4,2} < n_{4,2}$, $\lambda_1 > 0$ and $\lambda_4 < 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_4(t,s)h(s)\,ds.$$

Example 3.14. Consider the equation

$$u'''' - 0.29u''' + 0.027u'' - 0.0007u' - 0.00001u = h(t),$$

here h is the given continuous and 2π-periodic function. The characteristic equation $(\lambda - 0.1)^3(\lambda + 0.01) = 0$ has the roots $\lambda_1 = 0.1$ and $\lambda_4 = -0.01$. Since $p_{4,2} = 465.49 < n_{4,2} = 15472$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_4(t,s)h(s)\,ds$ with $\int_t^{t+w} G_4(t,s)\,ds = 10^6$ and $-16824 < G_4(t,s) < -15006 < 0$.

If all roots are negative, we suppose that $\lambda_1 < \lambda_4 < 0$ (the situation when $\lambda_4 < \lambda_1 < 0$ can be proved by using the same method), and we have

Theorem 3.19. *If $p_{4,3} > n_{4,3}$ and $\lambda_1 < \lambda_4 < 0$, then*

$$0 < A_{4,3} \le G_4(t,s) \le B_{4,3}.$$

Proof. We have $g_{4,1}(t,s) = h_{4,3}(t,s) + h_{4,4}(t,s)$. If $\lambda_1 < \lambda_4 < 0$, the study of the derivatives with respect to s gives $\frac{\partial}{\partial s} h_{4,3}(s,t) < 0$, $\frac{\partial}{\partial s} h_{4,4}(s,t) > 0$ and $\frac{\partial}{\partial s} g_{4,2}(s,t) > 0$. Similarly, as in the proof of Theorem 3.17, we obtain $0 < A_{4,3} \le G_4(t,s) \le B_{4,3}$. □

Corollary 3.15. *If $h \in \mathcal{C}_w^+$, $p_{4,3} > n_{4,3}$ and $\lambda_1 < \lambda_4 < 0$, then equation (1.1) has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_4(t,s)h(s)\,ds.$$

Example 3.15. Consider the equation

$$u'''' + 0.601u''' + 0.1206u'' + 0.00812u' + 8.0 \times 10^{-6}u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda+0.2)^3(\lambda+0.001) = 0$ has the roots $\lambda_1 = -0.2$, $\lambda_4 = -0.001$. Since $p_{4,3} = 20353 > n_{4,3} = 748.34$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_4(t,s)h(s)\,ds$ with $\int_t^{t+w} G_4(t,s)\,ds = 2.5\times 10^7$, $0 < 20134 < G_4(t,s) < 3.9784 \times 10^6$.

Case 5. *If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$. We use the following abbreviations:*

$$\begin{aligned}
A_{5,1} &= \frac{1}{6\lambda_1^3(e^{w\lambda_1}-1)^4}\Big(6w\lambda_1 e^{w\lambda_1}(e^{w\lambda_1}-1)^2 + w^3\lambda_1^3 e^{w\lambda_1}(e^{2(w\lambda_1)} + 4e^{w\lambda_1} + 1)\Big) \\
&\quad + \frac{1}{6\lambda_1^3(e^{w\lambda_1}-1)^4}\Big(3w^2\lambda_1^2 e^{w\lambda_1}(e^{2(w\lambda_1)} - 1) + 6(e^{w\lambda_1}-1)^3\Big) \\
&\quad - \frac{e^{w\lambda_1}}{2\lambda_1^3(e^{w\lambda_1}-1)^3}\Big(2(e^{w\lambda_1}-1)(e^{w\lambda_1} + w\lambda_1 - 1) + w^2\lambda_1^2(e^{w\lambda_1}+1)\Big), \\
A_{5,2} &= w^3\,\frac{e^{2(w\lambda_1)} + e^{w\lambda_1} + 4}{6(e^{w\lambda_1}-1)^3} + w^3 e^{w\lambda_1}\,\frac{e^{2(w\lambda_1)} + 4e^{w\lambda_1} + 1}{6(e^{w\lambda_1}-1)^4}\,;
\end{aligned}$$

$$\begin{aligned}
B_{5,1} &= \frac{1}{6\lambda_1^3(e^{w\lambda_1}-1)^4}\Big(6w\lambda_1 e^{w\lambda_1}(e^{w\lambda_1}-1)^2 + 3w^2\lambda_1^2 e^{w\lambda_1}(e^{2(w\lambda_1)} - 1)\Big) \\
&\quad + \frac{1}{6\lambda_1^3(e^{w\lambda_1}-1)^4}\Big(w^3\lambda_1^3 e^{w\lambda_1}(e^{2(w\lambda_1)} + 4e^{w\lambda_1} + 1) + 6e^{w\lambda_1}(e^{w\lambda_1}-1)^3\Big) \\
&\quad - \frac{2(e^{w\lambda_1}-1)(e^{w\lambda_1}(w\lambda_1+1) - 1) + w^2\lambda_1^2 e^{w\lambda_1}(e^{w\lambda_1}+1)}{2\lambda_1^3(e^{w\lambda_1}-1)^3}, \\
B_{5,2} &= w^3\,\frac{2e^{2(w\lambda_1)} - e^{w\lambda_1} + 2}{3(e^{w\lambda_1}-1)^4}\,;
\end{aligned}$$

$$n_{5,1} = \frac{e^{w\lambda_1}\big(2(e^{w\lambda_1}-1)(e^{w\lambda_1}+w\lambda_1-1)+w^2\lambda_1^2(e^{w\lambda_1}+1)\big)}{2\lambda_1^3(e^{w\lambda_1}-1)^3}\,,$$
$$n_{5,2} = -w^3\,\frac{e^{2(w\lambda_1)}+e^{w\lambda_1}+4}{6(e^{w\lambda_1}-1)^3}\,;$$

$$p_{5,1} = \frac{1}{6\lambda_1^3(e^{w\lambda_1}-1)^4}\Big(6w\lambda_1 e^{w\lambda_1}(e^{w\lambda_1}-1)^2 + w^3\lambda_1^3 e^{w\lambda_1}\big(e^{2(w\lambda_1)}+4e^{w\lambda_1}+1\big)\Big)$$
$$+\frac{1}{6\lambda_1^3(e^{w\lambda_1}-1)^4}\Big(3w^2\lambda_1^2 e^{w\lambda_1}(e^{2(w\lambda_1)}-1)+6(e^{w\lambda_1}-1)^3\Big),$$
$$p_{5,2} = w^3 e^{w\lambda_1}\,\frac{e^{2(w\lambda_1)}+4e^{w\lambda_1}+1}{6(e^{w\lambda_1}-1)^4}\,;$$

$$h_{5,1}(s,t) = \frac{e^{\lambda_1(t-s+w)}(s-t)^2(e^{w\lambda_1}+w\lambda_1-1)}{2\lambda_1(e^{w\lambda_1}-1)^2} - \frac{e^{\lambda_1(t-s+w)}(\lambda_1^3(s-t)^3)}{6\lambda_1^3-6\lambda_1^3 e^{w\lambda_1}}$$
$$+ e^{\lambda_1(t-s+w)}\,\frac{s\lambda_1-t\lambda_1+1}{2\lambda_1^3(e^{w\lambda_1}-1)^3}\Big(2(e^{w\lambda_1}-1)^2+2w\lambda_1(e^{w\lambda_1}-1)+w^2\lambda_1^2(e^{w\lambda_1}+1)\Big)$$
$$+\frac{w^3 e^{\lambda_1(t-s+w)}(e^{2(w\lambda_1)}+4e^{w\lambda_1}+1)}{6(e^{w\lambda_1}-1)^4}\,,$$
$$h_{5,2}(s,t) = -\frac{e^{\lambda_1(t-s+w)}}{2\lambda_1^3(e^{w\lambda_1}-1)^3}\Big((e^{w\lambda_1}-1)^2(\lambda_1^2(s-t)^2+2(\lambda_1(s-t)+1))\Big)$$
$$-\frac{e^{\lambda_1(t-s+w)}}{2\lambda_1^3(e^{w\lambda_1}-1)^3}\Big(2w\lambda_1(e^{w\lambda_1}-1)(\lambda_1(s-t)+1)+w^2\lambda_1^2(e^{w\lambda_1}+1)\Big),$$
$$h_{5,3}(s,t) = e^{\lambda_1(t-s+w)}\,\frac{s-t}{6(e^{w\lambda_1}-1)^3}\Big((s-t)^2(e^{w\lambda_1}-1)^2+3w^2(e^{w\lambda_1}+1)\Big),$$
$$h_{5,4}(s,t) = w\frac{e^{\lambda_1(t-s+w)}}{6(e^{w\lambda_1}-1)^4}\Big(3(s-t)^2(e^{w\lambda_1}-1)^2+w^2\big(e^{2(w\lambda_1)}+4e^{w\lambda_1}+1\big)\Big).$$

Theorem 3.20. *For all $t \in [0,w]$ and $s \in [t,t+w]$, we have*

$$\int_t^{t+w} G_5(t,s)\,ds = \frac{1}{\lambda_1^4}\,.$$

Proof. We have
$$\int_t^{t+w} G_5(t,s)\,ds = \int_t^{t+w} h_{5,1}(t,s)\,ds + \int_t^{t+w} h_{5,2}(t,s)\,ds.$$

So
$$\int_t^{t+w} G_5(t,s)\,ds = -\frac{3w^2\lambda_1^2(e^{w\lambda_1}+1)+w^3\lambda_1^3(e^{w\lambda_1}+2)-6(e^{w\lambda_1}-1)^2}{6\lambda_1^4(e^{w\lambda_1}-1)^2}$$
$$+\frac{w^3\lambda_1(e^{w\lambda_1}+2)}{6\lambda_1^2(e^{w\lambda_1}-1)^2}+\frac{3w^2(e^{w\lambda_1}+1)}{6\lambda_1^2(e^{w\lambda_1}-1)^2} = \frac{1}{\lambda_1^4}\,.$$ □

Theorem 3.21. *If $\lambda_1 > 0$ and $p_{5,1} > n_{5,1}$, then*

$$0 < A_{5,1} \le G_5(s,t) \le B_{5,1}.$$

Proof. We have $G_5(s,t) = h_{5,1}(t,s) + h_{5,2}(t,s)$. If $\lambda_1 > 0$, the study of the derivatives gives $\frac{\partial}{\partial s}h_{5,1}(t,s) < 0$ and $\frac{\partial}{\partial s}h_{5,2}(t,s) > 0$, so $h_{5,1}(t,t+w)+h_{5,2}(t,t) \le G_5(s,t) \le h_{5,1}(t,t)+h_{5,2}(t,t+w)$. If we use this double inequality together with the assumption $p_{5,1} > n_{5,1}$, we arrive at $0 < A_{5,1} \le G_5(s,t) \le B_{5,1}$. □

Corollary 3.16. *If $h \in \mathcal{C}_w^+$, $\lambda_1 > 0$ and $p_{5,1} > n_{5,1}$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_5(t,s)h(s)\,ds.$$

Example 3.16. Consider the equation

$$u'''' - 0.4u''' + +0.06u'' - 0.004u' + 0.0001u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda - 0.1)^4 = 0$ has the root $\lambda_1 = 0.1$. Since $p_{5,1} = 5866.2 > n_{5,1} = 5274.3$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_5(t,s)h(s)\,ds$ with $\int_t^{t+w} G_5(t,s)\,ds = 10^5$ and $0 < 591.86 < G_5(t,s) < 2591.9$.

Theorem 3.22. *If $\lambda_1 < 0$ and $p_{5,2} > n_{5,2}$, then*

$$0 < A_{5,2} \leq G_5(s,t) \leq B_{5,2}.$$

Proof. We have $G_5(s,t) = h_{5,3}(t,s) + h_{5,4}(t,s)$. If $\lambda_1 < 0$, the study of the derivatives gives $\frac{\partial}{\partial s} h_{5,3}(t,s) < 0$ and $\frac{\partial}{\partial s} h_{5,4}(t,s) > 0$. Similarly, as in the proof of Theorem 3.21, we obtain $0 < A_{5,2} \leq G_5(s,t) \leq B_{5,2}$. □

Corollary 3.17. *If $h \in \mathcal{C}_w^+$, $\lambda_1 < 0$ and $p_{5,2} > n_{5,2}$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_5(t,s)h(s)\,ds.$$

Example 3.17. Consider the equation

$$u'''' + 0.04u''' + 0.0006u'' + 4.0 \times 10^{-6}u' + 1.0 \times 10^{-8}u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda + 0.01)^4 = 0$ has the root $\lambda_1 = -0.01$. Since $p_{5,2} = 1.5915 \times 10^7 > n_{5,2} = 1.0655 \times 10^6$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_5(t,s)\,h(s)\,ds$ with $\int_t^{t+w} G_5(t,s)\,ds = 10^8$ and $0 < 1.4850 \times 10^7 < G_5(t,s) < 1.6981 \times 10^7$.

4 Positive periodic solutions

Lemma 4.1 ([10,11])**.** *Let X be a Banach space and let $K \subset X$ be a cone. Assume that Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$$

be a completely continuous operator such that either

(i) $\|Tu\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$,

or

(ii) $\|Tu\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Denote

$$f_0 = \lim_{u\to 0^+} \sup_{t\in[0,w]} \frac{f(t,u)}{u} \quad \text{and} \quad f_\infty = \lim_{u\to\infty} \inf_{t\in[0,w]} \frac{f(t,u)}{u}.$$

Theorem 4.1. *If $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$, then equation* (1.4) *has at least one positive periodic solution in the cases*

(i) $f_0 = 0$ *and* $f_\infty = \infty$,

or

(ii) $f_0 = \infty$ *and* $f_\infty = 0$.

Proof. To apply the Guo–Krasnosel'skiĭ's theorem, let

$$X = \big\{u \in \mathcal{C}(\mathbb{R},\mathbb{R}) : \ u(t+w) = u(t), \ \ t \in \mathbb{R}\big\}$$

with the norm $\|u\| = \sup\limits_{t\in[0,w]} |u(t)|$. Then $(X, \|\cdot\|)$ is a Banach space and we define the cone K by

$$K = \Big\{u \in X : \ u(t) \geq \frac{A_{1,1}}{B_{1,1}}\,\|u\| \ \text{ for all } \ t \in [0,w]\Big\}.$$

For $u \in K$, we define

$$Tu(t) = \int\limits_t^{t+w} G_1(t,s) f(s,u(s))\,ds.$$

In view of Theorem 3.2, we have

$$0 < Tu(t) = \int\limits_t^{t+w} G_1(t,s) f(s,u(s))\,ds \leq B_{1,1} \int\limits_t^{t+w} f(s,u(s))\,ds.$$

So $\|Tu\| \leq B_{1,1} \int\limits_t^{t+w} f(s,u(s))\,ds$. Also, we have

$$Tu(t) = \int\limits_t^{t+w} G_1(t,s) f(s,u(s))\,ds \geq A_{1,1} \int\limits_t^{t+w} f(s,u(s))\,ds \geq \frac{A_{1,1}}{B_{1,1}}\,\|Tu\|,$$

which shows that $T(K) \subset K$. Moreover, $T : K \to K$ is a completely continuous operator and the fixed point of T is a solution of (1.4).

(i) If $f_0 = 0$ and $f_\infty = \infty$.

Since $f_0 = 0$, we may choose $0 < r_1 < 1$ such that $f(t,u) \leq \varepsilon u$, for $0 \leq u \leq r_1$ and $t \in [0,w]$, where $\varepsilon > 0$ satisfies $w\varepsilon B_{1.1} \leq 1$.

Thus, if $u \in K$ and $\|u\| = r_1$, we have

$$Tu(t) = \int\limits_t^{t+w} G_1(t,s) f(s,u(s))\,ds \leq B_{1,1} \int\limits_t^{t+w} f(s,u(s))\,ds \leq w\varepsilon B_{1.1}\|u\| \leq r_1. \tag{4.1}$$

Now, if we set $\Omega_1 = \{u \in X : \ \|u\| < r_1\}$, then (4.1) shows that $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$.

Since $f_\infty = \infty$, there exists $r > r_1$ such that $f(t,u) \geq \eta u$ for $u \geq r$ and $t \in [0,w]$, where $\eta > 0$, so $\frac{A_{1,1}^2 w\eta}{B_{1,1}} \geq 1$.

Let

$$r_2 = \max\Big\{2r_1, \frac{B_{1,1} r}{A_{1,1}}\Big\},$$

and $\Omega_2 = \{u \in X : \ \|u\| < r_2\}$, then $u \in K$ and $\|u\| = r_2$ imply that

$$u(t) \geq \frac{A_{1,1}}{B_{1,1}}\,\|u\| = \frac{A_{1,1}}{B_{1,1}}\,r_2 \geq r,$$

and hence

$$Tu(t) = \int_t^{t+w} G_1(t,s) f(s,u(s))\, ds \ge A_{1.1} \int_t^{t+w} f(s,u(s))\, ds \ge \frac{(A_{1,1}^2 w\eta)}{B_{1.1}} \|u\| \ge \|u\|. \tag{4.2}$$

Thus (4.2) shows that $\|Tu\| \ge \|u\|$ for $u \in K \cap \partial\Omega_2$.

It follows from Lemma 4.1 that T has a fixed point $u^* \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$. Consequently, the equation has a positive w-periodic solution $0 < r_1 < u(t) < r_2$.

(ii) If $f_0 = \infty$ and $f_\infty = 0$.

We choose $r_3 > 0$ such that $f(u) \ge \lambda u$ for $0 \le u \le r_3$, where $\lambda > 0$ satisfies $\frac{\lambda A_{1,1}^2 w}{B_{1,1}} \ge 1$. Then for $u \in K$ and $\|u\| = r_3$, we have

$$Tu(t) = \int_t^{t+w} G_1(t,s) f(s,u(s))\, ds \ge A_{1,1} \int_t^{t+w} f(s,u(s))\, ds \ge \frac{\lambda A_{1,1}^2 w}{B_{1,1}} \|u\| \ge \|u\|. \tag{4.3}$$

If we put $\Omega_3 = \{u \in X : \ \|u\| < r_3\}$, (4.3) shows that $\|Tu\| \ge \|u\|$ for $u \in K \cap \partial\Omega_3$.

Since $f_\infty = 0$, there exists $M > 0$ such that $f(t,u) \le \xi u$ for $u \ge M$ and $\xi > 0$ satisfies $\xi B_{1,1} w < 1$. We choose

$$r_4 = \max\Big\{2r_3, \frac{B_{1,1}M}{A_{1,1}}\Big\},$$

then $u \in K$ and $\|u\| = r_4$, this implies that $u(t) \ge \frac{A_{1,1}}{B_{1,1}} \|u\| \ge M$, and so

$$\begin{aligned} Tu(t) &= \int_t^{t+w} G_1(t,s) f(s,u(s))\, ds \le B_{1,1} \int_t^{t+w} f(s,u(s)) rm\, ds \\ &\le B_{1,1}\xi \int_t^{t+w} u(s)\, ds \le B_{1,1} w \xi \|u\| \le \|u\|. \end{aligned} \tag{4.4}$$

We set $\Omega_4 = \{u \in X : \ \|u\| < r_4\}$, then for $u \in K \cap \partial\Omega_4$ we have $\|Tu\| \le \|u\|$.

In view of Lemma 4.1, equation (1.4) has at least one positive solution $0 < r_3 < u(t) < r_4$. □

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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THIRD-ORDER DIFFERENTIAL EQUATIONS WITH RAPIDLY VARYING NONLINEARITIES

Abstract. We obtain the existence conditions and asymptotic, as $t \uparrow \omega$ ($\omega \le +\infty$), representations of one class of solutions of a binomial nonautonomous third-order differential equation with rapidly varying nonlinearity and their derivatives of the first and second order.

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რეზიუმე. მესამე რიგის დიფერენციალური განტოლებისთვის სწრაფად ცვალებადი არაწრფივობით მიღებულია არსებობის პირობები და ასიმპტოტიკური წარმოდგენები ბინომიალური არაავტონომიური ამონახსნების ერთი კლასისთვის და მათი პირველი და მეორე რიგის წარმოებულებისთვის.

1 Introduction

Consider the differential equation

$$y''' = \alpha_0 p(t)\varphi(y), \tag{1.1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$, $\varphi : \Delta_{Y_0} \to]0, +\infty[$ is a twice continuously differentiable function such that

$$\varphi'(y) \neq 0 \ \text{ as } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{or} & 0, \\ \text{or} & +\infty, \end{cases} \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi(y)\varphi''(y)}{\varphi'^2(y)} = 1, \tag{1.2}$$

Y_0 is equal either to zero or to $\pm\infty$, Δ_{Y_0} is some one-sided neighborhood of the point Y_0.

From the identity

$$\frac{\varphi''(y)\varphi(y)}{\varphi'^2(y)} = \frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2} + 1 \ \text{ as } \ y \in \Delta_{Y_0}$$

and conditions (1.2) it follows that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)} \ \text{ as } \ y \to Y_0 \ \ (y \in \Delta_{Y_0}) \ \text{ and } \ \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty. \tag{1.3}$$

Hence, in the equation under consideration, the function φ and its first-order derivative are (see [10, Chapter 3, Section 3.4, Lemmas 3.2, 3.3, pp. 91–92]) rapidly varying as $y \to Y_0$.

The asymptotic properties of solutions of binomial second-order differential equations with nonlinearities satisfying condition (1.2) were studied in the works of M. Marić [10], V. M. Evtukhov and his students: N. G. Drik, V. M. Kharkov, A. G. Chernikova [4–6]. Moreover, in the monograph by M. Marić [10, Chapter 3, Section 3.4, pp. 90–99] in the particular case, where $\alpha_0 = 1$, $\omega = +\infty$, $Y_0 = 0$ and p is a properly varying function as $t \to +\infty$, the asymptotic representations of solutions that tend to zero as $t \to +\infty$ were obtained.

In the paper by V. M. Evtukhov and N. G. Drik [5], a special case, where $\varphi(y) = e^{\sigma y}$, $\sigma \neq 0$, was considered.

In [6], V. M. Evtukhov and V. M. Kharkov investigated a class of solutions, which is determined by using the function $\varphi(y)$.

In the paper by V. M. Evtukhov and A. G. Chernikova [4], for the second-order differential equation (1.1) in case φ is a rapidly varying function as $t \to +\infty$, the asymptotic properties of the so-called $P_\omega(Y_0, \lambda_0)$-solutions were completely investigated. It seems natural to try to extend these results to the third-order differential equations.

It should be noted that the results obtained by V. M. Evtukhov and V. N. Shinkarenko [9] on the asymptotic behavior of such solutions of differential equations of higher than the second order in the case, where $\varphi(y) = e^{\sigma y}$, $\sigma \neq 0$, are known.

Definition 1.1. A solution y of the differential equation (1.1) is called a $P_\omega(Y_0, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on the interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the conditions

$$y(t) \in \Delta_{Y_0} \ \text{ as } \ t \in [t_0, \omega[, \quad \lim_{t\uparrow\omega} y(t) = Y_0,$$

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{or} & 0, \\ \text{or} & \pm\infty, \end{cases} \quad k = 1, 2, \quad \lim_{t\uparrow\omega} \frac{y''^2(t)}{y'''(t)y'(t)} = \lambda_0.$$

The aim of the present paper is to obtain the necessary and sufficient existence conditions of $P_\omega(Y_0, \lambda_0)$-solutions of equation (1.1) in a non-particular case, where $\lambda_0 \in \mathbb{R} \setminus \{0, 1, \frac{1}{2}\}$, as well as asymptotic, as $t \uparrow \omega$, representations of such solutions and their derivatives of order up to two.

2 Functions from the Γ, $\Gamma_{Y_0}(Z_0)$ classes and their asymptotic properties

Without loss of generality, we will further assume that

$$\Delta_{Y_0} = \begin{cases} [y_0, Y_0[\,, & \text{if } \Delta_{Y_0} \text{ is a left neighborhood of the point } Y_0, \\ ]Y_0, y_0], & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of the point } Y_0, \end{cases} \tag{2.1}$$

where $y_0 \in \mathbb{R}$ such that $|y_0| < 1$ as $Y_0 = 0$ and $y_0 > 1$ $(y_0 < -1)$ as $Y_0 = +\infty$ (as $Y_0 = -\infty$).

The function $f : \Delta_{Y_0} \to \mathbb{R} \setminus \{0\}$ satisfying condition (1.2), as $Y_0 = \pm\infty$, and $\lim\limits_{y\to+\infty} f(y) = +\infty$, belongs to the class Γ introduced by L. Khan (see [1, Chapter 3, p. 3.10, p. 175]).

Definition 2.1. The class Γ consists of measurable nondecreasing and right continuous functions $f : [y_0, +\infty[\to]0, +\infty[$, for each of which there is a measurable function $g : [y_0, +\infty[\to]0, +\infty[$, which complements the function f, such that

$$\lim_{y\to+\infty} \frac{f(y + ug(y))}{f(y)} = e^u \text{ for any } u \in \mathbb{R}.$$

In [9], the asymptotic properties of functions from this class were investigated in sufficient detail.

Using the change of variables, the class Γ in the paper by of V. M. Evtukhov and A. G. Chernikova [4] was extended to the class $\Gamma_{Y_0}(Z_0)$ of functions $f : \Delta_{Y_0} \to]0, +\infty[$, where Y_0 is equal either to zero or to $\pm\infty$, and Δ_{Y_0} is a one-sided neighborhood of the point Y_0, for which

$$\lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} f(y) = Z_0 = \begin{cases} \text{or} & 0, \\ \text{or} & +\infty \end{cases}$$

Definition 2.2. We say that the function $f : \Delta_{Y_0} \to]0, +\infty[$ belongs to the class of functions $\Gamma_{Y_0}(Z_0)$, if:

(1) the function $f_0(y) = \frac{1}{f(y)}$, as $Y_0 = +\infty$ and $Z_0 = 0$;

(2) the function $f_0(y) = f(-y)$, as $Y_0 = -\infty$ and $Z_0 = +\infty$;

(3) the function $f_0(y) = f(\frac{1}{y})$, as $Y_0 = 0$, where Δ_{Y_0} is a right neighborhood of zero, and $Z_0 = +\infty$;

(4) the function $f_0(y) = \frac{1}{f(\frac{1}{y})}$, as $Y_0 = 0$, where Δ_{Y_0} is a right neighborhood of zero, and $Z_0 = 0$;

(5) the function $f_0(y) = f(-\frac{1}{y})$, as $Y_0 = 0$, where Δ_{Y_0} is a left neighborhood of zero, and $Z_0 = +\infty$;

(6) the function $f_0(y) = \frac{1}{f(-\frac{1}{y})}$, as $Y_0 = 0$, where Δ_{Y_0} is a left neighborhood of zero, and $Z_0 = 0$;

(7) the function $f_0(y) \equiv f(y)$, as $Y_0 = +\infty$ and $Z_0 = +\infty$ belongs to the class Γ.

Using these two definitions, we conclude that for the function $f \in \Gamma_{Y_0}(Z_0)$ the limit relation

$$\lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{f(y + ug(y))}{f(y)} = e^u \text{ for any } u \in \mathbb{R} \tag{2.2}$$

holds, in which the function g, that is complementary for f, in each of the cases 1) - 7) can be expressed through the function g_0, that is complementary for f_0, in the following way (respectively):

(1) $g(y) = -g_0(y)$;

(2) $g(y) = -g_0(-y)$;

(3) $g(y) = -y^2 g_0(\frac{1}{y})$;

(4) $g(y) = y^2 g_0(\frac{1}{y})$;

(5) $g(y) = y^2 g_0(-\frac{1}{y})$;

(6) $g(y) = -y^2 g_0(-\frac{1}{y})$;

(7) $g(y) = g_0(y)$.

Using the properties of the class Γ (see the monograph by Bingham [1]) the following statements were obtained in [4].

Lemma 2.1.

1. *If $f \in \Gamma_{Y_0}(Z_0)$ with the complementary function g, then* $\lim\limits_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{g(y)}{y} = 0$.

2. *If $f \in \Gamma_{Y_0}(Z_0)$ with the complementary function g, then for any function $u : \Delta_{Y_0} \to \mathbb{R}$, satisfying the conditions*
$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} u(y) = u_0 \in \mathbb{R}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} f(y + u(y)g(y)) = Z_0,$$
the limit relation
$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{f(y + u(y)g(y))}{f(y)} = e^{u_0}$$
holds.

If $f \in \Gamma_{Y_0}(Z_0)$ with the complementary function g and, moreover, is continuous and strictly monotone, then there exists a continuous strictly monotone inverse function $f^{-1} : \Delta_{Z_0} \to \Delta_{Y_0}$, where
$$\Delta_{Z_0} = \begin{cases} \text{or} & [z_0, Z_0[\,, \\ \text{or} &]Z_0, z_0], \end{cases} \quad z_0 = f(y_0), \quad Z_0 = \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} f(y).$$

By virtue of Theorems 3.10.4, 3.1.16 from the monograph [1, Chapter 3, p. 3.10, p. 176 and p. 3.1, p. 139] and Definition 2.2, this inverse function has the following properties.

Lemma 2.2. *If $f \in \Gamma_{Y_0}(Z_0)$ with the complementary function g and is a continuous strictly monotone function on the interval Δ_{Y_0}, then the inverse function $f^{-1} : \Delta_{Z_0} \to \Delta_{Y_0}$ is slowly varying as $z \to Z_0$ and satisfies the limit relation*
$$\lim_{\substack{z\to Z_0\\ z\in\Delta_{Z_0}}} \frac{f^{-1}(\lambda z) - f(z)}{g(f^{-1}(z))} = \ln\lambda \ \ \textit{for any} \ \ \lambda > 0.$$
Moreover, for any $\Lambda > 1$ this limit relation holds uniformly with respect to $\lambda \in [\frac{1}{\Lambda}, \Lambda]$.

We present some of the important properties of the class of twice continuously differentiable functions $f : \Delta_{Y_0} \to \mathbb{R} \setminus \{0\}$, where Y_0 is equal either to zero or to $\pm\infty$, and Δ_{Y_0} is some one-sided neighborhood of the point Y_0, each of which satisfies the conditions
$$f'(y) \neq 0 \ \text{ as } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} f(y) = \begin{cases} \text{or} & 0, \\ \text{or} & \pm\infty, \end{cases} \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{f(y)f''(y)}{f'^2(y)} = 1,$$
the proof of which is given in the work of V. M. Evtukhov and A. G. Chernikova [4].

Lemma 2.3. *If a twice continuously differentiable function $f:\Delta_{Y_0}\to]0,+\infty[$ satisfies conditions* (2.1)*, then it belongs to the class $\Gamma_{Y_0}(Z_0)$ with the complementary function $g:\Delta_{Y_0}\to\mathbb{R}$, which is uniquely determined up to the equivalent, as $y\to Y_0$, functions, which can, for example, be one of the following functions:*

$$\frac{\int\limits_Y^y\big(\int\limits_Y^t f(u)\,du\big)\,dt}{\int\limits_Y^y f(x)\,dx}\sim\frac{\int\limits_Y^y f(x)\,dx}{f(y)}\sim\frac{f(y)}{f'(y)}\sim\frac{f'(y)}{f''(y)}\quad as\;\; y\to Y_0,$$

where

$$Y=\begin{cases} y_0, & or\quad \lim\limits_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} f(y)=+\infty,\\ Y_0, & or\quad \lim\limits_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} f(y)=0.\end{cases}$$

Remark 2.1. The given Lemmas 2.1 and 2.2 refer to the case, where $f:\Delta_{Y_0}\to]0,+\infty[$ (i.e., it takes positive values). In the case of the function $f:\Delta_{Y_0}\to]-\infty,0[$ we will say that it belongs to the class $\Gamma_{Y_0}(Z_0)$, if $(-f)\in\Gamma_{Y_0}(-Z_0)$. Then it is not difficult to verify that Lemmas 2.1 and 2.2 also remain valid.

3 The main results

Let us introduce the necessary auxiliary notation. We assume that the domain of the function φ in equation (1.1) is determined by formula (2.2). Next, we set

$$\mu_0=\operatorname{sign}\varphi'(y),\quad \nu_0=\operatorname{sign}y_0,\quad \nu_1=\begin{cases}1, & \text{if }\; \Delta_{Y_0}=[y_0,Y_0[\,,\\ -1, & \text{if }\; \Delta_{Y_0}=]Y_0,y_0],\end{cases}$$

and introduce the following functions:

$$J(t)=\int\limits_A^t\pi_\omega^2(\tau)p(\tau)\,d\tau,\quad \Phi(y)=\int\limits_B^y\frac{ds}{\varphi(s)}\,,$$

where

$$\pi_\omega(t)=\begin{cases}t, & \text{if }\;\omega=+\infty,\\ t-\omega, & \text{if }\;\omega<+\infty,\end{cases}\tag{3.1}$$

$$A=\begin{cases}\omega, & \text{if }\displaystyle\int\limits_a^\omega\pi_\omega^2(\tau)p(\tau)\,d\tau=const,\\ a, & \text{if }\displaystyle\int\limits_a^\omega\pi_\omega^2(\tau)p(\tau)\,d\tau=\pm\infty,\end{cases}\qquad B=\begin{cases}Y_0, & \text{if }\displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{\varphi(s)}=const,\\ y_0, & \text{if }\displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{\varphi(s)}=\pm\infty.\end{cases}$$

Taking into account the definition of $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1.1), we note that the numbers ν_0, ν_1 determine the signs of any $P_\omega(Y_0,\lambda_0)$-solution, its first derivative (respectively) in some left neighborhood of ω. It is clear that the condition

$$\nu_0\nu_1<0\;\text{ if }\;Y_0=0,\quad \nu_0\nu_1>0\;\text{ if }\;Y_0=\pm\infty,$$

is necessary for the existence of such solutions.

Now we turn our attention to some properties of the function Φ. It retains a sign on the interval Δ_{Y_0}, tends either to zero or to $\pm\infty$, as $y\to Y_0$, and is increasing on Δ_{Y_0}, since on this interval

$\Phi'(y) = \frac{1}{\varphi(y)} > 0$. Therefore, there is an inverse function $\Phi^{-1} : \Delta_{Z_0} \to \Delta_{Y_0}$, where due to the second of conditions (1.2) and the monotone increase of Φ^{-1},

$$Z_0 = \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \Phi(y) = \begin{cases} \text{or} \quad 0, \\ \text{or} \quad +\infty, \end{cases} \qquad \Delta_{Z_0} = \begin{cases} [z_0, Z_0[\,, & \text{if } \Delta_{Y_0} = [y_0, Y_0[\,, \\ ]Z_0, z_0], & \text{if } \Delta_{Y_0} =]Y_0, y_0], \end{cases} \qquad z_0 = \varphi(y_0). \tag{3.2}$$

By virtue of the L'Hospital rule in the form of Stolz and the last of conditions (1.2), we get

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\Phi(y)}{\frac{1}{\varphi'(y)}} = \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\frac{1}{\varphi(y)}}{-\frac{\varphi''(y)}{\varphi'^2(y)}} = -\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi'^2(y)}{\varphi''(y)\varphi(y)} = -1.$$

Hence,

$$\Phi(y) \sim -\frac{1}{\varphi'(y)} \quad \text{as} \ \ y \to Y_0 \ \ \text{and} \ \ \operatorname{sign}\Phi(y) = -\mu_0 \ \ \text{as} \ \ y \in \Delta_{Y_0}. \tag{3.3}$$

From the first of these relations it also follows that

$$\frac{\Phi'(y)}{\Phi(y)} = \frac{\frac{1}{\varphi(y)}}{\Phi(y)} \sim -\frac{\varphi'(y)}{\varphi(y)}, \qquad \frac{\Phi''(y)\Phi(y)}{\Phi'^2(y)} = \frac{-\frac{\varphi'(y)}{\varphi^2(y)}\Phi(y)}{\frac{1}{\varphi^2(y)}} \sim 1 \ \ \text{as} \ \ y \to Y_0.$$

Therefore, according to Lemma 2.3, $\Phi \in \Gamma_{Y_0}(Z_0)$ with a complementary function, which can be selected as one of the equivalent functions

$$\frac{\Phi'(y)}{\Phi''(y)} \sim \frac{\Phi(y)}{\Phi'(y)} \sim -\frac{\varphi(y)}{\varphi'(y)} \quad \text{as} \ \ y \to Y_0. \tag{3.4}$$

In addition to the above notation, as $\lambda_0 \in \mathbb{R} \setminus \{0; 1; \frac{1}{2}\}$, we introduce the auxiliary functions

$$q(t) = \frac{\alpha_0(\lambda_0-1)^2\pi_\omega^3(t)p(t)\varphi\big(\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}(\lambda_0-1)J(t))\big)}{\lambda_0\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t))}\,,$$

$$H(t) = \frac{\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t))\varphi'\big(\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t))\big)}{\varphi\big(\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t))\big)}\,,$$

In addition to the above properties of the twice continuously differentiable functions $f : \Delta_{Y_0} \to \mathbb{R}\setminus\{0\}$ satisfying conditions (2.1), we will need one more auxiliary statement about a priori asymptotic properties of $P_\omega(Y_0, \lambda_0)$-solutions of the differential equation (1.1) which follows from Corollary 10.1 of [8].

Lemma 3.1. *If $\lambda_0 \in \mathbb{R} \setminus \{0; 1; \frac{1}{2}\}$, then for each $P_\omega(Y_0, \lambda_0)$-solution of differential equation* (1.1) *the asymptotic relations*

$$\frac{\pi_\omega(t)y'(t)}{y(t)} = \frac{2\lambda_0-1}{\lambda_0-1}\,[1+o(1)], \qquad \frac{\pi_\omega(t)y''(t)}{y'(t)} = \frac{\lambda_o}{\lambda_0-1}\,[1+o(1)], \qquad \frac{\pi_\omega(t)y'''(t)}{y''(t)} = \frac{1+o(1)}{\lambda_0-1} \tag{3.5}$$

as $t \uparrow \omega$ hold, where $\pi_\omega(t)$ is defined by (3.1).

For equation (1.1), the following assertions hold.

Theorem 3.1. *Let $\lambda_0 \in \mathbb{R}\setminus\{0; 1; \frac{1}{2}\}$. Then for the existence of $P_\omega(Y_0, \lambda_0)$-solutions of the differential equation* (1.1)*, it is necessary that the conditions*

$$\alpha_0\nu_1\lambda_0 > 0, \tag{3.6}$$

$$\nu_0\nu_1(2\lambda_0-1)(\lambda_0-1)\pi_\omega(t) > 0 \ \ \textit{as} \ \ t \in (a, \omega), \tag{3.7}$$

$$\alpha_0\mu_0\lambda_0 J(t) < 0 \ \ \textit{as} \ \ t \in (a, \omega), \tag{3.8}$$

$$\frac{\alpha_0}{\lambda_0}\lim_{t\uparrow\omega} J(t) = Z_0, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'(t)}{J(t)} = \pm\infty, \quad \lim_{t\uparrow\omega} q(t) = \frac{2\lambda_0-1}{\lambda_0-1} \tag{3.9}$$

hold. Moreover, each solution of that kind admits the asymptotic representations

$$y(t) = \Phi^{-1}\Big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\Big)\Big[1+\frac{o(1)}{H(t)}\Big] \quad \text{as } t \uparrow \omega, \tag{3.10}$$

$$y'(t) = \frac{(2\lambda_0-1)}{(\lambda_0-1)} \frac{\Phi^{-1}\big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\big)}{\pi_\omega(t)} [1+o(1)] \quad \text{as } t \uparrow \omega, \tag{3.11}$$

$$y''(t) = \frac{\lambda_0(2\lambda_0-1)}{(\lambda_0-1)^2} \frac{\Phi^{-1}\big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\big)}{\pi_\omega^2(t)} [1+o(1)] \quad \text{as } t \uparrow \omega. \tag{3.12}$$

Theorem 3.2. *Let* $\lambda_0 \in \mathbb{R} \setminus \{0;1;\frac{1}{2}\}$*, conditions* (3.6)–(3.9) *hold, there exist a limit*

$$\lim_{t\uparrow\omega} \Big[\frac{2\lambda_0-1}{\lambda_0-1} - q(t)\Big]|H(t)|^{\frac{2}{3}} = 0 \tag{3.13}$$

and a finite or equal to $\pm\infty$ *limit*

$$\lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2} \sqrt[3]{\Big(\frac{y\varphi'(y)}{\varphi(y)}\Big)^2}. \tag{3.14}$$

Then the differential equation (1.1) *has at least one* $P_\omega(Y_0,\lambda_0)$*-solution admitting the asymptotic, as* $t \uparrow \omega$*, representations*

$$y(t) = \Phi^{-1}\Big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\Big)\Big[1+\frac{o(1)}{H(t)}\Big], \tag{3.15}$$

$$y'(t) = \frac{2\lambda_0-1}{(\lambda_0-1)\pi_\omega(t)} \Phi^{-1}\Big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\Big)[1+o(1)H^{-\frac{2}{3}}], \tag{3.16}$$

$$y''(t) = \frac{\lambda_0(2\lambda_0-1)}{(\lambda_0-1)^2\pi_\omega^2(t)} \Phi^{-1}\Big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\Big)[1+o(1)H^{-\frac{1}{3}}]. \tag{3.17}$$

Moreover, there exist one-parameter family of such solutions in case $\mu_0\lambda_0\nu_1 < 0$*, and two-parameter family, when* $\mu_0\lambda_0\nu_1 > 0$*.*

Proof of Theorem 3.1. Let $y : [t_0,\omega[\to \mathbb{R}$ be an arbitrary $P_\omega(Y_0,\lambda_0)$-solution of the differential equation (1.1). Then, according to Lemma 3.1, the asymptotic relations (3.5) hold. By virtue of these relations and (1.1), this solution and its derivatives of the first, second and third order retain the signs on a certain interval $[t_1,\omega[\subset [t_0,\omega[$, and for these signs the asymptotic relations (3.5) hold, from which follow condition (3.6) and inequality (3.7). In addition, from (1.1), taking into account the second of the asymptotic relations (3.4), it follows that

$$\frac{y'(t)}{\varphi(y(t))} = \alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} \pi_\omega^2(t) p(t)[1+o(1)] \quad \text{as } t \uparrow \omega. \tag{3.18}$$

Integrating this relation from t_0 to t, we get

$$\int_{y(t_0)}^{y(t)} \frac{ds}{\varphi(s)} = \alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} \int_{t_0}^{t} \pi_\omega^2(\tau)p(\tau)[1+o(1)]\,d\tau \quad \text{as } t \uparrow \omega.$$

Since, according to the definition of $P_\omega(Y_0,\lambda_0)$-solution, $y(t) \to Y_0$ as $t \uparrow \omega$, it follows that the improper integrals

$$\int_{y(t_0)}^{Y_0} \frac{ds}{\varphi(s)} \quad \text{and} \quad \int_{t_0}^{\omega} \pi_\omega^2(\tau)p(\tau)\,d\tau$$

converge or diverge simultaneously. In view of this fact and the rule for choosing the integration limits A and B in the functions J and Φ, introduced at the beginning of this section, the aforementioned relation can be written as

$$\Phi(y(t)) = \alpha_0 \,\frac{(\lambda_0-1)^2}{\lambda_0}\, J(t)[1+o(1)] \ \text{ as } \ t\uparrow\omega. \tag{3.19}$$

From here, taking into account (3.2) and (3.3), it follows that inequality (3.8) and the first of conditions (3.9) are true. By virtue of the first of conditions (3.3), it follows from (3.18) and (3.19) that

$$\frac{y''(t)\varphi'(y'(t))}{\varphi(y(t))} = -\frac{\lambda_0\pi_\omega(t)p(t)}{(\lambda_0-1)J(t)}\,[1+o(1)] \ \text{ as } \ t\uparrow\omega,$$

and, therefore, taking into account the first and second of the asymptotic relations (3.4) and the asymptotic relations (3.5),

$$\frac{y(t)\varphi'(y(t))}{\varphi(y(t))} = -\frac{(\lambda_0-1)\pi_\omega^3(t)p(t)}{(2\lambda_0-1)J(t)} \ \text{ as } \ t\uparrow\omega.$$

From this relation, by virtue of (1.3) and the definition of the $P_\omega(Y_0,\lambda_0)$-solution, it directly follows that the second of the limit conditions (3.9) holds.

Now, from (3.19) we find that

$$y(t) = \Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\, J(t)[1+o(1)]\Big) \ \text{ as } \ t\uparrow\omega. \tag{3.20}$$

The function Φ, as is stated earlier, belongs to the class $\Gamma_{Y_0}(Z_0)$, where $Z_0 = \lim\limits_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi(y)$, and the function $g(y) = -\frac{\varphi(y)}{\varphi'(y)}$ can be chosen as its complementary function. Then, according to the conditions $\frac{\alpha_0}{\lambda_0}\lim\limits_{t\uparrow\omega} J(t) = Z_0$ and $\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\in\Delta_{Z_0}$ as $t\in[t_0,\omega[$, which follow from (3.8) and the first condition of (3.1), according to Lemma 2.2, we have

$$\lim_{t\uparrow\omega}\frac{\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}J(t)[1+o(1)]\big)-\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}J(t)\big)}{-\frac{\varphi\left(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\right)}{\varphi'\left(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\right)}} = \lim_{\substack{z\to Z_0\\ z\in\Delta_{Z_0}}}\frac{\Phi^{-1}(z(1+o(1)))-\Phi^{-1}(z)}{-\frac{\varphi(z)}{\varphi'(z)}} = 0,$$

whence it follows that

$$\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)[1+o(1)]\Big) = \Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big) + \frac{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}\,o(1) \ \text{ as } \ t\uparrow\omega.$$

By virtue of this relation, from (3.20) we obtain the asymptotic representation (3.10). If we consider that

$$\lim_{t\uparrow\omega}\frac{\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)\varphi'\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)} = \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{y\varphi'(y)}{\varphi(y)} = \pm\infty,$$

then (3.9) can be written as

$$y(t) = \Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)[1+o(1)] \ \text{ as } \ t\uparrow\omega$$

and, therefore, according to the first of the asymptotic relations (3.4), the asymptotic representations (3.11) and (3.12) hold.

It remains to establish the validity of the third of conditions (3.1). According to (3.10), from (3.1) we have

$$y'''(t) = \alpha_0 p(t)\varphi\Big(\Phi^{-1}\Big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\Big) + \frac{\varphi\big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\big)}{\varphi'\big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\big)}\, o(1)\Big) \ \text{ as } \ t \uparrow \omega. \tag{3.21}$$

Since $\varphi \in \Gamma_{Y_0}(Z_0)$, where $Z_0 = \lim\limits_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi(y)$, which according to the second conditions of (1.2) is equal either to zero or to $+\infty$, and the function $g(y) = \frac{\varphi(y)}{\varphi'(y)}$ can be chosen as its complementary function, on the basis of Lemma 2.1, taking into account the conditions $\lim\limits_{t\uparrow\omega} \Phi^{-1}\big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\big) = Y_0$ and $\Phi^{-1}\big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\big) \in \Delta_{Y_0}$ as $t \in [t_0, \omega[$, we obtain

$$\lim_{t\uparrow\omega} \frac{\varphi\Big(\Phi^{-1}(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\big) + \frac{\varphi\left(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\right)}{\varphi'\left(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\right)}\, o(1)\Big)}{\varphi\big(\Phi^{-1}(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t))\big)} = \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi\big(y + \frac{\varphi(y)}{\varphi'(y)}\, o(1)\big)}{\varphi(y)} = 1.$$

Hence,

$$\varphi\Big(\Phi^{-1}\Big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\Big) + \frac{\varphi\big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\big)}{\varphi'\big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\big)}\, o(1)\Big) = \varphi\Big(\Phi^{-1}\Big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\Big)\Big) \ \text{ as } \ t \uparrow \omega$$

and the asymptotic relation (3.21) can be written as

$$y'''(t) = \alpha_0 p(t)\varphi\Big(\Phi^{-1}\Big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\Big)\Big)[1+o(1)] \ \text{ as } \ t \uparrow \omega.$$

By virtue of this representation and (3.12),

$$\frac{\pi_\omega(t) y'''(t)}{y''(t)} = \frac{\alpha_0(\lambda_0-1)^2 \pi_\omega^3(t) p(t) \varphi\big(\Phi^{-1}(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t))\big)}{\lambda_0(2\lambda_0-1)\Phi^{-1}\big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\big)}\, [1+o(1)] \ \text{ as } \ t \uparrow \omega.$$

According to the third of the asymptotic relations (3.5), we obtain the validity of the third of conditions (3.9). □

Proof of Theorem 3.2. Suppose that there exists a limit (3.13) that is finite or equal to $\pm\infty$ and for some $\lambda_0 \in \mathbb{R} \setminus \{0, 1, \frac{1}{2}\}$ conditions (3.7), (3.8) and one of the conditions either (3.14) or (3.16) and (3.17) hold. Under these conditions, we establish the existence of $P_\omega(Y_0, \lambda_0)$-solutions of the differential equation (1.1) that admit asymptotic representations (3.9), (3.10), (3.11) and find the number of such solutions.

First, taking into account the existence of limit (3.13) that is finite or equal to $\pm\infty$, we show that this limit can only be zero. Assume the opposite. Then the relation

$$\frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^{\frac{4}{3}}} = \frac{z(y)}{y^{\frac{2}{3}}}$$

holds, where the function $z : \Delta_{Y_0} \to \mathbb{R}$ is continuous and such that

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} z(y) = \begin{cases} \text{or} & c = const \neq 0, \\ \text{or} & \pm\infty. \end{cases} \tag{3.22}$$

Integrating this relation on the interval from y_0 to y, we obtain

$$-3\Big(\frac{\varphi'(y)}{\varphi(y)}\Big)^{-\frac{1}{3}} = c_0 + \int_{y_0}^{y} \frac{z(s)}{s^{\frac{2}{3}}}\, ds, \tag{3.23}$$

where c_0 is some constant.

If $\int\limits_{y_0}^{Y_0} \frac{z(s)}{s^{\frac{2}{3}}}\, ds = \pm\infty$, then after dividing by $y^{\frac{1}{3}}$, we have

$$-3\Big(\frac{y\varphi'(y)}{\varphi(y)}\Big)^{-\frac{1}{3}} = \frac{\int\limits_{y_0}^{y} \frac{z(s)}{s^{\frac{2}{3}}}\, ds}{y^{\frac{1}{3}}}\,[1+o(1)] \ \text{ as } \ y \to Y_0.$$

Here, the expression on the left, by virtue of (1.3), tends to zero as $y \to Y_0$, and that of on the right, by virtue of condition (3.22), tends either to a nonzero constant or to $\pm\infty$, as according to the L'Hospital rule in the form of Stolz

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\int\limits_{y_0}^{y} \frac{z(s)}{s^{\frac{2}{3}}}\, ds}{y^{\frac{1}{3}}} = 3 \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} z(y),$$

which is impossible.

If $\int\limits_{y_0}^{Y_0} \frac{z(s)}{s^{\frac{2}{3}}}\, ds$ converges, which is possible only in the case $Y_0 = 0$, then we rewrite (3.23) in the form

$$-3\mu_0\Big(\frac{\varphi'(y)}{\varphi(y)}\Big)^{-\frac{1}{3}} = c_1 + \int\limits_{0}^{y} \frac{z(s)}{s^{\frac{2}{3}}}\, ds,$$

where $c_1 = c_0 + \int\limits_{y_0}^{0} \frac{z(s)}{s^{\frac{2}{3}}}\, ds$. Let us prove that $c_1 = 0$. Indeed, if $c_1 \neq 0$, then from this relation it follows that

$$\frac{\varphi'(y)}{\varphi(y)} = -\frac{27}{c_1^3} + o(1) \ \text{ as } \ y \to 0.$$

Hence, as a result of integration on the interval from y_0 to y, we get

$$\ln|\varphi(y)| = const + o(1) \ \text{ as } \ y \to 0,$$

which contradicts the second of conditions (1.2). Hence, $c_1 = 0$ and, therefore, we have

$$-3\Big(\frac{\varphi'(y)}{\varphi(y)}\Big)^{-\frac{1}{3}} = \int\limits_{0}^{y} \frac{z(s)}{s^{\frac{2}{3}}}\, ds.$$

Dividing both sides of this equality by $y^{\frac{1}{3}}$, we note that, by virtue of conditions (1.3), the left-hand side of the resulting relation tends to zero as $y \to 0$, and the right-hand side, by virtue of the L'Hospital rule and (3.22), tends either to a nonzero constant or to $\pm\infty$.

The contradictions obtained in each of the two possible cases lead to the conclusion that

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{(\frac{\varphi'(y)}{\varphi(y)})'}{(\frac{\varphi'(y)}{\varphi(y)})^2} \sqrt[3]{\Big(\frac{y\varphi'(y)}{\varphi(y)}\Big)^2} = 0. \tag{3.24}$$

Now, applying the transformation to equation (1.1),

$$\begin{aligned} y(t) &= \Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)\Big[1+\frac{y_1}{H(t)}\Big],\\ y'(t) &= \frac{2\lambda_0-1}{(\lambda_0-1)\pi_\omega(t)}\,\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)[1+y_2(t)],\\ y''(t) &= \frac{\lambda_0(2\lambda_0-1)}{(\lambda_0-1)^2\pi_\omega^2(t)}\,\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)[1+y_3(t)], \end{aligned} \tag{3.25}$$

we obtain a system of differential equations

$$
\begin{cases}
y_1' = \dfrac{H(t)}{\pi_\omega(t)} \Big[\dfrac{2\lambda_0 - 1}{\lambda_0 - 1} - q(t) + h(t)y_1 + \dfrac{2\lambda_0 - 1}{\lambda_0 - 1}\, y_2\Big], \\
y_2' = \dfrac{1}{\pi_\omega(t)} \Big[\Big(\dfrac{2\lambda_0 - 1}{\lambda_0 - 1} - q(t)\Big) + (1 - q(t))y_2 + \dfrac{\lambda_0}{\lambda_0 - 1}\, y_3\Big], \\
y_3' = \dfrac{1}{\pi_\omega(t)} \Big[2 - \dfrac{2q(t)(\lambda_0 - 1)}{2\lambda_0 - 1} + \dfrac{q(t)}{2\lambda_0 - 1}\, y_1 + (2 - q(t))y_3 + \dfrac{q(t)}{2\lambda_0 - 1}\, R(t, y_1)\Big],
\end{cases}
\tag{3.26}
$$

where

$$
h(t) = q(t)\, \frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2}\Bigg|_{y=\Phi^{-1}\left(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t)\right)},
$$

$$
R(t, y_1) = \frac{\varphi\Big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t)) + \frac{\varphi\left(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\right)}{\varphi'\left(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\right)}\, y_1\Big)}{\varphi\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)} - 1 - y_1.
$$

We consider this system of equations on the set

$$
\Omega = [t_0, \omega[\, \times D_1 \times D_2 \times D_3, \ \text{ where } \ D_i = \{y_i : \ |y_i| \le 1\} \ \ (i = 1, 2, 3),
$$

and the number $t_0 \in [a, \omega[$ is chosen, by taking into account conditions (3.2), (3.3), (3.8), the first two conditions (3.9) and (1.3), so that

$$
\alpha_0\, \frac{(\lambda_0 - 1)^2}{\lambda_0}\, J(t) \in \Delta_{Z_0} \ \text{ as } \ t \in [t_0, \omega[\,,
$$

$$
\Phi^{-1}\Big(\alpha_0\, \frac{(\lambda_0 - 1)^2}{\lambda_0}\, J(t)\Big) + \frac{\varphi\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}\, v_1 \in \Delta_{Y_0} \ \text{ as } \ t \in [t_0, \omega[\,, \ \text{ and } \ |v_1| \le 1.
$$

On this set, the right-hand sides of the system of differential equations (3.26) are continuous and the function R has on the set $[t_0, \omega[\, \times D_1$ continuous partial derivatives up to the second order inclusive with respect to the variable v_1. At the same time, we have

$$
R'_{y_1}(t, y_1) = \frac{\varphi'\Big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t)) + \frac{\varphi\left(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\right)}{\varphi'\left(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\right)}\, y_1\Big)}{\varphi'\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)} - 1.
$$

Here $\varphi' \in \Gamma_{Y_0}(Z_0)$ with the complementary function $g(y) = \frac{\varphi(y)}{\varphi'(y)}$. Therefore,

$$
\lim_{t \uparrow \omega} \frac{\varphi'\Big(\Phi^{-1}\big(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t)\big) + \frac{\varphi\left(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\right)}{\varphi'\left(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\right)}\, y_1\Big)}{\varphi'\Big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\Big)} = \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi'(y + y_1 \frac{\varphi(y)}{\varphi'(y)})}{\varphi'(y)} = e^{y_1}.
$$

If, for any fixed $t \in [t_0, \omega[$, the function R is expanded according to the Maclaurin formula with the residual Lagrange term to the second-order terms, then we obtain

$$
\begin{aligned}
R(t, v_1) = \frac{1}{2}\, &\frac{\varphi\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}{\varphi'^2\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)} \\
&\times \varphi''\Big(\Phi^{-1}\Big(\alpha_0\, \frac{(\lambda_0 - 1)^2}{\lambda_0}\, J(t)\Big) + \frac{\varphi\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}\, \xi\Big) y_1^2,
\end{aligned}
$$

where $|\xi| < |y_1|$. Here, by virtue of the last of conditions (1.2),

$$\varphi''\bigg(\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)+\frac{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}\,\xi\bigg)$$
$$=\frac{\varphi'^2\Big(\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)+\frac{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}\,\xi\Big)}{\varphi\Big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))+\frac{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}\,\xi\Big)}\,\big[1+r_1(t,y_1)\big],$$

where $\lim\limits_{t\uparrow\omega} r_1(t,y_1)=0$ uniformly with respect to $y_1\in D_1$. Therefore, considering that the functions $\varphi,\varphi'\in\Gamma_{Y_0}(Z_0)$ with the complementary function $g(y)=\frac{\varphi(y)}{\varphi'(y)}$, we have

$$\varphi''\bigg(\Phi^{-1}\big(\alpha_0(\lambda_0-1)J(t)\big)+\frac{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}\,\xi\bigg)$$
$$=\frac{\varphi'^2\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}\,e^{\xi}\big[1+r_2(t,y_1)\big],$$

where $\lim\limits_{t\uparrow\omega} r_2(t,y_1)=0$ uniformly with respect to $y_1\in D_1$. Therefore, (3.23) can be written as

$$R(t,y_1)=\frac{1}{2}\,e^{\xi}\big[1+r_1(t,y_1)\big]\big[1+r_2(t,y_1)\big]y_1^2.$$

It is clear from the above that for any $\varepsilon>0$ there are $\delta>0$ and $t_1\in[t_0,\omega[$ such that

$$|R(t,y_1)|\le(0.5+\varepsilon)|y_1|^2\ \text{ as }\ t\in[t_1,\omega[\ \text{ and }\ y_1\in D_{1\delta}=\big\{y_1:\ \ |y_1|\le\delta\big\}. \tag{3.27}$$

Choosing arbitrarily the number $\varepsilon>0$, we select for it, taking into account the aforementioned about the properties of the function R, the numbers $\delta>0$ and $t_1\in[t_0,\omega[$ such that inequality (3.27) holds, and consider system (3.30) on the set

$$\Omega_1=\Big\{(t,z_1,z_2,z_3)\in\mathbb{R}^4:\ \ t\in[t_1,\omega[,\ \ z_1\in[-\delta,\delta],\ \ z_2\in[-1,1],\ \ z_3\in[-1,1]\Big\}.$$

In addition, in the system of equations (3.26), due to conditions (3.6) – (3.8), (3.13), (1.2) and (1.3),

$$\lim_{t\uparrow\omega}q(t)=\frac{2\lambda_0-1}{\lambda_0-1}\,,\quad \lim_{t\uparrow\omega}h(t)=0,\quad \lim_{t\uparrow\omega}H(t)=\pm\infty. \tag{3.28}$$

To establish the existence of $P_\omega(Y_0,\lambda_0)$-solutions of equation (1.1) admitting asymptotic representations (3.10)–(3.12), it is necessary, according to transformation (3.25), to prove the existence of solutions that tend to zero, as $t\uparrow\omega$, of the system of differential equations (3.26). In order to use the well-known results on the existence of solutions of quasilinear systems of differential equations that disappear at a singular point, we reduce system (3.26) to the form that allows us to use such results.

Applying to system (3.26) an additional transformation

$$v_1=z_1,\quad v_2=H^{-\frac{2}{3}}(t)z_2,\quad v_3=H^{-\frac{1}{3}}(t)z_3, \tag{3.29}$$

we get a system of differential equations of the form

$$\begin{cases} z_1'=\dfrac{H^{\frac13}(t)}{\pi_\omega(t)}\,\big[f_1(t)+c_{11}(t)z_1+c_{12}(t)z_2+c_{13}(t)z_3\big],\\[2mm] z_2'=\dfrac{H^{\frac13}(t)}{\pi_\omega(t)}\,\big[f_2(t)+c_{21}(t)z_1+c_{22}(t)z_2+c_{23}(t)z_3\big],\\[2mm] z_3'=\dfrac{H^{\frac13}(t)}{\pi_\omega(t)}\,\Big[f_3(t)+c_{31}(t)z_1+c_{32}(t)z_2+c_{33}(t)z_3+\dfrac{q(t)}{2\lambda_0-1}\,V(t,z_1)\Big],\end{cases} \tag{3.30}$$

where

$$f_1(t)=\Big[\frac{2\lambda_0-1}{\lambda_0-1}-q(t)\Big]H^{\frac{2}{3}}(t),\quad f_2(t)=\Big[\frac{2\lambda_0-1}{\lambda_0-1}-q(t)\Big]H^{\frac{1}{3}}(t),\quad f_3(t)=2-\frac{2q(t)(\lambda_0-1)}{2\lambda_0-1}\,,$$
$$c_{11}(t)=h(t)H^{\frac{2}{3}}(t),\quad c_{12}(t)=\frac{2\lambda_0-1}{\lambda_0-1}\,,\quad c_{13}(t)=0,\quad c_{21}(t)=0,\quad c_{23}(t)=\frac{\lambda_0}{\lambda_0-1}\,,$$
$$c_{22}(t)=H^{-\frac{2}{3}}(t)\Big(1-\frac{1}{3}\,q(t)+\frac{2}{3}\,q(t)h(t)H(t)\Big),\quad c_{31}(t)=\frac{q(t)}{2\lambda_0-1}\,,\quad c_{32}(t)=0,$$
$$c_{33}(t)=H^{-\frac{2}{3}}(t)\Big(2-\frac{2}{3}\,q(t)+\frac{1}{3}\,q(t)h(t)H(t)\Big),\quad V(t,z_1)=\frac{q(t)}{2\lambda_0-1}\,R(t,z_1).$$

Choosing arbitrarily the number $\varepsilon>0$, we select for it, taking into account the aforementioned about the properties of the function R, the numbers $\delta>0$ and $t_1\in[t_0,\omega[$ such that inequality (3.27) holds, and consider system (3.30) on the set

$$\Omega_1=\Big\{(t,z_1,z_2,z_3)\in\mathbb{R}^4:\ \ t\in[t_1,\omega[\,,\ \ z_1\in[-\delta,\delta],\ \ z_2\in[-1,1],\ \ z_3\in[-1,1]\Big\}.$$

By virtue of (3.28), the replacement of y_1 by z_1 and the first of conditions (3.28),

$$\lim_{z_1\to0}\frac{V(t,z_1)}{z_1^2}=0\ \text{ uniformly with respect to } t\in[t_1,\omega[\,.$$

In addition, according to conditions (3.28), (3.24) and the notation introduced at the beginning of this section, we have $\operatorname{sign}H(t)\pi_\omega(t)=\mu_0\nu_0\pi_\omega(t)$ as $t\in(a,\omega)$ and

$$\lim_{t\uparrow\omega}f_1(t)=0,\quad \lim_{t\uparrow\omega}f_2(t)=0,$$
$$\lim_{t\uparrow\omega}f_3(t)=0,\quad \lim_{t\uparrow\omega}c_{11}(t)=0,\quad \lim_{t\uparrow\omega}c_{12}(t)=\frac{(2\lambda_0-1)}{\lambda_0-1}\,,$$
$$\lim_{t\uparrow\omega}c_{22}(t)=\frac{1}{\lambda_0-1}\,,\quad \lim_{t\uparrow\omega}c_{23}(t)=\frac{\lambda_0}{\lambda_0-1}\,,$$
$$\lim_{t\uparrow\omega}c_{31}(t)=\frac{1}{\lambda_0-1}\,,\quad \lim_{t\uparrow\omega}c_{33}(t)=0,$$
$$\int_{t_1}^{\omega}\frac{|H(\tau)|^{\frac{1}{3}}}{\pi_\omega(\tau)}\,d\tau=\pm\infty.$$

This, in particular, implies that the limit matrix of coefficients, standing at v_1, v_2 and v_3 in square brackets of system (3.30), has the form

$$C=\begin{pmatrix}0 & \dfrac{(2\lambda_0-1)}{\lambda_0-1} & 0\\ 0 & 0 & \dfrac{\lambda_0}{\lambda_0-1}\\ \dfrac{1}{\lambda_0-1} & 0 & 0\end{pmatrix}$$

and its characteristic equation is that of the form

$$\rho^3-\frac{\lambda_0(2\lambda_0-1)}{(\lambda_0-1)^3}=0. \tag{3.31}$$

If $\lambda_0(2\lambda_0-1)(\lambda_0-1)>0$, then in this case the algebraic equation (3.31) has two complex-conjugate roots with negative real part and one positive real root.

If $\lambda_0(2\lambda_0-1)(\lambda_0-1)<0$, then equation (3.31) has two complex-conjugate roots with a positive real part and one negative real root.

Suppose further that conditions (3.13) are satisfied. It follows that for the system of differential equations (3.30) all the conditions of Theorem 2.2 from [7] are satisfied. According to this theorem, we find that when $\mu_0\nu_1\lambda_0 > 0$, the system of differential equations (3.29) has a two-parameter family of solutions $(z_1, z_2, z_3) : [t_*, \omega[\to \mathbb{R}^3$ $(t_* \in [t_1, \omega[)$ that disappear at $t \uparrow \omega$. To each of them, due to substitutions (3.25) and (3.29), there corresponds a solution $y : [t_*, \omega[\to \mathbb{R}$ admitting asymptotic representations (3.10)–(3.12) and (3.15)–(3.17).

If $\mu_0\nu_1\lambda_<0$, the system of differential equations (3.30) has a one-parameter family of solutions $(z_1, z_2, z_3) : [t_*, \omega[\to \mathbb{R}^3$ $(t_* \in [t_1, \omega[)$ that disappear at $t \uparrow \omega$. To each of them, due to substitutions (3.25) and (3.29), there corresponds a solution $y : [t_*, \omega[\to \mathbb{R}$ admitting asymptotic representations (3.10)–(3.12) and (3.15)–(3.17). □

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MONOTONE ITERATIVE METHOD FOR SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS

Abstract. In this paper, we apply the monotone iteration method to establish the existence of a positive solution for the fractional differential equation

$$D_{0+}^{\alpha} u(t) + q(t) f(t, u(t)) = 0, \quad 0 < t < 1,$$

together with the boundary conditions (BCs)

$$u(0) = u'(0) = \cdots = u^{n-2}(0) = 0, \quad D_{0+}^{\beta} u(1) = \int\limits_0^1 h(s, u(s))\, dA(s),$$

where $n > 2$, $n - 1 < \alpha \le n$, $\beta \in [1, \alpha - 1]$, D_{0+}^{α} and D_{0+}^{β} are the standard Riemann–Liouville fractional derivatives of order α and β, respectively, and $f, h : [0,1] \times [0, \infty) \to [0, \infty)$ are continuous functions. The sufficient condition provided in this paper is new, interesting and easy to verify. Our conditions do not require the sublinearity or superlinearity on the nonlinear functions f and h at 0 or ∞. The paper is supplemented with examples illustrating the applicability of our result.

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Key words and phrases. Fractional differential equations, Riemann–Liouville derivative, boundary value problems, positive solutions, monotone iteration method.

რეზიუმე. სტატიაში გამოყენებულია მონოტონური იტერაციის მეთოდი, რათა დავადგინოთ დადებითი ამონახსნის არსებობა წილადური დიფერენციალური

$$D_{0+}^{\alpha} u(t) + q(t) f(t, u(t)) = 0, \quad 0 < t < 1,$$

განტოლებისთვის

$$u(0) = u'(0) = \cdots = u^{n-2}(0) = 0, \quad D_{0+}^{\beta} u(1) = \int\limits_0^1 h(s, u(s))\, dA(s)$$

სასაზღვრო პირობებით, სადაც $n > 2$, $n - 1 < \alpha \le n$, $\beta \in [1, \alpha - 1]$, D_{0+}^{α} და D_{0+}^{β}, შესაბამისად, α და β რიგის სტანდარტული რიმან-ლიუვილის წილადური წარმოებულებია და $f, h : [0,1] \times [0, \infty) \to [0, \infty)$ უწყვეტი ფუნქციებია. ამ სტატიაში წარმოდგენილი საკმარისი პირობა ახალი, საინტერესო და მარტივად შესამოწმებელია. ჩვენი პირობები არ მოითხოვს არაწრფივი f და h ფუნქციების ქვეწრფივობას ან ზეწრფივობას 0-ში ან ∞-ში. სტატიაში აგრეთვე მოყვანილია მაგალითები ჩვენი შედეგის გამოყენების საილუსტრაციოდ.

1 Introduction

The aim of the present paper is to demonstrate the applications of the monotone iteration method for studying the existence of at least one positive solution of the nonlinear fractional differential equation

$$D_{0+}^{\alpha}u(t)+q(t)f(t,u(t))=0,\ \ 0<t<1, \tag{1.1}$$

together with the boundary conditions (BCs)

$$u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\ \ D_{0+}^{\beta}u(1)=\int\limits_0^1 h(s,u(s))\,dA(s), \tag{1.2}$$

where $n-1<\alpha\le n$, $n>2$, $\beta\in[1,\alpha-1]$ is fixed, $q:(0,1)\to[0,\infty)$ is a continuous function, $f,h:(0,1)\times[0,\infty)\to[0,\infty)$ are continuous functions, $\int\limits_0^1 h(s,u(s))\,dA(s)$ is a Riemann–Stieltjes integral with A being nondecreasing and of bounded variation, and D_{0+}^{α}, D_{0+}^{β} are the standard Riemann–Liouville fractional derivatives of order α and β, respectively.

We define the fractional derivative and fractional integral for a function F of order γ, $\gamma\in[0,\infty)$ as follows.

Definition 1.1. The (left-sided) fractional integral of order $\gamma>0$ of a function $F:(0,\infty)\to\mathbb{R}$ is given by

$$(I_{0+}^{\gamma}F)(t)=\frac{1}{\Gamma(\gamma)}\int\limits_0^t (t-s)^{\gamma-1}F(s)\,ds,\ \ t>0,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$, where $\Gamma(\gamma)$ is the Euler Gamma function, defined by $\Gamma(\gamma)=\int\limits_0^\infty t^{\gamma-1}e^{-t}\,dt$, $\gamma>0$.

Definition 1.2. The Riemann–Liouville fractional derivative of order $\gamma>0$ of a function $F:(0,\infty)\to\mathbb{R}$ is given by

$$(D_{0+}^{\gamma}F)(t)=\Big(\frac{d}{dt}\Big)^n(I_{0+}^{n-\gamma}F)(t)=\frac{1}{\Gamma(n-\gamma)}\Big(\frac{d}{dt}\Big)^n\int\limits_0^t\frac{F(s)}{(t-s)^{\gamma-n+1}}\,ds$$

for $t>0$, where $n=[\![\gamma]\!]+1$ ($[\![\gamma]\!]$ is the largest integer, not greater than γ), provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 1.3. By a positive solution of (1.1), (1.2) we mean a function $u\in C[0,1]$ satisfying (1.1), (1.2) with $u(t)>0$ for all $t\in(0,1]$.

The fixed point theorems have been playing a crucial role in establishing the solutions of fractional differential equations. For instance, one may refer to [4–6, 8, 12, 15–20] on the use of a fixed point index property, Krasnoselskii's, Avery–Peterson's, Schauder's fixed point theorems, the Leray–Schauder alternative, and Guo–Krasnoselskii's fixed point theorem to study the existence of at least one, two or three positive solutions of fractional differential equations of form (1.1) with nonlinear BCs of form (1.2). For a system of fractional differential equations with integral boundary conditions of coupled or uncoupled type, one may refer to [1, 9–11, 13, 14].

In their recent work [16], Padhi et al. have used Schauder's fixed point theorem and the Leray–Schauder's alternative along with the Krasnoselskii's fixed point theorem to study the existence and uniqueness of positive solutions of (1.1), (1.2). Using the Avery–Peterson's fixed point theorem, the authors established the existence of at least three positive solutions of (1.1), (1.2).

In [16], Padhi et al. have shown that the boundary value problem (1.1), (1.2) is equivalent to the integral equation

$$u(t)=\int\limits_0^1 G(t,s)q(s)f(s,u(s))\,ds+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\,t^{\alpha-1}\int\limits_0^1 h(s,u(s))\,dA(s),$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1 \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1. \end{cases}$$

Further, it is proved in [16] that the Green's function $G(t,s)$ satisfies the inequality

$$t^{\alpha-1}G(1,s) = t^{\alpha-1} \max_{0\le t\le 1} G(t,s) \le G(t,s) \le \max_{0\le t\le 1} G(t,s) = G(1,s), \tag{1.3}$$

where

$$G(1,s) = \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-\beta-1}\big[1-(1-s)^{\beta}\big]. \tag{1.4}$$

To establish our results, we assume that the following conditions are satisfied:

(A1) $f, h \in C([0,1]\times[0,\infty),[0,\infty))$;

(A2) $q \in C((0,1),[0,\infty))$, and q does not vanish identically on any subinterval of $(0,1]$;

(A3) for any positive numbers r_1 and r_2 with $r_1 < r_2$, there exist continuous functions p_f and $p_h : (0,1) \to [0,\infty)$ such that

$$f(t,u) \le p_f(t), \quad h(t,u) \le p_h(t) \ \text{ for } \ 0 \le t \le 1, \ \ \frac{r_1}{2^{2(\alpha-1)}} \le u \le r_2,$$

and

$$\int_0^1 G(1,s)q(s)p_f(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 p_h(s)\,dA(s) < \infty,$$

where $G(1,s)$ is given in (1.4).

In this paper, we apply the monotone iterative method to obtain sufficient conditions on the existence of one positive solution and an iterative scheme for approximating the solutions. The following theorem states the main result of this paper.

Theorem 1.1. *Assume that there exist constants r and R with $0 < 2r < R$ such that the following conditions are satisfied:*

(A4)
$$\frac{r}{\int_0^1 G(1,s)q(s)\,ds} \le f(t,u) \le f(t,v) \le \frac{R}{2\int_0^1 G(1,s)q(s)\,ds}$$
$$\text{for } \ \mu^2 r \le u \le v \le R \ \text{ and } \ \frac{1}{2} \le t \le 1$$

and

(A5)
$$h(t,u) \le h(t,v) \le \frac{\Gamma(\alpha)R}{2\Gamma(\alpha-\beta)\int_0^1 dA(s)} \quad \text{for } \ \mu^2 r \le u \le v \le R \ \text{ and } \ \frac{1}{2} \le t \le 1.$$

Then problem (1.1), (1.2) *has at least one positive solution.*

2 Preliminaries

In this section, we provide some basic concepts on the cones in a Banach space and the monotone iteration method.

Definition 2.1. Let X be a real Banach space. A nonempty convex closed set $P \subset X$ is said to be a cone provided that

(i) $ku \in P$ for all $u \in P$ and all $k \geq 0$;

(ii) $u, -u \in P$ implies $u = 0$.

In order to prove Theorem 1.1, we use the following well known monotone iteration method imported from [2, 3, 7] or Theorem 7.A in [21].

Theorem 2.1. *Let X be a real Banach space and K be a cone in X. Assume that there exist constants v_0 and w_0 with $v_0 \leq w_0$ and $[v_0, w_0] \subset X$ such that*

(i) *$T : [v_0, w_0] \to X$ is completely continuous;*

(ii) *T is a monotonic increasing operator on $[v_0, w_0]$;*

(iii) *v_0 is a lower solution of T, that is, $v_0 \leq Tv_0$;*

(iv) *w_0 is an upper solution of T, that is, $Tw_0 \leq w_0$.*

Then T has a fixed point and the iterative sequences $v_{n+1} = Tv_n$ and $w_{n+1} = Tw_n$, $n = 1, 2, 3, \ldots,$ with

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq w_{n-1} \leq \cdots \leq w_1 \leq w_0$$

converges to v and w, respectively, which are the greatest and smallest fixed points of T in $[v_0, w_0]$.

In this paper, we let $X = C[0, 1]$ to be the Banach space endowed with the norm

$$\|u\| = \max_{0 \leq t \leq 1} |u(t)|.$$

Define a cone K on X as $K = \{u \in C[0, 1] : \ u(t) \geq 0, \ t \in [0, 1]\}$ and an operator $T : K \to X$ as

$$Tu(t) = \int_0^1 G(t, s)q(s)f(s, u(s))\, ds + \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)}\, t^{\alpha-1} \int_0^1 h(s, u(s))\, dA(s). \tag{2.1}$$

Then it is easy to verify that $u(t)$ is a positive solution of problem (1.1), (1.2) if and only if $u(t)$ is a fixed point of the operator T on the cone K.

Let $g(s) = G(1, s)$ with $\int_{1/2}^{1} g(s)\, ds > 0$ and $c(t) = t^{\alpha-1}$. Then (1.3) can be rewritten as

$$c(t)g(s) \leq G(t, s) \leq g(s) \ \text{ for } \ 0 \leq t, s \leq 1. \tag{2.2}$$

Since it is useful to work on a smaller cone than K, we consider a cone K_1 of the type

$$K_1 = \Big\{u \in X : \ u(t) \geq 0 \text{ and } \min_{t \in [a,b]} u(t) \geq c_{a,b}\|u\|\Big\},$$

where $[a, b]$ is some subinterval of $[0, 1]$ and $c_{a,b} > 0$. Condition (2.2) ensures that for $[a, b] \subset [0, 1]$, if $c_{a,b} = \min\{c(t) : \ t \in [a, b]\} > 0$, then T maps K into K_1. Since (2.2) is valid for any $t \in [0, 1]$, we can work on the subinterval $[1/2, 1] \subset [0,1]$ for which the inequality

$$\mu G(1, s) \leq G(t, s) \leq G(1, s)$$

replaces (1.3) or (2.2), where

$$\mu = \frac{1}{2^{\alpha-1}} = \min_{t\in[1/2,1]} c(t) = \min_{t\in[1/2,1]} t^{\alpha-1}.$$

In this case, the operator T, defined in (2.1), maps the cone K into the subcone P, where

$$P = \Big\{ u \in C[0,1] : \min_{t\in[1/2,1]} u(t) \ge \mu \|u\| \Big\}. \tag{2.3}$$

Also, $u(t)$ is a positive solution of problem (1.1), (1.2) if and only if $u(t)$ is a fixed point of the operator T on the subcone P.

3 Proof of Theorem 1.1

To prove our theorem, we consider the cone P, defined in (2.3). Let $u \in P$. Then

$$\|Tu\| \le \int_0^1 G(1,s)q(s)f(s,u(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 h(s,u(s))\,dA(s)$$

and

$$\begin{aligned}
\min_{t\in[1/2,1]} Tu(t) &\ge \Big(\min_{t\in[1/2,1]} t^{\alpha-1}\Big)\Big[\int_0^1 G(1,s)q(s)f(s,u(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 h(s,u(s))\,dA(s)\Big] \\
&= \mu\Big[\int_0^1 G(1,s)q(s)f(s,u(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 h(s,u(s))\,dA(s)\Big] \\
&\ge \mu \|Tu\|
\end{aligned}$$

implies that $T : P \to P$. Also, T is well defined.

Set $v_0 = \mu^2 r$ and $w_0 = R$; then $v_0 < w_0$. We now prove that $T : [v_0, w_0] \to P$ is completely continuous. Let $\{u_n\} \in [v_0, w_0]$ and $u \in [v_0, w_0]$ be such that $\lim\limits_{n\to\infty} u_n = u$. Then $\mu^2 r \le u_n \le R$ and $\mu^2 r \le u \le R$ for $t \in [0,1]$. Since f is continuous on $[0,1] \times [\mu^2 r, R]$, for $\varepsilon > 0$ there exists $\delta_1 > 0$ with $|u_1 - u_2| < \delta_1$ for $u_1, u_2 \in [\mu^2 r, R]$, and we have

$$|f(t,u_1) - f(t,u_2)| < \frac{\varepsilon}{2\int_0^1 G(1,s)q(s)\,ds}, \quad t \in [0,1].$$

Similarly, from the continuity of h on $[0,1] \times [\mu^2 r, R]$, we get

$$|h(t,u_1) - h(t,u_2)| < \frac{\Gamma(\alpha)\varepsilon}{2\Gamma(\alpha-\beta)\int_0^1 dA(s)}, \quad t \in [0,1],$$

for $\varepsilon > 0$ and $\delta_2 > 0$ with $|u_1 - u_2| < \delta_2$, $u_1, u_2 \in [\mu^2 r, R]$. Set $\delta = \min\{\delta_1, \delta_2\}$; then it follows from $\lim_{n\to\infty} u_n = u$ that there exists a positive number N such that for every $n \ge N$, we have $|u_n(t) - u(t)| < \delta$, $t \in [0,1]$. Then the inequality

$$\begin{aligned}
|Tu_n(t) - Tu(t)| \le{}& \int_0^1 G(1,s)q(s)\big|f(s,u_n(s)) - f(s,u(s))\big|\,ds \\
&+ \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 \big|h(s,u_n(s)) - h(s,u(s))\big|\,dA(s) < \varepsilon
\end{aligned}$$

shows that $T : [v_0, w_0] \to P$ is continuous.

Setting

$$f^* = \max_{t\in[0,1],u\in[\mu^2 r,R]} f(t,u) \ \text{ and } \ h^* = \max_{t\in[0,1],\, u\in[\mu^2 r,R]} h(t,u),$$

we have

$$|Tu(t)| \le f^* \int_0^1 G(1,s)q(s)\,ds + h^* \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 dA(s).$$

Thus, T is uniformly bounded on P.

Since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous there. Similarly, the function $t^{\alpha-1}$ is uniformly continuous on $[0,1]$, because it is continuous there. So, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|G(t_1,s) - G(t_2,s)| < \varepsilon$ and $|t_1^{\alpha-1} - t_2^{\alpha-1}| < \varepsilon$ for $|t_1 - t_2| < \delta$, $(t_1,s),(t_2,s) \in [0,1]\times[0,1]$. Consequently, for any $u \in [\mu^2 r, R] := [v_0, w_0]$ and $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}|Tu(t_1) - Tu(t_2)| &\le \int_0^1 |G(t_1,s) - G(t_2,s)|q(s)f(s,u(s))\,ds \\ &\quad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\,|t_1^{\alpha-1} - t_2^{\alpha-1}| \int_0^1 h(s,u(s))\,dA(s) < \varepsilon\Big[\int_0^1 q(s)p_f(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 p_h(s)\,dA(s)\Big].\end{aligned}$$

Hence the family $\{Tx : x \in [v_0, w_0]\}$ is equicontinuous on $[0,1]$, and so T is relatively compact. By the Arzela–Ascoli theorem, $T : [v_0, w_0] \to P$ is completely continuous.

Let $u, v \in [v_0, w_0]$ be such that $u \le v$. Then $v_0 \le u \le v \le w_0$. By (A4) and (A5), we have

$$\begin{aligned}Tu(t) &= \int_0^1 G(t,s)q(s)f(s,u(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\,t^{\alpha-1}\int_0^1 h(s,u(s))\,dA(s) \\ &\le \int_0^1 G(t,s)q(s)f(s,v(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\,t^{\alpha-1}\int_0^1 h(s,v(s))\,dA(s) \\ &= Tv(t).\end{aligned}$$

Thus, T is monotonic increasing in $[v_0, w_0]$.

Now we prove that $v_0 = \mu^2 r$ is a lower solution of T, that is, $v_0 \le Tv_0$. Indeed, for $v_0 \in P$, we have $Tv_0 \in P$ and so

$$\begin{aligned}Tv_0(t) &\ge \mu\|Tv_0(t)\| \ge \mu \min_{t\in[1/2,1]} Tv_0(t) \\ &= \mu\Big(\min_{t\in[1/2,1]} \int_0^1 G(t,s)q(s)f(s,v_0(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\,t^{\alpha-1}\int_0^1 h(s,v_0(s))\,dA(s)\Big) \\ &\ge \mu\int_0^1 \Big(\min_{t\in[1/2,1]} G(t,s)\Big)q(s)f(s,v_0(s))\,ds \ge \mu^2\int_0^1 G(1,s)q(s)f(s,u(s))\,ds \ge \mu^2 r = v_0(t).\end{aligned}$$

Finally, we show that $w_0 = R$ is an upper solution of T, that is, $Tw_0 \le w_0$. Clearly,

$$Tw_0(t) \le \int_0^1 G(1,s)q(s)f(s,w_0(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\,t^{\alpha-1}\int_0^1 h(s,w_0(s))\,dA(s) \le R = w_0(t),$$

so $w_0 = R$ is an upper solution of T.

If we construct the sequences $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ as

$$v_n = Tv_{n-1}, \quad w_n = Tw_{n-1}, \ \ n = 1, 2, \dots,$$

then it follows that

$$v_0 \le v_1 \le v_2 \le \cdots \le v_n \le \cdots \le w_n \le w_{n-1} \le \cdots \le w_1 \le w_0,$$

and $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ converge, respectively, to v and w, which are the greatest and smallest fixed points of T in $[v_0, w_0]$. Since $v \le w$, Theorem 2.1 guarantees that w is the positive solution of problem (1.1), (1.2). This completes the proof of the theorem. □

Remark. One may observe from the assumptions (A4) and (A5) that we do not require any superlinearity or sublinearity on f and h either at 0 or ∞. The only assumption we require on f and g is that they must be monotonically nondecreasing in the subinterval $[1/2, 1]$, which shows that the functions f and h may be decreasing or nonincreasing and also may be identically zero or zero at some points in $[0, 1/2)$. This fact is evident from Examples 4.1 and 4.2.

4 An Illustration

In this section, we provide two examples illustrating Theorem 1.1.

Example 4.1. Consider the fractional differential equation

$$D_{0+}^{5/2} u(t) + \Gamma\Big(\frac{5}{2}\Big)\big[1 - (1-t)^{3/2}\big]^{-1} f(t, u(t)) = 0, \ \ 0 < t < 1, \tag{4.1}$$

with the multipoint BCs

$$u(0) = u'(0) = 0, \quad D_{0+}^{3/2} u(1) = \int_0^1 h(s, u(s))\, dA(s), \tag{4.2}$$

where

$$A(t) = \begin{cases} t & \text{if } \ t \in \Big[0, \dfrac{4}{9}\Big) \cup \Big[\dfrac{5}{9}, \dfrac{8}{9}\Big), \\ \dfrac{4}{9} & \text{if } \ t \in \Big[\dfrac{4}{9}, \dfrac{5}{9}\Big), \\ \dfrac{8}{9} & \text{if } \ t \in \Big[\dfrac{8}{9}, 1\Big], \end{cases} \tag{4.3}$$

$$f(t, u) = \begin{cases} \dfrac{1}{2}\,(35 + e^{-\frac{1}{u-32}}) & \text{if } \ u > 32, \\ \dfrac{35}{2} & \text{if } \ u \le 32, \end{cases}$$

and

$$h(t, u) = \begin{cases} 28 + e^{-\frac{1}{u-2}} & \text{if } \ u > 2, \\ 28 & \text{if } \ u \le 2. \end{cases}$$

Here $\alpha = \frac{5}{2}$, $\beta = \frac{3}{2}$ and $q(t) = \Gamma(\frac{5}{2})[1 - (1-t)^{3/2}]^{-1}$. Clearly,

$$G(1, t) = \frac{1}{\Gamma(\frac{5}{2})}\,\big[1 - (1-t)^{3/2}\big], \ \ 0 < t \le 1,$$

implies that $q(t)G(1, t) \equiv 1$, hence

$$\int_0^1 G(1, t) q(t)\, dt \equiv 1.$$

Also,

$$\mu = \frac{1}{2^{\alpha-1}} = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}}.$$

For $u \le v$, we have $e^{-\frac{1}{u-32}} \le e^{-\frac{1}{v-32}}$, which implies that $f(t,u) \le f(t,v)$ for $u \le v$. In a similar way, we can prove that $h(t,u) \le h(t,v)$ for $u \le v$.

Set $r = 16$ and $R = 40$; then

$$f(t,u) \ge \frac{35}{2} = 17.5 > 16 = r$$

and

$$f(t,u) \le \frac{1}{2}(35 + e^{-\frac{1}{u-32}}) \le \frac{1}{2}(35 + e^{-\frac{1}{40-32}}) \le \frac{1}{2}(35 + e^{-\frac{1}{8}}) \le 18 < 20 = \frac{R}{2}$$

imply that

$$r \le f(t,u) \le f(t,v) \le \frac{R}{2} \ \text{ for } \ \frac{r}{8} \le u \le v \le R \ \text{ and } \ \frac{1}{2} \le t \le 1,$$

that is, condition (A4) is satisfied. Similarly, $h(t,u) \le 29 < \frac{135}{8}\sqrt{\pi}$ implies that condition (A5) is satisfied. Thus, by Theorem 1.1, problem (4.1), (4.2) has at least two positive solutions.

Example 4.2. Consider the fractional differential equation (4.1) together with the BCs (4.2) and $A(t)$ in (4.3) with $f(t,u(t)) = \frac{1}{2} + t\sin\frac{u}{3}$ and $h(t,u) = t + \frac{1}{2} + 0.88\sin u$. Set $r = \frac{1}{2}$ and $R = 3$. Since $\sin u$ is an increasing function for $\frac{1}{16} \le u \le 1$, then $f(t,u)$ and $h(t,u)$ satisfy the properties $f(t,u) \le f(t,v)$ and $h(t,u) \le h(t,v)$ for $u \le v$, $\frac{1}{2} \le t \le 1$ and $\frac{1}{16} = \mu^2 r \le u \le v \le R = 3$. Further, since $\sin u > 0$ for $\frac{1}{16} \le u \le 3$, we have

$$r \le \frac{1}{2} \le \frac{1}{2} + t\sin\frac{u}{3} = f(t,u) \le \frac{1}{2} + \sin 1 \le \frac{3}{2} = \frac{R}{2}$$

and

$$\begin{aligned} h(t,u) &\le 1 + \frac{1}{2} + 0.88\sin u \\ &\le 1 + \frac{1}{2} + (0.88)(0.8415) \\ &\le 2.24049 \\ &\le 2.243216 \\ &= \frac{27\sqrt{\pi}}{64} R, \end{aligned}$$

that is, conditions (A4) and (A5) are satisfied. Hence, by Theorem 1.1, problem (4.1), (4.2), with the considered $f(t,u(t))$ and $h(t,u)$, has at least two positive solutions.

5 Discussion and Conclusions

The fixed point theorems are playing a vital role in studying, analysing the systems of fractional differential equations and also in establishing positive solutions. These fixed point theorems are also helpful in examining the existence/non-existence conditions for various coexistence equilibria in many dynamical systems with applications to natural, biological and epidemiological sciences. Many of the existing fixed point theorems require the superlinearity and sublinearity conditions.

In [16], Padhi et al. applied Schauder's fixed point theorem (see [16, Theorems 4.2 and 4.4]) to prove the existence of a positive solution of (1.1), (1.2), where the function f is assumed to be either superlinear or sublinear at 0 or ∞. In another attempt, Theorem 4.5 in [16] requires the existence of two reals r_1 and r_2 with $0 < r_1 < r_2$ such that either one of the following conditions is required to prove the existence of a positive solution of (1.1), (1.2):

(A6)

$$r_1 \le \int_0^1 G(1,s)q(s)f_1(s,r_1)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 h_1(s,r_1)\,dA(s) < \infty,$$
$$\int_0^1 G(1,s)q(s)f_2(s,r_2)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 h_2(s,r_2)\,dA(s) \le r_2,$$

(A7)

$$\int_0^1 G(1,s)q(s)f_2(s,r_1)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 h_2(s,r_1)\,dA(s) < \infty,$$
$$r_2 \le \int_0^1 G(1,s)q(s)f_1(s,r_2)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 h_1(s,r_2)\,dA(s) < \infty,$$

where

$$\begin{aligned} f_1(t,r) &= \min\left\{f(t,u) : \ t^{\alpha-1}r \le u \le r\right\}, \ \ 0<t<1, \\ f_2(t,r) &= \max\left\{f(t,u) : \ t^{\alpha-1}r \le u \le r\right\}, \ \ 0<t<1, \\ h_1(t,r) &= \min\left\{h(t,u) : \ t^{\alpha-1}r \le u \le r\right\}, \ \ 0<t<1, \\ h_2(t,r) &= \max\left\{h(t,u) : \ t^{\alpha-1}r \le u \le r\right\}, \ \ 0<t<1. \end{aligned}$$

The present work proposes the fixed point theorem with the use of the monotone iterative method for establishing the existence of one positive solution and also the method for approximating the solution. In this process, the obtained sufficient conditions require no superlinearity and/or sublinearity on the functions under consideration at 0 or ∞. Thus, Theorem 1.1 cannot be comparable with Theorems 4.2 and 4.4 in [16]. Instead, the conditions in Theorem 1.1 require the only monotonic increase of the functions in the subinterval $[1/2, 1]$ and they may decrease or nonincrease or identically be zero in the other half of the interval $[0, 1/2)$. This shows that assumptions (A4) and (A5) are not comparable with (A6) and (A7). We strongly feel that Theorem 1.1 simplifies the calculations in establishing the existence of positive solutions of the boundary value fractional differential equations.

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ON THE REDUCTION OF THE DIFFERENTIAL MULTI-FREQUENCY SYSTEM WITH SLOWLY VARYING PARAMETERS TO A SPECIAL KIND

Abstract. For the multi-frequency system of the differential equations the right-hand sides of which are represented by a multiple Fourier series with slowly varying coefficients, the conditions are obtained under which there exists the transformation with the coefficients of similar structure leading this system to a system with slowly varying right-hand sides.

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რეზიუმე. დიფერენციალურ განტოლებათა მრავალსიხშირიანი სისტემისთვის, როცა მარჯვენა მხარეები წარმოიდგინება ფურიეს ჯერადი მწკრივებით ნელად ცვალებადი კოეფიციენტებით, მიღებულია პირობები, როდესაც არსებობს გარდაქმნა ანალოგიური სტრუქტურის მქონე კოეფიციენტებით, რომელსაც ეს სისტემა მიჰყავს სისტემამდე ნელად ცვალებადი მარჯვენა მხარეებით.

1 Introduction

In the nonlinear mechanics, the problem of reducing a multi-frequency system of differential equations

$$\frac{d\theta}{dt} = \omega + f(\theta), \tag{1.1}$$

where $\theta \in \mathbb{R}^m$, $\omega \in \mathbb{R}^m$, $f(\theta) \in \mathbb{R}^m$ is a 2π-periodic vector-function, by the transformation of kind

$$\theta = \varphi + w(\varphi), \tag{1.2}$$

where $w(\varphi)$ is also a 2π-periodic vector-function, to the form

$$\frac{d\varphi}{dt} = \nu, \tag{1.3}$$

ν is a constant vector, is well known.

This problem is the subject of numerous studies (see, e.g., [1,3,4]). As is known, the main difficulty here is the problem of small denominators: the scalar product (k,ω) ($k = \mathrm{colon}(k_1,\dots,k_m)$, $k_j \in \mathbb{Z}$) may be arbitrarily small and it turns out to be in the denominators of the expressions representing the solution in terms of some series or iterative processes. Therefore, the vector ω is imposed the condition

$$|(k,\omega)| \geq \frac{C}{\|k\|^{m+1}}\,, \tag{1.4}$$

C is a positive constant, $\|k\| = |k_1| + \dots + |k_m|$. The use of this condition in turn generates "large numerators" that can lead to the divergence of these series and processes. This difficulty is overcome by the method of accelerated convergence [1].

In this paper we consider the system of kind

$$\frac{dx}{dt} = (\Lambda(t) + A(t,\theta))x, \quad \frac{d\theta}{dt} = \omega(t) + b(t,\theta), \tag{1.5}$$

in which t belongs to a finite, but arbitrarily large interval, $\Lambda(t)$ is a diagonal matrix, and the elements of a small matrix $A(t,\theta)$ and a small vector $b(t,\theta)$ are represented by an absolutely and uniformly convergent multiple Fourier-series with respect to θ, with slowly varying coefficients, and the variable vector $\omega(t)$ is not subject to the condition of kind (1.4). For system (1.5), under certain conditions we have proved the existence of the transformation of kind

$$x = (E + W(t,\varphi))y, \quad \theta = \varphi + w(t,\varphi), \tag{1.6}$$

where the elements of the matrix $W(t,\theta)$ and vector $w(t,\theta)$ are of similar structure leading system (1.5) to the form

$$\frac{dy}{dt} = (\Lambda(t) + D(t))y, \quad \frac{d\varphi}{dt} = \omega(t) + \nu(t), \tag{1.7}$$

where the elements of the diagonal matrix $D(t)$ and vector $\nu(t)$ are slowly varying and do not depend on φ. The properties of $W(t,\varphi)$ and $w(t,\varphi)$ are investigated depending on the properties of $A(t,\theta)$ and $b(t,\theta)$. However, the ideas of the method of accelerated convergence are still used, because instead of the small denominators, due to the vector $\omega(t)$, here arise small denominators generated by another circumstances.

2 Basic notation and definitions

Let $\varepsilon \in (0,1]$, $\tau = \varepsilon t \in [0,L]$ ($L \in (0,+\infty)$), $G = [0,L] \times (0,1]$.

Definition 2.1. We say that a scalar function $p(\tau,\varepsilon)$, generally complex-valued, belongs to the class S, if it continuous with respect to $\tau \in [0,L]$ and bounded with respect to $\varepsilon \in (0,1]$.

Thus, $\sup\limits_G |p(\tau,\varepsilon)| < +\infty$.

Slowly variability of the function is understood here in the sense of its belonging to the class S.

Definition 2.2. We say that a vector-function $h(\tau,\varepsilon)=\mathrm{colon}(h_1(\tau,\varepsilon),\dots,h_m(\tau,\varepsilon))$ belongs to the class S_1, if $h_j(\tau,\varepsilon)\in S$ $(j=1,\dots,m)$.

Under the norm of a vector $h(\tau,\varepsilon)\in S_1$ is understood

$$\|h(\tau,\varepsilon)\|_0=\max_{1\le j\le N}\sup_{\tau\in[0,L]}|h_j(\tau,\varepsilon)|.$$

This norm may depend on ε.

Definition 2.3. We say that a scalar real function $f(\tau,\varepsilon,\theta)$ belongs to the class $F(M;\alpha;\theta)$, if

$$f(\tau,\varepsilon,\theta)=\sum_{k\in Z_m}f_n(\tau,\varepsilon)\exp(i(k,\theta)),$$

$Z_m=\{k=\mathrm{colon}(k_1,\dots,k_m),\ k_j\in\mathbb{Z}\}$, $\theta=\mathrm{colon}(\theta_1,\dots,\theta_m)$ is the real vector, $(k,\theta)=k_1\theta_1+\dots+k_m\theta_m$, $f_k(\tau,\varepsilon)\in S$, and

$$\sup_{\tau\in[0,L]}|f_k(\tau,\varepsilon)|\le M\exp\Big(-\frac{\alpha}{\varepsilon}\,\|k\|\Big),$$

$\|n\|=|k_1|+\dots+|k_m|$; $M\in(0,+\infty)$, $\alpha\in(0,1)$ is a constant not depending on ε.

Definition 2.4. We say that a real vector-function $h(\tau,\varepsilon,\theta)=\mathrm{colon}(h_1(\tau,\varepsilon,\theta),\dots,h_m(\tau,\varepsilon,\theta))$ belongs to the class $F_1(M;\alpha,\theta)$, if $h_j(\tau,\varepsilon,\theta)\in F(M;\alpha;\theta)$ $(j=1,\dots,m)$.

For the vector-function $h(\tau,\varepsilon,\theta)\in F_1(M;\alpha;\theta)$ and vector $k\in Z_m$ we denote

$$\Gamma_k[h(\tau,\varepsilon,\theta)]=\frac{1}{(2\pi)^m}\int\limits_0^{2\pi}\cdots\int\limits_0^{2\pi}h(\tau,\varepsilon,\theta)e^{-i(k,\theta)}\,d\theta_1\cdots\theta_m,$$

$$\overline{h(\tau,\varepsilon,\theta)}=\Gamma_{\vec 0}[h(\tau,\varepsilon,\theta)],\quad \widetilde{h(\tau,\varepsilon,\theta)}=h(\tau,\varepsilon,\theta)-\overline{h(\tau,\varepsilon,\theta)},$$

where $\vec 0$ is a null-vector of dimension m.

Definition 2.5. We say that a real matrix-function $A(\tau,\varepsilon,\theta)=(a_{jk}(\tau,\varepsilon,\theta))_{j,k=1,\dots,n}$ belongs to the class $F_2(M;\alpha,\theta)$, if $a_{jk}(\tau,\varepsilon,\theta)\in F(M;\alpha;\theta)$ $(j,k=1,\dots,n)$.

For the matrix $A(\tau,\varepsilon,\theta)\in F_2(M;\alpha;\theta)$ and vector $h(\tau,\varepsilon)\in S_1$ we denote

$$\Big(\frac{\partial A}{\partial\theta},h\Big)=\sum_{j=1}^m\frac{\partial A}{\partial\theta_j}\,h_j(\tau,\varepsilon).$$

3 Statement of the problem

Consider the following system of differential equations:

$$\frac{dx}{dt}=(\Lambda(\tau,\varepsilon)+A(\tau,\varepsilon,\theta))x,\quad \frac{d\theta}{dt}=\omega(\tau,\varepsilon)+b(\tau,\varepsilon,\theta),\tag{3.1}$$

where $\tau,\varepsilon\in G$, $x\in\mathbb{R}^n$, $\theta\in\mathbb{R}^m$, $\Lambda(\tau,\varepsilon)=\mathrm{diag}(\lambda_1(\tau,\varepsilon),\dots,\lambda_n(\tau,\varepsilon))$, the real functions $\lambda_j(\tau,\varepsilon)$ belong to the class S, $A(\tau,\varepsilon,\theta)\in F_2(M;\alpha;\theta)$, $\omega(\tau,\varepsilon)\in\mathbb{R}^m$, $\omega(\tau,\varepsilon)\in S_1$, $b(\tau,\varepsilon,\theta)\in F_1(M;\alpha;\theta)$ $(M\in(0,1))$.

We study the problem on the existence, construction and properties of the transformation of kind

$$x=(E_n+W(\tau,\varepsilon,\varphi))y,\quad \theta=\varphi+w(\tau,\varepsilon,\varphi),\tag{3.2}$$

where $y\in\mathbb{R}^n$, $\varphi\in\mathbb{R}^m$, E_n is a unit $(n\times n)$-matrix, $w(\tau,\varepsilon,\varphi)\in F_1(M_1^*;\alpha^*;\varphi)$, $W(\tau,\varepsilon,\varphi)\in F_2(M_2^*,\alpha^*,\varphi)$ $(M_1^*,M_2^*,\alpha^*$ are to be defined), whicht leads system (3.1) to the form

$$\frac{dy}{dt}=(\Lambda(\tau,\varepsilon)+D(\tau,\varepsilon))y,\quad \frac{d\varphi}{dt}=\omega(\tau,\varepsilon)+\Delta(\tau,\varepsilon),\tag{3.3}$$

where $D(\tau,\varepsilon)=\mathrm{diag}(d_1(\tau,\varepsilon),\dots,d_n(\tau,\varepsilon))$, $d_j(\tau,\varepsilon)\in S$ $(j=1,\dots,n)$, $\Delta(\tau,\varepsilon)\in S_1$.

4 Auxiliary results

Lemma 4.1. *Let the functions $p(\tau;\varepsilon)$, $q(\tau,\varepsilon)$ belong to the class S, $c = const$. Then the functions $cp(\tau,\varepsilon)$, $p(\tau,\varepsilon) \pm q(\tau,\varepsilon)$, $p(\tau,\varepsilon)q(\tau,\varepsilon)$ belong to the class S, as well.*

Lemma 4.2. *Let $0 < M_1 < M_2$, $0 < \alpha_1 < \alpha_2 < 1$. Then $F(M_1;\alpha;\theta) \subset F(M_2;\alpha;\theta)$, $F(M;\alpha_1,\theta) \supset F(M;\alpha_2;\theta)$.*

Lemma 4.3. *Let $f_j(\tau,\varepsilon;\theta) \in F(M_j;\alpha;\theta)$ $(j = 1,\dots,p)$, $c_1,\dots,c_p$ be the constants. Then*

$$\sum_{j=1}^{p} c_j f_j(\tau,\varepsilon,\theta) \in F\Big(\sum_{j=1}^{p} |c_j| M_j;\alpha;\theta\Big).$$

Lemma 4.4. *Let $p(\tau,\varepsilon) \in S$, $f(\tau,\varepsilon,\theta) \in F(M;\alpha;\theta)$, and $\sup\limits_{G} |p(\tau,\varepsilon)| \le P$. Then*

$$p(\tau,\varepsilon)f(\tau,\varepsilon,\theta) \in F(PM;\alpha;\theta).$$

The validity of Lemmas 4.1–4.4 is obvious.

Lemma 4.5. *Let $f(\tau,\varepsilon,\theta) \in F(M_1;\alpha;\theta)$, $g(\tau,\varepsilon,\theta) \in F(M_2;\alpha;\theta)$. Then*

$$f(\tau,\varepsilon,\theta)g(\tau,\varepsilon,\theta) \in F\Big(\frac{3^m M_1 M_2}{\delta^m};\alpha-\delta;\theta\Big),$$

where $\delta \in (0,\alpha)$.

Proof. We have

$$f(\tau,\varepsilon,\theta) = \sum_{k\in Z_m} f_k(\tau,\varepsilon)e^{i(k,\theta)}, \quad g(\tau,\varepsilon,\theta) = \sum_{k\in Z_m} g_k(\tau,\varepsilon)e^{i(k,\theta)},$$

and

$$\sup_{\tau\in[0,L]} |f_k(\tau,\varepsilon)| \le M_1 e^{-\frac{\alpha}{\varepsilon}\|k\|}, \quad \sup_{\tau\in[0,L]} |g_k(\tau,\varepsilon)| \le M_2 e^{-\frac{\alpha}{\varepsilon}\|k\|}.$$

Hence

$$f(\tau,\varepsilon,\theta)g(\tau,\varepsilon,\theta) = \sum_{k\in Z_m}\Big(\sum_{l\in Z_m} f_{k-l}(\tau,\varepsilon)g_l(\tau,\varepsilon)\Big)e^{i(k,\theta)},$$

where $l = \mathrm{colon}(l_1,\dots,l_m)$, $k-l = \mathrm{colon}(k_1-l_1,\dots,k_m-l_m)$.

We have

$$\begin{aligned}
&\sum_{l\in Z_m} \sup_{\tau\in[0,L]} |f_{k-l}(\tau,\varepsilon)| \sup_{\tau\in[0,L]} |g_l(\tau,\varepsilon)| \le M_1M_2 \sum_{l\in Z_m} \exp\Big(-\frac{\alpha}{\varepsilon}\,(\|k-l\|+\|l\|)\Big)\\
&\quad = M_1M_2 \sum_{l_1=-\infty}^{\infty}\cdots\sum_{l_m=-\infty}^{\infty} \exp\Big(-\frac{\alpha}{\varepsilon}\,(|k_1-l_1|+|l_1|+\cdots+|k_m-l_m|+|l_m|)\Big)\\
&\quad = M_1M_2\Big(\sum_{l_1=-\infty}^{\infty}\exp\Big(-\frac{\alpha}{\varepsilon}\,(|k_1-l_1|+|l_1|)\Big)\Big)\cdots\Big(\sum_{l_m=-\infty}^{\infty}\exp\Big(-\frac{\alpha}{\varepsilon}\,(|k_m-l_m|+|l_m|)\Big)\Big).
\end{aligned}$$

We denote

$$A(k_j) = \sum_{s=-\infty}^{\infty} \exp\Big(-\frac{\alpha}{\varepsilon}\,(|k_j-s|+|s|)\Big).$$

1. Let $k_j = 0$. We have

$$\begin{aligned}
A(0) &= \sum_{s=-\infty}^{\infty}\exp\Big(-\frac{2\alpha}{\varepsilon}\,|s|\Big) = 1 + 2\sum_{s=1}^{\infty}\exp\Big(-\frac{2\alpha}{\varepsilon}\,s\Big)\\
&\qquad = 1 + \frac{2e^{-\frac{2\alpha}{\varepsilon}}}{1-e^{-\frac{2\alpha}{\varepsilon}}} = 1 + \frac{2}{e^{\frac{2\alpha}{\varepsilon}}-1} < 1 + \frac{1}{\frac{\alpha}{\varepsilon}} = 1 + \frac{\varepsilon}{\alpha} < 1 + \frac{1}{\alpha} = \frac{\alpha+1}{\alpha} < \frac{2}{\alpha} < \frac{2}{\delta}\,.
\end{aligned}$$

2. Let $k_j > 0$. Then

$$A(k_j) = \sum_{s=-\infty}^{-1} \exp\Big(-\frac{\alpha}{\varepsilon}\,(|k_j - s| + |s|)\Big) + \sum_{s=0}^{k_j} \exp\Big(-\frac{\alpha}{\varepsilon}\,(|k_j - s| + |s|)\Big)$$

$$+ \sum_{s=k_j+1}^{\infty} \exp\Big(-\frac{\alpha}{\varepsilon}\,(|k_j - s| + |s|)\Big) = \sum_{s=-\infty}^{-1} \exp\Big(-\frac{\alpha}{\varepsilon}\,(k_j - s - s)\Big)$$

$$+ \sum_{s=0}^{k_j} \exp\Big(-\frac{\alpha}{\varepsilon}\,(k_j - s + s)\Big) + \sum_{s=k_j+1}^{\infty} \exp\Big(-\frac{\alpha}{\varepsilon}\,(s - k_j + s)\Big)$$

$$= e^{-\frac{\alpha}{\varepsilon} n_j} \sum_{s=1}^{\infty} e^{-\frac{2\alpha}{\varepsilon}\, s} + (k_j + 1) e^{-\frac{\alpha}{\varepsilon}\, k_j} + e^{\frac{\alpha}{\varepsilon}\, k_j} \sum_{s=k_j+1}^{\infty} e^{-\frac{2\alpha}{\varepsilon}\, s} = \frac{2e^{-\frac{\alpha}{\varepsilon} k_j}}{e^{\frac{2\alpha}{\varepsilon}} - 1} + (k_j + 1) e^{-\frac{\alpha}{\varepsilon}\, k_j}.$$

3. Let $k_j < 0$. Similarly to the previous arguments, we show that

$$A(k_j) = \frac{2e^{\frac{\alpha}{\varepsilon}\, k_j}}{e^{\frac{2\alpha}{\varepsilon}} - 1} + (1 - k_j) e^{\frac{\alpha}{\varepsilon}\, k_j}.$$

Thus, in case $k_j \neq 0$,

$$A(k_j) = \frac{2e^{-\frac{\alpha}{\varepsilon}\, |k_j|}}{e^{\frac{2\alpha}{\varepsilon}} - 1} + (1 + |k_j|) e^{-\frac{\alpha}{\varepsilon}\, |k_j|}.$$

Hence

$$A(k_j) < \frac{e^{-\frac{\alpha}{\varepsilon}\, |k_j|}}{\frac{\alpha}{\varepsilon}} + (1 + |k_j|) e^{-\frac{\alpha}{\varepsilon}\, |k_j|} = \Big(\frac{1}{\frac{\alpha}{\varepsilon}} + 1 + |k_j|\Big) e^{-\frac{\alpha}{\varepsilon}\, |k_j|}.$$

We choose a constant M_0 from the condition

$$\Big(\frac{\varepsilon}{\alpha} + 1 + |k_j|\Big) e^{-\frac{\alpha}{\varepsilon}\, |k_j|} \leq M_0 e^{-\frac{\alpha - \delta}{\varepsilon}\, |k_j|},$$

where $\delta \in (0, \alpha)$. We estimate

$$\max_{|k_j| \geq 1} \Big(\frac{\varepsilon}{\alpha} + 1 + |k_j|\Big) e^{-\frac{\delta}{\varepsilon}\, |k_j|}.$$

For the case $x \geq 1$, let us investigate the function

$$u(x) = \Big(\frac{\varepsilon}{\alpha} + 1 + x\Big) e^{-\frac{\delta}{\varepsilon}\, x}.$$

We have

$$u(1) = \Big(\frac{\varepsilon}{\alpha} + 2\Big) e^{-\frac{\delta}{\varepsilon}}, \quad u'(x) = \Big(1 - \frac{\delta}{\alpha} - \frac{\delta}{\varepsilon} - \frac{\delta}{\varepsilon}\, x\Big) e^{-\frac{\delta}{\varepsilon}\, x}.$$

The critical point is $x_0 = -1 + \varepsilon/\delta - \varepsilon/\alpha$. It is easy to establish that this is the maximum point of the function $u(x)$. In case $x_0 \leq 1$, i.e., $\varepsilon\alpha/(2\alpha + \varepsilon) \leq \delta < \alpha$, we get

$$\max_{[1, +\infty)} u(x) = u(1) = \Big(\frac{\varepsilon}{\alpha} + 2\Big)\, e^{-\frac{\delta}{\varepsilon}}.$$

In case $x_0 > 1$, i.e., $0 < \delta < \varepsilon\alpha/(2\alpha + \varepsilon)$, we obtain

$$\max_{[1, +\infty)} u(x) = u(x_0) = \frac{\varepsilon}{\delta}\, e^{-(1 - \frac{\delta}{\alpha} - \frac{\delta}{\varepsilon})}.$$

Anyway,

$$\max_{[1, +\infty)} u(x) < \frac{3}{\delta}\,,$$

therefore we can state that $M_0 = 3/\delta$.

Thus, if $k_j \neq 0$, then

$$A(k_j) < \frac{3}{\delta}\, e^{-\frac{\alpha-\delta}{\varepsilon}\,|k_j|}. \tag{4.1}$$

By virtue of the estimation for $A(0)$, we find that estimation (4.1) is true for all $k_j \in \mathbb{Z}$.

We now obtain

$$\sum_{l\in Z_m} \sup_{\tau\in[0,L]} |f_{k-l}(\tau,\varepsilon)| \sup_{\tau\in[0,L]} |g_l(\tau,\varepsilon)| \leq M_1 M_2\, \frac{3^m}{\delta^m}\, e^{-\frac{\alpha-\delta}{\varepsilon}\,\|k\|},$$

and thus Lemma 4.5 is proved. □

Lemma 4.6. *Let $f(\tau,\varepsilon,\theta) \in F(M_1;\alpha;\theta)$, $g(\tau,\varepsilon,\theta) \in F(M_2;\alpha-\delta;\theta)$, $\delta \in (0,\alpha)$. Then*

$$f(\tau,\varepsilon,\theta)g(\tau,\varepsilon,\theta) \in F\Big(\frac{4^m}{\delta^m}\, M_1 M_2; \alpha-\delta;\theta\Big).$$

The proof is analogous to that of Lemma 4.5.

Corollary of Lemmas 4.5 and 4.6. *Let $f_j(\tau,\varepsilon,\theta) \in F(M_j;\alpha;\theta)$ $(j = 1,\dots,p,\ p \geq 2)$. Then $f_1(\tau,\varepsilon,\theta)\cdots f_p(\tau,\varepsilon,\theta) \in F(V_p;\alpha-\delta;\theta)$, where*

$$V_p = \frac{4^{m(p-1)}}{\delta^{m(p-1)}}\, M_1\cdots M_p.$$

Lemma 4.7. *Let*

$$f(\tau,\varepsilon,\theta) = \sum_{k\in Z_m} f_k(\tau,\varepsilon) e^{i(k,\theta)} \in F(M;\alpha;\theta).$$

Then

$$\frac{\partial^s f(\tau,\varepsilon,\theta)}{\partial\theta^s} = \frac{\partial^s f(\tau,\varepsilon,\theta)}{\partial\theta_1^{s_1}\cdots\partial\theta_m^{s_m}} \in F\Big(\frac{s^s}{\delta^s e^s}\, M;\alpha-\delta;\theta\Big),$$

where $s = s_1+\dots+s_m \geq 1$, $\delta \in (0,\alpha)$.

Proof. We have

$$\frac{\partial^s f(\tau,\varepsilon,\theta)}{\partial\theta_1^{s_1}\cdots\partial\theta_m^{s_m}} = \sum_{\substack{k\in Z_m\\ (\|k\|\geq 1)}} (ik_1)^{s_1}\cdots(ik_m)^{s_m} f_k(\tau,\varepsilon)e^{i(k,\theta)},$$

$$\sum_{\substack{k\in Z_m\\ (\|k\|\geq 1)}} |k_1|^{s_1}\cdots|k_m|^{s_m} \sup_{\tau\in[0,L]} |f_k(\tau,\varepsilon)| \leq M \sum_{\substack{k\in Z_m\\ (\|k\|\geq 1)}} \|k\|^s e^{-\frac{\alpha}{\varepsilon}\,\|k\|}.$$

It is easy to show that if $x \geq 1$, $s \geq 1$, then

$$x^s e^{-\frac{\delta}{\varepsilon}\,x} < \frac{s^s}{\delta^s e^s}\,.$$

Hence

$$M\|k\| e^{-\frac{\alpha}{\varepsilon}\,\|k\|} < \frac{s^s}{\delta^s e^s}\, M e^{-\frac{\alpha-\delta}{\varepsilon}\,\|k\|},$$

and Lemma 4.7 is proved. □

Lemma 4.8. *Let the vector-function $w(\tau,\varepsilon,\theta) = \mathrm{colon}(w_1(\tau,\varepsilon,\theta),\dots,w_m(\tau,\varepsilon,\theta)) \in F_1(M_1;\alpha;\theta)$, and the vector-function $v(\tau,\varepsilon,\theta) = \mathrm{colon}(v_1(\tau,\varepsilon,\theta),\dots,v_m(\tau,\varepsilon,\theta)) \in F_1(M_2;\alpha-\delta;\theta)$, where $0<\delta<\alpha$. If $\delta \in (0,\alpha/2)$ and*

$$\mu = \frac{m\cdot 4^m}{\delta^{m+1}}\, M_2 < \frac{1}{2}\,, \tag{4.2}$$

then the vector-function $w(\tau,\varepsilon,\varphi+v(\tau,\varepsilon,\varphi)) - w(\tau,\varepsilon,\varphi)$, where $\varphi = \mathrm{colon}(\varphi_1,\dots,\varphi_m)$, belongs to the class

$$F_1\Big(\frac{m^2\cdot 4^m}{\delta^{2m+2}}\, M_1(M_2+M_2^2);\alpha-2\delta;\varphi\Big).$$

Proof. We expand the scalar functions

$$\begin{aligned}&w_j(\tau,\varepsilon,\varphi+v(\tau,\varepsilon,\varphi))\\&\qquad=w_j\big(\tau,\varepsilon,\varphi_1+v_1(\tau,\varepsilon,\varphi_1,\dots,\varphi_m),\dots,\varphi_m+v_m(\tau,\varepsilon,\varphi_1,\dots,\varphi_m)\big)\quad(j=1,\dots,N)\end{aligned}$$

with respect to $v_1,\dots,v_m$ in the Taylor series

$$w_j(\tau,\varepsilon,\varphi+v(\tau,\varepsilon,\varphi))-w_j(\tau,\varepsilon,\varphi)=dw_j(\tau,\varepsilon,\varphi)+\sum_{s=2}^{\infty}\frac{1}{s!}\,d^sw_j(\tau,\varepsilon,\varphi),\tag{4.3}$$

where

$$\begin{aligned}dw_j(\tau,\varepsilon,\varphi)&=\sum_{l=1}^{m}\frac{\partial w_j(\tau,\varepsilon,\varphi)}{\partial\varphi_l}\,v_l(\tau,\varepsilon,\varphi),\\d^sw_j(\tau,\varepsilon,\varphi)&=\sum_{\substack{s_1+\cdots+s_m=s\\(0\le s_\nu\le s)}}\frac{\partial^sw_j(\tau,\varepsilon,\varphi)}{\partial\varphi^s}\,(v(\tau,\varepsilon,\varphi))^s,\end{aligned}$$

and

$$\frac{\partial^sw_j(\tau,\varepsilon,\varphi)}{\partial\varphi^s}=\frac{\partial^sw_j(\tau,\varepsilon,\varphi)}{\partial\varphi_1^{s_1}\cdots\partial\varphi_m^{s_m}},$$
$$(v(\tau,\varepsilon,\varphi))^s=(v_1(\tau,\varepsilon,\varphi))^{s_1}\cdots(v_m(\tau,\varepsilon,\varphi))^{s_m},\quad s=2,3,\dots;\quad j=1,\dots,m.$$

By virtue of Lemma 4.7,

$$\frac{\partial w_j(\tau,\varepsilon,\varphi)}{\partial\varphi_\nu}\in F\Big(\frac{M_1}{\delta e}\,;\alpha-\delta;\varphi\Big),\quad\nu=1,\dots,m.$$

Due to Lemma 4.5,

$$\frac{\partial w_j(\tau,\varepsilon,\varphi)}{\partial\varphi_\nu}\,v_\nu(\tau,\varepsilon,\varphi)\in F\Big(\frac{3^NM_1M_2}{\delta^{N+1}e}\,;\alpha-2\delta;\varphi\Big),\quad\nu=1,\dots,m,$$

if $\delta\in(0,\alpha/2)$. Therefore

$$dw_j(\tau,\varepsilon,\varphi)\in F\Big(\frac{m3^mM_1M_2}{\delta^{m+1}e}\,;\alpha-2\delta;\varphi\Big).$$

By virtue of Lemma 4.7,

$$\frac{\partial^sw_j(\tau,\varepsilon,\varphi)}{\partial\varphi^s}\in F\Big(\frac{s^sM_1}{\delta^se^s}\,;\alpha-\delta;\varphi\Big),$$

if $s\ge2$, $\delta\in(0,\alpha)$.

By virtue of Corollary of Lemmas 4.5 and 4.6, if $s\ge2$, $\delta\in(0,\alpha/2)$, we have

$$(v(\tau,\varepsilon,\varphi))^s\in F\Big(\frac{4^{m(s-1)}}{\delta^{m(s-1)}}\,M_2^s;\alpha-2\delta;\varphi\Big).$$

Then by Lemma 4.6,

$$\frac{\partial^sw_j(\tau,\varepsilon,\varphi)}{\partial\varphi^s}\,(v(\tau,\varepsilon,\varphi))^s\in F\Big(\frac{4^{ms}s^s}{\delta^{(m+1)s}e^s}\,M_1M_2^s;\alpha-2\delta;\varphi\Big).$$

Hence

$$d^sw_j(\tau,\varepsilon,\varphi)\in F(W_s;\alpha-2\delta;\varphi),$$

where

$$W_s=\frac{m^s4^{ms}s^s}{\delta^{(m+1)s}e^s}\,M_1M_2^s.$$

We consider the expression

$$\sum_{s=2}^{\infty}\frac{W_s}{s!}=M_1\sum_{s=2}^{\infty}\frac{\mu^s s^s}{s!\,e^s}\,,$$

where μ is defined by formula (4.2). By virtue of the Stirling's formula [2, p. 371], we have

$$\frac{s^s}{s!\,e^s}<\frac{1}{\sqrt{2\pi s}}\,,$$

hence

$$\sum_{s=2}^{\infty}\frac{W_s}{s!}<M_1\sum_{s=2}^{\infty}\frac{\mu^s}{\sqrt{2\pi}\cdot\sqrt{s}}<\frac{M_1}{2\sqrt{\pi}}\sum_{s=2}^{\infty}\mu^s.$$

Due to inequality (4.2), this series is convergent, and we obtain

$$\sum_{s=2}^{\infty}\frac{W_s}{s!}<\frac{M_1}{2\sqrt{\pi}}\,\frac{\mu^2}{1-\mu}<\frac{m^2 4^{2m}}{\delta^{2m+2}}\,M_1M_2^2.$$

Hence

$$\sum_{s=2}^{\infty}\frac{1}{s!}\,d^s w_j(\tau,\varepsilon,\varphi)\in F\Big(\frac{m^2 4^{2m}}{\delta^{2m+2}}\,M_1M_2^2;\alpha-2\delta;\varphi\Big).$$

Now, by virtue of (4.3), we obtain

$$w_j(\tau,\varepsilon,\varphi+v(\tau,\varepsilon,\varphi))-w_j(\tau,\varepsilon,\varphi)\in F\bigg(M_1\Big(\frac{m3^m}{\delta^{m+1}e}\,M_2+\frac{m^2 4^{2m}}{\delta^{2m+2}}\,M_2^2\Big);\alpha-2\delta;\varphi\bigg),$$

and Lemma 4.8 is proved. □

Corollary. *If, in addition to the conditions of Lemma* 4.8*, the condition* $M_2<1$ *is satisfied, then*

$$w_j(\tau,\varepsilon,\varphi+v(\tau,\varepsilon,\varphi))-w_j(\tau,\varepsilon,\varphi)\in F\Big(\frac{2m^2 4^{2m}}{\delta^{2m+2}}\,M_1M_2;\alpha-2\delta;\varphi\Big).$$

Lemma 4.9. *Let the matrix-function* $A(\tau,\varepsilon,\theta)\equiv(a_{jk}(\tau,\varepsilon,\theta))_{j,k=1,\dots,m}\in F_2(M;\alpha;\theta)$*. Suppose that the conditions*

$$0<\delta<\alpha,\quad \frac{m\cdot 4^m}{\delta^m}\,M<\frac{1}{2} \tag{4.4}$$

hold. Then

$$(E_m+A(\tau,\varepsilon,\theta))^{-1}\in F_2(2;\alpha-\delta;\theta).$$

Proof. Let $A^p=(a_{jk}^{(p)})_{j,k=1,\dots,m}$, $p=2,3,\dots$. Then

$$a_{jk}^{(2)}=\sum_{s=1}^{m}a_{js}a_{sk},\quad j,k=1,\dots,m.$$

By virtue of Lemmas 4.3 and 4.5,

$$a_{jk}^{(2)}\in F\Big(\frac{m4^m}{\delta^m}\,M^2;\alpha-\delta;\theta\Big),\quad 0<\delta<\alpha.$$

Further,

$$a_{jk}^{(3)}=\sum_{s=1}^{m}a_{js}^{(2)}a_{sk},\quad j,k=1,\dots,m.$$

By virtue of Lemmas 4.3 and 4.6,

$$a_{jk}^{(3)}\in F\Big(\frac{m^2 4^{2m}}{\delta^{2m}}\,M^3;\alpha-\delta;\theta\Big),\quad 0<\delta<\alpha.$$

By the method of mathematical induction, we obtain

$$a_{jk}^{(p)} \in F\Big(\frac{m^{p-1}4^{m(p-1)}}{\delta^{m(p-1)}}\, M^p; \alpha-\delta; \theta\Big), \;\; 0<\delta<\alpha,$$

hence

$$A^p \in F_2\Big(\frac{m^{p-1}4^{m(p-1)}}{\delta^{m(p-1)}}\, M^p; \alpha-\delta; \theta\Big), \;\; 0<\delta<\alpha.$$

Consider the numerical series

$$1+\sum_{p=1}^{\infty}\frac{m^{p-1}4^{m(p-1)}}{\delta^{m(p-1)}}\, M^p = 1+M\sum_{p=1}^{\infty}\Big(\frac{m\cdot 4^m}{\delta^m}\, M\Big)^{p-1}.$$

By virtue of (4.4), this series is convergent, and its sum is less than $1+2M$. Since $2M<1$ (this also follows from (4.4)), thus we obtain what was required. □

5 The basic result

Theorem. *Let system* (3.1) *satisfy the following conditions:*

1)
$$|\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)| \geq \frac{\alpha}{L} > 0, \;\; j,s=1,\dots,n, \;\; j\neq s;$$

2)
$$r=\frac{H_1 M}{q^2}<1,$$

where

$$H_1 = 3^{5m+5}2n^2m^4 4^{4m+1}(L^2+L+1), \;\; q=\Big(\frac{\alpha}{\alpha+2}\Big)^{5m+5}.$$

Then there exists the transformation of kind (3.2) *in which*

$$W(\tau,\varepsilon,\varphi)\in F_2\Big(M_2^*; \frac{\alpha}{2}\,; \varphi\Big), \quad w(\tau,\varepsilon,\varphi)\in F_1\Big(M_1^*; \frac{\alpha}{2}\,; \varphi\Big),$$

where

$$M_1^* = Q(r,1)\exp\Big(\frac{H_1}{q}\, Q(r,q)\Big), \quad M_2^* = Q\Big(r,\frac{1}{4}\Big),$$

$Q(r,q)$ *is the sum of the numerical series*

$$\sum_{j=0}^{\infty}\frac{r^{2^j}}{q^j}\,,$$

convergent if $r,q\in(0,1)$*, which leads system* (3.1) *to kind* (3.3)*, in which*

$$\sup_G |d_j(\tau,\varepsilon)| \leq Q(r,1), \qquad \sup_{\varepsilon\in(0,1]} \|\Delta(\tau,\varepsilon)\|_0 \leq Q(r,1).$$

Proof. We denote

$$\beta_k = \Big(\frac{\alpha}{\alpha+2}\Big)^k, \quad \delta_k=\frac{\beta_k}{3}\,, \;\; k=1,2,\dots.$$

Obviously,

$$\delta_k^{5m+5} = \frac{1}{3^{5m+5}}\, q^k, \quad \sum_{k=1}^{\infty}\beta_k = \frac{\alpha}{2}\,,$$

and

$$\forall\, s\in\mathbb{N}: \;\; 0<\beta_1+\dots+\beta_s<\frac{\alpha}{2}\,, \;\; \frac{\alpha}{2}<\alpha-\beta_1-\dots-\beta_s<\alpha.$$

Following the method described in [1], we represent the process of reducing system (3.1) to form (3.3) as a sequence of steps. At the first step we make in system (3.1) the following substitution:

$$x = \big(E_n + U^{(1)}(\tau,\varepsilon,\varphi^{(1)})\big)y^{(1)}, \quad \theta = \varphi^{(1)} + v^{(1)}(\tau,\varepsilon,\varphi^{(1)}), \tag{5.1}$$

where $y^{(1)} \in \mathbb{R}^n$, $\varphi^{(1)} \in \mathbb{R}^m$, vector $v^{(1)}(\tau,\varepsilon,\varphi^{(1)})$ is defined from the vector partial differential equation

$$\frac{\partial v^{(1)}}{\partial \varphi^{(1)}}\big(\omega(\tau,\varepsilon) + \Delta^{(1)}(\tau,\varepsilon)\big) + \varepsilon\frac{\partial v^{(1)}}{\partial \tau} = b(\widetilde{\tau,\varepsilon,\varphi^{(1)}}), \tag{5.2}$$

where

$$\Delta^{(1)}(\tau,\varepsilon) = \overline{b(\tau,\varepsilon,\varphi^{(1)})}.$$

It is obvious that $\Delta^{(1)}(\tau,\varepsilon) \in \mathbb{R}^m$ and belongs to the class S_1.

The matrix $U^{(1)}(\tau,\varepsilon,\varphi^{(1)})$ is defined from the matrix partial differential equation

$$\Big(\frac{\partial U^{(1)}}{\partial \varphi^{(1)}}, \omega(\tau,\varepsilon) + \Delta^{(1)}(\tau,\varepsilon)\Big) + \varepsilon\,\frac{\partial U^{(1)}}{\partial \tau} = \big(\Lambda(\tau,\varepsilon) + D^{(1)}(\tau,\varepsilon)\big)U^{(1)} - U^{(1)}\big(\Lambda(\tau,\varepsilon) + D^{(1)}(\tau,\varepsilon)\big) + C^{(0)}(\tau,\varepsilon,\varphi^{(1)}), \tag{5.3}$$

where

$$D^{(1)}(\tau,\varepsilon) = \operatorname{diag}\Big(\overline{a_{11}(\tau,\varepsilon,\varphi^{(1)})}, \dots, \overline{a_{nn}(\tau,\varepsilon,\varphi^{(1)})}\Big),$$
$$C^{(0)}(\tau,\varepsilon,\varphi^{(1)}) = A(\tau,\varepsilon,\varphi^{(1)}) - D^{(1)}(\tau,\varepsilon).$$

As a result of substitution (5.1), system (3.1) is reduced to the form

$$\begin{aligned} \frac{dy^{(1)}}{dt} &= \Big(\Lambda(\tau,\varepsilon) + D^{(1)}(\tau,\varepsilon) + A^{(1)}(\tau,\varepsilon,\varphi^{(1)})\Big)y^{(1)}, \\ \frac{d\varphi^{(1)}}{dt} &= \omega(\tau,\varepsilon) + \Delta^{(1)}(\tau,\varepsilon) + b^{(1)}(\tau,\varepsilon,\varphi^{(1)}), \end{aligned} \tag{5.4}$$

where the vector $b^{(1)}(\tau,\varepsilon,\varphi^{(1)})$ is defined from the equation

$$\Big(E_m + \frac{\partial v^{(1)}}{\partial \varphi^{(1)}}\Big)b^{(1)} = b\big(\tau,\varepsilon,\varphi^{(1)} + v^{(1)}(\tau,\varepsilon,\varphi^{(1)})\big) - b(\tau,\varepsilon,\varphi^{(1)}), \tag{5.5}$$

and the matrix $A^{(1)}(\tau,\varepsilon,\varphi^{(1)})$ is defined from the equation

$$\begin{aligned} \big(E_n + U^{(1)}(\tau,\varepsilon,\varphi^{(1)})\big)A^{(1)} &= -\Big(\frac{\partial U^{(1)}}{\partial \varphi^{(1)}}, b^{(1)}(\tau,\varepsilon,\varphi^{(1)})\Big) + C^{(0)}(\tau,\varepsilon,\varphi^{(1)})U^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \\ &\quad + \Big[A\big(\tau,\varepsilon,\varphi^{(1)} + v^{(1)}(\tau,\varepsilon,\varphi^{(1)})\big) - A(\tau,\varepsilon,\varphi^{(1)})\Big]\big(E_n + U^{(1)}(\tau,\varepsilon,\varphi^{(1)})\big). \end{aligned} \tag{5.6}$$

Taking into account (5.2), we set

$$v^{(1)}(\tau,\varepsilon,\varphi^{(1)}) = \sum_{\substack{k\in Z_m \\ (\|k\|>0)}} v_k^{(1)}(\tau,\varepsilon)\exp(i(k,\varphi^{(1)})), \tag{5.7}$$

where

$$v_k^{(1)}(\tau,\varepsilon) = \frac{1}{\varepsilon}\exp\Big(-\frac{i}{\varepsilon}\int_0^\tau \big(k,\omega(\xi,\varepsilon) + \Delta^{(1)}(\xi,\varepsilon)\big)\,d\xi\Big) \times \int_0^\tau b_k(z,\varepsilon)\exp\Big(\frac{i}{\varepsilon}\int_0^z \big(k,\omega(\xi,\varepsilon) + \Delta^{(1)}(\xi,\varepsilon)\big)\,d\xi\Big)\,dz, \quad b_k(z,\varepsilon) = \Gamma_k[b(z,\varepsilon,\varphi^{(1)})]. \tag{5.8}$$

Thus $v_k^{(1)}(\tau,\varepsilon) \in S_1$, and

$$\|v_k^{(1)}(\tau,\varepsilon)\|_0 \le \frac{LM}{\varepsilon} \exp\Big(-\frac{\alpha}{\varepsilon}\,\|k\|\Big), \ \ \|k\| > 0.$$

We define the constant M_0 by the condition

$$\frac{1}{\varepsilon}\, e^{-\frac{\alpha}{\varepsilon}\,\|k\|} \le M_0 e^{-\frac{\alpha-\delta_1}{\varepsilon}\,\|k\|}, \ \ \|k\| > 0$$

$\forall\, \varepsilon \in (0,1]$, where $\delta_1 \in (0,\alpha)$ and does not depend on ε. Obviously, if $x \ge 1$, then

$$\frac{1}{\varepsilon}\, e^{-\frac{\delta_1}{\varepsilon}\, x} \le \frac{1}{\varepsilon}\, e^{-\frac{\delta_1}{\varepsilon}}.$$

Since

$$\lim_{\varepsilon\to+0} \frac{1}{\varepsilon}\, e^{-\frac{\delta_1}{\varepsilon}} = 0,$$

and $\forall\, \varepsilon \in (0,1]$

$$\frac{1}{\varepsilon}\, e^{-\frac{\delta_1}{\varepsilon}} \le \frac{1}{\delta_1 e} < \frac{1}{\delta_1}$$

is valid, we can state that $M_0 = 1/\delta_1$. Thus, if $\varepsilon \in (0,1]$ and $\|k\| \ge 1$, we obtain

$$\|v_k^{(1)}(\tau,\varepsilon)\|_0 \le \frac{LM}{\delta_1}\, e^{-\frac{\alpha-\delta_1}{\varepsilon}\,\|k\|}.$$

It follows that

$$v^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_1\Big(\frac{L}{\delta_1}\, M; \alpha-\delta_1; \varphi^{(1)}\Big).$$

By virtue of Lemma 4.7,

$$\frac{\partial v^{(1)}}{\partial \varphi^{(1)}} \in F_2\Big(\frac{L}{\delta_1^2}\, M; \alpha-2\delta_1; \varphi^{(1)}\Big),$$

if $\delta_1 \in (0,\alpha/2)$. In view of Lemma 4.9, we can conclude that if $\delta \in (0,\alpha/3)$ and

$$\frac{m4^m L}{\delta_1^{m+2}}\, M < \frac{1}{2}\,, \tag{5.9}$$

then the matrix $(E_m + \partial v^{(1)}/\partial\varphi^{(1)})^{-1}$ exists and belongs to the class $F_2(2; \alpha-3\delta_1; \varphi^{(1)})$.

From inequality (5.9), it follows that $Lm/\delta_1 < 1$, therefore, by virtue of Corollary from Lemma 4.8, we can conclude that

$$b\Big(\tau,\varepsilon,\varphi^{(1)} + v^{(1)}(\tau,\varepsilon,\varphi^{(1)})\Big) \in F_1\Big(\frac{2m^2 4^{2m} L}{\delta_1^{2m+3}}\, M^2; \alpha-2\delta_1; \varphi^{(1)}\Big).$$

Now, by virtue of Lemma 4.6 and equation (5.5),

$$b_1(\tau,\varepsilon,\varphi^{(1)}) \in F_1\Big(\frac{m^3 4^{3m+1} L}{\delta_1^{3m+3}}\, M^2; \alpha-\beta_1; \varphi^{(1)}\Big).$$

We now construct the matrix $U^{(1)}(\tau,\varepsilon,\varphi^{(1)}) = (u_{js}^{(1)}(\tau,\varepsilon,\varphi^{(1)}))_{j,s=1,\dots,n}$. We write equation (5.3) componentwise,

$$\sum_{l=1}^{n} \frac{\partial u_{js}^{(1)}}{\partial \varphi_l^{(1)}}\, (\omega(\tau,\varepsilon) + \Delta_l^{(1)}(\tau,\varepsilon)) + \varepsilon\, \frac{\partial u_{js}^{(1)}}{\partial \tau}$$
$$= \big(\lambda_j(\tau,\varepsilon) - \lambda_s(\tau,\varepsilon) + d_j^{(1)}(\tau,\varepsilon) - d_s^{(1)}(\tau,\varepsilon)\big) u_{js}^{(1)} + c_{js}^{(0)}(\tau,\varepsilon,\varphi^{(1)}), \ \ j,s = 1,\dots,n. \tag{5.10}$$

Consider first the case $j = s$. We set

$$u_{jj}^{(1)}(\tau,\varepsilon,\varphi^{(1)}) = \sum_{\substack{k\in Z_m \\ (\|k\|>0)}} u_{jj,k}^{(1)}(\tau,\varepsilon)\exp(i(k,\varphi^{(1)})), \tag{5.11}$$

where

$$\begin{aligned} u_{jj,k}^{(1)}(\tau,\varepsilon) &= \frac{1}{\varepsilon}\exp\bigg(-\frac{i}{\varepsilon}\int\limits_0^\tau \big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)\big)\,d\xi\bigg) \\ &\quad\times\int\limits_0^\tau c_{jj,k}^{(0)}(z,\varepsilon)\exp\bigg(\frac{i}{\varepsilon}\int\limits_0^z \big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)\big)\,d\xi\bigg)\,dz, \\ c_{jj,k}^{(0)}(z,\varepsilon) &= \frac{1}{(2\pi)^m}\int\limits_0^{2\pi}\cdots\int\limits_0^{2\pi} c_{jj}^{(0)}(z,\varepsilon,\varphi)e^{-i(k,\varphi)}\,d\varphi_1\cdots d\varphi_m, \quad j=1,\dots,n, \quad k\in Z_m. \end{aligned} \tag{5.12}$$

Hence

$$u_{jj}^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F\Big(\frac{L}{\delta_1}M;\alpha-\delta_1;\varphi^{(1)}\Big), \quad j=1,\dots,n,$$

where $\delta_1 \in (0,\alpha)$ and does not depend on ε.

Let now $j \neq s$. We choose M insomuch small that

$$2M < \frac{\alpha-\delta_1}{L}\,. \tag{5.13}$$

Then, by virtue of condition 1) of the theorem, we have

$$\big|\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)+d_j^{(1)}(\tau,\varepsilon)-d_s^{(1)}(\tau,\varepsilon)\big| \geq \frac{\alpha}{L}-\frac{\alpha-\delta_1}{L} = \frac{\delta_1}{L} > 0. \tag{5.14}$$

Here in turn, we consider two cases.

Case 1. Let $\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon) \leq -\alpha/L < 0$. Then

$$\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)+d_j^{(1)}(\tau,\varepsilon)-d_s^{(1)}(\tau,\varepsilon) \leq -\frac{\delta_1}{L} < 0.$$

We define the elements $u_{js}^{(1)}$ of matrix $U^{(1)}$ by the formulas

$$u_{js}^{(1)}(\tau,\varepsilon,\varphi^{(1)}) = \sum_{k\in Z_m} u_{js,k}^{(1)}(\tau,\varepsilon)\exp(i(k,\varphi^{(1)})),$$

where

$$\begin{aligned} u_{js,k}^{(1)}(\tau,\varepsilon) &= \frac{1}{\varepsilon}\exp\bigg(\frac{1}{\varepsilon}\int\limits_0^\tau (\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon)+d_j^{(1)}(\xi,\varepsilon)-d_s^{(1)}(\xi,\varepsilon)-i\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)\big)\,d\xi\bigg) \\ &\quad\times\int\limits_0^\tau c_{js,k}^{(0)}(z,\varepsilon)\exp\bigg(-\frac{1}{\varepsilon}\int\limits_0^z \Big(\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon) \\ &\quad+d_j^{(1)}(\xi,\varepsilon)-d_s^{(1)}(\xi,\varepsilon)-i\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)\big)\Big)\,d\xi\bigg)\,dz, \quad j,s=1,\dots,n, \quad k\in Z_m. \end{aligned} \tag{5.15}$$

We estimate

$$
\begin{aligned}
|u^{(1)}_{js,k}(\tau,\varepsilon)| &\le \frac{1}{\varepsilon}\int\limits_0^\tau |c^{(0)}_{js,k}(z,\varepsilon)| \exp\left(\frac{1}{\varepsilon}\int\limits_z^\tau (\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon)+d^{(1)}_j(\xi,\varepsilon)-d^{(1)}_s(\xi,\varepsilon))\,d\xi\right) dz \\
&\le \frac{1}{\varepsilon}\,M\exp\Big(-\frac{\alpha}{\varepsilon}\,\|k\|\Big)\int\limits_0^\tau \exp\Big(-\frac{\delta_1}{L\varepsilon}(\tau-z)\Big)\,dz \\
&= \frac{1}{\varepsilon}\,M\exp\Big(-\frac{\alpha}{\varepsilon}\,\|k\|\Big)\frac{1}{\frac{\delta_1}{L\varepsilon}}\Big(1-\exp\Big(-\frac{\delta_1}{L\varepsilon}\,\tau\Big)\Big) \le \frac{LM}{\delta_1}\exp\Big(-\frac{\alpha-\delta_1}{\varepsilon}\,\|k\|\Big).
\end{aligned}
$$

Hence

$$
u^{(1)}_{js}(\tau,\varepsilon,\varphi^{(1)}) \in F\Big(\frac{LM}{\delta_1}\,;\alpha-\delta_1;\varphi^{(1)}\Big).
$$

Case 2. Let $\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)\ge \alpha/L>0$. Then $\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)+d^{(1)}_j(\tau,\varepsilon)-d^{(1)}_s(\tau,\varepsilon)\ge \delta_1/L>0$. We define the elements $u^{(1)}_{js}$ of matrix $U^{(1)}$ by the formulas

$$
u^{(1)}_{js}(\tau,\varepsilon,\varphi^{(1)}) = \sum_{k\in Z_m} u^{(1)}_{js,k}(\tau,\varepsilon)\exp(i(k,\varphi^{(1)})),
$$

where

$$
\begin{aligned}
&u^{(1)}_{js,k}(\tau,\varepsilon) \\
&\quad= -\frac{1}{\varepsilon}\exp\left(\frac{1}{\varepsilon}\int\limits_0^\tau \Big(\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon)+d^{(1)}_j(\xi,\varepsilon)-d^{(1)}_s(\xi,\varepsilon)-i\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)\big)\Big)\,d\xi\right) \\
&\qquad\times\int\limits_\tau^L c^{(0)}_{js,k}(z,\varepsilon)\exp\Bigg(-\frac{1}{\varepsilon}\int\limits_0^z \Big(\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon) \\
&\qquad+d^{(1)}_j(\xi,\varepsilon)-d^{(1)}_s(\xi,\varepsilon)-i\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)\big)\Big)\,d\xi\Bigg)\,dz, \quad j,s=1,\dots,n, \quad k\in Z_m.
\end{aligned}
\tag{5.16}
$$

As in the first case, we show that

$$
u^{(1)}_{js}(\tau,\varepsilon,\varphi^{(1)}) \in F\Big(\frac{LM}{\delta_1}\,;\alpha-\delta_1;\varphi^{(1)}\Big).
$$

Thus

$$
U^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_2\Big(\frac{LM}{\delta_1}\,;\alpha-\delta_1;\varphi^{(1)}\Big).
$$

By virtue of Corollary from Lemma 4.8, we can conclude that under condition (5.9),

$$
A(\tau,\varepsilon,\varphi^{(1)}+v^{(1)}(\tau,\varepsilon,\varphi^{(1)}))-A(\tau,\varepsilon,\varphi^{(1)}) \in F_2\Big(\frac{2m^2 4^{2m}L}{\delta_1^{2m+3}}\,M^2;\alpha-2\delta_1;\varphi^{(1)}\Big).
$$

From (5.9) we have $LM/\delta_1<1$, hence

$$
E_n+U^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_2(2;\alpha-\delta_1;\varphi^{(1)}),
$$

and, by virtue of Lemma 4.6,

$$
\begin{aligned}
&\big(A(\tau,\varepsilon,\varphi^{(1)}+v^{(1)}(\tau,\varepsilon,\varphi^{(1)}))-A(\tau,\varepsilon,\varphi^{(1)})\big)(E_n+U^{(1)}(\tau,\varepsilon,\varphi^{(1)})) \\
&\qquad\qquad \in F_2\Big(\frac{nm^2 4^{3m+1}}{\delta_1^{3m+3}}\,M^2;\alpha-2\delta_1;\varphi^{(1)}\Big),
\end{aligned}
\tag{5.17}
$$

$$
C(\tau,\varepsilon,\varphi^{(1)})U^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_2\Big(\frac{n4^m L}{\delta_1^{m+1}}\,M^2;\alpha-\delta_1;\varphi^{(1)}\Big).
\tag{5.18}
$$

Due to Lemma 4.7,

$$\frac{\partial U^{(1)}}{\partial \varphi^{(1)}} \in F_2\Big(\frac{L}{\delta_1^2}\,M; \alpha - 2\delta_1; \varphi^{(1)}\Big),$$
$$\sum_{k=1}^{m} \frac{\partial U^{(1)}}{\partial \varphi_k^{(1)}}\, b_k^{(1)} \in F_2\Big(\frac{m^4 4^{4m+1} L^2}{\delta_1^{4m+5}}\,M^3; \alpha - 2\delta_1; \varphi^{(1)}\Big).$$

By virtue of Lemma 4.9 and condition (5.9), we can conclude that

$$\big(E_n + U^{(1)}(\tau,\varepsilon,\varphi^{(1)})\big)^{-1} \in F_2(2; \alpha - 2\delta_1; \varphi^{(1)}).$$

Hence, (5.6), (5.17) and (5.18) yield

$$A^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_2\Big(\frac{2n^2 m^4 4^{5m+1}}{\delta_1^{5m+5}}\,(L^2+L+1)M^2; \alpha - \beta_1; \varphi^{(1)}\Big).$$

Thus, under conditions (5.9) and (5.13), we have

$$v^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_1\Big(\frac{L}{\delta_1}\,; \alpha - \delta_1; \varphi^{(1)}\Big),$$
$$U^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_2\Big(\frac{L}{\delta_1}\,; \alpha - \delta_1; \varphi^{(1)}\Big),$$
$$b^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_1\Big(\frac{2n^2 m^4 4^{5m+1}}{\delta_1^{5m+5}}\,(L^2+L+1)M^2; \alpha - \beta_1; \varphi^{(1)}\Big),$$
$$A^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_2\Big(\frac{2n^2 m^4 4^{5m+1}}{\delta_1^{5m+5}}\,(L^2+L+1)M^2; \alpha - \beta_1; \varphi^{(1)}\Big).$$

This completes the first step of the process.

At the step with number $l-1$ of the process, we obtain the system

$$\begin{aligned} \frac{dy^{(l-1)}}{dt} &= \Big(\Lambda(\tau,\varepsilon) + D^{(1)}(\tau,\varepsilon) + \cdots + D^{(l-1)}(\tau,\varepsilon) + A^{(l-1)}(\tau,\varepsilon,\varphi^{(l-1)})\Big) y^{(l-1)}, \\ \frac{d\varphi^{(l-1)}}{dt} &= \omega(\tau,\varepsilon) + \Delta^{(1)}(\tau,\varepsilon) + \cdots + \Delta^{(l-1)}(\tau,\varepsilon) + b^{(l-1)}(\tau,\varepsilon,\varphi^{(l-1)}), \end{aligned} \tag{5.19}$$

where $D^{(1)},\dots,D^{(l-1)}$ are the diagonal $(n\times n)$-matrices with elements from the class S, the vectors $\Delta^{(1)},\dots,\Delta^{(l-1)}$ belong to the class S_1, $b^{(l-1)} \in F_1(K_{l-1}; \alpha-\beta_1-\cdots-\beta_{l-1}; \varphi^{(l-1)})$, $A^{(l-1)} \in F_2(K_{l-1}; \alpha-\beta_1-\cdots-\beta_{l-1}; \varphi^{(l-1)})$,

$$K_l = \frac{H^{2^l-1}}{\delta_l^{5m+5}(\delta_{l-1}^{5m+5})^2\cdots(\delta_1^{5m+5})^{2^l-1}}\,M^{2^l}, \quad H = 2n^2 m^4 4^{5m+1}(L^2+L+1).$$

At the step with number l, we make in system (5.19) the following substitution:

$$y^{(l-1)} = \big(E_n + U^{(l)}(\tau,\varepsilon,\varphi^{(l)})\big) y^{(l)}, \quad \varphi^{(l-1)} = \varphi^{(l)} + v^{(l)}(\tau,\varepsilon,\varphi^{(l)}), \tag{5.20}$$

where $y^{(l)} \in \mathbb{R}^n$, $\varphi^{(l)} \in \mathbb{R}^m$. The vector $v^{(1)}(\tau,\varepsilon,\varphi^{(1)})$ is defined from the vector partial differential equation

$$\frac{\partial v^{(l)}}{\partial \varphi^{(l)}}\Big(\omega(\tau,\varepsilon) + \Delta^{(1)}(\tau,\varepsilon) + \cdots + \Delta^{(l)}(\tau,\varepsilon)\Big) + \varepsilon\frac{\partial v^{(l)}}{\partial \tau} = b^{(l-1)}\widetilde{(\tau,\varepsilon,\varphi^{(1)})}, \tag{5.21}$$

where

$$\Delta^{(l)}(\tau,\varepsilon) = \overline{b^{(l-1)}(\tau,\varepsilon,\varphi^{(l)})}.$$

Obviously, $\Delta^{(l)}(\tau,\varepsilon) \in \mathbb{R}^m$ and belongs to the class S_1.

The matrix $U^{(l)}(\tau,\varepsilon,\varphi^{(l)})$ is defined from the matrix partial differential equation

$$\begin{aligned}\Big(\frac{\partial U^{(l)}}{\partial\varphi^{(l)}}\,,\omega(\tau,\varepsilon)+\Delta^{(1)}(\tau,\varepsilon)+\cdots+\Delta^{(l)}(\tau,\varepsilon)\Big)+\varepsilon\frac{\partial U^{(l)}}{\partial\tau}\\ =\Big(\Lambda(\tau,\varepsilon)+D^{(1)}(\tau,\varepsilon)+\cdots+D^{(l)}(\tau,\varepsilon)\Big)U^{(l)}\\ -U^{(l)}\Big(\Lambda(\tau,\varepsilon)+D^{(1)}(\tau,\varepsilon)+\cdots+D^{(l)}(\tau,\varepsilon)\Big)+C^{(l-1)}(\tau,\varepsilon,\varphi^{(l)}),\end{aligned}\tag{5.22}$$

where

$$D^{(l)}(\tau,\varepsilon)=\mathrm{diag}\Big(\overline{a_{11}^{(l-1)}(\tau,\varepsilon,\varphi^{(l)})},\ldots,\overline{a_{nn}^{(l-1)}(\tau,\varepsilon,\varphi^{(l)})}\Big),$$
$$C^{(l-1)}(\tau,\varepsilon,\varphi^{(1)})=A^{(l-1)}(\tau,\varepsilon,\varphi^{(1)})-D^{(l)}(\tau,\varepsilon).$$

Taking into account (5.21), we set

$$v^{(l)}(\tau,\varepsilon,\varphi^{(1)})=\sum_{\substack{k\in Z_m\\(\|k\|>0)}}v_k^{(l)}(\tau,\varepsilon)\exp(i(k,\varphi^{(l)})),\tag{5.23}$$

where

$$\begin{aligned}v_k^{(l)}(\tau,\varepsilon)=\frac{1}{\varepsilon}\exp\bigg(-\frac{i}{\varepsilon}\int\limits_0^\tau\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\big)\,d\xi\bigg)\\ \times\int\limits_0^\tau b_k^{(l-1)}(z,\varepsilon)\exp\bigg(\frac{i}{\varepsilon}\int\limits_0^z\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\big)\,d\xi\bigg)\,dz,\end{aligned}\tag{5.24}$$

$$b_k^{(l-1)}(z,\varepsilon)=\Gamma_k[b^{(l-1)}(z,\varepsilon,\varphi^{(l)})].$$

Taking into account (5.22), we set

$$U^{(l)}(\tau,\varepsilon,\varphi^{(1)})=(u_{js}^{(l)}(\tau,\varepsilon,\varphi^{(l)}))_{j,s=1,\ldots,n},$$
$$u_{jj}^{(l)}(\tau,\varepsilon,\varphi^{(l)})=\sum_{\substack{k\in Z_m\\(\|k\|>0)}}u_{jj,k}^{(l)}(\tau,\varepsilon)\exp(i(k,\varphi^{(1)})),$$

where

$$\begin{aligned}u_{jj,k}^{(l)}(\tau,\varepsilon)=\frac{1}{\varepsilon}\exp\bigg(-\frac{i}{\varepsilon}\int\limits_0^\tau\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\big)\,d\xi\bigg)\\ \times\int\limits_0^\tau c_{jj,k}^{(l-1)}(z,\varepsilon)\exp\bigg(\frac{i}{\varepsilon}\int\limits_0^z\big(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\big)\,d\xi\bigg)\,dz,\end{aligned}$$

$$c_{jj,k}^{(l-1)}(z,\varepsilon)=\frac{1}{(2\pi)^m}\int\limits_0^{2\pi}\cdots\int\limits_0^{2\pi}c_{jj}^{(l-1)}(z,\varepsilon,\varphi)e^{-i(k,\varphi)}d\varphi_1\cdots d\varphi_m,\quad j=1,\ldots,n,\quad k\in Z_m.$$

If $j\neq s$, then we set

$$u_{js}^{(l)}(\tau,\varepsilon,\varphi^{(l)})=\sum_{k\in Z_m}u_{js,k}^{(l)}(\tau,\varepsilon)\exp(i(k,\varphi^{(l)})),$$

where in case $\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)\le -\alpha/L<0$,

$$u^{(l)}_{js,k}(\tau,\varepsilon)=\frac{1}{\varepsilon}\exp\biggl(\frac{1}{\varepsilon}\int\limits_0^{\tau}\Bigl(\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon)+d^{(1)}_j(\xi,\varepsilon)-d^{(1)}_s(\xi,\varepsilon)+\cdots+d^{(l)}_j(\xi,\varepsilon)-d^{(l)}_s(\xi,\varepsilon)$$
$$-\,i\bigl(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\bigr)\Bigr)\,d\xi\biggr)$$
$$\times\int\limits_0^{\tau}c^{(l-1)}_{js,k}(z,\varepsilon)\exp\biggl(-\frac{1}{\varepsilon}\int\limits_0^{z}\Bigl(\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon)+d^{(1)}_j(\xi,\varepsilon)-d^{(1)}_s(\xi,\varepsilon)+\cdots+d^{(l)}_j(\xi,\varepsilon)-d^{(l)}_s(\xi,\varepsilon)$$
$$-\,i\bigl(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\bigr)\Bigr)\,d\xi\biggr)\,dz,\quad j,s=1,\dots,n,\quad k\in Z_m,$$

and in case $\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)\ge \alpha/L>0$,

$$u^{(l)}_{js,k}(\tau,\varepsilon)=-\frac{1}{\varepsilon}\exp\biggl(\frac{1}{\varepsilon}\int\limits_0^{\tau}\Bigl(\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon)+d^{(1)}_j(\xi,\varepsilon)-d^{(1)}_s(\xi,\varepsilon)+\cdots+d^{(l)}_j(\xi,\varepsilon)-d^{(l)}_s(\xi,\varepsilon)$$
$$-\,i\bigl(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\bigr)\Bigr)\,d\xi\biggr)$$
$$\times\int\limits_{\tau}^{L}c^{(l-1)}_{js,k}(z,\varepsilon)\exp\biggl(-\frac{1}{\varepsilon}\int\limits_0^{z}\Bigl(\lambda_j(\xi,\varepsilon)-\lambda_s(\xi,\varepsilon)+d^{(1)}_j(\xi,\varepsilon)-d^{(1)}_s(\xi,\varepsilon)+\cdots+d^{(l)}_j(\xi,\varepsilon)-d^{(l)}_s(\xi,\varepsilon)$$
$$-\,i\bigl(k,\omega(\xi,\varepsilon)+\Delta^{(1)}(\xi,\varepsilon)+\cdots+\Delta^{(l)}(\xi,\varepsilon)\bigr)\Bigr)\,d\xi\biggr)\,dz,\quad j,s=1,\dots,n,\quad k\in Z_m.$$

Here we suppose M insomuch small that

$$2K_{l-1}<\frac{\delta_{l-1}-\delta_l}{L}\,, \tag{5.25}$$
$$\frac{n4^mL}{\delta_l^{m+2}}\,K_{l-1}<\frac{1}{2}\,. \tag{5.26}$$

Then

$$\bigl|\lambda_j(\tau,\varepsilon)-\lambda_s(\tau,\varepsilon)+d^{(1)}_j(\tau,\varepsilon)-d^{(1)}_s(\tau,\varepsilon)+\cdots+d^{(l)}_j(\tau,\varepsilon)-d^{(l)}_s(\tau,\varepsilon)\bigr|\ge\frac{\delta_l}{L}\,.$$

We have

$$\delta_l^{5m+5}(\delta_{l-1}^{5m+5})^2\cdots(\delta_1^{5m+5})^{2^l-1}=\frac{1}{3^{5m+5}}\,q^l\,\frac{1}{(3^{5m+5})^2}\,(q^{l-1})^2\cdots\frac{1}{(3^{5m+5})^{2^l-2}}\,(q^2)^{2^{l-2}}$$
$$=\frac{1}{(3^{5m+5})^{1+2+\cdots+2^{l-1}}}\,q^{l+2(l-1)+\cdots+2\cdot2^{l-2}+2^{l-1}}=\frac{1}{(3^{5m+5})^{2^l-1}}\,q^{2^{l+1}-l-2},$$

where q is defined in the statement of the theorem. Therefore

$$K_l=\frac{H_1^{2^l-1}}{q^{2^{l+1}-l-2}}\,M^{2^l},$$

where $H_1=2n^2m^43^{5m+5}4^{5m+1}(L^2+L+1)$. Hence $K_l<r^{2^l}$, where $r=\frac{H_1}{q^2}\,M$.

The condition $r<1$ guarantees the convergence of the series $\sum\limits_{l=1}^{\infty}K_l$. It is easy to verify that this condition ensures that inequalities (5.25), (5.26) hold.

As a result of substitution (5.20), system (5.19) is reduced to the form

$$\begin{aligned}
\frac{dy^{(l)}}{dt} &= \Big(\Lambda(\tau,\varepsilon) + D^{(1)}(\tau,\varepsilon) + \cdots + D^{(l)}(\tau,\varepsilon) + A^{(l)}(\tau,\varepsilon,\varphi^{(l-1)})\Big) y^{(l)},\\
\frac{d\varphi^{(l)}}{dt} &= \omega(\tau,\varepsilon) + \Delta^{(1)}(\tau,\varepsilon) + \cdots + \Delta^{(l)}(\tau,\varepsilon) + b^{(l)}(\tau,\varepsilon,\varphi^{(l)}).
\end{aligned} \tag{5.27}$$

Carrying out the arguments analogous to those of the first step, we show that

$$\begin{aligned}
v^{(l)}(\tau,\varepsilon,\varphi^{(l)}) &\in F_1\Big(\frac{K_{l-1}}{\delta_l}\,; \alpha - \beta_1 - \cdots - \beta_{l-1} - \delta_l; \varphi^{(l)}\Big),\\
U^{(l)}(\tau,\varepsilon,\varphi^{(l)}) &\in F_2\Big(\frac{K_{l-1}}{\delta_l}\,; \alpha - \beta_1 - \cdots - \beta_{l-1} - \delta_l; \varphi^{(l)}\Big),\\
b^{(l)}(\tau,\varepsilon,\varphi^{(l)}) &\in F_1\Big(K_l; \alpha - \beta_1 - \cdots - \beta_l; \varphi^{(l)}\Big),\\
A^{(l)}(\tau,\varepsilon,\varphi^{(l)}) &\in F_2\Big(K_l; \alpha - \beta_1 - \cdots - \beta_l; \varphi^{(l)}\Big).
\end{aligned}$$

Hence, the iterative process

$$\begin{gathered}
x = \big(E_n + U^{(1)}(\tau,\varepsilon,\varphi^{(1)})\big) y^{(1)}, \quad \theta = \varphi^{(1)} + v^{(1)}(\tau,\varepsilon,\varphi^{(1)}),\\
y^{(l-1)} = \big(E_n + U^{(l)}(\tau,\varepsilon,\varphi^{(l)})\big) y^{(l)}, \quad \varphi^{(l-1)} = \varphi^{(l)} + v^{(l)}(\tau,\varepsilon,\varphi^{(l)}), \;\; l = 2,3,\dots,
\end{gathered} \tag{5.28}$$

in case it is convergent, leads system (3.1) to kind (3.3) in which

$$D(\tau,\varepsilon) = \sum_{l=1}^{\infty} D^{(l)}(\tau,\varepsilon), \quad \Delta(\tau,\varepsilon) = \sum_{l=1}^{\infty} \Delta^{(l)}(\tau,\varepsilon),$$

where

$$\begin{gathered}
\Delta^{(l)}(\tau,\varepsilon) \in S_1, \quad \|\Delta^{(l)}(\tau,\varepsilon)\|_0 \le K_{l-1}, \quad D^{(l)}(\tau,\varepsilon) = \operatorname{diag}\big(d_1^{(l)}(\tau,\varepsilon), \dots, d_n^{(l)}(\tau,\varepsilon)\big),\\
d_j^{(l)}(\tau,\varepsilon) \in S, \quad \sup_G |d_j^{(l)}(\tau,\varepsilon)| \le K_{l-1} \;\; (j = 1,\dots,n).
\end{gathered}$$

We prove the convergence of process (5.28). Towards this end, we represent process (5.28) in the form

$$x = \big(E_n + W^{(l)}(\tau,\varepsilon,\varphi^{(l)})\big) y^{(l)}, \quad \theta = \varphi^{(l)} + w^{(l)}(\tau,\varepsilon,\varphi^{(l)}), \;\; l = 1,2,\dots, \tag{5.29}$$

where

$$\begin{aligned}
W^{(1)}(\tau,\varepsilon,\varphi^{(1)}) &= U^{(1)}(\tau,\varepsilon,\varphi^{(1)}), \quad w^{(1)}(\tau,\varepsilon,\varphi^{(1)}) = v^{(1)}(\tau,\varepsilon,\varphi^{(1)}),\\
W^{(l)}(\tau,\varepsilon,\varphi^{(l)}) &= \big(E_n + W^{(l-1)}(\tau,\varepsilon,\varphi^{(l)} + v^{(l)}(\tau,\varepsilon,\varphi^{(l)}))\big) U^{(l)}(\tau,\varepsilon,\varphi^{(l)})\\
&\quad + W^{(l-1)}(\tau,\varepsilon,\varphi^{(l)} + v^{(l)}(\tau,\varepsilon,\varphi^{(l)})),
\end{aligned} \tag{5.30}$$

$$w^{(l)}(\tau,\varepsilon,\varphi^{(l)}) = v^{(l)}(\tau,\varepsilon,\varphi^{(l)}) + w^{(l-1)}(\tau,\varepsilon,\varphi^{(l)} + v^{(l)}(\tau,\varepsilon,\varphi^{(l)})), \;\; l = 2,3,\dots. \tag{5.31}$$

Then

$$w^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_1(r; \alpha - \beta_1, \varphi^{(1)}), \quad W^{(1)}(\tau,\varepsilon,\varphi^{(1)}) \in F_2(r; \alpha - \beta_1, \varphi^{(1)}).$$

By virtue of Corollary from Lemma 4.8, we successively obtain

$$\begin{gathered}
w^{(2)}(\tau,\varepsilon,\varphi^{2)}) \in F_1\Big(r^2 + r\Big(1 + \frac{H_1}{q^2}\, r^2\Big); \alpha - \beta_1 - \beta_2; \varphi^{(2)}\Big),\\
w^{(l)}(\tau,\varepsilon,\varphi^{l)}) \in F_1\big(w_l^*;\; \alpha - \beta_1 - \cdots - \beta_l; \varphi^{(l)}\big), \;\; l = 3,4,\dots,
\end{gathered}$$

where

$$\begin{aligned}
w_l^* = r^{2^{l-1}} &+ r^{2^{l-2}}\Big(1 + \frac{H_1}{q^l}\, r^{2^{l-1}}\Big) + r^{2^{l-3}}\Big(1 + \frac{H_1}{q^{l-1}}\, r^{2^{l-2}}\Big)\Big(1 + \frac{H_1}{q^l}\, r^{2^{l-1}}\Big) + \cdots\\
&+ r\Big(1 + \frac{H_1}{q^2}\, r^2\Big)\Big(1 + \frac{H_1}{q^3}\, r^4\Big) \cdots \Big(1 + \frac{H_1}{q^l}\, r^{2^{l-1}}\Big).
\end{aligned}$$

Consider

$$w^{(l+1)}(\tau,\varepsilon,\varphi)-w^{(l)}(\tau,\varepsilon,\varphi)=v^{(l+1)}(\tau,\varepsilon,\varphi)+w^{(l)}(\tau,\varepsilon,\varphi+v^{(l+1)}(\tau,\varepsilon,\varphi))-w^{(l)}(\tau,\varepsilon,\varphi).$$

By virtue of Corollary from Lemma 4.8, we have

$$w^{(l)}(\tau,\varepsilon,\varphi+v^{(l+1)}(\tau,\varepsilon,\varphi))-w^{(l)}(\tau,\varepsilon,\varphi)\in F_1\Big(\frac{H_1}{q^{l+1}}\,r^{2^l}w_l^*;\alpha-\beta_1-\cdots-\beta_l-2\delta_{l+1};\varphi\Big).$$

Hence,

$$w^{(l+1)}(\tau,\varepsilon,\varphi)-w^{(l)}(\tau,\varepsilon,\varphi)\in F_1\Big(r^{2^l}\Big(1+\frac{H_1}{q^{l+1}}\,w_l^*\Big);\alpha-\beta_1-\cdots-\beta_{l+1};\varphi\Big). \tag{5.32}$$

We estimate

$$\begin{aligned} w_l^* &\le \Big(\sum_{j=0}^{l-1} r^{2^j}\Big)\prod_{j=1}^{l-1}\Big(1+\frac{H_1}{q^{j+1}}\,r^{2^j}\Big)\\ &=\Big(\sum_{j=0}^{l-1} r^{2^j}\Big)\exp\Big[\ln\prod_{j=1}^{l-1}\Big(1+\frac{H_1}{q^{j+1}}\,r^{2^j}\Big)\Big]=\Big(\sum_{j=0}^{l-1} r^{2^j}\Big)\exp\Big[\sum_{j=1}^{l-1}\ln\Big(1+\frac{H_1}{q^{j+1}}\,r^{2^j}\Big)\Big]\\ &<\Big(\sum_{j=0}^{l-1} r^{2^j}\Big)\exp\Big(\sum_{j=1}^{l-1}\frac{H_1}{q^{j+1}}\,r^{2^j}\Big)<\Big(\sum_{j=0}^{l-1} r^{2^j}\Big)\exp\Big(\frac{H_1}{q}\sum_{j=0}^{l-1}\frac{r^{2^j}}{q^j}\Big). \end{aligned} \tag{5.33}$$

The numerical series

$$\sum_{j=0}^{\infty}\frac{r^{2^j}}{q^j}$$

under the condition $r,q\in(0,1)$ is convergent, we denote its sum by $Q(r,q)$. Then, by virtue of (5.33), we obtain

$$w_l^*<Q(r,1)\exp\Big(\frac{H_1}{q}\,Q(r,q)\Big). \tag{5.34}$$

Hence,

$$r^{2^l}\Big(1+\frac{H_1}{q^{l+1}}\,w_l^*\Big)<r^{2^l}\Big(1+\frac{H_1}{q^{l+1}}\,Q(r,1)\exp\Big(\frac{H_1}{q}\,Q(r,q)\Big)\Big),$$

from the latter inequality and (5.32) it follows that

$$w^{(l+1)}(\tau,\varepsilon,\varphi)-w^{(l)}(\tau,\varepsilon,\varphi)\in F_1(c_l^{(1)};\alpha-\beta_1-\cdots-\beta_{l+1};\varphi), \tag{5.35}$$

where $c_l^{(1)}$ is the element of a convergent positive sign numerical series.

Next, we consider the process defined by (5.30). Suppose that

$$W^{(l)}(\tau,\varepsilon,\varphi^{(l)})\in F_2(W_l^*;\alpha-\beta_1-\cdots-\beta_l;\varphi^{(l)}).$$

Then

$$\begin{aligned} &\big(E_n+W^{(l-1)}(\tau,\varepsilon,\varphi^{(l)}+v^{(l)}(\tau,\varepsilon,\varphi^{(l)}))\big)U^{(l)}(\tau,\varepsilon,\varphi^{(l)})\\ &\qquad\in F_2\Big(r^{2^{l-1}}(1+W_{l-1}^*(1+r^{2^{l-1}}));\alpha-\beta_1-\cdots-\beta_l;\varphi^{(l)}\Big). \end{aligned}$$

Hence,

$$\begin{aligned} &W^{(l)}(\tau,\varepsilon,\varphi^{(l)})\\ &\in F_2\Big(r^{2^{l-1}}(1+W_{l-1}^*(1+r^{2^{l-1}}))+W_{l-1}^*(1+r^{2^{l-1}});\alpha-\beta_1-\cdots-\beta_l;\varphi^{(l)}\Big),\quad l=2,3,\ldots. \end{aligned} \tag{5.36}$$

This implies

$$W_l^* \le r^{2^{l-1}}(1+W_{l-1}^*(1+r^{2^{l-1}}))+W_{l-1}^*(1+r^{2^{l-1}})$$
$$< r^{2^{l-1}}(1+2W_{l-1}^*)+r^{2^{l-1}}W_{l-1}^*+W_{l-1}^* = r^{2^{l-1}}(1+3W_{l-1}^*)+W_{l-1}^*,$$

whence, taking into account that $r<1$, we succesively obtain

$$W_1^* = r,$$
$$W_2^* < r^2(1+3r)+r < r+r^2+3r^3 < r+4r^2,$$
$$W_3^* < r^4(1+3(r+4r^2))+r+4r^2 < r+4r^2+16r^4.$$

Further, by the method of mathematical induction, we obtain

$$W_l^* < r+4r^2+\cdots+4^{l-1}r^{2^{l-1}},$$

from which we get

$$W_l^* < Q\Big(r, \frac{1}{4}\Big). \tag{5.37}$$

Consider

$$W^{(l+1)}(\tau,\varepsilon,\varphi)-W^{(l)}(\tau,\varepsilon,\varphi) = \big(E_n+W^{(l)}(\tau,\varepsilon,\varphi+v^{(l+1)}(\tau,\varepsilon,\varphi))\big)U^{(l+1)}(\tau,\varepsilon,\varphi)$$
$$+W^{(l)}(\tau,\varepsilon,\varphi+v^{(l+1)}(\tau,\varepsilon,\varphi))-W^{(l)}(\tau,\varepsilon,\varphi). \tag{5.38}$$

By virtue of Corollary from Lemma 4.8, we have

$$W^{(l)}(\tau,\varepsilon,\varphi+v^{(l+1)}(\tau,\varepsilon,\varphi))-W^{(l)}(\tau,\varepsilon,\varphi)$$
$$\in F_2\bigg(\frac{2m^2 4^{2m}}{\delta_{l+1}^{2m+2}}\,Q\Big(r,\frac{1}{4}\Big)\,\frac{K_l}{\delta_{l+1}}\,;\alpha-\beta_1-\cdots-\beta_l-2\delta_{l+1};\varphi\bigg),$$

hence,

$$W^{(l)}(\tau,\varepsilon,\varphi+v^{(l+1)}(\tau,\varepsilon,\varphi))-W^{(l)}(\tau,\varepsilon,\varphi) \in F_2\bigg(Q\Big(r,\frac{1}{4}\Big)f^{2^l};\alpha-\beta_1-\cdots-\beta_l-2\delta_{l+1};\varphi\bigg).$$

Next, taking into account (5.37),

$$(E_n+W^{(l)}(\tau,\varepsilon,\varphi+v^{(l+1)}(\tau,\varepsilon,\varphi)))U^{(l+1)}(\tau,\varepsilon,\varphi)$$
$$\in F_2\bigg(r^{2^l}\Big(1+2Q\Big(r,\frac{1}{4}\Big)\Big);\alpha-\beta_1-\cdots-\beta_{l+1};\varphi\bigg).$$

Hence, by virtue of (5.38),

$$W^{(l+1)}(\tau,\varepsilon,\varphi)-W^{(l)}(\tau,\varepsilon,\varphi)\in F_2(c_l^{(2)};\alpha-\beta_1-\cdots-\beta_{l+1};\varphi), \tag{5.39}$$

where $c_l^{(2)} = r^{2^l}(1+3Q(r,1/4))$ is the element of the convergent positive sign numerical series.

From formulas (5.35), (5.39) follows the convergence of process (5.29). From formulas (5.34) and (5.37) it follows that $w(\tau,\varepsilon,\varphi)\in F_1(M_1^*;\alpha/2;\varphi)$, $W(\tau,\varepsilon,\varphi)\in F_2(M_2^*;\alpha/2;\varphi)$, where

$$M_1^* = Q(r,1)\exp\Big(\frac{H_1}{q}\,Q(r,q)\Big), \quad M_2^* = Q\Big(r,\frac{1}{4}\Big). \qquad \square$$

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THE BOUNDARY VALUE PROBLEMS FOR THE BI-LAPLACE–BELTRAMI EQUATION

Abstract. The purpose of the present paper is to investigate the boundary value problems for the bi-Laplace–Beltrami equation $\Delta^2_{\mathscr{C}}\varphi = f$ on a smooth hypersurface $\mathscr{C}$ with the boundary $\Gamma = \partial\mathscr{C}$. The unique solvability of the BVP is proved on the basis of Green's formula and Lax–Milgram Lemma.

We also prove the invertibility of the perturbed operator in the Bessel potential spaces $\Delta^2_{\mathscr{C}} + \mathscr{H}\,I : \mathbb{H}^{s+2}_p(\mathscr{S}) \to \mathbb{H}^{s-2}_p(\mathscr{S})$ for a smooth closed hypersurface $\mathscr{S}$ without boundary for arbitrary $1 < p < \infty$ and $-\infty < s < \infty$, provided $\mathscr{H}$ is a smooth function, has non-negative real part $\operatorname{Re}\mathscr{H}(t) \geqslant 0$ for all $t \in \mathscr{S}$ and non-trivial support $\operatorname{mes}\operatorname{supp}\operatorname{Re}\mathscr{H} \neq 0$.

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Key words and phrases. Bi-Laplace–Beltrami equation, Günter's tangential derivatives, boundary value problems, mixed boundary condition, Bessel potential spaces.

რეზიუმე. სტატიის მიზანია გამოვიკვლიოთ სასაზღვრო ამოცანა ბი–ლაპლას–ბელტრამის $\Delta^2_{\mathscr{C}}\varphi = f$ განტოლებისთვის, როგორც ჩვეულებრივი, ასევე შერეული სასაზღვრო პირობებით გლუვ $\mathscr{C}$ ჰიპერზედაპირზე, რომლის საზღვარია $\Gamma = \partial\mathscr{C}$. მოცემული ამოცანის ამოხსნადობა და ამონახსნის ერთადერთობა დამტკიცებულია გრინის ფორმულისა და ლაქს–მილგრამის ლემის საშუალებით.

აგრეთვე დამტკიცებულია $\Delta^2_{\mathscr{C}} + \mathscr{H}\,I \ : \ \mathbb{H}^{s+2}_p(\mathscr{S}) \to \mathbb{H}^{s-2}_p(\mathscr{S})$ შეშფოთებული ოპერატორის შებრუნებადობა ბესელის პოტენციალთა სივრცეებში ჩაკეტილი გლუვი ჰიპერზედაპირისთვის $\mathscr{S}$ საზღვრის გარეშე, $1 < p < \infty$ და $-\infty < s < \infty$ პარამეტრებისთვის. ასევე დამტკიცებულია, რომ $\mathscr{H}$ არის გლუვი ფუნქცია, აქვს არაუარყოფითი $\operatorname{Re}\mathscr{H}(t) \geqslant 0$ ნამდვილი ნაწილი ყველა $t \in \mathscr{S}$-სთვის და $\operatorname{mes}\operatorname{supp}\operatorname{Re}\mathscr{H} \neq 0$.

1 Introduction

Let $\mathscr{C} \subset \mathscr{S}$ be a smooth subsurface of a closed hypersurface $\mathscr{S}$ in the Euclidean space $\mathbb{R}^n$ (see Section 2 for details) and $\Gamma = \partial\mathscr{C} \neq \varnothing$ be its smooth boundary. Let $\mathscr{D}_j := \partial_j - \nu_j \partial_{\boldsymbol{\nu}}$, $j = 1, \dots, n$, be Günter's tangential derivatives, and $\boldsymbol{\Delta}^2 := \sum\limits_{j,k=1}^{n} \mathscr{D}_j^2 \mathscr{D}_k^2$ be the bi-Laplace–Beltrami operator restricted to the surface $\mathscr{C}$.

The purpose of the present paper is to investigate the boundary value problems (BVPs) for the bi-Laplace–Beltrami equation

$$\begin{cases} \Delta^2_{\mathscr{C}} u(t) = f(t), & t \in \mathscr{C}, \\ (B_0 u)^+(s) = g(s), & \text{on } \Gamma, \\ (B_1 u)^+(s) = h(s), & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where the boundary operators can be chosen as follows:

$$\begin{gathered} B_0 = I \ \text{ and } \ B_1 = \partial_{\boldsymbol{\nu}_\Gamma}, \ \text{ or } \ B_1 = \Delta_{\mathscr{C}}, \\ B_0 = \partial_{\boldsymbol{\nu}_\Gamma} \ \text{ and } \ B_1 = \Delta_{\mathscr{C}}, \ \text{ or } \ B_1 = \partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}}. \end{gathered} \tag{1.2}$$

The BVP

$$\begin{cases} \Delta^2_{\mathscr{C}} u(t) = f(t), & t \in \mathscr{C}, \\ u^+(\tau) = 0, \quad (\partial_{\boldsymbol{\nu}_\Gamma} u)^+(\tau) = 0, & \tau \in \Gamma, \end{cases}$$

is called **a clamped surface equation** and is considered in the weak classical setting

$$u \in \mathbb{H}^2(\mathscr{C}), \ \ f \in \widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C}).$$

The BVP

$$\begin{cases} \Delta^2_{\mathscr{C}} u(t) = f(t), & t \in \mathscr{C}, \\ u^+(\tau) = g(\tau), \quad (\Delta_{\mathscr{C}} u)^+ + a\partial_{\boldsymbol{\nu}_\Gamma} u)^+(\tau) = h(\tau), & \tau \in \Gamma, \end{cases}$$

with **Steklov Boundary Conditions** is considered in the weak classical setting

$$u \in \mathbb{H}^2(\mathscr{C}), \ \ f \in \widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C}), \ \ g \in \mathbb{H}^{3/2}(\Gamma), \ \ h \in \mathbb{H}^{-3/2}(\Gamma).$$

Here a is a real-valued constant.

The BVP

$$\begin{cases} \Delta^2_{\mathscr{C}} u(t) = f(t), & t \in \mathscr{C}, \\ u^+(\tau) = g(\tau), \quad (\Delta_{\mathscr{C}} u)^+ = h(\tau), & \tau \in \Gamma \end{cases}$$

with **Navier Boundary Conditions** is considered in the weak classical setting

$$u \in \mathbb{H}^2(\mathscr{C}), \ \ f \in \widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C}), \ \ g \in \mathbb{H}^{3/2}(\Gamma), \ \ h \in \mathbb{H}^{-1/2}(\Gamma).$$

First we consider in detail the case

$$\begin{cases} \Delta^2 u(t) = f(t), & t \in \mathscr{C}, \\ (\partial_{\boldsymbol{\nu}_\Gamma} u)^+(s) = g(s), & \text{on } \Gamma, \\ (\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} u)^+(s) = h(s), & \text{on } \Gamma, \end{cases} \tag{1.3}$$

in the weak classical setting

$$u \in \mathbb{H}^2(\mathscr{C}), \ \ f \in \widetilde{\mathbb{H}}^{-2}_\Gamma\mathscr{C}), \ \ g \in \mathbb{H}^{1/2}(\Gamma), \ \ h \in \mathbb{H}^{-3/2}(\Gamma), \tag{1.4}$$

where

$$\widetilde{\mathbb{H}}^{-2}_\Gamma(\Omega) := \Big\{ f \in \widetilde{\mathbb{H}}^{-2}(\Omega) \ | \ \ (f, \varphi)_{\mathbb{L}^2(\Omega)} = 0, \ \ \varphi \in \mathbb{C}^\infty_0(\Omega) \Big\}. \tag{1.5}$$

Remark 1.1. Let us comment on the condition $f \in \widetilde{\mathbb{H}}_{\Gamma}^{-2}(\mathscr{C})$ in (1.4).

As is shown in [13, p. 196], the condition $f \in \widetilde{\mathbb{H}}^{-2}(\mathscr{C})$ does not ensure the uniqueness of a solution to the BVP (1.3). The right-hand side f needs additional constraint that it belongs to the subspace $\widetilde{\mathbb{H}}_0^{-2}(\Omega) \subset \widetilde{\mathbb{H}}^{-2}(\Omega)$ which is the orthogonal complement to the subspace $\widetilde{\mathbb{H}}_{\Gamma}^{-2}(\Omega)$ of those distributions from $\widetilde{\mathbb{H}}^{-2}(\Omega)$ which are supported only on the boundary $\Gamma = \partial\Omega$ of the domain (see (1.5)).

Another cases in (1.2) are considered analogously.

We will prove the unique solvability of the BVP (1.3) in the classical setting (1.4) by applying the Lax–Milgram Lemma.

We also consider the following BVP with the mixed boundary conditions: Let $\mathscr{C} \subset \mathscr{S}$ be a smooth subsurface of a closed hypersurface $\mathscr{S}$ in the Euclidean space $\mathbb{R}^n$ (see Section 2 for details) and its smooth boundary $\partial\mathscr{C} = \Gamma = \Gamma_1 \cup \Gamma_2 \neq \varnothing$ be decomposed into two non-intersecting connected parts. Consider the mixed BVP for the bi-Laplase-Beltrami equation

$$\begin{cases} \Delta^2 u(t) = f(t), & t \in \mathscr{C}, \\ (u)^+(s) = g_1(s), & \text{on } \Gamma_1, \\ (\partial_{\boldsymbol{\nu}_\Gamma} u)^+(s) = g_2(s), & \text{on } \Gamma_2, \\ (\Delta_{\mathscr{C}} u)^+(s) = h_1(s), & \text{on } \Gamma_1, \\ (\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} u)^+(s) = h_2(s), & \text{on } \Gamma_2, \end{cases} \tag{1.6}$$

in the weak classical setting

$$u \in \mathbb{H}^2(\mathscr{C}), \quad f \in \widetilde{\mathbb{H}}_{\Gamma}^{-2}(\mathscr{C}), \quad g_1 \in \mathbb{H}^{3/2}(\Gamma_1), \quad g_2 \in \mathbb{H}^{1/2}(\Gamma_2), \quad h_1 \in \mathbb{H}^{-1/2}(\Gamma_1), \quad h_2 \in \mathbb{H}^{-3/2}(\Gamma_2). \tag{1.7}$$

The following are the main theorems of the present paper. The proofs are exposed in Sections 3 and 4, below.

Prior formulating the theorems let us introduce the Hilbert spaces with detached constants $\mathbb{H}^2_{\#}(\mathscr{S}) := \mathbb{H}^2(\mathscr{S}) \setminus \{const\}$. Another description of the space $\mathbb{H}^2_{\#}(\mathscr{S})$ is that it consists of all functions $\varphi \in \mathbb{H}^2(\mathscr{S})$, which have the zero mean value, $(\varphi, 1)_{\mathscr{S}} = 0$.

Theorem 1.1. *The boundary value problem* (1.3) *in the weak classical setting* (1.4) *has a unique solution in the space* $\mathbb{H}^2_{\#}(\mathscr{C})$.

Theorem 1.2. *The mixed type boundary value problem* (1.6) *in the weak classical setting* (1.7) *has a unique solution in the space* $\mathbb{H}^2_{\#}(\mathscr{C})$.

The Bi-Laplace–Beltrami operator $\Delta^2 = \Delta \times \Delta$ is a model of a fourth-order operator. The BVPs on hypersurfaces arise in a variety of situations and have many practical applications. They appear in various problems of linear elasticity, for example, when looking for small displacements of a plate, whereas the Laplacian describes the behavior of a membrane.

A hypersurface $\mathscr{S}$ in $\mathbb{R}^n$ has the natural structure of an $(n-1)$-dimensional Riemannian manifold and the aforementioned partial differential equations (PDEs) are not the immediate analogues of the ones corresponding to the flat, Euclidean case, since they have to take into consideration geometric characteristics of $\mathscr{S}$ such as curvature. Inherently, these PDEs are originally written in local coordinates, intrinsic to the manifold structure of $\mathscr{S}$.

Another problem we encounter in considering BVPs (1.1) is the existence of a fundamental solution for the bi-Laplace–Beltrami operator. An essential difference between the differential operators on hypersurfaces and the Euclidean space $\mathbb{R}^n$ lies in the existence of the fundamental solution: In $\mathbb{R}^n$, a fundamental solution exists for all partial differential operators with constant coefficients if it is not trivially zero. On a hypersurface, the bi-Laplace–Beltrami operator has no fundamental solution because it has a non-trivial kernel, constants, in all Bessel potential spaces. Therefore we consider the bi-Laplace–Beltrami operator in the Hilbert spaces with detached constants $\Delta^2_{\mathscr{C}} : \mathbb{H}^2_{\#}(\mathscr{S}) \to \mathbb{H}^{-2}(\mathscr{S})$ and prove that it is an invertible operator. The established invertibility implies the existence of a certain fundamental solution, which can be used to define the volume (Newtonian), single layer and double layer potentials.

2 Auxiliary material

We commence with the definition of a hypersurface. There exist other equivalent definitions, but they are most convenient for us. Equivalence of these definitions and some other properties of hypersurfaces are discussed, e.g., in [7,8].

Definition 2.1. A subset $\mathscr{S} \subset \mathbb{R}^n$ of the Euclidean space is said to be a **hypersurface** if it has a covering $\mathscr{S} = \bigcup\limits_{j=1}^{M} \mathscr{S}_j$ and coordinate mappings

$$\Theta_j : \omega_j \longrightarrow \mathscr{S}_j := \Theta_j(\omega_j) \subset \mathbb{R}^n, \;\; \omega_j \subset \mathbb{R}^{n-1}, \;\; j = 1, \dots, M, \tag{2.1}$$

such that the corresponding differentials

$$D\Theta_j(p) := \operatorname{matr}\left[\partial_1\Theta_j(p), \dots, \partial_{n-1}\Theta_j(p)\right]$$

have the full rank

$$\operatorname{rank} D\Theta_j(p) = n - 1, \;\; \forall\, p \in Y_j, \;\; k = 1, \dots, n, \;\; j = 1, \dots, M,$$

i.e., all points of ω_j are regular for Θ_j for all $j = 1, \dots, M$.

Such a mapping is called an **immersion** as well.

Here and in what follows, $\operatorname{matr}[x_1, \dots, x_k]$ refers to the matrix with the listed vectors $x_1, \dots, x_k$ as columns.

A hypersurface is called **smooth** if the corresponding coordinate diffeomorphisms Θ_j in (2.1) are smooth (C^∞-smooth). Similarly is defined a μ**-smooth** hypersurface.

The next definition of a hypersurface is called **implicit**.

Definition 2.2. Let $k \geqslant 1$ and $\omega \subset \mathbb{R}^n$ be a compact domain. An implicit C^k-smooth hypersurface in $\mathbb{R}^n$ is defined as the set

$$\mathscr{S} = \left\{\mathscr{X} \in \omega : \;\; \Psi_{\mathscr{S}}(\mathscr{X}) = 0\right\},$$

where $\Psi_{\mathscr{S}} : \omega \to \mathbb{R}$ is a C^k-mapping, which has the non-vanishing gradient $\nabla\,\Psi(\mathscr{X}) \neq 0$.

The most important role in the calculus of tangential differential operators that we are going to apply belongs to the unit normal vector field $\boldsymbol{\nu}(y)$, $t \in \mathscr{C}$. The **unit normal vector field** to the surface $\mathscr{C}$, known also as the **Gauß mapping**, is defined by the vector product of the covariant basis

$$\boldsymbol{\nu}(\mathscr{X}) := \pm\frac{\boldsymbol{g}_1(\mathscr{X}) \wedge \cdots \wedge \boldsymbol{g}_{n-1}(\mathscr{X})}{|\boldsymbol{g}_1(\mathscr{X}) \wedge \cdots \wedge \boldsymbol{g}_{n-1}(\mathscr{X})|}, \quad \mathscr{X} \in \mathscr{C}.$$

The system of tangential vectors $\{\boldsymbol{g}_k\}_{k=1}^{n-1}$ to $\mathscr{C}$ is, by the definition, linearly independent and is known as the **covariant basis**. There exists the unique system $\{\boldsymbol{g}^k\}_{k=1}^{n-1}$ biorthogonal to it, i.e., the **contravariant basis**

$$\langle \boldsymbol{g}_j, \boldsymbol{g}^k\rangle = \delta_{jk}, \;\; j, k = 1, \dots, n-1.$$

The contravariant basis is defined by the formula

$$\boldsymbol{g}^k = \frac{1}{\det G_{\mathscr{S}}}\, \boldsymbol{g}_1 \wedge \cdots \wedge \boldsymbol{g}_{k-1} \wedge \boldsymbol{\nu} \wedge \boldsymbol{g}_{k+1} \wedge \cdots \wedge \boldsymbol{g}_{n-1}, \;\; k = 1, \dots, n-1,$$

where

$$G_{\mathscr{S}}(\mathscr{X}) := \left[\left\langle \boldsymbol{g}_k(\mathscr{X}), \boldsymbol{g}_m(\mathscr{X})\right\rangle\right]_{n-1\times n-1}, \;\; p \in \mathscr{S},$$

is the **Gram matrix**.

Günter's derivatives are the simplest examples of tangential differential operators

$$\mathscr{D}_j := \partial_j - \nu_j\partial_\nu = \partial_j - \nu_j \sum_{k=1}^{n} \nu_k\partial_k.$$

The surface divergence $\mathbf{div}_{\mathscr{S}}$ and the surface gradient $\nabla_{\mathscr{S}}$ are defined as follows:

$$\mathbf{div}_{\mathscr{S}}\mathbf{U} = \sum_{k=1}^{n} \partial_k U_k, \quad \nabla_{\mathscr{S}}\varphi := (\mathscr{D}_1\varphi, \dots, \mathscr{D}_n\varphi_n)^\top, \quad \mathbf{U} := (U_1, \dots, U_n)^\top,$$

and the surface Laplace–Beltrami operator $\boldsymbol{\Delta}_{\mathscr{S}}$ is their superposition

$$\boldsymbol{\Delta}_{\mathscr{S}}\psi = \mathbf{div}_{\mathscr{S}}\nabla_{\mathscr{S}}\psi = \sum_{j=1}^{n} \mathscr{D}_j\psi. \tag{2.2}$$

In contrast to the classical differential geometry, the **surface gradient**, the **surface divergence** and the surface Laplace–Beltrami operator $\boldsymbol{\Delta}_{\mathscr{S}}$ are defined by Günter's derivatives much simpler, with the help of only normal vector field ν, while definitions in the classical differential geometry are based on the **Christoffel symbols** Γ^j_{km}, the covariant and the contravariant $G^{-1} := [g^{jk}]$ Riemann metric tensors and are rather complicated.

It is well known that $\mathbf{div}_{\mathscr{S}}$ is the negative dual to the surface gradient

$$\langle \mathbf{div}_{\mathscr{S}}\boldsymbol{V}, f\rangle := -\langle \boldsymbol{V}, \nabla_{\mathscr{S}} f\rangle, \ \ \forall\, \boldsymbol{V} \in \mathscr{V}(\mathscr{S}), \ \ \forall\, f \in C^1(\mathscr{S}).$$

Let $\mathscr{M}$ be a non-trivial, $\operatorname{mes}\mathscr{M} \neq \varnothing$, smooth closed hypersurface, $s \in \mathbb{R}$ and $1 < p < \infty$. For the definitions of Bessel's potential $\mathbb{H}^s_p(\mathscr{M})$ and Sobolev–Slobodeckii $\mathbb{W}^s_p(\mathscr{M})$ spaces for a closed smooth manifold $\mathscr{M}$ we refer to [22] (see also [6, 12, 13]). For $p = 2$, the Sobolev–Slobodetski $\mathbb{W}^s(\mathscr{M}) := \mathbb{W}^s_2(\mathscr{M})$ and the Bessel potential $\mathbb{H}^s(\mathscr{M}) := \mathbb{H}^s_2(\mathscr{M})$ spaces coincide (i.e., the norms are equivalent).

Let $\mathscr{C}$ be a subsurface of a smooth closed surface $\mathscr{M}$, $\mathscr{C} \subset \mathscr{M}$, with the smooth boundary $\Gamma := \partial\mathscr{C}$. The space $\widetilde{\mathbb{H}}^s_p(\mathscr{C})$ is defined as the subspace of those functions $\varphi \in \mathbb{H}^s_p(\mathscr{M})$, which are supported in the closure of the subsurface, $\operatorname{supp}\varphi \subset \overline{\mathscr{C}}$, whereas $\mathbb{H}^s_p(\mathscr{C})$ denotes the quotient space $\mathbb{H}^s_p(\mathscr{C}) = \mathbb{H}^s_p(\mathscr{M})/\widetilde{\mathbb{H}}^s_p(\mathscr{C}^c)$ and $\mathscr{C}^c := \mathscr{M} \setminus \overline{\mathscr{C}}$ is the complementary subsurface to $\mathscr{C}$. The space $\mathbb{H}^s_p(\mathscr{C})$ can be identified with the space of distributions φ on $\mathscr{C}$ which have an extension to a distribution $\ell\varphi \in \mathbb{H}^s_p(\mathscr{M})$. Therefore $r_{\mathscr{C}}\mathbb{H}^s_p(\mathscr{M}) = \mathbb{H}^s_p(\mathscr{C})$, where $r_{\mathscr{C}}$ denotes the restriction operator of functions (distributions) from the surface $\mathscr{M}$ to the subsurface $\mathscr{C}$.

The spaces $\widetilde{\mathbb{W}}^s_p(\mathscr{C})$ and $\mathbb{W}^s_p(\mathscr{C})$ are defined similarly (see [22] and also [6, 12, 13]).

By $\mathbb{X}^s_p(\mathscr{C})$ we denote one of the spaces: $\mathbb{H}^s_p(\mathscr{C})$ or $\mathbb{W}^s_p(\mathscr{C})$, and by $\widetilde{\mathbb{X}}^s_p(\mathscr{C})$ one of the spaces: $\widetilde{\mathbb{H}}^s_p(\mathscr{C})$ and $\widetilde{\mathbb{W}}^s_p(\mathscr{C})$ (if $\mathscr{C}$ is open).

The bi-Laplace–Beltrami operator has the finite dimensional kernel $\dim\operatorname{Ker}\Delta_{\mathscr{C}} \leq \infty$, and its kernel consists only of constants. Hence the space $\mathbb{X}^s(\mathscr{C})$ decomposes into the direct sum

$$\mathbb{X}^s_p(\mathscr{C}) = \mathbb{X}^s_{p,\#}(\mathscr{C}) + \{const\},$$

where

$$\mathbb{X}^s_{p,\#}(\mathscr{C}) := \big\{\varphi \in \mathbb{X}^s_2(\mathscr{C}) : \ (\varphi, 1) = 0\big\} \tag{2.3}$$

is the space without constants.

Lemma 2.1. *The bi-Laplace–Beltrami operator* $\Delta^2_{\mathscr{S}}\varphi := (\mathbf{div}_{\mathscr{S}}\nabla_{\mathscr{S}})^2\varphi : \mathbb{H}^2(\mathscr{S}) \to \mathbb{H}^{-2}(\mathscr{S})$ *is elliptic, self-adjoint* $(\Delta^2_{\mathscr{S}})^* = \Delta^2_{\mathscr{S}}$*, non-negative*

$$(\Delta^2_{\mathscr{S}}\varphi, \varphi) = (\Delta_{\mathscr{S}}\varphi, \Delta_{\mathscr{S}}\varphi) = \big\|\Delta_{\mathscr{S}}\varphi\big|\mathbb{L}_2(\mathscr{S})\big\|^2 \geq 0, \ \ \varphi \in \mathbb{H}^2(\mathscr{S})$$

and the homogenous equation has only a constant solution

$$(\Delta^2_{\mathscr{S}}\varphi, \varphi) = 0, \ \ \textit{only for} \ \ \varphi = const. \tag{2.4}$$

Proof. $\Delta^2_{\mathscr{S}}$ is elliptic and self-adjoint since $\Delta_{\mathscr{S}}$ is elliptic and self-adjoint (see [7]).

Due to (2.2) and (2.4), we get

$$0 = (\Delta^2_{\mathscr{S}}\varphi, \varphi) = (\Delta_{\mathscr{S}}\varphi, \Delta_{\mathscr{S}}\varphi) = \big\|\Delta_{\mathscr{S}}\varphi\big|\mathbb{L}_2(\mathscr{S})\big\|^2$$

which gives $\Delta_{\mathscr{S}}\varphi = 0$ and, consequently, $\varphi = const$ (see [7]).

Corollary 2.1. *The space $\mathbb{X}^s(\mathscr{C})$ decomposes into the direct sum*

$$\mathbb{X}^s(\mathscr{C}) = \mathbb{X}^s_{\#}(\mathscr{C}) + \{const\},$$

where $\mathbb{X}^s_{\#}(\mathscr{C})$ is the space with detached constants and the operator $\Delta^2_{\mathscr{S}}$ is invertible between the spaces with detached constants (see (2.3)*)*

$$\Delta^2_{\mathscr{S}} : \mathbb{X}^{s+2}_{\#}(\mathscr{S}) \longrightarrow \mathbb{X}^{s-2}_{\#}(\mathscr{S}). \tag{2.5}$$

Therefore $\Delta^2_{\mathscr{S}}$ has the fundamental solution in the setting (2.5)*.*

Proof. The boundedness in (2.5) follows from that of the operator

$$\Delta_{\mathscr{S}} : \mathbb{X}^{s+1}_{\#}(\mathscr{S}) \longrightarrow \mathbb{X}^{s-1}_{\#}(\mathscr{S})$$

proved in [10].

Since $\Delta^2_{\mathscr{S}}$ has the trivial kernel in the setting (2.5) and is self-adjoint (see the foregoing Lemma 2.1), it has the trivial co-kernel as well and is invertible. □

Corollary 2.2. *For the bi-Laplace–Beltrami operator $\Delta^2_{\mathscr{C}}$ on the open hypersurface $\mathscr{C}$ the following* I *and* II *Green's formulae are valid:*

$$(\Delta^2_{\mathscr{C}}\varphi,\psi)_{\mathscr{C}} - (\Delta_{\mathscr{C}}\varphi,\Delta_{\mathscr{C}}\psi)_{\mathscr{C}} = -((\partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}}\varphi)^+,\psi^+)_\Gamma + ((\Delta_{\mathscr{C}}\varphi)^+,(\partial_{\boldsymbol{\nu}_\Gamma}\psi)^+)_\Gamma, \tag{2.6}$$

$$\begin{aligned}(\Delta^2_{\mathscr{C}}\varphi,\psi)_{\mathscr{C}} + ((\partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}}\varphi)^+,\psi^+)_\Gamma - ((\Delta_{\mathscr{C}}\varphi)^+,(\partial_{\boldsymbol{\nu}_\Gamma}\psi)^+)_\Gamma \\ = (\varphi,\Delta^2_{\mathscr{C}}\psi)_{\mathscr{C}} + (\varphi^+,(\partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}}\psi)^+)_\Gamma - ((\partial_{\boldsymbol{\nu}_\Gamma}\varphi)^+,(\Delta_{\mathscr{C}}\psi)^+)_\Gamma\end{aligned}$$

for arbitrary $\varphi,\psi \in \mathbb{X}^2(\mathscr{C})$ (see [4]*).*

Lemma 2.2 (see [14] (Lax–Milgram))**.** *Let $\mathfrak{B}$ be a Banach space, $A(\varphi,\psi)$ be a continuous, bilinear form*

$$A(\,\cdot\,,\,\cdot\,) : \mathfrak{B} \times \mathfrak{B} \longrightarrow \mathbb{R}$$

and positive definite

$$A(\varphi,\varphi) \geq C\big\|\varphi|\mathfrak{B}\big\|^2, \ \ \forall\,\varphi \in \mathfrak{B}, \ \ C > 0.$$

Let $L(\,\cdot\,) : \mathfrak{B} \to \mathbb{R}$ be a continuous linear functional.

A linear equation

$$A(\varphi,\psi) = L(\psi)$$

has a unique solution $\varphi \in \mathfrak{B}$ for an arbitrary $\psi \in \mathfrak{B}$.

3 The solvability of BVPs for the bi-Laplace–Beltrami equation

Let again $\mathscr{C} \subset \mathscr{S}$ be a smooth subsurface of a closed hypersurface $\mathscr{S}$ and $\Gamma = \partial\mathscr{C} \neq \varnothing$ be its smooth boundary.

To prove the forthcoming theorem about the unique solvability we will need more properties of the trace operators (called retractions) and their inverses, called co-retractions (see [22, § 2.7]).

To keep the exposition simpler we recall a very particular case of Lemma 4.8 from [6], which we apply to the present investigation.

Lemma 3.1. *Let $s > 0$, $s \notin \mathbb{N}$, $p = 2$, $\boldsymbol{B}(D)$ be a normal differential operator of the third order defined in the vicinity of the boundary $\Gamma = \partial\mathscr{C}$ and $\boldsymbol{A}(D)$ be a normal differential operator of the fourth order defined on the surface $\mathscr{C}$. Then there exists a continuous linear operator*

$$\mathscr{B} : \mathbb{H}^s(\Gamma) \otimes \mathbb{H}^{s-1}(\Gamma) \otimes \mathbb{H}^{s-2}(\Gamma) \otimes \mathbb{H}^{s-3}(\Gamma) \longrightarrow \mathbb{H}^{s+\frac{1}{2}}(\mathscr{C})$$

such that

$$(\mathscr{B}\Phi)^+ = \varphi_0, \quad (\boldsymbol{B}_1(D)\mathscr{B}\Phi)^+ = \varphi_1, \quad (\boldsymbol{B}_2(D)\mathscr{B}\Phi)^+ = \varphi_2,$$
$$(\boldsymbol{B}_3(D)\mathscr{B}\Phi)^+ = \varphi_3, \quad \boldsymbol{A}(D)\mathscr{B}\Phi \in \widetilde{\mathbb{H}}^{s-4+\frac{1}{2}}(\mathscr{C})$$

for an arbitrary quadruple of the functions $\Phi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3)^\top$*, where* $\varphi_0 \in \mathbb{H}^s(\Gamma)$*,* $\varphi_1 \in \mathbb{H}^{s-1}(\Gamma)$*,* $\varphi_2 \in \mathbb{H}^{s-2}(\Gamma)$ *and* $\varphi_3 \in \mathbb{H}^{s-3}(\Gamma)$*.*

Corollary 3.1. *Let* u *be a solution of the equation* $\Delta^2_{\mathscr{C}} u = f$*. Then it has the traces* $u^+ \in \mathbb{H}^{\frac{3}{2}}$*,* $(\partial_{\boldsymbol{\nu}_\Gamma} u)^+ \in \mathbb{H}^{\frac{1}{2}}$*,* $(\Delta_{\mathscr{C}} u)^+ \in \mathbb{H}^{-\frac{1}{2}}$*,* $(\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} u)^+ \in \mathbb{H}^{-\frac{3}{2}}$*.*

Proof. The existence of the traces $u^+ \in \mathbb{H}^{\frac{3}{2}}$, $(\partial_{\boldsymbol{\nu}_\Gamma} u)^+ \in \mathbb{H}^{\frac{1}{2}}$ is a direct consequence of the general trace theorem (see [22] for details). Let us prove the existence of the rest traces. Concerning the existence of the trace $(\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} \varphi)^+$ in (1.3) for a solution $u \in \mathbb{H}^2(\mathscr{C})$ is not guaranteed by the general trace theorem, but, according to Lemma 3.1, there exists a function $\psi \in \mathbb{H}^2(\mathscr{C})$ such that $(\partial_{\boldsymbol{\nu}_\Gamma} \psi)^+ = 0$. Then the first Green's formula (2.6) ensures the existence of the trace. Indeed, by setting $\varphi = u$ and inserting the data $\Delta^2_{\mathscr{C}} \varphi = f(t)$ into the first Green's formula (2.6), we get

$$-((\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} u)^+, \psi^+)_\Gamma = (f, \psi)_{\mathscr{C}} - (\Delta_{\mathscr{C}} u, \Delta_{\mathscr{C}} \psi)_{\mathscr{C}}. \tag{3.1}$$

The scalar product $(\Delta_{\mathscr{C}} u, \Delta_{\mathscr{C}} \psi)$ in the right-hand side of equality (3.1) is correctly defined and defines correct duality in the left-hand side of the equality. Since $\psi^+ \in \mathbb{H}^{3/2}(\Gamma)$ is arbitrary, by the duality argument this implies that $(\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} u)^+$ should be in the dual space, i.e., in $\mathbb{H}^{-3/2}(\Gamma)$.

Let us now prove the existence of the trace $(\Delta_{\mathscr{C}} \varphi)^+$. Taking an arbitrary $\psi \in \mathbb{H}^2(\mathscr{C})$ and rewriting the first Green's formula (2.6) in the form

$$((\Delta_{\mathscr{C}} u)^+, (\partial_{\boldsymbol{\nu}_\Gamma} \psi)^+)_\Gamma = (f, \psi)_{\mathscr{C}} - (\Delta_{\mathscr{C}} u, \Delta_{\mathscr{C}} \psi)_{\mathscr{C}} + ((\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} u)^+, \psi^+)_\Gamma, \tag{3.2}$$

we note that the right-hand side of equality (3.2) is correctly determined and defines correct duality in the left-hand side. Since $(\partial_{\boldsymbol{\nu}_\Gamma} \psi)^+ \in \mathbb{H}^{1/2}(\Gamma)$ is arbitrary, by the duality argument this implies that $(\Delta_{\mathscr{C}} u)^+$ should be in the dual space, i.e., in $\mathbb{H}^{-1/2}(\Gamma)$. □

Proof of Theorem 1.1. We commence with the reduction of the BVP (1.3) to an equivalent one with the homogeneous condition and apply Lemma 3.1: there exists a function $\Phi \in \mathbb{H}^2(\mathscr{C})$ such that $(\partial_{\boldsymbol{\nu}_\Gamma} \Phi)^+(t) = g(t)$ for $t \in \Gamma$ and $\Delta^2_{\mathscr{C}} \Phi \in \widetilde{\mathbb{H}}^{-2}_0(\overline{\mathscr{C}})$.

For a new unknown function $v := u - \Phi$ we have the following equivalent reformulation of the BVP (1.3):

$$\begin{cases} \Delta^2_{\mathscr{C}} v(t) = f_0(t), & t \in \mathscr{C}, \\ (\partial_{\boldsymbol{\nu}_\Gamma} v)^+(s) = 0, & \text{on } \Gamma, \\ (\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} v)^+(s)\rangle = h_0(s), & \text{on } \Gamma, \end{cases} \tag{3.3}$$

where

$$f_0 := f + \Delta^2_{\mathscr{C}} \Phi \in \widetilde{\mathbb{H}}^{-2}_0(\mathscr{C}), \quad h_0 := h + (\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} \Phi)^+ \in \mathbb{H}^{-3/2}(\Gamma), \quad v^+ \in \widetilde{\mathbb{H}}^{3/2}(\Gamma).$$

By inserting the data from the reformulated boundary value problem (3.3) into the first Green's identity (2.6), where $\varphi = \psi = v$, we get

$$(\Delta_{\mathscr{C}} v, \Delta_{\mathscr{C}} v)_{\mathscr{C}} = (\Delta^2_{\mathscr{C}} v, v)_{\mathscr{C}} + ((\partial_{\boldsymbol{\nu}_\Gamma} \Delta_{\mathscr{C}} v)^+, v^+)_\Gamma - ((\Delta_{\mathscr{C}} v)^+, (\partial_{\boldsymbol{\nu}_\Gamma} v)^+)_\Gamma = (f_0, v)_{\mathscr{C}} + (h_0, v^+)_\Gamma. \tag{3.4}$$

In the left-hand side of equality (3.4) we have a symmetric bilinear form, which is positive definite:

$$(\Delta_{\mathscr{S}} \varphi, \Delta_{\mathscr{S}} \varphi) = \big\| \Delta_{\mathscr{S}} \varphi \mid \mathbb{L}_2(\mathscr{S}) \big\|^2 \geq 0, \ \ \varphi \in \mathbb{H}^2_\#(\mathscr{S}).$$

$(h_0, v^+)_\Gamma$ and $(f_0, v)_{\mathscr{C}}$ from equality (3.4) are the correctly defined continuous functionals, since $h_0 \in \mathbb{H}^{-3/2}(\Gamma)$, $f_0 \in \widetilde{\mathbb{H}}^{-2}(\mathscr{C})$, while their counterparts in the functional belong to the dual spaces $v^+ \in \widetilde{\mathbb{H}}^{3/2}(\Gamma)$ and $v \in \widetilde{\mathbb{H}}^2(\Gamma, \mathscr{C}) \subset \mathbb{H}^2(\mathscr{C})$.

The Lax–Milgram Lemma 2.2 accomplishes the proof. □

4 The solvability of mixed BVPs for the bi-Laplace–Beltrami equation

Proof of Theorem 1.2. We commence with the reduction of the BVP (1.6) to an equivalent one with the homogeneous conditions. Towards this end, we extend the boundary data $g_1 \in \mathbb{H}^{3/2}(\Gamma_1)$, $g_2 \in \mathbb{H}^{1/2}(\Gamma_2)$ and $h_1 \in \mathbb{H}^{-1/2}(\Gamma_1)$ up to some functions $\widetilde{g_1} \in \mathbb{H}^{3/2}(\Gamma)$, $\widetilde{g_2} \in \mathbb{H}^{1/2}(\Gamma)$ and $\widetilde{h_1} \in \mathbb{H}^{-1/2}(\Gamma)$ on the entire boundary Γ and apply Lemma 3.1: there exists a function $\Phi \in \mathbb{H}^2(\mathscr{C})$ such that

$$\Phi^+ = \widetilde{g_1}, \ \ (\partial_{\boldsymbol{\nu}_\Gamma}\Phi)^+ = \widetilde{g_2}, \ \ (\Delta_{\mathscr{C}}\Phi)^+ = h_1, \ \text{ and } \ \Delta^2_{\mathscr{C}}\Phi \in \widetilde{\mathbb{H}}^{-2}_0(\overline{\mathscr{C}}).$$

For a new unknown function $v := u - \Phi$ we have the following equivalent reformulation of the BVP (1.6):

$$\begin{cases} \Delta^2 v(t) = f_0(t), & t \in \mathscr{C}, \\ (v)^+(s) = 0, & \text{on } \Gamma_1, \\ (\partial_{\boldsymbol{\nu}_\Gamma} v)^+(s) = 0, & \text{on } \Gamma_2, \\ (\Delta_{\mathscr{C}} v)^+(s) = 0, & \text{on } \Gamma_1, \\ (\partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}} v)^+(s) = h_0(s), & \text{on } \Gamma_2, \end{cases} \tag{4.1}$$

where

$$\begin{gathered} f_0 := f + \Delta^2_{\mathscr{C}}\Phi \in \widetilde{\mathbb{H}}^{-2}_0(\mathscr{C}), \quad h_0 := h_2 + (\partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}}\Phi)^+ \in \mathbb{H}^{-3/2}(\Gamma_2), \\ v^+ \in \widetilde{\mathbb{H}}^{3/2}(\Gamma_2), \quad (\partial_{\boldsymbol{\nu}_\Gamma} v)^+ \in \widetilde{\mathbb{H}}^{1/2}(\Gamma_1), \quad (\Delta_{\mathscr{C}} v)^+ \in \widetilde{\mathbb{H}}^{-1/2}(\Gamma_2) \end{gathered} \tag{4.2}$$

To justify the last inclusion $v^+ \in \widetilde{\mathbb{H}}^{3/2}(\Gamma_2)$, $(\partial_{\boldsymbol{\nu}_\Gamma} v)^+ \in \widetilde{\mathbb{H}}^{1/2}(\Gamma_1)$ and $(\Delta_{\mathscr{C}} v)^+ \in \widetilde{\mathbb{H}}^{-1/2}(\Gamma_2)$, note that, due to our construction, the traces of a solution vanish: $v^+ \,|_{\Gamma_1} = 0$, $(\partial_{\boldsymbol{\nu}_\Gamma} v)^+ \,|_{\Gamma_2} = 0$ and $(\Delta_{\mathscr{C}} v)^+ \,|_{\Gamma_1} = 0$. By inserting the data from the reformulated boundary value problem (4.1) into the first Green's identity (2.6), where $\varphi = \psi = v$, we get

$$\begin{aligned} (\Delta_{\mathscr{C}} v, \Delta_{\mathscr{C}} v)_{\mathscr{C}} = (\Delta^2_{\mathscr{C}} v, v)_{\mathscr{C}} + ((\partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}} v)^+, v^+)_{\Gamma_1} + ((\partial_{\boldsymbol{\nu}_\Gamma}\Delta_{\mathscr{C}} v)^+, v^+)_{\Gamma_2} \\ - ((\Delta_{\mathscr{C}} v)^+, (\partial_{\boldsymbol{\nu}_\Gamma} v)^+)_{\Gamma_1} - ((\Delta_{\mathscr{C}} v)^+, (\partial_{\boldsymbol{\nu}_\Gamma} v)^+)_{\Gamma_2} = (f_0, v)_{\mathscr{C}} + (h_0, v^+)_{\Gamma_2} \end{aligned} \tag{4.3}$$

In the left-hand side of equality (4.3) we have a symmetric bilinear form, which is positive definite:

$$(\Delta_{\mathscr{S}}\varphi, \Delta_{\mathscr{S}}\varphi) = \|\Delta_{\mathscr{S}}\varphi \mid \mathbb{L}_2(\mathscr{S})\|^2 \geq 0, \ \ \varphi \in \mathbb{H}^2_{\#}(\mathscr{S}),$$

$(h_0, v^+)_{\Gamma_2}$ and $(f_0, v)_{\mathscr{C}}$ from equality (4.3) are the correctly defined continuous functionals, since $h_0 \in \mathbb{H}^{-3/2}(\Gamma_2)$, $f_0 \in \widetilde{\mathbb{H}}^{-2}(\mathscr{C})$, while their counterparts in the functional belong to the dual spaces $v^+ \in \widetilde{\mathbb{H}}^{3/2}(\Gamma_2)$ and $v \in \widetilde{\mathbb{H}}^2(\Gamma, \mathscr{C}) \subset \mathbb{H}^2(\mathscr{C})$.

The Lax–Milgram Lemma 2.2 accomplishes the proof. □

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Short Communication

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ON THE WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR SYSTEMS OF LINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. The modified criterion of the Opial type condition is given for the well-posedness of the Cauchy problem for the systems of linear generalized ordinary differential equations. Moreover, there are established the sufficient conditions guaranteeing the nearness of the left and right limits of the solutions of the perturbed problems to the left and right limits of the solution of the limit problem, respectively.

რეზიუმე. განზოგადებულ ჩვეულებრივ წრფივ დიფერენციალურ განტოლებათა სისტემებისთვის მოცემულია კოშის ამოცანის კორექტულობის ოპიალის ტიპის პირობის მოდიფიცირებული კრიტერიუმი. გარდა ამისა, დადგენილია საკმარისი პირობები, რომლებიც უზრუნველყოფს შეშფოთებული ამოცანების ამონახსნების მარჯვენა და მარცხენა ზღვრების სიახლოვეს ზღვრული ამოცანის ამონახსნის მარჯვენა და მარცხენა ზღვრებთან, შესაბამისად.

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Let $A_0 \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $f_0 \in BV_{loc}(I;\mathbb{R}^n)$ and $t_0 \in I$, where $I \subset \mathbb{R}$ is an arbitrary interval, non-degenerated at the point. Consider the system

$$dx = dA_0(t)\cdot x + df_0(t) \ \text{ for } \ t \in I \tag{1}$$

under the Cauchy condition

$$x(t_0) = c_0, \tag{2}$$

where $c_0 \in \mathbb{R}^n$ is an arbitrary constant vector.

Let x_0 be a unique solution of problem $(1),(2)$.

Along with the Cauchy problem $(1),(2)$, consider the sequence of the Cauchy problems

$$dx = dA_k(t)\cdot x + df_k(t), \tag{1_k}$$

$$x(t_k) = c_k \tag{2_k}$$

$(k=1,2,\dots)$, where $A_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $f_k \in BV_{loc}(I;\mathbb{R}^n)$, $t_k \in I$ and $c_k \in \mathbb{R}^n$ $(k=1,2,\dots)$.

Without loss of generality, we assume that either **(a)** $t_k < t_0$ $(k=1,2,\dots)$, or **(b)** $t_k > t_0$ $(k=1,2,\dots)$, or **(c)** $t_k = t_0$ $(k=1,2,\dots)$.

In the paper we establish:

1. the sufficient conditions for the Cauchy problem $(1_k),(2_k)$ to have a unique solution x_k for any sufficiently large k and

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \|x_k(t) - x_0(t)\| = 0 \tag{3}$$

in the case, where

$$\lim_{k\to+\infty} c_{kj} = c_{0j} \ \text{ if } \ j\in\{1,2\} \ \text{ is such that } \ (-1)^j(t_k-t_0)\geq 0 \ \ (k=0,1,\dots), \tag{3_j}$$

where

$$\begin{aligned} c_{k1} &= x_k(t_k-) = c_k - \big(d_1A_k(t_k)\cdot c_k + d_1f_k(t_k)\big), \\ c_{k2} &= x_k(t_k+) = c_k + \big(d_2A_k(t_k)\cdot c_k + d_2f_k(t_k)\big) \end{aligned} \quad (j=1,2;\ \ k=0,1,\dots); \tag{4}$$

2. the sufficient conditions for the Cauchy problem $(1_k),(2_k)$ to have a unique solution x_k for any sufficiently large k and

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \big\|x_k(t)-x_0(t)-x_{0j}(t)\big\| = 0 \tag{5}$$

in the case, where

$$\lim_{k\to+\infty} c_{kj} = c_{*j} \ \text{ if } \ j\in\{1,2\} \ \text{ is such that } \ (-1)^j(t_k-t_0)\geq 0 \ \ (k=0,1,\dots), \tag{5_j}$$

where c_{kj} $(j=1,2;\ k=1,2,\dots)$ are defined by (4), $c_{*j}\in\mathbb{R}^n$ $(j=1,2)$ are arbitrary vectors, differing from c_{0j} $(j=1,2)$, in general; the function x_{01} is a solution of the homogeneous system

$$dx = dA_0(t)\cdot x \tag{1_0}$$

on the set $\{t\in I:\ t<t_0\}$ under the condition

$$x_{01}(t_0-) = c_{*1} - x_0(t_0-),$$

and the function x_{02} is a solution of the homogeneous system (1_0) on the set $\{t\in I:\ t>t_0\}$ under the condition

$$x_{02}(t_0+) = c_{*2} - x_0(t_0+).$$

We note that the condition

$$\det\big(I_n + (-1)^j d_jA_0(t)\big) \neq 0 \ \text{ for } \ t\in I, \ \ (-1)^j(t-t_0)<0 \ \ (j=1,2)$$

guarantees the unique solvability of the Cauchy problem $(1),(2)$ for every $f_0\in BV_{loc}(I;\mathbb{R}^n)$ and $c_0\in\mathbb{R}^n$. Therefore, the vector functions x_{01} and x_{02} defined above exist and are uniquely defined.

In earlier works (see [3–5]) there are investigated the analogous question for the convergence in a general case, i.e., without any restrictions on the sequence t_k $(k=1,2,\dots)$, when

$$\lim_{k\to+\infty} x_k(t) = x_0(t) \quad \text{uniformly on I}, \tag{6}$$

under the condition

$$\lim_{k\to+\infty} c_k = c_0, \tag{7}$$

and some condition guaranteeing the equalities

$$\lim_{k\to+\infty} d_jA_k(t_k) = d_jA_0(t_0), \quad \lim_{k\to+\infty} d_jf_k(t_k) = d_jf_0(t_0) \ \ (j=1,2). \tag{7_j}$$

Note that if $j\in\{1,2\}$ is such that (7_j) holds, then condition (3_j) follows from (4) and (7)

In the present paper we assume that (7) holds, but the fulfilment of conditions (7_j) $(j=1,2)$ is not required.

Analogous and some related questions for the initial and general linear boundary value problems are investigated e.g. in [1, 2, 9, 10, 12, 14] (see also the references therein) for systems of ordinary differential equations, in [3, 4, 8, 11, 13] for systems of generalized ordinary differential equations, and in [6] for systems of linear impulsive differential equations.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate linear ordinary differential, impulsive and difference equations from a unified point of view; in particular, these different type equations (linear) can be rewritten in form (1). Moreover, the convergence conditions for difference schemes corresponding to systems of ordinary differential and impulsive equations can be obtained from the results on the well-posedness of the corresponding problems for systems of generalized ordinary differential equations (see [5, 14, 15] and the references therein).

In the paper the use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$, $[a,b]$ and $]a,b[$ $(a,b \in \mathbb{R})$ are, respectively, closed and open intervals.

I is an arbitrary finite or infinite interval from $\mathbb{R}$. We say that some property is valid in I if it is valid on every closed interval from I.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$O_{n\times m}$ (or O) is the zero $n \times m$ matrix. We designate the zero n vector by 0, as well.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

If $X \in \mathbb{R}^{n\times n}$, then $\det(X)$ is the determinant of X.

I_n is the identity $n \times n$-matrix.

$\bigvee\limits_a^b(x)$ is the total variation of the function $x : [a,b] \to \mathbb{R}$; $\bigvee\limits_b^a(x) = -\bigvee\limits_a^b(x)$.

If $x : I \to \mathbb{R}$, then $\bigvee\limits_I(x)$ is the total variation of x on I, i.e. $\bigvee\limits_I(x) = \lim\limits_{a\to\alpha+,\, b\to\beta-} \bigvee\limits_a^b(x)$, where $\alpha = \inf I$ and $\beta = \sup I$.

$\bigvee\limits_a^b(X)$ is the sum of the total variations of the components x_{ij} $(i = 1,\dots,m;\ j = 1,\dots,m)$ of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$.

$\bigvee\limits_b^a(X) = -\bigvee\limits_a^b(X)$, $\bigvee\limits_I(X) = \lim\limits_{a\to\alpha+,\, b\to\beta-} \bigvee\limits_a^b(X)$, where $\alpha = \inf I$ and $\beta = \sup I$, $\bigvee\limits_{(b,a)}(X) = -\bigvee\limits_{(b,a)}(X)$.

If $X : I \to \mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_I(X)$ is the sum of total variations on I of its components x_{ij} $(i = 1,\dots,m;\ j = 1,\dots,m)$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t ($X(\alpha-) = X(\alpha)$ if $\alpha \in I$ and $X(\beta+) = X(\beta)$ if $\beta \in I$; if α or β do not belong to I, then $X(t)$ is defined by the continuity outside of I).

$d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\mathrm{BV}(I;\mathbb{R}^{n\times m})$ is the set of all bounded variation matrix-functions $X : I \to \mathbb{R}^{n\times m}$ (i.e. such that $\bigvee\limits_I(X) < \infty$).

$\mathrm{BV}(I;D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all bounded variation matrix-functions $X : I \to D$.

$\mathrm{BV}_{loc}(I;D)$ is the set of all $X : I \to D$ for which the restriction on $[a,b]$ belong to $\mathrm{BV}([a,b];D)$ for every closed interval $[a,b]$ from I.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

We introduce the operators. Let $a \in I$ be a fixed point, and $X \in \mathrm{BV}_{loc}(I,\mathbb{R}^{l\times n})$ and $Y \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m})$. Then we put

$$\mathcal{B}(X,Y)(t) = X(t)Y(t) - X(a)Y(a) - \int_a^t dX(\tau)\cdot Y(\tau) \ \text{ for } \ t \in I,$$

$$\mathcal{I}(X,Y)(t) = \int_a^t d\big(X(\tau) + \mathcal{B}(X,Y)(\tau)\big)\cdot X^{-1}(\tau) \ \text{ for } \ t \in I,$$

$$\mathcal{D}_{\mathcal{B}}(Y_1,X_1;Y_2,X_2)(t)=\mathcal{B}(X_1,Y_1)(t)-\mathcal{B}(X_2,Y_2)(t) \text{ for } t\in I,$$
$$\mathcal{D}_{\mathcal{I}}(Y_1,X_1;Y_2,X_2)(t)=\mathcal{I}(X_1,Y_1)(t)-\mathcal{I}(X_2,Y_2)(t) \text{ for } t\in I.$$

Definition 1. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}(A_0,f_0;t_0)$ if for every $c_0\in\mathbb{R}^n$ and a sequence $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ satisfying condition (7), the problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and condition (6) holds.

In [4,7], the necessary and sufficient (effectively sufficient) conditions are established that guarantee the inclusion

$$\big((A_k,f_k;t_k)\big)_{k=1}^{+\infty}\in S(A_0,f_0;t_0). \tag{8}$$

Analogous results are established for the general linear boundary value problems in [3,4].

Definition 2. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}_{loc}(A_0,f_0;t_0-)$ if $t_k<t_0$ $(k=1,2,\dots)$ and for every $c_0\in\mathbb{R}^n$ and the sequence $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ satisfying condition (3_1), the problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and condition (3) holds.

Definition 3. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}_{loc}(A_0,f_0;t_0+)$ if $t_k>t_0$ $(k=1,2,\dots)$ and for every $c_0\in\mathbb{R}^n$ and the sequence $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ satisfying condition (3_2), the problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and condition (3) holds.

Definition 4. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}_{loc}(A_0,f_0;t_0\pm)$ if $t_k=t_0$ $(k=1,2,\dots)$ and for every $c_0\in\mathbb{R}^n$, the sequence $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ and $j\in\{1,2\}$ satisfying condition (3_j), the problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and condition (3) holds.

Definition 5. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}^*_{loc}(A_0,f_0;t_0-)$ if $t_k<t_0$ $(k=1,2,\dots)$ and for every $c_{*1}\in\mathbb{R}^n$ and the sequence $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ satisfying condition (5_1), the problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and condition (3_1) holds.

Definition 6. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}^*_{loc}(A_0,f_0;t_0+)$ if $t_k>t_0$ $(k=1,2,\dots)$ and for every $c_{*2}\in\mathbb{R}^n$ and the sequence $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ satisfying condition (5_2), the problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and condition (3_2) holds.

Definition 7. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}^*_{loc}(A_0,f_0;t_0\pm)$ if $t_k=t_0$ $(k=1,2,\dots)$ and for every $c_{*j}\in\mathbb{R}^n$ $(j=1,2)$ and the sequences $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ satisfying conditions (5_j) $(j=1,2)$, the problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and conditions (3_j) $(j=1,2)$ hold.

(A) The results concerning the sets $\mathcal{S}(A_0,f_0;t_0)$, $\mathcal{S}_{loc}(A_0,f_0;t_0-)$, $\mathcal{S}_{loc}(A_0,f_0;t_0+)$ and $\mathcal{S}_{loc}(A_0,f_0;t_0\pm)$

Theorem 1. *Let $A_0\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0\in BV(I;\mathbb{R}^n)$, $t_0\in I$, and the sequence of points $t_k\in I$ $(k=1,2,\dots)$ be such that*

$$\det\big(I_n+(-1)^jd_jA_0(t)\big)\neq 0 \;\; \textit{for} \;\; t\in I, \;\; (-1)^j(t-t_0)<0 \;\; \textit{and for} \;\; t=t_0$$
$$\textit{if} \;\; j\in\{1,2\} \;\; \textit{is such that} \;\; (-1)^j(t_k-t_0)>0 \;\; \textit{for every} \;\; k\in\{1,2,\dots\} \tag{9}$$

and

$$\lim_{k\to+\infty}t_k=t_0. \tag{10}$$

Then inclusion (8) holds if and only if there exists a sequence of matrix-functions $H_k\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ such that

$$\inf\big\{|\det(H_0(t))|:\; t\in I\big\}>0, \tag{11}$$

and the conditions

$$\lim_{k\to+\infty} H_k(t) = H_0(t), \tag{12}$$

$$\lim_{k\to+\infty}\left\{\left\|\mathcal{D}_{\mathcal{I}}(A_k,H_k;A_0,H_0)(\tau)\Big|_{t_k}^{t}\right\|\left(1+\Big|\bigvee_{t_k}^{t}\left(\mathcal{D}_{\mathcal{I}}(A_k,H_k;A_0,H_0)\right)\Big|\right)\right\}=0$$

and

$$\lim_{k\to+\infty}\left\{\left\|\mathcal{D}_{\mathcal{B}}(f_k,H_k;f_0,H_0)(\tau)\Big|_{t_k}^{t}\right\|\left(1+\Big|\bigvee_{t_k}^{t}\left(\mathcal{D}_{\mathcal{I}}(A_k,H_k;A_0,H_0)\right)\Big|\right)\right\}=0$$

hold uniformly on I.

The last two conditions are analogy to the Opial conditions concerning to the well-posed question for the ordinary differential case (see [14]). Note that, the Opial condition has only the sufficient character for the last case.

We offer another form of criterion for inclusion (8), differing from Theorem 1.

Theorem 1′. *Let $A_0\in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0\in BV(I;\mathbb{R}^n)$, $t_0\in I$, and the sequence of points $t_k\in I$ $(k=1,2,\dots)$ be such that conditions* (9) *and* (10) *hold. Then inclusion* (8) *holds if and only if there exists a sequence of matrix-functions $H_k\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ such that conditions* (11) *and*

$$\limsup_{k\to+\infty}\bigvee_I\left(H_k+\mathcal{B}(H_k,A_k)\right)<+\infty$$

hold, and conditions (12),

$$\lim_{k\to+\infty}\left(\mathcal{B}(H_k,A_k)(t)-\mathcal{B}(H_k,A_k)(t_k)\right)=\mathcal{B}(H_0,A_0)(t)-\mathcal{B}(H_0,A_0)(t_0)$$

and

$$\lim_{k\to+\infty}\left(\mathcal{B}(H_k,f_k)(t)-\mathcal{B}(H_k,f_k)(t_k)\right)=\mathcal{B}(H_0,f_0)(t)-\mathcal{B}(H_0,f_0)(t_0)$$

hold uniformly on I.

Remark 1. Without loss of generality, we can assume that $H_0(t)\equiv I_n$ in Theorems 1 and 1′. So

$$\mathcal{B}(I_n,Y)(t)-\mathcal{B}(I_n,Y)(s)=Y(t)-Y(s)\ \text{ and }$$
$$\mathcal{I}(I_n,Y)(t)-\mathcal{I}(I_n,Y)(s)=Y(t)-Y(s)\ \text{ for }\ Y\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m}).$$

Theorem 2. *Let $A_0\in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0\in BV(I;\mathbb{R}^n)$, $t_0\in I$, and the sequence of points $t_k\in I$ $(k=1,2,\dots)$ be such that conditions* (9) *and* (10) *hold. Let, moreover, the sequences of matrix– and vector-functions $A_k\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ and $f_k\in BV_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ be such that the conditions*

$$\lim_{k\to+\infty}\sup_{t\in I,\,t\neq t_k}\left\{\|A_{kj}(t)-A_{0j}(t)\|\Big(1+\Big|\bigvee_{t_k}^{t}(A_k-A_0)\Big|\Big)\right\}=0 \tag{13}$$

and

$$\lim_{k\to+\infty}\sup_{t\in I,\,t\neq t_k}\left\{\|f_{kj}(t)-f_{0j}(t)\|\Big(1+\Big|\bigvee_{t_k}^{t}(A_k-A_0)\Big|\Big)\right\}=0 \tag{14}$$

hold if $j\in\{1,2\}$ is such that $(-1)^j(t_k-t_0)\ge 0$ for every $k\in\{1,2,\dots\}$, where

$$A_{kj}(t)\equiv(-1)^j\left(A_k(t)-A_k(t_k)\right)-d_jA_k(t_k)\ \ (j=1,2;\ \ k=0,1,\dots), \tag{15}$$

$$f_{kj}(t)\equiv(-1)^j\left(f_k(t)-f_k(t_k)\right)-d_jf_k(t_k)\ \ (j=1,2;\ \ k=0,1,\dots). \tag{16}$$

Then

$$\left((A_k,f_k;t_k)\right)_{k=1}^{+\infty}\in S_{loc}(A_0,f_0;t_0-)\ \text{ if }\ j=1,$$

$$((A_k, f_k; t_k))_{k=1}^{+\infty} \in S_{loc}(A_0, f_0; t_0+) \ \ \textit{if} \ \ j = 2$$

and

$$((A_k, f_k; t_k))_{k=1}^{+\infty} \in S_{loc}(A_0, f_0; t_0\pm) \ \ \textit{if} \ \ j \in \{1,2\}.$$

Remark 2. In Theorem 2, the sequence $x_k(t)$ $(k = 1, 2, \dots)$ converges to x_0 uniformly on the set $\{t \in I,\ t \le t_0\}$ if $t_k > t_0$ $(k = 1, 2, \dots)$, and on the set $\{t \in I,\ t \ge t_0\}$ if $t_k < t_0$ $(k = 1, 2, \dots)$; as for the case, where $t_k = t_0$ $(k = 1, 2, \dots)$, the sequence $x_k(t)$ $(k = 1, 2, \dots)$ converges to x_0 uniformly in both intervals $\{t \in I:\ t < t_0\}$ and $\{t \in I:\ t > t_0\}$. Moreover, if conditions (13) and (14) hold uniformly on the set I, then these conditions are equivalent to the conditions

$$\lim_{k\to+\infty} \left\{ \big\|(A_k(t) - A_k(t_k)) - (A_0(t) - A_0(t_0))\big\| \Big(1 + \Big|\bigvee_{t_k}^{t}(A_k - A_0)\Big|\Big) \right\} = 0 \tag{17}$$

and

$$\lim_{k\to+\infty} \left\{ \big\|(f_k(t) - f_k(t_k)) - (f_0(t) - f_0(t_0))\big\| \Big(1 + \Big|\bigvee_{t_k}^{t}(A_k - A_0)\Big|\Big) \right\} = 0 \tag{18}$$

uniformly on I, respectively, since (17) and (18) imply that

$$\lim_{k\to+\infty} d_j A_k(t) = d_j A_0(t) \ \ \text{and} \ \ \lim_{k\to+\infty} d_j f_k(t) = d_j f_0(t)$$

uniformly on I for every $j \in \{1, 2\}$. In addition, equalities (7_j) $(j = 1, 2)$ hold and,therefore, as above, conditions (3_j) $(j = 1, 2)$ hold, as well. Thus, in the case under consideration, condition (3) holds uniformly on I, i.e., condition (6) holds, as well.

Theorem 3. *Let $A_0^* \in \mathrm{BV}(I; \mathbb{R}^{n\times n})$, $f_0^* \in BV(I; \mathbb{R}^n)$, $c_0^* \in \mathbb{R}^n$, $t_0 \in I$, and the sequence of points $t_k \in I$ $(k = 1, 2, \dots)$ be such that condition* (10) *holds,*

$$\det\big(I_n + (-1)^j d_j A_0^*(t)\big) \ne 0 \ \ \textit{for} \ \ t \in I,\ \ (-1)^j (t - t_0) < 0 \ \ \textit{and for} \ \ t = t_0$$
$$\textit{if} \ \ j \in \{1, 2\} \ \ \textit{is such that} \ \ (-1)^j (t_k - t_0) > 0 \ \ \textit{for every} \ \ k \in \{1, 2, \dots\},$$

and the Cauchy problem

$$dx = dA_0^*(t) \cdot x + df_0^*(t),$$
$$x(t_0) = c_0^*$$

has a unique solution x_0^*. Let, moreover, the sequences of matrix– and vector-functions $A_k, H_k \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$ and $f_k, h_k \in BV_{loc}(I; \mathbb{R}^n)$ $(k = 1, 2, \dots)$ and of constant vectors $c_k^* \in \mathbb{R}^n$ $(k = 1, 2, \dots)$ be such that the conditions

$$\inf\big\{|\det(H_k(t)|:\ t \in I,\ t \ne t_k\big\} > 0 \ \ \textit{for every sufficiently large } k,$$
$$\lim_{k\to+\infty} c_k^* = c_0^*, \quad \lim_{k\to+\infty} c_{kj}^* = c_{0j}^*, \tag{19}$$
$$\lim_{k\to+\infty} \sup_{t\in I,\, t\ne t_k} \left\{ \|A_{kj}^*(t) - A_{0j}^*(t)\| \Big(1 + \Big|\bigvee_{t_k}^{t}(A_k^* - A_0^*)\Big|\Big) \right\} = 0$$

and

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\ne t_k} \left\{ \|f_{kj}^*(t) - f_{0j}^*\| \Big(1 + \Big|\bigvee_{t_k}^{t}(A_k^* - A_0^*)\Big|\Big) \right\} = 0$$

hold for some $j \in \{1, 2\}$, where

$$A_{kj}^*(t) = (-1)^j (A_k^*(t) - A_k^*(t_k)) - d_j A_k^*(t_k),$$

$$f^*_{kj}(t) = (-1)^j(f^*_k(t) - f^*_k(t_k)) - d_j f^*_k(t_k) \ \ for \ \ t \in I \ \ (j = 1, 2; \ k = 0, 1, \dots);$$
$$A^*_k(t) = \mathcal{I}(H_k, A_k)(t),$$
$$f^*_k(t) = h_k(t) - h_k(t_k) + \mathcal{B}(H_k, f_k)(t) - \mathcal{B}(H_k, f_k)(t_k) - \int_{t_k}^{t} dA^*_k(s) \cdot h_k(s) \ \ for \ \ t \in I \ \ (k = 1, 2, \dots);$$
$$c^*_k = H_k(t_k)c_k + h_k(t_k) \ \ (k = 1, 2, \dots),$$
$$c^*_{kj} = c^*_k + (-1)^j \big(d_j A^*_k(t_k)c^*_k + d_j f^*_k(t_k)\big) \ \ \ (j = 1, 2; \ \ k = 0, 1, \dots).$$

Then problem $(1_k), (2_k)$ *has a unique solution* x_k *for any sufficiently large* k *and*

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \big\|H_k(t)x_k(t) + h_k(t) - x^*_0(t)\big\| = 0.$$

Remark 3. In Theorem 3, the vector-function $x^*_k(t) \equiv H_k(t)\, x_k(t) + h_k(t)$ is a solution of the problem

$$dx = dA^*_k(t) \cdot x + df^*_k(t),$$
$$x(t_k) = c^*_k$$

for every sufficiently large k.

Corollary 1. *Let* $A_0 \in \mathrm{BV}(I; \mathbb{R}^{n\times n})$*,* $f_0 \in BV(I; \mathbb{R}^n)$*,* $c_0 \in \mathbb{R}^n$*,* $t_0 \in I$*, and the sequences* $A_k \in \mathrm{BV}(I; \mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$*,* $f_k \in BV(I; \mathbb{R}^n)$ $(k = 1, 2, \dots)$*,* $c_k \in \mathbb{R}^n$ $(k = 1, 2, \dots)$ *and* $t_k \in I$ $(k = 1, 2, \dots)$ *be such that conditions* (9)*,* (10)*,* (11)*,*

$$\lim_{k\to+\infty} (c_{kj} - \varphi_k(t_k)) = c_{0j},$$
$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \|H_k(t) - H_0(t)\| = 0,$$
$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \left\{ \left\| \mathcal{D}_{\mathcal{I}}(A_k, H_k; A_0, H_0)(\tau)\Big|_{t_k}^{t} \right\| \left(1 + \Big|\bigvee_{t_k}^{t}(\mathcal{D}_{\mathcal{I}}(A_k, H_k; A_0, H_0))\Big|\right) \right\} = 0$$

and

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \left\{ \left\| \mathcal{D}_{\mathcal{B}}(f_k - \varphi_k, H_k; f_0, H_0)(\tau)\Big|_{t_k}^{t} + \int_{t_k}^{t} d\mathcal{I}(H_k, A_k)(\tau) \cdot \varphi_k(\tau) \right\| \left(1 + \Big|\bigvee_{t_k}^{t}(\mathcal{D}_{\mathcal{I}}(A_k, H_k; A_0, H_0))\Big|\right) \right\} = 0$$

hold for some $j \in \{1, 2\}$*, where* $H_k \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times n})$ $(k = 0, 1, \dots)$*,* $\varphi_k \in \mathrm{BV}_{loc}(I; \mathbb{R}^n)$ $(k = 1, 2, \dots)$ *and* c_{kj} $(k = 0, 1, \dots)$ *are defined by* (4)*. Then problem* $(1_k), (2_k)$ *has a unique solution* x_k *for any sufficiently large* k *and*

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \|x_k(t) - \varphi_k(t) - x_0(t)\| = 0.$$

(B) The results concerning the sets $\mathcal{S}^*_{loc}(A_0, f_0; t_0-)$, $\mathcal{S}^*_{loc}(A_0, f_0; t_0+)$ and $\mathcal{S}^*_{loc}(A_0, f_0; t_0\pm)$

For the goal, we will use the following easy lemma.

Lemma 1. *Let* $j \in \{1, 2\}$ *be such that condition* (5_j) *hold, where* $c_{*j} \in \mathbb{R}^n$*, and the vectors* $c_{kj} (k = 1, 2, \dots)$ *are defined by* (4)*. Then the vector-function* $x_{*1}(t) \equiv x_0(t) + x_{01}(t)$ *will be a solution of system* (1) *under the condition* $x(t_0-) = c_{*1}$*, and the vector-function* $x_{*2}(t) \equiv x_0(t) + x_{02}(t)$ *will be a solution of system* (1) *under the condition* $x(t_0+) = c_{*2}$*.*

Theorem 2*. *Let $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0 \in BV(I;\mathbb{R}^n)$, $t_0 \in I$, and the sequence of points $t_k \in I$ $(k = 1, 2, \dots)$ be such that conditions* (9) *and* (10) *hold. Let, moreover, the sequences of matrix– and vector-functions $A_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$ and $f_k \in BV_{loc}(I;\mathbb{R}^n)$ $(k = 1, 2, \dots)$ be such that conditions* (13) *and* (14) *hold if $j \in \{1, 2\}$ is such that $(-1)^j(t_k - t_0) \ge 0$ for every $k \in \{1, 2, \dots\}$, where $A_{kj}(t)$ $(j = 1, 2;\ k = 0, 1, \dots)$ and $f_{kj}(t)$ $(j = 1, 2;\ k = 0, 1, \dots)$ are defined by* (15) *and* (16)*, respectively. Then*

$$((A_k, f_k; t_k))_{k=1}^{+\infty} \in S^*_{loc}(A_0, f_0; t_0-) \ \text{ if } j = 1,$$
$$((A_k, f_k; t_k))_{k=1}^{+\infty} \in S^*_{loc}(A_0, f_0; t_0+) \ \text{ if } j = 2$$

and

$$((A_k, f_k; t_k))_{k=1}^{+\infty} \in S^*_{loc}(A_0, f_0; t_0\pm) \ \text{ if } j \in \{1, 2\}.$$

Theorem 3*. *Let the conditions of Theorem* 3 *be fulfilled, with the exclusion of* (19)*, instead of which the condition*

$$\lim_{k\to+\infty} c^*_{kj} = c^*_j, \tag{20}$$

holds, where the vectors $c^*_{kj} \in \mathbb{R}^n$ $(k = 1, 2, \dots)$ are defined as in Theorem 3*, and $c^*_j \in \mathbb{R}^n$ is a vector differing from c^*_{0j}, in general. Then problem $(1_k), (2_k)$ has a unique solution x_k for any sufficiently large k and*

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \big\| H_k(t)x_k(t) + h_k(t) - x^*_0(t) - x^*_j(t) \big\| = 0,$$

where the function x^*_1 is a solution of the homogeneous system

$$dx = dA^*_0(t) \cdot x$$

on the set $\{t \in I : t < t_0\}$ under the condition

$$x^*_1(t_0-) = c^*_1 - x^*_0(t_0-),$$

and the function x^*_2 is a solution of the homogeneous system on the set $\{t \in I :\ t > t_0\}$ under the condition

$$x^*_2(t_0+) = c^*_2 - x^*_0(t_0+).$$

Corollary 1*. *Let the conditions of Corollary* 1 *be fulfilled with the exclusion of* (20)*, instead of which the condition*

$$\lim_{k\to+\infty} (c_{kj} - \varphi_k(t_k)) = c_{*j}$$

holds for some $j \in \{1, 2\}$, where the vectors $c^*_{kj} \in \mathbb{R}^n$ $(k = 1, 2, \dots)$ are defined as in Theorem 3*, and $c^*_j \in \mathbb{R}^n$ is a vector differing from c^*_{0j}, in general. Then problem $(1_k), (2_k)$ has a unique solution x_k for any sufficiently large k and*

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \big\| x_k(t) - \varphi_k(t) - x_0(t) - x_{0j} \big\| = 0,$$

where the vector-function x_{0j} is defined as above.

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