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ITERATIVE TECHNIQUES WITH INITIAL TIME DIFFERENCE AND COMPUTER REALIZATION FOR THE INITIAL VALUE PROBLEM FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

Abstract. Some algorithms are given and applied in an appropriate computer environment to solve approximately the initial value problem for scalar nonlinear Caputo fractional differential equations on a finite interval. Various schemes for constructing successive approximations are suggested. They do not use Mittag–Leffler functions and as a result the practical application of the algorithms is easier. Several particular initial value problems for Caputo fractional differential equations are given to illustrate the advantages of the iterative techniques.*

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1 Introduction

Various processes with anomalous dynamics in science and engineering can be formulated mathematically using fractional differential operators because of its memory and hereditary properties [6, 13]. There is only a small number of fractional differential equations, including linear equations with variable coefficients, which can be solved in closed form and this causes some problems in practical applications.

This paper considers an initial value problem for a nonlinear scalar Caputo fractional differential equation on a closed interval. Several iterative techniques combined with the method of lower and upper solutions are applied to find the approximate solution of the given problem. Mild lower and mild upper solutions are defined. Several algorithms for constructing two convergent monotone functional sequences are given and we prove that both sequences converge and their limits are minimal and maximal solutions of the problem. When the right-hand side of the equations are monotone functions with respect to the time variable, the elements of these sequences do not depend on Mittag–Leffler functions and they can be obtained in closed form with the help of an appropriate software such as Wolfram Mathematica.

We note that iterative techniques combined with lower and upper solutions are applied in the literature to approximately solve various problems in ordinary differential equations [11], second order periodic boundary value problems [5], differential equations with maxima [1, 7], difference equations with maxima [3], impulsive integro-differential equations [8], impulsive differential equations with supremum [9], differential equations of mixed type [10], and Riemann–Liouville fractional differential equations [4, 16].

2 Preliminary and auxiliary results

The Caputo fractional derivative of order $q \in (0,1)$ is defined by (see, for example, [13])

$$ {}_{t_0}^{c}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \int\limits_{t_0}^{t} (t-s)^{-q} m'(s)\,ds, \quad t \geq t_0. \tag{2.1}$$

Let t_0 be an arbitrary initial time. Usually we think of the independent variable t as time in differential equations, so we will assume $t_0 \in \mathbb{R}_+$.

Definition 2.1 ([15]). We say $m(t) \in C^q([t_0,T],\mathbb{R}^n)$ if $m(t)$ is differentiable (i.e., $m'(t)$ exists), the Caputo derivative ${}_{t_0}^{C}D^q m(t)$ exists and satisfies (1) for $t \in [t_0,T]$.

Consider the initial value problem (IVP) for the nonlinear *Caputo-type fractional differential equation* (FrDE)

$$\begin{aligned} {}_{t_0}^{C}D_t^q x(t) &= f(t,x(t)) \ \text{ for } \ t \in [t_0, t_0+T], \\ x(t_0) &= x_0, \end{aligned} \tag{2.2}$$

where $q \in (0,1)$, $x_0 \in \mathbb{R}$, $f : [t_0,t_0+T] \times \mathbb{R} \to \mathbb{R}$, $x : [t_0,t_0+T] \to \mathbb{R}$.

Any solution $x = x(t)$ of the IVP for FrDE (2.2) satisfies $x \in C^q([t_0,t_0+T],\mathbb{R})$.

If $x(t)$ is a solution of the IVP for FrDE (2.2), then it satisfies the following Volterra integral equation

$$ x(t) = x_0 + \frac{1}{\Gamma(q)} \int\limits_{t_0}^{t} (t-s)^{q-1} f(s,x(s))\,ds \ \text{ for } \ t \in [t_0,t_0+T] \tag{2.3}$$

and, conversely, if $x \in C^q([t_0,t_0+T],\mathbb{R})$ is a solution of (2.3), then it is a solution of the IVP for FrDE (2.2).

Definition 2.2. We say that the function $x \in C([t_0,t_0+T],\mathbb{R})$ is *a mild solution* of the IVP for FrDE (2.2) if it satisfies equation (2.3).

Remark 2.1. The mild solution $x(t)$ of the IVP for FrDE (2.2) might not have a fractional derivative ${}_{t_0}^{C}D_t^q x(t)$.

Let $\tau_0 \in \mathbb{R}_+$, $\tau_0 \neq t_0$ be a different initial time. Consider the following IVP for FrDE similar to (2.2) but with a different initial time (ITD):

$$ {}_{\tau_0}^{c}D_t^q x(t) = f(t, x(t)) \ \text{ for } \ t \in [\tau_0, \tau_0 + T], \quad x(\tau_0) = x_0. \tag{2.4} $$

The change of the initial time reflects not only the initial condition but also the fractional derivative on the solution.

Example 2.1. Let $t_0 = 0$, $\tau_0 = 1$, $q \in (0, 1)$ and consider two initial value problems with initial time difference for scalar Caputo fractional differential equations

$$ {}_{0}^{c}D_t^q x(t) = \frac{t^{1-q}}{\Gamma(2-q)} \ \text{ for } \ t > 0, \quad x(0) = x_0 \tag{2.5} $$

and

$$ {}_{1}^{c}D_t^q x(t) = x\,\frac{t^{-q}}{\Gamma(2-q)} \ \text{ for } \ t > 1, \quad x(1) = x_0. \tag{2.6} $$

Let $f(t, x) = x\,\frac{t^{-q}}{\Gamma(2-q)}$.

The solution of (2.5) with $x_0 = 0$ is $x(t) = t$, $t \geq 0$. The solution of (2.6) with $x_0 = 0$ is given by

$$ \widetilde{x}(t) = \frac{1}{\Gamma(q)} \int_1^t (t-s)^{q-1} s\, ds \neq t = x(t), \ \ t \geq 1. $$

Therefore, the shift of the fractional derivative changes the solution.

Note that for $y(t) = t - 1 = x(t-1)$ we get ${}_{1}^{c}D_t^q y(t) = \frac{(t-1)^{1-q}}{\Gamma(2-q)}$, i.e., ${}_{1}^{c}D_t^q y(t) = f(t-1, y(t))$. This result is theoretically proved in the following Lemma.

Lemma 2.1 ([2, Lemma 3.1] (Shift solutions in FrDE))**.** *Let the function $x \in C^q(\mathbb{R}_+, \mathbb{R}^n)$, $a \geq 0$, be a solution of the initial value problem for FrDE*

$$ {}_{a}^{c}D_t^q x(t) = f(t, x(t)) \ \textit{ for } \ t > a, \quad x(a) = x_0. \tag{2.7} $$

Then the function $\widetilde{x}(t) = x(t + \eta)$ satisfies the initial value problem for the FrDE

$$ {}_{b}^{c}D_t^q \widetilde{x}(t) = f(t + \eta, \widetilde{x}(t)) \ \textit{ for } \ t > b, \quad \widetilde{x}(b) = x_0, \tag{2.8} $$

where $b \geq 0$, $\eta = a - b$.

Remark 2.2. Let $y(t)$ be a solution of the IVP for FrDE (2.4) for $t \geq \tau_0$. Then according to Lemma 2.1, ${}_{t_0}^{c}D_t^q y(t + \eta) = f(t + \eta, y(t + \eta))$ with $\eta = \tau_0 - t_0$.

Let $\theta_0 \in \mathbb{R}_+$, $\theta_0 \neq t_0$, $\theta_0 \neq \tau_0$, be a different initial time. Consider the following IVP for FrDE

$$ {}_{\theta_0}^{c}D_t^q x(t) = f(t, x(t)) \ \text{ for } \ t \in [\theta_0, \theta_0 + T], \quad x(\theta_0) = x_0. \tag{2.9} $$

Note that the IVP for FrDE (2.9) is similar to (2.2) and (2.4) but with different initial times and, proceeding from the above, they may have different solutions in spite of the same initial value.

3 Mild lower and mild upper solutions of FrDE

Following the ideas in [12], we present various types of lower/upper solutions of FrDEs.

Definition 3.1. We say that the function $v \in C([t_0, t_0 + T], \mathbb{R})$ is a minimal (maximal) solution of the IVP for FrDE (2.2) if it is a solution of (2.2) and for any solution $u \in C([t_0, t_0 + T], \mathbb{R})$ of (2.2) the inequality $v(t) \leq u(t)$ $(v(t) \geq u(t))$ holds on $[t_0, t_0 + T]$.

For any point $t_0 \geq 0$ and any function $\xi \in C([t_0, t_0 + T])$ we define the operator Δ by

$$\Delta(t_0, \xi)(t) = \xi(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, \xi(s))\, ds, \quad t \in [t_0, t_0 + T]. \tag{3.1}$$

Remark 3.1. According to Definition 2.2, any mild solution $x(t)$ of the IVP for FrDE (2.2) is a fixed point of the operator Δ. Any fixed point $\xi \in C([t_0, t_0 + T])$ of the operator Δ is a mild solution the IVP for FrDE (2.2) if $\xi(t_0) = x_0$.

Similar to Definition 2.7 [12], we present the following definition.

Definition 3.2. We say that the function $v \in C([t_0, t_0 + T], \mathbb{R})$ is a mild lower (a mild upper) solution in $[t_0, t_0 + T]$ of the IVP for FrDE (2.2) if

$$\begin{aligned} v(t) \leq \ (\geq)\ \Delta(t_0, v)(t) \ \text{ for } \ t \in [t_0, t_0 + T], \\ v(t_0) \leq \ (\geq)\ x_0. \end{aligned} \tag{3.2}$$

Remark 3.2. A mild lower/upper solution of the IVP for FrDE (2.4) and (2.9), respectively, are defined by Definition 3.2 where the initial time point t_0 is replaced by τ_0 and θ_0, respectively.

Lemma 3.1. *Let the function $f \in C([t_0, t_0 + T] \times \mathbb{R}, \mathbb{R})$ be nondecreasing in its second argument, $x(t)$ be a mild solution of the IVP for FrDE (2.2) and $v(t)$ be a mild lower solution on $[t_0, t_0 + T]$ of (2.2) such that $v(t_0) < x_0$. Then $v(t) < x(t)$ on $[t_0, t_0 + T]$.*

Proof. Assume that the claim is not true. Therefore, there exists a point $t^* \in (t_0, t_0 + T)$ such that

$$v(t) < x(t), \ \ t \in [t_0, t^*), \quad v(t^*) = x(t^*) \ \text{ and } v(t) \geq x(t), \ \ t \in (t^*, t^* + \delta),$$

where δ is a small enough positive number. Using the monotonic property of the function f, we obtain

$$\begin{aligned} x(t^*) = v(t^*) \leq v(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t^*} (t-s)^{q-1} f(s, v(s))\, ds \\ < x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t^*} (t-s)^{q-1} f(s, x(s))\, ds = x(t^*), \end{aligned} \tag{3.3}$$

which is a contradiction. Therefore, $v(t) < x(t)$ on $[t_0, t_0 + T]$. □

Similar to Lemma 3.1, we have the following result.

Lemma 3.2. *Let the function $f \in C([t_0, t_0 + T] \times \mathbb{R}, \mathbb{R})$ be nondecreasing in its second argument, $x(t)$ be a mild solution of the IVP for FrDE (2.2) and $w(t)$ be a mild upper solution on $[t_0, t_0 + T]$ of (2.2) such that $w(t_0) > x_0$. Then $w(t) > x(t)$ on $[t_0, t_0 + T]$.*

Lemma 3.3. *Let $\theta_0 < t_0$ and the function $f \in C(([\theta_0, \theta_0 + T] \cup [t_0, t_0 + T]) \times \mathbb{R}, \mathbb{R})$ be nondecreasing in both its arguments, $x(t)$ be a mild solution of the IVP for FrDE (2.2) and $v(t)$ be a mild lower solution on $[\theta_0, \theta_0 + T]$ of (2.9) such that $v(\theta_0) < x_0$. Then $v(t - \eta) < x(t)$ on $[t_0, t_0 + T]$, where $\eta = t_0 - \theta_0 > 0$.*

Proof. From Definition 3.2 and Remark 3.2, we have

$$v(t) \leq v(\theta_0) + \frac{1}{\Gamma(q)} \int_{\theta_0}^{t} (t-s)^{q-1} f(s, v(s))\, ds, \quad t \in [\theta_0, \theta_0 + T], \tag{3.4}$$

or

$$v(t-\eta) \le v(\theta_0) + \frac{1}{\Gamma(q)} \int\limits_{\theta_0}^{t-\eta} (t-\eta-s)^{q-1} f(s, v(s))\, ds, \quad t \in [t_0, t_0+T]. \tag{3.5}$$

Applying the substitution $\nu = s + \eta$ to equation (3.5), we obtain

$$v(t-\eta) \le v(t_0-\eta) + \frac{1}{\Gamma(q)} \int\limits_{t_0}^{t} (t-\nu)^{q-1} f(\nu-\eta, v(\nu-\eta))\, d\nu \;\text{ for } t \in [t_0, t_0+T].$$

Define $\widetilde{v}(t) = v(t-\eta) \in C([t_0, t_0+T], \mathbb{R})$. Therefore $\widetilde{v}(t_0) = v(t_0-\eta) = v(\theta_0) < x_0$ and

$$\widetilde{v}(t) \le \widetilde{v}(t_0) + \frac{1}{\Gamma(q)} \int\limits_{t_0}^{t} (t-\nu)^{q-1} f(\nu-\eta, \widetilde{v}(\nu))\, d\nu \le \widetilde{v}(t_0) + \frac{1}{\Gamma(q)} \int\limits_{t_0}^{t} (t-\nu)^{q-1} f(\nu, \widetilde{v}(\nu))\, d\nu,$$

i.e., the function $\widetilde{v}(t)$ is a mild lower solution on $[t_0, t_0+T]$ of the IVP for FrDE (2.2).

According to Lemma 3.1, the inequality $\widetilde{v}(t) < x(t)$ holds on $[t_0, t_0+T]$. □

The proof of the following result is similar to that in Lemma 3.3 so we omit it.

Lemma 3.4. *Let $t_0 < \theta_0$ and the function $f \in C(([\theta_0, \theta_0+T] \cup [t_0, t_0+T]) \times \mathbb{R}, \mathbb{R})$ be nonincreasing in its first argument and nondecreasing in its second argument, $x(t)$ be a mild solution of the IVP for FrDE* (2.2) *and $v(t)$ be a mild lower solution on $[\theta_0, \theta_0+T]$ of* (2.4) *such that $v(\theta_0) > x_0$. Then $v(t-\eta) > x(t)$ on $[t_0, t_0+T]$, where $\eta = t_0 - \theta_0 < 0$.*

Similar to Lemma 3.3, using Lemma 3.2 instead of Lemma 3.1, we have the following results.

Lemma 3.5. *Let $t_0 < \tau_0$ and the function $f \in C(([\tau_0, \tau_0+T] \cup [t_0, t_0+T]) \times \mathbb{R}, \mathbb{R})$ be nondecreasing in both its arguments, $x(t)$ be a mild solution of the IVP for FrDE* (2.2) *and $w(t)$ be a mild upper solution on $[\tau_0, \tau_0+T]$ of* (2.4) *such that $w(\tau_0) > x_0$. Then $w(t+\xi) > x(t)$ on $[t_0, t_0+T]$, where $\xi = \tau_0 - t_0 > 0$.*

Lemma 3.6. *Let $t_0 > \tau_0$ and the function $f \in C(([\tau_0, \tau_0+T] \cup [t_0, t_0+T]) \times \mathbb{R}, \mathbb{R})$ be nonincreasing in its first argument and nondecreasing in its second argument, $x(t)$ be a mild solution of the IVP for FrDE* (2.2) *and $w(t)$ be a mild upper solution on $[\tau_0, \tau_0+T]$ of* (2.4) *such that $w(\tau_0) > x_0$. Then $w(t+\xi) > x(t)$ on $[t_0, t_0+T]$, where $\xi = \tau_0 - t_0 < 0$.*

4 Main results

We study the case when the IVP for FrDE (2.2), defined on $[t_0, t_0+T]$, has a mild lower and a mild upper solutions defined on different intervals. We will call this case the initial time difference (ITD). In the case when the right-hand side of the FrDE is a monotonic function we present two algorithms for constructing successive approximations to the solution of the IVP for FrDE (2.2).

Case 1. The initial time of the mild lower solution is less than the initial time of the mild upper solution.

Theorem 4.1. *Let the following conditions be fulfilled:*

(1) *Let the points $\theta_0, t_0, \tau_0 :\; 0 \le \theta_0 \le t_0 \le \tau_0$ be given and the function $v \in C([\theta_0, \theta_0+T])$ be a mild lower solution of the IVP for FrDE* (2.9) *on the interval $[\theta_0, \theta_0+T]$ such that $v(\theta_0) < x_0$ and the function $w \in C([\tau_0, \tau_0+T])$ be a mild upper solution of the IVP for FrDE* (2.4) *on the interval $[\tau_0, \tau_0+T]$ such that $w(\tau_0) > x_0$. Let, additionally, $v(t-\eta) \le w(t+\xi)$ for $t \in [t_0, t_0+T]$, where $\eta = t_0 - \theta_0 \ge 0,\; \xi = \tau_0 - t_0 > 0$.*

(2) *The function $f \in C(([\theta_0, \theta_0+T] \cup [t_0, t_0+T] \cup [\tau_0, \tau_0+T]) \times \mathbb{R}, \mathbb{R})$ and it is nondecreasing in both its arguments.*

Then there exist two sequences of functions $\{v^{(n)}(t)\}_0^\infty$ *and* $\{w^{(n)}(t)\}_0^\infty$, $t \in [t_0, t_0+T]$, *such that:*

(a) *The sequences are defined by* $v^{(0)}(t) = v(t-\eta)$, $w^{(0)}(t) = w(t+\xi)$ *and for* $n \geq 1$

$$v^{(n)}(t) = \Delta(t_0, v^{(n-1)}) + x_0 - v^{(n-1)}(t_0)$$
$$\equiv x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(n-1)}(s))\, ds \ \ \textit{for} \ \ t \in [t_0, t_0+T] \tag{4.1}$$

and

$$w^{(n)}(t) = \Delta(t_0, w^{(n-1)}) + x_0 - w^{(n-1)}(t_0)$$
$$\equiv x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, w^{(n-1)}(s))\, ds \ \ \textit{for} \ \ t \in [t_0, t_0+T]. \tag{4.2}$$

(b) *The functions* $v^{(n)}(t)$, $n = 0, 1, 2 \ldots$, *are mild lower solutions of the IVP for FrDE* (2.2).

(c) *The functions* $w^{(n)}(t)$, $n = 0, 1, 2 \ldots$, *are mild upper solutions of the IVP for FrDE* (2.2).

(d) *The sequence* $\{v^{(n)}(t)\}$ *is increasing on* $[t_0, t_0+T]$, *i.e.,* $v^{(k-1)}(t) \leq v^{(k)}(t)$ *for* $t \in [t_0, t_0+T]$, $k = 1, 2, \ldots$.

(e) *The sequence* $\{w^{(n)}(t)\}$ *is decreasing on* $[t_0, t_0+T]$, *i.e.,* $w^{(k-1)}(t) \geq w^{(k)}(t)$ *for* $t \in [t_0, t_0+T]$, $k = 1, 2, \ldots$.

(f) *The inequality*

$$v^{(k)}(t) \leq w^{(k)}(t) \ \ \textit{for} \ \ t \in [t_0, t_0+T], \ \ k = 1, 2, \ldots, \tag{4.3}$$

holds.

(g) *Both sequences converge on* $[t_0, t_0+T]$ *and*

$$V(t) = \lim_{k\to\infty} v^{(n)}(t), \quad W(t) = \lim_{k\to\infty} w^{(n)}(t), \ \ t \in [t_0, t_0+T].$$

(h) *The limit functions* $V(t)$ *and* $W(t)$ *are mild solutions of the IVP for FrDE* (2.2) *on* $[t_0, t_0+T]$.

(i) *For any mild solution* $x(t)$ *of IVP for FrDE* (2.2) *the inequalities* $V(t) \leq x(t) \leq W(t)$ *for* $t \in [t_0, t_0+T]$ *hold, i.e., the functions* $V(t)$, $W(t)$ *are mild minimal and maximal solutions.*

Proof. According to Lemma 3.3 and Lemma 3.5, if there exists a solution $x(t)$ in $[t_0, t_0+T]$ of the IVP for FrDE (2.2), then $v(t-\eta) < x(t) < w(t-\xi)$ for $t \in [t_0, t_0+T]$. We now prove the existence of the solution and will give an algorithm for obtaining it.

Define $v^{(0)}(t) = v(t-\eta)$ and $w^{(0)}(t) = w(t+\xi)$ for $t \in [t_0, t_0+T]$. Then $v^{(0)}(t_0) = v(\theta_0) < x_0$ and $w^{(0)}(t_0) = w(\tau_0) > x_0$.

Then applying the substitution $\nu = s + \eta$, we get

$$v^{(0)}(t) = v(t-\eta) \leq v(\theta_0) + \frac{1}{\Gamma(q)} \int_{\theta_0}^{t-\eta} (t-\eta-s)^{q-1} f(s, v(s))\, ds$$
$$= v(t_0-\eta) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-\nu)^{q-1} f(\nu-\eta, v(\nu-\eta))\, d\nu = v^{(0)}(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-\nu)^{q-1} f(\nu-\eta, v^{(0)}(\nu))\, d\nu$$
$$\leq v^{(0)}(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-\nu)^{q-1} f(\nu, v^{(0)}(\nu))\, d\nu, \ \ t \in [t_0, t_0+T], \tag{4.4}$$

Therefore, the function $v^{(0)}(t)$ is a mild lower solution on $[t_0, t_0+T]$ of the IVP for FrDE (2.2). Similarly, we prove that the function $w^{(0)}(t)$ is a mild upper solution on $[t_0, t_0+T]$ of the IVP for FrDE (2.2).

We use induction to prove the properties of sequences of successive approximations.

Let $n = 1$. From equation (4.1) we get $v^{(1)}(t_0) = x_0$ and applying the monotonic properties of the function f, we obtain

$$\begin{aligned} v^{(1)}(t) &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(0)}(s))\, ds \\ &\le v^{(1)}(\theta_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(1)}(s))\, ds, \quad t \in [t_0, t_0+T], \end{aligned} \tag{4.5}$$

i.e., the function $v^{(1)}(t)$ is a mild lower solution of the IVP for FrDE (2.9). Also,

$$\begin{aligned} v^{(0)}(t) = v(t-\eta) &= v^{(0)}(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-\nu)^{q-1} f(\nu-\eta, v^{(0)}(\nu))\, d\nu \\ &\le x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-\nu)^{q-1} f(\nu, v^{(0)}(\nu))\, d\nu = v^{(1)}(t), \quad t \in [t_0, t_0+T]. \end{aligned} \tag{4.6}$$

Assume $v^{(k-1)}(t) \le v^{(k)}(t),\ k \ge 1$ and $v^{(k)}(t)$ is a mild lower solution of the IVP for FrDE (2.9). Then

$$\begin{aligned} v^{(k)}(t) &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(k-1)}(s))\, ds \\ &\le x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(k)}(s))\, ds = v^{(k+1)}(t), \quad t \in [t_0, t_0+T], \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} v^{(k+1)}(t) &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(k)}(s))\, ds \\ &\le v^{(k+1)}(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(k+1)}(s))\, ds, \quad t \in [t_0, t_0+T], \end{aligned} \tag{4.8}$$

i.e., the claims (b)–(e) are true.

By induction we prove the claim (f).

The sequences $\{v^{(n)}(t)\}_0^\infty$ and $\{w^{(n)}(t)\}_0^\infty$ being monotonic and bounded are uniformly convergent on the interval $[t_0, t_0+T]$. Let for $t \in [t_0, t_0+T]$ we denote

$$V(t) = \lim_{n\to\infty} v^{(n)}(t) \ \text{ and } \ W(t) = \lim_{n\to\infty} w^{(n)}(t).$$

According to (b), (c) and (d), the inequalities

$$\begin{gathered} v^{(n)}(t) \le V(t), \ \ t \in [t_0, t_0+T], \quad W(t) \le w^{(n)}(t), \ \ t \in [t_0, t_0+T], \ \ n = 0, 1, 2, \dots, \\ V(t) \le W(t), \ \ t \in [t_0, t_0+T], \end{gathered} \tag{4.9}$$

hold.

Taking the limit in (4.1) and (4.2) we obtain the Volterra fractional integral equations

$$\begin{aligned} V(t) &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, V(s))\, ds, \quad t \in [t_0, t_0+T], \\ W(t) &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, W(s))\, ds, \quad t \in [t_0, t_0+T]. \end{aligned} \tag{4.10}$$

Therefore, the limit functions $V(t)$ and $W(t)$ are mild solutions of the IVP for FrDE (2.2).

Let $x(t)$ be an arbitrary mild solution of the IVP for FrDE (2.2). According to Lemma 3.3 and Lemma 3.5, it follows that $v^{(0)} = v(t-\eta) < x(t) < w(t+\xi) = w^{(0)}(t)$ on $[t_0, t_0+T]$. Therefore, there exists a number $N \in \mathbb{N} \cup \{0\}$ such that $v^{(N)}(t) \le x(t) \le w^{(0)}(t)$ for $t \in [t_0, t_0+T]$. Then applying the monotonicity property of the function f and the choice of N, we obtain

$$\begin{aligned} v^{(N+1)}(t) &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, v^{(N)}(s))\, ds \\ &\le x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))\, ds = x(t), \quad t \in [t_0, t_0+T]. \end{aligned} \tag{4.11}$$

Therefore, $V(t) \le x(t)$, $t \in [t_0, t_0+T]$. The rest of the proof is similar and we omit it. □

In the special case when the right side part of the FrDE (2.2) does not depend on x, i.e., $f(t,x) \equiv f(t)$, we have the following result.

Corollary 4.1. *Let condition (1) of Theorem 4.1 be satisfied and the function f be nondecreasing. Then for all $n \ge 1$ the equalities $v^{(n)}(t) \equiv w^{(n)}(t) \equiv x(t)$, $t \in [t_0, t_0+T]$, hold, where the successive approximations $v^{(n)}(t)$ and $w^{(n)}(t)$ are defined by (4.1) and (4.2).*

Example 4.1. Let $\theta_0 = 0, t_0 = 0, \tau_0 = 2,\ T = 0.9$ and consider the IVP for the scalar Caputo FrDE

$$ {}_0^cD_t^{0.5} x(t) = x + 1 \ \text{ for } \ t \in [0, 0.9], \quad x(t_0) = x_0. \tag{4.12}$$

Its solution is given by

$$ x(t) = x_0 E_{0.5}(t^{0.5}) + \int_0^t (t-s)^{-0.5} E_{0.5,0.5}((t-s)^{0.5})\, ds \ \text{ for } \ t \in [0, 0.9], \tag{4.13}$$

where $E_q(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}$ and $E_{\alpha,\beta}(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$ are the Mittag–Leffler functions with one and two parameters, respectively. Note that the integral in (4.13) cannot be solved in closed form and the solution cannot be obtained as an expression of classical functions.

Now we apply the above technique to find approximately the solution as a limit of a sequence of explicit functions.

Let $x_0 = 0$.

The function $v(t) = t^7 - 0.01$ is a mild lower solution on $[0, 0.9]$ of the IVP for FrDE (4.12), since the inequality

$$ t^7 - 0.01 \le \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5} (1 + s^7 - 0.01)\, ds \ \text{ for } \ t \in [0, 0.9] \tag{4.14}$$

is satisfied (see Figure 1).

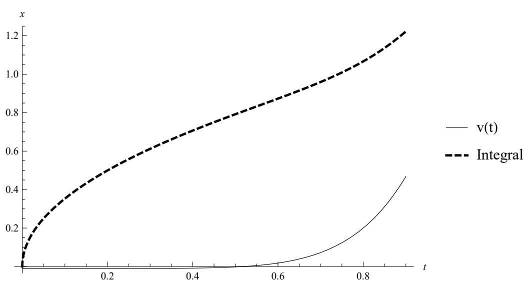

Figure 1. Example 2. Graph of the mild lower solution $t^7 - 0.01$ on $[0, 0.9]$.

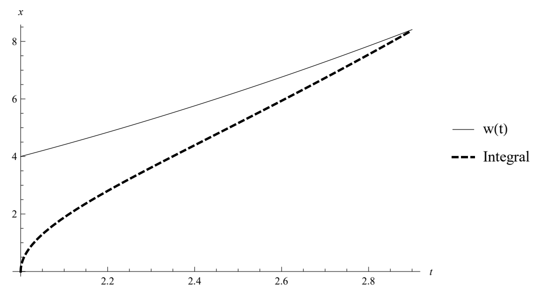

Figure 2. Example 2. Graph of the mild upper solution t^2 on $[2, 2.9]$.

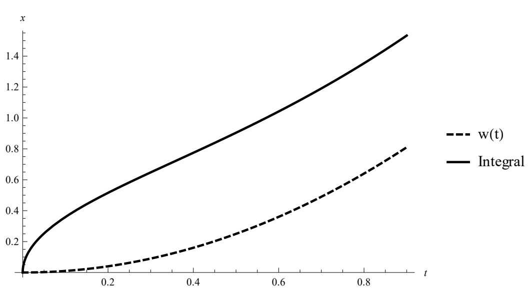

Figure 3. Example 2. Graph of the function t^2 on $[0, 0.9]$.

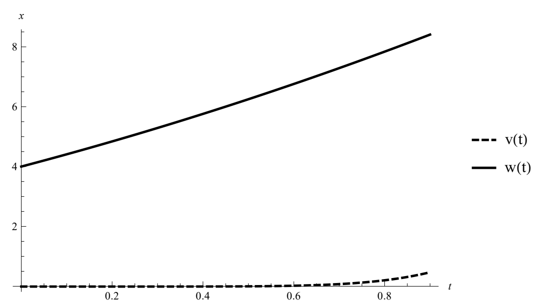

Figure 4. Exampe 2. Graph of the mild lower and mild upper solutions on $[0, 0.9]$.

Consider equation (4.12) with replaced $t_0 = 0$ by $\tau_0 = 2$. The function $w(t) = t^2$ is a mild upper solution on $[2, 2.9]$ of the IVP for FrDE (4.12) with $t_0 = 2$, since the inequality

$$t^2 \geq \frac{1}{\Gamma(0.5)} \int_2^t (t-s)^{-0.5}(1+s^2)\,ds \quad \text{for} \quad t \in [2, 2.9] \tag{4.15}$$

is satisfied (see Figure 2). Note that the function $w(t) = t^2$ is not a mild upper solution on $[0, 0.9]$ of the IVP for FrDE (4.12) (see Figure 3). At the same time, the inequalities $v(t) = t^7 - 0.01 \leq w(t+2) = (t+2)^2$ for $t \in [0, 0.9]$ hold (see Figure 4).

Define the zero lower and upper approximations by $v^{(0)}(t) = v(t) = t^7 - 0.01$ and $w^{(0)}(t) = w(t+2) = (t+2)^2$ for $t \in [0, 0.9]$.

From equation (4.1) we get

$$\begin{aligned} v^{(1)}(t) &= 0 + \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5}(1+s^7-0.01)\,ds = \frac{1}{\Gamma(0.5)} \Big(0.99\,\frac{2}{3}\,t^{1.5} + \frac{4096}{109395}\,t^{8.5}\Big), \\ v^{(2)}(t) &= \frac{1}{(\Gamma(0.5))^2} \int_0^t (t-s)^{-0.5} \Big(\Gamma(0.5) + 0.99\,\frac{2}{3}\,s^{1.5} + \frac{4096}{109395}\,s^{8.5}\Big)\,ds \\ &= \frac{1}{(\Gamma(0.5))^2} \Big(\Gamma(0.5)\,\frac{2}{3}\,t^{1.5} + 0.129591t^3 + 0.00109083t^{10}\Big), \quad t \in [0, 0.9], \end{aligned}$$

i.e., the successive approximations are polynomial functions and there is no problem in obtaining them in a closed form with the corresponding integrals.

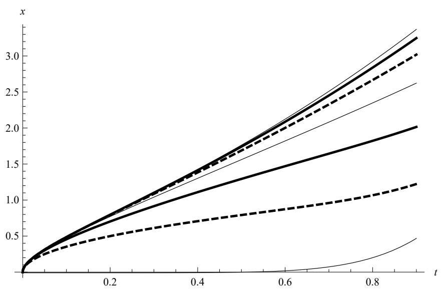

Figure 5. Example 2. Graph of the lower approximations defined by (4.1).

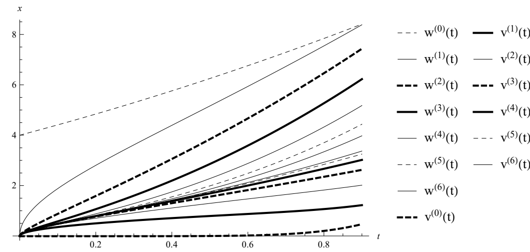

Figure 6. Example 2. Graph of the lower and upper approximations defined by (4.1) and (4.2).

Similarly, we get

$$\begin{aligned} w^{(1)}(t) &= 0 + \frac{1}{\Gamma(0.5)} \int\limits_0^t (t-s)^{-0.5}(1+(s+2)^2)\, ds \\ &= \frac{1}{\Gamma(0.5)} \Big(\frac{10}{3}\, t^{1.5} + \frac{16}{105}\, t^{3.5} + \frac{16}{15}\, t^{2.5} \Big), \quad t \in [0, 0.9]. \end{aligned}$$

Using the software Wolftam Mathematica 10.0 we obtain the successive approximations with the graphs given in Figures 5 and 6.

Then the approximate solution of the IVP for FrDE (4.12) is defined as

$$\begin{aligned} qx(t) = \frac{v^{(6)}(t) + w^{(6)}(t)}{2} &= 1.12838t^{0.5} + t^1 + 0.752253t^{1.5} + 0.5t^2 \\ &+ 0.300901t^{2.5} + 0.499167t^3 + 0.0833333t^4 + 0.00833333t^5 + 0.000694444t^{10}. \end{aligned}$$

Case 2. The initial time of the mild lower solution is greater than the initial time of the mild upper solution.

Theorem 4.2. *Let the following conditions be fulfilled:*

(1) *Let the points θ_0, t_0, τ_0: $0 \le \tau_0 \le t_0 \le \theta_0$ be given and the functions v, w: $v \in C([\theta_0, \theta_0 + T])$, $w \in C([\tau_0, \tau_0 + T])$ be lower and upper solutions of the IVP for NIFrDE* (2.2) *on the intervals $[\theta_0, \theta_0 + T]$ and $[\tau_0, \tau_0 + T]$, respectively. Let, additionally, $v(t + \eta) \le w(t)$ for $t \in [\tau_0, \tau_0 + T]$, where $\eta = \theta_0 - \tau_0 \ge 0$.*

(2) *The function $f \in C(([\theta_0, \theta_0 + T] \cup [t_0, t_0 + T] \cup [\tau_0, \tau_0 + T]) \times \mathbb{R}, \mathbb{R})$ is nonincreasing in its first argument t and it is nondecreasing in its second argument x.*

Then there exist two sequences of functions $\{v^{(n)}(t)\}_0^\infty$ and $\{w^{(n)}(t)\}_0^\infty$, defined by recurrence formulas (4.1) *and* (4.2)*, where $v^{(0)}(t) = v(t + \eta)$, $w^{(0)}(t) = w(t - \xi)$ and the claims* (b)–(i) *of Theorem* 4.1 *are true.*

Example 4.2. Consider the IVP for the scalar Caputo FrDE

$$ {}_1^cD_t^{0.3} x(t) = x^3 - \frac{t}{20} \ \text{ for } \ t \in [1, 3], \quad x(1) = 0. \tag{4.16}$$

Now we apply the above technique to find approximately the solution as a limit of two sequences of explicit functions.

In this case we can find mild lower and mild upper solutions of the IVP for FrDE (4.16) on the interval $[1, 3]$.

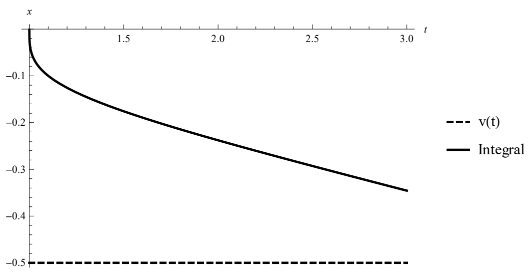

Figure 7. Example 3. Graph of the mild lower solution $v(t) = -0.5$ on $[1,3]$.

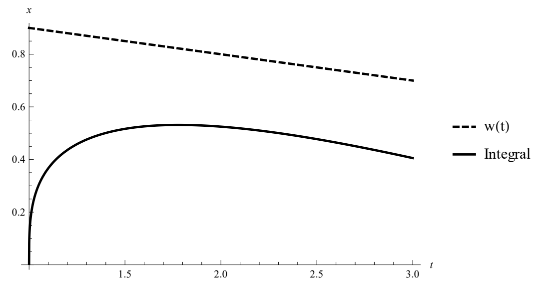

Figure 8. Example 3. Graph of the mild upper solution $w(t) = 1 - 0.1t$ on $[1,3]$.

The function $v(t) = -0.5$ is a mild lower solution on $[1,3]$ of the IVP for FrDE (4.16), since the inequality

$$-0.5 \le \frac{1}{\Gamma(0.3)} \int_1^t (t-s)^{-0.7}\Big((-0.5)^3 - \frac{s}{20}\Big)\, ds \;\text{ for }\; t \in [1,3] \tag{4.17}$$

is satisfied (see Figure 7).

The function $w(t) = 1 - 0.1t$ is a mild upper solution on $[1,3]$ of the IVP for FrDE (4.16), since the inequality

$$1 - 0.1t \ge \frac{1}{\Gamma(0.3)} \int_1^t (t-s)^{-0.7}\Big((1-0.1s)^3 - \frac{s}{20}\Big)\, ds \;\text{ for }\; t \in [1,3] \tag{4.18}$$

is satisfied (see Figure 8).

Therefore, $\theta_0 = t_0 = \tau_0 = 1,\ T = 2$ and $\eta = \xi = 0$.

Define the zero approximation by $v^{(0)}(t) = v(t) = -0.5$, $w^{(0)}(t) = w(t) = 1 - 0.1t$, $t \in [1,3]$.

From equation (4.1) for $t \in [1,3]$ we get

$$v^{(1)}(t) = 0 + \frac{1}{\Gamma(0.3)} \int_1^t (t-s)^{-0.7}\Big((-0.5)^3 - \frac{s}{20}\Big)\, ds$$
$$= 0.334273(-2. + t)^{0.3}\big(-0.998772 + t\big(0.703571 + t(-0.113519 + (0.00599603 - 0.000141416t)t)\big)\big),$$

i.e., the successive approximations are the polynomial functions and there is no problem solving in a closed form with the corresponding integrals.

Similarly, we get

$$w^{(1)}(t) = 0 + \frac{1}{\Gamma(0.3)} \int_1^t (t-s)^{-0.7}\Big((1-0.1s)^3 - \frac{s}{20}\Big)\, ds, \quad t \in [1,3].$$

The graphs of the mild lower and upper solutions, obtained by Wolfram Mathematica, which are successive approximations, are given in Figure 9.

Remark 4.1. Both algorithms given above in Theorems 4.1 and 4.2 and illustrated in Examples 4.1, 4.2 could be applied in the special case when both mild upper and mild lower solutions have one and the same initial time, i.e., $\eta = \xi = 0$.

Remark 4.2. From the proof of Theorem 4.1 we see that we use the monotonic property of the function $f(t,x)$ only when $x \in \mathbb{R}:\ v(t) \le x \le w(t)$ for $t \in [t_0, t_0+T]$. Therefore, if a function f satisfies the monotonic property in condition (2) of Theorem 4.1 and Theorem 4.2 when $x \in \mathbb{R}$: $v(t) \le x \le w(t)$ for $t \in [t_0, t_0+T]$, then we may be able to modify f so that Condition 2 is satisfied for all $x \in \mathbb{R}$.

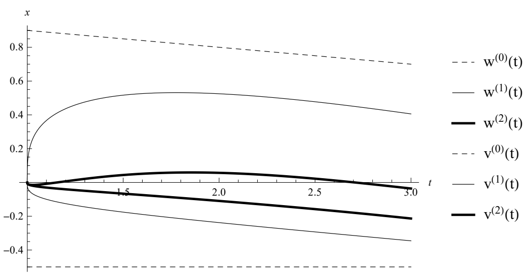

Figure 9. Example 3. Graph of the lower and upper approximations.

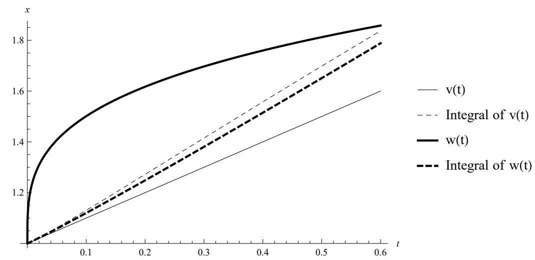

Figure 10. Example 4. Graphs of the mild lower/upper solutions and the correspong integrals on $[0, 0.6]$.

Example 4.3. Consider the IVP for the scalar Caputo FrDE

$$ {}_0^cD_t^{0.3}x(t) = \frac{t}{x(t)} + t^{0.7} \text{ for } t \in [0, 0.6], \quad x(0) = 1. \tag{4.19} $$

The function $f(t, x) = \frac{t}{x} + t^{0.7}$ is not defined when $x = 0$.

Consider the functions $v(t) = t + 1$, $w(t) = t^{0.3} + 1$, $t \in [0, 0.6]$ and note that

$$ \begin{aligned} &v(t) = t + 1 \le 1 + \frac{1}{\Gamma[0.3]} \int_0^t (t-s)^{0.3-1}\Big(\frac{s}{s+1} + s^{0.7}\Big)\, ds, \quad t \in [0, 0.6],\\ &w(t) = t^{0.3} + 1 \le 1 + \frac{1}{\Gamma[0.3]} \int_0^t (t-s)^{0.3-1}\Big(\frac{s}{s^{0.3}+1} + s^{0.7}\Big)\, ds, \quad t \in [0, 0.6],\\ &v(t) \le w(t), \quad t \in [0, 0.6], \end{aligned} \tag{4.20} $$

hold (see Figure 10).

Therefore, the functions $v(t)$, $w(t)$ are mild lower and upper solutions of the FrDE (4.19) and condition (1) of Theorem 4.2 is satisfied with $t_0 = \tau_0 = \theta_0 = 1$. We can define a function $f_1(t, x) \in C([0, 0.6] \times \mathbb{R}, \mathbb{R})$ by

$$ f_1(t, x) = \begin{cases} f(t, x) = \dfrac{t}{x} + t^{0.7}, & t \in [0.0.6], \quad x \ge 1,\\ f(t, 1) = t + t^{0.7}, & t \in [0, 0.6], \quad x < 1, \end{cases} $$

The function $f_1(t, x)$ is nondecreasing in t and nonincreasing in x, i.e., condition (2) of Theorem 4.2 is satisfied and we can construct two sequences of successive approximations to the solution $x(t)$ of the IVP for FrDE (4.19) with $v(t) \le x(t) \le w(t)$. We apply formulas (4.1) and (4.2) with $t_0 = \tau_0 = \theta_0 = 1$ and replace $f(t, x)$ by $f_1(t, x)$.

Remark 4.3. An appropriate iterative scheme for monotonic right side parts for the periodic boundary value problem for the Caputo fractional differential equation is given in [14].

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ON THE STABILITY OF INVARIANT TORI OF A CLASS OF DYNAMICAL SYSTEMS WITH THE LAPPO–DANILEVSKII CONDITION

Abstract. The sufficient conditions for the existence of an asymptotically stable invariant toroidal manifolds of linear extensions of dynamical system on torus are obtained in the case where the matrix of the system commutes with its integral. New theorem requires the conditions to hold only in a nonwandering set of the corresponding dynamical system in order to guarantee the existence and stability of the invariant manifold. Additionally, the proposed approach is applied to the investigation of invariant sets of a certain class of discontinuous dynamical systems.

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რეზიუმე. დადგენილია დინამიკური სისტემის წრფივი გაფართოების ასიმპტოტურად მდგრადი ინვარიანტული ტოროიდული მრავალსახეობების არსებობის საკმარისი პირობები ტორზე იმ შემთხვევაში, როცა სისტემის მატრიცა მისი ინტეგრალის კომუტატიურია. ინვარიანტული მრავალსახეობის არსებობისა და მდგრადობისთვის ახალ თეორემაში მოითხოვება გარკვეული პირობების შესრულება შესაბამისი დინამიკური სისტემის მხოლოდ არამოხეტიალე სიმრავლის შემთხვევაში. შემოთავაზებული მიდგომა გამოყენებულია წყვეტილი დინამიკური სისტემების გარკვეული კლასის ინვარიანტული სიმრავლეების გამოკვლევისთვის.

1 Introduction and preliminaries

One of the important questions within mathematical theory of multi-frequency oscillations is the problem of the existence and stability of invariant toroidal manifolds of the systems of differential equations that are defined in the direct product of a torus and Euclidean space. Such manifolds serve as carriers of multi-frequency oscillations in the system. The basics of this theory are systematically developed in [8, 13].

In this paper, we establish new sufficient conditions for the existence of an asymptotically stable invariant torus of a particular class of dynamical systems subjected to Lappo–Danilevskii condition [1, Chap. II, § 13]. We propose an approach that relaxes conventional constraints and requires the conditions to hold only in nonwandering set of the corresponding dynamical system in order to guarantee the existence and stability of invariant manifold. This extends the result in [3], where the analogous conditions are being imposed on the whole surface of the torus. In the last section, we extend this approach to a certain class of discontinuous dynamical system [4] defined in the direct product of a torus and Euclidean space.

We consider the following system of ordinary differential equations defined in the direct product of a torus $\mathcal{T}_m$ and Euclidean space $\mathbb{R}^n$

$$\frac{d\varphi}{dt} = a(\varphi)\,, \quad \frac{dx}{dt} = P(\varphi)x + f(\varphi), \tag{1.1}$$

where $\varphi = (\varphi_1, \dots, \varphi_m)^T \in \mathcal{T}_m$, $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, $P(\varphi), f(\varphi) \in C(\mathcal{T}_m)$; $C(\mathcal{T}_m)$ stands for a space of continuous 2π-periodic with respect to each of the component φ_v, $v = 1, \dots, m$, functions defined on the m-dimensional torus $\mathcal{T}_m$. The function $a(\varphi) \in C(\mathcal{T}_m)$ and satisfies the Lipschitz condition

$$\|a(\varphi'') - a(\varphi')\| \leq L\|\varphi'' - \varphi'\| \tag{1.2}$$

for any $\varphi', \varphi'' \in \mathcal{T}_m$ and some positive constant $L > 0$.

Condition (1.2) guarantees that the system

$$\frac{d\varphi}{dt} = a(\varphi) \tag{1.3}$$

generates a dynamical system on the torus $\mathcal{T}_m$, which we will denote as $\varphi_t(\varphi)$.

Along with system (1.1), we consider a linear system of equations

$$\frac{dx}{dt} = P(\varphi_t(\varphi))x + f(\varphi_t(\varphi)), \tag{1.4}$$

that depends on $\varphi \in \mathcal{T}_m$ as a parameter.

Definition 1.1 ([13])**.** A manifold $\mathcal{M}$ is called an invariant manifold of system (1.1) if $\mathcal{M}$ is defined by $x = u(\varphi)$, $\varphi \in \mathcal{T}_m$, with the function $u(\varphi) \in C(\mathcal{T}_m)$ such that $x(t, \varphi) = u(\varphi_t(\varphi))$ is a solution to (1.4) for any $\varphi \in \mathcal{T}_m$.

The main approach to investigate the properties of invariant toroidal manifolds of system (1.1) is based on the notion of a Green–Samoilenko function [13]. Consider the homogeneous system of differential equations

$$\frac{dx}{dt} = P(\varphi_t(\varphi))x, \tag{1.5}$$

that depends on $\varphi \in \mathcal{T}_m$ as a parameter and denote by $\Omega^t_\tau(\varphi)$ the fundamental matrix of (1.5) satisfying $\Omega^\tau_\tau(\varphi) \equiv E$.

Let $C(\varphi)$ be a matrix from the space $C(\mathcal{T}_m)$. Denote

$$G_0(\tau, \varphi) = \begin{cases} \Omega^0_\tau(\varphi)C(\varphi_\tau(\varphi)), & \tau \leq 0, \\ -\Omega^0_\tau(\varphi)(E - C(\varphi_\tau(\varphi))), & \tau > 0. \end{cases}$$

Definition 1.2 ([13])**.** The function $G_0(\tau,\varphi)$ is called a Green–Samoilenko function of the system

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = P(\varphi)x,$$

if $\int\limits_{-\infty}^{+\infty} \|G_0(\tau,\varphi)\|\, d\tau$ is bounded uniformly with respect to φ,

$$\sup_{\varphi\in\mathcal{T}_m} \int\limits_{-\infty}^{+\infty} \|G_0(\tau,\varphi)\|\, d\tau < \infty.$$

The existence of the Green–Samoilenko function guarantees the existence of an invariant toroidal manifold of system (1.1) for any inhomogeneity $f(\varphi) \in C(\mathcal{T}_m)$ and can be presented as [13]

$$x = u(\varphi) = \int\limits_{-\infty}^{+\infty} G_0(\tau,\varphi) f(\varphi_\tau(\varphi))\, d\tau, \quad \varphi \in \mathcal{T}_m.$$

2 Main results

Consider system (1.1) for the case when the matrix $P(\varphi_t(\varphi))$ commutes with its integral (the so-called Lappo–Danilevskii case [1, Chap. II, § 13]): for any $t \ge \tau$,

$$P(\varphi_t(\varphi)) \int\limits_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1 = \int\limits_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1 \cdot P(\varphi_t(\varphi)). \tag{2.1}$$

Then the equality

$$\Omega_\tau^t(\varphi) = e^{\int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1}$$

is actually the fundamental matrix of the homogeneous system (1.5) that depends on $\varphi \in \mathcal{T}_m$ as a parameter. Really, taking into account that

$$\frac{d}{dt} \int\limits_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1 = P(\varphi_t(\varphi)),$$

we have

$$\frac{d}{dt}\,\Omega_\tau^t(\varphi) = e^{\int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1} P(\varphi_t(\varphi)) = P(\varphi_t(\varphi)) e^{\int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1} = P(\varphi_t(\varphi))\Omega_\tau^t(\varphi).$$

Additionally, $\Omega_\tau^\tau(\varphi) = E$.

Note also [2] that a (2×2)-matrix of the form

$$P(\varphi_t(\varphi)) = \begin{bmatrix} p(\varphi_t(\varphi)) & q(\varphi_t(\varphi)) \\ q(\varphi_t(\varphi)) & p(\varphi_t(\varphi)) \end{bmatrix}$$

satisfies the Lappo–Danilevskii condition (2.1).

Definition 2.1 ([9])**.** A point $\varphi \in \mathcal{T}_m$ is called a wandering point of the dynamical system (1.3) if there exist a neighbourhood $U(\varphi)$ and a time $T = T(\varphi) > 0$ such that

$$U(\varphi) \cap \varphi_t(U(\varphi)) = \varnothing \quad \forall\, t \ge T.$$

Let W be a set of all wandering points of the dynamical system (1.3) and let $M = \mathcal{T}_m \setminus W$ be a set of all nonwandering points. Since $\mathcal{T}_m$ is a compact set, it is known [9] that $M \neq \varnothing$ is invariant and compact subset of $\mathcal{T}_m$. Moreover, the following theorem holds.

Theorem 2.1 ([9]). *For any $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that for any $\varphi \notin M$, the corresponding trajectory $\varphi_t(\varphi)$ remains outside the ε-neighbourhood of nonwandering set M for a time, not exceeding $T(\varepsilon)$.*

Now we are in position to state the main result of the paper.

Theorem 2.2. *Let the Lappo–Danilevskii condition* (2.1) *hold and uniformly with respect to $\varphi \in \mathcal{T}_m$ there exist*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t P(\varphi_s(\varphi))\, ds := A(\varphi). \tag{2.2}$$

If for every $\varphi \in M$

$$\operatorname{Re} \lambda(A(\varphi)) < 0 \tag{2.3}$$

for all eigenvalues $\lambda(A(\varphi))$ of the matrix $A(\varphi)$, then system (1.1) *has an asymptotically stable invariant toroidal manifold for any $f(\varphi) \in C(\mathcal{T}_m)$.*

Proof. From condition (2.1) it follows that for $t \ge s \ge 0$,

$$P(\varphi_t(\varphi)) \int_s^t P(\varphi_{t_1}(\varphi))\, dt_1 = \int_s^t P(\varphi_{t_1}(\varphi))\, dt_1 \cdot P(\varphi_t(\varphi)). \tag{2.4}$$

After differentiating equality (2.4) by s, we get $P(\varphi_t(\varphi))P(\varphi_s(\varphi)) = P(\varphi_s(\varphi))P(\varphi_t(\varphi))$. Hence,

$$\begin{aligned}\int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1 \cdot \frac{1}{s}\int_\tau^s P(\varphi_{t_2}(\varphi))\, dt_2 &= \frac{1}{s}\int_\tau^t \int_\tau^s P(\varphi_{t_1}(\varphi))P(\varphi_{t_2}(\varphi))\, dt_2\, dt_1 \\ &= \frac{1}{s}\int_\tau^t \int_\tau^s P(\varphi_{t_2}(\varphi))P(\varphi_{t_1}(\varphi))\, dt_2\, dt_1 = \frac{1}{s}\int_\tau^s P(\varphi_{t_2}(\varphi))\, dt_2 \cdot \int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1.\end{aligned}$$

Taking the limit $s \to \infty$ in the last equality, we get

$$\int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1 \cdot A(\varphi) = A(\varphi) \cdot \int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1. \tag{2.5}$$

It means that the limit matrix $A(\varphi)$ commutes with its integral $\int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1$.

Due to (2.2), we may introduce the matrix B such that

$$\frac{1}{t}\int_0^t P(\varphi_{t_1}(\varphi))\, dt_1 = A(\varphi) + B(t,\varphi),$$

where

$$\sup_{\varphi\in\mathcal{T}_m} \|B(t,\varphi)\| \longrightarrow 0, \quad t \to \infty.$$

The next step of the proof is to prove that the matrices $A(\varphi)$ and $B(t,\varphi)$ commute. Indeed, from (2.5) we get

$$\begin{aligned}A(\varphi)\cdot B(t,\varphi) &= A(\varphi)\cdot\left[\frac{1}{t}\int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1 - A(\varphi)\right] \\ &= \frac{1}{t}\int_\tau^t P(\varphi_{t_1}(\varphi))\, dt_1 \cdot A(\varphi) - A^2(\varphi) = B(t,\varphi)A(\varphi).\end{aligned}$$

Then from the equality

$$\int_0^t P(\varphi_s(\varphi))\,ds = A(\varphi)\cdot t + B(t,\varphi)\cdot t$$

we derive that the fundamental matrix of the homogeneous system (1.5) has a representation

$$\Omega_0^t(\varphi) = e^{\int_0^t P(\varphi_{t_1}(\varphi))\,dt_1} = e^{tA(\varphi)+tB(t,\varphi)} = e^{tA(\varphi)}\cdot e^{tB(t,\varphi)}. \tag{2.6}$$

The aim of the further steps of the proof is to prove that condition (2.3) guarantees the following estimate

$$\|\Omega_0^t(\varphi)\| \le Ke^{-\eta t} \quad \forall\, t\ge 0, \tag{2.7}$$

for any $\varphi\in\mathcal{T}_m$ and for some positive constants $K,\eta>0$ which do not depend on $\varphi\in\mathcal{T}_m$.

Due to the uniformity of the limit in (2.2), we find that

$$\text{map}\,\varphi\longmapsto A(\varphi)\ \text{ is continuous on } \mathcal{T}_m.$$

Then, from [5], the eigenvalues of $A(\varphi)$ depend continuously on φ. Hence, from (2.3), it follows that there exist $\gamma>0$ and $\varepsilon\in(0,\gamma)$ such that

$$\forall\,\varphi\in\overline{O_\varepsilon(M)},\ \ \operatorname{Re}\lambda(A(\varphi)) < -2\gamma,$$

where $O_\varepsilon(M)$ is an ε-neighbourhood of M.

By a picked $\varepsilon>0$, we choose $T_1 = T_1(\varepsilon)$ such that

$$\sup_{\varphi\in\mathcal{T}_m}\|B(t,\varphi\| < \varepsilon \quad \forall\, t\ge T_1. \tag{2.8}$$

Next we prove that there exists $K_1>0$ such that for any $\varphi\in\overline{O_\varepsilon(M)}$ and for any $t\ge 0$ the inequality

$$\|e^{A(\varphi)t}\| \le K_1\cdot e^{-\gamma t} \tag{2.9}$$

holds.

Choose some $\varphi_0\in\overline{O_\varepsilon(M)}$. Due to the properties of the exponent, there exists $C(\varphi_0)>0$ such that for any $t\ge 0$,

$$\|e^{A(\varphi_0)t}\| \le C(\varphi_0)e^{-\frac{3\gamma}{2}\,t}. \tag{2.10}$$

Due to the continuity of $A(\varphi)$, there exists $\delta=\delta(\varphi_0)>0$ such that for any $\varphi\in O_\delta(\varphi_0)$,

$$\|A(\varphi)-A(\varphi_0)\| < \frac{\gamma}{2C(\varphi_0)}\,. \tag{2.11}$$

The matrix $X(t) = e^{A(\varphi)t}$ is a solution to the Cauchy problem

$$\begin{cases}\dot X = A(\varphi_0)X + (A(\varphi)-A(\varphi_0))X,\\ X(0)=E.\end{cases}$$

Using the variation of the constant method, we obtain

$$X(t) = e^{A(\varphi_0)t} + \int_0^t e^{(t-s)A(\varphi_0)}\cdot(A(\varphi)-A(\varphi_0))X(s)\,ds.$$

Then from (2.10), (2.11) we get

$$\|X(t)\| \le C(\varphi_0)e^{-\frac{3\gamma}{2}\,t} + \int_0^t e^{-\frac{3\gamma}{2}\,(t-s)}\cdot\frac{\gamma}{2}\cdot\|X(s)\|\,ds,$$

$$\|X(t)\|\cdot e^{\frac{3\gamma}{2}\,t} \le C(\varphi_0) + \int_0^t \frac{\gamma}{2}\cdot e^{\frac{3\gamma}{2}\,s}\|X(s)\|\,ds.$$

Applying the Gronwall inequality to the last inequality, we finally get

$$\forall\, t \geq 0, \ \ \forall\, \varphi \in O_\delta(\varphi_0) \quad \|e^{A(\varphi)t}\| \leq C(\varphi_0)e^{-\gamma t}.$$

From a cover $\{O_{\delta(\varphi_0)}(\varphi_0)\}_{\varphi_0 \in \overline{O_\varepsilon(M)}}$ of the compact set $\overline{O_\varepsilon(M)}$ let us pick a finite subcover $\{O_{\delta(\varphi_i)}(\varphi_i)\}_{i=1}^N$. Letting $K_1 = \max\limits_{1\leq i \leq N} C(\varphi_i)$, we get (2.9).

Finally, for $\varphi \in \overline{O_\varepsilon(M)}$, due to equality (2.6) and inequalities (2.8), (2.9), we get: for all $t \geq T_1$,

$$\big\|e^{\int_0^t P(\varphi_s(\varphi))\,ds}\big\| = \|e^{A(\varphi)t + B(t,\varphi)t}\| \leq K_1 e^{(-\gamma+\varepsilon)t}. \tag{2.12}$$

In the case for $\varphi \notin O_\varepsilon(M)$, we use Theorem 2.1, which says that there exists $\tau = \tau(\varphi,\varepsilon) < T(\varepsilon)$ such that $\varphi_\tau(\varphi) \in O_\varepsilon(M)$. Hence, for $t > T(\varepsilon) + T_1$,

$$\begin{aligned}\big\|e^{\int_0^t P(\varphi_s(\varphi))\,ds}\big\| &= \big\|e^{\int_0^\tau P(\varphi_s(\varphi))\,ds} \cdot e^{\int_\tau^t P(\varphi_s(\varphi))\,ds}\big\| \leq e^{d\cdot T(\varepsilon)} \cdot \big\|e^{\int_0^{t-\tau} P(\varphi_s(\varphi_\tau(\varphi)))\,ds}\big\| \\ &\leq e^{d\cdot T(\varepsilon)} K_1 e^{(-\gamma+\varepsilon)(t-\tau)} \leq e^{(d+\gamma)\cdot T(\varepsilon)} K_1 e^{(-\gamma+\varepsilon)t},\end{aligned} \tag{2.13}$$

where $d = \max\limits_{\varphi \in T_m} \|P(\varphi)\|$.

From estimates (2.12), (2.13) we derive the desired inequality (2.7).

From (2.7) it directly follows that the function $G_0(\tau,\varphi) = \begin{cases}\Omega_\tau^0(\varphi), & \tau \leq 0, \\ 0, & \tau > 0\end{cases}$ satisfies the estimate $\|G_0(\tau,\varphi)\| \leq Ke^{-\eta|\tau|}$, $\tau \in \mathbb{R}$, and it is a Green–Samoilenko function of the invariant tori problem. Moreover, estimate (2.7) is sufficient for the existence of an asymptotically stable invariant toroidal manifold of system (1.1) of the form

$$x = u(\varphi) = \int\limits_{-\infty}^{0} \Omega_\tau^0(\varphi) f(\varphi_\tau(\varphi))\,d\tau, \quad \varphi \in \mathcal{T}_m.$$

This completes the proof. □

Remark 2.1. From the proof of Theorem 2.2 it follows that the constant $\eta > 0$ in (2.7) can be chosen as

$$\eta = -\max_{\varphi \in M} \max_{j=1,\dots,n} \operatorname{Re} \lambda_j(A(\varphi)) - \varepsilon,$$

where $\varepsilon > 0$ is arbitrarily small.

Remark 2.2. Since $\forall\, t \geq \tau$, $\forall\, \theta \in \mathbb{R}$ $\Omega_\tau^t(\varphi_\theta(\varphi)) = \Omega_{\tau+\theta}^{t+\theta}(\varphi)$, from (2.7) it follows that

$$\forall\, t \geq \tau, \ \ \forall\, \varphi \in \mathcal{T}_m \quad \|\, \Omega_\tau^t(\varphi)\, \| \leq Ke^{-\eta(t-\tau)}.$$

Example 2.1. Consider the following system:

$$\begin{aligned}\frac{d\varphi}{dt} &= -\sin^2\Big(\frac{\varphi}{2}\Big), \\ \begin{pmatrix}\dfrac{dx_1}{dt} \\[2mm] \dfrac{dx_2}{dt}\end{pmatrix} &= \begin{pmatrix}-\cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi\end{pmatrix} x + \begin{pmatrix} f_1(\varphi) \\ f_2(\varphi)\end{pmatrix},\end{aligned} \tag{2.14}$$

where $\varphi \in \mathcal{T}_1$, $x = (x_1,x_2) \in \mathbb{R}^2$, $f(\varphi) = (f_1(\varphi), f_2(\varphi)) \in C(\mathcal{T}_1)$.

Note that the symmetric matrix $P(\varphi) = \begin{pmatrix}-\cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi\end{pmatrix}$ satisfies the Lappo–Danilevskii condition (2.1).

For the dynamical system $\frac{d\varphi}{dt} = -\sin^2(\frac{\varphi}{2})$ on the torus $\mathcal{T}_1$, the set M of nonwandering points consists of a single point $\varphi = 0$. The point $\varphi = 0$ is a fixed point, and all other trajectories tend to 0 as $t \to +\infty$. Hence, uniformly with respect to $\varphi \in \mathcal{T}_1$,

$$\lim_{t\to\infty} P(\varphi_t(\varphi)) = \lim_{t\to\infty} \begin{pmatrix} -\cos(\varphi_t(\varphi)) & \sin(\varphi_t(\varphi)) \\ \sin(\varphi_t(\varphi)) & -\cos(\varphi_t(\varphi)) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$A = \lim_{t\to\infty} \frac{1}{t} \int_{\tau}^{t} P(\varphi_{t_1}(\varphi))\, dt_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since

$$\operatorname{Re} \lambda_j A = \operatorname{Re} \lambda_j \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1 < 0, \quad j = 1, 2,$$

system (2.14) satisfies the conditions of Theorem 2.2 and has an asymptotically stable invariant toroidal manifold.

3 Application to discontinuous dynamical systems

Let us apply the proposed approach to a certain class of discontinuous dynamical systems [6, 7, 12, 14]

$$\begin{aligned} \frac{d\varphi}{dt} &= a(\varphi), \quad \varphi \in \mathcal{T}_m, \\ \frac{dx}{dt} &= P(\varphi)x + f(\varphi), \quad \varphi \notin \Gamma, \\ \Delta x\big|_{\varphi\in\Gamma} &= I(\varphi)x + g(\varphi), \end{aligned} \tag{3.1}$$

where $\Gamma \subset \mathcal{T}_m$, $a(\varphi) \in C(\mathcal{T}_m)$ satisfies (1.2), $P(\varphi), f(\varphi) \in C(\mathcal{T}_m)$, $I(\varphi), g(\varphi) \in C(\Gamma)$.

We assume that the set Γ is a smooth submanifold of a torus $\mathcal{T}_m$ of dimension $m-1$ and is defined by the equation $\Phi(\varphi) = 0$, where $\Phi(\varphi)$ is a continuous scalar 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$, function.

Denote by $t_i(\varphi)$, $i \in \mathbb{Z}$, the solutions of the equation $\Phi(\varphi_t(\varphi)) = 0$, which are the moments of impulsive perturbations in system (3.1). We assume that for every $\varphi \in \mathcal{T}_m$ the corresponding solutions $t = t_i(\varphi)$ exist, $\lim\limits_{i\to\pm\infty} t_i(\varphi) = \pm\infty$, and uniformly with respect to $t \in \mathbb{R}$ and $\varphi \in \mathcal{T}_m$,

$$\lim_{T\to\pm\infty} \frac{i(t, t+T)}{T} = p < \infty, \tag{3.2}$$

where $i(a, b)$ is the number of points $t_i(\varphi)$ in the interval (a, b).

Along with system (3.1), we consider a linear system

$$\begin{aligned} \frac{dx}{dt} &= P(\varphi_t(\varphi))x + f(\varphi_t(\varphi)), \quad t \neq t_i(\varphi), \\ \Delta x\big|_{t=t_i(\varphi)} &= I(\varphi_{t_i(\varphi)}(\varphi))x + g(\varphi_{t_i(\varphi)}(\varphi)), \end{aligned} \tag{3.3}$$

that depends on $\varphi \in \mathcal{T}_m$ as a parameter.

Let $C_\Gamma(\mathcal{T}_m)$ be a space of piecewise continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$, functions that are defined on the m-dimensional torus $\mathcal{T}_m$.

Definition 3.1. A set $\mathcal{M}$ is called an invariant set of system (3.1) if $\mathcal{M}$ is defined by $x = u(\varphi)$, $\varphi \in \mathcal{T}_m$, where a piecewise continuous function $u(\varphi) \in C_\Gamma(\mathcal{T}_m)$ is such that $x(t, \varphi) = u(\varphi_t(\varphi))$ is a solution to (3.3) for any $\varphi \in \mathcal{T}_m$.

The problems of the existence and stability of invariant toroidal sets of (3.1) have been studied in [10, 11]. Consider the homogeneous system of equations

$$\begin{aligned} \frac{dx}{dt} &= P(\varphi_t(\varphi))x, \quad t \neq t_i(\varphi), \\ \Delta x\big|_{t=t_i(\varphi)} &= I(\varphi_{t_i(\varphi)}(\varphi))x. \end{aligned} \tag{3.4}$$

Let $X^t_\tau(\varphi)$ be a fundamental matrix of (3.4) with $X^\tau_\tau(\varphi) \equiv E$.

Let $C(\varphi)$ be some matrix from the space $C_\Gamma(\mathcal{T}_m)$. Denote

$$G_0(\tau,\varphi) = \begin{cases} X^0_\tau(\varphi)C(\varphi_\tau(\varphi)), & \tau \leq 0, \\ -X^0_\tau(\varphi)(E - C(\varphi_\tau(\varphi))), & \tau > 0. \end{cases}$$

Definition 3.2. A function $G_0(\tau,\varphi)$ is called a Green–Samoilenko function of the impulsive system

$$\begin{aligned} \frac{d\varphi}{dt} &= a(\varphi), \quad \varphi \in \mathcal{T}_m, \\ \frac{dx}{dt} &= P(\varphi)x, \quad \varphi \notin \Gamma, \\ \Delta x\big|_{\varphi\in\Gamma} &= I(\varphi)x, \end{aligned}$$

if

$$\|X^t_\tau(\varphi)\| \leq Ke^{-\eta|t-\tau|}, \quad t,\tau \in \mathbb{R}, \tag{3.5}$$

for some $K \geq 1$, $\eta > 0$, not depending on $\varphi \in \mathcal{T}_m$.

Then the invariant toroidal set of system (3.1) can be presented as

$$x = u(\varphi) = \int\limits_{-\infty}^{+\infty} G_0(\tau,\varphi)f(\varphi_\tau(\varphi))\,d\tau + \sum_{-\infty<t_i(\varphi)<\infty} G_0(t_i(\varphi)+0,\varphi)g(\varphi_{t_i(\varphi)}(\varphi)), \quad \varphi \in \mathcal{T}_m.$$

Conditions (3.2), (3.5) guarantee the convergence of the integral and sum. Hence, the existence of the Green–Samoilenko function $G_0(\tau,\varphi)$ is a sufficient condition for the existence of invariant toroidal set of system (3.1).

Theorem 3.1. *Let for system* (3.1) *conditions* (3.2) *hold, the matrix $P(\varphi)$ satisfy conditions* (2.1) *and* (2.2)*, the matrices $A(\varphi)$ and $I(\varphi)$ commute $\forall\,\varphi \in \mathcal{T}_m$ and, additionally,*

$$\gamma + p\ln\alpha < 0,$$

where

$$\gamma = \max_{\varphi\in M}\max_{j=1,\dots,n} \operatorname{Re}\lambda_j(A(\varphi)), \quad \alpha = \sup_{\varphi\in\Gamma}\|E + I(\varphi)\|.$$

Then system (3.1) *has an asymptotically stable invariant toroidal set.*

Proof. Choose $\varepsilon > 0$ such that $\gamma + p\ln\alpha + 3\varepsilon < 0$. The fundamental matrix of the impulsive system (3.4) can be presented in the form [14]

$$X^t_0(\varphi) = \Omega^t_{t_i(\varphi)}(\varphi) \prod_{0<t_j(\varphi)<t_i(\varphi)} \big(E + I(\varphi_{t_j(\varphi)}(\varphi))\big)\Omega^{t_j(\varphi)}_{t_{j-1}(\varphi)}(\varphi), \tag{3.6}$$

where $t_0(\varphi) = 0$, $t_i(\varphi) < t \leq t_{i+1}(\varphi)$, $\Omega^t_\tau(\varphi)$ is the fundamental matrix of unperturbed system for which the estimate

$$\sup_{\varphi\in\mathcal{T}_m}\|\Omega^t_\tau(\varphi)\| \leq K_1e^{(\gamma+\varepsilon)(t-\tau)} \ \text{ for } \ t \geq \tau \tag{3.7}$$

holds with an arbitrarily small ε and some constant $K_1 = K_1(\varepsilon) > 0$ (see Remark 2.1). Due to (2.6), we have

$$\Omega_\tau^t(\varphi) = e^{\int_\tau^t P(\varphi_{t_1}(\varphi))\,dt_1} = e^{(t-\tau)A(\varphi_\tau(\varphi))} \cdot e^{(t-\tau)B(t-\tau,\varphi_\tau(\varphi))}.$$

From a commutativity of the matrices $A(\varphi)$ and $I(\varphi)$ it follows that matrices $E + I(\varphi_{t_{j-1}}(\varphi))$ and $\Omega_{t_{j-1}(\varphi)}^{t_j(\varphi)}(\varphi)$ commute. Then from representation (3.6) and estimates (3.7), (2.8) we get the estimate

$$\|X_0^t(\varphi)\| \le K_2 e^{(\gamma+2\varepsilon)t} \alpha^{i(0,t)} \ \text{ for } \ t \ge 0,$$

where $K_2 = K_2(\varepsilon) > 0$ does not depend on φ.

From the existence of the uniform limit (3.2) it follows that there exists some $K_3 = K_3(\varepsilon) \ge 1$, not depending on φ, such that $\alpha^{i(0,t)} \le K_3 e^{(\varepsilon + p\ln\alpha)t}$. Then for the fundamental matrix we get the estimate

$$\|X_0^t(\varphi)\| \le K \cdot e^{(3\varepsilon+\gamma+p\ln\alpha)t} \ \text{ for } \ t \ge 0,$$

where $K = K(\varepsilon) > 0$ does not depend on φ. This means that the function $G_0(\tau,\varphi) = \begin{cases} X_\tau^0(\varphi), & \tau \le 0, \\ 0, & \tau > 0 \end{cases}$ is a Green–Samoilenko function of the invariant tori problem. Hence, system (3.1) has an asymptotically stable invariant toroidal set defined by

$$x = u(\varphi) = \int_{-\infty}^{0} X_\tau^0(\varphi) f(\varphi_\tau(\varphi))\,d\tau + \sum_{-\infty < t_i(\varphi) < 0} X_{t_i(\varphi)+0}^0(\varphi) g(\varphi_{t_i(\varphi)}(\varphi)), \ \ \varphi \in \mathcal{T}_m.$$

This completes the proof. □

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BOUNDARY VALUE PROBLEMS FOR FAMILIES OF FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. We consider boundary value problems for all equations from a family of linear functional differential equations. The necessary and sufficient conditions for the unique solvability and existence of non-negative (non-positive) solutions are obtained.*

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1 Introduction

In the recent years, the boundary value problems for functional differential equations have been investigated in many works (for example, [1, 6–12]). We offer new conditions for a unique solvability of boundary value problems and the existence of solutions with a given sign. It turns out, these conditions are sharp in some family of equations.

Here we use the following notation: $\mathbf{AC}^{n-1}[0,1]$ is the space of functions $x:[0,1]\to\mathbb{R}$ for which there exist absolutely continuous derivatives of order less than n; $\mathbf{C}[0,1]$ is the space of continuous functions $x:[0;1]\to\mathbb{R}$ with the norm $\|x\|_{\mathbf{C}}=\max\limits_{t\in[0,1]}|x(t)|$; $\mathbf{L}[0,1]$ is the space of integrable functions $z:[0;1]\to\mathbb{R}$ with the norm $\|z\|_{\mathbf{L}}=\int\limits_0^1|z(s)|\,ds$.

We consider general boundary value problems for linear functional differential equations

$$\begin{cases} x^{(n)}(t)=(Tx)(t)+f(t), \quad t\in[0,1],\\ \ell_i x=\alpha_i, \quad i=1,\dots,n, \end{cases} \tag{1.1}$$

where $T:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ is a linear bounded operator; $f\in\mathbf{L}[0,1]$; $\ell_i:\mathbf{AC}^{n-1}[0,1]\to\mathbb{R}$, $i=1,\dots,n$, are linear bounded functionals with the representation

$$\ell_i x=\sum_{j=0}^{n-1}a_{ij}x^{(j)}(0)+\int\limits_0^1\varphi_i(s)x^{(n)}(s)\,ds, \ \ i=1,\dots,n,$$

$\varphi_i:[0,1]\to\mathbb{R}$, $i=1,\dots,n$, are measurable bounded functions, $a_{ij}\in\mathbb{R}$, $i,j=1,\dots,n$; $\alpha_i\in\mathbb{R}$, $i=1,\dots,n$. A solution of (1.1) is a function from the space $\mathbf{AC}^{n-1}[0,1]$ which satisfies for almost all $t\in[0,1]$ the functional differential equation from problem (1.1) and the boundary value conditions from (1.1).

Such problem (1.1) has the Fredholm property (see, for example, [2]), therefore problem (1.1) is uniquely solvable if and only if the homogeneous boundary value problem

$$\begin{cases} x^{(n)}(t)=(Tx)(t), \quad t\in[0,1],\\ \ell_i x=0, \quad i=1,\dots,n, \end{cases} \tag{1.2}$$

has only the trivial solution.

We will use the notation $\ell\equiv\{\ell_1,\ell_2,\dots,\ell_n\}$, $\alpha\equiv\{\alpha_1,\alpha_2,\dots,\alpha_n\}$.

An operator $T:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ is called positive if for every non-negative function $x\in\mathbf{C}[0,1]$ the inequality $(Tx)(t)\ge 0$ holds for a.a. $t\in[0,1]$.

Here we suppose that p^+, $p^-\in\mathbf{L}[0,1]$ are the given non-negative functions.

Definition 1.1. Denote by $\mathbb{S}(p^+,p^-)$ the family of all operators $T:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ such that

$$T=T^+-T^-,$$

where T^+, $T^-:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ are linear positive operators satisfying the conditions

$$T^+\mathbf{1}=p^+, \quad T^-\mathbf{1}=p^-.$$

Definition 1.2. We say that the pair (p^+,p^-) belongs to the set $\mathbb{A}_{n,\ell}$ if problem (1.1) is uniquely solvable for every operator $T\in\mathbb{S}(p^+,p^-)$.

Definition 1.3. We say that the pair (p^+,p^-) belongs to the set $\mathbb{B}^+_{n,\ell}(\alpha,f)$ if $(p^+,p^-)\in\mathbb{A}_{n,\ell}$ and a unique solution of problem (1.1) is non-negative for every operator $T\in\mathbb{S}(p^+,p^-)$.

Definition 1.4. We say that the pair (p^+,p^-) belongs to the set $\mathbb{B}^-_{n,\ell}(\alpha,f)$ if $(p^+,p^-)\in\mathbb{A}_{n,\ell}$ and a unique solution of problem (1.1) is non-positive for every operator $T\in\mathbb{S}(p^+,p^-)$.

In this paper, we give an effective description of the sets $\mathbb{A}_{n,\ell}$, $\mathbb{B}^+_{n,\ell}(\alpha, f)$, $\mathbb{B}^-_{n,\ell}(\alpha, f)$ under the following condition. We suppose that the boundary value problem

$$\begin{cases} x^{(n)}(t) = f(t), \quad t \in [0,1], \\ \ell_i x = \alpha_i, \quad i = 1, \dots, n, \end{cases} \tag{1.3}$$

is uniquely solvable. Then its solution w has a representation

$$w(t) \equiv \sum_{i=1}^{n} \alpha_i x_i(t) + (Gf)(t), \;\; t \in [0,1],$$

where the functions x_1, $x_2, \dots, x_n$ form a fundamental system of solutions to the equation $x^{(n)} = \mathbf{0}$; $G : \mathbf{L}[0,1] \to \mathbf{AC}^{n-1}[0,1]$ is the Green operator defined by the equality

$$(Gf)(t) = \int_0^1 G(t,s) f(s)\, ds, \;\; t \in [0,1];$$

$G(t,s)$ is the Green function of problem (1.3). Note, that the Green function $G(t,s)$ has a representation

$$G(t,s) = C(t,s) + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i(t) \varphi_j(s), \;\; t, s \in [0,1],$$

where

$$C(t,s) = \begin{cases} \dfrac{(t-s)^{n-1}}{(n-1)!}, & 0 \le s \le t \le 1, \\ 0, & 0 \le t < s \le 1, \end{cases}$$

$c_{ij} \in \mathbb{R}$, $i, j \in \{1, 2, \dots, n\}$.

2 The unique solvability for all equations with operators from the family $\mathbb{S}(p^+, p^-)$

Denote

$$p(t) \equiv p^+(t) - p^-(t), \quad v(t) \equiv 1 - (Gp)(t), \;\; t \in [0,1],$$
$$g_{t_2,t_1,v}(s) \equiv G(t_2,s)v(t_1) - G(t_1,s)v(t_2), \quad s \in [0,1], \;\; 0 \le t_1 \le t_2 \le 1,$$
$$[a]^+ \equiv \frac{|a| + a}{2}, \quad [a]^- \equiv \frac{|a| - a}{2} \;\text{ for any } a \in \mathbb{R}.$$

Theorem 2.1. *The pair (p^+, p^-) belongs to the set $\mathbb{A}_{n,\ell}$ if and only if one of the following conditions holds:*

(1) $v(t) > 0$ *for all* $t \in [0,1]$ *and*

$$\int_0^1 \left(p^+(s)[g_{t_2,t_1,v}(s)]^- + p^-(s)[g_{t_2,t_1,v}(s)]^+ \right) ds < v(t_2) \;\text{ for all } 0 \le t_1 \le t_2 \le 1;$$

(2) $v(t) < 0$ *for all* $t \in [0,1]$ *and*

$$\int_0^1 \left(p^+(s)[g_{t_2,t_1,v}(s)]^+ + p^-(s)[g_{t_2,t_1,v}(s)]^- \right) ds < -v(t_2) \;\text{ for all } 0 \le t_1 \le t_2 \le 1.$$

For proving Theorem 2.1, we need the following lemma (see [3,4]).

Lemma 2.1. *Boundary value problem* (1.2) *has only the trivial solution for every operators* $T \in \mathbb{S}(p^+, p^-)$ *if and only if the boundary value problem*

$$\begin{cases} x^{(n)}(t) = p_1(t)x(t_1) + p_2(t)x(t_2), \quad t \in [0,1], \\ \ell_i x = 0, \quad i = 1, \dots, n, \end{cases} \tag{2.1}$$

has only the trivial solution for every functions p_1, p_2 *and points* t_1, t_2 *such that*

$$p_1, p_2 \in \mathbf{L}[0,1], \tag{2.2}$$

$$p_1 + p_2 = p^+ - p^-, \tag{2.3}$$

$$-p^-(t) \le p_i(t) \le p^+(t), \ \ t \in [0,1], \ \ i = 1,2, \tag{2.4}$$

$$0 \le t_1 \le t_2 \le 1. \tag{2.5}$$

Proof of Theorem 2.1. Boundary value problem (2.1) is equivalent to the equation

$$x(t) = (Gp_1)(t)x(t_1) + (Gp_2)(t)x(t_2), \ \ t \in [0,1].$$

This equation has only the trivial solution if and only if the algebraic system

$$x(t_1) = (Gp_1)(t_1)x(t_1) + (Gp_2)(t_1)x(t_2), \quad x(t_2) = (Gp_1)(t_2)x(t_1) + (Gp_2)(t_2)x(t_2)$$

with respect to $x(t_1)$, $x(t_2)$ has only the trivial solution, that is, when

$$\begin{aligned} \Delta(t_1, t_2, p_1, p_2) &\equiv \begin{vmatrix} 1 - (Gp_1)(t_1) & -(Gp_2)(t_1) \\ -(Gp_1)(t_2) & 1 - (Gp_2)(t_2) \end{vmatrix} \\ &= \begin{vmatrix} 1 - (Gp_1)(t_1) & v(t_1) \\ -(Gp_1)(t_2) & v(t_2) \end{vmatrix} = v(t_2) + \int_0^1 p_1(s) g_{t_2,t_1,v}(s)\, ds \ne 0, \end{aligned} \tag{2.6}$$

We use Lemma 2.1. From the form of the set of admissible function p_i (2.4), it follows that $\Delta(t_1, t_2, p_1, p_2)$ does not equal to zero for every t_i, p_i, $i = 1,2$, if and only if the conditions of Theorem 2.1 are fulfilled. It guarantees the unique solvability of all problems (2.1) under the conditions (2.2)–(2.5). □

3 Examples

Consider the Cauchy problem

$$\begin{cases} \dot{x}(t) = (Tx)(t) + f(t), \quad t \in [0,1], \\ x(0) = \alpha_1. \end{cases}$$

As an immediate result from Theorem 2.1, we have

Corollary 3.1. *The pair* (p^+, p^-) *belongs to the set* $\mathbb{A}_{1,\{x(0)\}}$ *if and only if the inequality*

$$1 + \int_0^{t_1} p^-(s)\, ds \left(1 - \int_{t_1}^{t_2} p^-(s)\, ds \right) - \int_0^{t_2} p^+(s)\, ds + \int_0^{t_1} p^+(s)\, ds \int_{t_1}^{t_2} p^+(s)\, ds > 0$$

holds for all $0 \le t_1 \le t_2 \le 1$.

Now we can easily get the following known assertion.

Corollary 3.2 ([5]).

$(p^+, \mathbf{0}\,) \in \mathbb{A}_{1,\{x(0)\}}$ *if and only if* $\int\limits_0^1 p^+(s)\,ds < 1$*;*

$(\mathbf{0}\,, p^-) \in \mathbb{A}_{1,\{x(0)\}}$ *if and only if* $\int\limits_0^1 p^-(s)\,ds < 3$*.*

Set $p^+(t) \equiv \mathcal{T}^+ t$, $p^-(t) \equiv \mathcal{T}^- t$, $t \in [0,1]$, where $\mathcal{T}^+ \geq 0$, $\mathcal{T}^- \geq 0$.

Corollary 3.3. *The pair* (p^+, p^-) *belongs to the set* $\mathbb{A}_{1,\{x(0)\}}$ *if and only if*

$$0 \leq \mathcal{T}^+ < 2, \;\; 0 \leq \mathcal{T}^- < 1 + \sqrt{5}$$

or

$$0 \leq \mathcal{T}^+ < 2, \;\; \mathcal{T}^- > 1 + \sqrt{5},$$
$$(\mathcal{T}^-)^2\,(6 - \mathcal{T}^-)(\mathcal{T}^- + 2) - (\mathcal{T}^+)^2\,(4 - \mathcal{T}^+)^2 + 2\mathcal{T}^+\mathcal{T}^-(\mathcal{T}^+\mathcal{T}^- - 2\mathcal{T}^+ - 4\mathcal{T}^-) > 0.$$

Consider the Cauchy problem for the second order functional differential equation

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t), \quad t \in [0,1], \\ x(a) = \alpha_1, \quad \dot{x}(a) = \alpha_2, \end{cases}$$

From Theorem 2.1, we have

Corollary 3.4.

$(\mathbf{0}\,, \mathcal{T}^-) \in \mathbb{A}_{2,\{x(0),\dot{x}(0)\}}$ *if and only if* $\mathcal{T}^- < 16$*;*

$(\mathbf{0}\,, p^-) \in \mathbb{A}_{2,\{x(0),\dot{x}(0)\}}$ *if* $p^-(t) \leq 16$ *for all* $t \in [0,1]$, $p^- \not\equiv 16$*.*

Consider the Dirichlet boundary value problem

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t), \quad t \in [0,1], \\ x(0) = \alpha_1, \quad x(1) = \alpha_2, \end{cases}$$

Corollary 3.5.

$(\mathcal{T}^+, \mathbf{0}\,) \in \mathbb{A}_{2,\{x(0),x(1)\}}$ *if and only if* $\mathcal{T}^+ < 32$*;*

$(p^+, \mathbf{0}\,) \in \mathbb{A}_{2,\{x(0),x(1)\}}$ *if* $p^+(t) \leq 32$ *for all* $t \in [0,1]$, $p^+ \not\equiv 32$*.*

4 Non-negative (non-positive) solutions for all equations with operators from the family $\mathbb{S}(p^+, p^-)$

Suppose $\alpha_i \in \mathbb{R}$, $i = 1, \dots, n$, $f \in \mathbf{L}$ and

$$\sum_{i=1}^{n} |\alpha_i| + \int\limits_0^1 |f(s)|\,ds > 0.$$

For every $0 \leq t_1 \leq t_2 \leq 1$, define

$$g_{t_2,t_1,w}(s) \equiv G(t_2, s)w(t_1) - G(t_1, s)w(t_2), \;\; s \in [0,1],$$

$$R_1(t_1,t_2) \equiv w(t_1) + \int_0^1 \left(p^+(s)[g_{t_2,t_1,w}(s)]^- + p^-(s)[g_{t_2,t_1,w}(s)]^+\right) ds,$$

$$R_2(t_1,t_2) \equiv w(t_2) + \int_0^1 \left(p^+(s)[g_{t_2,t_1,w}(s)]^+ + p^-(s)[g_{t_2,t_1,w}(s)]^-\right) ds,$$

$$R_3(t_1,t_2) \equiv w(t_1) - \int_0^1 \left(p^+(s)[g_{t_2,t_1,w}(s)]^+ + p^-(s)[g_{t_2,t_1,w}(s)]^-\right) ds,$$

$$R_4(t_1,t_2) \equiv w(t_2) - \int_0^1 \left(p^+(s)[g_{t_2,t_1,w}(s)]^- + p^-(s)[g_{t_2,t_1,w}(s)]^+\right) ds.$$

Theorem 4.1. *Suppose* $(p^+,p^-) \in \mathbb{A}_{n,\ell}$.

The pair (p^+,p^-) *belongs to the set* $\mathbb{B}^+_{n,\ell}(\alpha,f)$ *if and only if one of the following conditions holds:*

(1) $v(t) > 0$, $w(t) \geq 0$ *for all* $t \in [0,1]$ *and* $R_3(t_1,t_2) \geq 0$, $R_4(t_1,t_2) \geq 0$ *for all* $0 \leq t_1 \leq t_2 \leq 1$*;*

(2) $v(t) < 0$, $w(t) \leq 0$ *for all* $t \in [0,1]$ *and* $R_1(t_1,t_2) \leq 0$, $R_2(t_1,t_2) \leq 0$ *for all* $0 \leq t_1 \leq t_2 \leq 1$.

The pair (p^+,p^-) *belongs to the set* $\mathbb{B}^-_{n,\ell}(\alpha,f)$ *if and only if one of the following conditions holds:*

(1) $v(t) < 0$, $w(t) \geq 0$ *for all* $t \in [0,1]$ *and* $R_3(t_1,t_2) \geq 0$, $R_4(t_1,t_2) \geq 0$ *for all* $0 \leq t_1 \leq t_2 \leq 1$*;*

(2) $v(t) > 0$, $w(t) \leq 0$ *for all* $t \in [0,1]$ *and* $R_1(t_1,t_2) \leq 0$, $R_2(t_1,t_2) \leq 0$ *for all* $0 \leq t_1 \leq t_2 \leq 1$.

Lemma 4.1. *Let* $(p^+,p^-) \in \mathbb{A}_{n,\ell}$. *Then the set of all solutions of problems* (1.1) *for all operators* $T \in \mathbb{S}(p^+,p^-)$ *coincides with the set of solutions of the boundary value problem*

$$\begin{cases} x^{(n)}(t) = p_1(t)x(t_1) + p_2(t)x(t_2) + f(t), & t \in [0,1], \\ \ell_i x = \alpha_i, & i = 1,\ldots,n, \end{cases} \tag{4.1}$$

for all functions p_1, p_2 *and points* t_1, t_2 *satisfying conditions* (2.2)–(2.5).

Proof. Let y be a solution of problem (4.1) for some functions p_1, p_2 and for some points t_1, t_2 satisfying conditions (2.2)–(2.5). Then y is a solution of problem (1.1), where $T = T^+ - T^-$ and the positive operators T^+, T^- are defined by the equalities

$$(T^+x)(t) = p^+(t)\zeta(t)x(t_1) + p^+(t)(1-\zeta(t))x(t_2), \ \ t \in [0,1],$$
$$(T^-x)(t) = p^-(t)(1-\zeta(t))x(t_1) + p^-(t)\zeta(t)x(t_2), \ \ t \in [0,1],$$

$\zeta : [0,1] \to [0,1]$ is a measurable function such that

$$p_1(t) = p^+(t)\zeta(t) - p^-(t)(1-\zeta(t)), \ \ t \in [0,1].$$

Therefore, $T \in \mathbb{S}(p^+,p^-)$.

Conversely, let y be a solution of problem (1.1) with $T \in \mathbb{S}(p^+,p^-)$. Let

$$\min_{t\in[0,1]} y(t) = y(t_1), \quad \max_{t\in[0,1]} y(t) = y(t_2).$$

Then for positive operators T^+, T^- such that $T^+\mathbf{1} = p^+$, $T^-\mathbf{1} = p^-$ the following inequalities hold:

$$p^+(t)y(t_1) \leq (T^+y)(t) \leq p^+(t)y(t_2), \ \ t \in [0,1],$$
$$p^-(t)y(t_1) \leq (T^-y)(t) \leq p^-(t)y(t_2), \ \ t \in [0,1].$$

Therefore, there exist measurable functions $\zeta, \xi : [0,1] \to [0,1]$ such that

$$(T^+y)(t) = p^+(t)(1-\zeta(t))y(t_1) + p^+(t)\zeta(t)y(t_2), \ \ t \in [0,1],$$

$$(T^{-}y)(t) = p^{-}(t)(1-\xi(t))y(t_1) + p^{-}(t)\xi(t)y(t_2), \quad t \in [0,1].$$

So, the function y satisfies problem (4.1) for the functions

$$p_1(t) = (T^{+}\mathbf{1})(t)(1-\zeta(t)) - (T^{-}\mathbf{1})(t)(1-\xi(t)), \quad t \in [0,1],$$
$$p_2(t) = (T^{+}\mathbf{1})(t)\zeta(t) - (T^{-}\mathbf{1})(t)\xi(t), \quad t \in [0,1].$$

It is clear that equality (2.3) and inequalities (2.4) hold. If $t_1 > t_2$, then by renumbering p_1, p_2, t_1, t_2, condition (2.5) will be valid. □

Proof of Theorem 4.1. Find when solutions of (1.1) retain their sign for all $T \in \mathbb{S}(p^{+}, p^{-})$. Use Lemma 4.1. The maximal and minimal values $x_1 \equiv x(t_1)$, $x_2 \equiv x(t_2)$ of a unique solution of problem (1.1) satisfy the system

$$\begin{cases} x_1 = w(t_1) + (Gp_1)(t_1)x_1 + (Gp_2)(t_1)x_2, \\ x_2 = w(t_2) + (Gp_1)(t_2)x_1 + (Gp_2)(t_2)x_2 \end{cases} \tag{4.2}$$

for some p_1, $p_2 \in \mathbf{L}[0,1]$ such that conditions (2.3), (2.4) are fulfilled.

Note that $w \not\equiv \mathbf{0}$.

From (4.2), we obtain

$$x_1 = \frac{\Delta_1(t_1, t_2, p_1, p_2)}{\Delta(t_1, t_2, p_1, p_2)}, \quad x_2 = \frac{\Delta_2(t_1, t_2, p_1, p_2)}{\Delta(t_1, t_2, p_1, p_2)},$$

where the functional $\Delta(t_1, t_2, p_1, p_2)$ is defined by equality (2.6) and retains its sign (the conditions of Theorem 2.1 are fulfilled, therefore $\operatorname{sgn}(\Delta(t_1, t_2, p_1, p_2)) = \operatorname{sgn}(1 - Gp)$); the functionals $\Delta_1(t_1, t_2, p_1, p_2)$ and $\Delta_2(t_1, t_2, p_1, p_2)$ are defined by the equalities

$$\begin{aligned} \Delta_1(t_1, t_2, p_1, p_2) &\equiv \begin{vmatrix} w(t_1) & -(Gp_2)(t_1) \\ w(t_2) & 1-(Gp_2)(t_2) \end{vmatrix} = w(t_1) - \int_0^1 p_2(s) g_{t_2,t_1,w}(s)\,ds, \\ \Delta_2(t_1, t_2, p_1, p_2) &\equiv \begin{vmatrix} 1-(Gp_1)(t_1) & w(t_1) \\ -(Gp_1)(t_2) & w(t_2) \end{vmatrix} = w(t_2) + \int_0^1 p_1(s) g_{t_2,t_1,w}(s)\,ds. \end{aligned} \tag{4.3}$$

Find the maximum and the minimum of $\Delta_1(t_1, t_2, p_1, p_2)$, $\Delta_2(t_1, t_2, p_1, p_2)$ with respect to p_1, p_2 at the fixed rest arguments. From representations (4.3) we have

$$R_1(t_1, t_2) = \max_{-p^{-} \le p_2 \le p^{+}} \Delta_1(t_1, t_2, p_1, p_2), \quad R_2(t_1, t_2) = \max_{-p^{-} \le p_1 \le p^{+}} \Delta_2(t_1, t_2, p_1, p_2),$$
$$R_3(t_1, t_2) = \min_{-p^{-} \le p_2 \le p^{+}} \Delta_1(t_1, t_2, p_1, p_2), \quad R_4(t_1, t_2) = \min_{-p^{-} \le p_1 \le p^{+}} \Delta_2(t_1, t_2, p_1, p_2),$$

that proves the theorem. □

5 Example

As an illustrative example, consider the Dirichlet problem

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + 1, \quad t \in [0,1], \\ x(0) = 0, \quad x(1) = 0. \end{cases} \tag{5.1}$$

From Theorem 4.1 we immediately obtain a sharp condition for the existence of non-positive solutions of (5.1).

Corollary 5.1. *If $p^{+}(t) \le 11 + 5\sqrt{5}$ for all $t \in [0,1]$, then $(p^{+}, \mathbf{0}) \in \mathbb{B}^{-}_{2,\{x(0),x(1)\}}((0,0), \mathbf{1})$. The constant $11 + 5\sqrt{5}$ is sharp.*

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ON A RESOLVENT APPROACH IN A MIXED PROBLEM FOR THE WAVE EQUATION ON A GRAPH

Abstract. We study a mixed problem for the wave equation with integrable potential on the simplest geometric graph consisting of two ring edges that touch at a point. We use a new resolvent approach in the Fourier method. We do not use refined asymptotic formulas for the eigenvalues and any information on the eigenfunctions.*

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Key words and phrases. Wave equation, geometric graph, Fourier method, resolvent approach.

რეზიუმე. ტალღის განტოლებისათვის ინტეგრებადი პოტენციალით შესწავლილია შერეული ამოცანა უმარტივეს გრაფზე, რომელიც შედგება ერთ წერტილში მხები რგოლის ორი კიდესგან. გამოყენებულია ახალი რეზოლვენტური მიდგომა ფურიეს მეთოდში. ამასთან არ არის გამოყენებული დაზუსტებული ასიმპტოტური ფორმულები საკუთრივი მნიშვნელობებისთვის და რაიმე ინფორმაცია საკუთრივი ფუნქციების მიმართ.

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We consider the simplest geometric graph consisting of two ring edges that touch at a point (at the node of the graph). Parametrizing each edge by the interval $[0,1]$, we study the following mixed problem for the wave equation on this graph:

$$\frac{\partial^2 u_j(x,t)}{\partial t^2} = \frac{\partial^2 u_j(x,t)}{\partial x^2} - q_j(x)u_j(x,t), \quad x \in [0,1], \quad t \in (-\infty, +\infty) \ \ (j=1,2), \tag{1}$$

$$u_1(0,t) = u_1(1,t) = u_2(0,t) = u_2(1,t), \tag{2}$$

$$u'_{1x}(0,t) - u'_{1x}(1,t) + u'_{2x}(0,t) - u'_{2x}(1,t) = 0, \tag{3}$$

$$u_1(x,0) = \varphi_1(x), \quad u_2(x,0) = \varphi_2(x), \quad u'_{1t}(x,0) = u'_{2t}(x,0) = 0. \tag{4}$$

Conditions (2), (3) are generated by the structure of the graph.

In this problem the application of the Fourier method causes difficulties associated with the fact that the eigenvalues of the corresponding spectral problem might be multiple. These difficulties can be coped with by applying the resolvent approach [1]. Note that we do not use refined asymptotic formulas for the eigenvalues and any information on the eigenfunctions. Besides, we use Krylov's idea [2, Chapter VI] concerning the convergence acceleration of Fourier-like series.

The following result was obtained in [3]:

Theorem 1. *If $q_j(x) \in C[0,1]$ are complex-valued, $\varphi_j(x) \in C^2[0,1]$ and are complex-valued, $\varphi_1(0) = \varphi_1(1) = \varphi_2(0) = \varphi_2(1)$, $\varphi'_1(0) - \varphi'_1(1) + \varphi'_2(0) - \varphi'_2(1) = 0$, $\varphi''_1(0) = \varphi''_1(1) = \varphi''_2(0) = \varphi''_2(1)$, then the formal solution by Fourier method is a classical solution of problem* (1)–(4)*.*

Now, we assume that $q_j(x) \in L[0,1]$ are complex-valued. Then a classical solution is defined as a function $u(x,t)$ such that $u(x,t)$ and its first derivatives with respect to x and t are absolutely continuous, and satisfies the boundary and initial conditions (2)–(4) and the differential equation (1) almost everywhere. Here we use the scheme of analysis given in [4–6].

We assume that the vector functions $\varphi(x)$ and $\varphi'(x)$ are absolutely continuous and such that satisfy the following conditions:

$$\varphi_1(0) = \varphi_1(1) = \varphi_2(0) = \varphi_2(1), \quad \varphi'_1(0) - \varphi'_1(1) + \varphi'_2(0) - \varphi'_2(1) = 0, \ \ L\varphi \in L_2^2[0,1]. \tag{5}$$

Everywhere, by $L_2^2[0,1]$ we denote the space of vector functions $f(x) = (f_1(x), f_2(x))^T$ such that $f_k(x) \in L_2[0,1]$ $(k=1,2)$, T denotes the transpose.

1 The transformation of a formal solution

The Fourier method is related to the spectral problem $Ly = \lambda y$ for the operator

$$Ly = \left(-y''_1(x) - q_1(x)y_1(x), -y''_2(x) - q_2(x)y_2(x)\right)^T, \quad y = y(x) = (y_1(x), y_2(x))^T$$

with the boundary conditions

$$y_1(0) = y_1(1) = y_2(0) = y_2(1), \quad y'_1(0) - y'_1(1) + y'_2(0) - y'_2(1) = 0.$$

By $R_\lambda = (L - \lambda E)^{-1}$, $R_\lambda^0 = (L^0 - \lambda E)^{-1}$ are denoted the resolvents of the operators L and L^0, where L^0 is L with $q_j(x) \equiv 0$ (E is the identity operator, and λ is the spectral parameter). In the sequel the notation corresponding to L^0 is marked with a zero index.

The formal solution $u(x,t) = (u_1(x,t), u_2(x,t))^T$ of problem (1)–(4) produced by the Fourier method can be represented as

$$u(x,t) = -\frac{1}{2\pi i}\Bigg(\int\limits_{|\lambda|=r} + \sum_{n \ge n_0} \int\limits_{\gamma_n}\Bigg)(R_\lambda \varphi)(x) \cos \rho t \, d\lambda,$$

where $r > 0$ is fixed and such that all the eigenvalues λ_n, with $n < n_0$, belong to the disk $|\lambda| < r$, and there are no eigenvalues of L on the contour $|\lambda| = r$; γ_n are the contours of sufficiently small radius in λ-plane such that all the eigenvalues of operators L and L^0 with $n \ge n_0$ are only inside γ_n.

Proceeding as in [1], we obtain the following result.

Theorem 2. *The formal solution can be represented as*

$$u(x,t) = u_0(x,t) + u_1(x,t),$$

where

$$u_0(x,t) = -\frac{1}{2\pi i}\left(\int\limits_{|\lambda|=r} + \sum_{n\ge n_0}\int\limits_{\gamma_n}\right)\frac{R^0_\lambda g}{\lambda-\mu_0}\cos\rho t\, d\lambda,$$
$$u_1(x,t) = -\frac{1}{2\pi i}\left(\int\limits_{|\lambda|=r} + \sum_{n\ge n_0}\int\limits_{\gamma_n}\right)\frac{1}{\lambda-\mu_0}\left[R_\lambda g - R^0_\lambda g\right]\cos\rho t\, d\lambda,$$

$g = (L-\mu_0 E)\varphi$, *μ_0 is not an eigenvalue of L or L^0, $|\mu_0| > r$, and μ_0 lies outside γ_n for $n \ge n_0$.*

2 Spectral problem and resolvent

Let $\lambda = \rho^2$, where $\operatorname{Re}\rho \ge 0$. Denote by $\{y_{j1}(x), y_{j2}(x)\}$ $(j = 1,2)$, the fundamental systems of solutions of the equations

$$y_j''(x) - q_j(x)y_j(x) + \rho^2 y_j(x) = 0, \ \ (j = 1,2)$$

with initial conditions

$$y_{j1}(0) = 1, \quad y_{j1}'(0) = 0,$$
$$y_{j2}(0) = 0, \quad y_{j2}'(0) = 1.$$

Then $y_{ij}(x)$ are entire functions of ρ and λ. If $q_j(x) \equiv 0$, then

$$y^0_{j1}(x) = \cos\rho x, \quad (y^0_{j1}(x))' = -\rho\sin\rho x,$$
$$y^0_{j2}(x) = \frac{\sin\rho x}{\rho}, \quad (y^0_{j2}(x))' = \cos\rho x.$$

From [7] it follows that all ρ for which $\lambda = \rho^2$ are the eigenvalues of the operator L belong to the semi-infinite strip $S = \{\rho \mid \operatorname{Re}\rho \ge 0, |\operatorname{Im}\rho| \le h\}$, where $h > 0$ is sufficiently large.

Just as in [6, Lemma 7] we obtain

Lemma 1. *If $|\operatorname{Im}\rho| \le h$, then*

$$y_{j1}(x,\rho) = \cos\rho x + \frac{1}{2\rho}\sin\rho x\int\limits_0^x q_j(\tau)\,d\tau$$
$$+ \frac{1}{4\rho}\int\limits_0^x\left[q_j\left(\frac{x-\tau}{2}\right) + q_j\left(\frac{x+\tau}{2}\right)\right]\sin\rho\tau\, d\tau + O(\rho^{-2}),$$
$$y_{j2}(x,\rho) = \frac{\sin\rho x}{\rho} + \frac{1}{2\rho^2}\cos\rho x\int\limits_0^x q_j(\tau)\,d\tau$$
$$+ \frac{1}{4\rho^2}\int\limits_0^x\left[q_j\left(\frac{x-\tau}{2}\right) + q_j\left(\frac{x+\tau}{2}\right)\right]\cos\rho\tau\, d\tau + O(\rho^{-3}),$$

where the $O(\dots)$ estimates are uniform with respect to $x \in [0,1]$.

The eigenvalues of operator L are the zeros of the determinant

$$\Delta(\rho) = \begin{vmatrix} 1-y_{11}(1) & -y_{12}(1) & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1-y_{21}(1) & -y_{22}(1) \\ -y_{11}'(1) & 1-y_{12}'(1) & -y_{21}'(1) & 1-y_{22}'(1) \end{vmatrix}$$

The eigenvalues of L^0 (the zeros of $\Delta^0(\rho)$) are $\lambda_n^0 = (\rho_n^0)^2$, where $\rho_n^0 = n\pi$ ($n = 0, 1, 2, \dots$). If n is even, then eigenvalues are multiple. The eigenvalues λ_n of the operator L asymptotically approach λ_n^0 for large n.

Theorem 3. *For the resolvent $R_\lambda = (R_{1\lambda}, R_{2\lambda})^T$, the formula*

$$R_{j\lambda} f(x) = (M_{j\rho} f_j)(x) + \Omega_{j\lambda}(x, f), \quad f = (f_1, f_2)^T \ \ (j = 1, 2) \tag{6}$$

holds, where

$$(M_{j\rho} f_j)(x) = \int_0^x M_j(x, \xi, \rho) f_j(\xi)\, d\xi, \quad M_j(x, \xi, \rho) = \begin{vmatrix} y_{j1}(\xi) & y_{j2}(\xi) \\ y_{j1}(x) & y_{j2}(x) \end{vmatrix},$$

$$\Omega_{j\lambda}(x, f) = v_{j1}(x)(f_1, y_{11}) + v_{j2}(x)(f_1, y_{12}) + v_{j3}(x)(f_2, y_{21}) + v_{j4}(x)(f_2, y_{22}) \ \ (j = 1, 2), \tag{7}$$

$$v_{11}(x) = \sum_{k=1}^{2} \frac{y_{1k}(x)}{\Delta(\rho)} \big[\Delta_{1k}(\rho) y_{12}(1) + \Delta_{4k}(\rho) y_{12}'(1)\big],$$

$$v_{12}(x) = \sum_{k=1}^{2} \frac{y_{1k}(x)}{\Delta(\rho)} \big[-\Delta_{1k}(\rho) y_{11}(1) - \Delta_{4k}(\rho) y_{11}'(1)\big],$$

$$v_{13}(x) = \sum_{k=1}^{2} \frac{y_{1k}(x)}{\Delta(\rho)} \big[\Delta_{3k}(\rho) y_{22}(1) + \Delta_{4k}(\rho) y_{22}'(1)\big],$$

$$v_{14}(x) = \sum_{k=1}^{2} \frac{y_{1k}(x)}{\Delta(\rho)} \big[-\Delta_{3k}(\rho) y_{21}(1) - \Delta_{4k}(\rho) y_{21}'(1)\big],$$

$\Delta_{k,s}(\rho)$ are algebraic adjuncts of $\Delta(\rho)$, and $v_{2j}(x)$ are obtained by replacing Δ_{k1}, Δ_{k2} by Δ_{k3}, Δ_{k4}, and $y_{11}(x)$, $y_{12}(x)$ by $y_{21}(x)$, $y_{22}(x)$; $(f, g) = \int_0^1 f(x) g(x)\, dx$.

Proof. For $y = (y_1, y_2)^T = R_\lambda f$, we have

$$y_j''(x) - q_j(x) y_j(x) + \rho^2 y_j(x) = f_j(x), \quad j = 1, 2,$$

whence

$$y_k(x) = c_{k1} y_{k1}(x) + c_{k2} y_{k2}(x) + (M_{k\rho} f_k)(x), \quad k = 1, 2.$$

From the boundary conditions for operator L follows (6), where

$$\Omega_{1\lambda}(x, f) = \frac{y_{11}(x)}{\Delta(\rho)} \sum_{k=1}^{4} d_j \Delta_{k,1}(\rho) + \frac{y_{12}(x)}{\Delta(\rho)} \sum_{k=1}^{4} d_j \Delta_{k,2}(\rho),$$

$$\Omega_{2\lambda}(x, f) = \frac{y_{21}(x)}{\Delta(\rho)} \sum_{k=1}^{4} d_j \Delta_{k,3}(\rho) + \frac{y_{22}(x)}{\Delta(\rho)} \sum_{k=1}^{4} d_j \Delta_{k,4}(\rho),$$

$$d_1 = (M_{1\rho} f_1)\Big|_{x=1}, \quad d_2 = 0, \quad d_3 = (M_{2\rho} f_2)\Big|_{x=1},$$

$$d_4 = \int_0^1 \frac{d}{dx} M_1(x, \xi, \rho)\Big|_{x=1} f_1(\xi)\, d\xi + \int_0^1 \frac{d}{dx} M_2(x, \xi, \rho)\Big|_{x=1} f_2(\xi)\, d\xi.$$

Calculating the coefficients d_k in an explicit form, we get (7). $\square$

Define $\widetilde{\gamma}_n = \{\rho \,|\, |\rho - \pi n| = \delta\}$, where $\delta > 0$ is sufficiently small, $n \geq n_0$, and n_0 is chosen so that all λ_n with $n \geq n_0$ lie inside $\widetilde{\gamma}_n$. Let γ_n be the image of $\widetilde{\gamma}_n$ in the λ-plane ($\lambda = \rho^2$).

Lemma 2. *If $\rho \in \widetilde{\gamma}_n$, then*

$$\begin{aligned} v_{k1}^{(j)}(x,\rho) &= {v_{k1}^0}^{(j)}(x,\rho) + O(\rho^{j-2}) \quad (j=0,1),\\ v_{k2}^{(j)}(x,\rho) &= {v_{k2}^0}^{(j)}(x,\rho) + O(\rho^{j-1}) \quad (j=0,1),\\ v_{k1}''(x,\rho) - q_1(x)v_{k1}(x,\rho) - {v_{k1}^0}''(x,\rho) &= O(1),\\ v_{k2}''(x,\rho) - q_2(x)v_{k2}(x,\rho) - {v_{k2}^0}''(x,\rho) &= O(\rho) \end{aligned}$$

$(k=1,2)$, where the derivatives are taken with respect to x and the $O(\dots)$ estimates are uniform with respect to $x \in [0,1]$ (in the last two relations $O(\dots)$ stands for $\|O(\omega)\|_\infty \le c|\omega|$).

Proof. Since $v_j''(x,\rho) - q(x)v_j(x,\rho) = -\rho^2 v_j(x,\rho)$, this lemma follows from Lemma 2 in [4]. $\square$

Just as in [6], we can prove the following assertions.

Lemma 3. *By $p(x)$ denote the functions $\int\limits_x^1 m(\xi)q((\xi-x)/2)\,d\xi$ or $\int\limits_x^1 m(\xi)q((\xi+x)/2)\,d\xi$, where $m(\xi)$ is $g_1(\xi)$ or $g_2(\xi)$ $(g=(g_1,g_2)^T = (L-\mu_0 E)\varphi)$, and $q(x)$ is $q_1(x)$ or $q_2(x)$. Then*

$$\|p\|_{L_2} \le 2\|m\|_{L_2} \cdot \|q\|_{L_1},$$

where $\|\cdot\|_{L_s}$ is the norm on $L_s[0,1]$.

Lemma 4. *Let $\psi(x)$ denote the function $\cos x$ or $\sin x$. Let $m(x) \in L_2[0,1]$ and $m(x,\mu) = m(x)\psi(\mu x)$, for $\mu \in \gamma_0$, and $\beta_n(\mu) = (m(x,\mu), \psi(\pi n x))$. Further, by $\widetilde{\beta}_n(\mu)$ we denote the sum of all $|\beta_n(\mu)|$, where $m(x)$ is one of the functions $g_j(x)$, $g_j(x)\int\limits_0^x q_s(\xi)\,d\xi$, $p(x)$ ($p(x)$ is one of the functions from Lemma 3). Then*

$$\sum_{n=n_1}^{n_2} \frac{1}{n}\,\widetilde{\beta}_n(\mu) \le c\sqrt{\sum_{n=n_1}^{n_2} \frac{1}{n^2}}\,\|g\|_2,$$

where $c>0$ is a constant independent of n_1, n_2, and $\mu \in \gamma_0$, and by $\|g\|_2$ is denoted the norm of vector function $g(x) = (g_1(x), g_2(x))^T$ on $L_2^2[0,1]$.

Lemma 5. *If $g(x) = (g_1(x), g_2(x))^T \in L_2^2[0,1]$, $\rho \in \widetilde{\gamma}_n$, and $\rho = \pi n + \mu$, then*

$$\begin{aligned} (g_s, y_{j1}) &= O(\widetilde{\beta}_n(\mu)) + O(\rho^{-1}\widetilde{\beta}_n(\mu)) + O(\rho^{-2}\|g\|_2),\\ (g_s, y_{j1} - y_{j1}^0) &= O(\rho^{-1}\widetilde{\beta}_n(\mu)) + O(\rho^{-2}\|g\|_2),\\ (g_s, y_{j2}) &= O(\rho^{-1}\widetilde{\beta}_n(\mu)) + O(\rho^{-2}\widetilde{\beta}_n(\mu)) + O(\rho^{-3}\|g\|_2),\\ (g_s, y_{j2} - y_{j2}^0) &= O(\rho^{-2}\widetilde{\beta}_n(\mu)) + O(\rho^{-3}\|g\|_2), \end{aligned}$$

where $j=1,2$, $s=1,2$.

From Lemmas 2–5 follows

Lemma 6. *If $\rho = \pi n + \mu$, $\mu \in \widetilde{\gamma}_0$, $\Omega_\lambda(x,g) = (\Omega_{1\lambda}(x,g), \Omega_{2\lambda}(x,g))^T$, then*

$$\begin{aligned} \frac{d^j}{dx^j}\left(\Omega_\lambda(x,g)\right) &= O(\rho^{j-1}\widetilde{\beta}_n(\mu)) + O(\rho^{j-2}\|g\|_2) \quad (j=0,1),\\ \frac{d^j}{dx^j}\left(\Omega_\lambda(x,g) - \Omega_\lambda^0(x,g)\right) &= O(\rho^{j-2}\widetilde{\beta}_n(\mu)) + O(\rho^{j-3}\|g\|_2) \quad (j=0,1). \end{aligned}$$

3 Investigation of the function $u_0(x,t)$

Since $(M_{j\rho}g_j)(x)$, $(M^0_{j\rho}g_j)(x)$ are entire functions, it follows that

$$u_0(x,t) = -\frac{1}{2\pi i}\left(\int\limits_{|\lambda|=r} + \sum_{n\ge n_0}\int\limits_{\gamma_n}\right)\frac{\Omega^0_\lambda(x,g)}{\lambda-\mu_0}\cos\rho t\, d\lambda.$$

From [3, Lemmas 3, 4] we have

Lemma 7. *It is true that*

$$u_0(x,t) = \frac{1}{2}\left(F(x+t)+F(x-t)\right),$$

where

$$F(x) = -\frac{1}{2\pi i}\left(\int\limits_{|\lambda|=r} + \sum_{n\ge n_0}\int\limits_{\gamma_n}\right)\frac{1}{\lambda-\mu_0}\Omega^0_\lambda(x,g)\, d\lambda.$$

Lemma 8. *For $F(x) = (F_1(x), F_2(x))^T$, the relations*

$$\begin{aligned}
F_1(-x) &= \frac{1}{2}\left[F_1(1-x)+F_2(1-x)-F_1(x)+F_2(x)\right],\\
F_2(-x) &= \frac{1}{2}\left[F_1(1-x)+F_2(1-x)+F_1(x)-F_2(x)\right],\\
F_1(1+x) &= \frac{1}{2}\left[F_1(x)-F_1(1-x)+F_2(x)+F_2(1-x)\right],\\
F_2(1+x) &= \frac{1}{2}\left[F_1(x)+F_1(1-x)+F_2(x)-F_2(1-x)\right]
\end{aligned}$$

hold, and $F(x)=\widetilde{\varphi}(x)=R^0_{\mu_0}g$ for $x\in[0,1]$.

Therefore, as in [6], we get

Lemma 9. *The vector functions $F(x)$, $F'(x)$ are absolutely continuous, $F''(x)\in L^2_2[-A,A]$ for all $A>0$, and $F(x)=F(x+2)$.*

Theorem 4. *The function $u_0(x,t)$ is a classical solution of the reference problem obtained from (1)–(4) by setting $q_j(x)\equiv 0$ with initial conditions (4), where $\varphi(x)$ is replaced by $\widetilde{\varphi}(x)=R^0_{\mu_0}g$, and equation (1) is satisfied almost everywhere.*

4 Investigation of the function $u_1(x,t)$

For $u_1(x,t)$ we have

$$u_1(x,t) = -\frac{1}{2\pi i}\left(\int\limits_{|\lambda|=r} + \sum_{n\ge n_0}\int\limits_{\gamma_n}\right)\frac{1}{\lambda-\mu_0}\left[\Omega_\lambda(x,g)-\Omega^0_\lambda(x,g)\right]\cos\rho t d\lambda.$$

By the methods in [6], we obtain the following assertions.

Lemma 10. *The series $u_1(x,t)$ and the series obtained by differentiating $u_1(x,t)$ term by term with respect to x once and with respect to t twice is convergent absolutely and uniformly in $Q_T=[0,1]\times[-T,T]$, where $T>0$ is any fixed number.*

Lemma 11. *The function $u'_{1,x}(x,t)$ is absolutely continuous with respect to x, and the relation*

$$u''_{1,x^2}(x,t) = Q(x)u_1(x,t)+d(x,t)$$

holds for almost all x and t in the rectangle Q_T. Here $Q(x)=\operatorname{diag}(q_1(x),q_2(x))$,

$$d(x,t) = -\frac{1}{2\pi i}\left(\int\limits_{|\lambda|=r} + \sum_{n\ge n_0}\int\limits_{\gamma_n}\right)\frac{\lambda}{\lambda-\mu_0}\left[\Omega_\lambda(x,g)-\Omega^0_\lambda(x,g)\right]\cos\rho t d\lambda,$$

and the series $d(x,t)$ is convergent absolutely and uniformly in Q_T.

Using Theorem 4 and Lemmas 10 and 11, we obtain

Theorem 5. *If $q_j(x) \in L[0,1]$, the vector functions $\varphi(x)$ and $\varphi'(x)$ are absolutely continuous and such that they satisfy the conditions* (5)*, then the sum $u(x,t)$ of the formal solution has the following properties: the function $u(x,t)$ is continuously differentiable with respect to x and t; the function $u'_x(x,t)$* (*respectively, $u'_t(x,t)$*) *is absolutely continuous with respect to x* (*respectively, with respect to t*)*; and the function $u(x,t)$ satisfies equation* (1) *almost everywhere and conditions* (2)–(4)*; i.e., $u(x,t)$ is a classical solution of problem* (1)–(4) *with* (1) *satisfied almost everywhere.*

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RELIABLE SOLUTIONS OF PROBLEMS WITH UNCERTAIN HYSTERESIS OPERATORS

Abstract. Problems in technology lead to initial boundary value problems for partial differential equations. Material properties which appear in constitutive relations are obtained by measurements. These data are uncertain and thus are known to some extent only. Using their mean values in numerical modelling cause several serious failures in technology.

The problem of finding a reliable solution by uncertain data is solved by the so-called worst scenario method introduced by Ivo Babuška and Ivan Hlaváček. The method consists in looking for the worst scenario that may appear in the case of any admissible data, the badness of situation is estimated by means of a criterion-functional evaluating critical parts of the body.

In the contribution, the worst scenario method is applied to boundary value problems for nonlinear equation with a scalar hysteresis operator $\mathcal{F}$ or its inverse $\mathcal{G}$ of Prandtl–Ishlinskii type. The method demands special construction of admissible data and estimates the hysteresis operators. The existence of a reliable solution for the initial boundary value problem for the heat conduction or the diffusion equation $c\,u_t = (\mathcal{F}_\eta[u_x])_x + f$ with various types of criterion-functionals is proved.*

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Key words and phrases. Prandtl–Ishlinskii hysteresis operator, reliable solution, uncertain data, worst scenario method, heat conduction equation, diffusion equation.

რეზიუმე. ტექნოლოგიის ამოცანებს მივყავართ კერძოწარმოებულიანი დიფერენციალური განტოლებებისთვის დასმულ საწყის-სასაზღვრო ამოცანებამდე. მატერიალური მახასიათებლები, რომლებიც მონაწილეობს კონსტიტუციურ თანაფარდობებში, მიღებულია გაზომვის საშუალებით. ეს მონაცემები მხოლოდ გარკვეული სიზუსტით არის ცნობილი. მათი საშუალო მნიშვნელობების გამოყენებამ რიცხვით მოდელირებაში გამოიწვია რამდენიმე ტექნოლოგიური მარცხი.

განუსაზღვრელ მონაცემის საშუალებით საიმედო ამონახსნის მოძებნის ამოცანა გადაწყვეტილია ივო ბაბუშკას და ივან ჰლავაჩეკის მიერ შემოტანილი ე.წ. უარესი სცენარის მეთოდით. მეთოდი მდგომარეობს იმ უარესი სცენარის მოძებნაში, რომელიც შეიძლება გამოჩნდეს ნებისმიერი დასაშვები მონაცემების შემთხვევაში. სიტუაციის უარესობა შეფასებულია სხეულის კრიტიკული ნაწილების კრიტერიუმ-ფუნქციონალური შეფასების საშუალებით.

უარესი სცენარის მეთოდი გამოყენებულია სასაზღვრო ამოცანებში არაწრფივი განტოლებისთვის სკალარული ჰისტერეზისის $\mathcal{F}$ ოპერატორით ან მისი პრანდტლ-იშლინსკის ტიპის $\mathcal{G}$ შებრუნებულით. მეთოდი მდგომარეობს შესაძლო მონაცემების სპეციალურ კონსტრუქციაში და ჰისტერეზისის ოპერატორების შეფასებაში. დამტკიცებულია სითბოგამტარებლობის ან დიფუზიის $c\,u_t = (\mathcal{F}_\eta[u_x])_x + f$ განტოლებისთვის საწყისი ამოცანის საიმედო ამონახსნის არსებობა სხვადასხვა ტიპის კრიტერიუმ-ფუნქციონალისთვის.

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1 Introduction

Many problems in technology can be modelled by the initial boundary value problems for partial differential equations with a hysteresis operator. Among them let us consider a scalar one-dimensional equation

$$c\,u_t = q_x + f\,, \quad q = k\,u_x,\ \ x \in (0,\ell),\ \ t \in (0,T),$$

which can be physically interpreted as the heat conduction in a one-dimensional body, particularly in a bar $(0,\ell)$. The unknown $u(x,t)$ is the temperature, $q(x,t)$ is a negative heat flow, c is specific heat capacity and k is thermal conductivity. We take a negative heat flow q in order to obtain the linear Fourier law $q = k\,u_x$ with positive $k > 0$ instead of the usual Fourier law $q = -k\,u_x$ with (positive) heat flow q. We replace this Fourier law by the relation $q = \mathcal{F}[u_x]$ with a hysteresis operator $\mathcal{F}$ which describes behavior of a rate-independent material with memory or phase transition. In this way we obtain the equation

$$c\,u_t = q_x + f\,, \quad q = \mathcal{F}[u_x],\ \ x \in (0,\ell),\ t \in (0,T).$$

The equation contains material parameters, which are not known exactly, since they are obtained by measurements. They are uncertain, i.e., they are known to some extent only. In the past, using mean values of the data in the process of mathematical modelling caused several serious failures in technology. This problem with uncertain data has been solved by I. Babuška and I. Hlaváček in a series of papers, see [6, 7]. They proposed the so-called *worst scenario method*.

The method takes into account all data, i.e., all material parameters from their range of uncertainty. Using a criterion-functional which measures the badness of the situation, we seek for the worst scenario that may appear. The method is used in engineering for its simplicity: the model is deterministic (no need to deal with stochastic models), and optimization tools can be used for computing the maximum: theory, numeric analysis and the corresponding software.

The problem of longitudinal vibration of a nonhomogeneous elasto-plastic rod including homogenization problem was solved in [2]. The one-dimensional diffusion equation with a scalar hysteresis operator was solved in [3] and a higher space dimensional heat equation with a scalar hysteresis operator including homogenization problem was studied in [4]. Reliable solutions of the problem of periodic oscillations of an elasto-plastic beam was studied in [9]. Reliable solutions of a homogenization problem with monotone operators was studied in [5].

In the contribution, we study the initial boundary value problem for a nonlinear heat conduction equation (or diffusion equation) with a hysteresis operator of Prandtl–Ishlinskii type. These hysteresis operators are described and studied in e.g. [1, 8, 10]. The aim of the contribution is to propose sets for admissible data, criterion-functionals and to prove the existence of the worst scenario solution.

The paper is organized as follows. Section 2 contains the survey of hysteresis operators and their properties, in Section 3, the existence of a solution of the initial boundary value problem is proved, and the worst scenario method applied to the problem is considered in Section 4 including the setting of a set of admissible data and proposals of various criterion-functionals.

2 Hysteresis operators

In this section we deal with the one-dimensional hysteresis operators. These operators acting in a space of real functions on an interval $I = \langle 0,T\rangle$ representing time can be simply characterized by the following properties. The hysteresis operators $\mathcal{T}$ are:

- *rate independent* – the output $\mathcal{T}[v]$ is independent of speed of the input v: $\mathcal{T}[v\circ\varphi](t) = \mathcal{T}[v](\varphi(t))$ for any increasing mapping φ from I onto I,
- *causal* – the output is independent of future input, i.e., if $u(s) = v(s)$ for all $s \le t$, then $\mathcal{T}[u](t) = \mathcal{T}[v](t)$,
- *locally monotone* – a locally non-decreasing input yields a locally non-decreasing output and also a non-increasing input provides a non-increasing output, i.e.,

$$\mathcal{T}[v]'(t) \cdot v'(t) \ge 0 \ \text{ for a.e. } \ t \in I.$$

For more detailed study of hysteresis operators we can recommend [1, 8, 10].

2.1 Stop and play operators

Here we deal with hysteresis operators of Prandtl–Ishlinskii type. These operators are defined by means of operators called as a stop and a play operator with one parameter $r > 0$. Their definition is based on the solution of the following variational inequality. Let $v \in W^{1,1}(I)$ be an input function and $s_r^0 \in \langle -r, r\rangle$ be an initial state. We look for a function $s \in W^{1,1}(I)$ satisfying:

$$\begin{gathered} |s(t)| \le r \ \ \forall\, t \in I, \quad s(0) = s_r^0, \\ (s'(t) - v'(t))(\widetilde{s} - s(t)) \ge 0 \ \ \forall\, |\widetilde{s}| \le r, \ \text{ a.e. } \ t \in I. \end{gathered} \tag{2.1}$$

It should be noted that the above inequality yields $s'(t) = v'(t)$ provided $s(t)$ is inside the interval $(-r, r)$. If $s(t) = r$ and v is increasing, then $s'(t) = 0$ and, also, if $s(t) = -r$ and v is decreasing, then likewise $s'(t) = 0$.

The inequality admits a unique solution $s \in W^{1,1}(I)$ which defines the elementary hysteresis operators:

Definition 2.1. The solution $s(t)$ of the variational inequality (2.1) defines two complementary operators: the stop operator $\mathcal{S}_r$ and the play operator $\mathcal{P}_r$:

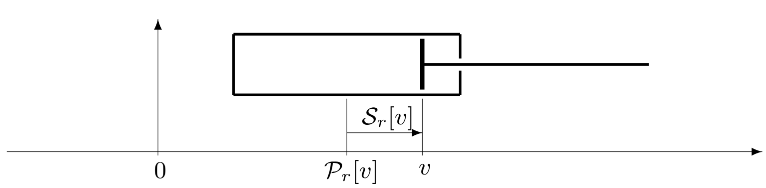
$$\mathcal{S}_r[v](t) := s(t), \quad \mathcal{P}_r[v](t) := v(t) - s(t), \ \ t \in \langle 0, T\rangle. \tag{2.2}$$

To simplify the notation, we have taken for the input $v(t)$ the so-called virgin initial state $s_r^0 = \min\{r, \max\{-r, v(0)\}\}$ and omit s_r^0 in the notation of the operators. We also put the input v into square brackets to indicate that the dependence is not local: the value at time t depends on values on the whole interval $\langle 0, t\rangle$. Let us note that the stop operator can be equivalently introduced on each interval of monotonicity $\langle t_a, t_b\rangle$ of the input $v(t)$ by the relation

$$\mathcal{S}_r[v](t) = \min\Big\{r, \max\big\{-r, \mathcal{S}_r[v](t_a) + v(t) - v(t_a)\big\}\Big\} \ \ \forall\, t \in (t_a, t_b\rangle.$$

Both stop and play operators are rate independent, causal and locally monotone, and in addition, they satisfy $\mathcal{S}_r[v]'(t) \cdot \mathcal{P}_r[v]' = 0$ for a.e. $t \in I$.

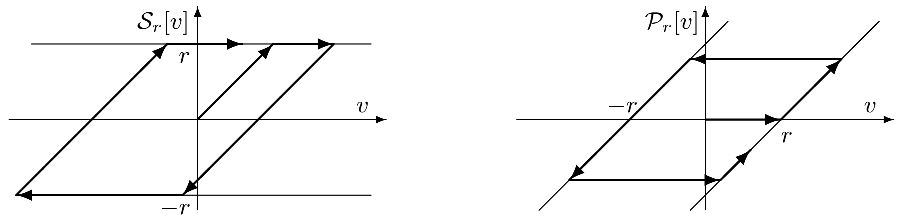
Values of the stop and play operators can be visualized by the so-called "piston in cylinder model". Let us consider a piston freely moving in a cylinder of length $2r$. Position of the piston is the input $v(t)$, position of the cylinder center is the value of the play operator $\mathcal{P}_r[v](t)$, while the position of the piston with respect to the cylinder center is the value of the stop operator $\mathcal{S}_r[v](t)$.



Piston in the cylinder model for the stop and play operators.

In mechanics, the stop operator $\mathcal{S}_r$ can be interpreted as the output stress $\mathcal{S}_r[v](t) = s(t)$ of an elasto-plastic material caused by the input strain (deformation) $v(t)$. Its rheological element consists of an elastic and a friction element combined in series. On the other hand, the play operator $\mathcal{P}_r$ can be interpreted as the output strain $\mathcal{P}_r[v](t) = v(t) - s(t)$ of an elasto-plastic material caused by the input stress $v(t)$. Its rheological element consists of an elastic and a friction elements combined in parallel. In both cases the elasticity modulus is 1 and plasticity limit is r.

Plane diagram $[v, \mathcal{S}_r[v]]$ of the stop operator has straight line segments with slope 0 or 1 with concave increasing branches and convex decreasing branches, while the plane diagram $[v, \mathcal{P}_r[v]]$ of the play operator has also straight line segments with slope 0 or 1, whereas the increasing branches are convex and decreasing branches are concave:



Diagrams of the stop operator $v \mapsto \mathcal{S}_r[v]$ *and the play operator* $v \mapsto \mathcal{P}_r[v]$.

Let us consider the properties of the stop and play operators (for proofs see, e.g., [2]).

Proposition 2.2. *Let* $v_1, v_2 \in W^{1,1}(I)$ *and put* $s_i(t) = \mathcal{S}_r[v_i](t)$, $p_i(t) = \mathcal{P}_r[v_i](t)$, $i = 1, 2$. *Then we have*

$$(p_1'(t) - p_2'(t))(s_1(t) - s_2(t)) \geq 0, \quad \textit{for a.e. } t \in I, \tag{2.3}$$

$$|p_1(t) - p_2(t)| \leq \max\left\{|p_1(0) - p_2(0)|, \|v_1 - v_2\|_{\langle 0,t\rangle}\right\} \ \textit{for } t \in I, \tag{2.4}$$

$$|s_1(t) - s_2(t)| \leq \|v_1 - v_2\|_{\langle 0,t\rangle} \ \textit{for } t \in I. \tag{2.5}$$

2.2 Prandtl–Ishlinskii operators

Diagrams of the stop and play operators consist of straight line segments with two slopes. But diagrams of the real elasto-plastic materials have curved changing slope branches. To obtain such diagrams we combine the operators with different parameters r of various weights.

The Prandtl–Ishlinskii operator $\mathcal{F}$ of stop type is defined as a parallel combination of the stop operators with increasing parameters r_i and various weights c_i

$$\mathcal{F}[v] = c_1\mathcal{S}_{r_1}[v] + c_2\mathcal{S}_{r_2}[v] + \cdots + c_n\mathcal{S}_{r_n}[v] + c_\infty v.$$

The combination can be rewritten with Stieltjes integral

$$\mathcal{F}[v] = \eta(\infty)\, v - \int_0^\infty \mathcal{S}_r[v]\, \mathrm{d}\eta(r)$$

by a non-increasing distribution function $\eta(r)$, where $\eta(r) = c_\infty$ for $r \in \langle r_n, \infty)$, $\eta(r) = c_i + c_{i+1} + \cdots + c_n + c_\infty$ for $r \in \langle r_{i-1}, r_i)$, $i = 1, 2, \ldots, n-1$, where $r_0 = 0$.

The Stieltjes integral enables us to cover both the discrete combination of stop operators $\mathcal{S}_{r_i}$, when η is piecewise constant, and a continuous combination of stop operators $\mathcal{S}_r$ if η is a continuous function.

Definition 2.3. Let $\alpha, \beta \in \mathbb{R}$ be positive constants, $\alpha\beta < 1$ and let $\eta : \langle 0, \infty) \to \langle 0, \infty)$ be a non-increasing right continuous function satisfying $\alpha \leq \eta(r) \leq \frac{1}{\beta}$ for all r. Then the Prandtl–Ishlinskii operator of stop type is given by

$$\mathcal{F}_\eta[v](t) := \eta(\infty)\, v(t) - \int_0^\infty \mathcal{S}_r[v](t)\, \mathrm{d}\eta(r)\, . \tag{2.6}$$

In the case of elasto-plastic material, the dependence of the stress q on strain $e = u_x$ can be modelled by this operator as

$$q(t) = \mathcal{F}_\eta[e](t). \tag{2.7}$$

The corresponding diagram of dependence of $q(t)$ on $e(t)$ is an oriented continuous curve with concave increasing and convex decreasing parts.

Similarly, the Prandtl–Ishlinskii operator of play type is defined as a serial combination of the play operators with increasing r_i. Again, we use the Stieltjes integral by a non-decreasing function η which enables us to describe both discrete and continuous combinations:

Definition 2.4. Let $\alpha, \beta \in \mathbb{R}$ be positive constants, $\alpha\beta < 1$ and let $\zeta : \langle 0, \infty) \to \langle 0, \infty)$ be a non-decreasing right continuous function satisfying $\beta \le \eta(r) \le \frac{1}{\alpha}$ for all r. Then the Prandtl–Ishlinskii operator of play type is given by

$$\mathcal{G}_\zeta[v](t) := \zeta(0)\, v(t) + \int_0^\infty \mathcal{P}_r[v](t)\, \mathrm{d}\zeta(r). \tag{2.8}$$

In the case of elasto-plastic material, the dependence of the strain e on the stress q can be modelled by this operator as

$$e(t) = \mathcal{G}_\zeta[q](t). \tag{2.9}$$

The corresponding diagram of the dependence of $e(t)$ on $q(t)$ is an oriented continuous curve with convex increasing and concave decreasing parts.

For the increasing input $v(s) = s$, $s \in \langle 0, \infty)$ and the Prandtl–Ishlinskii operator of stop type we obtain the so-called virgin curve $\varphi(s) = \mathcal{F}[v](s)$ which is a continuous increasing concave unbounded function on $\mathbb{R}^+$. Similarly, for the input $v(t) = t$, $t \in \langle 0, \infty)$ and the Prandtl–Ishlinskii operator of play type we obtain the curve $\psi(t) = \mathcal{G}[v](t)$ which is a continuous increasing convex unbounded function on $\mathbb{R}^+$.

Let these functions φ, ψ be a pair of increasing mutually inverse functions, i.e.,

$$t = \varphi(s) \iff s = \psi(t), \quad s, t \in \langle 0, \infty). \tag{2.10}$$

Moreover, the function φ is concave if and only if ψ is convex.

Definition 2.5. Let $\alpha, \beta > 0$, $\alpha\beta < 1$ be positive constants. We say that the functions $[\eta, \zeta]$ defined on $\mathbb{R}^+$ form a pair of Prandtl–Ishlinskii distribution functions if they are right continuous, η non-increasing, ζ non-decreasing, they satisfy

$$\alpha \le \eta(r) \le \frac{1}{\beta} \ \text{ and } \ \beta \le \zeta(r) \le \frac{1}{\alpha} \tag{2.11}$$

and their primitive functions

$$\varphi(s) = \int_0^s \eta(r)\, \mathrm{d}r, \quad \psi(t) = \int_0^t \zeta(r)\, \mathrm{d}r$$

are mutually inverse, i.e., they satisfy (2.10). The set of all such pairs of distribution functions will be denoted by $PI(\alpha, \beta)$ and the set of ζ by $PI^+(\alpha, \beta)$.

These pairs of distribution functions define mutually inverse operators:

Proposition 2.6. *Let $(\eta, \zeta) \in PI(\alpha, \beta)$. Then the corresponding Prandtl–Ishlinskii operators are mutually inverse, i.e., for each inputs e, q*

$$q(t) = \mathcal{F}_\eta[e](t) \ \textit{ if and only if } \ e(t) = \mathcal{G}_\zeta[q](t). \tag{2.12}$$

2.3 Properties of the operators

Let $\alpha, \beta > 0$, $\alpha\beta < 1$ and $(\eta, \zeta) \in PI(\alpha, \beta)$. Let us consider the properties of the corresponding Prandtl–Ishlinskii operators. They are locally monotone and Lipschitz continuous (for proofs, see [2, Propositions 2.7–2.12].

Proposition 2.7. *Let $\alpha, \beta > 0$, $\alpha\beta < 1$, $(\eta, \zeta) \in LP(\alpha, \beta)$. Then the corresponding operators $\mathcal{F}$ and $\mathcal{G}$ map $W^{1,\infty}(I)$ into $W^{1,\infty}(I)$ and also $W^{1,1}(I)$ into $W^{1,1}(I)$.*

Let further $q \in W^{1,1}(I)$ and put $e = \mathcal{G}_\zeta[q]$ or, equivalently, $q = \mathcal{F}_\eta[e]$. Then for a.e. $t \in I$, the derivatives exist and the following estimates hold:

$$\alpha (e'(t))^2 \le e'(t)\, q'(t) \le \frac{1}{\beta}\, (e'(t))^2, \tag{2.13}$$

$$\beta (q'(t))^2 \le e'(t)\, q'(t) \le \frac{1}{\alpha}\, (q'(t))^2. \tag{2.14}$$

The following estimates ensure the Lipschitz continuity of the operators:

Proposition 2.8. *Let $(\eta, \zeta) \in LP(\alpha, \beta)$ and $q_1, q_2, e_1, e_2 \in W^{1,1}(I)$. Then for $t \in I$, we have*

$$\left|\mathcal{F}_\eta[e_1](t) - \mathcal{F}_\eta[e_2](t)\right| \leq \left(\frac{2}{\beta} - \alpha\right) \cdot \|e_1 - e_2\|_{\langle 0,t\rangle}, \tag{2.15}$$

$$\left|\mathcal{G}_\zeta[q_1](t) - \mathcal{G}_\zeta[q_2](t)\right| \leq \frac{1}{\alpha} \cdot \|q_1 - q_2\|_{\langle 0,t\rangle}. \tag{2.16}$$

The following estimate is a consequence of (2.3) (see also [2,3]):

Proposition 2.9. *Let $\zeta \in LP^+(\alpha, \beta)$ and $q_1, q_2 \in W^{1,1}(I)$. Then for a.e. $t \in I$, we have*

$$\left(\mathcal{G}_\zeta[q_1](t) - \mathcal{G}_\zeta[q_2](t)\right)_t (q_1 - q_2) \geq \frac{\beta}{2} \frac{\mathrm{d}}{\mathrm{d}t}[(q_1 - q_2)^2]. \tag{2.17}$$

Finally, the following estimate yields the dependence of the operator $\mathcal{G}_\zeta$ on the distribution functions ζ (for proof see, e.g., [2, Proposition 2.10]).

Proposition 2.10. *Let $\zeta_1, \zeta_2 \in PI^+(\alpha, \beta)$ be two distribution functions, $\mathcal{G}_{\zeta_1}$, $\mathcal{G}_{\zeta_2}$ be the corresponding operators and $q_1, q_2 \in W^{1,1}(I)$ be arbitrary input functions. Then*

$$\|\mathcal{G}_{\zeta_1}[q_1] - \mathcal{G}_{\zeta_2}[q_2]\|_{[0,t]} \leq \zeta_1(\infty)\|q_1 - q_2\|_{[0,t]} + \int_0^{\|q_2\|_{[0,t]}} |\zeta_1(r) - \zeta_2(r)| \,\mathrm{d}r. \tag{2.18}$$

2.4 Space dependent case

In case of nonhomogeneous materials the material properties depend even on the space variable x. Thus both function η and ζ are not only the functions of r, but in addition, they depend on the space variable x, i.e., $\eta = \eta(x, r)$ and $\zeta = \zeta(x, r)$.

3 Heat conduction and diffusion equation with hysteresis operator

We deal with the following equations:

$$c\, u_t = q_x + f, \quad q = \mathcal{F}_\eta[u_x] \text{ or, equivalently, } u_x = \mathcal{G}_\zeta[q] \tag{3.1}$$

on $x \in \Omega \equiv (0, \ell)$ and $t \in I \equiv (0, T)$ with a pair of mutually inverse hysteresis operators $\mathcal{G}_\zeta$ or $\mathcal{F}_\eta$. The equations are completed with the boundary conditions, e.g.,

$$u(0, t) = 0, \quad q(\ell, t) = 0 \text{ for } t \in I \tag{3.2}$$

and the standard initial condition

$$u(x, 0) = u^0(x) \text{ for } x \in \Omega. \tag{3.3}$$

The problem can be physically interpreted as the heat conduction or the diffusion problem in some materials with a changing phase in a bar $(0, \ell)$ and time $(0, T)$. In the case of heat conduction, the variable u stands for temperature and q for a negative heat flow, and in the case of diffusion problem, u denotes concentration and q negative mass flow.

The boundary condition $u = 0$ prescribes zero temperature or zero concentration on the left end of the bar, while $q = 0$ means thermal or mass insulated right end of the bar. The hysteresis operator describes the relation between the negative heat or mass flow q and the temperature or concentration gradient $e = u_x$.

3.1 Solvability of the problem

Hypotheses 3.1. *We adopt the following hypotheses for the data of the problem:*

- $c \in L^\infty(\Omega)$ *and* $c_m \le c(x) \le c_M$ *for a.e.* $x \in \Omega$ *for some* $0 < c_m < c_M$,
- $f \in W^{1,1}(I, L^2(\Omega))$,
- $\eta, \zeta \in L^\infty(\Omega \times I)$ *such that* $(\eta, \zeta)(x, \cdot) \in PI(\alpha, \beta)$ *for a.e.* $x \in \Omega$ *for some constants* $\alpha, \beta > 0$, $\alpha\beta < 1$,
- $u^0 \in W^{1,2}(\Omega)$ *and it satisfies the compatibility condition with the boundary conditions, i.e.,* $u^0(0) = 0$ *and* $\mathcal{F}[u^0_x](\ell) = 0$.

Theorem 3.2. *Let Hypotheses* 3.1 *hold. Then the problem has a unique solution, namely, there exist the functions* $u, q \in C(\Omega \times I)$ *and* $e = u_x \in L^2(\Omega, C(I))$ *such that*

$$u_t, e_t, q_t, q_x, \in L^\infty(I, L^2(\Omega)),$$

and equalities (3.1)–(3.3) *hold almost everywhere.*

The solution is unique. Moreover, all unknowns and their derivatives are bounded in the corresponding norms by the constants depending on α, β, c_m, c_M *and the norm of* f *in* $W^{1,1}(I, L^2(\Omega))$ *and* u^0 *in* $W^{1,2}(\Omega)$.

Let us briefly sketch the proof of the theorem (the details can be found in [3]). The proof will be done in several steps.

3.2 Semidiscretized problem

First we convert the partial differential equation into a system of ordinary differential equations in t. We divide the interval $\Omega = (0, \ell)$ into n parts $\Omega_k = (x_{k-1}, x_k)$, $k = 1, 2, \dots, n$, of length $h = \ell/n$, where $x_k = kh$. In the semidiscretized problem, the space derivative is replaced by the difference, the unknowns u_k, e_k, q_k are the function of time $t \in I$ approximating the value at $x_k = kh$, $k = 0, 1, \dots, n$. In this way, we obtain the following system of equations, $k = 1, 2, \dots, n-1$ and $t \in I$,

$$c_k\, u_k' = \frac{1}{h}\,(q_{k+1} - q_k) + f_k, \tag{3.4}$$

$$e_k = \frac{1}{h}\,(u_k - u_{k-1}), \tag{3.5}$$

$$e_k = \mathcal{G}_k[q_k] \ \text{ or equivalently } \ q_k = \mathcal{F}_k[e_k], \tag{3.6}$$

$$u_k(0) = u_k^0\,, \tag{3.7}$$

where c_k, u_k^0, $f_k(t)$, $\zeta_k(r)$ are the integral means of the corresponding functions over the space interval Ω_k, e.g., $f_k(t) = \frac{1}{h} \int\limits_{\Omega_k} f(x,t)\,\mathrm{d}x$. The operator $\mathcal{G}_k$ is determined by the averaged distribution function $\zeta_k(r)$ and $\mathcal{F}_k$ is the operator, inverse to the operator $\mathcal{G}_k$.

We have obtained a system of ordinary differential equations (3.4) with the initial conditions (3.7) with additional equations (3.5), (3.6). Taking $q_k = \mathcal{F}_k[\frac{1}{h}(u_k - u_{k-1})]$ and the properties of the Prandtl–Ishlinskii operator of stop type, the right-hand side of the ODE (3.4) are Lipschitz continuous in u_k, and thus by the Piccard theorem, the system admits unique solutions $u_k \in W^{2,1}(I)$, $e_k \in W^{2,1}(I)$ and $q_k \in W^{1,\infty}(I)$.

3.3 Estimates

We use the following estimates.

Lemma 3.3. *The solutions* $\{u_k, e_k, q_k\}$ *to system* (3.4)–(3.7) *satisfies*

$$h\sum_{k=1}^{n-1}\Big[(u_k')^2+\Big(\frac{q_{k+1}-q_k}{h}\Big)^2\Big](t)\le C \quad \forall\, t\in I, \tag{3.8}$$

$$\int_I h\sum_{k=1}^{n-1}\Big[(q_k')^2+(e_k')^2+\Big(\frac{u_k-u_{k-1}}{h}\Big)^2\Big](\tau)\,\mathrm{d}\tau\le C, \tag{3.9}$$

where the constant C *is independent of* n*,* h*.*

Proof. To derive the estimate, we differentiate equation (3.4), multiply it by u_k' and sum it with equation (3.5) differentiated and multiplied by q_k':

$$c_k\,u_k''u_k'+e_k'q_k'=\frac{1}{h}\,(q_{k+1}'\,u_k'-q_k'\,u_{k-1}')+f_k'\,u_k'.$$

Further, summing up the equation for $k=1,2,\dots,n-1$, we obtain for a.e. $t\in I$

$$\sum_k c_k\,u_k''u_k'+\sum_k e_k'q_k'=\frac{1}{h}\,(q_n'\,u_{n-1}'-q_1'\,u_0')+\sum_k f_k'\,u_k'.$$

Owing to the boundary conditions, we have $q_n'=0$ and $u_0'=0$. We multiply the equality by h. Since $u_k''\,u_k'=\frac{1}{2}[(u_k')^2]'$, integration of the last equality from 0 to a fixed $t\le T$ yields

$$h\sum_k\frac{c_k}{2}\,(u_k'(t))^2+\int_0^t h\sum_k e_k'\,q_k'\,\mathrm{d}\tau=h\sum_k\frac{c_k}{2}\,(u_k'(0))^2+\int_0^t h\sum_k f_k'\,u_k'\,\mathrm{d}\tau. \tag{3.10}$$

Using equation (3.4), initial condition (3.7) and properties of the operator $\mathcal{G}_k$, the first term with $u_k'(0)$ can be estimated by a constant. Using the inequalities

$$\int_I |f(t)g(t)|\,\mathrm{d}t\le\max_I|f(t)|\int_I g(t)\,\mathrm{d}t,\qquad\Big(\int_I f(t)\,\mathrm{d}t\Big)^2\le|I|\int_I f^2(t)\,\mathrm{d}t$$

and $|ab|\le\varepsilon a^2+\frac{1}{4\varepsilon}b^2$, we estimate the term with f_k'

$$\Big|\int_0^t h\sum_k f_k'u_k'\,\mathrm{d}\tau\Big|\le\varepsilon h\max_{t\in\langle 0,t\rangle}\sum_k(u_k')^2+\int_0^t h\sum_k(f_k')^2\,\mathrm{d}\tau.$$

Since f_k is the integral mean of $f(x)$ over the interval Ω_k, we have $h\sum_k f_k=\int_\Omega f(x)\,\mathrm{d}x$. Thus, by (2.8) $e_k'q_k'\ge 0$, for sufficiently small $\varepsilon>0$, we obtain the estimate of the terms on the left-hand side of (3.10). Using inequalities (2.14), (2.15) and equations (3.4), (3.5), we obtain the remaining estimates of Lemma 3.3. □

3.4 Approximate solutions and passage to the limit

For a fixed n, using the solutions u_k, e_k, q_k and the data c_k, f_k, ζ_k, we construct approximated solutions

- $\overline{c}^{(n)}$, $\overline{f}^{(n)}$, $\overline{u}^{(n)}$ are "forward" piecewise constant approximate solutions defined by the relation $\overline{\varphi}^{(n)}=\varphi_{k-1}$ for $x\in\Omega_k$,
- $\overline{e}^{(n)}$, $\overline{q}^{(n)}$, $\overline{\zeta}^{(n)}$ are "backward" piecewise constant approximate solutions defined by the relation $\overline{\varphi}^{(n)}=\varphi_k$ for $x\in\Omega_k$,
- $\widehat{u}^{(n)}$, $\widehat{q}^{(n)}$ are continuous piecewise linear on each Ω_k approximation satisfying $\widehat{\varphi}^{(n)}(x_k)=\varphi_k$.

The above approximations satisfy the system of equations for all t and a.e. $x \in \Omega$:

$$\overline{c}^{(n)}\overline{u}_t^{(n)} = \widehat{q}_x^{(n)} + \overline{f}^{(n)}, \quad \overline{e}_t^{(n)} = \widehat{u}_x^{(n)}, \quad \overline{e}^{(n)} = \overline{\mathcal{G}}^{(n)}[\overline{q}^{(n)}]. \tag{3.11}$$

Estimates (3.8), (3.9) yield the estimates of the corresponding approximate solutions $\overline{u}^{(n)}$, $\overline{e}^{(n)}$, $\overline{q}^{(n)}$, $\widehat{u}^{(n)}$ and $\widehat{q}^{(n)}$. By the compactness, these sequences contain converging subsequences which converge to the functions u, e, q satisfying the problem. Thus the solution to the problem exists.

The proof of the uniqueness of a solution can be found in [3]. Since the uniqueness for the worst scenario method is not necessary, we omit the proof. We have also proved that the unknowns in the corresponding spaces are bounded by the constants depending on the constants ℓ, T, c_m, c_M, α, β and the norms of f and u^0 only.

4 Problems with uncertain data and reliable solutions

Mathematical models of particular problems in engineering contain data, mainly material constants or constitutive relation dependence. These data are obtained by measurements and thus are not known exactly, they are uncertain, their values are known to some extent only. Since using the mean value of the data by modelling already caused several failures of a construction in engineering practice, Ivo Babuška has proposed the so-called worst scenario method.

The method consists in considering the problems with all data admissible by the measurements and the corresponding solutions. According to the character of the problem, a criterion-functional on data and solutions is chosen. This functional should evaluate a rate of danger of the situation. The method thus looks for the data yielding the worst situation, i.e., what the worst can happen within the given uncertain data, although the probability may be very low. The worst scenario method for obtaining reliable solutions was further developed by Ivan Hlaváček and others (see, e.g., [7]) and also applied to many particular problems. The survey paper [6] can be recommended for a brief introduction.

The advantage of the worst scenario method consists in the possibility to use numerical methods, algorithms and software developed for optimization problems. For its deterministic character the method is much more simpler and effective than probabilistic approaches.

4.1 Worst Scenario Method

Here we describe the method. Let us denote by P_a the state problem with data a and the corresponding solution by u_a. The data a may contain coefficients of the equation, right-hand side, values in the boundary conditions, etc. The set of all admissible values of data a will be denoted by $\mathcal{U}_{\rm ad}$. It should be chosen such that for each $a \in \mathcal{U}_{\rm ad}$ the problem P_a admits a solution u_a.

The criterion-functional $\Phi = \Phi(a, u_a)$ "evaluating" danger of the situation will be defined on the data $a \in \mathcal{U}_{\rm ad}$ and the corresponding solutions u_a of the problem P_a. Then the worst scenario problem reads:

Problem. Find the data $a^* \in \mathcal{U}_{\rm ad}$ which maximize the functional Φ, i.e.,

$$\text{Look for } a^* \in \mathcal{U}_{\rm ad} \text{ s.t. } \Phi(a^*, u_{a^*}) \geq \Phi(a, u_a) \text{ for all } a \in \mathcal{U}_{\rm ad}. \tag{4.1}$$

In the case if the solution of the problem P_a exists but is not unique, the problem is modified to:

$$\text{Find } a^* \in \mathcal{U}_{\rm ad} \text{ and } u^* \in U_{a^*} \text{ s.t. } \Phi(a^*, u^*) \geq \Phi(a, u) \ \forall\, a \in \mathcal{U}_{\rm ad} \ \forall\, u \in U_a,$$

where U_a is the set of all solutions u_a to the problem P_a with data a.

The aim of the contribution is to prove that the problem admits a solution, i.e., the functional Φ is bounded and attains its maximum. Let us note that this maximum can be attained for more than one data a^* and in the case of a nonlinear problem the maximum can be reached for the data a^* in the interior of the set $\mathcal{U}_{\rm ad}$ of admissible data, i.e., not on the boundary of $\mathcal{U}_{\rm ad}$ as in the case of a linear problem.

Knowing that for each $a \in \mathcal{U}_{\rm ad}$ the problem P_a admits a solution, the procedure continues with the following steps:

- we choose the set $\mathcal{U}_{\rm ad}$ which is compact, i.e., each sequence $\{a_n\} \subset \mathcal{U}_{\rm ad}$ contains a subsequence converging to an element $a^* \in \mathcal{U}_{\rm ad}$,
- we prove that the mapping $a \mapsto u_a$ is continuous, i.e.,

$$a_n \to a^* \implies u_{a_n} \to u_{a^*}$$

- and we verify the continuous dependence of the functional Φ on the data a and the solution u_a, i.e.,

$$a_n \to a^*, \;\; u_{a_n} \to u_{a^*} \implies \Phi(a_n, u_{a_n}) \to \Phi(a^*, u_{a^*}).$$

Then the worst scenario problem admits a reliable solution. Indeed, let the sequence of data $\{a_n\}_{n=1}^{\infty}$ maximize the functional Φ on $\mathcal{U}_{\rm ad}$, i.e., $\Phi(a_n, u_{a_n})$ tends to a supremum of Φ on $\mathcal{U}_{\rm ad}$. Since the set $\mathcal{U}_{\rm ad}$ is compact, the sequence $\{a_n\}$ contains a subsequence $\{a_{n'}\}$ converging to $a^* \in \mathcal{U}_{\rm ad}$. The continuity of the mapping $a \to u_a$ yields $u_{a_{n'}} \to u_{a^*}$ and the continuity of Φ yields $\Phi(a_{n'}, u_{a_{n'}})$ tends to $\Phi(a^*, u_{a^*})$. Thus the supremum is a real number and Φ admits a maximum on $\mathcal{U}_{\rm ad}$.

4.2 Admissible data

The data in our problem are of several types: the material constant c, the pair of distribution functions (η, ζ) of the hysteresis operator, the right-hand side f, the initial condition u^0 and the boundary conditions for the unknown at $x = 0$ and $x = \ell$; for the sake of simplicity, they were chosen to be zero. For simplicity, we also take the initial condition u^0 and right-hand side to be certain.

We consider a nonhomogeneous medium composed of two or more homogeneous materials occupying the parts $\Omega_1, \dots, \Omega_k$. For the sake of simplicity, we assume that the parts Ω_i are known, only the constant c on each Ω_i is uncertain, i.e., its real values are within the intervals $\langle c_{im}, c_{iM}\rangle$. Thus $c(x)$ will be piecewise constant functions from the admissible set $\mathcal{C}_{\rm ad}$

$$\mathcal{C}_{\rm ad} = \big\{c:\ \Omega \to \mathbb{R},\ c(x) = c_i \in \langle c_{im}, c_{iM}\rangle \ \text{ for } \ x \in \Omega_i,\ i = 1, \dots, k\big\}, \tag{4.2}$$

where the constants $0 < c_{im} \le c_{iM}$ are given. Since the set $\mathcal{C}_{\rm ad}$ is "equivalent" to the cartesian product of compact intervals $\langle c_{im}, c_{iM}\rangle$, we have arrived to

Lemma 4.1. *The set $\mathcal{C}_{\rm ad}$ is compact in the maximum norm.*

4.3 Admissible data for hysteresis operators

Mutually inverse Prandtl–Ishlinskii operators $\mathcal{F}$ and G are fully determined by their distribution function $\eta \in PI(\alpha,\beta)^-$ or $\zeta \in PI(\alpha,\beta)^+$. Since the use is made of the operator $\mathcal{G}$ of play type, we define the set of admissible functions for ζ as a subset of $PI(\alpha,\beta)^+$ which are constant outside of the interval $\langle 0, R\rangle$ with some $R > 0$. Thus for $\alpha, \beta > 0$, $\alpha\beta < 1$ and $R > 0$ let $\mathcal{Z}(\alpha,\beta,R)$ be the set of all functions $\zeta = \zeta(r)$ satisfying:

(a) ζ is right-continuous nondecreasing function $\langle 0,\infty) \to \langle 0,\infty)$,

(b) $\beta \le \zeta(r) \le 1/\alpha \ \forall\, r \in \langle 0,\infty)$ (i.e., $\zeta \in PI(\alpha,\beta)^+$),

(c) $\zeta(r)$ is constant on $\langle R,\infty)$.

The set $\mathcal{Z}(\alpha,\beta,R)$ is compact in the following sense: Each sequence of $\{\zeta_n\}_{n=1}^{\infty}$ in the set $\mathcal{Z}(\alpha,\beta,R)$ contains a subsequence $\{\zeta_{n'}\}$ and there exists a function ζ^* in $\mathcal{Z}(\alpha,\beta,R)$ such that $\int_0^R |\zeta_{n'}(r) - \zeta^*(r)|\,\mathrm{d}r$ tends to zero as $n' \to \infty$.

This compactness can be proved by constructing finite ε-nets of piecewise constant functions. Let us divide the interval $\langle 0, R\rangle$ by the points $r_i = i\,R/n_r$ into n_r parts. Further, let us divide the interval of values $\langle \beta, \frac{1}{\alpha}\rangle$ by $z_j = \beta + (\frac{1}{\alpha} - \beta)j/n_z$ into n_z equal parts. Then the functions $\zeta(r)$ which are nondecreasing and take the values z_j on each part $\langle r_{j-1}, r_j)$ make for sufficiently large n_r and n_z a finite ε-net in $\mathcal{Z}(\alpha,\beta,R)$ which proves the compactness of $\mathcal{Z}(\alpha,\beta,R)$.

Let $\{q_n(t)\}$ be a bounded sequence $\|q_n\|_{\langle 0,T\rangle} \leq R$ of functions uniformly converging to $q^*(t)$, i.e., $\|q_n(t) - q^*(t)\|_{\langle 0,T\rangle} \to 0$. By (2.18), the convergence of $\zeta_n(r)$ ensures that of the corresponding $e_n = \mathcal{G}_{\zeta_n}[q_n]$.

Since the medium consists of k homogeneous materials, we take the functions $\zeta(r,x)$ constant in x on each $x \in \Omega_i$. For $\alpha_i, \beta_i > 0$, $\alpha_i\beta_i < 1$ and $R_i > 0$, $i = 1, 2, \dots, k$, we put

$$\mathcal{Z}_{\text{ad}} = \big\{\zeta(r,x), \ \text{ s.t. } \ \zeta(x,r) \in \mathcal{Z}(\alpha_i, \beta_i, R_i) \ \text{ for } \ x \in \Omega_i, \ i = 1, 2, \dots, k\big\}. \tag{4.3}$$

Thus we have arrived to

Lemma 4.2. *Each sequence of $\{\zeta_n\}_{n=1}^{\infty}$ in the set $\mathcal{Z}_{\text{ad}}$ of admissible distribution functions contains a subsequence $\{\zeta_{n'}\}$, and there exists a function ζ^* in $\mathcal{Z}_{\text{ad}}$ such that for any $q_n \in C(\Omega \times I)$ converging uniformly to a $q^* \in C(\Omega \times I)$ the sequence of $e_n = \mathcal{G}_{\zeta_n}[q_n]$ converges uniformly to $e^* = \mathcal{G}_{\zeta^*}[q^*]$.*

Thus the set for admissible data for our problem will be

$$\mathcal{U}_{\text{ad}} = \mathcal{C}_{\text{ad}} \times \mathcal{Z}_{\text{ad}}.$$

4.4 Continuity of the mapping $a \mapsto u_a$.

It remains to prove that the convergence of data $a_n \to a^*$ implies that of solutions $u_{a_n} \to u_{a^*}$, where $u_a = (u, e, q)$. Let us denote by $u_n = (u^n, e^n, q^n)$ the solution of problem P_{a_n} with the data $a_n = (c^n, \zeta^n)$ and by $u^* = (u^*, e^*, q^*)$ the solution of problem P_{a^*} with the data $a^* = (c^*, \zeta^*)$, i.e.,

$$c^n u_t^n = q_x^n + f, \quad e^n = u_x^n, \quad e^n = \mathcal{G}_{\zeta^n}[q^n],$$
$$c^* u_t^* = q_x^* + f, \quad e^* = u_x^*, \quad e^* = \mathcal{G}_{\zeta^*}[q^*].$$

Comparing the first pair of equations and splitting the left-hand side, we obtain

$$c^n u_t^n - c^* u_t^* \equiv (c^n - c^*) u_t^n + c^*(u_t^n - u_t^*) = q_x^n - q_x^*.$$

Multiplying the equation with $(u_t^n - u_t^*)$, we obtain

$$(c^n - c^*) u_t^n (u_t^n - u_t^*) + c^*(u_t^n - u_t^*)^2 = (q_x^n - q_x^*)(u_t^n - u_t^*). \tag{4.4}$$

The second pair of equalities yields $e^n - e^* = u_x^n - u_x^*$. Differentiating it by t and multiplying it by $(q^n - q^*)$, we obtain

$$(e_t^n - e_t^*)(q^n - q^*) = (u_{xt}^n - u_{xt}^*)(q^n - q^*). \tag{4.5}$$

Summing up (4.4) and (4.5) and using formula $f_x g + f g_x = (fg)_x$, we obtain

$$(c^n - c^*) u_t^n (u_t^n - u_t^*) + c^*(u_t^n - u_t^*)^2 + (e_t^n - e_t^*)(q^n - q^*) = ((u_t^n - u_t^*)(q^n - q^*))_x.$$

We integrate the equation over G. Formula $\int_0^{\ell} f_x \,\mathrm{d}x = f(\ell) - f(0)$ and the zero boundary conditions for $x = 0$ and $x = \ell$ give zero in the right-hand side. Finally, integrating the equality over $(0,t)$, we obtain

$$\int_{\Omega\times(0,t)} \big[(c^n - c^*) u_t^n (u_t^n - u_t^*) + c^*(u_t^n - u_t^*)^2 + (e_t^n - e_t^*)(q^n - q^*)\big] \,\mathrm{d}x \,\mathrm{d}\tau = 0. \tag{4.6}$$

Splitting $e^n - e^* = \mathcal{G}_{\zeta^n}[q^n] - \mathcal{G}_{\zeta^n}[q^*] + \mathcal{G}_{\zeta^n}[q^*] - \mathcal{G}_{\zeta^*}[q^*]$ and using the inequality

$$\big(\mathcal{G}_{\zeta^n}[q^n] - \mathcal{G}_{\zeta_n}[q^*]\big)_t (q^n - q^*) \geq \frac{\beta}{2} \frac{\mathrm{d}}{\mathrm{d}t} [q^n - q^*]^2$$

(see Proposition 2.9), for sufficiently large R, we obtain

$$\begin{aligned}\int_{\Omega\times(0,t)} & c^*(u_t^n - u_t^*)^2 \,\mathrm{d}x \,\mathrm{d}\tau + \frac{\beta}{2} \int_{\Omega} (q^n(t) - q^*(t))^2 \,\mathrm{d}x \\ &\leq \int_{\Omega\times(0,t)} \Big[\big|(c^n - c^*) u_t^n (u_t^n - u_t^*)\big| + \int_0^R |\zeta^n - \zeta^*| \,\mathrm{d}r\Big] |q^n - q^*| \,\mathrm{d}x \,\mathrm{d}\tau. \end{aligned} \tag{4.7}$$

Since by the compactness of $\mathcal{U}_{\rm ad}$ we have $c^n \rightrightarrows c^*$ and $\zeta^n - \zeta^* \to 0$, the right-hand side tends to zero which proves the convergence of the solutions. In this way even the uniqueness of the solution in Theorem 3.2 is proved. □

4.5 Criterion-functional

In mechanics, the dangerous situations are extremes of the deformation or stress. In our heat conduction problem or the diffusion problem the extremes of the temperature or concentration can be critical, the extremes of the temperature or concentration gradient can be critical, as well.

From the mathematical point of view, a continuous function on a compact (i.e., closed bounded) set attains its maximum. If the function is integrable only, say in the L^p space, then its values are determined except for measure zero sets and thus the value of the function at a point x has no sense. Instead of it we have to take integral mean of a small part G of Ω

$$\Phi(a,u_a) = \frac{1}{|G|}\int_G u_a(x)\,\mathrm{d}x,$$

where G is the small part of Ω, where the failure of the construction may be expected.

Following the existence Theorem 3.2, the solutions u, q are the continuous functions on $C(\Omega \times I)$. Thus for any point $x_0 \in \Omega$ and any time t_0 the criterion-functional for the data $a \in \mathcal{U}_{\rm ad}$ and the corresponding solution $u_a = (u, q, e)$ can be defined as the value of u or q for (x_0, t_0) or its maximum at the point x_0 or time t_0, for example,

$$\begin{aligned}
&\Phi_1(a,u_a) = u(x_0,t_0), && \Phi_2(a,u_a) = q(x_0,t_0),\\
&\Phi_3(a,u_a) = \max_{x\in G} u(x,t_0), && \Phi_4(a,u_a) = \max_{t\in J} q(x_0,t),\\
&\Phi_5(a,u_a) = \max_{(x,t)\in G\times J} u(x,t), && \Phi_6(a,u_a) = \max_{(x,t)\in G\times J} q(x,t),
\end{aligned}$$

where G is a closed subset of Ω and J is a closed subinterval of I.

Following Theorem 3.2, the unknown $e = u_x \in L^2(\Omega, C(I))$. Then the criterion-functional may be be the integral mean of u_x e.g.

$$\Phi_7(a,u_a) = \frac{1}{|G|}\int_G |u_x(x,t_0)|\,\mathrm{d}x, \quad \Phi_8(a,u_a) = \frac{1}{|G|}\int_G \max_{t\in J}|u_x(x,t)|\,\mathrm{d}x,$$

where G is an open subset of Ω and J a closed subinterval of I.

Finally, following Theorem 3.2, the gradients u_t, e_t, q_t, q_x are in $L^\infty(I, L^2(\Omega)$. Thus the criterion-functional may be the integral mean over a closed $G \subset \Omega$ and J a subinterval of I, e.g.,

$$\Phi_9(a,u_a) = \frac{1}{|G|\cdot|J|}\int_{G\times J} |v(x,t)|\,\mathrm{d}t\,\mathrm{d}x,$$

where v stands for any of u_t, e_t, q_t, q_x.

4.6 Main result

Since the set of admissible data $\mathcal{U}_{\rm ad}$ is compact with respect to the corresponding norms, the mapping $a \mapsto u_a$ is continuous and also each functional of type $\Phi_1, \dots, \Phi_9$ is continuous, we have arrived at the main result:

Theorem 4.3. *Let Hypothesis* 3.1 *be satisfied, the set of admissible data* $\mathcal{U}_{\rm ad} = \mathcal{C}_{\rm ad} \times \mathcal{Z}_{\rm ad}$ *be defined by* (4.2), (4.3).

Then the worst scenario problem for the problem (3.1)–(3.3) *with any criterion-functionals of type* $\Phi_1, \dots, \Phi_9$ *or their combination admits the solution.*

5 Concluding remarks

For the sake of simplicity, we have assumed certain zero boundary conditions for $u(0,t) = 0$, and $q(\ell,t) = 0$, certain initial condition and certain right-hand side $f(x,t)$. The result can be extended even to uncertain both boundary conditions $u(0,t) = u_0(t)$, or uncertain $q(0,t) = q_0(t)$ and similar uncertain data on the right end $x = \ell$. Also, the initial condition $u^0(t)$ and right-hand side $f(x,t)$ may be uncertain. One should define the corresponding convenient compact sets for these uncertain functions, and their difference will appear in the right-hand side of (4.7).

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GLOBAL ATTRACTOR OF A WEAKLY NONLINEAR PARABOLIC SYSTEM WITH DISCONTINUOUS TRAJECTORIES

Abstract. In the paper, we prove the existence of a global attractor for an impulsive dynamical system, which is generated by a weakly nonlinear parabolic system, when its trajectories have jumps at moments of intersection with certain surface of the phase space.

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რეზიუმე. ნაშრომში დამტკიცებულია იმპულსური დინამიკური სისტემის გლობალური ატრაქტორის არსებობა, რომელიც წარმოქმნილია სუსტად არაწრფივი პარაბოლური სისტემით, როდესაც მის ტრაექტორიებს გააჩნიათ ნახტომები ფაზური სივრცის გარკვეულ ზედაპირთან გადაკვეთის მომენტებში.

1 Introduction

One of the possible ways to describe the qualitative behaviour of evolutionary processes with instant impulsive perturbations is the theory of impulsive differential equations [14, 20, 21]. Autonomous equations, which trajectories have impulsive perturbations at moments of intersection with certain subset of the phase space, form an important subclass of impulsive differential equations and called impulsive (or discontinuous) dynamical systems (DS). Some aspects of the long-time behavior of such finite-dimensional systems have been studied in [1, 6, 13, 14, 17, 19–21]. For infinite-dimensional dissipative systems one of the important qualitative characteristics of their behavior is the concept of a global attractor [23]. In [8, 11, 18, 22, 24], the theory of global attractors has been investigated in the case, where the moments of impulsive perturbations are fixed.

First results of applying this theory to impulsive DS with a finite number of discontinuities along the trajectories arose in [4]. In further works [2, 3], using a priori estimates on the behavior of the trajectories in the neighborhood of impulsive set, the authors managed to transfer the basic constructions of the classical DS theory to an impulsive case and obtain abstract theorems on the existence and properties of the global attractor. However, the question of verifying the imposed conditions on the impulsive DS for special infinite-dimensional nonlinear evolution problems remains open. In [12], the authors offered another approach, based on the concept of a uniform attractor and applied it to scalar impulsive parabolic equations with a small nonlinearity. In [7], this approach was generalized to the multi-valued impulsive DS, generated by the solutions of evolution inclusions.

In this paper, using the methods of [12], we investigate the existence of a global attractor of impulsive DS generated by the two-dimensional parabolic system with a small nonlinearity, which solutions have impulsive perturbation at moments of intersection with a certain subset of the phase space. Moreover, the conditions on the parameters of the problem do not guarantee the uniqueness of the solution of the Cauchy problem, which requires to use the theory of global attractors of multi-valued DS [10, 15, 16].

2 Formulation of the problem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain. For the unknown functions $u(t,x), v(t,x)$ in $(0,+\infty)\times\Omega$, we consider the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u + \varepsilon f_1(u,v), \\ \dfrac{\partial v}{\partial t} = a\Delta v + 2b\Delta u + \varepsilon f_2(u,v), \\ u\big|_{\partial\Omega} = v\big|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $\varepsilon > 0$ is a small parameter,

$$a > 0, \quad |b| < a, \tag{2.2}$$

continuous linear functions $f_i : \mathbb{R}^2 \longmapsto \mathbb{R},\ i = 1, 2$, satisfy the following condition:

$$\exists\, C > 0 \quad \forall\, u, v \in \mathbb{R} \quad |f_1(u,v)| + |f_2(u,v)| \leq C. \tag{2.3}$$

The space $H = L^2(\Omega) \times L^2(\Omega)$ with the norm $\|z\|_H = \sqrt{\|u\|^2 + \|v\|^2}$ is the phase space of problem (2.1). Here and in the sequel, $\|\cdot\|$ and $(\,\cdot\,,\,\cdot\,)$ are the norm and the scalar product in $L^2(\Omega)$, respectively, $\{\lambda_i\}_{i=1}^{\infty} \subset (0,+\infty)$, $\{\psi_i\}_{i=1}^{\infty} \subset H_0^1(\Omega)$ are solutions of the spectral problem $\Delta\psi = -\lambda\psi$, $\psi \in H_0^1(\Omega)$.

Under conditions (2.2), (2.3), for every $\varepsilon > 0$, $z_0 \in H$, there exists at least one solution $z = \begin{pmatrix} u \\ v \end{pmatrix} \in C([0,+\infty); H)$ of problem (2.1), where $z(0) = z_0$ [5].

For the fixed $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\mu > 0$, we consider the following impulsive problem:

When the phase point $z(t)$ meets *the impulsive set*

$$M = \Big\{ z = \begin{pmatrix} u \\ v \end{pmatrix} \in H \;\Big|\; |(u,\psi_1)| \leq \gamma,\ \alpha(u,\psi_1) + \beta(v,\psi_1) = 1 \Big\}, \tag{2.4}$$

then *the impulsive map* $I : M \longmapsto M'$ maps it into a new position $Iz \in M'$, where

$$M' = \left\{ z = \begin{pmatrix} u \\ v \end{pmatrix} \in H \mid \ |(u,\psi_1)| \le \gamma,\ \alpha(u,\psi_1) + \beta(v,\psi_1) = 1+\mu \right\}. \tag{2.5}$$

We choose the set M due to the results from [12], where for a scalar parabolic equation, the impulsive set $M = \{u \in L^2(\Omega) \mid \ (u,\psi_1) = 1\}$.

The main purpose of this work is to establish the existence and investigate the properties of the global attractor of impulsive DS, generated by the solution of problem (2.1)–(2.5), for some class of compact-valued impulsive maps I, which have the following form:

$$\text{for } \ z = \sum_{i=1}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \in M$$

$$Iz \subseteq I_0 z = \left\{ \begin{pmatrix} c_1' \\ d_1' \end{pmatrix} \psi_1 + \sum_{i=2}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \mid \ |c_1'| \le \gamma,\ \alpha c_1' + \beta d_1' = 1+\mu \right\}. \tag{2.6}$$

In a particular case, the single-valued map $I : M \longmapsto M'$

$$I\left(\sum_{i=1}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \right) = \begin{pmatrix} c_1 \\ d_1 + \dfrac{\mu}{\beta} \end{pmatrix} \psi_1 + \sum_{i=2}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i,$$

and the compact-valued map $I \equiv I_0$ are the partial cases of formula (2.6).

The main result of the work is to prove the fact that for an arbitrary compact-valued map I, which satisfies (2.6), and for a sufficiently small $\varepsilon > 0$, in the phase space H the impulsive problem (2.1), (2.4), (2.6) generates impulsive (multi-valued) DS $\widetilde{G}_\varepsilon$, which has the global attractor Θ_ε and

$$\operatorname{dist}(\Theta_\varepsilon, \Theta) \longrightarrow 0, \ \ \varepsilon \to 0, \tag{2.7}$$

where

$$\Theta = \bigcup_{t \in [0,\tau],\, |c_1| \le \gamma} \left\{ \begin{pmatrix} c_1 \\ \frac{1+\mu-\alpha c_1}{\beta} - 2bc_1 t \end{pmatrix} e^{-a\lambda_1 t} \psi_1 \mid \ (1+\mu - 2b\beta c_1 \tau) e^{-a\lambda_1 \tau} = 1 \right\} \cup \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

3 Construction of impulsive MDS

Let $P(H)$ ($\beta(H)$) be a set of all non-empty (non-empty bounded) subsets of H.

Definition 3.1 ([15]). A multi-valued map $G : \mathbb{R}_+ \times H \to P(H)$ is called a multi-valued dynamical system (MDS), if the following conditions are satisfied:

$$\forall\, x \in H \ \ G(0,x) = x \quad \text{and} \quad \forall\, t,s \ge 0 \ \ G(t+s,x) \subseteq G(t, G(s,x)).$$

MDS G is called strict if $\forall\, x \in H\ \forall\, t,s \ge 0\ G(t+s,x) = G(t,G(s,x))$.

Remark 3.1. If G is a single-valued map, then we obtain the definition of a semigroup. However, we do not impose any continuity conditions on it, which is important when we consider impulsive systems.

Definition 3.2 ([18]). A subset $\Theta \subset H$ is called a global attractor of MDS G, if

(1) Θ is a compact set;

(2) Θ is a uniformly attracting set, i.e.,

$$\forall\, B \in \beta(H) \ \ \operatorname{dist}(G(t,B), \Theta) \longrightarrow 0, \ \ t \to \infty;$$

(3) Θ is minimal among all closed uniformly attracting sets.

The next result follows from [18] and provides a criterion of the existence of a global attractor for dissipative MDS.

Lemma 3.1. *Assume that* MDS G *satisfies the dissipativity condition*

$$\exists\, B_0 \in \beta(H) \;\; \forall\, B \in \beta(H) \;\; \exists\, T = T(B) > 0 \;\; \forall\, t \geq T \;\; G(t, B) \subset B_0.$$

Then the following conditions are equivalent:

(1) MDS G *has global attractor* Θ*;*

(2) MDS G *is asymtotically compact, i.e.,* $\forall\, t_n \nearrow \infty \;\forall\, B \in \beta(H)$

$$\forall\, \xi_n \in G(t_n, B) \;\; \textit{the sequence } \{\xi_n\} \textit{ is precompact in } H.$$

Moreover,

$$\Theta = \omega(B_0) := \bigcap_{\tau>0} \overline{\bigcup_{t\geq\tau} G(t, B_0)}.$$

Now let us consider a special subclass of MDS called impulsive MDS. Impulsive MDS $\widetilde{G}$ consists of non-empty closed set $M \subset H$ (impulsive set), compact-valued map $I : M \to P(H)$ (impulsive map) and some set K of continuous maps $\varphi : [0, +\infty) \to H$, which satisfy the following assumptions:

(K1) $\forall\, x \in H \;\exists\, \varphi \in K$: $\varphi(0) = x$;

(K2) $\forall\, \varphi \in K \;\forall\, s \geq 0 \;\varphi(\,\cdot\, + s) \in K$.

We denote $K_x = \{\varphi \in K \mid \;\; \varphi(0) = x\}$.

Remark 3.2. If in assumption (K1), for every $x \in H$, there exists a unique $\varphi \in K$ such that $\varphi(0) = x$, then K_x consists of a single trajectory φ, and the equality $V(t, x) = \varphi(t)$ defines a classical semigroup $V : \mathbb{R}_+ \times H \longmapsto H$.

A phase point of impulsive MDS moves along the trajectories of K, and at the moment of meeting the set M, it immediately jumps onto a new position from the set IM. For the "well-posedness" of the impulsive problem we assume the following conditions [4]:

$$M \cap IM = \varnothing, \tag{3.1}$$

$$\forall\, x \in M \;\; \forall\, \varphi \in K_x \;\; \exists\, \tau = \tau(\varphi) > 0 \;\; \forall\, t \in (0, \tau) \;\; \varphi(t) \notin M. \tag{3.2}$$

We denote

$$\forall\, \varphi \in K \;\; M^+(\varphi) = \Big(\bigcup_{t>0} \varphi(t)\Big) \cap M.$$

If $M^+(x) \neq \varnothing$, then there exists a moment of time $s := s(\varphi) > 0$ such that $\forall\, t \in (0, s)$ $\varphi(t) \notin M$, $\varphi(s) \in M$. Therefore, we can define the following function $s : K \to (0, +\infty]$:

$$s(\varphi) = \begin{cases} s & \text{if } \; M^+(\varphi) \neq \varnothing, \\ +\infty & \text{if } \; M^+(\varphi) = \varnothing. \end{cases}$$

Let us construct impulsive trajectory $\widetilde{\varphi}$, which starts from the point $x_0 \in H$. Let $\varphi_0 \in K_{x_0}$.

If $M^+(\varphi_0) = \varnothing$, then define $\widetilde{\varphi}$ on $[0, +\infty)$ as

$$\widetilde{\varphi}(t) = \varphi_0(t) \;\; \forall\, t \geq 0.$$

If $M^+(\varphi_0) \neq \varnothing$, then for $s_0 = s(\varphi_0) > 0$, $x_1 = \varphi_0(s_0) \in M$ and $x_1^+ \in Ix_1$ define $\widetilde{\varphi}$ on $[0, s_0]$ as

$$\widetilde{\varphi}(t) = \begin{cases} \varphi_0(t), & t \in [0, s_0), \\ x_1^+, & t = s_0. \end{cases}$$

Let $\varphi_1 \in K_{x_1^+}$. If $M^+(\varphi_1) = \varnothing$, then define $\widetilde{\varphi}$ on $[0,+\infty)$ as

$$\widetilde{\varphi}(t) = \varphi_1(t - s_0) \;\; \forall\, t \geq s_0.$$

If $M^+(\varphi_1) \neq \varnothing$, then for $s_1 = s(\varphi_1) > 0$, $x_2 = \varphi_1(s_1) \in M$ and $x_2^+ \in Ix_2$, we define $\widetilde{\varphi}$ on $[s_0, s_0+s_1]$ as

$$\widetilde{\varphi}(t) = \begin{cases} \varphi_1(t - s_0), & t \in [s_0, s_0 + s_1), \\ x_2^+, & t = s_0 + s_1. \end{cases}$$

Continuing this procedure, we obtain the impulsive trajectory $\widetilde{\varphi}$ with a finite or infinite number of impulsive points $\{x_n^+\}_{n\geq 1} \subset IM$, corresponding moments of time $\{s_n\}_{n\geq 0} \subset (0,\infty)$ and the functions $\{\varphi_n\}_{n\geq 0} \subset K$.

Let

$$t_0 = 0, \quad t_{n+1} := \sum_{k=0}^{n} s_k.$$

If $\widetilde{\varphi}$ has an infinite number of jumps, then it is defined by the formula

$$\forall\, n \geq 0 \;\; \forall\, t \geq 0 \;\; \widetilde{\varphi}(t) = \begin{cases} \varphi_n(t - t_n), & t \in [t_n, t_{n+1}), \\ x_{n+1}^+, & t = t_{n+1}. \end{cases} \tag{3.3}$$

By $\widetilde{K_x}$ we denote the set of all impulsive trajectories, which start from the point x.

Let us assume that

$$\forall\, x \in H \;\; \text{every } \widetilde{\varphi} \in \widetilde{K}_x \text{ is defined on } [0,+\infty). \tag{3.4}$$

Remark 3.3. Due to the construction, every impulsive trajectory is right continuous. Moreover, from (3.1) and (3.3) we obtain: $\forall\, x \in H \;\forall\, \widetilde{\varphi} \in \widetilde{K}_x$, $\forall\, t > 0 \;\widetilde{\varphi}(t) \notin M$.

Lemma 3.2 ([7]). *Assume that the conditions* (K1), (K2), (3.1), (3.2), (3.4) *are satisfied. Then the formula* $\widetilde{G}(t,x) = \{\widetilde{\varphi}(t) \mid \widetilde{\varphi} \in \widetilde{K}_x\}$ *defines MDS* $\widetilde{G} : \mathbb{R}_+ \times H \to P(H)$, *which we call impulsive MDS.*

4 Existence of global attractor of impulsive MDS, generated by problem (2.1), (2.4), (2.6)

Problem (2.1) generates a family of continuous maps:

$$K^\varepsilon = \big\{ z : [0,+\infty) \to H \mid \; z \text{ is a solution of } (2.1) \big\},$$

which due to the autonomy of the problem (2.1) satisfies the conditions (K1), (K2).

Lemma 4.1. *Under conditions* (2.2), (2.3) *and the inequality*

$$2\beta\gamma \leq 1 \tag{4.1}$$

for a sufficiently small ε, *problem* (2.1), (2.4), (2.6) *generates impulsive MDS, and every impulsive trajectory, which starts from the set* M', *has an infinite number of impulsive perturbations.*

Remark 4.1. Here and in the sequel, under the expression *for a sufficiently small* ε we mean that some property holds for all $\varepsilon \in (0, \varepsilon_0)$, where ε_0 depends only of the parameters of problem (2.1).

Proof of Lemma 4.1. Let us verify conditions (3.1), (3.2) and (3.4). Condition (3.1) follows from the definition of the sets M and M'. Due to conditions (2.2), (2.3) and Poincaré inequality, there exists $\delta > 0$ such that for every solution z of the problem (2.1) and for almost all $t > 0$ we get the inequality

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|_H^2 + \delta\|z(t)\|_H^2 \leq \varepsilon\sqrt{2}\,C\|z(t)\|_H. \tag{4.2}$$

Then, for a sufficiently small ε, we obtain

$$\forall z \in K^\varepsilon \ \ \forall t \geq 0 \ \ \|z(t)\|_H^2 \leq \|z(0)\|_H^2 e^{-\delta t} + 1. \tag{4.3}$$

Moreover, for every $z = \begin{pmatrix} u \\ v \end{pmatrix} \in K^\varepsilon$ and for every $i \geq 1$ we get the following equalities:

$$(u(t), \psi_i) = (u(0), \psi_i) e^{-a\lambda_i t} + \varepsilon \int_0^t e^{-a\lambda_i (t-s)} (f_1(u(s), v(s)), \psi_i)\, ds, \tag{4.4}$$

$$\begin{aligned} (v(t), \psi_i) = \big((v(0), \psi_i) - 2b\lambda_i (u(0), \psi_i) t\big) e^{-a\lambda_i t} + \varepsilon \int_0^t e^{-a\lambda_i (t-s)} (f_2(u(s), v(s)), \psi_i)\, ds \\ - \varepsilon 2b\lambda_i \int_0^t (t-s) e^{-a\lambda_i (t-s)} (f_1(u(s), v(s)), \psi_i)\, ds. \end{aligned} \tag{4.5}$$

Further, for the sake of simplicity, we denote $\psi := \psi_1$, $\lambda := \lambda_1$ and for $z \in K^\varepsilon$ consider the function

$$g_\varepsilon(t) = \alpha(u(t), \psi) + \beta(v(t), \psi).$$

From (4.4), (4.5), for $z(0) \in M$, we deduce

$$g_\varepsilon(t) = e^{-a\lambda t}(1 - 2\beta b\lambda t(u(0), \psi)) + \varepsilon F_\varepsilon(t),$$

where the function $F_\varepsilon \in C^1([0, \infty))$, $F_\varepsilon(0) = 0$, depends on $z \in K^\varepsilon$, however

$$\exists C_1 > 0 \ \ \forall \varepsilon \in (0,1) \ \ \sup_{t \geq 0} \big(|F_\varepsilon(t)| + |F_\varepsilon'(t)|\big) \leq C_1. \tag{4.6}$$

From (2.2) and (2.3) we get

$$g_\varepsilon'(0) = -a\lambda - 2\beta b\lambda(u(0), \psi) + \varepsilon F_\varepsilon'(0).$$

Since $|(u(0), \psi)| \leq \gamma$, from (4.1) and (4.6) for a sufficiently small ε there exists $\tau = \tau(z(0), \varepsilon) > 0$ such that $\forall t \in (0, \tau)$ $g_\varepsilon(t) < 1$. Thus we get property (3.2).

Let us prove (3.4). Due to estimation (4.3), condition (3.4) is satisfied if z do not intersect the set M. Thus, let us take arbitrarily $z \in K^\varepsilon$ from $z(0) = z_0 \in M'$ and consider the function $g_\varepsilon(t)$, which has the form

$$g_\varepsilon(t) = e^{-a\lambda t}(1 + \mu - 2\beta b\lambda t(u_0, \psi)) + \varepsilon F_\varepsilon(t).$$

Since $g_\varepsilon(0) = 1 + \mu$, $\limsup_{t \to \infty} g_\varepsilon(t) \leq \varepsilon C_1$, for a sufficient small $\varepsilon > 0$, there exists $s_\varepsilon > 0$ such that

$$\forall t \in (0, s_\varepsilon) \ \ g_\varepsilon(t) > 1, \ \ g_\varepsilon(s_\varepsilon) = 1. \tag{4.7}$$

Let us show that for a sufficiently small $\varepsilon > 0$ the inequality

$$|(u(s_\varepsilon), \psi)| \leq \gamma \tag{4.8}$$

is fulfilled, i.e., $z(s_\varepsilon) \in M$. Indeed, for a sufficiently small ε, from (4.7) we have the next inequality

$$\Big(1 + \frac{\mu}{2}\Big) e^{a\lambda s_\varepsilon} \geq 1 + \mu - 2\beta b\lambda s_\varepsilon (u_0, \psi). \tag{4.9}$$

As $|(u_0, \psi)| \leq \gamma$, from (4.9) we obtain

$$s_\varepsilon \geq \widetilde{s}, \tag{4.10}$$

where $\widetilde{s} > 0$ does not depend on ε, z_0 and is the root of the equation

$$\Big(1 + \frac{\mu}{2}\Big) e^{a\lambda \widetilde{s}} = 1 + \mu - 2\beta |b| \lambda \widetilde{s} \gamma.$$

Then from (4.4) we deduce the estimation

$$|(u(s_\varepsilon),\psi)| \le \gamma e^{-a\lambda\widetilde{s}} + \varepsilon C_1,$$

from which for a sufficiently small $\varepsilon > 0$ we obtain (4.8). Thus, every impulsive trajectory, which starts from the set M', has an infinite number of impulsive points and, due to estimation (4.10), we have (3.4). □

Therefore, for a sufficiently small ε, the impulsive multi-valued dynamical system $\widetilde{G}_\varepsilon : R_+ \times H \to P(H)$,

$$\forall\, t \ge 0 \ \ \forall\, z_0 \in H \ \ \widetilde{G}_\varepsilon(t,z_0) = \big\{z(t) \mid \ \ z(\,\cdot\,) \in \widetilde{K}^\varepsilon_{z_0}\big\}, \tag{4.11}$$

is correctly defined, where $\widetilde{K}^\varepsilon_{z_0}$ is the set of all impulsive trajectories of problem (2.1), (2.4), (2.6), which start from the point z_0.

The main result of this paper is the following

Theorem. *For a sufficiently small* $\varepsilon > 0$, *under conditions* (2.2), (2.3), (4.1), *the impulsive* MDS (4.11) *has a global attractor* Θ_ε. *Moreover, the limit equality* (2.7) *is fulfilled.*

Proof. Let us verify the dissipativity property. If for $\|z_0\| \le R$ the impulsive trajectory $z \in \widetilde{K}^\varepsilon_{z_0}$ does not have impulsive points, then from (4.3) it follows that

$$\|z(t)\| \le \sqrt{2} \ \ \forall\, t \ge T = \frac{1}{\delta}\,\ln R^2.$$

Otherwise, for a sufficiently small ε, using the function g_ε, for the moment $s_\varepsilon = s(z) > 0$, we obtain the inequality

$$e^{-a\lambda s_\varepsilon}\big(\alpha(u_0,\psi) + \beta(v_0,\psi) - 2\beta b\lambda s_\varepsilon(u_0,\psi)\big) \ge \frac{1}{2}\,,$$

thus, we deduce $s_\varepsilon \le s(R)$, where $s(R) > 0$ is a solution of the equation

$$\frac{1}{2}\,e^{a\lambda s_\varepsilon} = \sqrt{\alpha^2+\beta^2}\,R + 2\beta|b|\lambda s_\varepsilon R.$$

After this, the phase point jumps into the point $z_1^+ = z(s_\varepsilon) \in I(z(s_\varepsilon - 0))$. Due to the form of the impulsive map (2.6), we deduce the estimation

$$\forall\, z \in H\ \forall\, z^+ \in I(z) \ \ \|z^+\|_H^2 \le \kappa^2 + \|z\|_H^2, \tag{4.12}$$

where $\kappa^2 := \gamma^2 + (\frac{1+\mu+\alpha\gamma}{\beta})^2$. In particular,

$$\|z(s_\varepsilon)\|_H^2 \le \kappa^2 + R^2 + 1.$$

Therefore, it suffices to prove the dissipativity condition only for those impulsive trajectories, which start from the set IM, i.e., for a sufficiently small ε, it suffices to prove that

$$\begin{gathered}\exists\, R_0 > 0 \ \ \forall\, R > 0 \ \ \exists\, T = T(R) > 0 \ \ \forall\, z_0 \in IM, \ \ \|z_0\|_H \le R,\\ \forall\, z \in \widetilde{K}^\varepsilon_{z_0} \ \ \forall\, t \ge T \ \ \|z(t)\|_H \le R_0.\end{gathered} \tag{4.13}$$

But if $\{s^i_\varepsilon\}_{i=0}^\infty$ are the moments of the impulsive perturbation for $z \in \widetilde{K}^\varepsilon_{z_0}$, then from (4.3), (4.12) and inequality (4.10) we find that for $k \ge 0$,

$$\Big\| z\Big(\sum_{i=0}^k s^i_\varepsilon\Big)\Big\|_H^2 \le e^{-\delta(k+1)\overline{s}}R^2 + \frac{\kappa^2}{1-e^{-\delta\overline{s}}}\,. \tag{4.14}$$

Thus, from the last inequality and formula (4.3) follows (4.13), where $R_0 = 2 + \frac{\kappa^2}{1-e^{-\delta\overline{s}}}$.

Let us prove that $\widetilde{G}_\varepsilon$ is asymptotically compact. Towards this end, we fix an arbitrary solution $z = \begin{pmatrix} u \\ v \end{pmatrix}$ of problem (2.1). Considering every equation in (2.1) as a linear equation with right-hand side $h_1(t) = \varepsilon f_1(u(t), v(t))$, $h_2(t) = 2b\Delta u(t) + \varepsilon f_2(u(t), v(t))$, from the regularity lemma [23] we deduce that there exists a constant $C_2 > 0$, which depends only on the parameters of problem (2.1) and does not depend on ε, such that for almost all $t > 0$,

$$\frac{d}{dt}\,\|u(t)\|^2_{H^1_0} + a\|\Delta u(t)\|^2 \le C_2, \tag{4.15}$$

$$\frac{d}{dt}\,\|u(t)\|^2 + a\|u(t)\|^2_{H^1_0} \le C_2, \tag{4.16}$$

$$\frac{d}{dt}\,\|v(t)\|^2_{H^1_0} + a\|\Delta v(t)\|^2 \le \frac{4b^2}{a}\,\|\Delta u(t)\|^2 + C_2, \tag{4.17}$$

$$\frac{d}{dt}\,\|v(t)\|^2 + a\|v(t)\|^2_{H^1_0} \le \frac{4b^2}{a}\,\|u(t)\|^2_{H^1_0} + C_2. \tag{4.18}$$

From (4.15), (4.16) and the Uniform Gronwall Lemma [23] we obtain

$$\forall\, t > 0 \;\; \|u(t)\|^2_{H^1_0} \le C_2 t + \frac{\|u(0)\|^2}{at} + \frac{2C_2}{a}\,. \tag{4.19}$$

Then from (4.17)–(4.19) and the Uniform Gronwall Lemma we have

$$\forall\, t > 0\; \forall\, r \in (0, t)\; \|v(t)\|^2_{H^1_0} \le \Big(\frac{4b^2}{a^2} + 1\Big)\Big(\frac{\|u(0)\|^2 + \|v(0)\|^2}{ar} + \frac{2C_2}{a}\Big) + \frac{C_2 r + \|u(t-r)\|^2_{H^1_0}}{a}\,. \tag{4.20}$$

Assume that $r = \frac{t}{2}$ and from (4.19), (4.20) we get the following estimation:

$$\begin{aligned}\forall\, t > 0 \;\; \|v(t)\|^2_{H^1_0} \le \Big(\frac{4b^2}{a^2} + 1\Big)&\Big(\frac{2(\|u(0)\|^2 + \|v(0)\|^2)}{at} + \frac{2C_2}{a}\Big)\\ &+ \frac{C_2 t}{2a}\Big(1 + \frac{1}{2a}\Big) + \frac{2\|u(0)\|^2}{ta^2} + \frac{2C_2}{a^2}\,.\end{aligned} \tag{4.21}$$

Now, let $z_0^{(n)} = \sum\limits_{i=1}^{\infty} \begin{pmatrix} c_i^{(n)} \\ d_i^{(n)} \end{pmatrix} \cdot \psi_i$, $\|z_0^{(n)}\|_H \le R$, be an arbitrary bounded sequence of initial data, $\xi_n \in \widetilde{G}_\varepsilon(t_n, z_0^{(n)})$, $t_n \nearrow +\infty$. Then $\xi_n = z_n(t_n)$, where $z_n \in \widetilde{K}^\varepsilon_{z_0^{(n)}}$. If z_n does not have impulsive points, then for the function $y_n(t) = z_n(t + t_n - 1)$, $t \ge 0$ we obtain

$$y_n \in \widetilde{K}^\varepsilon_{z_n(t_n - 1)}, \;\; \xi_n = z_n(t_n) = y_n(1).$$

From (4.3) we find that $\|z_n(t_n - 1)\| \le \sqrt{2}\; \forall\, n \ge N(R)$. Therefore, from estimates (4.19), (4.21), the sequence $\{y_n(1) = \xi_n\}$ is bounded in $H^1_0(\Omega) \times H^1_0(\Omega)$ and, hence, is precompact in H.

Otherwise, without loss of generality, from the previous arguments we can assume that $z_0^{(n)} \in IM$, $\|z_0^{(n)}\|_H \le R$. Let $\{T_{i+1}^{(n)} = \sum\limits_{k=0}^{i} s_k^{(n)}\}_{i=0}^{\infty}$ be the moments of impulsive perturbation for $z_n(\,\cdot\,) = \begin{pmatrix} u_n(\,\cdot\,) \\ v_n(\,\cdot\,) \end{pmatrix}$, $\{\eta_i^{(n)+} = z_n(T_i^{(n)})\}_{i=1}^{\infty} \subset IM$ be the corresponding impulsive points. Let us prove the precompactness of the sequence $\{\eta_i^{(n)+}\}$. From the dissipativity condition (4.13), the estimation

$$\overline{s} \le s_k^{(n)} \le \widehat{s}, \tag{4.22}$$

and the estimates (4.19), (4.21), we get the existence of the constant $C(R)$, independent of ε, such that

$$\forall\, i \ge 1 \;\; \forall\, n \ge 1 \;\; \big\|u_n(T_i^{(n)} - 0)\big\|^2_{H^1_0} + \big\|v_n(T_i^{(n)} - 0)\big\|^2_{H^1_0} \le C(R). \tag{4.23}$$

Then from (2.6) and (4.23) for all $i \geq 1$, $n \geq 1$ we deduce the estimation

$$\left\|u_n(T_i^{(n)})\right\|_{H_0^1}^2 + \left\|v_n(T_i^{(n)})\right\|_{H_0^1}^2 \leq C(R) + 2\lambda\gamma^2. \tag{4.24}$$

Therefore, due to (4.24) and the compactness of the embedding $H_0^1(\Omega) \subset L^2(\Omega)$, there follows the required precompactness of the set $\{\eta_i^{(n)+} \mid i \geq 1,\ n \geq 1\}$ in H. Then for the sequence $\xi_n \in \widetilde{G}_\varepsilon(t_n, z_0^{(n)})$, for every $n \geq 1$, there exists a number $i = i(n)$, $i(n) \to \infty$, $n \to \infty$, such that $t_n \in [T_{i(n)}^{(n)}, T_{i(n)+1}^{(n)})$. Thus, from the inclusion

$$\xi_n = z_n(t_n) \in \widetilde{G}_\varepsilon\big(t_n - T_{i(n)}^{(n)}, \eta_{i(n)}^{(n)+}\big) \tag{4.25}$$

it follows that $\xi_n = y_n(\tau_n)$, where $\tau_n := t_n - T_{i(n)}^{(n)}$, $y_n \in K^\varepsilon$ is a sequence of solutions of the (non-perturbed) problem (2.1), where $y_n(0) = \eta_{i(n)}^{(n)+}$. Since from the previous arguments on some subsequence we have $\eta_{i(n)}^{(n)+} \to \eta$ in H, and from the inclusion $\tau_n \in [0, \widehat{s}]$ on some subsequence we have $\tau_n \to \tau \in [0, \widehat{s}]$, from the regularity results [9] of the solutions of the problem (2.1) we deduce the following result:

$$y_n(\tau_n) \longrightarrow y(\tau) \text{ in } H, \text{ where } y \in K^\varepsilon,\ y(0) = \eta. \tag{4.26}$$

Thus, the sequence $\{\xi_n\}$ is precompact in H, and from Lemma 3.1 we deduce the existence of the global attractor

$$\Theta_\varepsilon = \bigcap_{s>0} \overline{\bigcup_{t \geq s} \widetilde{G}_\varepsilon(t, B_0)}, \tag{4.27}$$

where the dissipative set B_0 is defined from (4.14) and does not depend on ε.

Let us prove convergence (2.7). It suffices to show that for $\varepsilon_k \to 0$, $\xi^{(k)} \in \Theta_{\varepsilon_k}$, on the subsequence

$$\xi^{(k)} \longrightarrow \xi \in \Theta \ \text{ in } \ H, \quad k \to \infty.$$

From (4.27), there exist the sequences $\{t_k \nearrow \infty\}$, $\{z_k^0\} \subset B_0$, $z_k \in \widetilde{K}_{z_k^0}^{\varepsilon_k}$, such that $\forall\, k \geq 1$ $\|\xi^{(k)} - z_k(t_k)\| \leq 1/k$. If z_k do not have impulsive perturbations, then using (4.2) we obtain the estimation

$$\forall\, t \geq 0 \ \ \|z_k(t)\|_H^2 \leq \|z_k^0\|_H^2 e^{-\delta t} + \frac{2\varepsilon_k^2 C^2}{\delta^2}\,,$$

from which it follows that $\xi^{(k)} \to 0$ in H.

Otherwise, if z_k have impulsive perturbations, then under conditions (4.25) for $\xi_k = z_k(t_k)$, we obtain the equality

$$\xi_k = y_k(\tau_k), \quad y_k \in K_{\eta_k^+}^{\varepsilon_k},$$

whence, using the notation from the previous part of the proving, it follows that

$$\tau_k := t_k - T_{i(k)}^{(k)} \longrightarrow \tau, \ \ \eta_k^+ := \eta_{i(k)}^{(k)+} \longrightarrow \eta, \ \ i(k) \to \infty, \ \ k \to \infty.$$

Since $\tau_k \in [0, s_{i(k)}^{(k)}]$ and the point $s_k := s_{i(k)}^{(k)}$ satisfies inequality (4.22) and is a solution of the equation

$$e^{-a\lambda s_k}(1 + \mu - 2\beta b\lambda s_k(u_0^k, \psi)) + \varepsilon_k F_{\varepsilon_k}(s_k) = 1,$$

where u_0^k, v_0^k are the components of the vector $\eta_k^+ \in IM$, for $k \to \infty$ we obtain that on the subsequence $s_k \to s$, where $\tau \in [0, s]$ and s is a solution of the equation

$$e^{-a\lambda s}(1 + \mu - 2\beta b\lambda s(u_0, \psi)) = 1, \tag{4.28}$$

$$|(u_0, \psi_1)| \leq \gamma, \quad \alpha(u_0, \psi_1) + \beta(v_0, \psi_1) = 1 + \mu. \tag{4.29}$$

From (4.4), (4.5) we deduce that $\forall\, i \geq 1$

$$(u_k(\tau_k), \psi_i) = (u_0^k, \psi_i)e^{-a\lambda_i\tau_k} + \varepsilon_k \int_0^{\tau_k} e^{-a\lambda_i(\tau_k - s)}(f_1(u_k(s), v_k(s)), \psi_i)\, ds,$$

$$
\begin{aligned}
(v_k(\tau_k), \psi_i) = \big((v_0^k, \psi_i) - 2b\lambda_i (u_0^k, \psi_i)\tau_k\big)e^{-a\lambda_i\tau_k} + \varepsilon_k \int\limits_0^{\tau_k} e^{-a\lambda_i(\tau_k - s)} (f_2(u_k(s), v_k(s)), \psi_i)\, ds \\
- \varepsilon_k 2b\lambda_i \int\limits_0^{\tau_k} (\tau_k - s) e^{-a\lambda_i(\tau_k - s)} (f_1(u_k(s), v_k(s)), \psi_i)\, ds.
\end{aligned}
$$

Analogously to (4.26), we can assume that

$$
\xi_k = \begin{pmatrix} u_k(\tau_k) \\ v_k(\tau_k) \end{pmatrix} \longrightarrow \xi = y(\tau) = \begin{pmatrix} u(\tau) \\ v(\tau) \end{pmatrix} \text{ in } H, \text{ where } y \in K^\varepsilon,\ y(0) = \eta.
$$

Then, as $k \to \infty$, we obtain

$$
(u(\tau), \psi_1) = (u_0, \psi_1) e^{-a\lambda_i \tau}, \tag{4.30}
$$

$$
(v(\tau), \psi_1) = \big((v_0, \psi_1) - 2b\lambda_1 (u_0, \psi_1)\tau\big) e^{-a\lambda_1 \tau}, \tag{4.31}
$$

where $\tau \in [0, s]$, s is a unique root of equation (4.28) under fixed u_0, v_0 from (4.29).

Taking into account a "non-impulsive" character of the coordinates $j \geq 2$ along each impulsive trajectory, from (2.2) we get

$$
\forall\, j \geq 2 \ \ |(u_0^k, \psi_j)| + |(v_0^k, \psi_j)| \longrightarrow 0, \ \ k \to \infty. \tag{4.32}
$$

Then from (4.30)–(4.32) we obtain that $\xi \in \Theta$ and (2.7) takes place. □

Remark 4.2. As is shown in [12], for the impulsive DS the global attractor Θ is, generally speaking, not invariant set of the semiflow $\widetilde{G}$. However, the set $\Theta \setminus M$ [4] may have such a property. The invariance property can be obtained from the explicit formula of Θ, when $\varepsilon = 0$. It turns out that if, additionally, the map I is upper-semicontinuous, this fact is valid for a sufficiently small $\varepsilon > 0$, i.e., the equality

$$
\forall\, t \geq 0 \ \ \widetilde{G}_\varepsilon(t, \Theta_\varepsilon \setminus M) = \Theta_\varepsilon \setminus M
$$

is satisfied. It will be done in the forthcoming papers.

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ON ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF A LINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH A VARIABLE COEFFICIENT

Abstract. The paper deals with qualitative analysis of solutions of a test linear differential equation involving variable coefficient and derivative of non-integer order. We formulate upper and lower estimates for these solutions depending on boundedness of the variable coefficient. In the special case of asymptotically constant coefficient, we present the sufficient (and nearly necessary) conditions for the convergence of solutions to zero.*

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რეზიუმე. ნაშრომში განხილულია ცვლადკოეფიციენტიანი წილად-წარმოებულიანი მოდელური წრფივი დიფერენციალური განტოლების ამონახსნების ხარისხობრივი ანალიზი. ამ ამონახსნებისთვის ჩამოყალიბებულია ცვლადი კოეფიციენტის შემოსაზღვრულობაზე დამოკიდებული ზედა და ქვედა შეფასებები. კერძო შემთხვევაში, როცა კოეფიციენტი ასიმპტოტურად მუდმივია, წარმოდგენილია ამონახსნის ნულისკენ კრებადობის საკმარისი (და თითქმის აუცილებელი) პირობები.

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1 Introduction

During several last decades, derivatives and integrals of non-integer orders, the so-called fractional derivatives and integrals, serve as an effective tool for modelling of many interesting technical and physical problems originating, e.g., in control theory, rheology, anomalous diffusion, chemistry (see, e.g., [4,7]). The extensive applications of this theory bring the need to understand well basic behaviour of the solutions of differential equations containing fractional derivatives.

Starting point for introductory investigation of the qualitative properties of fractional differential equations is the test equation of the form

$$\mathrm{D}_0^{\alpha} y(t) = \lambda y(t), \quad \alpha \in (0,1), \ \lambda \in \mathbb{R}, \tag{1.1}$$

$$\mathrm{D}_0^{\alpha-1} y(0) = y_0, \quad y_0 \in \mathbb{R}. \tag{1.2}$$

The asymptotic behaviour of (1.1), (1.2) was extensively studied by many authors (see, e.g., [6–8]) and their results can be summarized as

Theorem 1.1. *Let $\alpha \in (0,1)$, $\lambda \in \mathbb{R}$. Then the following statements hold:*

(i) *All solutions of* (1.1) *eventually tend to zero if and only if $\lambda \le 0$.*

(ii) *All non-trivial solutions of* (1.1) *are eventually unbounded if and only if $\lambda > 0$.*

Analogous results were obtained for the modifications of (1.1) including vector cases [6,8], delay [1] or discretized operators [2].

Although the statement of Theorem 1.1 seems to be quite similar to the results known from the classical analysis of the equation $y'(t) = \lambda y(t)$, fractional differential equations show several distinguish properties. Most apparent difference occurs for $\lambda = 0$, where in the integer-order case the solutions are known to be bounded but they do not tend to zero. Theorem 1.1 does not discuss the decay rate of solutions. If $\lambda < 0$, unlike for the integer-order differential equations, the solutions of (1.1) do not tend to zero exponentially, but algebraically (this decay depends on the derivative order α).

The goal of this paper is to generalize Theorem 1.1 for the linear fractional differential equation with variable coefficient, i.e.,

$$\mathrm{D}_0^{\alpha} y(t) = f(t) y(t), \quad \alpha \in (0,1), \quad \lambda \in \mathbb{R}, \tag{1.3}$$

where f is a continuous bounded real function and (1.2) is supplied as the initial condition.

Fractional differential equations with variable coefficients are usually studied in the literature from the viewpoint of constructing the solutions with no particular stress put on qualitative properties of such solutions (see, e.g., [10]). In [8,9], the authors considered (1.3) in the vector form and attempted to employ Grönwall's inequality to perform qualitative analysis, however the resulting assertions and proof techniques contain some unfeasible conditions and incorrect assumptions.

This paper is organized as follows. Section 2 presents basic definitions and preliminary results. Main results are contained in Section 3 including the corresponding proofs. Section 4 concludes the paper by some comments and remarks.

2 Preliminaries

Throughout this paper, we employ the Riemann–Liouville derivative of order α. It is introduced as follows: First, let y be a real scalar function defined on $(0,\infty)$. For $\gamma \in (0,\infty)$, the fractional integral of y is defined as

$$\mathrm{D}_0^{-\gamma} y(t) = \int_0^t \frac{(t-\xi)^{\gamma-1}}{\Gamma(\gamma)}\, y(\xi)\, \mathrm{d}\xi, \quad t \in (0,\infty),$$

and, for $\alpha \in (0,\infty)$, the Riemann–Liouville fractional derivative of y is defined as

$$\mathrm{D}_0^{\alpha} y(t) = \frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}t^{\lceil\alpha\rceil}} \left(\mathrm{D}_0^{-(\lceil\alpha\rceil-\alpha)} y(t)\right), \quad t \in (0,\infty),$$

where $\lceil \cdot \rceil$ denotes the ceiling function (also called upper integer part). We put $\mathrm{D}_0^0 y(t) = y(t)$ (for more on fractional calculus see, e.g., [5,7]).

It is well-known that the solution of (1.1), (1.2) is given by

$$y(t) = y_0 t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha),$$

where $E_{\alpha,\alpha}$ denotes the two-parameter Mittag–Leffler function introduced generally via the series

$$E_{\eta,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\eta j + \beta)}, \quad z \in \mathbb{C}, \quad \eta, \beta \in (0,\infty). \tag{2.1}$$

The Mittag–Leffler function is known to play a role of generalized exponential function within fractional calculus. Hence, asymptotic behaviour of (2.1) is essential with respect to the qualitative analysis of fractional differential equations. For some of these properties relevant for this paper see, e.g., [3,7,11].

Lemma 2.1. *Let $\eta, \beta \in (0,\infty)$. Then $E_{\eta,\beta}(z)$ is positive and increasing for $z \in \mathbb{R}$.*

Lemma 2.2. *Let $\eta, \beta \in (0,\infty)$, $\lambda \in \mathbb{R}$.*

(i) *If $\lambda > 0$, then*

$$t^{\beta-1} E_{\eta,\beta}(\lambda t^\eta) = \frac{\lambda^{(1-\beta)/\eta}}{\eta} \exp(\lambda^{1/\eta} t) + \mathcal{O}(t^{\beta-2\eta-1}) \quad \text{as} \quad t \to \infty.$$

(ii) *If $\lambda = 0$, then*

$$t^{\beta-1} E_{\eta,\beta}(\lambda t^\eta) = \frac{t^{\eta-1}}{\Gamma(\eta)}.$$

(iii) *If $\lambda < 0$, then*

$$t^{\beta-1} E_{\eta,\beta}(\lambda t^\eta) = \begin{cases} \dfrac{-t^{\beta-\eta-1}}{\lambda\Gamma(\beta-\eta)} + \mathcal{O}(t^{\beta-3\eta-1}), & \beta \neq \eta, \\[2ex] \dfrac{-t^{-\eta-1}}{\lambda^2\Gamma(-\eta)} + \mathcal{O}(t^{-2\eta-1}), & \beta = \eta \end{cases} \quad \text{as} \quad t \to \infty.$$

We note that the $\mathcal{O}$-symbol for any functions g, h is introduced as $g(t) = \mathcal{O}(h(t))$ as $t \to \infty$ if and only if there exist reals t_0, M such that $|g(t)| \leq M|h(t)|$ for all $t \geq t_0$.

3 Main results

In this section we study asymptotic properties of solutions of (1.3) based on the boundedness of the variable coefficient f. We supply (1.3) with the initial condition (1.2) where, without loss of generality, we assume $y_0 \in (0,\infty)$ throughout this section.

Lemma 3.1. *Let $\alpha \in (0,1)$, $U, L \in \mathbb{R}$ and let f be a continuous real function such that*

$$L \leq f(t) \leq U \quad \text{for all} \quad t \in (0,\infty).$$

Then every solution y of (1.3), (1.2) satisfies

$$y_0 t^{\alpha-1} E_{\alpha,\alpha}(L t^\alpha) \leq y(t) \leq y_0 t^{\alpha-1} E_{\alpha,\alpha}(U t^\alpha) \quad \text{for all} \quad t \in (0,\infty).$$

Proof. First we show that $y(t) \geq y_0 t^{\alpha-1} E_{\alpha,\alpha}(Lt^\alpha)$ for all $t > 0$. We introduce an auxiliary function ε^- via the relation $\varepsilon^-(t) = f(t) - L$, i.e. $L = f(t) - \varepsilon^-(t)$. Clearly, ε^- is non-negative and bounded by $U - L$. This enables us to rewrite (1.3) as

$$\mathrm{D}_0^\alpha y(t) = Ly(t) + \varepsilon^-(t)y(t).$$

We denote by y_h^L the solution of $\mathrm{D}_0^\alpha y(t) = Ly(t)$, $\mathrm{D}_0^\alpha y(0) = 1$. Hence, based on the variation of constants formula, the solution y of (1.3), (1.2) satisfies

$$y(t) = y_0 y_h^L(t) + \int_0^t y_h^L(t-\xi)\varepsilon^+(\xi)y(\xi)\,\mathrm{d}\xi. \tag{3.1}$$

Due to Lemma 2.1 we have $0 < y_h^L(t)$ for all $t \in (0,\infty)$. Assume that there exists $\widehat{t}$ such that $y(\widehat{t}) < y_0 y_h^L(\widehat{t})$. Relation (3.1) implies that there exists $t_0 \in (0,\widehat{t})$ such that

$$y(t) > y_0 y_h^L(t) \ \text{ for all } \ t \in (0, t_0).$$

Since y is a continuous function, t_0 can be chosen so that $y(t_0) = y_0 y_h^L(t_0)$. Therefore, by (3.1), we get

$$\int_0^{t_0} y_h^L(t_0-\xi)\varepsilon^-(\xi)y(\xi)\,\mathrm{d}\xi = 0. \tag{3.2}$$

Since y_h^L and ε^- are non-negative functions, (3.2) implies that there exists a subset of non-zero measure of $(0,t_0)$ where y is negative, which leads to a contradiction. Hence, $y(t) > y_0 y_h^L(t) = y_0 t^{\alpha-1}E_{\alpha,\alpha}(Lt^\alpha)$ for all $t > 0$.

The second part of the inequality, i.e., $y(t) \leq y_0 t^{\alpha-1}E_{\alpha,\alpha}(Ut^\alpha)$ for all $t > 0$, is proved analogously by using the auxiliary non-negative function ε^+ defined via the relation $\varepsilon^+(t) = U - f(t)$. That concludes the proof. □

This enables us to formulate

Theorem 3.2. *Let $\alpha \in (0,1)$, $U, L \in \mathbb{R}$, $t_0 \in (0,\infty)$ and let f be a bounded continuous function. Further, let $L < f(t) < U$ for all $t \in (t_0,\infty)$.*

- (i) *If $U < 0$, then all solutions of (1.3) tend to zero. Moreover, every non-trivial solution y of (1.3), (1.2) satisfies $\widehat{K}^L t^{-\alpha-1} \leq y(t) \leq \widehat{K}^U t^{-\alpha-1}$ as $t \to \infty$ for suitable positive real constants $\widehat{K}^L$, $\widehat{K}^U$.*
- (ii) *If $U = 0$, then all solutions of (1.3) tend to zero. Moreover, every non-trivial solution y of (1.3), (1.2) satisfies $\widehat{K}^L t^{-\alpha-1} \leq y(t) \leq \widehat{K}^U t^{\alpha-1}$ as $t \to \infty$ for suitable positive real constants $\widehat{K}^L$, $\widehat{K}^U$.*
- (iii) *If $L > 0$, then all non-trivial solutions of (1.3) are unbounded.*

Proof. (i) Since f is bounded, Lemma 3.1 implies that the solution y of (1.3) is positive.

First let us prove that $y(t) \leq \widehat{K}^U t^{-\alpha-1}$ as $t \to \infty$ for suitable real $\widehat{K}^U$. We denote $\varepsilon^+(t) = U - f(t)$ and, using similar approach as in the proof of Lemma 3.1, rewrite the solution of (1.3) as

$$y(t) = y_0 y_h^U(t) - \int_0^{t_0} y_h^U(t-\xi)\varepsilon^+(\xi)y(\xi)\,\mathrm{d}\xi - \int_{t_0}^t y_h^U(t-\xi)\varepsilon^+(\xi)y(\xi)\,\mathrm{d}\xi. \tag{3.3}$$

Now, we investigate each term of (3.3) separately. The asymptotic behaviour of the first term is known, indeed, due to $U < 0$ and Lemma 2.2, we have

$$y_0 y_h^U(t) = y_0 t^{\alpha-1}E_{\alpha,\alpha}(Ut^\alpha) = \frac{-y_0 t^{-\alpha-1}}{U^2\Gamma(-\alpha)} + \mathcal{O}(t^{-2\alpha-1}) \ \text{ as } \ t \to \infty. \tag{3.4}$$

The middle term of (3.3) contains positive functions y_h^U, y and the function ε^+ which is allowed to change its sign on $(0, t_0)$, but is bounded, i.e., there exists m such that $|\varepsilon^+(t)| < m$ for all $t \in (0, t_0)$. Thus, we get

$$\begin{aligned} \left| -\int_0^{t_0} y_h^U(t-\xi)\varepsilon^+(\xi)y(\xi)\,\mathrm{d}\xi \right| &\le \int_0^{t_0} y_h^U(t-\xi)|\varepsilon^+(\xi)|y(\xi)\,\mathrm{d}\xi \\ &\le m\Big(\frac{-y_0 t^{-\alpha-1}}{U^2\Gamma(-\alpha)} + \mathcal{O}(t^{-2\alpha-1})\Big)\int_0^{t_0} y(\xi)\,\mathrm{d}\xi \le K t^{-\alpha-1} \ \text{ as } \ t\to\infty, \end{aligned} \tag{3.5}$$

where we have used the fact that the solution y of (1.3) is integrable (see, e.g., [5, 7]).

The third term of (3.3) contains only positive functions y_h^U, y and ε^+ (more precisely, ε^+ is non-negative for $t \in (0, \infty)$). Considering this along with (3.4), (3.5), we can estimate (3.3) as

$$y(t) \le \widehat{K}^U t^{-\alpha-1} \ \text{ as } \ t \to \infty,$$

where $\widehat{K}^U$ is a suitable positive real constant.

The second part of the inequality, i.e., $y(t) \ge \widehat{K}^L t^{-\alpha-1}$, can be proved analogously.

The assertions (ii), (iii) can be proved by using similar steps as for (i). □

Theorem 3.2 directly implies the following results for f being asymptotically constant.

Corollary 3.3. *Let $\alpha \in (0,1)$, $P \in \mathbb{R}$ and let f be a bounded continuous function such that*

$$\lim_{t\to\infty} f(t) = P.$$

Then the following statements hold:

(i) *All solutions of* (1.3) *eventually tend to zero if $P < 0$.*

(ii) *All non-trivial solutions of* (1.3) *are eventually unbounded if $P > 0$.*

We can see that Corollary 3.3 is nearly in the effective form. The only case holding us from formulating not only sufficient but also necessary conditions, is $P = 0$. Theorem 1.1 indicates that $\lambda = 0$ plays a role of stability boundary for (1.1), (1.2). Corollary 3.3 therefore further highlights the special importance of the zero right-hand side of fractional differential equations.

Lemma 3.1 implies that if f is allowed to change its sign, the solutions of (1.3) can tend to zero and be unbounded. In particular, we can see that if f is non-positive and tends to zero, the solutions of (1.3) tend to zero (see Theorem 3.2(ii)). None of Theorems 1.1, 3.2 discusses situations when f tends to zero and is positive or oscillates. To illustrate the range of possible behaviours of solutions (1.3) in such cases, we consider the following examples:

(A) Let f be a bounded continuous function satisfying

$$\lim_{t\to\infty} f(t) = 0 \ \text{ and } \ f(t) > K t^{-\gamma},$$

where $\gamma \in (0, \infty)$, $\alpha \in (0, 1)$ and K is a positive real. Then the solution y of (1.3), (1.2) can be estimated as

$$\begin{aligned} y(t) &= y_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \int_0^t \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(\xi) y(\xi)\,\mathrm{d}\xi \\ &\ge y_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \int_0^t \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)} \frac{K\xi^{\alpha-\gamma-1}}{\Gamma(\alpha)}\,\mathrm{d}\xi = y_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{K\Gamma(\alpha-\gamma)}{\Gamma(\alpha)} \frac{t^{2\alpha-\gamma-1}}{\Gamma(2\alpha-\gamma)}. \end{aligned}$$

Obviously, if $\gamma \in (0, 2\alpha - 1)$ and $\alpha \in (1/2, 1)$, then y is eventually unbounded.

(B) Let f be a bounded continuous function such that

$$f(t) \geq 0 \ \text{ for } \ t \in (0,\infty) \ \text{ and } \ f(t) = 0 \ \text{ for } \ t \in (t_0,\infty),$$

where $t_0 \in (0,\infty)$, $\alpha \in (0,1)$. As in the proof of Theorem 3.2, the solution y of (1.3) can be estimated as

$$y(t) = y_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \int\limits_0^{t_0} \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(\xi) y(\xi)\,\mathrm{d}\xi$$

$$\leq y_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{K_1 (t-t_0)^{\alpha-1}}{\Gamma(\alpha)} \int\limits_0^{t_0} y(\xi)\,\mathrm{d}\xi \leq y_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{K_2 (t-t_0)^{\alpha-1}}{\Gamma(\alpha)}\,,$$

where K_1, K_2 are suitable positive reals. Obviously, y tends to zero.

Remark. The assumption of $y_0 \in (0,\infty)$ made throughout this section is not essential. Clearly, if $y_0 \in (-\infty,0)$, then the resulting inequalities only change their orientation.

4 Conclusions

We have studied asymptotic properties of solutions of the linear fractional differential equation with variable coefficient (1.3)).

Lemma 3.1 implies that if f is bounded, then the corresponding solution of (1.3) is bounded by the solutions of (1.1) for particular choices of λ depending on the bounds of f. Consequently, Theorem 1.1 shows that the solutions of (1.3) pose algebraic decay or exponential growth if f is bounded and non-positive or positive, respectively.

The assumptions on the sign of f needed in Theorem 1.1 were weakened in Theorem 3.2 where the fixed sign of f is required only for sufficiently large t. Finally, Corollary 3.3 outlines the specific case of asymptotically constant coefficient f. In particular, if f tends to a non-zero constant, the full discussion of asymptotic behaviour is presented. If f tends to zero, solutions can be eventually unbounded or tending to zero depending on decay rate of f as illustrated by the examples.

Possible future research directions are a deeper analysis of the case when f asymptotically approaches zero, and various generalizations of (1.3) to multi-term equations or vector forms.

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THE SHARPNESS OF SOME EXISTENCE STATEMENTS FOR DIFFERENTIAL SYSTEMS WITH NONLOCAL BOUNDARY CONDITIONS

Abstract. Recently, some extensions of results of M. A. Krasnosel'skii and Gustafson–Schmitt for systems of the type $x' = f(t,x)$ with periodic boundary conditions $x(0) = x(1)$ have been obtained for nonlocal boundary conditions of the type $x(1) = \int\limits_0^1 dh(s)\,x(s)$ or $x(0) = \int\limits_0^1 dh(s)\,x(s)$, where h is a real non-decreasing function satisfying some conditions, and containing the periodic boundary conditions as special cases. The situations with periodic and nonlocal boundary conditions are compared through the use of counterexamples, exhibiting the special character of the periodic case. Similar counterexamples also show, in the case of second order systems with some nonlocal boundary conditions, that the sense of some inequalities in the assumptions cannot be reversed.*

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Key words and phrases. Nonlocal boundary value problem, boundary value problem at resonance, periodic solutions, Leray–Schauder degree, convex sets.

რეზიუმე. პერიოდული ამოცანის $x' = f(t,x)$, $x(0) = x(1)$ ამოხსნადობის კარგად ცნობილი კრასნოსელსკის და გუსტავსონ-შმიტის შედეგები განზოგადებული იქნა $x(1) = \int\limits_0^1 dh(s)\,x(s)$ და $x(0) = \int\limits_0^1 dh(s)\,x(s)$ სახის არალოკალური სასაზღვრო პირობების შემთხვევაში (h ნამდვილი არაკლებადი ფუნქციაა), რომლებიც მოიცავენ პერიოდულ სასაზღვრო პირობებს, როგორც კერძო შემთხვევებს. კონტრმაგალითების საშუალებით შედარებულია პერიოდული და არალოკალური სასაზღვრო პირობების შემთხვევები, რომლებიც ავლენენ პერიოდული შემთხვევის სპეციალურ ხასიათს. ანალოგიური კონტრმაგალითები აგრეთვე გვიჩვენებენ, რომ მეორე რიგის სისტემების შემთხვევაში გარკვეული არალოკალური სასაზღვრო პირობებით ზოგიერთი უტოლობის ნიშნის შებრუნება არ შეიძლება.

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1 Introduction

Let $\langle \cdot \mid \cdot \rangle$ denote the usual inner product in $\mathbb{R}^n$, $|\cdot|$ the corresponding Euclidian norm, and $B_R \subset \mathbb{R}^n$ the open ball of center 0 and radius R. Throughout the paper, *let* $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous.*

Let us first consider the *periodic boundary value problem*

$$x' = f(t,x), \quad x(0) = x(1). \tag{1.1}$$

A classical existence theorem for problem (1.1), more than fifty years old, is the following one.

Existence theorem. *If there exists $R > 0$ such that either*

$$\langle u \mid f(t,u) \rangle \geq 0 \ \ \forall\, (t,u) \in [0,1] \times \partial B_R,$$

or

$$\langle u \mid f(t,u) \rangle \leq 0 \ \ \forall\, (t,u) \in [0,1] \times \partial B_R,$$

then problem (1.1) *has at least one solution such that* $x([0,1]) \subset \overline{B}_R$.

The two results are indeed equivalent, the second one being deduced from the first one through the change of variable $\tau = 1 - t$. They are a nonlinear counterpart to the elementary result that, for each $e \in C([0,1], \mathbb{R}^n)$ and each $\lambda \in \mathbb{R} \setminus \{0\}$, the problem

$$x' = \lambda x + e(t), \quad x(0) = x(1)$$

has a solution, a consequence of the fact that 0 is the unique real eigenvalue of the operator $\frac{d}{dt}$ with periodic boundary conditions.

Although the existence theorem above is a special case of a result given by M. A. Krasnosel'skii in 1966 ([6, Theorem 3.2]), and was surely known to him, its explicit statement is not contained in [6], and we did not find an earlier reference. One can just mention that in 1965, F. E. Browder [1] proved the existence of a solution of (1.1) with $\mathbb{R}^n$ replaced by an arbitrary real Hilbert space with inner product $\langle \cdot \mid \cdot \rangle$ when $f : [0,1] \times H \to H$ is continuous, $-f(t,\cdot)$ is monotone for each $t \in [0,1]$ and there exists $R > 0$ such that $\langle u \mid f(t,u) \rangle < 0$ for $(t,u) \in [0,1] \times \partial B_R$.

Krasnosel'skii's theorem. *If there exists a bounded open convex set $C \subset \mathbb{R}^n$, and functions $\Phi_i \in C^1(\mathbb{R}^n, \mathbb{R})$ $(i = 1, \dots, r)$ such that $\overline{C} = \{u \in \mathbb{R}^n : \ \Phi_i(u) \leq 0 \ (i = 1, \dots, r)\}$, $\nabla \Phi_i(u) \neq 0$ when $\Phi_i(u) = 0$ for some $u \in \partial C$, and either*

$$\big\langle \nabla \Phi_i(u) \mid f(t,u) \big\rangle \geq 0 \ \ \forall\, (t,u) \in [0,1] \times \partial C \ \ \textit{and} \ \ \forall\, i \in \alpha(u),$$

or

$$\big\langle \nabla \Phi_i(u) \mid f(t,u) \big\rangle \leq 0 \ \ \forall\, (t,u) \in [0,1] \times \partial C \ \ \textit{and} \ \ \forall\, i \in \alpha(u),$$

where $\alpha(u) := \{i \in \{1, \dots, r\} : \ \Phi_i(u) = 0\}$, then problem (1.1) *has at least one solution such that* $x([0,1]) \subset \overline{C}$.

The existence theorem above corresponds to the choice of $C = B_R$, $r = 1$ and $\Phi_1(u) = \frac{1}{2}\,(|u|^2 - R^2)$. A more direct proof of Krasnosel'skii's theorem based upon coincidence degree arguments has been given in 1974 in [7, Corollary 3.2].

Now, if $C \subset \mathbb{R}^n$ is an open convex neighborhood of $0 \in \mathbb{R}^n$, then, for each $u \in \partial C$, there exists some $\nu(u) \in \mathbb{R}^n \setminus \{0\}$ such that $\langle \nu(u) \mid u \rangle > 0$ and $C \subset \{v \in \mathbb{R}^n : \ \langle \nu(u) \mid v - u \rangle < 0\}$. $\nu(u)$ is called an *outer normal* to ∂C at u, and $\nu : \partial C \to \mathbb{R}^n \setminus \{0\}$ an *outer normal field* on ∂C. Notice that ν needs not to be continuous. The second condition means that $\nu(u)$ is orthogonal to a *supporting hyperplane* of C at u [2,5]. In 1974, using arguments similar to those of [7], Gustafson and Schmitt [4] introduced the following elegant existence condition.

Gustafson–Schmitt's theorem. *If there exists a bounded convex open neighborhood C of 0 in $\mathbb{R}^n$, and an outer normal field ν on ∂C such that either*

$$\big\langle \nu(u) \mid f(t,u) \big\rangle > 0 \ \ \forall\, (t,u) \in [0,1] \times \partial C, \tag{1.2}$$

or

$$\big\langle \nu(u) \mid f(t,u) \big\rangle < 0 \ \ \forall\, (t,u) \in [0,1] \times \partial C, \tag{1.3}$$

then problem (1.1) *has at least one solution such that* $x([0,1]) \in C$.

Notice that the monograph [6] is not quoted in [4], and that the special case where $C = B_R$ is explicitly stated there. The relation between [6] and [4] was explicited in [7, 8], where it was also shown that inequalities need not to be strict in Gustafson–Schmitt's assumptions (1.2), (1.3) if one replaces C by $\overline{C}$ in the conclusion. See also [3] for further generalizations. Krasnosel'skii's theorem follows from extended Gustafson–Schmitt's condition because if, without loss of generality, we assume that $0 \in C$ in Krasnosel'skii's statement, then, for $u \in \partial C$ and $i \in \alpha(u)$, $\nabla \Phi_i(u)$ is an outer normal to ∂C at u.

In [10], the following generalizations of problem (1.1)

$$x' = f(t,x), \quad x(1) = \int_0^1 dh(s)\, x(s), \tag{1.4}$$

and

$$x' = f(t,x), \quad x(0) = \int_0^1 dh(s)\, x(s) \tag{1.5}$$

(sometimes called *nonlocal terminal value problem*, and *nonlocal initial value problem*, respectively), have been considered, where

$$h : [0,1] \to \mathbb{R} \ \ \textit{is non-decreasing and} \ \ \int_0^1 dh(s) = 1.$$

Both boundary conditions in (1.4) and (1.5) can be seen as generalizations of the periodic boundary conditions $x(0) = x(1)$, where either $x(0)$ or $x(1)$ is replaced by some average of x over the interval $[0,1]$.

The following theorems are special cases of the results proved in [10] by reduction to a fixed point problem and the use of some version of Leray–Schauder continuation theorem.

Theorem 1.1. *If* $h(0) < h(\alpha)$ *for some* $\alpha \in (0,1)$ *and if there exists an open, bounded, convex neighborhood* C *of* 0 *in* $\mathbb{R}^n$ *and an outer normal field* ν *on* ∂C *such that*

$$\big\langle \nu(u) \mid f(t,u) \big\rangle \geq 0 \ \ \forall\, (t,u) \in [0,1] \times \partial C, \tag{1.6}$$

then problem (1.4) *has at least one solution* x *such that* $x([0,1]) \in \overline{C}$.

Theorem 1.2. *If* $h(\alpha) < h(1)$ *for some* $\alpha \in (0,1)$ *and if there exists an open, bounded, convex neighborhood* C *of* 0 *in* $\mathbb{R}^n$ *and an outer normal field* ν *on* ∂C *such that*

$$\big\langle \nu(u) \mid f(t,u) \big\rangle \leq 0 \ \ \forall\, (t,u) \in [0,1] \times \partial C, \tag{1.7}$$

then problem (1.5) *has at least one solution* x *such that* $x([0,1]) \subset \overline{C}$.

The following consequences of Theorems 1.1 and 1.2, corresponding to $C = B_R$, are also given in [10].

Corollary 1.1. *If* $h(0) < h(\alpha)$ *for some* $\alpha \in (0,1)$ *and if there exists* $R > 0$ *such that*

$$\langle u \mid f(t,u) \rangle \geq 0 \ \ \forall\, (t,u) \in [0,1] \times \partial B_R, \tag{1.8}$$

Then problem (1.4) *has at least one solution* x *such that* $x([0,1]) \in \overline{B}_R$.

Corollary 1.2. *If $h(\alpha) < h(1)$ for some $\alpha \in (0,1)$ and if there exists $R > 0$ such that*

$$\langle u \mid f(t,u)\rangle \leq 0 \ \ \forall\,(t,u) \in [0,1] \times \partial B_R, \tag{1.9}$$

then problem (1.5) *has at least one solution x such that $x([0,1]) \in \overline{B}_R$.*

Comparing those statements with our first existence theorem for the periodic problem, we see that the sense of the inequality in conditions (1.6) or (1.8) and (1.7) or (1.9) depends upon the boundary condition. On the other hand, as it is easily verified by direct computation, the system

$$x' = \lambda x + e(t),$$

with each of the three-point boundary conditions

$$x(1) = \frac{1}{2}\Big[x\Big(\frac{1}{2}\Big) + x(0)\Big], \ \ x'(0) = \frac{1}{2}\Big[x\Big(\frac{1}{2}\Big) + x(1)\Big],$$

has a solution for each $e \in C([0,1],\mathbb{R}^n)$ and each $\lambda \in \mathbb{R} \setminus \{0\}$. This is again a consequence of the fact that the only real eigenvalue of $\frac{d}{dt}$ with each boundary condition is 0. Hence a natural question is to know if the conclusion of the above corollaries still holds when the sense of the corresponding inequality upon f is reversed.

The aim of this paper is to show by some counterexamples that the answer is negative in general, which of course implies that the same negative answer holds for Theorems 1.1 and 1.2. In this sense, the existence conditions given in [10] are sharp.

The construction of our counterexamples in Section 4 depends upon the study of the associated complex eigenvalue problem in Section 2 and of the corresponding Fredholm alternative in Section 3 for some special three-point boundary conditions.

In Section 4, we exhibit a (complex) eigenvalue λ and show the existence of a function $e \in C([0,1],\mathbb{C})$ such that the equation

$$z' = \lambda z + e(t),$$

with the corresponding multipoint boundary conditions, has no solution $z : [0,1] \to \mathbb{C}$ and such that, for the equivalent 2-dimensional system obtained by letting

$$x_1 = \Re z, \ \ x_2 = \Im z, \ \ f_1(t,x) = \Re(\lambda z + e(t)), \ \ f_2(t,x) = \Im(\lambda z + e(t)),$$

$\langle u \mid f(t,u)\rangle$ has the opposite sign to the one in the corresponding corollary, for all $t \in [0,1]$ and all sufficiently large $|u|$. We complete this 2-dimensional counterexample by a 3-dimensional one, from which counterexamples can easily be obtained in all dimensions $n \geq 2$.

In Section 4, we also give an example of periodic problem (1.1) having no solution and such that $\langle x, f(t,x)\rangle$ changes sign when $|x| = R$ and $R > 0$ is sufficiently large. Hence, the assumptions of the existence theorem for periodic problems are sharp as well.

Finally, in Section 5, we construct in a similar way a counterexample related to the following *nonlocal boundary value problem for a second order system* considered in [9]

$$x'' = g(t,x,x'), \ \ x(0) = 0, \ \ x'(1) = \int_0^1 dh(s)\,x'(s), \tag{1.10}$$

where $g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and* $h : [0,1] \to \mathbb{R}$ *is non-decreasing and* $\int_0^1 dh(s) = 1$. The following existence result is proved in [9].

Theorem 1.3. *If $h(0) < h(\alpha)$ for some $\alpha \in (0,1)$ and if there exists an open, bounded, convex neighborhood C of 0 in $\mathbb{R}^n$ and an outer normal field ν on ∂C such that*

$$\big\langle \nu(v) \mid g(t,u,v)\big\rangle \geq 0 \ \ \forall\,(t,u,v) \in [0,1] \times \overline{C} \times \partial C,$$

then problem (1.10) *has at least one solution x such that $x([0,1]) \subset \overline{C}$ and $x'([0,1]) \subset \overline{C}$.*

Its special case where $C = B_R$ goes as follows.

Corollary 1.3. *If $h(0) < h(\alpha)$ for some $\alpha \in (0,1)$ and if there exists $R > 0$ such that*

$$\langle v \mid g(t,u,v)\rangle \geq 0 \ \ \forall\, (t,u,v) \in [0,1] \times \overline{B}_R \times \partial B_R, \tag{1.11}$$

then problem (1.10) *has at least one solution x such that $x([0,1]) \subset \overline{B}_R$ and $x'([0,1]) \subset \overline{B}_R$.*

In Section 5, we exhibit a counterexample showing that a statement like Corollary 1.3 does not hold if the sense of inequality (1.11) is reversed.

Notice that one could consider as well the problem

$$x'' = g(t,x,x'), \ \ x(0) = 0, \ \ x'(0) = \int_0^1 dh(s)\, x'(s), \tag{1.12}$$

when $h(\alpha) < h(1)$ for some $\alpha \in (0,1)$ and the assumptions on g, and prove, mimicking the approach of [9], the following existence result.

Theorem 1.4. *If $h(\alpha) < h(1)$ for some $\alpha \in (0,1)$ and if there exists an open, bounded, convex neighborhood C of 0 in $\mathbb{R}^n$ and an outer normal field ν on ∂C such that*

$$\big\langle \nu(v) \mid g(t,u,v)\big\rangle \leq 0 \ \ \forall\, (t,u,v) \in [0,1] \times \overline{C} \times \partial C,$$

then problem (1.12) *has at least one solution x such that $x([0,1]) \subset \overline{C}$ and $x'([0,1]) \subset \overline{C}$.*

We leave to the reader the task of stating the corresponding corollary analog to Corollary 1.3 and of constructing a counterexample to an existence statement with reversed inequalities.

In analogy with the periodic case for first order differential systems, the two-point boundary value problem

$$x'' = g(t,x,x'), \ \ x(0) = 0, \ \ x'(0) = x'(1) \tag{1.13}$$

is a special case of both problems (1.10) and (1.12). Hence, the existence of a solution to problem (1.13) is insured if there exists $R > 0$ such that either

$$\langle v \mid g(t,u,v)\rangle \geq 0 \ \ \forall\, (t,u,v) \in [0,1] \times \overline{B}_R \times \partial B_R,$$

or

$$\langle v \mid g(t,u,v)\rangle \leq 0 \ \ \forall\, (t,u,v) \in [0,1] \times \overline{B}_R \times \partial B_R.$$

2 First order eigenvalue problems

We consider the eigenvalue problem

$$z'(t) = \lambda z(t), \ \ z(1) = \frac{1}{2}\left[z(0) + z\Big(\frac{1}{2}\Big)\right], \tag{2.1}$$

where $\lambda \in \mathbb{C}$, $z : [0,1] \to \mathbb{C}$. Its three-point boundary condition is a special case of the one in Corollary 1.1 with

$$h(s) = \begin{cases} 0 & \text{if } \ s = 0, \\ 1/2 & \text{if } \ s \in (0,1/2] \\ 1 & \text{if } \ s \in (1/2,1]. \end{cases}$$

Proposition 2.1. *The eigenvalues of problem* (2.1) *are $\lambda_{tc,1,k} = 2k(2\pi i)$ and $\lambda_{tc,2,k} = -\log 4 + (2k+1)(2\pi i)$ $(k \in \mathbb{Z})$. They are located in the left part of the complex plane.*

Proof. The eigenvalue problem (2.1) has a nontrivial solution if and only if $\lambda \in \mathbb{C}$ is such that

$$e^{\lambda} = \frac{1}{2} + \frac{1}{2}\, e^{\lambda/2}. \tag{2.2}$$

Set $\mu := e^{\lambda/2}$, so that equation (2.2) becomes the equation in μ

$$\mu^2 - \frac{1}{2}\,\mu - \frac{1}{2} = 0,$$

whose solutions are $\mu_{tc,1} = 1$, $\mu_{tc,2} = -\frac{1}{2}$. The equation $e^{\lambda/2} = \mu_{tc,1} = 1$ is satisfied for $\frac{\lambda}{2} = 2k\pi i$ $(k \in \mathbb{Z})$ which gives the eigenvalues

$$\lambda_{tc,1,k} = 2k(2\pi i) \ \ (k \in \mathbb{Z}).$$

The equation $e^{\lambda/2} = \mu_{tc,2} = -\frac{1}{2}$ is satisfied for $\frac{\lambda}{2} = -\log 2 + \pi i + 2k\pi i = -\log 2 + (2k+1)\pi i$ $(k \in \mathbb{Z})$, which gives the eigenvalues

$$\lambda_{tc,2,k} = -\log 4 + (2k+1)(2\pi i) \ \ (k \in \mathbb{Z}).$$ □

Similarly, we consider the eigenvalue problem

$$z'(t) = \lambda z(t), \ \ z(0) = \frac{1}{2}\left[z\Big(\frac{1}{2}\Big) + z(1)\right], \tag{2.3}$$

where $\lambda \in \mathbb{C}$, $z : [0,1] \to \mathbb{C}$. Its multi-point boundary condition is a special case of the one in Corollary 1.2 with

$$h(s) = \begin{cases} 0 & \text{if } s \in [0, 1/2), \\ 1/2 & \text{if } s \in [1/2, 1), \\ 1 & \text{if } s = 1. \end{cases}$$

Proposition 2.2. *The eigenvalues of problem* (2.3) *are* $\lambda_{ic,1,k} = 2k(2\pi i)$ *and* $\lambda_{ic,2,k} = \log 4 + (2k+1)(2\pi i)$ $(k \in \mathbb{Z})$*. They are located in the left right part of the complex plane.*

Proof. The eigenvalue problem (2.3) has a nontrivial solution if and only if $\lambda \in \mathbb{C}$ is such that

$$1 = \frac{1}{2}\, e^{\lambda/2} + \frac{1}{2}\, e^{\lambda}. \tag{2.4}$$

Set $\mu := e^{\lambda/2}$, so that equation (2.4) becomes the equation in μ

$$\frac{1}{2}\,\mu^2 + \frac{1}{2}\,\mu - 1 = 0$$

whose solutions are $\mu_{ic,1} = 1$ and $\mu_{ic,2} = -2$. Consequently, we obtain, as above,

$$\lambda_{ic,1,k} = 2k(2\pi i) \ \ (k \in \mathbb{Z})$$

and

$$\lambda_{ic,2,k} = \log 4 + (2k+1)(2\pi i) \ \ (k \in \mathbb{Z}).$$ □

Remark 2.1. The situation can be compared with the spectrum for the periodic boundary conditions

$$z' = \lambda z, \ \ z(0) = z(1)$$

which, as easily seen, is made of the eigenvalues $\lambda_{p,k} = k(2\pi i)$ $(k \in \mathbb{Z})$. One can see that, in the case of (2.1), half of the eigenvalues of the periodic problem move to the line $\Re z = -\log 4$, and, in the case of (2.3), the same half moves to the line $\Re z = \log 4$. The spectra have lost their symmetry with respect to the imaginary axis.

3 Fredholm alternative

The construction of our counterexamples requires the use of the Fredholm alternative for the corresponding forced boundary value problems.

Proposition 3.1. *λ is an eigenvalue of* (2.1) (*resp.* (2.3)) *if and only if there exists a continuous function e such that the nonhomogeneous problem* (3.1) (*resp.* (3.2)) *has no solution.*

Proof. It is shown in [10] (or by direct verification) that the non-homogeneous problems

$$Lz := z' - z = e(t), \;\; z(0) = \frac{1}{2}\,z\Big(\frac{1}{2}\Big) + \frac{1}{2}\,z(1)$$

and

$$Mz := z' + z = e(t), \;\; z(0) = \frac{1}{2}\,z\Big(\frac{1}{2}\Big) + \frac{1}{2}\,z(1)$$

have a unique solution $z = L^{-1}e$ and $z = M^{-1}e$ for every $e \in C([0,1],\mathbb{C})$, and that the linear mappings L^{-1} and M^{-1} are compact in the space $C([0,1],\mathbb{C})$. As a consequence, each problem

$$z' - \lambda z = e(t), \;\; z(1) = \frac{1}{2}\,z(0) + \frac{1}{2}\,z\Big(\frac{1}{2}\Big) \tag{3.1}$$

and

$$z' + \lambda z = e(t), \;\; z(0) = \frac{1}{2}\,z\Big(\frac{1}{2}\Big) + \frac{1}{2}\,z(1) \tag{3.2}$$

can be written equivalently

$$z = (\lambda - 1)L^{-1}z + L^{-1}e, \;\; z = (\lambda + 1)M^{-1}z + M^{-1}e,$$

so that the Fredholm alternative follows from Riesz theory of linear compact operators. □

4 Counterexamples to Corollaries 1.1 and 1.2 with opposite vector fields sign conditions

We now finalize the construction of our counterexamples.

We first consider the case of a three-point boundary condition of terminal type, and apply Proposition 3.1 to the case of the eigenvalue $\lambda_{tc,2,0} = -\log 4 + (4k+2)\pi i$ of (2.1). Let $e : [0,1] \to \mathbb{C}$ be a continuous function such that the problem

$$z'(t) = (-\log 4 + 2\pi i)z(t) + e(t), \;\; z(1) = \frac{1}{2}\,z(0) + \frac{1}{2}\,z\Big(\frac{1}{2}\Big) \tag{4.1}$$

has no solution. Setting $z(t) = x_1(t) + ix_2(t)$, $e(t) = e_1(t) + ie_2(t)$, problem (4.1) is equivalent to the planar real system

$$\begin{cases} x_1'(t) = -(\log 4)x_1(t) - 2\pi x_2(t) + e_1(t), \\ x_2'(t) = 2\pi x_1(t) - (\log 4)x_2(t) + e_2(t), \\ x_1(1) = \dfrac{1}{2}\,x_1(0) + \dfrac{1}{2}\,x_1\Big(\dfrac{1}{2}\Big), \\ x_2(1) = \dfrac{1}{2}\,x_2(0) + \dfrac{1}{2}\,x_2\Big(\dfrac{1}{2}\Big). \end{cases} \tag{4.2}$$

Let

$$f(t,u) := \big(-(\log 4)u_1 - 2\pi u_2 + e_1(t), 2\pi u_1 - (\log 4)u_2 + e_2(t)\big).$$

For (4.2), we have

$$\begin{aligned} \langle u \mid f(t,u)\rangle &= u_1\big[-(\log 4)u_1 - 2\pi u_2 + e_1(t)\big] + u_2\big[2\pi u_1 - (\log 4)u_2 + e_2(t)\big] \\ &= -(\log 4)(u_1^2 + u_2^2) + u_1e_1(t) + u_2e_2(t) \\ &\le -(\log 4)|u|^2 + |e(t)|\,|u| < 0, \end{aligned} \tag{4.3}$$

when $|u| \geq R$ for some sufficiently large R.

Conclusion. *For problem* (1.4) *with the conditions of Corollary* 1.1 *on* f *and the existence of some* $R>0$ *such that*

$$\langle u \mid f(t,u)\rangle \leq 0 \ \ \forall\,(t,u) \in [0,1]\times \partial B_R,$$

there is no existence theorem similar to Corollary 1.1.

In the case of the three-point conditions of initial type, we similarly apply Proposition 3.1 to the case of the eigenvalue $\lambda_{ic,2,0} = \log 4 + 2\pi i$ of (2.3). Let $e : [0,1] \to \mathbb{C}$ be a continuous function such that the problem

$$z'(t) = (\log 4 + 2\pi i)z(t) + e(t), \quad z(1) = \frac{1}{2}\,z(0) + \frac{1}{2}\,z\Big(\frac{1}{2}\Big) \tag{4.4}$$

has no solution. Setting $z(t) = x_1(t) + ix_2(t)$, $e(t) = e_1(t) + ie_2(t)$, problem (4.4) is equivalent to the planar real system

$$\begin{cases} x_1'(t) = (\log 4)x_1(t) - 2\pi x_2(t) + e_1(t), \\ x_2'(t) = 2\pi x_1(t) + (\log 4)x_2(t) + e_2(t), \\ x_1(0) = \dfrac{1}{2}\,x_1\Big(\dfrac{1}{2}\Big) + \dfrac{1}{2}\,x_1(1), \\ x_2(0) = \dfrac{1}{2}\,x_2\Big(\dfrac{1}{2}\Big) + \dfrac{1}{2}\,x_2(1). \end{cases} \tag{4.5}$$

Let

$$f(t,u) := \big((\log 4)u_1 - 2\pi u_2 + e_1(t), 2\pi u_1 + (\log 4)u_2 + e_2(t)\big).$$

For (4.5), we have

$$\begin{aligned} \langle u \mid f(t,u)\rangle &= u_1[(\log 4)u_1 - 2\pi u_2 + e_1(t)] + u_2\big[2\pi u_1 + (\log 4)u_2 + e_2(t)\big] \\ &= (\log 4)(u_1^2 + u_2^2) + u_1e_1(t) + u_2e_2(t) \\ &\geq (\log 4)|u|^2 - |e(t)|\,|u| > 0, \end{aligned}$$

when $|u| \geq R$ for some sufficiently large R.

Conclusion. *For problem* (1.5) *with the conditions of Corollary* 1.2 *on* f *and the existence of some* $R>0$ *such that*

$$\langle u \mid f(t,u)\rangle \geq 0 \ \ \forall\,(t,u) \in [0,1]\times \partial B_R,$$

there is no existence theorem similar to Corollary 1.2.

Remark 4.1. The symmetry-breaking for the spectra of the three-point boundary value problems of terminal or initial type explains the difference in the existence conditions for the nonlinear problems with the three-point boundary conditions and with the periodic conditions. The presence of the complex spectrum in the left or the right half-plane influences like a ghost the existence of solutions of the real nonlinear systems. Maybe extra conditions upon f could provide existence results with the sign conditions of the counterexamples.

Remark 4.2. Strictly speaking, our counterexamples do not cover the case of $n=1$ or of n odd. For $n=3$, if one adds the equations

$$x_3' = -(\log 4)x_3 + \frac{\log 4}{4}\,(x_1 + x_2), \quad x_3(1) = \frac{1}{2}\,\Big[x_3(0) + x_3\Big(\frac{1}{2}\Big)\Big]$$

or

$$x_3' = (\log 4)x_3 + \frac{\log 4}{4}\,(x_1 + x_2), \quad x_3(0) = \frac{1}{2}\,\Big[x_3\Big(\frac{1}{2}\Big) + x_3(1)\Big]$$

to (4.2) or to (4.5), respectively, the corresponding boundary value problems have no solutions and the nonlinear parts verify the opposite sign conditions to Corollaries 1.1 and 1.2, respectively. Of course, the counterexamples for $n=2$ and $n=3$ easily provide counterexamples in any dimension $n \geq 2$. The case $n=1$ remains open.

Remark 4.3. The periodic problem

$$z' = 2\pi i z + e^{2\pi i t}, \quad z(0) = z(1) \tag{4.6}$$

has no solution. Indeed, if z is a possible solution, then

$$(e^{-2\pi i t} z)' = 1,$$

which gives a contradiction, by integration over $[0,1]$ and use of the boundary conditions.

Letting $z = x_1 + i x_2$, the following problem

$$x_1' = -2\pi x_2 + \cos(2\pi t), \quad x_2' = 2\pi x_1 + \sin(2\pi t), \quad x_1(0) = x_1(1), \quad x_2(0) = x_2(1),$$

equivalent to (4.6), has no solution. On the other hand, letting

$$\begin{gathered} f_1(t,x_1,x_2) = -2\pi x_2 + \cos(2\pi t), \quad f_2(t,x_1,x_2) = 2\pi x_1 + \sin(2\pi t), \\ x = (x_1,x_2), \quad f(t,x) = \big(f_1(t,x_1,x_2), f_2(t,x_1,x_2)\big), \end{gathered}$$

we have

$$\begin{aligned} \langle x, f(t,x)\rangle &= -2\pi x_2 x_1 + \cos(2\pi t) x_1 + 2\pi x_1 x_2 + \sin(2\pi t) x_2 \\ &= \cos(2\pi t) x_1 + \sin(2\pi t) x_2. \end{aligned}$$

For $x = R[\cos(2\pi\theta), \sin(2\pi\theta)] \in \partial B_R$ $(\theta \in [0,1])$, we have

$$\begin{aligned} \langle x, f(t,x)\rangle &= R\big[\cos(2\pi t)\cos(2\pi\theta) + \sin(2\pi t)\sin(2\pi\theta)\big] \\ &= R\cos[2\pi(t-\theta)] \quad (t,\theta \in [0,1]), \end{aligned}$$

which implies that, for each $t \in [0,1]$, $\langle x, f(t,x)\rangle$ takes both positive and negative values on ∂B_R, and shows that, for n even, the assumptions of the existence theorems for periodic problems given at the beginning of the Introduction are sharp.

5 Second order differential systems

As in Section 2, we start with the following "eigenvalue problem"

$$z''(t) = \lambda z'(t), \quad z(0) = 0, \quad z'(1) = \frac{1}{2} z'(0) + \frac{1}{2} z'\Big(\frac{1}{2}\Big), \tag{5.1}$$

where $\lambda \in \mathbb{C}$, $x : [0,1] \to \mathbb{C}$. Notice that it is not the classical eigenvalue associated to z'' in which $\lambda z'$ must be replaced by λz.

Proposition 5.1. *All the "eigenvalues" $\lambda_{bc,j,k}$ $(j = 1,2;\ k \in \mathbb{Z})$ of the multipoint boundary value problem* (5.1) *have real part equal to* 0 *or* $-\log 4$*, and hence are located in the left part of the complex plane.*

Proof. Setting $w(t) = z'(t)$, so that, using $z(0) = 0$, $z(t) = \int\limits_0^t w(s)\,ds$, problem (5.1) is equivalent to the eigenvalue problem

$$w'(t) = \lambda w(t), \quad w(1) = \frac{1}{2} w(0) + \frac{1}{2} w\Big(\frac{1}{2}\Big),$$

i.e., to the eigenvalue problem (2.3). Hence the result follows from Proposition 2.1. □

We now deduce, from the first order case, the Fredholm alternative.

Proposition 5.2. *λ is an "eigenvalue" of* (5.1) *if and only if there exists a continuous function e such that the nonhomogeneous problem* (5.2) *has no solution.*

Proof. In a similar way as in Proposition 5.1, the non-homogeneous problem

$$z'' - \lambda z' = e(t), \;\; z(0) = 0, \;\; z'(1) = \frac{1}{2}\, z'(0) + \frac{1}{2}\, z'\Big(\frac{1}{2}\Big) \tag{5.2}$$

is equivalent, with $w = z'$, to the non-homogeneous problem

$$w' - \lambda w = e(t), \;\; w(1) = \frac{1}{2}\left[w(0) + w\Big(\frac{1}{2}\Big)\right],$$

and then the conclusion follows from Proposition 3.1. □

To construct the counterexample, we apply Proposition 5.2 to the case of the "eigenvalue" $\lambda_{bc,2,0} = -\log 4 + 2\pi i$ of (5.1). Let $e : [0,1] \to \mathbb{C}$ be a continuous function such that the problem

$$z''(t) = (-\log 4 + 2\pi i) z'(t) + e(t), \;\; z(0) = 0, \;\; z'(1) = \frac{1}{2}\, z'(0) + \frac{1}{2}\, z'\Big(\frac{1}{2}\Big) \tag{5.3}$$

has no solution. Setting $z(t) = x_1(t) + i x_2(t)$, $e(t) = e_1(t) + i e_2(t)$, problem (5.3) is equivalent to the planar real system

$$\begin{cases} x_1''(t) = -(\log 4) x_1'(t) - 2\pi x_2'(t) + e_1(t), \\ x_2''(t) = 2\pi x_1'(t) - (\log 4) x_2'(t) + e_2(t), \\ x_1(0) = 0, \;\; x_1'(1) = \dfrac{1}{2}\, x_1'(0) + \dfrac{1}{2}\, x_1'\Big(\dfrac{1}{2}\Big), \\ x_2(0) = 0, \;\; x_2'(1) = \dfrac{1}{2}\, x_2'(0) + \dfrac{1}{2}\, x_2'\Big(\dfrac{1}{2}\Big). \end{cases}$$

Let

$$g(t,v) := \big(-(\log 4) v_1(t) - 2\pi v_2(t) + e_1(t), 2\pi v_1(t) - (\log 4) v_2(t) + e_2(t)\big).$$

By (4.3), we obtain $\langle v, g(t,v)\rangle < 0$, when $|v| \geq R$ for some sufficiently large R.

Conclusion. *For problem* (1.10) *with the conditions of Corollary* 1.3 *on g and the existence of some $R > 0$ such that*

$$\langle v, g(t,u,v)\rangle \leq 0 \;\; \forall\, (t,u,v) \in [0,1] \times \overline{B}_R \times \partial B_R,$$

there is no existence theorem similar to Corollary 1.3.

Similar conclusions hold for problem (1.12).

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ON THE SOLVABILITY OF A NONLOCAL BOUNDARY VALUE PROBLEM FOR THE FIRST ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. We study a nonlocal boundary value problem for nonlinear functional differential equations. New effective conditions are found for the solvability and unique solvability of the problem under consideration. General results are applied to differential equations with deviating arguments.*

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Introduction

On the interval $[a,b]$, we consider the functional differential equation

$$u'(t) = F(u)(t), \tag{0.1}$$

where $F : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is a continuous (in general) nonlinear operator. As usually, by a solution of this equation we understand an absolutely continuous function $u : [a,b] \to \mathbb{R}$ satisfying equality (0.1) almost everywhere on $[a,b]$. Along with equation (0.1), we consider the nonlocal boundary condition

$$h(u) = \varphi(u), \tag{0.2}$$

where $\varphi : C([a,b];\mathbb{R}) \to \mathbb{R}$ is a continuous (in general) nonlinear functional and $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ is a (nonzero) linear bounded functional.

The following notation is used in the sequel.

- $\mathbb{R}$ is the set of all real numbers. $\mathbb{R}_+ = [0,+\infty[$.
- $C([a,b];\mathbb{R})$ is the Banach space of continuous functions $v : [a,b] \to \mathbb{R}$ with the norm $\|v\|_C = \max\{|v(t)| : t \in [a,b]\}$.
- $AC([a,b];\mathbb{R})$ is the set of absolutely continuous functions $v : [a,b] \to \mathbb{R}$.
- $L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to \mathbb{R}$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\,\mathrm{d}s$.
- $L([a,b];\mathbb{R}_+) = \big\{p \in L([a,b];\mathbb{R}) : p(t) \geq 0 \text{ for almost all } t \in [a,b]\big\}$.
- $\mathcal{L}_{ab}$ is the set of linear operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ for which there exists a function $\eta \in L([a,b];\mathbb{R}_+)$ such that
$$|\ell(v)(t)| \leq \eta(t)\|v\|_C \ \text{ for a.e. } t \in [a,b] \text{ and all } v \in C([a,b];\mathbb{R}).$$
- P_{ab} is the set of so-called *positive* operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$.
- F_{ab} is the set of linear bounded functionals $h : C([a,b];\mathbb{R}) \to \mathbb{R}$.
- PF_{ab} is the set of so-called *positive* functionals $h \in F_{ab}$ transforming the set $C([a,b];\mathbb{R}_+)$ into the set $\mathbb{R}_+$.
- $\mathcal{B}^i_{hc} = \{u \in C([a,b];\mathbb{R}) : h(u)\operatorname{sgn}\big((2-i)u(a)+(i-1)u(b)\big) \leq c\}$, where $h \in F_{ab}$, $c \in \mathbb{R}$, $i = 1,2$.
- $K([a,b] \times A; B)$, where $A, B \subseteq \mathbb{R}$, is the set of function $f : [a,b] \times A \to B$ satisfying the Carathéodory conditions, i.e., $f(\,\cdot\,,x) : [a,b] \to B$ is a measurable function for all $x \in A$, $f(t,\cdot\,) : A \to B$ is a continuous function for almost every $t \in [a,b]$, and for every $r > 0$, there exists $q_r \in L([a,b];\mathbb{R}_+)$ such that
$$|f(t,x)| \leq q_r(t) \ \text{ for a.e. } t \in [a,b] \text{ and all } x \in A,\ |x| \leq r.$$

As usual, throughout the paper we suppose the following assumptions on a nonlinear operator F and a functional φ:

$$\begin{gathered} F : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R}) \ \text{ is a continuous operator such that the relation} \\ \sup\big\{|F(v)(\cdot)| : v \in C([a,b];\mathbb{R}),\ \|v\|_C \leq r\big\} \in L([a,b];\mathbb{R}_+) \ \text{ holds for every } r > 0. \end{gathered} \tag{H1}$$

and

$$\begin{gathered} \varphi : C([a,b];\mathbb{R}) \to \mathbb{R} \ \text{ is a continuous functional such that the condition} \\ \sup\big\{|\varphi(v)| : v \in C([a,b];\mathbb{R}),\ \|v\|_C \leq r\big\} < +\infty \ \text{ holds for every } r > 0. \end{gathered} \tag{H2}$$

The solvability of boundary value problems for functional differential equations is being studied intensively. There are many interesting results in the literature (see, e.g., [1–6, 8] and references therein). But in the case, where a nonlocal boundary condition is considered, there are still many open problems.

In this paper, we generalize the results stated in [4] in such a way that the boundary condition (0.2) is considered as a nonlocal perturbation of the two point condition

$$u(a) + \lambda u(b) = \varphi(u). \tag{0.3}$$

where $\lambda \in \mathbb{R}_+$. Consequently, in what follows, we consider the linear functional h in the form

$$h(v) \stackrel{\text{def}}{=} v(a) + \lambda v(b) - h_0(v) + h_1(v) \ \text{ for } \ v \in C([a,b];\mathbb{R}), \tag{0.4}$$

where $\lambda \in \mathbb{R}_+$ and $h_0, h_1 \in PF_{ab}$. There is no loss of generality to assume h like the above one, because an arbitrary linear functional h can be represented in this form.

One can see that a particular case of equation (0.1) is, for example, the differential equation with deviating arguments

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\sigma(t)) + f\big(t, u(t), u(\mu(t))\big), \tag{0.5}$$

where $p, g \in L([a,b];\mathbb{R}_+)$, $\tau, \sigma, \mu : [a,b] \to [a,b]$ are measurable functions, and $f \in K([a,b] \times \mathbb{R}^2;\mathbb{R})$. We mention that the conditions for the solvability and unique solvability of boundary value problems for this equation are presented in Section 2.

On the other hand, the boundary condition (0.2) covers, for example, the Cauchy problem, anti-periodic problem, condition (0.3) and an integral condition of the form $\int\limits_a^b u(s)\,\mathrm{d}s = c$.

The statements formulated below generalize some results stated in [7] concerning the linear case, as well as, some results presented in [4] concerning problem (0.1), (0.3).

1 Main results

In this section, new effective conditions are found for the solvability and unique solvability of problem (0.1), (0.2).

Theorem 1.1. *Let $c \in \mathbb{R}_+$, h be defined by (0.4), where $\lambda \in]0,1]$ and*

$$h_0(1) < \lambda. \tag{1.1}$$

Let, moreover,

$$\varphi(v)\operatorname{sgn} v(b) \le c \ \text{ for } \ v \in C([a,b];\mathbb{R}) \tag{1.2}$$

and there exist

$$\ell_0, \ell_1 \in P_{ab} \tag{1.3}$$

such that on the set $\mathcal{B}^2_{hc}([a,b];\mathbb{R})$ the inequality

$$\big(F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)\big)\operatorname{sgn} v(t) \ge -q(t, \|v\|_C) \ \text{ for a.e. } \ t \in [a,b] \tag{1.4}$$

holds, where the function $q \in K([a,b] \times \mathbb{R}_+;\mathbb{R}_+)$ satisfies

$$\lim_{x \to +\infty} \frac{1}{x} \int\limits_a^b q(s,x)\,\mathrm{d}s = 0. \tag{1.5}$$

If, in addition,

$$(1 + h_1(1))\|\ell_0(1)\|_L + \lambda\|\ell_1(1)\|_L < \lambda - h_0(1), \tag{1.6}$$

then problem (0.1), (0.2) has at least one solution.

Remark 1.1. Let the operator $\psi : L([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ be defined by the formula

$$\psi(z)(t) \stackrel{\text{def}}{=} z(a+b-t) \ \text{ for a.e. } t \in [a,b] \text{ and all } z \in L([a,b];\mathbb{R}).$$

Let, moreover, $\lambda \in [1,+\infty[$, ω be the restriction of ψ to the space $C([a,b];\mathbb{R})$, and

$$\widehat{F}(z)(t) \stackrel{\text{def}}{=} -\psi(F(\omega(z)))(t) \ \text{ for a.e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}),$$
$$\widehat{h}(z) \stackrel{\text{def}}{=} z(a) + \frac{1}{\lambda}\, z(b) - \frac{1}{\lambda}\, h_0(\omega(z)) + \frac{1}{\lambda}\, h_1(\omega(z)) \ \text{ for } \ z \in C([a,b];\mathbb{R}),$$
$$\widehat{\varphi}(z) \stackrel{\text{def}}{=} \frac{1}{\lambda}\, \varphi(\omega(z)) \ \text{ for } \ z \in C([a,b];\mathbb{R}).$$

It is not difficult to verify that if u is a solution of problem (0.1), (0.2), then the function $v \stackrel{\text{def}}{=} \omega(u)$ is a solution of the problem

$$v'(t) = \widehat{F}(v)(t), \quad \widehat{h}(v) = \widehat{\varphi}(v), \tag{1.7}$$

and vice versa, if v is a solution of problem (1.7), then the function $u \stackrel{\text{def}}{=} \omega(v)$ is a solution of problem (0.1), (0.2).

Using the transformation described in the previous remark, we can immediately derive from Theorem 1.1 the following statement.

Theorem 1.2. *Let $c \in \mathbb{R}_+$, h be defined by* (0.4)*, where $\lambda \in [1,+\infty[$ and*

$$h_0(1) < 1. \tag{1.8}$$

Let, moreover, the condition

$$\varphi(v)\operatorname{sgn} v(a) \le c \ \text{ for } \ v \in C([a,b];\mathbb{R}) \tag{1.9}$$

be fulfilled and there exist $\ell_0, \ell_1 \in P_{ab}$ such that on the set $\mathcal{B}^1_{hc}([a,b];\mathbb{R})$ the inequality

$$\big(F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)\big)\operatorname{sgn} v(t) \le q(t, \|v\|_C) \ \text{ for a.e. } \ t \in [a,b]$$

hold, where the function $q \in K([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$ satisfies (1.5)*. If, in addition,*

$$\|\ell_0(1)\|_L + (\lambda + h_1(1))\|\ell_1(1)\|_L < 1 - h_0(1), \tag{1.10}$$

then problem (0.1), (0.2) *has at least one solution.*

The next theorems deal with the unique solvability of problem (0.1), (0.2).

Theorem 1.3. *Let h be defined by* (0.4)*, where $\lambda \in [0,1[$ and $h_0(1)$ satisfies* (1.1)*. Let, moreover, the condition*

$$(\varphi(v) - \varphi(w))\operatorname{sgn}(v(b) - w(b)) \le 0 \tag{1.11}$$

hold for every $v, w \in C([a,b];\mathbb{R})$ and there exist $\ell_0, \ell_1 \in P_{ab}$ such that on the set $\mathcal{B}^2_{hc}([a,b];\mathbb{R})$ with $c = |\varphi(0)|$ the inequality

$$\big(F(v)(t) - F(w)(t) - \ell_0(v-w)(t) + \ell_1(v-w)(t)\big)\operatorname{sgn}(v(t) - w(t)) \ge 0 \tag{1.12}$$

is fulfilled for a.e. $t \in [a,b]$. If, in addition, condition (1.6) *is satisfied, then problem* (0.1), (0.2) *is uniquely solvable.*

Theorem 1.4. *Let h be defined by* (0.4)*, where $\lambda \ge 1$ and $h_0(1)$ satisfies* (1.8)*. Let, moreover, the condition*

$$(\varphi(v) - \varphi(w))\operatorname{sgn}(v(a) - w(a)) \le 0 \tag{1.13}$$

hold for every $v, w \in C([a,b];\mathbb{R})$ and there exist $\ell_0, \ell_1 \in P_{ab}$ such that, on the set $\mathcal{B}^1_{hc}([a,b];\mathbb{R})$ with $c = |\varphi(0)|$, the inequality

$$\big(F(v)(t) - F(w)(t) - \ell_0(v-w)(t) + \ell_1(v-w)(t)\big)\operatorname{sgn}(v(t) - w(t)) \le 0 \tag{1.14}$$

is fulfilled for a.e. $t \in [a,b]$. If, in addition, condition (1.10) *is satisfied, then problem* (0.1), (0.2) *is uniquely solvable.*

2 Corollaries for nonlinear delay differential equations

In this section, corollaries of the main theorems are presented. We formulate the conditions guaranteeing the solvability and the unique solvability of the problem

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\sigma(t)) + f(t, u(t), u(\mu(t))), \tag{0.5}$$
$$\varphi(u) = h(u), \tag{0.2}$$

where a linear functional h is considered by formula (0.4).

Corollary 2.1. *Let $c \in \mathbb{R}_+$ and h be defined by* (0.4)*, where $\lambda \in [0, 1[$ and $h_0(1)$ satisfies* (1.1)*. Let, moreover,* (1.2) *and*

$$f(t, x, y)\operatorname{sgn} x \geq -q(t) \ \textit{ for a.e. } t \in [a, b] \textit{ and all } x, y \in \mathbb{R}$$

be satisfied, where $q \in L([a, b]; \mathbb{R}_+)$. If, in addition,

$$(1 + h_1(1)) \int_a^b p(s)\,\mathrm{d}s + \lambda \int_a^b g(s)\,\mathrm{d}s < \lambda - h_0(1), \tag{2.1}$$

then problem (0.5), (0.2) *has at least one solution.*

Corollary 2.2. *Let $c \in \mathbb{R}_+$ and h be defined by* (0.4)*, where $\lambda \geq 1$ and $h_0(1)$ satisfies* (1.8)*. Let, moreover,* (1.9) *and*

$$f(t, x, y)\operatorname{sgn} x \leq q(t) \ \textit{ for a.e. } t \in [a, b] \textit{ and all } x, y \in \mathbb{R}$$

be satisfied, where $q \in L([a, b]; \mathbb{R}_+)$. If, in addition,

$$\int_a^b p(s)\,\mathrm{d}s + (\lambda + h_1(1)) \int_a^b g(s)\,\mathrm{d}s < 1 - h_0(1), \tag{2.2}$$

then problem (0.5), (0.2) *has at least one solution.*

Corollary 2.3. *Let h be defined by* (0.4)*, where $\lambda \in [0, 1[$ and $h_0(1)$ satisfies* (1.1)*. Let, moreover, conditions* (2.1) *and*

$$\big[f(t, x_1, y_1) - f(t, x_2, y_2)\big]\operatorname{sgn}(x_1 - x_2) \geq 0 \ \textit{ for a.e. } t \in [a, b] \textit{ and all } x_1, x_2, y_1, y_2 \in \mathbb{R}$$

hold. If, in addition, condition (1.11) *is fulfilled for every $v, w \in C([a, b]; \mathbb{R})$, then problem* (0.5), (0.2) *is uniquely solvable.*

Corollary 2.4. *Let h be defined by* (0.4)*, where $\lambda \geq 1$ and $h_0(1)$ satisfies* (1.8)*. Let, moreover, conditions* (2.2) *and*

$$\big[f(t, x_1, y_1) - f(t, x_2, y_2)\big]\operatorname{sgn}(x_1 - x_2) \leq 0 \ \textit{ for a.e. } t \in [a, b] \textit{ and all } x_1, x_2, y_1, y_2 \in \mathbb{R}$$

hold. If, in addition, condition (1.13) *is fulfilled for every $v, w \in C([a, b]; \mathbb{R})$, then problem* (0.5), (0.2) *is uniquely solvable.*

3 Auxiliary propositions

We use the lemma on a priory estimate stated in [6] to prove main results of the paper. It can be formulated as follows.

Lemma 3.1 ([6, Corollary 2])**.** *Let there exist a positive number ρ and an operator $\ell \in \mathcal{L}_{ab}$ such that the homogeneous problem*

$$u'(t) = \ell(u)(t), \quad h(u) = 0 \tag{3.1}$$

has only the trivial solution, and for every $\delta \in]0,1[$ an arbitrary function $u \in AC([a,b];\mathbb{R})$ satisfyings the relation

$$u'(t) = \ell(u)(t) + \delta\big[F(u)(t) - \ell(u)(t)\big] \ \text{ for a.e. } \ t \in [a,b], \quad h(u) = \delta\varphi(u) \tag{3.2}$$

admits the estimate

$$\|u\|_C \le \rho. \tag{3.3}$$

Then problem (0.1), (0.2) *has at least one solution.*

Definition 3.1. Let $h \in F_{ab}$. We say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $\mathcal{U}(h)$, if there exists $r > 0$ such that for arbitrary $q^* \in L([a,b];\mathbb{R}_+)$ and $c \in \mathbb{R}_+$ every function $u \in AC([a,b];\mathbb{R})$ satisfying the inequalities

$$h(u)\operatorname{sgn} u(b) \le c, \tag{3.4}$$

$$-(u'(t) - \ell(u)(t))\operatorname{sgn} u(t) \le q^*(t) \ \text{ for a.e. } \ t \in [a,b] \tag{3.5}$$

admits the estimate

$$\|u\|_C \le r(c + \|q^*\|_L). \tag{3.6}$$

Lemma 3.2. *Let $c \in \mathbb{R}_+$ and* (1.2) *hold. Let, moreover, there exists $\ell \in \mathcal{U}(h)$ such that on the set $\mathcal{B}^2_{hc}([a,b];\mathbb{R})$ the inequality*

$$-(F(v)(t) - \ell(v)(t))\operatorname{sgn} v(t) \le q(t, \|v\|_C) \ \text{ for a.e. } \ t \in [a,b] \tag{3.7}$$

is fulfilled, where the function $q \in K([a,b] \times \mathbb{R}_+;\mathbb{R}_+)$ satisfies (1.5)*. Then problem* (0.1), (0.2) *has at least one solution.*

Proof. Since $\ell \in \mathcal{U}(h)$, it is not difficult to show that the homogeneous problem (3.1) has only the trivial solution.

Assume that a function $u \in AC([a,b];\mathbb{R})$ satisfies (3.2) with some $\delta \in]0,1[$. By virtue of (1.2), inequality (3.4) is fulfilled, i.e., $u \in \mathcal{B}^2_{hc}([a,b];\mathbb{R})$. Moreover, from relations (3.2) and (3.7) we obtain that (3.5) holds with $q^* \equiv q(\,\cdot\,, \|u\|_C)$. Therefore, in view of (3.4), (3.5) and the assumption $\ell \in \mathcal{U}(h)$, there exist $r > 0$ such that estimate (3.6) holds.

On the other hand, according to (1.5), there exists $\rho > 2rc$ such that

$$\frac{1}{x}\int\limits_a^b q(s,x)\,\mathrm{d}s < \frac{1}{2r} \ \text{ for } \ x > \rho.$$

The last inequality, together with (3.6), yields that estimate (3.3) is satisfied. Since ρ depends neither on u nor on δ, it follows from Lemma 3.1 that problem (0.1), (0.2) has at least one solution. ☐

4 Proofs of main theorems

Proof of Theorem 1.1. Put $\ell = \ell_0 - \ell_1$, where ℓ_0, $\ell_1 \in P_{ab}$ are such that condition (1.6) holds. Firstly, we show that ℓ belongs to the set $\mathcal{U}(h)$.

Let $c \in \mathbb{R}_+$, $q^* \in L([a,b];\mathbb{R}_+)$, and $u \in AC([a,b];\mathbb{R})$ satisfy (3.4) and (3.5). We prove that estimate (3.6) holds, where the number r depends only on $\|\ell_0(1)\|_L$, $\|\ell_1(1)\|_L$, λ, $h_0(1)$, and $h_1(1)$.

It is obvious that

$$u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + \widetilde{q}(t) \ \text{ for a.e. } \ t \in [a,b], \tag{4.1}$$

where

$$\widetilde{q}(t) = u'(t) - \ell(u)(t) \ \text{ for a.e. } \ t \in [a,b].$$

Hence, in view of (0.4), (3.4) and (3.5), we get

$$\big(u(a)+\lambda u(b)-h_0(u)+h_1(u)\big)\operatorname{sgn} u(b)\le c \tag{4.2}$$

and

$$-\widetilde{q}(t)\operatorname{sgn} u(t)\le q^*(t) \ \text{ for a.e. } \ t\in[a,b]. \tag{4.3}$$

First suppose that the function u does not change its sign. Then from (4.2) it follows that

$$|u(a)|+\lambda|u(b)|-h_0(|u|)+h_1(|u|)\le c \ \text{ if } \ u(b)\ne 0. \tag{4.4}$$

Put

$$M_0=\max\{|u(t)|:\ t\in[a,b]\} \tag{4.5}$$

and choose $t_{M_0}\in[a,b]$ such that

$$|u(t_{M_0})|=M_0. \tag{4.6}$$

Clearly, $M_0\ge 0$ and, in view of (1.3), (4.3) and (4.6), from relation (4.1) we get

$$-|u(t)|'\le M_0\ell_1(1)(t)+q^*(t) \ \text{ for a.e. } \ t\in[a,b].$$

The integration of the last inequality from t_{M_0} to b with respect to (1.3), (4.4), (4.6), $\lambda\in\,]0,1]$ and $h_0,\,h_1\in PF_{ab}$, results in

$$M_0\big(\lambda-h_0(1)-\lambda\|\ell_1(1)\|_L\big)\le\|q^*\|_L+c. \tag{4.7}$$

Moreover, it follows from condition (1.6) that $\lambda-h_0(1)-\lambda\|\ell_1(1)\|_L>0$ and thus, relations (4.5) and (4.7) yield

$$\|u\|_C\le\big(\|q^*\|_L+c\big)\big(\lambda-h_0(1)-\lambda\|\ell_1(1)\|_L\big)^{-1}.$$

Consequently, estimate (3.6) holds with $r=(\lambda-h_0(1)-\lambda\|\ell_1(1)\|_L)^{-1}$.

Suppose now that the function u changes its sign. Put

$$m=-\min\{u(t):\ t\in[a,b]\},\quad M=\max\{u(t):\ t\in[a,b]\} \tag{4.8}$$

and choose $t_m,t_M\in[a,b]$ such that

$$-m=u(t_m),\quad M=u(t_M). \tag{4.9}$$

Obviously, $m>0$, $M>0$, and either

$$t_m>t_M \tag{4.10}$$

or

$$t_m<t_M. \tag{4.11}$$

Suppose that relation (4.10) holds. Then there exists $a_1\in\,]t_M,t_m[$ such that

$$u(a_1)=0,\quad u(t)>0 \ \text{ for } \ t_M\le t<a_1. \tag{4.12}$$

Let

$$a_2=\sup\big\{t\in[t_m,b]:\ u(s)<0 \ \text{ for } \ t_m\le s\le t\big\}.$$

Obviously,

$$u(t)<0 \ \text{ for } \ t_m\le t<a_2 \ \text{ and if } \ a_2<b \ \text{ then } \ u(a_2)=0. \tag{4.13}$$

Hence, in view of (4.2) and (4.9), we obtain

$$\lambda u(a_2)\ge -M(1+h_1(1))-mh_0(1)-c. \tag{4.14}$$

Integrating (4.1) from t_M to a_1 and from t_m to a_2, with respect to (1.3), (4.3), (4.8), (4.9), (4.12), (4.13), and (4.14), one gets

$$M\le M\int_{t_M}^{a_1}\ell_1(1)(s)\,\mathrm{d}s+m\int_{t_M}^{a_1}\ell_0(1)(s)\,\mathrm{d}s+\int_{t_M}^{a_1}q^*(s)\,\mathrm{d}s$$

and

$$\lambda m - M\big(1+h_1(1)\big) - mh_0(1) - c \le \lambda M \int\limits_{t_m}^{a_2} \ell_0(1)(s)\,\mathrm{d}s + \lambda m \int\limits_{t_m}^{a_2} \ell_1(1)(s)\,\mathrm{d}s + \lambda \int\limits_{t_m}^{a_2} q^*(s)\,\mathrm{d}s.$$

Hence, we have

$$\begin{gathered} M(1-A) \le mC + \|q^*\|_L, \\ m(\lambda - h_0(1) - \lambda B) \le M(1 + h_1(1) + \lambda D) + \lambda\|q^*\|_L + c, \end{gathered} \tag{4.15}$$

where

$$A = \int\limits_{t_M}^{a_1} \ell_1(1)(s)\,\mathrm{d}s, \quad B = \int\limits_{t_m}^{a_2} \ell_1(1)(s)\,\mathrm{d}s, \quad C = \int\limits_{t_M}^{a_1} \ell_0(1)(s)\,\mathrm{d}s, \quad D = \int\limits_{t_m}^{a_2} \ell_0(1)(s)\,\mathrm{d}s.$$

By virtue of (1.6) and $\lambda \in]0,1]$, it is clear that $\lambda - h_0(1) - \lambda B > 0$ and $1 - A > 0$. Consequently, inequalities (4.15) imply

$$\begin{gathered} 0 < M(1-A)(\lambda - h_0(1) - \lambda B) \le C\big(M(1+h_1(1)+\lambda D) + \lambda\|q^*\|_L + c\big) + \|q^*\|_L(\lambda - h_0(1) - \lambda B), \\ 0 < m(1-A)(\lambda - h_0(1) - \lambda B) \le (mC + \|q^*\|_L)\big(1 + h_1(1) + \lambda D\big) + (1-A)(\lambda\|q^*\|_L + c). \end{gathered} \tag{4.16}$$

Observe that

$$(1-A)(\lambda - h_0(1) - \lambda B) \ge \lambda - \lambda(A+B) - h_0(1) \ge \lambda - \lambda\|\ell_1(1)\|_L - h_0(1). \tag{4.17}$$

Moreover, from (1.6) and $\lambda \in]0,1]$ we get

$$C(1 + h_1(1) + \lambda D) \le (1 + h_1(1))(C + D) \le (1 + h_1(1))\|\ell_0(1)\|_L. \tag{4.18}$$

In view of inequalities (1.6), (4.17), and (4.18), it follows from (4.16) that

$$\begin{gathered} M \le r_0\big(1 + \lambda + h_1(1) + \lambda\|\ell_0(1)\|_L\big)(c + \|q^*\|_L), \\ m \le r_0\big(1 + \lambda + h_1(1) + \lambda\|\ell_0(1)\|_L\big)(c + \|q^*\|_L), \end{gathered} \tag{4.19}$$

where

$$r_0 = \Big(\lambda - h_0(1) - \lambda\|\ell_1(1)\|_L - (1 + h_1(1))\|\ell_0(1)\|_L\Big)^{-1}. \tag{4.20}$$

Consequently, estimate (3.6) holds, where the number r is given by

$$r = r_0\big(1 + \lambda + h_1(1) + \lambda\|\ell_0(1)\|_L\big). \tag{4.21}$$

Let now (4.11) hold. Then there exists $a_3 \in]t_M, t_m[$ such that

$$u(a_3) = 0, \quad u(t) < 0 \ \text{ for } \ t_m \le t < a_3. \tag{4.22}$$

Put

$$a_4 = \sup\big\{t \in [t_M, b] : u\,(s) > 0 \ \text{ for } \ t_M \le s \le t\big\}.$$

It is clear that $u(t) > 0$ for $t_M \le t < a_4$ and if $a_4 < b$ then $u(a_4) = 0$. Hence, by virtue of (4.2), we obtain

$$\lambda u(a_4) \le m + Mh_0(1) + mh_1(1) + c. \tag{4.23}$$

Integrating (4.1) from t_m to a_3 and from t_M to a_4 and taking into account (1.3), (4.3), (4.8), (4.9), (4.22) and (4.23), one gets

$$m \le M \int\limits_{t_m}^{a_3} \ell_0(1)(s)\,\mathrm{d}s + m \int\limits_{t_m}^{a_3} \ell_1(1)(s)\,\mathrm{d}s + \int\limits_{t_m}^{a_3} q^*(s)\,\mathrm{d}s$$

and

$$-m - Mh_0(1) - mh_1(1) - c + \lambda M \leq \lambda m \int\limits_{t_M}^{a_4} \ell_0(1)(s)\,\mathrm{d}s + \lambda M \int\limits_{t_M}^{a_4} \ell_1(1)(s)\,\mathrm{d}s + \lambda \int\limits_{t_M}^{a_4} q^*(s)\,\mathrm{d}s.$$

Hence,

$$\begin{aligned} m(1-\widetilde{A}) &\leq M\widetilde{C} + \|q^*\|_L, \\ M\big(\lambda - h_0(1) - \lambda\widetilde{B}\big) &\leq m\big(1 + h_1(1) + \lambda\widetilde{D}\big) + c + \lambda\|q^*\|_L, \end{aligned} \tag{4.24}$$

where

$$\widetilde{A} = \int\limits_{t_m}^{a_3} \ell_1(1)(s)\,\mathrm{d}s, \quad \widetilde{B} = \int\limits_{t_M}^{a_4} \ell_1(1)(s)\,\mathrm{d}s, \quad \widetilde{C} = \int\limits_{t_m}^{a_3} \ell_0(1)(s)\,\mathrm{d}s, \quad \widetilde{D} = \int\limits_{t_M}^{a_4} \ell_0(1)(s)\,\mathrm{d}s.$$

In view of $\lambda \in]0,1]$ and (1.6), we have $\lambda - h_0(1) - \lambda\widetilde{B} > 0$ and $1 - \widetilde{A} > 0$. Therefore, inequalities (4.24) yield

$$\begin{aligned} 0 < m(1-\widetilde{A})\big(\lambda - h_0(1) - \lambda\widetilde{B}\big) &\leq m\widetilde{C}(1 + h_1(1) + \lambda\widetilde{D}) + (\|q^*\|_L + c)\big(1 + \lambda + h_1(1) + \|\ell_0(1)\|_L\big), \\ 0 < M(1-\widetilde{A})\big(\lambda - h_0(1) - \lambda\widetilde{B}\big) &\leq M\widetilde{C}(1 + h_1(1) + \lambda\widetilde{D}) + (\|q^*\|_L + c)\big(1 + \lambda + h_1(1) + \|\ell_0(1)\|_L\big). \end{aligned}$$

Now, analogously as in case (4.10), we show that relations (4.19) hold with r_0 given by (4.20). Consequently, estimate (3.6) is fulfilled, where the number r is defined by (4.21).

We have proved that estimate (3.6) holds in all possible cases and therefore, the operator $\ell = \ell_0 - \ell_1$ belongs to the set $\mathcal{U}(h)$. Therefore, it follows from Lemma 3.2 that problem (0.1), (0.2) has at least one solution. □

Proof of Theorem 1.2. According to Remark 1.1, the assertion of the theorem follows immediately from Theorem 1.1. □

Proof of Theorem 1.3. It follows from assumption (1.11) that inequality (1.2) is fulfilled on the set $C([a,b];\mathbb{R})$, where $c = |\varphi(0)|$. On the other hand, from (1.12) we get that inequality (1.4) holds on the set $\mathcal{B}^2_{hc}([a,b];\mathbb{R})$, where $q \equiv |F(0)|$. Consequently, according to Theorem 1.1, problem (0.1), (0.2) has at least one solution. Moreover, it follows from the proof of Theorem 1.1 that the operator $\ell = \ell_0 - \ell_1$ belongs to the set $\mathcal{U}(h)$.

It remains to prove that problem (0.1), (0.2) has at most one solution. Let u_1, u_2 be solutions of problem (0.1), (0.2). Put

$$u(t) = u_1(t) - u_2(t) \ \text{ for } \ t \in [a,b].$$

From relations (1.11) and (1.12), we get $u_1, u_2 \in \mathcal{B}^2_{hc}([a,b];\mathbb{R})$ with $c \equiv |\varphi(0)|$,

$$h(u)\operatorname{sgn} u(b) \leq 0,$$

and

$$-(u'(t) - \ell(u)(t))\operatorname{sgn} u(t) \leq 0 \ \text{ for a.e. } \ t \in [a,b].$$

Consequently, the last inequalities, together with $\ell \in \mathcal{U}(h)$, result in $u \equiv 0$, which yields $u_1 \equiv u_2$. □

Proof of Theorem 1.4. The assertion can be proved analogously to Theorem 1.3. We only use Theorem 1.2 instead of Theorem 1.1 and relations (1.13), (1.14) instead of (1.11), (1.12). □

Proofs of Corollaries 2.1–2.4. The assertions of corollaries follow from Theorems 1.1–1.4. □

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SOME REMARKS ON FUNCTIONAL DIFFERENTIAL EQUATIONS IN ABSTRACT SPACES

Abstract. The aim of this paper is to present some remarks concerning the functional differential equation

$$v'(t) = G(v)(t)$$

in a Banach space $\mathbb{X}$, where $G : C([a,b];\mathbb{X}) \to B([a,b];\mathbb{X})$ is a continuous operator and $C([a,b];\mathbb{X})$, resp. $B([a,b];\mathbb{X})$, denotes the Banach space of continuous, resp. Bochner integrable, abstract functions.

It is proved, in particular, that both initial value problems (Darboux and Cauchy) for the hyperbolic functional differential equation

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} = F(u)(t,x)$$

with a Carathéodory right-hand side on the rectangle $[a,b] \times [c,d]$ can be rewritten as initial value problems for abstract functional differential equation with a suitable operator G and $\mathbb{X} = C([c,d];\mathbb{R})$.*

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რეზიუმე. ნაშრომში მოყვანილია რამდენიმე შენიშვნა, რომელიც ეხება ფუნქციონალურ-დიფერენციალურ განტოლებას

$$v'(t) = G(v)(t)$$

ბანახის $\mathbb{X}$ სივრცეში, სადაც $G : C([a,b];\mathbb{X}) \to B([a,b];\mathbb{X})$ უწყვეტი ოპერატორია, ხოლო $C([a,b];\mathbb{X})$ და $B([a,b];\mathbb{X})$, შესაბამისად, უწყვეტ და ბოხნერის აზრით ინტეგრებად აბსტრაქტულ ფუნქციათა სივრცეებია.

კერძოდ, დადგენილია, რომ დარბუსა და კოშის ამოცანები ჰიპერბოლური ტიპის ფუნქციონალურ-დიფერენციალური განტოლებისთვის

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} = F(u)(t,x)$$

კარათეოდორის მარჯვენა მხარით $[a,b] \times [c,d]$ მართკუთხედზე შეიძლება გადაიწეროს როგორც საწყისი ამოცანა აბსტრაქტული ფუნქიონალურ-დიფერენციალური განტოლებისთვის შესაფერისი G ოპერატორითა და $\mathbb{X} = C([c,d];\mathbb{R})$ სივრცით.

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1 Statement of problem

On the interval $[a,b]$, we consider the functional differential equation

$$v'(t) = G(v)(t) \tag{1.1}$$

in a Banach space $\langle \mathbb{X}, \|\cdot\|_{\mathbb{X}}\rangle$, where $G : C([a,b];\mathbb{X}) \to B([a,b];\mathbb{X})$ is a continuous operator[1] satisfying the local Carathéodory condition (see Definition 2.9).

Definition 1.1. By a *solution* of equation (1.1) we understand an abstract function $v : [a,b] \to \mathbb{X}$ which is strongly absolutely continuous on $[a,b]$ (see Definition 2.1), differentiable a.e. on $[a,b]$ (see Definition 2.2), and satisfies equality (1.1) almost everywhere on $[a,b]$.

Remark 1.2. In Definition 1.1:

(a) Differentiability a.e. on $[a,b]$ has to be assumed, because it does not follow from the strong absolute continuity (in general). Indeed, let $\mathbb{X} = L([0,1];\mathbb{R})$ and

$$v(t)(x) = \begin{cases} 1 & \text{if } 0 \le x \le t \le 1, \\ 0 & \text{if } 0 \le t < x \le 1. \end{cases}$$

Then v is strongly absolutely continuous on $[0,1]$, but not differentiable a.e. on $[0,1]$ (see [3, Example 7.3.9]).

(b) Solutions of equation (1.1) are understood as global and strong ones, the notions like local existence and extendability of solutions have no sense in our concept.

Remark 1.3. In the existing literature, several kinds of abstract differential equations can be found and for each of them, a solution is defined in a different way. For instance, equation (1.1) differs from frequently studied abstract differential equations of the type

$$v' = A(t)v + f(t, v_t),$$

where $A(t)$ are usually densely closed linear operators with values in $\mathbb{X}$ that generate a semigroup etc. In those cases the so-called mild solutions are considered, i.e., the solutions of the corresponding integral equation

$$v(t) = \widehat{V}(t,0)v(0) + \int_0^t \widehat{V}(t,s) f(s, v_s)\,\mathrm{d}s,$$

where $\widehat{V}(t,s)$ denotes an evolution operator for $A(t)$.

We mention here two natural and straightforward particular cases of equation (1.1):

(A) $\mathbb{X} = \mathbb{R}$ – scalar first-order functional differential equations, for example,

- differential equation with an argument deviation

$$v'(t) = f(t, v(t), v(\tau(t))),$$

  where $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function and $\tau : [a,b] \to [a,b]$ is a measurable function,

- integro-differential equation

$$v'(t) = \int_a^b K(t,s) v(\tau(s))\,\mathrm{d}s,$$

  where $K : [a,b] \times [a,b] \to \mathbb{R}$ and $\tau : [a,b] \to [a,b]$ are suitable functions,

[1] For definition of the spaces $C([a,b];\mathbb{X})$ and $B([a,b];\mathbb{X})$, see Section 2.

- differential equation with a maximum

$$v'(t) = p(t) \max\big\{v(s) : \ \tau_1(t) \leq s \leq \tau_2(s)\big\} + q(s),$$

  where $p, q \in L([a,b];\mathbb{R})$ and $\tau_1, \tau_2 : [a,b] \to [a,b]$ are measurable functions.

(B) $\mathbb{X} = \mathbb{R}^n$ – systems of first-order functional differential equations and scalar higher-order functional differential equations.

For both cases $\mathbb{R}$ and $\mathbb{R}^n$, there are plenty of results concerning solvability as well as unique solvability of various boundary value problems, theorems on differential inequalities (maximum principles in other terminology), oscillations, etc. In order to extend our results from those topics (as well as our methodology) for functional differential equations in abstract spaces, some additional operations and structures are needed in $\mathbb{X}$ (like ordering, positivity, monotonicity, unit element, ...). Therefore, we are interested in other particular cases of (1.1) besides (A) and (B) that can help one to find out what operations and structures a Banach space $\mathbb{X}$ should be endowed with. We will show in Section 4 that the hyperbolic functional differential equation

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} = F(u)(t,x)$$

with a Carathéodory right-hand side on the rectangle $[a,b] \times [c,d]$ can be regarded as a particular case of the abstract equation (1.1) with $\mathbb{X} = C([c,d];\mathbb{R})$.

2 Notation and definitions

The following notation is used throughout the paper:

(1) $\langle \mathbb{X}, \| \cdot \|_{\mathbb{X}} \rangle$ is a Banach space.

(2) $C([a,b];\mathbb{X})$ is the Banach space of continuous abstract functions $v : [a,b] \to \mathbb{X}$ endowed with the norm $\|v\|_{C([a,b];\mathbb{X})} = \max\{\|v(t)\|_{\mathbb{X}} : \ t \in [a,b]\}$.

(3) $AC([a,b];\mathbb{X})$ is the set of strongly absolutely continuous abstract functions $v : [a,b] \to \mathbb{X}$ (see Definition 2.1 below).

(4) $B([a,b];\mathbb{X})$ is the Banach space of Bochner integrable abstract functions $g : [a,b] \to \mathbb{X}$ endowed with the norm $\|g\|_{B([a,b];\mathbb{X})} = \int\limits_a^b \|g(t)\|_{\mathbb{X}}\, \mathrm{d}t$.

(5) $L([a,b];\mathbb{R}) = B([a,b];\mathbb{R})$, see Lemma 2.7 below.

(6) $\mathcal{D} = [a,b] \times [c,d]$.

(7) $C(\mathcal{D};\mathbb{R})$ is the Banach space of continuous functions $u : \mathcal{D} \to \mathbb{R}$ endowed with the norm $\|u\|_{C(\mathcal{D};\mathbb{R})} = \max\{|u(t,x)| : \ (t,x) \in \mathcal{D}\}$.

(8) The first- and the second-order partial derivatives of the function $v : \mathcal{D} \to \mathbb{R}$ at the point $(t,x) \in \mathcal{D}$ are denoted by $v'_{[1]}(t,x)$ (or $v'_t(t,x)$, $\frac{\partial v(t,x)}{\partial t}$), $v'_{[2]}(t,x)$ (or $v'_x(t,x)$, $\frac{\partial v(t,x)}{\partial x}$), $v''_{[1,2]}(t,x)$ (or $v''_{tx}(t,x)$, $\frac{\partial v(t,x)}{\partial t\, \partial x}$), and $v''_{[2,1]}(t,x)$ (or $v''_{xt}(t,x)$, $\frac{\partial v(t,x)}{\partial x\, \partial t}$).

(9) $AC(\mathcal{D};\mathbb{R})$ is the set of functions $u : \mathcal{D} \to \mathbb{R}$ absolutely continuous in the sense of Carathéodory (see Definition 2.4 and Proposition 2.5 below).

(10) $L(\mathcal{D};\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : \mathcal{D} \to \mathbb{R}$ endowed with the norm $\|p\|_{L(\mathcal{D};\mathbb{R})} = \iint\limits_{\mathcal{D}} |p(t,x)|\, \mathrm{d}t\, \mathrm{d}x$.

(11) $\operatorname{meas} E$ denotes the Lebesgue measure of a (measurable) set $E \subset \mathbb{R}$.

Definition 2.1 ([3, Definition 7.1.7])**.** A function $v : [a, b] \to \mathbb{X}$ is said to be *strongly absolutely continuous*, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum\limits_i \|v(b_i) - v(a_i)\|_{\mathbb{X}} < \varepsilon$ whenever $\{[a_i, b_i]\}$ is a finite system of mutually non-overlapping subintervals of $[a, b]$ that satisfies $\sum\limits_i (b_i - a_i) < \delta$.

Definition 2.2 ([3, Definition 7.3.2])**.** A function $v : [a, b] \to \mathbb{X}$ is said to be *differentiable at the point* $t \in [a, b]$, if there is $\chi \in \mathbb{X}$ such that

$$\lim_{\delta \to 0} \left\| \frac{v(t+\delta) - v(t)}{\delta} - \chi \right\|_{\mathbb{X}} = 0.$$

We denote $\chi = v'(t)$ the derivative of v at t. If v is differentiable at every point $t \in E \subseteq [a, b]$ with $\operatorname{meas} E = b - a$, then v is called *differentialbe almost everywhere (a.e.) on* $[a, b]$.

Definition 2.3 ([1, § 7.3])**.** Let $\mathcal{S}(\mathcal{D})$ denote the system of rectangles $[t_1, t_2] \times [x_1, x_2]$ contained in $\mathcal{D}$. A mapping $\Phi : \mathcal{S}(\mathcal{D}) \to \mathbb{R}$ is said to be *absolutely continuous function of rectangles*, if it is additive and for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite system $\{[a_i, b_i] \times [c_i, d_i]\}$ of mutually non-overlapping rectangles contained in $\mathcal{D}$, the implication

$$\sum_i (b_i - a_i)(d_i - c_i) < \delta \implies \sum_i \big|\Phi([a_i, b_i] \times [c_i, d_i])\big| < \varepsilon$$

holds.

Definition 2.4. We say that a function $u : \mathcal{D} \to \mathbb{R}$ is *absolutely continuous in the sense of Carathéodory* if the following conditions hold:

(a) the function of rectangles

$$\Phi_u([t_1, t_2] \times [x_1, x_2]) := u(t_1, x_1) - u(t_1, x_2) - u(t_2, x_1) + u(t_2, x_2) \text{ for } [t_1, t_2] \times [x_1, x_2] \subseteq \mathcal{D}$$

associated with u is absolutely continuous.

(b) the functions $u(\,\cdot\,, c) : [a, b] \to \mathbb{R}$ and $u(a, \,\cdot\,) : [c, d] \to \mathbb{R}$ are absolutely continuous.

Proposition 2.5 ([4, Theorem 3.1])**.** *The following assertions are equivalent:*

(1) *The function $u : \mathcal{D} \to \mathbb{R}$ is absolutely continuous in the sense of Carathéodory.*

(2) *The function $u : \mathcal{D} \to \mathbb{R}$ admits the integral representation*

$$u(t, x) = e + \int_a^t f(s)\,\mathrm{d}s + \int_c^x q(\eta)\,\mathrm{d}\eta + \iint\limits_{[a,t]\times[c,x]} p(s, \eta)\,\mathrm{d}s\,\mathrm{d}\eta \ \text{ for } \ (t, x) \in \mathcal{D}, \tag{2.1}$$

where $e \in \mathbb{R}$, $f \in L([a, b]; \mathbb{R})$, $q \in L([c, d]; \mathbb{R})$, and $p \in L(\mathcal{D}; \mathbb{R})$.

(3) *The function $u : \mathcal{D} \to \mathbb{R}$ satisfies the following conditions:*

(a) *$u(\,\cdot\,, x) \in AC([a, b]; \mathbb{R})$ for every $x \in [c, d]$, $u(a, \,\cdot\,) \in AC([c, d]; \mathbb{R})$,*

(b) *$u'_{[1]}(t, \,\cdot\,) \in AC([c, d]; \mathbb{R})$ for almost all $t \in [a, b]$,*

(c) *$u''_{[1,2]} \in L(\mathcal{D}; \mathbb{R})$.*

(4) *The function $u : \mathcal{D} \to \mathbb{R}$ satisfies the following conditions:*

(A) *$u(t, \,\cdot\,) \in AC([c, d]; \mathbb{R})$ for every $t \in [a, b]$, $u(\,\cdot\,, c) \in AC([a, b]; \mathbb{R})$,*

(B) *$u'_{[2]}(\,\cdot\,, x) \in AC([a, b]; \mathbb{R})$ for almost all $x \in [c, d]$,*

(C) *$u''_{[2,1]} \in L(\mathcal{D}; \mathbb{R})$.*

Lemma 2.6 ([4, Proposition 3.5]). *Let a function u be defined by formula* (2.1)*, where $e \in \mathbb{R}$, $f \in L([a,b];\mathbb{R})$, $q \in L([c,d];\mathbb{R})$, and $p \in L(\mathcal{D};\mathbb{R})$. Then there exists a measurable set $E \subseteq [a,b]$ such that* $\operatorname{meas} E = b - a$ *and*

$$u'_{[1]}(t,x) = f(t) + \int_c^x p(t,\eta)\,\mathrm{d}\eta \quad \textit{for} \ \ t \in E, \quad x \in [c,d].$$

Lemma 2.7 ([3, Remark 1.3.14]). *A function $g : [a,b] \to \mathbb{R}$ is Bochner integrable if and only if it is Lebesgue integrable and the two integrals of g have the same value.*

Lemma 2.8 ([3, Theorem 1.4.3]). *If $g \in B([a,b];\mathbb{X})$, then the function $\|g(\,\cdot\,)\|_{\mathbb{X}} : [a,b] \to \mathbb{R}$ is Lebesgue integrable.*

Definition 2.9. We say that an operator $G : C([a,b];\mathbb{X}) \to B([a,b];\mathbb{X})$ satisfies the *local Carathéodory condition* if for each $r > 0$ there exists a function $q_r \in L([a,b];\mathbb{R})$ such that

$$\|G(w)(t)\|_{\mathbb{X}} \leq q_r(t) \ \text{ for a.e. } t \in [a,b] \text{ and all } w \in C([a,b];\mathbb{X}), \ \ \|w\|_{C([a,b];\mathbb{X})} \leq r.$$

Definition 2.10. We say that an operator $F : C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ satisfies the *local Carathéodory condition* if for each $r > 0$ there exists a function $\zeta_r \in L(\mathcal{D};\mathbb{R})$ such that

$$|F(z)(t,x)| \leq \zeta_r(t,x) \ \text{ for a.e. } (t,x) \in \mathcal{D} \text{ and all } z \in C(\mathcal{D};\mathbb{R}), \ \ \|z\|_{C(\mathcal{D};\mathbb{R})} \leq r.$$

3 Hyperbolic functional differential equation

On the rectangle $\mathcal{D} = [a,b] \times [c,d]$, we consider the hyperbolic functional differential equation

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = F(u)(t,x), \tag{3.1}$$

where $F : C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ is a continuous operator satisfying the local Carathéodory condition (see Definition 2.10).

Definition 3.1. By a *solution* of equation (3.1) we understand a function $u : \mathcal{D} \to \mathbb{R}$ which is absolutely continuous in the sense of Carathéodory and satisfies equality (3.1) almost everywhere on $\mathcal{D}$.

Two main initial value problems for equation (3.1) are studied in the literature.

Darboux problem

The values of the solution u are prescribed on both characteristics $t = a$ and $x = c$, i.e., the initial conditions are

$$u(t,c) = \alpha(t) \ \text{ for } \ t \in [a,b], \quad u(a,x) = \beta(x) \ \text{ for } \ x \in [c,d], \tag{3.2}$$

where $\alpha \in AC([a,b];\mathbb{R})$, $\beta \in AC([c,d];\mathbb{R})$ are such that $\alpha(a) = \beta(c)$.

The following statement follows from the proof of [5, Theorem 4.1].

Proposition 3.2. *The function u is a solution of problem* (3.1), (3.2) *if and only if it is a solution of the integral equation*

$$u(t,x) = -\alpha(a) + \alpha(t) + \beta(x) + \int_a^t \int_c^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s$$

in the space $C(\mathcal{D};\mathbb{R})$.

Cauchy problem

Let $\mathcal{H}$ be a curve, which is defined as the graph of a decreasing continuous (not absolutely continuous, in general) function $h : [a,b] \to [c,d]$ such that $h(a) = d$ and $h(b) = c$. The values of the solution u and its partial derivative $u'_{[2]}$ are prescribed on $\mathcal{H}$ as follows:

$$u(t,h(t)) = \gamma(t) \ \text{ for } \ t \in [a,b], \quad u'_{[2]}(h^{-1}(x),x) = \psi(x) \ \text{ for a.e. } \ x \in [c,d], \tag{3.3}$$

where $\gamma \in C([a,b];\mathbb{R})$, $\psi \in L([c,d];\mathbb{R})$ are such that

$$\text{the function } \ t \longmapsto \gamma(t) + \int\limits_{h(t)}^{d} \psi(\eta)\,\mathrm{d}\eta \ \text{ is absolutely continuous on } [a,b] \tag{3.4}$$

(in other words, the pair (γ,ψ) is h-consistent, see [2, Section 3]).

The following statement follows from [2, Lemmas 3.3 and 3.4].

Proposition 3.3. *The function u is a solution of problem* (3.1), (3.3) *if and only if it is a solution of the integral equation*

$$u(t,x) = \gamma(t) + \int\limits_{h(t)}^{x} \psi(\eta)\,\mathrm{d}\eta + \int\limits_{h^{-1}(x)}^{t} \int\limits_{h(s)}^{x} F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s$$

in the space $C(\mathcal{D};\mathbb{R})$.

4 Main results

In this section, we formulate main results of the paper, namely, Theorems 4.1 and 4.4 showing that both Darboux and Cauchy problems for the hyperbolic equation (3.1) can be rewritten as initial value problems for the abstract equation (1.1) in the Banach space $C([c,d];\mathbb{R})$. Consequently, the hyperbolic equation (3.1) can be regarded as a particular case of (1.1) with $\mathbb{X} = C([c,d];\mathbb{R})$.

Theorem 4.1. *Let $\alpha \in AC([a,b];\mathbb{R})$, $\beta \in AC([c,d];\mathbb{R})$ be such that $\alpha(a) = \beta(c)$ and let $F : C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ be a continuous operator satisfying the local Carathéodory condition.*

If u is a solution of problem (3.1), (3.2), *then the function v defined by the formula*

$$v(t)(x) := u(t,x) \ \textit{ for } \ t \in [a,b], \ \ x \in [c,d], \tag{4.1}$$

is a solution of the problem

$$\begin{aligned} v'(t) &= G(v)(t), \\ v(a) &= \beta \end{aligned} \tag{4.2}$$

in the Banach space $C([c,d];\mathbb{R})$, where

$$\left.\begin{aligned} G(w)(t) &:= \widetilde{w}(t) \ \textit{ for a.e. } \ t \in [a,b] \ \textit{ and all } w \in C([a,b];C([c,d];\mathbb{R})), \\ \widetilde{w}(t)(x) &:= \alpha'(t) + \int\limits_{c}^{x} F(z)(t,\eta)\,\mathrm{d}\eta \ \textit{ for a.e. } \ t \in [a,b] \ \textit{ and all } x \in [c,d], \\ z(t,x) &:= w(t)(x) \ \textit{ for } \ (t,x) \in \mathcal{D}. \end{aligned}\right\} \tag{4.3}$$

Conversely, if v is a solution of problem (1.1), (4.2) *with G given by* (4.3), *then the function u defined by the formula*

$$u(t,x) := v(t)(x) \ \textit{ for } \ (t,x) \in \mathcal{D} \tag{4.4}$$

is a solution of problem (3.1), (3.2).

Remark 4.2. It follows from Propositions 5.1, 5.2, and 5.9 below that the formulation of Theorem 4.1 is correct.

Remark 4.3. Theorem 4.1 can be easily extended to a "more general" Darboux problem for equation (3.1), where the values of the solution u are prescribed on characteristics $t = t_0$ and $x = x_0$, i.e., the initial conditions are

$$u(t, x_0) = \alpha(t) \ \text{ for } \ t \in [a, b], \quad u(t_0, x) = \beta(x) \ \text{ for } \ x \in [c, d],$$

where $t_0 \in [a, b]$, $x_0 \in [c, d]$, $\alpha \in AC([a, b]; \mathbb{R})$, $\beta \in AC([c, d]; \mathbb{R})$ are such that $\alpha(t_0) = \beta(x_0)$.

Theorem 4.4. *Let $h \in C([a, b]; \mathbb{R})$ be a decreasing function such that $h(a) = d$ and $h(b) = c$. Let, moreover, $\gamma \in C([a, b]; \mathbb{R})$ and $\psi \in L([c, d]; \mathbb{R})$ be such that condition* (3.4) *holds and $F : C(\mathcal{D}; \mathbb{R}) \to L(\mathcal{D}; \mathbb{R})$ be a continuous operator satisfying the local Carathéodory condition.*

If u is a solution of problem (3.1), (3.3)*, then the function v defined by formula* (4.1) *is a solution of the problem*

$$\begin{gathered} v'(t) = G(v)(t), \\ v(t)(h(t)) = \gamma(t) \ \text{ for } \ t \in [a, b] \end{gathered} \tag{4.5}$$

in the Banach space $C([c, d]; \mathbb{R})$, where

$$\left.\begin{aligned} &G(w)(t) := \widetilde{w}(t) \ \text{ for a.e. } t \in [a, b] \text{ and all } w \in C([a, b]; C([c, d]; \mathbb{R})), \\ &\widetilde{w}(t)(x) := \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma(t) + \int_{h(t)}^{d} \psi(\eta)\,\mathrm{d}\eta\right) + \int_{h(t)}^{x} F(z)(t, \eta)\,\mathrm{d}\eta \ \text{ for a.e. } t \in [a, b] \text{ and all } x \in [c, d], \\ &z(t, x) := w(t)(x) \ \text{ for } \ (t, x) \in \mathcal{D}. \end{aligned}\right\} \tag{4.6}$$

Conversely, if v is a solution of problem (1.1), (4.5) *with G given by* (4.6)*, then the function u defined by formula* (4.4) *is a solution of problem* (3.1), (3.3).

Remark 4.5. It follows from Propositions 5.1, 5.2, and 5.10 below that the formulation of Theorem 4.4 is correct.

5 Proofs of main results

5.1 Auxiliary statements

We first show the properties of the relationship between abstract functions and the functions of two variables given by formulae (4.1) and (4.4).

Proposition 5.1. *Let $u \in C(\mathcal{D}; \mathbb{R})$ and the function v be defined by formula* (4.1)*. Then $v \in C([a, b]; C([c, d]; \mathbb{R}))$.*

Proof. It follows easily from the definitions of continuity. □

Proposition 5.2. *Let $v \in C([a, b]; C([c, d]; \mathbb{R}))$ and the function u be defined by formula* (4.4)*. Then $u \in C(\mathcal{D}; \mathbb{R})$.*

Proof. Let $(t_0, x_0) \in \mathcal{D}$ be arbitrary and let $\{(t_n, x_n)\}_{n=1}^{+\infty}$ be a sequence of points from the rectangle $\mathcal{D}$ such that $(t_n, x_n) \to (t_0, x_0)$ as $n \to +\infty$. Then, clearly,

$$\lim_{n \to +\infty} t_n = t_0, \quad \lim_{n \to +\infty} x_n = x_0.$$

Let $\varepsilon > 0$ be arbitrary. Since $v \in C([a, b]; C([c, d]; \mathbb{R}))$ and $v(t_0) \in C([c, d]; \mathbb{R})$, there exists $n_0 \in \mathbb{N}$ such that

$$\|v(t_n) - v(t_0)\|_{C([c,d];\mathbb{R})} < \frac{\varepsilon}{2}, \quad |v(t_0)(x_n) - v(t_0)(x_0)| < \frac{\varepsilon}{2} \ \text{ for } \ n \geq n_0$$

which yields
$$|v(t_n)(x) - v(t_0)(x)| < \frac{\varepsilon}{2} \text{ for } x \in [c,d], \quad n \geq n_0.$$
Consequently, we get
$$|u(t_n, x_n) - u(t_0, x_0)| \leq |v(t_n)(x_n) - v(t_0)(x_n)| + |v(t_0)(x_n) - v(t_0)(x_0)| \leq \varepsilon$$
for $n \geq n_0$ and thus, $\lim\limits_{n\to+\infty} u(t_n, x_n) = u(t_0, x_0)$. □

Proposition 5.3. *Let $u \in AC(\mathcal{D};\mathbb{R})$. Then the function v defined by formula* (4.1) *is strongly absolutely continuous, i.e., $v \in AC([a,b]; C([c,d];\mathbb{R}))$.*

Proof. It follows from Proposition 2.5 that the function u admits the integral representation (2.1), where $e \in \mathbb{R}$, $f \in L([a,b];\mathbb{R})$, $q \in L([c,d];\mathbb{R})$, and $p \in L(\mathcal{D};\mathbb{R})$.

Let $\varepsilon > 0$ be arbitrary. Since the function
$$t \longmapsto |f(t)| + \int_c^d |p(t,\eta)| \,\mathrm{d}\eta$$
is Lebesgue integrable on $[a,b]$, there exists $\delta > 0$ such that
$$\int_E \left(|f(s)| + \int_c^d |p(s,\eta)| \,\mathrm{d}\eta \right) \mathrm{d}s < \varepsilon \text{ for } E \subseteq [a,b], \text{ meas}\, E < \delta. \tag{5.1}$$
Let $\{[a_k, b_k]\}_{k=1}^n$ be an arbitrary system of mutually non-overlapping subintervals of $[a,b]$ such that
$$\sum_{k=1}^n (b_k - a_k) < \delta.$$
By virtue of (2.1) and (4.1), it is clear that
$$v(b_k)(x) - v(a_k)(x) = \int_{a_k}^{b_k} \left(f(s) + \int_c^x p(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s \text{ for } x \in [c,d], \quad k = 1, \dots, n,$$
and thus, we get
$$\begin{aligned} \sum_{k=1}^n \|v(b_k) - v(a_k)\|_{C([c,d];\mathbb{R})} &= \sum_{k=1}^n \max \left\{ \left| \int_{a_k}^{b_k} \left(f(s) + \int_c^x p(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s \right| : x \in [c,d] \right\} \\ &\leq \int_A \left(|f(s)| + \int_c^d |p(s,\eta)|\,\mathrm{d}\eta \right) \mathrm{d}s, \end{aligned} \tag{5.2}$$
where $A := \bigcup\limits_{k=1}^n [a_k, b_k]$. Since $\text{meas}\, A = \sum\limits_{k=1}^n (b_k - a_k) < \delta$, it follows from (5.1) and (5.2) that
$$\sum_{k=1}^n \|v(b_k) - v(a_k)\|_{C([c,d];\mathbb{R})} < \varepsilon.$$
□

Lemma 5.4. *Let $q \in L(\mathcal{D};\mathbb{R})$ be such that*
$$q(t,x) \geq 0 \ \textit{for a.e.} \ (t,x) \in \mathcal{D}. \tag{5.3}$$

Then there exists a measurable set $E \subseteq [a,b]$ *such that* $\operatorname{meas} E = b-a$ *and for each* $t \in E$ *the condition*

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_t^{t+\delta} \left(\int_c^x q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s = \int_c^x q(t,\eta)\,\mathrm{d}\eta \quad \textit{uniformly on } [c,d] \tag{5.4}$$

holds.

Proof. It follows from Lemma 2.6 that there exists a measurable set $E \subseteq \,]a,b[$ such that $\operatorname{meas} E = b-a$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \int_c^x q(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s = \int_c^x q(t,\eta)\,\mathrm{d}\eta \quad \text{for} \quad t \in E, \quad x \in [c,d],$$

i.e.,

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_t^{t+\delta} \left(\int_c^x q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s = \int_c^x q(t,\eta)\,\mathrm{d}\eta \quad \text{for} \quad (t,x) \in E \times [c,d]. \tag{5.5}$$

Let $t_0 \in E$ and $\varepsilon > 0$ be arbitrary. Since $q(t_0,\cdot\,) \in L([c,d];\mathbb{R})$, there exists $\zeta > 0$ such that

$$\int_{c_1}^{d_1} q(t_0,\eta)\,\mathrm{d}\eta < \frac{\varepsilon}{2} \quad \text{for} \quad c_1, d_1 \in [c,d], \quad |d_1 - c_1| < \zeta. \tag{5.6}$$

It is clear that there is a collection $x_1, x_2, \dots, x_n \in [c,d]$ such that $c = x_1 < x_2 < \dots < x_n = d$ and

$$\max\{x_{k+1} - x_k : k = 1, \dots, n-1\} < \zeta.$$

Condition (5.5) yields that for each $k \in \{1, \dots, n\}$, there exists $\zeta_k > 0$ such that

$$\left| \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \left(\int_c^{x_k} q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s - \int_c^{x_k} q(t_0,\eta)\,\mathrm{d}\eta \right| < \frac{\varepsilon}{2} \quad \text{for} \quad 0 < |\delta| < \zeta_k. \tag{5.7}$$

Put $\zeta_0 := \min\{\zeta_k : k = 1, \dots, n\}$ and let $x_0 \in [c,d]$ be arbitrary. It is clear that there exists $m \in \{1, \dots, n-1\}$ such that $x_m \le x_0 \le x_{m+1}$. According to assumption (5.3), the function

$$x \longmapsto \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \left(\int_c^x q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s \quad \text{is non-decreasing on } [c,d]$$

and thus, by virtue of (5.6) and (5.7), for any $\delta \in \mathbb{R}$ satisfying $0 < |\delta| < \zeta_0$, we get

$$\begin{aligned} &\frac{1}{\delta} \int_{t_0}^{t_0+\delta} \left(\int_c^{x_0} q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s - \int_c^{x_0} q(t_0,\eta)\,\mathrm{d}\eta \\ &\qquad \le \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \left(\int_c^{x_{m+1}} q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s - \int_c^{x_{m+1}} q(t_0,\eta)\,\mathrm{d}\eta + \int_{x_0}^{x_{m+1}} q(t_0,\eta)\,\mathrm{d}\eta \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and

$$\begin{aligned} &\int_c^{x_0} q(t_0,\eta)\,\mathrm{d}\eta - \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \left(\int_c^{x_0} q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s \\ &\qquad \le \int_{x_m}^{x_0} q(t_0,\eta)\,\mathrm{d}\eta + \int_c^{x_m} q(t_0,\eta)\,\mathrm{d}\eta - \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \left(\int_c^{x_m} q(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

However, it means that

$$\left|\frac{1}{\delta}\int\limits_{t_0}^{t_0+\delta}\left(\int\limits_{c}^{x_0} q(s,\eta)\,\mathrm{d}\eta\right)\mathrm{d}s-\int\limits_{c}^{x_0} q(t_0,\eta)\,\mathrm{d}\eta\right|<\varepsilon \ \text{ for } \ 0<|\delta|<\zeta_0.$$

Since x_0 is arbitrary and ζ_0 does not depend on x_0, we have

$$\left|\frac{1}{\delta}\int\limits_{t_0}^{t_0+\delta}\left(\int\limits_{c}^{x} q(s,\eta)\,\mathrm{d}\eta\right)\mathrm{d}s-\int\limits_{c}^{x} q(t_0,\eta)\,\mathrm{d}\eta\right|<\varepsilon \ \text{ for } \ 0<|\delta|<\zeta_0, \ \ x\in[c,d],$$

i.e., the desired condition (5.4) holds for every $t \in E$. □

Proposition 5.5. *Let $u \in AC(\mathcal{D};\mathbb{R})$. Then the function v defined by formula* (4.1) *is differentiable a.e. on $[a,b]$ and*

$$v'(t)=u'_{[1]}(t,\,\cdot\,) \ \textit{ for a.e. } \ t\in[a,b]. \tag{5.8}$$

Proof. It follows from Proposition 2.5 that the function u admits the integral representation (2.1), where $e \in \mathbb{R}$, $f \in L([a,b];\mathbb{R})$, $q \in L([c,d];\mathbb{R})$, and $p \in L(\mathcal{D};\mathbb{R})$.

By virtue of Lemma 2.6, there exists a measurable set $E_1 \subseteq [a,b]$ such that $\operatorname{meas} E_1 = b-a$ and

$$u'_{[1]}(t,x)=f(t)+\int\limits_{c}^{x} p(t,\eta)\,\mathrm{d}\eta \ \text{ for } \ t\in E_1, \ \ x\in[c,d], \tag{5.9}$$

$$\lim_{\delta\to 0}\frac{1}{\delta}\int\limits_{t}^{t+\delta} f(s)\,\mathrm{d}s=\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{a}^{t} f(s)\,\mathrm{d}s=f(t) \ \text{ for } \ t\in E_1. \tag{5.10}$$

Moreover, it follows from Lemma 5.4 with $q := \frac{|p|+p}{2}$ and $q := \frac{|p|-p}{2}$ that there exists a measurable set $E_2 \subseteq [a,b]$ such that $\operatorname{meas} E_2 = b-a$ and for every $t \in E_2$, relation (5.4) holds.

Put $E = E_1 \cap E_2$ and let $t_0 \in E$ be arbitrary. In view of (4.1) and (5.9), from (2.1) we get

$$\begin{aligned}
&\left|\frac{v(t_0+\delta)(x)-v(t_0)(x)}{\delta}-u'_{[1]}(t_0,x)\right|\\
&\qquad\le\left|\frac{1}{\delta}\int\limits_{t_0}^{t_0+\delta} f(s)\,\mathrm{d}s+\frac{1}{\delta}\int\limits_{t_0}^{t_0+\delta}\left(\int\limits_{c}^{x} p(s,\eta)\,\mathrm{d}\eta\right)\mathrm{d}s-f(t_0)-\int\limits_{c}^{x} p(t_0,\eta)\,\mathrm{d}\eta\right|\\
&\qquad\qquad\le\left|\frac{1}{\delta}\int\limits_{t_0}^{t_0+\delta} f(s)\,\mathrm{d}s-f(t_0)\right|+\left|\frac{1}{\delta}\int\limits_{t_0}^{t_0+\delta}\left(\int\limits_{c}^{x} p(s,\eta)\,\mathrm{d}\eta\right)\mathrm{d}s-\int\limits_{c}^{x} p(t_0,\eta)\,\mathrm{d}\eta\right|
\end{aligned}$$

for $x \in [c,d]$ and $\delta \neq 0$ small enough which, together with (5.10) and (5.4) with $t := t_0$, guarantees that

$$\lim_{\delta\to 0}\left\|\frac{v(t_0+\delta)-v(t_0)}{\delta}-u'_{[1]}(t_0,\,\cdot\,)\right\|_{C([c,d];\mathbb{R})}=0.$$

However, it means that the abstract function v is differentiable at t_0 and, moreover, $v'(t_0) = u'_{[1]}(t_0,\,\cdot\,)$. To conclude the proof it is sufficient to mention that $t_0 \in E$ was arbitrary and $\operatorname{meas} E = b-a$. □

Now we provide several statements concerning Bochner integrable abstract functions and their primitives.

Lemma 5.6. *Let $g \in B([a,b];\mathbb{X})$ and $F(t) := \int\limits_a^t g(s)\,\mathrm{d}s$ for $t \in [a,b]$. Then $F \in AC([a,b];\mathbb{X})$, F is differentiable a.e. on $[a,b]$, and*

$$F'(t)=g(t) \ \textit{ for a.e. } \ t\in[a,b]. \tag{5.11}$$

Proof. The assertion of the lemma follows from Theorems 7.4.9, 7.4.11, and 5.3.4 stated in [3]. □

Lemma 5.7 ([3, Theorem 7.4.13 and 5.3.4])**.** *Let $F \in AC([a,b];\mathbb{X})$ be differentiable a.e. on $[a,b]$ and condition* (5.11) *hold. Then $g \in B([a,b];\mathbb{X})$ and*

$$F(t) = F(a) + \int_a^t g(s)\,\mathrm{d}s \quad \textit{for} \ \ t \in [a,b]. \tag{5.12}$$

Proposition 5.8. *Let $f \in L([a,b];\mathbb{R})$ and $p \in L(\mathcal{D};\mathbb{R})$. For a.e. $t \in [a,b]$, we put*

$$g(t)(x) := f(t) + \int_c^x p(t,\eta)\,\mathrm{d}\eta \quad \textit{for} \ \ x \in [c,d]. \tag{5.13}$$

Then $g \in B([a,b];C([c,d];\mathbb{R}))$ and for each $t \in [a,b]$, the equality

$$\Big(\int_a^t g(s)\,\mathrm{d}s\Big)(x) = \int_a^t f(s)\,\mathrm{d}s + \int_a^t \int_c^x p(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \textit{for} \ \ x \in [c,d]\,^{2} \tag{5.14}$$

holds.

Proof. Observe that the abstract function $g : [a,b] \to C([c,d];\mathbb{R})$ is defined a.e. on $[a,b]$. Put

$$u(t,x) := \int_a^t f(s)\,\mathrm{d}s + \int_a^t \int_c^x p(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for} \ \ (t,x) \in \mathcal{D} \tag{5.15}$$

and define the function v by formula (4.1). It follows from Proposition 2.5 that $u \in AC(\mathcal{D};\mathbb{R})$ and, in view of Lemma 2.6, we have

$$u'_{[1]}(t,x) = f(t) + \int_c^x p(t,\eta)\,\mathrm{d}\eta \quad \text{for a.e. } t \in [a,b] \text{ and all } x \in [c,d]. \tag{5.16}$$

On the other hand, Proposition 5.5 yields that the abstract function $v : [a,b] \to C([c,d];\mathbb{R})$ is differentiable a.e. on $[a,b]$ and condition (5.8) holds which, together with (5.13) and (5.16), guarantees that

$$v'(t) = g(t) \quad \text{for a.e. } \ t \in [a,b]. \tag{5.17}$$

Moreover, according to Proposition 5.3, $v \in AC([a,b];C([c,d];\mathbb{R}))$ and thus, it follows from (5.17) and Lemma 5.7 that $g \in B([a,b];C([c,d];\mathbb{R}))$ and

$$v(t) = v(a) + \int_a^t g(s)\,\mathrm{d}s \quad \text{for} \ \ t \in [a,b].$$

However, in view of (4.1) and (5.15), it means that for each $t \in [a,b]$, equality (5.14) holds. □

At the end of this section, we provide two statements guaranteeing that formulations of Theorems 4.1 and 4.4 are correct.

Proposition 5.9. *Let $\alpha \in AC([a,b];\mathbb{R})$ and $F : C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ be a continuous operator satisfying the local Carathéodory condition* (*see Definition* 2.10)*. Then the operator G defined by formula* (4.3) *maps the set $C([a,b];C([c,d];\mathbb{R}))$ into the set $B([a,b];C([c,d];\mathbb{R}))$, it is continuous and satisfies the local Carathéodory condition* (*see Definition* 2.9)*.*

[2] Observe that the integral on the left-hand side of the equality is Bochner one, whereas both integrals on its righ-hand side are Lebesgue ones.

Proof. Let $w \in C([a,b];C([c,d];\mathbb{R}))$ be arbitrary and put $z(t,x) := w(t)(x)$ for $(t,x) \in \mathcal{D}$. Observe that, in view of Proposition 5.2, we have $z \in C(\mathcal{D};\mathbb{R})$. It follows from Proposition 5.8 with $f := \alpha'$ and $p := F(z)$ that $G(w) \in B([a,b];C([c,d];\mathbb{R}))$ and thus, the operator G maps $C([a,b];C([c,d];\mathbb{R}))$ into $B([a,b];C([c,d];\mathbb{R}))$.

Now let $v_n, v \in C([a,b];C([c,d];\mathbb{R}))$, $n \in \mathbb{N}$, be such that $\lim\limits_{n\to+\infty} \|v_n - v\|_{C([a,b];C([c,d];\mathbb{R}))} = 0$. Put

$$u_n(t,x) := v_n(t)(x), \quad u(t,x) := v(t)(x) \text{ for } (t,x) \in \mathcal{D}, \quad n \in \mathbb{N}. \tag{5.18}$$

Then, by virtue of Proposition 5.2, we have $u_n, u \in C(\mathcal{D};\mathbb{R})$ for $n \in \mathbb{N}$ and, moreover, it is not difficult to verify that

$$\lim_{n\to+\infty} \|u_n - u\|_{C(\mathcal{D};\mathbb{R})} = 0.$$

Since the operator F is supposed to be continuous, the latter relation yields

$$\lim_{n\to+\infty} \iint\limits_{\mathcal{D}} |F(u_n)(s,\eta) - F(u)(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta = 0. \tag{5.19}$$

According to (4.3) and (5.18), it is clear that

$$\begin{aligned}\|G(v_n)(t) - G(v)(t)\|_{C([c,d];\mathbb{R})} &= \max\left\{\left|\int_c^x F(u_n)(t,\eta)\,\mathrm{d}\eta - \int_c^x F(u_n)(t,\eta)\,\mathrm{d}\eta\right| :\ x \in [c,d]\right\} \\ &\leq \int_c^d |F(u_n)(t,\eta) - F(u_n)(t,\eta)|\,\mathrm{d}\eta \text{ for a.e. } t \in [a,b], \quad n \in \mathbb{N},\end{aligned}$$

whence we get

$$\begin{aligned}&\|G(v_n) - G(v)\|_{B([a,b];C([c,d];\mathbb{R}))} \\ &\quad = \int_a^b \|G(v_n)(t) - G(v)(t)\|_{C([c,d];\mathbb{R})}\,\mathrm{d}t \leq \iint\limits_{\mathcal{D}} |F(u_n)(s,\eta) - F(u)(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta \text{ for } n \in \mathbb{N}.\end{aligned}$$

However, the latter inequality and (5.19) guarantee that $\lim\limits_{n\to+\infty} \|G(v_n) - G(v)\|_{B([a,b];C([c,d];\mathbb{R}))} = 0$, i.e., the operator G is continuous.

Finally, let $r > 0$ be arbitrary and $\zeta_r \in L(\mathcal{D};\mathbb{R})$ be the function appearing in the Carathéodory condition for the operator F (see Definition 2.10). Let, moreover, $w \in C([a,b];C([c,d];\mathbb{R}))$ be an arbitrary function such that $\|w\|_{C([a,b];C([c,d];\mathbb{R}))} \leq r$ and put $z(t,x) := w(t)(x)$ for $(t,x) \in \mathcal{D}$. In view of Proposition 5.2, we have $z \in C(\mathcal{D};\mathbb{R})$ and, moreover, it is not difficult to verify that $\|z\|_{C(\mathcal{D};\mathbb{R})} \leq r$.

Then

$$\begin{aligned}\|G(w)(t)\|_{C([c,d];\mathbb{R})} &= \max\left\{\left|\alpha'(t) + \int_c^x F(z)(t,\eta)\,\mathrm{d}\eta\right| :\ x \in [c,d]\right\} \\ &\leq |\alpha'(t)| + \int_c^d |F(z)(t,\eta)|\,\mathrm{d}\eta \leq |\alpha'(t)| + \int_c^d \zeta_r(t,\eta)\,\mathrm{d}\eta\end{aligned}$$

for a.e. $t \in [a,b]$. Since the function

$$t \longmapsto |\alpha'(t)| + \int_c^d \zeta_r(t,\eta)\,\mathrm{d}\eta \text{ is Lebesgue integrable on } [a,b],$$

the operator G satisfies the local Carathéodory condition with the function $q_r = |\alpha'| + \int_c^d \zeta_r(\,\cdot\,,\eta)\,\mathrm{d}\eta$ (see Definition 2.9). $\square$

Proposition 5.10. *Let $h \in C([a,b];\mathbb{R})$ be a decreasing function such that $h(a) = d$ and $h(b) = c$. Let, moreover, $\gamma \in C([a,b];\mathbb{R})$ and $\psi \in L([c,d];\mathbb{R})$ be such that condition* (3.4) *holds and $F : C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ be a continuous operator satisfying the local Carathéodory condition* (*see Definition* 2.10). *Then the operator G defined by formula* (4.6) *maps $C([a,b];C([c,d];\mathbb{R}))$ into $B([a,b];C([c,d];\mathbb{R}))$, it is continuous and satisfies the local Carathéodory condition* (*see Definition* 2.9).

Proof. Let $w \in C([a,b];C([c,d];\mathbb{R}))$ be arbitrary. Put $z(t,x) := w(t)(x)$ for $(t,x) \in \mathcal{D}$ and

$$H := \big\{(s,\eta) \in \mathcal{D} :\ a \le s \le b,\ c \le \eta \le h(s)\big\}.$$

Observe that, in view of Proposition 5.2, we have $z \in C(\mathcal{D};\mathbb{R})$. Since $F(z) \in L(\mathcal{D};\mathbb{R})$, it is easy to see that

$$\iint\limits_{H} F(z)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = \int\limits_a^b \Bigg(\int\limits_c^{h(s)} F(z)(s,\eta)\,\mathrm{d}\eta\Bigg)\,\mathrm{d}s$$

and thus,

$$\text{the function } t \longmapsto \int\limits_c^{h(t)} F(z)(t,\eta)\,\mathrm{d}\eta \text{ is Lebesgue integrable on } [a,b]. \tag{5.20}$$

Now we put

$$\varphi(t) := \frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\gamma(t) + \int\limits_{h(t)}^{d} \psi(\eta)\,\mathrm{d}\eta\Bigg). \tag{5.21}$$

Clearly, $\varphi \in L([a,b];\mathbb{R})$ because we assume that condition (3.4) holds. Therefore, it follows from Proposition 5.8 with $f := \varphi - \int\limits_c^{h(\cdot)} F(z)(\cdot,\eta)\,\mathrm{d}\eta$ and $p := F(z)$ that $G(w) \in B([a,b];C([c,d];\mathbb{R}))$ and thus, the operator G maps $C([a,b];C([c,d];\mathbb{R}))$ into $B([a,b];C([c,d];\mathbb{R}))$.

Analogously to the proof of Proposition 5.9, we show that the operator G is continuous and satisfies the local Carathéodory condition with the function $q_r = \varphi + \int\limits_c^d \zeta_r(\cdot,\eta)\,\mathrm{d}\eta$ (see Definition 2.9), where $\zeta_r \in L(\mathcal{D};\mathbb{R})$ is the function appearing in the Carathéodory condition for the operator F (see Definition 2.10). ☐

5.2 Proofs of Theorems 4.1 and 4.4

Proof of Theorem 4.1. Let u be a solution of problem (3.1), (3.2) and let the function v be defined by formula (4.1). In view of Proposition 3.2, it follows from Lemma 2.6 that

$$u'_{[1]}(t,x) = \alpha'(t) + \int\limits_c^x F(u)(t,\eta)\,\mathrm{d}\eta \text{ for a.e. } t \in [a,b] \text{ and all } x \in [c,d]. \tag{5.22}$$

On the other hand, Propositions 5.3 and 5.5 yield that the abstract function $v : [a,b] \to C([c,d];\mathbb{R})$ is strongly absolutely continuous, differentiable a.e. on $[a,b]$, and satisfies condition (5.8). Hence, from (5.8) and (5.22) we get

$$v'(t) = \alpha'(t) + \int\limits_c^{\cdot} F(u)(t,\eta)\,\mathrm{d}\eta = G(v)(t) \text{ for a.e. } t \in [a,b],$$

where the operator G is defined by formula (4.3). Moreover, $v(a) = u(a,\cdot) = \beta$ and thus, the function v is a solution of problem (1.1), (4.2) in the Banach space $C([c,d];\mathbb{R})$.

Conversely, assume that v is a solution of problem (1.1), (4.2) with G given by (4.3) and define the function u by formula (4.4). Since the function v is strongly absolutely continuous, according to Proposition 5.2, we have $u \in C(\mathcal{D};\mathbb{R})$. It follows immediately from Lemma 5.7 that

$$v(t) = v(a) + \int_a^t G(v)(s)\,\mathrm{d}s \quad \text{for} \quad t \in [a,b], \tag{5.23}$$

i.e.,

$$v(t) = \beta + \int_a^t g(s)\,\mathrm{d}s \quad \text{for} \quad t \in [a,b], \tag{5.24}$$

where the function $g : [a,b] \to C([c,d];\mathbb{R})$ is for a. a. $t \in [a,b]$ defined by formula (5.13) with $f := \alpha'$ and $p := F(u)$. Therefore, by virtue of Proposition 5.8, we get

$$\Bigg(\int_a^t g(s)\,\mathrm{d}s\Bigg)(x) = \int_a^t \alpha'(s)\,\mathrm{d}s + \int_a^t \int_c^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for} \quad (t,x) \in \mathcal{D}$$

which, together with (4.4) and (5.24), yields that

$$u(t,x) = -\alpha(a) + \alpha(t) + \beta(x) + \int_a^t \int_c^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Consequently, according to Proposition 3.2, the function u is a solution of problem (3.1), (3.2). $\square$

Proof of Theorem 4.4. Define the function φ by formula (5.21). It is clear that $\varphi \in L([a,b];\mathbb{R})$ because we assume that condition (3.4) holds.

Let u be a solution of problem (3.1), (3.3) and let the function v be defined by formula (4.1). In view of Proposition 3.3, it follows from Lemma 2.6 that

$$u'_{[1]}(t,x) = \varphi(t) + \int_{h(t)}^x F(u)(t,\eta)\,\mathrm{d}\eta \quad \text{for a.e. } t \in [a,b] \text{ and all } x \in [c,d]. \tag{5.25}$$

On the other hand, Propositions 5.3 and 5.5 yield that the abstract function $v : [a,b] \to C([c,d];\mathbb{R})$ is strongly absolutely continuous, differentiable a.e. on $[a,b]$, and satisfies condition (5.8). Hence, from (5.8) and (5.25) we get

$$v'(t) = \varphi(t) + \int_{h(t)}^{\cdot} F(u)(t,\eta)\,\mathrm{d}\eta = G(v)(t) \quad \text{for a.e.} \quad t \in [a,b],$$

where the operator G is defined by formula (4.6). Moreover, $v(t)(h(t)) = u(t,h(t)) = \gamma(t)$ for $t \in [a,b]$ and thus, the function v is a solution of problem (1.1), (4.5) in the Banach space $C([c,d];\mathbb{R})$.

Conversely, assume that v is a solution of problem (1.1), (4.5) with G given by (4.6) and define the function u by formula (4.4). Since the function v is strongly absolutely continuous, according to Proposition 5.2, we have $u \in C(\mathcal{D};\mathbb{R})$. Analogously to the proof of Proposition 5.10, we show that

$$\text{the function} \quad t \longmapsto \int_c^{h(t)} F(u)(t,\eta)\,\mathrm{d}\eta \quad \text{is Lebesgue integrable on } [a,b].$$

It follows immediately from Lemma 5.7 that condition (5.23) holds, i.e.,

$$v(t) = v(a) + \int_a^t g(s)\,\mathrm{d}s \quad \text{for} \quad t \in [a,b], \tag{5.26}$$

where the function $g : [a,b] \to C([c,d];\mathbb{R})$ is for a.a. $t \in [a,b]$ defined by formula (5.13) with $f := \varphi - \int_c^{h(\cdot)} F(u)(\cdot,\eta)\,\mathrm{d}\eta$ and $p := F(u)$. Therefore, by virtue of Proposition 5.8, we get

$$\begin{aligned}\Big(\int_a^t g(s)\,\mathrm{d}s\Big)(x) &= \int_a^t \Big(\varphi(s) - \int_c^{h(s)} F(u)(s,\eta)\,\mathrm{d}\eta\Big)\,\mathrm{d}s + \int_a^t\int_c^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \\ &= \gamma(t) + \int_{h(t)}^d \psi(\eta)\,\mathrm{d}\eta - \gamma(a) + \int_a^t\int_{h(s)}^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s\end{aligned}$$

for $(t,x) \in \mathcal{D}$ which, together with (4.4) and (5.26), yields that

$$u(t,x) = u(a,x) + \gamma(t) - \gamma(a) + \int_{h(t)}^d \psi(\eta)\,\mathrm{d}\eta + \int_a^t\int_{h(s)}^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D}. \tag{5.27}$$

It follows from the initial condition (4.5) that $u(h^{-1}(x),x) = \gamma(h^{-1}(x))$ for $x \in [c,d]$. Therefore, substituting $h^{-1}(x)$ for t in equality (5.27), we get

$$u(a,x) = \gamma(a) - \int_x^d \psi(\eta)\,\mathrm{d}\eta - \int_a^{h^{-1}(x)}\int_{h(s)}^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } x \in [c,d].$$

Hence, (5.27) implies

$$u(t,x) = \gamma(t) + \int_{h(t)}^x \psi(\eta)\,\mathrm{d}\eta + \int_{h^{-1}(x)}^t\int_{h(s)}^x F(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D}.$$

Consequently, according to Proposition 3.3, the function u is a solution of problem (3.1), (3.3). $\square$

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EXISTENCE RESULTS FOR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

Abstract. We discuss the existence of solutions to the implicit fractional differential equation ${}^c\mathcal{D}^\alpha u = f(t, u, u', {}^c\mathcal{D}^\beta u, {}^c\mathcal{D}^\alpha u)$ satisfying nonlocal boundary conditions. Here $1 < \beta < \alpha \leq 2$, f is continuous and ${}^c\mathcal{D}$ is the Caputo fractional derivative. The existence results are proved by the Leray–Schauder degree method.*

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Key words and phrases. Implicit fractional differential equation, nonlocal condition, Leray–Schauder degree, Caputo fractional derivative.

რეზიუმე. შესწავლილია არალოკალური სასაზღვრო ამოცანის ამოხსნადობის საკითხი არაცხადი წილად-წარმოებულიანი ${}^c\mathcal{D}^\alpha u = f(t, u, u', {}^c\mathcal{D}^\beta u, {}^c\mathcal{D}^\alpha u)$ განტოლებისთვის, სადაც $1 < \beta < \alpha \leq 2$, f უწყვეტია და ${}^c\mathcal{D}$ კაპუტოს წილადი წარმოებულია. ამონახსნის არსებობა დადგენილია ლერე-შაუდერის ხარისხოვანი მეთოდის გამოყენებით.

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1 Introduction

Let $T > 0$ be given, $J = [0, T]$, $\|x\| = \max\{|x(t)| : t \in J\}$ be the norm in $C(J)$, while $\|x\|_1 = \|x\| + \|x'\|$ is the norm in $C^1(J)$.

In accordance with [12, 13], let $\mathcal{M}$ be the set of (generally nonlinear) functionals $\phi : C(J) \to \mathbb{R}$ which are

(i) continuous, $\phi(0) = 0$,

(ii) increasing, that is, $x, y \in C(J)$, $x(t) < y(t)$ for $t \in J \Longrightarrow \phi(x) < \phi(y)$.

Examples of functionals belonging to the set $\mathcal{M}$ were given in [12, 13].

We are interested in the implicit fractional differential equation

$$ {}^c\!D^\alpha u(t) = f\big(t, u(t), u'(t), {}^c\!D^\beta u(t), {}^c\!D^\alpha u(t)\big), \tag{1.1} $$

where $1 < \beta < \alpha \le 2$, $f \in C(J \times \mathbb{R}^4)$ and ${}^c\!D$ denotes the Caputo fractional derivative. Further conditions on f will be specified later.

Together with (1.1), we consider the nonlocal boundary condition

$$ u(0) = u(T), \quad \phi(u) = 0, \ \ \phi \in \mathcal{M}. \tag{1.2} $$

Example 1.1. The special cases of (1.2) are the boundary conditions:

$x(0) = 0$, $x(T) = 0$;

$x(0) = -x(\xi) = x(T)$, where $\xi \in (0, T)$;

$x(0) = x(T)$, $\min\{x(t) : t \in J\} = 0$;

$x(0) = x(T) = -\max\{x(t) : t \in J\}$.

Definition 1.1. We say that $u : J \to \mathbb{R}$ is *a solution of equation* (1.1) if $u', {}^c\!D^\alpha u \in C(J)$ and u satisfies (1.1) for $t \in J$. A solution u of (1.1) satisfying condition (1.2) is called *a solution of problem* (1.1), (1.2).

If $x, {}^c\!D^\alpha x \in C(J)$, then it is not difficult to verify that ${}^c\!D^\beta x(t) = I^{\alpha-\beta}\, {}^c\!D^\alpha x(t)$ for $t \in J$. Hence, if u is a solution of equation (1.1), then the equality

$$ {}^c\!D^\alpha u(t) = f\big(t, u(t), u'(t), I^{\alpha-\beta}\, {}^c\!D^\alpha u(t), {}^c\!D^\alpha u(t)\big), \ \ t \in J, $$

holds, that is, $w = {}^c\!D^\alpha u$ satisfies the equality

$$ w(t) = f\big(t, u(t), u'(t), I^{\alpha-\beta} w(t), w(t)\big) \ \text{ for } t \in J. \tag{1.3} $$

The special case of equation (1.1) (for $\alpha = 2$, $a \in C(J)$, $f(t, x, y, v, z) = a(t)v + f_1(t, x, y, z)$) is the implicit generalized Bagley–Torvik fractional differential equation

$$ u''(t) = a(t)\, {}^c\!D^\beta u(t) + f\big(t, u(t), u'(t), u''(t)\big). \tag{1.4} $$

For more details on the generalized Bagley–Torvik fractional differential equation one can see [13–15] and the references therein.

We recall the definitions of the Riemann–Liouville fractional integral and the Caputo fractional derivative [8, 9, 11].

The Riemann–Liouville fractional integral $I^\gamma x$ of order $\gamma > 0$ of a function $x : J \to \mathbb{R}$ is defined as

$$ I^\gamma x(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\, x(s)\, \mathrm{d}s, $$

where Γ is the Euler gamma function. I^0 is the identical operator.

The Caputo fractional derivative ${}^c\!D^\gamma x$ of order $\gamma > 0$, $\gamma \notin \mathbb{N}$, of a function $x : J \to \mathbb{R}$ is given as

$$ {}^c\!D^\gamma x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)} \Big(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} s^k \Big) \mathrm{d}s, $$

where $n = [\gamma] + 1$, $[\gamma]$ means the integral part of the fractional number γ. If $\gamma \in \mathbb{N}$, then ${}^c\!D^\gamma x(t) = x^{(\gamma)}(t)$.

In particular,

$$ \begin{aligned} {}^c\!D^\gamma x(t) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_0^t \frac{(t-s)^{1-\gamma}}{\Gamma(2-\gamma)} \left(x(s) - x(0) - x'(0)s \right) \mathrm{d}s \\ &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} I^{2-\gamma}(x(t) - x(0) - x'(0)t), \quad \gamma \in (1,2). \end{aligned} $$

It is well known that $I^\gamma : C(J) \to C(J)$ for $\gamma \in (0,1)$; $I^\gamma I^\mu x(t) = I^{\gamma+\mu} x(t)$ for $x \in C(J)$ and $\gamma, \mu \in (0, \infty)$; ${}^c\!D^\gamma I^\gamma x(t) = x(t)$ for $x \in C(J)$ and $\gamma > 0$; if $x, {}^c\!D^\gamma x \in C(J)$ and $\gamma \in (0,1)$, then $I^\gamma\, {}^c\!D^\gamma x(t) = x(t) - x(0)$.

The boundary value problems for implicit fractional differential equations were considered in the papers [1, 2, 4–6, 10] and the references therein. For instance, the problem

$$ \begin{gathered} {}^c\!D^\alpha u(t) = f\big(t, u(t), {}^c\!D^\alpha u(t)\big), \quad \alpha \in (0,1], \\ \sum_{k=1}^n a_k u(t_k) = u_0 \end{gathered} $$

was discussed in [6], while the problem

$$ \begin{gathered} {}^c\!D^\alpha u(t) = f\big(t, u(t), {}^c\!D^\alpha u(t)\big), \quad \alpha \in (1,2], \\ u(0) = u_0, \quad u(T) = u_1 \end{gathered} $$

was considered in [4].

The aim of this paper is to discuss the existence of solutions to problem (1.1), (1.2). The existence result is proved by the following procedure. We first show that for each $x \in C^1(J)$ there exists a unique solution $w \in C(J)$ of the equation $w = f(t, x(t), x'(t), I^{\alpha-\beta} w, w)$. Then we put $w = \mathcal{F}x$ and obtain an operator $\mathcal{F} : C^1(J) \to C(J)$ and prove that if u is a solution of the problem ${}^c\!D^\alpha u = \mathcal{F}u$, (1.2), then u is a solution of problem (1.1), (1.2). In order to prove that this problem has a solution, we introduce an operator $\mathcal{Q} : C^1(J) \times \mathbb{R} \to C^1(J) \times \mathbb{R}$ having the property that if (u, c) is its fixed point, then u is a solution of problem ${}^c\!D^\alpha u = \mathcal{F}u$, (1.2). The existence of a fixed point of $\mathcal{Q}$ is proved by the Leray–Schauder degree method [7].

We work with the following conditions on the function f in (1.1).

(H_1) There exist positive constants L_1 and L_2 such that

$$ \Delta = \frac{L_1 T^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)} + L_2 < 1 $$

and the estimate

$$ \big| f(t, x, y, v_1, z_1) - f(t, x, y, v_2, z_2) \big| \le L_1 |v_1 - v_2| + L_2 |z_1 - z_2| $$

is fulfilled for $t \in J$ and $x, y, v_j, z_j \in \mathbb{R}$.

(H_2) There exist $\rho, \mu \in (0,1)$ and $A > 0$ such that

$$ |f(t, x, y, 0, 0)| \le A\big(1 + |x|^\rho + |y|^\mu\big) \ \text{ for } t \in J,\ x, y \in \mathbb{R}. $$

(H_3) There exist positive constants A, B and C such that

$$|f(t,x,y,0,0)| \le A + B|x| + C|y| \ \text{ for } t \in J,\ x, y \in \mathbb{R}.$$

The paper is organized as follows. In Section 2, an operator $\mathcal{F}$ is introduced and its properties are given. In Section 3, the operators $\mathcal{Q}, \mathcal{K}$ and $\mathcal{H}$ are defined and their properties are stated. The main existence results for problem (1.1), (1.2) are given and proved in Section 4. Examples demonstrate our results.

2 Operator $\mathcal{F}$ and its properties

Keeping in mind (1.3), we need the following result.

Lemma 2.1. *Let (H_1) hold and let $x \in C^1(J)$. Then there exists a unique solution w of the equation*

$$w = f\big(t, x(t), x'(t), I^{\alpha-\beta} w, w\big) \tag{2.1}$$

in the set $C(J)$.

Proof. Let an operator $\mathcal{S} : C(J) \to C(J)$ be defined as

$$(\mathcal{S}w)(t) = f\big(t, x(t), x'(t), I^{\alpha-\beta} w(t), w(t)\big).$$

We show that $\mathcal{S}$ is a contractive operator. To this end, let $w_1, w_2 \in C(J)$. Then

$$\begin{aligned}
\big|(\mathcal{S}w_1)(t) - (\mathcal{S}w_2)(t)\big| &= \Big| f\big(t, x(t), x'(t), I^{\alpha-\beta} w_1(t), w_1(t)\big) - f\big(t, x(t), x'(t), I^{\alpha-\beta} w_2(t), w_2(t)\big)\Big| \\
&\le L_1 \big| I^{\alpha-\beta}(w_1(t) - w_2(t))\big| + L_2 |w_1(t) - w_2(t)| \\
&\le \frac{L_1 T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \|w_1 - w_2\| + L_2 \|w_1 - w_2\| \\
&\le \Delta \|w_1 - w_2\|, \quad t \in J.
\end{aligned}$$

In particular,

$$\|\mathcal{S}w_1 - \mathcal{S}w_2\| \le \Delta \|w_1 - w_2\|.$$

Due to $\Delta < 1$, the operator $\mathcal{S}$ is contractive and therefore there exists a unique fixed point w of $\mathcal{S}$. It is clear that w is a unique solution of (2.1) in $C(J)$. □

By Lemma 2.1, for each $x \in C^1(J)$ there exists a unique solution $w \in C(J)$ of equation (2.1). We put $w = \mathcal{F}x$ and obtain an operator $\mathcal{F} : C^1(J) \to C(T)$ satisfying

$$(\mathcal{F}x)(t) = f\big(t, x(t), x'(t), I^{\alpha-\beta}(\mathcal{F}x)(t), (\mathcal{F}x)(t)\big) \ \text{ for } t \in J \text{ and } x \in C^1(J). \tag{2.2}$$

The properties of $\mathcal{F}$ are collected in the following result.

Lemma 2.2. *Let (H_1) hold. Then $\mathcal{F} : C^1(J) \to C(J)$ is a continuous operator and*

$$\|\mathcal{F}x\| \le \frac{1}{1-\Delta} \max\big\{|f(t, x(t), x'(t), 0, 0)| : \ t \in J\big\}, \ \ x \in C^1(J). \tag{2.3}$$

Proof. Let $\{x_n\} \subset C^1(J)$ be a convergent sequence and let $x \in C^1(J)$ be its limit. Let (for $t \in J$, $n \in \mathbb{N}$)

$$d_n(t) = f\big(t, x_n(t), x_n'(t), I^{\alpha-\beta}\mathcal{F}x(t), \mathcal{F}x(t)\big) - f\big(t, x(t), x'(t), I^{\alpha-\beta}\mathcal{F}x(t), \mathcal{F}x(t)\big).$$

Then $\lim_{n\to\infty} \|d_n\| = 0$. It follows from the relation (see (2.2))

$$\begin{aligned}
\big|\mathcal{F}x_n(t) - \mathcal{F}x(t)\big| &\le \Big|f\big(t, x_n(t), x_n'(t), I^{\alpha-\beta}\mathcal{F}x_n(t), \mathcal{F}x_n(t)\big) - f\big(t, x_n(t), x_n'(t), I^{\alpha-\beta}\mathcal{F}x(t), \mathcal{F}x_n(t)\big)\Big| \\
&\quad + \Big|f\big(t, x_n(t), x_n'(t), I^{\alpha-\beta}\mathcal{F}x(t), \mathcal{F}x_n(t)\big) - f\big(t, x_n(t), x_n'(t), I^{\alpha-\beta}\mathcal{F}x(t), \mathcal{F}x(t)\big)\Big| \\
&\qquad + |d_n(t)| \\
&\le L_1\big|I^{\alpha-\beta}(\mathcal{F}x_n(t) - \mathcal{F}x(t))\big| + L_2\big|\mathcal{F}x_n(t) - \mathcal{F}x(t)\big| + |d_n(t)| \\
&\le \Big(\frac{L_1 T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + L_2\Big)\|\mathcal{F}x_n - \mathcal{F}x\| + \|d_n\|, \ \ t \in J, \ n \in \mathbb{N},
\end{aligned}$$

that

$$\|\mathcal{F}x_n - \mathcal{F}x\| \le \Delta\|\mathcal{F}x_n - \mathcal{F}x\| + \|d_n\|, \ \ n \in \mathbb{N}.$$

Therefore

$$\|\mathcal{F}x_n - \mathcal{F}x\| \le \frac{\|d_n\|}{1-\Delta}, \ \ n \in \mathbb{N},$$

and so $\lim_{n\to\infty} \|\mathcal{F}x_n - \mathcal{F}x\| = 0$. Hence $\mathcal{F}$ is continuous.

It remains to prove that estimate (2.3) is valid. Let $x \in C^1(J)$. Then (2.2) and (H_1) give

$$\begin{aligned}
|\mathcal{F}x(t)| &\le \big|f(t, x(t), x'(t), I^{\alpha-\beta}\mathcal{F}x(t), \mathcal{F}x(t)) - f(t, x(t), x'(t), 0, \mathcal{F}x(t))\big| \\
&\quad + \big|f(t, x(t), x'(t), 0, \mathcal{F}x(t) - f(t, x(t), x'(t), 0, 0)| + |f(t, x(t), x'(t), 0, 0)\big| \\
&\le L_1\big|I^{\alpha-\beta}\mathcal{F}x(t)\big| + L_2|\mathcal{F}x(t)| + |f(t, x(t), x'(t), 0, 0)| \\
&\le \Delta\|\mathcal{F}x\| + |f(t, x(t), x'(t), 0, 0)|, \ \ t \in J.
\end{aligned}$$

In particular,

$$\|\mathcal{F}x\| \le \Delta\|\mathcal{F}x\| + \max\big\{|f(t, x(t), x'(t), 0, 0)| : \ t \in J\big\}$$

and (2.3) follows. □

3 Auxiliary results

We investigate the fractional differential equation

$$^{c}D^{\alpha}u(t) = (\mathcal{F}u)(t). \tag{3.1}$$

The following result gives the relation between the solutions of problems (3.1), (1.2) and (1.1), (1.2).

Lemma 3.1. *Let (H_1) hold. If u is a solution of problem* (3.1), (1.2)*, then u is a solution of problem* (1.1), (1.2).

Proof. Let u be a solution of problem (3.1), (1.2). In view of (2.2), we see that

$$^{c}D^{\alpha}u(t) = f\big(t, u(t), u'(t), I^{\alpha-\beta}\,{}^{c}D^{\alpha}u(t), {}^{c}D^{\alpha}u(t)\big) \ \text{ for } t \in J.$$

Hence u is a solution of equation (1.1), because $I^{\alpha-\beta}\,{}^{c}D^{\alpha}u = {}^{c}D^{\beta}u$. Since u satisfies the boundary condition (1.2), u is a solution of problem (1.1), (1.2). □

In order to prove that problem (3.1), (1.2) has a solution, we introduce an operator $\mathcal{Q} : C^1(J)\times\mathbb{R} \to C^1(J)\times\mathbb{R}$ by the formula

$$Q(x, c) = \Big(c + I^{\alpha}(\mathcal{F}x)(t) - \frac{t}{T}\, I^{\alpha}(\mathcal{F}x)(t)\Big|_{t=T}, \ c + \phi(x)\Big),$$

where ϕ is from (1.2).

Lemma 3.2. *Let (H_1) hold. If (x,c) is a fixed point of the operator $\mathcal{Q}$, then x is a solution of problem* $(3.1),(1.2)$ *and* $c = x(0)$.

Proof. Let (x,c) be a fixed point of $\mathcal{Q}$. Then

$$x(t) = c + I^\alpha(\mathcal{F}x)(t) - \frac{t}{T}\,I^\alpha(\mathcal{F}x)(t)\Big|_{t=T}, \quad t \in J, \tag{3.2}$$

$$\phi(x) = 0. \tag{3.3}$$

It follows from (3.2) that $x(0) = c$, $x(T) = c$, $x \in C^1(J)$ and

$$^{c}\!D^\alpha x(t) = {}^{c}\!D^\alpha I^\alpha(\mathcal{F}x)(t) = (\mathcal{F}x)(t), \quad t \in J.$$

These facts together with (3.3) imply that x is a solution of (3.1), (1.2) and $c = x(0)$, □

Lemmas 3.1 and 3.2 show that for the solvability of problem $(1.1),(1.2)$ we need to prove that the operator $\mathcal{Q}$ admits a fixed point. Really, if (x,c) is a fixed point of $\mathcal{Q}$, then x is a solution of (1.1), (1.2). To this end, we first define an operator $\mathcal{K} : C^1(J) \times \mathbb{R} \times [0,1] \to C^1(J) \times \mathbb{R}$ as

$$\mathcal{K}(x,c,\lambda) = \big(c, c + \phi(x) + (\lambda - 1)\phi(-x)\big).$$

Let

$$\Omega_1 = \big\{(x,c) \in C^1(J) \times \mathbb{R} : \ \|x\|_1 < M, \ |c| < M\big\}.$$

where M is a positive constant.

Lemma 3.3. *The relation*

$$\deg(\mathcal{I} - \mathcal{K}(\,\cdot\,,\,\cdot\,,1), \Omega_1, 0) \neq 0$$

*is valid, where "*deg*" stands for the Leray–Schauder degree and $\mathcal{I}$ is the identical operator on* $C^1(J)\times\mathbb{R}$.

Proof. It is not difficult to show that $\mathcal{K}$ is a completely continuous operator and since

$$\mathcal{K}(-x,-c,0) = (-c, -c + \phi(-x) - \phi(x)) = -(c, c + \phi(x) - \phi(-x)) = -\mathcal{K}(x,c,0)$$

for $x \in C^1(J)$ and $c \in \mathbb{R}$, $\mathcal{K}(\,\cdot\,,\,\cdot\,,0)$ is an odd operator.

Assume that $\mathcal{K}(x,c,\lambda) = (x,c)$ for some $(x,c) \in C^1(J) \times \mathbb{R}$ and $\lambda \in [0,1]$. Then

$$x(t) = c, \quad t \in J, \tag{3.4}$$

$$\phi(x) + (\lambda - 1)\phi(-x) = 0. \tag{3.5}$$

In view of (3.4), it follows from (3.5) that $\phi(c) + (\lambda - 1)\phi(-c) = 0$. If $c \neq 0$, then properties (i) and (ii) of $\phi \in \mathcal{M}$ give $\phi(c)\phi(-c) < 0$, which contradicts $\phi(c) + (\lambda-1)\phi(-c) = 0$. Hence $c = 0$, and so $x = 0$. We have proved that $\mathcal{K}(x,c,\lambda) \neq (x,c)$ for $(x,c) \in \partial\Omega_1$ and $\lambda \in [0,1]$. By the Borsuk antipodal theorem and the homotopy property,

$$\deg\big(\mathcal{I} - \mathcal{K}(\,\cdot\,,\,\cdot\,,0), \Omega_1, 0\big) \neq 0,$$

$$\deg\big(\mathcal{I} - \mathcal{K}(\,\cdot\,,\,\cdot\,,0), \Omega_1, 0\big) = \deg\big(\mathcal{I} - \mathcal{K}(\,\cdot\,,\,\cdot\,,1), \Omega_1, 0\big).$$

Combining these relations we give the conclusion of Lemma 3.3. □

Finally, let an operator $\mathcal{H} : C^1(J) \times \mathbb{R} \times [0,1] \to C^1(J) \times \mathbb{R}$ be defined as

$$\mathcal{H}(x,c,\lambda) = (\mathcal{H}_1(x,c,\lambda), \mathcal{H}_2(x,c)),$$

where $\mathcal{H}_1 : C^1(J) \times \mathbb{R} \times [0,1] \to C^1(J)$, $\mathcal{H}_2(x,c) : C^1(J) \times \mathbb{R} \to \mathbb{R}$,

$$\mathcal{H}_1(x,c,\lambda)(t) = c + \lambda\Big(I^\alpha(\mathcal{F}x)(t) - \frac{t}{T}\,I^\alpha(\mathcal{F}x)(t)\Big|_{t=T}\Big),$$

$$\mathcal{H}_2(x,c) = c + \phi(x).$$

It is clear that

$$\mathcal{H}(x,c,0) = \mathcal{K}(x,c,1), \quad \mathcal{H}(x,c,1) = \mathcal{Q}(x,c) \tag{3.6}$$

for $(x,c) \in C^1(J) \times \mathbb{R}$.

The following result states that $\mathcal{H}$ is completely continuous.

Lemma 3.4. *Let (H_1) hold. Then $\mathcal{H}$ is a completely continuous operator.*

Proof. Step 1. $\mathcal{H}$ is continuous.

Let $\{x_n\} \subset C^1(J)$, $\{c_n\} \subset \mathbb{R}$, $\{\lambda_n\} \subset [0,1]$ be convergent sequences and let $\lim_{n\to\infty} \|x_n - x\|_1 = 0$, $\lim_{n\to\infty} c_n = c$, $\lim_{n\to\infty} \lambda_n = \lambda$, where $x \in C^1(J)$, $c \in \mathbb{R}$, $\lambda \in [0,1]$.

By Lemma 2.2, $\lim_{n\to\infty} \|\mathcal{F}x_n - \mathcal{F}x\| = 0$. Since

$$\begin{aligned}&\left| I^\alpha(\mathcal{F}x_n)(t) - \frac{t}{T}\, I^\alpha(\mathcal{F}x_n)(t)\Big|_{t=T} - I^\alpha(\mathcal{F}x)(t) + \frac{t}{T}\, I^\alpha(\mathcal{F}x)(t)\Big|_{t=T} \right|\\ &\qquad \le \|\mathcal{F}x_n - \mathcal{F}x\| \bigg(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, \mathrm{d}s + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \bigg)\, \mathrm{d}s \le \frac{2T^\alpha}{\Gamma(\alpha+1)}\, \|\mathcal{F}x_n - \mathcal{F}x\|\end{aligned}$$

and

$$\begin{aligned}&\left| I^{\alpha-1}(\mathcal{F}x_n)(t) - \frac{1}{T}\, I^\alpha(\mathcal{F}x_n)(t)\Big|_{t=T} - I^{\alpha-1}(\mathcal{F}x)(t) + \frac{1}{T}\, I^\alpha(\mathcal{F}x)(t)\Big|_{t=T} \right|\\ &\qquad \le \|\mathcal{F}x_n - \mathcal{F}x\| \bigg(\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, \mathrm{d}s + \frac{1}{T} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \bigg)\, \mathrm{d}s \le \frac{T^{\alpha-1}}{\Gamma(\alpha)} \Big(1 + \frac{1}{\alpha}\Big) \|\mathcal{F}x_n - \mathcal{F}x\|,\end{aligned}$$

it is easy to verify that $\lim_{n\to\infty} \|\mathcal{H}_1(x_n, c_n, \lambda_n) - \mathcal{H}_1(x, c, \lambda)\|_1 = 0$. This fact together with $\lim_{n\to\infty} \mathcal{H}_2(x_n, c_n) = \mathcal{H}_2(x, c)$ gives $\lim_{n\to\infty} \mathcal{H}(x_n, c_n, \lambda_n) = \mathcal{H}(x, c, \lambda)$ in $C^1(J) \times \mathbb{R}$. Hence $\mathcal{H}$ is continuous.

Step 2. $\mathcal{H}$ takes bounded sets into bounded sets.

Let $\mathcal{U} \subset C^1(J)$ and $\mathcal{V} \subset \mathbb{R}$ be bounded, $\|x\|_1 \le V$ for $x \in \mathcal{U}$, $|c| \le V$ for $c \in \mathcal{V}$, where V is a positive constant. Then $M_1 = \sup\{|f(t, x(t), x'(t), 0, 0)| : t \in J,\ x \in \mathcal{U}\} < \infty$. In view of (2.3), we have $\|\mathcal{F}x\| \le M$ for $x \in \mathcal{U}$, where $M = M_1/(1-\Delta)$. Hence (for $u \in \mathcal{U}$, $c \in \mathcal{V}$, $\lambda \in [0,1]$, $t \in J$)

$$|\mathcal{H}_1(x, c, \lambda)(t)| \le V + M \bigg(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, \mathrm{d}s + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\, \mathrm{d}s \bigg) \le V + \frac{2MT^\alpha}{\Gamma(\alpha+1)}\,,$$

$$\Big| \frac{\mathrm{d}}{\mathrm{d}t}\, \mathcal{H}_1(x, c, \lambda)(t) \Big| \le M \bigg(\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, \mathrm{d}s + \frac{1}{T} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\, \mathrm{d}s \bigg) \le \frac{MT^{\alpha-1}}{\Gamma(\alpha)} \Big(1 + \frac{1}{\alpha}\Big),$$

and therefore

$$\|\mathcal{H}_1(x, c, \lambda)\|_1 \le V + \frac{MT^{\alpha-1}}{\Gamma(\alpha)} \Big(1 + \frac{2T+1}{\alpha}\Big). \tag{3.7}$$

Due to the properties (i) and (ii) of ϕ and $-V \le x(t) \le V$ for $t \in J$, $x \in \mathcal{U}$, we see that $\phi(-V) \le \phi(x) \le \phi(V)$, and therefore

$$|\mathcal{H}_2(x, c)| = |c + \phi(x)| \le W \ \text{ for } u \in \mathcal{U},\ c \in \mathcal{V}, \tag{3.8}$$

where $W = V + \max\{|\phi(-V)|, \phi(V)\}$.

From (3.7) and (3.8) we conclude that $\mathcal{H}$ maps $\mathcal{U} \times \mathcal{V} \times [0,1]$ into a bounded set in $C^1(J) \times \mathbb{R}$.

Step 3. For each bounded $\mathcal{U} \subset C^1(J)$ the family $\{I^{\alpha-1}(\mathcal{F}x) : x \in \mathcal{U}\}$ is equicontinuous on J.

Let $\mathcal{U}$ be a bounded set in $C^1(J)$. As in Step 2, $\|\mathcal{F}x\| \le M$ for $x \in \mathcal{U}$, where $M > 0$. Let $x \in \mathcal{U}$

and $0 \le t_1 < t_2 \le T$. Then

$$\Big|I^{\alpha-1}(\mathcal{F}x)(t)\big|_{t=t_2} - I^{\alpha-1}(\mathcal{F}x)(t)\big|_{t=t_1}\Big| = \Big|\int\limits_0^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}(\mathcal{F}x)(s)\,\mathrm{d}s - \int\limits_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}(\mathcal{F}x)(s)\,\mathrm{d}s\Big|$$
$$= \Big|\int\limits_0^{t_1} \frac{(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}(\mathcal{F}x)(s)\,\mathrm{d}s + \int\limits_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}(\mathcal{F}x)(s)\,\mathrm{d}s\Big|$$
$$\le M\Big(\int\limits_0^{t_1} \frac{(t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,\mathrm{d}s + \int\limits_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,\mathrm{d}s\Big)$$
$$= \frac{M}{\Gamma(\alpha)}\left(t_1^{\alpha-1} + 2(t_2-t_1)^{\alpha-1} - t_2^{\alpha-1}\right) < \frac{2M}{\Gamma(\alpha)}(t_2-t_1)^{\alpha-1}.$$

Since $t^{\alpha-1}$ is a continuous function on J, we see that the family $\{I^{\alpha-1}(\mathcal{F}x) :\ x \in \mathcal{U}\}$ is equicontinuous on J.

To summarize, $\mathcal{H}$ is continuous by Step 1 and it follows from Steps 2 and 3 and the Arzelà-Ascoli theorem that $\mathcal{H}_1$ is relatively compact in $C^1(J)$. Besides, (3.8) implies that $\mathcal{H}_2$ is relatively compact in $\mathbb{R}$. Consequently, $\mathcal{H}$ is completely continuous. □

The following two results give bounds for fixed points of $\mathcal{H}$.

Lemma 3.5. *Let (H_1) and (H_2) hold. Then there exists $S > 0$ such that the estimate*

$$\|x\|_1 < S, \quad |c| < S, \tag{3.9}$$

holds for fixed points (x, c) of the operator $\mathcal{H}(\,\cdot\,,\,\cdot\,,\lambda)$ with $\lambda \in [0,1]$.

Proof. Let $\mathcal{H}(x,c,\lambda) = (x,c)$ for some $(x,c) \in C^1(J) \times \mathbb{R}$ and $\lambda \in [0,1]$. Then

$$x(t) = c + \lambda\Big(I^\alpha(\mathcal{F}x)(t) - \frac{t}{T}\, I^\alpha(\mathcal{F}x)(t)\Big|_{t=T}\Big), \quad t \in J, \tag{3.10}$$
$$\phi(x) = 0. \tag{3.11}$$

By (H_2),

$$\big|f(t,x(t),x'(t),0,0)\big| \le A\big(1 + |x(t)|^\rho + |x'(t)|^\mu\big) \le A\big(1 + \|x\|_1^\rho + \|x\|_1^\mu\big), \quad t \in J,$$

and therefore (see (2.3)))

$$\|\mathcal{F}x\| \le \frac{A(1 + \|x\|_1^\rho + \|x\|_1^\mu)}{1-\Delta}. \tag{3.12}$$

Due to (3.11), we have $x(\xi) = 0$ for some $\xi \in J$ [12]. Hence (3.10) gives

$$c = -\lambda\bigg(I^\alpha(\mathcal{F}x)(t)\Big|_{t=\xi} - \frac{\xi}{T}\, I^\alpha(\mathcal{F}x)(t)\Big|_{t=T}\bigg),$$

and therefore

$$x(t) = \lambda\bigg(I^\alpha(\mathcal{F}x)(t) - I^\alpha(\mathcal{F}x)(t)\Big|_{t=\xi} - \frac{t-\xi}{T}\, I^\alpha(\mathcal{F}x)(t)\Big|_{t=T}\bigg), \quad t \in J.$$

Then

$$|x(t)| \le \|\mathcal{F}x\|\Big(\int\limits_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s + \int\limits_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s + \int\limits_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s\Big) \le \frac{3T^\alpha}{\Gamma(\alpha+1)}\|\mathcal{F}x\|,$$
$$|x'(t)| \le \|\mathcal{F}x\|\Big(\int\limits_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,\mathrm{d}s + \frac{1}{T}\int\limits_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s\Big) \le \frac{T^{\alpha-1}}{\Gamma(\alpha)}\Big(1 + \frac{1}{\alpha}\Big)\|\mathcal{F}x\|, \quad t \in J.$$

In particular,

$$\|x\| \le \frac{3T^\alpha}{\Gamma(\alpha+1)}\,\|\mathcal{F}x\|, \quad \|x'\| \le \frac{T^{\alpha-1}}{\Gamma(\alpha)}\Big(1+\frac{1}{\alpha}\Big)\|\mathcal{F}x\|.$$

Hence

$$\|x\|_1 \le \frac{T^{\alpha-1}}{\Gamma(\alpha)}\Big(1+\frac{3T+1}{\alpha}\Big)\|\mathcal{F}x\| \tag{3.13}$$

and (see (3.12)))

$$\|x\|_1 \le K\big(1+\|x\|_1^\rho+\|x\|_1^\mu\big), \tag{3.14}$$

where

$$K = \frac{AT^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}\Big(1+\frac{3T+1}{\alpha}\Big).$$

Since (note that $\rho,\mu\in(0,1)$) $\lim_{v\to\infty}\frac{v}{K(1+v^\rho+v^\mu)}=\infty$, there exists $S>0$ such that

$$v > K(1+v^\rho+v^\mu) \ \text{ for all } v\ge S.$$

The last inequality together with (3.14) gives $\|x\|_1<S$. In view of $c=x(0)$, we get $|c|<S$. Since S is independent of x, c, λ, estimate (3.9) follows. □

Lemma 3.6. *Let (H_1) and (H_3) hold and let*

$$W = \frac{(B+C)T^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}\Big(1+\frac{3T+1}{\alpha}\Big) < 1.$$

Then the estimate

$$\|x\|_1 \le S_1, \ \ |c| \le S_1,$$

holds for fixed points (x,c) of the operator $\mathcal{H}(\,\cdot\,,\,\cdot\,,\lambda)$ with $\lambda\in[0,1]$, where

$$S_1 = \frac{AT^{\alpha-1}}{(1-\Delta)(1-W)\Gamma(\alpha)}\Big(1+\frac{3T+1}{\alpha}\Big).$$

Proof. Let $\mathcal{H}(x,c,\lambda)=(x,c)$ for some $(x,c)\in C^1(J)\times\mathbb{R}$ and $\lambda\in[0,1]$. Analysis similar to that in the proof of Lemma 3.5 shows that $c=x(\xi)$ for some $\xi\in J$ and estimate (3.13) is valid. From (H_3) and (2.3) we have

$$\|\mathcal{F}x\| \le \frac{A+B\|x\|+C\|x'\|}{1-\Delta} \le \frac{A+(B+C)\|x\|_1}{1-\Delta}\,,$$

and therefore

$$\|x\|_1 \le \frac{T^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}\Big(1+\frac{3T+1}{\alpha}\Big)\big(A+(B+C)\|x\|_1\big) = \frac{AT^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}\Big(1+\frac{3T+1}{\alpha}\Big)+W\|x\|_1.$$

Hence

$$(1-W)\|x\|_1 \le \frac{AT^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}\Big(1+\frac{3T+1}{\alpha}\Big),$$

which implies $\|x\|_1\le S_1$ and $|c|\le S_1$ because $c=x(0)$. □

4 The main results and examples

Theorem 4.1. *Let (H_1) and (H_2) hold. Then problem* (1.1), (1.2) *has at least one solution.*

Proof. Let $S>0$ be from Lemma 3.5 and let

$$\Omega = \big\{(x,c)\in C^1(J)\times\mathbb{R}:\ \|x\|_1<S,\ |c|<S\big\}.$$

By Lemma 3.4, the restriction of $\mathcal{H}$ to $\overline{\Omega} \times [0,1]$ is a compact operator and Lemma 3.5 shows that $\mathcal{H}(x,c,\lambda) \neq (x,c)$ for $(x,c) \in \partial\Omega$ and $\lambda \in [0,1]$. Hence it follows from the homotopy property that

$$\deg\big(\mathcal{I} - \mathcal{H}(\,\cdot\,,\,\cdot\,,0),\Omega,0\big) = \deg\big(\mathcal{I} - \mathcal{H}(\,\cdot\,,\,\cdot\,,1),\Omega,0\big).$$

In view of (3.6) and Lemma 3.3 (for $M = S$ in Ω_1), we have

$$\begin{aligned}
\deg\big(\mathcal{I} - \mathcal{H}(\,\cdot\,,\,\cdot\,,0),\Omega,0\big) &= \deg\big(\mathcal{I} - \mathcal{K}(\,\cdot\,,\,\cdot\,,1),\Omega,0\big) \neq 0,\\
\deg\big(\mathcal{I} - \mathcal{H}(\,\cdot\,,\,\cdot\,,1),\Omega,0\big) &= \deg\big(\mathcal{I} - \mathcal{Q}(\,\cdot\,,\,\cdot\,),\Omega,0\big),
\end{aligned}$$

and so

$$\deg\big(\mathcal{I} - \mathcal{Q}(\,\cdot\,,\,\cdot\,),\Omega,0\big) \neq 0. \tag{4.1}$$

Consequently, there exists a fixed point (u,c) of $\mathcal{Q}$ and, by Lemmas 3.1 and 3.2, u is a solution of problem (1.1), (1.2). ☐

Theorem 4.2. *Let (H_1) and (H_3) hold and let $W < 1$, where W is from Lemma 3.6. Then problem (1.1), (1.2) has at least one solution.*

Proof. Let S_1 be from Lemma 3.6 and let

$$\Omega = \big\{(x,c) \in C^1(J) \times \mathbb{R} :\ \|x\|_1 < S_1 + 1,\ |c| < S_1 + 1\big\}.$$

By Lemma 3.6, $\mathcal{H}(x,c,\lambda) \neq (x,c)$ for $(x,c) \in \partial\Omega$ and $\lambda \in [0,1]$. Analysis similar to that in the proof of Theorem 4.1 shows that relation (4.1) holds. Hence there exists a fixed point (u,c) of $\mathcal{Q}$ and u is a solution of problem (1.1), (1.2). ☐

Example 4.1. Let $r \in C(J)$, $\rho,\mu \in (0,1)$ and $k > \sqrt{2T^{\alpha-\beta}/\Gamma(\alpha-\beta+1)}$. Then the function

$$f(t,x,y,v,z) = r(t) + |x|^\rho + |y|^\mu \arctan y + \frac{1}{k+|v|} + \frac{(x+y)\ln(1+|z|)}{2+x^2+y^2}$$

satisfies condition (H_1) for $L_1 = 1/k^2$, $L_2 = 1/2$ and condition (H_2) for $A = \max\{\|r\|, \pi/2, 1/k\}$. By Theorem 4.1 there exists at least one solution u of the equation

$${}^cD^\alpha u = r(t) + |u|^\rho + |u'|^\mu \arctan u' + \frac{1}{k + {}^cD^\beta u} + \frac{(u+u')\ln(1+|{}^cD^\alpha u|)}{2+u^2+(u')^2} \tag{4.2}$$

satisfying the boundary condition (1.2).

For instance, if $\phi(u) = \min\{u(t) :\ t \in J\}$, then there exists at least one solution u of (4.2) fulfilling

$$u(0) = u(T), \quad \min\{u(t) :\ t \in J\} = 0.$$

Example 4.2. Let $T = 1$, $\alpha = 3/2$, $\beta \in (1,3/2)$, $|k| < \Gamma(5/2-\beta)/4$ and $r, r_1, r_2 \in C[0,1]$, $\|r_1\| + \|r_2\| < 3\sqrt{\pi}/44$. Then the function

$$f(t,x,y,v,z) = r(t) + r_1(t)x + r_2(t)y + kv + \frac{y\ln(1+|z|)}{1+4y^2}$$

satisfies condition (H_1) for $L_1 = |k|$, $L_2 = 1/4$ (note that $\Delta < 1/2$) and condition (H_3) for $A = \|r\|$, $B = \|r_1\|$, $C = \|r_2\|$. Since

$$W = \frac{(B+C)T^{\alpha-1}}{(1-\Delta)\Gamma(\alpha)}\Big(1 + \frac{3T+1}{\alpha}\Big) = \frac{22(\|r_1\| + \|r_2\|)}{3(1-\Delta)\sqrt{\pi}} \leq \frac{44(\|r_1\| + \|r_2\|)}{3\sqrt{\pi}} < 1,$$

by Theorem 4.2 there exists at least one solution u of the equation

$${}^cD^{3/2}u = r(t) + r_1(t)u + r_2(t)u' + k\,{}^cD^\beta u + \frac{u'\ln(1+|{}^cD^{3/2}u|)}{1+4(u')^2}$$

satisfying the boundary condition (1.2).

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VISUALIZATION AND ANALYSIS OF STABILITY REGIONS OF CERTAIN DISCRETIZATION OF DIFFERENTIAL EQUATION WITH CONSTANT DELAY

Abstract. The paper discusses the asymptotic stability regions of multistep discretization of linear delay differential equation with a constant delay. Different location of delay dependent parts of stability regions with respect to parity of number of steps clarifies unexpected changes in numerical solution's behaviour under various settings of the equation's parameters and stepsize.*

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Key words and phrases. Delay differential equation, constant delay, difference equation, numerical solution, necessary and sufficient conditions for asymptotic stability.

რეზიუმე. ნაშრომში განხილულია მრავალსაფეხურიანი დისკრეტიზაციის ასიმპტოტურად მდგრადი არეები წრფივი დაგვიანებულ-არგუმენტიანი დიფერენციალური განტოლებისთვის მუდმივი დაგვიანებით. მდგრადობის არეების დაგვიანებაზე დამოკიდებული ნაწილების განსხვავებული ადგილმდებარეობა ბიჯთა რაოდენობის ლუწ-კენტობასთან მიმართებაში ხსნის რიცხვითი ამონახსნების ქცევის მოულოდნელ ცვლილებებს განტოლების სხვადასხვა პარამეტრების და სხვადასხვა ბიჯის შემთხვევაში.

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1 Introduction

For the recent decades the delay differential equations theory has made great achievements. Consequently, appropriate numerical methods and corresponding theoretical background are being developed since 1970's. A valuable monograph, which summarize numerical methods for various delay differential equations and introduce comparison with the methods known for ordinary differential equations, is due to Bellen and Zennaro [2]. A various phenomena were observed as differences between the both mentioned classes of differential equations and their numerical discrete counterparts.

The concepts of asymptotic stability in numerical analysis are usually related to the numerical solution behaviour of the studied method applied to a certain test equation. Such equations in the delay differential case are, e.g.,

$$\begin{aligned} y'(t) &= by(t-\tau), \quad t>0, \\ y'(t) &= ay(t)+by(t-\tau), \quad t>0, \end{aligned} \tag{1.1}$$

where $a, b, \tau \in \mathbb{R}$, $\tau > 0$. In general, the coefficients a, b are considered as complex ones in various types of stability manner. In this paper, we constrain our considerations to the case of equation (1.1) with real coefficients a, b. This restriction arises from the studied numerical discretization and visualization purposes.

The numerical scheme, that we are going to analyse, can be captured by the linear difference equation

$$y_{n+2} + \alpha y_n + \gamma y_{n-\ell} = 0, \quad n = 0, 1, 2, \dots, \tag{1.2}$$

where $\alpha, \gamma \in \mathbb{R}$ and $\ell \in \mathbb{N}$. We recall that equations (1.1) and (1.2) are said to be asymptotically stable if for any of their solutions $y(t) \to 0$ as $t \to \infty$ and $y_n \to 0$ as $n \to \infty$, respectively. This terminology is usual in the theory of homogeneous linear differential (difference) equations with a constant delay.

In the case of linear difference equations with constant coefficients the asymptotic stability coincides with affiliation of all roots of a characteristic polynomial to the open unit disk in the complex plane. There exist several valuable criteria for checking this property, but these are suitable just for a computational verification for concrete given values of equation's (polynomial's) parameters. These criteria are mostly based on the analysis of signs of certain determinant sequences (see [9] or [12]). In several particular cases the necessary and sufficient conditions for asymptotic stability were derived in a closed effective form, i.e., a few conditions should be verified instead of a huge number of computations depending on the order of difference equation in the case of algebraic criterion. The asymptotic stability conditions for (1.2) in necessary and sufficient manner are introduced in [6]. We recall them in Section 2 for our consideration purposes. In addition to the previous, we remark that there are several results introducing closed form of necessary and sufficient conditions for asymptotic stability of certain difference equations, which cover many numerical schemes intended for delay differential equations, e.g.,

$$\begin{aligned} y_{n+1} + \alpha y_n + \gamma y_{n-\ell} &= 0, \quad n = 0, 1, 2, \dots, \\ y_{n+1} + \alpha y_{n-m} + \gamma y_{n-\ell} &= 0, \quad n = 0, 1, 2, \dots, \\ y_{n+1} + \alpha y_n + \beta y_{n-\ell+1} + \gamma y_{n-\ell} &= 0, \quad n = 0, 1, 2, \dots, \\ y_{n+2} + \alpha y_n + \beta y_{n-\ell+2} + \gamma y_{n-\ell} &= 0, \quad n = 0, 1, 2, \dots, \end{aligned}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $m, \ell \in \mathbb{N}$, $m < \ell$. The results can be found in [3, 4, 11] and [5], respectively.

The structure of the paper is as follows. In Section 2, we recall the necessary and sufficient asymptotic stability conditions for equations (1.1) and (1.2). In Section 3, we introduce the analysed numerical scheme, description and visualization of its stability regions and discussion of some unexpected situations arising at numerical computations. We conclude the paper by final remarks in Section 4.

2 Preliminaries

Any asymptotic stability property of a numerical scheme is usually connected with asymptotic stability properties of a certain test differential equation. In this paper, we are going to analyse asymptotic stability regions (i.e., the sets of pairs $(a, b) \in \mathbb{R}^2$ such that the studied discretization is asymptotically stable considering fixed stepsize) of numerical scheme applied to delay differential equation (1.1). Therefore we recall the necessary and sufficient conditions for asymptotic stability of (1.1) itself introduced in [1] and [7].

Theorem 1. *Any solution of equation* (1.1) *is asymptotically stable if and only if one of the following two conditions holds:*

$$a \le b < -a \ \text{ for any } \ \tau > 0; \tag{2.1}$$

$$|a| + b < 0 \ \text{ for } \ \tau < \frac{\arccos(-a/b)}{(b^2 - a^2)^{1/2}}\,. \tag{2.2}$$

As we can see, the first condition is valid for any positive delay τ. We call such case as delay independent asymptotic stability region, which is depicted in Figure 1 as S_{DI}. Condition (2.2) contains a restriction on delay τ. This condition forms a dependent stability region S_{DD} (see Figure 1). The greater value of τ, the closer the most right point of S_{DD} to the origin of the plane (a, b) is.

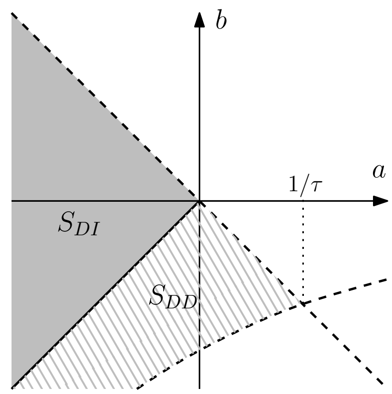

Figure 1. Asymptotic stability region of (1.1): delay independent (S_{DI}) and delay dependent (S_{DD}) case.

Next, we recall the necessary and sufficient conditions for asymptotic stability of difference equation (1.2) introduced in [6]:

Theorem 2. *Let α, γ be arbitrary reals such that $\alpha\gamma \neq 0$.*

(i) *Let ℓ be even and $\gamma(-\alpha)^{\ell/2+1} < 0$. Then* (1.2) *is asymptotically stable if and only if*

$$|\alpha| + |\gamma| < 1. \tag{2.3}$$

(ii) *Let ℓ be even and $\gamma(-\alpha)^{\ell/2+1} > 0$. Then* (1.2) *is asymptotically stable if and only if either*

$$|\alpha| + |\gamma| \le 1, \tag{2.4}$$

or

$$\big||\alpha| - |\gamma|\big| < 1 < |\alpha| + |\gamma|, \quad \ell < 2\arccos\frac{\alpha^2 + \gamma^2 - 1}{2|\alpha\gamma|}\Big/\arccos\frac{\alpha^2 - \gamma^2 + 1}{2|\alpha|}$$

holds.

(iii) *Let ℓ be odd and $\alpha < 0$. Then* (1.2) *is asymptotically stable if and only if* (2.3) *holds.*

(iv) *Let ℓ be odd and $\alpha > 0$. Then* (1.2) *is asymptotically stable if and only if either* (2.4)*, or*

$$\gamma^2 < 1 - \alpha < |\gamma|, \quad \ell < 2\arcsin\frac{1 - \alpha^2 - \gamma^2}{2|\alpha\gamma|}\Big/\arccos\frac{\alpha^2 - \gamma^2 + 1}{2|\alpha|}$$

holds.

Actually, equivalent description of asymptotic stability regions can be found, e.g., in [10] and [13], where another proving procedures naturally lead to another form of the conditions. In the first mentioned paper, the boundary of stability region was described by straight lines and parametric curves, while in the second one the conditions contained an auxiliary nonlinear equation, which should be solved for certain choice of differential equation parameters α, γ, ℓ.

A comparison of conditions for asymptotic stability for (1.1) (see Theorem 1) and conditions for its discrete counterpart (1.2) in Theorem 2 leads us to a conclusion that such asymptotic stability analysis is much more complicated in the case of difference equation.

3 Numerical discretization and its properties

We consider an equidistant mesh with stepsize h satisfying the property $\tau = kh$ with $k \in \mathbb{N}$, $k > 2$. We denote the nodal points of the mesh as $t_n = nh$, $n = 0, 1, 2, \ldots$.

By integration of both sides of (1.1) from t_n to t_{n+2} we obtain

$$y(t_{n+2}) = y(t_n) + \int_{t_n}^{t_{n+2}} ay(s)\,\mathrm{d}s + \int_{t_n}^{t_{n+2}} by(s-\tau)\,\mathrm{d}s. \tag{3.1}$$

Numerical scheme we obtain by applying trapezoidal rule and midpoint rule to the integrals in (3.1), respectively. Denoting by y_n the approximation of value $y(t_n)$, we have

$$y_{n+2} = y_n + ah(y_n + y_{n+2}) + 2bhy_{n-k+1}. \tag{3.2}$$

The obtained formula is a $(k+1)$-step numerical method. We emphasize that there is no need of interpolation dealing with delayed term due to the appropriate stepsize $h = \tau/k$ and integration of (1.1) over two steps. Since we are going to utilize Theorem 2, we rewrite (3.2) in the form of linear difference equation

$$y_{n+2} - \frac{1+ah}{1-ah} y_n - \frac{2bh}{1-ah}\, y_{n-k+1} = 0, \quad n = 0, 1, \ldots, \tag{3.3}$$

where the stepsize h satisfies $ah \neq 1$.

3.1 Asymptotic stability conditions

Now we state the necessary and sufficient conditions for asymptotic stability of (3.3). The analysis of (3.3) falls naturally into two parts according to the parity of k. For an effective and clear formulation of the main result we introduce the symbols

$$\tau_1^*(h) = h + 2h \arcsin \frac{a + b^2 h}{(1+ah)|b|} \Big/ \arccos \frac{1 + a^2h^2 - 2b^2h^2}{a^2h^2 - 1}\,,$$
$$\tau_2^*(h) = h + 2h \arccos \frac{a + b^2 h}{|(1+ah)b|} \Big/ \arccos \frac{1 + a^2h^2 - 2b^2h^2}{|a^2h^2 - 1|}\,,$$

which are utilized in these two parts, respectively.

Theorem 3. *The asymptotic stability conditions for* (3.3) *are formulated below in two cases, considering k even and k odd, respectively.*

1. *Let $k \geq 2$ be even. Then* (3.3) *is asymptotically stable if and only if one of the following conditions holds:*

$$|bh| \leq 1, \quad |b| + a < 0, \tag{3.4}$$
$$2 < 2b^2h^2 < 1 - ah, \quad \tau < \tau_1^*(h). \tag{3.5}$$

2. *Let $k \geq 3$ be odd and $m = (k-1)/2$. Then* (3.3) *is asymptotically stable if and only if one of the following conditions holds:*

$$a \leq b < -a, \quad |bh| < 1, \tag{3.6}$$

$$|b| + a < 0, \quad (-1)^m bh = 1, \tag{3.7}$$

$$b + |a| < 0, \quad bh > -1, \quad \tau < \tau_2^*(h), \tag{3.8}$$

$$(-1)^m b + a < 0, \quad (-1)^m bh > 1, \quad \tau < \tau_2^*(h), \tag{3.9}$$

$$(-1)^m b + a > 0, \quad (-1)^{m+1} bh > 1, \quad \tau < \tau_2^*(h). \tag{3.10}$$

Proof. The necessary and sufficient conditions stated above follow from the application of Theorem 2 to (3.3). Considering

$$\alpha = -\frac{1+ah}{1-ah}, \quad \gamma = -\frac{2bh}{1-ah}, \quad \ell = k-1,$$

the difference equation (3.3) turns into (1.2). The complete proof (with detailed analysis) can be found in [8]. □

The above asymptotic stability conditions define in the plane (a, b) the asymptotic stability regions. Analogously to the continuous counterpart, the delay independent ((3.4), (3.6), (3.7)) and delay dependent ((3.5), (3.8)–(3.10)) stability regions can be distinguished. Figures 2 and 3 illustrate these stability region in the case of k even and odd, respectively. Moreover, in the case of k odd a position of delay dependent stability regions (in figures hatched ones) depends also on a parity of $m = (k-1)/2$. We emphasize that in the case k even the delay dependent part for $b < 0$, $b < a$ is missing.

The next part illustrates by numerical examples consequences of stability regions location diversity with respect to the change of k.

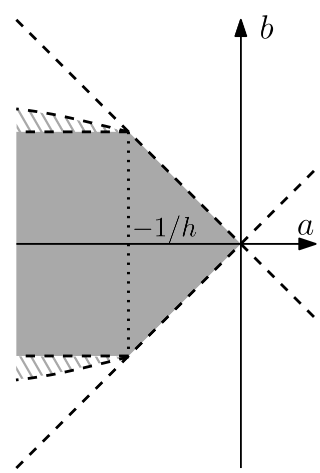

Figure 2. Asymptotic stability region for k even

3.2 Asymptotic stability discussion

Numerical solutions of delay differential equations can have some unexpected properties with respect to one's experience with numerical solving of ordinary differential equations. Several numerical phenomena are introduced in [2]. One of them is related to the following discussion:

We consider the initial value problem for (1.1) with $\tau = 1$

$$y'(t) = ay(t) + by(t-1), \quad t > 0, \tag{3.11}$$

$$y(t) = 1 \ \text{ for } \ t \in [-1, 0] \tag{3.12}$$

and we decide to use formula (3.3) to obtain numerical solution.

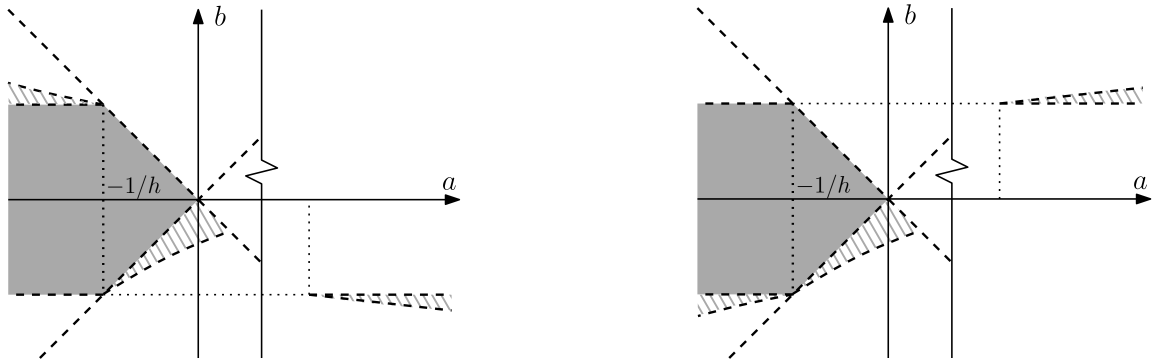

Figure 3. Asymptotic stability region for k odd and m even; k odd and m odd

Example 4. First we point the attention to the situation arising by the choice of $a = 30$, $b = -51/10$. As we can see from Theorem 1 (and as well as from Figure 1), the solution cannot be asymptotically stable. On the contrary, numerical solution with $k = 5$, i.e., $h = 0.2$, evinces asymptotically stable behaviour (see Figure 4) and the numerical formula really is asymptotically stable according to Theorem 3. Moreover, numerical solutions for any integer $k \geq 2$, $k \neq 5$, do not have this property. This extraordinary case $k = 5$ of asymptotic stable solution for given $(a, b) = (30, -51/10)$ is the only occurrence of (a, b) in delay dependent stability region within the fourth quadrant (see Figure 3(1)).

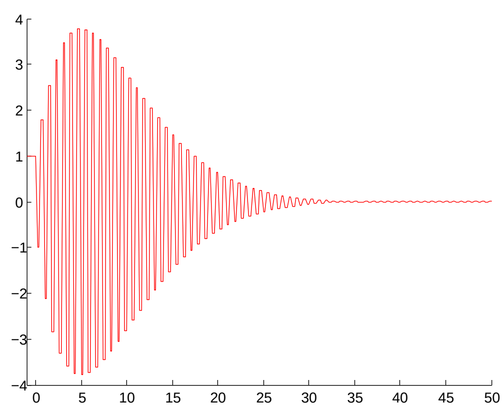

Figure 4. $a = 30$, $b = -51/10$, $k = 5$

Example 5. We consider $a = -1$, $b = -3/2$. The solution of (3.11), (3.12) is asymptotically stable in accordance with Theorem 1 (see Figure 1). The numerical solution for $k = 50$, $k = 51$ and $k = 52$, $k = 53$ is depicted on Figures 5 and 6, respectively.

As we can see, for this fixed pair of (a, b) there occurs switching of asymptotically stable (k even) and unstable (k odd) solutions for several values of k in sequence. This can be explained by Figures 3(1) and 3(2): (a, b) is included in a delay dependent stability region and (a, b) is not included in a delay dependent stability region, by rotation.

Finally, we discuss a limit form of Theorem 3 considering $h \to 0$. In the case of even k, the asymptotic stability region of (3.3) becomes $|b| + a < 0$. It corresponds to (2.1) with the exception of the boundary. In the case of odd k, the asymptotic stability conditions turn into (2.1), (2.2) letting $h \to 0$. These conditions are equivalent to the ones defining the asymptotic stability region of (1.1).

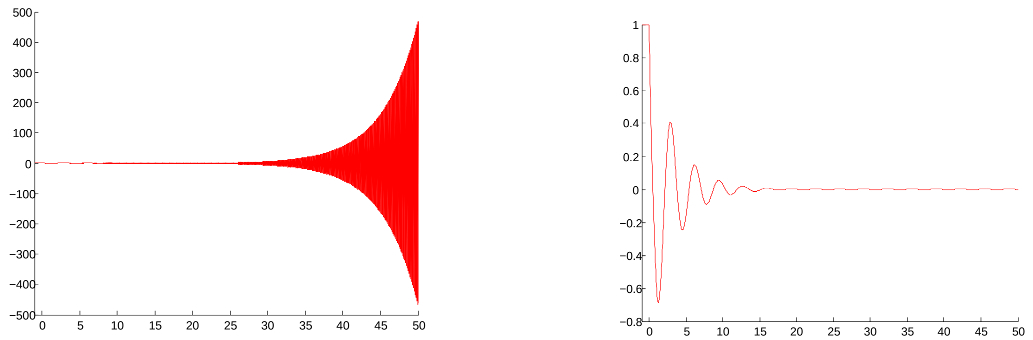

Figure 5. $a = -1,\ b = -3/2,\ k = 50$; $a = -1,\ b = -3/2,\ k = 51$

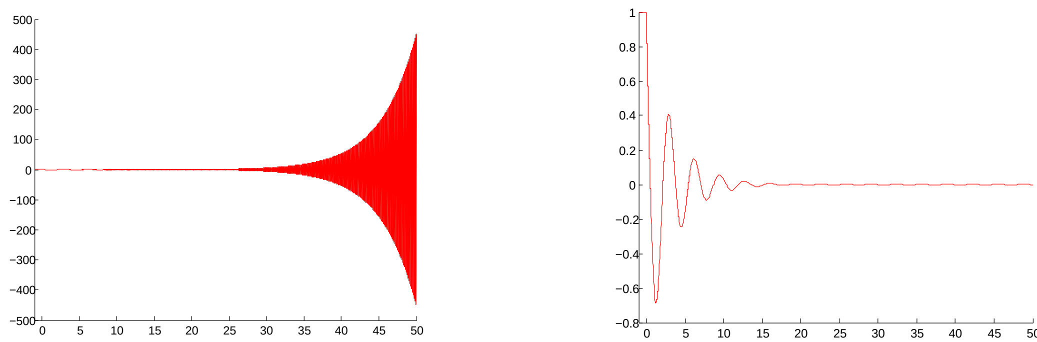

Figure 6. $a = -1,\ b = -3/2,\ k = 52$; $a = -1,\ b = -3/2,\ k = 53$

4 Conclusions and remarks

To summarize the previous, Theorem 3 describes the asymptotic stability regions of difference equation (3.3). This equation actually represents a discretization of delay differential equation (1.1) by modified midpoint rule. It was shown that the asymptotic stability regions depend not only on the value of stepsize h, but also on parity of k. We had provided the discussion with two examples, where specific situations occurred with respect to the position of delay dependent asymptotic stability regions. Deeper analysis for more complicated numerical methods is a great call because of the absence of effective form of the appropriate necessary and sufficient conditions for asymptotic stability.

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A STRING OSCILLATIONS SIMULATION WITH NONLINEAR CONDITIONS

Abstract. In the present paper we investigate the initial-boundary value problems describing oscillation processes with hysteresis type conditions. Analogues of the d'Alembert formula are obtained.*

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1 Introduction

There are many papers devoted to the control problems for distributed systems, not to mention the works of V. A. Il'in, E. I. Moiseev, L. N. Znamenskaya, A. I. Egorov, A. V. Borovskikh (see [1–4]). In these works, a control, allowing to govern by the oscillation process was obtained. The d'Alembert formula is very important for the search of the control. In the present paper, we investigate the initial-boundary value problems describing oscillation processes with conditions of hysteresis type. This kind of problems arise in a simulation of string oscillations, where the movement is restricted by a sleeve concentrated at one point. We consider the cases when the sleeve is located at the end of a segment and at the node of a graph-star. Analogues of the d'Alembert formula are obtained.

2 Preliminaries

In this section, we recall some notions and definitions which we will need in the sequel (details can be found in [5]).

Let H be a Hilbert space. The inner product in H is denoted by $\langle \cdot\,, \cdot \rangle$.

For a closed convex $C \subset H$ and $x \in C$, the set

$$N_C(x) = \big\{\xi \in H : \ \langle \xi, c - x\rangle \le 0 \ \ \forall\, c \in C\big\}$$

denotes the outward normal cone to C at x.

Note that we always have $0 \in N_C(x)$, $N_{\{x\}}(x) = H$, and $N_C(x) = \{0\}$ for $x \in \operatorname{int} C$, the interior of C, if provided $\operatorname{int} C \ne \varnothing$. The last relation shows that the outward normal cone is non-trivial only for $x \in \partial C$, the boundary of C.

Recall that the Hausdorff distance $d_H(C_1, C_2)$ between closed sets C_1 and C_2 is given by the formula

$$d_H(C_1, C_2) = \max\Big\{ \sup_{x\in C_2} \operatorname{dist}(x, C_1),\ \sup_{x\in C_1} \operatorname{dist}(x, C_2)\Big\}.$$

Consider the so-called "sweeping process"

$$-u'(t) \in N_{C(t)}(u(t)) \ \text{ for a.e. } \ t \in [0, T], \tag{2.1}$$

$$u(0) = u_0 \in C(0). \tag{2.2}$$

A function $u : [0, T] \to H$ is called a solution of the initial problem (2.1), (2.2) if

(a) $u(0) = u_0$;

(b) $u(t) \in C(t)$ for all $t \in [0, T]$;

(c) u is differentiable for almost every point $t \in [0, T]$;

(d) $-u'(t) \in N_{C(t)}(u(t))$ for almost every $t \in [0, T]$.

There are many papers devoted to sweeping processes (see, e.g., [5–13]).

Later we will use the next theorems.

Theorem 2.1 (Existence [5, Theorem 2]). *Assume that the map $t \to C(t)$ satisfies*

$$d_H(C(t), C(s)) \le L|t - s|$$

and $C(t) \subset H$ is nonempty, closed and convex for every $t \in [0, T]$. Let $u_0 \in C(0)$. Then there exists a solution $u : [0, T] \to H$ of (2.1), (2.2) which is Lipschitz continuous with the constant L. In particular, $|u'(t)| \le L$ for almost every $t \in [0, T]$.

Theorem 2.2 (Uniqueness [5, Theorem 3]). *The solution of (2.1), (2.2) is unique in the class of absolutely continuous functions.*

3 A string with a hysteresis type boundary value condition

Suppose a string is located along the segment $[0,l]$. Let $u(x,t)$ be a deviation from the equilibrium position at the time t. Assume that the left end of the string has the elastic support (a spring), so we have $u'_x(0,t)=\gamma u(0,t)$. The right end of the string moves along a vertical needle (without friction) inside a sleeve, represented by $[-h,h]$, where $h>0$. While $|u(l,t)|<h$, the right end of the string inside of the sleeve remains free, i.e., $u'_x(l,t)=0$. If the string reaches the boundary points of the sleeve, then the conditions $u(l,t)=h$ or $u(l,t)=-h$, respectively, are satisfied at a certain moment. Notice that we consider the case where the sleeve move in perpendicular to the axis Ox and its movement is given by

$$C(t)=[-h,h]+\xi(t). \tag{3.1}$$

Suppose that the string velocity is zero at the initial time $t=0$ and the string form is determined by a function $\varphi(x)\in W_2^1[0,l]$, where $\varphi'_x(0)=\gamma\varphi(0)$, $\varphi(l)\in C(0)$.

The mathematical model of such problem can be described as

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial t^2}, \quad 0<x<l, \ \ 0<t<T, \\ u(x,0)=\varphi(x), \\ \dfrac{\partial u}{\partial t}(x,0)=0, \\ u'_x(0,t)=\gamma u(0,t), \\ u(l,t)\in C(t) \\ -u'_x(l,t)\in N_{C(t)}(u(l,t)), \end{cases} \tag{3.2}$$

where the set $N_C(a)$ is an outward normal cone to C at a defined by

$$N_C(a)=\big\{\xi\in R^1:\ \xi\cdot(c-a)\le 0\ \ \forall\, c\in C\big\}.$$

Notice that if a is an interior point of C, then $N_C(a)=\{0\}$; if $a=-h+\xi(t)$, then $N_C(a)=(-\infty,0]$; if $a=h+\xi(t)$, then $N_C(a)=[0,+\infty)$.

The condition $-u'_x(l,t)\in N_{C(t)}(u(l,t))$ means that if $u(l,t)$ is an interior point of $C(t)$, then $u'_x(l,t)=0$, i.e., the oscillation process is the same as for a string with a free right end (see [14]); when the right end of the string is tangent to the boundary sleeve point, the right end of the string is not free anymore: there is a force $f(t)$, which blocks this end, so $-u'_x(l,t)=-f(t)\in N_{C(t)}(u(l,t))$.

We consider a solution of (3.2) belonging to a special class of functions introduced for the first time by V. A. Il'in in [15,16]. Let Q_T be the rectangle $Q_T=[0\le x\le l]\times[0\le t\le T]$. As in [15,16], we suppose that u belongs to the class $\widehat{W}_2^1(Q_T)$ if the function $u(x,t)$ is continuous in the closed rectangle Q_T and in this rectangle has both generalized partial derivatives $u'_x(x,t)$ and $u'_t(x,t)$, which belong to the class $L_2(Q_T)$ and, moreover, $u'_x(\,\cdot\,,t)$ belongs to the class $L_2[0\le x\le l]$ for every fixed t of the segment $[0,T]$, and $u'_t(x,\,\cdot\,)$ belongs to the class $L_2[0\le t\le T]$, for any fixed x of the segment $[0,l]$. By a solution of (3.2) we call a function $u(x,t)\in\widehat{W}_2^1(Q_T)$ such that $u(l,t)\in C(t)$ for all t, the condition $-u'_x(l,t)\in N_{C(t)}(u(l,t))$ holds for almost every t, and the integral identity

$$\int_0^l\int_0^T u(x,t)\big[\Psi_{tt}(x,t)-\Psi_{xx}(x,t)\big]\,dx\,dt+\int_0^l \Psi'_t(x,0)\varphi(x)\,dx \\ -\int_0^T \Psi(l,t)u'_x(l,t)\,dt+\int_0^T \Psi'_x(l,t)u(l,t)\,dt=0 \tag{3.3}$$

holds for any function $\Psi(x,t)\in C^2(Q_T)$, which satisfies the conditions $\Psi'_x(0,t)=\gamma\Psi(0,t)$, $\Psi(x,T)=0$, $\Psi'_t(x,T)=0$.

Theorem 3.1. *Assume that the function $\xi(t)$ satisfies the Lipschitz condition and the function $\varphi \in W_2^1[0,l]$. If $0 \le t \le l$, then the solution of problem (3.2) can be represented as*

$$u(x,t) = \frac{\Phi(x-t)+\Phi(x+t)}{2}\,, \tag{3.4}$$

where

$$\Phi(x) = \begin{cases} \varphi(x), & x \in [0,l], \\ 2g(x-l)+\varphi(2l-x)-2\varphi(l), & x \in [l,2l], \\ \varphi(-x) - 2\gamma e^{\gamma x} \int\limits_0^{-x} \varphi(s) e^{\gamma s}\, ds, & x \in [-l,0], \end{cases}$$

and g is a solution of the problem

$$-v_1'(t) \in N_{D(t)}(v_1(t)), v_1(0) = \varphi(l) \in D(0), \tag{3.5}$$

$$D(t) = C(t) + \int\limits_0^t \varphi'(l-s)\, ds.$$

Proof. Consider the problem

$$-v_1'(t) \in N_{D(t)}(v_1(t)), \quad v_1(0) = \varphi(l) \in D(0).$$

We will use Theorems 2.1 and 2.2. Since $D(t)$ is a nonempty, closed, convex set, and the mapping $t \mapsto D(t)$ satisfies the Lipschitz condition with a constant L^*, i.e.,

$$d_H(D(t), D(s)) \le L^* |t-s|, \quad t,s \in [0,T],$$

there is a unique absolutely continuous function $g(t)$ defined on all $[0,l]$, which is the solution of problem (3.5), and $|g'(t)| \le L^*$ for almost all $t \in [0,l]$. Since $g(t) \in D(t)$, where $D(t) = C(t)+\varphi(l)-\varphi(l-t)$ and $u(l,t) = g(t)+\varphi(l-t)-\varphi(l)$, we have $u(l,t) \in C(t)$.

Let us show that $-u_x'(l,t) \in N_{C(t)}(u(l,t))$. Notice that $u_x'(l,t) = g'(t)$. Let us show that $-g'(t) \in N_{C(t)}(g(t)+\varphi(l-t)-\varphi(l))$. Since $-g'(t) \in N_{D(t)}(g(t))$, we get $-g'(t)(c(t)-\varphi(l-t)+\varphi(l)-g(t)) \le 0$ for all $c(t) \in C(t)$. So, $-g'(t) \in N_{C(t)}(g(t)+\varphi(l-t)-\varphi(l))$.

Our aim now is to prove equality (3.3). We have

$$\begin{aligned}
&\int\limits_0^l \Big(\int\limits_0^T u(x,t)\Psi_{tt}(x,t)\, dt \Big)\, dx - \int\limits_0^T \Big(\int\limits_0^l u(x,t)\Psi_{xx}(x,t)\, dx \Big)\, dt \\
&\qquad + \int\limits_0^l \Psi_t'(x,0)\varphi(x)\, dx - \int\limits_0^T \Psi(l,t)u_x'(l,t)\, dt + \int\limits_0^T \Psi_x'(l,t)u(l,t)\, dt \\
&= \int\limits_0^l \big(u(x,T)\Psi_t'(x,T) - u(x,0)\Psi_t'(x,0) \big)\, dx - \int\limits_0^l \int\limits_0^T u_t'\Psi_t'\, dt\, dx - \int\limits_0^T \big(\Psi_x'(l,t)u(l,t) - \Psi_x'(0,t)u(0,t) \big)\, dt \\
&\qquad + \int\limits_0^T \int\limits_0^l u_x'\Psi_x'\, dx\, dt + \int\limits_0^l \Psi_t'(x,0)\varphi(x)\, dx - \int\limits_0^T \Psi(l,t)u_x'(l,t)\, dt + \int\limits_0^T \Psi_x'(l,t)u(l,t)\, dt.
\end{aligned}$$

We have to prove that

$$\int\limits_0^T \int\limits_0^l u_x'\Psi_x'\, dx\, dt - \int\limits_0^T \int\limits_0^l u_t'\Psi_t'\, dx\, dt = \int\limits_0^T \Psi(l,t)u_x'(l,t)\, dt - \gamma \int\limits_0^T \Psi(0,t)u(0,t)\, dt.$$

According to (3.4), we obtain

$$\frac{1}{2}\int_0^T\int_0^l (\Phi'(x-t)+\Phi'(x+t))\Psi'_x\,dx\,dt - \frac{1}{2}\int_0^l\int_0^T (\Phi'(x+t)-\Phi'(x-t))\Psi'_t\,dt\,dx$$

$$=\frac{1}{2}\int_0^l \big(\Psi'_x(x,T)(\Phi(x+T)-\Phi(x-T))-\Psi'_x(x,0)(\Phi(x)-\Phi(x))\big)\,dx-\frac{1}{2}\int_0^l\int_0^T(\Phi(x+t)-\Phi(x-t))\Psi_{xt}\,dt\,dx$$

$$-\frac{1}{2}\int_0^T \big(\Psi'_t(l,t)(\Phi(l+t)-\Phi(l-t))-\Psi'_t(0,t)(\Phi(t)-\Phi(-t))\big)\,dt+\frac{1}{2}\int_0^l\int_0^T(\Phi(x+t)-\Phi(x-t))\Psi_{xt}\,dt\,dx$$

$$=-\frac{1}{2}\int_0^T \Psi'_t(l,t)(\Phi(l+t)-\Phi(l-t))\,dt+\frac{1}{2}\int_0^T \Psi'_t(0,t)(\Phi(t)-\Phi(-t))\,dt$$

$$=\int_0^T \Psi'_t(l,t)(\varphi(l)-g(t)))\,dt+\gamma\int_0^T \Psi'_t(0,t)e^{-\gamma t}\int_0^t \varphi(s)e^{\gamma s}\,ds\,dt$$

$$=\int_0^T \Psi'_t(l,t)(\varphi(l)-g(t)))\,dt-\gamma\int_0^T \Psi(0,t)\Big(\varphi(t)-\gamma e^{-\gamma t}\int_0^t \varphi(s)e^{\gamma s}\,ds\Big)\,dt.$$

On the other hand,

$$\int_0^T \Psi(l,t)u'_x(l,t)\,dt-\gamma\int_0^T \Psi(0,t)u(0,t)\,dt$$

$$=\int_0^T \Psi'_t(l,t)(\varphi(l)-g(t))\,dt-\gamma\int_0^T \Psi(0,t)\Big(\varphi(t)-\gamma e^{-\gamma t}\int_0^t \varphi(s)e^{\gamma s}\,ds\Big)\,dt.$$

This completes the proof of the theorem. □

Remark 3.1. The presentation of the solution in the form of (3.4) is true for all T. The initial condition $u(x,0)=\Phi(x)=\varphi(x)$ determines the value of Φ on the interval $[0,l]$. According to the boundary condition at the point l, we can represent the derivative $u'_t(l,t)$ through $u'_x(l,t)$. We obtain

$$-u'_t(l,t)\in N_{C(t)}(u(l,t))+\Phi'(l-t),$$

where the function $\Phi'(l-t)$ for all $0\le t\le l$ is known, namely, $\Phi'(l-t)=\varphi'(l-t)$. Denote $w(t)=u(l,t)$, $\Phi'(l-t)=\eta(t)$. So we get the problem

$$-w'(t)\in N_{C(t)}(w(t))+\eta(t),\quad w(0)=\varphi(l)\in C(0).$$

Put the function $v_1(t)=w(t)+\int_0^t\eta(s)\,ds$, and the set $D(t)=C(t)+\int_0^t\eta(s)\,ds$. Notice that $N_{C(t)}(w(t))=N_{D(t)}(v_1(t))$. We get the problem

$$-v_1'(t)\in N_{D(t)}(v_1(t)),\quad v_1(0)=\varphi(l)\in D(0).$$

According to Theorems 2.1 and 2.2, this problem has a unique solution $v_1(t)$, which is defined on the whole interval $[0,l]$. This function $v_1(t)$ is absolutely continuous, and its derivative is bounded almost everywhere. Hence we have that the function $\Phi(x)$ on the interval $[l,2l]$ should be defined as

$$\Phi(x)=2v_1(x-l)+\varphi(2l-x)-2\varphi(l).$$

Using the condition $u'_x(0,t)=0$, we can extend $\Phi(x)$ on the interval $[-l,0]$. So, if $T\le l$, then the problem is solved. Otherwise, the function $\Phi(x)$ is known for all $x\in[-l,2l]$. Repeating the above scheme several times, we obtain the required representation.

Remark 3.2. Notice that problem (3.2) has a unique solution. Assume that $\varphi(l)\in(-h+\xi(0),h+\xi(0))$. Then the oscillation process occurs likewise for the string with a free end for all $t\in[0,t_1]$, and the string form is the solution of the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial t^2}, \quad 0<x<l, \ \ 0<t<t_1, \\ u(x,0)=\varphi(x), \\ \dfrac{\partial u}{\partial t}(x,0)=0, \\ u'_x(0,t)=\gamma u(0,t), \\ u'_x(l,t)=0. \end{cases}$$

Notice that the last problem has a unique solution $u(x,t)$. If $t_1<T$, then the relation $u(l,t_1)=\pm h+\xi(t)$ holds at the moment t_1, and for all $t\in[t_1,t_2]$ a string form is a solution of the problem

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2}=\dfrac{\partial^2 v}{\partial t^2}, \quad 0<x<l, \ \ t_1<t<t_2, \\ v(x,t_1)=u(x,t_1), \\ \dfrac{\partial v}{\partial t}(x,t_1)=u'_t(x,t_1), \\ v'_x(0,t)=\gamma v(0,t), \\ v(l,t)=-h+\xi(t) \end{cases}$$

or

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2}=\dfrac{\partial^2 v}{\partial t^2}, \quad 0<x<l, \ \ t_1<t<t_2, \\ v(x,t_1)=u(x,t_1), \\ \dfrac{\partial v}{\partial t}(x,t_1)=u'_t(x,t_1), \\ v'_x(0,t)=\gamma v(0,t), \\ v(l,t)=h+\xi(t). \end{cases}$$

Each of the above problems have a unique solution for every $t\in[t_1,t_2]$. By a similar reasoning, we find that the original problem has a unique solution.

4 A problem on a geometric graph

Let the points O, $A_1,A_2,\dots,A_n$ belong to the horizontal plane π. Consider a mechanical system consisting of n strings, which in equilibrium are the segments $OA_1,OA_2,\dots,OA_n$. The ends of the strings have elastic supports (springs) at the points $A_1,A_2,\dots,A_n$ and interconnected at the point O. There is a perpendicular needle inside a sleeve passing through the point O. The graph Γ consists of edges (intervals) $OA_1,OA_2,\dots,OA_n$ and vertices $O,A_1,A_2,\dots,A_n$. We will use the notions and the terminology from [17]. Under the influence of a distributed force perpendicular to the plane π, the strings deviate from the equilibrium position. We assume that a deviation of all points is parallel to the same straight line, which is perpendicular to the plane, and consider small deviations from the equilibrium position. Take the system of coordinates to describe string deformations. The X-axis Ox_i

for the i-th string ($i = 1, 2, \ldots, n$) contains the segment OA_i and is directed from A_i to O. Thus the graph is directed to the node. The Y-axis OY passes perpendicularly to the plane π. Consider the oscillation process. Let $u_i(x,t)$ be the deviation of i-th string from the equilibrium position at time t. We assume that the length of all strings equals l, i.e. $0 \le x \le l$.

Thus, the oscillations of each string of the system can be described by the wave equation

$$\frac{\partial^2 u_i}{\partial x^2} = \frac{\partial^2 u_i}{\partial t^2}\,.$$

The connection between the strings at the node means that $u_1(l,t) = u_i(l,t)$ $(i = 1, 2, \ldots, n)$. The conditions of the elastic fixing mean that $\frac{\partial u_i}{\partial x}(0,t) = \gamma u_i(0,t)$ $(i = 1, 2, \ldots, n)$. At the point $x = l$ there is the sleeve, whose movement in the perpendicular direction to the plane π is given by (3.1), where the function $\xi(t)$ satisfies the Lipschitz condition.

Assume that at the initial moment the initial form and the initial velocity of strings are $u_i(x,0) = \varphi_i(x)$, $\frac{\partial u_i}{\partial t}(x,0) = 0$, where $\varphi_i \in W_2^1[0,l]$, $\varphi_1(l) = \varphi_2(l) = \cdots = \varphi_n(l) = \varphi(l)$, $\frac{\partial \varphi_i}{\partial x}(0) = \gamma\varphi_i(0)$, $\varphi(l) \in C(0)$.

Then a mathematical model of such a problem can be described as

$$\begin{cases} \dfrac{\partial^2 u_i}{\partial x^2} = \dfrac{\partial^2 u_i}{\partial t^2}\,, \quad 0 < x < l,\ \ 0 < t < T \ \ (i = 1, 2, \ldots, n), \\ u_i(x,0) = \varphi_i(x), \\ \dfrac{\partial u_i}{\partial t}(x,0) = 0, \\ -\displaystyle\sum_{i=1}^{n} \dfrac{\partial u_i}{\partial x}(l-0,t) \in N_{C(t)}(u(l,t)), \\ u(l,t) = u_1(l,t) = u_2(l,t) = \cdots = u_n(l,t), \\ u(l,t) \in C(t), \\ \dfrac{\partial u_i}{\partial x}(0,t) = \gamma u_i(0,t). \end{cases}$$

By a solution of this problem we mean the function $u(x,t)$, whose restrictions to the edges coincide with $u_i(x,t)$ $(i = 1, 2, \ldots, n)$. The functions $u_i(x,t) \in \widehat{W}_2^1(Q_T)$ satisfy the conditions $u_1(l,t) = u_2(l,t) = \cdots = u_n(l,t) = u(l,t)$, $u(l,t) \in C(t)$ for all t. The condition

$$-\sum_{i=1}^{n} \frac{\partial u_i}{\partial x}(l-0,t) \in N_{C(t)}(u(l,t))$$

holds for almost all $t \in [0,T]$. The integral equalities

$$\sum_{i=1}^{n} \int_0^l \int_0^T u_i(x,t)\Big[\frac{\partial^2 \Psi_i}{\partial t^2}(x,t) - \frac{\partial^2 \Psi_i}{\partial x^2}(x,t)\Big]\,dx\,dt + \sum_{i=1}^{n} \int_0^l \frac{\partial \Psi_i}{\partial t}(x,0)\varphi_i(x)\,dx$$
$$+ \sum_{i=1}^{n} \int_0^T \Big(u(l,t)\,\frac{\partial \Psi_i}{\partial x}(l-0,t) - \Psi(l,t)\frac{\partial u_i}{\partial x}(l-0,t)\Big)\,dt = 0, \quad i = 1, \ldots, n,$$

hold for all $\Psi_i(x,t) \in C^2(Q_T)$, such as

$$\frac{\partial \Psi_i}{\partial x}(0,t) = \gamma\Psi_i(0,t), \quad \Psi_i(x,T) = 0, \quad \frac{\partial \Psi_i}{\partial t}(x,T) = 0, \quad \Psi_1(l,t) = \cdots = \Psi_n(l,t) = \Psi(l,t).$$

An analogue of the d'Alembert formula for the representation of the solution of this problem can be obtained by bringing the problem on the segment. Suppose that the solution of the last problem exists. Denote it by $v(x,t)$, where on each edge of the graph Γ the function $v(x,t)$ is defined as $v_i(x,t)$.

Let $\widetilde{u}(x,t) = \frac{1}{n}\sum\limits_{i=1}^{n} v_i(x,t)$. Then $\widetilde{u}(x,t)$ is the solution of the problem which has been studied above

$$\begin{cases} \dfrac{\partial^2 \widetilde{u}}{\partial x^2} = \dfrac{\partial^2 \widetilde{u}}{\partial t^2}, \quad 0 < x < l, \;\; 0 < t < T, \\ \widetilde{u}(x,0) = \frac{1}{n}\sum\limits_{i=1}^{n} \varphi_i(x), \\ \dfrac{\partial \widetilde{u}}{\partial t}(x,0) = 0, \\ -\dfrac{\partial \widetilde{u}}{\partial x}(l,t) \in N_{C(t)}(\widetilde{u}(l,t)), \\ \widetilde{u}(l,t) \in C(t), \\ \widetilde{u}'_x(0,t) = \gamma \widetilde{u}(0,t). \end{cases}$$

So,

$$\widetilde{u}(x,t) = \frac{\Phi(x-t) + \Phi(x+t)}{2}.$$

Suppose $\omega_i(x,t) = v_i(x,t) - \widetilde{u}(x,t)$ $(i = 1,2,\ldots,n)$. Notice that $\omega_i(x,t)$ $(i = 1,2,\ldots,n)$ are solutions of the problem

$$\begin{cases} \dfrac{\partial^2 \omega_i}{\partial x^2} = \dfrac{\partial^2 \omega_i}{\partial t^2}, \quad 0 < x < l, \;\; 0 < t < T \;\; (i = 1,2,\ldots,n), \\ \omega_i(x,0) = \varphi_i(x) - \dfrac{1}{n}\sum\limits_{j=1}^{n} \varphi_j(x), \\ \dfrac{\partial \omega_i}{\partial t}(x,0) = 0, \\ \dfrac{\partial \omega_i}{\partial x}(0,t) = \gamma \omega_i(0,t), \\ \omega_i(l,t) = 0. \end{cases}$$

For ω_i, we have

$$\omega_i(x,t) = \frac{\Phi_i(x-t) + \Phi_i(x+t)}{2},$$

where

$$\Phi_i(x) = \begin{cases} \varphi_i(x) - \dfrac{1}{n}\sum\limits_{j=1}^{n} \varphi_j(x), & x \in [0,l] \\ \varphi_i(-x) - \dfrac{1}{n}\sum\limits_{j=1}^{n} \varphi_j(-x) - 2\gamma e^{\gamma x} \int\limits_0^{-x} (\varphi_i(s) - \dfrac{1}{n}\sum\limits_{j=1}^{n} \varphi_j(s)) e^{\gamma s}\, ds, & x \in [-l,0], \\ \dfrac{1}{n}\sum\limits_{j=1}^{n} \varphi_j(2l-x) - \varphi_i(2l-x), & x \in [l,2l]. \end{cases}$$

Notice that $v_i(x,t) = \omega_i(x,t) + \widetilde{u}(x,t)$. So,

$$v_i(x,t) = \frac{\Phi(x-t) + \Phi(x+t)}{2} + \frac{\Phi_i(x-t) + \Phi_i(x+t)}{2}.$$

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