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Sergo Kharibegashvili

SOME LOCAL AND NONLOCAL MULTIDIMENSIONAL PROBLEMS FOR A CLASS OF SEMILINEAR HYPERBOLIC EQUATIONS AND SYSTEMS

Abstract. Multidimensional versions of the Cauchy characteristic problem, the Darboux problems, and the Sobolev problem for a class of second order semilinear hyperbolic systems are investigated. Depending on the type of nonlinearity, spatial dimension and structure of the hyperbolic system, the cases for which these problems are globally solvable, are singled out. Moreover, the cases of the absence of solutions of these problems are also considered. The questions of the solvability of some nonlocal in time problems for multidimensional second order semilinear hyperbolic equations are studied. The particular cases of the above-mentioned problems are the periodic and antiperiodic problems.

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Preface

The present work consists of five chapters. The first three chapters are devoted to the investigation of multidimensional versions of the Cauchy characteristic problem, the Darboux problems, and the Sobolev problem for one class of the second order semilinear hyperbolic systems. Depending on the type of nonlinearity, spatial dimension and structure of hyperbolic system, the cases for which these problems are globally solvable, are singled out. Moreover, the cases of the absence of solutions of the above-mentioned problems are also considered [56–59].

The questions of the solvability of some nonlocal in time problems for multidimensional second order semilinear hyperbolic equations are studied in the remaining two chapters [53, 60, 61]. The particular cases of these problems are the periodic and antiperiodic problems.

Chapter 1

The Cauchy characteristic problem for one class of the second order semilinear hyperbolic systems

1.1 Statement of the problem

In the space $\mathbb{R}^{n+1}$ of variables $x = (x_1, \dots, x_n)$ and t, we consider the second order semilinear hyperbolic system of the form

$$\square\, u_i + f_i(u_1, \dots, u_N) = F_i(x,t), \quad i = 1, \dots, N, \tag{1.1.1}$$

where $f = (f_1, \dots, f_N)$, $F = (F_1, \dots, F_N)$ are the given, and $u = (u_1, \dots, u_N)$ is an unknown real vector function, $n \geq 2$, $N \geq 2$, $\square := \frac{\partial^2}{\partial t^2} - \Delta$, $\Delta := \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

For the system of equations (1.1.1), let us consider the Cauchy characteristic problem of finding a solution $u(x,t)$ in the frustum of a light cone of the future $D_T:\ |x| < t < T$, $T = const > 0$, by the boundary condition

$$u\big|_{S_T} = g, \tag{1.1.2}$$

where $S_T:\ t = |x|, t \leq T$, is the conic surface, characteristic to the system (1.1.1), and $g = (g_1, \dots, g_N)$ is a given vector function on S_T. For $T = \infty$, we assume that $D_\infty:\ t > |x|$ and $S_\infty = \partial D_\infty:\ t = |x|$.

The questions on the existence or absence of a global solution of the Cauchy problem for semilinear scalar equations of the type (1.1.1) with the initial conditions of the form $u|_{t=0} = u_0$, $\frac{\partial u}{\partial t}\big|_{t=0} = u_1$ were the subject of investigation in many works (see, e.g., [17–19, 23, 25, 31, 33, 35, 36, 39–41, 62, 64–66, 69–72, 77, 80, 83, 84, 87–89, 94, 96–98]. The Cauchy characteristic problem (1.1.1), (1.1.2) in the light cone of the future for scalar semilinear equations has been studied in [44–47, 49, 50, 52, 54]. As is known, this problem in the linear case is well-posed in the corresponding function spaces (see, e.g., [5,16,30,43,63,73]). A particular case of the system (1.1.1), when $f(u) = \nabla G(u)$, i.e., $f_i(u) = \frac{\partial}{\partial u_i}\, G(u)$, $i = 1, \dots, N$, where $G = G(u)$ is a scalar function satisfying some conditions of smoothness and growth as $|u| \to \infty$, is studied in [57].

In the present chapter we consider a more general case of nonlinearity as compared with that presented in [57]; we impose certain conditions on the nonlinear vector function $f = f(u)$ from (1.1.1) which fulfilment implies that the problem (1.1.1), (1.1.2) is locally or globally solvable, while in some cases it does not have global solution.

1.2 Definition of a generalized solution of the problem (1.1.1), (1.1.2) on D_T and D_∞

Let $\overset{\circ}{C}{}^2(\overline{D}_T, S_T) := \{u \in C^2(\overline{D}_T) :\ u|_{S_T} = 0\}$ and $\overset{\circ}{W}{}^1_2(D_T, S_T) := \{u \in W^1_2(D_T) :\ u|_{S_T} = 0\}$, where $W^k_2(\Omega)$ is the Sobolev space, consisting of the elements of $L_2(\Omega)$, the generalize derivatives of which up to the k-th order inclusive belong to $L_2(\Omega)$, and the equality $u|_{S_T} = 0$ is understood in the sense of the trace theory [68, p. 71].

We rewrite the system of equations (1.1.1) in the form of one vectorial equation

$$Lu := \Box\, u + f(u) = F(x,t). \tag{1.2.1}$$

Together with the boundary condition (1.1.2), we consider the corresponding homogeneous boundary condition, i.e.,

$$u\big|_{S_T} = 0. \tag{1.2.2}$$

Below, on the nonlinear vector function $f = (f_1, \dots, f_N)$ from (1.1.1) we impose the following requirement

$$f \in C(\mathbb{R}^N),\ \ |f(u)| \le M_1 + M_2|u|^\alpha,\ \ \alpha = const \ge 0,\ \ u \in \mathbb{R}^N, \tag{1.2.3}$$

where $|\cdot|$ is the norm of the space $\mathbb{R}^N$ and $M_i = const \ge 0$, $u \in \mathbb{R}^N$.

Remark 1.2.1. The embedding operator $I : W^1_2(D_T) \to L_q(D_T)$ is a linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$ and $n > 1$ [68, p. 86]. At the same time, the Nemitsky operator $K : L_q(D_T) \to L_2(D_T)$, acting according to the formula $K(u) = f(u)$, where $u = (u_1, \dots, u_N) \in L_q(D_T)$ and the vector function $f = (f_1, \dots, f_N)$ satisfies the condition (1.2.3), is continuous and bounded for $q \ge 2\alpha$ [67, p. 349], [22, pp. 66,67]. Therefore, if $\alpha < \frac{n+1}{n-1}$, then there exists a number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \ge 2\alpha$. Thus in this case the operator

$$K_0 = KI : [W^1_2(D_T)]^N \to [L_2(D_T)]^N \tag{1.2.4}$$

is continuous and compact. Moreover, from $u \in W^1_2(D_T)$ it follows that $f(u) \in L_2(D_T)$ and, if $u^m \to u$ in the space $W^1_2(D_T)$, then $f(u^m) \to f(u)$ in the space $L_2(D_T)$.

Here and henceforth, the belonging of the vector $v = (v_1, \dots, v_N)$ to some space X means that each component v_i, $i \le i \le N$, of that vector belongs to the space X.

Definition 1.2.1. Let $f = (f_1, \dots, f_N)$ satisfy the condition (1.2.3), where $0 \le \alpha < \frac{n+1}{n-1}$, $F = (F_1, \dots, F_N) \in L_2(D_T)$ and $g = (g_1, \dots, g_N) \in W^1_2(S_T)$. We call a vector function $u = (u_1, \dots, u_N) \in W^1_2(D_T)$ a strong generalized solution of the problem (1.1.1), (1.1.2) of the class W^1_2 in the domain D_T if there exists a sequence of vector functions $u^m \in C^2(\overline{D}_T)$ such that $u^m \to u$ in the space $W^1_2(D_T)$, $Lu^m \to F$ in the space $L_2(D_T)$, and $u^m|_{S_T} \to g$ in the space $W^1_2(S_T)$. The convergence of the sequence $\{f(u^m)\}$ to $f(u)$ in the space $L_2(D_T)$, as $u^m \to u$ in the space $W^1_2(D_T)$, is provided by Remark 1.2.1. In the case $g = 0$, i.e., in the case of the homogeneous boundary condition (1.2.2), we assume that $u^m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$. Then it is obvious that $u \in \overset{\circ}{W}{}^1_2(D_T, S_T)$.

Obviously, the classical solution $u \in C^2(\overline{D}_T)$ of the problem (1.1.1), (1.1.2) is likewise a strong generalized solution of this problem of the class W^1_2 in the domain D_T in the sense of Definition 1.2.1.

Remark 1.2.2. It is easy to verify that if $u \in W^1_2(D_T)$ is the strong generalized solution of the problem (1.1.1), (1.1.2) of the class W^1_2 in the domain D_T in the sense of Definition 1.2.1, then for every test vector function $\varphi = (\varphi_1, \dots, \varphi_N) \in C^1(\overline{D}_T)$ such that $\varphi|_{t=T} = 0$, the equality

$$\int\limits_{D_T} \big[-u_t\varphi_t + \nabla u \nabla \varphi \big]\, dx\, dt = -\int\limits_{D_T} f(u)\varphi\, dx\, dt + \int\limits_{D_T} F\varphi\, dx\, dt - \int\limits_{S_T} \frac{\partial g}{\partial N}\, \varphi\, ds \tag{1.2.5}$$

is valid; here, $\frac{\partial}{\partial N} = \nu_{n+1}\frac{\partial}{\partial t} - \sum\limits_{i=1}^{n} \nu_i \frac{\partial}{\partial x_i}$ is the derivative along the conormal, $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ is the unit vector of the outer normal to ∂D, $\nabla = \big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\big)$.

Indeed, let $u^m \in C^2(\overline{D}_T)$ be the vector functions mentioned in Definition 1.2.1. Let $F^m := Lu^m$, where L is the operator from (1.2.1). Taking into account the fact that on the characteristic conic surface $S_T: \ t = |x|, \ t \le T$, the derivative along the conormal $\frac{\partial}{\partial N}$ represents an inner differential operator, and by integration by parts of the equality $Lu^m = F^m$, we obtain

$$\int\limits_{D_T} \big[- u_t^m \varphi_t + \nabla u^m \nabla \varphi \big] \, dx \, dt = - \int\limits_{D_T} f(u^m) \varphi \, dx \, dt + \int\limits_{D_T} F^m \varphi \, dx \, dt - \int\limits_{S_T} \frac{\partial g^m}{\partial N} \, \varphi \, ds, \tag{1.2.6}$$

where $g^m := u^m|_{S_T}$. Since, by Definition 1.2.1, $u^m \to u$ in the space $W_2^1(D_T)$, $F^m = Lu^m \to F$ in the space $L_2(D_T)$, $g^m = u^m|_{S_T} \to g$ in the space $W_2^1(S_T)$, and according to Remark 1.2.1 $f(u^m) \to f(u)$ in the space $L_2(D_T)$, passing to the limit in the equality (1.2.6) as $m \to \infty$ we obtain (1.2.5).

Note that the equality (1.2.5), valid for every $\varphi \in C^2(\overline{D_T})$, $\varphi|_{t=T} = 0$, may be put in the basis of the definition of a weak generalized solution u of the problem (1.1.1), (1.1.2) of the class W_2^1 in the domain D_T.

Definition 1.2.2. Let f satisfy the condition (1.2.3), where $0 \le \alpha < \frac{n+1}{n-1}$; $F \in L_{2,loc}(D_\infty)$, $g \in W_{2,loc}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. We say that the problem (1.1.1), (1.1.2) is locally solvable in the class W_2^1 if there exists a number $T_0 = T_0(F, g) > 0$ such that for $T < T_0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition 1.2.1.

Definition 1.2.3. Let f satisfy the condition (1.2.3), where $0 \le \alpha < \frac{n+1}{n-1}$; $F \in L_{2,loc}(D_\infty)$, $g \in W_{2,loc}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. We say that the problem (1.1.1), (1.1.2) is globally solvable in the class W_2^1 if for every $T > 0$ the problem has a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition 1.2.1.

Definition 1.2.4. Let f satisfy the condition (1.2.3), where $0 \le \alpha < \frac{n+1}{n-1}$; $F \in L_{2,loc}(D_\infty)$, $g \in W_{2,loc}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. We call the vector function $u = (u_1, \dots, u_N) \in W_{2,loc}^1(D_\infty)$ a global strong generalized solution of the problem (1.1.1), (1.1.2) of the class W_2^1 in the light cone of the future D_∞ if for every $T > 0$ the vector function $u|_{D_T}$ belongs to the space $W_2^1(D_T)$ and is a strong generalized solution of this problem of the class W_2^1 in the domain D_T in the sense of Definition 1.2.1.

Remark 1.2.3. Reasoning from the proof of the equality (1.2.5) allows us to conclude that a global strong generalized solution $u = (u_1, \dots, u_N)$ of the problem (1.1.1), (1.1.2) of the class W_2^1 in the domain D_∞ in the sense of Definition 1.2.4 satisfies the integral equality

$$\int\limits_{D_\infty} \big[- u_t \varphi_t + \nabla u \nabla \varphi \big] \, dx \, dt = - \int\limits_{D_\infty} f(u) \varphi \, dx \, dt + \int\limits_{D_\infty} F \varphi \, dx \, dt - \int\limits_{S_\infty} \frac{\partial g}{\partial N} \, \varphi \, ds \tag{1.2.7}$$

for any vector function $\varphi = (\varphi_1, \dots, \varphi_N) \in C^1(\overline{D}_\infty)$, finite with respect to the variable $r = (t^2 + |x|^2)^{1/2}$, i.e., $\varphi = 0$ for $r > r_0 = const > 0$. It is easy to see that the solution $u \in W_{2,loc}^1(D_\infty)$ satisfies the boundary condition (1.1.2) in the sense of the trace theory for $T = \infty$, i.e., $u|_{S_\infty} = g$.

1.3 Some cases of local and global solvability of the problem (1.1.1), (1.1.2) in the class W_2^1

For the sake of simplicity, we consider the case in which the boundary condition (1.1.2) is homogeneous. In this case the problem (1.1.1), (1.1.2) takes the form of the problem (1.2.1), (1.2.2).

Remark 1.3.1. First, let us consider the solvability of the problem (1.2.1), (1.2.2), when the vector function $f = 0$ in (1.2.1), i.e., the linear problem

$$L_0 u := \square \, u = F(x,t), \ \ (x,t) \in D_T, \tag{1.3.1}$$

$$u(x,t) = 0, \ \ (x,t) \in S_T. \tag{1.3.2}$$

For the problem (1.3.1), (1.3.2), just as for the problem (1.1.1), (1.1.2) in Definition 1.2.1, we introduce the notion of a strong generalized solution $u = (u_1, \dots, u_N)$ of the class W_2^1 in the domain D_T for $F = (F_1, \dots, F_N) \in L_2(D_T)$, i.e., of the vector function $u = (u_1, \dots, u_N) \in \overset{\circ}{W}{}_2^1(D_T, S_T) := \{u \in W_2^1(D_T) :\ u|_{S_T} = 0\}$ for which there exists a sequence of vector functions $u^m = \{u_1^m, \dots, u_N^m) \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T) := \{u \in C^2(\overline{D}_T) :\ u|_{S_T} = 0\}$ such that

$$\lim_{m\to\infty} \|u^m - u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0, \quad \lim_{m\to\infty} \|L_0 u^m - F\|_{L_2(D_T)} = 0. \tag{1.3.3}$$

For the solution $u \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ of the problem (1.3.1), (1.3.2) the following a priori estimate

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \le c(T)\|F\|_{L_2(D_T)}, \quad c(T) = \sqrt{T}\, \exp \frac{1}{2}\,(T + T^2) \tag{1.3.4}$$

is valid. Indeed, multiplying scalarly both parts of the equation (1.3.1) by $2\,\frac{\partial u}{\partial t}$ and integrating in the domain D_τ, $0 < \tau \le T$, after simple transformations, with the use of the equality (1.3.2) and integration by parts, we have the equality [45, p. 116]

$$\int\limits_{\Omega_\tau} \bigg[\Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2\bigg]\, dx = 2 \int\limits_{D_\tau} F\, \frac{\partial u}{\partial t}\, dx\, dt\,, \tag{1.3.5}$$

where $\Omega_\tau := D_T \cap \{t = \tau\}$. Since $S_T :\ t = |x|$, $t \le T$, due to (1.3.2), we have

$$u(x,\tau) = \int\limits_{|x|}^{\tau} \frac{\partial}{\partial t}\, u(x,s)\, ds, \quad (x,\tau) \in \Omega.$$

Squaring scalarly both parts of the obtained equation, integrating it in the domain Ω_τ and using the Schwartz inequality, we get

$$\begin{aligned} \int\limits_{\Omega_\tau} u^2\, dx &= \int\limits_{\Omega_\tau} \bigg(\int\limits_{|x|}^{\tau} \frac{\partial}{\partial t}\, u(x,s)\, ds\bigg)^2 dx \le \int\limits_{\Omega_\tau} (\tau - |x|)\bigg(\int\limits_{|x|}^{\tau} \Big(\frac{\partial u}{\partial t}\Big)^2 ds\bigg)\, dx \\ &\le T \int\limits_{\Omega_\tau} \bigg(\int\limits_{|x|}^{\tau} \Big(\frac{\partial u}{\partial t}\Big)^2 ds\bigg)\, dx = T \int\limits_{D_\tau} \Big(\frac{\partial u}{\partial t}\Big)^2\, dx\, dt. \end{aligned} \tag{1.3.6}$$

Denoting

$$w(\tau) = \int\limits_{\Omega_\tau} \bigg[u^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2\bigg]\, dx,$$

taking into account the inequality $2F\,\frac{\partial u}{\partial t} \le (\frac{\partial u}{\partial t})^2 + F^2$ and (1.3.5), (1.3.6), we have

$$\begin{aligned} w(\tau) &\le (1+T) \int\limits_{D_\tau} \Big(\frac{\partial u}{\partial t}\Big)^2\, dx\, dt + \int\limits_{D_\tau} F^2\, dx\, dt \\ &\le (1+T) \int\limits_{D_\tau} \bigg[u^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2\bigg]\, dx\, dt \\ &= (1+T) \int\limits_0^{\tau} w(s)\, ds + \|F\|^2_{L_2(D_\tau)}, \quad 0 < \tau \le T. \end{aligned} \tag{1.3.7}$$

According to the Gronwall lemma, from (1.3.7) it follows that

$$w(\tau) \le \|F\|^2_{L_2(D_\tau)} \exp(1+T)\tau \le \|F\|^2_{L_2(D_T)} \exp(1+T)T, \quad 0 < \tau \le T. \tag{1.3.8}$$

Further, according to (1.3.8), we have

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = \int\limits_{D_T} \left[u^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2\right] dx\,dt = \int\limits_0^T w(\tau)\,d\tau \le T\|F\|^2_{L_2(D_T)} \exp(1+T)T,$$

which ensures the a priori estimate (1.3.4).

Remark 1.3.2. Due to (1.3.3), for the strong generalized solution of the problem (1.3.1), (1.3.2) of the class W_2^1 in the domain D_T the a priori estimate (1.3.4) is also valid.

Since the space $C_0^\infty(D_T)$ of finite infinitely differentiable in D_T functions are dense in $L_2(D_T)$, for the given $F=(F_1,\dots,F_N)\in L_2(D_T)$ there exists a sequence of vector functions $F^m=(F_1^m,\dots,F_N^m)\in C_0^\infty(D_T)$ such that $\lim\limits_{m\to\infty}\|F^m-F\|_{L_2(D_T)}=0$. For the fixed m, extending F^m by zero beyond the domain D_T and retaining the same notation, we have $F^m\in C^\infty(\mathbb{R}_+^{n+1})$ with the support $\operatorname{supp}F^m\subset D_\infty$, where $\mathbb{R}_+^{n+1}:=\mathbb{R}^{n+1}\cap\{t\ge 0\}$. Denote by $u^m=(u_1^m,\dots,u_N^m)$ the solution of the Cauchy problem: $L_0u^m=F^m$, $u^m|_{t=0}=0$, $\frac{\partial u^m}{\partial t}\big|_{t=0}=0$, which exists, is unique and belongs to the space $C^\infty(\mathbb{R}_+^{n+1})$ [32, p. 192]. Since $\operatorname{supp}F^m\subset D_\infty$, $u^m|_{t=0}=0$, $\frac{\partial u}{\partial t}\big|_{t=0}=0$, in view of the geometry of the domain of dependence of the solution of the linear wave equation $L_0u^m=F^m$, we have $\operatorname{supp}u^m\subset D_\infty$ [32, p. 191]. Retaining the same notation, for the restriction of the vector function u^m on the domain D_T, one can see that $u^m\in\overset{\circ}{C}{}^2(\overline{D}_T,S_T)$ and, according to Remark 1.3.1 and (1.3.4),

$$\|u^m-u^k\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \le c(T)\|F^m-F^k\|_{L_2(D_T)}. \tag{1.3.9}$$

The sequence $\{F^m\}$ is fundamental in $L_2(D_T)$ and, due to (1.3.9), the sequence $\{u^m\}$ is likewise fundamental in the complete space $\overset{\circ}{W}{}_2^1(D_T,S_T)$. Therefore, there exists the vector function $u\in\overset{\circ}{W}{}_2^1(D_T,S_T)$ such that $\lim\limits_{m\to\infty}\|u^m-u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}=0$, and since $L_0u^m=F^m\to F$ in the space $L_2(D_T)$, according to Remark 1.3.1, this vector function will be the strong generalized solution of the problem (1.3.1), (1.3.2) of the class W_2^1 in the domain D_T. The uniqueness of this solution from the space $\overset{\circ}{W}{}_2^1(D_T,S_T)$ follows, in view of Remark 1.3.2, from the a priori estimate (1.3.4). Therefore, for the solution u of the problem (1.3.1), (1.3.2) we have $u=L_0^{-1}F$, where $L_0^{-1}:[L_2(D_T)]^N\to[\overset{\circ}{W}{}_2^1(D_T,S_T)]^N$ is a linear continuous operator, whose norm, according to Remark 1.3.2 and (1.3.4), has the following estimate:

$$\|L_0^{-1}\|_{[L_2(D_T)]^N\to[\overset{\circ}{W}{}_2^1(D_T,S_T)]^N} \le \sqrt{T}\,\exp\frac{1}{2}\,(T+T^2). \tag{1.3.10}$$

Remark 1.3.3. Due to (1.3.10), if the condition (1.2.3) is fulfilled, where $0\le\alpha<\frac{n+1}{n-1}$ and $F\in L_2(D_T)$, then in view of Remark 1.2.1, it is easy to see that the vector function $u=(u_1,\dots,u_N)\in\overset{\circ}{W}{}_2^1(D_T,S_T)$ is a strong generalized solution of the problem (1.2.1), (1.2.2) of the class W_2^1 in the domain D_T if and only if u is a solution of the functional equation

$$u=L_0^{-1}(-f(u)+F) \tag{1.3.11}$$

in the space $\overset{\circ}{W}{}_2^1(D_T,S_T)$.

Remark 1.3.4. Let the condition (1.2.3), where $0\le\alpha<\frac{n+1}{n-1}$, be fulfilled. We rewrite the equation (1.3.11) in the form

$$u=Au:=L_0^{-1}(-K_0u+F), \tag{1.3.12}$$

where the operator $K_0:[\overset{\circ}{W}{}_2^1(D_T,S_T)]^N\to[L_2(D_T)]^N$ from (1.2.4) is, due to Remark 1.2.1, a continuous compact operator. Therefore, in view of (1.3.10), (1.3.12), the operator $A:[\overset{\circ}{W}{}_2^1(D_T,S_T)]^N\to$

$[\overset{\circ}{W}{}_2^1(D_T,S_T)]^N$ is likewise continuous and compact. Denote by $B(0,r_0) := \{u = (u_1,\dots,u_N) \in \overset{\circ}{W}{}_2^1(D_T,S_T) :\ \|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq r_0\}$ a closed convex ball of radius r_0 with center at the origin in the Hilbert space $\overset{\circ}{W}{}_2^1(D_T,S_T)$. Since the operator A from (1.3.12), acting in the space $\overset{\circ}{W}{}_2^1(D_T,S_T)$, is continuous and compact, according to the Schauder principle, for the solvability of (1.3.12) in $\overset{\circ}{W}{}_2^1(D_T,S_T)$ it suffices to prove that the operator A maps the ball $B(0,r_0)$ into itself for some $r_0 > 0$ [90, p. 370].

Theorem 1.3.1. *Let f satisfy the condition* (1.2.3), *where* $1 \leq \alpha < \frac{n+1}{n-1}$; $g = 0$, $F \in L_{2,loc}(D_T)$ *and* $F_{D_T} \in L_2(D_T)$ *for every* $T > 0$. *Then the problem* (1.1.1), (1.1.2) *is locally solvable in the class* W_2^1, *i.e., there exists a number* $T_0 = T_0(F) > 0$ *such that for* $T < T_0$ *this problem has a strong generalized solution of the class* W_2^1 *in the domain* D_T *in the sense of Definition* 1.2.1.

Proof. Taking into account Remark 1.3.4, it suffices to prove the existence of the numbers $T_0 = T_0(F) > 0$ and $r_0 = r_0(T,F)$ such that for $T < T_0$, the operator A from (1.3.12) maps the ball $B(0,r_0)$ into itself. For this purpose, we find the needed estimate of $\|Au\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}$ for $u \in \overset{\circ}{W}{}_2^1(D_T,S_T)$.

For $u = (u_1,\dots,u_N) \in \overset{\circ}{W}{}_2^1(D_T,S_T)$, we denote by $\widetilde{u}$ the vector function representing the even continuation of u through the plane $t = T$ in the domain $D_T^* :\ T < t < 2T - |x|$, symmetric to D_T with respect to the same plane, i.e.,

$$\widetilde{u} = \begin{cases} u(x,t), & (x,t) \in D_T, \\ u(x,2T-t), & (x,t) \in D_T^*, \end{cases}$$

and $\widetilde{u}(x,t) = u(x,t)$ for $t = T$, $t = T$ in the sense of the trace theory. It is obvious that $\widetilde{u} \in \overset{\circ}{W}{}_2^1(\widetilde{D}_T) := \{v \in W_2^1(\widetilde{D}_T) :\ v|_{\partial \widetilde{D}_T} = 0\}$, where $\widetilde{D}_T :\ |x| < t < 2T - |x|$. Clearly, $\widetilde{D}_T = D_T \cup \Omega_T \cup D_T^*$, $\Omega_T := D_\infty \cap \{t = T\}$.

Using the inequality [93, p. 258]

$$\int\limits_\Omega |v|\, d\Omega \leq (\operatorname{mes}\Omega)^{1-\frac{1}{p}} \|v\|_{p,\Omega}, \quad p \geq 1,$$

and taking into account the equalities $\|\widetilde{u}\|^p_{L_p(\widetilde{D}_T)} = 2\|u\|^p_{L_p(D_T)}$, $\|\widetilde{u}\|^2_{\overset{\circ}{W}{}_2^1(\widetilde{D}_T)} = 2\|u\|^2_{\overset{\circ}{W}{}_2^1(D_T,S_T)}$, from the known multiplicative inequality [68, p. 78]

$$\|v\|_{p,\Omega} \leq \beta \|\nabla_{x,t} v\|^{\widetilde{\alpha}}_{m,\Omega} \|v\|^{1-\widetilde{\alpha}}_{r,\Omega} \quad \forall\, v \in \overset{\circ}{W}{}_2^1(\Omega), \quad \Omega \subset \mathbb{R}^{n+1},$$
$$\nabla x,t = \Big(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial t}\Big), \quad \widetilde{\alpha} = \Big(\frac{1}{r} - \frac{1}{p}\Big)\Big(\frac{1}{r} - \frac{1}{\widetilde{m}}\Big)^{-1}, \quad \widetilde{m} = \frac{(n+1)m}{n+1-m}$$

for $\Omega = \widetilde{D}_T \subset \mathbb{R}^{n+1}$, $v = \widetilde{u}$, $r = 1$, $m = 2$ and $1 < p \leq \frac{2(n+1)}{n-1}$, where $\beta = const > 0$ does not depend on v and T, follows the inequality

$$\|u\|_{L_p(D_T)} \leq c_0 (\operatorname{mes} D_T)^{\frac{1}{p}+\frac{1}{n+1}-\frac{1}{2}} \|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \quad \forall\, u \in \overset{\circ}{W}{}_2^1(D_T,S_T), \tag{1.3.13}$$

where $c_0 = const > 0$ does not depend on u and T. Taking into account the fact that $\operatorname{mes} D_T = \frac{\omega_n}{n+1} T^{n+1}$, where ω_n is the volume of a unit ball in $\mathbb{R}^n$, for $p = 2\alpha$, from (1.3.13), we obtain

$$\|u\|_{L_{2\alpha}(D_T)} \leq C_T \|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \quad \forall\, u \in \overset{\circ}{W}{}_2^1(D_T,S_T), \tag{1.3.14}$$

where

$$C_T = c_0 \Big(\frac{\omega_n}{n+1}\Big)^{\alpha_1} T^{(n+1)\alpha_1}, \quad \alpha_1 = \frac{1}{2\alpha} + \frac{1}{n+1} - \frac{1}{2}\,. \tag{1.3.15}$$

Note that $\alpha_1 = \frac{1}{2\alpha} + \frac{1}{n+1} - \frac{1}{2} > 0$ for $\alpha < \frac{n+1}{n-1}$ and, consequently, $\lim\limits_{t\to 0} C_T = 0$.

For the value of $\|K_0 u\|_{L_2(D_T)}$, where $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ and the operator K_0 acts according to the formula (1.2.4), in view of (1.2.3) and (1.3.14), we have the estimate

$$\|K_0 u\|^2_{L_2(D_T)} \leq \int\limits_{D_T} (M_1 + M_2 |u|^\alpha)^2 \, dx\, dt \leq 2M_1^2 \operatorname{mes} D_T + 2M_2^2 \int\limits_{D_T} |u|^{2\alpha} \, dx\, dt$$
$$= 2M_1^2 \operatorname{mes} D_T + 2M_2^2 \|u\|^{2\alpha}_{L_{2\alpha}(D_T} \leq 2M_1^2 \operatorname{mes} D_T + 2M_2^2 C_T^{2\alpha} \|u\|^{2\alpha}_{\overset{\circ}{W}{}_2^1(D_T,S_T)},$$

whence we obtain

$$\|K_0 u\|_{L_2(D_T)} \leq M_1 (2 \operatorname{mes} D_T)^{\frac{1}{2}} + \sqrt{2}\, M_2 C_T^\alpha \|u\|^\alpha_{\overset{\circ}{W}{}_2^1(D_T,S_T)}. \tag{1.3.16}$$

Further, from (1.3.10), (1.3.12) and (1.3.16), it follows that

$$\begin{aligned}
\|Au\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} &= \big\|L_0^{-1}(-K_0 u + F)\big\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \\
&\leq \|L_0^{-1}\|_{[L_2(D_T)]^N \to [\overset{\circ}{W}{}_2^1(D_T,S_T)]^N} \|(-K_0 u + F)\|_{L_2(D_T)} \\
&\leq \Big[\sqrt{T} \exp \frac{1}{2}\,(T + T^2)\Big] \big(\|K_0 u\|_{L_2(D_T)} + \|F\|_{L_2(D_T)}\big) \\
&\leq \Big[\sqrt{T} \exp \frac{1}{2}\,(T + T^2)\Big] \Big(M_1 (2 \operatorname{mes} D_T)^{\frac{1}{2}} + \sqrt{2}\, M_2 C_T^\alpha \|u\|^\alpha_{\overset{\circ}{W}{}_2^1(D_T,S_T)} + \|F\|_{L_2(D_T)}\Big) \\
&= a(T) \|u\|^\alpha_{\overset{\circ}{W}{}_2^1(D_T,S_T)} + b(T).
\end{aligned} \tag{1.3.17}$$

Here,

$$a(T) = \sqrt{2}\, M_2 C_T^\alpha \sqrt{T} \exp \frac{1}{2}\,(T + T^2), \tag{1.3.18}$$
$$b(T) = \Big[\sqrt{T} \exp \frac{1}{2}\,(T + T^2)\Big] \big(M_1 (2 \operatorname{mes} D_T)^{\frac{1}{2}} + \|F\|_{L_2(D_T)}\big). \tag{1.3.19}$$

For the fixed $T > 0$, with respect to the variable z we consider the equation

$$a z^\alpha + b = z, \tag{1.3.20}$$

where $a = a(T)$ and $b = b(T)$ are defined by (1.3.18) and (1.3.19), respectively.

First, consider the case $\alpha > 1$. A simple analysis, analogous to that given in the work [90, pp. 373, 374] for $\alpha = 3$, shows that:

(1) if $b = 0$, then the equation (1.3.20) has a unique positive root $z_2 = a^{-\frac{1}{\alpha-1}}$ besides the trivial root $z_1 = 0$;

(2) if $b > 0$, then for $0 < b < b_0$, where

$$b_0 = b_0(T) = \big[\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha}{\alpha-1}}\big] a^{-\frac{1}{\alpha-1}}, \tag{1.3.21}$$

the equation (1.3.20) has two positive roots z_1 and z_2, $0 < z_1 < z_2$; moreover, for $b = b_0$, these roots coincide and we have one positive root $z_1 = z_2 = z_0 = (\alpha a)^{-\frac{1}{\alpha-1}}$;

(3) for $b > b_0$, the equation (1.3.20) does not have nonnegative roots. Note that for $0 < b < b_0$, we have the inequalities $z_1 < z_0 = (\alpha a)^{-\frac{1}{\alpha-1}} < z_2$.

Due to the absolute continuity of the Lebesgue integral, we have

$$\lim_{T\to 0} \|F\|_{L_2(D_T)} = 0.$$

Therefore, taking into account that $\operatorname{mes} D_T = \frac{\omega_n}{n+1} T^{n+1}$,from (1.3.19) it follows that $\lim\limits_{T\to 0} b(T) = 0$. Besides, since $-\frac{1}{\alpha-1} < 0$ for $\alpha > 1$ and $\lim\limits_{t\to 0} C_T = 0$, from (1.3.18) and (1.3.21) we find that $\lim\limits_{T\to 0} b_0 = +\infty$. Therefore, there exists a number $T_0 = T_0(F) > 0$ such that for $0 < T < T_0$, due to (1.3.18)–(1.3.21), the condition $0 < b < b_0$ will be fulfilled and hence the equation (1.3.20) will have at least one positive root; we denote it by $r_0 = r_0(T, F)$.

When $\alpha = 1$, the equation (1.3.20) is linear, and $\lim\limits_{T\to 0} a(T) = 0$. Therefore, for $0 < T < T_0$, where $T_0 = T(F)$ is a sufficiently small positive number, this equation will have a unique positive root $z(T, F) = b(a - a)^{-1}$ which is also denoted by $r_0 = r_0(T, F)$.

Let us now show that the operator A from (1.3.12) maps the ball $B(0, r) \subset \overset{\circ}{W}{}^1_2(D_T, S_T)$ into itself. indeed, in view of (1.3.17) and the equality $ar_0^\alpha + b = r_0$, for every $u \in B(0, r_0)$ we have

$$\|Au\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} \leq a\|u\|^\alpha_{\overset{\circ}{W}{}^1_2(D_T,S_T)} + b \leq ar_0^\alpha + b = r_0. \tag{1.3.22}$$

According to Remark 1.3.4, the above reasoning proves Theorem 1.3.1. □

Theorem 1.3.2. *Let f satisfy the condition* (1.2.3)*, where $0 \leq \alpha < 1$; $g = 0$, $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for every $T > 0$. Then the problem* (1.1.1), (1.1.2) *is globally solvable in the class W_2^1, i.e., for any $T > 0$, the problem has a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition* 1.2.1.

Proof. According to Remark 1.3.4, it suffices to show that for any $T > 0$ there exists a number $r_0 = r_0(T, F) > 0$ such that the operator A from (1.3.12) maps the ball $B(0, r_0) \subset \overset{\circ}{W}{}^1_w(D_T, S_T)$ into itself. First, let $\frac{1}{2} < \alpha < 1$. Since $2\alpha > 1$, the inequality (1.3.14) is valid and thereby the estimate (1.3.17), as well. For the fixed $T > 0$, owing to $\alpha < 1$, there exists a number $r_0 = r_0(T, F) > 0$ such that

$$a(T)s^\alpha + b(T) \leq r_0 \ \ \forall\, s \in [0, r_0]. \tag{1.3.23}$$

Indeed, the function $\frac{\lambda(s)}{s}$, where $\lambda(s) = a(T)s^\alpha + b(T)$, is a monotonically decreasing continuous function, and $\lim\limits_{s\to+0} \frac{\lambda(s)}{s} = +\infty$ and $\lim\limits_{s\to+\infty} \frac{\lambda(s)}{s} = 0$. Therefore, there exists a number $s = r_0(T, F) > 0$ such that $\frac{\lambda(s)}{s}\big|_{s=r_0} = 1$. Hence, since the function $\lambda(s)$ for $s \geq 0$ is monotonically increasing, we immediately arrive at (1.3.23). Further, in view of (1.3.17) and (1.3.23), for every $u \in B(0, r_0)$ we have the inequality (1.3.22), i.e., $A(B(0, r_0)) \subset B(0, r_0)$.

The case $0 \leq \alpha \leq \frac{1}{2}$ can be reduced to the previous case $\frac{1}{2} < \alpha < 1$, since the vector function, satisfying the condition (1.2.3) for $0 \leq \alpha \leq \frac{1}{2}$, satisfies the same condition (1.2.3) for a certain fixed $\alpha = \alpha_1 \in (\frac{1}{2}, 1)$ with other positive constants M_1 and M_2 (it is easy to see that $M_1 + M_2|u|^\alpha \leq (M_1 + M_2) + M_2|u|^{\alpha_1}$ $\forall\, u \in \mathbb{R}$, $\alpha < \alpha_1$). This proves Theorem 1.3.2. □

1.4 The uniqueness and existence of the global solution of the problem (1.1.1), (1.1.2) of the class W_2^1

Below, we impose on the nonlinear vector function $f = (f_1, \dots, f_n)$ from (1.1.1) the additional requirements

$$f \in C^1(\mathbb{R}^N), \ \ \Big|\frac{\partial f_i(u)}{\partial u_j}\Big| \leq M_3 + M_4|u|^\gamma, \ \ 1 \leq i, j \leq N, \tag{1.4.1}$$

where M_3, M_4, $\gamma = const \geq 0$. For the sake of simplicity, we assume that the vector function $g = 0$ in the boundary condition (1.1.2), i.e., we consider the problem (1.2.1), (1.2.2).

Obviously, (1.4.1) results in the condition (1.2.3) for $\alpha = \gamma + 1$, and in the case for $\gamma < \frac{2}{n-1}$, we have $\alpha = \gamma + 1 < \frac{n+1}{n-1}$.

Theorem 1.4.1. *Let the condition* (1.4.1) *be fulfilled, where $0 \leq \gamma < \frac{2}{n-1}$, $F \in L_2(D_T)$, $g = 0$. Then the problem* (1.1.1), (1.1.2) *cannot have more than one strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition* 1.2.1.

Proof. Let $F \in L_2(D_T)$, $g = 0$ and the problem (1.1.1), (1.1.2) have two strong generalized solutions u^1 and u^2 of the class W_2^1 in the domain D_T in the sense of Definition 1.2.1, i.e., there exist two sequences of vector functions $u^{im} \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$, $i = 1, 2$; $m = 1, 2, \dots$, such that

$$\lim_{m\to\infty} \|u^{im} - u^i\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0, \quad \lim_{m\to\infty} \|Lu^{im} - F\|_{L_2(D_T)} = 0, \ \ i = 1, 2. \tag{1.4.2}$$

Let

$$w = u^2 - u^1, \quad w^m = u^{2m} - u^{1m}, \quad F^m = Lu^{2m} - Lu^{1m}. \tag{1.4.3}$$

According to (1.4.2), (1.4.3), we have

$$\lim_{m\to\infty} \|w^m - w\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0, \quad \lim_{m\to\infty} \|F^m\|_{L_2(D_T)} = 0. \tag{1.4.4}$$

In accordance with (1.2.1), (1.2.2) and (1.4.3), we consider the vector function $w^m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ as a solution of the following problem

$$\Box\, w^m = -\big[f(u^{2m}) - f(u^{1m})\big] + F^m, \tag{1.4.5}$$

$$w^m\big|_{S_T} = 0. \tag{1.4.6}$$

Multiplying scalarly both parts of the vector equality (1.4.5) by the vector $\frac{\partial w^m}{\partial t}$ in the space $\mathbb{R}^N$ and integrating by parts in the domain D_τ, $0 < \tau \le T$, due to (1.4.6), in the same way as that for obtaining the equality (1.3.5), we have

$$\int\limits_{\Omega_\tau} \bigg[\Big(\frac{\partial w^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial w^m}{\partial x_i}\Big)^2\bigg]\, dx$$
$$= 2\int\limits_{D_\tau} F^m\, \frac{\partial w^m}{\partial t}\, dx\, dt - 2\int\limits_{D_\tau} \big[f(u^{2m}) - f(u^{1m})\big]\, \frac{\partial u^m}{\partial t}\, dx\, dt, \ \ 0 < \tau \le T. \tag{1.4.7}$$

Taking into account the equality

$$f_i(u^{2m}) - f_i(u^{1m}) = \sum_{j=1}^N \int\limits_0^1 \frac{\partial}{\partial u_j}\, f_i\big(u^{1m} + s(u^{2m} - u^{1m})\big)\, ds\, (u_j^{2m} - u_j^{1m})$$

we obtain

$$\big[f(u^{2m}) - f(u^{1m})\big]\, \frac{\partial w^m}{\partial t} = \sum_{i,j=1}^N \bigg[\int\limits_0^1 \frac{\partial}{\partial u_j}\, f_i\big(u^{1m} + s(u^{2m} - u^{1m})\big)\, ds\bigg](u_j^{2m} - u_j^{1m})\, \frac{\partial w_i^m}{\partial t}\,. \tag{1.4.8}$$

From (1.4.1) and the obvious inequality

$$|D_1 + d_2|^\gamma \le 2^\gamma \max\big(|d_1|^\gamma, |d_2|^\gamma\big) \le 2^\gamma\big(|d_1|^\gamma + |d_2|^\gamma\big)$$

for $\gamma \ge 0$, $d_1, d_2 \in \mathbb{R}$, we have

$$\bigg|\int\limits_0^1 \frac{\partial}{\partial u_j}\, f_i\big(u^{1m} + s(u^{2m} - u^{1m})\big)\, ds\bigg|$$
$$\le \int\limits_0^1 \Big[M_3 + M_4\big|(1-s)u^{1m} + su^{2m}\big|^\gamma\Big]\, ds \le M_3 + 2^\gamma M_4\big(|u^{1m}|^\gamma + |u^{2m}|^\gamma\big). \tag{1.4.9}$$

From (1.4.8) and (1.4.9), with regard for (1.4.3), it follows that

$$\begin{aligned}\Big|\big[f(u^{2m})-f(u^{1m})\big]\,\frac{\partial w^m}{\partial t}\Big| &\le \sum_{i,j=1}^{n}\Big[M_3+2^{\gamma}M_4\big(|u^{1m}|^{\gamma}+|u^{2m}|^{\gamma}\big)\Big]\,|w_j^m|\,\Big|\frac{\partial w_i^m}{\partial t}\Big| \\ &\le N^2\Big[M_3+2^{\gamma}M_4\big(|u^{1m}|^{\gamma}+|u^{2m}|^{\gamma}\big)\Big]\,|w^m|\,\Big|\frac{\partial w^m}{\partial t}\Big| \\ &\le \frac{1}{2}\,N^2M_3\Big[(w^m)^2+\Big(\frac{\partial w^m}{\partial t}\Big)^2\Big]+2^{\gamma}N^2M_4\big(|u^{1m}|^{\gamma}+|u^{2m}|^{\gamma}\big)|w^m|\,\Big|\frac{\partial w^m}{\partial t}\Big|. \qquad (1.4.10)\end{aligned}$$

In view of (1.4.7) and (1.4.10), we have

$$\begin{aligned}\int\limits_{\Omega_\tau}\Big[\Big(\frac{\partial w^m}{\partial t}\Big)^2+\sum_{i=1}^{n}\Big(\frac{\partial w^m}{\partial x_i}\Big)^2\Big]\,dx \\ \le \int\limits_{D_\tau}\Big[\Big(\frac{\partial w^m}{\partial t}\Big)^2+(F^m)^2\Big]\,dx\,dt+N^2M_3\int\limits_{\Omega_\tau}\Big[(w^m)^2+\Big(\frac{\partial w^m}{\partial t}\Big)^2\Big]\,dx\,dt \\ +\,2^{\gamma+1}N^2M_4\int\limits_{D_\tau}\big(|u^{1m}|^{\gamma}+|u^{2m}|^{\gamma}\big)|w^m|\,\Big|\frac{\partial w^m}{\partial t}\Big|\,dx\,dt. \qquad (1.4.11)\end{aligned}$$

The last integral in the right-hand side of (1.4.11) can be estimated by means of Hölder's inequality

$$\begin{aligned}\int\limits_{D_\tau}\big(|u^{1m}|^{\gamma}+|u^{2m}|^{\gamma}\big)|w^m|\,\Big|\frac{\partial w^m}{\partial t}\Big|\,dx\,dt \\ \le \Big(\big\|\,|u^{1m}|^{\gamma}\big\|_{L_{n+1}(D_T)}+\big\|\,|u^{2m}|^{\gamma}\big\|_{L_{n+1}(D_T)}\Big)\|w^m\|_{L_p(D_\tau)}\Big\|\frac{\partial w^m}{\partial t}\Big\|^2_{L_2(D_\tau)}. \qquad (1.4.12)\end{aligned}$$

Here $\frac{1}{n+1}+\frac{1}{p}+\frac{1}{2}=1$, i.e.,

$$p=\frac{2(n+1)}{n-1}\,. \qquad (1.4.13)$$

For $1<q\le\frac{2(n+1)}{n-1}$, due to (1.3.13), we have

$$\|v\|_{L_q(D_\tau)}\le C_q(T)\|v\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}\quad \forall\,v\in \overset{\circ}{W}{}_2^1(D_T,S_T),\ \ 0<\tau<T, \qquad (1.4.14)$$

with the positive constant $C_q(T)$, not depending on $v\in \overset{\circ}{W}{}_2^1(D_T,S_T)$ and $\tau\in(0,T]$.

According to the conditions of the theorem $\gamma<\frac{2}{n-1}$, and hence $\gamma(n+1)<\frac{2(n+1)}{n-1}$. Thus, from (1.4.13) and (1.4.14), we get

$$\big\|\,|u^{im}|^{\gamma}\big\|_{L_{n+1}(D_T)}=\|u^{im}\|^{\gamma}_{L_{\gamma(n+1)}(D_T)}\le C^{\gamma}_{\gamma(n+1)}(T)\|u^{im}\|^{\gamma}_{\overset{\circ}{W}{}_2^1(D_T,S_T)},\ \ i=1,2;\ \ m\ge 1, \qquad (1.4.15)$$

$$\|w^m\|_{L_p(D_\tau)}\le C_p(T)\|w^m\|_{\overset{\circ}{W}{}_2^1(D_\tau)},\ \ m\ge 1. \qquad (1.4.16)$$

According to the first equality of (1.4.2), there exists a natural number m_0 such that for $m\ge m_0$, we have

$$\|u^{im}\|^{\gamma}_{\overset{\circ}{W}{}_2^1(D_T,S_T)}\le \|u^{i}\|^{\gamma}_{\overset{\circ}{W}{}_2^1(D_T,S_T)}+1,\ \ i=1,2;\ \ m\ge 1. \qquad (1.4.17)$$

Taking into account the above equalities, from (1.4.12)–(1.4.17) it follows that

$$2^{\gamma+1}N^2M_4\int\limits_{D_\tau}\big(|u^{1m}|^\gamma+|u^{2m}|^\gamma\big)|w^m|\,\Big\|\frac{\partial w^m}{\partial t}\Big\|\,dx\,dt$$

$$\leq 2^{\gamma+1}N^2M_4C^\gamma_{\gamma(n+1)}(T)\Big(\|u^1\|^\gamma_{\overset{\circ}{W}{}^1_2(D_T,S_T)}+\|u^2\|^\gamma_{\overset{\circ}{W}{}^1_2(D_T,S_T)}+2\Big)C_p(T)\|w^m\|_{\overset{\circ}{W}{}^1_2(D_\tau,S_\tau)}\Big\|\frac{\partial w^m}{\partial t}\Big\|^2_{L_2(D_\tau)}$$

$$\leq M_5\Big(\|w^m\|^2_{\overset{\circ}{W}{}^1_2(D_\tau)}+\Big\|\frac{\partial w^m}{\partial t}\Big\|^2_{L_2(D_\tau)}\Big)$$

$$\leq 2M_5\|w^m\|^2_{W^1_2(D_\tau)}=2M_5\int\limits_{D_\tau}\bigg[(w^m)^2+\Big(\frac{\partial w^m}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial w^m}{\partial x_i}\Big)^2\bigg]\,dx\,dt, \quad (1.4.18)$$

where

$$M_5=2^\gamma N^2M_4C^\gamma_{\gamma(n+1)}(T)\Big(\|u^1\|^\gamma_{\overset{\circ}{W}{}^1_2(D_T,S_T)}+\|u^2\|^\gamma_{\overset{\circ}{W}{}^1_2(D_T,S_T)}+2\Big)C_p(T).$$

In view of (1.4.18), from (1.4.11) we have

$$\int\limits_{\Omega_\tau}\bigg[\Big(\frac{\partial w^m}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial w^m}{\partial x_i}\Big)^2\bigg]\,dx$$

$$\leq M_6\int\limits_{\Omega_\tau}\bigg[(w^m)^2+\Big(\frac{\partial w^m}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial w^m}{\partial x_i}\Big)^2\bigg]\,dx\,dt+\int\limits_{D_T}(F^m)^2\,dx\,dt,\quad 0<\tau\leq T, \quad (1.4.19)$$

where $M_6=1+M_3N^2+2M_5$.

Note that the inequality (1.3.6) is valid for w^m, as well, and therefore,

$$\int\limits_{\Omega_\tau}(w^m)^2\,dx\leq T\int\limits_{D_\tau}\Big(\frac{\partial w^m}{\partial t})^2\,dx\,dt\leq T\int\limits_{D_\tau}\bigg[(w^m)^2+\Big(\frac{\partial w^m}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial w^m}{\partial x_i}\Big)^2\bigg]\,dx\,dt. \quad (1.4.20)$$

Putting

$$\lambda_m(\tau):=\int\limits_{\Omega_\tau}\bigg[(w^m)^2+\Big(\frac{\partial w^m}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial w^m}{\partial x_i}\Big)^2\bigg]\,dx\,dt \quad (1.4.21)$$

and adding up the inequalities (1.4.19) and (1.4.20), we obtain

$$\lambda_m(\tau)\leq(M_6+T)\int\limits_0^\tau\lambda_m(s)\,ds+\|F^m\|^2_{L_2(D_T)}.$$

Hence, in view of the Gronwall lemma, it follows that

$$\lambda_m(\tau)\leq\|F^m\|^2_{L_2(D_T)}\exp(M_6+T)\tau. \quad (1.4.22)$$

From (1.4.21) and (1.4.22) we have

$$\|w^m\|^2_{W^1_2(D_T)}=\int\limits_0^T\lambda_m(\tau)\,d\tau\leq T\|F^m\|^2_{L_2(D_T)}\exp(M_6+T)T. \quad (1.4.23)$$

Due to (1.4.3) and (1.4.4), from (1.4.23) it follows that

$$\|w\|_{W^1_2(D_T)}=\lim_{m\to\infty}\|w-w^m+w^m\|^2_{W^1_2(D_T)}\leq\lim_{m\to\infty}\|w-w^m\|_{W^1_2(D_T)}+\lim_{m\to\infty}\|w^m\|_{W^1_2(D_T)}$$

$$=\lim_{m\to\infty}\|w-w^m\|_{W^1_2(D_T)}=\lim_{m\to\infty}\|w-w^m\|_{\overset{\circ}{W}{}^1_2(D_T)}=0.$$

Therefore, $w=u_2-u_1=0$, i.e., $u_2=u_1$, which proves Theorem 1.4.1. □

From Theorems 1.3.2 and 1.4.1 the following existence and uniqueness theorem immediately follows.

Theorem 1.4.2. *Let the vector function f satisfy the condition* (1.2.3) *for $\alpha < 1$, and the condition* (1.4.1) *for $\gamma < \frac{2}{n-1}$. Then for every $F \in L_2(D_T)$ and $g = 0$, the problem* (1.1.1), (1.1.2) *has a unique strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ of the class W_2^1 in the domain D_T in the sense of Definition* 1.2.1.

The theorem below on the existence of a global solution of the problem (1.1.1), (1.1.2) follows from Theorem 1.4.2.

Theorem 1.4.3. *Let the vector function f satisfy the condition* (1.2.3) *for $\alpha < 1$, and the condition* (1.4.1) *for $\gamma < \frac{2}{n-1}$; $g = 0$ and $F \in L_{2,loc}(D_\infty)$ for every $F|_{D_T} \in L_2(D_T)$. Then the problem* (1.1.1), (1.1.2) *has a unique strong generalized solution $u \in W_{2,loc}^1(D_\infty)$ of the class W_2^1 in the cone of the future D_∞ in the sense of Definition* 1.2.4.

Proof. According to Theorem 1.4.2, under the fulfilment of the conditions of Theorem 1.4.3 for $T = m$, where m is a natural number, there exists a unique strong generalized solution $u^m \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ of the problem (1.1.1), (1.1.2) of the class W_2^1 in the domain $D_{T=m}$ in the sense of Definition 1.2.1. Since $u^{m+1}|_{D_{T=m}}$ is likewise a strong generalized solution of the problem (1.1.1), (1.1.2) of the class W_2^1 in the domain $D_{T=m}$, according to Theorem 1.4.2, we have $u^m = u^{m+1}|_{D_{T=m}}$, from which we obtain the following scheme of constructing a unique global strong generalized solution $u \in \overset{\circ}{W}{}_{2,loc}^1(D_\infty, S_\infty)$ of the problem (1.1.1), (1.1.2) of the class W_2^1 in the cone of the future D_∞ in the sense of Definition 1.2.4:

$$u(x,t) = u^m(x,t), \quad (x,t) \in D_\infty, \quad m = [t] + 1,$$

where $[t]$ is an integer part of the number. Thus Theorem 1.4.3 is proved. □

1.5 The cases of nonexistence of a global solution of the problem (1.1.1), (1.1.2) of the class W_2^1. Blow-up solutions of the problem (1.1.1), (1.1.2) of the class W_2^1

Theorem 1.5.1. *Let the vector function $f = (f_1, \dots, f_N)$ satisfy the condition* (1.2.3), *when $1 < \alpha < \frac{n+1}{n-1}$, and there exist the numbers $\ell_1, \ell_2, \dots, \ell_N$, $\sum\limits_{i=1}^N |\ell_i| \neq 0$ such that*

$$\sum_{i=1}^N \ell_i f_i(u) \le c_0 - c_1 \Big| \sum_{i=1}^N \ell_i u_i \Big|^\beta \ \ \forall\, u \in \mathbb{R}^N, \ \ 1 < \beta = const < \frac{n+1}{n-1}, \tag{1.5.1}$$

where $c_0, c_1 = const$, $c_1 > 0$. Let $F \in L_{2,loc}(D_\infty)$, $g \in W_{2,loc}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2(S_T)$ for every $T > 0$. Suppose that at least one of the functions $F_0 = \sum\limits_{i=1}^N \ell_i F_i - c_0$ or $\frac{\partial g_0}{\partial \mathcal{N}}\big|_{S_\infty}$, where $g_0 = \sum\limits_{i=1}^N \ell_i g_i$, is nontrivial (*i.e., differs from zero on a subset of positive measure in D_∞ or S_∞, respectively*). *If*

$$g_0 \ge 0, \ \ \frac{\partial g_0}{\partial \mathcal{N}}\Big|_{S_\infty} \le 0, \quad F_0\big|_{D_\infty} \ge 0, \tag{1.5.2}$$

then there exists a finite positive number $T_0 = T_0(F, g)$ such that for $T > T_0$, the problem (1.1.1), (1.1.2) *does not have a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition* 1.2.1. *Here, $\frac{\partial}{\partial \mathcal{N}}$ is a derivative along the conormal to S_∞, i.e., $\frac{\partial}{\partial \mathcal{N}} = \nu_{n+1} \frac{\partial}{\partial t} - \sum\limits_{i=1}^n \nu_i \frac{\partial}{\partial x_i}$, where $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ is a unit vector of the outer normal to $\partial D_\infty = S_\infty$.*

Proof. Let $u=(u_1,\dots,u_N)$ be a strong generalized solution of the problem (1.1.1), (1.1.2) of the class W_2^1 in the domain D_T. Here we apply the method of test functions [77, pp. 10–12]. According to Remark 1.2.3, the solution u of this problem satisfies the integral equality (1.2.5) in which we take as a test function $\varphi=(\ell_1\psi,\ell_2\psi,\dots,\ell_N\psi)$, where $\psi=\psi_0[2T^{-2}(t^2+|x|^2)]$ and the scalar function $\psi_0\in C^2((-\infty,\infty))$ satisfies the conditions $\psi_0\geq 0$, $\psi_0'\leq 0$; $\psi_0(\sigma)=1$ for $0\leq\sigma\leq 1$ and $\psi(\sigma)=0$ for $\sigma\geq 2$ [77, p. 22]. For such a test function φ with notations $v=\sum\limits_{i=1}^N \ell_i u_i$, $g_0=\sum\limits_{i=1}^N \ell_i g_i$, $F_*=\sum\limits_{i=1}^N \ell_i F_i$, $f_0=\sum\limits_{i=1}^N \ell_i f_i$, the integral equality (1.2.5) takes the form

$$\int\limits_{D_T}\big[-v_t\psi_t+\nabla v\nabla\psi\big]\,dx\,dt=-\int\limits_{D_T} f_0(u)\psi\,dx\,dt+\int\limits_{D_T} F_*\psi\,dx\,dt-\int\limits_{S_T}\frac{\partial g_0}{\partial\mathcal{N}}\,\psi\,ds. \tag{1.5.3}$$

Since $\psi|_{t\geq T}=0$ and the equality $v|_{S_T}=g_0$ holds in the sense of the trace theory, integrating by parts the left-hand side of the equality (1.5.3), we get

$$\int\limits_{D_T}\big[-v_t\psi_t+\nabla v\nabla\psi\big]\,dx\,dt=\int\limits_{D_T} v\,\square\,\psi\,dx\,dt-\int\limits_{S_T} v\,\frac{\partial\psi}{\partial\mathcal{N}}\,ds=\int\limits_{D_T} v\,\square\,\psi\,dx\,dt-\int\limits_{S_T} g_0\,\frac{\partial\psi}{\partial\mathcal{N}}\,ds. \tag{1.5.4}$$

From (1.5.3) and (1.5.4), due to (1.5.1) and $\psi\geq 0$, we obtain the inequality

$$\begin{aligned}\int\limits_{D_T} v\,\square\,\psi\,dx\,dt&\geq\int\limits_{D_T}\big[c_1|v|^\beta-c_0\big]\psi\,dx\,dt+\int\limits_{D_T}F_*\psi\,dx\,dt+\int\limits_{S_T} g_0\,\frac{\partial\psi}{\partial\mathcal{N}}\,ds-\int\limits_{S_T}\frac{\partial g_0}{\partial\mathcal{N}}\,\psi\,ds\\&=c_1\int\limits_{D_T}|v|^\beta\psi\,dx\,dt+\int\limits_{D_T}(F_*-c_0)\psi\,dx\,dt+\int\limits_{S_T} g_0\,\frac{\partial\psi}{\partial\mathcal{N}}\,ds-\int\limits_{S_T}\frac{\partial g_0}{\partial\mathcal{N}}\,\psi\,ds.\end{aligned} \tag{1.5.5}$$

According to the properties of the function ψ and the inequalities (1.5.2), the inequalities

$$\frac{\partial\psi}{\partial\mathcal{N}}\Big|_{S_T}\geq 0,\ \ \int\limits_{S_T} g_0\,\frac{\partial\psi}{\partial\mathcal{N}}\,ds\geq 0,\ \ \int\limits_{S_T}\frac{\partial g_0}{\partial\mathcal{N}}\,\psi\,ds\leq 0,\ \ \int\limits_{D_T}F_0\psi\,dx\,dt\geq 0, \tag{1.5.6}$$

where $F_0=F_*-c_0=\sum\limits_{i=1}^N\ell_iF_i-c_0$, are obvious.

Assuming that the functions F, g and ψ are fixed, we introduce the function of one variable

$$\gamma(T)=\int\limits_{D_T}F_0\psi\,dx\,dt+\int\limits_{S_T} g_0\,\frac{\partial\psi}{\partial\mathcal{N}}\,ds-\int\limits_{S_T}\frac{\partial g_0}{\partial\mathcal{N}}\,\psi\,ds,\ \ T>0. \tag{1.5.7}$$

Due to the absolute continuity of the integral and the inequalities (1.5.6), the function $\gamma(T)$ from (1.5.7) is nonnegative, continuous and nondecreasing; besides,

$$\lim_{T\to 0}\gamma(T)=0, \tag{1.5.8}$$

and since, according to our supposition, one of the functions $\frac{\partial g_0}{\partial\mathcal{N}}\big|_{S_\infty}$ or F_0 is nontrivial, we get

$$\lim_{T\to\infty}\gamma(T)>0. \tag{1.5.9}$$

In view of (1.5.7), the inequality (1.5.5) can be rewritten as follows:

$$c_1\int\limits_{D_T}|v|^\beta\psi\,dx\,dt\leq\int\limits_{D_T} v\,\square\,\psi\,dx\,dt-\gamma(T). \tag{1.5.10}$$

If in Young's inequality with the parameter $\varepsilon > 0$: $ab \le (\varepsilon/\beta)a^\beta + (\beta'\varepsilon^{\beta'-1})^{-1}b^{\beta'}$, where $\beta' = \frac{\beta}{\beta-1}$, we take $a = |v|\psi^{1/\beta}$, $b = |\square\,\psi|/\psi^{1/\beta}$, then, in view of the equality $\beta'/\beta = \beta' - 1$, we have

$$|v\,\square\,\psi| = |v|\psi^{1/\beta}\,\frac{|\square\,\psi|}{\psi^{1/\beta}} \le \frac{\varepsilon}{\beta}\,|v|^\beta\psi + \frac{|\square\,\psi|^{\beta'}}{\beta'\varepsilon^{\beta'-1}\psi^{\beta'-1}}\,. \tag{1.5.11}$$

Due to (1.5.11), from (1.5.10) we have the inequality

$$\Big(c_1 - \frac{\varepsilon}{\beta}\Big)\int\limits_{D_T}|v|^\beta\psi\,dx\,dt \le \frac{1}{\beta'\varepsilon^{\beta'-1}}\int\limits_{D_T}\frac{|\square\,\psi|^{\beta'}}{\psi^{\beta'-1}}\,dx\,dt - \gamma(T),$$

whence for $\varepsilon < c_1\beta$, we get

$$\int\limits_{D_T}|v|^\beta\psi\,dx\,dt \le \frac{\beta}{(c_1\beta-\varepsilon)\beta'\varepsilon^{\beta'-1}}\int\limits_{D_T}\frac{|\square\,\psi|^{\beta'}}{\psi^{\beta'-1}}\,dx\,dt - \frac{\beta}{c_1\beta-\varepsilon}\,\gamma(T). \tag{1.5.12}$$

Since $\beta' = \frac{\beta'}{\beta-1}$, $\beta = \frac{\beta'}{\beta'-1}$, due to the equality

$$\min_{0<\varepsilon<c_\beta}\frac{\beta}{(c_1\beta-\varepsilon)\beta'\varepsilon^{\beta'-1}} = \frac{1}{c_1^{\beta'}}\,,$$

which is achieved for $\varepsilon = c_1$, it follows from (1.5.12) that

$$\int\limits_{D_T}|v|^\beta\psi\,dx\,dt \le \frac{1}{c_1^{\beta'}}\int\limits_{D_T}\frac{|\square\,\psi|^{\beta'}}{\psi^{\beta'-1}}\,dx\,dt - \frac{\beta'}{c_1}\,\gamma(T). \tag{1.5.13}$$

According to the properties of the function ψ_0, the test function

$$\psi(x,t) = \psi_0\big[2T^{-2}(t^2+|x|^2)\big] = 0$$

for $r = (t^2+|x|^2)^{1/2} > T$. Therefore, after changing of variables $t = \sqrt{2}\,T\xi_0$, $x = \sqrt{2}\,T\xi$, it is not difficult to verify that

$$\int\limits_{D_T}\frac{|\square\,\psi|^{\beta'}}{\psi^{\beta'-1}}\,dx\,dt = \int\limits_{r=(t^2+|x|^2)^{1/2}\le T}\frac{|\square\,\psi|^{\beta'}}{\psi^{\beta'-1}}\,dx\,dt = (\sqrt{2}\,T)^{n+1-2\beta'}\,\varkappa_0. \tag{1.5.14}$$

Here,

$$\varkappa_0 = \int\limits_{1\le|\xi_0|^2+|\xi|^2\le 2} 2^{\frac{|2(1-n)\psi_0'+4(\xi_0^2-|\xi|^2)\psi_0''|^{\beta'}}{\psi_0^{\beta'-1}}}\,d\xi\,d\xi_0 < +\infty. \tag{1.5.15}$$

As is known, the test function $\psi(x,t) = \psi_0[2T^{-2}(t^2+|x|^2)]$ with the aforementioned properties, for which the condition (1.5.15) is fulfilled, exists [77, p. 22].

Due to (1.5.14), from (1.5.13), in view of $\psi_0(\sigma) = 1$ for $0 \le \sigma \le 1$, we have

$$\int\limits_{r\le\frac{T}{\sqrt{2}}}|v|^\beta\,dx\,dt \le \int\limits_{D_T}|v|^\beta\psi\,dx\,dt \le \frac{|\sqrt{2}\,T)^{n+1-2\beta'}}{c_1^{\beta'}}\,\varkappa_0 - \frac{\beta'}{c_1}\,\gamma(T). \tag{1.5.16}$$

In the case if $\beta < \frac{n+1}{n-1}$, i.e., if $n+1-2\beta' < 0$, the equation

$$\lambda(T) = \frac{(\sqrt{2}\,T)^{n+1-2\beta'}}{c_1^{\beta'}}\,\varkappa_0 - \frac{\beta'}{c_1}\,\gamma(T) = 0 \tag{1.5.17}$$

has a unique positive root $T = T_0(F, g)$, since the function

$$\lambda_1(T) = \frac{(\sqrt{2}\,T)^{n+1-2\beta'}}{c_1^{\beta'}}\,\varkappa_0$$

is a positive, continuous, strictly decreasing in $(0, +\infty)$, besides,

$$\lim_{T\to 0} \lambda_1(T) = +\infty \ \text{ and } \lim_{T\to+\infty} \lambda_1(T) = 0$$

and the function $\gamma(T)$ is, as noted above, nonnegative, continuous and nondecreasing, satisfying the conditions (1.5.8) and (1.5.9). Besides, $\lambda(T) < 0$ for $T > T_0$ and $\lambda(T) > 0$ for $0 < T < T_0$. Therefore, for $T > T_0$, the right-hand side of the inequality (1.5.16) is a negative value, which is impossible. Thus this contradiction proves Theorem 1.5.1. □

Remark 1.5.1. Let us consider one class of vector functions f satisfying the condition (1.5.1):

$$f_i(u_1, \dots, u_N) = \sum_{j=1}^{N} a_{ij}|u_j|^{\beta_{ij}} + b_i, \ \ i = 1, \dots, N, \tag{1.5.18}$$

where $a_{ij} = const > 0$, $b_i = const$, $1 < b_{ij} = const < \frac{n+1}{n-1}$, $i, j = 1, \dots, N$. In this case we can take $\ell_1 = \ell_2 = \dots = \ell_N = -1$. Indeed, we choose $\beta = const$ such that $1 < \beta < \beta_{ij}$, $i, j = 1, \dots, N$. It is easy to verify that $|s|^{\beta_{ij}} \geq |s|^{\beta} - 1\ \forall\, s \in (\infty, \infty)$. Using the inequality [21, p. 302]

$$\sum_{i=1}^{N} |y_i|^{\beta} \geq N^{1-\beta}\Big|\sum_{i=1}^{N} y_i\Big|^{\beta} \ \ \forall\, y = (y_1, \dots, y_N) \in \mathbb{R}^N, \ \ \beta = const > 1,$$

we get

$$\begin{aligned}\sum_{i=1}^{N} f_i(u_1, \dots, u_N) &\geq a_0 \sum_{i,j=1}^{N} |u_j|^{\beta_{ij}} + \sum_{i=1}^{N} b_i \geq a_0 \sum_{i,j=1}^{N} \big(|u_j|^{\beta} - 1\big) + \sum_{i=1}^{N} b_i \\ &= a_0 N \sum_{j=1}^{N} |u_j|^{\beta} - a_0 N^2 + \sum_{i=1}^{N} b_i \geq a_0 N^{2-\beta}\Big|\sum_{j=1}^{N} u_j\Big|^{\beta} + \sum_{i=1}^{N} b_i - a_0 N^2, \ \ a_0 = \min_{i,j} a_{ij} > 0.\end{aligned}$$

Hence we have the inequality (1.5.1) in which

$$\ell_1 = \ell_2 = \dots = \ell_N = -1, \ \ c_0 = a_0 N^2 - \sum_{i=1}^{N} b_i, \ \ c_1 = a_0 N^{2-\beta} > 0.$$

Note that the vector function f, represented by the equalities (1.5.18), likewise satisfies the condition (1.5.1) with $\ell_1 = \ell_2 = \dots = \ell_N = -1$ for less restrictive conditions when: $a_{ij} = const \geq 0$, but $a_{ik_i} > 0$, where $k_1, \dots, k_N$ is any fixed permutation of numbers $1, 2 \dots, N$; $i, j = 1, \dots, N$.

Remark 1.5.2. From Theorem 1.5.1 it follows that in the conditions of this theorem the problem (1.1.1), (1.1.2) cannot have a global strong generalized solution of the class W_2^1 in the domain D_∞ in the sense of Definition 1.2.4.

Remark 1.5.3. Let the vector function $f = (f_1, \dots, f_N)$ satisfy the condition (1.2.3) for $1 < \alpha < \frac{n+1}{n-1}$, the condition (1.4.1) for $\gamma < \frac{2}{n-1}$ and also the condition (1.5.1). Let $g = 0$, $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for every $T > 0$ and, moreover, let F satisfy the third condition of (1.5.2). Then, taking into account the fact that a strong generalized solution u of the problem (1.1.1), (1.1.2) of the class W_2^1 in the domain D_T in the sense of Definition 1.2.1 is also a solution of that problem in a smaller domain D_{T_1} for $T_1 < T$, from Theorems 1.3.1, 1.4.1 and 1.5.1 follows the existence of a finite positive number $T_* = T_*(F)$ such that for $T > T_*$, the problem (1.1.1), (1.1.2) does not have a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition 1.2.1. There exists a unique vector function $u = (u_1, \dots, u_N) \in W_{2,loc}^1(D_{T_*})$ such that for any $T < T_*$, the vector function u is a strong generalized solution of the problem (1.1.1), (1.1.2) of the class W_2^1 in the domain D_T. This vector function can be considered as a blow-up solution of the problem (1.1.1), (1.1.2) of the class W_2^1 in the sense that $\|u\|_{W_2^1(D_T)} < +\infty$ for $T < T_*$ and $\lim\limits_{T\to T_*-0} \|u\|_{W_2^1(D_T)} = +\infty$.

Chapter 2

One multidimensional version of the Darboux first problem for one class of semilinear second order hyperbolic systems

2.1 Statement of the Problem

In the Euclidean space $\mathbb{R}^{n+1}$ of independent variables $x = (x_1, \dots, x_n)$ and t consider a second order semilinear hyperbolic system of the form

$$\frac{\partial^2 u_i}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u_i}{\partial x_i^2} + f_i(u_1, \dots, u_N) = F_i, \quad i = 1, \dots, N, \tag{2.1.1}$$

where $f = (f_1, \dots, f_N)$, $F = (F_1, \dots, F_N)$ are the given, and $u = (u_1, \dots, u_N)$ is an unknown vector function, $n \geq 2$, $N \geq 2$.

Denote by $D: \ t > |x|, \ x_n > 0$, the half of the light cone of the future bounded by the part $S^0: \ \partial D \cap \{x_n = 0\}$ of a hyperplane $x_n = 0$ and the half $S: \ t = |x|, \ x_n \geq 0$, of the characteristic conoid $\Lambda: \ t = |x|$ of the system (2.1.1). Let $D_T := \{(x,t) \in D: \ t < T\}$, $S_T^0 := \{(x,t) \in S^0: \ t \leq T\}$, $S_T := \{(x,t) \in S: \ t \leq T\}$, $T > 0$.

For the system of equations (2.1.1) consider the problem on finding a solution $u(x,t)$ of this system by the following boundary conditions

$$\frac{\partial u}{\partial x_n}\Big|_{S_T^0} = 0, \quad u\big|_{S_T} = g, \tag{2.1.2}$$

where $g = (g_1, \dots, g_N)$ is a given vector function on S_T. In the case $T = \infty$, we have $D_\infty = D$, $S_\infty^0 = S^0$ and $S_\infty = S$.

The problem (2.1.1), (2.1.2) represents a multidimensional version of the Darboux first problem for the system (2.1.1), when one part of the problem data support is a characteristic manifold, while another part is of time type manifold [5, pp. 228, 233].

The questions on the existence and nonexistence of a global solution of the Cauchy problem for semilinear scalar equations of the form (2.1.1) with the initial conditions $u|_{t=0} = u_0$, $\frac{\partial u}{\partial t}\big|_{t=0} = u_1$ have been considered by many authors (see the corresponding references in Chapter 1). As is known, for the second order scalar linear hyperbolic equations, the multidimensional versions of the Darboux first problem are well-posed and they are globally solvable in suitable function spaces [5, 42, 43, 81, 91, 92]. In regard to the multidimensional problem (2.1.1), (2.1.2) for a scalar case, i.e., when $N = 1$, in the case of nonlinearity of the form $f(u) = \lambda |u|^p u$, in [51] it is shown that depending on the sign of the parameter λ and the values of the power exponent p, the problem (2.1.1), (2.1.2) is globally solvable in some cases and not globally solvable in other cases. Another multidimensional version of the Darboux first problem for a scalar semilinear equation of the form (2.1.1), where instead of the boundary condition $\frac{\partial u}{\partial x_n}\big|_{S_T^0} = 0$ in (2.1.2) is taken $u|_{S_T^0} = 0$, is considered in [9]. Noteworthy is the fact that the multidimensional version of the Darboux second problem for a scalar semilinear equation of the form (2.1.1) is studied in [56].

In the present chapter we introduce certain conditions for the nonlinear vector function $f = f(u)$ from (2.1.1) the fulfilment of which ensures local or global solvability of the problem (2.1.1), (2.1.2), while in some cases it will not have global solution, though it will be locally solvable.

2.2 Definition of a generalized solution of the problem (2.1.1), (2.1.2) in D_T and D_∞

Let

$$\overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T) := \Big\{ u \in C^2(\overline{D}_T) : \ \frac{\partial u}{\partial x_n}\Big|_{S_T^0} = 0, \ u\big|_{S_T} = 0 \Big\}.$$

Let, moreover, $\overset{\circ}{W}{}_2^1(D_T, S_T) := \{u \in W_2^1(D_T) : \ u|_{S_T} = 0\}$, where $W_2^k(\Omega)$ is the Sobolev space consisting of the elements of $L_2(\Omega)$ having up to the k-th order generalized derivatives from $L_2(\Omega)$, inclusive. Here, the equality $u|_{S_T} = 0$ should be understood in the sense of the trace theory [68, p. 71].

Below, under the belonging of the vector $v = (v_1, \dots, v_N)$ to some space X we mean the belonging of each component v_i, $1 \le i \le N$, of that vector to the same space X. In accordance with the above-said, for the sake of simplicity of our writing and to avoid misunderstanding, instead of $v = (v_1, \dots, v_N) \in [X]^N$, we write $v \in X$.

Rewrite the system of equations (2.1.1) in the form of one vector equation

$$Lu := \Box\, u + f(u) = F_1, \tag{2.2.1}$$

where $\Box := \frac{\partial^2}{\partial t^2} - \Delta$, $\Delta := \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

Together with the boundary conditions (2.1.2), we consider the corresponding homogeneous boundary conditions

$$\frac{\partial u}{\partial x_n}\Big|_{S_T^0} = 0, \quad u\big|_{S_T} = 0. \tag{2.2.2}$$

Below, on the nonlinear vector function $f = (f_1, \dots, f_N)$ in (2.1.1) we impose the following requirement

$$f \in C(\mathbb{R}^N), \ \ |f(u)| \le M_1 + M_2|u|^\alpha, \ \ \alpha = const \ge 0, \ \ u \in \mathbb{R}^N, \tag{2.2.3}$$

where $|\cdot|$ is a norm in the space $\mathbb{R}^N$, $M_i = const \ge 0$, $i = 1, 2$.

Remark 2.2.1. The embedding operator $I : [W_2^1(D_T)]^N \to [L_q(D_T)]^N$ is a linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$, when $n > 1$ [68, p. 86]. At the same time, Nemitski's operator $\mathcal{K} : [L_q(D_T)]^n \to [L_2(D_T)]^N$ acting by the formula $\mathcal{K}u = f(u)$, where $u = (u_1, \dots, u_N) \in [L_q(D_T)]^N$, and the vector function $f = (f_1, \dots, f_N)$ satisfies the condition (2.2.3), is continuous and bounded for $q \ge 2\alpha$ [67, p. 349], [22, pp. 66, 67]. Thus, if $\alpha < \frac{n+1}{n-1}$, i.e., $2\alpha < \frac{2(n+1)}{n-1}$, then there exists a number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \ge 2\alpha$. Therefore, in this case, the operator

$$\mathcal{K}_0 = \mathcal{K}I : [W_2^1(D_T)]^N \to [L_2(D_T)]^N \tag{2.2.4}$$

is continuous and compact. Clearly, from $u = (u_1, \dots, u_N) \in W_2^1(D_T)$ it follows that $f(u) \in L_2(D_T)$ and, if $u^m \to u$ in the space $W_2^1(D_T)$, then $f(u^m) \to f(u)$ in the space $L_2(D_T)$.

Definition 2.2.1. Let $f = (f_1, \dots, f_N)$ satisfy the condition (2.2.3), where $0 \le \alpha < \frac{n+1}{n-1}$, $F = (F_1, \dots, F_N) \in L_2(D_T)$ and $g = (g_1, \dots, g_N) \in W_2^1(S_T)$. We call the vector function $u = (u_1, \dots, u_N)$

$\in W_2^1(D_T)$ a strong generalized solution of the problem (2.1.1), (2.1.2) of the class W_2^1 in the domain D_T if there exists a sequence of vector functions $u^m \in C^2(\overline{D}_T)$ such that $\frac{\partial u^m}{\partial t}\big|_{S_T^0} = 0$, $u^m \to u$ in the space $W_2^1(D_T)$, $Lu^m \to F$ in the space $L_2(D_T)$ and $u^m|_{S_T} \to g$ in the space $W_2^1(S_T)$. Convergence of the sequence $\{f(u^m)\}$ to $f(u)$ in the space $L_2(D_T)$ as $u^m \to u$ in the space $W_2^1(D_T)$ follows from Remark 2.2.1. When $g = 0$, i.e., in the case of homogeneous boundary conditions (2.2.2), we assume that $u^m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T)$. Then it is clear that $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$.

It is obvious that the classical solution $u \in C^2(\overline{D}_T)$ of the problem (2.1.1), (2.1.2) is a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition 2.2.1.

Remark 2.2.2. It is easy to verify that if $u \in C^2(\overline{D}_T)$ is a classical solution of the problem (2.1.1), (2.1.2), then multiplying scalarly both sides of the system (2.2.1) by any test vector function $\varphi = (\varphi_1, \dots, \varphi_N) \in C^2(\overline{D}_T)$ satisfying the condition $\varphi|_{t=T} = 0$, after integration by parts, we obtain the equality

$$\int\limits_{D_T} \big[-u_t\varphi_t + \nabla u \nabla\varphi\big]\, dx\, dt = -\int\limits_{D_T} f(u)\varphi\, dx\, dt + \int\limits_{D_T} F\varphi\, dx\, dt - \int\limits_{S_T^0 \cup S_T} \frac{\partial u}{\partial \mathcal{N}}\, \varphi\, ds, \tag{2.2.5}$$

where $\frac{\partial}{\partial \mathcal{N}} = \nu_{n+1}\frac{\partial}{\partial t} - \sum\limits_{i=1}^{n} \nu_i \frac{\partial}{\partial x_i}$ is the derivative with respect to the conormal, $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ is the unit vector of the outer normal to ∂D_T, and $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. Taking into account that $\frac{\partial}{\partial \mathcal{N}}\big|_{S_T^0} = \frac{\partial}{\partial x_n}$ and S_T is a characteristic manifold on which the operator $\frac{\partial}{\partial \mathcal{N}}$ is an inner differential operator, from (2.1.2) we have

$$\frac{\partial u}{\partial \mathcal{N}}\Big|_{S_T^0} = 0, \quad \frac{\partial u}{\partial \mathcal{N}}\Big|_{S_T} = \frac{\partial g}{\partial \mathcal{N}}\Big|_{S_T}.$$

Therefore, the equality (2.2.5) takes the form

$$\int\limits_{D_T} \big[-u_t\varphi_t + \nabla u \nabla\varphi\big]\, dx\, dt = -\int\limits_{D_T} f(u)\varphi\, dx\, dt + \int\limits_{D_T} F\varphi\, dx\, dt - \int\limits_{S_T} \frac{\partial g}{\partial \mathcal{N}}\, \varphi\, ds. \tag{2.2.6}$$

It can be easily seen that the equality (2.2.6) is valid also for any vector function $\varphi = (\varphi_1, \dots, \varphi_N) \in W_2^1(D_T)$ such that $\varphi|_{t=T} = 0$ in the sense of the trace theory. Note that the equality (2.2.6) is also valid for a strong generalized solution $u \in W_2^1(D_T)$ of the problem (2.1.1), (2.1.2) of the class W_2^1 in the domain D_T in the sense of Definition 2.2.1. Indeed, if $u^m \in C^2(\overline{D}_T)$ is a sequence of vector functions from Definition 2.2.1, then writing the equality (2.2.6) for $u = u^m$ and passing to the limit as $m \to \infty$, we obtain (2.2.6). It should be noted that the equality (2.2.6), valid for any test vector function $\varphi \in W_2^1(D_T)$ satisfying the condition $\varphi|_{t=T} = 0$, can be put in the basis of the definition of a weak generalized solution $u \in W_2^1(D_T)$ of the problem (2.1.1), (2.1.2) of the class W_2^1 in the domain D_T.

Definition 2.2.2. Let f satisfy the condition (2.2.3), where $0 \le \alpha < \frac{n+1}{n-1}$, $F \in L_{2,loc}(D_\infty)$, $g \in W_{2,loc}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. We say that the problem (2.1.1), (2.1.2) is locally solvable in the class W_2^1 if there exists a number $T_0 = T_0(F, g) > 0$ such that for any $T < T_0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition 2.2.1.

Definition 2.2.3. Let f satisfy the condition (2.2.3), where $0 \le \alpha < \frac{n+1}{n-1}$, $F \in L_{2,loc}(D_\infty)$, $g \in W_{2,loc}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. We say that the problem (2.1.1), (2.1.2) is globally solvable in the class W_2^1 if for any $T > 0$ this problem has a strong generalized solution of the class in the domain D_T in the sense of Definition 2.2.1.

Definition 2.2.4. Let f satisfy the condition (2.2.3), where $0 \le \alpha < \frac{n+1}{n-1}$, $F \in L_{2,loc}(D_\infty)$, $g \in W_{2,loc}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. The vector function $u =$

$(u_1,\dots,u_N)\in W^1_{2,loc}(D_\infty)$ is called a global strong generalized solution of the problem (2.1.1), (2.1.2) of the class W^1_2 in the domain D_∞ if for any $T>0$ the vector function $u|_{D_T}$ belongs to the space $W^1_2(D_T)$ and is a strong generalized solution of the problem (2.1.1), (2.1.2) of the class W^1_2 in the domain D_T in the sense of Definition 2.2.1.

Remark 2.2.3. Reasoning used in the proof of the equation (2.2.6) makes it possible to conclude that the global strong generalized solution $u=(u_1,\dots,u_N)$ of the problem (2.1.1), (2.1.2) of the class W^1_2 in the domain D_∞ in the sense of Definition 2.2.4 satisfies the following integral equality

$$\int\limits_{D_\infty}\big[-u_t\varphi_t+\nabla u\nabla\varphi\big]\,dx\,dt=-\int\limits_{D_\infty}f(u)\varphi\,dx\,dt+\int\limits_{D_\infty}F\varphi\,dx\,dt-\int\limits_{S_\infty}\frac{\partial g}{\partial\mathcal N}\,\varphi\,ds$$

for any test vector function $\varphi=(\varphi_1,\dots,\varphi_N)\in C^1(D_\infty)$, which is finite with respect to the variable $r=(t^2+|x|^2)^{1/2}$, i.e., $\varphi=0$ for $r>r_0=const>0$.

2.3 Some cases of local and global solvability of the problem (2.1.1), (2.1.2) in the class W_2^1

For the sake of simplicity, we consider the case where the boundary conditions (2.1.2) are homogeneous. In this case the problem (2.1.1), (2.1.2) can be rewritten in the form (2.2.1), (2.2.2).

Remark 2.3.1. Before we proceed to considering the solvability of the problem (2.1.1), (2.1.2), let us consider the same question for the linear case, when the vector function $f=0$ in (2.2.1), i.e., for the problem

$$L_0u:=\square\,u=F(x,t),\ \ (x,t)\in D_T, \tag{2.3.1}$$

$$\frac{\partial u}{\partial x_n}\Big|_{S_T^0}=0,\quad u\big|_{S_T}=0. \tag{2.3.2}$$

For the problem (2.3.1), (2.3.2), by analogy to that in Definition 2.2.1 for the problem (2.1.1), (2.1.2), we introduce the notion of a strong generalized solution $u=(u_1,\dots,u_N)$ of the class W^1_2 in the domain D_T for $F=(F_1,\dots,F_N)\in L_2(D_T)$, i.e., for the vector function $u=(u_1,\dots,u_N)\in\overset{\circ}{W}{}^1_2(D_T,S_T)$, for which there exists a sequence of vector functions $u^m=(u^m_1,\dots,u^m_N)\in\overset{\circ}{C}{}^1_2(\overline D_T,S^0_T,S_T)$ such that

$$\lim_{m\to\infty}\|u^m-u\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)}=0,\quad \lim_{m\to\infty}\|L_0u^m-F\|_{L_2(D_T)}=0. \tag{2.3.3}$$

For the solution $u\in\overset{\circ}{C}{}^1_2(\overline D_T,S^0_T,S_T)$ of the problem (2.3.1), (2.3.2) the estimate

$$\|u\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)}\le c(T)\|F\|_{L_2(D_T)},\ \ c(T)=\sqrt T\,\exp\frac12\,(T+T^2), \tag{2.3.4}$$

is valid. Indeed, multiplying scalarly both parts of the vector equation (2.3.2) by $2\,\frac{\partial u}{\partial t}$ and integrating in the domain D_τ, $0<\tau\le T$, after simple transformations with the use of the equalities (2.3.2) and integration by parts, we arrive at the equality [51], [45, p. 116]

$$\int\limits_{\Omega_\tau}\Big[\Big(\frac{\partial u}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial u}{\partial x_i}\Big)^2\Big]\,dx=2\int\limits_{D_\tau}F\,\frac{\partial u}{\partial t}\,dx\,dt, \tag{2.3.5}$$

where $\Omega_\tau:=D_T\cap\{t=\tau\}$. Since $S_\tau:\ t=|x|,\ x_n\ge0,\ t\le\tau$, due to (2.3.2), we get

$$u(x,\tau)=\int\limits_{|x|}^{\tau}\frac{\partial}{\partial t}\,u(x,s)\,ds,\ \ (x,s)\in\Omega_\tau.$$

Squaring scalarly both parts of the obtained equality, integrating it in the domain Ω_τ and using the Schwartz inequality, we have

$$\int\limits_{\Omega_\tau} u^2\,dx = \int\limits_{\Omega_\tau}\Big(\int\limits_{|x|}^{\tau}\frac{\partial}{\partial t}\,u(x,s)\Big)^2 dx \le \int\limits_{\Omega_\tau}(\tau-|x|)\Big(\int\limits_{|x|}^{\tau}\Big(\frac{\partial u}{\partial t}\Big)^2 ds\Big)\,dx$$
$$\le T\int\limits_{\Omega_\tau}\Big(\int\limits_{|x|}^{\tau}\Big(\frac{\partial u}{\partial t}\Big)^2 ds\Big)^2 dx = T\int\limits_{D_\tau}\Big(\frac{\partial u}{\partial t}\Big)^2 dx\,dt. \tag{2.3.6}$$

Let

$$w(\tau) := \int\limits_{\Omega_\tau}\Big[u^2+\Big(\frac{\partial u}{\partial t}\Big)^2+\sum_{i=1}^{n}\Big(\frac{\partial u}{\partial x_i}\Big)^2\Big]\,dx.$$

Taking into account the inequality $2F\,\frac{\partial u}{\partial t} \le (\frac{\partial u}{\partial t})^2 + F^2$, due to (2.3.5) and (2.3.6), we have

$$w(\tau) \le (1+T)\int\limits_{D_T}\Big(\frac{\partial u}{\partial t}\Big)^2 dx\,dt + \int\limits_{D_\tau} F^2\,dx\,dt$$
$$\le (1+T)\int\limits_{D_T}\Big[u^2+\Big(\frac{\partial u}{\partial t}\Big)^2+\sum_{i=1}^{n}\Big(\frac{\partial u}{\partial x_i}\Big)^2\Big]\,dx\,dt + \|f\|^2_{L_2(D_T)}$$
$$= (1+T)\int\limits_{0}^{\tau} w(s)\,ds + \|F\|^2_{L_2(D_T)},\ \ 0<\tau\le T. \tag{2.3.7}$$

According to the Gronwall lemma, from (2.3.7) it follows that

$$w(\tau) \le \|F\|^2_{L_2(D_T)}\exp(1+T)T,\ \ 0<\tau\le T. \tag{2.3.8}$$

Using (2.3.8), we get

$$\|u\|^2_{\overset{\circ}{W}{}^1_2(D_T,S_T)} = \int\limits_{D_\tau}\Big[u^2+\Big(\frac{\partial u}{\partial t}\Big)^2+\sum_{i=1}^{n}\Big(\frac{\partial u}{\partial x_i}\Big)^2\Big]\,dx\,dt = \int\limits_{0}^{T} w(\tau)\,d\tau \le T\|F\|^2_{L_2(D_T)}\exp(1+T)T,$$

which results in the estimate (2.3.4).

Remark 2.3.2. Due to (2.3.3), a priori estimate (2.3.4) is also valid for a strong generalized solution of the problem (2.3.1), (2.3.2) of the class W_2^1 in the domain D_T.

Since the space $C_0^\infty(D_T)$ of finite infinitely differentiable in D_T functions is dense in $L_2(D_T)$, for the given $F=(F_1,\dots,F_N)\in L_2(D_T)$ there exists a sequence of vector functions $F^m=(F_1^m,\dots,F_N^m)\in C_0^\infty(D_T)$ such that

$$\lim_{m\to\infty}\|F^m-F\|_{L_2(D_T)}=0.$$

For the fixed m, extending F^m evenly with respect to the variable x_n in the domain $D_T^- := \{(x,t)\in\mathbb{R}^{n+1}:\ x_n<0,\ |x|<t<T\}$ and then by zero beyond the domain $D_T\cup D_T^-$ and retaining the same notation, we have $F^m\in C^\infty(\mathbb{R}^{n+1}_+)$, for which the support $\operatorname{supp} F^m\subset D_\infty\cup D_\infty^-$, where $\mathbb{R}^{n+1}_+ := \mathbb{R}^{n+1}\cap\{t\ge 0\}$. Denote by u^m the solution of the Cauchy problem

$$L_0u^m := \Box\,u^m = F^m,\quad u^m\big|_{t=0}=0,\quad \frac{\partial u^m}{\partial t}\Big|_{t=0}=0, \tag{2.3.9}$$

which, as is well-known [32, p. 192], exists, is unique and belongs to the space $C^\infty(\mathbb{R}^{n+1}_+)$. Since $\operatorname{supp} F^m\subset D_\infty\cup D_\infty^-\subset\{(x,t)\in\mathbb{R}^{n+1}:\ t>|x|\}$, $u^m|_{t=0}=0$ and $\frac{\partial u^m}{\partial t}\big|_{t=0}=0$, taking into account the geometry of the domain of dependence of the solution of the linear wave equation $L_0 u^m = F^m$, we have $\operatorname{supp} u^m \subset \{(x,t) \in \mathbb{R}^{n+1} : t > |x|\}$ [32, p. 191] and, in particular, $u^m|_{S_T} = 0$. On the other hand, the vector function $\widetilde{u}^m(x_1, \dots, x_n, t) = u^m(x_1, \dots, -x_n, t)$ is likewise a solution of the same Cauchy problem (2.3.9), since the vector function F^m is even with respect to the variable x_n. Therefore, due to the uniqueness of the solution of the Cauchy problem, we have $\widetilde{u}^m = u^m$, i.e., $u^m(x_1, \dots, -x_n, t) = u^m(x_1, \dots, x_n, t)$, and hence the vector function u^m is likewise an even function with respect to the variable x_n. This, in turn, implies that $\frac{\partial u^m}{\partial x_n}\big|_{x_n=0} = 0$, which under the condition $u^m|_{S_T} = 0$ indicates that if we retain the same notation for the restriction of the vector function u^m in the domain D_T, then it is obvious that $u^m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T)$. Further, due to (2.3.4) and (2.3.9), the inequality

$$\|u^m - u^k\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq c(T)\|F^m - F^k\|_{L_2(D_T)} \tag{2.3.10}$$

is valid.

Since the sequence $\{F^m\}$ is fundamental in $L_2(D_T)$, due to (2.3.10), the sequence $\{u^m\}$ is also fundamental in the complete space $\overset{\circ}{W}{}_2^1(D_T, S_T)$. Therefore, there exists a vector function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ such that

$$\lim_{m\to\infty} \|u^m - u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0,$$

and since $L_0 u^m = F^m \to F$ in the space $L_2(D_T)$, this vector function is, according to Remark 2.3.1, a strong generalized solution of the problem (2.3.1), (2.3.2) of the class W_2^1 in the domain D_T. The uniqueness of that solution from the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ follows, due to Remark 2.3.2, from the a priori estimate (2.3.4). Therefore, for the solution u of the problem (2.3.1), (2.3.2) we can write $u = L_0^{-1}F$, where $L_0^{-1} : [L_2(D_T)]^N \to [\overset{\circ}{W}{}_2^1(D_T, S_T)]^N$ is a linear continuous operator with a norm admitting, in view of (2.3.4), the following estimate:

$$\|L_0^{-1}\|_{[L_2(D_T)]^N \to [\overset{\circ}{W}{}_2^1(D_T,S_T)]^N} \leq \sqrt{T}\, \exp\frac{1}{2}\,(T + T^2). \tag{2.3.11}$$

Remark 2.3.3. Taking into account (2.3.11), when the condition (2.2.3) is fulfilled, where $0 \leq \alpha < \frac{n+1}{n-1}$ and $F \in L_2(D_T)$, due to Remark 2.2.1, it is easy to see that the vector function $u = (u_1, \dots, u_N) \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ is a strong generalized solution of the problem (2.2.1), (2.2.2) of the class W_2^1 in the domain D_T if and only if u is a solution of the functional equation

$$u = L_0^{-1}(-f(u) + F) \tag{2.3.12}$$

in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$.

Remark 2.3.4. Let the condition (2.2.3) be fulfilled and $0 \leq \alpha < \frac{n+1}{n-1}$. We rewrite the equation (2.3.12) in the form

$$u = Au := L_0^{-1}(-\mathcal{K}_0 u + F), \tag{2.3.13}$$

where the operator $\mathcal{K}_0 : [\overset{\circ}{W}{}_2^1(D_T, S_T)]^N \to [L_2(D_T)]^N$ from (2.2.4) is, due to Remark 2.2.1, continuous and compact. Therefore, according to (2.3.11) and (2.3.13), the operator $\mathcal{A} : [\overset{\circ}{W}{}_2^1(D_T, S_T)]^N \to [\overset{\circ}{W}{}_2^1(D_T, S_T)]^N$ is also continuous and compact. Denote by $B(0, r_0) := \{u = (u_1, \dots, u_N) \in \overset{\circ}{W}{}_2^1(D_T, S_T) : \|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq r_0\}$ a closed convex ball of radius $r_0 > 0$ with center in a null element in the Hilbert space $\overset{\circ}{W}{}_2^1(D_T, S_T)$.

Since the operator $\mathcal{A}$ from (2.3.13), acting in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$, is a compact continuous operator, according to the Schauder principle, for the solvability of the equation (2.3.13) in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ it suffices to prove that the operator $\mathcal{A}$ maps the ball $B(0, r_0)$ into itself for some $r_0 > 0$ [90, p. 370].

Theorem 2.3.1. *Let f satisfy the condition* (2.2.3)*, where $1 \le \alpha < \frac{n+1}{n-1}$, $g = 0$, $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for any $T > 0$. Then the problem* (2.1.1), (2.1.2) *is locally solvable in the class W_2^1, i.e., there exists a number $T_0 = T_0(F) > 0$ such that for any $T < T_0$, this problem has a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition* 2.2.1.

Proof. Due to Remark 2.3.4, it suffices to prove the existence of positive numbers $T_0 = T_0(F)$ and $r_0 = r_0(T, F)$ such that for $T < T_0$, the operator $\mathcal{A}$ from (2.3.13) maps the ball $B(0, r_0)$ into itself. Towards this end, let us evaluate $\|\mathcal{A}u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}$ for $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$. If $u = (u_1, \dots, u_N) \in \overset{\circ}{W}{}_2^1(D_T, S_T)$, we denote by $\widetilde{u}$ the vector function which represents an even extension of u through the planes $x_n = 0$ and $t = T$. Obviously, $\widetilde{u} \in \overset{\circ}{W}{}_2^1(D_T^*) := \{v \in W_2^1(D_T^* :\ v|_{\partial D_T^*} = 0\}$, where $D_T^* :\ |x| < t < 2T - |x|$.

Using the inequality [93, p. 258]

$$\int\limits_{\Omega} |v|\, d\Omega \le (\operatorname{mes}\Omega)^{1-\frac{1}{p}} \|v\|_{p,\Omega}, \ \ p \ge 1,$$

and taking into account the equalities

$$\|\widetilde{u}\|^p_{L_p(D_T^*)} = 2\|u\|^p_{L_p(D_T)}, \quad \|\widetilde{u}\|^2_{\overset{\circ}{W}{}_2^1(D_T^*)} = 2\|u\|^2_{\overset{\circ}{W}{}_2^1(D_T,S_T)},$$

from the known multiplicative inequality [68, p. 78]

$$\|v\|_{p,\Omega} \le \beta \|\nabla_{x,t} v\|^{\widetilde{\alpha}}_{m,\Omega} \|v\|^{1-\widetilde{\alpha}}_{r,\Omega} \ \ \forall\, v \in \overset{\circ}{W}{}_2^1(\Omega), \ \ \Omega \subset \mathbb{R}^{n+1},$$
$$\nabla_{x,t} = \Big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\Big), \quad \widetilde{\alpha} = \Big(\frac{1}{r} - \frac{1}{p}\Big)\Big(\frac{1}{r} - \frac{1}{\widetilde{m}}\Big)^{-1}, \ \ \widetilde{m} = \frac{(n+1)m}{n+1-m}$$

for $\Omega = D_T^* \subset \mathbb{R}^{n+1}$, $v = \widetilde{u}$, $r = 1$, $m = 2$ and $1 < p \le \frac{2(n+1)}{n+1-m}$, where $\beta = const > 0$ does not depend on v and T, we obtain the following inequality:

$$\|u\|_{L_p(D_T)} \le c_0 (\operatorname{mes} D_T))^{\frac{1}{p} + \frac{1}{n+1} - \frac{1}{2}} \|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \ \ \forall\, u \in \overset{\circ}{W}{}_2^1(D_T, S_T), \tag{2.3.14}$$

where $c_0 = const > 0$ does not depend on u and T.

Since $\operatorname{mes} D_T = \frac{\omega_n}{n+1}\, T^{n+1}$, where ω_n is the volume of a unit ball in $\mathbb{R}^n$, from (2.3.14) for $p = 2\alpha$ we get

$$\|u\|_{L_{2\alpha}(D_T)} \le C_T \|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \ \ \forall\, u \in \overset{\circ}{W}{}_2^1(D_T, S_T), \tag{2.3.15}$$

where $C_T = c_0 (\frac{\omega_n}{n+1})^{\alpha_1} T^{(n+1)\alpha_1}$, $\alpha_1 = \frac{1}{2\alpha} + \frac{1}{n+1} - \frac{1}{2}$.

Note that $\alpha_1 = \frac{1}{2\alpha} + \frac{1}{n+1} - \frac{1}{2} > 0$ for $\alpha < \frac{n+1}{n-1}$, and hence

$$\lim_{T \to 0} C_T = 0. \tag{2.3.16}$$

For $\|\mathcal{K}_0 u\|_{L_2(D_T)}$, where $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ and the operator $\mathcal{K}_0$ acts according to the formula (2.2.4), due to (2.2.3) and (2.3.15), we have the estimate

$$\|\mathcal{K}_0 u\|^2_{L_2(D_T)} \le \int\limits_{D_T} (M_1 + M_2 |u|^\alpha)^2\, dx\, dt \le 2M_2^1 \operatorname{mes} D_T + 2M_2^2 \int\limits_{D_T} |u|^{2\alpha}\, dx\, dt$$
$$= 2M_1^2 \operatorname{mes} D_T + 2M_2^2 \|u\|^{2\alpha}_{L_{2\alpha}(D_T)} \le 2M_1^2 \operatorname{mes} D_T + 2M_2^2 C_T^{2\alpha} \|u\|^{2\alpha}_{\overset{\circ}{W}{}_2^1(D_T,S_T)},$$

whence

$$\|\mathcal{K}_0 u\|_{L_{2\alpha}(D_T)} \le M_1 (2 \operatorname{mes} D_T)^{\frac{1}{2}} + \sqrt{2}\, M_2 C_T^\alpha \|u\|^\alpha_{\overset{\circ}{W}{}_2^1(D_T,S_T)}. \tag{2.3.17}$$

It follows from (2.3.11), (2.3.13) and (2.3.17) that

$$\begin{aligned}\|Au\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} &= \big\|L_0^{-1}(-\mathcal{K}_0u+F)\big\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}\\ &\le \|L_0^{-1}\|_{[L_2(D_T)]^N\to[\overset{\circ}{W}{}_2^1(D_T,S_T)]^N}\|(-\mathcal{K}_0u+F)\|_{L_2(D_T)}\\ &\le \Big[\sqrt{T}\,\exp\frac{1}{2}\,(T+T^2)\Big]\big(\|\mathcal{K}_0u\|_{L_2(D_T)}+\|F\|_{L_2(D_T)}\big)\\ &\le \Big[\sqrt{T}\,\exp\frac{1}{2}\,(T+T^2)\Big]\Big(M_1(2\,\mathrm{mes}\,D_T)^{\frac{1}{2}}+\sqrt{2}\,M_2C_T^\alpha\|u\|^\alpha_{\overset{\circ}{W}{}_2^1(D_T,S_T)}+\|F\|_{L_2(D_T)}\Big)\\ &= a(T)\|u\|^\alpha_{\overset{\circ}{W}{}_2^1(D_T,S_T)}+b(T). \qquad (2.3.18)\end{aligned}$$

Here,

$$a(T)=\sqrt{2}\,M_2C_T^\alpha\sqrt{T}\,\exp\frac{1}{2}\,(T+T^2), \qquad (2.3.19)$$

$$b(T)=\Big[\sqrt{T}\,\exp\frac{1}{2}\,(T+T^2)\Big]\big(M_1(2\,\mathrm{mes}\,D_T)^{\frac{1}{2}}+\|F\|_{L_2(D_T)}\big). \qquad (2.3.20)$$

For the fixed $T>0$, consider the equation

$$az^\alpha+b=z \qquad (2.3.21)$$

with respect to the unknown $z\in\mathbb{R}$, where $a=a(T)$ and $b=b(T)$ are defined by (2.3.19) and (2.3.20).

First, consider the case $\alpha>1$. A simple analysis, analogous to that performed for $\alpha=3$ in [90, pp. 373, 374], shows that:

(1) for $b=0$, together with a trivial root $z_1=0$, the equation (2.3.21) has a unique positive root $z_2=a^{-\frac{1}{\alpha-1}}$;

(2) if $b>0$, then for $0<b<b_0$, where

$$b_0=b_0(T)=\big[\alpha^{-\frac{1}{\alpha-1}}-\alpha^{-\frac{\alpha}{\alpha-1}}\big]a^{-\frac{1}{\alpha-1}}, \qquad (2.3.22)$$

the equation (2.3.21) has two positive roots z_1 and z_2, $0<z_1<z_2$, and for $b=b_0$, these roots merge, and we have one positive root $z_1=z_2=z_0=(\alpha a)^{-\frac{1}{\alpha-1}}$;

(3) for $b>b_0$, the equation (2.3.21) does not have nonnegative roots. Note that for $0<b<b_0$, the inequality $z_1<z_0=(\alpha a)^{-\frac{1}{\alpha-1}}<z_2$ is valid.

In view of the absolute continuity of the Lebesgue integral, we have $\lim\limits_{T\to0}\|F\|_{L_2(D_T)}=0$. Therefore, taking into account that $\mathrm{mes}\,D_T=\frac{\omega_n}{n+1}\,T^{n+1}$, it follows from (2.3.20) that $\lim\limits_{T\to0}b(T)=0$. At the same time, since $-\frac{1}{\alpha-1}<0$ for $\alpha>1$, due to (2.3.16), from (2.3.19) and (2.3.22), we get $\lim\limits_{T\to0}b_0(T)=\infty$. Therefore, there exists a number $T_0=T_0(F)>0$ such that for $0<T<T_0$, in view of (2.3.19)–(2.3.22), the condition $0<b<b_0$ holds and hence the equation (2.3.21) has at least one positive root, we denote it by $r_0=r_0(T,F)$.

In case $\alpha=1$, the equation (2.3.21) is linear, where $\lim\limits_{T\to0}a(T)=0$. Therefore, for $0<T<T_0$, where $T_0=T_0(F)$ is a sufficiently small positive number, this equation will have a unique positive root $z(T,F)=b(1-a)^{-1}$, which we also denote by $r_0=r_0(T,F)$.

Now, we will show that the operator $\mathcal{A}$ from (2.3.13) maps the ball $B(0,r_0)\subset\overset{\circ}{W}{}_2^1(D_T,S_T)$ into itself. Indeed, due to (2.3.18) and the equality $ar_0^\alpha+b=r_0$, for any $u\in B(0,r_0)$, we have

$$\|\mathcal{A}u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}\le a\|u\|^\alpha_{\overset{\circ}{W}{}_2^1(D_T,S_T)}+b\le ar_0^\alpha+b=r_0. \qquad (2.3.23)$$

In view of Remark 2.3.4, the above reasoning proves Theorem 2.3.1. □

Theorem 2.3.2. *Let f satisfy the condition* (2.2.3)*, where $0 \le \alpha < 1$, $g = 0$, $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for any $T > 0$. Then the problem* (2.1.1), (2.1.2) *is globally solvable in the class W_2^1, i.e., for any $T > 0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition* 2.2.1.

Proof. According to Remark 2.3.4, to prove Theorem 2.3.2, it suffices to show that for any $T > 0$ there exists a number $r_0 = r_0(T, F) > 0$ such that the operator $\mathcal{A}$ from (2.3.13) maps the ball $B(0, r_0) \subset \overset{\circ}{W}{}_2^1(D_T, S_T)$ into itself. Let $\frac{1}{2} < \alpha < 1$, then since $2\alpha > 1$, the equality (2.3.15) is valid and hence the estimate (2.3.18) is also valid. For the fixed $T > 0$, since $\alpha < 1$, there exists a number $r_0 = r_0(T, F) > 0$ such that

$$a(T)s^\alpha + b(T) \le r_0 \ \ \forall\, s \in [0, r_0]. \tag{2.3.24}$$

Indeed, the function $\frac{\lambda(s)}{s}$, where $\lambda(s) = a(T)s^\alpha + b(T)$, is a continuous decreasing function and

$$\lim_{s\to 0+} \frac{\lambda(s)}{s} = +\infty, \quad \lim_{s\to +\infty} \frac{\lambda(s)}{s} = 0.$$

Therefore, there exists a number $s = r_0(T, F) > 0$ such that $\frac{\lambda(s)}{s}\big|_{s=r_0} = 1$. This implies that since the function $\lambda(s)$ for $s \ge 0$ is a monotonic increasing function, (2.3.24) follows immediately. Now, in view of (2.3.18) and (2.3.24), for any $u \in B(0, r_0)$, the inequality (2.3.23) is valid, i.e., $A(B(0, r_0)) \subset B(0, r_0)$.

The case $0 \le \alpha \le \frac{1}{2}$ can be reduced to the previous case $\frac{1}{2} < \alpha < 1$, since the vector function f satisfying the condition (2.2.3) for $0 \le \alpha \le \frac{1}{2}$ satisfies the same condition (2.2.3) for a certain fixed $\alpha = \alpha \in (\frac{1}{2}, 1)$ with other positive constants M_1 and M_2 (it is easy to see that $M_1 + M_2\|u\|^\alpha \le (M_1 + M_2) + M_2|u|^{\alpha_1}$ $\forall\, u \in \mathbb{R}$, $\alpha < \alpha_1$). This proves Theorem 2.3.2 completely. □

Remark 2.3.5. The global solvability of the problem (2.1.1), (2.1.2) in Theorem 2.3.2 is proved for the case in which the function f satisfies the condition (2.2.3), where $0 \le \alpha < 1$. In the case $1 \le \alpha < \frac{n+1}{n-1}$, the local solvability of this problem is proved in Theorem 2.3.1, although in this case, for the additional conditions imposed on f the problem (2.1.1), (2.1.2) is globally solvable as is shown in the following theorem.

Theorem 2.3.3. *Let f satisfy the condition* (2.2.3)*, where $1 \le \alpha < \frac{n+1}{n-1}$ and $f = \nabla G$, i.e., $f_i(u) = \frac{\partial}{\partial u_i} G(u)$, $u \in \mathbb{R}^N$, $i = 1, \dots, N$, where $G = G(u) \in C^1(\mathbb{R}^N)$ is a scalar function satisfying the conditions $G(0) = 0$ and $G(u) \ge 0$ $\forall\, u \in \mathbb{R}^N$. Let $g = 0$, $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for any $T > 0$. Then the problem* (2.1.1), (2.1.2) *is globally solvable in the class W_2^1, i.e., for any $T > 0$, this problem has a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition* 2.2.1.

Proof. First, let us show that for any fixed $T > 0$, when the conditions of Theorem 2.3.3 are fulfilled, for a strong generalized solution u of the problem (2.1.1), (2.1.2) of the class W_2^1 in the domain D_T the a priori estimate (2.3.4) is valid. Indeed, due to Definition 2.2.1, there exists a sequence of vector functions $u^m \in \overset{\circ}{C}(\overline{D}_T, S_T^0, S_T)$ such that

$$\lim_{m\to\infty} \|u^m - u\|_{W_2^1(D_T)} = 0, \quad \lim_{m\to\infty} \|Lu^m - F\|_{L_2(D_T)} = 0. \tag{2.3.25}$$

Let

$$F^m := Lu^m, \tag{2.3.26}$$

then due to the equality (2.3.5), we have

$$\int\limits_{\Omega_\tau} \Big[\Big(\frac{\partial u^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u^m}{\partial x_i}\Big)^2\Big]\, dx = 2 \int\limits_{D_T} (F^m - f(u^m))\, \frac{\partial u^m}{\partial t}\, dx\, dt. \tag{2.3.27}$$

Since $f = \nabla G$, we have $f(u^m)\,\frac{\partial u^m}{\partial t} = \frac{\partial}{\partial t}\,G(u^m)$ and, taking into account that $u^m|_{S_T} = 0$, $\nu_{n+1}|_{S_T^0} = 0$, $\nu_{n+1}|_{\Omega_\tau} = 1$, $G(0) = 0$, by integration by parts we get

$$\int\limits_{D_\tau} f(u^m)\,\frac{\partial u^m}{\partial t}\,dx\,dt = \int\limits_{D_\tau} \frac{\partial}{\partial t}\,G(u^m)\,dx\,dt = \int\limits_{\partial D_\tau} G(u^m)\nu_{n+1}\,ds = \int\limits_{S_\tau^0\cup S_\tau\cup\Omega_\tau} G(u^m)\nu_{n+1}\,ds = \int\limits_{\Omega_\tau} G(u^m)\,dx. \tag{2.3.28}$$

In view of (2.3.28) and $G \ge 0$, from (2.3.27) we have

$$\int\limits_{\Omega_\tau} \Big[\Big(\frac{\partial u^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u^m}{\partial x_i}\Big)^2\Big]\,dx = 2\int\limits_{D_\tau} F^m\,\frac{\partial u^m}{\partial t}\,dx\,dt - 2\int\limits_{\Omega_\tau} G(u^m)\,dx \le 2\int\limits_{D_\tau} F^m\,\frac{\partial u^m}{\partial t}\,dx\,dt. \tag{2.3.29}$$

Using the same reasonings as those for finding the estimate (2.3.4), from (2.3.29) we get the following inequality

$$\|u^m\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \le c(T)\|F^m\|_{L_2(D_T)},\ \ c(T) = \sqrt{T}\,\exp\frac{1}{2}\,(T+T^2),$$

whence, due to (2.3.25) and (2.3.26), we have (2.3.4).

According to Remarks 2.3.3 and 2.3.4, under the fulfilment of the conditions of Theorem 2.3.3, the vector function $u \in \overset{\circ}{W}{}_2^1(D_T,S_T)$ represents a strong generalized solution of the problem (2.1.1), (2.1.2) of the class W_2^1 if and only if u represents a solution of the functional equation $u = \mathcal{A}u$ from (2.3.13) in the space $\overset{\circ}{W}{}_2^1(D_T,S_T)$, where the operator $\mathcal{A} : [\overset{\circ}{W}{}_2^1(D_T,S_T)]^N \to [\overset{\circ}{W}{}_2^1(D_T,S_T)]^N$ is continuous and compact. At the same time, as is shown above, for any $\mu \in [0,1]$ and any solution of equation $u = \mu\mathcal{A}u$ with the parameter μ, in the space $\overset{\circ}{W}{}_2^1(D_T,S_T)$ the following a priori estimate

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \le \mu c(T)\|F\|_{L_2(D_T)} \le c(T)\|F\|_{L_2(D_T)}$$

with the positive constant $c(T)$, independent of u, μ and F, is valid. Therefore, according to the Leray–Schauder's theorem [90, p. 375], the equation (2.3.13) and hence the problem (2.1.1), (2.1.2) has at least one strong generalized solution of the class W_2^1 in the domain D_T for any $T > 0$. Thus Theorem 2.3.3 is proved. □

2.4 The uniqueness and existence of a global solution of the problem (2.1.1), (2.1.2) in the class W_2^1

Below, we impose on the nonlinear vector function $f = (f_1,\dots,f_N)$ from (2.1.1) the following additional requirements

$$f \in C^1(\mathbb{R}^N),\ \ \Big|\frac{\partial f_i(u)}{\partial u_j}\Big| \le M_3 + M_4|u|^\gamma\ \ \forall\,u \in \mathbb{R}^N,\ \ 1 \le i,j \le N, \tag{2.4.1}$$

where M_3, M_4, $\gamma = const \ge 0$. For the sake of simplicity, we assume that the vector function $g = 0$ in the boundary condition (2.1.2).

Remark 2.4.1. It is obvious that from (2.4.1) follows the condition (2.2.3) for $\gamma = \alpha - 1$, and in the case $\gamma < \frac{2}{n-1}$, we have $1 \le \alpha = \gamma + 1 < \frac{n+1}{n-1}$.

Theorem 2.4.1. *Let the condition* (2.4.1) *be fulfilled, where* $0 \le \gamma < \frac{2}{n-1}$, $F \in L_2(D_T)$ *and* $g = 0$. *Then the problem* (2.1.1), (2.1.2) *cannot have more than one strong generalized solution of the class* W_2^1 *in the domain* D_T *in the sense of Definition* 2.2.1.

Proof. Let $F \in L_2(D_T)$, $g = 0$, and assume that the problem (2.1.1), (2.1.2) has two strong generalized solutions u^1 and u^2 of the class W_2^1 in the domain D_T in the sense of Definition 2.2.1, i.e., there exist two sequences of vector functions $u^{im} \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T)$, $i = 1, 2$; $m = 1, 2, \dots$, such that

$$\lim_{m\to\infty} \|u^{im} - u^i\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0, \quad \lim_{m\to\infty} \|Lu^{im} - F\|_{L_2(D_T)} = 0, \;\; i = 1, 2. \tag{2.4.2}$$

Let

$$w = u^2 - u^1, \;\; w^m = u^{2m} - u^{1m}, \;\; F^m = Lu^{2m} - Lu^{1m}. \tag{2.4.3}$$

In view of (2.4.2) and (2.4.3), we have

$$\lim_{m\to\infty} \|w^m - w\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0, \quad \lim_{m\to\infty} \|F^m\|_{L_2(D_T)} = 0. \tag{2.4.4}$$

In accordance with (2.4.3), consider the vector function $w^m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T)$ as a solution of the following problem:

$$\Box\, w^m = -\big[f(u^{2m}) - f(u^{1m})\big] + F^m, \tag{2.4.5}$$

$$\frac{\partial w^m}{\partial x_n}\Big|_{S_T^0} = 0, \quad w^m\big|_{S_T} = 0. \tag{2.4.6}$$

From (2.4.5), (2.4.6) and in view of the equality (2.3.5), it follows that

$$\int\limits_{\Omega_\tau} \Big[\Big(\frac{\partial w^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial w^m}{\partial x_i}\Big)^2\Big]\,dx$$

$$= 2\int\limits_{D_\tau} F^m \,\frac{\partial w^m}{\partial t}\,dx\,dt - 2\int\limits_{D_\tau} \big[f(u^{2m}) - f(u^{1m})\big]\,\frac{\partial w^m}{\partial x_i}\,dx\,dt, \;\; 0 < \tau \le T. \tag{2.4.7}$$

Taking into account the equality

$$f_i(u^{2m}) - f_i(u^{1m}) = \sum_{j=1}^N \int\limits_0^1 \frac{\partial}{\partial u_j}\, f_i\big(u^{1m} + s(u^{2m} - u^{1m})\big)\,ds\,(u_j^{2m} - u_j^{1m}),$$

we obtain

$$\big[f(u^{2m}) - f(u^{1m})\big]\,\frac{\partial w^m}{\partial t} = \sum_{i,j=1}^N \Big[\int\limits_0^1 \frac{\partial}{\partial u_j}\, f_i\big(u^{1m} + s(u^{2m} - u^{1m})\big)\,ds\Big](u_j^{2m} - u_j^{1m})\,\frac{\partial w_i^m}{\partial t}\,. \tag{2.4.8}$$

From (2.4.1) and the obvious inequality $|d_1 + d_2|^\gamma \le 2^\gamma \max(|d_1|^\gamma, |d_2|^\gamma) \le 2^\gamma(|d_1|^\gamma + |d_2|^\gamma)$ for $\gamma \ge 0$, $d_i \in \mathbb{R}$, we have

$$\Big|\int\limits_0^1 \frac{\partial}{\partial u_j}\, f_i\big(u^{1m} + s(u^{2m} - u^{1m})\big)\,ds\Big|$$

$$\le \int\limits_0^1 \Big[M_3 + M_4\big|(1-s)u^{1m} + su^{2m}\big|^\gamma\Big]\,ds \le M_3 + 2^\gamma M_4\big(|u^{1m}|^\gamma + |u^{2m}|^\gamma\big). \tag{2.4.9}$$

From (2.4.8) and (2.4.9), taking into account (2.4.3), we obtain

$$\begin{aligned}\Big|\big[f(u^{2m})-f(u^{1m})\big]\,\frac{\partial w^m}{\partial x_i}\Big| &\le \sum_{i,j=1}^{N}\Big[M_3+2^\gamma M_4\big(|u^{1m}|^\gamma+|u^{2m}|^\gamma\big)\Big]\,|w_j^m|\,\Big|\frac{\partial w_i^m}{\partial t}\Big|\\ &\le N^2\Big[M_3+2^\gamma M_4\big(|u^{1m}|^\gamma+|u^{2m}|^\gamma\big)\Big]\,|w^m|\,\Big|\frac{\partial w^m}{\partial t}\Big|\\ &\le \frac{1}{2}\,N^2M_3\Big[(w^m)^2+\Big(\frac{\partial w^m}{\partial t}\Big)^2\Big]+2^\gamma N^2M_4\big(|u^{1m}|^\gamma+|u^{2m}|^\gamma\big)|w^m|\,\Big|\frac{\partial w^m}{\partial t}\Big|. \qquad (2.4.10)\end{aligned}$$

Due to (2.4.7) and (2.4.10), we get

$$\begin{aligned}\int\limits_{\Omega_\tau}\Big[\Big(\frac{\partial w^m}{\partial t}\Big)^2+\sum_{i=1}^{n}\Big(\frac{\partial w^m}{\partial x_i}\Big)^2\Big]\,dx &\le \int\limits_{D_\tau}\Big[\Big(\frac{\partial w^m}{\partial t}\Big)^2+(F^m)^2\Big]\,dx\,dt+N^2M_3\int\limits_{D_\tau}\Big[(w^m)^2+\Big(\frac{\partial w^m}{\partial t}\Big)^2\Big]\,dx\,dt\\ &\quad+2^{\gamma+1}N^2M_4\int\limits_{D_\tau}\big(|u^{1m}|^\gamma+|u^{2m}|^\gamma\big)|w^m|\,\Big|\frac{\partial w^m}{\partial t}\Big|\,dx\,dt. \qquad (2.4.11)\end{aligned}$$

The latter integral in the right-hand side of (2.4.11) can be estimated by Hölder's inequality

$$\begin{aligned}\int\limits_{D_\tau}\big(|u^{1m}|^\gamma+|u^{2m}|^\gamma\big)|w^m|\,\Big|\frac{\partial w^m}{\partial t}\Big|\,dx\,dt &\le \big(\||u^{1m}|^\gamma\|_{L_{n+1}(D_T)}+\||u^{2m}|^\gamma\|_{L_{n+1}(D_T)}\big)\|w^m\|_{L_p(D_\tau)}\Big\|\frac{\partial w^m}{\partial t}\Big\|_{L_2(D_\tau)}. \qquad (2.4.12)\end{aligned}$$

Here, $\frac{1}{n+1}+\frac{1}{p}+\frac{1}{2}=1$, i.e.,

$$p=\frac{2(n+1)}{n-1}\,. \qquad (2.4.13)$$

In view of (2.3.14), for $q\le\frac{2(n+1)}{n-1}$ we have

$$\|v\|_{L_q(D_T)}\le C_q(T)\|v\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}\quad \forall\, v\in \overset{\circ}{W}{}_2^1(D_\tau,S_\tau),\ \ 0<\tau\le T, \qquad (2.4.14)$$

with the positive constant $C_q(T)$ not depending on $v\in \overset{\circ}{W}{}_2^1(D_\tau,S_\tau)$ and $\tau\in[0,T]$.

According to our theorem, $\gamma<\frac{2}{n-1}$ and hence $\gamma(n+1)<\frac{2(n+1)}{n-1}$. Thus, from (2.4.13) and (2.4.14), we obtain

$$\||u^{im}|^\gamma\|_{L_{n+1}(D_T)}=\|u^{im}\|^\gamma_{L_{\gamma(n+1)}(D_T)}\le C^\gamma_{\gamma(n+1)}(T)\|u^{im}\|^\gamma_{\overset{\circ}{W}{}_2^1(D_T,S_T)},\ \ i=1,2;\ \ m\ge 1, \qquad (2.4.15)$$

$$\|w^m\|_{L_p(D_\tau)}\le C_p(T)\|w^m\|_{W_2^1(D_\tau)},\ \ m\ge 1. \qquad (2.4.16)$$

In view of the first equality from (2.4.2), there exists a natural number m_0 such that for $m\ge m_0$, we have

$$\|u^{im}\|^\gamma_{\overset{\circ}{W}{}_2^1(D_T,S_T)}\le \|u^i\|^\gamma_{\overset{\circ}{W}{}_2^1(D_T,S_T)}+1,\ \ i=1,2;\ \ m\ge m_0. \qquad (2.4.17)$$

In view of the above inequalities, from (2.4.12)–(2.4.16) it follows that

$$
\begin{aligned}
2^{\gamma+1}N^2M_4 \int\limits_{D_\tau} \big(|u^{1m}|^\gamma + |u^{2m}|^\gamma\big)|w^m| \Big|\frac{\partial w^m}{\partial t}\Big|\,dx\,dt \le 2^{\gamma+1}N^2M_4C^\gamma_{\gamma(n+1)}(T) \\
\times \Big(\|u^1\|^\gamma_{\overset{\circ}{W}{}^1_2(D_T,S_T)} + \|u^2\|^\gamma_{\overset{\circ}{W}{}^1_2(D_T,S_T)} + 2\Big)C_p(T)\|w^m\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} \Big\|\frac{\partial w^m}{\partial t}\Big\|_{L_2(D_\tau)} \\
\le M_5\Big(\|w^m\|^2_{W^1_2(D_\tau)} + \Big\|\frac{\partial w^m}{\partial t}\Big\|_{L_2(D_\tau)}\Big) \le 2M_5\|w^m\|_{W^1_2(D_\tau)^2} \\
= 2M_5 \int\limits_{D_\tau} \Big[(w^m)^2 + \Big(\frac{\partial w^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial w^m}{\partial x_i}\Big)^2\Big]\,dx\,dt, \quad (2.4.18)
\end{aligned}
$$

where

$$M_5 = 2^\gamma N^2M_4C^\gamma_{\gamma(n+1)}(T)\Big(\|u^1\|^\gamma_{\overset{\circ}{W}{}^1_2(D_T,S_T)} + \|u^2\|^\gamma_{\overset{\circ}{W}{}^1_2(D_T,S_T)} + 2\Big)C_p(T).$$

Due to (2.4.17), from (2.4.11) we get

$$
\begin{aligned}
\int\limits_{\Omega_\tau} \Big[\Big(\frac{\partial w^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial w^m}{\partial x_i}\Big)^2\Big]\,ds \\
\le M_6 \int\limits_{D_\tau} \Big[(w^m)^2 + \Big(\frac{\partial w^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial w^m}{\partial x_i}\Big)^2\Big]\,dx\,dt + \int\limits_{D_\tau} (F^m)^2\,dx\,dt, \quad 0 < \tau \le T, \quad (2.4.19)
\end{aligned}
$$

where $M_6 = 1 + M_3N^2 + 2M_5$.

Note that the inequality (2.3.6) is likewise valid for w^m and, therefore,

$$\int\limits_{\Omega_\tau} (w^m)^2\,dx \le T \int\limits_{D_\tau} \Big(\frac{\partial w^m}{\partial t}\Big)^2\,dx\,dt \le T \int\limits_{D_\tau} \Big[(w^m)^2 + \Big(\frac{\partial w^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial w^m}{\partial x_i}\Big)^2\Big]\,dx\,dt. \quad (2.4.20)$$

Putting

$$\lambda_m(\tau) := \int\limits_{\Omega_\tau} \Big[(w^m)^2 + \Big(\frac{\partial w^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial w^m}{\partial x_i}\Big)^2\Big]\,dx \quad (2.4.21)$$

and adding (2.4.18) to (2.4.19), we obtain

$$\lambda_m(\tau) \le (M_6 + T) \int\limits_0^\tau \lambda_m(s)\,ds + \|F^m\|^2_{L_2(D_T)}.$$

whence, by the Gronwall lemma, it follows that

$$\lambda_m(\tau) \le \|F^m\|^2_{L_2(D_T)} \exp(M_6 + T)\tau. \quad (2.4.22)$$

From (2.4.20) and (2.4.21) we have

$$\|w^m\|^2_{W^1_2(D_T)} = \int\limits_0^T \lambda(\tau)\,d\tau \le T\|F^m\|^2_{L_2(D_T)} \exp(M_6 + T)T. \quad (2.4.23)$$

In view of (2.4.3) and (2.4.4), from (2.4.22) it follows that

$$
\begin{aligned}
\|w\|_{W^1_2(D_T)} = \lim_{m\to\infty} \|w - w^m + w^m\|_{W^1_2(D_T)} \le \lim_{m\to\infty} \|w - w^m\|_{W^1_2(D_T)} + \lim_{m\to\infty} \|w^m\|_{W^1_2(D_T)} \\
= \lim_{m\to\infty} \|w - w^m\|_{W^1_2(D_T)} = \lim_{m\to\infty} \|w - w^m\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} = 0.
\end{aligned}
$$

Therefore, $w = u_2 - u_1 = 0$, i.e., $u_2 = u_1$. Thus Theorem 2.4.1 is proved. □

From Theorems 2.3.2, 2.3.3, 2.4.1 and Remark 2.4.1 follows the next theorem on the existence and uniqueness.

Theorem 2.4.2. *Let the vector function f satisfy the condition* (2.4.1)*, where $0 \leq \gamma < \frac{2}{n-1}$, and either f satisfy the condition* (2.2.3) *for $\alpha < 1$, or $f = \nabla G$, where $G \in C^1(\mathbb{R}^N)$, $G(0) = 0$ and $G(u) \geq 0$ $\forall\, u \in \mathbb{R}^N$. Then for any $F \in L_2(D_T)$ and $g = 0$, the problem* (2.1.1), (2.1.2) *has a unique strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ of the class W_2^1 in the domain D_T in the sense of Definition* 2.2.1*.*

The theorem below on the existence of a global solution of this problem follows from Theorem 2.4.2.

Theorem 2.4.3. *Let the vector function f satisfy the condition* (2.4.1)*, where $0 \leq \gamma < \frac{2}{n-1}$, and either f satisfy the condition* (2.2.3) *for $\alpha < 1$ or $f = \nabla G$, where $G \in C^1(\mathbb{R}^N)$, $G(0) = 0$ and $G(u) \geq 0$ $\forall\, u \in \mathbb{R}^N$. Then the problem* (2.1.1), (2.1.2) *has a unique global strong generalized solution $u \in \overset{\circ}{W}{}_{2,loc}^1(D_\infty, S_\infty)$ of the class W_2^1 in the domain D_∞ in the sense of Definition* 2.2.4*.*

Proof. According to Theorem 2.4.2, when the conditions of Theorem 2.4.3 are fulfilled for $T = k$, where k is a natural number, there exists a unique strong generalized solution $u^k \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ of the problem (2.1.1), (2.1.2) of the class W_2^1 in the domain $D_{T=k}$ in the sense of Definition 2.2.1. Since $u^{k+1}|_{D_{T=k}}$ is also a strong generalized solution of the problem (2.1.1), (2.1.2) of the class W_2^1 in the domain $D_{T=k}$, in view of Theorem 2.4.2, we have $u^k = u^{k+1}|_{D_{T=k}}$. Thus one can construct a unique global generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ of the problem (2.1.1), (2.1.2) of the class W_2^1 in the domain D_∞ in the sense of Definition 2.2.4 as follows:

$$u(x,t) = u^k(x,t), \ \ (x,t) \in D_\infty, \ \ k = [t] + 1,$$

where $[t]$ is an integer part of the number t. Thus Theorem 2.4.3 is proved. □

2.5 The cases of the absence of a global solution of the problem (2.1.1), (2.1.2) of the class W_2^1

Theorem 2.5.1. *Let the vector function $f = (f_1, \dots, f_N)$ satisfy the condition* (2.2.3)*, where $1 < \alpha < \frac{n+1}{n-1}$, and there exist the numbers $\ell_1, \dots, \ell_N$, $\sum_{i=1}^N |\ell_i| \neq 0$, such that*

$$\sum_{i=1}^N \ell_i f_i(u) \leq c_0 - c_1 \Big| \sum_{i=1}^N \ell_i u_i \Big|^\beta \ \ \forall\, u \in \mathbb{R}^N, \ \ 1 < \beta = const < \frac{n+1}{n-1}\,, \tag{2.5.1}$$

where $c_0, c_1 = const$, $c_1 > 0$. Let $F \in L_{2,loc}(D_\infty)$, $g \in W_{2,loc}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. Let at least one of the functions $F_0 = \sum\limits_{i=1}^N \ell_i F_i - c_0$ or $\frac{\partial g_0}{\partial N}\big|_{S_\infty}$, where $g_0 = \sum\limits_{i=1}^N \ell_i g_i$, be nontrivial (*i.e., different from zero on a subset of positive measure in D_∞ or S_∞, respectively*)*. Then if*

$$g_0 \geq 0, \quad \frac{\partial g_0}{\partial N}\Big|_{S_\infty} \leq 0, \quad F_0\big|_{D_\infty} \geq 0, \tag{2.5.2}$$

there exists a finite positive number $T_0 = T_0(F, g)$ such that for $T > T_0$ the problem (2.1.1), (2.1.2) *does not have a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition* 2.2.1*. Here, $\frac{\partial}{\partial N}$ is a derivative with respect to the conormal to S_∞, i.e., $\frac{\partial}{\partial N} = \nu_{n+1} \frac{\partial}{\partial t} - \sum\limits_{i=1}^N \nu_i \frac{\partial}{\partial x_i}$, where $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ is a unit vector of the outer normal to $\partial D_\infty = S_\infty$, which is an inner differential operator on the characteristic manifold S_∞.*

Proof. Let $G_T: \ |x|<t<T$, $G_T^- = G_T \cap \{x_n<0\}$, $S_T^-: \ t=|x|, \ x_n \le 0, \ t \le T$. Obviously, $D_T = G_T^+: \ G_T \cap \{x_n>0\}$ and $G_T = G_T^- \cup (S_T^0 \setminus \partial S_T^0) \cup G_T^+$, where $S_T^0 = \partial D_T \cap \{x_n=0\}$. Let $u=(u_1,\dots,u_n)$ be a strong generalized solution of the problem (2.1.1), (2.1.2) of the class W_2^1 in the domain D_T in the sense of Definition 2.2.1. We extend the vector functions u, F and g evenly with respect to the variable x_n in G_T^- and S_T^-, respectively. For simplicity, we retain the same notations u, F and g to the extended functions defined in G_T and $S_T^- \cup S_T$. Let us show that the vector function $u=(u_1,\dots,u_N)$, defined in the domain G_T, satisfy the following integral equality

$$\int\limits_{G_T} \big[-u_t w_t + \nabla u \nabla w\big]\,dx\,dt = -\int\limits_{G_T} f(u)w\,dx\,dt + \int\limits_{G_T} Fw\,dx\,dt - \int\limits_{S_T^- \cup S_T} \frac{\partial g}{\partial \mathcal{N}}\,w\,ds \tag{2.5.3}$$

for any vector function $w=(w_1,\dots,w_N) \in W_2^1(G_T)$ such that $w|_{t=T}=0$ in the sense of the trace theory. Indeed, if $w \in W_2^1(G_T)$ and $w|_{t=T}=0$, then it is obvious that $w|_{D_T} \in W_2^1(D_T)$ and $\widetilde{w} \in W_2^1(D_T)$, where, by definition, $\widetilde{w}(x_1,\dots,x_n,t) = w(x_1,\dots,-x_n,t)$, $(x_1,\dots,x_n,t) \in D_T$ and $\widetilde{w}|_{t=T}=0$. Therefore, according to the equality (2.2.6), from Remark 2.2.2, for $\varphi=w$ and $\varphi=\widetilde{w}$, we have

$$\int\limits_{D_T} \big[-u_t w_t + \nabla u \nabla w\big]\,dx\,dt = -\int\limits_{D_T} f(u)w\,dx\,dt + \int\limits_{D_T} Fw\,dx\,dt - \int\limits_{S_T} \frac{\partial g}{\partial \mathcal{N}}\,w\,ds \tag{2.5.4}$$

and

$$\int\limits_{D_T} \big[-u_t \widetilde{w}_t + \nabla u \nabla \widetilde{w}\big]\,dx\,dt = -\int\limits_{D_T} f(u)\widetilde{w}\,dx\,dt + \int\limits_{D_T} F\widetilde{w}\,dx\,dt - \int\limits_{S_T} \frac{\partial g}{\partial \mathcal{N}}\,\widetilde{w}\,ds, \tag{2.5.5}$$

respectively. Since u, F and g are the even vector functions with respect to the variable x_n, and $\widetilde{w}(x_1,\dots,x_n,t) = w(x_1,\dots,-x_n,t)$, $(x_1,\dots,x_n,t) \in D_T$, we have

$$\int\limits_{D_T} \big[-u_t \widetilde{w}_t + \nabla u \nabla \widetilde{w}\big]\,dx\,dt = \int\limits_{G_T-} \big[-u_t w_t + \nabla u \nabla w\big]\,dx\,dt, \tag{2.5.6}$$

$$-\int\limits_{D_T} f(u)\widetilde{w}\,dx\,dt + \int\limits_{D_T} F\widetilde{w}\,dx\,dt - \int\limits_{S_T} \frac{\partial g}{\partial \mathcal{N}}\,\widetilde{w}\,ds$$

$$= -\int\limits_{G_T^-} f(u)w\,dx\,dt + \int\limits_{G_T^-} Fw\,dx\,dt - \int\limits_{S_T^-} \frac{\partial g}{\partial \mathcal{N}}\,w\,ds. \tag{2.5.7}$$

It follows from (2.5.5)–(2.5.7) that

$$\int\limits_{G_T^-} \big[-u_t w_t + \nabla u \nabla w\big]\,dx\,dt = -\int\limits_{G_T^-} f(u)w\,dx\,dt + \int\limits_{G_T^-} Fw\,dx\,dt - \int\limits_{S_T^-} \frac{\partial g}{\partial \mathcal{N}}\,w\,ds. \tag{2.5.8}$$

Finally, summing up the equalities (2.5.4) and (2.5.8), we obtain (2.5.3).

Let us apply the method of test functions [77, pp. 10–12].

In the integral equality (2.5.3), for the test function w we choose $w=(\ell_1\psi,\dots,\ell_N\psi)$, where $\psi = \psi_0[2T^{-2}(t^2+|x|^2)]$, while a scalar function $\psi_0 \in C^2(\mathbb{R})$ satisfies the following conditions: $\psi_0 \ge 0$, $\psi_0' \le$; $\psi(\sigma)=1$ for $0 \le \sigma \le 1$ and $\psi(\sigma)=0$ for $\sigma \ge 2$ [77, p. 22]. For the chosen test function w, using the notations $v=\sum\limits_{i=1}^N \ell_i u_i$, $g_0=\sum\limits_{i=1}^N \ell_i g_i$, $F_*=\sum\limits_{i=1}^N \ell_i F_i$, $f_0=\sum\limits_{i=1}^N \ell_i f_i$, the integral equality (2.5.3) takes the form

$$\int\limits_{G_T} \big[-v_t\psi_t + \nabla v \nabla \psi\big]\,dx\,dt = -\int\limits_{G_T} f_0(u)\psi\,dx\,dt + \int\limits_{G_T} F_*\psi\,dx\,dt - \int\limits_{S_T^- \cup S_T} \frac{\partial g_0}{\partial \mathcal{N}}\,\psi\,ds. \tag{2.5.9}$$

Due to $\psi|_{t\geq T}=0$ and the equality $v|_{S_T^-\cup S_T}=g_0$ in the sense of the trace theory, integrating by parts the left-hand side of the equality (2.5.9), we get

$$\int\limits_{G_T}\big[-v_t\psi_t+\nabla v\nabla\psi\big]\,dx\,dt = \int\limits_{G_T} v\,\Box\psi\,dx\,dt-\int\limits_{S_T^-\cup S_T} v\,\frac{\partial\psi}{\partial\mathcal N}\,ds=\int\limits_{G_T} v\,\Box\psi\,dx\,dt-\int\limits_{S_T^-\cup S_T} g_0\,\frac{\partial\psi}{\partial\mathcal N}\,ds. \tag{2.5.10}$$

From (2.5.9) and (2.5.10), in view of (2.5.1) and $\psi\geq 0$, we have

$$\begin{aligned}\int\limits_{G_T} v\,\Box\psi\,dx\,dt &\geq \int\limits_{G_T}[c_1|v|^\beta-c_0]\psi\,dx\,dt+\int\limits_{G_T}F_*\psi\,dx\,dt+\int\limits_{S_T^-\cup S_T} g_0\,\frac{\partial\psi}{\partial\mathcal N}\,ds-\int\limits_{S_T^-\cup S_T}\frac{\partial g_0}{\partial\mathcal N}\,\psi\,ds\\ &= c_1\int\limits_{G_T}|v|^\beta\psi\,dx\,dt+\int\limits_{G_T}(F_*-c_0)\psi\,dx\,dt+\int\limits_{S_T^-\cup S_T} g_0\,\frac{\partial\psi}{\partial\mathcal N}\,ds-\int\limits_{S_T^-\cup S_T}\frac{\partial g_0}{\partial\mathcal N}\,\psi\,ds.\end{aligned} \tag{2.5.11}$$

In view of the properties of the function ψ and the inequalities (2.5.2), we have

$$\begin{gathered}\frac{\partial\psi}{\partial\mathcal N}\Big|_{S_T^-\cup S_T}\geq 0,\qquad \int\limits_{S_T^-\cup S_T} g_0\,\frac{\partial\psi}{\partial\mathcal N}\,ds\geq 0,\\ \int\limits_{S_T^-\cup S_T}\frac{\partial g_0}{\partial\mathcal N}\,\psi\,ds\leq 0,\qquad \int\limits_{G_T}F_0\psi\,dx\,dt\geq 0,\end{gathered} \tag{2.5.12}$$

where $F_0=F_*-c_0=\sum\limits_{i=1}^N \ell_iF_i-c_0$. Upon derivation of the inequality (2.5.12), we have taken into account the fact that $\nu_{n+1}|_{S_T^-\cup S_T}<0$.

Assuming that the functions F, g and ψ are fixed, we introduce into consideration a function of one variable

$$\gamma(T)=\int\limits_{G_T}F_0\psi\,dx\,dt+\int\limits_{S_T^-\cup S_T} g_0\,\frac{\partial\psi}{\partial\mathcal N}\,ds-\int\limits_{S_T^-\cup S_T}\frac{\partial g_0}{\partial\mathcal N}\,\psi\,ds,\ \ T>0. \tag{2.5.13}$$

Due to the absolute continuity of the integral and the inequalities (2.5.12), the function $\gamma(T)$ from (2.5.13) is nonnegative, continuous and nondecreasing, and

$$\lim_{T\to 0}\gamma(T)=0. \tag{2.5.14}$$

Besides, since according to the supposition, at least one of the function $\frac{\partial g_0}{\partial\mathcal N}\big|_{S_\infty^-\cup S_\infty}$ or F_0 is nontrivial, we have

$$\lim_{T\to+\infty}\gamma(T)>0. \tag{2.5.15}$$

In view of (2.5.13), the inequality (2.5.11) can be rewritten as follows:

$$c_1\int\limits_{G_T}|v|^\beta\psi\,dx\,dt\leq\int\limits_{G_T}v\,\Box\psi\,dx\,dt-\gamma(T). \tag{2.5.16}$$

If in Young's inequality with the parameter $\varepsilon>0$

$$ab\leq\frac{\varepsilon}{\beta}\,a^\beta+(\beta' c^{\beta'-1})^{-1}b^{\beta},$$

where $\beta' = \frac{\beta}{\beta-1}$, we take $a = |v|\psi^{1/\beta}$, $b = \frac{|\square\,\psi|}{\psi^{1/\beta}}$, then taking into account the equality $\frac{\beta'}{\beta} = \beta' - 1$, we have

$$|v\,\square\,\psi| = |v|\psi^{1/\beta}\,\frac{|\square\,\psi|}{\psi^{1/\beta}} \leq \frac{\varepsilon}{\beta}\,|v|^\beta\psi + \frac{1}{\beta'\varepsilon^{\beta'-1}}\,\frac{|\square\,\psi|^{\beta'}}{\psi^{\beta'-1}}\,. \tag{2.5.17}$$

In view of (2.5.17), from (2.5.16) we have

$$\Big(c_1 - \frac{\varepsilon}{\beta}\Big)\int\limits_{G_T}|v|^\beta\psi\;dx\,dt \leq \frac{1}{\beta'\varepsilon^{\beta'-1}}\int\limits_{G_T}\frac{|\square\,\psi|^{\beta'}}{\psi^{\beta'-1}}\;dx\,dt - \gamma(T),$$

whence for $\varepsilon < c_1\beta$, we obtain

$$\int\limits_{G_T}|v|^\beta\psi\;dx\,dt \leq \frac{\beta}{(c_1\beta-\varepsilon)\beta'\varepsilon^{\beta'-1}}\int\limits_{G_T}\frac{|\square\,\psi|^{\beta'}}{\psi^{\beta'-1}}\;dx\,dt - \frac{\beta}{c_1\beta-\varepsilon}\,\gamma(T). \tag{2.5.18}$$

Taking into account the equalities $\beta' = \frac{\beta}{\beta-1}$, $\beta = \frac{\beta'}{\beta'-1}$ and also the equality

$$\lim_{0<\varepsilon<c_1\beta}\frac{\beta}{(c_1\beta-\varepsilon)\beta'\varepsilon^{\beta'-1}} = \frac{1}{c_1^{\beta'}}$$

obtained for $\varepsilon = c_1$, from (2.5.18) it follows that

$$\int\limits_{G_T}|v|^\beta\psi\;dx\,dt \leq \frac{1}{c_1^{\beta'}}\int\limits_{G_T}\frac{|\square\,\psi|^{\beta'}}{\psi^{\beta'-1}}\;dx\,dt - \frac{\beta'}{c_1}\,\gamma(T). \tag{2.5.19}$$

According to the properties of the function ψ_0, the test function $\psi(x,t) = \psi_0[2T^{-2}(t^2+|x|^2)] = 0$ for $r = (t^2+|x|^2)^{1/2} > T$.

Therefore, after substitution of variables $t = \frac{1}{\sqrt{2}}\,T\xi_0$, $x = \frac{1}{\sqrt{2}}\,T\xi$, we have

$$\int\limits_{G_T}\frac{|\square\,\psi|^{\beta'}}{\psi^{\beta'-1}}\;dx\,dt = \int\limits_{\substack{r=(t^2+|x|^2)^{1/2}<T,\\ t>|x|}}\frac{|\square\,\psi|^{\beta'}}{\psi^{\beta'-1}}\,dx = \Big(\frac{1}{\sqrt{2}}\,T\Big)^{n+1-2\beta'}\varkappa_0. \tag{2.5.20}$$

Here,

$$\varkappa_0 := \int\limits_{\substack{1<|\xi_0|^2+|\xi|^2<2,\\ \xi_0>|\xi|}}\frac{|2(1-n)\psi_0' + 4(\xi_0^2-|\xi|^2)\psi_0''|^{\beta'}}{\psi_0^{\beta'-1}}\;d\xi\,d\xi_0 < +\infty. \tag{2.5.21}$$

As is know, the test function $\psi(x,t) = \psi_0[2T^{-2}(t^2+|x|^2)]$ with the properties mentioned above, for which the condition (2.5.21) is valid, does exist [77, p. 22].

Due to (2.5.20), from the equality (2.5.19) and the fact that $\psi_0(\sigma) = 1$, for $0 \leq \sigma \leq 1$, we have

$$\int\limits_{r\leq\frac{T}{\sqrt{2}}}|v|^\beta\;dx\,dt \leq \int\limits_{D_T}|v|^\beta\psi\;dx\,dt \leq \frac{(\frac{1}{\sqrt{2}}\,T)^{n+1-2\beta'}}{c_1^{\beta'}}\,\varkappa_0 - \frac{\beta'}{c_1}\,\gamma(T). \tag{2.5.22}$$

When $\beta < \frac{n+1}{n-1}$, i.e., when $n+1-2\beta' < 0$, the equation

$$\lambda(T) = \frac{(\frac{1}{\sqrt{2}}\,T)^{n+1-2\beta'}}{c_1^{\beta'}}\,\varkappa_0 - \frac{\beta'}{c_1}\,\gamma(T) = 0$$

has a unique positive root $T = T_0(F,g)$, since the function

$$\lambda_1(T) = \Big(\frac{(\frac{1}{\sqrt{2}}\,T)^{n+1-2\beta'}}{c_1^{\beta'}}\Big)\varkappa_0$$

is positive, continuous, strictly decreasing on the interval $(0,+\infty)$ and, besides, $\lim\limits_{T\to 0}\lambda_1(T)=+\infty$ and $\lim\limits_{T\to+\infty}\lambda_1(T)=0$, and the function $\gamma(T)$ is, as stated above, nonnegative, continuous and nondecreasing, satisfying the conditions (2.5.14) and (2.5.15). Moreover, $\lambda(T)<0$ for $T>T_0$ and $\lambda(T)>0$ for $0<T<T_0$. Therefore, for $T>T_0$, the right-hand side of the inequality (2.5.22) is a negative value, which is impossible. This contradiction proves Theorem 2.5.1. □

Remark 2.5.1. As is shown in Chapter 1, the following class of vector functions $f=(f_1,\dots,f_N)$:

$$f_i(u_1,\dots,u_N)=\sum_{j=1}^{N} a_{ij}|u_j|^{\beta_{ij}}+b_i, \quad i=1,\dots,N, \tag{2.5.23}$$

where $a_{ij}=const>0$, $b_i=const$, $1<b_{ij}=const<\frac{n+1}{n-1}$, $i,j=1,\dots,N$, satisfies the condition (2.5.1). Note that the vector function f, given by the equality (2.5.23), likewise satisfies the condition (2.5.1) for $\ell=\ell_2=\cdot=\ell_N=-1$ for less restrictive conditions, when $a_{ij}=cons\geq 0$, but $a_{ik_i}>0$, where $k_1,\dots,k_N$ is any arbitrary fixed permutation of numbers $1,2,\dots,N$; $i,j=1,\dots,N$.

Remark 2.5.2. From Theorem 2.5.1 it follows that if its conditions are fulfilled, then the problem $(2.1.1),(2.1.2)$ fails to have a global strong generalized solution of the class W_2^1 in the domain D_∞ in the sense of Definition 2.2.4.

Chapter 3

One multidimensional version of the Darboux second problem for one class of semilinear second order hyperbolic systems

3.1 Statement of the problem

In the space $\mathbb{R}^{n+1}$ of the independent variables $x = (x_1, \dots, x_n)$ and t consider a second order semilinear hyperbolic system of the form

$$\square u_i + f_i(u_1, \dots, u_N) = F_i, \ \ i = 1, \dots, N, \tag{3.1.1}$$

where $f = (f_1, \dots, f_N)$, $F = (F_1, \dots, F_N)$ are the given, and $u = (u_1, \dots, u_N)$ is an unknown real vector function, $n \geq 2$, $N \geq 2$, $\square := \frac{\partial^2}{\partial t^2} - \Delta$, $\Delta := \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

Let D be a conic domain in the space $\mathbb{R}^{n+1}$, i.e., D contains, along with the point $(x,t) \in D$, the whole ray $\ell : (\tau x, \tau t)$, $0 < \tau < \infty$. Denote by S the conic surface ∂D. Suppose that D is homeomorphic to the conic domain $\omega : \ t > |x|$, and $S \setminus 0$ is a connected n-dimensional manifold of the class C^∞, where $O = (0, \dots, 0, 0)$ is the vertex of S. Suppose also that D lies in the half-space $t > 0$ and $D_T := \{(x,t) \in D : \ t < T\}$, $S_T := \{(x,t) \in S : \ t \leq T\}$, $T > 0$. It is clear that if $T = \infty$, then $D_\infty = D$ and $S_\infty = S$.

For the system (3.1.1), we consider the problem on finding a solution $u(x,t)$ of this system in the domain D_T by the boundary condition

$$u\big|_{S_T} = g, \tag{3.1.2}$$

where $g = (g_1, \dots, g_N)$ is the given vector function on S_T.

In the linear case, in which $f = 0$, $N = 1$, and the conic manifold $S = \partial D$ is time-oriented, i.e.,

$$\Big(\nu_0^2 - \sum_{i=1}^{n} \nu_i^2\Big)\Big|_S < 0, \quad \nu_0\big|_S < 0, \tag{3.1.3}$$

where $\nu = (\nu_1, \dots, \nu_n, \nu_0)$ is the unit vector of the outer normal to $S \setminus O$, the problem (3.1.1), (3.1.2) was posed by S. L. Sobolev [86], where the unique solvability of this problem in the corresponding functional spaces is proved. At the end of the above-mentioned work the author suggests that the obtained results will likewise be valid for a scalar nonlinear wave equation. In [52], for the scalar case ($N = 1$) and power nonlinearity $f(u) = \lambda |u|^p u$ ($\lambda = const$, $0 < p = const < \frac{2}{n-1}$), the global solvability of this problem for $\lambda > 0$ and the absence of a global solution for $\lambda < 0$ are shown when the space dimension of the wave equation $n = 2$. A more general nonlinearity case than in [52] for the scalar hyperbolic equation was considered in [56] in which the questions of existence, uniqueness, and the absence of a global solution to this problem were also investigated. Besides, the restriction here is omitted. It is noteworthy mentioning that this problem can be considered as a multidimensional version of the Darboux second problem, since the problem's data support S represents a conic time type manifold. In the case when one part of the boundary of the conic domain D is of time type, while the other part is a characteristic manifold, the boundary value problem can be considered as a multidimensional version of the Darboux first problem. For example, when $D:\ t > |x|,\ x_n > 0$ and the boundary conditions have the form

$$u\big|_{\Gamma_0} = 0, \quad u\big|_{\Gamma_1} = 0$$

or

$$\frac{\partial u}{\partial x_n}\Big|_{\Gamma_0} = 0, \quad u\big|_{\Gamma_1} = 0,$$

where $\Gamma_0 = \partial D \cap \{x_n = 0\}$ is a plane part of the time type boundary ∂D and $\Gamma_1 = \partial D \setminus \Gamma_0 :\ t = |x|$, $x_n > 0$ is a characteristic part of the boundary, we have a multidimensional version of the first Darboux problem.

Investigation of the multidimensional version of the Darboux second problem faces great difficulties as compared with the first problem. More detailed consideration of these problems in the linear case is given in A. B. Bitsadze's monograph [5].

This chapter is organized as follows. Section 3.2 provides us with the notion of a strong generalized solution of the problem (3.1.1), (3.1.2) of the class W_2^1 in the domain D_T and with a definition of a global solution of this problem of the class W_2^1 in the domain D_∞. In Section 3.3, we consider the cases of local and global solvability of the problem (3.1.1), (3.1.2) in the class W_2^1. We suppose that the growth of nonlinearity of the system (3.1.1) does not exceed power nonlinearity with exponent $\alpha = const \geq 0$. When $\alpha \leq 1$, for the solution of the boundary value problem the a priori estimate (Lemma 3.3.1) is valid, and no restrictions are imposed on the structure of the vector function $f = f(u)$. As it turned out, when $1 < \alpha < \frac{n+1}{n-1}$, the only constraint on the growth of nonlinearity of the vector function $f = f(u)$ is not sufficient for the existence of an a priori estimate for the solution of the boundary value problem. Here we need structural constraints on the vector function $f = f(u)$. For example, when $f = \nabla G$, i.e., $f_i(u) = \frac{\partial}{\partial u_i} G(u)$, $u \in \mathbb{R}^N$, $i = 1, \dots, N$, where $G = G(u) \in C^1(\mathbb{R}^N)$ is a scalar function satisfying the conditions $G(0) = 0$ and $G(u) \geq 0\ \forall\, u \in \mathbb{R}^N$, the a priori estimate of the solution of the boundary value problem and, therefore, a global solvability of this problem (Theorem 3.3.3) are valid. If the vector function f cannot be represented in the form $f = \nabla G$, where the scalar function G satisfies the conditions given above, then the boundary value problem may be globally unsolvable. For example, when $N = n = 2$ and $f = (f_1, f_2)$, where $f_1 = u_1^2 - 2u_2^2$, $f_2 = -2u_1^2 + u_2^2$, the exponent of the nonlinearity $\alpha = 2$ and $1 < \alpha < \frac{n+1}{n-1}$, and f is not representable in the form $f = \nabla G$, then from Theorem 3.5.1 we find that for $F_1 + F_2 \geq \frac{c}{t^\gamma}$, $t \geq 1$, where $c = const > 0$, $\gamma = const \leq 3$, $g = 0$, the problem under consideration is not globally solvable (see Remark 3.5.1). The conditions on the vector function f providing the uniqueness and existence of a global solution of this problem of the class W_2^1 are given in Section 3.4. Finally, in Section 3.5, for certain additional conditions on the vector functions f, F and g, we prove nonexistence of a global solution of the problem (3.1.1), (3.1.2) of the class W_2^1 in D_∞.

Below, it will be assumed that the condition (3.1.3) is satisfied.

3.2 Definition of a generalized solution of the problem (3.1.1), (3.1.2) in D_T and D_∞

We rewrite the system (3.1.1) in the form of one vector equation

$$Lu := \Box\, u + f(u) = F. \tag{3.2.1}$$

Below, we will assume that the condition (3.1.3) is fulfilled and the nonlinear vector function from (3.2.1) satisfies the following inequality

$$f \in C(\mathbb{R}^N), \ \ |f(u)| \leq M_1 + M_2|u|^\alpha, \ \ \alpha = const \geq 0, \ \ u \in \mathbb{R}^N, \tag{3.2.2}$$

where $|\cdot|$ is the norm in the space $\mathbb{R}^N$, $M_i = const \geq 0$, $i = 1, 2$.

Let $\overset{\circ}{C}{}^2(\overline{D}_T, S_T) := \{u \in \overset{\circ}{C}{}^2(\overline{D}_T) : \ u|_{S_T} = 0\}$. Denote by $W_2^k(\Omega)$ the Sobolev space consisting of the elements $L_2(\Omega)$, having generalized derivatives up to the k-order inclusive from $L_2(\Omega)$. Let $\overset{\circ}{W}{}_2^1(D_T, S_T) := \{u \in W_2^1(D_T) : \ u|_{S_T} = 0\}$, where the equality $u|_{S_T} = 0$ is understood in the sense of the trace theory [68].

Here and below we say that the vector $v = (v_1, \dots, v_N)$ belongs to the space X if each component v_i, $1 \leq i \leq N$, of that vector belongs to the same X. In accordance with the above-said, to simplify our writing and avoid misunderstanding, instead of $v = (v_1, \dots, v_N) \in X^N$ we will write $v \in X$.

Remark 3.2.1. The embedding operator $I : [W_2^1(D_T)]^N \to [L_q(D_T)]^N$ is a linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$, when $n > 1$ [68]. At the same time, Nemitski's operator $\mathcal{K} : [L_q(D_T)]^N \to [L_q(D_T)]^N$, acting by the formula $\mathcal{K}u = f(u)$, where $u = (u_1, \dots, u_N) \in [L_q(D_T)]^N$, and the vector function $f = (f_1, \dots, f_N)$ satisfies the condition (3.2.2), is continuous and bounded for $q \geq 2\alpha$ [22]. Thus, if $\alpha < \frac{n+1}{n-1}$, i.e., $2\alpha < \frac{2(n+1)}{n-1}$, then there exists a number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q > 2\alpha$. Therefore, in this case the operator

$$\mathcal{K}_0 = \mathcal{K}I : [W_2^1(D_T)]^N \to [L_q(D_T)]^N \tag{3.2.3}$$

is continuous and compact. It is clear that from $u = (u_1, \dots, u_N) \in W_2^1(D_T)$ it follows that $f(u) \in L_2(D_T)$ and, if $u^m \to u$ in the space $W_2^1(D_T)$, then $f(u^m) \to f(u)$ in the space $L_2(D_T)$.

Definition 3.2.1. Let $f = (f_1, \dots, f_N)$ satisfy the condition (3.2.2), where $0 \leq \alpha < \frac{n+1}{n-1}$, $F = (F_1, \dots, F_N) \in L_2(D_T)$ and $g = (g_1, \dots, g_n) \in W_2^1(S_T)$. We call a vector function $u = (u_1, \dots, u_N) \in W_2^1(D_T)$ a strong generalized solution of the problem (3.1.1), (3.1.2) of the class W_2^1 in the domain D_T if there exists a sequence of vector functions $u^m \in C^2(\overline{D}_T)$ such that $u^m \to u$ in the space $W_2^1(D_T)$, $Lu^m \to F$ in the space $L_2(D_T)$, and $u^m|_{S_T} \to g$ in the space $W_2^1(S_T)$. The convergence of the sequence $\{f(u^m)\}$ to the function $f(u)$ in the space $L_2(D_T)$ as $u^m \to u$ in the space $W_2^1(D_T)$ follows from Remark 3.2.1. When $g = 0$, i.e., in the case of the homogeneous boundary conditions (3.1.2), we assume that $u^m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$. Then it is clear that $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$.

Obviously, a classical solution $u \in C^2(\overline{D}_T)$ of the problem (3.1.1), (3.1.2) represents a strong generalized solution of that problem of the class W_2^1 in the domain D_T in the sense of Definition 3.2.1.

Definition 3.2.2. Let f satisfy the condition (3.2.2), where $0 \leq \alpha < \frac{n+1}{n-1}$, $F \in L_{2,loc}(D_\infty)$, $g \in W_{2,loc}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. We say that the problem (3.1.1), (3.1.2) is locally solvable in the class W_2^1, if there exists a number $T_0 = T_0(F, g) > 0$ such that for $T < T_0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition 3.2.1.

Definition 3.2.3. Let f satisfy the condition (3.2.2), where $0 \leq \alpha < \frac{n+1}{n-1}$, $F \in L_{2,loc}(D_\infty)$, $g \in W_{2,loc}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. We say that the problem (3.1.1), (3.1.2) is globally solvable in the class W_2^1 if for any $T > 0$ this problem has a strong generalized solution of the class in the domain D_T in the sense of Definition 3.2.1.

Definition 3.2.4. Let f satisfy the condition (3.2.2), where $0 \leq \alpha < \frac{n+1}{n-1}$, $F \in L_{2,loc}(D_\infty)$, $g \in W_{2,loc}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. A vector function $u = (u_1, \dots, u_N) \in W_{2,loc}^1(D_\infty)$ is called a global strong generalized solution of the problem (3.1.1), (3.1.2) of the class W_2^1 in the domain D_∞ if for any $T > 0$ the vector function $u|_{D_T}$ belongs to the space $W_2^1(D_T)$ and represents a strong generalized solution of the problem (3.1.1), (3.1.2) of the class W_2^1 in the domain D_T in the sense of Definition 3.2.1.

3.3 Some cases of global and local solvability of the problem (3.1.1), (3.1.2) in the class W_2^1

Lemma 3.3.1. *Let f satisfy the condition* (3.2.2)*, where $0 \le \alpha \le 1$, $F \in L_2(D_T)$ and $g \in W_2^1(S_T)$. Then for any strong generalized solution u of the problem* (3.1.1), (3.1.2) *of the class W_2^1 in the domain D_T in the sense of Definition* 3.2.1 *the a priori estimate*

$$\|u\|_{W_2^1(D_T)} \le c_1\|F\|_{L_2(D_T)} + c_2\|g\|_{W_2^1(S_T)} + c_3 \tag{3.3.1}$$

with the nonnegative constants $c_i = c_i(S,f,T)$, $i=1,2,3$, independent of u, g and F, with $c_j > 0$, $j=1,2$, is valid.

Proof. Let $u \in W_2^1(D_T)$ be a strong generalized solution of the problem (3.1.1), (3.1.2) of the class W_2^1 in the domain D_T. Then, due to Definition 3.2.1, there exists a sequence of vector functions $u^m = (u_1^m, \dots, u_N^m) \in C^2(\overline{D}_T)$ such that

$$\lim_{m\to\infty} \|u^m - u\|_{W_2^1(D_T)} = 0, \quad \lim_{m\to\infty} \|Lu^m - F\|_{L_2(D_T)} = 0, \tag{3.3.2}$$

$$\lim_{m\to\infty} \big\|u^m\big|_{S_T} - g\big\|_{W_2^1(D_T)} = 0. \tag{3.3.3}$$

Consider the vector function $u^m \in C^2(\overline{D}_T)$ as a solution of the following problem:

$$Lu^m = F^m, \tag{3.3.4}$$

$$u^m\big|_{S_T} = g^m. \tag{3.3.5}$$

Here,

$$F^m := Lu^m, \quad g^m := u^m|_{S_T}. \tag{3.3.6}$$

Multiplying scalarly both sides of the vector equation (3.3.4) by $\frac{\partial u^m}{\partial t}$ and integrating in the domain D_τ, $0 < \tau \le T$, we obtain

$$\frac{1}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u^m}{\partial t}\Big)^2 dx\,dt - \int\limits_{D_\tau} \Delta u^m \frac{\partial u^m}{\partial t}\,dx\,dt + \int\limits_{D_\tau} f(u^m)\frac{\partial u^m}{\partial t}\,dx\,dt = \int\limits_{D_\tau} F^m \frac{\partial u^m}{\partial t}\,dx\,dt. \tag{3.3.7}$$

Let $\Omega_\tau := D \cap \{t = \tau\}$ and denote by $\nu = (\nu_1, \dots, \nu_n, \nu_0)$ the unit vector of the outer normal to $S_T \setminus \{(0,\dots,0,0)\}$. Integrating by parts, by virtue of the equality (3.3.5) and $\nu|_{\Omega_\tau} = (0,\dots,0,1)$, we have

$$\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u^m}{\partial t}\Big)^2 dx\,dt = \int\limits_{\partial D_\tau} \Big(\frac{\partial u^m}{\partial t}\Big)^2 \nu_0\,ds = \int\limits_{\Omega_\tau} \Big(\frac{\partial u^m}{\partial t}\Big)^2 dx + \int\limits_{S_\tau} \Big(\frac{\partial u^m}{\partial t}\Big)^2 \nu_0\,ds,$$

$$\begin{aligned}
\int\limits_{D_\tau} \frac{\partial^2 u^m}{\partial x_i^2}\frac{\partial u^m}{\partial t}\,dx\,dt &= \int\limits_{\partial D_\tau} \frac{\partial u^m}{\partial x_i}\frac{\partial u^m}{\partial t}\,\nu_i\,ds - \frac{1}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u^m}{\partial x_i}\Big)^2 dx\,dt \\
&= \int\limits_{\partial D_\tau} \frac{\partial u^m}{\partial x_i}\frac{\partial u^m}{\partial t}\,\nu_i\,ds - \frac{1}{2}\int\limits_{\partial D_\tau} \Big(\frac{\partial u^m}{\partial x_i}\Big)^2 \nu_0\,ds \\
&= \int\limits_{\partial D_\tau} \frac{\partial u^m}{\partial x_i}\frac{\partial u^m}{\partial t}\,\nu_i\,ds - \frac{1}{2}\int\limits_{S_\tau} \Big(\frac{\partial u^m}{\partial x_i}\Big)^2 \nu_0\,ds - \frac{1}{2}\int\limits_{\Omega_\tau} \Big(\frac{\partial u^m}{\partial x_i}\Big)^2 dx,
\end{aligned}$$

whence, in view of (3.3.7), it follows that

$$\begin{aligned}
\int\limits_{D_\tau} F^m \frac{\partial u^m}{\partial t}\,dx\,dt &= \int\limits_{S_\tau} \frac{1}{2\nu_0}\bigg[\sum_{i=1}^n \Big(\frac{\partial u^m}{\partial x_i}\,\nu_0 - \frac{\partial u^m}{\partial t}\,\nu_i\Big)^2 + \Big(\frac{\partial u^m}{\partial t}\Big)^2\Big(\nu_0^2 - \sum_{j=1}^n \nu_j^2\Big)\bigg] ds \\
&\quad + \frac{1}{2}\int\limits_{\Omega_\tau} \bigg[\Big(\frac{\partial u^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u^m}{\partial x_i}\Big)^2\bigg] dx + \int\limits_{D_\tau} f(u^m)\frac{\partial u^m}{\partial t}\,dx\,dt.
\end{aligned} \tag{3.3.8}$$

From (3.2.2), when $0 \le \alpha \le 1$, we find that $|f(u)| \le M_1 + M_2 + M_2|u|\ \forall\, u \in \mathbb{R}^N$, therefore,

$$\Big|f(u^m)\,\frac{\partial u^m}{\partial t}\Big| \le \frac{1}{2}\Big[f^2(u^m) + \Big(\frac{\partial u^m}{\partial t}\Big)^2\Big]$$
$$\le \frac{1}{2}\Big[2(M_1+M_2)^2 + 2M_2^2|u^m|^2 + \Big(\frac{\partial u^m}{\partial t}\Big)^2\Big] = (M_1+M_2)^2 + M_2^2|u^m|^2 + \frac{1}{2}\Big(\frac{\partial u^m}{\partial t}\Big)^2. \quad (3.3.9)$$

Due to (3.1.3), (3.3.9) and $|F^m\,\frac{\partial u^m}{\partial t}| \le \frac{1}{2}\,[(\frac{\partial u^m}{\partial t})^2 + (F^m)^2]$, from (3.3.8) we have

$$\frac{1}{2}\int\limits_{\Omega_\tau}\Big[\Big(\frac{\partial u^m}{\partial t}\Big)^2 + \sum_{i=1}^n\Big(\frac{\partial u^m}{\partial x_i}\Big)^2\Big]\,dx \le \int\limits_{S_\tau}\frac{1}{2|\nu_0|}\Big[\sum_{i=1}^n\Big(\frac{\partial u^m}{\partial x_i}\,\nu_0 - \frac{\partial u^m}{\partial t}\,\nu_i\Big)^2\Big]\,ds$$
$$+ (M_1+M_2)^2\,\mathrm{mes}\,D_\tau + M_2^2\int\limits_{D_\tau}|u^m|^2\,dx\,dt + \int\limits_{D_\tau}\Big(\frac{\partial u^m}{\partial t}\Big)^2\,dx\,dt + \frac{1}{2}\int\limits_{D_\tau}(F^m)^2\,dx\,dt. \quad (3.3.10)$$

Since S is a conic surface, we have $\sup\limits_{S\setminus O}|\nu_0|^{-1} = \sup\limits_{S\cap\{t=1\}}|\nu_0|^{-1}$. At the same time, $S\setminus O$ is a smooth manifold, $S\cap\{t=1\} = \partial\Omega_{\tau=1}$ is also a compact manifold. Thus, noting that ν_0 is a continuous function on $S\setminus O$, we get

$$M_0 := \sup_{S\setminus O}|\nu_0|^{-1} = \sup_{S\cap\{t=1\}}|\nu_0|^{-1} < +\infty,\ \ |\nu_0| \le |\nu| = 1. \quad (3.3.11)$$

Taking into account that $(\nu_0\,\frac{\partial}{\partial x_i} - \nu_i\,\frac{\partial}{\partial t})$ $(i=1,\dots,n)$ is an inner differential operator on S_τ, due to (3.3.5), we have

$$\int\limits_{S_\tau}\Big[\sum_{i=1}^n\Big(\frac{\partial u^m}{\partial x_i}\,\nu_0 - \frac{\partial u^m}{\partial t}\,\nu_i\Big)^2\Big] \le \big\|u^m\big|_{S_T}\big\|^2_{W_2^1(S_t)} = \|g^m\|^2_{W_2^1(S_T)}. \quad (3.3.12)$$

It follows from (3.3.11) and (3.3.12) that

$$\int\limits_{S_\tau}\frac{1}{2|\nu_0|}\Big[\sum_{i=1}^n\Big(\frac{\partial u^m}{\partial x_i}\,\nu_0 - \frac{\partial u^m}{\partial t}\,\nu_i\Big)^2\Big] \le \frac{1}{2}\,M_0\|g^m\|^2_{W_2^1(S_T)}. \quad (3.3.13)$$

By virtue of (3.3.13), from (3.3.10) we obtain

$$\int\limits_{\Omega_\tau}\Big[\Big(\frac{\partial u^m}{\partial t}\Big)^2 + \sum_{i=1}^n\Big(\frac{\partial u^m}{\partial x_i}\Big)^2\Big]\,dx \le M_0\|g^m\|^2_{W_2^1(S_T)} + 2(M_1+M_2)^2\,\mathrm{mes}\,D_T$$
$$+ 2M_2^2\int\limits_{D_\tau}|u^m|^2\,dx\,dt + 2\int\limits_{D_\tau}\Big(\frac{\partial u^m}{\partial t}\Big)^2\,dx\,dt + \int\limits_{D_T}(F^m)^2\,dx\,dt,\ \ 0<\tau\le T. \quad (3.3.14)$$

If $t=\gamma(x)$ is the equation of the conic surface S, then, in view of (3.3.5), we have

$$u^m(x,\tau) = u^m(x,\gamma(x)) + \int\limits_{\gamma(x)}^{\tau}\frac{\partial}{\partial t}\,u^m(x,s)\,ds = g^m(x) + \int\limits_{\gamma(x)}^{\tau}\frac{\partial}{\partial t}\,u^m(x,s)\,ds,\ \ (x,\tau)\in\Omega_\tau.$$

Squaring scalarly both parts of the obtained equality, integrating in the domain Ω_τ and using the

Schwartz inequality, we get

$$\int\limits_{\Omega_\tau}(u^m)^2\,dx \le 2\int\limits_{\Omega_\tau}(g^m(x,\gamma(x))^2\,dx + 2\int\limits_{\Omega_\tau}\Big(\int\limits_{\gamma(x)}^{\tau}\frac{\partial}{\partial t}\,u^m(x,s)\,ds\Big)^2\,dx$$

$$\le 2\int\limits_{S_\tau}(g^m)^2\,ds + 2\int\limits_{\Omega_\tau}(\tau-\gamma(x))\Big[\int\limits_{\gamma(x)}^{\tau}\Big(\frac{\partial u^m}{\partial t}\Big)^2\,ds\Big]\,dx$$

$$\le 2\int\limits_{S_\tau}(g^m)^2\,ds + 2T\int\limits_{\Omega_\tau}\Big[\int\limits_{\gamma(x)}^{\tau}\Big(\frac{\partial u^m}{\partial t}\Big)^2\,ds\Big]\,dx = 2\int\limits_{S_\tau}(g^m)^2\,ds + 2T\int\limits_{D_\tau}\Big(\frac{\partial u^m}{\partial t}\Big)^2\,dx\,dt. \quad (3.3.15)$$

From (3.3.14) and (3.3.15) it follows

$$\int\limits_{\Omega_\tau}\Big[(u^m)^2+\Big(\frac{\partial u^m}{\partial t}\Big)^2+\sum_{i=1}^{n}\Big(\frac{\partial u^m}{\partial x_i}\Big)^2\Big]\,dx \le (M_0+2)\|g^m\|^2_{W_2^1(S_T)} + 2(M_1+M_2)^2\operatorname{mes}D_\tau$$

$$+\,2M_2^2\int\limits_{D_\tau}|u^m|^2\,dx\,dt + 2(T+1)\int\limits_{D_\tau}\Big(\frac{\partial u^m}{\partial t}\Big)^2\,dx\,dt + \|F^m\|^2_{L_2(D_T)}$$

$$\le (2M_2^2+2(T+1))\int\limits_{D_\tau}\Big[(u^m)^2+\Big(\frac{\partial u^m}{\partial t}\Big)^2+\sum_{i=1}^{n}\Big(\frac{\partial u^m}{\partial x_i}\Big)^2\Big]\,dx\,dt$$

$$+\Big[\|F^m\|^2_{L_2(D_T)} + (M_0+2)\|g^m\|^2_{W_2^1(S_T)} + 2(M_1+M_2)^2\operatorname{mes}D_T\Big]. \quad (3.3.16)$$

Putting

$$w(\tau) := \int\limits_{\Omega_\tau}\Big[(u^m)^2+\Big(\frac{\partial u^m}{\partial t}\Big)^2+\sum_{i=1}^{n}\Big(\frac{\partial u^m}{\partial x_i}\Big)^2\Big]\,dx, \quad (3.3.17)$$

from (3.3.16) we have

$$w(\tau) \le (2M_2^2+2T+2)\int\limits_0^{\tau} w(s)\,ds$$

$$+\Big[\|F^m\|^2_{L_2(D_T)} + (M_0+2)\|g^m\|^2_{W_2^1(S_T)} + 2(M_1+M_2)^2\operatorname{mes}D_T\Big],\ \ 0<\tau\le T, \quad (3.3.18)$$

whence by the Gronwall lemma it follows that

$$w(\tau) \le A_m\exp(2M_2^2+2T+2)\tau,\ \ 0<\tau\le T, \quad (3.3.19)$$

Here,

$$A_m = \|F^m\|^2_{L_2(D_T)} + (M_0+2)\|g^m\|^2_{W_2^1(S_T)} + 2(M_1+M_2)^2\operatorname{mes}D_T. \quad (3.3.20)$$

In view of (3.3.17) and (3.3.19), we find that

$$\|u^m\|^2_{W_2^1(D_T)} = \int\limits_0^{T} w(\tau)\,d\tau \le A_m T\exp(2M_2^2+2T+2)T. \quad (3.3.21)$$

Due to (3.3.2)–(3.3.5) and (3.3.20), passing to the limit in (3.3.21) as $m\to\infty$, we have

$$\|u\|^2_{W_2^1(D_T)} \le AT\exp(2M_2^2+2T+2)T. \quad (3.3.22)$$

Here,

$$A = \|F\|^2_{L_2(D_T)} + (M_0+2)\|g\|^2_{W_2^1(S_T)} + 2(M_1+M_2)^2\operatorname{mes}D_T. \quad (3.3.23)$$

Taking a square root from both sides of the inequality (3.3.22) and using the obvious inequality $(\sum\limits_{i=1}^{k} a_i^2)^{1/2} \leq \sum\limits_{i=1}^{k} |a_i|$, due to (3.3.23), we finally have

$$\|u\|_{W_2^1(D_T)} \leq c_1\|F\|_{L_2(D_T)} + c_2\|g\|_{W_2^1(S_T)} + c_3.$$

Here,

$$\begin{cases} c_1 = \sqrt{T}\,\exp(M_2^2+T+1)T, \\ c_2 = \sqrt{T}\,(M_0+2)^{1/2}\exp(M_2^2+T+1)T, \\ c_3 = \sqrt{2T}\,(M_1+M_2)(\operatorname{mes} D_T)^{1/2}\exp(M_2^2+T+1)T. \end{cases} \tag{3.3.24}$$

Thus Lemma 3.3.1 is proved completely. □

Before passing to the question of solvability of the problem (3.1.1), (3.1.2), let us consider the same question for the linear case of the needed form, when in (3.1.1) the vector function $f=0$, i.e., for the problem

$$L_0 u := \square\, u = F(x,t), \ \ (x,t)\in D_T, \tag{3.3.25}$$

$$u\big|_{S_T} = g. \tag{3.3.26}$$

For the problem (3.3.25), (3.3.26), analogously to Definition 3.2.1 for the problem (3.1.1), (3.1.2), we introduce the notion of a strong generalized solution $u=(u_1,\dots,u_N)\in W_2^1(D_T)$ of the class W_2^1 in the domain D_T with $F=(F_1,\dots,F_N)\in L_2(D_T)$ and $g=(g_1,\dots,g_N)\in W_2^1(D_T)$, for which there exists a sequence of vector functions $u^m\in C^2(\overline{D}_T)$ such that

$$\lim_{m\to\infty}\|u^m-u\|_{W_2^1(D_T)}=0, \quad \lim_{m\to\infty}\|L_0u^m-F\|_{L_2(D_T)}=0, \tag{3.3.27}$$

$$\lim_{m\to\infty}\big\|u^m\big|_{S_T}-g\big\|_{W_2^1(S_T)}=0. \tag{3.3.28}$$

Note that, as is easily seen from the proof of Lemma 3.3.1, by virtue of (3.3.24), when $f=0$, i.e., when $M_1=M_2=0$, for a strong generalized solution $u\in W_2^1(D_T)$ of the problem (3.3.25), (3.3.26) of the class W_2^1 in the domain D_T the following a priori estimate is valid:

$$\|u\|_{W_2^1(D_T)} \leq c\big(\|F\|_{L_2(D_T)} + \|g\|_{W_2^1(S_T)}\big), \tag{3.3.29}$$

where

$$c = \sqrt{T}\,(M_0+2)^{1/2}\exp(T+1)T. \tag{3.3.30}$$

Consider the Sobolev weight space $W_{2,\alpha}^{*}(D)$, $0<\alpha<\infty$, $k=1,2,\dots$, consisting of the functions belonging to that class $W_{2,loc}^{k}(D)$ for which the norm

$$\|w\|^2_{W_{2,\alpha}^k} = \sum_{i=0}^{k}\int\limits_D r^{-2\alpha-2(k-i)}\Big|\frac{\partial^i w}{\partial x^{i'}\partial t^{i_0}}\Big|^2\,dx\,dt$$

is finite [52], where

$$r=\Big(\sum_{j=1}^{n}x_j^2+t^2\Big)^{1/2}, \quad \frac{\partial^i w}{\partial x^{i'}\partial t^{i_0}} := \frac{\partial^i w}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}\partial t^{i_0}}\,, \ \ i=i_1+\cdots+i_n+i_0.$$

Analogously we introduce the space $W_{2,\alpha}^{k}(S)$, $S=\partial D$ [52].

Together with the problem (3.3.25), (3.3.26), consider in an infinite cone $D=D_\infty$ the analogous problem:

$$L_0 u = F(x,t), \ \ (x,t)\in D, \tag{3.3.31}$$

$$u\big|_{S} = g. \tag{3.3.32}$$

Due to (3.1.3), according to the result obtained in [43], there exists a constant $\alpha_0 = \alpha_0(k) > 1$ such that for $\alpha \geq \alpha_0$, the problem (3.3.31), (3.3.32) has a unique solution $u = (u_1, \dots, u_N) \in W^2_{2,\alpha}(D)$ for each $F = (F_1, \dots, F_N) \in W^{k-1}_{2,\alpha-1}(D)$ and $g = (g_1, \dots, g_N) \in W^k_{2,\alpha-\frac{1}{2}}(S)$, $k \geq 2$.

Since the space $C_0^\infty(D_T)$ of finite infinitely differentiable in D_T functions is dense in $L_2(D_T)$, for the given $F = (F_1, \dots, F_N) \in L_2(D_T)$, there exists a sequence of vector functions $F^m = (F_1^m, \dots, F_N^m) \in C_0^\infty(D_T)$ such that $\lim\limits_{m\to\infty} \|F^m - F\|_{L_2(D_T)} = 0$. For the fixed m, extending the vector function F^m by zero beyond the domain D_T and keeping the same notation, we have $F^m \in C_0^\infty(D)$. Obviously, $F^m \in W^{k-1}_{2,\alpha-1}(D)$ for any $k \geq 2$ and $\alpha > 1$, and also for $\alpha \geq \alpha_0 = \alpha_0(k)$. If $g \in W_2^1(S_T)$, then there exists $\widetilde{g} \in W_2^1(S)$ such that $g = \widetilde{g}|_{S_T}$ and $\operatorname{diam}\operatorname{supp}\widetilde{g} < +\infty$ [68]. Besides, the space $C_*^\infty(S) := \{g \in C^\infty(S) : \ \operatorname{diam}\operatorname{supp} g < +\infty,\ 0 \notin \operatorname{supp} g\}$ is dense in $W_2^1(S)$ [56]. Therefore, there exists a sequence $g^m \in C_*^\infty(S)$ such that $\lim\limits_{m\to\infty} \|g^m - g\|_{W_2^1(S)} = 0$. It is easy to see that $g^m \in W^k_{2,\alpha-\frac{1}{2}}(S)$ for any $k \geq 2$ and $\alpha > 1$ and, therefore, for $\alpha \geq \alpha_0 = \alpha(k)$. According to what has been mentioned above, there exists a solution $\widetilde{u}^m \in W^k_{2,\alpha}(D)$ of the problem (3.3.31), (3.3.32) for $F = F^m$ and $g = g^m$. Let $u^m = \widetilde{u}^m|_{D_T}$. Since $u^m \in W_2^k(D_T)$, taking the number k sufficiently large, namely, $k > \frac{n+1}{2} + 2$, we have $u^m \in C^2(\overline{D}_T)$. By virtue of the estimate (3.3.29), we have

$$\|u^m - u^{m'}\|_{W_2^1(D_T)} \leq c\big(\|F^m - F^{m'}\|_{L_2(D_T)} + \|g^m - g^{m'}\|_{W_2^1(S_T)}\big). \tag{3.3.33}$$

Since the sequences $\{F^m\}$ and $\{g^m\}$ are fundamental in the spaces $L_2(D_T)$ and $W_2^1(S_T)$, respectively, the sequence $\{u^m\}$ is, due to (3.3.33), fundamental in the space $W_2^1(D_T)$. Therefore, in view of the completeness of the space $W_2^1(D_T)$, there exists a vector function $u \in W_2^1(D_T)$ such that $\lim\limits_{m=\infty} \|u^m - u\|_{W_2^1(D_T)} = 0$, and since $L_0 u^m = F^m \to F$ in the space $L_2(D_T)$ and $g^m = u^m|_{S_T} \to g$ in the space $W_2^1(S_T)$, i.e., the limit equalities (3.3.27) and (3.3.28) are fulfilled, the vector function u is a strong generalized solution of the problem (3.3.25), (3.3.26) of the class W_2^1 in the domain D_T. The uniqueness of the solution of the problem (3.3.25), (3.3.26) of the class W_2^1 in the domain D_T follows from the a priori estimate (3.3.29). Thus for the solution u of the problem (3.3.25), (3.3.26) we have $u = L_0^{-1}(F, g)$, where $L_0^{-1} : [L_2(D_T)]^N \times [W_2^1(S_T)]^N \to [W_2^1(D_T)]^N$ is a linear continuous operator with a norm admitting, in view of (3.3.29), the following estimate

$$\|L_0^{-1}\|_{[L_2(D_T)]^N \times [W_2^1(S_T)]^N \to [W_2^1(D_T)]^N} \leq c, \tag{3.3.34}$$

where the constant c is determined from (3.3.30).

Owing to the linearity of the operator

$$L_0^{-1} : [L_2(D_T)]^N \times [W_2^1(S_T)]^N \to [W_2^1(D_T)]^N$$

we have a representation

$$L_0^{-1}(F, g) = L_{01}^{-1}(F) + L_{02}^{-1}(g), \tag{3.3.35}$$

where $L_{01}^{-1} : [L_2(D_T)]^N \to [W_2^1(D_T)]^N$ and $L_{02}^{-1} : [W_2^1(S_T)]^N \to [W_2^1(D_T)]^N$ are the linear continuous operators and, in view of (3.3.34), we have

$$\|L_{01}^{-1}\|_{[L_2(D_T)]^N \to [L_2(D_T)]^N} \leq c, \quad \|L_{02}^{-1}\|_{[W_2^1(S_T)]^N \to [W_2^1(D_T)]^N} \leq c. \tag{3.3.36}$$

Remark 3.3.1. Note that for $F \in L_2(D_T)$, $g \in W_2^1(S_T)$ and (3.2.2), where $0 \leq \alpha < \frac{n+1}{n-1}$, in view of (3.3.34), (3.3.35), (3.3.36) and Remark 3.2.1, the vector function $u = (u_1, \dots, u_N) \in W_2^1(D_T)$ is a strong generalized solution of the problem (3.1.1), (3.1.2) of the class W_2^1 in the domain D_T if and only if u is a solution of the following functional equation

$$u = L_{01}^{-1}(-f(u)) + L_{01}^{-1}(F) + L_{02}^{-1}(g) \tag{3.3.37}$$

in the space $W_2^1(D_T)$.

Rewrite the equation (3.3.37) in the form

$$u = A_0 u := -L_{01}^{-1}(\mathcal{K}_0 u) + L_{01}^{-1}(F) + L_{02}^{-1}(g), \tag{3.3.38}$$

where the operator $\mathcal{K}_0 : [W_2^1(D_T)]^N \to [L_2(D_T)]^N$ from (3.2.2) is, due to Remark 3.2.1, continuous and compact. Therefore, according to (3.3.36), the operator $\mathcal{A}_0 : [W_2^1(D_T)]^N \to [W_2^1(D_T)]^N$ is also continuous and compact. At the same time, according to Lemma 3.3.1 and the equalities (3.3.24), for any parameter $\tau \in [0,1]$ and any solution u of the equation $u = \tau \mathcal{A}_0 u$ with parameter τ, the same a priori estimate (3.3.1) with the constants c_i from (3.3.24), independent of u, F, g and τ, is valid. Therefore, due to Schaefer's fixed point theorem [20], the equation (3.3.38) and hence, according to Remark 3.3.1, the problem (3.1.1), (3.1.2) has at least one solution $u \in W_2^1(D_T)$.

Thus we have proved the following

Theorem 3.3.1. *Let f satisfy the condition* (3.2.2), *where $0 \le \alpha \le 1$. Then for any $F \in L_2(D_T)$ and $g \in W_2^1(S_T)$, the problem* (3.1.1), (3.1.2) *has at least one strong generalized solution u of the class W_2^1 in the domain D_T in the sense of Definition* 3.2.1.

A global solvability of the problem (3.1.1), (3.1.2) in the class W_2^1 in the sense of Definition 3.2.3 follows immediately from Theorem 3.3.1, when the conditions of this theorem are fulfilled.

Remark 3.3.2. In Theorem 3.3.1, a global solvability of the problem (3.1.1), (3.1.2) is proved for the case in which f satisfies the condition (3.2.2), where $0 \le \alpha \le 1$. In case $1 < \alpha < \frac{n+1}{n-1}$, the problem (3.1.1), (3.1.2) is, generally speaking, not globally solvable, as it will be shown in Section 3.5. At the same time, it will be proved below that when $1 < \alpha < \frac{n+1}{n-1}$, the problem (3.1.1), (3.1.2) is locally solvable in the sense of Definition 3.2.2.

Theorem 3.3.2. *Let f satisfy the condition* (3.2.2), *where $1 < \alpha < \frac{n+1}{n-1}$, $g = 0$, $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for any $T > 0$. Then the problem* (3.1.1), (3.1.2) *is locally solvable in the class W_2^1, i.e., there exists a number $T_0 = T_0(F) > 0$ such that for $T < T_0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition* 3.2.1.

Proof. According to Definition 3.2.1 and Remark 3.3.1, the vector function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T) := \{v \in W_2^1(D_T) : \ v|_{S_T} = 0\}$ is a strong generalized solution of the problem (3.1.1), (3.1.2) of the class W_2^1 in the domain D_T for $g = 0$ if and only if u is a solution of the functional equation (3.3.38) for $g = 0$, i.e.,

$$u = A_0 u := -L_{01}^{-1}(\mathcal{K}_0 u) + L_{01}^{-1}(F) \tag{3.3.39}$$

in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$. Denote by $B(0, r_0) := \{u = (u_1, \dots, u_N) \in \overset{\circ}{W}{}_2^1(D_T, S_T) : \ \|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \le r_0\}$ a closed convex ball in the Hilbert space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ of radius $r_0 > 0$ and with center in a null element. Since the operator A_0 from (3.3.39), acting in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$, is a continuous compact operator, according to Schauder's theorem, for the solvability of the equation (3.3.39) in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ it suffices to prove that the operator $\mathcal{A}_0$ maps the ball $B(0, r_0)$ into itself for certain $r_0 > 0$ [20]. Below we will show that for any fixed $r_0 > 0$, there exists a number $T_0 = T_0(r_0, F) > 0$ such that for $T < T_0$, the operator $\mathcal{A}_0$ from (3.3.39) maps the ball $B(0, r_0)$ into itself. Towards this end, we evaluate $\|\mathcal{A}_0 u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}$ for $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$.

When $u = (u_1, \dots, u_N) \in \overset{\circ}{W}{}_2^1(D_T, S_T)$, we denote by $\widetilde{u}$ the vector function which is an even extension of u through the plane $t = T$ in the domain D_T^*, symmetric to the domain D_T with respect to the same plane, i.e.,

$$\widetilde{u} = \begin{cases} u(x,t), & (x,t) \in D_T, \\ u(x, 2T - t), & (x,t) \in D_T^*, \end{cases}$$

and $\widetilde{u}(x,t) = u(x,t)$ for $t = T$ in the sense of the trace theory. It is obvious that $\widetilde{u} \in \overset{\circ}{W}{}_2^1(\widetilde{D}_T) : \{v \in W_2^1(D_T) : \ v|_{\partial \widetilde{D}_T} = 0\}$, where $\widetilde{D}_T = D_T \cup \Omega_T \cup D_T^*$, $\Omega_T := D \cap \{t = T\}$.

Using the inequality [93]

$$\int_\Omega |v| \, d\Omega \le (\operatorname{mes} \Omega)^{1 - \frac{1}{p}} \|v\|_{p,\Omega}, \ \ p \ge 1,$$

and taking into account the equalities

$$\|\widetilde{u}\|^p_{L_p(\widetilde{D}_T)} = 2\|u\|^p_{L_p(D_T)}, \quad \|\widetilde{u}\|^2_{\overset{\circ}{W}{}^1_2(\widetilde{D}_T)} = 2\|u\|^2_{\overset{\circ}{W}{}^1_2(D_T,S_T)},$$

from the known multiplicative inequality [68]

$$\|v\|_{p,\Omega} \le \beta \|\nabla_{x,t} v\|^{\widetilde{\alpha}}_{m,\Omega} \|v\|^{1-\widetilde{\alpha}}_{r,\Omega} \ \ \forall\, v \in \overset{\circ}{W}{}^1_2(\Omega), \Omega \subset \mathbb{R}^{n+1},$$
$$\nabla_{x,t} = \Big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\Big), \ \ \widetilde{\alpha} = \Big(\frac{1}{r} - \frac{1}{p}\Big)\Big(\frac{1}{r} - \frac{1}{\widetilde{m}}\Big)^{-1}, \ \ \widetilde{m} = \frac{(n+1)m}{n+1-m}$$

for $\Omega = \widetilde{D}_T \subset \mathbb{R}^{n+1}$, $v = \widetilde{v}$, $r = 1$, $m = 2$ and $1 < p \le \frac{2(n+1)}{n+1-m}$, where $\beta = const > 0$ does not depend on v and T, it follows the inequality

$$\|u\|_{L_p(D_T)} \le c_0 (\operatorname{mes} D_T)^{\frac{1}{p}+\frac{1}{p+1}-\frac{1}{2}} \|u\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} \ \ \forall\, u \in \overset{\circ}{W}{}^1_2(D_T,S_T), \tag{3.3.40}$$

where $c_0 = const > 0$ does not depend on u and T.

Since $\operatorname{mes} D_T = \frac{\omega}{n+1}\, T^{n+1}$, where ω is the n-dimensional measure of the section $\Omega_1 := D \cap \{t = 1\}$, for $p = 2\alpha$ from (3.3.40) we have

$$\|u\|_{L_{2\alpha}(D_T)} \le C_T \|u\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} \ \ \forall\, u \in \overset{\circ}{W}{}^1_2(D_T,S_T), \tag{3.3.41}$$

where

$$C_T = c_0 \Big(\frac{\omega}{n+1}\Big)^{\alpha_1} T^{(n+1)\alpha_1}, \ \ \alpha_1 = \frac{1}{2\alpha} + \frac{1}{n+1} - \frac{1}{2}\,. \tag{3.3.42}$$

Since $\alpha < \frac{n+1}{n-1}$, we have $\alpha_1 = \frac{1}{2\alpha} + \frac{1}{n+1} - \frac{1}{2} > 0$, and due to (3.3.41), and (3.3.42), for any $u \in \overset{\circ}{W}{}^1_2(D_T,S_T)$ we get

$$\|u\|_{L_{2\alpha}(D_T)} \le C_{T_1} \|u\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} \ \ \forall\, T \le T_1, \tag{3.3.43}$$

where T_1 is a fixed positive number.

For $\|\mathcal{K}_0 u\|_{L_2(D_T)}$, where $u \in \overset{\circ}{W}{}^1_2(D_T,S_T)$, $T \le T_1$, and the operator $\mathcal{K}_0$ acts according to the formula (3.2.3), due to (3.2.2) and (3.3.43), we have the following estimate

$$\begin{aligned}\|\mathcal{K}_0 u\|^2_{L_2(D_T)} &\le \int_{D_T} (M_1 + M_2|u|^\alpha)^2\, dx\, dt \le 2M_1^2 \operatorname{mes} D_T + 2M_2^2 \int_{D_T} |u|^{2\alpha}\, dx\, dt \\ &= 2M_1^2 \operatorname{mes} D_T + 2M_2^2 \|u\|^{2\alpha}_{L_{2\alpha}(D_T)} \le 2M_1^2 \operatorname{mes} D_T + 2M_2^2 C^{2\alpha}_{T_1} \|u\|^{2\alpha}_{\overset{\circ}{W}{}^1_2(D_T,S_T)},\end{aligned}$$

whence we obtain

$$\|\mathcal{K}_0 u\|_{L_{2\alpha}(D_T)} \le M_1 (2 \operatorname{mes} D_{T_1})^{1/2} + \sqrt{2}\, M_2 C^{\alpha}_{T_1} \|u\|^{\alpha}_{\overset{\circ}{W}{}^1_2(D_T,S_T)}. \tag{3.3.44}$$

From (3.3.30), (3.3.36), (3.3.39) and (3.3.44), it follows that

$$\begin{aligned}&\|\mathcal{A}_0 u\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} \\ &\quad \le \|L_{01}^{-1}\|_{[L_2(D_T)]^N \to [\overset{\circ}{W}{}^1_2(D_T,S_T)]^N} \|\mathcal{K}_0 u\|_{L_2(D_T)} + \|L_{01}^{-1}\|_{[L_2(D_T)]^N \to [\overset{\circ}{W}{}^1_2(D_T,S_T)]^N} \|F\|_{L_2(D_T)} \\ &\quad \le c\Big[\sqrt{2 \operatorname{mes} D_{T_1}}\, M_1 + \sqrt{2}\, M_2 C^{\alpha}_{T_1} \|u\|^{\alpha}_{\overset{\circ}{W}{}^1_2(D_T,S_T)} + \|F\|_{L_2(D_{T_1})}\Big] \\ &\quad \le \sqrt{T}\,(M_0+2)^{1/2} \exp(T_1+1) T_1 \\ &\qquad\qquad \times \Big[\sqrt{2 \operatorname{mes} D_{T_1}}\, M_1 + \sqrt{2}\, M_2 C^{\alpha}_{T_1} \|u\|^{\alpha}_{\overset{\circ}{W}{}^1_2(D_T,S_T)} + \|F\|_{L_2(D_{T_1})}\Big] \\ &\qquad\qquad\qquad\qquad \forall\, T \le T_1 \ \ \forall\, u \in \overset{\circ}{W}{}^1_2(D_T,S_T).\end{aligned} \tag{3.3.45}$$

Since the right-hand side of the inequality (3.3.45) contains $\sqrt{T}$ as a factor vanishing as $T\to 0$, there exists a positive number $T_0\le T_1$ such that for $T<T_0$ and $\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}\le r_0$, due to (3.3.45), we have $\|\mathcal{A}_0 u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}\le r_0$, i.e., the operator $\mathcal{A}_0:\overset{\circ}{W}{}_2^1(D_T,S_T)\to \overset{\circ}{W}{}_2^1(D_T,S_T)$ from (3.3.39) maps the ball $B(0,r_0)$ into itself. Thus Theorem 3.3.2 is proved completely. □

Remark 3.3.3. In the case if f satisfies the condition (3.2.2), where $1<\alpha<\frac{n+1}{n-1}$, Theorem 3.3.2 ensures a local solvability of the problem (3.1.1), (3.1.2), although in this case, with the additional conditions imposed on f, this problem is, as it will be shown in the theorem below, globally solvable.

Theorem 3.3.3. *Let f satisfy the condition* (3.2.2), *where $1<\alpha<\frac{n+1}{n-1}$, and $f=\nabla G$, i.e., $f_i(u)=\frac{\partial}{\partial u_i}G(u)$, $u\in\mathbb{R}^N$, $i=1,\dots,N$, where $G=G(u)\in C^1(\mathbb{R}^N)$ is a scalar function satisfying the conditions $G(0)=0$ and $G(u)\ge 0$ $\forall\, u\in\mathbb{R}^N$. Let $g=0$, $F\in L_{2,loc}(D_\infty)$ and $F|_{D_T}\in L_2(D_T)$ for any $T>0$. Then the problem* (3.1.1), (3.1.2) *is globally solvable in the class W_2^1, i.e., for any $T>0$, this problem has a strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition* 3.2.1.

Proof. First, let us show that for any fixed $T>0$, with the conditions of Theorem 3.3.3, for a strong generalized solution u of the problem (3.1.1), (3.1.2) of the class W_2^1 in the domain D_T, the estimate

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}\le c(T)\|F\|_{L_2(D_T)},\quad c(T)=\sqrt{T}\,\exp\frac{1}{2}\,(T+T^2) \tag{3.3.46}$$

is valid.

Indeed, according to Definition 3.2.1, in the case $g=0$, there exists a sequence of vector functions $u^m\in \overset{\circ}{C}{}^2(\overline{D}_T,S_T):=\{v\in C^2(\overline{D}_T):\ v|_{S_T}=0\}$ such that

$$\lim_{m\to\infty}\|u^m-u\|_{W_2^1(D_T)}=0,\qquad \lim_{m\to\infty}\|Lu^m-F\|_{L_2(D_T)}=0. \tag{3.3.47}$$

Putting

$$F^m:=Lu^m \tag{3.3.48}$$

and taking into account that $u^m|_{S_T}=0$ and the operator $\nu_0\,\frac{\partial}{\partial x_i}-\nu_i\,\frac{\partial}{\partial t}$ is an inner differential operator on S_T and, hence $\big(\frac{\partial u^m}{\partial x_i}\,\nu_0-\frac{\partial u^m}{\partial t}\,\nu_i\big)\big|_{S_T}=0$, $i=1,\dots,n$, due to (3.1.3), from (3.3.8) we get

$$\int\limits_{D_\tau}F^m\,\frac{\partial u^m}{\partial t}\,dx\,dt\ge\frac{1}{2}\int\limits_{\Omega_\tau}\Big[\Big(\frac{\partial u^m}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial u^m}{\partial x_i}\Big)^2\Big]\,dx+\int\limits_{D_\tau}f(u^m)\,\frac{\partial u^m}{\partial t}\,dx\,dt. \tag{3.3.49}$$

Since $f=\nabla G$, we have $f(u^m)\,\frac{\partial u^m}{\partial t}=\frac{\partial}{\partial t}\,G(u^m)$, and taking into account that $u^m|_{S_T}=0$, $\nu_0|_{\Omega_\tau}=1$, $G(0)=0$, and integrating by parts, we obtain

$$\begin{aligned}\int\limits_{D_\tau}f(u^m)\,\frac{\partial u^m}{\partial t}\,dx\,dt&=\int\limits_{D_\tau}\frac{\partial}{\partial t}\,G(u^m)\,dx\,dt\\&=\int\limits_{\partial D_\tau}G(u^m)\nu_0\,ds=\int\limits_{S_\tau\cup\Omega_\tau}G(u^m)\nu_0\,ds=\int\limits_{\Omega_\tau}G(u^m)\,dx.\end{aligned} \tag{3.3.50}$$

Owing to $G(u)\ge 0$ $\forall\, u\in\mathbb{R}^N$, due to (3.3.50), from (3.3.49), we get

$$\begin{aligned}&\int\limits_{\Omega_\tau}\Big[\Big(\frac{\partial u^m}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial u^m}{\partial x_i}\Big)^2\Big]\,dx\\&\qquad\le 2\int\limits_{D_T}F^m\,\frac{\partial u^m}{\partial t}\,dx\,dt\le\int\limits_{D_T}\Big(\frac{\partial u^m}{\partial t}\Big)^2\,dx\,dt+\int\limits_{D_T}(F^m)^2\,dx\,dt,\quad 0<\tau\le T.\end{aligned} \tag{3.3.51}$$

Since $u^m|_{S_T}=0$, we have $u(x,\tau)=\int\limits_{\gamma(x)}^{\tau}\frac{\partial}{\partial t}\,u^m(x,s)\,ds$, where $t=\gamma(x)$ is the equation of the conic surface S. Thus just as in obtaining the inequality (3.3.15), we get

$$\int\limits_{\Omega_\tau}(u^m)^2\,dx=\int\limits_{\Omega_\tau}\Big(\int\limits_{\gamma(x)}^{\tau}\frac{\partial}{\partial t}\,u^m(x,s)\,ds\Big)^2\,dx\le\int\limits_{\Omega_\tau}(\tau-|x|)\Big[\int\limits_{\gamma(x)}^{\tau}\Big(\frac{\partial}{\partial t}\,u^m\Big)^2\,ds\Big]\,dx$$
$$\le T\int\limits_{\Omega_\tau}\Big[\int\limits_{\gamma(x)}^{\tau}\Big(\frac{\partial u^m}{\partial t}\Big)^2\,ds\Big]\,dx=T\int\limits_{D_\tau}\Big(\frac{\partial u^m}{\partial t}\Big)^2\,dx\,dt. \tag{3.3.52}$$

Denoting

$$w(\tau):=\int\limits_{\Omega_\tau}\Big[(u^m)^2+\Big(\frac{\partial u^m}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial u^m}{\partial x_i}\Big)^2\Big]\,dx,$$

in view of (3.3.51) and (3.3.52), we have

$$\begin{aligned}w(\tau)&\le(1+T)\int\limits_{D_\tau}\Big(\frac{\partial u^m}{\partial t}\Big)^2\,dx\,dt+\int\limits_{D_\tau}(F^m)^2\,dx\,dt\\&\le(1+T)\int\limits_{D_\tau}\Big[(u^m)^2+\Big(\frac{\partial u^m}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial u^m}{\partial x_i}\Big)^2\Big]\,dx\,dt+\|F^m\|^2_{L_2(D_\tau)}\\&=(1+T)\int\limits_0^{\tau}w(s)\,ds+\|F^m\|^2_{L_2(D_\tau)},\quad 0<\tau\le T.\end{aligned} \tag{3.3.53}$$

By virtue of the Gronwall lemma, it follows from (3.3.53) that

$$w(\tau)\le\|F\|^2_{L_2(D_\tau)}\exp(1+T)\tau\le\|F\|^2_{L_2(D_T)}\exp(1+T)T,\quad 0<\tau\le T. \tag{3.3.54}$$

According to (3.3.54), we have

$$\|u^m\|^2_{\overset{\circ}{W}{}^1_2(D_T,S_T)}=\int\limits_{D_T}\Big[(u^m)^2+\Big(\frac{\partial u^m}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial u^m}{\partial x_i}\Big)^2\Big]\,dx\,dt$$
$$=\int\limits_0^T w(\tau)\,d\tau\le T\|F^m\|_{L_2(D_T)}\exp(1+T)T,$$

whence, due to the limit equalities (3.3.47), we arrive at the estimate (3.3.46).

According to Remark 3.3.1, when the conditions of Theorem 3.3.3 are fulfilled, the vector function $u\in\overset{\circ}{W}{}^1_2(D_T,S_T)$ is a strong generalized solution of the problem (3.1.1), (3.1.2) of the class W_2^1 if and only if u is a solution of the functional equation $u=\mathcal{A}_0u$ from (3.3.39) in the space $\overset{\circ}{W}{}^1_2(D_T,S_T)$, where the operator $\mathcal{A}_0$, acting in the space $\overset{\circ}{W}{}^1_2(D_T,S_T)$, is continuous and compact. At the same time, due to (3.3.46), for any solution of the equation $u=\mu\mathcal{A}_0u$, an a priori estimate

$$\|u\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)}\le\mu c(T)\|F\|_{L_2(D_T)}\le c(T)\|F\|_{L_2(D_T)}$$

with the positive constant $c(T)$, independent of u, μ and F, is valid. Thus, according to Schaefer's fixed point theorem [20], the equation (3.3.46), and hence the problem (3.1.1), (3.1.2), has at least one strong generalized solution of the class W_2^1 in the domain D_T for any $T>0$. Thus Theorem 3.3.3 is proved completely. □

3.4 The uniqueness and existence of a global solution of the problem (3.1.1), (3.1.2) of the class W_2^1

Below, we impose on the nonlinear vector function $f = (f_1, \dots, f_N)$ from (3.1.1) the additional requirements

$$f \in C^1(\mathbb{R}^N), \ \ \Big|\frac{\partial f_i(u)}{\partial u_j}\Big| \le M_3 + M_4|u|^\gamma \ \ \forall\, u \in \mathbb{R}^N, \ \ 1 \le i, j \le N, \tag{3.4.1}$$

where $M_3, M_4, \gamma = const \ge 0$. To simplify our reasoning, we suppose that the vector function $g = 0$ in the boundary condition (3.1.2).

Remark 3.4.1. It is obvious that from (3.4.1) follows the condition (3.2.2) for $\alpha = \gamma + 1$, and in the case $\gamma < \frac{2}{n-1}$, we have $\alpha < \frac{n+1}{n-1}$.

Theorem 3.4.1. *Let the condition* (3.4.1) *be fulfilled, where* $0 \le \gamma < \frac{2}{n-1}$, $F \in L_2(D_T)$ *and* $g = 0$. *Then the problem* (3.1.1), (3.1.2) *cannot have more than one strong generalized solution of the class* W_2^1 *in the domain* D_T *in the sense of Definition* 3.2.1.

Proof. Let $F \in L_2(D_T)$, $g = 0$, and the problem (3.1.1), (3.1.2) have two strong generalized solutions u^1 and u^2 of the class W_2^1 in the domain D_T in the sense of Definition 3.2.1, i.e., there exist two sequences of vector functions $u^{im} \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T) := \{u \in C^2(\overline{D}_T) : \ u|_{S_T} = 0\}$, $i = 1, 2$; $m = 1, 2, \dots$, such that

$$\lim_{m\to\infty} \|u^{im} - u^i\|_{\overset{\circ}{W}{}_2^1(D_T, S_T)} = 0, \quad \lim_{m\to\infty} \|Lu^m - F\|_{L_2(D_T)} = 0, \ \ i = 1, 2. \tag{3.4.2}$$

Let

$$w = u^2 - u^1, \quad w^m = u^{2m} - u^{1m}, \quad F^m = Lu^{2m} - Lu^{1m}. \tag{3.4.3}$$

In view of (3.4.2) and (3.4.3), we have

$$\lim_{m\to\infty} \|w^m - w\|_{\overset{\circ}{W}{}_2^1(D_T, S_T)} = 0, \quad \lim_{m\to\infty} \|F^m\|_{L_2(D_T)} = 0. \tag{3.4.4}$$

In accordance with (3.4.3), consider the vector function $w^m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ as a solution of the following problem:

$$\square\, w^m = -[f(u^{2m}) - f(u^{1m})] + F^m, \tag{3.4.5}$$

$$w^m\big|_{S_T} = 0. \tag{3.4.6}$$

In the same way as the inequality (3.3.49) was obtained, from (3.4.5) and (3.4.6) we arrive at

$$\begin{aligned} \int\limits_{\Omega_\tau} \Big[\Big(\frac{\partial w^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial w^m}{\partial x_i}\Big)^2\Big]\, dx \\ \le 2\int\limits_{D_\tau} F^m \frac{\partial w^m}{\partial t}\, dx\, dt - 2\int\limits_{D_\tau} [f(u^{2m}) - f(u^{1m})]\, \frac{\partial w^m}{\partial t}\, dx\, dt, \ \ 0 < \tau \le T. \end{aligned} \tag{3.4.7}$$

Taking into account the equality

$$f_i(u^{2m}) - f_i(u^{1m}) = \sum_{j=1}^N \int\limits_0^1 \frac{\partial}{\partial u_j}\, f_i\big(u^{1m} + s(u^{2m} - u^{1m})\big)\, ds\, (u_j^{2m} - u_j^{1m}),$$

we obtain

$$[f(u^{2m}) - f(u^{1m})]\, \frac{\partial w^m}{\partial t} = \sum_{i,j=1}^N \Big[\int\limits_0^1 \frac{\partial}{\partial u_j}\, f_i\big(u^{1m} + s(u^{2m} - u^{1m})\big)\, ds\Big] (u_j^{2m} - u_j^{1m})\, \frac{\partial w_i^m}{\partial t}\,. \tag{3.4.8}$$

By virtue of (3.4.1) and the obvious inequality $|d_1+d_2|^\gamma \le 2^\gamma \max(|d_1|^\gamma, |d_2|^\gamma) \le 2^\gamma(|d_1|^\gamma+|d_2|^\gamma)$ for $\gamma \ge 0$, $d_i \in \mathbb{R}$, we have

$$\Big|\int\limits_0^1 \frac{\partial}{\partial u_j}\, f_i\big(u^{1m}+s(u^{2m}-u^{1m})\big)\,ds\Big| \le \int\limits_0^1 \big[M_3+M_4|(1-s)u^{1m}+su^{2m}|^\gamma\big]\,ds \le M_3+2^\gamma M_4\big(|u^{1m}|^\gamma+|u^{2m}|^\gamma\big). \tag{3.4.9}$$

From (3.4.8) and (3.4.9), with regard for (3.4.3), we get

$$\begin{aligned}\Big|\big[f(u^{2m})-f(u^{1m})\big]\,\frac{\partial w^m}{\partial t}\Big| &\le \sum_{i,j=1}^N \big[M_3+2^\gamma M_4\big(|u^{1m}|^\gamma+|u^{2m}|^\gamma\big)\big]\,|w_j^m|\,\Big|\frac{\partial w_i^m}{\partial t}\Big| \\ &\le N^2\big[M_3+2^\gamma M_4\big(|u^{1m}|^\gamma+|u^{2m}|^\gamma\big)\big]\,|w^m|\,\Big|\frac{\partial w^m}{\partial t}\Big| \\ &\le \frac12\,N^2M_3\Big[(w^m)^2+\Big(\frac{\partial w^m}{\partial t}\Big)^2\Big]+2^\gamma N^2M_4\big(|u^{1m}|^\gamma+|u^{2m}|^\gamma\big)|w^m|\,\Big|\frac{\partial w^m}{\partial t}\Big|.\end{aligned} \tag{3.4.10}$$

Due to (3.4.7) and (3.4.10), we have

$$\begin{aligned}\int\limits_{\Omega_\tau}\Big[\Big(\frac{\partial w^m}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial w^m}{\partial x_i}\Big)^2\Big]\,dx &\le \int\limits_{D_\tau}\Big[\Big(\frac{\partial w^m}{\partial t}\Big)^2+(F^m)^2\Big]\,dx\,dt+N^2M_3\int\limits_{D_\tau}\Big[(w^m)^2+\Big(\frac{\partial w^m}{\partial t}\Big)^2\Big]\,dx\,dt \\ &\quad+2^{\gamma+1}N^2M_4\int\limits_{D_\tau}\big(|u^{1m}|^\gamma+|u^{2m}|^\gamma\big)|w^m|\,\Big|\frac{\partial w^m}{\partial t}\Big|\,dx\,dt.\end{aligned} \tag{3.4.11}$$

The last integral in the right-hand side of (3.4.11) can be estimated by Hölder's inequality

$$\int\limits_{D_\tau}\big(|u^{1m}|^\gamma+|u^{2m}|^\gamma\big)|w^m|\,\Big|\frac{\partial w^m}{\partial t}\Big|\,dx\,dt \le \big(\big\|\,|u^{1m}|^\gamma\big\|_{L_{n+1}(D_T)}+\big\|\,|u^{2m}|^\gamma\big\|_{L_{n+1}(D_T)}\big)\|w^m\|_{L_p(D_\tau)}\Big\|\frac{\partial w^m}{\partial t}\Big\|_{L_2(D_\tau)}. \tag{3.4.12}$$

Here, $\frac{1}{n+1}+\frac1p+\frac12=1$, i.e.,

$$p=\frac{2(n+1)}{n-1}\,. \tag{3.4.13}$$

By virtue of (3.3.40), for $q \le \frac{2(n+1)}{n-1}$, we have

$$\|v\|_{L_q(D_\tau)} \le C_q(T)\|v\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \quad \forall\, v\in \overset{\circ}{W}{}_2^1(D_\tau,S_\tau),\ \ 0<\tau\le T, \tag{3.4.14}$$

with the positive constant $C_q(T)$, not depending on $v\in \overset{\circ}{W}{}_2^1(D_\tau,S_\tau)$ and $\tau\in(0,T]$.

According to the theorem, $\gamma<\frac{1}{n-1}$ and, therefore, $\gamma(n+1)<\frac{2(n+1)}{n-1}$. Thus from (3.4.13) and (3.4.14) we obtain

$$\big\|\,|u^{im}|^\gamma\big\|_{L_{n+1}(D_T)}=\|u^{im}\|^\gamma_{L_{\gamma(n+1)}(D_T)} \le C^\gamma_{\gamma(n+1)}(T)\|u^{im}\|^\gamma_{\overset{\circ}{W}{}_2^1(D_T,S_T)},\ \ i=1,2;\ \ m\ge1, \tag{3.4.15}$$

$$\|w^m\|_{L_p(D_\tau)} \le C_p(T)\|w^m\|_{W_2^1(D_\tau)},\ \ m\ge m_0. \tag{3.4.16}$$

In view of the first limit equality from (3.4.2), there exists a natural number m_0 such that for $m \geq m_0$, we have

$$\|u^{im}\|^{\gamma}_{\overset{\circ}{W}{}^1_2(D_T,S_T)} \leq \|u^j\|^{\gamma}_{\overset{\circ}{W}{}^1_2(D_T,S_T)} + 1, \ \ i = 1, 2; \ \ m \geq m_0.$$

In view of the above inequalities, it follows from (3.4.12)–(3.4.16) that

$$\begin{aligned}
&2^{\gamma+1} N^2 M_4 \int\limits_{D_\tau} \big(|u^{1m}|^{\gamma} + |u^{2m}|^{\gamma}\big)|w^m| \left|\frac{\partial w^m}{\partial t}\right| dx\,dt \\
&\leq 2^{\gamma+1} N^2 M_4 C^{\gamma}_{\gamma(n+1)}(T)\Big(\|u^1\|^{\gamma}_{\overset{\circ}{W}{}^1_2(D_T,S_T)} + \|u^2\|^{\gamma}_{\overset{\circ}{W}{}^1_2(D_T,S_T)} + 2\Big) C_p(T) \|w^m\|_{\overset{\circ}{W}{}^1_2(D_\tau,S_\tau)} \left\|\frac{\partial w^m}{\partial t}\right\|_{L_2(D_\tau)} \\
&\qquad\qquad \leq M_5\Big(\|w^m\|^2_{W^1_2(D_\tau)} + \left\|\frac{\partial w^m}{\partial t}\right\|^2_{L_2(D_\tau)}\Big) \\
&\qquad\qquad \leq 2M_5 \|w^m\|^2_{W^1_2(D_\tau)} = 2M_5 \int\limits_{D_\tau} \Big[(w^m)^2 + \Big(\frac{\partial w^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial w^m}{\partial x_i}\Big)^2\Big]\,dx\,dt, \quad (3.4.17)
\end{aligned}$$

where

$$M_5 = 2^{\gamma} N^2 M_4 C^{\gamma}_{\gamma(n+1)}(T)\Big(\|u^1\|^{\gamma}_{\overset{\circ}{W}{}^1_2(D_T,S_T)} + \|u^2\|^{\gamma}_{\overset{\circ}{W}{}^1_2(D_T,S_T)} + 2\Big) C_p(T).$$

Due to (3.4.17), from (3.4.11) we have

$$\begin{aligned}
&\int\limits_{\Omega_\tau} \Big[\Big(\frac{\partial w^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial w^m}{\partial x_i}\Big)^2\Big]\,dx \\
&\qquad \leq M_6 \int\limits_{D_\tau} \Big[(w^m)^2 + \Big(\frac{\partial w^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial w^m}{\partial x_i}\Big)^2\Big]\,dx\,dt + \int\limits_{D_\tau} (F^m)^2\,dx\,dt, \ \ 0 < \tau \leq T, \quad (3.4.18)
\end{aligned}$$

where $M_6 = 1 + M_3 N^2 + 2M_5$.

Note that the inequality (3.3.52) is likewise valid for w^m and, therefore,

$$\int\limits_{\Omega_\tau} (w^m)^2\,dx \leq T \int\limits_{D_\tau} \Big(\frac{\partial w^m}{\partial t}\Big)^2\,dx\,dt \leq T \int\limits_{D_T} \Big[(w^m)^2 + \Big(\frac{\partial w^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial w^m}{\partial x_i}\Big)^2\Big]\,dx\,dt. \quad (3.4.19)$$

Putting

$$\lambda_m(\tau) := \int\limits_{\Omega_\tau} \Big[(w^m)^2 + \Big(\frac{\partial w^m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial w^m}{\partial x_i}\Big)^2\Big]\,dx \quad (3.4.20)$$

and adding (3.4.18) to (3.4.19), we obtain

$$\lambda_m(\tau) \leq (M_6 + T) \int\limits_0^{\tau} \lambda_m(s)\,ds + \|F^m\|^2_{L_2(D_T)},$$

whence by the Gronwall lemma, it follows that

$$\lambda_m(\tau) \leq \|F^m\|^2_{L_2(D_T)} \exp(M_6 + T)\tau. \quad (3.4.21)$$

From (3.4.20) and (3.4.21) we have

$$\|w^m\|^2_{W^1_2(D_T)} = \int\limits_0^T \lambda_m(\tau)\,d\tau \leq T\|F^m\|^2_{L_2(D_T)} \exp(M_6 + T)T. \quad (3.4.22)$$

In view of (3.4.3) and (3.4.4), it follows from (3.4.22) that

$$\|w\|_{W_2^1(D_T)} = \lim_{m\to\infty} \|w - w^m + w^m\|_{W_2^1(D_T)} \le \lim_{m\to\infty} \|w - w^m\|_{W_2^1(D_T)} + \lim_{m\to\infty} \|w^m\|_{W_2^1(D_T)}$$
$$= \lim_{m\to\infty} \|w - w^m\|_{W_2^1(D_T)} = \lim_{m\to\infty} \|w - w^m\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0.$$

Therefore, $w = u_2 - u_1 = 0$, i.e., $u_2 = u_1$. Thus Theorem 3.4.1 is proved completely. □

Theorems 3.3.1, 3.3.3, 3.4.1 and Remark 3.4.1 result in the following theorem of the existence and uniqueness.

Theorem 3.4.2. *Let the vector function f satisfy the condition* (3.4.1)*, where $0 \le \gamma < \frac{2}{n-1}$, and either f satisfy the condition* (3.2.2) *for $\alpha \le 1$ or $f = \nabla G$, where $G \in C^1(\mathbb{R}^N)$, $G(0) = 0$ and $G(u) \ge 0$ $\forall\, u \in \mathbb{R}^N$. Then for any $F \in L_2(D_T)$ and $g = 0$, the problem* (3.1.1), (3.1.2) *has a unique strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ of the class W_2^1 in the domain D_T in the sense of Definition* 3.2.1*.*

The following theorem on the existence of a global solution of this problem follows from Theorem 3.4.2.

Theorem 3.4.3. *Let the vector function f satisfy the condition* (3.4.1)*, where $0 \le \gamma < \frac{2}{n-1}$, and either f satisfy the condition* (3.2.2) *for $\alpha \le 1$ or $f = \nabla G$, where $G \in C^1(\mathbb{R}^N)$, $G(0) = 0$ and $G(u) \ge 0$ $\forall\, u \in \mathbb{R}^N$. Let $g = 0$, $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for each $T > 0$. Then the problem* (3.1.1), (3.1.2) *has a unique global strong generalized solution $u \in W_{2,loc}^1(D_\infty)$ of the class W_2^1 in the domain D_∞ in the sense of Definition* 3.2.4*.*

Proof. According to Theorem 3.4.2, when the conditions of Theorem 3.4.3 are fulfilled, for $T = k$, where k is a natural number, there exists a unique strong generalized solution $u^k \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ of the problem (3.1.1), (3.1.2) of the class W_2^1 in the domain $D_{T=k}$ in the sense of Definition 3.2.1. Since $u^{k+1}|_{D_{T=k}}$ is also a strong generalized solution of the problem (3.1.1), (3.1.2) of the class W_2^1 in the domain $D_{T=k}$, in view of Theorem 3.4.2 we have $u^k = u^{k+1}|_{D_{T=k}}$. Therefore, one can construct a unique generalized solution $u \in \overset{\circ}{W}{}_{2,loc}^1(D_\infty)$ of the problem (3.1.1), (3.1.2) of the class W_2^1 in the domain D_∞ in the sense of Definition 3.2.4 as follows:

$$u(x,t) = u^k(x,t), \quad (x,t) \in D_\infty, \quad k = [t] + 1,$$

where $[t]$ is an integer part of the number t. Thus Theorem 3.4.3 is proved completely. □

3.5 The cases of the nonexistence of a global solution of the problem (3.1.1), (3.1.2) of the class W_2^1

Theorem 3.5.1. *Let the vector function $f = (f_1, \dots, f_N)$ satisfy the condition* (3.2.2)*, where $1 < \alpha < \frac{n+1}{n-1}$, and there exist the numbers $\ell_1, \dots, \ell_N$, $\sum\limits_{i=1}^{N} |\ell_i| \ne 0$, such that*

$$\sum_{i=1}^{N} \ell_i f(u) \le c_0 - c_1 \Big| \sum_{i=1}^{N} \ell_i u_i \Big|^{\beta} \quad \forall\, u \in \mathbb{R}^N, \ \ 1 < \beta = const < \frac{n+1}{n-1}, \tag{3.5.1}$$

where $c_0, c_1 = const$, $c_1 > 0$. Let $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for any $T > 0$, $g = 0$. Let the scalar function $F_0 = \sum\limits_{i=1}^{N} \ell_i F_i - c_0$ in the domain D_∞ satisfy the following conditions:

$$F_0 \ge 0, \quad \lim_{t\to+\infty} \inf t^{\gamma} F_0(x,t) \ge c_2 = const > 0, \ \ \gamma = const \le n + 1. \tag{3.5.2}$$

Then there exists a finite positive number $T_0 = T_0(F)$ such that for $T > T_0$ the problem (3.1.1), (3.1.2) *does not have a strong generalized solution of the class W_2^1 in the sense of Definition* 3.2.1*.*

Proof. Let $u = (u_1, \dots, u_N)$ be a strong generalized solution of the problem (3.1.1), (3.1.2) of the class W_2^1 in the domain D_T in the sense of Definition 3.2.1. It is easy to verify that

$$\int\limits_{D_T} u \,\square\, \varphi \, dx\, dt = -\int\limits_{D_T} f(u)\varphi \, dx\, dt + \int\limits_{D_T} F\varphi \, dx\, dt \tag{3.5.3}$$

for any test vector function $\varphi = (\varphi_1, \dots, \varphi_N)$ such that

$$\varphi \in C^2(\overline{D}_T), \quad \varphi\big|_{\partial D_T} = \frac{\partial \varphi}{\partial \nu}\Big|_{\partial D_T} = 0, \tag{3.5.4}$$

where ν is the unit vector of the outer normal to ∂D_T. Indeed, according to the definition of the strong generalized solution of the problem (3.1.1), (3.1.2) of the class W_2^1 in the domain D_T, there exists a sequence of vector functions $u^m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ for which the limit equalities (3.3.47) are valid. Taking into account (3.3.48) and multiplying scalarly both parts of the equality $Lu^m = F^m$ by the test vector function $\varphi = (\varphi_1, \dots, \varphi_N)$, due to (3.5.4), after integrating by parts, we obtain

$$\int\limits_{D_T} u^m \,\square\, \varphi \, dx\, dt = -\int\limits_{D} f(u^m)\varphi \, dx\, dt + \int\limits_{D_T} F^m\varphi \, dx\, dt. \tag{3.5.5}$$

By virtue of (3.3.47) and Remark 3.2.1, passing in the equality (3.5.5) to the limit as $m \to \infty$, we get (3.5.3).

Let us apply the method of test functions [77]. Consider a scalar function $\varphi^0 = \varphi^0(x,t)$ such that

$$\varphi^0 \in C^2(\overline{D}_\infty), \ \ \varphi^0\big|_{D_{T=1}} > 0, \ \ \varphi^0\big|_{t \geq 1} = 0, \ \ \varphi^0\big|_{\partial D_{T=1}} = \frac{\partial \varphi^0}{\partial \nu}\Big|_{\partial D_{T=1}} = 0 \tag{3.5.6}$$

and

$$\varkappa_0 := \int\limits_{D_{T=1}} \frac{|\square\, \varphi^0|^{\beta'}}{|\varphi^0|^{\beta'-1}} \, dx\, dt < +\infty, \ \ \frac{1}{\beta} + \frac{1}{\beta'} = 1. \tag{3.5.7}$$

It is not difficult to see that in the capacity of the function φ^0, satisfying the conditions (3.5.6) and (3.5.7), we can choose the function

$$\varphi^0(x,t) = \begin{cases} \omega^m\Big(\dfrac{x}{t}\Big)(1-t)^m t^k, & (x,t) \in D_{T=1}, \\ 0, & t \geq 1, \end{cases}$$

for sufficiently large positive m and k, where the function $\omega \in C^\infty(\mathbb{R}^n)$ defines the equation of conic section $\partial\Omega_1 = S \cap \{t = 1\} : \ \omega(x) = 0$, $\nabla\, \omega|_{\partial\Omega_1} \neq 0$, and $\omega|_{\Omega_1} > 0$, $\Omega_1 : \ D \cap \{t = 1\}$.

Putting

$$\varphi_T(x,t) := \varphi^0\Big(\frac{x}{T}, \frac{t}{T}\Big), \ \ T > 0, \tag{3.5.8}$$

due to (3.5.6), it is easy to see that

$$\varphi_T \in C^2(\overline{D}_T), \quad \varphi_T\big|_{D_T} > 0, \quad \varphi_T\big|_{\partial D_T} = \frac{\partial \varphi_T}{\partial \nu}\Big|_{\partial D_T} = 0. \tag{3.5.9}$$

In the integral equality (3.5.3), for the test vector function φ we choose $\varphi = (\ell_1\varphi_T, \ell_2\varphi_T, \dots, \ell_N\varphi_T)$. For the chosen test vector function φ, using the notation

$$v = \sum_{i=1}^N \ell_i u_i, \quad F_* = \sum_{i=1}^N \ell_i F_i, \quad f_0 = \sum_{i=1}^N \ell_i f_i, \tag{3.5.10}$$

the integral equality (3.5.3) takes the form

$$\int\limits_{D_T} v \,\square\, \varphi_T \, dx\, dt = -\int\limits_{D_T} f_0(u)\varphi_T \, dx\, dt + \int\limits_{D_T} F_*\varphi_T \, dx\, dt. \tag{3.5.11}$$

From (3.5.1), (3.5.9) and (3.5.11), it follows that

$$\int\limits_{D_T} v\,\Box\,\varphi_T\,dx\,dt \geq \int\limits_{D_T} [c_1|v|^\beta - c_0]\varphi_T\,dx\,dt + \int\limits_{D_T} F_*\varphi_T\,dx\,dt = c_1\int\limits_{D_T} |v|^\beta\varphi_T\,dx\,dt + \chi(T), \tag{3.5.12}$$

where

$$\chi(T) = \int\limits_{D_T} (F_* - c_0)\varphi_T\,dx\,dt = \int\limits_{D_T} F_0\varphi_T\,dx\,dt \geq 0, \tag{3.5.13}$$

due to (3.5.2) and (3.5.9).

In view of (3.5.2), there exists a number $T_1 = T_1(F) > 0$ such that

$$F_0(x,t) \geq \frac{c_2}{2}\,t^{-\gamma},\ \ t > T_1. \tag{3.5.14}$$

By virtue of (3.5.8) and (3.5.14), after the substitution of variables $t = Tt'$, $x = Tx'$ in the integral (3.5.13), for $T > 2T_1$ we have

$$\begin{aligned}
\chi(T) &= T^{n+1}\int\limits_{D_{T=1}} F_0(Tx',Tt')\varphi^0(x',t')\,dx'\,dt' \\
&\geq T^{n+1}\int\limits_{D_{T=1}\cap\{\frac{1}{2}<t'<1\}} F_0(Tx',Tt')\varphi^0(x',t')\,dx'\,dt' \\
&\geq T^{n+1}\int\limits_{D_{T=1}\cap\{\frac{1}{2}<t'<1\}} \frac{c_2}{2}\,(Tt')^{-\gamma}\varphi^0(x',t')\,dx'\,dt' \\
&= \frac{c_2}{2}\,T^{n+1-\gamma}\int\limits_{D_{T=1}\cap\{\frac{1}{2}<t'<1\}} (t')^{-\gamma}\varphi^0(x',t')\,dx'\,dt' \\
&= c_3T^{n+1-\gamma},\ \ T > 2T_1,
\end{aligned} \tag{3.5.15}$$

where, due to $\varphi^0|_{D_{T=1}} > 0$,

$$c_3 = \frac{c_2}{2}\int\limits_{D_{T=1}\cap\{\frac{1}{2}<t'<1\}} (t')^{-\gamma}\varphi^0(x',t')\,dx'\,dt'\,dx'\,dt' = const > 0. \tag{3.5.16}$$

Since according to the conditions of Theorem 3.5.1, the constant $\gamma \leq n+1$, it follows from (3.5.15) and (3.5.16) that

$$\lim_{T\to+\infty}\inf\chi(T) \geq c_3. \tag{3.5.17}$$

Further, in view of (3.5.13), the inequality (3.5.12) can be rewritten in the form

$$c_1\int\limits_{D_T} |v|^\beta\varphi_T\,dx\,dt \leq \int\limits_{D_T} v\,\Box\,\varphi_T\,dx\,dt - \chi(T). \tag{3.5.18}$$

If in Young's inequality with the parameter $\varepsilon > 0$: $ab \leq (\varepsilon/\beta)a^\beta + (\beta'\varepsilon^{\beta'-1})^{-1}b^\beta$, where $\beta' = \beta/(\beta-1)$, we take $a = |u|\varphi_T^{1/\beta}$, $b = |\Box\,\varphi_T|/\varphi_T^{1/\beta}$, then taking into account the equality $\beta'/\beta = \beta'-1$, we obtain

$$|v\,\varphi_T| = |v|\varphi_T^{1/\beta}\,\frac{|\Box\,\varphi_T|}{\varphi_T^{1/\beta}} \leq \frac{\varepsilon}{\beta}\,|v|^\beta\varphi_T + \frac{1}{\beta'\varepsilon^{\beta'-1}}\,\frac{|\Box\,\varphi_T|^{\beta'}}{\varphi_T^{\beta'-1}}\,. \tag{3.5.19}$$

In view of (3.5.19), from (3.5.18) we get

$$\Big(c_1 - \frac{\varepsilon}{\beta}\Big)\int\limits_{D_T} |v|^\beta\varphi_T\,dx\,dt \leq \frac{1}{\beta'\varepsilon^{\beta'-1}}\int\limits_{D_T} \frac{|\Box\,\varphi_T|^{\beta'}}{\varphi_T^{\beta'-1}}\,dx\,dt - \chi(T),$$

whence for $\varepsilon < c_1\beta$, we obtain

$$\int\limits_{D_T} |v|^\beta \varphi_T \, dx\, dt \leq \frac{\beta}{(c_1\beta-\varepsilon)\beta'\varepsilon^{\beta'-1}} \int\limits_{D_T} \frac{|\Box\,\varphi_T|^{\beta'}}{\varphi_T^{\beta'-1}} \, dx\, dt - \frac{\beta}{c_1\beta-\varepsilon}\,\chi(T). \tag{3.5.20}$$

Taking into account the equalities $\beta' = \frac{\beta}{\beta-1}$, $\beta' = \frac{\beta'}{\beta'-1}$ and also the equality

$$\min_{0<\varepsilon<c_1\beta} \frac{\beta}{(c_1\beta-\varepsilon)\beta'\varepsilon^{\beta'-1}} = \frac{1}{c_1^{\beta'}}\,,$$

which is achieved for $\varepsilon = c_1$, it follows from (3.5.20) that

$$\int\limits_{D_T} |v|^\beta \varphi_T \, dx\, dt \leq \frac{1}{c_1^{\beta'}} \int\limits_{D_T} \frac{|\Box\,\varphi_T|^{\beta'}}{\varphi_T^{\beta'-1}} \, dx\, dt - \frac{\beta'}{c_1}\,\chi(T). \tag{3.5.21}$$

By virtue of (3.5.6)–(3.5.8), after the substitution of variables $x = Tx'$, $t = Tt'$, it can be easily verified that

$$\int\limits_{D_T} \frac{|\Box\,\varphi_T|^{\beta'}}{\varphi_T^{\beta'-1}} \, dx\, dt = T^{n+1-2\beta'} \int\limits_{D_{T=1}} \frac{|\Box\,\varphi^0|^{\beta'}}{(\varphi^0)^{\beta'-1}} \, dx'\, dt' = T^{n+1-2\beta'}\varkappa_0 < +\infty,$$

whence, due to (3.5.9), from the equality (3.5.21) we obtain

$$0 \leq \int\limits_{D_T} |v|^\beta \varphi_T \, dx\, dt \leq \frac{1}{c_1^{\beta'}}\, T^{n+1-2\beta'}\varkappa_0 - \frac{\beta'}{c_1}\,\chi(T). \tag{3.5.22}$$

Since, by supposition, $\beta < \frac{n+1}{n-1}$, we have $n+1-2\beta' < 0$ and hence

$$\lim_{T\to+\infty} \frac{1}{c_1^{\beta'}}\, T^{n+1-2\beta'}\varkappa_0 = 0. \tag{3.5.23}$$

From (3.5.16), (3.5.17) and (3.5.23) it follows that there exists a positive number $T_0 = T_0(F)$ such that for $T > T_0$, the right-hand side of the inequality (3.5.22) will be a negative value, which is impossible. This implies that if for the conditions of Theorem 3.5.1 there exists a strong generalized solution of the problem (3.5.1), (3.5.2) of the class W_2^1 in the domain D_T, then $T \leq T_0$ necessarily, which proves Theorem 3.5.1. □

Remark 3.5.1. As is shown in the first chapter, the following class of vector functions $f = (f_1, \dots, f_N)$:

$$f_i(u_1,\dots,u_N) = \sum_{j=1}^N a_{ij}|u_j|^{\beta_{ij}} + b_i, \quad i = 1,\dots,N, \tag{3.5.24}$$

where $a_{ij} = const > 0$, $b_i = const$, $1 < \beta_{ij} = const < \frac{n+1}{n-1}$; $i,j = 1,\dots,N$, satisfies the condition (3.5.1). Note that the vector function f represented by the equalities (3.5.24), satisfies likewise the condition (3.5.1) for $\ell_1 = \ell_2 = \cdots = \ell_N = -1$ for less restrictive conditions, when $a_{ij} = const \geq 0$, but $a_{ik_i} > 0$, where $k_1,\dots,k_N$ is any arbitrary fixed permutation of numbers $1,2,\dots,N$; $i,j = 1,\dots,N$.

When $N = n = 2$, $f_1 = a_{11}|u_1|^\gamma + a_{12}|u_2|^\beta$, $f_2 = a_{21}|u_1|^\gamma + a_{22}|u_2|^\beta$, $1 < \gamma, \beta < 3$, the restrictions $a_{ij} > 0$ can be omitted and replaced by the condition $\det(a_{ij}) \neq 0$. For example, for $f_1 = u_1^2 - 2u_2^2$, $f_2 = -2u_1^2 + u_2^2$, the condition (3.5.1) for $\ell_1 = \ell_2 = 1$, $\beta = 2$, $c_0 = 0$ and $c_1 = \frac{1}{2}$ will be valid, since in this case, $\ell_1 f_1(u) + \ell_2 f_2(u) = -(|u_1|^2 + |u_2|^2) \leq -\frac{1}{2}\,|u_1+u_2|^2$, and from Theorem 3.5.1 we find that for $F_1 + F_2 \geq \frac{c}{t^\gamma}$, where $c = const > 0$ and $\gamma = const \leq 3$, $g = 0$, the boundary value problem under consideration is not globally solvable. More precisely, from (3.5.17) and (3.5.22) it follows that

$$0 \leq \int\limits_{D_T} |v|^\beta \varphi_T \, dx\, dt \leq \frac{1}{c_1^{\beta'}}\, T^{n+1-2\beta'}\varkappa_0 - \frac{\beta'}{c_1}\, c_3,$$

the right-hand side of which becomes negative for $T > T_0 = \max([\varkappa_0^{-1}\beta' c_1^{\beta'-1} c_3]^{\frac{1}{n+1-2\beta'}}, 1)$ and, therefore, for $T > T_0$, the problem $(3.1.1), (3.1.2)$ does not have a solution. But for this concrete example, $n = 2$, $\beta = \beta' = 2$; $\varkappa_0$ is determined from (3.5.7). The constants c_1, c_2 and c_3 are determined from $(3.5.1), (3.5.2)$ and (3.5.16), respectively, and therefore, in this case $c_1 = \frac{1}{2}$ and $T_0 = \frac{\varkappa_0}{c_3}$. Further, due to Theorem 3.3.2 on the local solvability and Theorem 3.4.1 on the uniqueness of the solution of the problem, there exist a finite positive number $T_* = T_*(F)$ and a unique vector function $u = (u_1, u_2) \in W^1_{2,loc}(D_{T_*})$ such that u is a strong generalized solution of this problem of the class W^1_2 in the domain D_T for $T < T_*$. From the aforesaid it follows that for the life-span T_* of this solution we have the upper estimate $T_* \leq T_0 = \max(\frac{\varkappa_0}{c_3}, 1)$. The lower estimate for T_* can be obtained from the reasonings given in the proof of Theorem 3.3.2 on the local solvability.

Remark 3.5.2. From Theorem 3.5.1 it follows that when its conditions are fulfilled, the problem $(3.1.1), (3.1.2)$ fails to have a global strong generalized solution of the class W^1_2 in the domain D_∞ in the sense of Definition 3.2.4.

Chapter 4

Multidimensional problem with one nonlinear in time condition for some semilinear hyperbolic equations with the Dirichlet boundary condition

4.1 Statement of the problem

In the space $\mathbb{R}^{n+1}$ of variables $x = (x_1, \dots, x_n)$ and t, in the cylindrical domain $D_T = \Omega \times (0, T)$, where Ω is a Lipschitz domain in $\mathbb{R}^n$, consider a nonlocal problem of finding a solution $u(x,t)$ of the equation

$$L_\lambda u := \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \lambda f(x,t,u) = F(x,t), \quad (x,t) \in D_T, \tag{4.1.1}$$

satisfying the Dirichlet homogeneous boundary condition on a part of the boundary $\Gamma : \partial\Omega \times (0,T)$ of the cylinder D_T

$$u\big|_\Gamma = 0, \tag{4.1.2}$$

the initial condition

$$u(x,0) = \varphi(x), \quad x \in \Omega, \tag{4.1.3}$$

and the nonlocal condition

$$\mathcal{K}_\mu u_t : \ u_t(x,0) - \mu u_t(x,T) = \psi(x), \quad x \in \Omega, \tag{4.1.4}$$

where f, F, φ and ψ are the given functions; λ and μ are the given nonzero constants, and $n \geq 2$.

A great number of works have been devoted to the study of nonlocal problems for partial differential equations. When a nonlocal problem is posed for abstract evolution equations and hyperbolic partial differential equations, we suggest the reader to refer to the works [1–8, 10, 11, 13, 14, 26–29, 34, 37, 38, 53, 60, 61, 63–65, 74, 78, 82, 85, 95] and to the references therein.

In this chapter, the problem (4.1.1)–(4.1.4) in the multidimensional case is studied in the Sobolev space $W_2^1(D_T)$, basing on the expansions of functions from the space $\overset{\circ}{W}{}_2^1(\Omega)$ in the basis, consisting of eigenfunctions of the spectral problem $\Delta w = \widetilde{\lambda} w$, $w|_{\partial\Omega} = 0$, and using the embedding theorems in the Sobolev spaces. It should also be noted that if for $n = 1$ there is no need in any restriction on the behavior of the function $f(x,t,u)$ with respect to the variable u, as $u \to \infty$, whereas in the case for $n > 1$, we require of the function $f(x,t,u)$, as $u \to \infty$, to have a growth not exceeding a polynomial.

Moreover, for using the embedding theorems in the Sobolev spaces, it is additionally required for the order of polynomial growth to be less than a certain value that depends on the dimension of the space.

Below, on the function $f = f(x,t,u)$ we impose the following requirements:

$$f \in C(\overline{D}_T \times \mathbb{R}), \quad |f(x,t,u)| \leq M_1 + M_2|u|^\alpha, \ \ (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{4.1.5}$$

where

$$0 \leq \alpha = const < \frac{n+1}{n-1}\,. \tag{4.1.6}$$

Remark 4.1.1. The embedding operator $I : W_2^1(D_T) \to L_1(D_T)$ is a linear continuous operator for $1 < q < \frac{2(n+1)}{n-1}$, when $n > 1$ [68]. At the same time, Nemitski's operator $\mathcal{N} : L_q(D_T) \to L_2(D_T)$, acting by the formula $\mathcal{N}u = f(x,t,u)$, is, due to (4.1.5), continuous and bounded if $q \geq 2\alpha$ [22]. Thus, since due to (4.1.6) we have $2\alpha < \frac{2(n+1)}{n-1}$, there exists a number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \geq 2\alpha$. Therefore, in this case the operator

$$\mathcal{N}_0 = \mathcal{N}I : \overset{\circ}{W}{}_2^1(D_T,\Gamma) \to L_2(D_T), \tag{4.1.7}$$

where $\overset{\circ}{W}{}_2^1(D_T,\Gamma) := \{w \in W_2^1(D_T) : \ w|_\Gamma = 0\}$, is continuous and compact. Besides, it follows from $u \in \overset{\circ}{W}{}_2^1(D_T,\Gamma)$ that $f(x,t,u) \in L_2(D_T)$, and if $u_m \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T,\Gamma)$, then $f(x,t,u_m) \to f(x,t,u)$ in the space $L_2(D_T)$.

Definition 4.1.1. Let the function f satisfy the conditions (4.1.5) and (4.1.6), $F \in L_2(D_T)$, $\varphi \in \overset{\circ}{W}{}_2^1(\Omega) := \{v \in W_2^1(\Omega) : \ v|_{\partial\Omega} = 0\}$, $\psi \in L_2(\Omega)$. We call a function u a generalized solution of the problem (4.1.1)–(4.1.4) if $u \in \overset{\circ}{W}{}_2^1(D_T,\Gamma)$ and there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2(\overline{D}_T,\Gamma) := \{w \in C^2(\overline{D}_T) : \ w|_\Gamma = 0\}$ such that

$$\lim_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} = 0, \quad \lim_{m\to\infty} \|L_\lambda u_m - F\|_{L_2(D_T)} = 0, \tag{4.1.8}$$

$$\lim_{m\to\infty} \big\|u_m\big|_{t=0} - \varphi\big\|_{\overset{\circ}{W}{}_2^1(\Omega)} = 0, \quad \lim_{m\to\infty} \|\mathcal{K}_\mu u_m - \psi\|_{L_2(\Omega)} = 0. \tag{4.1.9}$$

Obviously, a classical solution $u \in C^2(\overline{D}_T)$ of the problem (4.1.1)–(4.1.4) is a generalized solution of this problem. It is easy to verify that a generalized solution of the problem (4.1.1)–(4.1.4) is a solution of the equation (4.1.1) in the sense of the theory of distributions. Indeed, let $F_m := L_\lambda u_m$, $\varphi_m := u_m|_{t=0}$, $\psi_m := \mathcal{K}_\mu u_{mt}$. Multiplying both sides of the equality $L_\lambda u_m = F_m$ by a test function $w \in V := \{v \in \overset{\circ}{W}{}_2^1(D_T,\Gamma) : \ v(x,T) - \mu v(x,0) = 0, \ x \in \Omega\}$ and integrating in the domain D_T, after simple transformations connected with integration by parts and the equality $w|_\Gamma = 0$, we get

$$\int\limits_\Omega \big[u_{mt}(x,T)w(x,T) - u_{mt}(x,0)w(x,0)\big]\,dx \\ + \int\limits_{D_T} \Big[-u_{mt}w_t + \sum_{i=1}^n u_{mx_i}w_{x_i} + \lambda f(x,t,u_m)w\Big]\,dx\,dt = \int\limits_{D_T} F_m w\,dx\,dt \ \ \forall\, w \in V. \tag{4.1.10}$$

Due to $\mathcal{K}_\mu u_{mt} = \psi_m(x)$ and $w(x,T) - \mu w(x,0) = 0$, $x \in \Omega$, it can be easily seen that $u_{mt}(x,T)w(x,T) - u_{mt}(x,0)w(x,0) = u_{mt}(x,T)(w(x,T) - \mu w(x,0)) - \psi_m(x)w(x,0) = -\psi_m(x)w(x,0)$, $x \in \Omega$. Therefore, the equality (4.1.10) takes the form

$$-\int\limits_\Omega \psi_m(x)w(x,0)\,dx \\ + \int\limits_\Omega \Big[-u_{mt}w_t + \sum_{i=1}^n u_{mx_i}w_{x_i} + \lambda f(x,t,u_m)w\Big]\,dx\,dt = \int\limits_{D_T} F_m w\,dx\,dt \ \ \forall\, w \in V. \tag{4.1.11}$$

In view of (4.1.5), (4.1.6), according to Remark 4.1.1, we have $f(x,t,u_m)\to f(x,t,u)$ in the space $L_2(D_T)$ as $u_m\to u$ in the space $\overset{\circ}{W}{}_2^1(D_T,\Gamma)$. Therefore, due to (4.1.8) and (4.1.9), passing in the equality (4.1.11) to the limit as $m\to\infty$, we get

$$-\int\limits_{\Omega}\psi(x)w(x,0)\,dx+\int\limits_{D_T}\Big[-u_tw_t+\sum_{i=1}^{n}u_{x_i}w_{x_i}+\lambda f(x,t,u)w\Big]\,dx\,dt=\int\limits_{D_T}Fw\,dx\,dt\ \ \forall\,w\in V. \quad (4.1.12)$$

Since $C_0^\infty(D_T)\subset V$, from (4.1.12), integrating by parts, we have

$$\int\limits_{D_T}\big[u\,\Box\,w+\lambda f(x,t,u)w\big]\,dx\,dt=\int\limits_{D_T}Fw\,dx\,dt\ \ \forall\,w\in C_0^\infty(D_T), \quad (4.1.13)$$

where $\Box:=\frac{\partial^2}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$, and $C_0^\infty(D_T)$ is a space of finite infinitely differentiable functions on D_T. The equality (4.1.13), which is valid for any $w\in C_0^\infty(D_T)$, implies that a generalized solution u of the problem (4.1.1)–(4.1.4) is a solution of the equation (4.1.1) in the sense of the theory of distributions, besides, since the trace operator $u\to u|_{t=0}$ is well defined in the space $\overset{\circ}{W}{}_2^1(D_T,\Gamma)$ and, particularly, is continuous from the space $\overset{\circ}{W}{}_2^1(D_T,\Gamma)$ into the space $L_2(\Omega\times\{t=0\})$, we find, due to (4.1.8) and (4.1.9), that the initial condition (4.1.3) is fulfilled in the sense of the trace theory, while the nonlocal condition (4.1.4) in the integral sense is taken into account in the equality (4.1.12), which is valid for all $w\in V$. Note also that if a generalized solution u belongs to the class $C^2(\overline{D}_T)$, then due to the standard reasoning connected with the integral equality (4.1.12), which is valid for any $w\in V$ [68], we find that u is a classical solution of the problem (4.1.1)–(4.1.4), satisfying the equation (4.1.1), the boundary condition (4.1.2), the initial condition (4.1.3) and the nonlinear condition (4.1.4) pointwise.

Note that even in the linear case, i.e., for $\lambda=0$, the problem (4.1.1)–(4.1.4) is not always well-posed. For example, when $\lambda=0$ and $|\mu|=1$, the corresponding to (4.1.1)–(4.1.4) homogeneous problem may have an infinite number of linearly independent solutions (see Remark 4.3.2).

4.2 An a priori estimate of a solution of the problem (4.1.1)–(4.1.4)

Let

$$g(x,t,u)=\int\limits_0^u f(x,t,s)\,ds,\ \ (x,t,u)\in\overline{D}_T\times\mathbb{R}. \quad (4.2.1)$$

Consider the following conditions imposed on the function $g=g(x,t,u)$:

$$g(x,t,u)\geq -M_3,\ \ (x,t,u)\in\overline{D}_T\times\mathbb{R}, \quad (4.2.2)$$

$$g_t\in C(\overline{D}_T\times\mathbb{R},\ \ g_t(x,t,u)\in M_4,\ \ (x,t,u)\in\overline{D}_T\times\mathbb{R}, \quad (4.2.3)$$

where $M_i=const\geq 0$, $i=3,4$.

Let us consider some classes of frequently encountered in applications functions $f=f(x,t,u)$ satisfying the conditions (4.1.5), (4.2.2) and (4.2.3):

1. $f(x,t,u)=f_0(x,t)\beta(u)$, where $f_0,\frac{\partial}{\partial t}f_0\in C(\overline{D}_T)$ and $\beta\in C(\mathbb{R})$, $|\beta(u)|\leq\widetilde{M}_1+\widetilde{M}_2|u|^\alpha$, $\widetilde{M}_i=const\geq 0$, $\alpha=const\geq 0$. In this case, $g(x,t,u)=f_0(x,t)\int\limits_0^u\beta(s)\,ds$ and when $f_0\geq 0$, $\frac{\partial}{\partial t}f_0\leq 0$, $\int\limits_0^u\beta(s)\,ds\geq -M$, $M=const\geq 0$, the conditions (4.1.5), (4.2.2) and (4.2.3) are fulfilled.

2. $f(x,t,u)=f_0(x,t)|u|^\alpha\operatorname{sign}u$, where $f_0,\frac{\partial}{\partial t}f_0\in C(\overline{D}_T)$ and $\alpha>1$. In this case, $g(x,t,u)=f_0(x,t)\,\frac{|u|^\alpha}{\alpha+1}$, and when $f_0\geq 0$, $\frac{\partial}{\partial t}f_0\leq 0$, the conditions (4.1.5), (4.2.2) and (4.2.3) are also fulfilled.

Lemma 4.2.1. *Let $\lambda > 0$, $|\mu| < 1$, $F \in L_2(D_T)$, $\varphi \in \overset{\circ}{W}{}_2^1(\Omega)$, $\psi \in L_2(\Omega)$ and the conditions* (4.1.5), (4.2.2) *and* (4.2.3) *be fulfilled. Then for a generalized solution u of the problem* (4.1.1)–(4.1.4) *the following a priori estimate*

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} \leq c_1\|F\|_{L_2(D_T)} + c_2\|\varphi\|_{\overset{\circ}{W}{}_2^1(\Omega)} + c_3\|\psi\|_{L_2(\Omega)} + c_4\|\varphi\|^{\frac{\alpha+1}{2}}_{\overset{\circ}{W}{}_2^1(\Omega)} + c_5 \tag{4.2.4}$$

is valid with nonnegative constants $c_i = c_i(\lambda, \mu, \Omega, T, M_1, M_2, M_3, M_4)$, not depending on u, F, φ, ψ, and $c_i > 0$ for $i < 4$, whereas in the linear case, i.e., when $\lambda = 0$, the constants $c_4 = c_5 = 0$, and in this case, due to (4.2.4)*, we have the uniqueness of the solution of the problem* (4.1.1)–(4.1.4).

Proof. Let u be a generalized solution of the problem (4.1.1)–(4.1.4). In view of Definition 4.1.1, there exists a sequence of the functions $u_m \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma)$ such that the limit equalities (4.1.8), (4.1.9) are fulfilled.

Set

$$L_\lambda u_m = F_m, \ \ (x,t) \in D_T, \tag{4.2.5}$$

$$u_m\big|_\Gamma = 0, \tag{4.2.6}$$

$$u_m(x,0) = \varphi_m(x), \ \ x \in \Omega, \tag{4.2.7}$$

$$\mathcal{K}_\mu u_{mt} = \psi_m(x), \ \ x \in \Omega. \tag{4.2.8}$$

Multiplying both sides of the equation (4.2.5) by $2u_{mt}$ and integrating in the domain $D_\tau := D_T \cap \{t < \tau\}$, $0 < \tau \leq T$, due to (4.2.1), we obtain

$$\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u_m}{\partial t}\Big)^2\,dx\,dt - 2\int\limits_{D_\tau}\sum_{i=1}^n \frac{\partial^2 u_m}{\partial x_i^2}\,\frac{\partial u_m}{\partial t}\,dx\,dt + 2\lambda\int\limits_{D_\tau}\frac{\partial}{\partial t}\,g(x,t,u_m(x,t))\,dx\,dt$$
$$- 2\lambda\int\limits_{D_\tau} g_t(x,t,u_m(x,t))\,dx\,dt = 2\int\limits_{D_\tau} F_m\,\frac{\partial u_m}{\partial t}\,dx\,dt. \tag{4.2.9}$$

Let $\omega_\tau := \{(x,t) \in \overline{D}_T : \ x \in \Omega, \ t = \tau\}$, $0 < \tau \leq T$. Denote by $\nu := (\nu_{x_1}, \dots, \nu_{x_n}, \nu_t)$ the unit vector of the outer normal to ∂D_τ. Since $\nu_{x_i}|_{\omega_\tau\cup\omega_0} = 0$, $i = 1, \dots, n$, $\nu_t|_{\Gamma_\tau = \Gamma\cap\{t\leq\tau\}} = 0$, $\nu_t|_{\omega_\tau} = 1$, $\nu_t|_{\omega_0} = -1$, taking into account the equalities (4.2.6) and integrating by parts, we have

$$\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_m}{\partial t}\Big)^2\,dx\,dt = \int\limits_{\partial D_\tau}\Big(\frac{\partial u_m}{\partial t}\Big)^2\nu_t\,ds = \int\limits_{\omega_\tau} u_{mt}^2\,dx - \int\limits_{\omega_0} u_{mt}^2\,dx, \tag{4.2.10}$$

$$-2\int\limits_{D_\tau}\frac{\partial^2 u_m}{\partial x_i^2}\,\frac{\partial u_m}{\partial t}\,dx\,dt = \int\limits_{D_\tau}\big[(u_{mx_i}^2)_t - 2(u_{mx_i}u_{mt})_{x_i}\big]\,dx\,dt$$
$$= \int\limits_{\omega_\tau} u_{mx_i}^2\,dx - \int\limits_{\omega_0} u_{mx_i}^2\,dx, \ \ i = 1, \dots, n, \tag{4.2.11}$$

$$2\lambda\int\limits_{D_\tau}\frac{\partial}{\partial t}\,g(x,t,u_m(x,t))\,dx\,dt = 2\lambda\int\limits_{\partial D_\tau} g(x,t,u_m(x,t))\nu_t\,ds$$
$$= 2\lambda\int\limits_{\omega_\tau} g(x,t,u_m(x,t))\,dx - 2\lambda\int\limits_{\omega_0} g(x,t,u_m(x,t))\,dx. \tag{4.2.12}$$

In view of (4.2.10), (4.2.11) and (4.2.12), from (4.2.9) we get

$$\int\limits_{\omega_\tau}\Big[u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2\Big]\,dx = \int\limits_{\omega_0}\Big[u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2\Big]\,dx - 2\lambda\int\limits_{\omega_\tau} g(x,t,u_m(x,t))\,dx$$
$$+ 2\lambda\int\limits_{\omega_0} g(x,t,u_m(x,t))\,dx + 2\lambda\int\limits_{D_\tau} g_t(x,t,u_m(x,t))\,dx\,dt + 2\int\limits_{D_\tau} F_m u_{mt}\,dx\,dt. \tag{4.2.13}$$

Let

$$w_m(\tau) := \int\limits_{\omega_\tau} \Big[u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2\Big]\,dx. \tag{4.2.14}$$

Since $2F_m u_{mt} \le \varepsilon^{-1}F_m^2 + \varepsilon u_{mt}^2$ for any $\varepsilon = const > 0$, due to (4.2.2), (4.2.3) and (4.2.14), it follows from (4.2.13) that

$$\begin{aligned} w_m(\tau) &\le w_m(0) + 2\lambda M_3 \operatorname{mes}\Omega \\ &\quad + 2\lambda \int\limits_{\omega_0} |g(x,t,u_m(x,t))|\,dx + 2\lambda M_4\tau \operatorname{mes}\Omega + \varepsilon \int\limits_{D_T} u_{mt}^2\,dx\,dt + \varepsilon^{-1}\int\limits_{D_T} F_m^2\,dx\,dt. \end{aligned} \tag{4.2.15}$$

Taking into account that

$$\int\limits_{D_\tau} u_{mt}^2\,dx\,dt = \int\limits_0^\tau \Big[\int\limits_{\omega_s} u_{mt}^2\,dx\Big]\,ds \le \int\limits_0^\tau \Big[\int\limits_{\omega_s} \Big[u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2\Big]\,dx\Big]\,ds = \int\limits_0^\tau w_m(s)\,ds,$$

from (4.2.15) we obtain

$$\begin{aligned} w_m(\tau) &\le \varepsilon \int\limits_0^\tau w_m(s)\,ds + w_m(0) + 2\lambda(M_3 + M_4\tau)\operatorname{mes}\Omega \\ &\quad + 2\lambda \int\limits_{\omega_0} |g(x,t,u_m(x,t))|\,dx + \varepsilon^{-1}\int\limits_{D_\tau} F_m^2\,dx\,dt, \quad 0 < \tau \le T. \end{aligned} \tag{4.2.16}$$

Because of $D_\tau \subset D_T$, $0 < \tau \le T$, according to the Gronwall lemma, it follows from (4.2.16) that

$$\begin{aligned} w_m(\tau) \le \Big[&w_m(0) + \lambda(M_3 + M_4 T)\operatorname{mes}\Omega \\ &+ 2\lambda \int\limits_{\omega_0} |g(x,t,u_m(x,t))|\,dx + \varepsilon^{-1}\int\limits_{D_T} F_m^2\,dx\,dt\Big]e^{\varepsilon\tau}, \quad 0 < \tau \le T. \end{aligned} \tag{4.2.17}$$

Using the obvious inequality

$$|a+b|^2 = a^2 + b^2 + 2ab \le a^2 + b^2 + \varepsilon_1 a^2 + \varepsilon_1^{-1}b^2 = (1+\varepsilon_1)a^2 + (1+\varepsilon_1^{-1})b^2,$$

that is valid for any $\varepsilon_1 > 0$, from (4.2.8) we have

$$|u_{mt}(x,0)|^2 = \big|\mu u_{mt}(x,T) + \psi_m(x)\big|^2 \le |\mu|^2(1+\varepsilon_1)u_{mt}^2(x,T) + (1+\varepsilon_1^{-1})\psi_m^2(x). \tag{4.2.18}$$

From (4.2.18) we obtain

$$\begin{aligned} \int\limits_{\omega_0} u_{mt}^2\,dx &= \int\limits_\Omega |u_{mt}(x,0)|^2\,dx \le |\mu|^2(1+\varepsilon_1)\int\limits_\Omega u_{mt}^2(x,T)\,dx + (1+\varepsilon_1^{-1})\int\limits_\Omega \psi_m^2(x)\,dx \\ &= |\mu|^2(1+\varepsilon_1)\int\limits_{\omega_T} u_{mt}^2(x,T)\,dx + (1+\varepsilon_1^{-1})\|\psi_m\|_{L_2(\Omega)}^2. \end{aligned} \tag{4.2.19}$$

In view of (4.2.7) and (4.2.14), from (4.2.17) we get

$$\int\limits_{\omega_T} u_{mt}^2(x,T)\,dx \le w_m(T) \le \Big[\int\limits_{\omega_0}\sum_{i=1}^n \varphi_{mx_i}^2\,dx + \int\limits_{\omega_T} u_{mt}^2(x,T)\,dx + M_5\Big]e^{\varepsilon T}, \tag{4.2.20}$$

where

$$M_5=2\lambda(M_3+M_4T)\operatorname{mes}\Omega+2\lambda\int\limits_{\omega_0}|g(x,t,u_m(x,t))|\,dx+\varepsilon^{-1}\int\limits_{D_T}F_m^2\,dx\,dt. \tag{4.2.21}$$

From (4.2.19) and (4.2.20) it follows that

$$\int\limits_{\omega_0}u_{mt}^2\,dx\le|\mu|^2(1+\varepsilon_1)\Big[\int\limits_{\omega_0}\sum_{i=1}^n\varphi_{mx_i}^2\,dx+\int\limits_{\omega_0}u_{mt}^2\,dx+M_5\Big]e^{\varepsilon T}+(1+\varepsilon_1^{-1})\|\psi_m\|^2_{L_2(\Omega)}. \tag{4.2.22}$$

Since $|\mu|<1$, the positive constants ε and ε_1 can be chosen insomuch small that

$$\mu_1=|\mu|^2(1+\varepsilon_1)e^{\varepsilon T}<1. \tag{4.2.23}$$

Due to (4.2.23), from (4.2.22) we obtain

$$\begin{aligned}\int\limits_{\omega_0}u_{mt}^2\,dx&\le(1-\mu_1)^{-1}\Big[|\mu|^2(1+\varepsilon_1)\Big(\int\limits_{\omega_0}\sum_{i=1}^n\varphi_{mx_i}^2\,dx+M_5\Big)e^{\varepsilon T}+(1+\varepsilon_1^{-1})\|\psi_m\|^2_{L_2(\Omega)}\Big]\\&\le(1-\mu_1)^{-1}\Big[|\mu|^2(1+\varepsilon_1)\big(\|\varphi_m\|^2_{\overset{\circ}{W}{}_2^1(\Omega)}+M_5\big)e^{\varepsilon T}+(1+\varepsilon_1^{-1})\|\psi_m\|^2_{L_2(\Omega)}\Big].\end{aligned} \tag{4.2.24}$$

It follows from (4.2.7), (4.2.14) and (4.2.24) that

$$\begin{aligned}w_m(0)&=\int\limits_{\omega_0}\Big[u_{mt}^2+\sum_{i=1}^n\varphi_{mx_i}^2\Big]\,dx\\&\le\|\varphi_m\|^2_{\overset{\circ}{W}{}_2^1(\Omega)}+(1-\mu_1)^{-1}\Big[|\mu|^2(1+\varepsilon_1)\big(\|\varphi_m\|^2_{\overset{\circ}{W}{}_2^1(\Omega)}+M_5\big)e^{\varepsilon T}+(1+\varepsilon_1^{-1})\|\psi_m\|^2_{L_2(\Omega)}\Big].\end{aligned} \tag{4.2.25}$$

In view of (4.2.21) and (4.2.25), from (4.2.17) we get

$$\begin{aligned}w_m(\tau)\le\Bigg\{&\|\varphi_m\|^2_{\overset{\circ}{W}{}_2^1(\Omega)}+(1-\mu_1)^{-1}\Bigg[|\mu|^2(1+\varepsilon_1)\Bigg(\|\varphi_m\|^2_{\overset{\circ}{W}{}_2^1(\Omega)}+2\lambda(M_3+M_4T)\operatorname{mes}\Omega\\&+2\lambda\int\limits_{\omega_0}|g(x,t,u_m(x,t))|\,dx+\varepsilon^{-1}\int\limits_{D_T}F_m^2\,dx\,dt\Bigg)e^{\varepsilon T}+(1+\varepsilon_1^{-1})\|\psi_m\|^2_{L_2(\Omega)}\Bigg]\\&+2\lambda(M_3+M_4T)\operatorname{mes}\Omega+2\lambda\int\limits_{\omega_0}|g(x,t,u_m(x,t))|\,dx+\varepsilon^{-1}\int\limits_{D_T}F_m^2\,dx\,dt\Bigg\}e^{\varepsilon T}\\=\widetilde\gamma_1\|F_m\|^2_{L_2(D_T)}&+\widetilde\gamma_2\|\varphi_m\|^2_{\overset{\circ}{W}{}_2^1(\Omega)}+\widetilde\gamma_3\|\psi_m\|^2_{L_2(\Omega)}+\widetilde\gamma_4\int\limits_{\omega_0}|g(x,t,u_m(x,t))|\,dx+\widetilde\gamma_5.\end{aligned} \tag{4.2.26}$$

Here,

$$\begin{aligned}&\widetilde\gamma_1=\varepsilon^{-1}e^{\varepsilon T}\big[(1-\mu_1)^{-1}(1+\varepsilon_1)e^{\varepsilon T}+1\big],\\&\widetilde\gamma_2=e^{\varepsilon T}\big[1+(1-\mu_1)^{-1}|\mu|^2(1+\varepsilon_1)\big],\\&\widetilde\gamma_3=(1-\mu_1)^{-1}(1+\varepsilon_1^{-1})e^{\varepsilon T},\\&\widetilde\gamma_4=2\lambda\big[(1-\mu_1)^{-1}|\mu|^2(1+\varepsilon_1)+1\big]e^{\varepsilon T},\\&\widetilde\gamma_5=2\lambda(M_3+M_4T)\operatorname{mes}\Omega\big[(1-\mu_1)^{-1}|\mu|^2(1+\varepsilon_1)e^{\varepsilon T}+1\big]e^{\varepsilon T}.\end{aligned} \tag{4.2.27}$$

Since for the fixed τ the function $u_m(x,\tau)\in\overset{\circ}{W}{}_2^1(\Omega)$, due to the Friedrichs inequality [68], we have

$$\int\limits_{\omega_\tau}\Big[u_m^2+u_{mt}^2+\sum_{i=1}^n u_{mx_i}^2\Big]\,dx\le c_0w_m(\tau)=c_0\int\limits_{\omega_\tau}\Big[u_{mt}^2+\sum_{i=1}^n u_{mx_i}^2\Big]\,dx, \tag{4.2.28}$$

where the positive constant $c_0 = c_0(\Omega)$ does not depend on u_m.

From (4.2.26) and (4.2.28) follows

$$\|u_m\|^2_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} = \int\limits_0^T \Big[\int\limits_{\omega_\tau} \Big(u_m^2 + u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2\Big)\,dx\Big]\,d\tau \le \int\limits_0^T c_0 w_m(\tau)\,d\tau \le c_0T\widetilde{\gamma}_1\|F_m\|^2_{L_2(D_T)} + c_0T\widetilde{\gamma}_2\|\varphi_m\|^2_{\overset{\circ}{W}{}_2^1(\Omega)} + c_0T\widetilde{\gamma}_3\|\psi_m\|^2_{L_2(\Omega)} + c_0T\widetilde{\gamma}_4\int\limits_\Omega |g(x,0,u_m(x,0))|\,dx + c_0T\widetilde{\gamma}_5. \tag{4.2.29}$$

Due to (4.2.1) and (4.1.5), we have

$$|g(x,0,s)| \le M_6 + M_7|s|^{\alpha+1}, \tag{4.2.30}$$

where M_6 and M_7 are some nonnegative constants. Taking into account (4.2.30), from (4.2.29) we get

$$\|u_m\|^2_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} \le c_0T\widetilde{\gamma}_1\|F_m\|^2_{L_2(D_T)} + c_0T\widetilde{\gamma}_2\|\varphi_m\|^2_{\overset{\circ}{W}{}_2^1(\Omega)} + c_0T\widetilde{\gamma}_3\|\psi_m\|^2_{L_2(\Omega)} + c_0T\widetilde{\gamma}_4M_6\operatorname{mes}\Omega + c_0T\widetilde{\gamma}_4M_7\int\limits_\Omega |u_m(x,0)|^{\alpha+1}\,dx + c_0T\widetilde{\gamma}|_5. \tag{4.2.31}$$

Reasoning from Remark 4.1.1 concerning the space $W_2^1(\Omega)$, in view of the equality $\dim\Omega = \dim D_T - 1 = n$ shows that the embedding operator $I : W_2^1(\Omega) \to L_q(\Omega)$ is a linear continuous compact operator for $1 < q < \frac{2n}{n-2}$, when $n > 2$, and for any $q > 1$, when $n = 2$ [68]. At the same time, Nemitski's operator $\mathcal{N}_1 : L_q(\Omega) \to L_2(\Omega)$, acting by the formula $\mathcal{N}_1 u = |u|^{\frac{\alpha+1}{2}}$, is continuous and bounded if $q \ge 2\frac{\alpha+1}{2} = \alpha+1$ [22]. Thus, if $\alpha+1 < \frac{2n}{n-2}$, i.e., $\alpha < \frac{n+2}{n-2}$, which, due to (4.1.6), is fulfilled since $\frac{n+1}{n-1} < \frac{n+2}{n-2}$, there exists a number q such that $1 < q < \frac{2n}{n-2}$ and $q \ge \alpha+1$. Therefore, in this case the operator

$$\mathcal{N}_2 = \mathcal{N}_1 I : W_2^1(\Omega) \to L_2(\Omega)$$

is continuous and compact. Thus, due to (4.1.9) and (4.2.7), it follows that

$$\lim_{m\to\infty} \int\limits_\Omega |u_m(x,0)|^{\alpha+1}\,dx = \int\limits_\Omega |\varphi(x)|^{\alpha+1}\,dx, \tag{4.2.32}$$

and also [68]

$$\int\limits_\Omega |\varphi(x)|^{\alpha+1}\,dx \le C_1\|\varphi\|^{\alpha+1}_{\overset{\circ}{W}{}_2^1(\Omega)} \tag{4.2.33}$$

with the positive constant C_1, not depending on $\varphi \in \overset{\circ}{W}{}_2^1(\Omega)$.

In view of (4.1.8), (4.1.9), (4.2.5)–(4.2.8), (4.2.32) and (4.2.33), passing in (4.2.31) to the limit as $m \to \infty$ we obtain

$$\|u\|^2_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} \le c_0T\widetilde{\gamma}_1\|F\|^2_{L_2(D_T)} + c_0T\widetilde{\gamma}_2\|\varphi\|^2_{\overset{\circ}{W}{}_2^1(\Omega)} + c_0T\widetilde{\gamma}_3\|\psi\|^2_{L_2(\Omega)} + c_0T\widetilde{\gamma}_4M_7C_1\|\varphi\|^{\alpha+1}_{\overset{\circ}{W}{}_2^1(\Omega)} + c_0T(\widetilde{\gamma}_5 + \widetilde{\gamma}_4M_6\operatorname{mes}\Omega). \tag{4.2.34}$$

Taking the square root from both sides of the inequality (4.2.34) and using the obvious inequality $\big(\sum\limits_{i=1}^k a_i^2\big)^{1/2} \le \sum\limits_{i=1}^k |a_i|$, we finally get

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} \le c_1\|F\|_{L_2(D_T)} + c_2\|\varphi\|_{\overset{\circ}{W}{}_2^1(\Omega)} + c_3\|\psi\|_{L_2(\Omega)} + c_4\|\varphi\|^{\frac{\alpha+1}{2}}_{\overset{\circ}{W}{}_2^1(\Omega)} + c_5. \tag{4.2.35}$$

Here,

$$c_1=(c_0T\widetilde{\gamma}_1)^{1/2},\quad c_2=(c_0T\widetilde{\gamma}_2)^{1/2},\quad c_3=(c_0T\widetilde{\gamma}_3)^{1/2},$$
$$c_4=(c_0T\widetilde{\gamma}_4M_7C_1)^{1/2},\quad c_5=\big[c_0T(\widetilde{\gamma}_5+\widetilde{\gamma}_4M_6\operatorname{mes}\Omega)\big]^{1/2}, \tag{4.2.36}$$

where $\widetilde{\gamma}_i$, $1\le i\le 5$, are defined in (4.2.27). In the linear case, i.e., for $\widetilde{\gamma}_4=\widetilde{\gamma}_5=0$, it follows from (4.2.35) that in the estimate (4.2.4) the constants $c_4=c_5=0$, whence it follows that the solution of the problem (4.1.1)–(4.1.4) is unique in the linear case. Thus, Lemma 4.2.1 is proved completely. □

4.3 The existence of a solution of the problem (4.1.1)–(4.1.4)

For the existence of a solution of the problem (4.1.1)–(4.1.4) in the case $|\mu|<1$, we will use the well-known facts dealing with the solvability of the following linear mixed problem [68]:

$$L_0u:=\frac{\partial^2u}{\partial t^2}-\sum_{i=1}^n\frac{\partial^2u}{\partial x_i^2}=F(x,t),\ \ (x,t)\in D_T, \tag{4.3.1}$$

$$u\big|_\Gamma=0,\ \ u(x,0)=\varphi(x),\ \ u_t(x,0)=\widetilde{\psi}(x),\ \ x\in\Omega, \tag{4.3.2}$$

where F, φ and $\widetilde{\psi}$ are the given functions.

For $F\in L_2(D_T)$, $\varphi\in\overset{\circ}{W}{}_2^1(\Omega)$, $\widetilde{\psi}\in L_2(\Omega)$, the unique generalized solution u of the problem (4.3.1), (4.3.2) (in the sense of the equality (4.1.12), where $f=0$, and the number $\mu=0$ in the definition of the space V) from the class $E_{2,1}(D_T)$ with the norm [68]

$$\|u\|^2_{E_{2,1}(D_T)}=\sup_{0\le\tau\le T}\int\limits_{\omega_\tau}\Big[u^2+u_t^2+\sum_{i=1}^n u_{x_i}^2\Big]\,dx$$

is given by the formula [68]

$$u=\sum_{k=1}^\infty\bigg(a_k\cos\mu_kt+b_k\sin\mu_kt+\frac{1}{\mu_k}\int\limits_0^t F_k(\tau)\sin\mu_k(t-\tau)\,d\tau\bigg)\varphi_k(x), \tag{4.3.3}$$

where $\widetilde{\lambda}_k=-\mu_k^2$, $0<\mu_1\le\mu_2\le\cdots$, $\lim\limits_{k\to\infty}\mu_k=\infty$ are the eigenvalues, while $\varphi_k\in\overset{\circ}{W}{}_2^1(\Omega)$ are the corresponding eigenfunctions of the spectral problem $\Delta w=\widetilde{\lambda}w$, $w|_{\partial\Omega}=0$ in the domain Ω ($\Delta:=\sum\limits_{i=1}^n\frac{\partial^2}{\partial x_i^2}$), forming simultaneously orthonormal basis in $L_2(\Omega)$ and orthogonal basis in $\overset{\circ}{W}{}_2^1(\Omega)$ in the sense of the scalar product $(v,w)_{\overset{\circ}{W}{}_2^1(\Omega)}=\int\limits_\Omega\sum_{i=1}^n v_{x_i}w_{x_i}\,dx$, i.e.,

$$(\varphi_k,\varphi_l)_{L_2(\Omega)}=\delta_k^l,\quad (\varphi_k,\varphi_l)_{\overset{\circ}{W}{}_2^1(\Omega)}=-\lambda_k\delta_k^l,\ \ \delta_k^l=\begin{cases}1, & l=k,\\ 0, & l\neq k.\end{cases} \tag{4.3.4}$$

Here,

$$a_k=(\varphi,\varphi_k)_{L_2(\Omega)},\quad b_k=\mu_k^{-1}(\widetilde{\psi},\varphi_k)_{L_2(\Omega)},\ \ k=1,2,\dots, \tag{4.3.5}$$

$$F(x,t)=\sum_{k=1}^\infty F_k(t)\varphi_k(x),\ \ F_k(t)=(F,\varphi_k)_{L_2(\omega_t)},\ \ \omega_\tau:D_T\cap\{t=\tau\}, \tag{4.3.6}$$

and, besides, for the solution u from (4.3.3), the estimate [68, 75]

$$\|u\|_{E_{2,1}(D_T)}\le\gamma\big(\|F\|_{L_2(D_T)}+\|\varphi\|_{\overset{\circ}{W}{}_2^1(\Omega)}+\|\widetilde{\psi}\|_{L_2(\Omega)}\big) \tag{4.3.7}$$

with the positive constant γ, independent of F, φ and $\widetilde{\psi}$, is valid.

Let us consider the linear problem corresponding to (4.1.1)–(4.1.4), i.e., the case for $\lambda = 0$:

$$L_0 u := \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = F(x,t), \quad (x,t) \in D_T, \tag{4.3.8}$$

$$u\big|_\Gamma = 0, \quad u(x,0) = \varphi(x), \quad \mathcal{K}_\mu u_t = \psi(x), \quad x \in \Omega, \tag{4.3.9}$$

Let us show that when $|\mu| < 1$ for any $F \in L_2(D_T)$, $\varphi \in \overset{\circ}{W}{}_2^1(\Omega)$ and $\psi \in L_2(\Omega)$, there exists a unique generalized solution of the problem (4.3.8), (4.3.9) in the sense of Definition 4.1.1 for $\lambda = 0$. Indeed, for $\varphi \in \overset{\circ}{W}{}_2^1(\Omega)$ and $\psi \in L_2(\Omega)$, the expansions $\varphi = \sum\limits_{k=1}^{\infty} a_k \varphi_k$ and $\psi = \sum\limits_{k=1}^{\infty} d_k \varphi_k$ in the spaces $\overset{\circ}{W}{}_2^1(\Omega)$ and $L_2(\Omega)$, respectively, are valid; here, $a_k = (\varphi, \varphi_k)_{L_2(\Omega)}$ and $d_k = (\psi, \varphi_k)_{L_2(\Omega)}$ [68]. Therefore, setting

$$\varphi_m = \sum_{k=1}^{m} a_k \varphi_k, \quad \psi_m = \sum_{k=1}^{m} d_k \varphi_k, \tag{4.3.10}$$

we have

$$\lim_{m\to\infty} \|\varphi_m - \varphi\|_{\overset{\circ}{W}{}_2^1(\Omega)} = 0, \quad \lim_{m\to\infty} \|\psi_m - \psi\|_{L_2(\Omega)} = 0. \tag{4.3.11}$$

Since the space of infinitely differentiable functions $C_0^\infty(D_T)$ is dense in the space $L_2(D_T)$, for $F \in L_2(D_T)$ and any natural number m there exists a function $F_m \in C_0^\infty(D_T)$ such that

$$\|F_m - F\|_{L_2(D_T)} < \frac{1}{m}\,. \tag{4.3.12}$$

On the other hand, for the function F_m in the space $L_2(D_T)$ the expansion [68]

$$F_m(X,T) = \sum_{k=1}^{\infty} F_{m,k}(t)\varphi_k(x), \quad F_{m,k}(t) = (F_m, \varphi_k)_{L_2(\Omega)} \tag{4.3.13}$$

is valid. Therefore, there exists a natural number ℓ_m such that $\lim\limits_{m\to\infty} \ell_m = \infty$, and for

$$\widetilde{F}_m(x,t) = \sum_{k=1}^{\ell_m} F_{m,k}(t)\varphi_k(x) \tag{4.3.14}$$

the inequality

$$\|\widetilde{F}_m - F_m\|_{L_2(D_T)} < \frac{1}{m} \tag{4.3.15}$$

is valid. From (4.3.12) and (4.3.15) it follows that

$$\lim_{m\to\infty} \|\widetilde{F}_m - F_m\|_{L_2(D_T)} = 0. \tag{4.3.16}$$

The solution $u = u_m$ of the problem (4.3.1), (4.3.2) for $\varphi = \varphi_{\ell_m}$, $\widetilde{\psi} = \sum\limits_{k=1}^{\ell_m} \widetilde{d}_k \varphi_k$ and $F = \widetilde{F}_m$, where φ_{ℓ_m} and $\widetilde{F}_m$ are defined in (4.3.10) and (4.3.14), is given by the formula (4.3.3) which, due to (4.3.4)–(4.3.6), takes the form

$$u_m = \sum_{k=1}^{\ell_m} \left(a_k \cos \mu_k t + \frac{\widetilde{d}_k}{\mu_k} \sin \mu_k t + \frac{1}{\mu_k} \int_0^t F_{m,k}(\tau) \sin \mu_k (t-\tau)\, d\tau \right) \varphi_k(x), \tag{4.3.17}$$

To determine the coefficients $\widetilde{d}_k$ we substitute the right-hand side of the expression (4.3.17) into the equality $\mathcal{K}_\mu u_{mt} = \psi_{\ell_m}(x)$, where ψ_{ℓ_m} is defined in (4.3.10). Consequently, taking into account that the system of functions $\{\varphi_k(x)\}$ represents a basis in $L_2(\Omega)$ and $1-\mu\cos\mu_k T\neq 0$ for $|\mu|<1$, we obtain the following formulas:

$$\widetilde{d}_k=\frac{1}{1-\mu\cos\mu_k T}\bigg[(\varphi_{\ell_m},\varphi_k)_{L_2(\Omega)}-a_k\mu\mu_k\sin\mu_k T+\mu\int\limits_0^T F_{m,k}(\tau)\cos\mu_k(T-\tau)\,d\tau\bigg],\quad k=1,\dots,\ell_m. \tag{4.3.18}$$

Below, we assume that the Lipschitz domain Ω is such that the eigenfunctions $\varphi_k\in C^2(\overline{\Omega})$, $k\geq 1$. For example, this will take place if $\partial\Omega\in C^{[\frac{n}{2}]+3}$ [75]. This fact will also take place in the case of a piecewise smooth Lipschitz domain, e.g., for the parallelepiped $\Omega:=\{x\in\mathbb{R}^n:\ |x_i|<a_i,\ i=1,\dots,n\}$, the corresponding eigenfunctions $\varphi_k\in C^\infty(\overline{\Omega})$ [76]. Therefore, since $F_m\in C_0^\infty(D_T)$, due to (4.3.13), the function $F_{m,k}\in C^2([0,T])$ and, consequently, the function u_m from (4.3.17) belongs to the space $C^2(\overline{D}_T)$. Further, since $\varphi_k|_{\partial\Omega}=0$, due to (4.3.17), we have $u_m|_\Gamma=0$, and thereby, $u_m\in \overset{\circ}{C}{}^2(\overline{D}_T,\Gamma)$, $m=1,2,\dots$.

According to the construction, the function u_m from (4.3.17) satisfies

$$u_m\big|_\Gamma=0,\ \ L_0u_m=\widetilde{F}_m,\ \ u_m(x,0)=\varphi_{\ell_m}(x),\ \ \mathcal{K}_\mu u_{mt}=\psi_{\ell_m}(x),\ \ x\in\Omega, \tag{4.3.19}$$

and hence

$$(u_m-u_k)\big|_\Gamma=0,\ \ L_0(u_m-u_k)=\widetilde{F}_m-\widetilde{F}_k,\ \ (u_m-u_k)(x,0)=(\varphi_{\ell_m}-\varphi_{\ell_k})(x),$$
$$\mathcal{K}_\mu(u_{mt}-u_{kt})=(\psi_{\ell_m}-\psi_{\ell_k}),\ \ x\in\Omega.$$

Therefore, from a priori estimate (4.2.4), where $\lambda=0$, the coefficients $c_4=c_5=0$, we obtain

$$\|u_m-u_k]\|_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)}\leq c_1\|\widetilde{F}_m-\widetilde{F}_k\|_{L_2(D_T)}+c_2\|\varphi_{\ell_m}-\varphi_{\ell_k}\|_{\overset{\circ}{W}{}_2^1(\Omega)}+c_3\|\psi_{\ell_m}-\psi_{\ell_k}\|_{L_2(\Omega)}. \tag{4.3.20}$$

In view of (4.3.11) and (4.3.16), from (4.3.20) it follows that the sequence $u_m\in\overset{\circ}{C}{}^2(\overline{D}_T,\Gamma)$ is fundamental in the complete space $\overset{\circ}{W}{}_2^1(D_T,\Gamma)$. Therefore, there exists a function $u\in\overset{\circ}{W}{}_2^1(D_T,\Gamma)$ such that due to (4.3.11), (4.3.16) and (4.3.19), the limit equalities (4.3.8), (4.3.9) are valid. The uniqueness of this solution follows from the a priori estimate (4.2.4), where the constants $c_4=c_5=0$ for $\lambda=0$. Therefore, for the solution u of the problem (4.3.8), (4.3.9), we have $u=L_0^{-1}(F,\varphi,\psi)$, where $L_0^{-1}:L_2(D_T)\times\overset{\circ}{W}{}_2^1(\Omega)\times L_2(\Omega)\to\overset{\circ}{W}{}_2^1(D_T,\Gamma)$, whose norm, due to (4.2.4), can be estimated as follows:

$$\|L_0^{-1}\|_{L_2(D_T)\times\overset{\circ}{W}{}_2^1(\Omega)\times L_2(\Omega)\to\overset{\circ}{W}{}_2^1(D_T,\Gamma)}\leq\gamma_0=\max(c_1,c_2,c_3). \tag{4.3.21}$$

Owing to the linearity of the operator

$$L_0^{-1}:L_2(D_T)\times\overset{\circ}{W}{}_2^1(\Omega)\times L_2(\Omega)\to\overset{\circ}{W}{}_2^1(D_T,\Gamma)$$

we have the representation

$$L_0^{-1}(F,\varphi,\psi)=L_0^{-1}(F,0,0)+L_0^{-1}(0,\varphi,0)+L_0^{-1}(0,0,\psi)=L_{01}^{-1}(F)+L_{02}^{-1}(\varphi)+L_{03}^{-1}(\psi), \tag{4.3.22}$$

where $L_{01}^{-1}:L_2(D_T)\to\overset{\circ}{W}{}_2^1(D_T,\Gamma)$, $L_{02}^{-1}:\overset{\circ}{W}{}_2^1(\Omega)\to\overset{\circ}{W}{}_2^1(D_T,\Gamma)$ and $L_{03}^{-1}:L_2(\Omega)\to\overset{\circ}{W}{}_2^1(D_T,\Gamma)$ are the linear continuous operators and, besides, according to (4.3.21),

$$\|L_{01}^{-1}\|_{L_2(D_T)\to\overset{\circ}{W}{}_2^1(D_T,\Gamma)}\leq\gamma_0,\quad \|L_{02}^{-1}\|_{\overset{\circ}{W}{}_2^1(\Omega)\to\overset{\circ}{W}{}_2^1(D_T,\Gamma)}\leq\gamma_0,\quad \|L_{03}^{-1}\|_{L_2(\Omega)\to\overset{\circ}{W}{}_2^1(D_T,\Gamma)}\leq\gamma_0. \tag{4.3.23}$$

Remark 4.3.1. Note that for $F\in L_2(D_T)$, $\varphi\in\overset{\circ}{W}{}_2^1(\Omega)$, $\psi\in L_2(\Omega)$, due to (4.1.5), (4.1.6), (4.3.21)–(4.3.23) and Remark 4.1.1, the function $u\in\overset{\circ}{W}{}_2^1(D_T,\Gamma)$ is a generalized solution of the problem (4.1.1)–(4.1.4) if and only if u is a solution of the following functional equation

$$u=L_{01}^{-1}(-\lambda f(x,t,u))+L_{01}^{-1}(F)+L_{02}^{-1}(\varphi)+L_{03}^{-1}(\psi) \tag{4.3.24}$$

in the space $\overset{\circ}{W}{}_2^1(D_T,\Gamma)$.

We rewrite the equation (4.3.24) in the form

$$u = A_0 u := -\lambda L_{01}^{-1}(\mathcal{N}_0 u) + L_{01}^{-1}(F) + L_{02}^{-1}(\varphi) + L_{03}^{-1}(\psi), \tag{4.3.25}$$

where the operator $\mathcal{N}_0 : \overset{\circ}{W}{}_2^1(D_T,\Gamma) \to L_2(D_T)$ from (4.1.7), is, according to Remark 4.1.1, continuous and compact. Therefore, due to (4.3.23), the operator $\mathcal{A}_0 : \overset{\circ}{W}{}_2^1(D_T,\Gamma) \to \overset{\circ}{W}{}_2^1(D_T,\Gamma)$ from (4.3.25) is also continuous and compact. At the same time, according to Lemma 4.2.1 and (4.2.36), for any parameter $\tau \in [0,1]$ and for any solution u of the equation $u = \tau\mathcal{A}_0 u$ with the parameter τ, the same a priori estimate (4.2.4) with nonnegative constants c_i, independent of u, F, φ, ψ and τ, is valid. Therefore, due to Schaefer's fixed point theorem [20], the equation (4.3.25) and hence, by Remark 4.3.1, the problem (4.1.1)–(4.1.4) has at least one solution $u \in \overset{\circ}{W}{}_2^1(D_T,\Gamma)$. Thus, we have proved the following theorem.

Theorem 4.3.1. *Let $\lambda > 0$, $|\mu| < 1$, $F \in L_2(D_T)$, $\varphi \in \overset{\circ}{W}{}_2^1(\Omega)$, $\psi \in L_2(\Omega)$ and the conditions* (4.1.5), (4.1.6), (4.2.2) *and* (4.2.3) *be fulfilled. Then the problem* (4.1.1)–(4.1.4) *has at least one generalized solution.*

Remark 4.3.2. Note that for $|\mu| = 1$, even in the liner case, i.e., for $f = 0$, the homogeneous problem corresponding to (4.1.1)–(4.1.4) may have a finite or even infinite number of linearly independent solutions. Indeed, in the case $\mu = 1$, we denote by $\Lambda(1)$ a set of points μ_k from (4.3.3), for which the ratio $\frac{\mu_k T}{2\pi}$ is a natural number, i.e., $\Lambda(1) = \{\mu_k : \frac{\mu_k T}{2\pi} \in \mathbb{N}\}$. If we seek for a solution of the problem (4.3.8), (4.3.9) in the form of the representation (4.3.3), then for determination of unknown coefficients b_k contained in it, we substitute the right-hand side of this representation into the equality $\mathcal{K}_\mu u_t = \psi(x)$. As a result, we have

$$\mu_k(1 - \mu\cos\mu_k T)b_k = (\psi,\varphi_k)_{L_2(\Omega)} - a_k\mu_k \sin\mu_k T + \int\limits_0^T F_k(\tau)\cos\mu_k(T-\tau)\,d\tau. \tag{4.3.26}$$

It is obvious that when $\Lambda(1) \neq \varnothing$ and $\mu_k \in \Lambda(1)$, $\mu = 1$ we have $1 - \cos\mu_k T = 0$, and for $F = 0$, $\varphi = \psi = 0$ and thereby for $a_k = 0$, $F_k(\tau) = 0$, the equality (4.3.26) will be satisfied by any number b_k. Therefore, in accordance with (4.3.3), the function $u_k(x,t) = C\sin\mu_k t\varphi_k(x)$, $C = const \neq 0$, satisfies the homogeneous problem corresponding to (4.3.8), (4.3.9). Analogously, in the case $\mu = -1$, we denote by $\Lambda(-1)$ the set of points from (4.3.3) for which the ratio $\frac{\mu_k T}{\pi}$ is an odd integer. In the case $1 - \mu\cos\mu_k T = 0$ for $\mu_k \in \Lambda(-1)$, $\mu = -1$ and the function $u_k(x,t) = C\sin\mu_k t\varphi_k(x)$, $C = const \neq 0$, is a nontrivial solution of the homogeneous problem corresponding to (4.3.8), (4.3.9). For example, when $n = 2$, $\Omega = (0,1)\times(0,1)$, the eigenvalues and eigenfunctions of the Laplace operator Δ are [76]

$$\lambda_k = -\pi^2(k_1^2 + k_2^2), \quad \varphi_k(x_1,x_2) = \sin k_1\pi x_1 \sin k_2\pi x_2, \;\; k = (k_1,k_2),$$

i.e., $\mu_k = \pi\sqrt{k_1^2 + k_2^2}$. For $k_1 = p^2 - q^2$, $k_2 = 2pq$, where p and q are any integers, we obtain $\mu_k = \pi(p^2+q^2)$. In this case, for $\frac{T}{2} \in \mathbb{N}$, we have $\frac{\mu_k T}{2\pi} = \frac{(p^2+q^2)T}{2} \in \mathbb{N}$, and according to the abovesaid, when $\mu = 1$, the homogeneous problem corresponding to (4.3.8), (4.3.9) has an infinite number of linearly independent solutions

$$u_{p,q}(x,t) = \sin\pi(p^2+q^2)t\sin\pi(p^2-q^2)x_1\sin 2\pi pqx_2 \;\; \forall\, p,q \in \mathbb{N}. \tag{4.3.27}$$

Analogously, when $\mu = -1$, the solutions of the homogeneous problem corresponding to (4.3.8), (4.3.9) are the functions from (4.3.27) if and only if p is an even number, while q and T are odd numbers.

4.4 The uniqueness of a solution of the problem (4.1.1)–(4.1.4)

On the function f in the equation (4.1.1) let us impose the following requirements:

$$f, f_u' \in C(\overline{D}_T \times \mathbb{R}), \;\; |f_u'(x,t,u)| \le a + b|u|^\gamma, \;\; (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{4.4.1}$$

where $a, b, \gamma = const \geq 0$.

It is obvious that from (4.4.1) we have the condition (4.1.5) for $\alpha = \gamma + 1$, and when $\gamma < \frac{2}{n-1}$, we have $\alpha = \gamma + 1 < \frac{n+1}{n-1}$, hence the condition (4.1.6) is fulfilled.

Theorem 4.4.1. *Let $|\mu| < 1$, $F \in L_2(D_T)$, $\varphi \in \overset{\circ}{W}{}_2^1(\Omega)$, $\psi \in L_2(\Omega)$ and the condition (4.4.1) be fulfilled, $\gamma < \frac{2}{n-1}$; and also, the conditions (4.2.2), (4.2.3) hold. Then there exists a positive number $\lambda_0 = \lambda_0(F, f, \varphi, \psi, \mu, D_T)$ such that for $0 < \lambda < \lambda_0$ the problem (4.1.1)–(4.1.4) cannot have more than one generalized solution.*

Proof. Indeed, suppose that the problem (4.1.1)–(4.1.4) has two different generalized solutions u_1 and u_2. According to Definition 4.1.1, there exist sequences of functions $u_{jk} \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma)$, $j = 1, 2$, such that

$$\lim_{k\to\infty} \|u_{jk} - u_j\|_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} = 0, \quad \lim_{k\to\infty} \|L_\lambda u_{jk} - F\|_{L_2(D_T)} = 0, \tag{4.4.2}$$

$$\lim_{k\to\infty} \big\|u_{jk}\big|_{t=0} - \varphi\big\|_{\overset{\circ}{W}{}_2^1(\Omega)} = 0, \quad \lim_{k\to\infty} \|\mathcal{K}_\mu u_{jkt} - \psi\|_{L_2(\Omega)} = 0, j = 1, 2. \tag{4.4.3}$$

Let

$$w := u_2 - u_1, \quad w_k := u_{2k} - u_{1k}, \quad F_k : L_\lambda u_{2k} - L_\lambda u_{1k}, \tag{4.4.4}$$

$$g_k : \lambda\big(f(x,t,u_{1k}) - f(x,t,u_{2k})\big). \tag{4.4.5}$$

In view of (4.4.2), (4.4.3) and (4.4.4), it is easy to see that

$$\lim_{k\to\infty} \|w_k - w\|_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} = 0, \quad \lim_{k\to\infty} \|F_k\|_{L_2(D_T)} = 0, \tag{4.4.6}$$

$$\lim_{k\to\infty} \big\|w_k\big|_{t=0}\big\|_{\overset{\circ}{W}{}_2^1(\Omega)} = 0, \quad \lim_{k\to\infty} \|\mathcal{K}_\mu w_{kt}\|_{L_2(\Omega)} = 0. \tag{4.4.7}$$

Owing to (4.4.4), (4.4.5), the function $w_k \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma)$ satisfies the following equalities:

$$\frac{\partial^2 w_k}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2 w_k}{\partial x_i^2} = (F_k + g_k)(x,t), \ \ (x,t) \in D_T, \tag{4.4.8}$$

$$w_k\big|_\Gamma = 0, \tag{4.4.9}$$

$$w_k(x,0) = \widetilde{\varphi}_k(x), \ \ x \in \Omega, \tag{4.4.10}$$

$$\mathcal{K}_\mu w_{kt}: \ w_{kt}(x,0) - \mu w_{kt}(x,T) = \widetilde{\psi}_k(x), \ \ x \in \Omega, \tag{4.4.11}$$

where $\widetilde{\varphi}_k(x) := u_{2k}(x,0) - u_{1k}(x,0)$, $\widetilde{\psi}_k(x) := \mathcal{K}_\mu u_{2kt} - \mathcal{K}_\mu u_{1kt}$.

First, let us estimate the function g_k from (4.4.5). Taking into account the obvious inequality $|d_1 + d_2|^\gamma \leq 2^\gamma \max(|d_1|^\gamma, |d_2|^\gamma) \leq 2^\gamma(|d_1|^\gamma + |d_2|^\gamma)$ for $\gamma \geq 0$, due to (4.4.1), we have

$$\begin{aligned} &\big|f(x,t,u_{2k}) - f(x,t,u_{1k})\big| \\ &= \bigg|(u_{2k} - u_{1k}) \int_0^1 f'_u(x,t,u_{1k} + \tau(u_{2k} - u_{1k}))\, d\tau\bigg| \leq |u_{2k} - u_{1k}| \int_0^1 \big(a + b|(1-\tau)u_{1k} + \tau u_{2k}|^\gamma\big)\, d\tau \\ &\qquad \leq a|u_{2k} - u_{1k}| + 2^\gamma b|u_{2k} - u_{1k}|\big(|u_{1k}|^\gamma + |u_{2k}|^\gamma\big) = a|w_k| + 2^\gamma b|w_k|\big(|u_{1k}|^\gamma + |u_{2k}|^\gamma\big). \end{aligned} \tag{4.4.12}$$

In view of (4.4.5), from (4.4.12) we obtain

$$\begin{aligned} \|g_k\|_{L_2(D_T)} &\leq \lambda a\|w_k\|_{L_2(D_T)} + \lambda 2^\gamma b\big\|\,|w_k|\big(|u_{1k}|^\gamma + |u_{2k}|^\gamma\big)\big\|_{L_2(D_T)} \\ &\leq \lambda a\|w_k\|_{L_2(D_T)} + \lambda_2 2^\gamma b\|w_k\|_{L_p(D_T)}\big\|\big(|u_{1k}|^\gamma + |u_{2k}|^\gamma\big)\big\|_{L_q(D_T)}. \end{aligned} \tag{4.4.13}$$

Here we have used Hölder's inequality [24]

$$\|v_1 v_2\|_{L_r(D_T)} \le \|v_1\|_{L_p(D_T)} \|v_2\|_{L_q(D_T)},$$

where $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, and in the capacity of p, q and r we take

$$p = 2\,\frac{n+1}{n-1} \quad q=n+1, \quad r=2. \tag{4.4.14}$$

Since $\dim D_T = n+1$, according to the Sobolev embedding theorem [22], for $1 \le p \le \frac{2(n+1)}{n-1}$, we get

$$\|v\|_{L_p(D_T)} \le C_p \|v\|_{W_2^1(D_T)} \ \ \forall\, v \in W_2^1(D_T) \tag{4.4.15}$$

with the positive constant C_p, not depending on $n \in W_2^1(D_T)$.

Due to the condition of the theorem, $\gamma < \frac{2}{n-1}$, and therefore, $\gamma(n+1) < \frac{2(n+1)}{n-1}$. Thus, due to (4.4.14) from (4.4.15), we have

$$\|w_k\|_{L_p(D_T)} \le C_p \|w_k\|_{W_2^1(D_T)}, \ \ p = \frac{2(n+1)}{n-1}, \ \ k \ge 1, \tag{4.4.16}$$

$$\begin{aligned} &\big\|\big(|u_{1k}|^\gamma + |u_{2k}|^\gamma\big)\big\|_{L_q(D_T)} \le \big\|\,|u_{1k}|^\gamma\big\|_{L_q(D_T)} + \big\|\,|u_{2k}|^\gamma\big\|_{L_q(D_T)} \\ &= \|u_{1k}\|^\gamma_{L_{\gamma(n+1)}(D_T)} + \|u_{2k}\|^\gamma_{L_{\gamma(n+1)}(D_T)} \le C^\gamma_{\gamma(n+1)}\big(\|u_{1k}\|^\gamma_{W_2^1(D_T)} + \|u_{2k}\|^\gamma_{W_2^1(D_T)}\big). \end{aligned} \tag{4.4.17}$$

In view of the first inequality of (4.4.2), there exists a natural number k_0 such that for $k \ge k_0$, we obtain

$$\|u_{ik}\|^\gamma_{W_2^1(D_T)} \le \|u_i\|^\gamma_{W_2^1(D_T)} + 1, \ \ i=1,2, \ \ k \ge k_0. \tag{4.4.18}$$

Further, in view of (4.4.16), (4.4.17) and (4.4.18), from (4.4.13) we get

$$\begin{aligned} &\|g_k\|_{L_2(D_T)} \le \lambda a \|w_k\|_{L_2(D_T)} \\ &\quad + \lambda 2^\gamma b C_p C^\gamma_{\gamma(n+1)}\big(\|u_1\|^\gamma_{W_2^1(D_T)} + \|u_2\|^\gamma_{W_2^1(D_T)} + 2\big)\|w_k\|_{L_2(D_T)} \le \lambda M_8 \|w_k\|_{W_2^1(D_T)}, \end{aligned} \tag{4.4.19}$$

where we have used the inequality $\|w_k\|_{L_2(D_T)} \le \|w_k\|_{W_2^1(D_T)}$,

$$M_8 = a + 2^\gamma b C_p C^\gamma_{\gamma(n+1)}\big(\|u_1\|^\gamma_{W_2^1(D_T)} + \|u_2\|^\gamma_{W_2^1(D_T)} + 2\big), \ \ p = \frac{2(n+1)}{n-1}\,. \tag{4.4.20}$$

Since the a priori estimate (4.2.4) is valid for $\lambda = 0$, due to (4.2.27) and (4.2.36), in this estimate $c_4 = c_5 = 0$ and, hence, for the solution w_k of the problem (4.4.8)–(4.4.11) the estimate

$$\|w_k\|_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} \le c_1^0 \|F_k + g_k\|_{L_2(D_T)} + c_2^0 \|\widetilde{\varphi}_k\|_{\overset{\circ}{W}{}_2^1(\Omega)} + c_3^0 \|\widetilde{\psi}_k\|_{L_2(\Omega)} \tag{4.4.21}$$

is valid, where the constants c_1^0, c_2^0, c_3^0 do not depend on λ.

Because of $\|w_k\|_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} = \|w_k\|_{W_2^1(D_T)}$ and due to (4.4.19), from (4.4.21) we have

$$\|w_k\|_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} \le c_1^0 \|F_k\|_{L_2(D_T)} + \lambda c_1^0 M_8 \|w_k\|_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} + c_2^0 \|\widetilde{\varphi}_k\|_{\overset{\circ}{W}{}_2^1(\Omega)} + c_3^0 \|\widetilde{\psi}\|_{L_2(\Omega)}. \tag{4.4.22}$$

Note that since for u_1 and u_2 the a priori estimate (4.2.4) is valid, the constant M_8 from (4.4.20) will depend on λ, F, f, φ, ψ, D_T; besides, due to (4.2.27) and (4.2.36), the value of M_8 depends continuously on λ for $\lambda \ge 0$, and

$$0 \le \lim_{\lambda \to 0+} M_8 = M_8^0 < +\infty. \tag{4.4.23}$$

Due to (4.4.23), there exists a positive number $\lambda_0 = \lambda_0(F, f, \varphi, \psi, \mu, D_T)$ such that for

$$0 < \lambda < \lambda_0, \tag{4.4.24}$$

we obtain $\lambda c_1^0 M_8 < 1$. Indeed, let us fix arbitrarily a positive number ε_1. Then, due to (4.4.23), there exists a positive number λ_1 such that $0 \le M_8 < M_8^0 + \varepsilon_1$ for $0 \le \lambda < \lambda_1$. It is obvious that for $\lambda_0 = \min(\lambda_1, (c_1^0(M_8^0 + \varepsilon_1))^{-1})$ the condition $\lambda c_1^0 M_8 < 1$ will be fulfilled.

Therefore, in the case (4.4.24), from (4.4.22) we get

$$\|w_k\|_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} \le (1 - \lambda c_1^0 M_8)^{-1} \Big[c_1^0 \|F_k\|_{L_2(D_T)} + c_2^0 \|\widetilde{\varphi}_k\|_{\overset{\circ}{W}{}_2^1(\Omega)} + c_3^0 \|\widetilde{\psi}_k\|_{L_2(\Omega)} \Big] \tag{4.4.25}$$

for $k \ge k_0$.

From (4.4.2) and (4.4.4), it follows that $\lim\limits_{k\to\infty} \|w_k\|_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} = \|u_2 - u_1\|_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)}$. On the other hand, due to (4.4.6), (4.4.7) and (4.4.10), (4.4.11), from (4.4.25) we have $\lim\limits_{k\to\infty} \|w_k\|_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} = 0$. Thus, $\|u_2 - u_1\|_{\overset{\circ}{W}{}_2^1(D_T,\Gamma)} = 0$, i.e., $u_2 = u_1$, which leads to the contradiction. Thus Theorem 4.4.1 is proved. □

Chapter 5

Multidimensional problem with two nonlocal in time conditions for some semilinear hyperbolic equations with the Dirichlet or Robin condition

5.1 Statement of the problem

In the space $\mathbb{R}^{n+1}$ of variables $x = (x_1, \ldots, x_n)$ and t, in the cylindrical domain $D_T = \Omega \times (0, T)$, where Ω is an open Lipschitz domain in $\mathbb{R}^n$, we consider a nonlocal problem of finding a solution $u(x, t)$ of the equation

$$L_\lambda u : \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \lambda f(x, t, u) = F(x, t), \ \ (x, t) \in D_T, \tag{5.1.1}$$

satisfying the Dirichlet homogeneous boundary condition

$$u\big|_\Gamma = 0 \tag{5.1.2}$$

on the lateral face $\Gamma := \partial\Omega \times (0, T)$ of the cylinder D_T and the homogeneous nonlocal conditions

$$\mathcal{K}_\mu u := u(x, 0) - \mu u(x, T) = 0, \ \ x \in \Omega, \tag{5.1.3}$$

$$\mathcal{K}_\mu u_t := u_t(x, 0) - \mu u_t(x, T) = 0, \ \ x \in \Omega, \tag{5.1.4}$$

where f and F are the given functions, λ and μ are the given nonzero constants, and $n \geq 2$.

Remark 5.1.1. Note that for $|\mu| \neq 1$, it suffices to consider the case $|\mu| < 1$, since the case $|\mu| > 1$ can be reduced to the latter one by passing from the variable t to the variable $t' = T - t$. The case for $|\mu| = 1$ will be considered at the end of this chapter. In particular, when $\mu = 1$ (-1), the problem (5.1.1)–(5.1.4) can be studied as a periodic (antiperiodic) problem.

We further impose on the function $f = f(x, t, u)$ the following restrictions:

$$f \in C(\overline{D}_T \times \mathbb{R}), \ \ |f(x, t, u)| \leq M_1 + M_2 |u|^\alpha, \ \ (x, t, u) \in \overline{D}_T \times \mathbb{R}, \tag{5.1.5}$$

where

$$0 \leq \alpha = const < \frac{n+1}{n-1}\,. \tag{5.1.6}$$

We consider the following functional spaces

$$\overset{\circ}{C}{}^2_\mu(\overline{D}_T) := \{v \in C^2(\overline{D}_T) : \ v\big|_\Gamma = 0, \ \mathcal{K}_\mu v = 0, \ \mathcal{K}_\mu v_t = 0\}, \tag{5.1.7}$$

$$\overset{\circ}{W}{}^1_{2,\mu}(D_T) := \{v \in W^1_2(D_T) : \ v\big|_\Gamma = 0, \ \mathcal{K}_\mu v = 0\}, \tag{5.1.8}$$

where $W^1_2(D_T)$ is an unknown Sobolev space, and the equalities $v|_\Gamma = 0$, $\mathcal{K}_\mu v = 0$ should be understood in the sense of the trace theory [68].

Remark 5.1.2. The embedding operator $I : W^1_2(D_T) \to L_q(D_T)$ represents a linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$, when $n > 1$ [68]. At the same time, Nemitski's operator $\mathcal{N} : L_q(D_T) \to L_2(D_T)$, acting by the formula $\mathcal{N}u = f(x,t,u)$, is continuous by (5.1.5) and bounded if $q \geq 2\alpha$ [22]. Thus, since by (5.1.6) we have $2\alpha < \frac{2(n+1}{n-1}$, there exists a number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \geq 2\alpha$. Therefore, in this case, the operator

$$\mathcal{N}_0 = \mathcal{N} I : \overset{\circ}{W}{}^1_{2,\mu}(D_T) \to L_2(D_T) \tag{5.1.9}$$

is continuous and compact. Besides, from $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ it follows that $f(x,t,u) \in L_2(D_T)$ and also, if $u_m \to u$ in the space $\overset{\circ}{W}{}^1_{2,\mu}(D_T)$, then $f(x,t,u_m) \to f(x,t,u)$ in the space $L_2(D_T)$.

Definition 5.1.1. Let the function f satisfy the conditions (5.1.5) and (5.1.6), and $F \in L_2(D_T)$. We call a function u a generalized solution of the problem (5.1.1)–(5.1.4) if $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ and there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2_\mu(\overline{D}_T)$ such that

$$\lim_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} = 0, \quad \lim_{m\to\infty} \|L_\lambda u_m - F\|_{L_2(D_T)} = 0. \tag{5.1.10}$$

Note that the above definition of a generalized solution of the problem (5.1.1)–(5.1.4) remains valid in the linear case, that is, for $\lambda = 0$.

It is obvious that a classical solution $u \in C^2(\overline{D}_T)$ of the problem (5.1.1)–(5.1.4) represents a generalized solution of this problem. It is easily seen that a generalized solution of the problem (5.1.1)–(5.1.4) is a solution of the equation (5.1.1) in the sense of the theory of distributions. Indeed, let $F_m := L_\lambda u_m$. Multiplying both sides of the equality $L_\lambda u_m = F_m$ by a test function $w \in V_\mu := \{v \in W^1_2(D_T) : \ v|_\Gamma = 0, \ v(x,T) - \mu v(x,0) = 0, \ x \in \Omega\}$ and integrating in the domain D_T, after simple transformations connected with the integration by parts and the equality $w|_\Gamma = 0$, we get

$$\begin{aligned}\int\limits_\Omega \big[u_{mt}(x,T)w(x,T) - u_{mt}(x,0)w(x,0)\big]\,dx \\ + \int\limits_\Omega \Big[-u_{mt}w_t + \sum_{i=1}^n u_{mx_i}w_{x_i} + \lambda f(x,t,u_m)w\Big]\,dx\,dt = \int\limits_{D_T} F_m w\,dx\,dt \ \ \forall\, w \in V_\mu.\end{aligned} \tag{5.1.11}$$

Since $\mathcal{K}_\mu u_{mt} = 0$ and $w(x,T) - \mu w(x,0) = 0$, $x \in \Omega$, it is not difficult to see that

$$\begin{aligned}&u_{mt}(x,T)w(x,T) - u_{mt}(x,0)w(x,0) \\ &\qquad = u_{mt}(x,T)(w(x,T) - \mu w(x,0)) - w(x,0)(u_{mt}(x,0) - \mu u_{mt}(x,T)) = 0.\end{aligned}$$

Therefore, the equation (5.1.11) takes the form

$$\int\limits_{D_T} \Big[-u_{mt}w_t + \sum_{i=1}^n u_{mx_i}w_{x_i} + \lambda f(x,t,u_m)w\Big]\,dx\,dt = \int\limits_{D_T} F_m w\,dx\,dt \ \ \forall\, w \in V_\mu. \tag{5.1.12}$$

In view of (5.1.5), (5.1.6) and Remark 5.1.2, we have $f(x,t,u_m) \to f(x,t,u)$ in the space $L_2(D_T)$ as $u_m \to u$ in the space $\overset{\circ}{W}{}^1_{2,\mu}(D_T)$. Therefore, by (5.1.10), passing to the limit in the equation (5.1.12) as $m \to \infty$, we get

$$\int\limits_{D_T} \Big[-u_t w_t + \sum_{i=1}^{n} u_{x_i} w_{x_i} + \lambda f(x,t,u) w \Big] \, dx \, dt = \int\limits_{D_T} F w \, dx \, dt \ \ \forall \, w \in V_\mu. \tag{5.1.13}$$

Since $C_0^\infty(D_T) \subset V_\mu$, from (5.1.13), integrating by parts, we have

$$\int\limits_{D_T} u \, \Box \, w \, dx \, dt + \lambda \int\limits_{D_T} f(x,t,u) w \, dx \, dt = \int\limits_{D_T} F w \, dx \, dt \ \ \forall \, w \in C_0^\infty(D_T), \tag{5.1.14}$$

where $\Box := \frac{\partial^2}{\partial t^2} - \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, and $C_0^\infty(D_T)$ is the space of finite infinitely differentiable functions in D_T. The equality (5.1.14), valid for any $w \in C_0^\infty(D_T)$, implies that a generalized solution u of the problem (5.1.1)–(5.1.4) is a solution of the equation (5.1.1) in the sense of the theory of distributions. Besides, since the trace operators $u \to u|_{t=0}$ and $u \to u_{t=T}$ are continuous, acting from the space $W_2^1(D_T)$ into the spaces $L_2(\Omega \times \{t=0\})$ and $L_2(\Omega \times \{t=T\})$, respectively, owing to (5.1.10), the generalized solution u of the problem (5.1.1)–(5.1.4) satisfies the nonlocal condition (5.1.3) in the sense of the trace theory. As for the nonlocal condition (5.1.4), we have taken it into account in the integral sense in the equality (5.1.13), which is valid for all $w \in V_\mu$. Note also that if a generalized solution u belongs to the class $C^2(\overline{D}_T)$, then by the standard reasoning combined with the integral identity (5.1.13) [68], we have that u is a classical solution of the problem (5.1.1)–(5.1.4), satisfying the pointwise equation (5.1.1), the boundary condition (5.1.2) and the nonlocal conditions (5.1.3) and (5.1.4).

Remark 5.1.3. Note that even in the linear case, that is, for $\lambda = 0$, the problem (5.1.1)–(5.1.4) is not always well-posed. For example, when $\lambda = 0$ and $|\mu| = 1$, the corresponding to (5.1.1)–(5.1.4) homogeneous problem may have an infinite number of linearly independent solutions (see Remark 5.3.2).

5.2 A priori estimate of a solution of the problem (5.1.1)–(5.1.4)

Let

$$g(x,t,u) = \int\limits_0^u f(x,t,s) \, ds, \ \ (x,t,u) \in \overline{D}_T \times \mathbb{R}. \tag{5.2.1}$$

Consider the following conditions imposed on the function $g = g(x,t,u)$:

$$g(x,t,u) \geq 0, \ \ (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{5.2.2}$$

$$g_t \in C(\overline{D}_T \times \mathbb{R}), \ \ g_t(x,t,u) \leq M_3, \ \ (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{5.2.3}$$

$$g(x,0,\mu u) \leq \mu^2 g(x,T,u), \ \ (x,u) \in \overline{\Omega} \times \mathbb{R}, \tag{5.2.4}$$

where $M_3 = \mathit{const} \geq 0$, and μ is the fixed constant from (5.1.3)–(5.1.4).

Remark 5.2.1. Let us consider the class of functions f from (5.1.1) satisfying the conditions (5.1.5), (5.2.2), (5.2.3) and (5.2.4). For $\alpha = \beta + 1$, consider the function $f = f_0(t)|u|^\beta u$, where $f_0 \in C^1([0,T])$, $f_0 \geq 0$, $\frac{df_0}{dt} \leq 0$, $f_0(0)\mu^\beta \leq f_0(T)$, $\beta \geq 0$, and $\mu > 0$ is the fixed constant from (5.1.3)–(5.1.4). In particular, these conditions are satisfied if $f_0 = \mathit{const} > 0$ and $0 < \mu \leq 1$. Indeed, using these conditions, by (5.2.1), we have

$$g = \frac{f_0(t)|u|^{\beta+2}}{\beta+2}, \ \ g \geq 0, \ \ g_t \leq 0$$

and

$$g(x,0,\mu v) = \frac{f_0(0)|\mu v|^{\beta+2}}{\beta+2} = \frac{\mu^2 (f_0(0)\mu^\beta)|v|^{\beta+2}}{\beta+2} \leq \mu^2 f_0(T) \, \frac{|v|^{\beta+2}}{\beta+2} = \mu^2 g(x,T,v).$$

Lemma 5.2.1. *Let $\lambda > 0$, $|\mu| < 1$, $f \in C(\overline{D}_T \times \mathbb{R})$, $F \in L_2(D_T)$, and the conditions* (5.2.2)–(5.2.4) *be satisfied. Then for a generalized solution u of the problem* (5.1.1)–(5.1.4)*, we have the a priori estimate*

$$\|u\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} \leq c_1 \|F_1\|_{L_2(D_T)} + c_2 \tag{5.2.5}$$

with nonnegative constants $c_i = c_i(\lambda, \mu, \Omega, T, M_1, M_2, M_3)$, not depending on u and F, $c_1 > 0$, whereas in the linear case ($\lambda = 0$), the constant $c_2 = 0$, and in this case, by (5.2.5)*, we have the uniqueness of the generalized solution of the problem* (5.1.1)–(5.1.4).

Proof. Let u be a generalized solution of the problem (5.1.1)–(5.1.4). By Definition 5.1.1, there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2_\mu(D_T)$ such that the limit equalities (5.1.10) are satisfied.

Set

$$L_\lambda u_m = F_m, \ \ (x,t) \in D_T. \tag{5.2.6}$$

Multiplying both sides of the equation (5.2.6) by $2u_{mt}$ and integrating in the domain $D_\tau := D_T \cap \{t < \tau\}$, $0 < \tau \leq T$, by (5.2.1) we obtain

$$\int\limits_{D_\tau} \frac{\partial}{\partial t} \Big(\frac{\partial u_m}{\partial t}\Big)^2 \, dx\,dt - 2 \int\limits_{D_\tau} \sum_{i=1}^n \frac{\partial^2 u_m}{\partial x_i^2} \, \frac{\partial u_m}{\partial t} \, dx\,dt + 2\lambda \int\limits_{D_\tau} \frac{\partial}{\partial t} \, (g(x,t,u_m(x,t)) \, dx\,dt \\ - 2\lambda \int\limits_{D_\tau} g_t(x,t,u_m(x,t)) \, dx\,dt = 2 \int\limits_{D_\tau} F_m \, \frac{\partial u_m}{\partial t} \, dx\,dt. \tag{5.2.7}$$

Let $\omega_\tau := \{(x,t) \in \overline{D}_T : \ x \in \Omega, \ t = \tau\}$, $0 \leq \tau \leq T$, where ω_0 and ω_T are the upper and lower bases of the cylindrical domain D_T, respectively. Denote by $\nu := (\nu_{x_1}, \dots, \nu_{x_n}, \nu_t)$ the unit vector of the outer normal to ∂D_τ. Since

$$\nu_{x_i}\big|_{\omega_\tau \cup \omega_0} = 0, \quad i = 1, \dots, n, \\ \nu_t\big|_{\Gamma_\tau := \Gamma \cap \{t \leq \tau\}} = 0, \quad \nu_t\big|_{\omega_\tau} = 1, \quad \nu_t\big|_{\omega_0} = -1,$$

taking into account that $u_m \in \overset{\circ}{C}{}^2_\mu(D_T)$ and, therefore, by (5.1.7),

$$u_m\big|_\Gamma = 0, \quad \mathcal{K}_\mu u_m = 0, \quad \mathcal{K}_\mu u_{mt} = 0, \tag{5.2.8}$$

after integrating by parts we obtain

$$\int\limits_{D_\tau} \frac{\partial}{\partial t} \Big(\frac{\partial u_m}{\partial t}\Big)^2 \, dx\,dt = \int\limits_{\partial D_\tau} \Big(\frac{\partial u_m}{\partial t}\Big)^2 \nu_t \, ds = \int\limits_{\omega_\tau} u_{mt}^2 \, dx - \int\limits_{\omega_0} u_{mt}^2 \, dx, \tag{5.2.9}$$

$$-2 \int\limits_{D_\tau} \frac{\partial^2 u_m}{\partial x_i^2} \, \frac{\partial u_m}{\partial t} \, dx\,dt = \int\limits_{D_\tau} \big[(u_{mx_i}^2)_t - 2(u_{mx_i} u_{mt})_{x_i}\big] \, dx\,dt \\ = \int\limits_{\omega_\tau} u_{mx_i}^2 \, dx - \int\limits_{\omega_0} u_{mx_i}^2 \, dx, \ \ i = 1, \dots, n, \tag{5.2.10}$$

$$2\lambda \int\limits_{D_\tau} \frac{\partial}{\partial t} \, (g(x,t,u_m(x,t))) \, dx\,dt = 2\lambda \int\limits_{\partial D_\tau} g(x,t,u_m(x,t)) \nu_t \, ds \\ = 2\lambda \int\limits_{\omega_\tau} g(x,t,u_m(x,t)) \, dx - 2\lambda \int\limits_{\omega_0} g(x,t,u_m(x,t)) \, dx. \tag{5.2.11}$$

In view of (5.2.9)–(5.2.11), from (5.2.7) we get

$$\int\limits_{\omega_\tau}\Big[u_{mt}^2+\sum_{i=1}^n u_{mx_i}^2\Big]\,dx=\int\limits_{\omega_0}\Big[u_{mt}^2+\sum_{i=1}^n u_{mx_i}^2\Big]\,dx-2\lambda\int\limits_{\omega_\tau} g(x,t,u_m(x,t))\,dx \\ +2\lambda\int\limits_{\omega_0} g(x,t,u_m(x,t))\,dx+2\lambda\int\limits_{\omega_\tau} g_t(x,t,u_m(x,t))\,dx\,dt+2\int\limits_{D_\tau} F_m u_{mt}\,dx\,dt. \quad (5.2.12)$$

Let

$$w_m(\tau):=\int\limits_{\omega_\tau}\Big[u_{mt}^2+\sum_{i=1}^n u_{mx_i}^2+2\lambda g(x,t,u_m(x,t))\Big]\,dx. \quad (5.2.13)$$

Since $2F_m u_{mt}\le \varepsilon^{-1}F_m^2+\varepsilon u_{mt}^2$ for any $\varepsilon=const>0$ and also since $\lambda>0$, by (5.2.3) and (5.2.13), from (5.2.12) it follows that

$$w_m(\tau)=w_m(0)+2\lambda\int\limits_{D_\tau} g_t(x,t,u_m(x,t))\,dx\,dt+2\int\limits_{D_\tau}F_m u_{mt}\,dx\,dt \\ \le w_m(0)+2\lambda M_3\tau\operatorname{mes}\Omega+\varepsilon\int\limits_{D_\tau}u_{mt}^2\,dx\,dt+\varepsilon^{-1}\int\limits_{D_\tau}F_m^2\,dx\,dt. \quad (5.2.14)$$

Since $\lambda>0$, taking into account (5.2.2) and the inequality

$$\int\limits_{D_\tau}u_{mt}^2\,dx\,dt=\int\limits_0^\tau\Big[\int\limits_{\omega_s}u_{mt}^2\,dx\Big]\,ds \\ \le\int\limits_0^\tau\Big[\int\limits_{\omega_s}\Big[u_{mt}^2+\sum_{i=1}^n u_{mx_i}^2+2\lambda g(x,t,u_m(x,t))\Big]\,dx\Big]\,ds=\int\limits_0^\tau w_m(s)\,ds,$$

from (5.2.14) we obtain

$$w_m(\tau)\le\varepsilon\int\limits_0^\tau w_m(s)\,ds+w_m(0)+2\lambda M_3\tau\operatorname{mes}\Omega+\varepsilon^{-1}\int\limits_{D_\tau}F_m^2\,dx\,dt,\ \ 0<\tau\le T. \quad (5.2.15)$$

Because of $D_\tau\subset D_T$, $0<\tau\le T$, the right-hand side of the inequality (5.2.15) is a nondecreasing function of the variable τ, and by the Gronwall lemma, it follows from (5.2.15) that

$$w_m(\tau)\le\Big[w_m(0)+2\lambda M_3T\operatorname{mes}\Omega+\varepsilon^{-1}\int\limits_{D_\tau}F_m^2\,dx\,dt\Big]e^{\varepsilon\tau},\ \ 0<\tau\le T. \quad (5.2.16)$$

In view of $\lambda>0$, by (5.2.4) and (5.2.8), from (5.2.13) follows

$$w_m(0)=\int\limits_\Omega\Big[u_{mt}^2(x,0)+\sum_{i=1}^n u_{mx_i}^2(x,0)+2\lambda g(x,0,u_m(x,0))\Big]\,dx \\ =\int\limits_\Omega\Big[\mu^2u_{mt}^2(x,T)+\mu^2\sum_{i=1}^n u_{mx_i}^2(x,T)+2\lambda g(x,0,\mu u_m(x,T))\Big]\,dx \\ \le\mu^2\int\limits_\Omega\Big[u_{mt}^2(x,T)+\sum_{i=1}^n u_{mx_i}^2(x,T)+2\lambda g(x,T,u_m(x,T))\Big]\,dx=\mu^2 w_m(T). \quad (5.2.17)$$

Using the inequality (5.2.16) for $\tau = T$, from (5.2.17) we obtain

$$w_m(0) \le \mu^2 w_m(T) \le \mu^2 \Big[w_m(0) + 2\lambda M_3 T \operatorname{mes}\Omega + \varepsilon^{-1} \int\limits_{D_T} F_m^2 \, dx\, dt \Big] e^{\varepsilon T}$$
$$= \mu^2 e^{\varepsilon T} w_m(0) + M_4 + \mu^2 \varepsilon^{-1} e^{\varepsilon T} \|F_m\|^2_{L_2(D_T)}, \quad (5.2.18)$$

where

$$M_4 := \mu^2 2\lambda M_3 T e^{\varepsilon T} \operatorname{mes}\Omega. \quad (5.2.19)$$

Since $|\mu| < 1$, a positive constant $\varepsilon = \varepsilon(\mu, T)$ can be chosen insomuch small that

$$\mu_1 = \mu^2 e^{\varepsilon T} < 1. \quad (5.2.20)$$

For example, we can set $\varepsilon = \frac{1}{T} \ln \frac{1}{|\mu|}$.

By (5.2.20), from (5.2.18), we have

$$w(0) \le (1-\mu_1)^{-1} M_4 + (1-\mu_1)^{-1} \mu^2 \varepsilon^{-1} e^{\varepsilon T} \|F_m\|^2_{L_2(D_T)}. \quad (5.2.21)$$

From (5.2.16) and (5.2.21) it follows that

$$w_m(\tau) \le \Big[(1-\mu_1)^{-1} M_4 + (1-\mu_1)^{-1} \mu^2 \varepsilon^{-1} e^{\varepsilon T} \|F_m\|^2_{L_2(D_T)}$$
$$+ 2\lambda M_3 T \operatorname{mes}\Omega + \varepsilon^{-1} \|F\|^2_{L_2(D_T)} \Big] e^{\varepsilon T} \le \sigma_1 \|F_m\|^2_{L_2(D_T)} + \sigma_2, \;\; 0 < \tau \le T, \quad (5.2.22)$$

where

$$\sigma_1 = \big[(1-\mu_1)^{-1} \mu^2 e^{\varepsilon T} + 1 \big] \varepsilon^{-1} e^{\varepsilon T}, \quad \sigma_2 = \big[(1-\mu_1)^{-1} M_4 + 2\lambda M_3 T \operatorname{mes}\Omega \big] e^{\varepsilon T}. \quad (5.2.23)$$

Since, for the fixed τ, the function $u_m(x,\tau)$ belongs to the space $\overset{\circ}{W}{}^1_2(\Omega) := \{ v \in W^1_2(\Omega) : \; v|_{\partial\Omega} = 0\}$, by the Friedrichs inequality [68], taking into account (5.2.2) and $\lambda > 0$, we have

$$\int\limits_{\omega_\tau} \Big[u_m^2 + u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2 \Big] dx$$
$$\le c_0 \int\limits_{\omega_\tau} \Big[u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2 \Big] dx \le c_0 \int\limits_{\omega_\tau} \Big[u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2 + \lambda g(x,t,u_m(x,t)) \Big] dx = c_0 w_m(\tau), \quad (5.2.24)$$

where the positive constant $c_0 = c_0(\Omega)$ does not depend on u_m.

From (5.2.22) and (5.2.24) it follows that

$$\|u_m\|^2_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} = \int\limits_0^T \Big[\int\limits_{\omega_\tau} \Big(u_m^2 + u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2 \Big) dx \Big] d\tau$$
$$\le c_0 \int\limits_0^T w_m(\tau)\, d\tau \le c_0 \int\limits_0^T \big[\sigma_1 \|F\|^2_{L_2(D_T)} + \sigma_2 \big] d\tau = c_0 \sigma_1 T \|F_m\|^2_{L_2(D_T)} + c_0 \sigma_2 T. \quad (5.2.25)$$

Extracting the square root from both sides of the inequality (5.2.25) and using the inequality $(a^2+b^2)^{1/2} \le |a| + |b|$, we get

$$\|u_m\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} \le c_1 \|F_m\|_{L_2(D_T)} + c_2, \quad (5.2.26)$$

where

$$c_1 = \Big(c_0 T \big[(1-\mu_1)^{-1} \mu^2 e^{\varepsilon T} + 1 \big] \varepsilon^{-1} e^{\varepsilon T} \Big)^{1/2},$$
$$c_2 = \Big(c_0 T \big[(1-\mu_1)^{-1} \mu^2 2\lambda M_3 T e^{\varepsilon T} \operatorname{mes}\Omega + 2\lambda M_3 T \operatorname{mes}\Omega \big] e^{\varepsilon T} \Big)^{1/2}. \quad (5.2.27)$$

In view of the limit equalities (5.1.10), passing to the limit in the inequality (5.2.26) as $m \to \infty$, we obtain (5.2.5). This proves Lemma 5.2.1. □

5.3 The existence of a solution of the problem (5.1.1)–(5.1.4)

For the existence of a solution of the problem (5.1.1)–(5.1.4) in the case $|\mu| < 1$, we will use the well-known facts on the solvability of the following linear mixed problem [68]:

$$L_\theta u := \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = F(x,t), \ \ (x,t) \in D_T, \tag{5.3.1}$$

$$u\big|_\Gamma = 0, \quad u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \ \ x \in \Omega, \tag{5.3.2}$$

where F, φ and ψ are the given functions.

For $F \in L_2(D_T)$, $\varphi \in \overset{\circ}{W}{}_2^1(\Omega)$ and $\psi \in L_2(\Omega)$, the unique generalized solution u of the problem (5.3.1), (5.3.2) (in the sense of the integral identity

$$-\int_\Omega \psi w(x,0)\,dx + \int_{D_T} \Big[-u_t w_t + \sum_{i=1}^{n} u_{x_i} w_{x_i} \Big]\,dx\,dt = \int_{D_T} Fw\,dx\,dt \ \ \forall\, w \in V_0,$$

where $V_0 := \{v \in W_2^1(D_T) :\ v|_\Gamma = 0,\ v(x,T) = 0,\ x \in \Omega\}$ and $u|_{t=0} = \varphi$) from the space $E_{2,1}(D_T)$ with the norm

$$\|v\|^2_{E_{2,1}(D_T)} = \sup_{0\le\tau\le T} \int_{\omega_\tau} \Big[v^2 + v_t^2 + \sum_{i=1}^{n} v_{x_i}^2 \Big]\,dx$$

is given by the formula [68]

$$u = \sum_{k=1}^{\infty} \bigg(\widetilde{a}_k \cos\mu_k t + \widetilde{b}_k \sin\mu_k t + \frac{1}{\mu_k} \int_0^t F_k(\tau) \sin\mu_k(t-\tau)\,d\tau \bigg) \varphi_k(x), \tag{5.3.3}$$

where $\widetilde{\lambda}_k = -\mu_k^2$ $(0 < \mu_1 \le \mu_2 \le \cdots,\ \lim\limits_{k\to\infty} \mu_k = \infty)$ and $\varphi_k \in \overset{\circ}{W}{}_2^1(\Omega)$ are the eigenvalues and the corresponding eigenfunctions of the spectral problem $\Delta w = \widetilde{\lambda} w$, $w|_{\partial\Omega} = 0$ in the domain Ω $(\Delta := \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2})$, forming simultaneously an orthonormal basis in $L_2(\Omega$ and an orthogonal basis in $\overset{\circ}{W}{}_2^1(\Omega)$ with respect to the scalar product $(v,w)_{\overset{\circ}{W}{}_2^1(\Omega)} = \int\limits_\Omega \sum\limits_{i=1}^{n} v_{x_i} w_{x_i}\,dx$ [68], that is,

$$(\varphi_k, \psi_l)_{L_2(\Omega)} = \delta_k^l, \ \ (\varphi_k, \varphi_l)_{\overset{\circ}{W}{}_2^1(\Omega)} = -\widetilde{\lambda}_k \delta_k^l, \ \ \delta_k^l = \begin{cases} 1, & l = k, \\ 0, & l \ne k. \end{cases} \tag{5.3.4}$$

Here,

$$\widetilde{a}_k = (\varphi, \varphi_k)_{L_2(\Omega)}, \ \ \widetilde{b}_k = \mu_k^{-1} (\psi, \varphi_k)_{L_2(\Omega)}, \ \ k = 1, 2, \dots, \tag{5.3.5}$$

$$F(x,t) = \sum_{k=1}^{\infty} F_k(t) \varphi_k(x), \ \ F_k(t) = (F, \varphi_k)_{L_2(\omega_t)}, \ \ \omega_t := D_T \cap \{t = \tau\}. \tag{5.3.6}$$

Besides, for the solution u from (5.3.3), we have the following estimate

$$\|u\|_{E_{2,1}(D_T)} \le \gamma \big(\|F\|_{L_2(D_T)} + \|\varphi\|_{\overset{\circ}{W}{}_2^1(\Omega)} + \|\psi\|_{L_2(\Omega)} \big) \tag{5.3.7}$$

with the positive constant γ, independent of F, φ and ψ [68, 75].

Let us consider the linear problem corresponding to (5.1.1)–(5.1.4), that is, the case $\lambda = 0$:

$$L_0 u := \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = F(x,t), \ \ (x,t) \in D_T, \tag{5.3.8}$$

$$u\big|_\Gamma = 0, \tag{5.3.9}$$

$$u(x,0) - \mu u(x,T) = 0, \quad u_t(x,0) - \mu u_t(x,T) = 0, \ \ x \in \Omega. \tag{5.3.10}$$

Let us show that when $|\mu| < 1$, for any $F \in L_2(D_T)$, there exists a unique generalized solution of the problem (5.3.8)–(5.3.10). Indeed, since the space of finite infinitely differentiable functions $C_0^\infty(D_T)$ is dense in the space $L_2(D_T)$, for $F \in L_2(D_T)$ and any natural number m, there exists a function $F_m \in C_0^\infty(D_T)$ such that

$$\|F_m - F\|_{L_2(D_T)} < \frac{1}{m}\,. \tag{5.3.11}$$

On the other hand, for a function F_m in the space $L_2(D_T)$, we have the following expansions [68]:

$$F_m(X,t) = \sum_{k=1}^{\infty} F_{m,k}(t)\varphi_k(x), \quad F_{m,k}(t) = (F_m, \varphi_k)_{L_2(\Omega)}. \tag{5.3.12}$$

Therefore, there exists a natural number ℓ_m such that $\lim\limits_{m\to\infty} \ell_m = \infty$ and, for

$$\widetilde{F}_m(x,t) = \sum_{k=1}^{\ell_m} F_{m,k}(t)\varphi_k(x), \tag{5.3.13}$$

we have

$$\|\widetilde{F}_m - F_m\|_{L_2(D_T)} < \frac{1}{m}\,. \tag{5.3.14}$$

From (5.3.11) and (5.3.14) it follows that

$$\lim_{m\to\infty} \|\widetilde{F}_m - F\|_{L_2(D_T)} = 0. \tag{5.3.15}$$

The solution $u = u_m$ of the problem (5.3.1), (5.3.2) for

$$\varphi = \sum_{k=1}^{\ell_m} \widetilde{a}_k\varphi_k, \quad \psi = \sum_{k=1}^{\ell_m} \mu_k\widetilde{b}_k\varphi_k, \quad F = \widetilde{F}_m,$$

is given by the formula (5.3.3), which by (5.3.4)–(5.3.6) and (5.3.13) can be rewritten as follows:

$$u_m = \sum_{k=1}^{\ell_m} \bigg(\widetilde{a}_k \cos\mu_k t + \widetilde{b}_k \sin\mu_k t + \frac{1}{\mu_k}\int_0^t F_{mk}(\tau)\sin\mu_k(t-\tau)\,d\tau\bigg)\varphi_k(x). \tag{5.3.16}$$

By the construction, the function u_m from (5.3.16) satisfies the equation (5.3.8) and the boundary condition (5.3.9) for $F = \widetilde{F}_m$ from (5.3.13). Let us define unknown coefficients $\widetilde{a}_k$ and $\widetilde{b}_k$ such that the function u_m from (5.3.16) would satisfy the nonlocal conditions (5.3.10), too. Towards this end, let us substitute the right-hand side of the expression (5.3.16) into the equalities (5.3.10). As a result, since the system of functions $\{\varphi_k(x)\}$ forms a basis in $L_2(\Omega)$, for defining the coefficients $\widetilde{a}_k$ and $\widetilde{b}_k$, we have the following system of linear algebraic equations:

$$\begin{gathered}
(1-\mu\cos\mu_k T)\widetilde{a}_k - (\mu\sin\mu_k T)\widetilde{b}_k = \frac{\mu}{\mu_k}\int_0^T F_{m,k}(\tau)\sin\mu_k(T-\tau)\,d\tau, \\
(\mu\mu_k\sin\mu_k T)\widetilde{a}_k + \mu_k(1-\mu\cos\mu_k T)\widetilde{b}_k = \mu\int_0^T F_{m,k}(\tau)\cos\mu_k(T-\tau)\,d\tau,
\end{gathered} \tag{5.3.17}$$

$k = 1, 2, \dots, \ell_m$. Its solution is

$$\widetilde{a}_k = \big[d_{1k}\mu\mu_k\sin\mu_k T - d_{2k}(1-\mu\cos\mu_k T)\big]\Delta_k^{-1}, \quad k = 1,\dots,\ell_m, \tag{5.3.18}$$

$$\widetilde{b}_k = \big[d_{2k}(1-\mu\cos\mu_k T) - d_{1k}\mu\mu_k\sin\mu_k T\big]\Delta_k^{-1}, \quad k = 1,\dots,\ell_m. \tag{5.3.19}$$

Here,

$$d_{1k} = \frac{\mu}{\mu_k} \int\limits_0^T F_{m,k}(\tau) \sin \mu_k (T-\tau)\, d\tau, \quad d_{2k} = \mu \int\limits_0^T F_{m,k}(\tau) \cos \mu_k (T-\tau)\, d\tau,$$

and since $|\mu| < 1$, for the determinant Δ_k of the system (5.3.17) we have

$$\Delta_k = \mu_k \big[(1-\mu \cos \mu_k T)^2 + \mu^2 \sin^2 \mu_k T \big] \geq \mu_k (1-|\mu|)^2 > 0. \tag{5.3.20}$$

Below, we assume that the Lipschitz domain Ω is such that the eigenfunctions $\varphi_k \in C^2(\overline{\Omega})$, $k \geq 1$. For example, this will take place if $\partial\Omega \in C^{[\frac{n}{2}]+3}$ [75]. This fact will also take place in the case of a piecewise smooth Lipschitz domain, e.g., for the parallelepiped $\Omega = \{x \in \mathbb{R}^n : |x_i| < a_i,\ i = 1, \dots, n\}$ the corresponding eigenfunctions $\varphi_k \in C^\infty(\Omega)$ [76] (see also Remark 5.3.2). Therefore, since $F_m \in C_0^\infty(D_T)$, due to (5.3.12), the function $F_{m,k} \in C^2([0,T])$ and, consequently, the function u_m from (5.3.16) belongs to the space $C^2(\overline{D}_T)$. Further, according to the construction, the function u_m from (5.3.16) will belong to the space $\overset{\circ}{C}{}^2_\mu(D_T)$ which is defined in (5.1.7), besides,

$$L_0 u_m = \widetilde{F}_m, \quad L_0(u_m - u_k) = \widetilde{F}_m - \widetilde{F}_k. \tag{5.3.21}$$

From (5.3.21) and the a priori estimate (5.2.5), when $\lambda = 0$, and due to Lemma 5.2.1, the coefficient $c_2 = 0$, we have

$$\|u_m - u_k\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} \leq c_1 \|\widetilde{F}_m - \widetilde{F}_k\|_{L_2(D_T)}. \tag{5.3.22}$$

In view of (5.3.15), from (5.3.22) it follows that the sequence $u_m \in \overset{\circ}{C}{}^2_\mu(D_T)$ is fundamental in the complete space $\overset{\circ}{W}{}^1_{2,\mu}(D_T)$. Therefore, there exists a function $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ such that, due to (5.3.15) and (5.3.21), the limit equalities (5.1.10) are valid for $\lambda = 0$. This implies that the function u is a generalized solution of the problem (5.3.8)–(5.3.10). The uniqueness of this solution follows from the a priori estimate (5.2.5), where the constant $c_2 = 0$ for $\lambda = 0$, i.e.,

$$\|u\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} \leq c_1 \|F\|_{L_2(D_T)}. \tag{5.3.23}$$

Therefore, for the solution u of the problem (5.3.8)–(5.3.10), we have $u = L_0^{-1}(F)$, where $L_0^{-1} : L_2(D_T) \to \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ is a linear continuous operator whose norm, due to (5.2.23), can be estimated as follows:

$$\|L_0^{-1}\|_{L_2(D_T) \to \overset{\circ}{W}{}^1_{2,\mu}(D_T)} \leq c_1. \tag{5.3.24}$$

Remark 5.3.1. Note that when the conditions (5.1.5), (5.1.6) are fulfilled and $F \in L_2(D_T)$, due to (5.3.24) and Remark 5.1.2, the function $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ is a generalized solution of the problem (5.1.1)–(5.1.4) in the sense of Definition 5.1.1 if and only if u is a solution of the following functional equation

$$u = L_0^{-1}(-\lambda f(x,t,u)) + L_0^{-1}(F) \tag{5.3.25}$$

in the space $\overset{\circ}{W}{}^1_{2,\mu}(D_T)$.

Rewrite the equation (5.3.25) in the form

$$u = A_0 u := -\lambda L_0^{-1}(\mathcal{N}_0 u) + L_0^{-1}(F), \tag{5.3.26}$$

where the operator $\mathcal{N}_0 : \overset{\circ}{W}{}^1_{2,\mu}(D_T) \to L_2(D_T)$ from (5.1.9) is, according to Remark 5.1.2, continuous and compact. Therefore, due to (5.3.24), the operator $\mathcal{A}_0 : \overset{\circ}{W}{}^1_{2,\mu}(D_T) \to \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ from (5.3.26) is also continuous and compact for $0 \leq \alpha < \frac{n+1}{n-1}$. At the same time, according to Lemma 5.2.1 and (5.2.27), for any parameter $\tau \in [0,1]$ and for any solution u of the equation $u = \tau \mathcal{A}_0 u$ with the parameter τ, the same a priori estimate (5.2.5) with nonnegative constants c_i, independent of u, F and τ, is valid. Therefore, due to Schaefer's fixed point theorem [20], the equation (5.3.26) and hence, due to Remark 5.3.1, the problem (5.1.1)–(5.1.4) has at least one solution $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$. Thus, we have proved the following

Theorem 5.3.1. *Let $\lambda > 0$, $|\mu| < 1$ and the conditions* (5.1.5), (5.1.6), (5.2.2)–(5.2.4) *be fulfilled. Then for any $F \in L_2(D_T)$, the problem* (5.1.1)–(5.1.4) *has at least one generalized solution $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ in the sense of Definition* 5.1.1.

Remark 5.3.2. Note that for $|\mu| = 1$, even in the linear case, i.e., for $f = 0$, the homogeneous problem corresponding to (5.1.1)–(5.1.4) may have a finite or even an infinite number of linearly independent solutions, while for the solvability of this problem the function $F \in L_2(D_T)$ must satisfy a finite or an infinite number of conditions of the form $\ell(F) = 0$, respectively, where ℓ is a continuous functional in $L_2(D_T)$. Indeed, in the case $\mu = 1$, denote by $\Lambda(1)$ a set of those numbers μ_k from (5.3.3) for which the ratio $\frac{\mu_k T}{2\pi}$ is a natural number, i.e., $\Lambda(1) = \{\mu_k : \frac{\mu_k T}{2\pi} \in \mathbb{N}\}$. The formulas (5.3.18), (5.3.19) for determination of unknown coefficients $\widetilde{a}_k$ and $\widetilde{b}_k$ in the representation (5.3.16) are obtained from the system of linear algebraic equations (5.3.17). In the case $\Lambda(1) \neq \varnothing$ and $\mu_k \in \Lambda(1)$, $\mu = 1$, the determinant Δ_k of the system (5.3.17), given by (5.3.20), equals zero. Moreover, in this case, all coefficients in front of the unknowns $\widetilde{a}_k$ and $\widetilde{b}_k$ in the left-hand side of the system (5.3.17) equal zero. Therefore, due to (5.3.16), the homogeneous problem corresponding to (5.3.8)–(5.3.10) will be satisfied by the function

$$u_k(x,t) = (C_1 \cos \mu_k t + C_2 \sin \mu_k t)\varphi_k(x), \tag{5.3.27}$$

where C_1 and C_2 are arbitrary constant numbers, and besides, in view of (5.3.17), the necessary conditions for the solvability of the nonhomogeneous problem (5.3.8)–(5.3.10) corresponding to $\mu_k \in \Lambda(1)$, are the following conditions

$$\begin{aligned} \ell_{k,1}(F) &= \int\limits_{D_T} F(x,t)\varphi_k(x) \sin \mu_k(T-t)\, dx\, dt = 0, \\ \ell_{k,2}(F) &= \int\limits_{D_T} F(x,t)\varphi_k(x) \cos \mu_k(T-t)\, dx\, dt = 0. \end{aligned} \tag{5.3.28}$$

Analogously, in the case $\mu = -1$, we denote by $\Lambda(-1)$ the set of points μ_k from (5.3.3) for which the ratio $\frac{\mu_k T}{\pi}$ is an odd integer. For $\mu_k \in \Lambda(-1)$, $\mu = -1$, the function u_k from (5.3.27) is also a solution of the homogeneous problem, corresponding to (5.3.8)–(5.3.10), and the conditions (5.3.28) are the corresponding necessary conditions for the solvability of this problem. For example, when $n = 2$, $\Omega = (0,1) \times (0,1)$, the eigenvalues and eigenfunctions of the Laplace operator Δ are [76]

$$\lambda_k = -\pi^2(k_1^2 + k_2^2), \quad \varphi_k(x_1,x_2) = 2 \sin k_1 \pi x_1 \cdot \sin k_2 \pi x_2, \ \ k = (k_1,k_2),$$

that is, $\mu_k = \pi\sqrt{k_1^2 + k_2^2}$. For $k_1 = p^2 - q^2$, $k_2 = 2pq$, where p and q are any integers, we obtain $\mu_k = \pi(p^2+q^2)$. In this case, for $\frac{T}{2} \in \mathbb{N}$, we have $\frac{\mu_k T}{2\pi} = \frac{(p^2+q^2)T}{2} \in \mathbb{N}$, and according to the above-said, when $\mu = 1$, the homogeneous problem, corresponding to (5.3.8)–(5.3.10), has an infinite number of linearly independent solutions

$$u_{p,q}(x,t) = \big[C_1 \cos \pi(p^2+q^2)t + C_2 \sin \pi(p^2+q^2)t\big] \sin(p^2-q^2)\pi x_1 \cdot \sin 2pq\pi x_2$$

for any integers p and q. Analogously, when $\mu = -1$, the solutions of the homogeneous problem corresponding to (5.3.8)–(5.3.10) in case p is even, while q and T are odd, are the functions from (5.3.27).

5.4 The uniqueness of a solution of the problem (5.1.1)–(5.1.4)

On the function f in the equation (5.1.1) we impose the following additional requirements:

$$f, f'_u \in C(\overline{D}_T \times \mathbb{R}), \quad |f'_u(x,t,u)| \leq a + b|u|^\gamma, \ \ (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{5.4.1}$$

where $a, b, \gamma = const \geq 0$.

It is obvious that from (5.4.1) we have the condition (5.1.5) for $\alpha = \gamma + 1$, and when $\gamma < \frac{2}{n-1}$, we have $\alpha = \gamma + 1 < \frac{n+1}{n-1}$.

Theorem 5.4.1. *Let $\lambda > 0$, $|\mu| < 1$, $F \in L_2(D_T)$ and the condition* (5.4.1) *be fulfilled for $\gamma < \frac{2}{n-1}$, and also the conditions* (5.2.2)–(5.2.4) *hold. Then there exists a positive number $\lambda_0 = \lambda_0(F, f, \mu, D_T)$ such that for $0 < \lambda < \lambda_0$, the problem* (5.1.1)–(5.1.4) *has no more than one generalized solution in the sense of Definition* 5.1.1.

Proof. Indeed, suppose that the problem (5.1.1)–(5.1.4) has two different generalized solutions u_1 and u_2. According to Definition 5.1.1, there exist sequences of functions $\mu_{jk} \in \overset{\circ}{C}{}^2_\mu(D_T)$, $j = 1, 2$, such that

$$\lim_{k\to\infty} \|u_{jk} - u_j\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} = 0, \;\; j = 1, 2, \quad \lim_{k\to\infty} \|L_\lambda u_{jk} - F\|L_{2(D_T)} = 0. \tag{5.4.2}$$

Let

$$w := u_2 - u_1, \quad w_k := u_{2k} - u_{1k}, \quad F_k := L_\lambda u_{2k} - L_\lambda u_{1k}, \tag{5.4.3}$$

$$g_k := \lambda\big(f(x,t,u_{2k}) - f(x,t,u_{1k})\big). \tag{5.4.4}$$

From (5.4.2) and (5.4.3), it is easy to see that

$$\lim_{k\to\infty} \|w_k - w\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} = 0, \quad \lim_{k\to\infty} \|F_k\|_{L_2(D_T)} = 0. \tag{5.4.5}$$

In view of (5.4.3) and (5.4.4), the function $w_k \in \overset{\circ}{C}{}^2_\mu(\overline{D}_T)$ satisfies the following equalities:

$$\frac{\partial^2 w_k}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2 w_k}{\partial x_i^2} = (F_k + g_k)(x,t), \;\; (x,t) \in D_T, \tag{5.4.6}$$

$$w_k\big|_\Gamma = 0, \;\; w_k(x,0) - \mu w_k(x,T) = 0, \;\; w_{kt}(x,0) - \mu w_{kt}(x,T) = 0, \;\; x \in \Omega. \tag{5.4.7}$$

First, let us estimate the function g_k from (5.4.4). Taking into account the obvious inequality $|d_1 + d_2|^\gamma \leq 2^\gamma \max(|d_1|^\gamma, |d_2|^\gamma) \leq 2^\gamma(|d_1|^\gamma + |d_2|^\gamma)$ for $\gamma > 0$, due to (5.4.1), we have

$$\begin{aligned}
&\big|f(x,t,u_{2k}) - f(x,t,u_{1k})\big| \\
&= \bigg|(u_{2k} - u_{1k} \int_0^1 f'_u\big(x,t,u_{1k} + \tau(u_{2k} - u_{1k})\big)\,d\tau\bigg| \leq |u_{2k} - u_{1k}| \int_0^1 \big(a + b|(1-\tau)u_{1k} + \tau u_{2k}|^\gamma\big)\,d\tau \\
&\leq a|u_{2k} - u_{1k}| + 2^\gamma b|u_{2k} - u_{1k}|\big(|u_{1k}|^\gamma + |u_{2k}|^\gamma\big) = a|w_k| + 2^\gamma b|w_k|\big(|u_{1k}|^\gamma + |u_{2k}|^\gamma\big).
\end{aligned} \tag{5.4.8}$$

In view of (5.4.4), from (5.4.8) we have

$$\begin{aligned}
\|g_k\|_{L_2(D_T)} &\leq \lambda a\|w_k\|_{L_2(D_T)} + \lambda\, 2^\gamma b\big\|\, |w_k|\big(|u_{1k}|^\gamma + |u_{2k}|^\gamma\big)\big\|_{L_2(D_T)} \\
&\leq \lambda a\|w_k\|_{L_2(D_T)} + \lambda\, 2^\gamma b\|w_k\|_{L_p(D_T)}\big\|\big(|u_{1k}|^\gamma + |u_{2k}|^\gamma\big)\big\|_{L_q(D_T)}.
\end{aligned} \tag{5.4.9}$$

Here we have used Hölder's inequality [24]

$$\|v_1 v_2\|_{L_r(D_T)} \leq \|v_1\|_{L_p(D_T)}\|v_2\|_{L_q(D_T)},$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and in the capacity of p, q and r we took

$$p = 2\,\frac{n+1}{n-1}, \;\; q = n+1, \;\; r = 2. \tag{5.4.10}$$

Since $\dim D_T = n + 1$, according to Sobolev's embedding theorem [22], for $1 \leq p \leq \frac{2(n+1)}{n-1}$, we have

$$\|v\|_{L_p(D_T)} \leq C_p\|v\|_{W_2^1(D_T)} \;\; \forall\, v \in W_2^1(D_T) \tag{5.4.11}$$

with the positive constant C_p, not depending on $v \in W_2^1(D_T)$.

Due to the condition of the theorem, $\gamma < \frac{2}{n-1}$, and therefore, $\gamma(n+1) < \frac{2(n+1)}{n-1}$. Thus, due to (5.4.10), from (5.4.11) we have

$$\|w_k\|_{L_p(D_T)} \leq C_p \|w_k\|_{W_2^1(D_T)}, \quad p = \frac{2(n+1)}{n-1} \quad k \geq 1, \tag{5.4.12}$$

$$\begin{aligned} \big\| \big(|u_{1k}|^\gamma + |u_{2k}|^\gamma\big) \big\|_{L_q(D_T)} &\leq \big\| \, |u_{1k}|^\gamma \big\|_{L_q(D_T)} + \big\| \, |u_{2k}|^\gamma \big\|_{L_q(D_T)} \\ &= \|u_{1k}\|^\gamma_{L_{\gamma(n+1)}(D_T)} + \|u_{2k}\|^\gamma_{L_{\gamma(n+1)}(D_T)} \leq C^\gamma_{\gamma(n+1)} \big(\|u_{1k}\|^\gamma_{W_2^1(D_T)} + \|u_{2k}\|^\gamma_{W_2^1(D_T)}\big). \end{aligned} \tag{5.4.13}$$

In view of the first equality of (5.4.2), there exists a natural number k_0 such that for $k \geq k_0$, we have

$$\|u_{ik}\|^\gamma_{W_2^1(D_T)} \leq \|u_i\|^\gamma_{W_2^1(D_T)} + 1, \quad i = 1, 2; \quad k \geq k_0. \tag{5.4.14}$$

Further, in view of (5.4.12), (5.4.13) and (5.4.14), from (5.4.9), we have

$$\begin{aligned} \|g_k\|_{L_2(D_T)} &\leq \lambda a \|w_k\|_{L_2(D_T)} \\ &\quad + \lambda\, 2^\gamma b C_p C^\gamma_{\gamma(n+1)} \big(\|u_1\|^\gamma_{W_2^1(D_T)} + \|u_2\|^\gamma_{W_2^1(D_T)} + 2\big) \|w_k\|_{W_2^1(D_T)} \leq \lambda M_5 \|w_k\|_{W_2^1(D_T)}, \end{aligned} \tag{5.4.15}$$

where we have used the inequality $\|w_k\|_{L_2(D_T)} \leq \|w_k\|_{W_2^1(D_T)}$,

$$M_5 = a + 2^\gamma b C_p C^\gamma_{\gamma(n+1)} \big(\|u_1\|^\gamma_{W_2^1(D_T)} + \|u_2\|^\gamma_{W_2^1(D_T)} + 2\big), \quad p = 2\,\frac{n+1}{n-1}\,. \tag{5.4.16}$$

Since the a priori estimate (5.2.5) is valid for $\lambda = 0$, due to (5.2.27), in this estimate $c_2 = 0$, and hence, for the solution w_k of the problem (5.4.6), (5.4.7), the estimate

$$\|w_k\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} \leq c_1^0 \|F_k + g_k\|_{L_2(D_T)} \tag{5.4.17}$$

is valid, where the constant c_1^0 does not depend on λ, F_k and g_k.

Because of $\|w_k\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} = \|w_k\|_{W_2^1(D_T)}$ and due to (5.4.15) and (5.4.17), we have

$$\|w_k\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} \leq c_1^0 \|F_k\|_{L_2(D_T)} + \lambda c_1^0 M_5 \|w_k\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)}. \tag{5.4.18}$$

It should be noted that since for u_1 and u_2 the a priori estimate (5.2.5) is valid, the constant M_5 from (5.4.16) depends on F, f, μ, D_T and λ. Moreover, due to (5.2.19), (5.2.23) and (5.2.27), the value of M_5 continuously depends on λ for $\lambda \geq 0$, and

$$0 \leq \lim_{\lambda \to 0+} M_5 = M_5^0 < +\infty. \tag{5.4.19}$$

Due to (5.4.19), there exists a positive number $\lambda_0 = \lambda_0(F, f, \mu, D_T)$ such that for

$$0 < \lambda < \lambda_0 \tag{5.4.20}$$

we have $\lambda c_1^0 M_5 < 1$. Indeed, let us fix arbitrarily a positive number ε_1. Then, due to (5.4.19), there exists a positive number λ_1 such that $0 \leq M_5 < M_5^0 + \varepsilon_1$ for $0 \leq \lambda < \lambda_1$. Obviously, for

$$\lambda_0 = \min\big(\lambda_1, (c_1^0(M_5^0 + \varepsilon_1))^{-1}\big),$$

the condition $\lambda c_1^0 M_5 < 1$ is fulfilled. Therefore, in the case (5.4.20), from (5.4.18) we get

$$\|w_k\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} \leq c_1^0 (1 - \lambda c_1^0 M_5)^{-1} \|F_k\|_{L_2(D_T)}, \quad k \geq k_0. \tag{5.4.21}$$

From (5.4.2) and (5.4.3) it follows that $\lim\limits_{k\to\infty} \|w_k\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} = \|u_2 - u\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)}$. On the other hand, due to (5.4.5), from (5.4.21) we obtain $\lim\limits_{k\to\infty} \|w_k\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} = 0$. Thus, $\|u_2 - u_1\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} = 0$, i.e., $u_2 = u_1$, which leads to the contradiction. This proves Theorem 5.4.1. □

5.5 The cases of absence of a solution of the problem (5.1.1)–(5.1.4)

In this section, using the test function [77], we show that when the condition (5.2.2) is violated, the problem (5.1.1)–(5.1.4) may not have a generalized solution in the sense of Definition 5.1.1.

Lemma 5.5.1. *Let u be a generalized solution of the problem* (5.1.1)–(5.1.4) *in the sense of Definition* 5.1.1 *and the conditions* (5.1.5) *and* (5.1.6) *be fulfilled. Then the following integral equality*

$$\int\limits_{D_T} u\,\Box\, v\,dx\,dt = -\lambda \int\limits_{D_T} f(x,t,u)v\,dx\,dt + \int\limits_{D_T} Fv\,dx\,dt \tag{5.5.1}$$

is valid for every test function v satisfying the conditions

$$v \in C^2(\overline{D}_T), \quad v\big|_{\partial D_T} = 0, \quad \nabla_{x,t} v\big|_{\partial D_T} = 0, \tag{5.5.2}$$

where $\Box := \frac{\partial^2}{\partial t^2} - \sum\limits_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, $\nabla_{x,t} := (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t})$.

Proof. According to the definition of a generalized solution of the problem (5.1.1)–(5.1.4), there exists the sequence $u_m \in \overset{\circ}{C}{}^2_\mu(D_T)$ such that the equalities (5.1.10), (5.2.8) are valid. We multiply both sides of the equality (5.2.6) by the function v and integrate the obtained equality in the domain D_T. Due to (5.5.2), integration by parts of the left-hand side of this equation yields

$$\int\limits_{D_T} u_m\,\Box\, v\,dx\,dt + \lambda \int\limits_{D_T} f(x,t,u_m)v\,dx\,dt = \int\limits_{D_T} F_m v\,dx\,dt. \tag{5.5.3}$$

Passing in the equation (5.5.3) to the limit as $m \to \infty$ and taking into account (5.2.6), the limit equalities (5.1.10) and Remark 5.1.2, we obtain the equality (5.5.2). Thus Lemma 5.5.1 is proved. $\square$

Consider the following condition imposed on the function f:

$$f(x,t,u) \le -|u|^p, \ \ (x,t,u) \in \overline{D}_T \times \mathbb{R}; \ \ p = const > 1. \tag{5.5.4}$$

Note that when the condition (5.5.4) is fulfilled, the condition (5.5.2) is violated. Let us introduce into consideration the function $v_0 = c_0(x,t)$ such that

$$v_0 \in C^2(\overline{D}_T), \ \ v_0\big|_{D_T} > 0, \ \ v_0\big|_{\partial D_T} = 0, \ \ \nabla_{x,t} v_0\big|_{\partial D_T} = 0, \tag{5.5.5}$$

and

$$\varkappa_0 := \int\limits_{D_T} \frac{|\Box\, v_0|^{p'}}{|v_0|^{p'-1}}\,dx\,dt < +\infty, \ \ \frac{1}{p} + \frac{1}{p'} = 1. \tag{5.5.6}$$

Below, we assume that $\partial D \in C^2$ and hence there exists a function $\omega \in C^2(\mathbb{R}^n)$ such that $\partial\Omega$: $\omega(x) = 0$, $\nabla_x \omega|_{\partial\Omega} \ne 0$, and $\omega|_\Omega > 0$ [24].

Simple verification shows that in the capacity of the function v_0, satisfying the conditions (5.5.5) and (5.5.6), can be chosen the function

$$v_0(x,t) = [t(T-t)\omega(x)]^k, \ \ (x,t) \in D_T,$$

for a sufficiently large $k = const > 0$.

In view of (5.5.4) and (5.5.5), from (5.5.1), where v_0 is taken instead of v, it follows that when $\lambda > 0$,

$$\lambda \int\limits_{D_T} |u|^p v_0\,dx\,dt \le \int\limits_{D_T} |u|\,|\Box v_0|\,dx\,dt - \int\limits_{D_T} Fv_0\,dx\,dt. \tag{5.5.7}$$

Theorem 5.5.1. *Let the function $f \in C(\overline{D}_T \times \mathbb{R})$ satisfy the conditions* (5.1.5), (5.1.6) *and* (5.5.4)*; $\lambda > 0$, $\partial\Omega \in C^2$, $F^0 \in L_2(D_T)$, $F^0 \geq 0$, $\|F^0\|_{L_2(D_T)} \neq 0$. Then there exists a number $\gamma_0 = \gamma_0(F^0, \alpha, p, \lambda) > 0$ such that for $\gamma > \gamma_0$, the problem* (5.1.1)–(5.1.4) *does not have a generalized solution in the sense of Definition* 5.1.1 *for $F = \gamma F^0$.*

Proof. If in Young's inequality with the parameter $\varepsilon > 0$,

$$ab \leq \frac{\varepsilon}{p}\,a^p + \frac{1}{p'\varepsilon^{p'-1}}\,b^{p'}, \ \ a, b \geq 0, \ \ \frac{1}{p} + \frac{1}{p'} = 1, \ \ p > 1,$$

we take $a = |u|v_0^{1/p}$, $b = \frac{|\Box v_0|}{v^{1/p}}$, then taking into account the equality $\frac{p'}{p} = p' - 1$, we have

$$|u|\,|\Box v_0| = |u|v_0^{1/p}\,\frac{|\Box\, v_0|}{v_0^{1/p}} \leq \frac{\varepsilon}{p}\,|u|^p v_0 + \frac{1}{p'\varepsilon^{p'-1}}\,\frac{|\Box\, v_0|^{p'}}{v_0^{p'-1}}\,. \tag{5.5.8}$$

Since $F = \gamma F^0$, using (5.5.8), from (5.5.7) we get

$$\Big(\lambda - \frac{\varepsilon}{p}\Big)\int\limits_{D_T} |u|^p v_0\;dx\,dt \leq \frac{1}{p'\varepsilon^{p'-1}}\int\limits_{D_T}\frac{|\Box v_0|^{p'}}{v_0^{p'-1}}\;dx\,dt - \gamma\int\limits_{D_T} F^0 v\;dx\,dt,$$

whence for $\varepsilon < \lambda p$, we obtain

$$\int\limits_{D_T} |u|^p v_0\;dx\,dt \leq \frac{p}{(\lambda p - \varepsilon)p'\varepsilon^{p'-1}}\int\limits_{D_T}\frac{|\Box\, v_0|^{p'}}{v_0^{p'-1}}\;dx\,dt - \frac{p\gamma}{\lambda p - \varepsilon}\int\limits_{D_T} F^0 v_0\;dx\,dt. \tag{5.5.9}$$

Since $p' = \frac{p}{p-1}$, $p = \frac{p'}{p'-1}$ and

$$\min_{0<\varepsilon<\lambda p}\frac{p}{(\lambda p - \varepsilon)p'\varepsilon^{p'-1}} = \frac{1}{\lambda^p}\,,$$

which is achieved for $\varepsilon = \lambda$, it follows from (5.5.9) that

$$\int\limits_{D_T} |u|^p v_0\;dx\,dt \leq \frac{1}{\lambda^{p'}}\int\limits_{D_T}\frac{|\Box\, v_0|^{p'}}{v_0^{p'-1}}\;dx\,dt - \frac{p'\gamma}{\lambda}\int\limits_{D_T} F^0 v_0\;dx\,dt. \tag{5.5.10}$$

Because of the conditions imposed on the function F^0, and $v_0|_{D_T} > 0$, we have

$$0 < \varkappa_1 := \int\limits_{D_T} F^0 v_0\;dx\,dt < +\infty. \tag{5.5.11}$$

Denoting by $\chi = \chi(\gamma)$ the right-hand side of the inequality (5.5.10), which is a linear function with respect to the parameter γ, due to (5.5.6) and (5.5.11), we have

$$\chi(\gamma) < 0 \ \text{ for } \ \gamma > \gamma_0 \ \text{ and } \ \chi(\gamma) > 0 \ \text{ for } \ \gamma < \gamma_0, \tag{5.5.12}$$

where

$$\chi(\gamma) = \frac{\varkappa_0}{\lambda^{p'}} - \frac{p'\gamma}{\lambda}\,\varkappa_1, \ \ \gamma_0 = \frac{\varkappa_0}{\lambda^{p'-1}p'\varkappa_1}\,.$$

It remains only to note that the left-hand side of the inequality (5.5.10) is nonnegative for $\gamma > \gamma_0$. Thus, for $\gamma > \gamma_0$, the problem (5.1.1)–(5.1.4) does not have a generalized solution in the sense of Definition 5.1.1. Thus Theorem 5.5.1 is proved. □

5.6 The case $|\mu| = 1$

As is mentioned at the end of the third section, for $|\mu| = 1$, the problem (5.1.1)–(5.1.4) may turn out to be ill-posed. Below, we will show that in the presence of additional terms $2au_t$ and cu in the left-hand side of the equation (5.1.1) the problem will be solvable for any $F \in L_2(D_T)$.

Consider the equation

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + 2au_t + cu + f_1(x,t,u) = F(x,t), \quad (x,t) \in D_T, \tag{5.6.1}$$

with the constant real coefficients a and c, where f_1 and F are the given real functions.

For the equation (5.6.1), consider a problem of finding u in the domain D_T satisfying the boundary condition (5.1.2) and the nonlocal conditions (5.1.3), (5.1.4) for $|\mu| = 1$. For the problem (5.6.1), (5.1.2)–(5.1.4), when $f_1 \in C(\overline{D}_T \times \mathbb{R})$ and $F \in L_2(D_T)$, analogously to what we have done in Definition 5.1.1, let us introduce the notion of a generalized solution $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$.

With respect to a new unknown function

$$v := \sigma^{-1}(t)u, \quad \text{where } \sigma(t) := \exp(-at), \quad 0 \le t \le T, \tag{5.6.2}$$

the problem (5.6.1), (5.1.2)–(5.1.4) can be rewritten as follows:

$$\frac{\partial^2 v}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2} + (c-a^2)v + \sigma^{-1}(t) f_1\big(x,t,\sigma(t)v(x,t)\big) = \sigma^{-1}(t)F(x,t), \quad (x,t) \in D_T, \tag{5.6.3}$$

$$v\big|_{\Gamma} = 0, \tag{5.6.4}$$

$$(\mathcal{K}_{\mu_1} v)(x) = 0, \quad (\mathcal{K}_{\mu_1} v_t)(x) = 0, \quad x \in \Omega, \tag{5.6.5}$$

where $\mu_1 = \mu\sigma(T)$, $|\mu| = 1$.

In the case $a > 0$, due to (5.6.2) and $|\mu| = 1$, it is obvious that $|\mu_1| < 1$.

It is not difficult to see that for $c - a^2 \ge 0$, the functions $f(x,t,u) = (c-a^2)u$ and $g(x,t,u) = \int\limits_0^u f(x,t,s)\,ds = \frac{1}{2}(c-a^2)u^2$ satisfy (5.1.5), (5.2.2)–(5.2.4).

For $f(x,t,u) = \sigma^{-1}(t) f_1(x,t,\sigma(t)u)$, we have

$$\begin{aligned} g(x,t,u) &= \int_0^u f(x,t,s)\,ds = \int_0^u \sigma^{-1}(t) f_1(x,t,\sigma(t)s)\,ds \\ &= \sigma^{-1}(t) \int_0^{\sigma(t)u} f_1(x,t,s')\,ds' = \sigma^{-2}(t) g_1(x,t,\sigma(t)u). \end{aligned} \tag{5.6.6}$$

Here,

$$g_1(x,t,u) = \int_0^u f_1(x,t,s)\,ds. \tag{5.6.7}$$

Let us show that if the function $g_1(x,t,u)$ from (5.6.7) satisfies the condition

$$g_1(x,0,\mu_1 u) \le g_1(x,T,|\mu_1|u), \quad (x,t) \in \overline{\Omega} \times \mathbb{R}, \tag{5.6.8}$$

for the fixed constant μ_1 from (5.6.5), then the function $g(x,t,u)$ from (5.6.6) satisfies the condition (5.2.4) for $\mu = \mu_1$. Indeed, in view of (5.6.2), (5.6.6) and (5.6.8), since $\mu_1 = \mu\sigma(T)$, $|\mu| = 1$, $\sigma(T) = |\mu_1|$, we have

$$g(x,0,\mu_1 u) = \sigma^{-2}(0) g_1(x,0,\sigma(0)\mu_1 u) = g_1(x,0,\mu_1 u),$$
$$\mu_1^2 g(x,T,u) = \mu_1^2 \sigma^{-2}(T) g_1(x,T,\sigma(T)u) = g_1(x,T,|\mu_1|u),$$

whence, due to (5.6.8), follows (5.2.4) for $\mu=\mu_1$.

Since $\sigma'(t)=-a\sigma(t)$, $(\sigma^{-2}(t))'=2a\sigma^{-2}(t)$, according to (5.6.6) and supposing that $f_1,f_{1t},f_{1u}\in C(\overline{D}_T\times\mathbb{R})$, we have

$$g_t(x,t,u)=2a\sigma^{-2}(t)g_1(x,t,\sigma(t)u)+\sigma^{-2}(t)g_{1t}(x,t,\sigma(t)u)-a\sigma^{-1}g_{1u}(x,t,\sigma(t)u).$$

Therefore, the condition

$$2a\sigma^{-2}(t)g_1(x,t,\sigma(t)u)+\sigma^{-2}(t)g_{1t}(x,t,\sigma(t)u)-a\sigma^{-1}(t)g_{1u}(x,t,\sigma(t)u)\le M_3,\quad (x,t,u)\in\overline{D}_T\times\mathbb{R}, \tag{5.6.9}$$

results in the condition (5.2.3).

Note that due to (5.6.6), from the condition

$$g_1(x,t,u)\ge 0,\quad (x,t,u)\in\overline{D}_T\times\mathbb{R}, \tag{5.6.10}$$

follows the condition (5.2.2).

It is easily seen that if the function $f_1(x,t,u)$ satisfies the condition of type (5.1.5), i.e.,

$$|f_1(x,t,u)|\le\widetilde{M}_1+\widetilde{M}_2|u|^{\alpha},\quad (x,t,u)\in\overline{D}_T\times\mathbb{R},\quad \widetilde{M}_i=const\ge 0, \tag{5.6.11}$$

then the function $f(x,t,u)=\sigma^{-1}(t)f_1(x,t,\sigma(t)u)$ from the left-hand side of the equation (5.6.3) satisfies the condition (5.1.5) for some nonnegative constants M_1 and M_2.

It should be noted that in the concrete case $f_1(x,t,u)=|u|^{\beta}u$, $\beta=const\ge 0$, the function $g_1(x,t,u)=\frac{|u|^{\beta+2}}{\beta+2}$, and

$$f(x,t,u)=\sigma^{-1}(t)f_1(x,t,\sigma(t)u)=\sigma^{\beta}(t)|u|^{\beta}u, \tag{5.6.12}$$

$$g(x,t,u)=\int\limits_0^u f(x,t,s)\,ds=\sigma^{\beta}(t)\,\frac{|u|^{\beta+2}}{\beta+2}\,. \tag{5.6.13}$$

Therefore, taking into account that $\sigma'(t)\le 0$, $g(x,0,\mu_1u)=|\mu_1|^{\beta+2}\,\frac{|u|^{\beta+2}}{\beta+2}$, $\mu_1^2g(x,T,u)=\mu_1^2\sigma^{\beta}(T)\,\frac{|u|^{\beta+2}}{\beta+2}$, $\sigma(T)=|\mu_1|$, it is easy to see that the functions $f(x,t,u)$ and $g(x,t,u)$ from (5.6.12) and (5.6.13) satisfy the conditions (5.1.5), (5.2.2)–(5.2.4) for $\mu=\mu_1$, $\alpha=\beta+1$, $M_3=0$.

Further, since the problems (5.6.1), (5.1.2)–(5.1.4) and (5.6.3), (5.6.4), (5.6.5) are equivalent, from Theorem 5.3.1 follows the theorem of the existence of the solution of the problem (5.6.1), (5.1.2)–(5.1.4).

Theorem 5.6.1. *Let $|\mu|=1$, $a>0$, $c-a^2\ge 0$, the function $f_1(x,t,u)$ from the left-hand side of the equation* (5.6.1) *and the function $g_1(x,t,u)$ from* (5.6.7) *satisfy the conditions $f_1,f_{1t},f_{1u}\in C(\overline{D}_T\times\mathbb{R})$,* (5.6.8)–(5.6.11). *Then if in the condition* (5.6.11) *the order of nonlinearity α satisfies the inequality $\alpha<\frac{n+1}{n-1}$, then the problem* (5.6.1), (5.1.2)–(5.1.4) *for any $F\in L_2(D_T)$ has at least one generalized solution.*

Remark 5.6.1. In the case when Robin's boundary condition

$$\Big(\frac{\partial u}{\partial\nu}+\sigma u\Big)\Big|_{\Gamma}=0 \tag{5.6.14}$$

is considered instead of the Dirichlet boundary condition (5.1.2), analogous results for the nonlocal problem (5.1.1), (5.6.14), (5.1.3), (5.1.4) can be found in [53].

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Author's address:

Sergo Kharibegashvili

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia.

E-mail: kharibegashvili@yahoo.com

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R. P. Agarwal, A. Aghajani, M. Mirafzal

EXACT CONDITIONS FOR THE EXISTENCE OF HOMOCLINIC ORBITS IN THE LIÉNARD SYSTEMS

Abstract. We consider the Liénard system $\dot{x} = y - F(x)$ and $\dot{y} = -g(x)$. Under the assumptions that the origin is a unique equilibrium, we investigate the existence of homoclinic orbits of this system which is closely related to the stability of the zero solution, center problem, global attractively of the origin, and oscillation of solutions of the system. We present the necessary and sufficient conditions for this system to have a positive orbit which starts at a point on the vertical isocline $y = F(x)$ and approaches the origin without intersecting the x-axis. Our results solve the problem completely in some sense.

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Key words and phrases. Homoclinic orbit, Liénard system, oscillation.

რეზიუმე. განვიხილავთ ლიენარდის სისტემას $\dot{x} = y - F(x)$ და $\dot{y} = -g(x)$. იმ დაშვებით, რომ სათავე არის ერთადერთი წონასწორობის წერტილი, ვსწავლობთ ამ სისტემის ჰომოკლინური ორბიტების არსებობას, რაც მჭიდროდ არის დაკავშირებული ნულოვანი ამოცანის მდგრადობას-თან, ცენტრის პრობლემასთან, სათავის გლობალურ მიზიდულობასთან და სისტემის ამონახსნთა რხევადობასთან. მოყვანილია აუცილებელი და საკმარისი პირობები, რათა ამ სისტემას გააჩნდეს დადებითი ორბიტები, რომლებიც იწყება $y = F(x)$ ვერტიკალური იზოკლინის წერ-ტილში და უახლოვდება სათავეს ისე, რომ არ გადაკვეთს x ღერძს. შედეგები გარკვეული თვალსაზრისით სრულად ხსნის დასმულ ამოცანას.

1 Introduction

It is well known that the Liénard system

$$\begin{aligned}\frac{dx}{dt} &= y - F(x),\\ \frac{dy}{dt} &= -g(x),\end{aligned} \tag{1.1}$$

is of great importance in various applications. Hence, asymptotic and qualitative behavior of this system and some of its extensions have been widely studied by many authors; results can be found in many books and papers [1–22]. In system (1.1), a trajectory is said to be a homoclinic orbit if its α- and ω-limit sets are the origin. The existence of homoclinic orbits in the Liénard-type systems (see [5]) is closely connected with the stability of the zero solution and the center problem. If system (1.1) has a homoclinic orbit, then the zero solution is no longer stable. A homoclinic orbit and a center cannot exist together in system (1.1). Our subject also has a near relation to the global attractivity of the origin and oscillation of solutions (see [9, 11]).

Taking the vector field of (1.1) into account, we see that every homoclinic orbit is in the upper or in the lower half-plane. In other words, no homoclinic orbit crosses the x-axis. When a homoclinic orbit appears in the upper (resp. lower) half-plane, all other homoclinic orbits exist in the same half-plane.

We say that system (1.1) has property (Z_1^+) (resp. (Z_3^+)) if there exists a point $P(x_0, y_0)$ with $y_0 = F(x_0)$ and $x_0 > 0$ (resp. $x_0 < 0$) such that the positive semitrajectory of (1.1) starting at P approaches the origin through only the first (resp. third) quadrant. We also say that system (1.1) has property (Z_2^-) (resp. (Z_4^-)) if there exists a point $P(x_0, y_0)$ with $y_0 = F(x_0)$ and $x_0 < 0$ (resp. $x_0 > 0$) such that the negative semitrajectory of (1.1) starting at P approaches the origin through only the second (resp. fourth) quadrant. If system (1.1) has both properties (Z_1^+) and (Z_2^-), then a homoclinic orbit exists in the upper half-plane. Similarly, if system (1.1) has both properties (Z_3^+) and (Z_4^-), then a homoclinic orbit exists in the lower half-plane. Notice that by the transformation $x \to -x$ and $t \to -t$, we can transfer any result for property (Z_1^+) to an analogous result with respect to property (Z_2^-). Also, by the transformation $x \to -x$ and $y \to -y$, we can transfer any result for property (Z_1^+) (resp. (Z_2^-)) to an analogous result with respect to property (Z_3^+) (resp. (Z_4^-)).

In this paper, we intend to give some conditions on $F(x)$ and $g(x)$ under which system (1.1) has properties (Z_1^+), (Z_2^-), (Z_3^+), or (Z_4^-). We assume that F and g are continuous on an open interval I which contains 0 and satisfy smoothness conditions for uniqueness of solutions of the initial value problems. We also assume that $F(0) = 0$ and

$$xg(x) > 0 \;\text{ for }\; x \neq 0,$$

which guarantee that the origin is the unique equilibrium of (1.1). Throughout this paper, in the results related to property (Z_1^+) (resp. (Z_2^-)), we assume that $F(x) > 0$ for $x > 0$ (resp. $x < 0$),$|x|$ sufficiently small. Because if $F(x)$ has an infinite number of positive (resp. negative) zeroes clustering at $x = 0$, then the system (1.1) fails to have property (Z_1^+) (resp. (Z_2^-)). Similarly, in the results related to property (Z_3^+) (resp. (Z_4^-)), we assume that $F(x) < 0$ for $x < 0$ (resp. $x > 0$), $|x|$ sufficiently small.

T. Hara and T. Yoneyama [10] considered system (1.1) and proved that if there exists $\delta > 0$ such that

$$F(x) > 0, \qquad \frac{1}{F(x)} \int_0^x \frac{g(\eta)}{F(\eta)}\, d\eta \le \frac{1}{4}$$

for $0 < x < \delta$, then system (1.1) has property (Z_1^+). They also proved that if there exist $a > 0$ such that $F(x) > 0$ for $0 < x \le a$ and some $\alpha > \frac{1}{4}$ such that

$$\frac{1}{F(x)} \int_0^x \frac{g(\eta)}{F(\eta)}\, d\eta \ge \alpha,$$

then system (1.1) fails to have property (Z_1^+) (see also [6, 9, 15, 19]).

In this paper, we present an implicit necessary and sufficient condition for system (1.1) to have property (Z_1^+). Then we drive sharp explicit conditions and solve this problem completely in some sense. We formulate similar results for properties (Z_2^-), (Z_3^+), and (Z_4^-).

The paper is organized as follows. In Section 2, we give implicit conditions for system (1.1) to have property (Z_1^+). In Section 3, we use our results obtained in Section 2 and present sufficient conditions for properties (Z_1^+), (Z_2^-), (Z_3^+), and (Z_4^-). In Section 4, we present the necessary conditions for properties (Z_1^+), (Z_2^-), (Z_3^+), and (Z_4^-) and show that the sufficient conditions presented in Section 3 are best possible.

2 Implicit conditions for property (Z_1^+)

In this section we present implicit conditions for system (1.1) to have property (Z_1^+). First, we introduce a system which is equivalent to (1.1). Let the function $\lambda(x)$ be defined by

$$\lambda(x) = \begin{cases} \sqrt{2G(x)} & \text{for } x \geq 0, \\ -\sqrt{2G(x)} & \text{for } x < 0 \end{cases}$$

and the mapping $\Lambda : R^2 \to R^2$ by

$$\Lambda(x, y) = (\lambda(x), y) \equiv (u, v).$$

Consider the canonical form of the Liénard systems

$$\begin{aligned} \frac{du}{d\tau} &= v - F^*(u), \\ \frac{dv}{d\tau} &= -u, \end{aligned} \tag{2.1}$$

in which $d\tau = [g(x)\,\mathrm{sgn}(x)/\sqrt{2G(x)}]\,dt$ and a continuous function F^* is defined by

$$F^*(u) = \begin{cases} F(G^{-1}\Big(\frac{1}{2}\,u^2\Big) & \text{if } u \geq 0, \\ F(G^{-1}\Big(-\frac{1}{2}\,u^2\Big) & \text{if } u < 0, \end{cases}$$

where $G^{-1}(w)$ is the inverse function to $G(x)\,\mathrm{sgn}(x)$. Then the mapping Λ is a homeomorphism of the (x, y)-plane onto an open subset of the (u, v)-plane which contains zero. It is obvious that Λ maps the x-axis into the u-axis. Consequently, we have only to determine whether system (2.1), instead of (1.1), has property (Z_1^+) or not. Hereafter we denote τ by t again.

Theorem 2.1. *Let $F^* \in C^1([0, \alpha])$ for some $\alpha > 0$. Then system* (2.1) *has property (Z_1^+) if and only if there exist a constant $b \leq \alpha$ and a function $\varphi \in C^1([0, b])$ such that $\varphi(0) = 0$,*

$$\varphi(u) > 0, \quad (F^*)'(u) \geq \frac{u}{\varphi(u)} + \varphi'(u) \ \text{ for } \ 0 < u \leq b. \tag{2.2}$$

Proof. Sufficiency. Consider the positive semitrajectory of (2.1) starting at a point $(b, F^*(b))$. This trajectory is considered as a solution $v(u)$ of

$$\frac{dv}{du} = -\frac{u}{v - F^*(u)} \tag{2.3}$$

with $v(b) = F^*(b)$. Suppose that the positive semitrajectory $v(u)$ crosses the negative y-axis. Then it also meets the curve $v = F^*(u) - \varphi(u)$ at a point $(s, F^*(s) - \varphi(s))$ with $s < b$ such that

$$\frac{dv}{du}\,(s) = \frac{-s}{(F^*(s) - \varphi(s)) - F^*(s)} > (F^*)'(s) - \varphi'(s).$$

Thus

$$(F^*)'(s) < \frac{s}{\varphi(s)} + \varphi'(s).$$

This is a contradiction. Hence, the trajectory $v(u)$ does not cross the negative y-axis, and, therefore, system (2.1) has property (Z_1^+).

Necessity. Suppose that system (2.1) has property (Z_1^+). Then there exists a positive semitrajectory of (2.1) starting at a point $(b, F^*(b))$ with $b > 0$, which does not meet the negative y-axis. This trajectory can be regarded as the graph of a continuously differentiable function $\psi(u)$ which is a solution of (2.3). Let $\varphi(u) = F^*(u) - \psi(u)$. Then it is clear that $\varphi(0) = 0$,

$$\varphi(u) > 0, \quad (F^*)'(u) = \frac{u}{\varphi(u)} + \varphi'(u) \text{ for } 0 < u \le b.$$

Hence, the condition (2.2) is verified. □

Theorem 2.2. *Suppose that system* (2.1) *with* F_1 *has property* (Z_1^+). *If*

$$F_2(u) \ge F_1(u) \tag{2.4}$$

for $u > 0$ *sufficiently small, then system* (2.1) *corresponding to* F_2 *has property* (Z_1^+).

Proof. Since system (2.1) with $F_1(u)$ has property (Z_1^+), there exists a positive semitrajectory of (2.1) starting at a point (u_0, v_0) with $u_0 > 0$, which approaches the origin through only the first quadrant. This trajectory can be regarded as the graph of a function $v = \psi_1(u)$ which is a solution of (2.3). Let $v = \psi_2(u)$ be the graph of the solution of system (2.3) corresponding to F_2 such that $(u(0), v(0)) = (u_0, v_0)$. We can assume that u_0 is sufficiently small, thus from (2.4) we have

$$\psi_2'(u) = \frac{-u}{v - F_2(u)} \le \frac{-u}{v - F_1(u)} = \psi_1'(u) \text{ for } 0 < u \le u_0.$$

Hence, $\psi_2(u) \ge \psi_1(u) > 0$ for $0 < u \le u_0$. Therefore, system (2.1) corresponding to F_2 has property (Z_1^+). □

3 Explicit sufficient conditions for property (Z_1^+)

In this section we use our implicit conditions to drive explicit sufficient conditions for properties (Z_1^+), (Z_2^-), (Z_3^+), and (Z_4^-). To this end, for $u > 0$ sufficiently small we define

$$L_1(u) = \log ku$$

and

$$L_n(u) = \log ku \times \log(b|\log ku|) \times \cdots \times \underbrace{\log\log\cdots\log}_{(n-1)\text{-times}}\big(b|\log ku|\big) \text{ for } n \ge 2,$$

where $k, b > 0$. Notice that $L_n(u) < 0$ for $u > 0$ sufficiently small.

Theorem 3.1. *Let* $k, b > 0$. *If*

$$F^*(u) \ge 2u - \frac{1}{4}\sum_{j=1}^{n-1} \frac{u}{(L_j(u))^2}$$

for some $n \ge 2$ *and* $u > 0$ *sufficiently small, then system* (2.1) *has property* (Z_1^+).

Proof. By Theorem 2.2, it suffices to prove the theorem when

$$F^*(u) = 2u - \frac{1}{4}\sum_{j=1}^{n-1} \frac{u}{(L_j(u))^2}\,.$$

Let

$$M_n(u) = \sum_{j=1}^{n-1} \Big(\frac{1}{L_j(u)} \sum_{i=1}^{j} \frac{1}{L_i(u)} \Big), \tag{3.1}$$

$$N_n(u) = \sum_{j=1}^{n-1} \frac{1}{L_j(u)}, \quad \varphi_n(u) = u + \frac{1}{2}\, u N_{n+1}(u). \tag{3.2}$$

We have

$$u\, \frac{d}{du}\, (L_n(u)) = N_n(u)L_n(u) + 1, \quad 2M_n(u) - (N_n(u))^2 = \sum_{j=1}^{n-1} \frac{1}{(L_j(u))^2}$$

and

$$\frac{d}{du}\, (N_n(u)) = -\frac{M_n(u)}{u}\,.$$

Thus

$$\frac{u}{\varphi_n(u)} + \varphi_n'(u) = 2 - \frac{1}{4(1+\frac{1}{2}N_{n+1}(u))} \Big(\sum_{j=1}^{n} \frac{1}{(L_j(u))^2} + N_{n+1}(u) M_{n+1}(u) \Big),$$

or

$$\frac{u}{\varphi_n(u)} + \varphi_n'(u) = 2 - \frac{1}{4} \sum_{j=1}^{n} \frac{1}{(L_j(u))^2} - \frac{(N_{n+1}(u))^3}{8(1-\frac{1}{2}N_{n+1}(u))} \tag{3.3}$$

for $u > 0$ sufficiently small. On the other hand,

$$(F^*)'(u) = 2 - \frac{1}{4} \sum_{j=1}^{n-1} \frac{1}{(L_j(u))^2} + \frac{1}{2} \sum_{j=1}^{n-1} \frac{N_j(u)L_j(u) + 1}{(L_j(u))^3}\,. \tag{3.4}$$

It is easy to check that

$$(F^*)'(u) > \frac{u}{\varphi_n(u)} + \varphi_n'(u)$$

for $u > 0$ sufficiently small. Hence, (2.2) holds and, by Theorem 2.1, system (2.1) has property (Z_1^+). □

Recall defining the function $F^*(u)$ as follows:

$$F^*(u) = F\Big(G^{-1}\Big(\frac{1}{2}u^2\Big)\Big) \ \text{ for } \ u \geq 0.$$

Put $x = G^{-1}(\frac{1}{2}u^2)$. Then for system (1.1) to have property (Z_1^+) we have the following sufficient condition.

Theorem 3.2. *Assume $k, b > 0$. If*

$$F(x) \geq \sqrt{8G(x)} - \frac{1}{4} \sum_{j=1}^{n-1} \frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2}$$

for some $n \geq 2$ and $x > 0$ sufficiently small, then system (1.1) has property (Z_1^+).

Similarly, for system (1.1) to have properties (Z_2^-), (Z_3^+), and (Z_4^-), we have the following sufficient conditions.

Theorem 3.3. *Assume $k, b > 0$. If*

$$F(x) \geq \sqrt{8G(x)} - \frac{1}{4} \sum_{j=1}^{n-1} \frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2}$$

for some $n \geq 2$ and $x < 0$, $|x|$ sufficiently small, then system (1.1) has property (Z_2^-).

Theorem 3.4. *Assume $k, b > 0$. If*

$$F(x) \le -\sqrt{8G(x)} + \frac{1}{4}\sum_{j=1}^{n-1} \frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2},$$

for some $n \ge 2$ and $x < 0$, $|x|$ sufficiently small, then system (1.1) *has property* (Z_3^+).

Theorem 3.5. *Assume $k, b > 0$. If*

$$F(x) \le -\sqrt{8G(x)} + \frac{1}{4}\sum_{j=1}^{n-1} \frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2},$$

for some $n \ge 2$ and $x > 0$ sufficiently small, then system (1.1) *has property* (Z_4^-).

4 Explicit necessary conditions for property (Z_1^+)

In this section we drive explicit necessary conditions for properties (Z_1^+), (Z_2^-), (Z_3^+), and (Z_4^-) and show that the sufficient conditions presented in Section 2 are best possible.

Definition 4.1. Let $f_1(u)$ and $f_2(u)$ be real-valued functions. By $f_1(u) \preceq f_2(u)$ we mean that there exists $b > 0$ such that $f_1(u) \le f_2(u)$ for $0 < u \le b$.

In proving Theorem 4.1 we will need the following

Lemma 4.1. *Suppose that $\varphi \in C^1([0,\alpha])$ for some $\alpha > 0$, $\varphi(0) = 0$, and $\varphi(u) > 0$ for $u > 0$ sufficiently small. If*

$$\frac{d}{du}\left(2u - \frac{1}{4}\sum_{j=1}^{n-1} \frac{u}{(L_j(u))^2} - \frac{\lambda u}{(L_n(u))^2}\right) \ge \frac{u}{\varphi(u)} + \varphi'(u), \quad \lambda \ge \frac{1}{4}, \tag{4.1}$$

for some $n \ge 2$, $k > 0$, $b > 0$, and $u > 0$ sufficiently small, then

(i) $\lim\limits_{u\to 0^+} \frac{\varphi(u)}{u} = 1,$

(ii) $|\frac{\varphi(u)-u}{u}| \le \frac{1}{|\log ku|}$ *for every $k > 0$ and $u > 0$ sufficiently small.*

Proof. It is easy to check that the left-hand side of inequality (4.1) tends to 2 as $u \to 0^+$. Thus, from (4.1) we get

$$\lim_{u\to 0^+}\Big(\frac{u}{\varphi(u)} + \varphi'(u)\Big) = \frac{1}{\varphi'(0^+)} + \varphi'(0^+) \le 2.$$

Hence,

$$\lim_{u\to 0^+} \frac{\varphi(u)}{u} = \varphi'(0^+) = 1.$$

This completes the proof of (i). Now let $\varphi(u) = u + h(u)$. Then we have

$$-\Big(\frac{u}{\varphi(u)} + \varphi'(u)\Big) = -2 + \frac{h(u)}{u + h(u)} - h'(u). \tag{4.2}$$

From (4.1) and (4.2) we conclude that

$$\frac{h(u)}{u + h(u)} - h'(u) > 0 \tag{4.3}$$

for u sufficiently small. Suppose that $\{u_n\}$ tends to zero and $h(u_n) = 0$, then there exists a sequence $\{c_n\}$ such that c_n tends to zero as $n \to \infty$, $h'(c_n) = 0$, and $h(c_n) \le 0$. This contradicts (4.3). Hence,

$h(u)$ is positive or negative for $u > 0$ sufficiently small, and we can let $h(u) = \frac{u}{f(u)}$ for $0 < u \le c$ with c sufficiently small. Notice that, by (i), $|f(u)| \to \infty$ as $u \to 0$. Since $\varphi(u) > 0$ for u sufficiently small,

$$\frac{f(u)+1}{f(u)} = \frac{\varphi(u)}{u} > 0. \tag{4.4}$$

Thus, from (4.3) and (4.4) we have

$$f'(u)\Big(\frac{f(u)+1}{f(u)}\Big) > \frac{1}{u}$$

for $0 < u \le b$ with b sufficiently small. Integration of the above leads to

$$f(u) + \log(|f(u)|) - f(b) - \log(|f(b)|) \le \log(u) - \log(b)$$

for $0 < u \le b$. Hence, $f(u) \to -\infty$ as $u \to 0^+$, and $|f(u)| > |\log ku|$ for every $k > 0$ and $u > 0$ sufficiently small. □

Theorem 4.1. *Suppose that there exist $\lambda > 1/4$, $n \ge 2$, and $k, b > 0$ such that*

$$F^*(u) \le 2u - \frac{1}{4}\sum_{j=1}^{n-1} \frac{u}{(L_j(u))^2} - \frac{\lambda u}{(L_n(u))^2}$$

for $u > 0$ sufficiently small. Then system (2.1) *fails to have property* (Z_1^+).

Proof. By Theorem 2.2, it suffices to prove the theorem when

$$F^*(u) = 2u - \frac{1}{4}\sum_{j=1}^{n-1} \frac{u}{(L_j(u))^2} - \frac{\lambda u}{(L_n(u))^2}\,, \quad \lambda > \frac{1}{4}\,,$$

for $u > 0$ sufficiently small. We prove the theorem by contradiction. Suppose that there exists a continuously differentiable function φ such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ sufficiently small, and

$$(F^*)'(u) \succeq \frac{u}{\varphi(u)} + \varphi'(u). \tag{4.5}$$

Let

$$h(u) = \varphi(u) - \varphi_{n-1}(u) = \varphi(u) - u\Big(1 + \frac{1}{2}\,N_n(u)\Big).$$

From (4.5), (3.3), and (3.4) we have

$$\begin{aligned}\frac{u}{\varphi_{n-1}(u)} - \frac{u}{\varphi_{n-1}(u)+h(u)} - h'(u) &\succeq \frac{u}{\varphi_{n-1}(u)} + \varphi'_{n-1}(u) - (F^*)'(u)\\ &= \frac{\lambda}{(L_n(u))^2} - \Big(2\lambda + \frac{1}{2}\Big)\sum_{j=1}^{n-1}\frac{N_j(u)L_j(u)+1}{(L_j(u))^3} - \frac{(N_{n+1}(u))^3}{8(1-\frac{1}{2}N_{n+1}(u))}\,.\end{aligned}$$

Then

$$\frac{\lambda'}{(L_n(u))^2} \preceq \frac{u}{\varphi_{n-1}(u)} - \frac{u}{\varphi_{n-1}(u)+h(u)} - h'(u), \tag{4.6}$$

where $1/4 < \lambda' < \lambda$. Suppose that $\{u_n\}$ tends to zero and $h(u_n) = 0$, then there exists a sequence $\{c_n\}$ such that c_n tends to zero as $n \to \infty$, $h'(c_n) = 0$, and $h(c_n) \le 0$. This contradicts (4.6). Hence, $h(u) \neq 0$ for $x > 0$ sufficiently small, and we can let $f(u) = \frac{u}{h(u)}$ for $0 < u \le c$ with c sufficiently small. From (4.5), Lemma 4.1, and the fact that $|N_n(u)| \preceq \frac{2}{|\log ku|}$, we conclude that

$$\frac{1}{|f(u)|} = |\frac{\varphi(u)-u}{u} - \frac{N_n(u)}{2}| \le \frac{2}{|\log ku|} \tag{4.7}$$

for $u > 0$ sufficiently small.

Let

$$T_n(u) = \Big(1 + \frac{N_n(u)}{2}\Big)\Big(1 + \frac{N_n(u)}{2} + \frac{1}{f(u)}\Big)$$

and

$$g(u) = \frac{f(u)}{L_n(u)}\,.$$

Then from (3.2) and (4.6) we have

$$\frac{\lambda'}{(L_n(u))^2} \preceq \frac{1}{1+\frac{1}{2}N_n(u)} - \frac{1}{1+\frac{1}{2}\,N_n(u)+\frac{1}{f(u)}} - \frac{f(u)-f'(u)u}{f^2(u)} = \frac{1}{f(u)T_n(u)} - \frac{1}{f(u)} + \frac{f'(u)u}{f^2(u)}\,.$$

Hence,

$$\lambda' \preceq \frac{L_n(u)}{g(u)T_n(u)} - \frac{L_n(u)}{g(u)} + \frac{(g(u)L_n(u))'u}{g^2(u)}\,. \tag{4.8}$$

Notice that $u(L_n(u))' = N_n(u)L_n(u) + 1$, thus, from (4.8),

$$\lambda' g^2(u) \preceq g'(u)uL_n(u) + g(u)L_n(u)\Big(\frac{1 - T_n(u) + N_n(u)T_n(u)}{T_n(u)}\Big) + g(u),$$

or

$$\Big(\lambda' - \frac{1}{4}\Big)g^2(u) + \Big(\frac{g(u)}{2} - 1\Big)^2$$
$$\preceq g'(u)uL_n(u) + \Big(1 - \frac{1}{T_n(u)}\Big) - \frac{N_n(u)}{2T_n(u)} - \frac{g(u)\big(N_n(u)L_n(u)(1-T_n(u)) + \frac{(N_n(u))^2}{4}L_n(u)\big)}{T_n(u)}\,.$$

Now, let

$$A(u) = -\frac{(N_n(u)L_n(u)(1-T_n(u)) + \frac{(N_n(u))^2}{4}\,L_n(u))}{T_n(u)}$$

and

$$B(u) = 1 - \frac{1}{T_n(u)} - \frac{N_n(u)}{2T_n(u)}\,.$$

It is easy to check that

$$\lim_{u\to 0^+}(1 - T_n(u)) = \lim_{u\to 0^+}(N_n(u))^2 L_n(u) = 0.$$

Also, by (4.7), we conclude that

$$\lim_{u\to 0^+} N_n(u)L_n(u)(1 - T_n(u)) = 0,$$

thus, $A(u)$ and $B(u)$ tend to 0 as $u \to 0^+$, and we have

$$\Big(\lambda' - \frac{1}{4}\Big)g^2(u) + \Big(\frac{g(u)}{2} - 1\Big)^2 \preceq g'(u)uL_n(u) + A(u)g(u) + B(u), \quad \lambda' > \frac{1}{4}\,, \tag{4.9}$$

and

$$\Big(\frac{g(u)}{2} - 1\Big)^2 \preceq g'(u)uL_n(u) + A(u)g(u) + B(u). \tag{4.10}$$

We now prove that if (4.10) holds, then

$$\lim_{u\to 0^+} g(u) = 2. \tag{4.11}$$

Suppose $u_n > 0$ tends to zero and $g'(u_n) = 0$. Then from (4.10) we conclude that

$$\lim_{n\to\infty} g(u_n) = 2.$$

Since g' vanishes at the extremum points, if $g(u)$ is not increasing or decreasing for $u > 0$ sufficiently small, then

$$\liminf_{u\to 0^+} g(u) = \limsup_{u\to 0^+} g(u) = 2,$$

and (4.11) holds. Suppose now that $g(u)$ is increasing or decreasing for $u > 0$ sufficiently small. If $\lim_{u\to 0^+} g(u) \neq 2$, then from (4.10) we conclude that there exists $c > 0$ such that

$$\frac{c}{uL_n(u)} > \frac{g'(u)}{(\frac{g(u)}{2} - 1)^2}$$

for $0 < u \le l$ with l sufficiently small. Integration of the above leads to

$$c\Big(\underbrace{\log\log\cdots\log}_{(n-1)\text{-times}}\big(b|\log kl|\big) - \underbrace{\log\log\cdots\log}_{(n-1)\text{-times}}\big(b|\log ku|\big)\Big) > \frac{-2}{\frac{g(l)}{2} - 1} + \frac{2}{\frac{g(u)}{2} - 1}$$

and, therefore, $\lim_{u\to 0^+} g(u) = 2$. This is a contradiction, thus $\lim_{u\to 0^+} g(u) = 2$. But if $\lim_{u\to 0^+} g(u) = 2$, then from (4.9) we conclude that there exists $d > 0$ such that

$$g'(u) \le \frac{d}{uL_n(u)}$$

for $u > 0$ sufficiently small. Hence, $\lim_{u\to 0^+} g(u) = -\infty$. This is a contradiction and condition (2.2) does not hold. Thus, by Theorem 2.1, system (2.1) fails to have property (Z_1^+). □

The following theorem gives a necessary condition for system (1.1) to have property (Z_1^+).

Theorem 4.2. *If there exist $\lambda > 1/4$, $n \ge 2$, and $k, b > 0$ such that*

$$F(x) \le \sqrt{8G(x)} - \frac{1}{4}\sum_{j=1}^{n-1} \frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2} - \frac{\lambda\sqrt{2G(x)}}{(L_n)(\sqrt{2G(x)})^2}$$

for $x > 0$ sufficiently small, then system (1.1) fails to have property (Z_1^+).

Similarly, we have the following necessary conditions for the properties (Z_2^-), (Z_3^+), and (Z_4^-).

Theorem 4.3. *If there exist $\lambda > 1/4$, $n \ge 2$, and $k, b > 0$ such that*

$$F(x) \le \sqrt{8G(x)} - \frac{1}{4}\sum_{j=1}^{n-1} \frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2} - \frac{\lambda\sqrt{2G(x)}}{(L_n)(\sqrt{2G(x)})^2}$$

for $x < 0$, $|x|$ sufficiently small, then system (1.1) fails to have property (Z_2^-).

Theorem 4.4. *If there exist $\lambda > 1/4$, $n \ge 2$, and $k, b > 0$ such that*

$$F(x) \ge -\sqrt{8G(x)} + \frac{1}{4}\sum_{j=1}^{n-1} \frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2} + \frac{\lambda\sqrt{2G(x)}}{(L_n)(\sqrt{2G(x)})^2}$$

for $x < 0$, $|x|$ sufficiently small, then system (1.1) fails to have property (Z_3^+).

Theorem 4.5. *If there exist $\lambda > 1/4$, $n \ge 2$, and $k, b > 0$ such that*

$$F(x) \ge -\sqrt{8G(x)} + \frac{1}{4}\sum_{j=1}^{n-1} \frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2} + \frac{\lambda\sqrt{2G(x)}}{(L_n)(\sqrt{2G(x)})^2}$$

for $x > 0$ sufficiently small, then system (1.1) fails to have property (Z_4^-).

Remark 4.1. Paying attention to the explicit sufficient and necessary conditions presented for properties (Z_1^+), (Z_2^-), (Z_3^+), and (Z_4^-), it seems that these results have solved the problem of the existence of homoclinic orbits in system (1.1) completely in some sense.

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Authors' addresses:

R. P. Agarwal

Department of Mathematics, Texas A&M University – Kingsville, 700 University Blvd., Kingsville, TX 78363-8202.

E-mail: agarwal@tamuk.edu

A. Aghajani, M. Mirafzal

School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16844-13114, Iran.

E-mail: aghajani@iust.ac.ir; mirafzal@iust.ac.ir

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V. M. Evtukhov, N. P. Kolun

ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

Abstract. The existence conditions and asymptotic representations as $t \uparrow \omega$ ($\omega \leq +\infty$) of one class of monotonous solutions of the n-th order differential equations containing on the right-hand side a sum of terms with regularly varying nonlinearities are established.

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Key words and phrases. Second-order differential equations, regularly varying nonlinearities, rapidly varying nonlinearities, asymptotics of solutions.

რეზიუმე. n-ური რიგის დიფერენციალური განტოლებებისთვის, რომელიც მარჯვენა მხარეში შეიცავს რეგულარულად ცვლადი არაწრფივი წევრების ჯამს, დადგენილია გარკვეული კლასის მონოტონური ამონახსნების არსებობის პირობები და ასიმპტოტური წარმოდგენები, როცა $t \uparrow \omega$ ($\omega \leq +\infty$).

1 Introduction

In the recent decades asymptotic properties of solutions of binomial essentially nonlinear second-order differential equations with a nonlinearity which differs from a power function have been actively studied (for the Emden–Fowler type not generalized equations see the monograph by I. T. Kiguradze and T. A. Chanturiya [13]). The case where the nonlinearity is a regularly varying function was investigated in [9,12,15,16,18], and the case where the nonlinearity is a rapidly varying function can be found in [1,3–5,8]. It should be noted here that the second-order equations containing in the right-hand side a sum of terms with nonlinearities that differ from power functions were considered only in the case when all nonlinearities are regularly varying functions (see, e.g., [6,7]). In this paper, we study the asymptotic properties of solutions of a second-order differential equation in the right-hand side of which, apart from the terms with regularly varying nonlinearities, there are also terms with rapidly varying nonlinearities.

Consider the differential equation

$$y'' = \sum_{i=1}^{m} \alpha_i p_i(t)\varphi_i(y), \tag{1.1}$$

where $\alpha_i \in \{-1,1\}$ $(i=\overline{1,m})$, $p_i : [a,\omega[\, \to]0,+\infty[$ $(i=\overline{1,m})$ are continuous functions, $-\infty < a < \omega \le +\infty$; $\varphi_i : \Delta_{Y_0} \to]0,+\infty[$ $(i=\overline{1,m})$, where Δ_{Y_0} is a one-sided neighborhood of the point Y_0, Y_0 is equal either to 0 or to $\pm\infty$, are continuous functions for $i=\overline{1,l}$ and twice continuously differentiable for $i=\overline{l+1,m}$, such that for each $i\in\{1,\dots,l\}$ as some $\sigma_i\in\mathbb{R}$

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi_i(\lambda y)}{\varphi_i(y)} = \lambda^{\sigma_i} \ \text{ for each } \ \lambda>0, \tag{1.2}$$

and for each $i\in\{l+1,\dots,m\}$,

$$\varphi_i'(y)\neq 0 \ \text{ as } \ y\in\Delta_{Y_0}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi_i(y)\in\{0,+\infty\}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi_i''(y)\varphi_i(y)}{\varphi_i'^2(y)} = 1. \tag{1.3}$$

The functions φ_i $(i=\overline{1,l})$ that satisfy conditions (1.2) are called regularly varying functions as $y\to Y_0$ of orders σ_i $(i=\overline{1,l})$ (see the monograph by E. Seneta [17, Ch. 1, § 1, pp. 9–10]). For each of them the representations of the form

$$\varphi_i(y) = |y|^{\sigma_i} L_i(y) \ \ (i=\overline{1,l}) \tag{1.4}$$

hold, where L_i are the slowly varying functions as $y\to Y_0$, i.e., such that

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{L_i(\lambda y)}{L_i(y)} = 1 \ \ (i=\overline{1,l}) \ \text{ for each } \ \lambda>0.$$

We also say that a function L_i $(i\in\{1,\dots,l\})$ satisfies the condition S_0 if

$$L_i(\nu e^{[1+o(1)]\ln|y|}) = L_i(y)[1+o(1)] \ \text{ as } \ y\to Y_0 \ \ (y\in\Delta_{Y_0}),$$

where $\nu=\operatorname{sign} y$.

Examples of functions slowly varying as $y\to Y_0$ are as follows:

$$|\ln|y||^{\gamma_1}, \ \ |\ln|y||^{\gamma_1}\big|\ln|\ln|y||\big|^{\gamma_2} \ \ (\gamma_1,\gamma_2\neq 0), \ \ e^{\sqrt{|\ln|y||}}.$$

The first two functions satisfy the condition S_0.

From conditions (1.3) it immediately follows that

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{y\varphi_i'(y)}{\varphi_i(y)} = \pm\infty \ \ (i=\overline{l+1,m}),$$

due to which each of the functions φ_i for $i\in\{l+1,\dots,m\}$ and its first derivative are rapidly varying as $y\to Y_0$ (see the monograph by M. Maric [14, Ch. 3, § 3.4, Lemmas 3.2, 3.3, pp. 91–92]).

Definition 1.1. A solution y of the differential equation (1.1) is called a $P_\omega(Y_0,\lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on some interval $[t_0,\omega[\subset [a,\omega[$ and satisfies the following conditions:

$$\lim_{t\uparrow\omega} y(t)=Y_0, \qquad \lim_{t\uparrow\omega} y'(t)=\begin{cases}\text{either} & 0,\\ \text{or} & \pm\infty,\end{cases} \qquad \lim_{t\uparrow\omega}\frac{y'^2(t)}{y''(t)y(t)}=\lambda_0. \tag{1.5}$$

In [10], $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1.1) were studied in the case $\lambda_0\in\mathbb{R}\backslash\{0;1\}$.

In this paper, for $\lambda_0=\pm\infty$, we establish the conditions for the existence of $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1.1) and give asymptotic representations, as $t\uparrow\omega$, of such solutions and their first-order derivatives when in each of such solutions the right-hand side of equation is equivalent, as $t\uparrow\omega$, to the s-th item, i.e., when for some $s\in\{1,\dots,l\}$,

$$\lim_{t\uparrow\omega}\frac{p_i(t)\varphi_i(y(t))}{p_s(t)\varphi_s(y(t))}=0 \ \text{ for all } \ i\in\{1,\dots,m\}\setminus\{s\}. \tag{1.6}$$

Upon studying the $P_\omega(Y_0,\pm\infty)$-solutions of equation (1.1), some of their a priori asymptotic properties will be used.

We set

$$\pi_\omega(t)=\begin{cases} t & \text{if } \ \omega=+\infty,\\ t-\omega & \text{if } \ \omega<+\infty.\end{cases}$$

Lemma 1.1. *Let $y:[t_0,\omega[\to\mathbb{R}$ be an arbitrary $P_\omega(Y_0,\pm\infty)$-solution of equation* (1.1). *Then*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)y'(t)}{y(t)}=1, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)y''(t)}{y'(t)}=0. \tag{1.7}$$

The validity of this assertion follows directly from [2] (see Corollary 10.1).

2 Statement of the main results

Here and in the sequel, without loss of generality, we assume that

$$\Delta_{Y_0}=\Delta_{Y_0}(b),$$

where

$$\Delta_{Y_0}(b)=\begin{cases}[b,Y_0[\,, & \text{if } \ \Delta_{Y_0} \ \text{ is a left neighborhood of } Y_0,\\ ]Y_0,b], & \text{if } \ \Delta_{Y_0} \ \text{ is a right neighborhood of } Y_0,\end{cases}$$

and the number b satisfies the inequalities

$$|b|<1 \ \text{ as } \ Y_0=0 \ \text{ and } \ b>1 \ \ (b<-1) \ \text{ as } \ Y_0=+\infty \ \ (Y_0=-\infty).$$

In addition, let us introduce two numbers

$$\nu_0=\operatorname{sign} b, \quad \nu_1=\begin{cases}1, & \text{if } \ \Delta_{Y_0}(b)=[b,Y_0[\,,\\ -1, & \text{if } \ \Delta_{Y_0}(b)=]Y_0,b].\end{cases}$$

According to the definition of the $P_\omega(Y_0,\lambda_0)$-solution of the differential equation (1.1), note that the numbers ν_0 and ν_1 determine the signs of any $P_\omega(Y_0,\lambda_0)$-solution and its first derivative (respectively) in some left neighborhood of ω. The conditions

$$\nu_0\nu_1=-1 \ \text{ if } \ Y_0=0, \quad \nu_0\nu_1=1 \ \text{ if } \ Y_0=\pm\infty$$

are necessary for the existence of $P_\omega(Y_0,\lambda_0)$-solutions.

Moreover, if for such solutions of (1.1) conditions (1.6) hold, then

$$y''(t)=\alpha_s p_s(t)\varphi_s(y(t))[1+o(1)] \ \text{ as } \ t\uparrow\omega, \tag{2.1}$$

from which it is clear that $\operatorname{sign} y''(t) = \alpha_s$ in some left neighborhood of ω, and in this case

$$\nu_1\alpha_s = -1 \ \text{ if } \ \lim_{t\uparrow\omega} y'(t) = 0, \quad \nu_1\alpha_s = 1 \ \text{ if } \ \lim_{t\uparrow\omega} y'(t) = \pm\infty.$$

In the case where $\nu_0 \lim_{t\uparrow\omega} |\pi_\omega(t)| = Y_0$, we choose the number $a_1 \in [a,\omega[$ so that $\nu_0|\pi_\omega(t)| \in \Delta_{Y_0}(b)$ as $t \in [a_1,\omega[$, and for $s \in \{1,\dots,l\}$ set

$$J_s(t) = \int_{A_s}^{t} p_s(\tau)\varphi_s(\nu_0|\pi_\omega(\tau)|)\,d\tau,$$

where

$$A_s = \begin{cases} a_1 & \text{if } \displaystyle\int_{a_1}^{\omega} p_s(\tau)\varphi_s(\nu_0|\pi_\omega(\tau)|)\,d\tau = \pm\infty, \\ \omega & \text{if } \displaystyle\int_{a_1}^{\omega} p_s(\tau)\varphi_s(\nu_0|\pi_\omega(\tau)|)\,d\tau = const. \end{cases}$$

Theorem 2.1. *Let $\sigma_s \neq 1$ for some $s \in \{1,\dots,l\}$ and the function L_s satisfy the condition S_0. Then for the existence of $P_\omega(Y_0,\pm\infty)$-solutions satisfying condition* (1.6) *of the differential equation* (1.1) *it is necessary that*

$$\nu_0 \lim_{t\uparrow\omega} |\pi_\omega(t)| = Y_0, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_s'(t)}{J_s(t)} = 0, \tag{2.2}$$

the inequalities

$$\alpha_s\nu_1(1-\sigma_s)J_s(t) > 0, \quad \nu_0\nu_1\pi_\omega(t) > 0 \ \text{ for } \ t \in]a_1,\omega[\,, \tag{2.3}$$

as well as the conditions

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})}{p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})} = 0 \tag{2.4}$$

for all $i \in \{1,\dots,l\} \setminus \{s\}$ and

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}(1+\delta_i))}{p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})} = 0 \tag{2.5}$$

for all $i \in \{l+1,\dots,m\}$ hold, where δ_i are arbitrary numbers of some one-sided neighborhood of zero. Moreover, for each of such solutions the following asymptotic representations are valid:

$$y(t) = \nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}[1+o(1)] \ \text{ as } \ t\uparrow\omega, \tag{2.6}$$

$$y'(t) = \nu_1|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}[1+o(1)] \ \text{ as } \ t\uparrow\omega. \tag{2.7}$$

Proof. Let $y : [t_0,\omega[\to \mathbb{R}$ be an arbitrary $P_\omega(Y_0,\pm\infty)$-solution for some $s \in \{1,\dots,l\}$ satisfying conditions (1.6) of equation (1.1). Then by virtue of (1.1) and (1.6), the asymptotic relation (2.1) holds.

According to Lemma 1.1, the limit relations (1.7) are valid, from which, in particular, it follows that the function y is regularly varying, as $t\uparrow\omega$, function of first order. Therefore, by virtue of the function L_s satisfying the condition S_0, representations (1.4) and the first of the limit relations (1.7), we have

$$\begin{aligned}\varphi_s(y(t)) = |y(t)|^{\sigma_s}L_s(y(t)) &= |y(t)|^{\sigma_s}L_s(\nu_0 e^{[1+o(1)]\ln|\pi_\omega(t)|}) \\ &= |\pi_\omega(t)y'(t)|^{\sigma_s}L_s(\nu_0|\pi_\omega(t)|)[1+o(1)] \ \text{ as } \ t\uparrow\omega.\end{aligned}$$

Taking into account this asymptotic relation, from (2.1) we obtain

$$\frac{y''(t)}{|y'(t)|^{\sigma_s}} = \alpha_s p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|)[1+o(1)] \ \text{ for } \ t\uparrow\omega. \tag{2.8}$$

Integrating (2.8) on the interval from t_1 ($t_1 \in [t_0, \omega[$) to t and using the second of conditions (1.5), we get

$$\nu_1 |y'(t)|^{1-\sigma_s} = \alpha_s (1-\sigma_s) J_s(t)[1+o(1)] \ \text{ as } \ t \uparrow \omega,$$

which implies representation (2.7) and the equality

$$\nu_1 = \alpha_s \operatorname{sign}[(1-\sigma_s) J_s(t)]. \tag{2.9}$$

From the first relation of (1.7) follows the second of inequalities (2.3), so taking into account (2.9), the first of inequalities (2.3) holds. Taking into account the first of limiting relations (1.7), the second inequality of (2.3) and (2.7), we obtain the asymptotic representation (2.6). The validity of the first limit relation of (2.2) follows from Definition 1.1 and the first equality of (1.7) of Lemma 1.1. The second limit relation of (2.2) follows immediately from (2.8) if we use the above-mentioned representation (2.7) and the second of conditions (1.7).

Since the functions φ_i ($i = \overline{1, l}$) are regularly varying as $y \to Y_0$, we have

$$\begin{aligned} &\varphi_i\big(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)} [1+o(1)]\big) \\ &\qquad = \varphi_i\big(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)}\big)[1+o(1)] \ \text{ as } \ t \uparrow \omega. \end{aligned}$$

Then, by virtue of (2.6),

$$\begin{aligned} \lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(y(t))}{p_s(t)\varphi_s(y(t))} &= \lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})[1+o(1)]}{p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})[1+o(1)]} \\ &= \lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})}{p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})} \ \ (i = \overline{1, l}) \end{aligned}$$

hence, taking into account (1.6), we find that conditions (2.4) are valid.

For $i \in \{l+1, \dots, m\}$, from (2.6) we have

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(y(t))}{p_s(t)\varphi_s(y(t))} = \lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}[1+o(1)])}{p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})}. \tag{2.10}$$

By the monotony of function φ_i ($i \in \{l+1, \dots, m\}$) on the interval $\Delta_{Y_0}(b)$ for each of δ_i from some one-sided neighborhood of zero there exists $t_2 \in [t_1, \omega[$ such that for $t \in [t_2, \omega[$, we have

$$\begin{aligned} &\frac{p_i(t)\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}[1+o(1)])}{p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})} \\ &\qquad \geq \frac{p_i(t)\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}[1+\delta_i])}{p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})} > 0. \end{aligned}$$

Thus, by virtue of (1.6) and (2.10), we find that conditions (2.5) are valid. The proof of the theorem is complete. □

Now we clarify the question of the actual existence of $P_\omega(Y_0, \pm\infty)$-solutions with the asymptotic representations (2.6) and (2.7) for equation (1.1).

Theorem 2.2. *Let for some $s \in \{1, \dots, l\}$ the function L_s satisfy the condition S_0, the inequality $\sigma_s \neq 1$ and conditions (2.2)–(2.4) hold, and for any $i \in \{l+1, \dots, m\}$,*

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}(1+u))}{p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})} = 0 \tag{2.11}$$

uniformly with respect to $u \in [-\delta, \delta]$ for some $0 < \delta < 1$. Then the differential equation (1.1) has at least one $P_\omega(Y_0, \pm\infty)$-solution that admits asymptotic representations (2.6) and (2.7). Moreover, if $\omega = +\infty$ and $A_s = +\infty$, there exists a one-parameter family with such representations, and if $A_s = a_1$, there is a two-parameter family.

Proof. By virtue of conditions (2.2) and (2.3), the function

$$Y(t)=\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}$$

is a first-order function that varies regularly as $t\uparrow\omega$,

$$\lim_{t\uparrow\omega}Y(t)=Y_0$$

and there exists a number $t_0\in[a_1,\omega[$ such that

$$Y(t)[1+u]\in\Delta_{Y_0}(b)\ \text{ for }\ t\in[t_0,\omega[\ \text{ and }\ |u|\le\delta.$$

By virtue of the properties of slowly varying functions, taking into account the fact that the function L_s satisfies the condition S_0, we have

$$\varphi_s(Y(t)(1+u))=|Y(t)(1+u)|^{\sigma_s}L_s(\nu_0|\pi_\omega(t)|)[1+R(t,u)],$$

where the function R is such that

$$\lim_{t\uparrow\omega}R(t,u)=0\ \text{ uniformly with respect to }\ u\in[-\delta,\delta].$$

Now applying to equation (1.1) the transformation

$$\begin{aligned}y(t)&=\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}[1+u_1(t)],\\ y'(t)&=\nu_1|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}[1+u_2(t)],\end{aligned}\tag{2.12}$$

taking into account inequalities (2.3), we obtain a system of differential equations

$$\begin{cases}u_1'=h_1(t)[f_1(t,u_1)-u_1+u_2],\\ u_2'=h_2(t)\big[f_2(t,u_1)+\sigma_su_1-u_2+V(u_1)\big],\end{cases}\tag{2.13}$$

where

$$h_1(t)=\frac{1}{\pi_\omega(t)}\,,\quad h_2(t)=\frac{J_s'(t)}{(1-\sigma_s)J_s(t)}\,,$$

$$\begin{aligned}f_1(t,u_1)&=-\frac{\pi_\omega(t)J_s'(t)}{(1-\sigma_s)J_s(t)}\,(1+u_1),\\ f_2(t,u_1)&=(1+u_1)^{\sigma_s}R(t,u_1)+(1+u_1)^{\sigma_s}(1+R(t,u_1))R_1(t,u_1),\end{aligned}$$

$$R_1(t,u_1)=\sum_{\substack{i=1\\ i\ne s}}^{m}\frac{\alpha_ip_i(t)\varphi_i(Y(t)(1+u_1))}{\alpha_sp_s(t)\varphi_s(Y(t)(1+u_1))}\,,\quad V(u_1)=(1+u_1)^{\sigma_s}-1-\sigma_su_1.$$

We consider system (2.13) on the set

$$\Omega=[t_0,\omega[\times D,\ \text{ where }\ D=\big\{(u_1,u_2):\ |u_i|\le\delta,\ i=1,2\big\}.$$

We show that the function R_1 is such that

$$\lim_{t\uparrow\omega}R_1(t,u_1)=0\ \text{ uniformly with respect to }\ u_1\in[-\delta,\delta].\tag{2.14}$$

Since the functions φ_i with $i\in\{1,\dots,l\}$ are regularly varying of orders σ_i as $y\to Y_0$, by virtue of (1.4), taking into account the properties of slowly varying functions, we have

$$\begin{aligned}\varphi_i(Y(t)(1+u_1))&=\varphi_i\big(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}(1+u_1)\big)\\ &=|\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}(1+u_1)|^{\sigma_i}L_i\big(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}(1+u_1)\big)\\ &=|\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}(1+u_1)|^{\sigma_i}L_i\big(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})(1+r_i(t,u_1)\big)\\ &=\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})(1+u_1)^{\sigma_i}(1+r_i(t,u_1))\ \ (i=\overline{1,l})\end{aligned}$$

where the functions r_i are such that

$$\lim_{t\uparrow\omega} r_i(t,u_1)=0 \ \text{ uniformly with respect to } \ u_1\in[-\delta,\delta].$$

By virtue of the above conditions,

$$\lim_{t\uparrow\omega}\sum_{\substack{i=1\\ i\neq s}}^{l}\frac{\alpha_i p_i(t)\varphi_i(Y(t)(1+u_1))}{\alpha_s p_s(t)\varphi_s(Y(t)(1+u_1))}=0 \tag{2.15}$$

uniformly with respect to $u_1\in[-\delta,\delta]$, since due to (2.4),

$$\begin{aligned}
&\lim_{t\uparrow\omega}\sum_{\substack{i=1\\ i\neq s}}^{l}\frac{\alpha_i p_i(t)\varphi_i(Y(t)(1+u_1))}{\alpha_s p_s(t)\varphi_s(Y(t)(1+u_1))}\\
&\qquad=\lim_{t\uparrow\omega}\sum_{\substack{i=1\\ i\neq s}}^{l}\frac{\alpha_i p_i(t)\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})(1+r_i(t,u_1))}{\alpha_s p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})(1+r_s(t,u_1))}\\
&=\lim_{t\uparrow\omega}\sum_{\substack{i=1\\ i\neq s}}^{l}\frac{\alpha_i p_i(t)\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})}{\alpha_s p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})}=0 \ \text{ uniformly with respect to } \ u_1\in[-\delta,\delta].
\end{aligned}$$

From (2.11) and (2.15), by virtue of the form of function R_1, we find that (2.14) is valid. In the system of equations (2.13) the functions h_1, $h_2:[t_0,\omega[\to\mathbb{R}$ are continuous and are such that

$$h_1(t)h_2(t)\neq 0 \ \text{ for } \ t\in[t_0,\omega[\,,$$

$$\int_{t_0}^{\omega} h_2(\tau)\,d\tau=\frac{1}{1-\sigma_s}\int_{t_0}^{\omega}\frac{J_s'(\tau)}{J_s(\tau)}\,d\tau=\frac{1}{1-\sigma_s}\,\ln|J_s(\tau)|\,\Big|_{t_0}^{\omega}=\pm\infty.$$

In addition, by virtue of the second of conditions (2.2), we have

$$\lim_{t\uparrow\omega}\frac{h_2(t)}{h_1(t)}=\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_s'(t)}{(1-\sigma_s)J_s(t)}=0.$$

Further, by the form of the functions V, f_k $(k=1,2)$, we have

$$\frac{h_1(t)}{h_2(t)}\,f_1(t,u_1) \ \text{ is bounded on the set } \Omega,$$

$$\lim_{u_1\to 0}\frac{dV(u_1)}{du_1}=0,$$

$$\lim_{t\uparrow\omega} f_2(t,u_1)=0 \ \text{ uniformly with respect to } \ u_1\in[-\delta,\delta].$$

Coefficient at u_1 in square brackets of the first equation of system (2.13) is nonzero. In addition, the sum of the coefficients of u_1 and u_2 in the square brackets of the first equation of system (2.13) is zero, and in the second equation is equal to the number σ_s-1, which is nonzero. This implies that system (2.13) satisfies all the assumptions of Theorem 2.7 of [11]. According to this theorem, the system of differential equations (2.13) has at least one solution $u=(u_1,u_2):[t_*,\omega[\to\mathbb{R}^2$ $(t_*\ge t_0)$, tending to zero as $t\uparrow\omega$. Each solution of this kind of system (2.13), by virtue of transformations (2.12), corresponds to the solution of the differential equation (1.1) that admits, as $t\uparrow\omega$, asymptotic representations (2.6), (2.7), and this solution is the $P_\omega(Y_0,\pm\infty)$-solution of equation (1.1). Moreover, if $\omega=+\infty$, then there exists a one-parameter family of such solutions if $\frac{J_s'(t)}{J_s(t)}<0$ on $]a_1,+\infty[$ (this inequality holds when J_s is chosen for the integration limit of A_s to be equal to $+\infty$), and a two-parameter family if the inequality $\frac{J_s'(t)}{J_s(t)}>0$ holds (i.e., when $A_s=a_1$). The proof of the theorem is complete. □

Remark. In the case when there are no terms in equation (1.1) with rapidly varying nonlinearity, i.e., when $m = l$, the assertion of Theorems 2.1 and 2.2 remains true without conditions (2.5) and (2.11).

3 Example

As an example illustrating the results obtained in this paper, we consider a differential equation of the form

$$y'' = \alpha_1 p_1(t)|y|^\sigma + \alpha_2 p_2(t)e^{\mu y}, \tag{3.1}$$

in which $\alpha_i \in \{-1,1\}$ $(i=1,2)$, $p_i : [a,\omega[\to]0,+\infty[$ $(i=1,2)$ are continuous functions, $-\infty < a < \omega \le +\infty$, $\mu \neq 0$.

For equation (3.1) let us clarify the existence of $P_\omega(Y_0,\pm\infty)$-solutions for which

$$\lim_{t\uparrow\omega} y(t) = \pm\infty \ \ (Y_0 = \pm\infty), \quad \lim_{t\uparrow\omega} \frac{p_2(t)e^{\mu y(t)}}{p_1(t)|y(t)|^\sigma} = 0. \tag{3.2}$$

From Theorems 2.1 and 2.2 we have

Corollary 3.1. *Suppose that inequality $\sigma \neq 1$ holds. Then for the existence of $P_\omega(Y_0,\pm\infty)$-solutions of the differential equation* (3.1) *satisfying conditions* (3.2) *it is necessary, and if*

$$p_2(t) = o\Big(\frac{p_1(t)t^\sigma|(1-\sigma)J_1(t)|^{\frac{\sigma}{1-\sigma}}}{e^{\mu\nu_0 t|(1-\sigma)J_1(t)(1+u)|^{\frac{1}{1-\sigma}}}}\Big) \ \ as \ \ t \to +\infty$$

uniformly with respect to $u \in [-\delta,\delta]$ for some $0<\delta<1$, it is sufficient that the conditions

$$\omega = +\infty, \quad \lim_{t\to+\infty} \frac{tJ_1'(t)}{J_1(t)} = 0,$$
$$\nu_0\nu_1 > 0, \quad \alpha_1\nu_1(1-\sigma)J_1(t) > 0 \ \ for \ \ t \in]a_1,+\infty[$$

hold. Moreover, each solution of that kind admits the asymptotic representations

$$y(t) = \nu_0 t|(1-\sigma)J_1(t)|^{\frac{1}{1-\sigma}}[1+o(1)] \ \ as \ \ t \to +\infty,$$
$$y'(t) = \nu_1|(1-\sigma)J_1(t)|^{\frac{1}{1-\sigma}}[1+o(1)] \ \ as \ \ t \to +\infty.$$

Moreover, if $A_s = +\infty$, there exists a one-parameter family with such representations, and in case $A_s = a_1$, there is a two-parameter family.

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Authors' address:

Odessa I. I. Mechnikov National University, 2 Dvoryanska St., Odessa 65082, Ukraine.

E-mail: `emden@farlep.net`; `nataliiakolun@ukr.net`

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Ivan Kiguradze, Nino Partsvania

SOME OPTIMAL CONDITIONS FOR THE SOLVABILITY AND UNIQUE SOLVABILITY OF THE TWO–POINT NEUMANN PROBLEM

Abstract. For second order ordinary differential equations, unimprovable sufficient conditions are established for the solvability and unique solvability of the Neumann boundary value problem.

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Key words and phrases. Second order ordinary differential equation, linear, nonlinear, the Neumann problem, existence theorem, uniqueness theorem.

რეზიუმე. მეორე რიგის ჩვეულებრივი დიფერენციალური განტოლებებისათვის დადგენილია ნეიმანის სასაზღვრო ამოცანის ამოხსნადობისა და ცალსახად ამოხსნადობის არაგაუმჯობესებადი საკმარისი პირობები.

1 Formulation of the main results

On a finite interval $[a,b]$, we consider the differential equation

$$u'' = f(t,u) \tag{1.1}$$

with the Neumann two-point boundary conditions

$$u'(a) = c_1, \ \ u'(b) = c_2, \tag{1.2}$$

where $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a function satisfying the local Carathéodory conditions, while c_1 and c_2 are real constants.

A number of interesting and unimprovable in a certain sense results concerning the existence and uniqueness of a solution of problem (1.1), (1.2) are known (see, e.g., [1–3, 5–8, 12] and the references therein). In the present paper, general theorems on the existence and uniqueness of a solution of that problem are proved which are nonlinear analogues of the first Fredholm theorem. Based on these theorems, unimprovable sufficient conditions, different from the above mentioned results, for the solvability and unique solvability of problem (1.1), (1.2) are obtained.

We use the following notation.

$\mathbb{R}$ is the set of real numbers; $\mathbb{R}_+ = [0,+\infty[$; $\mathbb{R}_- =]-\infty, 0]$;

$$[x]_- = \frac{|x| - x}{2};$$

$L([a,b])$ is the space of Lebesgue integrable functions.

Definition 1.1. Let $p_i \in L([a,b])$ $(i = 1,2)$ and

$$p_1(t) \le p_2(t) \ \text{ for almost all } \ t \in [a,b]. \tag{1.3}$$

We say that the vector function (p_1, p_2) belongs to the set $\boldsymbol{\mathcal{N}\mathrm{eum}([a,b])}$ if for any measurable function $p : [a,b] \to \mathbb{R}$, satisfying the inequality

$$p_1(t) \le p(t) \le p_2(t) \ \text{ for almost all } \ t \in [a,b], \tag{1.4}$$

the homogeneous Neumann problem

$$u'' = p(t)u, \tag{1.5}$$

$$u'(a) = 0, \ \ u'(b) = 0 \tag{1.6}$$

has only the trivial solution.

Theorem 1.1. *Let there exist* $(p_1, p_2) \in \boldsymbol{\mathcal{N}\mathrm{eum}([a,b])}$ *and an integrable in the first and non-decreasing in the second argument function* $q : [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\lim_{x \to +\infty} \int_a^b \frac{q(t,x)}{x}\, dt = 0, \tag{1.7}$$

and on the set $[a,b] \times \mathbb{R}$ *the inequality*

$$p_1(t)|x| - q(t,|x|) \le f(t,x)\operatorname{sgn}(x) \le p_2(t)|x| + q(t,|x|) \tag{1.8}$$

holds. Then problem (1.1), (1.2) *has at least one solution.*

Corollary 1.1. *Let on the set* $[a,b] \times \mathbb{R}$ *inequality* (1.8) *be satisfied, where* $p_i \in L([a,b])$ $(i = 1,2)$ *are the functions satisfying inequality* (1.3), *and* $q : [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ *is an integrable in the first and non-decreasing in the second argument function satisfying condition* (1.7). *Let, moreover,*

$$\int_a^b p_2(t)\, dt \le 0, \ \ \operatorname{mes}\{[t \in [a,b] : \ p_2(t) < 0\} > 0, \tag{1.9}$$

and there exist a number $\lambda \geq 1$ *such that*

$$\int\limits_a^b [p_1(t)]_-^\lambda \, dt \leq \frac{4(b-a)}{\pi^2} \left(\frac{\pi}{b-a}\right)^{2\lambda}. \tag{1.10}$$

Then problem (1.1), (1.2) *has at least one solution.*

Corollary 1.2. *Let on the set* $[a,b] \times \mathbb{R}$ *inequality* (1.8) *be satisfied, where* $p_1 : [a,b] \to \mathbb{R}_-$ *and* $p_2 : [a,b] \to \mathbb{R}$ *are integrable functions satisfying inequalities* (1.3) *and* (1.9)*, while* $q : [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ *is an integrable in the first and non-decreasing in the second argument function satisfying condition* (1.7)*. Let, moreover, there exist* $t_0 \in]a,b[$ *such that the function* p_1 *is non-increasing and non-decreasing in the intervals* $]a,t_0[$ *and* $]t_0,b[$*, respectively, and*

$$\int\limits_a^{t_0} \sqrt{|p_1(t)|} \, dt \leq \frac{\pi}{2}, \quad \int\limits_{t_0}^b \sqrt{|p_1(t)|} \, dt \leq \frac{\pi}{2}, \quad \int\limits_a^b \sqrt{|p_1(t)|} \, dt < \pi. \tag{1.11}$$

Then problem (1.1), (1.2) *has at least one solution.*

Theorem 1.2. *Let on the set* $[a,b] \times \mathbb{R}$ *the inequality*

$$p_1(t)|x-y| \leq (f(t,x) - f(t,y)) \operatorname{sgn}(x-y) \leq p_2(t)|x-y| \tag{1.12}$$

be satisfed, where $(p_1,p_2) \in \boldsymbol{\mathcal{N}\mathrm{eum}([a,b])}$*. Then problem* (1.1), (1.2) *has one and only one solution.*

Corollary 1.3. *Let on the set* $[a,b] \times \mathbb{R}$ *condition* (1.12) *hold, where* $p_i \in L([a,b])$ $(i=1,2)$ *are the functions satisfying inequalities* (1.3) *and* (1.9)*. If, moreover, for some* $\lambda \geq 1$ *inequality* (1.10) *is satisfied, then problem* (1.1), (1.2) *has one and only one solution.*

Corollary 1.4. *Let on the set* $[a,b] \times \mathbb{R}$ *inequality* (1.12) *hold, where* $p_1 : [a,b] \to \mathbb{R}_-$ *and* $p_2 : [a,b] \to \mathbb{R}$ *are integrable functions satisfying inequalities* (1.3) *and* (1.9)*. Let, moreover, there exist* $t_0 \in]a,b[$ *such that the function* p_2 *is non-increasing and non-decreasing in the intervals* $]a,t_0[$ *and* $]t_0,b[$*, respectively, and satisfies inequality* (1.11)*. Then problem* (1.1), (1.2) *has one and only one solution.*

The following two corollaries of Theorem 1.2 concern the linear differential equation

$$u'' = p(t)u + q(t), \tag{1.13}$$

where p and $q \in L([a,b])$.

Corollary 1.5. *Let*

$$\int\limits_a^b p(t) \, dt \leq 0, \quad \operatorname{mes}\{t \in [a,b] : \ p(t) < 0\} > 0, \tag{1.14}$$

and let there exist a number $\lambda \geq 1$ *such that*

$$\int\limits_a^b [p(t)]_-^\lambda \, dt \leq \frac{4(b-a)}{\pi^2} \left(\frac{\pi}{b-a}\right)^{2\lambda}. \tag{1.15}$$

Then problem (1.13), (1.2) *has one and only one solution.*

Corollary 1.6. *Let there exist a number* $t_0 \in]a,b[$ *such that the function* p *along with* (1.14) *satisfies the conditions*

$$p_0(t) = \operatorname{ess\,sup}\big\{[p(s)]_- : \ a < s < t\big\} < +\infty \quad \text{for } a < t < t_0, \tag{1.16}$$

$$p_0(t) = \operatorname{ess\,sup}\big\{[p(s)]_- : \ t < s < b\big\} < +\infty \quad \text{for } t_0 < t < b, \tag{1.17}$$

$$\int\limits_a^{t_0} \sqrt{p_0(t)} \, dt \leq \frac{\pi}{2}, \quad \int\limits_{t_0}^b \sqrt{p_0(t)} \, dt \leq \frac{\pi}{2}, \quad \int\limits_a^b \sqrt{p_0(t)} \, dt < \pi. \tag{1.18}$$

Then problem (1.13), (1.2) *has one and only one solution.*

Remark 1.1. In the case, where instead of (1.14) the more hard condition

$$p(t) \le 0 \quad \text{for} \;\; a < t < b, \quad \operatorname{mes}\{t \in [a,b] : \; p(t) < 0\} > 0 \tag{1.19}$$

is satisfied, the results analogous to Corollary 1.5 previously were obtained in [5,6,12]. More precisely, in [12] it is required that along with (1.19) the inequalities

$$\int_a^b |p(t)|\,dt \le \frac{4}{b-a}, \quad \operatorname{ess\,sup}\{|p(t)| : \; a \le t \le b\} < +\infty$$

be satisfied (see [12, Theorem 3]), while in [5] and [6] it is assumed, respectively, that

$$\int_a^b |p(t)|\,dt \le \frac{4}{b-a}$$

(see [5, Corollary 1.2]), and

$$\int_a^b |p(t)|^\lambda\,dt \le \frac{4(b-a)}{\pi^2}\left(\frac{\pi}{b-a}\right)^{2\lambda},$$

where $\lambda \equiv const \ge 1$ (see [6, Corollary 1.3]).

Example 1.1. Suppose

$$p(t) \equiv -\left(\frac{\pi}{b-a}\right)^2,$$

ε is arbitrarily small positive number, while λ is so large that

$$\left(1 + \frac{\varepsilon}{\pi}\right)^\lambda > \frac{\pi}{2}.$$

Then instead of (1.15) the inequality

$$\int_a^b [p(t)]_-^\lambda\,dt < \frac{4(b-a)}{\pi^2}\left(\frac{\pi+\varepsilon}{b-a}\right)^{2\lambda} \tag{1.20}$$

is satisfied. On the other hand, the homogeneous problem (1.5), (1.6) has a nontrivial solution $u_0(t) = \cos\frac{\pi(t-a)}{b-a}$, and the nonhomogeneous problem (1.13), (1.2) has no solution if only

$$c_1 + c_2 + \int_a^b u_0(t)q(t)\,dt \ne 0.$$

Consequently, condition (1.15) in Corollary 1.5 is unimprovable and it cannot be replaced by condition (1.20).

The above example shows also that condition (1.10) in Corollaries 1.1 and 1.3 is unimprovable in the sense that it cannot be replaced by the condition

$$\int_a^b [p_1(t)]_-^\lambda\,dt < \frac{4(b-a)}{\pi^2}\left(\frac{\pi+\varepsilon}{b-a}\right)^{2\lambda},$$

where ε is a positive constant independent of λ.

Note that condition (1.10) in the above mentioned corollaries is unimprovable also in the case where $\lambda = 1$, and it cannot be replaced by the condition

$$\int\limits_a^b [p_1(t)]_- \, dt < \frac{4+\varepsilon}{b-a}$$

no matter how small $\varepsilon > 0$ would be (see [5, p. 357, Remark 1.1]).

Example 1.2. Suppose $t_0 \in \,]a,b[$ and

$$p(t) = \begin{cases} -\dfrac{\pi^2}{4(t_0-a)^2} & \text{for } \ a \le t \le t_0, \\[2ex] -\dfrac{\pi^2}{4(b-t_0)^2} & \text{for } \ t_0 < t \le b. \end{cases}$$

Then inequalities (1.16), (1.17) hold, and instead of (1.18) we have

$$\int\limits_a^{t_0} \sqrt{p_0(t)} \, dt = \frac{\pi}{2}, \quad \int\limits_{t_0}^b \sqrt{p_0(t)} \, dt = \frac{\pi}{2}.$$

On the other hand, the homogeneous problem (1.5), (1.6) has a nontrivial solution

$$u_0(t) = \begin{cases} (t_0-a)\cos\dfrac{\pi(t-a)}{2(t_0-a)} & \text{for } \ a \le t \le t_0, \\[2ex] (t_0-b)\cos\dfrac{\pi(b-t)}{2(b-t_0)} & \text{for } \ t_0 < t \le b, \end{cases}$$

while the nonhomogeneous problem (1.13), (1.2) has no solution if only

$$(t_0-a)c_1 + (b-t_0)c_2 + \int\limits_a^b u_0(t)q(t) \, dt \neq 0.$$

Consequently, condition (1.18) in Corollary 1.6 is unimprovable in the sense that it cannot be replaced by the condition

$$\int\limits_a^{t_0} \sqrt{p_0(t)} \, dt \le \frac{\pi}{2}, \quad \int\limits_{t_0}^b \sqrt{p_0(t)} \, dt \le \frac{\pi}{2}.$$

From the above said it is also clear that condition (1.11) in both Corollary 1.2 and Corollary 1.4 is unimprovable and it cannot be replaced by the condition

$$\int\limits_a^{t_0} \sqrt{|p_1(t)|} \, dt \le \frac{\pi}{2}, \quad \int\limits_{t_0}^b \sqrt{|p_1(t)|} \, dt \le \frac{\pi}{2}\,.$$

2 Auxiliary propositions

2.1. Lemma on a priori estimate. In the segment $[a,b]$, we consider the differential inequality

$$p_1(t)|u(t)| - q(t) \le u''(t)\operatorname{sgn}(u(t)) \le p_2(t)|u(t)| + q(t), \tag{2.1}$$

where

$$(p_1,p_2) \in \mathcal{N}\mathbf{eum}([a,b]), \tag{2.2}$$

and $q \in L([a,b])$ is a non-negative function.

A function $u : [a,b] \to \mathbb{R}$ is said to be a solution of the differential inequality (2.1) if it is continuously differentiable, has an absolutely continuous on $[a,b]$ first derivative, and almost everywhere on this segment satisfies inequality (2.1).

Lemma 2.1. *If condition* (2.2) *holds, then there exists a positive constant* r_0 *such that for any non-negative function* $q \in L([a,b])$ *every solution of the differential inequality* (2.1) *admits the estimate*

$$|u(t)| \le r_o\bigg(|u'(a)| + |u'(b)| + \int_a^b q(s)\,ds\bigg) \quad \text{for } a \le t \le b. \tag{2.3}$$

Proof. Assume the contrary that the lemma is not true. Then for any natural number k there exist a non-negative function $q_k \in L([a,b])$ and a solution u_k of the differential inequality (2.1) such that

$$\|u_k\| > k^2\bigg(|u_k'(a)| + |u_k'(b)| + \int_a^b q_k(s)\,ds\bigg),$$

where $\|u_k\| = \max\{|u_k(t)| : \ t \in [a,b]\}$.

Let I_k be the set of all $t \in [a,b]$ at which there exists $u_k''(t)$,

$$u_{0k}(t) = u_k(t)/\|u_k\| \ \text{ for } t \in [a,b], \quad q_{0k}(t) = kq(t)/\|u_k\| \ \text{ for } t \in I_k.$$

Then

$$p_1(t)|u_{0k}(t)| - q_{0k}(t)/k \le u_{0k}''(t)\operatorname{sgn}(u_{0k}(t)) \le p_2(t)|u_{0k}(t)| + q_{0k}(t)/k \ \text{ for } t \in I_k, \tag{2.4}$$

$$|u_{0k}'(a)| + |u_{0k}'(b)| < \frac{1}{k}, \quad \|u_{0k}\| = 1, \tag{2.5}$$

$$\int_a^b q_{0k}(s)\,ds < \frac{1}{k}. \tag{2.6}$$

Put

$$I_{1k} = \Big\{t \in I_k : \ |u_{0k}(t)| \ge \frac{1}{k}\Big\}, \quad I_{2k} = I_k \setminus I_{1k},$$

$$p_{0k}(t) = \begin{cases} \dfrac{u_{0k}''(t)}{u_{0k}(t)} & \text{for } t \in I_{1k}, \\[2mm] p_1(t) & \text{for } t \in I_{2k}, \end{cases}$$

$$q_{1k}(t) = \begin{cases} 0 & \text{for } t \in I_{1k}, \\[2mm] u_{0k}''(t) - p_1(t)u_{0k}(t) & \text{for } t \in I_{2k}, \end{cases}$$

$$P_k(t) = \int_a^t p_{0k}(s)\,ds.$$

Then

$$u_{0k}''(t) = p_{0k}(t)u_{0k}(t) + q_{1k}(t) \ \text{ for } t \in I_k. \tag{2.7}$$

On the other hand, according to conditions (2.4) and (2.5) we have

$$|u_{0k}''(t)| \le \ell(t) + q_{0k}(t) \ \text{ for } t \in I_k,$$
$$p_1(t) - q_{0k}(t) \le p_{0k}(t) \le p_2(t) + q_{0k}(t) \ \text{ for } t \in I_k,$$
$$|q_{1k}(t)| \le \left(|p_1(t)| + \ell(t) + q_{0k}(t)\right)/k \ \text{ for } t \in I_k,$$

where $\ell(t) = |p_1(t)| + |p_2(t)|$.

If along with these estimates we take into account inequality (2.6), then it becomes evident that

$$|u'_{0k}(t) - u'_{0k}(\tau)| \leq \int_\tau^t \ell(s)\,ds + \frac{1}{k} \quad \text{for } a \leq \tau < t \leq b, \tag{2.8}$$

$$P_k(a) = 0, \ \int_\tau^t p_1(s)\,ds - \frac{1}{k} < P_k(t) - P_k(\tau) < \int_\tau^t p_2(s)\,ds + \frac{1}{k} \quad \text{for } a \leq \tau < t \leq b, \tag{2.9}$$

$$\int_a^b |p_{0k}(s)|\,ds < \ell_0, \tag{2.10}$$

$$\int_a^b |q_{1k}(s)|\,ds < \frac{\ell_0}{k}, \tag{2.11}$$

where

$$\ell_0 = 1 + \int_a^b \big(|p_1(s)| + \ell(s)\big)\,ds.$$

By virtue of conditions (2.5), (2.8) and(2.9), the sequences $(u_k)_{k=1}^{+\infty}$, $(u'_k)_{k=1}^{+\infty}$, $(P_k)_{k=1}^{+\infty}$ are uniformly bounded and equicontinuous on $[a,b]$. By the Arzelà–Ascoli lemma, without loss of generality we can assume that these sequences are uniformly convergent.

Put

$$u(t) = \lim_{k\to+\infty} u_{0k}(t), \quad P(t) = \lim_{k\to+\infty} P_k(t). \tag{2.12}$$

If we pass to the limit in inequality (2.9) as $k \to +\infty$, then we get

$$P(a) = 0, \quad \int_\tau^t p_1(s)\,ds \leq P(t) - P_(\tau) \leq \int_\tau^t p_2(s)\,ds \quad \text{for } a \leq \tau < t \leq b.$$

Hence it is clear that the function P is absolutely continuous and admits the representation

$$P(t) = \int_a^t p(s)\,ds \quad \text{for } a \leq t \leq b, \tag{2.13}$$

where $p \in L([a,b])$ is a function satisfying inequality (1.4).

By Lemma 1.1 from [4], conditions (2.10), (2.12) and (2.13) guarantee the validity of the equality

$$\lim_{k\to+\infty} \int_a^t p_{0k}(s)u_{0k}(s)\,ds = \int_a^t p(s)u(s)\,ds \quad \text{for } a \leq t \leq b. \tag{2.14}$$

In view of (2.7) we have

$$u'_{0k}(t) = u'_{0k}(a) + \int_a^t \big(p_{0k}(s)u_{0k}(s) + q_{1k}(s)\big)\,ds \quad \text{for } a \leq t \leq b.$$

If along with this identity we take into account conditions (2.5), (2.11) and (2.14), then we find

$$u'(t) = \int_a^t p(s)u(s)\,ds \quad \text{for } a \leq t \leq b$$

$$u'(a) = u'(b) = 0, \quad \|u\| = 1.$$

Consequently, u is a nontrivial solution of the homogeneous problem (1.5), (1.6). On the other hand, due to conditions (1.4) and (2.2), this problem has no nontrivial solution. The contradiction obtained proves the lemma. □

2.2. Lemmas on two-point boundary value problems for equation (1.5). Let $p \in L([a,b])$. We consider the differential equation (1.5) with the boundary conditions

$$u'(a) = 0, \quad u(b) = 0, \tag{2.15}$$

or

$$u(a) = 0, \quad u'(b) = 0. \tag{2.16}$$

Lemma 2.2 (T. Kiguradze)**.** *Let*

$$p(t) \geq -p_0(t) \quad \text{for almost all } t \in [a,b], \tag{2.17}$$

where $p_0 \in L([a,b])$ is a non-negative function. If, moreover, for some $\lambda \geq 1$ the inequality

$$\int_a^b (b-t)p_0^\lambda(t)\,dt \leq \left(\frac{\pi}{2(b-a)}\right)^{2\lambda-2}$$

holds, then problem (1.5), (2.15) *has only the trivial solution. And if*

$$\int_a^b (t-a)p_0^\lambda(t)\,dt \leq \left(\frac{\pi}{2(b-a)}\right)^{2\lambda-2},$$

then problem (1.5), (2.16) *has only the trivial solution.*

This lemma is a corollary of Theorem 1.3 from [10].

Lemma 2.3. *Let inequality* (2.17) *hold where $p_0 \in L([a,b])$ is a non-negative non-decreasing* (*non-increasing*) *function such that*

$$\int_a^b \sqrt{p_0(t)}\,dt < \frac{\pi}{2}. \tag{2.18}$$

Then problem (1.5), (2.15) (*problem* (1.5), (2.16)) *has only the trivial solution.*

Proof. We consider only problem (1.5), (2.15) since problem (1.5), (2.16) can be considered analogously.

Assume that problem (1.5), (2.15) has a nontrivial solution u. Without loss of generality we can assume that $u'(b) < 0$. Then there exists $a_0 \in [a,b[$ such that

$$\begin{gathered} u(t) > 0, \quad u'(t) < 0 \quad \text{for } a_0 < t < b, \\ u'(a_0) = 0. \end{gathered} \tag{2.19}$$

By virtue of conditions (2.17) and (2.19), almost everywhere on $[a_0,b]$ the inequality

$$u''(t)u'(t) \leq -p_0(t)u'(t)u(t)$$

is satisfied. If along with this we take into account the fact that p_0 is a non-decreasing function, then we obtain

$$u'^{\,2}(t) \leq -2\int_{a_0}^t p_0(s)u'(s)u(s)\,ds \leq p_0(t)\bigg(-\int_{a_0}^t u'(s)u(s)\,ds\bigg) = p_0(t)(u^2(a_0) - u^2(t)) \quad \text{for } a_0 \leq t \leq b.$$

Consequently,

$$\sqrt{p_0(t)} \geq \frac{-u'(t)}{\sqrt{u^2(a_0) - u^2(t)}} \quad \text{for } a_0 < t \leq b.$$

Integrating this inequality from a_0 to b, we get

$$\int\limits_{a_0}^{b} \sqrt{p_0(t)}\,dt \geq -\int\limits_{a_0}^{b} \frac{-u'(t)\,dt}{\sqrt{u^2(a_0) - u^2(t)}} = \int\limits_{0}^{1} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2},$$

which contradicts inequality (2.18). The contradiction obtained provers the lemma. □

Remark 2.1. From Lemma 2.3 it follows, in particular, that if $p : [a,b] \to \mathbb{R}_-$ is a non-decreasing (a non-increasing) function and for some $t_0 \in\,]a,b[$ the inequalities

$$\int\limits_{a}^{t_0} \sqrt{|p(s)|}\,ds \leq \frac{\pi}{2}, \quad p(t_0) > -\frac{\pi^2}{4(b-t_0)^2} \quad \bigg(p(t_0) > -\frac{\pi^2}{4(t_0-a)^2}, \quad \int\limits_{t_0}^{b} \sqrt{|p(s)|}\,ds \leq \frac{\pi}{2}\bigg)$$

hold, then the Dirichlet problem

$$u'' = p(t)u, \quad u(a) = u(b) = 0$$

has only the trivial solution. This result generalizes Z. Nehari's theorem [11, Theorem 1], where it is assumed that

$$\int\limits_{a}^{b} \sqrt{|p(s)|}\,ds \leq \frac{\pi}{2}.$$

Along with Lemmas 2.2 and 2.3, below we need Lemma 2.4 as well, concerning problem (1.5), (1.6).

Lemma 2.4. *If condition* (1.14) *holds, then every solution of problem* (1.5), (1.6) *has at least one zero in the interval* $]a,b[$.

Proof. Assume the contrary that problem (1.5), (1.6) has a solution u not having a zero in $]a,b[$. Then by (1.6),

$$u(t) \neq 0 \quad \text{for } a \leq t \leq b,$$

and almost everywhere on $[a,b]$ the equality

$$\frac{u''(t)}{u(t)} = p(t)$$

holds. If we integrate this identity from a to b, then by conditions (1.6) and (1.14) we get

$$0 < \int\limits_{a}^{b} \frac{u'^{\,2}(t)}{u^2(t)}\,dt = \int\limits_{a}^{b} p(t)\,dt \leq 0.$$

The contradiction obtained provers the lemma. □

2.3. Lemmas on the set $\mathcal{N}$eum$([a,b])$.

Lemma 2.5. *Let* $p_i \in L([a,b])$ $(i=1,2)$ *be functions satisfying inequalities* (1.3), (1.9) *and* (1.10), *where* $\lambda \geq 1$. *Then*

$$(p_1,p_2) \in \mathcal{N}\mathbf{eum}([a,b]).$$

Proof. Assume the contrary that

$$(p_1, p_2) \notin \mathcal{N}\mathbf{eum}([a,b]).$$

Then there exists a function $p \in L([a,b])$, satisfying condition (1.4), such that problem (1.5), (1.6) has a nontrivial solution u.

Inequalities (1.4) and (1.9) imply inequalities (1.14). Hence by Lemma 2.4 follows the existence of $t_1 \in]a,b[$ such that

$$u(t_1) = 0. \tag{2.20}$$

On the other hand, by Lemma 2.2 inequality (1.4) and equalities (1.6) and (2.20) result in

$$\Big(\frac{\pi}{2}\Big)^{2\lambda-2} < (t_1-a)^{2\lambda-2} \int_a^{t_1} (t_1-t)[p_1(t)]_-^\lambda\,dt < (t_1-a)^{2\lambda-1} \int_a^{t_1} [p_1(t)]_-^\lambda\,dt,$$

$$\Big(\frac{\pi}{2}\Big)^{2\lambda-2} < (b-t_1)^{2\lambda-2} \int_{t_1}^{b} (t-t_1)[p_1(t)]_-^\lambda\,dt < (b-t_1)^{2\lambda-1} \int_{t_1}^{b} [p_1(t)]_-^\lambda\,dt.$$

Thus

$$\Big(\frac{\pi}{2}\Big)^{4\lambda-4} < ((t_1-a)(b-t_1))^{2\lambda-1} \Big(\int_a^{t_1} [p_1(t)]_-^\lambda\,dt\Big)\Big(\int_{t_1}^{b} [p_1(t)]_-^\lambda\,dt\Big).$$

Hence, in view of the inequalities

$$(t_1-a)(b-t_1) \le \frac{1}{4}(b-a)^2,$$

$$\Big(\int_a^{t_1} [p_1(t)]_-^\lambda\,dt\Big)\Big(\int_{t_1}^{b} [p_1(t)]_-^\lambda\,dt\Big) \le \frac{1}{4}\Big(\int_a^{b} [p_1(t)]_-^\lambda\,dt\Big)^2,$$

it follows that

$$\Big(\frac{\pi}{2}\Big)^{4\lambda-4} < 2^{-4\lambda}(b-a)^{4\lambda-2}\Big(\int_a^{b} [p_1(t)]_-^\lambda\,dt\Big)^2.$$

Consequently,

$$\int_a^{b} [p_1(t)]_-^\lambda\,dt > \frac{4(b-a)}{\pi^2}\Big(\frac{\pi}{b-a}\Big)^{2\lambda},$$

which contradicts inequality (1.10). The contradiction obtained provers the lemma. □

Lemma 2.6. *Let $p_1 : [a,b] \to \mathbb{R}_-$ and $p_2 : [a,b] \to \mathbb{R}$ be integrable functions satisfying inequalities* (1.3) *and* (1.9)*. Let, moreover, there exist $t_0 \in]a,b[$ such that the function p_1 is non-increasing and non-decreasing in the intervals $]a,t_0[$ and $]t_0,b[$, respectively, and inequalities* (1.11) *are satisfied. Then*

$$(p_1, p_2) \in \mathcal{N}\mathbf{eum}([a,b]).$$

Proof. Let $p \in L([a,b])$ be an arbitrary function satisfying inequality (1.4), and let u be an arbitrary solution of problem (1.5), (1.6).

Inequalities (1.4) and (1.9) result in inequalities (1.14). Hence by Lemma 2.4 follows the existence at least one zero of the function u in $]a,b[$. Consequently, there exists $t_1 \in]a,b[$ such that

$$u'(a) = 0, \quad u(t_1) = 0, \tag{2.21}$$

$$u(t_1) = 0, \quad u'(b) = 0. \tag{2.22}$$

If along with (1.11) we take into account the monotonicity of the function p_1 in the intervals $]a, t_0[$ and $]t_0, b[$, then it becomes clear that either

$$a < t_1 \le t_0, \quad \int\limits_a^{t_1} \sqrt{|p_1(t)|}\, dt < \frac{\pi}{2}, \tag{2.23}$$

or

$$t_0 \le t_1 < b, \quad \int\limits_{t_1}^{b} \sqrt{|p_1(t)|}\, dt < \frac{\pi}{2}. \tag{2.24}$$

However, if condition (2.23) (condition (2.24)) holds, then by Lemma 2.3 problem (1.5), (2.21) (problem (1.5), (2.22)) has only the trivial solution. Thus we have proved that $u(t) \equiv 0$. Hence, in view of the arbitrariness of a solution u of problem (1.5), (1.6) and a function p, we have $(p_1, p_2) \in$ $\boldsymbol{\mathcal{N}}\mathbf{eum}([\boldsymbol{a}, \boldsymbol{b}])$. □

2.4. Lemma on the solvability of problem (1.1), (1.2). Along with problem (1.1), (1.2) we consider the auxiliary problem

$$u'' = (1-\lambda)p(t)u + \lambda f(t,u), \tag{2.25}$$

$$u'(a) = \lambda c_1, \quad u'(b) = \lambda c_2, \tag{2.26}$$

where $p \in L([a,b])$, and λ is a parameter.

According to Corollary 2 from [9], the following lemma is valid.

Lemma 2.7. *Let problem* (1.5), (1.6) *have only the trivial solution and let there exist a positive constant r such that for any $\lambda \in]0,1[$ an arbitrary solution u of problem* (2.25), (2.26) *admits the estimate*

$$|u(t)| + |u'(t)| < r \quad \text{for } a \le t \le b. \tag{2.27}$$

Then problem (1.1), (1.2) *has at least one solution.*

3 Proof of the main results

Proof of Theorem 1.1. By Lemma 2.1, there exists a positive constant r_0 such that every solution u of the differential inequality

$$p_1(t)|u(t)| - q(t,|u(t)|) \le u''(t)\operatorname{sgn}(u(t)) \le p_2(t)|u(t)| + q(t,|u(t)|) \tag{3.1}$$

admits the estimate

$$\|u\| \le r_0\bigg(|u'(a)| + |u'(b)| + \int\limits_a^b q(s,\|u\|)\, ds\bigg), \tag{3.2}$$

where

$$\|u\| = \max\{|u(t)| : a \le t \le b\}.$$

On the other hand, according to equality (1.7), there exists a number r_1 such that

$$r_0\bigg(|c_1| + |c_2| + \int\limits_a^b q(s,x)\, ds\bigg) < x \quad \text{for } x \ge r_1. \tag{3.3}$$

Put

$$r_2 = \bigg(\frac{1}{r_0} + \int\limits_a^b (|p_1(s)| + |p_2(s)|)\, ds\bigg) r_1, \quad r = r_1 + r_2.$$

Let $p \in L([a,b])$ be an arbitrary function satisfying inequality (1.4), $\lambda \in]0,1[$, and u be an arbitrary solution of problem (2.25), (2.26). By Lemma 2.7 and condition (2.2), it suffices to state that u admits estimate (2.27).

By virtue of inequality (1.8), the function u is a solution of problem (3.1), (2.26). Thus it admits the estimate

$$\|u\| \le r_0\bigg(|c_1| + |c_2| + \int\limits_a^b q(s, \|u\|)\, ds\bigg).$$

Hence in view of (3.3) we have

$$\|u\| \le r_1.$$

If along with this inequality we take into account conditions (2.26) and (3.3), we find

$$|u'(t)| \le |u'(a)| + \int\limits_a^b |u''(s)|\, ds \le |c_1| + \int\limits_a^b q(s, r_1)\, ds + \int\limits_a^b (|p_1(s)| + |p_2(s)|)\, |u(s)|\, ds$$
$$\le r_1/r_0 + r_1 \int\limits_a^b (|p_1(s)| + |p_2(s)|)\; ds = r_2 \quad \text{for } a \le t \le b.$$

Therefore estimate (2.27) is valid. □

Proof of Theorem 1.2. Inequality (1.12) yields inequality (1.8), where $q(t,|x|) \equiv |f(t,0)|$. Consequently, all the conditions of Theorem 1.1 are fulfilled which guarantees the solvability of problem (1.1), (1.2).

Let u_1 and u_2 be arbitrary solutions of the above mentioned problem. Put

$$u(t) = u_1(t) - u_2(t).$$

In view of condition (1.12), the function u is a solution of the differential inequality

$$p_1(t)|u(t)| \le u''(t)\operatorname{sgn}(u(t)) \le p_2(t)|u(t)|,$$

satisfying the boundary conditions (1.6). Hence by Lemma 2.1 it follows that $u(t) \equiv 0$. Consequently, problem (1.1), (1.2) has one and only one solution. □

By Lemma 2.5, Theorems 1.1 and 1.2 yield Corollaries 1.1 and 1.3, respectively. By Lemma 2.6, Theorems 1.1 and 1.2 yield Corollaries 1.2 and 1.4, respectively.

In the case, where $f(t,x) \equiv p(t)x + q(t)$, Corollary 1.3 results in Corollary 1.5, and Corollary 1.4 results in Corollary 1.6.

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Authors' address:

Ivan Kiguradze, **Nino Partsvania**

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia.

E-mail: `ivane.kiguradze@tsu.ge`; `nino.partsvania@tsu.ge`

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Kemal Özen

SOLVABILITY OF A NONLOCAL PROBLEM BY A NOVEL CONCEPT OF FUNDAMENTAL FUNCTION

Abstract. Cauchy function, Green function and Riemann function are the several of the fundamental functions used frequently in the expression of a fundamental solution in the literature. In order to construct such functions, various ideas can be considered. The lesser-known one of these ideas is contained in the papers [1–4] by Seyidali S. Akhiev. Inspired by these papers, the solvability of some problems [12, 14, 15, 17–19] has been investigated. In this work, a novel kind of adjoint problem for a generally nonlocal problem, and also Green's functional via the solvability of that adjoint problem are constructed [21]. By means of the obtained Green's functional, an integral representation for the solution of the nonlocal problem is established.[1]

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რეზიუმე. კოშის ფუნქცია, გრინის ფუნქცია და რიმანის ფუნქცია ძირითადი ფუნქციებია, რომლებიც ლიტერატურაში ხშირად გამოიყენება ფუნდამენტური ამონახსნის წარმოსადგენად. ამ ფუნქციების ასაგებად არსებობს რამდენიმე მიდგომა. მათ შორის ერთ-ერთი ნაკლებად ცნობილი მოყვანილია ს. ს. ახიევის ნაშრომებში [1–4]. ამ სტატიებზე დაყრდნობით გამოკვლეულ იქნა ზოგიერთი ამოცანის ამოხსნადობა [12, 14, 15, 17–19]. ნაშრომში ზოგადი არალოკალური ამოცანისთვის აგებულია ახალი ტიპის შეუღლებული ამოცანა, რომლის ამოხსნადობაზე დაყრდნობით აგებულია გრინის ფუნქციონალი [21]. მიღებული გრინის ფუნქციონალის საშუალებით დადგენილია არალოკალური ამოცანის ამონახსნის ინტეგრალური წარმოდგენა.

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1 Introduction

There are various papers related to the investigations on the differential systems involving general boundary conditions [7,8,20,23]. To the best of our knowledge, there is no paper on the construction of Green's functional for an uncoupled system of linear ordinary differential equations with the exception the abstract of conference [13]. This work deals with the construction of Green's functional for such a system with a general nonlocal condition. The main aim at this dealing is to identify the Green function for the above-said system.

The rest of the work is organized as follows. In Section 2, the problem considered throughout the work is stated in detail. In Section 3, the solution space and its adjoint space are introduced. In Section 4, the adjoint operator, adjoint system and solvability conditions for the completely nonhomogeneous problem are given. In Section 5, Green's functional is defined. In the last section, the conclusions are emphasized.

2 Statement of the problem

Let $\mathbb{R}$ be the space of all real numbers, consider a bounded open interval $G=(0,1)$ in $\mathbb{R}$. The problem under consideration is stated as follows:

$$(V_1U)(x)\equiv U'(x)+A(x)U(x)=Z^1(x),\quad x\in G=(0,1), \tag{2.1}$$

$$V_0U\equiv aU(0)+\int\limits_0^1 g(\xi)U'(\xi)\,d\xi=Z^0, \tag{2.2}$$

where $U(x)=\begin{bmatrix}u_1(x)\\u_2(x)\end{bmatrix}$, $Z^1(x)=\begin{bmatrix}z_1^1(x)\\z_2^1(x)\end{bmatrix}$, $A(x)=\begin{bmatrix}A_1(x)&0\\0&A_2(x)\end{bmatrix}$, $g(\xi)=\begin{bmatrix}g_1(\xi)&0\\0&g_2(\xi)\end{bmatrix}$ are 2-vectors and 2-square matrices defined on G, respectively; $Z^0=\begin{bmatrix}z_1^0\\z_2^0\end{bmatrix}$ and $a=\begin{bmatrix}a_1&0\\0&a_2\end{bmatrix}$ are 2-vector and 2-square matrix with real entries, respectively. The symbol $'$ denotes the ordinary derivative of order one. Here $A_1(x),A_2(x),z_1^1(x),z_2^1(x)\in L_p(G)$ with $1\le p<\infty$ and $g_1(\xi),g_2(\xi)\in L_q(G)$ ($\frac{1}{p}+\frac{1}{q}=1$). $L_p(G)$ with $1\le p<\infty$ denotes the space of Lebesgue p-integrable functions on G. $L_\infty(G)$ denotes the space of measurable and essentially bounded functions on G, and $W_p^{(1)}(G)$ with $1\le p\le\infty$ denotes the space of all functions $u(x)\in L_p(G)$ having derivative $du/dx\in L_p(G)$ [12,16,19]. The space $W_p^{(1)}(G)$ is equipped with the norm

$$\|u\|_{W_p^{(1)}(G)}=\sum_{k=0}^{1}\Big\|\frac{d^ku}{dx^k}\Big\|_{L_p(G)}.$$

The characteristic feature of this problem is that, instead of an ordinary boundary condition, it involves a more comprehensive nonlocal boundary condition. The stated problem is investigated for a solution vector U such that its entries u_1 and u_2 belong to the space $W_p^{(1)}(G)$.

Problem (2.1), (2.2) is a linear problem which can be considered as an operator equation

$$VU=Z \tag{2.3}$$

with the linear operator $V=(V_1,V_0)$ and $Z=(Z^1(x),Z^0)$.

From the considerations given above, we have that V is bounded from $W_p^{(1)}(G)^2$ into the Banach space $E_p^2\equiv L_p(G)^2\times\mathbb{R}^2$ of the elements $Z=(Z^1(x),Z^0)$ with

$$\|z_1\|_{E_p}=\|z_1^1(x)\|_{L_p(G)}+|z_1^0|,\quad \|z_2\|_{E_p}=\|z_2^1(x)\|_{L_p(G)}+|z_2^0|,\ \ 1\le p\le\infty.$$

If, for a given $Z\in E_p^2$, problem (2.1), (2.2) has a unique solution $U\in W_p^{(1)}(G)^2$ with $\|u_1\|_{W_p^{(1)}(G)}\le c_0\|z_1\|_{E_p}$ and $\|u_2\|_{W_p^{(1)}(G)}\le c_1\|z_2\|_{E_p}$, then this problem is called a well-posed problem, where c_0 and c_1 are constants independent of z_1 and z_2, respectively. Problem (2.1), (2.2) is well-posed if and only if $V:W_p^{(1)}(G)^2\to E_p^2$ is a (linear) homeomorphism.

3 Adjoint space of the solution space

Problem (2.1), (2.2) is investigated by means of a novel concept of the adjoint problem which is introduced in [2, 5]. Some isomorphic decompositions of the solution space $W_p^{(1)}(G)^2$ and its adjoint space $W_p^{(1)}(G)^{2*}$ are employed. Some of the principal features concerning with the solution space can be given as follows: any function $u \in W_p^{(1)}(G)$ can be represented as

$$u(x) = u(\alpha) + \int_{\alpha}^{x} u'(\xi)\, d\xi, \tag{3.1}$$

where α is a given point in $\overline{G}$ which is the set of closure points for G [12, 16, 19]. Furthermore, the trace or the value operator $D_0 u = u(\gamma)$ is bounded and surjective from $W_p^{(1)}(G)$ onto $\mathbb{R}$ for a given point γ of $\overline{G}$. In addition, the value $u(\alpha)$ and the derivative $u'(x)$ are unrelated elements of the function $u \in W_p^{(1)}(G)$ such that for any real number ν_0 and any function $\nu_1 \in L_p(G)$, there exists one and only one $u \in W_p^{(1)}(G)$ such that $u(\alpha) = \nu_0$ and $u'(x) = \nu_1(x)$. Therefore, there exists a linear homeomorphism between $W_p^{(1)}(G)^2$ and E_p^2. In other words, the space $W_p^{(1)}(G)^2$ has the isomorphic decomposition $W_p^{(1)}(G)^2 = L_p(G)^2 \times \mathbb{R}^2$. The structure of the adjoint space is determined by the following theorem.

Theorem 3.1 ([1, 2, 4, 12, 16, 19]). *If $1 \le p < \infty$, then any linear bounded functional $F \in W_p^{(1)}(G)^{2*}$ can be represented as*

$$F(U) = \begin{bmatrix} F^1(u_1) \\ F^2(u_2) \end{bmatrix} = \begin{bmatrix} \int_0^1 u_1'(x)\varphi_1^1(x)\, dx + u_1(0)\varphi_0^1 \\ \int_0^1 u_2'(x)\varphi_1^2(x)\, dx + u_2(0)\varphi_0^2 \end{bmatrix} \tag{3.2}$$

with a unique element $\varphi = (\varphi_1(x), \varphi_0) \in E_q^2$, where $\frac{1}{p} + \frac{1}{q} = 1$. Any linear bounded functional $F \in W_\infty^{(1)}(G)^{2*}$ can be represented as

$$F(U) = \begin{bmatrix} F^1(u_1) \\ F^2(u_2) \end{bmatrix} = \begin{bmatrix} \int_0^1 u_1'(x)\, d\varphi_1^1 + u_1(0)\varphi_0^1 \\ \int_0^1 u_2'(x)\, d\varphi_1^2 + u_2(0)\varphi_0^2 \end{bmatrix} \tag{3.3}$$

with a unique element $\varphi = (\varphi_1(e), \varphi_0) \in \widehat{E}_1 = (BA(\Sigma, \mu))^2 \times \mathbb{R}^2$, where μ is Lebesgue measure on $\mathbb{R}$, Σ is σ-algebra of the μ-measurable subsets $e \subset G$ and $BA(\Sigma, \mu)$ is the space of all bounded additive functions $\varphi_1(e)$ defined on Σ with $\varphi_1(e) = 0$ when $\mu(e) = 0$ [9]. The inverse is also valid, that is, if $\varphi \in E_q^2$, then (3.2) is bounded on $W_p^{(1)}(G)^{2*}$ for $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\varphi \in \widehat{E}_1$, then (3.3) is bounded on $W_\infty^{(1)}(G)^{2*}$.

Proof. The operator $NU \equiv (U'(x), U(0)) : W_p^{(1)}(G)^2 \to E_p^2$ is bounded and has a bounded inverse N^{-1} represented by

$$U(x) = (N^{-1}h)(x) \equiv \int_0^x h_1(\xi)\, d\xi + h_0, \quad h = (h_1(x), h_0) \in E_p^2.$$

The kernel $\operatorname{Ker} N$ of N is trivial and the image $\operatorname{Im} N$ of N is equal to E_p^2. Hence, there exists a bounded adjoint operator $N^* : E_p^{2*} \to W_p^{(1)}(G)^{2*}$ with $\operatorname{Ker} N^* = \{0\}$ and $\operatorname{Im} N^* = W_p^{(1)}(G)^{2*}$. In other words, for a given $F \in W_p^{(1)}(G)^{2*}$, there exists a unique $\psi \in E_p^{2*}$ such that

$$F = N^*\psi \ \text{ or } \ F(U) = \psi(NU), \ \ U \in W_p^{(1)}(G)^2. \tag{3.4}$$

If $1 \le p < \infty$, then $E_p^{2*} = E_q^2$ in the sense of an isomorphism [9]. Hence, the functional ψ can be represented by

$$\psi(h) = \int_0^1 \varphi_1(x)h_1(x)\,dx + \varphi_0 h_0, \ \ h \in E_p^2, \tag{3.5}$$

with a unique element $\varphi = (\varphi_1(x), \varphi_0) \in E_q^2$. Due to expressions (3.4) and (3.5), any $F \in W_p^{(1)}(G)^{2*}$ can uniquely be written by (3.2). For a given $\varphi \in E_q^2$, the functional F written by (3.2) is bounded on $W_p^{(1)}(G)^2$. Hence, (3.2) is a general form for the functional $F \in W_p^{(1)}(G)^{2*}$.

The proof is complete due to the fact that the case $p = \infty$ can likewise be shown [4,12,16,19]. □

Theorem 3.1 guarantees that $W_p^{(1)}(G)^{2*} = E_q^2$ for all $1 \le p < \infty$, and $W_\infty^{(1)}(G)^{2*} = E_\infty^{2*} = \widehat{E}_1$. The space E_1 can also be considered as a subspace of the space $\widehat{E}_1$ [4,12,16,19].

4 Adjoint operator, adjoint system and solvability conditions

In this section, an explicit form for the adjoint operator V^* of V is investigated. To this end, any $f = (f_1(x), f_0) \in E_q^2$ is taken as a linear bounded functional on E_p^2 and also we assume

$$f(VU) \equiv \int_0^1 f_1(x)(V_1U)(x)\,dx + f_0(V_0U), \ \ U \in W_p^{(1)}(G)^2. \tag{4.1}$$

By substituting expressions (2.1) and (2.2), and expression (3.1) for all entries of $U \in W_p^{(1)}(G)^2$ (for $\alpha = 0$) into (4.1), we have

$$f(VU) \equiv \begin{bmatrix} \displaystyle\int_0^1 f_1^1(x)\big\{u_1'(x) + A_1(x)u_1(x)\big\}\,dx + f_0^1\left(a_1u_1(0) + \int_0^1 g_1(\xi)u_1'(\xi)\,d\xi\right) \\ \displaystyle\int_0^1 f_1^2(x)\big\{u_2'(x) + A_2(x)u_2(x)\big\}\,dx + f_0^2\left(a_2u_2(0) + \int_0^1 g_2(\xi)u_2'(\xi)\,d\xi\right) \end{bmatrix}.$$

Hence, we obtain

$$\begin{aligned} f(VU) \equiv \int_0^1 f_1(x)(V_1U)(x)\,dx + f_0(V_0U) &= \int_0^1 (w_1f)(\xi)U'(\xi)\,d\xi + (w_0f)U(0) \\ &\equiv (wf)(U) \ \ \forall f \in E_q^2, \ \ \forall U \in W_p^{(1)}(G)^2, \ \ 1 \le p \le \infty, \end{aligned} \tag{4.2}$$

where

$$w_1 = \begin{bmatrix} w_1^1 \\ w_1^2 \end{bmatrix}, \quad w_0 = \begin{bmatrix} w_0^1 \\ w_0^2 \end{bmatrix},$$

$$\begin{aligned} (w_1^1f^1)(\xi) &= f_1^1(\xi) + \int_\xi^1 f_1^1(s)A_1(s)\,ds + f_0^1g_1(\xi), \quad w_0^1f^1 = \int_0^1 f_1^1(x)A_1(x)\,dx + f_0^1a_1, \\ (w_1^2f^2)(\xi) &= f_1^2(\xi) + \int_\xi^1 f_1^2(s)A_2(s)\,ds + f_0^2g_2(\xi), \quad w_0^2f^2 = \int_0^1 f_1^2(x)A_2(x)\,dx + f_0^2a_2. \end{aligned} \tag{4.3}$$

The operators w_1^1, w_0^1, w_1^2 and w_0^2 are linear and bounded from the space E_q of the pairs $f = (f_1(x), f_0)$ into the spaces $L_q(G), \mathbb{R}, L_q(G)$ and $\mathbb{R}$, respectively. Therefore, the operator $w = (w_1, w_0) : E_q^2 \to E_q^2$ represented by $wf = (w_1 f, w_0 f)$ is linear and bounded. By (4.2) and Theorem 3.1, the operator w is an adjoint operator for the operator V, when $1 \le p < \infty$, in other words, $V^* = w$. When $p = \infty$, $w : E_1^2 \to E_1^2$ is bounded; in this case, the operator w is the restriction of the adjoint operator $V^* : E_\infty^{2*} \to W_\infty^{(1)}(G)^{2*}$ of V onto $E_1^2 \subset E_\infty^{2*}$.

Equation (2.3) can always be transformed into the following equivalent equation

$$VSh = Z \tag{4.4}$$

with an unknown $h = (h_1, h_0) \in E_p^2$ by the transformation $U = Sh$, where $S = N^{-1}$. If $U = Sh$, then $U'(x) = h_1(x)$, $U(0) = h_0$. Hence, (4.2) can be rewritten as

$$\begin{aligned} f(VSh) &\equiv \int_0^1 f_1(x)(V_1 Sh)(x)\,dx + f_0(V_0 Sh) \\ &= \int_0^1 (w_1 f)(\xi) h_1(\xi)\,d\xi + (w_0 f) h_0 \equiv (wf)(h) \ \ \forall\, f \in E_q^2,\ \ \forall\, h \in E_p^2,\ \ 1 \le p \le \infty. \end{aligned}$$

Therefore, one of the operators VS and w becomes an adjoint operator for the other one. Consequently, the equation

$$wf = \varphi \tag{4.5}$$

with an unknown function $f = (f_1(x), f_0) \in E_q^2$ and a given function $\varphi = (\varphi_1(x), \varphi_0) \in E_q^2$ can be considered as an adjoint equation of (4.4) (or of (2.3)) for all $1 \le p \le \infty$, where

$$\varphi_1 = \begin{bmatrix} \varphi_1^1 \\ \varphi_1^2 \end{bmatrix}, \quad \varphi_0 = \begin{bmatrix} \varphi_0^1 \\ \varphi_0^2 \end{bmatrix}.$$

Equation (4.5) can be written in explicit form as the system of equations

$$\begin{aligned} (w_1^1 f^1)(\xi) &= \varphi_1^1(\xi), \ \ \xi \in G, \\ w_0^1 f^1 &= \varphi_0^1, \\ (w_1^2 f^2)(\xi) &= \varphi_1^2(\xi), \ \ \xi \in G, \\ w_0^2 f^2 &= \varphi_0^2. \end{aligned} \tag{4.6}$$

By expressions (4.3), the first and third equations in (4.6) are the integral equations for $f_1^1(\xi), f_1^2(\xi)$, respectively, and include f_0^1, f_0^2, respectively, as parameters; on the other hand, the second and fourth equations in (4.6) are the algebraic equations for the unknowns f_0^1, f_0^2, respectively, and they include some integral functionals defined on $f_1^1(\xi), f_1^2(\xi)$, respectively. In other words, (4.6) is a system of four integro-algebraic equations. This system called the adjoint system for (4.4) (or (2.3)) is constructed by using (4.2) which is actually a formula of integration by parts in a nonclassical form. The traditional type of an adjoint problem is defined by the classical Green's formula of integration by parts [22], therefore, has a sense only for some restricted class of problems [4, 12, 16, 19].

The following theorem concerning with the solvability of the problem can be derived.

Theorem 4.1 ([4, 12, 16, 19]). *If $1 < p < \infty$, then $VU = 0$ has either only the trivial solution or a finite number of linearly independent solutions in $W_p^{(1)}(G)^2$:*

(1) *If $VU = 0$ has only the trivial solution in $W_p^{(1)}(G)^2$, then also $wf = 0$ has only the trivial solution in E_q^2. Then the operators $V : W_p^{(1)}(G)^2 \to E_p^2$ and $w : E_q^2 \to E_q^2$ become linear homeomorphisms.*

(2) *If* $VU = 0$ *has* m *linearly independent solutions* $U_1, U_2, \dots, U_m$ *in* $W_p^{(1)}(G)^2$, *then* $wf = 0$ *has also* m *linearly independent solutions*

$$f^{\star\,1\,\star} = \big(f_1^{\star\,1\,\star}(x), f_0^{\star\,1\,\star}\big), \dots, f^{\star\,m\,\star} = \big(f_1^{\star\,m\,\star}(x), f_0^{\star\,m\,\star}\big)$$

in E_q^2. *In this case,* (2.3) *and* (4.5) *have solutions* $U \in W_p^{(1)}(G)^2$ *and* $f \in E_q^2$ *for the given* $Z \in E_p^2$ *and* $\varphi \in E_q^2$ *if and only if the conditions*

$$\int\limits_0^1 f_1^{\star\,i\,\star}(\xi)Z^1(\xi)\,d\xi + f_0^{\star\,i\,\star}Z^0 = 0, \ \ i = 1, \dots, m,$$

and

$$\int\limits_0^1 \varphi_1(\xi)U_i'(\xi)\,d\xi + \varphi_0 U_i(0) = 0, \ \ i = 1, \dots, m,$$

are satisfied, respectively.

5 Green's functional

Consider the equation in the form of a functional identity

$$(wf)(U) = U(x) \ \ \forall\, U \in W_p^{(1)}(G)^2, \tag{5.1}$$

where $f = (f_1(\xi), f_0) \in E_q^2$ is an unknown pair and $x \in \overline{G}$ is a parameter [4,12,16,19].

Definition 5.1 ([4,12,16,19])**.** Let $f(x) = (f_1(\xi, x), f_0(x)) \in E_q^2$ be a pair with parameter $x \in \overline{G}$. If $f = f(x)$ is a solution of (5.1) for a given $x \in \overline{G}$, then $f(x)$ is called Green's functional of V (or of (2.3)).

Theorem 5.1 ([4,12,16,19])**.** *If Green's functional* $f(x) = (f_1(\xi, x), f_0(x))$ *of* V *exists, then any solution* $U \in W_p^{(1)}(G)^2$ *of* (2.3) *can be represented by*

$$U(x) = \int\limits_0^1 f_1(\xi, x)Z^1(\xi)\,d\xi + f_0(x)Z^0.$$

Additionally, $\operatorname{Ker} V = \{0\}$.

6 Conclusion

The proposed approach principally differs from the known classical construction methods of Green's function, it is based on the use of the structural properties of the space of solutions instead of the classical Green's formula of integration by parts, and it has a natural property which can be easily applied to a very wide class of linear and some nonlinear boundary value problems involving linear nonlocal nonclassical multi-point conditions with also integral-type terms. Because of these properties, it is one of the scarce methods which are aimed at the derivation of a solution to such problems by reducing to an integral equation in general. The proposed approach can successfully be employed also for the functional differential problems resulting from the addition of some delayed, loaded (forced) or neutral terms to the main operator as long as its linearity is conserved [6]. The work emphasizes as a significant result that the unique solvability of the stated problem arises in the unique solvability of the stated adjoint systems of integro-algebraic equations.

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Authors' address:

Department of Mathematics, Namık Kemal University, Değirmenaltı, Tekirdağ 59030, Turkey.

E-mail: kozen@nku.edu.tr

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Short Communication

MALKHAZ ASHORDIA AND VALIDA SESADZE

ON THE SOLVABILITY AND THE WELL-POSEDNESS OF THE MODIFIED CAUCHY PROBLEM FOR LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS WITH SINGULARITIES

Abstract. Effective sufficient conditions are given for the unique solvability and for the so-called H-well-posedness of the modified Cauchy problem for linear systems of generalized ordinary differential equations with singularities.

რეზიუმე. მოცემულია სინგულარობებიან განზოგადებულ ჩვეულებრივ დიფერენციალურ განტოლებათა წრფივი სისტემებისთვის კოშის სახეშეცვლილი ამოცანის ცალსახად ამოხსნადობისა და ე.წ. H-კორექტულობის ეფექტური საკმარისი პირობები.

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1 Statement of the problem and basic notation

Let $I \subset \mathbb{R}$ be an interval non-degenerate at the point, $t_0 \in I$, and

$$I_{t_0} = I \setminus \{t_0\}, \quad I_{t_0}^- =]-\infty, t_0[\cap I, \quad I_{t_0}^+ =]t_0, +\infty[\cap I.$$

Consider the linear system of generalized ordinary differential equations

$$dx = dA(t) \cdot x + df(t) \text{ for } t \in I_{t_0}, \tag{1.1}$$

where

$$A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(I_{t_0}, \mathbb{R}^{n\times n}), \quad f = (f_k)_{k=1}^n \in \mathrm{BV}_{loc}(I_{t_0}, \mathbb{R}^n).$$

Let $H = \operatorname{diag}(h_1, \dots, h_n) : I_{t_0} \to \mathbb{R}^{n\times n}$ be arbitrary diagonal matrix-functions with continuous diagonal elements

$$h_k : I_{t_0} \to]0, +\infty[\ \ (k = 1, \dots, n).$$

We consider the problem of finding a solution $x \in \mathrm{BV}_{loc}(I_{t_0}, \mathbb{R}^n)$ of system (1.1) satisfying the modified Cauchy condition

$$\lim_{t\to t_0-} (H^{-1}(t)\, x(t)) = 0 \text{ and } \lim_{t\to t_0+} (H^{-1}(t)\, x(t)) = 0. \tag{1.2}$$

Along with system (1.1), consider the perturbed singular system

$$dy = d\widetilde{A}(t) \cdot y + d\widetilde{f}(t) \text{ for } t \in I_{t_0}, \tag{1.3}$$

where

$$\widetilde{A}=(\widetilde{a}_{ik})_{i,k=1}^{n}\in \mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n}),\quad \widetilde{f}=(\widetilde{f}_k)_{k=1}^{n}\in \mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n})$$

are, as above, the matrix- and vector-functions, respectively.

In the present paper, we give sufficient conditions for the unique solvability of problem (1.1), (1.2). Moreover, we investigate the question when the unique solvability of problem (1.1), (1.2) guarantees unique solvability of problem (1.3), (1.2) and, as well, the nearness of their solutions in the definite sense if the matrix-functions A and $\widetilde{A}$ and the vector-functions f and $\widetilde{f}$ are near, respectively.

The analogous problems for system of ordinary differential equations with singularities

$$\frac{dx}{dt}=P(t)\,x+q(t)\ \text{ for }\ t\in I, \tag{1.4}$$

where

$$P\in L_{loc}(I_{t_0},\mathbb{R}^{n\times n}),\quad q\in L_{loc}(I_{t_0},\mathbb{R}^{n}),$$

have been investigated in the papers [6–8].

The singularity of system (1.4) is considered in the sense that the matrix-function P and the vector-function q are, in general, not integrable at the point t_0. In general, a solution of problem (1.4), (1.2) is not continuous at the point t_0 and, therefore, it cannot be a solution in the classical sense. But its restriction on every interval from I_{t_0} is a solution of system (1.4). In this connection we give the example from [8].

Let $\alpha>0$ and $\varepsilon\in\,]0,\alpha[$. Then the problem

$$\frac{dx}{dt}=-\frac{\alpha\,x}{t}+\varepsilon|t|^{\varepsilon-1-\alpha},\quad \lim_{t\to 0}(t^{\alpha}x(t))=0$$

has the unique solution $x(t)=|t|^{\varepsilon-\alpha}\operatorname{sgn} t$. This function is not a solution of the equation in the set $I=\mathbb{R}$, but its restrictions on $]-\infty,0[$ and $]0,+\infty[$ are the solutions of these equation.

The singularity of system (1.1) is considered in the sense that the matrix-function A and the vector-function f may have non-bounded total variation at the point t_0, i.e., on some closed interval $[a,b]$ from I such that $t_0\in[a,b]$.

As is known, such a problem for generalized differential system (1.1) has not been studied. So, the problem remains actual.

Some singular two-point boundary problems for generalized differential system (1.1) are investigated in [3–5].

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to study ordinary differential, impulsive and difference equations from a unified point of view (see [2–5, 10, 11] and the references therein).

In the paper the use will be made of the following notation and definitions.

$\mathbb{R}=\,]-\infty,+\infty[$, $\mathbb{R}_+=[0,+\infty[$, $[a,b]$ and $]a,b[$ $(a,b\in\mathbb{R})$ are, respectively, the closed and open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X=(x_{ik})_{i,k=1}^{n,m}$ with the norm $\|X\|=\max\limits_{k=1,\dots,m}\sum\limits_{i=1}^{n}|x_{ik}|$.

If $X=(x_{ik})_{i,k=1}^{n,m}\in\mathbb{R}^{n\times m}$, then $|X|=(|x_{ik}|)_{i,k=1}^{n,m}$, $[X]_+=\frac{|X|+X}{2}$, $[X]_-=\frac{|X|-X}{2}$.

$\mathbb{R}_+^{n\times m}=\{(x_{ik})_{i,k=1}^{n,m}:\ x_{ik}\ge 0\ (i=1,\dots,n;\ k=1,\dots,m)\}$.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^{n}$; $\mathbb{R}_+^n=\mathbb{R}_+^{n\times 1}$.

If $X\in\mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n\times n$-matrix.

The inequalities between the matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

If $X:\mathbb{R}\to\mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_a^b(X)$ is the sum of total variations on $[a,b]$ of its components x_{ik} $(i=1,\dots,n;\ k=1,\dots,m)$; if $a>b$, then we assume $\bigvee\limits_a^b(X)=-\bigvee\limits_b^a(X)$;

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$ at the point t ($X(a-) = X(a)$, $X(b+) = X(b)$).

$d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all bounded variation matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X) < \infty$).

$\mathrm{BV}_{loc}(J;D)$, where $J \subset \mathbb{R}$ is an interval and $D \subset \mathbb{R}^{n\times m}$, is the set of all $X : J \to D$ whose restriction on $[a,b]$ belongs to $\mathrm{BV}([a,b];D)$ for every closed interval $[a,b]$ from J.

$\mathrm{BV}_{loc}(I_{t_0};D)$ is the set of all $X : I \to D$ whose restriction on $[a,b]$ belongs to $\mathrm{BV}([a,b];D)$ for every closed interval $[a,b]$ from I_{t_0}.

Everywhere we assume that $a_1 \in I_{t_0}^-$ and $a_2 \in I_{t_0}^+$ are some fixed points.

If $X \in \mathrm{BV}_{loc}(I_{t_0};\mathbb{R}^{n\times m})$, then $V(X)(t) = (v(x_{ik})(t))_{i,k=1}^{n,m}$ for $t \in I_{t_0}$, where $v(x_{ik})(a_j) = 0$, $v(x_{ik})(t) \equiv \bigvee\limits_{a_j}^t(x_{ik})$ for $(t-t_0)(a_j-t_0) > 0$ $(j=1,2)$.

$[X(t)]_+^v \equiv \frac{V(X)(t)+X(t)}{2}$, $[X(t)]_-^v \equiv \frac{V(X)(t)-X(t)}{2}$.

s_1, s_2, s_c and $\mathcal{J} : \mathrm{BV}_{loc}(I_{t_0};\mathbb{R}) \to \mathrm{BV}_{loc}(I_{t_0};\mathbb{R})$ are the operators defined, respectively, by

$$s_1(x)(a_j) = s_2(x)(a_j) = 0, \quad s_c(x)(a_j) = x(a_j);$$

$$s_1(x)(t) = s_1(x)(s) + \sum_{s<\tau\le t} d_1x(\tau), \quad s_2(x)(t) = s_2(x)(s) + \sum_{s\le\tau<t} d_2x(\tau)$$

$$s_c(x)(t) = s_c(x)(s) + x(t) - x(s) - \sum_{j=1}^2 (s_j(x)(t) - s_j(x)(s))$$

$$\text{for } s<t<t_0 \text{ if } a_j<t_0 \text{ and for } t_0<s<t \text{ if } a_j>t_0 \ (j=1,2)$$

and

$$\mathcal{J}(x)(a_j) = x(a_j),$$

$$\mathcal{J}(x)(t) = \mathcal{J}(x)(s) + s_c(x)(t) - s_c(x)(s) - \sum_{s<\tau\le t} \ln|1-d_1x(\tau)| + \sum_{s\le\tau<t} \ln|1+d_2x(\tau)|$$

$$\text{for } s<t<t_0 \text{ if } a_j<t_0 \text{ and for } t_0<s<t<t_0 \text{ if } a_j>t_0 \ (j=1,2).$$

If $X \in \mathrm{BV}_{loc}(I_{t_0};\mathbb{R}^{n\times n})$, $\det(I_n+(-1)^jd_jX(t)) \ne 0$ for $t \in I_{t_0}$ $(j=1,2)$, and $Y \in \mathrm{BV}_{loc}(I_{t_0};\mathbb{R}^{n\times m})$, then

$$\mathcal{A}(X,Y)(a_j) = O_{n\times m},$$

$$\mathcal{A}(X,Y)(t) - \mathcal{A}(X,Y)(s) = Y(t) - Y(s) + \sum_{s<\tau\le t} d_1X(\tau)\cdot(I_n - d_1X(\tau))^{-1}\, d_1Y(\tau)$$

$$- \sum_{s\le\tau<t} d_2X(\tau)\cdot(I_n + d_2X(\tau))^{-1}\, d_2Y(\tau)$$

$$\text{for } s<t<t_0 \text{ if } a_j<t_0 \text{ and for } t_0<s<t<t_0 \text{ if } a_j>t_0 \ (j=1,2).$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\, dg(\tau) = \int_{]s,t[} x(\tau)\, ds_c(g)(\tau) + \sum_{s<\tau\le t} x(\tau)\, d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)\, d_2g(\tau),$$

where $\int\limits_{]s,t[} x(\tau)\, ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_c(g))$ corresponding to the function $s_c(g)$. If $a = b$, then we assume $\int\limits_a^b x(t)\, dg(t) = 0$, and if $a > b$, then $\int\limits_a^b x(t)\, dg(t) = -\int\limits_b^a x(t)\, dg(t)$. So, $\int\limits_s^t x(\tau)\, dg(\tau)$ is the Kurzweil integral [9–11].

Moreover, we put

$$\int_s^{t+} x(\tau)\,dg(\tau)=\lim_{\delta\to 0+}\int_s^{t+\delta} x(\tau)\,dg(\tau),\quad \int_s^{t-} x(\tau)\,dg(\tau)=\lim_{\delta\to 0+}\int_s^{t-\delta} x(\tau)\,dg(\tau).$$

If $g(t)\equiv g_1(t)-g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_s^t x(\tau)\,dg_1(\tau)-\int_s^t x(\tau)\,dg_2(\tau)\ \text{ for }\ s,t\in\mathbb{R}.$$

If $G=(g_{ik})_{i,k=1}^{l,n}:[a,b]\to\mathbb{R}^{l\times n}$ is a nondecreasing matrix-function and $X=(x_{kj})_{k,j=1}^{n,m}:[a,b]\to\mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau)=\bigg(\sum_{k=1}^n\int_s^t x_{kj}(\tau)\,dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}\ \text{ for }\ a\le s\le t\le b,$$
$$S_c(G)(t)\equiv(s_c(g_{ik})(t))_{i,k=1}^{l,n},\ \ S_j(G)(t)\equiv\big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n}\ \ (j=1,2).$$

If $G_j:[a,b]\to\mathbb{R}^{l\times n}$ $(j=1,2)$ are nondecreasing matrix-functions, $G=G_1-G_2$ and $X:[a,b]\to\mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau)=\int_s^t dG_1(\tau)\cdot X(\tau)-\int_s^t dG_2(\tau)\cdot X(\tau)\ \text{ for }\ s,\ t\in\mathbb{R},$$
$$S_c(G)=S_c(G_1)-S_c(G_2),\ \ S_j(G)=S_j(G_1)-S_j(G_2)\ \ (j=1,2).$$

A vector-function $x:I_{t_0}\to\mathbb{R}^n$ is said to be a solution of system (1.1) if $x\in\mathrm{BV}([a,b],\mathbb{R}^n)$ for every closed interval $[a,b]$ from I_{t_0} and

$$x(t)=x(s)+\int_s^t dA(\tau)\cdot x(\tau)+f(t)-f(s)\quad\text{for}\quad a\le s<t\le b.$$

We assume that

$$\det(I_n+(-1)^j d_jA(t))\neq 0\quad\text{for}\quad t\in I_{t_0}\quad (j=1,2).$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems (see [9–11]), i.e., for the case when $A\in\mathrm{BV}_{loc}(I,\mathbb{R}^{n\times n})$ and $f\in\mathrm{BV}_{loc}(I,\mathbb{R}^n)$.

Let the matrix-function $A_0\in\mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n})$ be such that

$$\det\big(I_n+(-1)^j d_jA_0(t)\big)\neq 0\ \text{ for }\ t\in I_{t_0}\ \ (j=1,2). \tag{1.5}$$

Then a matrix-function $C_0:I_{t_0}\times I_{t_0}\to\mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the generalized differential system

$$dx=dA_0(t)\cdot x, \tag{1.6}$$

if for every interval and $J\subset I$ and $\tau\in J$, the restriction of the matrix-function $C_0(.,\tau):I_{t_0}\to\mathbb{R}^{n\times n}$ on J is the fundamental matrix of system (1.6) satisfying the condition

$$C_0(\tau,\tau)=I_n.$$

Therefore, C_0 is the Cauchy matrix of system (1.6) if and only if the restriction of C_0 on every interval $J\times J$ is the Cauchy matrix of the system in the sense of definition given in [11].

We assume

$$I_{t_0}^-(\delta)=[t_0-\delta,t_0[\,\cap I_{t_0},\ \ I_{t_0}^+(\delta)=]t_0,t_0+\delta]\cap I_{t_0},\ \ I_{t_0}(\delta)=I_{t_0}^-(\delta)\cup I_{t_0}^+(\delta)$$

for every $\delta>0$.

2 Existence and uniqueness of solutions of the Cauchy problem

In this section we give sufficient conditions for the unique solvability of problem (1.1), (1.2).

Theorem 2.1. *Let there exist a matrix-function $A_0 \in \mathrm{BV}_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$ and constant matrices B_0 and B from $\mathbb{R}_+^{n\times n}$ such that conditions* (1.5) *and*

$$r(B) < 1 \tag{2.1}$$

hold, and the estimates

$$|C_0(t,\tau)| \le H(t)\, B_0\, H^{-1}(\tau) \quad \text{for} \quad t \in I_{t_0}(\delta), \quad (t-t_0)(\tau-t_0) > 0, \ \ |\tau - t_0| \le |t-t_0| \tag{2.2}$$

and

$$\bigg|\int\limits_{t_0\mp}^{t} |C_0(t,\tau)|\, dV(\mathcal{A}(A_0, A-A_0)(\tau)) \cdot H(\tau)\bigg| \le H(t)\, B$$
$$\text{for} \ \ t \in I_{t_0}^{-}(\delta) \ \ \text{and} \ \ t \in I_{t_0}^{+}(\delta), \ \ \text{respectively,} \tag{2.3}$$

are valid for some $\delta > 0$, where C_0 is the Cauchy matrix of system (1.4)*. Let, moreover, respectively,*

$$\lim_{t\to t_0\mp} \bigg\|\int\limits_{t_0\mp}^{t} H^{-1}(\tau)\, |C_0(t,\tau)|\, dV(\mathcal{A}(A_0, f))(\tau)\bigg\| = 0. \tag{2.4}$$

Then problem (1.1), (1.2) *has the unique solution.*

Theorem 2.2. *Let there exist a constant matrix $B = (b_{ik})_{i,k=1}^{n} \in \mathbb{R}_+^{n\times n}$ such that conditions* (2.1) *and*

$$\begin{aligned} &\big[(-1)^j d_j a_{ii}(t)\big]_+ > -1 \ \ \text{for} \ \ t < t_0 \ \ (j=1,2;\ i=1,\dots,n),\\ &\big[(-1)^j d_j a_{ii}(t)\big]_- < 1 \ \ \text{for} \ \ t > t_0 \ \ (j=1,2;\ i=1,\dots,n) \end{aligned} \tag{2.5}$$

hold, and the estimates

$$|c_i(t,\tau)| \le b_0 \frac{h_i(t)}{h_i(\tau)} \ \ \text{for} \ \ t \in I_{t_0}(\delta), \ \ (t-t_0)(\tau-t_0) > 0, \ \ |\tau-t_0| \le |t-t_0| \ \ (i=1,\dots,n), \tag{2.6}$$

$$\bigg|\int\limits_{t_0\mp}^{t} c_i(t,\tau) h_i(\tau)\, d\big[a_{ii}(\tau)\,\mathrm{sgn}(\tau-t_0)\big]_+^{v}\bigg|$$
$$\le b_{ii}(t)\, h_i(t) \ \ \text{for} \ \ t \in I_{t_0}^{-}(\delta) \ \ \text{and} \ \ t \in I_{t_0}^{+}(\delta), \ \ \text{respectively} \ \ (i=1,\dots,n) \tag{2.7}$$

and

$$\bigg|\int\limits_{t_0\mp}^{t} c_i(t,\tau) h_k(\tau)\, dV(\mathcal{A}(a_{0ii}, a_{ik}))(\tau)\bigg| \le b_{ik}(t)\, h_i(t)$$
$$\text{for} \ \ t \in I_{t_0}^{-}(\delta) \ \ \text{and} \ \ t \in I_{t_0}^{+}(\delta), \ \ \text{respectively} \ \ (i \ne k;\ i,k=1,\dots,n) \tag{2.8}$$

are valid for some $b_0 > 0$ and $\delta > 0$. Let, moreover, respectively,

$$\lim_{t\to t_0\mp} \int\limits_{t_0\mp}^{t} \frac{c_i(t,\tau)}{h_i(t)}\, dV(\mathcal{A}(a_{0ii}, f_i))(\tau) = 0 \ \ (i=1,\dots,n), \tag{2.9}$$

where $a_{0ii}(t) \equiv -[a_{ii}(t)\,\mathrm{sgn}(t-t_0)]_-^{v}\,\mathrm{sgn}(t-t_0)$ $(i=1,\dots,n)$ and c_i is the Cauchy function of the equation $dx = x\, da_{0ii}(t)$ for $i \in \{1,\dots,n\}$. Then problem (1.1), (1.2) *has the unique solution.*

Remark 2.1. The Cauchy functions $c_i(t,\tau)$ $(i=1,\dots,n)$, mentioned in the theorem, for $t,\tau\in I_{t_0}^-$ and $t,\tau\in I_{t_0}^+$, have the form

$$c_i(t,\tau)=\begin{cases}\exp\big(s_0(a_{0ii})(t)-s_0(a_{0ii})(\tau)\big)\prod\limits_{\tau<s\le t}(1-d_1a_{0ii}(s))^{-1}\prod\limits_{\tau\le s<t}(1+d_2a_{0ii}(s)) & \text{for } \ t>\tau,\\ \exp\big(s_0(a_{0ii})(t)-s_0(a_{0ii})(\tau)\big)\prod\limits_{t<s\le \tau}(1-d_1a_{0ii}(s))\prod\limits_{t\le s<\tau}(1+d_2a_{0ii}(s))^{-1} & \text{for } \ t<\tau,\\ 1 & \text{for } \ t=\tau.\end{cases}$$

Corollary 2.1. *Let there exist a constant matrix $B=(b_{ik})_{i,k=1}^n\in\mathbb{R}_+^{n\times n}$ such that conditions* (2.1) *and* (2.5) *hold, and the estimates*

$$\bigg|\int\limits_{t_0\mp}^{t}|\tau-t_0|\,d\big[a_{ii}(\tau)\operatorname{sgn}(\tau-t_0)\big]_+^v\bigg| \le b_{ii}\,|t-t_0| \ \text{ for } \ t\in I_{t_0}^-(\delta) \ \text{ and } \ t\in I_{t_0}^+(\delta), \ \text{ respectively } \ (i=1,\dots,n) \tag{2.10}$$

and

$$\bigg|\int\limits_{t_0\mp}^{t}|\tau-t_0|\,dV(\mathcal{A}(a_{0ii},a_{ik}))(\tau)\bigg| \le b_{ik}\,|t-t_0| \ \text{ for } \ t\in I_{t_0}^-(\delta) \ \text{ and } \ t\in I_{t_0}^+(\delta), \ \text{ respectively } \ (i\ne k;\ i,k=1,\dots,n) \tag{2.11}$$

are valid for some $\delta>0$. Let, moreover, respectively,

$$\lim_{t\to t_0\mp}\frac{1}{|t-t_0|}\,\bigg|\bigvee_{t_0}^{t}(\mathcal{A}(a_{0ii},f_i))(\tau)\bigg|=0 \ \ (i=1,\dots,n), \tag{2.12}$$

where $a_{0ii}(t)\equiv-[a_{ii}(t)\operatorname{sgn}(t-t_0))]_-^v\operatorname{sgn}(t-t_0)$ $(i=1,\dots,n)$. Then system (1.1) *has the unique solution satisfying the initial condition*

$$\lim_{t\to t_0\mp}\frac{\|x(t)\|}{t-t_0}=0. \tag{2.13}$$

Remark 2.2. In Corollary 2.2, if the estimates

$$\bigg|\int\limits_{s}^{t}|\tau-t_0|\,d\big[a_{ii}(\tau)\operatorname{sgn}(\tau-t_0)\big]_+^v\bigg|\le b_{ii}|t-s|$$
$$\text{for } \ t,s\in I_{t_0}(\delta), \ \ (t-t_0)(s-t_0)>0, \ \ |s-t_0|\le|t-t_0| \ \ (i=1,\dots,n)$$

and

$$\bigg|\int\limits_{s}^{t}|\tau-t_0|\,dV(\mathcal{A}(a_{0ii},a_{ik}))(\tau)\bigg|\le b_{ik}\,|t-s|$$
$$\text{for } \ t,s\in I_{t_0}(\delta), \ \ (t-t_0)(s-t_0)>0, \ \ |s-t_0|\le|t-t_0| \ \ (i\ne k;\ i,k=1,\dots,n)$$

hold instead of (2.10) and (2.11), respectively, then the solution of problem (1.1), (2.13) belongs to $\mathrm{BV}_{loc}(I,\mathbb{R}^n)$.

Corollary 2.2. *Let conditions* (2.5) *and*

$$\mathcal{J}(a_{0ii})(t)-\mathcal{J}(a_{0ii})(\tau)\le-\lambda_i\ln\frac{t-t_0}{\tau-t_0}+a_{ii}^*(t)-a_{ii}^*(\tau)$$
$$\text{for } \ t,\tau\in I_{t_0}, \ \ (t-t_0)(\tau-t_0)>0, \ \ |\tau-t_0|\le|t-t_0| \ \ (i=1,\dots,n) \tag{2.14}$$

hold, where $a_{0ii}(t) \equiv -[a_{ii}(t)\operatorname{sgn}(t-t_0)]^v_- \operatorname{sgn}(t-t_0)$ $(i=1,\dots,n)$, $\lambda_i \ge 0$ $(i=1,\dots,n)$, a^*_{ii} $(i=1,\dots,n)$ *are nondecreasing functions on the intervals* $I^-_{t_0}$ *and* $I^+_{t_0}$*. Let, moreover,*

$$\bigg|\int\limits_{t_0\mp}^{t} |\tau-t_0|^{\lambda_i-\lambda_k}\,dV(\mathcal{A}(a_{0ii},a_{ik}))(\tau)\bigg| < +\infty$$

$$\text{for } t\in I^-_{t_0} \text{ and } t\in I^+_{t_0}, \text{ respectively } (i\neq k;\ i,k=1,\dots,n), \tag{2.15}$$

and

$$\bigg|\int\limits_{t_0\mp}^{t} |\tau-t_0|^{\lambda_i}\,dV(\mathcal{A}(a_{0ii},f_i))(\tau)\bigg| < +\infty$$

$$\text{for } t\in I^-_{t_0} \text{ and } t\in I^+_{t_0}, \text{ respectively } (i=1,\dots,n). \tag{2.16}$$

Then system (1.1) *has the unique solution satisfying the initial condition*

$$\lim_{t\to t_0\mp}\big(|t-t_0|^{\lambda_i}\,x_i(t)\big)=0 \ \ (i=1,\dots,n). \tag{2.17}$$

3 Well-posedness of the Cauchy problem

Let $I_{t_0t} =]\min\{t_0,t\},\max\{t_0,t\}[$ for $t\in I$.

Definition 3.1. Problem (1.1), (1.2) is said to be H-well-posed if it has the unique solution x and for every $\varepsilon>0$ there exists $\eta>0$ such that problem (1.3), (1.2) has the unique solution y and the estimate

$$\|H(t)\,(x(t)-y(t))\| < \varepsilon \ \text{ for } t\in I$$

holds for every $\widetilde{A}\in \mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n})$ and $\widetilde{f}\in \mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n})$ such that

$$\det\big(I_n+(-1)^j d_j\widetilde{A}(t)\big)\neq 0 \ \text{ for } t\in I_{t_0} \ \ (j=1,2);$$

$$\bigg\|\int\limits_{t_0\mp}^{t} H^{-1}(s)\,dV(\widetilde{A}-A)(s)\cdot H(s)\bigg\| + \sum_{j=1}^{2}\bigg\|\sum_{\tau\in I_{t_0t}} H^{-1}(\tau)|d_j(\widetilde{A}-A)(\tau)|H(\tau)\bigg\| < \eta$$

$$\text{for } t\in I^-_{t_0} \text{ and } t\in I^+_{t_0}, \text{ respectively } \ (j=1,2),$$

and

$$\bigg\|\int\limits_{t_0\mp}^{t} H^{-1}(s)\,dV(\widetilde{f}-f)(s)\cdot H(s)\bigg\| + \sum_{j=1}^{2}\bigg\|\sum_{\tau\in I_{t_0t}} H^{-1}(\tau)|d_j(\widetilde{f}-f)(\tau)|H(\tau)\bigg\| < \eta$$

$$\text{for } t\in I^-_{t_0} \text{ and } t\in I^+_{t_0}, \text{ respectively } \ (j=1,2).$$

Theorem 3.1. *Let* I *be a closed interval and there exist a matrix-function* $A_0\in \mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n})$ *and constant matrices* B_0 *and* B *from* $\mathbb{R}^{n\times n}_+$ *such that conditions* (1.5), (2.1) *hold and estimates* (2.2),

$$|C_0(t,\tau)|\,|d_jA_0(\tau)(I_n+(-1)^j d_jA_0(\tau))^{-1}| \le H(t)\,B_0\,H^{-1}(\tau)$$

$$\text{for } t\in I_{t_0}(\delta),\ \ (t-t_0)(\tau-t_0)>0,\ \ |\tau-t_0|\le|t-t_0| \ \ (j=1,2)$$

and

$$\Big\|\int\limits_{t_0\mp}^{t}|C_0(t,\tau)|\,dV(A)(s)\cdot H(s)\Big\| + \sum_{j=1}^{2}\Big\|\sum_{l\in I_{t_0t}}|C_0(t,\tau)||d_jA_0(\tau)\cdot(I_n+(-1)^jd_jA_0(\tau))^{-1}|\,|d_jA(\tau)|H(\tau)\Big\|<\eta$$
$$\text{for } t\in I_{t_0}^{-} \text{ and } t\in I_{t_0}^{+}, \text{ respectively,}$$

are valid for some $\delta>0$, *where* C_0 *is the Cauchy matrix of system* (1.6). *Let, moreover, respectively,*

$$\lim_{t\to t_0\mp}\bigg(\Big\|\int\limits_{t_0\mp}^{t}H^{-1}(t)\,|C_0(t,\tau)|\,dV(f)(\tau)\Big\| + \sum_{j=1}^{2}\Big\|\sum_{l\in I_{t_0t}}H^{-1}(t)|C_0(t,\tau)|\,|d_jA_0(\tau)\cdot(I_n+(-1)^jd_jA_0(\tau))^{-1}|\,|d_jf(\tau)|\Big\|\bigg)=0.$$

Then problem (1.1), (1.2) *is* H*-well-posed.*

Theorem 3.2. *Let* I *be a closed interval and there exist a constant matrix* $B=(b_{ik})_{i,k=1}^n\in\mathbb{R}_+^{n\times n}$ *such that conditions* (2.1), (2.5) *hold and estimates* (2.6), (2.7),

$$|c_i(t,\tau)|\,|d_ja_{0ii}(\tau)\cdot(1+(-1)^jd_ja_{0ii}(\tau))^{-1}|\le b_0\frac{h_i(t)}{h_i(\tau)}$$
$$\text{for } t\in I_{t_0}(\delta),\ \ (t-t_0)(\tau-t_0)>0,\ \ |\tau-t_0|\le|t-t_0|\ \ (i=1,\dots,n;\ j=1,2)$$

and

$$\Big|\int\limits_{t_0\mp}^{t}|c_i(t,\tau)|h_k(\tau)\,dv(a_{ik})(\tau)\Big| + \sum_{j=1}^{2}\Big|\sum_{\tau\in I_{t_0t}}|c_i(t,\tau)||d_ja_{0ii}(\tau)\cdot(1+(-1)^jd_ja_{0ii}(\tau))^{-1}|\,|d_ja_{ik}(\tau)|h_i(\tau)\Big|\le b_{ik}\,h_i(t)$$
$$\text{for } t\in I_{t_0}^{-}(\delta) \text{ and } t\in I_{t_0}^{+}(\delta), \text{ respectively } (i\ne k;\ i,k=1,\dots,n)$$

are valid for some $b_0>0$ *and* $\delta>0$. *Let, moreover, respectively,*

$$\lim_{t\to t_0\mp}\bigg(\Big|\int\limits_{t_0\mp}^{t}\frac{|c_i(t,\tau)|}{h_i(t)}\,dv(f_i)(\tau)\Big| + \sum_{j=1}^{2}\sum_{\tau\in I_{t_0t}}\frac{|c_i(t,\tau)|}{h_i(t)}|d_ja_{0ii}(\tau)\cdot(1+(-1)^jd_ja_{0ii}(\tau))^{-1}|\,|d_jf_i(\tau)|\bigg)=0\ \ (i=1,\dots,n),$$

where $a_{0ii}(t)\equiv-[a_{ii}(t)\operatorname{sgn}(t-t_0)]_-^v\operatorname{sgn}(t-t_0)$ $(i=1,\dots,n)$, *and* c_i *is the Cauchy function of the equation* $dx=x\,da_{0ii}(t)$ *for* $i\in\{1,\dots,n\}$. *Then problem* (1.1), (1.2) *is* H*-well-posed.*

Corollary 3.1. *Let* I *be a closed interval and there exist a constant matrix* $B=(b_{ik})_{i,k=1}^n\in\mathbb{R}_+^{n\times n}$

such that conditions (2.1) *and* (2.5) *hold, and the estimates*

$$\mathcal{J}(a_{0ii})(t) - \mathcal{J}(a_{0ii})(\tau) \le \mu_i \ln \frac{t-t_0}{\tau - t_0}$$
$$\text{for } t,\tau \in I_{t_0},\ \ (t-t_0)(\tau-t_0) > 0,\ \ |\tau - t_0| \le |t-t_0|\ \ (i=1,\dots,n), \tag{3.1}$$
$$\lim_{\tau \to t_0 \mp} \Big| \big[a_{ii}(t)\operatorname{sgn}(t-t_0)\big]_+^v - \big[a_{ii}(\tau)\operatorname{sgn}(\tau-t_0)\big]_+^v \Big|$$
$$\le b_{ii}\ \text{ for } t \in I_{t_0}^-(\delta) \text{ and } t \in I_{t_0}^+(\delta), \text{ respectively } (i=1,\dots,n)$$

and

$$\lim_{\tau \to t_0 \mp} \Big| v(a_{ik})(t) - v(a_{ik})(\tau) + \sum_{j=1}^{2} \sum_{s \in I_{t_0\tau}} |d_j a_{0ii}(s) \cdot (1 + (-1)^j d_j a_{0ii}(s))^{-1}|\, |d_j a_{ik}(s)| \le b_{ik}$$
$$\text{for } t \in I_{t_0}^-(\delta) \text{ and } t \in I_{t_0}^+(\delta), \text{ respectively } (i \ne k;\ i,k=1,\dots,n)$$

are valid for some $\mu_i \ge 0$ $(i=1,\dots,n)$ *and* $\delta > 0$, *where* $a_{0ii}(t) \equiv -[a_{ii}(t)\operatorname{sgn}(t-t_0)]_-^v \operatorname{sgn}(t-t_0)$ $(i=1,\dots,n)$. *Let, moreover, respectively,*

$$\lim_{t \to t_0 \mp} \bigg(\Big| \int_{t_0 \mp}^{t} \frac{1}{|\tau - t_0|^{\mu_i}}\, dv(f_i)(\tau) \Big|$$
$$+ \sum_{j=1}^{2} \sum_{\tau \in I_{t_0\tau}} \frac{1}{|\tau - t_0|^{\mu_i}} |d_j a_{0ii}(\tau) \cdot (1 + (-1)^j d_j a_{0ii}(\tau))^{-1}|\, |d_j f_i(\tau)| \bigg) = 0\ \ (i=1,\dots,n).$$

Then system (1.1) *under the condition*

$$\lim_{t \to t_0 \mp} \frac{x_i(t)}{|t-t_0|^{\mu_i}} = 0\ \ (i=1,\dots,n) \tag{3.2}$$

is H-well-posed.

Remark 3.1. Let, in addition to the conditions of Corollary 3.1, the condition

$$\lim_{t \to t_0 \mp} \sup \xi_{ji}(t) < +\infty\ \ (j=1,2;\ i=1,\dots,n) \tag{3.3}$$

hold, where

$$\xi_{ji}(t) = \sum_{\tau \in I_{tj}} \sum_{k=1}^{n} |\tau - t_0|^{\mu_k} |d_j a_{ik}(\tau)| + |d_j f_i(\tau)| \ \text{ for } t \in I_{t_0} \cap]a_1, a_2[\ \ (j=1,2;\ i=1,\dots,n), \tag{3.4}$$

$I_{t1} =]a_1, t]$ and $I_{t2} = [a_1, t[$ for $a_1 < t < t_0$, $I_{t1} =]t, a_2]$ and $I_{t2} = [t, a_2[$ for $t_0 < t < a_2$. Then the solution of problem (1.1), (3.2) belongs to $\mathrm{BV}_{loc}(I, \mathbb{R}^n)$.

Corollary 3.2. *Let I be a closed interval and there exist a constant matrix $B = (b_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n\times n}$ such that conditions* (2.1) *and* (2.5) *hold, and estimates* (2.10), (3.1) *for* $\mu_i = 0$ $(i=1,\dots,n)$ *and*

$$\Big| \int_{t_0 \mp}^{t} |\tau - t_0|\, dv(a_{ik}))(\tau) \Big| + \sum_{j=1}^{2} \sum_{\tau \in I_{t_0 t}} |\tau - t_0| |d_j a_{0ii}(\tau) \cdot (1 + (-1)^j d_j a_{0ii}(\tau))^{-1}|\, |d_j a_{ik}(\tau)| \le b_{ik} |t - t_0|$$
$$\text{for } t \in I_{t_0}^-(\delta) \text{ and } t \in I_{t_0}^+(\delta), \text{ respectively } (i \ne k;\ i,k=1,\dots,n)$$

are valid for some $\delta > 0$, *where* $a_{0ii}(t) \equiv -[a_{ii}(t)\operatorname{sgn}(t-t_0)]_-^v \operatorname{sgn}(t-t_0)$ $(i=1,\dots,n)$. *Let, moreover, respectively,*

$$\lim_{t\to t_0\mp}\frac{1}{|t-t_0|}\bigg(|v(f_i)(t)-v(f_i)(t_0\mp)| + \sum_{j=1}^{2}\sum_{\tau\in I_{t_0\tau}}|d_ja_{0ii}(\tau)\cdot(1+(-1)^jd_ja_{0ii}(\tau))^{-1}|\,|d_jf_i(\tau)|\bigg)=0 \ \ (i=1,\dots,n).$$

Then problem (1.1), (2.13) *is* H*-well-posed.*

Remark 3.2. Let, in addition to the conditions of Corollary 3.2, condition (3.3) hold, where the functions ξ_{ji} $(j=1,2;\ i=1,\dots,n)$ are defined by (3.4), $\mu_i=1$ $(i=1,\dots,n)$, and the intervals I_{tj} $(j=1,2)$ are defined as in Remark 3.1. Then the solution of problem (1.1), (2.13) belongs to $\mathrm{BV}_{loc}(I,\mathbb{R}^n)$.

Corollary 3.3. *Let* I *be a closed interval and let conditions* (2.5) *and* (2.14) *hold, where* $a_{0ii}(t)\equiv -[a_{ii}(t)\operatorname{sgn}(t-t_0)]_-^v\operatorname{sgn}(t-t_0)$ $(i=1,\dots,n)$, $\lambda_i\geq 0$ $(i=1,\dots,n)$, *and the functions* $a_{ii}^*(t)\operatorname{sgn}(t-t_0)$ $(i=1,\dots,n)$ *are nondecreasing on the interval* I. *Let, moreover,*

$$\bigg|\int_{t_0\mp}^{t}|\tau-t_0|^{\lambda_i-\lambda_k}\,dv(a_{ik}))\tau)\bigg| + \sum_{j=1}^{2}\bigg|\sum_{\tau\in I_{t_0t}}|\tau-t_0|^{\lambda_i-\lambda_k}|d_ja_{0ii}(\tau)\cdot(1+(-1)^jd_ja_{0ii}(\tau))^{-1}|\,|d_ja_{ik}(\tau)|\bigg|<+\infty$$
$$\text{for } t\in I_{t_0}^+ \text{ and } t\in I_{t_0}^-, \text{ respectively } (i\neq k;\ i,k=1,\dots,n)$$

and

$$\bigg|\int_{t_0\mp}^{t}|\tau-t_0|^{\lambda_i}\,dv(f_i))(\tau)\bigg| + \sum_{j=1}^{2}\sum_{\tau\in I_{t_0t}}|\tau-t_0|^{\lambda_i-\lambda_k}|d_ja_{0ii}(\tau)\cdot(1+(-1)^jd_ja_{0ii}(\tau))^{-1}|\,|d_jf_i(\tau)|<+\infty$$
$$\text{for } t\in I_{t_0}^- \text{ and } t\in I_{t_0}^+, \text{ respectively } (i=1,\dots,n).$$

Then system (1.1) *under the condition*

$$\lim_{t\to t_0\mp}\big(|t-t_0|^{\lambda_i}\,x_i(t)\big)=0 \ \ (i=1,\dots,n) \tag{3.5}$$

is H*-well-posed.*

Remark 3.3. Let the conditions of Corollary (3.3) hold, where $\lambda_i=0$ $(i=1,\dots,n)$. Let, in addition, condition (3.3) hold, where the functions ξ_{ji} $(j=1,2;\ i=1,\dots,n)$ are defined by (3.4), $\mu_i=0$ $(i=1,\dots,n)$, and the intervals I_{tj} $(j=1,2)$ are defined as in Remark 3.1. Then the solution of problem (1.1), (3.5) belongs to $\mathrm{BV}_{loc}(I,\mathbb{R}^n)$.

Remark 3.4. In Remarks 3.1–3.3, condition (3.3) is essential, i.e., if the condition is violated, then the conclusion of our remarks are not true. Below, we reduce the corresponding example. Let $I=[0,1]$, $n=1$, $t_0=0$, $t_n=1/\sqrt{n}$ $(n=1,2,\dots)$, the function $a:I\to\mathbb{R}$ is defined by

$$a(0)=0,\ \ a(1)=-\ln 2,\ \ a(t)=\ln\Big(k_n(t-t_n)+\frac{1}{n}\Big)\ \text{ for }\ t_n\leq t<t_{n-1}\ \ (n=2,3,\dots),$$

where $k_n=(n-2)(2n(n-1)(t_n-t_{n-1}))^{-1}$ $(n=2,3,\dots)$. It is evident that the singular Cauchy problem

$$dx=xda(t),\ \ \lim_{t\to 0}t^{-1}|x(t)|=0$$

has the unique solution x defined by the equalities

$$x(t) = k_n(t - t_n) + \frac{1}{n} \text{ for } t_n \le t < t_{n-1} \ (n = 2, 3, \dots), \ x(1) = -\ln 2.$$

Moreover, we have $d_2x(t) \equiv 0$ and $d_1x(t_n) = 1/2$ $(n = 2, 3, \dots)$. Thus we conclude that $x \in \mathrm{BV}_{loc}(I_{t_0}; \mathbb{R})$, but $x \notin \mathrm{BV}_{loc}(I; \mathbb{R})$. Besides, taking into account that the function $a(t)$ is non-increasing on the intervals $t_n \le t < t_{n-1}$ $(n = 2, 3, \dots)$, we conclude that $[a(t)]^v_+ = 0$ on these intervals. Therefore, due to the equalities $d_2a(t) \equiv 0$ and $d_1a(t_n) = 1/2$ $(n = 2, 3, \dots)$, all the conditions of our remarks are fulfilled with the exclusion of (3.3).

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Authors' addresses:

Malkhaz Ashordia

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia;

2. Sokhumi State University, 9 A. Politkovskaia St., Tbilisi 0186, Georgia.

E-mail: malkhaz.ashordia@tsu.ge, ashord@rmi.ge

Valida Sesadze

Georgian Technical University, 77 M. Kostava Str., Tbilisi 0175, Georgia.

E-mail: v.sesadze@gtu.ge

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