Memoirs on Differential Equations and Mathematical Physics

Volume 74, 2018, 1–6

Tengiz Gegelia

(1928–1994)

This year we celebrate the 90th anniversary of the birth of prominent Georgian mathematician Professor Tengiz Gegelia.

Tengiz Gegelia was born on January 28, 1928, in Patara Jikhaishi, a village in Georgia near the city of Kutaisi. In 1945 he entered the Faculty of Physics and Mathematics of Tbilisi State University and completed his university education in 1950. In 1950–1954 he was a post-graduate student, and in 1954–1956 an assistant at the chair of differential and integral equations of Tbilisi State University. In 1956–1966 Gegelia worked as a senior researcher at A. Razmadze Mathematical Institute of the Georgian Academy of Sciences. In 1966 he headed Department of Continuum Mechanics of Institute of Applied Mathematics. In 1980 this department was moved to A. Razmadze Mathematical Institute and Tengiz Gegelia was at its head until his death in 1994.

T. Gegelia defended Candidate of Science thesis in 1954 and his doctoral thesis in 1964. Since 1967, he was a professor at the Tbilisi State University. In 1981–1994, he held the chair of differential and integral equations at Tbilisi State University. In 1974, T. Gegelia was elected a corresponding member of the Georgian Academy of Sciences.

Tengiz Gegelia's mathematical activity covered several fundamental areas: problems of the potential theory and singular integral equations, problems of the classical elasticity theory, as well as the theories of other models of elastic medium such as couple-stress and thermomoment elasticity and electroelasticity. In his first papers published in 1952-1954 T. Gegelia considered singular integral equations with the Cauchy kernel and boundary value problems of the theory of holomorphic functions. He studied these problems for much wider classes of lines than those of straight or piecewise-smooth ones which where considered before. These lines can have an infinite number of angular points, cusp points and points of more complicated structure. To accomplish such an extension, he generalized the notion of the integral in the sense of the Cauchy principal value and investigated the so-called loaded singular integral operator. The results he obtained then formed the basis of his Candidate of Science thesis.

In 1955–1963 Tengiz Gegelia published a series of papers on multidimensional singular integral operators. He investigated differential properties of functions represented by singular integrals as well as of solutions of the corresponding singular integral equations. He also considered singular potentials in various spaces of smooth functions. Other noteworthy results obtained by T. Gegelia in this field include a formula for the difierentiation of singular integrals, a formula for the change of integration order in iterated singular integrals, as well as an estimate of the continuity modulus of the multidimensional singular integral by means of the continuity modulus of the density and the main smoothness characteristics of the kernel and the integration surface. In particular, for a Cauchy type integral, the latter estimate yields the well-known Zygmund inequalities. These papers made an important contribution to the investigation of boundary value problems of elasticity. Victor Kupradze and he were the first scientists who investigated the solvability of the system of boundary integral equations corresponding to the Neumann boundary value problem of elasticity. Together with his associates T. Gegelia investigated boundary value problems of various nonclassical models of elastic medium, which take into account couple and thermal stresses, electric, diffusive and other fields. It also should be mentioned the study of the asymptotic behaviour of solutions of various systems of elasticity in the neighbourhood of isolated singular points. These results significantly stimulated application of the potential method and the theory of singular integral equations to investigation of three-dimensional problems of elasticity. Most of the above-mentioned results of T. Gegelia were included into the well-known monographs *"Three-dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity"* by V. Kupradze, T. Gegelia, M. Basheleishvili, and T. Burchuladze and *"Development of the Potential Method in the Theory of Elasticity"* by T. Burchuladze and T. Gegelia and into his other monographs and papers.

The scientific activities of T. Gegelia won him a wide recognition. He was a member of many national and international scientific organizations and societies. In 1976 he was elected a member of the Bureau of the Scientific Council on Solidity and Plasticity of the USSR Academy of Sciences, and from 1982 he was chairman of the elasticity theory sector of the said Council. From 1984 T. Gegelia was a member of the International Society of Interaction of Mathematics and Mechanics (ISIMM), and, from 1985, a member of the USSR National Committee on Theoretical and Applied Mechanics.

T. Gegelia made a great contribution to the search and development of young talented mathematicians in Georgia. In spite of constant intensive work, he yet managed to find time for teaching at a mathematical secondary school. Tengiz Gegelia was the author of many original textbooks for university and secondary school curricula. He showed interest in teaching mathematics and was regarded as a commonly acknowledged authority in this field. For many years he headed the Methodics Council of the Georgian Public Education Ministry and chaired the organizing committee for holding mathematical olympiads in Georgia. He was the initiator of founding the specialized mathematical school under Tbilisi State University. which is still successfully functioning.

Tengiz Buchukuri

List of Main Publications

(i) Monographs

[1] Three-dimensional Problems of the Mathematical Theory of Elasticity (with V. D. Kupradze, M. O. Basheleishvili, T. V. Burchuladze). (Russian) *Izdat. Tbilis. Univ., Tbilisi*, 1968.

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Memoirs on Differential Equations and Mathematical Physics

Volume 74, 2018, 7–26

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ON THE SOLVABILITY OF THE PERIODIC PROBLEM FOR SYSTEMS OF LINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. A periodic problem for systems of linear generalized differential equations is considered. The Green type theorem on the unique solvability of the problem and the representation of its solution are established. Effective necessary and sufficient conditions (of spectral type) for the unique solvability of the problem are also given.

2010 Mathematics Subject Classification. 34K06, 34K10, 34K13.

Key words and phrases. Nonlocal boundary value problem, periodic problem, linear systems, generalized ordinary differential equations, unique solvability, effective conditions.

რეზიუმე. განზოგადებულ წრფივ დიფერენციალურ განტოლებათა სისტემებისთვის განხილულია პერიოდული ამოცანა. დამტკიცებულია გრინის ტიპის თეორემა ამ ამოცანის ცალსახად ამოხსნადობისა და ამონახსნის წარმოდგენის შესახებ. დადგენილია ცალსახად ამოხსნადობის აუცილებელი და საკმარისი სპექტრალური ტიპის პირობები.

1 Statement of the problem and formulation of the results

In the present paper, we investigate the solvability for the system of linear generalized ordinary differential equations

$$dx(t) = dA(t) \cdot x(t) + df(t) \tag{1.1}$$

with the ω-periodic ($\omega > 0$) condition

$$x(t + \omega) = x(t) \ \text{ for } \ t \in \mathbb{R}, \tag{1.2}$$

where $A = (a_{ik})_{i,k=1}^{n} : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $f = (f_i)_{i=1}^{n} : \mathbb{R} \to \mathbb{R}^n$ are, respectively, the matrix- and the vector-functions with bounded variation components on every closed interval from $\mathbb{R}$, and ω is a fixed positive number.

We establish the Green type theorem on the solvability of problem (1.1), (1.2) and the representation of a solution of the problem. In addition, we give effective necessary and sufficient conditions (of spectral type) for the unique solvability of the problem.

The general linear boundary value problem for system (1.1) has been investigated sufficiently well (see, e.g., [6, 7, 15] and the references therein, where the Green type theorems are obtained for the unique solvability). Some questions related to the periodic problem for system (1.1) are investigated in [2–5, 8, 14] (see also the references therein), but in these works no attention is given to the investigation of specific properties analogous to the already established ones for the ordinary differential case (see, e.g., [11]). But some questions concerning the results obtained in [11] for the periodic problem for linear ordinary differential case is not investigated for the periodic problem for the generalized differential case. So, the problem considered in the paper is quite topical.

We establish some special conditions for the unique solvability of the problem.

To a considerable extent, the interest in the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive differential and difference equations from the unified point of view (see [1, 7, 9, 10, 13, 14] and the references therein).

The theory of generalized ordinary differential equations was introduced by J. Kurzweil [13] in connection with the investigation of the well-posed problem for the Cauchy problem for ordinary differential equations.

In the paper, the use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[\,$, $\mathbb{R}_+ = [0, +\infty[\,$.

$\mathbb{R}^{n\times m}$ is the space of all $n \times m$ real matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$\mathbb{R}_+^{n\times m} = \{(x_{ij})_{i,j=1}^{n,m} :\ x_{ij} \geq 0\ (i = 1, \dots, n;\ j = 1, \dots, m)\}$.

$O_{n\times m}$ (or O) is the zero $n \times m$ matrix.

If $X = (x_{ij})_{i,j=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = (|x_{ij}|)_{i,j=1}^{n,m}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^{n}$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

$x * y$ is the scalar product of the vectors x and $y \in \mathbb{R}^n$.

If $X \in \mathbb{R}^{n\times n}$, then: X^{-1} is the matrix, inverse to X; $\det X$ is the determinant of X; $r(X)$ is the spectral radius of X; X^T is the matrix transposed to X; $\lambda_0(X)$ and $\lambda^0(X)$ are, respectively, the minimal and maximal eigenvalues of the symmetric matrix X.

I_n is the identity $n \times n$-matrix.

The inequalities between the real matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

If $X : [a, b] \to \mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_a^b(X)$ is the sum of variations on $[a, b]$ of its components x_{ij} $(i = 1, \dots, n;\ j = 1, \dots, m)$; $V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(a) = 0$, $v(x_{ij})(t) = \bigvee\limits_a^t(x_{ij})$ for $a < t \leq b$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t ($X(a-)=X(a)$, $X(b+)=X(b)$); $d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

$\|X\|_s=\sup\{\|X(t)\|:\ t\in[a,b]\}$, $|X|_s=(\|x_{ij}\|_s)_{i,j=1}^{n,m}$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the normed space of all bounded variation matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X)<\infty$) with the norm $\|X\|_s$.

$\mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ whose restrictions on every closed interval $[a,b]$ from $\mathbb{R}$ belong to $\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$.

$\mathrm{BV}_{\omega}(\mathbb{R},\mathbb{R}^{n\times m})$ is the set of all matrix-functions $G:\mathbb{R}\to\mathbb{R}^{n\times m}$ whose restrictions on $[0,\omega]$ belong to $\mathrm{BV}([0,\omega],\mathbb{R}^{n\times m})$ and there exists a constant matrix $C\in\mathbb{R}^{n\times m}$ such that

$$G(t+\omega)=G(t)+C\ \text{ for }\ t\in\mathbb{R}.$$

$\mathrm{BV}([a,b],\mathbb{R}_+^{n\times m})=\{X\in\mathrm{BV}([a,b],\mathbb{R}^{n\times m}):\ X(t)\geq O_{n\times m}\text{ for }t\in[a,b]\}$.

$s_c,s_1,s_2:\mathrm{BV}([a,b],\mathbb{R})\to\mathrm{BV}([a,b],\mathbb{R})$ are the operators defined, respectively, by

$$s_1(x)(a)=s_2(x)(a)=0,$$
$$s_1(x)(t)=\sum_{a<\tau\leq t}d_1x(\tau),\quad s_2(x)(t)=\sum_{a\leq\tau<t}d_2x(\tau)\ \text{ for }\ a<t\leq b,$$

and

$$s_c(x)(t)=x(t)-s_1(x)(t)-s_2(x)(t)\ \text{ for }\ t\in[a,b].$$

If $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, $x:[a,b]\to\mathbb{R}$ and $a\leq s<t\leq b$, then

$$\int\limits_s^t x(\tau)\,dg(\tau)=\int\limits_{]s,t[}x(\tau)\,ds_c(g)(\tau)+\sum_{s<\tau\leq t}x(\tau)d_1g(\tau)+\sum_{s\leq\tau<t}x(\tau)d_2g(\tau),$$

where $\int\limits_{]s,t[}x(\tau)\,ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu(s_c(g))$ corresponding to the function $s_c(g)$.

If $a=b$, then we assume

$$\int\limits_a^b x(t)\,dg(t)=0,$$

and if $a>b$, then we assume

$$\int\limits_a^b x(t)\,dg(t)=-\int\limits_b^a x(t)\,dg(t).$$

So, $\int\limits_a^b x(\tau)\,dg(\tau)$ is the Kurzweil–Stieltjes integral (see [13, 14]).

If $g(t)\equiv g_1(t)-g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int\limits_s^t x(\tau)\,dg(\tau)=\int\limits_s^t x(\tau)\,dg_1(\tau)-\int\limits_s^t x(\tau)\,dg_2(\tau)\ \text{ for }\ s\leq t.$$

$L([a,b],\mathbb{R};g)$ is the set of all functions $x:[a,b]\to\mathbb{R}$, measurable and integrable with respect to the measures $\mu(g_i)$ $(i=1,2)$, i.e., such that

$$\int\limits_a^b|x(t)|\,dg_i(t)<+\infty\ \ (i=1,2).$$

If $G=(g_{ik})_{i,k=1}^{l,n}\in \mathrm{BV}([a,b],\mathbb{R}^{l\times n})$ and $X=(x_{kj})_{k,j=1}^{n,m}:[a,b]\to\mathbb{R}^{n\times m}$, then

$$S_c(G)(t)\equiv (s_c(g_{ik})(t))_{i,k=1}^{l,n},\quad S_j(G)(t)\equiv (s_j(g_{ik})(t))_{i,k=1}^{l,n}\ \ (j=1,2)$$

and

$$\int_a^b dG(\tau)\cdot X(\tau)=\bigg(\sum_{k=1}^n\int_a^b x_{kj}(\tau)\,dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}.$$

We introduce the operator $\mathcal{A}$ as follows. If the matrix-function $X\in \mathrm{BV}_{loc}(\mathbb{R},;\mathbb{R}^{n\times n})$ is such that $\det(I_n+(-1)^j d_jX(t))\neq 0$ for $t\in\mathbb{R}$ $(j=1,2)$, and $Y\in \mathrm{BV}_{loc}(\mathbb{R};\mathbb{R}^{n\times m})$, then

$$\begin{aligned}
&\mathcal{A}(X,Y)(0)=O_{n\times m},\\
&\mathcal{A}(X,Y)(t)=Y(t)-Y(0)+\sum_{0<\tau\le t} d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)\\
&\qquad\qquad -\sum_{0\le\tau<t} d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau)\ \text{ for } t>0,\\
&\mathcal{A}(X,Y)(t)=-\mathcal{A}(X,Y)(t)\ \text{ for } t<0.
\end{aligned}$$

Here, the use will be made of the following formulas:

$$\int_a^b f(t)\,d\bigg(\int_a^t h(s)\,dg(s)\bigg)=\int_a^b f(t)h(t)\,dg(t)\ \text{ (substitution formula)};$$

$$\begin{aligned}
\int_a^b f(t)\,dg(t)+\int_a^b g(t)\,df(t)=f(b)g(b)-f(a)g(a)+\sum_{a<t\le b} d_1f(t)\cdot d_1g(t)\\
-\sum_{a\le t<b} d_2f(t)\cdot d_2g(t)\ \text{ (integration by parts formula)},
\end{aligned}$$

$$\begin{aligned}
\int_a^b h(t)\,d(f(t)g(t))=\int_a^b h(t)f(t)\,dg(t)+\int_a^b h(t)g(t)\,df(t)-\sum_{a<t\le b} h(t)d_1f(t)\cdot d_1g(t)\\
-\sum_{a\le t<b} h(t)d_2f(t)\cdot d_2g(t)\ \text{ (general integration by parts formula)}
\end{aligned}$$

and

$$d_j\bigg(\int_a^t f(s)\,dg(s)\bigg)=f(t)\,d_jg(t)\ \text{ for } t\in[a,b]\ \ (j=1,2),$$

where f, g and $h\in \mathrm{BV}([a,b],\mathbb{R})$ (see Theorems I.4.25 and I.4.33 in [14]). Further, we use these formulas without special indication.

We say that the matrix-function $X\in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ satisfies the Lappo–Danilevskiĭ condition if the matrices $S_c(X)(t)$, $S_1(X)(t)$ and $S_2(X)(t)$ are pairwise permutable for every $t\in[a,b]$ and there exists $t_0\in[a,b]$ such that

$$\int_{t_0}^t S_c(X)(\tau)\,dS_c(X)(\tau)=\int_{t_0}^t dS_c(X)(\tau)\cdot S_c(X)(\tau)\ \text{ for } t\in[a,b].$$

A vector-function $x\in \mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^{n\times m})$ is said to be a solution of system (1.1) if

$$x(t)-x(s)=\int_s^t dA(\tau)\cdot x(\tau)+f(t)-f(s)\ \text{ for } s<t,\ \ s,t\in\mathbb{R}.$$

We assume that $A \in \mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times n})$ and $f \in \mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^n)$, i.e.,

$$A(t+\omega) = A(t) + C \ \text{ and } \ f(t+\omega) = f(t) + c \ \text{ for } \ t \in \mathbb{R}, \tag{1.3}$$

where $C \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}^n$ are, respectively, some constant matrix and vector. Moreover, we assume that

$$\det\big(I_n + (-1)^j d_j A(t)\big) \neq 0 \ \text{ for } \ t \in \mathbb{R} \ \ (j=1,2). \tag{1.4}$$

If a matrix-function $X \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ is such that $\det(I_n - d_1X(t)) \neq 0$ for $t \in [0,\omega]$, then we put

$$[X(t)]_0 = (I_n - d_1X(t))^{-1}, \quad [X(t)]_i = (I_n - d_1X(t))^{-1}\int_0^t dX_-(\tau)\cdot[X(\tau)]_{i-1}$$
$$\text{for } \ t \in [0,\omega] \ \ (i=1,2,\dots), \tag{1.5$_1$}$$

$$(X(t))_0 = O_{n\times n}, \ \ (X(t))_1 = X(t), \ \ (X(t))_{i+1} = \int_0^t dX_-(\tau)\cdot(X(\tau))_i$$
$$\text{for } \ t \in [0,\omega] \ \ (i=1,2,\dots), \tag{1.6$_1$}$$

and

$$\begin{aligned} V_1(X)(t) &= \big|(I_n - d_1X(t))^{-1}\big|V(X_-)(t),\\ V_{i+1}(X)(t) &= \big|(I_n - d_1X(t))^{-1}\big|\int_0^t dV(X_-)(\tau)\cdot V_i(X)(\tau) \ \text{ for } \ t \in [0,\omega] \ \ (i=1,2,\dots), \end{aligned} \tag{1.7$_1$}$$

where $X_-(t) \equiv X(t-)$; and if $X \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ is such that $\det(I_n + d_2X(t)) \neq 0$ for $t \in [0,\omega]$, then we put

$$[X(t)]_0 = (I_n + d_2X(t))^{-1}, \ \ [X(t)]_i = (I_n + d_2X(t))^{-1}\int_\omega^t dX_+(\tau)\cdot[X(\tau)]_{i-1}$$
$$\text{for } \ t \in [0,\omega] \ \ (i=1,2,\dots), \tag{1.5$_2$}$$

$$(X(t))_0 = O_{n\times n}, \ \ (X(t))_1 = X(t), \ \ (X(t))_{i+1} = \int_\omega^t dX_+(\tau)\cdot(X(\tau))_i$$
$$\text{for } \ t \in [0,\omega] \ \ (i=1,2,\dots) \tag{1.6$_2$}$$

and

$$\begin{aligned} V_1(X)(t) &= \big|(I_n + d_2X(t))^{-1}\big|\,\big|(V(X_+)(t)(\omega) - V(X_+)(t)\big|,\\ V_{i+1}(X)(t) &= \big|(I_n + d_2X(t))^{-1}\big|\,\bigg|\int_\omega^t dV(X_+)(\tau)\cdot V_i(X)(\tau)\bigg| \ \text{ for } \ t \in [0,\omega] \ \ (i=1,2,\dots), \end{aligned} \tag{1.7$_2$}$$

where $X_+(t) \equiv X(t+)$.

Alongside with system (1.1), we consider the corresponding homogeneous system

$$dx(t) = dA(t)\cdot x(t). \tag{1.1$_0$}$$

Moreover, along with condition (1.2) we consider the condition

$$x(0) = x(\omega). \tag{1.8}$$

Definition 1.1. Let condition (1.4) hold and let there exist a fundamental matrix Y of problem $(1.1_0), (1.8)$ such that

$$\det(D) \neq 0, \tag{1.9}$$

where $D = Y(\omega) - Y(0)$. A matrix-function $\mathcal{G} : [0,\omega] \times [0,\omega] \to \mathbb{R}^{n\times n}$ is said to be the Green matrix of problem $(1.1_0), (1.8)$ if:

(a) the matrix-function $\mathcal{G}(\,\cdot\,, s)$ satisfies the matrix equation

$$dX(t) = dA(t) \cdot X(t)$$

on both $[0, s[$ and $]s, \omega]$ for every $s \in]0, \omega[$;

(b) for $t \in]a, b[$,

$$\mathcal{G}(t, t+) - \mathcal{G}(t, t-) = Y(t)D^{-1}\big\{Y(\omega)Y^{-1}(t)(I_n + d_2A(t))^{-1} - Y(0)Y^{-1}(t)(I_n - d_1A(t))^{-1}\big\};$$

(c) $\mathcal{G}(t, \,\cdot\,) \in BV([0,\omega], \mathbb{R}^{n\times n})$ for every $t \in [0,\omega]$;

(d) the equality

$$\int_0^\omega d_s\big(\mathcal{G}(\omega, s) - \mathcal{G}(0, s)\big) \cdot f(s) = 0$$

holds for every $f \in BV([0,\omega], \mathbb{R}^n)$.

The Green matrix of problem $(1.1_0), (1.8)$ exists and is unique in the following sense (see [6,15]). If $\mathcal{G}(t, s)$ and $\mathcal{G}_1(t, s)$ are two matrix-functions satisfying conditions (a)–(d) of Definition 1.1, then

$$\mathcal{G}(t, s) - \mathcal{G}_1(t, s) \equiv Y(t)H_*(s),$$

where $H_* \in \mathrm{BV}([0,\omega], \mathbb{R}^{n\times n})$ is a matrix-function such that

$$H_*(s+) = H_*(s-) = C = const \ \text{ for } \ s \in [0,\omega],$$

and $C \in \mathbb{R}^{n\times n}$ is a constant matrix.

In particular,

$$\mathcal{G}(t, s) = \begin{cases} Y(t)D^{-1}Y(0)Y^{-1}(s) & \text{for } \ 0 \le s < t \le \omega, \\ Y(t)D^{-1}Y(\omega)Y^{-1}(s) & \text{for } \ 0 \le t < s \le \omega, \\ \text{arbitrary} & \text{for } \ t = s. \end{cases}$$

Theorem 1.1. *System* (1.1) *has a unique ω-periodic solution x if and only if the corresponding homogeneous system* (1.1_0) *has only the trivial solution satisfying condition* (1.8)*, i.e., when condition* (1.9) *holds, where Y is a fundamental matrix of system* (1.1_0)*. If the last condition holds, then the solution x can be written in the form*

$$x(t) = \int_0^\omega d_s\mathcal{G}(t, s) \cdot f(s) \ \text{ for } \ t \in [0,\omega], \tag{1.10}$$

where $\mathcal{G} : [a, b] \times [a, b] \to \mathbb{R}^{n\times n}$ is the Green matrix of problem $(1.1_0), (1.8)$*.*

Corollary 1.1. *Let conditions* (1.3) *and* (1.4) *hold, and the matrix-function A satisfy the Lappo–Danilevskiĭ condition. Then system* (1.1) *has a unique ω-periodic solution if and only if*

$$\det\bigg(\exp(S_0(A)(\omega)) \prod_{0\le\tau<\omega} (I_n + d_2A(\tau)) \prod_{a<\tau\le\omega} (I_n - d_1A(\tau))^{-1} - I_n\bigg) \neq 0. \tag{1.11}$$

Note that if the matrix-function A satisfies the Lappo–Danilevskiĭ condition, then the matrix-function Y defined by $Y(0)=I_n$ and

$$Y(t)\equiv \exp(S_0(A)(t))\prod_{0\le\tau<t}(I_n+d_2A(\tau))\prod_{0<\tau\le t}(I_n-d_1A(\tau))^{-1}\ \text{ for }\ t\in[0,\omega] \tag{1.12}$$

is the fundamental matrix of system (1.1_0).

Remark 1.1. Let system (1.1_0) have a nontrivial ω-periodic solution. Then there exists $f\in \mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^n)$ such that system (1.1) has no ω-periodic solution (see [6]).

In general, it is rather difficult to verify condition (1.9) directly even in the case where one is able to write the fundamental matrix of system (1.1_0) explicitly. Therefore, it is important to find effective conditions which would guarantee the absence of nontrivial ω-periodic solutions of the homogeneous system (1.1_0). Below, we will give the results concerning this question. Analogous results have been obtained by T. Kiguradze for ordinary differential equations (see [11, 12]).

Theorem 1.2. *System* (1.1) *has a unique ω-periodic solution if and only if there exist natural numbers k and m such that the matrix*

$$M_k=-\sum_{i=0}^{k-1}\big([A(\omega)]_i-[A(0)]_i\big) \tag{1.13}$$

is nonsingular and

$$r(M_{k,m})<1, \tag{1.14}$$

where

$$M_{k,m}=V_m(A)(c)+\Big(\sum_{i=0}^{m-1}\big|[A(\cdot)]_i\big|_s\Big)\cdot|M_k^{-1}|\cdot\big(V_k(A)(\omega)-V_k(A)(0)\big), \tag{1.15}$$

$[A(t)]_i$ $(i=0,\dots,m-1)$ *and* $V_i(A)(t)$ $(i=0,\dots,m-1)$ *are defined, respectively, by* (1.5_l) *and* (1.7_l) *for some* $l\in\{1,2\}$, *and* $c=(2-l)\omega$.

Corollary 1.2. *System* (1.1) *has a unique ω-periodic solution if and only if there exist natural numbers k and m such that the matrix*

$$M_k=-\sum_{i=0}^{k-1}\big((A(\omega))_i-(A(0))_i\big) \tag{1.16}$$

is nonsingular and inequality (1.14) *holds, where*

$$M_{k,m}=(V(A)(c))_m+\Big(I_n+\sum_{i=0}^{m-1}\big|(A(\cdot))_i\big|_s\Big)\cdot|M_k^{-1}|\cdot\big[(V(A)(\omega))_k-(V(A)(0))_k\big], \tag{1.17}$$

$(A(t))_i$ $(i=0,\dots,m-1)$ *and* $(V(A)(t))_i$ $(i=0,\dots,m-1)$ *are defined by* (1.6_l) *for some* $l\in\{1,2\}$, *and* $c=(2-l)\omega$.

Corollary 1.3. *Let there exist a natural j such that*

$$(A(0))_i=(A(\omega))_i\ \ (i=1,\dots,j-1) \tag{1.18}$$

and

$$\det\big((A(\omega))_j-(A(0))_j\big)\ne 0, \tag{1.19}$$

where $(A(t))_i$ $(i=0,\dots,j)$ *are defined by* (1.6_l) *for some* $l\in\{1,2\}$. *Then there exists* $\varepsilon_0>0$ *such that the system*

$$dx(t)=\varepsilon dA(t)\cdot x(t)+df(t) \tag{1.20}$$

has one and only one ω-periodic solution for every $\varepsilon\in]0,\varepsilon_0[$.

Theorem 1.3. *Let a matrix-function $A_0 \in \mathrm{BV}_\omega(\mathbb{R}, \mathbb{R}^{n\times n})$ be such that*

$$\det\big(I_n + (-1)^j d_j A_0(t)\big) \neq 0 \ \ \text{for} \ \ t \in [0,\omega] \ \ (j=1,2) \tag{1.21}$$

and the homogeneous system

$$dx(t) = dA_0(t)\cdot x(t) \tag{1.22}$$

has only the trivial ω-periodic solution. Let, moreover, the matrix-function $A \in \mathrm{BV}_\omega(\mathbb{R}, \mathbb{R}^{n\times n})$ admit the estimate

$$\int\limits_0^\omega |\mathcal{G}_0(t,\tau)|\, dV(S_0(A-A_0))(\tau) \\ + \sum_{0<\tau\le\omega} \Big|\mathcal{G}_0(t,\tau-)\cdot d_1\big(A(\tau)-A_0(\tau)\big)\Big| + \sum_{0\le\tau<\omega} \Big|\mathcal{G}_0(t,\tau+)\cdot d_2\big(A(\tau)-A_0(\tau)\big)\Big| \le M, \tag{1.23}$$

where $\mathcal{G}_0(t,\tau)$ is the Green matrix of problem (1.22), (1.8), *and $M \in \mathbb{R}_+^{n\times n}$ is a constant matrix such that*

$$r(M) < 1. \tag{1.24}$$

Then system (1.1) *has one and only one ω-periodic solution.*

Formula (1.10) can be written in a simpler form if we introduce the concept of the Green matrix for problem $(1.1_0), (1.2)$.

Definition 1.2. A matrix-function $\mathcal{G}_\omega : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$ is said to be the Green matrix of problem $(1.1_0), (1.2)$ if:

(a)

$$\mathcal{G}_\omega(t+\omega,\tau+\omega) = \mathcal{G}_\omega(t,\tau), \ \ \mathcal{G}_\omega(t,t+\omega) - \mathcal{G}_\omega(t,t) = I_n \ \ \text{for} \ \ t,\tau\in\mathbb{R}; \tag{1.25}$$

(b) the matrix-function $\mathcal{G}_\omega(\,\cdot\,,\tau) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a fundamental matrix of system (1.1_0) for every $\tau \in \mathbb{R}$.

Theorem 1.4. *Let condition* (1.3) *hold,*

$$\det\big(I_n \pm d_j A(t)\big) \neq 0 \ \ \text{for} \ \ t\in\mathbb{R} \ \ (j=1,2), \tag{1.26}$$

and system (1.1_0) have a unique ω-periodic solution. Then system (1.1) *has likewise a unique ω-periodic solution x which is written in the form*

$$x(t) = \int\limits_t^{t+\omega} \mathcal{G}_\omega(t,\tau)\, d\mathcal{A}\big(A, \mathcal{A}(-A,f)\big)(\tau) \ \ \text{for} \ \ t\in\mathbb{R}, \tag{1.27}$$

where $\mathcal{G}_\omega$ is the Green matrix of problem $(1.1_0), (1.2)$.

We introduce the following class of matrix-functions.

Let $m, r_1, \dots, r_m$ and $n_1, \dots, n_m$ $(0 = n_0 < n_1 < \dots < n_m = n)$ be natural numbers; $\sigma_j \in \{-1, 1\}$ $(j = 1,\dots,m)$; $g_{lj} : [0,\omega] \to \mathbb{R}$ $(l = 1,\dots,r_j;\ j = 1,\dots,m)$ be nondecreasing functions; $\alpha_{lj} \in L([0,\omega],\mathbb{R}; g_{lj})$ $(l = 1,\dots,r_j;\ j = 1,\dots,m)$, and let matrix-functions $\mathcal{P}_{lj} = (p_{ljik})_{i,k=1}^n$ $(l = 1,\dots,r_j;$ $j = 1,\dots,m)$ be such that $p_{ljik} \in L([0,\omega],\mathbb{R}; g_{lj})$ $(i,k = n_{j-1}+1,\dots,n_j)$ and

$$\sigma_j \sum_{i,k=n_{j-1}+1}^{n_j} p_{ljik}(t) x_i x_k \ge \alpha_{lj}(t) \sum_{i,k=n_{j-1}+1}^{n_j} x_i^2 \ \ \text{for} \ \ \mu(g_{lj})\text{-almost all} \ \ t\in[0,\omega], \ \ (x_i)_{i=1}^n \in \mathbb{R}^n \\ (l = 1,\dots,r_j;\ \ j = 1,\dots,m). \tag{1.28}$$

Then by $Q_\omega((r_j,n_j,\sigma_j)_j^m,(g_{lj},\alpha_{lj},\mathcal{P}_{lj})_{l=1,j=1}^{r_j,m})$ we denote the set of all matrix-functions $A\in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ such that

$$a_{ik}(t)\equiv 0\ \ (i=n_{j-1}+1,\dots,n_j;\ \ k=n_j+1,\dots,n;\ \ j=1,\dots,m), \tag{1.29}$$

$$\sigma_j\bigg(b_{jii}(t)-b_{jii}(s)-\sum_{l=1}^{r_j}\int\limits_s^t p_{ljii}(\tau)dg_{lj}(\tau)\bigg)\geq 0\ \text{ for }\ 0\leq s\leq t\leq\omega$$
$$(i=n_{j-1}+1,\dots,n_j;\ \ j=1,\dots,m) \tag{1.30}$$

and

$$b_{jik}(t)=\sum_{l=1}^{r_j}\int\limits_s^t p_{ljik}(\tau)\,dg_{lj}(\tau)\ \text{ for }\ t\in[0,\omega]\ \ (i\neq k;\ \ i,k=n_{j-1}+1,\dots,n_j;\ \ j=1,\dots,m), \tag{1.31}$$

where

$$b_{jik}(t)\equiv a_{ik}(t)-\bigg(\frac{1}{2}\sum_{0<\tau\leq t}\ \sum_{r=n_{j-1}+1}^{n_j}d_1a_{ri}(\tau)\cdot d_1a_{rk}(\tau)-\sum_{0\leq\tau<t}\ \sum_{r=n_{j-1}+1}^{n_j}d_2a_{ri}(\tau)\cdot d_2a_{rk}(\tau)\bigg)$$
$$(i,k=n_{j-1}+1,\dots,n_j;\ \ j=1,\dots,m). \tag{1.32}$$

If $s\in\mathbb{R}$ and $\beta\in \mathrm{BV}([0,\omega],\mathbb{R})$ are such that

$$1+(-1)^jd_j\beta(t)\neq 0\ \text{ for }\ (-1)^j(t-s)<0\ \ (j=1,2),$$

then by $\gamma_s(\beta)$ we write a unique solution of the Cauchy problem

$$d\gamma(t)=\gamma(t)\,d\beta(t),\ \ \gamma(s)=1.$$

Notice that condition (1.4) guarantees the unique solvability of the Cauchy problem for system (1.1) (see, e.g., [13,14]).

It is known (see [9,10]) that

$$\gamma_s(\beta)(t)=\begin{cases}\exp\big(s_0(\beta)(t)-s_0(\beta)(s)\big)\displaystyle\prod_{s<\tau\leq t}(1-d_1\beta(\tau))^{-1}\prod_{s\leq\tau<t}(1+d_2\beta(\tau)) & \text{for } s<t\leq\omega,\\ \exp\big(s_0(\beta)(t)-s_0(\beta)(s)\big)\displaystyle\prod_{t<\tau\leq s}(1-d_1\beta(\tau))\prod_{t\leq\tau<s}(1+d_2\beta(\tau))^{-1} & \text{for } 0\leq t<s.\end{cases} \tag{1.33}$$

Theorem 1.5. *Let there exist natural numbers m, $r_1,\dots,r_m$ and $n_1,\dots,n_m$ $(0=n_0<n_1<\cdots<n_m=n)$, $\sigma_j\in\{-1,1\}$ $(j=1,\dots,m)$, nondecreasing functions $g_{lj}:[0,\omega]\to\mathbb{R}$ $(l=1,\dots,r_j;$ $j=1,\dots,m)$, functions $\alpha_{lj}\in L([0,\omega],\mathbb{R};g_{lj})$ $(l=1,\dots,r_j;\ j=1,\dots,m)$ and matrix-functions $\mathcal{P}_{lj}=(p_{ljik})_{i,k=1}^n$ $(l=1,\dots,r_j;\ j=1,\dots,m)$, $p_{ljik}\in L([0,\omega],\mathbb{R};g_{lj})$ $(i,k=n_{j-1}+1,\dots,n_j)$ such that*

$$A\in Q_\omega\big((r_j,n_j,\sigma_j)_j^m,(g_{lj},\alpha_{lj},\mathcal{P}_{lj})_{l=1,j=1}^{r_j,m}\big). \tag{1.34}$$

Let, moreover,

$$(1-\sigma_j)d_1g_j(t)+(1+\sigma_j)d_2g_j(t)\neq -2\ \text{ for }\ t\in[0,\omega]\ \ (j=1,\dots,m) \tag{1.35}$$

and

$$\gamma_{t_j}(\sigma_jg_j)(\omega-t_j)<1\ \ (j=1,\dots,m), \tag{1.36}$$

where $t_j=\frac{1}{2}(1+\sigma_j)\omega$, the functions $\gamma_{t_j}(\sigma_jg_j)$ $(j=1,\dots,m)$ are defined by (1.33), and

$$g_j(t)\equiv 2\sum_{l=1}^{r_j}\int\limits_0^t\alpha_{lj}(\tau)\,dg_{lj}(\tau).$$

Then system (1.1) has a unique ω-periodic solution.

Remark 1.2. In the above theorem, if in addition to condition (1.35), the condition

$$(1+\sigma_j)\,d_1g_j(t)+(1-\sigma_j)\,d_2g_j(t)<2 \tag{1.37}$$

holds, then, by (1.33), inequality (1.36) is equivalent to

$$\exp\big(s_0(g_j)(\omega)\big) > -\frac{1}{2}\bigg((1+\sigma_j)\prod_{0<\tau\le\omega}(1-d_1g_j(\tau))\prod_{0\le\tau<\omega}(1+d_1g_j(\tau))^{-1} \\ +(1-\sigma_j)\prod_{0<\tau\le\omega}(1+d_1g_j(\tau))^{-1}\prod_{0\le\tau<\omega}(1-d_2g_j(\tau))\bigg) \ \text{ for } \ t\in[0,\omega] \ \ (j=1,\dots,m).$$

Let $g:[0,\omega]\to\mathbb{R}$ be a nondecreasing function and $P=(p_{ik})_{i,k=1}^n$, where $p_{ik}\in L([0,\omega],\mathbb{R};g)$ $(i,k=1,\dots,n)$. Then we denote by $Q_\omega(P;g)$ the set of all matrix-functions $A=(a_{ik})_{i,k=1}^n\in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ such that

$$b_{ik}(t)=\int\limits_0^t p_{ik}(\tau)\,dg(\tau) \ \text{ for } \ t\in[0,\omega] \ \ (i,k=1,\dots,n), \tag{1.38}$$

where

$$b_{ik}(t)\equiv a_{ik}(t)-\frac{1}{2}\bigg(\sum_{l=1}^n\sum_{0<\tau\le t}d_1a_{li}(\tau)\cdot d_1a_{lk}(\tau)-\sum_{0\le\tau<t}d_2a_{li}(\tau)\cdot d_2a_{lk}(\tau)\bigg) \ \ (i,k=1,\dots,n). \tag{1.39}$$

Theorem 1.6. *Let $A\in Q_\omega(P;g)$. Let, moreover, either*

(a)

$$\sum_{i,k=1}^n p_{ik}(t)x_ix_k\ge\alpha(t)\sum_{i=1}^n x_i^2 \ \textit{ for } \ \mu(g)-a.a. \ t\in[0,\omega], \ \ (x_i)_{i=1}^n\in\mathbb{R}^n, \tag{1.40}$$

$$1-2\alpha(t)d_1g(t)>0, \quad 1+2\alpha(t)d_2g(t)\ne0 \ \textit{ for } \ 0\le t<\omega, \tag{1.41}$$

$$\gamma_\omega(\alpha)(0)<1 \tag{1.42}$$

or

(b)

$$\sum_{i,k=1}^n p_{ik}(t)x_ix_k\le\beta(t)\sum_{i=1}^n x_i^2 \ \textit{ for } \ \mu(g)-a.a. \ t\in[0,\omega], \ \ (x_i)_{i=1}^n\in\mathbb{R}^n, \tag{1.43}$$

$$1+2\beta(t)d_2g(t)>0, \quad 1-2\beta(t)d_1g(t)\ne0 \ \textit{ for } \ 0<t\le\omega, \tag{1.44}$$

$$\gamma_0(\beta)(\omega)<1, \tag{1.45}$$

where $\alpha,\beta\in L([0,\omega],\mathbb{R};g)$, the function $\gamma_0(\beta)$ is defined by (1.33), *and*

$$g_\alpha(t)\equiv2\int\limits_0^t\alpha(\tau)\,dg(\tau) \ \textit{ and } \ g_\beta(t)\equiv2\int\limits_0^t\beta(\tau)\,dg(\tau). \tag{1.46}$$

Then system (1.1) *has a unique ω-periodic solution.*

Corollary 1.4. *Let $A\in Q_\omega(P;g)$. Let, moreover, either* (a) *conditions* (1.41) *and* (1.42) *hold, or* (b) *conditions* (1.44) *and* (1.45) *hold, where the functions g_α and g_β are defined by* (1.46), *$\alpha(t)\equiv\lambda_0(P^*(t))$, $\beta(t)\equiv\lambda^0(P^*(t))$, and $P^*(t)\equiv P(t)+P^T(t)$. Then system* (1.1) *has a unique ω-periodic solution.*

2 Auxiliary propositions

Lemma 2.1. *The following statements are valid:*

(a) *if x is a solution of system* (1.1)*, then the vector-function $y(t) = x(t+\omega)$ $(t \in \mathbb{R})$ will be a solution of system* (1.1)*, as well;*

(b) *problem* (1.1), (1.2) *is solvable if and only if system* (1.1) *has on the closed interval $[0,\omega]$ a solution satisfying the boundary condition* (1.8)*. Moreover, the set of restrictions of solutions of problem* (1.1), (1.2) *on $[0,\omega]$ coincides with the set of solutions of problem* (1.1), (1.8)*.*

Proof. Let x be an arbitrary solution of system (1.1). Assume $y(t) = x(t+\omega)$ for $t \in \mathbb{R}$. Then, by (1.3), we have

$$\begin{aligned}
y(t) &= x(0) + \int_0^{t+\omega} dA(\tau)\cdot x(\tau) + f(t+\omega) - f(0)\\
&= x(0) + \int_0^{\omega} dA(\tau)\cdot x(\tau) + f(\omega) - f(0) + \int_{\omega}^{t+\omega} dA(\tau)\cdot x(\tau) + f(t+\omega) - f(\omega)\\
&= x(\omega) + \int_0^{t} dA(\tau+\omega)\cdot x(\tau+\omega) + f(t+\omega) - f(\omega)\\
&= y(0) + \int_0^{t} dA(\tau)\cdot y(\tau) + f(t) - f(0) \ \text{ for } \ t \in \mathbb{R}.
\end{aligned}$$

Therefore, y is likewise a solution of system (1.1). Thus statement (a) is proved.

Let us show statement (b). It is evident that the restrictions of every solution of problem (1.1), (1.2) on the interval $[0,\omega]$ will be a solution of problem (1.1), (1.8). Consider now an arbitrary solution x of problem (1.1), (1.8). Any continuation of this solution we again denote by x. According to statement (a), the vector-function $y(t) = x(t+\omega)$ will be a solution of system (1.1), as well. On the other hand, in view of (1.8), we have

$$y(0) = x(\omega) = x(0).$$

This implies that the functions x and y are the solutions of system (1.1) under the common initial value condition. So, $x(t) \equiv y(t)$. Therefore, x is a solution of problem (1.1), (1.2). □

Lemma 2.2. *An arbitrary fundamental matrix Y of system* (1.1_0) *satisfies the identity*

$$Y(t+\omega) = Y(t)Y^{-1}(0)Y(\omega) \ \ \textit{for} \ \ t \in \mathbb{R}. \tag{2.1}$$

Proof. By Lemma 2.1, the columns of the matrix-function $Z(t) = Y(t+\omega)$ are the solutions of system (1.1_0). Therefore, there exists a constant matrix $C \in \mathbb{R}$ such that

$$Z(t) = Y(t)C \ \text{ for } \ t \in \mathbb{R}.$$

Thus it is clear that

$$C = Y^{-1}(0)Z(0) = Y^{-1}(0)Y(\omega).$$

Hence equality (2.1) holds. □

Lemma 2.3. *Let problem* (1.1_0), (1.2) *have only the trivial solution. Then there exists a unique Green matrix of the problem having the following form:*

$$\mathcal{G}_\omega(t,\tau) = Y(t)\big(Y^{-1}(\omega)Y(0) - I_n\big)^{-1}Y^{-1}(\tau) \ \ \textit{for} \ \ t,\tau \in \mathbb{R}, \tag{2.2}$$

where Y is a fundamental matrix of system (1.1_0)*.*

Proof. Let Y be an arbitrary fundamental matrix of system (1.1_0). Then, by Lemma 2.1, condition (1.9) holds because the lemma guarantees the validity of Theorem 1.1 (see the proof of Theorem 1.1 below). According to Definition 1.2, the matrix-function $\mathcal{G}_\omega : \mathbb{R}\times\mathbb{R}\to\mathbb{R}^{n\times n}$ is the Green matrix if and only if

$$\mathcal{G}_\omega(t,\tau) = Y(t)C(\tau) \text{ for } t,\tau\in\mathbb{R}, \tag{2.3}$$

where the matrix-function $C:\mathbb{R}\to\mathbb{R}^{n\times n}$ is such that equalities (1.25) hold, i.e.,

$$Y(t+\omega)C(\tau+\omega) = Y(t)C(\tau), \quad Y(t)\big(C(t+\omega)-C(t)\big) = I_n \text{ for } t,\tau\in\mathbb{R}. \tag{2.4}$$

By equality (2.1), equalities (2.4) hold if and only if

$$Y^{-1}(0)Y(\omega)C(\tau+\omega) = C(\tau), \quad C(\tau+\omega)-C(\tau) = Y^{-1}(\tau) \text{ for } \tau\in\mathbb{R}.$$

Clearly,

$$\big(I_n - Y^{-1}(0)Y(\omega)\big)C(\tau) = Y^{-1}(0)Y(\omega)Y^{-1}(\tau) \text{ for } \tau\in\mathbb{R}.$$

Therefore, taking into account condition (1.9), we conclude that

$$C(\tau) = \big(Y^{-1}(\omega)Y(0)-I_n\big)^{-1}Y^{-1}(\tau) \text{ for } \tau\in\mathbb{R}.$$

Putting the obtained value of $C(t)$ in (2.4), we obtain equality (2.2). □

Lemma 2.4. *If $X\in \mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times n})$ and $Y\in \mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times m})$, then*

(a)

$$d_jX(t+\omega) = d_jX(t) \textit{ for } t\in\mathbb{R} \ (j=1,2); \tag{2.5}$$

(b)

$$\mathcal{A}(X,Y)\in \mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times m}), \textit{ i.e., } \mathcal{A}(X,Y)(t+\omega) = \mathcal{A}(X,Y)(t)+C \textit{ for } t\in\mathbb{R}, \tag{2.6}$$

where C is some constant $n\times n$-matrix.

Proof. Consider equality (2.5). Let $j=1$. Then by the definition of the set $\mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^{n\times m})$, we have

$$\begin{aligned} d_1X(t+\omega) &= \lim_{\varepsilon\to 0,\,\varepsilon>0}\big(X(t+\omega)-X(t+\omega-\varepsilon)\big)\\ &= \lim_{\varepsilon\to 0,\,\varepsilon>0}\big(X(t)-X(t-\varepsilon)\big) = d_1X(t) \text{ for } t\in\mathbb{R}. \end{aligned}$$

Analogously, we show equality (2.5) for $j=2$.

Let us show (2.6). From the definition of the operator $\mathcal{A}$ and equalities (2.5), we conclude that

$$\begin{aligned} \mathcal{A}(X,Y)(t+\omega) &= Y(t+\omega)-Y(0)+\sum_{0<\tau\le t+\omega} d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)\\ &\qquad -\sum_{0\le\tau<t+\omega} d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau)\\ &= Y(t+\omega)-Y(0)+C_0+\sum_{\omega<\tau\le t+\omega} d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)\\ &\qquad -\sum_{0\le\tau<t} d_2X(\tau+\omega)\cdot(I_n+d_2X(\tau+\omega))^{-1}\,d_2Y(\tau+\omega)\\ &= Y(t+\omega)-Y(0)+C_0+\sum_{0<\tau\le t} d_1X(\tau+\omega)\cdot(I_n-d_1X(\tau+\omega))^{-1}\,d_1Y(\tau+\omega)\\ &\qquad -\sum_{0\le\tau<t} d_2X(\tau)\cdot(I_n+d_2X(\tau+\omega))^{-1}\,d_2Y(\tau+\omega) = \mathcal{A}(X,Y)(t)+C, \end{aligned}$$

where

$$C_0 = \sum_{0<\tau\le\omega} d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau) - \sum_{0\le\tau<\omega} d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau),$$

and C is some constant matrix. □

Lemma 2.5. *Let $g : [a,b] \to \mathbb{R}$ be a nondecreasing function, $t_0 \in [a,b]$ and $c_0 \in \mathbb{R}$. Let, in addition, $z \in \mathrm{BV}([a,b],\mathbb{R})$ be such that*

$$\begin{gathered}\big(dz(t) - z(t)\,dg(t)\big)\operatorname{sgn}(t-t_0) \le 0 \ \ \textit{for}\ \ t \in [a,b],\\ 1 - d_1 g(t) > 0, \quad 1 + d_2 g(t) \ne 0 \ \ \textit{for}\ \ a \le t < t_0,\\ 1 + d_2 g(t) > 0, \quad 1 - d_1 g(t) \ne 0 \ \ \textit{for}\ \ t_0 < t \le b,\\ (-1)^j\big(d_j z(t_0) - c_0 d_j g(t_0)\big) \le 0 \ \ (j=1,2)\end{gathered}$$

and $z(t_0) \le c_0$. Then

$$z(t) \le x(t) \ \ \textit{for}\ \ t \in [a,b],$$

where x is a unique solution of the problem

$$\begin{gathered}dx(t) = x(t)\,dg(t) \ \ \textit{for}\ \ t \in [a,b],\\ x(t_0) = c_0.\end{gathered}$$

The above lemma is a particular case of Lemma 2.4 from the paper [1].

Lemma 2.6. *If C is a symmetric matrix, then the inequalities*

$$\lambda_0(C)(x * x) \le Cx * x \le \lambda^0(C)(x * x)$$

hold for every $x \in \mathbb{R}$.

The lemma is proved in [11, Lemma 1.9].

3 Proof of the results

By Lemma 2.1, Theorem 1.1 follows immediately from the corresponding results of the papers [6, 15], and Theorems 1.2, 1.3 and Corollaries 1.1–1.3 follow immediately from Theorems 2.1, 2.2 and Corollaries 2.2–2.4 of [7], respectively, if we assume that the linear operator l appearing there has the form $l(x) \equiv x(0) - x(\omega)$. Note that condition (1.9) has form (1.11) when the fundamental matrix of system (1.1_0) is given by (1.12) in Corollary 1.1.

Proof of Theorem 1.4. By Theorem 1.1 and Lemma 2.3, problem (1.1), (1.2) is uniquely solvable, and problem (1.1_0), (1.2) has the unique Green matrix $\mathcal{G}_\omega$. Therefore, for the proof it is sufficient to verify that the vector-function given by (1.27) is the ω-periodic solution of system (1.1).

Assume

$$\varphi(t) = \mathcal{A}(-A,f)(t) \ \ \text{for}\ \ t \in \mathbb{R}.$$

Let us show that the vector-function x defined by (1.27) satisfies condition (1.2). By Lemma 2.4, it is evident that $\mathcal{A}(A,\varphi) \in \mathrm{BV}_\omega(\mathbb{R},\mathbb{R}^n)$ and, therefore,

$$\mathcal{A}(A,\varphi)(t+\omega) = \mathcal{A}(A,\varphi)(t) + c \ \ \text{for}\ \ t \in \mathbb{R}, \tag{3.1}$$

where c is some constant n-vector. Taking into account (3.1) and (1.27), due to (1.25) we have

$$x(t+\omega) = \int\limits_{t+\omega}^{t+2\omega} \mathcal{G}_\omega(t+\omega,\tau)\,d\mathcal{A}(A,\varphi)(\tau) = \int\limits_{t}^{t+\omega} \mathcal{G}_\omega(t+\omega,\tau+\omega)\,d\mathcal{A}(A,\varphi)(\tau+\omega) = x(t).$$

Let us verify that the vector-function x satisfies system (1.1). By equality (2.2),

$$\mathcal{G}_\omega(t,\tau) = Y(t)C_\omega Y^{-1}(\tau) \ \ \text{for}\ \ t,\tau \in \mathbb{R},$$

where Y is a fundamental matrix of system (1.1_0), and

$$C_\omega = \big(Y^{-1}(\omega)Y(0) - I_n\big)^{-1}.$$

Thus, using the general integration by parts formula, we find that

$$\begin{aligned}
x(t)-x(s) &= \int_s^t dx(\tau) = \int_s^t d\bigg(\int_\tau^{\tau+\omega} \mathcal{G}_\omega(\tau,\eta)\,d\mathcal{A}(A,\varphi)(\eta)\bigg) = \int_s^t d\bigg(Y(\tau)C_\omega \int_\tau^{\tau+\omega} Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)\bigg) \\
&= \int_s^t dY(\tau)\cdot C_\omega \int_\tau^{\tau+\omega} Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta) + \int_s^t Y(\tau)C_\omega\,d\bigg(\int_\tau^{\tau+\omega} Y^{-1}(\eta)d\mathcal{A}(A,\varphi)(\eta)\bigg) \\
&\quad - \sum_{s<\eta\le t} d_1Y(\tau)\cdot C_\omega\,d_1\bigg(\int_\tau^{\tau+\omega} Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)\bigg) \\
&\quad + \sum_{s\le\eta<t} d_2Y(\tau)\cdot C_\omega\,d_2\bigg(\int_\tau^{\tau+\omega} Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta)\bigg) \ \text{ for } \ s<t. \qquad (3.2)
\end{aligned}$$

On the other hand, due to (2.1),

$$Y^{-1}(t+\omega)-Y^{-1}(t) \equiv C_\omega^{-1}Y^{-1}(t). \qquad (3.3)$$

By (3.1), for $\tau\in\mathbb{R}$, we conclude that

$$\begin{aligned}
\int_\tau^{\tau+\omega} Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta) &= \int_\tau^{\omega} Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta) + \int_\omega^{\tau+\omega} Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta) \\
&= \int_\tau^{\omega} Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta) + \int_0^{\tau} Y^{-1}(\eta+\omega)\,d\mathcal{A}(A,\varphi)(\eta+\omega) \\
&= \int_0^{\omega} Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta) + \int_0^{\tau} \big(Y^{-1}(\eta+\omega)-Y^{-1}(\eta)\big)\,d\mathcal{A}(A,\varphi)(\eta).
\end{aligned}$$

Hence, taking into account (3.3), we get

$$\int_\tau^{\tau+\omega} Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta) \equiv \int_0^{\omega} Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta) + C_\omega^{-1}\int_0^{\tau} Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta).$$

Due to the last equality and the general integration-by-parts formula, taking into account the equalities

$$dY(t)=dA(t)\cdot Y(t) \ \text{ and } \ d_jY(t)=d_jA(t)\cdot Y(t) \ \text{ for } \ t\in\mathbb{R} \ \ (j=1,2),$$

it follows from (3.2) that

$$\begin{aligned}
x(t)-x(s) &= \int_s^t dA(\tau)\cdot Y(\tau)C_\omega \int_\tau^{\tau+\omega} Y^{-1}(\eta)\,d\mathcal{A}(A,\varphi)(\eta) + F(s,t) \\
&= \int_s^t dA(\tau)\cdot x(\tau) + F(s,t) \ \text{ for } \ s,t\in\mathbb{R}, \ \ s<t, \qquad (3.4)
\end{aligned}$$

where

$$\begin{aligned}
F(s,t) &= \mathcal{A}(A,\varphi)(t)-\mathcal{A}(A,\varphi)(s) \\
&\quad - \sum_{s<\tau\le t} d_1A(\tau)\cdot d_1\mathcal{A}(A,\varphi)(\tau) + \sum_{s\le\tau<t} d_2A(\tau)\cdot d_2\mathcal{A}(A,\varphi)(\tau) \ \text{ for } \ s,t\in\mathbb{R}, \ \ s<t.
\end{aligned}$$

Moreover, taking into account condition (1.26), according to the definition of the operator $\mathcal{A}$ and the function φ, we conclude that

$$d_1\varphi(\tau) = d_1 f(\tau) - \sum_{s<\tau\le t} d_1A(\tau)\cdot\big(I_n + d_1A(\tau)\big)^{-1} d_1 f(\tau) \ \text{ for } \ \tau\in\mathbb{R},$$

and

$$d_2\varphi(\tau) = d_2 f(\tau) + \sum_{s\le\tau<t} d_2A(\tau)\cdot\big(I_n - d_2A(\tau)\big)^{-1} d_2 f(\tau) \ \text{ for } \ \tau\in\mathbb{R}.$$

Using the last equalities, we can easily show that

$$\begin{aligned}
F(s,t) = \varphi(t)-\varphi(s) &+ \sum_{s<\tau\le t} d_1A(\tau)\cdot(I_n - d_1A(\tau))^{-1} d_1\varphi(\tau) \\
&- \sum_{s\le\tau<t} d_2A(\tau)\cdot(I_n + d_2A(\tau))^{-1} d_2\varphi(\tau) \\
&- \sum_{s<\tau\le t} (d_1A(\tau))^2\cdot(I_n - d_1A(\tau))^{-1} d_1\varphi(\tau) \\
&- \sum_{s\le\tau<t} (d_2A(\tau))^2\cdot(I_n + d_2A(\tau))^{-1} d_2\varphi(\tau) \\
= \varphi(t)-\varphi(s) &+ \sum_{s<\tau\le t} d_1A(\tau)\cdot d_1\varphi(\tau) - \sum_{s\le\tau<t} d_2A(\tau)\cdot d_2\varphi(\tau) \\
= f(t)-f(s) &\quad \text{for } \ s,t\in\mathbb{R}, \ \ s<t.
\end{aligned}$$

Consequently, due to (3.4), the vector-function x satisfies equation (1.1). □

Proof of Theorem 1.5. According to Theorem 1.1, to prove the theorem, it suffices to show that the homogeneous system (1.1_0) has only the trivial ω-periodic solution. Let $x=(x_i)_{i=1}^n$ be an arbitrary solution of the latter problem. Assume

$$u_j(t) = \sum_{i=n_{j-1}+1}^{n_j} x_i^2(t) \ \text{ for } \ t\in[0,\omega] \ \ (j=1,\dots,m).$$

By condition (1.34), conditions (1.28)–(1.32) are fulfilled. In view of (1.29) and the formula of integration by parts, we find that

$$\begin{aligned}
\sigma_1(u_1(t)-u_1(s)) &= \sigma_1\sum_{i=1}^{n_1}\bigg(2\int_s^t x_i(\tau)\,dx_i(\tau) - \sum_{s<\tau\le t}(d_1x_i(t))^2 + \sum_{s\le\tau<t}(d_2x_i(t))^2\bigg) \\
&= \sigma_1\sum_{i=1}^{n_1}\bigg(2\int_s^t x_i(\tau)x_k(\tau)\,da_{ik}(\tau) + \sum_{s<\tau\le t}\big(x_i^2(\tau)-x_i^2(\tau-) - 2x_i(\tau)\,d_1x_i(\tau)\big) \\
&\qquad + \sum_{s\le\tau<t}\big(x_i^2(\tau+)-x_i^2(\tau) - 2x_i(\tau)\,d_2x_i(\tau)\big)\bigg) \\
&= 2\sigma_1\sum_{i,k=1}^{n_1}\bigg(\int_s^t x_i(\tau)x_k(\tau)\,da_{ik}(\tau) - \sum_{s<\tau\le t} x_i(\tau)x_k(\tau)\,d_1a_{ik}(\tau) - \sum_{s\le\tau<t} x_i(\tau)x_k(\tau)\,d_2a_{ik}(\tau)\bigg) \\
&\qquad + \sigma_1\sum_{j=1}^{2}\big(s_j(u_1)(t)-s_j(u_1)(s)\big) \ \text{ for } \ 0\le s\le t\le\omega.
\end{aligned}$$

Hence,

$$\sigma_1\big(u_1(t)-u_1(s)\big)=2\sigma_1\sum_{i,k=1}^{n_1}\int\limits_s^t x_i(\tau)x_k(\tau)\,ds_0(a_{ik})(\tau)\\ +\sigma_1\sum_{j=1}^{2}\big(s_j(u_1)(t)-s_j(u_1)(s)\big)\ \text{ for }\ 0\le s\le t\le\omega.$$

On the other hand,

$$\sigma_1\sum_{j=1}^{2}\big(s_j(u_1)(t)-s_j(u_1)(s)\big)\\ =\sum_{i,k=1}^{n_1}\bigg\{\sum_{s<\tau\le t}d_1x_i(\tau)\cdot\big(2x_i(\tau)-d_1x_i(\tau)\big)+\sum_{s\le\tau<t}d_2x_i(\tau)\cdot\big(2x_i(\tau)+d_2x_i(\tau)\big)\bigg\}\\ =2\sum_{i,k=1}^{n_1}\bigg\{\sum_{s<\tau\le t}x_i(\tau)x_k(\tau)\Big(d_1a_{ik}(\tau)-\frac12\sum_{r=1}^{n_1}d_1a_{ri}(\tau)\cdot d_1a_{rk}(\tau)\Big)\\ +\sum_{s\le\tau<t}x_i(\tau)x_k(\tau)\Big(d_2a_{ik}(\tau)-\frac12\sum_{r=1}^{n_1}d_2a_{ri}(\tau)\cdot d_2a_{rk}(\tau)\Big)\bigg\}\ \text{ for }\ 0\le s\le t\le\omega.$$

From this and (1.32), we obtain

$$\sigma_1(u_1(t)-u_1(s))=2\sigma_1\sum_{i,k=1}^{n_1}\int\limits_s^t x_i(\tau)x_k(\tau)\,db_{1ik})(\tau)\ \text{ for }\ 0\le s\le t\le\omega. \tag{3.5}$$

With regard for (1.28)–(1.31), it follows from (3.5) that

$$\sigma_1(u_1(t)-u_1(s))=2\sigma_1\sum_{i=1}^{n_1}\int\limits_s^t x_i^2(\tau)\,db_{1ii}(\tau)+2\sigma_1\sum_{i\neq k;i,k=1}^{n_1}\int\limits_s^t x_i(\tau)x_k(\tau)\,db_{1ik}(\tau)\\ \ge 2\sigma_1\sum_{l=1}^{r_1}\sum_{i,k=1}^{n_1}\int\limits_s^t p_{l1ik}(\tau)x_i(\tau)x_k(\tau)\,dg_{l1}(\tau)\ge 2\sum_{l=1}^{r_1}\int\limits_s^t \alpha_{l1}(\tau)\sum_{i=1}^{n_1}x_i^2(\tau)\,dg_{l1}(\tau),$$

i.e.,

$$\sigma_1(u_1(t)-u_1(s))\ge\int\limits_s^t u_1(\tau)\,dg_1(\tau)\ \text{ for }\ 0\le s\le t\le\omega.$$

Moreover, by (1.35), the conditions of Lemma 2.5 are fulfilled for $t_0=t_1$, $c_0=u_1(t_0)$ and $g(t)\equiv g_1(t)$. In addition, by (1.35),

$$1+(-1)^jd_jg_1(t)\neq 0\ \text{ for }\ t\in[0,\omega]\ \ (j=1,2)$$

and, therefore, the problem

$$dx(t)=\sigma_1x(t)\,dg_1(t),\quad x(t_0)=c_0$$

has a unique solution x given by

$$x(t)=c_0\gamma_{t_0}(\sigma_1g_1)(t)\ \text{ for }\ t\in[0,\omega],$$

where the function $\gamma_{t_0}(\sigma_1g_1)(t)$ is defined by (1.33). According to Lemma 2.5, we have

$$u_1(t)\le c_0\gamma_{t_0}(\sigma_1g_1)(t)\ \text{ for }\ t\in[0,\omega]. \tag{3.6}$$

Due to (1.8), we have $u_1(0) = u_1(\omega)$. Hence, it follows from (3.6) that

$$u_1(\omega - t_1) \le u_1(t_1)\gamma_{t_1}(\sigma_1 g_1)(\omega - t_1) = u_1(\omega - t_1)\,\gamma_{t_1}(\sigma_1 g_1)(\omega - t_1).$$

Therefore, due to (1.36),

$$c_0 = u_1(0) = u_1(\omega) = 0,$$

and thus, by (3.6), we have

$$u_1(t) \equiv 0.$$

Using this identity and (1.28)–(1.32), by induction, we prove $u_j(t) \equiv 0$ $(j = 1, \dots, m)$. Consequently, $x_i(t) = 0$ for $t \in [0, \omega]$ $(i = 1, \dots, n)$. □

Proof of Theorem 1.6. According to Theorem 1.1, to prove the theorem, it suffices to show that the homogeneous system (1.1_0) has only the trivial ω-periodic solution. Let $x = (x_i)_{i=1}^n$ be an arbitrary solution of the latter problem. Assume

$$u(t) = \sum_{i=1}^n x_i^2(t) \ \text{ for } \ t \in [0, \omega].$$

Consider the case (a). Analogously to the proof of equality (3.5) in Theorem 1.5, using (1.39), we can show that the equality

$$u(t) - u(s) = 2\sum_{i=1}^n \int_s^t x_i(\tau)x_k(\tau)\,db_{ik}(\tau) \ \text{ for } \ 0 \le s \le t \le \omega$$

is valid. Thus, by (1.38), we have

$$u(t) - u(s) = 2\sum_{i=1}^n \int_s^t p_{ik}(\tau)x_i(\tau)x_k(\tau)\,dg(\tau) \ \text{ for } \ 0 \le s \le t \le \omega.$$

Therefore, due to (1.40), we find

$$u(t) - u(s) \ge \int_s^t u(\tau)\,dg_\alpha(\tau) \ \text{ for } \ 0 \le s \le t \le \omega,$$

i.e.,

$$\big(du(t) - u(t)\,dg_\alpha(t)\big)\operatorname{sgn}(t - \omega) \le 0 \ \text{ for } \ 0 \le s \le t \le \omega$$

and

$$d_1 u(\omega) - u(\omega)\,d_1 g_\alpha(\omega) \ge 0.$$

Now, taking into account condition (1.41), due to Lemma 2.5, we find

$$u(t) \le u(\omega)\gamma_\omega(\alpha)(t) \ \text{ for } \ t \in [0, \omega], \tag{3.7}$$

whence, by equality $u(0) = u(\omega)$ and (1.42), we have

$$u(\omega) = u(0) \le u(\omega)\gamma_\omega(\alpha)(0)$$

and $u(\omega) = 0$. Hence by (3.7) we find $u(t) \equiv 0$ and $x_i(t) \equiv 0$ $(i = 1, \dots, n)$.

In a similar way we can prove the theorem in the case (b) as well. It should only be noted that due to (1.43), (1.44) and Lemma 2.5, we have the estimate

$$u(t) \le u(0)\gamma_0(\beta)(t) \ \text{ for } \ t \in [0, \omega]$$

instead of (3.7). Thus

$$u(\omega) = u(0) \le u(0)\gamma_0(\beta)(\omega)$$

and, therefore, by (1.45), we get $u(0) = 0$, $u(t) \equiv 0$ and $x_i(t) \equiv 0$ $(i = 1, \dots, n)$. □

Proof of Corollary 1.4. It is evident that

$$\sum_{i,k=1}^{n} p_{ik}(t)x_i x_k \equiv \frac{1}{2}\sum_{i,k=1}^{n}\big(p_{ik}(t)+p_{ki}(t)\big) \ \text{ for } \ \mu(g)\text{-almost all } \ t\in[0,\omega], \ \ (x_i)_{i=1}^n\in\mathbb{R}^n,$$

from which by Lemma 2.6, we have

$$\lambda_0(P^*(t))\sum_{i=1}^{n} x_i^2 \le \sum_{i,k=1}^{n} p_{ik}(t)x_i x_k \le \lambda^0(P^*(t))\sum_{i=1}^{n} x_i^2 \ \text{ for } \ \mu(g)\text{-almost all } \ t\in[0,\omega], \ \ (x_i)_{i=1}^n\in\mathbb{R}^n.$$

Therefore, the corollary follows immediately from Theorem 1.6. □

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(Received 05.05.2017)

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Memoirs on Differential Equations and Mathematical Physics

Volume 74, 2018, 27–38

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EXISTENCE RESULTS FOR A NEW CLASS OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH STATE DEPENDENT DELAY

Abstract. In this paper we investigate the existence and uniqueness of solutions on a compact interval for non-linear fractional integro-differential equations with state-dependent delay and non-instantaneous impulses. Our results are based on the Banach contraction principle and the Krasnoselkii fixed point theorem. For the illustration of the results, an example is also discussed.

2010 Mathematics Subject Classification. 26A33, 34A37, 34G20.

Key words and phrases. Fractional integro-differential equations, sectorial operators, state-dependent delay, resolvent of operators, non-instantaneous impulse conditions.

რეზიუმე. სტატიაში ჩვენ შევისწავლით კომპაქტურ ინტერვალზე განსაზღვრულ ამონახსნთა არსებობასა და ერთადერთობას არაწრფივი ფრაქციული ინტეგროდიფერენციალური განტოლებებისათვის შინაგან მდგომარეობაზე დამოკიდებული დაგვიანებითა და არამყისიერი იმპულსებით. ჩვენი შედეგები ეფუძნება ბანახის კუმშვის პრინციპს და კრასნოსელსკის უძრავი წერტილის თეორემას. შედეგების ილუსტრაციისთვის განხილულია შესაბამისი მაგალითი.

1 Introduction

This paper is concerned with the existence of solutions defined on a compact real interval for semilinear integro-differential equations of fractional order for which impulses are not instantaneous of the form

$$y'(t) - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Ay(s)\,ds = f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in (s_i, t_{i+1}] \subset J, \quad i = 0, \ldots, N, \tag{1.1}$$

$$y(t) = g_i(t, y_{\rho(t,y_t)}), \quad t \in (t_i, s_i], \quad i = 1, \ldots, N, \tag{1.2}$$

$$y_0 = \phi \in \mathcal{B}, \tag{1.3}$$

where $1 < \alpha < 2$, $J = [0, b]$, $b > 0$, $A : D(A) \subset E \to E$ is a closed linear operator of sectorial type on a complex Banach space $(E, \| \cdot \|_E)$, the convolution integral in the equation is known as the Riemann–Liouville fractional integral, $f : J \times \mathcal{B} \to E$ and $\rho : J \times \mathcal{B} \to (-\infty, b]$ are suitable functions. For any function y defined on $(-\infty, b]$ and any $t \geq 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t+\theta)$ for $\theta \in (-\infty, 0]$. Here, $y_t(\,\cdot\,)$ represents the history of the state from each time $\theta \in (-\infty, 0]$ up to the present time t. We assume that the histories y_t belong to some abstract phase space $\mathcal{B}$, to be specified later, let $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_{N-1} \leq s_N \leq t_N \leq t_{N+1} = b$ be pre-fixed numbers, and $g_i \in C((t_i, s_i] \times \mathcal{B}, E)$, for all $i = 1, 2, \ldots, N$, stand for the impulsive conditions.

Fractional differential equations have been of great interest recently, in both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, economy and so on (see [20, 21, 23]). In particular, the question on the existence of solutions of the Cauchy problem for fractional integro-differential equations was studied in numerous works; we refer the reader to the books by Abbas *et al.* [1, 2], Kilbas *et al.* [16], Lakshmikantham *et al.* [18], and the papers by Anguraj *et al.* [3], Balachandran *et al.* [5]. Cuevas *et al.* [6, 8, 9], studying S-asymptotically w-periodic solutions of some classes of semilinear differential and integro-differential equations. Recently, Wang and Chen [24] considered a class of retarded integro-differential equations with nonlocal initial conditions where the existence results of solutions are given over the half-line $[0, \infty)$. In [11], Gautam and Dabas studied the existence of local and global mild solution for an impulsive fractional integro-differential equation with state dependent delay.

Recently, Hernández and O'Regan [13] initiated the study on the Cauchy problems for a new type of first order evolution equations with non instantaneous impulses. In the model analyzed in [13], the impulses start abruptly at the points t_i and their action continue on a finite time interval $[t_i, s_i]$. This type of problem motivates to study certain dynamical changes of evolution processes in pharmacotherapy. For example, as in [13], we note the following simplified situation concerning the hemodynamical equilibrium of a person. In the case of decompensation (for example, high or low levels of glucose) one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorbtion for the body are gradual and continuous processes, we can interpret this situation as an impulsive action which starts abruptly and stays active on a finite time interval.

In this paper, we provide sufficient conditions for the existence of solutions for problem (1.1)–(1.3). Our approach is based on the Banach contraction principle and on the Krasnoselskii fixed point theorem.

2 Preliminaries

We introduce notations, definitions and theorems which are used throughout this paper.

Let $C(J, E)$ be the Banach space of continuous functions from J into E with the norm

$$\|y\|_\infty = \sup \big\{ \|y(t)\|_E : \ t \in J \big\}.$$

Let $B(E)$ denote the Banach space of bounded linear operators from E into E.

A measurable function $y : J \to E$ is Bochner integrable if and only if $\|y\|_E$ is Lebesgue integrable.

Let $L^1(J,E)$ denote the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b \|y(t)\|_E \, dt.$$

We define

$$\mathrm{PC}(J,E) = \big\{y :\ J \to E;\ y \in C((t_k,t_{k+1}],E),\ k = 0,1,\dots,N \\ \text{and } y(t_k^+), y(t_k^-) \text{ exist with, } y(t_k^-) = y(t_k),\ k = 1,\dots,N\big\}.$$

Obviously, $\mathrm{PC}(J,E)$ is a Banach space with the norm

$$\|y\|_{\mathrm{PC}} = \sup_{t\in J} \|y(t)\|_E.$$

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [12] and follow the terminology used in [15]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty,0]$ into E, and satisfying the following axioms:

(A_1) If $y : (-\infty,b) \to E$, $b > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ we have the following conditions:

- (i) $y_t \in \mathcal{B}$;
- (ii) there exists a positive constant H such that $\|y(t)\|_E \le H\|y_t\|_{\mathcal{B}}$;
- (iii) there exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that

$$\|y_t\|_{\mathcal{B}} \le K(t) \sup\big\{\|y(s)\|_E :\ 0 \le s \le t\big\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Denote

$$K_b = \sup\big\{K(t) :\ t \in J\big\} \ \text{ and } \ M_b = \sup\big\{M(t) :\ t \in J\big\}.$$

Remark 2.1.

1. (A_1)(ii) is equivalent to $\|\phi(0)\| \le H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.
2. Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \le 0$.
3. From the equivalence in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi-\psi\|_{\mathcal{B}} = 0$, we necessarily have $\phi(0) = \psi(0)$.

Definition 2.2. A function $f : J \times \mathcal{B} \to E$ is said to be a Carathéodory function if

- (i) for each $t \in J$, the function $f(t,\cdot) : \mathcal{B} \to E$ is continuous;
- (ii) for each $y \in \mathcal{B}$, the function $f(\cdot,y) : J \to E$ is measurable.

Definition 2.3. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space E. We recall that A is the generator of a solution operator if there exists $\mu \in \mathbb{R}$ and a strongly continuous function $S : \mathbb{R}^+ \to B(E)$ such that

$$\big\{\lambda^\alpha :\ \mathrm{Re}(\lambda) > \mu\big\} \subset \rho(A)$$

and

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S(t)x \, dt, \ \ \mathrm{Re}\,\lambda > \mu, \ \ x \in E.$$

In this case, $S_\alpha(t)$ is called the solution operator generated by A.

Remark 2.4. The concept of a solution operator, as defined above, is closely related to the concept of a resolvent family (see Prüss [22]). Because of the uniqueness of the Laplace transform, in the border case $\alpha = 1$, the family $S(t)$ corresponds to a C_0 semigroup (see [10]), whereas in the case $\alpha = 2$, a solution operator corresponds to the concept of a cosine family (see [4]). We note that solution operators, as well as resolvent families, are a particular case of (a, k)-regularized families introduced in [19]. According to [19], a solution operator $S_\alpha(t)$ corresponds to a $(1, \frac{t^{\alpha-1}}{\Gamma(\alpha)})$-regularized family. The following result is a direct consequence of [19, Proposition 3.1 and Lemma 2.2].

Proposition 2.5. *Let $S_\alpha(t)$ be a solution operator on E with generator A. Then the following conditions are satisfied:*

(a) *$S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$, $t \geq 0$.*

(b) *Let $x \in D(A)$ and $t \geq 0$,*

$$S_\alpha(t)x = x + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} AS_\alpha(s)\, ds.$$

(c) *Let $x \in E$. Then $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(s)x\, ds \in D(A)$ and*

$$S_\alpha(t)x = x + A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(s)\, ds.$$

Definition 2.6. Let $A : D(A) \subset E \to E$ be a closed linear operator. A is said to be sectorial of the type (M, θ, μ) if there exist $\mu \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$ and $M > 0$ such that the resolvent of A exists outside the sector and following two conditions are satisfied:

(1) $\mu + S_\theta = \{\mu + s : \ \lambda \in \mathbb{C}, \ |\arg(-\lambda)| < \theta\}$;

(2) $\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda-\mu|}$, $\lambda \notin \mu + S_\theta$.

In this paper, we assume that in problem (1.1)–(1.3) the operator A is sectorial of type μ with $0 \leq \theta < \pi(\frac{1-\alpha}{2})$. Then A is the generator of a solution operator given by

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\gamma \exp^{\lambda t} \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\, d\lambda,$$

where γ is a suitable path lying outside the sector $\mu + S_\theta$.

Cuesta [7] has proved that if A is a sectorial operator of type μ, for some $M > 0$ and $0 < \theta < \pi(1 - \frac{\alpha}{2})$, there is $C > 0$ such that

$$\|S_\alpha(t)\|_{B(E)} \leq \frac{CM}{1 + |\mu|t^\alpha} \ \text{ if } \ \mu < 0$$

and

$$\|S_\alpha(t)\|_{B(E)} \leq CM(1 + \mu t^\alpha)e^{\mu^{\frac{1}{\alpha}} t} \ \text{ if } \ \mu \geq 0.$$

Note that $S_\alpha(t)$ is, in fact, integrable on $[0, b]$.

Theorem 2.7 (Krasnoselkii's fixed point theorem [17])**.** *Let B be a closed convex and nonempty subset of a Banach space X. Let P and Q be two operators such that*

(i) *$Px + Qy \in B$, whenever $x, y \in B$;*

(ii) *P is compact and continuous;*

(iii) *Q is a contraction mapping.*

Then there exists $z \in B$ such that $z = Pz + Qz$.

3 Main results

Motivated by [9], we give the following definition of a mild solution of (1.1)–(1.3).

Definition 3.1. We say that the function $y : (-\infty, b] \to E$ is a mild solution of (1.1)–(1.3) if $y_0 = \phi \in \mathcal{B}$ on $(-\infty, b]$, $y|_{[0,b]} \in \mathrm{PC}([0, b], E)$ and

$$y(t) = \begin{cases} S_\alpha(t)\phi(0) + \int\limits_0^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)})\,ds, & t \in [0, t_1], \\ g_i(t, y_{\rho(t,y_t)}), & t \in (t_i, s_i], \quad i = 1, 2, \ldots, N, \\ S_\alpha(t - s_i) g_i(s_i, y_{\rho(s_i, y_{s_i})}) + \int\limits_{s_i}^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)})\,ds, & t \in (s_i, t_{i+1}]. \end{cases}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s, \varphi) : \ (s, \varphi) \in J \times \mathcal{B}, \rho(s, \varphi) \le 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to (-\infty, b]$ is continuous. Additionally, we introduce the following hypothesis:

(H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \le L^\phi(t) \|\phi\|_{\mathcal{B}} \ \text{ for every } \ t \in \mathcal{R}(\rho^-).$$

Remark 3.2. The condition (H_φ) is frequently verified by the functions, continuous and bounded. For more details, see, e.g., [15].

Lemma 3.3 ([14, Lemma 2.4])**.** *If $y : (-\infty, b] \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \le (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b \sup\{|y(\theta)| : \ \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^\phi = \sup\limits_{t \in \mathcal{R}(\rho^-)} L^\phi(t)$.

Proposition 3.4. *From (H_φ), $(A1)$ and Lemma 3.3, for all $t \in [0, b]$ we have*

$$\|y_{\rho(t,y_t)}\|_{\mathcal{B}} \le (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b \|y(t)\|.$$

Our first result is based on the Banach contraction principle.

Theorem 3.5. *Assume:*

$(H1)$ *The solution operator $S_\alpha(t)$ is compact for $t > 0$, and there exists $M > 0$ such that $\|S_\alpha(t)\| \le M$ for every $t \in J$.*

$(H2)$ *There exists $l > 0$ such that*

$$\|f(t, u) - f(t, v)\|_E \le l_f \|u - v\|_{\mathcal{B}} \ \text{ for all } \ u, v \in \mathcal{B}.$$

$(H3)$ *The functions $g_i : (t_i, s_i] \times \mathcal{B} \to E$, $i = 1, \ldots, N$, are continuous and there exist the constants $h_i > 0$, $i = 1, \ldots, N$ such that*

$$\|g_i(t, u) - g_i(t, v)\|_E \le l'_g \|u - v\|_{\mathcal{B}} \ \text{ for all } \ u, v \in \mathcal{B}.$$

If

$$C = M K_b (l'_g + l_f b) < 1,$$

then there exists a unique solution of problem (1.1)–(1.3)*.*

Proof. Let $Y = \{u \in \mathrm{PC}(E): \ u(0) = \phi(0) = 0\}$ be endowed with the uniform convergence topology and $P: Y \to Y$ be the operator defined by

$$(Py)(t) = \begin{cases} S_\alpha(t)\phi(0) + \int\limits_0^t S_\alpha(t-s) f(s, \overline{y}_{\rho(s,\overline{y}_s)})\, ds, & t \in [0, t_1], \ \ i = 0, \\ g_i(t, \overline{y}_{\rho(t,\overline{y}_t)}), & t \in (t_i, s_i], \ \ i \geq 1, \\ S_\alpha(t-s_i) g_i(s_i, \overline{y}_{\rho(s_i,\overline{y}_{s_i})}) + \int\limits_{s_i}^t S_\alpha(t-s) f(s, \overline{y}_{\rho(s,\overline{y}_s)})\, ds, & t \in (s_i, t_{i+1}], \ \ i \geq 1, \end{cases}$$

where $\overline{y}: (-\infty, b] \to E$ is such that $\overline{y}_0 = \phi$ and $\overline{y} = y$ on J. Let $\overline{\phi}: (-\infty, b] \to E$ be the extension of ϕ to $(-\infty, b]$ such that $\overline{\phi}(\theta) = \phi(0) = 0$ on J. It is clear that P is a well-defined operator from Y into Y. We show that P has a fixed point which is, in turn, a mild solution of problem (1.1)–(1.3).

For any $t \in [0, t_1]$ and $y, y^* \in Y$, from $(H1)$–$(H2)$ we have

$$\begin{aligned} \|(Py)(t) - (Py^*)(t)\|_E &\leq \int\limits_0^t \|S_\alpha(t-s)\|_{B(E)} \big\| f(s, \overline{y}_{\rho(s,\overline{y}_s)}) - f(s, \overline{y}^*_{\rho(s,\overline{y}^*_s)}) \big\|_E\, ds \\ &\leq \int\limits_0^t Ml_f \big\| \overline{y}_{\rho(s,\overline{y}_s)} - \overline{y}^*_{\rho(s,\overline{y}^*_s)} \big\|_{\mathcal{B}}\, ds. \end{aligned}$$

Using Proposition 3.4, we get

$$\begin{aligned} \|(Py)(t) - (Py^*)(t)\|_E &\leq \int\limits_0^t Ml_f K_b \big\| \overline{y}(s) - \overline{y}^*(s) \big\|_E\, ds \leq Ml_f K_b \int\limits_0^t \big\| \overline{y}(s) - \overline{y}^*(s) \big\|_E\, ds \\ &= Ml_f K_b \int\limits_0^t \|y(s) - y^*(s)\|_E\, ds \qquad (\text{since } \overline{y} = y \text{ on } [0,b]) \\ &\leq Ml_f K_b b \|y - y^*\|_{\mathrm{PC}}. \end{aligned}$$

For any $t \in (t_i, s_i]$, $i = 1, \dots, N$, we have

$$\|(Py)(t) - (Py^*)(t)\|_E = \big\| g_i(t, \overline{y}_{\rho(t,\overline{y}_t)}) - g_i(t, \overline{y}^*_{\rho(t,\overline{y}^*_t)}) \big\|_E \leq l'_g K_b \|y - y^*\|_{\mathrm{PC}}.$$

Similarly, for any $t \in (s_i, t_{i+1}]$, $i = 1, \dots, N$, we have

$$\begin{aligned} \|(Py)(t) - (Py^*)(t)\|_E &\leq \Big\| S_\alpha(t-s_i) \big[g_i(s_i, \overline{y}_{\rho(s_i,\overline{y}_{s_i})}) - g_i(s_i, \overline{y}^*_{\rho(s_i,\overline{y}^*_{s_i})}) \big] \Big\|_E \\ &\quad + \int\limits_{s_i}^t \|S_\alpha(t-s)\|_{B(E)} \big\| f(s, \overline{y}_{\rho(s,\overline{y}_s)}) - f(s, \overline{y}^*_{\rho(s,\overline{y}^*_s)}) \big\|_E\, ds \\ &\leq Ml'_g K_b \|y - y^*\|_{\mathrm{PC}} + Ml_f K_b b \|y - y^*\|_{\mathrm{PC}} \leq (Ml'_g K_b + Ml_f K_b b) \|y - y^*\|_{\mathrm{PC}}. \end{aligned}$$

Thus, for all $t \in [0, b]$, we obtain $\|(Py) - (Py^*)\|_{\mathrm{PC}} \leq C \|y - y^*\|_{\mathrm{PC}}$. Hence, P is a contraction on Y and has a unique fixed point $y \in P$, which is, obviously, a unique mild solution of problem (1.1)–(1.3) on $[0, b]$. □

To obtain an existence result via Krasnoselskii's fixed point theorem, we need the following assumptions.

$(H4)$ The function $f: J \times \mathcal{B} \to E$ is Carathéodory one.

(*H*5) There exist a function $p \in L^1(J;\mathbb{R}_+)$ and a continuous nondecreasing function $\psi : \mathbb{R}_+ \to (0,\infty)$ such that
$$\|f(t,u)\|_E \le p(t)\psi(\|u\|_{\mathcal{B}}) \ \text{ for a.e. } \ t \in J \ \text{ and each } \ u \in \mathcal{B}.$$

(*H*6) The functions $t \to g_i(t,0)$ are bounded with
$$G^* = \max_{i=1,\dots,N} \|g_i(t,0)\|_E.$$

Theorem 3.6. *Assume that* $(H1)$, $(H3)$*–*$(H6)$ *and* (H_φ) *hold. Then problem* (1.1)*–*(1.3) *has a mild solution.*

Proof. Let P be the operator introduced in the proof of Theorem 3.5. We introduce the decomposition $Py(t) = P^1y(t) + P^2y(t)$, where

$$(P^1y)(t) = \begin{cases} S_\alpha(t-s_i)g_i(s_i,\overline{y}_{\rho(s_i,\overline{y}_{s_i})}) + \displaystyle\int_{s_i}^{t} S_\alpha(t-s)f(s,\overline{y}_{\rho(s,\overline{y}_s)})\,ds, & \text{if } \ t \in (s_i,t_{i+1}], \ \ i \ge 1,\\ S_\alpha(t)\phi(0) + \displaystyle\int_0^t S_\alpha(t-s)f(s,\overline{y}_{\rho(s,\overline{y}_s)})\,ds, & \text{if } \ t \in [0,t_1],\\ 0, & \text{if } \ t \in (t_i,s_i], \ \ i \ge 1, \end{cases}$$

and
$$(P^2y)(t) = \begin{cases} g_i(t,\overline{y}_{\rho(t,\overline{y}_t)}), & \text{if } \ t \in (t_i,s_i], \ \ i \ge 1,\\ 0, & \text{if } \ t \in (s_i,t_{i+1}], \ \ i \ge 1,\\ 0, & \text{if } \ t \in [0,t_1]. \end{cases}$$

Let $B_r = \{y \in Y : \ \|y\|_{\mathrm{PC}} \le r\}$. The proof of the theorem will be given in a couple of steps.

Step 1: For any $y \in B_r$, we have $P^1y + P^2y \in B_r$.

Case 1. For each $t \in [0,t_1]$, we obtain

$$\begin{aligned} \|(P^1y+P^2y)(t)\|_E &\le \|S_\alpha(t)\|_{B(E)}\|\phi(0)\|_{\mathcal{B}} + \int_0^t \|S_\alpha(t-s)f(s,\overline{y}_{\rho(s,\overline{y}_s)})\|_E\,ds\\ &\le M\|\phi\|_{\mathcal{B}} + M\int_0^t \|f(s,\overline{y}_{\rho(s,\overline{y}_s)}\|_E\,ds \le M\|\phi\|_{\mathcal{B}} + M\int_0^t p(s)\psi\big(\|\overline{y}_{\rho(s,\overline{y}_s)}\|_{\mathcal{B}}\big)\,ds. \end{aligned}$$

Set
$$d = (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b r.$$

Then we have
$$\|(P^1y+P^2y)(t)\|_E \le M\|\phi\|_{\mathcal{B}} + M\psi(d)\int_0^t p(s)\,ds.$$

Thus,
$$\|P^1(y)+P^2(y)\| \le M\|\phi\|_{\mathcal{B}} + M\psi(d)\|p\|_{L^1[0,t_1]} \le r.$$

Case 2. For each $t \in [t_i,s_i)$, $i = 1,2,\dots,N$, we have

$$\begin{aligned} \|(P^1y+P^2y)(t)\|_E &\le \|g_i(t,\overline{y}_{\rho(t,\overline{y}_t)})\|_E\\ &\le \big\|g_i(t,\overline{y}_{\rho(t,\overline{y}_t)}) - g_i(t,0)\big\|_E + \|g_i(t,0)\|_E \le l_g'\|\overline{y}_{\rho(t,\overline{y}_t)}\|_{\mathcal{B}} + G^* \le l_g'd + G^*. \end{aligned}$$

Then
$$\|P^1y+P^2y\| \le l_g'd + G^* \le r.$$

Case 3. For each $t \in (s_i, t_{i+1})$, $i = 1, 2, \dots, N$, we obtain

$$\|(P^1 y + P^2 y)(t)\|_E \leq \left\| S_\alpha(t - s_i) g_i(s_i, \overline{y}_{\rho(s_i, \overline{y}_{s_i})}) \right\|_E + \int_{s_i}^{t} \left\| S_\alpha(t-s) f(s, \overline{y}_{\rho(s,\overline{y}_s)}) \right\|_E ds \leq M(l'_g d + G^*) + M\psi(d) \int_{s_i}^{t} p(s)\, ds.$$

Then

$$\|P^1 y + P^2 y\| \leq M\left(l'_g d + G^* + \psi(d) \int_{s_i}^{t} p(s)\, ds \right) \leq r.$$

Thus, we obtain $P^1 y + P^2 y \in B_r$ for any $y \in B_r$.

Step 2: We show that P^2 is a contraction on B_r.

Case 1. For $y_1, y_2 \in B_r$ and for $t \in [0, t_1]$, we have

$$\left\| (P^2 y_1)(t) - (P^2 y_2)(t) \right\|_E = 0.$$

Case 2. For $y_1, y_2 \in B_r$ and for $t \in [t_i, s_i)$, $i = 1, 2, \dots, N$, we have

$$\left\| (P^2 y_1)(t) - (P^2 y_2)(t) \right\|_E \leq l'_g d.$$

Case 3. For $y_1, y_2 \in B_r$ and for $t \in (s_i, t_{i+1}]$, $i = 1, 2, \dots, N$, we obtain

$$\left\| (P^2 y_1)(t) - (P^2 y_2)(t) \right\|_E = 0.$$

Thus, we obtain

$$\|P^2 y_1 - P^2 y_2\|_{\mathrm{PC}} \leq l'_g d = L,$$

which implies that P^2 is a contraction due to $L < 1$.

Step 3: P^1 is continuous.

Let y^n be a sequence such that $y^n \to y$ in B_r.

Case 1. For each $t \in [0, t_1]$, we have

$$\left\| P^1(y^n)(t) - P^1(y)(t) \right\|_E = \left\| S_\alpha(t)\phi(0) + \int_0^t S_\alpha(t-s) \left[f(s, \overline{y}^n_{\rho(s,\overline{y}^n_s)}) - f(s, \overline{y}_{\rho(s,\overline{y}_s)}) \right] ds \right\|_E \leq M \int_0^t \left\| f(s, \overline{y}^n_{\rho(s,\overline{y}^n_s)}) - f(s, \overline{y}_{\rho(s,\overline{y}_s)}) \right\|_E ds.$$

Case 2. For each $t \in [t_i, s_i)$, $i = 1, 2, \dots, N$, we have

$$\left\| P^1(y^n)(t) - P^1(y)(t) \right\|_E = 0.$$

Case 3. For each $t \in (s_i, t_{i+1}]$, $i = 1, 2, \dots, N$, we obtain

$$\left\| P^1(y^n)(t) - P^1(y)(t) \right\|_E = \left\| S_\alpha(t)\phi(0) + \int_0^t S_\alpha(t-s) \left[f(s, \overline{y}^n_{\rho(s,\overline{y}^n_s)}) - f(s, \overline{y}_{\rho(s,\overline{y}_s)}) \right] ds \right\|_E \leq M \int_0^t \left\| f(s, \overline{y}^n_{\rho(s,\overline{y}^n_s)}) - f(s, \overline{y}_{\rho(s,\overline{y}_s)}) \right\|_E ds.$$

Then by $(H4)$, by the Lebesgue dominated convergence theorem, we have

$$\|P^1 y^n - P^1 y\|_{\mathrm{PC}} \to 0 \ \text{ as } \ n \to +\infty.$$

Step 4: P^1 is compact.

I. $P^1(B_r) \subset B_r$ because $\|P^1 y\| \le r$.

II. We show that P^1 maps a bounded set into a equicontinuous set of B_r.

Case 1. For the interval $t \in [0, t_1]$, $0 \le \tau_1 \le \tau_2 \le t_1$, any $y \in B_r$, one has

$$\begin{aligned}\big\|(P^1y)(\tau_2) - (P^1y)(\tau_1)\big\|_E &\le \|S_\alpha(\tau_2) - S_\alpha(\tau_1)\|_{B(E)}\|\phi\|_{\mathcal{B}} \\ &\quad + \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}\|f(s, \overline{y}_{\rho(s,\overline{y}_s)})\|_E\, ds + \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s) f(s, \overline{y}_{\rho(s,\overline{y}_s)})\|\|_E\, ds \\ &\le \|S_\alpha(\tau_2) - S_\alpha(\tau_1)\|_{B(E)}\|\phi\|_{\mathcal{B}} + \psi(d)\int_0^{\tau_1} \big\|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\big\|_{B(E)} p(s)\, ds + M\psi(d)\int_{\tau_1}^{\tau_2} p(s)\, ds.\end{aligned}$$

Case 2. For the interval $t \in [t_i, s_i)$, $i = 1, 2, \dots, N$, $t_i \le \tau_1 \le \tau_2 \le s_i$, any $y \in B_r$, one has

$$\big\|(P^1y)(\tau_2) - (P^1y)(\tau_1)\big\|_E = 0.$$

Case 3. For the interval $t \in (s_i, t_{i+1}]$, $i = 1, 2, \dots, N$, $s_i \le \tau_1 \le \tau_2 \le t_i + 1$, any $y \in B_r$, one has

$$\begin{aligned}\big\|(P^1y)(\tau_2) - (P^1y)(\tau_1)\big\|_E &\le \big\|S_\alpha(\tau_2 - s_i) - S_\alpha(\tau_1 - s_i)\big\|_{B(E)}\|g_i(s_i, \overline{y}_{\rho(s_i,\overline{y}_{s_i})})\|_E \\ &\quad + \int_0^{\tau_1} \big\|S_\alpha(\tau_2 - s_i) - S_\alpha(\tau_1 - s_i)\big\|_{B(E)}\|f(s, \overline{y}_{\rho(s,\overline{y}_s)})\|_E\, ds + \int_{\tau_1}^{\tau_2} \big\|S_\alpha(\tau_2 - s_i) f(s, \overline{y}_{\rho(s,\overline{y}_s)})\big\|_E\, ds \\ &\le \big\|S_\alpha(\tau_2 - s_i) - S_\alpha(\tau_1 - s_i)\big\|_{B(E)}(l'_g d + G^*) \\ &\quad + \psi(d)\int_0^{\tau_1} \big\|S_\alpha(\tau_2 - s_i) - S_\alpha(\tau_1 - s_i)\big\|_{B(E)} p(s)\, ds + M\psi(d)\int_{\tau_1}^{\tau_2} p(s)\, ds.\end{aligned}$$

From the aforementioned equation, we find that $\|(P^1y)(\tau_2) - (P^1y)(\tau_1)\| \to 0$ as $\tau_2 \to \tau_1$, since $S_\alpha(t)$ is continuous in the uniform operator topology. So, P^1 is equicontinuous. As a consequence of Steps 3–4, together with the Arzelà–Ascoli theorem, we can conclude that $P^1 : B_r \to B_r$ is continuous and completely continuous. By using Krasnoselskii's fixed point theorem, the operator $P = P^1 + P^2$ has a fixed point, which is a solution of problem (1.1)–(1.3). □

4 An example

We consider the fractional differential equation with a state-dependent delay of the form

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \dfrac{1}{\Gamma(\alpha - 1)}\displaystyle\int_{-\infty}^{t} (t-s)^{\alpha-2} L_x u(s,x)\, ds \\ \qquad = \dfrac{e^{-\gamma t + t}|u(t - \sigma(u(t,0)), x)|}{3(e^{-t} + e^t)(1 + |u(t - \sigma(u(t,0), x))|)}, \quad (t,x) \in \displaystyle\bigcup_{i=1}^{N} [s_i, t_{i+1}] \times [0, \pi], \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0,b], \\ u(\tau, x) = u_0(\tau, x), \quad \theta \in (-\infty, 0], \quad x \in [0,\pi], \\ u(t,x) = G_i\big(t, u\big(t - \sigma(u(t,0)), x\big)\big), \quad (t,x) \in (t_i, s_i] \times [0,\pi], \quad i = 1, 2, \dots, N, \end{cases} \tag{4.1}$$

where $1 < \alpha < 2$, $0 = t_0 = s_0 < t_1 < t_2 < \cdots < t_N - 1 \le s_N \le t_N \le t_N + 1 = b$ are prefixed real numbers, $\sigma \in C(\mathbb{R}, [0, \infty))$, $\gamma > 0$, L_x stands for the operator with respect to the spatial variable x which is given by $L_x = \frac{\partial^2}{\partial x^2} - r$, with $r > 0$. Take $E = L^2([0,\pi],\mathbb{R})$ and the operator $A := L_x : D(A) \subset E \to E$ with domain $D(A) := \{u \in E : \ u'' \in E, \ u(0) = u(\pi) = 0\}$. Clearly, A is densely defined in E and is sectorial. Hence A is a generator of a solution operator on E. For the phase space, we choose $\mathcal{B} = \mathcal{B}_\gamma$ defined by

$$\mathcal{B}_\gamma = \Big\{\phi \in C((-\infty,0],\mathbb{R}) : \ \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists}\Big\}$$

with the norm

$$\|\phi\|_\gamma = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}|\phi(\theta)|.$$

Notice that the phase space $\mathcal{B}_\gamma$ satisfies axioms (A_1), (A_2) and (A_3) (see [15] for more details). Set

$$\begin{aligned} y(t)(x) &= u(t,x), \\ \phi(\theta)(x) &= u_0(\theta,x), \\ f(t,\phi)(x) &= \frac{e^{-\gamma t+t}|\phi(0,x)|}{3(e^{-t}+e^t)(1+|\phi(0,x)|)}, \\ g_i(t,\phi)(x) &= G_i\big(t, u\big(t-\sigma(u(t,0)),x\big)\big), \\ \rho(t,\phi) &= t-\sigma(\phi(0,0)). \end{aligned}$$

Let $\phi \in \mathcal{B}_\gamma$ be such that (H_ϕ) holds, and let $t \to \phi_t$ be continuous on $\mathcal{R}(\rho^-)$. Then by Theorem 3.5, there exists at least one mild solution of (4.1).

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(Received 18.09.2017)

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Memoirs on Differential Equations and Mathematical Physics

Volume 74, 2018, 39–78

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MIXED AND CRACK TYPE PROBLEMS OF THE THERMOPIEZOELECTRICITY THEORY WITHOUT ENERGY DISSIPATION

Abstract. In this paper, we study mixed and crack type boundary value problems of the linear theory of thermopiezoelectricity for homogeneous isotropic bodies possessing the inner structure and containing interior cracks. The model under consideration is based on the Green–Naghdi theory of thermopiezoelectricity without energy dissipation. This theory permits propagation of thermal waves at finite speed. Using the potential method and the theory of pseudodifferential equations on manifolds with boundary we prove existence and uniqueness of solutions and analyze their smoothness and asymptotic properties. We describe an efficient algorithm for finding the singularity exponents of the thermo-mechanical and electric fields near the crack edges and near the curves where different types of boundary conditions collide. By explicit calculations it is shown that the stress singularity exponents essentially depend on the material parameters, in general.

2010 Mathematics Subject Classification. 35B65, 35S15, 45M05, 47G30, 74A15, 74F05, 74F15, 74G40, 74G70.

Key words and phrases. Thermopiezoelectricity without energy dissipation, bodies with microstructure, mixed boundary value problem, crack problem, potential method, pseudodifferential equations, stress singularities.

რეზიუმე. ამ სტატიაში ჩვენ შევისწავლით თერმოპიეზოელექტროობის წრფივი თეორიის შერეულ და ბზარის ტიპის სასაზღვრო ამოცანებს შინაგანი სტრუქტურის მქონე ერთგვაროვანი იზოტროპული სხეულებისთვის, რომელთაც გააჩნია შინაგანი სტრუქტურა და შეიცავს შიდა ბზარებს. განხილული მოდელი ეფუძნება გრინ-ნახდის თერმოპიეზოელექტროობის თეორიას ენერგიის დისიპაციის გარეშე. ამ თეორიაში დასაშვებია თერმული ტალღების გავრცელება სასრული სიჩქარით. პოტენციალთა მეთოდისა და საზღვრიან მრავალსახეობებზე გავრცელებული ფსევდოდიფერენციულ განტოლებათა თეორიის გამოყენებით ჩვენ ვამტკიცებთ ამოცანების ამონახსნთა არსებობასა და ერთადერთობას, შევისწავლით მათ სიგლუვესა და ასიმპტოტურ თვისებებს. ჩვენ აღვწერთ ეფექტურ ალგორითმს თერმომექანიკური და ელექტრული ველების სინგულარობის ექსპონენტების გამოსათვლელად ბზარის კიდეების მახლობლობაში და ისეთი წირების მიდამოში, სადაც სხვადასხვა ტიპის სასაზღვრო პირობები ერთმანეთს ხვდება. პირდაპირი გამოთვლებით დგინდება, რომ ძაბვის სინგულარობის ექსპონენტები საზოგადოდ არსებითად არის დამოკიდებული მატერიალურ პარამეტრებზე.

1 Introduction

Theories of thermo-mechanics of continua consistent with a finite speed propagation of heat recently are attracting increasing attention. In contrast to the conventional heat transfer theory, these non-classical refined theories involve a hyperbolic-type heat transport equation, and are motivated by experiments exhibiting the actual occurrence of wave-type heat transport (second sound). Several authors have formulated these theories on different grounds, and a wide variety of problems revealing characteristic features of the theories has been investigated.

Green and Naghdi [13, 14] in 1993 developed a thermo-mechanical theory of thermoelastic bodies based on an entropy balance law rather than an entropy inequality (hereinafter we refer this theory as Green–Naghdi theory). The linearized form of this theory does not sustain energy dissipation and permits the transmission of heat as thermal waves at finite speed. Moreover, the heat flux vector is determined by the same potential function that determines the stress. The thermal waves propagate with finite speeds and the solution has no dissipative term.

Almost complete historical and bibliographical notes to this direction can be found in the reference [16] where the dynamical equations of the thermopiezoelectricity without energy dissipation are derived on the basis of the Green–Naghdi theory established in [13,14] and Eringen's results obtained in [9,10].

In the present paper we consider the pseudo-oscillation equations obtained by the Laplace transform from the dynamical equations derived by Ieşan in [16] for homogeneous isotropic solids possessing thermopiezoelectricity properties without energy dissipation. We formulate the basic, mixed and crack type boundary value problems (BVP) and prove existence and uniqueness of solutions. Our main tools are the potential method and the theory of pseudodifferential equations. Solutions to the mixed and crack type boundary value problems have singularities near the crack edges and near the lines where the different types of boundary conditions collide, regardless of the smoothness of the boundary surfaces and given boundary data. Throughout the paper we shall refer to such lines as *exceptional curves*. We carry out a detailed theoretical investigation of regularity and asymptotic properties of thermo-mechanical and electric fields near the exceptional curves. By explicit calculations we show that the stress singularity exponents essentially depend on the material parameters, in general. We describe an efficient algorithm for finding the singularity exponents of the thermo-mechanical and electric fields. The obtained asymptotic formulas allow us to establish optimal regularity results for solutions.

2 Basic equations

Let $\Omega = \Omega^+$ be a bounded 3-dimensional domain in $\mathbb{R}^3$ with a simply connected piecewise smooth Lipschitz boundary $S = \partial\Omega$, and $\overline{\Omega} = \Omega \cup S$. Throughout the paper $n(x)$ stands for the outward unit normal vector to S at the point $x \in S$. We assume also that the origin of the co-ordinate system belongs to Ω.

By $C^k(\overline{\Omega})$ we denote the subspace of functions from $C^k(\Omega)$ whose derivatives up to the order k are continuously extendable to S from Ω and by $C_0^\infty(\Omega)$ the space of infinitely differentiable test functions with compact supports in $\Omega \subset \mathbb{R}^3$.

The symbols $\{\,\cdot\,\}_S^+$ and $\{\,\cdot\,\}_S^-$ designate one sided limits on S from Ω and $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega}$, respectively. We drop the subscript S if it does not lead to misunderstanding.

By L_p, $L_{p,loc}$, W_p^r, $W_{p,loc}^r$, H_p^s, and $B_{p,q}^s$ (with $r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) are denoted the Lebesgue, Sobolev–Slobodetskii, Bessel potential, and Besov function spaces, respectively (see, e.g., [23]). Recall that $H_2^r = W_2^r = B_{2,2}^r$, $H_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$, and $H_p^k = W_p^k$, for any $r \geq 0$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

We use the notation $v_{i_1 \dots i_m}$ for the components of tensor v of order m and employ the usual Einstein summation convention where the subscripts range over the integers $\{1, 2, 3\}$. Partial derivatives with respect to spatial variable x_j we denote by $\partial_j = \partial/\partial x_j$, $j = 1, 2, 3$, while a superposed dot denotes partial differentiation with respect to the time variable t.

We consider an elastic body that at some instant occupies the region Ω of the Euclidean three-dimensional space and is bounded by a piecewise smooth Lipschitz surface S.

We restrict our consideration to the linear theory of homogeneous isotropic thermoelastic bodies developed by Green and Naghdi [13,14]. According to this theory the system of the governing equations consists of the following field equations [16]:

- The local form of the conservation law of linear momentum

$$\partial_j t_{ji} + \rho_0 f_i = \rho_0 \ddot{u}_i, \tag{2.1}$$

where t_{ji} is the stress tensor, $u = (u_1, u_2, u_3)^\top$ is the displacement vector, f_i is the external body force per unit mass, and ρ_0 is the density in the reference configuration.

- The local form of the conservation law of the moment of momentum

$$\partial_j m_{ji} + \varepsilon_{ijk} t_{jk} + \rho_0 X_i = I_{ij} \ddot{\phi}_j, \tag{2.2}$$

where m_{ij} is the couple stress tensor, ε_{ijk} is the alternating Levi-Civita symbol, X_i is the external body couple per unit mass, I_{ij} are the coefficients of inertia, and ϕ_i is the microrotation vector.

- Maxwell's equations for the quasi-static electric fields

$$\partial_j D_j = f \ \text{ and } \ E_k = -\partial_k \psi, \tag{2.3}$$

where D is the electric displacement field, f is the density of free charge, E is the electric intensity, and ψ is the electric potential.

- The local form of energy balance

$$\rho_0 \dot{e} = t_{ij} \dot{e}_{ij} + m_{ij} \dot{\varkappa}_{ij} + \pi_i \dot{\zeta}_i + \epsilon \dot{\varphi} + \rho_0 s \theta + \partial_i (\Phi_i \theta) + E_i \dot{D}_i,$$

where e is the internal energy per unit mass, φ is the microstretch function, π_i is the microstretch stress vector, s is the external rate of supply of entropy per unit mass, θ is the absolute temperature, Φ_i are components of the entropy flux vector,

$$e_{ij} = \partial_i u_j + \varepsilon_{jik} \phi_k, \quad \varkappa_{ij} = \partial_i \phi_j, \quad \zeta_i = \partial_i \varphi \tag{2.4}$$

and

$$\epsilon = \partial_j \pi_j + \rho_0 \mathcal{F} - j_0 \ddot{\varphi}, \tag{2.5}$$

where j_0 is the microstretch inertia, and $\mathcal{F}$ is the microstretch body force.

- The equation of entropy

$$\rho_0 T_0 \dot{\eta} = q_{j,j} + \rho_0 Q, \tag{2.6}$$

where η is the entropy per unit mass and unit time, T_0 is the initial reference temperature, that is, the temperature in the natural state in the absence of deformation and electromagnetic field, q_i is the heat flux vector

$$q_i = T_0 \Phi_i,$$

and Q is the external rate of supply of heat per unit mass.

The quantities t_{ij}, m_{ij}, π_i, ϵ, D_i, q_i and $\rho_0\eta$ for homogeneous isotropic media can be expressed via u_i, ϕ_i, φ, ψ, ϑ by the following *constitutive relations* [16]:

$$t_{ij} = \lambda e_{rr} \delta_{ij} + (\mu + \varkappa) e_{ij} + \mu e_{ji} + \lambda_0 \varphi \delta_{ij} - \beta_0 T \delta_{ij}, \tag{2.7}$$

$$m_{ij} = \alpha \varkappa_{rr} \delta_{ij} + \beta \varkappa_{ji} + \gamma \varkappa_{ij} + b_0 \varepsilon_{ijk} \zeta_k + \lambda_1 \varepsilon_{jik} E_k + \nu_2 \varepsilon_{ijk} \partial_k \vartheta, \tag{2.8}$$

$$\pi_i = a_0 \zeta_i + \lambda_2 E_i + b_0 \varepsilon_{rsi} \varkappa_{rs} + \nu_1 \partial_i \vartheta, \tag{2.9}$$

$$\epsilon = \lambda_0 e_{rr} + \xi_0 \varphi - c_0 T, \tag{2.10}$$

$$D_i = -\lambda_1 \varepsilon_{ijk} \varkappa_{kj} - \lambda_2 \zeta_i - \nu_3 \partial_i \vartheta + \chi E_i, \tag{2.11}$$

$$q_i = T_0 (\nu_2 \varepsilon_{rsi} \varkappa_{rs} + \nu_1 \zeta_i + k \partial_i \vartheta + \nu_3 E_i), \tag{2.12}$$

$$\rho_0 \eta = \beta_0 e_{rr} + c_0 \varphi + a T, \tag{2.13}$$

where ϑ is the temperature change to a reference temperature T_0,

$$T = \theta - T_0, \quad \vartheta = \int_{t_0}^{t} T\,dt,$$

δ_{ij} is the Kronecker delta and λ, μ, $\varkappa$, λ_0, β_0, α, β, γ, λ_1, ν_1, a_0, λ_2, ν_2, ξ_0, c_0, a, k, ν_3, and χ, are constitutive constants, then the field equations (2.1)–(2.3), (2.5), (2.6), read as [16]

$$(\mu+\varkappa)\partial_j\partial_j u_i + (\lambda+\mu)\partial_j\partial_i u_j + \varkappa\varepsilon_{ijk}\partial_j\phi_k + \lambda_0\partial_i\varphi - \beta_0\partial_i\dot{\vartheta} + \rho_0 f_i = \rho_0\ddot{u}_i, \tag{2.14}$$

$$\gamma\partial_j\partial_j\phi_i + (\alpha+\beta)\partial_j\partial_i\phi_j + \varkappa\varepsilon_{ijk}\partial_j u_k - 2\varkappa\phi_i + \rho_0 X_i = I_0\ddot{\phi}_i, \tag{2.15}$$

$$(a_0\partial_j\partial_j - \xi_0)\varphi - \lambda_2\partial_j\partial_j\psi + \nu_1\partial_j\partial_j\vartheta - \lambda_0\partial_j u_j + c_0\dot{\vartheta} + \rho_0\mathcal{F} = j_0\ddot{\varphi}, \tag{2.16}$$

$$\lambda_2\partial_j\partial_j\varphi + \chi\partial_j\partial_j\psi + \nu_3\partial_j\partial_j\vartheta = -f, \tag{2.17}$$

$$k\partial_j\partial_j\vartheta - \beta_0\partial_j\dot{u}_j - a\ddot{\vartheta} - c_0\dot{\varphi} + \nu_1\partial_j\partial_j\varphi - \nu_3\partial_j\partial_j\psi = -\frac{1}{T_0}\,\rho_0 Q, \tag{2.18}$$

Let $v = (e_{ij}, \varkappa_{ij}, \zeta_i, \varphi, T, \vartheta_i, E_i)$ and $v' = (e'_{ij}, \varkappa'_{ij}, \zeta'_i, \varphi', T', \vartheta'_i, E'_i)$. Introduce a symmetric bilinear form

$$\begin{aligned} B(v,v') := {} & \lambda e_{ii}e'_{jj} + (\mu+\varkappa)e_{ij}e'_{ij} + \mu e_{ji}e'_{ij} + \lambda_0(e_{jj}\varphi' + e'_{jj}\varphi) + \xi_0\varphi\varphi' \\ & + k\vartheta_j\vartheta'_j + \alpha\varkappa_{ii}\varkappa'_{jj} + \beta\varkappa_{ji}\varkappa'_{ij} + \gamma\varkappa_{ij}\varkappa'_{ij} + b_0\varepsilon_{ijk}(\varkappa_{ij}\zeta'_k + \varkappa'_{ij}\zeta_k) \\ & + \nu_2\varepsilon_{ijk}(\varkappa_{ij}\vartheta'_k + \varkappa'_{ij}\vartheta_k) + a_0\zeta_i\zeta'_i + \nu_1(\vartheta_i\zeta'_i + \vartheta'_i\zeta_i) + \chi E_iE'_i + aTT'. \end{aligned} \tag{2.19}$$

The corresponding quadratic form $B(v,v)$ can be represented as follows:

$$\begin{aligned} B(v,v) = {} & F_1(e_{11}, e_{22}, e_{33}, \varphi) + F_2(e_{12}, e_{21}, e_{13}, e_{31}, e_{23}, e_{32}) + F_3(\varkappa_{11}, \varkappa_{22}, \varkappa_{33}) \\ & + F_4(\varkappa_{12}, \varkappa_{13}, \varkappa_{21}, \varkappa_{23}, \varkappa_{31}, \varkappa_{32}, \zeta_1, \zeta_2, \zeta_3, \vartheta_1, \vartheta_2, \vartheta_3) + F_5(E_1, E_2, E_3, T), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} F_1(e_{11}, e_{22}, e_{33}, \varphi) = {} & (\lambda+2\mu+\varkappa)e_{11}e_{11} + \lambda e_{11}e_{22} + \lambda e_{11}e_{33} + \lambda_0 e_{11}\varphi + \lambda e_{22}e_{11} \\ & + (\lambda+2\mu+\varkappa)e_{22}e_{22} + \lambda e_{22}e_{33} + \lambda_0 e_{22}\varphi + \lambda e_{33}e_{11} + \lambda e_{33}e_{22} \\ & + (\lambda+2\mu+\varkappa)e_{33}e_{33} + \lambda_0 e_{33}\varphi + \lambda_0\varphi e_{11} + \lambda_0\varphi e_{22} + \lambda_0\varphi e_{33} + \xi_0\varphi^2, \\ F_2(e_{12}, e_{21}, e_{13}, e_{31}, e_{23}, e_{32}) = {} & (\mu+\varkappa)e_{12}e_{12} + \mu e_{12}e_{21} + (\mu+\varkappa)e_{13}e_{13} \\ & + \mu e_{13}e_{31} + \mu e_{21}e_{12} + (\mu+\varkappa)e_{21}e_{21} + \mu e_{23}e_{32} + (\mu+\varkappa)e_{23}e_{23} \\ & + \mu e_{31}e_{13} + (\mu+\varkappa)e_{31}e_{31} + \mu e_{32}e_{23} + (\mu+\varkappa)e_{32}e_{32}, \\ F_3(\varkappa_{11}, \varkappa_{22}, \varkappa_{33}) = {} & (\alpha+\beta+\gamma)\varkappa_{11}\varkappa_{11} + \alpha\varkappa_{11}\varkappa_{22} + \alpha\varkappa_{11}\varkappa_{33} + \alpha\varkappa_{22}\varkappa_{11} \\ & + (\alpha+\beta+\gamma)\varkappa_{22}\varkappa_{22} + \alpha\varkappa_{22}\varkappa_{33} + \alpha\varkappa_{33}\varkappa_{11} + \alpha\varkappa_{33}\varkappa_{22} + (\alpha+\beta+\gamma)\varkappa_{33}\varkappa_{33}, \\ F_4(\varkappa_{12}, \varkappa_{21}, \varkappa_{13}, \varkappa_{31}, \varkappa_{23}, \varkappa_{32}, \zeta_1, \zeta_2, \zeta_3, \vartheta_1, \vartheta_2, \vartheta_3) = {} & \varkappa_{12}(\gamma\varkappa_{12} + \beta\varkappa_{21} + b_0\zeta_3 + \nu_2\vartheta_3) \\ & + \varkappa_{21}(\beta\varkappa_{12} + \gamma\varkappa_{21} - b_0\zeta_3 - \nu_2\vartheta_3) + \varkappa_{13}(\gamma\varkappa_{13} + \beta\varkappa_{31} - b_0\zeta_2 - \nu_2\vartheta_2) \\ & + \varkappa_{31}(\beta\varkappa_{13} + \gamma\varkappa_{31} + b_0\zeta_2 + \nu_2\vartheta_2) + \varkappa_{23}(\gamma\varkappa_{23} + \beta\varkappa_{32} + b_0\zeta_1 + \nu_2\vartheta_1) \\ & + \varkappa_{32}(\beta\varkappa_{23} + \gamma\varkappa_{32} - b_0\zeta_1 - \nu_2\vartheta_1) + \zeta_1(b_0\varkappa_{23} - b_0\varkappa_{32} + a_0\zeta_1 + \nu_1\vartheta_1) \\ & + \zeta_2(-b_0\varkappa_{13} + b_0\varkappa_{31} + a_0\zeta_2 + \nu_1\vartheta_2) + \zeta_3(b_0\varkappa_{12} - b_0\varkappa_{21} + a_0\zeta_3 + \nu_1\vartheta_3) \\ & + \vartheta_1(\nu_2\varkappa_{23} - \nu_2\varkappa_{32} + \nu_1\zeta_1 + k\vartheta_1) + \vartheta_2(-\nu_2\varkappa_{13} + \nu_2\varkappa_{31} + \nu_1\zeta_2 + k\vartheta_2) \\ & + \vartheta_3(\nu_2\varkappa_{12} - \nu_2\varkappa_{21} + \nu_1\zeta_3 + k\vartheta_3), \\ F_5(E_1, E_2, E_3, T) = {} & \chi E_iE_i + aT^2. \end{aligned}$$

Throughout the paper we assume that $B(v,\overline{v})$ is a positive definite form with respect to the vector $v = (e_{ij}, \varkappa_{ij}, \zeta_j, \varphi, T, \vartheta_i, E_i)$,

$$B(v,\overline{v}) > 0 \ \text{ for all } \ v \neq 0. \tag{2.21}$$

From the positive-definiteness of the forms F_1, F_2, F_3, F_4, and F_5, by Sylvester's criterion we derive the following necessary and sufficient conditions for form (2.20) to be positive definite:

$$\begin{gathered}\varkappa>0,\ \ \varkappa+2\mu>0,\ \ \varkappa+2\mu+3\lambda>0,\ \ \xi_0\,(\varkappa+2\mu+3\lambda)>3\lambda_0^2,\\ \gamma>|\beta|,\ \ a_0k-\nu_1^2>0,\ \ \beta+\gamma+3\alpha>0,\ \ \chi>0,\ \ a>0,\ \ k>0,\ \ a_0>0,\\ a_0(\gamma-\beta)>2b_0^2,\ \ (\gamma-\beta)(a_0k-\nu_1^2)+4b_0\nu_1\nu_2-2a_0\nu_2^2-2kb_0^2>0.\end{gathered}\tag{2.22}$$

Further, we assume also that

$$\rho_0>0,\ \ I_0>0,\ \ j_0>0.\tag{2.23}$$

3 Equations of pseudo-oscillations

Let the sought functions u_i, ϕ_i, φ, ψ, ϑ, as well as the sources f_i, X_i, $\mathcal{F}$, f, Q involved in the system of equations (2.14)–(2.18), be harmonic time dependent, i.e.

$$\begin{gathered}u_i(x,t)=e^{\tau t}u_i(x),\ \ \phi_i(x,t)=e^{\tau t}\phi_i(x),\ \ \varphi(x,t)=e^{\tau t}\varphi(x),\ \ \psi(x,t)=e^{\tau t}\psi(x),\ \ \vartheta(x,t)=e^{\tau t}\vartheta(x),\\ f_i(x,t)=e^{\tau t}f_i(x),\ \ X_i(x,t)=e^{\tau t}X_i(x),\ \ \mathcal{F}(x,t)=e^{\tau t}\mathcal{F}(x),\ \ f(x,t)=e^{\tau t}f(x),\ \ Q(x,t)=e^{\tau t}Q(x),\end{gathered}$$

where $\tau=\sigma+\mathrm{i}\,\omega$ is a complex parameter, $\sigma,\omega\in\mathbb{R}$. Then equations (2.14)–(2.18) lead to the system

$$(\mu+\varkappa)\partial_j\partial_ju_i+(\lambda+\mu)\partial_j\partial_iu_j-\tau^2\rho_0u_i+\varkappa\varepsilon_{ijk}\partial_j\phi_k+\lambda_0\partial_i\varphi-\tau\beta_0\partial_i\vartheta=-\rho_0f_i,\tag{3.1}$$
$$\gamma\partial_j\partial_j\phi_i+(\alpha+\beta)\partial_j\partial_i\phi_j-\tau^2I_0\phi_i+\varkappa\varepsilon_{ijk}\partial_ju_k-2\varkappa\phi_i=-\rho_0X_i,\tag{3.2}$$
$$(a_0\partial_j\partial_j-\xi_0)\varphi-\tau^2j_0\varphi-\lambda_2\partial_j\partial_j\psi+\nu_1\partial_j\partial_j\vartheta+\tau c_0\vartheta-\lambda_0\partial_ju_j=-\rho_0\mathcal{F},\tag{3.3}$$
$$\chi\partial_j\partial_j\psi+\lambda_2\partial_j\partial_j\varphi+\nu_3\partial_j\partial_j\vartheta=-f,\tag{3.4}$$
$$k\partial_j\partial_j\vartheta-\tau^2a\vartheta-\tau\beta_0\partial_ju_j-\tau c_0\varphi+\nu_1\partial_j\partial_j\varphi-\nu_3\partial_j\partial_j\psi=-\frac{1}{T_0}\,\rho_0Q.\tag{3.5}$$

If τ is a pure imaginary number, we obtain *the steady state oscillation equations*, and if $\tau=0$, then we get *the equations of statics*.

Constitutive relations (2.7)–(2.13) for pseudo-oscillation state read as

$$t_{ij}=\lambda\partial_ku_k\delta_{ij}+(\mu+\varkappa)\partial_iu_j+\varkappa\varepsilon_{jik}\phi_k+\mu\partial_ju_i+\lambda_0\varphi\delta_{ij}-\tau\beta_0\vartheta\delta_{ij},\tag{3.6}$$
$$m_{ij}=\alpha\partial_k\phi_k\delta_{ij}+\beta\partial_j\phi_i+\gamma\partial_i\phi_j+b_0\varepsilon_{ijk}\partial_k\varphi+\lambda_1\varepsilon_{ijk}\partial_k\psi+\nu_2\varepsilon_{ijk}\partial_k\vartheta,\tag{3.7}$$
$$\pi_i=a_0\partial_i\varphi-\lambda_2\partial_i\psi+b_0\varepsilon_{kli}\partial_k\phi_l+\nu_1\partial_i\vartheta,\tag{3.8}$$
$$\epsilon=\lambda_0\partial_ku_k+\xi_0\varphi-\tau c_0\vartheta,\tag{3.9}$$
$$D_i=-\lambda_1\varepsilon_{kli}\partial_k\phi_l-\lambda_2\partial_i\varphi-\nu_3\partial_i\vartheta-\chi\partial_i\psi,\tag{3.10}$$
$$q_i=T_0(\nu_2\varepsilon_{lki}\partial_l\phi_k+\nu_1\partial_i\varphi+k\partial_i\vartheta-\nu_3\partial_i\psi),\tag{3.11}$$
$$\rho_0\eta=\beta_0\partial_ku_k+c_0\varphi+\tau a\vartheta,\ \ i,j=1,2,3.\tag{3.12}$$

Denote by

$$A(\partial,\tau)=\big[A_{ij}(\partial,\tau)\big]_{9\times9}$$

the matrix differential operator generated by the left hand side expressions in (3.1)–(3.5),

$$\begin{gathered}A_{ij}(\partial,\tau)=\delta_{ij}(\mu+\varkappa)\partial_l\partial_l+(\lambda+\mu)\partial_i\partial_j-\tau^2\rho_0\delta_{ij},\quad A_{i,j+3}(\partial,\tau)=-\varkappa\varepsilon_{ijl}\partial_l,\\ A_{i7}(\partial,\tau)=\lambda_0\partial_i,\quad A_{i8}(\partial,\tau)=0,\quad A_{i9}(\partial,\tau)=-\tau\beta_0\partial_i,\quad A_{i+3,j}(\partial,\tau)=-\varkappa\varepsilon_{ijl}\partial_l,\\ A_{i+3,j+3}(\partial,\tau)=\delta_{ij}\gamma\partial_l\partial_l+(\alpha+\beta)\partial_i\partial_j-(2\varkappa+\tau^2I_0)\delta_{ij},\quad A_{i+3,j+6}(\partial,\tau)=0,\quad A_{7,j}(\partial,\tau)=-\lambda_0\partial_j,\\ A_{7,j+3}(\partial,\tau)=0,\ \ A_{77}(\partial,\tau)=a_0\partial_l\partial_l-(\xi_0+\tau^2j_0),\ \ A_{78}(\partial,\tau)=-\lambda_2\partial_l\partial_l,\ \ A_{79}(\partial,\tau)=\nu_1\partial_l\partial_l+\tau c_0,\\ A_{8j}(\partial,\tau)=0,\quad A_{8,j+3}(\partial,\tau)=0,\quad A_{87}(\partial,\tau)=\lambda_2\partial_l\partial_l,\quad A_{88}(\partial,\tau)=\chi\partial_l\partial_l,\\ A_{89}(\partial,\tau)=\nu_3\partial_l\partial_l,\quad A_{9j}(\partial,\tau)=-\tau\beta_0\partial_j,\quad A_{9,j+3}(\partial,\tau)=0,\\ A_{97}(\partial,\tau)=\nu_1\partial_l\partial_l-\tau c_0,\quad A_{98}(\partial,\tau)=-\nu_3\partial_l\partial_l,\quad A_{99}(\partial,\tau)=k\partial_l\partial_l-\tau^2a,\ \ i,j=1,2,3.\end{gathered}$$

Then we can rewrite system (3.1)–(3.5) in the matrix form

$$A(\partial,\tau)U=\Phi, \tag{3.13}$$

where

$$U=(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3,\varphi,\psi,\vartheta)^\top,$$
$$\Phi=-\Big(\rho_0 f_1,\rho_0 f_2,\rho_0 f_3,\rho_0 X_1,\rho_0 X_2,\rho_0 X_3,\rho_0\mathcal{F},f,\frac{1}{T_0}\,\rho_0 Q\Big)^\top.$$

4 Generalized stress operator and Green's formulae

Let n be a unit vector field on $\overline{\Omega}$ coinciding with the outward unit normal vector to $\partial\Omega$. Introduce the generalized stress operator $\mathcal{T}(\partial,n,\tau)=[\mathcal{T}_{jk}(\partial,n,\tau)]_{9\times 9}$ defined by the relation

$$\mathcal{T}(\partial,n,\tau)(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3,\varphi,\psi,\vartheta)^\top$$
$$=(t_{l1}n_l,t_{l2}n_l,t_{l3}n_l,m_{l1}n_l,m_{l2}n_l,m_{l3}n_l,\pi_l n_l,-D_l n_l,T_0^{-1}q_l n_l)^\top,$$

where $t_{ij},m_{ij},\pi_j,D_j,q_i$ are defined in (2.7)–(2.13). Entries of the matrix $\mathcal{T}(\partial,n,\tau)$ read as

$$\mathcal{T}_{ij}(\partial,n,\tau)=\lambda n_i\partial_j+\mu n_j\partial_i+\delta_{ij}(\mu+\varkappa)n_k\partial_k,\quad \mathcal{T}_{i,j+3}(\partial,n,\tau)=-\varkappa\varepsilon_{ijk}n_k,$$
$$\mathcal{T}_{i7}(\partial,n,\tau)=\lambda_0 n_i,\quad \mathcal{T}_{i8}(\partial,n,\tau)=0,\quad \mathcal{T}_{i,9}(\partial,n,\tau)=-\tau\beta_0 n_i,\quad \mathcal{T}_{i+3,j}(\partial,n)=0,$$
$$\mathcal{T}_{i+3,j+3}(\partial,n,\tau)=\alpha n_i\partial_j+\beta n_j\partial_i+\delta_{ij}\gamma n_k\partial_k,\quad \mathcal{T}_{i+3,7}(\partial,n,\tau)=b_0\varepsilon_{lik}n_l\partial_k,$$
$$\mathcal{T}_{i+3,8}(\partial,n,\tau)=\lambda_1\varepsilon_{lik}n_l\partial_k,\quad \mathcal{T}_{i+3,9}(\partial,n,\tau)=\nu_2\varepsilon_{lik}n_l\partial_k,\quad \mathcal{T}_{7j}(\partial,n,\tau)=0,$$
$$\mathcal{T}_{7,j+3}(\partial,n,\tau)=-b_0\varepsilon_{ljk}n_l\partial_k,\quad \mathcal{T}_{77}(\partial,n,\tau)=a_0 n_k\partial_k,\quad \mathcal{T}_{78}(\partial,n,\tau)=-\lambda_2 n_k\partial_k,$$
$$\mathcal{T}_{79}(\partial,n,\tau)=\nu_1 n_k\partial_k,\quad \mathcal{T}_{8j}(\partial,n,\tau)=0,\quad \mathcal{T}_{8,j+3}(\partial,n,\tau)=-\lambda_1\varepsilon_{ljk}n_l\partial_k,\quad \mathcal{T}_{87}(\partial,n,\tau)=\lambda_2 n_k\partial_k,$$
$$\mathcal{T}_{88}(\partial,n,\tau)=\chi n_k\partial_k,\quad \mathcal{T}_{89}(\partial,n,\tau)=\nu_3 n_k\partial_k,\quad \mathcal{T}_{9j}(\partial,n,\tau)=0,\quad \mathcal{T}_{9,j+3}(\partial,n,\tau)=-\nu_2\varepsilon_{ljk}n_l\partial_k,$$
$$\mathcal{T}_{97}(\partial,n,\tau)=\nu_1 n_k\partial_k,\quad \mathcal{T}_{98}(\partial,n,\tau)=-\nu_3 n_k\partial_k,\quad \mathcal{T}_{99}(\partial,n,\tau)=kn_l\partial_l,\ \ i,j=1,2,3.$$

For a domain with smooth boundary and smooth complex valued vector functions

$$U=(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3,\varphi,\psi,\vartheta)^\top\in[C^2(\overline{\Omega})]^9,$$
$$U'=(u_1',u_2',u_3',\phi_1',\phi_2',\phi_3',\varphi',\psi',\vartheta')^\top\in[C^2(\overline{\Omega})]^9$$

the following Green formula holds

$$\int\limits_{\Omega}A(\partial,\tau)U\cdot U'\,dx=\int\limits_{\partial\Omega}\{\mathcal{T}(\partial,n,\tau)U\}^+\cdot\{U'\}^+\,dS-\int\limits_{\Omega}E(U,\overline{U'})\,dx, \tag{4.1}$$

where the overbar denotes complex conjugation operation, the central dot designates the scalar product in the complex space $\mathbb{C}^9$,

$$\begin{aligned}E(U,U')=&(\mu+\varkappa)\partial_j u_i\partial_j u_i'+\tau^2\rho_0 u_i u_i'+\lambda\partial_j u_j\partial_i u_i'+\mu\partial_i u_j\partial_j u_i'+\varkappa\varepsilon_{ijk}\phi_k\partial_j u_i'+\lambda_0\varphi\partial_i u_i'\\&-\tau\beta_0\vartheta\partial_i u_i'+\gamma\partial_j\phi_i\partial_j\phi_i'+(2\varkappa+\tau^2 I_0)\phi_i\phi_i'+\alpha\partial_j\phi_j\partial_i\phi_i'+\beta\partial_i\phi_j\partial_j\phi_i'\\&+\varkappa\varepsilon_{ijk}\partial_j u_i\phi_k'+b_0\varepsilon_{ijk}\partial_k\varphi\partial_i\phi_j'+\lambda_1\varepsilon_{ijk}\partial_k\psi\partial_i\phi_j'+\nu_2\varepsilon_{ijk}\partial_k\vartheta\partial_i\phi_j'+a_0\partial_j\varphi\partial_j\varphi'\\&+(\xi_0+\tau^2 j_0)\varphi\varphi'-\lambda_2\partial_j\psi\partial_j\varphi'+\nu_1\partial_j\vartheta\partial_j\varphi'-\tau c_0\vartheta\varphi'+\lambda_0\partial_j u_j\varphi'+b_0\varepsilon_{ijk}\partial_i\phi_j\partial_k\varphi'\\&+\chi\partial_j\psi\partial_j\psi'\lambda_2\partial_j\varphi\partial_j\psi'+\nu_3\partial_j\vartheta\partial_j\psi'-\lambda_1\varepsilon_{ijk}\partial_j\phi_k\partial_i\psi'+k\partial_j\vartheta\partial_j\vartheta'+\tau^2 a\vartheta\vartheta'\\&+\tau\beta_0\partial_j u_j\vartheta'+\nu_1\partial_j\varphi\partial_j\vartheta'+\tau c_0\varphi\vartheta'-\nu_3\partial_j\psi\partial_j\vartheta'+\nu_2\varepsilon_{ijk}\partial_j\phi_k\partial_i\vartheta'.\end{aligned} \tag{4.2}$$

By standard limiting procedure Green's formula (4.1) can be extended to Lipschitz domains and to vector-functions $U\in[W_p^1(\Omega)]^9$ and $U'\in[W_{p'}^1(\Omega)]^9$ with $A(\partial,\tau)U\in[L_p(\Omega)]^9$ $1<p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$.

With the help of Green's formula (4.1) we can correctly determine *a generalized trace vector* $\{\mathcal{T}(\partial,n,\tau)U\}^+ \in [B_{p,p}^{-1/p}(\partial\Omega)]^9$ for a vector function $U\in[W_p^1(\Omega)]^9$ with $A(\partial,\tau)U\in[L_p(\Omega)]^9$ by the relation (cf. [20])

$$\big\langle\{\mathcal{T}(\partial,n,\tau)U\}^+,\{U'\}^+\big\rangle_{\partial\Omega} := \int\limits_{\Omega}\big[A(\partial,\tau)U\cdot U' + E(U,\overline{U'})\big]\,dx, \tag{4.3}$$

where $U'\in[W_{p'}^1(\Omega)]^9$ is an arbitrary vector function. Here the symbol $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega}$ denotes the duality between the function spaces $[B_{p,p}^{-1/p}(\partial\Omega)]^9$ and $[B_{p',p'}^{1/p}(\partial\Omega)]^9$ which extends the conventional L_2 inner product for complex valued vector functions,

$$\langle f,g\rangle_{\partial\Omega} = \int\limits_{\partial\Omega}\sum_{j=1}^{9} f_j(x)\overline{g_j(x)}\,dS \;\text{ for }\; f,g\in[L_2(\partial\Omega)]^9.$$

Introduce the boundary operator $\widetilde{\mathcal{T}}(\partial,n,\tau)=[\widetilde{\mathcal{T}}(\partial,n,\tau)_{ij}]_{9\times 9}$ associated with the formally adjoint differential operator $A^*(\partial,\tau)=A^\top(-\partial,\tau)$,

$$\begin{gathered}
2\widetilde{\mathcal{T}}_{ij}(\partial,n,\tau)=\lambda n_i\partial_j+\mu n_j\partial_i+\delta_{ij}(\mu+\varkappa)n_k\partial_k, \quad \widetilde{\mathcal{T}}_{i,j+3}(\partial,n,\tau)=-\varkappa\varepsilon_{ijk}n_k,\\
\widetilde{\mathcal{T}}_{i7}(\partial,n,\tau)=\lambda_0 n_i, \quad \widetilde{\mathcal{T}}_{i8}(\partial,n,\tau)=0, \quad \widetilde{\mathcal{T}}_{i,9}(\partial,n,\tau)=\tau\beta_0 n_i, \quad \widetilde{\mathcal{T}}_{i+3,j}(\partial,n,\tau)=0,\\
\widetilde{\mathcal{T}}_{i+3,j+3}(\partial,n,\tau)=\alpha n_i\partial_j+\beta n_j\partial_i+\delta_{ij}\gamma n_k\partial_k, \quad \widetilde{\mathcal{T}}_{i+3,7}(\partial,n,\tau)=b_0\varepsilon_{ilk}n_l\partial_k,\\
\widetilde{\mathcal{T}}_{i+3,8}(\partial,n,\tau)=\lambda_1\varepsilon_{lik}n_l\partial_k, \quad \widetilde{\mathcal{T}}_{i+3,9}(\partial,n,\tau)=\nu_2\varepsilon_{ilk}n_l\partial_k, \quad \widetilde{\mathcal{T}}_{7j}(\partial,n,\tau)=0,\\
\widetilde{\mathcal{T}}_{7,j+3}(\partial,n,\tau)=0, \quad \widetilde{\mathcal{T}}_{77}(\partial,n,\tau)=a_0 n_k\partial_k, \quad \widetilde{\mathcal{T}}_{78}(\partial,n,\tau)=\lambda_2 n_k\partial_k, \quad \widetilde{\mathcal{T}}_{79}(\partial,n,\tau)=\nu_1 n_k\partial_k,\\
\widetilde{\mathcal{T}}_{8j}(\partial,n,\tau)=0, \quad \widetilde{\mathcal{T}}_{8,j+3}(\partial,n,\tau)=0, \quad \widetilde{\mathcal{T}}_{87}(\partial,n,\tau)=-\lambda_2 n_k\partial_k, \quad \widetilde{\mathcal{T}}_{88}(\partial,n,\tau)=\chi n_k\partial_k,\\
\widetilde{\mathcal{T}}_{89}(\partial,n,\tau)=-\nu_3 n_k\partial_k, \quad \widetilde{\mathcal{T}}_{9j}(\partial,n,\tau)=0, \quad \widetilde{\mathcal{T}}_{9,j+3}(\partial,n,\tau)=0,\\
\widetilde{\mathcal{T}}_{97}(\partial,n,\tau)=\nu_1 n_k\partial_k, \quad \widetilde{\mathcal{T}}_{98}(\partial,n,\tau)=\nu_3 n_k\partial_k, \quad \widetilde{\mathcal{T}}_{99}(\partial,n,\tau)=k n_l\partial_l, \;\; i,j=1,2,3.
\end{gathered}$$

From (4.1) we deduce Green's second formula,

$$\begin{aligned}
&\int\limits_{\Omega}\big[A(\partial,\tau)U\cdot U' - U\cdot A^*(\partial,\tau)U'\big]\,dx\\
&\qquad = \int\limits_{\partial\Omega}\Big[\big\{\mathcal{T}(\partial,n,\tau)U\big\}^+\cdot\{U'\}^+ - \big\{\widetilde{\mathcal{T}}(\partial,n,\tau)U'\big\}^+\cdot\{U'\}^+\Big]\,dS. \qquad (4.4)
\end{aligned}$$

From Green's formulae (4.3) and (4.4) by standard limiting procedure we derive similar formulae for the exterior domain Ω^- provided vector functions $U,U'\in[W^1_{p,loc}(\Omega^-)]^9\cap\mathbf{Z}(\Omega^-)$ and $A(\partial,\tau)U$ is compactly supported. The class $\mathbf{Z}(\Omega^-)$ is defined as a set of functions U possesing the following asymptotic properties as $|x|\to\infty$:

$$\begin{gathered}
u_k(x)=\mathcal{O}(|x|^{-2}), \;\; \partial_j u_k(x)=\mathcal{O}(|x|^{-2}), \;\; \phi_k(x)=\mathcal{O}(|x|^{-2}), \;\; \partial_j\phi_k(x)=\mathcal{O}(|x|^{-2}),\\
\varphi(x)=\mathcal{O}(|x|^{-1}), \;\; \partial_j\varphi(x)=\mathcal{O}(|x|^{-2}), \;\; \psi(x)=\mathcal{O}(|x|^{-1}), \;\; \partial_j\psi(x)=\mathcal{O}(|x|^{-2}),\\
\vartheta(x)=\mathcal{O}(|x|^{-2}), \;\; \partial_j\vartheta(x)=\mathcal{O}(|x|^{-2}), \;\; k,j=1,2,3.
\end{gathered} \tag{4.5}$$

Note that the fundamental matrix of the operator $A(\partial_x,\tau)$ with $\tau=\sigma+i\omega$, $\sigma>\sigma_0\geq 0$, possesses the decay properties (4.5) (see Appendix B).

If $A^*(\partial_x,\tau)\,U'$ is compactly supported as well and U' satisfies the decay conditions (4.5), then the following Green formulae hold for the exterior domain Ω^-:

$$\big\langle\{\mathcal{T}(\partial,n,\tau)U\}^-,\{U'\}^-\big\rangle_{\partial\Omega} = -\int\limits_{\Omega^-}\big[A(\partial,\tau)U\cdot U' + E(U,\overline{U'})\big]\,dx, \tag{4.6}$$

$$\int\limits_{\Omega^-} \big[A(\partial,\tau)U\cdot U' - U\cdot A^*(\partial,\tau)U'\big]\,dx = -\int\limits_{\partial\Omega} \Big[\big\{\mathcal{T}(\partial,n,\tau)U\big\}^-\cdot\{U'\}^- - \{U\}^-\cdot\big\{\widetilde{\mathcal{T}}(\partial,n,\tau)U'\big\}^-\Big]\,dS.$$

We recall that the direction of the unit normal vector to $S=\partial\Omega$ is outward with respect to the domain $\Omega=\Omega^+$.

Denote by $\mathcal{E}(U,V)$ the sesquilinear form on $[H_2^1(\Omega)]^9\times[H_2^1(\Omega)]^9$

$$\mathcal{E}(U,V) := \int\limits_{\Omega} E(U,\overline{V})\,dx, \tag{4.7}$$

where $E(U,\overline{V})$ is defined by (4.2).

For $U=(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3,\varphi,\psi,\vartheta)^\top$, $v=(e_{ij},\varkappa_{ij},\zeta_j,\varphi,T,\vartheta_i,E_i)$, where $e_{ij}=\partial_i u_j+\varepsilon_{jik}\phi_k$, $\varkappa_{ij}=\partial_i\phi_j$, $\zeta_i=\partial_i\varphi$, $T=\tau\vartheta$, $\vartheta_i=\partial_i\vartheta$, $E_i=-\partial_i\psi$, we have

$$\begin{aligned}E(U,\overline{U}) = B(v,\overline{v}) &+ 2i\lambda_1\varepsilon_{ijk}\operatorname{Im}(\partial_i\phi_j\partial_k\overline{\psi}) + 2i\lambda_2\operatorname{Im}(\partial_j\varphi\partial_j\overline{\psi}) + 2i\nu_3\operatorname{Im}(\partial_j\psi\partial_j\overline{\vartheta})\\ &+ 2i\tau\beta_0\operatorname{Im}(\partial_j u_j\overline{\vartheta}) + 2i\tau c_0\operatorname{Im}(\varphi\overline{\vartheta}) + \tau^2(\rho_0 u_i\overline{u}_i + I_0\phi\overline{\phi} + j_0\varphi\overline{\varphi} + a\vartheta\overline{\vartheta}).\end{aligned} \tag{4.8}$$

Therefore from (4.7), (4.8), (2.21), and (2.22) it follows that

$$\operatorname{Re}\mathcal{E}(U,U) \geq c_1\|U\|^2_{[H_2^1(\Omega)]^9} - c_2\|U\|^2_{[H_2^0(\Omega)]^9} \ \text{ for all } \ U\in[H_2^1(\Omega)]^9 \tag{4.9}$$

with some positive constants c_1 and c_2 depending on the material parameters and on the complex parameter τ, which shows that the sesquilinear form $\mathcal{E}(U,V)$ defined in (4.7) is coercive.

5 Boundary value problems and uniqueness theorems

Here we preserve the notation introduced in the previous subsections and formulate the boundary value problems for the pseudo-oscillation equation (3.13) assuming that

$$\tau=\sigma+i\omega,\quad \sigma>\sigma_0\geqslant 0,\ \ \omega\in\mathbb{R}.$$

Further, let S_m $(m=1,2,\dots,10)$ be proper sub-manifolds of $\partial\Omega$ such that $\overline{S}_1\cup S_2=\overline{S}_3\cup S_4=\overline{S}_5\cup S_6=\overline{S}_7\cup S_8=\overline{S}_9\cup S_{10}=\partial\Omega$, $S_1\cap S_2=S_3\cap S_4=S_5\cap S_6=S_7\cap S_8=S_9\cap S_{10}=\varnothing$.

We consider the following boundary value problems.

The general mixed boundary value problem $(\mathbf{G})^+_\tau$: *Find a solution*

$$U=(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3,\varphi,\psi,\vartheta)^\top\in[W_p^1(\Omega)]^9$$

to the pseudo-oscillation equation (3.13) *with* $\Phi\in[L_p(\Omega)]^9$, $1<p<\infty$, *satisfying the boundary conditions*

$$\begin{gathered}u_i=\widetilde{u}_i \ \text{ on } \ S_1,\quad t_{ji}n_j=\widetilde{\epsilon}_i \ \text{ on } \ S_2,\quad \phi_i=\widetilde{\phi}_i \ \text{ on } \ S_3,\quad m_{ji}n_j=\widetilde{m}_i \ \text{ on } \ S_4,\\ \varphi=\widetilde{\varphi} \ \text{ on } \ S_5,\quad \pi_k n_k=\widetilde{\pi} \ \text{ on } \ S_6,\quad \psi=\widetilde{\psi} \ \text{ on } \ S_7,\quad D_j n_j=\widetilde{D}_i \ \text{ on } \ S_8,\\ \vartheta=\widetilde{\vartheta} \ \text{ on } \ S_9,\quad q_j n_j=\widetilde{q} \ \text{ on } \ S_{10},\quad i=1,2,3,\end{gathered} \tag{5.1}$$

where $\widetilde{u}_i$, $\widetilde{\phi}_i$, $\widetilde{\varphi}$, $\widetilde{\psi}$, $\widetilde{\vartheta}$, $\widetilde{\epsilon}_i$, $\widetilde{m}_i$, $\widetilde{\pi}$, $\widetilde{D}$ *and* $\widetilde{q}$ *are given functions. Here equation* (3.13) *is understood in the distributional sense, the Dirichlet type conditions are understood in the usual trace sense and the corresponding data belong to the space* $B^{1-1/p}_{p,p}$, *while the Neumann type conditions are understood in the generalized functional trace sense and the corresponding data belong to the space* $B^{-1/p}_{p,p}$.

The Dirichlet problem $(\mathbf{D})^+_\tau$: *Find a solution*

$$U=(u,\phi,\varphi,\psi,\vartheta)^\top\in[W_p^1(\Omega)]^9$$

to the pseudo-oscillation equation (3.13) *with* $\Phi \in [L_p(\Omega)]^9$, $1 < p < \infty$, *satisfying the Dirichlet type boundary condition*

$$\{U\}^+ = f \ \text{ on } \ S, \tag{5.2}$$

where $f \in [B_{p,p}^{1-1/p}(S)]^9$ *is a given vector function.*

In the case when U satisfies the homogeneous equation

$$A(\partial_x, \tau)U = 0 \ \text{ in } \ \Omega, \tag{5.3}$$

we denote the corresponding problem by $\mathbf{(D)}_{\tau,0}^+$.

The Neumann problem $\mathbf{(N)}_\tau^+$: *Find a solution*

$$U = (u, \phi, \varphi, \psi, \vartheta)^\top \in [W_p^1(\Omega)]^9$$

to the pseudo-oscillation equation (3.13) *with* $\Phi \in [L_p(\Omega)]^9$, $1 < p < \infty$, *satisfying the Neumann type boundary condition*

$$\{\mathcal{T}(\partial_x, n, \tau)U\}^+ = F \ \text{ on } \ S, \tag{5.4}$$

where $F \in [B_{p,p}^{-1/p}(S)]^9$ *is a given vector function.*

In the case when U satisfies the homogeneous equation (5.3) we denote the corresponding problem by $\mathbf{(N)}_{\tau,0}^+$.

Mixed boundary value problem for solids with interior cracks. Let us assume that a solid under consideration contains an interior crack. We identify the crack surface as a two-dimensional, two-sided manifold Σ with the crack edge $\ell_c := \partial\Sigma$. We assume that Σ is a proper part of a closed surface $S_0 \subset \Omega$ surrounding a domain $\overline{\Omega}_0 \subset \Omega$ and that Σ, S_0, and ℓ_c are C^∞-smooth. Denote $\Omega_\Sigma := \Omega \setminus \overline{\Sigma}$.

We write $v \in W_p^1(\Omega_\Sigma)$ if $v \in W_p^1(\Omega_0)$, $v \in W_p^1(\Omega \setminus \overline{\Omega_0})$, and $r_{S_0\setminus\overline{\Sigma}}\{v\}^+ = r_{S_0\setminus\overline{\Sigma}}\{v\}^-$.

Recall that throughout the paper $n = (n_1, n_2, n_3)$ stands for the exterior unit normal vector to $\partial\Omega$ and $S_0 = \partial\Omega_0$. This agreement defines the positive direction of the normal vector on the crack surface Σ.

Further, we assume that S is dissected into two smooth subsurfaces, the Dirichlet part S_D and the Neumann part S_N, $S = \bar{S_D} \cap \bar{S_N}$, and consider the following mixed BVP $\mathsf{(MC)}_\tau^+$:

(i) on the subsurface S_D there are given the displacement and the microrotation vectors, the microstretch function, the temperature and the electric potential functions (i.e., on S_D there are given the components of the vector $\{U\}^+$ - the Dirichlet data);

(ii) on the subsurface S_N there are prescribed the mechanical stress vector, the normal components of the microstretch stress vector, the heat flux, and the electric displacement vector (i.e., on S_N there are given the components of the vector $\{\mathcal{T}U\}^+$ – the Neumann data);

(iii) the crack surface Σ is mechanically traction free and we assume that the microstretch function, temperature, electric potential, and the normal components of the microstretch stress vector, heat flux, and the electric displacement vector are continuous across the crack surface.

Reducing the nonhomogeneous differential equation (3.13) to the corresponding homogeneous one, we can formulate the above mixed problem mathematically as follows: Find a vector function

$$U = (u, \phi, \varphi, \psi, \theta)^\top = (u_1, \dots, u_9)^\top \in [W_p^1(\Omega_\Sigma)]^9 \ \text{ with } \ 1 < p < \infty,$$

satisfying the homogeneous differential equation

$$A(\partial_x, \tau)\,U = 0 \ \text{ in } \ \Omega_\Sigma, \tag{5.5}$$

the crack conditions on Σ,

$$\big\{[\mathcal{T}U]_j\big\}^+ = F_j^+ \ \text{ on } \ \Sigma, \ \ j = \overline{1,6}, \tag{5.6}$$

$$\big\{[\mathcal{T}U]_j\big\}^- = F_j^- \ \text{ on } \ \Sigma, \quad j=\overline{1,6}, \tag{5.7}$$

$$\{u_7\}^+ - \{u_7\}^- = f_7 \ \text{ on } \ \Sigma, \tag{5.8}$$

$$\big\{[\mathcal{T}U]_7\big\}^+ - \big\{[\mathcal{T}U]_7\big\}^- = F_7 \ \text{ on } \ \Sigma, \tag{5.9}$$

$$\{u_8\}^+ - \{u_8\}^- = f_8 \ \text{ on } \ \Sigma, \tag{5.10}$$

$$\big\{[\mathcal{T}U]_8\big\}^+ - \big\{[\mathcal{T}U]_8\big\}^- = F_8 \ \text{ on } \ \Sigma, \tag{5.11}$$

$$\{u_9\}^+ - \{u_9\}^- = f_9 \ \text{ on } \ \Sigma, \tag{5.12}$$

$$\big\{[\mathcal{T}U]_9\big\}^+ - \big\{[\mathcal{T}U]_9\big\}^- = F_9 \ \text{ on } \ \Sigma, \tag{5.13}$$

and the mixed boundary conditions on $S = \overline{S}_D \cup \overline{S}_N$,

$$\{U\}^+ = g^{(D)} \ \text{ on } \ S_D, \tag{5.14}$$

$$\big\{\mathcal{T}U\big\}^+ = g^{(N)} \ \text{ on } \ S_N. \tag{5.15}$$

We require that the boundary data belong to the natural spaces,

$$f_7, f_8, f_9 \in \widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma), \ \ F_7, F_8, F_9 \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma), \ \ g^{(D)} \in [B_{p,p}^{1-\frac{1}{p}}(S_D)]^9, \ \ g^{(N)} \in [B_{p,p}^{-\frac{1}{p}}(S_N)]^9, \tag{5.16}$$

and the compatibility conditions

$$F_j^+ - F_j^- \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma), \quad j=\overline{1,6},$$

are satisfied.

Remark that if $U \in [W_p^1(\Omega_\Sigma)]^9$ solves the homogeneous differential equation (5.5) then actually we have the inclusion $U \in [C^\infty(\Omega_\Sigma)]^9$ due to the ellipticity of the corresponding differential operator. In fact, U is a complex valued analytic vector function of spatial real variables (x_1, x_2, x_3) in Ω_Σ.

Now we prove the uniqueness theorem (cf. [16, Theorem 3.1]).

Theorem 5.1. *Let conditions* (2.22) *and* (2.23) *be satisfied and let $U = (u, \phi, \varphi, \psi, \vartheta)$ be a solution of the problem $(G)_\tau^+$ for the homogeneous equation* (5.3) *satisfying the homogeneous boundary conditions* (5.1) *for $p = 2$. Then, $u = \phi = \varphi = \vartheta = 0$, and $\psi = const$. Moreover, if $S_7 \neq \varnothing$, then $\psi = 0$ as well.*

Proof. Due to (2.1), (2.2), we have the system of equations

$$\partial_j t_{ji} - \tau^2 \rho_0 u_i = 0, \quad i = 1,2,3, \tag{5.17}$$

$$\partial_j m_{ji} + \varepsilon_{ijk} t_{jk} - \tau^2 I_0 \phi_i = 0, \quad i = 1,2,3, \tag{5.18}$$

$$\partial_j \pi_j - \epsilon - \tau^2 j_0 \varphi = 0, \tag{5.19}$$

$$\partial_j q_j - \tau \rho_0 T_0 \eta = 0, \tag{5.20}$$

$$\partial_j D_j = 0, \tag{5.21}$$

where t_{ji}, m_{ji}, π_j, ϵ, q_j, η, D_j are defined from (3.6)–(3.11).

Multiply (5.17), (5.18), (5.19), (5.20), and (5.21) by $\overline{u}_i$, $\overline{\phi}_i$, $\overline{\varphi}$, $\overline{\vartheta}$, and $\overline{\psi}$, respectively, and integrate over Ω. In view of (2.4) and homogeneous boundary conditions we find

$$\int\limits_\Omega \Big(t_{ij}\overline{e}_{ij} + \varepsilon_{ijk} t_{ij} \overline{\phi}_k + \tau^2 \rho_0 u_i \overline{u}_i\Big)\, dx = \int\limits_{\partial\Omega} n_j t_{ji} \overline{u}_i \, dS = 0, \tag{5.22}$$

$$\int\limits_\Omega \Big(m_{ij}\overline{\varkappa}_{ij} - \varepsilon_{ijk} t_{ij} \overline{\phi}_k + \tau^2 I_0 \phi_i \overline{\phi}_i\Big)\, dx = \int\limits_{\partial\Omega} n_j m_{ji} \overline{\phi}_i \, dS = 0, \tag{5.23}$$

$$\int\limits_\Omega \Big(\pi_i \overline{\zeta}_i + \epsilon \overline{\varphi} + \tau^2 j_0 \varphi \overline{\varphi}\Big)\, dx = \int\limits_{\partial\Omega} n_i \pi_i \overline{\varphi} \, dS = 0, \tag{5.24}$$

$$\frac{1}{T_0}\int\limits_{\Omega}\Big(q_i\partial_i\overline{\vartheta}+\tau\rho_0T_0\eta\overline{\vartheta}\Big)\,dx=\int\limits_{\partial\Omega}n_iq_i\overline{\vartheta}\,dS=0, \tag{5.25}$$

$$\int\limits_{\Omega}D_i\overline{E}_i\,dx=\int\limits_{\partial\Omega}n_iD_i\overline{\psi}\,dS=0. \tag{5.26}$$

By summing equalities (5.22)–(5.25) and complex conjugate of (5.26) we obtain

$$\int\limits_{\Omega}\Big(t_{ij}\overline{e}_{ij}+m_{ij}\overline{\varkappa}_{ij}+\pi_i\overline{\zeta}_i+\epsilon\overline{\varphi}+\frac{1}{T_0}\,q_i\partial_i\overline{\vartheta}+\tau\rho_0\eta\overline{\vartheta}+\overline{D}_iE_i+\tau^2\big(\rho_0u_i\overline{u}_i+I_0\phi\overline{\phi}+j_0\varphi\overline{\varphi}\big)\Big)\,dx=0. \tag{5.27}$$

By virtue of (2.19) the integrand in (5.27) can be rewritten as

$$\begin{aligned}
&\lambda e_{ii}\overline{e}_{jj}+(\mu+\varkappa)e_{ij}\overline{e}_{ij}+\mu e_{ji}\overline{e}_{ij}+\lambda_0\varphi\overline{e}_{jj}-\beta_0T\overline{e}_{jj}+\alpha\varkappa_{ii}\overline{\varkappa}_{jj}+\beta\varkappa_{ji}\overline{\varkappa}_{ij}+\gamma\varkappa_{ij}\overline{\varkappa}_{ij}\\
&\quad+b_0\varepsilon_{ijk}\zeta_k\overline{\varkappa}_{ij}+\lambda_1\varepsilon_{jik}\overline{\varkappa}_{ij}E_k+\nu_2\varepsilon_{ijk}\overline{\varkappa}_{ij}\partial_k\vartheta+a_0\zeta_i\overline{\zeta}_i+\lambda_2E_i\overline{\zeta}_i\\
&\quad+b_0\varepsilon_{ijk}\varkappa_{ij}\overline{\zeta}_k+\nu_1\partial_i\vartheta\overline{\zeta}_i+\lambda_0e_{jj}\overline{\varphi}+\xi_0\varphi\overline{\varphi}-c_0T\overline{\varphi}-\lambda_1\varepsilon_{jik}\overline{\varkappa}_{ij}E_k\\
&\quad-\lambda_2\overline{\zeta}_iE_i-\nu_3\partial_i\overline{\vartheta}E_i+\chi\overline{E}_iE_i+\nu_2\varepsilon_{ijk}\varkappa_{ij}\partial_k\overline{\vartheta}+\nu_1\zeta_i\partial_i\overline{\vartheta}+k\partial_i\vartheta\partial_i\overline{\vartheta}\\
&\quad+\nu_3E_i\partial_i\overline{\vartheta}+\tau\beta_0e_{jj}\overline{\vartheta}+\tau c_0\varphi\overline{\vartheta}+\tau aT\overline{\vartheta}+\tau^2(\rho_0u_i\overline{u}_i+I_0\phi\overline{\phi}+j_0\varphi\overline{\varphi})\\
&=B(v,\overline{v})+\tau\beta_0(e_{jj}\overline{\vartheta}-\overline{e}_{jj}\vartheta)+\tau c_0(\varphi\overline{\vartheta}-\overline{\varphi}\vartheta)+\tau^2(\rho_0u_i\overline{u}_i+I_0\phi\overline{\phi}+j_0\varphi\overline{\varphi}+a\vartheta\overline{\vartheta}),
\end{aligned}$$

where $B(v,v')$ is the bilinear form with respect to the variables $v=(e_{ij},\varkappa_{ij},\zeta_i,\varphi,T,\partial_i\vartheta,E_i)$ and $v'=(e'_{ij},\varkappa'_{ij},\zeta'_i,\varphi',T',\partial_i\vartheta',E'_i)$ defined in (2.19),

$$\begin{aligned}
B(v,v')=\lambda e_{ii}e'_{jj}&+(\mu+\varkappa)e_{ij}e'_{ij}+\mu e_{ji}e'_{ij}+\lambda_0(e_{jj}\varphi'+e'_{jj}\varphi)+\xi_0\varphi\varphi'+\alpha\varkappa_{ii}\varkappa'_{jj}\\
&+\beta\varkappa_{ji}\varkappa'_{ij}+\gamma\varkappa_{ij}\varkappa'_{ij}+b_0\varepsilon_{ijk}(\varkappa_{ij}\zeta'_k+\varkappa'_{ij}\zeta_k)+\nu_2\varepsilon_{ijk}(\varkappa_{ij}\partial_k\vartheta'+\varkappa'_{ij}\partial_k\vartheta)\\
&+a_0\zeta_i\zeta'_i+\nu_1(\partial_i\vartheta\zeta'_i+\partial_i\vartheta'\zeta_i)+\chi E_i\overline{E}_i+k\partial\vartheta\,\partial\vartheta'.
\end{aligned}$$

Due to (2.22) we have $B(v,\overline{v})>0$ for any complex valued vector $v\neq0$.

Let $\tau=\sigma+i\,\omega$, $\sigma>0$. Separating the real and imaginary parts of (5.27) we get

$$\int\limits_{\Omega}\Big(B(v,\overline{v})-2\omega\beta_0\,\mathrm{Im}(e_{jj}\overline{\vartheta})-2\omega c_0\,\mathrm{Im}(\varphi\overline{\vartheta})$$

$$+(\sigma^2-\omega^2)\big(\rho_0|u|^2+I_0|\phi|^2+j_0|\varphi|^2+a|\vartheta|^2\big)\Big)\,dx=0, \tag{5.28}$$

$$\int\limits_{\Omega}\Big(2\sigma\beta_0\,\mathrm{Im}(e_{jj}\overline{\vartheta})+2\sigma c_0\,\mathrm{Im}(\varphi\overline{\vartheta})+2\sigma\omega\big(\rho_0|u|^2+I_0|\phi|^2+j_0|\varphi|^2+a|\vartheta|^2\big)\Big)\,dx=0. \tag{5.29}$$

Multiply (5.29) by ω/σ and add to (5.28) to obtain

$$\int\limits_{\Omega}\Big(B(v,\overline{v})+(\sigma^2+\omega^2)\big(\rho_0|u|^2+I_0|\phi|^2+j_0|\varphi|^2+a|\vartheta|^2\big)\Big)\,dx=0,$$

implying $|u|=|\phi|=|\varphi|=|\vartheta|=0$ and $\int\limits_{\Omega}\chi|E|^2\,dx=0$. Whence $E=-\operatorname{grad}\psi=0$ and thus $\psi=const$. Evidently, if $S_7\neq\varnothing$, then $\psi=0$ follows, which completes the proof. □

From Theorem 5.1 the following uniqueness theorem follows directly.

Theorem 5.2. *Let S be Lipschitz surface and $\tau=\sigma+i\omega$ with $\sigma>\sigma_0\geqslant0$ and $\omega\in\mathbb{R}$.*

(i) *The basic Dirichlet BVP $(D)^+_\tau$ has at most one solution in the space $[W^1_2(\Omega)]^9$.*

(ii) *Solutions to the Neumann type BVP $(N)^+_\tau$ in the space $[W^1_2(\Omega)]^9$ are defined modulo a vector of type $U^{(N)}=(0,0,0,0,0,0,0,b,0)^\top$, where b is an arbitrary constant.*

(iii) *Mixed type boundary value problem $(MC)^+_\tau$ has at most one solution in the space $[W^1_2(\Omega_\Sigma)]^9$.*

Similar uniqueness result for $p\neq2$ will be proved later.

6 Properties of potentials and boundary operators

The full symbol of the pseudo-oscillation differential operator $A(\partial_x,\tau)$ with $\operatorname{Re}\tau\neq 0$ is non-singular, i.e.,

$$\det A(-i\,\xi,\tau)\neq 0\ \ \forall\,\xi\in\mathbb{R}^3\setminus\{0\}.$$

Moreover, the entries of the inverse matrix $A^{-1}(-i\,\xi,\tau)$ are locally integrable functions decaying at infinity as $\mathcal{O}(|\xi|^{-2})$. Therefore, we can construct the fundamental matrix $\Gamma(x,\tau)=[\Gamma_{kj}(x,\tau)]_{9\times 9}$ of the operator $A(\partial_x,\tau)$ with the help of the Fourier transform technique,

$$\Gamma(x,\tau)=\mathcal{F}^{-1}_{\xi\to x}[A^{-1}(-i\xi,\tau)].$$

The structure of the matrix $A^{-1}(-i\xi,\tau)$ allows to represent the fundamental matrix $\Gamma(x,\tau)$ in terms of elementary functions (see Appendix B). These explicit formulas imply that in a neighbourhood of the origin the fundamental matrix possesses the property $\Gamma(x,\tau)=\mathcal{O}(|x|^{-1})$, while the columns of $\Gamma(x,\tau)$ satisfy the decay conditions (4.5) at infinity.

Here we collect some necessary results for our analysis. Proofs of the theorems below are similar to the proofs of their counterparts in [2,3,8,17,18].

Let us introduce the single and double layer potentials:

$$V(h)(x)=V_S(h)=\int\limits_S \Gamma(x-y,\tau)\,h(y)\,d_yS,$$

$$W(h)(x)=W_S(h)=\int\limits_S \Big[\widetilde{\mathcal{T}}(\partial_y,n(y),\tau)[\Gamma(x-y,\tau)]^\top\Big]^\top h(y)\,d_yS,$$

where $h=(h_1,h_2,\dots,h_9)^\top$ is a density vector function.

Theorem 6.1. *Let $1<p<\infty$, $1\le q\le\infty$, $s\in\mathbb{R}$. Then the single and double layer potentials can be extended to the continuous operators*

$$\begin{aligned}
V &: \big[B^s_{p,q}(S)\big]^9\to\big[B^{s+1+\frac1p}_{p,q}(\Omega)\big]^9, & W &: [B^s_{p,q}(S)]^9\to[B^{s+\frac1p}_{p,q}(\Omega)]^9,\\
&: [B^s_{p,q}(S)]^9\to[B^{s+1+\frac1p}_{p,q,loc}(\Omega^-)]^9, & &: [B^s_{p,q}(S)]^9\to[B^{s+\frac1p}_{p,q,loc}(\Omega^-)]^9,\\
&: [B^s_{p,p}(S)]^9\to[H^{s+1+\frac1p}_p(\Omega)]^9, & &: [B^s_{p,p}(S)]^9\to[H^{s+\frac1p}_p(\Omega)]^9,\\
&: [B^s_{p,p}(S)]^9\to[H^{s+1+\frac1p}_{p,loc}(\Omega^-)]^9, & &: [B^s_{p,p}(S)]^9\to[H^{s+\frac1p}_{p,loc}(\Omega^-)]^9.
\end{aligned}$$

Theorem 6.2. *Let $h^{(1)}\in[B^{-\frac1p}_{p,q}(S)]^9$, $h^{(2)}\in[B^{1-\frac1p}_{p,q}(S)]^9$, $1<p<\infty$, $1\le q\le\infty$. Then*

$$\big\{V(h^{(1)})(z)\big\}^\pm=\int\limits_S \Gamma(z-y,\tau)h^{(1)}(y)\,d_yS\ \ \text{on}\ \ S,$$

$$\big\{W(h^{(2)})(z)\big\}^\pm=\pm\frac12\,h^{(2)}(z)+\int\limits_S \Big[\widetilde{\mathcal{T}}(\partial_y,n(y),\tau)[\Gamma(z-y,\tau)]^\top\Big]^\top h^{(2)}(y)\,d_yS\ \ \text{on}\ \ S.$$

The equalities are understood in the sense of the space $[B^{1-1/p}_{p,q}(S)]^9$ (cf. [21])

Theorem 6.3. *Let $h^{(1)}\in[B^{-\frac1p}_{p,q}(S)]^9$, $h^{(2)}\in[B^{1-\frac1p}_{p,q}(S)]^9$, $1<p<\infty$, $1\le q\le\infty$. Then*

$$\big\{\mathcal{T}V(h^{(1)})(z)\big\}^\pm=\mp\frac12\,h^{(1)}(z)+\int\limits_S \mathcal{T}(\partial_z,n(z),\tau)\Gamma(z-y,\tau)\,h^{(1)}(y)\,d_yS\ \ \text{on}\ \ S,$$

$$\big\{\mathcal{T}W(h^{(2)})(z)\big\}^+=\big\{\mathcal{T}W(h^{(2)})(z)\big\}^-\ \ \text{on}\ \ S,$$

where the equalities are understood in the sense of the space $[B^{-\frac1p}_{p,q}(S)]^9$.

We introduce the following notation for the boundary operators generated by the single and double layer potentials:

$$\mathcal{H}(h)(z)=\int\limits_S \Gamma(z-y,\tau)h(y)\,d_yS,\quad z\in S, \tag{6.1}$$

$$\mathcal{K}(h)(z)=\int\limits_S \mathcal{T}(\partial_z,n(z),\tau)\Gamma(z-y,\tau)h(y)\,d_yS,\quad z\in S, \tag{6.2}$$

$$\mathcal{N}(h)(z)=\int\limits_S \Big[\widetilde{\mathcal{T}}(\partial_y,n(y),\tau)[\Gamma(z-y,\tau)]^\top\Big]^\top h(y)\,d_yS,\quad z\in S, \tag{6.3}$$

$$\mathcal{L}(h)(z)=\big\{\mathcal{T}W(h)(z)\big\}^+=\big\{\mathcal{T}W(h)(z)\big\}^-,\quad z\in S. \tag{6.4}$$

Note that $\mathcal{H}$ is a weakly singular integral operator (pseudodifferential operator of order -1), $\mathcal{K}$ and $\mathcal{N}$ are singular integral operators (pseudodifferential operator of order 0), and $\mathcal{L}$ is a singular integro-differential operator (pseudodifferential operator of order 1). These operators possess the following mapping and Fredholm properties.

Theorem 6.4. *Let $1<p<\infty$, $1\le q\le\infty$, $s\in\mathbb{R}$. Then the operators*

$$\begin{aligned}
\mathcal{H}&:[B^s_{p,q}(S)]^9\to[B^{s+1}_{p,q}(S)]^9, &\quad \mathcal{H}&:[H^s_p(S)]^9\to[H^{s+1}_p(S)]^9,\\
\mathcal{K},\mathcal{N}&:[B^s_{p,q}(S)]^9\to[B^{s}_{p,q}(S)]^9, &\quad \mathcal{K},\mathcal{N}&:[H^s_p(S)]^9\to[H^{s}_p(S)]^9,\\
\mathcal{L}&:[B^s_{p,q}(S)]^9\to[B^{s-1}_{p,q}(S)]^9, &\quad \mathcal{L}&:[H^s_p(S)]^9\to[H^{s-1}_p(S)]^9,
\end{aligned}$$

are continuous.

The operators $\mathcal{H}$ and $\mathcal{L}$ are strongly elliptic pseudodifferential operators, while the operators $\pm\frac12\,I_9+\mathcal{K}$ and $\pm\frac12\,I_9+\mathcal{N}$ are elliptic, where I_9 stands for the 9×9 unit matrix.

Moreover, the operators $\mathcal{H}$, $\frac12\,I_9+\mathcal{N}$, and $\frac12\,I_9+\mathcal{K}$ are invertible, whereas the operators $-\frac12\,I_9+\mathcal{K}$, $-\frac12\,I_9+\mathcal{N}$, and $\mathcal{L}$ are Fredholm operators with zero index.

The following operator equalities hold in appropriate function spaces

$$\mathcal{L}\mathcal{H}=-\frac14\,I_9+\mathcal{K}^2,\quad \mathcal{H}\mathcal{L}=-\frac14\,I_9+\mathcal{N}^2. \tag{6.5}$$

7 Existence and regularity of solutions to mixed BVP $(\mathbf{MC})_\tau$

Before we start analysis of the mixed problem we present here existence results for the basic Dirichlet and Neumann boundary value problems. Using Theorem 6.4 and the fact that the null spaces of strongly elliptic pseudodifferential operators acting in Bessel potential $H^s_p(S)$ and Besov $B^s_{p,q}(S)$ spaces actually do not depend on the parameters s, p, and q, by quite the same arguments as in [3], we arrive at the following existence results.

Theorem 7.1. *Let $1<p<\infty$ and $f\in[B^{1-\frac1p}_{p,p}(S)]^9$. Then the pseudodifferential operator*

$$2^{-1}I_9+\mathcal{N}:[B^{1-\frac1p}_{p,p}(S)]^9\to[B^{1-\frac1p}_{p,p}(S)]^9$$

is continuously invertible, the interior Dirichlet BVP (5.3), (5.2) is uniquely solvable in the space $[W^1_p(\Omega)]^9$ and the solution is representable in the form of double layer potential $U=W(h)$ with the density vector function $h\in[B^{1-\frac1p}_{p,p}(S)]^9$ being a unique solution of the singular integral equation

$$[2^{-1}I_9+\mathcal{N}]h=f\ \text{ on } S.$$

Theorem 7.2. *Let $1<p<\infty$ and a vector function $U\in[W^1_p(\Omega)]^9$ solves the homogeneous differential equation $A(\partial,\tau)U=0$ in Ω. Then it is uniquely representable in the form*

$$U(x)=V(\mathcal{H}^{-1}\{U\}^+)(x),\quad x\in\Omega,$$

where $\{U\}^+$ *is the trace of* U *on* S *from* Ω *and belongs to the space* $[B^{1-\frac{1}{p}}_{p,p}(S)]^9$. *Here* $\mathcal{H}^{-1}$ *is the inverse to the operator* $\mathcal{H} : B^{-\frac{1}{p}} \to B^{1-\frac{1}{p}}$.

Theorem 7.3. *Let* $1 < p < \infty$ *and* $F = (F_1, \dots, F_9)^\top \in [B^{-\frac{1}{p}}_{p,p}(S)]^9$.

(i) *The operator*

$$-2^{-1}I_9 + \mathcal{K} : [B^{-\frac{1}{p}}_{p,p}(S)]^9 \to [B^{-\frac{1}{p}}_{p,p}(S)]^9 \tag{7.1}$$

is an elliptic pseudodifferential operator with zero index and has a one-dimensional null space spanned by the vector function $h_0 = \mathcal{H}^{-1}\Psi$, *where*

$$\Psi := (0,0,0,0,0,0,0,1,0)^\top \ \text{ on } \ S. \tag{7.2}$$

(ii) *The null space of the operator adjoint to* (7.1),

$$-2^{-1}I_9 + \mathcal{K}^* : [B^{\frac{1}{p}}_{p',p'}(S)]^9 \to [B^{\frac{1}{p}}_{p',p'}(S)]^9, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

is the linear span of the vector $(0,0,0,0,0,0,0,1,0)^\top$.

(iii) *The equation*

$$[-2^{-1}I_9 + \mathcal{K}]h = F \ \text{ on } \ S, \tag{7.3}$$

is solvable if and only if

$$\int\limits_S F_8(x)\, dS = 0. \tag{7.4}$$

(iv) *If condition* (7.4) *holds, then solutions to equation* (7.3) *are defined modulo constant times* $h_0 = \mathcal{H}^{-1}\Psi$ *with* Ψ *defined in* (7.2).

(v) *If condition* (7.4) *holds, then the interior Neumann type boundary value problem* (5.3), (5.4) *is solvable in the space* $[W^1_p(\Omega)]^9$ *and its solution is representable in the form of single layer potential* $U = V(h)$, *where the density vector function* $h \in [B^{-\frac{1}{p}}_{p,p}(S)]^9$ *is defined by equation* (7.3). *A solutions to the interior Neumann BVP in* Ω *is defined modulo summand* $C\,\Psi$ *with arbitrary constant* C *and* Ψ *given by* (7.2).

Now we start investigation of the mixed boundary value problem $(MC)_\tau$.

First let us note that the boundary conditions on the crack faces Σ, (5.6) and (5.7), can be transformed equivalently as

$$\big\{[\mathcal{T}U]_j\big\}^+ - \big\{[\mathcal{T}U]_j\big\}^- = F_j^+ - F_j^- \in \widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma), \ \ j = \overline{1,6},$$
$$\big\{[\mathcal{T}U]_j\big\}^+ + \big\{[\mathcal{T}U]_j\big\}^- = F_j^+ + F_j^- \in B^{-\frac{1}{p}}_{p,p}(\Sigma), \ \ j = \overline{1,6}.$$

Therefore the boundary conditions (5.6)–(5.15) of the problem under consideration can be rewritten as

$$\big\{\mathcal{T}U\big\}^+ = g^{(N)} \ \text{ on } \ S_N, \tag{7.5}$$
$$\{U\}^+ = g^{(D)} \ \text{ on } \ S_D, \tag{7.6}$$
$$\big\{[\mathcal{T}U]_j\big\}^+ + \big\{[\mathcal{T}U]_j\big\}^- = F_j^+ + F_j^- \ \text{ on } \ \Sigma, \ \ j = \overline{1,6}, \tag{7.7}$$
$$\{u_7\}^+ - \{u_7\}^- = f_7 \ \text{ on } \ \Sigma, \tag{7.8}$$
$$\{u_8\}^+ - \{u_8\}^- = f_8 \ \text{ on } \ \Sigma, \tag{7.9}$$
$$\{u_9\}^+ - \{u_9\}^- = f_9 \ \text{ on } \ \Sigma, \tag{7.10}$$
$$\big\{[\mathcal{T}U]_j\big\}^+ - \big\{[\mathcal{T}U]_j\big\}^- = F_j^+ - F_j^- \ \text{ on } \ \Sigma, \ \ j = \overline{1,6}, \tag{7.11}$$
$$\big\{[\mathcal{T}U]_7\big\}^+ - \big\{[\mathcal{T}U]_7\big\}^- = F_7 \ \text{ on } \ \Sigma, \tag{7.12}$$

$$\big\{[\mathcal{T}U]_8\big\}^+ - \big\{[\mathcal{T}U]_8\big\}^- = F_8 \ \text{ on } \ \Sigma, \tag{7.13}$$

$$\big\{[\mathcal{T}U]_9\big\}^+ - \big\{[\mathcal{T}U]_9\big\}^- = F_9 \ \text{ on } \ \Sigma. \tag{7.14}$$

We look for a solution of the boundary value problem (5.5), (7.5)–(7.14) in the form

$$U = V(\mathcal{H}^{-1}\, h) + W_c(h^{(2)}) + V_c(h^{(1)}) \ \text{ in } \ \Omega_\Sigma, \tag{7.15}$$

where

$$V_c(h^{(1)})(x) := \int\limits_\Sigma \Gamma(x-y,\tau) h^{(1)}(y)\, d_yS,$$

$$W_c(h^{(2)})(x) := \int\limits_\Sigma \big[\widetilde{\mathcal{T}}(\partial_y, n(y), \tau)[\Gamma(x-y,\tau)]^\top\big]^\top h^{(2)}(y)\, d_yS,$$

$$V(\mathcal{H}^{-1}h)(x) := \int\limits_S \Gamma(x-y,\tau)(\mathcal{H}^{-1}h)(y)\, d_yS,$$

$h^{(i)} = (h_1^{(i)}, \dots, h_9^{(i)})^\top$, $i=1,2$, and $h=(h_1,\dots,h_9)^\top$ are unknown densities,

$$h^{(1)} \in \big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma)\big]^9, \quad h^{(2)} \in \big[\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)\big]^9, \quad h \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^9. \tag{7.16}$$

Due to the above inclusions, clearly, in the potentials V_c and W_c we can take the closed surface S_0 as an integration manifold instead of the crack surface Σ. Recall that Σ is assumed to be a proper part of $S_0 = \partial\Omega_0 \subset \Omega$ (see Section 5).

The boundary and transmission conditions (7.5)–(7.14) lead to the equations:

$$r_{S_N}\mathcal{A}h + r_{S_N}\big[\mathcal{T}W_c(h^{(2)})\big] + r_{S_N}\big[\mathcal{T}V_c(h^{(1)})\big] = g^{(N)} \ \text{ on } \ S_N, \tag{7.17}$$

$$r_{S_D}h + r_{S_D}\big[W_c(h^{(2)})\big] + r_{S_D}V_c(h^{(1)}) = g^{(D)} \ \text{ on } \ S_D, \tag{7.18}$$

$$r_\Sigma\big[\mathcal{T}V(\mathcal{H}^{-1}h)\big]_j + r_\Sigma\big[\mathcal{L}_c h^{(2)}\big]_j + r_\Sigma\big[\mathcal{K}_c(h^{(1)})\big]_j = 2^{-1}(F_j^+ + F_j^-) \ \text{ on } \ \Sigma, \ \ j=\overline{1,6}, \tag{7.19}$$

where

$$h_7^{(2)} = f_7, \ \ h_8^{(2)} = f_8, \ \ h_9^{(2)} = f_9, \ \ h_j^{(1)} = F_j^- - F_j^+, \ \ j=\overline{1,6},$$
$$h_7^{(1)} = -F_7, \ \ h_8^{(1)} = -F_8, \ \ h_9^{(1)} = -F_9 \ \text{ on } \ \Sigma,$$

and $\mathcal{A} := (-2^{-1}I_9 + \mathcal{K})\mathcal{H}^{-1}$ is the Steklov–Poincaré type operator on S, and

$$\mathcal{L}_c(h^{(2)})(z) := \big\{\mathcal{T}W_c(h^{(2)})(z)\big\}^+ = \big\{\mathcal{T}W_c(h^{(2)})(z)\big\}^- \ \text{ on } \ \Sigma,$$

$$\mathcal{K}_c(h^{(1)})(z) := \int\limits_\Sigma \mathcal{T}(\partial_z, n(z), \tau)\Gamma(z-y,\tau)h^{(1)}(y)\, d_yS \ \text{ on } \ \Sigma.$$

As we see the sought for density $h^{(1)}$ and the last three components of the vector $h^{(2)}$ are determined explicitly by the data of the problem. Hence, it remains to find the density h and the first six components $\widetilde{h}^{(2)} = (h_1^{(2)}, \dots, h_6^{(2)})^\top$ of the vector $h^{(2)}$.

The operator generated by the left hand side expressions of the above simultaneous equations (7.17)–(7.19), acting upon the unknown vector $(h, \widetilde{h}^{(2)})$, reads as

$$\mathcal{Q} := \begin{bmatrix} r_{S_N}\mathcal{A} & r_{S_N}[\mathcal{T}W_c]_{9\times 6} \\ r_{S_D}I_9 & [r_{S_D}W_c]_{9\times 6} \\ r_\Sigma[\mathcal{T}V(\mathcal{H}^{-1})]_{6\times 9} & r_\Sigma[\mathcal{L}_c]_{6\times 6} \end{bmatrix}_{24\times 15},$$

where $[M]_{m\times n}$ denotes the upper left $m\times n$ dimensional block of a matrix M of dimension $m_0\times n_0$ with $m_0\geqslant m$ and $n_0\geqslant n$. This operator possesses the following mapping properties:

$$\begin{aligned}\mathcal{Q}:[H^s_p(S)]^9\times[\widetilde{H}^s_p(\Sigma)]^6\to[H^{s-1}_p(S_N)]^9\times[H^s_p(S_D)]^9\times[H^{s-1}_p(\Sigma)]^6,\\ \mathcal{Q}:[B^s_{p,q}(S)]^9\times[\widetilde{B}^s_{p,q}(\Sigma)]^6\to[B^{s-1}_{p,q}(S_N)]^9\times[B^s_{p,q}(S_D)]^9\times[B^{s-1}_{p,q}(\Sigma)]^6,\\ 1<p<\infty,\ \ 1\le q\le\infty,\ \ s\in\mathbb{R}.\end{aligned}\tag{7.20}$$

Our main goal is to establish invertibility of the operators (7.20). To this end, by introducing a new additional unknown vector we extend equation (7.18) from S_D onto the whole of S. We will do this in the following way. Denote by $g_0^{(D)}$ some fixed extension of $g^{(D)}$ from S_D onto the whole of S preserving the space. In particular, for the zero vector $g^{(D)}=0$ on S_D we always choose the fixed extension vector $g_0^{(D)}=0$ on S.

Introduce a new unknown vector w on S

$$w=h+r_{_S}[W_c(h^{(2)})]+r_{_S}V_c(h^{(1)})-g_0^{(D)}.\tag{7.21}$$

It is evident that $w\in[\widetilde{B}^{1-\frac1p}_{p,p}(S_N)]^9$ in accordance with (7.18), (7.16), (5.16), and the imbedding $g_0^{(D)}\in[B^{1-\frac1p}_{p,p}(S)]^9$. Moreover, the restriction of equation (7.21) on S_D coincides with equation (7.18). Therefore, we can replace equation (7.18) in system (7.17)–(7.19) by equation (7.21). Finally, we arrive at the following simultaneous equations with respect to unknowns h, w, and $\widetilde{h}^{(2)}$:

$$r_{_{S_N}}\mathcal{A}h+r_{_{S_N}}[\mathcal{T}W_c]_{9\times6}(\widetilde{h}^{(2)})=g^{(1)}\ \text{ on }\ S_N,\tag{7.22}$$

$$h-w+r_{_S}[W_c]_{9\times6}(\widetilde{h}^{(2)})=g^{(2)}\ \text{ on }\ S,\tag{7.23}$$

$$r_\Sigma\big[\mathcal{T}V(\mathcal{H}^{-1})\big]_{6\times9}(h)+r_\Sigma[\mathcal{L}_c]_{6\times6}(\widetilde{h}^{(2)})=g^{(3)}\ \text{ on }\ \Sigma,\tag{7.24}$$

where

$$\begin{aligned}g^{(1)}&=g^{(N)}-r_{_{S_N}}\big[\mathcal{T}V_c(h^{(1)})\big]-r_{_{S_N}}\Big[\mathcal{T}W_c\big(\big([0]_{1\times6},h_7^{(2)},h_8^{(2)},h_9^{(2)}\big)^\top\big)\Big],\\ g^{(2)}&=g_0^{(D)}-r_{_S}\big[V_c(h^{(1)})\big]-r_{_S}\Big[W_c\big(\big([0]_{1\times6},h_7^{(2)},h_8^{(2)},h_9^{(2)}\big)^\top\big)\Big],\\ g^{(3)}&=2^{-1}(F^++F^-)-r_{_\Sigma}[\mathcal{K}_c]_{6\times9}(h^{(1)})-r_{_\Sigma}\Big[\mathcal{L}_c\big(\big([0]_{1\times6},h_7^{(2)},h_8^{(2)},h_9^{(2)}\big)^\top\big)\Big],\end{aligned}$$

with $F^\pm=(F_1^\pm,\dots,F_6^\pm)^\top$.

Rewrite system (7.22)–(7.24) in the equivalent form

$$r_{_{S_N}}\mathcal{A}w+r_{_{S_N}}[\mathcal{T}W_c]_{9\times6}(\widetilde{h}^{(2)})-r_{_{S_N}}\mathcal{A}[r_{_S}W_c]_{9\times6}(\widetilde{h}^{(2)})=g^{(1)}-r_{_{S_N}}\mathcal{A}g^{(2)}\ \text{ on }\ S_N,\tag{7.25}$$

$$-w+h+r_{_S}[W_c]_{9\times6}(\widetilde{h}^{(2)})=g^{(2)}\ \text{ on }\ S,\tag{7.26}$$

$$r_\Sigma\big[\mathcal{T}V(\mathcal{H}^{-1})\big]_{6\times9}(h)+r_\Sigma[\mathcal{L}_c]_{6\times6}(\widetilde{h}^{(2)})=g^{(3)}\ \text{ on }\ \Sigma.\tag{7.27}$$

Remark 7.4. Systems (7.17)–(7.19) and (7.25)–(7.27) are equivalent in the following sense:

(i) if $(h,\widetilde{h}^{(2)})^\top$ solves system (7.17)–(7.19), then $(w,h,\widetilde{h}^{(2)})^\top$ with w given by (7.21) where $g_0^{(D)}$ is some fixed extension of the vector $g^{(D)}$ from S_D onto the whole of S involved in the right hand side of equation (7.26), solves system (7.25)–(7.27);

(ii) if $(w,h,\widetilde{h}^{(2)})^\top$ solves system (7.25)–(7.27), then $(h,\widetilde{h}^{(2)})^\top$ solves system (7.17)–(7.19).

The operator generated by the left hand sides of system (7.25)–(7.27) reads as

$$\mathcal{M}:=\begin{bmatrix}r_{_{S_N}}\mathcal{A} & [0]_{9\times9} & r_{_{S_N}}\mathcal{R}\\ -r_{_S}I_9 & r_{_S}I_9 & [r_{_S}W_c]_{9\times6}\\ [0]_{6\times9} & r_{_\Sigma}\big[\mathcal{T}V(\mathcal{H}^{-1})\big]_{6\times9} & r_{_\Sigma}[\mathcal{L}_c]_{6\times6}\end{bmatrix}_{24\times24},$$

where

$$\mathcal{R} = [\mathcal{T}W_c]_{9\times 6} - \mathcal{A}[r_{_S} W_c]_{9\times 6}.$$

This operator has the following mapping properties:

$$\begin{gathered}
\mathcal{M} : [\widetilde{H}^s_p(S_N)]^9 \times [H^s_p(S)]^9 \times [\widetilde{H}^s_p(\Sigma)]^6 \to [H^{s-1}_p(S_N)]^9 \times [H^s_p(S)]^9 \times [H^{s-1}_p(\Sigma)]^6, \\
\mathcal{M} : [\widetilde{B}^s_{p,q}(S_N)]^9 \times [B^s_{p,q}(S)]^9 \times [\widetilde{B}^s_{p,q}(\Sigma)]^6 \to [B^{s-1}_{p,q}(S_N)]^9 \times [B^s_{p,q}(S)]^9 \times [B^{s-1}_{p,q}(\Sigma)]^6, \\
1 < p < \infty, \ \ 1 \le q \le \infty, \ \ s \in \mathbb{R}.
\end{gathered} \tag{7.28}$$

Due to the above agreement about the extension of the zero vector we see that if the right hand side functions of the system (7.17)–(7.19) vanish then the same holds for the system (7.25)–(7.27) and vice versa.

The uniqueness Theorem 5.2 and properties of the single and double layer potentials imply the following assertion.

Lemma 7.5. *The null spaces of the operators $\mathcal{Q}$ and $\mathcal{M}$ are trivial for $s = 1/2$ and $p = 2$.*

Now we start to analyse Fredholm properties of the operator $\mathcal{M}$.

For the principal part $\mathcal{M}_0$ of the operator $\mathcal{M}$ we have

$$\mathcal{M}_0 := \begin{bmatrix} r_{_{S_N}}\mathcal{A} & [0]_{9\times 9} & [0]_{9\times 6} \\ -r_{_S} I_9 & r_{_S} I_9 & [0]_{9\times 6} \\ [0]_{6\times 9} & [0]_{6\times 9} & r_{_\Sigma}\mathcal{L}^{(1)} \end{bmatrix}_{24\times 24}, \tag{7.29}$$

where $\mathcal{L}^{(1)} := [\mathcal{L}_c]_{6\times 6}$.

Clearly, the operator $\mathcal{M}_0$ has the same mapping properties as $\mathcal{M}$ and the difference $\mathcal{M} - \mathcal{M}_0$ is compact.

By the same arguments as in [3], we can establish that the operators $\mathcal{L}_c$ and $\mathcal{A}$ are strongly elliptic pseudodifferential operators of order 1, therefore $\mathcal{L}^{(1)}$ is a strongly elliptic pseudodifferential operator as well. Moreover, we have the following invertibility results.

Theorem 7.6. *Let $1 < p < \infty$, $1 \le q \le \infty$, $1/p - 1/2 < s < 1/p + 1/2$. Then the operators*

$$r_{_\Sigma}\mathcal{L}^{(1)} : \big[\widetilde{H}^s_p(\Sigma)\big]^6 \to \big[H^{s-1}_p(\Sigma)\big]^6, \quad r_{_\Sigma}\mathcal{L}^{(1)} : \big[\widetilde{B}^s_{p,q}(\Sigma)\big]^6 \to \big[B^{s-1}_{p,q}(\Sigma)\big]^6 \tag{7.30}$$

are invertible.

Proof. With the help of the first equality in (6.5) we find that the principal homogeneous symbol matrix of the strongly elliptic pseudodifferential operator $\mathcal{L}_c$ reads as

$$\begin{aligned}
\mathfrak{S}(\mathcal{L}_c; x, \xi) = \mathfrak{S}(\mathcal{L}_{S_0}; x, \xi) &:= \big[-4^{-1} I_9 + \mathfrak{S}^2(\mathcal{K}_{S_0}; x, \xi)\big]\big[\mathfrak{S}(\mathcal{H}_{S_0}; x, \xi)\big]^{-1} \\
&= \big[-4^{-1} I_9 + \mathfrak{S}^2(\mathcal{K}_c; x, \xi)\big]\big[\mathfrak{S}(\mathcal{H}_c; x, \xi)\big]^{-1}, \ \ x \in \overline{\Sigma}, \ \ \xi \in \mathbb{R}^2 \setminus \{0\},
\end{aligned}$$

where $\mathcal{H}_{S_0}$ and $\mathcal{K}_{S_0}$ are integral operators given by (6.1) and (6.2) with S_0 for S.

One can show that the principal homogeneous symbol matrix of the operator $\mathcal{K}_c$ is an odd matrix function in ξ, whereas the principal homogeneous symbol matrix of the operator $\mathcal{H}_c$ is an even matrix function in ξ. Consequently, the matrix $\mathfrak{S}(\mathcal{L}_c; x, \xi)$ is even in ξ (for details see [3, Lemma C.2]).

From these results it follows that $\mathcal{L}^{(1)}$ is a strongly elliptic pseudodifferential operator with even principal homogeneous symbol. Therefore the matrix $[\mathfrak{S}(\mathcal{L}^{(1)}; x, 0, +1)]^{-1}\mathfrak{S}(\mathcal{L}^{(1)}; x, 0, -1)$ is the unit matrix and the corresponding eigenvalues equal to 1. Now, from Theorem A.1 in Appendix A it follows that the operators (7.30) are Fredholm with zero index for $1 < p < \infty$, $1 \le q \le \infty$ and $1/p - 1/2 < s < 1/p + 1/2$. It remains to show that the corresponding null spaces are trivial. In turn, due to the same Theorem A.1, it suffices to prove that the operator $r_{_\Sigma}\,\mathcal{L}^{(1)} : [\widetilde{H}^{\frac{1}{2}}_2(\Sigma)]^6 \to [H^{-\frac{1}{2}}_2(\Sigma)]^6$ is injective, i.e, we have to prove that the homogeneous equation

$$r_{_\Sigma}\mathcal{L}^{(1)} g = 0 \ \text{ on } \ \Sigma \tag{7.31}$$

possesses only the trivial solution in the space $[\widetilde{H}_2^{\frac{1}{2}}(\Sigma)]^6$.

Let $g \in [\widetilde{H}_2^{\frac{1}{2}}(\Sigma)]^6$ solve equation (7.31) and construct the double layer potential

$$U = (u_1, \dots, u_9)^\top = W_c(\widetilde{g}), \quad \widetilde{g} = (g, 0, 0, 0)^\top.$$

In view of properties of the double layer potential and equation (7.31), it can easily be verified that the vector $U \in [W_2^1(\mathbb{R}^3 \setminus \overline{\Sigma})]^9$ is a solution to the following crack type boundary transmission problem:

$$\begin{aligned} A(\partial_x, \tau)U = 0 \ \text{ in } \ \mathbb{R}^3 \setminus \overline{\Sigma}, \\ \big\{[\mathcal{T}U]_j\big\}^+ = \big\{[\mathcal{T}U]_j\big\}^- = 0, \ \ j = \overline{1,6} \ \text{ on } \ \Sigma, \\ \{u_k\}^+ - \{u_k\}^- = 0, \ \ k = 7, 8, 9 \ \text{ on } \ \Sigma, \\ \big\{[\mathcal{T}U]_k\big\}^+ - \big\{[\mathcal{T}U]_k\big\}^- = 0, \ \ k = 7, 8, 9 \ \text{ on } \ \Sigma \end{aligned}$$

and satisfies the decay conditions (4.5) at infinity, i.e., $U \in \mathbf{Z}(\mathbb{R}^3 \setminus \overline{\Sigma})$.

Applying Green's identities (4.1), (4.6) by standard arguments we can show that $U = 0$ in $\mathbb{R}^3 \setminus \overline{\Sigma}$. Whence $g = (g_1, \dots, g_6)^\top = 0$ on Σ follows due to the equalities $\{u_j\}^+ - \{u_j\}^- = g_j$ on Σ, $j = \overline{1,6}$. This completes the proof. □

Due to (4.9) the operator $\mathcal{A}$ is coercive and consequently is elliptic. Moreover, it is strongly elliptic. Indeed, let $\mathcal{A}_x$ be the operator $\mathcal{A}$ written in some local coordinate system with origin at the frozen point $x \in S$. Denote by $\mathcal{A}_x^{(0)}$ the principal part of the operator $\mathcal{A}_x$ and let $\mathbb{R}^3(n)$ be the half-space $y_1 n_1(x) + y_2 n_2(x) + y_3 n_3(x) < 0$ with plane boundary $\mathbb{R}^2(n) = \partial\mathbb{R}^3(n)$. Evidently, $n(x)$ is the unit outward normal vector to $\mathbb{R}^3(n)$. From Green's formula (4.1) with $\Omega = \mathbb{R}^3(n)$, equality (4.8), and positive definiteness of form (4.1) it follows that for all $\varphi \in [C_0^\infty(\mathbb{R}^2)]^9$, $\varphi \neq 0$,

$$\operatorname{Re} \int\limits_{\mathbb{R}^2(n)} \mathcal{A}_x^{(0)} \varphi(y) \cdot \varphi(y)\, dy = \int\limits_{\mathbb{R}^2(n)} \operatorname{Re} \mathfrak{S}(\mathcal{A}; x, \xi) \psi(\xi) \cdot \psi(\xi)\, d\xi \geq 0, \quad \psi(\xi) = \mathcal{F}_{y \to \xi}(\varphi)(y),$$

(cf. [19, Theorem 17]) which ensures strong ellipticity property of the symbol $\mathfrak{S}(\mathcal{A}; x, \xi)$, that is, there exists a positive constant c such that $\operatorname{Re} \mathfrak{S}(\mathcal{A}; x, \xi)\zeta \cdot \zeta \geq c|\xi|\,|\zeta|^2$ for $x \in S$, $\xi \in \mathbb{R}^2$, $\zeta \in \mathbb{C}^9$.

Let $\widetilde{\lambda}_k$, $k = \overline{1,9}$, be the eigenvalues of the matrix $a_0(x) := [\mathfrak{S}(\mathcal{A}; x, 0, +1)]^{-1} \mathfrak{S}(\mathcal{A}; x, 0, -1)$, $x \in \ell_m = \partial S_D = \partial S_N$, where $\mathfrak{S}(\mathcal{A}; x, \xi)$ with $x \in \overline{S}_N$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ is the principal homogeneous symbol of the Steklov–Poincaré operator $\mathcal{A}$. As we will see below one of the eigenvalues ($\widetilde{\lambda}_9$ say) of the matrix $a_0(x)$ equals to 1.

Let us introduce the notation

$$\delta' = \inf_{\substack{1 \leq j \leq 9 \\ x \in \ell_m}} \frac{1}{2\pi} \arg \widetilde{\lambda}_j(x), \quad \delta'' = \sup_{\substack{1 \leq j \leq 9 \\ x \in \ell_m}} \frac{1}{2\pi} \arg \widetilde{\lambda}_j(x). \tag{7.32}$$

Due to strong ellipticity of the operator $\mathcal{A}$ and since one eigenvalue equals to 1, we deduce that $-1/2 < \delta' \leq 0 \leq \delta'' < 1/2$. Theorem A.1 in Appendix A implies the following assertion (cf. [3, Theorem 5.19]).

Theorem 7.7. *Let $1 < p < \infty$, $1 \leq q \leq \infty$, $1/p - 1/2 + \delta'' < s < 1/p + 1/2 + \delta'$ with δ' and δ'' given by* (7.32)*. Then the Steklov–Poincaré operators*

$$r_{_{S_N}} \mathcal{A} : \big[\widetilde{H}_p^s(S_N)\big]^9 \to [H_p^{s-1}(S_N)]^9, \quad r_{_{S_N}} \mathcal{A} : \big[\widetilde{B}_{p,q}^s(S_N)\big]^9 \to [B_{p,q}^{s-1}(S_N)]^9$$

are invertible.

In turn, Theorem 7.7 leads to the following invertibility result.

Theorem 7.8. *Let*

$$1 < p < \infty, \ \ 1 \leq q \leq \infty, \ \ \frac{1}{p} - \frac{1}{2} + \delta'' < s < \frac{1}{p} + \frac{1}{2} + \delta'. \tag{7.33}$$

Then operators (7.28) *are invertible.*

Proof. From Theorems 7.6 and 7.7 we conclude that for arbitrary p, q, and s satisfying conditions (7.33), the operators

$$\mathcal{M}_0 : [\widetilde{H}^s_p(S_N)]^9 \times [H^s_p(S)]^9 \times [\widetilde{H}^s_p(\Sigma)]^6 \to [H^{s-1}_p(S_N)]^9 \times [H^s_p(S)]^9 \times [H^{s-1}_p(\Sigma)]^6,$$
$$\mathcal{M}_0 : [\widetilde{B}^s_{p,q}(S_N)]^9 \times [B^s_{p,q}(S)]^9 \times [\widetilde{B}^s_{p,q}(\Sigma)]^6 \to [B^{s-1}_{p,q}(S_N)]^9 \times [B^s_{p,q}(S)]^9 \times [B^{s-1}_{p,q}(\Sigma)]^6,$$

with $\mathcal{M}_0$ defined in (7.29) are invertible. Therefore the operators (7.28) are Fredholm operators with index 0.

By Lemma 7.5 we conclude then that for $s = 1/2$ and $p = 2$ operator (7.28) is invertible. The null-spaces and indices of the operators (7.28) are the same for all values of the parameter $q \in [1, +\infty]$, provided p and s satisfy the inequalities (7.33) (see [1, Chapter 3, Proposition 10.6]). Therefore, for such values of the parameters p and s they are invertible. In particular, the nonhomogeneous system (7.25)–(7.27) is uniquely solvable in the corresponding spaces. Moreover, it can be easily shown that the solution vectors h, $\widetilde{h}^{(2)}$ do not depend on the extension of the vector $g^{(D)}$, while w does. However, the sum $w + g^{(D)}_0$ is defined uniquely. □

Due to Remark 7.4 we conclude that the operators (7.20) are invertible if p, q and s satisfy conditions (7.33).

With the help of this theorem we arrive at the following existence result for the original mixed BVP.

Theorem 7.9. *Let*

$$\frac{4}{3 - 2\delta''} < p < \frac{4}{1 - 2\delta'} \tag{7.34}$$

with δ' and δ'' given by (7.32)*. Then the BVP* (5.5)*–*(5.15) *has a unique solution U in the space $[W^1_p(\Omega_\Sigma)]^9$, which can be represented as $U = V(\mathcal{H}^{-1}h) + W_c(h^{(2)}) + V_c(h^{(1)})$ in Ω_Σ, where h, $h^{(2)}$, and $h^{(1)}$ are defined by the system* (7.17)*–*(7.19)*.*

Proof. The condition (7.34) follows from the inequality (7.33) with $s = 1 - 1/p$. Now existence of a solution $U \in [W^1_p(\Omega_\Sigma)]^9$ with p satisfying (7.34) follows from Theorem 7.8 and Remark 7.4. Due to the inequalities $-1/2 < \delta' \le \delta'' < 1/2$ we have $p = 2 \in (\frac{4}{3-2\delta''}, \frac{4}{1-2\delta'})$. Therefore the unique solvability for $p = 2$ is a consequence of Theorem 5.2.

To show the uniqueness result for all other values of p from the interval (7.34) we proceed as follows. Let a vector $U \in [W^1_p(\Omega_\Sigma)]^9$ with p satisfying (7.34) be a solution to the homogeneous boundary value problem (5.5)–(5.15).

Then it is evident that

$$\{U\}^+_S \in [B^{1-\frac{1}{p}}_{p,p}(S)]^9, \ \ \{\mathcal{T}U\}^+_S \in [B^{-\frac{1}{p}}_{p,p}(S)]^9, \ \ \{U\}^\pm_\Sigma \in [B^{1-\frac{1}{p}}_{p,p}(\Sigma)]^9, \ \ \{\mathcal{T}U\}^\pm_\Sigma \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^9,$$
$$\{U\}^+_\Sigma - \{U\}^-_\Sigma \in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^9, \quad \{\mathcal{T}U\}^+_\Sigma - \{\mathcal{T}U\}^-_\Sigma = 0 \ \text{ on } \ \Sigma.$$

By the general integral representation formula the vector U can be represented in Ω_Σ as

$$U = W_c(\{U\}^+_\Sigma - \{U\}^-_\Sigma)V_c(\{\mathcal{T}U\}^+_\Sigma - \{\mathcal{T}U\}^-_\Sigma) + W(\{U\}^+_S) - V(\{\mathcal{T}U\}^+_S),$$

i.e.,

$$U = U^* + W_c(h^{(2)}) + V_c(h^{(1)}) \ \text{ in } \ \Omega_\Sigma, \tag{7.35}$$

where

$$h^{(1)} = \{\mathcal{T}U\}^+_\Sigma - \{\mathcal{T}U\}^-_\Sigma, \quad h^{(2)} := \{U\}^+_\Sigma - \{U\}^-_\Sigma \ \text{ on } \Sigma,$$
$$U^* := W(\{U\}^+_S) - V\big(\{\mathcal{T}U\}^+_S\big) \in [W^1_p(\Omega)]^9.$$

Note that U^* solves the homogeneous equation

$$A(\partial, \tau)U^* = 0 \ \text{ in } \ \Omega.$$

Denote $h := \{U^*\}^+_S$. Clearly, $h \in [B^{1-1/p}_{p,p}(S)]^9$. Since the Dirichlet problem possesses a unique solution in the space $[W^1_p(\Omega)]^9$ for arbitrary $p \in [1, +\infty)$, due to Theorem 7.2 we can represent U^*

uniquely in the form of a single layer potential, $U^* = V(\mathcal{H}^{-1}h)$ in Ω (for details see [3, Chapter 5, Section 5.6]). Therefore from (7.35) we get

$$U = V(\mathcal{H}^{-1}h) + W_c(h^{(2)}) + V_c(h^{(1)}) \ \text{ in } \ \Omega_\Sigma.$$

Now, the homogeneous boundary and transmission conditions for U lead to the homogeneous system (cf. (7.17)–(7.19)) $\mathcal{Q}\Psi = 0$, where $\Psi = (h, h^{(2)}, h^{(1)})^\top$. Whence, $\Psi = 0$ follows immediately due to invertibility of $\mathcal{Q}$ (see Theorem 7.8 and Remark 7.4). Consequently, $U = 0$ in Ω_Σ. □

Let us now present some regularity results for solutions of the mixed boundary value problem (5.5)–(5.15).

Theorem 7.10. *Let* $1 < t < \infty$, $1 \le q \le \infty$,

$$\frac{4}{3-2\delta''} < p < \frac{4}{1-2\delta'}\,, \quad \frac{1}{t} - \frac{1}{2} + \delta'' < s < \frac{1}{t} + \frac{1}{2} + \delta'$$

with δ' *and* δ'' *given by* (7.32), *and let* $U \in [W^1_p(\Omega_\Sigma)]^9$ *be the solution of the boundary value problem* (5.5)–(5.15). *Then the following regularity results hold:*

(i) *If*

$$F_j^+, F_j^- \in B^{s-1}_{t,t}(\Sigma), \ \ F_j^+ - F_j^- \in \widetilde{B}^{s-1}_{t,t}(\Sigma), \ \ j = \overline{1,6},$$
$$F_k \in \widetilde{B}^{s-1}_{t,t}(\Sigma), \ \ f_k \in \widetilde{B}^{s}_{t,t}(\Sigma), \ \ k = 7,8,9,$$
$$g^{(D)} \in [B^{s}_{t,t}(S_D)]^9, \ \ g^{(N)} \in [B^{s-1}_{t,t}(S_N)]^9,$$

then

$$U \in [H^{s+\frac{1}{t}}_t(\Omega_\Sigma)]^9;$$

(ii) *If*

$$F_j^+, F_j^- \in B^{s-1}_{t,q}(\Sigma), \ \ F_j^+ - F_j^- \in \widetilde{B}^{s-1}_{t,q}(\Sigma), \ \ j = \overline{1,6},$$
$$F_k \in \widetilde{B}^{s-1}_{t,q}(\Sigma), \ \ f_k \in \widetilde{B}^{s}_{t,q}(\Sigma), \ \ k = 7,8,9,$$
$$g^{(D)} \in [B^{s}_{t,q}(S_D)]^9, \ \ g^{(N)} \in [B^{s-1}_{t,q}(S_N)]^9,$$

then

$$U \in [B^{s+\frac{1}{t}}_{t,q}(\Omega_\Sigma)]^9;$$

(iii) *If* $\alpha > 0$ *and*

$$F_j^+, F_j^- \in B^{\alpha-1}_{\infty,\infty}(\Sigma), \ \ F_j^+ - F_j^- \in \widetilde{B}^{\alpha-1}_{\infty,\infty}(\Sigma), \ \ j = \overline{1,6},$$
$$F_k \in \widetilde{B}^{\alpha-1}_{\infty,\infty}(\Sigma), \ \ f_k \in C^\alpha(\overline{\Sigma}), \ \ r_{\ell_c} f_k = 0, \ \ k = 7,8,9,$$
$$g^{(D)} \in [C^\alpha(\overline{S}_D)]^9, \ \ g^{(N)} \in [B^{\alpha-1}_{\infty,\infty}(S_N)]^9,$$

then

$$U \in \bigcap_{\alpha'<\gamma} C^{\alpha'}(\overline{\Omega}_j), \ \ j = 0,1,$$

where $\gamma = \min\{\alpha, 1/2 + \delta'\}$, $-1/2 < \delta' \le 0$ *and* Ω_0 *is an arbitrary proper subdomain of* Ω *such that* $\Sigma \subset \partial\Omega_0 = S_0 \in C^\infty$ *and* $\Omega_1 = \Omega \setminus \overline{\Omega}_0$.

Moreover, in one-sided interior and exterior neighbourhoods of the surface S_0 *the vector* U *has* $C^{\gamma'-\varepsilon}$*-smoothness with* $\gamma' = \min\{\alpha, 1/2\}$, *while in a one-sided interior neighbourhood of the surface* S *the vector* U *possesses* $C^{\gamma''-\varepsilon}$*-smoothness with* $\gamma'' = \min\{\alpha, 1/2+\delta'\}$; *here* ε *is an arbitrarily small positive number.*

Proof. The proof is exactly the same as that of Theorem 5.22 in [3]. □

8 Asymptotic expansion of solutions

Here we investigate the asymptotic behaviour of solutions to the problem (5.5)–(5.15) near the exceptional curves ℓ_c and ℓ_m. For simplicity of description of the method applied below, we assume that the boundary data of the problem are infinitely smooth, $F_j^+, F_j^- \in C^\infty(\overline{\Sigma})$, $F_j^+ - F_j^- \in C_0^\infty(\overline{\Sigma})$, $j=\overline{1,6}$, $f_k, F_k \in C_0^\infty(\overline{\Sigma})$, $k=7,8,9$, $g^{(D)} \in [C^\infty(\overline{S}_D)]^9$, $g^{(N)} \in [C^\infty(\overline{S}_N)]^9$, where $C_0^\infty(\overline{\Sigma})$ denotes a space of functions vanishing along with all tangential (to Σ) derivatives at $\ell_c = \partial\Sigma$.

In Section 7, we have shown that the boundary value problem (5.5)–(5.15) is uniquely solvable and the solution U can be represented by (7.15), where the densities are defined by equations (7.17)–(7.19) or by the equivalent system (7.25)–(7.27).

Let $\Phi := (w, h, \widetilde{h}^{(2)})^\top$ be a solution of the system (7.25)–(7.27): $\mathcal{M}\Phi = G$, where G is the vector constructed by the right hand sides of the system, $G \in [C^\infty(\overline{S}_N)]^9 \times [C^\infty(S)]^9 \times [C^\infty(\overline{\Sigma})]^6$. To establish the asymptotic behaviour of the vector U near the curves ℓ_c and ℓ_m, we rewrite (7.15) as follows:

$$U = V(\mathcal{H}^{-1}w) + W_c(\widetilde{g}) + \mathcal{R}, \tag{8.1}$$

where

$$\mathcal{R} := -V\big(\mathcal{H}^{-1}\big[r_{_S} W_c(h^{(2)}) + r_{_S} V_c(h^{(1)}) - g_0^{(D)}\big]\big) + W_c(f_0) + V_c(h^{(1)}),$$

with $f_0 = (0,0,0,0,0,0,f_7,f_8,f_9)^\top$.

Due to the relations

$$\begin{gathered}
r_{_S} W_c(h^{(2)}) + r_{_S} V_c(h^{(1)}) - g_0^{(D)} \in [C^\infty(S)]^9,\\
h^{(1)} = \big(F_1^- - F_1^+, \dots, F_6^- - F_6^+, -F_7, -F_8, -F_9\big) \in [C_0^\infty(\overline{\Sigma})]^6,\\
h_7^{(2)} = f_7 \in C_0^\infty(\overline{\Sigma}),\ \ h_8^{(2)} = f_8 \in C_0^\infty(\overline{\Sigma}),\ \ h_9^{(2)} = f_9 \in C_0^\infty(\overline{\Sigma}).
\end{gathered}$$

we deduce $r_{_{\overline{\Omega}_j}} \mathcal{R} \in [C^\infty(\overline{\Omega}_j)]^6$, where Ω_j, $j=0,1$, are as in Theorem 7.10(iii).

The vector $\widetilde{g}$ involved in (8.1) is defined as follows: $\widetilde{g} = (\widetilde{h}^{(2)}, 0,0,0)^\top$, where $\widetilde{h}^2$ solves the pseudodifferential equation

$$r_{_\Sigma} \mathcal{L}^{(1)}\, \widetilde{h}^{(2)} = \Psi^{(1)} \ \text{ on } \ \Sigma \tag{8.2}$$

with $\Psi^{(1)} = (\Psi_1^{(1)}, \dots, \Psi_6^{(1)})^\top$. Evidently,

$$\Psi^{(1)} = g^{(3)} - r_{_\Sigma}[\mathcal{T}V(\mathcal{H}^{-1})]_{6\times 9}(h).$$

Finally, the vector w involved in (8.1) solves the pseudodifferential equation

$$r_{_{S_N}} \mathcal{A}w = \Psi^{(2)} \ \text{ on } \ S_N, \tag{8.3}$$

where

$$\Psi^{(2)} = g^{(1)} - r_{_{S_N}} \mathcal{A}g^{(2)} - r_{_{S_N}}\big([\mathcal{T}W_c]_{9\times 6}(\widetilde{h}^{(2)}) - \mathcal{A}[r_{_S} W_c]_{9\times 6}(\widetilde{h}^{(2)})\big) \in [C^\infty(\overline{S}_N)]^9.$$

As we have already mentioned, the principal homogeneous symbol $\mathfrak{S}(\mathcal{L}^{(1)}; x, \xi)$, $x \in \overline{\Sigma}$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ of the pseudodifferential operator $\mathcal{L}^{(1)}$ is even with respect to the variable ξ and therefore the matrix

$$\big[\mathfrak{S}(\mathcal{L}^{(1)}; x, 0, +1)\big]^{-1} \mathfrak{S}(\mathcal{L}^{(1)}; x, 0, -1), \ \ x \in \ell_c,$$

is the unit matrix I_6. Consequently, all eigenvalues of this matrix equal to one, $\widetilde{\lambda}_j(x) = 1$, $j=\overline{1,6}$, $x \in \ell_c$. Applying a partition of unity, natural local coordinate systems and local diffeomorphisms, we can rectify ℓ_c and Σ locally in a standard way. For simplicity, let us denote the local rectified images of ℓ_c and Σ under this diffeomorphisms by the same symbols. Then we identify a one-sided neighbourhood (on Σ) of an arbitrary point $\widetilde{x} \in \ell_c$ as a part of the half-plane $x_2 > 0$. Thus, we assume that $(x_1, 0) \in \ell_c$ and $(x_1, x_{2,+}) \in \Sigma$ for $0 < x_{2,+} < \varepsilon$. Clearly, $x_{2,+} = \operatorname{dist}(x, \ell_c)$.

Applying the results obtained in the references [6] and [7] we can derive the following asymptotic expansion for the solution $\widetilde{h}^{(2)}$ of the strongly elliptic pseudodifferential equation (8.2),

$$\widetilde{h}^{(2)}(x_1,x_{2,+})=c_0(x_1)x_{2,+}^{\frac12}+\sum_{k=1}^{M}c_k(x_1)x_{2,+}^{\frac12+k}+\widetilde{h}^{(2)}_{M+1}(x_1,x_{2,+}), \tag{8.4}$$

where M is an arbitrary natural number, $c_k\in[C^\infty(\ell_c)]^6$, $k=0,1,\dots,M$, and the remainder term satisfies the inclusion

$$\widetilde{h}^{(2)}_{M+1}\in[C^{M+1}(\ell^+_{c,\varepsilon})]^6,\ \ \ell^+_{c,\varepsilon}=\ell_c\times[0,\varepsilon].$$

Note that, according to [7], the terms in expansion (8.4) do not contain logarithms, since the principal homogeneous symbol $\mathfrak{S}(\mathcal{L}^{(1)};x,\xi)$ of the pseudodifferential operator $\mathcal{L}^{(1)}$ is even in ξ.

To derive analogous asymptotic expansion for the solution vector w of equation (8.3), we apply the same local technique as above to a one-sided neighbourhood (in S_N) of the curve ℓ_m and preserve the same notation for the local coordinates.

Consider a 9×9 matrix $a_0(x_1)$ constructed by means of the principal homogeneous symbol of the Steklov–Poincaré operator $\mathcal{A}$,

$$a_0(x_1):=\big[\mathfrak{S}(\mathcal{A};x_1,0,+1)\big]^{-1}\,\mathfrak{S}(\mathcal{A};x_1,0,-1),\ \ (x_1,0)\in\ell_m. \tag{8.5}$$

Note that unlike to the above considered case, now (8.5) is not the unit matrix and therefore we proceed as follows.

Denote by $\widetilde{\lambda}_1(x_1),\dots,\widetilde{\lambda}_9(x_1)$ the eigenvalues of the matrix a_0. Let μ_j, $j=1,\dots,l$, $1\le l\le 9$, be the distinct eigenvalues and m_j be their algebraic multiplicities: $m_1+\dots+m_l=9$. It is well known that the matrix $a_0(x_1)$ admits the decomposition (see, e.g., [12, Chapter 7, Section 7]) $a_0(x_1)=\mathcal{D}(x_1)\mathcal{J}_{a_0}(x_1)\mathcal{D}^{-1}(x_1)$, $(x_1,0)\in\ell_m$, where $\mathcal{D}$ is 9×9 nondegenerate matrix with infinitely differentiable entries and $\mathcal{J}_{a_0}$ has a block diagonal structure $\mathcal{J}_{a_0}(x_1):=\operatorname{diag}\{\mu_1(x_1)B^{(m_1)}(1),\dots,\mu_l(x_1)B^{(m_l)}(1)\}$. Here $B^{(\nu)}(t)$, $\nu\in\{m_1,\dots,m_l\}$, are upper triangular matrices:

$$B^{(\nu)}(t)=\|b^{(\nu)}_{jk}(t)\|_{\nu\times\nu},\quad b^{(\nu)}_{jk}(t)=\begin{cases}\dfrac{t^{k-j}}{(k-j)!}, & j<k,\\ 1, & j=k,\\ 0, & j>k,\end{cases}$$

i.e.,

$$B^{(\nu)}(t)=\begin{bmatrix}1 & t & \dfrac{t^2}{2!} & \cdots & \dfrac{t^{\nu-2}}{(\nu-2)!} & \dfrac{t^{\nu-1}}{(\nu-1)!}\\ 0 & 1 & t & \cdots & \dfrac{t^{\nu-3}}{(\nu-3)!} & \dfrac{t^{\nu-2}}{(\nu-2)!}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 1 & t\\ 0 & 0 & 0 & \cdots & 0 & 1\end{bmatrix}_{\nu\times\nu}.$$

Denote

$$B_0(t):=\operatorname{diag}\big\{B^{(m_1)}(t),\dots,B^{(m_l)}(t)\big\}.$$

Again, applying the results from the reference [6] we derive the following asymptotic expansion for the solution ω of the strongly elliptic pseudodifferential equation (8.3):

$$\begin{aligned}\omega(x_1,x_{2,+})=\mathcal{D}(x_1)x_{2,+}^{\frac12+\Delta(x_1)}B_0\Big(-\frac{1}{2\pi i}\,\log x_{2,+}\Big)\mathcal{D}^{-1}(x_1)b_0(x_1)\\ +\sum_{k=1}^{M}\mathcal{D}(x_1)x_{2,+}^{\frac12+\Delta(x_1)+k}B_k(x_1,\log x_{2,+})+\omega_{M+1}(x_1,x_{2,+}),\end{aligned} \tag{8.6}$$

where $b_0 \in [C^\infty(\ell_m)]^9$, $\omega_{M+1} \in [C^\infty(\ell^+_{m,\varepsilon})]^9$, $\ell^+_{m,\varepsilon} = \ell_m \times [0,\varepsilon]$, and

$$B_k(x_1,t) = B_0\Big(-\frac{t}{2\pi i}\Big)\sum_{j=1}^{k(2m_0-1)} t^j d_{kj}(x_1).$$

Here $m_0 = \max\{m_1,\dots,m_9\}$, the coefficients $d_{kj} \in [C^\infty(\ell_m)]^9$, $\Delta := (\Delta_1,\dots,\Delta_9)$, and

$$\Delta_j(x_1) = \frac{1}{2\pi i}\log\widetilde{\lambda}_j(x_1) = \frac{1}{2\pi}\arg\widetilde{\lambda}_j(x_1) + \frac{1}{2\pi i}\log|\widetilde{\lambda}_j(x_1)|,$$
$$-\pi < \arg\widetilde{\lambda}_j(x_1) < \pi, \ \ (x_1,0)\in\ell_m, \ \ j=\overline{1,9}.$$

Furthermore,
$$x_{2,+}^{\frac12+\Delta(x_1)} := \operatorname{diag}\Big\{x_{2,+}^{\frac12+\Delta_1(x_1)},\dots,x_{2,+}^{\frac12+\Delta_9(x_1)}\Big\}.$$

Now, having at hand formulae (8.4) and (8.6) with the help of the asymptotic expansion of potential-type functions obtained in [5] we can write the following spatial asymptotic expansions for the solution vector U of the boundary value problem (5.5)–(5.15) near the crack edge ℓ_c and near the collision curve ℓ_m.

(a) ***Asymptotic expansion near the crack edge ℓ_c:***

$$U(x) = \sum_{\mu=\pm1}\Bigg[\sum_{s=1}^{l_0}\sum_{j=0}^{n_s-1} x_3^j z_{s,\mu}^{\frac12-j} d_{sj}^{(c)}(x_1,\mu) + \sum_{\substack{k,l=0\\ k+l+j+p\ge1}}^{M+2}\ \sum_{j+p=0}^{M+2-l} x_2^l x_3^j z_{s,\mu}^{\frac12+p+k} d_{slkjp}^{(c)}(x_1,\mu)\Bigg] + U_{M+1}^{(c)}(x) \quad (8.7)$$

with the coefficients $d_{sj}^{(c)}(\,\cdot\,,\mu), d_{slkjp}^{(c)}(\,\cdot\,,\mu) \in [C^\infty(\ell_c)]^9$ and $U_{M+1}^{(c)} \in [C^{M+1}(\overline{\Omega}_j)]^9$, $j=0,1$. Here Ω_j, $j=0,1$, are as in Theorem 7.10(iii), and

$$z_{s,+1} = -(x_2+x_3\zeta_{s,+1}), \ \ z_{s,-1} = x_2 - x_3\zeta_{s,-1}, \ \ -\pi < \arg z_{s,\pm1} < \pi, \ \ \zeta_{s,\pm1}\in C^\infty(\ell_c), \quad (8.8)$$

where $\{\zeta_{s,\pm1}\}_{s=1}^{l_0}$ are the different roots in ζ of multiplicity n_s, $s=1,\dots,l_0$, of the polynomial $\det A^{(0)}([J_\varkappa^\top(x_1,0,0)]^{-1}\eta_\pm)$ with $\eta_\pm = (0,\pm1,\zeta)^\top$, satisfying the condition $\operatorname{Re}\zeta_{s,\pm1} < 0$. The matrix $J_\varkappa$ stands for the Jacobian matrix corresponding to the canonical diffeomorphism $\varkappa$ related to the local co-ordinate system. Under this diffeomorphism ℓ_c and Σ are locally rectified and we assume that $(x_1,0,0)\in\ell_c$, $x_2 = \operatorname{dist}(x^{(\Sigma)},\ell_c)$, $x_3 = \operatorname{dist}(x,\Sigma)$, where $x^{(\Sigma)}$ is the projection of the reference point $x\in\Omega_\Sigma$ onto the plane corresponding to the image of Σ under the diffeomorphism $\varkappa$.

Note that the coefficients $d_{sj}^{(c)}(\,\cdot\,,\mu)$ can be expressed by the first coefficient c_0 in the asymptotic expansion (8.4) (for details see [5, Theorem 2.3]).

(b) ***Asymptotic expansion near the collision curve ℓ_m:***

$$U(x) = \sum_{\mu=\pm1}\Bigg\{\sum_{s=1}^{l_0}\sum_{j=0}^{n_s-1} x_3^j\Big[d_{sj}^{(m)}(x_1,\mu) z_{s,\mu}^{\frac12+\Delta(x_1)-j} B_0\Big(-\frac{1}{2\pi i}\log z_{s,\mu}\Big)\Big]\widetilde{c}_j(x_1)$$
$$+ \sum_{\substack{k,l=0\\ k+l+j+p\ge1}}^{M+2}\ \sum_{j+p=0}^{M+2-l} x_2^l x_3^j d_{sljp}^{(m)}(x_1,\mu) z_{s,\mu}^{\frac12+\Delta(x_1)+p+k} B_{skjp}(x_1,\log z_{s,\mu})\Bigg\} + U_{M+1}^{(m)}(x), \quad (8.9)$$

where $d_{sj}^{(m)}(\,\cdot\,,\mu)$ and $d_{sljp}^{(m)}(\,\cdot\,,\mu)$ are matrices with entries belonging to the space $C^\infty(\ell_m)$, $\widetilde{c}_j \in [C^\infty(\ell_m)]^9$, $U_{M+1}^{(m)} \in [C^{M+1}(\overline{\Omega}_1)]^9$, and

$$z_{s,\mu}^{\kappa+\Delta(x_1)} := \operatorname{diag}\big\{z_{s,\mu}^{\kappa+\Delta_1(x_1)},\dots,z_{s,\mu}^{\kappa+\Delta_9(x_1)}\big\}, \ \ \kappa\in\mathbb{R}, \ \ \mu=\pm1, \ \ x_1\in\ell_m;$$

$B_{skjp}(x_1,t)$ are polynomials with respect to the variable t with vector coefficients which depend on the variable x_1 and have the order $\nu_{kjp} = k(2m_0-1)+m_0-1+j+p$, in general, where $m_0 = \max\{m_1,\dots,m_l\}$ and $m_1+\cdots+m_l = 9$.

Note that the coefficients $d_{sj}^{(m)}(\,\cdot\,,\mu)$ can be calculated explicitly, whereas the coefficients $\widetilde{c}_j$ can be expressed by means of the first coefficient b_0 in the asymptotic expansion (8.6) (for details see [5, Theorem 2.3]).

9 Analysis of singularities of solutions

Let $x' \in \ell_c$ and $\Pi^{(c)}_{x'}$ be the plane passing through the point x' and orthogonal to the curve ℓ_c. We introduce the polar coordinates (r,α), $r \ge 0$, $-\pi \le \alpha \le \pi$, in the plane $\Pi^{(c)}_{x'}$ with pole at the point x'. Denote by $\Sigma^{\pm}$ the two different faces of the crack surface Σ. It is clear that $(r, \pm\pi) \in \Sigma^{\pm}$.

Denote the similar orthogonal plane to the curve ℓ_m by $\Pi^{(m)}_{x'}$ at the point $x' \in \ell_m$ and introduce there the polar coordinates (r,α), with pole at the point x'. The intersection of the plane $\Pi^{(m)}_{x'}$ and Ω_Σ can be identified with the half-plane $r \ge 0$ and $0 \le \alpha \le \pi$.

In these coordinate systems, the functions $z_{s,\pm 1}$ given by (8.8) read as follows:

$$z_{s,+1} = -r(\cos\alpha + \zeta_{s,+1}(x')\sin\alpha), \quad z_{s,-1} = r(\cos\alpha - \zeta_{s,-1}(x')\sin\alpha),$$

where $x' \in \ell_c \cup \ell_m$, $s = 1, \dots, l_0$. We can rewrite asymptotic expansions (8.7) and (8.9) in more convenient forms, in terms of the variables r and α. Moreover, we establish more refined asymptotic properties of the solution vector $U = (u, \phi, \varphi, \psi, \vartheta)^\top \in [C^\infty(\Omega_\Sigma)]^9$ near the exceptional curves.

(i) ***Asymptotic analysis of solutions near the crack edge ℓ_c.***

The asymptotic expansion (8.7) yields

$$U = (u, \phi, \varphi, \psi, \vartheta)^\top = a_0(x', \alpha)\, r^{1/2} + a_1(x', \alpha)\, r^{3/2} + \cdots, \tag{9.1}$$

where r is the distance from the reference point $x \in \Pi^{(c)}_{x'}$ to the curve ℓ_c, and $a_j = (a_{j1}, \dots, a_{j9})^\top$, $j = 0, 1, \dots$, are smooth vector functions of $x' \in \ell_c$.

From this representation it follows that in one-sided interior and exterior neighbourhoods of the surface $S_0 = \partial\Omega_0$ the vector $U = (u, \phi, \varphi, \psi, \vartheta)^\top$ has $C^{\frac{1}{2}}$-smoothness.

(ii) ***Asymptotic analysis of solutions near the curve ℓ_m.***

The asymptotic expansion (8.9) yields

$$U(x) = \sum_{\mu=\pm 1} \sum_{s=1}^{l_0} \sum_{j=0}^{n_s - 1} c_{sj\mu}(x', \alpha) r^{\gamma + i\delta} B_0\Big(-\frac{1}{2\pi i}\, \log r\Big) \widetilde{c}_{sj\mu}(x', \alpha) + \cdots, \tag{9.2}$$

where $x' \in \ell_m$,

$$\begin{gathered} r^{\gamma+i\delta} := \operatorname{diag}\big\{r^{\gamma_1 + i\delta_1}, \dots, r^{\gamma_9 + i\delta_9}\big\}, \\ \gamma_j = \frac{1}{2} + \frac{1}{2\pi}\, \arg\widetilde{\lambda}_j(x'), \quad \delta_j = \frac{1}{2\pi}\, \log|\widetilde{\lambda}_j(x')|, \quad j = \overline{1,9}, \end{gathered} \tag{9.3}$$

and $\widetilde{\lambda}_j$, $j = \overline{1,9}$, are eigenvalues of the matrix

$$a_0(x') = \big[\mathfrak{S}(\mathcal{A}; x', 0, +1)\big]^{-1} \mathfrak{S}(\mathcal{A}; x', 0, -1), \quad x' \in \ell_m.$$

Recall that here $\mathfrak{S}(\mathcal{A}; x', \xi)$ is the principal homogeneous symbol of the Steklov–Poincaré operator $\mathcal{A} = (-2^{-1} I_9 + \mathcal{K})\mathcal{H}^{-1}$. Moreover, the eigenvalues $\widetilde{\lambda}_j$, $j = \overline{1,9}$, can be expressed in terms of the eigenvalues β_j, $j = \overline{1,9}$, of the matrix $\mathfrak{S}(\mathcal{K}; x', 0, +1)$, where $\mathfrak{S}(\mathcal{K}; x', \xi)$ is the principal homogeneous symbol matrix of the singular integral operator $\mathcal{K}$ (see [4, Theorem 6.3]),

$$\widetilde{\lambda}_j = \frac{1 + 2\beta_j}{1 - 2\beta_j}, \quad j = \overline{1,9}. \tag{9.4}$$

The symbol matrix $\mathfrak{S}(\mathcal{K};x',0,+1)$ is calculated explicitly

$$\mathfrak{S}(\mathcal{K};x',0,+1)=\begin{bmatrix}0&0&0&0&0&0&0&0&0\\0&0&-ia&0&0&0&0&0&0\\0&ia&0&0&0&0&0&0&0\\0&0&0&0&0&0&ic&ip&iq\\0&0&0&0&0&-ib&0&0&0\\0&0&0&0&ib&0&0&0&0\\0&0&0&-\dfrac{ib_0}{2\gamma}&0&0&0&0&0\\0&0&0&-\dfrac{i\lambda_1}{2\gamma}&0&0&0&0&0\\0&0&0&-\dfrac{i\nu_2}{2\gamma}&0&0&0&0&0\end{bmatrix}_{9\times 9},$$

where

$$a=\frac{1}{4}\Big(\frac{\lambda}{\lambda+2\mu+\varkappa}-\frac{\mu}{\mu+\varkappa}\Big),\quad b=\frac{1}{4}\Big(\frac{\alpha}{\alpha+\beta+\gamma}-\frac{\beta}{\gamma}\Big),$$

$$c=b_0b_{11}+\lambda_1b_{21}+\nu_2b_{31},\quad p=b_0b_{12}+\lambda_1b_{22}+\nu_2b_{32},\quad q=b_0b_{13}+\lambda_1b_{23}+\nu_2b_{33},$$

$$\big[b_{jk}\big]_{3\times3}=\begin{bmatrix}a_0&-\lambda_2&\nu_1\\\lambda_2&\chi&\nu_3\\\nu_1&-\nu_3&k\end{bmatrix}^{-1}=(k\chi a_0+k\lambda_2^2-\chi\nu_1^2-2\lambda_2\nu_1\nu_3+a_0\nu_3^2)^{-1}$$

$$\times\begin{bmatrix}k\chi+\nu_3^2&k\lambda_2-\nu_1\nu_3&\chi\nu_1+\lambda_2\nu_3\\-k\lambda_2+\nu_1\nu_3&ka_0-\nu_1^2&-\nu a_0+\lambda_2\nu_1\\\chi\nu_1+\lambda_2\nu_3&-\lambda_2\nu_1+a_0\nu_3&\chi a_0+\lambda_2^2\end{bmatrix}.$$

The characteristic polynomial of the matrix $\mathfrak{S}(\mathcal{K};x',0,+1)$ can be represented as

$$\det\big(\mathfrak{S}(\mathcal{K};x',0,+1)-\beta I\big)=-\frac{\beta^3(\beta^2-a^2)(\beta^2-b^2)(2\gamma\beta^2-cb_0-p\lambda_1-q\nu_2)}{2\gamma}.$$

Therefore we have the following expressions for eigenvalues of the matrix $\mathfrak{S}(\mathcal{K};x',0,+1)$:

$$\beta_{1,2}=\mp\sqrt{d}\,,\ \ \beta_{3,4}=\mp a,\ \ \beta_{5,6}=\mp b,\ \ \beta_7=\beta_8=\beta_9=0,$$

where

$$|a|<\frac{1}{2}\,,\ \ |b|<\frac{1}{2}\,,\ \ d=\frac{cb_0+p\lambda_1+q\nu_2}{2\gamma}\,,\ \ \gamma>0. \tag{9.5}$$

Then due to (9.4) we have

$$\widetilde{\lambda}_1=\frac{1}{\widetilde{\lambda}_2}=\begin{cases}\dfrac{1-2i\sqrt{-d}}{1+2i\sqrt{-d}} & \text{if } d<0,\\[2mm] \dfrac{1-2\sqrt{d}}{1+2\sqrt{d}} & \text{if } d\geqslant 0,\end{cases}$$

$$\widetilde{\lambda}_3=\frac{1-2a}{1+2a}\,,\ \ \widetilde{\lambda}_4=\frac{1}{\widetilde{\lambda}_3}\,,\ \ \widetilde{\lambda}_5=\frac{1-2b}{1+2b}\,,\ \ \widetilde{\lambda}_6=\frac{1}{\widetilde{\lambda}_5}\,,\ \ \widetilde{\lambda}_7=\widetilde{\lambda}_8=\widetilde{\lambda}_9=1.$$

Note, that $\widetilde{\lambda}_3,\dots,\widetilde{\lambda}_9$ are positive eigenvalues, whereas $\widetilde{\lambda}_1$, and $\widetilde{\lambda}_2$ are positive if $d>0$ (see Appendix A) and $|\widetilde{\lambda}_1|=|\widetilde{\lambda}_2|=1$ if $d<0$.

Applying the above results we can explicitly write the exponents of the dominant terms in the asymptotic expansion (9.2)–(9.3):

$$\gamma_1=\frac{1}{2}-\frac{1}{\pi}\,\arctan 2\sqrt{-d}\,,\ \ \gamma_2=\frac{1}{2}+\frac{1}{\pi}\,\arctan 2\sqrt{-d}\,,\ \ \delta_1=\delta_2=0\ \text{ if }\ d<0, \tag{9.6}$$

$$\gamma_1 = \gamma_2 = \frac{1}{2}\,, \quad \delta_1 = \frac{1}{2\pi}\,\ln\frac{1-2\sqrt{d}}{1+2\sqrt{d}}\,, \quad \delta_2 = -\delta_1 \ \text{ if } \ d \geqslant 0, \tag{9.7}$$

and

$$\gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = \gamma_8 = \gamma_9 = \frac{1}{2}\,,$$
$$\delta_3 = \frac{1}{2\pi}\,\ln\frac{1-2a}{1+2a}\,, \quad \delta_4 = -\delta_3, \quad \delta_5 = \frac{1}{2\pi}\,\ln\frac{1-2b}{1+2b}\,, \quad \delta_6 = -\delta_5, \quad \delta_7 = \delta_8 = \delta_9 = 0.$$

Note, that $B_0(t)$ has the form

$$B_0(t) = \begin{bmatrix} I_6 & [0]_{6\times 3} \\ [0]_{3\times 6} & B^{(3)}(t) \end{bmatrix}, \quad B^{(3)}(t) = \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \ \text{ if } \ d < 0,$$

and

$$B_0(t) = I_9 \ \text{ if } \ d \geqslant 0.$$

Now we can draw the conclusions concerning the asymptotic behaviour of solution U to the mixed problem near the exceptional curve ℓ_m:

- If $d < 0$, then the asymptotic expansion has the form

$$U = c_1 r^{\gamma_1} + c_2 r^{1/2+i\delta_3} + c_3 r^{1/2-i\delta_3} + c_4 r^{1/2+i\delta_5} + c_5 r^{1/2-i\delta_5} + c_6 r^{1/2}\ln r + c_7 r^{1/2}\ln^2 r + c_8 r^{1/2} + c_9 r^{\gamma_2} + \cdots\,.$$

  As we see from (9.5) and (9.6), the exponent γ_1 characterizing the behaviour of the solution near the line ℓ_m depends on the material constants and may take an arbitrary value from the interval $(0, \frac{1}{2})$. In this case the solution possesses C^{γ_1} smoothness in a neighbourhood of the line ℓ_m and since $\gamma_1 < \frac{1}{2}$ the first order derivatives of solutions have non-oscillating singularities near the exceptional curve ℓ_m.

- If $d \geqslant 0$, then

$$U = d_1 r^{1/2} + d_2 r^{1/2+i\delta_1} + d_3 r^{1/2-i\delta_1} + d_4 r^{1/2+i\delta_3} + d_5 r^{1/2-i\delta_3} + d_6 r^{1/2+i\delta_5} + d_7 r^{1/2-i\delta_5} + \mathcal{O}(r^{3/2-\varepsilon}),$$

  where ε is a sufficiently small positive number. In this case the solution possesses $C^{\frac{1}{2}}$-smoothness in a neighbourhood of the line ℓ_m.

10 Appendix A: Fredholm properties of strongly elliptic pseudodifferential operators on manifolds with boundary

Here we collect some results describing the Fredholm properties of strongly elliptic pseudodifferential operators on a compact manifold with boundary. They can be found in [1, 11, 15, 22]. We essentially use these results in Section 7 to prove the existence and regularity of solutions to the mixed boundary value problem for a solid with an interior crack.

Let $\overline{\mathcal{M}} \in C^\infty$ be a compact, n-dimensional, nonselfintersecting manifold with boundary $\partial\mathcal{M} \in C^\infty$ and let $\mathcal{A}$ be a strongly elliptic $N\times N$ matrix pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\mathfrak{S}(\mathcal{A}; x, \xi)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system $(x \in \overline{\mathcal{M}}, \xi \in \mathbb{R}^n \setminus \{0\})$.

Let $\widetilde{\lambda}_1(x), \dots, \widetilde{\lambda}_N(x)$ be the eigenvalues of the matrix

$$\big[\mathfrak{S}(\mathcal{A}; x, 0, \dots, 0, +1)\big]^{-1}\mathfrak{S}(\mathcal{A}; x, 0, \dots, 0, -1), \quad x \in \partial\overline{\mathcal{M}},$$

and let

$$\delta_j(x) = \operatorname{Re}\big[(2\pi i)^{-1}\ln\widetilde{\lambda}_j(x)\big], \quad j = 1,\dots,N.$$

Here $\ln\zeta$ denotes the branch of the logarithm analytic in the complex plane cut along $(-\infty, 0]$. Due to the strong ellipticity of $\mathcal{A}$ we have the strict inequality $-1/2 < \delta_j(x) < 1/2$ for $x \in \overline{\mathcal{M}}$. The numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system at the point x. In particular, if the eigenvalue $\widetilde{\lambda}_j$ is real, then it is positive and consequently the corresponding $\delta_j = 0$.

Note that when $\mathfrak{S}(\mathcal{A}, x, \xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ or when it is an even matrix in ξ we have $\delta_j(x) = 0$ for $j = 1,\dots,N$, since all the eigenvalues $\widetilde{\lambda}_j(x)$ $(j = \overline{1,N})$ are positive numbers for any $x \in \overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudodifferential operators are characterized by the following theorem.

Theorem A.1. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, and let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator of order $\nu \in \mathbb{R}$, that is, there is a positive constant c_0 such that*

$$\operatorname{Re}\big(\mathfrak{S}(\mathcal{A}; x,\xi)\zeta\cdot\zeta\big) \ge c_0|\zeta|^2 \ \ \text{for} \ \ x \in \overline{\mathcal{M}}, \ \ \xi \in \mathbb{R}^n$$

with $|\xi| = 1$, and $\zeta \in \mathbb{C}^N$. Then

$$\mathcal{A} : \widetilde{H}^s_p(\mathcal{M}) \to H^{s-\nu}_p(\mathcal{M}), \quad \mathcal{A} : \widetilde{B}^s_{p,q}(\mathcal{M}) \to B^{s-\nu}_{p,q}(\mathcal{M}), \tag{A.1}$$

are Fredholm operators with index zero if

$$\frac{1}{p} - 1 + \sup_{x\in\partial\mathcal{M},\ 1\le j\le N}\delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{x\in\partial\mathcal{M},\ 1\le j\le N}\delta_j(x). \tag{A.2}$$

Moreover, the null-spaces and indices of the operators (A.1) *are the same* (*for all values of the parameter $q \in [1, +\infty]$*) *provided p and s satisfy the inequality* (A.2).

11 Appendix B: Fundamental solution

Let Γ be the fundamental solution of the operator $A(\partial, \tau)$,

$$A(\partial,\tau)\Gamma(x) = \delta(x)I_9, \tag{B.1}$$

where $\delta(x)$ is Dirac's delta function and I_9 is the 9×9 unite matrix.

Denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the direct and inverse Fourier transform operators in $\mathbb{R}^3$,

$$\mathcal{F}_{x\to\xi}[f] \equiv \widehat{f}(\xi) = \int\limits_{\mathbb{R}^3} e^{ix\cdot\xi}f(x)\,dx, \ \ \xi \in \mathbb{R}^3,$$

$$\mathcal{F}^{-1}_{\xi\to x}[g] = \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} e^{-i\xi\cdot x}g(\xi)\,d\xi, \ \ x \in \mathbb{R}^3.$$

Applying the Fourier operator $\mathcal{F}$ to both sides of equation (B.1) we get

$$A(-i\xi,\tau)\widehat{\Gamma}(\xi) = I_9,$$

whence

$$\widehat{\Gamma}(\xi) = [A(-i\xi,\tau)]^{-1}. \tag{B.2}$$

From (B.2) it follows that $\widehat{\Gamma} = (X^{(1)},\dots,X^{(9)})$, where $X^{(k)} = (X^{(k)}_1,\dots,X^{(k)}_9)^\top$, $k = 1,\dots,9$, is a solution of the equation

$$A(-i\xi,\tau)X^{(k)} = B^{(k)} \tag{B.3}$$

with the right side $B^{(k)} = ((C^{(k)})^\top, (F^{(k)})^\top, G^{(k)}, H^{(k)}, L^{(k)})^\top$, where

$$C^{(k)} = (\delta_{1k},\delta_{2k},\delta_{3k})^\top, \ \ F^{(k)} = (\delta_{4k},\delta_{5k},\delta_{6k})^\top, \ \ G^{(k)} = \delta_{7k}, \ \ H^{(k)} = \delta_{8k}, \ \ L^{(k)} = \delta_{9k}.$$

Introduce the notations

$$\widehat{u}^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)})^\top, \ \ \widehat{\Phi}^{(k)} = (x_4^{(k)}, x_5^{(k)}, x_6^{(k)})^\top, \ \ \widehat{\varphi}^{(k)} = x_7^{(k)}, \ \ \widehat{\psi}^{(k)} = x_8^{(k)}, \ ; \ \widehat{\vartheta}^{(k)} = x_9^{(k)}. \quad \text{(B.4)}$$

Then equation (B.3) can be rewritten as

$$\begin{aligned}
\big[(\mu+\varkappa)|\xi|^2+\tau^2\rho_0\big]\widehat{u}^{(k)}+(\lambda+\mu)\xi(\xi\cdot\widehat{u}^{(k)})+i\varkappa[\xi\times\widehat{\Phi}^{(k)}]+i\lambda_0\xi\widehat{\varphi}^{(k)}-i\tau\beta_0\xi\widehat{\vartheta}^{(k)}&=-C^{(k)},\\
\big[\gamma|\xi|^2+(2\varkappa+\tau^2I_0)\big]\widehat{\Phi}^{(k)}+(\alpha+\beta)\xi(\xi\cdot\widehat{\Phi}^{(k)})+i\varkappa\big[\xi\times\widehat{u}^{(k)}\big]&=-F^{(k)},\\
\big(a_0|\xi|^2+\xi_0+\tau^2j_0\big)\widehat{\varphi}^{(k)}+\lambda_2|\xi|^2\widehat{\psi}^{(k)}-\big(\nu_1|\xi|^2-\tau c_0\big)\widehat{\vartheta}^{(k)}-i\lambda_0(\xi\cdot\widehat{u}^{(k)})&=-G^{(k)},\\
\lambda_2|\xi|^2\widehat{\varphi}^{(k)}+\chi|\xi|^2\widehat{\psi}^{(k)}+\nu_3|\xi|^2\widehat{\vartheta}^{(k)}&=-H^{(k)},\\
\big(k|\xi|^2+\tau^2a\big)\widehat{\vartheta}^{(k)}-i\tau\beta_0(\xi\cdot\widehat{u}^{(k)})+\big(\nu_1|\xi|^2+\tau c_0\big)\widehat{\varphi}^{(k)}-\nu_3|\xi|^2\widehat{\psi}^{(k)}&=-L^{(k)},
\end{aligned}\quad \text{(B.5)}$$

Multiplying the first and second equations of (B.5) by $i\xi$ and denoting $\eta^{(k)} := i\xi\cdot\widehat{u}^{(k)}$, $\zeta^{(k)} := i\xi\cdot\widehat{\Phi}^{(k)}$, we get

$$\zeta^{(k)} = -\frac{i\xi_{k-3}}{(\alpha+\beta+\gamma)(|\xi|^2-k_1^2)}\,, \quad k_1^2 = -\frac{\tau^2 I_0+2\varkappa}{\alpha+\beta+\gamma}\,,$$

for $k=4,5,6$ and $\zeta^{(k)}=0$ otherwise, whereas the remaining equations constitute a system of four equations for unknowns $\eta^{(k)}$, $\widehat{\varphi}^{(k)}$, $\widehat{\psi}^{(k)}$, $\widehat{\vartheta}^{(k)}$,

$$\begin{aligned}
\big[(\lambda+2\mu+\varkappa)|\xi|^2+\tau^2\rho_0\big]\eta^{(k)}-\lambda_0|\xi|^2\widehat{\varphi}^{(k)}+\tau\beta_0|\xi|^2\widehat{\vartheta}^{(k)}&=-i\xi\cdot C^{(k)},\\
\big(a_0|\xi|^2+\xi_0+\tau^2j_0\big)\widehat{\varphi}^{(k)}+\lambda_2|\xi|^2\widehat{\psi}^{(k)}-\big(\nu_1|\xi|^2-\tau c_0\big)\widehat{\vartheta}^{(k)}-\lambda_0\eta^{(k)}&=-G^{(k)},\\
\lambda_2|\xi|^2\widehat{\varphi}^{(k)}+\chi|\xi|^2\widehat{\psi}^{(k)}+\nu_3|\xi|^2\widehat{\vartheta}^{(k)}&=-H^{(k)},\\
\big(k|\xi|^2+\tau^2a\big)\widehat{\vartheta}^{(k)}-\tau\beta_0\eta+\big(\nu_1|\xi|^2+\tau c_0\big)\widehat{\varphi}^{(k)}-\nu_3|\xi|^2\widehat{\psi}^{(k)}&=-L^{(k)}.
\end{aligned}\quad \text{(B.6)}$$

Denote by $\widetilde{A}(|\xi|^2)$ the matrix of coefficients of system (B.6)

$$\widetilde{A}(|\xi|^2) := \begin{bmatrix}
\big[(\lambda+2\mu+\varkappa)|\xi|^2+\tau^2\rho_0\big] & -\lambda_0|\xi|^2 & 0 & \tau\beta_0|\xi|^2\\
-\lambda_0 & \big(a_0|\xi|^2+\xi_0+\tau^2j_0\big) & \lambda_2|\xi|^2 & -\big(\nu_1|\xi|^2-\tau c_0\big)\\
0 & \lambda_2|\xi|^2 & \chi|\xi|^2 & \nu_3|\xi|^2\\
-\tau\beta_0 & \big(\nu_1|\xi|^2+\tau c_0\big) & -\nu_3|\xi|^2 & \big(k|\xi|^2+\tau^2a\big)
\end{bmatrix}.$$

Note, that

$$D(|\xi|^2) := \det\big(\widetilde{A}(|\xi|^2)\big)$$

can be factorized as

$$D(|\xi|^2) = d_0|\xi|^2\big(|\xi|^2-k_4^2\big)\big(|\xi|^2-k_5^2\big)\big(|\xi|^2-k_6^2\big),$$

where

$$d_0 = (\lambda+2\mu+\varkappa)(a_0k\chi+a_0\nu_3^2+\chi\nu_1^2+2\lambda_2\nu_1\nu_3-k\lambda_2^2)$$

and k_4^2, k_5^2, k_6^2 are the roots of the polynomial

$$P(z) = z^3+p_1z^2+p_2z+p_3 \quad \text{(B.7)}$$

with

$$\begin{aligned}
p_1 = \frac{\alpha+\beta+\gamma}{d_0}\Big\{ &-k\chi\lambda_0^2-\tau^2\big[a(\varkappa+\lambda+2\mu)+\beta_0^2\big]\lambda_2^2-2\tau\chi\beta_0\lambda_0\nu_1-2\tau\beta_0\lambda_0\lambda_2\nu_3\\
&-\lambda_0^2\nu_3^2+(\varkappa+\lambda+2\mu)\tau^2j_0(k\chi+\nu_3^2)+k\varkappa\chi\xi_0+k\lambda\chi\xi_0+2k\mu\chi\xi_0+\varkappa\nu_3^2\xi_0+\lambda\nu_3^2\xi_0\\
&+2\mu\nu_3^2\xi_0+\tau^2(-k\lambda_2^2+\chi\nu_1^2+2\lambda_2\nu_1\nu_3)\rho_0+\tau^2a_0\big[a(\varkappa+\lambda+2\mu)\chi+\chi\beta_0^2+(k\chi+\nu_3^2)\rho_0\big]\Big\},\\
p_2 = \frac{2(\alpha+\beta+\gamma)}{d_0}\,&(a\varkappa\tau^2\chi a_0+a\lambda\tau^2\chi a_0+2a\mu\tau^2\chi a_0+k\varkappa\tau^2\chi j_0+k\lambda\tau^2\chi j_0+2k\mu\tau^2\chi j_0
\end{aligned}$$

$$+\tau^2\chi a_0\beta_0^2-k\chi\lambda_0^2-a\varkappa\tau^2\lambda_2^2-a\lambda\tau^2\lambda_2^2-2a\mu\tau^2\lambda_2^2-\tau^2\beta_0^2\lambda_2^2-2\tau\chi\beta_0\lambda_0\nu_1-2\tau\beta_0\lambda_0\lambda_2\nu_3$$
$$+\varkappa\tau^2 j_0\nu_3^2+\lambda\tau^2 j_0\nu_3^2+2\mu\tau^2 j_0\nu_3^2-\lambda_0^2\nu_3^2+k\varkappa\chi\xi_0+k\lambda\chi\xi_0+2k\mu\chi\xi_0+\varkappa\nu_3^2\xi_0+\lambda\nu_3^2\xi_0$$
$$+2\mu\nu_3^2\xi_0+k\tau^2\chi a_0\rho_0-k\tau^2\lambda_2^2\rho_0+\tau^2\chi\nu_1^2\rho_0+2\tau^2\lambda_2\nu_1\nu_3\rho_0+\tau^2 a_0\nu_3^2\rho_0),$$
$$p_3=\frac{\alpha+\beta+\gamma}{d_0}\,\tau^4\chi\big[-c_0^2+a(\tau^2 j_0+\xi_0)\big]\rho_0, \tag{B.8}$$

From (B.6) for $\eta^{(k)}$, $\widehat{\varphi}^{(k)}$, $\widehat{\psi}^{(k)}$, $\widehat{\vartheta}^{(k)}$ we have

$$\big(\eta^{(k)},\widehat{\varphi}^{(k)},\widehat{\psi}^{(k)},\widehat{\vartheta}^{(k)}\big)^\top=-\widetilde{A}^{-1}\big(|\xi|^2\big)\big(iC^{(k)}\cdot\xi,G^{(k)},H^{(k)},L^{(k)}\big)^\top,$$

implying

$$\eta^{(1)}=-i|\xi|^2\Big(\chi\Big(-\tau^2c_0^2+\big(|\xi|^2k+a\tau^2\big)\big(\xi_0+\tau^2 j_0\big)\Big)-|\xi|^2\Big(|\xi|^2k\chi a_0+a\tau^2\chi a_0+|\xi|^2k\lambda_0\lambda_2$$
$$-a\tau^2\lambda_0\lambda_2+|\xi|^2\chi\nu_1^2+\big(\tau c_0\big(\lambda_0-\lambda_2\big)+|\xi|^2(\lambda_0+\lambda_2)\nu_1\big)\nu_3+\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\nu_3^2\Big)\Big)\,\frac{\xi_1}{D(|\xi|^2)}\,,$$
$$\eta^{(2)}=-i|\xi|^2\Big(\chi\Big(-\tau^2c_0^2+\big(|\xi|^2k+a\tau^2\big)(\xi_0+\tau^2 j_0)\Big)-|\xi|^2\Big(|\xi|^2k\chi a_0+a\tau^2\chi a_0+|\xi|^2k\lambda_0\lambda_2$$
$$-a\tau^2\lambda_0\lambda_2+|\xi|^2\chi\nu_1^2+\big(\tau c_0(\lambda_0-\lambda_2)+|\xi|^2(\lambda_0+\lambda_2)\nu_1\big)\nu_3+\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\nu_3^2\Big)\Big)\,\frac{\xi_2}{D(|\xi|^2)}\,,$$
$$\eta^{(3)}=-i|\xi|^2\Big(\chi\Big(-\tau^2c_0^2+\big(|\xi|^2k+a\tau^2\big)(\xi_0+\tau^2 j_0)\Big)-|\xi|^2\Big(|\xi|^2k\chi a_0+a\tau^2\chi a_0+|\xi|^2k\lambda_0\lambda_2$$
$$-a\tau^2\lambda_0\lambda_2+|\xi|^2\chi\nu_1^2+\big(\tau c_0(\lambda_0-\lambda_2)+|\xi|^2(\lambda_0+\lambda_2)\nu_1\big)\nu_3+(\xi_0+|\xi|^2a_0+\tau^2 j_0)\nu_3^2\Big)\Big)\,\frac{\xi_3}{D(|\xi|^2)}\,,$$
$$\eta^{(4)}=\eta^{(5)}=\eta^{(6)}=0,$$
$$\eta^{(7)}=-|\xi|^4\Big(\tau^2\chi c_0\beta_0+|\xi|^2\tau\beta_0(\chi\nu_1+\lambda_2\nu_3)+\lambda_0\big(|\xi|^2k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$
$$\eta^{(8)}=|\xi|^4\Big(\big(|\xi|^2k+a\tau^2\big)\lambda_0^2+|\xi|^2\lambda_0\nu_1(\tau\beta_0-\nu_3)$$
$$+\tau\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\beta_0\nu_3+\tau c_0\lambda_0(\tau\beta_0+\nu_3)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$
$$\eta^{(9)}=|\xi|^4\Big(\tau\beta_0\Big(\chi\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)-|\xi|^2\lambda_0\lambda_2-\lambda_0(-\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_0\nu_3)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$
$$\widehat{\varphi}^{(1)}=-|\xi|^2\Big(-\tau^2\chi c_0\beta_0+|\xi|^2\tau\chi\beta_0\nu_1+\lambda_0\big(\big(|\xi|^2k+a\tau^2\big)\chi+|\xi|^2\nu_3(\tau\beta_0+\nu_3)\big)\Big)\,\frac{\xi_1}{D(|\xi|^2)}\,,$$
$$\widehat{\varphi}^{(2)}=-|\xi|^2\Big(-\tau^2\chi c_0\beta_0+|\xi|^2\tau\chi\beta_0\nu_1+\lambda_0\Big(\big(|\xi|^2k+a\tau^2\big)\chi+|\xi|^2\nu_3(\tau\beta_0+\nu_3)\Big)\Big)\,\frac{\xi_2}{D(|\xi|^2)}\,,$$
$$\widehat{\varphi}^{(3)}=-|\xi|^2\Big(-\tau^2\chi c_0\beta_0+|\xi|^2\tau\chi\beta_0\nu_1+\lambda_0\Big(\big(|\xi|^2k+a\tau^2\big)\chi+|\xi|^2\nu_3(\tau\beta_0+\nu_3)\Big)\Big)\,\frac{\xi_3}{D(|\xi|^2)},$$
$$\widehat{\varphi}^{(4)}=\widehat{\varphi}^{(5)}=\widehat{\varphi}^{(6)}=0,$$
$$\widehat{\varphi}^{(7)}=-|\xi|^2\Big(|\xi|^2\tau^2\chi\beta_0^2+\big(|\xi|^2k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$
$$\widehat{\varphi}^{(8)}=|\xi|^2\Big(\big(\tau c_0-|\xi|^2\nu_1\big)\nu_3\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$+\lambda_0\Big(|\xi|^2\tau^2\beta_0^2+|\xi|^2\tau\beta_0\nu_3+\big(|\xi|^2k+a\tau^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$
$$\widehat{\varphi}^{(9)}=|\xi|^2\Big(\tau\chi c_0\Big(|\xi|^2(\varkappa+\lambda+2\mu)-\tau^2\rho_0-\tau\chi\beta_0\lambda_0$$
$$+(\chi\nu_1+\lambda_0\nu_3)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$
$$\widehat{\psi}^{(1)}=i|\xi|^2\Big(\lambda_2\Big(|\xi|^2k\lambda_0+\tau\big(-\tau c_0\beta_0+a\tau\lambda_0+|\xi|^2\beta_0\nu_1\big)\Big)$$

$$
\begin{aligned}
&\qquad + \Big(\tau\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\beta_0-\lambda_0\big(\tau c_0+|\xi|^2\nu_1\big)\Big)\nu_3\Big)\,\frac{\xi_1}{D(|\xi|^2)}\,,\\
\widehat{\psi}^{(2)} &= i|\xi|^2\Big(\lambda_2\Big(|\xi|^2k\lambda_0+\tau\big(-\tau c_0\beta_0+a\tau\lambda_0+|\xi|^2\beta_0\nu_1\big)\Big)\\
&\qquad + \Big(\tau\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\beta_0-\lambda_0\big(\tau c_0+|\xi|^2\nu_1\big)\Big)\nu_3\Big)\,\frac{\xi_2}{D(|\xi|^2)}\,,\\
\widehat{\psi}^{(3)} &= i|\xi|^2\Big(\lambda_2\Big(|\xi|^2k\lambda_0+\tau\big(-\tau c_0\beta_0+a\tau\lambda_0+|\xi|^2\beta_0\nu_1\big)\Big)\\
&\qquad + \Big(\tau\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\beta_0-\lambda_0\big(\tau c_0+|\xi|^2\nu_1\big)\Big)\nu_3\Big)\,\frac{\xi_3}{D(|\xi|^2)}\,,\\
\widehat{\psi}^{(4)} &= \widehat{\psi}^{(5)}=\widehat{\psi}^{(6)}=0,\\
\widehat{\psi}^{(7)} &= |\xi|^2\Big(\lambda_2\Big(|\xi|^2\tau^2\beta_0^2+\big(|\xi|^2k+a\tau^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\\
&\qquad -\nu_3\Big(-|\xi|^2\tau\beta_0\lambda_0+\big(\tau c_0+|\xi|^2\nu_1\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,\\
\widehat{\psi}^{(8)} &= -|\xi|^2\tau\beta_0\Big(\tau\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\beta_0+\lambda_0\big(\tau c_0-|\xi|^2\nu_1\big)\Big)\\
&\qquad -\big(|\xi|^2k+a\tau^2\big)\Big(-|\xi|^2\lambda_0^2+\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\\
&\qquad +\big(\tau c_0+|\xi|^2\nu_1\big)\Big(|\xi|^2\tau\beta_0\lambda_0+\big(\tau c_0-|\xi|^2\nu_1\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\,\frac{1}{D(|\xi|^2)}\,,\\
\widehat{\psi}^{(9)} &= \widehat{\vartheta}^{(8)}=|\xi|^2\Big(\nu_3\Big(-|\xi|^2\lambda_0^2+\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\\
&\qquad -\lambda_2\Big(|\xi|^2\tau\beta_0\lambda_0+\big(\tau c_0-|\xi|^2\nu_1\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,\\
\widehat{\vartheta}^{(1)} &= -i|\xi|^2\Big(\tau\beta_0\big(\chi\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)-|\xi|^2\lambda_0\lambda_2\big)-\lambda_0\big(\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_2\nu_3\big)\Big)\,\frac{\xi_1}{D(|\xi|^2)}\,,\\
\widehat{\vartheta}^{(2)} &= -i|\xi|^2\Big(\tau\beta_0\big(\chi\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)-|\xi|^2\lambda_0\lambda_2\big)-\lambda_0\big(\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_2\nu_3\big)\Big)\,\frac{\xi_2}{D(|\xi|^2)}\,,\\
\widehat{\vartheta}^{(3)} &= -i|\xi|^2\Big(\tau\beta_0\big(\chi\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)-|\xi|^2\lambda_0\lambda_2\big)-\lambda_0\big(\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_2\nu_3\big)\Big)\,\frac{\xi_3}{D(|\xi|^2)}\,,\\
\widehat{\vartheta}^{(4)} &= \widehat{\vartheta}^{(5)}=\widehat{\vartheta}^{(6)}=0,\\
\widehat{\vartheta}^{(7)} &= -|\xi|^2\Big(-\tau\chi c_0\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\\
&\qquad -|\xi|^2\Big(-\tau\chi\beta_0\lambda_0+(\chi\nu_1+\lambda_2\nu_3)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,,\\
\widehat{\vartheta}^{(8)} &= -\nu_3\Big(-|\xi|^2\lambda_0^2+\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\big(|\xi|^2(\kappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\\
&\qquad -\lambda_0\Big(-|\xi|^2\tau\beta_0\lambda_0+\big(\tau c_0+|\xi|^2\nu_1\big)\big(|\xi|^2(\kappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\,\frac{|\xi|^2}{D(|\xi|^2)}\,,\\
\widehat{\vartheta}^{(9)} &= -|\xi|^2\Big(-|\xi|^2\lambda_0\lambda_2\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\\
&\qquad +\chi\Big(-|\xi|^2\lambda_0^2+\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)(|\xi|^2)}\,.
\end{aligned}
$$

Rewrite the first two equations of (B.5) as follows:

$$
\big[(\mu+\varkappa)|\xi|^2+\tau^2\rho_0\big]\widehat{u}^{(k)}+i\varkappa\big[\xi\times\widehat{\Phi}^{(k)}\big]=-C^{(k)}+i(\lambda+\mu)\eta^{(k)}\xi-i\lambda_0\xi\widehat{\varphi}^{(k)}+i\tau\beta_0\xi\widehat{\vartheta}^{(k)}, \tag{B.9}
$$

$$
\big[\gamma|\xi|^2+(2\varkappa+\tau^2I_0)\big]\widehat{\Phi}^{(k)}+i\varkappa\big[\xi\times\widehat{u}^{(k)}\big]=-F^{(k)}+i(\alpha+\beta)\zeta^{(k)}\xi. \tag{B.10}
$$

Taking cross product of ξ with both sides of (B.9) and employ the identity

$$
\big[\xi\times[\xi\times a]\big]=(\xi\cdot a)\xi-|\xi|^2a
$$

we get

$$\big[(\mu+\varkappa)|\xi|^2+\tau^2\rho_0\big]\,\big[\xi\times\widehat{u}^{(k)}\big]-i\varkappa|\xi|^2\widehat{\Phi}^{(k)}=-\big[C^{(k)}\times\xi\big]-\varkappa\zeta^{(k)}\xi,$$
$$\big[\gamma|\xi|^2+(2\varkappa+\tau^2I_0)\big]\widehat{\Phi}^{(k)}+i\varkappa\big[\xi\times\widehat{u}^{(k)}\big]=-F^{(k)}+i(\alpha+\beta)\zeta^{(k)}\xi.$$

Hence

$$\widehat{\Phi}^{(k)}(\xi)=\frac{i\varkappa(\zeta^{(k)}\varkappa\xi+[C^{(k)}\times\xi])-(F^{(k)}-i(\alpha+\beta)\zeta^{(k)}\xi)((\varkappa+\mu)|\xi|^2+\tau^2\rho_0)}{\Theta(\xi)}$$

with

$$\Theta(\xi)=\big(2\varkappa+\gamma|\xi|^2+\tau^2I_0\big)\big((\varkappa+\mu)|\xi|^2+\tau^2\rho_0\big)-\varkappa^2|\xi|^2.$$

Similarly, if we take cross product of ξ with both sides of (B.10),

$$\big[(\mu+\varkappa)|\xi|^2+\tau^2\rho_0\big]\widehat{u}^{(k)}+i\varkappa\big[\xi\times\widehat{\Phi}^{(k)}\big]=-C^{(k)}+i(\lambda+\mu)\eta^{(k)}\xi-i\lambda_0\xi\widehat{\varphi}^{(k)}+i\tau\beta_0\xi\widehat{\vartheta}^{(k)},$$
$$\big[\gamma|\xi|^2+(2\varkappa+\tau^2I_0)\big]\,\big[\xi\times\widehat{\Phi}^{(k)}\big]-i\varkappa|\xi|^2\widehat{u}^{(k)}=-[F^{(k)}\times\xi]-\varkappa\eta^{(k)}\xi,$$

we find

$$\widehat{u}^{(k)}(\xi)=\frac{1}{\Theta(\xi)}\,\Big[\Big(i(\lambda+\mu)\eta^{(k)}\xi-C^{(k)}-i\lambda_0\xi\widehat{\varphi}^{(k)}+i\tau\beta_0\xi\widehat{\vartheta}^{(k)}\Big)$$
$$\times\big(\gamma|\xi|^2+2\varkappa+\tau^2I_0\big)+i\varkappa\big([F^{(k)}\times\xi]+\varkappa\eta^{(k)}\xi\big)\Big].$$

Let k_2^2 and k_3^2 be the roots of the quadratic polynomial

$$Q(z)=(2\varkappa+\gamma z+\tau^2I_0)\big((\varkappa+\mu)z+\tau^2\rho_0\big)-\varkappa^2z=\gamma(\varkappa+\mu)z^2+q_1z+q_2, \tag{B.11}$$

where

$$q_1=\gamma\tau^2\rho_0+(\varkappa+\mu)(2\varkappa+\tau^2I_0)-\varkappa^2,\ \ q_2=\tau^4\rho_0I_0,$$

then

$$k_2^2=\frac{-q_1-\sqrt{q_1^2-4\gamma(\varkappa+\mu)q_2}}{2\gamma(\varkappa+\mu)}\,,\quad k_3^2=\frac{-q_1+\sqrt{q_1^2-4\gamma(\varkappa+\mu)q_2}}{2\gamma(\varkappa+\mu)}\,,$$
$$\frac{1}{Q(|\xi|^2)}=\frac{1}{\gamma(\varkappa+\mu)(k_2^2-k_3^2)}\Big(\frac{1}{|\xi|^2-k_2^2}-\frac{1}{|\xi|^2-k_3^2}\Big),$$

and

$$\widehat{\Phi}^{(k)}(\xi)=\frac{1}{Q(|\xi|^2)}\,\Big[i\varkappa\big(\zeta^{(k)}\varkappa\xi+[C^{(k)}\times\xi]\big)-\big(F^{(k)}-i(\alpha+\beta)\zeta^{(k)}\xi\big)\big((\varkappa+\mu)|\xi|^2+\tau^2\rho_0\big)\Big], \tag{B.12}$$

$$\widehat{u}^{(k)}(\xi)=\frac{1}{Q(|\xi|^2)}\,\Big[\big(-C^{(k)}+i(\lambda+\mu)\eta^{(k)}\xi-i\lambda_0\xi\widehat{\varphi}^{(k)}+i\tau\beta_0\xi\widehat{\vartheta}^{(k)}\big)\big(\gamma|\xi|^2+2\varkappa+\tau^2I_0\big)$$
$$+i\varkappa\big([F^{(k)}\times\xi]+\varkappa\eta^{(k)}\xi\big)\Big]. \tag{B.13}$$

From (B.12)–(B.13) we obtain

$$\widehat{\Phi}_j^{(m)}=i\varkappa\varepsilon_{jmk}\,\frac{\xi_k}{Q(|\xi|^2)}+\frac{(\varkappa^2+(\alpha+\beta)((\varkappa+\mu)|\xi|^2+\tau^2\rho_0))}{\alpha+\beta+\gamma}\cdot\frac{\xi_j\xi_m}{(|\xi|^2-k_1^2)Q(|\xi|^2)}\,,\ \ j,m=1,2,3,$$
$$\widehat{\Phi}_j^{(m+3)}=-\delta_{mj}\big[(\varkappa+\mu)|\xi|^2+\tau^2\rho_0\big]\,\frac{1}{Q(|\xi|^2)}\,,\ \ m,j=1,2,3,$$
$$\widehat{\Phi}_j^{(m)}=0,\ \ j=1,2,3;\ \ m=7,8,9,$$
$$\widehat{u}_j^{(m)}=\Big[\big(\gamma|\xi|^2+2\varkappa+\tau^2I_0\big)\Big(-1+|\xi|^2\xi_j\xi_m\Big(-\lambda_0^2\big((k|\xi|^2+a\tau^2)\chi+|\xi|^2\nu_3(\tau\beta_0+\nu_3)\big)$$
$$+\tau\chi\big(\xi_0+|\xi|^2a_0+\tau^2j_0\big)\beta_0\tau\beta_0-|\xi|^2\lambda_0\big(\tau\chi\beta_0\nu_1+\big(\chi\nu_1+\lambda_2(\tau\beta_0+\nu_3)\big)\tau\beta_0\big)\Big)\Big)\Big]\,\frac{1}{Q(|\xi|^2)}$$

$$+\Big[\Big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big)\Big(|\xi|^2k\xi_0\chi+a\xi_0\tau^2\chi-\tau^2\chi c_0^2+|\xi|^2k\tau^2\chi j_0$$
$$+a\tau^4\chi j_0-|\xi|^4k\lambda_0\lambda_2-a|\xi|^2\tau^2\lambda_0\lambda_2+|\xi|^4\chi\nu_1^2$$
$$+|\xi|^2\big(\tau c_0(\lambda_0-\lambda_2)+|\xi|^2(\lambda_0+\lambda_2)\nu_1\big)\nu_3+|\xi|^2(\xi_0+\tau^2 j_0)\nu_3^2$$
$$+|\xi|^2a_0\big(|\xi|^2k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)\Big)\Big]\,\frac{|\xi|^2\xi_j\xi_m}{D(|\xi|^2)Q(|\xi|^2)}\,,\quad j,m=1,2,3,$$

$$\widehat{u}_j^{(m+3)}=i\varkappa\varepsilon_{jmk}\,\frac{\xi_k}{Q(|\xi|^2)}\,,\quad j,m=1,2,3,$$

$$\widehat{u}_j^{(7)}=i\Big[\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big(\big(\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_2\nu_3\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\tau\beta_0$$
$$+\lambda_0\Big(|\xi|^2\tau^2\chi\beta_0^2+\big(|\xi|^2k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)$$
$$\times\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)-|\xi|^2\tau\chi\beta_0\tau\beta_0\Big)\Big)\Big]\,\frac{|\xi|^2\xi_j}{Q(|\xi|^2)}$$
$$-i\Big[\big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\big)\Big(\tau^2\chi c_0\beta_0+|\xi|^2\tau\beta_0(\chi\nu_1+\lambda_2\nu_3)$$
$$+\lambda_0\big(|\xi|^2k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)\Big)\Big]\,\frac{|\xi|^4\xi_j}{D(|\xi|^2)Q(|\xi|^2)}\,,\quad j=1,2,3,$$

$$\widehat{u}_j^{(8)}=-i\Big[\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big(\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\nu_3\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\tau\beta_0$$
$$+\lambda_0^2\Big(|\xi|^2\tau^2\beta_0^2+\big(|\xi|^2k+a\tau^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$+|\xi|^2\tau\beta_0(\nu_3-\tau\beta_0)-|\xi|^2\nu_3\tau\beta_0\Big)$$
$$-\lambda_0\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big(|\xi|^2\nu_1(\nu_3-\tau\beta_0-\tau c_0(\nu_3+\tau\beta_0))\big)\Big)\Big]\,\frac{|\xi|^2\xi_j}{Q(|\xi|^2)}$$
$$+i\Big[\big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\big)\Big(\big(|\xi|^2k+a\tau^2\big)\lambda_0^2+|\xi|^2\lambda_0\nu_1(\tau\beta_0-\nu_3)$$
$$+\tau\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\beta_0\nu_3+\tau c_0\lambda_0(\tau\beta_0+\nu_3)\Big)\Big]\,\frac{|\xi|^4\xi_j}{D(|\xi|^2)Q(|\xi|^2)},\quad j=1,2,3,$$

$$\widehat{u}_j^{(9)}=i\Big[\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big(\lambda_0\Big(-\tau\chi c_0\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$+|\xi|^2\big(-\tau\chi\beta_0\lambda_0+(\chi\nu_1+\lambda_0\nu_3)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big)\Big)$$
$$-\Big(-|\xi|^2\lambda_0\lambda_2\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$+\chi\Big(-|\xi|^2\lambda_0^2+(\xi_0+|\xi|^2a_0+\tau^2 j_0)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\tau\beta_0\Big)\Big]\,\frac{|\xi|^2\xi_j}{Q(|\xi|^2)}$$
$$+i\Big[\big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\big)\Big(\tau\beta_0\big(\chi(\xi_0+|\xi|^2a_0+\tau^2 j_0)-|\xi|^2\lambda_0\lambda_2\big)$$
$$-\lambda_0\big(-\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_0\nu_3\big)\Big)\Big]\,\frac{|\xi|^4\xi_j}{D(|\xi|^2)Q(|\xi|^2)}\,,\quad j=1,2,3.$$

From (B.4) it follows that the Fourier transform of the entries of the fundamental solution matrix have the form

$$\widehat{\Gamma}_{jm}=\bigg\{\bigg[\big(\gamma|\xi|^2+2\varkappa+\tau^2 I_0\big)\Big(-1+|\xi|^2\xi_j\xi_m\Big(-\lambda_0^2\big((k|\xi|^2+a\tau^2)\chi+|\xi|^2\nu_3(\tau\beta_0+\nu_3)\big)$$
$$+\tau\chi\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\beta_0\tau\beta_0-|\xi|^2\lambda_0\big(\tau\chi\beta_0\nu_1+\big(\chi\nu_1+\lambda_2(\tau\beta_0+\nu_3)\big)\tau\beta_0\big)\Big)\Big)\bigg]$$
$$+\Big[\big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\big)\Big(|\xi|^2k\xi_0\chi+a\xi_0\tau^2\chi-\tau^2\chi c_0^2+|\xi|^2k\tau^2\chi j_0$$
$$+a\tau^4\chi j_0-|\xi|^4k\lambda_0\lambda_2-a|\xi|^2\tau^2\lambda_0\lambda_2+|\xi|^4\chi\nu_1^2$$

$$+|\xi|^2\big(\tau c_0(\lambda_0-\lambda_2)+|\xi|^2(\lambda_0+\lambda_2)\nu_1\big)\nu_3+|\xi|^2(\xi_0+\tau^2 j_0)\nu_3^2$$
$$+|\xi|^2 a_0\big(|\xi|^2 k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)\Big)\Big]\,\frac{|\xi|^2\xi_j\xi_m}{D(|\xi|^2)}\Big\}\,\frac{1}{Q(|\xi|^2)}\,,\ \ j,m=1,2,3,$$

$$\widehat{\Gamma}_{j(m+3)}=i\varkappa\varepsilon_{jmk}\,\frac{\xi_k}{Q(|\xi|^2)}\,,\ \ j,m=1,2,3,$$

$$\widehat{\Gamma}_{j7}=i\Big[\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big(\big(\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_2\nu_3\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\tau\beta_0$$
$$+\lambda_0\Big(|\xi|^2\tau^2\chi\beta_0^2+\big(|\xi|^2k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)$$
$$\times\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)-|\xi|^2\tau\chi\beta_0\tau\beta_0\Big)\Big)\Big]\,\frac{|\xi|^2\xi_j}{Q(|\xi|^2)}$$
$$-i\Big[\big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\big)\Big(\tau^2\chi c_0\beta_0+|\xi|^2\tau\beta_0(\chi\nu_1+\lambda_2\nu_3)$$
$$+\lambda_0\big(|\xi|^2k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)\Big)\Big]\,\frac{|\xi|^4\xi_j}{D(|\xi|^2)Q(|\xi|^2)}\,,\ \ j=1,2,3,$$

$$\widehat{\Gamma}_{j8}=-i\Big[\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big(\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\nu_3\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\tau\beta_0$$
$$+\lambda_0^2\Big(|\xi|^2\tau^2\beta_0^2+\big(|\xi|^2k+a\tau^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$+|\xi|^2\tau\beta_0(\nu_3-\tau\beta_0)-|\xi|^2\nu_3\tau\beta_0\Big)$$
$$-\lambda_0\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big(|\xi|^2\nu_1(\nu_3-\tau\beta_0)-\tau c_0(\nu_3+\tau\beta_0)\big)\Big)\Big]\,\frac{|\xi|^2\xi_j}{Q(|\xi|^2)}$$
$$+i\Big[\Big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big)\Big(\big(|\xi|^2k+a\tau^2\big)\lambda_0^2+|\xi|^2\lambda_0\nu_1(\tau\beta_0-\nu_3)$$
$$+\tau\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\beta_0\nu_3+\tau c_0\lambda_0(\tau\beta_0+\nu_3)\Big)\Big]\,\frac{|\xi|^4\xi_j}{D(|\xi|^2)Q(|\xi|^2)}\,,\ \ j=1,2,3,$$

$$\widehat{\Gamma}_{j9}=i\Big[\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big(\lambda_0\Big(-\tau\chi c_0\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$+|\xi|^2\big(-\tau\chi\beta_0\lambda_0+(\chi\nu_1+\lambda_0\nu_3)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big)\Big)$$
$$-\Big(-|\xi|^2\lambda_0\lambda_2\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$+\chi\Big(-|\xi|^2\lambda_0^2+\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\tau\beta_0\Big)\Big]\,\frac{|\xi|^2\xi_j}{Q(|\xi|^2)}$$
$$+i\Big[\Big(\varkappa^2+(\lambda+\mu)\big(|\xi|^2\gamma+2\varkappa+\tau^2 I_0\big)\Big)\Big(\tau\beta_0\big(\chi\big(\xi_0+|\xi|^2a_0+\tau^2 j_0\big)-|\xi|^2\lambda_0\lambda_2\big)$$
$$-\lambda_0\big(-\tau\chi c_0+|\xi|^2\chi\nu_1+|\xi|^2\lambda_0\nu_3\big)\Big)\Big]\,\frac{|\xi|^4\xi_j}{D(|\xi|^2)Q(|\xi|^2)}\,,\ \ j=1,2,3,$$

$$\widehat{\Gamma}_{j+3,m}=i\varkappa\varepsilon_{jmk}\,\frac{\xi_k}{Q(|\xi|^2)}+\frac{(\varkappa^2+(\alpha+\beta)((\varkappa+\mu)|\xi|^2+\tau^2\rho_0))}{\alpha+\beta+\gamma}\cdot\frac{\xi_j\xi_m}{(|\xi|^2-k_1^2)Q(|\xi|^2)}\,,\ \ j,m=1,2,3,$$

$$\widehat{\Gamma}_{j+3,m+3}=-\delta_{mj}\big[(\varkappa+\mu)|\xi|^2+\tau^2\rho_0\big]\,\frac{1}{Q(|\xi|^2)}\,,\ \ j=1,2,3,\ \ m=1,\dots,6,$$

$$\widehat{\Gamma}_{7j}=-|\xi|^2\Big(-\tau^2\chi c_0\beta_0+|\xi|^2\tau\chi\beta_0\nu_1$$
$$+\lambda_0\Big(\big(|\xi|^2k+a\tau^2\big)\chi+|\xi|^2\nu_3(\tau\beta_0+\nu_3)\Big)\Big)\,\frac{\xi_j}{D(|\xi|^2)}\,,\ \ j=1,2,3,$$

$$\widehat{\Gamma}_{74}=\widehat{\Gamma}_{75}=\widehat{\Gamma}_{76}=0,$$

$$\widehat{\Gamma}_{77}=-|\xi|^2\Big(|\xi|^2\tau^2\chi\beta_0^2+\big(|\xi|^2k\chi+a\tau^2\chi+|\xi|^2\nu_3^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\,\frac{1}{D(|\xi|^2)}\,,$$

$$\begin{aligned}
\widehat{\Gamma}_{78} &= |\xi|^2\Big(\big(\tau c_0 - |\xi|^2\nu_1\big)\nu_3\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big) \\
&\quad + \lambda_0\Big(|\xi|^2\tau^2\beta_0^2 + |\xi|^2\tau\beta_0\nu_3 + \big(|\xi|^2k + a\tau^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,, \\
\widehat{\Gamma}_{79} &= |\xi|^2\Big(\tau\chi c_0\Big(|\xi|^2(\varkappa+\lambda+2\mu) - \tau^2\rho_0 - \tau\chi\beta_0\lambda_0 \\
&\quad + (\chi\nu_1+\lambda_0\nu_3)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}, \\
\widehat{\Gamma}_{8j} &= i|\xi|^2\Big(\lambda_2\Big(|\xi|^2k\lambda_0 + \tau\big(-\tau c_0\beta_0 + a\tau\lambda_0 + |\xi|^2\beta_0\nu_1\big)\Big) \\
&\quad + \Big(\tau\big(\xi_0 + |\xi|^2a_0 + \tau^2 j_0\big)\beta_0 - \lambda_0\big(\tau c_0 + |\xi|^2\nu_1\big)\Big)\nu_3\Big)\,\frac{\xi_j}{D(|\xi|^2)},\quad j=1,2,3, \\
\widehat{\Gamma}_{84} &= \widehat{\Gamma}_{85} = \widehat{\Gamma}_{86} = 0, \\
\widehat{\Gamma}_{87} &= \widehat{\psi}^{(7)} = |\xi|^2\Big(\lambda_2\Big(|\xi|^2\tau^2\beta_0^2 + \big(|\xi|^2k + a\tau^2\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big) \\
&\quad - \nu_3\Big(-|\xi|^2\tau\beta_0\lambda_0 + \big(\tau c_0 + |\xi|^2\nu_1\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,, \\
\widehat{\Gamma}_{88} &= \widehat{\psi}^{(8)} = -|\xi|^2\tau\beta_0\Big(\tau\big(\xi_0 + |\xi|^2a_0 + \tau^2 j_0\big)\beta_0 + \lambda_0\big(\tau c_0 - |\xi|^2\nu_1\big)\Big) \\
&\quad - \big(|\xi|^2k + a\tau^2\big)\Big(-|\xi|^2\lambda_0^2 + \big(\xi_0 + |\xi|^2a_0 + \tau^2 j_0\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big) \\
&\quad + \big(\tau c_0 + |\xi|^2\nu_1\big)\Big(|\xi|^2\tau\beta_0\lambda_0 + \big(\tau c_0 - |\xi|^2\nu_1\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\,\frac{1}{D(|\xi|^2)}\,, \\
\widehat{\Gamma}_{89} &= \widehat{\psi}^{(9)} = |\xi|^2\Big(\nu_3\big(-|\xi|^2\lambda_0^2 + \big(\xi_0 + |\xi|^2a_0 + \tau^2 j_0\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big) \\
&\quad - \lambda_2\Big(|\xi|^2\tau\beta_0\lambda_0 + \big(\tau c_0 - |\xi|^2\nu_1\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,, \\
\widehat{\Gamma}_{9j} &= \widehat{\vartheta}^{(1)} = -i|\xi|^2\Big(\tau\beta_0\big(\chi\big(\xi_0 + |\xi|^2a_0 + \tau^2 j_0\big) - |\xi|^2\lambda_0\lambda_2\big) \\
&\quad - \lambda_0\Big(\tau\chi c_0 + |\xi|^2\chi\nu_1 + |\xi|^2\lambda_2\nu_3\big)\Big)\,\frac{\xi_j}{D(|\xi|^2)}\,,\quad j=1,2,3, \\
\widehat{\Gamma}_{94} &= \widehat{\Gamma}_{95} = \widehat{\Gamma}_{96} = 0, \\
\widehat{\Gamma}_{97} &= -|\xi|^2\Big(-\tau\chi c_0\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big) \\
&\quad - |\xi|^2\Big(-\tau\chi\beta_0\lambda_0 + (\chi\nu_1+\lambda_2\nu_3)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,, \\
\widehat{\Gamma}_{98} &= -\nu_3\Big(-|\xi|^2\lambda_0^2 + \big(\xi_0 + |\xi|^2a_0 + \tau^2 j_0\big)\big(|\xi|^2(\kappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big) \\
&\quad - \lambda_0\Big(-|\xi|^2\tau\beta_0\lambda_0 + \big(\tau c_0 + |\xi|^2\nu_1\big)\big(|\xi|^2(\kappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\,\frac{|\xi|^2}{D(|\xi|^2)}\,, \\
\widehat{\Gamma}_{99} &= -|\xi|^2\Big(-|\xi|^2\lambda_0\lambda_2\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big) \\
&\quad + \chi\big(-|\xi|^2\lambda_0^2 + \big(\xi_0 + |\xi|^2a_0 + \tau^2 j_0\big)\big(|\xi|^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\Big)\Big)\,\frac{1}{D(|\xi|^2)}\,.
\end{aligned}$$

Remark B.1. To perform the inverse Fourier thransform, for simplicity, now we assume that the polynomials $P(z) = z^3 + p_1z^2 + p_2z + p_3$ and $Q(z) = \gamma(\varkappa+\mu)z^2 + q_1z + q_2$ defined in (B.7) and (B.11) respectively have distinct non-negative roots in z. Note that this assumption does not follow from conditions (2.22) and (2.23). Indeed, let $\tau > 0$ and choose λ_2 and c_0, which are not involved in conditions (2.22) and (2.23), sufficiently large. We will have $p_3 > 0$ in view of (B.8) and therefore the polynomial $P(z)$ will have at least one negative root without violating conditions (2.22) and (2.23).

In what follows we will find an explicit representation of the fundamental matrix in terms of elementary functions by inverting the Fourier transform

$$\Gamma(x)=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} e^{-i\xi\cdot x}\,\widehat{\Gamma}(\xi)\,d\xi. \tag{B.14}$$

To this end, let us note that the functions

$$\frac{1}{Q(|\xi|^2)}\,,\quad \frac{1}{D(|\xi|^2)}\,,\quad \frac{1}{D(|\xi|^2)Q(|\xi|^2)}\,,\quad \frac{1}{(|\xi|^2-k_1^2)Q(|\xi|^2)}$$

can be expanded as follows:

$$\begin{gathered}
\frac{1}{Q(|\xi|^2)}=\sum_{\alpha=2}^{3}\frac{c_\alpha^{(1)}}{|\xi|^2-k_\alpha^2}\,,\quad \frac{1}{(|\xi|^2-k_1^2)Q(|\xi|^2)}=\sum_{\alpha=1}^{3}\frac{c_\alpha^{(2)}}{|\xi|^2-k_\alpha^2}\,,\\
\frac{1}{D(|\xi|^2)}=c_0^{(3)}\,\frac{1}{|\xi|^2}+\sum_{\alpha=4}^{6}\frac{c_\alpha^{(3)}}{|\xi|^2-k_\alpha^2}\,,\quad \frac{1}{D(|\xi|^2)Q(|\xi|^2)}=c_0^{(4)}\,\frac{1}{|\xi|^2}+\sum_{\alpha=2}^{6}\frac{c_\alpha^{(4)}}{|\xi|^2-k_\alpha^2}\,,
\end{gathered} \tag{B.15}$$

where

$$\begin{gathered}
c_2^{(1)}=-c_3^{(1)}=\big(\gamma(\varkappa+\mu)(k_2^2-k_3^2)\big)^{-1},\quad c_1^{(2)}=\big(\gamma(\varkappa+\mu)(k_1^2-k_2^2)(k_1^2-k_3^2)\big)^{-1},\\
c_3^{(2)}=\big(\gamma(\varkappa+\mu)(k_3^2-k_1^2)(k_3^2-k_2^2)\big)^{-1},\quad c_0^{(3)}=-\big(d_0k_4^2k_5^2k_6^2\big)^{-1},\\
c_4^{(3)}=\big(d_0k_4^2(k_4^2-k_5^2)(k_4^2-k_6^2)\big)^{-1},\quad c_5^{(3)}=\big(d_0k_5^2(k_5^2-k_4^2)(k_5^2-k_6^2)\big)^{-1},\\
c_6^{(3)}=\big(d_0k_6^2(k_6^2-k_4^2)(k_6^2-k_5^2)\big)^{-1},\quad c_0^{(4)}=\Big(\gamma(\varkappa+\mu)d_0\prod_{j=2}^{6}k_j^2\Big)^{-1},\\
c_2^{(4)}=\Big(\gamma(\varkappa+\mu)d_0k_2^2(k_2^2-k_3^2)(k_2^2-k_4^2)(k_2^2-k_5^2)(k_2^2-k_6^2)\Big)^{-1},\\
c_3^{(4)}=\Big(\gamma(\varkappa+\mu)d_0k_3^2(k_3^2-k_4^2)(k_3^2-k_5^2)(k_3^2-k_5^2)(k_3^2-k_6^2)\Big)^{-1},\\
c_4^{(4)}=\Big(\gamma(\varkappa+\mu)d_0k_4^2(k_4^2-k_2^2)(k_4^2-k_3^2)(k_4^2-k_5^2)(k_4^2-k_6^2)\Big)^{-1},\\
c_5^{(4)}=\Big(\gamma(\varkappa+\mu)d_0k_5^2(k_5^2-k_2^2)(k_5^2-k_3^2)(k_5^2-k_4^2)(k_5^2-k_6^2)\Big)^{-1},\\
c_6^{(4)}=\Big(\gamma(\varkappa+\mu)d_0k_6^2(k_6^2-k_2^2)(k_6^2-k_3^2)(k_6^2-k_4^2)(k_6^2-k_5^2)\Big)^{-1}.
\end{gathered}$$

Let $k_0=0$. Choose k_p, $p=1,\dots,6$ so, that $-\pi<\arg(k_p)\leq 0$ and denote by K_p, $p=0,\dots,6$, the functions

$$K_p(x)=\frac{\exp(-ik_p|x|)}{4\pi|x|}\,,\ \ p=0,\dots,6. \tag{B.16}$$

Then K_p belongs to the space $\mathcal{S}'(\mathbb{R}^3)$ of tempered distributions in $\mathbb{R}^3$ and

$$(\Delta+k_p^2)K_p(x)=-\delta(x),\ \ \big(|\xi|^2-k_p^2\big)\widehat{K}_p(\xi)=1,\ \ \mathcal{F}^{-1}_{\xi\to x}\big(|\xi|^2\widehat{K}_p(\xi)\big)=\delta(x)+k_p^2K_p(x),\ \ p=0,\dots,6,$$

where $\widehat{K}_p(\xi)=\mathcal{F}_{x\to\xi}(K_p)(\xi)$.

From (B.15) we get

$$\begin{gathered}
\mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{Q(|\xi|^2)}\Big)=\sum_{p=2}^{3}c_p^{(1)}K_p(x),\quad \mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{(|\xi|^2-k_1^2)Q(|\xi|^2)}\Big)=\sum_{p=1}^{3}c_p^{(2)}K_p(x),\\
\mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{D(|\xi|^2)}\Big)=\sum_{p=0}^{6}c_p^{(3)}K_p(x),\quad \mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{D(|\xi|^2)Q(|\xi|^2)}\Big)=\sum_{p=0}^{6}c_p^{(4)}K_p(x),
\end{gathered} \tag{B.17}$$

where $c_p^{(3)}=0$, $p=1,2,3$, $c_1^{(4)}=0$.

To obtain the expression of the fundamental solution Γ, we have to evaluate the inverse Fourier transform (B.14) of the Fourier image $\widehat{\Gamma}$. Note that due to ellipticity of the operator $A(\partial,\tau)$ its fundamental solution Γ belongs to $C^\infty(\mathbb{R}^3\setminus\{0\})\cap L^1_{loc}(\mathbb{R}^3)$ and therefore terms containing $\delta(x)$ are canceled. Taking into consideration relations (B.16)–(B.17) and properties of the inverse Fourier transform operator we arrive at the following expressions for the components of the fundamental solution matrix:

$$\begin{aligned}
\Gamma_{jm}(x) &= \sum_{p=2}^{3} c_p^{(1)}\Big[(\gamma k_p^2+2\varkappa+\tau^2 I_0)\Big(-1+k_p^2\xi_j\xi_m\Big(-\lambda_0^2\big((kk_p^2+a\tau^2)\chi+k_p^2\nu_3(\tau\beta_0+\nu_3)\big)\\
&\qquad +\tau\chi(\xi_0+k_p^2a_0+\tau^2 j_0)\beta_0\tau\beta_0-k_p^2\lambda_0\big(\tau\chi\beta_0\nu_1+\big(\chi\nu_1+\lambda_2(\tau\beta_0+\nu_3)\big)\tau\beta_0\big)\Big)\Big)\Big]K_p(x)\\
&\quad -\sum_{p=0}^{6} c_p^{(1)}\Big[\big(\varkappa^2+(\lambda+\mu)(k_p^2\gamma+2\varkappa+\tau^2 I_0)\big)\Big(k_p^2k\xi_0\chi+a\xi_0\tau^2\chi-\tau^2\chi c_0^2+k_p^2k\tau^2\chi j_0\\
&\qquad +a\tau^4\chi j_0-k_p^4k\lambda_0\lambda_2-ak_p^2\tau^2\lambda_0\lambda_2+k_p^4\chi\nu_1^2+k_p^2\big(\tau c_0(\lambda_0-\lambda_2)+k_p^2(\lambda_0+\lambda_2)\nu_1\big)\nu_3\\
&\qquad +k_p^2(\xi_0+\tau^2 j_0)\nu_3^2+k_p^2a_0(k_p^2k\chi+a\tau^2\chi+k_p^2\nu_3^2)\Big)\Big]k_p^2\partial_j\partial_m K_p(x),\quad j,m=1,2,3,\\
\Gamma_{j\,(m+3)}(x) &= \varkappa\varepsilon_{jmk}\sum_{p=2}^{3} c_p^{(1)}\partial_k K_p(x),\quad j,m=1,2,3,\\
\Gamma_{j7}(x) &= \sum_{p=2}^{3} c_p^{(1)}\Big[(k_p^2\gamma+2\varkappa+\tau^2 I_0)\Big((\tau\chi c_0+k_p^2\chi\nu_1+k_p^2\lambda_2\nu_3)\\
&\qquad\quad \times\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\tau\beta_0+\lambda_0\Big(k_p^2\tau^2\chi\beta_0^2+(k_p^2k\chi+a\tau^2\chi+k_p^2\nu_3^2)\\
&\qquad\quad \times\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)-k_p^2\tau\chi\beta_0\tau\beta_0\Big)\Big)\Big]k_p^2\partial_j K_p(x)\\
&\qquad +\sum_{p=0}^{6} c_p^{(4)}\Big[\big(\varkappa^2+(\lambda+\mu)(k_p^2\gamma+2\varkappa+\tau^2 I_0)\big)\Big(\tau^2\chi c_0\beta_0+k_p^2\tau\beta_0(\chi\nu_1+\lambda_2\nu_3)\\
&\qquad\quad +\lambda_0(k_p^2k\chi+a\tau^2\chi+k_p^2\nu_3^2)\Big)\Big]k_p^4\partial_j K_p(x),\quad j=1,2,3,\\
\Gamma_{j8}(x) &= -\sum_{p=2}^{3} c_p^{(1)}\Big[(k_p^2\gamma+2\varkappa+\tau^2 I_0)\Big((\xi_0+k_p^2a_0+\tau^2 j_0)\nu_3\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\tau\beta_0\\
&\qquad +\lambda_0^2\Big(k_p^2\tau^2\beta_0^2+(k_p^2k+a\tau^2)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)+k_p^2\tau\beta_0(\nu_3-\tau\beta_0)-k_p^2\nu_3\tau\beta_0\Big)\\
&\qquad\quad -\lambda_0\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big(k_p^2\nu_1(\nu_3-\tau\beta_0)-\tau c_0(\nu_3+\tau\beta_0)\big)\Big)\Big]k_p^2\partial_j K_p(x)\\
&\qquad +\sum_{p=0}^{6} c_p^{(4)}\Big[\big(\varkappa^2+(\lambda+\mu)(k_p^2\gamma+2\varkappa+\tau^2 I_0)\big)\Big((k_p^2k+a\tau^2)\lambda_0^2+k_p^2\lambda_0\nu_1(\tau\beta_0-\nu_3)\\
&\qquad\quad +\tau(\xi_0+k_p^2a_0+\tau^2 j_0)\beta_0\nu_3+\tau c_0\lambda_0(\tau\beta_0+\nu_3)\Big)\Big]k_p^4\partial_j K_p(x),\quad j=1,2,3,\\
\Gamma_{j9}(x) &= \sum_{p=2}^{3} c_p^{(1)}\Big[(k_p^2\gamma+2\varkappa+\tau^2 I_0)\Big(\lambda_0\Big(-\tau\chi c_0\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\\
&\qquad\quad +k_p^2\big(-\tau\chi\beta_0\lambda_0+(\chi\nu_1+\lambda_0\nu_3)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big)\Big)\\
&\qquad -\Big(-k_p^2\lambda_0\lambda_2\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)
\end{aligned}$$

$$+\chi\Big(-k_p^2\lambda_0^2+(\xi_0+k_p^2a_0+\tau^2j_0)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)\Big)\tau\beta_0\Big)\Big]k_p^2\partial_jK_p(x)$$

$$+\sum_{p=0}^{6}c_p^{(4)}\Big[\big(\varkappa^2+(\lambda+\mu)(k_p^2\gamma+2\varkappa+\tau^2I_0)\big)\Big(\tau\beta_0\big(\chi(\xi_0+k_p^2a_0+\tau^2j_0)-k_p^2\lambda_0\lambda_2\big)$$

$$-\lambda_0(-\tau\chi c_0+k_p^2\chi\nu_1+k_p^2\lambda_0\nu_3)\Big)\Big]k_p^4\partial_jK_p(x),\ \ j=1,2,3,$$

$$\Gamma_{j+3,m}(x)=\sum_{p=2}^{3}c_p^{(1)}\varkappa\varepsilon_{jmk}\partial_kK_p(x)$$

$$+\sum_{p=1}^{3}c_p^{(2)}\,\frac{(\varkappa^2+(\alpha+\beta)((\varkappa+\mu)k_p^2+\tau^2\rho_0))}{\alpha+\beta+\gamma}\,\partial_j\partial_mK_p(x),\ \ j,m=1,2,3,$$

$$\Gamma_{j+3,m+3}(x)=-\sum_{p=2}^{3}c_p^{(1)}\delta_{mj}\big[(\varkappa+\mu)k_p^2+\tau^2\rho_0\big]K_p(x),\ \ j=1,2,3,\ \ m=1,\dots,6,$$

$$\Gamma_{7j}(x)=\sum_{p=0}^{6}c_p^{(3)}\Big[k_p^2\Big(-\tau^2\chi c_0\beta_0+k_p^2\tau\chi\beta_0\nu_1$$

$$+\lambda_0\big((k_p^2k+a\tau^2)\chi+k_p^2\nu_3(\tau\beta_0+\nu_3)\big)\Big)\Big]\partial_jK_p(x),\ \ j=1,2,3,$$

$$\Gamma_{74}(x)=\Gamma_{75}(x)=\Gamma_{76}(x)=0,$$

$$\Gamma_{77}(x)=-\sum_{p=0}^{6}c_p^{(3)}\Big[k_p^2\big(k_p^2\tau^2\chi\beta_0^2+(k_p^2k\chi+a\tau^2\chi+k_p^2\nu_3^2)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)\Big]K_p(x),$$

$$\Gamma_{78}(x)=\sum_{p=0}^{6}c_p^{(3)}\Big[k_p^2\Big((\tau c_0-k_p^2\nu_1)\nu_3\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$

$$+\lambda_0\big(k_p^2\tau^2\beta_0^2+k_p^2\tau\beta_0\nu_3+(k_p^2k+a\tau^2)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)\Big)\Big]K_p(x),$$

$$\Gamma_{79}(x)=\sum_{p=0}^{6}c_p^{(3)}\Big[k_p^2\Big(\tau\chi c_0\big(k_p^2(\varkappa+\lambda+2\mu)-\tau^2\rho_0-\tau\chi\beta_0\lambda_0$$

$$+(\chi\nu_1+\lambda_0\nu_3)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)\Big)\Big]K_p(x),$$

$$\Gamma_{8j}(x)=\sum_{p=0}^{6}c_p^{(3)}\Big[k_p^2\Big(\lambda_2\big(k_p^2k\lambda_0+\tau(-\tau c_0\beta_0+a\tau\lambda_0+k_p^2\beta_0\nu_1)\big)$$

$$+\big(\tau(\xi_0+k_p^2a_0+\tau^2j_0)\beta_0-\lambda_0(\tau c_0+k_p^2\nu_1)\big)\nu_3\Big)\Big]\partial_jK_p(x),\ \ j=1,2,3,$$

$$\Gamma_{84}(x)=\Gamma_{85}(x)=\Gamma_{86}(x)=0,$$

$$\Gamma_{87}(x)=\sum_{p=0}^{6}c_p^{(3)}\Big[k_p^2\Big(\lambda_2\big(k_p^2\tau^2\beta_0^2+(k_p^2k+a\tau^2)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)$$

$$-\nu_3\big(-k_p^2\tau\beta_0\lambda_0+(\tau c_0+k_p^2\nu_1)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)\Big)\Big]K_p(x),$$

$$\Gamma_{88}(x)=\sum_{p=0}^{6}c_p^{(3)}\Big[-k_p^2\tau\beta_0\big(\tau(\xi_0+k_p^2a_0+\tau^2j_0)\beta_0+\lambda_0(\tau c_0-k_p^2\nu_1)\big)$$

$$-(k_p^2k+a\tau^2)\big(-k_p^2\lambda_0^2+(\xi_0+k_p^2a_0+\tau^2j_0)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)$$

$$+(\tau c_0+k_p^2\nu_1)\big(k_p^2\tau\beta_0\lambda_0+(\tau c_0-k_p^2\nu_1)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)\Big]K_p(x),$$

$$\Gamma_{89}(x)=\sum_{p=0}^{6}c_p^{(3)}\Big[k_p^2\Big(\nu_3\big(-k_p^2\lambda_0^2+(\xi_0+k_p^2a_0+\tau^2j_0)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)\big)$$

$$- \lambda_2\big(k_p^2\tau\beta_0\lambda_0 + (\tau c_0 - k_p^2\nu_1)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big)\Big)\Big]K_p(x),$$

$$\Gamma_{9j}(x) = \sum_{p=0}^{6} c_p^{(3)}\Big[k_p^2\big(\tau\beta_0\big(\chi(\xi_0+k_p^2a_0+\tau^2j_0)-k_p^2\lambda_0\lambda_2\big)-\lambda_0\big(\tau\chi c_0+k_p^2\chi\nu_1+k_p^2\lambda_2\nu_3\big)\big)\Big]\partial_j K_p(x),$$
$$j=1,2,3,$$

$$\Gamma_{94}(x) = \Gamma_{95}(x) = \Gamma_{96}(x) = 0,$$

$$\Gamma_{97}(x) = \sum_{p=0}^{6} c_p^{(3)}\Big[-k_p^2\big(-\tau\chi c_0(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0)$$
$$- k_p^2(-\tau\chi\beta_0\lambda_0 + (\chi\nu_1+\lambda_2\nu_3)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big))\big)\Big]K_p(x),$$

$$\Gamma_{98}(x) = \sum_{p=0}^{6} c_p^{(3)}\Big[-\nu_3\big(-k_p^2\lambda_0^2 + (\xi_0+k_p^2a_0+\tau^2j_0)\big(k_p^2(\kappa+\lambda+2\mu)+\tau^2\rho_0\big)\big)$$
$$- \lambda_0\big(-k_p^2\tau\beta_0\lambda_0 + (\tau c_0 + k_p^2\nu_1)\big(k_p^2(\kappa+\lambda+2\mu)+\tau^2\rho_0\big)\big)\Big]k_p^2K_p(x),$$

$$\Gamma_{99}(x) = \sum_{p=0}^{6} c_p^{(3)}\Big[-k_p^2\Big(-k_p^2\lambda_0\lambda_2\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)$$
$$+ \chi\big(-k_p^2\lambda_0^2 + (\xi_0+k_p^2a_0+\tau^2j_0)\big(k_p^2(\varkappa+\lambda+2\mu)+\tau^2\rho_0\big)\big)\Big)\Big]K_p(x).$$

Acknowledgement

This research was supported by the Shota Rustaveli National Science Foundation (SRNSF) Grant # FR/286/5-101/13.

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(Received 17.02.2017)

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Memoirs on Differential Equations and Mathematical Physics

Volume 74, 2018, 79–92

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ASYMPTOTIC REPRESENTATIONS OF A CLASS OF REGULARLY VARYING SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH RAPIDLY AND REGULARLY VARYING NONLINEARITIES

Abstract. The asymptotic representations of solutions of a class of differential equations of the second order with rapidly and regularly varying nonlinearities are established.

2010 Mathematics Subject Classification. 34C41, 34 10.

Key words and phrases. Asymptotic representations of solutions, rapidly varying functions, regularly varying functions, $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the equation, regularly varying solutions.

რეზიუმე. მიღებულია გარკვეული კლასის სწრაფად და რეგულარულად ცვლადი არაწრფივობის მქონე მეორე რიგის დიფერენციალური განტოლებების ამონახსნთა ასიმპტოტური წარმოდგენები.

1 Introduction

We consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y'), \tag{1.1}$$

where $\alpha_0 \in \{-1; 1\}$, the functions $p : [a;\omega[\to]0;+\infty[$ $(-\infty < a < \omega \leq +\infty)$, and $\varphi_i : \Delta_{Y_i} \to]0;+\infty[$ $(i \in \{0,1\})$ are continuous, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either an interval $[y_i^0, Y_i[$ [1] or an interval $]Y_i; y_i^0]$. We suppose that φ_1 is a regularly varying function of index σ_1 as $y \to Y_1$ $(y \in \Delta_{Y_1})$ [7, pp. 10–15], and the function φ_0 is strongly monotonous on Δ_{Y_0}, twice continuously differentiable on Δ_{Y_0} and satisfies the following conditions:

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi_0(y) \in \{0, +\infty\}, \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi_0(y)\varphi_0''(y)}{(\varphi_0'(y))^2} = 1. \tag{1.2}$$

The second order differential equations with both power-type and exponential-type nonlinearities in the right-hand side play an important role in the qualitative theory of differential equations. Such equations have a lot of applications in practice. The fact takes place, for example, during investigations of distribution of electrostatic potential in a cylindrical plasma volume of combustion products. The corresponding equation can be reduced to the following one:

$$y'' = \alpha_0 p(t) e^{\sigma y} |y'|^{\lambda}.$$

This equation is of type (1.1), in which $\varphi_1(z) = |z|^{\lambda}$, $\varphi_0(z) = e^{\sigma z}$. Under some restrictions on the function $p(t)$, certain results for the asymptotic behavior of all regular solutions of that equation have been obtained in the papers by V. M. Evtukhov and N. G. Dric (see, for example, [2]).

The differential equation

$$y'' = \alpha_0 p(t)\varphi(y)$$

with a rapidly varying function φ has been considered in the paper by V. M. Evtukhov and V. M. Kharkov [3]. But in that paper the introduced class of solutions of the equation depends on the function φ that in most cases not useful for practical applications.

Equation (1.1) is a natural generalization of two previous ones.

The solution y of equation (1.1) defined on the interval $[t_0, \omega[\subset [a, \omega[$ is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution $(-\infty \leq \lambda_0 \leq +\infty)$ if the conditions

$$y^{(i)} : [t_0, \omega[\to \Delta_{Y_i}, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0 \tag{1.3}$$

are satisfied.

The goal of the present paper is to find for $\lambda_0 \in R \setminus \{0; 1\}$ the necessary and sufficient conditions for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equation (1.1) together with asymptotic representations of those solutions and their first order derivatives as $t \uparrow \omega$. According to the definition, such solutions are the regularly varying functions as $t \uparrow \omega$ of index $\frac{1}{\lambda_0 - 1}$.

2 Main results

First of all, we introduce some notations that will be necessary in the sequel. We consider

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty, \\ t - \omega, & \text{if } \omega < +\infty, \end{cases} \qquad \theta_1(y) = \varphi_1(y)|y|^{-\sigma_1},$$

[1] If $Y_i = +\infty$ (resp. $Y_i = -\infty$), we will take $y_i^0 > 0$ (resp. $y_i^0 < 0$).

$$\Phi_0(y)=\int\limits_{A_\omega}^{y}|\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\,dz,\quad A_\omega=\begin{cases} y_0^0, & \text{if } \int\limits_{y_0^0}^{Y_0}|\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\,dz=\pm\infty,\\ Y_0, & \text{if } \int\limits_{y_0^0}^{Y_0}|\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\,dz=const,\end{cases}$$

$$Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\Phi_0(y)}{y},\quad \Phi_1(y)=\int\limits_{A_\omega}^{y}\frac{\Phi_0(\tau)}{\tau}\,d\tau,\quad Z_1=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi_1(y),$$

$$F(t)=\frac{\Phi_1^{-1}(I_1(t))\Phi_1'(\Phi_1^{-1}(I_1(t)))}{\pi_\omega(t)I_1'(t)}.$$

If $y_1^0\lim\limits_{t\uparrow\omega}|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}=Y_1$, we put

$$I(t)=|\lambda_0-1|^{\frac{1}{1-\sigma_1}}\cdot y_1^0\cdot\int\limits_{B_\omega^0}^{t}\left|\pi_\omega(\tau)p(\tau)\theta_1\Big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}y_1^0\Big)\right|^{\frac{1}{1-\sigma_1}}d\tau,$$

$$B_\omega^0=\begin{cases} b, & \text{if } \int\limits_{b}^{\omega}\left|\pi_\omega(\tau)p(\tau)\theta_1\Big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}y_1^0\Big)\right|^{\frac{1}{1-\sigma_1}}d\tau=+\infty,\\ \omega, & \text{if } \int\limits_{b}^{\omega}\left|\pi_\omega(\tau)p(\tau)\theta_1\Big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}y_1^0\Big)\right|^{\frac{1}{1-\sigma_1}}d\tau=<+\infty,\end{cases}$$

$$I_1(t)=\int\limits_{B_\omega^1}^{t}\frac{\lambda_0 I(\tau)}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau,\quad B_\omega^1=\begin{cases} b, & \text{if } \int\limits_{b}^{\omega}\frac{\lambda_0 I(\tau)}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau=\pm\infty,\\ \omega, & \text{if } \int\limits_{b}^{\omega}\frac{\lambda_0|I(\tau)|}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau=const.\end{cases}$$

Here, the number $b\in[a,\omega[$ is chosen in such a way that $y_1^0|\pi_\omega(t))|^{\frac{1}{\lambda_0-1}}\in\Delta_{Y_1}$ as $t\in[b;\omega]$.

Note 2.1. From conditions (1.2) it follows that $Z_0,Z_1\in\{0,+\infty\}$ and

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\Phi_0''(y)\cdot\Phi_0(y)}{(\Phi_0'(y))^2}=1,\quad \lim_{\substack{\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\Phi_1''(y)\cdot\Phi_1(y)}{(\Phi_1'(y))^2}=1. \tag{2.1}$$

Note 2.2. The following statements are valid:

1)

$$\Phi_0(y)=(\sigma_1-1)\,\frac{\varphi_0^{\frac{\sigma_1}{\sigma_1-1}}(y)}{\varphi_0'(y)}\,[1+o(1)]\ \text{ when } \ y\to Y_0\quad(y\in\Delta_{Y_0})$$

and therefore

$$\operatorname{sign}(\varphi_0'(y)\Phi_0(y))=\operatorname{sign}(\sigma_1-1),\ \text{ when } \ y\in\Delta_{Y_0}.$$

2)

$$\Phi_1(y)=\frac{\Phi_0^2(y)}{y\Phi_0'(y)}\,[1+o(1)],\ \text{ when } \ y\to Y_0\quad(y\in\Delta_{Y_0})$$

and therefore

$$\operatorname{sign}(\Phi_1(y))=y_0^0\ \text{ when } \ y\in\Delta_{Y_0}.$$

Note that, by (2.1), the relation

$$\lim_{z\to Z_0}\frac{\Phi''(\Phi_1^{-1}(z))z}{(\Phi'(\Phi_1^{-1}(z)))^2}=\lim_{y\to Y_0}\frac{\Phi_1''(\Phi_1^{-1}(\Phi_1(y)))\Phi_1(y)}{(\Phi_1'(\Phi_1^{-1}(\Phi_1(y))))^2}=\lim_{y\to Y_0}\frac{\Phi_1''(y)\Phi_1(y)}{(\Phi_1'(y))^2}=1$$

is valid, and from the latter it follows that

$$\lim_{z\to Z_0}\frac{z\cdot\big(\frac{\Phi_1'(\Phi_1^{-1}(z))}{\Phi_1(\Phi_1^{-1}(z))}\big)'}{\frac{\Phi_1'(\Phi_1^{-1}(z))}{\Phi_1(\Phi_1^{-1}(z))}}=\lim_{y\to Z_0}\frac{\Phi_1''(\Phi_1^{-1}(z))z}{(\Phi_1'(\Phi_1^{-1}(z)))^2}-1=0.$$

Thus the function $\frac{\Phi_1'(\Phi_1^{-1}(z))}{\Phi_1(\Phi_1^{-1}(z))}$ is slowly varying as $z\to Z_0$. The function $\Phi_1^{-1}(z)$ is also slowly varying as an inverse to the rapidly varying function. So, we have the following

Note 2.3. The function $\Phi^{-1}(z)\cdot\frac{\Phi_1'(\Phi_1^{-1}(z))}{z}$ is slowly varying as $z\to Z_1$.

Let $Y\in\{0,\infty\}$, Δ_Y be some one-sided neighborhood of Y. The continuously differentiable function $L:\Delta_Y\to]0;+\infty[$ is called [6, p. 2–3] normalized slowly varying as $z\to Y$ $(z\in\Delta_Y)$, if

$$\lim_{\substack{y\to Y\\ y\in\Delta_Y}}\frac{yL'(y)}{L(y)}=0. \tag{2.2}$$

We say that a slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $\theta:\Delta_Y\to]0;+\infty[$ satisfies the condition S as $z\to Y$, if for any normalized slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $L:\Delta_{Y_i}\to]0;+\infty[$ the following equality takes place: $z\to Y$ $(z\in\Delta_Y)$

$$\theta(zL(z))=\theta(z)(1+o(1)).$$

We will consider that a slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $L_0:\Delta_Y\to]0;+\infty[$ satisfies the condition S_1 as $z\to Y$, if for any finite segment $[a;b]\subset]0;+\infty[$ the inequality

$$\limsup_{\substack{z\to Y\\ z\in\Delta_Y}}\Big|\ln|z|\cdot\Big(\frac{L(\lambda z)}{L(z)}-1\Big)\Big|<+\infty\ \text{ for all }\ \lambda\in[a;b]$$

is true.

Conditions S and S_1 are satisfied by the functions $\ln|y|$, $|\ln|y||^\mu$ $(\mu\in R)$, $\ln|\ln|y||$ and by many others.

The following theorem has been obtained.

Theorem 2.1. *Let for equation* (1.1) $\sigma_1\neq1$, *the function* $\theta_1(z)$ *satisfy the condition* S *as* $z\to Y_1$ $(z\in\Delta_{Y_1})$, *and the function* $\Phi_1^{-1}(z)\cdot\frac{\Phi_1'(\Phi_1^{-1}(z))}{z}$ *satisfy the condition* S_1 *as* $z\to Z_1$. *Then for the existence of* $P_\omega(Y_0,Y_1,\lambda_0)$*-solutions of equation* (1.1), *where* $\lambda_0\in\mathbb{R}\setminus\{0,1\}$, *it is necessary and, if*

$$I(t)I_1(t)\lambda_0(\sigma_1-1)>0\quad\textit{as}\ \ t\in]b,\omega[\,, \tag{2.3}$$

and the finite or infinite limits

$$\lim_{t\uparrow\omega}\pi_\omega(t)F'(t)\ \ \textit{and}\ \ \lim_{t\uparrow\omega}\frac{\sqrt{|\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}|}}{\ln|I_1(t)|}\quad\textit{exist,} \tag{2.4}$$

sufficient the fulfilment of the following conditions:

$$\pi_\omega(t)y_1^0y_0^0\lambda_0(\lambda_0-1)>0;\quad \pi_\omega(t)y_1^0\alpha_0(\lambda_0-1)>0\quad\textit{as}\ \ t\in[a;\omega[\,, \tag{2.5}$$

$$y_1^0\cdot\lim_{t\uparrow\omega}|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}=Y_1,\quad \lim_{t\uparrow\omega}I_1(t)=Z_1, \tag{2.6}$$

$$\lim_{t\uparrow\omega}\frac{I_1''(t)I_1(t)}{(I_1'(t))^2}=1,\quad \lim_{t\uparrow\omega}F(t)=\frac{\lambda_0-1}{\lambda_0}. \tag{2.7}$$

Moreover, for each such solution there take place the following asymptotic representations as $t\uparrow\omega$*:*

$$\Phi_1(y(t))=I_1(t)[1+o(1)],\quad \frac{\pi_\omega(t)y'(t)}{y(t)}=\frac{\lambda_0}{\lambda_0-1}\,[1+o(1)]. \tag{2.8}$$

Proof. Necessity. Let the function $y : [t_0, \omega[\to \Delta_{Y_0}$ be a $P_\omega(Y_0, Y_1, \lambda_0)$-solution of equation (1.1), for which $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$. Then, according to the properties of such solutions established by V. M. Evtukhov (see, e.g., [4]), we have

$$\frac{y(t)}{y'(t)} = \frac{\lambda_0}{(\lambda_0 - 1)\pi_\omega(t)} [1 + o(1)], \quad \frac{y''(t)}{y'(t)} = \frac{1}{(\lambda_0 - 1)\pi_\omega(t)} [1 + o(1)] \quad \text{as} \ \ t \in [a; \omega[\,. \tag{2.9}$$

Thus we obtain (2.5).

From (2.9), it also follows that $y'(t)$ as $t \in [a; \omega[$ is a regularly varying function of index $\frac{1}{\lambda_0 - 1}$. It can be represented in the form

$$y'(t) = |\pi_\omega(t)|^{\frac{1}{\lambda_0 - 1}} L_1(t) \quad \text{as} \ \ t \uparrow \omega, \tag{2.10}$$

where $L_1(t)$ is a regularly varying function as $t \uparrow \omega$ (see [7, p. 10]).

Hence, taking into account the properties of regularly varying functions [7, p. 10–15], we obtain the first of conditions (2.6).

From (1.1) and (2.9), it follows that as $t \uparrow \omega$

$$\frac{|y'(t)|^{1-\sigma_1} \operatorname{sign} y_1^0}{\varphi_0(y(t))} = \alpha_0(\lambda_0 - 1)\pi_\omega(t)\varphi_1(y'(t))|y'(t)|^{-\sigma_1} p(t)[1 + o(1)]. \tag{2.11}$$

Substituting (2.10) into (2.11), we get as $t \uparrow \omega$ the equality

$$\frac{y'(t)}{|\varphi_0(y(t))|^{\frac{1}{1-\sigma_1}}} = y_1^0 |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \Big|\pi_\omega(t)\theta_1\Big(|\pi_\omega(t)|^{\frac{1}{\lambda_0 - 1}} L_1(t) y_1^0\Big) p(t)\Big|^{\frac{1}{1-\sigma_1}} [1 + o(1)]. \tag{2.12}$$

In (2.10), the function L_1 is a slowly varying when its argument tends to Y_1. The function θ_1 satisfies the condition S. So, from (2.12), we have as $t \uparrow \omega$

$$\frac{y'(t)}{|\varphi_0(y(t))|^{\frac{1}{1-\sigma_1}}} = y_1^0 |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \Big|\pi_\omega(t)\theta_1\Big(|\pi_\omega(t)|^{\frac{1}{\lambda_0 - 1}} y_1^0\Big) p(t)\Big|^{\frac{1}{1-\sigma_1}} [1 + o(1)]. \tag{2.13}$$

Integrating the relation from t_0 to t, we get as $t \uparrow \omega$

$$\int\limits_{y(t_0)}^{y(t)} \frac{dz}{|\varphi_0(z)|^{\frac{1}{1-\sigma_1}}} = y_1^0 |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \int\limits_{t_0}^{t} \Big|\pi_\omega(\tau)\theta_1\Big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0 - 1}} y_1^0\Big) p(\tau)\Big|^{\frac{1}{1-\sigma_1}} [1 + o(1)]\, d\tau.$$

Taking into account the choice of A_ω, and that $y \to Y_0$ ($Y_0 \in \Delta_{Y_0}$), we have

$$\Phi_0(y(t)) = I(t)[1 + o(1)] \quad \text{as} \ \ t \uparrow \omega. \tag{2.14}$$

From (2.13) and (2.14), according to (2.9), we get

$$\frac{\pi_\omega(t) y'(t)}{y(t)} \cdot \frac{y(t)\Phi_0'(y(t))}{\Phi_0(y(t))} = \frac{\pi_\omega(t) I'(t)}{I(t)} [1 + o(1)] \quad \text{as} \ \ t \uparrow \omega. \tag{2.15}$$

By conditions (1.2), the function $\Phi_0(y)$ is rapidly varying as $y \to Y_0$ ($Y_0 \in \Delta_{Y_0}$). Thus from (2.15) it follows that

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) I'(t)}{I(t)} = \infty. \tag{2.16}$$

Taking into account equalities (2.14) and (2.9), we get

$$\frac{y'(t)\Phi_0(y(t))}{y(t)} = \frac{\lambda_0 I(t)}{(\lambda_0 - 1)\pi_\omega(t)} [1 + o(1)] \quad \text{as} \ \ t \uparrow \omega. \tag{2.17}$$

From here in the same way as equality (2.14) was obtained, we get the equality

$$\Phi_1(y(t)) = I_1(t)[1 + o(1)] \quad \text{as} \ \ t \uparrow \omega. \tag{2.18}$$

Thus, the correctness of the first representation of (2.8) and the first equality of (2.6) are justified. We get the correctness of the second representation of (2.8) as a result of division (2.17) by (2.18).

The second representation of (2.8) can be rewritten in the form

$$\frac{\pi_\omega(t)y'(t)}{y(t)}\cdot\frac{y(t)\Phi_1'(y(t))}{\Phi_1(y(t))}=\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}\,[1+o(1)]\quad\text{as}\ \ t\uparrow\omega.$$

With the help of (2.9), from the above representation we get

$$\frac{\lambda_0}{\lambda_0-1}\cdot\frac{y(t)\Phi_1'(y(t))}{\Phi_1(y(t))}=\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}\,[1+o(1)]\quad\text{as}\ \ t\uparrow\omega. \tag{2.19}$$

From conditions (1.2) imposed on the function $\varphi_0(y(t))$ and Note 2.2, we find that $\Phi_1(y)$ is a rapidly varying function as $y\to Y_0$ $(Y_0\in\Delta_{Y_0})$. Then, taking into account (2.19), we get

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}=\infty. \tag{2.20}$$

By (2.1), (2.15), (2.16) and (2.19), we have

$$\lim_{t\uparrow\omega}\frac{I_1''(t)I_1(t)}{(I_1'(t))^2}=\lim_{t\uparrow\omega}\frac{\frac{\pi_\omega(t)I'(t)}{I(t)}}{\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}}=\lim_{t\uparrow\omega}\frac{\frac{y(t)\Phi_0'(y(t))}{\Phi_0(y(t))}}{\frac{y(t)\Phi_1'(y(t))}{\Phi_1(y(t))}}=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\Phi_1''(y)\cdot\Phi_1(y)}{(\Phi_1'(y))^2}=1. \tag{2.21}$$

It means that the first of conditions (2.7) holds.

Note that the function $\Phi_1^{-1}(y)$ is slowly varying as $y\to Z_0$, since it is inverse to a rapidly varying as $y\to Y_0$ $(Y_0\in\Delta_{Y_0})$ function Φ_1. Taking into account this fact and (2.18), we get as $t\uparrow\omega$

$$y(t)=\Phi_1^{-1}(I_1(t))[1+o(1)].$$

The correctness of the second of conditions (2.6) follows from this fact.

Note that (2.19) can be written in the form

$$\frac{\lambda_0}{\lambda_0-1}\cdot\Phi_1^{-1}(I_1(t))\cdot\frac{\Phi_1'(\Phi_1^{-1}(I_1(t)))}{\Phi_1(\Phi_1^{-1}(I_1(t)))}=\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}\,[1+o(1)]\quad\text{as}\ \ t\uparrow\omega.$$

The validity the second of conditions (2.7) is justified, and hence the necessity is proved.

Sufficiency. Let us suppose that conditions (2.3)–(2.7) of the theorem take place.

We apply to equation (1.1) the transformation

$$\begin{cases}\Phi_1(y(t))=I_1(t)[1+v_1(x)],\\ \dfrac{y'(t)}{y(t)}=\dfrac{\lambda_0}{\lambda_0-1}\cdot\dfrac{1}{\pi_\omega(t)}\,[1+v_2(x)]\end{cases} \tag{2.22}$$

and reduce system (2.22) to the following system of differential equations:

$$\begin{cases}v_1'=\dfrac{I_1'(t)}{I_1(t)}\,[1+v_1]\cdot\Big(\dfrac{\lambda_0}{\lambda_0-1}\cdot F(t)\cdot M(t,v_1)[1+v_2]-1\Big),\\ v_2'=\dfrac{1}{\pi_\omega(t)}\,[1+v_2]\cdot\Big[Q(t,v_1,v_2)(1+v_1)^{\sigma_1-1}(1+v_2)^{\sigma_1-1}-\dfrac{1}{\lambda_0}-v_2\Big].\end{cases} \tag{2.23}$$

Here,

$$M(t,v_1)=\frac{Y(t,v_1)\,\frac{\Phi_1'}{\Phi_1}\,(\Phi_1^{-1}(Y(t,v_1)))}{\Phi_1^{-1}(I_1(t))\,\frac{\Phi_1'}{\Phi_1}\,(\Phi_1^{-1}(I_1(t)))}\,,\quad Y(t,v_1)=\Phi_1^{-1}\big(I_1(t)[1+v_1]\big),$$

$$Q(t,v_1,v_2)=\frac{N(t,v_1,v_2)}{\lambda_0}\,\bigg(F(t)\Big(\frac{\lambda_0}{\lambda_0-1}\Big)^2\cdot M(t,v_1)$$

$$\times\Big(\frac{L(t)}{1+L(t)}+F(t)M(t,v_1)\cdot\frac{\Phi_1''(Y(t,v_1))\Phi_1(Y(t,v_1))}{(\Phi_1(Y(t,v_1)))^2}\cdot\frac{1}{\frac{I_1(t)I_1''(t)}{(I_1'(t))^2}+G(t)}\Big)\Big)^{\sigma_1-1},$$

$$N(t,v_1,v_2)=\frac{\theta_1\Big(\frac{\lambda_0 Y(t,v_1)}{(\lambda_0-1)\pi_\omega(t)}\cdot[1+v_2]\Big)}{\theta_1(|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}\operatorname{sign}y_1^0)},\quad G(t)=\frac{I_1(t)}{\pi_\omega(t)I_1'(t)},\quad L(t)=\frac{I_1'(t)}{\pi_\omega(t)I_1''(t)}.$$

From the first of conditions (2.7) we have

$$\lim_{t\uparrow\omega}G(t)=0. \tag{2.24}$$

We have already proved that the function $\Phi_1^{-1}(z)$ is slowly varying as $z\to Z_1$. So, taking into account the second of conditions (2.6), we have

$$\lim_{t\uparrow\omega}Y(t,v_1)=Y_0 \ \text{ uniformly by } \ v_1:|v_1|<\frac{1}{2}\,. \tag{2.25}$$

By Note 2.3, we have

$$\lim_{t\uparrow\omega}M(t,v_1)=1 \ \text{ uniformly by } \ v_1:|v_1|<\frac{1}{2}\,. \tag{2.26}$$

From the second of conditions (2.7), we get

$$\lim_{t\uparrow\omega}F(t)=\frac{\lambda_0}{\lambda_0-1}\,. \tag{2.27}$$

Now, we can prove that

$$\lim_{t\uparrow\omega}N(t,v_1,v_2)=1 \ \text{ uniformly by } \ v_1:|v_1|<\frac{1}{2} \ \text{ and uniformly by } \ v_2:|v_2|<\frac{1}{2}\,. \tag{2.28}$$

From (2.26) and (2.27), it follows that

$$\lim_{t\uparrow\omega}\frac{\Bigg(\frac{\Phi_1^{-1}(I_1(t))}{|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}}}\Bigg)'\cdot\pi_\omega(t)}{\frac{\Phi_1^{-1}(I_1(t))}{|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}}}}=\lim_{t\uparrow\omega}\frac{1}{F(t)M(t,v_1)}-\frac{\lambda_0}{(1-\gamma_0)(\lambda_0-1)}=0.$$

Hence

$$\Bigg(\frac{\Phi_1^{-1}(I_1(t))}{|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}}}\Bigg)$$

is a normalized slowly varying function as $t\uparrow\omega$. Statement (2.28) follows from the above according to the fact that the function Φ_1^{-1} is slowly variable as its argument tends to Z_1, and the function θ_1 satisfies condition S.

Taking into account the first of conditions (2.7), we have

$$\lim_{t\uparrow\omega}L(t)=0. \tag{2.29}$$

From (2.24)–(2.29), it follows that

$$\lim_{t\uparrow\omega}Q(t,v_1,v_2)=\frac{1}{\lambda_0} \ \text{ uniformly by } \ v_1:|v_1|<\frac{1}{2} \ \text{ and uniformly by } \ v_2:|v_2|<\frac{1}{2}\,. \tag{2.30}$$

By (2.6), from the fact that the function Φ_1^{-1} is slowly varying as the argument tends to Z_1, it follows that there exists a number $t_0\in[a,\omega[$ such that

$$\Phi_1^{-1}\big(I_1(t)(1+v_1)\big)\in\Delta_{Y_0} \quad \text{as} \ \ t\in[t_0,\omega[\,, \ \ |v_1|\le\frac{1}{2}\,.$$

Further, we consider the system of differential equations (2.23) on the set

$$\Omega = [t_0, \omega[\times D, \quad D = \Big\{(v_1, v_2): \ |v_i| \le \frac{1}{2}, \ i = 1, 2\Big\}$$

and rewrite the system in the form

$$\begin{cases} v_1' = \dfrac{I_1'(t)}{I_1(t)} \Big[A_{11}(t)v_1 + A_{12}(x)v_2 + R_1(x, v_1, v_2) + R_2(x, v_1, v_2)\Big], \\ v_2' = \dfrac{1}{\pi_\omega(t)} \Big[A_{21}v_1 + A_{22}v_2 + R_3(x, v_1, z_2) + R_4(x, v_1, v_2)\Big], \end{cases} \tag{2.31}$$

where

$$A_{11}(t) = \frac{\lambda_0}{\lambda_0 - 1} F(t) - 1, \quad A_{12}(t) = \frac{\lambda_0}{\lambda_0 - 1} F(t),$$

$$R_1(t, v_1, v_2) = \frac{\lambda_0}{\lambda_0 - 1} F(t) - 1 + \frac{\lambda_0}{\lambda_0 - 1} F(t)(M(t, v_1) - 1)(1 + v_1 + v_2),$$

$$R_2(t, v_1, v_2) = \frac{\lambda_0}{\lambda_0 - 1} F(t)M(t, v_1)v_1v_2,$$

$$A_{21} = \frac{\sigma_1 - 1}{\lambda_0}, \quad A_{22} = \frac{\sigma_1 - 1 - \lambda_0}{\lambda_0},$$

$$R_3(t, v_1, z_2) = \frac{1}{\lambda_0}\big(1 + (\sigma_1 - 1)v_1 + \sigma_1 v_2\big) \cdot \big(\lambda_0 Q(t, v_1, v_2) - 1\big),$$

$$\begin{aligned} R_4(t, v_1, v_2) = Q(t, v_1, v_2)\Big[&(1 + \sigma_1 v_2)\big((1 + v_1)^{\sigma_1 - 1} - 1 - (\sigma_1 - 1)v_1\big) \\ &+ \sigma_1(\sigma_1 - 1)v_1v_2 + \big((1 + v_2)_1^{\sigma} - 1 - \sigma_1 v_2\big)(1 + v_1)_1^{\sigma}\Big] - v_2^2. \end{aligned}$$

By virtue of equalities (2.24)–(2.29), for $k \in \{2, 4\}$, we get

$$\lim_{|v_1| + |v_2| \to 0} \frac{R_k(t, v_1, v_2)}{|v_1| + |v_2|} = 0 \ \text{ uniformly by } t \text{ as } \ t \in [t_0, \omega[\,, \tag{2.32}$$

and for $k \in \{1, 3\}$,

$$\lim_{t \uparrow \omega} R_k(t, z_1, z_2) = 0 \ \text{ uniformly by } v_1, \ v_2 \text{ as } \ (v_1, v_2) \in D. \tag{2.33}$$

At the next stage of the proof we apply to system (2.31) the following transformation:

$$\begin{cases} v_1 = r_1, \\ v_2 = r_2 - H(t). \end{cases} \tag{2.34}$$

Here,

$$H(t) = \frac{\frac{\lambda_0}{\lambda_0 - 1} F(t) - 1}{\frac{\lambda_0}{\lambda_0 - 1} F(t)}.$$

By (2.27), we have

$$\lim_{t \uparrow \omega} H(t) = 0. \tag{2.35}$$

Thus get a system

$$\begin{cases} r_1' = \dfrac{I_1'(t)}{I_1(t)} \dfrac{\lambda_0}{\lambda_0 - 1} F(t)\big[r_2 + r_1r_2 + R(t; r_1; r_2)\big], \\ r_2' = \dfrac{1}{\pi_\omega(t)} \big[A_{21}r_1 + A_{22}r_2 + V_3(t, r_1, r_2) + V_4(t, r_1, r_2)\big], \end{cases} \tag{2.36}$$

where

$$R(t,r_1,r_2)=(M(t,r_1)-1)(1+r_1)(1+r_2-H(t)),$$
$$V_3(t,r_1,r_2)=R_4(t,r_1,r_2-H(t))-R_4(t,r_1,r_2)+\pi_\omega(t)H'(t)-A_{22}H(t)+R_3(t,r_1,r_2-H(t)),$$
$$V_4(t,r_1,r_2)=R_4(t,r_1,r_2).$$

Let us show that

$$\lim_{t\uparrow\omega}\pi_\omega(t)H'(t)=0. \tag{2.37}$$

According to condition (2.4) of the theorem, there exists the following finite or infinite limit

$$\lim_{t\uparrow\omega}\pi_\omega(t)H'(t).$$

Let

$$\pi_\omega(t)H'(t)=q(t) \ \text{ and } \ \lim_{t\uparrow\omega}q(t)\neq 0. \tag{2.38}$$

Then

$$H'(t)=\frac{q(t)}{\pi_\omega(t)}\,.$$

As a result of integration of the above equality from t_0 to t, we have

$$H(t)-H(t_0)=\int\limits_{t_0}^{t}\frac{q(\tau)}{\pi_\omega(\tau)}\,d\tau. \tag{2.39}$$

From (2.35) and (2.39), it follows that the integral $\int\limits_{t_0}^{\omega}\frac{q(\tau)}{\pi_\omega(\tau)}\,d\tau$ must be finite. But this is possible only if

$$\lim_{t\uparrow\omega}q(t)=0.$$

Thus, taking into account (2.38), we have proved the correctness of statement (2.37).

Owing to the properties of the function R_4, by (2.28) and (2.35), it follows that

$$\lim_{t\uparrow\omega}\big[R_4(t,r_1,r_2-H(t))-R_4(t,r_1,r_2)\big]=0 \ \text{ uniformly by } r_1 \text{ and } r_2 \text{ as } |r_i|<\frac{1}{2}\,, \ \ i=1,2. \tag{2.40}$$

Applying the transformation

$$\begin{cases} r_1=w_1,\\ r_2=\sqrt{|G(t(x))|}\,w_2, \end{cases} \tag{2.41}$$

where

$$x=\beta\ln|I_1(t)|, \quad \beta=\begin{cases} 1, & \text{if } \lim\limits_{t\uparrow\omega}I_1(t)=\infty,\\ -1, & \text{if } \lim\limits_{t\uparrow\omega}I_1(t)=0, \end{cases} \tag{2.42}$$

to system (2.31) and taking into account (2.3), we obtain the system

$$\begin{cases} w_1'=\beta\sqrt{|G(t(x))|}\,\Big[\dfrac{\lambda_0}{\lambda_0-1}\,F(t(x))w_2+\dfrac{\lambda_0}{\lambda_0-1}\,F(t(x))w_1w_2+W(x;w_1;w_2)\Big],\\ w_2'=\beta\sqrt{|G(t(x))|}\,\Big[\operatorname{sign}G(t(x))A_{21}w_1\\ \qquad +\Big(\sqrt{|G(t(x))|}\operatorname{sign}G(t(x))A_{22}(x)-\widetilde{N}(x)\Big)w_2+W_3(x,w_1,w_2)+W_4(x,w_1,w_2)\Big], \end{cases} \tag{2.43}$$

where

$$W(x;w_1;w_2)=\frac{\lambda_0}{\lambda_0-1}\,F(t(x))\cdot\frac{(M(t(x),w_1)-1)}{\sqrt{|G(t(x))|}}\,(1+w_1)\Big(1+\sqrt{|G(t(x))|}\,w_2-H(t(x))\Big),$$

$$W_3(x, w_1, w_2) = V_3\Big(t(x), w_1, \sqrt{|G(t(x))|}\, w_2\Big),$$
$$W_4(x, w_1, w_2) = V_4\Big(t(x), w_1, \sqrt{|G(t(x))|}\, w_2\Big),$$
$$\widetilde{N}(x) = \frac{\operatorname{sign}(G(t(x)))G'(t(x))I(t(x))}{2G(t(x))\sqrt{|G(t(x))|}\, I'(t(x))}\,.$$

Note that

$$\widetilde{N}(x) = \frac{\operatorname{sign}(G(t(x)))G'(t(x))I(t(x))}{2G(t(x))\sqrt{|G(t(x))|}\, I'(t(x))} = \frac{\operatorname{sign}(G(t(x)))G'(t(x))\pi_\omega(t(x))}{2\sqrt{|G(t(x))|}}\,.$$

At the same time, the equality

$$\frac{(M(t, w_1) - 1)}{\sqrt{|G(t(x))|}} = \ln|I_1(t)| \cdot \Big(\frac{\Phi_1^{-1}(I_1(t)[1+v_1])\psi(\Phi_1^{-1}(I_1(t)[1+v_1]))}{\Phi^{-1}(I_1(t))\psi(\Phi^{-1}(I_1(t)))} - 1\Big) \cdot \frac{\sqrt{|\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}|}}{\ln|I_1(t)|}$$

is true. Next, let us prove that

$$\lim_{t\uparrow\omega} \frac{\sqrt{|\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}|}}{\ln|I_1(t)|} = 0. \tag{2.44}$$

By de L'Hospital rule we have

$$\lim_{t\uparrow\omega} \frac{\sqrt{|\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}|}}{\ln|I_1(t)|} = -\frac{1}{2} \lim_{t\uparrow\omega} \frac{G'(t)\pi_\omega(t)}{\sqrt{|G(t)|}}\,.$$

The last limit has a finite or infinite boundary, since the second limit in (2.4) exists.
Now let us prove that

$$\lim_{t\uparrow\omega} \frac{G'(t)\pi_\omega(t)}{\sqrt{|G(t)|}} = 0. \tag{2.45}$$

According to condition (2.4), there exists the following finite or infinite limit

$$\lim_{t\uparrow\omega} \frac{G'(t)\pi_\omega(t)}{\sqrt{|G(t)|}}\,.$$

Suppose that

$$\frac{G'(t)\pi_\omega(t)}{\sqrt{|G(t)|}} = q_1(t) \quad \text{and} \quad \lim_{t\uparrow\omega} q_1(t) \neq 0. \tag{2.46}$$

Then

$$\frac{G'(t)}{\sqrt{|G(t)|}} = \frac{q_1(t)}{\pi_\omega(t)}\,.$$

As a result of integration of this equality from t_0 to t, we have

$$2\sqrt{|G(t)|} - 2\sqrt{|G(t_0)|} = \int_{t_0}^{t} \frac{q_1(\tau)}{\pi_\omega(\tau)}\, d\tau. \tag{2.47}$$

From (2.24) and (2.47), it follows that the integral $\int_{t_0}^{\omega} \frac{q_1(\tau)}{\pi_\omega(\tau)}\, d\tau$ must be finite. But this is possible only if

$$\lim_{t\uparrow\omega} q_1(t) = 0. \tag{2.48}$$

The last one is in contradiction with assumption (2.46). So, statement (2.44) is true.

Let us now prove that

$$\lim_{x\to+\infty}\widetilde{N}(x)=0. \tag{2.49}$$

The function $\Phi^{-1}(z)\cdot\frac{\Phi_1'(\Phi^{-1}(z))}{z}$ satisfies condition B, hence

$$\left|\ln|I_1(t(x))|\cdot\Big(\frac{\Phi_1^{-1}(I_1(t)[1+v_1])\psi(\Phi_1^{-1}(I_1(t)[1+v_1]))}{\Phi^{-1}(I_1(t))\psi(\Phi^{-1}(I_1(t)))}-1\Big)\right|<\infty.$$

From the above equality and statement (2.49), it follows that

$$\lim_{x\to+\infty}W(x;w_1;w_2)=0 \ \text{ uniformly towards } w_1 \text{ and } w_2 \text{ if } \ |w_i|<\frac{1}{2}\,, \ \ i=1,2. \tag{2.50}$$

Note that the characteristic equation of a matrix

$$\begin{pmatrix} 0 & \beta \\ \beta\,\mathrm{sign}(\lambda_0(\sigma_1-1))A_{21} & 0\end{pmatrix}$$

has the form

$$\mu^2-\frac{|\sigma_1-1|}{|\lambda_0|}=0.$$

This equation has no roots with real part equal to zero. Let us consider $\int\limits_{x_0}^{\infty}G(t(x))\,dx$. Taking into account the presentation $G(t(x))=\frac{I(t(x))}{\pi_\omega(t(x))I'(t(x))}$, we have

$$\int\limits_{x_0}^{\infty}G(t(x))\,dx=\int\limits_{x_0}^{\infty}\frac{I_1(t(x))}{\pi_\omega(t(x))I_1'(t(x))}\,dx=\int\limits_{t(x_0)}^{\omega}\frac{I_1(t)}{\pi_\omega(t)I_1'(t)}\,\frac{I_1'(t)}{I_1(t)}\,dt=\ln|\pi_\omega(t)|_{d_1}^{\omega}\longrightarrow\infty \quad \text{as } \ t\to\omega.$$

Since in some neighborhood of zero the inequality

$$\int\limits_{x_0}^{\infty}\sqrt{|G(t(x))|}\,dx\ge\mathrm{sign}(G(t(x)))\int\limits_{x_0}^{\infty}G(t(x))\,dx$$

takes place, it is true that

$$\int\limits_{x_0}^{\infty}\sqrt{|G(t(x))|}\,dx\longrightarrow+\infty.$$

We have got that for the system of differential equations (2.43) all conditions of Theorem 2.2 from [5] are fulfilled. According to this theorem, system (2.43) has a one-parameter family of solutions $\{w_i\}_{i=1}^2:[x_1,+\infty[\,\to\mathbb{R}^2$ $(x_1\ge x_0,\ x_0=\beta\ln|I_1(t_0)|)$ that tend to zero as $x\to+\infty$. By (2.42), (2.22) these solutions correspond to those solutions y of equation (1.1) that admit asymptotic representations (2.8) as $t\uparrow\omega$.

By representations (2.8) and inequality (2.3) it is clear that the obtained solutions are indeed the $P_\omega(Y_0,Y_1,\lambda_0)$-solutions. The theorem is proved completely. □

3 Illustration of the results

To illustrate the results obtained above, we consider the following differential equation for $t\in[2,+\infty[$

$$y''=\psi(t)\exp\big(\exp(|y|^a)-\exp(t^d)\big)|y|^{\sigma_0}|y'|^{\sigma_1}. \tag{3.1}$$

Here, $\sigma_0,\sigma_1\in R$, $\sigma_1>1$, $a,d\in\,]0,+\infty[$, the function $\psi:[2,+\infty[\,\to\,]0,+\infty[$ is continuous, regularly varying at infinity of index γ, $\gamma\in R$.

This equation is of type (1.1) for which

$$\alpha_0 = 1, \quad p(t) = \psi(t)\exp\big(-\exp(t^d)\big), \quad \varphi_0(y) = |y|^{\sigma_0}\exp\big(\exp(|y|^a)\big), \quad \varphi_1(y') = |y'|^{\sigma_1}.$$

Using the above proven theorem, let us investigate the question of the existence and asymptotic behavior as $t \to +\infty$ of $P_{+\infty}(\infty, Y_1, \lambda_0)$-solutions of equation (3.1) for which $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$.

In our case,

$$\pi_\omega(t) = t, \quad \theta_1(y) = 1.$$

Thus the function θ_1 satisfies condition S.

Taking into account the choice of $B^0_{+\infty}$, as $t \to +\infty$, we have

$$I(t) = |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \cdot y_0^1 \cdot \frac{\sigma_1 - 1}{d} \cdot t^{1-d+\frac{1}{1-\sigma_1}} \cdot |\psi(t)|^{\frac{1}{1-\sigma_1}} \cdot \exp\Big(\frac{\exp(t^d)}{\sigma_1 - 1} - t^d\Big)[1 + o(1)].$$

In the same way, as $t \to +\infty$, we have

$$I_1(t) = |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \cdot y_0^1 \cdot \Big(\frac{\sigma_1 - 1}{d}\Big)^2 \cdot t^{1-2d+\frac{1}{1-\sigma_1}} \cdot |\psi(t)|^{\frac{1}{1-\sigma_1}} \cdot \exp\Big(\frac{\exp(t^d)}{\sigma_1 - 1} - 2t^d\Big)[1 + o(1)].$$

In addition, in our case, since $Y_0 = \infty$, taking into account the choice of A^0_∞, we get

$$\Phi_0(y) = \frac{\sigma_1 - 1}{a} \cdot y^{\frac{\sigma_0}{\sigma_1 - 1}+1-a} \cdot \exp\Big(\frac{\exp(|y|^a)}{\sigma_1 - 1} - |y|^a\Big)[1 + o(1)] \quad \text{as } y \to \infty.$$

Similarly, we have

$$\Phi_1(y) = \Big(\frac{\sigma_1 - 1}{a}\Big)^2 \cdot y^{\frac{\sigma_0}{\sigma_1 - 1}+1-2a} \cdot \exp\Big(\frac{\exp(|y|^a)}{\sigma_1 - 1} - 2|y|^a\Big)[1 + o(1)] \quad \text{as } y \to \infty. \tag{3.2}$$

We have

$$\lim_{t\uparrow+\infty} F(t) = \frac{a}{d}\,. \tag{3.3}$$

From (3.3) and the second condition of (2.7), it follows that equation (3.1) may have only $P_{+\infty}(\infty, Y_1, \lambda_0)$-solutions with

$$\lambda_0 = \frac{d}{d-a}\,.$$

Taking into account asymptotic representations for functions I, I_1, Φ_0, Φ_1, Φ_1^{-1}, we get

$$\lim_{t\to+\infty} tF'(t) = 0.$$

So, the first condition of (2.4) is valid.

Note that

$$\frac{\sqrt{|\frac{\pi_\omega(t) I_1'(t)}{I_1(t)}|}}{\ln|I_1(t)|} = \sqrt{d(\sigma_1 - 1)}\,\frac{t^{\frac{d}{2}}}{\exp(\frac{t^d}{2})}\,[1 + o(1)] \quad \text{as } t \to \infty,$$

from which the second condition of (2.4) takes place.

At the same time,

$$\Phi_1^{-1}(y) \cdot \frac{\Phi_1'(\Phi_1^{-1}(y))}{y} = \frac{(\sigma_1 - 1)^2}{a}\ln y \cdot \big(\ln((\sigma_1 - 1)\ln y)\big)^{\frac{\frac{\sigma_0}{\sigma_1 - 1} - 2a + 1}{a}}[1 + o(1)] \quad \text{as } y \to \infty.$$

This means that condition S_1 is satisfied.

Thus, all conditions of Theorem 2.1 are satisfied. By virtue of this theorem, equation (3.1) may have only $P_{+\infty}(+\infty, +\infty, \frac{d}{d-a})$-solutions. From Theorem 2.1 it also follows that equation (3.1) has one-parameter family of $P_{+\infty}(+\infty, +\infty, \frac{d}{d-a})$-solutions.

Also, using the known asymptotic behavior of the function Φ_1^{-1}, it is easy to find that every $P_{+\infty}(+\infty,+\infty,\frac{d}{d-a})$-solution of equation (3.1) and its derivative satisfy the following asymptotic representations

$$(y(t))^{\frac{\sigma_0}{\sigma_1-1}+1-2a}\cdot\exp\Big(\frac{\exp(|y(t)|^a)}{\sigma_1-1}-2|y(t)|^a\Big)$$
$$=\Big|\frac{a}{d-a}\Big|^{\frac{1}{1-\sigma_1}}\cdot\Big(\frac{a}{d}\Big)^2\cdot t^{1-2d+\frac{1}{1-\sigma_1}}\cdot\psi^{\frac{1}{1-\sigma_1}}(t)\cdot\exp\Big(\frac{\exp(t^d)}{\sigma_1-1}-2t^d\Big)[1+o(1)]\quad\text{as }\ t\to+\infty,$$
$$y'(t)=\frac{y(t)}{t}\,[1+o(1)]\quad\text{as }\ t\to+\infty.$$

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(Received 31.10.2017)

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Memoirs on Differential Equations and Mathematical Physics

Volume 74, 2018, 93–111

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UNILATERAL CONTACT PROBLEMS FOR HOMOGENEOUS HEMITROPIC ELASTIC SOLIDS WITH A FRICTION

Abstract. In the present paper, we study a one-sided contact problem for a micropolar homogeneous elastic hemitropic medium with a friction. Here, on a part of the elastic medium surface with a friction, instead of a normal component of force stress there is prescribed the normal component of the displacement vector. We consider two cases, the so-called coercive case (when the elastic medium is fixed along some part of the boundary) and noncoercive case (without fixed parts). By using the Steklov–Poincaré operator, we reduce this problem to an equivalent boundary variational inequality. Based on our variational inequality approach, we prove the existence and uniqueness theorems for the weak solution. In the coercive case, the problem is unconditionally solvable, and the solution depends continuously on the data of the original problem. In the noncoercive case, we present in a closed-form the necessary condition for the existence of a solution of the contact problem. Under additional assumptions, this condition is also sufficient for the existence of a solution.

2010 Mathematics Subject Classification. 35J86, 49J40, 74M10, 74M15.

Key words and phrases. Elasticity theory, hemitropic solids, contact problem with a friction, boundary variational inequality.

რეზიუმე. წარმოდგენილ ნაშრომში შესწავლილია ცალმხრივი საკონტაქტო ამოცანა მიკროპოლარული, ერთგვაროვანი, ჰემიტროპული დრეკადი სხეულისთვის ხახუნის გათვალისწინებით. ამ შემთხვევაში დრეკადი სხეულის საზღვრის იმ ნაწილზე, სადაც ხახუნის ეფექტია გათვალისწინებული, ნაცვლად ძაბვის ნორმალური მდგენელისა მოცემულია გადაადგილების ნორმალური მდგენელი. განხილულია ორი შემთხვევა, კოერციტიული (როდესაც სხეული საზღვრის დადებითი ზომის გარკვეული ნაწილით ჩამაგრებულია) და არაკოერციტიული (როდესაც ასეთი ჩამაგრებები არ გვაქვს). სტეკლოვ-პუანკარეს ოპერატორის გამოყენებით განსახილველი ფიზიკური ამოცანა ეკვივალენტურად დაიყვანება სასაზღვრო ვარიაციულ უტოლობაზე. ვარიაციულ უტოლობათა ზოგადი თეორიის საფუძველზე შესწავლილია სუსტი ამონახსნების არსებობისა და ერთადერთობის საკითხი. კერძოდ, დადგენილია, რომ კოერციტიულ შემთხვევაში ამოცანა ამოხსნადია ცალსახად და უპირობოდ, ხოლო არაკოერციტიულ შემთხვევაში ცხადი სახით იწერება ამონახსნის არსებობის აუცილებელი პირობა, რომელიც გარკვეულ დამატებით მოთხოვნებში წარმოადგენს ამონახსნის არსებობის საკმარის პირობასაც.

1 Introduction

In the present paper, we investigate the one-sided contact problem for a homogeneous hemitropic elastic medium with a friction. In the considered model of the theory of elasticity, as distinct from the classical theory, every elementary medium particle undergoes both displacement and rotation. In this case, all mechanical values are expressed in terms of the displacement and rotation vectors.

In their works [2] and [3], E. Cosserat and F. Cosserat created and presented the model of a solid medium in which every material point has six degrees of freedom, three of which are defined by displacement components and the other three by the components of rotation (for the history of the model of elasticity see [5,28,30,31,34,39,40] and the references therein).

A micropolar medium, not possessing symmetry with respect to the inversion, is called a hemitropic or noncentrosymmetric medium.

Improved mathematical models describing hemitropic properties of elastic materials have been obtained and considered in [29] and [1]. The main equations of that model are interrelated and generate a matrix second order differential operator of dimension 6×6. Particular problems for solid media of the hemitropic theory of elasticity have been considered in [35,36,39] and [40]. The basic boundary value problems and also the transmission problems of the hemitropic theory of elasticity with the use of the potential method for smooth and non-smooth Lipschitz domains were studied in [35], the one-sided contact problems of statics of the hemitropic theory of elasticity free from friction were investigated in [16, 18, 20], and the contact problems of statics and dynamics with a friction were considered in [8–15,17,19,21–24]. Analogous one-sided problems of classical linear theory of elasticity have been considered in many works and monographs (see [4,6,7,26,27,41] and the references therein).

In the present work, we present the basic equations of statics of the elasticity theory for homogeneous hemitropic media in a vector-matrix form, introduce the generalized stress operator and a quadratic form of potential energy. Then we describe mathematical model of boundary conditions which show the contact between a hemitropic medium and a solid body with regard for the friction effect. We will consider the case, where some part of the elastic medium boundary is fixed mechanically. The problem is reduced equivalently to the boundary variational inequality, the question on the existence and uniqueness of a weak solution of the initial problem is treated, and a continuous Lipschitz dependence of the solution on the data of the problem is investigated. Further, we will investigate more complicated cases, where friction is considered on the whole medium boundary. In such cases, the corresponding mathematical problem is, in general, unsolvable. The necessary conditions of solvability are established and the sufficient conditions for the existence of a solution are formulated explicitly.

2 Basic equations and Green's formulas

2.1 Basic equations

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with a C^∞-smooth boundary $S = \partial\Omega$, $\overline{\Omega} = \Omega \cup S$. The domain Ω is assumed to be filled with a homogeneous hemitropic material.

The basic equilibrium equations in the hemitropic theory of elasticity written in components of the displacement and rotation vectors are of the form

$$\begin{aligned}
&(\mu+\alpha)\Delta u(x) + (\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div} u(x) + (\varkappa+\nu)\Delta\omega(x)\\
&\qquad + (\delta+\varkappa-\nu)\operatorname{grad}\operatorname{div}\omega(x) + 2\alpha\operatorname{curl}\omega(x) + \rho F(x) = 0,\\
&(\varkappa+\nu)\Delta u(x) + (\delta+\varkappa-\nu)\operatorname{grad}\operatorname{div} u(x) + 2\alpha\operatorname{curl} u(x) + (\gamma+\varepsilon)\Delta\omega(x)\\
&\qquad + (\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\omega(x) + 4\nu\operatorname{curl}\omega(x) - 4\alpha\omega(x) + \rho\Psi(x) = 0,
\end{aligned} \tag{2.1}$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator, $\partial_j = \partial/\partial x_j$, $u = (u_1, u_2, u_3)^\top$ is the displacement vector, $\omega = (\omega_1, \omega_2, \omega_3)^\top$ is the vector of rotation, $F = (F_1, F_2, F_3)^\top$ and $\Psi = (\Psi_1, \Psi_2, \Psi_3)^\top$ are the mass force and mass moment calculated per unit of mass, ρ is density of the elastic medium, and α, β, γ, δ, λ, μ, ν, $\varkappa$ and ε are elastic constants (see [1,36]). Here and in what follows, the symbol $(\,\cdot\,)^\top$ denotes transposition.

We introduce a matrix differential operator corresponding to the left-hand side of system (2.1):

$$L(\partial)=\begin{bmatrix} L^{(1)}(\partial) & L^{(2)}(\partial)\\ L^{(3)}(\partial) & L^{(4)}(\partial)\end{bmatrix}_{6\times 6},$$

$$L^{(1)}(\partial):=(\mu+\alpha)\Delta I_3+(\lambda+\mu-\alpha)Q(\partial),$$

$$L^{(2)}(\partial)=L^{(3)}(\partial):=(\varkappa+\nu)\Delta I_3+(\delta+\varkappa-\nu)Q(\partial)+2\alpha R(\partial),$$

$$L^{(4)}(\partial):=\big[(\gamma+\varepsilon)\Delta-4\alpha\big]I_3+(\beta+\gamma-\varepsilon)Q(\partial)+4\nu R(\partial),$$

where I_k is the unit $k\times k$-matrix and

$$Q(\partial)=\big[Q_{kj}(\partial)\big]_{3\times 3},\quad Q_{kj}(\partial)=\partial_k\,\partial_j,\quad R(\partial)=\begin{bmatrix} 0 & -\partial_3 & \partial_2\\ \partial_3 & 0 & -\partial_1\\ -\partial_2 & \partial_1 & 0\end{bmatrix}.$$

The system of equations (2.1) can be rewritten in the matrix form

$$L(\partial)U(x)+\mathcal{G}(x)=0,\ \ x\in\Omega,$$

where $U=(u,\omega)^\top$ and $\mathcal{G}=(\rho F,\rho\Psi)^\top$.

By $T(\partial,n)$ we denote the generalized stress operator of dimension 6×6 (see [36]):

$$T(\partial,n)=\begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n)\\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n)\end{bmatrix},\quad T^{(j)}=\big[T^{(j)}_{pq}\big]_{3\times 3},\ \ j=\overline{1,4},$$

where

$$T^{(1)}_{pq}(\partial,n):=(\mu+\alpha)\delta_{pq}\partial_n+(\mu-\alpha)n_q\partial_p+\lambda n_p\partial_q,$$

$$T^{(2)}_{pq}(\partial,n):=(\varkappa+\nu)\delta_{pq}\partial_n+(\varkappa-\nu)n_q\partial_p+\delta n_p\partial_q-2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}n_k,$$

$$T^{(3)}_{pq}(\partial,n):=(\varkappa+\nu)\delta_{pq}\partial_n+(\varkappa-\nu)n_q\partial_p+\delta n_p\partial_q,$$

$$T^{(4)}_{pq}(\partial,n):=(\gamma+\varepsilon)\delta_{pq}\partial_n+(\gamma-\varepsilon)n_q\partial_p+\beta n_p\partial_q-2\nu\sum_{k=1}^{3}\varepsilon_{pqk}n_k.$$

Here, $n(x)=(n_1(x),n_2(x),n_3(x))$ denotes the outward (with respect to Ω) unit normal vector at the point $x\in S$, and $\partial_n=\partial/\partial n$ is the normal derivative in the direction of the vector n. The six-component generalized stress vector has the form

$$T(\partial,n)U=(\mathcal{T}U,\mathcal{M}U)^\top,$$

where $\mathcal{T}U:=T^{(1)}u+T^{(2)}\omega$ is the force stress vector and $\mathcal{M}U:=T^{(3)}u+T^{(4)}\omega$ is the moment stress vector.

2.2 Green's formulas

For the real-valued vector functions $U=(u,\omega)^\top$ and $U'=(u',\omega')^\top$ of the class $[C^2(\overline{\Omega})]^6$ the following Green's formula [36]

$$\int\limits_{\Omega}\big[L(\partial)U\cdot U'+E(U,U')\big]\,dx=\int\limits_{S}\big\{T(\partial,n)U\big\}^+\cdot\{U'\}^+\,dS \tag{2.2}$$

is valid, where $\{\,\cdot\,\}^+$ denotes the trace operator on S from Ω, and $E(\,\cdot\,,\,\cdot\,)$ is a bilinear form defined by the equality

$$\begin{aligned}&E(U,U')=E(U',U)\\&=\sum_{p,q=1}^{3}\Big\{(\mu+\alpha)u'_{pq}u_{pq}+(\mu-\alpha)u'_{pq}u_{qp}+(\varkappa+\nu)\big(u'_{pq}\omega_{pq}+\omega'_{pq}u_{pq}\big)+(\varkappa-\nu)\big(u'_{pq}\omega_{qp}+\omega'_{pq}u_{qp}\big)\\&\qquad+(\gamma+\varepsilon)\omega'_{pq}\omega_{pq}+(\gamma-\varepsilon)\omega'_{pq}\omega_{qp}+\delta\big(u'_{pp}\omega_{qq}+\omega'_{qq}u_{pp}\big)+\lambda u'_{pp}u_{qq}+\beta\omega'_{pp}\omega_{qq}\Big\},\end{aligned}$$

where u_{pq} and ω_{pq} are the so-called tensors of deformation and torsion-bending for the hemitropic media,

$$u_{pq}=u_{pq}(U)=\partial_p u_q-\sum_{k=1}^{3}\varepsilon_{pqk}\omega_k,\quad \omega_{pq}=\omega_{pq}(U)=\partial_p\omega_q,\quad p,q=1,2,3. \tag{2.3}$$

Here and in the sequel, by $a\cdot b$ we denote the scalar product of two vectors $a,b\in\mathbb{R}^m: a\cdot b=\sum\limits_{j=1}^{m}a_jb_j$.

Under certain assumptions on elastic constants (see [1, 10, 23]), specific energy of deformation $E(U,U)$ is a positive definite quadratic form with respect to $u_{pq}(U)$ and $\omega_{pq}(U)$, i.e., there exists a positive number $C_0>0$, depending only on the elastic constants, such that

$$E(U,U)\geq C_0\sum_{p,q=1}^{3}\big[u_{pq}^2+\omega_{pq}^2\big].$$

The following assertion describes the null space of the energy quadratic form $E(U,U)$ (see [36]).

Lemma 2.1. *Let $U=(u,\omega)^\top\in[C^1(\overline{\Omega})]^6$ and $E(U,U)=0$ in Ω. Then*

$$u(x)=[a\times x]+b,\ \ \omega(x)=a,\ \ x\in\Omega,$$

where a and b are arbitrary three-dimensional constant vectors and $[\,\cdot\,\times\,\cdot\,]$ denotes the cross product of two vectors.

Vectors of the type $([a\times x]+b,a)$ are called generalized rigid vectors. We observe that a generalized rigid displacement vector vanishes, i.e., $a=b=0$ if it is zero at a single point.

Throughout the paper, $L_p(\Omega)$ $(1\leq p\leq\infty)$, $L_2(\Omega)=H^0(\Omega)$ and $H^s(\Omega)=H^s_2(\Omega)$, $s\in\mathbb{R}$, denote, respectively, the Lebesgue and Bessel potential spaces (see, e.g., [32,42]). The corresponding norms we denote by the symbols $\|\cdot\|_{L_p(\Omega)}$ and $\|\cdot\|_{H^s(\Omega)}$. By $\mathcal{D}(\Omega)$ we denote the class of $C^\infty(\Omega)$ functions with support in the domain Ω. If M is an open proper part of the manifold $\partial\Omega$, i.e., $M\subset\partial\Omega$, $M\neq\partial\Omega$, then by $H^s(M)$ we denote the restriction of the space $H^s(\partial\Omega)$ on M, $H^s(M):=\{r_{_M}\varphi:\ \varphi\in H^s(\partial\Omega)\}$, where $r_{_M}$ stands for the restriction operator on the set M. Further, let $\widetilde{H}^s(M):=\{\varphi\in H^s(\partial\Omega):\operatorname{supp}\varphi\subset\overline{M}\}$.

From the positive definiteness of the energy form $E(\,\cdot\,,\,\cdot\,)$ with respect to the variables (2.3) it follows that

$$B(U,U):=\int\limits_{\Omega}E(U,U)\,dx\geq 0. \tag{2.4}$$

Moreover, there exist positive constants c_1 and c_2, depending only on the material parameters, such that the following Korn's type inequality (see [7, Part I, § 12])

$$B(U,U)\geq c_1\|U\|^2_{[H^1(\Omega)]^6}-c_2\|U\|^2_{[H^0(\Omega)]^6} \tag{2.5}$$

holds for an arbitrary real-valued vector function $U\in[H^1(\Omega)]^6$.

Remark 2.2. If $U\in[H^1(\Omega)]^6$ and on some part $S^*\subset\partial\Omega$ the trace $\{U\}^+$ vanishes, i.e., $r_{_{S^*}}\{U\}^+=0$, we have the strict Korn's inequality $B(U,U)\geq C\|U\|^2_{[H^1(\Omega)]^6}$ with some positive constant $C>0$ which does not depend on the vector U. This follows from (2.5) and the fact that in this case $B(U,U)>0$ for $U\neq 0$ (see, e.g., [33, Ch. 2]; [37, Ch. 3, p. 193]).

Remark 2.3. By the standard limiting arguments, Green's formula (2.2) can be extended to the Lipschitz domains and to the vector function $U \in [H^1(\Omega)]^6$ with $L(\partial)U \in [L_2(\Omega)]^6$ and $U' \in [H^1(\Omega)]^6$ (see [32, 37]),

$$\int\limits_{\Omega} \big[L(\partial)U \cdot U' + E(U,U')\big]\,dx = \big\langle \{T(\partial,n)U\}^+, \{U'\}^+ \big\rangle_{\partial\Omega}\,, \tag{2.6}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega}$ denotes duality between the spaces $[H^{-1/2}(\partial\Omega)]^6$ and $[H^{1/2}(\partial\Omega)]^6$ which generalizes the usual inner product in the space $[L_2(\partial\Omega)]^6$. By virtue of this relation, the generalized trace of the stress operator $\{T(\partial,n)U\}^+ \in [H^{-1/2}(\partial\Omega)]^6$ is determined correctly.

3 Contact problems with a friction

3.1 Pointwise and variational formulation of the contact problem

Let the boundary S of the domain Ω be divided into two open, connected and non overlapping parts S_1 and S_2 of positive measure, $S = \overline{S}_1 \cup \overline{S}_2$, $S_1 \cap S_2 = \varnothing$. Assume that the hemitropic elastic body occupying the domain Ω is in contact with another rigid body along the subsurface S_2.

Definition 3.1. A vector function $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ is said to be a weak solution of the equation

$$L(\partial)U + \mathcal{G} = 0, \ \ \mathcal{G} \in [L_2(\Omega)]^6 \tag{3.1}$$

in the domain Ω if

$$B(U,\Phi) = \int\limits_{\Omega} \mathcal{G}\cdot\Phi\;dx \quad \forall\,\Phi \in [\mathcal{D}(\Omega)]^6,$$

where the bilinear form $B(\,\cdot\,,\,\cdot\,)$ is given by formula (2.4).

For the normal and tangential components of the force stress vector we will use, respectively, the following notation:

$$(\mathcal{T}U)_n := \mathcal{T}U\cdot n, \quad (\mathcal{T}U)_s := \mathcal{T}U - n(\mathcal{T}U)_n.$$

Further, let $\mathcal{G} = (\rho F, \rho\Psi)^\top \in [L_2(\Omega)]^6$, $\varphi \in [H^{-1/2}(S_2)]^3$, $f \in H^{1/2}(S_2)$, $g \in L_\infty(S_2)$, $g \geq 0$.

Consider the following contact problem of statics with a friction.

Problem A. Find a vector function $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ which is a weak solution of equation (3.1) and satisfies the inclusion $r_{S_2}\{(\mathcal{T}U)_s\}^+ \in [L_\infty(S_2)]^3$ and the following conditions:

$$r_{S_1}\{U\}^+ = 0 \ \text{ on } \ S_1, \tag{3.2}$$

$$r_{S_2}\{\mathcal{M}U\}^+ = \varphi \ \text{ on } \ S_2, \tag{3.3}$$

$$r_{S_2}\{u_n\}^+ = f \ \text{ on } \ S_2, \tag{3.4}$$

if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| < g$, then $r_{S_2}\{u_s\}^+ = 0$, if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| = g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, and $\lambda_1 r_{S_2}\{u_s\}^+ = -\lambda_2 r_{S_2}\{(\mathcal{T}U)_s\}^+$, where the symbol $\{\,\cdot\,\}^+$ stands for the trace operator on S_i $(i=1,2)$ from Ω. Conditions (3.2) and (3.4) are understood in the usual trace sense, whereas (3.3) is understood in the generalized functional sense described in Remark 2.3.

To reduce Problem A to a boundary variational inequality, we first reduce the inhomogeneous equation (3.1) to a homogeneous one. In this connection, we consider the following auxiliary linear boundary value problem.

Find a vector function $U_0 = (u_0,\omega_0)^\top \in [H^1(\Omega)]^6$ that is a weak solution of equation (3.1) and satisfies the conditions

$$\begin{gathered} r_{S_1}\{U_0\}^+ = 0 \ \text{ on } \ S_1, \quad r_{S_2}\{\mathcal{M}U_0\}^+ = 0 \ \text{ on } \ S_2, \\ r_{S_2}\{u_{0n}\}^+ = f \ \text{ on } \ S_2, \quad r_{S_2}\{(\mathcal{T}U_0)_s\}^+ = 0 \ \text{ on } \ S_2. \end{gathered} \tag{3.5}$$

It is well known (see [36]) that this problem is uniquely solvable, because S is neither a surface of revolution, nor a ruled surface. Let $V \in [H^1(\Omega)]^6$ be a solution of Problem A, and let $U_0 \in [H^1(\Omega)]^6$ be a solution of the auxiliary problem (3.5); then the difference $U := V - U_0$ is a solution of the following problem.

Problem A$_0$. Find a weak solution $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ of the equation

$$L(\partial)U = 0 \ \text{ in } \ \Omega \tag{3.6}$$

satisfying the inclusion $r_{S_2}\{(\mathcal{T}U)_s\}^+ \in [L_\infty(S_2)]^3$ and the following conditions:

$$r_{S_1}\{U\}^+ = 0 \ \text{ on } \ S_1, \tag{3.7}$$

$$r_{S_2}\{\mathcal{M}U\}^+ = \varphi \ \text{ on } \ S_2, \tag{3.8}$$

$$r_{S_2}\{u_n\}^+ = 0 \ \text{ on } \ S_2, \tag{3.9}$$

$$\text{if } \ \big|r_{S_2}\{(\mathcal{T}U)_s\}^+\big| < g, \ \text{ then } \ r_{S_2}\{u_s\}^+ = \psi_0, \tag{3.10}$$

if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| = g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, such that

$$\lambda_1\big[r_{S_2}\{u_s\}^+ - \psi_0\big] = -\lambda_2 r_{S_2}\{(\mathcal{T}U)_s\}^+, \tag{3.11}$$

where the symbol $\{\,\cdot\,\}^+$ stands for the trace operator on S_i (i=1,2) from Ω and $\psi_0 = -r_{S_2}\{u_{0s}\}^+ \in [H^{1/2}(S_2)]^3$.

In what follows, we will study Problem A$_0$. Obviously, if a vector function $U \in [H^1(\Omega)]^6$ is a solution of Problem A$_0$, then the sum $U + U_0$ is a solution of Problem A.

3.2 Reduction of Problem A$_0$ to a boundary variational inequality

To reduce Problem A$_0$ to an equivalent boundary variational inequality, we recall that the vector $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ is a solution of equation (3.6) satisfying the Dirichlet boundary condition $\{U\}^+ = h$ on S with $h \in [H^{1/2}(S)]^6$ and hence can be uniquely represented by the simple layer potential (see [35])

$$U(x) = V(\mathcal{H}^{-1}h)(x) := \int_S \Gamma(x-y)(\mathcal{H}^{-1}h)(y)\, d_yS, \ \ x \in \Omega,$$

where Γ is the fundamental solution matrix of the operator $L(\partial)$ and $\mathcal{H}$ is the boundary integral operator generated by the trace of the simple layer potential on the boundary S (see the closed-form representation of Γ in [35, 36]),

$$\mathcal{H}(h)(x) = \lim_{\Omega \ni z \to x \in S} \int_S \Gamma(z-y)h(y)\, d_yS = \{V(h)\}^+.$$

Note that the simple layer potential V and the integral operator $\mathcal{H}$ have the following properties (see [35, 36]):

$$V : [H^r(S)]^6 \to [H^{r+3/2}(\Omega)]^6, \quad \mathcal{H} : [H^r(S)]^6 \to [H^{r+1}(S)]^6, \quad r \in \mathrm{R}. \tag{3.12}$$

These operators are continuous. Moreover, $\mathcal{H}$ is an invertible operator and

$$\mathcal{H}^{-1} : [H^r(S)]^6 \to [H^{r-1}(S)]^6, \ \ r \in \mathrm{R}. \tag{3.13}$$

The relation

$$\big\{T(\partial, n)V(h)\big\}^+ = (-2^{-1}I_6 + \mathcal{K})h \ \text{ on } \ S \tag{3.14}$$

holds for an arbitrary $h \in [H^{-1/2}(S)]^6$, where $\mathcal{K}$ is the singular integral operator,

$$\mathcal{K}h(x) = \int_S [T(\partial, n)\Gamma(x-y)]h(y)\, d_yS.$$

Note that

$$-\frac{1}{2}\,I_6+\mathcal{K}:[H^{-1/2}(S)]^6\to[H^{-1/2}(S)]^6$$

is a continuous singular operator of normal type with zero index (for details, see [35, 36]).

Next, for the Dirichlet problem we introduce the so-called Green's operator $G:[H^{1/2}(S)]^6\to[H^1(\Omega)]^6$ which is defined by the relation

$$Gh:=V(\mathcal{H}^{-1}h). \tag{3.15}$$

Obviously, $L(\partial)(Gh)=0$ in Ω and $\{Gh\}^+=h$ on S. Taking into account the properties of the trace operator and mappings (3.12), we find that there exist positive numbers C_1 and C_2 such that

$$C_1\|h\|_{[H^{1/2}(S)]^6}\le\|Gh\|_{[H^1(\Omega)]^6}\le C_2\|h\|_{[H^{1/2}(S)]^6} \tag{3.16}$$

for all $h\in[H^{1/2}(S)]^6$.

Now we introduce a generalized operator of the Steklov–Poincaré type by the relation

$$\mathcal{A}h:=\big\{T(\partial,n)(Gh)\big\}^+=\big\{T(\partial,n)V(\mathcal{H}^{-1}h)\big\}^+=(-2^{-1}I_6+\mathcal{K})(\mathcal{H}^{-1}h). \tag{3.17}$$

By $\Lambda(S)$ we denote the set of restrictions of rigid displacement vectors to S, i.e.,

$$\Lambda(S):=\Big\{\chi(x)=(\rho,a)^\top=\big([a\times x]+b,a\big)^\top,\ x\in S\ |\ \ a,b\in\mathrm{R}\Big\}. \tag{3.18}$$

By using the Green's formula (2.6) for $U=U'=V(\mathcal{H}^{-1}h)$, relations (3.14), (3.17) and (3.18), and the uniqueness theorems for the Dirichlet boundary value problem, we obtain $\ker\mathcal{A}=\Lambda(S)$.

Now we state the following lemma describing the properties of the Steklov–Poincaré operator.

Lemma 3.1. *Let $h,\eta\in[H^{1/2}(S)]^6$ and $g\in[\widetilde{H}^{1/2}(S^*)]^6$, where S^* is a regular open subset of the boundary $S=\partial\Omega$. Then the following assertions hold:*

(i) $\langle\mathcal{A}h,\eta\rangle_S=\langle\mathcal{A}\eta,h\rangle_S$;

(ii) $\mathcal{A}:[H^{1/2}(S)]^6\to[H^{-1/2}(S)]^6$ *is a continuous operator;*

(iii) $\langle\mathcal{A}h,h\rangle_S\ge C_1\|h\|^2_{[H^{1/2}(S)]^6}-C_2\|h\|^2_{[L_2(S)]^6}$;

(iv) $\langle\mathcal{A}g,g\rangle_S\ge C\|g\|^2_{[H^{1/2}(S)]^6}$;

(v) $\langle\mathcal{A}h,h\rangle_S\ge C\|h-\mathcal{P}h\|^2_{[H^{1/2}(S)]^6}$.

Here, $\mathcal{P}$ is the operator of orthogonal projection (in the sense of $L_2(S)$) of the space $[H^{1/2}(S)]^6$ onto the space $\Lambda(S)$; the positive constants C, C_1, and C_2 depend on the elasticity constants and on the geometric properties of the surface S and are independent of h and g.

Proof. Let $h,\eta\in[H^{1/2}(S)]^6$. Since the vector Gh is a weak solution of the homogeneous equation $L(\partial)(Gh)=0$, it follows from the Green's formula (2.6) that

$$\begin{aligned}\langle\mathcal{A}h,\eta\rangle_S&=\big\langle\{T(\partial,n)(Gh)\}^+,\{G\eta\}^+\big\rangle_S=B(Gh,G\eta)=B(G\eta,Gh)\\&=\big\langle\{T(\partial,n)(G\eta)\}^+,\{Gh\}^+\big\rangle_S=\langle\mathcal{A}\eta,h\rangle_S.\end{aligned}$$

This implies assertion (i). Assertion (ii) is obvious, because the operator $\mathcal{A}$ is the composition of the continuous operator $\mathcal{H}^{-1}$ and operator $-2^{-1}I_6+\mathcal{K}$ (see relations (3.14) and (3.17)). The proof of (iii) can be carried out as follows. By using condition (2.5), for an arbitrary $h\in[H^{1/2}(S)]^6$, we obtain the inequality

$$\langle\mathcal{A}h,h\rangle_S=B\big(V(\mathcal{H}^{-1}h),V(\mathcal{H}^{-1}h)\big)\ge c_1\|V(\mathcal{H}^{-1}h)\|^2_{[H^1(\Omega)]^6}-c_2\|V(\mathcal{H}^{-1}h)\|^2_{[L_2(\Omega)]^6}.$$

Relations (3.15) and (3.16) imply the inequalities $\|V(\mathcal{H}^{-1}h)\|_{[H^1(\Omega)]^6} \geq C_1\|h\|_{[H^{1/2}(S)]^6}$. On the other hand, since the space $[L_2(S)]^6$ is compactly embedded in $[H^{-1/2}(S)]^6$, it follows from the continuity of operators (3.12) and (3.13) that

$$\|V(\mathcal{H}^{-1}h)\|_{[L_2(\Omega)]^6} \leq C_1^*\|\mathcal{H}^{-1}h\|_{[H^{-3/2}(S)]^6} \leq C_2^*\|h\|_{[H^{-1/2}(S)]^6} \leq C_3^*\|h\|_{[L_2(S)]^6}$$

with some positive constants C_1^*, C_2^* and C_3^* independent of h.

We finally obtain the inequality

$$\langle \mathcal{A}h, h\rangle_S \geq c_1 C_1^2 \|h\|^2_{[H^{1/2}(S)]^6} - c_2 (C_3^*)^2 \|h\|^2_{[L_2(S)]^6},$$

which implies assertion (iii).

Now, assertion (v) follows from assertion (iii) and the nonnegativity of the operator $\mathcal{A}$, and assertion (iv) is a consequence of (iii). The proof of the lemma is complete. □

Our aim is to reduce Problem A_0 to an equivalent boundary variational inequality. To this end, on the space $[H^{1/2}(S_2)]^3$ we introduce a convex continuous functional

$$j(v) = \int\limits_{S_2} g|v_s - \psi_0|\, dS, \quad v \in [H^{1/2}(S_2)]^3 \tag{3.19}$$

and the convex closed set

$$\mathcal{K}_0 = \big\{h = (h^{(1)}, h^{(2)})^\top \in [H^{1/2}(S)]^6 :\ r_{s_1} h = 0,\ r_{s_2} h_n^{(1)} = 0\big\}. \tag{3.20}$$

On the set $\mathcal{K}_0$, we consider the following boundary variational inequality.

Find a function $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}_0$ such that the boundary variational inequality

$$\langle \mathcal{A}h_0, h - h_0\rangle_S + j(h^{(1)}) - j(h_0^{(1)}) \geq \big\langle \varphi, r_{s_2}(h^{(2)} - h_0^{(2)})\big\rangle_{S_2} \tag{3.21}$$

holds for all $h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}_0$.

4 Equivalence theorem

Let us prove the equivalence of the boundary variational inequality (3.21) and the contact Problem A_0.

Theorem 4.1. *The boundary variational inequality* (3.21) *and the contact* Problem A_0 *are equivalent in the following sense: if* $U \in [H^1(\Omega)]^6$ *is a solution of* Problem A_0*, then* $h_0 = \{U\}^+ \in [H^{1/2}(S)]^6$ *is a solution of the variational inequality* (3.21) *and vice versa, if* $h_0 \in \mathcal{K}_0$ *is a solution of the variational inequality* (3.21)*, then* $U := Gh_0 \in [H^1(\Omega)]^6$ *is a solution of* Problem A_0.

Proof. Let $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ be a solution of Problem A_0, and let $h_0 = (h_0^{(1)}, h_0^{(2)})^\top := \{U\}^+$. Since $U \in [H^1(\Omega)]^6$ is a solution of Problem A_0, it readily follows from conditions (3.7) and (3.9) that $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}_0$, and by virtue of the definition of the operator G (see relation (3.15)), the solution U in the domain Ω can be uniquely represented in the form $U = Gh_0$. By taking into account the definition of the Steklov–Poincaré operator, we obtain

$$\begin{aligned}\langle \mathcal{A}h_0, h - h_0\rangle_S &+ j(h^{(1)}) - j(h_0^{(1)}) - \big\langle \varphi, r_{s_2}(h^{(2)} - h_0^{(2)})\big\rangle_{S_2}\\ &= \big\langle \{T(\partial, n)(Gh_0)\}^+, h - h_0\big\rangle_S + j(h^{(1)}) - j(h_0^{(1)}) - \big\langle \varphi, r_{s_2}(h^{(2)} - h_0^{(2)})\big\rangle_{S_2}\end{aligned}$$

for each $h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}_0$. Since h and h_0 are elements of the set $\mathcal{K}_0$ and conditions (3.7) and (3.8) are satisfied, we have

$$\begin{aligned}\langle \mathcal{A}h_0, h - h_0\rangle_S &+ j(h^{(1)}) - j(h_0^{(1)}) - \big\langle \varphi, r_{s_2}(h^{(2)} - h_0^{(2)})\big\rangle_{S_2}\\ &= \big\langle \{T(\partial, n)(Gh_0)\}^+, r_{s_2}(h - h_0)\big\rangle_{S_2} + \big\langle g, r_{s_2}(|h^{(1)} - \psi_0| - |h_0^{(1)} - \psi_0|)\big\rangle_{S_2}\\ &= \big\langle \{(\mathcal{T}(Gh_0))_s\}^+, r_{s_2}(h_s^{(1)} - h_{0s}^{(1)})\big\rangle_{S_2} + \big\langle g, |r_{s_2} h_s^{(1)} - \psi_0| - |r_{s_2} h_{0s}^{(1)} - \psi_0|\big\rangle_{S_2} := I.\end{aligned} \tag{4.1}$$

Let

$$\left|r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+\right| < g,$$

then $r_{S_2}\{h_{0s}^{(1)}\}^+ = \psi_0$ and it is obvious that $I \geq 0$. If

$$\left|\{(\mathcal{T}(Gh_0))_s\}^+\right| = g,$$

then

$$\lambda_1\big[r_{S_2}\{h_{0s}^{(1)}\}^+ - \psi_0\big] = -\lambda_2 r_{S_2}\big\{(\mathcal{T}(Gh_0))_s\big\}^+$$

and when $\lambda_1 \neq 0$, we obtain

$$\begin{aligned} I = \int\limits_{S_2} (\mathcal{T}(Gh_0))_s \cdot \big(h_s^{(1)} - \psi_0 - (h_{0s}^{(1)} - \psi_0)\big)\, ds \\ + \int\limits_{S_2} g\big(|h_s^{(1)} - \psi_0| - |h_{0s}^{(1)} - \psi_0|\big)\, ds = \int\limits_{S_2} (\mathcal{T}(Gh_0))_s \cdot (h_s^{(1)} - \psi_0)\, ds \\ + \int\limits_{S_2} g|h_s^{(1)} - \psi_0|\, ds - \bigg\{ \int\limits_{S_2} \Big[-\frac{\lambda_2}{\lambda_1}\, |(\mathcal{T}(Gh_0))_s|^2 + \frac{\lambda_2}{\lambda_1} g^2 \Big] \bigg\}\, ds \geq 0. \end{aligned}$$

The case $\lambda_2 \neq 0$ is proved similarly.

Therefore, the right-hand side of equation (4.1) is non-negative and, consequently, we find that inequality (3.21) is satisfied. The proof of the first part of Theorem 4.1 is thereby complete.

Now assume that $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}_0$ is a solution of the variational inequality (3.21). Let us show that the vector function $U = (u, \omega)^\top := Gh_0 \in [H^1(\Omega)]^6$ is a solution of Problem A_0. By the definition of Green's operator G, the vector Gh_0 is a weak solution of the equation $L(\partial)U = 0$ in Ω; since $h_0 \in \mathcal{K}_0$, we have $r_{S_1}\{U\}^+ = r_{S_1}\{Gh_0\}^+ = r_{S_1} h_0 = 0$; i.e., condition (3.7) is satisfied. Condition (3.9) is automatically satisfied, since $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}_0$ and $r_{S_2}\{u_n\}^+ = r_{S_2} h_{0n}^{(1)} = 0$.

Let $h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}_0$, $h^{(1)} = h_0^{(1)}$, and $h^{(2)} = h_0^{(2)} \pm \chi$, where $\chi \in [\widetilde{H}^{1/2}(S_2)]^3$ is an arbitrary vector function. Since $r_{S_1}(h - h_0) = 0$, it follows from inequality (3.21) that

$$\langle \{\mathcal{M}(Gh_0)\}^+ - \varphi, r_{S_2}\chi\rangle_{S_2} = 0 \quad \forall\, \chi \in [\widetilde{H}^{1/2}(S_2)]^3,$$

so $\{\mathcal{M}(Gh_0)\}^+ = \varphi$; i.e., condition (3.8) is satisfied. Therefore, inequality (3.21) can be represented in the form

$$\big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}(h_s^{(1)} - h_{0s}^{(1)})\big\rangle_{S_2} + j(h^{(1)}) - j(h_0^{(1)}) \geq 0 \;\; \forall\, h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}_0,$$

i.e.,

$$\big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}(h_s^{(1)} - \psi_0 - (h_{0s}^{(1)} - \psi_0))\big\rangle_{S_2} + \big\langle g, r_{S_2}\big(|h_s^{(1)} - \psi_0| - |h_{0s}^{(1)} - \psi_0|\big)\big\rangle_{S_2} \geq 0.$$

Let $\chi \in [\widetilde{H}^{1/2}(S_2)]^3$. Since

$$\big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}\chi_s\big\rangle_{S_2} = \big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}\chi\big\rangle_{S_2}$$

and $|r_{S_2}\chi_s| \leq |r_{S_2}\chi|$, taking $r_{S_2}(h_s^{(1)} - \psi_0)$ instead of $r_{S_2}\chi_s$, we obtain

$$\begin{aligned} &\big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}\chi\big\rangle_{S_2} + \big\langle g, r_{S_2}|\chi|\big\rangle_{S_2} \\ &\quad - \Big\{\big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}(h_{0s}^{(1)} - \psi_0)\big\rangle_{S_2} + \big\langle g, r_{S_2}|h_{0s}^{(1)} - \psi_0|\big\rangle\Big\} \geq 0 \;\; \forall\, \chi \in [\widetilde{H}^{1/2}(S_2)]^3. \end{aligned} \tag{4.2}$$

Further, let $t \geq 0$ be an arbitrary number and take $\pm t\chi$ for χ in (4.2)

$$\begin{aligned} &t\Big\{ \pm \big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}\chi\big\rangle_{S_2} + \big\langle g, r_{S_2}|\chi|\big\rangle_{S_2}\Big\} \\ &\quad - \Big\{\big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}(h_{0s}^{(1)} - \psi_0)\big\rangle_{S_2} + \big\langle g, r_{S_2}|h_{0s}^{(1)} - \psi_0|\big\rangle_{S_2}\Big\} \geq 0 \;\; \forall\, \chi \in [\widetilde{H}^{1/2}(S_2)]^3, \end{aligned}$$

whence, by making t tending first to $+\infty$ and then to 0, we easily derive

$$\big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}(h^{(1)}_{0s}-\psi_0)\big\rangle_{S_2} + \big\langle g, r_{S_2}|h^{(1)}_{0s}-\psi_0|\big\rangle_{S_2} \le 0, \tag{4.3}$$

$$\big|\big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}\chi\big\rangle_{S_2}\big| \le \big\langle g, r_{S_2}|\chi|\big\rangle_{S_2} \ \ \forall\,\chi\in[\widetilde{H}^{1/2}(S_2)]^3. \tag{4.4}$$

Now we prove that $r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+\in[L_\infty(S_2)]^3$. To this end, on the space $[\widetilde{H}^{1/2}(S_2)]^3$ we consider the linear functional

$$\Phi(\chi) = \big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}\chi\big\rangle_{S_2} \ \ \forall\,\chi\in[\widetilde{H}^{1/2}(S_2)]^3.$$

Inequality (4.4) shows that the functional Φ is continuous on the space $r_{S_2}[\widetilde{H}^{1/2}(S_2)]^3$ with respect to the topology induced by the space $[L_1(S_2)]^3$. Since the space $r_{S_2}[\widetilde{H}^{1/2}(S_2)]^3$ is dense in $[L_1(S_2)]^3$, the functional Φ can be continuously extended to the space $[L_1(S_2)]^3$ preserving the norm. Therefore, by the Riesz theorem, there is a functional $\Phi^*\in[L_\infty(S_2)]^3$ such that

$$\Phi(\chi) = \int\limits_{S_2}\Phi^*\cdot\chi\,dS \ \ \forall\,\chi\in[L_1(S_2)]^3.$$

Thus,

$$\big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}\chi\big\rangle_{S_2} = \int\limits_{S_2}\Phi^*\cdot\chi\,dS \ \ \forall\,\chi\in[L_1(S_2)]^3,$$

i.e.,

$$\big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+ - \Phi^*, r_{S_2}\chi\big\rangle_{S_2} = 0 \ \ \forall\,\chi\in[\widetilde{H}^{1/2}(S_2)]^3,$$

which implies

$$r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+ = \Phi^*\in[L_\infty(S_2)]^3.$$

It is well known that for an arbitrary essentially bounded function $\widetilde{\psi}\in L_\infty(S_2)$ there is a sequence $\widetilde{\varphi}_l\in C^\infty(S_2)$ with $\operatorname{supp}\widetilde{\varphi}_l\subset S_2$ such that (see, e.g., [38, Lemma 1.4.2])

$$\lim_{l\to\infty}\widetilde{\varphi}_l(x) = \widetilde{\psi}(x) \ \text{ for almost all } \ x\in S_2,$$
$$|\widetilde{\varphi}_l(x)| \le \operatorname*{ess\,sup}_{y\in S_2}|\widetilde{\psi}(y)| \ \text{ for almost all } \ x\in S_2.$$

Therefore, by the Lebesgue dominated convergence theorem, it follows from inequality (4.4) that

$$\int\limits_{S_2}\big[\pm\{(\mathcal{T}(Gh_0))_s\}^+\cdot\chi - g|\chi|\big]\,dS \le 0 \ \ \forall\,\chi\in[L_\infty(S_2)]^3.$$

Instead of χ we can put $\gamma(S^*)\chi$, where $\chi\in[L_\infty(S_2)]^3$ and $\gamma(S^*)$ is the characteristic function of an arbitrary measurable subset $S^*\subset S_2$. As a result, we arrive at the inequality $\pm\{(\mathcal{T}(Gh_0))_s\}^+\cdot\chi\le g|\chi|$ on S_2 for all $\chi\in[L_\infty(S_2)]^3$ and, by choosing $\chi=\{(\mathcal{T}(Gh_0))_s\}^+$, we finally get

$$\big|r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+\big| \le g \ \text{ on } \ S_2. \tag{4.5}$$

In view of (4.3) and (4.5), we obtain

$$r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+\cdot r_{S_2}(h^{(1)}_{0s}-\psi_0) + g|r_{S_2}(h^{(1)}_{0s}-\psi_0)| = 0 \ \text{ on } \ S_2. \tag{4.6}$$

Now, it is evident that if $|r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+| < g$, then (4.6) implies $r_{S_2}h^{(1)}_{0s}=\psi_0$. Also, if $|r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+| = g$, then (4.6) can be rewritten as

$$g|r_{S_2}(h^{(1)}_{0s}-\psi_0)|(\cos\alpha+1) = 0 \ \text{ on } \ S_2,$$

where α is the angle between the vectors $r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+$ and $r_{S_2}(h_{0s}^{(1)} - \psi_0)$ at a point $x \in S_2$. Therefore, there exist the functions $\lambda_1(x) \geq 0$ and $\lambda_2(x) \geq 0$ such that$\lambda_1(x) + \lambda_2(x) > 0$ and

$$\lambda_1(x) r_{S_2}(h_{0s}^{(1)} - \psi_0) = -\lambda_2(x) r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+ \text{ on } S_2.$$

Moreover, we can assume that λ_1 belongs to the same class as $\{(\mathcal{T}(Gh_0))_s\}^+$ and λ_2 belongs to the same class as $r_{S_2}(h_{0s}^{(1)} - \psi_0)$.

Thus, conditions (3.10) and (3.11) of Problem A_0 hold as well, and the proof of Theorem 4.1 is complete. □

5 The existence and uniqueness of a solution

5.1 Uniqueness

Let us prove the following uniqueness theorem.

Theorem 5.1. Problem A_0 *has at most one solution.*

Proof. Let $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}_0$ and $\widetilde{h}_0 = (\widetilde{h}_0^{(1)}, \widetilde{h}_0^{(2)})^\top \in \mathcal{K}_0$ be two arbitrary solutions of the variational inequality (3.21). Then

$$\begin{aligned} &\big\langle \mathcal{A}h_0, \widetilde{h}_0 - h_0\big\rangle_S + j(\widetilde{h}_0^{(1)}) - j(h_0^{(1)}) \geq \big\langle \varphi, r_{S_2}(\widetilde{h}_0^{(2)} - h_0^{(2)})\big\rangle_{S_2}, \\ &\big\langle \mathcal{A}\widetilde{h}_0, h_0 - \widetilde{h}_0\big\rangle_S + j(h_0^{(1)}) - j(\widetilde{h}_0^{(1)}) \geq \big\langle \varphi, r_{S_2}(h_0^{(2)} - \widetilde{h}_0^{(2)})\big\rangle_{S_2}. \end{aligned}$$

By summing these inequalities, we obtain $\langle \mathcal{A}(h_0 - \widetilde{h}_0), h_0 - \widetilde{h}_0\rangle_S \leq 0$. Since $\mathcal{A}$ is a positive definite operator, it follows that $\langle \mathcal{A}(h_0 - \widetilde{h}_0), h_0 - \widetilde{h}_0\rangle_S = 0$. By virtue of relation (3.17) and Lemma 2.1, we have

$$\begin{aligned} 0 &= \big\langle \mathcal{A}(h_0 - \widetilde{h}_0), h_0 - \widetilde{h}_0\big\rangle_S \\ &= \big\langle \{T(\partial, n) V(\mathcal{H}^{-1}(h_0 - \widetilde{h}_0))\}^+, h_0 - \widetilde{h}_0\big\rangle_S = \big\langle \{T(\partial, n) G(h_0 - \widetilde{h}_0)\}^+, h_0 - \widetilde{h}_0\big\rangle_S \\ &= \big\langle \{T(\partial, n) G(h_0 - \widetilde{h}_0)\}^+, \{G(h_0 - \widetilde{h}_0)\}^+\big\rangle_S = B\big(G(h_0 - \widetilde{h}_0), G(h_0 - \widetilde{h}_0)\big). \end{aligned}$$

Hence we derive the relation $G(h_0 - \widetilde{h}_0) = V(\mathcal{H}^{-1}(h_0 - \widetilde{h}_0)) = \big([a \times x] + b, a\big)^\top$ in Ω. Since $h_0, \widetilde{h}_0 \in \mathcal{K}_0$, we have $r_{S_1}\{G(h_0 - \widetilde{h}_0)\}^+ = r_{S_1}(h_0 - \widetilde{h}_0) = 0$; i.e., $([a \times x] + b, a)^\top = 0$ on S_1. Consequently, $a = b = 0$ and $V(\mathcal{H}^{-1}(h_0 - \widetilde{h}_0)) = 0$ in Ω. Therefore, $h_0 = \widetilde{h}_0$ on S. □

5.2 Existence of a solution

To prove the existence of a solution, on the set $\mathcal{K}_0$ we introduce the functional

$$\mathcal{I}(h) = \frac{1}{2}\langle \mathcal{A}h, h\rangle_S + j(h^{(1)}) - \langle \varphi, r_{S_2} h^{(2)}\rangle_{S_2} \quad \forall\, h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}_0. \tag{5.1}$$

Since $\mathcal{A}$ is a symmetric operator (see Lemma 3.1(i)), it follows that the existence of a solution of the variational inequality (3.21) is equivalent to the existence of an element of the set $\mathcal{K}_0$ minimising the functional (5.1); i.e., the variational inequality (3.21) is equivalent to the following minimization problem:

$$\mathcal{I}(h_0) = \inf_{h \in \mathcal{K}} \mathcal{I}(h). \tag{5.2}$$

By the general theory of variational inequalities (see [4,25]), the solvability of the minimization problem (5.2) readily follows from the coerciveness of the functional $\mathcal{I}$, i.e., from the property

$$\mathcal{I}(h) \to \infty \text{ as } \|h\|_{[H^{1/2}(S)]^6} \to \infty, \quad h \in \mathcal{K}_0.$$

Since $\mathcal{A}$ is a coercive operator on the set $\mathcal{K}_0$ (see Lemma 3.1(iv)) and $j(h^{(1)}) \geq 0$, we find that the coerciveness of the consequence of the obvious estimate

$$\mathcal{I}(h) \geq C_1 \|h\|^2_{[H^{1/2}(S)]^6} - C_2 \|h\|_{[H^{1/2}(S)]^6}, \ \ h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}_0,$$

where C_1 and C_2 are the positive constants independent of h. Consequently, functional (5.1) is coercive on the closed set $\mathcal{K}_0$. In addition, $\mathcal{I}$ is a convex continuous functional. By the general theory of variational inequalities (see [4, 25]), we find that the variational inequality (3.21) has a unique solution. Therefore, from Theorem 4.1 we obtain the following assertion of the existence of the solution of Problem A_0.

Theorem 5.2. *Let* $\operatorname{mes} S_1 > 0$, $\varphi \in [H^{-1/2}(S_2)]^3$, $g \in L_\infty(S_2)$ *and* $g \geq 0$. *Then the variational inequality* (3.21) *has a unique solution* $h_0 \in [H^{1/2}(S)]^6$, *and* $U = G h_0$ *is a solution of* Problem A_0.

Remark 5.3. Let $\operatorname{mes} S_1 > 0$, $\mathcal{G} \in [L_2(\Omega)]^6$, $\varphi \in [H^{-1/2}(S_2)]^3$, $f \in H^{1/2}(S_2)$, $g \in L_\infty(S_2)$ and $g \geq 0$. Then Problem A has a unique solution which can be represented in the form $U + U_0$, where U is a solution of Problem A_0 and U_0 is a solution of the auxiliary problem (3.5).

5.3 Lipschitz continuous dependence of the solution on the problem data

Let $U \in [H^1(\Omega)]^6$ and $\widetilde{U} \in [H^1(\Omega)]^6$ be two solutions of Problem A_0 corresponding to the data φ, g and $\widetilde{\varphi}$, $\widetilde{g}$, respectively. Further, let $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}_0$ and $\widetilde{h}_0 = (\widetilde{h}_0^{(1)}, \widetilde{h}_0^{(2)})^\top \in \mathcal{K}_0$ be the traces of the vector functions U and $\widetilde{U}$, respectively, on the boundary S. By Theorem 4.1, the vectors h_0 and $\widetilde{h}_0$ are the solutions of the corresponding variational inequalities (3.21) for the above-introduced data. Therefore, we have two variational inequalities of form (3.21), one for h_0 and another for $\widetilde{h}_0$. By substituting $\widetilde{h}_0$ for h into the first inequality and h_0 into the second one, we obtain the inequalities:

$$\langle \mathcal{A} h_0, \widetilde{h}_0 - h_0 \rangle_S + \int\limits_{S_2} g\big(|\widetilde{h}_{0s}^{(1)} - \psi_0| - |h_{0s}^{(1)} - \psi_0|\big)\, dS \geq \big\langle \varphi, r_{s_2}(\widetilde{h}_0^{(2)} - h_0^{(2)})\big\rangle_{S_2},$$

$$\langle \mathcal{A} \widetilde{h}_0, h_0 - \widetilde{h}_0 \rangle_S + \int\limits_{S_2} \widetilde{g}\big(|h_{0s}^{(1)} - \psi_0| - |\widetilde{h}_{0s}^{(1)} - \psi_0|\big)\, dS \geq \big\langle \widetilde{\varphi}, r_{s_2}(h_0^{(2)} - \widetilde{h}_0^{(2)})\big\rangle_{S_2}.$$

By summing these inequalities, we obtain

$$\big\langle \mathcal{A}(h_0 - \widetilde{h}_0), \widetilde{h}_0 - h_0 \big\rangle_S + \int\limits_{S_2} \big(g - \widetilde{g}\big)\big(|\widetilde{h}_{0s}^{(1)} - \psi_0| - |h_{0s}^{(1)} - \psi_0|\big)\, dS \geq \big\langle \varphi - \widetilde{\varphi}, r_{s_2}(\widetilde{h}_0^{(2)} - h_0^{(2)})\big\rangle_{S_2},$$

i.e.,

$$\big\langle \mathcal{A}(h_0 - \widetilde{h}_0), h_0 - \widetilde{h}_0 \big\rangle_S \leq \int\limits_{S_2} (g - \widetilde{g})\big(|\widetilde{h}_{0s}^{(1)} - \psi_0| - |h_{0s}^{(1)} - \psi_0|\big)\, dS + \big\langle \widetilde{\varphi} - \varphi, r_{s_2}(\widetilde{h}_0^{(2)} - h_0^{(2)})\big\rangle_{S_2}.$$

This inequality, together with (3.16), property (iv) of the operator $\mathcal{A}$ (see Lemma 3.1(iv)), and the continuous inclusion $H^{1/2}(S) \subset L_2(S)$ implies the Lipschitz estimate

$$\|U - \widetilde{U}\|_{[H^1(\Omega)]^6} \leq C_1 \|h_0 - \widetilde{h}_0\|_{[H^{1/2}(S)]^6} \leq C_2 \big(\|\varphi - \widetilde{\varphi}\|_{[H^{-1/2}(S)]^6} + \|g - \widetilde{g}\|_{L_2(S)}\big),$$

where C_1 and C_2 are the positive constants independent of U and $\widetilde{U}$ and the data of the problem under consideration.

6 The semicoercive case

Let $S_1 = \varnothing$, $S_2 = S$, $\mathcal{G} \in [L_2(\Omega)]^6$, $\varphi \in [H^{-1/2}(S)]^3$, $g \in L_\infty(S)$ and $g \geq 0$. Consider the boundary contact problem.

Problem B. Find a vector function $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ which is a weak solution of equation (3.1) in the domain Ω, satisfying the inclusion $\{(\mathcal{T}U)_s\}^+ \in [L_\infty(S)]^3$ and the following boundary conditions on the surface S:

$$\{\mathcal{M}U\}^+ = \varphi, \ \ \{u_n\}^+ = 0,$$

if $|\{(\mathcal{T}U)_s\}^+| < g$, then $\{u_s\}^+ = 0$, if $|\{(\mathcal{T}U)_s\}^+| = g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, and $\lambda_1\{u_s\}^+ = -\lambda_2\{(\mathcal{T}U)_s\}^+$.

To reduce Problem B to an equivalent boundary variational inequality, we first reduce the inhomogeneous equation (3.1) to a homogeneous one. In this connection, we consider the following auxiliary linear boundary value problem.

Find a weak solution $U_0 = (u_0,\omega_0)^\top \in [H^1(\Omega)]^6$ of equation (3.1) in the domain Ω under the conditions

$$\{u_0\}^+ = 0, \quad \{\mathcal{M}U_0\}^+ = 0 \tag{6.1}$$

on S. It is well known (see [23]) that the problem is uniquely solvable. Let $W \in [H^1(\Omega)]^6$ be a solution of Problem B, and let $U_0 \in [H^1(\Omega)]^6$ be a solution of the auxiliary problem (6.1), then the difference $U := W - U_0$ is a solution of the following problem.

Problem $\mathbf{B_0}$. Find a vector function $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ that is a weak solution of the homogeneous equation

$$L(\partial)U = 0 \ \text{ in } \ \Omega$$

satisfying the inclusion $\{(\mathcal{T}U)_s\}^+ \in [L_\infty(S)]^3$ and the following conditions on S:

$$\{\mathcal{M}U\}^+ = \varphi, \ \ \{u_n\}^+ = 0;$$

if $|\{(\mathcal{T}U)_s\}^+ + \varphi_0| < g$, then $\{u_s\}^+ = 0$, if $|\{(\mathcal{T}U)_s\}^+ + \varphi_0| = g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, and

$$\lambda_1\{u_s\}^+ = -\lambda_2\big(\{(\mathcal{T}U)_s\}^+ + \varphi_0\big),$$

where $\varphi_0 = \{(\mathcal{T}U_0)_s\}^+$.

By analogy with the preceding coercive case (see Theorem 4.1), one can show that Problem B_0 is equivalent to the following boundary variational inequality.

Find a vector $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}$ such that the inequality

$$\langle \mathcal{A}h_0, h - h_0\rangle_S + j_1(h^{(1)}) - j_1(h_0^{(1)}) \ge \langle \varphi, h^{(2)} - h_0^{(2)}\rangle_S \tag{6.2}$$

holds for all $h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}$, where

$$j_1(v) = \int\limits_S g|v_s|\,dS + \langle \varphi_0, v_s\rangle_S, \ \ v \in [H^{1/2}(S)]^3,$$

$$\mathcal{K} = \big\{h = (h^{(1)}, h^{(2)})^\top \in [H^{1/2}(S)]^6 : \ h_n^{(1)} = 0\big\}. \tag{6.3}$$

Note that the variational inequality (6.2) is equivalent to Problem B_0 in the following sense: if $U \in [H^1(\Omega)]^6$ is a solution of Problem B_0, then $h_0 = \{U\}^+ \in \mathcal{K}$ is a solution of the variational inequality (6.2); conversely, if $h_0 \in \mathcal{K}$ is a solution of the variational inequality (6.2), then $Gh_0 \in [H^1(\Omega)]^6$ is a weak solution of Problem B_0 (here the operator G is defined by relation (3.15)).

Let $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}$ be a solution of the variational inequality (6.2). By substituting first $h = 0$ and then $h = 2h_0$ into inequality (6.2), we obtain the relation $\langle \mathcal{A}h_0, h_0\rangle_S + j_1(h_0^{(1)}) = \langle \varphi, h_0^{(2)}\rangle_S$ which, together with (6.2), implies that

$$\langle \mathcal{A}h_0, h\rangle_S + j_1(h^{(1)}) \ge \langle \varphi, h^{(2)}\rangle_S. \tag{6.4}$$

Let $\xi = (\rho, a)^\top \in \Lambda(S)$ and $\rho_n = 0$ on S. By substituting $\pm\xi \in \Lambda(S)$ for h into inequality (6.4) ($\Lambda(S)$ is defined by relation (3.18)) and taking into account the relation $\ker\mathcal{A} = \Lambda(S)$, we obtain the inequality

$$\int\limits_S g|\rho_s|\,dS - \big|\langle \varphi, a\rangle_S - \langle \varphi_0, \rho_s\rangle_S\big| \ge 0. \tag{6.5}$$

Inequality (6.5) is a necessary condition for the solvability of the variational inequality (6.2).

Consider the case in which inequality (6.5) is strict. Taking into account the fact that the space $\Lambda(S)$ has finite dimension ($\dim\Lambda(S)=6$), one can readily see that inequality (6.5) is equivalent to the relation

$$\int\limits_S g|\rho_s|\,dS-\big|\langle\varphi,a\rangle_S-\langle\varphi_0,\rho_s\rangle_S\big|\geq\|\xi\|_{[L_2(\Omega)]^6} \tag{6.6}$$

with some positive constant C and with an arbitrary $\xi=(\rho,a)^\top\in\Lambda(S)$. Let $\mathcal{P}$ be the operator of orthogonal projection of the space $[H^{1/2}(S)]^6$ onto $\Lambda(S)$ in the sense of $[L_2(S)]^6$; i.e., any function $h\in[H^{1/2}(S)]^6$ can be represented in the form $h=\xi+\chi$, where $\xi=(\rho,a)^\top=\mathcal{P}h\in\Lambda(S)$ and $\chi=(\eta,\zeta)^\top\in\Lambda^\perp(S):=\{h\in[H^{1/2}(S)]^6:(h,\xi)_{[L_2(S)]^6}=0\;\forall\,\xi\in\Lambda(S)\}$.

One can readily see that the norm $\|h\|_{[H^{1/2}(S)]^6}$ is equivalent to the norm $\|\chi\|_{[H^{1/2}(S)]^6}+\|\xi\|_{[L_2(S)]^6}$. On the convex closed set $\mathcal{K}$ we introduce the continuous convex functional

$$\mathcal{I}_1(h)=\frac12\langle\mathcal{A}h,h\rangle_S+j_1(h^{(1)})-\langle\varphi,h^{(2)}\rangle_S,\;\;h=(h^{(1)},h^{(2)})^\top\in\mathcal{K}$$

$\forall\,h=\chi+\xi\in[H^{1/2}(S)]^6$ with $\chi=(\eta,\zeta)^\top$ and $\xi=(\rho,a)^\top$, we obtain

$$\begin{aligned}\mathcal{I}_1(h)=\mathcal{I}_1(\chi+\xi)&=\frac12\,\big\langle\mathcal{A}(\chi+\xi),\chi+\xi\big\rangle_S+j_1(\eta+\rho)-\langle\varphi,\zeta+a\rangle_S\\&=\frac12\,\langle\mathcal{A}\chi,\chi\rangle_S-\langle\varphi,\zeta\rangle_S+j_1(\rho)-\langle\varphi,a\rangle_S+j_1(\eta+\rho)-j_1(\rho)\\&\qquad\geq C_1\|\chi\|^2_{[H^{1/2}(S)]^6}-C_2\|\chi\|_{[H^{1/2}(S)]^6}+C\|\xi\|_{[L_2(S)]^6}+j_1(\eta+\rho)-j_1(\rho),\end{aligned}$$

with some positive constants C, C_1 and C_2. Now let us estimate the difference $j_1(\eta+\rho)-j_1(\rho)$. We have

$$\begin{aligned}j_1(\eta+\rho)-j_1(\rho)&=\int\limits_S g|\eta_s+\rho_s|\,dS+\langle\varphi_0,\eta_s+\rho_s\rangle_S-\int\limits_S g|\rho_s|\,dS-\langle\varphi_0,\rho_s\rangle_S\\&=\int\limits_S g\big(|\eta_s+\rho_s|-|\rho_s|\big)\,dS+\langle\varphi_0,\eta_s\rangle_S\geq-\int\limits_S g|\eta_s|\,dS-C_3\|\chi\|_{[H^{1/2}(S)]^6}\geq-C_4\|\chi\|_{[H^{1/2}(S)]^6},\end{aligned}$$

where C_4 is a positive constant independent of η and ρ. By taking into account this inequality, we finally obtain the estimate

$$\mathcal{I}_1(h)\geq C_1\|\chi\|^2_{[H^{1/2}(S)]^6}+C\|\xi\|_{[L_2(S)]^6}-C_5\|\chi\|_{[H^{1/2}(S)]^6},$$

which implies that

$$\mathcal{I}_1(h)\to+\infty\;\text{ as }\;\|h\|_{[H^{1/2}(S)]^6}\to\infty,\;\;h\in\mathcal{K}.$$

We have thereby shown that the functional $\mathcal{I}_1$ is coercive and the minimization problem is solvable for this functional. Consequently, the corresponding variational inequality (6.2) is solvable (see [4,25]). By virtue of the symmetry of the operator $\mathcal{A}$, the problem of minimization of the functional $\mathcal{I}_1$ on the space $[H^{1/2}(S)]^6$ is equivalent to the solvability of the variational inequality (6.2). Next, note that $\langle\mathcal{A}(h_0-\widetilde{h}_0),h_0-\widetilde{h}_0\rangle_S=0$ for two possible solutions h_0 and $\widetilde{h}_0$ of the variational inequality (6.2) in the set $\mathcal{K}$. Hence it follows that $h_0-\widetilde{h}_0=([a\times x]+b,a)^\top$, $a,b\in\mathrm{R}^3$. We have thereby proved the following theorem on the existence and uniqueness of the solution.

Theorem 6.1. *Let $S_1=\varnothing$, $\varphi\in[H^{-1/2}(S)]^3$, $g\in L_\infty(S)$, $g\geq0$ and let inequality* (6.6) *be satisfied. Then the variational inequality* (6.2)*is solvable and if $h_0\in\mathcal{K}$ is a solution of inequality* (6.2)*, then $U=Gh_0$ is a solution of* Problem B_0*. Moreover, two solutions can differ from each other only by a rigid displacement vector.*

Remark 6.2. Let $S_1=\varnothing$, $\mathcal{G}\in[L_2(\Omega)]^6$, $\varphi\in[H^{-1/2}(S)]^3$, $g\in L_\infty(S)$, $g\geq0$ and let inequality (6.6) be satisfied. Then Problem B has a solution which can be represented in the form $U+U_0$, where U is a solution of Problem B_0 and U_0 is a solution of the auxiliary problem (6.1).

Remark 6.3. Let the boundary $S = \partial\Omega$ fall into three mutually disjoint parts S_1, S_T and S_2 such that $\overline{S}_1 \cup \overline{S}_T \cup \overline{S}_2 = S$, $\overline{S}_1 \cap \overline{S}_2 = \varnothing$. By analogy with the coercive case, we can study the problem, when on S_T the traction boundary condition $r_{S_T}\{T(\partial,n)U\}^+ = Q$ is assigned, where $Q \in [H^{-1/2}(S_T)]^6$. The conditions on the parts S_1 and S_2 in this case remain the same as in Problem A.

To reduce this problem to a boundary variational inequality, we first consider the following auxiliary problem.

Find a vector function $U_0 = (u_0, \omega_0)^\top \in [H^1(\Omega)]^6$ that is a weak solution of equation (3.1) in the domain Ω and satisfies the boundary conditions

$$r_{S_2}\{U_0\}^+ = 0, \quad r_{S_T}\{T(\partial,n)U_0\}^+ = 0,$$
$$r_{S_2}\{\mathcal{M}U_0\}^+ = 0, \quad r_{S_2}\{u_{0n}\}^+ = f, \quad r_{S_2}\{(\mathcal{T}U_0)_s\}^+ = 0.$$

It is well known that this problem has a unique weak solution (see [4,25]), because S is neither a surface of revolution, nor a ruled surface. Obviously, if V is a solution of the above-considered problem and U_0 is a solution of the auxiliary problem, then the difference $U := V - U_0$ is a solution of the following problem.

Find a weak solution $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ of the equation

$$L(\partial)U = 0 \ \text{ on } \ \Omega,$$

which satisfies the inclusion $r_{S_2}\{(\mathcal{T}U)_s\}^+ \in [L_\infty(S_2)]^3$ and the following conditions:

$$r_{S_1}\{U\}^+ = 0 \ \text{ on } \ S_1, \quad r_{S_T}\big\{T(\partial,n)U\big\}^+ = Q \ \text{ on } \ S_T,$$
$$r_{S_2}\{\mathcal{M}U\}^+ = \varphi \ \text{ on } \ S_2, \quad r_{S_2}\{u_n\}^+ = 0 \ \text{ on } \ S_2,$$

if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| < g$, then $r_{S_2}\{u_s\}^+ = \psi_0$, whereas if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| = g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, and $\lambda_1(r_{S_2}\{u_s\}^+ - \psi_0) = -\lambda_2 r_{S_2}\{(\mathcal{T}U)_s\}^+$, where $\psi_0 = -r_{S_2}\{u_{os}\}^+$. Just as above, this problem can be reduced to an equivalent boundary variational inequality.

Find a vector $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}_0$ such that the inequality

$$\langle \mathcal{A}h_0, h - h_0\rangle_S + j(h^{(1)}) - j(h_0^{(1)}) \geq \langle Q, r_{S_T}(h - h_0)\rangle_{S_T} + \langle \varphi, h^{(2)} - h_0^{(2)}\rangle_S$$

holds for all $h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}_0$, where the functional j and the convex set $\mathcal{K}_0$ are defined by relations (3.19) and (3.20), respectively. The proof of the existence, uniqueness and Lipschitz continuous dependence of the solution on the problem data in this case can be carried out just as in Problem A_0 in the coercive case.

Remark 6.4. By analogy with the non-coercive case, we can study the problem when on the part S_1 of the boundary instead of the Dirichlet condition (3.7) there is assigned the tractional boundary condition $r_{S_1}\{T(\partial,n)U\}^+ = Q$, where $Q \in [\widetilde{H}^{-1/2}(S_1)]^6$. Moreover, we assume that the vector φ appearing in condition (3.8) belongs to the space $[\widetilde{H}^{-1/2}(S_2)]^3$ and the conditions imposed on the part S_2 are the same as in Problem A_0.

To reduce the above problem to the equivalent boundary variational inequality, we preliminarily reduce the inhomogeneous equation (3.1) to a homogeneous one. In this connection, we consider the following auxiliary problem.

In the domain Ω, find a weak solution $U_0 = (u_0, \omega_0)^\top \in [H^1(\Omega)]^6$ of equation (3.1) with the following condition on S:

$$r_{S_1}\big\{T(\partial,n)U_0\big\}^+ = 0, \quad r_{S_2}\{u_0\}^+ = 0, \quad r_{S_2}\{\mathcal{T}U_0\}^+ = 0.$$

By [23], this problem is uniquely solvable. In this regard, we also consider the following problem.

Problem C_0. Find a vector function $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ which is a weak solution of the homogeneous equation

$$L(\partial)U = 0 \ \text{ in } \ \Omega$$

satisfying the inclusion $\{(\mathcal{T}U)_s\}^+ \in [L_\infty(S)]^3$ and the following conditions on S:

$$r_{_{S_1}}\{T(\partial,n)U\}^+ = Q, \ \ r_{_{S_2}}\{\mathcal{M}U\}^+ = \varphi - \varphi_0,$$
$$\varphi_0 = r_{_{S_2}}\{\mathcal{M}U_0\}^+ \in [H^{-1/2}(S_2)]^3, \ \ r_{_{S_2}}\{u_n\}^+ = 0;$$

if $|\{(\mathcal{T}U)_s\}^+| < g$, then $r_{_{S_2}}\{u_s\}^+ = 0$, if $|\{(\mathcal{T}U)_s\}^+| = g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, and $\lambda_1 r_{_{S_2}}\{u_s\}^+ = -\lambda_2 r_{_{S_2}}\{(\mathcal{T}U)_s\}^+$. In this case, we obtain the following boundary variational inequality.

Find a function $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}$ such that the inequality

$$\langle \mathcal{A}h_0, h - h_0\rangle_S + j_1(h^{(1)}) - j_1(h_0^{(1)}) \geq \big\langle r_{_{S_1}}Q, r_{_{S_1}}(h - h_0)\big\rangle_{S_1} + \big\langle r_{_{S_2}}(\varphi - \varphi_0), r_{_{S_2}}(h^{(2)} - h_0^{(2)})\big\rangle_{S_2} \quad (6.7)$$

holds for all $h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}$, where $j_1(h^{(1)}) = \int\limits_{S_2} g|h_s^{(1)}|\,dS$ and $\mathcal{K}$ defined by formula (6.3).

Now the necessary condition for the solvability of the variational inequality acquires the form

$$\int\limits_{S_2} g|\rho_s|\,dS - \Big|\big\langle r_{_{S_2}}(\varphi - \varphi_0), a\big\rangle_{S_2} + \big\langle r_{_{S_1}}Q, r_{_{S_1}}\xi\big\rangle_{S_1}\Big| \geq 0 \quad (6.8)$$

for all $\xi = (\rho, a)^\top \in \Lambda(S)$, $r_{_{S_2}}\rho_n = 0$. When inequality (6.8) is strict, then, just as in the non-coercive case, one can show that condition (6.8) is sufficient for the solvability of inequality (6.7).

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(Received 13.03.2017)

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Memoirs on Differential Equations and Mathematical Physics

Volume 74, 2018, 113–124

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ON THE EXISTENCE OF ANALYTIC SOLUTIONS OF CERTAIN TYPES OF SYSTEMS, PARTIALLY RESOLVED RELATIVELY TO THE DERIVATIVES IN THE CASE OF A POLE

Abstract. For the systems of ordinary differential equations which are partially resolved relatively to the derivatives in the case of a pole, the theorems on the existence of at least one analytic in the complex domain solution of the Cauchy problem with an additional condition are established. Moreover, the asymptotic behavior of these solutions in this domain is studied.

2010 Mathematics Subject Classification. 39A45.

Key words and phrases. Ordinary differential equation, pole, Cauchy's problem, complex domain, singularity, asymptotic behavior, analytic solution.

რეზიუმე. ჩვეულებრივი დიფერენციალური განტოლებების სისტემებისთვის, რომლებიც ნაწილობრივ ამოხსნადია წარმოებულების მიმართ პოლუსის შემთხვევაში, დამტკიცებულია კომპლექსურ არეში ერთი მაინც ანალიზური ამონახსნის არსებობის თეორემები კოშის ამოცანისთვის დამატებითი პირობით. გარდა ამისა, შესწავლილია ასეთი ამონახსნების ასიმპტოტური ყოფაქცევა ამ არეში.

Introduction

R. Fuchs, Ch. Beriot, J. Bouquet, A. Lyapunov, H. Poincare, P. Painleve are the founders of the theory that investigates the behavior of solutions of systems of ordinary differential equations in the neighborhood of the singularity.

A separate class of problems in this area is the study of the existence and asymptotic behavior of solutions of systems of differential equations that are not resolved relatively to the derivatives. Certain types of systems not resolved relatively to the derivatives in a complex domain were investigated by such scientists as M. Jwano [4], O Song Guk, Pak Ponk, Chol Permissible [12], V. Gromak and many others.

One of the methods studying the systems of differential equations that are not resolved relatively to the derivatives in the real-valued domain was suggested by R. Grabovskaya [3] and J. Diblic [1, 2]. Later, this method in the case of a complex domain was developed by G. Samkova [7,8], N. Sharay [10], E. Michalenko, D. Limanska [5, 6] and others. The present article is a continuation of the research devoted to the systems of differential equations that are not resolved relatively to the derivatives in a complex domain.

Let us consider the system of ordinary differential equations

$$A(z)Y' = B(z)Y + f(z, Y, Y'), \tag{0.1}$$

where the matrices $A, B : D_1 \to \mathbb{C}^{p\times n}$, $D_1 = \{z \in \mathbb{C} : \ |z| < R_1, \ R_1 > 0\}$, the matrices $A(z)$, $B(z)$ are analytic in the domain D_{10}, $D_{10} = D_1 \setminus \{0\}$, the pencil of matrices $A(z)\lambda - B(z)$ is singular as $z \to 0$, the vector-function $f : D_1 \times G_1 \times G_2 \to \mathbb{C}^p$, where domains $G_k \subset \mathbb{C}^n$, $0 \in G_k$, $k = 1, 2$, the function $f(z, Y, Y')$ is analytic in the domain $D_{10} \times G_{10} \times G_{20}$, $G_{k0} = G_k \setminus \{0\}$, $k = 1, 2$.

The main goal of our paper is to establish the existence and to study the asymptotic behavior of solutions of the system of differential equations (0.1) in the domain with the point $z = 0$ on its border, under the conditions that $p < n$, the matrix $A(z)$ is analytic in the domain D_1 and $\operatorname{rank} A(z) = p$ in this domain.

1 On some singular Cauchy problem for a system of ordinary differential equations, not resolved relatively to the derivatives

Let us consider the system of differential equations

$$z^l Y_1' = z^l P(z)Y_1 + F(z, Y_1, Y_1'), \tag{1.1}$$

where $l \in \mathbb{Z}$, $Y_1 = \operatorname{col}(Y_{11}(z), \dots, Y_{1p}(z))$, $Y_1 : D_1 \to \mathbb{C}^p$, the matrix $P(z)$ is analytic in the domain D_1, $F : D_1 \times G_{11} \times G_{21} \to \mathbb{C}^p$, $G_{j1} \subset \mathbb{C}^p$, $j = 1, 2$, $F(z, Y_1, Y_1')$ is analytic vector-function in the domain $D_1 \times G_{11} \times G_{21}$, $F(0, 0, 0) = 0$.

We study the questions of the existence of analytic solutions of system (1.1) that satisfy the initial condition

$$Y_1(z) \to 0 \ \text{ for } \ z \to 0, \ \ z \in D_{10}, \tag{1.2}$$

and the additional condition

$$Y_1'(z) \to 0 \ \text{ for } \ z \to 0, \ \ z \in D_{10}. \tag{1.3}$$

According to the method of analytic continuation of solutions [3], system (1.1) will be investigated over two sets of curves. We analytically continue solutions from the curve of the first set to some domain by using the curves of the second set.

1.1 Introduction of some intermediary notations

For arbitrary fixed $t_1 > 0$, $v_1, v_2 \in \mathbb{R}$, $v_1 < v_2$, let us introduce the following intermediary sets:

$$\check{I} = \big\{(t, v) \in \mathbb{R}^2 : \ t \in (0, t_1), \ v \in (v_1, v_2)\big\},$$
$$L_{v_0}(t_1) = \big\{(t, v) \in \mathbb{R}^2 : \ t \in (0, t_1), \ v = v_0 \in (v_1, v_2)\big\},$$

v_0 is a fixed number.

For arbitrary fixed $t_0 \in (0, t_1)$, $O_{t_1}(t_0) = \{(t, v) \in \mathbb{R}^2 : \ t = t_0, \ v \in (v_1, v_2)\}$.

For $z = z(t, v) = te^{iv}$, let us assign for set $\check{I} \subset \mathbb{R}^2$ the set $I \subset \mathbb{C}$, $I = \{z = te^{iv} \in \mathbb{C} : \ t \in (0, t_1), \ v \in (v_1, v_2)\}$.

Definition 1.1. Let $p, g : \check{I} \to [0, +\infty)$. We say that the function $p(t, v)$ possesses property Q_1 relative to the function $g(t, v)$ for $v = v_0 \in (v_1, v_2)$, if the function $p(t, v_0)$ is of higher-order of smallness relative to the function $g(t, v_0)$ as $t \to +0$.

Definition 1.2. Let $p, g : \check{I} \to [0, +\infty)$. We say that the function $p(t, v)$ possesses property Q_2 relative to the function $g(t, v)$, if there exist $C_1 \geq 0$, $C_2 \geq 0$ such that in the set $\check{I}$ the inequalities

$$C_1 \cdot g(t, v) \leq p(t, v) \leq C_2 \cdot g(t, v)$$

hold.

Let us introduce the following intermediary vector-functions:

$$\varphi^{(0)}(z) = (\varphi_1^{(0)}(z), \ldots, \varphi_p^{(0)}(z)), \ \ \varphi^{(0)} : I \to \mathbb{C}^p,$$
$$\psi^{(0)}(t, v) = (\psi_1^{(0)}(t, v), \ldots, \psi_p^{(0)}(t, v)), \ \ \psi_j^{(0)} : \check{I} \to [0; +\infty), \ \ j = \overline{1, p}.$$

For $z = z(t, v) = te^{iv}$, we have

$$\psi_j^{(0)}(t, v) = |\varphi_j^{(0)}(z(t, v))|, \ \ j = \overline{1, p}.$$

Definition 1.3. We say that the analytic on the set I vector-function $\varphi^{(0)}(z)$ possesses the property T_0, if for any $z \in I$, for the counterpart vector-functions $\psi_j^{(0)}(t, v)$ the conditions

$$\psi_j^{(0)}(t, v) > 0, \ \ (\psi_j^{(0)}(t, v))'_t > 0, \ \ (\psi_j^{(0)}(t, v))'_v \geq 0,$$
$$\psi_j^{(0)}(+0, v) = 0, \ \ (\psi_j^{(0)}(+0, v))'_t = 0, \ \ j = \overline{1, p} \ \text{ uniformly in } \ v \in (v_1, v_2)$$

are fulfilled.

1.2 System (1.1) on the set $L_{v_0}(t_1)$

Let us consider system (1.1) over the interval $L_{v_0}(t_1)$ for an arbitrary fixed $v_0 \in (v_1, v_2)$.

For $z = z(t, v_0) = te^{iv_0}$, in system (1.1) we write each vector-function and matrix in the algebraic form and separate real and imaginary parts. Introduce the following designations:

$$Y_1(z(t, v_0)) = \widetilde{Y}_1(t), \ \ \widetilde{Y}_1(t) = \widetilde{Y}_{11}(t) + i\widetilde{Y}_{12}(t); \ \ \widetilde{Y}_{1j}(t) = \mathrm{col}(\widetilde{Y}_{1j1}(t), \ldots, \widetilde{Y}_{1jp}(t)), \ \ j = 1, 2,$$
$$P(z(t, v_0)) = \|\widetilde{p}_{jk}(t)\|_{j,k=1}^p = \widetilde{P}_1(t) + i\widetilde{P}_2(t), \ \ \widetilde{P}_s(t) = \|\widetilde{p}_{jks}(t)\|_{j,k=1}^p, \ \ s = 1, 2,$$

where

$$\widetilde{p}_{jk}(t) = \widetilde{p}_{jk1}(t) + i\widetilde{p}_{jk2}(t), \ \ j, k = \overline{1, p},$$
$$F\big(z(t, v_0), Y_1(z(t, v_0)), Y_1'(z(t, v_0))\big) = \widetilde{F}(t, \widetilde{Y}_1, \widetilde{Y}_1'),$$
$$\widetilde{F}\big(t, \widetilde{Y}_1, \widetilde{Y}_1'\big) = \mathrm{col}\left(\widetilde{F}_1(t, \widetilde{Y}_1, \widetilde{Y}_1'), \ldots, \widetilde{F}_p(t, \widetilde{Y}_1, \widetilde{Y}_1')\right),$$
$$\widetilde{F}_j(t, \widetilde{Y}_1, \widetilde{Y}_1') = \widetilde{F}_{1j}(t, \widetilde{Y}_1, \widetilde{Y}_1') + i\widetilde{F}_{2j}(t, \widetilde{Y}_1, \widetilde{Y}_1'), \ \ j = \overline{1, p}.$$

Due to the fact that for each $v \in [v_1, v_2]$ we have the equality

$$\widetilde{Y}_1'(t) = \big(Y_1(z(t, v))\big)'_t = \frac{dY_1}{dz} \cdot \frac{dz}{dt} = Y_1'(z) \cdot e^{iv},$$

then for $z = z(t, v_0) = te^{iv_0}$ system (1.1) takes the form

$$t^l(\widetilde{Y}_{11}' + i\widetilde{Y}_{12}') = t^l(\widetilde{P}_1 + i\widetilde{P}_2)(\widetilde{Y}_{11} + i\widetilde{Y}_{12})e^{iv_0} + e^{(1-l)iv_0}\big(\mathrm{Re}\,\widetilde{F}(t, \widetilde{Y}_1, \widetilde{Y}_1') + i\,\mathrm{Im}\,\widetilde{F}(t, \widetilde{Y}_1, \widetilde{Y}_1')\big). \quad (1.4)$$

Let us introduce the matrices and the vector-function

$$\widetilde{P}(t)=\begin{pmatrix}\widetilde{P}_1(t) & -\widetilde{P}_2(t)\\ \widetilde{P}_2(t) & \widetilde{P}_1(t)\end{pmatrix},$$

$$\widetilde{Q}_1(v_0)=\begin{pmatrix}\cos(v_0)E & -\sin(v_0)E\\ \sin(v_0)E & \cos(v_0)E\end{pmatrix},\quad \widetilde{Q}_2(v_0)=\begin{pmatrix}\cos((l-1)v_0)E & \sin((l-1)v_0)E\\ -\sin((l-1)v_0)E & \cos((l-1)v_0)E\end{pmatrix},$$

$$\widetilde{f}\big(t,\widetilde{Y}_{11},\widetilde{Y}_{12},\widetilde{Y}'_{11},\widetilde{Y}'_{12}\big)=\operatorname{col}\big(\widetilde{F}_{11}\cdots\widetilde{F}_{1p}\widetilde{F}_{21}\cdots\widetilde{F}_{2p}\big),$$

where E is the $p\times p$ identity matrix.

Equating the real and imaginary parts of the vector-functions from the left– and right-hand sides of system (1.4), system (1.4) reduces to

$$t^l\begin{pmatrix}\widetilde{Y}'_{11}(t)\\ \widetilde{Y}'_{12}(t)\end{pmatrix}=t^l\widetilde{P}(t)\widetilde{Q}_1(v_0)\begin{pmatrix}\widetilde{Y}_{11}(t)\\ \widetilde{Y}_{12}(t)\end{pmatrix}+\widetilde{Q}_2(v_0)\widetilde{f}\big(t,\widetilde{Y}_{11},\widetilde{Y}_{12},\widetilde{Y}'_{11},\widetilde{Y}'_{12}\big). \tag{1.5}$$

This implies that system (1.1) over the interval $L_{v_0}(t_1)$ for an arbitrary fixed $v_0\in(v_1,v_2)$ reduces to the system of real differential equations (1.5).

1.3 System (1.1) on the set $O_{t_1}(t_0)$

Let us consider system (1.1) over the arc of circle $O_{t_1}(t_0)$ for an arbitrary fixed $t_0\in(0,t_1)$.

For $z=z(t,v_0)=te^{iv_0}$, in system (1.1) we write each vector-function and matrix in the algebraic form and separate real and imaginary parts. Let us introduce the following designations:

$$Y_1(z(t_0,v))=\widehat{Y}_1(v),\ \ \widehat{Y}_1(v)=\widehat{Y}_{11}(v)+i\widehat{Y}_{12}(v);$$

$$\widehat{Y}_{1j}(v)=\operatorname{col}\big(\widehat{Y}_{1j1}(v),\dots,\widehat{Y}_{1jp}(v)\big),\ \ j=1,2,$$

$$P(z(t_0,v))=\|\widehat{p}_{jk}(v)\|^p_{j,k=1}=\widehat{P}_1(v)+i\widehat{P}_2(v),\ \ \widehat{P}_s(v)=\|\widehat{p}_{jks}(v)\|^p_{j,k=1},\ \ s=1,2,$$

where

$$\widehat{p}_{jk}(v)=\widehat{p}_{jk1}(v)+i\widehat{p}_{jk2}(v),\ \ j,k=\overline{1,p},$$

$$F\big(z(t_0,v),Y_1(z(t_0,v)),Y'_1(z(t_0,v))\big)=\widehat{F}(v,\widehat{Y}_1,\widehat{Y}'_1),$$

$$\widehat{F}(v,\widehat{Y}_1,\widehat{Y}'_1)=\operatorname{col}\big(\widehat{F}_1(v,\widehat{Y}_1,\widehat{Y}'_1),\dots,\widehat{F}_p(v,\widehat{Y}_1,\widehat{Y}'_1)\big),$$

$$\widehat{F}_j(v,\widehat{Y}_1,\widehat{Y}'_1)=\widehat{F}_{1j}(v,\widehat{Y}_1,\widehat{Y}'_1)+i\widehat{F}_{2j}(v,\widehat{Y}_1,\widehat{Y}'_1),\ \ j=\overline{1,p}.$$

Due to the fact that for each $t\in(0,t_1)$ we have the equality

$$\widehat{Y}'_1(v)=\big(Y_1(z(t,v))\big)'_t=\frac{dY_1}{dz}\cdot\frac{dz}{dv}=Y'_1(z)\cdot ite^{iv},$$

then for $z=z(t_0,v)=t_0e^{iv}$, system (1.1) reduces to the form

$$t_0^{l-1}(\widehat{Y}'_{11}+i\widehat{Y}'_{12})=it_0^l(\widehat{P}_1+i\widehat{P}_2)(\widehat{Y}_{11}+i\widehat{Y}_{12})e^{iv}+e^{(1-l)iv}\big(\operatorname{Re}\widehat{F}(v,\widehat{Y}_1,\widehat{Y}'_1)+i\operatorname{Im}\widehat{F}(v,\widehat{Y}_1,\widehat{Y}'_1)\big). \tag{1.6}$$

Let us introduce matrices and the vector-function

$$\widehat{P}(v)=\begin{pmatrix}\widehat{P}_1(v) & -\widehat{P}_2(v)\\ \widehat{P}_2(v) & \widehat{P}_1(v)\end{pmatrix},$$

$$\widehat{Q}_1(v)=\begin{pmatrix}-\sin(v)E & -\cos(v)E\\ \cos(v)E & -\sin(v)E\end{pmatrix},\quad \widehat{Q}_2(v)=\begin{pmatrix}\sin((l-1)v)E & -\cos((l-1)v)E\\ \cos((l-1)v)E & \sin((l-1)v)E\end{pmatrix},$$

$$\widehat{f}\big(v,\widehat{Y}_{11},\widehat{Y}_{12},\widehat{Y}'_{11},\widehat{Y}'_{12}\big)=\operatorname{col}(\widehat{F}_{11}\cdots\widehat{F}_{1p}\widehat{F}_{21}\cdots\widehat{F}_{2p}),$$

where E is the $p\times p$ identity matrix.

Equating the real and imaginary parts of the vector-functions from the left– and right-hand sides of system (1.6), system (1.6) reduces to

$$t_0^{l-1}\begin{pmatrix}\widehat{Y}'_{11}(v)\\ \widehat{Y}'_{12}(v)\end{pmatrix} = t_0^l\widehat{P}(v)\widehat{Q}_1(v)\begin{pmatrix}\widehat{Y}_{11}(v)\\ \widehat{Y}_{12}(v)\end{pmatrix} + \widehat{Q}_2(v)\widehat{f}\big(v,\widehat{Y}_{11},\widehat{Y}_{12},\widehat{Y}'_{11},\widehat{Y}'_{12}\big). \tag{1.7}$$

This implies that system (1.1) over the arc of the circle $O_{t_1}(t_0)$ for an arbitrary fixed $t_0 \in (0,t_1)$ reduces to the system of real differential equations (1.7).

1.4 On some classes of systems of form (1.1)

Definition 1.4. We say that the matrix $P(z)$ possesses property S_{2l} relative to the vector-function $\varphi^{(0)}(z)$, if the following conditions are fulfilled:

(1) For each $v_0 \in (v_1,v_2)$, the functions $(\psi_j^{(0)}(t,v))'_t$ possess property Q_1 relative to the functions $|\widetilde{p}_{jj}(t)|\psi_j^{(0)}(t,v)$, $j=\overline{1,p}$, for $v=v_0\in(v_1,v_2)$.

(2) The functions $t^l(\psi_j^{(0)}(t,v))'_v$ possess property Q_2 relative to the functions $t^{l-1}|\widehat{p}_{jj}(v)|\psi_j^{(0)}(t,v)$, $j=\overline{1,p}$.

(3) For each $v_0\in(v_1,v_2)$, the functions $|\widetilde{p}_{jk}(t)|\psi_k^{(0)}(t,v)$ possess property Q_1 relative to the functions $|\widetilde{p}_{jj}(t)|\psi_j^{(0)}(t,v)$, $j,k=\overline{1,p}$, $j\neq k$, for $v=v_0\in(v_1,v_2)$.

(4) The functions $t^l|\widehat{p}_{jk}(v)|\psi_k^{(0)}(t,v))$ possess property Q_2 relative to the functions $t^{l-1}(\psi_j^{(0)}(t,v))'_v$, $j,k=\overline{1,p}$, $j\neq k$.

Let us define the sets

$$\widetilde{\Omega}\big(\delta,\varphi^{(0)}(z(t,v_0))\big) = \Big\{(t,\widetilde{Y}_{11},\widetilde{Y}_{12}):\ t\in(0,t_1),\ \widetilde{Y}^2_{11j}+\widetilde{Y}^2_{12j} < \delta_j^2\big(\psi_j^{(0)}(t,v_0)\big)^2,\ j=\overline{1,p}\Big\},$$

v_0 is fixed on (v_1,v_2),

$$\widehat{\Omega}\big(\sigma,\varphi^{(0)}(z(t_0,v))\big) = \Big\{(v,\widehat{Y}_{11},\widehat{Y}_{12}):\ v\in(v_1,v_2),\ \widehat{Y}^2_{11j}+\widehat{Y}^2_{12j} < \sigma_j^2(\psi_j^{(0)}(t_0,v))^2,\ j=\overline{1,p}\Big\},$$

t_0 is fixed on $(0,t_1)$, where $\delta=(\delta_1,\dots,\delta_p)$, $\sigma=(\sigma_1,\dots,\sigma_p)$, $\delta_j,\sigma_j\in\mathbb{R}\setminus\{0\}$, $j=\overline{1,p}$.

Definition 1.5. We say that the vector-function $F(z,Y_1,Y'_1)$ possesses property M_{2l} relative to the vector-function $\varphi^{(0)}(z)$, if the following conditions hold:

(1) For each $v_0 \in (v_1,v_2)$, when $(t,\widetilde{Y}_{11},\widetilde{Y}_{12}) \in \widehat{\Omega}(\sigma,\varphi^{(0)}(z(t,v_0)))$, the functions $\widetilde{F}_{kj}(t,\widetilde{Y}_{11},\widetilde{Y}_{12},\widetilde{Y}'_{11},\widetilde{Y}'_{12})$ possess property Q_1 relative to the vector-functions $t^l|\widetilde{p}_{jj}(t)|\psi_j^{(0)}(t,v)$, $j=\overline{1,p}$, $k=1,2$, for $v=v_0\in(v_1,v_2)$.

(2) For each $(v,\widehat{Y}_{11},\widehat{Y}_{12})\in\widehat{\Omega}(\sigma,\varphi^{(0)}(z(t_0,v)))$ the functions $\widehat{F}_{kj}(v,\widehat{Y}_{11},\widehat{Y}_{12},\widehat{Y}'_{11},\widehat{Y}'_{12})$ possess property Q_2 relative to vector-functions $t^l|\widehat{p}_{jj}(v)|\psi_j^{(0)}(t,v)$, $j=\overline{1,p}$, $k=1,2$.

Let us introduce intermediary functions $\widetilde{\alpha}_{jk}(t)$, $\widehat{\alpha}_{jk}(v)$, $j,k=\overline{1,p}$,

$$\left.\begin{aligned}\cos(\widetilde{\alpha}_{jk}(t)) &= \frac{\widetilde{p}_{jk1}(t)}{\sqrt{(\widetilde{p}_{jk1}(t))^2+(\widetilde{p}_{jk2}(t))^2}}\,,\\ \sin(\widetilde{\alpha}_{jk}(t)) &= \frac{\widetilde{p}_{jk2}(t)}{\sqrt{(\widetilde{p}_{jk1}(t))^2+(\widetilde{p}_{jk2}(t))^2}}\,,\end{aligned}\right.\quad j,k=\overline{1,p}, \tag{1.8}$$

$$\left.\begin{aligned}\cos(\widehat{\alpha}_{jk}(t)) &= \frac{\widehat{p}_{jk1}(t)}{\sqrt{(\widehat{p}_{jk1}(t))^2+(\widehat{p}_{jk2}(t))^2}}\,,\\ \sin(\widehat{\alpha}_{jk}(t)) &= \frac{\widehat{p}_{jk2}(t)}{\sqrt{(\widehat{p}_{jk1}(t))^2+(\widehat{p}_{jk2}(t))^2}}\,,\end{aligned}\right.\quad j,k=\overline{1,p}. \tag{1.9}$$

Without loss of generality, let us suppose that $t_1 \le R_1$ and introduce the domains $\Lambda_{+.k}(t_1)$, $k \in \{+,-\}$ defined as follows:

$$\Lambda_{+.+}(t_1) = \Big\{(t,v):\ \cos\big((l-1)v + \widetilde{\alpha}_{jj}(t)\big) > 0,\ \sin\big((l-1)v + \widehat{\alpha}_{jj}(v)\big) > 0,$$
$$j = \overline{1,p},\ t \in (0,t_1),\ v \in (v_1,v_2)\Big\};$$
$$\Lambda_{+.-}(t_1) = \Big\{(t,v):\ \cos((l-1)v + \widetilde{\alpha}_{jj}(t)) > 0,\ \sin((l-1)v + \widehat{\alpha}_{jj}(v)) < 0,$$
$$j = \overline{1,p},\ t \in (0,t_1),\ v \in (v_1,v_2)\Big\}.$$

Definition 1.6. We say that system (1.1) belongs to the class $C_{+.k}$, $k \in \{+,-\}$, if for the matrix $P(z) = P(te^{iv})$ the condition $(t,v) \in \Lambda_{+.k}(t_1)$, $k \in \{+,-\}$ is true.

1.5 On the existence of a solution of problem (1.1), (1.2), (1.3)

Let us introduce the domains $G_{+.k}(t_1) = \{z = z(t,v):\ 0 < |z| < t_1,\ (t,v) \in \Lambda_{+.k}(t_1)\}$, $k \in \{+,-\}$.

Theorem 1.1. *For system* (1.1)*, let the following conditions be fulfilled:*

(1) *The matrix $P(z)$ is analytic in the domain D_1 and possesses property S_{2l} relative to the analytic vector-function $\varphi^{(0)}(z)$.*

(2) *The vector-function $F(z,Y_1,Y_1')$ is analytic in the domain $D_1 \times G_{11} \times G_{21}, F(0,0,0) = 0$ and possesses property M_{2l} relative to the analytic vector-function $\varphi^{(0)}(z)$.*

(3) *System* (1.1) *belongs to one of the classes $C_{+.k}$, $k \in \{+,-\}$.*

Then for each $k \in \{+,-\}$ and for some $t^* \in (0,t_1)$ there exist analytic solutions $Y_1(z)$ of system (1.1) *that satisfy the initial condition $Y_1(z_0) = Y_{10}$ for $z_0 \in G_{+.k}(t^*)$, $Y_{10} \in \{Y_1 :\ |Y_{1j}(z_0)| < \delta_j|\varphi_j^{(0)}(z_0)|,\ \delta_j > 0,\ j = \overline{1,p}\}$. These solutions are analytic in the domain $D_1 \cap G_{+.k}(t^*)$ and satisfy the inequalities*

$$|Y_{1j}(z)|^2 < \delta_j^2|\varphi_j^{(0)}(z)|^2,\ \ j = \overline{1,p}. \tag{1.10}$$

Proof. (1) Let us consider system (1.1) over the interval $L_{v_0}(t_1)$ for an arbitrary fixed $v_0 \in (v_1,v_2)$.

We introduce the sets

$$\widetilde{\Omega}_j\big(\delta,\varphi^{(0)}(z(t,v_0))\big) = \Big\{(t,\widetilde{Y}_{11},\widetilde{Y}_{12}):\ \widetilde{Y}_{11j}^2 + \widetilde{Y}_{12j}^2 < \delta_j^2(\psi_j^{(0)}(t,v_0))^2,\ t \in (0,t_1)\Big\},\ \ j = \overline{1,p}.$$

Thus the set $\widetilde{\Omega}(\delta,\varphi^{(0)}(z(t,v_0)))$ can be considered as intersection of the sets $\widetilde{\Omega}_j$ of the form

$$\widetilde{\Omega}\big(\delta,\varphi^{(0)}(z(t,v_0))\big) = \bigcap_{j=1}^{p} \widetilde{\Omega}_j\big(\delta,\varphi^{(0)}(z(t,v_0))\big).$$

A part of the boundary of the set $\widetilde{\Omega}_j$, $j \in \{1,2,\dots,p\}$, will be denoted by

$$\partial\widetilde{\Omega}_j\big(\delta,\varphi^{(0)}(z(t,v_0))\big) = \Big\{(t,\widetilde{Y}_{11},\widetilde{Y}_{12}):\ \widetilde{Y}_{11j}^2 + \widetilde{Y}_{12j}^2 = \delta_j^2(\psi_j^{(0)}(t,v_0))^2,$$
$$\widetilde{Y}_{11k}^2 + \widetilde{Y}_{12k}^2 < \delta_k^2(\psi_j^{(0)}(t,v_0))^2,\ k = \overline{1,p},\ k \ne j,\ t \in (0,t_1)\Big\}.$$

Assume

$$\widetilde{\Phi}_j(t,\widetilde{Y}(t)) = \widetilde{Y}_{11j}^2(t) + \widetilde{Y}_{12j}^2(t) - \delta_j^2(\psi_j^{(0)}(t,v_0))^2,\ \ j \in \{1,2,\dots,p\}.$$

Then the outward normal vector for the surface $\partial(\widetilde{\Omega}_j)(\delta,\psi(z(t,v_0)))$, for the fixed $j \in \{1,\dots,p\}$, will take the form

$$\frac{\overline{N_j}}{2} = (-\delta_j^2\psi_j^{(0)}(t,v_0))\Big((\psi_j^{(0)}(t,v_0))_t', 0,\dots,0,\widetilde{Y}_{11j},0,\dots,0,\widetilde{Y}_{12j},0,\dots,0\Big).$$

Let $\overline{T}$ be a slope-field vector of system (1.5) at an arbitrary fixed point $(t^*, \widetilde{Y}_{11}(t^*), \widetilde{Y}_{12}(t^*)) \in \partial\widetilde{\Omega}_j(\delta, \varphi^{(0)}(z(t, v_0)))$, $j \in \{1, \dots, p\}$.

Consider the dot product

$$\Big(t^l\overline{T}, \frac{\overline{N}_j}{2}\Big) = -t^l\delta_j^2\psi_j^{(0)}(t, v_0)(\psi_j^{(0)}(t, v_0))'_t$$
$$+ t^l\Big(\widetilde{p}_{jj1}(t)\cos((l-1)v_0) - \widetilde{p}_{jj2}(t)\sin((l-1)v_0)\Big)\delta_j^2\big(\psi_j^{(0)}(t, v_0)\big)^2$$
$$+ t^l\sum_{k=1,\ k\neq j}^{p}\Big(\widetilde{p}_{jk1}(t)\cos\big(((l-1)v_0) - \widetilde{p}_{jk2}(t)\sin((l-1)v_0)\big)\big(\widetilde{Y}_{11k}\widetilde{Y}_{11j} + \widetilde{Y}_{12k}\widetilde{Y}_{12j})\Big)$$
$$+ t^l\sum_{k=1,\ k\neq j}^{p}\Big(\widetilde{p}_{jk1}(t)\sin\big(((l-1)v_0) + \widetilde{p}_{jk2}(t)\cos((l-1)v_0)\big)\big(\widetilde{Y}_{11k}\widetilde{Y}_{12j} - \widetilde{Y}_{12k}\widetilde{Y}_{11j})\Big)$$
$$+\big(\widetilde{F}_{1j}\cos((l-1)v_0)+\widetilde{F}_{2j}\sin((l-1)v_0)\big)\widetilde{Y}_{11j}+\big(-\widetilde{F}_{1j}\sin((l-1)v_0)+\widetilde{F}_{2j}\cos((l-1)v_0)\big)\widetilde{Y}_{12j},\quad j=\overline{1,p}.$$

Since by condition the matrix $P(z)$ possesses property S_{2l} and the vector-function $F(z, Y_1, Y_1')$ possesses property M_{2l} relative to the vector-function $\varphi^{(0)}(z)$, we have

$$\Big(t^l\overline{T}, \frac{\overline{N}_j}{2}\Big) \sim \sqrt{(\widetilde{p}_{jj1}(t))^2 + (\widetilde{p}_{jj2}(t))^2}\,\big(\cos((l-1)v_0 + \widetilde{\alpha}_{jj}(t))\big),\quad j = \overline{1,p},$$

as $t \to +0$, where the functions $\widetilde{\alpha}_{jj}(t)$ are defined by equalities (1.8).

According to the fact that system (1.1) pertains to one of the classes $C_{+.k}(t, v)$, $k \in \{+, -\}$, there exists t^* such that for $t \in (0, t^*)$ the inequality $(t^l\overline{T}, \frac{\overline{N}_j}{2}) > 0$, $j = \overline{1,p}$, holds true. Thus, for $t \in (0, t^*)$, $\partial\widetilde{\Omega}(\delta, \varphi^{(0)}(z(t, v_0)))$ is the surface without contact for system (1.5). Moreover, the integral curve enters the domain $\widetilde{\Omega}(\delta, \varphi^{(0)}(z(t, v_0)))$ as the variable t decreases.

According to the topological principle of T. Wazewski [13], at least one smooth integral curve of system (1.5) goes through every point of the set $\widetilde{\Omega}(\delta, \varphi^{(0)}(z(t, v_0))) \cup \partial\widetilde{\Omega}(\delta, \varphi^{(0)}(z(t, v_0))) \cap (t = t^*)$. All integral curves of this system going through the points $\widetilde{\Omega}(\delta, \varphi^{(0)}(z(t, v_0))) \cup \partial\widetilde{\Omega}(\delta, \varphi^{(0)}(z(t, v_0))) \cap (t = t^*)$, remain in the domain $\widetilde{\Omega}(\delta, \varphi^{(0)}(z(t, v_0)))$ for $(t, v_0) \in \Lambda_{+.k}(t^*)$, $k \in \{+, -\}$, $v_0 \in (v_1, v_2)$. Moreover, the inequalities

$$|Y_{1sj}(z(t, v_0))|^2 < \delta_j^2\big(\psi_j^{(0)}(t, v_0)\big)^2,\quad j = \overline{1,p},\quad s = 1, 2, \tag{1.11}$$

are fulfilled for $(t, v_0) \in \Lambda_{+.k}(t^*)$, $k \in \{+, -\}$.

(2) Consider system (1.1) over the arc of circle $O_{t_1}(t_0)$ for an arbitrary fixed $t_0 \in (0, t_1)$.

Let us introduce the sets

$$\widehat{\Omega}_j\big(\sigma, \varphi^{(0)}(z(t_0, v))\big) = \Big\{(v, \widehat{Y}_{11}, \widehat{Y}_{12}):\ \widehat{Y}_{11j}^2 + \widehat{Y}_{12j}^2 < \sigma_j^2(\psi_j^{(0)}(t_0, v))^2,\ v \in (v_1, v_2)\Big\},\quad j = \overline{1,p}.$$

Thus the set $\widehat{\Omega}(\sigma, \varphi^{(0)}(z(t_0, v)))$ can be considered as the intersection of sets $\widehat{\Omega}_j$ of the form

$$\widehat{\Omega}(\sigma, \varphi^{(0)}(z(t_0, v))) = \bigcap_{j=1}^{p}\widehat{\Omega}_j\big(\sigma, \varphi^{(0)}(z(t_0, v))\big).$$

A part of the boundary of the set $\widehat{\Omega}_j$, $j \in \{1, 2, \dots, p\}$ is denoted by

$$\partial\widehat{\Omega}_j\big(\sigma, \varphi^{(0)}(z(t_0, v))\big) = \Big\{(v, \widehat{Y}_{11}, \widehat{Y}_{12}):\ \widehat{Y}_{11j}^2 + \widehat{Y}_{12j}^2 = \sigma_j^2(\psi_j^{(0)}(t_0, v))^2,$$
$$\widehat{Y}_{11k}^2 + \widehat{Y}_{12k}^2 < \sigma_k^2(\psi_j^{(0)}(t_0, v))^2,\ k = \overline{1,p},\ k \neq j,\ t \in (0, t_1)\Big\}.$$

Let $\overline{T}$ be a slope-field vector of system (1.7) at an arbitrary fixed point $(t^*, \widehat{Y}_{11}(t^*), \widehat{Y}_{12}(t^*)) \in \partial\widehat{\Omega}_j(\sigma, \varphi(z(t_0, v)))$, for the fixed $j \in \{1, \dots, p\}$,

$$\begin{aligned}\Big(t_0^{l-1}\overline{T}, \frac{\overline{N}_j}{2}\Big) &= -t_0^{l-1}\sigma_j^2\psi_j^{(0)}(t_0, v)(\psi_j^{(0)}(t_0, v))'_v \\ &\quad + t_0^l\Big(\widehat{p}_{jj1}(v)\cos((l-1)v) - \widehat{p}_{jj2}(v)\sin((l-1)v)\Big)\sigma_j^2(\psi_j^{(0)}(t_0, v))^2 \\ &\quad + t_0^l \sum_{k=1,\, k\neq j}^{p} \Big(\widehat{p}_{jk1}(v)\cos\big(((l-1)v) - \widehat{p}_{jk2}(v)\sin((l-1)v)\big)\big(\widehat{Y}_{11k}\widehat{Y}_{12j} - \widehat{Y}_{12k}\widehat{Y}_{11j}\big)\Big) \\ &\quad + t_0^l \sum_{k=1,\, k\neq j}^{p} \Big(-\widehat{p}_{jk1}(v)\sin\big(((l-1)v) - \widehat{p}_{jk2}(v)\cos((l-1)v)\big)\big(\widehat{Y}_{12k}\widehat{Y}_{11j} + \widehat{Y}_{12k}\widehat{Y}_{12j}\big)\Big) \\ &\quad + \big(\widehat{F}_{1j}\sin((l-1)v) + \widehat{F}_{2j}\cos((l-1)v)\big)\widehat{Y}_{11j} + \big(\widehat{F}_{1j}\cos((l-1)v) + \widehat{F}_{2j}\sin((l-1)v)\big)\widehat{Y}_{12j}, \quad j = \overline{1, p}.\end{aligned}$$

Since by the condition the matrix $P(z)$ possesses property S_{2l} and the vector-function $F(z, Y_1, Y_1^{'})$ possesses property M_{2l} relative to the vector-function $\varphi^{(0)}(z)$, we have

$$\Big(t_0^{l-1}\overline{T}, \frac{\overline{N}_j}{2}\Big) \sim \sqrt{(\widehat{p}_{jj1}(v))^2 + (\widehat{p}_{jj2}(v))^2}\,\big(\sin((l-1)v) + \widehat{\alpha}_{jj}(v)\big), \quad j = \overline{1, p},$$

as $t \to +0$, $v \in (v_1, v_2)$, where the functions $\widehat{\alpha}_{jj}(v)$ are defined by equalities (1.9). Thus

$$\operatorname{sign}\Big(t_0^{l-1}\overline{T}, \frac{\overline{N}_j}{2}\Big) = \operatorname{sign}\big(\sin((l-1)v) + \widehat{\alpha}_{jj}(v)\big), \quad j = \overline{1, p}, \quad v \in (v_1, v_2).$$

Without loss of generality, we suppose that for each fixed $t_0 \in (0, t^*)$, $\partial\widehat{\Omega}(\sigma, \varphi^{(0)})(z(t_0, v)) \in \Lambda_{+.k}(t^*)$, $k \in \{+, -\}$ is the surface without contact for system (1.7).

According to the fact that system (1.1) belongs to one of the classes $C_{+.k}(t, v)$, $k \in \{+, -\}$, any integral curve of system (1.7) going through the point of the set $\widehat{\Omega}(\sigma, \varphi^{(0)}(z(t_0, v))) \cap (v = v_0)$, $v_0 \in (v_1, v_2)$, remains in the domain $\widehat{\Omega}(\sigma, \varphi^{(0)}(z(t_0, v)))$ under the condition that variable v decreases if $(t_0, v_0) \in \Lambda_{+.+}(t^*)$, and v increases if $(t_0, v_0) \in \Lambda_{+.-}(t^*)$.

Moreover, the inequalities

$$|Y_{1sj}(z(t_0, v))|^2 < \sigma_j^2\big(\psi_j^{(0)}(t_0, v)\big)^2, \quad j = \overline{1, p}, \quad s = 1, 2, \tag{1.12}$$

hold true for $(t_0, v) \in \Lambda_{+.k}(t^*)$, $k \in \{+, -\}$.

(3) Let us use the method of analytic continuation of solutions for the problems that are solved relatively to the derivatives, i.e., the method suggested by R. Grabovskaya [3] and developed by G. Samkova [7, 8] for the problems that are not solved relatively to the derivatives and also used by D. Limanska and G. Samkova [6] in the proof of the third point of Theorem 2 [6].

Let us suppose that for vectors $\delta, \sigma \in \mathbb{C}^p$, $\delta_j \neq 0$, $\sigma_j \neq 0$, $j = \overline{1, p}$, the inequalities

$$(\delta_j)^2 < (\sigma_j)^2, \quad j = \overline{1, p}, \tag{1.13}$$

are true.

In the proof of item (1) of the theorem, we have got the fact that there are infinitely many continuously differentiable solutions of system (1.5) over the interval $v_0 \in (v_1, v_2)$ for $t \in (0, t^*)$, and these solutions satisfy inequality (1.11). We denote a set of such solutions by $\{Y_1(z(t, v_0))\}$.

Any solution $Y_1(z(t, v_0))$ from the set $\{Y_1(z(t, v_0))\}$ is analytically continuable from the interval $L_{v_0}(t_1)$, where $(t, v) \in \Lambda_{+.k}(t^*)$, for fixed $v_0 \in (v_1, v_2)$, to the domain containing this interval, with preservation of inequalities (1.12).

From the proof of item 2 of the theorem it follows that if inequalities (1.13) are fulfilled, then the solution $Y_1(z(t, v))$ for fixed $v = v_0$ can be continued from the interval $L_{v_0}(t_1)$ over the curves $O_{t_1}(t_0)$ to the set $\widehat{\Omega}(\sigma, \varphi^{(0)}(z(t^*, v)))$ for $t \in (0, |z(t_0, v)|\,]$. We denote the obtained analytic continuation by $Y_1(z)$. The set of solutions of system (1.4) is $\{Y_1(z)\}$.

As a result, the solutions $Y_1(z)$ of system (1.1) are analytically continuable to the domain $G_{+.k}(t^*) \times \{Y : |Y_{1j}| < \delta_j|\varphi_j^{(0)}(z(t_0, v))|,\ j = \overline{1, p}\}$, and, moreover, in this domain solutions $Y_1(z)$ satisfy inequality (1.10). □

2 The main results for system (0.1)

Let us consider the system of ordinary differential equations (0.1) under the conditions that $p < n$, $A(z)$ is an analytic matrix in the domain D_1, and $\operatorname{rank} A(z) = p$ for $z \in D_1$. Let us introduce the function $Y = \operatorname{col}\begin{pmatrix} Y_1 & Y_2 \end{pmatrix}$, $Y_1 = \operatorname{col}(Y_{11}(z), \dots, Y_{1p}(z))$, $Y_2 = \operatorname{col}(Y_{21}(z), \dots, Y_{2n-p}(z))$, $Y_1 : D_1 \to \mathbb{C}^p$, $Y_2 : D_1 \to \mathbb{C}^{n-p}$. Without loss of generality, we assume that the matrices $A(z)$, $B(z)$ and the vector-function $f(z, Y, Y')$ take the forms

$$A(z) = \begin{pmatrix} A_1(z) & A_2(z) \end{pmatrix}, \quad B(z) = \begin{pmatrix} B_1(z) & B_2(z) \end{pmatrix}, \quad f(z, Y, Y') = f^*(z, Y_1, Y_2, Y_1', Y_2'),$$

$A_1 : D_1 \to \mathbb{C}^{p\times p}$, $A_2 : D_1 \to \mathbb{C}^{p\times(n-p)}$, $B_1 : D_1 \to \mathbb{C}^{p\times p}$, $B_2 : D_1 \to \mathbb{C}^{p\times(n-p)}$, $\det A_1(z) \neq 0$ for $z \in D_1$, $f^* : D_1 \times G_{11} \times G_{12} \times G_{21} \times G_{22} \to C^p$, $G_{j1} \times G_{j2} = G_j$, $G_{j1} \subset C^p$, $G_{j2} \subset C^{n-p}$, $j = 1, 2$.

Due to the above-said, system (0.1) can be written as

$$Y_1' = A_1^{-1}(z)B_1(z)Y_1 + A_1^{-1}(z)B_2(z)Y_2 - A_1^{-1}(z)A_2(z)Y_2' + A_1^{-1}(z)f^*(z, Y_1, Y_2, Y_1', Y_2') \tag{2.1}$$

Suppose that the matrices $A_1^{-1}(z)B_1(z)$, $A_1^{-1}(z)A_2(z)$, $A_1^{-1}(z)B_2(z)$ are analytic in the domain D_{10} and have removable singularity at the point $z = 0$.

Let us introduce

$$\begin{gathered} P(z) = A_1^{-1}(z)B_1(z), \\ F^*(z, Y_1, Y_2, Y_1', Y_2') = A_1^{-1}(z)B_2(z)Y_2 - A_1^{-1}(z)A_2(z)Y_2' + A_1^{-1}f^*(z, Y_1, Y_2, Y_1', Y_2'), \end{gathered} \tag{2.2}$$

then system (1.1) can be written as

$$Y_1' = P(z)Y_1 + F^*(z, Y_1, Y_2, Y_1', Y_2'), \tag{2.3}$$

where $P(z)$ is the matrix, analytic in the domain D_{10} having removable singularity at the point $z = 0$, and $P : D_{10} \times \mathbb{C}^{p\times p}$, $F^*(z, Y_1, Y_2, Y_1', Y_2')$ is the vector-function, analytic in the domain $D_{10} \times G_{110} \times G_{120} \times G_{210} \times G_{220}$, $G_{jk0} = G_{jk} \setminus \{0\}$, $j, k = 1, 2$. Therefore, the vector-function $F^*(z, Y_1, Y_2, Y_1', Y_2')$ has isolated singularity at the point $(0, 0, 0, 0, 0)$. This means that according to the theorem on the isolated singularity of the function of several complex variables, the point $(0, 0, 0, 0, 0)$ is a removable singular point of that function.

Let us define the vector-function $F^*(z, Y_1, Y_2, Y_1', Y_2')$ at the point $(0, 0, 0, 0, 0)$ in such a way that it becomes analytic in the domain $D_1 \times G_{11} \times G_{12} \times G_{21} \times G_{22}$. Without loss of generality, assume that $F^*(0, 0, 0, 0, 0) = 0$.

By H_r^{n-p} we basically mean a class of $(n-p)$-dimensional analytic in the domain D_{10} functions that have pole of r-order at the point $z = 0$.

Let us consider system (2.3) for an arbitrary fixed vector-function $Y_2 \in H_r^{n-p}$. Then the function $Y_2 = Y_2(z)$ can be written as

$$Y_2(z) = z^{-r}Y_2^*(z), \tag{2.4}$$

where $r \in \mathbb{N}$, $Y_2^*(z)$ is an analytic vector-function in the domain D_1 such that $Y_2^*(0) \neq 0$. Moreover, the function $Y_2^*(z)$ is represented as a convergent power series for $z \in D_1$. Therefore, (2.4) in the domain D_{10} takes the form

$$Y_2(z) = \sum_{k=0}^{\infty} C_k z^{k-r},$$

where $C_k \in \mathbb{C}^{n-p}$, $k = 0, 1, 2, \dots, _0 \neq 0$.

Since $C_0 \neq 0$, the vector-function $Y_2'(z)$ has a pole of $r+1$-order at the point $z = 0$.

Since the vector-function $F^*(z, Y_1, Y_2, Y_1', Y_2')$ is analytic in the domain $D_1 \times G_{11} \times G_{12} \times G_{21} \times G_{22}$ and $F^*(0, 0, 0, 0, 0) = 0$, we get that $F^*(z, Y_1, Y_2, Y_1', Y_2')$ can be represented as a convergent power series

$$F^*(z, Y_1, Y_2, Y_1', Y_2') = \sum_{a+|j|+|k|+|b|+|d|=1}^{\infty} C_a jkbd z^a Y_1^j Y_2^k (Y_1')^b (Y_2')^d$$

near the point (0,0,0,0,0), where $C_{ajkbd} \in \mathbb{C}^p$, $j = (j_1, \dots, j_p)$, $(Y_1)^j = (Y_{11})^{j_1} \cdots (Y_{1p})^{j_p}$, $|j| = j_1 + \dots + j_p$, $k = (k_1, \dots, k_{n-p})$, $(Y_2)^k = (Y_{21})^{k_1} \cdots (Y_{2(n-p)})^{k_{n-p}}$, $|k| = k_1 + \dots + k_{n-p}$, $b = (b_1, \dots, b_p)$, $(Y_1')^b = (Y_{11}')^{b_1} \cdots (Y_{1p}')^{b_p}$, $|b| = b_1 + \dots + b_p$, $d = (d_1, \dots, d_{n-p})$, $(Y_2')^d = (Y_{21}')^{d_1} \cdots (Y_{2(n-p)}')^{d_{n-p}}$, $|d| = d_1 + \dots + d_{n-p}$.

Assume that there exist $q \in \mathbb{N}$ and $s \in \mathbb{N}$ such that

(1) for some $a_0 \in \mathbb{N}$, $j_0 = (j_{01}, \dots, j_{0p})$, $j_{1h} \in \mathbb{N} \cup \{0\}$, $b_0 = (b_{01}, \dots, b_{0p})$, $b_{0h} \in \mathbb{N} \cup \{0\}$, $h = \overline{1,p}$, we have $C_{a_0 j_0 k b_0 d} \neq 0$ for $|k| = q$, $|d| = s$;

(2) for any $h, m \in \mathbb{N}$ and $u = 1, 2, \dots, n-p$, $c = 1, 2, \dots, n-p$, we have $C_{aj(k+he_u)b(d+me_c)} = 0$,

where e_u is the $(n-p)$-dimensional uth orthogonal unit vector, and e is the $(n-p)$-dimensional cth orthogonal unit vector.

Consequently, the summands in the power series expansion of function F^* in the neighbourhood of the point $(0,0,0,0,0)$, containing the maximum powers of vector-functions Y_2 and Y_2' with non-zero coefficients, take the form

$$\begin{aligned} C_{ajkbd} z^a Y_1^j Y_2^k (Y_1')^b (Y_2')^d &= C_{ajkbd} z^a Y_1^j (z^{-r} Y_2^*)^k (Y_1')^b (z^{-r} Y_2^{*\prime} - r z^{-r-1} Y_2^*)^d \\ &= C_{ajkbd} z^{a - rq - (r+1)s} Y_1^j (Y_2^*)^k (Y_1')^b (z Y_2^{*\prime} - r Y_2^*)^d, \end{aligned}$$

for $a = 0, 1, 2 \dots$, $|j| = 0, 1, 2, \dots$, $|b| = 0, 1, 2, \dots$, $|k| = q$, $|d| = s$ and, at least, if $a = a_0$, $j = j_0$, $b = b_0$.

Two logical cases are possible:

(1) $a - rq - (r+1)s \geq 0$. Then for an arbitrary fixed function $Y_2 \in H_r^{n-p}$, we have $F^*(z, Y_1, Y_2, Y_1', Y_2') = F(z, Y_1, Y_1')$, where $F(z, Y_1, Y_1')$ is analytic at the point (0,0,0), and system (2.1) is reduced to the system

$$Y_1' = P(z) Y_1 + F(z, Y_1, Y_1'). \tag{2.5}$$

According to Theorem 1.1 of [6, p. 22], the sufficient conditions for the existence of analytic solutions of the Cauchy problem (2.5), (1.2) with the additional condition (1.3) are found.

(2) $a - rq - (r+1)s < 0$. Let us introduce $l = rq + (r+1)s - a$, then the vector-function $F^*(z, Y_1, Y_2, Y_1', Y_2')$ may take the form

$$\begin{aligned} F^*(z, Y_1, Y_2, Y_1', Y_2') &= z^{-l} \sum_{a+|j|+|k|+|b|+|d|=0}^{\infty} C_{ajkbd} z^a Y_1^j (Y_2^*)^k (Y_1')^b (z Y_2^{*\prime} - r Y_2^*)^d \\ &= z^{-l} \dot{F}(z, Y_1, Y_2^*, Y_1', Y_2^{*\prime}), \end{aligned}$$

where $\dot{F}(z, Y_1, Y_2^*, Y_1', Y_2^{*\prime})$ is the analytic vector-function in the domain $D_1 \times G_{11} \times G_{12} \times G_{21} \times G_{22}$. Without loss of generality, we assume that $\dot{F}(z, Y_1, Y_2^*, Y_1', Y_2^{*\prime}) = F(z, Y_1, Y_1')$, and $F(0,0,0) = 0$.

According to (2.2), system (0.1) takes form (1.1). Let us consider the problem on the existence and asymptotic behavior of the solutions of system (0.1) that satisfy the initial condition (1.2) and the additional condition (1.3).

Theorem 2.1. *Let $p < n$, $A(z)$ be an analytic matrix in the domain D_1,* rank $A(z) = p$ *for $z \in D_1$. Moreover, let system* (0.1) *take form* (2.3)*, and for $Y_2 \in H_r^{n-p}$, conditions* (1)–(3) *of Theorem* 1.1 *be true for the associate system* (1.1).

Then for each $k \in \{+, -\}$, some $t^* \in (0, t_1)$ and for each $Y_2 \in H_r^{n-p}$ there exist analytic solutions $Y(z) = (Y_{11}(z), \dots, Y_{1p}(z), Y_{21}(z), \dots, Y_{2n-p}(z))$ of system (1.1). *The first p-elements of these solutions are analytic in the domain $D_1 \cap G_{+.k}(t^*)$ and satisfy inequality* (1.10).

Proof. According to Theorem 1.1, the solution $Y_1(z)$ of system (1.1) is analytically continuable on $G_{+.k}(t^*) \times \{Y : \ |Y_{1j}| < \delta_j |\varphi_j(z)|, \ j = \overline{1,p}\}$. Moreover, the solution satisfies inequality (1.10) in this domain. Therefore, system (0.1), for an arbitrary fixed function $Y_2 \in H_r^{n-p}$, has solutions $Y = (Y_1(z), Y_2(z))$, the first p-elements of which are analytic in the domain $G_{+.k}(t^*) \times \{Y : \ |Y_{1j}| < \delta_j |\varphi_j(z)|, \ j = \overline{1,p}\}$ and satisfy inequality (1.10) for $z \in D_1 \cap G_{+.k}(t^*)$. □

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(Received 14.12.2017)

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Memoirs on Differential Equations and Mathematical Physics

Volume 74, 2018, 125–140

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VARIATION FORMULAS OF SOLUTIONS FOR CONTROLLED FUNCTIONAL DIFFERENTIAL EQUATIONS WITH THE CONTINUOUS INITIAL CONDITION WITH REGARD FOR PERTURBATIONS OF THE INITIAL MOMENT AND SEVERAL DELAYS

Abstract. Variation formulas of solutions for nonlinear controlled functional differential equations are proved which show the effect of perturbations of the initial moment, constant delays and also that of the continuous initial condition.

2010 Mathematics Subject Classification. 34K99.

Key words and phrases. Delay controlled functional differential equation, variation formula of solution, effect of the initial moment perturbation, effect of delay perturbation, effect of the continuous initial condition.

რეზიუმე. არაწრფივი სამართი ფუნქციონალურ-დიფერენციალური განტოლებებისთვის დამტკიცებულია ამონახსნის ვარიაციის ფორმულები, რომლებშიც გამოვლენილია საწყისი მომენტისა და დაგვიანებების შეშფოთების ეფექტი, აგრეთვე უწყვეტი საწყისი პირობის ეფექტი.

1 Introduction and formulation of main results

The term "variation formula of a solution" has been introduced by R. V. Gamkrelidze and proved in [2] for the ordinary differential equation. The effects of perturbation of the initial moment and the discontinuous initial condition in the variation formulas of solutions (shortly, variation formulas) were revealed by T. A. Tadumadze in [4] for the delay differential equation.

In the present paper, for the controlled functional differential equation

$$\dot{x}(t)=f\big(t,x(t),x(t-\tau_1),\dots,x(t-\tau_s),u(t)\big)$$

with the continuous initial condition

$$x(t)=\varphi(t),\ \ t\le t_0,$$

the variation formulas are proved in the framework of new wide classes of variations of the initial data. The continuity of the initial condition means that the values of the initial function and the trajectory always coincide at the initial moment, i.e., $x(t_0)=\varphi(t_0)$. In [5,9], the variation formulas were proved for the equations

$$\dot{x}(t)=f\big(t,x(t),x(t-\tau)\big),\ \ t\in[t_0,t_1],$$
$$\dot{x}(t)=f\big(t,x(t),x(t-\tau),u(t)\big),\ \ t\in[t_0,t_1],$$

respectively, in the case where the initial moment and delay variations had the same signs. In this paper, the essential novelty is that here we consider the equation with several delays, the variation formulas are proved for the controlled functional differential equations with several delays and the variations of the initial moment and delays are, in general, of different signs.

The variation formula plays the basic role in proving of the necessary conditions of optimality [2,3]. The variation formulas for various classes of controlled functional differential equations without perturbation of delays are derived in [1,3,7,8].

Let $I=[a,b]$ be a finite interval and $0<\theta_{i1}<\theta_{i2}$, $i=1,\dots,s$, be the given numbers; suppose that $O\subset\mathbb{R}^n$ and $U_0\subset\mathbb{R}^r$ are the open sets. Let the n-dimensional function $f(t,x,x_1,\dots,x_s,u)$ satisfy the following conditions: for almost all fixed $t\in I$, the function $f(t,\cdot\,):O^{1+s}\times U_0\to\mathbb{R}^n$ is continuously differentiable; for each fixed $(x,x_1,\dots,x_s,u)\in O^{1+s}\times U_0$, the functions $f(t,x,x_1,\dots,x_s,u)$, $f_x(t,\cdot\,)$, $f_{x_i}(t,\cdot\,)$, $i=1,\dots,s$, and $f_u(t,\cdot\,)$ are measurable on I; for arbitrary compact sets $K\subset O$, $U\subset U_0$, there exists a function $m_{K,U}(t)\in L_1(I,\mathbb{R}_+)$, $\mathbb{R}_+=[0,\infty)$ such that

$$|f(t,x,x_1,\dots,x_s,u)|+|f_x(t,\cdot\,)|+\sum_{i=1}^{s}|f_{x_i}(t,\cdot\,)|+|f_u(t,\cdot\,)|\le m_{K,U}(t)$$

for all $(x,x_1,\dots,x_s,u)\in K^{1+s}\times U$ and for almost all $t\in I$.

Let Φ be a set of continuous functions $\varphi:I_1=[\widehat{\tau},b]\to O$, where $\widehat{\tau}=a-\max\{\theta_{12},\dots,\theta_{s2}\}$ and let Ω be a set of measurable functions $u(t)$, $t\in I$, satisfying the condition $\operatorname{clu}(I)\subset U_0$ and be compact in $\mathbb{R}^r$.

To each element $\mu=(t_0,\tau_1,\dots,\tau_s,\varphi,u)\in\Lambda=[a,b)\times[\theta_{11},\theta_{12}]\times\cdots\times[\theta_{s1},\theta_{s2}]\times\Phi\times\Omega$ we assign the delay controlled functional differential equation

$$\dot{x}(t)=f\big(t,x(t),x(t-\tau_1),\dots,x(t-\tau_s),u(t)\big) \tag{1.1}$$

with the continuous initial condition

$$x(t)=\varphi(t),\ \ t\in[\widehat{\tau},t_0]. \tag{1.2}$$

Definition 1.1. Let $\mu=(t_0,\tau_1,\dots,\tau_s,\varphi,u)\in\Lambda$. A function $x(t)=x(t;\mu)\in O$, $t\in[\widehat{\tau},t_1]$, $t_1\in(t_0,b]$, is called a solution of equation (1.1) with the initial condition (1.2) or a solution corresponding to the element μ and defined on the interval $[\widehat{\tau},t_1]$ if it satisfies condition (1.2) and is absolutely continuous on the interval $[t_0,t_1]$, and satisfies equation (1.1) almost everywhere (a.e.) on $[t_0,t_1]$.

Let us introduce a set of variations:

$$V=\Big\{\delta\mu=(\delta t_0,\delta\tau_1,\dots,\delta\tau_s,\delta\varphi,\delta u):\ \ |\delta t_0|\le\alpha,|\delta\tau_i|\le\alpha,\ \ i=1,\dots,s,\\ \delta\varphi=\sum_{i=1}^k\lambda_i\delta\varphi_i,\ \ |\lambda_i|\le\alpha,\ \ \|\delta u\|\le\alpha,\ \ i=1,\dots,k\Big\},\tag{1.3}$$

where $\delta\varphi_i\in\Phi-\varphi_0$, $i=1,\dots,k$, and $\varphi_0\in\Phi$ are fixed functions; $\alpha>0$ is a fixed number and $\|\delta u\|=\sup\{|\delta u(t)|:\ t\in I\}$.

Let $x_0(t)$ be a solution corresponding to the element $\mu_0=(t_{00},\tau_{10},\dots,\tau_{s0},\varphi_0,u_0)\in\Lambda$ and defined on the interval $[\widehat{\tau},t_{10}]$, where $t_{00},t_{10}\in(a,b)$, $t_{00}<t_{10}$ and $\tau_{i0}\in(\theta_{i1},\theta_{i2})$, $i=1,\dots,s$.

There exist the numbers $\delta_1>0$ and $\varepsilon_1>0$ such that for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_1)\times V$ we have $\mu_0+\varepsilon\delta\mu\in\Lambda$ and a solution $x(t;\mu_0+\varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau},t_{10}+\delta_1]\subset I_1$ corresponds to it (see Lemma 2.2).

Due to the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau},t_{10}+\delta_1]$. Therefore, in the sequel, the solution $x_0(t)$ is assumed to be defined on the interval $[\widehat{\tau},t_{10}+\delta_1]$.

Let us define the increment of the solution $x_0(t)=x(t;\mu_0)$:

$$\Delta x(t)=\Delta x(t;\varepsilon\delta\mu)=x(t;\mu_0+\varepsilon\delta\mu)-x_0(t),\ \ (t,\varepsilon,\delta\mu)\in[\widehat{\tau},t_{10}+\delta_1]\times(0,\varepsilon_1)\times V.\tag{1.4}$$

Theorem 1.1. *Let the function $\varphi_0(t)$, $t\in I_1$, be absolutely continuous. Let the functions $\dot\varphi_0(t)$ and $f(w,u),(w,u)\in I\times O^{1+s}\times U_0$, be bounded, where $w=(t,x,x_1,\dots,x_s)$. Moreover, there exist the finite limits*

$$\lim_{t\to t_{00}-}\dot\varphi_0(t)=\dot\varphi_0^-,\quad \lim_{w\to w_0}f(w,u_0(t))=f^-,\ \ w\in(a,t_{00}]\times O^{1+s},$$

where $w_0=(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_{10}),\dots,\varphi_0(t_{00}-\tau_{s0}))$. Then there exist the numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[t_{00},t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^-$, we have

$$\Delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x(t;\delta\mu)+o(t;\varepsilon\delta\mu),[1]\tag{1.5}$$

where $V^-=\{\delta\mu\in V:\delta t_0\le 0\}$ and

$$\delta x(t;\delta\mu)=Y(t_{00};t)(\dot\varphi_0^--f^-)\delta t_0+\beta(t;\delta\mu),\tag{1.6}$$

$$\beta(t;\delta\mu)=Y(t_{00};t)\delta\varphi(t_{00})+\sum_{i=1}^s\int_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi\\ -\int_{t_{00}}^tY(\xi;t)\Big[\sum_{i=1}^sf_{x_i}[\xi]\dot x_0(\xi-\tau_{i0})\delta\tau_i\Big]\,d\xi+\int_{t_{00}}^tY(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi,\tag{1.7}$$

where $Y(\xi;t)$ is the $n\times n$-matrix function satisfying the equation

$$Y_\xi(\xi;t)=-Y(\xi;t)f_x[\xi]-\sum_{i=1}^sY(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}],\ \ \xi\in[t_{00},t],\tag{1.8}$$

and the condition

$$Y(\xi;t)=\begin{cases}\Upsilon & \text{for }\ \xi=t,\\ \Theta & \text{for }\ \xi>t.\end{cases}\tag{1.9}$$

Here,

$$f_{x_i}=\frac{\partial}{\partial x_i}f,\quad f_{x_i}[\xi]=f_{x_i}\big(\xi,x_0(\xi),x_0(\xi-\tau_{10}),\dots,x_0(\xi-\tau_{s0}),u_0(\xi)\big),$$

Υ is the identity matrix and Θ is the zero matrix.

[1]Here and throughout the paper, the symbols $O(t;\varepsilon\delta\mu),o(t;\varepsilon\delta\mu)$ stand for quantities (scalar or vector) having the corresponding order of smallness with respect to ε uniformly with respect to $(t,\delta\mu)$.

Some comments. The function $\delta x(t;\delta\mu)$ is called the first variation of the solution $x_0(t)$, $t\in[t_{00},t_{10}+\delta_2]$, and expression (1.6) is called the variation formula. On the basis of the Cauchy formula for solutions of the linear delay functional differential equation, we conclude that the function

$$\delta x(t)=\begin{cases}\delta\varphi(t), & t\in[\widehat{\tau},t_{00}),\\ \delta x(t;\delta\mu), & t\in[t_{00},t_{10}+\delta_2],\end{cases}$$

is a solution of the equation

$$\dot{\delta x}(t)=f_x[t]\delta x(t)+\sum_{i=1}^{s}f_{x_i}[t]\delta x(t-\tau_{i0})-\sum_{i=1}^{s}f_{x_i}[t]\dot{x}_0(t-\tau_{i0})\delta\tau_i+f_u[t]\delta u(t)$$

with the initial condition

$$\delta x(t)=\delta\varphi(t),\ \ t\in[\widehat{\tau},t_{00}),\ \ \delta x(t_{00})=(\dot{\varphi}_0^- - f^-)\delta t_0+\delta\varphi(t_{00}).$$

The addend $-\int\limits_{t_{00}}^{t}Y(\xi;t)\big[\sum\limits_{i=1}^{s}f_{x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\delta\tau_i\big]\,d\xi$ in formula (1.7) is the effect of perturbations of the delays τ_{i0}, $i=1,\dots,s$.

The expression $Y(t_{00};t)(\dot{\varphi}_0^- - f^-)\delta t_0$ is the effect of the continuous initial condition (1.2) and of the perturbation of the initial moment t_{00}.

The expression $Y(t_{00};t)\delta\varphi(t_{00})+\sum\limits_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi$ in formula (1.6) is the effect of perturbation of the initial function $\varphi_0(t)$.

The expression $\int\limits_{t_{00}}^{t}Y(\xi;t)\delta u[\xi]\,d\xi$ in formula (1.7) is the effect of perturbation of the control function $u_0(t)$.

Theorem 1.2. *Let the function $\varphi_0(t)$, $t\in I_1$, be absolutely continuous. Let the functions $\dot{\varphi}_0(t)$ and $f(w,u),(w,u)\in I\times O^{1+s}\times U_0$, be bounded. Moreover, there exist the finite limits*

$$\lim_{t\to t_{00}+}\dot{\varphi}_0(t)=\dot{\varphi}_0^+,\quad \lim_{w\to w_0}f(w)=f^+,\ \ w\in[t_{00},b)\times O^{1+s}.$$

Then for each $\widehat{t}_0\in(t_{00},t_{10})$, there exist the numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[\widehat{t}_0,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^+$, where $V^+=\{\delta\mu\in V:\ \delta t_0\geq 0\}$, formula (1.5) *holds, where*

$$\delta x(t;\delta\mu)=Y(t_{00};t)(\dot{\varphi}_0^+ - f^+)\delta t_0+\beta(t;\delta\mu). \tag{1.10}$$

The following assertion is a corollary to Theorems 1.1 and 1.2.

Theorem 1.3. *Let the assumptions of Theorems* 1.1 *and* 1.2 *be fulfilled. Moreover, $\dot{\varphi}_0^- - f^- = \dot{\varphi}_0^+ - f^+ := \widehat{f}$. Then for each $\widehat{t}_0\in(t_{00},t_{10})$, there exist the numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[\widehat{t}_0,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V$ formula* (1.5) *holds, where $\delta x(t;\delta\mu)=Y(t_{00};t)\widehat{f}\delta t_0+\beta(t;\delta\mu)$.*

All assumptions of Theorem 1.3 are satisfied if the function $f(t,x,x_1,\dots,x_s,u)$ is continuous and bounded, the function $\varphi_0(t)$ is continuously differentiable and the function $u_0(t)$ is continuous at the point t_{00}. Clearly, in this case,

$$\widehat{f}=\dot{\varphi}_0(t_{00})-f\big(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_{10}),\dots,\varphi_0(t_{00}-\tau_{s0}),u_0(t_{00})\big).$$

2 Auxiliary assertions

To each element $\mu=(t_0,\tau_1,\dots,\tau_s,\varphi,u)\in\Lambda$ we assign the controlled functional differential equation

$$\dot{y}(t)=f(t_0,\tau_1,\dots,\tau_s,\varphi,y,u)(t) \tag{2.1}$$

with the initial condition

$$y(t_0) = \varphi(t_0), \tag{2.2}$$

where

$$f(t_0, \tau_1, \dots, \tau_s, \varphi, y, u)(t) = f\big(t, y(t), h(t_0, \varphi, y)(t - \tau_1), \dots, h(t_0, \varphi, y)(t - \tau_s), u(t)\big)$$

and $h(t_0, \varphi, y)(t)$ is the operator given by the formula

$$h(t_0, \varphi, y)(t) = \begin{cases} \varphi(t), & t \in [\widehat{\tau}, t_0), \\ y(t), & t \in [t_0, b]. \end{cases} \tag{2.3}$$

Definition 2.1. Let $\mu = (t_0, \tau_1, \dots, \tau_s, \varphi, u) \in \Lambda$. An absolutely continuous function $y(t) = y(t; \mu) \in O$, $t \in [r_1, r_2] \subset I$, is called a solution of equation (2.1) with the initial condition (2.2) or a solution corresponding to the element μ and defined on the interval $[r_1, r_2]$ if $t_0 \in [r_1, r_2], y(t_0) = \varphi(t_0)$ and the function $y(t)$ satisfies equation (2.1) (a.e.) on $[r_1, r_2]$.

Remark 2.1. Let $y(t; \mu)$, $t \in [r_1, r_2]$, be a solution corresponding to the element $\mu = (t_0, \tau_1, \dots, \tau_s, \varphi, u) \in \Lambda$. Then the function

$$x(t; \mu) = h(t_0, \varphi, y(\,\cdot\,; \mu))(t), \ \ t \in [\widehat{\tau}, r_2], \tag{2.4}$$

is the solution of equation (1.1) with the initial condition (1.2) (see Definition 1.1 and (2.3)).

Lemma 2.1. *Let $y_0(t)$ be a solution corresponding to the element $\mu_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0, u_0) \in \Lambda$ and defined on $[r_1, r_2] \subset (a, b)$; let $t_{00} \in [r_1, r_2)$, $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = 1, \dots, s$, and let $K_1 \subset O$ be a compact set containing a neighborhood of the set $\varphi_0(I_1) \cup y_0([r_1, r_2])$. Then there exist the numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that, for any $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times V$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$. In addition, to this element there corresponds a solution $y(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[r_1 - \delta_1, r_2 + \delta_1] \subset I$. Moreover,*

$$\begin{cases} \varphi(t) = \varphi_0(t) + \varepsilon\delta\varphi(t) \in K_1, & t \in I_1, \\ y(t; \mu_0 + \varepsilon\delta\mu) \in K_1, & t \in [r_1 - \delta_1, r_2 + \delta_1], \end{cases} \tag{2.5}$$

$$\lim_{\varepsilon \to 0} y(t; \mu_0 + \varepsilon\delta\mu) = y(t; \mu_0) \ \ \textit{uniformly for} \ \ (t, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times V.$$

This lemma is a result of Theorem 3.1 in [6].

Lemma 2.2. *Let $x_0(t)$ be a solution corresponding to the element $\mu_0 = (t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0, u_0) \in \Lambda$ and defined on $[\widehat{\tau}, t_{10}]$ (see Definition 1.1), let $t_{00}, t_{10} \in (a, b)$, $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = 1, \dots, s$, and let $K_1 \subset O$ be a compact set containing a neighborhood of the set $\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then there exist the numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that, for any $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times V$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$. In addition, to this element there corresponds a solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$. Moreover,*

$$x(t; \mu_0 + \varepsilon\delta\mu) \in K_1, \ \ t \in [\widehat{\tau}, t_{10} + \delta_1]. \tag{2.6}$$

It is easy to see that if in Lemma 2.1 one put $r_1 = t_{00}$, $r_2 = t_{10}$, then $x_0(t) = y_0(t)$, $t \in [t_{00}, t_{10}]$, and $x(t; \mu_0 + \varepsilon\delta\mu) = h(t_0, \varphi, y(\,\cdot\,; \mu_0 + \varepsilon\delta\mu))(t)$, $(t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times V$ (see (2.4)). Thus, Lemma 2.2 is a simple corollary of Lemma 2.1 (see (2.5)).

Remark 2.2. Due to the uniqueness, the solution $y(t; \mu_0)$ on the interval $[r_1 - \delta_1, r_2 + \delta_1]$ is a continuation of the solution $y_0(t)$. Therefore, we can assume that the solution $y_0(t)$ is defined on the interval $[r_1 - \delta_1, r_2 + \delta_1]$.

Lemma 2.1 allows one to define the increment of the solution $y_0(t) = y(t; \mu_0)$:

$$\Delta y(t) = \Delta y(t; \varepsilon\delta\mu) = y(t; \mu_0 + \varepsilon\delta\mu) - y_0(t), \ \ (t, \varepsilon, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times (0, \varepsilon_1) \times V. \tag{2.7}$$

Obviously,

$$\lim_{\varepsilon \to 0} \Delta y(t; \varepsilon\delta\mu) = 0 \tag{2.8}$$

uniformly with respect to $(t, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times V$ (see Lemma 2.1).

Lemma 2.3. *Let the conditions of Theorem* 1.1 *hold. Then there exist the numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *such that*

$$\max_{t\in[t_{00},r_2+\delta_2]} |\Delta y(t)| \le O(\varepsilon\delta\mu) \tag{2.9}$$

for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times V^-$*. Moreover,*

$$\Delta y(t_{00}) = \varepsilon\big[\delta\varphi(t_{00}) + (\dot\varphi_0^- - f^-)\delta t_0\big] + o(\varepsilon\delta\mu). \tag{2.10}$$

Proof. Let $\varepsilon_2' \in (0, \varepsilon_1)$ be so small that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2') \times V^-$ the inequalities

$$t_0 + \tau_i > t_{00}, \quad i = 1, \dots, s, \tag{2.11}$$

hold, where $t_0 = t_{00}+\varepsilon\delta t_0$, $\tau_i = \tau_{i0}+\varepsilon\delta\tau_i$. On the interval $[t_{00}, r_2+\delta_1]$, the function $\Delta y(t) = y(t)-y_0(t)$ satisfies the equation

$$\Delta\dot y(t) = a(t; \varepsilon\delta\mu), \tag{2.12}$$

where

$$\begin{aligned} a(t;\varepsilon\delta\mu) = f\big(t, y_0(t) + \Delta y(t), h(t_0, \varphi, y_0 + \Delta y)(t-\tau_1), \dots, h(t_0, \varphi, y_0 + \Delta y)(t-\tau_s), u(t)\big) \\ - f\big(t, y_0(t), h(t_{00}, \varphi_0, y_0)(t-\tau_{10}), \dots, h(t_{00}, \varphi_0, y_0)(t-\tau_{s0}), u_0(t)\big). \end{aligned} \tag{2.13}$$

We rewrite equation (2.12) in the integral form

$$\Delta y(t) = \Delta y(t_{00}) + \int_{t_{00}}^{t} a(\xi; \varepsilon\delta\mu)\, d\xi.$$

Hence it follows that

$$|\Delta y(t)| \le |\Delta y(t_{00})| + a_1(t; t_{00}, \varepsilon\delta\mu), \tag{2.14}$$

where

$$a_1(t; t_{00}, \varepsilon\delta\mu) = \int_{t_{00}}^{t} |a(\xi; \varepsilon\delta\mu)|\, d\xi, \quad t \in [t_{00}, r_2 + \delta_1].$$

Let us prove formula (2.10). We have

$$\begin{aligned} \Delta y(t_{00}) &= y(t_{00}; \mu_0 + \varepsilon\delta\mu) - y_0(t_{00}) \\ &= \varphi_0(t_0) + \varepsilon\delta\varphi(t_0) + \int_{t_0}^{t_{00}} f\big(t, y_0(t) + \Delta y(t), \varphi(t-\tau_1), \dots, \varphi(t-\tau_s), u(t)\big)\, dt - \varphi_0(t_{00}) \end{aligned} \tag{2.15}$$

(see (2.11) and (2.3)). Since

$$\int_{t_{00}}^{t_0} \dot\varphi_0(t)\, dt = \varepsilon\dot\varphi_0^-\delta t_0 + o(\varepsilon\delta\mu),$$

$$\lim_{\varepsilon\to 0} \delta\varphi(t_0) = \delta\varphi(t_{00}) \ \text{ uniformly with respect to } \ \delta\mu \in V^-$$

(see (1.3)), we get

$$\begin{aligned} \varphi_0(t_0) + \varepsilon\delta\varphi(t_0) - \varphi_0(t_{00}) &= \int_{t_{00}}^{t_0} \dot\varphi_0(t)\, dt + \varepsilon\delta\varphi(t_{00}) + \varepsilon\big[\delta\varphi(t_0) - \delta\varphi(t_{00})\big] \\ &= \varepsilon\big[\dot\varphi^-\delta t_0 + \delta\varphi(t_{00})\big] + o(\varepsilon\delta\mu). \end{aligned} \tag{2.16}$$

It is clear that if $t \in [t_0, t_{00}]$, then

$$\lim_{\varepsilon\to 0}\big(t, y_0(t)+\Delta y(t), \varphi(t-\tau_1), \dots, \varphi(t-\tau_s)\big) = \lim_{t\to t_{00}-}\big(t, y_0(t), \varphi_0(t-\tau_{10}), \dots, \varphi_0(t-\tau_{s0})\big) = w_0$$

(see (2.8)). Consequently,

$$\lim_{\varepsilon\to 0}\sup_{t\in[t_0,t_{00}]}\big|f(t, y_0(t)+\Delta y(t), \varphi(t-\tau_1), \dots, \varphi(t-\tau_s), u(t)) - f^-\big| = 0.$$

This relation implies that

$$\begin{aligned}\int_{t_0}^{t_{00}} & f\big(t, y_0(t)+\Delta y(t), \varphi(t-\tau_1), \dots, \varphi(t-\tau_s), u(t)\big)\,dt \\ &= -\varepsilon f^-\delta t_0 + \int_{t_0}^{t_{00}}\big[f(t, y_0(t)+\Delta y(t), \varphi(t-\tau_1), \dots, \varphi(t-\tau_s), u(t)) - f^-\big]\,dt \\ &= -\varepsilon f^-\delta t_0 + o(\varepsilon\delta\mu). \qquad (2.17)\end{aligned}$$

From (2.15), by virtue of (2.16) and (2.17), we obtain (2.10).

Now, let us prove inequality (2.9). First, we note that for any compact set $K_1 \subset O$ and $U_1 \subset U_0$, there exists a function $L_{K_1,U_1}(t) \in L_1(I, R_+)$ such that

$$\big|f(t, x, x_1, \dots, x_s, u_1) - f(t, y, y_1, \dots, y_s, u_2)\big| \le L_{K_1,U_1}(t)\Big(|x-y| + \sum_{i=1}^{s}|x_i - y_i| + |u_1 - u_2|\Big)$$

for almost all $t \in I$ and for any $(x, y) \in K^2$, $(x_i, y_i) \in K^2$, $i = 1, \dots, s$, $u_1, u_2 \in U_1$.

Now, we estimate $a_1(t; t_{00}, \varepsilon\delta\mu)$, $t \in [t_{00}, r_2 + \delta_1]$. Obviously,

$$a_1(t; t_{00}, \varepsilon\delta\mu) \le \int_{t_{00}}^{t} L_{K_1,U_1}(\xi)|\Delta y(\xi)|\,d\xi + \sum_{i=1}^{s} a_{2i}(t; t_{00}, \varepsilon\delta\mu) + \varepsilon\int_{t_{00}}^{t} L_{K_1,U_1}(\xi)|\delta u(\xi)|\,d\xi, \qquad (2.18)$$

where

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) = \int_{t_{00}}^{t} L_{K_1,U_1}(\xi)\big|h(t_0, \varphi, y_0 + \Delta y)(\xi - \tau_i) - h(t_{00}, \varphi_0, y_0)(\xi - \tau_{i0})\big|\,d\xi$$

(see (2.13)).

Evidently,

$$\varepsilon\int_{t_{00}}^{t} L_{K_1,U_1}(\xi)|\delta u(\xi)|\,d\xi \le \varepsilon\alpha\int_{I} L_{K_1,U_1}(t)\,dt = O(\varepsilon).$$

Let $t_{00} + \tau_{i0} \le r_2$ and let ε_2' be so small that $t_{00} + \tau_i < r_2 + \delta_1$. Furthermore, let $\rho_{i1} = \min\{t_0 + \tau_i, t_{00} + \tau_{i0}\}$, $\rho_{i2} = \max\{t_{00} + \tau_i, t_{00} + \tau_{i0}\}$. It is easy to see that $\rho_{i2} \ge \rho_{i1} > t_{00}$ and $\rho_{i2} - \rho_{i1} = O(\varepsilon\delta\mu)$. Let $t \in [t_{00}, \rho_{i1})$. Then for $\xi \in [t_{00}, t]$, we have $\xi - \tau_i < t_0$ and $\xi - \tau_{i0} < t_{00}$. Therefore,

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) = \int_{t_{00}}^{t} L_{K_1,U_1}(\xi)|\varphi(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})|\,d\xi.$$

From the boundedness of the function $\dot\varphi_0(t)$, $t \in I_1$, it follows that

$$\begin{aligned}|\varphi(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})| &= \big|\varphi_0(\xi - \tau_i) + \varepsilon\delta\varphi(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})\big| \\ &= O(\varepsilon\delta\mu) + \Big|\int_{\xi-\tau_{i0}}^{\xi-\tau_i}\dot\varphi_0(t)\,dt\Big| \le O(\varepsilon\delta\mu). \qquad (2.19)\end{aligned}$$

Thus, for $t \in [t_{00}, \rho_{i1}]$, we have

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) \le O(\varepsilon\delta\mu), \quad i = 1, \dots, s. \tag{2.20}$$

Let $t \in [\rho_{i1}, \rho_{i2}]$, then

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) \le a_{2i}(\rho_{i1}; t_{00}, \varepsilon\delta\mu) + a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu).$$

Let $\rho_{i1} = t_0 + \tau_i$ and $\rho_{i2} = t_{00} + \tau_i$, i.e. $t_0 + \tau_i < t_{00} + \tau_{i0} < t_{00} + \tau_i$. We have

$$\begin{aligned}
a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) \le \int\limits_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{K_1,U_1}(\xi)\big|y(\xi-\tau_i; \mu_0+\varepsilon\delta\mu) - \varphi_0(\xi-\tau_{i0})\big|\,d\xi \\
+ \int\limits_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{K_1,U_1}(\xi)\big|y(\xi-\tau_i; \mu_0+\varepsilon\delta\mu) - y_0(\xi-\tau_{i0})\big|\,d\xi \\
\le \int\limits_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{K_1,U_1}(\xi)\big|y(\xi-\tau_i; \mu_0+\varepsilon\delta\mu) - \varphi(\xi-\tau_i)\big|\,d\xi \\
+ \int\limits_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{K_1,U_1}(\xi)|\varphi(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0})|\,d\xi + \int\limits_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{K_1,U_1}(\xi)\big|y(\xi-\tau_i; \mu_0+\varepsilon\delta\mu) - \varphi(\xi-\tau_i)\big|\,d\xi \\
+ \int\limits_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{K_1,U_1}(\xi)|\varphi(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0})|\,d\xi + \int\limits_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{K_1,U_1}(\xi)|\varphi_0(\xi-\tau_{i0}) - y_0(\xi-\tau_{i0})|\,d\xi \\
\le o(\varepsilon\delta\mu) + \int\limits_{t_0+\tau_i}^{t_{00}+\tau_i} L_{K_1,U_1}(\xi)\big|y(\xi-\tau_i; \mu_0+\varepsilon\delta\mu) - \varphi(\xi-\tau_i)\big|\,d\xi \\
+ \int\limits_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{K_1,U_1}(\xi)|\varphi_0(\xi-\tau_{i0}) - y_0(\xi-\tau_{i0})|\,d\xi \\
= o(\varepsilon\delta\mu) + \int\limits_{t_0}^{t_{00}} L_{K_1,U_1}(\xi+\tau_i)|y(\xi; \mu_0+\varepsilon\delta\mu) - \varphi(\xi)|\,d\xi + \int\limits_{t_{00}}^{t_{00}+\tau_i-\tau_{i0}} L_{K_1,U_1}(\xi+\tau_{i0})|\varphi_0(\xi) - y_0(\xi)|\,d\xi
\end{aligned}$$

(see (2.19)) with $t_{00} + \tau_i - \tau_{i0} > t_{00} + \tau_{i0} - \tau_{i0} = t_{00}$. The functions $f(w,u), (w,u) \in I \times O^{1+s} \times U_0$, and $\dot\varphi_0(t)$, $t \in I_1$, are bounded; therefore, we have

$$\begin{aligned}
&|y(\xi; \mu_0+\varepsilon\delta\mu) - \varphi(\xi)| \\
&\quad = \left|\varphi(t_0) + \int\limits_{t_0}^{\xi} f(t_0, \tau_1, \dots, \tau_s, \varphi, y_0+\Delta y, u)(t)\,dt - \varphi(\xi)\right| \le O(\varepsilon\delta\mu), \quad \xi \in [t_0, t_{00}],
\end{aligned} \tag{2.21}$$

$$|\varphi_0(\xi) - y_0(\xi)| = \left|\varphi_0(\xi) - \varphi_0(t_{00}) - \int\limits_{t_{00}}^{\xi} f(t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0, y_0, u_0)(t)\,dt\right| \le O(\varepsilon\delta\mu),$$

$$\xi \in [t_{00}, t_{00} + \tau_i - \tau_{i0}].$$

Thus, $a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) = o(\varepsilon\delta\mu)$. Let $\rho_{i1} = t_0 + \tau_i$ and $\rho_{i2} = t_{00} + \tau_{i0}$, then

$$a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) = \int\limits_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{K_1,U_1}(\xi)\big|y(\xi-\tau_i; \mu_0+\varepsilon\delta\mu) - \varphi_0(\xi-\tau_{i0})\big|\,d\xi = o(\varepsilon\delta\mu).$$

Let $\rho_{i1} = t_{00} + \tau_{i0}$ and $\rho_{i2} = t_{00} + \tau_i$, i.e., $t_{00} + \tau_{i0} < t_0 + \tau_i < t_{00} + \tau_i$. We have

$$a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) \le \int\limits_{t_{00}+\tau_{i0}}^{t_0+\tau_i} L_{K_1,U_1}(\xi)|\varphi(\xi-\tau_i) - y_0(\xi-\tau_{i0})|\,d\xi \\ + \int\limits_{t_0+\tau_i}^{t_{00}+\tau_i} L_{K_1,U_1}(\xi)\big|y(\xi-\tau_i;\mu_0+\varepsilon\delta\mu) - y_0(\xi-\tau_{i0})\big|\,d\xi = o(\varepsilon\delta\mu).$$

Consequently, for $t \in [t_{00}, \rho_{i2}]$, inequality (2.20) holds.

Let $t \in [\rho_{i2}, r_2 + \delta_1]$, then $t - \tau_i \ge t_0$ and $t - \tau_{i0} \ge t_{00}$. Therefore,

$$\begin{aligned} a_{2i}(t; t_{00}, \varepsilon\delta\mu) &= a_{2i}(\rho_{i2}; t_{00}, \varepsilon\delta\mu) + \int\limits_{\rho_{i2}}^{t} L_{K_1,U_1}(\xi)\big|y_0(\xi-\tau_i) + \Delta y(\xi-\tau_i) - y_0(\xi-\tau_{i0})\big|\,d\xi \\ &\le O(\varepsilon\delta\mu) + \int\limits_{\rho_{i2}-\tau_i}^{t-\tau_i} L_{K_1,U_1}(\xi+\tau_i)|\Delta y(\xi)|\,d\xi + \int\limits_{\rho_{i2}}^{t} L_{K_1,U_1}(\xi)|y_0(\xi-\tau_i) - y_0(\xi-\tau_{i0})|\,d\xi \\ &\le O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t} \chi(\xi+\tau_i)L_{K_1,U_1}(\xi+\tau_i)|\Delta y(\xi)|\,d\xi + \int\limits_{\rho_{i2}}^{r_2+\delta_1} L_{K_1,U_1}(\xi)|y_0(\xi-\tau_i) - y_0(\xi-\tau_{i0})|\,d\xi, \end{aligned}$$

where $\chi(\xi)$ is the characteristic function of the interval I.

Further, for $\xi \in [\rho_{i2}, r_2 + \delta_1]$,

$$|y_0(\xi-\tau_i) - y_0(\xi-\tau_{i0})| \le \int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i} \big|f(t_{00}, \tau_{10}, \dots, \tau_{s0}, y_0, u_0)(t)\big|\,dt \le O(\varepsilon\delta\mu).$$

Thus, for $t \in [t_{00}, r_2 + \delta_1]$, we get

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t} \chi(\xi+\tau_i)L_{K_1,U_1}(\xi+\tau_i)|\Delta y(\xi)|\,d\xi. \tag{2.22}$$

We now consider the case where $t_{00} + \tau_{i0} > r_2$. Let $\delta_2 \in (0, \delta_1)$ and $\varepsilon_2'' \in (0, \varepsilon_1)$ be so small numbers that $t_{00} + \tau_{i0} > r_2 + \delta_2$ and $t_0 + \tau_i > r_2 + \delta_2$ for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2'') \times V^-$.

It is easy to see that

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) \le \int\limits_{t_{00}}^{t} L_{K_1,U_1}(\xi)|\varphi(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0})|\,dt \le O(\varepsilon\delta\mu).$$

Thus, for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{00}, r_2 + \delta_2] \times (0, \varepsilon_2) \times V^-$ and $i = 1, \dots, s$, where $\varepsilon_2 = \min(\varepsilon_2', \varepsilon_2'')$, inequality (2.22) holds.

Consequently, we have

$$a_1(t; t_{00}, \varepsilon\delta\mu) \le O(\varepsilon\delta\mu) \\ + \int\limits_{t_{00}}^{t} \Big[L_{K_1,U_1}(\xi) + \sum_{i=1}^{s} \chi(\xi+\tau_i)L_{K_1,U_1}(\xi+\tau_i)\Big]\,|\Delta y(\xi)|\,d\xi, \quad t \in [t_{00}, r_2 + \delta_1] \tag{2.23}$$

(see (2.18)).

According to (2.10) and (2.23), inequality (2.14) directly implies

$$|\Delta y(t)| \le O(\varepsilon\delta\mu) + \int_{t_{00}}^{t} \Big[L_{K_1,U_1}(\xi) + \sum_{i=1}^{s} \chi(\xi+\tau_i) L_{K_1,U_1}(\xi+\tau_i)\Big] |\Delta y(\xi)|\, d\xi, \ \ t \in [t_{00}, r_2+\delta_2]$$

from which, by the Gronwall lemma, we get (2.9). □

The following lemma, with a minor modification can be proved analogously to Lemma 2.3.

Lemma 2.4. *Let the conditions of Theorem* 1.2 *hold. Then there exist the numbers* $\varepsilon_2 \in (0,\varepsilon_1)$ *and* $\delta_2 \in (0,\delta_1)$ *such that* $\max\limits_{t\in[t_0,r_2+\delta_2]} |\Delta y(t)| \le O(\varepsilon\delta\mu)$ *for arbitrary* $(\varepsilon,\delta\mu) \in (0,\varepsilon_2) \times V^+$*. Moreover,*

$$\Delta y(t_0) = \varepsilon\big[\delta\varphi(t_{00}) + (\dot{\varphi}_0^{+} - f^{+})\delta t_0\big] + o(\varepsilon\delta\mu).$$

3 Proof of Theorem 1.1

Let $r_1 = t_{00}$ and $r_2 = t_{10}$ in Lemma 2.1, then

$$x_0(t) = \begin{cases} \varphi_0(t), & t \in [\widehat{\tau}, t_{00}), \\ y_0(t), & t \in [t_{00}, t_{10}], \end{cases}$$

and for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_1) \times V^-$,

$$x(t;\mu_0+\varepsilon\delta\mu) = \begin{cases} \varphi(t) := \varphi_0(t) + \varepsilon\delta\varphi(t), & t \in [\widehat{\tau}, t_0), \\ y(t;\mu_0+\varepsilon\delta\mu), & t \in [t_0, t_{10}+\delta_1] \end{cases}$$

(see (2.4)).

We note that $\delta\mu \in V^-$, i.e., $t_0 < t_{00}$, therefore, we have

$$\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t) & \text{for } \ t \in [\widehat{\tau}, t_0), \\ y(t;\mu_0+\varepsilon\delta\mu) - \varphi_0(t) & \text{for } \ t \in [t_0, t_{00}), \\ \Delta y(t) & \text{for } \ t \in [t_{00}, t_{10}+\delta_1] \end{cases}$$

(see (1.4) and (2.7)). By Lemma 2.3 and the relation $|y(t;\mu_0+\varepsilon\delta\mu) - \varphi_0(t)| \le O(\varepsilon\delta\mu)$, $t \in [t_0, t_{00}]$, we have

$$|\Delta x(t)| \le O(\varepsilon\delta\mu) \ \ \forall\, (t,\varepsilon,\delta\mu) \in [\widehat{\tau}, t_{10}+\delta_2] \times (0,\varepsilon_2) \times V^-, \tag{3.1}$$

$$\Delta x(t_{00}) = \varepsilon\big[\delta\varphi(t_{00}) + (\dot{\varphi}_0^{-} - f^{-})\delta t_0\big] + o(\varepsilon\delta\mu). \tag{3.2}$$

The function $\Delta x(t)$ satisfies the equation

$$\begin{aligned} \dot{\Delta} x(t) &= f\Big(t, x_0(t)+\Delta x(t), x_0(t-\tau_1)+\Delta x(t-\tau_1), \dots, x_0(t-\tau_s)+\Delta x(t-\tau_s), u(t)\Big) - f[t] \\ &= f_x[t]\Delta x(t) + \sum_{i=1}^{s} f_{x_i}[t]\Delta x(t-\tau_{i0}) + \varepsilon f_u[t]\delta u(t) + r(t;\varepsilon\delta\mu) \end{aligned} \tag{3.3}$$

on the interval $[t_{00}, t_{10}+\delta_2]$, where

$$\begin{aligned} r(t;\varepsilon\delta\mu) = f\Big(t, x_0(t)+\Delta x(t), x_0(t-\tau_1)+\Delta x(t-\tau_1), \dots, x_0(t-\tau_s)+\Delta x(t-\tau_s), u(t)\Big) \\ -f[t] - f_x[t]\Delta x(t) - \sum_{i=1}^{s} f_{x_i}[t]\Delta x(t-\tau_{i0}) - \varepsilon f_u[t]\delta u(t), \end{aligned} \tag{3.4}$$

$$f[t] = f\big(t, x_0(t), x_0(t-\tau_{10}), \dots, x_0(t-\tau_{s0}), u_0(t)\big),$$

By using the Cauchy formula, one can represent the solution of equation (3.3) in the form

$$\Delta x(t) = Y(t_{00};t)\Delta x(t_{00}) + \varepsilon \int_{t_{00}}^{t} Y(\xi;t) f_u[t]\delta u(t)\,dt + \sum_{p=0}^{1} R_p(t;t_{00},\varepsilon\delta\mu), \quad t \in [t_{00}, t_{10}+\delta_2], \tag{3.5}$$

where

$$\begin{cases} R_0(t;t_{00},\varepsilon\delta\mu) = \displaystyle\sum_{i=1}^{s} R_{i0}(t;t_{00},\varepsilon\delta\mu), \\ R_{i0}(t;t_{00},\varepsilon\delta\mu) = \displaystyle\int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t) f_{x_i}[\xi+\tau_{i0}]\Delta x(\xi)\,d\xi, \\ R_1(t;t_{00},\varepsilon\delta\mu) = \displaystyle\int_{t_{00}}^{t} Y(\xi;t) r(\xi;\varepsilon\delta\mu)\,d\xi \end{cases} \tag{3.6}$$

and $Y(\xi;t)$ is the matrix function satisfying equation (1.8) and condition (1.9). The function $Y(\xi;t)$ is continuous on the set $\Pi = \{(\xi,t):\ t_{00}-\delta_2 \le \xi \le t,\ t \in [t_{00}, t_{10}+\delta_2]\}$ by Lemma 2.1.7 in [3, p. 22]. Therefore,

$$Y(t_{00};t)\Delta x(t_{00}) = \varepsilon Y(t_{00};t)\big[\delta\varphi(t_{00}) + (\dot\varphi_0^- - f^-)\delta t_0\big] + o(t;\varepsilon\delta\mu) \tag{3.7}$$

(see (3.2)), where $o(t;\varepsilon\delta\mu) = Y(t_{00};t)o(\varepsilon\delta\mu)$. One can readily see that

$$\begin{aligned} R_{i0}(t;t_{00},\varepsilon\delta\mu) &= \varepsilon \int_{t_{00}-\tau_{i0}}^{t_0} Y(\xi+\tau_{i0};t) f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi + \int_{t_0}^{t_{00}} Y(\xi+\tau_{i0};t) f_{x_i}[\xi+\tau_{i0}]\Delta x(\xi)\,d\xi \\ &= \varepsilon \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t) f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi + o(t;\varepsilon\delta\mu) \end{aligned} \tag{3.8}$$

(see (3.1)). Thus,

$$R_0(t;t_{00},\varepsilon\delta\mu) = \varepsilon \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t) f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi + o(t;\varepsilon\delta\mu).$$

We introduce the notations:

$$\begin{aligned} f[t;\theta,\varepsilon\delta\mu] = f\Big(&t, x_0(t)+\theta\Delta x(t), x_0(t-\tau_{10}) + \theta\big(x_0(t-\tau_1) - x_0(t-\tau_{10}) + \Delta x(t-\tau_1)\big), \dots, \\ & x_0(t-\tau_{s0}) + \theta\big(x_0(t-\tau_s) - x_0(t-\tau_{s0}) + \Delta x(t-\tau_s)\big), u_0(t) + \theta\varepsilon\delta u(t)\Big), \end{aligned}$$

$\sigma(t;\theta,\varepsilon\delta\mu) = f_x[t;\theta,\varepsilon\delta\mu] - f_x[t]$, $\varrho_i(t;\theta,\varepsilon\delta\mu) = f_{x_i}[t;\theta,\varepsilon\delta\mu] - f_{x_i}[t]$,
$\vartheta(t;\theta,\varepsilon\delta\mu) = f_u[t;\theta,\varepsilon\delta\mu] - f_u[t]$.

It is easy to see that

$$\begin{aligned} f\Big(t, x_0(t)+\Delta x(t), x_0(t-\tau_1)+\Delta x(t-\tau_1), \dots, x_0(t-\tau_s)+\Delta x(t-\tau_s), u_0(t)+\varepsilon\delta u(t)\Big) - f[t] \\ = \int_0^1 \frac{d}{d\theta} f[t;\theta,\varepsilon\delta\mu]\,d\theta \end{aligned}$$

$$= \int_0^1 \Big\{ f_x[t;\theta,\varepsilon\delta\mu]\Delta x(t) + \sum_{i=1}^{s} f_{x_i}[t;\theta,\varepsilon\delta\mu]\big(x_0(t-\tau_i) - x_0(t-\tau_{i0}) + \Delta x(t-\tau_i)\big) + \varepsilon f_u[t;\theta,\varepsilon\delta\mu]\delta u(t) \Big\}\,d\theta$$

$$= \Big[\int_0^1 \sigma(t;\theta,\varepsilon\delta\mu)\,d\theta\Big]\Delta x(t) + \sum_{i=1}^{s}\Big[\int_0^1 \varrho_i(t;\theta,\varepsilon\delta\mu)\,d\theta\Big]\big(x_0(t-\tau_i)-x_0(t-\tau_{i0})+\Delta x(t-\tau_i)\big)$$

$$+\varepsilon\Big[\int_0^1 \vartheta(t;\theta,\varepsilon\delta\mu)\,d\theta\Big]\delta u(t)+f_x[t]\Delta x(t)+\sum_{i=1}^{s} f_{x_i}[t]\big(x_0(t-\tau_i)-x_0(t-\tau_{i0})+\Delta x(t-\tau_i)\big)+\varepsilon f_u[t]\delta u(t).$$

Taking into account the last relation for $t\in[t_{00},t_{10}+\delta_2]$, we have

$$R_1(t;t_{00},\varepsilon\delta\mu)=\sum_{p=2}^{6} R_p(t;t_{00},\varepsilon\delta\mu),$$

where

$$R_2(t;t_{00},\varepsilon\delta\mu)=\int_{t_{00}}^{t} Y(\xi;t)\sigma_1(\xi;\varepsilon\delta\mu)\Delta x(\xi)\,d\xi,\quad \sigma_1(\xi;\varepsilon\delta\mu)=\int_0^1 \sigma(\xi;\theta,\varepsilon\delta\mu)\,d\theta,$$

$$R_3(t;t_{00},\varepsilon\delta\mu)=\sum_{i=1}^{s}\int_{t_{00}}^{t} Y(\xi;t)\varrho_{i1}(\xi;\varepsilon\delta\mu)\big[x_0(\xi-\tau_i)-x_0(\xi-\tau_{i0})+\Delta x(\xi-\tau_i)\big]\,d\xi,$$

$$\varrho_{i1}(\xi;\varepsilon\delta\mu)=\int_0^1 \varrho_i(\xi;\theta,\varepsilon\delta\mu)\,d\theta,$$

$$R_4(t;t_{00},\varepsilon\delta\mu)=\sum_{i=1}^{s}\int_{t_{00}}^{t} Y(\xi;t)f_{x_i}[\xi]\big[x_0(\xi-\tau_i)-x_0(\xi-\tau_{i0})\big]\,d\xi,$$

$$R_5(t;t_{00},\varepsilon\delta\mu)=\sum_{i=1}^{s}\int_{t_{00}}^{t} Y(\xi;t)f_{x_i}[\xi]\big[\Delta x(\xi-\tau_i)-\Delta x(\xi-\tau_{i0})\big]\,d\xi,$$

$$R_6(t;t_{00},\varepsilon\delta\mu)=\varepsilon\int_{t_{00}}^{t} Y(\xi;t)\vartheta_1(\xi;\varepsilon\delta\mu)\delta u(\xi)\,d\xi,\quad \vartheta_1(\xi;\varepsilon\delta\mu)=\int_0^1 \vartheta(\xi;\theta,\varepsilon\delta\mu)\,d\theta$$

(see (3.4)). The function $x_0(t)$, $t\in[\widehat{\tau},t_{10}+\delta_2]$, is absolutely continuous, then for each fixed Lebesgue point $\xi_i\in(t_{00},t_{10}+\delta_2)$ of function $\dot x_0(\xi-\tau_{i0})$, we get

$$x_0(\xi_i-\tau_i)-x_0(\xi_i-\tau_{i0})=\int_{\xi_i}^{\xi_i-\varepsilon\delta\tau_i}\dot x_0(\varsigma-\tau_{i0})\,d\varsigma=-\varepsilon\dot x_0(\xi_i-\tau_{i0})\delta\tau_i+\gamma_i(\xi_i;\varepsilon\delta\mu) \tag{3.9}$$

where

$$\lim_{\varepsilon\to 0}\frac{\gamma_i(\xi_i;\varepsilon\delta\mu)}{\varepsilon}=0 \ \text{ uniformly for } \ \delta\mu\in V^-. \tag{3.10}$$

Thus, (3.9) is valid for almost all points of the interval $(t_{00},t_{10}+\delta_2)$. From (3.9), taking into account the boudedness of the function

$$\dot x_0(t)=\begin{cases}\dot\varphi_0(t), & t\in[\widehat{\tau},t_{00}],\\ f\big(t,x_0(t),x_0(t-\tau_{10}),\dots,x_0(t-\tau_{s0}),u_0(t)\big), & t\in(t_{00},t_{10}+\delta_2],\end{cases}$$

it follows that

$$|x_0(\xi_i-\tau_i)-x_0(\xi_i-\tau_{i0})|\le O(\varepsilon\delta\mu) \ \text{ and } \ \Big|\frac{\gamma_i(\xi_i;\varepsilon\delta\mu)}{\varepsilon}\Big|\le const. \tag{3.11}$$

Clearly,

$$|\Delta x(\xi-\tau_i)-\Delta x(\xi-\tau_{i0})| = \begin{cases} o(\xi;\varepsilon\delta\mu) & \text{for } \ \xi\in[t_{00},\rho_{i1}],\\ O(\xi;\varepsilon\delta\mu) & \text{for } \ \xi\in[\rho_{i1},\rho_{i2}] \end{cases} \tag{3.12}$$

(see (3.1)).

Let $\xi\in[\rho_{i2},t_{10}+\delta_1]$, then $\xi-\tau_i\ge t_{00}$, $\xi-\tau_{i0}\ge t_{00}$. Therefore,

$$\begin{aligned} \big|\Delta x(\xi-\tau_i)-\Delta x(\xi-\tau_{i0})\big| &\le \int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i} |\dot{\Delta} x(s)|\,ds \le \int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i} L_{K_1,U_1}(\varsigma)\Big[|\Delta x(\varsigma)| \\ &+\sum_{i=1}^{s}|x_0(\varsigma-\tau_i)-x_0(\varsigma-\tau_{i0})|+|\Delta x(\varsigma-\tau_i)|\Big]\,d\varsigma+\varepsilon\alpha\int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i} L_{K_1,U_1}(\varsigma)\,d\varsigma = o(\xi;\varepsilon\delta\mu) \end{aligned} \tag{3.13}$$

(see (2.6), (3.1), (3.3) and (3.11)). According to (3.1), (3.9) and (3.11)–(3.13) for the expressions $R_p(t;t_{00},\varepsilon\delta\mu)$, $p=2,\dots,6$, we have

$$|R_2(t;t_{00},\varepsilon\delta\mu)|\le\|Y\|O(\varepsilon\delta\mu)\sigma_2(\varepsilon\delta\mu),\quad \sigma_2(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{10}+\delta_1}|\sigma_1(\xi;\varepsilon\delta\mu)|\,d\xi,$$

$$|R_3(t;t_{00},\varepsilon\delta\mu)|\le\|Y\|O(\varepsilon\delta\mu)\sum_{i=1}^{s}\rho_{i2}(\varepsilon\delta\mu),\quad \rho_{i2}(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{10}+\delta_1}|\rho_{i1}(\xi;\varepsilon\delta\mu)|\,d\xi,$$

$$R_4(t;t_{00},\varepsilon\delta\mu)=-\varepsilon\sum_{i=1}^{s}\bigg[\int\limits_{t_{00}}^{t}Y(\xi;t)f_{x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i+\sum_{i=1}^{s}\gamma_{i1}(t;\varepsilon\delta\mu),$$

$$|R_5(t;t_{00},\varepsilon\delta\mu)|=o(t;\varepsilon\delta\mu),$$

$$|R_6(t;t_{00},\varepsilon\delta\mu)|\le\varepsilon\|Y\|\vartheta_2(\varepsilon\delta\mu),\quad \vartheta_2(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{10}+\delta_1}|\vartheta_1(\xi;\varepsilon\delta\mu)|\,d\xi,$$

where

$$\|Y\|=\sup\big\{|Y(\xi;t)|:\ (\xi,t)\in\Pi\big\},\quad \gamma_{i1}(t;\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t}Y(\xi;t)f_{x_i}[\xi]\gamma_i(\xi;\varepsilon\delta\mu)\,d\xi.$$

Obviously,

$$\Big|\frac{\gamma_{i1}(t;\varepsilon\delta\mu)}{\varepsilon}\Big|\le\|Y\|\int\limits_{t_{00}}^{t_{10}+\delta_1}\big|f_{x_i}[\xi]\big|\,\Big|\frac{\gamma_i(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big|\,d\xi.$$

By the Lebesgue theorem on the passage to the limit under the integral sign, we have

$$\lim_{\varepsilon\to0}\sigma_2(\varepsilon\delta\mu)=0,\quad \lim_{\varepsilon\to0}\rho_{i2}(\varepsilon\delta\mu)=0,\quad \lim_{\varepsilon\to0}\vartheta_2(\varepsilon\delta\mu)=0,\quad \lim_{\varepsilon\to0}\Big|\frac{\gamma_{i1}(t;\varepsilon\delta\mu)}{\varepsilon}\Big|=0$$

uniformly for $(t,\delta\mu)\in[t_{00},t_{10}+\delta_2]\times V^-$ (see (3.10)). Thus,

$$R_p(t;t_{00},\varepsilon\delta\mu)=o(t;\varepsilon\delta\mu),\ \ p=2,3,5,6, \tag{3.14}$$

$$R_4(t;t_{00},\varepsilon\delta\mu)=-\varepsilon\sum_{i=1}^{s}\bigg[\int\limits_{t_{00}}^{t}Y(\xi;t)f_{x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i \tag{3.15}$$

On the basis of (3.14), (3.15), we obtain

$$R_1(t;t_{00},\varepsilon\delta\mu)=-\varepsilon\sum_{i=1}^{s}\bigg[\int\limits_{t_{00}}^{t}Y(\xi;t)f_{x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i+o(t;\varepsilon\delta\mu). \tag{3.16}$$

From (3.5), by virtue of (3.7), (3.8) and (3.16), we obtain (1.5), where $\delta x(t;\delta\mu)$ has the form (1.6).

4 Proof of Theorem 1.2

First of all, we note that $\delta\mu\in V^+$, i.e., $t_{00}<t_0$, therefore, we have

$$\Delta x(t)=\begin{cases}\varepsilon\delta\varphi(t) & \text{for } t\in[\widehat{\tau},t_{00}),\\ \varphi(t)-y_0(t) & \text{for } t\in[t_{00},t_0),\\ \Delta y(t) & \text{for } t\in[t_0,t_{10}+\delta_1].\end{cases}$$

In a similar way (see (2.21)), one can prove $|\varphi(t)-y_0(t)|=O(t;\varepsilon\delta\mu)$, $t\in[t_{00},t_0]$. According to the last relation and Lemma 2.4, we have

$$|\Delta x(t)|\le O(\varepsilon\delta\mu)\ \ \forall\,(t,\varepsilon,\delta\mu)\in[\widehat{\tau},t_{10}+\delta_2]\times[0,\varepsilon_2]\times V^+,$$
$$\Delta x(t_0)=\varepsilon\big[\delta\varphi(t_{00})+(\dot{\varphi}_0^+-f^+)\delta t_0\big]+o(\varepsilon\delta\mu).$$

Let $\widehat{t}\in(t_{00},t_{10})$ be a fixed point, and let $\varepsilon_2\in(0,\varepsilon_1)$ be so small that $t_0<\widehat{t}$ for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_2)\times V^+$. The function $\Delta x(t)$ satisfies equation (3.3) on the interval $[\widehat{t},t_{10}+\delta_2]$; therefore, by using the Cauchy formula, we can represent it in the form

$$\Delta x(t)=Y(t_0;t)\Delta x(t_0)+\varepsilon\int\limits_{t_0}^{t}Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi+\sum_{i=0}^{1}R_i(t;t_0,\varepsilon\delta\mu) \tag{4.1}$$

(see (3.6)). The matrix function $Y(\xi;t)$ is continuous on $[t_{00},\widehat{t}]\times[\widehat{t},t_{10}+\delta_2]$; therefore,

$$Y(t_0;t)\Delta x(t_0)=\varepsilon Y(t_{00};t)[\delta\varphi(t_{00})+(\dot{\varphi}_0^+-f^+)\delta t_0]+o(t;\varepsilon\delta\mu), \tag{4.2}$$

where $o(t;\varepsilon\delta\mu)=Y(t_0,t)o(\varepsilon\delta\mu)$. Let us now transform

$$\begin{aligned}R_{i0}(t;t_0,\varepsilon\delta\mu)&=\varepsilon\int\limits_{t_0-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi+\int\limits_{t_{00}}^{t_0}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\Delta x(\xi)\,d\xi\\&=\varepsilon\int\limits_{t_{00}-\tau_0}^{t_{00}}Y(\xi+\tau_0;t)f_{x_i}[\xi+\tau_{i0};t]\delta\varphi(\xi)\,d\xi+o(t;\varepsilon\delta\mu).\end{aligned}$$

Thus,

$$R_0(t;t_0,\varepsilon\delta\mu)=\varepsilon\int\limits_{t_{00}-\tau_0}^{t_{00}}Y(\xi+\tau_0;t)f_{x_i}[\xi+\tau_{i0};t]\delta\varphi(\xi)\,d\xi+o(t;\varepsilon\delta\mu). \tag{4.3}$$

In a similar way, with nonessential changes, for $t\in[\widehat{t},t_{10}+\delta_2]$ one can prove

$$R_1(t;t_0,\varepsilon\delta\mu)=-\varepsilon\sum_{i=1}^{s}\int\limits_{t_{00}}^{t}Y(\xi;t)\big[f_{x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\delta\tau_i\big]\,d\xi+o(t;\varepsilon\delta\mu). \tag{4.4}$$

Finally, note that

$$\varepsilon \int\limits_{t_0}^{t} Y(\xi;t)\delta f_u[\xi]\delta u(\xi)\,d\xi = \varepsilon \int\limits_{t_{00}}^{t} Y(\xi;t)\delta f_u[\xi]\delta u(\xi)\,d\xi + o(t;\varepsilon\delta\mu) \tag{4.5}$$

for $t \in [\widehat{t}, t_{10} + \delta_2]$. Taking into account (4.2)–(4.5), from (4.1), we obtain (1.5), where $\delta x(t;\varepsilon\delta\mu)$ has the form (1.10).

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(Received 05.04.2017)

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Memoirs on Differential Equations and Mathematical Physics

Volume 74, 2018, 141–152

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THE NEUMANN BOUNDARY VALUE PROBLEM OF THERMO-ELECTRO-MAGNETO ELASTICITY FOR HALF SPACE

Abstract. We prove the uniqueness theorem for the Neumann boundary value problem of statics of the thermo-electro-magneto-elasticity theory in the case of a half-space. The corresponding unique solution is represented explicitly by means of the inverse Fourier transform under some natural restrictions imposed on the boundary vector function.

2010 Mathematics Subject Classification. 35J57, 74F05, 74F15, 74E10, 74G05, 74G25.

Key words and phrases. Thermo-electro-magneto-elasticity, piezoelectricity, boundary value problem.

რეზიუმე. ნახევარსივრცის შემთხვევაში დამტკიცებულია თერმო-ელექტრო-მაგნეტო დრეკადობის თეორიის ნეიმანის სასაზღვრო ამოცანისათვის ერთადერთობის თეორემა. გარკვეულ ბუნებრივ შეზღუდვებში, რომლებსაც ვადებთ სასაზღვრო ვექტორ-ფუნქციას, შესაბამისი ნეიმანის სასაზღვრო ამოცანის ერთადერთი ამონახსნი წარმოდგენილია ცხადი სახით შებრუნებული ფურიეს გარდაქმნის მეშვეობით.

1 Introduction

In the study of active material systems, there is significant interest in the coupling effects between elastic, electric, magnetic and thermal fields.

Although natural materials rarely show full coupling between elastic, electric, magnetic and thermal fields, some artificial materials do. In [16] it is reported that the fabrication of $BaTiO_3$-$CoFe_2O_4$ composite had the magnetoelectric effect not existing in either constituent. Other examples of similar complex coupling can be found in the references [1–7, 9–11, 14, 17].

The mathematical model of the thermo-electro-magneto-elasticity theory is described by the non-self-adjoint 6×6 system of second order partial differential equations with the appropriate boundary and initial conditions. The problem is to determine three components of the elastic displacement vector, the electric and magnetic scalar potential functions and the temperature distribution. Other field characteristics (e.g., mechanical stresses, electric and magnetic fields, electric displacement vector, magnetic induction vector, heat flux vector and entropy density) can be then determined by the gradient equations and the constitutive equations.

In the paper we prove the uniqueness theorem of solutions for Neumann boundary value problems of statics for half-space.

Under some natural restriction on the boundary vector functions the corresponding unique solution is represented by the inverse Fourier transform.

2 Basic equations and formulation of boundary value problems

2.1 Field equations

Throughout the paper $u = (u_1, u_2, u_3)^\top$ denotes the displacement vector, σ_{ij} is the mechanical stress tensor, $\varepsilon_{kj} = 2^{-1}(\partial_k u_j + \partial_j u_k)$ is the strain tensor, $E = (E_1, E_2, E_3)^\top = -\operatorname{grad}\varphi$ and $H = (H_1, H_2, H_3) = -\operatorname{grad}\psi$ are electric and magnetic fields, respectively, $D = (D_1, D_2, D_3)^\top$ is the electric displacement vector and $B = (B_1, B_2, B_3)^\top$ is the magnetic induction vector, φ and ψ stand for the electric and magnetic potentials, ϑ is the temperature increment, $q = (q_1, q_2, q_3)^\top$ is the heat flux vector, and S is the entropy density. We employ the notation $\partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial_j$, $\partial_t = \partial/\partial_t$; the superscript $(\,\cdot\,)^\top$ denotes transposition operation; the summation over the repeated indices is meant from 1 to 3, unless stated otherwise.

In this subsection we collect the field equations of the linear theory of thermo-electro-magneto-elasticity for a general anisotropic case and introduce the corresponding matrix partial differential operators [12].

Constitutive relations:

$$\begin{aligned}
\sigma_{rj} = \sigma_{jr} &= c_{rjkl}\varepsilon_{kl} - e_{lrj}E_l - q_{lrj}H_l - \lambda_{rj}\vartheta, \quad r, j = 1, 2, 3, \\
D_j &= e_{jkl}\varepsilon_{kl} + \varkappa_{jl}E_l + a_{jl}H_l + p_j\vartheta, \quad j = 1, 2, 3, \\
B_j &= q_{jkl}\varepsilon_{kl} + a_{jl}E_l + \mu_{jl}H_l + m_j\vartheta, \quad j = 1, 2, 3, \\
S &= \lambda_{kl}\varepsilon_{kl} + p_k E_k + m_k H_k + \gamma\vartheta.
\end{aligned}$$

Fourier Law: $q_j = -\eta_{jl}\partial_l\vartheta$, $j = 1, 2, 3$.

Equations of motion: $\partial_j\sigma_{rj} + X_r = \varrho\partial_t^2 u_r$, $r = 1, 2, 3$.

Quasi-static equations for electro-magnetic fields where the rate of magnetic field is small (electric field is curl free) and there is no electric current (magnetic field is curl free): $\partial_j D_j = \varrho_e$, $\partial_j B_j = 0$.

Linearised equation of the entropy balance: $T_0\partial_t S - Q = -\partial_j q_j$,

Here ϱ is the mass density, ϱ_e is the electric density, c_{rjki} are the elastic constants, e_{jki} are the piezoelectric constants, q_{jki} are the piezomagnetic constants, $\varkappa_{jk}$ are the dielectric (permittivity) constants, μ_{jk} are the magnetic permeability constants, a_{jk} are the coupling coefficients connecting electric and magnetic fields, p_j and m_j are constants characterizing the relation between thermodynamic processes and electro-magnetic effects, λ_{jk} are the thermal strain constants, η_{jk} are the heat conductivity coefficients, $\gamma = \varrho c T_0^{-1}$ is the thermal constant, T_0 is the initial reference temperature, c is the specific heat per unit mass, $X = (X_1, X_2, X_3)^\top$ is a mass force density, Q is a heat source intensity. The constants involved in these equations satisfy the symmetry conditions

$$\begin{gathered} c_{rjkl} = c_{jrkl} = c_{klrj}, \quad e_{klj} = e_{kjl}, \quad q_{klj} = q_{kjl}, \quad \varkappa_{kj} = \varkappa_{jk}, \\ \lambda_{kj} = \lambda_{jk}, \quad \mu_{kj} = \mu_{jk}, \quad \eta_{kj} = \eta_{jk}, \quad a_{kj} = a_{jk}, \ \ r, j, k, l = 1, 2, 3. \end{gathered} \tag{2.1}$$

From physical considerations it follows (see, e.g., [8, 13])

$$c_{rjkl}\xi_{rj}\xi_{kl} \geq c_0 \xi_{kl}\xi_{kl}, \quad \varkappa_{kj}\xi_k\xi_j \geq c_1|\xi|^2, \quad \mu_{kj}\xi_k\xi_j \geq c_2|\xi|^2, \quad \eta_{kj}\xi_k\xi_j \geq c_3|\xi|^2, \tag{2.2}$$

for all $\xi_{kj} = \xi_{jk} \in \mathbb{R}$ and for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, where c_0, c_1, c_2 and c_3 are positive constants. More careful analysis related to the positive definiteness of the potential energy and thermodynamical laws insure positive definiteness of the matrix

$$\Xi = \begin{bmatrix} [\varkappa_{kj}]_{3\times3} & [a_{kj}]_{3\times3} & [p_j]_{3\times1} \\ [a_{kj}]_{3\times3} & [\mu_{kj}]_{3\times3} & [m_j]_{3\times1} \\ [p_j]_{1\times3} & [m_j]_{1\times3} & \gamma \end{bmatrix}_{7\times7}. \tag{2.3}$$

Further we introduce the following generalised stress operator

$$\mathcal{T}(\partial, n) := \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times3} & [e_{lrj}n_j\partial_l]_{3\times3} & [q_{lrj}n_j\partial_l]_{3\times1} & [-\lambda_{rj}n_j]_{3\times1} \\ [-e_{jkl}n_j\partial_l]_{1\times3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & -p_jn_j \\ [-q_{jkl}n_j\partial_l]_{1\times3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & -m_jn_j \\ [0]_{1\times3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6\times6}.$$

Evidently, for a six vector $U := (u, \varphi, \psi, \vartheta)^\top$ we have

$$\mathcal{T}(\partial, n)U = (\sigma_{1j}n_j, \sigma_{2j}n_j, \sigma_{3j}n_j, -D_jn_j, -B_jn_j, -q_jn_j)^\top. \tag{2.4}$$

The components of the vector $\mathcal{T}U$ given by (2.4) have the physical sense: the first three components correspond to the mechanical stress vector in the theory of thermo-electro-magneto-elasticity, the forth, fifth and sixth ones are respectively the normal components of the electric displacement vector, magnetic induction vector and heat flux vector with opposite sign.

From the above equations of dynamics, in the case of statics, we get the following equations

$$A(\partial)U(x) = \Phi(x),$$

where $U = (u_1, \dots, u_6)^\top := (u, \varphi, \psi, \vartheta)^\top$ is the sought for vector function and $\Phi = (\Phi_1, \dots, \Phi_6)^\top := (-X_1, -X_2, -X_3, -\varrho_e, 0, -Q)^\top$ is a given vector function; $A(\partial) = [A_{pq}(\partial)]_{6\times6}$ is the matrix differential operator

$$A(\partial) = \begin{bmatrix} [c_{rjkl}\partial_j\partial_l]_{3\times3} & [e_{lrj}\partial_j\partial_l]_{3\times3} & [q_{lrj}\partial_j\partial_l]_{3\times1} & [-\lambda_{rj}\partial_j]_{3\times1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -p_j\partial_j \\ [-q_{jkl}\partial_j\partial_l]_{1\times3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -m_j\partial_j \\ [0]_{1\times3} & 0 & 0 & \eta_{jl}\partial_j\partial_l \end{bmatrix}_{6\times6}.$$

From the symmetry conditions (2.1), inequalities (2.2) and positive definiteness of the matrix (2.3) it follows that $A(\partial)$ is a formally non-self adjoint strongly elliptic operator.

2.2 Formulation of boundary value problems

Let $\mathbb{R}^3$ be divided by some plane into two half-spaces. Without loss of generality we can assume that these half-spaces are

$$\begin{aligned} \mathbb{R}^3_1 &:= \big\{x \mid \ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \ \text{ and } \ x_3 > 0\big\}, \\ \mathbb{R}^3_2 &:= \big\{x \mid \ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \ \text{ and } \ x_3 < 0\big\}; \end{aligned}$$

$n = (n_1, n_2, n_3) = (0, 0, -1)$ is the outward unit normal vector with respect to $\mathbb{R}^3_1$; $S := \partial\mathbb{R}^3_{1,2}$.

Now we formulate the **Neumann type boundary-value problems** $(\mathbf{N})^{\pm}$ of the thermo-electro-magnetoelasticity theory for a half-space:

Find a solution vector $U = (u, \varphi, \psi, \vartheta)^\top \in [C^1(\overline{\mathbb{R}^3_{1,2}})]^6 \cap [C^2(\mathbb{R}^3_{1,2})]^6$ to the system of equations

$$A(\partial)U = 0 \ \text{ in } \ \mathbb{R}^3_{1,2} \tag{2.5}$$

satisfying the Neumann type boundary condition

$$\{\mathcal{T}(\partial, n)U\}^{\pm} = F \ \text{ on } \ S. \tag{2.6}$$

The symbols $\{\,\cdot\,\}^{\pm}$ denote the one-sided limits on S from $\mathbb{R}^3_1$ (sign "+") and $\mathbb{R}^3_2$ (sign "−").

We require that the boundary data involved in the above setting possess the following smoothness property: $F \in \overset{\circ}{C}{}^{\infty}(\mathbb{R}^2)$, where $\overset{\circ}{C}{}^{\infty}(\mathbb{R}^2)$ is the space of infinitely differentiable functions with compact support.

Let $\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}$ and $\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}$ denote the direct and inverse generalized Fourier transforms in the space of tempered distributions (the Schwartz space $\mathcal{S}'(\mathbb{R}^2)$) which for regular summable functions f and g read as follows

$$\begin{aligned} \mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[f] &= \int\limits_{\mathbb{R}^2} f(\widetilde{x})\, e^{i\widetilde{x}\cdot\widetilde{\xi}}\, d\widetilde{x}, \\ \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[g] &= \frac{1}{4\pi^2}\int\limits_{\mathbb{R}^2} g(\widetilde{\xi})\, e^{-i\widetilde{x}\cdot\widetilde{\xi}}\, d\widetilde{\xi}, \end{aligned} \tag{2.7}$$

where $\widetilde{x} = (x_1, x_2)$, $\widetilde{\xi} = (\xi_1, \xi_2)$, $d\widetilde{x} = dx_1\, dx_2$, $\widetilde{x}\cdot\widetilde{\xi} = x_1\xi_1 + x_2\xi_2$.

Note that if $f(x) = f(x_1, x_2, x_3) = f(\widetilde{x}, x_3)$, then

$$\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[\partial_{x_j} f(x)] = -i\xi_j \mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[f] = -i\xi_j \widehat{f}(\widetilde{\xi}, x_3), \ \ j = 1, 2,$$

and hence

$$\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[\nabla_x f(x)] = \begin{bmatrix} -i\xi_1 \\ -i\xi_2 \\ \partial_{x_3} \end{bmatrix} \mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[f(x)] = P(-i\widetilde{\xi}, \partial x_3)\widehat{f}(\widetilde{\xi}, x_3) \tag{2.8}$$

with $\widehat{f}(\widetilde{\xi}, x_3) = \mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[f]$ and

$$P = P(-i\widetilde{\xi}, \partial_{x_3}) = (-i\xi_1, -i\xi_2, \partial_{x_3})^\top. \tag{2.9}$$

Applying Fourier transform (2.7) in (2.5)–(2.6) and taking into account (2.9) we arrive at the problems:

$$A(P)\widehat{U}(\widetilde{\xi}, x_3) = 0, \quad x_3 \in (0; +\infty) \ \text{ or } \ x_3 \in (-\infty; 0), \tag{2.10}$$

$$\big\{\mathcal{T}(\partial, n)\widehat{U}(\widetilde{\xi}, x_3)\big\}^{\pm}_{(x_3\to 0\pm)} = \widehat{F}(\widetilde{\xi}). \tag{2.11}$$

We see that (2.10) is the system of ordinary differential equations of second order for each $\widetilde{\xi} \in \mathbb{R}^2$. We denote these problems by $\widehat{N}^{\pm}$.

3 Uniqueness theorems

We start with constructing a system of linear independent solutions to system (2.10).

Let us denote by $k_j = k_j(\widetilde{\xi})$, $j = \overline{1, 12}$, the roots of the equation

$$\det A(-i\xi) = 0 \tag{3.1}$$

with respect to ξ_3, where $A(-i\xi)$ is the symbol matrix of the operator $A(\partial)$.

Note that $\det A(-i\xi)$ is a homogeneous polynomial of order 12 and the equation (3.1) has no real roots, $\operatorname{Im} k_j \neq 0$, $j = \overline{1,12}$. These roots are continuously dependent on the coefficients of (3.1) and the number of roots with positive and negative imaginary parts are equal. Denote by $k_1, k_2, \dots, k_6$ roots with positive imaginary parts and by $k_7, \dots, k_{12}$ with negative ones.

Let us construct the following matrices:

$$\Phi^{(+)}(\widetilde{\xi}, x_3) = \int\limits_{\ell^+} A^{-1}(-i\xi)\, e^{-i\xi_3 x_3}\, d\xi_3, \tag{3.2}$$

$$\Phi^{(-)}(\widetilde{\xi}, x_3) = \int\limits_{\ell^-} A^{-1}(-i\xi)\, e^{-i\xi_3 x_3}\, d\xi_3, \tag{3.3}$$

where ℓ^+ (respectively, ℓ^-) is a closed simple curve of positive counterclockwise orientation (respectively, negative clockwise orientation) in the upper (respectively, lower) complex half-plane $\operatorname{Re}\xi_3 > 0$ (respectively, $\operatorname{Re}\xi_3 < 0$) enclosing all the roots with respect to ξ_3 of the equation $\det A(-i\xi) = 0$ with positive (respectively, negative) imaginary parts (see Fig. 1). Clearly, (3.2) and (3.3) do not depend on the shape of ℓ^+ (respectively, ℓ^-).

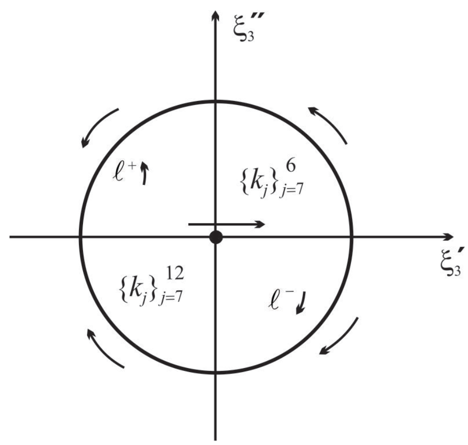

Figure 1.

With the help of the Cauchy integral theorem for analytic functions, we conclude that the entries of the matrix $\Phi^{(+)}(\widetilde{\xi}, x_3) = [\Phi^{(+)}_{kj}(\widetilde{\xi}, x_3)]_{6\times 6}$ are increasing exponentially as $x_3 \to +\infty$ and are decreasing exponentially as $x_3 \to -\infty$ (since $-i\xi_3 x_3 = -i(\xi_3' + i\xi_3'')x_3 = -i\xi_3' x_3 + \xi_3'' x_3$).

Analogously, the entries of the matrix $\Phi^{(-)}(\widetilde{\xi}, x_3) = [\Phi^{(-)}_{kj}(\widetilde{\xi}, x_3)]_{6\times 6}$ are increasing exponentially as $x_3 \to -\infty$ and vanish exponentially as $x_3 \to +\infty$.

Due to Lemma 3.1 in [15] the columns of $\Phi^{(\pm)}(\widetilde{\xi}, x_3)$ are linearly independent solutions to system (2.10).

Theorem 3.1. *The boundary value problems* $\widehat{N}^{\pm}$ (2.10)–(2.11) *have only one solution in the space of functions vanishing at infinity.*

Proof. If $x_3 \in (0; +\infty)$, then we look for a solution of the Neumann problem in the following form

$$\widehat{U}(\widetilde{\xi}, x_3) = \Phi^{(-)}(\widetilde{\xi}, x_3)C, \quad x_3 > 0,$$

where $C = (C_1, \dots, C_6)$ is unknown vector depending only on $\widetilde{\xi}$.

From (2.11) we have

$$\mathcal{T}(-i\xi, n)\Phi^{(-)}(\widetilde{\xi}, 0)C = \widehat{F}(\widetilde{\xi})$$

and since $\det[\mathcal{T}(-i\xi, n)\Phi^{(-)}(\widetilde{\xi}, 0)] \neq 0$, $|\widetilde{\xi}| \neq 0$, due to Lemma 3.1 in [15], we obtain

$$C = \big[\mathcal{T}(-i\xi, n)\Phi^{(-)}(\widetilde{\xi}, 0)\big]^{-1}\widehat{F}(\widetilde{\xi}).$$

Therefore the unique solution of $\widehat{N}^{+}$ has the following form

$$\widehat{U}(\widetilde{\xi},x_3)=\Phi^{(-)}(\widetilde{\xi},x_3)\big[\mathcal{T}(-i\xi,n)\Phi^{(-)}(\widetilde{\xi},0)\big]^{-1}\widehat{F}(\widetilde{\xi}),\ \ x_3>0. \tag{3.4}$$

Similarly, if $x_3\in(-\infty;0)$, then the unique solution of $\widehat{N}^{-}$ has the form

$$\widehat{U}(\widetilde{\xi},x_3)=\Phi^{(+)}(\widetilde{\xi},x_3)\big[\mathcal{T}(-i\xi,n)\Phi^{(+)}(\widetilde{\xi},0)\big]^{-1}\widehat{F}(\widetilde{\xi}),\ \ x_3<0. \tag{3.5}$$

The theorem is proved. □

Lemma 3.2. *There hold the following relations*

$$\big[\mathcal{T}(-i\xi,n)\Phi^{(-)}(\widetilde{\xi},0)\big]^{-1}=\begin{bmatrix}[\mathcal{O}(1)]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(1)\end{bmatrix}_{6\times6}. \tag{3.6}$$

Proof. Note that

$$\mathcal{T}(-i\xi,n):=\begin{bmatrix}[c_{rjkl}n_j(-i\xi_l)]_{3\times3} & [e_{lrj}n_j(-i\xi_l)]_{3\times3} & [q_{lrj}n_j(-i\xi_l)]_{3\times1} & [-\lambda_{rj}n_j]_{3\times1}\\ [-e_{jkl}n_j(-i\xi_l)]_{1\times3} & \varkappa_{jl}n_j(-i\xi_l) & a_{jl}n_j(-i\xi_l) & -p_jn_j\\ [-q_{jkl}n_j(-i\xi_l)]_{1\times3} & a_{jl}n_j(-i\xi_l) & \mu_{jl}n_j(-i\xi_l) & -m_jn_j\\ [0]_{1\times3} & 0 & 0 & \eta_{jl}n_j(-i\xi_l)\end{bmatrix}_{6\times6}.$$

It is clear (see Theorem 3.1) that

$$\det\mathcal{T}(-i\xi,n)\neq0,\ \ |\xi|\neq0,$$

and

$$\mathcal{T}(-i\xi,n)=\begin{bmatrix}[\mathcal{O}(|\xi|)]_{5\times5} & [\mathcal{O}(1)]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(|\xi|)\end{bmatrix}_{6\times6}. \tag{3.7}$$

It can easily be checked that $\det\mathcal{T}(-i\xi,n)=\mathcal{O}(|\xi|^6)$ and there exist constants $c_1^*>0$ and $c_2^*>0$ such that

$$c_1^*|\xi|^6\le|\det\mathcal{T}(-i\xi,n)|\le c_2^*|\xi|^6. \tag{3.8}$$

If $\mathcal{T}_c(-i\xi,n)$ is the corresponding matrix of cofactors, then

$$[\mathcal{T}(-i\xi,n)]^{-1}=\frac{1}{\det\mathcal{T}(-i\xi,n)}\,\mathcal{T}_c(-i\xi,n).$$

Taking into account (3.7) and (3.8) we can write

$$[\mathcal{T}(-i\xi,n)]^{-1}=\frac{1}{\det\mathcal{T}(-i\xi,n)}\begin{bmatrix}[\mathcal{O}(|\xi|^5)]_{5\times5} & [\mathcal{O}(|\xi|^4)]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(|\xi|^5)\end{bmatrix}_{6\times6}.$$

For arbitrary $|\widetilde{\xi}|\neq0$ we obtain

$$[\mathcal{T}(-i\xi,n)]^{-1}=\begin{bmatrix}[\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-2})]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(|\widetilde{\xi}|^{-1})\end{bmatrix}_{6\times6}. \tag{3.9}$$

Note that (see Lemma 3.3 in [15])

$$\big[\Phi^{(-)}(\widetilde{\xi},0)\big]^{-1}=\begin{bmatrix}[\mathcal{O}(|\widetilde{\xi}|)]_{5\times5} & [\mathcal{O}(1)]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(|\widetilde{\xi}|)\end{bmatrix}_{6\times6}. \tag{3.10}$$

Taking into account (3.9) and (3.10) we derive the following relations

$$\begin{aligned}\big[\mathcal{T}(-i\xi,n)\Phi^{(-)}(\widetilde{\xi},0)\big]^{-1} &= \big[\Phi^{(-)}(\widetilde{\xi},0)\big]^{-1}[\mathcal{T}(-i\xi,n)]^{-1}\\ &= \begin{bmatrix}[\mathcal{O}(|\widetilde{\xi}|)]_{5\times5} & [\mathcal{O}(1)]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(|\widetilde{\xi}|)\end{bmatrix}_{6\times6}\begin{bmatrix}[\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-2})]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(|\widetilde{\xi}|^{-1})\end{bmatrix}_{6\times6}\\ &= \begin{bmatrix}[\mathcal{O}(1)]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(1)\end{bmatrix}_{6\times6}.\end{aligned}$$ □

Remark 3.3. For arbitrary $x_3 > 0$ (see [15])

$$\Phi^{(-)}(\widetilde{\xi},x_3) = \begin{bmatrix}[\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-2})]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(|\widetilde{\xi}|^{-1})\end{bmatrix}_{6\times6}$$

and due to (3.6)

$$\Phi^{(-)}(\widetilde{\xi},x_3)\big[\mathcal{T}(-i\xi,n)\Phi^{(-)}(\widetilde{\xi},0)\big]^{-1} = \begin{bmatrix}[\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-2})]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(|\widetilde{\xi}|^{-1})\end{bmatrix}_{6\times6}. \tag{3.11}$$

Similarly, for arbitrary $x_3 < 0$

$$\Phi^{(+)}(\widetilde{\xi},x_3)\big[\mathcal{T}(-i\xi,n)\Phi^{(+)}(\widetilde{\xi},0)\big]^{-1} = \begin{bmatrix}[\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-2})]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(|\widetilde{\xi}|^{-1})\end{bmatrix}_{6\times6}. \tag{3.12}$$

Theorem 3.4. *The Neumann boundary value problems* (2.5)–(2.6) *have at most one solution* $U = (u,\varphi,\psi,\vartheta)^\top$ *in the space* $[C^1(\overline{\mathbb{R}^3_{1,2}})]^6 \cap [C^2(\mathbb{R}^3_{1,2})]^6$ *provided*

$$\vartheta(x) = \mathcal{O}(|x|^{-1}), \tag{3.13}$$

$$\partial^\alpha \widetilde{U}(x) = \mathcal{O}\big(|x|^{-1-|\alpha|}\ln|x|\big) \ \ \text{as} \ \ |x|\to\infty \tag{3.14}$$

for arbitrary multi-index $\alpha = (\alpha_1,\alpha_2,\alpha_3)$*. Here* $\widetilde{U} = (u,\varphi,\psi)^\top$*.*

Proof. Let $U^{(1)} = (u^{(1)},\varphi^{(1)},\psi^{(1)},\vartheta^{(1)})^\top$ and $U^{(2)} = (u^{(2)},\varphi^{(2)},\psi^{(2)},\vartheta^{(2)})$ be two solutions of the problem under consideration with properties indicated in the theorem for $\mathbb{R}^3_1$. It is evident that the difference

$$V = (u',\varphi',\psi',\vartheta') = U^{(1)} - U^{(2)}$$

solves the corresponding homogeneous problem.

Therefore for the temperature function we get the separated homogeneous Neumann problem

$$[A(\partial)V]_6 = \eta_{jl}\partial_j\partial_l\vartheta' = 0 \ \text{ in } \ \mathbb{R}^3_1, \tag{3.15}$$

$$\{\eta_{jl}n_j\partial_l\vartheta'\}^+ = 0 \ \text{ on } \ S. \tag{3.16}$$

By Green's formula (see (2.83) in [12]) for $B^+(0;R) := \{(x_1,x_2,x_3) \mid \ x_1^2+x_2^2+x_3^2 \le R^2$ and $x_3 > 0\}$ and (3.15)–(3.16) we have

$$\int\limits_{B^+(0;R)} \eta_{jl}\partial_l\vartheta'\partial_j\vartheta'\,dx = \int\limits_{\partial B^+(0;R)} \{\eta_{jl}n_j\partial_l\vartheta'\}^+\{\vartheta'\}^+\,dS = \int\limits_{\Sigma^+(0;R)} \{\eta_{jl}n_j\partial_l\vartheta'\}^+\{\vartheta'\}^+\,d\Sigma. \tag{3.17}$$

Here $\Sigma^+(0;R)$ is the upper half sphere.

Taking the limit as $R\to\infty$ in (3.17) according to (3.13)–(3.14) we get

$$\int\limits_{\mathbb{R}^3_1} \eta_{jl}\partial_l\vartheta'\partial_j\vartheta'\,dx = 0.$$

Due to (2.2) $\vartheta' = const$ and from (3.13) we conclude that $\vartheta' = 0$.

Therefore the five dimensional vector $\widetilde{V} = (u', \varphi', \psi')^\top$ constructed by the first five components of the solution vector V, solves the following homogeneous boundary value problem

$$\begin{aligned} \widetilde{A}(\partial)\widetilde{V} &= 0 \ \text{ in } \ \mathbb{R}^3_1, \\ \{\widetilde{\mathcal{T}}(\partial, n)\widetilde{V}\}^+ &= 0 \ \text{ on } \ S, \end{aligned} \tag{3.18}$$

where $\widetilde{A}(\partial)$ is the 5×5 differential operator of statics of the electro-magneto-elasticity theory without taking into account thermal effects (see [12]):

$$\widetilde{A}(\partial) = [\widetilde{A}_{pq}(\partial)]_{5\times 5} := \begin{bmatrix} [c_{rjkl}\partial_j\partial_l]_{3\times 3} & [e_{lrj}\partial_j\partial_l]_{3\times 1} & [q_{lrj}\partial_j\partial_l]_{3\times 1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l \\ [-q_{jkl}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l \end{bmatrix}_{5\times 5}$$

and $\widetilde{\mathcal{T}}(\partial, n)$ is the corresponding 5×5 generalized stress operator

$$\widetilde{\mathcal{T}}(\partial, n) = [\widetilde{\mathcal{T}}_{pq}(\partial, n)]_{5\times 5} := \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [e_{lrj}n_j\partial_l]_{3\times 1} & [q_{lrj}n_j\partial_l]_{3\times 1} \\ [-e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l \\ [-q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l \end{bmatrix}_{5\times 5}.$$

Using the limiting procedure as above in the corresponding Green's identity for vectors satisfying decay conditions (3.14) we obtain

$$\int\limits_{\mathbb{R}^3_1} \big[\widetilde{A}(\partial)\widetilde{V}\cdot\widetilde{V} + \widetilde{\mathcal{E}}(\widetilde{V}, \widetilde{V})\big]\, dx = \lim_{R\to\infty} \int\limits_{\Sigma^+(0;R)} [\widetilde{\mathcal{T}}\widetilde{V}]^+ \cdot [\widetilde{V}]^+\, d\Sigma, \tag{3.19}$$

where $\widetilde{\mathcal{E}}(\widetilde{V}, \widetilde{V})$ has the following form:

$$\widetilde{\mathcal{E}}(\widetilde{V}, \widetilde{V}) = c_{rjkl}\partial_l u'_k \partial_j u'_r + \varkappa_{jl}\partial_l\varphi'\partial_j\varphi' + a_{jl}(\partial_l\varphi'\partial_j\psi' + \partial_j\psi'\partial_l\varphi') + \mu_{jl}\partial_l\psi'\partial_j\psi'. \tag{3.20}$$

If $\widetilde{V}$ is a solution of (3.18) satisfying (3.14), then from (3.19) we have

$$\int\limits_{\mathbb{R}^3_1} \widetilde{\mathcal{E}}(\widetilde{V}, \widetilde{V})\, dx = 0. \tag{3.21}$$

From (3.18), (3.20) and (3.21) along with (2.2) we get

$$u'(x) = a\times x + b, \quad \varphi'(x) = b_4, \quad \psi' = b_5,$$

where $a = (a_2, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are arbitrary constant vectors and b_4, b_5 are arbitrary constants. Now, in view of (3.14) we arrive at the equalities $u'(x) = 0$, $\varphi'(x) = 0$, $\psi'(x) = 0$ for all $x \in \mathbb{R}^3_1$, consequently, $U^{(1)} = U^{(2)}$ in $\mathbb{R}^3_1$.

The proof is similar for the domain $\mathbb{R}^3_2$. □

Theorem 3.5. *Let* $F \in \overset{\circ}{C}{}^\infty(\mathbb{R}^2)$ *and for arbitrary multi-index* $\beta = (\beta_1, \beta_2)$

$$\int\limits_{\mathbb{R}^2} F(\widetilde{x})\widetilde{x}^\beta\, d\widetilde{x} = 0, \ \ |\beta| = 0, 1, 2.$$

Then the Neumann boundary value problems (2.5)–(2.6) *possess unique solutions which can be represented in the following form*

$$U(x) = \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}} \Big[\Phi^{(-)}(\widetilde{\xi}, x_3)\big[\mathcal{T}(-i\xi, n)\Phi^{(-)}(\widetilde{\xi}, 0)\big]^{-1}\widehat{F}(\widetilde{\xi})\Big], \ \ x_3 > 0, \tag{3.22}$$

or

$$U(x) = \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}} \Big[\Phi^{(+)}(\widetilde{\xi}, x_3)\big[\mathcal{T}(-i\xi, n)\Phi^{(+)}(\widetilde{\xi}, 0)\big]^{-1}\widehat{F}(\widetilde{\xi})\Big], \ \ x_3 < 0. \tag{3.23}$$

Proof. It suffices to show that the vector functions (3.22) and (3.23) satisfy the conditions (3.13)–(3.14). This will be done if we prove that the following relations hold for all $x \in \mathbb{R}^3$:

$$x_j \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[\widehat{U}(\widetilde{\xi}, x_3)] = \mathcal{O}(1), \quad j = 1, 2, 3, \tag{3.24}$$

and

$$x_j^2 \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[\widehat{U}(\widetilde{\xi}, x_3)] = \mathcal{O}(1), \quad j = 1, 2, 3, \tag{3.25}$$

where $\widehat{U}(\widetilde{\xi}, x_3)$ is defined by (3.4) or (3.5).

Under the restriction on F we conclude that $\widehat{F} \in \mathcal{S}(\mathbb{R}^2)$ and $\widehat{F}(\widetilde{\xi}) = \mathcal{O}(|\widetilde{\xi}|^3)$ as $|\widetilde{\xi}| \to 0$, where $\mathcal{S}$ is the space of rapidly decreasing functions. Therefore in view of (3.11)–(3.12) we have

$$\begin{aligned} &\frac{\partial \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j} = \mathcal{O}(1), \quad |\widetilde{\xi}| \to 0, \\ &\frac{\partial \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j} = \mathcal{O}(|\widetilde{\xi}|^{-k}), \quad |\widetilde{\xi}| \to \infty, \quad k \geq 2, \end{aligned} \tag{3.26}$$

uniformly for all $x \in \mathbb{R}^3$.

For $j = 1$ or $j = 2$, we find

$$\begin{aligned} x_j \int\limits_{\mathbb{R}^2} \widehat{U}(\widetilde{\xi}, x_3)\, e^{-i\widetilde{\xi}\cdot\widetilde{x}}\, d\widetilde{\xi} &= i \int\limits_{\mathbb{R}^2} \widehat{U}(\widetilde{\xi}, x_3)\, \frac{\partial e^{-i\widetilde{\xi}\cdot\widetilde{x}}}{\partial \xi_j}\, d\widetilde{\xi} = i \lim_{R\to\infty} \int\limits_{K(0;R)} \widehat{U}(\widetilde{\xi}, x_3)\, \frac{\partial e^{-i\widetilde{\xi}\cdot\widetilde{x}}}{\partial \xi_j}\, d\widetilde{\xi} \\ &= -i \lim_{R\to\infty} \Bigg(\int\limits_{K(0;R)} \frac{\partial \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j}\, e^{-i\widetilde{\xi}\cdot\widetilde{x}}\, d\widetilde{\xi} - \int\limits_{\partial K(0;R)} \widehat{U}(\widetilde{\xi}, x_3)\, e^{-i\widetilde{\xi}\cdot\widetilde{x}}\, \frac{\xi_j}{R}\, ds \Bigg) \\ &= -i \lim_{R\to\infty} \int\limits_{K(0;R)} \frac{\partial \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j}\, e^{-i\widetilde{\xi}\cdot\widetilde{x}}\, d\widetilde{\xi} = -i \int\limits_{\mathbb{R}^2} \frac{\partial \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j}\, e^{-i\widetilde{\xi}\cdot\widetilde{x}}\, d\widetilde{\xi}, \end{aligned} \tag{3.27}$$

where $K(0, R)$ is the circle of radius R centered at the origin.

It is clear that the relations (3.26) and (3.27) imply (3.24). The condition (3.25) can be proved similarly if we note that

$$\begin{aligned} &\frac{\partial^2 \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j^2} = \mathcal{O}\big(|\widetilde{\xi}|^{-1}\big), \quad |\widetilde{\xi}| \to 0, \\ &\frac{\partial^2 \widehat{U}(\widetilde{\xi}, x_3)}{\partial \xi_j^2} = \mathcal{O}(|\widetilde{\xi}|^{-k-1}), \quad |\widetilde{\xi}| \to \infty, \quad k \geq 2, \end{aligned}$$

uniformly for all $x \in \mathbb{R}^3$.

For arbitrary $x_3 > 0$ we can write

$$x_3 \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[\widehat{U}(\widetilde{\xi}, x_3)] = x_3 \int\limits_{\mathbb{R}^2} \Bigg(\int\limits_{\ell^-} A^{-1}(-i\xi)\, e^{-i\xi_3 x_3}\, d\xi_3 \Bigg) [\mathcal{T}(-i\xi, n)\Phi^{(-)}(\widetilde{\xi}, 0)]^{-1} \widehat{F}(\widetilde{\xi})\, e^{-i\widetilde{\xi}\cdot\widetilde{x}}\, d\widetilde{\xi}. \tag{3.28}$$

Due to Lemma 3.3 in [15] the entries of the matrix $A^{-1}(-i\xi)$ are homogeneous functions in ξ and

$$A^{-1}(-i\xi) = \begin{bmatrix} [\mathcal{O}(|\xi|^{-2})]_{5\times5} & [\mathcal{O}(|\xi|^{-3})]_{5\times1} \\ [0]_{1\times5} & \mathcal{O}(|\xi|^{-2}) \end{bmatrix}_{6\times6}. \tag{3.29}$$

Using the Cauchy integral theorem for analytic functions and the relations (3.6), (3.29), from

(3.28) we get

$$\begin{aligned}
&x_3\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[\widehat{U}(\widetilde{\xi},x_3)]\\
&\quad= x_3\int\limits_{\mathbb{R}^2} e^{-|\widetilde{\xi}|x_3}\begin{bmatrix}[\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-2})]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(|\widetilde{\xi}|^{-1})\end{bmatrix}\begin{bmatrix}[\mathcal{O}(1)]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(1)\end{bmatrix}\widehat{F}(\widetilde{\xi})\,d\widetilde{\xi}\\
&\quad= x_3\int\limits_{\mathbb{R}^2} e^{-|\widetilde{\xi}|x_3}\begin{bmatrix}[\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-2})]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(|\widetilde{\xi}|^{-1})\end{bmatrix}\widehat{F}(\widetilde{\xi})\,d\widetilde{\xi} = I_1 + I_2,
\end{aligned}$$

where

$$I_1 = x_3\int\limits_{|\xi|\le M} e^{-|\widetilde{\xi}|x_3}\begin{bmatrix}[\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-2})]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(|\widetilde{\xi}|^{-1})\end{bmatrix}\widehat{F}(\widetilde{\xi})\,d\widetilde{\xi},$$

$$I_2 = x_3\int\limits_{|\xi|> M} e^{-|\widetilde{\xi}|x_3}\begin{bmatrix}[\mathcal{O}(|\widetilde{\xi}|^{-1})]_{5\times5} & [\mathcal{O}(|\widetilde{\xi}|^{-2})]_{5\times1}\\ [0]_{1\times5} & \mathcal{O}(|\widetilde{\xi}|^{-1})\end{bmatrix}\widehat{F}(\widetilde{\xi})\,d\widetilde{\xi}$$

for some positive number M.

Since $\widehat{F}(\widetilde{\xi}) \in S(\mathbb{R}^2)$, it is easy to check that $I_1 = \mathcal{O}(1)$ and $I_2 = \mathcal{O}(1)$ and hence (3.24) holds.

We can prove the boundedness of the vector function $x_3^2\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[\widehat{U}(\widetilde{\xi},x_3)]$ quite similarly taking into account that $\widehat{F}(\widetilde{\xi}) = \mathcal{O}(|\widetilde{\xi}|^3)$ as $|\widetilde{\xi}| \to 0$. $\square$

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(Received 19.05.2016)

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Memoirs on Differential Equations and Mathematical Physics

VOLUME 74, 2018, 153–164

Short Communication

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ON THE WELL-POSEDNESS OF ANTIPERIODIC PROBLEM FOR SYSTEMS OF NONLINEAR IMPULSIVE EQUATIONS WITH FIXED IMPULSES POINTS

Abstract. The antiperiodic problem for systems of nonlinear impulsive equations with fixed points of impulses actions is considered. The sufficient (among them effective) conditions for the well-posedness of this problem are given.

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2010 Mathematics Subject Classification: 34K10, 34K45.

Key words and phrases: Antiperiodic problem, nonlinear systems, impulsive equations, fixed impulses points, well-posedness, effective conditions.

Let m_0 be a fixed natural number, ω be a fixed positive real one, and $0 < \tau_1 < \cdots < \tau_{m_0} < \omega$ be fixed points (we assume $\tau_0 = 0$ and $\tau_{m_0+1} = \omega$, if necessary). Let $T = \{\tau_l + m\omega : \ l = 1, \dots, m_0; \ m = 0, \pm 1, \pm 2, \dots\}$.

Consider the system of nonlinear impulsive equations with fixed impulses points

$$\frac{dx}{dt} = f(t,x) \ \text{ almost everywhere on } \ \mathbb{R} \setminus T,$$
$$x(\tau+) - x(\tau-) = I(\tau, x(\tau)) \ \text{ for } \ \tau \in T$$

with the ω-antiperiodic condition

$$x(t+\omega) = -x(t) \ \text{ for } \ t \in \mathbb{R},$$

where $f = (f_i)_{i=1}^n$ is a vector-function belonging to the Carathéodory class $Car(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, and $I = (I_i)_{i=1}^n : T \times R^n \to \mathbb{R}^n$ is a vector-function such that $I(\tau, \cdot\,)$ is continuous for every $\tau \in T$.

We assume that

$$f(t+\omega, x) = -f(t, -x) \ \text{ and } \ I(\tau+\omega, x) = -I(\tau, -x) \ \text{ for } \ t \in \mathbb{R}, \ \ \tau \in T, \ \ x \in \mathbb{R}^n.$$

Due to the above condition, if $x : \mathbb{R} \to \mathbb{R}^n$ is a solution of the given system, then the vector-function $y(t) = -x(t+\omega)$ $(t \in \mathbb{R})$ will likewise be a solution of that system. Moreover, it is evident that if $x : \mathbb{R} \to \mathbb{R}^n$ is a solution of the given ω-antiperiodic problem, then its restriction on the closed interval $[0, \omega]$ will be a solution of the problem

$$\frac{dx}{dt} = f(t,x) \ \text{ almost everywhere on } \ [0,\omega] \setminus \{\tau_1, \dots, \tau_{m_0}\}, \tag{1}$$
$$x(\tau_l+) - x(\tau_l-) = I(\tau_l, x(\tau_l)) \ \ (l = 1, \dots, m_0); \tag{2}$$
$$x(0) = -x(\omega). \tag{3}$$

Let now $x : [0,\omega] \to \mathbb{R}^n$ be a solution of system (1), (2) on $[0,\omega]$. By x we designate the continuation of this function on the whole R just as a solution of system (1), (2), as well. As above, the vector-function $y(t) = -x(t+\omega)$ $(t \in \mathbb{R})$ will be a solution of system (1), (2). On the other hand, according to equality (3), we have $y(0) = -x(\omega) = x(0)$. So, if we assume that system (1), (2) under the Cauchy condition $x(0) = c$ is uniquely solvable for every $c \in \mathbb{R}^n$, then $x(t+\omega) = -x(t)$ for $t \in \mathbb{R}$, i.e., x is ω-antiperiodic. This means that the set of restrictions of the ω-antiperiodic solutions of system (1), (2) on $[0,\omega]$ coincides with the set of solutions of problem (1), (2); (3).

In this connection, we consider the boundary value problem (1), (2); (3) on the closed interval $[0,\omega]$. Below, we will give the sufficient conditions guaranteeing the well-posedness of this problem.

Consider a sequence of vector-functions $f_k \in Car([0,\omega] \times \mathbb{R}^n, \mathbb{R}^n)$ $(k = 1,2,\dots)$, sequences of points τ_{lk} $(k = 1,2,\dots;\ l = 1,\dots,m_0)$, $0 < \tau_{1k} < \dots < \tau_{m_0k} < \omega$, and sequences of operators $I_k : \{\tau_{1k},\dots,\tau_{m_0k}\} \times R^n \to \mathbb{R}^n$ $(k = 1,2,\dots)$ such that $I_k(\tau_{lk},\cdot)$ $(k = 1,2,\dots;\ l = 1,\dots,m_0)$ are continuous.

In this paper, we establish the sufficient conditions guaranteeing both the solvability of the impulsive systems

$$\frac{dx}{dt} = f_k(t,x) \ \text{ almost everywhere on } \ [0,\omega] \setminus \{\tau_{1k},\dots,\tau_{m_0k}\}, \tag{1_k}$$

$$x(\tau_{lk}+) - x(\tau_{lk}-) = I_k(\tau_{lk}, x(\tau_{lk})) \ \ (l = 1,\dots,m_0) \tag{2_k}$$

$(k = 1,2,\dots)$ under condition (3) for any sufficiently large k and the convergence of their solutions to a solution of problem (1), (2); (3), as $k \to +\infty$.

We assume that the above-described concept is fulfilled for problems $(1_k), (2_k); (3)$ $(k = 1,2,\dots)$, as well.

The well-posed problem for the linear boundary value problem for impulsive systems with a finite number of impulses points has been investigated in [5], where the necessary and sufficient conditions were given for the case. Analogous problems are investigated in [1, 11–13] (see also the references therein) for the linear and nonlinear boundary value problems for ordinary differential systems.

A good many issues on the theory of systems of differential equations with impulsive effect (both linear and nonlinear) have been studied sufficiently well (for a survey of the results on impulsive systems see, e.g., [2–4, 6–9, 14–16] and the references therein). But the above-mentioned works do not, as we know, contain the results obtained in the present paper.

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R} =]-\infty,+\infty[$, $\mathbb{R}_+ = [0,+\infty[$; $[a,b]$ $(a,b \in R)$ is a closed interval.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max\limits_{j=1,\dots,m} \sum\limits_{i=1}^{n} |x_{ij}|$.

$|X| = (|x_{ij}|)_{i,j=1}^{n,m}$, $[X]_+ = \frac{|X|+X}{2}$.

$\mathbb{R}_+^{n\times m} = \big\{(x_{ij})_{i,j=1}^{n,m} :\ x_{ij} \ge 0\ (i = 1,\dots,n;\ j = 1,\dots,m)\big\}$.

$\mathbb{R}^{(n\times n)\times m} = \mathbb{R}^{n\times n} \times \dots \times \mathbb{R}^{n\times n}$ $(m-\text{times})$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = R_+^{n\times 1}$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix, inverse to X, the determinant of X and the spectral radius of X; $I_{n\times n}$ is the identity $n \times n$-matrix.

$\bigvee\limits_a^b(X)$ is the total variation of the matrix-function $X : [a,b] \to R^{n\times m}$, i.e., the sum of total variations of components of X; $V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(a) = 0$, $v(x_{ij})(t) = \bigvee\limits_a^t(x_{ij})$ for $a < t \le b$.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X : [a,b] \to R^{n\times m}$ at the point t (we will assume $X(t) = X(a)$ for $t \le a$ and $X(t) = X(b)$ for $t \ge b$, if necessary).

$\mathrm{BV}([a,b], R^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a,b] \to R^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X) < +\infty$).

$C([a,b], D)$, where $D \subset R^{n\times m}$, is the set of all continuous matrix-functions $X : [a,b] \to D$.

Let $T_{m_0} = \{\tau_1,\dots,\tau_{m_0}\}$.

$C([a,b],D;T_{m_0})$ is the set of all matrix-functions $X:[a,b]\to D$, having the one-sided limits $X(\tau_l-)$ $(l=1,\dots,m_0)$ and $X(\tau_l+)$ $(l=1,\dots,m_0)$, whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b]\setminus T_{m_0}\}$ belong to $C([c,d],D)$.

$C_s([a,b],\mathbb{R}^{n\times m};T_{m_0})$ is the Banach space of all $X\in C([a,b],\mathbb{R}^{n\times m};T_{m_0})$ with the norm $\|X\|_s=\sup\{\|X(t)\|:\ t\in[a,b]\}$.

If $y\in C_s([a,b],\mathbb{R};T_{m_0})$ and $r\in]0,+\infty[$, then $U(y;r)=\{x\in C_s([a,b],\mathbb{R}^n;T_{m_0}):\ \|x-y\|_s<r\}$.

$D(y,r)$ is the set of all $x\in\mathbb{R}^n$ such that $\inf\{\|x-y(t)\|:\ t\in[a,b]\}<r$.

$\widetilde{C}([a,b],D)$, where $D\subset R^{n\times m}$, is the set of all absolutely continuous matrix-functions $X:[a,b]\to D$.

$\widetilde{C}([a,b],D;T_{m_0})$ is the set of all matrix-functions $X:[a,b]\to D$, having the one-sided limits $X(\tau_l-)$ $(l=1,\dots,m_0)$ and $X(\tau_l+)$ $(l=1,\dots,m_0)$, whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b]\setminus T_{m_0}$ belong to $\widetilde{C}([c,d],D)$.

If B_1 and B_2 are normed spaces, then an operator $g:B_1\to B_2$ (nonlinear, in general) is positive homogeneous if $g(\lambda x)=\lambda g(x)$ for every $\lambda\in R_+$ and $x\in B_1$.

An operator $\varphi: C([a,b],\mathbb{R}^{n\times m};T_{m_0})\to R^n$ is called nondecreasing if the inequality $\varphi(x)(t)\le\varphi(y)(t)$ for $t\in[a,b]$ holds for every $x,y\in C([a,b],\mathbb{R}^{n\times m};T_{m_0})$ such that $x(t)\le y(t)$ for $t\in[a,b]$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

$L([a,b],D)$, where $D\subset R^{n\times m}$, is the set of all measurable and integrable matrix-functions $X:[a,b]\to D$.

If $D_1\subset\mathbb{R}^n$ and $D_2\subset R^{n\times m}$, then $Car([a,b]\times D_1,D_2)$ is the Carathéodory class, i.e., the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that for each $i\in\{1,\dots,l\}$, $j\in\{1,\dots,m\}$ and $k\in\{1,\dots,n\}$:

(a) the function $f_{kj}(\,\cdot\,,x):[a,b]\to D_2$ is measurable for every $x\in D_1$;

(b) the function $f_{kj}(t,\,\cdot\,):D_1\to D_2$ is continuous for almost every $t\in[a,b]$, and $\sup\{|f_{kj}(\,\cdot\,,x)|:\ x\in D_0\}\in L([a,b],R;g_{ik})$ for every compact $D_0\subset D_1$.

$Car^0([a,b]\times D_1,D_2)$ is the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that the functions $f_{kj}(\,\cdot\,,x(\,\cdot\,))$ $(k=1,\dots,n;\ j=1,\dots,m;)$ are measurable for every vector-function $x:[a,b]\to\mathbb{R}^n$ with a bounded total variation.

We say that the pair $\{X;\{Y_l\}_{l=1}^m\}$, consisting of a matrix-function $X\in L([a,b],\mathbb{R}^{n\times n})$ and of a sequence of constant $n\times n$ matrices $\{Y_l\}_{l=1}^m\}$, satisfies the Lappo–Danilevskiĭ condition if the matrices $Y_1,\dots,Y_m$ are pairwise permutable and there exists $t_0\in[a,b]$ such that

$$\int\limits_{t_0}^{t}X(\tau)\,dX(\tau)=\int\limits_{t_0}^{t}dX(\tau)\cdot X(\tau)\ \text{ for }\ t\in[a,b],$$
$$X(t)Y_l=Y_lX(t)\ \text{ for }\ t\in[a,b]\ \ (l=1,\dots,m).$$

$M([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ is the set of all functions $\omega\in Car([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ such that the function $\omega(t,\,\cdot\,)$ is nondecreasing and $\omega(t,0)=0$ for every $t\in[a,b]$.

By a solution of the impulsive system (1), (2) we understand a continuous from the left vector-function $x\in\widetilde{C}([0,\omega],\mathbb{R}^n;T_{m_0})$ satisfying both system (1) for a.e. on $[0,\omega]\setminus T_{m_0}$ and relation (2) for every $l\in\{1,\dots,m_0\}$.

Definition 1. Let $\ell: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ be, respectively, a linear continuous and a positive homogeneous operators. We say that a pair (P,J), consisting of a matrix-function $P\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and a continuous with respect to the last n-variables operator $J:T_{m_0}\times R^n\to\mathbb{R}^n$, satisfies the Opial condition with respect to the pair (ℓ,ℓ_0) if:

(a) there exist a matrix-function $\Phi\in L([0,\omega],\mathbb{R}^{n\times n}_+)$ and constant matrices $\Psi_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ such that

$$|P(t,x)|\le\Phi(t)\ \text{ a.e. on }\ [0,\omega],\ \ x\in\mathbb{R}^n,$$
$$|J(\tau_l,x)|\le\Psi_l\ \text{ for }\ x\in\mathbb{R}^n\ \ (l=1,\dots,m_0);$$

(b)

$$\det(I_{n\times n}+G_l)\neq 0 \ \ (l=1,\dots,m_0) \tag{4}$$

and the problem

$$\frac{dx}{dt}=A(t)x \ \text{ a.e. on } \ [0,\omega]\setminus T_{m_0}, \tag{5}$$

$$x(\tau_l+)-x(\tau_l-)=G_l\,x(\tau_l) \ \ (l=1,\dots,m_0), \tag{6}$$

$$|\ell(x)|\leq \ell_0(x) \tag{7}$$

has only the trivial solution for every matrix-function $A\in L([0,\omega],\mathbb{R}^{n\times n})$ and constant matrices $G_l,\dots,G_{m_0}$ for which there exists a sequence $y_k\in \widetilde{C}([0,\omega],\mathbb{R}^n;T_{m_0})$ $(k=1,2,\dots)$ such that

$$\lim_{k\to+\infty}\int\limits_0^t P(\tau,y_k(\tau))\,d\tau=\int\limits_0^t A(\tau)\,d\tau \ \text{ uniformly on } \ [0,\omega],$$

$$\lim_{k\to+\infty} J(\tau_l,y_k(\tau_l))=G_l \ \ (l=1,\dots,m_0).$$

Remark 1. In particular, condition (4) holds if $\|\Psi_l\|<1$ $(l=1,\dots,m_0)$.

As above, we assume that $f=(f_i)_{i=1}^n\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and, in addition, $f(\tau_l,x)$ is arbitrary for $x\in\mathbb{R}^n$ $(l=1,\dots,m_0)$.

Let x^0 be a solution of problem $(1),(2);(3)$, and r be a positive number. Let us introduce the following definition.

Definition 2. The solution x^0 is said to be strongly isolated in the radius r if there exist matrix- and vector-functions $P\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and $q\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n)$, continuous with respect to the last n-variables operators $J,H:T_{m_0}\times R^n\to\mathbb{R}^n$, linear continuous ℓ and $\widetilde{\ell}$ and a positive homogeneous ℓ_0 operators acting from $C_s([0,\omega],\mathbb{R}^n;T_{m_0})$ into $\mathbb{R}^n$ such that

(a) the equalities

$$f(t,x)=P(t,x)x+q(t,x) \ \text{ for } \ t\in[0,\omega]\setminus T_{m_0}, \ \ \|x-x^0(t)\|<r,$$

$$I(\tau_l,x)=J(\tau_l,x)x+H(\tau_l,x) \ \text{ for } \ \|x-x^0(\tau_l)\|<r \ \ (l=1,\dots,m_0),$$

$$x(0)+x(\omega)=\ell(x)+\widetilde{\ell}(x) \ \text{ for } \ x\in U(x^0;r)$$

are valid;

(b) the functions $\alpha(t,\rho)=\max\{\|q(t,x)\|:\|x\|\leq\rho\}$, $\beta(\tau_l,\rho)=\max\{\|H(\tau_l,x)\|:\|x\|\leq\rho\}$ $(l=1,\dots,m_0)$ and $\gamma(\rho)=\sup\{[|\widetilde{l}(x)|-l_0(x)]_+:\|x\|_s\leq\rho\}$ satisfy the condition

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\gamma(\rho)+\int\limits_0^\omega \alpha(t,\rho)\,dt+\sum_{l=1}^{m_0}\beta(\tau_l,\rho)\bigg)=0; \tag{8}$$

(c) the problem

$$\frac{dx}{dt}=P(t,x)x+q(t,x) \ \text{ a.e. on } \ [0,\omega]\setminus T_{m_0},$$

$$x(\tau_l+)-x(\tau_l-)=J(\tau_l,x(\tau_l))\,x(\tau_l)+H(\tau_l,x(\tau_l)) \ \ (l=1,\dots,m_0);$$

$$\ell(x)+\widetilde{\ell}(x)=0$$

has no solution different from x^0;

(d) the pair (P,J) satisfies he Opial condition with respect to the pair (ℓ,ℓ_0).

Remark 2. If $\ell(x) \equiv x(0) + x(\omega)$ and $\ell_0(x) \equiv 0$, then we say that the pair (P, J) satisfies the Opial ω-antiperiodic condition. In this case, condition (7) coincides with condition (3), and $\widetilde{\ell}(x) \equiv 0$ and $\gamma(\rho) \equiv 0$ in Definitions 1 and 2.

Definition 3. We say that a sequence (f_k, I_k) $(k = 1, 2, \dots)$ belongs to the set $W_r(f, I; x^0)$ if:

(a) the equalities

$$\lim_{k\to+\infty} \int_0^t f_k(\tau, x)\, d\tau = \int_0^t f(\tau, x)\, d\tau \ \text{ uniformly on } \ [0, \omega],$$

$$\lim_{k\to+\infty} I_k(\tau_{lk}, x) = I(\tau_l, x) \ \ (l = 1, \dots, m_0)$$

are valid for every $x \in D(x^0; r)$;

(b) there exist a sequence of functions $\omega_k \in M([a, b] \times \mathbb{R}_+, \mathbb{R}_+)$ $(k = 1, 2, \dots)$ such that

$$\sup\bigg\{ \int_0^\omega \omega_k(t, r)\, dt : \ k = 1, 2, \dots \bigg\} < +\infty, \tag{9}$$

$$\sup\bigg\{ \sum_{l=1}^{m_0} \omega_k(\tau_{lk}, r) : \ k = 1, 2, \dots \bigg\} < +\infty; \tag{10}$$

$$\lim_{s\to0+} \sup\bigg\{ \int_0^\omega \omega_k(t, s)\, dt : \ k = 1, 2, \dots \bigg\} = 0, \tag{11}$$

$$\lim_{s\to0+} \sup\bigg\{ \sum_{l=1}^{m_0} \omega_k(\tau_{lk}, s) : \ k = 1, 2, \dots \bigg\} = 0; \tag{12}$$

$$\|f_k(t, x) - f_k(t, y)\| \le \omega_k(t, \|x - y\|) \ \text{ for } \ t \in [0, \omega] \setminus T_{m_0}, \ x, y \in D(x^0; r) \ \ (k = 1, 2, \dots),$$

$$\|I_k(\tau_{lk}, x) - I_k(\tau_{lk}, y)\| \le \omega_k(\tau_{lk}, \|x - y\|) \ \text{ for } \ x, y \in D(x^0; r) \ \ (l = 1, \dots, m_0; k = 1, 2, \dots).$$

Remark 3. If for every natural m there exists a positive number ν_m such that $\omega_k(t, m\delta) \le \nu_m \omega_k(t, \delta)$ for $\delta > 0$, $t \in [0, \omega] \setminus T_{m_0}$ $(k = 1, 2, \dots)$, then estimate (9) follows from condition (11); analogously, if $\omega_k(\tau_{lk}, m\delta) \le \nu_m \omega_k(\tau_{lk}, \delta)$ for $\delta > 0$ $(l = 1, \dots, m_0;\ k = 1, 2, \dots)$, then estimate (10) follows from condition (12). In particular, the sequences of functions

$$\omega_k(t, \delta) = \max\big\{ \|f_k(t, x) - f_k(t, y)\| : \ x, y \in U(0, \|x^0\| + r), \ \|x - y\| \le \delta \big\} \quad \text{for } \ t \in [0, \omega] \setminus T_{m_0} \ \ (k = 1, 2, \dots),$$

$$\omega_k(\tau_{lk}, \delta) = \max\big\{ \|I_k(\tau_{lk}, x) - I_k(\tau_{lk}, y)\| : x, y \in U(0, \|x^0\| + r), \ \|x - y\| \le \delta \big\} \quad (l = 1, \dots, m_0;\ k = 1, 2, \dots)$$

have the latters properties, respectively.

Definition 4. Problem $(1), (2); (3)$ is said to be $(x^0; r)$-correct if for every $\varepsilon \in]0, r[$ and $(f_k, I_k)_{k=1}^{+\infty} \in W_r(f, I; x^0)$ there exists a natural number k_0 such that problem $(1_k), (2_k)$ has at last one ω-antiperiodic solution contained in $U(x^0; r)$, and any such solution belongs to the ball $U(x^0; \varepsilon)$ for every $k \ge k_0$.

Definition 5. Problem $(1), (2); (3)$ is said to be correct if it has a unique solution x^0 and is $(x^0; r)$-correct for every $r > 0$.

Theorem 1. *If problem* $(1), (2); (3)$ *has a solution* x^0 *strongly isolated in the radius* r*, then it is* $(x^0; r)$*-correct.*

Theorem 2. *Let the conditions*

$$\|f(t,x)-P(t,x)\,x\| \le \alpha(t,\|x\|) \;\; a.e. \;\; on \;\; [0,\omega]\setminus T_{m_0}, \;\; x\in\mathbb{R}^n, \tag{13}$$
$$\|I(\tau_l,x)-J(\tau_l,x)x\| \le \beta(\tau_l,\|x\|) \;\; for \;\; x\in\mathbb{R}^n \;\; (l=1,\dots,m_0), \tag{14}$$
$$|x(0)+x(\omega)-\ell(x)| \le \ell_0(x)+\ell_1(\|x\|_s) \;\; for \;\; x\in \mathrm{BV}([0,\omega],\mathbb{R}^n) \tag{15}$$

hold, where $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0 : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ are, respectively, a linear continuous and a positive homogeneous operators, the pair (P,J) satisfies the Opial condition with respect to the pair (ℓ,ℓ_0); $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ and $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ are the functions, nondecreasing in the second variable, and $\ell_1\in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\left(\|\ell_1(\rho)\|+\int\limits_0^{\omega}\alpha(t,\rho)\,dt+\sum_{l=1}^{m_0}\beta(\tau_l,\rho)\right)=0. \tag{16}$$

Then problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Theorem 3. *Let conditions* (13)–(15),

$$P_1(t)\le P(t,x)\le P_2(t) \;\; a.e. \;\; on \;\; [0,\omega]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \;\; x\in\mathbb{R}^n, \tag{17}$$
$$J_{1l}\le J(\tau_l,x)\le J_{2l} \;\; for \;\; x\in\mathbb{R}^n \;\; (l=1,\dots,m_0) \tag{18}$$

hold, where $P\in Car^0([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, $P_i\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_{il}\in\mathbb{R}^{n\times n}$ $(i=1,2;\ l=1,\dots,m_0)$; $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0 : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ are, respectively, a linear continuous and a positive homogeneous operators; $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ and $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ are the functions, nondecreasing in the second variable, and $\ell_1\in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that condition (16) *holds. Let, moreover, condition* (4) *hold and problem* (5), (6); (7) *have only the trivial solution for every matrix-function $A\in L([0,\omega],\mathbb{R}^{n\times n})$ and constant matrices $G_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ such that*

$$P_1(t)\le A(t)\le P_2(t) \;\; a.e. \;\; on \;\; [0,\omega]\setminus T_{m_0}, \;\; x\in\mathbb{R}^n, \tag{19}$$
$$J_{1l}\le G_l\le J_{2l} \;\; for \;\; x\in\mathbb{R}^n \;\; (l=1,\dots,m_0). \tag{20}$$

Then problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Remark 4. Theorem 3 is interesting only in the case where $P\notin Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, because the theorem follows immediately from Theorem 2 in the case where $P\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$.

Theorem 4. *Let conditions* (15),

$$|f(t,x)-P(t)\,x|\le Q(t)\,|x|+q(t,\|x\|) \;\; a.e. \;\; on \;\; [0,\omega]\setminus T_{m_0}, \;\; x\in\mathbb{R}^n, \tag{21}$$
$$|I_l(x)-J_l\,x|\le H_l\,|x|+h(\tau_l,\|x\|) \;\; for \;\; x\in\mathbb{R}^n \;\; (l=1,\dots,m_0) \tag{22}$$

hold, where $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $Q\in L([0,\omega],\mathbb{R}^{n\times n}_+)$, $J_l\in\mathbb{R}^{n\times n}$ and $H_l\in\mathbb{R}^{n\times n}_+$ $(l=1,\dots,m_0)$ are the constant matrices, $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0 : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ are, respectively, a linear continuous and a positive homogeneous operators; $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ and $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ are the vector-functions, nondecreasing in the second variable, and $\ell_1\in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that the condition

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\left(\|\ell_1(\rho)\|+\int\limits_0^{\omega}\|q(t,\rho)\|\,dt+\sum_{l=1}^{m_0}\|h(\tau_l,\rho)\|\right)=0 \tag{23}$$

holds. Let, moreover, the conditions

$$\det(I_{n\times n}+J_l)\ne 0 \;\; (l=1,\dots,m_0) \tag{24}$$
$$\|H_l\|\cdot\|(I_{n\times n}+J_l)^{-1}\|<1 \;\; (j=1,2;\ l=1,\dots,m_0) \tag{25}$$

hold and the system of impulsive inequalities

$$\left|\frac{dx}{dt} - P(t)x\right| \le Q(t)\,x \ \text{ a.e. on } [0,\omega] \setminus T_{m_0}, \tag{26}$$

$$\big|x(\tau_l+) - x(\tau_l-) - J_l x(\tau_l)\big| \le H_l|x(\tau_l)| \ \ (l = 1,\dots,m_0) \tag{27}$$

have only the trivial solution satisfying condition (7). *Then problem* (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 1. *Let the conditions*

$$|f(t,x) - P(t)x| \le q(t,\|x\|) \ \text{ a.e. on } [0,\omega] \setminus T_{m_0}, \ \ x \in \mathbb{R}^n, \tag{28}$$

$$|I(\tau_l,x) - J_l x| \le h(\tau_l,\|x\|) \ \text{ for } x \in \mathbb{R}^n \ \ (l = 1,\dots,m_0), \tag{29}$$

$$|x(0) + x(\omega) - \ell(x)| \le \ell_1(\|x\|_s) \ \text{ for } x \in \mathrm{BV}([0,\omega],\mathbb{R}^n) \tag{30}$$

hold, where $P \in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ are the constant matrices satisfying condition (24), *$\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}^n$ is the linear continuous operator; $q \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ and $h \in C(T_{m_0} \times \mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ are the vector-functions, nondecreasing in the second variable, and $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that condition* (23) *holds. Let, moreover, the problem*

$$\frac{dx}{dt} = P(t)\,x \ \text{ a.e. on } [0,\omega] \setminus T_{m_0}, \tag{31}$$

$$x(\tau_l+) - x(\tau_l-) = J_l\,x(\tau_l) \ \ (l = 1,\dots,m_0); \tag{32}$$

$$\ell(x) = 0. \tag{33}$$

have only the trivial solution. Then problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Remark 5. Let $Y = (y_1,\dots,y_n)$ be a fundamental matrix, with columns $y_1,\dots,y_n$, of system (31), (32). Then the homogeneous boundary value problem (31), (32); (33) has only the trivial solution if and only if

$$\det(\ell(Y)) \ne 0, \tag{34}$$

where $\ell(Y) = (\ell(y_1),\dots,\ell(y_n))$.

If the pair $\{P;\{J_l\}_{l=1}^{m_0}\}$ satisfies the Lappo–Danilevskiĭ condition, then the fundamental matrix Y $(Y(0) = I_{n\times n})$ of the homogeneous system (31), (32) has the form

$$Y(t) \equiv \exp\bigg(\int\limits_0^t P(\tau)\,d\tau\bigg) \cdot \prod_{0\le\tau_l<t} (I_{n\times n} + J_l).$$

Theorem 5. *Let the conditions*

$$\big|f(t,x) - f(t,y) - P(t)(x-y)\big| \le Q(t)|x-y| \ \text{ a.e. on } [0,\omega] \setminus T_{m_0}, \ \ x,y \in \mathbb{R}^n, \tag{35}$$

$$\big|I(\tau_l,x) - I(\tau_l,y) - J_l\cdot(x-y)\big| \le H_l|x-y| \ \text{ for } x,y \in \mathbb{R}^n \ \ (k = l,\dots,m_0), \tag{36}$$

$$\big|x(0) - y(0) + x(\omega) - y(\omega) - \ell(x-y)\big| \le \ell_0(x-y) \ \text{ for } x,y \in \mathrm{BV}([0,\omega],\mathbb{R}^n)$$

hold, where $P \in L([0,\omega],\mathbb{R}^{n\times n})$, $Q \in L([0,\omega],\mathbb{R}^{n\times n}_+)$, $J_l \in \mathbb{R}^{n\times n}$ and $H_l \in \mathbb{R}^{n\times n}_+$ $(l = 1,\dots,m_0)$ are the constant matrices satisfying conditions (24) *and* (25), *$\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}^n$ and $\ell_0 : C_s([0,\omega],\ \mathbb{R}^n;T_{m_0}) \to \mathbb{R}^n_+$ are, respectively, linear continuous and positive homogeneous continuous operators. Let, moreover, problem* (26), (27); (7) *have only the trivial solution. Then problem* (1), (2); (3) *is correct.*

Corollary 2. *Let there exist a solution x^0 of problem* (1), (2); (3) *and a positive number $r > 0$ such that the conditions*

$$\big|f(t,x) - f(t,x^0(t)) - P(t)(x - x^0(t))\big| \le Q(t)|x - x^0(t)| \ \text{ a.a. } [0,\omega] \setminus T_{m_0}, \ \ \|x - x^0(t)\| < r,$$

$$\big|I(\tau_l,x) - I(\tau_l,x^0(\tau_l)) - J_l\cdot(x - x^0(\tau_l))\big| \le H_l|x - x^0(\tau_l)| \ \text{ for } \|x - x^0(\tau_l)\| < r \ \ (l = l,\dots,m_0),$$

$$\big|x(0) - x^0(0) + x(\omega) - x^0(\omega) - \ell(x - x^0)\big| \le \ell^*(|x - x^0|) \ \text{ for } x \in U(x^0,r)$$

hold, where $P \in L([0,\omega],\mathbb{R}^{n\times n})$, $Q \in L([0,\omega],\mathbb{R}_+^{n\times n})$, J_l *and* $H_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are the constant matrices satisfying conditions* (24) *and* (25), $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}^n$ *and* $\ell^* : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}_+^n$ *are, respectively, linear continuous and positive homogeneous continuous operators. Let, moreover, the system of impulsive inequalities*

$$\Big|\frac{dx}{dt} - P(t)x\Big| \le Q(t)x \ \text{ a.e. on } [0,\omega]\setminus T_{m_0},$$
$$\big|x(\tau_l+) - x(\tau_l-) - J_l\cdot x(\tau_l)\big| \le H_l\cdot x(\tau_l) \ \ (l = 1,\dots,m_0)$$

have only the trivial solution under the condition $|\ell(x)| \le \ell^*(|x|)$. *Then problem* (1), (2); (3) *is* $(x^0;r)$*-correct.*

Corollary 3. *Let the components of the vector-functions* f *and* I_l $(l = 1,\dots,n)$ *have partial derivatives by the last* n *variables belonging to the Carathéodory class* $Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n)$. *Let, moreover,* x^0 *be a solution of problem* (1), (2); (3) *such that the condition*

$$\det\big(I_{n\times n} + G_l(x^0(\tau_l))\big) \ne 0 \ \ (l = 1,\dots,m_0)$$

hold and the system

$$\frac{dx}{dt} = F(t,x^0(t))x \ \text{ almost everywhere on } [0,\omega]\setminus T_{m_0},$$
$$x(\tau_l+) - x(\tau_l-) = G_l(x^0(\tau_l))\cdot x(\tau_l) \ \ (l = 1,\dots,m_0);$$
$$\ell(x) = 0,$$

where $F(t,x) \equiv \frac{\partial f(t,x)}{\partial x}$ *and* $G_l(x) \equiv \frac{\partial I_l(x)}{\partial x}$, *have only the trivial solution under condition* (3). *Then problem* (1), (2); (3) *is* $(x^0;r)$*-correct for any sufficiently small* r.

In general, it is rather difficult to verify condition (34) directly even in the case if one is able to write out the fundamental matrix of system (31), (32); (33). Therefore, it is important to seek for effective conditions which would guarantee the absence of nontrivial ω-antiperiodic solutions of the homogeneous system (31), (32); (33). Below, we will give the results concerning the question. Analogous results have been obtained in [2] for the general linear boundary value problems for impulsive systems, and in [12] by T. Kiguradze for the case of ordinary differential equations.

In this connection, we introduce the operators. For every matrix-function $X \in L([0,\omega],\mathbb{R}^{n\times n})$ and a sequence of constant matrices $Y_k \in \mathbb{R}^{n\times n}$ $(k = 1,\dots,m_0)$ we put

$$\big[(X,Y_1,\dots,Y_{m_0})(t)\big]_0 = I_n \ \text{ for } 0 \le t \le \omega,$$
$$\big[(X,Y_1,\dots,Y_{m_0})(0)\big]_i = O_{n\times n} \ \ (i = 1,2,\dots),$$
$$\big[(X,Y_1,\dots,Y_{m_0})(t)\big]_{i+1} = \int\limits_0^t X(\tau)\cdot\big[(X,Y_1,\dots,Y_{m_0})(\tau)\big]_i\,d\tau$$
$$+ \sum_{0\le\tau_l<t} Y_l\cdot\big[(X,Y_1,\dots,Y_{m_0})(\tau_l)\big]_i \ \text{ for } 0 < t \le \omega \ \ (i = 1,2,\dots). \quad (37)$$

Corollary 4. *Let conditions* (28)–(30) *hold, where*

$$\ell(x) \equiv \int\limits_0^\omega d\mathcal{L}(t)\cdot x(t),$$

$P \in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are the constant matrices satisfying condition* (24), $\mathcal{L} \in L([0,\omega],\mathbb{R}^{n\times n})$; $q \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n)$ *and* $h \in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}_+^{n\times n})$ *are the vector-functions nondecreasing in the second variable, and* $\ell_1 \in C(\mathbb{R},\mathbb{R}_+^n)$ *is a vector-function such that condition* (23) *holds. Let, moreover, there exist natural numbers* k *and* m *such that the matrix*

$$M_k = -\sum_{i=0}^{k-1}\int\limits_0^\omega d\mathcal{L}(t)\cdot\big[(P,J_l,\dots,J_{m_0})(t)\big]_i$$

is nonsingular and

$$r(M_{k,m}) < 1, \tag{38}$$

where the operators $[(P, J_1, \dots, J_{m_0})(t)]_i$ $(i = 0, 1, \dots)$ *are defined by* (37)*, and*

$$M_{k,m} = \big[\big(|P|, |J_1|, \dots, |J_{m_0}|\big)(\omega)\big]_m + \sum_{i=0}^{m-1} \big[\big(|P|, |J_1|, \dots, |J_{m_0}|\big)(\omega)\big]_i \int_0^\omega dV(M_k^{-1}\mathcal{L})(t) \cdot \big[\big(|P|, |J_1|, \dots, |J_{m_0}|\big)(t)\big]_k.$$

Then problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 5. *Let conditions* (28)–(30) *hold, where*

$$\ell(x) \equiv \sum_{j=1}^{n_0} \mathcal{L}_j x(t_j), \tag{39}$$

$P \in L([0,\omega], \mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1, \dots, m_0)$ *are the constant matrices satisfying condition* (24)*,* $t_j \in [0,\omega]$ *and* $\mathcal{L}_j \in \mathbb{R}^{n\times n}$ $(j = 1, \dots, n_0)$*,* $\mathcal{L} \in L([0,\omega], \mathbb{R}^{n\times n})$*,* $\ell : C_s([0,\omega], \mathbb{R}^n; T_{m_0}) \to \mathbb{R}^n$ *is the linear continuous operator;* $q \in Car([0,\omega] \times \mathbb{R}_+, \mathbb{R}^n_+)$ *and* $h \in C(T_{m_0} \times \mathbb{R}_+; \mathbb{R}^{n\times n}_+)$ *are the vector-functions nondecreasing in the second variable, and* $\ell_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ *is a vector-function such that condition* (23) *holds. Let, moreover, there exist natural numbers* k *and* m *such that the matrix*

$$M_k = \sum_{j=1}^{n_0} \sum_{i=0}^{k-1} \mathcal{L}_j \big[(P, J_l, \dots, J_{m_0})(t_j)\big]_i$$

is nonsingular and inequality (38) *holds, where*

$$M_{k,m} = \big[\big(|P|, |J_l|, \dots, |J_{m_0}|\big)(\omega)\big]_m + \Big(\sum_{i=0}^{m-1} \big[\big(|P|, |J_l|, \dots, |J_{m_0}|\big)(\omega)\big]_i\Big) \sum_{j=1}^{n_0} |M_k^{-1}\mathcal{L}_j| \cdot \big[\big(|P|, |J_l|, \dots, |J_{m_0}|\big)(t_j)\big]_k.$$

Then problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 5 for $k = 1$ and $m = 1$ has the following form.

Corollary 6. *Let conditions* (28)–(30) *hold, where the operator* ℓ *is defined by* (39)*,* $P \in L([0,\omega], \mathbb{R}^{n\times n})$*,* $J_l \in \mathbb{R}^{n\times n}$ $(l = 1, \dots, m_0)$ *are the constant matrices satisfying condition* (24)*,* $t_j \in [0,\omega]$ *and* $\mathcal{L}_j \in \mathbb{R}^{n\times n}$ $(j = 1, \dots, n_0)$*;* $q \in Car([0,\omega] \times \mathbb{R}_+, \mathbb{R}^n_+)$ *and* $h \in C(T_{m_0} \times \mathbb{R}_+; \mathbb{R}^{n\times n}_+)$ *are the vector-functions, nondecreasing in the second variable, and* $\ell_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ *is the vector-function such that condition* (23) *holds. Let, moreover,*

$$\det\Big(\sum_{j=1}^{n_0} \mathcal{L}_j\Big) \neq 0 \ \text{ and } \ r(\mathcal{L}_0\, A_0) < 1,$$

where

$$\mathcal{L}_0 = I_{n\times n} + \Big|\Big(\sum_{j=1}^{n_0} \mathcal{L}_j\Big)^{-1}\Big| \cdot \sum_{j=1}^{n_0} |\mathcal{L}_j| \ \text{ and } \ A_0 = \int_0^\omega |P(t)|\, dt + \sum_{l=1}^{m_0} |J_l|.$$

Then problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Remark 6. If the pair $\{P;\{J_l\}_{l=1}^{m_0}\}$ satisfies the Lappo–Danilevskiĭ condition, then condition (34) has the forms

$$\det\bigg(\int\limits_0^{\omega} d\mathcal{L}(t)\cdot\exp\bigg(\int\limits_0^t P(\tau)\,d\tau\bigg)\cdot\prod_{0\le\tau_l<t}(I_{n\times n}+J_l)\bigg)\neq 0,$$

$$\det\bigg(\sum_{j=1}^{n_0} L_j\exp\bigg(\int\limits_0^{t_j} P(\tau)\,d\tau\bigg)\cdot\prod_{0\le\tau_l<t_j}(I_{n\times n}+J_l)\bigg)\neq 0$$

for the operators ℓ defined, respectively, in Corollary 4 and Corollary 5.

By Remark 2, in the case if $\ell(x)\equiv x(0)+x(\omega)$ and $\ell_0(x)\equiv 0$, the results given above have, respectively, the following forms.

Theorem 2′. *Let conditions* (13) *and* (14) *hold, where the pair* (P,J) *satisfies the Opial* ω*-antiperiodic condition;* $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ *is a function, nondecreasing in the second variable, and* $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ *is nondecreasing in the second variable function such that*

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\int\limits_0^{\omega}\alpha(t,\rho)\,dt+\sum_{l=1}^{m_0}\beta(\tau_l,\rho)\bigg)=0. \tag{40}$$

Then problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Theorem 3′. *Let conditions* (13), (14), (17), (18) *and* (40) *hold, where* $P\in Car^0([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, $P_i\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_{il}\in\mathbb{R}^{n\times n}$ $(i=1,2;\ l=1,\dots,m_0)$*;* $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ *is a function, nondecreasing in the second variable, and* $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ *is nondecreasing in the second variable function. Let, moreover, condition* (4) *hold and problem* (5), (6); (3) *have only the trivial solution for every matrix-function* $A\in L([0,\omega],\mathbb{R}^{n\times n})$ *and constant matrices* $G_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ *satisfying conditions* (19) *and* (20)*. Then problem* (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Theorem 4′. *Let conditions* (21) *and* (22) *hold, where* $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $Q\in L([0,\omega],\mathbb{R}_+^{n\times n})$, $J_l\in\mathbb{R}^{n\times}$ *and* $H_l\in\mathbb{R}_+^{n\times n}$ $(l=1,\dots,m_0)$ *are the constant matrices satisfying conditions* (24) *and* (25), $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n)$*, and* $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}_+^{n\times n})$ *are the vector-functions, nondecreasing in the second variable, such that*

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\int\limits_0^{\omega}\|q(t,\rho)\|\,dt+\sum_{l=1}^{m_0}\|h(\tau_l,\rho)\|\bigg)=0. \tag{41}$$

Let, moreover, the system of impulsive inequalities (26), (27) *have only the trivial solution satisfying condition* (3)*. Then problem* (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 1′. *Let conditions* (28), (29) *and* (40) *hold, where* $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ *are the constant matrices satisfying condition* (24), $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n)$ *and* $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}_+^{n\times n})$ *are the vector-functions, nondecreasing in the second variable. Let, moreover, problem* (31), (32), (3) *have only the trivial solution. Then problem* (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Theorem 5′. *Let conditions* (35) *and* (36) *hold, where* $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $Q\in L([0,\omega],\mathbb{R}_+^{n\times n})$, $J_l\in\mathbb{R}^{n\times n}$ *and* $H_l\in\mathbb{R}_+^{n\times n}$ $(l=1,\dots,m_0)$ *are the constant matrices satisfying conditions* (24) *and* (25)*. Let, moreover, problem* (26), (27); (7) *have only the trivial solution. Then problem* (1), (2); (3) *is correct.*

Corollary 5′. *Let conditions* (28), (29) *and* (41) *hold, where* $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ *are the constant matrices satisfying condition* (24)*;* $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n)$ *and* $h\in$

$C(T_{m_0} \times \mathbb{R}_+; \mathbb{R}_+^{n\times n})$ are the vector-functions, nondecreasing in the second variable. Let, moreover, there exist natural numbers k and m such that the matrix

$$M_k = \sum_{i=0}^{k-1} \big[(P, J_l, \dots, J_{m_0})(\omega)\big]_i$$

is nonsingular and inequality (38) holds, where

$$M_{k,m} = \big[(|P|, |J_l|, \dots, |J_{m_0}|)(\omega)\big]_m + \Big(\sum_{i=0}^{m-1} \big[(|P|, |J_l|, \dots, |J_{m_0}|)(\omega)\big]_i \Big) |M_k^{-1}| \cdot \big[(|P|, |J_l|, \dots, |J_{m_0}|)(\omega)\big]_k .$$

Then problem (1), (2); (3) is solvable. If, moreover, the problem has a unique solution, then it is correct.

Corollary 5′ for $k = 1$ and $m = 1$ has the following form.

Corollary 6′. *Let conditions (28), (29) and (41) hold, where $P \in L([0,\omega], \mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ ($l = 1, \dots, m_0$) are the constant matrices satisfying condition (24); $q \in Car([0,\omega] \times \mathbb{R}_+, \mathbb{R}_+^n)$ and $h \in C(T_{m_0} \times \mathbb{R}_+; \mathbb{R}_+^{n\times n})$ are the vector-functions, nondecreasing in the second variable. Let, moreover,*

$$r(A_0) < \frac{1}{2}\,,$$

where

$$A_0 = \int_0^\omega |P(t)|\,dt + \sum_{l=1}^{m_0} |J_l|.$$

Then problem (1), (2); (3) is solvable. If, moreover, the problem has a unique solution, then it is correct.

Remark 7. In the conditions of Corollary 6′, if the pair $\{P; \{J_l\}_{l=1}^{m_0}\}$ satisfies the Lappo–Danilevskiĭ condition, then condition (34) has the form

$$\det \bigg(I_{n\times n} + \exp\bigg(\int_0^\omega P(\tau)\,d\tau \bigg) \cdot \prod_{l=1}^{m_0} (I_{n\times n} + J_l) \bigg) \neq 0.$$

The analogous questions are investigated in [7] for system (1), (2) under the general nonlinear boundary condition $h(x) = 0$, where $h : C([0,\omega], \mathbb{R}^n; T_{m_0}) \to \mathbb{R}^n$ is a continuous vector-functional, nonlinear, in general. The results given in the paper are the particular cases of the results obtained in [7] for $h(x) \equiv x(0) + x(\omega)$.

Acknowledgement

The present work was supported by the Shota Rustaveli National Science Foundation (Grant # FR/182/5-101/11).

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(Received 24.04.2016)

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Memoirs on Differential Equations and Mathematical Physics

Volume 74, 2018

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