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CONTROLLABILITY FOR IMPULSIVE FRACTIONAL EVOLUTION EQUATIONS WITH STATE-DEPENDENT DELAY

Abstract. In this paper, we prove the controllability for a class of impulsive fractional evolution equations with state-dependent delay in a Banach space. Our study is based on the Sadovskii's fixed point theorem. For the illustration of the main result, an example is given.

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1 Introduction

Fractional order differential equations are generalizations of classical integer order differential equations. These are increasingly used to model problems in fluid flow, mechanics, viscoelasticity, biology, physics, engineering and other applications. In recent years, there has been a significant development in ordinary and partial fractional differential equations; see the monographs by Abbas *et al.* [1, 2], Baleanu *et al.* [9], Diethelm [14], Hilfer [22], Kilbas *et al.* [25], Miller and Ross [28], Podlubny [30], Samko *et al.* [33], Tarasov [38], and Zhou [41, 42] and the references therein.

Functional differential equations with state-dependent delay appear frequently in applications as a model of equations and for this reason the study of this type of equations has received great attention in the last years (see [3, 4, 6, 11, 17–21, 24, 27, 35, 39, 40]).

The problem of controllability of linear and nonlinear systems represented by ordinary differential equations in finite dimensional space has been extensively studied. Several authors have extended the controllability concept to infinite dimensional systems in Banach space. Mophou *et al.* [29] studied the controllability of semilinear neutral fractional functional evolution equations with infinite delay, whereas Tai and Wang [37] discussed the controllability of fractional-order impulsive neutral functional infinite delay integrodifferential systems. Controllability of impulsive fractional differential equations with infinite delay is studied by Aissani and Benchohra [5].

Motivated by the previous literature, the purpose of this work is to establish the controllability for a class of impulsive fractional equations with state-dependent delay described by

$$\begin{gathered} D_t^\alpha x(t) = Ax(t) + Bu(t) + f(t, x_{\rho(t,x_t)}, x(t)), \ \ t \in J_k = (t_k, t_{k+1}], \ \ k = 0, 1, \dots, m, \\ \Delta x(t_k) = I_k(x(t_k^-)), \ \ k = 1, 2, \dots, m, \\ x(t) = \phi(t), \ \ t \in (-\infty, 0], \end{gathered} \tag{1.1}$$

where D_t^α is the Caputo fractional derivative of order α, $0 < \alpha < 1$, $A : D(A) \subset E \to E$ is the infinitesimal generator of an α-resolvent family $(S_\alpha(t))_{t\geq 0}$, $f : J \times \mathcal{B} \times E \to E$ is a given function, $J = [0, T]$, $T > 0$, and $\rho : J \times \mathcal{B} \to (-\infty, T]$ is an appropriate function, B is a bounded linear operator from E into E, the control $u \in L^2(J; E)$, the Banach space of admissible controls. Here, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k : E \to E$, $k = 1, 2, \dots, m$, are the given functions, $(E, \|\cdot\|)$ is a complex Banach space, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim\limits_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim\limits_{h\to 0} x(t_k - h)$ denotes the right and the left limit of $x(t)$ at $t = t_k$, respectively. We denote by x_t the element of $\mathcal{B}$ defined by $x_t(\theta) = x(t + \theta)$, $\theta \in (-\infty, 0]$. Here x_t represents the history of the state up to the present time t. We assume that the histories x_t belong to some abstract phase space $\mathcal{B}$, to be specified later, and $\phi \in \mathcal{B}$.

2 Preliminaries

In what follows, we recall some notations, definitions, and results that we will need in the sequel.

Let $C = C(J, E)$ be the Banach space of continuous functions from J into E with the norm

$$\|y\|_C = \sup\big\{\|y(t)\| : \ t \in J\big\}.$$

$L(E)$ is the Banach space of all linear and bounded operators on E.

A measurable function $y : J \to E$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable.

$L^1(J, E)$ is the Banach space of measurable functions $y : J \to E$ that are Bochner integrable, with the norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\|\, dt \ \text{ for all } \ y \in L^1(J, E).$$

$B_r(x, E)$ represents the closed ball in E with the center at x and of radius r.

We need some basic definitions and properties of the fractional calculus theory which will be used further in this paper.

Definition 2.1. Let $\alpha > 0$ and $f : \mathbb{R}_+ \to E$ be in $L^1(\mathbb{R}_+, E)$. Then the Riemann–Liouville integral is given by

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds.$$

For more details on the Riemann–Liouville fractional derivative, we refer the reader to [13].

Definition 2.2 ([30])**.** The Caputo derivative of order α for a function $f : [0, +\infty) \to E$ can be written as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\, ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, \quad n-1 \le \alpha < n.$$

If $0 \le \alpha < 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}}\, ds.$$

Obviously, the Caputo derivative of a constant is equal to zero.

In order to define a mild solution of problem (1.1), we recall the following

Definition 2.3. A closed linear operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in [\frac{\pi}{2}, \pi]$, $M > 0$, such that the following two conditions are satisfied:

1. $\sum\limits_{(\theta,\omega)} := \{\lambda \in C : \ \lambda \neq \omega, \ |arg(\lambda - \omega)| < \theta\} \subset \rho(A)$ ($\rho(A)$ being the resolvent set of A).
2. $\|R(\lambda, A)\|_{L(E)} \le \frac{M}{|\lambda-\omega|}$, $\lambda \in \sum\limits_{(\theta,\omega)}$.

Sectorial operators are well studied in the literature. For details see [15].

Definition 2.4 ([8])**.** Let A be a closed linear operator with domain $D(A)$ defined on a Banach space E and $\alpha > 0$. We say that A is the generator of an α-resolvent family if there exist $\omega \ge 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(E)$ such that $\{\lambda^\alpha : \ \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$ and

$$(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x\, dt, \quad \mathrm{Re}\,\lambda > \omega, \quad x \in E.$$

In this case, $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 2.5 (see Definition 2.1 in [7])**.** Let A be a closed linear operator with domain $D(A)$ defined on a Banach space E and $\alpha > 0$. We say that A is the generator of a solution operator if there exist $\omega \ge 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(E)$ such that $\{\lambda^\alpha : \ \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x\, dt, \quad \mathrm{Re}\,\lambda > \omega, \quad x \in E.$$

In this case, $S_\alpha(t)$ is called the solution operator generated by A. For more details see [26, 31].

In this paper, we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced by Hale and Kato [16]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into E endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, and satisfying the following axioms:

(A1) If $x : (-\infty, T] \to E$ is such that $x_0 \in \mathcal{B}$, then for every $t \in J$, $x_t \in \mathcal{B}$ and

$$\|x(t)\| \le C \|x_t\|_{\mathcal{B}},$$

where $C > 0$ is a constant.

(A2) There exist a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ in $t \geq 0$ such that

$$\|x_t\|_{\mathcal{B}} \leq C_1(t) \sup_{s\in[0,t]} \|x(s)\| + C_2(t)\|x_0\|_{\mathcal{B}},$$

for $t \in J$ and x as in $(A1)$.

(A3) The space $\mathcal{B}$ is complete.

Example 2.6. The phase space $C_r \times L^p(g, X)$.

Let $r \geq 0$, $1 \leq p < \infty$, and let $g : (-\infty, -r) \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions $(g-5), (g-6)$ in the terminology of [23]. Briefly, this means that g is locally integrable and there exists a nonnegative locally bounded function Λ on $(-\infty, 0]$ such that $g(\xi+\theta) \leq \Lambda(\xi)g(\theta)$ for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subset (-\infty, -r)$ is a set with Lebesgue measure zero.

The space $C_r \times L^p(g, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $g\|\varphi\|^p$ on $(-\infty, -r)$. The seminorm in $\|\cdot\|_{\mathcal{B}}$ is defined by

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta\in[-r,0]} \|\varphi(\theta)\| + \left(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p \, d\theta \right)^{\frac{1}{p}}.$$

The space $\mathcal{B} = C_r \times L^p(g, X)$ satisfies axioms (A1), (A2), (A3). Moreover, for $r = 0$ and $p = 2$, this space coincides with

$$C_0 \times L^2(g, X), \quad H = 1, \quad M(t) = \Lambda(-t)^{\frac{1}{2}}, \quad K(t) = 1 + \left(\int_{-r}^{0} g(\tau) \, d\tau \right)^{\frac{1}{2}}.$$

For more details see [23, Theorem 1.3.8].

Definition 2.7. A function $f : J \times \mathcal{B} \times E \to E$ is said to be a Carathéodory function if it satisfies:

(i) for each $t \in J$, the function $f(t, \cdot, \cdot) : \mathcal{B} \times E \to E$ is continuous;

(ii) for each $(v, w) \in \mathcal{B} \times E$, the function $f(\cdot, v, w) : J \to E$ is measurable.

Definition 2.8. Problem (1.1) is said to be controllable on the interval J if for every initial function $\phi \in \mathcal{B}$ and $x_1 \in E$ there exists a control $u \in L^2(J, E)$ such that the mild solution $x(\cdot)$ of (1.1) satisfies $x(T) = x_1$.

Next, we give the concept of a measure of noncompactness [10].

Definition 2.9. Let B be a bounded subset of a seminormed linear space Y. The Kuratowski's measure of noncompactness of B is defined as

$$\alpha(B) = \inf\{d > 0 : \ B \text{ has a finite cover by sets of diameter } \leq d\}.$$

We need to use the following basic properties of α measure and Sadovskii's fixed point theorem (see [34]).

Lemma 2.10. *Let A and B be two bounded sets of the Banach space E. Then:*

1. *If $A \subseteq B$, then $\alpha(A) \leq \alpha(B)$;*
2. *$\alpha(A) = 0 \Longleftrightarrow \overline{A}$ is compact (A is relatively compact);*
3. *$\alpha(A + B) \leq \alpha(A) + \alpha(B)$.*

Theorem 2.11 (Sadovskii's fixed point Theorem)**.** *Let $\mathcal{N}$ be a condensing operator on the Banach space X, i.e., $\mathcal{N}$ is continuous and takes bounded sets into bounded sets, and $\alpha(\mathcal{N}(D)) < \alpha(D)$ for every bounded set D of E with $\alpha(D) > 0$. If $\mathcal{N}(S) \subset S$ for a convex, closed and bounded set S of X, then $\mathcal{N}$ has a fixed point in S.*

3 Controllability results

Before going further, we need the following lemma [36].

Lemma 3.1. *Consider the Cauchy problem*

$$\begin{gathered} D_t^\alpha x(t) = Ax(t) + Bu(t) + f(t), \quad 0 < \alpha < 1, \\ x(0) = x_0, \end{gathered} \tag{3.1}$$

if f satisfies the uniform Hölder condition with exponent $\beta \in (0,1]$ and A is a sectorial operator, then the unique solution of the Cauchy problem (3.1) *is given by*

$$x(t) = T_\alpha(t)x_0 + \int_0^t S_\alpha(t-s)Bu(s)\,ds + \int_0^t S_\alpha(t-s)f(s)\,ds,$$

where

$$T_\alpha(t) = \frac{1}{2\pi i}\int_{\widehat{B_r}} e^{\lambda t}\,\frac{\lambda^{\alpha-1}}{\lambda^\alpha - A}\,d\lambda, \quad S_\alpha(t) = \frac{1}{2\pi i}\int_{\widehat{B_r}} e^{\lambda t}\,\frac{1}{\lambda^\alpha - A}\,d\lambda,$$

$\widehat{B_r}$ denotes the Bromwich path, $S_\alpha(t)$ is called the α-resolvent family and $T_\alpha(t)$ is the solution operator generated by A.

Theorem 3.2 ([12,36])**.** *If $\alpha \in (0,1)$ and $A \in \mathbb{A}^\alpha(\theta_0, \omega_0)$, then for any $x \in E$ and $t > 0$, we have*

$$\|T_\alpha(t)\|_{L(E)} \le Me^{\omega t} \ \text{ and } \ \|S_\alpha(t)\|_{L(E)} \le Ce^{\omega t}(1 + t^{\alpha-1}), \ \ t > 0, \ \ \omega > \omega_0.$$

Let

$$\widetilde{M}_T = \sup_{0\le t\le T}\|T_\alpha(t)\|_{L(E)}, \quad \widetilde{M}_s = \sup_{0\le t\le T} Ce^{\omega t}(1 + t^{\alpha-1}),$$

hence we have

$$\|T_\alpha(t)\|_{L(E)} \le \widetilde{M}_T, \quad \|S_\alpha(t)\|_{L(E)} \le t^{\alpha-1}\widetilde{M}_s.$$

Let us consider the set of functions

$$\mathcal{B}_1 = \Big\{x :\ (-\infty, T] \to E \text{ such that } x\big|_{J_k} \in C(J_k, E) \text{ and there exist} \\ x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k) = x(t_k^-),\ x_0 = \phi,\ k = 1, 2, \ldots, m\Big\}.$$

Endowed with the seminorm,

$$\|x\|_{\mathcal{B}_1} = \sup\big\{\|x(s)\| :\ s \in [0, T]\big\} + \|\phi\|_{\mathcal{B}}, \ \ x \in \mathcal{B}_1,$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$.

From Lemma 3.1 we can define a mild solution of system (1.1) as follows.

Definition 3.3. A function $x \in \mathcal{B}_1$ is called a mild solution of (1.1) if it satisfies the following integral equation:

$$x(t)=\begin{cases}\phi(t), & t\in(-\infty,0],\\ \displaystyle\int_0^t S_\alpha(t-s)Bu(s)\,ds+\int_0^t S_\alpha(t-s)f\big(s,x_{\rho(s,x_s)},x(s)\big)\,ds, & t\in[0,t_1],\\ \displaystyle T_\alpha(t-t_1)\big(x(t_1^-)+I_1(x(t_1^-))\big)+\int_{t_1}^t S_\alpha(t-s)Bu(s)\,ds & \\ \displaystyle\qquad+\int_{t_1}^t S_\alpha(t-s)f\big(s,x_{\rho(s,x_s)},x(s)\big)\,ds, & t\in(t_1,t_2],\\ \qquad\vdots & \\ \displaystyle T_\alpha(t-t_m)\big(x(t_m^-)+I_m(x(t_m^-))\big)+\int_{t_m}^t S_\alpha(t-s)Bu(s)\,ds & \\ \displaystyle\qquad+\int_{t_m}^t S_\alpha(t-s)f\big(s,x_{\rho(s,x_s)},x(s)\big)\,ds, & t\in(t_m,T].\end{cases}\tag{3.2}$$

Set

$$\mathcal{R}(\rho^-)=\big\{\rho(s,\varphi):\ (s,\varphi)\in J\times\mathcal{B},\ \rho(s,\varphi)\le 0\big\}.$$

We always assume that $\rho: J\times\mathcal{B}\to(-\infty,T]$ is continuous. Additionally, we introduce the following hypothesis:

(H_φ) The function $t\to\varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi:\mathcal{R}(\rho^-)\to(0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}}\le L^\phi(t)\|\phi\|_{\mathcal{B}}\ \text{ for every }\ t\in\mathcal{R}(\rho^-).$$

Remark 3.4. Condition (H_φ) is frequently verified by the continuous and bounded functions. For more details see, e.g., [23].

Remark 3.5. In the rest of this section, C_1^* and C_2^* are the constants

$$C_1^*=\sup_{s\in J}C_1(s)\ \text{ and }\ C_2^*=\sup_{s\in J}C_2(s).$$

Lemma 3.6 ([21])**.** *If $x:(-\infty,T]\to X$ is a function such that $x_0=\phi$, then*

$$\|x_s\|_{\mathcal{B}}\le(C_2^*+L^\phi)\|\phi\|_{\mathcal{B}}+C_1^*\sup\big\{\|y(\theta)\|:\ \theta\in[0,max\{0,s\}]\big\},\quad s\in\mathcal{R}(\rho^-)\cup J,$$

where $L^\phi=\sup\limits_{t\in\mathcal{R}(\rho^-)}L^\phi(t)$.

Let us introduce the following hypotheses:

(H1) The semigroup $S(t)$ is compact for $t>0$.

(H2) $f: J\times\mathcal{B}\times E\to E$ satisfies the Carathéodory conditions.

(H3) There exist a continuous function $\mu\in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\psi:\mathbb{R}^+\to(0,+\infty)$ such that

$$\|f(t,x,y)\|\le\mu(t)\psi\big(\|x\|_{\mathcal{B}}+\|y\|\big),\ \ (t,x,y)\in J\times\mathcal{B}\times E.$$

(H4) The function $I_k : E \to E$ is continuous, and there exists $\Omega > 0$ such that

$$\Omega = \max_{1 \le k \le m} \left\{ \|I_k(x)\| : \ x \in B_r \right\}.$$

(H5) The linear operator $W : L^2(J, E) \to E$ defined by

$$Wu = \int_0^T S_\alpha(T-s) Bu(s)\, ds,$$

has an inverse operator $\widetilde{W}^{-1}$, which takes values in $L^2(J,E)/\ker W$ and there exist two positive constants M_1 and M_2 such that

$$\|B\|_{L(E)} \le M_1, \ \ \|\widetilde{W}^{-1}\|_{L(E)} \le M_2.$$

Remark 3.7. The construction of the operator $\widetilde{W}^{-1}$ and its properties are discussed in [32].

Theorem 3.8. *Assume that Hypotheses* (H_φ), $(H1)$–$(H5)$ *are satisfied with* $\widetilde{M_T} < 1$, *then the IVP* (1.1) *is controllable on* $(-\infty, T]$.

Proof. We transform problem (1.1) into a fixed-point problem. Consider the operator $N : \mathcal{B}_1 \to \mathcal{B}_1$ defined by:

$$Nx(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \displaystyle\int_0^t S_\alpha(t-s)Bu(s)\,ds + \int_0^t S_\alpha(t-s) f\big(s, x_{\rho(s,x_s)}, x(s)\big)\,ds, & t \in [0, t_1], \\ \displaystyle T_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) + \int_{t_1}^t S_\alpha(t-s)Bu(s) & \\ \displaystyle\qquad + \int_{t_1}^t S_\alpha(t-s) f\big(s, x_{\rho(s,x_s)}, x(s)\big)\,ds, & t \in (t_1, t_2], \\ \qquad \vdots & \\ \displaystyle T_\alpha(t-t_m)\big(x(t_m^-) + I_m(x(t_m^-))\big) + \int_{t_m}^t S_\alpha(t-s)Bu(s)\,ds & \\ \displaystyle\qquad + \int_{t_m}^t S_\alpha(t-s) f\big(s, x_{\rho(s,x_s)}, x(s)\big)\,ds, & t \in (t_m, T]. \end{cases}$$

Using hypothesis $(H5)$, for an arbitrary function $x(\cdot)$, we define the control

$$u(t) = \begin{cases} \widetilde{W}^{-1}\Big[x_1 - \int\limits_0^T S_\alpha(t-s) f\big(s, x_{\rho(s,x_s)}, x(s)\big)\, ds\Big](t), & t \in [0, t_1], \\ \widetilde{W}^{-1}\Big[x_1 - T_\alpha(T-t_1)\big(x(t_1^-) + I_1(x(t_1^-))\big) \\ \qquad - \int\limits_{t_1}^T S_\alpha(t-s) f\big(s, x_{\rho(s,x_s)}, x(s)\big)\, ds\Big](t), & t \in (t_1, t_2], \\ \quad \vdots \\ \widetilde{W}^{-1}\Big[x_1 - T_\alpha(T-t_m)\big(x(t_m^-) + I_m(x(t_m^-))\big) \\ \qquad - \int\limits_{t_m}^T S_\alpha(t-s) f\big(s, x_{\rho(s,x_s)}, x(s)\big)\, ds\Big](t), & t \in (t_m, T]. \end{cases} \tag{3.3}$$

Clearly, fixed points of the operator N are mild solutions of problem (1.1).

Let us define $y(\cdot) : (-\infty, T] \to E$ as

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ 0, & t \in J. \end{cases}$$

Then $y_0 = \phi$. For each $z \in C(J, E)$ with $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ z(t), & t \in J. \end{cases}$$

If $x(\cdot)$ satisfies (3.2), we can decompose it as $x(t) = y(t) + \overline{z}(t)$ for $t \in J$, which implies $x_t = y_t + \overline{z}_t$ for every $t \in J$, the expression of the control given by (3.3) becomes

$$u(t) = \begin{cases} \widetilde{W}^{-1}\Big[x_1 - \int\limits_0^T S_\alpha(t-s) f\big(s, y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}, y(s) + \overline{z}(s)\big)\, ds\Big](t), & t \in [0, t_1], \\ \widetilde{W}^{-1}\Big[x_1 - T_\alpha(T-t_1)\big[y(t_1^-) + \overline{z}(t_1^-) + I_1(y(t_1^-) + \overline{z}(t_1^-))\big] \\ \qquad - \int\limits_{t_1}^T S_\alpha(t-s) f\big(s, y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}, y(s) + \overline{z}(s)\big)\, ds\Big](t), & t \in (t_1, t_2], \\ \quad \vdots \\ \widetilde{W}^{-1}\Big[x_1 - T_\alpha(T-t_m)\big[y(t_m^-) + \overline{z}(t_m^-) + I_m(y(t_m^-) + \overline{z}(t_m^-))\big] \\ \qquad - \int\limits_{t_m}^T S_\alpha(t-s) f\big(s, y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}, y(s) + \overline{z}(s)\big)\, ds\Big](t), & t \in (t_m, T], \end{cases}$$

and

$$z(t)=\begin{cases}\displaystyle\int_0^t S_\alpha(t-s)Bu(s)\,ds\\ \qquad\displaystyle+\int_0^t S_\alpha(t-s)f\big(s,y_{\rho(s,y_s+\overline{z}(s))}+\overline{z}_{\rho(s,y_s+\overline{z}(s))},y(s)+\overline{z}(s)\big)\,ds, & t\in[0,t_1],\\ \displaystyle T_\alpha(t-t_1)\big[y(t_1^-)+\overline{z}(t_1^-)+I_1(y(t_1^-)+\overline{z}(t_1^-))\big]+\int_{t_1}^t S_\alpha(t-s)Bu(s)\,ds\\ \qquad\displaystyle+\int_{t_1}^t S_\alpha(t-s)f\big(s,y_{\rho(s,y_s+\overline{z}(s))}+\overline{z}_{\rho(s,y_s+\overline{z}(s))},y(s)+\overline{z}(s)\big)\,ds, & t\in(t_1,t_2],\\ \qquad\vdots\\ \displaystyle T_\alpha(t-t_m)\big[y(t_m^-)+\overline{z}(t_m^-)+I_m(y(t_m^-)+\overline{z}(t_m^-))\big]+\int_{t_m}^t S_\alpha(t-s)Bu(s)\,ds\\ \qquad\displaystyle+\int_{t_m}^t S_\alpha(t-s)f\big(s,y_{\rho(s,y_s+\overline{z}(s))}+\overline{z}_{\rho(s,y_s+\overline{z}(s))},y(s)+\overline{z}(s)\big)\,ds, & t\in(t_m,T].\end{cases}$$

Moreover, $z_0=0$.

Let

$$\mathcal{B}_2=\big\{z\in\mathcal{B}_1:\ z_0=0\big\}.$$

For any $z\in\mathcal{B}_2$, we have

$$\|z\|_{\mathcal{B}_2}=\sup_{t\in J}\|z(t)\|+\|z_0\|_{\mathcal{B}}=\sup_{t\in J}\|z(t)\|.$$

Thus $(\mathcal{B}_2,\|\cdot\|_{\mathcal{B}_2})$ is a Banach space. We define the operator $P:\mathcal{B}_2\to\mathcal{B}_2$ by

$$P(z)(t)=\begin{cases}\displaystyle\int_0^t S_\alpha(t-s)Bu(s)\,ds\\ \qquad\displaystyle+\int_0^t S_\alpha(t-s)f\big(s,y_{\rho(s,y_s+\overline{z}(s))}+\overline{z}_{\rho(s,y_s+\overline{z}(s))},y(s)+\overline{z}(s)\big)\,ds, & t\in[0,t_1],\\ \displaystyle T_\alpha(t-t_1)\big[y(t_1^-)+\overline{z}(t_1^-)+I_1(y(t_1^-)+\overline{z}(t_1^-))\big]+\int_{t_1}^t S_\alpha(t-s)Bu(s)\,ds\\ \qquad\displaystyle+\int_{t_1}^t S_\alpha(t-s)f\big(s,y_{\rho(s,y_s+\overline{z}(s))}+\overline{z}_{\rho(s,y_s+\overline{z}(s))},y(s)+\overline{z}(s)\big)\,ds, & t\in(t_1,t_2],\\ \qquad\vdots\\ \displaystyle T_\alpha(t-t_m)\big[y(t_m^-)+\overline{z}(t_m^-)+I_m(y(t_m^-)+\overline{z}(t_m^-))\big]+\int_{t_m}^t S_\alpha(t-s)Bu(s)\,ds\\ \qquad\displaystyle+\int_{t_m}^t S_\alpha(t-s)f\big(s,y_{\rho(s,y_s+\overline{z}(s))}+\overline{z}_{\rho(s,y_s+\overline{z}(s))},y(s)+\overline{z}(s)\big)\,ds, & t\in(t_m,T].\end{cases}$$

Obviously, the operator N has a fixed point is equivalent to P to have a fixed point, so it remains to prove that P has a fixed point. Let

$$B_r=\big\{z\in\mathcal{B}_2:\ \|z\|_{\mathcal{B}_2}\le r\big\},$$

where r is any fixed finite real number which satisfies the inequality

$$r \geq \frac{\widetilde{M}_T \Omega}{1-\widetilde{M}_T} + \frac{\widetilde{M}_S}{1-\widetilde{M}_T}\, \psi\big((C_2^* + L^{\phi})\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big)\|\mu\|_{L^1}.$$

Clearly, the subset B_r is a closed, bounded and convex set of $\mathcal{B}_2$. We need the following

Lemma 3.9. *If $x \in B_r$, then we have*

$$\|y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}\|_{\mathcal{B}} \leq (C_2^* + L^{\phi})\|\phi\|_{\mathcal{B}} + C_1^* r,$$

and

$$\|u(s)\| \leq \begin{cases} M_2\Big[\|x_1\| + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1}\mu(\tau) \\ \qquad \times\psi\Big(\big\|y_{\rho(\tau,y_\tau+\overline{z}(\tau))} + \overline{z}_{\rho(\tau,y_\tau+\overline{z}(\tau))}\big\|_{\mathcal{B}} + \|y(\tau)+\overline{z}(\tau)\|\Big)\, d\tau\Big], & t \in [0,t_1], \\ M_2\Big[\|x_1\| + \widetilde{M}_T(r+\Omega) + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1}\mu(\tau) \\ \qquad \times\psi\Big(\big\|y_{\rho(\tau,y_\tau+\overline{z}(\tau))} + \overline{z}_{\rho(\tau,y_\tau+\overline{z}(\tau))}\big\|_{\mathcal{B}} + \|y(\tau)+\overline{z}(\tau)\|\Big)\, d\tau\Big], & t \in (t_1,t_2], \\ \qquad \vdots \\ M_2\Big[\|x_1\| + \widetilde{M}_T(r+\Omega) + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1}\mu(\tau) \\ \qquad \times\psi\Big(\big\|y_{\rho(\tau,y_\tau+\overline{z}(\tau))} + \overline{z}_{\rho(\tau,y_\tau+\overline{z}(\tau))}\big\|_{\mathcal{B}} + \|y(\tau)+\overline{z}(\tau)\|_E\Big)\, d\tau\Big], & t \in (t_m,T]. \end{cases} \tag{3.4}$$

Proof. Using Lemma 3.6, (H3) and (H5), we obtain

$$\begin{aligned} &\big\|y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}\big\|_{\mathcal{B}} \\ &\qquad \leq (C_2^* + L^{\phi})\|\phi\|_{\mathcal{B}} + C_1^* \sup\big\{|y(\theta)| :\ \theta \in [0, max\{0,t\}]\big\} \leq (C_2^* + L^{\phi})\|\phi\|_{\mathcal{B}} + C_1^* r. \end{aligned}$$

Also, we get

$$\|u(s)\| \leq \begin{cases} \|\widetilde{W}^{-1}\| \Big[\|x_1\| + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1} \\ \qquad \times \Big\| f\big(\tau, y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau + \overline{z}(\tau))}, y(\tau) + \overline{z}(\tau)\big) \Big\| \, d\tau \Big], \quad t \in [0, t_1], \\ \|\widetilde{W}^{-1}\| \Big[\|x_1\| + \widetilde{M}_T(r + \Omega) + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1} \\ \qquad \times \Big\| f\Big(\tau, y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau + \overline{z}(\tau))}, y(\tau) + \overline{z}(\tau)\Big) \Big\| \, d\tau \Big], \quad t \in (t_1, t_2], \\ \qquad \vdots \\ \|\widetilde{W}^{-1}\| \Big[\|x_1\| + \widetilde{M}_T(r + \Omega) + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1} \\ \qquad \times \Big\| f\big(\tau, y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau + \overline{z}(\tau))}, y(\tau) + \overline{z}(\tau)\big) \Big\| \, d\tau \Big], \quad t \in (t_m, T] \end{cases}$$

$$\leq \begin{cases} M_2 \Big[\|x_1\| + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1} \mu(\tau) \\ \qquad \times \psi\Big(\big\| y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau + \overline{z}(\tau))} \big\|_{\mathcal{B}} + \big\| y(\tau) + \overline{z}(\tau) \big\|_E \Big) \, d\tau \Big], \quad t \in [0, t_1], \\ M_2 \Big[\|x_1\| + \widetilde{M}_T(r + \Omega) + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1} \mu(\tau) \\ \qquad \times \psi\Big(\big\| y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau + \overline{z}(\tau))} \|_{\mathcal{B}} + \| y(\tau) + \overline{z}(\tau) \big\| \Big) \, d\tau \Big], \quad t \in (t_1, t_2], \\ \qquad \vdots \\ M_2 \Big[\|x_1\| + \widetilde{M}_T(r + \Omega) + \widetilde{M}_S \int\limits_0^T (t-\tau)^{\alpha-1} \mu(\tau) \\ \qquad \times \psi\Big(\big\| y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau + \overline{z}(\tau))} \|_{\mathcal{B}} + \| y(\tau) + \overline{z}(\tau) \big\|_E \Big) \, d\tau \Big], \quad t \in (t_m, T]. \end{cases}$$

Thus the lemma is proved. $\square$

Now, we define two operators P_1 and P_2 on B_r as

$$P_1(z)(t) = \begin{cases} \int_0^t S_\alpha(t-s) f\big(s, y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}, y(s)+\overline{z}(s)\big)\, ds, & t \in [0,t_1], \\ T_\alpha(t-t_1)\Big[y(t_1^-) + \overline{z}(t_1^-) + I_1\big(y(t_1^-)+\overline{z}(t_1^-)\big)\Big] & \\ \qquad + \int_{t_1}^t S_\alpha(t-s) f\big(s, y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}, y(s)+\overline{z}(s)\big)\, ds, & t \in (t_1,t_2], \\ \qquad \vdots & \\ T_\alpha(t-t_m)\big[y(t_m^-) + \overline{z}(t_m^-) + I_m(y(t_m^-)+\overline{z}(t_m^-))\big] & \\ \qquad + \int_{t_m}^t S_\alpha(t-s) f\big(s, y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}, y(s)+\overline{z}(s)\big)\, ds, & t \in (t_m,T], \end{cases}$$

$$P_2(z)(t) = \begin{cases} \int_0^t S_\alpha(t-s) Bu(s)\, ds, \quad t \in [0,t_1], \\ \int_{t_1}^t S_\alpha(t-s) Bu(s)\, ds, \quad t \in (t_1,t_2], \\ \qquad \vdots \\ \int_{t_m}^t S_\alpha(t-s) Bu(s)\, ds, \quad t \in (t_m,T]. \end{cases}$$

Firstly, we show that the operator P_1 maps B_r into itself, next, we prove that P_2 is completely continuous.

Step 1: Let $z \in B_r$, then show that $P_1 z \in B_r$. For $t \in [0,t_1]$, we have

$$\begin{aligned} \|P_1(z)(t)\| &\le \int_0^t \|S_\alpha(t-s)\|_{L(E)} \Big\| f\big(s, y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}, y(s)+\overline{z}(s)\big)\Big\|\, ds \\ &\le \widetilde{M}_S \int_0^t (t-s)^{\alpha-1}\mu(s)\psi\Big(\big\|y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}\|_{\mathcal{B}} + \|y(s)+\overline{z}(s)\big\|\Big)\, ds \\ &\le \widetilde{M}_S \int_0^t (t-s)^{\alpha-1}\mu(s)\psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + C_1^* r + r\big)\, ds \\ &\le \widetilde{M}_S \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big) \int_0^t \mu(s)\, ds \\ &\le \widetilde{M}_S \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big)\|\mu\|_{L^1} \\ &\le r. \end{aligned}$$

Moreover, when $t \in (t_i, t_{i+1}]$, $i = 1, \ldots, m$, we have the estimate

$$\begin{aligned}\|P_1(z)(t)\| &\leq T_\alpha(t-t_i)\big[z(t_i^-) + I_i(z(t_i^-))\big] \\ &\quad + \int_0^t \|S_\alpha(t-s)\|_{L(E)} \Big\| f\big(s, y_{\rho(s,y_s+\overline{z}(s))} + \overline{z}_{\rho(s,y_s+\overline{z}(s))}, y(s) + \overline{z}(s)\big)\Big\|\, ds \\ &\leq \widetilde{M}_T(r+\Omega) + \widetilde{M}_S \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big)\|\mu\|_{L^1} \\ &\leq r.\end{aligned}$$

Step 2: P_2 is completely continuous. This will be given in several claims.

Claim 1: P_2 is continuous.

Let $\{z^n\}_{n\in\mathbb{N}}$ be a sequence such that $z^n \to z$ in $\mathcal{B}_2$ as $n \to \infty$. Since f satisfies $(H2)$, we get

$$f\big(\tau, y_\tau + \overline{z}_\tau^n, y(\tau) + \overline{z}^n(\tau)\big) \longrightarrow f\big(\tau, y_\tau + \overline{z}_\tau, y(\tau) + \overline{z}(\tau)\big) \text{ as } n \to \infty.$$

Now for all $t \in [0, t_1]$, we have

$$\begin{aligned}\big\|P_2(z^n)(t) - P_2(z)(t)\big\| &\leq \int_0^t \big\|S_\alpha(t-s)B(u_n(s) - u(s))\big\|_{L(E)}\, ds \\ &\leq \int_0^t \|S_\alpha(t-s)\|_{L(E)}\|B\|_{L(E)}\|u_n(s)-u(s)\|\, ds \leq M_1\widetilde{M}_S \int_0^t (t-s)^{\alpha-1}\|(u_n(s)-u(s))\|\, ds \\ &\leq M_1M_2\widetilde{M}_S^2 \int_0^t (t-s)^{\alpha-1}\int_0^T (T-\tau)^{\alpha-1}\Big\|f\big(\tau, y_{\rho(\tau,y_\tau+\overline{z}^n(\tau))} + \overline{z}^n_{\rho(\tau,y_\tau+\overline{z}^n(\tau))}, y(\tau)+\overline{z}^n(\tau)\big) \\ &\qquad - f\big(\tau, y_{\rho(\tau,y_\tau+\overline{z}(\tau))} + \overline{z}_{\rho(\tau,y_\tau+\overline{z}(\tau))}, y(\tau)+\overline{z}(\tau)\big)\Big\|\, d\tau\, ds \leq M_1M_2\widetilde{M}_S^2\,\frac{T^{2\alpha}}{\alpha^2}\,\varepsilon,\end{aligned}$$

where $\varepsilon > 0$, $\varepsilon \to 0$ as $n \to \infty$. Moreover,

$$\begin{aligned}\big\|P_2(z^n)(t) - P_2(z)(t)\big\| &\leq M_1M_2\widetilde{M}_S\int_{t_i}^t (t-s)^{\alpha-1}\Bigg[\widetilde{M}_T\big\|z^n(t_i^-) - z(t_i^-)\big\| + \big\|I_i(z^n(t_i^-)) - I_i(z(t_i^-))\big\| \\ &\quad + \widetilde{M}_S\int_{t_i}^T (T-\tau)^{\alpha-1}\Big\|f\big(\tau, y_{\rho(\tau,y_\tau+\overline{z}^n(\tau))} + \overline{z}^n_{\rho(\tau,y_\tau+\overline{z}^n(\tau))}, y(\tau)+\overline{z}^n(\tau)\big) \\ &\quad - f\big(\tau, y_{\rho(\tau,y_\tau+\overline{z}(\tau))} + \overline{z}_{\rho(\tau,y_\tau+\overline{z}(s))}, y(s)+\overline{z}(s)\big)\Big\|\, d\tau\Bigg]\, ds \\ &\leq M_1M_2\widetilde{M}_S\widetilde{M}_T\,\frac{T^\alpha}{\alpha}\Big[\big\|z^n(t_i^-) - z(t_i^-)\big\| + \big\|I_i(z^n(t_i^-)) - I_i(z(t_i^-))\big\|\Big] + M_1M_2\widetilde{M}_S^2\,\frac{T^{2\alpha}}{\alpha^2}\,\varepsilon,\end{aligned}$$

where $\varepsilon > 0$, $\varepsilon \to 0$ as $n \to \infty$, for all $t \in (t_i, t_{i+1}]$, $i = 1, \ldots, m,$. The impulsive functions I_k, $k = 1, \ldots, m$, are continuous, and we get

$$\lim_{n\to\infty}\|P_2z^n - P_2z\|_{\mathcal{B}_2} = 0.$$

This means that P_2 is continuous.

Claim 2: P_2 maps bounded sets of $\mathcal{B}_2$ into bounded sets in $\mathcal{B}_2$. So, let us prove that for any $r > 0$, there exists $\xi > 0$ such that for each $z \in B_r = \{z \in \mathcal{B}_2 : \ \|z\|_{\mathcal{B}_2} \leq r\}$, $\|P_2z\|_{\mathcal{B}_2} \leq \xi$. Indeed, for any

$z \in B_r$, $t \in [0, t_1]$, we have

$$
\begin{aligned}
\|P_2(z)(t)\| &\le \int_0^t \|S_\alpha(t-s)\|_{L(E)} \|B\|_{L(E)} \|u(s)\|\, ds \\
&\le M_1 M_2 \widetilde{M}_S \int_0^t (t-s)^{\alpha-1} \Big[\|x_1\| + \widetilde{M}_S \int_0^T (T-\tau)^{\alpha-1} \mu(\tau) \\
&\qquad \times \psi\Big(\big\| y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(s, y_\tau + \overline{z}(\tau))} \|_{\mathcal{B}} + \|y(\tau) + \overline{z}(\tau)\big\| \Big)\, d\tau \Big]\, ds \\
&\le M_1 M_2 \widetilde{M}_S \int_0^t (t-s)^{\alpha-1} \Big[\|x_1\| + \widetilde{M}_S \int_0^T (T-\tau)^{\alpha-1} \mu(\tau) \\
&\qquad \times \psi\big((C_2^* + L^\phi) \|\phi\|_{\mathcal{B}} + C_1^* r + r \big)\, d\tau \Big]\, ds \\
&\le M_1 M_2 \widetilde{M}_S \,\frac{T^\alpha}{\alpha}\, \|x_1\| + M_1 M_2 \widetilde{M}_S^2 \,\frac{T^{2\alpha}}{\alpha^2}\, \psi\big((C_2^* + L^\phi) \|\phi\|_{\mathcal{B}} + (C_1^* + 1) r \big) \int_0^t \mu(s)\, ds \\
&\le M_1 M_2 \widetilde{M}_S \,\frac{T^\alpha}{\alpha}\, \|x_1\| + M_1 M_2 \widetilde{M}_S^2 \,\frac{T^{2\alpha}}{\alpha^2}\, \psi\big((C_2^* + L^\phi) \|\phi\|_{\mathcal{B}} + (C_1^* + 1) r \big) \|\mu\|_{L^1}.
\end{aligned}
$$

Moreover, when $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$, we have the estimate

$$
\begin{aligned}
\|P_2(z)(t)\| \le M_1 M_2 \widetilde{M}_S \,\frac{T^\alpha}{\alpha}\, \|x_1\| &+ M_1 M_2 \widetilde{M}_S \widetilde{M}_T (r + \Omega)\, \frac{T^\alpha}{\alpha} \\
&+ M_1 M_2 \widetilde{M}_S^2 \,\frac{T^{2\alpha}}{\alpha^2}\, \psi\big((C_2^* + L^\phi) \|\phi\|_{\mathcal{B}} + (C_1^* + 1) r \big) \|\mu\|_{L^1}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|P_2 z\|_{\mathcal{B}_2} \le M_1 M_2 \widetilde{M}_S \,\frac{T^\alpha}{\alpha}\, \|x_1\| &+ M_1 M_2 \widetilde{M}_S \widetilde{M}_T (r + \Omega)\, \frac{T^\alpha}{\alpha} \\
&+ M_1 M_2 \widetilde{M}_S^2 \,\frac{T^{2\alpha}}{\alpha^2}\, \psi\big((C_2^* + L^\phi) \|\phi\|_{\mathcal{B}} + (C_1^* + 1) r \big) \|\mu\|_{L^1}.
\end{aligned}
$$

Claim 3: $P_2(B_r)$ is bounded and equicontinuous. Letting $u, v \in [0, T]$, with $u < v$, we have

$$
\begin{aligned}
\|P_2(z)(v) - P_2(z)(u)\| &\le Q_1 + Q_2, \\
Q_1 &= \int_u^v \|S_\alpha(v-s)\|_{L(E)} \|B\|_{L(E)} \|u(s)\|\, ds, \\
Q_2 &= \int_0^u \|S_\alpha(v-s) - S_\alpha(u-s)\|_{L(E)} \|B\|_{L(E)} \|u(s)\|\, ds.
\end{aligned}
$$

In view of (3.4), for $t \in [0, t_1]$, we have

$$
\begin{aligned}
Q_1 &= \int_u^v \|S_\alpha(v-s)\|_{L(E)} \|B\|_{L(E)} \|u(s)\|\, ds \\
&\le M_1 M_2 \widetilde{M}_S \int_u^v (v-s)^{\alpha-1} \\
&\qquad \times \Big[\|x_1\| + \widetilde{M}_S \int_0^T (T-\tau)^{\alpha-1} f\big(\tau, y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau+\overline{z}(\tau))}, y(\tau) + \overline{z}(\tau)\big)\, d\tau\Big]\, ds \\
&\le M_1 M_2 \widetilde{M}_S \frac{(v-u)^\alpha}{\alpha} \Big[\|x_1\| + \widetilde{M}_S \frac{T^\alpha}{\alpha} \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big)\|\mu\|_{L^1}\Big].
\end{aligned}
$$

Hence, $\lim\limits_{u\to v} Q_1 = 0$. Similarly, for $u, v \in (t_i, t_{i+1}]$, with $u < v$, $i = 1, \ldots, m$, we get

$$
\begin{aligned}
Q_1 &= \int_u^v \|S_\alpha(v-s)\|_{L(E)} \|B\|_{L(E)} \|u(s)\|\, ds \\
&\le M_1 M_2 \widetilde{M}_S \int_u^v (v-s)^{\alpha-1} \Big[\|x_1\| + \widetilde{M_T}(r+\Omega) \\
&\quad + \widetilde{M}_S \int_0^T (T-\tau)^{\alpha-1} f\big(\tau, y_{\rho(\tau, y_\tau + \overline{z}(\tau))} + \overline{z}_{\rho(\tau, y_\tau+\overline{z}(\tau))}, y(\tau) + \overline{z}(\tau)\big)\, d\tau\Big]\, ds \\
&\le M_1 M_2 \widetilde{M}_S \frac{(v-u)^\alpha}{\alpha} \Big[\|x_1\| + \widetilde{M_T}(r+\Omega) + \widetilde{M}_S \frac{T^\alpha}{\alpha} \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big)\|\mu\|_{L^1}\Big].
\end{aligned}
$$

Hence, we deduce that $\lim\limits_{u\to v} Q_1 = 0$.

Using (3.4), for all $t \in [0, t_1]$ we get

$$
\begin{aligned}
Q_2 &= \int_0^u \big\|S_\alpha(v-s) - S_\alpha(u-s)\big\|_{L(E)} \|B\|_{L(E)} \|u(s)\|\, ds \\
&\le M_1 M_2 \Big[\|x_1\| + \widetilde{M}_S \frac{T^\alpha}{\alpha} \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big)\|\mu\|_{L^1}\Big] \\
&\qquad \times \int_0^u \big\|S_\alpha(v-s) - S_\alpha(u-s)\big\|_{L(E)}\, ds.
\end{aligned}
$$

Similarly, when $u, v \in (t_i, t_{i+1}]$, $i = 1, \ldots, m$, we have the estimate

$$
\begin{aligned}
Q_2 &= \int_0^u \big\|S_\alpha(v-s) - S_\alpha(u-s)\big\|_{L(E)} \|B\|_{L(E)} \|u(s)\|\, ds \\
&\le M_1 M_2 \Big[\|x_1\| + \widetilde{M_T}(r+\Omega) + \widetilde{M}_S \frac{T^\alpha}{\alpha} \psi\big((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\big)\|\mu\|_{L^1}\Big] \\
&\qquad \times \int_0^u \big\|S_\alpha(v-s) - S_\alpha(u-s)\big\|_{L(E)}\, ds.
\end{aligned}
$$

Since

$$
\big\|S_\alpha(v-s) - S_\alpha(u-s)\big\|_{L(E)} \le 2\widetilde{M}_s (t_i - s)^{\alpha-1},
$$

which belongs to $L^1(J,\mathbb{R}_+)$ and $S_\alpha(v-s)-S_\alpha(u-s)\to 0$ as $u\to v$, S_α is strongly continuous. This implies that $\lim\limits_{u\to v} Q_2=0$. Thus, from the above inequalities, we have

$$\lim_{u\to v}\|P(z)(v)-P(z)(u)\|=0.$$

So, $P_2(B_r)$ is equicontinuous.

Finally, combining Claims 1 and 3 together with the Arzelà–Ascoli's theorem, we conclude that the operator P_2 is compact. In fact, by Step 1–Step 2 and Lemma 2.10, one can conclude that $P=P_1+P_2$ is continuous and takes bounded sets into bounded sets. Meanwhile, it is easy to see that $\alpha(P_2(B_r))=0$, since $P_2(B_r)$ is relatively compact. It comes from $P_1(B_r)\subseteq B_r$ and $\alpha(P_2(B_r))=0$ that

$$\alpha(P(B_r))\le\alpha(P_1(B_r))+\alpha(P_2(B_r))\le\alpha(B_r)$$

for every bounded set B_r of $\mathcal{B}_2$ with $\alpha(B_r)>0$.

Since $P(B_r)\subset B_r$ for a convex, closed and bounded set B_r of $\mathcal{B}_2$, using Theorem 2.11, P has a fixed point z in $B_r\subset\mathcal{B}_2$. It is easy to see that x is a fixed point of the operator N which is a mild solution of (1.1) satisfying $x(T)=x_1$. Thus, system (1.1) is controllable on $(-\infty,T]$. □

4 An example

To apply our abstract results, we consider the impulsive fractional integro-differential system:

$$\begin{aligned}
&\frac{\partial_t^q}{\partial t^q}v(t,\zeta)=\frac{\partial^2}{\partial\zeta^2}v(t,\zeta)+\omega\mu(t,\zeta)\\
&+\int_{-\infty}^{t}a_1(s-t)v\big(s-\rho_1(t)\rho_2(|v(t)|),\xi\big)\,ds+t^2\cos|v(t,\zeta)|,\quad t\in[0,T],\quad \zeta\in[0,\pi],\\
&v(t,0)=v(t,\pi)=0,\quad t\in[0,T],\\
&v(t,\zeta)=v_0(\theta,\zeta),\quad \theta\in(-\infty,0],\quad \zeta\in[0,\pi],\\
&\Delta v(t_k)(\zeta)=\int_{-\infty}^{t_k}p_k(t_k-y)\,dy\cos(v(t_k)(\zeta)),\quad k=1,2,\dots,m,
\end{aligned}\tag{4.1}$$

where $0<q<1$, $\omega>0$, $\mu:[0,T]\times[0,\pi]\to[0,\pi]$, $p_k:\mathbb{R}\to\mathbb{R}$, $k=1,2,\dots,m$, and $a_1:(-\infty,0]\to\mathbb{R}$, $\rho_i:[0,+\infty)\to[0,+\infty)$, $i=1,2$, $v_0:(-\infty,0]\times[0,\pi]\to\mathbb{R}$ are continuous functions.

Set $E=L^2([0,\pi])$ and let $D(A)\subset E\to E$ be the operator $A\omega=\omega''$ with the domain

$$D(A)=\big\{\omega\in E:\ \omega,\ \omega' \text{ are absolutely continuous},\ \omega''\in E,\ \omega(0)=\omega(\pi)=0\big\},$$

then

$$A\omega=\sum_{n=1}^{\infty}n^2(\omega,\omega_n)\omega_n,\quad \omega\in D(A),$$

where $\omega_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$, $n\in\mathbb{N}$, is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\ge0}$ in E and is given by

$$T(t)\omega=\sum_{n=1}^{\infty}e^{-n^2t}(\omega,\omega_n)\omega_n \ \text{ for all }\ \omega\in E\ \text{ and all }\ t>0.$$

From these expressions it follows that $\{T(t)\}_{t\ge0}$ is a uniformly bounded compact semigroup such that $R(\lambda,A)=(\lambda-A)^{-1}$ is a compact operator for all $\lambda\in\rho(A)$, that is, $A\in\mathbb{A}^\alpha(\theta_0,\omega_0)$.

For the phase space, we choose $\mathcal{B}=C_0\times L^2(g,X)$ (for details, see Example 2.6).

Set

$$x(t)(\zeta) = v(t,\zeta), \quad t \in [0,T], \quad \zeta \in [0,\pi];$$
$$\phi(\theta)(\zeta) = v_0(\theta,\zeta), \quad \theta \in (-\infty,0], \quad \zeta \in [0,\pi];$$
$$f(t,\varphi,x(t))(\zeta) = \int_{-\infty}^{0} a_1(s)\varphi(s,\xi)\,ds + t^2 \cos|x(t)(\zeta)|, \quad t \in [0,T], \quad \zeta \in [0,\pi];$$
$$\rho(s,\varphi) = s - \rho_1(s)\rho_2(|\varphi(0)|);$$
$$I_k(x(t_k^-))(\zeta) = \int_{-\infty}^{0} p_k(t_k - y)\,dy\, \cos(x(t_k)(\zeta)), \quad k = 1,2,\dots,m;$$
$$Bu(t)(\zeta) = \omega\mu(t,\zeta).$$

Under the above conditions, we can represent system (4.1) in the abstract form (1.1). Assume that the operator $W : L^2(J,E) \to X$ defined by

$$Wu(\,\cdot\,) = \int_0^T S_\alpha(T-s)\omega\mu(s,\,\cdot\,)\,ds$$

has a bounded invertible operator $\widetilde{W}^{-1}$ in $L^2(J,E)/\ker W$.

The following result is a direct consequence of Theorem 3.8.

Proposition 4.1. *Let $\varphi \in \mathcal{B}$ be such that (H_φ) holds, and assume that the above conditions are fulfilled, then system* (4.1) *is controllable on* $(-\infty, T]$.

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APPROXIMATING SOLUTION OF DISTRIBUTED DELAY DIFFERENTIAL EQUATION USING GAMMA SERIES OF DELAY DENSITY FUNCTION

Abstract. The linear chain trick can be used to solve differential equations with distributed delays of gamma type. In this paper we show that other densities of delay can be expressed as a sum of gamma densities, which can then be used to find approximate solution of differential equation with distributed delay.[1]

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Key words and phrases. Distributed delay, gamma series, Laguerre polynomials, linear chain trick.

რეზიუმე. ნაშრომში ჯაჭვური კანონის გამოყენებით ამოხსნილია γ-ტიპის დაგვიანებულ არგუმენტიანი განტოლება. ნაჩვენებია, რომ დაგვიანების სიმკვრივე შეიძლება გამოისახოს როგორც γ-ტიპის სიმკვრივეთა ჯამი. აღნიშნული გამოსახვა შეიძლება გამოყენებულ იქნას დაგვიანებულ არგუმენტიანი განტოლების მიახლოებითი ამონახსნის მოსაძებნად.

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1 Introduction

In this paper we are looking for an approximate solution of a differential equation with distributed delay, i.e.,

$$\begin{aligned} \dot{x}(t) &= f\Big(t, x(t), \int_0^\infty x(t-s)g(s)\,\mathrm{d}s\Big), \\ x(t) &= \phi(t), \quad t \le t_0, \end{aligned} \tag{1.1}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is a Lipschitz function (to ensure the existence and uniqueness of the solution), ϕ is an initial function (we usually need it to be continuous and bounded on its domain) and $g : [0,\infty) \to [0,\infty)$ is a weight function which describes how past states of x are affecting present rate of change. We can presume that g is normed, that is, $\int_0^\infty g(s)\,\mathrm{d}s = 1$. This means that g is a density of some nonnegative random variable which we interpret as a delay.

Problem (1.1) is a generalization of a differential equation with a constant delay, i.e.,

$$\begin{aligned} \dot{x}(t) &= f\big(t, x(t), x(t-\tau)\big), \\ x(t) &= \phi(t), \quad t \in [t_0-\tau, t_0], \end{aligned} \tag{1.2}$$

where f, ϕ are the same as in (1.2) and $\tau > 0$ is a constant delay. We can find a solution of (1.2) by the method of steps. However, the method of steps can be used to transform (1.1) to an ordinary differential equation only if $0 \notin \operatorname{supp}(g)$. This restriction may be quite problematic, not often describing the modelled phenomena well.

Another possible way to solve (1.1) is the use of the Laplace transform. This is not a versatile method, since it entails several nontrivial steps, such as finding the Laplace transform of g, solving an algebraic equation and, finally, applying an inverse Laplace transform on a possibly complicated function.

In a special case, where g is a density of gamma distribution, that is,

$$g_a^p(t) = \begin{cases} \dfrac{a^p t^{p-1} \mathrm{e}^{-at}}{\Gamma(p)}, & t \ge 0, \\ 0, & \text{otherwise}, \end{cases} \tag{1.3}$$

where $a > 0$, $p \in \mathbb{N}$ and $\Gamma(t)$ denotes the gamma function at t, we can transform (1.1) to a system of ordinary differential equations. This process is called the linear chain trick and is in detail explained in [3]. We will briefly describe it for the case of a scalar equation with one distributed delay of gamma type, but it can be easily generalized to the case of a vector equation or multiple distributed delays of gamma type.

Consider (1.1) with $g = g_a^p$. We can introduce new variables $y_1, y_2, \ldots, y_p$,

$$y_k = \int_0^\infty x(t-s) g_a^k(s)\,\mathrm{d}s, \quad k = 1, 2, \ldots, p. \tag{1.4}$$

Since

$$\begin{aligned} \dot{g}_a^p &= a(g_a^{p-1} - g_a^p), \quad p > 1, \\ \dot{g}_a^1 &= -a g_a^1, \end{aligned} \tag{1.5}$$

new variables y_k satisfy the system of ordinary differential equations

$$\begin{aligned} \dot{y}_p(t) &= a\big(y_{p-1}(t) - y_p(t)\big) \\ \dot{y}_{p-1}(t) &= a\big(y_{p-2}(t) - y_{p-1}(t)\big) \\ &\ \ \vdots \\ \dot{y}_2(t) &= a\big(y_1(t) - y_2(t)\big) \\ \dot{y}_1(t) &= a\big(x(t) - y_1(t)\big). \end{aligned} \tag{1.6}$$

Together with the original equation

$$\dot{x}(t) = f\big(t, x(t), y_p(t)\big), \tag{1.7}$$

we obtain a system of $p+1$ ordinary differential equations. The initial values are given by

$$\begin{aligned} x(t_0) &= \phi(t_0), \\ y_k(t_0) &= \int_0^\infty \phi(t_0 - s) g_a^k(s)\,\mathrm{d}s, \quad k = 1, 2, \dots, p. \end{aligned} \tag{1.8}$$

System (1.6) is in itself an autonomous linear system with constant coefficients, therefore if (1.1) is autonomous or linear (with constant coefficients), the same is true for the new system.

2 The main result

The linear chain trick can only be used for gamma densities. However, if we could express other densities of nonnegative random variables as a sum of gamma densities, we could apply linear chain trick on each element of the sum. In other words, we are interested in describing the linear span of $\{g_a^p,\ a > 0,\ p \in \mathbb{N}\}$.

The method of expanding density of a nonnegative random variable into a sum of gamma densities is described in [1]. We perform similar construction for $a = 1$. The choice of the value of parameter a is not important at the moment, so the optimal value is to be discussed.

Consider the space $L^2_\gamma(\mathbb{R}_0^+)$, i.e., the linear space of real functions $f : [0, \infty) \to \mathbb{R}$ satisfying

$$\int_0^\infty \mathrm{e}^{-x} f^2(x)\,\mathrm{d}x < \infty. \tag{2.1}$$

This is a Hilbert space with the inner product $\langle f, g\rangle_\gamma$ given by

$$\langle f, g\rangle_\gamma = \int_0^\infty \mathrm{e}^{-x} f(x) g(x)\,\mathrm{d}x. \tag{2.2}$$

Lemma 2.1. *The set of Laguerre polynomials*

$$\Big\{L_n(x) = \frac{\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,(x^n \mathrm{e}^{-x}),\ n \in \mathbb{N}\Big\} = \Big\{L_n(x) = \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j!}\,x^j,\ n \in \mathbb{N}\Big\} \tag{2.3}$$

is a complete orthonormal set in $L^2_\gamma(\mathbb{R}_0^+)$.

Proof. See [2]. □

Let f be a density of a nonnegative random variable. We want to express it as a series

$$f(x) = \mathrm{e}^{-x} \sum_{k=0}^\infty a_k L_k(x). \tag{2.4}$$

For $n \in \mathbb{N}$ (using orthonormality of Laguerre polynomials),

$$\int_0^\infty f(x) L_n(x)\,\mathrm{d}x = \int_0^\infty L_n(x) \mathrm{e}^{-x} \sum_{k=0}^\infty a_k L_k(x)\,\mathrm{d}x = a_n \tag{2.5}$$

holds. This is well-defined if the first n raw moments of the density function f are finite. Series (2.4) is, in fact, a sum of gamma densities, since

$$\begin{aligned}\mathrm{e}^{-x}\sum_{k=0}^{\infty}a_kL_k(x) &= \sum_{k=0}^{\infty}\Big(\mathrm{e}^{-x}a_k\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j}{j!}x^j\Big)\\ &= \sum_{k=0}^{\infty}\Big(\mathrm{e}^{-x}a_k\sum_{j=0}^{k}\binom{k}{j}(-1)^j g_1^{j+1}(x)\Big) = \sum_{k=0}^{\infty}\alpha_k g_1^{k+1}(x), \qquad (2.6)\end{aligned}$$

where α_k contains all coefficients at the corresponding gamma density g_1^{k+1}. We call series (2.4) the gamma series of the function f.

We have assumed that (2.4) held. Regarding that, there arises the important question for what densities f the corresponding gamma series converges to the original function f.

Theorem 2.1. *Let f be a density of a nonnegative random variable with all of the raw moments finite and let there exist $x_0 > 0$ and constants $c > 0$, $\delta > 0$ such that for all $x \geq x_0$, the inequality $f(x) \leq c\mathrm{e}^{-x\frac{1+\delta}{2}}$ is satisfied. Then the gamma series of f converges to f in the sense of*

$$\lim_{n\to\infty}\int_0^{\infty}\mathrm{e}^{x}\Big(f(x)-\mathrm{e}^{-x}\sum_{k=0}^{n}a_kL_k(x)\Big)^2\,\mathrm{d}x = 0. \qquad (2.7)$$

Proof. Denote $h(x) = \mathrm{e}^{x}f(x)$. The gamma series of h is $\sum_{k=0}^{\infty}a_kL_k(x)$. The Laguerre polynomials are a complete orthonormal set in $L^2_\gamma(\mathbb{R}_0^+)$, therefore $h \in L^2_\gamma(\mathbb{R}_0^+)$ must hold. To prove this, we calculate its norm in $L^2_\gamma(\mathbb{R}_0^+)$, i.e.,

$$\int_0^{\infty}\mathrm{e}^{-x}h^2(x)\,\mathrm{d}(x) = \int_0^{\infty}\mathrm{e}^{x}f^2(x)\,\mathrm{d}x = \underbrace{\int_0^{x_0}\mathrm{e}^{x}f^2(x)\,\mathrm{d}x}_{I_1} + \underbrace{\int_{x_0}^{\infty}\mathrm{e}^{x}f^2(x)\,\mathrm{d}x}_{I_2}. \qquad (2.8)$$

The first term I_1 is an integral of a bounded function over a finite interval, so $I_1 < \infty$. Using the theorem's assumptions, we can show $I_2 < \infty$ as well, since

$$I_2 \leq c^2\int_{x_0}^{\infty}\mathrm{e}^{x-x(1+\delta)}\,\mathrm{d}x = \frac{c^2}{\delta}\mathrm{e}^{x_0} < \infty. \qquad (2.9)$$

This means that $h \in L^2_\gamma(\mathbb{R}_0^+)$ and the rest of the theorem is a consequence of the Fourier series theory, specifically the Riesz–Fischer theorem. $\square$

Corollary 2.1. *Any density of a nonnegative random variable with a compact support can be expressed as a gamma series.*

This result can be used to find an approximate solution of equation (1.1):

- Find the first n terms of gamma series of g.
- Apply the linear chain trick to each term.
- Solve (analyze) the resulting system of ordinary differential equations.

Remark 2.1. To find the first n terms in the gamma series of a function we need only the first n raw moments. This is useful in the case where delays are measured experimentally and we need to estimate the probability density function, since we can use sample raw moments instead of theoretical ones and thus obtain an estimation in the form of a gamma series.

Table 1. The first n coefficients of gamma series (2.6) of hat distribution (3.2).

	$n=1$	$n=3$	$n=5$	$n=10$	$n=20$
α_0	1	0.083308	−0.591735	−0.154871	0.152195
α_1	0	2.333400	5.291708	1.046663	−0.978461
α_2		−1.916725	−7.011521	11.170907	6.456673
α_3		0.500017	4.766992	−40.563154	71.573716
α_4			−1.719577	71.475719	−598.149055
α_5			0.261133	−80.029644	2354.843929
α_6				60.821780	−6267.150889
α_7				−31.534065	12544.162081
α_8				10.745646	−19803.452473
α_9				−2.178752	25272.614335
α_{10}				0.1997700	−26411.951554
α_{11}					22735.938215
α_{12}					−16129.161624
α_{13}					9390.531536
α_{14}					−4446.376297
α_{15}					1686.572512
α_{16}					−500.587669
α_{17}					112.058428
α_{18}					−17.798574
α_{19}					1.788478
α_{20}					−0.085504
Σ	1	1	1	1	1

3 Example

Consider the initial value problem

$$\begin{aligned} \dot{x}(t) &= -2\int_0^\infty x(t-s)g_h(s)\,\mathrm{d}s, \\ x(t) &= 1, \quad t \le 0, \end{aligned} \tag{3.1}$$

where g_h is the probability density function of the hat distribution, to be specific,

$$g_h(t) = \begin{cases} t, & t \in [0,1], \\ 2-t, & t \in [1,2], \\ 0, & \text{otherwise}. \end{cases} \tag{3.2}$$

We will compare an approximate solution $\widehat{x}_n$ obtained by expanding g_h into the gamma series with the first n terms and another approximate solution x_h, where x_h is obtained by a discretization

$$x_h(t+h) = x(t) - 2h\sum_{k=0}^{200} x(t-hk)g_h(hk) \tag{3.3}$$

with step size $h = 0.01$.

To illustrate the method, we compute the approximate solution of problem (3.1) for different values of n, in particular, for $n = 1, 3, 5, 10, 20$.

First, we compute the first n coefficients of gamma series of the hat distribution by numerically integrating (2.5) and then sum the results according to (2.6). Numerical values of coefficients α_k, $k = 0, 1, \dots, n$, are given in Table 1. Notice that the sum of coefficients for each n is 1.

Instead of the initial value problem (3.1), we can solve the problem

$$\dot{\widehat{x}}(t) = -2\int_0^\infty \Big(\widehat{x}(t-s)\sum_{k=0}^{n} g_1^{k+1}(s)\Big)\,\mathrm{d}s = -2\sum_{k=0}^{n}\alpha_k\int_0^\infty \widehat{x}(t-s)g_1^{k+1}(s)\,\mathrm{d}s, \qquad (3.4)$$
$$x(t) = 1,\quad t \le 0.$$

Using the linear chain trick, we obtain a system of $n+1$ ordinary differential equations with constant coefficients

$$\begin{pmatrix}\dot{\widehat{x}}(t)\\ \dot{y}_1(t)\\ \dot{y}_2(t)\\ \vdots\\ \dot{y}_n(t)\end{pmatrix} = \begin{pmatrix}0 & -2\alpha_0 & -2\alpha_1 & \cdots & -2\alpha_n\\ 1 & -1 & 0 & \cdots & 0\\ 0 & 1 & -1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & -1\end{pmatrix}\begin{pmatrix}\widehat{x}(t)\\ y_1(t)\\ y_2(t)\\ \vdots\\ y_n(t)\end{pmatrix}. \qquad (3.5)$$

Initial values are $\widehat{x}(0) = y_1(0) = \cdots = y_n(0) = 1$.

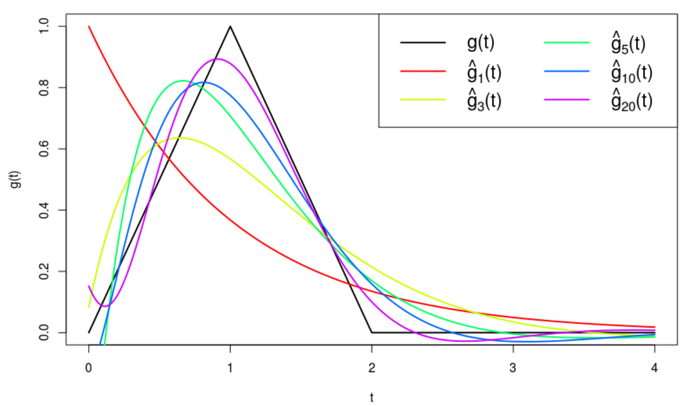

Figure 1. Approximation of the hat probability density function g_h by the first n terms of its gamma series.

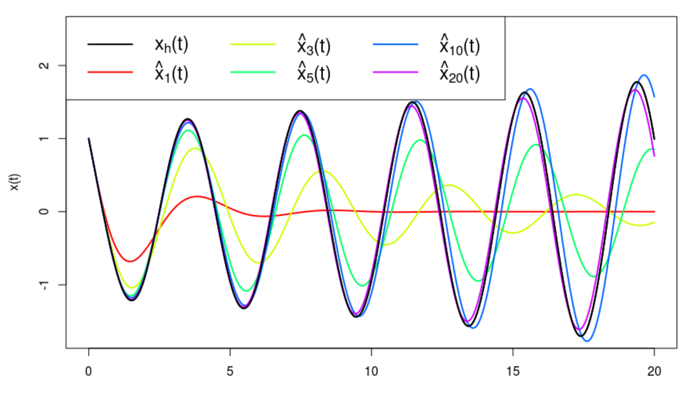

Figure 2. Approximate solutions $\widehat{x}_n$ of (3.1) obtained by approximating g_h by the first n terms of its gamma series and approximate solution x_h computed by discretization.

We denote by $\widehat{x}_n$ the solution of (3.4) obtained by using the gamma series of order n. Solutions x_h and $\widehat{x}_n$ are computed by using R. Approximation of the hat density is given in Figure 1 and the corresponding solutions are given in Figure 2. Since we do not know the exact solution, we do not know how precise our solutions are. To our knowledge, there is no distributed delay differential equation (except for a delay of gamma type) with a known exact solution that could by used as a test case.

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EXISTENCE RESULTS OF A SINGULAR FRACTIONAL DIFFERENTIAL EQUATION WITH PERTURBED TERM

Abstract. The boundary value problem

$$D^{\alpha}u(t) + \mu a(t)f(t, u(t)) - q(t) = 0,$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \lambda \int\limits_0^1 u(s)\,ds$$

is studied, where μ is a positive parameter, $f : [0,1] \times [0;+\infty) \to [0;+\infty)$ and $a : (0,1) \to [0,+\infty)$ are continuous functions, while $q : (0,1) \to [0,+\infty)$ is a measurable function. The case, where the function a has singularities at the points $t = 0$ and $t = 1$, is admissible.

Conditions are found guaranteeing, respectively, the existence of at least one and at least two positive solutions. Examples are gives.[1]

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Key words and phrases. Fractional differential equation, positive solution, integral boundary conditions, Green's function, dependence on a parameter, perturbed term.

რეზიუმე. შესწავლილია დადებით μ პარამეტრზე დამოკიდებული სასაზღვრო ამოცანა

$$D^{\alpha}u(t) + \mu a(t)f(t, u(t)) - q(t) = 0,$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \lambda \int\limits_0^1 u(s)\,ds$$

სადაც $n \in \mathbb{N}$, $n \geq 3$, $n-1 < \alpha \leq n$, $0 < \lambda < \alpha$, D^{α} არის α რიგის რიმან-ლიუვილის წარმოებული, $f : [0,1] \times [0;+\infty) \to [0;+\infty)$ და $a : (0,1) \to [0,+\infty)$ უწყვეტი, ხოლო $q : (0,1) \to [0,+\infty)$ ზომადი ფუნქციებია. დასაშვებია შემთხვევა, როცა a ფუნქციას გააჩნია სინგულარობები $t = 0$ და $t = 1$ წერტილებში.

ნაპოვნია პირობები, რომლებიც სათანადოდ უზრუნველყოფენ ერთი მაინც და ორი მაინც დადებითი ამონახსნის არსებობას. მოყვანილია მაგალითები.

[1] Reported on Conference "Differential Equation and Applications", September 4–7, 2017, Brno

1 Introduction

Fractional differential equations have applications in various fields of science and engineering and have been a focus of research for decades (see [6, 9, 10, 12] and the references therein). There is a large number of important subjects in various fields of fractional calculus and related applications such as the solvability, existence and multiplicity of positive solutions for the given boundary value problems of fractional differential equations. For more details see [1, 3, 4, 11].

Namely, A. Cabada and Z. Hamdi [3] presented the existence results for the following boundary value problem

$$\begin{cases} D^\alpha u(t) + \mu g(t) f(u(t)) = 0 \text{ in } [0,1], \\ u(0) = u'(0) = 0, \quad u(1) = \lambda \int\limits_0^1 u(s)\,ds, \end{cases}$$

where μ is a positive parameter, $2 < \alpha \le 3$, $0 < \lambda < \alpha$ and f, g are continuous functions. Under the conditions $g \in L^1([0,1])$ and $\int\limits_{1/2}^{1} g(t)\,dt > 0$, they derived various existence and multiplicity results of positive solutions depending on the parameter $\mu > 0$.

However, all of the above mentioned works are based on a key assumption, that is, the nonlinear term is required to be nonnegative. When nonlinear fractional differential equations involve a sign-changing term, J. Henderson and R. Luca [5] investigated the existence of a positive solution for the nonlinear fractional problem, and then under the similar conditions X. Zhang, L. Liu and Y. Wu [13] studied the existence of positive solutions of the boundary value problem for a singular fractional differential equation with a negatively perturbed term. More precisely, the authors considered the following problem

$$\begin{cases} -D^\alpha u(t) = p(t) f(t, u(t)) - q(t) \text{ in } (0,1), \\ u(0) = u'(0) = u(1) = 0, \end{cases}$$

where $2 < \alpha \le 3$. The function p is continuous nonnegative on $(0,1)$ and f is in $\mathcal{C}([0,1] \times [0,+\infty), [0,+\infty))$. The perturbed term $q : (0,1) \to [0,+\infty)$ is Lebesgue integrable and may be singular at some zero measure sets of $[0,1]$.

Under other boundary conditions, X. Zhou, J.-G. Peng and Y.-D. Chu [14] studied the following problem

$$\begin{cases} D^\alpha u(t) = p(t) f(t, u(t)) - q(t) \text{ in } (0,1), \\ u(0) = u(1) = u'(1) = 0, \end{cases}$$

where $2 < \alpha \le 3$. The functions p and q are Lebesgue integrable on $(0,1)$ and f is in $\mathcal{C}([0,1] \times [0,+\infty), [0,+\infty))$.

The existence of positive solutions of a fractional differential equation with a perturbed term, integral boundary and parametric dependence, however, has not been studied previously. In this paper, motivated by [2, 3, 13, 14], we give sufficient conditions for the existence and multiplicity of positive solutions for problem

$$\begin{cases} D^\alpha u(t) + \mu a(t) f(t, u(t)) - q(t) = 0 \text{ in } (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \lambda \int\limits_0^1 u(s)\,ds. \end{cases} \tag{1.1}$$

The results derived depend on the positive parameter μ.

The outline of this paper is as follows. In Section 2, we present some preliminaries and lemmas that will be used for the proofs of our main results. The main theorems are presented in Section 3. The final section of the paper contains examples to illustrate our results.

2 Preliminaries and lemmas

In this section, we introduce definitions and preliminary facts that will be used throughout this paper. We refer the reader to [2, 6, 8] for more details.

Definition 2.1. The Riemann–Liouville fractional integral of order $\alpha > 0$ for a measurable function $f : (0, +\infty) \to \mathbb{R}$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t > 0,$$

where Γ is the Euler Gamma function, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2. The Riemann–Liouville fractional derivative of order $\alpha > 0$ for a measurable function $f : (0, +\infty) \to \mathbb{R}$ is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int\limits_0^t (t-s)^{n-\alpha-1} f(s)\, ds = \left(\frac{d}{dt}\right)^n I^{n-\alpha} f(t),$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$. Here $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.3. *Let $\alpha > 0$. Let $u \in \mathcal{C}(0,1) \cap L^1(0,1)$. Then*

(i) $D^\alpha I^\alpha u = u$.

(ii) *For $\delta > \alpha - 1$, $D^\alpha t^\delta = \frac{\Gamma(\delta+1)}{\Gamma(\delta-\alpha+1)} t^{\delta-\alpha}$. Moreover, we have $D^\alpha t^{\alpha-i} = 0$, $i = 1, 2, \ldots, n$.*

(iii) *$D^\alpha u(t) = 0$ if and only if $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$.*

(iv) *Assume that $D^\alpha u \in \mathcal{C}(0,1) \cap L^1(0,1)$, then we have*

$$I^\alpha D^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad c_i \in \mathbb{R}, \quad i = 1, 2, \ldots, n.$$

Now, we give the explicit expression of the Green function for the linear fractional differential equation associated to the problem (1.1).

Lemma 2.4 ([2])**.** *Let $n \geq 3$, $n - 1 < \alpha \leq n$ and $\lambda \in (0, \alpha)$. Let $y \in \mathcal{C}([0,1])$. Then the unique solution of the linear fractional differential problem*

$$\begin{cases} D^\alpha u(t) + y(t) = 0 \ \ \text{in} \ \ (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \lambda \displaystyle\int\limits_0^1 u(s)\, ds, \end{cases} \tag{2.1}$$

is given by

$$u(t) = \int\limits_0^1 G(t,s) y(s)\, ds,$$

where for all $t, s \in [0,1]$,

$$G(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha - \lambda + \lambda s) - (\alpha-\lambda)((t-s)^*)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}, \tag{2.2}$$

$G(t,s)$ is called the Green function of the boundary value problem (2.1)*. Here, for $x \in \mathbb{R}$, $x^* = \max(x, 0)$.*

Now we recall some properties of the Green function.

Proposition 2.5. *Let $n \in \mathbb{N}$, $n \geq 3$, $n-1 < \alpha \leq n$, and $\lambda \in [0, \alpha)$. Then the function G defined by* (2.2) *satisfies the following properties:*

(i) *G is a nonnegative continuous function on $[0,1] \times [0,1]$ and $G(t,s) > 0$ for all $t, s \in (0,1)$.*

(ii) *$G(t,s) \leq \eta K(s)$ for all $t, s \in [0,1]$, where $K(s) = \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)}$ and $\eta = \frac{\alpha}{\alpha-\lambda}$.*

(iii) *$G(t,s) \leq \eta t^{\alpha-1} K_1(s)$ for all $t, s \in [0,1]$, where $K_1(s) = \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}$.*

(iv) *$G(t,s) \geq \eta \lambda^* t^{\alpha-1} K(s)$ $\forall\, t, s \in [0,1]$, where $\lambda^* = \dfrac{\lambda}{\alpha}$.*

(v) *If $\theta \in (0, \frac{1}{2})$, $s \in [0,1]$, then $\min\limits_{t \in [\theta, 1-\theta]} G(t,s) \geq \gamma K(s)$, where $\gamma = (\frac{\theta}{\alpha-1} + \frac{\lambda}{\alpha-\lambda})\theta^{\alpha-1}$.*

Proof. The proofs of (i), (ii) and (v) are given in [2]. To prove (iii), we use Lemmas 2.5 and 2.6 in [2]. Assertion (iv) follows immediately from Proposition 2.7 in [2]. □

Using assertion (ii) of Proposition 2.5, we have the following

Proposition 2.6. *Let q be a nonnegative measurable function on $(0,1)$. Then $w(t) = \int\limits_0^1 G(t,s)q(s)\,ds$ is continuous on $[0,1]$ if and only if $\int\limits_0^1 (1-t)^{\alpha-1} q(t)\,dt$ converges.*

Now we state the following key lemma.

Lemma 2.7. *Let $n \geq 3$, $n-1 < \alpha \leq n$ and $0 < \lambda < \alpha$. Assume that $(1-t)^{\alpha-1}q(t) \in \mathcal{C}(0,1) \cap L(0,1)$. Then the boundary value problem*

$$\begin{cases} D^\alpha w(t) + q(t) = 0 \ \ in \ \ (0,1), \\ w(0) = w'(0) = \cdots = w^{(n-2)}(0) = 0, \quad w(1) = \lambda \displaystyle\int_0^1 w(s)\,ds, \end{cases} \tag{2.3}$$

has a unique nonnegative solution $w(t) = \int\limits_0^1 G(t,s)q(s)\,ds \in \mathcal{C}([0,1])$ satisfying

$$w(t) \leq \eta \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |q(s)|\,ds$$

on $[0,1]$.

Proof. First, we will prove that $D^\alpha w(t) + q(t) = 0$ on $(0,1)$. By Proposition 2.6, we have that w is continuous on $[0,1]$ and so $I^{n-\alpha}|w|$ is bounded on $[0,1]$. Thus, using Fubini's theorem, for each $t \in (0,1)$ we obtain

$$I^{n-\alpha} w(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^1 \int_0^t (t-s)^{n-\alpha-1} G(s,\xi) q(\xi)\,ds\,d\xi = \int_0^1 H(t,\xi) q(\xi)\,d\xi, \tag{2.4}$$

where

$$H(t,\xi) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} G(s,\xi)\,ds.$$

Now, let us find an explicit form of $H(t,\xi)$. Let $t,\xi\in(0,1)$ and $c=(\alpha-\lambda)\Gamma(\alpha)\Gamma(n-\alpha)$, then

$$cH(t,\xi)=\begin{cases}(\alpha-\lambda+\lambda\xi)(1-\xi)^{\alpha-1}\int\limits_0^t (t-s)^{n-\alpha-1}s^{\alpha-1}\,ds, & 0<t\le\xi<1,\\ (\alpha-\lambda+\lambda\xi)(1-\xi)^{\alpha-1}\int\limits_0^t (t-s)^{n-\alpha-1}s^{\alpha-1}\,ds & \\ \qquad -(\alpha-\lambda)\int\limits_\xi^t (t-s)^{n-\alpha-1}(s-\xi)^{\alpha-1}\,ds, & 0<\xi\le t<1.\end{cases}$$

Using the fact that for each $a,b\ge 0$ and $p,q>0$,

$$\int\limits_a^b (b-\theta)^p(\theta-a)^q\,d\theta=\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}\,(b-a)^{p+q+1},$$

we get

$$H(t,\xi)=\frac{1}{(\alpha-\lambda)(n-1)!}\begin{cases}(\alpha-\lambda+\lambda\xi)(1-\xi)^{\alpha-1}t^{n-1}, & 0<t\le\xi<1,\\ (\alpha-\lambda+\lambda\xi)(1-\xi)^{\alpha-1}t^{n-1}-(\alpha-\lambda)(t-\xi)^{n-1}, & 0<\xi\le t<1.\end{cases}\tag{2.5}$$

Thus, by (2.4) and (2.5), we obtain

$$\begin{aligned}(\alpha-\lambda)(n-1)!I^{n-\alpha}w(t)&=\int\limits_0^t \big((1-\xi)^{\alpha-1}(\alpha-\lambda+\lambda\xi)t^{n-1}-(\alpha-\lambda)(t-\xi)^{n-1}\big)q(\xi)\,d\xi\\ &\quad+\int\limits_t^1 (1-\xi)^{\alpha-1}(\alpha-\lambda+\lambda\xi)t^{n-1}q(\xi)\,d\xi\\ &:=I_1(t)+I_2(t).\end{aligned}$$

From the hypothesis, we deduce that the function $\xi\to q(\xi)$ is continuous and integrable near 0 and the function $\xi\to(1-\xi)^{\alpha-1}q(\xi)$ is continuous and integrable near 1. Hence, I_1 and I_2 are integrable on $(0,1)$. So we get, I_1 and I_2 are differentiable on $(0,1)$ and for each $t\in(0,1)$ we have

$$\begin{aligned}&\frac{d}{dt}((n-1)!(\alpha-\lambda)I^{n-\alpha}w(t))\\ &\quad=(n-1)\int\limits_0^t \big((1-\xi)^{\alpha-1}(\alpha-\lambda+\lambda\xi)t^{n-2}-(\alpha-\lambda)(t-\xi)^{n-2}\big)q(\xi)\,d\xi\\ &\qquad+(n-1)\int\limits_t^1 (1-\xi)^{\alpha-1}(\alpha-\lambda+\lambda\xi)t^{n-2}q(\xi)\,d\xi.\end{aligned}$$

Analogously, using the same arguments as above, we prove that $I^{n-\alpha}w(t)$ is differentiable on $(0,1)$ and for each $t\in(0,1)$ we have

$$\Big(\frac{d}{dt}\Big)^n\big((n-1)!(\alpha-\lambda)I^{n-\alpha}w(t)\big)=-(n-1)!(\alpha-\lambda)q(t).$$

Thus

$$\Big(\frac{d}{dt}\Big)^n I^{n-\alpha}w(t)=-q(t).$$

So, $D^\alpha w(t) + q(t) = 0$ for all $t \in (0,1)$.

Next, let us verify the boundary conditions. Using Proposition 2.5(iii), for each $t \in [0,1]$, we have

$$|w(t)| \le \eta t^{\alpha-1} \int_0^1 K_1(s)|q(s)|\,ds$$

which implies that $w(0) = 0$.

On the other hand, for each $t \in (0,1)$, we have

$$\begin{aligned}(\alpha-\lambda)\Gamma(\alpha)w(t) &= \int_0^t \big((\alpha-\lambda+\lambda s)(1-s)^{\alpha-1}t^{\alpha-1} - (\alpha-\lambda)(t-s)^{\alpha-1}\big)q(s)\,ds \\ &\quad + \int_t^1 (\alpha-\lambda+\lambda s)t^{\alpha-1}(1-s)^{\alpha-1}q(s)\,ds \\ &:= J_1(t) + J_2(t). \end{aligned} \tag{2.6}$$

It is clear that $\lim\limits_{t\to 0} \frac{|J_1(t)|}{t} = 0$ and $\lim\limits_{t\to 0} \frac{|J_2(t)|}{t} = 0$. Thus $\lim\limits_{t\to 0} \frac{w(t)}{t} = 0$ and hence $w'(0) = 0$. Now, using the fact that J_1 is continuous and integrable near 0 and J_2 is continuous and integrable near 1, we deduce that J_1 and J_2 are differentiable on $(0,1)$ and thus we can take derivatives from both sides of (2.6). So for each $t \in (0,1)$, we have

$$\begin{aligned}(\alpha-\lambda)\Gamma(\alpha)w'(t) &= (\alpha-1)\int_0^t \big((\alpha-\lambda+\lambda s)(1-s)^{\alpha-1}t^{\alpha-2} - (\alpha-\lambda)(t-s)^{\alpha-2}\big)q(s)\,ds \\ &\quad + (\alpha-1)\int_t^1 (\alpha-\lambda+\lambda s)t^{\alpha-2}(1-s)^{\alpha-1}q(s)\,ds \\ &= L_1(t) + L_2(t). \end{aligned}$$

Since $\lim\limits_{t\to 0} \frac{|L_1(t)|}{t} = 0$ and $\lim\limits_{t\to 0} \frac{|L_2(t)|}{t} = 0$, we deduce that $\lim\limits_{t\to 0} \frac{w'(t)}{t} = 0$ and then $w''(0) = 0$.

In a similar way as above, we prove that $w^{(3)}(0) = \cdots = w^{(n-2)}(0) = 0$.

Now, using Fubini's theorem, a simple calculus yields

$$\begin{aligned}(\alpha-\lambda)\Gamma(\alpha)\int_0^1 w(t)\,dt &= \int_0^1 \Bigg((\alpha-\lambda+\lambda s)(1-s)^{\alpha-1}\int_0^s t^{\alpha-1}\,dt \\ &\quad + \int_s^1 \big((\alpha-\lambda+\lambda s)(1-s)^{\alpha-1}t^{\alpha-1} - (\alpha-\lambda)(t-s)^{\alpha-1}\big)\,dt\Bigg)q(s)\,ds \\ &= \int_0^1 s(1-s)^{\alpha-1}q(s)\,ds = \frac{(\alpha-\lambda)\Gamma(\alpha)}{\lambda}\int_0^1 G(1,s)q(s)\,ds, \end{aligned}$$

which implies that $w(1) = \lambda\int_0^1 w(t)\,dt$.

Finally, let us prove the uniqueness of the solution. Suppose w_1 and w_2 are two continuous solutions on $[0,1]$ of the boundary value problem (2.3). Then we have $D^\alpha(w_2(t) - w_1(t)) = 0$ on $(0,1)$. Thus, by Lemma 2.3(iii), there exist $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$w_2(t) - w_1(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}.$$

Using the boundary conditions, we find $c_n = \cdots = c_2 = 0$. So we get

$$w_2(t) - w_1(t) = c_1 t^{\alpha-1}. \tag{2.7}$$

On the other hand, using (2.7), we get

$$w_2(1) - w_1(1) = \lambda \int_0^1 w_2(t) - w_1(t)\, dt = \frac{\lambda}{\alpha} c_1.$$

This implies that $c_1 = 0$. Then $w_1 = w_2$. □

In the proofs of our main results we shall use the Guo–Krasnosel'skii fixed point theorem presented below.

Lemma 2.8 ([7]). *Let P be the cone of a real Banach space E and let Ω_1, Ω_2 be two bounded open balls of E centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Tx\| \geq \|x\|$, $x \in P \cap \partial\Omega_1$, *and* $\|Tx\| \leq \|x\|$, $x \in P \cap \partial\Omega_2$,

or

(ii) $\|Tx\| \leq \|x\|$, $x \in P \cap \partial\Omega_1$, *and* $\|Tx\| \geq \|x\|$, $x \in P \cap \partial\Omega_2$,

hold. Then the operator T has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let $E = \mathcal{C}([0,1])$, the Banach space endowed with the supremum norm $\|u\| = \sup\limits_{t\in[0,1]} |u(t)|$. Let $\theta \in [0, \frac{1}{2})$, and set $J_\theta = [\theta, 1-\theta]$. For a function $b : (0,1) \to (0, +\infty)$, we denote

$$\sigma_b^\theta = \int_\theta^{1-\theta} b(t) K(t)\, dt.$$

Next, define the cone

$$\Omega = \big\{u \in E :\ u(t) \geq 0 \text{ on } [0,1],\ u(t) \geq \lambda^* t^{\alpha-1}\|u\|\big\},$$

and for $r > 0$, let

$$\Omega_r = \big\{u \in \Omega :\ \|x\| < r\big\}.$$

In the rest of the paper, we suppose that the following assumptions hold:

(H$_1$) $q : (0,1) \to [0, +\infty)$ and $0 < \sigma < \infty$, where $\sigma = \int\limits_0^1 q(t) K_1(t)\, dt$.

(H$_2$) $a \in \mathcal{C}((0,1), [0 + \infty))$ and $0 < \sigma_a^0 < \infty$.

(H$_3$) $f \in \mathcal{C}([0,1] \times [0, +\infty), [0, +\infty))$.

(H$_4$) There exists $t_0 \in (0,1)$ such that $f(t_0, u) > 0$ for each $u \in (0, +\infty)$.

Remark. We note that (H_1) implies $0 < \sigma_q^0 < \infty$.

In this work we are concerned with a positive solution of problem (1.1). By a positive solution we mean a function $u \in \mathcal{C}([0,1])$ satisfying (1.1) with $u(t) \geq 0$ for all $t \in [0,1]$ and $u(t) > 0$ for all $t \in (0,1]$.

Now, we introduce the following intermediary boundary value problem

$$\begin{cases} D^\alpha x(t) + \mu a(t) f\big(t, [x(t) - w(t)]^*\big) + q(t) = 0 \text{ in } (0,1), \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \quad x(1) = \lambda \int\limits_0^1 x(s)\,ds, \end{cases} \tag{2.8}$$

$$\begin{cases} D^\alpha w(t) + 2q(t) = 0 \text{ in } (0,1), \\ w(0) = w'(0) = \cdots = w^{(n-2)}(0) = 0, \quad w(1) = \lambda \int\limits_0^1 w(s)\,ds, \end{cases} \tag{2.9}$$

where $[x(t) - w(t)]^* = \max\{x(t) - w(t), 0\}$ for each $t \in [0,1]$ and w is the unique solution of problem (2.9) given by $w(t) = 2\int\limits_0^1 G(t,s)q(s)\,ds$.

By Lemma 2.7, the solution w of problem (2.9) satisfies

$$w(t) \le 2\eta\sigma t^{\alpha-1} \ \ \forall\, t \in [0,1]. \tag{2.10}$$

We shall prove that there exists a solution $x(t)$ for the boundary value problem (2.8) with $x(t) \ge w(t)$ for any $t \in [0,1]$ and $x(t) > w(t)$ for any $t \in (0,1)$. In this case, $x(t) - w(t)$ represents a positive solution of the boundary value problem (1.1).

Next, we define the operator $T : E \to E$ as follows:

$$Tx(t) = \int\limits_0^1 G(t,s)\Big(\mu a(s) f\big(s, [x(s) - w(s)]^*\big) + q(s)\Big)\,ds \ \ \forall\, t \in [0,1]. \tag{2.11}$$

Lemma 2.9. *Suppose that* (H_1)–(H_4) *hold. Then* $x \in \mathcal{C}([0,1])$ *is a solution of the boundary value problem* (2.8) *if and only if* $x \in \mathcal{C}([0,1])$ *is a solution of the integral equation*

$$x(t) = \int\limits_0^1 G(t,s)\Big(\mu a(s) f\big(s, [x(s) - w(s)]^*\big) + q(s)\Big)\,ds.$$

That is, x *is a fixed point of the operator* T *defined by* (2.11).

Proof. The proof is immediate from Lemma 2.4, so we omit it here. □

Lemma 2.10. *Suppose that* (H_1)–(H_4) *hold. Then* $T : \Omega \to \Omega$ *is completely continuous.*

Proof. Since G, f are nonnegative continuous functions, using (H_1), (H_2) we conclude that $T : \Omega \to E$ is continuous. Let $x \in \Omega$, then by Proposition 2.5(iv), for all $t \in [0,1]$, it follows that

$$\begin{aligned} Tx(t) &\ge \eta\lambda^* t^{\alpha-1} \int\limits_0^1 K(s)\Big(\mu a(s) f\big(s, [x(s) - w(s)]^*\big) + q(s)\Big)\,ds \\ &\ge \lambda^* t^{\alpha-1} \int\limits_0^1 G(\tau,s)\Big(\mu a(s) f\big(s, [x(s) - w(s)]^*\big) + q(s)\Big)\,ds \ \ \forall\, \tau \in [0,1]. \end{aligned}$$

So, for each $t \in [0,1]$, we have

$$\begin{aligned} Tx(t) &\ge \lambda^* t^{\alpha-1} \max_{\tau \in [0,1]} \bigg\{ \int\limits_0^1 G(\tau,s)\Big(\mu a(s) f\big(s, [x(s) - w(s)]^*\big) + q(s)\Big)\,ds \bigg\} \\ &= \lambda^* t^{\alpha-1} \|Tx\|. \end{aligned}$$

Then $T(\Omega) \subset \Omega$. Now, let S be a bounded set of Ω, then there exists a positive constant $M > 0$ such that $\|x\| \leq M$ for all $x \in S$. Therefore, $[x(s) - w(s)]^* \leq \|x\| \leq M$.

Let $M_1 := \max\big\{1, \max\limits_{t\in[0,1],\ x\in[0,M]} f(t,x)\big\}$.

From hypotheses (H_1), (H_2) and Proposition 2.5(ii), for all $t \in [0,1]$ and for all $x \in S$, we have

$$Tx(t) \leq \eta \int_0^1 K(s)\Big(\mu a(s) f\big(s, [x(s) - w(s)]^*\big) + q(s)\Big)\, ds \leq M_1 \eta(\mu\sigma_a^0 + \sigma_q^0).$$

So we obtain $\|Tx\| \leq M_1\eta(\mu\sigma_a^0 + \sigma_q^0)$. Hence, $T(S)$ is uniformly bounded.

Now, let us prove that $T(S)$ is equicontinuous on $[0,1]$.

Using Proposition 2.5, we obtain that G is uniformly continuous on $[0,1] \times [0,1]$. Then for $t_1, t_2 \in [0,1]$ and for all $s \in [0,1]$, we get

$$|G(t_2, s) - G(t_1, s)| \to 0 \ \text{ as } \ t_2 \to t_1$$

and

$$|G(t_2, s) - G(t_1, s)| \leq 2\eta M_1(a(s)K(s) + q(s)K(s)).$$

By (H_1) and (H_2), $2\eta M_1(a(s)K(s) + q(s)K(s))$ is a nonnegative integrable function on $(0,1)$. Thus by the Lebesgue control convergence theorem, we obtain

$$|Tx(t_2) - Tx(t_1)| \to 0 \ \text{ as } \ |t_2 - t_2| \to 0,$$

and so $T(S)$ is equicontinuous. Consequently, by Ascoli's theorem, we conclude that $T(S)$ is relatively compact in E. Hence, $T : \Omega \to \Omega$ is completely continuous. This completes the proof. $\square$

3 Main results

We shall give the existence results of positive solutions for the nonlinear boundary value problem (1.1).

Theorem 3.1. *Suppose that conditions* (H_1)–(H_4) *hold. In addition, suppose that there exists* $\theta \in (0, \frac{1}{2})$ *such that*

$$f_\infty := \lim_{x\to\infty} \Big\{\min_{t\in J_\theta} \frac{f(t,x)}{x}\Big\} = \infty.$$

Then there exists $\mu^* > 0$ *such that for every* $0 < \mu < \mu^*$, *problem* (1.1) *has at least one positive solution.*

Proof. Choose

$$r > \frac{2\eta\sigma}{\lambda^*}\,.$$

Define $\mu^* = \frac{r - 2\eta\sigma_q^0}{M\eta\sigma_a^0}$, where $M = \max\limits_{t\in[0,1],\ x\in[0,r]} f(t,x)$, and let $0 < \mu < \mu^*$.

Then for each $x \in \partial\Omega_r$ and $s \in [0,1]$, we have

$$[x(s) - w(s)]^* \leq x(s) \leq \|x\| = r.$$

Therefore, by Proposition 2.5(ii), for any $x \in \partial\Omega_r$, we have

$$T(x)(t) \leq \eta\mu \int_0^1 K(s)a(s)f\big(s, [x(s) - w(s)]^*\big)\, ds + 2\eta\sigma_q^0 \leq \mu\eta M\sigma_a^0 + 2\eta\sigma_q^0 \leq \mu^*\eta M\sigma_a^0 + 2\eta\sigma_q^0 = r.$$

So we get

$$\|Tx\| \leq \|x\| \ \text{ for } \ x \in \partial\Omega_r. \tag{3.1}$$

Now, if the condition $f_\infty = \infty$ holds, then for $A = \frac{2}{\mu\gamma\lambda^*\sigma_a^\theta\theta^{\alpha-1}}$, there exists $B > 0$ such that $f(t,x) \geq Ax \; \forall\, t \in J_\theta, \forall\, x \geq B$.

Define $R = \max\{2r, \frac{2B}{\lambda^*\theta^{\alpha-1}}\}$. Then, using (2.10), for any $x \in \partial\Omega_R$ and $t \in [0,1]$, we obtain

$$x(t) - w(t) \geq x(t) - 2\eta\sigma t^{\alpha-1} \geq x(t) - 2\eta\sigma\,\frac{x(t)}{\|x\|} \geq x(t)\Big(1 - \frac{2\eta\sigma}{\lambda^* R}\Big) \geq \frac{1}{2}\,x(t) \geq 0.$$

Therefore, we conclude that for all $t \in J_\theta$,

$$[x(t) - w(t)]^* \geq \frac{\lambda^*}{2}\,Rt^{\alpha-1} \geq \frac{\lambda^*}{2}\,R\theta^{\alpha-1} \geq B,$$

and so for any $x \in \partial\Omega_R$ and $t \in J_\theta$, we have

$$f\big(t, [x(t) - w(t)]^*\big) \geq A[x(t) - w(t)]^* \geq \frac{A}{2}\,x(t). \tag{3.2}$$

By (3.2) and Proposition 2.5(v), it follows that for any $x \in \partial\Omega_R$ and $t \in J_\theta$,

$$Tx(t) \geq \mu\gamma \int\limits_\theta^{1-\theta} K(s)a(s)f\big(s, [x(s) - w(s)]^*\big)\,ds \geq \frac{\mu\gamma\lambda^*}{2}\,\sigma_a^\theta\theta^{\alpha-1}AR = R.$$

Then we have

$$\|Tx\| \geq \|x\| \;\; \forall x \in \partial\Omega_R. \tag{3.3}$$

Thus, using (3.1) and (3.3), we deduce by Lemma 2.8 that the operator T has a fixed point in $\overline{\Omega_R}\setminus\Omega_r$. Therefore, by Lemma 2.9, x is a nonnegative continuous solution of problem (2.8) satisfying

$$r < \|x\| \leq R. \tag{3.4}$$

So we deduce that $x - w$ is a nonnegative continuous solution of problem (1.1).

Now, let us prove that $x - w$ is a positive solution of (1.1), that is, $x(t) - w(t) > 0$ for all $t \in (0,1]$. Since x satisfies (3.4), using (2.10) we obtain

$$x(t) - w(t) \geq t^{\alpha-1}(\lambda^* r - 2\eta\sigma) > 0 \;\; \forall\, t \in (0,1].$$

Hence, $x - w$ is a positive solution of problem (1.1). This completes the proof. □

Theorem 3.2. *Suppose that conditions* (H$_1$)–(H$_4$) *hold. In addition, assume that the following assertions hold:*

(A$_1$) *there exits $\theta \in (0, \frac{1}{2})$ such that $f_\infty^* := \lim\limits_{x\to\infty} \big\{ \min\limits_{t\in J_\theta} f(t,x)\big\} = \infty$;*

(A$_2$) $f^\infty := \lim\limits_{x\to\infty} \big\{ \max\limits_{t\in[0,1]} \frac{f(t,x)}{x}\big\} = 0$.

Then there exists $\mu^* > 0$ such that problem (1.1) *has at least one positive solution for every $\mu > \mu^*$.*

Proof. First, suppose that (A$_1$) holds, then there exists $R_0 > 0$ such that

$$f(t,x) \geq \frac{f_\infty^*}{2} \;\; \forall\, t \in J_\theta, \;\; \forall\, x \geq R_0.$$

Now, fix $R_1 > \max\big\{\frac{2R_0}{\lambda^*\theta^{\alpha-1}}, \frac{4\eta\sigma}{\lambda^*}\big\}$. Define $\mu^* = \frac{2R_1}{\gamma\sigma_a^\theta f_\infty^*} > 0$ and let $\mu > \mu^*$. Then, for each $x \in \partial\Omega_{R_1}$ and $t \in [0,1]$, we have

$$x(t) - w(t) \geq x(t) - 2\eta\sigma t^{\alpha-1} \geq x(t) - \frac{2\eta}{\lambda^*}\,\sigma\,\frac{x(t)}{\|x\|} \geq x(t)\Big(1 - \frac{2\eta\sigma}{\lambda^* R_1}\Big) \geq \frac{1}{2}\,x(t) \geq 0.$$

So, for $x \in \partial\Omega_{R_1}$ and $t \in J_\theta$, we get

$$[x(t) - w(t)]^* \geq \frac{1}{2}\, x(t) \geq \frac{1}{2}\, \lambda^* \theta^{\alpha-1} R_1 > R_0.$$

Then for any $x \in \partial\Omega_{R_1}$ and $t \in J_\theta$, we obtain

$$f\big(t, [x(t) - w(t)]^*\big) \geq \frac{f_\infty^*}{2}\,.$$

It follows that for any $x \in \partial\Omega_{R_1}$ and $t \in J_\theta$,

$$Tx(t) \geq \mu\gamma \int\limits_{\theta}^{1-\theta} K(s)a(s) f\big(s, [x(s) - w(s)]^*\big)\, ds \geq \mu\gamma\, \frac{f_\infty^*}{2} \int\limits_{\theta}^{1-\theta} K(s)a(s)\, ds \geq \mu^* \gamma\, \frac{f_\infty^*}{2}\, \sigma_a^\theta = R_1.$$

Thus

$$\|Tx\| \geq \|x\| \;\; \forall\, x \in \partial\Omega_{R_1}.$$

On the other hand, since $f^\infty = 0$, for $\varepsilon = \frac{1}{\mu\eta\sigma_a^0} > 0$, there exists $B > 0$ such that for each $t \in [0,1]$, $x \geq B$, we have $f(t,x) \leq \varepsilon x$. Therefore, we obtain

$$f(t,x) \leq M + \varepsilon x \;\; \forall\, t \in [0,1], \;\; \forall\, x \geq 0,$$

where $M = \max\limits_{t\in[0,1],\; x\in[0,B]} f(t,x)$. Let $M_1 = \max\{1, M\}$ and choose

$$R_2 > \max\left\{2R_1, \mu\eta\sigma_a^0 M_1 \Big(\frac{1}{2} - \mu\sigma_a^0\eta\varepsilon\Big)^{-1}, 2\eta M_1 \sigma_q^0\right\}.$$

It follows that for any $x \in \partial\Omega_{R_2}$ and $t \in [0,1]$,

$$\begin{aligned} Tx(t) &\leq \mu\eta \int\limits_0^1 K(s)a(s) f\big(s, [x(s) - w(s)]^*\big)\, ds + \eta\sigma_q^0 \\ &\leq \mu\eta M \sigma_a^0 + \mu\eta\varepsilon \int\limits_0^1 K(s)a(s)[x(s) - x(s)]^*\, ds + \eta\sigma_q^0 \leq \mu\eta M_1 \sigma_a^0 + \mu\eta\sigma_a^0 \varepsilon R_2 + \eta M_1 \sigma_q^0 \\ &\leq R_2 \Big(\frac{1}{2} - \mu\sigma_a^0 \eta\varepsilon\Big) + \mu\eta\sigma_a^0 \varepsilon R_2 + \eta M_1 \sigma_q^0 = \|x\|. \end{aligned}$$

So, we get

$$\|Tx\| \leq \|x\| \;\; \forall\, x \in \partial\Omega_{R_2}.$$

Thus, by Lemma 2.8, we deduce that the operator T has a fixed point in $\overline{\Omega_{R_2}} \setminus \Omega_{R_1}$. Therefore, by Lemma 2.9, x is a solution of problem (2.8). Thus, we deduce that $x - w$ is a nonnegative solution of problem (1.1).

The positivity of the solution is shown as in the proof of the previous theorem. □

Now we state the multiple existence result.

Theorem 3.3. *Assume that $\mu = 1$ and (H_1)–(H_4) hold. In addition, suppose that the following conditions are satisfied:*

(A_1) *there exists $R_1 > \frac{4\eta\sigma}{\lambda^*}$ such that $f(t,x) \leq \frac{R_1 - \eta\sigma_q^0}{\eta\sigma_a^0}$ $\forall\, t \in [0,1]$, $x \in [0, R_1]$;*

(A_2) *there exists $\theta \in (0, \frac{1}{2})$ such that the following assertion holds: $\exists\, R_2 > 2R_1 :\;\; \gamma\sigma_a^\theta f(t,x) \geq R_2$ $\forall\, t \in J_\theta$, $\forall\, x \in [\frac{3}{4}\lambda^* \theta^{\alpha-1} R_2, R_2]$;*

(A_3) $f^\infty = \lim\limits_{x\to\infty} \big\{ \max\limits_{t\in[0,1]} \frac{f(t,x)}{x} \big\} = 0.$

Then problem (1.1) *has two positive solutions.*

Proof. First, suppose that condition (A_1) holds, then for each $x \in \partial\Omega_{R_1}$ and $t \in [0,1]$, we have

$$[x(s)-w(s)]^* \le x(s) \le R_1 \ \text{ and } \ [x(s)-w(s)]^* \ge \frac{1}{2}\,x(s) \ge 0.$$

So, for each $x \in \partial\Omega_{R_1}$ and $t \in [0,1]$,

$$f\big(t,[x(t)-w(t)]^*\big) \le \frac{R_1-\eta\sigma_q^0}{\eta\sigma_a^0}\,.$$

Therefore, for any $x \in \partial\Omega_{R_1}$ and $t \in [0,1]$, we get

$$Tx(t) \le \eta\int_0^1 K(s)a(s)f\big(s,[x(s)-w(s)]^*\big)\,ds + \eta\sigma_q^0 \le \eta\sigma_a^0\Big(\frac{R_1-\eta\sigma_q^0}{\eta\sigma_a^0}\Big) + \eta\sigma_q^0 = \|x\|.$$

Thus, we have

$$\|Tx\| \le \|x\| \ \ \forall\, x \in \partial\Omega_{R_1}. \tag{3.5}$$

On the other hand, if (A_2) holds, it follows that for $R_2 > 2R_1$ and $x \in \partial\Omega_{R_2}$, $t \in [0,1]$,

$$x(t)-w(t) \ge \lambda^* t^{\alpha-1}R_2 - 2\eta\sigma t^{\alpha-1} \ge \lambda^* t^{\alpha-1}R_2 - \frac{1}{2}\,\lambda^* t^{\alpha-1}R_1 \ge \frac{3\lambda^*}{4}\,t^{\alpha-1}R_2.$$

Thus, for all $x \in \partial\Omega_{R_2}$ and $t \in J_\theta$, we have

$$x(t)-w(t) \ge \frac{3}{4}\,\lambda^*\theta^{\alpha-1}R_2.$$

Therefore, for all $x \in \partial\Omega_{R_2}$ and $t \in J_\theta$, we get

$$\gamma\sigma_a^\theta f\big(s,[x(s)-w(s)]\big) \ge R_2.$$

So, for any $x \in \partial\Omega_{R_2}$ and $t \in J_\theta$, we obtain

$$Tx(t) \ge \gamma\int_\theta^{1-\theta} K(s)a(s)f\big(s,[x(s)-w(s)]^*\big)\,ds \ge \gamma\sigma_a^\theta\,\frac{R_2}{\gamma\sigma_a^\theta} = R_2.$$

Thus,

$$\|Tx\| \ge \|x\| \ \ \forall\, x \in \partial\Omega_{R_2}. \tag{3.6}$$

Now, hypothesis (A_3) implies that for $\varepsilon = \frac{1}{\eta\sigma_a^0}$, there exists $B > 0$ such that $f(t,x) \le \varepsilon x \ \forall\, x \ge B$. Therefore, we obtain

$$f(t,x) \le M + \varepsilon x \ \ \forall\, t \in [0,1], \ \ x \ge 0,$$

where $M = \max\limits_{t\in[0,1],\ x\in[0,B]} f(t,x)$. Put $M_1 = \max\{1, M\}$ and choose

$$R_3 > \max\Big\{2R_2, \eta\sigma_a^0 M_1\Big(\frac{1}{2}-\sigma_a^0\eta\varepsilon\Big)^{-1}, 2\eta M_1\sigma_q^0\Big\}.$$

Then for any $x \in \partial\Omega_{R_3}$ and $t \in [0,1]$, we have

$$\begin{aligned}
Tx(t) &\le \eta\int_0^1 K(s)a(s)f\big(s,[x(s)-w(s)]^*\big)\,ds + \eta\sigma_q^0\\
&\le \eta M\sigma_a^0 + \mu\eta\varepsilon\int_0^1 K(s)a(s)[x(s)-x(s)]^*\,ds + \eta\sigma_q^0 \le \eta M_1\sigma_a^0 + \eta\sigma_a^0\varepsilon R_3 + \eta M_1\sigma_q^0\\
&\le R_3\Big(\frac{1}{2}-\sigma_a^0\eta\varepsilon\Big) + \eta\sigma_a^0\varepsilon R_3 + \eta M_1\sigma_q^0 = \|x\|.
\end{aligned}$$

So, we get

$$\|Tx\| \le \|x\| \ \ \forall x \in \partial\Omega_{R_3}. \tag{3.7}$$

Therefore, due to Lemma 2.8 and using (3.5), (3.6) and (3.7), we deduce that the operator T has two fixed points x_1 and x_2, respectively, in $\overline{\Omega_{R_2}} \setminus \Omega_{R_1}$ and $\overline{\Omega_{R_3}} \setminus \Omega_{R_2}$. Therefore, by Lemma 2.9, problem (2.8) admits two nonnegative solutions $R_1 < \|x_1\| < R_2 < \|x_2\| < R_3$. Thus, problem (1.1) has two nonnegative solutions. The positivity of the solutions is shown in the same manner as in proving Theorem 3.1. □

4 Examples

In this section, we present some examples illustrating our results. We remark that by the following examples it can immediately be verified that conditions (H_1)–(H_4) hold.

Example 4.1. We consider the following nonlinear fractional differential equations

$$\begin{cases} D^{\frac{5}{2}}u(t) + \mu \,\dfrac{1}{t}\,(u(t))^2 - \dfrac{1}{1-t} = 0 \ \text{ in } \ (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = \displaystyle\int_0^1 u(s)\,ds. \end{cases} \tag{4.1}$$

Let $f(t,u) = u^2$, $a(t) = \frac{1}{t}$, $\lambda = 1$ and $q(t) = \frac{1}{1-t}$. By a direct calculation, we obtain $f_\infty = \infty$ for any $\theta \in (0, \frac{1}{2})$. We also get $\sigma_a^0 \approx 0.3009$, $\sigma_q^0 \approx 0.2006$ and $\sigma = 0.5015$. Choose $r = 5$, then by a simple calculation we get $\mu^* = 0.34547$. Then by Theorem 3.1, problem (4.1) has at least one positive solution for every $0 < \mu < 0.34547$.

Example 4.2. Consider the following boundary value problem

$$\begin{cases} D^{\frac{7}{3}}u(t) + \mu \,\dfrac{1}{t}\left(100 + \dfrac{1}{1+\sqrt{u}}\right) - \dfrac{1}{1-t} = 0 \ \text{ in } \ (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = \displaystyle\int_0^1 u(s)\,ds, \end{cases} \tag{4.2}$$

Let $f(t,u) = 100 + \frac{1}{\sqrt{u}+1}$, $a(t) = \frac{1}{t}$ and $q(t) = \frac{1}{1-t}$. By a direct calculation, we obtain $f^\infty = 0$ and for $\theta = \frac{1}{4}$ we have $f_\infty^* = 100$. We also obtain $\sigma_a^0 \approx 0.35995$, $\sigma_q^0 \approx 0.26996$, $\sigma \approx 0.62991$ and $\sigma_a^\theta \approx 0.16979$. Choose $R_1 = 50$ and $R_2 = 102$. A simple calculation yields $\mu^* = 39.889$. So Theorem 3.2 ensures the existence of a solution of problem (4.2) such that $50 < \|u + w\| < 102$ for every $\mu > 39.889$.

Example 4.3. Consider the following boundary value problem:

$$\begin{cases} D^{\frac{7}{3}}u(t) + \mu \,\dfrac{1}{t}\, f(t,u) - \dfrac{1}{1-t} = 0 \ \text{ in } \ (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = \displaystyle\int_0^1 u(s)\,ds, \end{cases} \tag{4.3}$$

where

$$f(t,u) = \begin{cases} \dfrac{1}{3}\,u, & 0 \le u \le 12, \\ 10000u - 119996, & 12 < u \le 13.78, \\ u + 17790.3, & 13.78 < u \le 50, \\ 2523u^{\frac{1}{2}}, & u > 50. \end{cases}$$

Then problem (4.3) admits two positive solutions. In fact, let $a(t) = \frac{1}{t}$ and $q(t) = \frac{1}{1-t}$. By a direct calculation, we get $\sigma_a^0 \approx 0.35995$, $\sigma_q^0 \approx 0.26996$ and $\sigma \approx 0.62991$. Choose $R_1 = 12 > \frac{4\eta\sigma}{\lambda^*}$, then for any $t \in [0,1]$, $u \in [0,12]$, $f(t,u) \leq \frac{R_1 - \eta\sigma_q^0}{\eta\sigma_a^0} \approx 18.3$. Thus condition (A_1) is satisfied. On the other hand, for $\theta = \frac{1}{4}$, we have $\sigma_a^\theta \approx 0.16979$. Take $R_2 = 50$, then $R_2 > 2R_1$ and for any $t \in J_\theta$ and for all $u \in [\frac{3}{4}\lambda^*\theta^{\alpha-1}R_2, R_2]$, we have $f(t,u) \geq \frac{R_2}{\gamma\sigma_a^\theta} \approx 1994.5$ which implies that condition (A_2) is satisfied.

Finally, since $f^\infty = 0$, the assertion (A_3) is satisfied. Consequently, by Theorem 3.3, problem (4.3) admits two positive solutions u_1 and u_2 satisfying

$$R_1 \leq \|u_1 + w\| \leq R_2 \leq \|u_2 + w\| \leq R_3.$$

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ON BEHAVIOR OF OSCILLATING SOLUTIONS TO SECOND-ORDER EMDEN–FOWLER TYPE DIFFERENTIAL EQUATIONS

Abstract. The second-order Emden–Fowler type differential equation with positive bounded potential is considered. Asymptotic behavior of maximally extended oscillating solutions to the equation is described.[1]

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1 Introduction

Consider the second-order Emden–Fowler type differential equation

$$y'' + p(x, y, y')|y|^k \operatorname{sgn} y = 0, \quad k > 0, \quad k \neq 1, \tag{1.1}$$

with continuous in x and Lipschitz continuous in u, v positive function $p(x, u, v)$ defined on $\mathbb{R} \times \mathbb{R}^2$. The asymptotic behavior of all solutions to equation (1.1) in the case $p = p(x)$ was described by I. T. Kiguradze and T. A. Chanturia (see [11]). The results on asymptotic classification of maximally extended solutions to third- and fourth-order similar differential equations for $k > 0$, $k \neq 1$, were given by I. V. Astashova (see [1–5]). The asymptotic classification of solutions to equation (1.1) with negative function $p(x, u, v)$ for regular ($k > 1$) and singular ($0 < k < 1$) nonlinearities is contained in [6, 7].

Using the methods described in [2], we investigate the behavior of solutions to equation (1.1) in the case $p(x, u, v) > 0$ (see [8]). Further, suppose that the function $p(x, u, v)$ additionally satisfies the inequalities

$$0 < m \leq p(x, u, v) \leq M < +\infty. \tag{1.2}$$

2 Oscillation of solutions and their first derivatives

Consider the trajectories $\{(y(x), y'(x))\} \subset \mathbb{R}^2$ generated by nontrivial solutions to equation (1.1). Divide $\mathbb{R}^2$ by four closed sets crossing over the boundaries only

$$\begin{bmatrix} + \\ + \end{bmatrix}, \quad \begin{bmatrix} + \\ - \end{bmatrix}, \quad \begin{bmatrix} - \\ - \end{bmatrix}, \quad \begin{bmatrix} - \\ + \end{bmatrix}. \tag{2.1}$$

For the sets boundaries we use the following notation:

$$\begin{bmatrix} + \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ - \end{bmatrix}, \quad \begin{bmatrix} - \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ + \end{bmatrix}.$$

For example,

$$\begin{bmatrix} + \\ - \end{bmatrix} = \big\{(y_0, y_1) \in \mathbb{R}^2 : \ y_0 \geq 0, \ y_1 \leq 0\big\},$$

$$\begin{bmatrix} 0 \\ + \end{bmatrix} = \big\{(y_0, y_1) \in \mathbb{R}^2 : \ y_0 = 0, \ y_1 \geq 0\big\}.$$

Lemma 2.1. *Suppose $k \in (0, 1) \cup (1, +\infty)$, the function $p(x, u, v)$ is continuous in x, Lipschitz continuous in u, v, satisfies inequalities (1.2) and $y(x)$ is a nontrivial maximally extended solution to equation (1.1). Then neither $y(x)$ nor its first derivative $y'(x)$ can be constant-sign functions in the neighborhood of domain boundaries.*

Proof. Using the substitutions $x \mapsto -x$, $y(x) \mapsto -y(x)$, we obtain an equation of the same type as (1.1). That is why we further investigate behavior of nontrivial solutions to equation (1.1) and their first derivatives near the right-side boundary of the domain only.

Prove the statement for solution $y(x)$, the proof for its first derivative $y'(x)$ is similar. Assume that a solution $y(x)$ to equation (1.1) is defined on a finite or on an infinite interval (a, b) and is positive in some neighborhood of b. According to the type of equation (1.1), the second derivative is negative in this neighborhood, therefore the first derivative decreases monotonously and has a finite or an infinite limit as $x \to b - 0$. It means that the first derivative is a constant-sign function in the neighborhood of b. That is why $y(x)$ is monotonous in the neighborhood of b and tends to a finite or an infinite value as $x \to b - 0$.

Let $b < +\infty$. If a solution $y(x)$ (and hence $y''(x)$) or its first derivative has a finite limit, then integrating the second derivative or the first derivative, respectively, on a finite interval, we obtain the finite limits in both cases. So, we get a contradiction with the right-maximally extension of a solution. If solution and first derivative limits are infinite, then they must have the same sign. So, we get a contradiction with equation (1.1).

Let $b = +\infty$. If a solution $y(x)$ (and hence $y''(x)$) or its first derivative has a nontrivial limit, then integrating the second derivative or the first derivative, respectively, on the whole domain, we obtain infinite limits in both cases. Thus, they must be of the same sign, and therefore we get a contradiction with equation (1.1). If solution and first derivative limits are equal to zero, then the solution is positive in a neighborhood of $+\infty$, monotonously decreases to zero and its first derivative is negative and monotonously increases to zero as $x \to +\infty$. It means that the second derivative (as the solution is positive) decreases to zero at infinity. So, we get a contradiction with equation (1.1). $\square$

Theorem 2.1. *Suppose $k \in (0,1) \cup (1,+\infty)$, the function $p(x,u,v)$ is continuous in x, Lipschitz continuous in u, v and satisfies inequalities* (1.2). *Then all nontrivial maximally extended solutions and their first derivatives to equation* (1.1) *are oscillating at the left– and right-hand sides, zeroes x_j of solutions and zeroes x'_j of their first derivatives alternate, i.e.,*

$$\cdots < x_{j-1} < x'_j < x_j < x'_{j+1} < \cdots, \quad j \in \mathbb{Z}.$$

Moreover, for any $j \in \mathbb{Z}$, the following inequalities hold:

$$-\sqrt{\frac{M}{m}} \le \frac{y'(x_{j+1})}{y'(x_j)} \le -\sqrt{\frac{m}{M}}, \quad -\Big(\frac{M}{m}\Big)^{\frac{1}{k+1}} \le \frac{y(x'_{j+1})}{y(x'_j)} \le -\Big(\frac{m}{M}\Big)^{\frac{1}{k+1}}.$$

Proof. As mentioned above, it suffices to investigate the asymptotic behavior of nontrivial maximally extended solutions at the right-hand side.

Prove that a trajectory generated by any nontrivial maximally extended solution $y(x)$ to equation (1.1) moves between the introduced sets (2.1) at the right-hand side only by the following scheme:

$$\begin{array}{ccc} \begin{bmatrix} + \\ + \end{bmatrix} & \longrightarrow & \begin{bmatrix} + \\ - \end{bmatrix} \\ \uparrow & & \downarrow \\ \begin{bmatrix} - \\ + \end{bmatrix} & \longleftarrow & \begin{bmatrix} - \\ - \end{bmatrix} \end{array}. \tag{2.2}$$

Indeed, suppose that $(y(x), y'(x))$ is an internal point for the set $\begin{bmatrix} + \\ + \end{bmatrix}$ at some moment. It means that $y(x) > 0$, $y'(x) > 0$ and $y''(x) < 0$. Therefore, $y(x)$ is positive and increases, $y'(x)$ is positive and decreases, while the trajectory generated by the solution $y(x)$ is located in the interior of $\begin{bmatrix} + \\ + \end{bmatrix}$. Then either $y'(x)$ is equal to zero and the corresponding trajectory will get to the boundary $\begin{bmatrix} + \\ 0 \end{bmatrix}$ of $\begin{bmatrix} + \\ + \end{bmatrix}$ or $y'(x)$ is nontrivial and have a nonnegative limit at the right-hand side, i.e., the first derivative will be a constant-sign function. So, we get a contradiction with Lemma 2.1. Thus, the case is possible if and only if the trajectory generated by the solution $y(x)$ gets to the boundary $\begin{bmatrix} + \\ 0 \end{bmatrix}$, i.e., the solution $y(x)$ is positive and has a local extremum at some point x'_0, moreover, $y''(x'_0) < 0$. Then there exists a constant $\delta > 0$ such that $y(x) > 0$, $y'(x) < 0$ for $x \in (x'_0, x'_0 + \delta)$. So, the trajectory will get to the interior of the set $\begin{bmatrix} + \\ - \end{bmatrix}$.

Further, we have $y(x) > 0$, $y'(x) < 0$ and $y''(x) < 0$. Therefore, $y(x)$ is positive and decreases, $y'(x)$ is positive and increases, while the corresponding trajectory is located in the interior of $\begin{bmatrix} + \\ - \end{bmatrix}$. According to Lemma 2.1, the solution $y(x)$ cannot be positive at the right-hand side, that is why it will be equal to zero at some point $x_0 > x'_0$, and the trajectory generated by this solution will get to

$\begin{bmatrix} 0 \\ - \end{bmatrix}$. As $y'(x_0) < 0$, there exists a constant $\widetilde{\delta} > 0$ such that $y(x) < 0$, $y'(x) < 0$ for $x \in (x_0, x_0 + \widetilde{\delta})$. Thus, the trajectory will get to the interior of the set $\begin{bmatrix} - \\ - \end{bmatrix}$.

Now, we have $y(x) < 0$ and $y'(x) < 0$. Similarly, prove that the trajectory generated by $y(x)$ at the right-hand side gets to the boundary $\begin{bmatrix} - \\ 0 \end{bmatrix}$, i.e., $y(x)$ has a local minimum at some point $x_1' > x_0 > x_0'$. It moves further towards the interior of the set $\begin{bmatrix} - \\ + \end{bmatrix}$, and according to Lemma 2.1, tends to the boundary $\begin{bmatrix} 0 \\ + \end{bmatrix}$ for $x_1 > x_1' > x_0 > x_0'$. Thereafter the trajectory goes to the interior of the set $\begin{bmatrix} + \\ + \end{bmatrix}$.

So, we have proved that the trajectory generated by any nontrivial maximally extended solution $y(x)$ to equation (1.1) can move between the introduced sets (2.1) at the right-hand side only by the scheme (2.2).

Besides, according to Lemma 2.1, it cannot stay in any set (2.1) at the left- and right-hand sides. Therefore, the solution $y(x)$ to equation (1.1) and its first derivative $y'(x)$ are oscillating at the left- and right-hand sides, zeroes x_j of solutions and zeroes x_j' of their first derivatives alternate, i.e.,

$$\cdots < x_{j-1} < x_j' < x_j < x_{j+1}' < \cdots, \quad j \in \mathbb{Z}.$$

Further, without any restrictions, we assume $y'(x_j) < 0$. Note

$$0 = |y(x_j)|^{k+1} - |y(x_{j+1})|^{k+1} = -(k+1) \int\limits_{y(x_j)}^{y(x_{j+1})} |y|^{k-1} y \, dy,$$

and from equation (1.1) we have

$$\begin{aligned} 0 = -(k+1) \int\limits_{y(x_j)}^{y(x_{j+1})} |y|^{k-1} y \, dy &= (k+1) \int\limits_{x_j}^{x_{j+1}} \frac{y''\, y'}{p(x, y, y')} \, dx \\ &= (k+1) \int\limits_{x_j}^{x_{j+1}'} \frac{y''\, y'}{p(x, y, y')} \, dx + (k+1) \int\limits_{x_{j+1}'}^{x_{j+1}} \frac{y''\, y'}{p(x, y, y')} \, dx. \end{aligned} \tag{2.3}$$

As $y'(x_j) < 0$, we have $y'(x) < 0$ and $y''(x) > 0$ for $x \in (x_j, x_{j+1}')$. Also, for $x \in (x_{j+1}', x_{j+1})$, we have $y'(x) > 0$ and $y''(x) > 0$. So, $\frac{y''\, y'}{p(x,y,y')} < 0$ for $x \in (x_j, x_{j+1}')$ and $\frac{y''\, y'}{p(x,y,y')} > 0$ for $x \in (x_{j+1}', x_{j+1})$.

Estimate expression (2.3):

$$\begin{aligned} &(k+1) \int\limits_{x_j}^{x_{j+1}'} \frac{y''\, y'}{p(x, y, y')} \, dx + (k+1) \int\limits_{x_{j+1}'}^{x_{j+1}} \frac{y''\, y'}{p(x, y, y')} \, dx \\ &\le \frac{k+1}{M} \int\limits_{x_j}^{x_{j+1}'} y''\, y' \, dx + \frac{k+1}{m} \int\limits_{x_{j+1}'}^{x_{j+1}} y''\, y' \, dx = \frac{k+1}{M} \int\limits_{y'(x_j)}^{y'(x_{j+1}')} y' \, dy' + \frac{k+1}{m} \int\limits_{y'(x_{j+1}')}^{y'(x_{j+1})} y' \, dy' \\ &\qquad = \frac{k+1}{2M} \, (y')^2 \Big|_{x_j}^{x_{j+1}'} + \frac{k+1}{2m} \, (y')^2 \Big|_{x_{j+1}'}^{x_{j+1}} = -\frac{k+1}{2M} \, (y'(x_j))^2 + \frac{k+1}{2m} \, (y'(x_{j+1}))^2, \end{aligned}$$

whence

$$\frac{k+1}{2M} \, (y'(x_j))^2 \le \frac{k+1}{2m} \, (y'(x_{j+1}))^2.$$

Obtain another estimate for (2.3):

$$(k+1)\int\limits_{x_j}^{x'_{j+1}} \frac{y''\,y'}{p(x,y,y')}\,dx + (k+1)\int\limits_{x'_{j+1}}^{x_{j+1}} \frac{y''\,y'}{p(x,y,y')}\,dx$$

$$\geq \frac{k+1}{m}\int\limits_{x_j}^{x'_{j+1}} y''\,y'\,dx + \frac{k+1}{M}\int\limits_{x'_{j+1}}^{x_{j+1}} y''\,y'\,dx = \frac{k+1}{m}\int\limits_{y'(x_j)}^{y'(x'_{j+1})} y'\,dy' + \frac{k+1}{M}\int\limits_{y'(x'_{j+1})}^{y'(x_{j+1})} y'\,dy'$$

$$= \frac{k+1}{2m}\,(y')^2\Big|_{x_j}^{x'_{j+1}} + \frac{k+1}{2M}\,(y')^2\Big|_{x'_{j+1}}^{x_{j+1}} = -\frac{k+1}{2m}\,(y'(x_j))^2 + \frac{k+1}{2M}\,(y'(x_{j+1}))^2,$$

whence

$$\frac{k+1}{2m}\,(y'(x_j))^2 \geq \frac{k+1}{2M}\,(y'(x_{j+1}))^2.$$

Therefore,

$$\sqrt{\frac{m}{M}}\,|y'(x_j)| \leq |y'(x_{j+1})| \leq \sqrt{\frac{M}{m}}\,|y'(x_j)| \tag{2.4}$$

and

$$\sqrt{\frac{m}{M}} \leq \Big|\frac{y'(x_{j+1})}{y'(x_j)}\Big| \leq \sqrt{\frac{M}{m}}\,.$$

Since zeroes x_j and extremum points x'_j of a nontrivial maximally extended solution to equation (1.1) alternate, for any $j \in \mathbb{Z}$ we have $y'(x_{j+1})\,y'(x_j) < 0$ and

$$-\sqrt{\frac{M}{m}} \leq \frac{y'(x_{j+1})}{y'(x_j)} \leq -\sqrt{\frac{m}{M}}\,.$$

Obtain the second estimate. We have $y(x'_j) > 0$. Note

$$|y(x'_j)|^{k+1} = |y(x'_j)|^{k+1} - |y(x_j)|^{k+1} = -(k+1)\int\limits_{y(x'_j)}^{y(x_j)} |y|^{k-1}y\,dy = (k+1)\int\limits_{x'_j}^{x_j} \frac{y''\,y'}{p(x,y,y')}\,dx.$$

As $y(x'_j) > 0$, we have $y'(x) < 0$ and $y''(x) < 0$ for $x \in (x'_j, x_j)$, i.e., $\frac{y''\,y'}{p(x,y,y')} > 0$ for $x \in (x'_j, x_j)$. So,

$$\frac{k+1}{M}\int\limits_{x'_j}^{x_j} y''\,y'\,dx \leq (k+1)\int\limits_{x'_j}^{x_j} \frac{y''\,y'}{p(x,y,y')}\,dx \leq \frac{k+1}{m}\int\limits_{x'_j}^{x_j} y''\,y'\,dx,$$

then

$$\frac{k+1}{M}\int\limits_{y'(x'_j)}^{y'(x_j)} y'\,dy' \leq |y(x'_j)|^{k+1} \leq \frac{k+1}{m}\int\limits_{y'(x'_j)}^{y'(x_j)} y'\,dy'$$

and

$$\frac{k+1}{2M}\,(y'(x_j))^2 \leq |y(x'_j)|^{k+1} \leq \frac{k+1}{2m}\,(y'(x_j))^2. \tag{2.5}$$

Analogously, on the interval (x_j, x'_{j+1}) we obtain the estimates similar to (2.5):

$$\frac{k+1}{2M}\,(y'(x_j))^2 \leq |y(x'_{j+1})|^{k+1} \leq \frac{k+1}{2m}\,(y'(x_j))^2$$

and, therefore,

$$\frac{2m}{k+1}\,|y(x'_{j+1})|^{k+1} \le (y'(x_j))^2 \le \frac{2M}{k+1}\,|y(x'_{j+1})|^{k+1}.$$

So,

$$\frac{m}{M}\,|y(x'_{j+1})|^{k+1} \le |y(x'_j)|^{k+1} \le \frac{M}{m}\,|y(x'_{j+1})|^{k+1},$$
$$\Big(\frac{m}{M}\Big)^{\frac{1}{k+1}}|y(x'_{j+1})| \le |y(x'_j)| \le \Big(\frac{M}{m}\Big)^{\frac{1}{k+1}}|y(x'_{j+1})|$$

and

$$\Big(\frac{m}{M}\Big)^{\frac{1}{k+1}} \le \Big|\frac{y(x'_{j+1})}{y(x'_j)}\Big| \le \Big(\frac{M}{m}\Big)^{\frac{1}{k+1}}.$$

Since zeroes x_j and extremum point x'_j of a nontrivial maximally extended solution to equation (1.1) alternate, for any $j \in \mathbb{Z}$ we have $y(x'_{j+1})\,y(x'_j) < 0$ and

$$-\Big(\frac{M}{m}\Big)^{\frac{1}{k+1}} \le \frac{y(x'_{j+1})}{y(x'_j)} \le -\Big(\frac{m}{M}\Big)^{\frac{1}{k+1}}. \qquad \square$$

Repeating the steps described in the proof of Theorem 2.1, T. Korchemkina has obtained the following

Corollary ([9])**.** *Introduce the notation*

$$m_j = \min_{x\in[x_j,x_{j+1}]} p(x,y(x),y'(x)), \quad M_j = \max_{x\in[x_j,x_{j+1}]} p(x,y(x),y'(x)), \ \ j \in \mathbb{Z}.$$

Then, for any $j \in \mathbb{Z}$, the following inequalities hold:

$$-\sqrt{\frac{M_j}{m_j}} \le \frac{y'(x_{j+1})}{y'(x_j)} \le -\sqrt{\frac{m_j}{M_j}} - \Big(\frac{M_j^2}{m_j m_{j-1}}\Big)^{\frac{1}{k+1}} \le \frac{y(x'_j)}{y(x'_{j+1})} \le -\Big(\frac{m_j^2}{M_j\,M_{j-1}}\Big)^{\frac{1}{k+1}}.$$

3 Asymptotic behavior of maximally extended solutions

I. T. Kiguradze and T. A. Chanturia in [11] proved that if $p = p(x)$ is a positive locally integrable function of locally bounded variation, then for both regular ($k > 1$) and singular ($0 < k < 1$) nonlinearities, any nontrivial right-maximally extended solution to equation (1.1) is proper, i.e., is defined in the neighborhood of $+\infty$.

For $k > 1$, an example is given [10] of a continuous function $p = p(x)$ satisfying inequalities (1.2) such that there exists a solution to (1.1) with a resonance asymptote $x = x^*$ ($\overline{\lim\limits_{x\to x^*-0}}\, y(x) = +\infty$, $\underline{\lim\limits_{x\to x^*-0}}\, y(x) = -\infty$), i.e., a non-proper solution. Step by step we construct a continuous function $p(x)$ and an oscillating solution $y(x)$ to equation (1.1). On each step we define p, construct a solution to equation (1.1) and estimate the distance between consecutive zeros $x_{j+1} - x_j$.

Moreover, the sufficient conditions on the function $p = p(x)$ are obtained under which all nontrivial maximally extended solutions are defined on the whole axis.

Theorem 3.1. *Suppose $k \in (0,1) \cup (1,+\infty)$, $p = p(x)$ is a continuous function of a globally bounded variation satisfying inequalities* (1.2)*. Then for any nontrivial maximally extended solution $y(x)$ to* (1.1) *there exist the finite positive limits $\lim\limits_{j\to\pm\infty} |y'(x_j)|$, $\lim\limits_{j\to\pm\infty} |y(x'_j)|$ and $\lim\limits_{j\to\pm\infty} (x_{j+1} - x_j)$.*

Proof. Let $y(x)$ be a nontrivial maximally extended solution to equation (1.1). Now we investigate an asymptotic behavior of $y(x)$ at the right-side boundary of the domain ($j \to +\infty$), the case $j \to -\infty$ is similar.

Let us use the following notation:

$$m_j = \min_{x\in[x_j,x_{j+1}]} p(x), \quad M_j = \max_{x\in[x_j,x_{j+1}]} p(x), \quad j \in \mathbb{Z},$$
$$m'_j = \min_{x\in[x'_j,x'_{j+1}]} p(x), \quad M'_j = \max_{x\in[x'_j,x'_{j+1}]} p(x), \quad j \in \mathbb{Z}.$$

By repeating the steps described in the proof of Theorem 2.1, for any $j \in \mathbb{N}$, we obtain similar to (2.5) estimates:

$$\frac{k+1}{2M'_j}\,(y'(x_j))^2 \le |y(x'_j)|^{k+1} \le \frac{k+1}{2m'_j}\,(y'(x_j))^2,$$
$$\frac{k+1}{2M'_j}\,(y'(x_j))^2 \le |y(x'_{j+1})|^{k+1} \le \frac{k+1}{2m'_j}\,(y'(x_j))^2,$$

whence

$$\Big(\frac{m'_j}{M'_j}\Big)^{\frac{1}{k+1}} \le \Big|\frac{y(x'_j)}{y(x'_{j+1})}\Big| \le \Big(\frac{M'_j}{m'_j}\Big)^{\frac{1}{k+1}}.$$

Moreover, due to the above estimate and estimate (2.4), for any $j \in \mathbb{N}$, we have

$$\big|\ln|y'(x_{j+1})| - \ln|y'(x_j)|\big| \le \frac{1}{2}\,(\ln M_j - \ln m_j) \le \frac{1}{2}\,V_{[x_j,x_{j+1}]}\ln p(x),$$
$$\big|\ln|y(x'_{j+1})| - \ln|y(x'_j)|\big| \le \frac{1}{k+1}\,(\ln M'_j - \ln m'_j) \le \frac{1}{k+1}\,V_{[x'_j,x'_{j+1}]}\ln p(x),$$
$$\sum_{j=1}^{+\infty} V_{[x_j,x_{j+1}]}\ln p(x) = V_{[x_1,+\infty)}\ln p(x) < +\infty,$$
$$\sum_{j=1}^{+\infty} V_{[x'_j,x'_{j+1}]}\ln p(x) = V_{[x'_1,+\infty)}\ln p(x) < +\infty,$$

where $V_{[a,b]}\ln p(x)$, $V_{[c,+\infty)}\ln p(x)$ are variations of the function $\ln p(x)$ on $[a,b]$ and $[c,+\infty)$, respectively. Due to the Weierstrass test, the series $\sum\limits_{j=1}^{+\infty}(\ln|y'(x_{j+1})| - \ln|y'(x_j)|)$ converges.

Therefore, there exists a finite $\lim\limits_{j\to+\infty}\ln|y'(x_j)|$, hence there exists a finite $\lim\limits_{j\to+\infty}|y'(x_j)|$. Analogously, we obtain the existence of a finite positive $\lim\limits_{j\to+\infty}|y(x'_j)|$.

Further, let us show that the distance between consecutive zeros $(x_{j+1}-x_j)$ has a limit as $j\to+\infty$. Multiplying equation (1.1) by y', integrating it on $[x'_{j+1},x]$, $x \le x_{j+1}$, and assuming without any restrictions that $y(x) \ge 0$ on $[x'_{j+1},x_{j+1}]$, we obtain

$$(y'(x))^2 = -2\int_{x'_{j+1}}^{x} p(s)y'(s)y^k(s)\,ds = 2\int_{x'_{j+1}}^{x} p(s)|y'(s)|y^k(s)\,ds \le \frac{2M'_{j+1}}{k+1}\,\big(H_{j+1}^{k+1} - y^{k+1}(x)\big).$$

Analogously, we obtain the estimate

$$(y'(x))^2 \ge \frac{2m'_{j+1}}{k+1}\,\big(H_{j+1}^{k+1} - y^{k+1}(x)\big),$$

so,

$$\sqrt{\frac{2m'_{j+1}}{k+1}}\,\sqrt{H_{j+1}^{k+1} - y^{k+1}(x)} \le |y'(x)| \le \sqrt{\frac{2M'_{j+1}}{k+1}}\,\sqrt{H_{j+1}^{k+1} - y^{k+1}(x)}\,.$$

Note that

$$x_{j+1} - x'_{j+1} = \int_{x_{j+1}}^{x'_{j+1}} \frac{y'(x)}{|y'(x)|}\,dx = \int_0^{H_{j+1}} \frac{dy}{|y'|} \le \sqrt{\frac{k+1}{2m'_{j+1}}}\int_0^{H_{j+1}} \frac{dy}{\sqrt{H_{j+1}^{k+1} - y^{k+1}}}$$

and making the replacement $y = uH_{j+1}$ in the last integral, we obtain

$$x_{j+1} - x'_{j+1} \le \sqrt{\frac{k+1}{2m'_{j+1}}}\, H_{j+1}^{-\frac{k-1}{2}} \int\limits_0^1 \frac{du}{\sqrt{1-u^{k+1}}}\,.$$

Analogously, the inequality

$$x_{j+1} - x'_{j+1} \ge \sqrt{\frac{k+1}{2M'_{j+1}}}\, H_{j+1}^{-\frac{k-1}{2}} \int\limits_0^1 \frac{du}{\sqrt{1-u^{k+1}}}$$

holds. Due to the assumptions of the theorem, the function $p(x)$ has a finite positive limit p_+ as $x \to +\infty$ and we have proved that there exists a finite positive $\lim\limits_{j\to+\infty} H_j$. Thus, passing to the limit in last inequalities, we can conclude that the distance between the extremum point and zero $(x_{j+1} - x'_{j+1})$ has a finite positive limit as $j \to +\infty$. Analogously, both the distance $(x'_{j+1} - x_j)$ and hence their sum $(x_{j+1} - x_j)$ have finite positive limits as $j \to +\infty$. □

Remark 3.1. Note that the theorem assumption of a globally bounded variation for the function $p(x)$ is essential for the existence of finite positive limits $\lim\limits_{j\to\pm\infty} |y'(x_j)|$, $\lim\limits_{j\to\pm\infty} |y(x'_j)|$ and $\lim\limits_{j\to\pm\infty} (x_{j+1} - x_j)$. An example of a continuous function $p(x) > 0$ (satisfying inequalities (1.2) but not of a globally bounded variation) is given [10] such that there exists an unbounded proper solution $\lim\limits_{j\to+\infty} |y'(x_j)| = \lim\limits_{j\to+\infty} |y(x'_j)| = +\infty$. Also, an example of a continuous function $p(x) > 0$ (satisfying inequalities (1.2) but not of a globally bounded variation) is given [10] such that there exists a nontrivial proper oscillating solution tending at $+\infty$ to zero with its first derivative.

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ON ONE UPPER ESTIMATE FOR THE FIRST EIGENVALUE OF A STURM–LIOUVILLE PROBLEM WITH DIRICHLET BOUNDARY CONDITIONS AND A WEIGHTED INTEGRAL CONDITION

Abstract. We consider a Sturm–Liouville problem on the interval $(0,1)$ with Dirichlet boundary conditions and a weighted integral condition on the potential which may have singularities of different orders at the end-points of the interval $(0,1)$. One upper estimate for the first eigenvalue for some values of parameters in the integral condition is obtained.[1]

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Key words and phrases. Sturm–Liouville problem, first eigenvalue, Dirichlet boundary conditions, weighted integral condition, minimization of the functional, variational principle, boundary-value problem.

რეზიუმე. $(0,1)$ შუალედში განხილულია შტურმ-ლიუვილის ამოცანა დირიხლეს სასაზღვრო პირობებითა და პოტენციალზე დადებული წონიანი ინტეგრალური შეზღუდვით; ამასთან დასაშვებია, რომ პოტენციალს გააჩნდეს სინგულარობები $(0,1)$ შუალედის ბოლო წერტილებში. ინტეგრალურ შეზღუდვაში მონაწილე პარამეტრთა ზოგიერთი მნიშვნელობისთვის დადგენილია პირველი საკუთრივი მნიშვნელობის ზემოდან შეფასება.

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1 Introduction

We consider a problem whose origin was the Lagrange problem of finding the form of the firmest column of the given volume. The Lagrange problem was the source for different extremal eigenvalue problems for second-order differential equations with integral conditions on the potential.

We develop the methods used in Yu. V. Egorov and V. A. Kondratiev's works (see, e.g., [1]) devoted to estimation of eigenvalues for Sturm–Liouville problems. The Sturm–Liouville problem for the equation $y'' + \lambda Q(x)y = 0$ with Dirichlet boundary conditions and a non-negative summable on $[0,1]$ function Q satisfying the condition $\int\limits_0^1 Q^\gamma(x)\,dx = 1$ as $\gamma \in \mathbb{R}$, $\gamma \neq 0$, was considered by Yu. V. Egorov and V. A. Kondratiev in [1]. The Sturm–Liouville problem for the equation $y'' - Q(x)y + \lambda y = 0$ with Dirichlet boundary conditions and a real Lebesgue integrable on $(0,1)$ function Q satisfying the condition $\int\limits_0^1 Q^\gamma(x)\,dx = 1$ as $\gamma \geqslant 1$, was considered by V. A. Vinokurov, V. A. Sadovnichii in [2]. In the present article we consider a problem of that kind in case the integral condition contains a weight function. Some results devoted to the Sturm–Liouville problems with weighted integral conditions can be found in [6]– [9].

Consider the Sturm–Liouville problem

$$y'' + Q(x)y + \lambda y = 0, \quad x \in (0,1), \tag{1.1}$$

$$y(0) = y(1) = 0, \tag{1.2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all real–valued measurable on $(0,1)$ functions with non–negative values such that the following integral condition holds:

$$\int\limits_0^1 x^\alpha (1-x)^\beta Q^\gamma(x)\,dx = 1, \quad \alpha,\beta,\gamma \in \mathbb{R}, \quad \gamma \neq 0. \tag{1.3}$$

A function y is a *solution* to problem $(1.1), (1.2)$ if it is absolutely continuous on the segment $[0,1]$, satisfies (1.2), its derivative y' is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and equality (1.1) holds almost everywhere in the interval $(0,1)$.

We give estimates for

$$M_{\alpha,\beta,\gamma} = \sup_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q).$$

For any function $Q \in T_{\alpha,\beta,\gamma}$, by H_Q we denote the closure of the set $C_0^\infty(0,1)$ with respect to the norm

$$\|y\|_{H_Q} = \left(\int\limits_0^1 {y'}^2\,dx + \int\limits_0^1 Q(x)y^2\,dx \right)^{\frac{1}{2}}.$$

For any function $Q \in T_{\alpha,\beta,\gamma}$, we can prove (see, e.g., [3,6]) that

$$\lambda_1(Q) = \inf_{y \in H_Q \setminus \{0\}} R[Q,y], \quad \text{where} \quad R[Q,y] = \frac{\int_0^1 ({y'}^2 - Q(x)y^2)\,dx}{\int_0^1 y^2\,dx}.$$

2 One upper estimate for the first eigenvalue for $\gamma < 0$

Theorem 2.1. *If $\gamma < 0$ and $\alpha, \beta > 3\gamma - 1$, then $M_{\alpha,\beta,\gamma} < \pi^2$. If $\gamma < -1$, $\alpha, \beta > -1$, then there exist a function $Q_* \in T_{\alpha,\beta,\gamma}$ and a positive on the interval $(0,1)$ function $u \in H_{Q_*}$ such that $M_{\alpha,\beta,\gamma} = R[Q_*, u]$, moreover, u satisfies the equation*

$$u'' + mu = -x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}} u^{\frac{\gamma+1}{\gamma-1}}$$

and the integral condition

$$\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}} u^{\frac{2\gamma}{\gamma-1}}\,dx = 1.$$

Proof. Suppose that $\gamma < 0$. For any $Q \in T_{\alpha,\beta,\gamma}$ and $y \in H_Q$, by the Hölder inequality we have

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx \leqslant \Big(\int_0^1 x^{\alpha}(1-x)^{\beta}Q^{\gamma}(x)\,dx\Big)^{\frac{1}{1-\gamma}}\Big(\int_0^1 Q(x)y^2\,dx\Big)^{\frac{\gamma}{\gamma-1}}.$$

Then

$$\int_0^1 Q(x)y^2\,dx \geqslant \Big(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{\gamma-1}{\gamma}} \tag{2.1}$$

and

$$\inf_{y\in H_Q\setminus\{0\}} R[Q,y] \leqslant \inf_{y\in H_Q\setminus\{0\}} G[y],$$

where

$$G[y] = \frac{\int_0^1 y'^2\,dx - \big(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx\big)^{\frac{\gamma-1}{\gamma}}}{\int_0^1 y^2\,dx}.$$

Consider the function

$$u_\varepsilon(x) = \begin{cases} 0, & 0<x<\varepsilon,\\ 1, & \varepsilon\leqslant x\leqslant 1-\varepsilon,\\ 0, & 1-\varepsilon<x<1,\end{cases}$$

where $0<\varepsilon<\frac{1}{3}$. By the average processing for $\rho = \frac{\varepsilon}{2}$ we obtain the function

$$u_{\varepsilon_\rho}(x) = \int_{-\infty}^{+\infty}\omega_\rho(x-y)u_\varepsilon(y)\,dy = \int_{-\infty}^{+\infty}\omega_\rho(y-x)u_\varepsilon(y)\,dy = \int_{-\rho}^{\rho}\omega_\rho(z)u_\varepsilon(z+x)\,dz.$$

For the function $y_\varepsilon(x) = u_{\varepsilon\rho}(x)\cdot\sin\pi x$ of $C_0^\infty(0,1)$ it is true that for any $Q\in T_{\alpha,\beta,\gamma}$ the function y_ε belongs to H_Q and

$$\|y_\varepsilon(x)-\sin\pi x\|_{H_0^1(0,1)} \longrightarrow 0 \ \text{ as } \ \varepsilon\to 0.$$

For $\gamma<0$, $\alpha,\beta>3\gamma-1$, the integral $\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}(\sin\pi x)^{\frac{2\gamma}{\gamma-1}}\,dx$ converges. Then for any $Q\in T_{\alpha,\beta,\gamma}$ we have

$$\inf_{y\in H_Q\setminus\{0\}} R[Q,y] \leqslant \inf_{y\in H_Q\setminus\{0\}} G[y] \leqslant \lim_{\varepsilon\to 0} G[y_\varepsilon] = G[\sin\pi x] < \pi^2$$

and $M_{\alpha,\beta,\gamma} < \pi^2$.

Let us show the method of finding sharp estimates for $M_{\alpha,\beta,\gamma}$ for $\gamma<-1$, $\alpha,\beta>-1$. For any function $y\in H_0^1(0,1)$, the inequalities $y^2<Cx$ and $y^2<C(1-x)$ hold, where $C=\int_0^1 y'^2\,dx$. If the integral $\int_0^1 Q(x)x(1-x)\,dx$ converges, then $\int_0^1 Q(x)y^2\,dx$ also converges. Consequently, for $\gamma<0$, $\alpha,\beta>2\gamma-1$, the sets of functions of H_Q and $H_0^1(0,1)$ coincide.

Let us prove that for $\gamma<0$, $\alpha,\beta>2\gamma-1$ the functional G is bounded from below in $H_0^1(0,1)$. By the Hölder inequality, for $x\in(0,\frac{1}{2})$ we have

$$y^2(x) = \Big(\int_0^x y'(t)\,dt\Big)^2 \leqslant x\int_0^x y'^2(t)\,dt \leqslant x\int_0^{\frac{1}{2}} y'^2(t)\,dt$$

and for $x\in(\frac{1}{2},1)$ we have

$$y^2(x) = \Big(-\int_x^1 y'(t)\,dt\Big)^2 \leqslant (1-x)\int_x^1 y'^2(t)\,dt \leqslant (1-x)\int_{\frac{1}{2}}^1 y'^2(t)\,dt.$$

Then

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx \leqslant \int_0^{\frac{1}{2}} x^{\frac{\alpha-\gamma}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}\left(\int_0^{\frac{1}{2}} y'^2(t)\,dt\right)^{\frac{\gamma}{\gamma-1}} dx$$
$$+\int_{\frac{1}{2}}^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta-\gamma}{1-\gamma}}\left(\int_{\frac{1}{2}}^1 y'^2(t)\,dt\right)^{\frac{\gamma}{\gamma-1}} dx \leqslant \left(\int_0^1 y'^2(t)\,dt\right)^{\frac{\gamma}{\gamma-1}}(C_1+C_2),$$

where

$$C_1=\int_0^{\frac{1}{2}} x^{\frac{\alpha-\gamma}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}\,dx, \quad C_2=\int_{\frac{1}{2}}^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta-\gamma}{1-\gamma}}\,dx.$$

Note that for $\gamma<0$, $\alpha,\beta>2\gamma-1$, the integrals $\int\limits_0^{\frac{1}{2}} x^{\frac{\alpha-\gamma}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}\,dx$ and $\int\limits_{\frac{1}{2}}^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta-\gamma}{1-\gamma}}\,dx$ converge. Then for any $y\in H_0^1(0,1)$, we have

$$G[y]\geqslant \pi^2\big(1-(C_1+C_2)^{\frac{\gamma-1}{\gamma}}\big).$$

Thus, the functional G is bounded from below in $H_0^1(0,1)$ and

$$m=\inf_{y\in H_0^1(0,1)\setminus\{0\}} G[y]. \tag{2.2}$$

For any function $Q\in T_{\alpha,\beta,\gamma}$,

$$\lambda_1(Q)=\inf_{y\in H_Q\setminus\{0\}} R[Q,y]\leqslant \inf_{y\in H_Q\setminus\{0\}} G[y]=\inf_{y\in H_0^1(0,1)\setminus\{0\}} G[y]=m.$$

Then

$$M_{\alpha,\beta,\gamma}=\sup_{Q\in T_{\alpha,\beta,\gamma}}\lambda_1(Q)\leqslant \inf_{y\in H_0^1(0,1)\setminus\{0\}} G[y]=m.$$

Consequently, $M_{\alpha,\beta,\gamma}\leqslant m$.

Let us prove that for $\gamma<-1$, $\alpha,\beta>-1$ there exist a function $Q_*\in T_{\alpha,\beta,\gamma}$ and a positive on the interval $(0,1)$ function $u\in H_{Q_*}$ such that $M_{\alpha,\beta,\gamma}=R[Q_*,u]=m$.

Put

$$\Gamma_*=\left\{y\in H_0^1(0,1)\ \Big|\ \int_0^1 y^2\,dx=1\right\}$$

and

$$I[y]=\int_0^1 {y'}^2\,dx-\left(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx\right)^{\frac{\gamma-1}{\gamma}}.$$

Lemma 2.1. *There exists a function $u_*\in\Gamma_*$ such that $I[u_*]=m$, where m is defined by* (2.2).

Proof. Let $\{\widetilde{y}_k\}$ be a minimizing sequence of the functional G in $H_0^1(0,1)$. Then $y_k=\frac{\widetilde{y}_k}{C_k^{1/2}}$, where $C_k=\int\limits_0^1 \widetilde{y}_k^2\,dx$, is a minimizing sequence of the functional I in Γ_*, i.e., $I[y_k]\to m$ as $k\to\infty$. Then

$$m=\inf_{y\in H_0^1(0,1)\setminus\{0\}} G[y]=\inf_{y\in\Gamma_*} I[y].$$

Let us show that for $\alpha,\beta>-1$, the sequence $\{y_k\}$ is bounded in $H_0^1(0,1)$. Since $m=\inf\limits_{y\in\Gamma_*} I[y]$, for all sufficiently large values of k we have

$$I[y_k]=\int_0^1 {y_k'}^2\,dx-\Big(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y_k|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{\gamma-1}{\gamma}}<m+1.$$

For $\alpha,\beta\geqslant 0$, by the Hölder inequality, we have

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y_k|^{\frac{2\gamma}{\gamma-1}}\,dx\leqslant\Big(\int_0^1 x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}y_k^2\,dx\Big)^{\frac{\gamma}{\gamma-1}}\leqslant\Big(\int_0^1 y_k^2\,dx\Big)^{\frac{\gamma}{\gamma-1}}=1$$

and

$$\int_0^1 {y_k'}^2\,dx=I[y_k]+\Big(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y_k|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{\gamma-1}{\gamma}}\leqslant m+2.$$

For $\alpha,\beta<0$, by the Hölder inequality, we have

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y_k|^{\frac{2\gamma}{\gamma-1}}\,dx\leqslant\Big(\int_0^1 (x^{-p}(1-x)^{-p})^{1-\gamma}\,dx\Big)^{\frac{1}{1-\gamma}}$$
$$\times\Big(\int_0^1 x^{-\frac{\alpha}{\gamma}}x^{p\cdot\frac{\gamma-1}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}(1-x)^{p\cdot\frac{\gamma-1}{\gamma}}y_k^2\,dx\Big)^{\frac{\gamma}{\gamma-1}}\leqslant M\cdot\Big(\int_0^1 y_k^2\,dx\Big)^{\frac{\gamma}{\gamma-1}}=M,$$

where $M=\big(\int_0^1(x^{-p}(1-x)^{-p})^{1-\gamma}\,dx\big)^{\frac{1}{1-\gamma}}$ and p is a number such that $-\frac{\alpha}{\gamma}+p\cdot\frac{\gamma-1}{\gamma}>0$, $-\frac{\beta}{\gamma}+p\cdot\frac{\gamma-1}{\gamma}>0$, $-p(1-\gamma)>-1$. Consequently, p satisfies the inequalities

$$\frac{\alpha}{\gamma-1}<p<\frac{1}{1-\gamma}\,,\quad \frac{\beta}{\gamma-1}<p<\frac{1}{1-\gamma}\,,$$

which hold for $\alpha,\beta>-1$. The proofs for the cases $\alpha\geqslant 0>\beta>-1$ and $\beta\geqslant 0>\alpha>-1$ are similar.

Since for $\alpha,\beta>-1$ the sequence $\{y_k\}$ is bounded in $H_0^1(0,1)$, it contains a subsequence $\{z_k\}$ which converges weakly in $H_0^1(0,1)$ to some function u_*, moreover,

$$\|u_*\|^2_{H_0^1(0,1)}\leqslant\max\big\{m+3,m+2+M^{\frac{\gamma-1}{\gamma}}\big\}.$$

Since the space $H_0^1(0,1)$ is compactly embedded in the space $C[0,1]$, there exists a subsequence $\{s_k\}$ of $\{z_k\}$ which converges in $C[0,1]$. Since the space $C[0,1]$ is embedded in $L_2(0,1)$, the sequence $\{s_k\}$ converges in $L_2(0,1)$ to the function u_*. Consequently, for the functional G we have

$$\int_0^1 s_k^2\,dx\longrightarrow\int_0^1 u_*^2\,dx\ \text{ as }\ k\to\infty$$

and

$$\int_0^1 u_*^2\,dx=1. \tag{2.3}$$

Since for $\alpha,\beta>-1$ the sequence $\{s_k\}$ is bounded in $H_0^1(0,1)$, by the definition of the norm $\|s_k\|_{H_0^1(0,1)}$ the sequence $\{s_k'\}$ is bounded in $L_2(0,1)$. Then there exists a subsequence $\{w_k\}$ of $\{s_k\}$ such that the sequence $\{w_k'\}$ converges weakly to the function u_*' in $L_2(0,1)$. Then ([10, p. 217])

$$\|u_*'\|^2_{L_2(0,1)}\leqslant\varliminf_{k\to\infty}\|w_k'\|^2_{L_2(0,1)}=A.$$

Thus, we have

$$\|u_*'\|^2_{L_2(0,1)} \leqslant A. \tag{2.4}$$

Let $\{v_k\}$ be a subsequence of $\{w_k\}$ such that

$$\lim_{k\to\infty} \int\limits_0^1 {v_k'}^2\,dx = \varliminf_{k\to\infty} \int\limits_0^1 {w_k'}^2\,dx = A.$$

Since m is a limit of the sequence $\{I[v_k]\}$, $m - A$ is a limit of the sequence

$$\Big\{-\Big(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|v_k|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{\gamma-1}{\gamma}}\Big\}.$$

Then, for any $\varepsilon > 0$, there exists a number K such that for any $k \geqslant K$ the inequality

$$-\Big(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|v_k|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{\gamma-1}{\gamma}} < m - A + \varepsilon$$

holds. Then

$$\Big(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|v_k|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{\gamma-1}{\gamma}} > A - m - \varepsilon$$

and

$$\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|v_k|^{\frac{2\gamma}{\gamma-1}}\,dx > (A - m - \varepsilon)^{\frac{\gamma}{\gamma-1}}. \tag{2.5}$$

Let us use the Lebesgue theorem. For the sequence $\{x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|r_k|^{\frac{2\gamma}{\gamma-1}}\}$, we have

$$x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|r_k|^{\frac{2\gamma}{\gamma-1}} \longrightarrow x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u_*|^{\frac{2\gamma}{\gamma-1}} \text{ as } k\to\infty \text{ almost everywhere on } [0,1].$$

We have proved the existence of a constant $V = \max\{1, M\}$ such that for any sufficiently large value of k we have

$$\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|r_k|^{\frac{2\gamma}{\gamma-1}}\,dx \leqslant V.$$

Then

$$x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u_*|^{\frac{2\gamma}{\gamma-1}} \in L_1(0,1)$$

and

$$\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|r_k|^{\frac{2\gamma}{\gamma-1}}\,dx \longrightarrow \int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u_*|^{\frac{2\gamma}{\gamma-1}}\,dx \text{ as } k\to\infty.$$

If for any $k \geqslant K$ inequality (2.5) holds and

$$\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|r_k|^{\frac{2\gamma}{\gamma-1}}\,dx \longrightarrow \int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u_*|^{\frac{2\gamma}{\gamma-1}}\,dx \text{ as } k\to\infty,$$

then we have

$$\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u_*|^{\frac{2\gamma}{\gamma-1}}\,dx \geqslant (A - m - \varepsilon)^{\frac{\gamma}{\gamma-1}}.$$

Since ε may be sufficiently small, we obtain

$$\left(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u_*|^{\frac{2\gamma}{\gamma-1}}\,dx\right)^{\frac{\gamma-1}{\gamma}} \geqslant A-m$$

and

$$-\left(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u_*|^{\frac{2\gamma}{\gamma-1}}\,dx\right)^{\frac{\gamma-1}{\gamma}} \leqslant m-A. \tag{2.6}$$

By virtue of (2.4) and (2.6), we obtain

$$I[u_*] \leqslant m. \tag{2.7}$$

Since $m = \inf\limits_{y\in\Gamma_*} I[y]$, we have $I[u_*] = m$. By (2.3), we obtain $u_* \in \Gamma_*$. $\square$

Let us consider the set

$$\Gamma = \left\{ y \in H_0^1(0,1) \mid \int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx = 1 \right\}.$$

The function $u = Cu_*$, where

$$C = \left(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u_*|^{\frac{2\gamma}{\gamma-1}}\,dx\right)^{\frac{1-\gamma}{2\gamma}},$$

is non-negative on $[0,1]$ and belongs to Γ. Then $G[u] = G[u_*] = I[u_*] = m$.

Let us fix the argument u of the functional G and fix some variation $z \in H_0^1(0,1)$ of the argument u and let us consider a set of functions $u + tz$, where t is an arbitrary parameter. On the functions $u + tz$ the functional G turns to the function of $t \in \mathbb{R}$:

$$g(t) = \frac{\int_0^1 (u'(x)+tz'(x))^2\,dx - \left(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u(x)+tz(x)|^{\frac{2\gamma}{\gamma-1}}\,dx\right)^{\frac{\gamma-1}{\gamma}}}{\int_0^1 (u(x)+tz(x))^2\,dx}.$$

Since the functional G reaches an extremum at $y = u$ and for $\gamma < -1$ the function $g(t)$ is differentiable at zero, we have $g'(0) = 0$. Since $u \in \Gamma$ and $G[u] = m$, we obtain

$$\int\limits_0^1 u'z'\,dx - \int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u|^{\frac{\gamma+1}{\gamma-1}}\operatorname{sgn} u\, z\,dx = m\int\limits_0^1 uz\,dx. \tag{2.8}$$

For $\gamma < -1$, $\alpha,\beta > -1$, equality (2.8) holds for any function $z \in H_0^1(0,1)$, because by virtue of the Hölder inequality, we have

$$\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u|^{\frac{\gamma+1}{\gamma-1}}|z|\,dx$$
$$\leqslant \left(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u|^{\frac{2\gamma}{\gamma-1}}\,dx\right)^{\frac{\gamma+1}{2\gamma}} \left(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|z|^{\frac{2\gamma}{\gamma-1}}\,dx\right)^{\frac{\gamma-1}{2\gamma}}.$$

If $z \in C_0^\infty(0,1)$, then u' has a generalized derivative equal to

$$u'' = -x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u|^{\frac{\gamma+1}{\gamma-1}}\operatorname{sgn} u - mu.$$

Since $G[y] = G[|y|]$, we can assume that the sequence $\{y_k\}$ is non-negative and $u \geqslant 0$. Similarly, to the case $\alpha = \beta = 0$ we can prove (see, e.g., [3]) that the function u is convex upward. Thus on the interval $(0,1)$ we have $u(x) > 0$.

Since $u \in AC[0,1]$, for $\gamma < -1$, the function $x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u|^{\frac{\gamma+1}{\gamma-1}} \operatorname{sgn} u$ is continuous on the segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and $u'' \in L_p(\rho, 1-\rho)$. Let v be a generalized derivative of u of second order. The Corollary 2.6.1 of Theorem 2.6.1 (see [12, p. 41]) implies that if $u, v \in L_p(\rho, 1-\rho)$, $p \geqslant 1$, then the function u is continuously differentiable on $[\rho, 1-\rho]$ and almost everywhere on it has the classical derivative of the second order $u'' = v$. Thus,

$$u'' + x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|u|^{\frac{2}{\gamma-1}}u + mu = 0 \text{ for } x \in [\rho, 1-\rho]. \tag{2.9}$$

Since the number ρ may be sufficiently small and the function u is continuous and positive on $(0,1)$, the function u'' is also continuous on $(0,1)$ and equality (2.9) holds everywhere on $(0,1)$.

On $(0,1)$, let us consider the function

$$Q_*(x) = x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{2}{\gamma-1}}.$$

Since $Q_*(x)$ satisfies the integral condition (1.3):

$$\int_0^1 x^\alpha (1-x)^\beta Q_*^\gamma(x)\, dx = \int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{2\gamma}{\gamma-1}}\, dx = 1,$$

the function u belongs to H_{Q_*}.

Since u satisfies equation (2.9) and conditions (1.2), for $Q = Q_*$ it satisfies equation (1.1) and conditions (1.2). Therefore, since u is continuous on $[0,1]$ and its derivative u' is continuous on $(0,1)$, the function u is the first eigenfunction of problem (1.1)–(1.3) with $Q = Q_*$ and the first eigenvalue $\lambda_1(Q_*) = m$.

Then

$$\inf_{y \in H_{Q_*} \setminus \{0\}} R[Q_*, y] = R[Q_*, u] = G[u] = m$$

and

$$M_{\alpha,\beta,\gamma} = \sup_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q) \geqslant \lambda_1(Q_*) = \inf_{y \in H_{Q_*} \setminus \{0\}} R[Q_*, y] = R[Q_*, u] = G[u] = m.$$

Consequently, we obtain $M_{\alpha,\beta,\gamma} = m$. □

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A VARIANT OF THE METHOD OF STEP ALGORITHM FOR A DELAY DIFFERENTIAL EQUATION

Abstract. In this paper we develop a new method to obtain explicit solutions for a first order linear delay differential equation based upon the generating function concept. The advantage of this new method as regards the traditional Method of Step Algorithm (MSA) is also showed through an example.[1]

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1 Introduction

In this work[2] we present the detailed proofs of the results reached in [5,6], as referred in [3], concerning the solutions of the Basic Initial Problem (BIP),

$$\begin{cases} x'(t) = f(\phi(t-r)), \\ x(0) = \phi(0), \end{cases}$$

where B and r are constants, $r > 0$ is the delay, and $\phi(t)$ is a given continuous function on $[-r, 0]$.

Delay differential equations (DDEs) are well studied in [1,4,7] from a point of view of the existence, uniqueness and properties of solutions. Here we point out that when the MSA is applied to the BIP, there appears combinatorial structure on the solutions.

This kind of structure that we designated by tree combinatorial structure led us to conjecture that there is a generating function defined over a specific class of polynomials with a single delay that solves the initial problem. As far as we know, the approach via a generating function is new to the relevant literature.

Assuming $\phi(t)$ is constant on $[-r, 0]$, and applying the MSA to the BIP, the solutions $x_n(t)$ defined on $A_n = ((n-1)r, nr]$, $n \geq 1$, showed one tree structure effect for the solution $x(t)$ of the problem.

That $x(t)$ *is the generating function for a sequence of polynomials with a single delay,* $P_j^n(rB)$, is our starting point.

Using the MSA, each solution $x_n(t)$ would depend upon the solution $x_{n-1}(t)$ defined on the previous intervals A_{n-1}. In order to provide an explicite formula for $x_n(t)$ on the interval A_n without the knowledge of all back solutions $x_{n-1}(t)$ defined on the previous intervals A_{n-1}, we introduce the polynomials $P_j^n(rB)$, which we refer to as delay polynomials, and the main theorem proves that this is possible.

The present paper is organized as follows. Section 2 describes the MSA and presents the conjecture. Section 3 constructs the alternative method to obtain the BIP's solution. Section 4 contains the two fundamental propositions allowing us to obtain the calculating formulas for any solution $x_n(t)$ defined on A_n with $n \geq 2$. Section 5 is devoted to the Main Theorem. Section 6 consists of the proof of the lemma, which is the basis of the new solution's method. An example is given to illustrate the theorem in Section 7.

2 Preliminaries

2.1 The method of step algorithm

Consider the Basic Initial Problem

$$\begin{cases} x'(t) = Bx(t-r), & t \geq 0, \\ x(t) = \phi(t), & t \in [-r, 0], \end{cases} \tag{2.1}$$

where $B \in \Re$, $r > 0$ is the delay, and $\phi(t)$ is a given continuous function on $[-r, 0]$.

The Method of Step Algorithm (MSA) can be described as follows.

Step 1

Consider $x'(t) = f(x(t-r))$. Given $\phi(t)$ on $[-r, 0]$, we can determine $x(t)$ on the interval $[0, r]$ by solving the ODE

$$\begin{cases} x'(t) = f(\phi(t-r)), \\ x(0) = \phi(0). \end{cases}$$

Denote its solution by $x_1(t)$.

[2]See [2] for a previous version of this paper

Step n

For each integer $n \geq 2$, given the solution $x_{n-1}(t)$ on $[(n-2)r, (n-1)r]$, we can determine $x(t)$ on the interval $[(n-1)r, nr]$ by solving the ODE

$$\begin{cases} x'(t) = f(x_{n-1}(t-r)), \\ x((n-1)r) = x_{n-1}((n-1)r). \end{cases}$$

Denote its solution by $x_n(t)$.

Conclusion

We can define the solution of

$$\begin{cases} x'(t) = f(x(t-r)), & t \geq 0, \\ x(t) = \phi(t), & t \in [-r, 0], \end{cases}$$

on each interval $A_n = [(n-1)r, nr]$, $n \geq 1$, by

$$x_n(t) = x_{n-1}((n-1)r) + \int\limits_{(n-1)r}^{t} f(x_{n-1}(s-r))\, ds,$$

where $x_0(\cdot) \equiv \phi(\cdot)$.

For $j = 1, \ldots, 5$, let $x_j(t)$ be the solution of (2.1) defined on the interval A_j obtained with the MSA. Assuming $\phi(t) = \phi(0)$ is a constant,

$$\begin{aligned}
x_1(t) &= \phi(0)[1 + Bt], \\
x_2(t) &= \phi(0)\Big[\frac{(Bt)^2}{2!} + Bt(1 - Br) + \frac{(Br)^2}{2!} + 1\Big], \\
x_3(t) &= \phi(0)\Bigg[\frac{(Bt)^3}{3!} + \frac{(Bt)^2}{2!}\Big(1 - \frac{3.2}{3}Br\Big) + Bt\Big(1 - Br + \frac{3.2^2}{3!}(Br)^2\Big) - \frac{(Br)^3}{3!}2^3 + \frac{(Br)^2}{2!} + 1\Bigg], \\
x_4(t) &= \phi(0)\Bigg[\frac{(Bt)^4}{4!} + \frac{(Bt)^3}{3!}\Big(1 - \frac{4.3}{4}Br\Big) + \frac{(Bt)^2}{2!}\Big(1 - \frac{3.2}{3}Br + \frac{6.3^2}{4.3}(Br)^2\Big) \\
&\qquad + Bt\Big(1 - Br + \frac{3.2^2}{3!}(Br)^2 - \frac{4.3^3}{4!}(Br)^3\Big) + \frac{(Br)^4}{4!}3^4 - \frac{(Br)^3}{3!}2^3 + \frac{(Br)^2}{2!} + 1\Bigg], \\
x_5(t) &= \phi(0)\Bigg[\frac{(Bt)^5}{5!} + \frac{(Bt)^4}{4!}\Big(1 - \frac{5.4}{5}Br\Big) + \frac{(Bt)^3}{3!}\Big(1 - \frac{4.3}{4}Br + \frac{10.4^2}{5.4}(Br)^2\Big) \\
&\qquad + \frac{(Bt)^2}{2!}\Big(1 - \frac{3.2}{3}Br + \frac{6.3^2}{4.3}(Br)^2 - \frac{10.4^3}{5.4.3}(Br)^3\Big) \\
&\qquad + Bt\Big(1 - Br + \frac{3.2^2}{3!}(Br)^2 - \frac{4.3^3}{4!}(Br)^3 + \frac{5.4^4}{5!}(Br)^4\Big) \\
&\qquad - \frac{(Br)^5}{5!}4^5 + \frac{(Br)^4}{4!}3^4 - \frac{(Br)^3}{3!}2^3 + \frac{(Br)^2}{2!} + 1\Bigg].
\end{aligned}$$

Analysing the form of these first iterates, we observe a tree structure effect, which allow us to formulate the following conjecture.

2.2 The conjecture

Definition 2.1 (Rainville, [8])**.** Let c_j, $j \in \mathbb{N}_0$, be a specified sequence independent of r and t. We say that $X(r,t)$ is a generating function of the set $g_j(r)$ if

$$X(r,t) = \sum_{j \geq 0} c_j g_j(r) t^j.$$

Conjecture 2.2. *If $x(t)$, $t \geq 0$, is a solution of BIP, then*

$$x(t) \equiv X(r,t) = \sum_{j\geq 0} v_j(r)t^j,$$

i.e., $x(t)$ is a generating function for some sequence $(v_j(r))_{j\geq 0}$ in the delay r.

3 Construction of a new solution's method

3.1 A new solution's formalization

In order to prove our claim, we will proceed in the following way. Consider the decomposition of $(0,\infty)$ in disjoint subintervals of equal length r. We will consider the restriction of the solution to each of these subintervals, as a generating function of some family of polynomials in r. That is,

$$(0,\infty) = \bigcup_{n\geq 1} A_n, \quad \text{where for each } \ n \geq 1, \quad A_n = ((n-1)r, nr],$$

$$\varphi(t) = \begin{cases} \phi(t) & \text{if } \ t \in A_0 = [-r,0], \\ \sum_{j\geq 0} w_j^1(r)t^j & \text{if } \ t \in A_1 = (0,r], \\ \sum_{j\geq 0} w_j^2(r)t^j & \text{if } \ t \in A_2 = (r,2r], \\ \quad\vdots & \\ \sum_{j\geq 0} w_j^n(r)t^j & \text{if } \ t \in A_n = ((n-1)r,nr], \\ \quad\vdots & \end{cases}$$

Hence, we have $\varphi(t)$ defined on each interval A_n, $n \geq 1$, as

$$\varphi(t)\big|_{t\in A_n} \equiv x_n(t) = \sum_{j\geq 0} w_j^n(r)t^j. \tag{3.1}$$

If our conjecture is valid, we must have $\varphi'(t) = B\varphi(t-r)$ for $t \geq 0$, where the derivative at $t = 0$ represents the right-hand derivative. Two different types of conditions must hold. On the one hand, we are concerned with the differentiability at each point $t = nr$, which will guarantee the continuity of the solution. This will be treated in conditions **(2.A)**.

On the other hand, we want $x'(t) = Bx(t-r)$ to be satisfied at any interior point of A_n. This will be treated in conditions **(2.B)**. To do this, we determine which of the conditions should the iterates $w_j^n(r)$ in equation (3.1) satisfy in terms of $\varphi(t)$. Meaning

$$\varphi'(0) = B\varphi(-r), \tag{3.2}$$

$$x_n'(t) = Bx_{n-1}(t-r) \ \text{ for } \ t \in A_n, \quad n \geq 1, \quad \text{where } \ x_0 \equiv \phi. \tag{3.3}$$

3.2 The constructive process

(2.A.1) $\varphi'(0) = B\phi(-r)$**.** At $t = 0$, we have

$$\varphi'(0) = \lim_{h\to 0^+} \frac{\sum\limits_{j\geq 0} w_j^1(r)h^j - \phi(0)}{h} = \lim_{h\to 0^+} \frac{w_0^1(r) + w_1^1(r)h + w_2^1(r)h^2 + \cdots - \phi(0)}{h}.$$

A sufficient condition for (2.A.1) to hold, is

(2.a.1) $\boxed{w_0^1(r) = \phi(0) \text{ and } w_1^1(r) = B\phi(-r) \text{ and } w_j^1(r) \text{ takes any value for } j \geq 2\,.}$

(2.B.1) $\varphi'(t) = B\phi(t-r)$**,** $t \in (0,r)$**.** Since

$$\varphi'(t) = \sum_{j\geq 0}(j+1)w^1_{j+1}(r)t^j,$$

we can establish the following statement. A sufficient condition for having **(2.B.1)** is

(2.b.1) $$\boxed{B\phi(-r) + \sum_{j\geq 2} jw^1_j(r)t^{j-1} = B\phi(t-r)\,.}$$

We want to emphasize an important statement that later will lead us to the Main Theorem. If the initial function $\phi(t)$ is constant, combining **(2.a.1)** and **(2.b.1)** we can choose $w^1_j(r) = (\phi(0), B\phi(-r), 0, 0, \dots)$. In fact, condition **(2.b.1)** implies $\phi(-r) = \phi(t-r)$ since $t-r \in (-r, 0)$, and on this interval the function is constant.

Therefore, the solution on the interval A_1 can be defined as

$$x_1(t) = \sum_{j\geq 0} w^1_j(r)t^j = \phi(0)[1+Bt],$$

where $\phi(t) = \phi(0)$ for $t \in [-r, 0]$, which is exactly the solution obtained by the MSA.

Returning to a continuous initial function $\phi(t)$, we can state the following proposition.

Proposition 3.1. *If there exists* $w^1_j(r)$ *satisfying* **(2.a.1)** *and* **(2.b.1)***, then* $x_1(t) = \sum_{j\geq 0} w^1_j(r)t^j$ *satisfies* (3.3) *on the interval* $(0,r)$*.*

Proof. For $t \in (0,r)$, let

$$x_1(t) = \sum_{j\geq 0} w^1_j(r)t^j = w^1_0(r) + w^1_1(r)t + \sum_{j\geq 2} w^1_j(r)t^j.$$

If $w^1_j(r)$ satisfies **(2.a.1)**, then

$$x_1(t) = \phi(0) + B\phi(-r)t + \sum_{j\geq 2} w^1_j(r)t^j.$$

By differentiation we obtain

$$x_1'(t) = B\phi(-r) + \sum_{j\geq 2} jw^1_j(r)t^{j-1},$$

and if **(2.b.1)** holds, then

$$x_1'(t) = B\phi(t-r).$$ □

From now on we will use the following lemma whose proof can be seen in Section 6.

Lemma 3.2. *For* $t \neq 0$ *and* $t \neq r$*,*

$$\sum_{j\geq 0} f_j(r)(t-r)^j = \sum_{j\geq 0} \frac{t^j}{j!}\Big(\sum_{i\geq 0} f_{j+i}(r)\,\frac{(-r)^i}{i!}\,(j+i)!\Big).$$

(2.A.2) $\varphi'(r) = B\phi(0)$**.** This equality requires: the existence of the derivative at $t = r$, the derivative has the value $B\phi(0)$.

(ai) To prove the existence of $\varphi'(r)$, notice that

$$\begin{aligned}
\varphi'(r^-) &= \lim_{h\to 0^-} \frac{\varphi(r+h)-\varphi(r)}{h}\\
&= \lim_{h\to 0^-} \frac{\sum\limits_{j\geq 0} w_j^1(r)(r+h)^j - \sum\limits_{j\geq 0} w_j^1(r)r^j}{h}\\
&= \lim_{h\to 0^-} \frac{\sum\limits_{j\geq 0} \frac{h^j}{j!}\big(\sum\limits_{i\geq 0} w_{j+i}^1(r)\frac{r^i}{i!}(j+i)!\big) - \sum\limits_{j\geq 0} w_j^1(r)r^j}{h}\\
&= \lim_{h\to 0^-} \frac{\sum\limits_{i\geq 0} w_i^1(r)r^i + h\sum\limits_{i\geq 0} w_{1+i}^1(r)r^i(1+i) + \sum\limits_{j\geq 2}\frac{h^j}{j!}\big(\sum\limits_{i\geq 0} w_{j+i}^1(r)\frac{r^i}{i!}(j+i)!\big) - \sum\limits_{j\geq 0} w_j^1(r)r^j}{h}\\
&= \sum_{i\geq 0} w_{1+i}^1(r)r^i(1+i) + \lim_{h\to 0^-}\sum_{j\geq 2}\frac{h^{j-1}}{j!}\Big(\sum_{i\geq 0} w_{j+i}^1(r)\,\frac{r^i}{i!}\,(j+i)!\Big).
\end{aligned}$$

If convergence of the series is ensured, then the left-hand derivative at $t=r$ is equal to

$$\varphi'(r^-) = \sum_{i\geq 0} w_{1+i}^1(r)r^i(1+i).$$

Proceeding in a similar way, we have

$$\begin{aligned}
\varphi'(r^+) &= \lim_{h\to 0^+} \frac{\varphi(r+h)-\varphi(r)}{h}\\
&= \lim_{h\to 0^+} \frac{\sum\limits_{j\geq 0} w_j^2(r)(r+h)^j - \sum\limits_{j\geq 0} w_j^1(r)r^j}{h}\\
&= \lim_{h\to 0^+} \frac{\sum\limits_{j\geq 0} \frac{h^j}{j!}\big(\sum\limits_{i\geq 0} w_{j+i}^2(r)\frac{r^i}{i!}(j+i)!\big) - \sum\limits_{j\geq 0} w_j^1(r)r^j}{h}\\
&= \lim_{h\to 0^+} \frac{\sum\limits_{i\geq 0} w_i^2(r)r^i + h\sum\limits_{i\geq 0} w_{1+i}^2(r)r^i(1+i) + \sum\limits_{j\geq 2}\frac{h^j}{j!}\big(\sum\limits_{i\geq 0} w_{j+i}^2(r)\frac{r^i}{i!}(j+i)!\big) - \sum\limits_{j\geq 0} w_j^1(r)r^j}{h}.
\end{aligned}$$

The right-hand derivative of φ at $t=r$ exists, if

$$\sum_{i\geq 0} w_i^2(r)r^i = \sum_{j\geq 0} w_j^1(r)r^j,$$
$$\sum_{i\geq 0} w_{1+i}^2(r)r^i(1+i) = \sum_{i\geq 0} w_{1+i}^1(r)r^i(1+i),$$

and the series $\sum\limits_{i\geq 0} w_{j+i}^k(r)\,\frac{r^i}{i!}\,(j+i)!$, $k=1,2$, converge.

(aii) $\varphi'(r) = B\phi(0)$.

We notice that the second condition

$$\sum_{i\geq 0} w_{1+i}^2(r)r^i(1+i) = \sum_{i\geq 0} w_{1+i}^1(r)r^i(1+i)$$

represents the equality between, respectively, $\varphi'(r^+)$ and $\varphi'(r^-)$. In order to have $\varphi'(r) = B\phi(0)$, it suffices to have

$$\sum_{i\geq 0} w_{1+i}^2(r)r^i(1+i) = B\phi(0).$$

The next proposition tells us the behaviour $w_j^2(r)$ must be such that the delay differential equation is satisfied at $t = r$. We note that in Proposition 3.1, we have established an equivalent result for the interior points of A_1.

Proposition 3.3. *If there exists $w_j^2(r)$ satisfying*

$$\text{(2.2}i\text{)} \qquad \boxed{\begin{cases} \sum_{j\geq 0} w_j^2(r)r^j = \sum_{j\geq 0} w_j^1(r)r^j, \\ \sum_{j\geq 0} w_{1+j}^2(r)r^j(1+j) = \sum_{j\geq 0} w_{1+j}^1(r)r^j(1+j) \end{cases}}$$

and

$$\text{(2.2}ii\text{)} \qquad \boxed{\sum_{j\geq 0} w_{1+j}^2(r)r^j(1+j) = B\phi(0),}$$

then $\varphi'(r) = B\phi(0)$ and equality (2.b.1) *holds at $t = r$.*

Proof. We have already seen that (2.2i) and (2.2ii) imply $\varphi'(r) = B\phi(0)$. We have to prove

$$B\phi(-r) + \sum_{j\geq 2} jw_j^1(r)r^{j-1} - B\phi(0) = 0.$$

If (2.2ii) holds, then we can write the first term as

$$\begin{aligned} B\phi(-r) &+ \sum_{i\geq 0}(2+i)w_{2+i}^1(r)r^{i+1} - \sum_{j\geq 0} w_{1+j}^2(r)r^j(1+j) \\ &= B\phi(-r) + \sum_{i\geq 0}(2+i)w_{2+i}^1(r)r^{i+1} - \sum_{j\geq 0} w_{1+j}^1(r)r^j(1+j) \\ &= \sum_{j\geq 0} w_{1+j}^1(r)r^j(1+j) - \sum_{j\geq 0} w_{1+j}^1(r)r^j(1+j), \end{aligned}$$

taking into account that $B\phi(-r) = w_1^1(r)$ and associating the terms in an appropriate way. □

The procedure we have just described, can be repeated in an inductive manner. Hence we can proceed in the following way;

(2.B.2) $\varphi'(t) = B\varphi(t-r)$**,** $t \in (r, 2r)$**.** Since

$$\varphi'(t) = \sum_{j\geq 0}(j+1)w_{j+1}^2(r)t^j$$

and

$$\varphi(t-r) = \sum_{j\geq 0} w_j^1(r)(t-r)^j = \sum_{j\geq 0} \frac{t^j}{j!}\Big(\sum_{i\geq 0} w_{j+i}^1(r)\,\frac{(-r)^i}{i!}\,(j+i)!\Big),$$

we can state that a sufficient condition for having **(2.B.2)** is

$$\textbf{(2.b.2)} \qquad \boxed{w_{j+1}^2(r) = \frac{B}{(j+1)!}\sum_{i\geq 0} w_{j+i}^1(r)\,\frac{(-r)^i}{i!}\,(j+i)! \text{ for each } j \geq 0.}$$

Thus we have finished the analysis of the solution defined on interior points of A_2.

4 The fundamental propositions

From a structural point of view, conditions (2.2i), (2.2ii) and **(2.b.2)** are identical in each interval A_n, for $n \geq 2$. Then we can state two fundamental propositions which establish sufficient conditions on $w_j^n(r)$, $n \geq 2$, in order for (3.3) to hold. The first one concerns with interior points, and the second one concerns with end points.

Proposition 4.1. *If for each $n \geq 2$ and $j \geq 0$, there exist a sequence $w_j^n(r)$ satisfying*

$$w_{j+1}^n(r) = \frac{B}{(j+1)!} \sum_{i \geq 0} w_{j+i}^{n-1}(r) \frac{(-r)^i}{i!} (j+i)!, \tag{4.1}$$

then $x_n'(t) = Bx_{n-1}(t-r)$ for $t \in int\ A_n$.

Proof. Let $x_n(t) = \sum_{j \geq 0} w_j^n(r) t^j$. If $t \in int\, A_n$ and $n \geq 2$, then

$$\begin{aligned} x_n'(t) &= \sum_{j \geq 0} (1+j) w_{1+j}^n(r) t^j = \sum_{j \geq 0} (1+j) t^j \Big(\frac{B}{(j+1)!} \sum_{i \geq 0} w_{j+i}^{n-1}(r) \frac{(-r)^i}{i!} (j+i)! \Big) \\ &= B \sum_{j \geq 0} \frac{t^j}{j!} \Big(\sum_{i \geq 0} w_{j+i}^{n-1}(r) \frac{(-r)^i}{i!} (j+i)! \Big) = B \sum_{j \geq 0} w_j^{n-1}(r)(t-r)^j = Bx_{n-1}(t-r), \end{aligned}$$

where we have considered (4.1) and Lemma 3.2. $\square$

Proposition 4.2. *If for each $n \geq 2$ there exist a sequence $w_j^n(r)$ satisfying the conditions*

$$\begin{cases} \sum_{j \geq 0} (nr)^j w_j^{n+1}(r) = \sum_{j \geq 0} (nr)^j w_j^n(r), \\ \sum_{j \geq 0} (1+j)(nr)^j w_{1+j}^{n+1}(r) = \sum_{j \geq 0} (1+j)(nr)^j w_{1+j}^n(r) \end{cases} \tag{4.2}$$

and

$$\sum_{j \geq 0} (1+j)(nr)^j w_{1+j}^{n+1}(r) = B \sum_{j \geq 0} w_j^{n-1}(r)[(n-1)r]^j, \tag{4.3}$$

then $x_n'(nr) = Bx_{n-1}((n-1)r)$.

Proof. Let $n \geq 2$ and $t = nr$.

We start by showing the existence of a derivative at the points $t = nr$ for $n \geq 2$.

- The left-hand derivative:

$$\begin{aligned} x_n'(nr^-) &= \lim_{h \to 0^-} \frac{x_n(nr+h) - x_n(nr)}{h} \\ &= \lim_{h \to 0^-} \frac{\sum_{j \geq 0} w_j^n(r)(nr+h)^j - \sum_{j \geq 0} w_j^n(r)(nr)^j}{h} \\ &= \lim_{h \to 0^-} \frac{\sum_{j \geq 0} \frac{h^j}{j!} \big(\sum_{i \geq 0} w_{j+i}^n(r) \frac{(nr)^i}{i!} (j+i)! \big) - \sum_{j \geq 0} w_j^n(r)(nr)^j}{h} \\ &= \lim_{h \to 0^-} \Bigg\{ \frac{\sum_{i \geq 0} w_i^n(r)(nr)^i + h \sum_{i \geq 0} w_{1+i}^n(r)(nr)^i(1+i)}{h} \\ &\qquad + \frac{\sum_{j \geq 2} \frac{h^j}{j!} \big(\sum_{i \geq 0} w_{j+i}^n(r) \frac{(nr)^i}{i!} (j+i)! \big) - \sum_{j \geq 0} w_j^n(r)(nr)^j}{h} \Bigg\} \\ &= \sum_{i \geq 0} w_{1+i}^n(r)(nr)^i(1+i), \end{aligned}$$

assuming that $\sum_{i\geq 0} w_{j+i}^n(r)\,\frac{(nr)^i}{i!}\,(j+i)!$ converges for $j \geq 1$.

- The right-hand derivative:

$$\begin{aligned}
x_n'(nr^+) &= \lim_{h\to 0^+} \frac{x_n(nr+h) - x_n(nr)}{h} \\
&= \lim_{h\to 0^+} \frac{\sum_{j\geq 0} w_j^{n+1}(r)(nr+h)^j - \sum_{j\geq 0} w_j^n(r)(nr)^j}{h} \\
&= \lim_{h\to 0^+} \frac{\sum_{j\geq 0} \frac{h^j}{j!}\left(\sum_{i\geq 0} w_{j+i}^{n+1}(r)\,\frac{(nr)^i}{i!}\,(j+i)!\right) - \sum_{j\geq 0} w_j^n(r)(nr)^j}{h} \\
&= \lim_{h\to 0^+} \left\{ \frac{\sum_{i\geq 0} w_i^{n+1}(r)(nr)^i + h\sum_{i\geq 0} w_{1+i}^{n+1}(r)(nr)^i(1+i)}{h} \right. \\
&\qquad\left. + \frac{\sum_{j\geq 2} \frac{h^j}{j!}\left(\sum_{i\geq 0} w_{j+i}^{n+1}(r)\,\frac{(nr)^i}{i!}\,(j+i)!\right) - \sum_{j\geq 0} w_j^n(r)(nr)^j}{h} \right\} \\
&= \sum_{i\geq 0} w_{1+i}^{n+1}(r)(nr)^i(1+i),
\end{aligned}$$

assuming that (4.2) holds, and $\sum_{i\geq 0} w_{j+i}^{n+1}(r)\,\frac{(nr)^i}{i!}\,(j+i)!$ converges for $n \geq 1$ and $j \geq 1$.

We have proved the existence of derivative of $x_n(t)$ at $t = nr$, $n \geq 2$, and

$$x_n'(nr) = \sum_{j\geq 0} w_{1+j}^{n+1}(r)(nr)^j(1+j) = \sum_{j\geq 0} w_{1+j}^n(r)(nr)^j(1+j).$$

Next, we will show that $x_n'(t) = Bx_{n-1}(t-r)$ at $t = nr$, $n \geq 2$.

$$x_n'(nr) = \sum_{j\geq 0} w_{1+j}^{n+1}(r)(nr)^j(1+j) = B\sum_{j\geq 0} w_j^{n-1}(r)[(n-1)r]^j = Bx_{n-1}((n-1)r),$$

where we have considered (4.3). □

We point out that equalities (4.1), (4.2) and (4.3) provide calculating formulas for all terms of the sequences $w_j^n(r)$ for $n \geq 2$.

Corollary 4.3. *Equality* (4.3) *is a direct consequence of* (4.1) *and* (4.2).

Proof.

$$\begin{aligned}
\sum_{j\geq 0}(1+j)(nr)^j w_{1+j}^{n+1}(r) &\underset{(4.2)}{=} \sum_{j\geq 0}(1+j)(nr)^j w_{1+j}^n(r) \\
&= \sum_{j\geq 0}(1+j)(nr)^j \left(\frac{B}{(j+1)!}\sum_{i\geq 0} w_{i+j}^{n-1}(r)\,\frac{(-r)^i}{i!}\,(i+j)!\right) \\
&= B\sum_{j\geq 0}\frac{(nr)^j}{j!}\left(\sum_{i\geq 0} w_{i+j}^{n-1}(r)\,\frac{(-r)^i}{i!}\,(i+j)!\right) = B\sum_{j\geq 0} w_j^{n-1}(r)(nr-r)^j,
\end{aligned}$$

where we have considered (4.1) and Lemma 3.2. □

We also have a correspondent result to (2.2ii), which refers to $n = 1$.

Corollary 4.4. *Equality* (2.2*ii*) *is a direct consequence of* (2.2*i*) *and* **(2.b.1)**, *being the last one applied to* $t = r$.

Proof. As a consequence of Proposition 3.3, **(2.b.1)** is verified at $t = r$, so

$$B\phi(-r) + \sum_{j\geq 2} j w_j^1(r) r^{j-1} = B\phi(0).$$

Then

$$\begin{aligned}\sum_{j\geq 0} w_{1+j}^2(r) r^j (1+j) &= \sum_{j\geq 0} w_{1+j}^1(r) r^j (1+j) \\ &= w_1^1(r) + \sum_{j\geq 1} w_{1+j}^1(r) r^j (1+j) = w_1^1(r) + B\phi(0) - B\phi(-r) = B\phi(0),\end{aligned}$$

since $w_1^1(r) = B\phi(-r)$. □

These two corollaries suggest that during the constructive process of the solution, some conditions with distinct functions emerge.

5 The main theorem

From now on, we consider $\phi(t) = \phi(0) = C$ for $t \in [-r, 0]$, where C is a real constant.

Proposition 5.1. *If*

$$w_0^1(r) = C, \quad w_1^1(r) = BC \ \textit{ and } \ w_j^1(r) = 0 \ \textit{ for } \ j \geq 2 \tag{5.1}$$

then

$$x_1(t) = \sum_{j\geq 0} w_j^1(r) t^j = C(1 + Bt)$$

is the solution of problem (2.1) *defined on* $A_1 = (0, r]$.

Proof. Equalities (5.1) verify **(2.a.1)** and **(2.b.1)**, implying $x_1'(t) = BC$ for $t \in [0, r)$. According to (3.1), we can then write

$$x_1(t) = \sum_{j\geq 0} w_j^1(r) t^j = C(1 + Bt) \ \text{ for } \ t \in (0, r).$$

By Proposition 3.3, the result is also true at $t = r$. □

The main result of this paper is the following theorem.

Theorem 5.2. *The solution of problem* (2.1) *with* $\phi(t) = C$ *if* $t \in [-r, 0]$ *can be written as*

$$X(r, t) = \sum_{j\geq 0} v_j(r) t^j$$

for $t \geq 0$. *The sequence* $v_j(r)$ *is defined by*

$$v_j(r) = C\,\frac{B^j}{j!}\,P_j^n(rB),$$

where the polynomials $P_j^n(rB)$ *are defined by*

$$P_j^n(rB) = \begin{cases} 1 + \displaystyle\sum_{i=0}^{n-(j+1)} \frac{(-rB)^{i+1}}{(i+1)!}\,(i+j)^{i+1} & \textit{if } \ j \leq n-1, \\ 1 & \textit{if } \ j = n, \\ 0 & \textit{if } \ j \geq n+1. \end{cases}$$

The proof of this Theorem is divided into four stages: Propositions 5.3 and 5.4, and Corollaries 5.5 and 5.6. In Proposition 5.3, we will obtain the calculating formulas to get all terms of the sequences, $w_j^n(r)$, $n \geq 2$. Moreover, we will show that these formulas do not depend on the fact that the initial function is constant. This fact makes this procedure an alternative method to solve problem (2.1). In Proposition 5.4 and its Corollaries, we will show the consequences of taking the initial function as a constant one.

Proposition 5.3. *For $n \geq 2$, the solution $x_n(t) = \sum_{j\geq 0} w_j^n(r)t^j$, defined on each interval A_n, is obtained through the application of the following formulas, in the following order*

$$w_{j+1}^n(r) = \frac{B}{(j+1)!} \sum_{i\geqslant 0} w_{i+j}^{n-1}(r)\, \frac{(-r)^i}{i!}\, (i+j)! \tag{5.2}$$

and

$$w_0^{n+1}(r) = w_0^n(r) - \sum_{j\geqslant 1} \left[w_j^{n+1}(r) - w_j^n(r)\right](nr)^j. \tag{5.3}$$

Proof. Equality (5.2) is the sufficient condition (4.1) in Proposition 4.1, and equality (5.3) is obtained from the first equality of (4.2). □

From now on, we consider $w_j^1(r) = 0$ for $j \geq 2$.

Combining equalities (5.2) and (5.3), we obtain the sequence $w_j^2(r)$.

According to (5.2), for $n \geq 2$,

$$w_1^2(r) = B \sum_{i\geq 0} w_i^1(r)(-r)^i = B \sum_{i=0}^{1} w_i^1(r)(-r)^i,$$

and since $w_j^1(r) = 0$ for $j \geq 2$,

$$w_1^2(r) = B(C - rBC).$$

On the other hand,

$$w_2^2(r) = \frac{B}{2!} \sum_{i\geq 0} w_{i+1}^1(r)(-r)^i(i+1) = \frac{B}{2!}\, BC.$$

It is easy to check that $w_j^2(r) = 0$ for $j \geq 3$. This will lead us to assume that for $j \geq n+1$, $w_j^n(r) = 0$. We will prove this fact in the next proposition.

It remains to calculate the first term. According to (5.3),

$$w_0^2(r) = w_0^1(r) - \sum_{j\geq 1}(w_j^2(r) - w_j^1(r))r^j = C - \sum_{j=1}^{2}(w_j^2(r) - w_j^1(r))r^j,$$

and since $w_j^2(r) = 0$ for $j \geq 3$,

$$w_0^2(r) = C - r(w_1^2(r) - w_1^1(r)) - r^2 w_2^2(r) = C\Big(1 + \frac{r^2B^2}{2}\Big).$$

Finally, we obtain the solution $x_2(t)$ defined on $A_2 = (r, 2r]$,

$$x_2(t) = \sum_{i=0}^{2} w_j^2(r)t^j = C\Big(1 + \frac{(rB)^2}{2} + t(B - rB^2) + \frac{t^2B^2}{2!}\Big).$$

We can verify that

$$x_1(r) = x_2(r) = C(1 + rB).$$

Also, notice that the form of $x_2(t)$, obtained with our calculating formulas, has exactly the same form like the one obtained with MSA.

Proposition 5.4. *Consider problem* (2.1)*, where* $\phi(t) = C$ *for* $t \in [-r, 0]$*. If* $w_j^1(r) = 0$ *for* $j \geq 2$*, then*

$$w_j^n(r) = 0 \quad \text{for} \quad j \geq n+1. \tag{5.4}$$

Proof. We will use the induction on n. The case $n = 1$ is obviously true. Assuming $w_j^n(r) = 0$ for $j \geq n+1$, consider $j \geq n+2$. As a consequence of (5.2),

$$w_j^{n+1}(r) = \frac{B}{j!} \sum_{i \geq 0} w_{i+j-1}^n(r) \, \frac{(-r)^i}{i!} \, (i+j-1)! \, .$$

By the induction step, $w_j^{n+1}(r) = 0$ if $i+j-1 \geq n+1$. Since $i \geq 0$, we can conclude $w_j^{n+1}(r) = 0$ for $j \geq n+2$ as required. □

Corollary 5.5. *In the above conditions, if* $n \geq 1$*, then*

$$w_n^n(r) = C \, \frac{B^n}{n!} \, . \tag{5.5}$$

Proof. We will use the induction on n. Since $w_1^1(r) = BC$, the result holds for $n = 1$. Assuming $w_n^n(r) = C \, \frac{B^n}{n!}$, we have

$$\begin{aligned} w_{n+1}^{n+1}(r) &= \frac{B}{(n+1)!} \sum_{i \geq 0} w_{i+n}^n(r) \, \frac{(-r)^i}{i!} \, (i+n)! \\ &= \frac{B}{(n+1)!} \, w_n^n(r) n! = \frac{B}{(n+1)!} \, C \, \frac{B^n}{n!} \, n! = C \, \frac{B^{n+1}}{(n+1)!} \, , \end{aligned}$$

where we used (5.2), (5.4) and the induction step. □

Corollary 5.6. *In the conditions of Proposition 5.4, if* $j \leq n-1$*, then*

$$w_j^n(r) = C \, \frac{B^j}{j!} \Big(1 + \sum_{i=0}^{n-(j+1)} \frac{(-rB)^{i+1}}{(i+1)!} \, (i+j)^{i+1} \Big). \tag{5.6}$$

Proof. To prove (5.6), we will use the induction reasoning applied to $j = n-k$ for the successive values $k = 1, 2, \ldots, n$. So, we will do it, first considering $k = 1$, then $k = 2$ and, finally, by an induction reasoning.

1. If $j = n-1$, $n \geq 1$, we have to prove that

$$w_{n-1}^n(r) = C \, \frac{B^{n-1}}{(n-1)!} \big[1 + (-rB)(n-1) \big].$$

Using the induction on n, the case $n = 1$ is valid as a consequence of **(2.a.1)**. Assuming $w_{n-1}^n(r) = C \, \frac{B^{n-1}}{(n-1)!} \, [1 + (-rB)(n-1)]$, we have

$$\begin{aligned} w_n^{n+1}(r) &= \frac{B}{n!} \sum_{i \geq 0} w_{i+n-1}^n(r) \, \frac{(-r)^i}{i!} \, (i+n-1)! \\ &= \frac{B}{n!} \sum_{i=0}^{1} w_{i+n-1}^n(r) \, \frac{(-r)^i}{i!} \, (i+n-1)! = \frac{B}{n!} \, \big\{ w_{n-1}^n(r)(n-1)! + w_n^n(r)(-r) n! \big\} \\ &= B \Big\{ \frac{1}{n} \, \frac{C B^{n-1}}{(n-1)!} \, \big[1 + (-rB)(n-1) \big] + (-r) C \, \frac{B^n}{n!} \Big\} \\ &= C \, \frac{B^n}{n!} \, [1 - rBn + rB - rB] = C \, \frac{B^n}{n!} \, [1 + (-rB) n], \end{aligned}$$

where we have used (5.2) for $j = n-1$, (5.4), (5.5) and the induction step.

2. If $j = n-2$ for $n \geq 2$, we have to prove that

$$w_{n-2}^{n}(r) = C\,\frac{B^{n-2}}{(n-2)!}\left[1 + (-rB)(n-2) + \frac{(-rB)^2}{2!}\,(n-1)^2\right].$$

For $n = 2$, using (5.3), we have

$$\begin{aligned} w_0^2(r) = w_0^1(r) - \sum_{j\geq 1}\left[w_j^2(r) - w_j^1(r)\right]r^j &= C - [w_1^2(r) - w_1^1(r)]r - [w_2^2(r) - 0]r^2 - 0 \\ &= C - \left[B(C - rBC) - CB\right]r - C\,\frac{B^2}{2!}\,r^2 = C\Big(1 + \frac{(-rB)^2}{2!}\Big). \end{aligned}$$

Assuming

$$w_{n-2}^{n}(r) = C\,\frac{B^{n-2}}{(n-2)!}\left[1 + (-rB)(n-2) + \frac{(-rB)^2}{2!}\,(n-1)^2\right],$$

we have

$$\begin{aligned} w_{n-1}^{n+1}(r) &= \frac{B}{(n-1)!}\sum_{i\geq 0} w_{i+n-2}^{n}(r)\,\frac{(-r)^i}{i!}\,(i+n-2)! \\ &= \frac{B}{(n-1)!}\Big\{w_{n-2}^{n}(r)(n-2)! + w_{n-1}^{n}(r)(-r)(n-1)! + w_n^n(r)\,\frac{(-r)^2}{2!}\,n!\Big\} \\ &= C\,\frac{B^{n-1}}{(n-1)!}\left[1 + (-rB)(n-1) + \frac{(-rB)^2}{2!}\,n^2\right], \end{aligned}$$

where we have used (5.2) for $j = n-2$, (5.4), (5.5) and the induction step.

It can be assumed by induction that

$$\text{for } \; k = 1, 2, \ldots, n \; \text{ and } \; n \geq k,$$

$$w_{n-k}^{n}(r) = C\,\frac{B^{n-k}}{(n-k)!}\Big(1 + \sum_{i=0}^{k-1}\frac{(-rB)^{i+1}}{(i+1)!}\,(i+n-k)^{i+1}\Big).$$

The proof of the theorem is now complete. □

Hence, if we fix $n \in \mathbb{N}$, we can calculate $x_n(t)$ with $t \in A_n = ((n-1)r, nr]$ using

$$x_n(t) = \sum_{j\geq 0} w_j^n(r)t^j = C\sum_{j\geq 0}\frac{B^j}{j!}\,P_j^n(rB)t^j, \tag{5.7}$$

where $P_j^n(rB)$ are defined in Theorem 5.2.

The solution found by this new method coincides with the one obtained by the method of steps, the recurrences formulas (5.2) and (5.3) can be replaced by (5.7), whenever $\phi(t)$ is a constant, and, finally, the solution of problem (2.1) is the generating function for $\{w_j^n(r)\}_{j=0,1,\ldots}$.

6 Proof of Lemma 3.2

For $r > 0$, $t \neq 0$ and $t \neq r$,

$$\sum_{j\geq 0} f_j(r)(t-r)^j = \sum_{j\geq 0}\frac{t^j}{j!}\Big(\sum_{i\geq 0} f_{j+i}(r)\,\frac{(-r)^i}{i!}(j+i)!\Big).$$

Proof.

$$
\begin{aligned}
&\sum_{j\geq 0} f_j(r)(t-r)^j \\
&\quad = f_0(r)+f_1(r)(t-r)+f_2(r)(t-r)^2+f_3(r)(t-r)^3+\cdots+f_p(r)(t-r)^p+f_{p+1}(r)(t-r)^{p+1}+\cdots \\
&\quad = \Big[f_0(r)+f_1(r)(-r)+f_2(r)(-r)^2+f_3(r)(-r)^3+\cdots+f_p(r)(-r)^p+f_{p+1}(r)(-r)^{p+1}+\cdots\Big] \\
&\qquad + t\Big[f_1(r)-2rf_2(r)+3r^2f_3(r)-4r^3f_4(r)+\cdots+\binom{p}{p-1}(-r)^{p-1}f_p(r) \\
&\qquad\qquad + f_{p+1}(r)(-r)^p\Big(1+\binom{p}{p-1}\Big)+\cdots\Big] \\
&\qquad + t^2\Big[f_2(r)-3rf_3(r)+6r^2f_4(r)+\cdots+\binom{p}{p-2}(-r)^{p-2}f_p(r) \\
&\qquad\qquad + f_{p+1}(r)(-r)^{p-1}\Big(\binom{p}{p-1}+\binom{p}{p-2}\Big)+\cdots\Big] \\
&\qquad + t^3\Big[f_3(r)-4rf_4(r)+\cdots+\binom{p}{p-3}(-r)^{p-3}f_p(r) \\
&\qquad\qquad + f_{p+1}(r)(-r)^{p-2}\Big(\binom{p}{p-2}+\binom{p}{p-3}\Big)+\cdots\Big] \\
&\qquad + \cdots + t^{p-2}\Big[\binom{p}{2}(-r)^2f_p(r)+f_{p+1}(r)(-r)^3\Big(\binom{p}{3}+\binom{p}{2}\Big)+\cdots\Big] \\
&\qquad + t^{p-1}\Big[\binom{p}{1}(-r)f_p(r)+f_{p+1}(r)(-r)^2\Big(\binom{p}{2}+\binom{p}{1}\Big)+\cdots\Big] \\
&\qquad + t^p\Big[f_p(r)+f_{p+1}(r)(-r)\Big(\binom{p}{1}+1\Big)+\cdots\Big]+\cdots.
\end{aligned}
$$

Define $g(z)=\sum_{j\geqslant 0}(-1)^j f_j(r)z^j$, where $g(r)=\sum_{j\geqslant 0} f_j(r)(-r)^j$ represents

$$f_0(r)+f_1(r)(-r)+f_2(r)(-r)^2+f_3(r)(-r)^3+\cdots+f_p(r)(-r)^p+f_{p+1}(r)(-r)^{p+1}+\cdots$$

Claim 6.1.

$$
\begin{aligned}
t\Big[&f_1(r)-2rf_2(r)+3r^2f_3(r)-4r^3f_4(r)+\cdots \\
&+\binom{p}{p-1}(-r)^{p-1}f_p(r)+f_{p+1}(r)(-r)^p\Big(1+\binom{p}{p-1}\Big)+\cdots\Big]
\end{aligned}
$$

can be written as

$$-tg'(z)\big|_{z=r}.$$

Indeed, $g'(z)=\sum_{j\geqslant 0}(-1)^{j+1}(j+1)f_{j+1}(r)z^j$. So,

$$
\begin{aligned}
-tg'(z)_{z=r} &= t\Big[\sum_{j\geqslant 0}(-z)^j(j+1)f_{j+1}(r)\Big]_{z=r} \\
&= t\big[f_1(r)-2rf_2(r)+3r^2f_3(r)+\cdots+p(-r)^{p-1}f_p(r)+\cdots\big].
\end{aligned}
$$

Claim 6.2.

$$
\begin{aligned}
t^2\Big[&f_2(r)-3rf_3(r)+6r^2f_4(r)+\cdots \\
&+\binom{p}{p-2}(-r)^{p-2}f_p(r)+f_{p+1}(r)(-r)^{p-1}\Big(\binom{p}{p-1}+\binom{p}{p-2}\Big)+\cdots\Big]
\end{aligned}
$$

can be written as

$$\frac{t^2}{2!}\, g''(z)\Big|_{z=r}.$$

Indeed, $g''(z) = \sum_{j\geqslant 0}(-1)^{j+2}(j+1)(j+2)f_{j+2}(r)z^j$. So,

$$\begin{aligned}\frac{t^2}{2!}\, g''(z)_{z=r} = \frac{t^2}{2!}\Big[\sum_{j\geqslant 0}(-z)^j(j+1)(j+2)f_{j+2}(r)\Big]_{z=r}\\ = t^2\Bigg[f_2(r) - 3rf_3(r) + 6r^2f_4(r) + \cdots + \binom{p}{2}(-r)^{p-2}f_p(r) + \cdots\Bigg].\end{aligned}$$

Repeating the process,

Claim 6.3.

$$t^n\Bigg[f_n(r) + f_{n+1}(r)(-r)\Bigg(\binom{n}{1}+1\Bigg) + \cdots\Bigg]$$

can be written as

$$(-1)^n\,\frac{t^n}{n!}\, g^{(n)}(z)\Big|_{z=r},$$

where

$$g^{(n)}(z) = \sum_{j\geqslant 0}(-1)^{j+n}(j+1)(j+2)\cdots(j+n)f_{j+n}(r)z^j.$$

We prove this fact by the induction on n.

Proof. As we have already seen,

$$g'(z) = \sum_{j\geqslant 0}(-1)^{j+1}(j+1)f_{j+1}(r)z^j,$$

so the case $n = 1$ is verified.

Assuming

$$g^{(n-1)}(z) = \sum_{j\geqslant 0}(-1)^{j+n-1}(j+1)(j+2)\cdots(j+n-1)f_{j+n-1}(r)z^j,$$

we have to prove that

$$g^{(n)}(z) = \sum_{j\geqslant 0}(-1)^{j+n}(j+1)(j+2)\cdots(j+n)f_{j+n}(r)z^j.$$

We have

$$\begin{aligned}g^{(n)}(z) &= \frac{d}{dz}\Big(\sum_{j\geqslant 0}(-1)^{j+n-1}(j+1)(j+2)\cdots(j+n-1)f_{j+n-1}(r)z^j\Big)\\ &= \sum_{j\geqslant 0}(-1)^{j+n}(j+1)(j+2)\cdots(j+n)f_{j+n}(r)z^j.\end{aligned}$$

Hence, we can write

$$\begin{aligned}\sum_{j\geq 0} f_j(r)(t-r)^j &= g(r) - tg'(r) + \frac{t^2}{2!}\, g''(r) - \frac{t^3}{3!}\, g'''(r) + \cdots + (-1)^p\,\frac{t^p}{p!}\, g^{(p)}(r) + \cdots\\ &= \sum_{j\geqslant 0}(-1)^j\,\frac{t^j}{j!}\, g^{(j)}(r) = \sum_{j\geqslant 0}(-1)^j\,\frac{t^j}{j!}\Big(\sum_{i\geqslant 0}(-1)^{i+j}(i+1)\cdots(i+j)f_{i+j}(r)r^i\Big)\\ &= \sum_{j\geqslant 0}\frac{t^j}{j!}\Big(\sum_{i\geqslant 0}(-r)^i\,\frac{(i+j)!}{i!}\, f_{i+j}(r)\Big) = \sum_{j\geqslant 0}\frac{t^j}{j!}\Big(\sum_{i\geqslant 0} f_{i+j}(r)\,\frac{(-r)^i}{i!}\,(i+j)!\Big),\end{aligned}$$

where we have used the equality $(i+1)(i+2)(i+3)\cdots(i+j) = \frac{(i+j)!}{i!}$. □

7 An application

This example is presented in [5].

Suppose we wish to study a population $P(t) = (x(t), y(t))$, where $x(t)$ denotes the average height and $y(t)$ the average weight. It was observed that $x(t)$ depends on the height of the previous generation through $x'(t) = Bx(t-r)$, where r is the size (per units of time) of a generation.

We can determine explicitly the behaviour of this variable regarding the fourth generation. This means that we wish to compute $x_4(t)$, given $x(t) = C$ for $t \in [-r, 0]$, where C is the average height.

Using equation (5.7), we can determine it directly, without having to compute the height for the previous generations,

$$x_4(t) = \sum_{j\geq 0} w_j^4(r)t^j = C\sum_{j\geq 0} \frac{B^j}{j!}\, P_j^4(rB)t^j,$$

where $P_j^4(rB)$ are computed by applying Theorem 5.2. From this theorem, since

$$\begin{aligned}
&P_4^4(rB) = 1,\\
&P_3^4(rB) = 1 + \sum_{i=0}^{0} \frac{(-rB)^{i+1}}{(i+1)!}\,(i+3)^{i+1} = 1 + 3(-rB),\\
&P_2^4(rB) = 1 + \sum_{i=0}^{1} \frac{(-rB)^{i+1}}{(i+1)!}\,(i+2)^{i+1} = 1 + 2(-rB) + \frac{3^2}{2!}\,(-rB)^2,\\
&P_1^4(rB) = 1 + \sum_{i=0}^{2} \frac{(-rB)^{i+1}}{(i+1)!}\,(i+1)^{i+1} = 1 + (-rB) + \frac{2^2}{2!}\,(-rB)^2 + \frac{3^3}{3!}\,(-rB)^3,\\
&P_0^4(rB) = 1 + \sum_{i=0}^{3} \frac{(-rB)^{i+1}}{(i+1)!}\,i^{i+1} = 1 + \frac{(-rB)^2}{2!} + \frac{2^3}{3!}\,(-rB)^3 + \frac{3^4}{4!}\,(-rB)^4,
\end{aligned}$$

for $j = 0, 1, 2, 3, 4$, we have

$$\begin{aligned}
x_4(t) &= C\Big\{P_0^4(rB) + BP_1^4(rB)t + \frac{B^2}{2!}\,P_2^4(rB)t^2 + \frac{B^3}{3!}\,P_3^4(rB)t^3 + \frac{B^4}{4!}\,P_4^4(rB)t^4\Big\}\\
&= C\Bigg\{1 + \frac{(-rB)^2}{2!} + \frac{2^3}{3!}\,(-rB)^3 + \frac{3^4}{4!}\,(-rB)^4 + Bt\Big(1 - rB + \frac{2^2}{2!}\,(-rB)^2 + \frac{3^3}{3!}\,(-rB)^3\Big)\\
&\qquad + \frac{B^2}{2!}\,t^2\Big(1 - 2rB + \frac{3^2}{2!}\,(-rB)^2\Big) + \frac{B^3}{3!}\,t^3(1 - 3rB) + \frac{B^4}{4!}\,t^4\Bigg\}.
\end{aligned}$$

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SOME PROPERTIES OF A SOLUTION AND FINITE DIFFERENCE SCHEME FOR ONE NONLINEAR PARTIAL DIFFERENTIAL MODEL BASED ON THE MAXWELL SYSTEM

Abstract. Linear stability and Hoph bifurcation of a solution of the initial-boundary value problem as well as the finite difference scheme for one system of nonlinear partial differential equations are investigated. The blow up case is fixed. The mentioned system is based on the Maxwell equations which describe the process of electromagnetic field penetration into a substance. Numerous computer experiments are carried out and relying on the obtained results, some graphical illustrations are presented.

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რეზიუმე. ერთი არაწრფივი კერძოწარმოებულებიან დიფერენციალურ განტოლებათა სისტემისთვის გამოკვლეულია საწყის-სასაზღვრო ამოცანის ამონახსნის წრფივი მდგრადობა და ჰოფის ბიფურკაცია. დაფიქსირებულია ფეთქებადი ამონახსნის შემთხვევა. აღნიშნული სისტემა დაფუძნებულია მაქსველის განტოლებებზე, რომელიც აღწერს ელექტრომაგნიტური ველის გარემოში გავრცელების პროცესს. ჩატარებულია მრავალი რიცხვითი ექსპერიმენტი და მიღებულ შედეგებზე დაყრდნობით წარმოდგენილია გრაფიკული ილუსტრაციები.

1 Introduction

The aim of the present paper is to study the linear stability and Hoph bifurcation of a solution of the initial-boundary value problem and finite difference scheme for one diffusion system of nonlinear partial differential equations. Such systems arise in mathematical modeling of the process of penetration of an electromagnetic field into a substance. Upon penetrating through a material, the variable magnetic field induces in it a variable electronic field, which causes the appearance of currents. The currents lead to the heating of the material and arise its temperature that affects the diffusion process. For large oscillations of temperature, the dependence on it should be taken into consideration. In a quasistationary case, the corresponding system of Maxwell equations has the form [10]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}(\nu_m \operatorname{rot} H), \tag{1.1}$$

$$c_\nu \frac{\partial \theta}{\partial t} = \nu_m (\operatorname{rot} H)^2, \tag{1.2}$$

where $H = (H_1, H_2, H_3)$ is the vector of a magnetic field, θ is temperature, c_ν and ν_m characterize the thermal heat capacity and electroconductivity of the substance. System (1.1) defines the process of diffusion of the magnetic field and equation (1.2) describes the change of temperature. As a rule, the coefficients c_ν and ν_m depend on temperature θ, $c_\nu = c_\nu(\theta)$, $\nu_m = \nu_m(\theta)$.

Many authors are studying models (1.1), (1.2) and their different variations and generalizations (see, e.g., [1–3, 8, 14, 16, 17] and the references therein). In [7], the reduction to the integro-differential model of system (1.1), (1.2) was proposed and investigated. As for the investigation and approximation solution of various versions of Maxwell system and the corresponding to it integro-differential models, one can find, for example, in [8] (see also the references therein). The existence of the corresponding initial-boundary value problems for such kind of integro-differential models can be proved by using Galerkin's modified method and compactness arguments as in [11, 15] for nonlinear parabolic equations and, as it is carried out in [5–7], for the case of one-component magnetic field.

The rest of the present paper is organized as follows. In Section 2, the problem is stated and the linear stability of a solution of the initial-boundary value problem with nonhomogeneous boundary conditions on the right side of the lateral boundary is studied. The possibility of appearance of Hoph bifurcation and the blow up case are fixed, as well. In Section 3, the finite difference scheme for the problem considered in Section 2 is constructed and its convergence is investigated. At the end of this section, some graphical illustrations, confirming theoretical findings are given. The final Section 4 contains brief conclusion.

2 Linear stability and Hoph bifurcation

The model of Maxwell equations (1.1), (1.2) is complex enough for theoretical investigations and practical applications.

In some physical assumptions, if the vector of a magnetic field has the form $H = (0, U, V)$, where $U = U(x,t)$ and $V = V(x,t)$, then in the cylinder $[0,1] \times [0,\infty)$ we consider the initial-boundary value problem

$$\begin{gathered}
\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(S^\alpha \frac{\partial U}{\partial x}\Big), \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\Big(S^\alpha \frac{\partial V}{\partial x}\Big), \\
\frac{\partial S}{\partial t} = -aS^\beta + bS^\gamma \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big], \\
U(0,t) = V(0,t) = 0, \quad U(1,t) = \psi_1, \quad V(1,t) = \psi_2, \\
U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \\
S(x,0) = S_0(x) > s_0 = const > 0,
\end{gathered} \tag{2.1}$$

where a, b, ψ_1, ψ_2 are positive constants and α, β, γ are real numbers which will be specified later; $U_0(x)$, $V_0(x)$, $S_0(x)$ are the known functions of their arguments.

Stabilization of the stationary solution and the finite difference scheme for the special cases of the above model were investigated in [4, 6, 9].

It is not difficult to show that if $\beta \neq \gamma$, then the stationary solution of problem (2.1) has the form

$$U_s = \psi_1 x, \quad V_s = \psi_2 x, \quad S_s = \Big(\frac{(\psi_1^2+\psi_2^2)b}{a}\Big)^{\frac{1}{\beta-\gamma}}.$$

The following statement holds.

Theorem 2.1. *Let $2\alpha+\beta-\gamma>0$, $\beta\neq\gamma$, then the stationary solution $(U_s, V_s S_s)$ of problem (2.1) is linearly stable if and only if the inequality*

$$a(\gamma-\beta)\Big[\frac{b}{a}\,(\psi_1^2+\psi_2^2)\Big]^{\frac{\beta-\alpha-1}{\beta-\gamma}} < \pi^2$$

is fulfilled.

Proof. Assume that a solution of problem (2.1) has the form

$$U(x,t) = U_s + u(x,t), \quad V(x,t) = V_s + v(x,t), \quad S(x,t) = S_s + s(x,t), \tag{2.2}$$

where $u(x,t)$, $v(x,t)$, $s(x,t)$ are small perturbations.

Introducing the notations

$$\begin{aligned}
&\alpha_s = \alpha\psi_1\Big(\frac{(\psi_1^2+\psi_2^2)b}{a}\Big)^{-\frac{\alpha-1}{\beta-\gamma}}, &&\beta_s = \Big(\frac{(\psi_1^2+\psi_2^2)b}{a}\Big)^{\frac{\alpha}{\beta-\gamma}},\\
&\gamma_s = \alpha\psi_2\Big(\frac{(\psi_1^2+\psi_2^2)b}{a}\Big)^{\frac{\alpha-1}{\beta-\gamma}}, &&\nu_s = (\gamma-\beta)\,\frac{b^{\frac{\beta-1}{\beta-\gamma}}}{a^{\frac{\gamma-1}{\beta-\gamma}}}\,(\psi_1^2+\psi_2^2)^{\frac{\beta-1}{\beta-\gamma}},\\
&\eta_s = 2\psi_1 b\Big[\frac{b}{a}\,(\psi_1^2+\psi_2^2)\Big]^{\frac{\gamma}{\beta-\gamma}}, &&\mu_s = 2\psi_2 b\Big[\frac{b}{a}(\psi_1^2+\psi_2^2)\Big]^{\frac{\gamma}{\beta-\gamma}},
\end{aligned}$$

after linearization of the system of problem (2.1) we get the following system of partial differential equations:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \alpha_s\,\frac{\partial s}{\partial x} + \beta_s\,\frac{\partial^2 u}{\partial x^2},\\
\frac{\partial v}{\partial t} &= \gamma_s\,\frac{\partial s}{\partial x} + \beta_s\,\frac{\partial^2 v}{\partial x^2},\\
\frac{\partial s}{\partial t} &= \nu_s s + \eta_s\,\frac{\partial u}{\partial x} + \mu_s\,\frac{\partial v}{\partial x}\,.
\end{aligned}\tag{2.3}$$

We seek for a solution of system (2.3) in the form

$$u(x,t) = u(x)e^{\omega t}, \quad v(x,t) = v(x)e^{\omega t}, \quad s(x,t) = s(x)e^{\omega t}, \tag{2.4}$$

and get the problem on eigenvalues for the following system of ordinary differential equations:

$$\begin{gathered}
\omega u = \alpha_s\,\frac{ds}{dx} + \beta_s\,\frac{d^2u}{dx^2}\,, \quad \omega v = \gamma_s\,\frac{ds}{dx} + \beta_s\,\frac{d^2v}{dx^2}\,,\\
\omega s = \nu_s s + \eta_s\,\frac{du}{dx} + \mu_s\,\frac{dv}{dx}\,.
\end{gathered}\tag{2.5}$$

Assume now that a solution of system (2.5) is of the form

$$u(x) = u_0 e^{ikx}, \quad v(x) = v_0 e^{ikx}, \quad s(x) = s_0 e^{ikx}.$$

Substituting these functions in (2.5), we get

$$\begin{gathered}
\omega u_0 e^{ikx} = \alpha_s ike^{ikx} s_0 - \beta_s e^{ikx}k^2 u_0, \quad \omega v_0 e^{ikx} = \gamma_s ike^{ikx}s_0 - \beta_s e^{ikx}k^2 v_0,\\
\omega s_0 e^{ikx} = \nu_s s_0 e^{ikx} + \eta_s iku_0 e^{ikx} + \mu_s ikv_0 e^{ikx}
\end{gathered}$$

from which we obtain

$$u_0(\omega+\beta_s k^2)-\alpha_s iks_0=0, \quad v_0(\omega+\beta_s k^2)-\gamma_s iks_0=0,$$
$$u_0ik\eta_s+v_0\mu_s ik+s_0(\nu_s-\omega)=0.$$

It is clear that this system has a nontrivial solution if and only if the condition

$$\Delta(\omega,k)=\begin{vmatrix}\omega+\beta_s k^2 & 0 & -ik\alpha_s\\ 0 & \omega+\beta_s k^2 & -ik\gamma_s\\ ik\eta_s & ik\mu_s & \nu_s-\omega\end{vmatrix}$$
$$=(\omega+\beta_s k^2)^2(\nu_s-\omega)-(\omega+\beta_s k^2)k^2\alpha_s\eta_s-(\omega+\beta_s k^2)\mu_s\gamma_s k^2=0,$$

or

$$(\omega+\beta_s k^2)[(\omega+\beta_s k^2)(\nu_s-\omega)-k^2\alpha_s\eta_s-k^2\mu_s\gamma_s]=0$$

is fulfilled. This implies that

$$k^2(\beta_s\nu_s-\beta_s\omega-\alpha_s\eta_s-\mu_s\gamma_s)-\omega^2+\omega\nu_s=0. \tag{2.6}$$

Since the case $\omega+\beta_s k^2=0$ is trivial, the latest equality gives two values of the parameter k such as $k_1=-k_2$.

It is easy to show that the solution of system (2.5) has the form

$$\begin{gathered}u(x)=\frac{ik_1\alpha_s}{\omega+\beta_s k_1^2}\,(S_1e^{ik_1x}-S_2e^{-ik_1x}), \quad v(x)=\frac{ik_1\gamma_s}{\omega+\beta_s k_1^2}\,(S_1e^{ik_1x}-S_2e^{-ik_1x}),\\ s(x)=S_1e^{ik_1x}+S_2e^{-ik_1x},\end{gathered} \tag{2.7}$$

where S_1 and S_2 are the constants.

Taking into account the boundary conditions (2.1), from (2.2) and (2.4) we get

$$u(0)=u(1)=0.$$

From this, taking into account (2.7), we get the following system:

$$S_1-S_2=0,$$
$$S_1e^{ik_1}-S_2e^{-ik_1}=0,$$

which above has a nontrivial solution when

$$\Delta=\begin{vmatrix}1 & -1\\ e^{ik_1} & -e^{-ik_1}\end{vmatrix}=e^{ik_1}-e^{-ik_1}=2i\sin k_1=0,$$

or

$$k_{1n}=\pi n, \ \ n\in Z.$$

Let us rewrite equation (2.6) in the form

$$\omega_n^2+P_n(\beta_s,k_n,\nu_s)\omega_n+L_n(\beta_s,k_n,\nu_s,\eta_s,\mu_s,\gamma_s)=0,$$

where

$$P_n(\beta_s,k_n,\nu_s)=\beta_s k_n^2-\nu_s,$$
$$L_n(\beta_s,k_n,\nu_s,\eta_s,\mu_s,\gamma_s)=-\beta_s\nu_s k_n^2+\alpha_s\eta_s k_n^2+\mu_s\gamma_s k_n^2.$$

It should be noted that the solution of problem (2.1) is linearly stable if and only if for all n the inequality $\mathrm{Re}(\omega_n)<0$ holds. It is easy to show that if $2\alpha+\beta-\gamma>0$, then $L_n(\beta_s,k_n,\nu_s,\eta_s,\mu_s,\gamma_s)>0$.

Therefore, for the solution to be linearly stable, it is necessary and sufficient that the inequality

$$P_n=\beta_s k_n^2-\nu_s=\Big(\frac{(\psi_1^2+\psi_2^2)b}{a}\Big)^{\frac{\alpha}{\beta-\gamma}}\pi^2n^2-(\gamma-\beta)\,\frac{b^{\frac{\beta-1}{\beta-\gamma}}}{a^{\frac{\gamma-1}{\beta-\gamma}}}(\psi_1^2+\psi_2^2)^{\frac{\beta-1}{\beta-\gamma}}>0,$$

or

$$a(\gamma-\beta)\Big[\frac{b}{a}\,(\psi_1^2+\psi_2^2)\Big]^{\frac{\beta-\alpha-1}{\beta-\gamma}}<\pi^2 \ \ (n=1)$$

holds. Thus, the proof of Theorem 2.1 is complete. □

Remark. As we can see from the above inequality, when $\gamma < \beta$, the solution of problem (2.1) is always linearly stable.

Assume that $\gamma > \beta$, $\beta - \alpha - 1 \neq 0$ and consider the value

$$\psi_s = \Big[\frac{\pi^2}{\gamma - \beta} a^{\frac{\gamma-\alpha-1}{\beta-\gamma}} b^{\frac{\alpha-\beta+1}{\beta-\gamma}}\Big]^{\frac{\beta-\gamma}{\beta-\alpha-1}},$$

for which

$$P_1(\psi_s, \alpha, \beta, \gamma) = 0, \quad P_n(\psi_s, \alpha, \beta, \gamma) > 0, \quad n = 2, 3, \dots .$$

In addition, if we assume that $\beta - \alpha - 1 < 0$, then for $\psi \in (0, \psi_s)$, $\psi = \psi_1^2 + \psi_2^2$, we have $P_n(\psi, \alpha, \beta, \gamma) > 0$, $n \in Z_0$.

Therefore, if $\psi \in (0, \psi_s)$, then the solution of problem (2.1) is linearly stable, and if $\psi > \psi_s$, then it is unstable. For $\psi = \psi_s$, we have $\mathrm{Re}(\omega_1) = 0$ and $\mathrm{Im}(\omega_1) \neq 0$, i.e., there appears the possibility of Hoph bifurcation. The small perturbations may cause transformation of a solution into a periodic oscillations [12].

Consider the problem

$$\begin{gathered}
\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(S^\alpha \frac{\partial U}{\partial x}\Big), \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\Big(S^\alpha \frac{\partial V}{\partial x}\Big), \\
\frac{\partial S}{\partial t} = S^\alpha \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big], \\
U(0,t) = V(0,t) = 0, \quad U(1,t) = \psi_1, \quad V(1,t) = \psi_2, \\
U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad S(x,0) = S_0(x) \geq s_0 = const > 0.
\end{gathered} \tag{2.8}$$

It is not difficult to verify that if $\alpha \neq 1$ and $S_0(x) = s_0$, then the functions

$$\begin{gathered}
U(x,t) = \psi_1 x, \quad V(x,t) = \psi_2 x, \\
S(x,t) = \big[s_0^{1-\alpha} + (1-\alpha)(\psi_1^2 + \psi_2^2)t\big]^{\frac{1}{1-\alpha}}
\end{gathered}$$

are the solutions of problem (2.8). But if $\alpha > 1$ at a finite time $t_0 = s_0^{1-\alpha}/[(\psi_1^2 + \psi_2^2)(\alpha - 1)]$, the function $S(x,t)$ becomes infinity. This example shows that the solution of problem (2.8) with smooth initial and boundary conditions can be blown up at a finite time.

3 Convergence of finite difference scheme

In the rectangle $[0,1] \times [0,T]$, where T is a positive number, let us consider the initial-boundary value problem (2.1).

Now, we study a numerical approximation of problem (2.1). If we introduce the notation $W = S^{1/2}$, then problem (2.1) takes the form

$$\begin{gathered}
\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\Big(W^{2\alpha} \frac{\partial U}{\partial x}\Big) = 0, \quad \frac{\partial V}{\partial t} - \frac{\partial}{\partial x}\Big(W^{2\alpha} \frac{\partial V}{\partial x}\Big) = 0, \\
\frac{\partial W}{\partial t} = -\frac{a}{2} W^{2\beta-1} + \frac{b}{2} W^{2\gamma-1}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big], \\
U(0,t) = V(0,t) = 0, \quad U(1,t) = \psi_1, \quad V(1,t) = \psi_2, \\
U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad W(x,0) = [S_0(x)]^{1/2}.
\end{gathered} \tag{3.1}$$

Let us discretize the domain $[0,1] \times [0,T]$ and apply the following known notations [13]

$$\begin{gathered}
h = \frac{1}{M}, \quad \tau = \frac{T}{N}, \quad x_i = ih, \quad t_j = j\tau, \quad r(x_i, t_j) = r_i^j, \\
\overline{\omega}_h = \big\{x_i, \ i = 0, 1, \dots, M\big\}, \quad \omega_h^* = \Big\{x_i = \Big(i - \frac{1}{2}\Big)h, \ i = 1, 2, \dots, M\Big\}, \\
\omega_\tau = \big\{t_j = j\tau, \ j = 0, 1, \dots, N\big\}, \quad \overline{\omega}_{h\tau} = \overline{\omega}_h \times \omega_\tau, \quad \omega_{h\tau}^* = \omega_h^* \times \omega_\tau, \\
r_{x,i}^j = \frac{r_{i+1}^j - r_i^j}{h}, \quad r_{\overline{x},i}^j = \frac{r_i^j - r_{i-1}^j}{h}, \quad r_{t,i}^j = \frac{r_i^{j+1} - r_i^j}{\tau}
\end{gathered}$$

and the corresponding inner products and norms

$$(r^j, g^j) = h \sum_{i=1}^{M-1} r_i^j g_i^j, \quad (r^j, g^j] = h \sum_{i=1}^{M} r_i^j g_i^j,$$
$$\|r^j\| = (r^j, r^j)^{1/2}, \quad \|r^j]| = (r^j, r^j]^{1/2}.$$

For problem (3.1), consider the following finite difference scheme:

$$\begin{gathered} u_t^j = (w^{2\alpha} u_{\overline{x}})_x, \quad v_t^j = (w^{2\alpha} v_{\overline{x}})_x, \\ w_t^j = -\frac{a}{2}\, w^{2\beta-1} + \frac{b}{2}\, w^{2\gamma-1}(u_{\overline{x}}^2 + v_{\overline{x}}^2), \\ u_0^j = v_0^j = 0, \quad u_M^j = \psi_1, \quad v_M^j = \psi_2, \quad j = 0, 1, \dots, N, \\ u_i^0 = U_0(x_i), \quad v_i^0 = V_0(x_i), \quad w_i^0 = \left[S_0(x_{i+1/2})\right]^{1/2}, \ \ i = 0, 1, \dots, M-1, \end{gathered} \tag{3.2}$$

where the grid functions u and v are defined on $\overline{\omega}_{h\tau}$, while the grid function w is defined on $\omega_{h\tau}^*$. Note that here and below, if the grid functions are taken without indices of time level, it assumed that they are considered at t_{j+1}.

It is not difficult to show that an approximation error of scheme (3.2) on smooth solutions of problem (3.1) is $O(\tau + h^2)$.

The following statement holds.

Theorem 3.1. *An approximation error of scheme* (3.2) *on smooth solutions of problem* (3.1) *is* $O(\tau + h^2)$ *and if* $\beta \geq 1/2$, $\alpha = \gamma$, $|\alpha| \leq 1/2$, *then a solution of scheme* (3.2) *converges to the solution of problem* (3.1) *in discrete analogues of the norms of the space* $L_2(0,1)$ *and the rate of convergence is the same as an approximation error.*

Proof. For the errors $X = u - U$, $Y = v - V$ and $Z = w - W$, we have

$$X_t^j = (w^{2\alpha} u_{\overline{x}} - W^{2\alpha} U_{\overline{x}})_x + \varphi_1, \tag{3.3}$$

$$Y_t^j = (w^{2\alpha} v_{\overline{x}} - W^{2\alpha} V_{\overline{x}})_x + \varphi_2, \tag{3.4}$$

$$Z_t^j = -\frac{a}{2}\,(w^{2\beta-1} - W^{2\beta-1}) + \frac{b}{2}\,(w^{2\gamma-1} u_{\overline{x}}^2 - W^{2\gamma-1} U_{\overline{x}}^2 + w^{2\gamma-1} v_{\overline{x}}^2 - W^{2\gamma-1} V_{\overline{x}}^2) + \varphi_3, \tag{3.5}$$

where $\varphi_k = O(\tau + h^2)$, $k = 1, 2, 3$.

Assume $\alpha = \gamma$ and $|\alpha| \leq \frac{1}{2}$. Let us multiply scalarly equations (3.3)–(3.5) by $2\tau X$, $2\tau Y$ and $\frac{2}{b}\,\tau Z$, respectively. Using the discrete analogue of integration by parts and the identities [13]

$$2\tau(X_t, X) = \|X\|^2 - \|X^j\|^2 + \tau^2 \|X_t\|^2, \quad 2\tau(Y_t, Y) = \|Y\|^2 - \|Y^j\|^2 + \tau^2 \|Y_t\|^2,$$
$$2\tau(Z_t, Z] = ||Z]|^2 - ||Z^j]|^2 + \tau^2 ||Z_t]|^2,$$

we get

$$\begin{aligned} \|X\|^2 - \|X^j\|^2 + \tau^2 \|X_t\|^2 &= -\,2\tau\big[(w^\delta, u_{\overline{x}}^2] - (w^\delta + W^\delta, u_{\overline{x}} U_{\overline{x}}] + (W^\delta, U_{\overline{x}}^2] - (\varphi_1, X)\big], \\ \|Y\|^2 - \|Y^j\|^2 + \tau^2 \|Y_t\|^2 &= -\,2\tau[(w^\delta, v_{\overline{x}}^2] - (w^\delta + W^\delta, v_{\overline{x}} V_{\overline{x}}] + (W^\delta, V_{\overline{x}}^2] - (\varphi_2, Y)], \\ \frac{1}{b}\left(||Z]|^2 - ||Z^j]|^2 + \tau^2 ||Z_t]|^2\right) &= -\,\frac{a}{b}\,\tau(w^{2\beta-1} - W^{2\beta-1})(w - W) \\ &\quad + \tau\Big((w^\delta - w^{\delta-1} W, u_{\overline{x}}^2] - (W^{\delta-1} w - W^\delta, U_{\overline{x}}^2] \\ &\quad + (w^\delta - w^{\delta-1} W, v_{\overline{x}}^2] - (W^{\delta-1} w - W^\delta, V_{\overline{x}}^2]\Big) + \frac{2\tau}{b}\,(\varphi_3, Z]. \end{aligned}$$

Here we introduced the notation $2\alpha = \delta$.

Adding the above equalities and assuming $\beta \geq 1/2$, we get

$$
\begin{aligned}
\|X\|^2 &- \|X^j\|^2 + \|Y\|^2 - \|Y^j\|^2 + \frac{\tau}{b}\left(||Z]|^2 - ||Z^j]|^2\right) \\
&\leq -2\tau\bigg[\Big(\frac{w^\delta + w^{\delta-1}W}{2}\,u_{\overline{x}}^2 + \frac{W^\delta + W^{\delta-1}w}{2}\,U_{\overline{x}}^2, 1\Big] - (w^\delta + W^\delta, u_{\overline{x}} U_{\overline{x}}] \\
&+ \Big(\frac{w^\delta + w^{\delta-1}W}{2}\,v_{\overline{x}}^2 + \frac{W^\delta + W^{\delta-1}w}{2}\,V_{\overline{x}}^2, 1\Big] - (w^\delta + W^\delta, v_{\overline{x}} V_{\overline{x}}] - (\varphi_1, X) - (\varphi_2, Y) - \frac{1}{b}\,(\varphi_3, Z]\bigg] \\
&\leq -2\tau\Big\{\Big(\big[(w^\delta + w^{\delta-1}W)(W^\delta + W^{\delta-1}w)\big]^{\frac{1}{2}} - w^\delta - W^\delta, |u_{\overline{x}}|\,|U_{\overline{x}}|\Big] \\
&+ \Big(\big[(w^\delta + w^{\delta-1}W)(W^\delta + W^{\delta-1}w)\big]^{\frac{1}{2}} - w^\delta - W^\delta, |v_{\overline{x}}|\,|V_{\overline{x}}|\Big] - (\varphi_1, X) - (\varphi_2, Y) - \frac{1}{b}\,(\varphi_3, Z]\Big\}. \quad (3.6)
\end{aligned}
$$

Note that

$$
\begin{aligned}
(w^\delta &- w^{\delta-1}W)(W^\delta - W^{\delta-1}w) - (w^\delta + W^\delta)^2 \\
&= 2w^\delta W^\delta + w^{\delta+1}W^{\delta-1} + w^{\delta-1}W^{\delta+1} - w^{2\delta} - 2w^\delta W^\delta - W^{2\delta} \\
&= (w^{\delta+1} - W^{\delta+1})(W^{\delta-1} - w^{\delta-1}). \quad (3.7)
\end{aligned}
$$

Since $|\delta| \leq 1$, we have

$$(w^{\delta+1} - W^{\delta+1})(W^{\delta-1} - w^{\delta-1}) \geq 0.$$

Using relations (3.6) and (3.7) and taking into account the last inequality, we arrive at

$$\|X\|^2 + \|Y\|^2 + \frac{1}{b}\,||Z]|^2 \leq \|X^j\|^2 + \|Y^j\|^2 + \frac{1}{b}\,\|Z^j]|^2 + 2\tau\Big((\varphi_1, X) + (\varphi_2, Y) + \frac{1}{b}\,(\varphi_3, Z]\Big),$$

which yields

$$\|X\| + \|Y\| + ||Z]| = O(\tau + h^2).$$

Thus, the proof of Theorem 3.1 is complete. □

Using the approach on proving Theorem 3.1, it is not difficult to prove the stability of scheme (3.2).

Applying scheme (3.2) given in this section and the Newton iterative method, various numerical experiments have been carried out which fully agree with theoretical findings. Using the results obtained in Section 2, we get graphical illustrations for the stability of solution (see Fig. 1) and fix the bifurcation phenomena (see Fig. 2).

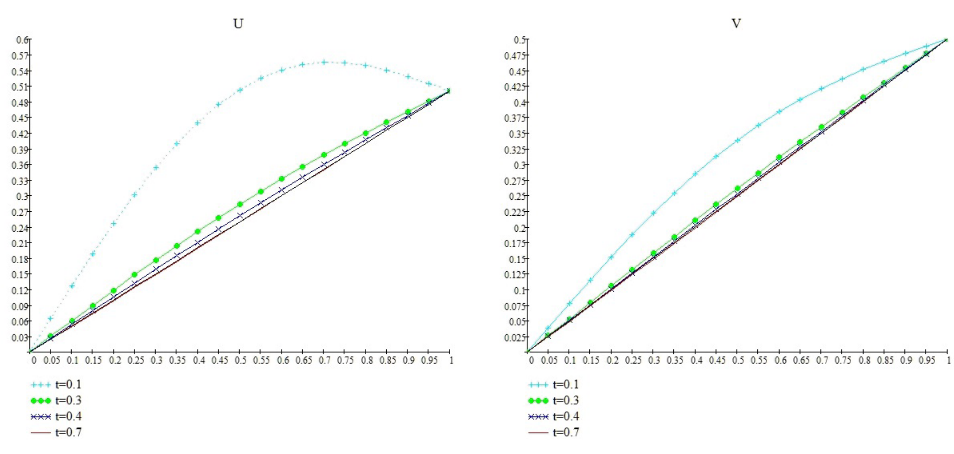

Figure 1. Stabilization of solution.

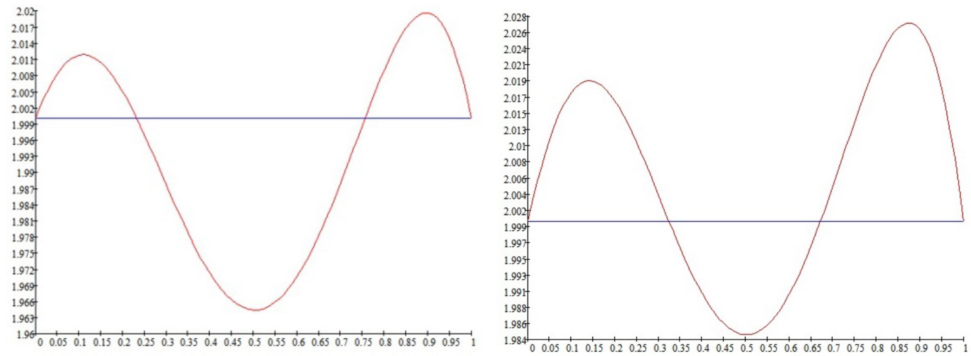

Figure 2. Hoph bifurcation.

4 Conclusion

For the system of nonlinear partial differential equations, which is based on the Maxwell equations describing the process of penetration of an electromagnetic field into a substance, the linear stability of a solution, as well as the possibility of Hoph bifurcation are studied. The blow up case is fixed, too. The corresponding finite difference scheme is constructed and its convergence is proved. The carried out various numerical experiments show the linear stability of a solution of the corresponding initial-boundary value problem and also Hopf type bifurcation for certain boundary data.

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ON A NUMERICAL SOLUTION OF TWO-DIMENSIONAL NONLINEAR MITCHISON MODEL

Abstract. In the paper, for the construction of a numerical solution of two-dimensional Mitchison nonlinear partial differential system, the variable directions difference scheme and the difference scheme corresponding to the average method are used. Practical realization of those algorithms and comparative analysis of the obtained results are carried out. Numerical experiments are in accordance with theoretical findings. On the basis of experiments the corresponding tables of data are given.

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Key words and phrases. Nonlinear partial differential equations, numerical methods, economic algorithms, finite difference scheme, variable directions method, average method.

რეზიუმე. ნაშრომში მიჩისონის ორგანზომილებიან არაწრფივ კერძოწარმოებულებიან დიფერენციალურ განტოლებათა სისტემის რიცხვითი ამონახსნის ასაგებად გამოყენებულია ცვალებადი მიმართულებისა და გასაშუალებული მეთოდის შესაბამისი სხვაობიანი სქემები. განხორციელებულია ამ ალგორითმების პრაქტიკული რეალიზაცია და ჩატარებულია მიღებული შედეგების შედარებითი ანალიზი. რიცხვითი ექსპერიმენტების შედეგები შესაბამისობაშია თეორიულ კვლევებთან. ექსპერიმენტებზე დაყრდნობით მოცემულია შესაბამისი მონაცემების ცხრილები.

1 Introduction

Using the nonlinear partial differential equations, a lot of natural processes are described. Among them there is one of the important mathematical model that describes vein formation in the leaves of higher plants. This model was proposed by J. Michison [15].

The model proposed by Michison has the form:

$$\begin{aligned} \frac{\partial S}{\partial t} &= \frac{\partial}{\partial x_1}\Big(D_1\,\frac{\partial S}{\partial x_1}\Big) + \frac{\partial}{\partial x_2}\Big(D_2\,\frac{\partial S}{\partial x_2}\Big), \\ \frac{\partial D_i}{\partial t} &= f_i\Big(D_i, D_i\,\frac{\partial S}{\partial x_i}\Big), \quad i = 1, 2, \end{aligned} \tag{1.1}$$

where $S(t, x_1, x_2)$ is concentration of signal, D_1 and D_2 are diffusion coefficients to the Ox_1- and Ox_2-axis, respectively.

Some qualitative and structural properties of solutions of system (1.1) are established in [15]. Investigations for one-dimensional analogue of system (1.1) with two unknown functions S and D_1 are carried out in [2]. In [2,15] and [16], the authors pointed out on theoretical and practical importance of the investigation and construction of approximate solutions of the initial boundary value problems for systems (1.1). In biological modeling there are many other works where this and many models of similar processes are also presented and discussed (see, e.g., [3,6,7,16,19,20] and the references therein).

The complexity of model (1.1), besides of the nonlinearity, is caused by its two-dimensionality. In general, a numerical solution of multi-dimensional problems is often carried out by applying decomposition methods.

Investigations for one-dimensional analogue of system (1.1) were carried out in [2].

Starting from the basic works [4,18], the methods of constructing the effective algorithms for the numerical solution of the multi-dimensional problems of mathematical physics and the class of problems solvable with the help of those algorithms were essentially extended [8,14,21]. Those algorithms belong mainly to the methods of splitting-up or sum approximation. Some schemes of the variable directions are constructed and studied in [1], too.

Some questions of construction and investigation of the schemes of variable directions and the average model of sum approximation as well as the difference schemes for one-dimensional case for the system of type (1.1) are discussed in [5, 9–13, 17].

The paper is organized as follows. In Section 2, the statement of the problem is given. In Section 3, two economic difference schemes are constructed and the theorem of stability and convergence for the variable direction scheme is stated. Section 4 contains some results of numerical experiments. The brief conclusion in Section 5 ends the paper.

2 Statement of the problem

In the domain $Q = \Omega \times [0, T]$, where $\Omega = (0, 1) \times (0, 1)$, let us consider the certain function f and pose the following initial boundary value problem for the special case of two-dimensional system (1.1):

$$\begin{aligned} \frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\Big(V_1\,\frac{\partial U}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(V_2\,\frac{\partial U}{\partial y}\Big) &= 0, \\ \frac{\partial V_1}{\partial t} + V_1 - g_1\Big(V_1\,\frac{\partial U}{\partial x}\Big) &= 0, \\ \frac{\partial V_2}{\partial t} + V_2 - g_2\Big(V_2\,\frac{\partial U}{\partial y}\Big) &= 0 \end{aligned} \tag{2.1}$$

with initial

$$\begin{aligned} U(x, y, 0) &= U_0(x, y), \quad (x, y) \in \overline{\Omega}, \\ V_1(x, y, 0) &= V_{10}(x, y), \quad (x, y) \in \overline{\Omega}, \\ V_2(x, y, 0) &= V_{20}(x, y), \quad (x, y) \in \overline{\Omega}, \end{aligned} \tag{2.2}$$

and boundary conditions

$$U(x,y,t)=0,\ \ (x,y,t)\in\partial\Omega\times[0,T]. \tag{2.3}$$

Here g_α, U_0, $V_{\alpha 0}$, $\alpha=1,2$, are the given sufficiently smooth functions such that

$$\begin{gathered} V_{\alpha 0}\geq\delta_0,\ \ \delta_0=const>0,\ \ (x,y)\in\overline{\Omega},\\ g_0\leq g_\alpha(\xi_\alpha)\leq G_0,\ \ |g'_\alpha(\xi_\alpha)|\leq G_1,\ \ \xi_\alpha\in R, \end{gathered} \tag{2.4}$$

where δ_0, g_0, G_0, G_1 are some positive constants.

3 Economic schemes

In the sequel, for the construction of the grid on the domain $\overline{Q}$ we follow the known notation:

$$\begin{gathered} \overline{\omega}_h=\big\{(x_i,y_j)=(ih,jh)\big\},\quad \overline{\omega}_{1h}=\Big\{(x_i,y_j)=\Big(\frac{i-1}{2}\,h,jh\Big)\Big\},\\ \overline{\omega}_{2h}=\Big\{(x_i,y_j)=\Big(ih,\Big(j-\frac{1}{2}\Big)h\Big)\Big\},\quad i,j=0,\dots,M,\ \ Mh=1,\\ \omega_h=\Omega\cap\overline{\omega}_h,\quad \gamma_h=\frac{\overline{\omega}_h}{\omega_h}\,,\quad \overline{\omega}_h=\omega_h\cup\gamma_h,\\ \omega_\tau=\big\{t_k=k\tau,\ k=0,\dots,N,\ N\tau=T\big\}. \end{gathered} \tag{3.1}$$

Following the known notation [21], let us correspond to problem (2.1)–(2.3) the following difference scheme of variable directions:

$$\begin{gathered} u_{1t}-(\widehat{v}_1\widehat{u}_{1\overline{x}})_x-(v_2u_{2\overline{y}})_y=0,\quad u_{2t}-(\widehat{v}_1\widehat{u}_{1\overline{x}})_x-(\widehat{v}_2\widehat{u}_{2\overline{y}})_y=0,\\ v_{1t}+\widehat{v}_1-g_1(v_1u_{1\overline{x}})=0,\quad v_{2t}+\widehat{v}_2-g_2(v_2u_{2\overline{y}})=0,\\ u_1(x,y,0)=U_0(x,y),\ \ (x,y)\in\overline{\omega}_h,\\ u_2(x,y,0)=U_0(x,y),\ \ (x,y)\in\overline{\omega}_h,\\ v_1(x,y,0)=V_{10},\ \ (x,y)\in\overline{\omega}_{1h},\\ v_2(x,y,0)=V_{20},\ \ (x,y)\in\overline{\omega}_{2h},\\ u_1(x,y,t)=u_2(x,y,t)=0,\\ (x,y,t)\in\gamma_h\times\omega_\tau. \end{gathered} \tag{3.2}$$

Using the continuous variant of the averaged model of sum approximation [5], we correspond to problem (2.1)–(2.3) the following decomposition finite difference scheme:

$$\begin{gathered} u_{1t}-(\widehat{v}_1\widehat{u}_{1\overline{x}})_x=0,\quad u_{2t}-(\widehat{v}_2\widehat{u}_{2\overline{y}})_y=0,\\ v_{1t}+\widehat{v}_1-g_1(v_1u_{1\overline{x}})=0,\quad v_{2t}+\widehat{v}_2-g_2(v_2u_{2\overline{y}})=0,\\ u_1(x,y,0)=U_0(x,y),\ \ (x,y)\in\overline{\omega}_h,\\ u_2(x,y,0)=U_0(x,y),\ \ (x,y)\in\overline{\omega}_h,\\ v_1(x,y,0)=V_{10},\ \ (x,y)\in\overline{\omega}_{1h},\\ v_2(x,y,0)=V_{20},\ \ (x,y)\in\overline{\omega}_{2h},\\ u_1(x,y,t)=u_2(x,y,t)=0,\\ (x,y,t)\in\gamma_h\times\omega_\tau,\\ u=\eta_1u_1+\eta_2u_2,\ \ \eta_1>0,\ \ \eta_2>0,\ \ \eta_1+\eta_2=1. \end{gathered} \tag{3.3}$$

Let us introduce the following notation for the errors: $Z_1=u_1-U$, $Z_2=u_2-U$, $S_1=v_1-V_1$, $S_2=v_2-V_2$.

Theorem. *If problem* (2.1)–(2.3) *has a sufficiently smooth solution, then the finite difference scheme* (3.2) *is stable, its solution converges to the exact solution of problem* (2.1)–(2.3) *as* $\tau\to 0$, $h\to 0$, *and the inequality*

$$\|Z_1\|_{\overline{\omega}_h}+\|Z_2\|_{\overline{\omega}_h}+\|S_1\|_{\overline{\omega}_{1h}}+\|S_2\|_{\overline{\omega}_{2h}}\leq C(\tau+h^2)$$

holds.

Table 1. CPU time and error for solution u, v_1, v_2 applying scheme of variable directions (3.2).

t	CPU time	Error u	Error v_1	Error v_2
0.2	0.074	0.00013912790131447	0.00000712766408961	0.00002916084998672
0.4	0.148	0.00022425859907783	0.00001730244454379	0.00009005618525060
0.6	0.224	0.00031286373416026	0.00004804529821700	0.00017715471240609
0.8	0.301	0.00040788793632886	0.00009668298990784	0.00028758192640277
1.0	0.378	0.00051151056363487	0.00016425091499817	0.00041715052893787

Table 2. CPU time and error for solution u, v_1, v_2 applying difference scheme (3.3) corresponding to averaged method.

t	CPU time	Error u	Error v_1	Error v_2
0.2	0.072	0.00006973950435170	0.00001634140038553	0.00001662571352523
0.4	0.146	0.00007422011594080	0.00003786305693865	0.00003781271060488
0.6	0.221	0.00007890208614024	0.00006202878270467	0.00005790906416947
0.8	0.295	0.00008480943243865	0.00008875495749039	0.00007978157763566
1.0	0.369	0.00009205402490850	0.00011818090303972	0.00010625023389577

Here C is a positive constant independent of τ and h, the norms are discrete analogous of the norm of space L_2.

4 Numerical experiments

Using the algorithms proposed in (3.2) and (3.3), let us carry out comparative analysis of the numerical results for the above schemes.

Let us take

$$g_1(\xi) = g_2(\xi) = \frac{1}{1 + (1+\xi)^2}$$

and choose the right-hand sides of the corresponding nonhomogeneous system (2.1) so that the solution of problem (2.1)–(2.3) is:

$$\begin{aligned} U &= xy(1-x)(1-y)(1+t), \\ V_1 &= 1 + xy(1-x)(1-y)(1+t+t^2), \\ V_2 &= 1 + xy(1-x)(1-y)(1+t+t^3). \end{aligned}$$

CPU time and errors for the variable directions difference scheme (3.2) are given in Table 1 and the CPU time and errors for scheme (3.3) are given in Table 2.

The approximation error for the variable direction difference scheme (3.2) is smaller compared with the scheme (3.3). However, CPU time is better for scheme (3.3) than for scheme (3.2).

Table 3. Absolute value of maximum errors and rate of convergence with respect to τ and h for the function u.

τ	h	Error	Rate of τ	Rate of h
0.00125	0.05	0.00024074087939129	0.99175505389520200	1.98351010779040000
0.0008	0.04	0.00015728407178949	0.98629676971885200	1.97259353943770000
0.0003125	0.025	0.00006418736860213	0.99204420486615900	1.98408840973232000
0.0002	0.02	0.00004172715815061	0.99421791633935300	1.98843583267871000
0.00005	0.01	0.00001084525005050		

Table 4. Absolute value of maximum errors and rate of convergence with respect to τ and h for the function v_1.

τ	h	Error	Rate of τ	Rate of h
0.00125	0.05	0.00015579938599405	0.99768312053704900	1.99536624107410000
0.0008	0.04	0.00009981476150336	0.99879576821464700	1.99759153642929000
0.0003125	0.025	0.00003903430252067	0.99941764475219500	1.99883528950439000
0.0002	0.02	0.00002498844720772	0.99974995294653000	1.99949990589306000
0.00005	0.01	0.00002498844720772		

Table 5. Absolute value of maximum errors and rate of convergence with respect to τ and h for the function v_2.

τ	h	Error	Rate of τ	Rate of h
0.00125	0.05	0.00015579938599405	0.99732714185756800	1.99465428371514000
0.0008	0.04	0.00009981476150336	0.99873489122313100	1.99746978244626000
0.0003125	0.025	0.00003903430252067	0.99935953799193100	1.99871907598386000
0.0002	0.02	0.00002498844720772	0.99972504350513300	1.99945008701027000
0.00005	0.01	0.00002498844720772		

In Tables 3–5 we also computed errors for different values of time and space steps applying scheme (3.2) for $T = 1$ and obtained the rates of convergence confirming the theoretical result in theorem from the previous section.

5 Conclusion

Numerous numerical experiments are performed for problem (2.1)–(2.3) by using schemes (3.2) and (3.3). The approximation errors for the variable direction difference scheme (3.2) are smaller compared with scheme (3.3), but CPU time is better for scheme (3.3) than for scheme (3.2). We have carried out various numerical experiments and calculated the absolute value of maximum errors for different time and space steps and obtained the rate of convergence of scheme (3.2). In all cases, the numerical results fully agree with the theoretical ones.

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ON THE BEHAVIOR OF SOLUTIONS TO SECOND-ORDER DIFFERENTIAL EQUATION WITH GENERAL POWER-LAW NONLINEARITY

Abstract. The second-order differential equation with general power-law nonlinearity with continuous potential bounded by positive constants is considered. The behavior of solutions to the equation is studied with respect to the values of nonlinearity. The necessary and sufficient conditions for the existence of a finite right-side boundary of the domain or horizontal asymptote are obtained. The distance to the right-side boundary of the domain and the limits of solutions with horizontal asymptotes near their boundaries are estimated. The continuous dependence of the right-side boundary of the domain and horizontal asymptotes on initial data is proved.[1]

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რეზიუმე. განხილულია მეორე რიგის დიფერენციალური განტოლება ზოგადი ხარისხობრივი არაწრფივობით და დადებითი მუდმივებით შემოსაზღვრული უწყვეტი პოტენციალით. ნაპოვნია სასრულ შუალედზე განსაზღვრული ვერტიკალური და ჰორიზონტალური ასიმპტოტების მქონე ამონახსნების არსებობის აუცილებელი და საკმარისი პირობები და დადგენილია მათი ასიმპტოტური შეფასებები.

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Consider the second-order Emden–Fowler type nonlinear equation

$$y'' = p(x, y, y')|y|^{k_0}|y'|^{k_1}\operatorname{sgn}(yy'), \quad k_0 > 0, \quad k_1 > 0, \quad k_0, k_1 \in \mathbb{R}, \tag{0.1}$$

with positive continuous in x and Lipschitz continuous in u, v function $p(x, u, v)$.

The asymptotic behavior of solutions to (0.1) in the case $k_1 = 0$ is described in [5]. Using the methods described in [1] by I. V. Astashova, the behavior of decreasing solutions to (0.1) near the right domain boundary is investigated with respect to the values k_0 and k_1.

In the case $p = p(x)$, the asymptotic behavior of solutions to (0.1) is obtained by V. M. Evtukhov [6]. Using the methods described in [2–4] by I. V. Astashova, the behavior of positive increasing solutions to (0.1) near the right endpoint of their domains is investigated with respect to the values k_0 and k_1.

1 Preliminary results

Consider the behavior of solutions according to initial data.

Lemma 1.1. *Suppose $k_0 > 0$, $k_1 > 0$. Let $p(x, u, v)$ be a positive continuous in x and Lipschitz continuous in u, v function. Then all maximally extended solutions to equation* (0.1) *can be divided into the following five types according to their behavior:*

0. *Constant solutions;*
1. *Increasing positive solutions;*
2. *Increasing negative solutions;*
3. *Increasing solutions negative near the left boundary of the domain and positive near the right boundary of the domain;*
4. *Decreasing solutions positive near the left boundary of the domain and negative near the right boundary of the domain.*

Proof. Let us show first that if there is a point x_0 such that $y'(x_0) = 0$, then $y(x) \equiv y(x_0)$. Indeed, from equation (0.1) we derive that $y''(x_0) = 0$ and since $y_0(x) \equiv y(x_0)$ is a solution to (0.1), by the theorem of the existence and uniqueness, $y(x) \equiv y_0(x) \equiv y(x_0)$.

Thus, every solution with an extremum at some point is a constant solution (type 0), and therefore every non-constant solution is either increasing or decreasing on its domain.

Consider increasing solutions. Assume that at some point x_0 we have $y(x_0) > 0$ and $y'(x_0) > 0$. Then, according to the equation, $\operatorname{sgn} y'' = \operatorname{sgn} y$, and therefore $y''(x) > 0$ and $y'(x)$ is positive and increasing, while $y(x) > 0$. This implies $y(x) > 0$, $y'(x) > 0$ and $y''(x) > 0$ for all $x > x_0$, so the solution is positive and increasing on its domain. Consider now $x < x_0$. Since $y'(x)$ is positive on the whole domain of the solution, either there is a point $\widetilde{x}$ such that $y(\widetilde{x}) = 0$ or $y(x) > 0$ (also $y'(x) > 0$, and therefore $y''(x) > 0$) for all $x < x_0$. Consider the first case. Since the first derivative of the solution is positive, $y'(x) > 0$ and $y(x) < 0$ (therefore, $y''(x) < 0$) for all $x < \widetilde{x}$. Thus, $y(x)$ is an increasing solution negative near the left boundary of the domain and positive near the right one.

Assume now that at some point x_0 we have $y(x_0) < 0$, $y'(x_0) > 0$. According to the equation, $\operatorname{sgn} y'' = \operatorname{sgn} y$, and therefore $y''(x) < 0$, $y'(x) > 0$ and $y(x) < 0$ for all $x < x_0$. Consider $x > x_0$: since $y'(x) > 0$, either the solution $y(x)$ is negative and increasing on the whole domain or there exists a point $\widetilde{x}$ such that $y(\widetilde{x}) = 0$. In the second case, for $x > \widetilde{x}$ we have $y(x) > 0$, $y'(x) > 0$, and thus $y(x)$ is an increasing solution, negative near the left boundary of domain and positive near the right one.

Consider decreasing solutions. Suppose at some point x_0 we have $y(x_0) > 0$ and $y'(x_0) < 0$. According to the equation, $\operatorname{sgn} y'' = -\operatorname{sgn} y$, and therefore $y''(x) < 0$ and $y'(x)$ is negative and decreasing, while $y(x) > 0$. Thus, $y'(x) < y'(x_0)$ and

$$y(x) < y(x_0) + y'(x_0)(x - x_0) = -|y'(x_0)|x + \big(y(x_0) - y'(x_0)x_0\big),$$

while $y(x)$ is positive. Since $y(x)$ is estimated from above by a linear function, it cannot be positive on its whole domain and therefore there exists a point $\widetilde{x}$ such that $y(\widetilde{x}) = 0$. Note that $y'(\widetilde{x})$ is negative and therefore in some neighbourhood $(\widetilde{x}, \widetilde{x}+\varepsilon)$, $\varepsilon > 0$, the solution $y(x)$ and its derivative $y'(x)$ are both negative and, due to equation (0.1), we have $y''(x) > 0$. Then for all $x > \widetilde{x}$, the solution is decreasing, and since its derivative is of a constant (negative) sign, we have $y(x) < 0$, $y'(x) < 0$, $y''(x) > 0$ for all $x > \widetilde{x}$ and $y(x) > 0$, $y'(x) < 0$, $y''(x) > 0$ at $x < \widetilde{x}$. Thus, $y(x)$ is a decreasing solution, positive near the left boundary of the domain and negative near the right one. □

Lemma 1.2. *Suppose $k_0 > 0$, $k_1 > 0$, $k_1 \neq 2$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying the inequalities*

$$0 < m \le p(x,u,v) \le M < +\infty. \tag{1.1}$$

Then for any solution $y(x)$ to equation (0.1)*, strictly monotonous and having a constant sign on $[x_1, x_2]$, the following inequalities hold:*

$$\begin{aligned} m\big(|y(x_2)|^{k_0+1} - |y(x_1)|^{k_0+1}\big)\operatorname{sgn}(yy') &\le \frac{k_0+1}{2-k_1}\left(|y'(x_2)|^{2-k_1} - |y'(x_1)|^{2-k_1}\right)\operatorname{sgn} y \\ &\le M\big(|y(x_2)|^{k_0+1} - |y(x_1)|^{k_0+1}\big)\operatorname{sgn}(yy'). \end{aligned} \tag{1.2}$$

Proof. Due to inequalities (1.1) and equation (0.1), we can estimate the absolute value of the second derivative as

$$m|y|^{k_0}|y'|^{k_1} \le |y''| = |p(x,y,y')|y|^{k_0}|y'|^{k_1}\operatorname{sgn}(yy')| \le M|y|^{k_0}|y'|^{k_1}.$$

Then

$$m|y|^{k_0}|y'| \le |y''|\,|y'|^{1-k_1} \le M|y|^{k_0}|y'|$$

and by integrating these inequalities on (x_1, x_2), we obtain

$$\begin{aligned} &\frac{m}{k_0+1}\left(|y(x_2)|^{k_0+1} - |y(x_1)|^{k_0+1}\right)\operatorname{sgn}(yy') \\ &\quad\le \frac{1}{2-k_1}\left(|y'|^{2-k_1} - |y'(x_1)|^{2-k_1}\right)\operatorname{sgn}(y'y'') \le \frac{M}{k_0+1}\left(|y(x_2)|^{k_0+1} - |y(x_1)|^{k_0+1}\right)\operatorname{sgn}(yy'), \end{aligned}$$

where $\operatorname{sgn}(yy')$ and $\operatorname{sgn}(y'y'')$ are constant and can be taken at any point from $[x_1, x_2]$. Therefore if $\operatorname{sgn} y' \neq 0$,

$$\begin{aligned} &m\big(|y(x_2)|^{k_0+1} - |y(x_1)|^{k_0+1}\big)\operatorname{sgn}(yy') \\ &\quad\le \frac{k_0+1}{2-k_1}\left(|y'(x_2)|^{2-k_1} - |y'(x_1)|^{2-k_1}\right)\operatorname{sgn} y \le M\big(|y(x_2)|^{k_0+1} - |y(x_1)|^{k_0+1}\big)\operatorname{sgn}(yy'). \end{aligned}$$ □

2 Increasing solutions

Theorem 2.1. *Suppose $k_0 > 0$, $k_1 > 0$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1)*. Let $y(x)$ be a maximally extended solution to* (0.1) *with $y(x_0) \ge 0$ and $y'(x_0) > 0$ at some point x_0. Then the existence of a finite point $x^* > x_0$ such that $\lim\limits_{x\to x^*-0} y'(x) = +\infty$ is equivalent to the condition $k_0 + k_1 > 1$. Moreover, there exists a positive constant $\xi = \xi(m, k_0)$ such that*

$$x^* - x_0 < \xi(y'(x_0))^{-\frac{k_0+k_1-1}{k_0+1}}.$$

Proof. Consider the case $k_0 + k_1 > 1$.

Denote $y_1 = y'(x_0) > 0$. According to Lemma 1.1, the solution $y(x)$ with positive initial data tends to infinity along with its derivative. This implies that for any $i \in \mathbb{N}$ there exists a point $x_i > x_{i-1}$ such that $y'(x_i) = 2y'(x_{i-1}) = 2^i y_1$. Let us estimate the difference $x_{i+1} - x_i$.

For $x \in [x_i, x_{i+1}]$, the inequalities

$$y'(x) \ge y_1, \quad y(x) - y(x_i) \ge y_1(x - x_i)$$

hold, and since $y(x_i) \geq y(x_0) \geq 0$, we have $y(x) \geq y_1(x - x_i)$, hence

$$y^{k_0}(x) \geq (y_1(x - x_i))^{k_0} \ \text{ and } \ (y'(x))^{k_1} \geq y_1^{k_1},$$
$$y''(x) = p(x, y, y')|y|^{k_0}|y'|^{k_1} \operatorname{sgn}(yy') \geq m y_1^{k_0+k_1}(x - x_i)^{k_0}.$$

Integrating this inequality on the segment $[x_i, x_{i+1}]$, we obtain

$$y'(x_{i+1}) - y'(x_i) \geq \frac{m}{k_0 + 1}\, y_1^{k_0+k_1}(x_{i+1} - x_i)^{k_0+1},$$

which means

$$2^i y_1 \geq \frac{m}{k_0 + 1}\, y_1^{k_0+k_1}(x_{i+1} - x_i)^{k_0+1},$$
$$(x_{i+1} - x_i)^{k_0+1} \leq 2^i\, \frac{k_0 + 1}{m}\, y_1^{-(k_0+k_1-1)},$$
$$x_{i+1} - x_i \leq 2^{\frac{i}{k_0+1}} \Big(\frac{k_0 + 1}{m}\Big)^{\frac{1}{k_0+1}} y_1^{-\frac{k_0+k_1-1}{k_0+1}}.$$

Thus, the distance $x_{i+1} - x_i$ is estimated from above by the term of a converging series multiplied by a positive constant. This implies that there exists a limit

$$x^* = \lim_{n\to+\infty} \sum_{i=0}^{n} (x_{i+1} - x_i) + x_0 = \lim_{n\to+\infty} x_n,$$

and since a solution to (0.1) is continuous, $\lim\limits_{x\to x^*-0} y'(x) = +\infty$. Moreover,

$$x^* - x_0 = \sum_{i=0}^{+\infty} (x_{i+1} - x_i) \leq \sum_{i=0}^{+\infty} 2^{\frac{i}{k_0+1}} \Big(\frac{k_0 + 1}{m}\Big)^{\frac{1}{k_0+1}} y_1^{-\frac{k_0+k_1-1}{k_0+1}},$$
$$x^* - x_0 \leq \Big(\frac{k_0 + 1}{m}\Big)^{\frac{1}{k_0+1}} y_1^{-\frac{k_0+k_1-1}{k_0+1}} \sum_{i=0}^{+\infty} 2^{\frac{i}{k_0+1}},$$

which implies

$$x^* - x_0 < \xi\, (y'(x_0))^{-\frac{k_0+k_1-1}{k_0+1}}$$

for

$$\xi = \xi(m, k_0) = \Big(\frac{k_0 + 1}{m}\Big)^{\frac{1}{k_0+1}} (1 - 2^{\frac{1}{k_0+1}})^{-1} > 0.$$

For the case $k_0 + k_1 \leq 1$, we can apply the following

Theorem (K. Dulina, T. Korchemkina [5])**.** *Suppose $k > 0$, $k \neq 1$. Let the function $P(x, u, v)$ be continuous in x, Lipschitz continuous in u, v. Let there exist the constants $u_0 > 0$, $v_0 > 0$ and $\alpha \leq 1 - k$ such that for $u > u_0$, $v > v_0$ the inequality $P(x, u, v) \leq C|v|^{-\alpha}$ holds. Then any non-extensible solution $y(x)$ to equation*

$$y'' - P(x, y, y')|y|^k \operatorname{sgn} y = 0$$

with initial data $y(x_0) \geq u_0$, $y'(x_0) \geq v_0$ can be extended on $(x_0, +\infty)$ and

$$\lim_{x\to+\infty} y(x) = \lim_{x\to+\infty} y(x) = +\infty.$$

Indeed, here we have $P(x, u, v) = p(x, u, v)|v|^{k_1} \leq M v^{k_1}$, so, the above theorem holds if $k_1 \leq 1 - k_0$, i.e., $k_0 + k_1 \leq 1$. ☐

Remark. It is sufficient that $p(x, u, v) \geq m$ for the solution to have a finite right-side boundary x^* of its domain.

Note that after the substitution $y(x) \mapsto -y(-x)$ we obtain an equation of the same type as (0.1), so the following statement is also true.

Theorem 2.2. *Suppose $k_0 > 0$, $k_1 > 0$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1). *Let $y(x)$ be a maximally extended solution to* (0.1) *with $y(x_0) \leq 0$ and $y'(x_0) > 0$ at some point x_0. Then the existence of a finite point $x_* < x_0$ such that $\lim\limits_{x\to x_*+0} y'(x) = -\infty$ is equivalent to the condition $k_0 + k_1 > 1$. Moreover, there exists a positive constant $\xi = \xi(m, k_0)$ such that*

$$x_0 - x_* < \xi \left(y'(x_0)\right)^{-\frac{k_0+k_1-1}{k_0+1}}.$$

It follows from [5, Theorem 3.4] that in the case $k_1 > 2$ all positive increasing solutions are *the black hole* solutions [7], i.e., $\lim\limits_{x\to x^*-0} y(x) < \infty$.

Applying now Lemma 1.2 for $x_1 = x_0$, $x_2 = x$ and considering inequalities (1.2) as $x \to x^* - 0$, we obtain the following estimates for the limit $\lim\limits_{x\to x^*-0} y(x)$.

Theorem 2.3. *Suppose $k_1 > 2$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1). *Let $y(x)$ be a maximally extended solution to* (0.1) *with $y(x_0) \geq 0$ and $y'(x_0) > 0$ at some point x_0. Then for the right-side boundary of the domain x^* which existence is stated in Theorem* 2.1, *the limit $\lim\limits_{x\to x^*-0} y(x) = y^*$ is finite and*

$$\frac{k_0+1}{2-k_1}\frac{1}{M}\left(y'(x_0)\right)^{2-k_1} \leq (y^*)^{k_0+1} - y_0^{k_0+1} \leq \frac{k_0+1}{2-k_1}\frac{1}{m}\left(y'(x_0)\right)^{2-k_1}.$$

Analogously, we obtain the similar statement for the limit $\lim\limits_{x\to x_*-0} y(x)$.

Theorem 2.4. *Suppose $k_1 > 2$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1). *Let $y(x)$ be a maximally extended solution to* (0.1) *with $y(x_0) \leq 0$ and $y'(x_0) > 0$ at some point x_0. Then for the left-side boundary of the domain x_* which existence is stated in Theorem* 2.2, *the limit $\lim\limits_{x\to x_*-0} y(x) = y_*$ is finite and*

$$\frac{k_0+1}{2-k_1}\frac{1}{M}\left(y'(x_0)\right)^{2-k_1} \leq |y_*|^{k_0+1} - |y_0|^{k_0+1} \leq \frac{k_0+1}{2-k_1}\frac{1}{m}\left(y'(x_0)\right)^{2-k_1}.$$

3 Decreasing solutions

Consider now decreasing solutions. Let us prove that every solution of such type has two horizontal asymptotes.

Theorem 3.1. *Suppose $k_0 > 0$, $k_1 \in (0,2)$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1). *Then any solution $y(x)$ to equation* (0.1) *with initial data $y(x_0) \leq 0$, $y'(x_0) < 0$ is defined on the whole axis and there exists a finite negative value $y_+ < y(x_0)$ such that $\lim\limits_{x\to+\infty} y(x) = y_+$. Moreover,*

$$\frac{k_0+1}{2-k_1}\frac{1}{M}|y'(x_0)|^{2-k_1} \leq |y_+|^{k_0+1} - |y(x_0)|^{k_0+1} \leq \frac{k_0+1}{2-k_1}\frac{1}{m}|y'(x_0)|^{2-k_1}.$$

Proof. According to the proof of Lemma 1.1, for any $x > x_0$, we have $y(x) < 0$, $y'(x) < 0$ and therefore $y''(x) > 0$. This implies that $y'(x) \to 0$ as $x \to \widetilde{x}$, where $\widetilde{x} > x_0$ is a right domain boundary of $y(x)$.

Denote $y_1 = |y'(x_0)| = -y'(x_0)$. While $y'(x) \neq 0$, from Lemma 1.2 with $x_1 = x_0$ and $x_2 = x > x_0$ we derive

$$\frac{k_0+1}{2-k_1}\frac{|y'(x_0)|^{2-k_1} - |y'(x)|^{2-k_1}}{M} \leq |y(x)|^{k_0+1} - |y(x_0)|^{k_0+1} \leq \frac{k_0+1}{2-k_1}\frac{|y'(x_0)|^{2-k_1} - |y'(x)|^{2-k_1}}{m}.$$

Denote $Y = \lim\limits_{x \to \widetilde{x}} y(x)$, then considering the above inequalities at $x \to \widetilde{x}$, we obtain

$$\frac{k_0+1}{2-k_1}\,\frac{|y'(x_0)|^{2-k_1}}{M} \le |Y|^{k_0+1} - |y(x_0)|^{k_0+1} \le \frac{k_0+1}{2-k_1}\,\frac{|y'(x_0)|^{2-k_1}}{m},$$

which implies $|Y| < +\infty$.

Consider now $\widetilde{x}$ in correspondence with k_1. Let $x^* > x_0$, $x^* \le +\infty$ be the closest to x_0 point such that $\lim\limits_{x \to x^*} y'(x) = 0$.

From equation (0.1), on the interval (x_0, x^*), we derive

$$y''|y'|^{-k_1} = p(x,y,y')|y|^{k_0}\operatorname{sgn}(yy'),$$

and since at $x > x_0$ we have $y(x) < 0$, $y'(x) < 0$, therefore

$$y''(-y')^{-k_1} = p(x,y,y')|y|^{k_0},$$

and for $k_1 \ne 1$,

$$\frac{1}{1-k_1}\left(|y'(x_0)|^{1-k_1} - |y'|^{1-k_1}\right) = \int\limits_{x_0}^{x} p(x,y,y')|y|^{k_0}\,dx.$$

In the case $k_1 \in (1,2)$, we get

$$\frac{1}{1-k_1}\left(|y'(x_0)|^{1-k_1} - |y'|^{1-k_1}\right) \le \int\limits_{x_0}^{x} M|Y|^{k_0}\,dx = M|Y|^{k_0}(x-x_0),$$

$$x - x_0 \ge \frac{1}{M|Y|^{k_0}(k_1-1)}\left(|y'(x)|^{1-k_1} - |y'(x_0)|^{1-k_1}\right).$$

Since $y'(x) \to 0$ as $x \to x^*$ and $1-k_1 < 0$, the right part of the above inequality tends to infinity as $x \to x^*$, which implies $x^* = +\infty$, and therefore the solution $y(x)$ is defined on $(x_0, +\infty)$, $y_+ = Y$ and the theorem for the case $k_1 \in (1,2)$ is proved.

Analogously, in the case $k_1 = 1$, we obtain

$$x - x_0 \ge \frac{1}{M|Y|^{k_0}}\left(\ln|y'(x_0)| - \ln|y'|\right).$$

Since $y'(x) \to 0$ as $x \to x^*$, the right part of the above inequality tends to infinity as $x \to x^*$, which implies $x^* = +\infty$, and therefore the solution $y(x)$ is defined on $(x_0, +\infty)$, $y_+ = Y$ and hence the theorem for the case $k_1 = 1$ is also proved.

In the case $k_1 \in (0,1)$, we denote $\widetilde{x}_0 = x_0$ if $y(x_0) \ne 0$ and otherwise $\widetilde{x}_0 = x_0 + \varepsilon$, where $\varepsilon > 0$ is such that $y(x) < 0$ and $y'(x) < 0$ on $(x_0, x_0+\varepsilon)$. Then $|y(x)|^{k_0} \ge |y(\widetilde{x}_0)|^{k_0}$ on $(\widetilde{x}_0, x^*)$, and analogously we obtain the estimate

$$\frac{1}{1-k_1}\left(|y'(\widetilde{x}_0)|^{1-k_1} - |y'|^{1-k_1}\right) \ge \int\limits_{\widetilde{x}_0}^{x} m|y(\widetilde{x}_0)|^{k_0}\,dx = m|y(\widetilde{x}_0)|^{k_0}(x-\widetilde{x}_0),$$

$$x - \widetilde{x}_0 \le \frac{1}{m|y(\widetilde{x}_0)|^{k_0}(1-k_1)}\left(|y'(\widetilde{x}_0)|^{1-k_1} - |y'(x)|^{1-k_1}\right).$$

Since $y'(x) \to 0$ as $x \to x^*$ and $1-k_1 > 0$, the right part of the above inequality tends to a constant value $\frac{|y'(\widetilde{x}_0)|^{1-k_1}}{m|y(\widetilde{x}_0)|^{k_0}(1-k_1)}$ as $x \to x^*$, which implies $x^* < +\infty$, and therefore the solution $y(x)$ is unique only on (x_0, x^*). Note that even though the uniqueness of solutions is not satisfied, there is only one possible way to extend the solution $y(x)$ to the right. Thus, $y(x) < 0$, is decreasing on (x_0, x^*) and is equal to a constant on $[x^*, +\infty)$. This implies $y_+ = \lim\limits_{x \to +\infty} y(x) = y(x^*) = Y$ and the theorem is proved. $\square$

Since the substitution $y(x) \mapsto -y(-x)$ gives an equation of the same type as (0.1), the following statement is also true.

Theorem 3.2. *Suppose $k_0 > 0$, $k_1 \in (0,2)$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1). *Then any solution $y(x)$ to equation* (0.1) *with initial data $y(x_0) \geq 0$, $y'(x_0) < 0$ is defined on the whole axis and there exists a finite positive value $y_- > y(x_0)$ such that $\lim\limits_{x\to-\infty} y(x) = y_-$. Moreover,*

$$\frac{k_0+1}{2-k_1}\,\frac{1}{M}\,|y'(x_0)|^{2-k_1} \leq |y_-|^{k_0+1} - |y(x_0)|^{k_0+1} \leq \frac{k_0+1}{2-k_1}\,\frac{1}{m}\,|y'(x_0)|^{2-k_1}\,.$$

Definition ([8])**.** $y(x)$ is a *white hole* solution to equation (0.1) if there exists a finite point $\widetilde{x}$ such that $\lim\limits_{x\to\widetilde{x}} y'(x) = 0$, but $\lim\limits_{x\to\widetilde{x}} y(x) \neq 0$.

Thus, all decreasing solutions to equation (0.1) in the case $k_1 \in (1,2)$ are the white hole solutions.

Lemma 3.1. *Suppose $k_0 > 0$, $k_1 \in (0,2)$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1). *Then any decreasing solution $y(x)$ to equation* (0.1) *is defined on the whole axis and there exist a finite positive value y_- and a finite negative value y_+ such that $\lim\limits_{x\to\pm\infty} y(x) = y_\pm$. Moreover,*

$$\left(\frac{m}{M}\right)^{\frac{1}{k_0+1}} \leq \left|\frac{y_+}{y_-}\right| \leq \left(\frac{M}{m}\right)^{\frac{1}{k_0+1}}.$$

Proof. Indeed, let x_0 be a zero of a decreasing solution $y(x)$ to equation (0.1). Then the limits $y_\pm = \lim\limits_{x\to\pm\infty} y(x)$ are finite and the estimates from Theorems 3.1 and 3.2 take the form

$$\frac{k_0+1}{2-k_1}\,\frac{1}{M}\,|y'(x_0)|^{2-k_1} \leq |y_+|^{k_0+1} \leq \frac{k_0+1}{2-k_1}\,\frac{1}{m}\,|y'(x_0)|^{2-k_1},$$
$$\frac{k_0+1}{2-k_1}\,\frac{1}{M}\,|y'(x_0)|^{2-k_1} \leq y_-^{k_0+1} \leq \frac{k_0+1}{2-k_1}\,\frac{1}{m}\,|y'(x_0)|^{2-k_1},$$

hence

$$\frac{m}{M} \leq \left|\frac{y_+}{y_-}\right|^{k_0+1} \leq \frac{M}{m}\,,$$

which implies the statement of the lemma. ☐

Applying Lemma 3.1 for the case $p(x,u,v) \equiv p_0 = const$, we obtain the following

Corollary. *Suppose $k_0 > 0$, $k_1 \in (0,2)$, $p(x,u,v) \equiv p_0 = const$. Then any solution $y(x)$ to* (0.1) *satisfying at some point x_0 the condition $y'(x_0) < 0$ is defined on the whole axis and the limits $y_\pm = \lim\limits_{x\to\pm\infty} y(x)$ are finite and satisfying the equality $y_- = -y_+$.*

Theorem 3.3. *Suppose $k_0 > 0$, $k_1 \geq 2$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1). *Then any solution $y(x)$ to equation* (0.1) *with initial data $y(x_0) \leq 0$, $y'(x_0) < 0$ is unbounded and defined on the whole axis.*

Proof. Let us prove the theorem for $x > x_0$. Consider first the case $k_1 > 2$.

According to the proof of Lemma 1.1, for any $x > x_0$ we have $y(x) < 0$, $y'(x) < 0$ and, therefore, $y''(x) > 0$. This implies that $y'(x) \to 0$ as $x \to \widetilde{x}$, where $\widetilde{x} > x_0$ is the right domain boundary of $y(x)$.

Denote $y_1 = |y'(x_0)| = -y'(x_0)$. While $y'(x) \neq 0$, from Lemma 1.2 with $x_1 = x_0$ and $x_2 = x > x_0$ we derive

$$m\big(|y(x)|^{k_0+1} - |y(x_0)|^{k_0+1}\big) \leq \frac{k_0+1}{k_1-2}\,\big(|y'(x)|^{2-k_1} - y_1^{2-k_1}\big) \leq M\big(|y(x)|^{k_0+1} - |y(x_0)|^{k_0+1}\big).$$

Denote $Y = \lim\limits_{x\to\widetilde{x}} y(x)$, then considering the above inequalities at $x \to \widetilde{x}$, we obtain

$$\frac{k_0+1}{k_1-2}\,\frac{|y'(x)|^{2-k_1}-y_1^{2-k_1}}{M} \le |Y|^{k_0+1} - |y(x_0)|^{k_0+1} \le \frac{k_0+1}{k_1-2}\,\frac{|y'(x)|^{2-k_1}-y_1^{2-k_1}}{m},$$

and since $y'(x) \to 0$ as $x \to \widetilde{x}$ and $2-k_1<0$, it follows that $|Y| = +\infty$.

Analogously, for $k_1 = 2$, we obtain

$$\frac{k_0+1}{M}\big(\ln y_1 - \ln|y'(x)|\big) \le |Y|^{k_0+1} - |y(x_0)|^{k_0+1} \le \frac{k_0+1}{m}\big(\ln y_1 - \ln|y'(x)|\big),$$

and since $y'(x) \to 0$ as $x \to \widetilde{x}$, it follows that $|Y| = +\infty$.

Consider now $\widetilde{x}$ in correspondence with k_1. Let $x^* > x_0$, $x^* \le +\infty$ be the closest to x_0 point such that $\lim\limits_{x\to x^*} y'(x) = 0$.

From equation (0.1), on the interval (x_0, x^*), we derive

$$y''|y'|^{-k_1} = p(x,y,y')|y|^{k_0}\operatorname{sgn}(yy'),$$

and since at $x > x_0$ there is $y(x) < 0$, $y'(x) < 0$, we have

$$y''(-y')^{-k_1} = p(x,y,y')|y|^{k_0},$$

$$\frac{1}{1-k_1}\big(|y'(x_0)|^{1-k_1} - |y'|^{1-k_1}\big) = \int\limits_{x_0}^{x} p(x,y,y')|y|^{k_0}\,dx,$$

therefore

$$\frac{1}{1-k_1}\big(|y'(x_0)|^{1-k_1} - |y'|^{1-k_1}\big) \le \int\limits_{x_0}^{x} M|Y|^{k_0}\,dx = M|Y|^{k_0}(x-x_0)$$

and

$$x - x_0 \ge \frac{1}{M|Y|^{k_0}(k_1-1)}\big(|y'(x)|^{1-k_1} - |y'(x_0)|^{1-k_1}\big).$$

Since $y'(x) \to 0$ as $x \to x^*$ and $1-k_1<0$, the right part of the above inequality tends to infinity as $x \to x^*$, which implies $x^* = +\infty$ and, therefore, the solution $y(x)$ is defined on $(x_0, +\infty)$, $y_+ = Y$ and the theorem is proved. □

4 Continuous dependence of boundaries of domain or horizontal asymptotes of solutions on initial data

Consider first continuous dependence of the right-side boundary of the domain on initial data.

Theorem 4.1. *Suppose $k_0 > 0$, $k_1 > 0$, $k_0 + k_1 > 1$. Let $p(x,u,v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequality $p(x,u,v) \ge m > 0$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any x_0, $\widetilde{x}_0$, y_0, z_0, y_1, z_1 satisfying $|\widetilde{x}_0 - x_0| < \delta$, $|z_0 - y_0| < \delta$, $|z_1 - y_1| < \delta$, $y_0 \ge 0$, $y_1 > 0$, $z_0 \ge 0$, $z_1 > 0$, the maximally extended solutions $y(x)$ and $z(x)$ to equation* (0.1) *with the initial data*

$$\begin{cases} y(x_0) = y_0, \\ y'(x_0) = y_1 \end{cases} \tag{4.1}$$

and

$$\begin{cases} y(\widetilde{x}_0) = z_0, \\ y'(\widetilde{x}_0) = z_1, \end{cases} \tag{4.2}$$

respectively, have finite right-side boundaries of the domains $x_1^* > x_0$ and $x_2^* > \widetilde{x}_0$, respectively, and $|x_2^* - x_1^*| < \varepsilon$.

Proof. From Theorem 2.1 it follows that $y'(x) \to +\infty$ as $x \to x_1^* - 0$, there exists a point x_1 such that $\widetilde{y}_1 = y'(x_1)$ satisfies

$$\widetilde{y}_1 > \left(\frac{\varepsilon}{2\xi}\right)^{-\frac{k_0+1}{k_0+k_1-1}}, \quad \xi\widetilde{y}_1^{\,-\frac{k_0+k_1-1}{k_0+1}} < \frac{\varepsilon}{2}\,,$$

where ξ is a constant from Theorem 2.1. Then

$$x_1^* - x_1 < \xi(y'(x_1))^{-\frac{k_0+k_1-1}{k_0+1}} < \frac{\varepsilon}{2}\,.$$

For any $\varepsilon > 0$, there exists $\widetilde{\delta} > 0$ such that if $|\widetilde{z}_1 - \widetilde{y}_1| < \widetilde{\delta}$, then $\xi\widetilde{z}_1^{\,-\frac{k_0+k_1-1}{k_0+1}} < \frac{\varepsilon}{2}$. Also for every $\widetilde{\delta} > 0$ there exists $\delta > 0$ such that for any x_0, $\widetilde{x}_0$, y_0, z_0, y_1, z_1 satisfying $|\widetilde{x}_0 - x_0| < \delta$, $|z_0 - y_0| < \delta$, $|z_1 - y_1| < \delta$, $y_0 \geq 0$, $y_1 > 0$, $z_0 \geq 0$, $z_1 > 0$ the inequality $|z'(x_1) - y'(x_1)| < \widetilde{\delta}$ holds. Then from Theorem 2.1 we derive that the solution $z(x)$ with initial data (4.2) has a finite right-side boundary of the domain x_2^* and

$$x_2^* - x_1 < \xi(z'(x_1))^{-\frac{k_0+k_1-1}{k_0+1}} < \frac{\varepsilon}{2}\,.$$

Thus, for any ε, there exists $\delta > 0$ such that

$$|x_2^* - x_1^*| \leq |x_2^* - x_1| + |x_1 - x_1^*| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon. \qquad \square$$

Analogously, continuous dependence of the left-side boundary of the domain on the initial data is obtained.

Theorem 4.2. *Suppose $k_0 > 0$, $k_1 > 0$, $k_0 + k_1 > 1$. Let $p(x, u, v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying the inequality $p(x, u, v) \geq m > 0$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any x_0, $\widetilde{x}_0$, y_0, z_0, y_1, z_1 satisfying $|\widetilde{x}_0 - x_0| < \delta$, $|z_0 - y_0| < \delta$, $|z_1 - y_1| < \delta$, $y_0 \leq 0$, $y_1 > 0$, $z_0 \leq 0$, $z_1 > 0$, the maximally extended solutions $y(x)$ and $z(x)$ to equation* (0.1) *with initial data* (4.1) *and* (4.2)*, respectively, have finite left-side boundaries of domains $x_{1*} < x_0$ and $x_{2*} < \widetilde{x}_0$, respectively, and $|x_{2*} - x_{1*}| < \varepsilon$.*

Analogously, with the help of the estimates from Theorems 3.1 and 3.2 the following results on the continuous dependence of solutions' limits on the initial data are obtained.

Theorem 4.3. *Suppose $k_0 > 0$, $k_1 \in (0, 2)$. Let $p(x, u, v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1)*. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any x_0, $\widetilde{x}_0$, y_0, z_0, y_1, z_1 satisfying $|\widetilde{x}_0 - x_0| < \delta$, $|z_0 - y_0| < \delta$, $|z_1 - y_1| < \delta$, $y_0 \leq 0$, $y_1 < 0$, $z_0 \leq 0$, $z_1 < 0$, the maximally extended solutions $y(x)$ and $z(x)$ to equation* (0.1) *with initial data* (4.1) *and* (4.2)*, respectively, have finite limits $y_+ < y(x_0)$ and $z_+ < z(\widetilde{x}_0)$, respectively, as $x \to +\infty$, and $|y_+ - z_+| < \varepsilon$.*

Theorem 4.4. *Suppose $k_0 > 0$, $k_1 \in (0, 2)$. Let $p(x, u, v)$ be a continuous in x and Lipschitz continuous in u, v function satisfying inequalities* (1.1)*. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any x_0, $\widetilde{x}_0$, y_0, z_0, y_1, z_1 satisfying $|\widetilde{x}_0 - x_0| < \delta$, $|z_0 - y_0| < \delta$, $|z_1 - y_1| < \delta$, $y_0 \geq 0$, $y_1 < 0$, $z_0 \geq 0$, $z_1 < 0$, the maximally extended solutions $y(x)$ and $z(x)$ to equation* (0.1) *with initial data* (4.1) *and* (4.2)*, respectively, have finite limits $y_- > y(x_0)$ and $z_- > z(\widetilde{x}_0)$, respectively, as $x \to -\infty$, and $|x_- - z_-| < \varepsilon$.*

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LOCALIZED LOCAL MAXIMA FOR NON-NEGATIVE GROUND STATE SOLUTION OF NONLINEAR SCHRÖDINGER EQUATION WITH NON-MONOTONE EXTERNAL POTENTIAL

Abstract. A non-negative ground state solution $u(x)$ of the nonlinear Schrödinger equation with non-monotone potential is studied. The existence of local maxima of $u(x)$ which are attained on the given intervals in one-dimensional space variable x is shown. Next, it is proved that the stationary point of $u(x)$ per one interval is unique. The co-existence of the local extrema of ground state solution and external potential on the same interval is considered, too.[1]

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რეზიუმე. შროდინგერის არაწრფივი დიფერენციალური განტოლებისთვის არამონოტონური პოტენციალით შესწავლილია არაუარყოფითი ძირითადი მდგომარეობის $u(x)$ ამონახსნი. ნაჩვენებია $u(x)$-ის ლოკალური მაქსიმუმების არსებობა, რომლებიც მიიღწევა ერთგანზომილებიანი სივრცითი x ცვლადის მოცემულ ინტერვალებზე. დამტკიცებულია, რომ $u(x)$-ის სტაციონარული წერტილი თითოეული ინტერვალისთვის არის ერთადერთი. განხილულია აგრეთვე ძირითადი მდგომარეობის ამონახსნის ლოკალური ექსტრემუმებისა და იმავე ინტერვალზე გარე პოტენციალის თანაარსებობის საკითხი.

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1 Introduction and mathematical setting

1.1 Localized local maxima

Let $[a,b] \subset \mathbb{R}$ be a bounded interval and $u : \mathbb{R} \to \mathbb{R}$, $u = u(x)$, be a C^1-function. Recall that $u(x)$ *attains a local maximum in a prescribed interval* $[a,b]$ if there exists a point $x_{\rm s} \in [a,b]$ such that $u'(x_{\rm s}) = 0$ (stationary point of $u(x)$) and $u'(x)$ changes sign at $x_{\rm s}$ such that $u'(x) > 0$ in $(x_{\rm s} - \varepsilon, x_{\rm s})$ and $u'(x) < 0$ in $(x_{\rm s}, x_{\rm s} + \varepsilon)$ for some $\varepsilon > 0$. One can say that $x_{\rm s}$ is *localized* on $[a,b]$.

For instance, if $[a,b] = [0,\pi]$ and $u(x) = \exp(\sin(x))$, then the differential equation $u'' + (\sin(x) - \cos^2(x))u = 0$ possesses a positive solution $u(x)$ having a local maximum at $x_{\rm s} = \pi/2$, which is localized and unique in $[a,b]$.

1.2 Time-independent nonlinear Schrödinger equation (NLSE)

In the paper, we consider C^2-solutions $u(x)$ of the following one-dimensional time-independent nonlinear Schrödinger equation:

$$u'' + \Big(\mu - \frac{2m}{\hbar^2}\, V(x)\Big)u + \frac{2m}{\hbar^2}\, f\big(x, |u|^2\big)u = 0, \tag{1.1}$$

where $\mu \in \mathbb{R}$ is the chemical potential, $\hbar$ is the Planck constant, m is the particle mass, $V(x)$ is a continuous the so-called *linear,* or *external,* or *trapping potential* and the *nonlinear potential* f satisfies:

$$f(x, s^2) \geq -g(x), \quad (x,s) \in \mathbb{R}^2, \tag{1.2}$$

where $g(x)$ is a continuous function. In the accordance with (1.2), the following two cases occur:

(1) if $g(x) \leq 0$, then $f(x,s^2)$ is an *attractive potential*: $f(x,s^2) \geq 0$, $(x,s) \in \mathbb{R}^2$; especially for $g(x) \equiv 0$, assumption (1.2) allows $f(x,s^2)$ to be a *classic attractive potential*: $f(x,s^2) = f_0(x)s^2$ with $f_0(x) \geq 0$; hence, in this case, our result can be interperted as the non-monotonic behaviour of particle density in the Bose–Einstein condensate (BEC);

(2) if $g(x) \geq 0$ and $g(x) \not\equiv 0$, then assumption (1.2) allows $f(x,s^2)$ to be a *repulsive potential*: $f(x,s^2) \leq 0$, $(x,s) \in \mathbb{R}^2$, but not a *classic repulsive potential*: $f(x,s^2) = f_0(x)s^2$ with $f_0(x) \leq 0$; an example of a repulsive potential satisfying (1.2) is $f(x,s^2) = -g_0(x)\arctan(s^2)$, where $g(x) = \frac{\pi}{2}\, g_0(x)$ with $g_0(x) \geq 0$.

1.3 Motivation for mathematical treatment of localized local maxima of ground state solution of NLSE

The so-called solitary wave $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ defined by

$$\psi(x,t) = e^{-{\rm i}\,\frac{\hbar\mu}{2m}\,t}\, u(x) \tag{1.3}$$

satisfies the time-dependent nonlinear Schrödinger equation

$${\rm i}\hbar\,\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\,\frac{\partial^2\psi}{\partial x^2} + V(x)\psi - f\big(x, |\psi|^2\big)\psi, \tag{1.4}$$

provided $u(x)$ is a solution of our main equation (1.1). In such a situation, $u(x)$ is called as the *ground state solution* of NLSE (1.1). If $f(x,s^2) = f_0(x)s^2$, equation (1.4) is known as the Gross–Pitaevski equation (GPE), which is a model for a wave function of the particles in an atomic cloud in BEC. The quantity $|\psi(x,t)|^2$ represents the particle density in BEC, which has the common stationary points in the variable x with a non-negative ground state solution, since

$$|\psi(x,t)|^2 = u^2(x) \;\text{ and }\; \frac{\partial}{\partial x}\,|\psi(x,t)|^2 = (u^2(x))' = 2u(x)u'(x). \tag{1.5}$$

Hence, the non-monotonic behaviour of particle density $|\psi(x,t)|^2$ is strictly related with the extrema of the ground state solution $u(x)$. Among all known numerical simulations in which we can see the non-monotonic behaviour of particle density in BEC (see [1–4] and [7–11]), we point out the next three:

- **BEC with spatially modulated parameters – Figure 1.** The exact ground state solution $u(x) = \rho(x)\Phi(\theta(x))$ of the main equation (1.1) especially for $f(x, s^2) = f_0(x)s^2$, where $\Phi(t)$ is a solution of the corresponding Duffing equation. The potential $V(x)$, the spatially modulation $f_0(x)$ and the frequency $\theta(x)$ are generated by the amplitude function $\rho(x)$ via certain differential relations derived by the similarity transformations (for details see [4]).

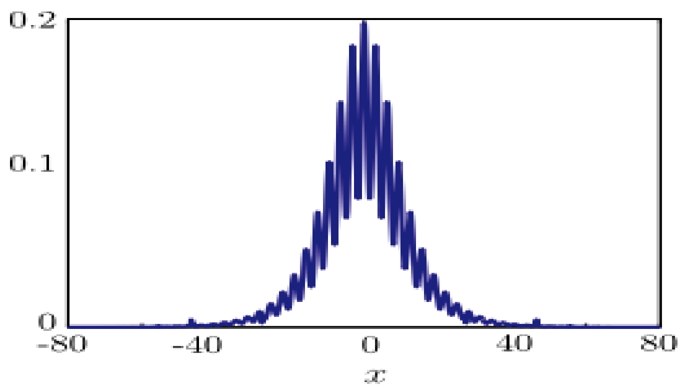

Figure 1. [4, Figure 2 – case (a)]

- **A spin-orbit coupled BEC – Figure 2.** The numerical simulation realized by a split-step Crank–Nicolson method for the stationary states $|\psi_1|$ and $|\psi_2|$ of an integrable system of coupled GPEs (1.4) solved by combining the Lax pair method and gauge transformation approach (for details see [11]).

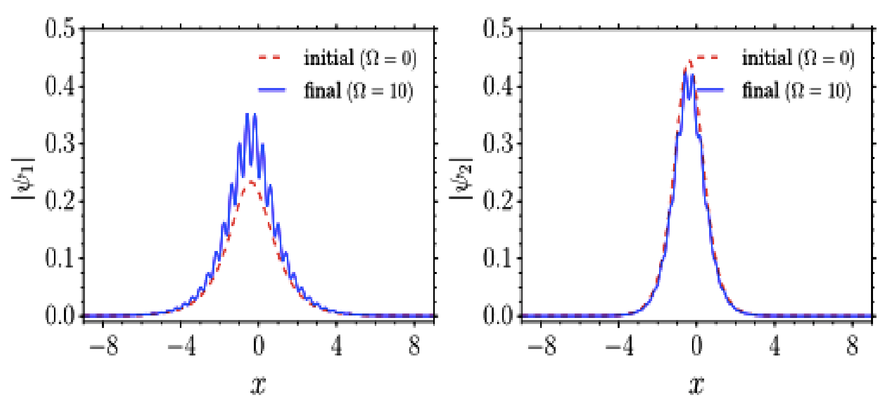

Figure 2. [11, Figure 7]

- **The ground and first excited states in BEC – Figure 3.** The numerically ground state solution $u(x)$ of the main equation (1.1), which is computed by the gradient flow with discrete normalization, where the discretizing has been made in two ways (the backward Euler sine-pseudospectral and backward/forward Euler sine-pseudospectral methods) (for details see [3]).

This numerical simulation is the most interesting for our consideration in the paper, because it visualizes the next two issues:

- *relation between non-monotonic behaviours of $u(x)$ and $V(x)$:* when $V(x)$ is non-monotonic, then $u(x)$ is non-monotonic too, although it is very well known that the classic theory for the

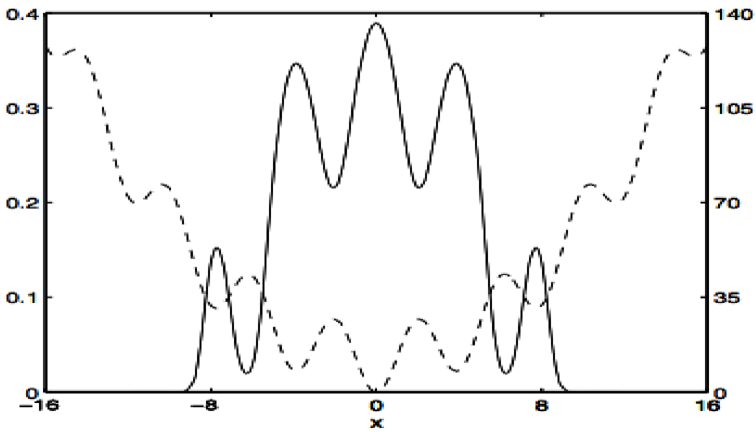

Figure 3. [3, Figure 1(b), $u(x)$ – solid line, $V(x)$ – dashed lines]

linear Schrödinger equation says that when $V(x)$ is a harmonic potential: $V(x) = A|x|^2$, $A > 0$, which is increasing on $(0, \infty)$, then $u(x)$ is of Gaussian type: $u(x) = Be^{-|x|^2}$, $B > 0$, which is decreasing on $(0, \infty)$, see in [8, Section 2.3: Density profile and velocity distribution];

- *the co-existence of local extrema on the same interval:* $u(x)$ attains the local maxima (resp., minima) in the intervals where the $V(x)$ attains its minima (resp., maxima).

In Section 2, we state and describe our main assumptions and results, which are proved in Section 3. The essential advantages of our method with respect to the method presented in the recently published paper [5] are: the assumption for strictly positivity of $u(x)$ is relaxed so that $u(x)$ is now a non-negative ground state solution having the most finite number of zeros per one interval; here, the nonlinear potential $f(x, s^2)$ is not only of attractive type but it can also be of a repulsive type, which is described above just after (1.2); our conditions on the external potential $V(x)$ is more general than related one considered in [6], which is shown below in Subsection 2.2.

2 Statement of the basic assumptions and main results

2.1 Basic assumptions

Let $[a, b] \subset \mathbb{R}$ be a bounded interval on which the ground state solution $u(x)$ satisfies:

$$u(x) \text{ possesses at most finite number of zeros in } [a, b], \tag{H_0}$$

and the potential difference between μ and $(V(x) + g(x))2m/\hbar^2$ satisfies:

$$\mu - \frac{2m}{\hbar^2}\,(V(x) + g(x)) > 0 \ \text{ in } \ [a, b]. \tag{H-basic}$$

The next consequence of the assumptions (H_0) and (H-basic) is worth to be pointed out.

Proposition 2.1. *Let* (1.2) *and* (H-basic) *hold. If the ground state solution* $u(x)$ *of* (1.1) *satisfies* (H_0) *and* $u(x) \geq 0$ *in* $[a, b]$*, then* $u(x)$ *has at most one stationary point in* $[a, b]$.

Indeed, if the ground state solution $u(x)$ is non-negative in $[a, b]$ and has two stationary points $x_1, x_2 \in [a, b]$, $x_1 \neq x_2$, then integrating (1.1) over $[x_1, x_2]$ together with assumptions (1.2), (H_0) and (H-basic), we have

$$0 = u'(x_2) - u'(x_1) \leq -\int\limits_{x_1}^{x_2} \Big[\mu - \frac{2m}{\hbar^2}\,(V(x) + g(x))\Big] u(x)\, dx < 0,$$

which is not possible. Thus, the stationary point of $u(x)$ in $[a, b]$ is unique if it exists of course.

Next, the assumption (H_0) is more general than the next one,

$$u(x) \neq 0, \quad x \in [a, b]. \tag{$H_{\neq 0}$}$$

Although $(H_{\neq 0})$ is involved in all preceding Figures 1–3, the general assumption (H_0) is also appearing in the context of particle density in BEC (see, for instance, [2]).

Remark 2.1. Especially for $g(x) \equiv 0$ (attractive case) or $g(x) \geq 0$ (repulsive case), the assumption (H-basic) implies

$$\mu - \frac{2m}{\hbar^2} V(x) > 0 \ \text{ in } \ [a, b]. \tag{2.1}$$

Since the chemical potential μ is a constant and $V(x)$ is a continuous potential in $\mathbb{R}$, thanks to (2.1) it is possible to take for $[a, b]$ such an interval in which $V(x)$ attains its minimum. This is in the accordance with the numerical simulation given in Figure 3 above. More accurate relation between the non-monotonic behaviours of $u(x)$ and $V(x)$ is considered in Subsection 2.3 below about the co-existence of local extrema of $u(x)$ and $V(x)$.

2.2 The existence of localized local extrema of $u(x)$

On a given interval $[a, b]$, we involve on the potentials μ, $V(x)$ and $g(x)$ the following additional assumption: for some $\varphi \in C^1(a, b)$, $\varphi(a) = \varphi(b) = 0$, $\varphi(x) \neq 0$ in (a, b), we have

$$\int_a^b |\varphi(x)|^2 \, dx > \int_a^b \frac{|\varphi'(x)|^2}{\mu - \frac{2m}{\hbar^2} (V(x) + g(x))} \, dx. \tag{H-general}$$

The condition (H-general) is particularly related with the eigenvalue problem for the one-dimensional Laplacian operator in (a, b) with respect to the first eigenvalue $\lambda_1 > 0$ and the corresponding eigenvalue vector $\varphi \in C^2(a, b)$ (let us remark that $\lambda_1 = (\pi/(b-a))^2$ and $\varphi(x) = \sin(\sqrt{\lambda_1}(x-a))$):

$$\varphi'' + \lambda_1 \varphi = 0 \ \text{ in } \ (a, b), \quad \varphi(a) = \varphi(b) = 0. \tag{2.2}$$

Indeed, if we suppose

$$\mu - \frac{2m}{\hbar^2} (V(x) + g(x)) > \lambda_1 \ \text{ in } \ [a, b], \tag{2.3}$$

which is a more concrete condition than (H-general), from (2.2) and (2.3) we get

$$\int_a^b |\varphi(x)|^2 \, dx = \frac{1}{\lambda_1} \int_a^b |\varphi'(x)|^2 \, dx > \int_a^b \frac{|\varphi'(x)|^2}{\mu - \frac{2m}{\hbar^2} (V(x) + g(x))} \, dx.$$

Thus, condition (2.3) is a particular case of (H-general) taking for $\varphi(x)$ the eigenfunction from (2.2).

The first main result is

Theorem 2.1. *Suppose that* (1.2) *is satisfied and let* $[a, b]$ *be an interval such that* (H-basic) *and* (H-general) *hold. Then every solution* $u(x)$ *of the nonlinear Schrödinger equation* (1.1) *has a stationary point in* $[a, b]$. *Furthermore, if* $u(x) \geq 0$ *in* $[a, b]$ *and satisfies* (H_0), *then the stationary point of* $u(x)$ *is unique in* $[a, b]$. *Moreover,* $u(x)$ *attains its local maximum in* $[a, b]$.

Since (2.3) is a particular case of (H-general), we have also derived the next interesting consequence of the main result.

Theorem 2.2. *Suppose that* (1.2) *holds and let* $[a, b]$ *be an interval such that the potentials* μ, $V(x)$ *and* $g(x)$ *satisfy* (2.3). *Then every solution* $u(x)$ *of the nonlinear Schrödinger equation* (1.1) *has a stationary point on* $[a, b]$. *Furthermore, if* $u(x) \geq 0$ *in* $[a, b]$ *and satisfies* (H_0), *then the stationary point of* $u(x)$ *is unique in* $[a, b]$. *Moreover,* $u(x)$ *attains its local maximum in* $[a, b]$.

Thus, Theorem 2.2 is a particular case of Theorem 2.1, and Theorem 2.1 is more general than [6, Theorem 3.1] even in the case $g(x) \equiv 0$, because the condition $(H_{\neq 0})$ is relaxed here with (H_0).

2.3 The co-existence of local extrema of ground state solution $u(x)$ and potential $V(x)+g(x)$

According to Theorem 2.1, we are able now to explain the case in which the ground state solution $u(x)$ attains a local minimum on an interval where the potential $V(x)+g(x)$ attains its local maximum. This is also visualized in the next figure:

Figure 4. $u(x)$ - solid line, $V(x)+g(x)$ – dashed lines.

For this purpose, we need to work with two disjoint intervals $[a_1,b_1]$ and $[a_2,b_2]$ such that

$$a_1 < b_1 < a_2 < b_2. \tag{2.4}$$

In order to simplify the notation, let

$$W(x) = \mu - \frac{2m}{\hbar^2}\,(V(x)+g(x)).$$

Let the assumptions (H-basic), (H-general) and $u(x) \geq 0$ with (H_0) be satisfied on both intervals $[a_k,b_k]$, $k \in \{1,2\}$. Firstly, it implies that $W(x) > 0$ on $[a_1,b_1] \cup [a_2,b_2]$. Since $W(x)$ is a continuous potential on $\mathbb{R}$, we have $W(x) > 0$ on $[a_1, b_1+\varepsilon) \cup (a_2-\varepsilon, b_2]$ for some small enough $\varepsilon > 0$. Secondly, from Theorem 2.1 applied to $[a_1,b_1]$ and $[a_2,b_2]$ simultaneously, we obtain that $u(x)$ has two points of local maximum $x_1 \in [a_1,b_1]$ and $x_2 \in [a_2,b_2]$ as well as x_1 (resp., x_2) is a unique stationary point on $[a_1,b_1]$ (resp., $[a_2,b_2]$). Hence, $u(x)$ attains its local minimum on $[b_1,a_2]$. On the other hand, we claim that

$$\text{there exists } x_0 \in (b_1+\varepsilon, a_2-\varepsilon) \text{ such that } W(x_0) < 0. \tag{2.5}$$

Indeed, if we suppose the contrary, then $W(x) \geq 0$ in $(b_1+\varepsilon, a_2-\varepsilon)$ and hence, $W(x) > 0$ on $J_\varepsilon := [x_1, b_1+\varepsilon) \cup (a_2-\varepsilon, x_2]$. Next, since $u'(x_1) = u'(x_2) = 0$, integrating equation (1.1) over $[x_1,x_2] \subset [a_1,b_2]$, as in the proof of Proposition 2.1, we obtain

$$0 \leq -\int_{x_1}^{x_2} W(x)u(x)\,dx. \tag{2.6}$$

Since $W(x) > 0$ on J_ε and $u(x) \geq 0$, from (H_0) and (2.6) it follows that $0 < 0$. Hence, $W(x)$ has to satisfy (2.5). Since $W(x)$ is supposed to be strictly positive on $[a_k,b_k]$, $k \in \{1,2\}$, this implies that $W(x)$ has a negative minimum on $[b_1,a_2]$ and hence, $V(x)+g(x)$ attains a local maximum on $[b_1,a_2]$. Thus, we have shown the next result.

Theorem 2.3. *Suppose that* (1.2) *is satisfied and let* $[a_k,b_k]$, $k \in \{1,2\}$ *be two disjoint intervals such that* (2.4) *hold. If* (H-basic) *and* (H-general) *are satisfied on* $[a_k,b_k]$, $k \in \{1,2\}$, *then on the interval* $[b_1,a_2]$ *the ground state solution* $u(x)$ *has a local minimum and the potential* $V(x)+g(x)$ *attains a local maximum.*

In particular, for $g(x) \equiv 0$, Theorem 2.3 shows that $V(x)$ has to be necessarily a non-monotonic potential on $[b_1,a_2]$.

3 Proofs of main results

3.1 Some propositions

Before stating two propositions used in the proof of Theorem 2.1, we first state and prove the next

Proposition 3.1. *Every solution $u(x)$ of NLSE* (1.1) *which satisfies* ($H_{\neq 0}$) *has a stationary point in* $[a,b]$ *if and only if there is no any solution* (v,R) *of the first-order system*

$$\begin{cases} R' = 1 + R^2\Big[\Big(\mu - \dfrac{2m}{\hbar^2}\,V(x)\Big) + \dfrac{2m}{\hbar^2}\,f\big(x,|v(x)|^2\big)\Big] & \textit{in } (a,b), \\ v' = \dfrac{1}{R(x)}\,v & \textit{in } (a,b), \end{cases} \tag{3.1}$$

such that $v,R \in C([a,b]) \cap C^1(a,b)$, $v(x) \neq 0$ *and* $R(x) \neq 0$, $\forall x \in [a,b]$.

Proof. (Direction $\Longrightarrow$) Arguing by contradiction, let there exist a function $v \in C([a,b]) \cap C^1(a,b)$, $v(x) \neq 0$ on $[a,b]$ and a function $R \in C([a,b]) \cap C^1(a,b)$, $R(x) \neq 0$ on $[a,b]$ which satisfy the first-order system (3.1). Then

$$\begin{aligned} v''(x) &= \frac{v'(x)}{R(x)} - \frac{v(x)}{R^2(x)}\,R'(x) \\ &= \frac{v(x)}{R^2(x)}\,(1 - R'(x)) = -\Big[\Big(\mu - \frac{2m}{\hbar^2}\,V(x)\Big) + \frac{2m}{\hbar^2}\,f\big(x,|v(x)|^2\big)\Big]v(x) \end{aligned}$$

and thus, $v(x)$ is a solution of NLSE (1.1) such that $v'(x) = v(x)/R(x) \neq 0$ on $[a,b]$. It contradicts the assumption that every solution of NLSE (1.1) has a stationary point in $[a,b]$.

(Direction $\Longleftarrow$) On the contrary, if $u(x)$ is a solution of NLSE (1.1) such that $u'(x) \neq 0$ on $[a,b]$, then the pair of functions $R(x) := u(x)/u'(x)$ and $v(x) := u(x)$ is the solution of system (3.1) such that $R(x) \neq 0$ and $u(x) \neq 0$ on $[a,b]$, because of ($H_{\neq 0}$) and

$$\begin{aligned} R'(x) &= 1 - \frac{u(x)}{u'^2(x)}\,u''(x) \\ &= 1 + \frac{u^2(x)}{u'^2(x)}\,\Big[\Big(\mu - \frac{2m}{\hbar^2}\,V(x)\Big) + \frac{2m}{\hbar^2}\,f\big(x,|u(x)|^2\big)\Big] \\ &= 1 + R^2(x)\Big[\Big(\mu - \frac{2m}{\hbar^2}\,V(x)\Big) + \frac{2m}{\hbar^2}\,f\big(x,|u(x)|^2\big)\Big]. \end{aligned}$$

This contradicts the assumption that (3.1) has no such a solution. It completes the proof of this proposition. □

In the absence of the strong assumption ($H_{\neq 0}$), we have the following essential proposition, which is weaker than Proposition 3.1, but it is used in the proof of the main result.

Proposition 3.2. *If for a function $v(x)$ there is no any solution $R \in C([a,b]) \cap C^1(a,b)$, $R = R(x)$ of the first-order differential equation*

$$R' = 1 + R^2\Big[\Big(\mu - \frac{2m}{\hbar^2}\,V(x)\Big) + \frac{2m}{\hbar^2}\,f\big(x,|v(x)|^2\big)\Big] \quad \textit{in } (a,b), \tag{3.2}$$

then every solution $u(x)$ of NLSE (1.1) *has a stationary point in* $[a,b]$.

Proof. By contradiction, let $u(x)$ be a solution of (1.1) such that $u'(x) \neq 0$ for all $x \in [a,b]$. Then the function $R(x) = u(x)/u'(x)$ is well defined on $[a,b]$, $R \in C([a,b]) \cap C^1(a,b)$ and satisfies equation (3.2) with $v(x) = u(x)$ (because we can use the similar computation as in the proof of Proposition 3.1). This contradicts the main assumption of this lemma and hence, there exists $x_{\mathrm{s}} \in [a,b]$ such that $u'(x_{\mathrm{s}}) = 0$, which proves the proposition. □

Now we give a condition ensuring that $u(x)$ attains its local maximum at a stationary point.

Proposition 3.3. *Suppose that* (1.2) *holds and let* $x_{\rm s} \in [a,b]$ *be a stationary point of a solution* $u(x)$ *of NLSE* (1.1)*. If* $u(x) \geq 0$ *on* $[a,b]$ *and satisfies* (H_0)*, and the potentials* μ*,* $V(x)$ *and* $g(x)$ *satisfy* (H-basic)*, then* $x_{\rm s}$ *is a unique stationary point of* $u(x)$*. Moreover,* $u(x)$ *attains a local maximum at* $x_{\rm s}$*.*

Proof. Let $u(x) \geq 0$ and satisfy (H_0). Since all potentials in (H-basic) are continuous, there exists $\varepsilon > 0$ such that

$$\mu - \frac{2m}{\hbar^2}\,(V(x) + g(x)) > 0 \;\text{ in }\; (a-\varepsilon, b+\varepsilon). \tag{3.3}$$

Integrating (1.1) over $[x, x_{\rm s}]$, where $x \in (a-\varepsilon, x_{\rm s})$, and using (1.2), (H_0) and (3.3), we obtain

$$\begin{aligned} -u'(x) &= -\int_x^{x_{\rm s}} \Big(\mu - \frac{2m}{\hbar^2}\,V(\sigma)\Big)u(\sigma)\,d\sigma - \frac{2m}{\hbar^2}\int_x^{x_{\rm s}} f\big(\sigma, |u(\sigma)|^2\big)u(\sigma)\,d\sigma \\ &\leq -\int_x^{x_{\rm s}} \Big[\mu - \frac{2m}{\hbar^2}\,(V(\sigma) + g(\sigma))\Big]u(\sigma)\,d\sigma < 0, \end{aligned}$$

which shows that $u'(x) > 0$ for all $x \in (a-\varepsilon, x_{\rm s})$. Analogously, integrating (1.1) over $[x_{\rm s}, x]$, where $x \in (x_{\rm s}, b+\varepsilon)$, we obtain

$$u'(x) \leq -\int_{x_{\rm s}}^{x} \Big[\mu - \frac{2m}{\hbar^2}\,(V(\sigma) + g(\sigma))\Big]u(\sigma)\,d\sigma < 0,$$

which shows that $u'(x) < 0$ for all $x \in (x_{\rm s}, b+\varepsilon)$. Thus, $u(x)$ has a local maximum at the given stationary point $x_{\rm s}$. The uniqueness of $x_{\rm s}$ immediately follows from Proposition 2.1. $\square$

3.2 Proof of Theorem 2.1

By Proposition 3.2 it is enough to show that the assumption (H-general) ensures that for any $v(x)$ there is no any solution $R(x)$, $R \in C([a,b]) \cap C^1(a,b)$ of equation (3.2). Indeed, if there exists such a solution, then multiplying (3.2) by $\varphi^2(x)$, where $\varphi \in C([a,b]) \cap C^1(a,b)$, $\varphi(x) \neq 0$ in (a,b), $\varphi(a) = \varphi(b) = 0$ and using (1.2), we obtain

$$\int_a^b \varphi^2(x)\,dx \leq -\int_a^b \bigg[\sqrt{Q(x)}\,\varphi(x)R(x) + \frac{\varphi'(x)}{\sqrt{Q(x)}}\bigg]^2 dx + \int_a^b \frac{\varphi'^2(x)}{Q(x)}\,dx,$$

where $Q(x) := \mu - \frac{2m}{\hbar^2}\,(V(x) + g(x))$ and $Q(x) > 0$ on $[a,b]$ due to the assumption (H-basic). Previous inequality contradicts the main assumption of this theorem and hence, there is no any solution $R(x)$, $R \in C([a,b]) \cap C^1(a,b)$ of equation (3.2). Therefore, Proposition 3.2 gives the existence of a stationary point of $u(x)$ in $[a,b]$. Now, the rest of this proof immediately follows from Proposition 3.3.

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V. V. Rogachev

ON THE EXISTENCE OF SOLUTIONS TO HIGHER-ORDER REGULAR NONLINEAR EMDEN–FOWLER TYPE EQUATIONS WITH GIVEN NUMBER OF ZEROS ON THE PRESCRIBED INTERVAL

Abstract. The existence of solutions with a given number of zeros to higher-order regular-nonlinear Emden–Fowler type equations is proven.[1]

2010 Mathematics Subject Classification. 34C11, 34E10

Key words and phrases. Higher-order Emden–Fowler type differential equations, regular nonlinearity, boundary value problem.

რეზიუმე. მაღალი რიგის ემდენ-ფაულერის ტიპის არაწრფივი, რეგულარული დიფერენციალური განტოლებებისთვის მტკიცდება ისეთი ამონახსნების არსებობა, რომლის ნულების რაოდენობა მოცემულ სასრულ შუალედში წინასწარ დასახელებული რიცხვის ტოლია.

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1 Introduction

Consider the equation

$$y^{(n)} + p(t, y, y', \dots, y^{(n-1)})|y|^k \operatorname{sgn} y = 0, \tag{1.1}$$

where $n \geq 2$, $k \in (1, +\infty)$, the function $p(t, y_1, y_2, y_3, \dots, y_n) \in C(\mathbb{R}^{n+1})$ is Lipschitz continuous in $(y_1, y_2, y_3, \dots, y_n)$ and for some m, $M > 0$ satisfies the inequalities

$$0 < m \leq p(t, y_1, y_2, \dots, y_n) \leq M < +\infty.$$

The problem of the existence of solutions to (1.1) with the given number of zeros on the prescribed domain is investigated.

Asymptotic classification of solutions to (1.1) with $n = 3, 4$, $k \in (1, +\infty)$, $p(t, y, y', \dots, y^{(n-1)}) \equiv$ const and with $n = 3$, $k \in (0, 1)$, $p(t, y, y', \dots, y^{(n-1)}) \equiv const$ is provided in [1, 3] by I. Astashova. Later, the existence of quasiperiodic solutions to the regular ($k \in (1, +\infty)$) higher-order Emden–Fowler type equations has been proved in [2].

Using [1], the existence of solutions with the given number of zeros was proved for the case of third- and fourth-order equations with the constant coefficient p and with $k \in (0, 1) \cup (1, +\infty)$ (see [4]). Later, the case of the higher-order differential equation (1.1) with the constant potential and regular nonlinearity ($k > 1$) was considered in [5]. In [6], the existence of solutions with the given number of zeros was proved for (1.1) with $n = 3$, $k \in (1, +\infty)$. In [7], the existence of such solutions was proved for the equation with $k \in (0, 1)$.

Now we generalize these results to the case of equation (1.1).

2 Main result

Theorem 2.1. *For any real a and b satisfying $-\infty < a < b < +\infty$ and any integer $S \geq 2$, equation* (1.1) *has a solution defined on the segment $[a, b]$, vanishing at its end points a, b and having exactly S zeros on $[a, b]$.*

3 Preliminary results

The following statements are used to prove the main theorem.

Lemma 3.1 (Generalization of 7.1 from [1])**.** *Let $y(t)$ be a solution to* (1.1)*. If for some t_0 the inequalities*

$$y(t_0) \geq 0,\ y'(t_0) > 0,\ y''(t_0) \geq 0,\ \dots,\ y^{(n-1)}(t_0) \geq 0$$

hold, then there is a local supremum of y at some point $t_0' > t_0$ satisfying the inequalities

$$t_0' - t_0 \leq (\mu y'(t_0))^{-\frac{k-1}{k+n-1}},$$
$$y(t_0') > (\mu y'(t_0))^{\frac{n}{k+n-1}},$$

where $\mu > 0$ is a constant depending only on n, k, m, M.

Lemma 3.2 (Generalization of 7.2 from [1])**.** *Let $y(t)$ be a solution to* (1.1)*. If for some t_0' the inequalities*

$$y(t_0') > 0,\ y'(t_0') \leq 0,\ \dots,\ y^{(n-1)}(t_0') \leq 0$$

hold, then y is equal to zero at some point $t_0 > t_0'$ satisfying the inequalities

$$t_0 - t_0' \leq (\mu y(t_0'))^{-\frac{k-1}{n}},$$
$$y'(t_0) < -(\mu y(t_0'))^{\frac{k+n-1}{n}},$$

where $\mu > 0$ is a constant depending only on n, k, m, M.

Lemma 3.3 (Generalization of 7.3 from [1]). *Under the conditions of Lemmas* 3.1*,* 3.2*, for any* $t_1 > t_0$ *such that* $y(t_0) = 0$*,* $y(t_1) = 0$*, the inequality*

$$|y'(t_1)| > Q|y'(t_0)|$$

holds, where $Q > 1$ *is a constant depending only on* k*,* m*,* M*.*

Lemma 3.4. *Let* D *be a subset of* $\mathbb{R}^n$ *and* $\widetilde{D}$ *be a subset of* $\mathbb{R}^{n+1}$*. Suppose that for any* $c \in D$ *there exists* $x_c > 0$ *such that* $\{c\} \times [0, x_c] \subset \widetilde{D}$*. Consider a continuous function* $f(c, x) : \widetilde{D} \to \mathbb{R}$ *and introduce the following conditions:*

- $f(c, 0) = 0$ *for any* $c \in D$*,*
- *for every* $c \in D$*, there exists a point* $x_1(c) \in (0, x_c)$ *such that* $f(c, x_1(c)) = 0$ *and* $f(c, x) \neq 0$ *whenever* $x \in (0, x_1(c))$*,*
- $f(c, x)$ *is differentiable in* x*, and* $\frac{df}{dx}(c, x_1(c)) \neq 0$ *for all* $c \in D$*.*

If these conditions hold, then $x_1(c) : D \to \mathbb{R}$ *is a continuous function.*

Proof. By definition, $x_1(c)$ describes the distance from 0 to the first zero of the function $f(c, \cdot\,)$. The existence of such a zero is stated in the second condition of the lemma. Therefore $x_1(c)$ is actually a function (its value is defined for every $c \in D$), but, perhaps, discontinuous. We intend to prove that $x_1(c)$ is a continuous function.

At every point $(c, x_1(c)) \in \widetilde{D}$, the function $f(c, x)$ fulfills the conditions of the Implicit Function Theorem. Therefore for any $\widetilde{c} \in D$ there exist rectangular neighborhoods $U \subset D$ of $\widetilde{c}$, $V \subset \mathbb{R}$ of $x_1(\widetilde{c})$, and a continuous function $g_{\widetilde{c}}(c) : U \to V$ such that for all $(c, x) \in U \times V$ the conditions $f(c, x) = 0$ and $x = g_{\widetilde{c}}(c)$ are equivalent.

It is clear that $x_1(\widetilde{c}) = g_{\widetilde{c}}(\widetilde{c})$, but we have to prove that $x_1(c) \equiv g_{\widetilde{c}}(c)$ in some neighborhood of $\widetilde{c}$. (We know that $f(c, g_{\widetilde{c}}(c)) = 0$, but the zeros of $f(c, \cdot\,)$ provided by $g_{\widetilde{c}}(c)$ may not be the zeros closest to the point $x = 0$.)

We will prove this by contradiction. Suppose that in any punctured neighborhood of some point $c^* \in D$ there exists a point c such that $g_{c^*}(c) \neq x_1(c)$. Then we have an infinite set $\{c_\alpha\}$ such that for every c_α the inequality $g_{c^*}(c_\alpha) \neq x_1(c_\alpha)$ holds. We can extract from $\{c_\alpha\}$ a sequence $\{c_n\}$ tending to the point c^*. The implicit function theorem for $f(c, x)$ takes place in a neighborhood $U \times V$ of the point $(c^*, x_1(c^*))$.

Now we look closely at the set $\{(c_n, x_1(c_n))\}$. It is a sequence in $\widetilde{D}$, which cannot enter $U \times V$, because otherwise the condition $f(c_n, x_1(c_n)) = 0$ inside $U \times V$ contradicts the very definition of $\{c_n\}$. At the same time, the points $(c_n, x_1(c_n))$ cannot be above the graph of $g_{c^*}(c)$ and above $U \times V$ by the definition of the function $x_1(c)$.

So, the sequence $\{x_1(c_n)\}$ is bounded by zero from below and by $\inf V < x_1(c^*)$ from above. Hence $\{x_1(c_n)\}$ has a limit inferior $x^* < x_1(c^*)$. We extract a subsequence $\{x_1(c_{n_i})\}$ tending to the above limit and then consider a sequence $\{(c_{n_i}, x_1(c_{n_i}))\}$. The function $f(c, x)$ is continuous, $f(c_{n_i}, x_1(c_{n_i})) = 0$, and $(c_{n_i}, x_1(c_{n_i})) \to (c^*, x^*)$ as $i \to \infty$. Therefore, $f(c^*, x^*) = 0$. But at the same time we have $x^* < x_1(c^*)$, and this contradicts the conditions of the lemma. Therefore, the point c^*, in fact, does not exist.

This means that for every point $\widetilde{c} \in D$ the equality $x_1(c) \equiv g_{\widetilde{c}}(c)$ is true in some neighborhood of $\widetilde{c}$. Every function $g_{\widetilde{c}}(c)$ is continuous near $\widetilde{c}$. Therefore, $x_1(c)$ is continuous at every point $c \in D$. □

3.1 Proof of the main result

Proof of Theorem 2.1. Consider a maximally extended solution $y(t)$ to (1.1) with initial data $y^{(i)}(a) = y_i$, $i \in \overline{0, n-1}$.

It follows from Lemmas 3.1–3.3 that if the inequalities

$$y(t_0) \geq 0,\ y'(t_0) > 0,\ y''(t_0) \geq 0,\ \ldots,\ y^{(n-1)}(t_0) \geq 0$$

hold at some point t_0, then there exists a point $t_1 > t_0$ such that

$$y(t_1) = 0,\ y'(t_1) < 0,\ y''(t_1) \leq 0,\ \ldots,\ y^{(n-1)}(t_1) \leq 0$$

and

$$t_1 - t_0 \le (\mu' y'(t_0))^{-\frac{k-1}{k+2}},$$

where $\mu' > 0$ and $Q > 1$ are constants depending only on k, m, and M.

The analogous statement takes place if

$$y(t_0) \le 0,\ y'(t_0) < 0,\ y''(t_0) \le 0,\ \dots,\ y^{(n-1)}(t_0) \le 0.$$

Hence, if $y_0 = 0$ and $y_i > 0$ for $i \in \overline{1, n-1}$, then $y(t)$ is an oscillating solution, i.e., it has an infinite sequence of zeros $\{a, t_1, t_2, \dots\}$. In the sequel, $y_0 = 0$ and $y_i > 0$ for $i \in \overline{1, n-1}$.

We denote the distance between zeros by $L_i = t_i - t_{i-1}$. The distance from a to the $(S-1)$st zero is a function

$$L(y_1, y_2, \dots, y_{n-1}) = \sum_{j=1}^{S-1} L_j(y_1, y_2, \dots, y_{n-1}),$$

and its value depends on the initial data of the solution $y(t)$.

If $L(y_1, y_2, \dots, y_{n-1}) = b - a$, then the solution $y(t)$ has exactly S zeros on $[a, b]$. To prove the theorem we have to prove that for any b and a the last equation has at least one solution.

First, notice that L is a continuous function. If we rewrite (1.1) as a system of first-order ODEs, that system will satisfy the conditions of the continuous dependence on initial data theorem [8, § 7, Theorem 6]. By $Y(t, a, y_0, y_1, y_2, \dots, y_{n-1})$ we denote a maximally extended solution to (1.1) with initial data $y^{(i)}(a) = y_i$, $i \in \overline{0, n-1}$. Therefore, $Y(t, a, y_0, y_1, y_2, \dots, y_{n-1})$ and n of its derivatives in t are continuous functions on their domains.

Are the conditions of Lemma 3.4 fulfilled? Put

$$D = \big\{(y_1, y_2, \dots, y_{n-1}) \mid \quad y_i > 0\big\} \subset \mathbb{R}^{n-1}.$$

For every such $(y_1, y_2, \dots, y_{n-1})$ we have already proved the existence of the first zero t_1, which satisfies $y'(t_1) \ne 0$. Further, there exists the second zero t_2, and for $\widetilde{D} \subset \mathbb{R}^n$ we take the area above $D \times \{0\}$ and under the graph of $t_2(y_1, y_2, \dots, y_{n-1})$. Obviously, $Y(a, a, y_0, y_1, y_2, \dots, y_{n-1}) = 0$, and $Y(t, a, y_0, y_1, y_2, \dots, y_{n-1})$ is defined on $\widetilde{D}$. (Here a is fixed and y_0 is equal to zero.)

The conditions of Lemma 3.4 are fulfilled, hence $t_1(y_1, y_2, \dots, y_{n-1})$, or L_1 is a continuous function on D. It is possible to prove by using Lemma 3.4 that all L_i, and therefore L are continuous. For L_2, for example, notice that $y(t_1(y_1, y_2, \dots, y_{n-1})), y'(t_1(y_1, y_2, \dots, y_{n-1})), \dots, y^{(n-1)}(t_1(y_1, y_2, \dots, y_{n-1}))$ are also continuous, because they are compositions of continuous functions $Y^{(i)}(\cdot, a, y_0, y_1, y_2, \dots, y_{n-1})$ and $t_1(y_1, \dots, y_{n-1})$.

Now we are to find an upper estimate of L. It is already proved that

$$L_1 \le (\mu' y_1)^{-\frac{k-1}{k+n-1}}.$$

It follows from Lemma 3.3 that

$$|y'(t_i)| \ge Q^i |y'(a)|.$$

Consider L_i. Since $-\frac{k-1}{k+n-1} < 0$, we have

$$L_i \le (\mu' Q^{i-1} y_1)^{-\frac{k-1}{k+n-1}} = (Q^{-\frac{k-1}{k+n-1}})^{i-1} (\mu' y_1)^{-\frac{k-1}{k+n-1}}.$$

Put $\widetilde{Q} = Q^{-\frac{k-1}{k+n-1}}$. Since $Q > 1$, $-\frac{k-1}{k+n-1} < 0$, and therefore $0 < \widetilde{Q} < 1$, the upper estimates of L_i form a decreasing geometric progression. Therefore,

$$L = L_1 + L_2 + \dots + L_{S-1} \le \frac{1 - \widetilde{Q}^S}{1 - \widetilde{Q}} (\mu' y'(a))^{-\frac{k-1}{k+n-1}} = c_1 y'(a)^{-\frac{k-1}{k+n-1}},$$

$$L < c_1 y'(a)^{-\frac{k-1}{k+n-1}}, \tag{3.1}$$

where c_1 is a constant depending on n, k, m, M, and S.

To get a lower estimate of L it is sufficient to make a lower estimation of L_1. Consider a point $t_0' \in [a, t_1]$ such that $y'(t_0') = 0$. On the segment $[t_0', t_1]$, the derivatives y', y'' are non-positive. Therefore,

$$Qy'(a) < |y'(t_1)| = |y'(t_1)| - |y'(t_0')| = \int\limits_{t_0'}^{t_1} |y''(\xi)|\, d\xi < |t_1 - t_0'| \max_{[t_0', t_1]} |y''|.$$

We must get an upper estimate of $\max\limits_{[t_0', t_1]} |y''|$. Notice the behaviour of the derivatives of $y(t)$ as t goes from a to t_1. On the segment $[a, t_1]$, the inequality $y(t) > 0$ holds. First, near a, every derivative, except $y^{(n)}$, is positive. It follows that $y^{(n-1)}$ is decreasing and after some point the inequality $y^{(n-1)} < 0$ holds, when $y^{(n)}$ is still negative. Hence, now $y^{(n-2)}$ starts to decrease, and we can repeat the same steps, until the solution y intersects the $0 - t$-axis, i.e., when we move t from a to t_1, the derivatives change their signs in order and higher-order derivatives change sign before low-order ones. Therefore, on $[t_0', t_1]$, the second derivative of the solution y is negative, because on the segment $[t_0', t_1]$ the first derivative $y'(t) < 0$.

Denote $|y|^k \operatorname{sgn} y$ by $|y|^k_{\pm}$. All initial data are positive, hence

$$\begin{aligned} 0 > y''(t) &= y_2 + y_3(t-a) + y_4 \frac{(t-a)^2}{2!} + \cdots + y_{n-1} \frac{(t-a)^{n-3}}{(n-3)!} \\ &\qquad - \int\limits_a^t \cdots \int\limits_a^t p(t, y, \ldots, y^{(n-1)}) |y|^k_{\pm} (d\,t)^{n-2} \\ &> - \int\limits_a^t \cdots \int\limits_a^t p(t, y, \ldots, y^{(n-1)}) |y|^k_{\pm} (d\,t)^{n-2} > -M|t-a|^{n-2} \max_{[a,t_1]} |y|^k, \end{aligned}$$

whence

$$\max_{[t_0', t_1]} |y''| < M|t_1 - a|^{n-2} \max_{[a,t_1]} |y|^k.$$

Now we get an upper estimation of $\max\limits_{[a,t_1]} |y|^k$. The inequality $y(t) > 0$ holds on $[a, t_1]$, whence

$$\begin{aligned} y(t) &= y_1(t-a) + y_2 \frac{(t-a)^2}{2!} + \cdots + y_{n-1} \frac{(t-a)^{n-1}}{(n-1)!} - \int\limits_a^t \cdots \int\limits_a^t p(\xi, y, \ldots, y^{(n-1)}) |y|^k_{\pm}\ (d\xi)^n \\ &< y_1(t-a) + y_2\, \frac{(t-a)^2}{2!} + \cdots + y_{n-1}\, \frac{(t-a)^{n-1}}{(n-1)!}\,. \end{aligned}$$

Therefore,

$$\max_{[a,t_1]} |y(t)|^k < \Big(y_1(t_1 - a) + \cdots + y_{n-1}\, \frac{(t_1-a)^{n-1}}{(n-1)!} \Big)^k.$$

Combining both estimates, we get

$$Qy_1 < M|t_1 - t_0'||t_1 - a|^{n-2} \Big(y_1(t_1 - a) + \cdots + y_{n-1}\, \frac{(t_1-a)^{n-1}}{(n-1)!} \Big)^k.$$

By definition, $t_1 - a = L_1$ and $|t_1 - t_0'| < L_1$, hence

$$Qy_1 < ML_1^{n-1} \Big(y_1 L_1 + \cdots + y_{n-1}\, \frac{L_1^{n-1}}{(n-1)!} \Big)^k.$$

Suppose $y_1 = y_2 = \cdots = y_{n-1}$ and y_1 is a variable. In this case,

$$ML_1^{n-1} \Big(L_1 + \cdots + \frac{L_1^{n-1}}{(n-1)!} \Big)^k > Qy_1^{1-k}.$$

In the left-hand side of the inequality we have the function of L_1 which is defined for every $L_1 > 0$, is equal to zero when $L_1 = 0$, and is monotonically increasing. The value of the right-hand side may be arbitrarily large as y_1 is arbitrarily small. Hence, for any $\lambda > 0$, we can choose initial data providing $L > \lambda$.

But, due to (3.1), for any $\lambda > 0$ we can choose initial data providing $0 < L < \lambda$. Therefore, the value of $L(y_1, y_2, \dots, y_{n-1})$ may be arbitrarily large, arbitrarily small, and, at the same time, $L(y_1, y_2, \dots, y_{n-1})$ is proven to be continuous. Thus, we conclude that the range of values of $L(y_1, y_2, \dots, y_{n-1})$ is $(0, +\infty)$. Therefore, the equation

$$L(y_1, y_2, \dots, y_{n-1}) = b - a$$

can be resolved for any $b > a$. This proves the theorem. □

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FRACTIONAL HARDY TYPE INEQUALITIES VIA CONFORMABLE CALCULUS

Abstract. α-fractional analogs of of Hardy's classical integral inequalities are established.

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1 Introduction

In 1925, Hardy [4] used the calculus of variations to prove the inequality

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)\,dt\Big)^p dx \le \Big(\frac{p}{p-1}\Big)^p \int_0^\infty f^p(x)\,dx, \tag{1.1}$$

where $f \ge 0$ is integrable over any finite interval $(0,x)$ and f^p is integrable and convergent over $(0,\infty)$ and $p>1$. The constant $(p/(p-1))^p$ is the best possible.

In 1928, Hardy [5] generalized inequality (1.1) and proved that if $p>1$ and f is non-negative for $x \ge 0$, then

$$\int_0^\infty x^{-c}\Big(\int_0^x f(t)\,dt\Big)^p dx \le \Big(\frac{p}{c-1}\Big)^p \int_0^\infty x^{p-c} f^p(x)\,dx \quad \text{for} \quad c>1, \tag{1.2}$$

and

$$\int_0^\infty x^{-c}\Big(\int_x^\infty f(t)\,dt\Big)^p dx \le \Big(\frac{p}{1-c}\Big)^p \int_0^\infty x^{p-c} f^p(x)\,dx \quad \text{for} \quad c<1. \tag{1.3}$$

The constants $(p/(c-1))^p$ and $(p/(1-c))^p$ are the best possible.

In recent years, fractional inequalities were studied by using the fractional Caputo and Riemann–Liouville derivative; for details, we refer the reader to [3] and [17]. In [1] and [7], the authors presented conformable calculus and classical inequalities with the use of conformable fractional calculus such as Opial's inequality (see [11] and [12]), Hermite–Hadamard's inequality (see [8] and [10]), Chebyshev's inequality (see [2]) and Steffensen's inequality (see [13]). In this paper, using a somewhat different approach we present new Hardy type inequalities via conformable fractional calculus. Also, one can see from our approach and presentation that the conformable fractional inequalities encountered in the literature are, in fact, special cases of weighted inequalities (for an appropriate weight function). Our goal in this paper is, first, to show how naturally weights work in inequalities and, second, to indicate and correct some slight mistakes (usually when one integrates by parts) in the literature.

The paper is organized as follows. In Section 2, we present some concepts on conformable fractional calculus and also Hölder's inequality for α-fractional differentiable functions which we will use to prove our main results. In Section 3, we prove some Hardy type inequalities for α-fractional differentiable functions and obtain the classical ones as special cases when $\alpha = 1$.

2 Basic concepts and lemmas

In this section, we present some basic definitions concerning conformable fractional calculus. For more details, we refer the reader to [1] and [7].

Definition 2.1. Let $f : [0,\infty) \to \mathbb{R}$. Then the conformable fractional derivative of order α of f is defined by

$$D_\alpha f(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for all $t>0$ and $0<\alpha\le 1$, and

$$D_\alpha f(0) = \lim_{t\to 0^+} D_\alpha f(t).$$

Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point t. Then

$$D_\alpha(fg) = fD_\alpha g + gD_\alpha f. \tag{2.1}$$

Further, let $\alpha \in (0,1]$ and f, g be α-differentiable at a point t, with $g(t) \ne 0$. Then

$$D_\alpha\Big(\frac{f}{g}\Big) = \frac{gD_\alpha f - fD_\alpha g}{g^2}\,. \tag{2.2}$$

Remark 2.1. If f is a differentiable function, then

$$D_\alpha f(t) = t^{1-\alpha}\,\frac{df(t)}{dt}\,.$$

Definition 2.2. Let $f : [0,\infty) \to \mathbb{R}$. Then the conformable fractional integral of order α of f is defined by

$$I_\alpha f(t) = \int_0^t f(x)\; d_\alpha x = \int_0^t x^{\alpha-1} f(x)\, dx \tag{2.3}$$

for all $t > 0$ and $0 < \alpha \le 1$.

Now, we state an integration by parts formula (see [1] and [7]) which is immediate.

Lemma 2.1. *Assume that $w, g : [0,\infty) \to \mathbb{R}$ are two functions such that w, g are differentiable and $0 < \alpha \le 1$. Then for any $b > 0$,*

$$\int_0^b w(x) D_\alpha g(x)\, d_\alpha x = w(x)g(x)|_0^b - \int_0^b g(x) D_\alpha w(x)\; d_\alpha x. \tag{2.4}$$

Next, we prove the Hölder type inequality needed in the next section (of course, it is the usual Hölder inequality for the functions under consideration (i.e., $x^{\frac{(\alpha-1)}{p}} f(x)$ and $x^{\frac{(\alpha-1)}{q}} g(x)$); for completeness we include its proof).

Lemma 2.2. *Let $f, g : [0,\infty) \to \mathbb{R}$ and $0 < \alpha \le 1$. Then for any $b > 0$,*

$$\int_0^b |f(x)g(x)|\, d_\alpha x \le \Big(\int_0^b |f(x)|^p\, d_\alpha x\Big)^{\frac{1}{p}} \Big(\int_0^b |g(x)|^q\, d_\alpha x\Big)^{\frac{1}{q}}, \tag{2.5}$$

where $1/p + 1/q = 1$ (provided the integrals exist (and are finite)).

Proof. For nonnegative real numbers β, γ, the classical Young inequality is

$$\beta^{\frac{1}{p}} \gamma^{\frac{1}{q}} \le \frac{\beta}{p} + \frac{\gamma}{p}\,.$$

Suppose now, without loss of generality, that

$$\int_0^b |f(x)|^p\, d_\alpha x \ne 0 \quad \text{and} \quad \int_0^b |g(x)|^q\, d_\alpha x \ne 0.$$

Applying Young's inequality with

$$\beta = \frac{|f(x)|^p}{\int_0^b |f(x)|^p\, d_\alpha x}, \qquad \gamma = \frac{|g(x)|^q}{\int_0^b |g(x)|^q\; d_\alpha x}\,,$$

and integrating the obtained inequality from 0 to b, we get

$$\begin{aligned}\int_0^b \frac{|f(x)|}{\big(\int_0^b |f(s)|^p\, d_\alpha s\big)^{\frac{1}{p}}}\; \frac{|g(x)|}{\big(\int_0^b |g(s)|^q\, d_\alpha s\big)^{\frac{1}{q}}}\, d_\alpha x \\ = \int_0^b \beta^{\frac{1}{p}}(x)\gamma^{\frac{1}{q}}(x)\, d_\alpha x \le \int_0^b \Big(\frac{\beta}{p} + \frac{\gamma}{q}\Big)\, d_\alpha x\end{aligned}$$

$$= \int_0^b \left(\frac{|f(x)|^p}{p\big(\int_0^b |f(s)|^p \, d_\alpha s\big)} + \frac{|g(x)|^q}{q\big(\int_0^b |g(s)|^q \, d_\alpha s\big)} \right) d_\alpha x$$

$$= \frac{\int_0^b |f(x)|^p \, d_\alpha x}{p\big(\int_0^b |f(s)|^p \, d_\alpha s\big)} + \frac{\int_0^b |g(x)|^q \, d_\alpha x}{q\big(\int_0^b |g(s)|^q \, d_\alpha s\big)} = \frac{1}{p} + \frac{1}{q} = 1,$$

which is the desired inequality (2.5). □

3 Hardy type inequalities of α-fractional order

In this section, we state and prove the main results of this paper and we begin with the fractional version of the classical Hardy type inequality. Throughout the paper, we will assume that the functions are nonnegative locally α-integrable and the integrals throughout are assumed to exist (and are finite, i.e., convergent).

Theorem 3.1. *Let f be a nonnegative function on $(0,\infty)$, and $0 < \alpha \le 1$ and $p > 1$. Also assume $x^{\alpha-1} f(x)$ is continuous on $[0,\infty)$. Then*

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(s) \, d_\alpha s \right)^p d_\alpha x \le \left(\frac{p}{p-\alpha} \right)^p \int_0^\infty (x^{\alpha-1} f(x))^p \, d_\alpha x. \tag{3.1}$$

Proof. Let

$$F(x) := \frac{1}{x} \int_0^x f(s) \, d_\alpha s. \tag{3.2}$$

Integrating by parts, see formula (2.4) with $w(x) = F^p(x)$ and $D_\alpha g(x) = 1$ (note here $g(x) = \frac{x^\alpha}{\alpha}$), and using Remark 2.1, we obtain (here $t > 0$)

$$\begin{aligned} \int_0^t F^p(x) \, d_\alpha x &= \frac{F^p(x) x^\alpha}{\alpha} \bigg|_0^t - \int_0^t \frac{x^\alpha}{\alpha} D_\alpha F^p(x) \, d_\alpha x = \frac{t^\alpha F^p(t)}{\alpha} - \frac{p}{\alpha} \int_0^t \frac{x^\alpha}{\alpha} x^{1-\alpha} F^{p-1}(x) F'(x) \, d_\alpha x \\ &= \frac{t^\alpha F^p(t)}{\alpha} - \frac{p}{\alpha} \int_0^t x F^{p-1}(x) F'(x) \, d_\alpha x; \end{aligned} \tag{3.3}$$

note

$$\lim_{x\to 0^+} x^{\frac{\alpha}{p}} F(x) = \lim_{x\to 0^+} \frac{\int_0^x s^{\alpha-1} f(s) \, ds}{x^{\frac{p-\alpha}{p}}} = \lim_{x\to 0^+} \frac{x^{\alpha-1} f(x)}{(\frac{p-\alpha}{p}) x^{-\frac{\alpha}{p}}} = \lim_{x\to 0^+} \Big(\frac{p}{p-\alpha} \Big) x^{\alpha-1} f(x) x^{\frac{\alpha}{p}} = 0.$$

From the definition of F, we see that

$$x F'(x) = x^{\alpha-1} f(x) - F(x),$$

and substituting it into (3.3), we obtain

$$\begin{aligned} \int_0^t F^p(x) \, d_\alpha x &= \frac{t^\alpha F^p(t)}{\alpha} - \frac{p}{\alpha} \int_0^t F^{p-1}(x) \big(x^{\alpha-1} f(x) - F(x) \big) \, d_\alpha x \\ &= \frac{t^\alpha F^p(t)}{\alpha} - \frac{p}{\alpha} \int_0^t x^{\alpha-1} F^{p-1}(x) f(x) \, d_\alpha x + \frac{p}{\alpha} \int_0^t F^p(x) \, d_\alpha x, \end{aligned}$$

and so

$$\Big(1-\frac{p}{\alpha}\Big)\int\limits_0^t F^p(x)\,d_\alpha x=\frac{t^\alpha F^p(t)}{\alpha}-\frac{p}{\alpha}\int\limits_0^t x^{\alpha-1}F^{p-1}(x)f(x)\,d_\alpha x.$$

Thus

$$\int\limits_0^t F^p(x)\,d_\alpha x=\frac{t^\alpha F^p(t)}{\alpha-p}+\frac{p}{p-\alpha}\int\limits_0^t x^{\alpha-1}F^{p-1}(x)f(x)\,d_\alpha x.$$

Applying Hölder's inequality with indices p and $p/(p-1)$, and using the fact that $t^\alpha F^p(x)/(\alpha-p)$ is negative, we get

$$\int\limits_0^t F^p(x)\,d_\alpha x\le\frac{p}{p-\alpha}\bigg(\int\limits_0^t F^p(x)\,d_\alpha x\bigg)^{\frac{p-1}{p}}\bigg(\int\limits_0^t \big(x^{\alpha-1}f(x)\big)^p\,d_\alpha x\bigg)^{\frac{1}{p}},$$

and so,

$$\bigg(\int\limits_0^t F^p(x)\,d_\alpha x\bigg)^{\frac{1}{p}}\le\frac{p}{p-\alpha}\bigg(\int\limits_0^t \big(x^{\alpha-1}f(x)\big)^p\,d_\alpha x\bigg)^{\frac{1}{p}}.$$

Hence,

$$\int\limits_0^t F^p(x)\,d_\alpha x\le\Big(\frac{p}{p-\alpha}\Big)^p\int\limits_0^t \big(x^{\alpha-1}f(x)\big)^p\,d_\alpha x.$$

Let $t\to\infty$, and then

$$\int\limits_0^\infty F^p(x)\,d_\alpha x\le\Big(\frac{p}{p-\alpha}\Big)^p\int\limits_0^\infty \big(x^{\alpha-1}f(x)\big)^p\,d_\alpha x,$$

which is the desired inequality (3.1). □

Remark 3.1. From the proof of Theorem 3.1 we see that if the condition "$x^{\alpha-1}f(x)$ is continuous on $[0,\infty)$" is replaced either by

(i) $x^{\alpha-1}f(x)$ is continuous on $(0,\infty)$ and $\lim\limits_{x\to0^+} x^{\alpha-1+\frac{\alpha}{p}}f(x)=0$,

or

(ii) $\lim\limits_{x\to0^+} x^\alpha F^p(x)=0$,

then (3.1) is again true.

Corollary 3.1. *In Theorem* 3.1, *if* $\alpha=1$, *then we obtain the classical Hardy inequality* (1.1).

Theorem 3.2. *Let f be a nonnegative function on $(0,\infty)$ and $0<\alpha\le1$. Let $c>1$ and $p>1$. Also assume that $x^{\alpha-1}f(x)$ is continuous on $[0,\infty)$ and $p>c-\alpha$. Then*

$$\int\limits_0^\infty x^{-c}\bigg(\int\limits_0^x f(t)\,d_\alpha t\bigg)^p d_\alpha x\le\Big(\frac{p}{c-\alpha}\Big)^p\int\limits_0^\infty (x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x. \tag{3.4}$$

Proof. Let

$$F(x):=\int\limits_0^x f(s)\,d_\alpha s.$$

Integrating by parts $\int\limits_0^t x^{-c} F^p(x)\, d_\alpha x$ (here $t > 0$) with

$$w(x) = x^{-c} F^p(x), \quad D_\alpha g(x) = 1 \quad \Big(g(x) = \frac{x^\alpha}{\alpha}\Big),$$
$$D_\alpha w(x) = x^{1-\alpha}\big(-cx^{-c-1}F^p(x) + px^{-c}F^{p-1}(x)F'(x)\big),$$

we obtain

$$\begin{aligned}\int\limits_0^t x^{-c}F^p(x)\, d_\alpha x &= \frac{x^{-c}F^p(x)x^\alpha}{\alpha}\Big|_0^t - \int\limits_0^t \frac{x^\alpha}{\alpha} x^{1-\alpha}\big(-cx^{-c-1}F^p(x) + px^{-c}F^{p-1}(x)F'(x)\big)\, d_\alpha x \\ &= \frac{t^{\alpha-c}F^p(t)}{\alpha} + \frac{c}{\alpha}\int\limits_0^t x^{-c}F^p(x)\, d_\alpha x - \frac{p}{\alpha}\int\limits_0^t x^{1-c}F^{p-1}(x)F'(x)\, d_\alpha x;\end{aligned}$$

note

$$\begin{aligned}\lim_{x\to 0^+} x^{\frac{\alpha-c}{p}} F(x) &= \lim_{x\to 0^+} \frac{\int\limits_0^x s^{\alpha-1} f(s)\, ds}{x^{\frac{c-\alpha}{p}}} \\ &= \lim_{x\to 0^+} \frac{x^{\alpha-1}f(x)}{(\frac{c-\alpha}{p})x^{\frac{c-\alpha}{p}-1}} = \lim_{x\to 0^+}\Big(\frac{p}{c-\alpha}\Big)x^{\alpha-1}f(x)x^{1+\frac{\alpha-c}{p}} = 0.\end{aligned}$$

Thus

$$\int\limits_0^t x^{-c}F^p(x)\, d_\alpha x = \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \frac{p}{c-\alpha}\int\limits_0^t x^{1-c}F^{p-1}(x)F'(x)\, d_\alpha x.$$

Since $F'(x) = x^{\alpha-1}f(x)$, we obtain

$$\begin{aligned}\int\limits_0^t x^{-c}F^p(x)\, d_\alpha x &= \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \frac{p}{c-\alpha}\int\limits_0^t x^{1-c}F^{p-1}(x)x^{\alpha-1}f(x)\, d_\alpha x \\ &\le \frac{p}{c-\alpha}\int\limits_0^t x^{\alpha-c}F^{p-1}(x)f(x)\, d_\alpha x \le \frac{p}{c-\alpha}\int\limits_0^t x^{\alpha-c}F^{p-1}(x)f(x)\, d_\alpha x \\ &\le \frac{p}{c-\alpha}\int\limits_0^t x^{\alpha-c}\frac{F^{p-1}(x)}{(x^{-c})^{\frac{p-1}{p}}(x^c)^{\frac{p-1}{p}}}f(x)\, d_\alpha x \le \frac{p}{c-\alpha}\int\limits_0^t \frac{x^{\alpha-c}}{(x^{-c})^{\frac{p-1}{p}}}\big((x^{-c}F^p(x))\big)^{\frac{p-1}{p}}f(x)\, d_\alpha x \\ &\le \frac{p}{c-\alpha}\int\limits_0^t x^{\alpha-\frac{c}{p}}\big(x^{-c}F^p(x)\big)^{\frac{p-1}{p}}f(x)\, d_\alpha x.\end{aligned}$$

Applying Hölder's inequality with indices p and $p/(p-1)$, we obtain (note $\alpha - c < 0$)

$$\begin{aligned}\int\limits_0^t x^{-c}F^p(x)\, d_\alpha x &\le \frac{p}{c-\alpha}\Bigg(\int\limits_0^t\Big((x^{-c}F^p(x))^{\frac{p-1}{p}}\Big)^{\frac{p}{p-1}} d_\alpha x\Bigg)^{\frac{p-1}{p}}\Bigg(\int\limits_0^t (x^{\alpha-\frac{c}{p}}f(x))^p\, d_\alpha x\Bigg)^{\frac{1}{p}} \\ &\le \frac{p}{c-\alpha}\Bigg(\int\limits_0^t x^{-c}F^p(x)\, d_\alpha x\Bigg)^{\frac{p-1}{p}}\Bigg(\int\limits_0^t (x^{\alpha-\frac{c}{p}}f(x))^p\, d_\alpha x\Bigg)^{\frac{1}{p}}.\end{aligned}$$

Thus

$$\Bigg(\int\limits_0^t x^{-c}F^p(x)\, d_\alpha x\Bigg)^{\frac{1}{p}} \le \Big(\frac{p}{c-\alpha}\Big)\Bigg(\int\limits_0^t (x^{\alpha-\frac{c}{p}}f(x))^p\, d_\alpha x\Bigg)^{\frac{1}{p}},$$

and so

$$\int_0^t x^{-c}F^p(x)\,d_\alpha x \le \Big(\frac{p}{c-\alpha}\Big)^p \int_0^t (x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x.$$

Let $t\to\infty$, and then

$$\int_0^\infty x^{-c}F^p(x)\,d_\alpha x \le \Big(\frac{p}{c-\alpha}\Big)^p \int_0^\infty (x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x,$$

which is the desired inequality (3.4). □

Remark 3.2. From the proof of Theorem 3.2 we see that if the condition "$x^{\alpha-1}f(x)$ is continuous on $[0,\infty)$" is replaced either by

(i) $x^{\alpha-1}f(x)$ is continuous on $(0,\infty)$ and $\lim\limits_{x\to0^+} x^{\alpha+\frac{\alpha-c}{p}}f(x)=0$,

or

(ii) $\lim\limits_{x\to0^+} x^{\alpha-c}F^p(x)=0$,

then (3.4) is again true.

Corollary 3.2. *In Theorem* 3.2*, if $\alpha=1$, then we have the weighted Hardy inequality* (1.2)*.*

Corollary 3.3. *In Theorem* 3.2*, if $c=p$ and $\alpha=1$, then we have the classical Hardy inequality* (1.1)*.*

Theorem 3.3. *Let f be a nonnegative function on $(0,\infty)$ and $0<c<\alpha\le1$. Let $p>1$. In addition, assume that $x^{\alpha-1}f(x)$ is continuous on $(0,\infty)$ and $\lim\limits_{t\to\infty} t^{\alpha+\frac{\alpha-c}{p}}f(t)=0$. Then*

$$\int_0^\infty x^{-c}\Big(\int_x^\infty f(t)\,d_\alpha t\Big)^p d_\alpha x \le \Big(\frac{p}{\alpha-c}\Big)^p \int_0^\infty (x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x. \tag{3.5}$$

Proof. Let $F(x):=\int_x^\infty f(s)\,d_\alpha s=\int_x^\infty s^{\alpha-1}f(s)\,ds$ and integrate by parts the term $\int_\epsilon^t x^{-c}F^p(x)\,d_\alpha x$ (here $t>0$ and $0<\epsilon<t$ small) with

$$w(x)=x^{-c}F^p(x),\quad D_\alpha g(x)=1\quad\Big(g(x)=\frac{x^\alpha}{\alpha}\Big),$$
$$D_\alpha w(x)=x^{1-\alpha}\big(-cx^{-c-1}F^p(x)+px^{-c}F^{p-1}(x)F'(x)\big).$$

Then we obtain

$$\begin{aligned}\int_\epsilon^t x^{-c}F^p(x)\,d_\alpha x &= \frac{x^{-c}F^p(x)x^\alpha}{\alpha}\Big|_\epsilon^t-\int_\epsilon^t \frac{x^\alpha}{\alpha}x^{1-\alpha}\big(-cx^{-c-1}F^p(x)+px^{-c}F^{p-1}(x)F'(x)\big)\,d_\alpha x\\ &=\frac{t^{\alpha-c}F^p(t)}{\alpha}-\frac{\epsilon^{\alpha-c}F^p(\epsilon)}{\alpha}+\frac{c}{\alpha}\int_\epsilon^t x^{-c}F^p(x)\,d_\alpha x-\frac{p}{\alpha}\int_\epsilon^t x^{1-c}F^{p-1}(x)F'(x)\,d_\alpha x\\ &\le \frac{t^{\alpha-c}F^p(t)}{\alpha}+\frac{c}{\alpha}\int_\epsilon^t x^{-c}F^p(x)\,d_\alpha x-\frac{p}{\alpha}\int_\epsilon^t x^{1-c}F^{p-1}(x)F'(x)\,d_\alpha x,\end{aligned}$$

and therefore (letting $\epsilon\to0^+$), since $\alpha-c>0$, we have

$$\int_0^t x^{-c}F^p(x)\,d_\alpha x\le\frac{t^{\alpha-c}F^p(t)}{\alpha-c}-\frac{p}{\alpha-c}\int_0^t x^{1-c}F^{p-1}(x)F'(x)\,d_\alpha x.$$

Since $F'(x) = -x^{\alpha-1}f(x)$, we obtain

$$\int_0^t x^{-c}F^p(x)\,d_\alpha x \le \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \frac{p}{\alpha-c}\int_0^t x^{1-c}F^{p-1}(x)x^{\alpha-1}f(x)\,d_\alpha x$$

$$= \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \frac{p}{\alpha-c}\int_0^t x^{\alpha-c}F^{p-1}(x)f(x)\,d_\alpha x = \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \int_0^t x^{\alpha-c}\frac{F^{p-1}(x)}{(x^{-c})^{\frac{p-1}{p}}.(x^c)^{\frac{p-1}{p}}}f(x)\,d_\alpha x$$

$$= \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \int_0^t \frac{x^{\alpha-c}}{(x^{-c})^{\frac{p-1}{p}}}\left((x^{-c}F(x))^p\right)^{\frac{p-1}{p}}f(x)\,d_\alpha x$$

$$= \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \int_0^t x^{\alpha-\frac{c}{p}}(x^{-c}F^p(x))^{\frac{p-1}{p}}f(x)\,d_\alpha x.$$

Applying Hölder's inequality with indices p and $p/(p-1)$, we obtain

$$\int_0^t x^{-c}F^p(x)\,d_\alpha x \le \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \frac{p}{\alpha-c}\left(\int_0^t \left((x^{-c}F^p(x))^{\frac{p-1}{p}}\right)^{\frac{p}{p-1}}d_\alpha x\right)^{\frac{p-1}{p}}\left(\int_0^t \left(x^{\alpha-\frac{c}{p}}f(x)\right)^p d_\alpha x\right)^{\frac{1}{p}}$$

$$\le \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \frac{p}{\alpha-c}\left(\int_0^t x^{-c}F^p(x)\,d_\alpha x\right)^{\frac{p-1}{p}}\left(\int_0^t \left(x^{\alpha-\frac{c}{p}}f(x)\right)^p d_\alpha x\right)^{\frac{1}{p}},$$

so,

$$\left(\int_0^t x^{-c}F^p(x)\,d_\alpha x\right)^{\frac{1}{p}} \le \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \left(\frac{p}{\alpha-c}\right)\left(\int_0^t (x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x\right)^{\frac{1}{p}}.$$

Thus

$$\int_0^t x^{-c}F^p(x)\,d_\alpha x \le \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \left(\frac{p}{\alpha-c}\right)^p\int_0^t (x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x.$$

Let $t\to\infty$ and note

$$\lim_{t\to\infty} t^{\frac{\alpha-c}{p}}F(t) = \lim_{t\to\infty}\frac{\int_t^\infty s^{\alpha-1}f(s)}{t^{\frac{c-\alpha}{p}}}\,ds = \lim_{t\to\infty} -\frac{t^{\alpha-1}f(t)}{(\frac{c-\alpha}{p})t^{\frac{c-\alpha}{p}-1}} = -\lim_{t\to\infty}\left(\frac{p}{c-\alpha}\right)f(t)t^{\alpha+\frac{\alpha-c}{p}} = 0,$$

so,

$$\int_0^\infty x^{-c}F^p(x)\,d_\alpha x \le \left(\frac{p}{\alpha-c}\right)^p\int_0^\infty (x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x,$$

which is the desired inequality (3.5). □

Corollary 3.4. *In Theorem* 3.3*, if* $\alpha = 1$*, then we have the weighted Hardy inequality* (1.3)*.*

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Short Communication

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ON THE ANTIPERIODIC PROBLEM FOR SYSTEMS OF NONLINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. A general theorem (principle of a priori boundedness) on the solvability of the antiperiodic problem for systems of nonlinear generalized ordinary differential equations is given.

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Let n be a natural number, $\omega > 0$ be a real number, $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a matrix-function with bounded total variation components on every closed interval of the real axis, and $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ be a vector-function belonging to the Carathéodory class corresponding to the matrix-function A on every closed interval of the real axis.

Consider the nonlinear system of generalized ordinary differential equations

$$dx = dA(t)\cdot f(t,x) \tag{1}$$

with the antiperiodic condition

$$x(t+\omega) = -x(t) \ \text{ for } \ t\in\mathbb{R}. \tag{2}$$

We will assume that

$$A(t+\omega) = A(t) + C \ \text{ and } \ f(t+\omega,x) = -f(t,-x) \ \text{ for } \ t\in\mathbb{R}, \ \ x\in\mathbb{R}^n, \tag{3}$$

or

$$A(t+\omega) = -A(t) + C \ \text{ and } \ f(t+\omega,x) = f(t,-x) \ \text{ for } \ t\in\mathbb{R}, \ \ x\in\mathbb{R}^n, \tag{4}$$

where $C\in\mathbb{R}^{n\times n}$ is a constant matrix.

The theorem on the existence of a solution of problem (1), (2), which is given below and called the principle of a priori boundedness, generalizes the well known Conti–Opial type theorems (see [6, 7, 12] for the case of ordinary differential equations) and supplements earlier known criteria for the solvability of nonlinear boundary value and initial problems for systems of generalized ordinary differential equations (see, e.g., [1–5, 11, 13, 14] and the references therein).

Analogous and related questions are investigated in [7–10] (see also the references therein) for the boundary value problems for linear and nonlinear systems of ordinary differential and functional differential equations.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see, e.g., [1–5, 11, 13, 14] and the references therein).

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R} =]-\infty, +\infty[\,$, $\mathbb{R}_+ = [0, +\infty[\,$, $[a,b]$ $(a, b \in \mathbb{R})$ is a closed interval.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X = (x_{il})_{i,l=1}^{n,m}$ with the norm $\|X\| = \sum\limits_{i,l=1}^{n,m} |x_{il}|$;

$\mathbb{R}_+^{n\times m} = \big\{(x_{il})_{i,l=1}^{n,m} :\ x_{il} \geq 0\ (i=1,\dots,n;\ l=1,\dots,m)\big\}$.

$O_{n\times m}$ (or O) is the zero $n\times m$-matrix.

If $X = (x_{il})_{i,l=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = (|x_{il}|)_{i,l=1}^{n,m}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

If $X \in \mathbb{R}^{n\times n}$, then $\det X$ is the determinant of X; I_n is the identity $n\times n$-matrix; $\operatorname{diag}(\lambda_1,\dots,\lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1,\dots,\lambda_n$.

$\operatorname{var}_a^b(X)$ is the total variation of the matrix-function $X : \mathbb{R} \to \mathbb{R}^{n\times m}$ on the closed interval $[a,b]$, i.e., the sum of total variations of its components x_{il} $(i=1,\dots,n;\, l=1,\dots,m)$; $V(X)(t) = (v(x_{il})(t))_{i,l=1}^{n,m}$, where $v(x_{il})(0) = 0$, $v(x_{il})(t) = \operatorname{var}_0^t(x_{il})$ for $t > 0$ and $v(x_{il})(t) = -\operatorname{var}_t^0(x_{il})$ for $t < 0$;

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$ at the point t (we will assume $X(t) = X(a)$ for $t \leq a$ and $X(t) = X(b)$ for $t \geq b$, if necessary); $\Delta^- X(t) = X(t) - X(t-)$, $\Delta^+ X(t) = X(t+) - X(t)$;

$\mathrm{BV}([a,b], \mathbb{R}^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a,b] \to \mathbb{R}^{n\times m}$ (i.e., such that $\operatorname{var}_a^b(X) < +\infty$);

$\mathrm{BV}_s([a,b], \mathbb{R}^{n\times m})$ is the normed space of all $X \in \mathrm{BV}([a,b], \mathbb{R}^{n\times m})$ with the norm $\|X\|_s = \sup\{\|X(t)\| :\ t \in [a,b]\}$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

$I \subset \mathbb{R}$ is an interval.

$C(I, \mathbb{R}^{n\times m})$ is the set of all continuous matrix-functions $X : I \to \mathbb{R}^{n\times m}$.

If B_1 and B_2 are normed spaces, then the operator $g : B_1 \to B_2$ (nonlinear, in general) is positive homogeneous if $g(\lambda x) = \lambda g(x)$ for every $\lambda \in \mathbb{R}_+$ and $x \in B_1$.

The operator $\varphi : \mathrm{BV}([a,b], \mathbb{R}^n) \to \mathbb{R}^n$ is called nondecreasing if for every $x, y \in \mathrm{BV}([a,b], \mathbb{R}^n)$ such that $x(t) \leq y(t)$ for $t \in [a,b]$ the inequality $\varphi(x)(t) \leq \varphi(y)(t)$ holds for $t \in [a,b]$.

If $\alpha : I \to \mathbb{R}$ is a nondecreasing function, then $D_\alpha = \{t \in I :\ \alpha(t+) - \alpha(t-) \neq 0\}$.

$s_1, s_2, s_c : \mathrm{BV}([a,b], \mathbb{R}) \to \mathrm{BV}([a,b], \mathbb{R})$ are the operators defined by

$$s_1(x)(a) = s_2(x)(a) = 0,$$
$$s_1(x)(t) = \sum_{a<\tau\leq t} \Delta^- x(\tau) \ \text{ and } \ s_2(x)(t) = \sum_{a\leq\tau<t} \Delta^+ x(\tau) \ \text{ for } \ a < t \leq b,$$

and

$$s_c(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \ \text{ for } \ t \in [a,b].$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$, then

$$\int_s^t x(\tau)\, dg(\tau) = \int_{]s,t[} x(\tau)\, ds_c(g)(\tau) + \sum_{s<\tau\leq t} x(\tau)\Delta^- g(\tau) + \sum_{s\leq\tau<t} x(\tau)\Delta^+ g(\tau) \quad \text{for} \quad a \leq s < t \leq b,$$

where $\int_{]s,t[} x(\tau)\, ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu(s_c(g))$ corresponding to the function $s_c(g)$; if $a = b$, then we assume $\int_a^b x(t)\, dg(t) = 0$; so, $\int_s^t x(\tau)\, dg(\tau)$ is the Kurzweil–Stieltjes integral (see [11, 13, 14]);

$L([a,b],\mathbb{R};g)$ is the space of all functions $x:[a,b]\to\mathbb{R}$, measurable and integrable with respect to the measure $\mu(g_c(g))$ for which

$$\sum_{a<t\le b}|x(t)|\Delta^- g(t)+\sum_{a\le t<b}|x(t)|\Delta^+ g(t)<+\infty,$$

with the norm $\|x\|_{L,g}=\int\limits_a^b |x(t)|\,dg(t)$.

If $g_j:[a,b]\to\mathbb{R}$ $(j=1,2)$ are nondecreasing functions, $g(t)\equiv g_1(t)-g_2(t)$, and $x:[a,b]\to\mathbb{R}$, then

$$\int\limits_s^t x(\tau)\,dg(\tau)=\int\limits_s^t x(\tau)\,dg_1(\tau)-\int\limits_s^t x(\tau)\,dg_2(\tau)\ \text{ for }\ a\le s\le t\le b.$$

If $G=(g_{ik})_{i,k=1}^{l,n}:[a,b]\to\mathbb{R}^{l\times n}$ is a nondecreasing matrix-function and $D\subset\mathbb{R}^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X=(x_{kj})_{k,j=1}^{n,m}:[a,b]\to D$ such that $x_{kj}\in L([a,b],R;g_{ik})$ $(i=1,\dots,l;\ k=1,\dots,n;\ j=1,\dots,m)$;

$$\int\limits_s^t dG(\tau)\cdot X(\tau)=\bigg(\sum_{k=1}^n\int\limits_s^t x_{kj}(\tau)\,dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}\ \text{ for }\ a\le s\le t\le b,$$

$$S_j(G)(t)\equiv\big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n}\ \ (j=1,2)\ \text{ and }\ S_c(G)(t)\equiv\big(s_c(g_{ik})(t)\big)_{i,k=1}^{l,n}.$$

If $D_1\subset\mathbb{R}^n$ and $D_2\subset\mathbb{R}^{n\times m}$, then $Car([a,b]\times D_1,D_2;G)$ is the Carathéodory class, i.e., the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that for each $i\in\{1,\dots,l\}$, $j\in\{1,\dots,m\}$ and $k\in\{1,\dots,n\}$:

(i) the function $f_{kj}(\,\cdot\,,x):I\to D_2$ is $\mu(s_c(g_{ik}))$-measurable for every $x\in D_1$;

(ii) the function $f_{kj}(t,\,\cdot\,):D_1\to D_2$ is continuous for $\mu(s_c(g_{ik}))$-almost every $t\in I$ and for every $t\in D_{g_{ik}}$, and

$$\sup\big\{|f_{kj}(\,\cdot\,,x)|:\ x\in D_0\big\}\in L([a,b],\mathbb{R};g_{ik})$$

for every compact $D_0\subset D_1$.

If $G_j:[a,b]\to\mathbb{R}^{l\times n}$ $(j=1,2)$ are nondecreasing matrix-functions, $G(t)\equiv G_1(t)-G_2(t)$, and $X:[a,b]\to\mathbb{R}^{n\times m}$, then

$$\int\limits_s^t dG(\tau)\cdot X(\tau)=\int\limits_s^t dG_1(\tau)\cdot X(\tau)-\int\limits_s^t dG_2(\tau)\cdot X(\tau)\ \text{ for }\ a\le s\le t\le b,$$

$$S_k(G)(t)\equiv S_k(G_1)(t)-S_k(G_2)(t)\ \ (k=1,2),\quad S_c(G)(t)\equiv S_c(G_1)(t)-S_c(G_2)(t);$$

If $G_1(t)\equiv V(G)(t)$ and $G_2(t)\equiv V(G)(t)-G(t)$, then

$$L([a,b],D;G)=\bigcap_{j=1}^2 L([a,b],D;G_j),$$

$$Car([a,b]\times D_1,D_2;G)=\bigcap_{j=1}^2 Car([a,b]\times D_1,D_2;G_j).$$

If $G(t)\equiv\operatorname{diag}(t,\dots,t)$, then we omit G in the notation containing G.

The inequalities between the vectors and between the matrices are understood componentwise.

Below we assume that

$$A_1(t)\equiv V(A)(t)\ \text{ and }\ A_2(t)\equiv V(A)(t)-A(t).$$

A vector-function $x:\mathbb{R}\to\mathbb{R}^n$ is said to be a solution of system (1) if its restriction on every closed interval $[a,b]\subset\mathbb{R}$ belongs to $\mathrm{BV}([a,b],\mathbb{R}^n)$, and

$$x(t)=x(s)+\int\limits_s^t dA(\tau)\cdot f(\tau,x(\tau)) \ \text{ for } \ s\le t.$$

Under the solution of problem (1), (2) we mean a solutions of system (1) satisfying the condition (2).

Let $B\in\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$, $\eta:[a,b]\to\mathbb{R}^n$ and $q:\mathrm{BV}([a,b],\mathbb{R}^n)\to\mathrm{BV}([a,b],\mathbb{R}^n)$ be a matrix-function, a vector-function and an operator, respectively. Then by a solution of the system of generalized ordinary differential inequalities

$$dx-dB(t)\cdot x\le d\eta(t)+dq(x)\ (\ge) \ \text{ for } \ t\in[a,b]$$

we mean a vector-function $x\in\mathrm{BV}([a,b],\mathbb{R}^n)$ such that

$$x(t)-x(s)-\int\limits_s^t dB(\tau)\cdot x(\tau)\le\eta(t)-\eta(s)+q(x)(t)-q(x)(s)\ (\ge) \ \text{ for } \ a\le s\le t\le b.$$

In addition, if the vector-function $\eta:[a,b]\to\mathbb{R}^n$ is nondecreasing and $g:\mathrm{BV}([a,b],\mathbb{R}^n)\to\mathrm{BV}([a,b],\mathbb{R}^n_+)$ is a positive homogeneous nondecreasing operator, then by $\Omega_{B,\eta,g}$ we denote a set of all solutions of the system

$$|dx-dB(t)\cdot x|\le d\eta(t)+dg(|x|).$$

If $\eta(t)\equiv0$ and q is the trivial operator, then we omit η and q in the notations containing ones. So, Ω_B is the set of all solutions of the homogeneous system of generalized differential equations

$$dx=dB(t)\cdot x.$$

We define

$$\alpha_l(t)=\sum_{i=1}^n v(a_{il})(t)\ \ (l=1,\dots,n) \ \text{ and } \ \alpha(t)=\sum_{i=1}^n\alpha_i(t) \ \text{ for } \ t\in\mathbb{R}.$$

Under conditions (3) or (4), it is not difficult to verify that if a vector-function x is a solution of system (1), then the vector-function $y(t)=-x(t+\omega)$ $(t\in\mathbb{R})$ will be the solution of system (1), as well. Indeed, by definition of the solution of the system, using (3) or (4), we have

$$\begin{aligned}y(t)-y(s)&=-\big(x(t+\omega)-x(s+\omega)\big)\\&=-\int\limits_{s+\omega}^{t+\omega}dA(\tau)\cdot f(\tau,x(\tau))=\int\limits_s^t dA(\tau+\omega)\cdot f(\tau+\omega,x(\tau+\omega))\\&=\int\limits_s^t dA(\tau)\cdot f(\tau,y(\tau)) \ \text{ for } \ s<t.\end{aligned}$$

Therefore, if $x\in\mathrm{BV}([a,b],\mathbb{R}^n)$ is a solution of system (1) on the closed interval $[0,\omega]$ satisfying the condition

$$x(\omega)=-x(0), \tag{5}$$

then its ω-antiperiodic continuation, i.e. the vector-function $y(t)=(-1)^kx(t-k\omega)$ for $k\omega\le t<(k+1)\omega$ $(k=0,\pm1,\pm2,\dots)$ will be a solution of the ω-antiperiodic problem (1), (2).

In connection with this fact, we consider the boundary value problem (1), (5) on the closed interval $[0,\omega]$. Below we will give the sufficient conditions guaranteing the solvability of the latter and hence of problem (1), (2), as well.

Definition 1. The pair (P,l) of a matrix-function $P \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n};A)$ and a continuous operator $l:\mathrm{BV}_s([0,\omega],\mathbb{R}^n)\times\mathrm{BV}_s([0,\omega],\mathbb{R}^n)\to\mathbb{R}^n$ is said to be consistent if:

(i) for any fixed $x\in\mathrm{BV}_s([0,\omega],\mathbb{R}^n)$ the operator $l(x,\cdot\,):\mathrm{BV}_s([0,\omega],\mathbb{R}^n)\to\mathbb{R}^n$ is linear;

(ii) for any $z\in\mathbb{R}^n$, x and $y\in\mathrm{BV}_s([0,\omega],\mathbb{R}^n)$, the inequalities

$$\|P(t,z)\|\le\xi(t,\|z\|),\ \ \|l(x,y)\|\le\xi_0(\|x\|_s)\cdot\|y\|_s$$

are fulfilled for $\mu(g_c(\alpha))$-almost all $t\in[0,\omega]$ and for $t\in D_\alpha$, where $\xi_0:\mathbb{R}_+\to\mathbb{R}_+$ is a nondecreasing function, and $\xi:[0,\omega]\times\mathbb{R}_+\to\mathbb{R}_+$ is a nondecreasing in the second variable function such that $\xi(\cdot\,,s)\in L([0,\omega],\mathbb{R}_+;\alpha)$ for every $s\in\mathbb{R}_+$;

(iii) there exists a positive number β such that for any $x\in\mathrm{BV}_s([0,\omega],\mathbb{R}^n)$, $q\in L([0,\omega],\mathbb{R}^n;A)$ and $c_0\in\mathbb{R}^n$, for which the conditions

$$\det\big(I_n-\Delta^-A(t)\cdot P(t,x(t))\big)\ne0\ \text{ for }\ t\in[0,\omega]$$

and

$$\det\big(I_n+\Delta^+A(t)\cdot P(t,x(t))\big)\ne0\ \text{ for }\ t\in[0,\omega]$$

hold, an arbitrary solution x of the boundary value problem

$$dy=dA(t)\cdot\big(P(t,x(t))y+q(t)\big),\ \ l(x,y)=c_0$$

admits the estimate

$$\|y\|_s\le\beta\big(\|c_0\|+\|q\|_{L,\alpha}\big).$$

Theorem 1. *Let $A\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f\in Car([0,\omega]\times R^n,R^n;A)$ and let there exist a positive number ρ and a consistent pair (P,l) of a matrix-function $P\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n};A)$ and a continuous operator $l:\mathrm{BV}_s([0,\omega],\mathbb{R}^n)\times\mathrm{BV}_s([0,\omega],\mathbb{R}^n)\to\mathbb{R}^n$ such that an arbitrary solution of the problem*

$$dx=dA(t)\cdot\big(P(t,x)x+\lambda[f(t,x)-P(t,x)]x\big),\tag{6}$$
$$\lambda(x(0)+x(\omega))+(1-\lambda)l(x,x)=0\tag{7}$$

admits the estimate

$$\|x\|_s\le\rho\tag{8}$$

for any $\lambda\in\,]0,1[\,$. Then problem (1), (2) is solvable.

Definition 2. Let $\mathcal{S}\subset\mathrm{BV}_s([0,\omega],\mathbb{R}^{n\times n})$, $\mathcal{L}$ be a subset of the set of all bounded vector-functionals $l:\mathrm{BV}_s([0,\omega],\mathbb{R}^n)\to\mathbb{R}^n$, and $y\in\mathrm{BV}([0,\omega],\mathbb{R}^n)$. We say that

(i) a matrix-function $B_0\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ belongs to the set $\mathcal{E}^n_{\mathcal{S}}$ if the condition

$$\det\big(I_n-\Delta^-B_0(t)\big)\ne0\ \text{ and }\ \det\big(I_n+\Delta^+B_0(t)\big)\ne0\ \text{ for }\ t\in[0,\omega]\tag{9}$$

holds and there exists a sequence $B_k\in\mathcal{S}$ $(k=1,2,\dots)$ such that

$$\lim_{k\to+\infty}\|B_k-B_0\|_s=0;$$

(ii) a vector-functional $l_0:\mathrm{BV}_s([0,\omega],\mathbb{R}^n)\to\mathbb{R}^n$ belongs to the set $\mathcal{E}^n_{\mathcal{L}}(y)$ if there exists a sequence $l_k\in\mathcal{L}$ $(k=1,2,\dots)$ such that

$$\lim_{k\to+\infty}l_k(y)=l_0(y).$$

Definition 3. Let $g_0:\mathrm{BV}([0,\omega],\mathbb{R}^n_+)\to\mathrm{BV}([0,\omega],\mathbb{R}^n)$ be a positive homogeneous nondecreasing operator, and $h_0:\mathrm{BV}_s([0,\omega],\mathbb{R}^n_+)\to\mathbb{R}^n_+$ be a positive homogeneous operator. We say that the pair $(\mathcal{S},\mathcal{L})$ of the set $\mathcal{S}\subset\mathrm{BV}_s([0,\omega],\mathbb{R}^{n\times n})$ and the set $\mathcal{L}$ of some vector-functionals $l:\mathrm{BV}_s([0,\omega],\mathbb{R}^n)\to\mathbb{R}^n$ belongs to the Opial class $\mathcal{O}^n_{g_0,h_0}$ if:

(i) every operator $l \in \mathcal{L}$ is linear and continuous with respect to the norm $\|\cdot\|_s$;

(ii) there exist the numbers $r_0,\ \xi_0 \in \mathbb{R}_+$ and a nondecreasing function $\varphi : [0,\omega] \to \mathbb{R}$ such that the inequalities
$$\|B(0)\| \le r_0, \quad \|B(t) - B(s)\| \le \varphi(t) - \varphi(s) \ \text{ for } \ 0 \le s < t \le \omega$$
and
$$\|l(y)\| \le \xi_0 \|y\|_s$$
are fulfilled for any $B \in \mathcal{S}$, $l \in \mathcal{L}$ and $y \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$;

(iii) if for $B_0 \in \mathcal{E}^n_{\mathcal{S}}$ the function $y \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$ is a solution of the system
$$|dy - dB_0(t) \cdot y| \le dg_0(|y|)$$
under the condition
$$|l_0(y)| \le h_0(|y|),$$
where $l_0 \in \mathcal{E}^n_{\mathcal{L}}(y)$, then $y(t) \equiv 0$.

If
$$g_0(y)(t) \equiv \int\limits_0^t dG_0(\tau) \cdot q_0(y)(\tau) \ \text{ for } \ y \in \mathrm{BV}([0,\omega],\mathbb{R}^n_+),$$
where $G_0 : [0,\omega] \to \mathbb{R}^n$ is a nondecreasing matrix-function, and $q_0 : \mathrm{BV}_s([0,\omega],\mathbb{R}^n_+) \to \mathrm{BV}_s([0,\omega],\mathbb{R}^n_+)$ is a positive homogeneous operator, then we write $\mathcal{O}^n_{G_0,q_0,h_0}$ instead of $\mathcal{O}^n_{g_0,h_0}$.

Definition 4. Let $P \in Car([0,\omega] \times \mathbb{R}^n, \mathbb{R}^{n\times n}; A)$ and let $l : \mathrm{BV}_s([0,\omega],\mathbb{R}^n) \times \mathrm{BV}_s([0,\omega],\mathbb{R}^n) \to \mathbb{R}^n$ be a continuous vector-functional. We say that the pair (B_0, l_0) of the matrix-function $B_0 \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ and the vector-functional $l_0 : \mathrm{BV}_s([0,\omega],\mathbb{R}^n) \to \mathbb{R}^n$ belongs to the set $\mathcal{E}^n_{A,P,l}$ if there exists a sequence $x_k \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$ $(k = 1,2,\dots)$ such that the conditions
$$\lim_{k\to+\infty} \int\limits_a^t dA(\tau) \cdot P(\tau, x_k(\tau)) = B_0(t) \ \text{ uniformly on } [0,\omega] \tag{10}$$
and
$$\lim_{k\to+\infty} l(x_k, y) = l_0(y) \ \text{ for } \ y \in \Omega_{B_0}$$
are valid.

Definition 5. We say that the pair (P, l) of the matrix-function $P \in Car([0,\omega] \times \mathbb{R}^n, \mathbb{R}^{n\times n}; A)$ and the continuous operator $l : \mathrm{BV}_s([0,\omega],\mathbb{R}^n) \times \mathrm{BV}_s([0,\omega],\mathbb{R}^n) \to \mathbb{R}^n$ belongs to the Opial class $\mathcal{O}^n_A$ with respect to the matrix-function A if:

(i) for any fixed $x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$, the operator $l(x,\cdot\,) : \mathrm{BV}_s([0,\omega],\mathbb{R}^n) \to \mathbb{R}^n$ is linear;

(ii) for any $z \in \mathbb{R}^n$, x and $y \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$, the inequalities
$$\|P(t,z)\| \le \xi(t), \quad \|l(x,y)\| \le \xi_0 \|y\|_s \tag{11}$$
are fulfilled for $\mu(g_c(\alpha))$-almost all $t \in [0,\omega]$ and for $t \in D_\alpha$, where $\xi_0 \in R_+$, and $\xi \in L([0,\omega],\mathbb{R}_+;\alpha)$;

(iii) the problem
$$dy = dB_0(t) \cdot y, \quad l_0(y) = 0$$
has only the trivial solution for every pair $(B_0, l_0) \in \mathcal{E}^n_{A,P,l}$.

Remark 1. By (10) and (11), the condition

$$\|\Delta^- A(t)\| \cdot \xi(t) < 1 \ \text{ and } \ \|\Delta^+ A(t)\| \cdot \xi(t) < 1 \ \text{ for } \ t \in [0,\omega]$$

guarantees condition (9).

Corollary 1. *Let $A \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n;A)$ and let there exist a positive number ρ and a pair $(P,l) \in \mathcal{O}_A^n$ such that an arbitrary solution of problem* (6), (7) *admits estimate* (8) *for any $\lambda \in]0,1[$. Then problem* (1), (2) *is solvable.*

Corollary 2. *Let $A \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n;A)$, $P \in L([0,\omega],\mathbb{R}^{n\times n};A)$, and let $l : \mathrm{BV}_s([0,\omega],\mathbb{R}^n) \to \mathbb{R}^n$ be a bounded linear operator such that*

$$\det\big(I_n - \Delta^- A(t)\cdot P(t)\big) \neq 0 \ \textit{ and } \ \det\big(I_n + \Delta^+ A(t)\cdot P(t)\big) \neq 0 \ \textit{ for } \ t \in [0,\omega]$$

and the problem

$$dy = dA(t)\cdot P(t)y, \quad l(y) = 0$$

has only the trivial solution. Let, moreover, there exists a positive number ρ such that an arbitrary solution of the problem

$$dx = dA(t)\cdot\big(P(t)x + \lambda[f(t,x) - P(t)x]\big),$$
$$\lambda\big(x(0)+x(\omega)\big) + (1-\lambda)l(x) = 0$$

admits estimate (8) *for any $\lambda \in]0,1[$. Then problem* (1), (2) *is solvable.*

The following result is analogous to the well-known one belonging to R. Conti and Z. Opial for the boundary value problems for ordinary nonlinear differential equations (see [6, 7, 12]).

Corollary 3. *Let $A \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n;A)$ and let a pair $(P,l) \in \mathcal{O}_A^n$ be such that*

$$|f(t,x) - P(t,x)x| \le \beta(t,\|x\|) \ \textit{ for } \ t \in [0,\omega], \ \ x \in \mathbb{R}^n, \tag{12}$$

and

$$|x(0)+x(\omega) - l(x,x)| \le l_0(|x|) + l_1(\|x\|_s) \ \textit{ for } \ x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n), \tag{13}$$

where $\beta \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n;A)$ is a nondecreasing in the second variable vector-function, $l_0 : \mathrm{BV}_s([0,\omega],\mathbb{R}_+^n) \to \mathbb{R}_+^n$ is a positive homogeneous continuous operator, and $l_1 \in C(\mathbb{R}_+,\mathbb{R}_+^n)$. Let, moreover,

$$\lim_{k\to+\infty} \frac{1}{\rho}\int_a^b dV(A)(\tau)\cdot\beta(\tau,\rho) = 0_n, \quad \lim_{\rho\to+\infty}\frac{l_1(\rho)}{\rho} = 0_n. \tag{14}$$

Then problem (1), (2) *is solvable.*

By $Y_P(x)$ we denote the fundamental matrix of the system

$$dy = dA(t)\cdot P(t,x(t))\,y$$

for every $x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$, satisfying the condition $Y_P(x)(a) = I_n$.

Corollary 4. *Let $A \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n;A)$, $P \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n};A)$ and a continuous operator $l : \mathrm{BV}_s([0,\omega],\mathbb{R}^n)\times\mathrm{BV}_s([0,\omega],\mathbb{R}^n) \to \mathbb{R}^n$, satisfying conditions* (i) *and* (ii) *of Definition* 5, *be such that conditions* (12)–(14) *hold, where $\beta \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n;A)$ is a nondecreasing in the second variable vector-function, $l_0 : \mathrm{BV}_s([0,\omega],\mathbb{R}_+^n) \to \mathbb{R}_+^n$ is a positive homogeneous continuous operator, and $l_1 \in C(\mathbb{R}_+,\mathbb{R}_+^n)$. Let, moreover,*

$$\inf\big\{|\det(l(x,Y_P(x)))| : \ x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)\big\} > 0. \tag{15}$$

Then problem (1), (2) *is solvable.*

Remark 2. In Corollary 4, condition (15) cannot be replaced by the condition

$$\det(l(x, Y_P(x))) \neq 0 \ \text{ for } \ x \in \mathrm{BV}_s([0,\omega], \mathbb{R}^n). \tag{16}$$

The corresponding example for the ordinary differential systems, i.e., for the case where $A(t) \equiv \operatorname{diag}(t, \dots, t)$, has been constructed in [8]. Basing on these example, it is not difficult to construct analogous examples for the case where $A(t) \not\equiv \operatorname{diag}(t, \dots, t)$. Consider the scalar boundary value problem

$$dx = \left(\frac{|x|x}{1+|x|} + 1\right) d\alpha(t), \quad x(0) = -x(\omega),$$

where $\alpha(t) = 0$ for $0 \le t \le c$ and $\alpha(t) = -2$ for $c < t \le \omega$, and $c = \omega/2$. Every solution of the system has the form

$$x(t) = \begin{cases} x(0) & \text{for } \ 0 \le t \le c, \\ x(0) - 2\Big(\dfrac{|x(0)|x(0)}{1+|x(0)|} + 1\Big) & \text{for } \ c < t \le \omega. \end{cases}$$

This problem is not solvable because the equation $x(0) + x(\omega) = 0$ is not solvable with respect to the $x(0)$. On the other hand, if we assume $P(t, x) = \frac{|x|}{1+|x|}$ and $l(x, y) = y(0) + y(\omega)$ in this case, then

$$Y(t) = \begin{cases} 1 & \text{for } \ 0 \le t \le c, \\ 1 - \dfrac{2|x(c)|}{1+|x(c)|} & \text{for } \ c < t \le \omega \end{cases}$$

for $x \in \mathrm{BV}_s([0,\omega], \mathbb{R}^n)$ and, therefore,

$$\det(l(x, Y_P(x))) = \frac{2}{1+|x(c)|} \ \text{ for } \ x \in \mathrm{BV}_s([0,\omega], \mathbb{R}^n).$$

Thus, all conditions of Corollary 4 are fulfilled except of condition (15), instead of which condition (16) holds.

Remark 3. In particular, we can assume that $l(x, y) \equiv x(0) + x(\omega)$ and $l(x) = l(x, x) \equiv x(0) + x(\omega)$ in the results given above. So, for example, the second estimate in condition (ii) of Definition 1 is fulfilled. Condition (7) in Theorem 1 and Corollary 1 as well as the analogous condition in Corollary 2 coincides to condition (3). Condition (13) is valid for the $l_0 \equiv 0$ and $l_1 \equiv 0$ operators.

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