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ASYMPTOTIC REPRESENTATIONS OF ONE CLASS OF SOLUTIONS OF n-th ORDER NONAUTONOMOUS ORDINARY DIFFERENTIAL EQUATIONS

Abstract. Asymptotic representations of some classes of solutions of non-autonomous ordinary differential n-th order equations which somewhat are close to linear equations are established.

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1 Introduction

Consider the differential equation

$$y^{(n)} = \alpha_0 p(t) y |\ln |y||^{\sigma}, \tag{1.1}$$

where $\alpha_0 \in \{-1, 1\}$, $\sigma \in \mathbb{R}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$[1].

A solution y of the equation (1.1), which is nonzero on the interval $[t_y, \omega[\subset [a, \omega[$, is said to be a $P_\omega(\lambda_0)$-solution if it satisfies the following conditions:

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{either } \ 0, \\ \text{or } \ \ \pm\infty \end{cases} \quad (k = \overline{0, n-1}), \qquad \lim_{t\uparrow\omega} \frac{(y^{(n-1)}(t))^2}{y^{(n)}(t) y^{(n-2)}(t)} = \lambda_0. \tag{1.2}$$

We notice that the differential equation (1.1) is a special case of the differential equation of a more general form

$$y^{(n)} = \alpha_0 p(t) \varphi(y),$$

where α_0 and p are the same as in the equation (1.1) and $\varphi : \Delta_{Y_0} \to]0, +\infty[$ is a continuous and regularly varying function as $y \to Y_0$ of the order γ, Y_0 is equal either to zero or to $\pm\infty$, Δ_{Y_0} is some one-sided neighborhood of Y_0.

The differential equation (1.1) belongs to the class of two-term non-autonomous equations with regularly varying nonlinear function $\varphi(y)$ as $y \to 0$ and $y \to \pm\infty$. In recent decades, the asymptotic theory of such equations has been studied by many authors (see, e.g., monograph by V. Maric [8] and the references therein concerning the second order equation; see also the papers by V. M. Evtukhov, A. M. Samoilenko [6] and by V. M. Evtukhov, A. M. Klopot [4] for differential equations of order n).

In [6] and [4], for the two-term differential equations of n-th order with regularly varying nonlinear function $\varphi(y)$ as $y \to 0$ and $y \to \pm\infty$, the authors obtained asymptotic representation for all possible types of $P_\omega(\lambda_0)$-solutions and their derivatives up to the order $n-1$, inclusive. However, the results of these works do not cover the case where $\varphi(y) = y|\ln|y||^{\sigma}$ is a regularly varying function of order one. By such nonlinearity of the equation (1.1), not being a substantially non-linear, and due to the asymptotic relation $\varphi(y) = y^{1+o(1)}$ as $y \to 0$ $(\pm\infty)$, the differential equation is asymptotically close to the linear differential equation

$$y^{(n)} = \alpha_0 p(t) y, \tag{1.3}$$

and therefore is of theoretical interest.

In [3], for the equation (1.1), the asymptotic behavior of $P_\omega(\lambda_0)$-solutions as $t \uparrow \omega$ was investigated when $\lambda_0 \in R \setminus \{0, \frac{1}{2}, \dots, \frac{n-2}{n-1}\}$.

The aim of the present paper is to establish the existence conditions of $P_\omega(\lambda_0)$-solutions of the equation (1.1) in case $\lambda_0 = 0$, and to obtain asymptotic representations as $t \uparrow \omega$ for all such solutions and their derivatives up to order $n-1$, inclusive.

2 Auxiliary statements

To obtain our main results we need two lemmas, the first one is related to a priori asymptotic properties of $P_\omega(0)$-solutions and the other is about the existence of vanishing at a singular point solutions of a system of quasi-linear differential equations.

To state the first one, we introduce the function

$$\pi_\omega(t) = \begin{cases} t & \text{if } \ \omega = +\infty, \\ t - \omega & \text{if } \ \omega < +\infty. \end{cases}$$

From Lemma 10.6 introduced in [2, Ch. 3, § 10, pp. 143–144] we get the following statement.

[1]We assume that $a > 1$ for $\omega = +\infty$ and $\omega - a < 1$ for $\omega < +\infty$.

Lemma 2.1. *If $n \geq 2$, then each $P_\omega(0)$-solution of the differential equation* (1.1) *satisfies the following asymptotic relation as $t \uparrow \omega$:*

$$y^{(k-1)}(t) \sim \frac{[\pi_\omega(t)]^{n-k-1}}{(n-k-1)!}\, y^{(n-2)}(t) \ \ (k=1,\dots,n-2), \quad y^{(n-1)}(t) = o\Big(\frac{y^{(n-2)}(t)}{\pi_\omega(t)}\Big), \tag{2.1}$$

and in case $\lim\limits_{t\uparrow\omega} \frac{\pi_\omega(t) y^{(n)}(t)}{y^{(n-1)}(t)}$ (finite or equal to $\pm\infty$) exists, the following relation holds:

$$y^{(n)}(t) \sim -\frac{y^{(n-1)}(t)}{\pi_\omega(t)} \ \ \textit{as} \ \ t \uparrow \omega. \tag{2.2}$$

Next, we consider a system of quasi-linear differential equations

$$\begin{cases} v_k' = h(t)\Big[f_k(t, v_1, \dots, v_n) + \sum\limits_{i=1}^{n} c_{ki} v_i\Big] \qquad (k = \overline{1, n-1}), \\ v_n' = H(t)\Big[f_n(t, v_1, \dots, v_n) + \sum\limits_{i=1}^{n} c_{ni} v_i\Big], \end{cases} \tag{2.3}$$

in which $c_{ki} \in \mathbb{R}$ $(k, i = \overline{1,n})$, $h, H : [t_0, \omega[\to \mathbb{R} \setminus \{0\}$ are continuously differentiable functions, and $f_k : [t_0, \omega[\times \mathbb{R}^n_{\frac{1}{2}}$ $(k = \overline{1,n})$ are continuous functions satisfying the condition

$$\lim_{t\uparrow\omega} f_k(t, v_1, \dots, v_n) = 0 \ \text{ uniformly in } \ (v_1, \dots, v_n) \in \mathbb{R}^n_{\frac{1}{2}}, \tag{2.4}$$

where

$$\mathbb{R}^n_{\frac{1}{2}} = \Big\{(v_1, \dots, v_n) \in \mathbb{R}^n : \ |v_i| \leq \frac{1}{2} \ \ (i = \overline{1,n})\Big\}.$$

By Theorem 2.6 from [5] for the system of differential equations (2.3) the following lemma holds.

Lemma 2.2. *Let the functions h and H satisfy the conditions*

$$\lim_{t\uparrow\omega} \frac{H(t)}{h(t)} = 0, \quad \int_{t_0}^{\omega} H(\tau)\, d\tau = \pm\infty, \quad \lim_{t\uparrow\omega} \frac{1}{H(t)} \Big(\frac{H(t)}{h(t)}\Big)' = 0.$$

Moreover, suppose the matrices $C_n = (c_{ki})_{k,i=1}^n$ and $C_{n-1} = (c_{ki})_{k,i=1}^{n-1}$ are such that $\det C_n \neq 0$ and C_{n-1} has no eigenvalues with zero real part. Then the system of differential equations (2.3) *has at least one solution $(v_k)_{k=1}^n : [t_1, \omega[\, [\mathbb{R}^n_{\frac{1}{2}}$ $(t_1 \in [t_0, \omega[)$ that tends to zero as $t \uparrow \omega$. Furthermore, if among the eigenvalues of matrix C_{n-1} there are m eigenvalues (taking into account the multiplicity) whose real parts have a sign opposite to that of the function $h(t)$ on the interval $[t_0, \omega[$, then if the inequality $H(t)(\det C_n)(\det C_{n-1}) > 0$ holds on $[t_0, \omega[$, there exist m-parameter solutions of the system* (2.3), *and there exists an $m+1$-parameter family when the opposite inequality holds.*

3 Main results

In order to formulate the main results, let us introduce the following auxiliary functions:

$$P_1(t) = \int_{A_1}^{t} p(\tau)\, d\tau, \quad P_2(t) = \int_{A_2}^{t} P_1(\tau)\, d\tau,$$

$$J_A(t) = \int_{A}^{t} p(\tau) \pi_\omega^{n-2}(\tau) |\ln|\pi_\omega(\tau)||^\sigma\, d\tau, \quad I(t) = \int_{a}^{t} J_A(\tau)\, d\tau,$$

where

$$A_1 = \begin{cases} a, & \text{if } \int\limits_a^\omega p(\tau)\,d\tau = +\infty, \\ \omega, & \text{if } \int\limits_a^\omega p(\tau)\,d\tau < \infty, \end{cases} \qquad A_2 = \begin{cases} a, & \text{if } \int\limits_a^\omega |P_1(\tau)|\,d\tau = +\infty, \\ \omega, & \text{if } \int\limits_a^\omega |P_1(\tau)|\,d\tau < \infty. \end{cases}$$

$$A = \begin{cases} a, & \text{if } \int\limits_a^\omega p(\tau)|\pi_\omega(\tau)|^{n-2}|\ln|\pi_\omega(\tau)||^\sigma\,d\tau = +\infty, \\ \omega, & \text{if } \int\limits_a^\omega p(\tau)|\pi_\omega(\tau)|^{n-2}|\ln|\pi_\omega(\tau)||^\sigma\,d\tau < +\infty. \end{cases}$$

When $n = 2$, i.e., in the case of a second order differential equation, the conditions of the existence and asymptotic behavior of $P_\omega(0)$-solutions were obtained in [1].

Theorem 3.1. *Let $n = 2$ and $\sigma \neq 1$, then the differential equation* (1.1) *has $P_\omega(0)$-solutions if and only if the following conditions hold:*

$$\lim_{t\uparrow\omega} |P_2(t)|^{\frac{1}{1-\sigma}} = +\infty, \quad \lim_{t\uparrow\omega} \frac{P_1^2(t)|P_2(t)|^{\frac{\sigma}{1-\sigma}}}{p(t)} = 0, \tag{3.1}$$

Moreover, each of these solutions admits the following asymptotic representations as $t \uparrow \omega$:

$$\ln|y(t)| = \mu|(1-\sigma)P_2(t)|^{\frac{1}{1-\sigma}}[1+o(1)], \quad \frac{y'(t)}{y(t)} = \alpha_0 P_1(t)|(1-\sigma)P_2(t)|^{\frac{\sigma}{1-\sigma}}[1+o(1)], \tag{3.2}$$

where $\mu = \alpha_0 \operatorname{sign}[(1-\sigma)P_2(t)]$. Furthermore, if the conditions (3.1) *are valid, then the differential equation* (1.1) *has a one-parametric (two-parametric) family of such solutions in the case where $A_1 = \omega$ ($A_1 = a$).*

For the case $n > 2$, the following theorem holds.

Theorem 3.2. *Let $n \geq 3$ and suppose that*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_A'(t)}{J_A(t)} \tag{3.3}$$

exists (finite or equal to $\pm\infty$). Then the differential equation (1.1) *has $P_\omega(0)$-solutions if and only if the following conditions hold:*

$$\lim_{t\uparrow\omega} \pi_\omega(t)J_A(t) = 0, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_A'(t)}{J_A(t)} = -1, \quad \lim_{t\uparrow\omega} I(t) = \pm\infty, \tag{3.4}$$

and each of these solutions admits the following asymptotic representations as $t \uparrow \omega$:

$$\frac{y^{(k-1)}(t)}{y^{(n-2)}(t)} = \frac{[\pi_\omega(t)]^{n-k-1}}{(n-k-1)!}[1+o(1)] \quad (k = \overline{1, n-2}), \tag{3.5}$$

$$\ln|y^{(n-2)}(t)| = \frac{\alpha_0|n-2|^\sigma}{(n-2)!} I(t)[1+o(1)], \tag{3.6}$$

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = \frac{\alpha_0|n-2|^\sigma}{(n-2)!} J_A(t)[1+o(1)]. \tag{3.7}$$

Moreover, when the conditions (3.4) *are satisfied, the differential equation* (1.1) *has an $n-1$-parametric family of solutions that admits asymptotic representations* (3.5)–(3.7) *as $t \uparrow \omega$ in case $\omega = +\infty$, and it has two-parametric family of solutions with such representations in case $\omega < +\infty$.*

Proof. Necessity. Let $y : [t_y, \omega[\to \mathbb{R}$ be an arbitrary $P_\omega(0)$-solution of the equation (1.1). Then by the definition of $P_\omega(\lambda_0)$-solution there exists $t_0 \in [t_y, \omega[$ such that $\ln|y(t)| \neq 0$ on the interval $[t_0, \omega[$ and, by Lemma 2.1, the asymptotic relations (2.1) hold. According to the first asymptotic relation of (2.1), we have the asymptotic representations (3.4) from which, in particular, we get

$$y(t) \sim \frac{\pi_\omega^{n-2}(t)}{(n-2)!}\, y^{(n-2)}(t), \quad y'(t) \sim \frac{\pi_\omega^{n-3}(t)}{(n-3)!}\, y^{(n-2)}(t) \ \text{ as } \ t \uparrow \omega.$$

This implies that

$$\frac{y'(t)}{y(t)} \sim \frac{n-2}{\pi_\omega(t)} \ \text{ as } \ t \uparrow \omega$$

and therefore

$$\ln|y(t)| \sim (n-2)\ln|\pi_\omega(t)| \ \text{ as } \ t \uparrow \omega.$$

By virtue of these asymptotic relations, from (1.1) we get

$$y^{(n)}(t) = \frac{\alpha_0}{(n-2)!} p(t)\pi_\omega^{n-2}(t)|(n-2)\ln|\pi_\omega(t)||^\sigma y^{(n-2)}(t)[1+o(1)] \ \text{ as } \ t \uparrow \omega,$$

i.e.,

$$\frac{y^{(n)}(t)}{y^{(n-2)}(t)} = \frac{\alpha_0 |n-2|^\sigma p(t)\pi_\omega^{n-2}(t)}{(n-2)!}\, |\ln|\pi_\omega(t)||^\sigma [1+o(1)] \ \text{ as } \ t \uparrow \omega. \tag{3.8}$$

Since

$$\Big(\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}\Big)' = \frac{y^{(n)}(t)}{y^{(n-2)}(t)}\Big[1 - \frac{[y^{(n-1)}(t)]^2}{y^{(n)}(t)y^{(n-2)}(t)}\Big]$$

and, by the definition of $P_\omega(0)$-solution,

$$\lim_{t\uparrow\omega} \frac{[y^{(n-1)}(t)]^2}{y^{(n)}(t)y^{(n-2)}(t)} = 0,$$

we have

$$\Big(\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}\Big)' \sim \frac{y^{(n)}(t)}{y^{(n-2)}(t)} \ \text{ as } \ t \uparrow \omega.$$

Therefore, the asymptotic relation (3.8) can be written as

$$\Big(\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}\Big)' = \frac{\alpha_0 |n-2|^\sigma p(t)\pi_\omega^{n-2}(t)}{(n-2)!}\, |\ln|\pi_\omega(t)||^\sigma [1+o(1)] \ \text{ as } \ t \uparrow \omega.$$

Integrating this relation from t_0 to t, we obtain

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = c_0 + \frac{\alpha_0 |n-2|^\sigma}{(n-2)!} \int_{t_0}^{t} p(\tau)\pi_\omega^{n-2}(\tau)|\ln|\pi_\omega(\tau)||^\sigma [1+o(1)]\, d\tau, \tag{3.9}$$

where c_0 is a constant, or taking into account the choice of limit integration A in the function J_A, we get

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = c + \frac{\alpha_0 |n-2|^\sigma}{(n-2)!} J_A(t)[1+o(1)] \ \text{ as } \ t \uparrow \omega,$$

where

$$c = c_0 + \frac{\alpha_0 |n-2|^\sigma}{(n-2)!} \int_{t_0}^{A} p(\tau)\pi_\omega^{n-2}(\tau)|\ln|\pi_\omega(\tau)||^\sigma [1+o(1)]\, d\tau.$$

In the case where $A = a$, the integral on the right-hand side of (3.9) tends to $\pm\infty$ as $t \uparrow \omega$, and then (3.9) can be written as

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = \frac{\alpha_0 |n-2|^\sigma}{(n-2)!} J_A(t)[1+o(1)] \ \text{ as } \ t \uparrow \omega. \tag{3.10}$$

We will show that in case $A = \omega$, when the integral on the right-hand side of (3.9) tends to zero as $t \uparrow \omega$, the relation (3.10) also holds, i.e., $c = 0$. Indeed, if $c \neq 0$, then from (3.9) we have

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = c + o(1) \text{ as } t \uparrow \omega.$$

This representation for $\omega = +\infty$ (i.e., $\pi_\omega(t) = t$) contradicts the last relation of (2.1), and if $\omega < +\infty$, by integration we obtain

$$\ln |y^{(n-2)}(t)| = c_1 + o(1) \text{ as } t \uparrow \omega \quad (c_1 = const),$$

which is in contradiction with the first condition of (2.1) (when $k = n - 2$).

Therefore, in each of two possible cases under consideration the asymptotic relation (3.10) holds, that is, (3.7) holds, and by the use of the last asymptotic relation of (2.1), the first condition of (3.4) is satisfied.

Moreover, from (3.10) and (3.8) it follows that

$$\frac{y^{(n)}(t)}{y^{(n-1)}(t)} = \frac{J'_A(t)}{J_A(t)}\,[1 + o(1)] \text{ as } t \uparrow \omega.$$

Then

$$\frac{\pi_\omega(t)y^{(n)}(t)}{y^{(n-1)}(t)} = \frac{\pi_\omega(t)J'_A(t)}{J_A(t)}\,[1 + o(1)] \text{ as } t \uparrow \omega \tag{3.11}$$

and, by virtue of the existence of the limit (3.3) (finite or equal to $\pm\infty$) and using Lemma 2.1, we conclude that (2.2) holds, whereby from (3.11) follows the validity of the second condition of (3.4).

Finally, integrating (3.10) from t_0 to t we get

$$\ln |y^{(n-2)}(t)| = c + \frac{\alpha_0 |n-2|^\sigma}{(n-2)!} \int_{t_0}^{t} J_A(\tau)[1 + o(1)]\,d\tau.$$

Since, by the definition of $P_\omega(0)$-solutions, $\lim_{t\uparrow\omega} \ln |y^{(n-2)}(t)| = \pm\infty$, the third condition of (3.4) is fulfilled and it can be written as (3.6).

Sufficiency. Let $n \geq 3$ and the conditions (3.4) hold. We will show that in this case the differential equation (1.1) has $P_\omega(0)$-solutions admitting asymptotic representations (3.5)–(3.7) as $t \uparrow \omega$, and we find out the quantities of solutions with such representations.

Since

$$\pi_\omega(t)J_A(t) = \frac{\pi_\omega(t)J_A(t)}{I(t)}\, I(t),$$

from the conditions (3.4) we get

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_A(t)}{I(t)} = 0. \tag{3.12}$$

Moreover, by the L'Hospital rule,

$$\lim_{t\uparrow\omega} \frac{I(t)}{\ln |\pi_\omega(t)|} = \lim_{t\uparrow\omega} \pi_\omega(t)J_A(t) = 0. \tag{3.13}$$

Applying now to the equation (1.1) transformations

$$\begin{aligned}
\frac{y^{(k-1)}(t)}{y^{(n-2)}(t)} &= \frac{[\pi_\omega(t)]^{n-k-1}}{(n-k-1)!}\,[1 + v_k(t)] \quad (k = \overline{1, n-2}),\\
\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} &= \frac{\alpha_0 |n-2|^\sigma}{(n-2)!}\, J_A(t)[1 + v_{n-1}(t)],\\
\ln |y^{(n-2)}(t)| &= \frac{\alpha_0 |n-2|^\sigma}{(n-2)!}\, I(t)[1 + v_n(t)],
\end{aligned} \tag{3.14}$$

we obtain the system of differential equations

$$
\begin{aligned}
v_k' &= \frac{n-k-1}{\pi_\omega(t)}\,(v_{k+1}-v_k) - \frac{\alpha_0|n-2|^\sigma}{(n-2)!}\,J_A(t)(1+v_k)(1+v_{n-1}) \ \ (k=\overline{1,n-3}),\\
v_{n-2}' &= -\frac{v_{n-2}}{\pi_\omega(t)} - \frac{\alpha_0|n-2|^\sigma}{(n-2)!}\,J_A(t)(1+v_{n-2})(1+v_{n-1}),\\
v_{n-1}' &= -\frac{J_A'(t)}{J_A(t)}\,(1+v_{n-1}) - \frac{\alpha_0|n-2|^\sigma}{(n-2)!}J_A(t)(1+v_{n-1})^2\\
&\quad + \frac{J_A'(t)}{J_A(t)}\,(1+v_1)\,\frac{|\ln|\frac{\pi_\omega^{n-2}(t)}{(n-2)!}\,(1+v_1)||^\sigma}{|n-2|^\sigma|\ln|\pi_\omega(t)||^\sigma}\,\Big|1+\frac{\alpha_0|n-2|^\sigma}{(n-2)!}\,\frac{I(t)(1+v_n)}{\ln|\frac{\pi_\omega^{n-2}(t)}{(n-2)!}\,(1+v_1)|}\Big|^\sigma,\\
v_n' &= \frac{J_A(t)}{I(t)}\,(1+v_{n-1}) - \frac{J_A(t)}{I(t)}\,(1+v_n).
\end{aligned}
$$

We set

$$
\begin{gathered}
h(t) = \frac{1}{\pi_\omega(t)}, \quad H(t) = \frac{J_A(t)}{I(t)}\,,\\
\delta_1(t) = \frac{\alpha_0|n-2|^\sigma}{(n-2)!}\,\pi_\omega(t)J_A(t), \quad \delta_2(t) = \frac{\pi_\omega(t)J_A'(t)}{J_A(t)}+1,\\
\delta_3(t) = \frac{\alpha_0|n-2|^\sigma}{(n-2)!(n-2)}\,\frac{I(t)}{\ln|\pi_\omega(t)|}\,, \quad \delta_4(t,v_1) = \frac{\ln|\frac{1+v_1}{(n-2)!}|}{(n-2)\ln|\pi_\omega(t)|}\,,
\end{gathered}
$$

and rewrite this system in the form

$$
\begin{cases}
v_k' = h(t)\big[f_k(t,v_1,\dots,v_n) - (n-k-1)v_k + (n-k-1)v_{k+1}\big] \quad (k=\overline{1,n-3}),\\
v_{n-2}' = h(t)\big[f_{n-2}(t,v_1,\dots,v_n) - v_{n-2}\big],\\
v_{n-1}' = h(t)[f_{n-1}(t,v_1,\dots,v_n) - v_1 + v_{n-1}],\\
v_n' = H(t)[v_{n-1}-v_n],
\end{cases} \tag{3.15}
$$

where

$$
\begin{aligned}
f_k(t,v_1,\dots,v_n) &= \delta_2(t)(1+v_k)(1+v_{n-1}) \ \ (k=\overline{1,n-3}),\\
f_{n-2}(t,v_1,\dots,v_n) &= \delta_1(t)(1+v_{n-1})^2 - \delta_2(t)(1+v_{n-1}),\\
f_{n-1}(t,v_1,\dots,v_n) &= \delta_1(t)(1+v_{n-1})(1+v_{n-1}) - \delta_2(t)(1+v_{n-1})\\
&\qquad + (1+v_1)\Big[1+\frac{\pi_\omega(t)J_A'(t)}{J_A(t)}|1+\delta_4(t,v_1)|^\sigma\Big|1+\frac{\delta_3(t)(1+v_n)}{1+\delta_4(t,v_1)}\Big|^\sigma\Big].
\end{aligned}
$$

Here, by the conditions (3.4) and (3.13),

$$
\lim_{t\uparrow\omega}\delta_i(t) = 0 \ \ (i=1,2,3) \tag{3.16}
$$

and

$$
\lim_{t\uparrow\omega}\delta_4(t,v_1) = 0 \ \text{ uniformly in } \ v_1 \in \Big[-\frac{1}{2},\frac{1}{2}\Big]. \tag{3.17}
$$

Taking into account these limit relations, we choose a number $t_0 \in]a,\omega[$ such that for $t \in [t_0,\omega[$ and $|v_1| \le \frac{1}{2}$, $|v_n| \le \frac{1}{2}$ the inequalities

$$
|\delta_4(t,v_1)| \le \frac{1}{2}\,, \quad \Big|\frac{\delta_3(t)(1+v_n)}{1+\delta_4(t,v_1)}\Big| \le \frac{1}{2}
$$

hold. Next, we consider the system (3.15) on the set

$$
\Omega = [t_0,\omega[\,\times\mathbb{R}^n_{\frac{1}{2}}, \ \text{ where } \ \mathbb{R}^n_{\frac{1}{2}} = \Big\{(v_1,\dots,v_n)\in\mathbb{R}^n: \ |v_i| \le \frac{1}{2}\,, \ i=\overline{1,n}\Big\}.
$$

The right-hand sides of (3.15) are continuous on this set, the functions h, H are continuously differentiable on the interval $[t_0,\omega[$, and by the conditions (3.16), (3.17),

$$\lim_{t\uparrow\omega} f_k(t,v_1,\dots,v_n)=0 \ \text{ uniformly in } \ (v_1,\dots,v_n)\in\mathbb{R}^n_{\frac12}.$$

Hence, the system of differential equations (3.15) is a quasilinear system of differential equations of the type (2.3).

We show that for (3.15) all conditions of Lemma 2.2 are satisfied.

By virtue of the definition of functions I and J_A,

$$\int\limits_{t_0}^{t} H(\tau)\,d\tau \sim \ln|J_A(t)| \longrightarrow \pm\infty \ \text{ as } \ t\uparrow\omega.$$

Moreover,

$$\frac{H(t)}{h(t)}=\frac{\pi_\omega(t)J_A(t)}{I(t)},\qquad \frac{1}{H(t)}\Big(\frac{H(t)}{h(t)}\Big)'=1+\frac{\pi_\omega(t)J_A'(t)}{J_A(t)}-\frac{\pi_\omega(t)J_A(t)}{I(t)}$$

and therefore, in view of the second conditions of (3.4) and (3.12), we obtain

$$\lim_{t\uparrow\omega}\frac{H(t)}{h(t)}=0,\quad \lim_{t\uparrow\omega}\frac{1}{H(t)}\Big(\frac{H(t)}{h(t)}\Big)'=0.$$

Thus the conditions (2.4) of Lemma 2.2 are satisfied for the system (3.15).

The matrices C_{n-1} and C_n of dimension $(n-1)\times(n-1)$ and $n\times n$ (respectively) from Lemma 2.2, in the case of the system of differential equations (3.15), have the form

$$C_{n-1}=\begin{pmatrix} -(n-2) & n-2 & 0 & \dots & 0 & 0 & 0\\ 0 & -(n-3) & n-3 & \dots & 0 & 0 & 0\\ 0 & 0 & -(n-4) & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & -2 & 2 & 0\\ 0 & 0 & 0 & \dots & 0 & -1 & 0\\ -1 & 0 & 0 & \dots & 0 & 0 & 1\end{pmatrix},\quad C_n=\begin{pmatrix} C_{n-1} & 0_{n-1}\\ e_{n-1} & -1\end{pmatrix},$$

where 0_{n-1} is a zero column vector of dimension $n-1$ and e_{n-1} is a unit row vector of dimension $n-1$ with the last component equal to one.

These matrices are such that

$$\det C_{n-1}=(-1)^{n-2}(n-2)!,\quad \det C_n=(-1)^{n-1}(n-2)!$$

and

$$\det[C_{n-1}-\rho E_{n-1}]=(-1)^{n-1}(\rho+n-2)(\rho+n-3)\cdots(\rho+1)(\rho-1),$$

where E_{n-1} is the identity matrix of dimension $(n-1)\times(n-1)$. Hence, in particular, we get that the matrix C_{n-1} has $n-1$ nonzero real eigenvalues from which $n-2$ are negative and one is positive.

Thus, for (3.15) the conditions of Lemma 2.2 are satisfied. According to this lemma, (3.15) has at least one solution $(v_k)_{k=1}^n:[t_1,\omega[\to R^n$ $(t_1\in[t_0,\omega[)$, which tends to zero as $t\uparrow\omega$. Moreover, among the eigenvalues of the matrix C_{n-1} we have $n-2$ positive and one negative, and $\det C_n\det C_{n-1}<0$. By Lemma 2.2, if the inequality $h(t)>0$ (resp., $h(t)<0$) holds on the interval $[t_0,\omega[$, then (3.15) has $(n-2)$-parametric (resp., one-parametric) family of solutions vanishing at ω in case $H(t)<0$ on $[t_0,\omega[$, and $n-1$-parametric (resp., two-parametric) family of solutions in case $H(t)>0$ on $[t_0,\omega[$.

For the final conclusion on a number of vanishing solutions, as $t\uparrow\omega$, of the system (3.15) it is necessary to determine the signs of functions h and H on $[t_0,\omega[$.

Since $h(t)=\pi_\omega^{-1}(t)$, by the definition of π_ω we have

$$\operatorname{sign} h(t)=\begin{cases}1 & \text{if } \ \omega=+\infty,\\ -1 & \text{if } \ \omega<+\infty.\end{cases}$$

For the function H, according to the definition of I we have

$$H(t) = \frac{J_A(t)}{I(t)} = \frac{|J_A(t)|}{\int_a^t |J_A(\tau)|\, d\tau} > 0 \quad \text{if } \ t \in [t_0, \omega[\,.$$

Using the obtained sign conditions for the functions h and H, we arrive at the following final conclusions about a number of vanishing solutions as $t \uparrow \omega$ for the system of differential equations (3.15):

(1) if $\omega = +\infty$, then the system of differential equations (3.15) has $n-1$-parametric family of vanishing solutions as $t \to +\infty$;

(2) if $\omega < +\infty$, then the system of differential equations (3.15) has two-parametric family of vanishing solutions as $t \uparrow \omega$.

Using the substitution (3.14), every solution $(v_k)_{k=1}^n : [t_1, \omega[\to \mathbb{R}^n$ of (3.15) which tends to zero corresponds to a solution $y : [t_1, \omega[\to \mathbb{R}$ of the differential equation (1.1) which admits as $t \uparrow \omega$ the asymptotic representations (3.5)–(3.7). Using these representations and the condition (3.4), it is not difficult to see that each such solution is $P_\omega(\frac{n-i-1}{n-i})$-solution of (1.1). □

Remark 3.3. When checking the fulfillment of the conditions (3.4), we may consider that owing to the first of these conditions, the second and third conditions are equivalent, respectively, to

$$\lim_{t\uparrow\omega} p(t)\pi_\omega^n(t)|\ln|\pi_\omega(t)||^\sigma = 0 \quad \text{and} \quad \int\limits_a^\omega p(t)|\pi_\omega(t)|^{n-1}|\ln|\pi_\omega(t)||^\sigma\, dt = +\infty.$$

Finally, pay attention to the fact that Theorem 3.2 covers the case $\sigma = 0$, that is, when the equation (1.1) is a linear differential equation of the form (1.3).

For (1.3), by Theorem 3.2 and with regard for Remark 3.3, the following corollary holds.

Corollary 3.4. *Let $n \geq 3$ and suppose that the limit* (3.3) *exists* (*finite or equal to* $\pm\infty$). *Then the linear differential equation* (1.3) *has $P_\omega(0)$-solutions if and only if the following conditions hold:*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega^{n-1}(t)p(t)}{\int\limits_A^t \pi_\omega^{n-2}(\tau)p(\tau)\, d\tau} = -1, \quad \int\limits_a^\omega |\pi_\omega(\tau)|^{n-1}p(\tau)\, d\tau = +\infty, \quad \lim_{t\uparrow\omega} \pi_\omega^n(t)p(t) = 0, \tag{3.18}$$

and for each such solution the following asymptotic representations take place as $t \uparrow \omega$:

$$\frac{y^{(k-1)}(t)}{y^{(n-2)}(t)} = \frac{[\pi_\omega(t)]^{n-k-1}}{(n-k-1)!}\,[1+o(1)] \quad (k = \overline{1, n-2}), \tag{3.19}$$

$$\ln|y^{(n-2)}(t)| = -\frac{\alpha_0}{(n-2)!} \int\limits_a^t p(\tau)\pi_\omega^{n-1}(\tau)\, d\tau[1+o(1)], \tag{3.20}$$

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = -\frac{\alpha_0}{(n-2)!}\, p(t)\pi_\omega^{n-1}(t)[1+o(1)]. \tag{3.21}$$

Moreover, when the conditions (3.18) *are satisfied, the differential equation* (1.3) *has $n-1$-parametric family of $P_\omega(0)$-solutions with the representations* (3.19)–(3.21) *in case $\omega = +\infty$, and in case $\omega < \infty$* (1.3) *has two-parametric family.*

This corollary in case $\omega = +\infty$ complements the results for linear differential equations with asymptotically small coefficients given in [7, Ch. 1, Section 6, pp. 184–186].

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GRAND CONFLUENT HYPERGEOMETRIC FUNCTION APPLYING REVERSIBLE THREE-TERM RECURRENCE FORMULA

Abstract. In this paper, by applying a reversible three-term recurrence formula (R3TRF) (see [13, Chapter 1]), we construct:

(1) power series expansions in closed forms of the grand confluent hypergeometric (GCH) equation,

(2) its integral forms for an infinite series and a polynomial which makes the leading non-constant coefficient on the RHS of the recurrence relation terminated,

(3) generating functions for GCH polynomials which makes the leading coefficient on the RHS terminated.

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რეზიუმე. ნაშრომში შექცევადი სამწევრა რეკურენტული ფორმულის (R3TRF) (იხ. [13, Chapter 1]) გამოყენებით აგებულია:

(1) GCH განტოლების წარმოდგენა ხარისხოვანი მწკრივის სახით;

(2) ინტეგრალური წარმოდგენები;

(3) GCH პოლინომთა მაწარმოებელი ფუნქციები.

1 Introduction

The equation

$$x \frac{d^2y}{dx^2} + (\mu x^2 + \varepsilon x + \nu) \frac{dy}{dx} + (\Omega x + \varepsilon \omega) y = 0 \tag{1.1}$$

is the grand confluent hypergeometric (GCH) differential equation where μ, ε, ν, Ω and ω are real or complex parameters [9, 11]. The GCH ordinary differential equation is of Fuchsian types with two singular points: one regular singular point which is zero with exponents $\{0, 1-\nu\}$, and another irregular singular point which is infinity with an exponent Ω/μ. In contrast, the Heun equation of Fuchsian types has four regular singularities. The Heun equation has four kinds of confluent forms [20]: (1) confluent Heun (two regular and one irregular singularities), (2) doubly confluent Heun (two irregular singularities), (3) biconfluent Heun (one regular and one irregular singularities), (4) triconfluent Heun equations (one irregular singularity).

The BCH equation is derived from the GCH equation by changing all coefficients* [36]. The GCH (or BCH) equation is applicable in the modern physics [1,21,22,35,37]. The BCH equation appears in the radial Schrödinger equation with those potentials such as the rotating harmonic oscillator [30], the doubly anharmonic oscillator [6,7,23], a three-dimensional anharmonic oscillator [17,18,23], Coulomb potential with a linear confining potential [23,34] and other kinds of potentials [24,25].

The fundamental solutions of the BCH equation for an infinite series and the BCH spectral polynomials about $x = 0$ in the canonical form were obtained by applying the power series expansion [2,15,19,39]. For the case of the irregular singular point $x = \infty$, the three-term recurrence of the power series in the BCH equation was derived [26,31], and the analytic solution of the BCH equation was left as solutions of recurrences due to a 3-term recursive relation between successive coefficients in its power series expansion of the BCH equation.† In comparison with the two term recursion relation of the power series in a linear differential equation, analytic solutions in closed forms on the three-term recurrence relation of the power series are unknown currently because of their complex mathematical calculations.

As is known, there are no examples for analytic solutions of the BCH equation about $x = 0$ and $x = \infty$ in the form of definite or contour integrals containing the well-known special functions such as ${}_2F_1$ or ${}_1F_1$, consisting of two-term recursion relation in their power series of linear differential equations. In place of describing the integral representation of the BCH equation involving only simple functions, especially for confluent hypergeometric functions, the BCH equation is obtained by means of Fredholm-type integral equations; such integral relationships express one analytic solution in terms of another analytic solution [3–5, 8, 27–29].

2 The GCH equation about a regular singular point at zero

Assume that the solution of (1.1) is

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+\lambda}, \tag{2.1}$$

where λ is an indicial root. Substitute (2.1) into (1.1). We obtain a three-term recurrence relation for the coefficients c_n:

$$c_{n+1} = A_n c_n + B_n c_{n-1}, \quad n \geq 1, \tag{2.2}$$

*For the canonical form of the BCH equation [36], replace μ, ε, ν, Ω and ω by -2, $-\beta$, $1+\alpha$, $\gamma-\alpha-2$ and $1/2(\delta/\beta+1+\alpha)$ in (1.1). For DLFM version ([32] or [38]), replace μ and ω by 1 and $-q/\varepsilon$ in (1.1).

†For the special case, the explicit solutions of the BCH equation in the canonical form was constructed when one of the coefficients $\beta = 0$ [16].

where

$$A_n = \frac{-\varepsilon(n+\omega+\lambda)}{(n+1+\lambda)(n+\nu+\lambda)}\,, \tag{2.3a}$$

$$B_n = -\frac{\Omega+\mu(n-1+\lambda)}{(n+1+\lambda)(n+\nu+\lambda)}\,, \tag{2.3b}$$

$$c_1 = A_0 c_0. \tag{2.3c}$$

We have two indicial roots which are $\lambda = 0$ and $1-\nu$.

2.1 Power series

2.1.1 Polynomial of type 2

By putting a power series $y(x) = \sum_{n=0}^{\infty} c_n x^{n+\lambda}$ into a linear ordinary differential equation (ODE), the recurrence relation between successive coefficients starts to appear. In general, the recurrence relation for a 3-term is given by (2.2) where $c_1 = A_0 c_0$ and $c_0 \neq 0$. As is known, there are two types of power series expansions for the two-term recurrence relation in a linear ODE such as a polynomial and an infinite series. In contrast, there are an infinite series and three types of polynomials in the three term recurrence relation of a linear ODE:

(1) polynomial which makes B_n term terminated: A_n term is not terminated, designated as 'a polynomial of type 1',

(2) polynomial which makes A_n term terminated: B_n term is not terminated, denominated as 'a polynomial of type 2',

(3) polynomial which makes A_n and B_n terms terminated simultaneously.

For $n = 0, 1, 2, 3, \ldots$ in (2.2), the sequence c_n is expanded to combinations of A_n and B_n terms. It is suggested that a sub-power series $y_l(x)$, where $l \in \mathbb{N}_0$, is constructed by observing the term of sequence c_n which includes l terms of $A_n^{\prime} s$ [10]. The power series solution is described by sums of each $y_l(x)$ such as $y(x) = \sum_{n=0}^{\infty} y_n(x)$. By allowing for A_n in the sequence c_n to be the leading term of each sub-power series $y_l(x)$, the general summation formulas of the 3-term recurrence relation in a linear ODE are constructed for an infinite series and a polynomial of type 1, designated as 'three-term recurrence formula (3TRF)'.

Similarly, by allowing for B_n in the sequence c_n to be the leading term of each sub-power series in a function $y(x)$ [13, Chapter 1], we have obtained the general summation formulas of the 3-term recurrence relation in a linear ODE for an infinite series and a polynomial of type 2: the term of the sequence c_n which includes zero term of B_n's, one term of B_n's, two terms of B_n's, three terms of B_n's, etc. is observed. These general summation expressions are denominated as 'reversible three-term recurrence formula (R3TRF)'.

In general, the GCH polynomial is defined as type 3 polynomial where A_n and B_n terms terminated. For the type 3 GCH polynomial about $x = 0$, it has a fixed integer value of Ω, just as it has a fixed value of ω. In the three-term recurrence relation, a polynomial of type 3 is categorized as a complete polynomial. In Chapters 9 and 10 of [14], general solutions in series for the GCH polynomial of type 3 around $x = 0$ and $x = \infty$ are constructed.

For type 1, the GCH polynomial about $x = 0$, μ, ε, ν and ω are treated as free variables and Ω as a fixed value. In [11,12], the analytic solutions of the GCH equation about the regular singular point at $x = 0$ are constructed by applying the three-term recurrence formula (3TRF) [10]:

(1) power series expansions in closed forms for an infinite series and a polynomial of type 1,

(2) their integral forms,

(3) generating functions for GCH polynomials of type 1.

Four examples of the analytic wave functions and their eigenvalues in the radial Schrödinger equation with certain potentials are presented:

(1) Schrödinger equation with the rotating harmonic oscillator and a class of confinement potentials,

(2) the spin free Hamiltonian involving only scalar potential for the $q-\bar{q}$ system,

(3) the radial Schrödinger equation with confinement potentials,

(4) two interacting electrons in a uniform magnetic field and a parabolic potential.

The Frobenius solutions in closed forms and their combined definite and contour integrals of these four quantum mechanical wave functions are derived analytically.

For the GCH polynomial of type 2 about $x=0$, μ, ε, ν and Ω are treated as free variables and ω as a fixed value. In this paper, by applying R3TRF in Chapter 1 of [13], the power series expansions are constructed in closed forms of the GCH equation about the regular singular point at $x=0$ for an infinite series and a polynomial of type 2. The integral forms of the GCH equation and their generating functions for GCH polynomials of type 2 are derived analytically. Also, the Frobenius solutions of the GCH equation about the irregular singular point at $x=\infty$ by applying 3TRF [10] are obtained analytically including their integral representations and generating functions for the GCH polynomials of type 1.

In Chapter 1 of [13], the general expression of a power series of $y(x)$ for a polynomial of type 2 is defined by

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} y_n(x) = y_0(x)+y_1(x)+y_2(x)+y_3(x)+\cdots \\
&= c_0 x^{\lambda}\left\{ \sum_{i_0=0}^{\alpha_0}\left(\prod_{i_1=0}^{i_0-1} A_{i_1}\right) x^{i_0} + \sum_{i_0=0}^{\alpha_0}\left\{ B_{i_0+1}\prod_{i_1=0}^{i_0-1} A_{i_1}\sum_{i_2=i_0}^{\alpha_1}\left(\prod_{i_3=i_0}^{i_2-1} A_{i_3+2}\right)\right\} x^{i_2+2} \right.\\
&\quad + \sum_{N=2}^{\infty}\left\{ \sum_{i_0=0}^{\alpha_0}\left\{ B_{i_0+1}\prod_{i_1=0}^{i_0-1} A_{i_1}\prod_{k=1}^{N-1}\left(\sum_{i_{2k}=i_{2(k-1)}}^{\alpha_k} B_{i_{2k}+2k+1}\prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1} A_{i_{2k+1}+2k}\right)\right.\right.\\
&\quad \left.\left.\left.\times \sum_{i_{2N}=i_{2(N-1)}}^{\alpha_N}\left(\prod_{i_{2N+1}=i_{2(N-1)}}^{i_{2N}-1} A_{i_{2N+1}+2N}\right)\right\}\right\} x^{i_{2N}+2N}\right\}.
\end{aligned} \tag{2.4}$$

Here $\alpha_i \leq \alpha_j$ only if $i \leq j$, where $i, j, \alpha_i, \alpha_j \in \mathbb{N}_0$.

For a polynomial, we need the condition

$$A_{\alpha_i+2i} = 0 \quad \text{where} \quad i, \alpha_i = 0, 1, 2, \ldots . \tag{2.5}$$

In this paper, the Pochhammer symbol $(x)_n$ is used to represent the rising factorial: $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$. In the above, α_i is an eigenvalue that makes A_n term terminated at certain value of the index n. (2.5) makes each $y_i(x)$ where $i=0,1,2,\ldots$ as the polynomial in (2.4). Replace α_i by ω_i in (2.5) and put $n=\omega_i+2i$ in (2.3a) with the condition $A_{\omega_i+2i}=0$. Then we obtain eigenvalues ω such that

$$\omega = -(\omega_i + 2i + \lambda).$$

In (2.3a), we replace ω by $-(\omega_i+2i+\lambda)$ and insert it and (2.3b) in (2.4), where the index α_i is replaced by ω_i. After the replacement process, the general expression of a power series of the GCH equation for a polynomial of type 2 is given by

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} y_n(x) = y_0(x)+y_1(x)+y_2(x)+y_3(x)+\cdots \\
&= c_0 x^{\lambda}\left\{ \sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\eta^{i_0} + \left\{\sum_{i_0=0}^{\omega_0}\frac{(i_0+\Omega/\mu+\lambda)}{(i_0+2+\lambda)(i_0+1+\nu+\lambda)}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\right.\right.
\end{aligned}$$

$$\begin{aligned}
&\times \sum_{i_1=i_0}^{\omega_1} \frac{(-\omega_1)_{i_1}(3+\lambda)_{i_0}(2+\nu+\lambda)_{i_0}}{(-\omega_1)_{i_0}(3+\lambda)_{i_1}(2+\nu+\lambda)_{i_1}}\, \eta^{i_1}\Bigg\}\rho \\
&+ \sum_{n=2}^{\infty} \Bigg\{ \sum_{i_0=0}^{\omega_0} \frac{(i_0+\Omega/\mu+\lambda)}{(i_0+2+\lambda)(i_0+1+\nu+\lambda)} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \\
&\times \prod_{k=1}^{n-1} \Bigg\{ \sum_{i_k=i_{k-1}}^{\omega_k} \frac{(i_k+2k+\Omega/\mu+\lambda)}{(i_k+2k+2+\lambda)(i_k+2k+1+\nu+\lambda)} \\
&\times \frac{(-\omega_k)_{i_k}(2k+1+\lambda)_{i_{k-1}}(2k+\nu+\lambda)_{i_{k-1}}}{(-\omega_k)_{i_{k-1}}(2k+1+\lambda)_{i_k}(2k+\nu+\lambda)_{i_k}} \Bigg\} \\
&\times \sum_{i_n=i_{n-1}}^{\omega_n} \frac{(-\omega_n)_{i_n}(2n+1+\lambda)_{i_{n-1}}(2n+\nu+\lambda)_{i_{n-1}}}{(-\omega_n)_{i_{n-1}}(2n+1+\lambda)_{i_n}(2n+\nu+\lambda)_{i_n}}\, \eta^{i_n} \Bigg\}\rho^n \Bigg\}, \qquad (2.6)
\end{aligned}$$

where

$$\begin{cases} \eta = -\varepsilon x, \\ \rho = -\mu x^2, \\ \omega = -(\omega_j+2j+\lambda) \ \text{ as } j,\omega_j \in \mathbb{N}_0, \\ \omega_i \le \omega_j \ \text{ only if } \ i \le j \ \text{ where } \ i,j \in \mathbb{N}_0. \end{cases}$$

Put $c_0 = 1$ as $\lambda = 0$ for the first kind of independent solution of the GCH equation and as $\lambda = 1-\nu$ for the second one in (2.6).

Remark 2.1. The power series expansion of the first kind GCH equation for a polynomial of type 2 about $x = 0$ as $\omega = -(\omega_j + 2j)$, where $j, \omega_j \in \mathbb{N}_0$, is

$$\begin{aligned}
y(x) &= QW^R_{\omega_j}\Big(\mu,\varepsilon,\nu,\Omega,\omega=-(\omega_j+2j);\rho=-\mu x^2,\eta=-\varepsilon x\Big) \\
&= \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1)_{i_0}(\nu)_{i_0}}\eta^{i_0} + \Bigg\{ \sum_{i_0=0}^{\omega_0} \frac{(i_0+\Omega/\mu)}{(i_0+2)(i_0+1+\nu)} \frac{(-\omega_0)_{i_0}}{(1)_{i_0}(\nu)_{i_0}} \sum_{i_1=i_0}^{\omega_1} \frac{(-\omega_1)_{i_1}(3)_{i_0}(2+\nu)_{i_0}}{(-\omega_1)_{i_0}(3)_{i_1}(2+\nu)_{i_1}}\, \eta^{i_1}\Bigg\}\rho \\
&+ \sum_{n=2}^{\infty} \Bigg\{ \sum_{i_0=0}^{\omega_0} \frac{(i_0+\Omega/\mu)}{(i_0+2)(i_0+1+\nu)} \frac{(-\omega_0)_{i_0}}{(1)_{i_0}(\nu)_{i_0}} \\
&\times \prod_{k=1}^{n-1} \Bigg\{ \sum_{i_k=i_{k-1}}^{\omega_k} \frac{(i_k+2k+\Omega/\mu)}{(i_k+2k+2)(i_k+2k+1+\nu)} \frac{(-\omega_k)_{i_k}(2k+1)_{i_{k-1}}(2k+\nu)_{i_{k-1}}}{(-\omega_k)_{i_{k-1}}(2k+1)_{i_k}(2k+\nu)_{i_k}} \Bigg\} \\
&\times \sum_{i_n=i_{n-1}}^{\omega_n} \frac{(-\omega_n)_{i_n}(2n+1)_{i_{n-1}}(2n+\nu)_{i_{n-1}}}{(-\omega_n)_{i_{n-1}}(2n+1)_{i_n}(2n+\nu)_{i_n}}\, \eta^{i_n} \Bigg\}\rho^n. \qquad (2.7)
\end{aligned}$$

For the minimum value of the first kind GCH equation for a polynomial of type 2 around $x = 0$, we put $\omega_0 = \omega_1 = \omega_2 = \cdots = 0$ in (2.7).

$$\begin{aligned}
y(x) &= QW^R_0\Big(\mu,\varepsilon,\nu,\Omega,\omega=-2j;\rho=-\mu x^2,\eta=-\varepsilon x\Big) \\
&= {}_1F_1\Big(\frac{\Omega}{2\mu},\frac{\nu}{2}+\frac{1}{2},-\frac{1}{2}\mu x^2\Big), \ \text{ where } \ -\infty < x < \infty.
\end{aligned}$$

As in the above, ${}_1F_1(a,b,x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{x^n}{n!}$.

Remark 2.2. The power series expansion of the second kind GCH equation for a polynomial of type 2 about $x = 0$ as $\omega = -(\omega_j + 2j + 1 - \nu)$, where $j, \omega_j \in \mathbb{N}_0$, is

$$y(x) = RW^R_{\omega_j}\Big(\mu,\varepsilon,\nu,\Omega,\omega=-(\omega_j+2j+1-\nu);\rho=-\mu x^2,\eta=-\varepsilon x\Big)$$

$$
\begin{aligned}
&= x^{1-\nu}\Bigg\{ \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(2-\nu)_{i_0}(1)_{i_0}}\,\eta^{i_0} \\
&\quad + \Bigg\{ \sum_{i_0=0}^{\omega_0} \frac{(i_0+1+\Omega/\mu-\nu)}{(i_0+3-\nu)(i_0+2)}\,\frac{(-\omega_0)_{i_0}}{(2-\nu)_{i_0}(1)_{i_0}} \sum_{i_1=i_0}^{\omega_1} \frac{(-\omega_1)_{i_1}(4-\nu)_{i_0}(3)_{i_0}}{(-\omega_1)_{i_0}(4-\nu)_{i_1}(3)_{i_1}}\,\eta^{i_1}\Bigg\}\rho \\
&\quad + \sum_{n=2}^{\infty} \Bigg\{ \sum_{i_0=0}^{\omega_0} \frac{(i_0+1+\Omega/\mu-\nu)}{(i_0+3-\nu)(i_0+2)}\,\frac{(-\omega_0)_{i_0}}{(2-\nu)_{i_0}(1)_{i_0}} \\
&\quad \times \prod_{k=1}^{n-1} \Bigg\{ \sum_{i_k=i_{k-1}}^{\omega_k} \frac{(i_k+2k+1+\Omega/\mu-\nu)}{(i_k+2k+3-\nu)(i_k+2k+2)}\,\frac{(-\omega_k)_{i_k}(2k+2-\nu)_{i_{k-1}}(2k+1)_{i_{k-1}}}{(-\omega_k)_{i_{k-1}}(2k+2-\nu)_{i_k}(2k+1)_{i_k}}\Bigg\} \\
&\quad \times \sum_{i_n=i_{n-1}}^{\omega_n} \frac{(-\omega_n)_{i_n}(2n+2-\nu)_{i_{n-1}}(2n+1)_{i_{n-1}}}{(-\omega_n)_{i_{n-1}}(2n+2-\nu)_{i_n}(2n+1)_{i_n}}\,\eta^{i_n}\Bigg\}\rho^n\Bigg\}.
\end{aligned} \tag{2.8}
$$

For the minimum value of the second kind GCH equation, for a polynomial of type 2 about $x=0$, we put $\omega_0=\omega_1=\omega_2=\cdots=0$ in (2.8).

$$
\begin{aligned}
y(x) &= RW_0^R\Big(\mu,\varepsilon,\nu,\Omega,\omega=-(2j+1-\nu);\rho=-\mu x^2,\eta=-\varepsilon x\Big) \\
&= x^{1-\nu}\;{}_1F_1\Big(\frac{\Omega}{2\mu}-\frac{\nu}{2}+\frac{1}{2},-\frac{\nu}{2}+\frac{3}{2},-\frac{1}{2}\,\mu x^2\Big), \quad \text{where } -\infty<x<\infty.
\end{aligned}
$$

In [11,12], Ω is treated as a fixed value and μ, ε, ν, ω are treated as free variables to construct the GCH polynomials of type 1 around $x=0$: (1) if $\Omega=-\mu(2\beta_j+j)$, where $j,\beta_j\in\mathbb{N}_0$, an analytic solution of the GCH equation turns to be the first kind of independent solution of the GCH polynomial of type 1; (2) if $\Omega=-\mu(2\psi_j+j+1-\nu)$ where $j,\psi_j\in\mathbb{N}_0$, an analytic solution of the GCH equation turns to be the second kind of independent solution of the GCH polynomial of type 1.

In this paper, ω is treated as a fixed value and μ, ε, ν, Ω are treated as free variables to construct the GCH polynomials of type 2 around $x=0$: (1) if $\omega=-(\omega_j+2j)$, where $j,\omega_j\in\mathbb{N}_0$, an analytic solution of the GCH equation turns to be the first kind of independent solution of the GCH polynomial of type 2; (2) if $\omega=-(\omega_j+2j+1-\nu)$, the analytic solution of the GCH equation turns to be the second kind of independent solution of the GCH polynomial of type 2.

2.1.2 Infinite series

In Chapter 1 of [13], the general expression of a power series of $y(x)$ for an infinite series is defined by

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} y_n(x) = y_0(x)+y_1(x)+y_2(x)+y_3(x)+\cdots \\
&= c_0 x^{\lambda}\Bigg\{ \sum_{i_0=0}^{\infty}\Big(\prod_{i_1=0}^{i_0-1} A_{i_1}\Big)x^{i_0} + \sum_{i_0=0}^{\infty}\Big\{ B_{i_0+1}\prod_{i_1=0}^{i_0-1} A_{i_1} \sum_{i_2=i_0}^{\infty}\Big(\prod_{i_3=i_0}^{i_2-1} A_{i_3+2}\Big)\Big\}x^{i_2+2} \\
&\quad + \sum_{N=2}^{\infty}\Bigg\{ \sum_{i_0=0}^{\infty}\Big\{ B_{i_0+1}\prod_{i_1=0}^{i_0-1} A_{i_1} \prod_{k=1}^{N-1}\Big(\sum_{i_{2k}=i_{2(k-1)}}^{\infty} B_{i_{2k}+2k+1} \prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1} A_{i_{2k+1}+2k}\Big) \\
&\quad \times \sum_{i_{2N}=i_{2(N-1)}}^{\infty}\Big(\prod_{i_{2N+1}=i_{2(N-1)}}^{i_{2N}-1} A_{i_{2N+1}+2N}\Big)\Big\}\Bigg\} x^{i_{2N}+2N}\Bigg\}.
\end{aligned} \tag{2.9}
$$

Substitute (2.3a)–(2.3c) into (2.9). The general expression of a power series of the GCH equation for an infinite series about $x=0$ is given by

$$
y(x) = \sum_{n=0}^{\infty} y_n(x) = y_0(x)+y_1(x)+y_2(x)+y_3(x)+\cdots
$$

$$
\begin{aligned}
&= c_0 x^{\lambda}\Bigg\{ \sum_{i_0=0}^{\infty} \frac{(\omega+\lambda)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\,\eta^{i_0} + \Bigg\{ \sum_{i_0=0}^{\infty} \frac{\Xi^{(i_0)}(\omega+\lambda)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \\
&\quad \times \sum_{i_1=i_0}^{\infty} \frac{(\omega+2+\lambda)_{i_1}(3+\lambda)_{i_0}(2+\nu+\lambda)_{i_0}}{(\omega+2+\lambda)_{i_0}(3+\lambda)_{i_1}(2+\nu+\lambda)_{i_1}}\,\eta^{i_1}\Bigg\}\rho \\
&+ \sum_{n=2}^{\infty}\Bigg\{ \sum_{i_0=0}^{\infty} \frac{\Xi^{(i_0)}(\omega+\lambda)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \\
&\quad \times \prod_{k=1}^{n-1}\Bigg\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{\Xi^{(i_k)}(\omega+2k+\lambda)_{i_k}(2k+1+\lambda)_{i_{k-1}}(2k+\nu+\lambda)_{i_{k-1}}}{(\omega+2k+\lambda)_{i_{k-1}}(2k+1+\lambda)_{i_{k-1}}(2k+\nu+\lambda)_{i_k}}\Bigg\} \\
&\quad \times \sum_{i_n=i_{n-1}}^{\infty} \frac{(\omega+2n+\lambda)_{i_n}(2n+1+\lambda)_{i_{n-1}}(2n+\nu+\lambda)_{i_{n-1}}}{(\omega+2n+\lambda)_{i_{n-1}}(2n+1+\lambda)_{i_{n-1}}(2n+\nu+\lambda)_{i_n}}\,\eta^{i_n}\Bigg\}\rho^n\Bigg\}, \qquad (2.10)
\end{aligned}
$$

where

$$
\begin{cases}
\Xi^{(i_0)} = \dfrac{(i_0+\Omega/\mu+\lambda)}{(i_0+2+\lambda)(i_0+1+\nu+\lambda)}\,, \\
\Xi^{(i_k)} = \dfrac{(i_k+2k+\Omega/\mu+\lambda)}{(i_k+2k+2+\lambda)(i_k+2k+1+\nu+\lambda)}\,.
\end{cases}
$$

Put c_0= 1 as $\lambda = 0$ for the first kind of independent solution of the GCH equation and as $\lambda = 1-\nu$ for the second one in (2.10).

Remark 2.3. The power series expansion of the GCH equation of the first kind for an infinite series about $x = 0$ using R3TRF is

$$
\begin{aligned}
y(x) &= QW^R\big(\mu,\varepsilon,\nu,\Omega,\omega;\rho=-\mu x^2,\eta=-\varepsilon x\big) \\
&= \sum_{i_0=0}^{\infty} \frac{(\omega)_{i_0}}{(1)_{i_0}(\nu)_{i_0}}\eta^{i_0} + \Bigg\{ \sum_{i_0=0}^{\infty} \frac{(i_0+\Omega/\mu)}{(i_0+2)(i_0+1+\nu)}\,\frac{(\omega)_{i_0}}{(1)_{i_0}(\nu)_{i_0}} \sum_{i_1=i_0}^{\infty} \frac{(\omega+2)_{i_1}(3)_{i_0}(2+\nu)_{i_0}}{(\omega+2)_{i_0}(3)_{i_1}(2+\nu)_{i_1}}\,\eta^{i_1}\Bigg\}\rho \\
&\quad + \sum_{n=2}^{\infty}\Bigg\{ \sum_{i_0=0}^{\infty} \frac{(i_0+\Omega/\mu)}{(i_0+2)(i_0+1+\nu)}\frac{(\omega)_{i_0}}{(1)_{i_0}(\nu)_{i_0}} \\
&\quad \times \prod_{k=1}^{n-1}\Bigg\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(i_k+2k+\Omega/\mu)}{(i_k+2k+2)(i_k+2k+1+\nu)}\,\frac{(\omega+2k)_{i_k}(2k+1)_{i_{k-1}}(2k+\nu)_{i_{k-1}}}{(\omega+2k)_{i_{k-1}}(2k+1)_{i_{k-1}}(2k+\nu)_{i_k}}\Bigg\} \\
&\quad \times \sum_{i_n=i_{n-1}}^{\infty} \frac{(\omega+2n)_{i_n}(2n+1)_{i_{n-1}}(2n+\nu)_{i_{n-1}}}{(\omega+2n)_{i_{n-1}}(2n+1)_{i_{n-1}}(2n+\nu)_{i_n}}\,\eta^{i_n}\Bigg\}\rho^n. \qquad (2.11)
\end{aligned}
$$

Remark 2.4. The power series expansion of the GCH equation of the second kind for an infinite series about $x = 0$ using R3TRF is

$$
\begin{aligned}
y(x) &= RW^R\big(\mu,\varepsilon,\nu,\Omega,\omega;\rho=-\mu x^2,\eta=-\varepsilon x\big) \\
&= x^{1-\nu}\Bigg\{ \sum_{i_0=0}^{\infty} \frac{(\omega+1-\nu)_{i_0}}{(2-\nu)_{i_0}(1)_{i_0}}\,\eta^{i_0} \\
&\quad + \Bigg\{ \sum_{i_0=0}^{\infty} \frac{(i_0+1+\Omega/\mu-\nu)}{(i_0+3-\nu)(i_0+2)}\,\frac{(\omega+1-\nu)_{i_0}}{(2-\nu)_{i_0}(1)_{i_0}} \sum_{i_1=i_0}^{\infty} \frac{(\omega+3-\nu)_{i_1}(4-\nu)_{i_0}(3)_{i_0}}{(\omega+3-\nu)_{i_0}(4-\nu)_{i_1}(3)_{i_1}}\,\eta^{i_1}\Bigg\}\rho \\
&\quad + \sum_{n=2}^{\infty}\Bigg\{ \sum_{i_0=0}^{\infty} \frac{(i_0+1+\Omega/\mu-\nu)}{(i_0+3-\nu)(i_0+2)}\,\frac{(\omega+1-\nu)_{i_0}}{(2-\nu)_{i_0}(1)_{i_0}} \\
&\qquad \times \prod_{k=1}^{n-1}\Bigg\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(i_k+2k+1+\Omega/\mu-\nu)}{(i_k+2k+3-\nu)(i_k+2k+2)}
\end{aligned}
$$

$$\times \frac{(\omega+2k+1-\nu)_{i_k}(2k+2-\nu)_{i_{k-1}}(2k+1)_{i_{k-1}}}{(\omega+2k+1-\nu)_{i_{k-1}}(2k+2-\nu)_{i_{k-1}}(2k+1)_{i_k}}\Bigg\} \times \sum_{i_n=i_{n-1}}^{\infty} \frac{(\omega+2n+1-\nu)_{i_n}(2n+2-\nu)_{i_{n-1}}(2n+1)_{i_{n-1}}}{(\omega+2n+1-\nu)_{i_{n-1}}(2n+2-\nu)_{i_{n-1}}(2n+1)_{i_n}}\, \eta^{i_n}\Bigg\}\rho^n\Bigg\}. \tag{2.12}$$

It is required that $\nu \neq 0,-1,-2,\dots$ for the first kind of independent solutions of the GCH equation for an infinite series and a polynomial. But if it is not the case, its solutions will be divergent. And it is required that $\nu \neq 2,3,4,\dots$ for the second kind of independent solutions of the GCH equation for all cases.

Infinite series in this paper are equivalent to those in [11,12]. In this paper, B_n is the leading term in the sequence c_n of analytic function $y(x)$. In [11,12], A_n is the leading term in the sequence c_n of analytic function $y(x)$.*

2.2 Integral representation

2.2.1 Polynomial of type 2

Now I consider the combined definite and contour integral representation of the GCH equation by using R3TRF. There is a generalized hypergeometric function such as

$$\begin{aligned} I_l &= \sum_{i_l=i_{l-1}}^{\omega_l} \frac{(-\omega_l)_{i_l}(2l+1+\lambda)_{i_{l-1}}(2l+\nu+\lambda)_{i_{l-1}}}{(-\omega_l)_{i_{l-1}}(2l+1+\lambda)_{i_l}(2l+\nu+\lambda)_{i_l}}\, \eta^{i_l} \\ &= \sum_{j=0}^{\infty} \frac{B_{1,j}B_{2,j}(i_{l-1}-\omega_l)_j \eta^{i_{l-1}}}{(i_{l-1}+2l+\lambda)^{-1}(i_{l-1}+2l-1+\nu+\lambda)^{-1}(1)_j\, j!}\, \eta^j. \end{aligned} \tag{2.13}$$

By using integral form of the beta function,

$$B_{1,j} = B(i_{l-1}+2l+\lambda, j+1) = \int_0^1 dt_l\; t_l^{i_{l-1}+2l-1+\lambda}(1-t_l)^j, \tag{2.14a}$$

$$B_{2,j} = B(i_{l-1}+2l-1+\nu+\lambda, j+1) = \int_0^1 du_l\; u_l^{i_{l-1}+2l-2+\nu+\lambda}(1-u_l)^j. \tag{2.14b}$$

Substitute (2.14a) and (2.14b) into (2.13) and the result divide by $(i_{l-1}+2l+\lambda)(i_{l-1}+2l-1+\nu+\lambda)$. We get

$$\begin{aligned} &\frac{(i_{l-1}+2l+\lambda)^{-1}}{(i_{l-1}+2l-1+\nu+\lambda)} \sum_{i_l=i_{l-1}}^{\omega_l} \frac{(-\omega_l)_{i_l}(2l+1+\lambda)_{i_{l-1}}(2l+\nu+\lambda)_{i_{l-1}}}{(-\omega_l)_{i_{l-1}}(2l+1+\lambda)_{i_l}(2l+\nu+\lambda)_{i_l}}\, \eta^{i_l} \\ &= \int_0^1 dt_l\; t_l^{2l-1+\lambda} \int_0^1 du_l\; u_l^{2l-2+\nu+\lambda} (\eta t_l u_l)^{i_{l-1}} \sum_{j=0}^{\infty} \frac{(i_{l-1}-\omega_l)_j}{(1)_j\, j!} \big(\eta(1-t_l)(1-u_l)\big)^j. \end{aligned} \tag{2.15}$$

The integral form of the confluent hypergeometric function of the first kind is given by

$$\sum_{j=0}^{\infty} \frac{(-\alpha_0)_j}{(\gamma)_j j!} z^j = \frac{\Gamma(\alpha_0+1)\Gamma(\gamma)}{2\pi i\,\Gamma(\alpha_0+\gamma)} \oint dv_l\; \frac{\exp(-\frac{z v_l}{(1-v_l)})}{v_l^{\alpha_0+1}(1-v_l)^{\gamma}}. \tag{2.16}$$

*As $\Gamma(1/2+\nu/2-\Omega/(2\mu))/\Gamma(1/2+\nu/2)$ is multiplied by (2.11), the new (2.11) is equivalent to the first kind solution of the GCH equation for an infinite series using 3TRF [11]. Again, as $(-\mu/2)^{1/2(1-\nu)}\,\Gamma(1-\Omega/(2\mu))/\Gamma(3/2-\nu/2)$ is multiplied by (2.12), the new (2.12) corresponds to the second kind solution of the GCH equation for an infinite series using 3TRF [11].

Replacing α_0, γ and z in (2.16), respectively, by $\omega_l - i_{l-1}$, 1 and $\eta(1-t_l)(1-u_l)$, we obtain

$$\sum_{j=0}^{\infty} \frac{(i_{l-1}-\omega_l)_j}{(1)_j\, j!} \left(\eta(1-t_l)(1-u_l)\right)^j = \frac{1}{2\pi i} \oint dv_l\, \frac{\exp\left(-\frac{v_l}{(1-v_l)}\,\eta(1-t_l)(1-u_l)\right)}{v_l^{\omega_l+1-i_{l-1}}(1-v_l)}\,. \tag{2.17}$$

Substitute (2.17) into (2.15):

$$\begin{aligned} K_l &= \frac{(i_{l-1}+2l+\lambda)^{-1}}{(i_{l-1}+2l-1+\nu+\lambda)} \sum_{i_l=i_{l-1}}^{\omega_l} \frac{(-\omega_l)_{i_l}(2l+1+\lambda)_{i_{l-1}}(2l+\nu+\lambda)_{i_{l-1}}}{(-\omega_l)_{i_{l-1}}(2l+1+\lambda)_{i_l}(2l+\nu+\lambda)_{i_l}}\,\eta^{i_l} \\ &= \int_0^1 dt_l\, t_l^{2l-1+\lambda} \int_0^1 du_l\, u_l^{2l-2+\nu+\lambda}\, \frac{1}{2\pi i}\oint dv_l\, \frac{\exp\left(-\frac{v_l}{(1-v_l)}\,\eta(1-t_l)(1-u_l)\right)}{v_l^{\omega_l+1}(1-v_l)}\,(\eta t_l u_l v_l)^{i_{l-1}}. \end{aligned} \tag{2.18}$$

Substitute (2.18) into (2.6), where $l = 1, 2, 3, \ldots$: apply K_1 into the second summation of the sub-power series $y_1(x)$; apply K_2 into the third summation and K_1 into the second summation of the sub-power series $y_2(x)$; apply K_3 into the forth summation, K_2 into the third summation and K_1 into the second summation of the sub-power series $y_3(x)$, etc.*

Theorem 2.5. *The general representation in the form of an integral of the GCH polynomial of type* 2 *is given by*

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} y_n(x) = y_0(x)+y_1(x)+y_2(x)+y_3(x)+\cdots \\ &= c_0 x^{\lambda}\Bigg\{ \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\,\eta^{i_0} + \sum_{n=1}^{\infty}\Bigg\{ \prod_{k=0}^{n-1}\Bigg\{ \int_0^1 dt_{n-k}\, t_{n-k}^{2(n-k)-1+\lambda} \int_0^1 du_{n-k}\, u_{n-k}^{2(n-k-1)+\nu+\lambda} \\ &\qquad \times \frac{1}{2\pi i}\oint dv_{n-k}\, \frac{\exp\left(-\frac{v_{n-k}}{(1-v_{n-k})}\, w_{n-k+1,n}(1-t_{n-k})(1-u_{n-k})\right)}{v_{n-k}^{\omega_{n-k}+1}(1-v_{n-k})} \\ &\qquad \times w_{n-k,n}^{-(\Omega/\mu+2(n-k-1)+\lambda)}\left(w_{n-k,n}\partial_{w_{n-k,n}}\right) w_{n-k,n}^{\Omega/\mu+2(n-k-1)+\lambda}\Bigg\} \\ &\qquad \times \sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\, w_{1,n}^{i_0}\Bigg\}\rho^n\Bigg\}, \end{aligned} \tag{2.19}$$

where

$$w_{a,b} = \begin{cases} \eta \prod_{l=a}^{b} t_l u_l v_l, & \text{where } a \le b, \\ \eta & \text{only if } a > b. \end{cases}$$

Here the first sub-integral form contains one term of B_n's, the second one contains two terms of B_n's, the third one contains three terms of B_n's, etc.

Proof. In (2.6), the power series expansions of sub-summation terms $y_0(x)$, $y_1(x)$, $y_2(x)$ and $y_3(x)$ of the GCH polynomial of type 2 are

$$y(x) = \sum_{n=0}^{\infty} y_n(x) = y_0(x)+y_1(x)+y_2(x)+y_3(x)+\cdots\,, \tag{2.20}$$

where

$$y_0(x) = c_0 x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\,\eta^{i_0}, \tag{2.21a}$$

*$y_1(x)$ means the sub-power series in (2.6), contains one term of B_n's; $y_2(x)$ means the sub-power series in (2.6), contains two terms of B_n's; $y_3(x)$ means the sub-power series in (2.6), contains three terms of B_n's, etc.

$$y_1(x) = c_0 x^{\lambda}\left\{ \sum_{i_0=0}^{\omega_0} \frac{(i_0+\Omega/\mu+\lambda)}{(i_0+2+\lambda)(i_0+1+\nu+\lambda)} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \right.$$
$$\left. \times \sum_{i_1=i_0}^{\omega_1} \frac{(-\omega_1)_{i_1}(3+\lambda)_{i_0}(2+\nu+\lambda)_{i_0}}{(-\omega_1)_{i_0}(3+\lambda)_{i_1}(2+\nu+\lambda)_{i_1}} \eta^{i_1} \right\}\rho, \tag{2.21b}$$

$$y_2(x) = c_0 x^{\lambda}\left\{ \sum_{i_0=0}^{\omega_0} \frac{(i_0+\Omega/\mu+\lambda)}{(i_0+2+\lambda)(i_0+1+\nu+\lambda)} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \right.$$
$$\times \sum_{i_1=i_0}^{\omega_1} \frac{(i_1+2+\Omega/\mu+\lambda)}{(i_1+4+\lambda)(i_1+3+\nu+\lambda)} \frac{(-\omega_1)_{i_1}(3+\lambda)_{i_0}(2+\nu+\lambda)_{i_0}}{(-\omega_1)_{i_0}(3+\lambda)_{i_1}(2+\nu+\lambda)_{i_1}}$$
$$\left. \times \sum_{i_2=i_1}^{\omega_2} \frac{(-\omega_2)_{i_2}(5+\lambda)_{i_1}(4+\nu+\lambda)_{i_1}}{(-\omega_2)_{i_1}(5+\lambda)_{i_2}(4+\nu+\lambda)_{i_2}} \eta^{i_2} \right\}\rho^2, \tag{2.21c}$$

$$y_3(x) = c_0 x^{\lambda}\left\{ \sum_{i_0=0}^{\omega_0} \frac{(i_0+\Omega/\mu+\lambda)}{(i_0+2+\lambda)(i_0+1+\nu+\lambda)} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \right.$$
$$\times \sum_{i_1=i_0}^{\omega_1} \frac{(i_1+2+\Omega/\mu+\lambda)}{(i_1+4+\lambda)(i_1+3+\nu+\lambda)} \frac{(-\omega_1)_{i_1}(3+\lambda)_{i_0}(2+\nu+\lambda)_{i_0}}{(-\omega_1)_{i_0}(3+\lambda)_{i_1}(2+\nu+\lambda)_{i_1}}$$
$$\times \sum_{i_2=i_1}^{\omega_2} \frac{(i_2+4+\Omega/\mu+\lambda)}{(i_2+6+\lambda)(i_2+5+\nu+\lambda)} \frac{(-\omega_2)_{i_2}(5+\lambda)_{i_1}(4+\nu+\lambda)_{i_1}}{(-\omega_2)_{i_1}(5+\lambda)_{i_2}(4+\nu+\lambda)_{i_2}}$$
$$\left. \times \sum_{i_3=i_2}^{\omega_3} \frac{(-\omega_3)_{i_3}(7+\lambda)_{i_2}(6+\nu+\lambda)_{i_2}}{(-\omega_3)_{i_2}(7+\lambda)_{i_3}(6+\nu+\lambda)_{i_3}} \eta^{i_3} \right\}\rho^3. \tag{2.21d}$$

Put $l=1$ in (2.18) and insert it into (2.21b):

$$y_1(x) = c_0 x^{\lambda}\rho \int_0^1 dt_1\, t_1^{1+\lambda} \int_0^1 du_1\, \frac{u_1^{\nu+\lambda}}{2\pi i} \oint dv_1\, \frac{\exp\left(-\frac{v_1}{(1-v_1)}\eta(1-t_1)(1-u_1)\right)}{v_1^{\omega_1+1}(1-v_1)}$$
$$\times \left\{ \sum_{i_0=0}^{\omega_0} (i_0+\Omega/\mu+\lambda) \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} (\eta t_1 u_1 v_1)^{i_0} \right\}\rho$$
$$= c_0 x^{\lambda} \int_0^1 dt_1\, t_1^{1+\lambda} \int_0^1 du_1\, \frac{u_1^{\nu+\lambda}}{2\pi i} \oint dv_1 \frac{\exp\left(-\frac{v_1}{(1-v_1)}\eta(1-t_1)(1-u_1)\right)}{v_1^{\omega_1+1}(1-v_1)}$$
$$\times w_{1,1}^{-(\Omega/\mu+\lambda)}(w_{1,1}\partial_{w_{1,1}}) w_{1,1}^{\Omega/\mu+\lambda} \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} w_{1,1}^{i_0}, \tag{2.22}$$

where

$$w_{1,1} = \eta \prod_{l=1}^{1} t_l u_l v_l.$$

Put $l=2$ in (2.18) and insert it into (2.21c):

$$y_2(x) = c_0 x^{\lambda}\rho^2 \int_0^1 dt_2\, t_2^{3+\lambda} \int_0^1 du_2\, \frac{u_2^{2+\nu+\lambda}}{2\pi i} \oint dv_2\, \frac{\exp\left(-\frac{v_2}{(1-v_2)}\eta(1-t_2)(1-u_2)\right)}{v_2^{\omega_2+1}(1-v_2)}$$
$$\times w_{2,2}^{-(\Omega/\mu+2+\lambda)}(w_{2,2}\partial_{w_{2,2}}) w_{2,2}^{\Omega/\mu+2+\lambda} \sum_{i_0=0}^{\omega_0} \frac{(i_0+\Omega/\mu+\lambda)}{(i_0+2+\lambda)(i_0+1+\nu+\lambda)} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}$$
$$\times \sum_{i_1=i_0}^{\omega_1} \frac{(-\omega_1)_{i_1}(3+\lambda)_{i_0}(2+\nu+\lambda)_{i_0}}{(-\omega_1)_{i_0}(3+\lambda)_{i_1}(2+\nu+\lambda)_{i_1}} w_{2,2}^{i_1}, \tag{2.23}$$

where

$$w_{2,2} = \eta \prod_{l=2}^{2} t_l u_l v_l.$$

Put $l=1$ and $\eta = w_{2,2}$ in (2.18) and insert it into (2.23). We get

$$\begin{aligned}
y_2(x) &= c_0 x^{\lambda} \rho^2 \int_0^1 dt_2\ t_2^{3+\lambda} \int_0^1 du_2\ \frac{u_2^{2+\nu+\lambda}}{2\pi i} \\
&\times \oint dv_2\ \frac{\exp\left(-\frac{v_2}{(1-v_2)}\,\eta(1-t_2)(1-u_2)\right)}{v_2^{\omega_2+1}(1-v_2)}\, w_{2,2}^{-(\Omega/\mu+2+\lambda)}(w_{2,2}\partial_{w_{2,2}}) w_{2,2}^{\Omega/\mu+2+\lambda} \\
&\times \int_0^1 dt_1\ t_1^{1+\lambda} \int_0^1 du_1\ \frac{u_1^{\nu+\lambda}}{2\pi i} \oint dv_1\ \frac{\exp\left(-\frac{v_1}{(1-v_1)}\, w_{2,2}(1-t_1)(1-u_1)\right)}{v_1^{\omega_1+1}(1-v_1)} \\
&\times w_{1,2}^{-(\Omega/\mu+\lambda)}(w_{1,2}\partial_{w_{1,2}}) w_{2,2}^{\Omega/\mu+\lambda} \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\, w_{1,2}^{i_0},
\end{aligned} \tag{2.24}$$

where

$$w_{1,2} = \eta \prod_{l=1}^{2} t_l u_l v_l.$$

By using similar process as in the previous cases for integral forms of $y_1(x)$ and $y_2(x)$, we obtain the following integral form of the sub-power series expansion $y_3(x)$:

$$\begin{aligned}
y_3(x) &= c_0 x^{\lambda} \rho^3 \int_0^1 dt_3\ t_3^{5+\lambda} \int_0^1 du_3\ \frac{u_3^{4+\nu+\lambda}}{2\pi i} \\
&\times \oint dv_3\ \frac{\exp\left(-\frac{v_3}{(1-v_3)}\,\eta(1-t_3)(1-u_3)\right)}{v_3^{\omega_3+1}(1-v_3)}\, w_{3,3}^{-(\Omega/\mu+4+\lambda)}(w_{3,3}\partial_{w_{3,3}}) w_{3,3}^{\Omega/\mu+4+\lambda} \\
&\times \int_0^1 dt_2\ t_2^{3+\lambda} \int_0^1 du_2\ \frac{u_2^{2+\nu+\lambda}}{2\pi i} \\
&\times \oint dv_2\ \frac{\exp\left(-\frac{v_2}{(1-v_2)}\, w_{3,3}(1-t_2)(1-u_2)\right)}{v_2^{\omega_2+1}(1-v_2)}\, w_{2,3}^{-(\Omega/\mu+2+\lambda)}(w_{2,3}\partial_{w_{2,3}}) w_{2,3}^{\Omega/\mu+2+\lambda} \\
&\times \int_0^1 dt_1\ t_1^{1+\lambda} \int_0^1 du_1\ \frac{u_1^{\nu+\lambda}}{2\pi i} \\
&\times \oint dv_1\ \frac{\exp\left(-\frac{v_1}{(1-v_1)}\, w_{2,3}(1-t_1)(1-u_1)\right)}{v_1^{\omega_1+1}(1-v_1)}\, w_{1,3}^{-(\Omega/\mu+\lambda)}(w_{1,3}\partial_{w_{1,3}})\, w_{1,3}^{\Omega/\mu+\lambda} \\
&\times \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\, w_{1,3}^{i_0},
\end{aligned} \tag{2.25}$$

where

$$\begin{cases}
w_{3,3} = \eta \prod_{l=3}^{3} t_l u_l v_l, \\
w_{2,3} = \eta \prod_{l=2}^{3} t_l u_l v_l, \\
w_{1,3} = \eta \prod_{l=1}^{3} t_l u_l v_l\,.
\end{cases}$$

By repeating the above process, we obtain integral forms of all higher sub-summation terms $y_m(x)$, where $m \geq 4$. Substituting (2.21a), (2.22), (2.24), (2.25) and including integral forms of $y_m(x)$, $m \geq 4$, into (2.20), we obtain (2.19). □

Put $c_0 = 1$ as $\lambda = 0$ for the first kind of independent solution of the GCH equation and as $\lambda = 1-\nu$ for the second kind one in (2.19).

Remark 2.6. The integral representation of the first kind GCH equation for a polynomial of type 2 about $x = 0$ as $\omega = -(\omega_j + 2j)$, where $j, \omega_j = 0, 1, 2, \dots$, is

$$\begin{aligned} y(x) &= QW^R_{\omega_j}\Big(\mu,\varepsilon,\nu,\Omega,\omega=-(\omega_j+2j);\rho=-\mu x^2,\eta=-\varepsilon x\Big) \\ &= {}_1F_1(-\omega_0;\nu;\eta) + \sum_{n=1}^{\infty}\left\{\prod_{k=0}^{n-1}\left\{\int_0^1 dt_{n-k}\; t_{n-k}^{2(n-k)-1}\int_0^1 du_{n-k}\; u_{n-k}^{2(n-k-1)+\nu}\right.\right. \\ &\qquad \times \frac{1}{2\pi i}\oint dv_{n-k}\;\frac{\exp\left(-\frac{v_{n-k}}{(1-v_{n-k})}\,w_{n-k+1,n}(1-t_{n-k})(1-u_{n-k})\right)}{v_{n-k}^{\omega_{n-k}+1}(1-v_{n-k})} \\ &\qquad \left.\left.\times\, w_{n-k,n}^{-(\Omega/\mu+2(n-k-1))}(w_{n-k,n}\partial_{w_{n-k,n}})w_{n-k,n}^{\Omega/\mu+2(n-k-1)}\right\}{}_1F_1(-\omega_0;\nu;w_{1,n})\right\}\rho^n. \end{aligned} \tag{2.26}$$

Remark 2.7. The integral representation of the second kind GCH equation for a polynomial of type 2 about $x = 0$ as $\omega = -(\omega_j + 2j + 1 - \nu)$, where $j, \omega_j = 0, 1, 2, \dots$, is

$$\begin{aligned} y(x) &= RW^R_{\omega_j}\Big(\mu,\varepsilon,\nu,\Omega,\omega=-(\omega_j+2j+1-\nu);\rho=-\mu x^2,\eta=-\varepsilon x\Big) \\ &= x^{1-\nu}\left\{{}_1F_1(-\omega_0;2-\nu;\eta) + \sum_{n=1}^{\infty}\left\{\prod_{k=0}^{n-1}\left\{\int_0^1 dt_{n-k}\; t_{n-k}^{2(n-k)-\nu}\int_0^1 du_{n-k}\; u_{n-k}^{2(n-k)-1}\right.\right.\right. \\ &\qquad \times \frac{1}{2\pi i}\oint dv_{n-k}\;\frac{\exp\left(-\frac{v_{n-k}}{(1-v_{n-k})}\,w_{n-k+1,n}(1-t_{n-k})(1-u_{n-k})\right)}{v_{n-k}^{\omega_{n-k}+1}(1-v_{n-k})} \\ &\left.\left.\left.\times w_{n-k,n}^{-(\Omega/\mu+2(n-k)-1-\nu)}(w_{n-k,n}\partial_{w_{n-k,n}})w_{n-k,n}^{\Omega/\mu+2(n-k)-1-\nu}\right\}{}_1F_1(-\omega_0;2-\nu;w_{1,n})\right\}\rho^n\right\}. \end{aligned} \tag{2.27}$$

In the above equalities, ${}_1F_1(a;b;z)$ is a Kummer function of the first kind defined as

$$\begin{aligned} {}_1F_1(a;b;z) = M(a,b,z) &= \sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n n!}\;, z^n = e^z M(b-a,b,-z) \\ &= -\frac{1}{2\pi i}\;,\frac{\Gamma(1-a)\Gamma(b)}{\Gamma(b-a)}\oint dv_j\; e^{zv_j}(-v_j)^{a-1}(1-v_j)^{b-a-1} \\ &= \frac{\Gamma(a)}{2\pi i}\oint dv_j\; e^{v_j}v_j^{-b}(1-\frac{z}{v_j})^{-a} \\ &= \frac{1}{2\pi i}\;,\frac{\Gamma(1-a)\Gamma(b)}{\Gamma(b-a)}\oint dv_j\; e^{-\frac{z\,v_j}{1-v_j}}\;v_j^{a-1}(1-v_j)^{-b}. \end{aligned} \tag{2.28}$$

2.2.2 Infinite Series

Let us consider the integral representation of the GCH equation about $x = 0$ for an infinite series by applying R3TRF. There is a generalized hypergeometric function which is given by

$$\begin{aligned} M_l &= \sum_{i_l=i_{l-1}}^{\infty}\frac{(\omega+2l+\lambda)_{i_l}(2l+1+\lambda)_{i_{l-1}}(2l+\nu+\lambda)_{i_{l-1}}}{(\omega+2l+\lambda)_{i_{l-1}}(2l+1+\lambda)_{i_l}(2l+\nu+\lambda)_{i_l}}\,\eta^{i_l} \\ &= \sum_{j=0}^{\infty}\frac{B_{i_{l-1},j}(\omega+2l+\lambda+i_{l-1})_j\eta^{i_{l-1}}}{(i_{l-1}+2l+\lambda)^{-1}(i_{l-1}+2l-1+\nu+\lambda)^{-1}(1)_j\, j!}\,\eta^j, \end{aligned} \tag{2.29}$$

where

$$B_{i_{l-1},j} = B(i_{l-1}+2l+\lambda, j+1)B(i_{l-1}+2l-1+\nu+\lambda, j+1).$$

Substituting (2.14a) and (2.14b) into (2.29) and dividing the obtained equality by $(i_{l-1}+2l+\lambda)(i_{l-1}+2l-1+\nu+\lambda)$, we get

$$\begin{aligned}&\sum_{i_l=i_{l-1}}^{\infty} \frac{A_{i_{l-1}}(\omega+2l+\lambda)_{i_l}(2l+1+\lambda)_{i_{l-1}}(2l+\nu+\lambda)_{i_{l-1}}}{(\omega+2l+\lambda)_{i_{l-1}}(2l+1+\lambda)_{i_l}(2l+\nu+\lambda)_{i_l}}\,\eta^{i_l}\\ &= \int_0^1 dt_l\; t_l^{2l-1+\lambda}\int_0^1 du_l\; u_l^{2l-2+\nu+\lambda}(\eta t_l u_l)^{i_{l-1}}\sum_{j=0}^{\infty}\frac{(\omega+2l+\lambda+i_{l-1})_j}{(1)_j\, j!}\,\big(\eta(1-t_l)(1-u_l)\big)^j, \end{aligned} \tag{2.30}$$

where

$$A_{i_{l-1}} = \frac{1}{(i_{l-1}+2l+\lambda)(i_{l-1}+2l-1+\nu+\lambda)}\,.$$

In (2.28), replacing a, b and z, respectively, by $\omega+2l+\lambda+i_{l-1}$, 1 and $\eta(1-t_j)(1-u_j)$, and inserting the resulting equality into (2.30), we obtain

$$\begin{aligned}V_l &= \sum_{i_l=i_{l-1}}^{\infty} \frac{A_{i_{l-1}}(\omega+2l+\lambda)_{i_l}(2l+1+\lambda)_{i_{l-1}}(2l+\nu+\lambda)_{i_{l-1}}}{(\omega+2l+\lambda)_{i_{l-1}}(2l+1+\lambda)_{i_l}(2l+\nu+\lambda)_{i_l}}\,\eta^{i_l}\\ &= \int_0^1 dt_l\; t_l^{2l-1+\lambda}\int_0^1 du_l\; u_l^{2l-2+\nu+\lambda}\,\frac{1}{2\pi i}\oint dv_l\;\frac{\exp\left(-\frac{v_l}{(1-v_l)}\,\eta(1-t_l)(1-u_l)\right)}{v_l^{-(\omega+2l-1+\lambda)}(1-v_l)}(\eta t_l u_l v_l)^{i_{l-1}}\,. \end{aligned} \tag{2.31}$$

We substitute (2.31) into (2.10), where $l=1,2,3,\ldots$: apply V_1 into the second summation of the sub-power series $y_1(x)$; apply V_2 into the third summation and V_1 into the second summation of the sub-power series $y_2(x)$; apply V_3 into the forth summation, V_2 into the third summation and V_1 into the second summation of the sub-power series $y_3(x)$, etc.*

Theorem 2.8. *The general representation in the form of an integral of the GCH equation for an infinite series about $x=0$ using R3TRF is given by*

$$\begin{aligned}y(x) &= \sum_{n=0}^{\infty} y_n(x) = y_0(x)+y_1(x)+y_2(x)+y_3(x)+\cdots\\ &= c_0 x^{\lambda}\left\{\sum_{i_0=0}^{\infty}\frac{(\omega+\lambda)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\,\eta^{i_0}\right.\\ &\quad+\sum_{n=1}^{\infty}\left\{\prod_{k=0}^{n-1}\left\{\int_0^1 dt_{n-k}\; t_{n-k}^{2(n-k)-1+\lambda}\int_0^1 du_{n-k}\; u_{n-k}^{2(n-k-1)+\nu+\lambda}\right.\right.\\ &\qquad\times\frac{1}{2\pi i}\oint dv_{n-k}\;\frac{\exp\left(-\frac{v_{n-k}}{(1-v_{n-k})}\,w_{n-k+1,n}(1-t_{n-k})(1-u_{n-k})\right)}{v_{n-k}^{-(\omega+2(n-k)-1+\lambda)}(1-v_{n-k})}\\ &\qquad\quad\left.\times\, w_{n-k,n}^{-(\Omega/\mu+2(n-k-1)+\lambda)}(w_{n-k,n}\partial_{w_{n-k,n}})w_{n-k,n}^{\Omega/\mu+2(n-k-1)+\lambda}\right\}\\ &\qquad\qquad\left.\left.\times\sum_{i_0=0}^{\infty}\frac{(\omega+\lambda)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\,w_{1,n}^{i_0}\right\}\rho^n\right\}. \end{aligned} \tag{2.32}$$

*$y_1(x)$ means the sub-power series in (2.10), contains one term of B_n's; $y_2(x)$ means the sub-power series in (2.10), contains two terms of B_n's; $y_3(x)$ means the sub-power series in (2.10), contains three terms of B_n's, etc. Or we replace the finite summation with an interval $[0,\omega_0]$ by an infinite summation with an interval $[0,\infty]$ in (2.19). We also replace ω_0 and ω_{n-j} by $-(\omega+\lambda)$ and substitute $-(\omega+2(n-k)+\lambda)$ into the new (2.19). Its solution is likewise equivalent to (2.32).

Here the first sub-integral form contains one term of B_n's, the second one contains two terms of B_n's, the third one contains three terms of B_n's, etc. [*]

Put $c_0 = 1$ as $\lambda = 0$ for the first kind of independent solution of the GCH equation and as $\lambda = 1-\nu$, for the second kind one in (2.32).

Remark 2.9. The integral representation of the first kind GCH equation for an infinite series about $x = 0$ applying R3TRF is

$$\begin{aligned} y(x) &= QW^R\big(\mu,\varepsilon,\nu,\Omega,\omega;\rho=-\mu x^2,\eta=-\varepsilon x\big) \\ &= {}_1F_1(\omega;\nu;\eta) + \sum_{n=1}^{\infty}\left\{\prod_{k=0}^{n-1}\left\{\int_0^1 dt_{n-k}\; t_{n-k}^{2(n-k)-1}\int_0^1 du_{n-k}\; u_{n-k}^{2(n-k-1)+\nu}\right.\right. \\ &\quad\times\frac{1}{2\pi i}\oint dv_{n-k}\,\frac{\exp\left(-\frac{v_{n-k}}{(1-v_{n-k})}\,w_{n-k+1,n}(1-t_{n-k})(1-u_{n-k})\right)}{v_{n-k}^{-(\omega+2(n-k)-1)}(1-v_{n-k})} \\ &\quad\left.\left.\times\, w_{n-k,n}^{-(\Omega/\mu+2(n-k-1))}(w_{n-k,n}\partial_{w_{n-k,n}})w_{n-k,n}^{\Omega/\mu+2(n-k-1)}\right\}{}_1F_1(\omega;\nu;w_{1,n})\right\}\rho^n. \end{aligned} \tag{2.33}$$

Remark 2.10. The integral representation of the second kind GCH equation for an infinite series about $x = 0$ applying R3TRF is

$$\begin{aligned} y(x) &= RW^R\big(\mu,\varepsilon,\nu,\Omega,\omega;\rho=-\mu x^2,\eta=-\varepsilon x\big) \\ &= x^{1-\nu}\left\{{}_1F_1(\omega+1-\nu;2-\nu;\eta) + \sum_{n=1}^{\infty}\left\{\prod_{k=0}^{n-1}\left\{\int_0^1 dt_{n-k}\; t_{n-k}^{2(n-k)-\nu}\int_0^1 du_{n-k}\; u_{n-k}^{2(n-k)-1}\right.\right.\right. \\ &\quad\times\frac{1}{2\pi i}\oint dv_{n-k}\,\frac{\exp\left(-\frac{v_{n-k}}{(1-v_{n-k})}\,w_{n-k+1,n}(1-t_{n-k})(1-u_{n-k})\right)}{v_{n-k}^{-(\omega+2(n-k)-\nu)}(1-v_{n-k})} \\ &\left.\left.\left.\times w_{n-k,n}^{-(\Omega/\mu+2(n-k)-1-\nu)}(w_{n-k,n}\partial_{w_{n-k,n}})w_{n-k,n}^{\Omega/\mu+2(n-k)-1-\nu}\right\}{}_1F_1(\omega+1-\nu;2-\nu;w_{1,n})\right\}\rho^n\right\}. \end{aligned} \tag{2.34}$$

(2.33) multiplied by $\frac{\Gamma(1/2+\nu/2-\Omega/(2\mu))}{\Gamma(1/2+\nu/2)}$ is equivalent to the integral form of the first kind solution of the GCH equation for an infinite series applying 3TRF [11]. Also, (2.34) multiplied by $(-\mu/2)^{1/2(1-\nu)}\frac{\Gamma(1-\Omega/(2\mu))}{\Gamma(3/2-\nu/2)}$ corresponds to the integral representation of the second kind solution of the GCH equation for an infinite series applying 3TRF [11].

2.3 Generating function for the GCH polynomial of type 2

Now let us investigate generating functions for the type 2 GCH polynomials of the first and second kind around $x = 0$.

Definition 2.11. Define

$$\begin{cases} s_{a,b} = \begin{cases} s_a\cdot s_{a+1}\cdot s_{a+2}\cdots s_{b-2}\cdot s_{b-1}\cdot s_b, & \text{where } a<b, \\ s_a \text{ only if } a=b, \end{cases} \\ \widetilde{w}_{i,j} = \eta s_{i,\infty}\prod_{l=i}^{j} t_l u_l, \end{cases} \tag{2.35}$$

where $a, b, i, j \in \mathbb{N}_0$, $0 \le a \le b \le \infty$ and $1 \le i \le j \le \infty$.

[*] The method how to prove an integral for an infinite series is similar as an integral for a fixed value of ω at Subsection 2.2.1. Explicit proof for this integral is available on pages 250–253 in Chapter 6 [13].

We have

$$\sum_{\omega_i=\omega_j}^{\infty} s_i^{\omega_i} = \frac{s_i^{\omega_j}}{(1-s_i)} \quad \text{at } \ |s_i|<1. \tag{2.36}$$

Theorem 2.12. *The general expression of the generating function for the GCH polynomial of type 2 about $x=0$ is given by*

$$\begin{aligned}
&\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}y(x)=\prod_{k=1}^{\infty}\frac{1}{(1-s_{k,\infty})}\,\mathbf{\Upsilon}(\lambda;s_{0,\infty};\eta)\\
&\quad+\Bigg\{\prod_{k=1}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_1\;t_1^{1+\lambda}\int_0^1 du_1\;u_1^{\nu+\lambda}\exp\Big(-\frac{s_{1,\infty}}{(1-s_{1,\infty})}\,\eta(1-t_1)(1-u_1)\Big)\\
&\qquad\times\widetilde{w}_{1,1}^{-(\Omega/\mu+\lambda)}(\widetilde{w}_{1,1}\partial_{\widetilde{w}_{1,1}})\widetilde{w}_{1,1}^{\Omega/\mu+\lambda}\mathbf{\Upsilon}(\lambda;s_0;\widetilde{w}_{1,1})\Bigg\}\rho\\
&+\sum_{n=2}^{\infty}\Bigg\{\prod_{k=n}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_n\;t_n^{2n-1+\lambda}\int_0^1 du_n\;u_n^{2(n-1)+\nu+\lambda}\exp\Big(-\frac{s_{n,\infty}}{(1-s_{n,\infty})}\,\eta(1-t_n)(1-u_n)\Big)\\
&\qquad\times\widetilde{w}_{n,n}^{-(\Omega/\mu+2(n-1)+\lambda)}(\widetilde{w}_{n,n}\partial_{\widetilde{w}_{n,n}})\widetilde{w}_{n,n}^{\Omega/\mu+2(n-1)+\lambda}\\
&\times\prod_{j=1}^{n-1}\Bigg\{\int_0^1 dt_{n-j}\;t_{n-j}^{2(n-j)-1+\lambda}\int_0^1 du_{n-j}\;u_{n-j}^{2(n-j-1)+\nu+\lambda}\;\frac{\exp\big(-\frac{s_{n-j}}{(1-s_{n-j})}\,\widetilde{w}_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})\big)}{(1-s_{n-j})}\\
&\qquad\times\widetilde{w}_{n-j,n}^{-(\Omega/\mu+2(n-j-1)+\lambda)}(\widetilde{w}_{n-j,n}\partial_{\widetilde{w}_{n-j,n}})\widetilde{w}_{n-j,n}^{\Omega/\mu+2(n-j-1)+\lambda}\Bigg\}\mathbf{\Upsilon}(\lambda;s_0;\widetilde{w}_{1,n})\Bigg\}\rho^n,
\end{aligned}\tag{2.37}$$

where

$$\begin{cases}
\mathbf{\Upsilon}(\lambda;s_{0,\infty};\eta)=\displaystyle\sum_{\omega_0=0}^{\infty}\frac{s_{0,\infty}^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\Big\{c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\,\eta^{i_0}\Big\},\\
\mathbf{\Upsilon}(\lambda;s_0;\widetilde{w}_{1,1})=\displaystyle\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\Big\{c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\,\widetilde{w}_{1,1}^{i_0}\Big\},\\
\mathbf{\Upsilon}(\lambda;s_0;\widetilde{w}_{1,n})=\displaystyle\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\Big\{c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\,\widetilde{w}_{1,n}^{i_0}\Big\}.
\end{cases}$$

Proof. Applying the summation operator $\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}$ to the form of a general integral of type 2 GCH polynomial $y(x)$, we get

$$\begin{aligned}
&\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}y(x)\\
&\qquad=\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}(y_0(x)+y_1(x)+y_2(x)+\cdots).
\end{aligned}\tag{2.38}$$

Applying the summation operator $\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}$ to (2.21a) and using (2.35) and (2.36), we obtain

$$\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\} y_0(x) = \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \sum_{\omega_0=0}^{\infty} \frac{s_{0,\infty}^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} c_0 x^{\lambda} \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \eta^{i_0}. \quad (2.39)$$

Applying the summation operator $\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\}$ to (2.22) and using (2.35) and (2.36), we get

$$\begin{aligned} \sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\} y_1(x) &= \prod_{k=2}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1\; t_1^{1+\lambda} \int_0^1 du_1\; u_1^{\nu+\lambda} \\ &\times \frac{1}{2\pi i} \oint dv_1\; \frac{\exp\left(-\frac{v_1}{(1-v_1)}\,\eta(1-t_1)(1-u_1)\right)}{v_1(1-v_1)} \sum_{\omega_1=\omega_0}^{\infty} \left(\frac{s_{1,\infty}}{v_1}\right)^{\omega_1} w_{1,1}^{-(\Omega/\mu+\lambda)} (w_{1,1}\partial_{w_{1,1}}) w_{1,1}^{\Omega/\mu+\lambda} \\ &\times \sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \left\{ c_0 x^{\lambda} \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} w_{1,1}^{i_0} \right\} \rho. \end{aligned} \quad (2.40)$$

Replacing ω_i, ω_j and s_i, respectively, by ω_1, ω_0 and $\frac{s_{1,\infty}}{v_1}$ in (2.36) and inserting it into (2.40), we have

$$\begin{aligned} \sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\} y_1(x) &= \prod_{k=2}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1\; t_1^{1+\lambda} \int_0^1 du_1\; u_1^{\nu+\lambda} \\ &\times \frac{1}{2\pi i} \oint dv_1\; \frac{\exp\left(-\frac{v_1}{(1-v_1)}\,\eta(1-t_1)(1-u_1)\right)}{(1-v_1)(v_1-s_{1,\infty})} w_{1,1}^{-(\Omega/\mu+\lambda)} (w_{1,1}\partial_{w_{1,1}}) w_{1,1}^{\Omega/\mu+\lambda} \\ &\times \sum_{\omega_0=0}^{\infty} \frac{1}{\omega_0!} \left(\frac{s_{0,\infty}}{v_1}\right)^{\omega_0} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \left\{ c_0 x^{\lambda} \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} w_{1,1}^{i_0} \right\} \rho. \end{aligned} \quad (2.41)$$

By using Cauchy's integral formula, the contour integrand has poles at $v_1 = 1$ or $s_{1,\infty}$, where $s_{1,\infty}$ is only inside the unit circle. As we compute the residue in (2.41), we obtain

$$\begin{aligned} \sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\} y_1(x) &= \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1\; t_1^{1+\lambda} \int_0^1 du_1\; u_1^{\nu+\lambda} \\ &\times \exp\left(-\frac{s_{1,\infty}}{(1-s_{1,\infty})}\,\eta(1-t_1)(1-u_1)\right) \widetilde{w}_{1,1}^{-(\Omega/\mu+\lambda)} (\widetilde{w}_{1,1}\partial_{\widetilde{w}_{1,1}}) \widetilde{w}_{1,1}^{\Omega/\mu+\lambda} \\ &\times \sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \left\{ c_0 x^{\lambda} \sum_{i_0=0}^{\omega_0} \frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}} \widetilde{w}_{1,1}^{i_0} \right\} \rho, \end{aligned} \quad (2.42)$$

where

$$\widetilde{w}_{1,1} = \eta s_{1,\infty} \prod_{l=1}^{1} t_l u_l.$$

Applying the summation operator $\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\}$ to (2.24) and using (2.35) and (2.36), we obtain

$$\begin{aligned}
&\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}y_2(x)\\
&=\prod_{k=3}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_2\;t_2^{3+\lambda}\int_0^1 du_2\;u_2^{2+\nu+\lambda}\,\frac{1}{2\pi i}\oint dv_2\;\frac{\exp\left(-\frac{v_2}{(1-v_2)}\,\eta(1-t_2)(1-u_2)\right)}{v_2(1-v_2)}\\
&\times\sum_{\omega_2=\omega_1}^{\infty}\left(\frac{s_{2,\infty}}{v_2}\right)^{\omega_2}w_{2,2}^{-(\Omega/\mu+2+\lambda)}(w_{2,2}\partial_{w_{2,2}})w_{2,2}^{\Omega/\mu+2+\lambda}\int_0^1 dt_1\;t_1^{1+\lambda}\int_0^1 du_1\;u_1^{\nu+\lambda}\\
&\times\frac{1}{2\pi i}\oint dv_1\;\frac{\exp\left(-\frac{v_1}{(1-v_1)}\,w_{2,2}(1-t_1)(1-u_1)\right)}{v_1(1-v_1)}\sum_{\omega_1=\omega_0}^{\infty}\left(\frac{s_1}{v_1}\right)^{\omega_1}w_{1,2}^{-(\Omega/\mu+\lambda)}(w_{1,2}\partial_{w_{1,2}})w_{1,2}^{\Omega/\mu+\lambda}\\
&\times\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\left\{c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\,w_{1,2}^{i_0}\right\}\rho^2. \qquad (2.43)
\end{aligned}$$

Replacing in (2.36) ω_i, ω_j and s_i, respectively, by ω_2, ω_1 and $\frac{s_{2,\infty}}{v_2}$ and inserting the obtained formula into (2.43), we get

$$\begin{aligned}
&\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}y_2(x)=\prod_{k=3}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_2\;t_2^{3+\lambda}\int_0^1 du_2\;u_2^{2+\nu+\lambda}\\
&\times\frac{1}{2\pi i}\oint dv_2\;\frac{\exp\left(-\frac{v_2}{(1-v_2)}\,\eta(1-t_2)(1-u_2)\right)}{(1-v_2)(v_2-s_{2,\infty})}\,w_{2,2}^{-(\Omega/\mu+2+\lambda)}(w_{2,2}\partial_{w_{2,2}})w_{2,2}^{\Omega/\mu+2+\lambda}\\
&\times\int_0^1 dt_1\;t_1^{1+\lambda}\int_0^1 du_1\;u_1^{\nu+\lambda}\,\frac{1}{2\pi i}\oint dv_1\;\frac{\exp\left(-\frac{v_1}{(1-v_1)}\,w_{2,2}(1-t_1)(1-u_1)\right)}{v_1(1-v_1)}\\
&\times\sum_{\omega_1=\omega_0}^{\infty}\left(\frac{s_{1,\infty}}{v_1v_2}\right)^{\omega_1}w_{1,2}^{-(\Omega/\mu+\lambda)}(w_{1,2}\partial_{w_{1,2}})w_{1,2}^{\Omega/\mu+\lambda}\\
&\times\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\left\{c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}w_{1,2}^{i_0}\right\}\rho^2. \qquad (2.44)
\end{aligned}$$

By using Cauchy's integral formula, the contour integrand has poles at $v_2=1$ or $s_{2,\infty}$, where $s_{2,\infty}$ is only inside the unit circle. As we compute the residue in (2.44), we obtain

$$\begin{aligned}
&\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}y_2(x)=\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_2\;t_2^{3+\lambda}\int_0^1 du_2\;u_2^{2+\nu+\lambda}\\
&\times\exp\left(-\frac{s_{2,\infty}}{(1-s_{2,\infty})}\,\eta(1-t_2)(1-u_2)\right)\widetilde{w}_{2,2}^{-(\Omega/\mu+2+\lambda)}(\widetilde{w}_{2,2}\partial_{\widetilde{w}_{2,2}})\widetilde{w}_{2,2}^{\Omega/\mu+2+\lambda}\\
&\times\int_0^1 dt_1\;t_1^{1+\lambda}\int_0^1 du_1\;u_1^{\nu+\lambda}\,\frac{1}{2\pi i}\oint dv_1\;\frac{\exp\left(-\frac{v_1}{(1-v_1)}\,\widetilde{w}_{2,2}(1-t_1)(1-u_1)\right)}{v_1(1-v_1)}\\
&\times\sum_{\omega_1=\omega_0}^{\infty}\left(\frac{s_1}{v_1}\right)^{\omega_1}\ddot{w}_{1,2}^{-(\Omega/\mu+\lambda)}(\ddot{w}_{1,2}\partial_{\ddot{w}_{1,2}})\ddot{w}_{1,2}^{\Omega/\mu+\lambda}\\
&\times\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\left\{c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\,\ddot{w}_{1,2}^{i_0}\right\}\rho^2, \qquad (2.45)
\end{aligned}$$

where

$$\widetilde{w}_{2,2}=\eta s_{2,\infty}\prod_{l=2}^{2}t_lu_l,\quad \ddot{w}_{1,2}=\eta s_{2,\infty}v_1\prod_{l=1}^{2}t_lu_l.$$

Replace in (2.36) ω_i, ω_j and s_i, respectively, by ω_1, ω_0 and $\frac{s_1}{v_1}$ and insert the result into (2.45). We have

$$\begin{aligned}\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}y_2(x)&=\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_2\;t_2^{3+\lambda}\int_0^1 du_2\;u_2^{2+\nu+\lambda}\\
\times\exp\Big(-\frac{s_{2,\infty}}{(1-s_{2,\infty})}\eta(1-t_2)(1-u_2)\Big)\widetilde{w}_{2,2}^{-(\Omega/\mu+2+\lambda)}(\widetilde{w}_{2,2}\partial_{\widetilde{w}_{2,2}})\widetilde{w}_{2,2}^{\Omega/\mu+2+\lambda}&\int_0^1 dt_1\;t_1^{1+\lambda}\int_0^1 du_1\;u_1^{\nu+\lambda}\\
\times\frac{1}{2\pi i}\oint dv_1\;\frac{\exp\big(-\frac{v_1}{(1-v_1)}\widetilde{w}_{2,2}(1-t_1)(1-u_1)\big)}{(1-v_1)(v_1-s_1)}&\ddot{w}_{1,2}^{-(\Omega/\mu+\lambda)}(\ddot{w}_{1,2}\partial_{\ddot{w}_{1,2}})\ddot{w}_{1,2}^{\Omega/\mu+\lambda}\\
\times\sum_{\omega_0=0}^{\infty}\frac{1}{\omega_0!}\Big(\frac{s_{0,1}}{v_1}\Big)^{\omega_0}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}&\Big\{c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\ddot{w}_{1,2}^{i_0}\Big\}\rho^2. \qquad (2.46)\end{aligned}$$

By using Cauchy's integral formula, the contour integrand has poles at $v_1 = 1$ or s_1, where s_1 is only inside the unit circle. As we compute the residue in (2.46), we obtain

$$\begin{aligned}\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}y_2(x)&=\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_2\;t_2^{3+\lambda}\int_0^1 du_2\;u_2^{2+\nu+\lambda}\\
\times\exp\Big(-\frac{s_{2,\infty}}{(1-s_{2,\infty})}\eta(1-t_2)(1-u_2)\Big)&\widetilde{w}_{2,2}^{-(\Omega/\mu+2+\lambda)}(\widetilde{w}_{2,2}\partial_{\widetilde{w}_{2,2}})\widetilde{w}_{2,2}^{\Omega/\mu+2+\lambda}\\
\times\int_0^1 dt_1\;t_1^{1+\lambda}\int_0^1 du_1\;u_1^{\nu+\lambda}\frac{\exp\big(-\frac{s_1}{(1-s_1)}\widetilde{w}_{2,2}(1-t_1)(1-u_1)\big)}{(1-s_1)}&\widetilde{w}_{1,2}^{-(\Omega/\mu+\lambda)}(\widetilde{w}_{1,2}\partial_{\widetilde{w}_{1,2}})\widetilde{w}_{1,2}^{\Omega/\mu+\lambda}\\
\times\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}&\Big\{c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\widetilde{w}_{1,2}^{i_0}\Big\}\rho^2, \qquad (2.47)\end{aligned}$$

where

$$\widetilde{w}_{1,2}=\eta s_{1,\infty}\prod_{l=1}^{2}t_l u_l.$$

Applying the summation operator $\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}$ to (2.47) and using (2.35) and (2.36), we get

$$\begin{aligned}\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}y_3(x)&=\prod_{k=3}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_3\;t_3^{5+\lambda}\int_0^1 du_3\;u_3^{4+\nu+\lambda}\\
\times\exp\Big(-\frac{s_{3,\infty}}{(1-s_{3,\infty})}\eta(1-t_3)(1-u_3)\Big)&\widetilde{w}_{3,3}^{-(\Omega/\mu+4+\lambda)}(\widetilde{w}_{3,3}\partial_{\widetilde{w}_{3,3}})\widetilde{w}_{3,3}^{\Omega/\mu+4+\lambda}\\
\times\int_0^1 dt_2\;t_2^{3+\lambda}\int_0^1 du_2\;u_2^{2+\nu+\lambda}\frac{\exp\big(-\frac{s_2}{(1-s_2)}\widetilde{w}_{3,3}(1-t_2)(1-u_2)\big)}{(1-s_2)}&\widetilde{w}_{2,3}^{-(\Omega/\mu+2+\lambda)}(\widetilde{w}_{2,3}\partial_{\widetilde{w}_{2,3}})\widetilde{w}_{2,3}^{\Omega/\mu+2+\lambda}\\
\times\int_0^1 dt_1\;t_1^{1+\lambda}\int_0^1 du_1\;u_1^{\nu+\lambda}\frac{\exp\big(-\frac{s_1}{(1-s_1)}\widetilde{w}_{2,3}(1-t_1)(1-u_1)\big)}{(1-s_1)}&\widetilde{w}_{1,3}^{-(\Omega/\mu+\lambda)}(\widetilde{w}_{1,3}\partial_{\widetilde{w}_{1,3}})\widetilde{w}_{1,3}^{\Omega/\mu+\lambda}\\
\times\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')}&\Big\{c_0x^{\lambda}\sum_{i_0=0}^{\omega_0}\frac{(-\omega_0)_{i_0}}{(1+\lambda)_{i_0}(\nu+\lambda)_{i_0}}\widetilde{w}_{1,3}^{i_0}\Big\}\rho^3, \qquad (2.48)\end{aligned}$$

where

$$\widetilde{w}_{3,3} = \eta s_{3,\infty} \prod_{l=3}^{3} t_l u_l, \quad \widetilde{w}_{2,3} = \eta s_{2,\infty} \prod_{l=2}^{3} t_l u_l, \quad \widetilde{w}_{1,3} = \eta s_{1,\infty} \prod_{l=1}^{3} t_l u_l.$$

By repeating the above process for all integral forms of higher sub-summation terms $y_m(x)$, $m > 3$, we obtain every term $\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma')}{\Gamma(\gamma')} \prod_{n=1}^{\infty} \{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \} y_m(x)$. If into (2.38) along with (2.39), (2.42), (2.47), (2.48) we substitute all such terms, we obtain (2.37). □

Remark 2.13. The generating function for the first kind GCH polynomial of type 2 about $x = 0$ as $\omega = -(\omega_j + 2j)$, where $j, \omega_j = 0, 1, 2, \ldots$, is

$$\begin{aligned}
&\sum_{\omega_0=0}^{\infty} \frac{s_0^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\nu)}{\Gamma(\nu)} \prod_{n=1}^{\infty} \Big\{ \sum_{\omega_n=\omega_{n-1}}^{\infty} s_n^{\omega_n} \Big\} QW_{\omega_j}^R(\mu,\varepsilon,\nu,\Omega,\omega;\rho,\eta) = \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \mathbf{A}(s_{0,\infty};\eta) \\
&+ \Big\{ \prod_{k=1}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_1\, t_1 \int_0^1 du_1\, u_1^{\nu} \overleftrightarrow{\mathbf{\Gamma}}_1(s_{1,\infty};t_1,u_1,\eta) \widetilde{w}_{1,1}^{-\Omega/\mu} (\widetilde{w}_{1,1}\partial_{\widetilde{w}_{1,1}}) \widetilde{w}_{1,1}^{\Omega/\mu} \mathbf{A}(s_0;\widetilde{w}_{1,1}) \Big\} \rho \\
&+ \sum_{n=2}^{\infty} \Big\{ \prod_{k=n}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_n\, t_n^{2n-1} \int_0^1 du_n\, u_n^{2(n-1)+\nu} \overleftrightarrow{\mathbf{\Gamma}}_n(s_{n,\infty};t_n,u_n,\eta) \\
&\qquad \times \widetilde{w}_{n,n}^{-(\Omega/\mu+2(n-1))} (\widetilde{w}_{n,n}\partial_{\widetilde{w}_{n,n}}) \widetilde{w}_{n,n}^{\Omega/\mu+2(n-1)} \\
&\times \prod_{j=1}^{n-1} \Big\{ \int_0^1 dt_{n-j}\, t_{n-j}^{2(n-j)-1} \int_0^1 du_{n-j}\, u_{n-j}^{2(n-j-1)+\nu} \overleftrightarrow{\mathbf{\Gamma}}_{n-j}(s_{n-j};t_{n-j},u_{n-j},\widetilde{w}_{n-j+1,n}) \\
&\qquad \times \widetilde{w}_{n-j,n}^{-(\Omega/\mu+2(n-j-1))} (\widetilde{w}_{n-j,n}\partial_{\widetilde{w}_{n-j,n}}) \widetilde{w}_{n-j,n}^{\Omega/\mu+2(n-j-1)} \Big\} \mathbf{A}(s_0;\widetilde{w}_{1,n}) \Big\} \rho^n, \qquad (2.49)
\end{aligned}$$

where

$$\begin{cases}
\omega = -(\omega_j + 2j), \quad \rho = -\mu x^2, \quad \eta = -\varepsilon x; \\
\overleftrightarrow{\mathbf{\Gamma}}_1(s_{1,\infty};t_1,u_1,\eta) = \exp\Big(-\frac{s_{1,\infty}}{(1-s_{1,\infty})} \eta (1-t_1)(1-u_1) \Big); \\
\overleftrightarrow{\mathbf{\Gamma}}_n(s_{n,\infty};t_n,u_n,\eta) = \exp\Big(-\frac{s_{n,\infty}}{(1-s_{n,\infty})} \eta (1-t_n)(1-u_n) \Big); \\
\overleftrightarrow{\mathbf{\Gamma}}_{n-j}(s_{n-j};t_{n-j},u_{n-j},\widetilde{w}_{n-j+1,n}) = \dfrac{\exp\Big(-\frac{s_{n-j}}{(1-s_{n-j})} \widetilde{w}_{n-j+1,n}(1-t_{n-j})(1-u_{n-j}) \Big)}{(1-s_{n-j})}
\end{cases}$$

and

$$\begin{cases}
\mathbf{A}(s_{0,\infty};\eta) = (1-s_{0,\infty})^{-\nu} \exp\Big(-\frac{\eta s_{0,\infty}}{(1-s_{0,\infty})} \Big), \\
\mathbf{A}(s_0;\widetilde{w}_{1,1}) = (1-s_0)^{-\nu} \exp\Big(-\frac{\widetilde{w}_{1,1} s_0}{(1-s_0)} \Big), \\
\mathbf{A}(s_0;\widetilde{w}_{1,n}) = (1-s_0)^{-\nu} \exp\Big(-\frac{\widetilde{w}_{1,n} s_0}{(1-s_0)} \Big).
\end{cases}$$

Proof. The generating function for a confluent first kind Hypergeometric polynomial is given by

$$\sum_{\omega_0=0}^{\infty} \frac{t^{\omega_0}}{\omega_0!} \frac{\Gamma(\omega_0+\gamma)}{\Gamma(\gamma)} {}_1F_1(-\omega_0;\gamma;z) = (1-t)^{-\gamma} \exp\Big(-\frac{zt}{(1-t)} \Big). \qquad (2.50)$$

Replacing t, γ and z, respectively, by $s_{0,\infty}$, ν and η in (2.50), we get

$$\sum_{\omega_0=0}^{\infty}\frac{s_{0,\infty}^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\nu)}{\Gamma(\nu)}\;{}_1F_1(-\omega_0;\nu;\eta)=(1-s_{0,\infty})^{-\nu}\exp\Big(-\frac{\eta s_{0,\infty}}{(1-s_{0,\infty})}\Big). \tag{2.51}$$

Replacing t, γ and z, respectively, by s_0, ν and $\widetilde{w}_{1,1}$ in (2.50), we get

$$\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\nu)}{\Gamma(\nu)}\;{}_1F_1(-\omega_0;\nu;\widetilde{w}_{1,1})=(1-s_0)^{-\nu}\exp\Big(-\frac{\widetilde{w}_{1,1}s_0}{(1-s_0)}\Big). \tag{2.52}$$

Replacing t, γ and z, respectively, by s_0, ν and $\widetilde{w}_{1,n}$ in (2.50), we get

$$\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+\nu)}{\Gamma(\nu)}\;{}_1F_1(-\omega_0;\nu;\widetilde{w}_{1,n})=(1-s_0)^{-\nu}\exp\Big(-\frac{\widetilde{w}_{1,n}s_0}{(1-s_0)}\Big). \tag{2.53}$$

Taking $c_0=1$, λ=0 and $\gamma'=\nu$ in (2.37) and substituting (2.51), (2.52) and (2.53) into the obtained equality, we get the desired result. □

Remark 2.14. The generating function for the second kind GCH polynomial of type 2 about $x=0$ as $\omega=-(\omega_j+2j+1-\nu)$, where $j,\omega_j=0,1,2,\dots$, is

$$\begin{aligned}
&\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+2-\nu)}{\Gamma(2-\nu)}\prod_{n=1}^{\infty}\Big\{\sum_{\omega_n=\omega_{n-1}}^{\infty}s_n^{\omega_n}\Big\}RW^R_{\omega_j}(\mu,\varepsilon,\nu,\Omega,\omega;\rho,\eta)\\
&=x^{1-\nu}\Bigg\{\prod_{k=1}^{\infty}\frac{1}{(1-s_{k,\infty})}\mathbf{B}(s_{0,\infty};\eta)+\Bigg\{\prod_{k=1}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_1\;t_1^{2-\nu}\int_0^1 du_1\;u_1\overleftrightarrow{\mathbf{\Gamma}}_1(s_{1,\infty};t_1,u_1,\eta)\\
&\qquad\times\widetilde{w}_{1,1}^{-(\Omega/\mu+1-\nu)}(\widetilde{w}_{1,1}\partial_{\widetilde{w}_{1,1}})\widetilde{w}_{1,1}^{\Omega/\mu+1-\nu}\mathbf{B}(s_0;\widetilde{w}_{1,1})\Bigg\}\rho\\
&+\sum_{n=2}^{\infty}\Bigg\{\prod_{k=n}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_n\;t_n^{2n-\nu}\int_0^1 du_n\;u_n^{2n-1}\overleftrightarrow{\mathbf{\Gamma}}_n(s_{n,\infty};t_n,u_n,\eta)\\
&\qquad\times\widetilde{w}_{n,n}^{-(\Omega/\mu+2n-1-\nu)}(\widetilde{w}_{n,n}\partial_{\widetilde{w}_{n,n}})\widetilde{w}_{n,n}^{\Omega/\mu+2n-1-\nu}\\
&\times\prod_{j=1}^{n-1}\Bigg\{\int_0^1 dt_{n-j}\;t_{n-j}^{2(n-j)-\nu}\int_0^1 du_{n-j}\;u_{n-j}^{2(n-j)-1}\overleftrightarrow{\mathbf{\Gamma}}_{n-j}(s_{n-j};t_{n-j},u_{n-j},\widetilde{w}_{n-j+1,n})\\
&\qquad\times\widetilde{w}_{n-j,n}^{-(\Omega/\mu+2(n-j)-1-\nu)}(\widetilde{w}_{n-j,n}\partial_{\widetilde{w}_{n-j,n}})\widetilde{w}_{n-j,n}^{\Omega/\mu+2(n-j)-1-\nu}\Bigg\}\mathbf{B}(s_0;\widetilde{w}_{1,n})\Bigg\}\rho^n\Bigg\},
\end{aligned} \tag{2.54}$$

where

$$\begin{cases}
\omega=-(\omega_j+2j+1-\nu);\quad \rho=-\mu x^2,\quad \eta=-\varepsilon x;\\
\overleftrightarrow{\mathbf{\Gamma}}_1(s_{1,\infty};t_1,u_1,\eta)=\exp\Big(-\frac{s_{1,\infty}}{(1-s_{1,\infty})}\eta(1-t_1)(1-u_1)\Big);\\
\overleftrightarrow{\mathbf{\Gamma}}_n(s_{n,\infty};t_n,u_n,\eta)=\exp\Big(-\frac{s_{n,\infty}}{(1-s_{n,\infty})}\eta(1-t_n)(1-u_n)\Big);\\
\overleftrightarrow{\mathbf{\Gamma}}_{n-j}(s_{n-j};t_{n-j},u_{n-j},\widetilde{w}_{n-j+1,n})=\dfrac{\exp\big(-\frac{s_{n-j}}{(1-s_{n-j})}\widetilde{w}_{n-j+1,n}(1-t_{n-j})(1-u_{n-j})\big)}{(1-s_{n-j})}
\end{cases}$$

and

$$\begin{cases} \mathbf{B}(s_{0,\infty};\eta) = (1-s_{0,\infty})^{\nu-2}\exp\Big(-\dfrac{\eta s_{0,\infty}}{(1-s_{0,\infty})}\Big), \\ \mathbf{B}(s_0;\widetilde{w}_{1,1}) = (1-s_0)^{\nu-2}\exp\Big(-\dfrac{\widetilde{w}_{1,1}s_0}{(1-s_0)}\Big), \\ \mathbf{B}(s_0;\widetilde{w}_{1,n}) = (1-s_0)^{\nu-2}\exp\Big(-\dfrac{\widetilde{w}_{1,n}s_0}{(1-s_0)}\Big). \end{cases}$$

Proof. Replacing t, γ and z, respectively, by $s_{0,\infty}$, $2-\nu$ and η in (2.50), we get

$$\sum_{\omega_0=0}^{\infty}\frac{s_{0,\infty}^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+2-\nu)}{\Gamma(2-\nu)}\,{}_1F_1(-\omega_0;2-\nu;\eta) = (1-s_{0,\infty})^{\nu-2}\exp\Big(-\frac{\eta s_{0,\infty}}{(1-s_{0,\infty})}\Big). \tag{2.55}$$

Replacing t, γ and z, respectively, by s_0, $2-\nu$ and $\widetilde{w}_{1,1}$ in (2.50), we get

$$\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+2-\nu)}{\Gamma(2-\nu)}\,{}_1F_1(-\omega_0;2-\nu;\widetilde{w}_{1,1}) = (1-s_0)^{\nu-2}\exp\Big(-\frac{\widetilde{w}_{1,1}s_0}{(1-s_0)}\Big). \tag{2.56}$$

Replacing t, γ and z, respectively, by s_0, $2-\nu$ and $\widetilde{w}_{1,n}$ in (2.50), we get

$$\sum_{\omega_0=0}^{\infty}\frac{s_0^{\omega_0}}{\omega_0!}\frac{\Gamma(\omega_0+2-\nu)}{\Gamma(2-\nu)}\,{}_1F_1(-\omega_0;2-\nu;\widetilde{w}_{1,n}) = (1-s_0)^{\nu-2}\exp\Big(-\frac{\widetilde{w}_{1,n}s_0}{(1-s_0)}\Big). \tag{2.57}$$

Taking $c_0 = 1$, $\lambda = 1-\nu$ and $\gamma' = 2-\nu$ in (2.37) and substituting (2.55), (2.56) and (2.57) into the obtained equality, we get the desired result. □

3 GCH equation about an irregular singular point at infinity

Let $z = \frac{1}{x}$ in (1.1) in order to get an analytic solution of the GCH equation about $x = \infty$:

$$z^4\frac{d^2y}{dz^2} + ((2-\nu)z^3 - \varepsilon z^2 - \mu z)\frac{dy}{dz} + (\Omega + \varepsilon\omega z)y = 0. \tag{3.1}$$

Assume that its solution is

$$y(z) = \sum_{n=0}^{\infty} c_n z^{n+\lambda}, \tag{3.2}$$

where λ is indicial root. Substitute (3.2) into (3.1). For the coefficients c_n, we get the following three-term recurrence relation:

$$c_{n+1} = A_n c_n + B_n c_{n-1}, \quad n \geq 1, \tag{3.3}$$

where

$$A_n = -\frac{\varepsilon}{\mu}\frac{(n-\omega+\lambda)}{(n+1-\Omega/\mu+\lambda)}\,, \tag{3.4a}$$

$$B_n = \frac{1}{\mu}\frac{(n-1+\lambda)(n-\nu+\lambda)}{(n+1-\Omega/\mu+\lambda)}\,, \tag{3.4b}$$

$$c_1 = A_0 c_0. \tag{3.4c}$$

We have an indicial root $\lambda = \Omega/\mu$.

Now, let us test for the convergence of the analytic function $y(z)$. As $n \to \infty$, from (3.4a) and (3.4b), we get

$$\lim_{n\to\infty} A_n = -\frac{\varepsilon}{\mu}\,, \tag{3.5a}$$

$$\lim_{n\to\infty} B_n = \frac{n}{\mu} \to \infty. \tag{3.5b}$$

There are no analytic solutions for a polynomial of type 2 and infinite series. Since, by (3.5b), $y(z)$ is divergent as $n \to \infty$, there are only two types of analytic solutions of the GCH equation about $x = \infty$ such as polynomials of type 1 and of type 3. In Chapter 10 [14], the polynomial of type 3 about $x = \infty$ is derived: μ, ε, Ω are treated as free variables and ν, ω as fixed values. In this section, we have constructed the power series expansion, an integral form and the generating function for the GCH polynomial of type 1 about $x = \infty$: μ, ε, ω and Ω are treated as free variables and ν as a fixed value.

3.1 Power series for a polynomial of type 1

In [10], the general expression of a power series of $y(x)$ for a polynomial of type 1 is given by

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} y_n(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots \\
&= c_0 x^{\lambda} \Bigg\{ \sum_{i_0=0}^{\beta_0} \Big(\prod_{i_1=0}^{i_0-1} B_{2i_1+1} \Big) x^{2i_0} + \sum_{i_0=0}^{\beta_0} \Big\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=i_0}^{\beta_1} \Big(\prod_{i_3=i_0}^{i_2-1} B_{2i_3+2} \Big) \Big\} x^{2i_2+1} \\
&\quad + \sum_{N=2}^{\infty} \Big\{ \sum_{i_0=0}^{\beta_0} \Big\{ A_{2i_0} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \prod_{k=1}^{N-1} \Big(\sum_{i_{2k}=i_{2(k-1)}}^{\beta_k} A_{2i_{2k}+k} \prod_{i_{2k+1}=i_{2(k-1)}}^{i_{2k}-1} B_{2i_{2k+1}+(k+1)} \Big) \\
&\quad \times \sum_{i_{2N}=i_{2(N-1)}}^{\beta_N} \Big(\prod_{i_{2N+1}=i_{2(N-1)}}^{i_{2N}-1} B_{2i_{2N+1}+(N+1)} \Big) \Big\} \Big\} x^{2i_{2N}+N} \Bigg\}.
\end{aligned} \tag{3.6}
$$

Here $\beta_i \le \beta_j$ only if $i \le j$, where $i, j, \beta_i, \beta_j \in \mathbb{N}_0$.

For a polynomial we need the following condition:

$$B_{2\beta_i+(i+1)} = 0, \quad \text{where } \ i = 0, 1, 2, \ldots, \quad \beta_i = 0, 1, 2, \ldots. \tag{3.7}$$

Here β_i is an eigenvalue that makes B_n term terminated at a certain value of the index n. (3.7) turns each $y_i(x)$, where $i = 0, 1, 2, \ldots$, into the polynomial in (3.6). Replace β_i by ν_i in (3.7) and put $n = 2\nu_i + (i+1)$ in (3.4b) with the condition $B_{2\nu_i+(i+1)} = 0$. Then we obtain eigenvalues ν of the form

$$\nu = 2\nu_i + i + 1 + \lambda.$$

In (3.4b), we replace ν by $2\nu_i + i + 1 + \lambda$, and insert the obtained result and (3.4a) into (3.6), where a variable x and an index β_i are, respectively, replaced by z and ν_i. Hence the general expression of a power series of the GCH equation for a polynomial of type 1 about $x = \infty$ is given by

$$
\begin{aligned}
y(z) &= \sum_{n=0}^{\infty} y_n(z) = y_0(z) + y_1(z) + y_2(z) + y_3(z) + \cdots \\
&= c_0 z^{\lambda} \Bigg\{ \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}} \eta^{i_0} \\
&\quad + \Big\{ \sum_{i_0=0}^{\nu_0} \frac{(i_0 - \frac{\omega}{2} + \frac{\lambda}{2})}{(i_0 + \frac{1}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2})} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}} \sum_{i_1=i_0}^{\nu_1} \frac{(-\nu_1)_{i_1} (\frac{1}{2} + \frac{\lambda}{2})_{i_1} (\frac{3}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}}{(-\nu_1)_{i_0} (\frac{1}{2} + \frac{\lambda}{2})_{i_0} (\frac{3}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_1}} \eta^{i_1} \Big\} \xi \\
&\quad + \sum_{n=2}^{\infty} \Big\{ \sum_{i_0=0}^{\nu_0} \frac{(i_0 - \frac{\omega}{2} + \frac{\lambda}{2})}{(i_0 + \frac{1}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2})} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}} \\
&\quad \times \prod_{k=1}^{n-1} \Big\{ \sum_{i_k=i_{k-1}}^{\nu_k} \frac{(i_k + \frac{k}{2} - \frac{\omega}{2} + \frac{\lambda}{2})}{(i_k + \frac{k}{2} + \frac{1}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2})} \frac{(-\nu_k)_{i_k} (\frac{k}{2} + \frac{\lambda}{2})_{i_k} (\frac{k}{2} + 1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_{k-1}}}{(-\nu_k)_{i_{k-1}} (\frac{k}{2} + \frac{\lambda}{2})_{i_{k-1}} (\frac{k}{2} + 1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_k}} \Big\} \\
&\quad \times \sum_{i_n=i_{n-1}}^{\nu_n} \frac{(-\nu_n)_{i_n} (\frac{n}{2} + \frac{\lambda}{2})_{i_n} (\frac{n}{2} + 1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_{n-1}}}{(-\nu_n)_{i_{n-1}} (\frac{n}{2} + \frac{\lambda}{2})_{i_{n-1}} (\frac{n}{2} + 1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_n}} \eta^{i_n} \Big\} \xi^n \Bigg\},
\end{aligned} \tag{3.8}
$$

where

$$\begin{cases} \eta = \dfrac{2}{\mu}\, z^2, \\ \xi = -\dfrac{\varepsilon}{\mu}\, z, \\ \nu = 2\nu_j + j + 1 + \lambda, \\ z = \dfrac{1}{x}\,, \\ \nu_i \le \nu_j \ \text{ only if } \ i \le j, \ \text{ where } \ i, j, \nu_i, \nu_j \in \mathbb{N}_0 \ldots. \end{cases}$$

Put c_0= 1, as $\lambda = \Omega/\mu$ in (3.8).

Remark 3.1. The power series expansion of the first kind GCH equation for a polynomial of type 1 about $x = \infty$, as $\nu = 2\nu_j + j + 1 + \Omega/\mu$, where $j, \nu_j \in \mathbb{N}_0$, is

$$\begin{aligned} y(z) &= Q^{(i)} W_{\nu_j}\Big(\mu, \varepsilon, \Omega, \omega, \nu = 2\nu_j + j + 1 + \frac{\Omega}{\mu}; z = \frac{1}{x}, \xi = -\frac{\varepsilon}{\mu}\, z, \eta = \frac{2}{\mu}\, z^2\Big) \\ &= z^{\frac{\Omega}{\mu}} \Bigg\{ \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}}\, \eta^{i_0} \\ &\quad + \Bigg\{ \sum_{i_0=0}^{\nu_0} \frac{(i_0 - \frac{\omega}{2} + \frac{\Omega}{2\mu})}{(i_0 + \frac{1}{2})} \frac{(-\nu_0)_{i_0} (\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}} \sum_{i_1=i_0}^{\nu_1} \frac{(-\nu_1)_{i_1} (\frac{1}{2} + \frac{\Omega}{2\mu})_{i_1} (\frac{3}{2})_{i_0}}{(-\nu_1)_{i_0} (\frac{1}{2} + \frac{\Omega}{2\mu})_{i_0} (\frac{3}{2})_{i_1}}\, \eta^{i_1} \Bigg\} \xi \\ &\quad + \sum_{n=2}^{\infty} \Bigg\{ \sum_{i_0=0}^{\nu_0} \frac{(i_0 - \frac{\omega}{2} + \frac{\Omega}{2\mu})}{(i_0 + \frac{1}{2})} \frac{(-\nu_0)_{i_0} (\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}} \\ &\quad \times \prod_{k=1}^{n-1} \Bigg\{ \sum_{i_k=i_{k-1}}^{\nu_k} \frac{(i_k + \frac{k}{2} - \frac{\omega}{2} + \frac{\Omega}{2\mu})}{(i_k + \frac{k}{2} + \frac{1}{2})} \frac{(-\nu_k)_{i_k} (\frac{k}{2} + \frac{\Omega}{2\mu})_{i_k} (\frac{k}{2} + 1)_{i_{k-1}}}{(-\nu_k)_{i_{k-1}} (\frac{k}{2} + \frac{\Omega}{2\mu})_{i_{k-1}} (\frac{k}{2} + 1)_{i_k}} \Bigg\} \\ &\quad \times \sum_{i_n=i_{n-1}}^{\nu_n} \frac{(-\nu_n)_{i_n} (\frac{n}{2} + \frac{\Omega}{2\mu})_{i_n} (\frac{n}{2} + 1)_{i_{n-1}}}{(-\nu_n)_{i_{n-1}} (\frac{n}{2} + \frac{\Omega}{2\mu})_{i_{n-1}} (\frac{n}{2} + 1)_{i_n}}\, \eta^{i_n} \Bigg\} \xi^n \Bigg\}. \end{aligned} \tag{3.9}$$

For the minimum value of the first kind GCH equation for a polynomial of type 1 about $x = \infty$, in (3.9) we set $\nu_0 = \nu_1 = \nu_2 = \cdots = 0$ and get

$$\begin{aligned} y(z) &= Q^{(i)} W_0\Big(\mu, \varepsilon, \Omega, \omega, \nu = j + 1 + \frac{\Omega}{\mu}; z = \frac{1}{x}, \xi = -\frac{\varepsilon}{\mu}\, z, \eta = \frac{2}{\mu}\, z^2\Big) \\ &= z^{\frac{\Omega}{\mu}} \sum_{n=0}^{\infty} \frac{(\Omega/\mu - \omega)_n}{n!}\, \xi^n = z^{\frac{\Omega}{\mu}} \Big(1 + \frac{\varepsilon}{\mu}\, z\Big)^{-(\frac{\Omega}{\mu} - \omega)}. \end{aligned}$$

From the above it follows that a polynomial of type 1 requires $|\frac{\varepsilon}{\mu}\, z| < 1$ for the convergence of the radius.

3.2 Integral representation for a polynomial of type 1

There is a generalized hypergeometric function such that

$$\begin{aligned} L_l &= \sum_{i_l=i_{l-1}}^{\nu_l} \frac{(-\nu_l)_{i_l} (\frac{l}{2} + \frac{\lambda}{2})_{i_l} (\frac{l}{2} + 1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_{l-1}}}{(-\nu_l)_{i_{l-1}} (\frac{l}{2} + \frac{\lambda}{2})_{i_{l-1}} (\frac{l}{2} + 1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_l}}\, \eta^{i_l} \\ &= \eta^{i_{l-1}} \sum_{j=0}^{\infty} \frac{A_j (i_{l-1} - \nu_l)_j (i_{l-1} + \frac{l}{2} + \frac{\lambda}{2})_j}{(i_{l-1} + \frac{l}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2})^{-1} (1)_j}\, \eta^j, \end{aligned} \tag{3.10}$$

where

$$A_j = B\Big(i_{l-1} + \frac{l}{2} - \frac{\Omega}{2\mu} + \frac{\lambda}{2}, j + 1\Big).$$

By using integral form of the beta function, we have

$$B\Big(i_{l-1}+\frac{l}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2},j+1\Big)=\int_0^1 dt_l\; t_l^{i_{l-1}+\frac{l}{2}-1-\frac{\Omega}{2\mu}+\frac{\lambda}{2}}(1-t_l)^j. \tag{3.11}$$

Substitute (3.11) into (3.10), and divide L_l by $(i_{l-1}+\frac{l}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})$. We obtain

$$\begin{aligned}G_l &= \frac{1}{(i_{l-1}+\frac{l}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})}\sum_{i_l=i_{l-1}}^{\nu_l}\frac{(-\nu_l)_{i_l}(\frac{l}{2}+\frac{\lambda}{2})_{i_l}(\frac{l}{2}+1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_{l-1}}}{(-\nu_l)_{i_{l-1}}(\frac{l}{2}+\frac{\lambda}{2})_{i_{l-1}}(\frac{l}{2}+1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_l}}\eta^{i_l}\\ &= \int_0^1 dt_l\; t_l^{\frac{l}{2}-1-\frac{\Omega}{2\mu}+\frac{\lambda}{2}}(\eta t_l)^{i_{l-1}}\sum_{j=0}^{\infty}\frac{(i_{l-1}-\nu_l)_j(i_{l-1}+\frac{l}{2}+\frac{\lambda}{2})_j}{(1)_j}\,(\eta(1-t_l))^j. \end{aligned}\tag{3.12}$$

Tricomi's function is defined by

$$U(a,b,z)=\frac{\Gamma(1-b)}{\Gamma(a-b+1)}\,M(a,b,z)+\frac{\Gamma(b-1)}{\Gamma(a)}\,z^{1-b}M(a-b+1,2-b,z). \tag{3.13}$$

The contour integral form of (3.13) is given by (see [33])

$$U(a,b,z)=e^{-a\pi i}\,\frac{\Gamma(1-a)}{2\pi i}\int_{\infty}^{(0+)} dp_l\; e^{-zp_l}p_l^{a-1}(1+p_l)^{b-a-1}, \tag{3.14}$$

where

$$a\neq 1,2,3,\ldots,\quad |\mathrm{ph}\,z|<\frac{1}{2}\,\pi.$$

Also (3.13) is written as (see [33])

$$U(a,b,z)=z^{-a}\sum_{j=0}^{\infty}\frac{(a)_j(a-b+1)_j}{(1)_j}\,(-z^{-1})^j=z^{-a}\;{}_2F_0(a,a-b+1;-;-z^{-1}). \tag{3.15}$$

Replace a, b and z in (3.15), respectively, by $i_{l-1}-\nu_l$, $-\nu_l+1-\frac{l}{2}-\frac{\lambda}{2}$ and $\frac{-1}{\eta(1-t_l)}$. We get

$$\begin{aligned}&\sum_{j=0}^{\infty}\frac{(i_{l-1}-\nu_l)_j(i_{l-1}+\frac{l}{2}+\frac{\lambda}{2})_j}{(1)_j}\,(\eta(1-t_l))^j\\ &\qquad=\Big(\frac{-1}{\eta(1-t_l)}\Big)^{i_{l-1}-\nu_l}U\Big(i_{l-1}-\nu_l,-\nu_l+1-\frac{l}{2}-\frac{\lambda}{2},\varpi\Big),\end{aligned}\tag{3.16}$$

where

$$\varpi=\frac{-1}{\eta(1-t_l)}\,.$$

Replace a, b and z in (3.14), respectively, by $i_{l-1}-\nu_l$, $-\nu_l+1-\frac{l}{2}-\frac{\lambda}{2}$ and $\frac{-1}{\eta(1-t_l)}$ and insert the result into (3.16). We obtain

$$\begin{aligned}&\sum_{j=0}^{\infty}\frac{(i_{l-1}-\nu_l)_j(i_{l-1}+\frac{l}{2}+\frac{\lambda}{2})_j}{(1)_j}\,(\eta(1-t_l))^j=\frac{\Gamma(\nu_l-i_{l-1}+1)}{2\pi i}\\ &\quad\times\int_{\infty}^{(0+)} dp_l\;\exp\Big(\frac{p_l}{\eta(1-t_l)}\Big)p_l^{-1}(1+p_l)^{-\frac{1}{2}(l+\lambda)}\Big(\frac{\eta(1-t_l)}{p_l}\Big)^{\nu_l}\Big(\frac{p_l}{\eta(1-t_l)(1+p_l)}\Big)^{i_{l-1}}.\end{aligned}\tag{3.17}$$

The Gamma function $\Gamma(z)$ is defined as follows:

$$\Gamma(z)=\int_0^\infty du_l\; e^{-u_l}u_l^{z-1},\;\; \text{where}\;\; \mathrm{Re}\,(z)>0. \tag{3.18}$$

Put $z=\nu_l-i_{l-1}+1$ in (3.18). We have

$$\Gamma(\nu_l-i_{l-1}+1)=\int_0^\infty du_l\; e^{-u_l}u_l^{\nu_l-i_{l-1}}. \tag{3.19}$$

Substitute (3.19) in (3.17) and insert the result into (3.12). We get

$$\begin{aligned}G_l &= \frac{1}{(i_{l-1}+\frac{l}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})}\sum_{i_l=i_{l-1}}^{\nu_l}\frac{(-\nu_l)_{i_l}(\frac{l}{2}+\frac{\lambda}{2})_{i_l}(\frac{l}{2}+1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_{l-1}}}{(-\nu_l)_{i_{l-1}}(\frac{l}{2}+\frac{\lambda}{2})_{i_{l-1}}(\frac{l}{2}+1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_l}}\eta^{i_l}\\
&= \int_0^1 dt_l\; t_l^{\frac{1}{2}(l-2-\frac{\Omega}{\mu}+\lambda)}\int_0^\infty du_l\;\frac{1}{2\pi i}\\
&\quad\times\oint_\infty^{(0+)} dp_l\;\exp\left(\frac{p_l}{\eta(1-t_l)}\right)p_l^{-1}(1+p_l)^{-\frac{1}{2}(l+\lambda)}\left(\frac{\eta u_l(1-t_l)}{p_l}\right)^{\nu_l}\left(\frac{t_lp_l}{u_l(1-t_l)(1+p_l)}\right)^{i_{l-1}}.\end{aligned} \tag{3.20}$$

Substitute (3.20) into (3.8), where $l=1,2,3,\ldots$: apply G_1 into the second summation of the sub-power series $y_1(z)$; apply G_2 into the third summation and G_1 into the second summation of the sub-power series $y_2(z)$; apply G_3 into the forth summation, G_2 into the third summation and G_1 into the second summation of the sub-power series $y_3(z)$, etc.*

Theorem 3.2. *The general representation in the form of an integral of the GCH polynomial of type* 1 *about $x=\infty$ is given by*

$$\begin{aligned}y(z) &= \sum_{n=0}^\infty y_n(z)=y_0(z)+y_1(z)+y_2(z)+y_3(z)+\cdots\\
&= c_0z^\lambda\left\{\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\eta^{i_0}+\sum_{n=1}^\infty\left\{\prod_{k=0}^{n-1}\left\{\int_0^1 dt_{n-k}\; t_{n-k}^{\frac{1}{2}(n-k-2-\frac{\Omega}{\mu}+\lambda)}\int_0^\infty du_{n-k}\; e^{-u_{n-k}}\right.\right.\right.\\
&\quad\times\frac{1}{2\pi i}\oint_\infty^{(0+)} dp_{n-k}\; p_{n-k}^{-1}(1+p_{n-k})^{-\frac{1}{2}(n-k+\lambda)}\\
&\quad\times\exp\left(\frac{p_{n-k}}{w_{n-k+1,n}(1-t_{n-k})}\right)\left(\frac{w_{n-k+1,n}u_{n-k}(1-t_{n-k})}{p_{n-k}}\right)^{\nu_{n-k}}\\
&\quad\left.\left.\left.\times w_{n-k,n}^{-\frac{1}{2}(n-k-1-\omega+\lambda)}(w_{n-k,n}\partial_{w_{n-k,n}})w_{n-k,n}^{\frac{1}{2}(n-k-1-\omega+\lambda)}\right\}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}w_{1,n}^{i_0}\right\}\xi^n\right\},\end{aligned} \tag{3.21}$$

where

$$w_{i,j}=\begin{cases}\dfrac{t_ip_i}{u_i(1-t_i)(1+p_i)}, & \text{where}\;\; i\le j,\\ \eta\;\; \text{only if}\;\; i>j.\end{cases}$$

Here the first sub-integral form contains one term of A_n's, the second one contains two terms of A_n's, the third one contains three terms of A_n's, etc.

*$y_1(z)$ means the sub-power series in (3.8), contains one term of A_n's; $y_2(z)$ means the sub-power series in (3.8), contains two terms of A_n's; $y_3(z)$ means the sub-power series in (3.8), contains three terms of A_n's, etc.

Proof. In (3.8), the power series expansions of the sub-summation terms $y_0(z)$, $y_1(z)$, $y_2(z)$ and $y_3(z)$ of the GCH polynomial of type 1 about $x=\infty$ are

$$y(z)=\sum_{n=0}^{\infty} y_n(z) = y_0(z)+y_1(z)+y_2(z)+y_3(z)+\cdots , \tag{3.22}$$

where

$$y_0(z)=\sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\,\eta^{i_0}, \tag{3.23a}$$

$$\begin{aligned} y_1(z)=c_0z^{\lambda}\Bigg\{&\sum_{i_0=0}^{\nu_0} \frac{(i_0-\frac{\omega}{2}+\frac{\lambda}{2})}{(i_0+\frac{1}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})}\,\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}} \\ &\times\sum_{i_1=i_0}^{\nu_1} \frac{(-\nu_1)_{i_1}(\frac{1}{2}+\frac{\lambda}{2})_{i_1}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}{(-\nu_1)_{i_0}(\frac{1}{2}+\frac{\lambda}{2})_{i_0}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_1}}\,\eta^{i_1}\Bigg\}\xi, \end{aligned} \tag{3.23b}$$

$$\begin{aligned} y_2(z)=c_0z^{\lambda}\Bigg\{&\sum_{i_0=0}^{\nu_0} \frac{(i_0-\frac{\omega}{2}+\frac{\lambda}{2})}{(i_0+\frac{1}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})}\,\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}} \\ &\times\sum_{i_1=i_0}^{\nu_1} \frac{(i_1+\frac{1}{2}-\frac{\omega}{2}+\frac{\lambda}{2})}{(i_1+1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})}\,\frac{(-\nu_1)_{i_1}(\frac{1}{2}+\frac{\lambda}{2})_{i_1}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}{(-\nu_1)_{i_0}(\frac{1}{2}+\frac{\lambda}{2})_{i_0}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_1}} \\ &\times\sum_{i_2=i_1}^{\nu_2} \frac{(-\nu_2)_{i_2}(1+\frac{\lambda}{2})_{i_2}(2-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_1}}{(-\nu_2)_{i_1}(1+\frac{\lambda}{2})_{i_1}(2-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_2}}\,\eta^{i_2}\Bigg\}\xi^2, \end{aligned} \tag{3.23c}$$

$$\begin{aligned} y_3(z)=c_0z^{\lambda}\Bigg\{&\sum_{i_0=0}^{\nu_0} \frac{(i_0-\frac{\omega}{2}+\frac{\lambda}{2})}{(i_0+\frac{1}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})}\,\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}} \\ &\times\sum_{i_1=i_0}^{\nu_1} \frac{(i_1+\frac{1}{2}-\frac{\omega}{2}+\frac{\lambda}{2})}{(i_1+1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})}\,\frac{(-\nu_1)_{i_1}(\frac{1}{2}+\frac{\lambda}{2})_{i_1}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}{(-\nu_1)_{i_0}(\frac{1}{2}+\frac{\lambda}{2})_{i_0}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_1}} \\ &\times\sum_{i_2=i_1}^{\nu_2} \frac{(i_2+1-\frac{\omega}{2}+\frac{\lambda}{2})}{(i_2+\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})}\,\frac{(-\nu_2)_{i_2}(1+\frac{\lambda}{2})_{i_2}(2-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_1}}{(-\nu_2)_{i_1}(1+\frac{\lambda}{2})_{i_1}(2-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_2}} \\ &\times\sum_{i_3=i_2}^{\nu_3} \frac{(-\nu_3)_{i_3}(\frac{3}{2}+\frac{\lambda}{2})_{i_3}(\frac{5}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_2}}{(-\nu_3)_{i_2}(\frac{3}{2}+\frac{\lambda}{2})_{i_2}(\frac{5}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_3}}\,\eta^{i_3}\Bigg\}\xi^3. \end{aligned} \tag{3.23d}$$

Put $l=1$ in (3.20) and insert the result into (3.23b). We get

$$\begin{aligned} y_1(z) &= c_0z^{\lambda}\xi\int_0^1 dt_1\, t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)}\int_0^{\infty} du_1\, e^{-u_1}\,\frac{1}{2\pi i}\int_{\infty}^{(0+)} dp_1 \\ &\quad\times\exp\left(\frac{p_1}{\eta(1-t_1)}\right)p_1^{-1}(1+p_1)^{-\frac{1}{2}(1+\lambda)}\left(\frac{\eta u_1(1-t_1)}{p_1}\right)^{\nu_1} \\ &\quad\times\sum_{i_0=0}^{\nu_0}\left(i_0-\frac{\omega}{2}+\frac{\lambda}{2}\right)\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\left(\frac{t_1p_1}{u_1(1-t_1)(1+p_1)}\right)^{i_0} \\ &= c_0z^{\lambda}\xi\int_0^1 dt_1\, t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)}\int_0^{\infty} du_1\, e^{-u_1}\,\frac{1}{2\pi i}\int_{\infty}^{(0+)} dp_1 \\ &\quad\times\exp\left(\frac{p_1}{\eta(1-t_1)}\right)p_1^{-1}(1+p_1)^{-\frac{1}{2}(1+\lambda)}\left(\frac{\eta u_1(1-t_1)}{p_1}\right)^{\nu_1} \\ &\quad\times w_{1,1}^{-\frac{1}{2}(-\omega+\lambda)}(w_{1,1}\partial_{w_{1,1}})w_{1,1}^{\frac{1}{2}(-\omega+\lambda)}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}w_{1,1}^{i_0}, \end{aligned} \tag{3.24}$$

where $w_{1,1}=\frac{t_1p_1}{u_1(1-t_1)(1+p_1)}$. Put $l=2$ in (3.20) and insert the result into (3.23c). We get

$$
\begin{aligned}
y_2(z) &= c_0 z^{\lambda}\xi^2 \int_0^1 dt_2\; t_2^{\frac{1}{2}(-\frac{\Omega}{\mu}+\lambda)} \int_0^{\infty} du_2\; e^{-u_2}\, \frac{1}{2\pi i}\oint_{\infty}^{(0+)} dp_2 \\
&\times \exp\left(\frac{p_2}{\eta(1-t_2)}\right) p_2^{-1}(1+p_2)^{-\frac{1}{2}(2+\lambda)} \left(\frac{\eta u_2(1-t_2)}{p_2}\right)^{\nu_2} w_{2,2}^{-\frac{1}{2}(1-\omega+\lambda)} (w_{2,2}\partial_{w_{2,2}}) w_{2,2}^{\frac{1}{2}(1-\omega+\lambda)} \\
&\times \sum_{i_0=0}^{\nu_0} \frac{(i_0-\frac{\omega}{2}+\frac{\lambda}{2})}{(i_0+\frac{1}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})} \frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}} \sum_{i_1=i_0}^{\nu_1} \frac{(-\nu_1)_{i_1}(\frac{1}{2}+\frac{\lambda}{2})_{i_1}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}{(-\nu_1)_{i_0}(\frac{1}{2}+\frac{\lambda}{2})_{i_0}(\frac{3}{2}-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_1}}\, w_{2,2}^{i_1}.
\end{aligned}
\tag{3.25}
$$

where $w_{2,2}=\frac{t_2p_2}{u_2(1-t_2)(1+p_2)}$. Put $l=1$ and $\eta=w_{2,2}$ in (3.20) and insert the result into (3.25). We get

$$
\begin{aligned}
y_2(z) &= c_0 z^{\lambda}\xi^2 \int_0^1 dt_2\; t_2^{\frac{1}{2}(-\frac{\Omega}{\mu}+\lambda)} \int_0^{\infty} du_2\; e^{-u_2}\, \frac{1}{2\pi i}\oint_{\infty}^{(0+)} dp_2 \\
&\times \exp\left(\frac{p_2}{\eta(1-t_2)}\right) p_2^{-1}(1+p_2)^{-\frac{1}{2}(2+\lambda)} \left(\frac{\eta u_2(1-t_2)}{p_2}\right)^{\nu_2} w_{2,2}^{-\frac{1}{2}(1-\omega+\lambda)} (w_{2,2}\partial_{w_{2,2}}) w_{2,2}^{\frac{1}{2}(1-\omega+\lambda)} \\
&\times \int_0^1 dt_1\; t_1^{\frac{1}{2}(-1-\frac{\Omega}{\mu}+\lambda)} \int_0^{\infty} du_1\; e^{-u_1}\, \frac{1}{2\pi i}\oint_{\infty}^{(0+)} dp_1 \exp\left(\frac{p_1}{w_{2,2}(1-t_1)}\right) p_1^{-1}(1+p_1)^{-\frac{1}{2}(1+\lambda)} \\
&\times \left(\frac{w_{2,2}u_1(1-t_1)}{p_1}\right)^{\nu_1} w_{1,2}^{-\frac{1}{2}(-\omega+\lambda)} (w_{1,2}\partial_{w_{1,2}}) w_{1,2}^{\frac{1}{2}(-\omega+\lambda)} \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\, w_{1,2}^{i_0},
\end{aligned}
\tag{3.26}
$$

where $w_{1,2}=\frac{t_1p_1}{u_1(1-t_1)(1+p_1)}$.

By using similar process as for the previous cases of integral forms of $y_1(z)$ and $y_2(z)$, the integral form of the sub-power series expansion $y_3(z)$ takes the form

$$
\begin{aligned}
y_3(z) &= c_0 z^{\lambda}\xi^3 \int_0^1 dt_3\; t_3^{\frac{1}{2}(1-\frac{\Omega}{\mu}+\lambda)} \int_0^{\infty} du_3\; e^{-u_3}\, \frac{1}{2\pi i}\oint_{\infty}^{(0+)} dp_3 \\
&\times \exp\left(\frac{p_3}{\eta(1-t_3)}\right) p_3^{-1}(1+p_3)^{-\frac{1}{2}(3+\lambda)} \left(\frac{\eta u_3(1-t_3)}{p_3}\right)^{\nu_3} w_{3,3}^{-\frac{1}{2}(2-\omega+\lambda)} (w_{3,3}\partial_{w_{3,3}}) w_{3,3}^{\frac{1}{2}(2-\omega+\lambda)} \\
&\times \int_0^1 dt_2\; t_2^{\frac{1}{2}(-\frac{\Omega}{\mu}+\lambda)} \int_0^{\infty} du_2\; e^{-u_2}\, \frac{1}{2\pi i}\oint_{\infty}^{(0+)} dp_2 \\
&\times \exp\left(\frac{p_2}{w_{3,3}(1-t_2)}\right) p_2^{-1}(1+p_2)^{-\frac{1}{2}(2+\lambda)} \left(\frac{w_{3,3}u_2(1-t_2)}{p_2}\right)^{\nu_2} w_{2,3}^{-\frac{1}{2}(1-\omega+\lambda)} (w_{2,3}\partial_{w_{2,3}}) w_{2,3}^{\frac{1}{2}(1-\omega+\lambda)} \\
&\times \int_0^1 dt_1\; t_1^{\frac{1}{2}(-1-\frac{\Omega}{\mu}+\lambda)} \int_0^{\infty} du_1\; e^{-u_1}\, \frac{1}{2\pi i}\oint_{\infty}^{(0+)} dp_1 \exp\left(\frac{p_1}{w_{2,3}(1-t_1)}\right) p_1^{-1}(1+p_1)^{-\frac{1}{2}(1+\lambda)} \\
&\times \left(\frac{w_{2,3}u_1(1-t_1)}{p_1}\right)^{\nu_1} w_{1,3}^{-\frac{1}{2}(-\omega+\lambda)} (w_{1,3}\partial_{w_{1,3}}) w_{1,3}^{\frac{1}{2}(-\omega+\lambda)} \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\, w_{1,3}^{i_0},
\end{aligned}
\tag{3.27}
$$

where

$$
\begin{cases}
w_{3,3} = \dfrac{t_3p_3}{u_3(1-t_3)(1+p_3)}\,, \\[2ex]
w_{2,3} = \dfrac{t_2p_2}{u_2(1-t_2)(1+p_2)}\,, \\[2ex]
w_{1,3} = \dfrac{t_1p_1}{u_1(1-t_1)(1+p_1)}\,.
\end{cases}
$$

By repeating this process for all higher terms of integral forms of sub-summation terms $y_m(z)$, $m \geq 4$, we obtain their integral forms. If we substitute (3.23a), (3.24), (3.26), (3.27) and the integral forms of $y_m(z)$, $m \geq 4$, into (3.22), we obtain (3.21). □

Remark 3.3. The integral representation of the first kind GCH equation for a polynomial of type 1 about $x = \infty$ as $\nu = 2\nu_j + j + 1 + \Omega/\mu$ where $j, \nu_j \in \mathbb{N}_0$ is

$$\begin{aligned} y(z) &= Q^{(i)} W_{\nu_j}\Big(\mu, \varepsilon, \Omega, \omega, \nu = 2\nu_j + j + 1 + \frac{\Omega}{\mu}; z, \xi, \eta\Big) \\ &= z^{\frac{\Omega}{\mu}} \Bigg\{ (-\eta)^{\nu_0} U\Big(-\nu_0, -\nu_0 + 1 - \frac{\Omega}{\mu}, -\eta^{-1}\Big) + \sum_{n=1}^{\infty} \Bigg\{ \prod_{k=0}^{n-1} \Bigg\{ \int_0^1 dt_{n-k}\, t_{n-k}^{\frac{1}{2}(n-k-2)} \\ &\quad \times \int_0^\infty du_{n-k}\, e^{-u_{n-k}} \frac{1}{2\pi i} \int_\infty^{(0+)} dp_{n-k}\, p_{n-k}^{-1} (1 + p_{n-k})^{-\frac{1}{2}(n-k+\frac{\Omega}{\mu})} \\ &\quad \times \exp\Big(\frac{p_{n-k}}{w_{n-k+1,n}(1 - t_{n-k})}\Big)\Big(\frac{w_{n-k+1,n} u_{n-k}(1 - t_{n-k})}{p_{n-k}}\Big)^{\nu_{n-k}} w_{n-k,n}^{-\frac{1}{2}(n-k-1-\omega+\frac{\Omega}{\mu})} \\ &\quad \times (w_{n-k,n} \partial_{w_{n-k,n}}) w_{n-k,n}^{\frac{1}{2}(n-k-1-\omega+\frac{\Omega}{\mu})} \Bigg\} (-w_{1,n})^{\nu_0} U\Big(-\nu_0, -\nu_0 + 1 - \frac{\Omega}{\mu}, -w_{1,n}^{-1}\Big) \Bigg\} \xi^n \Bigg\}, \end{aligned} \tag{3.28}$$

where

$$\begin{cases} z = \dfrac{1}{x}, \\ \xi = -\dfrac{\varepsilon}{\mu} z, \\ \eta = \dfrac{2}{\mu} z^2. \end{cases}$$

Proof. Replace a, b and z, respectively, by $-\nu_0$, $-\nu_0 + 1 - \frac{\Omega}{2\mu}$ and $-\eta^{-1}$ into (3.15):

$$\sum_{j=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}} \eta^j = (-\eta)^{\nu_0} U\Big(-\nu_0, -\nu_0 + 1 - \frac{\Omega}{\mu}, -\eta^{-1}\Big). \tag{3.29}$$

Replace a, b and z, respectively, by $-\nu_0$, $-\nu_0 + 1 - \frac{\Omega}{2\mu}$ and $-w_{1,n}^{-1}$ into (3.15):

$$\sum_{j=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1 - \frac{\Omega}{2\mu} + \frac{\lambda}{2})_{i_0}} w_{1,n}^j = (-w_{1,n})^{\nu_0} U\Big(-\nu_0, -\nu_0 + 1 - \frac{\Omega}{\mu}, -w_{1,n}^{-1}\Big). \tag{3.30}$$

Putting $c_0 = 1$ and $\lambda = \Omega/\mu$ in (3.21) and substituting (3.29) and (3.30) into obtained equality we get the result. □

3.3 Generating function of the GCH polynomial of type 1

Let us investigate the generating function for the first kind GCH polynomial of type 1 about $x = \infty$.

Definition 3.4. Define

$$\begin{cases} s_{a,b} = \begin{cases} s_a \cdot s_{a+1} \cdot s_{a+2} \cdots s_{b-2} \cdot s_{b-1} \cdot s_b, & \text{where } a < b, \\ s_a \text{ only if } a = b, \end{cases} \\ \widetilde{w}_{i,j} = \begin{cases} \dfrac{s_i t_i \widetilde{w}_{i+1,j}}{1 + s_i u_i (1 - t_i) \widetilde{w}_{i+1,j}}, & \text{where } i < j, \\ \dfrac{s_{i,\infty} t_i \eta}{1 + s_{i,\infty} u_i (1 - t_i) \eta} & \text{only if } i = j, \end{cases} \end{cases} \tag{3.31}$$

where $a, b, i, j \in \mathbb{N}_0$, $0 \leq a \leq b \leq \infty$ and $1 \leq i \leq j \leq \infty$.

We have

$$\sum_{\nu_i=\nu_j}^{\infty} s_i^{\nu_i} = \frac{s_i^{\nu_j}}{(1-s_i)} \quad \text{at } |s_i|<1. \tag{3.32}$$

Theorem 3.5. *The general expression of the generating function for the GCH polynomial of type 1 about $x=\infty$ is given by*

$$\begin{aligned}
&\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y(z)=\prod_{k=1}^{\infty}\frac{1}{(1-s_{k,\infty})}\boldsymbol{\Upsilon}(\lambda;s_{0,\infty};\eta)\\
&+\left\{\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_1\; t_1^{\frac{1}{2}(-1-\frac{\Omega}{\mu}+\lambda)}\int_0^{\infty}du_1\;\exp(-(1-s_{1,\infty})u_1)(1+s_{1,\infty}u_1(1-t_1)\eta)^{-\frac{1}{2}(1+\lambda)}\right.\\
&\qquad\left.\times\,\widetilde{w}_{1,1}^{-\frac{1}{2}(-\omega+\lambda)}(\widetilde{w}_{1,1}\partial_{\widetilde{w}_{1,1}})\widetilde{w}_{1,1}^{\frac{1}{2}(-\omega+\lambda)}\boldsymbol{\Upsilon}(\lambda;s_0;\widetilde{w}_{1,1})\right\}\xi\\
&+\sum_{n=2}^{\infty}\left\{\prod_{k=n+1}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_n\; t_n^{\frac{1}{2}(n-2-\frac{\Omega}{\mu}+\lambda)}\int_0^{\infty}du_n\;\exp(-(1-s_{n,\infty})u_n)(1+s_{n,\infty}u_n(1-t_n)\eta)^{-\frac{1}{2}(n+\lambda)}\right.\\
&\qquad\times\,\widetilde{w}_{n,n}^{-\frac{1}{2}(n-1-\omega+\lambda)}(\widetilde{w}_{n,n}\partial_{\widetilde{w}_{n,n}})\widetilde{w}_{n,n}^{\frac{1}{2}(n-1-\omega+\lambda)}\prod_{j=1}^{n-1}\left\{\int_0^1 dt_{n-j}\; t_{n-j}^{\frac{1}{2}(n-j-2-\frac{\Omega}{\mu}+\lambda)}\int_0^{\infty}du_{n-j}\right.\\
&\qquad\times\exp(-(1-s_{n-j})u_{n-j})(1+s_{n-j}u_{n-j}(1-t_{n-j})\widetilde{w}_{n-j+1,n})^{-\frac{1}{2}(n-j+\lambda)}\\
&\qquad\left.\left.\times\,\widetilde{w}_{n-j,n}^{-\frac{1}{2}(n-j-1-\omega+\lambda)}(\widetilde{w}_{n-j,n}\partial_{\widetilde{w}_{n-j,n}})\widetilde{w}_{n-j,n}^{\frac{1}{2}(n-j-1-\omega+\lambda)}\right\}\boldsymbol{\Upsilon}(\lambda;s_0;\widetilde{w}_{1,n})\right\}\xi^n,
\end{aligned} \tag{3.33}$$

where

$$\begin{cases}
\boldsymbol{\Upsilon}(\lambda;s_{0,\infty};\eta)=\displaystyle\sum_{\nu_0=0}^{\infty}\frac{s_{0,\infty}^{\nu_0}}{\nu_0!}\left\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\eta^{i_0}\right\},\\
\boldsymbol{\Upsilon}(\lambda;s_0;\widetilde{w}_{1,1})=\displaystyle\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\left\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\widetilde{w}_{1,1}^{i_0}\right\},\\
\boldsymbol{\Upsilon}(\lambda;s_0;\widetilde{w}_{1,n})=\displaystyle\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\left\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\widetilde{w}_{1,n}^{i_0}\right\}.
\end{cases}$$

Proof. Applying the summation operator $\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}$ to the form of a general integral of type 1 GCH polynomial $y(z)$, we obtain

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y(z)=\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}\big(y_0(z)+y_1(z)+y_2(z)+\cdots\big). \tag{3.34}$$

Applying the summation operator $\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}$ to (3.23a) by using (3.31) and (3.32), we get

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_0(z)=\prod_{k=1}^{\infty}\frac{1}{(1-s_{k,\infty})}\sum_{\nu_0=0}^{\infty}\frac{s_{0,\infty}^{\nu_0}}{\nu_0!}\left\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\eta^{i_0}\right\} \tag{3.35}$$

Applying the summation operator $\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}$ to (3.24), by using (3.31) and (3.32), we get

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_1(z)=\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_1\;t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)}\int_0^\infty du_1\;e^{-u_1}\,\frac{1}{2\pi i}\int_\infty^{(0+)}dp_1$$
$$\times\exp\Big(\frac{p_1}{\eta(1-t_1)}\Big)\,\frac{(1+p_1)^{-\frac{1}{2}(1+\lambda)}}{p_1}\sum_{\nu_1=\nu_0}^{\infty}\Big(\frac{s_{1,\infty}\eta u_1(1-t_1)}{p_1}\Big)^{\nu_1}$$
$$\times\, w_{1,1}^{-\frac{1}{2}(-\omega+\lambda)}(w_{1,1}\partial_{w_{1,1}})w_{1,1}^{\frac{1}{2}(-\omega+\lambda)}\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\left\{c_0z^\lambda\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}\left(\frac{\lambda}{2}\right)_{i_0}}{\left(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2}\right)_{i_0}}\,w_{1,1}^{i_0}\right\}\xi. \quad (3.36)$$

Replace ν_i, ν_j and s_i by ν_1, ν_0 and $\frac{s_{1,\infty}\eta u_1(1-t_1)}{p_1}$ in (3.32). Substitute the new (3.32) into (3.36),

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_1(z)=\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_1\;t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)}\int_0^\infty du_1\;e^{-u_1}$$
$$\times\frac{1}{2\pi i}\int_\infty^{(0+)}dp_1\exp\Big(\frac{p_1}{\eta(1-t_1)}\Big)\,\frac{(1+p_1)^{-\frac{1}{2}(1+\lambda)}}{p_1-s_{1,\infty}\eta u_1(1-t_1)}\,w_{1,1}^{-\frac{1}{2}(-\omega+\lambda)}(w_{1,1}\partial_{w_{1,1}})w_{1,1}^{\frac{1}{2}(-\omega+\lambda)}$$
$$\times\sum_{\nu_0=0}^{\infty}\Big(\frac{s_{0,\infty}\eta u_1(1-t_1)}{p_1}\Big)^{\nu_0}\frac{1}{\nu_0!}\left\{c_0z^\lambda\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}\left(\frac{\lambda}{2}\right)_{i_0}}{\left(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2}\right)_{i_0}}\,w_{1,1}^{i_0}\right\}\xi. \quad (3.37)$$

By using Cauchy's integral formula, the contour integrand has poles at $p_1=s_{1,\infty}\eta u_1(1-t_1)$, where $s_{1,\infty}\eta u_1(1-t_1)$ is inside the unit circle. Computing the residue in (3.37), we obtain

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_1(z)=\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_1\;t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)}$$
$$\times\int_0^\infty du_1\;\exp(-(1-s_{1,\infty})u_1)(1+s_{1,\infty}u_1(1-t_1)\eta)^{-\frac{1}{2}(1+\lambda)}$$
$$\times\,\widetilde{w}_{1,1}^{-\frac{1}{2}(-\omega+\lambda)}(\widetilde{w}_{1,1}\partial_{\widetilde{w}_{1,1}})\widetilde{w}_{1,1}^{\frac{1}{2}(-\omega+\lambda)}\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\left\{c_0z^\lambda\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}\left(\frac{\lambda}{2}\right)_{i_0}}{\left(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2}\right)_{i_0}}\,\widetilde{w}_{1,1}^{i_0}\right\}\xi, \quad (3.38)$$

where

$$\widetilde{w}_{1,1}=\left.\frac{t_1p_1}{u_1(1-t_1)(1+p_1)}\right|_{p_1=s_{1,\infty}\eta u_1(1-t_1)}=\frac{s_{1,\infty}t_1\eta}{1+s_{1,\infty}u_1(1-t_1)\eta}\,.$$

Applying the summation operator $\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}$ to (3.26), by using (3.31) and (3.32), we have

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_2(z)=\prod_{k=3}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_2\;t_2^{\frac{1}{2}\left(-\frac{\Omega}{\mu}+\lambda\right)}\int_0^\infty du_2\;e^{-u_2}$$
$$\times\frac{1}{2\pi i}\int_\infty^{(0+)}dp_2\;\exp\Big(\frac{p_2}{\eta(1-t_2)}\Big)\,\frac{(1+p_2)^{-\frac{1}{2}(2+\lambda)}}{p_2}$$
$$\times\sum_{\nu_2=\nu_1}^{\infty}\Big(\frac{s_{2,\infty}\eta u_2(1-t_2)}{p_2}\Big)^{\nu_2}w_{2,2}^{-\frac{1}{2}(1-\omega+\lambda)}(w_{2,2}\partial_{w_{2,2}})w_{2,2}^{\frac{1}{2}(1-\omega+\lambda)}$$

$$\times \int_0^1 dt_1\, t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)} \int_0^\infty du_1\, e^{-u_1} \frac{1}{2\pi i} \int_\infty^{(0+)} dp_1\, \exp\left(\frac{p_1}{w_{2,2}(1-t_1)}\right) \frac{(1+p_1)^{-\frac{1}{2}(1+\lambda)}}{p_1}$$
$$\times \sum_{\nu_1=\nu_0}^{\infty} \left(\frac{s_1 w_{2,2} u_1 (1-t_1)}{p_1}\right)^{\nu_1} w_{1,2}^{-\frac{1}{2}(-\omega+\lambda)} (w_{1,2}\partial_{w_{1,2}}) w_{1,2}^{\frac{1}{2}(-\omega+\lambda)}$$
$$\times \sum_{\nu_0=0}^{\infty} \frac{s_0^{\nu_0}}{\nu_0!} \left\{ c_0 z^\lambda \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}} w_{1,2}^{i_0} \right\} \xi^2. \qquad (3.39)$$

Replace ν_i, ν_j and s_i, respectively, by ν_2, ν_1 and $\frac{s_{2,\infty}\,\eta u_2(1-t_2)}{p_2}$ in (3.32) and the insert the result into (3.39). We have

$$\sum_{\nu_0=0}^{\infty} \frac{s_0^{\nu_0}}{\nu_0!} \prod_{n=1}^{\infty} \left\{ \sum_{\nu_n=\nu_{n-1}}^{\infty} s_n^{\nu_n} \right\} y_2(z) = \prod_{k=3}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_2\, t_2^{\frac{1}{2}\left(-\frac{\Omega}{\mu}+\lambda\right)} \int_0^\infty du_2\, e^{-u_2}$$
$$\times \frac{1}{2\pi i} \int_\infty^{(0+)} dp_2\, \exp\left(\frac{p_2}{\eta(1-t_2)}\right) \frac{(1+p_2)^{-\frac{1}{2}(2+\lambda)}}{p_2 - s_{2,\infty}\eta u_2(1-t_2)} w_{2,2}^{-\frac{1}{2}(1-\omega+\lambda)} (w_{2,2}\partial_{w_{2,2}}) w_{2,2}^{\frac{1}{2}(1-\omega+\lambda)}$$
$$\times \int_0^1 dt_1\, t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)} \int_0^\infty du_1\, e^{-u_1} \frac{1}{2\pi i} \int_\infty^{(0+)} dp_1\, \exp\left(\frac{p_1}{w_{2,2}(1-t_1)}\right) \frac{(1+p_1)^{-\frac{1}{2}(1+\lambda)}}{p_1}$$
$$\times \sum_{\nu_1=\nu_0}^{\infty} \left(\frac{s_{1,\infty}\eta u_2(1-t_2) w_{2,2} u_1 (1-t_1)}{p_1 p_2}\right)^{\nu_1} w_{1,2}^{-\frac{1}{2}(-\omega+\lambda)} (w_{1,2}\partial_{w_{1,2}}) w_{1,2}^{\frac{1}{2}(-\omega+\lambda)}$$
$$\times \sum_{\nu_0=0}^{\infty} \frac{s_0^{\nu_0}}{\nu_0!} \left\{ c_0 z^\lambda \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}} w_{1,2}^{i_0} \right\} \xi^2. \qquad (3.40)$$

By using Cauchy's integral formula, the contour integrand has poles at $p_2 = s_{2,\infty}\eta u_2(1-t_2)$, where $s_{2,\infty}\eta u_2(1-t_2)$ is inside the unit circle. Computing the residue in (3.40), we obtain

$$\sum_{\nu_0=0}^{\infty} \frac{s_0^{\nu_0}}{\nu_0!} \prod_{n=1}^{\infty} \left\{ \sum_{\nu_n=\nu_{n-1}}^{\infty} s_n^{\nu_n} \right\} y_2(z) = \prod_{k=3}^{\infty} \frac{1}{(1-s_{k,\infty})} \int_0^1 dt_2\, t_2^{\frac{1}{2}\left(-\frac{\Omega}{\mu}+\lambda\right)}$$
$$\times \int_0^\infty du_2\, \exp\Big(-(1-s_{2,\infty})u_2\Big) \Big(1+s_{2,\infty}u_2(1-t_2)\eta\Big)^{-\frac{1}{2}(2+\lambda)} \widetilde{w}_{2,2}^{-\frac{1}{2}(1-\omega+\lambda)} (\widetilde{w}_{2,2}\partial_{\widetilde{w}_{2,2}}) \widetilde{w}_{2,2}^{\frac{1}{2}(1-\omega+\lambda)}$$
$$\times \int_0^1 dt_1\, t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)} \int_0^\infty du_1\, e^{-u_1} \frac{1}{2\pi i} \int_\infty^{(0+)} dp_1\, \exp\left(\frac{p_1}{\widetilde{w}_{2,2}(1-t_1)}\right) \frac{(1+p_1)^{-\frac{1}{2}(1+\lambda)}}{p_1}$$
$$\times \sum_{\nu_1=\nu_0}^{\infty} \left(\frac{s_1 \widetilde{w}_{2,2} u_1 (1-t_1)}{p_1}\right)^{\nu_1} w_{1,2}^{-\frac{1}{2}(-\omega+\lambda)} (w_{1,2}\partial_{w_{1,2}}) w_{1,2}^{\frac{1}{2}(-\omega+\lambda)}$$
$$\times \sum_{\nu_0=0}^{\infty} \frac{s_0^{\nu_0}}{\nu_0!} \left\{ c_0 z^\lambda \sum_{i_0=0}^{\nu_0} \frac{(-\nu_0)_{i_0} (\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}} w_{1,2}^{i_0} \right\} \xi^2, \qquad (3.41)$$

where

$$\widetilde{w}_{2,2} = \frac{t_2 p_2}{u_2(1-t_2)(1+p_2)}\bigg|_{p_2=s_{2,\infty}\eta u_2(1-t_2)} = \frac{s_{2,\infty} t_2 \eta}{1+s_{2,\infty}u_2(1-t_2)\eta}.$$

Replace ν_i, ν_j and s_i, respectively, by ν_1, ν_0 and $\frac{s_1 \widetilde{w}_{2,2} u_1(1-t_1)}{p_1}$ in (3.32) and insert the result in (3.41):

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_2(z)=\prod_{k=3}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_2\; t_2^{\frac{1}{2}\left(-\frac{\Omega}{\mu}+\lambda\right)}$$
$$\times\int_0^{\infty}du_2\;\exp(-(1-s_{2,\infty})u_2)\big(1+s_{2,\infty}u_2(1-t_2)\eta\big)^{-\frac{1}{2}(2+\lambda)}\widetilde{w}_{2,2}^{-\frac{1}{2}(1-\omega+\lambda)}(\widetilde{w}_{2,2}\partial_{\widetilde{w}_{2,2}})\widetilde{w}_{2,2}^{\frac{1}{2}(1-\omega+\lambda)}$$
$$\times\int_0^1 dt_1\; t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)}\int_0^{\infty}du_1\;e^{-u_1}\frac{1}{2\pi i}\int_{\infty}^{(0+)}dp_1\;\exp\Big(\frac{p_1}{\widetilde{w}_{2,2}(1-t_1)}\Big)$$
$$\times\frac{(1+p_1)^{-\frac{1}{2}(1+\lambda)}}{p_1-s_1\widetilde{w}_{2,2}u_1(1-t_1)}\,w_{1,2}^{-\frac{1}{2}(-\omega+\lambda)}(w_{1,2}\partial_{w_{1,2}})w_{1,2}^{\frac{1}{2}(-\omega+\lambda)}$$
$$\times\sum_{\nu_0=0}^{\infty}\Big(\frac{s_{0,1}\widetilde{w}_{2,2}u_1(1-t_1)}{p_1}\Big)^{\nu_0}\frac{1}{\nu_0!}\left\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\,w_{1,2}^{i_0}\right\}\xi^2. \tag{3.42}$$

By using Cauchy's integral formula, the contour integrand has poles at $p_1=s_1\widetilde{w}_{2,2}u_1(1-t_1)$, where $s_1\widetilde{w}_{2,2}u_1(1-t_1)$ is inside the unit circle. Computing the residue in (3.42), we obtain

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_2(z)=\prod_{k=3}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_2\; t_2^{\frac{1}{2}\left(-\frac{\Omega}{\mu}+\lambda\right)}$$
$$\times\int_0^{\infty}du_2\;\exp(-(1-s_{2,\infty})u_2)\big(1+s_{2,\infty}u_2(1-t_2)\eta\big)^{-\frac{1}{2}(2+\lambda)}\widetilde{w}_{2,2}^{-\frac{1}{2}(1-\omega+\lambda)}(\widetilde{w}_{2,2}\partial_{\widetilde{w}_{2,2}})\widetilde{w}_{2,2}^{\frac{1}{2}(1-\omega+\lambda)}$$
$$\times\int_0^1 dt_1\; t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)}\int_0^{\infty}du_1\;\exp(-(1-s_1)u_1)\big(1+s_1u_1(1-t_1)\widetilde{w}_{2,2}\big)^{-\frac{1}{2}(1+\lambda)}$$
$$\times\widetilde{w}_{1,2}^{-\frac{1}{2}(-\omega+\lambda)}(\widetilde{w}_{1,2}\partial_{\widetilde{w}_{1,2}})\widetilde{w}_{1,2}^{\frac{1}{2}(-\omega+\lambda)}\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\left\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\,\widetilde{w}_{1,2}^{i_0}\right\}\xi^2, \tag{3.43}$$

where

$$\widetilde{w}_{1,2}=\frac{t_1p_1}{u_1(1-t_1)(1+p_1)}\bigg|_{p_1=s_1\widetilde{w}_{2,2}u_1(1-t_1)}=\frac{s_1t_1\widetilde{w}_{2,2}}{1+s_1u_1(1-t_1)\widetilde{w}_{2,2}}\,.$$

Applying the summation operator $\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}$ to (3.27), by using (3.31) and (3.32), we have

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\Big\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}y_3(z)=\prod_{k=4}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_3\; t_3^{\frac{1}{2}\left(1-\frac{\Omega}{\mu}+\lambda\right)}$$
$$\times\int_0^{\infty}du_3\;\exp(-(1-s_{3,\infty})u_3)\big(1+s_{3,\infty}u_3(1-t_3)\eta\big)^{-\frac{1}{2}(3+\lambda)}\widetilde{w}_{3,3}^{-\frac{1}{2}(2-\omega+\lambda)}(\widetilde{w}_{3,3}\partial_{\widetilde{w}_{3,3}})\widetilde{w}_{3,3}^{\frac{1}{2}(2-\omega+\lambda)}$$
$$\times\int_0^1 dt_2\; t_2^{\frac{1}{2}\left(-\frac{\Omega}{\mu}+\lambda\right)}\int_0^{\infty}du_2\;\exp(-(1-s_2)u_2)(1+s_2u_2(1-t_2)\widetilde{w}_{3,3})^{-\frac{1}{2}(2+\lambda)}\widetilde{w}_{2,3}^{-\frac{1}{2}(1-\omega+\lambda)}$$
$$\times(\widetilde{w}_{2,3}\partial_{\widetilde{w}_{2,3}})\widetilde{w}_{2,3}^{\frac{1}{2}(1-\omega+\lambda)}\int_0^1 dt_1\; t_1^{\frac{1}{2}\left(-1-\frac{\Omega}{\mu}+\lambda\right)}\int_0^{\infty}du_1\;\exp(-(1-s_1)u_1)\big(1+s_1u_1(1-t_1)\widetilde{w}_{2,3}\big)^{-\frac{1}{2}(1+\lambda)}$$
$$\times\widetilde{w}_{1,3}^{-\frac{1}{2}(-\omega+\lambda)}(\widetilde{w}_{1,3}\partial_{\widetilde{w}_{1,3}})\widetilde{w}_{1,3}^{\frac{1}{2}(-\omega+\lambda)}\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\left\{c_0z^{\lambda}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\lambda}{2})_{i_0}}{(1-\frac{\Omega}{2\mu}+\frac{\lambda}{2})_{i_0}}\,\widetilde{w}_{1,3}^{i_0}\right\}\xi^3, \tag{3.44}$$

where

$$\begin{cases} \widetilde{w}_{3,3} = \left.\dfrac{t_3 p_3}{u_3(1-t_3)(1+p_3)}\right|_{p_3=s_{3,\infty}\eta u_3(1-t_3)} = \dfrac{s_{3,\infty} t_3 \eta}{1+s_{3,\infty} u_3(1-t_3)\eta}\,, \\ \widetilde{w}_{2,3} = \left.\dfrac{t_2 p_2}{u_2(1-t_2)(1+p_2)}\right|_{p_2=s_2\widetilde{w}_{3,3}u_2(1-t_2)} = \dfrac{s_2 t_2 \widetilde{w}_{3,3}}{1+s_2 u_2(1-t_2)\widetilde{w}_{3,3}}\,, \\ \widetilde{w}_{1,3} = \left.\dfrac{t_1 p_1}{u_1(1-t_1)(1+p_1)}\right|_{p_1=s_1\widetilde{w}_{2,3}u_1(1-t_1)} = \dfrac{s_1 t_1 \widetilde{w}_{2,3}}{1+s_1 u_1(1-t_1)\widetilde{w}_{2,3}}\,. \end{cases}$$

By repeating this process for all higher terms of integral forms of the sub-summation $y_m(z)$ terms, where $m>3$, we obtain every $\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\left\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\right\}y_m(z)$ terms. Since we substitute (3.35), (3.38), (3.43), (3.44) and include all $\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod_{n=1}^{\infty}\left\{\sum_{\nu_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\right\}y_m(z)$ terms, where $m>3$, into (3.34), we obtain (3.33). □

Remark 3.6. The generating function for the first kind GCH polynomial of type 1 about $x=\infty$ as $\nu = 2\nu_j + j + 1 + \Omega/\mu$, where $j,\nu_j\in\mathbb{N}_0$, is

$$\begin{aligned}
&\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\prod y_{n=1}^{\infty}\Big\{\sum_{\nu y_n=\nu_{n-1}}^{\infty}s_n^{\nu_n}\Big\}Q^{(i)}W_{\nu_j}(\mu,\varepsilon,\Omega,\omega,\nu;z,\xi,\eta)\\
&\quad = z^{\frac{\Omega}{\mu}}\Bigg\{\prod_{k=1}^{\infty}\frac{1}{(1-s_{k,\infty})}\mathbf{A}(s_{0,\infty};\eta)+\Bigg\{\prod_{k=2}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_1\;t_1^{-\frac{1}{2}}\\
&\quad\times\int_0^{\infty}du_1\;\overleftrightarrow{\mathbf{\Gamma}}_1(s_{1,\infty};t_1,u_1,\eta)\widetilde{w}_{1,1}^{-\frac{1}{2}\left(-\omega+\frac{\Omega}{\mu}\right)}(\widetilde{w}_{1,1}\partial_{\widetilde{w}_{1,1}})\widetilde{w}_{1,1}^{\frac{1}{2}\left(-\omega+\frac{\Omega}{\mu}\right)}\mathbf{A}(s_0;\widetilde{w}_{1,1})\Bigg\}\xi\\
&\quad+\sum_{n=2}^{\infty}\Bigg\{\prod_{k=n+1}^{\infty}\frac{1}{(1-s_{k,\infty})}\int_0^1 dt_n\;t_n^{\frac{1}{2}(n-2)}\\
&\quad\times\int_0^{\infty}du_n\;\overleftrightarrow{\mathbf{\Gamma}}_n(s_{n,\infty};t_n,u_n,\eta)\widetilde{w}_{n,n}^{-\frac{1}{2}\left(n-1-\omega+\frac{\Omega}{\mu}\right)}(\widetilde{w}_{n,n}\partial_{\widetilde{w}_{n,n}})\widetilde{w}_{n,n}^{\frac{1}{2}\left(n-1-\omega+\frac{\Omega}{\mu}\right)}\\
&\quad\times\prod_{j=1}^{n-1}\Bigg\{\int_0^1 dt_{n-j}\;t_{n-j}^{\frac{1}{2}(n-j-2)}\int_0^{\infty}du_{n-j}\;\overleftrightarrow{\mathbf{\Gamma}}_{n-j}(s_{n-j};t_{n-j},u_{n-j},\widetilde{w}_{n-j+1,n})\\
&\quad\times\widetilde{w}_{n-j,n}^{-\frac{1}{2}\left(n-j-1-\omega+\frac{\Omega}{\mu}\right)}(\widetilde{w}_{n-j,n}\partial_{\widetilde{w}_{n-j,n}})\widetilde{w}_{n-j,n}^{\frac{1}{2}\left(n-j-1-\omega+\frac{\Omega}{\mu}\right)}\Bigg\}\mathbf{A}(s_0;\widetilde{w}_{1,n})\Bigg\}\xi^n\Bigg\}, \qquad (3.45)
\end{aligned}$$

where

$$\begin{cases} \nu = 2\nu_j + j + 1 + \frac{\Omega}{\mu};\quad z=\frac{1}{x}\,,\quad \xi = -\frac{\varepsilon}{\mu}\,z,\quad \eta=\frac{2}{\mu}\,z^2;\\ \overleftrightarrow{\mathbf{\Gamma}}_1(s_{1,\infty};t_1,u_1,\eta) = \exp(-(1-s_{1,\infty})u_1)(1+s_{1,\infty}u_1(1-t_1)\eta)^{-\frac{1}{2}\left(1+\frac{\Omega}{\mu}\right)};\\ \overleftrightarrow{\mathbf{\Gamma}}_n(s_{n,\infty};t_n,u_n,\eta) = \exp(-(1-s_{n,\infty})u_n)(1+s_{n,\infty}u_n(1-t_n)\eta)^{-\frac{1}{2}\left(n+\frac{\Omega}{\mu}\right)};\\ \overleftrightarrow{\mathbf{\Gamma}}_{n-j}(s_{n-j};t_{n-j},u_{n-j},\widetilde{w}_{n-j+1,n})\\ \qquad = \exp(-(1-s_{n-j})u_{n-j})(1+s_{n-j}u_{n-j}(1-t_{n-j})\widetilde{w}_{n-j+1,n})^{-\frac{1}{2}\left(n-j+\frac{\Omega}{\mu}\right)} \end{cases}$$

and

$$\begin{cases} \mathbf{A}(s_{0,\infty};\eta) = \exp(s_{0,\infty})(1+s_{0,\infty}\eta)^{-\frac{\Omega}{2\mu}}, \\ \mathbf{A}(s_0;\widetilde{w}_{1,1}) = \exp(s_0)(1+s_0\widetilde{w}_{1,1})^{-\frac{\Omega}{2\mu}}, \\ \mathbf{A}(s_0;\widetilde{w}_{1,n}) = \exp(s_0)(1+s_0\widetilde{w}_{1,n})^{-\frac{\Omega}{2\mu}}. \end{cases}$$

Proof. Replace a, b, j and z, respectively, by $-\nu_0$, $-\nu_0+1-a$, i_0 and $-z^{-1}$ in (3.15). Applying the summation operator $\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}$ to the resulting equality, we have

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(a)_{i_0}}{(1)_{i_0}}\,z^{i_0} = \sum_{\nu_0=0}^{\infty}\frac{(-s_0z)^{\nu_0}}{\nu_0!}\,U(-\nu_0,-\nu_0+1-a,-z^{-1}). \tag{3.46}$$

Replace a, b, p_l and z, respectively, by $-\nu_0$, $-\nu_0+1-a$, p and $-z^{-1}$ in (3.14):

$$U(-\nu_0,-\nu_0+1-a,-z^{-1}) = e^{\nu_0\pi i}\,\frac{\nu_0!}{2\pi i}\int_{\infty}^{(0+)} dp\; e^{\frac{p}{z}}p^{-\nu_0-1}(1+p)^{-a}. \tag{3.47}$$

Insert (3.47) into (3.46):

$$\begin{aligned}\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(a)_{i_0}}{(1)_{i_0}}\,z^{i_0} &= \frac{1}{2\pi i}\int_{\infty}^{(0+)} dp\; e^{\frac{p}{z}}p^{-1}(1+p)^{-a}\sum_{\nu_0=0}^{\infty}\left(\frac{s_0z}{p}\right)^{\nu_0} \\ &= \frac{1}{2\pi i}\int_{\infty}^{(0+)} dp\; e^{\frac{p}{z}}\frac{(1+p)^{-a}}{(p-s_0z)} = \exp(s_0)(1+s_0z)^{-a}. \end{aligned}\tag{3.48}$$

Replace s_0, a and z, respectively, by $s_{0,\infty}$, $\frac{\Omega}{2\mu}$ and η in (3.48):

$$\sum_{\nu_0=0}^{\infty}\frac{s_{0,\infty}^{\nu_0}}{\nu_0!}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}}\,\eta^{i_0} = \exp(s_{0,\infty})(1+s_{0,\infty}\eta)^{-\frac{\Omega}{2\mu}}. \tag{3.49}$$

Replace a and z, respectively, by $\frac{\Omega}{2\mu}$ and $\widetilde{w}_{1,1}$ in (3.48):

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}}\,\widetilde{w}_{1,1}^{i_0} = \exp(s_0)(1+s_0\widetilde{w}_{1,1})^{-\frac{\Omega}{2\mu}}. \tag{3.50}$$

Replace a and z, respectively, by $\frac{\Omega}{2\mu}$ and $\widetilde{w}_{1,n}$ in (3.48):

$$\sum_{\nu_0=0}^{\infty}\frac{s_0^{\nu_0}}{\nu_0!}\sum_{i_0=0}^{\nu_0}\frac{(-\nu_0)_{i_0}(\frac{\Omega}{2\mu})_{i_0}}{(1)_{i_0}}\,\widetilde{w}_{1,n}^{i_0} = \exp(s_0)(1+s_0\widetilde{w}_{1,n})^{-\frac{\Omega}{2\mu}}. \tag{3.51}$$

Putting c_0= 1 and $\lambda = \frac{\Omega}{\mu}$ in (3.33) and substitute (3.49), (3.50) and (3.51) into obtained equality we get the result. □

4 Summary

The canonical form of the biconfluent Heun equation is defined by [26, 36]

$$x\,\frac{d^2y}{dx^2} + (1+\alpha-\beta x-2x^2)\,\frac{dy}{dx} + \Big((\gamma-\alpha-2)x - \frac{1}{2}\left[\delta+(1+\alpha)\beta\right]\Big)y = 0$$

in which $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$. This equation has two singular points: of a regular singularity at $x = 0$ and of an irregular singularity at ∞. This equation is derived from the GCH equation by replacing all coefficients μ, ε, ν, Ω and ω, respectively, by -2, $-\beta$, $1 + \alpha$, $\gamma - \alpha - 2$ and $1/2(\delta/\beta + 1 + \alpha)$ in (1.1) [9].

In previous two papers of the author [11, 12], it was shown the way of deriving power series expansions in closed forms of the GCH equation about $x = 0$ by applying 3TRF for an infinite series of a polynomial of type 1 including their integral forms (each sub-integral $y_m(x)$ of a general integral $y(x) = \sum_{m=0}^{\infty} y_m(x)$ is composed of $2m$ terms of the definite integrals and m terms of the contour integrals), and generating functions for the GCH polynomials of type 1 were analyzed.

In the present paper, it is shown how one can construct power series expansions in closed forms and their integral forms of the GCH equation about $x = 0$ for an infinite series and a polynomial of type 2 by applying R3TRF. This is performed by letting B_n in the sequence c_n be the leading term in the analytic function $y(x)$. For a polynomial of type 2, we treat ω as a fixed value and μ, ε, ν, Ω as free variables.

The power series expansions and integral representations of the GCH equation about $x = 0$ for an infinite series in the present paper are equivalent to an infinite series of the GCH equation in [11,12]. In this paper, B_n is the leading term in the sequence c_n in the analytic function $y(x)$. In [11,12], A_n is the leading term in the sequence c_n in the analytic function $y(x)$.

As we can see in [11, 12], the power series expansions of the GCH equation for an infinite series and a polynomial of type 1, the denominators and numerators in all B_n terms of each sub-power series expansion $y_m(x)$, where $m = 0, 1, 2, \ldots$, arise with the Pochhammer symbol. In this paper, the denominators and numerators in all A_n terms of each sub-power series expansion $y_m(x)$ arise likewise with the Pochhammer symbol. Since we construct the power series expansions with Pochhammer symbols in numerators and denominators, we are able to describe integral representations of the GCH equation analytically. As we consider representations in closed form integrals of the GCH equation about $x = 0$ by applying either 3TRF or R3TRF, a ${}_1F_1$ function (the Kummer function of the first kind) recurs in each of its sub-integral forms. It means that we are able to transform the GCH (or BCH) functions about $x = 0$ into any well-known special functions having two term recursive relation between successive coefficients in the power series of their ODEs, because a ${}_1F_1$ function arises in each of sub-integral forms on the GCH equation. Having replaced ${}_1F_1$ functions in their integral forms by other special functions, we can rebuild the Frobenius solutions of the GCH equation about $x = 0$ in a backward.

In [12] and in this paper, it is shown how to derive generating functions for type 1 and type 2 GCH polynomials from their analytic integral representations. We are able to derive orthogonal relations, recursion relations and expectation values of physical quantities from these two generating functions; the processes for obtaining orthogonal and recursion relations of the GCH polynomials are similar to the case of a normalized wave function for the hydrogen-like atoms.*

In Section 3, we construct the Frobenius solution of the GCH equation about $x = \infty$ for the type 1 polynomial by applying 3TRF analytically [10]. Its integral representation and the generating function for the GCH polynomial are likewise derived analytically. There are no such solutions for an infinite series and for the type 2 polynomial, since the B_n term is divergent in (3.5b) and the index $n \to \infty$. Therefore, there are only two types of the analytic solution of the GCH equation about $x = \infty$ such as the type 1 and type 3 polynomials. In comparison with integral forms of the GCH polynomials of the type 1 and 2 around $x = 0$, a Tricomi's function (Kummer's function of the second kind) recurs in each of sub-integral forms of the GCH polynomial of type 1 about $x = \infty$.

*For instance, in the quantum mechanical aspects, if the eigenenergy is contained in B_n term in a 3-term recursive relation between successive coefficients of the power series expansion, we have to apply the type 1 GCH polynomial. If the eigenenergy is included in A_n term in a 3-term recursive relation, we should apply the type 2 GCH polynomial. If the first eigenenergy (mathematically, it is denoted by a spectral parameter) is included in A_n term and the second one is involved in B_n terms, we must apply the type 3 GCH polynomial. In Chapters 9 and 10 of [14] we discuss about the type 3 GCH polynomials.

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ON THE GLOBAL SOLVABILITY OF THE FIRST DARBOUX PROBLEM FOR ONE CLASS OF NONLINEAR SECOND ORDER HYPERBOLIC SYSTEMS

Abstract. The first Darboux problem for one class of nonlinear second order hyperbolic systems is considered. The questions of the existence, uniqueness and smoothness of a global solution of this problem are considered.

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Key words and phrases. Nonlinear hyperbolic system, first Darboux problem, existence, uniqueness and smoothness of a global solution.

რეზიუმე. მეორე რიგის არაწრფივ ჰიპერბოლურ სისტემათა ერთი კლასისათვის განხილულია დარბუს პირველი ამოცანა. გამოკვლეულია ამ ამოცანის გლობალური ამონახსნის არსებობის, ერთადერთობის და სიგლუვის საკითხები.

1 Statement of the problem

In a plane of variables x and t we consider the hyperbolic second order system of the type

$$Lu := u_{tt} - u_{xx} + A(x,t)u_x + B(x,t)u_t + C(x,t)u + f(x,t,u) = F(x,t), \tag{1.1}$$

where A, B, C are the given square n-th order matrices, $f = (f_1, \dots, f_n)$ and $F = (F_1, \dots, F_n)$ are the given and $u = (u_1, \dots, u_n)$ is an unknown vector functions, $n \geq 2$.

By D_T we denote an angular domain lying in the characteristic angle $\{(x,t) \in \mathbb{R}^2 : \ t > |x|\}$ and bounded both by the characteristic segment $\gamma_{1,T} : \ x = t, \ 0 \leq t \leq T$, and by the noncharacteristic segments $\gamma_{2,T} : \ x = 0, \ 0 \leq t \leq T$, and $\gamma_{3,T} : \ t = T, \ 0 \leq x \leq T$.

For system (1.1) in the domain D_T, we consider the boundary value problem which is formulated as follows: find in the domain D_T a solution $u = u(x,t)$ of system (1.1) by the boundary conditions

$$u\big|_{\gamma_{i,T}} = \varphi_i, \ \ i = 1, 2, \tag{1.2}$$

where φ_i, $i = 1, 2$, are the given on $\gamma_{i,T}$ vector functions satisfying at their common point $O = O(0,0)$ the agreement condition $\varphi_1(O) = \varphi_2(O)$. When $T = \infty$, we have $D_\infty : \ \ t > |x|, \ x > 0$, and $\gamma_{1,\infty} : \ x = t, \ 0 \leq t < \infty$, $\gamma_{2,\infty} : \ x = 0, \ 0 \leq t < \infty$. In a scalar case, where $n = 1$, problem (1.1), (1.2) is known as the first Darboux problem.

If in a linear case for a scalar hyperbolic equation the boundary value problems, in particular, the Goursat and Darboux problems, are well studied [4, 6, 7, 10, 15, 16], there arise additional difficulties and new effects in passing to a hyperbolic system. First this has been observed by A. Bitsadze [5] who constructed examples of second order hyperbolic systems for which the corresponding homogeneous characteristic problem had a finite number, and in some cases, an infinite set of linearly independent solutions. Later on, these problems for linear second order hyperbolic systems became a subject of investigations (see [8,9]). In this direction, the work [3] is also noteworthy, in which by simple examples the effect of lowest terms on the well-posedness of the problems under consideration has been revealed. As is shown in [1,2,11–13], the presence of a nonlinear term in a scalar hyperbolic equation may affect the well-posedness of the Darboux problem, when in one case this problem is globally solvable and in other cases there may arise the so-called blow-up solutions. It should be noted that the above-mentioned works do not contain linear terms involving the first order derivatives, since their presence causes difficulties in investigating the problem, and not only of technical character.

In the present work, we investigate the Darboux problem for the nonlinear system (1.1) in the presence of lowest terms involving the first order derivatives. The results obtained here are new even in the case when (1.1) is a scalar hyperbolic equation.

Definition 1.1. Let $A, B, C, F \in C(\overline{D}_T)$, $f \in C(\overline{D}_T \times \mathbb{R}^n)$ and $\varphi_i \in C^1(\gamma_{i,T})$, $i = 1, 2$. The vector function u is said to be a generalized solution of problem (1.1), (1.2) of the class C in the domain D_T, if $u \in C(\overline{D}_T)$ and there exists a sequence of vector functions $u^m \in C^2(\overline{D}_T)$ such that $u^m \to u$ and $Lu^m \to F$ in the space $C(\overline{D}_T)$, and $u^m|_{\gamma_{i,T}} \to \varphi_i$ in the space $C^1(\gamma_{i,T})$, $i = 1, 2$, as $m \to \infty$.

Remark 1.1. Obviously, the classical solution $u \in C^2(\overline{D}_T)$ of problem (1.1), (1.2) is likewise a generalized solution of that problem of the class C in the domain D_T. Moreover, if a generalized solution of problem (1.1), (1.2) of the class C in the domain D_T belongs to the space $C^2(\overline{D}_T)$, then this solution will likewise be a classical solution of that problem. It should also be noted that a generalized solution of problem (1.1), (1.2) of the class C in the domain D_T satisfies the boundary conditions (1.2) in an ordinary classical sense. In case $\varphi_2 = 0$ in Definition 1.1, we will assume that $u^m \in C_0^2(\overline{D}_T; \gamma_{2,T}) := \{v \in C^2(\overline{D}_T) : \ v|_{\gamma_{2,T}} = 0\}$.

Definition 1.2. Let $A, B, C, F \in C(\overline{D}_\infty)$, $f \in C(\overline{D}_\infty \times \mathbb{R}^n)$ and $\varphi_i \in C^1(\gamma_{i,\infty})$, $i = 1, 2$. We say that problem (1.1), (1.2) is locally solvable in the class C, if there exists the number $T_0 = T_0(F, \gamma, \gamma_2) > 0$ such that for any $T < T_0$, problem (1.1), (1.2) has at least one generalized solution of the class C in the domain D_T.

Definition 1.3. Let $A, B, C, F \in C(\overline{D}_\infty)$, $f \in C(\overline{D}_\infty \times \mathbb{R}^n)$ and $\varphi_i \in C^1(\gamma_{i,\infty})$, $i = 1, 2$. We say that problem (1.1), (1.2) is globally solvable in the class C, if for any positive number T, problem (1.1), (1.2) has at least one generalized solution of the class C in the domain D_T.

Definition 1.4. Let $A,B,C,F\in C(\overline{D}_\infty)$, $f\in C(\overline{D}_\infty\times\mathbb{R}^n)$ and $\varphi_i\in C^1(\gamma_{i,\infty})$, $i=1,2$. The vector function $u\in C(\overline{D}_\infty)$ is said to be a global generalized solution of problem (1.1), (1.2) of the class C, if for any positive number T, the vector function $U|_{D_T}$ is a generalized solution of that problem of the class C in the domain D_T.

2 A priori estimate of a solution of problem (1.1), (1.2)

Let us consider the following conditions imposed on the vector function $f=f(x,t,u)$:

$$\|f_i(x,t,u)\|\le M_1+M_2\|u\|,\ \ (x,t,u)\in\overline{D}_T\times\mathbb{R}^n,\ \ i=1,2\dots,n, \tag{2.1}$$

where $M_j=M_j(T)=const\ge 0$, $j=1,2$, $\|u\|=\sum\limits_{i=1}^n|u_i|$.

Assume

$$M_0=\sup_{(x,t)\in\overline{D}_T}\max_{1\le i,j\le n}\Big(\max\big\{|A_{i,j}(x,t)|,|B_{i,j}(x,t)|,|C_{i,j}(x,t)|\big\}\Big).$$

Lemma 2.1. *Let $F\in C(\overline{D}_T)$, $\varphi_1\in C^1(\gamma_{1,T})$, $\varphi_2=0$, and the vector function $f\in C(\overline{D}_T\times\mathbb{R}^n)$ satisfy condition (2.1). Then for a generalized solution $u=u(x,t)$ of problem (1.1), (1.2) of the class C in the domain D_T the a priori estimate*

$$\|u\|_{C(\overline{D}_T)}\le c_1\|F\|_{C(\overline{D}_T)}+c_2\|\varphi_1\|_{C^1(\gamma_{1,T})}+c_3 \tag{2.2}$$

is valid, where the nonnegative constants $c_i=c_i(M_0,M_1,M_2,T)$, $i=1,2,3$, are independent of u, F and φ_1, where $c_i>0$, $i=1,2$, and

$$\|u\|_{C(\overline{D}_T)}=\sum_{i=1}^n\|u_i\|_{C(\overline{D}_T)},\quad \|F\|_{C(\overline{D}_T)}=\sum_{i=1}^n\|F_i\|_{C(\overline{D}_T)},$$
$$\|\varphi_1\|_{C^1(\gamma_{1,T})}=\sum_{i=1}^n\|\varphi_{1i}\|_{C^1(\gamma_{1,T})}.$$

Proof. Let $u=u(x,t)$ be a generalized solution of problem (1.1), (1.2) of the class C in the domain D_T. Then, according to Definition 1.1 and Remark 1.1, the vector function $u\in C(\overline{D}_T)$ and there exists a sequence of vector functions $u^m\in C_0^2(\overline{D}_T,\gamma_{2,T})$ such that

$$\lim_{m\to\infty}\|u^m-u\|_{C(\overline{D}_T)}=0,\quad \lim_{m\to\infty}\|Lu^m-F\|_{C(\overline{D}_T)}=0, \tag{2.3}$$
$$\lim_{m\to\infty}\big\|u^m\big|_{\gamma_{1,T}}-\varphi_1\big\|_{C^1(\gamma_{1,T})}=0. \tag{2.4}$$

Consider the vector function $u^m\in C_0^2(\overline{D}_T,\gamma_{2,T})$ as a solution of the problem

$$Lu^m=F^m, \tag{2.5}$$
$$u^m\big|_{\gamma_{1,T}}=\varphi_1^m,\quad u^m\big|_{\gamma_{2,T}}=0. \tag{2.6}$$

Here

$$F^m=Lu^m,\quad \varphi_1^m=u^m\big|_{\gamma_{1,T}}. \tag{2.7}$$

Multiplying both parts of system (2.5) scalarwise by $\frac{\partial u^m}{\partial t}$ and integrating over the domain $D_\tau:=\{(x,t)\in D_T:\ t<\tau\}$, $0<\tau\le T$, we have

$$\begin{aligned}\frac12\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u^m}{\partial t},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt-\int\limits_{D_\tau}\Big(\frac{\partial^2u^m}{\partial x^2},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt+\int\limits_{D_\tau}\Big(A(x,t)\,\frac{\partial u^m}{\partial x},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt\\ +\int\limits_{D_\tau}\Big(B(x,t)\,\frac{\partial u^m}{\partial t},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt+\int\limits_{D_\tau}\Big(C(x,t)u^m,\frac{\partial u^m}{\partial t}\Big)\,dx\,dt\\ +\int\limits_{D_\tau}\Big(f(x,t,u^m),\frac{\partial u^m}{\partial t}\Big)\,dx\,dt=\int\limits_{D_\tau}\Big(F^m,\frac{\partial u^m}{\partial t}\Big)\,dx\,dt,\end{aligned} \tag{2.8}$$

where $(v,w)=\sum\limits_{i=1}^{n} v_iw_i$ is a scalar product in the space $\mathbb{R}^n$, $v=(v_1,\dots,v_n), w=(w_1,\dots,w_n)\in\mathbb{R}^n$.

Integrating by parts and applying Green's formula, we obtain

$$\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u^m}{\partial t},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt=\frac{1}{2}\int\limits_{\partial D_\tau}\Big(\frac{\partial u^m}{\partial t},\frac{\partial u^m}{\partial t}\Big)\nu_t\,ds, \tag{2.9}$$

$$\begin{aligned}-\int\limits_{D_\tau}\Big(\frac{\partial^2 u^m}{\partial x^2},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt&=-\int\limits_{\partial D_\tau}\Big(\frac{\partial u^m}{\partial x},\frac{\partial u^m}{\partial t}\Big)\nu_x\,ds+\int\limits_{D_\tau}\Big(\frac{\partial u^m}{\partial x},\frac{\partial^2 u^m}{\partial t\partial x}\Big)\,dx\,dt\\&=-\int\limits_{\partial D_\tau}\Big(\frac{\partial u^m}{\partial x},\frac{\partial u^m}{\partial t}\Big)\nu_x\,ds+\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u^m}{\partial x},\frac{\partial u^m}{\partial x}\Big)\,dx\,dt\\&=-\int\limits_{\partial D_\tau}\Big(\frac{\partial u^m}{\partial x},\frac{\partial u^m}{\partial t}\Big)\nu_x\,ds+\frac{1}{2}\int\limits_{D_\tau}\Big(\frac{\partial u^m}{\partial x},\frac{\partial u^m}{\partial x}\Big)\nu_t\,ds,\end{aligned} \tag{2.10}$$

where $\nu=(\nu_x,\nu_t)$ is the unit vector of the outer normal to the boundary ∂D_τ of the domain D_τ.

Taking into account the fact that $\partial D_\tau=\gamma_{1,\tau}\cup\gamma_{2,\tau}\cup\omega_\tau$, where $\gamma_{i,\tau}=\gamma_{i,\tau}\cap\{t\le\tau\}$, $i=1,2$, and $\omega_\tau=\partial D_\tau\cap\{t=\tau\}=\{t=\tau,\ 0\le x\le\tau\}$, we have

$$(\nu_x,\nu_t)\big|_{\gamma_{1,\tau}}=\Big(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\Big), \tag{2.11}$$

$$(\nu_x,\nu_t)\big|_{\gamma_{2,\tau}}=(-1,0),\quad (\nu_x,\nu_t)\big|_{\omega_\tau}=(0,1), \tag{2.12}$$

$$(\nu_x^2-\nu_t^2)\big|_{\gamma_{1,\tau}}=0, \tag{2.13}$$

$$\nu_t\big|_{\gamma_{1,\tau}}<0. \tag{2.14}$$

In view of (2.11)–(2.14) and the fact that $u^m|_{\gamma_{2,T}}=0$, from (2.9) and (2.10) we arrive at

$$\begin{aligned}\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u^m}{\partial t},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt&=\frac{1}{2}\int\limits_{\omega_\tau}\Big(\frac{\partial u^m}{\partial t},\frac{\partial u^m}{\partial t}\Big)\,dx+\frac{1}{2}\int\limits_{\gamma_{1,\tau}}\Big(\frac{\partial u^m}{\partial t},\frac{\partial u^m}{\partial t}\Big)\nu_t\,ds\\&=\frac{1}{2}\int\limits_{\omega_\tau}\Big(\sum_{i=1}^{n}(u^m_{it})^2\Big)\,dx+\frac{1}{2}\int\limits_{\gamma_{1,\tau}}\Big(\sum_{i=1}^{n}(u^m_{it})^2\Big)\nu_t\,ds,\end{aligned} \tag{2.15}$$

$$\begin{aligned}-\int\limits_{D_\tau}\Big(\frac{\partial^2 u^m}{\partial x^2},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt&=\frac{1}{2}\int\limits_{\omega_\tau}\Big(\sum_{i=1}^{n}(u^m_{ix})^2\Big)\,dx\\&\quad+\frac{1}{2}\int\limits_{\gamma_{1,\tau}}\Big(\sum_{i=1}^{n}(u^m_{ix})^2\Big)\nu_t\,ds-\int\limits_{\gamma_{1,\tau}}\Big(\sum_{i=1}^{n}u^m_{ix}u^m_{it}\Big)\nu_x\,ds.\end{aligned} \tag{2.16}$$

By virtue of (2.13), it follows from (2.15) and (2.16) that

$$\begin{aligned}&\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u^m}{\partial t},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt-\int\limits_{D_\tau}\Big(\frac{\partial^2 u^m}{\partial x^2},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt\\&\quad=\frac{1}{2}\int\limits_{\omega_\tau}\Big(\sum_{i=1}^{n}\big((u^m_{ix})^2+(u^m_{it})^2\big)\Big)\,dx+\int\limits_{\gamma_{1,\tau}}\frac{1}{2\nu_t}\Big(\sum_{i=1}^{n}\big[(u^m_{ix}\nu_t-u^m_{it}\nu_x)^2+(u^m_{it})^2(\nu_t^2-\nu_x^2)\big]\Big)\,ds\\&\quad=\frac{1}{2}\int\limits_{\omega_\tau}\Big(\sum_{i=1}^{n}\big((u^m_{ix})^2+(u^m_{it})^2\big)\Big)\,dx+\int\limits_{\gamma_{1,\tau}}\frac{1}{2\nu_t}\Big(\sum_{i=1}^{n}(u^m_{ix}\nu_t-u^m_{it}\nu_x)^2\Big)\,ds.\end{aligned} \tag{2.17}$$

Since $(\nu_t\,\frac{\partial}{\partial x}-\nu_x\,\frac{\partial}{\partial t})$ is the derivative to the tangent, i.e., it is an inner differential operator on $\gamma_{1,\tau}$, taking into account (2.6), we find that

$$\big|(u^m_{ix}\nu_t-u^m_{it}\nu_x)\big|_{\gamma_{1,\tau}}\big|\le\|\varphi^m_{1i}\|_{C^1(\gamma_{1,\tau})}\le\|\varphi^m_{1i}\|_{C^1(\gamma_{1,T})}. \tag{2.18}$$

In view of (2.18) and the fact that $\nu_t|_{\gamma_{1,\tau}} = -\frac{1}{\sqrt{2}}$, (2.17) yields

$$\frac{1}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u^m}{\partial t}, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt - \int\limits_{D_\tau}\Big(\frac{\partial^2 u^m}{\partial x^2}, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt$$
$$\geq \frac{1}{2}\int\limits_{\omega_\tau}\Big(\sum_{i=1}^n \big((u^m_{ix})^2 + (u^m_{it})^2\big)\Big)\,dx - \frac{1}{\sqrt{2}}\int\limits_{\gamma_{1,\tau}} \sum_{i=1}^n \|\varphi^m_i\|^2_{C^1(\gamma_{1,\tau})}\,ds$$
$$\geq \frac{1}{2}\int\limits_{\omega_\tau}\Big(\sum_{i=1}^n \big((u^m_{ix})^2 + (u^m_{it})^2\big)\Big)\,dx - \frac{\operatorname{mes}\gamma_{1,T}}{\sqrt{2}}\sum_{i=1}^n \|\varphi^m_{1i}\|^2_{C^1(\gamma_{1,T})}. \tag{2.19}$$

Let $E = E(x,t) \in C(\overline{D}_T)$ be a square matrix of order n and $u, v \in \mathbb{R}^n$.
If $m_0 = \sup\limits_{(x,t)\in\overline{D}_T}\max\limits_{1\leq i,j\leq n}|E_{ij}(x,t)|$, then

$$|(E(x,t)u,v)| \leq m_0\Big(\sum_{i=1}^n |u_i|\Big)\Big(\sum_{i=1}^n |v_i|\Big)$$
$$\leq \frac{1}{2}\,m_0\Big(\sum_{i=1}^n |u_i|\Big)^2 + \frac{1}{2}\,m_0\Big(\sum_{i=1}^n |v_i|\Big)^2 \leq \frac{n}{2}\,m_0\sum_{i=1}^n |u_i|^2 + \frac{n}{2}\,m_0\sum_{i=1}^n |v_i|^2. \tag{2.20}$$

Analogously, in view of condition (2.1), we have

$$|(f(x,t,u),v)| \leq (M_1 + M_2\|u\|)\sum_{i=1}^n |v_i|$$
$$\leq \frac{1}{2}\,(M_1 + M_2\|u\|)^2 + \frac{1}{2}\Big(\sum_{i=1}^n |v_i|\Big)^2 \leq M_1^2 + M_2^2\Big(\sum_{i=1}^n |u_i|\Big)^2 + \frac{1}{2}\,\Big(\sum_{i=1}^n |v_i|\Big)^2$$
$$\leq M_1^2 + M_2^2 n\sum_{i=1}^n |u_i|^2 + \frac{n}{2}\,\Big(\sum_{i=1}^n |v_i|^2\Big). \tag{2.21}$$

Taking into account inequalities (2.20), (2.21) and the definition of the number M_0, we obtain

$$\bigg|\int\limits_{D_\tau}\Big(A(x,t)\,\frac{\partial u^m}{\partial x}, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt + \int\limits_{D_\tau}\Big(B(x,t)\,\frac{\partial u^m}{\partial t}, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt$$
$$+\int\limits_{D_\tau}\Big(C(x,t)u^m, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt + \int\limits_{D_\tau}\Big(f(x,t,u^m), \frac{\partial u^m}{\partial t}\Big)\,dx\,dt\bigg|$$
$$\leq \int\limits_{D_\tau}\bigg(\frac{n}{2}\,M_0\sum_{i=1}^n\Big|\frac{\partial u^m_i}{\partial x}\Big|^2 + \frac{n}{2}\,M_0\sum_{i=1}^n\Big|\frac{\partial u^m_i}{\partial t}\Big|^2\bigg)\,dx\,dt$$
$$+\int\limits_{D_\tau}\bigg(nM_0\sum_{i=1}^n\Big|\frac{\partial u^m_i}{\partial t}\Big|^2\bigg)\,dx\,dt + \int\limits_{D_\tau}\bigg(\frac{n}{2}\,M_0\sum_{i=1}^n|u^m_i|^2 + \frac{n}{2}\,M_0\sum_{i=1}^n\Big|\frac{\partial u^m_i}{\partial t}\Big|^2\bigg)\,dx\,dt$$
$$+\int\limits_{D_\tau}\bigg(M_1^2 + M_2^2 n\sum_{i=1}^n|u^m_i|^2 + \frac{n}{2}\sum_{i=1}^n\Big|\frac{\partial u^m_i}{\partial t}\Big|^2\bigg)\,dx\,dt$$
$$\leq M_1^2\operatorname{mes}D_\tau + \Big(M_2^2 n + \frac{n}{2}\,M_0\Big)\int\limits_{D_\tau}\sum_{i=1}^n|(u^m_i)^2|\,dx\,dt$$
$$+\frac{n}{2}\,M_0\int\limits_{D_\tau}\sum_{i=1}^n\Big|\frac{\partial u^m_i}{\partial x}\Big|^2\,dx\,dt + \Big(2nM_0 + \frac{n}{2}\Big)\int\limits_{D_\tau}\Big|\frac{\partial u^m_i}{\partial x}\Big|^2\,dx\,dt$$

$$\le M_1^2 \operatorname{mes} D_\tau + \Big(M_2^2 n + 2nM_0 + \frac{n}{2}\Big) \int\limits_{D_\tau} \sum_{i=1}^n \bigg((u_i^m)^2 + \Big|\frac{\partial u_i^m}{\partial x}\Big|^2 + \Big|\frac{\partial u_i^m}{\partial t}\Big|^2\bigg)\,dx\,dt$$

$$= M_3 + M_4 \int\limits_{D_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\,dx\,dt, \tag{2.22}$$

where

$$M_3 = M_1^2 \operatorname{mes} D_\tau, \quad M_4 = M_2^2 n + 2nM_0 + \frac{n}{2}\,. \tag{2.23}$$

By virtue of (2.19) and (2.22), it follows from (2.8) that

$$\int\limits_{D_\tau} \Big(F^m, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt$$

$$\ge \frac{1}{2} \int\limits_{\omega_\tau} \Big(\sum_{i=1}^n \big((u_{ix}^m)^2 + (u_{it}^m)^2\big)\,dx - \frac{1}{\sqrt{2}} \operatorname{mes}\gamma_{1,T} \sum_{i=1}^n \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,T})}$$

$$- M_3 - M_4 \int\limits_{D_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\,dx\,dt,$$

whence, owing to the fact that

$$\Big(F^m, \frac{\partial u^m}{\partial t}\Big) \le \frac{1}{2}\sum_{i=1}^n (F_i^m)^2 + \frac{1}{2}\sum_{i=1}^n (u_{it}^m)^2,$$

we get

$$\frac{1}{2} \int\limits_{\omega_\tau} \Big(\sum_{i=1}^n \big((u_{ix}^m)^2 + (u_{it}^m)^2\big)\,dx \le M_4 \int\limits_{D_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\,dx\,dt$$

$$+ \frac{1}{\sqrt{2}} \operatorname{mes}\gamma_{1,T} \sum_{i=1}^n \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,T})} + M_3 + \frac{1}{2} \int\limits_{D_\tau} \sum_{i=1}^n (u_{it}^m)^2\,dx\,dt + \frac{1}{2} \int\limits_{D_\tau} \sum_{i=1}^n (F_i^m)^2\,dx\,dt$$

$$\le \Big(M_4 + \frac{1}{2}\Big) \int\limits_{D_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\,dx\,dt$$

$$+ \frac{1}{2} \int\limits_{D_\tau} \sum_{i=1}^n (F_i^m)^2\,dx\,dt + \frac{1}{\sqrt{2}} \operatorname{mes}\gamma_{1,T} \sum_{i=1}^n \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,T})} + M_3. \tag{2.24}$$

Since $u_i^m(0,t) = 0$, $i = 1,\dots,n$, we have

$$u_i^m(x,\tau) = \int\limits_0^x u_{ix}^m(\sigma,\tau)\,d\sigma, \quad 0 \le x \le \tau.$$

Hence, taking into account the Schwartz inequality, we get

$$(u_i^m)^2(x,\tau) \le \int\limits_0^x 1^2\,d\sigma \int\limits_0^x (u_{ix}^m)^2(\sigma,\tau)\,d\sigma \le x \int\limits_0^\tau (u_{ix}^m)^2(\sigma,\tau)\,d\sigma \le T \int\limits_{\omega_\tau} (u_{ix}^m)^2\,d\sigma. \tag{2.25}$$

Arguing analogously and taking into account (2.6), we obtain

$$u_i^m(x,\tau) = \varphi_{1i}^m + \int\limits_x^\tau u_{it}^m(x,s)\,ds$$

and, consequently,

$$(u_i^m)^2(x,\tau) \le 2(\varphi_{1i}^m)^2(x) + 2\Big(\int\limits_x^\tau u_{it}^m(x,s)\,ds\Big)^2 \le 2(\varphi_{1i}^m)^2(x) + 2\int\limits_x^\tau 1^2\,dt \int\limits_x^\tau (u_{it}^m)^2(x,t)\,dt$$
$$= 2(\varphi_{1i}^m)^2(x) + 2(\tau - x)\int\limits_x^\tau (u_{it}^m)^2(x,t)\,dt \le 2(\varphi_{1i}^m)^2(x) + 2T\int\limits_x^\tau (u_{it}^m)^2(x,t)\,dt. \tag{2.26}$$

Integration of inequality (2.26) yields

$$\int\limits_{\omega_\tau} (u_i^m)^2\,dx = \int\limits_0^\tau (u_i^m)^2(x,\tau)\,dx$$
$$\le 2\int\limits_x^\tau (\varphi_{1i}^m)^2(x)\,dx + 2T\int\limits_0^\tau \Big[\int\limits_x^\tau (u_{it}^m)^2(x,t)\,dt\Big]\,dx = 2\int\limits_0^\tau (\varphi_{1i}^m)^2(x)\,dx + 2T\int\limits_{D_\tau} (u_{it}^m)^2\,dx\,dt$$
$$\le 2\tau\|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,\tau})} + 2T\int\limits_{D_\tau} (u_{it}^m)^2(x,t)\,dx\,dt \le 2T\|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,\tau})} + 2T\int\limits_{D_\tau} (u_{it}^m)^2\,dx\,dt,$$

from which it follows that

$$\frac{1}{2}\int\limits_{\omega_\tau} \Big(\sum_{i=1}^n (u_i^m)^2\Big)\,dx \le T\sum_{i=1}^n \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,T})} + T\int\limits_{D_\tau} \sum_{i=1}^n (u_{it}^m)^2\,dx\,dt. \tag{2.27}$$

Combining inequalities (2.24) and (2.27), we obtain

$$\frac{1}{2}\int\limits_{\omega_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\,dx$$
$$\le \Big(M_4 + T + \frac{1}{2}\Big)\int\limits_{D_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\,dx\,dt$$
$$+ \frac{1}{2}\int\limits_{D_\tau} \sum_{i=1}^n (F_i^m)^2\,dx\,dt + \Big(\frac{1}{\sqrt{2}}\,\mathrm{mes}\,\gamma_{1,T} + T\Big)\sum_{i=1}^n \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,T})} + M_3. \tag{2.28}$$

Assume

$$w(\tau) = \int\limits_{\omega_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\,dx. \tag{2.29}$$

Taking into account that

$$\int\limits_{D_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\,dx\,dt = \int\limits_0^\tau w(\sigma)\,d\sigma,$$
$$\int\limits_{D_\tau} \sum_{i=1}^n (F_i^m)^2\,dx\,dt \le \mathrm{mes}\,D_T \sum_{i=1}^n \|F_i^m\|^2_{C(D_T)},$$

from (2.28), in view of (2.29), we get

$$w(\tau) \le M_5\int\limits_0^\tau w(\sigma)\,d\sigma + M_6\sum_{i=1}^n \|F_i^m\|^2_{C(D_T)} + M_7\sum_{i=1}^n \|\varphi_i^m\|^2_{C^1(\gamma_{1,T})} + M_8, \tag{2.30}$$

where

$$M_5 = 2M_4 + 2T + 1, \quad M_6 = \operatorname{mes} D_T, \quad M_7 = \sqrt{2}\operatorname{mes}\gamma_{1,\tau} + 2T, \quad M_8 = 2M_3. \tag{2.31}$$

According to Gronwall's lemma, it follows from (2.30) that

$$w(\tau) \le \Big[M_6 \sum_{i=1}^n \|F_i^m\|^2_{C(D_T)} + M_7 \sum_{i=1}^n \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,T})} + M_8\Big] \exp M_5 T, \ \ 0 \le \tau \le T. \tag{2.32}$$

By virtue of (2.25), (2.29) and (2.32), it is not difficult to see that

$$\begin{aligned}(u_i^m)^2(x,\tau) &\le T \int\limits_{\omega_\tau} \sum_{i=1}^n (u_{ix}^m)^2\,dx \le T w(\tau)\\ &\le T\Big[M_6 \sum_{i=1}^n \|F_i^m\|^2_{C(D_T)} + M_7 \sum_{i=1}^n \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,T})} + M_8\Big] \exp M_5 T, \ \ 0 \le \tau \le T. \end{aligned} \tag{2.33}$$

Taking into account the obvious inequality $\big(\sum\limits_{i=1}^n a_i^2\big)^{\frac12} \le \sum\limits_{i=1}^n |a_i|$, from (2.33) we obtain

$$\begin{aligned}\|u^m\|_{C(\overline{D}_T)} &= \sum_{i=1}^n \|u_i^m\|_{C(\overline{D}_T)}\\
&\le n^{\frac12}\Big(\sum_{i=1}^n \|u_i^m\|^2_{C(\overline{D}_T)}\Big)^{\frac12} = n^{\frac12}\Big(\sum_{i=1}^n \sup_{(x,t)\in D_T} |u_i^m(x,\tau)|^2\Big)^{\frac12}\\
&\le n^{\frac12}\bigg(nT\Big[M_6 \sum_{i=1}^n \|F_i^m\|^2_{C(\overline{D}_T)} + M_7 \sum_{i=1}^n \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,\tau})} + M_8\Big] \exp M_5 T\bigg)^{\frac12}\\
&\le n^{\frac12}\bigg(n^{\frac12}(TM_6)^{\frac12}\sum_{i=1}^n \|F_i^m\|_{C(\overline{D}_T)} + n^{\frac12}(TM_7)^{\frac12}\sum_{i=1}^n \|\varphi_{1i}^m\|_{C^1(\gamma_{1,\tau})} + n^{\frac12}(TM_8)^{\frac12}\bigg) \exp \frac12\, M_5 T\\
&\le n(TM_6)^{\frac12} \exp \frac12\, M_5 T \sum_{i=1}^n \|F_i^m\|_{C(\overline{D}_T)}\\
&\qquad + n(TM_7)^{\frac12} \exp \frac12\, M_5 T \sum_{i=1}^n \|\varphi_{1i}^m\|_{C(\gamma_{1,T})} + n(TM_8)^{\frac12} \exp \frac12\, M_5 T\\
&= c_1 \|F^m\|_{C(\overline{D}_T)} + c_2 \|\varphi_1^m\|_{C^1(\gamma_{1,\tau})} + c_3. \end{aligned} \tag{2.34}$$

Here

$$c_1 = n(TM_6)^{\frac12} \exp \frac12\, M_5 T, \quad c_2 = n(TM_7)^{\frac12} \exp \frac12\, M_5 T, \quad c_3 = n(TM_8)^{\frac12} \exp \frac12\, M_5 T. \tag{2.35}$$

By (2.3) and (2.4), passing in inequality (2.34) to the limit, as $m \to \infty$, we obtain an a priori estimate (2.2) in which the constants c_1, c_2 and c_3 are given by equalities (2.35), and the constants M_5, M_6, M_7 and M_8 in (2.35) are defined from (2.1), (2.23) and (2.31). In addition, $c_i > 0$, $i = 1, 2$. □

3 Reduction of problem (1.1), (1.2) to a nonlinear system of integral Volterra type equations

As a result of our passage to new independent variables ξ and η:

$$\xi = \frac12\,(t + x), \quad \eta = \frac12\,(t - x), \tag{3.1}$$

the domain D_T turns into a triangle $G_T = OP_1P_2$ of the plane $O_{\xi\eta}$, where $O = O(0,0)$, $P_1 = P_1(T,0)$, $P_2 = P_2(\frac{1}{2}T, \frac{1}{2}T)$, and problem (1.1), (1.2) can now be rewritten in the form

$$L_1 v := v_{\xi\eta} + A_1(\xi,\eta)v_\xi + B_1(\xi,\eta)v_\eta + c_1(\xi,\eta)v + f_1(\xi,\eta,v) = F_1(\xi,\eta), \quad (\xi,\eta) \in G_T, \tag{3.2}$$

$$v\big|_{OP_1:\eta=0,\ 0\le\xi\le T} = \psi_1(\xi), \quad 0 \le \xi \le T, \tag{3.3}$$

$$v\big|_{OP_2:\xi=\eta,\ 0\le\eta\le\frac{1}{2}T} = \psi_2(\eta), \quad 0 \le \eta \le T, \tag{3.4}$$

with respect to a new unknown vector function $v(\xi,\eta) = u(\xi-\eta, \xi+\eta)$. Here

$$\begin{cases} A_1(\xi,\eta) = \dfrac{1}{2}\big(A(\xi-\eta,\xi+\eta) + B(\xi-\eta,\xi+\eta)\big), \\ B_1(\xi,\eta) = \dfrac{1}{2}\big(B(\xi-\eta,\xi+\eta) - A(\xi-\eta,\xi+\eta)\big), \\ C_1(\xi,\eta) = C(\xi-\eta,\xi+\eta), \\ F_1(\xi,\eta) = F(\xi-\eta,\xi+\eta), \\ f_1(\xi,\eta,v) = f(\xi-\eta,\xi+\eta,v), \end{cases} \tag{3.5}$$

$$\psi_1(\xi) = \varphi_1(\xi), \quad \psi_2(\eta) = \varphi_2(2\eta). \tag{3.6}$$

Below, it will be assumed that $u \in C^2(\overline{D}_T)$ is a classical solution of problem (1.1), (1.2), and according to this fact, $v \in C^2(\overline{D}_T)$ is a classical solution of problem (3.2)–(3.4).

Consider first the case when in equation (3.2)

$$f_1(\xi,\eta,v) = 0, \tag{3.7}$$

and the coefficients A_1, B_1 and C_1 of that equation satisfy the following condition:

$$B_{1\eta} + A_1B_1 - C_1 = 0. \tag{3.8}$$

When conditions (3.7) and (3.8) are fulfilled, equation (3.2) can be rewritten in the form

$$\Big(\frac{\partial}{\partial\eta} + A_1\Big)\Big(\frac{\partial v}{\partial\xi} + B_1 v\Big) = F_1, \quad (\xi,\eta) \in G_T. \tag{3.9}$$

If we adopt the notation

$$w = \frac{\partial v}{\partial \xi} + B_1 v, \tag{3.10}$$

then by virtue of (3.3) and (3.9), the vector function $w = w(\xi,\eta)$ for fixed ξ will be a solution of the Cauchy problem

$$w_\eta + A_1(\xi,\eta)w = F_1(\xi,\eta), \tag{3.11}$$

$$w(\xi,0) = \psi_{1\xi}(\xi) + B_1(\xi,0)\psi_1(\xi). \tag{3.12}$$

Since under the above assumptions $A_1 = A_1(\xi,\eta) \in C(\overline{G}_T)$, therefore, as is known, there exists the fundamental matrix $X_1 = X_1(\xi,\eta)$ of the corresponding to (3.11) homogeneous system satisfying both the following matrix equality [14]

$$X_{1\eta} + A_1X_1 = 0 \tag{3.13}$$

and the condition

$$\det X_1(\xi,\eta) \neq 0, \quad (\xi,\eta) \in G_T. \tag{3.14}$$

Denote by $K = K(\xi,\eta,\zeta)$ the Cauchy matrix of order n of system (3.13) which satisfies the conditions

$$K_\eta + A_1K = 0, \tag{3.15}$$

$$K(\xi,\zeta,\zeta) = I, \tag{3.16}$$

where I is the unit matrix of order n.

As is known, the Cauchy matrix K is given by the equality

$$K(\xi,\eta,\zeta)=X_1(\xi,\eta)X_1^{-1}(\xi,\zeta), \tag{3.17}$$

where $X_1=X_1(\xi,\eta)$ is the fundamental matrix satisfying conditions (3.13), (3.14) [14].

The Cauchy matrix K for the constant matrix A_1 is given by the equality [14]

$$K(\xi,\eta,\zeta)=\exp(A_1(\zeta-\eta)). \tag{3.18}$$

By virtue of (3.15) and (3.16), the unit solution of the Cauchy problem (3.11), (3.12) is defined by the formula [14]

$$w(\xi,\eta)=K(\xi,\eta,0)\big(\psi_{1\xi}(\xi)+B_1(\xi,0)\psi_1(\xi)\big)+\int\limits_0^\eta K(\xi,\eta,\zeta)F_1(\xi,\zeta)\,d\zeta. \tag{3.19}$$

Owing to (3.18), in case the matrix A_1 is constant, formula (3.19) takes the form

$$w(\xi,\eta)=\exp(-A_1\eta)\big(\psi_{1\xi}(\xi)+B_1(\xi,0)\psi_1(\xi)\big)+\int\limits_0^\eta \exp(A_1(\zeta-\eta))F_1(\xi,\zeta)\,d\zeta. \tag{3.20}$$

Taking into account equalities (3.9)–(3.12), it follows from the above reasoning that a solution v of problem (3.2)–(3.4) satisfies the Cauchy problem

$$\frac{\partial v}{\partial\xi}+B_1v=w(\xi,\eta),\quad \eta\le\xi\le T-\eta, \tag{3.21}$$

$$v(\xi,\eta)\big|_{\xi=\eta}=\psi_2(\eta),\quad 0\le\eta\le\frac{1}{2}\,T, \tag{3.22}$$

where the vector function $w=w(\xi,\eta)$ is given by formula (3.19).

Analogously to the matrix K, we denote by $\Lambda=\Lambda(\eta,\xi,\theta)$ the Cauchy matrix of the corresponding to (3.21) homogeneous system which satisfies the conditions

$$\Lambda_\xi+B_1\Lambda=0, \tag{3.23}$$

$$\Lambda(\eta,\theta,\theta)=1, \tag{3.24}$$

and which is given by the equality

$$\Lambda(\eta,\xi,\theta)=X_2(\eta,\xi)X_2^{-1}(\eta,\theta), \tag{3.25}$$

where $X_2(\eta,\xi)$ is the fundamental matrix for the corresponding to (3.21) homogeneous system.

When the matrix B_1 is constant, the Cauchy matrix Λ is given by the equality

$$\Lambda(\eta,\xi,\theta)=\exp(B_1(\theta-\xi)). \tag{3.26}$$

Owing to (3.23) and (3.24), the unique solution of the Cauchy problem (3.21), (3.22) is defined by the formula [14]

$$v(\xi,\eta)=\Lambda(\eta,\xi,\eta)\psi_2(\eta)+\int\limits_\eta^\xi \Lambda(\eta,\xi,\theta)w(\theta,\eta)\,d\theta. \tag{3.27}$$

By (3.26), when the matrix B_1 is constant, formula (3.27) takes the form

$$v(\xi,\eta)=\exp(B_1(\eta-\xi))\psi_2(\eta)+\int\limits_\eta^\xi \exp(B_1(\theta-\xi))w(\theta,\eta)\,d\theta. \tag{3.28}$$

Substituting (3.19) for the vector function $w(\xi,\eta)$ into the right-hand side of equality (3.27), we obtain

$$\begin{aligned} v(\xi,\eta) &= \Lambda(\eta,\xi,\eta)\psi_2(\eta) \\ &\quad + \int_\eta^\xi \Lambda(\eta,\xi,\theta)\Big[K(\theta,\eta,0)\big(\psi_{1\xi}(\theta)+B_1(\theta,0)\psi_1(\theta)\big) + \int_0^\eta K(\theta,\eta,\zeta)F_1(\theta,\zeta)\,d\zeta\Big]\,d\theta \\ &= \Lambda(\eta,\xi,\eta)\psi_2(\eta) + \int_\eta^\xi \Lambda(\eta,\xi,\theta)\big[K(\theta,\eta,0)\big(\psi_{1,\xi}(\theta)+B_1(\theta,0)\psi_1(\theta)\big)\big]\,d\theta+ \\ &\quad + \int_\eta^\xi\int_0^\eta \Lambda(\eta,\xi,\theta)K(\theta,\eta,\zeta)F_1(\theta,\zeta)\,d\zeta\,d\theta, \quad (\xi,\eta)\in G_T. \end{aligned} \tag{3.29}$$

We rewrite equality (3.29) in the form

$$v(\xi,\eta) = \int_\eta^\xi\int_0^\eta R(\xi,\eta;\theta,\zeta)F_1(\theta,\zeta)\,d\zeta\,d\theta + F_2(\xi,\eta), \quad (\xi,\eta)\in G_T. \tag{3.30}$$

where

$$R(\xi,\eta;\theta,\zeta) = \Lambda(\eta,\xi,\theta)K(\theta,\eta,\zeta), \tag{3.31}$$

$$F_2(\xi,\eta) = \Lambda(\eta,\xi,\eta)\psi_2(\eta) + \int_\eta^\xi \Lambda(\eta,\xi,\theta)\big[K(\theta,\eta,0)\big(\psi_{1\xi}(\theta)+B_1(\theta,0)\psi_1(\theta)\big)\big]\,d\theta. \tag{3.32}$$

In case the matrices A_1 and B_1 are constant, by virtue of (3.18) and (3.26), equalities (3.31) and (3.32) take the form

$$R(\xi,\eta;\theta,\zeta) = \exp\big(B_1(\theta-\xi)+A_1(\zeta-\eta)\big), \tag{3.33}$$

$$\begin{aligned} F_2(\xi,\eta) &= \exp(B_1(\eta-\xi))\psi_2(\eta) \\ &\quad + \int_\eta^\xi \exp(B_1(\theta-\xi))\big[\exp(A_1\eta)\big(\psi_{1\xi}(\theta)+B_1(\theta,0)\psi_1(\theta)\big)\big]\,d\theta. \end{aligned} \tag{3.34}$$

Consider now a general case when it is not necessary for conditions (3.7) and (3.8) to be fulfilled. We rewrite system (3.2) in the form

$$\Big(\frac{\partial}{\partial\eta}+A_1\Big)\Big(\frac{\partial v}{\partial\xi}+B_1 v\Big) = (B_{1\eta}+A_1B_1-C_1)v - f_1 + F_1. \tag{3.35}$$

Then, due to representation (3.30), the classical solution of problem (3.2)–(3.4) or, what comes to the same, of problem (3.35), (3.3), (3.4), is given by the formula

$$v(\xi,\eta) = \int_\eta^\xi\int_0^\eta R(\xi,\eta;\theta,\zeta)\big[(B_{1\eta}+A_1B_1-C_1)v(\theta,\zeta)-f_1(\theta,\zeta,v)\big]\,d\zeta\,d\theta+F_3(\xi,\eta), \quad (\xi,\eta)\in G_T, \tag{3.36}$$

where

$$F_3(\xi,\eta) = \int_\eta^\xi\int_0^\eta R(\xi,\eta;\theta,\zeta)F_1(\theta,\zeta)\,d\zeta\,d\theta + F_2(\xi,\eta). \tag{3.37}$$

Remark 3.1. Equality (3.36) can be considered as a nonlinear system of integral Volterra type equations which we rewrite as follows:

$$v = L_2 v + L_3 F_1 + l_0(\psi_0, \psi_2), \tag{3.38}$$

where the operator L_2 acts according to the formula

$$(L_2 v)(\xi,\eta) = \int\limits_{\eta}^{\xi}\int\limits_{0}^{\eta} R(\xi,\eta;\theta,\zeta)\Big[(B_{1\eta} + A_1 B_1 - C_1)v(\theta,\zeta) - f_1(\theta,\zeta,v)\Big]\,d\zeta\,d\theta,\ \ (\xi,\eta)\in G_T, \tag{3.39}$$

and the operators L_3 and l_0, by virtue of (3.32) and (3.37), act by the formulas

$$(L_3 F_1)(\xi,\eta) = \int\limits_{\eta}^{\xi}\int\limits_{0}^{\eta} R(\xi,\eta;\theta,\zeta) F_1(\theta,\zeta)\,d\zeta\,d\theta, \tag{3.40}$$

$$(l_0(\psi_1,\psi_2))(\xi,\eta) = \Lambda(\eta,\xi,\eta)\psi_2(\eta) + \int\limits_{\eta}^{\xi} \Lambda(\eta,\xi,\theta)\big[K(\theta,\eta,0)\big(\psi_{1\xi}(\theta) + B_1(\theta,0)\psi_1(\theta)\big)\big], \tag{3.41}$$

where $(\xi,\eta)\in G_T$.

4 Global solvability of problem (1.1), (1.2) in the class of continuous functions

Remark 4.1. If we impose on the coefficients and on the vector function f appearing in equation (1.1) the requirements of smoothness

$$A, B \in C^2(\overline{D}_T), \quad C \in C^1(\overline{D}_T), \quad f \in C^1(\overline{D}_T\times\mathbb{R}^n), \tag{4.1}$$

and along with equalities (3.17) and (3.25) take into account the properties dealt with the smoothness of solutions of the system of ordinary differential equations, we will have [14]

$$R(\xi,\eta;\theta,\zeta) \in C^2(\overline{G}_T\times\overline{G}_T). \tag{4.2}$$

Remark 4.2. Under conditions (4.1), in view of (4.2) for the operator L_2 acting according to formula (3.39), we have

$$L_2 v \in C^{k+1}(\overline{G}_T), \ \text{ if } \ v \in C^k(\overline{G}_T), \ \ k = 0,1, \tag{4.3}$$

and, hence, the operator $L_2 : C^k(\overline{G}_T)\to C^{k+1}(\overline{G}_T)$ will be continuous.

Arguing as above, we find that

$$L_3 F_1 \in C^{k+1}(\overline{G}_T), \ \text{ if } \ F_1 \in C^k(\overline{G}_T), \ \ k = 0,1, \tag{4.4}$$

and

$$l_0(\psi_1,\psi_2) \in C^{k+1}(\overline{G}_T), \ \text{ if } \ \psi_i \in C^k(OP_i), k = 0,1,2; \ \ i = 1,2. \tag{4.5}$$

In addition, the operators $L_3 : C^k(\overline{G}_T)\to C^{k+1}(\overline{G}_T)$ and $l_0 : C^k(OP_1)\times C^k(OP_2)\to C^k(\overline{G}_T)$ will be continuous.

Remark 4.3. It can be easily verified that if u is a generalized solution of problem (1.1), (1.2) of the class C in the domain D_T, then the vector function $v(\xi,\eta) = u(\xi-\eta,\xi+\eta)$ will be a generalized solution of problem (3.2)–(3.4) of the class C in the domain G_T in the following sense: $v \in C(\overline{G}_T)$, and there exists the sequence of vector functions $v^m \in C^2(\overline{G}_T)$ such that

$$\lim_{m\to\infty} \|v^m - v\|_{C(\overline{G}_T)} = 0, \quad \lim_{m\to\infty} \|L_1 v^m - F_1\|_{C(\overline{G}_T)} = 0, \tag{4.6}$$

$$\lim_{m\to\infty} \big\| v^m\big|_{OP_i} - \psi_i\big\|_{C^1(OP_i)} = 0, \ \ i = 1,2, \tag{4.7}$$

and the converse statement holds, too.

Lemma 4.1. *Let conditions* (4.1) *be fulfilled. Then the vector function will be a generalized solution of problem* (3.2)–(3.4) *of the class C in the domain G_T if and only if v is a solution of the nonlinear system of integral Volterra type equations* (3.38) *of the class $C(\overline{G}_T)$.*

Proof. Let $v \in C(\overline{G}_T)$ be a solution of system (3.38). Since the space $C^k(\overline{G}_T)$, $k = 1, 2$, is the dense in $C(\overline{G}_T)$ and the space $C^2(OP_i)$ is the dense in $C^1(OP_i)$, $i = 1, 2$, [17], there exists the sequence of vector functions $F_{1n} \in C^1(\overline{G}_T)$ ($\psi_{in} \in C^2(OP_i)$, $i = 1, 2$) such that

$$\lim_{n\to\infty} \|F_{1n} - F_1\|_{C(\overline{G}_T)} = 0 \quad \Big(\lim_{n\to\infty} \|\psi_{in} - \psi_i\|_{C^1(OP_i)} = 0, \ \ i = 1, 2 \Big). \tag{4.8}$$

Analogously, since $v \in C(\overline{G}_T)$, there exists the sequence of vector functions $w_n \in C^2(\overline{G}_T)$ such that

$$\lim_{n\to\infty} \|w_n - v\|_{C(\overline{G}_T)} = 0. \tag{4.9}$$

Let us now introduce the following sequence of vector functions:

$$v_n = L_2 w_n + L_3 F_{1n} + l_0(\psi_{1n}, \psi_{2n}). \tag{4.10}$$

By virtue of (4.1)–(4.5), the vector function $v_n \in C^2(\overline{G}_T)$, and owing to its construction, we will have

$$v_n\big|_{OP_i} = \psi_{in}, \ \ i = 1, 2. \tag{4.11}$$

Taking into account Remark 4.2 and the limiting equalities (4.8), (4.9), we find that

$$v_n \longrightarrow \big[L_2 v + L_3 F_1 + l_0(\psi_1, \psi_2)\big] \tag{4.12}$$

in the space $C(\overline{G}_T)$, as $n \to \infty$. At the same time, by equality (3.38), we have

$$L_2 v + L_3 F_1 + l_0(\psi_1, \psi_2) = v. \tag{4.13}$$

It follows from (4.12) and (4.13) that

$$\lim_{n\to\infty} \|v_n - v\|_{C(\overline{G}_T)} = 0. \tag{4.14}$$

In view of equality (4.10) and Remark 4.2, as well as of the fact how we have obtained equality (3.30), from the representation (3.9) we get

$$\Big(\frac{\partial}{\partial\eta} + A_1\Big)\Big(\frac{\partial v_n}{\partial\xi} + B_1 v_n\Big) = (B_{1\eta} + A_1 B_1 - C_1) w_n - (f_1(\,\cdot\,, w_n)) + F_{1n}, \tag{4.15}$$

$$v_n\big|_{OP_i} = \psi_{in}, \ \ i = 1, 2. \tag{4.16}$$

By virtue of the representation of equation (3.2) by equality (3.35), from (4.15) we obtain

$$L_1 v_n = (B_{1\eta} + A_1 B_1 - C_1)(w_n - v_n) + \big(f_1(\,\cdot\,, v_n) - f_1(\,\cdot\,, w_n)\big) + F_{1n},$$

whence, in view of (4.9) and (4.14), we get

$$\lim_{n\to\infty} \|L_1 v_n - F_1\|_{C(\overline{G}_T)} = 0.$$

It follows from (4.16) and (4.8) that

$$\lim_{n\to\infty} \big\|v_n\big|_{OP_i} - \psi_i\big\|_{C^1(OP_i)} = 0, \ \ i = 1, 2.$$

The last two limiting equalities show that if $v \in C(\overline{G}_T)$ is a solution of system (3.38), then the vector function v will be a generalized solution of problem (3.2)–(3.4) of the class C in the domain G_T. Thus Lemma 4.1 is proved, since the converse statement can be easily verified. □

As is known, the space $C^1(\overline{G}_T)$ is compactly imbedded in the space $C(\overline{G}_T)$. Therefore, taking into account Remark 4.2 and considering L_2 as the operator acting in the space $C(\overline{G}_T)$ by formula (3.39), the operator

$$L_2 : C(\overline{G}_T) \longrightarrow C(\overline{G}_T)$$

will be compact. In addition, for the fixed ψ_1, ψ_2 and F_1, the operators L_3 and l_0 acting by formulas (3.40) and (3.41) are constant, and hence their sum

$$L_0 := (L_2 + L_3F_1 + l_0(\psi_1, \psi_2)) : C(\overline{G}_T) \longrightarrow C(\overline{G}_T) \tag{4.17}$$

will likewise be compact. By (4.17), system (3.38) can be rewritten in the form

$$v = L_0 v. \tag{4.18}$$

Let $v \in C(\overline{D}_T)$ be a solution of equation (4.18), and $\psi_2 = 0$. Then, since v is connected with $u \in C(\overline{G}_T)$ by the equality $v(\xi, \eta) = u(\xi - \eta, \xi + \eta)$, and u satisfies a priori estimate (2.2), in view of Lemma 4.1 and Lemma 2.1, an a priori estimate of the same type will take place likewise for the vector function v,

$$\|v\|_{C(\overline{G}_T)} \leq c_1 \|F\|_{C(\overline{G}_T)} + c_2 \|\varphi_1\|_{C^1(\gamma_{1,T})} + c_3, \tag{4.19}$$

where the constants c_i, $i = 1, 2, 3$, are defined from equalities (2.35). It should now be noted that owing to Remark 4.3 and Lemma 4.1, if $v \in C(\overline{G}_T)$ is a solution of equation $v = \tau L_0 v$, where $\tau \in [0, 1]$, then the same a priori estimate (4.19) with the constants c_1, c_2 and c_3, independent in view of (2.1), (2.23), (2.31) and (2.35) of v, F, φ_1 and τ, will be valid. Therefore, taking into account that the operator $L_0 : C(\overline{G}_T) \to C(\overline{G}_T)$ is continuous and compact, it follows from the Lere–Schauder theorem [18] that equation (4.18) has at least one solution in the space $C(\overline{G}_T)$. This, in its turn, in view of the above remarks, implies that problem (1.1), (1.2) has at least one generalized solution of the class C in the domain D_T. Thus, the following theorem is valid.

Theorem 4.1. *Let conditions* (2.1), (4.1) *and* $F \in C(\overline{D}_T)$, $\varphi_1 \in C^1(\gamma_{1,T})$, $\varphi_2 = 0$, *be fulfilled. Then problem* (1.1), (1.2) *has at least one generalized solution of the class* C *in the domain* D_T.

5 The smoothness and uniqueness of a solution of problem (1.1), (1.2). The existence of a global solution in the domain D_∞

By virtue of (4.3), (4.4) and (4.5), from Remark 4.3 and Lemma 4.1 follows

Lemma 5.1. *Let the vector function* u *be a generalized solution of problem* (1.1), (1.2) *of the class* C *in the domain* D_T *in a sense of Definition* 1.1, *and in addition, the conditions of smoothness* (4.1) *and* $F \in C^1(\overline{D}_T)$, $\varphi_1 \in C^2(\gamma_{1,T})$, $i = 1, 2$, *hold. Then* u *belongs to the class* $C^2(\overline{D}_T)$ *and is a classical solution of problem* (1.1), (1.2).

We say that the vector function $f = f(x, t, u)$ satisfies the local Lipschitz condition on the set $\overline{D}_T \times \mathbb{R}$ if

$$\big\|f(x, t, u_2) - f(x, t, u_1)\big\| \leq M(T, R)\|u_2 - u_1\|, \ \ (x, t) \in \overline{D}_T, \ \ \|u_i\| \leq R, \ \ i = 1, 2, \tag{5.1}$$

where $M = M(T, R) = const \geq 0$. Note that if $f \in C^1(\overline{D}_T \times \mathbb{R}^n)$, then condition (5.1) will automatically be fulfilled.

Lemma 5.2. *If the vector function* $f \in C(\overline{D}_T \times \mathbb{R}^n)$ *satisfies condition* (5.1), *then problem* (1.1), (1.2) *fails to have more than one generalized solution of the class* C *in the domain* D_T.

Proof. Assume that problem (1.1), (1.2) has two generalized solutions u_1 and u_2 of the class C in the domain D_T. According to Remark 1.1 and Definition 1.1, there exists a sequence of vector functions $u_j^m \in C_0^2(\overline{D}_T, \gamma_{2,T})$ such that

$$\lim_{m\to\infty} \|u_j^m - u_j\|_{C(\overline{D}_T)} = \lim_{m\to\infty} \|Lu_j^m - F\|_{C(D_T)} = \lim_{m\to\infty} \big\|u^m\big|_{\gamma_{1,T}} - \varphi_1\big\|_{C^1(\gamma_{1,T})} = 0. \tag{5.2}$$

We introduce the notation $w^m = u_2^m - u_1^m$. It is easy to verify that $w^m \in C^2(\overline{D}_T)$ is a solution of the following problem:

$$w_{tt}^m - w_{xx}^m + A(x,t)w_x^m + B(x,t)w_t^m + C(x,t)w^m + g^m = F^m, \tag{5.3}$$

$$w^m\big|_{\gamma_{1,T}} = \varphi_1^m, \quad w^m\big|_{\gamma_{2,T}} = 0. \tag{5.4}$$

Here

$$g^m = f(x,t,u_2^m) - f(x,t,u_1^m), \tag{5.5}$$

$$F^m = Lu_2^m - Lu_1^m, \tag{5.6}$$

$$\varphi_1^m = (u_2^m - u_1^m)\big|_{\gamma_{1,T}}. \tag{5.7}$$

It follows from (5.2) that there exists a number $d = const > 0$ such that it does not depend on the indices j and m, and $\|u_j^m\|_{C(\overline{D}_T)} \leq d$. Hence, by virtue of (5.1) and (5.5), we have

$$\|g^m\| \leq M(T,d)\|u_2^m - u_1^m\| = M(T,d)\|w^m\|. \tag{5.8}$$

Reasoning now for the solution w^m of problem (5.3), (5.4) in the same way as for the solution u^m of problem (2.5), (2.6), owing to (5.8), we have to take in inequalities (2.1), (2.23), (2.28), (2.30) and (2.34) the constants, corresponding to M_1, M_3, M_8 and c_3, equal to zero. Consequently, instead of inequality (2.34) we will have

$$\|w_j^m\|_{C(\overline{D}_T)} \leq \widetilde{c}_1\|F^m\|_{C(\overline{D}_T)} + \widetilde{c}_2\|\varphi_1^m\|_{C^1(\gamma_{1,T})}. \tag{5.9}$$

Here, unlike (2.35), for the constants $\widetilde{c}_1$ and $\widetilde{c}_2$ we have

$$\widetilde{c}_1 = n(TM_6)^{\frac{1}{8}} \exp\frac{1}{2}\,\widetilde{M}_5T, \quad \widetilde{c}_2 = n(TM_7)^{\frac{1}{2}} \exp\frac{1}{2}\,\widetilde{M}_5T,$$

where M_6 and M_7 are defined from (2.31) and, in view of (2.23),

$$\widetilde{M}_5 = 2\widetilde{M}_4 + 2T + 1, \quad \widetilde{M}_4 = M^2(T,d)n + 2nM_0 + \frac{n}{2}\,.$$

It follows from (5.2), (5.6) and (5.7) that

$$\begin{gathered}\lim_{m\to\infty}\|w^m\|_{C(\overline{D}_T)} = \|u_2 - u_1\|_{C(\overline{D}_T)}, \quad \lim_{m\to\infty}\|F^m\|_{C(\overline{D}_T)} = 0,\\ \lim_{m\to\infty}\|\varphi_1^m\|_{C^1(\gamma_{1,T})} = 0.\end{gathered} \tag{5.10}$$

If now we pass in inequality (5.9) to the limit, as $m \to \infty$, then due to the limiting equalities (5.10) we get $\|u_2 - u_1\|_{C(\overline{D}_T)} \leq 0$, which implies that $u_2 = u_1$. □

The consequence of Theorem 4.1 and Lemmas 5.1 and 5.2 is the following

Theorem 5.1. *Let for any positive T conditions* (2.1), (4.1) *and $F \in C^1(\overline{D}_\infty)$, $\varphi_1 \in C^2(\gamma_{1,\infty})$, $\varphi_2 = 0$ be fulfilled. Then problem* (1.1), (1.2) *has the unique classical solution $u \in C^2(\overline{D}_\infty)$ in the domain D_∞.*

Proof. It follows from Theorem 4.1 and Lemmas 5.1 and 5.2 that in the domain D_T, where $T = k \in N$, there exists the unique classical solution $u_k \in C^2(\overline{D}_k$ of problem (1.1), (1.2). In addition, $u_{k+1}|_{D_k}$ is likewise the classical solution of problem (1.1), (1.2) in the domain D_k. Therefore, by Lemma 5.2, the equality $u_{k+1}|_{D_k} = u_k$ holds. This implies that the vector function u constructed in the domain D_∞ by the rule: $u(x,t) = u_k(x,t)$, where $k = [t] + 1$, $[t]$ is an integer part of the number, and $(x,t) \in D_\infty$, is the unique classical solution of problem (1.1), (1.2) in the domain D_∞. □

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REGULARIZING PROPERTIES OF THE DOUBLE LAYER POTENTIAL OF SECOND ORDER ELLIPTIC DIFFERENTIAL OPERATORS

Abstract. We prove the validity of regularizing properties of a double layer potential associated to the fundamental solution of a nonhomogeneous second order elliptic differential operator with constant coefficients in Schauder spaces by exploiting an explicit formula for the tangential derivatives of the double layer potential itself. We also introduce ad hoc norms for kernels of integral operators in order to prove continuity results of integral operators upon variation of the kernel, which we apply to the layer potentials.

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რეზიუმე. ორმაგი ფენის პოტენციალის მხები წარმოებულის ცხადი ფორმულის გამოყენებით დამტკიცებულია იმ ორმაგი ფენის პოტენციალის რეგულარული თვისებები, რომელიც დაკავშირებულია არაერთგვაროვანი მეორე რიგის მუდმივკოეფიციენტებიანი ელიფსური დიფერენციალური ოპერატორის ფუნდამენტურ ამონახსნთან შაუდერის სივრცეებში. აგრეთვე შემოღებულია სპეციალური ნორმები ინტეგრალური ოპერატორების გულებისთვის, რათა დამტკიცებულ იქნას ინტეგრალური ოპერატორების უწყვეტობა გულის ცვლილებისას, რომელიც გამოყენებულია ორმაგი ფენის პოტენციალებისთვის.

1 Introduction

In this paper, we consider the double layer potential associated to the fundamental solution of a second order differential operator with constant coefficients. Throughout the paper, we assume that

$$n \in \mathbb{N} \setminus \{0,1\},$$

where $\mathbb{N}$ denotes a set of natural numbers including 0. Let $\alpha \in]0,1[$, $m \in \mathbb{N}\setminus\{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of the class $C^{m,\alpha}$. Let $\nu \equiv (\nu_l)_{l=1,\dots,n}$ denote the external unit normal to $\partial\Omega$. Let N_2 denote the number of multi-indices $\gamma \in \mathbb{N}^n$ with $|\gamma| \leq 2$. For each

$$\mathbf{a} \equiv (a_\gamma)_{|\gamma|\leq 2} \in \mathbb{C}^{N_2}, \tag{1.1}$$

we set

$$a^{(2)} \equiv (a_{lj})_{l,j=1,\dots,n}, \quad a^{(1)} \equiv (a_j)_{j=1,\dots,n}, \quad a \equiv a_0,$$

with $a_{lj} \equiv 2^{-1}a_{e_l+e_j}$ for $j \neq l$, $a_{jj} \equiv a_{e_j+e_j}$, and $a_j \equiv a_{e_j}$, where $\{e_j : j = 1,\dots,n\}$ is the canonical basis of $\mathbb{R}^n$. We note that the matrix $a^{(2)}$ is symmetric. Then we assume that $\mathbf{a} \in \mathbb{C}^{N_2}$ satisfies the following ellipticity assumption

$$\inf_{\xi\in\mathbb{R}^n, |\xi|=1} \operatorname{Re}\Big\{\sum_{|\gamma|=2} a_\gamma \xi^\gamma\Big\} > 0, \tag{1.2}$$

and we consider the case in which

$$a_{lj} \in \mathbb{R} \quad \forall\, l,j = 1,\dots,n. \tag{1.3}$$

Introduce the operators

$$P[\mathbf{a},D]u \equiv \sum_{l,j=1}^{n} \partial_{x_l}(a_{lj}\partial_{x_j}u) + \sum_{l=1}^{n} a_l \partial_{x_l} u + au,$$
$$B^*_\Omega v \equiv \sum_{l,j=1}^{n} \overline{a}_{jl}\nu_l \partial_{x_j} v - \sum_{l=1}^{n} \nu_l \overline{a}_l v,$$

for all $u, v \in C^2(\overline{\Omega})$, a fundamental solution $S_{\mathbf{a}}$ of $P[\mathbf{a},D]$, and the double layer potential

$$w\big[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu\big](x) \equiv \int_{\partial\Omega} \mu(y)\overline{B^*_{\Omega,y}}\big(S_{\mathbf{a}}(x-y)\big)\,d\sigma_y$$
$$= -\int_{\partial\Omega} \mu(y) \sum_{l,j=1}^{n} a_{jl}\nu_l(y)\,\frac{\partial S_{\mathbf{a}}}{\partial x_j}(x-y)\,d\sigma_y - \int_{\partial\Omega} \mu(y) \sum_{l=1}^{n} \nu_l(y) a_l S_{\mathbf{a}}(x-y)\,d\sigma_y \quad \forall\, x \in \mathbb{R}^n, \tag{1.4}$$

where the density (or the moment) μ is a function from $\partial\Omega$ to $\mathbb{C}$. Here the subscript y of $\overline{B^*_{\Omega,y}}$ means that we take y as a variable of the differential operator $\overline{B^*_{\Omega,y}}$. The role of the double layer potential in the solution of boundary value problems for the operator $P[\mathbf{a},D]$ is well known (cf. e.g., Günter [14], Kupradze, Gegelia, Basheleishvili and Burchuladze [20], Mikhlin [23]).

The analysis of the continuity and compactness properties of the integral operator associated to the double layer potential is a classical topic. In particular, it has long been known that if μ is of the class $C^{m,\alpha}$, then the restriction of the double layer potential to the sets

$$\Omega^+ \equiv \Omega, \quad \Omega^- \equiv \mathbb{R}^n \setminus \operatorname{cl}\Omega$$

can be extended to a function of $C^{m,\alpha}(\operatorname{cl}\Omega^+)$ and to a function of $C^{m,\alpha}_{\mathrm{loc}}(\operatorname{cl}\Omega^-)$, respectively (cf., e.g., Miranda [24], Wiegner [36], Dalla Riva [3], Dalla Riva, Morais and Musolino [5]).

In case $n = 3$ and Ω is of the class $C^{1,\alpha}$ and $S_{\mathbf{a}}$ is the fundamental solution of the Laplace operator, it has long been known that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is a linear and compact operator in $C^{1,\alpha}(\partial\Omega)$ and is linear and continuous from $C^0(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$ (cf. Schauder [30], [31], Miranda [24].)

In case $n = 3$, $m \geq 1$ and Ω is of the class C^{m+1} and if $P[\mathbf{a}, D]$ is the Laplace operator, Günter [14, Ch. II, § 21, Thm. 3] has proved that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is bounded from $C^{m-1,\alpha'}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$ for $\alpha' \in]\alpha, 1[$ and, accordingly, is compact in $C^{m,\alpha}(\partial\Omega)$.

Fabes, Jodeit and Rivière [12] have proved that if Ω is of the class C^1 and if $P[\mathbf{a}, D]$ is the Laplace operator, then $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is compact in $L^p(\partial\Omega)$ for $p \in]1, +\infty[$. Later, Hofmann, M. Mitrea and Taylor [16] have proved the same compactness result under more general conditions on $\partial\Omega$.

In case $n = 2$ and Ω is of the class $C^{2,\alpha}$, and if $P[\mathbf{a}, D]$ is the Laplace operator, Schippers [32] has proved that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is continuous from $C^0(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$.

In case $n = 3$ and Ω is of the class C^2, and if $P[\mathbf{a}, D]$ is the Helmholtz operator, Colton and Kress [2] have developed works of Günter [14] and Mikhlin [23] and proved that the operator $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is bounded from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ and, accordingly, is compact in $C^{1,\alpha}(\partial\Omega)$.

Wiegner [36] has proved that if $\gamma \in \mathbb{N}^n$ has odd length and Ω is of the class $C^{m,\alpha}$, then the operator with kernel $(x-y)^\gamma |x-y|^{-(n-1)-|\gamma|}$ is continuous from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\mathrm{cl}\,\Omega)$ (and a corresponding result holds for the exterior of Ω).

In case $n = 3$, $m \geq 2$ and Ω is of the class $C^{m,\alpha}$, and if $P[\mathbf{a}, D]$ is the Helmholtz operator, Kirsch [18] has proved that the operator $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is bounded from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$ and, accordingly, is compact in $C^{m,\alpha}(\partial\Omega)$.

von Wahl [35] has considered the case of Sobolev spaces and proved that if Ω is of the class C^∞ and $S_{\mathbf{a}}$ is the fundamental solution of the Laplace operator, then the double layer improves the regularity of one unit on the boundary.

Later on, Heinemann [15] has developed the ideas of von Wahl in the frame of Schauder spaces and proved that if Ω is of the class C^{m+5} and $S_{\mathbf{a}}$ is the fundamental solution of the Laplace operator, then the double layer improves the regularity of one unit on the boundary, *i.e.*, $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is linear and continuous from $C^{m,\alpha}(\partial\Omega)$ to $C^{m+1,\alpha}(\partial\Omega)$.

Maz'ya and Shaposhnikova [22] have proved that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is continuous in fractional Sobolev spaces under sharp regularity assumptions on the boundary and if $P[\mathbf{a}, D]$ is the Laplace operator.

Mitrea [26] has proved that the double layer of second order equations and systems is compact in $C^{0,\beta}(\partial\Omega)$ for $\beta \in]0, \alpha[$ and bounded in $C^{0,\alpha}(\partial\Omega)$ under the assumption that Ω is of the class $C^{1,\alpha}$. Then by exploiting a formula for the tangential derivatives such results have been extended to the compactness and boundedness results in $C^{1,\beta}(\partial\Omega)$ and $C^{1,\alpha}(\partial\Omega)$, respectively.

Mitrea, Mitrea and Verdera [28] have proved that if q is a homogeneous polynomial of odd order, then the operator with kernel $q(x-y)|x-y|^{-(n-1)-\deg(q)}$ maps $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega)$.

In this paper, of special interest are the regularizing properties of the operator $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ in Schauder spaces under the assumption that Ω is of the class $C^{m,\alpha}$. We prove our statements by exploiting tangential derivatives and an inductive argument to reduce the problem to the case of the action of $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ on $C^{0,\alpha}(\partial\Omega)$ instead of flattening the boundary with parametrization functions as done by the other authors. We mention that the idea of exploiting an inductive argument together with the formula for the tangential gradient in order to prove the continuity and compactness properties of the double layer potential has been used by Kirsch [18, Thm. 3.2] in case $n = 3$, $P[\mathbf{a}, D]$ equals the Helmholtz operator and $S_{\mathbf{a}}$ is the fundamental solution satisfying the radiation condition. The tangential derivatives of $f \in C^1(\partial\Omega)$ are defined by the equality

$$M_{lr}[f] \equiv \nu_l \frac{\partial \widetilde{f}}{\partial x_r} - \nu_r \frac{\partial \widetilde{f}}{\partial x_l} \quad \text{on} \quad \partial\Omega$$

for all $l, r \in \{1, \dots, n\}$. Here $\widetilde{f}$ denotes an extension of f to an open neighbourhood of $\partial\Omega$, and one can easily verify that $M_{lr}[f]$ is independent of the specific choice of the extension $\widetilde{f}$ of f. Then we prove an explicit formula for

$$M_{lr}\big[w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]\big](x) - w\big[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, M_{lr}[\mu]\big](x) \quad \forall\, x \in \partial\Omega \tag{1.5}$$

for all $\mu \in C^1(\partial\Omega)$ and $l, r \in \{1, \ldots, n\}$ (see formula (9.1)).

We note that Günter [14, Ch. II, § 10, (42)] presents the formula for the partial derivatives of the double layer with respect to the variables in $\mathbb{R}^n$ in case $n = 3$ and $P[\mathbf{a}, D]$ equals the Laplace operator (see (7.1) for the case of the Laplace operator). A similar formula can be found in Kupradze, Gegelia, Basheleishvili and Burchuladze [20, Ch. V, § 6, (6.11)] for the elastic double layer potential in case $n = 3$. Schwab and Wendland [33] have proved that the difference in (1.5) can be written in terms of pseudodifferential operators of order -1. Dindoš and Mitrea have proved a number of properties of the double layer potential. In particular, [7, Prop. 3.2] proves the existence of integral operators such that the gradient of the double layer potential corresponding to the Stokes system can be written as a sum of such integral operators applied to the gradient of the moment of the double layer. Duduchava, Mitrea, and Mitrea [11] analyze various properties of the tangential deriatives. Duduchava [10] investigates partial differential equations on hypersurfaces and the Bessel potential operators. In particular, [10, point B of the proof of Lem. 2.1] analyzes the commutator properties both of the Bessel potential operator and of a tangential derivative. Hofmann, Mitrea and Taylor [16, (6.2.6)] prove a general formula for the tangential derivatives of the double layer potential corresponding to the second order elliptic *homogeneous* equations and systems in explicit terms.

Formula (9.1), we have computed here, extends the formula of [21] for the Laplace operator, which has been computed with arguments akin to those of Günter [14, Ch. II, § 10, (42)], and a formula of [8] for the Helmholtz operator, and can be considered as a variant of the formula of Hofmann, Mitrea and Taylor [16, (6.2.6)] for the second order *nonhomogeneous* elliptic differential operator $P[\mathbf{a}, D]$.

Formula (9.1) involves auxiliary operators, which we analyze in Section 8. We have based our analysis of the auxiliary operators involved in formula (9.1) on the introduction of boundary norms for weakly singular kernels and on the result of the joint continuity of weakly singular integrals both on the kernel of the integral and on the functional variable of the corresponding integral operator (see Section 6). For fixed choices of the kernel and for some choices of the parameters, such lemmas are known (cf. e.g., Kirsch and Hettlich [19, Thm. 3.17, p. 121]). The authors believe that the methods of Section 6 may be applied to simplify also the exposition of other classical proofs of properties of layer potentials.

By exploiting formula (9.1), we can prove that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ induces a linear and continuous operator from $C^m(\partial\Omega)$ to the generalized Schauder space $C^{m,\omega_\alpha}(\partial\Omega)$ of functions with m-th order derivatives which satisfy a generalized ω_α-Hölder condition with

$$\omega_\alpha(r) \sim r^\alpha |\ln r| \quad \text{as} \quad r \to 0,$$

and that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ induces a linear and continuous operator from $C^{m,\beta}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$ for all $\beta \in]0, \alpha]$. In particular, the double layer potential has a regularizing effect on the boundary if Ω is of the class $C^{m,\alpha}$. As a consequence of our result, $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ induces a compact operator from $C^m(\partial\Omega)$ to itself, and from $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$ to itself, and from $C^{m,\alpha}(\partial\Omega)$ to itself when Ω is of the class $C^{m,\alpha}$.

2 Notation

We denote the norm on a normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by $\|(x, y)\|_{\mathcal{X}\times\mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. For standard definitions of Calculus in normed spaces, we refer to Deimling [6]. If A is a matrix with real or complex entries, then A^t denotes the transpose matrix of A. The set $M_n(\mathbb{R})$ denotes the set of $n \times n$ matrices with real entries. Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\mathrm{cl}\,\mathbb{D}$ denotes the closure of $\mathbb{D}$, and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$, and $\mathrm{diam}(\mathbb{D})$ denotes the diameter of $\mathbb{D}$. The symbol $|\cdot|$ denotes the Euclidean modulus in $\mathbb{R}^n$ or in $\mathbb{C}$. For all $R \in]0, +\infty[$, $x \in \mathbb{R}^n$, x_j denotes the j-th coordinate of x, and $\mathbb{B}_n(x, R)$ denotes the ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. Let Ω be an open subset of $\mathbb{R}^n$. The space of m times continuously differentiable complex-valued functions on Ω is denoted by $C^m(\Omega, \mathbb{C})$ or, more simply, by $C^m(\Omega)$. Let $s \in \mathbb{N} \setminus \{0\}$, $f \in (C^m(\Omega))^s$. Then Df denotes the Jacobian matrix of f. Let $\eta \equiv (\eta_1, \ldots, \eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \cdots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \ldots \partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\eta f$ of order $|\eta| \le m$ can be extended with continuity to $\mathrm{cl}\,\Omega$ is denoted by $C^m(\mathrm{cl}\,\Omega)$.

The subspace of $C^m(\mathrm{cl}\ \Omega)$ whose derivatives up to order m are bounded is denoted by $C^m_b(\mathrm{cl}\ \Omega)$. Then $C^m_b(\mathrm{cl}\ \Omega)$ endowed with the norm $\|f\|_{C^m_b(\mathrm{cl}\ \Omega)} \equiv \sum_{|\eta|\le m} \sup_{\mathrm{cl}\ \Omega} |D^\eta f|$ is a Banach space.

Now, let ω be a function of $]0, +\infty[$ to itself such that

$$\omega \text{ is increasing and } \lim_{r\to 0^+} \omega(r) = 0\,, \tag{2.1}$$

$$\sup_{(a,t)\in[1,+\infty[\times]0,+\infty[} \frac{\omega(at)}{a\omega(t)} < +\infty,$$

and

$$\sup_{r\in\,]0,1[} \omega^{-1}(r) r < \infty\,. \tag{2.2}$$

If f is a function from a subset $\mathbb{D}$ of $\mathbb{R}^n$ to $\mathbb{C}$, we set

$$|f : \mathbb{D}|_{\omega(\,\cdot\,)} \equiv \sup\left\{ \frac{|f(x) - f(y)|}{\omega(|x-y|)} :\ x, y \in \mathbb{D},\ x \neq y \right\}.$$

If $|f : \mathbb{D}|_{\omega(\,\cdot\,)} < \infty$, we say that the function f is $\omega(\,\cdot\,)$-Hölder continuous. Sometimes we simply write $|f|_{\omega(\,\cdot\,)}$ instead of $|f : \mathbb{D}|_{\omega(\,\cdot\,)}$. If $\omega(r) = r$ and $|f : \mathbb{D}|_{\omega(\,\cdot\,)} < \infty$, then we say that f is Lipschitz continuous and we set $\mathrm{Lip}(f) \equiv |f : \mathbb{D}|_{\omega(\,\cdot\,)}$. The subspace of $C^0(\mathbb{D})$ whose functions are $\omega(\,\cdot\,)$-Hölder continuous is denoted by $C^{0,\omega(\,\cdot\,)}(\mathbb{D})$, and the subspace of $C^0(\mathbb{D})$ whose functions are Lipschitz continuous is denoted by $\mathrm{Lip}(\mathbb{D})$.

Let Ω be an open subset of $\mathbb{R}^n$. The subspace of $C^m(\mathrm{cl}\ \Omega)$ whose functions have m-th order derivatives that are $\omega(\,\cdot\,)$-Hölder continuous is denoted by $C^{m,\omega(\,\cdot\,)}(\mathrm{cl}\ \Omega)$. We set

$$C^{m,\omega(\,\cdot\,)}_b(\mathrm{cl}\ \Omega) \equiv C^{m,\omega(\,\cdot\,)}(\mathrm{cl}\ \Omega) \cap C^m_b(\mathrm{cl}\ \Omega)\,.$$

The space $C^{m,\omega(\,\cdot\,)}_b(\mathrm{cl}\,\Omega)$, equipped with its usual norm

$$\|f\|_{C^{m,\omega(\,\cdot\,)}_b(\mathrm{cl}\ \Omega)} = \|f\|_{C^m_b(\mathrm{cl}\ \Omega)} + \sum_{|\eta|=m} |D^\eta f :\, \Omega|_{\omega(\,\cdot\,)}\,,$$

is well-known to be a Banach space.

Obviously, $C^{m,\omega(\,\cdot\,)}_b(\mathrm{cl}\ \Omega) = C^{m,\omega(\,\cdot\,)}(\mathrm{cl}\ \Omega)$ if Ω is bounded (in this case, we shall always drop the subscript b). The subspace of $C^m(\mathrm{cl}\ \Omega)$ of those functions f such that $f_{|\mathrm{cl}\,(\Omega\cap\mathbb{B}_n(0,R))} \in C^{m,\omega(\,\cdot\,)}(\mathrm{cl}\,(\Omega\cap \mathbb{B}_n(0,R)))$ for all $R \in]0, +\infty[$ is denoted by $C^{m,\omega(\,\cdot\,)}_{\mathrm{loc}}(\mathrm{cl}\ \Omega)$. Clearly, $C^{m,\omega(\,\cdot\,)}_{\mathrm{loc}}(\mathrm{cl}\ \Omega) = C^{m,\omega(\,\cdot\,)}(\mathrm{cl}\ \Omega)$ if Ω is bounded.

Of particular importance is the case in which $\omega(\,\cdot\,)$ is the function r^α for some fixed $\alpha \in]0, 1]$. In this case, we simply write $|\cdot : \mathrm{cl}\,\Omega|_\alpha$ instead of $|\cdot : \mathrm{cl}\,\Omega|_{r^\alpha}$, $C^{m,\alpha}(\mathrm{cl}\ \Omega)$ instead of $C^{m,r^\alpha}(\mathrm{cl}\ \Omega)$, and $C^{m,\alpha}_b(\mathrm{cl}\ \Omega)$ instead of $C^{m,r^\alpha}_b(\mathrm{cl}\ \Omega)$. We observe that property (2.2) implies that

$$C^{m,1}_b(\mathrm{cl}\ \Omega) \subseteq C^{m,\omega(\,\cdot\,)}_b(\mathrm{cl}\ \Omega)\,.$$

For the definition of a bounded open Lipschitz subset of $\mathbb{R}^n$, we refer, e.g., to Nečas [29, §1.3]. Let $m \in \mathbb{N} \setminus \{0\}$. We say that a bounded open subset Ω of $\mathbb{R}^n$ is of the class $C^{m,\alpha}$ if for every $P \in \partial\Omega$ there exist an open neighborhood W of P in $\mathbb{R}^n$, and a diffeomorphism $\psi \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_n, \mathbb{R}^n)$ of $\mathbb{B}_n \equiv \{x \in \mathbb{R}^n :\ |x| < 1\}$ onto W such that $\psi(0) = P$, $\psi(\{x \in \mathbb{B}_n :\ x_n = 0\}) = W \cap \partial\Omega$, $\psi(\{x \in \mathbb{B}_n :\ x_n < 0\}) = W \cap \Omega$ (ψ is said to be a parametrization of $\partial\Omega$ around P). Now, let Ω be bounded and of class $C^{m,\alpha}$. By the compactness of $\partial\Omega$ and by definition of a set of the class $C^{m,\alpha}$, there exist $P_1,\dots,P_r \in \partial\Omega$, and parametrizations $\{\psi_i\}_{i=1,\dots,r}$, with $\psi_i \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_n, \mathbb{R}^n)$ such that $\bigcup_{i=1}^r \psi_i(\{x \in \mathbb{B}_n :\ x_n = 0\}) = \partial\Omega$. Let $h \in \{1, \dots, m\}$. Let ω be as in (2.1), (2.2). Let

$$\sup_{r\in\,]0,1[} \omega^{-1}(r) r^\alpha < \infty\,. \tag{2.3}$$

We denote by $C^{h,\omega(\cdot)}(\partial\Omega)$ the linear space of functions f of $\partial\Omega$ to $\mathbb{C}$ such that $f\circ\psi_i(\cdot,0)\in C^{h,\omega(\cdot)}(\mathrm{cl}\,\mathbb{B}_{n-1})$ for all $i=1,\dots,r$, and we set

$$\|f\|_{C^{h,\omega(\cdot)}(\partial\Omega)}\equiv\sup_{i=1,\dots,r}\|f\circ\psi_i(\cdot,0)\|_{C^{h,\omega(\cdot)}(\mathrm{cl}\,\mathbb{B}_{n-1})}\quad\forall f\in C^{h,\omega(\cdot)}(\partial\Omega).$$

It is well known that by choosing a different finite family of parametrizations as $\{\psi_i\}_{i=1,\dots,r}$, we would obtain an equivalent norm. In case $\omega(\cdot)$ is the function r^α, we have the spaces $C^{h,\alpha}(\partial\Omega)$.

It is known that $(C^{h,\omega(\cdot)}(\partial\Omega),\|\cdot\|_{C^{h,\omega(\cdot)}(\partial\Omega)})$ is complete. Moreover, condition (2.3) implies that the restriction operator is linear and continuous from $C^{h,\omega(\cdot)}(\mathrm{cl}\,\Omega)$ to $C^{h,\omega(\cdot)}(\partial\Omega)$.

We denote by $d\sigma$ the area element of a manifold imbedded in $\mathbb{R}^n$ and retain the standard notation for the Lebesgue spaces.

Remark 2.1. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of the class $C^{m,\alpha}$.

Let ω be as in (2.1), (2.2). If $h\in\{1,\dots,m\}$, $h<m$, then $m-1\geq 1$ and Ω is of the class $C^{m-1,1}$, and condition (2.2) implies the validity of condition (2.3) with α replaced by 1. Thus we can consider the space $C^{h,\omega(\cdot)}(\partial\Omega)$ even if we do not assume condition (2.3). If instead of h we take m, the definition we gave requires (2.3).

Remark 2.2. Let ω be as in (2.1), $\mathbb{D}$ be a subset of $\mathbb{R}^n$ and let f be a bounded function from $\mathbb{D}$ to $\mathbb{C}$, $a\in]0,+\infty[$. Then

$$\sup_{x,y\in\mathbb{D},\,|x-y|\geq a}\frac{|f(x)-f(y)|}{\omega(|x-y|)}\leq\frac{2}{\omega(a)}\sup_{\mathbb{D}}|f|\,.$$

Thus the difficulty of estimating the Hölder quotient $\frac{|f(x)-f(y)|}{\omega(|x-y|)}$ of a bounded function f lies entirely in case $0<|x-y|<a$. Then we have the following well known extension result. For a proof, we refer to Troianiello [34, Thm. 1.3, Lem. 1.5].

Lemma 2.1. *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$, $j\in\{0,\dots,m\}$, Ω be a bounded open subset of $\mathbb{R}^n$ of the class $C^{m,\alpha}$, and let $R\in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega\subseteq\mathbb{B}_n(0,R)$. Then there exists a linear and continuous extension operator '$\sim$' of $C^{j,\alpha}(\partial\Omega)$ to $C^{j,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R))$, which takes $\mu\in C^{j,\alpha}(\partial\Omega)$ to a map $\widetilde{\mu}\in C^{j,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R))$ such that $\widetilde{\mu}_{|\partial\Omega}=\mu$ and the support of μ is compact and contained in $\mathbb{B}_n(0,R)$. The same statement holds by replacing $C^{m,\alpha}$ by C^m and $C^{j,\alpha}$ by C^j.*

Let Ω be a bounded open subset of $\mathbb{R}^n$ of the class C^1. The tangential gradient $D_{\partial\Omega}f$ of $f\in C^1(\partial\Omega)$ is defined as

$$D_{\partial\Omega}f\equiv D\widetilde{f}-(\nu\cdot D\widetilde{f})\nu\quad\text{on}\quad\partial\Omega\,,$$

where $\widetilde{f}$ is an extension of f of the class C^1 in an open neighborhood of $\partial\Omega$, and we have

$$\frac{\partial\widetilde{f}}{\partial x_r}-(\nu\cdot D\widetilde{f})\nu_r=\sum_{l=1}^{n}M_{lr}[f]\nu_l\quad\text{on}\quad\partial\Omega$$

for all $r\in\{1,\dots,n\}$. If $\mathbf{a}$ is as in (1.1), (1.2), then we also set

$$D_{\mathbf{a}}f\equiv(D_{\mathbf{a},r}f)_{r=1,\dots,n}\equiv D\widetilde{f}-\frac{D\widetilde{f}a^{(2)}\nu}{\nu^t a^{(2)}\nu}\,\nu\quad\text{on}\quad\partial\Omega\,.$$

Since

$$D_{\mathbf{a},r}f=\frac{\partial\widetilde{f}}{\partial x_r}-\frac{D\widetilde{f}a^{(2)}\nu}{\nu^t a^{(2)}\nu}\,\nu_r=\sum_{l=1}^{r}M_{lr}[f]\left(\frac{\sum_{h=1}^{n}a_{lh}\nu_h}{\nu^t a^{(2)}\nu}\right)\quad\text{on}\quad\partial\Omega\tag{2.4}$$

for all $r\in\{1,\dots,n\}$, $D_{\mathbf{a}}f$ is independent of the specific choice of the extension $\widetilde{f}$ of f. We also need the following well known consequence of the Divergence Theorem.

Lemma 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of the class C^1. If φ, $\psi \in C^1(\partial\Omega)$, then*

$$\int_{\partial\Omega} M_{lj}[\varphi]\psi \, d\sigma = -\int_{\partial\Omega} \varphi M_{lj}[\psi] \, d\sigma$$

for all $l, j \in \{1, \dots, n\}$.

Next, we introduce the following auxiliary Lemmas, whose proof is based on the definition of the norm in a Schauder space.

Lemma 2.3. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1]$. Let ω be as in (2.1), (2.2), (2.3), and let Ω be a bounded open connected subset of $\mathbb{R}^n$ of the class $C^{m,\alpha}$. Then the following statements hold:*

(i) *A function $f \in C^1(\partial\Omega)$ belongs to $C^{m,\omega(\,\cdot\,)}(\partial\Omega)$ if and only if $M_{lr}[f] \in C^{m-1,\omega(\,\cdot\,)}(\partial\Omega)$ for all $l, r \in \{1, \dots, n\}$.*

(ii) *The norm $\| \cdot \|_{C^{m,\omega(\,\cdot\,)}(\partial\Omega)}$ is equivalent to the norm on $C^{m,\omega(\,\cdot\,)}(\partial\Omega)$ defined by*

$$\|f\|_{C^0(\partial\Omega)} + \sum_{l,r=1}^{n} \big\|M_{lr}[f]\big\|_{C^{m-1,\omega(\,\cdot\,)}(\partial\Omega)} \quad \forall\, f \in C^{m,\omega(\,\cdot\,)}(\partial\Omega)\,.$$

We have the following (see also Remark 2.1)

Lemma 2.4. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1]$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$, and let $h \in \{1, \dots, m\}$. Then the following statements hold:*

(i) *Let $h < m$ and ω be as in (2.1), (2.2). Then M_{lj} is linear and continuous from $C^{h,\omega(\,\cdot\,)}(\partial\Omega)$ to $C^{h-1,\omega(\,\cdot\,)}(\partial\Omega)$ for all $l, j \in \{1, \dots, n\}$. If we further assume that ω satisfies condition (2.3), then the same statement holds also for $h = m$.*

(ii) *Let $h < m$, ω be as in (2.1), (2.2), and let $\mathbf{a}$ be as in (1.1), (1.2). Then the function from $C^{h,\omega(\,\cdot\,)}(\partial\Omega)$ to $C^{h-1,\omega(\,\cdot\,)}(\partial\Omega, \mathbb{R}^n)$, which takes f to $D_{\mathbf{a}}f$ is linear and continuous. If we further assume that ω satisfies condition (2.3), then the same statement holds also for $h = m$.*

(iii) *Let $h < m$ and ω be as in (2.1), (2.2). Then the space $C^{h,\omega(\,\cdot\,)}(\partial\Omega)$ is continuously imbedded into $C^{h-1,1}(\partial\Omega)$. If we further assume that ω satisfies condition (2.3), then the same statement holds also for $h = m$.*

(iv) *Let $h < m$. Let ψ_1, ψ_2 be as in (2.1), (2.2), and let the condition $\sup_{r \in]0,1[} \psi_2^{-1}(r)\psi_1(r) < \infty$ hold. Then $C^{h,\psi_1(\,\cdot\,)}(\partial\Omega)$ is continuously imbedded into $C^{h,\psi_2(\,\cdot\,)}(\partial\Omega)$. If we further assume that ψ_j satisfies condition (2.3) for $j \in \{1, 2\}$, then the same statement holds also for $h = m$.*

(v) *Let $h < m$. Let ψ_1, ψ_2, ψ_3 be as in (2.1), (2.2), and let the conditions $\sup_{j=1,2} \sup_{r \in]0,1[} \psi_j(r)\psi_3^{-1}(r) < \infty$ hold. Then the pointwise product is bilinear and continuous from $C^{h,\psi_1(\,\cdot\,)}(\partial\Omega) \times C^{h,\psi_2(\,\cdot\,)}(\partial\Omega)$ to $C^{h,\psi_3(\,\cdot\,)}(\partial\Omega)$. If we further assume that ψ_j satisfies condition (2.3) for $j \in \{1, 2, 3\}$, then the same statement holds also for $h = m$.*

Lemma 2.5. *Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let ψ_1, ψ_2, ψ_3 be as in (2.1), (2.2), and let the conditions $\sup_{j=1,2} \sup_{r \in]0,1[} \psi_j(r)\psi_3^{-1}(r) < \infty$ hold. Then the pointwise product is bilinear and continuous from $C^{0,\psi_1(\,\cdot\,)}(\partial\Omega) \times C^{0,\psi_2(\,\cdot\,)}(\partial\Omega)$ to $C^{0,\psi_3(\,\cdot\,)}(\partial\Omega)$.*

3 Preliminary inequalities

We first introduce the following elementary lemma on matrices.

Lemma 3.1. *Let $\Lambda \in M_n(\mathbb{R})$ be invertible. Let $|\Lambda| \equiv \sup_{|x|=1} |\Lambda x|$. Then the following statements hold:*

(i) *If $\tau_\Lambda \equiv \max\{|\Lambda|, |\Lambda^{-1}|\}$, then*

$$\tau_\Lambda^{-1}|x| \leq |\Lambda x| \leq \tau_\Lambda |x| \quad \forall\, x \in \mathbb{R}^n\,.$$

(ii) *If $r \in]0, +\infty[$, then*

$$|\Lambda^{-1}x|^{-r} \leq |\Lambda|^r |x|^{-r} \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}\,.$$

Proof. Statement (i) is well known. We now consider statement (ii). Let $x \in \mathbb{R}^n \setminus \{0\}$. Then we have

$$|x| = |\Lambda(\Lambda^{-1}x)| \leq |\Lambda|\,|\Lambda^{-1}x|\,.$$

Hence, $|\Lambda^{-1}x| \geq |\Lambda|^{-1}|x|$ and the statement follows. □

Then we introduce the following elementary lemma, which collects either the known inequalities or the variants of the known inequalities, which we will need in the sequel.

Lemma 3.2. *Let $\gamma \in \mathbb{R}$ and $\Lambda \in M_n(\mathbb{R})$ be invertible. The following statements hold:*

(i)

$$\frac{1}{2}\,|x' - y| \leq |x'' - y| \leq 2|x' - y|\,,$$

$$\frac{1}{2\tau_\Lambda^2}\,|\Lambda x' - \Lambda y| \leq |\Lambda x'' - \Lambda y| \leq 2\tau_\Lambda^2 |\Lambda x' - \Lambda y|\,,$$

for all $x', x'' \in \mathbb{R}^n$, $x' \neq x''$, $y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x' - x''|)$.

(ii)

$$|x' - y|^\gamma \leq 2^{|\gamma|}|x'' - y|^\gamma\,, \quad |x'' - y|^\gamma \leq 2^{|\gamma|}|x' - y|^\gamma\,,$$

$$|\Lambda x' - \Lambda y|^\gamma \leq (2\tau_\Lambda^2)^{|\gamma|}|\Lambda x'' - \Lambda y|^\gamma\,, \quad |\Lambda x'' - \Lambda y|^\gamma \leq (2\tau_\Lambda^2)^{|\gamma|}|\Lambda x' - \Lambda y|^\gamma\,,$$

for all $x', x'' \in \mathbb{R}^n$, $x' \neq x''$, $y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x' - x''|)$.

(iii)

$$\big||x' - y|^\gamma - |x'' - y|^\gamma\big| \leq (2^{|\gamma|} - 1)|x' - y|^\gamma \quad \forall\, y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x' - x''|)\,,$$

for all $x', x'' \in \mathbb{R}^n$, $x' \neq x''$.

(iv) *There exist m_γ, $m_\gamma(\Lambda) \in]0, +\infty[$ such that*

$$\big||x' - y|^\gamma - |x'' - y|^\gamma\big| \leq m_\gamma |x' - x''|\,|x' - y|^{\gamma - 1},$$

$$\big||\Lambda x' - \Lambda y|^\gamma - |\Lambda x'' - \Lambda y|^\gamma\big| \leq m_\gamma(\Lambda)|\Lambda x' - \Lambda x''|\,|\Lambda x' - \Lambda y|^{\gamma - 1}$$

for all $x', x'' \in \mathbb{R}^n$, $x' \neq x''$, $y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x' - x''|)$.

(v)

$$\big|\ln|x' - y| - \ln|x'' - y|\big| \leq 2|x' - x''|\,|x' - y|^{-1} \quad \forall\, y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x' - x''|)\,,$$

for all $x', x'' \in \mathbb{R}^n$, $x' \neq x''$.

Proof. The first two inequalities of statement (i) follow by the triangular inequality. Further, we have

$$|\Lambda x' - \Lambda y| \leq \tau_\Lambda |x' - y| \leq \tau_\Lambda 2|x'' - y| \leq 2\tau_\Lambda^2 |\Lambda x'' - \Lambda y|\,,$$

and thus the first of the second two inequalities of statement (i) holds true. The second of the second two inequalities of statement (i) can be proved by interchanging the roles of x' and x''.

It now suffices to prove only the second inequalities in statements (ii), (iv). Indeed, the first inequalities follow by the second ones and by the equality $\tau_\Lambda = 1$ when Λ is the identity matrix.

The first of the second inequalities in (ii) for $\gamma \geq 0$ follows by raising the inequality $|\Lambda x' - \Lambda y| \leq (2\tau_\Lambda^2)|\Lambda x'' - \Lambda y|$ of statement (i) to the power γ. For $\gamma < 0$ the same inequality follows by raising the inequality $|\Lambda x'' - \Lambda y| \leq (2\tau_\Lambda^2)|\Lambda x' - \Lambda y|$ of statement (i) to the power γ. The second of the second inequalities of (ii) can be proved by interchanging the roles of x' and x''.

Statement (iii) follows by a direct application of (ii). To prove (iv) and (v), we follow Cialdea [1, § 8]. First consider (iv) and assume that $|\Lambda x' - \Lambda y| \leq |\Lambda x'' - \Lambda y|$. By the Lagrange Theorem, there exists $\zeta \in [|\Lambda x' - \Lambda y|, |\Lambda x'' - \Lambda y|]$ such that

$$\big||\Lambda x' - \Lambda y|^\gamma - |\Lambda x'' - \Lambda y|^\gamma\big| \leq \big|\gamma|\zeta^{\gamma-1}\big|\,|\Lambda x' - \Lambda y| - |\Lambda x'' - \Lambda y|\big|\,.$$

If $\gamma \geq 1$, then the inequality $\zeta \leq |\Lambda x'' - \Lambda y|$ and (i) imply

$$\zeta^{\gamma-1} \leq |\Lambda x'' - \Lambda y|^{\gamma-1} \leq (2\tau_\Lambda^2)^{|\gamma-1|}|\Lambda x' - \Lambda y|^{\gamma-1}\,.$$

If $\gamma < 1$, then the inequalities $\zeta \geq |\Lambda x' - \Lambda y|$ and $\tau_\Lambda \geq 1$ imply

$$\zeta^{\gamma-1} \leq |\Lambda x' - \Lambda y|^{\gamma-1} \leq (2\tau_\Lambda^2)^{|\gamma-1|}|\Lambda x' - \Lambda y|^{\gamma-1}\,.$$

Then we have

$$\big||\Lambda x' - \Lambda y|^\gamma - |\Lambda x'' - \Lambda y|^\gamma\big| \leq |\gamma|(2\tau_\Lambda^2)^{|\gamma-1|}\big||\Lambda x' - \Lambda y| - |\Lambda x'' - \Lambda y|\big|\,|\Lambda x' - \Lambda y|^{\gamma-1}\,, \tag{3.1}$$

which implies the validity of (iv). Similarly, in case $|\Lambda x' - \Lambda y| > |\Lambda x'' - \Lambda y|$, we can prove that (3.1) holds with x' and x'' interchanged. Thus (ii) implies the validity of (iv).

We now consider statement (v) and assume that $|x' - y| \leq |x'' - y|$. By the Lagrange Theorem, there exists $\zeta \in [|x' - y|, |x'' - y|]$ such that

$$\big|\ln|x' - y| - \ln|x'' - y|\big| \leq \zeta^{-1}\big|\,|x' - y| - |x'' - y|\,\big| \leq \zeta^{-1}|x' - x''|\,. \tag{3.2}$$

By the above assumption, $\zeta^{-1} \leq |x' - y|^{-1}$, and thus statement (v) follows. Similarly, if $|x' - y| > |x'' - y|$, we can prove that (3.2) holds with x' and x'' interchanged and (i) implies that $\zeta^{-1} \leq |x'' - y|^{-1} \leq 2|x' - y|^{-1}$, which yields the validity of (v). □

Lemma 3.3. *Let G be a nonempty bounded subset of $\mathbb{R}^n$. Then the following statements hold:*

(i) *Let $F \in \mathrm{Lip}(\partial\mathbb{B}_n \times [0, \mathrm{diam}\,(G)])$ with*

$$\mathrm{Lip}(F) \equiv \left\{\frac{|F(\theta', r') - F(\theta'', r'')|}{|\theta' - \theta''| + |r' - r''|} :\ (\theta', r'), (\theta'', r'') \in \partial\mathbb{B}_n \times [0, \mathrm{diam}\,(G)],\ (\theta', r') \neq (\theta'', r'')\right\}.$$

Then

$$\left|F\Big(\frac{x' - y}{|x' - y|}, |x' - y|\Big) - F\Big(\frac{x'' - y}{|x'' - y|}, |x'' - y|\Big)\right| \leq \mathrm{Lip}(F)(2 + \mathrm{diam}\,(G))\,\frac{|x' - x''|}{|x' - y|} \qquad \forall\, y \in G \setminus \mathbb{B}_n\big(x', 2|x' - x''|\big) \tag{3.3}$$

for all $x', x'' \in G$, $x' \neq x''$. In particular, if $f \in C^1(\partial\mathbb{B}_n \times \mathbb{R}, \mathbb{C})$, then

$$M_{f,G} \equiv \sup\left\{\left|f\Big(\frac{x' - y}{|x' - y|}, |x' - y|\Big) - f\Big(\frac{x'' - y}{|x'' - y|}, |x'' - y|\Big)\right|\frac{|x' - y|}{|x' - x''|} :\ x', x'' \in G,\ \ x' \neq x'',\ \ y \in G \setminus \mathbb{B}_n\big(x', 2|x' - x''|\big)\right\} < \infty\,.$$

(ii) *Let W be an open neighbourhood of $\mathrm{cl}\,(G - G)$. Let $f \in C^1(W, \mathbb{C})$. Then*

$$\widetilde{M}_{f,G} \equiv \sup\left\{\big|f(x' - y) - f(x'' - y)\big|\,|x' - x''|^{-1} :\ x', x'' \in G,\ x' \neq x'',\ y \in G\right\} < \infty\,.$$

Here $G - G \equiv \{y_1 - y_2 :\ y_1, y_2 \in G\}$.

Proof. First we consider statement (i). The Lipschitz continuity of F implies that the left-hand side of (3.3) is less or equal to

$$\begin{aligned}
&\mathrm{Lip}(F)\left\{\left|\frac{x'-y}{|x'-y|}-\frac{x''-y}{|x''-y|}\right|+\big||x'-y|-|x''-y|\big|\right\}\\
&\qquad\le \mathrm{Lip}(F)\left\{|x''-y|\left|\frac{1}{|x''-y|}-\frac{1}{|x'-y|}\right|+\frac{1}{|x'-y|}\big||x''-y|-|x'-y|\big|+|x'-x''|\right\}\\
&\le \mathrm{Lip}(F)\left\{|x''-y|\,\frac{|x'-x''|}{|x''-y|\,|x'-y|}+\frac{|x'-x''|}{|x'-y|}+|x'-x''|\right\}\le \mathrm{Lip}(F)|x'-x''|\left\{\frac{2+|x'-y|}{|x'-y|}\right\},
\end{aligned}$$

and thus inequality (3.3) holds true.

Since $\partial\mathbb{B}_n\times\mathbb{R}$ is a manifold of the class C^∞ imbedded into $\mathbb{R}^{n+1}$, there exists $F\in C^1(\mathbb{R}^{n+1})$ which extends f. Since $\partial\mathbb{B}_n\times[0,\mathrm{diam}\,(G)]$ is a compact subset of $\mathbb{R}^{n+1}$, F is Lipschitz continuous on $\partial\mathbb{B}_n\times[0,\mathrm{diam}\,(G)]$, and the second part of statement (i) follows by inequality (3.3).

We now consider statement (ii). Since $f\in C^1(W,\mathbb{C})$, f is Lipschitz continuous on the compact set $\mathrm{cl}\,(G-G)$, and statement (ii) follows. □

We have the following well-known statement.

Lemma 3.4. *Let $\alpha\in]0,1]$ and Ω be a bounded open connected subset of $\mathbb{R}^n$ of the class $C^{1,\alpha}$. Then there exists $c_{\Omega,\alpha}>0$ such that*

$$\big|\nu(y)\cdot(x-y)\big|\le c_{\Omega,\alpha}|x-y|^{1+\alpha}\quad\forall\,x,y\in\partial\Omega\,.$$

Next, we introduce a list of classical inequalities which can be verified by exploiting the local parametrizations of $\partial\Omega$.

Lemma 3.5. *Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Then the following statements hold:*

(i) *Let $\gamma\in]-\infty,n-1[$. Then*

$$c'_{\Omega,\gamma}\equiv\sup_{x\in\partial\Omega}\int_{\partial\Omega}\frac{d\sigma_y}{|x-y|^\gamma}<+\infty\,.$$

(ii) *Let $\gamma\in]-\infty,n-1[$. Then*

$$c''_{\Omega,\gamma}\equiv\sup_{x',x''\in\partial\Omega,\;x'\neq x''}|x'-x''|^{-(n-1)+\gamma}\int_{\mathbb{B}_n(x',3|x'-x''|)\cap\partial\Omega}\frac{d\sigma_y}{|x'-y|^\gamma}<+\infty\,.$$

(iii) *Let $\gamma\in]n-1,+\infty[$. Then*

$$c'''_{\Omega,\gamma}\equiv\sup_{x',x''\in\partial\Omega,\;x'\neq x''}|x'-x''|^{-(n-1)+\gamma}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^\gamma}$$

is finite.

(iv)

$$c^{iv}_\Omega\equiv\sup_{x',x''\in\partial\Omega,\;0<|x'-x''|<1/e}\big|\ln|x'-x''|\big|^{-1}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{n-1}}<+\infty\,.$$

4 Preliminaries on the fundamental solution

First we introduce a formula for the fundamental solution of $P[\mathbf{a}, D]$. For this, we follow a formulation of Dalla Riva [3, Thm. 5.2, 5.3] and Dalla Riva, Morais and Musolino [5, Thm. 3.1, 3.2] (see also John [17], Miranda [24] for homogeneous operators, and Mitrea and Mitrea [27, p. 203]).

Theorem 4.1. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2). *Let* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$. *Then there exist a real analytic function* A_0 *from* $\partial\mathbb{B}_n$ *to* $\mathbb{C}$, *a real analytic function* A_1 *from* $\partial\mathbb{B}_n \times \mathbb{R}$ *to* $\mathbb{C}$, $b_0 \in \mathbb{C}$, *a real analytic function* B_1 *from* $\mathbb{R}^n$ *to* $\mathbb{C}$, $B_1(0) = 0$, *and a real analytic function* C *from* $\mathbb{R}^n$ *to* $\mathbb{C}$ *such that*

$$S_{\mathbf{a}}(x) = |x|^{2-n} A_0\Big(\frac{x}{|x|}\Big) + |x|^{3-n} A_1\Big(\frac{x}{|x|}, |x|\Big) + b_0 \ln|x| + B_1(x) \ln|x| + C(x) \tag{4.1}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$, *and both* b_0 *and* B_1 *equal zero if* n *is odd. Moreover,*

$$|x|^{2-n} A_0\Big(\frac{x}{|x|}\Big) + \delta_{2,n} b_0 \ln|x|$$

is a fundamental solution for the principal part $\sum_{l,j=1}^{n} \partial_{x_l}(a_{lj}\partial_{x_j})$ *of* $P[\mathbf{a}, D]$. *Here* $\delta_{2,n}$ *denotes the Kronecker symbol. Namely,*

$$\delta_{2,n} = 1 \ \text{ if } \ n = 2, \quad \delta_{2,n} = 0 \ \text{ if } \ n > 2\,.$$

Corollary 4.1. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2). *Let* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$. *Then the following statements hold:*

(i) *If* $n \geq 3$, *then there exists one and only one fundamental solution of the principal part* $\sum_{l,j=1}^{n} \partial_{x_l}(a_{lj}\partial_{x_j})$ *of* $P[\mathbf{a}, D]$ *which is positively homogeneous of degree* $2 - n$ *in* $\mathbb{R}^n \setminus \{0\}$.

(ii) *If* $n = 2$, *then there exists one and only one fundamental solution* $S(x)$ *of the principal part* $\sum_{l,j=1}^{n} \partial_{x_l}(a_{lj}\partial_{x_j})$ *of* $P[\mathbf{a}, D]$ *such that*

$$\beta_0 \equiv \lim_{x\to 0} \frac{S(x)}{\ln|x|} \in \mathbb{C}\,, \qquad \int_{\partial\mathbb{B}_n} S\, d\sigma = 0\,,$$

and $S(x) - \beta_0 \ln|x|$ *is positively homogeneous of degree* 0 *in* $\mathbb{R}^n \setminus \{0\}$.

Proof. We retain the notation of Theorem 4.1. We first consider statement (i). By Theorem 4.1, the function $|x|^{2-n} A_0(\frac{x}{|x|})$ is a fundamental solution of the principal part of $P[\mathbf{a}, D]$ and is, clearly, positively homogeneous of degree $2-n$. Now assume that u is a fundamental solution of the principal part of $P[\mathbf{a}, D]$ and u is positively homogeneous of degree $2 - n$ in $\mathbb{R}^n \setminus \{0\}$. Then the difference

$$w(x) \equiv |x|^{2-n} A_0\Big(\frac{x}{|x|}\Big) - u(x)$$

defines an entire real analytic function in $\mathbb{R}^n$ and is positively homogeneous of degree $2-n$ in $\mathbb{R}^n \setminus \{0\}$. In particular,

$$\lambda^{n-2} w(\lambda x) = w(x) \quad \forall\, (\lambda, x) \in]0, +\infty[\, \times (\mathbb{R}^n \setminus \{0\})\,,$$

and, accordingly,

$$\lambda^{(n-2)+|\beta|} D^\beta w(\lambda x) = D^\beta w(x) \quad \forall\, (\lambda, x) \in]0, +\infty[\, \times (\mathbb{R}^n \setminus \{0\})$$

for all $\beta \in \mathbb{N}^n$. Then by letting λ tend to 0^+, we obtain $D^\beta w(0) = 0$ for all $\beta \in \mathbb{N}^n$. Since w is real analytic, we deduce that w is equal to 0 in $\mathbb{R}^n$ and thus statement (i) holds.

Now assume that $n = 2$. By Theorem 4.1, the function

$$S(x) \equiv A_0\Big(\frac{x}{|x|}\Big) - \frac{1}{2\pi}\int\limits_{\partial\mathbb{B}_n} A_0\, d\sigma + b_0 \ln|x|$$

is a fundamental solution of the principal part of $P[\mathbf{a}, D]$ and satisfies the conditions of statement (ii). Suppose that u is another fundamental solution. Then the difference

$$w(x) \equiv A_0\Big(\frac{x}{|x|}\Big) - \frac{1}{2\pi}\int\limits_{\partial\mathbb{B}_n} A_0\, d\sigma + b_0 \ln|x| - u(x)$$

defines an entire real analytic function in $\mathbb{R}^n$ and we have

$$0 = \lim_{x\to 0}\frac{w(x)}{\ln|x|} = \lim_{x\to 0}\frac{A_0(\frac{x}{|x|}) - \frac{1}{2\pi}\int_{\partial\mathbb{B}_n} A_0\, d\sigma}{\ln|x|} + b_0 - \lim_{x\to 0}\frac{u(x)}{\ln|x|}\,,$$

and, accordingly,

$$b_0 = \lim_{x\to 0}\frac{u(x)}{\ln|x|} \equiv \beta_0 \in \mathbb{C}\,.$$

Then our assumption implies that the real analytic function

$$u(x) - \beta_0 \ln|x| = u(x) - b_0 \ln|x|$$

is positively homogeneous of degree 0 in $\mathbb{R}^n \setminus \{0\}$. Hence, there exists a function g_0 from $\partial\mathbb{B}_n$ to $\mathbb{C}$ such that

$$u(x) - b_0 \ln|x| = g_0\Big(\frac{x}{|x|}\Big) \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}\,.$$

In particular, g_0 is real analytic and

$$\begin{aligned} w(x) &= A_0\Big(\frac{x}{|x|}\Big) - \frac{1}{2\pi}\int\limits_{\partial\mathbb{B}_n} A_0\, d\sigma + b_0\ln|x| - \Big(g_0\Big(\frac{x}{|x|}\Big) + b_0 \ln|x|\Big) \\ &= A_0\Big(\frac{x}{|x|}\Big) - \frac{1}{2\pi}\int\limits_{\partial\mathbb{B}_n} A_0\, d\sigma - g_0\Big(\frac{x}{|x|}\Big)\,. \end{aligned}$$

Moreover, w must be positively homogeneous of degree 0 in $\mathbb{R}^n \setminus \{0\}$. Since w is continuous at 0, w must be constant in the whole $\mathbb{R}^n$. Since

$$\int\limits_{\partial\mathbb{B}_n} w\, d\sigma = \int\limits_{\partial\mathbb{B}_n} S\, d\sigma - \int\limits_{\partial\mathbb{B}_n} u\, d\sigma = 0,$$

such a constant must equal 0 and thus

$$A_0\Big(\frac{x}{|x|}\Big) - \frac{1}{2\pi}\int\limits_{\partial\mathbb{B}_n} A_0\, d\sigma = g_0\Big(\frac{x}{|x|}\Big) \ \text{ for all } \ x \in \mathbb{R}^n \setminus \{0\}.$$

Hence,

$$u(x) = A_0\Big(\frac{x}{|x|}\Big) - \frac{1}{2\pi}\int\limits_{\partial\mathbb{B}_n} A_0\, d\sigma + b_0 \ln|x|$$

and statement (ii) follows. □

We can introduce the following

Definition 4.1. Let $\mathbf{a}$ be as in (1.1), (1.2). We define the normalized fundamental solution of the principal part of $P[\mathbf{a}, D]$, to be the only fundamental solution of Corollary 4.1.

By Theorem 4.1 and Corollary 4.1, the normalized fundamental solution of the principal part of $P[\mathbf{a}, D]$ equals

$$|x|^{2-n} A_0\Big(\frac{x}{|x|}\Big)$$

if $n \geq 3$, and

$$A_0\Big(\frac{x}{|x|}\Big) - \frac{1}{2\pi}\int_{\partial\mathbb{B}_n} A_0\, d\sigma + b_0 \ln|x|$$

if $n = 2$, where A_0 and b_0 are as in Theorem 4.1. We now see that if the principal coefficients of $P[\mathbf{a}, D]$ are real, then the normalized fundamental solution of the principal part of $P[\mathbf{a}, D]$ has a very specific form. To do so, we introduce the fundamental solution S_n of the Laplace operator. Namely, we set

$$S_n(x) \equiv \begin{cases} \frac{1}{s_n} \ln|x| & \forall\, x \in \mathbb{R}^n \setminus \{0\}, \text{ if } n = 2\,, \\ \frac{1}{(2-n)s_n} |x|^{2-n} & \forall\, x \in \mathbb{R}^n \setminus \{0\}, \text{ if } n > 2\,, \end{cases}$$

where s_n denotes the $(n-1)$-dimensional measure of $\partial\mathbb{B}_n$. Then we have the following elementary statement, which can be verified by the chain rule and by Corollary 4.1 (cf. e.g., Dalla Riva [4]).

Lemma 4.1. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3). *Then there exists an invertible matrix* $T \in M_n(\mathbb{R})$ *such that*

$$a^{(2)} = TT^t \tag{4.2}$$

and the function

$$S_{a^{(2)}}(x) \equiv \frac{1}{\sqrt{\det a^{(2)}}} S_n(T^{-1}x) \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}\,,$$

coincides with the normalized fundamental solution of the principal part of $P[\mathbf{a}, D]$ *if* $n \geq 3$, *and coincides with the normalized fundamental solution of the principal part of* $P[\mathbf{a}, D]$ *up to an additive constant if* $n = 2$.

Theorem 4.1, Corollary 4.1 and Lemma 4.1 imply the validity of the following

Corollary 4.2. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $T \in M_n(\mathbb{R})$ *be as in* (4.2) *and let* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$.

Then there exist a real analytic function A_1 *from* $\partial\mathbb{B}_n \times \mathbb{R}$ *to* $\mathbb{C}$, *a real analytic function* B_1 *from* $\mathbb{R}^n$ *to* $\mathbb{C}$, $B_1(0) = 0$, *and a real analytic function* C *from* $\mathbb{R}^n$ *to* $\mathbb{C}$ *such that*

$$S_{\mathbf{a}}(x) = \frac{1}{\sqrt{\det a^{(2)}}} S_n(T^{-1}x) + |x|^{3-n} A_1\Big(\frac{x}{|x|}, |x|\Big) + \big(B_1(x) + b_0(1 - \delta_{2,n})\big) \ln|x| + C(x)\,, \tag{4.3}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$, *and both* b_0 *and* B_1 *equal zero if* n *is odd. Moreover,*

$$\frac{1}{\sqrt{\det a^{(2)}}} S_n(T^{-1}x)$$

is a fundamental solution for the principal part of $P[\mathbf{a}, D]$.

Next we prove the following technical statement.

Lemma 4.2. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$ *and let* G *be a nonempty bounded subset of* $\mathbb{R}^n$.

(i) *Let* $\gamma \in [0, 1[$. *Then*

$$C_{0,S_{\mathbf{a}},G,n-1-\gamma} \equiv \sup_{0<|x|\leq \operatorname{diam}(G)} |x|^{n-1-\gamma} |S_{\mathbf{a}}(x)| < +\infty\,. \tag{4.4}$$

If $n > 2$, *then* (4.4) *holds also for* $\gamma = 1$.

(ii)

$$\widetilde{C}_{0,S_{\mathbf{a}},G} \equiv \sup\biggl\{ \frac{|x'-y|^{n-1}}{|x'-x''|} \bigl|S_{\mathbf{a}}(x'-y) - S_{\mathbf{a}}(x''-y)\bigr| : \\ x',x'' \in G, x' \neq x'',\ y \in G \setminus \mathbb{B}_n(x', 2|x'-x''|)\biggr\} < \infty\,.$$

Proof. Statement (i) is an immediate consequence of formula (4.1). Now prove statement (ii). For this, we resort to formula (4.1) and set

$$A(\theta,r) \equiv A_0(\theta) + rA_1(\theta,r) \quad \forall\,(\theta,r) \in \partial\mathbb{B}_n \times \mathbb{R},$$
$$B(x) \equiv b_0 + B_1(x) \quad \forall\, x \in \mathbb{R}^n\,.$$

Then Lemmas 3.2 and 3.3 imply

$$\begin{aligned}
\bigl|S_{\mathbf{a}}(x'-y) - S_{\mathbf{a}}(x''-y)\bigr| &\leq |x'-y|^{2-n}\left|A\Bigl(\frac{x'-y}{|x'-y|}, |x'-y|\Bigr) - A\Bigl(\frac{x''-y}{|x''-y|}, |x''-y|\Bigr)\right| \\
&\quad + \left|A\Bigl(\frac{x''-y}{|x''-y|}, |x''-y|\Bigr)\right| \bigl||x'-y|^{2-n} - |x''-y|^{2-n}\bigr| + \bigl|\ln|x'-y|\bigr|\,\bigl|B(x'-y) - B(x''-y)\bigr| \\
&\quad + |B(x''-y)|\,\bigl|\ln|x'-y| - \ln|x''-y|\bigr| + \bigl|C(x'-y) - C(x''-y)\bigr| \\
&\leq |x'-y|^{2-n} M_{A,G}\,\frac{|x'-x''|}{|x'-y|} + \Bigl(\sup_{\partial\mathbb{B}_n\times[0,\mathrm{diam}\,(G)]} |A|\Bigr) m_{2-n}\,\frac{|x'-x''|}{|x'-y|^{n-1}} \\
&\quad + \bigl|\ln|x'-y|\bigr|\widetilde{M}_{B,G}|x'-x''| + \sup_{G-G}|B|\,2\,\frac{|x'-x''|}{|x'-y|} + \widetilde{M}_{C,G}|x'-x''|\,.
\end{aligned}$$

Since A is continuous on the compact set $\partial\mathbb{B}_n \times [0, \mathrm{diam}\,(G)]$, and B and C are continuous on the compact set $\mathrm{cl}\,(G-G)$, there exists $c > 0$ such that

$$\begin{aligned}
\bigl|S_{\mathbf{a}}(x'-y) - S_{\mathbf{a}}(x''-y)\bigr| &\leq c|x'-x''|\Bigl\{|x'-y|^{1-n} + \frac{1}{|x'-y|} + \ln|x'-y| + 1\Bigr\} \\
&\leq c|x'-x''|\,|x'-y|^{1-n}\Bigl\{1 + |x'-y|^{n-2} + |x'-y|^{n-1}\ln|x'-y| + |x'-y|^{n-1}\Bigr\}\,,
\end{aligned}$$

and thus statement (ii) holds. □

Lemma 4.3. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $T \in M_n(\mathbb{R})$ *be as in* (4.2). *Let* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$, B_1, C *be as in Corollary* 4.2, *and let* G *be a nonempty bounded subset of* $\mathbb{R}^n$. *Then the following statements hold:*

(i) *There exists a real analytic function* A_2 *from* $\partial\mathbb{B}_n \times \mathbb{R}$ *to* $\mathbb{C}^n$ *such that*

$$DS_{\mathbf{a}}(x) = \frac{1}{s_n\sqrt{\det a^{(2)}}}\,|T^{-1}x|^{-n}x^t(a^{(2)})^{-1} \\ + |x|^{2-n}A_2\Bigl(\frac{x}{|x|}, |x|\Bigr) + DB_1(x)\ln|x| + DC(x) \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}\,. \tag{4.5}$$

(ii)

$$C_{1,S_{\mathbf{a}},G} \equiv \sup_{0<|x|\leq \mathrm{diam}\,(G)} |x|^{n-1}|DS_{\mathbf{a}}(x)| < +\infty\,.$$

(iii)

$$\widetilde{C}_{1,S_{\mathbf{a}},G} \equiv \sup\biggl\{ \frac{|x'-y|^{n}}{|x'-x''|} \bigl|DS_{\mathbf{a}}(x'-y) - DS_{\mathbf{a}}(x''-y)\bigr| : \\ x',x'' \in G, x' \neq x'',\ y \in G \setminus \mathbb{B}_n(x', 2|x'-x''|)\biggr\} < \infty\,.$$

Proof. By formula (4.3) and by the chain rule, we have

$$DS_{\mathbf{a}}(x) = \frac{1}{s_n\sqrt{\det a^{(2)}}}\,|T^{-1}x|^{-n}x^t(a^{(2)})^{-1} + (3-n)|x|^{2-n}\,\frac{x^t}{|x|}A_1\left(\frac{x}{|x|},|x|\right)$$
$$+ |x|^{3-n}\left\{DA_1\left(\frac{x}{|x|},|x|\right)\left[|x|I - x\otimes x|x|^{-1}\right]|x|^{-2} + \frac{\partial A_1}{\partial r}\left(\frac{x}{|x|},|x|\right)\frac{x^t}{|x|}\right\}$$
$$+ DB_1(x)\ln|x| + \big(B_1(x)+b_0(1-\delta_{2,n})\big)\,\frac{x^t}{|x|^2} + DC(x) \tag{4.6}$$

for all $x\in\mathbb{R}^n\setminus\{0\}$, where by A_1 we have still denote any real analytic extension of the function A_1 of Corollary 4.2 to an open neighbourhood of $\partial\mathbb{B}_n\times\mathbb{R}$ in $\mathbb{R}^{n+1}$ and where $x\otimes x$ denotes the matrix $(x_lx_j)_{l,j=1,\dots,n}$. Next, we consider the term $B_1(x)/|x|$. By the Fundamental Theorem of Calculus, we have

$$\frac{B_1(x)}{|x|} = \int_0^1 DB_1\Big(t\,\frac{x}{|x|}|x|\Big)\,\frac{x}{|x|}\,dt \quad \forall\, x\in\mathbb{R}^n\setminus\{0\}\,. \tag{4.7}$$

Thus, if we set

$$\beta(\theta,r) = \int_0^1 DB_1(t\theta r)\theta\,dt \quad \forall\,(\theta,r)\in\mathbb{R}^n\times\mathbb{R}\,,$$

the function β will be real analytic and will satisfy the equality

$$\frac{B_1(x)}{|x|} = \beta\Big(\frac{x}{|x|},|x|\Big) \quad \forall\, x\in\mathbb{R}^n\setminus\{0\}\,. \tag{4.8}$$

Define

$$A_2(\theta,r) \equiv (3-n)\theta^tA_1(\theta,r) + DA_1(\theta,r)[I-\theta\otimes\theta] + \frac{\partial A_1}{\partial r}(\theta,r)\theta^t r$$
$$+ \beta(\theta,r)r^{n-2}\theta^t + r^{n-3}\theta^t b_0(1-\delta_{2,n}) \quad \forall\,(\theta,r)\in\partial\mathbb{B}_n\times\mathbb{R}\,.$$

By the real analyticity of A_1 and β, and by the equality $r^{n-3}\theta^tb_0(1-\delta_{2,n})=0$ if $n=2$, the function A_2 is real analytic. Hence, equalities (4.6) and (4.8) imply the validity of statement (i).

Next, we turn to the proof of statement (ii). By Lemma 3.1(ii) and by the Schwartz inequality, we have

$$|T^{-1}x|^{-n}\,|x^t(a^{(2)})^{-1}| \le |x|^{1-n}|T|^n\left|(a^{(2)})^{-1}\right|.$$

Hence, formula (4.5) implies that

$$|x|^{n-1}|DS_{\mathbf{a}}(x)| \le \frac{1}{s_n\sqrt{\det a^{(2)}}}\,|T|^n\left|(a^{(2)})^{-1}\right|$$
$$+ \left\{|x|A_2\Big(\frac{x}{|x|},|x|\Big) + \big(|x|^{n-1}\ln|x|\big)DB_1(x) + |x|^{n-1}DC(x)\right\}$$

for all $x\in\mathbb{R}^n\setminus\{0\}$. Then the continuity of A_2 on the compact set $\partial\mathbb{B}_n\times[0,\operatorname{diam}(G)]$ and the continuity of DB_1 and DC on the compact set $\operatorname{cl}\mathbb{B}_n(0,\operatorname{diam}(G))$ imply the validity of statement (ii).

We now turn to statement (iii). Let $x',x''\in G$, $x'\neq x''$, $y\in G\setminus\mathbb{B}_n(x',2|x'-x''|)$. By statement (i), we have

$$\left|DS_{\mathbf{a}}(x'-y) - DS_{\mathbf{a}}(x''-y)\right|$$
$$\le \frac{1}{s_n\sqrt{\det a^{(2)}}}\left||T^{-1}(x'-y)|^{-n}(x'-y)^t(a^{(2)})^{-1} - |T^{-1}(x''-y)|^{-n}(x''-y)^t(a^{(2)})^{-1}\right|$$
$$+ \left||x'-y|^{2-n}A_2\Big(\frac{x'-y}{|x'-y|},|x'-y|\Big) - |x''-y|^{2-n}A_2\Big(\frac{x''-y}{|x''-y|},|x''-y|\Big)\right|$$
$$+ \Big|\ln|x'-y|DB_1(x'-y) - \ln|x''-y|DB_1(x''-y)\Big| + \left|DC(x'-y) - DC(x''-y)\right|. \tag{4.9}$$

We first estimate the first summand in the right-hand side of inequality (4.9). By the triangular inequality, we have

$$\begin{aligned}\Big||T^{-1}(x'-y)|^{-n}(x'-y)^t(a^{(2)})^{-1}-|T^{-1}(x''-y)|^{-n}(x''-y)^t(a^{(2)})^{-1}\Big|\\ \le |x'-y|\,\big|(a^{(2)})^{-1}\big|\,\Big||T^{-1}(x'-y)|^{-n}-|T^{-1}(x''-y)|^{-n}\Big|\\ +|x'-x''|\,\big|(a^{(2)})^{-1}\big|\,|T^{-1}(x''-y)|^{-n}\,.\end{aligned} \tag{4.10}$$

Thus Lemmas 3.1(ii), 3.2(ii),(iv) with $\gamma=-n$, $\Lambda=T^{-1}$ imply that

$$\begin{aligned}\big||T^{-1}(x'-y)|^{-n}-|T^{-1}(x''-y)|^{-n}\big| &\le m_{-n}(T^{-1})|T^{-1}x'-T^{-1}x''|\,|T^{-1}x'-T^{-1}y|^{-n-1}\\ &\le m_{-n}(T^{-1})|T^{-1}|\,|T|^{n+1}|x'-x''|\,|x'-y|^{-n-1}\,,\\ |T^{-1}(x''-y)|^{-n}\le |T|^n|x''-y|^{-n}\,,&\quad |x''-y|^{-n}\le 2^n|x'-y|^{-n}\,.\end{aligned} \tag{4.11}$$

Next, we estimate the second summand in the right-hand side of inequality (4.9). By Lemmas 3.2(iv) and 3.3(i), the second summand is less or equal to

$$\begin{aligned}&\big||x'-y|^{2-n}-|x''-y|^{2-n}\big|\,\Big|A_2\Big(\frac{x''-y}{|x''-y|},|x''-y|\Big)\Big|\\ &\qquad +|x'-y|^{2-n}\Big|A_2\Big(\frac{x'-y}{|x'-y|},|x'-y|\Big)-A_2\Big(\frac{x''-y}{|x''-y|},|x''-y|\Big)\Big|\\ &\le m_{2-n}|x'-x''|\,|x'-y|^{2-n-1}\sup_{\partial\mathbb{B}_n\times[0,\mathrm{diam}(G)]}|A_2|+|x'-y|^{2-n}\Big(\sum_{j=1}^n M_{A_{2,j},G}\Big)|x'-x''||x'-y|^{-1}\,.\end{aligned} \tag{4.12}$$

Further, we estimate the third summand in the right-hand side of inequality (4.9). By Lemmas 3.2(v) and 3.3(ii), the third summand is less or equal to

$$\begin{aligned}&\big|\ln|x'-y|-\ln|x''-y|\big|\,|DB_1(x''-y)|+\big|\ln|x'-y|\big|\,\big|DB_1(x'-y)-DB_1(x''-y)\big|\\ &\qquad\le 2|x'-x''|\,|x'-y|^{-1}\sup_{G-G}|DB_1|+\Big(\sum_{j=1}^n\widetilde{M}_{\frac{\partial B_1}{\partial x_j},G}\Big)|x'-x''|\,\big|\ln|x'-y|\big|\\ &\le |x'-x''|\,|x'-y|^{-n}\Big\{2|x'-y|^{n-1}\sup_{G-G}|DB_1|+\Big(\sum_{j=1}^n\widetilde{M}_{\frac{\partial B_1}{\partial x_j},G}\Big)|x'-y|^n\big|\ln|x'-y|\big|\Big\}\,.\end{aligned} \tag{4.13}$$

Finally, Lemma 3.3(ii) implies that

$$\begin{aligned}|DC(x'-y)-DC(x''-y)|&\le\Big(\sum_{j=1}^n\widetilde{M}_{\frac{\partial C}{\partial x_j},G}\Big)|x'-x''|\\ &\le |x'-x''|\,|x'-y|^{-n}\Big(\sum_{j=1}^n\widetilde{M}_{\frac{\partial C}{\partial x_j},G}\Big)\sup_{(x',y)\in G\times G}|x'-y|^n\,.\end{aligned} \tag{4.14}$$

Thus inequalities (4.9)–(4.14) imply the validity of statement (iii). □

5 Preliminary inequalities on the boundary operator

Let us turn to estimate the kernel $\overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x-y))$ of the double layer potential of (1.4). We will do it under assumption (1.3). For this, we introduce some basic inequalities for $\overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x-y))$ by means of the following

Lemma 5.1. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $T\in M_n(\mathbb{R})$ *be as in* (4.2) *and let* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a},D]$.

Let $\alpha\in\,]0,1]$ *and* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{1,\alpha}$. *Then the following statements hold:*

(i) *If* $\alpha \in]0,1[$ *, then*

$$b_{\Omega,\alpha} \equiv \sup \left\{ |x-y|^{n-1-\alpha} \left| \overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x-y)) \right| : \ x,y \in \partial\Omega, \ x \neq y \right\} < +\infty \,. \tag{5.1}$$

If $n > 2$*, then* (5.1) *holds also for* $\alpha = 1$*.*

(ii)

$$\begin{aligned} \widetilde{b}_{\Omega,\alpha} \equiv \sup \Bigg\{ \frac{|x'-y|^{n-\alpha}}{|x'-x''|} \left| \overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x'-y)) - \overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x''-y)) \right| : \\ x',x'' \in \partial\Omega, \ x' \neq x'', \ y \in \partial\Omega \setminus \mathbb{B}_n(x', 2|x'-x''|) \Bigg\} < +\infty \,. \end{aligned}$$

Proof. By Lemma 4.3(i), we have

$$\begin{aligned} \overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x-y)) =& - DS_{\mathbf{a}}(x-y)a^{(2)}\nu(y) - \nu^t(y)a^{(1)}S_{\mathbf{a}}(x-y) \\ =& - \frac{1}{s_n\sqrt{\det a^{(2)}}} |T^{-1}(x-y)|^{-n}(x-y)^t\nu(y) \\ & - |x-y|^{2-n}A_2\Big(\frac{x-y}{|x-y|}, |x-y|\Big)a^{(2)}\nu(y) - DB_1(x-y)a^{(2)}\nu(y)\ln|x-y| \\ & - DC(x-y)a^{(2)}\nu(y) - \nu^t(y)a^{(1)}S_{\mathbf{a}}(x-y) \quad \forall\, x,y \in \partial\Omega, \ \ x \neq y \,. \end{aligned} \tag{5.2}$$

By Lemmas 3.1(ii), 3.4, 4.2(i), and by the equality in (5.2), we have

$$\begin{aligned} |x-y|^{n-1-\alpha}\left|\overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x-y))\right| \leq & \frac{1}{s_n\sqrt{\det a^{(2)}}}\, c_{\Omega,\alpha}|T|^n|x-y|^{-n+1+\alpha+n-1-\alpha} \\ & + |x-y|^{2-1-\alpha}|a^{(2)}|\left|A_2\Big(\frac{x-y}{|x-y|}, |x-y|\Big)\right| + |x-y|^{n-1-\alpha}\big|\ln|x-y|\big|\,|a^{(2)}|\,|DB_1(x-y)| \\ & + |x-y|^{n-1-\alpha}|a^{(2)}|\,|DC(x-y)| + |a^{(1)}|C_{0,S_{\mathbf{a}},\partial\Omega,n-1-\alpha} \end{aligned}$$

for all $x,y \in \partial\Omega$, $x \neq y$. If either $\alpha \in]0,1[$ or $\alpha \in]0,1]$ and $n > 2$, then the right-hand side is bounded for $x,y \in \partial\Omega$, $x \neq y$. Hence, we conclude that statement (i) holds true.

Next, we consider statement (ii).

$$\begin{aligned} & \left|\overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x'-y)) - \overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x''-y))\right| \\ & \leq \frac{\left||T^{-1}(x'-y)|^{-n}(x'-y)^t\nu(y) - |T^{-1}(x''-y)|^{-n}(x''-y)^t\nu(y)\right|}{s_n\sqrt{\det a^{(2)}}} \\ & + |a^{(2)}|\left|A_2\Big(\frac{x'-y}{|x'-y|}, |x'-y|\Big) - A_2\Big(\frac{x''-y}{|x''-y|}, |x''-y|\Big)\right| |x'-y|^{2-n} \\ & + |a^{(2)}|\left|A_2\Big(\frac{x''-y}{|x''-y|}, |x''-y|\Big)\right| \left||x'-y|^{2-n} - |x''-y|^{2-n}\right| \\ & + |a^{(2)}|\,\left|DB_1(x'-y) - DB_1(x''-y)\right|\,\big|\ln|x'-y|\big| + |a^{(2)}|\,|DB_1(x''-y)|\,\big|\ln|x'-y| - \ln|x''-y|\big| \\ & + |a^{(2)}|\,\left|DC(x'-y) - DC(x''-y)\right| + |a^{(1)}|\,\left|S_{\mathbf{a}}(x'-y) - S_{\mathbf{a}}(x''-y)\right| \end{aligned} \tag{5.3}$$

for all $x',x'' \in \partial\Omega$, $x' \neq x''$, $y \in \partial\Omega \setminus \mathbb{B}_n(x', 2|x'-x''|)$. Denote by J_1 the first term in the right-hand side of (5.3). By Lemmas 3.1(ii), 3.2(ii),(iv) with $\gamma=-n$, $\Lambda=T^{-1}$, and by Lemma 3.4, we have

$$
\begin{aligned}
J_1 \le{}& \frac{1}{s_n\sqrt{\det a^{(2)}}}\\
&\times\Big\{\big||T^{-1}(x'-y)|^{-n}-|T^{-1}(x''-y)|^{-n}\big|\,\big|(x'-y)^t\nu(y)\big|+|T^{-1}(x''-y)|^{-n}\big|(x'-x'')^t\nu(y)\big|\Big\}\\
\le{}& \frac{1}{s_n\sqrt{\det a^{(2)}}}\\
&\times\Big\{m_{-n}(T^{-1})\big||T^{-1}x'-T^{-1}x''|\,|T^{-1}x'-T^{-1}y|^{-n-1}\big|\,|x'-y|^{1+\alpha}c_{\Omega,\alpha}\\
&\qquad+2^n|T|^n|x'-y|^{-n}\big|(x'-x'')^t\nu(y)\big|\Big\}
\end{aligned}
\tag{5.4}
$$

for all $x',x''\in\partial\Omega$, $x'\neq x''$, $y\in\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)$. Note that

$$
\begin{aligned}
\big|(x'-x'')^t\nu(y)\big| &\le \big|(x'-x'')^t(\nu(y)-\nu(x'))\big|+\big|(x'-x'')^t\nu(x')\big|\\
&\le |x'-x''|\,|\nu|_\alpha|x'-y|^\alpha+c_{\Omega,\alpha}|x'-x''|^{1+\alpha}\le|x'-x''|\,|x'-y|^\alpha(|\nu|_\alpha+c_{\Omega,\alpha})
\end{aligned}
$$

and, accordingly,

$$
\begin{aligned}
J_1\le \frac{|x'-x''|}{s_n\sqrt{\det a^{(2)}}}\Big\{&m_{-n}(T^{-1})|T^{-1}|\,|T|^{n+1}|x'-y|^{-n-1}|x'-y|^{1+\alpha}c_{\Omega,\alpha}\\
&+2^n|T|^n|x'-y|^{-n}|x'-y|^\alpha(|\nu|_\alpha+c_{\Omega,\alpha})\Big\}
\end{aligned}
\tag{5.5}
$$

for all $x',x''\in\partial\Omega$, $x'\neq x''$, $y\in\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)$. Next, we denote by J_2 the sum of the terms different from J_1 in the right-hand side of (5.3). Then Lemma 3.2(iv),(v) and Lemmas 3.3, 4.2(ii) imply that

$$
\begin{aligned}
J_2\le{}& |a^{(2)}|\Big(\sum_{j=1}^n M_{A_{2,j},\partial\Omega}\Big)\frac{|x'-x''|}{|x'-y|}\,|x'-y|^{2-n}+|a^{(2)}|\sup_{\partial\mathbb{B}_n\times[0,\mathrm{diam}(\partial\Omega)]}|A_2|m_{2-n}|x'-x''|\,|x'-y|^{1-n}\\
&+|a^{(2)}|\Big(\sum_{j=1}^n \widetilde{M}_{\frac{\partial B_1}{\partial x_j},\partial\Omega}\Big)|x'-x''|\,\big|\ln|x'-y|\big|+|a^{(2)}|\sup_{\partial\Omega-\partial\Omega}|DB_1|\,2\,\frac{|x'-x''|}{|x'-y|}\\
&+\widetilde{M}_C|x'-x''|+\widetilde{C}_{0,S_{\mathbf{a}},\partial\Omega}|a^{(1)}|\,\frac{|x'-x''|}{|x'-y|^{n-1}}
\end{aligned}
\tag{5.6}
$$

for all $x',x''\in\partial\Omega$, $x'\neq x''$, $y\in\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)$. By inequalities (5.3), (5.5), (5.6), we conclude that statement (ii) holds. □

6 Boundary norms for kernels

For each subset A of $\mathbb{R}^n$, we find it convenient to set

$$
\Delta_A\equiv\big\{(x,y)\in A\times A:\ x=y\big\}\,.
$$

We now introduce a class of functions on $(\partial\Omega)^2\setminus\Delta_{\partial\Omega}$ which may carry a singularity as the variable tends to a point of the diagonal, just as in the case of the kernels of integral operators corresponding to layer potentials defined on the boundary of an open subset Ω of $\mathbb{R}^n$.

Definition 6.1. Let G be a nonempty bounded subset of $\mathbb{R}^n$. Let $\gamma_1,\ \gamma_2,\ \gamma_3\in\mathbb{R}$. We denote by $\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(G)$ the set of continuous functions K from $(G\times G)\setminus\Delta_G$ to $\mathbb{C}$ such that

$$
\begin{aligned}
&\|K\|_{\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(G)}\equiv\sup\Big\{|x-y|^{\gamma_1}|K(x,y)|:\ x,y\in G,\ x\neq y\Big\}\\
&+\sup\Big\{\frac{|x'-y|^{\gamma_2}}{|x'-x''|^{\gamma_3}}\,|K(x',y)-K(x'',y)|:\ x',x''\in G,\ x'\neq x'',\ y\in G\setminus\mathbb{B}_n(x',2|x'-x''|)\Big\}<+\infty\,.
\end{aligned}
$$

One can easily verify that $(\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(G), \|\cdot\|_{\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(G)})$ is a Banach space.

Remark 6.1. Let $\mathbf{a}$ be as in (1.1), (1.2) and $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a}, D]$.

(i) Let G be a nonempty bounded subset of $\mathbb{R}^n$. Then Lemma 4.2 implies that $S_{\mathbf{a}}(x-y) \in \mathcal{K}_{n-1-\gamma,n-1,1}(G)$ for all $\gamma \in [0,1[$ and the same membership holds also for $\gamma = 1$ if $n > 2$. If we further assume that $\mathbf{a}$ satisfies (1.3), then Lemma 4.3 implies that $\frac{\partial}{\partial x_j} S_{\mathbf{a}}(x-y) \in \mathcal{K}_{n-1,n,1}(G)$ for all $j \in \{1, \ldots, n\}$.

(ii) Let $\mathbf{a}$ satisfy (1.3), $\alpha \in]0,1[$ and let Ω be a bounded open subset of $\mathbb{R}^n$ of the class $C^{1,\alpha}$. Then Lemma 5.1 implies that $\overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x-y)) \in \mathcal{K}_{n-1-\alpha,n-\alpha,1}(\partial\Omega)$.

For each $\theta \in]0,1]$, we define the function $\omega_\theta(\cdot)$ from $]0,+\infty[$ to itself by setting

$$\omega_\theta(r) \equiv \begin{cases} r^\theta |\ln r|, & r \in]0, r_\theta], \\ r_\theta^\theta |\ln r_\theta|, & r \in]r_\theta, +\infty[, \end{cases}$$

where

$$r_\theta \equiv \begin{cases} \min\left\{e^{-1/\theta}, e^{\frac{2\theta-1}{\theta(1-\theta)}}\right\} & \text{if } \theta \in]0,1[, \\ e^{-1} & \text{if } \theta = 1. \end{cases}$$

Obviously, $\omega_\theta(\cdot)$ is concave and satisfies (2.1), (2.2), and (2.3) with $\alpha = \theta$. We also note that if $\mathbb{D}$ is a subset of $\mathbb{R}^n$, then the continuous imbedding

$$C_b^{0,\omega_\theta(\cdot)}(\mathbb{D}) \subseteq C_b^{0,\theta'}(\mathbb{D})$$

holds for all $\theta' \in]0, \theta[$. We now consider the properties of an integral operator with a kernel in the class $\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega)$.

Proposition 6.1. *Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\gamma_1 \in]-\infty, n-1[$, γ_2, $\gamma_3 \in \mathbb{R}$. Then the following statements hold:*

(i) *If $(K,\mu) \in \mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega) \times L^\infty(\partial\Omega)$, then the function $K(x,\cdot)\mu(\cdot)$ is integrable in $\partial\Omega$ for all $x \in \partial\Omega$, and the function $u[\partial\Omega, K, \mu]$ from $\partial\Omega$ to $\mathbb{C}$ defined by*

$$u[\partial\Omega, K, \mu](x) \equiv \int_{\partial\Omega} K(x,y)\mu(y)\, d\sigma_y \quad \forall\, x \in \partial\Omega \tag{6.1}$$

is continuous. Moreover, the bilinear map from $\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega) \times L^\infty(\partial\Omega)$ to $C^0(\partial\Omega)$, which takes (K,μ) to $u[\partial\Omega, K, \mu]$, is continuous.

(ii) *If $\gamma_1 \in [n-2, n-1[$, $\gamma_2 \in]n-1, +\infty[$, $\gamma_3 \in]0,1]$, $(n-1) - \gamma_2 + \gamma_3 \in]0,1]$, then the bilinear map from $\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega) \times L^\infty(\partial\Omega)$ to the space $C^{0,\min\{(n-1)-\gamma_1,(n-1)-\gamma_2+\gamma_3\}}(\partial\Omega)$, which takes (K,μ) to $u[\partial\Omega, K, \mu]$, is continuous.*

(iii) *If $\gamma_1 \in [n-2, n-1[$, $\gamma_2 = n-1$, $\gamma_3 \in]0,1]$, then the bilinear map from $\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega) \times L^\infty(\partial\Omega)$ to the space $C^{0,\max\{r^{(n-1)-\gamma_1}, \omega_{\gamma_3}(r)\}}(\partial\Omega)$, which takes (K,μ) to $u[\partial\Omega, K, \mu]$ is continuous.*

Proof. By definition of the norm in $\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega)$, we have

$$\left|K(x,y)\mu(y)\right| \le \|K\|_{\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega)} \|\mu\|_{L^\infty(\partial\Omega)} \frac{1}{|x-y|^{\gamma_1}} \quad \forall\, (x,y) \in (\partial\Omega)^2 \setminus D_{\partial\Omega}\,.$$

Then the function $K(x,\cdot)\mu(\cdot)$ is integrable in $\partial\Omega$ for all $x \in \partial\Omega$, and the Vitali Convergence Theorem implies that $u[\partial\Omega, K, \mu]$ is continuous on $\partial\Omega$ (cf., e.g., Folland [13, (2.33), pp. 60, 180].) By Lemma 3.5(i), we also have

$$\left| \int_{\partial\Omega} K(x,y)\mu(y)\, d\sigma_y \right| \le \|K\|_{\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega)} \|\mu\|_{L^\infty(\partial\Omega)} c'_{\Omega,\gamma_1}\,. \tag{6.2}$$

Hence, statement (i) follows. Next, we turn to estimate the Hölder coefficient of $u[\partial\Omega, K, \mu]$ under the assumptions of statements (ii) and (iii). Let $x', x'' \in \partial\Omega$, $x' \neq x''$. By Remark 2.2, there is no loss of generality in assuming that $0 < |x' - x''| \leq r_{\gamma_3}$. Then the inclusion $\mathbb{B}_n(x', 2|x' - x''|) \subseteq \mathbb{B}_n(x'', 3|x' - x''|)$ and the triangular inequality imply that

$$\left|u[\partial\Omega, K, \mu](x') - u[\partial\Omega, K, \mu](x'')\right| \leq \|\mu\|_{L^\infty(\partial\Omega)}\Bigg\{ \int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} |K(x', y)|\, d\sigma_y + \int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} |K(x'', y)|\, d\sigma_y + \int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \left|K(x', y) - K(x'', y)\right| d\sigma_y \Bigg\}. \tag{6.3}$$

From Lemma 3.5(ii) it follows that

$$\begin{aligned}
&\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} |K(x', y)|\, d\sigma_y + \int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} |K(x'', y)|\, d\sigma_y \\
&\quad \leq \|K\|_{\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega)} \Bigg\{ \int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} \frac{d\sigma_y}{|x' - y|^{\gamma_1}} + \int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} \frac{d\sigma_y}{|x'' - y|^{\gamma_1}} \Bigg\} \\
&\qquad \leq \|K\|_{\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega)} 2c''_{\Omega,\gamma_1} |x' - x''|^{(n-1)-\gamma_1}\,.
\end{aligned} \tag{6.4}$$

Moreover, we have

$$\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \left|K(x', y) - K(x'', y)\right| d\sigma_y \leq \|K\|_{\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega)} \int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{|x' - x''|^{\gamma_3}}{|x' - y|^{\gamma_2}}\, d\sigma_y \tag{6.5}$$

both in case $\gamma_2 \in]n-1, +\infty[$ and $\gamma_2 = n-1$ and for all $\gamma_3 \in]0, 1]$.

Under the assumptions of statement (ii), Lemma 3.5(iii) yields

$$\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{|x' - x''|^{\gamma_3}}{|x' - y|^{\gamma_2}}\, d\sigma_y \leq c'''_{\Omega,\gamma_2} |x' - x''|^{(n-1)-\gamma_2+\gamma_3}\,. \tag{6.6}$$

Instead, under the assumptions of statement (iii), Lemma 3.5(iv) implies that

$$\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{|x' - x''|^{\gamma_3}}{|x' - y|^{\gamma_2}}\, d\sigma_y \leq c^{iv}_{\Omega} |x' - x''|^{\gamma_3} \left|\ln|x' - x''|\right|. \tag{6.7}$$

Inequalities (6.2)–(6.7) imply the validity of statements (ii), (iii). □

Note that Proposition 6.1(ii) for $n = 3$, $\gamma_1 = 2 - \alpha$, $\gamma_2 = 3 - \alpha$, $\gamma_3 = 1$ and for fixed K is known (see Kirsch and Hettlich [19, § 3.1.3, Thm. 3.17 (a)]). Next, we introduce two technical lemmas, which we need to define an auxiliary integral operator.

Lemma 6.1. *Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$, $\alpha, \beta \in]0, 1[$ and $\gamma_2 \in \mathbb{R}$, $\gamma_3 \in]0, 1]$.*

If $\gamma_2 - \beta > n - 1$, we further require that $\gamma_3 + (n-1) - (\gamma_2 - \beta) > 0$.

Then there exists $c > 0$ such that the function $u[\partial\Omega, K, \mu]$ defined by (6.1) satisfies the inequality

$$\begin{aligned}
\left|u[\partial\Omega, K, \mu](x') - u[\partial\Omega, K, \mu](x'')\right| &\leq c\|K\|_{\mathcal{K}_{(n-1)-\alpha,\gamma_2,\gamma_3}(\partial\Omega)} \|\mu\|_{C^{0,\beta}(\partial\Omega)} \omega(|x' - x''|) \\
&\quad + \|\mu\|_{C^0(\partial\Omega)} \left|u[\partial\Omega, K, 1](x') - u[\partial\Omega, K, 1](x'')\right| \quad \forall\, x', x'' \in \partial\Omega
\end{aligned} \tag{6.8}$$

for all $(K, \mu) \in \mathcal{K}_{(n-1)-\alpha,\gamma_2,\gamma_3}(\partial\Omega) \times C^{0,\beta}(\partial\Omega)$, where

$$\omega(r) \equiv \begin{cases} r^{\min\{\alpha+\beta,\gamma_3\}} & \text{if } \gamma_2 - \beta < n-1\,, \\ \max\left\{r^{\alpha+\beta}, \omega_{\gamma_3}(r)\right\} & \text{if } \gamma_2 - \beta = n-1\,, \\ r^{\min\{\alpha+\beta,\gamma_3+(n-1)-(\gamma_2-\beta)\}_-} & \text{if } \gamma_2 - \beta > n-1\,, \end{cases} \qquad \forall\, r \in]0, +\infty[\,.$$

Proof. By Remark 2.2 and Proposition 6.1(i), it suffices to consider the case $0 < |x' - x''| < r_{\gamma_3}$. By the triangular inequality, we have

$$\begin{aligned}
&\big|u[\partial\Omega, K, \mu](x') - u[\partial\Omega, K, \mu](x'')\big| \\
&\quad \le \left| \int_{\partial\Omega} \big[K(x', y) - K(x'', y)\big](\mu(y) - \mu(x'))\, d\sigma_y \right| + |\mu(x')| \left| \int_{\partial\Omega} \big[K(x', y) - K(x'', y)\big]\, d\sigma_y \right|. \qquad (6.9)
\end{aligned}$$

By exploiting the inclusion $\mathbb{B}_n(x', 2|x' - x''|) \subseteq \mathbb{B}_n(x'', 3|x' - x''|)$, the triangular inequality, Lemmas 3.2(i), 3.5(ii), and the inequality

$$|y - x'|^\beta \le |y - x''|^\beta + |x' - x''|^\beta ,$$

we have

$$\begin{aligned}
&\left| \int_{\partial\Omega} \big[K(x', y) - K(x'', y)\big](\mu(y) - \mu(x'))\, d\sigma_y \right| \\
&\le \int_{\mathbb{B}_n(x', 2|x'-x''|)\cap\partial\Omega} |K(x', y)|\, |y - x'|^\beta\, d\sigma_y \|\mu\|_{C^{0,\beta}(\partial\Omega)} \\
&\quad + \int_{\mathbb{B}_n(x'', 3|x'-x''|)\cap\partial\Omega} |K(x'', y)|\, |y - x'|^\beta\, d\sigma_y \|\mu\|_{C^{0,\beta}(\partial\Omega)} \\
&\quad + \int_{\partial\Omega\setminus\mathbb{B}_n(x', 2|x'-x''|)} \big|K(x', y) - K(x'', y)\big|\, |y - x'|^\beta\, d\sigma_y \|\mu\|_{C^{0,\beta}(\partial\Omega)} \\
&\le \|K\|_{\mathcal{K}_{(n-1)-\alpha,\gamma_2,\gamma_3}(\partial\Omega)} \|\mu\|_{C^{0,\beta}(\partial\Omega)} \Bigg\{ \int_{\mathbb{B}_n(x', 2|x'-x''|)\cap\partial\Omega} \frac{d\sigma_y}{|y - x'|^{(n-1)-(\alpha+\beta)}} \\
&\quad + \int_{\mathbb{B}_n(x'', 3|x'-x''|)\cap\partial\Omega} \frac{|x' - x''|^\beta}{|y - x''|^{(n-1)-\alpha}}\, d\sigma_y \\
&\quad + \int_{\mathbb{B}_n(x'', 3|x'-x''|)\cap\partial\Omega} \frac{d\sigma_y}{|y - x''|^{(n-1)-(\alpha+\beta)}} \\
&\quad + \int_{\partial\Omega\setminus\mathbb{B}_n(x', 2|x'-x''|)} \frac{|x' - x''|^{\gamma_3} |x' - y|^\beta}{|x' - y|^{\gamma_2}}\, d\sigma_y \Bigg\} \\
&\le \|K\|_{\mathcal{K}_{(n-1)-\alpha,\gamma_2,\gamma_3}(\partial\Omega)} \|\mu\|_{C^{0,\beta}(\partial\Omega)} \\
&\qquad \times \Bigg\{ 2c''_{\Omega,(n-1)-(\alpha+\beta)} |x' - x''|^{\alpha+\beta} + |x' - x''|^\beta c''_{\Omega,(n-1)-\alpha} |x' - x''|^\alpha \\
&\qquad\qquad + |x' - x''|^{\gamma_3} \int_{\partial\Omega\setminus\mathbb{B}_n(x', 2|x'-x''|)} \frac{d\sigma_y}{|x' - y|^{\gamma_2-\beta}} \Bigg\}. \qquad (6.10)
\end{aligned}$$

At this point we distinguish three cases. If $\gamma_2 - \beta < n - 1$, then by Lemma 3.5(i)

$$\int_{\partial\Omega\setminus\mathbb{B}_n(x', 2|x'-x''|)} \frac{d\sigma_y}{|x' - y|^{\gamma_2-\beta}} \le \int_{\partial\Omega} \frac{d\sigma_y}{|x' - y|^{\gamma_2-\beta}} \le c'_{\Omega,\gamma_2-\beta} ,$$

and thus inequalities (6.9) and (6.10) imply that there exists $c > 0$ such that inequality (6.8) holds with $\omega(r) = r^{\min\{\alpha+\beta,\gamma_3\}}$. If $\gamma_2 - \beta = n - 1$, then by Lemma 3.5(iv)

$$\int_{\partial\Omega\setminus\mathbb{B}_n(x', 2|x'-x''|)} \frac{d\sigma_y}{|x' - y|^{\gamma_2-\beta}} \le c^{iv}_\Omega \big| \ln |x' - x''| \big| ,$$

and thus inequalities (6.9) and (6.10) imply that there exists $c > 0$ such that inequality (6.8) holds with $\omega(r) = \max\{r^{\alpha+\beta}, \omega_{\gamma_3}(r)\}$. If $\gamma_2 - \beta > n-1$, then by Lemma 3.5(iii)

$$\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{d\sigma_y}{|x'-y|^{\gamma_2-\beta}} \leq c'''_{\Omega,\gamma_2-\beta}|x'-x''|^{(n-1)-(\gamma_2-\beta)} ,$$

and thus inequalities (6.9) and (6.10) imply that there exists $c > 0$ such that inequality (6.8) holds with $\omega(r) = r^{\min\{\alpha+\beta,\gamma_3+(n-1)-(\gamma_2-\beta)\}}$. □

We also point out the validity of the following 'folklore' Lemma.

Lemma 6.2. *Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$, $\gamma_1 \in]-\infty, n-1[$, G be a subset of $\mathbb{R}^n$. Let $K \in C^0((G \times \partial\Omega) \setminus \Delta_{\partial\Omega})$ be such that*

$$\kappa_{\gamma_1} \equiv \sup_{(x,y)\in(G\times\partial\Omega)\setminus\Delta_{\partial\Omega}} |x-y|^{\gamma_1}|K(x,y)| < +\infty .$$

Let $\mu \in L^\infty(\partial\Omega)$. Then the function $K(x,\cdot)\mu(\cdot)$ is integrable in $\partial\Omega$ for all $x \in G$ and the function $u^\sharp[\partial\Omega, K, \mu]$ from G to $\mathbb{C}$ defined by

$$u^\sharp[\partial\Omega, K, \mu](x) \equiv \int_{\partial\Omega} K(x,y)\mu(y)\, d\sigma_y \quad \forall\, x \in G$$

is continuous. If $\sup_{x\in G}\int_{\partial\Omega} \frac{d\sigma_y}{|x-y|^{\gamma_1}} < \infty$, then $u^\sharp[\partial\Omega, K, \mu]$ satisfies the inequality

$$\left|u^\sharp[\partial\Omega, K, \mu](x)\right| \leq \sup_{x\in G}\int_{\partial\Omega} \frac{d\sigma_y}{|x-y|^{\gamma_1}}\, \kappa_{\gamma_1} \|\mu\|_{L^\infty(\partial\Omega)} \quad \forall\, x \in G . \tag{6.11}$$

Proof. The integrability of $K(x,\cdot)\mu(\cdot)$ follows from the inequality

$$\left|K(x,y)\mu(y)\right| \leq \frac{\kappa_{\gamma_1}\|\mu\|_{L^\infty(\partial\Omega)}}{|x-y|^{\gamma_1}} \quad \text{a.a. } y \in \partial\Omega .$$

Since $\sup_{x\in G}\int_{\partial\Omega} \frac{d\sigma_y}{|x-y|^{\gamma_1}} < \infty$, inequality (6.11) follows and the Vitali Convergence Theorem implies that $u^\sharp[\partial\Omega, K, \mu]$ is continuous on G (cf., e.g., Folland [13, (2.33) pp. 60, 180]). □

We now introduce an auxiliary integral operator and deduce some properties which we will need in the sequel by applying Proposition 6.1 and Lemma 6.1.

Lemma 6.3. *Let $\theta \in]0,1]$ and Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Then the following statements hold:*

(i) *Let $Z \in C^0((\mathrm{cl}\,\Omega \times \partial\Omega) \setminus \Delta_{\partial\Omega})$ satisfy the inequality*

$$\kappa_{n-1}[Z] \equiv \sup_{(x,y)\in(\mathrm{cl}\,\Omega\times\partial\Omega)\setminus\Delta_{\partial\Omega}} |x-y|^{n-1}|Z(x,y)| < +\infty . \tag{6.12}$$

Let $(f,\mu) \in C^{0,\theta}(\mathrm{cl}\,\Omega) \times L^\infty(\partial\Omega)$ and $H^\sharp[Z,f]$ be the function from $(\mathrm{cl}\,\Omega \times \partial\Omega) \setminus \Delta_{\partial\Omega}$ to $\mathbb{C}$ defined by

$$H^\sharp[Z,f](x,y) \equiv (f(x) - f(y))Z(x,y) \quad \forall\, (x,y) \in (\mathrm{cl}\,\Omega \times \partial\Omega) \setminus \Delta_{\partial\Omega} .$$

If $x \in \mathrm{cl}\,\Omega$, then the function $H^\sharp[Z,f](x,\cdot)$ is Lebesgue integrable in $\partial\Omega$ and the function $Q^\sharp[Z,f,\mu]$ from $\mathrm{cl}\,\Omega$ to $\mathbb{C}$ defined by

$$Q^\sharp[Z,f,\mu](x) \equiv \int_{\partial\Omega} H^\sharp[Z,f](x,y)\mu(y)\, d\sigma_y \quad \forall\, x \in \mathrm{cl}\,\Omega$$

is continuous.

(ii) *The map H from $\mathcal{K}_{n-1,n,1}(\partial\Omega) \times C^{0,\theta}(\partial\Omega)$ to $\mathcal{K}_{n-1-\theta,n-1,\theta}(\partial\Omega)$, which takes (Z,g) to the function from $(\partial\Omega)^2 \setminus \Delta_{\partial\Omega}$ to $\mathbb{C}$ defined by*

$$H[Z,g](x,y) \equiv (g(x)-g(y))Z(x,y) \quad \forall\,(x,y) \in (\partial\Omega)^2 \setminus \Delta_{\partial\Omega}\,,$$

is bilinear and continuous.

(iii) *The map Q from $\mathcal{K}_{n-1,n,1}(\partial\Omega) \times C^{0,\theta}(\partial\Omega) \times L^\infty(\Omega)$ to $C^{0,\omega_\theta(\,\cdot\,)}(\partial\Omega)$, which takes (Z,g,μ) to the function from $\partial\Omega$ to $\mathbb{C}$ defined by*

$$Q[Z,g,\mu](x) \equiv \int_{\partial\Omega} H[Z,g](x,y)\mu(y)\,d\sigma_y \quad \forall\, x \in \partial\Omega\,,$$

is trilinear and continuous.

(iv) *Let $\alpha \in]0,1[$, $\beta \in]0,1]$. Then there exists $q \in]0,+\infty[$ such that*

$$\begin{aligned}\bigl|Q[Z,g,\mu](x') - Q[Z,g,\mu](x'')\bigr| &\le q\|Z\|_{\mathcal{K}_{n-1,n,1}(\partial\Omega)}\|g\|_{C^{0,\alpha}(\partial\Omega)}\|\mu\|_{C^{0,\beta}(\partial\Omega)}|x'-x''|^\alpha \\ &\quad + \|\mu\|_{C^0(\partial\Omega)}\bigl|Q[Z,g,1](x') - Q[Z,g,1](x'')\bigr| \quad \forall\, x',x'' \in \partial\Omega\end{aligned}$$

for all $(Z,g,\mu) \in \mathcal{K}_{n-1,n,1}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega) \times C^{0,\beta}(\partial\Omega)$.

Proof. By assumption (6.12) and by the Hölder continuity of f, we have

$$\bigl|H^\sharp[Z,f](x,y)\bigr| \le \frac{|f|_\theta}{|x-y|^{(n-1)-\theta}}\kappa_{n-1}[Z]$$

for all $(x,y) \in (\mathrm{cl}\,\Omega \times \partial\Omega) \setminus \Delta_{\partial\Omega}$. Thus Lemma 6.2 implies the validity of statement (i).

By the Hölder continuity of g, we have

$$\bigl|H[Z,g](x,y)\bigr| \le \frac{|g|_\theta}{|x-y|^{(n-1)-\theta}}\,\|Z\|_{\mathcal{K}_{n-1,n,1}(\partial\Omega)} \quad \forall\,(x,y) \in (\partial\Omega)^2 \setminus \Delta_{\partial\Omega}\,. \tag{6.13}$$

Now, let x', $x'' \in \partial\Omega$, $x' \neq x''$, $y \in \partial\Omega \setminus \mathbb{B}_n(x', 2|x'-x''|)$. Then we have

$$\begin{aligned}\bigl|H[Z,g](x',y) - H[Z,g](x'',y)\bigr| &\le |g(x')-g(y)|\,\bigl|Z(x',y)-Z(x'',y)\bigr| + |g(x')-g(x'')|\,|Z(x'',y)| \\ &\le \|g\|_{C^{0,\theta}(\partial\Omega)}\|Z\|_{\mathcal{K}_{n-1,n,1}(\partial\Omega)}\left\{\frac{|x'-y|^\theta|x'-x''|}{|x'-y|^n} + \frac{|x'-x''|^\theta}{|x''-y|^{n-1}}\right\}. \end{aligned}\tag{6.14}$$

Since $|x'-x''| \le |x'-y|$, we have $|x'-x''|^{1-\theta} \le |x'-y|^{1-\theta}$. Moreover, Lemma 3.2(i) implies that $|x''-y| \ge \frac{1}{2}|x'-y|$ and thus the term in braces in the right-hand side of (6.14) is less or equal to

$$\frac{|x'-y|\,|x'-x''|^\theta}{|x'-y|^n} + \frac{2^{n-1}|x'-x''|^\theta}{|x'-y|^{n-1}} \le (1+2^{n-1})\,\frac{|x'-x''|^\theta}{|x'-y|^{n-1}}\,. \tag{6.15}$$

Thus inequalities (6.13)–(6.15) imply that

$$\bigl\|H[Z,g]\bigr\|_{\mathcal{K}_{n-1-\theta,n-1,\theta}(\partial\Omega)} \le 2^n\|Z\|_{\mathcal{K}_{n-1,n,1}(\partial\Omega)}\|g\|_{C^{0,\theta}(\partial\Omega)}\,. \tag{6.16}$$

Hence statement (ii) holds true. We now turn to prove (iii). By Proposition 6.1(iii) with $\gamma_1 = n-1-\theta$, $\gamma_2 = n-1$, $\gamma_3 = \theta$, the map $u[\partial\Omega,\,\cdot\,,\,\cdot\,]$ is continuous from $\mathcal{K}_{n-1-\theta,n-1,\theta}(\partial\Omega) \times L^\infty(\partial\Omega)$ to $C^{0,\max\{r^{(n-1)-[(n-1)-\theta]},\omega_\theta(r)\}}(\partial\Omega) = C^{0,\omega_\theta(\,\cdot\,)}(\partial\Omega)$. Then statement (ii) implies that $u[\partial\Omega, H[\,\cdot\,,\,\cdot\,],\,\cdot\,]$ is continuous from $\mathcal{K}_{n-1,n,1}(\partial\Omega) \times C^{0,\theta}(\partial\Omega) \times L^\infty(\partial\Omega)$ to $C^{0,\omega_\theta(\,\cdot\,)}(\partial\Omega)$. Since

$$u\bigl[\partial\Omega, H[Z,g],\mu\bigr] = \int_{\partial\Omega} H[Z,g](x,y)\mu(y)\,d\sigma_y \quad \forall\, x \in \partial\Omega\,, \tag{6.17}$$

statement (iii) holds true. Since $C^{0,\beta_1}(\partial\Omega)$ is continuously imbedded into $C^{0,\beta_2}(\partial\Omega)$ whenever $0 < \beta_2 \le \beta_1 \le 1$, we can assume that $\alpha + \beta < 1$. Then by equality (6.17), by Lemma 6.1 with $\gamma_2 = n-1$, $\gamma_3 = \alpha$ and by statement (ii) with $\theta = \alpha$, statement (iv) holds true. □

7 Preliminaries on layer potentials

Let $\mathbf{a}$ be as in (1.1), (1.2), $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a},D]$ and let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. If $\mu \in L^\infty(\partial\Omega)$, Lemma 4.2(i) ensures the convergence of the integral

$$v[\partial\Omega, S_{\mathbf{a}}, \mu](x) \equiv \int_{\partial\Omega} S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \quad \forall\, x \in \mathbb{R}^n\,,$$

which defines the single layer potential relative to μ, $S_{\mathbf{a}}$. We collect in the following statement some known properties of the single layer potential which we will exploit in the sequel (cf. Miranda [24], Wiegner [36], Dalla Riva [3], Dalla Riva, Morais and Musolino [5] and the references therein).

Theorem 7.1. *Let $\mathbf{a}$ be as in (1.1), (1.2), $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a},D]$, $\alpha \in]0,1[$, $m \in \mathbb{N}\setminus\{0\}$ and let Ω be a bounded open subset of $\mathbb{R}^n$ of the class $C^{m,\alpha}$. Then the following statements hold:*

(i) *If $\mu \in C^{m-1,\alpha}(\partial\Omega)$, then the function $v^+[\partial\Omega, S_{\mathbf{a}}, \mu] \equiv v[\partial\Omega, S_{\mathbf{a}}, \mu]_{|\mathrm{cl}\,\Omega}$ belongs to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ and the function $v^-[\partial\Omega, S_{\mathbf{a}}, \mu] \equiv v[\partial\Omega, S_{\mathbf{a}}, \mu]_{|\mathrm{cl}\,\Omega^-}$ belongs to $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\,\Omega^-)$. Moreover, the map which takes μ to the function $v^+[\partial\Omega, S_{\mathbf{a}}, \mu]$ is continuous from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ and the map from the space $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)$ which takes μ to $v^-[\partial\Omega, S_{\mathbf{a}}, \mu]_{|\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega}$ is continuous for all $R \in]0,+\infty[$ such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$.*

(ii) *Let $l \in \{1,\dots,n\}$. If $\mu \in C^{0,\alpha}(\partial\Omega)$, then we have the following jump relation*

$$\frac{\partial}{\partial x_l} v^{\pm}[\partial\Omega, S_{\mathbf{a}}, \mu](x) = \mp \frac{\nu_l(x)}{2\nu(x)^t a^{(2)}\nu(x)}\,\mu(x) + \int_{\partial\Omega} \partial_{x_l} S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \quad \forall\, x \in \partial\Omega\,,$$

where the integral in the right-hand side exists in the sense of the principal value.

We now introduce the following refinement of a classical result for the homogeneous second order elliptic operators (cf. Miranda [25]).

Theorem 7.2. *Let $\mathbf{a}$ be as in (1.1), (1.2), $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a},D]$, Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$ and let $\gamma \in]0,1[$. Then the operator $v[\partial\Omega, S_{\mathbf{a}}, \cdot\,]_{|\partial\Omega}$ from $L^\infty(\partial\Omega)$ to $C^{0,\gamma}(\partial\Omega)$ which takes μ to $v[\partial\Omega, S_{\mathbf{a}}, \mu]_{|\partial\Omega}$ is continuous.*

If, in addition, we assume that $n > 2$, then $v[\partial\Omega, S_{\mathbf{a}}, \cdot\,]_{|\partial\Omega}$ is continuous from $L^\infty(\partial\Omega)$ to $C^{0,\omega_1(\,\cdot\,)}(\partial\Omega)$.

Proof. By Lemma 4.2, we have $S_{\mathbf{a}}(x-y) \in \mathcal{K}_{(n-1)-\gamma,n-1,1}(\partial\Omega)$, and also $S_{\mathbf{a}}(x-y) \in \mathcal{K}_{n-2,n-1,1}(\partial\Omega)$ if we assume that $n > 2$. Since

$$v[\partial\Omega, S_{\mathbf{a}}, \mu]_{|\partial\Omega} = u\big[\partial\Omega, S_{\mathbf{a}}(x-y), \mu\big]\,,$$

Proposition 6.1(iii) implies that $v[\partial\Omega, S_{\mathbf{a}}, \cdot\,]$ is continuous from $L^\infty(\partial\Omega)$ to $C^{0,\max\{r^\gamma,\omega_1(r)\}}(\partial\Omega) = C^{0,\gamma}(\partial\Omega)$, and also that $v[\partial\Omega, S_{\mathbf{a}}, \cdot\,]$ is continuous from $L^\infty(\partial\Omega)$ to $C^{0,\max\{r,\omega_1(r)\}}(\partial\Omega) = C^{0,\omega_1(r)}(\partial\Omega)$ if we assume that $n > 2$. □

Next, we turn to the double layer potential and introduce the following technical result (cf. Miranda [24], Wiegner [36], Dalla Riva [3], Dalla Riva, Morais and Musolino [5] and the references therein).

Theorem 7.3. *Let $\mathbf{a}$ be as in (1.1), (1.2), $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a},D]$, $\alpha \in]0,1[$, $m \in \mathbb{N}\setminus\{0\}$ and let Ω be a bounded open subset of $\mathbb{R}^n$ of the class $C^{m,\alpha}$. Then the following statements hold:*

(i) *If $\mu \in C^{0,\alpha}(\partial\Omega)$, then the restriction $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\Omega}$ can be extended uniquely to a continuous function $w^+[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]$ from $\mathrm{cl}\,\Omega$ to $\mathbb{C}$, and $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\Omega^-}$ can be extended uniquely to a continuous function $w^-[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]$ from $\mathrm{cl}\,\Omega^-$ to $\mathbb{C}$, and we have the following jump relation*

$$w^{\pm}[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu](x) = \pm\frac{1}{2}\,\mu(x) + w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu](x) \quad \forall\, x \in \partial\Omega\,.$$

(ii) *If* $\mu \in C^{m,\alpha}(\partial\Omega)$*, then* $w^+[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]$ *belongs to* $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ *and* $w^-[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]$ *belongs to* $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\,\Omega^-)$*. Moreover, the map from the space* $C^{m,\alpha}(\partial\Omega)$ *to* $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ *which takes* μ *to* $w^+[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]$ *is continuous and the map from the space* $C^{m,\alpha}(\partial\Omega)$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R) \setminus \Omega)$ *which takes* μ *to* $w^-[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega}$ *is continuous for all* $R \in]0, +\infty[$ *such that* $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$.

(iii) *Let* $r \in \{1, \ldots, n\}$*. If* $\mu \in C^{m,\alpha}(\partial\Omega)$ *and* U *is an open neighborhood of* $\partial\Omega$ *in* $\mathbb{R}^n$ *and* $\widetilde{\mu} \in C^m(U)$*,* $\widetilde{\mu}_{|\partial\Omega} = \mu$*, then the equality*

$$\frac{\partial}{\partial x_r} w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu](x) = \sum_{j,l=1}^{n} a_{lj} \frac{\partial}{\partial x_l} \Big\{ \int_{\partial\Omega} S_{\mathbf{a}}(x-y) \Big[\nu_r(y) \frac{\partial \widetilde{\mu}}{\partial y_j}(y) - \nu_j(y) \frac{\partial \widetilde{\mu}}{\partial y_r}(y) \Big] d\sigma_y \Big\}$$
$$+ \int_{\partial\Omega} \big[DS_{\mathbf{a}}(x-y) a^{(1)} + a S_{\mathbf{a}}(x-y) \big] \nu_r(y) \mu(y)\, d\sigma_y$$
$$- \int_{\partial\Omega} \partial_{x_r} S_{\mathbf{a}}(x-y) \nu^t(y) a^{(1)} \mu(y)\, d\sigma_y \quad \forall\, x \in \mathbb{R}^n \setminus \partial\Omega \tag{7.1}$$

holds.

Note that formula (7.1) for the Laplace operator with $n = 3$ can be found in Günter [14, Ch. 2, § 10, (42)]. By combining Theorems 7.1 and 7.3, we deduce that under the assumptions of Theorem 7.3(iii), the equality

$$\frac{\partial}{\partial x_r} w^+[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu] = \sum_{j,l=1}^{n} a_{lj} \frac{\partial}{\partial x_l} v^+\big[\partial\Omega, S_{\mathbf{a}}, M_{rj}[\mu]\big] + Dv^+[\partial\Omega, S_{\mathbf{a}}, \nu_r \mu] a^{(1)}$$
$$+ a v^+[\partial\Omega, S_{\mathbf{a}}, \nu_r \mu] - \frac{\partial}{\partial x_r} v^+[\partial\Omega, S_{\mathbf{a}}, (\nu^t a^{(1)})\mu] \ \text{ on } \ \mathrm{cl}\,\Omega \tag{7.2}$$

holds.

Next, we introduce a result proved by Schauder [30, Hilfsatz VII, p. 112] for the Laplace operator, which we extend here to the second order elliptic operators by exploiting Proposition 6.1.

Theorem 7.4. *Let* $\mathbf{a}$ *be as in* (1.1)*,* (1.2)*,* (1.3)*,* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$*,* $\alpha \in]0,1[$ *and let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{1,\alpha}$*. If* $\mu \in L^\infty(\partial\Omega)$*, then* $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega} \in C^{0,\alpha}(\partial\Omega)$*. Moreover, the operator from* $L^\infty(\partial\Omega)$ *to* $C^{0,\alpha}(\partial\Omega)$ *which takes* μ *to* $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega}$ *is continuous.*

Proof. By Lemma 5.1, the function $K_{\mathbf{a}}(x,y) \equiv \overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x-y))$ belongs to $\mathcal{K}_{(n-1)-\alpha, n-\alpha, 1}(\partial\Omega)$. Since

$$w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega} = u[\partial\Omega, K_{\mathbf{a}}, \mu]\,,$$

Proposition 6.1(ii) implies that the function $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is continuous from $L^\infty(\partial\Omega)$ to $C^{0,\min\{\alpha,(n-1)-(n-\alpha)+1\}}(\partial\Omega) = C^{0,\alpha}(\partial\Omega)$. □

8 Auxiliary integral operators

In order to compute the tangential derivatives of the double layer potential, we introduce the following two statements which concern two auxiliary integral operators. To shorten our notation, we define the function Θ from $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta_{\mathbb{R}^n}$ to $\mathbb{R}^n \setminus \{0\}$ as follows:

$$\Theta(x,y) \equiv x - y \quad \forall\, (x,y) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta_{\mathbb{R}^n}\,. \tag{8.1}$$

Theorem 8.1. *Let* $\mathbf{a}$ *be as in* (1.1)*,* (1.2)*,* (1.3)*,* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$ *and* $r \in \{1, \ldots, n\}$*. Then the following statements hold:*

(i) *Let* Ω *be a bounded open Lipschitz subset of* $\mathbb{R}^n$ *and* $\theta \in]0,1]$. *If* $(f,\mu) \in C^{0,\theta}(\mathrm{cl}\,\Omega) \times L^\infty(\partial\Omega)$, *then the function*

$$Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, f, \mu\Big](x) = \int_{\partial\Omega} (f(x) - f(y))\,\frac{\partial S_{\mathbf{a}}}{\partial x_r}(x-y)\mu(y)\,d\sigma_y \quad \forall\, x \in \mathrm{cl}\,\Omega$$

is continuous.

(ii) *Let* $\alpha \in]0,1[$, $\beta, \theta \in]0,1]$, $m \in \mathbb{N} \setminus \{0\}$ *and* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{m,\alpha}$. *Then the map* $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot\,, \cdot\,]$ *from* $C^{m-1,\theta}(\mathrm{cl}\,\Omega) \times C^{m-1,\beta}(\partial\Omega)$ *to* $C^{m-1,\min\{\alpha,\beta,\theta\}}(\mathrm{cl}\,\Omega)$ *which takes* (f,μ) *to* $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, f, \mu]$ *is bilinear and continuous.*

Proof. By Lemma 4.3(ii), statement (i) is an immediate consequence of Lemma 6.3(i). Consider statement (ii). By treating separately the cases $x \in \partial\Omega$ and $x \in \Omega$, and exploiting Theorem 7.1(ii), we have

$$Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, f, \mu\Big](x) = f(x)\,\frac{\partial}{\partial x_r} v^+[\partial\Omega, S_{\mathbf{a}}, \mu](x) - \frac{\partial}{\partial x_r} v^+[\partial\Omega, S_{\mathbf{a}}, f\mu](x),$$

for all $x \in \mathrm{cl}\,\Omega$. Then the statement follows by Theorem 7.1(i) and by the continuity of the pointwise product in Schauder spaces. □

Theorem 8.2. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3) *and* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$. *Then the following statement holds:*

(i) *Let* Ω *be a bounded open Lipschitz subset of* $\mathbb{R}^n$ *and* $\theta \in]0,1]$. *Then the bilinear map* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot\,, \cdot\,]$ *from* $C^{0,\theta}(\partial\Omega) \times L^\infty(\partial\Omega)$ *to* $C^{0,\omega_\theta(\cdot)}(\partial\Omega)$, *which takes* (g,μ) *to the function*

$$Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, \mu\Big](x) = \int_{\partial\Omega} (g(x) - g(y))\,\frac{\partial S_{\mathbf{a}}}{\partial x_r}\,(x-y)\mu(y)\,d\sigma_y \quad \forall\, x \in \partial\Omega\,, \tag{8.2}$$

is continuous.

(ii) *Let* $\alpha \in]0,1[$, $\beta \in]0,1]$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{1,\alpha}$. *Then the bilinear map* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot\,, \cdot\,]$ *from* $C^{0,\alpha}(\partial\Omega) \times C^{0,\beta}(\partial\Omega)$ *to* $C^{0,\alpha}(\partial\Omega)$, *which takes* (g,μ) *to* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, \mu]$, *is continuous.*

Proof. By Lemma 4.3, we have $\frac{\partial S_{\mathbf{a}}}{\partial x_r} \in \mathcal{K}_{n-1,n,1}(\partial\Omega)$. Then Lemma 6.3(iii) implies the validity of statement (i).

We now consider statement (ii). By statement (i) and by the continuity of the inclusion of $C^{0,\beta}(\partial\Omega)$ into $L^\infty(\partial\Omega)$, we already know that $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot\,, \cdot\,]$ is continuous from $C^{0,\alpha}(\partial\Omega) \times C^{0,\beta}(\partial\Omega)$ to $C^0(\partial\Omega)$. Then it suffices to show that $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot\,, \cdot\,]$ is continuous from $C^{0,\alpha}(\partial\Omega) \times C^{0,\beta}(\partial\Omega)$ to the semi-normed space $(C^{0,\alpha}(\partial\Omega), |\cdot : \partial\Omega|_\alpha)$. By Lemma 6.3(iv), there exists $q \in]0,+\infty[$ such that

$$\begin{aligned}
&\Big|Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, \mu\Big](x') - Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, \mu\Big](x'')\Big| \\
&\qquad \le q\Big\|\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta\Big\|_{\mathcal{K}_{n-1,n,1}(\partial\Omega)} \|g\|_{C^{0,\alpha}(\partial\Omega)} \|\mu\|_{C^{0,\beta}(\partial\Omega)} |x' - x''|^\alpha \\
&\qquad\qquad + \|\mu\|_{C^0(\partial\Omega)} \Big|Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, 1\Big](x') - Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, 1\Big](x'')\Big|
\end{aligned} \tag{8.3}$$

for all $x', x'' \in \partial\Omega$. Let $R \in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$. Let '$\sim$' be an extension operator as in Lemma 2.1, defined on $C^{0,\alpha}(\partial\Omega)$. Since

$$Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, 1\Big](x) = Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \widetilde{g}, 1\Big](x) \quad \forall\, x \in \partial\Omega\,,$$

Theorem 8.1(ii) implies that $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\,\cdot\,,1]$ is continuous from $C^{0,\alpha}(\partial\Omega)$ to itself and, accordingly, there exists $q'\in]0,+\infty[$ such that

$$\Big\|Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,1\Big]\Big\|_{C^{0,\alpha}(\partial\Omega)}\leq q'\|g\|_{C^{0,\alpha}(\partial\Omega)}\quad\forall\,g\in C^{0,\alpha}(\partial\Omega)\,. \tag{8.4}$$

Combining inequalities (8.3) and (8.4), we deduce that $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\,\cdot\,,\,\cdot\,]$ is continuous from $C^{0,\alpha}(\partial\Omega)\times C^{0,\beta}(\partial\Omega)$ to $(C^{0,\alpha}(\partial\Omega),|\cdot:\partial\Omega|_\alpha)$ and thus the proof is complete. □

In the next lemma, we introduce a formula for the tangential derivatives of $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu]$.

Lemma 8.1. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a},D]$, $\alpha\in]0,1[$, $\theta\in]0,1]$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{2,\alpha}$, $r\in\{1,\dots,n\}$ *and let* $g\in C^{1,\theta}(\partial\Omega)$, $\mu\in C^1(\partial\Omega)$. *Then* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu]\in C^1(\partial\Omega)$ *and the formula*

$$\begin{aligned}
M_{lj}\Big[Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu\Big]\Big]&=\nu_l(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,D_{\mathbf{a},j}g,\mu\Big](x)-\nu_j(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,D_{\mathbf{a},l}g,\mu\Big](x)\\
&+\nu_l(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\sum_{s=1}^n M_{sj}\Big[\sum_{h=1}^n\frac{a_{sh}\nu_h}{\nu^t a^{(2)}\nu}\mu\Big]\Big](x)\\
&-\nu_j(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\sum_{s=1}^n M_{sl}\Big[\sum_{h=1}^n\frac{a_{sh}\nu_h}{\nu^t a^{(2)}\nu}\mu\Big]\Big](x)\\
&+\sum_{s,h=1}^n a_{sh}\nu_l(x)\Big\{Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,\nu_j,\frac{M_{hr}[g]\mu}{\nu^t a^{(2)}\nu}\Big](x)+Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,g,M_{hr}\Big[\frac{\nu_j\mu}{\nu^t a^{(2)}\nu}\Big]\Big](x)\Big\}\\
&-\sum_{s,h=1}^n a_{sh}\nu_j(x)\Big\{Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,\nu_l,\frac{M_{hr}[g]\mu}{\nu^t a^{(2)}\nu}\Big](x)+Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,g,M_{hr}\Big[\frac{\nu_l\mu}{\nu^t a^{(2)}\nu}\Big]\Big](x)\Big\}\\
&-\sum_{s=1}^n a_s\Big\{\nu_l(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,g,\frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)-\nu_j(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,g,\frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)\Big\}\\
&-a\Big\{g(x)\Big[\nu_l(x)v[\partial\Omega,S_{\mathbf{a}},\frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)-\nu_j(x)v\Big[\partial\Omega,S_{\mathbf{a}},\frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)\Big]\\
&-\Big[\nu_l(x)v\Big[\partial\Omega,S_{\mathbf{a}},g\frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)-\nu_j(x)v\Big[\partial\Omega,S_{\mathbf{a}},g\frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)\Big]\Big\}
\end{aligned}\tag{8.5}$$

holds for all $x\in\partial\Omega$ *and* $l,j\in\{1,\dots,n\}$. (*For* Q *see* (8.2).)

Proof. Let $R\in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega\subseteq\mathbb{B}_n(0,R)$. Let '$\sim$' be an extension operator as in Lemma 2.1, defined either on $C^{1,\theta}(\partial\Omega)$ or on $C^{1,\alpha}(\partial\Omega)$ depending on whether it has been applied to $g\in C^{1,\theta}(\partial\Omega)$ or to $\nu_l\in C^{1,\alpha}(\partial\Omega)$ for $l=1,\dots,n$.

Now, fix $\beta\in]0,\min\{\theta,\alpha\}[$ and first prove the formula under the assumption $\mu\in C^{1,\beta}(\partial\Omega)$. By Theorem 8.1(ii), we already know that $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu]$ belongs to $C^1(\mathrm{cl}\,\Omega)$. Then we find it convenient to introduce the notation

$$M^\sharp_{lj}[f](x)\equiv\tilde{\nu}_l(x)\,\frac{\partial f}{\partial x_j}\,(x)-\tilde{\nu}_j(x)\,\frac{\partial f}{\partial x_l}\,(x)\quad\forall\,x\in\mathrm{cl}\,\Omega$$

for all $f\in C^1(\mathrm{cl}\,\Omega)$. If necessary, we write $M^\sharp_{lj,x}$ to emphasize that we are taking x as variable of the differential operator $M^\sharp_{lj}$. Next, we fix $x\in\Omega$ and compute

$$\tilde{\nu}_l(x)\,\frac{\partial}{\partial x_j}\,Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\tilde{g},\mu\Big](x)-\tilde{\nu}_j(x)\,\frac{\partial}{\partial x_l}\,Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\tilde{g},\mu\Big](x)\,.$$

Clearly,

$$\frac{\partial}{\partial x_l} Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\widetilde{g},\mu\Big](x) = \int_{\partial\Omega} \frac{\partial \widetilde{g}}{\partial x_l}(x)\,\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y + \int_{\partial\Omega}(\widetilde{g}(x)-\widetilde{g}(y))\,\frac{\partial^2}{\partial x_l\partial x_r}S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\,.$$

To shorten our notation, we set

$$J_1(x) \equiv \int_{\partial\Omega}\frac{\partial \widetilde{g}}{\partial x_l}(x)\,\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\,.$$

Then we have

$$\begin{aligned}
\frac{\partial}{\partial x_l} Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\widetilde{g},\mu\Big](x)
&= J_1(x) - \int_{\partial\Omega}(\widetilde{g}(x)-\widetilde{g}(y))\sum_{s,h=1}^n \frac{\nu_s(y)a_{sh}\nu_h(y)}{\nu^t(y)a^{(2)}\nu(y)}\,\frac{\partial}{\partial y_l}\Big[\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\Big]\mu(y)\,d\sigma_y\\
&= J_1(x) - \int_{\partial\Omega}(\widetilde{g}(x)-\widetilde{g}(y))\sum_{s=1}^n\Big(\nu_s(y)\,\frac{\partial}{\partial y_l}-\nu_l(y)\,\frac{\partial}{\partial y_s}\Big)\Big[\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\Big]\\
&\qquad\qquad\times\sum_{h=1}^n\frac{a_{sh}\nu_h(y)}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y\\
&\quad - \int_{\partial\Omega}(\widetilde{g}(x)-\widetilde{g}(y))\sum_{s=1}^n\frac{\partial}{\partial y_s}\Big[\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\Big]\sum_{h=1}^n a_{sh}\nu_h(y)\,\frac{\nu_l(y)}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y\,.
\end{aligned}$$

By Lemma 2.2, the second term in the right-hand side takes the form

$$\begin{aligned}
&\int_{\partial\Omega}(\widetilde{g}(x)-\widetilde{g}(y))\sum_{s=1}^n M_{sl,y}\Big[\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\Big]\,\frac{(a^{(2)}\nu(y))_s}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y\\
&\quad= -\int_{\partial\Omega}\sum_{s=1}^n M_{sl,y}\big[\widetilde{g}(x)-\widetilde{g}(y)\big]\,\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\,\frac{(a^{(2)}\nu(y))_s}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y\\
&\qquad\qquad - \int_{\partial\Omega}\sum_{s=1}^n(\widetilde{g}(x)-\widetilde{g}(y))\,\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)M_{sl}\Big[\frac{(a^{(2)}\nu)_s}{\nu^t a^{(2)}\nu}\,\mu\Big](y)\,d\sigma_y\,.
\end{aligned}$$

Since $M_{sl,y}[\widetilde{g}(x)-\widetilde{g}(y)] = -M_{sl}[\widetilde{g}](y)$, we have

$$\begin{aligned}
\frac{\partial}{\partial x_l} Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\widetilde{g},\mu\Big](x) &= \frac{\partial \widetilde{g}}{\partial x_l}(x)\int_{\partial\Omega}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\quad - \int_{\partial\Omega}\sum_{s=1}^n M_{sl}[\widetilde{g}](y)\,\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\,\frac{(a^{(2)}\nu(y))_s}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y\\
&\quad + \int_{\partial\Omega}\sum_{s=1}^n(\widetilde{g}(x)-\widetilde{g}(y))\,\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)M_{sl}\Big[\frac{(a^{(2)}\nu)_s}{\nu^t a^{(2)}\nu}\,\mu\Big](y)\,d\sigma_y\\
&\quad - \int_{\partial\Omega}(\widetilde{g}(x)-\widetilde{g}(y))\sum_{s=1}^n\frac{\partial}{\partial y_s}\Big[\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\Big](a^{(2)}\nu(y))_s\,\frac{\nu_l(y)}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y\,.
\end{aligned}$$

Accordingly, we have

$$
\begin{aligned}
M^\sharp_{lj}\Big[Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\widetilde{g},\mu\Big]\Big](x) &= M^\sharp_{lj}[\widetilde{g}](x)\int_{\partial\Omega}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&-\int_{\partial\Omega}\sum_{s=1}^n\Big\{\widetilde{\nu}_l(x)M_{sj}[\widetilde{g}](y)-\widetilde{\nu}_j(x)M_{sl}[\widetilde{g}](y)\Big\}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\,\frac{(a^{(2)}\nu(y))_s}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y \\
&+\int_{\partial\Omega}\sum_{s=1}^n(\widetilde{g}(x)-\widetilde{g}(y))\,\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\Big\{\widetilde{\nu}_l(x)M_{sj}\Big[\frac{(a^{(2)}\nu)_s}{\nu^t a^{(2)}\nu}\,\mu\Big](y)-\widetilde{\nu}_j(x)M_{sl}\Big[\frac{(a^{(2)}\nu)_s}{\nu^t a^{(2)}\nu}\,\mu\Big](y)\Big\}\,d\sigma_y \\
&-\int_{\partial\Omega}(\widetilde{g}(x)-\widetilde{g}(y))\sum_{s=1}^n\frac{\partial}{\partial y_s}\,\Big[\frac{\partial}{\partial x_r}S_{\mathbf{a}}(x-y)\Big](a^{(2)}\nu)_s(y)\,\frac{\widetilde{\nu}_l(x)\nu_j(y)-\widetilde{\nu}_j(x)\nu_l(y)}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y\,. \qquad (8.6)
\end{aligned}
$$

We now consider the first two terms in the right-hand side of formula (8.6). By the obvious identity

$$
M^\sharp_{lj}[\widetilde{g}]=\widetilde{\nu}_l\Big[\frac{\partial}{\partial x_j}\,\widetilde{g}-\frac{D\widetilde{g}a^{(2)}\widetilde{\nu}}{\widetilde{\nu}^t a^{(2)}\widetilde{\nu}}\,\widetilde{\nu}_j\Big]-\widetilde{\nu}_j\Big[\frac{\partial}{\partial x_l}\,\widetilde{g}-\frac{D\widetilde{g}a^{(2)}\widetilde{\nu}}{\widetilde{\nu}^t a^{(2)}\widetilde{\nu}}\,\widetilde{\nu}_l\Big]\quad\text{in }\ \mathrm{cl}\,\Omega\,,
$$

by the corresponding formula for $M_{lj}[\widetilde{g}]$ on $\partial\Omega$, by formula (2.4) and by straightforward computations, we obtain

$$
\begin{aligned}
&M^\sharp_{lj}[\widetilde{g}](x)\int_{\partial\Omega}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&\quad-\int_{\partial\Omega}\sum_{s=1}^n\Big\{\widetilde{\nu}_l(x)M_{sj}[\widetilde{g}](y)-\widetilde{\nu}_j(x)M_{sl}[\widetilde{g}](y)\Big\}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\,\frac{(a^{(2)}\nu(y))_s}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y \\
&\qquad=\widetilde{\nu}_l(x)\Big[\frac{\partial}{\partial x_j}\,\widetilde{g}(x)-\frac{D\widetilde{g}(x)a^{(2)}\widetilde{\nu}(x)}{\widetilde{\nu}^t(x)a^{(2)}\widetilde{\nu}(x)}\,\widetilde{\nu}_j(x)\Big]\int_{\partial\Omega}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&\qquad-\widetilde{\nu}_j(x)\Big[\frac{\partial}{\partial x_l}\,\widetilde{g}(x)-\frac{D\widetilde{g}(x)a^{(2)}\widetilde{\nu}(x)}{\widetilde{\nu}^t(x)a^{(2)}\widetilde{\nu}(x)}\,\widetilde{\nu}_l(x)\Big]\int_{\partial\Omega}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&-\widetilde{\nu}_l(x)\int_{\partial\Omega}\Big[\frac{\partial}{\partial y_j}\,\widetilde{g}(y)-\frac{D\widetilde{g}(y)a^{(2)}\widetilde{\nu}(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\,\widetilde{\nu}_j(y)\Big]\Big(\sum_{s,h=1}^n\widetilde{\nu}_s(y)\,\frac{a^{(2)}_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\Big)\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&\qquad+\widetilde{\nu}_l(x)\int_{\partial\Omega}\widetilde{\nu}_j(y)\Big\{\sum_{s,h=1}^n\frac{\partial}{\partial y_s}\,\widetilde{g}(y)\,\frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}-\frac{D\widetilde{g}(y)a^{(2)}\widetilde{\nu}(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)} \\
&\qquad\qquad\times\Big(\widetilde{\nu}_s(y)\frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\Big)\Big\}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&+\widetilde{\nu}_j(x)\int_{\partial\Omega}\Big[\frac{\partial}{\partial y_l}\,\widetilde{g}(y)-\frac{D\widetilde{g}(y)a^{(2)}\widetilde{\nu}(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\,\widetilde{\nu}_l(y)\Big]\Big(\sum_{s,h=1}^n\widetilde{\nu}_s(y)\,\frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\Big)\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&\qquad-\widetilde{\nu}_j(x)\int_{\partial\Omega}\widetilde{\nu}_l(y)\Big\{\sum_{s,h=1}^n\frac{\partial}{\partial y_s}\,\widetilde{g}(y)\,\frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}-\frac{D\widetilde{g}(y)a^{(2)}\widetilde{\nu}(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)} \\
&\qquad\qquad\times\Big(\widetilde{\nu}_s(y)\,\frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\Big)\Big\}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\,. \qquad (8.7)
\end{aligned}
$$

Since

$$
\widetilde{\nu}(y)=\nu(y)\,,\qquad\Big(\sum_{s,h=1}^n\widetilde{\nu}_s(y)\,\frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\Big)=1\quad\forall\,y\in\partial\Omega\,,
$$

we have

$$\Big\{ \sum_{s,h=1}^{n} \frac{\partial}{\partial y_s}\, \widetilde{g}(y)\, \frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)} - \frac{D\widetilde{g}(y)a^{(2)}\widetilde{\nu}(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)} \Big(\widetilde{\nu}_s(y)\, \frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\Big) \Big\} = 0$$

for all $y \in \partial\Omega$ and, accordingly, the right-hand side of (8.7) equals

$$\widetilde{\nu}_l(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \frac{\partial}{\partial x_j}\, \widetilde{g} - \frac{D\widetilde{g}a^{(2)}\widetilde{\nu}}{\nu^t a^{(2)}\nu}\, \widetilde{\nu}_j, \mu\Big](x) - \widetilde{\nu}_j(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \frac{\partial}{\partial x_l}\, \widetilde{g} - \frac{D\widetilde{g}a^{(2)}\widetilde{\nu}}{\nu^t a^{(2)}\nu}\, \widetilde{\nu}_l, \mu\Big](x)\,.$$

Consider the third term in the right-hand side of formula (8.6) and note that

$$\begin{aligned}\int_{\partial\Omega} \sum_{s=1}^{n} (\widetilde{g}(x) - \widetilde{g}(y))\, \frac{\partial}{\partial x_r}\, S_{\mathbf{a}}(x-y) \Big\{ \widetilde{\nu}_l(x) M_{sj}\Big[\frac{(a^{(2)}\nu)_s}{\nu^t a^{(2)}\nu}\, \mu\Big](y) - \widetilde{\nu}_j(x) M_{sl}\Big[\frac{(a^{(2)}\nu)_s}{\nu^t a^{(2)}\nu}\, \mu\Big](y) \Big\}\, d\sigma_y \\ = \widetilde{\nu}_l(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \widetilde{g}, \sum_{s=1}^{n} M_{sj}\Big[\frac{(a^{(2)}\nu)_s}{\nu^t a^{(2)}\nu}\, \mu\Big]\Big](x) \\ - \widetilde{\nu}_j(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \widetilde{g}, \sum_{s=1}^{n} M_{sl}\Big[\frac{(a^{(2)}\nu)_s}{\nu^t a^{(2)}\nu}\, \mu\Big]\Big](x)\,. \qquad (8.8)\end{aligned}$$

Next, we consider the last integral in the right-hand side of formula (8.6) and note that if $x \in \Omega$ and $y \in \partial\Omega$, we have

$$\sum_{s,h=1}^{n} \frac{\partial}{\partial x_h}\, \Big[a_{sh}\, \frac{\partial}{\partial x_s}\, S_{\mathbf{a}}(x-y)\Big] + \sum_{s=1}^{n} a_s\, \frac{\partial}{\partial x_s}\, S_{\mathbf{a}}(x-y) + aS_{\mathbf{a}}(x-y) = 0\,.$$

Thus we obtain

$$\begin{aligned}\sum_{s,h=1}^{n} a_{sh}\nu_h(y)\, \frac{\partial}{\partial x_r}\, \Big[\frac{\partial}{\partial y_s}\, S_{\mathbf{a}}(x-y)\Big] \\ = \sum_{s,h=1}^{n} a_{sh}\Big(\nu_h(y)\, \frac{\partial}{\partial y_r} - \nu_r(y)\, \frac{\partial}{\partial y_h}\Big)\Big[\frac{\partial}{\partial x_s}\, S_{\mathbf{a}}(x-y)\Big] + \nu_r(y) \sum_{s=1}^{n} a_s\, \frac{\partial}{\partial x_s}\, S_{\mathbf{a}}(x-y) + \nu_r(y)aS_{\mathbf{a}}(x-y)\,,\end{aligned}$$

and we note that the first parenthesis in the right-hand side equals $M_{hr,y}$. The last integral in the right-hand side of formula (8.6) equals

$$\begin{aligned}\int_{\partial\Omega} (\widetilde{g}(x) - \widetilde{g}(y)) \sum_{s,h=1}^{n} a_{sh}\nu_h(y)\, \frac{\partial}{\partial y_s}\, \Big[\frac{\partial}{\partial x_r}\, S_{\mathbf{a}}(x-y)\Big] \frac{\widetilde{\nu}_l(x)\nu_j(y) - \widetilde{\nu}_j(x)\nu_l(y)}{\nu^t(y)a^{(2)}\nu(y)}\, \mu(y)\, d\sigma_y \\ = \int_{\partial\Omega} (\widetilde{g}(x) - \widetilde{g}(y)) \Big\{ \sum_{s,h=1}^{n} a_{sh} M_{hr,y}\Big[\frac{\partial}{\partial x_s}\, S_{\mathbf{a}}(x-y)\Big] \\ + \nu_r(y) \sum_{s=1}^{n} a_s\, \frac{\partial}{\partial x_s}\, S_{\mathbf{a}}(x-y) + \nu_r(y)aS_{\mathbf{a}}(x-y) \Big\} \frac{\widetilde{\nu}_l(x)\nu_j(y) - \widetilde{\nu}_j(x)\nu_l(y)}{\nu^t(y)a^{(2)}\nu(y)}\, \mu(y)\, d\sigma_y \\ = \sum_{s,h=1}^{n} a_{sh} \int_{\partial\Omega} (\widetilde{g}(x) - \widetilde{g}(y)) M_{hr,y}\Big[\frac{\partial}{\partial x_s}\, S_{\mathbf{a}}(x-y)\Big] \\ \times \frac{\widetilde{\nu}_l(x)(\widetilde{\nu}_j(y) - \widetilde{\nu}_j(x)) + \widetilde{\nu}_j(x)(\widetilde{\nu}_l(x) - \widetilde{\nu}_l(y))}{\nu^t(y)a^{(2)}\nu(y)}\, \mu(y)\, d\sigma_y \\ + \int_{\partial\Omega} (\widetilde{g}(x) - \widetilde{g}(y)) \Big[\sum_{s=1}^{n} a_s\, \frac{\partial}{\partial x_s} S_{\mathbf{a}}(x-y) + aS_{\mathbf{a}}(x-y)\Big] \frac{\widetilde{\nu}_l(x)\nu_j(y) - \widetilde{\nu}_j(x)\nu_l(y)}{\nu^t(y)a^{(2)}\nu(y)}\, \nu_r(y)\mu(y)\, d\sigma_y\,. \qquad (8.9)\end{aligned}$$

We now consider separately each of the terms in the right-hand side of (8.9). By Lemma 2.2 and the equality $-M_{hr,y}[\widetilde{g}(x)-\widetilde{g}(y)] = M_{hr,y}[\widetilde{g}(y)]$, the first integral in the right-hand side of (8.9) equals

$$\begin{aligned}
&\int_{\partial\Omega} (\widetilde{g}(x)-\widetilde{g}(y)) M_{hr,y}\Big[\frac{\partial}{\partial x_s} S_{\mathbf{a}}(x-y)\Big] \\
&\qquad\qquad \times \frac{\widetilde{\nu}_l(x)(\widetilde{\nu}_j(y)-\widetilde{\nu}_j(x)) + \widetilde{\nu}_j(x)(\widetilde{\nu}_l(x)-\widetilde{\nu}_l(y))}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y \\
&= \int_{\partial\Omega} M_{hr}[\widetilde{g}]\,\frac{\partial}{\partial x_s} S_{\mathbf{a}}(x-y)\Big(-\widetilde{\nu}_l(x)\,\frac{\widetilde{\nu}_j(x)-\nu_j(y)}{\nu^t(y)a^{(2)}\nu(y)} + \widetilde{\nu}_j(x)\,\frac{\widetilde{\nu}_l(x)-\nu_l(y)}{\nu^t(y)a^{(2)}\nu(y)}\Big)\mu(y)\,d\sigma_y \\
&\quad + \int_{\partial\Omega} (\widetilde{g}(x)-\widetilde{g}(y))\,\frac{\partial}{\partial x_s} S_{\mathbf{a}}(x-y) \\
&\qquad \times \bigg(-\widetilde{\nu}_l(x) M_{hr}\Big[\frac{\nu_j\mu}{\nu^t a^{(2)}\nu}\Big](y) + \widetilde{\nu}_j(x) M_{hr}\Big[\frac{\nu_l\mu}{\nu^t a^{(2)}\nu}\Big](y)\bigg)\,d\sigma_y \\
&= -\widetilde{\nu}_l(x)\int_{\partial\Omega} (\widetilde{\nu}_j(x)-\nu_j(y))\,\frac{\partial}{\partial x_s} S_{\mathbf{a}}(x-y)\,\frac{M_{hr}[\widetilde{g}]}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y \\
&\quad + \widetilde{\nu}_j(x)\int_{\partial\Omega} (\widetilde{\nu}_l(x)-\nu_l(y))\,\frac{\partial}{\partial x_s} S_{\mathbf{a}}(x-y)\,\frac{M_{hr}[\widetilde{g}]}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y \\
&\quad - \widetilde{\nu}_l(x)\int_{\partial\Omega} (\widetilde{g}(x)-\widetilde{g}(y))\,\frac{\partial}{\partial x_s} S_{\mathbf{a}}(x-y) M_{hr}\Big[\frac{\nu_j\mu}{\nu^t a^{(2)}\nu}\Big](y)\,d\sigma_y \\
&\quad + \widetilde{\nu}_j(x)\int_{\partial\Omega} (\widetilde{g}(x)-\widetilde{g}(y))\,\frac{\partial}{\partial x_s} S_{\mathbf{a}}(x-y) M_{hr}\Big[\frac{\nu_l\mu}{\nu^t a^{(2)}\nu}\Big](y)\,d\sigma_y \\
&= -\widetilde{\nu}_l(x)\bigg\{Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{\nu}_j, \frac{M_{hr}[g]\mu}{\nu^t a^{(2)}\nu}\Big](x) + Q^\sharp\bigg[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{g}, M_{hr}\Big[\frac{\nu_j\mu}{\nu^t a^{(2)}\nu}\Big]\bigg](x)\bigg\} \\
&\quad + \widetilde{\nu}_j(x)\bigg\{Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{\nu}_l, \frac{M_{hr}[g]\mu}{\nu^t a^{(2)}\nu}\Big](x) + Q^\sharp\bigg[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{g}, M_{hr}\Big[\frac{\nu_l\mu}{\nu^t a^{(2)}\nu}\Big]\bigg](x)\bigg\}. \qquad (8.10)
\end{aligned}$$

Next, we note that the second integral in the right-hand side of (8.9) equals

$$\begin{aligned}
&\sum_{s=1}^n a_s\bigg\{\widetilde{\nu}_l(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{g}, \frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x) - \widetilde{\nu}_j(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{g}, \frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)\bigg\} \\
&\quad + a\bigg\{\widetilde{g}(x)\bigg[\widetilde{\nu}_l(x)v\Big[\partial\Omega, S_{\mathbf{a}}, \frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x) - \widetilde{\nu}_j(x)v\Big[\partial\Omega, S_{\mathbf{a}}, \frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)\bigg] \\
&\qquad - \bigg[\widetilde{\nu}_l(x)v\Big[\partial\Omega, S_{\mathbf{a}}, g\,\frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x) - \widetilde{\nu}_j(x)v\Big[\partial\Omega, S_{\mathbf{a}}, g\,\frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)\bigg]\bigg\}.
\end{aligned}$$

By combining formulas (8.6)–(8.10), we obtain

$$\begin{aligned}
&M^\sharp_{lj}\Big[Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \widetilde{g}, \mu\Big]\Big](x) = \widetilde{\nu}_l(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \frac{\partial}{\partial x_j}\widetilde{g} - \frac{D\widetilde{g}a^{(2)}\widetilde{\nu}}{\nu^t a^{(2)}\nu}\,\widetilde{\nu}_j, \mu\Big](x) \\
&- \widetilde{\nu}_j(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \frac{\partial}{\partial x_l}\widetilde{g} - \frac{D\widetilde{g}a^{(2)}\widetilde{\nu}}{\nu^t a^{(2)}\nu}\,\widetilde{\nu}_l, \mu\Big](x) + \widetilde{\nu}_l(x)Q^\sharp\bigg[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \widetilde{g}, \sum_{s=1}^n M_{sj}\Big[\sum_{h=1}^n \frac{a_{sh}\nu_h}{\nu^t a^{(2)}\nu}\,\mu\Big]\bigg](x) \\
&\qquad - \widetilde{\nu}_j(x)Q^\sharp\bigg[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \widetilde{g}, \sum_{s=1}^n M_{sl}\Big[\sum_{h=1}^n \frac{a_{sh}\nu_h}{\nu^t a^{(2)}\nu}\,\mu\Big]\bigg](x) \\
&+ \sum_{s,h=1}^n a_{sh}\widetilde{\nu}_l(x)\bigg\{Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \nu_j, \frac{M_{hr}[g]\mu}{\nu^t a^{(2)}\nu}\Big](x) + Q^\sharp\bigg[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{g}, M_{hr}\Big[\frac{\nu_j\mu}{\nu^t a^{(2)}\,\nu}\Big]\bigg](x)\bigg\}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{s,h=1}^{n} a_{sh}\widetilde{\nu}_j(x)\Big\{Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,\nu_l,\frac{M_{hr}[g]\mu}{\nu^t a^{(2)}\nu}\Big](x)+Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,\widetilde{g},M_{hr}\Big[\frac{\nu_l\mu}{\nu^t a^{(2)}\nu}\Big]\Big](x)\Big\}\\
&-\sum_{s=1}^{n} a_s\Big\{\widetilde{\nu}_l(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,\widetilde{g},\frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\,\mu\Big](x)-\widetilde{\nu}_j(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,\widetilde{g},\frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\,\mu\Big](x)\Big\}\\
&\quad - a\Big\{g(x)\Big[\widetilde{\nu}_l(x)v\Big[\partial\Omega,S_{\mathbf{a}},\frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\,\mu\Big](x)-\widetilde{\nu}_j(x)v\Big[\partial\Omega,S_{\mathbf{a}},\frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\,\mu\Big](x)\Big]\\
&\qquad -\Big[\widetilde{\nu}_l(x)v\Big[\partial\Omega,S_{\mathbf{a}},g\,\frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\,\mu\Big](x)-\widetilde{\nu}_j(x)v\Big[\partial\Omega,S_{\mathbf{a}},g\,\frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\,\mu\Big](x)\Big]\Big\}. \qquad (8.11)
\end{aligned}
$$

Under our assumptions, the first argument of the maps $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\cdot,\cdot]$ and $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,\cdot,\cdot]$, which appear in the right-hand side of (8.11) belongs to the space $C^{0,\min\{\alpha,\theta\}}(\mathrm{cl}\,\Omega)$ and the second argument of the maps $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\cdot,\cdot]$, $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,\cdot,\cdot]$, which appear in the right-hand side of (8.11) belongs to $C^0(\partial\Omega)$. By Theorem 7.1(i) with $m=1$, the single layer potentials in the right-hand side of (8.11) are continuous in $x\in\mathrm{cl}\,\Omega$. Then Theorem 8.1(i) implies that the right-hand side of (8.11) defines a continuous function of the variable $x\in\mathrm{cl}\,\Omega$. Since Ω is of the class $C^{2,\alpha}$ and $\widetilde{g}\in C^{1,\theta}(\mathrm{cl}\,\Omega)$ and since we are assuming that $\mu\in C^{1,\beta}(\partial\Omega)$, Theorem 8.1(ii) implies that $M^\sharp_{lj}[Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu]]$ belongs to $C^0(\mathrm{cl}\,\Omega)$. Hence, the equation of (8.11) must hold for all $x\in\mathrm{cl}\,\Omega$ and, in particular, for all $x\in\partial\Omega$. Since $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\cdot,\cdot]=Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\cdot,\cdot]$ and $M^\sharp_{lj}=M_{lj}$ on $\partial\Omega$, we conclude that (8.5) holds.

Next, we assume that $\mu\in C^1(\partial\Omega)$. We denote by $P_{ljr}[g,\mu]$ the right-hand side of (8.5). By Theorem 8.2(i), the operators $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\cdot]$, $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,D_{\mathbf{a},j}g,\cdot]$, $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\nu_l,\cdot]$ are linear and continuous from the space $C^0(\partial\Omega)$ to $C^0(\partial\Omega)$. By Theorem 7.2 and by the continuity of the pointwise product in $C^0(\partial\Omega)$, the operator $P_{ljr}[g,\cdot]$ is continuous from $C^0(\partial\Omega)$ to $C^0(\partial\Omega)$. In particular, $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu]$, $P_{ljr}[g,\mu]\in C^0(\partial\Omega)$.

We now show that the weak M_{lj}-derivative of $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\cdot]$ in $\partial\Omega$ coincides with $P_{ljr}[g,\mu]$.

Considering both an extension of μ of the class C^1 with a compact support in $\mathbb{R}^n$ and a sequence of mollifiers of such an extension, and then taking the restriction to $\partial\Omega$, we can conclude that there exists a sequence of functions $\{\mu_b\}_{b\in\mathbb{N}}$ in $C^2(\partial\Omega)$ converging to μ in $C^1(\partial\Omega)$. We note that if $\varphi\in C^1(\partial\Omega)$, then the validity of (8.5) for $\mu_b\in C^2(\partial\Omega)\subseteq C^{1,\beta}(\partial\Omega)$, the membership of $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu_b]$ in $C^1(\partial\Omega)$ (see Theorem 8.1(ii)) and Lemma 2.2 imply that

$$
\begin{aligned}
&\int_{\partial\Omega} Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu\Big]M_{lj}[\varphi]\,d\sigma=\lim_{b\to\infty}\int_{\partial\Omega} Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu_b\Big]M_{lj}[\varphi]\,d\sigma\\
&\qquad=-\lim_{b\to\infty}\int_{\partial\Omega} M_{lj}\Big[Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu_b\Big]\Big]\varphi\,d\sigma=-\lim_{b\to\infty}\int_{\partial\Omega} P_{ljr}[g,\mu_b]\varphi\,d\sigma=-\int_{\partial\Omega} P_{ljr}[g,\mu]\varphi\,d\sigma\,.
\end{aligned}
$$

Hence, $P_{ljr}[g,\mu]$ coincides with the weak M_{lj}-derivative of $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu]$ for all $l,j\in\{1,\dots,n\}$. Since both $P_{ljr}[g,\mu]$ and $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu]$ are the continuous functions, it follows that $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu]\in C^1(\partial\Omega)$ and $M_{lj}[Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu]]=P_{ljr}[g,\mu]$, classically. Hence (8.5) holds also for $\mu\in C^1(\partial\Omega)$. □

By exploiting formula (8.5), we can prove the following theorem.

Theorem 8.3. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a},D]$, $\alpha\in]0,1[$, $m\in\mathbb{N}\setminus\{0\}$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{m,\alpha}$ *and let* $r\in\{1,\dots,n\}$. *Then the following statements hold:*

(i) *Let* $\theta\in]0,1]$. *Then the bilinear map* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\cdot,\cdot]$ *from the space* $C^{m-1,\theta}(\partial\Omega)\times C^{m-1}(\partial\Omega)$ *to* $C^{m-1,\omega_\theta(\cdot)}(\partial\Omega)$, *which takes a pair* (g,μ) *to* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu]$, *is continuous.*

(ii) *Let* $\beta\in]0,1]$. *Then the bilinear map* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\cdot,\cdot]$ *from the space* $C^{m-1,\alpha}(\partial\Omega)\times C^{m-1,\beta}(\partial\Omega)$ *to* $C^{m-1,\alpha}(\partial\Omega)$, *which takes a pair* (g,μ) *to* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu]$, *is continuous.*

Proof. We first prove statement (i). We proceed by induction on m. Case $m = 1$ holds by Theorem 8.2(i). We now prove that if the statement holds for m, then it holds for $m+1$. Thus we now assume that Ω is of the class $C^{m+1,\alpha}$, and we turn to prove that $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$ is bilinear and continuous from $C^{m,\theta}(\partial\Omega) \times C^m(\partial\Omega)$ to $C^{m,\omega_\theta(\cdot)}(\partial\Omega)$. By Lemma 2.3(ii), it suffices to prove that the following two statements hold:

(j) $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$ is continuous from $C^{m,\theta}(\partial\Omega) \times C^m(\partial\Omega)$ to $C^0(\partial\Omega)$;

(jj) $M_{lj}[Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]]$ is continuous from $C^{m,\theta}(\partial\Omega) \times C^m(\partial\Omega)$ to the space $C^{m-1,\omega_\theta(\cdot)}(\partial\Omega)$ for all $l,\, j \in \{1, \dots, n\}$.

Statement (j) holds by the case $m = 1$, and by the imbedding of $C^{m,\theta}(\partial\Omega) \times C^m(\partial\Omega)$ into $C^{0,\theta}(\partial\Omega) \times C^0(\partial\Omega)$. We now prove statement (jj). Since $m + 1 \geq 2$, Lemma 8.1 and the inductive assumption imply that we can actually apply M_{lj} to $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$. We find it convenient to denote by $P_{ljr}[g, \mu]$ the right-hand side of formula (8.5). Then we have

$$M_{lj}\Big[Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, \mu\Big]\Big] = P_{ljr}[g, \mu] \quad \forall\, (g, \mu) \in C^{m,\theta}(\partial\Omega) \times C^m(\partial\Omega)\,.$$

By Lemma 2.4 and the membership of ν in $C^{m,\alpha}(\partial\Omega, \mathbb{R}^n)$, which is contained in $C^{m-1,1}(\partial\Omega, \mathbb{R}^n)$, by the continuity of the pointwise product in Schauder spaces, by the continuity of the imbedding of $C^m(\partial\Omega)$ into $C^{m-1}(\partial\Omega)$ and of $C^{m,\alpha}(\partial\Omega)$ into $C^{m-1,\theta}(\partial\Omega)$, by the inductive assumption on the continuity of $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$, by the continuity of $v[\partial\Omega, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega) \subseteq C^{m-1,\theta}(\partial\Omega)$, and by the continuity of the imbedding of $C^m(\partial\Omega)$ into $C^{m-1,\alpha}(\partial\Omega)$ and of $C^m(\partial\Omega)$ into $C^{m-1,\omega_\theta(\cdot)}(\partial\Omega)$, and by the continuity of $D_{\mathbf{a}}$ from $C^{m,\theta}(\partial\Omega)$ to $C^{m-1,\theta}(\partial\Omega)$, we conclude that $P_{ljr}[\cdot, \cdot]$ is bilinear and continuous from $C^{m,\theta}(\partial\Omega) \times C^m(\partial\Omega)$ to $C^{m-1,\omega_\theta(\cdot)}(\partial\Omega)$, and the proof of statement (jj) and, accordingly, of statement (i) is complete. The proof of statement (ii) follows the lines of the proof of statement (i), by replacing the use of Theorem 8.2(i) with that of Theorem 8.2(ii). □

Definition 8.1. Let $\mathbf{a}$ be as in (1.1), (1.2), (1.3), $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a}, D]$, $\alpha \in]0,1[$ and let Ω be a bounded open subset of $\mathbb{R}^n$ of the class $C^{1,\alpha}$. Then we set

$$\begin{aligned} R[g,h,\mu] \equiv \sum_{r=1} a_r \Big\{ Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, gh, \mu\Big] - gQ\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, h, \mu\Big] - Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, h, g\mu\Big] \Big\} \\ + a\Big\{ gv[\partial\Omega, S_{\mathbf{a}}, h\mu] - hv[\partial\Omega, S_{\mathbf{a}}, g\mu] \Big\} \end{aligned}$$

for all $(g, h, \mu) \in (C^{0,\alpha}(\partial\Omega))^2 \times C^0(\partial\Omega)$.

Since

$$g(x)h(y) - g(y)h(x) = \big[g(x)h(x) - g(y)h(y)\big] - g(x)[h(x) - h(y)] - g(y)[h(x) - h(y)] \quad \forall\, x, y \in \partial\Omega\,,$$

we have

$$R[g,h,\mu] = \int_{\partial\Omega} \Big\{ \sum_{r=1}^{n} a_r \frac{\partial}{\partial x_r} S_{\mathbf{a}}(x-y) + aS_{\mathbf{a}}(x-y) \Big\} \big[g(x)h(y) - g(y)h(x)\big] \mu(y)\, d\sigma_y \quad \forall\, x \in \partial\Omega\,.$$

Since R is a composition of the operator $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$ and of a single layer potential, Theorems 7.1, 7.2 and 8.3, the continuity of the product in Schauder spaces and also of the imbeddings of $C^{m-1}(\partial\Omega)$ into $C^{m-2,\alpha}(\partial\Omega)$ for $m \geq 2$, of $C^{m-1,\alpha}(\partial\Omega)$ into $C^{m-1,\omega_\alpha(\cdot)}(\partial\Omega)$ and also of $C^{m,\beta}(\partial\Omega)$ into $C^{m-1,\alpha}(\partial\Omega)$, imply that the following theorem is valid.

Theorem 8.4. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$, $\alpha \in]0,1[$, $m \in \mathbb{N} \setminus \{0\}$ *and let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{m,\alpha}$. *Then the following statements hold:*

(i) *The trilinear map R from the space $(C^{m-1,\alpha}(\partial\Omega))^2 \times C^{m-1}(\partial\Omega)$ to $C^{m-1,\omega_\alpha(\cdot)}(\partial\Omega)$, which takes a triple (g,h,μ) to $R[g,h,\mu]$, is continuous.*

(ii) *Let $\beta \in]0,1]$. Then the trilinear map R from the space $(C^{m-1,\alpha}(\partial\Omega))^2 \times C^{m-1,\beta}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$, which takes a triple (g,h,μ) to $R[g,h,\mu]$, is continuous.*

9 Tangential derivatives and regularizing properties of the double layer potential

We now exploit Theorems 7.3, 7.4, Lemma 8.1 and Theorems 8.3, 8.4 in order to prove a formula for the tangential derivatives of the double layer potential, which generalizes the corresponding formula of Hofmann, Mitrea and Taylor [16, (6.2.6)] for homogeneous operators. We do so by means of the following

Theorem 9.1. *Let $\mathbf{a}$ be as in* (1.1), (1.2), (1.3), *$S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a},D]$, $\alpha \in]0,1[$ and let Ω be a bounded open subset of $\mathbb{R}^n$ of the class $C^{1,\alpha}$. If $\mu \in C^1(\partial\Omega)$, then $w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]_{|\partial\Omega} \in C^1(\partial\Omega)$ and*

$$
\begin{aligned}
M_{lj}\big[w\big[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu\big]_{|\partial\Omega}\big] &= w\big[\partial\Omega,\mathbf{a},S_{\mathbf{a}},M_{lj}[\mu]\big]_{|\partial\Omega} \\
&\quad + \sum_{b,r=1}^{n} a_{br}\Big\{Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_b}\circ\Theta,\nu_l,M_{jr}[\mu]\Big] - Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_b}\circ\Theta,\nu_j,M_{lr}[\mu]\Big]\Big\} \\
&\qquad + \nu_l Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_j}\circ\Theta,\nu\cdot a^{(1)},\mu\Big] - \nu_j Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_l}\circ\Theta,\nu\cdot a^{(1)},\mu\Big] \\
&\quad + \nu\cdot a^{(1)}\Big\{Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_l}\circ\Theta,\nu_j,\mu\Big] - Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_j}\circ\Theta,\nu_l,\mu\Big]\Big\} \\
&\quad - \nu\cdot a^{(1)} v\big[\partial\Omega,S_{\mathbf{a}},M_{lj}[\mu]\big] + v\big[\partial\Omega,S_{\mathbf{a}},\nu\cdot a^{(1)}M_{lj}[\mu]\big] + R[\nu_l,\nu_j,\mu] \quad \text{on } \partial\Omega
\end{aligned}
\tag{9.1}
$$

for all $l,j \in \{1,\dots,n\}$. (For Q see (8.2).)

Proof. Fix $\beta \in]0,\alpha[$. First consider the specific case in which $\mu \in C^{1,\beta}(\partial\Omega)$. Let $R \in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$. Let '$\sim$' be an extension operator of $C^{1,\beta}(\partial\Omega)$ to $C^{1,\beta}(\mathrm{cl}\,\mathbb{B}_n(0,R))$ as in Lemma 2.1. By Theorem 7.3(i),(ii), we have $w^+[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu] \in C^{1,\beta}(\mathrm{cl}\,\Omega)$ and

$$
M_{lj}\big[w^+[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]_{|\partial\Omega}\big] = \frac{1}{2}M_{lj}[\mu] + M_{lj}\big[w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]_{|\partial\Omega}\big]\,. \tag{9.2}
$$

By the definition of M_{lj} and by equality (7.2), we obtain

$$
\begin{aligned}
M_{lj}\big[w^+[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]_{|\partial\Omega}\big] &= \nu_l\frac{\partial}{\partial x_j}w^+[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu] - \nu_j\frac{\partial}{\partial x_l}w^+[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu] \\
&= \nu_l\Big[\sum_{b,r=1}^{n} a_{br}\frac{\partial}{\partial x_b}v^+\big[\partial\Omega,S_{\mathbf{a}},M_{jr}[\mu]\big] + \sum_{b=1}^{n} a_b\frac{\partial}{\partial x_b}v^+[\partial\Omega,S_{\mathbf{a}},\nu_j\mu] \\
&\qquad - \frac{\partial}{\partial x_j}v^+[\partial\Omega,S_{\mathbf{a}},(\nu^t\cdot a^{(1)})\mu] + av^+[\partial\Omega,S_{\mathbf{a}},\nu_j\mu]\Big] \\
&\quad - \nu_j\Big[\sum_{b,r=1}^{n} a_{br}\frac{\partial}{\partial x_b}v^+\big[\partial\Omega,S_{\mathbf{a}},M_{lr}[\mu]\big] + \sum_{b=1}^{n} a_b\frac{\partial}{\partial x_b}v^+[\partial\Omega,S_{\mathbf{a}},\nu_l\mu] \\
&\qquad - \frac{\partial}{\partial x_l}v^+[\partial\Omega,S_{\mathbf{a}},(\nu^t\cdot a^{(1)})\mu] + av^+[\partial\Omega,S_{\mathbf{a}},\nu_l\mu]\Big] \\
&= \sum_{b,r=1}^{n} a_{br}\Big\{\nu_l\frac{\partial}{\partial x_b}v^+\big[\partial\Omega,S_{\mathbf{a}},M_{jr}[\mu]\big] - \nu_j\frac{\partial}{\partial x_b}v^+\big[\partial\Omega,S_{\mathbf{a}},M_{lr}[\mu]\big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{b=1}^{n} a_b \Big\{ \nu_l \frac{\partial}{\partial x_b} v^+[\partial\Omega, S_{\mathbf{a}}, \nu_j \mu] - \nu_j \frac{\partial}{\partial x_b} v^+[\partial\Omega, S_{\mathbf{a}}, \nu_l \mu] \Big\} \\
&- \Big\{ \nu_l \frac{\partial}{\partial x_j} v^+[\partial\Omega, S_{\mathbf{a}}, (\nu^t \cdot a^{(1)})\mu] - \nu_j \frac{\partial}{\partial x_l} v^+[\partial\Omega, S_{\mathbf{a}}, (\nu^t \cdot a^{(1)})\mu] \Big\} \\
&\qquad + a \Big\{ \nu_l v[\partial\Omega, S_{\mathbf{a}}, \nu_j \mu] - \nu_j v[\partial\Omega, S_{\mathbf{a}}, \nu_l \mu] \Big\} \quad \text{on } \partial\Omega . \qquad (9.3)
\end{aligned}
$$

We now consider the first term in braces in the right–hand side of (9.3) and note that

$$
\begin{aligned}
&\Big\{ \nu_l(x) \frac{\partial}{\partial x_b} v^+\big[\partial\Omega, S_{\mathbf{a}}, M_{jr}[\mu]\big](x) - \nu_j \frac{\partial}{\partial x_b} v^+\big[\partial\Omega, S_{\mathbf{a}}, M_{lr}[\mu]\big](x) \Big\} \\
&\qquad = -\frac{\nu_l(x)\nu_b(x)}{2\nu^t(x) a^{(2)} \nu(x)} M_{jr}[\mu](x) + \nu_l(x) \int_{\partial\Omega} \frac{\partial}{\partial x_b} S_{\mathbf{a}}(x-y) M_{jr}[\mu](y)\, d\sigma_y \\
&\quad + \frac{\nu_j(x)\nu_b(x)}{2\nu^t(x) a^{(2)} \nu(x)} M_{lr}[\mu](x) - \nu_j(x) \int_{\partial\Omega} \frac{\partial}{\partial x_b} S_{\mathbf{a}}(x-y) M_{lr}[\mu](y)\, d\sigma_y \\
&\qquad = \nu_b(x) \frac{-\nu_l(x) M_{jr}[\mu](x) + \nu_j(x) M_{lr}[\mu](x)}{2\nu^t(x) a^{(2)} \nu(x)} \\
&\qquad\qquad + \int_{\partial\Omega} \frac{\partial}{\partial x_b} S_{\mathbf{a}}(x-y) \big\{ \nu_l(x) M_{jr}[\mu](y) - \nu_j(x) M_{lr}[\mu](y) \big\}\, d\sigma_y . \qquad (9.4)
\end{aligned}
$$

Further, we note that

$$
\big[\nu_l M_{jr}[\mu] - \nu_j M_{lr}[\mu]\big] = \nu_l \nu_j \frac{\partial \mu}{\partial x_r} - \nu_l \nu_r \frac{\partial \mu}{\partial x_j} - \nu_j \nu_l \frac{\partial \mu}{\partial x_r} + \nu_j \nu_r \frac{\partial \mu}{\partial x_l} = -\nu_r M_{lj}[\mu] \quad \text{on } \partial\Omega . \qquad (9.5)
$$

Then we obtain

$$
\begin{aligned}
&\sum_{b,r=1}^{n} a_{br} \nu_b \frac{-\nu_l M_{jr}[\mu] + \nu_j M_{lr}[\mu]}{2\nu^t a^{(2)} \nu} \\
&\qquad = \sum_{b,r=1}^{n} a_{br} \nu_b \frac{\nu_r M_{lj}[\mu]}{2\nu^t a^{(2)} \nu} = \frac{\sum_{b,r=1}^{n} \nu_b a_{br} \nu_r}{2\nu^t a^{(2)} \nu} M_{lj}[\mu] = \frac{1}{2} M_{lj}[\mu] \quad \text{on } \partial\Omega . \qquad (9.6)
\end{aligned}
$$

Consider the term in braces in the argument of the integral in the right-hand side of (9.4) and note that equality (9.5) yields

$$
\begin{aligned}
&\nu_l(x) M_{jr}[\mu](y) - \nu_j(x) M_{lr}[\mu](y) \\
&\quad = [\nu_l(x) - \nu_l(y)] M_{jr}[\mu](y) + \big[\nu_l(y) M_{jr}[\mu](y) - \nu_j(y) M_{lr}[\mu](y)\big] - [\nu_j(x) - \nu_j(y)] M_{lr}[\mu](y) \\
&\quad = [\nu_l(x) - \nu_l(y)] M_{jr}[\mu](y) - \nu_r(y) M_{lj}[\mu](y) - [\nu_j(x) - \nu_j(y)] M_{lr}[\mu](y) \quad \forall\, x, y \in \partial\Omega . \qquad (9.7)
\end{aligned}
$$

We now consider the term in the second braces in the right-hand side of equality (9.3) and we note that

$$
\begin{aligned}
&\nu_l(x) \frac{\partial}{\partial x_b} v^+[\partial\Omega, S_{\mathbf{a}}, \nu_j \mu](x) - \nu_j(x) \frac{\partial}{\partial x_b} v^+[\partial\Omega, S_{\mathbf{a}}, \nu_l \mu](x) \\
&\qquad = -\nu_l(x) \frac{\nu_b(x)}{2\nu^t(x) a^{(2)} \nu(x)} \nu_j(x) \mu(x) + \nu_l(x) \int_{\partial\Omega} \frac{\partial}{\partial x_b} S_{\mathbf{a}}(x-y) \nu_j(y) \mu(y)\, d\sigma_y \\
&\qquad\quad + \nu_j(x) \frac{\nu_b(x)}{2\nu^t(x) a^{(2)} \nu(x)} \nu_l(x) \mu(x) - \nu_j(x) \int_{\partial\Omega} \frac{\partial}{\partial x_b} S_{\mathbf{a}}(x-y) \nu_l(y) \mu(y)\, d\sigma_y \\
&\qquad = \int_{\partial\Omega} \frac{\partial}{\partial x_b} S_{\mathbf{a}}(x-y) [\nu_l(x) \nu_j(y) - \nu_j(x) \nu_l(y)] \mu(y)\, d\sigma_y \quad \forall\, x \in \partial\Omega . \qquad (9.8)
\end{aligned}
$$

Next, we consider the term in the third braces in the right-hand side of equality (9.3) and we note that

$$
\begin{aligned}
&\nu_l(x)\,\frac{\partial}{\partial x_j}\,v^+[\partial\Omega, S_{\mathbf{a}}, (\nu^t\cdot a^{(1)})\mu](x) - \nu_j(x)\,\frac{\partial}{\partial x_l}v^+[\partial\Omega, S_{\mathbf{a}}, (\nu^t\cdot a^{(1)})\mu](x)\\
&\quad = -\nu_l(x)\,\frac{\nu_j(x)}{2\nu^t(x)a^{(2)}\nu(x)}\,(\nu^t(x)\cdot a^{(1)})\mu(x) + \nu_l(x)\int_{\partial\Omega}\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\nu^t(y)\cdot a^{(1)}\mu(y)\,d\sigma_y\\
&\quad + \nu_j(x)\,\frac{\nu_l(x)}{2\nu^t(x)a^{(2)}\nu(x)}\,(\nu^t(x)\cdot a^{(1)})\mu(x) - \nu_j(x)\int_{\partial\Omega}\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\nu^t(y)\cdot a^{(1)}\mu(y)\,d\sigma_y\\
&\quad = -\nu_l(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)}) - (\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\qquad + \nu_l(x)\int_{\partial\Omega}(\nu^t(x)\cdot a^{(1)})\,\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\qquad + \nu_j(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)}) - (\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\qquad - \nu_j(x)\int_{\partial\Omega}(\nu^t(x)\cdot a^{(1)})\,\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\quad = -\nu_l(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)}) - (\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\qquad + \nu_j(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)}) - (\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\qquad + (\nu^t(x)\cdot a^{(1)})\int_{\partial\Omega}\Big(\nu_l(x)\,\frac{\partial}{\partial x_j} - \nu_j(x)\,\frac{\partial}{\partial x_l}\Big)S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\quad = -\nu_l(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)}) - (\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\qquad + \nu_j(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)}) - (\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&+ (\nu^t(x)\cdot a^{(1)})\Big\{\int_{\partial\Omega}(\nu_l(x)-\nu_l(y))\,\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y - \int_{\partial\Omega}(\nu_j(x)-\nu_j(y))\,\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\Big\}\\
&\qquad + (\nu^t(x)\cdot a^{(1)})\int_{\partial\Omega}\Big(\nu_l(y)\,\frac{\partial}{\partial x_j} - \nu_j(y)\,\frac{\partial}{\partial x_l}\Big)S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y
\end{aligned}
\tag{9.9}
$$

for all $x\in\partial\Omega$. By Lemma 2.2, the last integral in the right-hand side of (9.9) equals

$$
-\int_{\partial\Omega}M_{lj,y}[S_{\mathbf{a}}(x-y)]\mu(y)\,d\sigma_y = \int_{\partial\Omega}S_{\mathbf{a}}(x-y)M_{lj}[\mu](y)\,d\sigma_y \quad \forall\, x\in\partial\Omega\,. \tag{9.10}
$$

Thus the last term in the right-hand side of (9.9) equals

$$
\begin{aligned}
(\nu^t(x)\cdot a^{(1)})\int_{\partial\Omega}S_{\mathbf{a}}(x-y)M_{lj}[\mu](y)\,d\sigma_y &= \int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)}) - (\nu^t(y)\cdot a^{(1)})\big]S_{\mathbf{a}}(x-y)M_{lj}[\mu](y)\,d\sigma_y\\
&+ \int_{\partial\Omega}(\nu^t(y)\cdot a^{(1)})S_{\mathbf{a}}(x-y)M_{lj}[\mu](y)\,d\sigma_y \quad \forall\, x\in\partial\Omega\,.
\end{aligned}
\tag{9.11}
$$

The last term in braces of equation (9.3) equals

$$\int_{\partial\Omega} S_{\mathbf{a}}(x-y)\big[\nu_l(x)\nu_j(y)-\nu_j(x)\nu_l(y)\big]\mu(y)\,d\sigma_y \quad \forall\, x\in\partial\Omega\,. \tag{9.12}$$

Combining (9.2)–(9.4), (9.6)–(9.12), we obtain

$$\begin{aligned}
M_{lj}\big[w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]\big](x) &= \sum_{b,r=1}^{n} a_{br}\Big\{\int_{\partial\Omega}(\nu_l(x)-\nu_l(y))\,\frac{\partial}{\partial x_b}\,S_{\mathbf{a}}(x-y)M_{jr}[\mu](y)\,d\sigma_y \\
&\quad -\int_{\partial\Omega}(\nu_j(x)-\nu_j(y))\,\frac{\partial}{\partial x_b}\,S_{\mathbf{a}}(x-y)M_{lr}[\mu](y)\,d\sigma_y-\int_{\partial\Omega}\nu_r(y)\,\frac{\partial}{\partial x_b}\,S_{\mathbf{a}}(x-y)M_{lj}[\mu](y)\,d\sigma_y\Big\} \\
&\quad +\sum_{b=1}^{n} a_b\int_{\partial\Omega}\frac{\partial}{\partial x_b}\,S_{\mathbf{a}}(x-y)\big[\nu_l(x)\nu_j(y)-\nu_j(x)\nu_l(y)\big]\mu(y)\,d\sigma_y \\
&\quad +\nu_l(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)})-(\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&\quad -\nu_j(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)})-(\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&\quad -(\nu^t(x)\cdot a^{(1)})\Big\{\int_{\partial\Omega}(\nu_l(x)-\nu_l(y))\,\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y-\int_{\partial\Omega}(\nu_j(x)-\nu_j(y))\,\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\Big\} \\
&\quad -\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)})-(\nu^t(y)\cdot a^{(1)})\big]S_{\mathbf{a}}(x-y)M_{lj}[\mu](y)\,d\sigma_y-\int_{\partial\Omega}(\nu^t(y)\cdot a^{(1)})S_{\mathbf{a}}(x-y)M_{lj}[\mu](y)\,d\sigma_y \\
&\quad +a\int_{\partial\Omega}S_{\mathbf{a}}(x-y)\big[\nu_l(x)\nu_j(y)-\nu_j(x)\nu_l(y)\big]\mu(y)\,d\sigma_y \quad \forall\, x\in\partial\Omega\,,
\end{aligned}$$

which we rewrite as

$$\begin{aligned}
M_{lj}\big[w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]\big](x) &= \sum_{b,r=1}^{n} a_{br}\Big\{Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_b}\circ\Theta,\nu_l,M_{jr}[\mu]\Big](x)-Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_b}\circ\Theta,\nu_j,M_{lr}[\mu]\Big](x)\Big\} \\
&\quad +\nu_l(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_j}\circ\Theta,\nu^t\cdot a^{(1)},\mu\Big](x)-\nu_j(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_l}\circ\Theta,\nu^t\cdot a^{(1)},\mu\Big](x) \\
&\quad +w\big[\partial\Omega,\mathbf{a},S_{\mathbf{a}},M_{lj}[\mu]\big](x)+(\nu^t(x)\cdot a^{(1)})\Big\{Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_l}\circ\Theta,\nu_j,\mu\Big](x)-Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_j}\circ\Theta,\nu_l,\mu\Big](x)\Big\} \\
&\quad -(\nu^t(x)\cdot a^{(1)})v\big[\partial\Omega,S_{\mathbf{a}},M_{lj}[\mu]\big](x)+v\big[\partial\Omega,S_{\mathbf{a}},(\nu^t\cdot a^{(1)})M_{lj}[\mu]\big](x)+R[\nu_l,\nu_j,\mu](x) \quad \forall\, x\in\partial\Omega\,.
\end{aligned}$$

Thus we have proved formula (9.1) for $\mu\in C^{1,\beta}(\partial\Omega)$.

Next, we assume that $\mu\in C^1(\partial\Omega)$. We denote by $T_{lj}[\mu]$ the right-hand side of (9.1). By the continuity of M_{lj} from $C^1(\partial\Omega)$ to $C^0(\partial\Omega)$, of $w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\cdot\,]_{|\partial\Omega}$ and $v[\partial\Omega,S_{\mathbf{a}},\cdot\,]_{|\partial\Omega}$ from $C^0(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$, of $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\,\cdot\,,\,\cdot\,]$ from $C^{0,\alpha}(\partial\Omega)\times C^0(\partial\Omega)$ to $C^{0,\omega_\alpha}(\partial\Omega)$, of R from $(C^{0,\alpha}(\partial\Omega))^2\times C^0(\partial\Omega)$ to $C^{0,\omega_\alpha}(\partial\Omega)$, and by the continuity of the pointwise product in Schauder spaces, we can conclude that the operators $w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\cdot\,]_{|\partial\Omega}$ and $T_{lj}[\,\cdot\,]$ are continuous from $C^1(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$ and from $C^1(\partial\Omega)$ to $C^{0,\omega_\alpha(\,\cdot\,)}(\partial\Omega)$, respectively. In particular, $T_{lj}[\mu]$ and $w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]_{|\partial\Omega}$ belong to $C^0(\partial\Omega)$. We now show that the weak M_{lj}-derivative of $w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]_{|\partial\Omega}$ coincides with $T_{lj}[\mu]$.

By arguing just as at the end of the proof of Lemma 8.1, there exists a sequence of functions $\{\mu_b\}_{b\in\mathbb{N}}$ in $C^{1,\alpha}(\partial\Omega)$, which converges to μ in $C^1(\partial\Omega)$. Note that if $\varphi\in C^1(\partial\Omega)$, then the validity of (9.1) for $\mu_b\in C^{1,\alpha}(\partial\Omega)$, the membership of $w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu_b]_{|\partial\Omega}$ in $C^{1,\alpha}(\partial\Omega)$, the above-mentioned continuity of $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot\,]_{|\partial\Omega}$, and also Lemma 2.2 imply that

$$\int_{\partial\Omega} w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega} M_{lj}[\varphi]\, d\sigma = \lim_{b\to\infty} \int_{\partial\Omega} w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu_b]_{|\partial\Omega} M_{lj}[\varphi]\, d\sigma$$

$$= -\lim_{b\to\infty} \int_{\partial\Omega} M_{lj}\left[w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu_b]_{|\partial\Omega}\right]\varphi\, d\sigma = -\lim_{b\to\infty}\int_{\partial\Omega} T_{lj}[\mu_b]\varphi\, dx = -\int_{\partial\Omega} T_{lj}[\mu]\varphi\, dx\,.$$

Hence, $T_{lj}[\mu]$ coincides with the weak M_{lj}-derivative of $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega}$ for all l, j in $\{1, \ldots, n\}$. Since both $T_{lj}[\mu]$ and $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega}$ are the continuous functions, it follows that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega} \in C^1(\partial\Omega)$ and $M_{lj}[w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega}] = T_{lj}[\mu]$, classically. Hence (9.1) holds also for $\mu \in C^1(\partial\Omega)$. □

Using formula (9.1), we now prove the following result, which says that the double layer potential on $\partial\Omega$ has a regularizing effect.

Theorem 9.2. *Let* $\mathbf{a}$ *be as in* (1.1)*,* (1.2)*,* (1.3)*,* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$*,* $\alpha \in]0,1[$*,* $m \in \mathbb{N}\setminus\{0\}$ *and let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{m,\alpha}$*. Then the following statements hold:*

(i) *The operator* $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot\,]_{|\partial\Omega}$ *is linear and continuous from* $C^m(\partial\Omega)$ *to* $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$*.*

(ii) *Let* $\beta \in]0, \alpha]$*. Then the operator* $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot\,]_{|\partial\Omega}$ *is linear and continuous from* $C^{m,\beta}(\partial\Omega)$ *to* $C^{m,\alpha}(\partial\Omega)$*.*

Proof. We prove statement (i) by induction on m. As in the previous proof, we denote by $T_{lj}[\mu]$ the right-hand side of formula (9.1). We first consider the case $m = 1$. By Lemma 2.3(ii) and formula (9.1), it suffices to prove that the following two statements hold:

(j) $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot\,]_{|\partial\Omega}$ is continuous from $C^1(\partial\Omega)$ to $C^0(\partial\Omega)$;

(jj) $T_{lj}[\,\cdot\,]$ is continuous from $C^1(\partial\Omega)$ to $C^{0,\omega_\alpha(\cdot)}(\partial\Omega)$ for all $l,\, j \in \{1, \ldots, n\}$.

Theorem 7.4 implies the validity of (j). Statement (jj) follows by the continuity of the pointwise product in Schauder spaces, by the continuity of M_{lj} from $C^1(\partial\Omega)$ to $C^0(\partial\Omega)$, by the continuity of $v[\partial\Omega, S_{\mathbf{a}}, \cdot\,]_{|\partial\Omega}$ and of $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot\,]_{|\partial\Omega}$ from $C^0(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$ (cf. Theorems 7.2, 7.4), and also by the continuity of $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot\,, \cdot\,]$ from $C^{0,\alpha}(\partial\Omega) \times C^0(\partial\Omega)$ to $C^{0,\omega_\alpha(\cdot)}(\partial\Omega)$ (cf. Theorem 8.2(i)) and by the continuity of R from $\left(C^{0,\alpha}(\partial\Omega)\right)^2 \times C^0(\partial\Omega)$ to $C^{0,\omega_\alpha(\cdot)}(\partial\Omega)$ (cf. Theorem 8.4(i).)

Next, we assume that Ω is of the class $C^{m+1,\alpha}$ and we turn to prove that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot\,]_{|\partial\Omega}$ is continuous from $C^{m+1}(\partial\Omega)$ to $C^{m+1,\omega_\alpha(\cdot)}(\partial\Omega)$. By Lemma 2.3(ii) and formula (9.1), it suffices to prove that the following two statements hold:

(a) $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot\,]_{|\partial\Omega}$ is continuous from $C^{m+1}(\partial\Omega)$ to $C^0(\partial\Omega)$;

(b) $T_{lj}[\,\cdot\,]$ is continuous from $C^{m+1}(\partial\Omega)$ to $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$. for all $l,\, j \in \{1, \ldots, n\}$.

Statement (a) holds by the inductive assumption. We now prove statement (b). Since Ω is of the class $C^{m+1,\alpha}$, then ν is of the class $C^{m,\alpha}(\partial\Omega)$. Theorem 8.3(i) ensures that $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \nu \cdot a^{(1)}, \cdot\,]$ and $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \nu_j, \cdot\,]$ are continuous from $C^m(\partial\Omega)$ to $C^{m,\omega_\alpha}(\partial\Omega)$ for all $l,\, j,\, r$ in $\{1, \ldots, n\}$. Since M_{lj} is continuous from $C^{m+1}(\partial\Omega)$ to $C^m(\partial\Omega)$, the inductive assumption implies that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, M_{lj}[\,\cdot\,]]_{|\partial\Omega}$ is continuous from $C^{m+1}(\partial\Omega)$ to $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$ for all $l,\, j$ in $\{1, \ldots, n\}$.

Since M_{lj} is continuous from $C^{m+1}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$ and $v[\partial\Omega, S_{\mathbf{a}}, \cdot\,]_{|\partial\Omega}$ is continuous from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$, $\nu \in (C^{m,\alpha}(\partial\Omega))^n$ and $C^{m,\alpha}(\partial\Omega)$ is continuously imbedded into $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$, we conclude that $v[\partial\Omega, S_{\mathbf{a}}, M_{lj}[\,\cdot\,]]_{|\partial\Omega}$ and $v[\partial\Omega, S_{\mathbf{a}}, \nu \cdot a^{(1)} M_{lj}[\,\cdot\,]]_{|\partial\Omega}$ are continuous from the space $C^{m+1}(\partial\Omega)$ to $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$ for all $l,\, j$ in $\{1, \ldots, n\}$. Moreover, R is continuous from $(C^{m,\alpha}(\partial\Omega))^2 \times C^m(\partial\Omega)$ to $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$ (cf. Theorem 8.4(i)). Then statement (b) holds true.

Statement (iii) can be proved by the same argument of the proof of statement (i) by exploiting Theorem 8.3(ii) instead of Theorem 8.3(i) and Theorem 8.4(ii) instead of Theorem 8.4(i). □

Since $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$ is compactly imbedded into $C^m(\partial\Omega)$ and $C^{m,\alpha}(\partial\Omega)$ is compactly imbedded into $C^{m,\beta}(\partial\Omega)$ for all $\beta\in]0,\alpha[$, we have the following immediate consequence of Theorem 9.2.

Corollary 9.1. *Under the assumptions of Theorem* 9.2*, the linear operator* $w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ *is compact from* $C^m(\partial\Omega)$ *to itself, from* $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$ *to itself and from* $C^{m,\alpha}(\partial\Omega)$ *to itself.*

10 Other layer potentials associated to $P[\mathbf{a},D]$

Another relevant layer potential operator associated to the analysis of boundary value problems for the operator $P[\mathbf{a},D]$ is the following

$$w_*[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu](x)\equiv\int_{\partial\Omega}\mu(y)DS_{\mathbf{a}}(x-y)a^{(2)}\nu(x)\,d\sigma_y\quad\forall\,x\in\partial\Omega\,,$$

which we now turn to consider.

Theorem 10.1. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a},D]$, $\alpha\in]0,1[$, $m\in\mathbb{N}\setminus\{0\}$ *and let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{m,\alpha}$*. Then the following statements hold:*

(i) *The operator* $w_*[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ *is linear and continuous from* $C^{m-1}(\partial\Omega)$ *to* $C^{m-1,\omega_\alpha(\cdot)}(\partial\Omega)$.

(ii) *Let* $\beta\in]0,\alpha]$*. Then the operator* $w_*[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ *is linear and continuous from* $C^{m-1,\beta}(\partial\Omega)$ *to* $C^{m-1,\alpha}(\partial\Omega)$.

Proof. First note that

$$\begin{aligned}
w_*[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu](x)&=\sum_{b,r=1}^n a_{br}\int_{\partial\Omega}\nu_r(x)\frac{\partial}{\partial x_b}S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&=\sum_{b,r=1}^n a_{br}Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_b}\circ\Theta,\nu_r,\mu\Big](x)+\sum_{b,r=1}^n a_{br}\int_{\partial\Omega}\nu_r(y)\,\frac{\partial}{\partial x_b}S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&=\sum_{b,r=1}^n a_{br}Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_b}\circ\Theta,\nu_r,\mu\Big](x)-\int_{\partial\Omega}\mu(y)\sum_{b,r=1}^n a_{br}\nu_r(y)\,\frac{\partial}{\partial y_b}\,S_{\mathbf{a}}(x-y)\,d\sigma_y\\
&=\sum_{b,r=1}^n a_{br}Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_b}\circ\Theta,\nu_r,\mu\Big](x)-w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu](x)-v[\partial\Omega,S_{\mathbf{a}},(a^{(1)}\nu)\mu](x)
\end{aligned}\tag{10.1}$$

for all $x\in\partial\Omega$ and $\mu\in C^0(\partial\Omega)$.

If $m=1$, then Theorem 7.2 implies that $v[\partial\Omega,S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ is linear and continuous from $C^{m-1}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$.

If $m>1$, then $C^{m-1}(\partial\Omega)$ is continuously imbedded into $C^{m-2,\alpha}(\partial\Omega)$ and Theorem 7.1 implies that $v[\partial\Omega,S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ is linear and continuous from $C^{m-2,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$. Hence, $v[\partial\Omega,S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ is continuous from the space $C^{m-1}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$ for all $m\geq 1$. Then formula (10.1), the continuity of the imbedding of $C^{m-1,\alpha}(\partial\Omega)$ into $C^{m-1,\omega_\alpha}(\partial\Omega)$ and Theorems 8.3(i), 9.2(i) imply the validity of statement (i).

We now consider statement (ii). Since $v[\partial\Omega,S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ is continuous from $C^{m-1,\beta}(\partial\Omega)$ to $C^{m,\beta}(\partial\Omega)$ and $C^{m,\beta}(\partial\Omega)$ is continuously imbedded into $C^{m-1,\alpha}(\partial\Omega)$, the operator $v[\partial\Omega,S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ is continuous from $C^{m-1,\beta}(\partial\Omega)$ into $C^{m-1,\alpha}(\partial\Omega)$. Then formula (10.1) and Theorems 8.3(ii), 9.2(ii) imply the validity of statement (ii). □

Since the space $C^{m-1,\omega_\alpha(\cdot)}(\partial\Omega)$ is compactly imbedded into $C^{m-1}(\partial\Omega)$, and $C^{m-1,\alpha}(\partial\Omega)$ is compactly imbedded into $C^{m-1,\beta}(\partial\Omega)$ for all $\beta\in]0,\alpha[$, we have the following immediate consequence of Theorem 10.1(ii).

Corollary 10.1. *Under the assumptions of Theorem* 10.1, $w_*[\partial\Omega,S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ *is compact from* $C^{m-1}(\partial\Omega)$ *to itself, from* $C^{m-1,\omega_\alpha(\cdot)}(\partial\Omega)$ *to itself and from* $C^{m-1,\alpha}(\partial\Omega)$ *to itself.*

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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF ONE CLASS OF n-th ORDER DIFFERENTIAL EQUATIONS

Abstract. We obtain the existence conditions and asymptotic representations of a certain class of power-mode solutions of a binomial non-autonomous n-th order ordinary differential equation with regularly varying nonlinearities and their derivatives of order up to $n-1$.

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რეზიუმე. n-ური რიგის ბინომიალური არაავტონომიური ჩვეულებრივი დიფერენციალური განტოლებებისათვის რეგულარულად ცვლადი არაწრფივობებით დადგენილია ამონახსნთა ერთი კლასის არსებობის პირობები და ნაპოვნია მათი ასიმპტოტური წარმოდგენები.

1 Introduction

Consider the differential equation

$$y^{(n)} = \alpha p(t) \prod_{j=0}^{n-1} \varphi_j(y^{(j)}), \tag{1.1}$$

where $n \geq 2$, $\alpha \in \{-1, 1\}$, $p : [a, +\infty[\to]0, +\infty[$ is a continuous function, $a \in \mathbb{R}$, $\varphi_j : \Delta Y_j \to]0, +\infty[$ are the continuous functions regularly varying, as $y^{(j)} \to Y_j$, of order σ_j, $j = \overline{0, n-1}$, ΔY_j is a one-sided neighborhood of the point Y_j, $Y_j \in \{0, \pm\infty\}$[1].

Equation (1.1) is a particular case of the equation

$$y^{(n)} = \sum_{k=1}^{m} \alpha_k p_k(t) \prod_{j=0}^{n-1} \varphi_{kj}(y^{(j)}),$$

which is comprehensively studied by V. M. Evtukhov and A. M. Klopot [1, 2], M. M. Klopot [3, 4]. Here $n \geq 2$, $\alpha_k \in \{-1, 1\}$ $(k = \overline{1, m})$, $p_k : [a, \omega[\to]0, +\infty[$ $(k = \overline{1, m})$ are continuous functions, $-\infty < a < \omega \leq +\infty$, $\varphi_{kj} : \Delta Y_j \to]0, +\infty[$ $(k = \overline{1, m}$, $j = \overline{0, n-1})$ are continuous functions regularly varying, as $y^{(j)} \to Y_j$, of order σ_j, ΔY_j is a one-sided neighborhood of the point Y_j, which is equal either to 0 or to $\pm\infty$.

From the above-mentioned results, the necessary and sufficient existence conditions of the so-called $\mathcal{P}_{+\infty}(Y_0, \dots, Y_{n-1}, \lambda_0)$-solutions of equation (1.1) can be obtained for all λ_0 $(-\infty \leq \lambda_0 \leq +\infty)$. Moreover, asymptotic representations as $t \to +\infty$ of such solutions and their derivatives of order up to $n-1$ can be established.

It follows directly from the definition of these solutions that the conditions

$$\lim_{t \to +\infty} y^{(j)}(t) = Y_j \ \ (j = \overline{0, n-1}), \quad \lim_{t \to +\infty} \frac{[y^{(n-1)}(t)]^2}{y^{(n-2)}(t) y^{(n)}(t)} = \lambda_0 \tag{1.2}$$

hold.

However, the set of monotonous solutions of equation (1.1), defined in some neighborhood of $+\infty$, can also have the solutions for each of which there exists a number $k \in \{1, \dots, n\}$ such that

$$y^{(n-k)}(t) = c + o(1) \ \ (c \neq 0) \ \text{ as } \ t \to +\infty. \tag{1.3}$$

When $k = 1, 2$, or the functions $\varphi_i(y^{(i)})$ $(i = \overline{n-k+1, n-2})$ tend to the positive constants, as $y^{(i)} \to Y_i$, a question on the existence of solutions of type (1.3) of equation (1.1) can be resolved without any assumption like the last condition in (1.2). Otherwise, we will not be able to get asymptotic formulas of these solutions and their derivatives of order up to $n-1$ directly from equation (1.1).

Some results concerning the existence of solutions of type (1.3) have been obtained in Corollary 8.2 of the monograph by I. T. Kiguradze and T. A. Chanturiya [5, Ch. II, § 8, p. 207] for the equations of general type. But these results provide for a considerably strict restriction to the $(n-k+1)$-st derivative of a solution. In order to get new results with less strict restrictions to the behaviour of this and the subsequent derivatives of order $\leq n-1$ in case $k \in \{3, \dots, n\}$ and not all $\varphi_i(y^{(i)})$ $(i = \overline{n-k+1, n-2})$ tend to a positive constant, as $y^{(i)} \to Y_i$, we formulate the following definition.

Definition 1.1. A solution y of the differential equation (1.1) is called (for $k \in \{3, \dots, n\}$) a $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution, where $-\infty \leq \lambda_0 \leq +\infty$, if it is defined on the interval $[t_0, +\infty[\subset [a, +\infty[$ and satisfies the conditions

$$\lim_{t \to +\infty} y^{(n-k)}(t) = c \ \ (c \neq 0), \quad \lim_{t \to +\infty} \frac{[y^{(n-1)}(t)]^2}{y^{(n-2)}(t) y^{(n)}(t)} = \lambda_0. \tag{1.4}$$

It is obvious that by virtue of the first relation in (1.4), for these solutions the following representations

$$y^{(l-1)}(t) = \frac{c t^{n-l-k+1}}{(n-l-k+1)!} \, [1 + o(1)] \ \ (l = \overline{1, n-k}) \ \text{ as } \ t \to +\infty \tag{1.5}$$

[1]For $Y_j = \pm\infty$ here and in the sequel, all numbers in the neighborhood of ΔY_j are assumed to have constant sign.

hold, and $c \in \Delta Y_{n-k}$.

It readily follows from the form of equation (1.1) that $y^{(n)}(t)$ has a constant sign in some neighborhood of $+\infty$. Then $y^{(n-l)}(t)$ $(l = \overline{1, k-1})$ are strictly monotone functions in the neighborhood of $+\infty$ and, by virtue of (1.3), can tend only to zero, as $t \to +\infty$. Therefore, it is necessary that

$$Y_{j-1} = 0 \ \text{ for } \ j = \overline{n-k+2, n}. \tag{1.6}$$

Let us introduce the numbers μ_j $(j = \overline{0, n-1})$,

$$\mu_j = \begin{cases} 1 & \text{if } Y_j = +\infty, \text{ or } Y_j = 0 \text{ and } \Delta Y_j \text{ is a right neighborhood of the point } 0, \\ -1 & \text{if } Y_j = -\infty, \text{ or } Y_j = 0 \text{ and } \Delta Y_j \text{ is a left neighborhood of the point } 0, \end{cases}$$

and assume that they satisfy the following conditions:

$$\begin{aligned} &\mu_j \mu_{j+1} > 0 \ \text{ for } \ j = \overline{0, n-k-1}, \\ &\mu_j \mu_{j+1} < 0 \ \text{ for } \ j = \overline{n-k+1, n-2}, \end{aligned} \tag{1.7}$$

$$\alpha \mu_{n-1} < 0. \tag{1.8}$$

These conditions on μ_j $(j = \overline{0, n-1})$ and α are necessary for the existence of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions of equation (1.1) as long as for each of them in some neighborhood of $+\infty$

$$\operatorname{sign} y^{(j)}(t) = \mu_j \ \ (j = \overline{0, n-1}), \ \ \operatorname{sign} y^{(n)}(t) = \alpha.$$

Besides, for such solutions it follows from (1.5) that

$$Y_{j-1} = \begin{cases} +\infty & \text{if } \ \mu_{n-k} > 0, \\ -\infty & \text{if } \ \mu_{n-k} < 0 \end{cases} \quad \text{for } \ j = \overline{1, n-k}. \tag{1.9}$$

The aim of the present paper is to obtain the necessary and sufficient existence conditions of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions $(k \in \{3, \dots, n\})$ of equation (1.1) for $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\}$, and to establish asymptotic, as $t \to +\infty$, formulas of their derivatives of order $\le n-1$. Moreover, a question on the quantity of the studied by us solutions will be solved.

It is significant to note that by virtue of the results obtained by V. M. Evtukhov [6], the solutions of equation (1.1) satisfy the following a priori asymptotic conditions.

Lemma 1.1. *Let $k \in \{3, \dots, n\}$ and $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\}$. Then for each $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution $y : [t_0, +\infty[\to \mathbb{R}$ of equation* (1.1) *the following asymptotic, as $t \to +\infty$, relations hold:*

$$y^{(l-1)}(t) \sim \frac{[(\lambda_0 - 1)t]^{n-l}}{\prod\limits_{i=l}^{n-1} [(n-i)\lambda_0 - (n-i-1)]} \, y^{(n-1)}(t) \ \ (l = \overline{n-k+2, n-1}). \tag{1.10}$$

2 Auxiliary notations and the main results

In equation (1.1), each of the functions φ_j $(j = \overline{0, n-1})$, being a regularly varying function of order σ_j, as $y^{(j)} \to Y_j$, can be represented (see [7, Ch. I, § 1, p. 10]) in the form

$$\varphi_j(y^{(j)}) = |y^{(j)}|^{\sigma_j} L_j(y^{(j)}) \ \ (j = \overline{0, n-1}), \tag{2.1}$$

where $L_j : \Delta Y_j \to]0, +\infty[$ $(j = \overline{0, n-1})$ is a slowly varying function, as $y^{(j)} \to Y_j$. According to the definition and properties of slowly varying functions,

$$\lim_{\substack{y^{(j)} \to Y_j \\ y^{(j)} \in \Delta Y_j}} \frac{L_j(\lambda y^{(j)})}{L_j(y^{(j)})} = 1 \ \text{ for each } \ \lambda > 0 \ \ (j = \overline{0, n-1}), \tag{2.2}$$

and these limit relations hold uniformly with respect to λ on an arbitrary interval $[c,d]\subset]0,+\infty[$. Moreover, by virtue of Theorem 1.2 (see [7, Ch. I, § 2, p. 10]), there exist continuously differentiable functions $L_{0j}:\Delta Y_j\to]0,+\infty[$ $(j=\overline{0,n-1})$, slowly varying as $y^{(j)}\to Y_j$, such that

$$\lim_{\substack{y^{(j)}\to Y_j\\ y^{(j)}\in\Delta Y_j}}\frac{L_j(y^{(j)})}{L_{0j}(y^{(j)})}=1,\qquad \lim_{\substack{y^{(j)}\to Y_j\\ y^{(j)}\in\Delta Y_j}}\frac{y^{(j)}L'_{0j}(y^{(j)})}{L_{0j}(y^{(j)})}=0. \tag{2.3}$$

Examples of functions, slowly varying as $y\to Y_0$, are the functions

$$|\ln|y||^{\gamma_1},\ \ \ln^{\gamma_2}|\ln|y||,\ \ \gamma_1,\gamma_2\in\mathbb{R},$$
$$\exp\big(|\ln|y||^{\gamma_3}\big),\ \ 0<\gamma_3<1,\quad \exp\Big(\frac{\ln|y|}{\ln|\ln|y||}\Big),$$

as well as the functions that have a nonzero finite limit as $y\to Y_0$, and others.

We say that a continuous function $L:\Delta Y_0\to]0,+\infty[$, slowly varying as $y\to Y_0$, satisfies the condition S_0 if

$$L(\mu e^{[1+o(1)]\ln|y|})=L(y)[1+o(1)]\ \text{ as }\ y\to Y_0\ \ (y\in\Delta Y_0),$$

where $\mu=\operatorname{sign}y$.

The condition S_0 is necessarily satisfied for functions L that have a nonzero finite limit, as $y\to Y_0$, for functions of the form

$$L(y)=|\ln|y||^{\gamma_1},\quad L(y)=|\ln|y||^{\gamma_1}\big|\ln|\ln|y||\big|^{\gamma_2},$$

where $\gamma_1,\gamma_2\neq 0$, and for many others.

Remark 2.1. If a function $L:\Delta Y_0\to]0,+\infty[$, slowly varying as $y\to Y_0$, satisfies the condition S_0, then for each function $l:\Delta Y_0\to]0,+\infty[$, slowly varying as $y\to Y_0$, we have

$$L(yl(y))=L(y)[1+o(1)]\ \text{ as }\ y\to Y_0\ \ (y\in\Delta Y_0).$$

Remark 2.2 (see [8])**.** If a function $L:\Delta Y_0\to]0,+\infty[$, slowly varying as $y\to Y_0$, satisfies the condition S_0 and $y:[t_0,+\infty[\to\Delta Y_0$ is a continuously differentiable function such that

$$\lim_{t\to+\infty}y(t)=Y_0,\qquad \frac{y'(t)}{y(t)}=\frac{\xi'(t)}{\xi(t)}\,[r+o(1)]\ \text{ as }\ t\to+\infty,$$

where r is a nonzero real constant, ξ is a real function, continuously differentiable in some neighborhood of $+\infty$ and such that $\xi'(t)\neq 0$, then

$$L(y(t))=L(\mu|\xi(t)|^r)[1+o(1)]\ \text{ as }\ t\to+\infty,$$

where $\mu=\operatorname{sign}y(t)$ in some neighborhood of $+\infty$.

Remark 2.3 (see [2])**.** If a function $L:\Delta Y_0\to]0,+\infty[$, slowly varying as $y\to Y_0$, satisfies the condition S_0 and a function $r:\Delta Y_0\times K\to\mathbb{R}$, where K is compact in $\mathbb{R}^n$, is such that

$$\lim_{\substack{y\to\Delta Y_0\\ y\in\Delta Y_0}}r(z,v)=0\ \text{ uniformly with respect to }\ v\in K,$$

then

$$\lim_{\substack{y\to\Delta Y_0\\ y\in\Delta Y_0}}\frac{L(ve^{[1+r(z,v)]\ln|z|})}{L(z)}=1\ \text{ uniformly with respect to }\ v\in K,$$

where $v=\operatorname{sign}z$.

Besides these facts about the functions, regularly and slowly varying as $y^{(j)} \to Y_j$ $(j=\overline{0,n-1})$, we need the following auxiliary notations:

$$\gamma = 1 - \sum_{j=n-k+1}^{n-1} \sigma_j, \quad \nu = \sum_{j=n-k+1}^{n-2} \sigma_j (n-j-1), \quad a_{0j} = (n-j)\lambda_0 - (n-j-1) \ \ (j=\overline{1,n}),$$

$$C = \prod_{j=n-k+1}^{n-2} \left| \frac{(\lambda_0 - 1)^{n-j-1}}{\prod\limits_{i=j+1}^{n-1} a_{0i}} \right|^{\sigma_j},^{2} \quad M(c) = \prod_{j=1}^{n-k} \left| \frac{c}{(n-j-k+1)!} \right|^{\sigma_{j-1}},$$

$$I(t) = \varphi_{n-k}(c) M(c) \int_A^t p(\tau) \tau^{\nu} \varphi_0(\mu_0 \tau^{n-k}) \cdots \varphi_{n-k-1}(\mu_{n-k-1}\tau)\, d\tau,$$

where

$$A = \begin{cases} a_1 & \text{if } \int\limits_{a_1}^{+\infty} p(\tau)\tau^{\nu} \varphi_0(\mu_0 \tau^{n-k}) \cdots \varphi_{n-k-1}(\mu_{n-k-1}\tau)\, d\tau = +\infty, \\ +\infty & \text{if } \int\limits_{a_1}^{+\infty} p(\tau)\tau^{\nu} \varphi_0(\mu_0 \tau^{n-k}) \cdots \varphi_{n-k-1}(\mu_{n-k-1}\tau)\, d\tau < +\infty, \end{cases}$$

$a_1 \ge a$ such that $\mu_{j-1} t^{n-k-j+1} \in \Delta Y_{j-1}$ $(j=\overline{1,n-k})$ for $t \ge a_1$.

The following assertions hold for equation (1.1).

Theorem 2.1. *Let $\gamma \neq 0$, $k \in \{3,\dots,n\}$ and $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\}$. Then, for the existence of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions of equation* (1.1)*, it is necessary that $c \in \Delta Y_{n-k}$ and along with* (1.6)–(1.9) *the conditions*

$$\lambda_0 < 1, \quad a_{0j+1} > 0 \ \ (j = \overline{n-k+1, n-2}), \tag{2.4}$$

$$\lim_{t\to+\infty} \frac{tI'(t)}{I(t)} = \frac{\gamma}{\lambda_0 - 1} \tag{2.5}$$

hold. Moreover, each solution of that kind admits along with (1.3) *and* (1.5) *the asymptotic representations* (1.10) *as $t \to +\infty$ and*

$$\frac{|y^{(n-1)}(t)|^{\gamma}}{\prod\limits_{j=n-k+1}^{n-1} L_j\Big(\frac{[(\lambda_0-1)t]^{n-j-1}}{\prod_{i=j+1}^{n-1} a_{0i}} y^{(n-1)}(t)\Big)} = \alpha \mu_{n-1} \gamma C I(t)[1+o(1)]. \tag{2.6}$$

Here we have the asymptotic, as $t \to +\infty$, representations (1.10) and (2.6), written out implicitly. Let us define conditions under which asymptotic, as $t \to +\infty$, representations of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions of equation (1.1) and their derivatives of order $\le n-1$ can be written out in explicit form.

Theorem 2.2. *Let $\gamma \neq 0$, $k \in \{3,\dots,n\}$, $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\}$ and the functions L_j $(j = \overline{n-k+1, n-1})$, slowly varying as $y^{(j)} \to Y_j$, satisfy the condition S_0. Then, in case of the existence of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions of equation* (1.1)*, the following condition*

$$\int_{a_2}^{+\infty} \tau^{k-2} |I(\tau) \prod_{j=n-k+1}^{n-1} L_j(\mu_j \tau^{\frac{a_{0j+1}}{\lambda_0 - 1}})|^{\frac{1}{\gamma}}\, d\tau < +\infty \tag{2.7}$$

[2]Here and in the sequel, it is assumed that $\prod\limits_m^l = 1$ if $m > l$.

holds, where $a_2 \ge a_1$ *such that* $\mu_{j-1} t^{\frac{a_{0j}}{\lambda_0 - 1}} \in \Delta Y_{j-1}$ $(j = \overline{n-k+2, n})$ *for* $t \ge a_2$*, and each solution of that kind admits along with* (1.5) *the following asymptotic, as* $t \to +\infty$*, representations:*

$$y^{(n-k)}(t) = c + \frac{\mu_{n-1}(\lambda_0 - 1)^{k-2}}{\prod\limits_{i=n-k+2}^{n-1} a_{0i}} W(t)[1 + o(1)], \tag{2.8$_1$}$$

$$y^{(l-1)}(t) = \frac{\mu_{n-1}(\lambda_0 - 1)^{n-l} t^{n-l-k+2}}{\prod\limits_{i=l}^{n-1} a_{0i}} W'(t)[1 + o(1)] \ \ (l = \overline{n-k+2, n-1}), \tag{2.8$_2$}$$

$$y^{(n-1)}(t) = \mu_{n-1} \frac{W'(t)}{t^{k-2}} [1 + o(1)], \tag{2.8$_3$}$$

where

$$W(t) = \int\limits_{+\infty}^{t} \tau^{k-2} \Big| \gamma C I(\tau) \prod_{j=n-k+1}^{n-1} L_j\Big(\mu_j \tau^{\frac{a_{0j+1}}{\lambda_0 - 1}}\Big) \Big|^{\frac{1}{\gamma}} d\tau.$$

Theorem 2.3. *Let* $\gamma \ne 0$, $k \in \{3, \dots, n\}$, $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\}$, $c \in \Delta Y_{n-k}$*, the conditions* (1.6)–(1.9), (2.4), (2.5), (2.7) *hold and the functions* L_j $(j = \overline{n-k+1, n-1})$*, slowly varying as* $y^{(j)} \to Y_j$*, satisfy the condition* S_0*. In addition, let the inequality* $\sigma_{n-1} \ne 1$ *hold and the algebraic relative to* ρ *equation*

$$\sum_{j=2}^{k-1} \frac{\sigma_{n-j}}{\lambda_0 - 1} \prod_{l=1}^{j-1} \frac{a_{0n-l}}{\lambda_0 - 1} \prod_{l=j}^{k-2} \Big(\rho + \frac{a_{0n-l}}{\lambda_0 - 1}\Big) = \Big(\rho - \frac{\sigma_{n-1} - 1}{\lambda_0 - 1}\Big) \prod_{l=1}^{k-2} \Big(\rho + \frac{a_{0n-l}}{\lambda_0 - 1}\Big) \tag{2.9}$$

have no roots with a zero real part. Then for $\lambda_0 \in \,]-\infty, \frac{k-2}{k-1}[\, \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}\}$ ($\lambda_0 \in [\frac{k-2}{k-1}, 1[$)*, equation* (1.1) *has a* $(n-k+m+1)$*-parameter* (($n-k+m$)*-parameter, respectively*) *family of* $\mathcal{P}_{+\infty}^{k}(\lambda_0)$*-solutions that admit asymptotic, as* $t \to +\infty$*, representations* (1.5) *and* (2.8$_i$) $(i = 1, 2, 3)$*, where* m *is a number of roots* (*taking into account divisible*) *with a negative real part of the algebraic equation* (2.9).

Proof of Theorems 2.1–2.2. Let $y : [t_0, +\infty[\to \Delta Y_0$ be an arbitrary $\mathcal{P}_{+\infty}^{k}(\lambda_0)$-solution of equation (1.1). Then, as it has been proved before formulations of the theorems, $c \in \Delta Y_{n-k}$, the conditions (1.6)–(1.9) hold and the asymptotic relations (1.3) and (1.5) are true. It follows from (1.5) that

$$\frac{y^{(j+1)}(t)}{y^{(j)}(t)} = \frac{n-j-k}{t} [1 + o(1)] \ \ (j = \overline{0, n-k-1}) \ \text{ as } \ t \to +\infty.$$

Now, by taking into account representations (2.1) of the functions $\varphi_j(y^{(j)})$ $(j = \overline{0, n-k-1})$, regularly varying as $t \to +\infty$, and the fact that relations (2.2) hold uniformly with respect to λ on an arbitrary interval $[d_1, d_2] \subset]0, +\infty[$, we have

$$\begin{aligned}
&\varphi_{j-1}\Big(\frac{c t^{n-j-k+1}}{(n-j-k+1)!} [1 + o(1)]\Big) \\
&\qquad = \Big| \frac{c t^{n-j-k+1}}{(n-j-k+1)!} [1 + o(1)] \Big|^{\sigma_{j-1}} L_{j-1}\Big(\frac{c t^{n-j-k+1}}{(n-j-k+1)!} [1 + o(1)]\Big) \\
&\qquad = \Big| \frac{c}{(n-j-k+1)!} \Big|^{\sigma_{j-1}} t^{n-j-k+1} L_{j-1}(\mu_{j-1} t^{n-j-k+1})[1 + o(1)] \\
&\qquad = \Big| \frac{c}{(n-j-k+1)!} \Big|^{\sigma_{j-1}} \varphi_{j-1}(\mu_{j-1} t^{n-j-k+1})[1 + o(1)] \ \ (j = \overline{1, n-k}) \ \text{ as } \ t \to +\infty.
\end{aligned}$$

Therefore, by virtue of (1.1), we obtain

$$\begin{aligned}
&\frac{y^{(n)}(t)}{\varphi_{n-1}(y^{(n-1)}(t)) \cdots \varphi_{n-k+1}(y^{(n-k+1)}(t))} \\
&\qquad = \alpha M(c) p(t) \varphi_0(\mu_0 t^{n-k}) \varphi_1(\mu_1 t^{n-k-1}) \cdots \varphi_{n-k}(c)[1 + o(1)] \ \text{ as } \ t \to +\infty.
\end{aligned} \tag{2.10}$$

It follows from the second relation in (1.4) that

$$\frac{y^{(n)}(t)}{y^{(n-1)}(t)} = \frac{1}{(\lambda_0 - 1)t}\,[1 + o(1)] \text{ as } t \to +\infty. \tag{2.11}$$

Then, by virtue of (1.7), the first inequality in (2.4) is true, namely, $\lambda_0 < 1$.

Furthermore, Lemma 1.1 implies that the asymptotic relations (1.10) hold, and therefore

$$\frac{y^{(j+1)}(t)}{y^{(j)}(t)} = \frac{a_{0j+1}}{(\lambda_0 - 1)t}\,[1 + o(1)] \;\; (j = \overline{n-k+1, n-2}) \text{ as } t \to +\infty. \tag{2.12}$$

Hence, by virtue of (1.7) and the first inequality in (2.4), the second one in (2.4) is true.

Taking into account (2.1) and (1.10), we rewrite (2.10) as

$$\frac{y^{(n)}(t)|y^{(n-1)}(t)|^{\gamma-1}}{\prod\limits_{j=n-k+1}^{n-1} L_j(y^{(j)}(t))} = \alpha M(c)Cp(t)t^{\nu}\varphi_{n-k}(c) \prod_{j=0}^{n-k-1} \varphi_j(\mu_j t^{n-k-j})[1 + o(1)]. \tag{2.13}$$

Integrating this relation from t_0 to t if $A = a_1$ and from t to $+\infty$ if $A = +\infty$, we have

$$\begin{aligned}
\int_B^t \frac{y^{(n)}(\tau)|y^{(n-1)}(\tau)|^{\gamma-1}}{\prod\limits_{j=n-k+1}^{n-1} L_j(y^{(j)}(\tau))}\,d\tau &= \alpha M(c)C\varphi_{n-k}(c)\int_B^t p(\tau)\tau^{\nu}\prod_{j=0}^{n-k-1}\varphi_j(\mu_j\tau^{n-k-j})[1 + o(1)]\,d\tau \\
&= \alpha M(c)C\varphi_{n-k}(c)\int_A^t p(\tau)\tau^{\nu}\prod_{j=0}^{n-k-1}\varphi_j(\mu_j\tau^{n-k-j})\,d\tau\,[1 + o(1)] \\
&= \alpha CI(t)[1 + o(1)] \text{ as } t \to +\infty,
\end{aligned} \tag{2.14}$$

where $B \in \{t_0, +\infty\}$.

Let us compare the integral occurring on the left-hand side with the expression $\dfrac{|y^{(n-1)}(t)|^{\gamma}}{\prod\limits_{j=n-k+1}^{n-1} L_{0j}(y^{(j)}(t))}$.

Taking into account (2.3), the second condition in (1.4) and (2.11), by the l'Hospital rule in the Stolz form, we have

$$\begin{aligned}
&\lim_{t\to+\infty} \frac{\dfrac{|y^{(n-1)}(t)|^{\gamma}}{\prod\limits_{j=n-k+1}^{n-1} L_{0j}(y^{(j)}(t))}}{\int\limits_B^t \dfrac{y^{(n)}(\tau)|y^{(n-1)}(\tau)|^{\gamma-1}}{\prod\limits_{j=n-k+1}^{n-1} L_j(y^{(j)}(\tau))}\,d\tau} \\
&= \mu_{n-1}\lim_{t\to+\infty}\frac{\prod\limits_{j=n-k+1}^{n-1} L_j(y^{(j)}(t))}{\prod\limits_{j=n-k+1}^{n-1} L_{0j}(y^{(j)}(t))}\left[\gamma - \sum_{j=n-k+1}^{n-1}\Big(\frac{y^{(j)}(t)L'_{0j}(y^{(j)}(t))}{L_{0j}(y^{(j)}(t))}\,\frac{y^{(j+1)}(t)}{y^{(j)}(t)}\,\frac{y^{(n-1)}(t)}{y^{(n)}(t)}\Big)\right] \\
&= \mu_{n-1}\gamma.
\end{aligned}$$

By virtue of this limit relation and (2.3), from (2.14) we obtain

$$\frac{|y^{(n-1)}(t)|^{\gamma}}{\prod\limits_{j=n-k+1}^{n-1} L_j(y^{(j)}(t))} = \alpha\mu_{n-1}\gamma CI(t)[1 + o(1)] \text{ as } t \to +\infty.$$

Hence, taking into account (1.10) and the properties of regularly varying functions, we establish the asymptotic representations (2.6), as $t \to +\infty$. In addition, they, together with (2.13), imply that

$$\frac{y^{(n)}(t)}{y^{(n-1)}(t)} = \frac{I'(t)}{\gamma I(t)}\,[1 + o(1)] \text{ as } t \to +\infty,$$

and, by virtue of (2.11), the limit relation (2.5) holds. Thus assertions of Theorem 2.1 are true.

Let us additionally suppose that the functions L_j ($j=\overline{n-k+1,n-1}$), slowly varying as $t\to+\infty$, satisfy the condition S_0. Then, by virtue of (2.11) and (2.12), the assertions

$$\frac{y^{(j+1)}(t)}{y^{(j)}(t)}=\frac{1}{t}\left[\frac{a_{0j+1}}{\lambda_0-1}+o(1)\right]\ \text{ as }\ t\to+\infty\ \ (j=\overline{n-k+1,n-1})$$

hold, and therefore, by Remark 2.2 and the second inequality in (2.4), we have

$$L_j\Bigg(\frac{[(\lambda_0-1)t]^{n-j-1}}{\prod\limits_{i=j+1}^{n-1}a_{0i}}\,y^{(n-1)}(t)\Bigg)=L_j(\mu_j t^{\frac{a_{0j+1}}{\lambda_0-1}})[1+o(1)]\ \text{ as }\ t\to+\infty\ \ (j=\overline{n-k+1,n-1}).$$

It follows from the obtained relations and (2.6) that for $t\to+\infty$

$$y^{(n-1)}(t)=\mu_{n-1}\bigg|\gamma CI(t)\prod_{j=n-k+1}^{n-1}L_j\Big(\mu_j t^{\frac{a_{0j+1}}{\lambda_0-1}}\Big)\bigg|^{\frac{1}{\gamma}}[1+o(1)].$$

This, together with (1.10), implies that

$$\begin{aligned}y^{(l-1)}(t)=&\ \frac{\mu_{n-1}[(\lambda_0-1)t]^{n-l}}{\prod\limits_{i=l}^{n-1}a_{0i}}\\&\times\bigg|\gamma CI(t)\prod_{j=n-k+1}^{n-1}L_j\Big(\mu_j t^{\frac{a_{0j+1}}{\lambda_0-1}}\Big)\bigg|^{\frac{1}{\gamma}}[1+o(1)]\ \ (l=\overline{n-k+2,n-1})\ \text{ as }\ t\to+\infty.\end{aligned}$$

Integrating this relation for $l=n-k+2$ from t_* to t, where $t_*=\max\{a_2,t_0\}$, we have

$$\begin{aligned}y^{(n-k)}(t)=&\ y^{(n-k)}(t_*)\\&+\frac{\mu_{n-1}[(\lambda_0-1)]^{k-2}}{\prod\limits_{i=n-k+2}^{n-1}a_{0i}}\int_{t_*}^{t}\tau^{k-2}\bigg|\gamma CI(\tau)\prod_{j=n-k+1}^{n-1}L_j\Big(\mu_j\tau^{\frac{a_{0j+1}}{\lambda_0-1}}\Big)\bigg|^{\frac{1}{\gamma}}[1+o(1)]\,d\tau.\end{aligned}$$

By virtue of the first condition in (1.4), we find that

$$\lim_{t\to+\infty}\int_{t_*}^{t}\tau^{k-2}\bigg|I(\tau)\prod_{j=n-k+1}^{n-1}L_j\Big(\mu_j\tau^{\frac{a_{0j+1}}{\lambda_0-1}}\Big)\bigg|^{\frac{1}{\gamma}}[1+o(1)]\,d\tau=const$$

and therefore, by the comparison criterion, the assertion (2.7) holds. Using Proposition 6 of the monograph [9, Ch. V, § 3, p. 293] on the asymptotic calculation of integrals, for the $(n-k)$-th derivative of a solution we get the representation form (2.8_1).

Consequently, the asymptotic relations (1.3), (1.10) and (2.6), as $t\to+\infty$, can be rewritten in the form (2.8_i) $(i=1,2,3)$. The proof of Theorems 2.1–2.2 is complete. □

Proof of Theorem 2.3. Let us show that, for this c from the hypothesis of the theorem, equation (1.1) has at least one $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution that is defined on some interval $[t_0,+\infty[\,\subset[a,+\infty[$ and admits the asymptotic representations (1.5) and (2.8_i) $(i=1,2,3)$, as $t\to+\infty$. Moreover, consider the problem on evaluating a number of such solutions. At the same time note that by virtue of the first inequality in (2.4), in case $\lambda_0>1$, the differential equation (1.1) does not have $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions.

Applying the transformation

$$\begin{aligned} y^{(l-1)}(t) &= \frac{ct^{n-l-k+1}}{(n-l-k+1)!}\,[1+v_l(t)] \quad (l=\overline{1,n-k}),\\ y^{(n-k)}(t) &= c+\frac{\mu_{n-1}(\lambda_0-1)^{k-2}}{\prod\limits_{i=n-k+2}^{n-1} a_{0i}}\,W(t)[1+v_{n-k+1}(t)],\\ y^{(l-1)}(t) &= \frac{\mu_{n-1}(\lambda_0-1)^{n-l}t^{n-l-k+2}}{\prod\limits_{i=l}^{n-1} a_{0i}}\,W'(t)[1+v_l(t)] \quad (l=\overline{n-k+2,n-1}),\\ y^{(n-1)}(t) &= \mu_{n-1}\,\frac{W'(t)}{t^{k-2}}\,[1+v_n(t)], \end{aligned} \tag{2.15}$$

to equation (1.1), we obtain the system of differential equations

$$\begin{cases} v_l' = \dfrac{n-l-k+1}{t}\,[-v_l+v_{l+1}] \quad (l=\overline{1,n-k-1}),\\[2mm] v_{n-k}' = \dfrac{1}{t}\left[\dfrac{\mu_{n-1}(\lambda_0-1)^{k-2}}{c\prod\limits_{i=n-k+2}^{n-1} a_{0i}}\,W(t)[1+v_{n-k+1}]-v_{n-k}\right],\\[2mm] v_{n-k+1}' = \dfrac{W'(t)}{W(t)}\big[-v_{n-k+1}+v_{n-k+2}\big],\\[2mm] v_l' = \dfrac{1}{t}\,\dfrac{a_{0l}}{\lambda_0-1}\,[1+v_{l+1}] \\ \qquad\quad -\dfrac{1}{t}\,(n-l-k+2)[1+v_l]-\dfrac{W''(t)}{W'(t)}\,[1+v_l] \quad (l=\overline{n-k+2,n-1}),\\[2mm] v_n' = \dfrac{1}{t}\bigg[\Big(-2+k-\dfrac{W''(t)t}{W'(t)}\Big)[1+v_n] \\ \qquad\quad +\dfrac{\alpha p(t)\varphi_0\big(\frac{ct^{n-k}}{(n-k)!}\,[1+v_1]\big)\cdots\varphi_{n-1}(\mu_{n-1}t^{2-k}W'(t)[1+v_n])}{\mu_{n-1}t^{1-k}W'(t)}\bigg]. \end{cases} \tag{2.16}$$

Consider the resulting system on the set $\Omega^n=[t_0,+\infty[\,\times\mathbb{R}^n_{\frac12}$, where $\mathbb{R}^n_{\frac12}=\{(v_1,\dots,v_n)\in\mathbb{R}^n:|v_j|\le\frac12\,,\ j=\overline{1,n}\}$ and $t_0\ge a_2$ is chosen, by virtue of (2.7), so that for $t>t_0$ and $(v_1,\dots,v_n)\in\mathbb{R}^n_{\frac12}$ the conditions hold:

$$\begin{gathered} \frac{ct^{n-j-k+1}}{(n-j-k+1)!}\,[1+v_j(t)]\in\Delta Y_{j-1} \quad (j=\overline{1,n-k}),\\ c+\frac{\mu_{n-1}(\lambda_0-1)^{k-2}}{\prod\limits_{i=n-k+2}^{n-1} a_{0i}}\,W(t)[1+v_{n-k+1}(t)]\in\Delta Y_{n-k},\\ \frac{\mu_{n-1}(\lambda_0-1)^{n-j}t^{n-j-k+2}}{\prod\limits_{i=j}^{n-1} a_{0i}}\,W'(t)[1+v_j(t)]\in\Delta Y_{j-1} \quad (j=\overline{n-k+2,n-1}),\\ \mu_{n-1}\frac{W'(t)}{t^{k-2}}\,[1+v_n(t)]\in\Delta Y_{n-1}. \end{gathered}$$

As the functions $\varphi_j(y^{(j)})$ $(j\in\{0,\dots,n-1\}\setminus\{n-k\})$ are representable as (2.1) and the relations (2.2) hold uniformly with respect to λ on an arbitrary interval $[d_1,d_2]\subset\,]0,+\infty[$, and in addition, by virtue of the continuity of the function $\varphi_{n-k}(y^{(n-k)})$, (2.7) and the fact that the functions L_j

($j=\overline{n-k+1,n-1}$), slowly varying as $t\to+\infty$, satisfy the condition S_0, we have

$$\begin{aligned}
&\varphi_j\Big(\frac{ct^{n-k-j}}{(n-k-j)!}\,[1+v_{j+1}]\Big)=\varphi_j\Big(\frac{ct^{n-k-j}}{(n-k-j)!}\Big)(1+v_{j+1})^{\sigma_j}\big(1+R_j(t,v_{j+1})\big)\\
&\qquad=\Big|\frac{c}{(n-k-j)!}\Big|^{\sigma_j}\varphi_j(\mu_j t^{n-k-j})(1+v_{j+1})^{\sigma_j}\big(1+R_j(t,v_{j+1})\big)\ \ (j=\overline{0,n-k-1}),\\
&\varphi_j\Bigg(\frac{\mu_{n-1}(\lambda_0-1)^{n-j-1}t^{n-j-k+1}}{\prod\limits_{i=j+1}^{n-1}a_{0i}}\,W'(t)[1+v_{j+1}]\Bigg)\\
&\qquad=\Bigg|\frac{(\lambda_0-1)^{n-j-1}}{\prod\limits_{i=j+1}^{n-1}a_{0i}}\Bigg|^{\sigma_j}\varphi_j\big(\mu_j t^{n-k-j+1}W'(t)\big)(1+v_{j+1})^{\sigma_j}\big(1+R_j(t,v_{j+1})\big)\\
&\qquad=\Bigg|\frac{(\lambda_0-1)^{n-j-1}}{\prod\limits_{i=j+1}^{n-1}a_{0i}}\Bigg|^{\sigma_j}\varphi_j(\mu_j t^{\frac{a_{0j+1}}{\lambda_0-1}})(1+v_{j+1})^{\sigma_j}\big(1+R_j(t,v_{j+1})\big)\ \ (j=\overline{n-k+1,n-2}),\\
&\varphi_{n-1}\big(\mu_{n-1}t^{2-k}W'(t)[1+v_n]\big)=\varphi_{n-1}(\mu_{n-1}t^{2-k}W'(t))(1+v_n)^{\sigma_{n-1}}\big(1+R_{n-1}(t,v_n)\big)\\
&\qquad=\varphi_{n-1}(\mu_{n-1}t^{\frac{1}{\lambda_0-1}})(1+v_n)^{\sigma_{n-1}}\big(1+R_{n-1}(t,v_n)\big),\\
&\varphi_{n-k}\Bigg(c+\frac{\mu_{n-1}(\lambda_0-1)^{k-2}}{\prod\limits_{i=n-k+2}^{n-1}a_{0i}}\,W(t)[1+v_{n-k+1}(t)]\Bigg)=\varphi_{n-k}(c)\big(1+R_{n-k}(t,v_{n-k+1})\big),
\end{aligned}$$

where the functions $R_j(t,v_{j+1})$ ($j=\overline{0,n-1}$) tend to zero, as $t\to+\infty$ uniformly with respect to $v_{j+1}\in[-\frac12,\frac12]$.

It follows from the form of $W(t)$ and (2.7) that

$$\lim_{t\to+\infty}\frac{W'(t)t}{W(t)}=k-1+\frac{1}{\lambda_0-1},\qquad \lim_{t\to+\infty}\frac{W''(t)t}{W'(t)}=k-2+\frac{1}{\lambda_0-1},$$

and both of these limits are nonzero in case $\lambda_0\in]-\infty,1[\setminus\{0,\frac12,\dots,\frac{k-2}{k-1}\}$. Therefore, using the aforementioned representations and (2.5), the system of equations (2.16) can be rewritten in the form

$$\begin{cases}
v_l'=\dfrac{n-l-k+1}{t}\,[-v_l+v_{l+1}]\ \ (l=\overline{1,n-k-1}),\\[2mm]
v_{n-k}'=\dfrac1t\,\big[-v_{n-k}+Y_{n-k,1}(t,v_1,\dots,v_n)\big],\\[2mm]
v_l'=\dfrac1t\Big[-\dfrac{a_{0l}}{\lambda_0-1}\,v_l+\dfrac{a_{0l}}{\lambda_0-1}\,v_{l+1}+Y_{l,1}(t,v_1,\dots,v_n)\Big]\ \ (l=\overline{n-k+1,n-1}),\\[2mm]
v_n'=\dfrac1t\Big[\displaystyle\sum_{j=1}^{n-k}\frac{\sigma_{j-1}}{\lambda_0-1}\,v_j+\sum_{j=n-k+2}^{n-1}\frac{\sigma_{j-1}}{\lambda_0-1}\,v_j+\frac{\sigma_{n-1}-1}{\lambda_0-1}\,v_n+\sum_{i=1}^{2}Y_{n,i}(t,v_1,\dots,v_n)\Big],
\end{cases}\tag{2.17}$$

where

$$\begin{aligned}
Y_{n-k,1}(t,v_1,\dots,v_n)&=\frac{\mu_{n-1}(\lambda_0-1)^{k-2}}{c\prod\limits_{i=n-k+2}^{n-1}a_{0i}}\,W(t)(1+v_{n-k+1}),\\
Y_{n-k+1,1}(t,v_1,\dots,v_n)&=\frac{W'(t)t}{W(t)}-k+1-\frac{1}{\lambda_0-1}\,,
\end{aligned}$$

$$Y_{l,1}(t,v_1,\dots,v_n)=\frac{W''(t)t}{W'(t)}-k+2-\frac{1}{\lambda_0-1}\quad (l=\overline{n-k+2,n-1}),$$
$$Y_{n1}(t,v_1,\dots,v_n)=\frac{1}{\lambda_0-1}\Big(\prod_{j=0}^{n-1}\big(1+R_j(t,v_{j+1})\big)-1\Big)\prod_{\substack{j=1\\ j\neq n-k+1}}^{n}(1+v_j)^{\sigma_{j-1}}$$
$$+\Big(-2+k-\frac{W''(t)t}{W'(t)}+\frac{1}{\lambda_0-1}\Big)[1+v_n],$$
$$Y_{n2}(t,v_1,\dots,v_n)=\frac{1}{\lambda_0-1}\Big(\prod_{\substack{j=1\\ j\neq n-k+1}}^{n}(1+v_j)^{\sigma_{j-1}}-\prod_{\substack{j=1\\ j\neq n-k+1}}^{n}v_j\sigma_{j-1}-1\Big).$$

At the same time we note here that

$$\lim_{t\to+\infty}Y_{j,1}(t,v_1,\dots,v_n)=0\quad (j=\overline{n-k,n})$$

uniformly with respect to $(v_1,\dots,v_n)\in\mathbb{R}^n_{\frac{1}{2}}$, and

$$\lim_{|v_1|+\dots+|v_n|\to 0}\frac{Y_{n,2}(t,v_1,\dots,v_n)}{|v_1|+\dots+|v_n|}=0$$

uniformly with respect to $t\in[t_0,+\infty[$.

The characteristic equation of the matrix consisting of coefficients of $v_1,\dots,v_n$ in system (2.17),

$$\prod_{l=k}^{n-1}(\rho+(n-l))\Big(\rho+\frac{a_{0n-k+1}}{\lambda_0-1}\Big)$$
$$\times\Big[\sum_{j=2}^{k-1}\frac{\sigma_{n-j}}{\lambda_0-1}\prod_{l=1}^{j-1}\frac{a_{0n-l}}{\lambda_0-1}\prod_{l=j}^{k-2}\Big(\rho+\frac{a_{0n-l}}{\lambda_0-1}\Big)-\Big(\rho-\frac{\sigma_{n-1}-1}{\lambda_0-1}\Big)\prod_{l=1}^{k-2}\Big(\rho+\frac{a_{0n-l}}{\lambda_0-1}\Big)\Big]=0,$$

has a zero root if $\frac{a_{0n-k+1}}{\lambda_0-1}=0$ (in case $\lambda_0=\frac{k-2}{k-1}$), $n-k$ negative roots $\rho_l=-(n-l)$ $(l=\overline{k,n-1})$ and $k-1$ roots of the algebraic equation (2.9), among which there are no any roots (according to the hypothesis of the theorem) with a zero real part.

Consequently, we get the system of differential equations that for $\lambda_0\in\,]-\infty,1[\,\setminus\{0,\frac{1}{2},\dots,\frac{k-2}{k-1}\}$ satisfies all assumptions of Theorem 2.2 in [10]. This theorem implies that the system (2.17) has at least one solution $(v_j)_{j=1}^n:[t_1,+\infty[\,\to\mathbb{R}^n_{\frac{1}{2}}$ $(t_1\in[t_0,+\infty[)$ that tends to zero as $t\to+\infty$. By virtue of the transformation (2.15), each solution of this kind corresponds to a $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution of equation (1.1) that admits the asymptotic representations (1.5) and (2.8$_i$) $(i=1,2,3)$ as $t\to+\infty$.

Moreover, in accordance with this theorem, if there are m (taking into account divisible) roots with a negative real part of the algebraic equation (2.9), then in case $\lambda_0\in\,]-\infty,\frac{k-2}{k-1}[\,\setminus\{0,\frac{1}{2},\dots,\frac{k-3}{k-2}\}$ $(\lambda_0\in\,]\frac{k-2}{k-1};1[)$ there exists an $(n-k+m+1)$-parameter ($(n-k+m)$-parameter, respectively) family of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions of equation (1.1) with the found representations.

Consider now the case $\lambda_0=\frac{k-2}{k-1}$. Applying the change of variables

$$\begin{cases}v_j=z_j\quad (j=1,n-k),\\ v_{n-k+1}=z_n,\\ v_{j+1}=z_j\quad (j=n-k+1,n-1),\end{cases}\tag{2.18}$$

we reduce (2.16) to the system of differential equations

$$\begin{cases} z_l' = \dfrac{n-l-k+1}{t}\,[-z_l + z_{l+1}] \ \ (l=\overline{1,n-k-1}), \\ z_{n-k}' = \dfrac{1}{t}\big[-z_{n-k} + Z_{n-k,1}(t,z_1,\dots,z_n)\big], \\ z_l' = \dfrac{1-k}{t}\big[-a_{0l}z_l + a_{0l}z_{l+1} + Z_{l,1}(t,z_1,\dots,z_n)\big] \ \ (l=\overline{n-k+1,n-2}), \\ z_{n-1}' = \dfrac{1-k}{t}\Big[\displaystyle\sum_{j=1}^{n-k}\sigma_{j-1}z_j + \sum_{j=n-k+2}^{n-1}\sigma_{j-1}z_{j-1} \\ \qquad\qquad + (\sigma_{n-1}-1)z_{n-1} + \displaystyle\sum_{i=1}^{2} Z_{n,i}(t,z_1,\dots,z_n)\Big], \\ z_n' = \dfrac{W'(t)}{W(t)}\,[-z_n + z_{n-k+1}], \end{cases} \tag{2.19}$$

where

$$Z_{j,m}(t,z_1,\dots,z_n) = Y_{j,m}(t,v_1,\dots,v_{n-k},v_{n-k+2},\dots,v_n,v_{n-k+1}) \ \ (m=1,2, \ \ j=\overline{n-k,n})$$

are such that

$$\lim_{t\to+\infty} Z_{j,1}(t,z_1,\dots,z_n) = 0$$

uniformly with respect to $(z_1,\dots,z_n)\in\mathbb{R}^n_{\frac12}$, and

$$\lim_{|z_1|+\dots+|z_n|\to 0} \frac{\partial Z_{n,2}(t,z_1,\dots,z_n)}{\partial z_k} = 0 \ \ (k=\overline{1,n})$$

uniformly with respect to $t\in[t_0,+\infty[$.

It follows from the form of $W(t)$ and (2.7) that $\lim\limits_{t\to+\infty} W(t)=0$,

$$\lim_{t\to+\infty}\frac{W'(t)t}{W(t)} = 0, \qquad \int_{t_0}^{+\infty}\frac{W'(t)dt}{W(t)} = \pm\infty \ \text{ and } \ \frac{W'(t)}{W(t)} < 0 \ \text{ as } \ t > t_0.$$

The characteristic equation of the matrix consisting of coefficients of $z_1,\dots,z_{n-1}$ (the coefficient of z_n differs from 0) in system (2.19),

$$\prod_{l=k}^{n-1}(\rho+(n-l))\Big[\sum_{j=2}^{k-1}(1-k)\sigma_{n-j}\prod_{l=1}^{j-1}((1-k)a_{0n-l})\prod_{l=j}^{k-2}(\rho+(1-k)a_{0n-l}) \\ - \big(\rho-(1-k)(\sigma_{n-1}-1)\big)\prod_{l=1}^{k-2}(\rho+(1-k)a_{0n-l})\Big] = 0,$$

has $n-k$ negative roots $\rho_l=-(n-l)$ $(l=\overline{k,n-1})$ and $k-1$ roots of the algebraic equation (2.9), as $\lambda_0=\frac{k-2}{k-1}$, among which there are no any roots (according to the hypothesis of the theorem) with a zero real part.

Consequently, system (2.19) satisfies all assumptions of Theorem 2.6 in [10]. Hence it has at least one solution $(z_j)_{j=1}^n : [t_1,+\infty[\,\to\mathbb{R}^n_{\frac12}$ $(t_1\in[t_0,+\infty[)$ that tends to zero as $t\to+\infty$. By virtue of transformations (2.15) and (2.18), each solution of this kind corresponds to the $\mathcal{P}^k_{+\infty}(\frac{k-2}{k-1})$-solution of equation (1.1) that admits asymptotic representations (1.5) and (2.8_i) $(i=1,2,3)$ as $t\to+\infty$.

As $\rho_l=-(n-l)$ $(l=\overline{k,n-1})$ are negative roots, then, in accordance with this theorem, there certainly exists an $(n-k)$-parameter family of such solutions. Moreover, there exists an $(n-k+m)$-parameter family of solutions with the above found representations, where m is a number of roots (taking into account divisible) with a negative real part of the algebraic equation (2.9), as $\lambda_0=\frac{k-2}{k-1}$. The proof of the theorem is complete. □

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ON THE BLOCK SEPARATION OF A LINEAR HOMOGENEOUS DIFFERENTIAL SYSTEM WITH OSCILLATING COEFFICIENTS IN A SPECIAL CASE

Abstract. For the linear homogeneous system of differential equations, coefficients of which are represented by an absolutely and uniformly convergent Fourier series with slowly varying coefficients and frequency, the conditions of existence of the linear transformation with coefficients of similar structure leading this system to a block-diagonal form in a special case are obtained.

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რეზიუმე. დიფერენციალურ განტოლებათა წრფივი ერთგვაროვანი სისტემისთვის, რომლის კოეფიციენტები წარმოიდგინება აბსოლუტურად და თანაბრად კრებადი ფურიეს მწკრივებით ნელა ცვლადი კოეფიციენტებით და სიხშირით, დადგენილია ანალოგიური სტრუქტურის კოეფიციენტების მქონე ისეთი წრფივი გარდაქმნის არსებობის პირობები, რომელსაც ერთ სპეციალურ შემთხვევაში ეს სისტემა დაჰყავს უჯრულ-დიაგონალურ ფორმამდე.

1 Introduction

This article continues the research started by the author in [1] on the problem of the block separation of the linear homogeneous system of differential equations, whose coefficients are represented by an absolutely and uniformly convergent Fourier series with slowly varying in some sense coefficients and frequency. Now we study a special case which by the conditions of the theorem proved in [1] is not covered.

2 Basic notations and definitions

Let $G = \{t, \varepsilon :\ t \in \mathbf{R},\ \varepsilon \in [0, \varepsilon_0],\ \varepsilon_0 \in \mathbf{R}^+\}$.

Definition 2.1. We say that a function $p(t, \varepsilon)$, generally complex-valued, belongs to the class $S(m; \varepsilon_0)$, $m \in \mathbf{N} \cup \{0\}$, if $t, \varepsilon \in G$ and

1) $p(t, \varepsilon) \in C^m(G)$ with respect to t;

2) $\dfrac{d^k p(t, \varepsilon)}{dt^k} = \varepsilon^k p_k^*(t, \varepsilon), \quad \sup\limits_G |p_k^*(t, \varepsilon)| < +\infty \ (0 \le k \le m)$.

Slowly variability of a function is understood in the sense of its belonging to the class $S(m; \varepsilon_0)$. As examples of functions of this class may serve, in general, complex-valued, bounded together with their derivatives up to and including the order m functions that depend on the "slow time" $\tau = \varepsilon t$: $\sin\tau$, $\operatorname{arctg}\tau$ etc.

Definition 2.2. We say that a function $f(t, \varepsilon, \theta(t, \varepsilon))$ belongs to the class $F(m; \varepsilon_0; \theta)$, $m \in \mathbf{N} \cup \{0\}$, if it can be represented as

$$f(t, \varepsilon, \theta(t, \varepsilon)) = \sum_{n=-\infty}^{\infty} f_n(t, \varepsilon) \exp(in\theta(t, \varepsilon)),$$

and

1) $f_n(t, \varepsilon) \in S(m; \varepsilon_0)$, $\dfrac{d^k f_n(t, \varepsilon)}{dt^k} = \varepsilon^k f_{nk}(t, \varepsilon)$ $(n \in \mathbf{Z},\ 0 \le k \le m)$;

2) $\|f\|_{F(m;\varepsilon_0;\theta)} \overset{def}{=} \sum\limits_{k=0}^{m} \sum\limits_{n=-\infty}^{\infty} \sup\limits_G |f_{nk}(t, \varepsilon)| < +\infty$,

3) $\theta(t, \varepsilon) = \int\limits_0^t \varphi(\tau, \varepsilon)\, d\tau$, $\varphi(t, \varepsilon) \in \mathbf{R}^+$, $\varphi(t, \varepsilon) \in S(m; \varepsilon_0)$, $\inf\limits_G \varphi(t, \varepsilon) > 0$.

Some properties of functions from the class $F(m; \varepsilon_0; \theta)$ are described in [1].

For any function $f(t, \varepsilon, \theta) \in F(m; \varepsilon_0; \theta)$ denote

$$\Gamma_n(f) = \frac{1}{2\pi} \int\limits_0^{2\pi} f(t, \varepsilon, u) \exp(-inu)\, du, \quad I(f) = f - \Gamma_0(f).$$

We say that the function $f(t, \varepsilon, \theta) \in F(m; \varepsilon_0; \theta)$ satisfies *condition* (A), if $\Gamma_0(f) \equiv 0$.

Let $A(t, \varepsilon, \theta) = (a_{js}(t, \varepsilon, \theta))_{j=\overline{1,M};\, s=\overline{1,K}}$, $a_{js} \in F(m; \varepsilon_0; \theta)$ $(j = \overline{1, M};\ s = \overline{1, K})$. Denote

$$\|A\|^*_{F(m;\varepsilon_0;\theta)} = \max_{1 \le j \le M} \sum_{l=1}^{K} \|a_{jl}(t, \varepsilon, \theta)\|_{F(m;\varepsilon_0;\theta)}.$$

3 Statement of the problem

We consider the system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= H_1(\varphi)x_1 + \mu(B_{11}(t,\varepsilon,\theta)x_1 + B_{12}(t,\varepsilon,\theta)x_2), \\ \frac{dx_2}{dt} &= H_2(\varphi)x_2 + \mu(B_{21}(t,\varepsilon,\theta)x_1 + B_{22}(t,\varepsilon,\theta)x_2), \end{aligned} \tag{3.1}$$

where $x_1 = \operatorname{colon}(x_{11}, \dots, x_{1N_1})$, $x_2 = \operatorname{colon}(x_{21}, \dots, x_{2N_2})$,

$$H_1(\varphi) = \begin{pmatrix} ip\varphi & 0 & \cdots & 0 & 0 \\ 1 & ip\varphi & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & ip\varphi & 0 \\ 0 & 0 & \cdots & 1 & ip\varphi \end{pmatrix}, \quad H_2(\varphi) = \begin{pmatrix} ir\varphi & 0 & \cdots & 0 & 0 \\ 1 & ir\varphi & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & ir\varphi & 0 \\ 0 & 0 & \cdots & 1 & ir\varphi \end{pmatrix}$$

are the Jordan blocks of dimensions N_1 and N_2, respectively ($N_1 + N_2 = N$); $p, r \in \mathbf{Z}$; $B_{jk}(t,\varepsilon,\theta)$ are the $(N_j \times N_k)$-matrices with elements from the class $F(m;\varepsilon;\theta)$; $\varphi(t,\varepsilon)$ is the function appearing in the definition of the class $F(m;\varepsilon;\theta)$; $\mu \in (0,1)$. In this sense, we are dealing with the resonance case.

Just as in [1], we study the question of the existence as well as the properties of the transformation of the form

$$x_j = L_{j1}(t,\varepsilon,\theta,\mu)\widetilde{x}_1 + L_{j2}(t,\varepsilon,\theta,\mu)\widetilde{x}_2, \quad j = 1,2, \tag{3.2}$$

where the elements L_{jk} $(j,k = 1,2)$ of $(N_j \times N_k)$-matrices belong to the class $F(m-1;\varepsilon_1;\theta)$ $(0 < \varepsilon_1 \le \varepsilon_0)$, reducing the system (3.1) to the form

$$\frac{d\widetilde{x}_1}{dt} = D_{N_1}(t,\varepsilon,\theta,\mu)\widetilde{x}_1, \quad \frac{d\widetilde{x}_2}{dt} = D_{N_2}(t,\varepsilon,\theta,\mu)\widetilde{x}_2, \tag{3.3}$$

where the elements D_{N_j} $(j = 1,2)$ of $(N_j \times N_j)$-matrices also belong to the class $F(m-1;\varepsilon^*;\theta)$.

Performing in the system (3.1) the transformation

$$x_1 = e^{ip\theta}y_1, \quad x_2 = e^{ir\theta}y_2,$$

where $y_1 = \operatorname{colon}(y_{11}, \dots, y_{1N_1})$, $y_2 = \operatorname{colon}(y_{21}, \dots, y_{2N_2})$, we obtain

$$\begin{aligned} \frac{dy_1}{dt} &= J_{N_1}y_1 + \mu(\widetilde{B}_{11}(t,\varepsilon,\theta)y_1 + \widetilde{B}_{12}(t,\varepsilon,\theta)y_2), \\ \frac{dy_2}{dt} &= J_{N_2}y_2 + \mu(\widetilde{B}_{21}(t,\varepsilon,\theta)y_1 + \widetilde{B}_{22}(t,\varepsilon,\theta)y_2), \end{aligned} \tag{3.4}$$

where

$$J_{N_1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad J_{N_2} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

are the Jordan blocks of dimensions N_1 and N_2, respectively, whose diagonal elements are equal to zero, and all elements of matrices $\widetilde{B}_{jk}(t,\varepsilon,\theta)$ belong to the class $F(m;\varepsilon_0;\theta)$.

Thus, the problem of the existence of transformation (3.2) reduces to the problem of the existence of the transformation

$$y_1 = z_1 + \mu Q_{12}(t,\varepsilon,\theta,\mu)z_2, \quad y_2 = \mu Q_{21}(t,\varepsilon,\theta,\mu)z_1 + z_2, \tag{3.5}$$

leading the system (3.4) to the form

$$\frac{dz_1}{dt} = D_{N_1}(t,\varepsilon,\theta,\mu)z_1, \quad \frac{dz_2}{dt} = D_{N_2}(t,\varepsilon,\theta,\mu)z_2,$$

where D_{N_1}, D_{N_2} are matrices of dimensions $(N_1 \times N_1)$ and $(N_2 \times N_2)$, respectively.

The matrices Q_{12}, Q_{21} must satisfy the system of matrix-equations

$$\begin{aligned}\frac{dQ_{jk}}{dt} &= J_{N_j} Q_{jk} - Q_{jk} J_{N_k} + \widetilde{B}_{jk}(t,\varepsilon,\theta) \\ &+ \mu\big(\widetilde{B}_{jj}(t,\varepsilon,\theta) Q_{jk} - Q_{jk}\widetilde{B}_{kk}(t,\varepsilon,\theta)\big) - \mu^2 Q_{jk}\widetilde{B}_{kj} Q_{jk}, \quad j,k = 1,2 \ \ (j \neq k).\end{aligned} \tag{3.6}$$

Then

$$\begin{aligned} D_{N_1} &= J_{N_1} + \mu \widetilde{B}_{11}(t,\varepsilon,\theta) + \mu^2 \widetilde{B}_{12}(t,\varepsilon,\theta) Q_{21}(t,\varepsilon,\theta,\mu), \\ D_{N_2} &= J_{N_1} + \mu \widetilde{B}_{22}(t,\varepsilon,\theta) + \mu^2 \widetilde{B}_{21}(t,\varepsilon,\theta) Q_{12}(t,\varepsilon,\theta,\mu). \end{aligned} \tag{3.7}$$

It is easy to see that the system (3.6) is divided into two independent matrix-equations, each of which has the form

$$\frac{dX}{dt} = J_M X - X J_K + F(t,\varepsilon,\theta) + \mu\big(A(t,\varepsilon,\theta)X - XB(t,\varepsilon,\theta)\big) - \mu^2 X R(t,\varepsilon,\theta) X, \tag{3.8}$$

where $X = (x_{js})_{j=\overline{1,M};\, s=\overline{1,K}}$,

$$J_M = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad J_K = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

are the Jordan blocks of dimensions M and K, respectively, whose diagonal elements are equal to zero, $F = (f_{js})_{j=\overline{1,M};\, s=\overline{1,K}}$, $A = (a_{js})_{j,s=\overline{1,M}}$, $B = (b_{js})_{j,s=\overline{1,K}}$, $R = (r_{js})_{j=\overline{1,K};\, s=\overline{1,M}}$. All elements of matrices F, A, B, R belong to the class $F(m;\varepsilon_0;\theta)$.

Therefore the problem of the existence of transformation (3.5), where all elements of matrices Q_{12}, Q_{21} belong to the class $F(m-1;\varepsilon^*;\theta)$ $(0 < \varepsilon^* < \varepsilon_0)$, reduces to the problem of the existence of a particular solution X of the equation (3.8) such that $x_{js} \in F(m-1;\varepsilon^*;\theta)$ $(j = \overline{1,M};\ s = \overline{1,K})$.

In [1], the conditions of the existence of such a solution are obtained when one of the sets of assumptions I, II, III is fulfilled.

I. (1) $M < K$;

(2) $V_1(F) \equiv 0$, where $V_1 = \operatorname{colon}(v_{11}(t,\varepsilon),\dots,v_{1M}(t,\varepsilon))$,

$$v_{1j}(t,\varepsilon) = \sum_{s=1}^{j} \Gamma_0(f_{s,K-j+s}(t,\varepsilon,\theta)) \ (j = \overline{1,M});$$

(3) $\inf\limits_G \big|\Gamma_0(b_{1K}(t,\varepsilon,\theta))\big| > 0$.

II. (1) $M = K$;

(2) $V_2(F) \equiv 0$, where $V_2 = \operatorname{colon}(v_{21}(t,\varepsilon),\dots,v_{2M}(t,\varepsilon))$,

$$v_{2j}(t,\varepsilon) = \sum_{s=1}^{j} \Gamma_0(f_{s,K-j+s}(t,\varepsilon,\theta)) \ (j = \overline{1,M});$$

(3) $\inf\limits_G \big|\Gamma_0(a_{1M}(t,\varepsilon,\theta) - b_{1M}(t,\varepsilon,\theta))\big| > 0$.

III. (1) $M > K$;

(2) $V_3(F) \equiv 0$, where $V_3 = \operatorname{colon}(v_{31}(t,\varepsilon),\dots,v_{3K}(t,\varepsilon))$,

$$v_{3j}(t,\varepsilon) = \sum_{s=1}^{j} \Gamma_0(f_{s,K-j+s}(t,\varepsilon,\theta)) \ (j = \overline{1,K});$$

(3) $\inf\limits_G |\Gamma_0(a_{1M}(t,\varepsilon,\theta))| > 0$.

In this paper it is assumed that the condition (2) in each of sets I, II, III is satisfied. But instead of the condition (3) it is accordingly supposed that

$$\begin{aligned}\Gamma_0(b_{1K}(t,\varepsilon,\theta)) &\equiv 0 \ \ (M<K);\\ \Gamma_0(a_{1M}(t,\varepsilon,\theta)-b_{1M}(t,\varepsilon,\theta)) &\equiv 0 \ \ (M=K);\\ \Gamma_0(a_{1M}(t,\varepsilon,\theta)) &\equiv 0 \ \ (M>K).\end{aligned}$$

4 Auxiliary results

As in [1], along with the equation (3.8) we consider an auxiliary matrix-equation

$$\varphi(t,\varepsilon)\,\frac{d\Xi}{d\theta}=J_M\Xi-\Xi J_K+F(t,\varepsilon,\theta)+\mu\big(A(t,\varepsilon,\theta)\Xi-\Xi B(t,\varepsilon,\theta)\big)-\mu^2\Xi R(t,\varepsilon,\theta)\Xi, \tag{4.1}$$

where t, φ are considered as constants, $\Xi=(\xi_{js})_{j=\overline{1,M};\,s=\overline{1,K}}$, F, A, B, R are the same as in the equation (3.8).

In accordance with the Poincaré method of small parameter [2], we construct an approximate 2π-periodic with respect to θ solution of the equation (4.1) in the form of the sum

$$\Xi=\sum_{\nu=0}^{2q-1}\Xi_\nu(t,\varepsilon,\theta)\mu^\nu, \tag{4.2}$$

where $\Xi_\nu=(\xi_{\nu,js})_{j=\overline{1,M};\,s=\overline{1,K}}$. The coefficients Ξ_ν are determined from the following chain of linear nonhomogeneous matrix differential equations:

$$\varphi(t,\varepsilon)\,\frac{d\Xi_0}{d\theta}=J_M\Xi_0-\Xi_0J_K+F(t,\varepsilon,\theta), \tag{4.3}$$

$$\varphi(t,\varepsilon)\,\frac{d\Xi_1}{d\theta}=J_M\Xi_1-\Xi_1J_K+A(t,\varepsilon,\theta)\Xi_0-\Xi_0B(t,\varepsilon,\theta), \tag{4.4}$$

$$\varphi(t,\varepsilon)\,\frac{d\Xi_2}{d\theta}=J_M\Xi_2-\Xi_2J_K+A(t,\varepsilon,\theta)\Xi_1-\Xi_1B(t,\varepsilon,\theta)-\Xi_0R(t,\varepsilon,\theta)\Xi_0, \tag{4.5}$$

$$\begin{aligned}\varphi(t,\varepsilon)\,\frac{d\Xi_\nu}{d\theta}&=J_M\Xi_\nu-\Xi_\nu J_K+A(t,\varepsilon,\theta)\Xi_{\nu-1}-\Xi_{\nu-1}B(t,\varepsilon,\theta)\\ &\qquad-\sum_{l=0}^{\nu-2}\Xi_lR(t,\varepsilon,\theta)\Xi_{\nu-2-l},\quad \nu=\overline{3,2q-1}.\end{aligned}$$

First, we consider the case $M<K$.

In scalar form, the equation (4.3) can be written as a following system of differential equations:

$$\begin{aligned}\varphi(t,\varepsilon)\,\frac{d\xi_{0,1K}}{d\theta}&=f_{1K}(t,\varepsilon,\theta),\\ \varphi(t,\varepsilon)\,\frac{d\xi_{0,jK}}{d\theta}&=\xi_{0,j-1,K}+f_{jK}(t,\varepsilon,\theta)\ \ (j=\overline{2,M}),\\ \varphi(t,\varepsilon)\,\frac{d\xi_{0,1s}}{d\theta}&=-\xi_{0,1,s+1}+f_{1s}(t,\varepsilon,\theta)\ \ (s=\overline{1,K-1}),\\ \varphi(t,\varepsilon)\,\frac{d\xi_{0,js}}{d\theta}&=-\xi_{0,j-1,s}-\xi_{0,j,s+1}+f_{js}(t,\varepsilon,\theta)\ \ (j=\overline{2,M};\ \ s=\overline{1,K-1}).\end{aligned} \tag{4.6}$$

The condition I(2) ensures the existence of a 2π-periodic with respect to θ solution of the equation (4.3) of the form

$$\Xi_0(t,\varepsilon,\theta)=C_0^{(1)}(t,\varepsilon)+L_1(F(t,\varepsilon,\theta)), \tag{4.7}$$

where the $(M\times K)$-matrix $C_0^{(1)}(t,\varepsilon)$ has the form

$$C_0^{(1)}(t,\varepsilon)=\begin{pmatrix}c_{01}^{(1)}(t,\varepsilon) & 0 & \cdots & 0 & 0 & \cdots & 0\\ c_{02}^{(1)}(t,\varepsilon) & c_{01}^{(1)}(t,\varepsilon) & \cdots & 0 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ c_{0M}^{(1)}(t,\varepsilon) & c_{0,M-1}^{(1)}(t,\varepsilon) & \cdots & c_{01}^{(1)}(t,\varepsilon) & 0 & \cdots & 0\end{pmatrix} \tag{4.8}$$

with $c_{01}^{(1)}(t,\varepsilon),\dots,c_{0M}^{(1)}(t,\varepsilon)$ as yet unknown scalar functions of the class $S(m;\varepsilon_0)$, $L_1(F(t,\varepsilon,\theta)) = (\widetilde{\xi}_{0,js}(t,\varepsilon,\theta))_{j=\overline{1,M};\,s=\overline{1,K}}$, and $\widetilde{\xi}_{0,js}$ are defined from the following equalities:

$$
\begin{aligned}
&\widetilde{\xi}_{0,1K}(t,\varepsilon,\theta) = I(f_{1K}(t,\varepsilon,\theta)) + p_{1K}(t,\varepsilon),\\
&\widetilde{\xi}_{0,jK}(t,\varepsilon,\theta) = I\big(\widetilde{\xi}_{0,j-1,K}(t,\varepsilon,\theta) + f_{jK}(t,\varepsilon,\theta)\big) + p_{jK}(t,\varepsilon) \ \ (j=\overline{2,M}),\\
&\widetilde{\xi}_{0,11}(t,\varepsilon,\theta) = I\big(f_{11}(t,\varepsilon,\theta) - \widetilde{\xi}_{0,12}(t,\varepsilon,\theta)\big) + p_{11}(t,\varepsilon),\\
&\widetilde{\xi}_{0,1s}(t,\varepsilon,\theta) = I\big(f_{1s}(t,\varepsilon,\theta) - \widetilde{\xi}_{0,1,s+1}(t,\varepsilon,\theta)\big) + p_{1s}(t,\varepsilon) \ \ (s=\overline{1,K-1}),\\
&\widetilde{\xi}_{0,js}(t,\varepsilon,\theta) = I\big(\widetilde{\xi}_{0,j-1,s}(t,\varepsilon,\theta) - \widetilde{\xi}_{0,j,s+1}(t,\varepsilon,\theta) + f_{js}(t,\varepsilon,\theta)\big) + p_{js}(t,\varepsilon) \ \ (j=\overline{2,M};\ \ s=\overline{1,K-1}),
\end{aligned}
$$

where $p_{js}(t,\varepsilon)$ are the functions from the class $S(m;\varepsilon_0)$ determined from the condition: all right-hand sides of the equations in (4.6) must satisfy condition (A). It is easy to verify that $p_{js}(t,\varepsilon)$ can be represented as some linear combinations of functions $\Gamma_0(f_{\alpha\beta}(t,\varepsilon,\theta))$ ($\alpha=\overline{1,M}$; $\beta=\overline{1,K}$).

We now define the matrix $C_0^{(1)}(t,\varepsilon)$ from the condition

$$V_1\big(A(t,\varepsilon,\theta)\Xi_0 - \Xi_0 B(t,\varepsilon,\theta)\big) = 0.$$

By virtue of (4.7), this condition can be rewritten as

$$V_1\big(A(t,\varepsilon,\theta)C_0^{(1)} - C_0^{(1)}B(t,\varepsilon,\theta)\big) = V_1\Big(L_1(F(t,\varepsilon,\theta))B(t,\varepsilon,\theta) - A(t,\varepsilon,\theta)L_1(F(t,\varepsilon,\theta))\Big). \tag{4.9}$$

In scalar form, the condition (4.9) can be written as a triangular with respect to $c_{01}^{(1)},\dots,c_{0M}^{(1)}$ system of linear algebraic equations:

$$\sum_{l=1}^{j} g_{jl}^{(1)}(t,\varepsilon)c_{0l}^{(1)} = h_j^{(1)}(t,\varepsilon), \ \ j=\overline{1,M},$$

where $g_{jl}^{(1)}(t,\varepsilon), h_j^{(1)}(t,\varepsilon) \in S(m;\varepsilon_0)$ and $g_{jj}^{(1)}(t,\varepsilon) = \Gamma_0(b_{1K}(t,\varepsilon,\theta))$ ($j=\overline{1,M}$) are the know functions.

Suppose that

$$g_{jl}^{(1)}(t,\varepsilon) \equiv 0 \ \ (j,l=\overline{1,M}, \ \ l\le j), \tag{4.10}$$

$$h_j^{(1)}(t,\varepsilon) \equiv 0 \ \ (j=\overline{1,M}). \tag{4.11}$$

Then

$$V_1\big(A(t,\varepsilon,\theta)C_0 - C_0B(t,\varepsilon,\theta)\big) = 0 \tag{4.12}$$

for any matrix C_0 of the form (4.8). Besides,

$$V_1\Big(A(t,\varepsilon,\theta)L_1(F(t,\varepsilon,\theta)) - L_1(F(t,\varepsilon,\theta))B(t,\varepsilon,\theta)\Big) = 0. \tag{4.13}$$

Therefore the equation (4.9) is satisfied for any matrix $C_0^{(1)}$ of the form (4.8).

The equalities (4.12), (4.13) ensure the existence of a 2π-periodic with respect to θ solution of the equation (4.4) having the form

$$\Xi_1(t,\varepsilon,\theta) = C_1^{(1)}(t,\varepsilon) + L_1\big(A(t,\varepsilon,\theta)\Xi_0 - \Xi_0B(t,\varepsilon,\theta)\big), \tag{4.14}$$

where

$$C_1^{(1)}(t,\varepsilon) = \begin{pmatrix} c_{11}^{(1)}(t,\varepsilon) & 0 & \cdots & 0 & 0 & \cdots & 0\\ c_{12}^{(1)}(t,\varepsilon) & c_{11}^{(1)}(t,\varepsilon) & \cdots & 0 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ c_{1M}^{(1)}(t,\varepsilon) & c_{1,M-1}^{(1)}(t,\varepsilon) & \cdots & c_{11}^{(1)}(t,\varepsilon) & 0 & \cdots & 0 \end{pmatrix}.$$

The solution (4.14) can be written as

$$\Xi_1(t,\varepsilon,\theta) = C_1^{(1)}(t,\varepsilon) + L_1\big(A(t,\varepsilon,\theta)C_0^{(1)} - C_0^{(1)}B(t,\varepsilon,\theta)\big) + F_1(t,\varepsilon,\theta), \tag{4.15}$$

where $F_1(t,\varepsilon,\theta) = L_1(AL_1(F) - L_1(F)B)$ does not depend on $C_0^{(1)}$.

We write down the conditions of the existence of a 2π-periodic with respect to θ solution of the equation (4.5):

$$V_1\Big(A(t,\varepsilon,\theta)\Xi_1 - \Xi_1 B(t,\varepsilon,\theta) - \Xi_0 R(t,\varepsilon,\theta)\Xi_0\Big) = 0.$$

Taking into account the equalities (4.7) and (4.15), this condition can be rewritten (for brevity, we omit the arguments t, ε, θ) as

$$V_1(AC_1^{(1)} - C_1^{(1)}B) + V_1\Big(AL_1(AC_0^{(1)} - C_0^{(1)}B) - L_1(AC_0^{(1)} - C_0^{(1)}B)B\Big) + V_1(AF_1 - F_1B) \\ - V_1(C_0^{(1)}RC_0^{(1)}) - V_1\big(L_1(F)RC_0^{(1)} + C_0^{(1)}RL_1(F)\big) - V_1(L_1(F)RL_1(F)) = 0. \quad (4.16)$$

Due to (4.12), the condition (4.16) can be rewritten as

$$V_1\Big(AL_1(AC_0^{(1)} - C_0^{(1)}B) - L_1(AC_0^{(1)} - C_0^{(1)}B)B\Big) \\ - V_1\big(L_1(F)RC_0^{(1)} + C_0^{(1)}RL_1(F)\big) - V_1(C_0^{(1)}RC_0^{(1)}) + U^{(1)} = 0, \quad (4.17)$$

where $U^{(1)} = U^{(1)}(t,\varepsilon)$ is the known M-vector that does not depend on $C_0^{(1)}$.

In scalar form, the equation (4.17) can be written as a nonlinear with respect to $c_{01}^{(1)}, \dots, c_{0M}^{(1)}$ system of algebraic equations

$$\Phi_j^{(1)}(t,\varepsilon,c_{01}^{(1)},\dots,c_{0M}^{(1)}) = 0, \quad j = \overline{1,M}, \quad (4.18)$$

with quadratic nonlinearities.

Suppose that the system (4.18) has a solution $c_{01}^{(1)}, \dots, c_{0M}^{(1)}$ such that

$$\inf_G \left| \det \frac{\partial(\Phi_1^{(1)},\dots,\Phi_M^{(1)})}{\partial(c_{01}^{(1)},\dots,c_{0M}^{(1)})} \right| > 0. \quad (4.19)$$

Then the equation (4.5) has a 2π-periodic with respect to θ solution $\Xi_2(t,\varepsilon,\theta)$ belonging to the class $F(m;\varepsilon_0;\theta)$.

We now consider the equation for the vector-function $\Xi_{\nu+2}$ and distinguish in it explicitly the terms which depend on $\Xi_{\nu+1}$, Ξ_ν:

$$\varphi(t,\varepsilon)\,\frac{d\Xi_{\nu+2}}{d\theta} = J_M\Xi_{\nu+2} - \Xi_{\nu+2}J_K + A(t,\varepsilon,\theta)\Xi_{\nu+1} - \Xi_{\nu+1}B(t,\varepsilon,\theta) \\ - \Xi_0R(t,\varepsilon,\theta)\Xi_\nu - \Xi_\nu R(t,\varepsilon,\theta)\Xi_0 - \sum_{l=1}^{\nu-1}\Xi_l R(t,\varepsilon,\theta)\Xi_{\nu-l}. \quad (4.20)$$

For $\alpha = \overline{0,\nu+1}$, we have

$$\Xi_\alpha(t,\varepsilon,\theta) = C_\alpha^{(1)}(t,\varepsilon) + \widetilde{\widetilde{\Xi}}_\alpha(t,\varepsilon,\theta), \quad (4.21)$$

where $C_\alpha^{(1)}(t,\varepsilon)$ is the $(M\times K)$-matrix of the form (4.8), and $\widetilde{\widetilde{\Xi}}_\alpha(t,\varepsilon,\theta)$ is the known vector-function belonging to the class $F(m;\varepsilon_0;\theta)$.

We suppose that the matrices $\Xi_0(t,\varepsilon,\theta), \Xi_1(t,\varepsilon,\theta), \dots, \Xi_{\nu-1}(t,\varepsilon,\theta)$ are completely defined, including the matrix $C_{\nu-1}^{(1)}(t,\varepsilon)$, and the matrix $C_\nu^{(1)}(t,\varepsilon)$, $C_{\nu+1}^{(1)}(t,\varepsilon)$ have to be defined.

We write down the conditions of the existence of a 2π-periodic with respect to θ solution of the equation (4.20) as follows:

$$V_1\big(A(t,\varepsilon,\theta)\Xi_{\nu+1} - \Xi_{\nu+1}B(t,\varepsilon,\theta) - \Xi_0R(t,\varepsilon,\theta)\Xi_\nu\big) \\ - V_1\Big(\Xi_\nu R(t,\varepsilon,\theta)\Xi_0 + \sum_{l=1}^{\nu-1}\Xi_l R(t,\varepsilon,\theta)\Xi_{\nu-l}\Big) = 0. \quad (4.22)$$

Represent the matrix $\widetilde{\Xi}_{\nu+1}$ as

$$\widetilde{\Xi}_{\nu+1} = \widetilde{\Xi}^{(*)}_{\nu+1} + \widetilde{\Xi}^{(**)}_{\nu+1}, \tag{4.23}$$

where $\widetilde{\Xi}^{(*)}_{\nu+1}$ is a 2π-periodic with respect to θ solution of the equation

$$\varphi(t,\varepsilon)\,\frac{d\Xi_{\nu+1}}{d\theta} = J_M\Xi_{\nu+1} - \Xi_{\nu+1}J_K + A(t,\varepsilon,\theta)C^{(1)}_\nu(t,\varepsilon) - C^{(1)}_\nu(t,\varepsilon)B(t,\varepsilon,\theta) \tag{4.24}$$

and $\widetilde{\Xi}^{(**)}_{\nu+1}$ is a 2π-periodic with respect to θ solution of the equation

$$\varphi(t,\varepsilon)\,\frac{d\Xi_{\nu+1}}{d\theta} = J_M\Xi_{\nu+1} - \Xi_{\nu+1}J_K + A(t,\varepsilon,\theta)\widetilde{\Xi}_\nu - \widetilde{\Xi}_\nu B(t,\varepsilon,\theta) - \sum_{l=1}^{\nu-1}\Xi_l R(t,\varepsilon,\theta)\Xi_{\nu-1-l}.$$

The condition of the existence of a 2π-periodic with respect to θ solution of the equation (4.24) has the form

$$V_1(A(t,\varepsilon,\theta)C^{(1)}_\nu - C^{(1)}_\nu B(t,\varepsilon,\theta)) = 0.$$

By (4.12), this equality holds for any matrix C_ν of the kind

$$C_\nu(t,\varepsilon) = \begin{pmatrix} c_{\nu1}(t,\varepsilon) & 0 & \cdots & 0 & 0 & \cdots & 0\\ c_{\nu2}(t,\varepsilon) & c_{\nu1}(t,\varepsilon) & \cdots & 0 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ c_{\nu M}(t,\varepsilon) & c_{\nu,M-1}(t,\varepsilon) & \cdots & c_{\nu1}(t,\varepsilon) & 0 & \cdots & 0 \end{pmatrix}.$$

Therefore the equation (4.24) has a 2π-periodic with respect to θ solution of the kind

$$\widetilde{\Xi}^{(1)}_{\nu+1} = L_1\big(A(t,\varepsilon,\theta)C^{(1)}_\nu - C^{(1)}_\nu B(t,\varepsilon,\theta)\big).$$

Taking into account (4.21) and (4.23), the condition (4.22) can be rewritten as

$$\begin{aligned} V_1\big(A(t,\varepsilon,\theta)C^{(1)}_{\nu+1} - C^{(1)}_{\nu+1}B(t,\varepsilon,\theta)\big) + V_1\Big(A(t,\varepsilon,\theta)(\widetilde{\Xi}^{(*)}_{\nu+1} + \widetilde{\Xi}^{(**)}_{\nu+1}) - (\widetilde{\Xi}^{(*)}_{\nu+1} + \widetilde{\Xi}^{(**)}_{\nu+1})B(t,\varepsilon,\theta)\Big)&\\ - V_1\big(\Xi_0 R(t,\varepsilon,\theta)\Xi_\nu + \Xi_\nu R(t,\varepsilon,\theta)\Xi_0\big) + V_1^*(t,\varepsilon) = 0,& \end{aligned} \tag{4.25}$$

where $V_1^*(t,\varepsilon)$ is the known M-vector belonging to the class $S(m;\varepsilon_0)$.

Based on (4.12), (4.21) and (4.23), we can rewrite (4.25) as

$$\begin{aligned} V_1\Big(A(t,\varepsilon,\theta)L_1\big(A(t,\varepsilon,\theta)C^{(1)}_\nu - C^{(1)}_\nu B(t,\varepsilon,\theta)\big) - L_1\big(A(t,\varepsilon,\theta)C^{(1)}_\nu - C^{(1)}_\nu B(t,\varepsilon,\theta)\big)B(t,\varepsilon,\theta)\Big)&\\ - V_1\big(L_1(F)R(t,\varepsilon,\theta)C^{(1)}_\nu + C^{(1)}_\nu R(t,\varepsilon,\theta)L_1(F)\big)&\\ - V_1\big(C_0R(t,\varepsilon,\theta)C^{(1)}_\nu + C^{(1)}_\nu R(t,\varepsilon,\theta)C_0\big) + Z^{(1)}(t,\varepsilon) = 0,& \end{aligned} \tag{4.26}$$

where $Z^{(1)}(t,\varepsilon)$ is the known M-vector belonging to the class $S(m;\varepsilon_0)$.

It is not difficult to establish the validity of the relations

$$(XRY)_{\alpha\beta} = \begin{cases} \displaystyle\sum_{j=1}^{\alpha} x_j \sum_{l=1}^{M+1-\beta} r_{\alpha+1-j,l+\beta-1}y_l, & \text{if } \beta \le M,\\ 0, & \text{if } \beta > M, \end{cases}$$

where X, Y are the $(M\times K)$-matrices of the kind (4.8). It follows that in a scalar form the equation (4.26) can be written as

$$\sum_{l=1}^{M}\frac{\partial\Phi^{(1)}_j(t,\varepsilon,c^{(1)}_{01},\dots,c^{(1)}_{0M})}{\partial c^{(1)}_{0l}}\,c^{(1)}_{\nu l} = z^{(1)}_j(t,\varepsilon),\quad j=\overline{1,M}, \tag{4.27}$$

where $u_j^{(1)}(t,\varepsilon)$ are the known functions belonging to the class $S(m;\varepsilon_0)$. By the condition (4.19), the system (4.27) has a unique solution $c_{\nu 1}^{(1)}(t,\varepsilon),\ldots,c_{\nu M}^{(1)}(t,\varepsilon)$ belonging to the class $S(m;\varepsilon_0)$.

Thus, all the matrices $\Xi_\nu(t,\varepsilon,\theta)$ $(\nu=\overline{0,2q-1})$ are completely defined and belong to the class $F(m;\varepsilon_0;\theta)$. Therefore, by (4.2), the matrix $\Xi(t,\varepsilon,\theta,\mu)$ is also completely defined $\forall\,\mu\in(0,1)$ and belongs to the class $F(m;\varepsilon_0;\theta)$.

Lemma 4.1. *Let the equation* (3.8) *satisfy the following conditions:*

(1) $M<K$*;*

(2) $V_1(F(t,\varepsilon,\theta))\equiv 0$*;*

(3) *the equalities* (4.10)*,* (4.11) *hold;*

(4) *the system* (4.18) *has a solution satisfying the condition* (4.19)*.*

Then there exists $\mu_1\in(0,1)$ *such that for any* $\mu\in(0,\mu_1)$ *there exists a transformation of the form*

$$X=\Xi(t,\varepsilon,\theta,\mu)+\Phi(t,\varepsilon,\theta,\mu)Y\Psi(t,\varepsilon,\theta,\mu), \tag{4.28}$$

where the matrix $\Xi(t,\varepsilon,\theta,\mu)$ *is defined by the equality* (4.2) *and the elements of the* $(M\times M)$*-matrix* Φ *and those of the* $(K\times K)$*-matrix* Ψ *belong to the class* $F(m;\varepsilon_0;\theta)$ $\forall\,\mu\in(0,\mu_1)$*, which reduces the equation* (3.8) *to the form*

$$\begin{aligned}\frac{dY}{dt}&=J_MY-YJ_K+\Big(\sum_{l=1}^{q}U_{l1}(t,\varepsilon)\mu^l\Big)Y-Y\Big(\sum_{l=1}^{q}U_{l2}(t,\varepsilon)\mu^l\Big)\\&\quad+\varepsilon\big(U_1(t,\varepsilon,\theta,\mu)Y-YU_2(t,\varepsilon,\theta,\mu)\big)+\mu^{q+1}\big(W_1(t,\varepsilon,\theta,\mu)Y-YW_2(t,\varepsilon,\theta,\mu)\big)\\&\quad+\varepsilon H_1(t,\varepsilon,\theta,\mu)+\mu^{2q}H_2(t,\varepsilon,\theta,\mu)+\mu YR_1(t,\varepsilon,\theta,\mu)Y,\end{aligned} \tag{4.29}$$

where the elements of matrices U_{l1}*,* U_{l2} $(l=\overline{1,q})$ *belong to the class* $S(m;\varepsilon_0)$*, and the elements of matrices* U_1*,* U_2*,* W_1*,* W_2*,* H_1*,* H_2*,* R_1 *of the corresponding dimensions belong to the class* $F(m-1;\varepsilon_0;\theta)$*.*

Proof. Substituting

$$X=\Xi(t,\varepsilon,\theta,\mu)+\widetilde{X}$$

in (3.8), where $\widetilde{X}$ is a new unknown matrix, we obtain

$$\begin{aligned}\frac{d\widetilde{X}}{dt}&=J_M\widetilde{X}-\widetilde{X}J_K+\varepsilon H_3(t,\varepsilon,\theta,\mu)+\mu^{2q}H_4(t,\varepsilon,\theta,\mu)\\&\quad+\Big(\sum_{l=1}^{q}P_l(t,\varepsilon,\theta)\mu^l\Big)\widetilde{X}-\widetilde{X}\Big(\sum_{l=1}^{q}Q_l(t,\varepsilon,\theta)\mu^l\Big)\\&\quad+\mu^{q+1}(W_1^*(t,\varepsilon,\theta,\mu)\widetilde{X}-\widetilde{X}W_2^*(t,\varepsilon,\theta,\mu))+\mu^2\widetilde{X}R(t,\varepsilon,\theta)\widetilde{X}.\end{aligned} \tag{4.30}$$

By Lemma 1 from [1], using the substitution of the kind

$$\widetilde{X}=\Big(E_M+\sum_{l=1}^{q}\Phi_l(t,\varepsilon,\theta)\mu^l\Big)Y\Big(E_K+\sum_{l=1}^{q}\Psi_l(t,\varepsilon,\theta)\mu^l\Big),$$

where E_M,E_K are the identity matrices of dimensions M and K, respectively, the elements of the $(M\times M)$-matrices Φ_l and those of $(K\times K)$-matrices Ψ_l $(l=\overline{1,q})$ belong to the class $F(m;\varepsilon_0;\theta)$, we reduce the equation (4.30) to the form (4.29). □

We now consider the case $M=K$. The condition II (2) ensures the existence of a 2π-periodic with respect to θ solution of the equation (4.3), which is of the form

$$\Xi_0(t,\varepsilon,\theta)=C_0^{(2)}(t,\varepsilon)+L_2(F(t,\varepsilon,\theta))$$

with

$$C_0^{(2)}(t,\varepsilon) = \begin{pmatrix} c_{01}^{(2)}(t,\varepsilon) & 0 & \cdots & 0 \\ c_{02}^{(2)}(t,\varepsilon) & c_{01}^{(2)}(t,\varepsilon) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ c_{0M}^{(2)}(t,\varepsilon) & c_{0,M-1}^{(2)}(t,\varepsilon) & \cdots & c_{01}^{(2)}(t,\varepsilon) \end{pmatrix}, \tag{4.31}$$

where the linear matrix-operator $L_2(F)$ can be constructed similarly to the operator $L_1(F)$. The matrix $C_0^{(2)}$ is defined from the equation

$$V_2\big(A(t,\varepsilon,\theta)C_0^{(2)} - C_0^{(2)}B(t,\varepsilon,\theta)\big) = V_2\Big(L_2(F(t,\varepsilon,\theta))B(t,\varepsilon,\theta) - A(t,\varepsilon,\theta)L_2(F(t,\varepsilon,\theta))\Big). \tag{4.32}$$

In scalar form, the condition (4.32) can be written as a triangular with respect to $C_{01}^{(2)},\dots,C_{0M}^{(2)}$ system of linear algebraic equations:

$$\sum_{l=1}^{j} g_{jl}^{(2)}(t,\varepsilon)c_{0l}^{(2)} = h_j^{(2)}(t,\varepsilon), \ \ j=\overline{1,M},$$

where $g_{jl}^{(2)}(t,\varepsilon), h_j^{(2)}(t,\varepsilon) \in S(m;\varepsilon_0)$ and $g_{jj}^{(2)}(t,\varepsilon) = \Gamma_0(a_{1M}(t,\varepsilon,\theta) - b_{1M}(t,\varepsilon,\theta))$ $(j=\overline{1,M})$ are the know functions.

Suppose that

$$g_{jl}^{(2)}(t,\varepsilon) \equiv 0 \ \ (j,l=\overline{1,M}, \ \ l\le j), \tag{4.33}$$

$$h_j^{(2)}(t,\varepsilon) \equiv 0 \ \ (j=\overline{1,M}). \tag{4.34}$$

Then

$$V_2(A(t,\varepsilon,\theta)C_0 - C_0B(t,\varepsilon,\theta)) = 0$$

for any C_0 of the kind (4.31), and

$$V_2\Big(L_2(F(t,\varepsilon,\theta))B(t,\varepsilon,\theta) - A(t,\varepsilon,\theta)L_2(F(t,\varepsilon,\theta))\Big) = 0.$$

Therefore the equation (4.32) is satisfied for any $C_0^{(2)}$ of the kind (4.31).

Similarly to the case $M<K$, we define the matrix $C_0^{(2)}(t,\varepsilon)$ from the equation

$$V_2\Big(AL_2(AC_0^{(2)} - C_0^{(2)}B) - L_2(AC_0^{(2)} - C_0^{(2)}B)B\Big) \\ - V_2\big(L_2(F)RC_0^{(2)} + C_0^{(2)}RL_2(F)\big) - V_2(C_0^{(2)}RC_0^{(2)}) + U^{(2)} = 0, \tag{4.35}$$

where $U^{(2)} = U^{(2)}(t,\varepsilon)$ is the known M-vector, which does not depend on $C_0^{(2)}$.

In scalar form, the equation (4.35) can be written as a nonlinear with respect to $c_{01}^{(2)},\dots,c_{0M}^{(2)}$ system of algebraic equations

$$\Phi_j^{(2)}(t,\varepsilon,c_{01}^{(2)},\dots,c_{0M}^{(2)}) = 0, \ \ j=\overline{1,M}, \tag{4.36}$$

with quadratic nonlinearities.

Suppose that the system (4.36) has a solution $c_{01}^{(2)},\dots,c_{0M}^{(2)}$ such that

$$\inf_G \left| \det \frac{\partial(\Phi_1^{(2)},\dots,\Phi_M^{(2)})}{\partial(c_{01}^{(2)},\dots,c_{0M}^{(2)})} \right| > 0. \tag{4.37}$$

Lemma 4.2. *Let the equation* (3.8) *satisfy the following conditions:*

(1) $M=K$*;*

(2) $V_2(F(t,\varepsilon,\theta)) \equiv 0$;

(3) *the equalities* (4.33), (4.34) *hold;*

(4) *the system* (4.36) *has a solution satisfying the condition* (4.37).

Then there exists $\mu_2 \in (0,1)$ *such that for any* $\mu \in (0,\mu_2)$ *there exists a transformation of the form* (4.28), *where the matrix* $\Xi(t,\varepsilon,\theta,\mu)$ *is defined by* (4.2) *and the elements of the* $(M \times M)$*-matrix* Φ *and those of the* $(K \times K)$*-matrix* Ψ *belong to the class* $F(m;\varepsilon_0;\theta)$ $\forall\, \mu \in (0,\mu_2)$, *which reduces the equation* (3.8) *to the form* (4.29).

Proof of Lemma 4.2 is similar to that of Lemma 4.1.

Finally, we consider the case $M > K$.

The condition III (2) ensures the existence of a 2π-periodic with respect to θ solution of the equation (4.3), which has the form

$$\Xi_0(t,\varepsilon,\theta) = C_0^{(3)}(t,\varepsilon) + L_3(F(t,\varepsilon,\theta))$$

with

$$C_0^{(3)}(t,\varepsilon) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \cdots & 0 \\ c_{01}^{(3)}(t,\varepsilon) & 0 & \cdots & 0 \\ c_{02}^{(3)}(t,\varepsilon) & c_{01}^{(3)}(t,\varepsilon) & \cdots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ c_{0K}^{(3)}(t,\varepsilon) & c_{0,K-1}^{(3)}(t,\varepsilon) & \cdots & c_{01}^{(3)}(t,\varepsilon) \end{pmatrix}, \tag{4.38}$$

where the linear matrix-operator $L_3(F)$ is constructed similarly to the operator $L_1(F)$. The matrix $C_0^{(3)}$ is defined from the equation

$$V_3\big(A(t,\varepsilon,\theta)C_0^{(3)} - C_0^{(3)}B(t,\varepsilon,\theta)\big) = V_3\Big(L_3(F(t,\varepsilon,\theta))B(t,\varepsilon,\theta) - A(t,\varepsilon,\theta)L_3(F(t,\varepsilon,\theta))\Big). \tag{4.39}$$

In scalar form, the condition (4.39) can be written as a triangular with respect to $c_{01}^{(3)},\dots,c_{0K}^{(3)}$ system of linear algebraic equations:

$$\sum_{l=1}^{j} g_{jl}^{(3)}(t,\varepsilon)c_{0l}^{(3)} = h_j^{(3)}(t,\varepsilon), \quad j=\overline{1,K},$$

where $g_{jl}^{(3)}(t,\varepsilon), h_j^{(3)}(t,\varepsilon) \in S(m;\varepsilon_0)$ and $g_{jj}^{(3)}(t,\varepsilon) = \Gamma_0(a_{1M}(t,\varepsilon,\theta))$ $(j=\overline{1,K})$ are the known functions.

Suppose that

$$g_{jl}^{(3)}(t,\varepsilon) \equiv 0 \ \ (j,l=\overline{1,K}, \ \ l \le j), \tag{4.40}$$

$$h_j^{(3)}(t,\varepsilon) \equiv 0 \ \ (j=\overline{1,K}). \tag{4.41}$$

Then

$$V_3\big(A(t,\varepsilon,\theta)C_0 - C_0B(t,\varepsilon,\theta)\big) = 0$$

for any C_0 of the kind (4.38) and

$$V_3\Big(L_3(F(t,\varepsilon,\theta))B(t,\varepsilon,\theta) - A(t,\varepsilon,\theta)L_3(F(t,\varepsilon,\theta))\Big) = 0.$$

Therefore the equation (4.39) is satisfied for any $C_0^{(3)}$ of the kind (4.38).

Define the matrix $C_0^{(3)}(t,\varepsilon)$ from the equation

$$V_3\big(AL_3(AC_0^{(3)} - C_0^{(3)}B) - L_3(AC_0^{(3)} - C_0^{(3)}B)B\big) \\ - V_3\big(L_3(F)RC_0^{(3)} + C_0^{(3)}RL_3(F)\big) - V_3(C_0^{(3)}RC_0^{(3)}) + U^{(3)} = 0, \quad (4.42)$$

where $U^{(3)} = U^{(3)}(t,\varepsilon)$ is the known M-vector, which does not depend on $C_0^{(3)}$.

In scalar form, the equation (4.42) can be written as a nonlinear with respect to $c_{01}^{(3)},\dots,c_{0K}^{(3)}$ system of algebraic equations

$$\Phi_j^{(3)}(t,\varepsilon,c_{01}^{(3)},\dots,c_{0K}^{(3)}) = 0, \quad j=\overline{1,K}, \tag{4.43}$$

with quadratic nonlinearities.

Suppose that the system (4.43) has a solution $c_{01}^{(3)},\dots,c_{0K}^{(3)}$ such that

$$\inf_G \left| \det \frac{\partial(\Phi_1^{(3)},\dots,\Phi_K^{(3)})}{\partial(c_{01}^{(3)},\dots,c_{0K}^{(2)})}\right| > 0. \tag{4.44}$$

Lemma 4.3. *Let the equation* (3.8) *satisfy the following conditions:*

(1) $M > K$;

(2) $V_3(F(t,\varepsilon,\theta)) \equiv 0$;

(3) *the equalities* (4.40), (4.41) *hold;*

(4) *the system* (4.43) *has a solution, which satisfy the condition* (4.44).

Then there exists $\mu_3 \in (0,1)$ *such that for any* $\mu \in (0,\mu_3)$ *there exists a transformation of the form* (4.28)*, where the matrix* $\Xi(t,\varepsilon,\theta,\mu)$ *is defined by* (4.2) *and the elements of the* $(M\times M)$*-matrix* Φ *and those of the* $(K\times K)$*-matrix* Ψ *belong to the class* $F(m;\varepsilon_0;\theta)$ $\forall\,\mu\in(0,\mu_3)$*, which reduces the equation* (3.8) *to the form* (4.29).

Proof of Lemma 4.3 is similar to that of Lemma 4.1, too.

Introduce the matrices

$$\widetilde{U}_1(t,\varepsilon,\mu) = \sum_{l=1}^{q} U_{l1}(t,\varepsilon)\mu^l, \quad \widetilde{U}_2(t,\varepsilon,\mu) = \sum_{l=1}^{q} U_{l2}(t,\varepsilon)\mu^l,$$

where U_{l1}, U_{l2} $(l=\overline{1,q})$ are defined in Lemma 4.1.

Lemma 4.4. *Let the equation* (4.29) *satisfy the following conditions:*

(1) *eigenvalues* $\lambda_{1j}(t,\varepsilon,\mu)$ $(j=\overline{1,M})$ *of the matrix* $J_M+\widetilde{U}_1(t,\varepsilon,\mu)$ *and* $\lambda_{2s}(t,\varepsilon,\mu)$ $(s=\overline{1,K})$ *of the matrix* $J_K+\widetilde{U}_2(t,\varepsilon,\mu)$ *are such that*

$$\inf_G \big|\operatorname{Re}\big(\lambda_{1j}(t,\varepsilon,\mu)-\lambda_{2s}(t,\varepsilon,\mu)\big)\big| \ge \gamma_0\mu^{q_0} \quad (\gamma_0>0,\ \ 0<q_0\le q;\ \ j=\overline{1,M};\ \ s=\overline{1,K});$$

(2) *there exist a* $(M\times M)$*-matrix* $P_1(t,\varepsilon,\mu)$ *and a* $(K\times K)$*-matrix* $P_2(t,\varepsilon,\mu)$ *such that*

  (a) *all the elements of these matrices belong to the class* $S(m;\varepsilon_0)\subset F(m;\varepsilon_0;\theta)$;

  (b) $\|P_j^{-1}(t,\varepsilon,\mu)\|^*_{F(m\varepsilon_0,\theta)} \le M_1\mu^{-\alpha}$, $M_1\in(0,+\infty)$, $\alpha\in[0,q]$, $j=1,2$;

  (c) $P_1^{-1}(J_M+\widetilde{U}_1)P_1=\Lambda_1(t,\varepsilon,\mu)$, $P_2(J_K+\widetilde{U}_2)P_2^{-1}=\Lambda_2(t,\varepsilon,\mu)$, *where* $\Lambda_1=\operatorname{diag}(\lambda_{11},\dots,\lambda_{1M})$, $\Lambda_2=\operatorname{diag}(\lambda_{21},\dots,\lambda_{2K})$;

(3) $q > q_0+\alpha-1/2$.

Then there exist $\mu_4 \in (0,1)$ *and* $K_4 \in (0,+\infty)$ *such that for any* $\mu \in (0,\mu_4)$ *the matrix differential equation* (4.29) *has a particular solution* $Y(t,\varepsilon,\theta,\mu)$ *all elements of which belong to the class* $F(m-1;\varepsilon_1(\mu);\theta)$*, where* $\varepsilon_1(\mu) = \min(\varepsilon_0, K_4\mu^{2q_0+2\alpha-1})$.

Proof of Lemma 4.4 is completely analogous to that of Lemma 3 in [1].

The following Lemma is an immediate consequence of the above ones.

Lemma 4.5. *Let the equation* (3.8) *satisfy all conditions of Lemma* 4.1 (*in case* $M < K$), *or Lemma* 4.2 (*in case* $M = K$), *or Lemma* 4.3 (*in case* $M > K$), *and the equation* (4.29), *obtained from* (3.8) *by means of the transformation* (4.28), *satisfy all conditions of Lemma* 4.4. *Then there exist* $\mu_5 \in (0,1)$ *and* $K_5 \in (0,+\infty)$ *such that for any* $\mu \in (0,\mu_5)$ *the equation* (3.8) *has a particular solution belonging to the class* $F(m-1;\varepsilon_2(\mu);\theta)$*, where* $\varepsilon_2(\mu) = K_5\mu^{2q_0+2\alpha-1}$ *and* q_0, α *are defined in Lemma* 4.4.

5 The basic result

Based on the above reasoning in Section 3 and Lemma 4.5 we obtain the following result.

Theorem. *Let each of the equations* (3.6) *satisfy all conditions of Lemma* 4.5. *Then there exist* $\mu_6 \in (0,1)$ *and* $K_6 \in (0,+\infty)$ *such that for any* $\mu \in (0,\mu_6)$ *there exists a transformation of the form* (3.2) *with coefficients from the class* $F(m-1;\varepsilon_3(\mu);\theta)$*, where* $\varepsilon_3(\mu) = K_6\mu^{2q_0+2\alpha-1}$ (q_0, α *are defined in Lemma* 4.4), *which reduces the system* (3.1) *to the block-diagonal form* (3.3). *The matrices* D_{N_1}, D_{N_2} *are defined by* (3.7).

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Short Communications

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ON THE WELL-POSEDNESS OF ANTIPERIODIC PROBLEM FOR SYSTEMS OF NONLINEAR IMPULSIVE DIFFERENTIAL EQUATIONS WITH FIXED IMPULSES POINTS

Abstract. The antiperiodic problem for systems of nonlinear impulsive equations with fixed points of impulses actions is considered. The sufficient (among them effective) conditions for the well-posedness of this problem are given.

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Let m_0 be a fixed natural number, ω be a fixed positive real number, and $0 < \tau_1 < \cdots < \tau_{m_0} < \omega$ be fixed points (we assume $\tau_0 = 0$ and $\tau_{m_0+1} = \omega$, if necessary). Let $T = \{\tau_l + m\omega : \ l = 1, \dots, m_0; \ m = 0, \pm 1, \pm 2, \dots\}$.

Consider the system of nonlinear impulsive differential equations with fixed impulses points

$$\frac{dx}{dt} = f(t,x) \ \text{ almost everywhere on } \ \mathbb{R} \setminus T,$$
$$x(\tau+) - x(\tau-) = I(\tau, x(\tau)) \ \text{ for } \ \tau \in T,$$

under the ω-antiperiodic problem

$$x(t+\omega) = -x(t) \ \text{ for } \ t \in \mathbb{R},$$

where $f = (f_i)_{i=1}^n$ is a vector-function belonging to the Carathéodory class $Car([\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, and $I = (I_i)_{i=1}^n : T \times R^n \to \mathbb{R}^n$ is a vector-function such that $I(\tau, \cdot\,)$ is continuous for every $\tau \in T_{m_0}$.

We assume that

$$f(t+\omega, x) = -f(t, -x) \ \text{ and } \ I(\tau+\omega, x) = -I(\tau, -x), \ \ t \in \mathbb{R}, \ \ \tau \in T, \ \ x \in \mathbb{R}^n.$$

In view of this condition, if $x : \mathbb{R} \to \mathbb{R}^n$ is a solution of the given system, then the vector-function $y(t) = -x(t+\omega)$ $(t \in \mathbb{R})$ will be a solution of the system, as well. Moreover, it is evident that if $x : \mathbb{R} \to \mathbb{R}^n$ is a solution of the given ω-antiperiodic problem, then its restriction on the closed interval $[0, \omega]$ will be a solution of the problem

$$\frac{dx}{dt} = f(t,x) \ \text{ almost everywhere on } \ [0,\omega] \setminus \{\tau_1, \dots, \tau_{m_0}\}, \tag{1}$$
$$x(\tau_l+) - x(\tau_l-) = I(\tau_l, x(\tau_l)) \ \ (l = 1, \dots, m_0); \tag{2}$$
$$x(0) = -x(\omega). \tag{3}$$

Let now $x : [0, \omega] \to \mathbb{R}^n$ be a solution of the system on $[0, \omega]$. By x we designate the continuation of this function on the whole R as a solution of the system $(1), (2)$. As above, the vector-function $y(t) = -x(t+\omega)$ $(t \in \mathbb{R})$ will be the solution of the system $(1), (2)$. On the other hand, according to the equality (3), we have $y(0) = -x(\omega) = x(0)$. Thus, if we assume that the system $(1), (2)$ under the Cauchy condition $x(0) = c$ is uniquely solvable for every $c \in \mathbb{R}^n$, then $x(t+\omega) = -x(t)$ for $t \in \mathbb{R}$, i.e., x is ω-antiperiodic. This means that the set of restrictions of the ω-antiperiodic solutions of the system $(1),(2)$ on $[0,\omega]$ coincides with the set of solutions of the problem $(1),(2);(3)$.

In this connection we consider the boundary value problem $(1),(2);(3)$ on the closed interval $[0,\omega]$. Below, we will give the sufficient conditions guaranteeing the well-posedness of this problem.

Consider a sequence of vector-functions $f_k \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n)$ $(k=1,2,\dots)$, the sequences of points τ_{lk} $(k=1,2,\dots;\ l=1,\dots,m_0)$, $a<\tau_{1k}<\dots<\tau_{m_0k}<b$, a sequences of operators $I_k:\{\tau_{1k},\dots,\tau_{m_0k}\}\times R^n\to\mathbb{R}^n$ $(k=1,2,\dots)$ such that $I_k(\tau_{lk},\cdot)$ $(k=1,2,\dots;\ l=1,\dots,m_0)$ are continuous.

In this paper the sufficient conditions are established which guarantee both the solvability of the impulsive systems $(k=1,2,\dots)$

$$\frac{dx}{dt}=f_k(t,x)\ \text{ almost everywhere on }\ [0,\omega]\setminus\{\tau_{1k},\dots,\tau_{m_0k}\}, \tag{1_k}$$

$$x(\tau_{lk}+)-x(\tau_{lk}-)=I_k(\tau_{lk},x(\tau_{lk}))\ \ (l=1,\dots,m_0) \tag{2_k}$$

under the condition (3) for any sufficient large k and the convergence of its solutions to a solution of the problem $(1),(2);(3)$ as $k\to+\infty$.

We assume that the circumscribed above concept is fulfilled for the problems $(1_k),(2_k);(3)$ $(k=1,2,\dots)$, as well.

The well-posed problem for the linear boundary value problem for impulsive systems with a finite number of impulses points is investigated in [5], where the necessary and sufficient conditions are given for the case. Analogous problems are investigated in [2,12–14] (see also the references therein) for the linear and nonlinear boundary value problems for ordinary differential systems.

Quite a number of issues on the theory of systems of differential equations with impulsive effect (both linear and nonlinear) have been studied sufficiently well (for a survey of the results on impulsive systems see, e.g., [1,3,4,6–10,15–17] and the references therein). But the above-mentioned works, as we know, do not contain the results obtained in the present paper.

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R}=]-\infty,+\infty[$, $\mathbb{R}_+=[0,+\infty[$, $[a,b]$ $(a,b\in R)$ is a closed segment.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\|=\max\limits_{j=1,\dots,m}\sum\limits_{i=1}^{n}|x_{ij}|$, $|X|=(|x_{ij}|)_{i,j=1}^{n,m}$, $[X]_+=\frac{|X|+X}{2}$.

$\mathbb{R}_+^{n\times m}=\big\{(x_{ij})_{i,j=1}^{n,m}:\ x_{ij}\ge 0\ (i=1,\dots,n;\ j=1,\dots,m)\big\}$.

$\mathbb{R}^{(n\times n)\times m}=\mathbb{R}^{n\times n}\times\dots\times\mathbb{R}^{n\times n}$ (m-times).

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$; $\mathbb{R}_+^n=R_+^{n\times 1}$.

If $X\in\mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; $I_{n\times n}$ is the identity $n\times n$-matrix.

$\bigvee\limits_a^b(X)$ is the total variation of the matrix-function $X:[a,b]\to R^{n\times m}$, i.e., the sum of total variations of the latter components; $V(X)(t)=(v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(a)=0$, $v(x_{ij})(t)=\bigvee\limits_a^t(x_{ij})$ for $a<t\le b$.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X:[a,b]\to R^{n\times m}$ at the point t (we assume $X(t)=X(a)$ for $t\le a$ and $X(t)=X(b)$ for $t\ge b$, if necessary).

$\mathrm{BV}([a,b],R^{n\times m})$ is the set of all matrix-functions of bounded variation $X:[a,b]\to R^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X)<+\infty$).

$C([a,b],D)$, where $D\subset R^{n\times m}$, is the set of all continuous matrix-functions $X:[a,b]\to D$.

Let $T_{m_0}=\{\tau_1,\dots,\tau_{m_0}\}$.

$C([a,b],D;T_{m_0})$, is the set of all matrix-functions $X:[a,b]\to D$ having the one-sided limits $X(\tau_l-)$ $(l=1,\dots,m_0)$ and $X(\tau_l+)$ $(l=1,\dots,m_0)$ whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b]\setminus T_{m_0}$ belong to $C([c,d],D)$.

$C_s([a,b],\mathbb{R}^{n\times m};T_{m_0})$ is the Banach space of all $X\in C([a,b],\mathbb{R}^{n\times m};T_{m_0})$ with the norm $\|X\|_s=\sup\{\|X(t)\|:\ t\in[a,b]\}$.

If $y \in C_s([a,b],\mathbb{R};T_{m_0})$ and $r \in]0,+\infty[$, then

$$U(y;r) = \Big\{x \in C_s([a,b],\mathbb{R}^n;T_{m_0}) : \|x-y\|_s < r\Big\}.$$

$D(y,r)$ is the set of all $x \in \mathbb{R}^n$ such that $\inf\{\|x-y(t)\| : \ t \in [a,b]\} < r$.

$\widetilde{C}([a,b],D)$, where $D \subset R^{n\times m}$, is the set of all absolutely continuous matrix-functions $X : [a,b] \to D$.

$\widetilde{C}([a,b],D;T_{m_0})$ is the set of all matrix-functions $X : [a,b] \to D$ having the one-sided limits $X(\tau_l-)$ $(l=1,\dots,m_0)$ and $X(\tau_l+)$ $(l=1,\dots,m_0)$ whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b]\setminus T_{m_0}$ belong to $\widetilde{C}([c,d],D)$.

If B_1 and B_2 are the normed spaces, then an operator $g : B_1 \to B_2$ (nonlinear, in general) is positive homogeneous if $g(\lambda x) = \lambda g(x)$ for every $\lambda \in R_+$ and $x \in B_1$.

An operator $\varphi : C([a,b],\mathbb{R}^{n\times m};T_{m_0}) \to R^n$ is called nondecreasing if the inequality $\varphi(x)(t) \le \varphi(y)(t)$ for $t \in [a,b]$ holds for every $x,y \in C([a,b],\mathbb{R}^{n\times m};T_{m_0})$ such that $x(t) \le y(t)$ for $t \in [a,b]$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

$L([a,b],D)$, where $D \subset R^{n\times m}$, is the set of all measurable and integrable matrix-functions $X : [a,b] \to D$.

If $D_1 \subset R^n$ and $D_2 \subset R^{n\times m}$, then $Car([a,b]\times D_1,D_2)$ is the Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that for each $i \in \{1,\dots,l\}$, $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$:

(a) the function $f_{kj}(\,\cdot\,,x) : [a,b] \to D_2$ is measurable for every $x \in D_1$;

(b) the function $f_{kj}(t,\,\cdot\,) : D_1 \to D_2$ is continuous for almost every $t \in [a,b]$, and

$$\sup\{|f_{kj}(\,\cdot\,,x)| : \ x \in D_0\} \in L([a,b],R;g_{ik}) \ \text{ for every compact } D_0 \subset D_1.$$

$Car^0([a,b]\times D_1,D_2)$ is the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that the functions $f_{kj}(\,\cdot\,,x(\,\cdot\,))$ $(i=1,\dots,l;\ k=1,\dots,n)$ are measurable for every vector-function $x : [a,b] \to \mathbb{R}^n$ with bounded total variation.

We say that the pair $\{X;\{Y_l\}_{l=1}^m\}$ consisting of the matrix-function $X \in L([a,b],\mathbb{R}^{n\times n})$ and of a sequence of constant $n\times n$ matrices $\{Y_l\}_{l=1}^m\}$ satisfies the Lappo–Danilevskiĭ condition if the matrices $Y_1,\dots,Y_m$ are pairwise permutable and there exists $t_0 \in [a,b]$ such that

$$\int_{t_0}^{t} X(\tau)\,dX(\tau) = \int_{t_0}^{t} dX(\tau)\cdot X(\tau) \ \text{ for } \ t \in [a,b]$$

and

$$X(t)Y_l = Y_lX(t) \ \text{ for } \ t \in [a,b] \ \ (l=1,\dots,m).$$

$M([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ is the set of all functions $\omega \in Car([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ such that the function $\omega(t,\,\cdot\,)$ is nondecreasing and $\omega(t,0)=0$ for every $t \in [a,b]$.

By a solution of the impulsive system (1), (2) we understand a continuous from the left vector-function $x \in \widetilde{C}([0,\omega],\mathbb{R}^n;T_{m_0})$ satisfying both the system (1) for a.e. on $[0,\omega]\setminus T_{m_0}$ and the relation (2) for every $l \in \{1,\dots,m_0\}$.

Definition 1. Let $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}^n$ and $\ell_0 : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}^n_+$ be, respectively, a linear continuous and a positive homogeneous operators. We say that a pair (P,J), consisting of a matrix-function $P \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and a continuous with respect to the last n-variables operator $J : T_{m_0}\times R^n \to \mathbb{R}^n$, satisfies the Opial condition with respect to the pair (ℓ,ℓ_0) if:

(a) there exist a matrix-function $\Phi \in L([0,\omega],\mathbb{R}^{n\times n}_+)$ and a constant matrices $\Psi_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ such that

$$|P(t,x)| \le \Phi(t) \ \text{ a.e. on } \ [0,\omega], \ \ x \in \mathbb{R}^n,$$

and

$$|J(\tau_l,x)| \le \Psi_l \ \text{ for } \ x \in \mathbb{R}^n \ \ (l=1,\dots,m_0);$$

(b)
$$\det(I_{n\times n}+G_l)\neq 0 \ \ (l=1,\dots,m_0) \tag{4}$$
and the problem
$$\frac{dx}{dt}=A(t)\,x \ \text{ a.e. on } \ [0,\omega]\setminus T_{m_0}, \tag{5}$$
$$x(\tau_l+)-x(\tau_l-)=G_l\,x(\tau_l) \ \ (l=1,\dots,m_0); \tag{6}$$
$$|\ell(x)|\leq \ell_0(x) \tag{7}$$
has only a trivial solution for every matrix-function $A\in L([0,\omega],\mathbb{R}^{n\times n})$ and constant matrices $G_l,\dots,G_{m_0}$ for which there exists a sequence $y_k\in \widetilde{C}([0,\omega],\mathbb{R}^n;T_{m_0})$ $(k=1,2,\dots)$ such that
$$\lim_{k\to+\infty}\int\limits_0^t P(\tau,y_k(\tau))\,d\tau=\int\limits_0^t A(\tau)\,d\tau \ \text{ uniformly on } \ [0,\omega]$$
and
$$\lim_{k\to+\infty} J(\tau_l,y_k(\tau_l))=G_l \ \ (l=1,\dots,m_0).$$

Remark 1. In particular, the condition (4) holds if
$$\|\Psi_l\|<1 \ \ (l=1,\dots,m_0).$$

As above, we assume that $f=(f_i)_{i=1}^n\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and, moreover, $f(\tau_l,x)$ is arbitrary for $x\in\mathbb{R}^n$ $(l=1,\dots,m_0)$.

Let x^0 be a solution of the problem $(1),(2);(3)$, and r be a positive number. We introduce the following

Definition 2. A solution x^0 is said to be strongly isolated in the radius r if there exist the matrix- and the vector-functions $P\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and $q\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n)$, a continuous with respect to the last n-variables operators J, $H:T_{m_0}\times R^n\to\mathbb{R}^n$, linear continuous operators ℓ and $\widetilde{\ell}$ and a positive homogeneous operator ℓ_0 acting from $C_s([0,\omega],\mathbb{R}^n;T_{m_0})$ into $\mathbb{R}^n$ such that:

(a) the equalities
$$f(t,x)=P(t,x)\,x+q(t,x) \ \text{ for } \ t\in[0,\omega]\setminus T_{m_0}, \ \ \|x-x^0(t)\|<r,$$
$$I(\tau_l,x)=J(\tau_l,x)\,x+H(\tau_l,x) \ \text{ for } \ \|x-x^0(\tau_l)\|<r \ \ (l=1,\dots,m_0)$$
and
$$x(0)+x(\omega)=\ell(x)+\widetilde{\ell}(x) \ \text{ for } \ x\in U(x^0;r)$$
are valid;

(b) the functions $\alpha(t,\rho)=\max\{\|q(t,x)\|: \ \|x\|\leq\rho\}$, $\beta(\tau_l,\rho)=\max\{\|H(\tau_l,x)\|: \ \|x\|\leq\rho\}$ $(l=1,\dots,m_0)$ and $\gamma(\rho)=\sup\{[|\widetilde{l}(x)|-l_0(x)]_+: \ \|x\|_s\leq\rho\}$ satisfy the condition
$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\gamma(\rho)+\int\limits_0^\omega \alpha(t,\rho)\,dt+\sum_{l=1}^{m_0}\beta(\tau_l,\rho)\bigg)=0; \tag{8}$$

(c) the problem
$$\frac{dx}{dt}=P(t,x)\,x+q(t,x) \ \text{ a.e. on } \ [0,\omega]\setminus T_{m_0},$$
$$x(\tau_l+)-x(\tau_l-)=J(\tau_l,x(\tau_l))\,x(\tau_l)+H(\tau_l,x(\tau_l)) \ \ (l=1,\dots,m_0);$$
$$\ell(x)+\widetilde{\ell}(x)=0$$
has no solution different from x^0.

(d) the pair (P,J) satisfies the Opial condition with respect to the pair (ℓ,ℓ_0).

Remark 2. If $\ell(x)\equiv x(0)+x(\omega)$ and $\ell_0(x)\equiv 0$, then we say that the pair (P,J) satisfies the Opial ω-antiperiodic condition. In this case, the condition (7) coincides with the condition (3), and $\widetilde{\ell}(x)\equiv 0$ and $\gamma(\rho)\equiv 0$ in Definitions 1 and 2.

Definition 3. We say that a sequence (f_k, I_k) $(k = 1, 2, \dots)$ belongs to the set $W_r(f, I; x^0)$ if:

(a) the equalities

$$\lim_{k\to+\infty}\int_0^t f_k(\tau,x)\,d\tau=\int_0^t f(\tau,x)\,d\tau \ \text{ uniformly on } \ [0,\omega]$$

and

$$\lim_{k\to+\infty} I_k(\tau_{lk},x)=I(\tau_l,x) \ \ (l=1,\dots,m_0)$$

are valid for every $x \in D(x^0; r)$;

(b) there exists a sequence of functions $\omega_k \in M([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ $(k = 1, 2, \dots)$ such that

$$\sup\bigg\{\int_0^\omega \omega_k(t,r)\,dt:\ k=1,2,\dots\bigg\}<+\infty, \tag{9}$$

$$\sup\bigg\{\sum_{l=1}^{m_0}\omega_k(\tau_{lk},r):\ k=1,2,\dots\bigg\}<+\infty; \tag{10}$$

$$\lim_{s\to0+}\sup\bigg\{\int_0^\omega \omega_k(t,s)\,dt:\ k=1,2,\dots\bigg\}=0, \tag{11}$$

$$\lim_{s\to0+}\sup\bigg\{\sum_{l=1}^{m_0}\omega_k(\tau_{lk},s):\ k=1,2,\dots\bigg\}=0; \tag{12}$$

$$\big\|f_k(t,x)-f_k(t,y)\big\|\le\omega_k\big(t,\|x-y\|\big) \ \text{ for } \ t\in[0,\omega]\setminus T_{m_0}, \ \ x,y\in D(x^0;r) \ \ (k=1,2,\dots),$$
$$\big\|I_k(\tau_{lk},x)-I_k(\tau_{lk},y)\big\|\le\omega_k\big(\tau_{lk},\|x-y\|\big) \ \text{ for } \ x,y\in D(x^0;r) \ \ (l=1,\dots,m_0;\ \ k=1,2,\dots).$$

Remark 3. If for every natural m there exists a positive number ν_m such that

$$\omega_k(t,m\delta)\le\nu_m\omega_k(t,\delta) \ \text{ for } \ \delta>0, \ \ t\in[0,\omega]\setminus T_{m_0} \ \ (k=1,2,\dots),$$

then the estimate (9) follows from the condition (11); analogously, if

$$\omega_k(\tau_{lk},m\delta)\le\nu_m\omega_k(\tau_{lk},\delta) \ \text{ for } \ \delta>0, \ \ (l=1,\dots,m_0;\ \ k=1,2,\dots),$$

then the estimate (10) follows from the condition (12). In particular, the sequences of functions

$$\omega_k(t,\delta)=\max\Big\{\big\|f_k(t,x)-f_k(t,y)\big\|:\ \ x,y\in U\big(0,\|x^0\|+r\big), \ \ \|x-y\|\le\delta\Big\}$$
$$\text{for } \ t\in[0,\omega]\setminus T_{m_0} \ \ (k=1,2,\dots)$$

and

$$\omega_k(\tau_{lk},\delta)=\max\Big\{\big\|I_k(\tau_{lk},x)-I_k(\tau_{lk},y)\big\|:\ \ x,y\in U\big(0,\|x^0\|+r\big), \ \ \|x-y\|\le\delta\Big\}$$
$$(l=1,\dots,m_0;\ k=1,2,\dots)$$

have the latters' properties, respectively.

Definition 4. The problem $(1),(2);(3)$ is said to be $(x^0; r)$-correct if for every $\varepsilon\in]0,r[$ and $(f_k, I_k)_{k=1}^{+\infty} \in W_r(f, I; x^0)$ there exists a natural number k_0 such that the problem $(1_k),(2_k)$ has at last one ω-antiperiodic solution contained in $U(x^0; r)$, and any such solution belongs to the ball $U(x^0; \varepsilon)$ for every $k \ge k_0$.

Definition 5. The problem $(1),(2);(3)$ is said to be correct if it has a unique solution x^0 and it is $(x^0; r)$-correct for every $r > 0$.

Theorem 1. *If the problem* $(1),(2);(3)$ *has a solution* x^0*, strongly isolated in the radius* r*, then it is* $(x^0; r)$*-correct.*

Theorem 2. *Let the conditions*

$$\big\|f(t,x)-P(t,x)\,x\big\| \le \alpha(t,\|x\|) \ \text{ a.e. on } \ [0,\omega]\setminus T_{m_0}, \ \ x\in\mathbb{R}^n, \tag{13}$$

$$\big\|I(\tau_l,x)-J(\tau_l,x)\,x\big\| \le \beta(\tau_l,\|x\|) \ \text{ for } \ x\in\mathbb{R}^n \ \ (l=1,\dots,m_0) \tag{14}$$

and

$$\big|x(0)+x(\omega)-\ell(x)\big| \le \ell_0(x)+\ell_1(\|x\|_s) \ \text{ for } \ x\in \mathrm{BV}([0,\omega],\mathbb{R}^n) \tag{15}$$

hold, where $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0 : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ are, respectively, a linear continuous and a positive homogeneous operators, the pair (P,J) satisfies the Opial condition with respect to the pair (ℓ,ℓ_0); $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ and $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ are the functions, nondecreasing in the second variable, and $\ell_1\in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\|\ell_1(\rho)\|+\int\limits_0^{\omega}\alpha(t,\rho)\,dt+\sum_{l=1}^{m_0}\beta(\tau_l,\rho)\bigg)=0. \tag{16}$$

Then the problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Theorem 3. *Let the conditions* (13)–(15),

$$P_1(t)\le P(t,x)\le P_2(t) \ \text{ a.e. on } \ [0,\omega]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ x\in\mathbb{R}^n, \tag{17}$$

and

$$J_{1l}\le J(\tau_l,x)\le J_{2l} \ \text{ for } \ x\in\mathbb{R}^n \ \ (l=1,\dots,m_0) \tag{18}$$

hold, where $P\in Car^0([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, $P_i\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_{il}\in\mathbb{R}^{n\times n}$ $(i=1,2;\ l=1,\dots,m_0)$; $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0 : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ are, respectively, a linear continuous and a positive homogeneous operators; $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ and $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ are the functions, nondecreasing in the second variable, and $\ell_1\in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that the condition (16) *holds. Let, moreover, the condition* (4) *hold and the problem* (5), (6), (7) *have only a trivial solution for every matrix-function $A\in L([0,\omega],\mathbb{R}^{n\times n})$ and constant matrices $G_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ such that*

$$P_1(t)\le A(t)\le P_2(t) \ \text{ a.e. on } \ [0,\omega]\setminus T_{m_0}, \ \ x\in\mathbb{R}^n, \tag{19}$$

and

$$J_{1l}\le G_l\le J_{2l} \ \text{ for } \ x\in\mathbb{R}^n \ \ (l=1,\dots,m_0). \tag{20}$$

Then the problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Remark 4. Theorem 3 is of interest only in the case $P\notin Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, because the theorem immediately follows from Theorem 2 in the case $P\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$.

Theorem 4. *Let the conditions* (15),

$$|f(t,x)-P(t)\,x|\le Q(t)\,|x|+q(t,\|x\|) \ \text{ a.e. on } \ [0,\omega]\setminus T_{m_0}, \ \ x\in\mathbb{R}^n, \tag{21}$$

and

$$|I_l(x)-J_l\,x|\le H_l\,|x|+h(\tau_l,\|x\|) \ \text{ for } \ x\in\mathbb{R}^n \ \ (l=1,\dots,m_0) \tag{22}$$

hold, where $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $Q\in L([0,\omega],\mathbb{R}^{n\times n}_+)$, $J_l\in\mathbb{R}^{n\times n}$ and $H_l\in\mathbb{R}^{n\times n}_+$ $(l=1,\dots,m_0)$ are constant matrices, $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0 : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ are, respectively, a linear continuous and a positive homogeneous operators; $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ and $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ are the vector-functions, nondecreasing in the second variable, and $\ell_1\in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that the condition

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\|\ell_1(\rho)\|+\int\limits_0^{\omega}\|q(t,\rho)\|\,dt+\sum_{l=1}^{m_0}\|h(\tau_l,\rho)\|\bigg)=0. \tag{23}$$

holds. Let, moreover, the conditions

$$\det(I_{n\times n}+J_l)\ne 0 \ \ (l=1,\dots,m_0) \tag{24}$$

and

$$\|H_l\| \cdot \big\|(I_{n\times n}+J_l)^{-1}\big\| < 1 \ \ (j=1,2;\ l=1,\dots,m_0) \tag{25}$$

hold and the system of impulsive inequalities

$$\Big|\frac{dx}{dt} - P(t)\,x\Big| \le Q(t)\,x \ \ \text{a.e. on } [0,\omega]\setminus T_{m_0}, \tag{26}$$

$$\big|x(\tau_l+) - x(\tau_l-) - J_l x(\tau_l)\big| \le H_l\,|x(\tau_l)| \ \ (l=1,\dots,m_0) \tag{27}$$

have only a trivial solution satisfying the condition (7)*. Then the problem* (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 1. *Let the conditions*

$$|f(t,x) - P(t)\,x| \le q(t,\|x\|) \ \ \text{a.e. on } [0,\omega]\setminus T_{m_0},\ \ x\in\mathbb{R}^n, \tag{28}$$

$$\big|I(\tau_l,x) - J_l\,x\big| \le h(\tau_l,\|x\|) \ \ \text{for } x\in\mathbb{R}^n \ \ (l=1,\dots,m_0) \tag{29}$$

and

$$\big|x(0)+x(\omega)-\ell(x)\big| \le \ell_1(\|x\|_s) \ \ \text{for } x\in \mathrm{BV}([0,\omega],\mathbb{R}^n) \tag{30}$$

hold, where $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ are constant matrices satisfying the condition (24)*, $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ is the linear continuous operator; $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ and $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ are the vector-functions, nondecreasing in the second variable, and $\ell_1\in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that the condition* (23) *holds. Let, moreover, the problem*

$$\frac{dx}{dt} = P(t)\,x \ \ \text{a.e. on } [0,\omega]\setminus T_{m_0}, \tag{31}$$

$$x(\tau_l+) - x(\tau_l-) = J_l\,x(\tau_l) \ \ (l=1,\dots,m_0); \tag{32}$$

$$\ell(x) = 0. \tag{33}$$

have only a trivial solution. Then the problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Remark 5. Let $Y=(y_1,\dots,y_n)$ be a fundamental matrix, with columns $y_1,\dots,y_n$, of the system (31), (32). Then the homogeneous boundary value problem (31), (32); (33) has only a trivial solution if and only if

$$\det(\ell(Y)) \ne 0, \tag{34}$$

where $\ell(Y) = (\ell(y_1),\dots,\ell(y_n))$.

If the pair $\{P;\{J_l\}_{l=1}^{m_0}\}$ satisfies the Lappo–Danilevskiĭ condition, then the fundamental matrix Y $(Y(0)=I_{n\times n})$ of the homogeneous system (31), (32) has the form

$$Y(t) \equiv \exp\bigg(\int_0^t P(\tau)\,d\tau\bigg)\cdot \prod_{0\le\tau_l<t}(I_{n\times n}+J_l).$$

Theorem 5. *Let the conditions*

$$\big|f(t,x) - f(t,y) - P(t)\,(x-y)\big| \le Q(t)|x-y| \ \ \text{a.e. on } [0,\omega]\setminus T_{m_0},\ \ x,y\in\mathbb{R}^n, \tag{35}$$

$$\big|I(\tau_l,x) - I(\tau_l,y) - J_l\,(x-y)\big| \le H_l|x-y| \ \ \text{for } x,y\in\mathbb{R}^n \ \ (k=l,\dots,m_0) \tag{36}$$

and

$$\big|x(0)-y(\omega)+x(\omega)-y(\omega)-\ell(x-y)\big| \le \ell_0(x-y) \ \ \text{for } x,y\in \mathrm{BV}([0,\omega],\mathbb{R}^n)$$

hold, where $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $Q\in L([0,\omega],\mathbb{R}^{n\times n}_+)$, $J_l\in\mathbb{R}^{n\times n}$ and $H_l\in\mathbb{R}^{n\times n}_+$ $(l=1,\dots,m_0)$ are constant matrices satisfying the conditions (24) *and* (25)*, $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0 : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ are, respectively, linear continuous and positive homogeneous continuous operators. Let, moreover, the problem* (26), (27); (7) *have only a trivial solution. Then the problem* (1), (2); (3) *is correct.*

Corollary 2. *Let there exist a solution x^0 of the problem* (1), (2); (3) *and a positive number $r > 0$ such that the conditions*

$$\big|f(t,x) - f(t,x^0(t)) - P(t)\,(x - x^0(t))\big| \le Q(t)\big|x - x^0(t)\big| \ \ a.a.\ [0,\omega]\setminus T_{m_0}, \ \ \|x - x^0(t)\| < r,$$

$$\Big|I\big(\tau_l, x\big) - I\big(\tau_l, x^0(\tau_l)\big) - J_l\,(x - x^0(\tau_l))\Big| \le H_l\big|x - x^0(\tau_l)\big| \ \ for\ \ \|x - x^0(\tau_l)\| < r \ \ (l = l,\dots,m_0)$$

and

$$\big|x(0) - x^0(0) + x(\omega) - x^0(\omega) - \ell(x - x^0)\big| \le \ell^*\big(|x - x^0|\big) \ \ for\ \ x \in U(x^0, r)$$

hold, where $P \in L([0,\omega],\mathbb{R}^{n\times n})$, $Q \in L([0,\omega],\mathbb{R}_+^{n\times n})$, $J_l, H_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ are constant matrices satisfying the conditions (24) *and* (25), *$\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}^n$ and $\ell^* : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}_+^n$ are, respectively, linear continuous and positive homogeneous continuous operators. Let, moreover, the system of impulsive inequalities*

$$\Big|\frac{dx}{dt} - P(t)\,x\Big| \le Q(t)\,x \ \ a.e.\ on\ \ [0,\omega]\setminus T_{m_0},$$

$$\big|x(\tau_l+) - x(\tau_l-) - J_l\,x(\tau_l)\big| \le H_l \cdot x(\tau_l) \ \ (l = 1,\dots,m_0)$$

have only a trivial solution under the condition

$$|\ell(x)| \le \ell^*(|x|).$$

Then the problem (1), (2); (3) *is $(x^0; r)$-correct.*

Corollary 3. *Let the components of the vector-functions f and I_l $(l = 1,\dots,n)$ have partial derivatives by the last n variables belonging to the Carathéodory class $Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n)$. Let, moreover, x^0 be a solution of the problem* (1), (2); (3) *such that the condition*

$$\det\big(I_{n\times n} + G_l(x^0(\tau_l))\big) \ne 0 \ \ (l = 1,\dots,m_0)$$

holds and the system

$$\frac{dx}{dt} = F(t, x^0(t))\,x \ \ almost\ everywhere\ on\ \ [0,\omega]\setminus T_{m_0},$$

$$x(\tau_l+) - x(\tau_l-) = G_l(x^0(\tau_l))\,x(\tau_l) \ \ (l = 1,\dots,m_0);$$

$$\ell(x) = 0,$$

where $F(t,x) \equiv \frac{\partial f(t,x)}{\partial x}$ and $G_l(x) \equiv \frac{\partial I_l(x)}{\partial x}$, have only a trivial solution under the condition (3). *Then the problem* (1), (2); (3) *is $(x^0; r)$-correct for any sufficiently small r.*

In general, it is quite difficult to verify the condition (34) directly even in the case where one is able to write out the fundamental matrix of the system (31), (32); (33). Therefore it is important to seek for effective conditions which would guarantee the absence of nontrivial ω-antiperiodic solutions of the homogeneous system (31), (32); (33). Below we will give the results concerning the question under consideration. Analogous results have been obtained in [3] for general linear boundary value problems for impulsive systems, and in [14] by T. Kiguradze for the case of ordinary differential equations.

In this connection, we introduce the following operators. For every matrix-function $X \in L([0,\omega],\mathbb{R}^{n\times n})$ and a sequence of constant matrices $Y_k \in \mathbb{R}^{n\times n}$ $(k = 1,\dots,m_0)$ we put

$$\big[(X, Y_1,\dots,Y_{m_0})(t)\big]_0 = I_n \ \ \text{for}\ \ 0 \le t \le \omega,$$

$$\big[(X, Y_1,\dots,Y_{m_0})(0)\big]_i = O_{n\times n} \ \ (i = 1,2,\dots),$$

$$\big[(X, Y_1,\dots,Y_{m_0})(t)\big]_{i+1} = \int\limits_0^t X(\tau)\,\big[(X, Y_1,\dots,Y_{m_0})(\tau)\big]_i\,d\tau + \sum_{0\le\tau_l<t} Y_l\,\big[(X, Y_1,\dots,Y_{m_0})(\tau_l)\big]_i \ \ \text{for}\ \ 0 < t \le \omega \ \ (i = 1,2,\dots). \qquad (37)$$

Corollary 4. *Let the conditions* (28)–(30) *hold, where*

$$\ell(x) \equiv \int\limits_0^\omega d\mathcal{L}(t)\cdot x(t),$$

$P \in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ are constant matrices satisfying the condition (24), *$\mathcal{L} \in L([0,\omega],\mathbb{R}^{n\times n})$; $q \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ and $h \in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ are the vector-functions, nondecreasing in the second variable, and $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that the condition* (23) *holds. Let, moreover, there exist natural numbers k and m such that the matrix*

$$M_k = -\sum_{i=0}^{k-1}\int_0^\omega d\mathcal{L}(t)\cdot\big[(P,J_l,\dots,J_{m_0})(t)\big]_i$$

is nonsingular and

$$r(M_{k,m}) < 1, \tag{38}$$

where the operators $[(P,J_1,\dots,J_{m_0})(t)]_i$ $(i = 0,1,\dots)$ are defined by (37), *and*

$$M_{k,m} = \big[\big(|P|,|J_1|,\dots,|J_{m_0}|\big)(\omega)\big]_m + \sum_{i=0}^{m-1}\big[\big(|P|,|J_1|,\dots,|J_{m_0}|\big)(\omega)\big]_i\int_0^\omega dV(M_k^{-1}\mathcal{L})(t)\cdot\big[\big(|P|,|J_1|,\dots,|J_{m_0}|\big)(t)\big]_k.$$

Then the problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 5. *Let the conditions* (28)–(30) *hold, where*

$$\ell(x) \equiv \sum_{j=1}^{n_0}\mathcal{L}_j x(t_j), \tag{39}$$

$P \in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ are constant matrices satisfying the condition (24), *$t_j \in [0,\omega]$ and $\mathcal{L}_j \in \mathbb{R}^{n\times n}$ $(j = 1,\dots,n_0)$, $\mathcal{L} \in L([0,\omega],\mathbb{R}^{n\times n})$, $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}^n$ is the linear continuous operator; $q \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ and $h \in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ are the vector-functions, nondecreasing in the second variable, and $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that the condition* (23) *holds. Let, moreover, there exist natural numbers k and m such that the matrix*

$$M_k = \sum_{j=1}^{n_0}\sum_{i=0}^{k-1}\mathcal{L}_j\big[(P,J_l,\dots,J_{m_0})(t_j)\big]_i$$

is nonsingular and the inequality (38) *holds, where*

$$M_{k,m} = \big[\big(|P|,|J_l|,\dots,|J_{m_0}|\big)(\omega)\big]_m + \Big(\sum_{i=0}^{m-1}\big[\big(|P|,|J_l|,\dots,|J_{m_0}|\big)(\omega)\big]_i\Big)\sum_{j=1}^{n_0}|M_k^{-1}\mathcal{L}_j|\cdot\big[\big(|P|,|J_l|,\dots,|J_{m_0}|\big)(t_j)\big]_k.$$

Then the problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 5 has the following form for $k = 1$ and $m = 1$.

Corollary 6. *Let the conditions* (28)–(30) *hold, where the operator ℓ is defined by* (39), *$P \in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ are constant matrices satisfying the condition* (24), *$t_j \in [0,\omega]$ and $\mathcal{L}_j \in \mathbb{R}^{n\times n}$ $(j = 1,\dots,n_0)$; $q \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ and $h \in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ are the vector-functions, nondecreasing in the second variable, and $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ is the vector-function such that the condition* (23) *holds. Let, moreover,*

$$\det\Big(\sum_{j=1}^{n_0}\mathcal{L}_j\Big) \neq 0 \ \text{ and } \ r(\mathcal{L}_0 A_0) < 1,$$

where

$$\mathcal{L}_0 = I_{n\times n} + \bigg|\Big(\sum_{j=1}^{n_0}\mathcal{L}_j\Big)^{-1}\bigg|\cdot\sum_{j=1}^{n_0}|\mathcal{L}_j| \ \text{ and } \ A_0 = \int_0^\omega |P(t)|\,dt + \sum_{l=1}^{m_0}|J_l|.$$

Then the problem $(1), (2); (3)$ *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Remark 6. If the pair $\{P; \{J_l\}_{l=1}^{m_0}\}$ satisfies the Lappo–Danilevskiĭ condition, then the condition (34) has the forms

$$\det\bigg(\int_0^\omega d\mathcal{L}(t)\cdot\exp\bigg(\int_0^t P(\tau)\,d\tau\bigg)\cdot\prod_{0\le\tau_l<t}(I_{n\times n}+J_l)\bigg)\neq 0$$

and

$$\det\bigg(\sum_{j=1}^{n_0} L_j\exp\bigg(\int_0^{t_j} P(\tau)\,d\tau\bigg)\cdot\prod_{0\le\tau_l<t_j}(I_{n\times n}+J_l)\bigg)\neq 0$$

for the operators ℓ defined, respectively, in Corollary 4 and Corollary 5.

By Remark 2, in the case where $\ell(x)\equiv x(0)+x(\omega)$ and $\ell_0(x)\equiv 0$, the results given above have the following forms, respectively.

Theorem 2′. *Let the conditions* (13) *and* (14) *hold, where the pair* (P, J) *satisfies the Opial* ω*-antiperiodic condition,* $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ *and* $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ *are the functions, nondecreasing in the second variable, such that*

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\int_0^\omega \alpha(t,\rho)\,dt+\sum_{l=1}^{m_0}\beta(\tau_l,\rho)\bigg)=0. \tag{40}$$

Then the problem $(1), (2); (3)$ *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Theorem 3′. *Let the conditions* (13), (14), (17), (18) *and* (40) *hold, where* $P\in Car^0([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, $P_i\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_{il}\in\mathbb{R}^{n\times n}$ $(i=1,2;\ l=1,\dots,m_0)$; $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ *and* $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ *are the functions, nondecreasing in the second variable. Let, moreover, the condition* (4) *hold and the problem* $(5), (6); (3)$ *have only a trivial solution for every matrix-function* $A\in L([0,\omega],\mathbb{R}^{n\times n})$ *and constant matrices* $G_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ *satisfying the conditions* (19) *and* (20). *Then the problem* $(1), (2); (3)$ *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Theorem 4′. *Let the conditions* (21) *and* (22) *hold, where* $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $Q\in L([0,\omega],\mathbb{R}_+^{n\times n})$, $J_l\in\mathbb{R}^{n\times}$ *and* $H_l\in\mathbb{R}_+^{n\times n}$ $(l=1,\dots,m_0)$ *are the constant matrices satisfying the conditions* (24) *and* (25), $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n)$, *and* $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}_+^{n\times n})$ *are the vector-functions, nondecreasing in the second variable, such that*

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\int_0^\omega \|q(t,\rho)\|\,dt+\sum_{l=1}^{m_0}\|h(\tau_l,\rho)\|\bigg)=0. \tag{41}$$

Let, moreover, the system of impulsive inequalities $(26), (27)$ *have only a trivial solution satisfying the condition* (3). *Then the problem* $(1), (2); (3)$ *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 1′. *Let the conditions* (28), (29) *and* (40) *hold, where* $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ *are constant matrices satisfying the condition* (24), $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n)$ *and* $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}_+^{n\times n})$ *are the vector-functions, nondecreasing in the second variable. Let, moreover, the problem* $(31), (32), (3)$ *have only a trivial solution. Then the problem* $(1), (2); (3)$ *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Theorem 5′. *Let the conditions* (35) *and* (36) *hold, where* $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $Q\in L([0,\omega],\mathbb{R}_+^{n\times n})$, $J_l\in\mathbb{R}^{n\times n}$ *and* $H_l\in\mathbb{R}_+^{n\times n}$ $(l=1,\dots,m_0)$ *are constant matrices satisfying the conditions* (24) *and* (25). *Let, moreover, the problem* $(26), (27); (7)$ *have only a trivial solution. Then the problem* $(1), (2); (3)$ *is correct.*

Corollary 5′. *Let the conditions* (28), (29) *and* (41) *hold, where* $P \in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are constant matrices satisfying the condition* (24)*;* $q \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ *and* $h \in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ *are the vector-functions, nondecreasing in the second variable. Let, moreover, there exist natural numbers* k *and* m *such that the matrix*

$$M_k = \sum_{i=0}^{k-1} \big[(P,J_l,\dots,J_{m_0})(\omega)\big]_i$$

is nonsingular and the inequality (38) *holds, where*

$$M_{k,m} = \big[\big(|P|,|J_l|,\dots,|J_{m_0}|\big)(\omega)\big]_m + \Big(\sum_{i=0}^{m-1}\big[\big(|P|,|J_l|,\dots,|J_{m_0}|\big)(\omega)\big]_i\Big)|M_k^{-1}|\cdot\big[\big(|P|,|J_l|,\dots,|J_{m_0}|\big)(\omega)\big]_k.$$

Then the problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 5′ has the following form for $k = 1$ and $m = 1$.

Corollary 6′. *Let the conditions* (28), (29) *and* (41) *hold, where* $P \in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are constant matrices satisfying the condition* (24)*;* $q \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ *and* $h \in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ *are the vector-functions, nondecreasing in the second variable. Let, moreover,*

$$r(A_0) < \frac{1}{2}\,,$$

where

$$A_0 = \int_0^\omega |P(t)|\,dt + \sum_{l=1}^{m_0}|J_l|.$$

Then the problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Remark 7. In the conditions of Corollary 6′, if the pair $\{P;\{J_l\}_{l=1}^{m_0}\}$ satisfies the Lappo–Danilevskiĭ condition, then the condition (34) has the form

$$\det\bigg(I_{n\times n} + \exp\bigg(\int_0^\omega P(\tau)\,d\tau\bigg)\cdot\prod_{l=1}^{m_0}(I_{n\times n}+J_l)\bigg) \neq 0.$$

The analogous questions have been investigated in [7, 8] for the system (1), (2) under the general nonlinear boundary condition $h(x) = 0$, where $h : C([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}^n$ is a continuous vector-functional which is nonlinear, in general. The results given in the paper are the particular cases of the results obtained in [7, 8] when $h(x) \equiv x(0) + x(\omega)$.

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ON THE SOLVABILITY OF A BOUNDARY VALUE PROBLEM WITH DIRICHLET AND POINCARE CONDITIONS IN THE ANGULAR DOMAIN FOR ONE CLASS OF NONLINEAR SECOND ORDER HYPERBOLIC SYSTEMS

Abstract. Darboux type problem with Dirichlet and Poincare boundary conditions for one class of nonlinear second order hyperbolic systems is considered. The questions of existence and nonexistence, uniqueness and smoothness of global solution of this problem are investigated.

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In the plane of the variables x and t we consider a nonlinear second order hyperbolic system of type

$$Lu: \ u_{tt} - u_{xx} + A(x,t)u_x + B(x,t)u_t + C(x,t)u + f(x,t,u) = F(x,t), \tag{1}$$

where A, B, C are given real $n\times n$-matrices, $f = (f_1,\dots,f_n)$ is a given nonlinear with respect to u real vector-function, $F = (F_1,\dots,F_n)$ is a given and $u = (u_1,\dots,u_n)$ is an unknown real vector-function, $n \geq 2$.

By D_T we denote a triangular domain lying inside the characteristic angle $\{(x,t)\in R^2: \ t > |x|\}$ and bounded by the characteristic segment $\gamma_{1,t}: \ x = t, \ 0 \leq t \leq T$, and segments $\gamma_{2,t}: \ x = 0, \ 0 \leq t \leq T$, $\gamma_{3,t}: \ t = T, \ 0 \leq x \leq T$, of time and spatial type, respectively.

For the system (1), we consider a boundary value problem: find in the domain D_T a solution $u = u(x,t)$ of that system, satisfying on segments $\gamma_{1,T}$ and $\gamma_{2,T}$ the Dirichlet and Poincare conditions, respectively,

$$u\big|_{\gamma_{1,T}} = \varphi, \tag{2}$$

$$(\mu_1 v_x + \mu_2 v_t)\big|_{\gamma_{2,T}} = 0, \tag{3}$$

where $\varphi = (\varphi_1,\dots,\varphi_n)$ is a given real vector-function and μ_i, $i = 1,2$, are given real $n\times n$-matrices. In the case of $T = \infty$ we have $D_\infty := t > |x|$, $x > 0$, and $\gamma_{1,\infty}: \ x = t, \ 0 \leq t \leq \infty$, $\gamma_{2,\infty}: \ x = 0, \ 0 \leq t \leq \infty$.

Definition 1. Let $A, B, C, F, f \in C(\overline{D}_T \times \mathbb{R}^n)$ and $\varphi \in C^1(\varphi_{1,T})$, $\mu_i \in C(\gamma_{2,T})$, $i = 1,2$. We call a vector-function u a generalized solution of the problem (1), (2), (3) of the class C in the domain D_T if $u \in C(\overline{D}_T)$ and there exists a sequence of vector-functions

$$u^m \in C^2_0(\overline{D}_T) := \Big\{ v \in C^2(\overline{D}_T): \ (\mu_1 v_x + \mu_2 v_t)\big|_{\gamma_{2,T}} = 0 \Big\}$$

such that $u^m \to u$ and $Lu^m \to F$ in the space $C(\overline{D}_T)$, $u^m|_{\gamma_{1,T}} \to \varphi$ in the space $C^1(\gamma_{1,T})$, as $m \to \infty$.

It is obvious that a classical solution $u \in C^2(\overline{D}_T)$ of the problem (1), (2), (3) represents a generalized solution of this problem of the class C in the domain D_T in the sense of Definition 1.

Definition 2. Let $A, B, C, F, f \in C(\overline{D}_\infty \times \mathbb{R}^n)$ and $\varphi \in C^1(\gamma_{1,\infty})$, $\mu_i \in C(\gamma_{2,\infty})$, $i = 1,2$. We say that the problem (1), (2), (3) is locally solvable in the class C if there exists a number $T_0 = T_0(F,\varphi) > 0$ such that for $T < T_0$ this problem has a generalized solution of the class C in the domain D_T in the sense of the Definition 1.

Definition 3. Let $A, B, C, F, f \in C(\overline{D}_\infty \times \mathbb{R}^n)$ and $\varphi \in C^1(\gamma_{1,\infty})$, $\mu_i \in C(\gamma_{2,\infty})$, $i = 1, 2$. We say that the problem $(1), (2), (3)$ is globally solvable in the class C if for any $T > 0$ this problem has a generalized solution of the class C in the domain D_T in the sense of Definition 1.

Definition 4. Let $A, B, C, F, f \in C(\overline{D}_\infty \times \mathbb{R}^n)$ and $\varphi \in C^1(\gamma_{1,\infty})$, $\mu_i \in C(\gamma_{2,\infty})$, $i = 1, 2$. A vector-function $u \in C(\overline{D}_\infty)$ is called a global generalized solution of the problem $(1), (2), (3)$ of the class C in the domain D_∞ if for any $T > 0$ the vector-function $u|_{D_T}$ is a generalized solution of the class C in the domain D_T in the sense of Definition 1.

If in the linear case for scalar hyperbolic equations the boundary value problems of Goursat and Darboux type are well studied (see [5–7, 9, 12, 16]), there arise additional difficulties and new effects in passing to hyperbolic systems. This has been first noticed by A. V. Bitsadze [3] who constructed examples of second order hyperbolic systems for which the corresponding homogeneous characteristic problem has a finite number, and in some cases, an infinite of number of linearly independent solutions. Later these problems for linear second order hyperbolic systems have become a subject of study in the works [10, 11]. In this direction it should also be noted the work [4], in which on the simple examples it is revealed the effect of lowest terms on the correctness of these problems. As shown in [1, 2, 13–15], the presence of the nonlinear term in the scalar hyperbolic equation may affect on the correctness of the Darboux problem, when in some cases this problem is globally solvable, and in other cases may arise the so-called blow up solutions. It should be noted that the above-mentioned works do not contain linear terms involving the first order derivatives, since their presence causes difficulties in investigating the problem, and not only of technical character. In this paper, we study the Darboux type problem for nonlinear system (1) with lowest terms of the first order. The results presented here are new in the case when (1) is a scalar hyperbolic equation.

Local solvability of the problem $(1), (2), (3)$ in sense of Definition 2 holds under the additional requirements

$$\det(\mu_2 - \mu_1)\big|_{\gamma_{2,\infty}} \neq 0 \tag{4}$$

and

$$A, B \in C^2(\overline{D_\infty}), \;\; C \in C^1(\overline{D_\infty}), \;\; f \in C^1(\overline{D}_\infty \times \mathbb{R}^n), \;\; \mu_i \in C^1(\gamma_{2,\infty}). \tag{5}$$

Under the conditions given in the Definition 2, if we additionally require that

$$\|f_i(x,t,u)\| \leq M_1 + M_2\|u\|, \;\; (x,t,u) \in \overline{D}_\infty \times \mathbb{R}^n, \;\; i = 1, \dots, n, \tag{6}$$

and

$$\det \mu_1\big|_{\gamma_{2,T}} \neq 0, \;\; (\mu_1^{-1}\mu_2\theta, \theta)\big|_{\gamma_{2,T}} \leq 0 \;\; \forall\, \theta \in \mathbb{R}^n, \tag{7}$$

where $M_j = M_j(T) = const \geq 0$, $j = 1, 2$, $\forall\, T > 0$; $\|u\| = \sum\limits_{i=1}^{n} |u_i|$, $(\,\cdot\,,\,\cdot\,)$ is scalar product in the Euclidean space $\mathbb{R}^n$, then for a generalized solution of the problem $(1), (2), (3)$ of the class C in the domain D_T the a priori estimate

$$\|u\|_{C(\overline{D_T})} \leq c_1\|F\|_{C(\overline{D_T})} + c_2\|\varphi\|_{C^1(\gamma_{1,T})} + c_3, \tag{8}$$

is valid with nonnegative constants $c_i = c_i(M_0, M_1, M_2, T)$, $i = 1, 2, 3$, not depending on u, F, φ and where $c_i > 0$, $i = 1, 2$. Here $M_0 = M_0(A, B, C) = const \geq 0$.

Under the conditions (4)–(7), from the a priori estimate (8) by virtue of Learay–Schauder's theorem there follows the global solvability of the problem $(1), (2), (3)$ in the class C in the sense of Definition 3.

Note also that in the above assumptions (4)–(7) there exists a unique global generalized solution of the problem $(1), (2), (3)$ of the class C in the domain D_∞ in the sense of Definition 4.

Now consider the case when the condition (5) is violated, i.e.,

$$\overline{\lim_{\|u\| \to \infty}} \frac{\|f(x,t,u)\|}{\|u\|} = \infty,$$

and the problem $(1), (2), (3)$ is not globally solvable, in particular, it does not have a global generalized solution of the class C in the domain D_∞ in the sense of Definition 4.

Theorem. *Let* $A = B = C = 0$, $f = f(u) \in C(\mathbb{R}^n)$, $F \in C(\overline{D}_\infty)$, $\varphi = 0$. *There exists numbers* $l_1, \dots, l_n$, $\sum\limits_{i=1}^n |l_i| \neq 0$ *such that*

$$\sum_{i=1}^n l_i f_i(u) \le c_0 - c_1 \Big| \sum_{i=1}^n l_i u_i \Big|^\beta, \quad \beta = const > 1, \tag{9}$$

where $c_0, c_1 = const$, $c_1 > 0$. *Let the function* $F_0 = \sum\limits_{i=1}^n l_i F_i - c_0$ *satisfies the following conditions:*

$$F_0 \ge 0, \quad F(x,t)\big|_{t \ge 1} \ge c_2 t^{-k}; \quad c_2 = const > 0, \quad 0 \le k = const \le 2.$$

Then there exists a finite positive number $T_0 = T_0(F)$ *such that for* $T > T_0$ *the problem* (1), (2), (3) *does not have a generalized solution of the class* C *in the domain* D_T.

Corollary. *Under the conditions of the theorem, although the problem is locally solvable, it does not have a global generalized solution of the class* C *in the domain* D_∞.

Now let us consider one class of vector-functions f satisfying the condition (9):

$$f_i(u_1, \dots, u_n) = \sum_{j=1}^n a_{ij} |u_j|^{\beta_{ij}} + b_i, \quad i = 1, \dots, n, \tag{10}$$

where $a_{ij} = const > 0$, $b_i = const$, $\beta_{ij} = const > 1$; $i, j = 1, \dots, n$. In this case we can take: $l_1 = l_2 = \dots = l_n = -1$. Indeed, let us choose $\beta = const$ such that $1 < \beta < \beta_{ij}$; $i, j = 1, \dots, n$. It is easy to verify that $|s|^{\beta_{ij}} \ge |s|^\beta - 1 \ \forall \, s \in (-\infty, \infty)$. Now, using well - known inequality [8]

$$\sum_{i=1}^n |y_i|^\beta \ge n^{1-\beta} \Big| \sum_{i=1}^n y_i \Big|^\beta \quad \forall \, y = (y_1, \dots, y_n) \in \mathbb{R}^n, \quad \beta = const > 1,$$

we receive

$$\sum_{i=1}^n f_i(u_1, \dots, u_n) \ge a_0 \sum_{i,j=1}^n |u_j|^{\beta_{ij}} + \sum_{i=1}^n b_i \ge a_0 \sum_{i,j=1}^n \big(|u_j|^\beta - 1\big) + \sum_{i=1}^n b_i$$
$$= a_0 n \sum_{j=1}^n |u_j|^\beta - a_0 n^2 + \sum_{i=1}^n b_i \ge a_0 n^{2-\beta} \Big| \sum_{j=1}^n u_j \Big|^\beta + \sum_{i=1}^n b_i - a_0 n^2, \quad a_0 = \min_{i,j} a_{ij} > 0.$$

Hence we have the inequality (9) in which: $l_1 = l_2 = \dots = l_n = -1$, $c_0 = a_0 n^2 - \sum\limits_{i=1}^n b_i$, $c_1 = a_0 n^{2-\beta} > 0$. Note that the vector-function f, represented by the equalities (10), also satisfies the condition (9) with $l_1 = l_2 = \dots = l_n = -1$ for less restrictive conditions when $a_{ij} = const \ge 0$, but $a_{ik_i} > 0$, where $k_1, \dots, k_n$ is any fixed permutation of numbers $1, 2, \dots, n$.

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