Revaz Gamkrelidze – 90

Revaz Gamkrelidze, the well-known Georgian mathematician, Academician of the Georgian National Academy of Sciences and the Russian Academy of Sciences, was born on February 4, 1927 in Kutaisi (Georgia). His father, Valerian Gamkrelidze, was one of the noted figures of Publishing house in Georgia. Upon graduation from the secondary school, R. Gamkrelidze continued his education at the Mechanics and Mathematics department of Tbilisi State University, where he won reputation of a talented and successful student. All the aforesaid gives every reason to send R. Gamkrelidze to Lomonosov Moscow State University for further studies. Here, for the first time, he met one of the leading figures in mathematical thought, the great Russian mathematician, Academician Lev Pontryagin, who has left an ineffaceable mark on further creation of scientific interests and demonstration of unique talent in young R. Gamkrelidze.

The first scientific result obtained by Revaz Gamkrelidze deals with Chern's cycles of complex algebraic manifolds, and his next scientific activity refers to the nonclassical variational calculus, i.e., the optimal control theory. R. Gamkrelidze, together with L. Pontryagin, V. Boltyanskii and E. Mishchenko, laid the foundation of the Mathematical Theory of Optimal Processes. He was the first who has proved the maximum principle, the necessary optimality condition of control for the linear time-optimal control problem. He studied the question of the existence of optimal control, introduced the notion of a general state of the system and showed that the maximum principle for such systems is likewise a sufficient optimality condition. Notable is the fact that the maximum principle, as a hypothesis, has been stated by Pontryagin, and in scientific literature is known as Pontryagin's maximum principle. R. Gamkrelidze's fundamental researches dealing with the optimal problems with bounded phase coordinates belong to the same period. Scientific achievements obtained by Pontryagin's school were published in 1961 as a monograph under the title "The mathematical theory of optimal processes" (Russian): L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko, *Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow*, 1961. The work was awarded with the State Lenin Prize in 1962. It

has been translated into many languages and made a great contribution to the development of the theory of optimal control in many countries. Later on, R. Gamkrelidze discovered and studied sliding optimal modes.

Investigations of these objects took him to the concepts of a quasi-convex set and a quasi-convex filter in vector topological spaces, which laid the foundation for creation of a general theory of extremal problems. In this theory, R. Gamkrelidze was the first who has proved maximum principles in the integral form. Besides, he has studied quasi-linear differential games and identified an escape strategy for them. Later, he has constructed exponential representation of flows and the chronological calculus. Fundamental results in geometrical theory of control also belong to him.

R. Gamkrelidze is occupied with profound scientific-research and pedagogical activities in different scientific centers of the world (USA, Germany, France, England, etc.). He takes part in many International scientific congresses and conferences. He is a member of editorial boards of many scientific journals. Among R. Gamkrelidze's pupils are many famous scientists all over the world.

Over a long period of time, R. Gamkrelidze headed the Department of Theory of Control at Tbilisi State University (TSU) and the Department of Differential Equations at Steklov Institute of Mathematics. R. Gamkrelidze, together with his pupil G. Kharatishvili, has made a great contribution to the development of the optimal control theory in Georgia, he is deemed a successor of A. Razmadze, the first Georgian mathematician. A course of lectures delivered by R. Gamkrelidze in TSU laid the foundation for writing the monograph, which was translated into Russian and published by Publishing house of Tbilisi State University. For his book, R. Gamkrelidze was endowed with A. Razmadze's Prize of the Georgian National Academy of Sciences in 1975. Later on, the monograph has been translated into English and published in the USA: R. V. Gamkrelidze, "Principles of Optimal Control Theory", Mathematical Concepts and Methods in Science and Engineering, Vol. 7. *Plenum Press, New York–London*, 1978.

Along with the fertile scientific-pedagogical activities, R. Gamkrelidze is engaged with editorial activity. He is editor-in-chief of many scientific publications, including abstract journal *"Matematika"*, serial publications: *"Contemporary Mathematics and Its Applications"*, *"Contemporary Mathematics, Fundamental Directions"*, *"Fundamental and Applied Mathematics"*.

The honored and famous scientist Revaz Gamkrelidze meets his ninetieth anniversary with inexhaustible endeavor and achievements, with unshakable faith in the future. Wish him a long life and creative success.

Tamaz Tadumadze

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Tamaz Tadumadze

VARIATION FORMULAS OF SOLUTIONS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH SEVERAL CONSTANT DELAYS AND THEIR APPLICATIONS IN OPTIMAL CONTROL PROBLEMS

Dedicated to the 90th birthday anniversary of academician Revaz Gamkrelidze

Abstract. For nonlinear functional differential equations with several constant delays, the theorems on the continuous dependence of solutions of the Cauchy problem on perturbations of the initial data and on the right-hand side of the equation are proved. Under the initial data we mean the collection of the initial moment, constant delays, initial vector and initial function. Perturbations of the initial data and of the right-hand side of the equation are small in a standard norm and in an integral sense, respectively. Variation formulas of a solution are derived for equations with a discontinuous initial and continuous initial conditions. In the variation formulas, the effects of perturbations of the initial moment and delays as well as the effects of continuous initial and discontinuous initial conditions are revealed. For the optimal control problems with delays, general boundary conditions and functional, the necessary conditions of optimality are obtained in the form of equality or inequality for the initial and final moments, for delays and an initial vector and also in the form of the integral maximum principle for the initial function and control.

რეზიუმე. არაწრფივი ფუნქციონალურ-დიფერენციალური განტოლებებისათვის მუდმივი დაგვი-ანებებით დამტკიცებულია კოშის ამოცანის ამონახსნების უწყვეტად დამოკიდებულების თეორე-მები საწყისი მონაცემებისა და განტოლების მარჯვენა მხარის შეშფოთების მიმართ. საწყისი მონაცემების ქვეშ იგულისხმება საწყისი მომენტის, მუდმივი დაგვიანებების, საწყისი ვექტორისა და ფუნქციის ერთობლიობა. საწყისი მონაცემებისა და განტოლების მარჯვენა მხარის შეშფო-თებები, შესაბამისად, მცირეა სტანდარტული ნორმით და ინტეგრალური აზით. გამოყვანილია ამონახსნის ვარიაციის ფორმულები განტოლებებისათვის, როგორც წყვეტილი, ასევე უწყვეტი საწყისი პირობით. ვარიაციის ფორმულებში გამოვლენილია საწყისი მომენტისა და დაგვიანებე-ბის შეშფოთების ეფექტები, წყვეტილი და უწყვეტი საწყისი პირობების ეფექტები. ოპტიმალური მართვის ამოცანებისთვის დაგვიანებებით, ზოგადი სასაზღვრო პირობებით და ფუნქციონალით მიღებულია ოპტიმალურობის აუცილებელი პირობები: უტოლობებისა და ტოლობების სახით საწყისი და საბოლოო მომენტებისათვის, დაგვიანებებისა და საწყისი ვექტორისათვის; ინტეგრა-ლური მაქსიმუმის პრინციპის ფორმით საწყისი და მართვის ფუნქციებისთვის.

Introduction

As is known, real economical, biological, physical and majority of processes contain an information about their behavior in the past, i.e., the processes that contain effects with delayed action and which are described by functional differential equations with delays. To illustrate this, below we will consider two simplest models of the economic growth and the immune response with several constant delays.

The economic growth model. Let $N(t)$ be a quantity of a product produced at the moment t expressed in money units. The fundamental principle of the economic growth is of the form

$$N(t) = C(t) + I_{inv}(t), \tag{0.1}$$

where $C(t)$ is the so-called apply function and $I_{inv}(t)$ is a quantity induced investment. We consider the case where the functions $C(t)$ and $I(t)$ have the form

$$C(t) = \alpha_0 N(t), \;\; \alpha_0 \in (0,1), \tag{0.2}$$

and

$$I_{inv}(t) = \sum_{i=1}^{s} \alpha_i N(t - \tau_i) + \alpha_{s+1} \dot{N}(t), \;\; \tau_i > 0, \;\; i = \overline{1,s}. \tag{0.3}$$

Formula (0.3) shows that the value of investment at the moment t depends on the quantity of money at the moments $t - \tau_i$, $i = \overline{1,s}$ (in the past), and on the velocity (production current) at the moment t. From the formulas (0.1)–(0.3) we get the equation with delays

$$\dot{N}(t) = \frac{1 - \alpha_0}{\alpha_{s+1}} N(t) - \sum_{i=1}^{s} \frac{\alpha_i}{\alpha_{s+1}} N(t - \tau_i).$$

The immune response Marchuk's model [26]. A simple model about viruses attack on an organism and its immune response is the following functional differential equation:

$$\begin{cases} \dot{x}_1(t) = p_1 x_1(t) - p_2 x_1(t) x_3(t), \\ \dot{x}_2(t) = \sum_{i=1}^{s} p_{i+2} x_1(t - \tau_i) x_3(t - \tau_i) - p_{s+3}(x_2(t) - x_2^*), \\ \dot{x}_3(t) = p_{s+4} x_2(t) - p_{s+5} x_3(t) - p_{s+6} x_1(t) x_3(t), \end{cases} \tag{0.4}$$

where $x_1(t)$ is the viruses concentration at time t; $x_2(t)$ is the plasma cells concentration producing antibodies. Plasma cells after a certain time period give the immune response which is characterized by the summand $\sum_{i=1}^{s} p_{i+2} x_1(t - \tau_i) x_3(t - \tau_i)$, where $\tau_i > 0$ are delays of immune reactions, i.e., this expression supports reproduction of antibodies; $x_3(t)$ is the antibodies concentration which kills viruses. The first equation of system (0.4) describes changes of $x_1(t)$, here the first term $p_1 x_1(t)$ supports reproduction of viruses and the second term $p_2 x_1(t) x_3(t)$ characterizes the struggle between viruses and antibodies and do not supports reproduction of viruses. x_2^* is the physiological level of plasma cells, i.e., this concentration of plasma cells is always in the organism, and in the absence of viruses in the organism, the plasma cells remain at a constant level. Finally, $p_1, p_2, \ldots, p_{s+6}$ are the positive constants.

A great deal of works (including, for example, [1–4, 12, 13, 19, 22]) are devoted to the investigation of functional differential equations with delay.

The present work consists of two parts, interconnected naturally in their meaning.

The first part considers the equation

$$\dot{x}(t) = f\big(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_s)\big) \tag{0.5}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \ \ t < t_0, \ \ x(t_0) = x_0. \tag{0.6}$$

The condition (0.6) is called a discontinuous initial condition since, in general, $x(t_0) \neq \varphi(t_0)$.

In the same part we study the continuous dependence of solutions of the problem (0.5), (0.6) on the initial data and on the right-hand side of the equation (0.5). Under the initial data we mean the collection of initial moment t_0, delays τ_i, $i = \overline{1,s}$, initial vector x_0 and initial function $\varphi(t)$. Moreover, we derive variation formulas of a solution (variation formulas) for the differential equation (0.5) with the discontinuous initial condition (0.6) and the continuous initial condition

$$x(t) = \varphi(t), \ \ t \leq t_0. \tag{0.7}$$

The condition (0.7) is called a continuous initial condition since, always, $x(t_0) = \varphi(t_0)$. The term "variation formula of solution" has been introduced by R. V. Gamkrelidze and proved in [6] for the ordinary differential equation.

In the second part, the optimization problems are investigated for the controlled equation

$$\dot{x}(t) = f\big(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_s), u(t)\big),$$

and the necessary optimality conditions are obtained.

In Section 1, we prove a theorem on the continuous dependence of a solution in the case where the perturbation of f is small in the integral sense and initial data are small in the standard norm. Theorems on the continuous dependence of solutions of the Cauchy problem and the boundary value problems for various classes of ordinary differential equations and delay functional differential equations when perturbations of the right-hand side are small in the integral sense, are given in [6, 7, 18–21, 23, 24, 33–35, 39].

In Sections 2 and 3, we prove the variation formulas in which the effects of perturbations of the initial moment and several delays and also the effects of discontinuous and continuous initial conditions are detected. The variation formula of a solution plays a basic role in proving the necessary conditions of optimality for sensitivity analysis of mathematical models. Moreover, the variation formula allows one to get an approximate solution of the perturbed equation. The variation formulas for various classes of differential equations are given in [6, 7, 18–20, 36–42].

In Section 4, we extend the central result of the axiomatic theory of extremal problems (R. V. Gamkrelidze and G. L. Kharatishvili's theorem on the necessary criticality condition [7–9]) to the mappings defined on a finitely locally convex set. This is stipulated by the fact that it is more convenient to treat the optimal problems with delays as the problems of finding the mappings, defined and critical on a finitely locally convex set and on a quasi-convex filter, respectively. The proof of the necessary criticality condition given in Subsection 4.1, is performed according to the scheme presented in [7–9] with nonessential changes.

In Subsection 4.3, we prove the quasiconvexity of the filter arising in the optimal control problem with delays. The concept of quasiconvexity of a filter was introduced by R. V. Gamkrelidze, as a result of studying slide modes [10, 11]. Of special interest is the finding of control systems with a quasiconvex filter, since the necessary optimality conditions for these systems are deduced from the necessary criticality condition. In Subsection 4.4, we consider optimal control problems with a general functional and boundary conditions, the discontinuous initial condition and the continuous condition. The necessary conditions are obtained: for the initial and final moments in the form of inequalities and equalities, for delays in the form of inequalities and equalities, for the initial vector in the form of equality, and for the initial function and control function in the form of integral maximum principle. Optimal control problems for various classes of functional differential equations are investigated in [5, 15–18, 20, 25, 27–31].

1 Continuous dependence of solutions

1.1 Notation and auxiliary assertions

Let $I = [a, b]$ be a finite interval and $\mathbb{R}^n$ be the n-dimensional vector space of points $x = (x^1, \ldots, x^n)^\top$ with $|x|^2 = \sum\limits_{i=1}^{n} |x^i|^2$, where T is the sign of transposition. Let $\theta_{i2} > \theta_{i1} > 0$, $i = \overline{1, s}$, be the given numbers; suppose that $O \subset \mathbb{R}^n$ is an open set, and E_f is a set of functions $f = (f^1, \ldots, f^n)^\top : I \times O^{s+1} \to \mathbb{R}^n$ satisfying the following conditions: for each fixed $(x, x_1, \ldots, x_s) \in O^{s+1}$, the function $f(t, x, x_1, \ldots, x_s)$ is measurable; for each $f \in E_f$ and compact set $K \subset O$, there exist functions $m_{f,K}(t), L_{f,K}(t) \in L_1(I, R_+)$, $R_+ = [0, \infty)$, such that for almost all $t \in I$

$$|f(t, x, x_1, \ldots, x_s)| \le m_{f,K}(t) \ \ \forall (x, x_1, \ldots, x_s) \in K^{s+1}$$

and

$$\left| f(t, x, x_1, \ldots, x_s) - f(t, y, y_1, \ldots, y_s) \right| \le L_{f,K}(t) \Big[|x - y| + \sum_{i=1}^{s} |x_i - y_i| \Big]$$

$$\forall (x, x_1, \ldots, x_s) \in K^{s+1}, \ \ \forall (y, y_1, \ldots, y_s) \in K^{s+1}.$$

Two functions $f_1, f_2 \in E_f$ are said to be equivalent, if for every fixed $(x, x_1, \ldots, x_s) \in O^{s+1}$ and for almost all $t \in I$

$$f_1(t, x, x_1, \ldots, x_s) - f_2(t, x, x_1, \ldots, x_s) = 0.$$

The equivalence classes of functions of the space E_f compose a vector space which is also denoted by E_f; these classes are called the functions and denoted by f again. In what follows, under $f \in E_f$ it is assumed any representative from the equivalence class of f.

Lemma 1.1 ([6, p. 56]). *Let $f \in E_f$. Then the function*

$$H(f; t', t'', x, x_1, \ldots, x_s) = \left| \int\limits_{t'}^{t''} f(t, x, x_1, \ldots, x_s) \, dt \right|$$

is continuous in $(t', t'', x, x_1, \ldots, x_s) \in I^2 \times O^{s+1}$

Lemma 1.2 ([6, p. 41]). *Let $K_0 \subset O$ and $K_1 \subset O$ be compact sets with $K_0 \subset \operatorname{int} K_1$. Then there exist a compact set $Q \subset O^{s+1}$ and a continuously differentiable function $\chi(x, x_1, \ldots, x_s), (x, x_1, \ldots, x_s) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ such that $K_0^{s+1} \subset Q \subset \operatorname{int} K_1^{s+1}$ and*

$$\chi(x, x_1, \ldots, x_s) = \begin{cases} 1, & (x, x_1, \ldots, x_s) \in Q, \\ 0, & (x, x_1, \ldots, x_s) \notin K_1^{s+1}. \end{cases} \tag{1.1}$$

Lemma 1.3. *Let $f \in E_f$. Then the function*

$$g(t, x, x_1, \ldots, x_s) = \begin{cases} \chi(x, x_1, \ldots, x_s) f(t, x, x_1, \ldots, x_s), & t \in I, \ (x, x_1, \ldots, x_s) \in K_1^{s+1}, \\ 0, & t \in I, \ (x, x_1, \ldots, x_s) \notin K_1^{s+1}, \end{cases} \tag{1.2}$$

satisfies for almost all $t \in I$ the following conditions:

$$|g(t, x, x_1, \ldots, x_s)| \le m_{f, K_1}(t) \ \ \forall (x, x_1, \ldots, x_s) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n \tag{1.3}$$

and

$$\left| g(t, x, x_1, \ldots, x_s) - g(t, y, y_1, \ldots, y_s) \right| \le L_f(t) \Big[|x - y| + \sum_{i=1}^{s} |x_i - y_i| \Big] \tag{1.4}$$

$$\forall (x, x_1, \ldots, x_s) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n, \ \ \forall (y, y_1, \ldots, y_s) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n,$$

where

$$L_f(t) = L_{f,K_1}(t) + \alpha_0 m_{f,K_1}(t), \tag{1.5}$$

$$\alpha_0 = \sup\left\{ |\chi_x(x, x_1, \ldots, x_s)| + \sum_{i=1}^{s} |\chi_{x_i}(x, x_1, \ldots, x_s)| : \ (x, x_1, \ldots, x_s) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n \right\}.$$

Proof. The inequality (1.3) follows from the definition of the function g. Let

$$(x, x_1, \ldots, x_s) \in K_1^{s+1} \ \text{ and } \ (y, y_1, \ldots, y_s) \in K_1^{s+1},$$

then (see (1.2)) we have

$$\begin{aligned}
\big| g(t, & x, x_1, \ldots, x_s) - g(t, y, y_1, \ldots, y_s) \big| \\
&= \big| \chi(x, x_1, \ldots, x_s) f(t, x, x_1, \ldots, x_s) - \chi(y, y_1, \ldots, y_s) f(t, y, y_1, \ldots, y_s) \big| \\
&= \Big| \chi(x, x_1, \ldots, x_s)\big(f(t, x, x_1, \ldots, x_s) - f(t, y, y_1, \ldots, y_s) \big) \\
&\qquad\qquad + \big(\chi(x, x_1, \ldots, x_s) - \chi(y, y_1, \ldots, y_s) \big) f(t, y, y_1, \ldots, y_s) \Big| \\
&\le L_{f,K_1}(t)\Big[|x - y| + \sum_{i=1}^{s} |x_i - y_i| \Big] + \big| \chi(x, x_1, \ldots, x_s) - \chi(y, y_1, \ldots, y_s)) \big| m_{f,K_1}(t). \tag{1.6}
\end{aligned}$$

It is not difficult to see that

$$\begin{aligned}
\big| \chi(x, & x_1, \ldots, x_s) - \chi(y, y_1, \ldots, y_s)) \big| \\
&= \left| \int_0^1 \frac{d}{d\xi}\, \chi\big(y + \xi(x - y), y_1 + \xi(x_1 - y_1), \ldots, y_s + \xi(x_s - y_s) \big)\, d\xi \right| \\
&\le \int_0^1 \Big[\big| \chi_x\big(y + \xi(x - y), y_1 + \xi(x_1 - y_1), \ldots, y_s + \xi(x_s - y_s) \big) \big|\, |x - y| + \sum_{i=1}^{s} |\chi_{x_i}(\,\cdot\,)|\, |x_i - y_i| \Big]\, d\xi \\
&\le \alpha_0 \Big[|x - y| + \sum_{i=1}^{s} |x_i - y_i| \Big].
\end{aligned}$$

Taking this relation into account, from (1.6) we obtain (1.4). Let

$$(x, x_1, \ldots, x_s) \in K_1^{s+1} \ \text{ and } \ (y, y_1, \ldots, y_s) \notin K_1^{s+1},$$

then $\chi(y, y_1, \ldots, y_s) = 0$, i.e., $g(y, y_1, \ldots, y_s) = 0$, therefore we have

$$\begin{aligned}
\big| g(t, & x, x_1, \ldots, x_s) - g(t, y, y_1, \ldots, y_s) \big| \\
&= |g(t, x, x_1, \ldots, x_s)| = \big| \chi(x, x_1, \ldots, x_s) - \chi(y, y_1, \ldots, y_s)) \big|\, |f(t, x, x_1, \ldots, x_s)| \\
&\le \alpha_0 m_{f,K_1}(t)\Big[|x - y| + \sum_{i=1}^{s} |x_i - y_i| \Big] \le L_f(t)\Big[|x - y| + \sum_{i=1}^{s} |x_i - y_i| \Big].
\end{aligned}$$

It is easily seen that the latter inequality also holds in the case

$$(x, x_1, \ldots, x_s) \notin K_1^{s+1} \ \text{ and } \ (y, y_1, \ldots, y_s) \in K_1^{s+1}. \qquad\qquad \square$$

Let $I_1 = [\widehat{\tau}, b]$, where $\widehat{\tau} = a - \max\{\theta_{12}, \ldots, \theta_{s2}\}$. By $\mathrm{PC}(I_1, \mathbb{R}^n)$ we denote the space of piecewise-continuous functions $\varphi : I_1 \to \mathbb{R}^n$ with finitely many discontinuities of the first kind equipped with the norm $\|\varphi\|_{I_1} = \sup\{|\varphi(t)| : \ t \in I_1\}$. By $\Phi = \{\varphi \in \mathrm{PC}(I_1, \mathbb{R}^n) : \ \mathrm{cl}\, \varphi(I_1) \subset O\}$ we denote a set of initial functions, where $\varphi(I_1) = \{\varphi(t) : \ t \in I_1\}$.

Let $\varphi_i \in \Phi$, $i = \overline{0,s}$, be fixed functions and let $t_\alpha \in (a,b)$, $\alpha = \overline{1,p}$, be discontinuity points of the function $\psi(t) = (\varphi_0(t), \varphi_1(t-\tau_1), \ldots, \varphi_s(t-\tau_s))$, where $\tau_i \in [\theta_{i1}, \theta_{i2}]$, $i = \overline{1,s}$, are the given numbers.

We now introduce the notation

$$\varphi_{ij}(t) = \begin{cases} \varphi_i(t_{j-1} - \tau_i+), & t = t_{j-1}, \\ \varphi_i(t-\tau_i), & t \in (t_{j-1}, t_j), \\ \varphi_i(t_j - \tau_i-), & t = t_j, \end{cases} \tag{1.7}$$

where $i = \overline{0,s}$, $j = \overline{1,p+1}$, $t_0 = a$, $t_{p+1} = b$, $\tau_0 = 0$. Clearly, the function $\varphi_{ij}(t)$ is continuous on the interval $[t_{j-1}, t_j]$. Next, let k be a fixed natural number,

$$w_j(k; \psi) = \sup \left\{ \sum_{i=0}^{s} |\varphi_{ij}(t') - \varphi_{ij}(t'')| : \ t', t'' \in [t_{j-1}, t_j], \ |t' - t''| \leq \frac{t_j - t_{j-1}}{k} \right\},$$

$$w(k; \psi) = \sup \left\{ w_j(k; \psi) : \ 1 \leq j \leq p+1 \right\}.$$

Lemma 1.4. *Let $\varphi_i \in \Phi$, $i = \overline{0,s}$, and let $\varphi_i(t) \in K$, where $K \subset O$ is a compact set. Then for an arbitrary $f \in E_f$ and a natural number k, the inequality*

$$\beta = \sup \left\{ \left| \int_{\xi_1}^{\xi_2} f\big(t, \varphi_0(t), \varphi_1(t-\tau_1), \ldots, \varphi_s(t-\tau_s)\big) \, dt \right| : \ \xi_1, \xi_2 \in I \right\}$$

$$\leq w(k; \psi) \int_I L_{f,K}(t) \, dt + k(p+1) H_0(f; K)$$

holds, where

$$H_0(f; K) = \sup \left\{ H(f; t', t'', x, x_1, \ldots, x_s) : \ (t', t'', x, x_1, \ldots, x_s) \in I^2 \times K^{s+1} \right\}$$

(see Lemma 1.1).

Proof. There exist the numbers $a_1, b_1 \in I$ such that

$$\beta = \left| \int_{a_1}^{b_1} f\big(t, \varphi_0(t), \varphi_1(t-\tau_1), \ldots, \varphi_s(t-\tau_s)\big) \, dt \right|.$$

Let $a_1 \in [t_{l-1}, t_l)$ and $b_1 \in [t_{q-1}, t_q)$ with $1 \leq l \leq q \leq p+1$. Divide each of the intervals $[a_1, t_l]$, $[t_{j-1}, t_j]$, $j = \overline{l+1, q-1}$, $[t_{q-1}, b_1]$, into k equal parts Δ_ρ^l, Δ_ρ^j, $j = \overline{l+1, q-1}$, Δ_ρ^q, $\rho = \overline{1,k}$, respectively. Obviously,

$$[a_1, b_1] = [a_1, t_l] \cup \Big(\bigcup_{j=l+1}^{q-1} [t_{j-1}, t_j] \Big) \cup [t_{q-1}, b_1] = \bigcup_{j=l}^{q} \bigcup_{\rho=1}^{k} \Delta_\rho^j.$$

Using this relation and the notation (1.7), we obtain

$$\beta \leq \sum_{j=l}^{q} \sum_{\rho=1}^{k} \left| \int_{\Delta_\rho^j} f\big(t, \varphi_{0j}(t), \varphi_{1j}(t), \ldots, \varphi_{sj}(t)\big) \, dt \right|.$$

Let $t_\rho^j \in \Delta_\rho^j$, $j = \overline{1,q}$, $\rho = \overline{1,k}$, be arbitrary fixed points. Then

$$\beta \leq \sum_{j=l}^{q} \sum_{\rho=1}^{k} \int_{\Delta_\rho^j} \left| f\big(t, \varphi_{0j}(t), \varphi_{1j}(t), \ldots, \varphi_{sj}(t)\big) - f\big(t, \varphi_{0j}(t_\rho^j), \varphi_{1j}(t_\rho^j), \ldots, \varphi_{sj}(t_\rho^j)\big) \right| dt$$

$$+ \sum_{j=l}^{q} \sum_{\rho=1}^{k} \left| \int_{\Delta_\rho^j} f\big(t, \varphi_{0j}(t_\rho^j), \varphi_{1j}(t_\rho^j), \ldots, \varphi_{sj}(t_\rho^j)\big) \, dt \right|$$

$$\leq \sum_{j=l}^{q} \sum_{\rho=1}^{k} \int_{\Delta_\rho^j} \left[L_{f,K}(t) \sum_{i=0}^{s} |\varphi_{ij}(t) - \varphi_{ij}(t_\rho^j)| \right] dt + k(q - l + 1) H_0(f; K)$$

$$\leq \sum_{j=l}^{q} \sum_{\rho=1}^{k} w_j(k; \psi) \int_{\Delta_\rho^j} L_{f,K}(t) \, dt + k(p + 1) H_0(f; K)$$

$$\leq w(k; \psi) \int_I L_{f,K}(t) \, dt + k(p + 1) H_0(f; K). \qquad \square$$

Lemma 1.5. *Let $\varphi_i \in \Phi$, $i = \overline{0, s}$, and let $\varphi_i(t) \in K$, where $K \subset O$ is a compact set. Further, let the sequence $\delta f_i \in E_f$, $i = 1, 2, \ldots$, satisfy the conditions*

$$\int_I L_{\delta f_i, K}(t) \, dt \leq \alpha_1 = const, \ \ i = 1, 2, \ldots, \ \ and \ \ \lim_{i \to \infty} H_0(\delta f_i; K) = 0.$$

Then $\lim\limits_{i \to \infty} \beta_i = 0$, where

$$\beta_i = \sup \left\{ \left| \int_{\xi_1}^{\xi_2} \delta f_i\big(t, \varphi_0(t), \varphi_1(t - \tau_1), \ldots, \varphi_s(t - \tau_s)\big) \, dt \right| : \ \xi_1, \xi_2 \in I \right\}.$$

Proof. Let $\varepsilon > 0$ be an arbitrary number. By Lemma 1.4, we have

$$\beta_i \leq w(k; \psi) \int_I L_{\delta f_i, K}(t) \, dt + k(p + 1) H_0(\delta f_i; K) \leq \alpha_1 w(k; \psi) + k(p + 1) H_0(\delta f_i; K). \qquad (1.8)$$

The functions $\varphi_{ij}(t)$, $t \in [t_{j-1}, t_j]$, are continuous. Therefore, $\lim\limits_{k \to \infty} w(k; \psi) = 0$. There exist natural numbers k_0 and i_0 such that

$$w(k_0; \psi) \leq \frac{\varepsilon}{2} \ \ and \ \ k_0(p + 1) H_0(\delta f_i; K) \leq \frac{\varepsilon}{2}, \ \ i \geq i_0. \qquad (1.9)$$

Taking into account the relations (1.9) in (1.8), we obtain $\beta_i \leq \varepsilon$ for $i \geq i_0$. By the arbitrariness of ε, we can conclude that $\beta_i \to 0$, as $i \to \infty$. $\qquad \square$

Lemma 1.6 ([6, p. 68]). *Let $m(t) \in L_1(I, \mathbb{R}_+)$. Then the formula*

$$\int_a^t m(\xi_1) \, d\xi_1 \int_a^{\xi_1} m(\xi_2) \, d\xi_2 \cdots \int_a^{\xi_{k-1}} m(\xi_k) \, d\xi_k = \frac{1}{k!} \left(\int_a^t m(\xi) \, d\xi \right)^k$$

holds.

Lemma 1.7. *Let $f_1, f_2 \in E_f$ be equivalent functions. Then for an arbitrary function $\varphi \in \Phi$, the relation*

$$\left| \int_{\xi_1}^{\xi_2} \widehat{f}\big(t, \varphi(t), \varphi(t - \tau_1), \ldots, \varphi(t - \tau_s)\big) \, dt \right| = 0 \ \ \forall \xi_1, \xi_2 \in I \qquad (1.10)$$

holds, where

$$\widehat{f}(t, x, x_1, \ldots, x_s) = f_1(t, x, x_1, \ldots, x_s) - f_2(t, x, x_1, \ldots, x_s).$$

Proof. It is clear that for almost all $t \in I$,

$$\widehat{f}(t, x, x_1, \ldots, x_s) = 0 \ \ \forall (x, x_1, \ldots, x_s) \in O^{1+s}.$$

Therefore

$$H_0(\widehat{f}; K) = 0, \ \text{ where } \ K = \text{cl}\,\varphi(I_1) \subset O.$$

Using Lemma 1.3, for an arbitrary natural number k and $\xi_1, \xi_2 \in I$, we get

$$\left| \int_{\xi_1}^{\xi_2} \widehat{f}\big(t, \varphi(t), \varphi(t - \tau_1), \ldots, \varphi(t - \tau_s)\big)\, dt \right| \leq w(k; \psi) \int_I L_{\widehat{f}; K}(t)\, dt,$$

where $\psi(t) = (\varphi(t), \varphi(t - \tau_1), \ldots, \varphi_s(t - \tau_s))$ and $w(k; \psi) \to 0$, as $k \to \infty$. Thus the relation (1.10) is valid. $\qquad\square$

Lemma 1.8. *Let $f \in E_f$. Then the mapping*

$$\varphi \longrightarrow \int_a^t f\big(\xi, \varphi(\xi), \varphi(\xi - \tau_1), \ldots, \varphi(\xi - \tau_s)\big)\, d\xi, \ \ \varphi \in \Phi,$$

is uniquely defined (see Lemma 1.7).

Let X be a metric space, ϱ be a distance function on X, and let

$$F(\,\cdot\,; \mu) : X \to X \tag{1.11}$$

be a family of mappings depending on the parameter $\mu \in \Lambda$, where Λ is a topological space. The family of the mappings (1.11) is said to be uniformly contractive if there exists a number $\alpha \in (0, 1)$ independent of μ such that the inequality

$$\varrho\big(F(y_1; \mu), F(y_2; \mu)\big) \leq \alpha \varrho(y_1, y_2) \ \ \forall y_1, y_2 \in X$$

holds for each $\mu \in \Lambda$.

Define the iteration of the mapping (1.11):

$$F^k(y; \mu) = F(F^{k-1}(y; \mu); \mu), \ \ k = 1, 2, \ldots, \ \ F^0(y; \mu) = y.$$

Obviously,

$$F^k(\,\cdot\,; \mu) : X \to X \ \ \forall \mu \in \Lambda. \tag{1.12}$$

Theorem 1.1 ([6, p. 61]; [14, p. 608]). *Let X be a complete metric space. If a certain iteration (1.12) is a uniformly contractive family, then for every $\mu \in \Lambda$ the mapping (1.11) has a unique fixed point $y_\mu \in X$, i.e., $F(y_\mu; \mu) = y_\mu$. Moreover, if for fixed $\mu_0 \in \Lambda$, a certain iteration $F^k(y_{\mu_0}; \,\cdot\,) : \Lambda \to X$ is continuous at the point μ_0, then the mapping $y_\mu : \Lambda \to X$ is likewise continuous at the point μ_0.*

1.2 Theorems on continuous dependence of solutions

To each element

$$\mu = (t_0, \tau_1, \ldots, \tau_s, x_0, \varphi, f) \in \Lambda = [a, b) \times [\theta_{11}, \theta_{12}] \times \cdots \times [\theta_{s1}, \theta_{s2}] \times O \times \Phi \times E_f$$

we assign the delay functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_s)\big) \tag{1.13}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0), \ \ x(t_0) = x_0. \tag{1.14}$$

Definition 1.1. Let $\mu = (t_0, \tau_1, \ldots, \tau_s, x_0, \varphi, f) \in \Lambda$. A function $x(t) = x(t; \mu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is called a solution of the equation (1.13) with the initial condition (1.14), or a solution corresponding to the element μ and defined on the interval $[\widehat{\tau}, t_1]$, if it satisfies the condition (1.14) and on the interval $[t_0, t_1]$ satisfies the integral equation

$$x(t) = x_0 + \int\limits_{t_0}^{t} f\big(\xi, x(\xi), x(\xi - \tau_1), \ldots, x(\xi - \tau_s)\big)\, d\xi$$

(see Lemma 1.7).

Obviously, the function $x(t; \mu)$, $t \in [t_0, t_1]$, is absolutely continuous and satisfies the equation (1.13) almost everywhere on $[t_0, t_1]$. If $t_1 - t_0$ is sufficiently small, then there exists a solution corresponding to μ [3, 13, 22].

In the space E_f, we introduce a family of subsets

$$\Re = \big\{ V_{K, \delta} : \ \ K \subset O, \ \ \delta > 0 \big\}.$$

Here, $K \subset O$ is a compact set, $\delta > 0$ is an arbitrary number, and

$$V_{K, \delta} = \big\{ \delta f \in E_f : \ \ H_0(\delta f; K) \le \delta \big\}.$$

The family $\Re$ can be taken as a basis of neighborhoods of zero in the space E_f [32]. Hence it defines a locally convex Hausdorff vector topology with which E_f becomes a topological vector space. Everywhere in what follows, we will assume that the space E_f is endowed precisely with that topology.

We introduce the set

$$W(K; \alpha) = \left\{ \delta f \in E_f : \ \ \exists\, m_{\delta f, K}(t), L_{\delta f, K}(t) \in L_1(I, R_+), \ \ \int\limits_{I} \big[m_{\delta f, K}(t) + L_{\delta f, K}(t) \big]\, dt \le \alpha \right\},$$

where $K \subset O$ is a compact set and $\alpha > 0$ is a fixed number independent of δf.

Let $\mu_0 = (t_{00}, \tau_{10}, \ldots, \tau_{s0}, x_{00}, \varphi_0, f_0) \in \Lambda$ be a fixed element,

$$B(t_{00}; \delta) = \big\{ t_0 \in I : \ |t_0 - t_{00}| < \delta \big\}, \quad B(\tau_{i0}; \delta) = \big\{ \tau_i \in [\theta_{i1}, \theta_{i2}] : \ |\tau_i - \tau_{i0}| < \delta \big\}, \ \ i = \overline{1, s},$$

$$B(x_{00}; \delta) = \big\{ x_0 \in O : \ |x_0 - x_{00}| < \delta \big\}, \quad B(\varphi_0; \delta) = \big\{ \varphi \in \Phi : \ \|\varphi - \varphi_0\|_{I_1} < \delta \big\},$$

Theorem 1.2. *Let $x_0(t)$ be a solution corresponding to $\mu_0 = (t_{00}, \tau_{10}, \ldots, \tau_{s0}, x_{00}, \varphi_0, f_0) \in \Lambda$ and defined on $[\widehat{\tau}, t_{10}]$, where $t_{10} < b$, and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \mathrm{cl}\, \varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following conditions hold:*

1.1. *There exist numbers $\delta_i > 0$, $i = 0, 1$, such that to each element*

$$\mu = (t_0, \tau_1, \ldots, \tau_s, x_0, \varphi, f_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha)$$
$$= B(t_{00}; \delta_0) \times B(\tau_{10}; \delta_0) \times \cdots \times B(\tau_{s0}; \delta_0) \times B(x_{00}; \delta_0) \times B(\varphi_0; \delta_0)$$
$$\times \big[f_0 + (W(K_1; \alpha) \cap V_{K_1, \delta_0}) \big]$$

there corresponds the solution $x(t; \mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ and satisfying the condition $x(t; \mu) \in K_1$.

1.2. *For an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the inequality*

$$|x(t; \mu) - x(t; \mu_0)| \le \varepsilon \ \ \forall\, t \in [\theta, t_{10} + \delta_1], \ \ \theta = \max\{t_0, t_{00}\}$$

holds for any $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$.

1.3. *For an arbitrary $\varepsilon > 0$, there exists a number $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ such that the inequality*

$$\int\limits_{\widehat{\tau}}^{t_{10}+\delta_1} |x(t; \mu) - x(t; \mu_0)|\, dt \le \varepsilon$$

holds for any $\mu \in V(\mu_0; K_1, \delta_3, \alpha)$.

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$.

In the space $E_{\delta\mu} = E_\mu - \mu_0$ with the elements $\delta\mu = (\delta t_0, \delta\tau_1, \ldots, \delta\tau_s, \delta x_0, \delta\varphi, \delta f)$, where $E_\mu = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{R}^n \times \mathrm{PC}(I_1, \mathbb{R}^n) \times E_f$, we introduce the set of variations

$$\Im = \Bigg\{ \delta\mu = (\delta t_0, \delta\tau_1, \ldots, \delta\tau_s, \delta x_0, \delta\varphi, \delta f) \in E_{\delta\mu} :$$

$$|\delta t_0| \leq \gamma, \ \ |\delta\tau_i| \leq \gamma, \ \ i = \overline{1, s}, \ \ |\delta x_0| \leq \gamma, \ \ \|\delta\varphi\|_I \leq \gamma, \ \ \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, \ \ |\lambda_i| \leq \gamma, \ \ i = \overline{1, k} \Bigg\},$$

where $\gamma > 0$ is a fixed number and $\delta f_i \in E_f - f_0$, $i = \overline{1, k}$, are fixed functions.

Theorem 1.3. *Let $x_0(t)$ be a solution corresponding to $\mu_0 = (t_{00}, \tau_{10}, \ldots, \tau_{s0}, x_{00}, \varphi_0, f_0) \in \Lambda$ and defined on $[\widehat{\tau}, t_{10}]$ with $t_{00}, t_{10} \in (a, b)$, $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = \overline{1, s}$, and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set K_0. Then the following conditions hold:*

1.4. *There exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times \Im$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$ and the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to this element. Moreover, $x(t; \mu_0 + \varepsilon\delta\mu) \in K_1$.*

1.5. *The following relations hold:*

$$\lim_{\varepsilon \to 0} \sup \Big\{ |x(t; \mu_0 + \varepsilon\delta\mu) - x(t; \mu_0)| : \ t \in [\theta, t_{10} + \delta_1] \Big\} = 0,$$

$$\lim_{\varepsilon \to 0} \int_{\widehat{\tau}}^{t_{10} + \delta_1} |x(t; \mu_0 + \varepsilon\delta\mu) - x(t; \mu_0)| \, dt = 0$$

uniformly in $\delta\mu \in \Im$, where $\theta = \max\{t_{00}, t_{00} + \varepsilon\delta t_0\}$.

Theorem 1.2 is the corollary of Theorem 1.1.

Let $E_u(I)$ be the space of measurable functions $u(t) \in \mathbb{R}^r$, $t \in I$, satisfying the condition: $\mathrm{cl}\, u(I)$ is a compact set in $\mathbb{R}^r$. Let $U_0 \subset \mathbb{R}^r$ be an open set and $\Omega(I, U_0) = \{u \in E_u(I) : \ \mathrm{cl}\, u(I) \subset U_0\}$.

To each element $w = (t_0, \tau_1, \ldots, \tau_s, x_0, \varphi, u) \in \Lambda_1 = [a, b) \times [\theta_{11}, \theta_{12}] \times \cdots \times [\theta_{s1}, \theta_{s2}] \times O \times \Phi \times \Omega(I, U_0)$ we assign the delay controlled functional differential equation

$$\dot{x}(t) = \phi\big(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_s), u(t)\big) \tag{1.15}$$

with the discontinuous initial condition (1.14). Here the function $\phi(t, x, x_1, \ldots, x_s, u)$ is defined on $I \times O^{s+1} \times U_0$ and satisfies the following conditions: for each fixed $(x, x_1, \ldots, x_s, u) \in O^{s+1} \times U_0$ the function $\phi(\cdot, x, x_1, \ldots, x_s, u) : I \to \mathbb{R}^n$ is measurable; for each compact sets $K \subset O$ and $U \subset U_0$ there exist the functions $m_{K,U}(t), L_{K,U}(t) \in L_1(I, R_+)$ such that for almost all $t \in I$,

$$\big|\phi(t, x, x_1, \ldots, x_s, u)\big| \leq m_{K,U}(t) \ \ \forall (x, x_1, \ldots, x_s, u) \in K^{s+1} \times U,$$

$$\big|\phi(t, x, x_1, \ldots, x_s, u_1) - \phi(t, y, y_1, \ldots, y_s, u_2)\big| \leq L_{f,K}(t) \Big[|x - y| + \sum_{i=1}^{s} |x_i - y_i| + |u_1 - u_2| \Big]$$

$$\forall (x, x_1, \ldots, x_s) \in K^{s+1}, \ \ \forall (y, y_1, \ldots, y_s) \in K^{s+1} \ \text{and} \ \forall (u_1, u_2) \in U^2.$$

Definition 1.2. *Let $w = (t_0, \tau_1, \ldots, \tau_s, x_0, \varphi, u) \in \Lambda_1$. A function $x(t) = x(t; w) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b)$, is called a solution of the equation (1.15) with the initial condition (1.14), or a solution corresponding to the element w and defined on the interval $[\widehat{\tau}, t_1]$, if it satisfies the condition (1.14) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (1.15) almost everywhere (a.e.) on $[t_0, t_1]$.*

Theorem 1.4. *Let $x_0(t)$ be a solution corresponding to $w_0 = (t_{00}, \tau_{10}, \ldots, \tau_{s0}, x_{00}, \varphi_0, u_0) \in \Lambda_1$ and defined on $[\widehat{\tau}, t_{10}]$, with $t_{10} < b$, and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \mathrm{cl}\, \varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following conditions hold:*

1.6. *There exist the numbers $\delta_i > 0$, $i = 0, 1$, such that to each element*

$$w = (t_0, \tau_1, \ldots, \tau_s, x_0, \varphi, u) \in \widehat{V}(w_0; \delta_0)$$
$$= B(t_{00}; \delta_0) \times B(\tau_{10}; \delta_0) \times \cdots \times B(\tau_{s0}; \delta_0) \times B(x_{00}; \delta_0) \times B(\varphi_0; \delta_0) \times B(u_0; \delta_0)$$

there corresponds a solution $x(t; w)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ and satisfying the condition $x(t; w) \in K_1$; here $B(u_0; \delta_0) = \{u \in \Omega(I, U_0) : \|u - u_0\|_I < \delta_0\}$.

1.7. *For an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the inequality*

$$|x(t; w) - x(t; w_0)| \leq \varepsilon \quad \forall t \in [\theta, t_{10} + \delta_1], \quad \theta = \max\{t_0, t_{00}\},$$

holds for any $w \in \widehat{V}(w_0; \delta_2)$.

1.8. *For an arbitrary $\varepsilon > 0$, there exists a number $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ such that the inequality*

$$\int\limits_{\widehat{\tau}}^{t_{10}+\delta_1} |x(t; w) - x(t; w_0)| \, dt \leq \varepsilon$$

holds for any $w \in \widehat{V}(w_0; \delta_3)$.

Due to the uniqueness, the solution $x(t; w_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$.

In the space $E_{\delta w} = E_w - w_0$ with the elements $\delta w = (\delta t_0, \delta \tau_1, \ldots, \delta \tau_s, \delta x_0, \delta \varphi, \delta u)$, where $E_w = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{R}^n \times \mathrm{PC}(I_1, \mathbb{R}^n) \times E_u(I)$, we introduce the set of variations

$$\Im_1 = \Big\{ \delta w = (\delta t_0, \delta \tau_1, \ldots, \delta \tau_s, \delta x_0, \delta \varphi, \delta u) \in E_{\delta w} :$$
$$|\delta t_0| \leq \gamma, \ \ |\delta \tau_i| \leq \gamma, \ \ i = \overline{1, s}, \ \ |\delta x_0| \leq \beta, \ \ \|\delta \varphi\|_{I_1} \leq \gamma, \ \ \|\delta u\|_I \leq \gamma \Big\},$$

where $\gamma > 0$ is a fixed number.

Theorem 1.5. *Let $x_0(t)$ be a solution corresponding to $w_0 = (t_{00}, \tau_{10}, \ldots, \tau_{s0}, x_{00}, \varphi_0, u_0) \in \Lambda_1$ and defined on $[\widehat{\tau}, t_{10}]$ with $t_{00}, t_{10} \in (a, b)$, $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = \overline{1, s}$, and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set K_0. Then the following conditions hold:*

1.9. *There exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for an arbitrary $(\varepsilon, \delta w) \in [0, \varepsilon_1] \times \Im_1$ we have $w_0 + \varepsilon \delta w \in \Lambda_1$ and the solution $x(t; w_0 + \varepsilon \delta w)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to this element. Moreover, $x(t; w_0 + \varepsilon \delta w) \in K_1$.*

1.10. *The following relations hold:*

$$\lim_{\varepsilon \to 0} \sup \Big\{ |x(t; w_0 + \varepsilon \delta w) - x(t; w_0)| : \ t \in [\theta, t_{10} + \delta_1] \Big\} = 0,$$
$$\lim_{\varepsilon \to 0} \int\limits_{\widehat{\tau}}^{t_{10}+\delta_1} \big| x(t; w_0 + \varepsilon \delta w) - x(t; w_0) \big| \, dt = 0$$

uniformly in $\delta w \in \Im_1$, where $\theta = \max\{t_{00}, t_{00} + \varepsilon \delta t_0\}$.

Theorem 1.5 is the corollary of Theorem 1.4.

Let $I_2 = [a, \widehat{\tau}_1]$, where $\widehat{\tau}_1 = b + \max\{\theta_{12}, \ldots, \theta_{s2}\}$. By $\Phi_1 = \{\varphi \in \mathrm{PC}(I_2, \mathbb{R}^n) : \ \mathrm{cl} \, \varphi(I_2) \subset O\}$ we denote a set of initial functions for the functional differential equation with advanced arguments. To each element

$$\vartheta = (t_1, \tau_1, \ldots, \tau_s, x_1, \varphi, f) \in \Lambda_2 = (a, b] \times [\theta_{11}, \theta_{12}] \times \cdots \times [\theta_{s1}, \theta_{s2}] \times O \times \Phi_1 \times E_f$$

we assign the functional differential equation with the advanced argument

$$\dot{x}(t) = f\big(t, x(t), x(t + \tau_1), \ldots, x(t + \tau_s)\big)$$

with the discontinuous initial condition

$$x(t_1) = x_1, \quad x(t) = \varphi(t), \quad t \in (t_1, \widehat{\tau}_1].$$

Definition 1.3. Let $\vartheta = (t_1, \tau_1, \ldots, \tau_s, x_1, \varphi, f) \in \Lambda_2$. A function $x(t) = x(t; \vartheta) \in O$, $t \in [t_0, \widehat{\tau}_1]$, $t_0 \in [a, t_1)$, is called a solution corresponding to the element ϑ and defined on the interval $[t_0, \widehat{\tau}_1]$ if it satisfies the initial condition and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the integral equation

$$x(t) = x_1 + \int\limits_t^{t_1} f\big(\xi, x(\xi), x(\xi + \tau_1), \ldots, x(\xi + \tau_s)\big)\, d\xi.$$

Theorem 1.6. Let $x_0(t)$ be a solution corresponding to $\vartheta_0 = (t_{10}, \tau_{10}, \ldots, \tau_{s0}, x_{10}, \varphi_0, f_0) \in \Lambda_2$ and defined on $[t_{00}, \widehat{\tau}_2]$, where $t_{00} > a$, and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $\mathrm{cl}\, \varphi_0(I_2) \cup x_0([t_{00}, t_{10}])$. Then the following conditions hold:

1.11. *There exist numbers $\delta_i > 0$, $i = 0, 1$, such that to each element*

$$\vartheta = (t_1, \tau_1, \ldots, \tau_s, x_1, \varphi, f_0 + \delta f) \in V(\vartheta_0; K_1, \delta_0, \alpha)$$
$$= B(t_{10}; \delta_0) \times B(\tau_{10}; \delta_0) \times \cdots \times B(\tau_{s0}; \delta_0) \times B(x_{10}; \delta_0) \times B_1(\varphi_0; \delta_0) \times \big[f_0 + (W(K_1; \alpha) \cap V_{K_1, \delta_0})\big]$$

there corresponds the solution $x(t; \vartheta)$ defined on the interval $[t_{00} - \delta_1, \widehat{\tau}_2] \subset I_2$ and satisfying the condition $x(t; \vartheta) \in K_1$.

1.12. *For an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the inequality*

$$|x(t; \vartheta) - x(t; \vartheta_0)| \leq \varepsilon \quad \forall\, t \in [t_{00} - \delta_1, \theta], \quad \theta = \min\{t_1, t_{10}\}$$

holds for any $\vartheta \in V(\vartheta_0; K_1, \delta_2, \alpha)$.

1.13. *For an arbitrary $\varepsilon > 0$, there exists a number $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ such that the inequality*

$$\int\limits_{t_{00} - \delta_1}^{\widehat{\tau}_2} |x(t; \vartheta) - x(t; \vartheta_0)|\, dt \leq \varepsilon$$

holds for any $\mu \in V(\mu_0; K_1, \delta_3, \alpha)$.

Here $B_1(\varphi_0; \delta) = \big\{\varphi \in \Phi_1 : \ \|\varphi - \varphi_0\|_{I_2} < \delta\big\}$.

Theorem 1.6 is proved analogously to Theorem 1.2.

1.3 Proof of Theorem 1.2 (on the continuous dependence of a solution for a class of functional differential equations)

To each element $\mu = (t_0, \tau_1, \ldots, \tau_s, x_0, \varphi, f) \in \Lambda$ we assign the functional differential equation

$$\dot{y}(t) = f(t_0, \tau_1, \ldots, \tau_s, \varphi, y)(t) = f\big(t, y(t), h(t_0, \varphi, y)(t - \tau_1), \ldots, h(t_0, \varphi, y)(t - \tau_s)\big) \qquad (1.16)$$

with the initial condition

$$y(t_0) = x_0, \qquad (1.17)$$

where $h : I \times \Phi \times C(I, \mathbb{R}^n) \to \mathrm{PC}(I_1, \mathbb{R}_n)$ is the operator given by the formula

$$h(t_0, \varphi, y)(t) = \begin{cases} \varphi(t) & \text{for } t \in [\widehat{\tau}, t_0), \\ y(t) & \text{for } t \in [t_0, b], \end{cases} \qquad (1.18)$$

and $C(I, \mathbb{R}^n)$ is the space of continuous function $y : I \to \mathbb{R}^n$ equipped with the distance $d(y_1, y_2) = \|y_1 - y_2\|_I$.

Definition 1.4. An absolutely continuous function $y(t) = y(t; \mu) \in O$, $t \in [r_1, r_2] \subset I$, is called a solution of the equation (1.16) with the initial condition (1.17), or a solution corresponding to the element $\mu \in \Lambda$ and defined on $[r_1, r_2]$, if $t_0 \in [r_1, r_2]$, $y(t_0) = x_0$ and it satisfies the equation (1.16) a.e. on the interval $[r_1, r_2]$.

Remark 1.1. Let $y(t; \mu)$, $t \in [r_1, r_2]$, $\mu \in A$, be a solution of the equation (1.16) with the initial condition (1.17). Then, as is easily seen, the function

$$x(t; \mu) = h(t_0, \varphi, y(\,\cdot\,; \mu))(t), \quad t \in [\widehat{\tau}, r_2],$$

is the solution of the equation (1.13) with the initial condition (1.14).

Theorem 1.7. *Let* $y_0(t) = y(t; \mu_0)$, $\mu_0 \in A$, *be a solution defined on* $[r_1, r_2] \subset (a, b)$, *and let* $K_1 \subset O$ *be a compact set containing a certain neighborhood of the set* $K_0 = \mathrm{cl}\,\varphi_0(I_1) \cup y_0([r_1, r_2])$. *Then the following conditions hold:*

1.14. *There exist the numbers* $\delta_i > 0$, $i = 0, 1$, *such that a solution* $y(t; \mu)$ *defined on* $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ *corresponds to each element*

$$\mu = (t_0, \tau_1, \ldots, \tau_s, x_0, \varphi, f_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha).$$

Moreover,
$$\varphi(t) \in K_1, \quad t \in I_1; \quad y(t; \mu) \in K_1, \quad t \in [r_1 - \delta_1, r_2 + \delta_1],$$
for arbitrary $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$.

1.15. *For an arbitrary* $\varepsilon > 0$, *there exists a number* $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0]$ *such that the inequality*

$$|y(t; \mu) - y(t; \mu_0)| \le \varepsilon \ \ \forall\, t \in [r_1 - \delta_1, r_2 + \delta_1] \tag{1.19}$$

holds for any $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$.

Proof. Let $\varepsilon_0 > 0$ be insomuch small that a closed ε_0-neighborhood of the set K_0

$$K(\varepsilon_0) = \left\{ x \in \mathbb{R}^n : \ \exists \widehat{x} \in K_0, \ \ |x - \widehat{x}| \le \varepsilon_0 \right\}$$

lies in int K_1. By Lemma 1.2, there exist a compact set $Q : K_0^{s+1}(\varepsilon_0) \subset Q \subset K_1^{s+1}$ and a continuously differentiable function $\chi : \mathbb{R}^{n(s+1)} \to [0, 1]$ of the form (1.1).

To each element $\mu \in \Lambda$, we assign the functional differential equation

$$\dot{z}(t) = g(t_0, \tau_1, \ldots, \tau_s, \varphi, z)(t) = g\big(t, z(t), h(t_0, \varphi, z)(t - \tau_1), \ldots, h(t_0, \varphi, z)(t - \tau_s)\big) \tag{1.20}$$

with the initial condition

$$z(t_0) = x_0, \tag{1.21}$$

where $g = \chi f$. The function $g(t, x, x, x_1, \ldots, x_s)$ satisfies the conditions (1.3) and (1.4).

The solution of the equation (1.20) with the initial condition (1.21) depends on the parameter

$$\mu \in \Lambda_0 = [a, b] \times [\theta_{11}, \theta_{12}] \times \cdots \times [\theta_{s1}, \theta_{s2}] \times O \times \Phi \times \Big(f_0 + W(K_1, \alpha)\Big) \subset E_\mu.$$

The topology in Λ_0 is inherited from the vector space E_μ.

On the complete metric space $C(I, \mathbb{R}^n)$ we introduce a family

$$F(\,\cdot\,; \mu) : C(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n) \tag{1.22}$$

of mapping depending on the parameter μ by the formula

$$\zeta(t) = \zeta(t; z, \mu) = x_0 + \int\limits_{t_0}^{t} g(t_0, \tau_1, \ldots, \tau_s, \varphi, z)(\xi)\, d\xi.$$

Clearly, every fixed point $z(t; \mu)$, $t \in I$, of the mapping (1.22) is a solution of the equation (1.20) with the initial condition (1.21).

Define the kth iteration $F^k(z; \mu)$ by

$$\zeta_k(t) = \zeta_k(t; z, \mu) = x_0 + \int_{t_0}^t g(t_0, \tau_1, \ldots, \tau_s, \varphi, \zeta_{k-1})(\xi) \, d\xi, \quad k = 1, 2, \ldots, \quad \zeta_0(t) = z(t).$$

Let us now prove that for a sufficiently large k, the family of mappings $F^k(z; \mu)$ is uniformly contractive. For this purpose, we estimate the difference

$$|\zeta_k'(t) - \zeta_k''(t)| = \left| \zeta_k(t; z', \mu) - \zeta_k(t; z'', \mu) \right|$$

$$\leq \int_a^t \left| g(t_0, \tau_1, \ldots, \tau_s, \varphi, \zeta_{k-1}')(\xi) - g(t_0, \tau_1, \ldots, \tau_s, \varphi, \zeta_{k-1}'')(\xi) \right| d\xi$$

$$\leq \int_a^t L_f(\xi) \left[|\zeta_{k-1}'(\xi) - \zeta_{k-1}''(\xi)| + \sum_{j=1}^s \left| h(t_0, \varphi, \zeta_{k-1}')(t - \tau_j) - h(t_0, \varphi, \zeta_{k-1}'')(t - \tau_j) \right| \right] d\xi, \quad (1.23)$$

$$k = 1, 2, \ldots$$

(see (1.4)), where the function $L_f(\xi)$ is of the form (1.5). Here it is assumed that $\zeta_0' = z'(t)$ and $\zeta_0'' = z''(t)$. It follows from the definition of the operator $h(\,\cdot\,)$ (see (1.18)) that

$$h(t_0, \varphi, \zeta_{k-1}')(\xi - \tau_j) - h(t_0, \varphi, \zeta_{k-1}'')(\xi - \tau_j) = h(t_0, 0, \zeta_{k-1}' - \zeta_{k-1}'')(\xi - \tau_j).$$

Hence, for $\xi \in [a, t_0 + \tau_j)$, we have

$$h(t_0, 0, \zeta_{k-1}' - \zeta_{k-1}'')(\xi - \tau_j) = 0. \quad (1.24)$$

Let $t_0 + \tau_j < b$; then for $\xi \in [t_0 + \tau_j, b]$ we obtain

$$\left| h(t_0, 0, \zeta_{k-1}' - \zeta_{k-1}'')(\xi - \tau_j) \right| = \left| \zeta_{k-1}'(\xi - \tau_j) - \zeta_{k-1}''(\xi - \tau_j) \right|$$

$$\leq \sup \left\{ \left| \zeta_{k-1}'(t - \tau_j) - \zeta_{k-1}''(t - \tau_j) \right| : \; t \in [t_0 + \tau_j, \xi] \right\}$$

$$\leq \sup \left\{ \left| \zeta_{k-1}'(t) - \zeta_{k-1}''(t) \right| : \; t \in [a, \xi] \right\}. \quad (1.25)$$

If $t_0 + \tau_j > b$, then (1.24) holds on the whole interval I. The relation (1.23), together with (1.24) and (1.25), implies that

$$|\zeta_k'(t) - \zeta_k''(t)| \leq \sup \left\{ |\zeta_k'(\xi) - \zeta_k''(\xi)| : \; \xi \in [a, t] \right\}$$

$$\leq (s + 1) \int_a^t L_f(\xi_1) \sup \left\{ |\zeta_{k-1}'(\xi) - \zeta_{k-1}''(\xi)| : \; \xi \in [a, \xi_1] \right\} d\xi_1, \quad k = 1, 2, \ldots.$$

Therefore,

$$|\zeta_k'(t) - \zeta_k''(t)| \leq (s + 1)^2 \int_a^t L_f(\xi_1) \, d\xi_1 \int_a^{\xi_1} L_f(\xi_2) \sup \left\{ |\zeta_{k-2}'(\xi) - \zeta_{k-2}''(\xi)| : \; \xi \in [a, \xi_2] \right\} d\xi_2.$$

Continuing this procedure, we obtain

$$|\zeta_k'(t) - \zeta_k''(t)| \leq (s + 1)^k \alpha_k(t) \|z' - z''\|_I,$$

where

$$\alpha_k(t) = \int_a^t L_f(\xi_1) \, d\xi_1 \int_a^{\xi_1} L_f(\xi_2) \, d\xi_2 \cdots \int_a^{\xi_{k-1}} L_f(\xi_k) \, d\xi_k = \frac{1}{k!} \left(\int_a^t L_f(\xi) \, d\xi \right)^k$$

(see Lemma 1.6). Thus

$$d\big(F^k(z';\mu), F^k(z'';\mu)\big) = \|\zeta'_k - \zeta''_k\|_I \le \frac{(s+1)^k}{k!}\left(\int_a^b L_f(\xi)\,d\xi\right)^k \|z' - z''\|_I = \alpha_k(b)\|z' - z''\|_I.$$

Let us prove the existence of a number $\alpha_2 > 0$ such that

$$\int_I L_f(t)\,dt \le \alpha_2 \quad \forall f \in f_0 + W(K_1;\alpha).$$

Indeed, let $(x, x_1, \ldots, x_s) \in K_1^{s+1}$ and $f \in f_0 + W(K_1;\alpha)$, then

$$|f(t, x, x_1, \ldots, x_2)| \le m_{f_0,K_1}(t) + m_{\delta f,K_1}(t) := m_{f,K_1}(t), \ \ t \in I.$$

Further, let $x'x'_i, x'', x_i'' \in K_1$, $i = \overline{1,s}$, then

$$\big|f(t, x'.x'_1, \ldots, x'_s) - f(t, x'', x''_1, \ldots, x''_s)\big|$$
$$\le \big|f_0(t, x', x'_1, \ldots, x'_s) - f_0(t, x'', x''_1, \ldots, x''_s)\big| + \big|\delta f(t, x', x'_1, \ldots, x'_s) - \delta f(t, x'', x''_1, \ldots, x''_s)\big|$$
$$\le \big(L_{f_0,K_1}(t) + L_{\delta f,K_1}(t)\big)\Big[|x' - x''| + \sum_{i=1}^s |x'_i - x''_i|\Big]$$
$$= L_{f,K_1}(t)\Big[|x' - x''| + \sum_{i=1}^s |x'_i - x''_i|\Big],$$

where $L_{f,K_1}(t) = L_{f_0,K_1}(t) + L_{\delta f,K_1}(t)$.

By (1.5),

$$\int_I L_f(t)\,dt = \int_I \big(L_{f,K_1}(t) + \alpha_0 m_{f,K_1}(t)\big)\,dt$$
$$= \int_I \big[L_{f_0,K_1}(t) + L_{\delta f,K_1}(t) + \alpha_0(m_{f_0,K_1}(t) + m_{\delta f,K_1}(t))\big]\,dt$$
$$\le \alpha(\alpha_0 + 1) + \int_I \big[L_{f_0,K_1}(t) + \alpha_0 m_{f_0,K_1}(t)\big]\,dt := \alpha_2.$$

Taking into account this estimate, we obtain $\alpha_k(b) \le ((s+1)\alpha_2)^k/k!$. Consequently, there exists a positive integer k_1 such that $\alpha_{k_1}(b) < 1$. Therefore, the k_1st iteration of the family (1.22) is contracting. By Theorem 1.1, the mapping (1.22) has a unique fixed point for each μ. Hence it follows that the equation (1.20) with the initial condition (1.21) has a unique solution $z(t;\mu)$, $t \in I$.

Let us prove that the mapping $F^k(z(\,\cdot\,;\mu_0);\,\cdot\,) : \Lambda_0 \to C(I, \mathbb{R}^n)$ is continuous at the point $\mu = \mu_0$ for an arbitrary $k = 1, 2, \ldots$. Towards this end, it suffices to show that if the sequence $\mu_i = (t_{0i}, \tau_{1i}, \ldots, \tau_{si}, x_{0i}, \varphi_i, f_i) \in A_0$, $i = 1, 2, \ldots$, where $f_i = f_0 + \delta f_i$, converges to $\mu_0 = (t_{00}, \tau_{10}, \ldots, \tau_{s0}, x_{00}, \varphi_0, f_0)$, i.e. if

$$\lim_{i \to \infty} \Big(|t_{0i} - t_{00}| + \sum_{j=1}^s |\tau_{ji} - \tau_{j0}| + |x_{0i} - x_{00}| + \|\varphi_i - \varphi_0\|_{1_1} + H_0(\delta f_i; K_1)\Big) = 0,$$

then

$$\lim_{i \to \infty} F^k(z(\,\cdot\,;\mu_0);\mu_i) = F^k(z(\,\cdot\,;\mu_0);\mu_0) = z(\,\cdot\,;\mu_0). \tag{1.26}$$

We now prove the relation (1.26) by induction. Let $k = 1$, then we have

$$|\zeta_1^i(t) - z_0(t)| \le |x_{0i} - x_{00}|$$
$$+ \left|\int_{t_{0i}}^t g_i(t_{0i}, \tau_{1i}, \ldots, \tau_{si}, \varphi_i, z_0)(\xi)\,d\xi - \int_{t_{00}}^t g_0(t_{00}, \tau_{10}, \ldots, \tau_{s0}, \varphi_0, z_0)(\xi)\,d\xi\right| \le \alpha_1^i + \alpha_2^i(t), \tag{1.27}$$

where

$$\zeta_1^i(t) = \zeta_1(t; z_0, \mu_i), \quad z_0(t) = z_0(t; \mu_0), \quad g_i = \chi f_i = g_0 + \delta g_i, \quad g_0 = \chi f_0, \quad \delta g_i = \chi \delta f_i;$$

$$\alpha_1^i = |x_{0i} - x_{00}| + \left| \int_{t_{0i}}^{t_{00}} \left| g_0(t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0, z_0)(\xi) \right| d\xi \right|,$$

$$\alpha_2^i(t) = \left| \int_{t_{0i}}^{t} \left[g_i(t_{0i}, \tau_{1i}, \dots, \tau_{si}, \varphi_i, z_0)(\xi) - g_0(t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0, z_0)(\xi) \right] d\xi \right|.$$

According to (1.3),

$$\alpha_1^i \leq |x_{0i} - x_{00}| + \left| \int_{t_{0i}}^{t_{00}} m_{f_0, K_1}(t) \, dt \right|,$$

therefore,

$$\lim_{i \to \infty} \alpha_1^i = 0. \tag{1.28}$$

After elementary transformation we obtain

$$\alpha_2^i(t) \leq \left| \int_{t_{0i}}^{t} \left[g_0(t_{0i}, \tau_{1i}, \dots, \tau_{si}, \varphi_i, z_0)(\xi) - g_0(t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0, z_0)(\xi) \right] d\xi \right|$$

$$+ \left| \int_{t_0}^{t} \left[\delta g_i(t_{0i}, \tau_{1i}, \dots, \tau_{si}, \varphi_i, z_0)(\xi) - \delta g_i(t_{0i}, \tau_{1i}, \dots, \tau_{si}, \varphi_0, z_0)(\xi) \right] d\xi \right|$$

$$+ \left| \int_{t_{0i}}^{t} \delta g_i(t_{0i}, \tau_{1i}, \dots, \tau_{si}, \varphi_0, z_0)(\xi) \, d\xi \right|$$

$$\leq \sum_{j=1}^{s} (\alpha_{2j}^i + \alpha_{3j}^i) + \alpha_4^i(t), \tag{1.29}$$

where

$$\alpha_{2j}^i = \int_I L_{f_0}(\xi) \left| h(t_{0i}, \varphi_i, z_0)(\xi - \tau_{ji}) - h(t_{00}, \varphi_0, z_0)(\xi - \tau_{j0}) \right| d\xi,$$

$$\alpha_{3j}^i = \int_I L_{\delta f_i}(\xi) \left| h(t_{0i}, \varphi_i, z_0)(\xi - \tau_{ji}) - h(t_{0i}, \varphi_0, z_0)(\xi - \tau_{j0}) \right| d\xi,$$

$$\alpha_4^i(t) = \left| \int_{t_{0i}}^{t} \delta g_i(t_{0i}, \tau_{0i}, \dots, \tau_{si}, \varphi_0, z_0)(\xi) \, d\xi \right|, \quad \delta g_i = g_i - g_0.$$

We now estimate α_{2j}^i, α_{3j}^i and $\alpha_4^i(t)$. We have

$$\alpha_{2j}^i \leq \int_I L_{f_0}(\xi) \left| h(t_{0i}, \varphi_i, z_0)(\xi - \tau_{ji}) - h(t_{0i}, \varphi_0, z_0)(\xi - \tau_{ji}) \right| d\xi$$

$$+ \int_I L_{f_0}(t) \left| h(t_{0i}, \varphi_0, z_0)(\xi - \tau_{ji}) - h(t_{00}, \varphi_0, z_0)(\tau_{j0}(t)) \right| d\xi$$

$$\leq \int_I L_{f_0}(\xi) \left| h(t_{0i}, \varphi_i - \varphi_0, 0)(\xi - \tau_{ji}) \right| d\xi$$

$$+ \int_I L_{f_0}(\xi)\big|h(t_{0i},\varphi_0,z_0)(\xi - \tau_{ji}) - h(t_{00},\varphi_0,z_0)(\xi - \tau_{ji})\big|\, d\xi$$

$$+ \int_I L_{f_0}(\xi)\big|h(t_{00},\varphi_0,z_0)(\xi - \tau_{ji}) - h(t_{00},\varphi_0,z_0)(\xi - \tau_{j0})\big|\, d\xi$$

$$\leq \|\varphi_i - \varphi_0\|_{I_1} \int_I L_{f_0}(\xi)\, d\xi + \alpha_{21j}^i + \alpha_{22j}^i.$$

Introduce the notation

$$\xi_{0ji} = \min\{t_{00} + \tau_{ji}, t_{0i} + \tau_{ji}\}, \quad \xi_{1ji} = \max\{t_{00} + \tau_{ji}, t_{0i} + \tau_{ji}\}.$$

It is easy to see that

$$\alpha_{21j}^i = \int_{\xi_{0ji}}^{\xi_{1ji}} L_{f_0}(\xi)\big|h(t_{0i},\varphi_0,z_0)(\xi - \tau_{ji}) - h(t_{00},\varphi_0,z_0)(\xi - \tau_{ji})\big|\, d\xi$$

and

$$\lim_{i\to\infty}(\xi_{1ji} - \xi_{0ji}) = 0.$$

Consequently, $\alpha_{21j}^i \to 0$.

Introduce the notation

$$\nu_{0ji} = \min\{t_{00} + \tau_{ji}, t_{00} + \tau_{j0}\}, \quad \nu_{1ji} = \max\{t_{00} + \tau_{ji}, t_{00} + \tau_{j0}\}.$$

For α_{22j}^i, we have

$$\alpha_{22j}^i = \int_{\nu_{0ji}}^{\nu_{1ji}} L_{f_0}(\xi)\big|h(t_{00},\varphi_0,z_0)(\xi - \tau_{ji}) - h(t_{00},\varphi_0,z_0)(\xi - \tau_{j0})\big|\, d\xi.$$

Thus, $\alpha_{22j}^i \to 0$. Consequently,

$$\alpha_{2j}^i \to 0. \tag{1.30}$$

Further,

$$\alpha_{3j}^i \leq \int_I L_{\delta f_i}(\xi)\big|\varphi_i(\xi - \tau_{ji}) - \varphi_0(\xi - \tau_{ji})\big|\, d\xi \leq \|\varphi_i - \varphi_0\|_{I_1} \int_I L_{\delta f_i}(\xi)\, d\xi \longrightarrow 0. \tag{1.31}$$

We now estimate $\alpha_4^i(t)$. The function $\varphi_0(\xi)$, $\xi \in I_1$, is piecewise-continuous with a finite number of discontinuity points of the first kind, i.e., there exist subintervals (θ_p, θ_{p+1}), $p = \overline{1,m}$, where the function $\varphi_0(\xi)$ is continuous, with

$$\theta_1 = \widehat{\tau}, \ \ \theta_{m+1} = b, \ \ I_1 = \bigcup_{p=1}^{m-1} [\theta_p, \theta_{p+1}) \cup [\theta_m, \theta_{m+1}].$$

On the interval I_1, we define the continuous functions $z_i(\xi)$, $i = \overline{1, m+1}$, as follows:

$$z_1(\xi) = \varphi_{01}(\xi), \dots, z_m(\xi) = \varphi_{0m}(\xi), \quad z_{m+1}(\xi) = \begin{cases} z_0(a), & \xi \in [\widehat{\tau}, a), \\ z_0(\xi), & \xi \in I, \end{cases}$$

where

$$\varphi_{0p}(\xi) = \begin{cases} \varphi_0(\theta_p+), & \xi \in [\widehat{\tau}, \theta_p], \\ \varphi_0(\xi), & \xi \in (\theta_p, \theta_{p+1}), \\ \varphi_0(\theta_{p+1}-), & \xi \in [\theta_{p+1}, b], \end{cases} \quad p = \overline{1,m},$$

One can readily see that $\alpha_4^i(t)$ satisfies the following estimation:

$$
\begin{aligned}
\alpha_4^i(t) &\leq \sum_{m_1=1}^{m+1} \cdots \sum_{m_s=1}^{m+1} \max_{t',t'' \in I} \left| \int_{t'}^{t''} \delta g_i\big(\xi, z_0(\xi), z_{m_1}(\xi - \tau_{1i}), \ldots, z_{m_s}(\xi - \tau_{si})\big)\, d\xi \right| \\
&\leq \sum_{m_1=1}^{m+1} \cdots \sum_{m_s=1}^{m+1} \max_{t',t'' \in I} \left| \int_{t'}^{t''} \delta g_i\big(\xi, z_0(\xi), z_{m_1}(\xi - \tau_{10}), \ldots, z_{m_s}(\xi - \tau_{s0})\big)\, d\xi \right| \\
&\quad + \sum_{m_1=1}^{m+1} \cdots \sum_{m_s=1}^{m+1} \max_{t',t'' \in I} \left| \int_{t'}^{t''} \left| \delta g_i\big(\xi, z_0(\xi), z_{m_1}(\xi - \tau_{1i}), \ldots, z_{m_s}(\xi - \tau_{si})\big) \right. \right. \\
&\qquad\qquad \left. \left. - \delta g_i\big(\xi, z_0(\xi), z_{m_1}(\xi - \tau_{10}), \ldots, z_{m_s}(\xi - \tau_{s0})\big) \right| d\xi \right| \\
&\leq \sum_{m_1=1}^{m+1} \cdots \sum_{m_s=1}^{m+1} \max_{t',t'' \in I} \left| \int_{t'}^{t''} \delta g_i\big(\xi, z_0(\xi), z_{m_1}(\xi - \tau_{10}), \ldots, z_{m_s}(\xi - \tau_{s0})\big)\, d\xi \right| \\
&\quad + \sum_{m_1=1}^{m+1} \cdots \sum_{m_s=1}^{m+1} \int_I L_{\delta f_i, K_1}(\xi) \sum_{j=1}^{s} \left| z_{m_j}(\xi - \tau_{ji}) - z_{m_j}(\xi - \tau_{j0}) \right| d\xi \\
&\leq \sum_{m_1=1}^{m+1} \cdots \sum_{m_s=1}^{m+1} \max_{t',t'' \in I} \left| \int_{t'}^{t''} \delta g_i\big(\xi, z_0(\xi), z_{m_1}(\xi - \tau_{10}), \ldots, z_{m_s}(\xi - \tau_{s0})\big)\, d\xi \right| \\
&\quad + \sum_{m_1=1}^{m+1} \cdots \sum_{m_s=1}^{m+1} \sum_{j=1}^{s} \max_{\xi \in I} \left| z_{m_j}(\xi - \tau_{ji}) - z_{m_j}(\xi - \tau_{j0}) \right| \int_I L_{\delta g_i, K_1}(\xi)\, d\xi. \qquad (1.32)
\end{aligned}
$$

Obviously,
$$
H_0(\delta g_i; K_1) = H_0(\chi \delta f_i; K_1) \leq H_0(\delta f_i; K_1)
$$
(see (1.1)). Since $H_0(\delta f_i; K_1) \to 0$, as $i \to \infty$, we have
$$
\lim_{i \to \infty} H_0(\delta g_i, K_1) = 0.
$$

This allows us to use Lemma 1.5 which, in its turn, implies that

$$
\lim_{i \to \infty} \max_{t',t'' \in I} \left| \int_{t'}^{t''} \delta g_i\big(\xi, z_0(\xi), z_{m_1}(\xi - \tau_{10}), \ldots, z_{m_s}(\xi - \tau_{s0})\big)\, d\xi \right| = 0 \ \ \forall m_k \in \{1, m+1\}, \ \ k = \overline{1, s}.
$$

Moreover, it is clear that
$$
\lim_{i \to \infty} \max_{t \in I} \left| z_{m_j}(\tau_{ji}(\xi)) - z_{m_j}(\tau_{j0}(\xi)) \right| = 0.
$$

The right-hand side of the inequality (1.32) consists of finitely many summands and, therefore,
$$
\lim_{i \to \infty} \alpha_4^i(t) = 0 \qquad (1.33)
$$
uniformly in $t \in I$.

The conditions (1.30), (1.31) and (1.33) yield
$$
\lim_{i \to \infty} \alpha_2^i(t) = 0 \qquad (1.34)
$$
uniformly in $t \in I$ (see (1.29)).

Taking into account (1.28) and (1.34), we see that (1.27) implies
$$
\|\zeta_1^i - z_0\|_I = 0.
$$

The relation (1.26) is proved for $k = 1$.

Let (1.26) hold for a certain $k > 1$; we will prove it for $k + 1$. Elementary transformations yield

$$|\zeta_{k+1}^i(t) - z_0(t)|$$

$$\leq |x_{0i} - x_{00}| + \left| \int\limits_{t_{0i}}^{t} g_i(t_{0i}, \tau_{1i}, \ldots, \tau_{si}, \varphi_i, \zeta_k^i)(\xi)\, d\xi - \int\limits_{t_{00}}^{t} g_0(t_{00}, \tau_{10}, \ldots, \tau_{s0}, \varphi_0, z_0)(\xi)\, d\xi \right|$$

$$\leq |x_{0i} - x_{00}| + \left| \int\limits_{t_{0i}}^{t_{00}} g_0(t_{00}, \tau_{10}, \ldots, \tau_{s0}, \varphi_0, z_0)(\xi)\, d\xi \right|$$

$$+ \left| \int\limits_{t_{0i}}^{t} \left[g_i(t_{0i}, \tau_{1i}, \ldots, \tau_{si}, \varphi_i, z_0)(\xi) - g_0(t_{00}, \tau_{10}, \ldots, \tau_{s0}, \varphi_0, z_0)(\xi) \right] d\xi \right|$$

$$+ \left| \int\limits_{t_{0i}}^{t} \left| g_i(t_{0i}, \tau_{1i}, \ldots, \tau_{si}, \varphi_i, \zeta_k^i)(\xi) - g_i(t_{0i}, \tau_{1i}, \ldots, \tau_{si}, \varphi_i, z_0)(\xi) \right| d\xi \right| = \alpha_1^i + \alpha_2^i(t) + \alpha_{4k}^i.$$

The quantities α_1^i and $\alpha_2^i(t)$ have been estimated previously, and it remains to estimate α_{4k}^i. We have

$$\alpha_{4k}^i \leq \int\limits_{I} L_{f_i}(\xi) \left[|\zeta_k^i(\xi) - z_0(\xi)| + \sum_{j=1}^{s} |h(t_{0i}, 0, \zeta_k^i - z_0)(\xi - \tau_{ji})| \right] d\xi$$

$$\leq (s+1)\|\zeta_k^i - z_0\|_I \int\limits_{I} L_{f_i}(\xi)\, d\xi \leq (s+1)\alpha_2 \|\zeta_k^i - z_0\|_I.$$

Since

$$\lim_{i \to \infty} \|\zeta_k^i - z_0\|_I = 0,$$

it follows that

$$\lim_{i \to \infty} \alpha_{4k}^i = 0. \tag{1.35}$$

According to (1.28), (1.34) and (1.35), we have

$$\lim_{i \to \infty} \|\zeta_{k+1}^i - z_0\|_I = 0.$$

The relation (1.26) is proved for every $k = 1, 2, \ldots$.

Let the number $\delta_1 > 0$ be insomuch small that $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ and $|z(t; \mu_0) - z(r_1; \mu_0)| \leq \varepsilon_0/2$ for $t \in [r_1 - \delta_1, r_1]$ and $|z(t; \mu_0) - z(r_2; \mu_0)| \leq \varepsilon_0/2$ for $t \in [r_2, r_2 + \delta_1]$.

From the uniqueness of the solution $z(t; \mu_0)$ we can conclude that $z(t; \mu_0) = y_0(t)$ for $t \in [r_1, r_2]$. Taking into account the above inequalities, we have

$$\left(z(t; \mu_0), h(t_{00}, \varphi_0, z(\,\cdot\,; \mu_0))(t - \tau_{10}), \ldots, h(t_{00}, \varphi_0, z(\,\cdot\,; \mu_0))(t - \tau_{s0}) \right) \in K^{s+1}\left(\frac{\varepsilon_0}{2} \right) \subset Q,$$

$$t \in [r_1 - \delta_1, r_2 + \delta_1].$$

Hence

$$\chi\left(z_0(t), h(t_{00}, \varphi_0, z(\,\cdot\,; \mu_0))(t - \tau_{10}), \ldots, h(t_{00}, \varphi_0, z(\,\cdot\,; \mu_0))(t - \tau_{s0}) \right) = 1, \quad t \in [r_1 - \delta_1, r_2 + \delta_1],$$

and the function $z(t; \mu_0)$ satisfies the equation

$$\dot{y}(t) = f_0\left(t, z_0(t), h(t_{00}, \varphi, y)(t - \tau_{10}), \ldots, h(t_{00}, \varphi, y)(t - \tau_{s0}) \right), \quad t \in [r_1 - \delta_1, r_2 + \delta_1],$$

and the initial condition

$$y(t_{00}) = x_{00}.$$

Therefore,

$$y(t; \mu_0) = z(t; \mu_0), \quad t \in [r_1 - \delta_1, r_2 + \delta_1].$$

According to the fixed point Theorem 1.1, for $\varepsilon_0/2$ there exists a number $\delta_0 \in (0, \varepsilon_0)$ such that a solution $z(t; \mu)$ satisfying the condition

$$|z(t; \mu) - z(t; \mu_0)| \le \frac{\varepsilon_0}{2}, \quad t \in I,$$

corresponds to each element $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$.

Therefore, for $t \in [r_1 - \delta_1, r_2 + \delta_1]$,

$$z(t; \mu) \in K(\varepsilon_0) \quad \forall \mu \in V(\mu_0; K_1, \delta_0, \alpha).$$

Taking into account that $\varphi(t) \in K(\varepsilon_0)$, we can see that for $t \in [r_1 - \delta_1, r_2 + \delta_1]$,

$$\chi\Big(z(t; \mu), h\big(t_0, \varphi, z(\,\cdot\,; \mu)\big)(t - \tau_1), \ldots, h\big(t_0, \varphi, z(\,\cdot\,; \mu)\big)(t - \tau_s)\Big) = 1 \quad \forall \mu \in V(\mu_0; K_1, \delta_0, \alpha).$$

Hence the function $z(t; \mu)$ satisfies the equation (1.16) and the condition (1.17), i.e.,

$$y(t; \mu) = z(t; \mu) \in K_1, \quad t \in [r_1 - \delta_1, r_2 + \delta_1], \quad \mu \in V(\mu_0; K_1, \delta_0, \alpha). \tag{1.36}$$

The first part of Theorem 1.7 is proved. By Theorem 1.1, for an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that for each $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$,

$$|z(t; \mu) - z(t; \mu_0)| \le \varepsilon, \quad t \in I,$$

whence, using (1.36), we obtain (1.19). $\qquad\square$

Proof of Theorem 1.2. In Theorem 1.7, let $r_1 = t_{00}$ and $r_2 = t_{10}$. Obviously, the solution $x_0(t) = x(t; \mu_0)$ on the interval $[t_{00}, t_{10}]$ satisfies the following equation:

$$\dot{y}(t) = f_0(t_0, \tau_{10}, \ldots, \tau_{s0}, \varphi_0, y)(t).$$

Therefore, in Theorem 1.7, in the capacity of the solution $y_0(t) = y(t; \mu_0)$ we can take the function $x_0(t)$, $t \in [t_{00}, t_{10}]$.

By Theorem 1.7, there exist the numbers $\delta_i > 0$, $i = 0, 1$, and for an arbitrary $\varepsilon > 0$ there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0]$ such that the solution $y(t; \mu)$, $t \in [t_{00} - \delta_1, t_{10} + \delta_1]$, corresponds to each $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$. Moreover, the following conditions hold:

$$\begin{cases} \varphi(t) \in K_1, & t \in I_1; \ y(t; \mu) \in K_1, \\ |y(t : \mu) - y(t; \mu_0)| \le \varepsilon, & t \in [t_{00} - \delta_1, t_{10} + \delta_1], \\ \mu \in V(\mu_0; K_1, \delta_2, \alpha). \end{cases} \tag{1.37}$$

For an arbitrary $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$, the function

$$x(t; \mu) = \begin{cases} \varphi(t), & t \in [\widehat{\tau}, t_0), \\ y(t; \mu), & t \in [t_0, t_1 + \delta_1], \end{cases}$$

is the solution corresponding to μ. Moreover, if $t \in [\theta, t_{10} + \delta_1]$, then $x(t; \mu_0) = y(t; \mu_0)$ and $x(t; \mu) = y(t; \mu)$. Taking into account (1.37), we see that this implies 1.1 and 1.2. It is not difficult to note that for an arbitrary $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$, we have

$$\int_{\widehat{\tau}}^{t_{10}+\delta_1} |x(t; \mu) - x(t; \mu_0)| \, dt = \int_{\widehat{\tau}}^{\theta_0} |\varphi(t) - \varphi_0(t)| \, dt + \int_{\theta_0}^{\theta} |x(t; \mu) - x(t; \mu_0)| \, dt + \int_{\theta}^{t_{10}+\delta_1} |x(t; \mu) - x(t; \mu_0)| \, dt$$

$$\le \|\varphi - \varphi_0\|_{I_1} (b - \widehat{\tau}) + N|t_0 - t_{00}| + \max_{t \in [\theta, t_{10}+\delta_1]} |x(t; \mu) - x(t; \mu_0)|((b - \widehat{\tau}),$$

where $\theta_0 = \min\{t_0, t_{00}\}$, $N = \sup\{|x' - x''| : x', x'' \in K_1\}$.

By 1.1 and 1.2, this inequality implies 1.3. $\qquad\square$

1.4   Proof of Theorem 1.4

To each element $w \in \Lambda_1$ we put in correspondence the functional differential equation

$$\dot{y}(t) = \phi(t_0, \varphi, \tau_1, \ldots, \tau_s, y, u)(t) = \phi\big(t, y(t), h(t_0, \varphi, y)(t - \tau_1), \ldots, h(t_0, \varphi, y)(t - \tau_s), u(t)\big) \quad (1.38)$$

with the initial condition (1.17).

Theorem 1.8. *Let* $y_0(t) = y(t; w_0)$, $w_0 = (t_{00}, \tau_{10}, \ldots, \tau_{s0}, x_{00}, \varphi_0, u_0) \in \Lambda_1$ *be defined on* $[r_1, r_2] \subset (a, b)$ *and let* $K_1 \subset O$ *be a compact set containing a certain neighborhood of the set* $\mathrm{cl}\, \varphi_0(I_1) \cup y_0([r_1, r_2])$. *Then the following conditions hold:*

1.16. *There exist numbers* $\delta_i > 0$, $i = 0, 1$, *such that to each element*

$$w = (t_0, \tau_1, \ldots, \tau_s, x_0, \varphi, u) \in \widehat{V}(w_0; \delta_0)$$
$$= B(t_{00}; \delta_0) \times V(\tau_{10}; \delta_0) \times \cdots \times B(\tau_{s0}; \delta_0) \times B(x_{00}; \delta_0) \times B(\varphi_0; \delta_0) \times V_2(u_0; \delta_0)$$

there corresponds the solution $y(t; w)$ *defined on the interval* $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ *and satisfying the condition* $y(t; w) \in K_1$.

1.17. *For an arbitrary* $\varepsilon > 0$, *there exists a number* $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ *such that the inequality*

$$|y(t; w) - y(t; w_0)| \leq \varepsilon \ \ \forall t \in [r_1 - \delta_1, r_2 + \delta_1]$$

holds for any $w \in \widehat{V}(w_0; \delta_2)$.

Proof. We rewrite the equation (1.38) in the form

$$\dot{y}(t) = \phi_0(t_0, \varphi, \tau_1, \ldots, \tau_s, y)(t) + \delta\phi_u(t_0, \varphi, \tau_1, \ldots, \tau_s, y)(t),$$

where

$$\phi_0(t, x, x_1, \ldots, x_s) = \phi(t, x, x_1, \ldots, x_s, u_0(t)) \in E_f,$$
$$\delta\phi_u(t, x, x_1, \ldots, x_s) = \phi(t, x, x_1, \ldots, x_s, u(t)) - \phi_0(t, x, x_1, \ldots, x_s) \in E_f.$$

Let $\widehat{\delta}_0 > 0$ be a number insomuch small that $B(u_0; \widehat{\delta}_0) \subset \Omega$. There exists a compact set $\widehat{U} \subset U_0$ such that any function from the neighborhood $B(u_0; \widehat{\delta}_0)$ takes its values in $\widehat{U}$.

Let $K \subset O$ be a compact set. There exists a function $L_K(t) \in L_1(I, \mathbb{R}_+)$ such that for almost all $t \in I$, the inequality

$$\left| \phi(t, x', x_1', \ldots, x_s'', u') - \phi(t, x'', x_1', \ldots, x_s'', u'') \right| \leq L_K(t) \left[|x' - x''| + \sum_{i=1}^{s} |x_i' - x_i''| + |u' - u''| \right]$$
$$\forall x', x'' \in K, \ \ \forall x_i', x_i'' \in K, \ \ i = \overline{1, s}, \ \ \forall u', u'' \in \widehat{U}$$

holds. Hence

$$|\delta\phi_u(t, x, x_1, \ldots, x_s)| \leq L_K(t)|u(t) - u_0(t)| \leq \widehat{\delta}_0 L_K(t) \ \ \forall x_i \in K, \ \ i = \overline{1, s}, \ \ \forall u \in B(u_0; \widehat{\delta}_0),$$

$$\left| \delta\phi_u(t, x', x_1', \ldots, x_s') - \delta\phi_u(t, x'', x_1'', \ldots, x_s'') \right| \leq 2L_K(t) \left[|x' - x''| + \sum_{i=1}^{s} |x_i' - x_i''| \right]$$
$$\forall x', x'' \in K, \ \ \forall x_i', x_i'' \in K, \ \ i = \overline{1, s}.$$

It is easy to see that the inclusions $\{\delta\phi_u(t, x, x_1, \ldots, x_s) : \ u \in B(u_0; \delta)\} \subset W(K; \alpha)$ and $\{\delta\phi_u(t, x, x_1, \ldots, x_s) : \ u \in B(u_0; \delta)\} \subset V_{K, \widehat{\delta}_1}$ hold for $\delta \in (0, \widehat{\delta}_0]$, where

$$\alpha = (2 + \widehat{\delta}_0) \int_I L_K(t)\, dt, \quad \widehat{\delta}_1 = \delta \int_I L_f(t)\, dt.$$

We can now apply Theorem 1.7 which, in its turn, proves Theorem 1.8. $\square$

Proof of Theorem 1.4. In Theorem 1.8, let $r_1 = t_{00}$ and $r_2 = t_{10}$. Obviously, the solution $x_0(t) = x(t; w_0)$ satisfies on the interval $[t_{00}, t_{10}]$ the following equation:

$$\dot{y}(t) = \phi(t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0, y, u_0)(t).$$

Therefore, in Theorem 1.8, we can take the function $x_0(t)$, $t \in [t_{00}, t_{10}]$ as the solution $y_0(t) = y(t; w_0)$. Then the proof of the theorem completely coincides with that of Theorem 1.2; for this purpose, it suffices to replace everywhere the element μ by the element w and the set $V(\mu_0; K_1, \delta_0, \alpha)$ by the set $\widehat{V}(w_0; \delta_0)$. □

2 Variation formulas of solutions for equations with the discontinuous initial condition

2.1 Auxiliary assertions

Consider the set of functions $f = (f^1, \dots, f^n)^\top : I \times O^{s+1} \to \mathbb{R}^n$ satisfying the following conditions: for almost all $t \in I$, the function $f(t, \cdot) : O^{s+1} \to \mathbb{R}^n$ is continuously differentiable; for every $(x, x_1, \dots, x_s) \in O^{s+1}$, the functions $f(t, x, x_1, \dots, x_s), f_x(t, \cdot), f_{x_i}(t, \cdot)$, $i = \overline{1, s}$, where $x = (x^1, \dots, x^n)^\top$, $x_i = (x_i^1, \dots, x_i^n)^\top$, are measurable on I; for any such function f and any compact set $K \subset O$, there exists a function $m_{f,K}(t) \in L_1(I, \mathbb{R}_+)$ such that for any $(x, x_1, \dots, x_s) \in K^{s+1}$ and for almost all $t \in I$,

$$|f(t, x, x_1, \dots, x_s)| + |f_x(t, \cdot)| + \sum_{i=1}^{s} |f_{x_i}(t, \cdot)| \le m_{f,K}(t).$$

The classes of such equivalent functions compose a vector space, which will be denoted by $E_f^{(1)}$; these classes are also called the functions and they will likewise be denoted by f.

Lemma 2.1 ([6, p. 80]). *Let $K \subset O$ be a compact set and let $f \in E_f^{(1)}$. Then*

$$\sup \left\{ |f(t, x, x_1, \dots, x_s)| + |f_x(t, \cdot)| + \sum_{i=1}^{s} |f_{x_i}(t, \cdot)| : (x, x_1, \dots, x_s) \in K^{s+1} \right\} \in L_1(I, \mathbb{R}_+).$$

Lemma 2.2. *The inclusion*

$$E_f^{(1)} \subset E_f \tag{2.1}$$

holds.

Proof. Let $f \in E_f^{(1)}$ and let $K_0 \subset O$ be an arbitrary compact set. To prove the inclusion (2.1), it suffices to show that there exists a function $L_{f,K_0}(t) \in L_1(I, \mathbb{R}_+)$ such that for almost all $t \in I$,

$$\left| f(t, x', x'_1, \dots, x'_s) - f(t, x'', x''_1, \dots, x''_s) \right| \le L_{f,K_0}(t) \left\{ |x' - x''| + \sum_{i=1}^{s} |x'_i - x''_i| \right\}$$

$$\forall\, x', x'' \in K_0, \quad \forall\, x'_i, x''_i \in K_0, \quad i = \overline{1, s}.$$

Introduce the function $g = (g^1, \dots, g^n)^\top = \chi f = (\chi f^1, \dots, \chi f^n)^\top$ (see (1.1) and (1.2)). Clearly, for $(x, x_1, \dots, x_s) \notin K_1^{s+1}$, we have

$$|g_x(t, x, x_1, \dots, x_s)| + \sum_{i=1}^{s} |g_{x_i}(t, \cdot)| = 0, \quad i = \overline{1, s}. \tag{2.2}$$

Let $(x, x_1, \dots, x_s) \in K_1^{s+1}$. It is not difficult to see that the relations

$$|g_x| = \Big[\sum_{k,j=1}^{n} |g_{x^j}^k|^2 \Big]^{\frac{1}{2}} \le \sum_{k,j=1}^{n} \left| (\chi f^k)_{x^j} \right|,$$

$$|g_{x_i}| = \Big[\sum_{k,j=1}^{n} |g_{x_i^j}^k|^2 \Big]^{\frac{1}{2}} \le \sum_{k,j=1}^{n} \left| (\chi f^k)_{x_i^j} \right|,$$

where

$$(\chi f^k)_{x^j} = \frac{\partial}{\partial x^j}\,(\chi f^k),$$

are valid. We have

$$|(\chi f^k)_{x^j}| \le |f^k|\,|\chi_{x^j}| + |f^k_{x^j}| \le |f|\,|\chi_{x^j}| + |f_x| \le m_{f,K_1}(t)(\alpha_0 + 1),$$
$$|(\chi f^k)_{x_i^j}| \le |f^k|\,|\chi_{x_i^j}| + |f^k_{x_i^j}| \le |f|\,|\chi_{x_i^j}| + |f_{x_i}| \le m_{f,K_1}(t)(\alpha_0 + 1), \ \ i = \overline{1,s},$$

where

$$\alpha_0 = \sup\Big\{|\chi_x(x,x_1,\dots,x_s)| + \sum_{i=1}^s |\chi_{x_i}(\,\cdot\,)| : \ (x,x_1,\dots,x_s) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n\Big\},$$

$$m_{f,K_1}(t) = \sup\Big\{|f(t,x,x_1,\dots,x_s)| + |f_x(t,\cdot)| + \sum_{i=1}^s f_{x_i}(t,\cdot)| : \ (x,x_1,\dots,x_s) \in K_1^{s+1}\Big\}$$

(see (2.1)). Thus, for $\forall\,(t,x,x_1,\dots,x_s) \in I \times K_1^{s+1}$, we get

$$|g_x(t,x,x_1,\dots,x_s)| + \sum_{i=1}^s |g_{x_i}(t,\,\cdot\,)| \le m_{g,K_1}(t), \ \ i = \overline{1,s}, \tag{2.3}$$

where

$$m_{g,K_1}(t) = n^2(s+1)(\alpha_0 + 1)m_{f,K_1}(t).$$

It is clear that (2.3) is valid for $(t,x,x_1,\dots,x_s) \in I \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$, as well (see (2.2)). Let $(x',x_1',\dots,x_s')$ and $(x'',x_1'',\dots,x_s'')$ be arbitrary points in K_0^{s+1}. Then (see (1.1)) we have

$$\big|f(t,x',x_1',\dots,x_s') - f(t,x'',x_1'',\dots,x_s'')\big| = \big|g(t,x',x_1',\dots,x_s') - g(t,x'',x_1'',\dots,x_s'')\big|$$

$$\le \Big|\int_0^1 \frac{d}{d\theta}\,g\big(t,x''+\theta(x'-x''),x_1''+\theta(x_1'-x_1''),\dots,x_s''+\theta(x_s'-x_s'')\big)\,d\theta\Big|$$

$$\le \int_0^1 \Big[\big|g_x\big(t,x''+\theta(x'-x''),x_1''+\theta(x_1'-x_1''),\dots,x_s''+\theta(x_s'-x_s'')\big)\big|\,|x'-x''|$$

$$+ \sum_{i=1}^s \big|g_{x_i}\big(t,x''+\theta(x'-x''),x_1''+\theta(x_1'-x_1''),\dots,x_s''+\theta(x_s'-x_s'')\big)\big|\,|x_i''-x_i''|\Big]\,d\theta$$

$$\le m_{g,K_1}(t)\Big[|x'-x''| + \sum_{i=1}^s |x_i''-x_i''|\Big].$$

Therefore, as $L_{f,K_0}(t)$ we can take $m_{g,K_1}(t)$.   □

Consider now the linear delay functional differential equation

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^s A_i(t)x_i(t-\tau_i) + f(t) \tag{2.4}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0), \ \ x(t_0) = x_0. \tag{2.5}$$

Here, $A(t)$, $A_i(t)$, $i = \overline{1,s}$, are the integrable matrix functions of dimension $n \times n$; $t_0 \in [a,b)$, $\tau_i \in [\theta_{i1},\theta_{i2}]$, $i = \overline{1,s}$, are fixed numbers; $\varphi \in \mathrm{PC}(I_1,\mathbb{R}^n)$ is a fixed initial function and $x_0 \in \mathbb{R}^n$ is a fixed initial vector.

The equation (2.4) with the initial condition (2.5) has a unique solution $x(t)$ defined on $[\widehat{\tau}, b]$ (see Definition 1.1).

For every $t \in (a, b)$, on the interval $[a, t]$, let us consider the following matrix functional differential equation with the advanced arguments:

$$Y_\xi(\xi; t) = -Y(\xi; t)A(\xi) - \sum_{i=1}^{s} Y(\xi + \tau_i; t)A_i(\xi + \tau_i), \ \ \xi \in [a, t], \tag{2.6}$$

with the initial condition

$$Y(t; t) = \Upsilon, \ \ Y(\xi; t) = \Theta, \ \ \xi \in (t, \widehat{\tau}_1], \tag{2.7}$$

where Υ is the identity matrix and Θ is the zero matrix.

For every $t \in (a, b)$, the equation (2.6) with the discontinuous initial condition (2.7) has a unique solution $Y(\xi; t)$ defined on $[a, \widehat{\tau}_1]$ (see Definition 1.3).

Lemma 2.3 (Cauchy formula). *The solution of the equation* (2.4) *with the initial condition* (2.5) *can be represented on the interval* $[t_0, b]$ *by the following formula:*

$$x(t) = Y(t_0; t)x_0 + \sum_{i=1}^{s} \int_{t_0 - \tau_i}^{t_0} Y(\xi + \tau_i; t)A_i(\xi + \tau_i)\varphi(\xi)\,d\xi + \int_{t_0}^{t} Y(\xi; t)f(\xi)\,d\xi, \tag{2.8}$$

where $Y(\xi; t)$ *is a solution of the equation* (2.6) *with the initial condition* (2.7).

Proof. On the interval $[t_0, t]$, where $\xi \in (t_0, b]$, consider the equation

$$\dot{x}(\xi) = A(\xi)x(\xi) + \sum_{i=1}^{s} A_i(\xi)x_i(\xi - \tau_i) + f(\xi) \tag{2.9}$$

with the initial condition

$$x(\xi) = \varphi(\xi), \ \ \xi \in [\widehat{\tau}, t_0), \ \ x(t_0) = x_0.$$

Multiplying the equation (2.9) by the matrix function $Y(\xi; t)$ and integrating in $\xi \in [t_0, t]$, we obtain

$$\int_{t_0}^{t} Y(\xi; t)\dot{x}(\xi)\,d\xi = \int_{t_0}^{t} Y(\xi; t)\Big[A(\xi)x(\xi) + \sum_{i=1}^{s} A_i(\xi)x(\xi - \tau_i)\Big]\,d\xi + \int_{t_0}^{t} Y(\xi; t)f(\xi)\,d\xi. \tag{2.10}$$

The integration by pats on the left-hand side of (2.10) with regard for $Y(t; t) = \Upsilon$ yields

$$\int_{t_0}^{t} Y(\xi; t)\dot{x}(\xi)\,d\xi = x(t) - Y(t_0; t)x_0 - \int_{t_0}^{t} Y_\xi(\xi; t)x(\xi)\,d\xi. \tag{2.11}$$

Further,

$$\int_{t_0}^{t} Y(\xi; t)A_i(\xi)x(\xi - \tau_i)\,d\xi = \int_{t_0 - \tau_i}^{t - \tau_i} Y(\xi + \tau_i; t)A_i(\xi + \tau_i)x(\xi)\,d\xi$$

$$= \int_{t_0 - \tau_i}^{t_0} Y(\xi + \tau_i; t)A_i(\xi + \tau_i)\varphi(\xi)\,d\xi + \int_{t_0}^{t} Y(\xi + \tau_i; t)A_i(\xi + \tau_i)x(\xi)\,d\xi \tag{2.12}$$

(see (2.7)). Taking into account (2.11) and (2.12), from (2.10) we find that

$$x(t) = Y(t_0; t)x_0 + \sum_{i=1}^{s} \int_{t_0 - \tau_i}^{t_0} Y(\xi + \tau_i; t)A_i(\xi + \tau_i)\varphi(\xi)\,d\xi$$

$$+ \int_{t_0}^{t} \Big[Y_\xi(\xi; t) + A(\xi)x(\xi) + \sum_{i=1}^{s} Y(\xi + \tau_i; t)A_i(\xi + \tau)\Big]x(\xi)\,d\xi + \int_{t_0}^{t} Y(\xi; t)f(\xi)d\xi.$$

$Y(\xi; t)$ satisfies the equation (2.6) and, therefore, the latter relation implies the formula (2.8). $\qquad\square$

Lemma 2.4 (Gronwall–Bellman's inequality)**.** *Let $v(t) \geq 0, t \in [t_0, b]$, be a continuous scalar-valued function, $m(t) \in L_1(I, \mathbb{R}_+)$, and let the inequality*

$$v(t) \leq c + \int\limits_{t_0}^{t} m(\xi)v(\xi)\, d\xi,$$

where $c \geq 0$, hold. Then

$$v(t) \leq c \exp\left(\int\limits_{t_0}^{t} m(\xi)\, d\xi \right), \quad t \in [t_0, b].$$

Lemma 2.5. *Let $t' \in (a, b]$. For an arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that the inequality*

$$|Y(\xi; t') - Y(\xi; t'')| \leq \varepsilon \ \ \forall \xi \in [a, \underline{t}],$$

holds for arbitrary $t'' \in [t' - \delta, t' + \delta] \cap I$, where $\underline{t} = \min\{t', t''\}$.

Lemma 2.5 is a simple consequence of Theorem 1.6.

Lemma 2.6. *The matrix function $Y(\xi; t)$ is continuous on the set $\Pi = \{(\xi, t) : a < \xi < t, \ t \in (a, b)\}$.*

Proof. Let $(\xi, t) \in \Pi$ be a fixed point. There exists $\delta_1 > 0$ such that $\xi + \Delta\xi < \min\{t + \Delta t, t\}$ and $t + \Delta t < b$ for $|\Delta\xi| \leq \delta_1, |\Delta t| \leq \delta_1$, i.e., $(\xi + \Delta\xi, t + \Delta t) \in \Pi$.

Using Lemma 2.5, we see that for each $\varepsilon > 0$, there exists $\delta_2 \in (0, \delta_1)$ such that for arbitrary $\Delta\xi$ and Δt satisfying the conditions $|\Delta\xi| \leq \delta_2$ and $|\Delta t| \leq \delta_2$, the inequality

$$\left| Y(\xi + \Delta\xi; t + \Delta t) - Y(\xi + \Delta\xi; t) \right| \leq \frac{\varepsilon}{2}$$

holds.

On the other hand, the function $Y(\varsigma; t)$ is continuous with respect to $\varsigma \in [a, t]$, i.e., there exists a number $\delta_3 \in (0, \delta_1)$ such that

$$\left| Y(\xi + \Delta\xi; t) - Y(\xi; t) \right| \leq \frac{\varepsilon}{2}, \ \ |\Delta\xi| \leq \delta_3.$$

Hence, for $|\Delta\xi| \leq \delta$, $|\Delta t| \leq \delta$, $\delta = \{\delta_2, \delta_3\}$, we have

$$\left| Y(\xi + \Delta\xi; t + \Delta t) - Y(\xi; t) \right|$$
$$\leq \left| Y(\xi + \Delta\xi; t + \Delta t) - Y(\xi + \Delta\xi; t) \right| + \left| Y(\xi + \Delta\xi; t) - Y(\xi; t) \right| \leq \varepsilon. \qquad \square$$

2.2   Formulation of main results

To each element

$$\mu = (t_0, \tau_1, \ldots, \tau_s, x_0, \varphi, f) \in \Lambda^{(1)} = [a, b) \times [\theta_{11}, \theta_{12}] \times \cdots \times [\theta_{s1}, \theta_{s2}] \times O \times \Phi \times E_f^{(1)}$$

we assign the delay functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_s)\big) \tag{2.13}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0), \ \ x(t_0) = x_0. \tag{2.14}$$

Definition 2.1. Let $\mu = (t_0, \tau_1, \ldots, \tau_s, x_0, \varphi, f) \in \Lambda^{(1)}$. A function $x(t) = x(t; \mu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is called a solution of the equation (2.13) with the initial condition (2.14), or a solution corresponding to the element μ and defined on the interval $[\widehat{\tau}, t_1]$, if it satisfies the condition (2.14) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (2.13) a.e. on $[t_0, t_1]$.

Let $x_0(t)$ be the solution corresponding to the element $\mu_0 = (t_{00}, \tau_{10}, \ldots, \tau_{s0}, x_0, \varphi_0, f_0) \in \Lambda^{(1)}$ and defined on the interval $[\widehat{\tau}, t_{10}]$, where $t_{00}, t_{10} \in (a, b)$, $t_{00} < t_{10}$ and $\tau_{i0} \in (\theta_{1i}, \theta_{2i})$, $i = \overline{1, s}$.

In the space $E_{\delta\mu}^{(1)} = E_\mu^{(1)} - \mu_0$ with the elements $\delta\mu = (\delta t_0, \delta\tau_1, \ldots, \delta\tau_s, \delta x_0, \delta\varphi, \delta f)$, where $E_\mu^{(1)} = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{R}^n \times \mathrm{PC}(I_1, \mathbb{R}^n) \times E_f^{(1)}$, we introduce the set of variations

$$\Im^{(1)} = \left\{ \delta\mu = (\delta t_0, \delta\tau_1, \ldots, \delta\tau_s, \delta x_0, \delta\varphi, \delta f) \in E_{\delta\mu}^{(1)} : \; |\delta t_0| \le \gamma, \; |\delta\tau_i| \le \gamma, \; i = \overline{1, s}, \right.$$

$$\left. |\delta x_0| \le \gamma, \; \|\delta\varphi\|_I \le \gamma, \; \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, \; |\lambda_i| \le \gamma, \; i = \overline{1, k} \right\},$$

where $\gamma > 0$ is a fixed number and $\delta f_i \in E_f - f_0$, $i = \overline{1, k}$, are fixed functions.

There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im^{(1)}$, to the element $\mu_0 + \varepsilon\delta\mu$ there corresponds the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ (see Lemma 2.8).

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$. Therefore, in the sequel, the solution $x_0(t)$ is assumed to be defined on the interval $[\widehat{\tau}, t_{10} + \delta_1]$.

Let us define the increment of the solution $x_0(t) = x(t; \mu_0)$:

$$\Delta x(t) = \Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t) \; \forall (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times \Im^{(1)}. \qquad (2.15)$$

Theorem 2.1. *Let the following conditions hold:*

2.1. $\tau_{s0} > \cdots > \tau_{10}$ *and* $t_{00} + \tau_{s0} < t_{10}$;

2.2. *the function* $\varphi_0(t)$ *is absolutely continuous and* $\dot\varphi_0(t)$ *is bounded;*

2.3. *the function* $f_0(w)$, $w = (t, x, x_1, \ldots, x_s) \in I \times O^{s+1}$, *is bounded;*

2.4. *there exists the finite limit*

$$\lim_{w \to w_0} f_0(w) = f_0^-, \; w \in (a, t_{00}] \times O^{s+1},$$

where $w_0 = (t_{00}, x_{00}, \varphi_0(t_{00} - \tau_{10}), \ldots, \varphi_0(t_{00} - \tau_{s0}))$;

2.5. *there exist the finite limits*

$$\lim_{(w_{1i}, w_{2i}) \to (w_{1i}^0, w_{2i}^0)} [f_0(w_{1i}) - f_0(w_{2i})] = f_{0i},$$

where $w_{1i}, w_{2i} \in (a, b) \times O^{s+1}$, $i = \overline{1, s}$,

$$w_{1i}^0 = \Big(t_{00} + \tau_{i0}, x_0(t_{00} + \tau_{i0}), x_0(t_{00} + \tau_{i0} - \tau_{10}), \ldots, x_0(t_{00} + \tau_{i0} - \tau_{i-10}),$$

$$x_{00}, x_0(t_{00} + \tau_{i0} - \tau_{i+10}), \ldots, x_0(t_{00} + \tau_{i0} - \tau_{s0}) \Big),$$

$$w_{2i}^0 = \Big(t_{00} + \tau_{i0}, x_0(t_{00} + \tau_{i0}), x_0(t_{00} + \tau_{i0} - \tau_{10}), \ldots, x_0(t_{00} + \tau_{i0} - \tau_{i-10}),$$

$$\varphi_0(t_{00}), x_0(t_{00} + \tau_{i0} - \tau_{i+10}), \ldots, x_0(t_{00} + \tau_{i0} - \tau_{s0}) \Big).$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$, *with* $t_{10} - \delta_2 > t_{00} + \tau_{s0}$, *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im_-^{(1)}$, *where* $\Im_-^{(1)} = \{\delta\mu \in \Im^{(1)} : \; \delta t_0 \le 0\}$, *we have*

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu). \qquad (2.16)$$

Here

$$\delta x(t; \delta\mu) = -Y(t_{00}; t) f_0^- \delta t_0 + \beta(t; \delta\mu) \qquad (2.17)$$

and

$$\beta(t;\delta\mu) = Y(t_{00};t)\delta x_0 - \Big[\sum_{i=1}^{s} Y(t_{00}+\tau_{i0};t)f_{0i}\Big]\delta t_0$$

$$-\sum_{i=1}^{s}\Big[Y(t_{00}+\tau_{i0};t)f_{0i} + \int_{t_{00}}^{t} Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\Big]\delta\tau_i$$

$$+\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi + \int_{t_{00}}^{t} Y(\xi;t)\delta f[\xi]\,d\xi,$$

where it is assumed that

$$\int_{t_{00}}^{t} Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi = \int_{t_{00}}^{t_{00}+\tau_{i0}} Y(\xi;t)f_{0x_i}[\xi]\dot{\varphi}_0(\xi-\tau_{i0})\,d\xi + \int_{t_{00}+\tau_{i0}}^{t} Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi.$$

Next, $Y(\xi;t)$ is the $n \times n$-matrix function satisfying the equation

$$Y_\xi(\xi;t) = -Y(\xi;t)f_{0x}[\xi] - \sum_{i=1}^{s} Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}], \quad \xi \in [t_{00},t], \tag{2.18}$$

and the condition

$$Y(t;t) = \Upsilon, \quad Y(\xi;t) = \Theta, \quad \xi > t; \tag{2.19}$$

where

$$f_{0x_i}[\xi] = f_{0x_i}\big(\xi, x_0(\xi), x_0(\xi-\tau_{10}), \ldots, x_0(\xi-\tau_{s0})\big),$$
$$\delta f[\xi] = \delta f\big(\xi, x_0(\xi), x_0(\xi-\tau_{10}), \ldots, x_0(\xi-\tau_{s0})\big).$$

Some comments. The expression (2.17) is called the variation formula of the solution.

The addend

$$-\Big[Y(t_{00};t)f_0^- + \sum_{i=1}^{s} Y(t_{00}+\tau_{i0};t)f_{0i}\Big]\delta t_0$$

in the formula (2.17) is the effect of the discontinuous initial condition (2.14) and perturbation of the initial moment t_{00}.

The addend

$$-\sum_{i=1}^{s}\Big[Y(t_{00}+\tau_{i0};t)f_{0i} + \int_{t_{00}}^{t} Y(\xi;t)f_{0x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\Big]\delta\tau_i$$

in the formula (2.17) is the effect of perturbations of the delays τ_{i0}, $i = \overline{1,s}$. The expression

$$Y(t_{00};t)\delta x_0 + \sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi$$

in the formula (2.17) is the effect of perturbations of the initial vector x_{00} and the initial function $\varphi_0(t)$.

The addend

$$\int_{t_{00}}^{t} Y(\xi;t)\delta f[\xi]\,d\xi$$

in the formula (2.17) is the effect of perturbation of the right-hand side of the equation

$$\dot{x}(t) = f_0\big(t, x(t), x(t - \tau_{10}), \ldots, x(t - \tau_{s0})\big).$$

Next, it is clear that

$$\delta x(t; \delta\mu) = \delta x_0(t; \delta\mu) - \sum_{i=1}^{s} \delta x_i(t; \delta\mu), \ \ t \in [t_{10} - \delta_2, t_{10} + \delta_2],$$

where

$$\delta x_0(t; \delta\mu) = Y(t_{00}; t)[\delta x_0 - f_0^- \delta t_0] + \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}] \delta\varphi(\xi) \, d\xi$$

$$+ \int_{t_{00}}^{t} Y(\xi; t) \Big[-\sum_{i=1}^{s} f_{0x_i}[\xi] \dot{x}_0(\xi - \tau_{i0}) \delta\tau_i + \delta f[\xi] \Big] d\xi,$$

$$\delta x_i(t; \delta\mu) = Y(t_{00} + \tau_{i0}; t) f_{0i}(\delta t_0 + \delta\tau_i).$$

On the basis of the Cauchy formula (see Lemma 2.3), the function

$$\delta x_0(t) = \begin{cases} \delta\varphi(t), & t \in [\widehat{\tau}, t_{00}), \\ \delta x_0(t, \delta\mu), & t \in [t_{00}, t_{10} + \delta_2], \end{cases}$$

is a solution of the equation

$$\dot{\delta x}(t) = f_{0x}[t]\delta x(t) + \sum_{i=1}^{s} f_{0x_i}[t]\delta x(t - \tau_i) - \sum_{i=1}^{s} f_{0x_i}[\xi]\dot{x}_0(\xi - \tau_{i0})\delta\tau_i + \delta f[\xi]$$

with the initial condition

$$\delta x(t) = \delta\varphi(t), \ \ t \in [\widehat{\tau}, t_{00}), \ \ \delta x(t_{00}) = \delta x_0 - f_0[t]\delta t_0,$$

and the function

$$\delta x_i(t) = \begin{cases} 0, & t \in [\widehat{\tau}, t_{00} + \tau_{i0}), \\ Y(t_{00} + \tau_{i0}; t) f_{0i}(\delta t_0 + \delta\tau_i), & t \in [t_{00} + \tau_{i0}, t_{10} + \delta_2], \end{cases}$$

is a solution of the equation

$$\dot{\delta x}(t) = f_{0x}[t]\delta x(t) + \sum_{i=1}^{s} f_{0x_i}[t]\delta x(t - \tau_i)$$

with the initial condition

$$\delta x(t) = 0, \ \ t \in [\widehat{\tau}, t_{00} + \tau_{i0}), \ \ \delta x(t_{00} + \tau_{i0}) = f_{0i}(\delta t_0 + \delta\tau_i).$$

Theorem 2.2. *Let the conditions* 2.1–2.3 *and* 2.5 *of Theorem* 2.1 *hold. Moreover, there exists the finite limit*

$$\lim_{w \to w_0} f_0(w) = f_0^+, \ \ w \in [t_{00}, b) \times O^{s+1}. \tag{2.20}$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$, *with* $t_{10} - \delta_2 > t_{00} + \tau_{s0}$ *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im_+^{(1)}$, *where* $\Im_+^{(1)} = \{\delta\mu \in \Im^{(1)} : \delta t_0 \geq 0\}$, *the formula* (2.16) *holds. Here,*

$$\delta x(t; \delta\mu) = -Y(t_{00}; t) f_0^+ \delta t_0 + \beta(t; \delta\mu).$$

Theorem 2.3. *Let the conditions* 2.1–2.5 *of Theorem* 2.1 *and the condition* (2.20) *hold. Moreover,* $f_0^- = f_0^+ := \widehat{f}_0$. *Then there exist numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$, *with* $t_{10} - \delta_2 > t_{00} + \tau_{s0}$, *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im^{(1)}$, *the formula* (2.16) *holds, where*

$$\delta x(t; \delta\mu) = -Y(t_{00}; t)\widehat{f}_0 \delta t_0 + \beta(t; \delta\mu).$$

Theorem 2.3 is a corollary to Theorems 2.1 and 2.2.

Theorem 2.4. *Let the conditions* 2.1–2.4 *of Theorem* 2.1 *hold. Moreover, there exist the finite limits*

$$\lim_{(w_{1i}, w_{2i}) \to (w_{1i}^0, w_{2i}^0)} \left[f_0(w_{1i}) - f_0(w_{2i}) \right] = f_{0i}^-, \quad w_{1i}, w_{2i} \in (a, t_{00} + \tau_{i0}] \times O^{s+1}, \quad i = \overline{1, s},$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$, *with* $t_{10} - \delta_2 > t_{00} + \tau_{s0}$, *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im_{--}^{(1)}$, *where* $\Im_{--}^{(1)} = \{\delta\mu \in \Im^{(1)} : \delta t_0 \leq 0, \ \delta\tau_i \leq 0, \ i = \overline{1, s}\}$, *the formula* (2.16) *holds. Here*

$$\delta x(t; \delta\mu) = -\left[Y(t_{00}; t)f_0^- + \sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t)f_{0i}^- \right]\delta t_0 - \sum_{i=1}^{s} \left[Y(t_{00} + \tau_{i0}; t)f_{0i}^- \right]\delta\tau_i + \beta_1(t; \delta\mu),$$

where

$$\beta_1(t; \delta\mu) = Y(t_{00}; t)\delta x_0 - \sum_{i=1}^{s} \left[\int_{t_{00}}^{t} Y(\xi; t)f_{0x_i}[\xi]\dot{x}_0(\xi - \tau_{i0}) \, d\xi \right]\delta\tau_i$$

$$+ \sum_{i=1}^{s} \int_{t_{00} - \tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t)f_{0x_i}[\xi + \tau_{i0}]\delta\varphi(\xi) \, d\xi + \int_{t_{00}}^{t} Y(\xi; t)\delta f[\xi] \, d\xi.$$

Theorem 2.4 can be proved by analogy to Theorem 2.1.

Theorem 2.5. *Let the conditions* 2.1–2.3 *of Theorem* 2.1 *and the condition* (2.20) *hold. Moreover, there exist the finite limits*

$$\lim_{(w_{1i}, w_{2i}) \to (w_{1i}^0, w_{2i}^0)} \left[f_0(w_{1i}) - f_0(w_{2i}) \right] = f_{0i}^+, \quad w_{1i}, w_{2i} \in [t_{00} + \tau_{i0}, b) \times O^{s+1}, \quad i = \overline{1, s},$$

Then there exist the numbers $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$, *with* $t_{10} - \delta_2 > t_{00} + \tau_{s0}$, *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im_{++}^{(1)}$, *where* $\Im_{++}^{(1)} = \{\delta\mu \in \Im^{(1)} : \delta t_0 \geq 0, \ \delta\tau_i \geq 0, \ i = \overline{1, s}\}$, *the formula* (2.16) *holds. Here*

$$\delta x(t; \delta\mu) = -\left[Y(t_{00}; t)f_0^+ + \sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t)f_{0i}^+ \right]\delta t_0 - \sum_{i=1}^{s} \left[Y(t_{00} + \tau_{i0}; t)f_{0i}^+ \right]\delta\tau_i + \beta_1(t; \delta\mu).$$

Theorem 2.5 can be proved by analogy to Theorem 2.2.

2.3 Lemma on estimation of the increment of a solution with respect to the variation set $\Im_-^{(1)}$

To each element $\mu = (t_0, \tau_1, \ldots, \tau_s, x_0, \varphi, f) \in \Lambda^{(1)}$, we assign the functional differential equation

$$\dot{y}(t) = f(t_0, \tau_1, \ldots, \tau_s, \varphi, y)(t) \tag{2.21}$$

with the initial condition

$$y(t_0) = x_0 \tag{2.22}$$

(see (1.16)).

Definition 2.2. Let $\mu = (t_0, \tau_1, \ldots, \tau_s, x_0, \varphi, f) \in \Lambda^{(1)}$. An absolutely continuous function $y(t) = y(t; \mu) \in O$, $t \in [r_1, r_2] \subset I$, is called a solution of the equation (2.21) with the initial condition (2.22), or a solution corresponding to the element μ and defined on the interval $[r_1, r_2]$, if $t_0 \in [r_1, r_2]$, $y(t_0) = x_0$, and the function $y(t)$ satisfies the equation (2.21) a.e. on $[r_1, r_2]$.

Remark 2.1. Let $y(t; \mu)$, $t \in [r_1, r_2]$, be a solution corresponding to the element $\mu = (t_0, \tau_1, \ldots, \tau_s, x_0, \varphi, f) \in \Lambda^{(1)}$. Then the function

$$x(t; \mu) = h(t_0, \varphi, y(\,\cdot\,; \mu))(t), \quad t \in [\widehat{\tau}, r_2], \tag{2.23}$$

is a solution of the equation (2.13) with the initial condition (2.14) (see Definition 2.1 and (1.18)).

Lemma 2.7. Let $y_0(t)$ be a solution corresponding to the element $\mu_0 = (t_{00}, \tau_{10}, \ldots, \tau_{s0}, x_0, \varphi_0, f_0) \in \Lambda^{(1)}$ and defined on $[r_1, r_2] \subset (a, b)$; let $t_{00} \in [r_1, r_2)$, $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = \overline{1, s}$, and let $K_1 \subset O$ be a compact set containing a neighborhood of the set $\varphi_0(I_1) \cup y_0([r_1, r_2])$. Then there exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for any $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im^{(1)}$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda^{(1)}$. In addition, to this element there corresponds a solution $y(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[r_1 - \delta_1, r_2 + \delta_1] \subset I$. Moreover,

$$\begin{cases} \varphi(t) := \varphi_0(t) + \varepsilon\delta\varphi(t) \in K_1, & t \in I_1, \\ y(t; \mu_0 + \varepsilon\delta\mu) \in K_1, & t \in [r_1 - \delta_1, r_2 + \delta_1], \end{cases} \tag{2.24}$$

and

$$\lim_{\varepsilon \to 0} y(t; \mu_0 + \varepsilon\delta\mu) = y(t; \mu_0)$$

uniformly for $(t, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times \Im^{(1)}$.

This lemma is a consequence of Theorem 1.7.

Lemma 2.8. Let $x_0(t)$ be a solution corresponding to the element $\mu_0 = (t_{00}, \tau_{10}, \ldots, \tau_{s0}, x_0, \varphi_0, f_0) \in \Lambda^{(1)}$ and defined on $[\widehat{\tau}, t_{10}]$ (see Definition 1.1), let $t_{00}, t_{10} \in (a, b)$, $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = \overline{1, s}$, and let $K_1 \subset O$ be a compact set containing a neighborhood of the set $\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then there exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for any $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im^{(1)}$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda^{(1)}$. In addition, to this element there corresponds a solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$. Moreover,

$$x(t; \mu_0 + \varepsilon\delta\mu) \in K_1, \quad t \in [\widehat{\tau}, t_{10} + \delta_1]. \tag{2.25}$$

It is easy to see that if in Lemma 2.7 one puts $r_1 = t_{00}$, $r_2 = t_{10}$, then $x_0(t) = y_0(t)$, $t \in [t_{00}, t_{10}]$, and

$$x(t; \mu_0 + \varepsilon\delta\mu) = h\big(t_0, \varphi, y(\,\cdot\,; \mu_0 + \varepsilon\delta\mu)\big)(t), \quad (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times \Im^{(1)}$$

(see (2.23)). Thus, Lemma 2.8 is a simple corollary of Lemma 2.6 (see (2.24)).

Due to the uniqueness, the solution $y(t; \mu_0)$ on the interval $[r_1 - \delta_1, r_2 + \delta_1]$ is a continuation of the solution $y_0(t)$. Therefore, we can assume that the solution $y_0(t)$ is defined on the interval $[r_1 - \delta_1, r_2 + \delta_1]$.

Lemma 2.7 allows one to define the increment of the solution $y_0(t) = y(t; \mu_0)$:

$$\begin{aligned} \Delta y(t) = \Delta y(t; \varepsilon\delta\mu) &= y(t; \mu_0 + \varepsilon\delta\mu) - y(t; \mu_0) \\ &= y(t) - y_0(t) \quad \forall (t, \varepsilon, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times (0, \varepsilon_1) \times \Im^{(1)}. \end{aligned} \tag{2.26}$$

Obviously,

$$\lim_{\varepsilon \to 0} \Delta y(t; \varepsilon\delta\mu) = 0 \tag{2.27}$$

uniformly with respect to $(t, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times \Im^{(1)}$ (see Lemma 2.7).

Lemma 2.9. *Let $\tau_{s0} > \cdots > \tau_{10}$ and $t_{00} + \tau_{s0} \leq r_2$. Moreover, the conditions 2.2–2.4 of Theorem 2.1 hold. Then there exists a number $\varepsilon_2 \in (0, \varepsilon_1)$ such that*

$$\max_{t \in [t_{00}, r_2 + \delta_1]} |\Delta y(t)| \leq O(\varepsilon \delta \mu) \tag{2.28}$$

for arbitrary $(\varepsilon, \delta \mu) \in (0, \varepsilon_2) \times \Im_{-}^{(1)}$. Moreover,

$$\Delta y(t_{00}) = \varepsilon[\delta x_0 - f_0^- \delta t_0] + o(\varepsilon \delta \mu). \tag{2.29}$$

Proof. Let $\varepsilon_2 \in (0, \varepsilon_1)$ be insomuch small that for arbitrary $(\varepsilon, \delta \mu) \in (0, \varepsilon_2) \times \Im_{-}^{(1)}$ the inequalities

$$t_0 + \tau_i > t_{00}, \quad i = \overline{1, s}, \tag{2.30}$$

hold, where $t_0 = t_{00} + \varepsilon \delta t_0$, $\tau_i = \tau_{i0} + \varepsilon \delta \tau_i$. On the interval $[t_{00}, r_2 + \delta_1]$, the function $\Delta y(t) = y(t) - y_0(t)$, where $y(t) = y(t; \mu_0 + \varepsilon \delta \mu)$, satisfies the equation

$$\dot{\Delta} y(t) = a(t; \varepsilon \delta \mu) + \varepsilon b(t; \varepsilon \delta \mu), \tag{2.31}$$

where

$$\begin{aligned}
a(t; \varepsilon \delta \mu) &= f_0(t_0, \tau_1, \ldots, \tau_s, \varphi, y_0 + \Delta y)(t) - f_0(t_{00}, \tau_{10}, \ldots, \tau_{s0}, \varphi_0, y_0)(t), \\
b(t; \varepsilon \delta \mu) &= \delta f(t_0, \tau_1, \ldots, \tau_s, \varphi, y_0 + \Delta y)(t).
\end{aligned} \tag{2.32}$$

We rewrite the equation (2.31) in the integral form

$$\Delta y(t) = \Delta y(t_{00}) + \int_{t_{00}}^{t} \left[a(\xi; \varepsilon \delta \mu) + \varepsilon b(\xi; \varepsilon \delta \mu) \right] d\xi.$$

Hence it follows that

$$|\Delta y(t)| \leq |\Delta y(t_{00})| + a_1(t; t_{00}, \varepsilon \delta \mu) + \varepsilon b_1(t_{00}; \varepsilon \delta \mu), \tag{2.33}$$

where

$$a_1(t; t_{00}, \varepsilon \delta \mu) = \int_{t_{00}}^{t} |a(\xi; \varepsilon \delta \mu)| \, d\xi, \quad b_1(t_{00}; \varepsilon \delta \mu) = \int_{t_{00}}^{r_2 + \delta_1} |b(\xi; \varepsilon \delta \mu)| \, d\xi.$$

Let us prove the formula (2.29). We have

$$\Delta y(t_{00}) = y(t_{00}; \mu_0 + \varepsilon \delta \mu) - y_0(t_{00})$$

$$= x_{00} + \varepsilon \delta x_0 + \int_{t_0}^{t_{00}} \left[f_0(t, y_0(t) + \Delta y(t), \varphi(t - \tau_1), \ldots, \varphi(t - \tau_s)) + \varepsilon b(t; \varepsilon \delta \mu) \right] dt - x_{00} \tag{2.34}$$

(see (2.30)). It is clear that if $t \in [t_0, t_{00}]$, then

$$\lim_{\varepsilon \to 0} \big(t, y_0(t) + \Delta y(t), \varphi(t - \tau_1), \ldots, \varphi(t - \tau_s)\big) = \lim_{t \to t_{00}-} \big(t, y_0(t), \varphi_0(t - \tau_{10}), \ldots, \varphi_0(t - \tau_{s0})\big) = w_0$$

(see (2.27)). Consequently,

$$\lim_{\varepsilon \to 0} \sup_{t \in [t_0, t_{00}]} \left| f_0(t, y_0(t) + \Delta y(t), \varphi(t - \tau_1), \ldots, \varphi(t - \tau_s)) - f_0^- \right| = 0.$$

This relation implies that

$$\int_{t_0}^{t_{00}} f_0\big(t, y_0(t) + \Delta y(t), \varphi(t - \tau_1), \ldots, \varphi(t - \tau_s)\big) \, dt$$

$$= -\varepsilon f_0^- \delta t_0 + \int_{t_0}^{t_{00}} \left[f_0(t, y_0(t) + \Delta y(t), \varphi(t - \tau_1), \ldots, \varphi(t - \tau_s)) - f_0^- \right] dt = -\varepsilon f_0^- \delta t_0 + o(\varepsilon \delta \mu). \tag{2.35}$$

Further, we have

$$\int\limits_{t_0}^{t_{00}} |b(t;\varepsilon\delta\mu)|\,dt \le \int\limits_{t_0}^{t_{00}} \sum_{i=1}^{k} |\lambda_i|\,\big|\delta f_i(t_0,\tau_1,\ldots,\tau_s,\varphi,y_0+\Delta y)(t)\big|\,dt \le \gamma \sum_{i=1}^{k} \int\limits_{t_0}^{t_{00}} m_{\delta f_i,K_1}(t)\,dt. \qquad (2.36)$$

From (2.34), by virtue of (2.35), (2.36), we obtain (2.29).

Let us now prove the inequality (2.28). Let

$$\rho_{i,1} = \min\{t_0+\tau_i, t_{00}+\tau_{i0}\}, \quad \rho_{i,2} = \max\{t_{00}+\tau_i, t_{00}+\tau_{i0}\}, \quad i=\overline{1,s}.$$

It is easy to see that

$$\rho_{i,2} \ge \rho_{i,1} > t_{00} \;\text{ and }\; \rho_{i,2} - \rho_{i,1} = O(\varepsilon\delta\mu).$$

Let ε_2 be insomuch small that for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_2) \times \Im_{-}^{(1)}$ the inequalities

$$\rho_{i,1} < \rho_{i+1,1}, \;\; i=\overline{1,s-1}, \;\; \rho_{s,2} < r_2+\delta_1$$

hold. Now we estimate $a_1(t;t_{00},\varepsilon\delta\mu)$, $t \in [t_{00}, r_2+\delta_1]$. Let $t \in [t_{00},\rho_{1,1}]$. Obviously,

$$a_1(t;t_{00},\varepsilon\delta\mu) \le \int\limits_{t_{00}}^{t} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi + \sum_{i=1}^{s} a_{2i}(t;t_{00},\varepsilon\delta\mu), \qquad (2.37)$$

where

$$a_{2i}(t;t_{00},\varepsilon\delta\mu) = \int\limits_{t_{00}}^{t} L_{f_0,K_1}(\xi)\big|h(t_0,\varphi,y_0+\Delta y)(\xi-\tau_i) - h(t_{00},\varphi_0,y_0)(\xi-\tau_{i0})\big|\,d\xi$$

(see Lemma 2.2). It is clear that if $t \in [t_{00},\rho_{11})$, then for $\xi \in [t_{00},t]$ and any $i=\overline{1,s}$, we have $\xi-\tau_i < t_0$ and $\xi-\tau_{i0} < t_{00}$, therefore,

$$a_{2i}(t;t_{00},\varepsilon\delta\mu) = \int\limits_{t_{00}}^{t} L_{f_0,K_1}(\xi)\big|\varphi(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0})\big|\,d\xi$$

$$\le O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{b} L_{f_0,K_1}(\xi)\big|\varphi_0(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0})\big|\,d\xi, \;\; i=\overline{1,s}.$$

The boundedness of the function $\dot\varphi_0(t)$, $t \in I_1$, yields

$$|\varphi_0(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0})| = \left|\int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i} \dot\varphi_0(t)\,dt\right| = O(\varepsilon\delta\mu).$$

Thus, for $t \in [t_{00},\rho_{1,1}]$, we have

$$a_{2i}(t;t_{00},\varepsilon\delta\mu) \le O(\varepsilon\delta\mu), \;\; i=\overline{1,s}.$$

Consequently, for $t \in [t_{00},\rho_{1,1}]$, we get

$$a_1(t;t_{00},\varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\,d\xi. \qquad (2.38)$$

Let $t \in [\rho_{1,1},\rho_{1,2}]$. Then

$$a_1(t;t_{00},\varepsilon\delta\mu) = a_1(\rho_{1,1};t_{00},\varepsilon\delta\mu) + a_1(t;\rho_{1,1},\varepsilon\delta\mu).$$

By the condition of Theorem 2.1, the function $|a(\xi; \varepsilon\delta\mu)|$, $\xi \in [t_{00}, r_2 + \delta_1]$, is bounded, i.e.,

$$\left|a_1(t; \rho_{1,1}, \varepsilon\delta\mu)\right| \leq O(\varepsilon\delta\mu), \quad t \in [\rho_{1,1}, \rho_{1,2}].$$

Therefore, for $t \in [\rho_{1,1}, \rho_{1,2}]$, we have

$$a_1(t; t_{00}, \varepsilon\delta\mu) \leq a_1(\rho_{1,1}; t_{00}, \varepsilon\delta\mu) + O(\varepsilon\delta\mu)$$

$$\leq O(\varepsilon\delta\mu) + \int_{t_{00}}^{\rho_{1,1}} L_{f_0, K_1}(\xi)|\Delta y(\xi)|\, d\xi \leq O(\varepsilon\delta\mu) + \int_{t_{00}}^{t} L_{f_0, K_1}(\xi)|\Delta y(\xi)|\, d\xi.$$

Thus on the interval $[t_{00}, \rho_{1,2}]$, the formula (2.38) is valid.

Let $t \in [\rho_{1,2}, \rho_{2,1}]$, then $t - \tau_1 > t_0$, $t - \tau_{10} > t_{00}$ and $t - \tau_i < t_0$, $t - \tau_{i0} < t_{00}$, $i = \overline{2, s}$. For this case we have

$$a_1(t; t_{00}, \varepsilon\delta\mu) = a_1(\rho_{1,2}; t_{00}, \varepsilon\delta\mu) + a_1(t; \rho_{1,2}, \varepsilon\delta\mu)$$

$$\leq O(\varepsilon\delta\mu) + \int_{t_{00}}^{\rho_{1,2}} L_{f_0, K_1}(\xi)|\Delta y(\xi)|\, d\xi + \int_{\rho_{1,2}}^{t} L_{f_0, K_1}(\xi)|\Delta y(\xi)|\, d\xi + \int_{\rho_{1,2}}^{t} L_{f_0, K_1}(\xi)|\Delta y(\xi - \tau_1)|\, d\xi$$

$$+ \int_{\rho_{1,2}}^{t} L_{f_0, K_1}(\xi)\big|y_0(\xi - \tau_1) - y_0(\xi - \tau_{10})\big|\, d\xi + \sum_{i=2}^{s} a_{2i}(t; \rho_{12}, \varepsilon\delta\mu)$$

(see (2.37)). It is clear that

$$\big|y_0(\xi - \tau_1) - y_0(\xi - \tau_{10})\big| \leq \left| \int_{\xi - \tau_1}^{\xi - \tau_{10}} \big|f_0(t_{00}, \tau_{10}, \ldots, \tau_{s0}, y_0)(t)\big|\, dt \right| \leq O(\varepsilon\delta\mu)$$

and

$$a_{2i}(t; \rho_{1,2}, \varepsilon\delta\mu = \int_{\rho_{1,2}}^{t} L_{f_0, K_1}(\xi)\big|\varphi(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})\big|\, d\xi$$

$$\leq O(\varepsilon\delta\mu) + \int_{t_{00}}^{b} L_{f_0, K_1}(\xi)\big|\varphi_0(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})\big|\, d\xi \leq O(\varepsilon\delta\mu), \quad i = \overline{2, s}.$$

Thus, for $t \in [t_{00}, \rho_{2,1}]$,

$$a_1(t; t_{00}, \varepsilon\delta\mu) \leq O(\varepsilon\delta\mu) + \int_{t_{00}}^{t} L_{f_0, K_1}(\xi)|\Delta y(\xi)|\, d\xi + \int_{\rho_{1,2} - \tau_1}^{t - \tau_1} L_{f_0, K_1}(\xi + \tau_1)|\Delta y(\xi)|\, d\xi$$

$$\leq O(\varepsilon\delta\mu) + \int_{t_{00}}^{t} \Big[L_{f_0, K_1}(\xi) + \chi_1(\xi + \tau_1)L_{f_0, K_1}(\xi + \tau_1)\Big] |\Delta y(\xi)|\, d\xi, \quad \rho_{1,2} - \tau_1 \geq t_{00},$$

where $\chi_1(\xi)$ is the characteristic function of the interval I. Continuing this process for $t \in [t_{00}, \rho_{s,2}]$, we can prove that

$$a_1(t; t_{00}, \varepsilon\delta\mu) \leq O(\varepsilon\delta\mu) + \int_{t_{00}}^{t} \Big[L_{f_0, K_1}(\xi) + \sum_{i=1}^{s-1}(s - i)\chi_1(\xi + \tau_i)L_{f_0, K_1}(\xi + \tau_i)\Big] |\Delta y(\xi)|\, d\xi. \qquad (2.39)$$

Let $t \in [\rho_{s,2}, r_2 + \delta_1]$, then

$$a_1(t; t_{00}, \varepsilon\delta\mu) = a_1(\rho_{s,2}; t_{00}, \varepsilon\delta\mu) + a_1(t; \rho_{s,2}, \varepsilon\delta\mu)$$

$$\leq O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{\rho_{s,2}} \left[L_{f_0,K_1}(\xi) + \sum_{i=1}^{s-1}(s-i)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i) \right] |\Delta y(\xi)|\, d\xi$$

$$+ \int\limits_{\rho_{s,2}}^{t} L_{f_0,K_1}(\xi)\Big[|\Delta y(\xi)| + \sum_{i=1}^{s} \big|y_0(\xi-\tau_i) - y_0(\xi-\tau_{i0})\big| + \sum_{i=1}^{s} |\Delta y(\xi-\tau_i)|\, d\xi \Big]$$

$$\leq O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t} L_{f_0,K_1}(\xi)|\Delta y(\xi)|\, d\xi + \int\limits_{t_{00}}^{t} \Big[\sum_{i=1}^{s-1}(s-i)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i) \Big] |\Delta y(\xi)|\, d\xi$$

$$+ \sum_{i=1}^{s} \int\limits_{\rho_{s,2}-\tau_i}^{t-\tau_i} \chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i)|\Delta y(\xi)|\, d\xi$$

$$\leq O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t} \Big[L_{f_0,K_1}(\xi) + \sum_{i=1}^{s}(s-i+1)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i) \Big] |\Delta y(\xi)|\, d\xi.$$

Consequently, for $t \in [t_{00}, r_2 + \delta_1]$, we have

$$a_1(t; t_{00}, \varepsilon\delta\mu) \leq O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t} \Big[L_{f_0,K_1}(\xi) + \sum_{i=1}^{s}(s-i+1)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i) \Big] |\Delta y(\xi)|\, d\xi \quad (2.40)$$

(see (2.39)). Obviously,

$$b_1(t_{00}, \varepsilon\delta\mu) \leq \gamma \int\limits_{t_{00}}^{r_2+\delta_1} \sum_{i=1}^{k} \big|\delta f_i(t_0, \tau_1, \ldots, \tau_s, y_0 + \Delta y)(t)\big|\, dt \leq \gamma \sum_{i=1}^{k} \int\limits_{I} m_{\delta f_i, K_1}(t)\, dt. \quad (2.41)$$

According to (2.29), (2.40) and (2.41), the inequality (2.33) directly implies

$$|\Delta y(t)| \leq O(\varepsilon\delta\mu)$$
$$+ \int\limits_{t_{00}}^{t} \Big[L_{f_0,K_1}(\xi) + \sum_{i=1}^{s}(s-i+1)\chi_1(\xi+\tau_i)L_{f_0,K_1}(\xi+\tau_i) \Big] |\Delta y(\xi)|\, d\xi, \ \ t \in [t_{00}, r_2 + \delta_1].$$

By the Gronwall–Bellman inequality, from the above we obtain (2.28). $\qquad\square$

2.4 Proof of Theorem 2.1

Let $r_1 = t_{00}$ and $r_2 = t_{10}$ in Lemma 2.8, then

$$x_0(t) = \begin{cases} \varphi_0(t), & t \in [\widehat{\tau}, t_{00}), \\ y_0(t), & t \in [t_{00}, t_{10}], \end{cases}$$

and for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im_{-}^{(1)}$,

$$x(t; \mu_0 + \varepsilon\delta\mu) = \begin{cases} \varphi(t) := \varphi_0(t) + \varepsilon\delta\varphi(t), & t \in [\widehat{\tau}, t_0), \\ y(t; \mu_0 + \varepsilon\delta\mu), & t \in [t_0, t_{10} + \delta_1] \end{cases}$$

(see (2.23)). Note that $\delta\mu \in \Im_{-}^{(1)}$, i.e., $t_0 < t_{00}$, therefore we have

$$
\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t) & \text{for } t \in [\widehat{\tau}, t_0), \\ y(t; \mu_0 + \varepsilon\delta\mu) - \varphi_0(t) & \text{for } t \in [t_0, t_{00}), \\ \Delta y(t) & \text{for } t \in [t_{00}, t_{10} + \delta_1] \end{cases}
$$

(see (2.15) and (2.26)). By Lemma 2.9, we have

$$
|\Delta x(t)| \le O(\varepsilon\delta\mu) \ \ \forall\, (t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_1] \times (0, \varepsilon_2) \times \Im_{-}^{(1)}, \tag{2.42}
$$

$$
\Delta x(t_{00}) = \varepsilon[\delta x_0 - f_0^- \delta t_0] + o(\varepsilon\delta\mu). \tag{2.43}
$$

The function $\Delta x(t)$ satisfies the equation

$$
\dot{\Delta} x(t) = f_0[t, x_0 + \Delta x] + \varepsilon\delta f[t, x_0 + \Delta x] - f_0[t]
$$

$$
= f_{0x}[t]\Delta x(t) + \sum_{i=1}^{s} f_{0x_i}[t]\Delta x(t - \tau_{i0}) + \varepsilon\delta f[t] + \sum_{i=1}^{2} \vartheta_i(t; \varepsilon\delta\mu) \tag{2.44}
$$

on the interval $[t_{00}, t_{10} + \delta_1]$, where

$$
f_0[t, x_0 + \Delta x] = f_0\Big(t, x_0(t) + \Delta x(t), x_0(t - \tau_1) + \Delta x(t - \tau_1), \dots, x_0(t - \tau_s) + \Delta x(t - \tau_s)\Big),
$$

$$
f_0[t] = f_0\big(t, x_0(t), x_0(t - \tau_{10}), \dots, x_0(t - \tau_{s0})\big),
$$

$$
\delta f[t, x_0 + \Delta x] = \delta f\Big(t, x_0(t) + \Delta x(t), x_0(t - \tau_1) + \Delta x(t - \tau_1), \dots, x_0(t - \tau_s) + \Delta x(t - \tau_s)\Big),
$$

$$
\delta f[t] = \delta f\big(t, x_0(t), x_0(t - \tau_1), \dots, x_0(t - \tau_s)\big),
$$

$$
\vartheta_1(t; \varepsilon\delta\mu) = f_0[t, x_0 + \Delta x] - f_0[t] - f_{0x}[t]\Delta x(t) - \sum_{i=1}^{s} f_{0x_i}[t]\Delta x(t - \tau_{i0}), \tag{2.45}
$$

$$
\vartheta_2(t; \varepsilon\delta\mu) = \varepsilon\big[\delta f([t, x_0 + \Delta x]) - \delta f[t]\big]. \tag{2.46}
$$

By using the Cauchy formula, one can represent the solution of the equation (2.44) in the form

$$
\Delta x(t) = Y(t_{00}; t)\Delta x(t_{00}) + \varepsilon \int_{t_{00}}^{t} Y(\xi; t)\delta f[\xi]\, d\xi + \sum_{p=0}^{2} R_p(t; t_{00}, \varepsilon\delta\mu), \ \ t \in [t_{00}, t_{10} + \delta_1], \tag{2.47}
$$

where

$$
\begin{cases} R_0(t; t_{00}, \varepsilon\delta\mu) = \displaystyle\sum_{i=1}^{s} R_{i0}(t; t_{00}, \varepsilon\delta\mu), \\[2ex] R_{i0}(t; t_{00}, \varepsilon\delta\mu) = \displaystyle\int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}]\Delta x(\xi)\, d\xi, \\[2ex] R_p(t; t_{00}, \varepsilon\delta\mu) = \displaystyle\int_{t_{00}}^{t} Y(\xi; t)\vartheta_p(\xi; \varepsilon\delta\mu)\, d\xi, & p = 1, 2, \end{cases} \tag{2.48}
$$

and $Y(\xi; t)$ is the matrix function satisfying the equation (2.18) and the condition (2.19). Let $\delta_2 \in (0, \delta_1)$ be insomuch small that the inequalities

$$
t_{00} - \delta_2 > a, \ \ t_{00} + \tau_{s0} < t_{10} - \delta_2
$$

hold. The function $Y(\xi; t)$ is continuous on the set $\{(\xi, t) : \ \xi \in [t_{00} - \delta_2, t_{00}], \ t \in [t_{10} - \delta_2, t_{10} + \delta_2]\} \subset \Pi$, by Lemma 2.6. Therefore,

$$
Y(t_{00}; t)\Delta x(t_{00}) = \varepsilon Y(t_{00}; t)[\delta x_0 - f_0^- \delta t_0] + o(t; \varepsilon\delta\mu) \tag{2.49}
$$

(see (2.29)). One can readily see that

$$R_{i0}(t; t_{00}, \varepsilon\delta\mu) = \varepsilon \int_{t_{00}-\tau_{i0}}^{t_0} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}] \delta\varphi(\xi) \, d\xi + \int_{t_0}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}] \Delta x(\xi) \, d\xi$$

$$= \varepsilon \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}] \delta\varphi(\xi) \, d\xi + \int_{t_0+\tau_{i0}}^{t_{00}+\tau_{i0}} Y(\xi; t) f_{0x_i}[\xi] \Delta x(\xi - \tau_{i0}) d\xi + o(t; \varepsilon\delta\mu).$$

Thus

$$R_0(t; t_{00}, \varepsilon\delta\mu) = \varepsilon \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}] \delta\varphi(\xi) \, d\xi$$

$$+ \sum_{i=1}^{s} \int_{t_0+\tau_{i0}}^{t_{00}+\tau_{i0}} Y(\xi; t) f_{0x_i}[\xi] \Delta x(\xi - \tau_{i0}) \, d\xi + o(t; \varepsilon\delta\mu). \qquad (2.50)$$

Let
$$\varrho_{i,1} = \min\{t_0 + \tau_i, t_{00} + \tau_{i0}\}, \quad \varrho_{i,2} = \max\{t_0 + \tau_i, t_{00} + \tau_{i0}\}, \quad i = \overline{1,s},$$

and let a number $\varepsilon_2 \in (0, \varepsilon_1)$ be insomuch small that

$$t_{00} < \varrho_{1,1}, \quad \varrho_{i,2} < \varrho_{i+1,1}, \quad i = \overline{1, s-1}, \quad \varrho_{s,2} < t_{10} - \delta_2.$$

For $t \in [t_{10} - \delta_2, t_{10} + \delta_2]$, we have

$$R_1(t; t_{00}, \varepsilon\delta\mu) = \sum_{i=0}^{s} w_i(t; \varepsilon\delta\mu),$$

where

$$w_0(t; \varepsilon\delta\mu) = \int_{t_{00}}^{t_{00}+\tau_{10}} \vartheta_{11}(\xi; t, \varepsilon\delta\mu) \, d\xi, \quad w_i(t; \varepsilon\delta\mu) = \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_{i+10}} \vartheta_{11}(\xi; t, \varepsilon\delta\mu) \, d\xi, \quad i = \overline{1, s-1},$$

$$w_s(t; \varepsilon\delta\mu) = \int_{t_{00}+\tau_{s0}}^{t} \vartheta_{11}(\xi; t, \varepsilon\delta\mu) \, d\xi, \quad \vartheta_{11}(\xi; t, \varepsilon\delta\mu) = Y(\xi; t)\vartheta_1(\xi; \varepsilon\delta\mu)$$

(see (2.45)). Let $\varrho_{1,1} = t_0 + \tau_1$ and $t_0 + \tau_1 < t_0 + \tau_{10}$, then we have

$$w_0(t; \varepsilon\delta\mu) = w_{01}(t; \varepsilon\delta\mu) + w_{02}(t; \varepsilon\delta\mu).$$

Here,

$$w_{01}(t; \varepsilon\delta\mu) = \int_{t_{00}}^{t_0+\tau_1} \vartheta_{11}(\xi; t, \varepsilon\delta\mu) \, d\xi,$$

$$w_{02}(t; \varepsilon\delta\mu) = \int_{t_0+\tau_1}^{t_{00}+\tau_{10}} Y(\xi; t)\{f_0[\xi, x_0 + \Delta x] - f_0[\xi]\} \, d\xi$$

$$- \int_{t_0+\tau_1}^{t_{00}+\tau_{10}} Y(\xi; t)\Big[f_{0x}[\xi]\Delta x(\xi) + \sum_{i=2}^{s} f_{0x_i}[\xi]\Delta x(\xi - \tau_{i0})\Big] \, d\xi$$

$$- \int\limits_{t_0+\tau_1}^{t_0+\tau_{10}} Y(\xi;t) f_{0x_1}[\xi] \Delta x(\xi - \tau_{10})\, d\xi - \int\limits_{t_0+\tau_{10}}^{t_{00}+\tau_{10}} Y(\xi;t) f_{0x_1}[\xi] \Delta x(\xi - \tau_{10})\, d\xi.$$

We introduce the notations:

$$f_0[\xi;\theta,\varepsilon\delta\mu] = f_0\Big(\xi, x_0(\xi) + \theta\Delta x(\xi), x_0(\xi - \tau_{10}) + \theta\big(x_0(\xi - \tau_1) - x_0(\xi - \tau_{10}) + \Delta x(\xi - \tau_1)\big), \dots,$$

$$x_0(\xi - \tau_{s0}) + \theta\big(x_0(\xi - \tau_s) - x_0(\xi - \tau_{s0}) + \Delta x(\xi - \tau_s)\big)\Big),$$

$$\sigma(\xi;\theta,\varepsilon\delta\mu) = f_{0x}[\xi;\theta,\varepsilon\delta\mu] - f_{0x}[\xi], \quad \sigma_i(\xi;\theta,\varepsilon\delta\mu) = f_{0x_i}[\xi;\theta,\varepsilon\delta\mu] - f_{0x_i}[\xi].$$

It is easy to see that

$$f_0[\xi, x_0 + \Delta x] - f_0[\xi] = \int\limits_0^1 \frac{d}{d\theta}\, f_0[\xi;\theta,\varepsilon\delta\mu]\, d\theta$$

$$= \int\limits_0^1 \left\{ f_{0x}[\xi;\theta,\varepsilon\delta\mu]\Delta x(\xi) + \sum_{i=1}^s f_{0x_i}[\xi;\theta,\varepsilon\delta\mu]\big(x_0(\xi - \tau_i) - x_0(\xi - \tau_{i0}) + \Delta x(\xi - \tau_i)\big) \right\} d\theta$$

$$= \sigma_1(\xi;\varepsilon\delta\mu)\Delta x(\xi) + \sum_{i=1}^s \sigma_{i1}(\xi;\varepsilon\delta\mu)\big(x_0(\xi - \tau_i) - x_0(\xi - \tau_{i0}) + \Delta x(\xi - \tau_i)\big)$$

$$+ f_{0x}[\xi]\Delta x(\xi) + \sum_{i=1}^s f_{0x_i}[\xi]\big(x_0(\xi - \tau_i) - x_0(\xi - \tau_{i0}) + \Delta x(\xi - \tau_i)\big),$$

where

$$\sigma_1(\xi;\varepsilon\delta\mu) = \int\limits_0^1 \sigma(\xi;\theta,\varepsilon\delta\mu)\, d\theta, \quad \sigma_{i1}(\xi;\varepsilon\delta\mu) = \int\limits_0^1 \sigma_i(\xi;\theta,\varepsilon\delta\mu)\, d\theta.$$

Taking into account the last relation for $t \in [t_{10} - \delta_2, t_{10} + \delta_2]$, we have

$$w_{01}(t;\varepsilon\delta\mu) = \sum_{p=1}^4 w_{01}^{(p)}(t;\varepsilon\delta\mu),$$

where

$$w_{01}^{(1)}(t;\varepsilon\delta\mu) = \int\limits_{t_{00}}^{t_0+\tau_1} Y(\xi;t)\sigma_1(\xi;\varepsilon\delta\mu)\Delta x(\xi)\, d\xi,$$

$$w_{01}^{(2)}(t;\varepsilon\delta\mu) = \sum_{i=1}^s \int\limits_{t_{00}}^{t_0+\tau_1} Y(\xi;t)\sigma_{i1}(\xi;\varepsilon\delta\mu)\big[x_0(\xi - \tau_i) - x_0(\xi - \tau_{i0}) + \Delta x(\xi - \tau_i)\big]\, d\xi$$

$$= \sum_{i=1}^s \int\limits_{t_{00}}^{t_0+\tau_1} Y(\xi;t)\sigma_{i1}(\xi;\varepsilon\delta\mu)\big[\varphi_0(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0}) + \varepsilon\delta\varphi(\xi - \tau_i)\big]\, d\xi,$$

$$w_{01}^{(3)}(t;\varepsilon\delta\mu) = \sum_{i=1}^s \int\limits_{t_{00}}^{t_0+\tau_1} Y(\xi;t) f_{0x_i}[\xi]\big[x_0(\xi - \tau_i) - x_0(\xi - \tau_{i0})\big]\, d\xi$$

$$= \sum_{i=1}^s \int\limits_{t_{00}}^{t_0+\tau_1} Y(\xi;t) f_{0x_i}[\xi]\big[\varphi_0(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})\big]\, d\xi,$$

$$w_{01}^{(4)}(t;\varepsilon\delta\mu) = \sum_{i=1}^{s} \int_{t_{00}}^{t_0+\tau_1} Y(\xi;t)f_{0x_i}[\xi]\left[\Delta x(\xi-\tau_i)-\Delta x(\xi-\tau_{i0})\right]d\xi$$

$$= \varepsilon\sum_{i=1}^{s} \int_{t_{00}}^{t_0+\tau_1} Y(\xi;t)f_{0x_i}[\xi]\left[\delta\varphi(\xi-\tau_i)-\delta\varphi(\xi-\tau_{i0})\right]d\xi.$$

The function $\varphi_0(\xi)$, $\xi \in I_1$ is absolutely continuous, therefore for each fixed Lebesgue point $\xi \in (t_{00}, t_{10}+\delta_2)$ of function $\dot{\varphi}_0(\xi-\tau_{i0})$ we get

$$\varphi_0(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0}) = \int_{\xi}^{\xi-\varepsilon\delta\tau_i} \dot{\varphi}_0(\varsigma-\tau_{i0})\,d\varsigma = -\varepsilon\dot{\varphi}_0(\xi-\tau_{i0})\delta\tau_i + \gamma_i(\xi;\varepsilon\delta\mu), \tag{2.51}$$

where

$$\lim_{\varepsilon\to 0}\frac{\gamma_i(\xi;\varepsilon\delta\mu)}{\varepsilon} = 0 \text{ uniformly for } \delta\mu \in \Im_-^{(1)}. \tag{2.52}$$

Thus, (2.51) is valid for almost all points of the interval $(t_{00}, t_{10}+\delta_2)$. From (2.51), taking into account the boundedness of the function $\dot{\varphi}_0(\xi)$, we have

$$\left|\varphi_0(\xi-\tau_i)-\varphi_0(\xi-\tau_{i0})\right| \leq O(\varepsilon\delta\mu) \text{ and } \left|\frac{\gamma_i(\xi;\varepsilon\delta\mu)}{\varepsilon}\right| \leq const. \tag{2.53}$$

According to (2.42) and (2.51), for the expression $w_{01}^{(p)}(t;\varepsilon\delta\mu)$, $p = \overline{1,4}$, we have

$$|w_{01}^{(1)}(t;\varepsilon\delta\mu)| \leq \|Y\|O(\varepsilon\delta\mu)\sigma_1(\varepsilon\delta\mu), \quad |w_{01}^{(2)}(t;\varepsilon\delta\mu)| \leq \|Y\|O(\varepsilon\delta\mu)\sum_{i=1}^{s}\sigma_{i1}(\varepsilon\delta\mu),$$

$$w_{01}^{(3)}(t;\varepsilon\delta\mu) = \sum_{i=1}^{s}\left[\gamma_{i1}(t;\varepsilon\delta\mu) - \varepsilon\left(\int_{t_{00}}^{t_0+\tau_1} Y(\xi;t)f_{0x_i}[\xi]\dot{\varphi}_0(\xi-\tau_{i0})\,d\xi\right)\delta\tau_i\right],$$

$$|w_{01}^{(4)}(t;\varepsilon\delta\mu)| \leq o(\varepsilon\delta\mu)\|Y\|\sum_{i=1}^{s}\int_{t_{00}}^{t_0+\tau_1}|f_{0x_i}[\xi]|\,d\xi.$$

Here

$$\sigma_1(\varepsilon\delta\mu) = \int_{t_{00}}^{t_{00}+\tau_{10}}|\sigma_1(\xi;\varepsilon\delta\mu)|\,d\xi, \quad \sigma_{i1}(\varepsilon\delta\mu) = \int_{t_{00}}^{t_{00}+\tau_{10}}|\sigma_{i1}(\xi;\varepsilon\delta\mu)|\,d\xi,$$

$$\|Y\| = \sup\left\{|Y\xi;t| : (\xi,t)\in\Pi\right\}, \quad \gamma_{i1}(t;\varepsilon\delta\mu) = \int_{t_{00}}^{t} Y(\xi;t)f_{0x_i}[\xi]\gamma_i(\xi;\varepsilon\delta\mu)\,d\xi.$$

Obviously,

$$\left|\frac{\gamma_{i1}(t;\varepsilon\delta\mu)}{\varepsilon}\right| \leq \|Y\|\int_{t_{00}}^{t_{00}+\tau_{10}}|f_{x_i}[\xi]|\left|\frac{\gamma_i(\xi;\varepsilon\delta\mu)}{\varepsilon}\right|d\xi.$$

By the Lebesgue theorem on the passage to the limit under the integral sign, we have

$$\lim_{\varepsilon\to 0}\sigma_1(\varepsilon\delta\mu) = 0, \quad \lim_{\varepsilon\to 0}\sigma_{i1}(\varepsilon\delta\mu) = 0, \quad \lim_{\varepsilon\to 0}\left|\frac{\gamma_{i1}(t;\varepsilon\delta\mu)}{\varepsilon}\right| = 0$$

uniformly for $(t,\delta\mu) \in [t_{00}, t_{00}+\tau_{10}] \times \Im_-^{(1)}$ (see (2.52) and (2.53)). Thus,

$$w_{01}^{(1)}(t;\varepsilon\delta\mu) = w_{01}^{(2)}(t;\varepsilon\delta\mu) = w_{01}^{(4)}(t;\varepsilon\delta\mu) = o(t;\varepsilon\delta\mu),$$

$$w_{01}^{(3)}(t;\varepsilon\delta\mu) = -\varepsilon\sum_{i=1}^{s}\left[\int_{t_{00}}^{t_0+\tau_1} Y(\xi;t)f_{0x_i}[\xi]\dot{\varphi}_0(\xi-\tau_{i0})\,d\xi\right]\delta\tau_i + o(t;\varepsilon\delta\mu). \tag{2.54}$$

Further,

$$\varepsilon \int\limits_{t_0+\tau_1}^{t_{00}+\tau_{10}} Y(\xi;t) f_{0x_i}[\xi] \dot{\varphi}_0(\xi - \tau_{i0}) d\xi = o(t;\varepsilon\delta\mu), \quad \dot{x}_0(\xi - \tau_{i0}) = \dot{\varphi}_0(\xi - \tau_{i0}), \quad \xi \in [t_{00}, t_{00} + \tau_{i0}),$$

therefore,

$$w_{01}^{(3)}(t;\varepsilon\delta\mu) = -\varepsilon \sum_{i=1}^{s} \left[\int\limits_{t_{00}}^{t_{00}+\tau_{10}} Y(\xi;t) f_{0x_i}[\xi] \dot{x}_0(\xi - \tau_{i0}) \, d\xi \right] \delta\tau_i + o(t;\varepsilon\delta\mu). \tag{2.55}$$

On the basis of (2.54) and (2.55), we obtain

$$w_{01}(t;\varepsilon\delta\mu) = -\varepsilon \sum_{i=1}^{s} \left[\int\limits_{t_{00}}^{t_{00}+\tau_{10}} Y(\xi;t) f_{0x_i}[\xi] \dot{x}_0(\xi - \tau_{i0}) \, d\xi \right] \delta\tau_i + o(t;\varepsilon\delta\mu).$$

Let us now transform $w_{02}(t;\varepsilon\delta\mu)$. We have

$$w_{02}(t;\varepsilon\delta\mu) = \int\limits_{t_0+\tau_1}^{t_{00}+\tau_{10}} Y(\xi;t) \big\{ f_0[\xi, x_0 + \Delta x] - f_0[\xi] \big\} \, d\xi - \int\limits_{t_0+\tau_{10}}^{t_{00}+\tau_{10}} Y(\xi;t) f_{0x_1}[\xi] \Delta x(\xi - \tau_{10}) \, d\xi + o(t;\varepsilon\delta\mu).$$

Since for $\xi \in [t_0 + \tau_1, t_{00} + \tau_{10}]$,

$$|\Delta x(\xi)| \le O(\varepsilon\delta\mu), \quad |\Delta x(\xi - \tau_i)| = \varepsilon|\delta\varphi(\xi - \tau_i)|, \quad x_0(\xi - \tau_i) = \varphi_0(\xi - \tau_i), \quad i = \overline{2, s},$$

and

$$x_0(\xi - \tau_1) + \Delta x(\xi - \tau_1) = x(\xi - \tau_1; \mu_0 + \varepsilon\delta\mu) = y(\xi - \tau_1; \mu_0 + \varepsilon\delta\mu) = y_0(\xi - \tau_1) + \Delta y(\xi - \tau_1; \varepsilon\delta\mu),$$

we get

$$\lim_{\varepsilon \to 0} \big(\xi, x_0(\xi) + \Delta x(\xi), x_0(\xi - \tau_1) + \Delta x(\xi - \tau_1), \ldots, x_0(\xi - \tau_s) + \Delta x(\xi - \tau_s) \big)$$

$$= \lim_{\xi \to t_{00}+\tau_{10}-} \big(\xi, x_0(\xi), y_0(\xi - \tau_{10}), x_0(\xi - \tau_{10}), \ldots, x_0(\xi - \tau_{s0}) \big) = w_{1i}^0,$$

$$\lim_{\xi \to t_{00}+\tau_{10}-} \big(\xi, x_0(\xi), x_0(\xi - \tau_{10}), \ldots, x_0(\xi - \tau_{s0}) \big) = w_0,$$

i.e.,

$$\lim_{\varepsilon \to 0} \sup_{\xi \in [t_0+\tau_1, t_{00}+\tau_{10}]} \big[f_0[\xi, x_0 + \Delta x] - f_0[\xi] \big] = f_{01}.$$

Moreover, the function $Y(\xi;t)$ is continuous at the set $[t_{00}, t_{00} + \tau_{10}] \times [t_{00} - \tau_2, t_{10} + \delta_2] \subset \Pi$. Thus,

$$\int\limits_{t_0+\tau_1}^{t_{00}+\tau_{10}} Y(\xi;t) \big\{ f_0[\xi, x_0 + \Delta x] - f_0[\xi] \big\} \, d\xi = -\varepsilon Y(t_{00} + \tau_{10}; t) f_{10}(\delta t_0 + \delta\tau_1) + o(t;\varepsilon\delta\mu).$$

The expression $-\varepsilon Y(t_{00} + \tau_{10}; t) f_{10}(\delta t_0 + \delta\tau_1)$ is the effect of discontinuity. Consequently,

$$w_0(t;\varepsilon\delta\mu) = -\varepsilon \sum_{i=1}^{s} \left[\int\limits_{t_{00}}^{t_{00}+\tau_{10}} Y(\xi;t) f_{0x_i}[\xi] \dot{x}_0(\xi - \tau_{i0}) \, d\xi \right] \delta\tau_i - \varepsilon Y(t_{00} + \tau_{10}; t) f_{10}(\delta t_0 + \delta\tau_1)$$

$$- \int\limits_{t_0+\tau_{10}}^{t_{00}+\tau_{10}} Y(\xi;t) f_{0x_1}[\xi] \Delta x(\xi - \tau_{10}) d\xi + o(t;\varepsilon\delta\mu). \tag{2.56}$$

Let $\varrho_{1,1} = t_0 + \tau_1$ again and $t_0 + \tau_{10} < t_0 + \tau_1$, then we have

$$w_0(t; \varepsilon\delta\mu) = \sum_{k=1}^{2} \widehat{w}_{0k}(t; \varepsilon\delta\mu),$$

where

$$\widehat{w}_{01} = \int_{t_{00}}^{t_0+\tau_{10}} Y(\xi; t)\vartheta_1(\xi; \varepsilon\delta\mu)\, d\xi,$$

$$\widehat{w}_{02} = \int_{t_0+\tau_{10}}^{t_0+\tau_1} Y(\xi; t)\big\{f_0[\xi; x_0 + \Delta x] - f_0[\xi]\big\}\, d\xi + \int_{t_0+\tau_1}^{t_{00}+\tau_{10}} Y(\xi; t)\big\{f_0[\xi; x_0 + \Delta x] - f_0[\xi]\big\}\, d\xi$$

$$- \int_{t_0+\tau_{10}}^{t_{00}+\tau_{10}} Y(\xi; t)\Big\{f_{0x}[\xi]\Delta x(\xi) + \sum_{i=2}^{s} f_{0x_i}[\xi]\Delta x(\xi - \tau_{i0})\Big\}\, d\xi$$

$$- \int_{t_0+\tau_{10}}^{t_{00}+\tau_{10}} Y(\xi; t)f_{x_1}[\xi]\Delta x(\xi - \tau_{10})\, d\xi.$$

For this case the formula (2.56) is valid and can be proved by the scheme described above.

Let $\varrho_{1,1} = t_{00} + \tau_{10}$, i.e., $t_{00} + \tau_{10} < t_0 + \tau_i$. In this case, by analogous transformations can be proved the formula

$$w_0(t; \varepsilon\delta\mu) = -\varepsilon \sum_{i=1}^{s} \left[\int_{t_{00}}^{t_{00}+\tau_{10}} Y(\xi; t)f_{0x_i}[\xi]\dot{x}_0(\xi - \tau_{i0})\, d\xi \right]\delta\tau_i$$

$$- \int_{t_0+\tau_{10}}^{t_{00}+\tau_{10}} Y(\xi; t)f_{0x_1}[\xi]\Delta x(\xi - \tau_{10})\, d\xi + o(t; \varepsilon\delta\mu)$$

without discontinuity effect $-\varepsilon f_{01}(\delta t_0 + \delta\tau_1)$. We notice that this effect appears under transformation of the addend $w_1(t; \varepsilon\delta\mu)$. For $R_1(t; t_{00}, \varepsilon\delta\mu)$, after transformations of $w_i(t; \varepsilon\delta\mu)$, $i = \overline{1, s}$, we obtain

$$R_1(t; t_{00}, \varepsilon\delta\mu) = -\varepsilon \sum_{i=1}^{s} \left[\int_{t_{00}}^{t} Y(\xi; t)f_{0x_i}[\xi]\dot{x}_0(\xi - \tau_{i0})\, d\xi \right]\delta\tau_i - \sum_{i=1}^{s} \int_{t_0+\tau_{i0}}^{t_{00}+\tau_{i0}} Y(\xi; t)f_{0x_1}[\xi]\Delta x(\xi - \tau_{i0})\, d\xi$$

$$- \varepsilon \sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t)f_{i0}(\delta t_0 + \delta\tau_i) + o(t; \varepsilon\delta\mu). \tag{2.57}$$

Finally, let us estimate $R_2(t; t_{00}, \varepsilon\delta\mu)$. We have

$$|R_2(t; t_{00}, \varepsilon\delta\mu)| \le \varepsilon\gamma v(\varepsilon\delta\mu),$$

where

$$v(\varepsilon\delta\mu) = \int_{t_{00}}^{t_{10}+\delta_2} \widehat{L}(t)\Big\{|\Delta x(t)| + \sum_{i=1}^{s} \big[|x_0(t - \tau_i) - x_0(t - \tau_{i0})| + |\Delta x(t - \tau_i)|\big]\Big\}\, dt,$$

$$\widehat{L}(t) = \sum_{j=1}^{s} L_{\delta f_j, K_1}(t)$$

(see (2.46)). It is clear that

$$v(\varepsilon\delta\mu) \leq O(\varepsilon\delta\mu) \int\limits_{t_{00}}^{t_{10}+\delta_2} \widehat{L}(t)\,dt + \sum_{i=1}^{s} \int\limits_{t_{00}}^{\varrho_{i,1}} \widehat{L}(t) \big[|\varphi_0(t-\tau_i) - \varphi_0(t-\tau_{i0})| + \varepsilon|\delta\varphi(t-\tau_i)|\big]\,dt$$

$$+ \sum_{i=1}^{s} \int\limits_{\varrho_{i,1}}^{\varrho_{i,2}} \widehat{L}(t) \big[|x_0(t-\tau_i) - x_0(t-\tau_{i0})| + |\Delta x(t-\tau_i)|\big]\,dt$$

$$+ \sum_{i=1}^{s} \int\limits_{\varrho_{i,2}}^{t_{00}+\tau_{i0}} \widehat{L}(t) \big[|x_0(t-\tau_i) - x_0(t-\tau_{i0})| + O(\varepsilon\delta\mu)\big]\,dt.$$

Further,

$$\varphi_0(t-\tau_i) - \varphi_0(t-\tau_{i0}) = \int\limits_{t-\tau_{i0}}^{t-\tau_i} \dot\varphi_0(\xi)\,d\xi,$$

$$x_0(t-\tau_i) - x_0(t-\tau_{i0}) = \int\limits_{t-\tau_{i0}}^{t-\tau_i} \dot x_0(\xi)\,d\xi = \int\limits_{t-\tau_{i0}}^{t-\tau_i} f_0[\xi]\,d\xi.$$

Taking into account the boundedness of the functions $\dot\varphi_0(\xi)$ and $f_0[\xi]$, we obtain

$$\big|\varphi_0(t-\tau_i) - \varphi_0(t-\tau_{i0})\big| = O(\varepsilon\delta\mu), \quad \big|x_0(t-\tau_i) - x_0(t-\tau_{i0})\big| = O(\varepsilon\delta\mu).$$

Moreover,

$$\big|x_0(t-\tau_i) - x_0(t-\tau_{i0})\big| + |\Delta x(t-\tau_i)|, \quad t \in [\varrho_{i,1}, \varrho_{i,2}],$$

is bounded. From these relations it follows that

$$\lim_{\varepsilon\to 0} v(\varepsilon\delta\mu) = 0$$

uniformly for $\delta\mu \in \Im_-^{(1)}$. Thus,

$$R_2(t; t_{00}, \varepsilon\delta\mu) = o(t; \varepsilon\delta\mu). \tag{2.58}$$

From (2.47), by virtue of (2.50), (2.57) and (2.58), we obtain (2.16), where $\delta x(t; \delta\mu)$ has the form (2.17).

2.5   Lemma on the estimation of the increment of a solution with respect to the variation set $\Im_+^{(1)}$

Lemma 2.10. *Let $\tau_{s0} > \cdots > \tau_{10}$ and $t_{00}+\tau_{s0} \leq r_2$. Moreover, the conditions 2.2–2.3 of Theorem 2.1 and the condition (2.20) hold. Then there exists a number $\varepsilon_2 \in (0, \varepsilon_1)$ such that*

$$\max_{t\in[t_0, r_2+\delta_1]} |\Delta y(t)| \leq O(\varepsilon\delta\mu) \tag{2.59}$$

for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times \Im_+^{(1)}$. Moreover,

$$\Delta y(t_0) = \varepsilon[\delta x_0 - f_0^+ \delta t_0] + o(\varepsilon\delta\mu). \tag{2.60}$$

Proof. Let $\varepsilon_2 \in (0, \varepsilon_1)$ be insomuch small that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times \Im_+^{(1)}$ the inequalities

$$t_{00} + \tau_i > t_0, \quad t_{00} + \tau_{i0} > t_0, \quad i = \overline{1, s}, \tag{2.61}$$

hold. On the interval $[t_0, r_2 + \delta_1]$, the function $\Delta y(t) = y(t) - y_0(t)$ satisfies the equation

$$\dot{\Delta} y(t) = a(t; \varepsilon\delta\mu) + \varepsilon b(t; \varepsilon\delta\mu) \tag{2.62}$$

(see (2.32)). We rewrite the equation (2.62) in the integral form

$$\Delta y(t) = \Delta y(t_0) + \int_{t_0}^{t} \big[a(\xi; \varepsilon\delta\mu) + \varepsilon b(\xi; \varepsilon\delta\mu) \big] \, d\xi.$$

Hence it follows that

$$|\Delta y(t)| \leq |\Delta y(t_0)| + a_1(t; t_0, \varepsilon\delta\mu) + \varepsilon b_1(t_0, \varepsilon\delta\mu). \tag{2.63}$$

Let us prove the formula (2.60). We have

$$\Delta y(t_0) = y(t_0; \mu_0 + \varepsilon\delta\mu) - y_0(t_0) = x_{00} + \varepsilon\delta x_0 - x_{00} - \int_{t_{00}}^{t_0} f_0\big(t, y_0(t), \varphi_0(t-\tau_{10}), \dots, \varphi_0(t-\tau_{s0})\big) \, dt \tag{2.64}$$

(see (2.61)). It is clear that if $t \in [t_{00}, t_0]$, then

$$\lim_{\varepsilon \to 0} \big(t, y_0(t), \varphi_0(t-\tau_{10}), \dots, \varphi(t-\tau_{s0})\big) = \lim_{t \to t_{00}+} \big(t, y_0(t), \varphi_0(t-\tau_{10}), \dots, \varphi_0(t-\tau_{s0})\big) = w_0.$$

Consequently,

$$\lim_{\varepsilon \to 0} \sup_{t \in [t_{00}, t_0]} \big| f_0\big(t, y_0(t), \varphi_0(t-\tau_{10}), \dots, \varphi_0(t-\tau_{s0})\big) - f_0^+ \big| = 0.$$

This relation implies

$$\int_{t_{00}}^{t_0} f_0\big(t, y_0(t), \varphi_0(t-\tau_{10}), \dots, \varphi_0(t-\tau_{s0})\big) \, dt$$

$$= -\varepsilon f_0^+ \delta t_0 + \int_{t_{00}}^{t_0} \big[f_0(t, y_0(t), \varphi_0(t-\tau_{10}), \dots, \varphi_0(t-\tau_{s0})) - f_0^+ \big] \, dt = -\varepsilon f_0^+ \delta t_0 + o(\varepsilon\delta\mu). \tag{2.65}$$

From (2.64), by virtue of (2.65), we obtain (2.60).

Now let us prove the inequality (2.59). Let

$$\rho_{i,1} = \min\{t_{00} + \tau_i, t_{00} + \tau_{i0}\}, \quad \rho_{i,2} = \max\{t_0 + \tau_i, t_{00} + \tau_{i0}\}, \quad i = \overline{1, s}.$$

It is easy to see that $\rho_{i,2} \geq \rho_{i,1} > t_0$ and $\rho_{i,2} - \rho_{i,1} = O(\varepsilon\delta\mu)$. Let ε_2 be insomuch small that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times \mathfrak{S}_+^{(1)}$ the inequalities $\rho_{i,1} < \rho_{i+1,1}$, $i = \overline{1, s-1}$, hold. We now estimate $a_1(t; t_0, \varepsilon\delta\mu)$, $t \in [t_0, r_2 + \delta_1]$. Let $t \in [t_0, \rho_{1,1}]$. Obviously,

$$a_1(t; t_0, \varepsilon\delta\mu) \leq \int_{t_0}^{t} L_{f_0, K_1}(\xi) |\Delta y(\xi)| \, d\xi + \sum_{i=1}^{s} a_{2i}(t; t_0, \varepsilon\delta\mu) \tag{2.66}$$

(see (2.37)). It is clear that if $t \in [t_0, \rho_{1,1})$, then for $\xi \in [t_0, t]$ and for any $i = \overline{1, s}$, we have $\xi - \tau_i < t_0$ and $\xi - \tau_{i0} < t_{00}$, hence,

$$a_{2i}(t; t_0, \varepsilon\delta\mu) = \int_{t_0}^{t} L_{f_0, K_1}(\xi) \big| \varphi(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0}) \big| \, d\xi$$

$$\leq O(\varepsilon\delta\mu) + \int_{t_0}^{b} L_{f_0, K_1}(\xi) \big| \varphi_0(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0}) \big| \, d\xi, \quad i = \overline{1, s}.$$

From the boundedness of the function $\dot\varphi_0(t)$, $t \in I_1$, follows

$$\left|\varphi_0(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})\right| = \left|\int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i} \dot\varphi_0(t)\, dt\right| = O(\varepsilon\delta\mu).$$

Thus, for $t \in [t_0, \rho_{1,1}]$, we have

$$a_{2i}(t; t_0, \varepsilon\delta\mu) \le O(\varepsilon\delta\mu), \quad i = \overline{1, s}.$$

Consequently, for $t \in [t_0, \rho_{1,1}]$, we get

$$a_1(t; t_0, \varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{t} L_{f_0, K_1}(\xi)|\Delta y(\xi)|\, d\xi. \tag{2.67}$$

Let $t \in [\rho_{1,1}, \rho_{1,2}]$, then $a_1(t; t_0, \varepsilon\delta\mu) = a_1(\rho_{1,1}; t_0, \varepsilon\delta\mu) + a_1(t; \rho_{1,1}, \varepsilon\delta\mu)$. The function $|a(\xi; \varepsilon\delta\mu)|$, $\xi \in [t_0, r_2 + \delta_1]$, is bounded, i.e., $|a_1(t; \rho_{1,1}, \varepsilon\delta\mu)| \le O(\varepsilon\delta\mu)$, $t \in [\rho_{1,1}, \rho_{1,2}]$. Therefore, for $t \in [\rho_{i,1}, \rho_{i,2}]$, we have

$$a_1(t; t_0, \varepsilon\delta\mu) \le a_1(\rho_{1,1}; t_0, \varepsilon\delta\mu) + O(\varepsilon\delta\mu)$$

$$\le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{\rho_{1,1}} L_{f_0, K_1}(\xi)|\Delta y(\xi)|\, d\xi \le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{t} L_{f_0, K_1}(\xi)|\Delta y(\xi)|\, d\xi.$$

Thus, on the interval $[t_0, \rho_{1,2}]$, the formula (2.67) is valid. Let $t \in [\rho_{1,2}, \rho_{2,1}]$, then $t - \tau_1 > t_0$, $t - \tau_{10} > t_{00}$ and $t - \tau_i < t_0$, $t - \tau_{i0} < t_{00}$, $i = \overline{2, s}$.

For this case, we have

$$a_1(t; t_0, \varepsilon\delta\mu) = a_1(\rho_{1,2}; t_0, \varepsilon\delta\mu) + a_1(t; \rho_{1,2}, \varepsilon\delta\mu)$$

$$\le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{\rho_{1,2}} L_{f_0, K_1}(\xi)|\Delta y(\xi)|\, d\xi + \int\limits_{\rho_{1,2}}^{t} L_{f_0, K_1}(\xi)|\Delta y(\xi)|\, d\xi + \int\limits_{\rho_{1,2}}^{t} L_{f_0, K_1}(\xi)|\Delta y(\xi - \tau_1)|\, d\xi$$

$$+ \int\limits_{\rho_{1,2}}^{t} L_{f_0, K_1}(\xi)|y_0(\xi - \tau_1) - y_0(\xi - \tau_{10})|\, d\xi + \sum_{i=2}^{s} a_{2i}(t; \rho_{12}, \varepsilon\delta\mu).$$

It is clear that

$$\left|y_0(\xi - \tau_1) - y_0(\xi - \tau_{10})\right| \le \left|\int\limits_{\xi-\tau_1}^{\xi-\tau_{10}} \left|f_0(t_{00}, \tau_{10}, \ldots, \tau_{s0}, y)(t)\right| dt\right| \le O(\varepsilon\delta\mu)$$

and

$$a_{2i}(t; \rho_{1,2}, \varepsilon\delta\mu) = \int\limits_{\rho_{1,2}}^{t} L_{f_0, K_1}(\xi)\left|\varphi(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})\right| d\xi$$

$$\le O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{b} L_{f_0, K_1}(\xi)\left|\varphi_0(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})\right| d\xi \le O(\varepsilon\delta\mu), \quad i = \overline{2, s}.$$

Thus, for $t \in [t_0, \rho_{2,1}]$,

$$a_1(t; t_0, \varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{t} L_{f_0, K_1}(\xi)|\Delta y(\xi)| \, d\xi + \int\limits_{\rho_{1,2} - \tau_1}^{t - \tau_1} L_{f_0, K_1}(\xi + \tau_1)|\Delta y(\xi)| \, d\xi$$

$$\le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{t} \left[L_{f_0, K_1}(\xi) + \chi_1(\xi + \tau_1) L_{f_0, K_1}(\xi + \tau_1) \right] |\Delta y(\xi)| \, d\xi, \ \ \rho_{1,2} - \tau_1 \ge t_0.$$

Continuing this process, we can prove that for $t \in [t_0, \rho_{s,2}]$,

$$a_1(t; t_0, \varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{t} \left[L_{f_0, K_1}(\xi) + \sum_{i=1}^{s-1}(s - i)\chi_1(\xi + \tau_i) L_{f_0, K_1}(\xi + \tau_i) \right] |\Delta y(\xi)| \, d\xi. \qquad (2.68)$$

Let $t \in [\rho_{s,2}, r_2 + \delta_1]$, then

$$a_1(t; t_0, \varepsilon\delta\mu) = a_1(\rho_{s,2}; t_0, \varepsilon\delta\mu) + a_1(t; \rho_{s,2}, \varepsilon\delta\mu)$$

$$\le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{\rho_{s,2}} \left[L_{f_0, K_1}(\xi) + \sum_{i=1}^{s-1}(s - i)\chi_1(\xi + \tau_i) L_{f_0, K_1}(\xi + \tau_i) \right] |\Delta y(\xi)| \, d\xi$$

$$+ \int\limits_{\rho_{s,2}}^{t} L_{f_0, K_1}(\xi) \left[|\Delta y(\xi)| + \sum_{i=1}^{s} |y_0(\xi - \tau_i) - y_0(\xi - \tau_{i0})| + \sum_{i=1}^{s} |\Delta y(\xi - \tau_i)| \right] d\xi$$

$$\le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{t} L_{f_0, K_1}(\xi)|\Delta y(\xi)| \, d\xi + \int\limits_{t_0}^{t} \left[\sum_{i=1}^{s-1}(s - i)\chi_1(\xi + \tau_i) L_{f_0, K_1}(\xi + \tau_i) \right] |\Delta y(\xi)| \, d\xi$$

$$+ \sum_{i=1}^{s} \int\limits_{\rho_{s,2} - \tau_i}^{t - \tau_i} \chi_1(\xi + \tau_i) L_{f_0, K_1}(\xi + \tau_i)|\Delta y(\xi)| \, d\xi$$

$$\le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{t} \left[L_{f_0, K_1}(\xi) + \sum_{i=1}^{s}(s - i + 1)\chi_1(\xi + \tau_i) L_{f_0, K_1}(\xi + \tau_i) \right] |\Delta y(\xi)| \, d\xi.$$

Consequently, for $t \in [t_0, r_2 + \delta_1]$, we have

$$a_1(t; t_0, \varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{t} \left[L_{f_0, K_1}(\xi) + \sum_{i=1}^{s}(s - i + 1)\chi_1(\xi + \tau_i) L_{f_0, K_1}(\xi + \tau_i) \right] |\Delta y(\xi)| \, d\xi \qquad (2.69)$$

(see (2.66) and (2.68)). Obviously,

$$b_1(t_0, \varepsilon\delta\mu) \le \gamma \int\limits_{t_0}^{r_2 + \delta_1} \sum_{i=1}^{k} \left| \delta f_i(t_0, \tau_1, \ldots, \tau_s, y_0 + \Delta y)(t) \right| dt \le \gamma \sum_{i=1}^{k} \int\limits_{I} m_{\delta f_i, K_1}(t) \, dt. \qquad (2.70)$$

According to (2.60), (2.69) and (2.70), the inequality (2.63) directly implies

$$|\Delta y(t)| \le O(\varepsilon\delta\mu) + \int\limits_{t_0}^{t} \left[L_{f_0, K_1}(\xi) + \sum_{i=1}^{s}(s - i + 1)\chi_1(\xi + \tau_i) L_{f_0, K_1}(\xi + \tau_i) \right] |\Delta y(\xi)| \, d\xi, \ \ t \in [t_0, r_2 + \delta_1].$$

By the Gronwall–Bellman inequality, from the above we obtain (2.59). $\qquad \square$

2.6   Proof of Theorem 2.2

First of all, we note that $\delta\mu \in \Im_+^{(1)}$ i.e., $t_{00} < t_0$, therefore we have

$$\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t) & \text{for } t \in [\widehat{\tau}, t_{00}), \\ \varphi(t) - y_0(t) & \text{for } t \in [t_{00}, t_0), \\ \Delta y(t) & \text{for } t \in [t_0, t_{10} + \delta_1]. \end{cases}$$

By Lemma 2.10, we get

$$|\Delta x(t)| \le O(\varepsilon\delta\mu) \ \ \forall (t, \varepsilon, \delta\mu) \in [t_0, t_{10} + \delta_1] \times (0, \varepsilon_2) \times \Im_+^{(1)} \tag{2.71}$$

and

$$\Delta x(t_0) = \varepsilon[\delta x_0 - f_0^+ \delta t_0] + o(\varepsilon\delta\mu). \tag{2.72}$$

The function $\Delta x(t)$ satisfies the equation (2.44) on the interval $[t_0, t_{10} + \delta_1]$; therefore, by using the Cauchy formula, we can represent it in the form

$$\Delta x(t) = Y(t_0; t)\Delta x(t_0) + \varepsilon \int_{t_0}^{t} Y(\xi; t)\delta f[\xi] \, d\xi + \sum_{p=0}^{2} R_p(t; t_0, \varepsilon\delta\mu), \ \ t \in [t_0, t_{10} + \delta_1] \tag{2.73}$$

(see (2.48)). Let $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ be insomuch small that the inequalities

$$t_0 + \tau_i < t_{10} - \delta_2, \ \ i = \overline{1, s}, \ \ t_{00} + \tau_{s0} < t_{10} - \delta_2$$

hold. The function $Y(\xi; t)$ is continuous on the set $[t_{00}, t_{00} + \tau_{s0}] \times [t_{10} - \delta_2, t_{10} + \delta_2] \subset \Pi$. Therefore,

$$Y(t_0; t)\Delta x(t_0) = \varepsilon Y(t_{00}; t)[\delta x_0 - f_0^+ \delta t_0] + o(t; \varepsilon\delta\mu) \tag{2.74}$$

(see (2.72)).

Let us transform $R(t; t_0, \varepsilon\delta\mu)$. It is not difficult to see that

$$R_0(t; t_0, \varepsilon\delta\mu) = \sum_{i=1}^{s} \int_{t_0 - \tau_i}^{t_0} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}]\Delta x(\xi) \, d\xi$$

$$= \sum_{i=1}^{s} \left[\varepsilon \int_{t_0 - \tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}]\delta\varphi(\xi) \, d\xi + \int_{t_{00}}^{t_0} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}]\Delta x(\xi) \, d\xi \right]$$

$$= \varepsilon \sum_{i=1}^{s} \int_{t_{00} - \tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}]\delta\varphi(\xi) \, d\xi$$

$$+ \sum_{i=1}^{s} \int_{t_{00} + \tau_{i0}}^{t_0 + \tau_{i0}} Y(\xi; t) f_{0x_i}[\xi]\Delta x(\xi - \tau_{i0}) \, d\xi + o(t; \varepsilon\delta\mu). \tag{2.75}$$

In a similar way, with nonessential changes, one can prove

$$R_1(t; t_0, \varepsilon\delta\mu) = -\varepsilon \sum_{i=1}^{s} \left[\int_{t_0}^{t} Y(\xi; t) f_{0x_i}[\xi]\dot{x}_0(\xi - \tau_{i0}) \, d\xi \right] \delta\tau_i$$

$$- \sum_{i=1}^{s} \int_{t_{00} + \tau_{i0}}^{t_0 + \tau_{i0}} Y(\xi; t) f_{0x_1}[\xi]\Delta x(\xi - \tau_{i0}) \, d\xi - \varepsilon \sum_{i=1}^{s} f_{i0}(\delta t_0 + \delta\tau_i) + o(t; \varepsilon\delta\mu), \tag{2.76}$$

$$R_2(t; t_0, \varepsilon\delta\mu) = o(t; \varepsilon\delta\mu). \tag{2.77}$$

Finally, note that

$$\varepsilon \int_{t_0}^{t} Y(\xi;t)\delta f[\xi]\,d\xi = \varepsilon \int_{t_{00}}^{t} Y(\xi;t)\delta f[\xi]d\xi + o(t;\varepsilon\delta\mu). \tag{2.78}$$

From (2.73), by virtue of (2.74)–(2.78), we obtain (2.16), where

$$\delta x(t;\delta\mu) = -Y(t_{00};t)f_0^+\delta t_0 + \beta(t;\delta\mu).$$

3 Variation formulas of solutions for equations with the continuous initial condition

3.1 Formulation of the main results

To each element

$$\mu = (t_0,\tau_1,\ldots,\tau_s,\varphi,f) \in \Lambda^{(2)} = [a,b] \times [\theta_{11},\theta_{12}] \times \cdots \times [\theta_{s1},\theta_{s2}] \times \Phi_2 \times E_f^{(1)}$$

we assign the functional differential equation

$$\dot{x}(t) = f\big(t,x(t),x(t-\tau_1),\ldots,x(t-\tau_s)\big) \tag{3.1}$$

with the continuous initial condition

$$x(t) = \varphi(t), \quad t \in [\widehat{\tau},t_0], \tag{3.2}$$

where $\Phi_2 = \{\varphi \in C(I_1,\mathbb{R}^n) : \varphi(t) \in O\}$.

Definition 3.1. Let $\mu = (t_0,\tau_1,\ldots,\tau_s,\varphi,f) \in \Lambda^{(2)}$. A function $x(t) = x(t;\mu) \in O$, $t \in [\widehat{\tau},t_1]$, $t_1 \in (t_0,b]$, is called a solution of the equation (3.1) with the initial condition (3.2), or a solution corresponding to the element μ and defined on the interval $[\widehat{\tau},t_1]$, if it satisfies the condition (3.2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies the equation (3.1) a.e. on $[t_0,t_1]$.

Let $x_0(t)$ be a solution corresponding to a fixed element

$$\mu_0 = (t_{00},\tau_{10},\ldots,\tau_{s0},\varphi_0,f_0) \in \Lambda^{(2)}$$

and defined on the interval $[\widehat{\tau},t_{10}]$, where $t_{00},t_{10} \in (a,b)$, $t_{00} < t_{10}$, and $\tau_{i0} \in (\theta_{1i},\theta_{2i})$, $i = \overline{1,s}$.

In the space $E_{\delta\mu}^{(2)} = E_\mu^{(2)} - \mu_0$ with the elements $\delta\mu = (\delta t_0,\delta\tau_1,\ldots,\delta\tau_s,\delta\varphi,\delta f)$, where $E_\mu^{(2)} = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times C(I_1,\mathbb{R}^n) \times E_f^{(2)}$, we introduce the set of variations

$$\Im^{(2)} = \Big\{\delta\mu = (\delta t_0,\delta\tau_1,\ldots,\delta\tau_s,\delta\varphi,\delta f) : \ |\delta t_0| \le \gamma, \ |\delta\tau_i| \le \gamma, \ i = \overline{1,s},$$

$$\delta\varphi = \sum_{i=1}^{k}\lambda_i\delta\varphi_i, \ \delta f = \sum_{i=1}^{k}\lambda_i\delta f_i, \ |\lambda_i| \le \gamma, \ i = \overline{1,k}\Big\}, \tag{3.3}$$

where $\delta\varphi_i \in \Phi_2 - \varphi_0$, $\delta f_i \in E_f^{(2)} - f_0$, $i = \overline{1,k}$, are fixed functions; $\gamma > 0$ is a fixed number.

There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_1) \times \Im^{(2)}$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda^{(2)}$, and to the element $\mu_0 + \varepsilon\delta\mu$ there corresponds the solution $x(t;\mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau},t_{10} + \delta_1] \subset I_1$.

Due to the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau},t_{10} + \delta_1]$. Therefore, in the sequel, the solution $x_0(t)$ is assumed to be defined on the interval $[\widehat{\tau},t_{10} + \delta_1]$.

Let us define the increment of the solution $x_0(t) = x(t;\mu_0)$:

$$\Delta x(t) = \Delta x(t;\varepsilon\delta\mu) = x(t;\mu_0 + \varepsilon\delta\mu) - x_0(t) \ \forall(t,\varepsilon,\delta\mu) \in [\widehat{\tau},t_{10} + \delta_1] \times (0,\varepsilon_1) \times \Im^{(2)}. \tag{3.4}$$

Theorem 3.1. *Let the function* $\varphi_0(t)$, $t \in I_1$, *be absolutely continuous. Let the functions* $\dot{\varphi}_0(t)$ *and* $f_0(w)$, $w = (t, x, x_1, \ldots, x_s) \in I \times O^{s+1}$ *be bounded. Moreover, there exist the finite limits*

$$\dot{\varphi}_0^- = \dot{\varphi}_0(t_{00}-), \quad \lim_{w \to w_0} f_0(w) = f_0^-, \ \ w \in (a, t_{00}] \times O^{s+1},$$

where

$$w_0 = \big(t_{00}, \varphi_0(t_{00}), \varphi_0(t_{00} - \tau_{10}), \ldots, \varphi_0(t_{00} - \tau_{s0})\big).$$

Then there exist the numbers $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im_-^{(2)}$,

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu) \tag{3.5}$$

where $\Im_-^{(2)} = \{\delta\mu \in \Im^{(2)} : \ \delta t_0 \leq 0\}$ *and*

$$\delta x(t; \delta\mu) = Y(t_{00}; t)(\dot{\varphi}_0^- - f_0^-)\delta t_0 + \beta(t; \delta\mu), \tag{3.6}$$

$$\beta(t; \delta\mu) = Y(t_{00}; t)\delta\varphi(t_{00}) + \sum_{i=1}^s \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t)f_{0x_i}[\xi + \tau_{i0}]\delta\varphi(\xi)\, d\xi$$

$$- \int_{t_{00}}^t Y(\xi; t)\Big[\sum_{i=1}^s f_{0x_i}[\xi]\dot{x}_0(\xi - \tau_{i0})\delta\tau_i\Big]\, d\xi + \int_{t_{00}}^t Y(\xi; t)\delta f[\xi]\, d\xi. \tag{3.7}$$

Here $Y(\xi; t)$ *is the* $n \times n$-*matrix function satisfying the equation*

$$Y_\xi(\xi; t) = -Y(\xi; t)f_{0x}[\xi] - \sum_{i=1}^s Y(\xi + \tau_{i0}; t)f_{0x_i}[\xi + \tau_{i0}], \ \ \xi \in [t_{00}, t], \tag{3.8}$$

and the condition

$$Y(t; t) = \Upsilon, \quad Y(\xi; t) = \Theta, \tag{3.9}$$

where

$$f_{0x_i}[\xi] = f_{0x_i}\big(\xi, x_0(\xi), x_0(\xi - \tau_{10}), \ldots, x_0(\xi - \tau_{s0})\big),$$
$$\delta f[\xi] = \delta f\big(\xi, x_0(\xi), x_0(\xi - \tau_{10}), \ldots, x_0(\xi - \tau_{s0})\big).$$

Some comments. On the basis of the Cauchy formula, we can conclude that the function

$$\delta x(t) = \begin{cases} \delta\varphi(t), & t \in [\hat{\tau}, t_{00}), \\ \delta x(t; \delta\mu), & t \in [t_{00}, t_{10} + \delta_2], \end{cases}$$

is a solution of the equation

$$\dot{\delta x}(t) = f_{0x}[t]\delta x(t) + \sum_{i=1}^s f_{0x_i}[t]\delta x(t - \tau_{i0}) - \sum_{i=1}^s f_{0x_i}[t]\dot{x}_0(t - \tau_{i0})\delta\tau_i + \delta f[t]$$

with the discontinuous initial condition

$$\delta x(t) = \delta\varphi(t), \ \ t \in [\hat{\tau}, t_{00}), \ \ \delta x(t_{00}) = (\dot{\varphi}_0^- - f_0^-)\delta t_0 + \delta\varphi(t_{00}).$$

The addend

$$- \int_{t_{00}}^t Y(\xi; t)\Big[\sum_{i=1}^s f_{0x_i}[\xi]\dot{x}_0(\xi - \tau_{i0})\delta\tau_i\Big]\, d\xi$$

in the formula (3.7) is the effect of perturbations of the delays τ_{i0}, $i = \overline{1, s}$.

The expression

$$Y(t_{00}; t)(\dot{\varphi}_0^- - f_0^-)\delta t_0$$

in the formula (3.6) is the effect of the continuous initial condition (3.2) and perturbation of the initial moment t_{00}.

The expression

$$Y(t_{00}; t)\delta\varphi(t_{00}) + \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(s + \tau_{i0}; t)f_{0x_i}[\xi + \tau_{i0}]\delta\varphi(\xi)\, d\xi$$

in the formula (3.7) is the effect of perturbation of the initial function $\varphi_0(t)$.

The addend

$$\int_{t_{00}}^{t} Y(\xi; t)\delta f[\xi]\, d\xi$$

in (3.7) is the effect of perturbation of the right-hand side of equation

$$\dot{x}(t) = f_0\big(t, x(t), x(t - \tau_{10}), \ldots, x(t - \tau_{s0})\big).$$

Theorem 3.2. *Let the function $\varphi_0(t)$, $t \in I_1$, be absolutely continuous. Let the functions $\dot{\varphi}_0(t)$ and $f_0(w)$, $w = (t, x, x_1, \ldots, x_s) \in I \times O^{s+1}$, be bounded. Moreover, there exist the finite limits*

$$\dot{\varphi}_0^+ = \dot{\varphi}_0(t_{00}+), \quad \lim_{w \to w_0} f_0(w) = f_0^+, \quad w \in [t_{00}, b) \times O^{s+1}.$$

Then for each $\widehat{t}_0 \in (t_{00}, t_{10})$ there exist the numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for arbitrary $(t, \varepsilon, \delta\mu) \in [\widehat{t}_0, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im_+^{(2)}$, where $\Im_+^{(2)} = \{\delta\mu \in \Im^{(2)} : \delta t_0 \geq 0\}$, the formula (3.5) holds, where

$$\delta x(t; \delta\mu) = Y(t_{00}; t)(\dot{\varphi}_0^+ - f_0^+)\delta t_0 + \beta(t; \delta\mu). \tag{3.10}$$

The following assertion is a corollary to Theorems 3.1 and 3.2.

Theorem 3.3. *Let the assumptions of Theorems 3.1 and 3.2 be fulfilled. Moreover, $\dot{\varphi}_0^- - f_0^- = \dot{\varphi}_0^+ - f_0^+ := \widehat{f}_0$. Then, for each $\widehat{t}_0 \in (t_{00}, t_{10})$, there exist the numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for arbitrary $(t, \varepsilon, \delta\mu) \in [\widehat{t}_0, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im^{(2)}$ the formula (3.5) holds, where*

$$\delta x(t; \delta\mu) = Y(t_{00}; t)\widehat{f}_0\delta t_0 + \beta(t; \delta\mu).$$

All the assumptions of Theorem 3.3 are satisfied if the function $f_0(t, x, x_1, \ldots, x_s)$ is continuous and bounded, and the function $\varphi_0(t)$ is continuously differentiable. Clearly, in this case,

$$\widehat{f}_0 = \dot{\varphi}_0(t_{00}) - f_0\big(t_{00}, \varphi_0(t_{00}), \varphi_0(t_{00} - \tau_{10}), \ldots, \varphi_0(t_{00} - \tau_{s0})\big).$$

Theorems 3.1–3.3 correspond to the cases where there exist the left-sided, right-sided and two-sided variations of the initial moment t_{00}, respectively.

3.2 Lemma on estimation of the increment of a solution with respect to the variation set $\Im_-^{(2)}$

To each element $\mu = (t_0, \tau_1, \ldots, \tau_s, \varphi, f) \in \Lambda^{(2)}$ we assign the functional differential equation

$$\dot{y}(t) = f(t_0, \tau_1, \ldots, \tau_s, \varphi, y)(t) \tag{3.11}$$

with the initial condition

$$y(t_0) = \varphi(t_0) \tag{3.12}$$

(see (1.16)).

Definition 3.2. Let $\mu = (t_0, \tau_1, \ldots, \tau_s, \varphi, f) \in \Lambda^{(2)}$. An absolutely continuous function $y(t) = y(t; \mu) \in O$, $t \in [r_1, r_2] \subset I$, is called a solution of the equation (3.11) with the initial condition (3.12), or a solution corresponding to the element μ and defined on the interval $[r_1, r_2]$, if $t_0 \in [r_1, r_2]$, $y(t_0) = \varphi(t_0)$ and the function $y(t)$ satisfies the equation (3.11) a.e. on $[r_1, r_2]$.

Remark 3.1. Let $y(t; \mu)$, $t \in [r_1, r_2]$, be a solution corresponding to the element $\mu = (t_0, \tau_1, \ldots, \tau_s, \varphi, f) \in \Lambda^{(2)}$. Then the function

$$x(t; \mu) = h(t_0, \varphi, y(\,\cdot\,; \mu))(t), \quad t \in [\widehat{\tau}, r_2], \tag{3.13}$$

is a solution of the equation (3.11) with the initial condition (3.12) (see Definition 3.1 and (1.18)).

Lemma 3.1. Let $y_0(t)$ be a solution corresponding to the element $\mu_0 = (t_{00}, \tau_{10}, \ldots, \tau_{s0}, \varphi_0, f_0) \in \Lambda^{(2)}$ and defined on $[r_1, r_2] \subset (a, b)$; let $t_{00} \in [r_1, r_2), \tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = \overline{1, s}$, and let $K_1 \subset O$ be a compact set containing a neighborhood of the set $\varphi_0(I_1) \cup y_0([r_1, r_2])$. Then there exist the numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for any $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im^{(2)}$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda^{(2)}$. In addition, to this element there corresponds a solution $y(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[r_1 - \delta_1, r_2 + \delta_1] \subset I$. Moreover,

$$\begin{cases} \varphi(t) := \varphi_0(t) + \varepsilon\delta\varphi(t) \in K_1, & t \in I_1, \\ y(t; \mu_0 + \varepsilon\delta\mu) \in K_1, & t \in [r_1 - \delta_1, r_2 + \delta_1], \end{cases} \tag{3.14}$$

and

$$\lim_{\varepsilon \to 0} y(t; \mu_0 + \varepsilon\delta\mu) = y(t; \mu_0)$$

uniformly for $(t, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times \Im^{(2)}$.

This lemma is a consequence of Theorem 1.7.

Lemma 3.2. Let $x_0(t)$ be a solution corresponding to the element $\mu_0 = (t_{00}, \tau_{10}, \ldots, \tau_{is}, \varphi_0, f_0) \in \Lambda^{(2)}$ and defined on $[\widehat{\tau}, t_{10}]$ (see Definition 3.1), let $t_{00}, t_{10} \in (a, b)$, $\tau_{i0} \in (\theta_{i1}, \theta_{i2})$, $i = \overline{1, s}$, and let $K_1 \subset O$ be a compact set containing a neighborhood of the set $\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then there exist the numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for any $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im^{(2)}$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda^{(2)}$. In addition, to this element there corresponds a solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$. Moreover,

$$x(t; \mu_0 + \varepsilon\delta\mu) \in K_1, \quad t \in [\widehat{\tau}, t_{10} + \delta_1]. \tag{3.15}$$

It is easy to see that if in Lemma 3.1 one puts $r_1 = t_{00}$, $r_2 = t_{10}$, then $x_0(t) = y_0(t)$, $t \in [t_{00}, t_{10}]$, and

$$x(t; \mu_0 + \varepsilon\delta\mu) = h\big(t_0, \varphi, y(\,\cdot\,; \mu_0 + \varepsilon\delta\mu)\big)(t) \quad \forall (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times \Im^{(2)}$$

(see (3.13)). Thus, Lemma 3.2 is a simple corollary of Lemma 3.1 (see (3.14)).

Due to the uniqueness, the solution $y(t; \mu_0)$ on the interval $[r_1 - \delta_1, r_2 + \delta_1]$ is a continuation of the solution $y_0(t)$. Therefore, we can assume that the solution $y_0(t)$ is defined on the interval $[r_1 - \delta_1, r_2 + \delta_1]$.

Lemma 3.1 allows one to define the increment of the solution $y_0(t) = y(t; \mu_0)$:

$$\Delta y(t) = \Delta y(t; \varepsilon\delta\mu) = y(t; \mu_0 + \varepsilon\delta\mu) - y_0(t) \quad \forall (t, \varepsilon, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times (0, \varepsilon_1) \times \Im^{(2)}. \tag{3.16}$$

Obviously,

$$\lim_{\varepsilon \to 0} \Delta y(t; \varepsilon\delta\mu) = 0 \tag{3.17}$$

uniformly with respect to $(t, \delta\mu) \in [r_1 - \delta_1, r_2 + \delta_1] \times \Im^{(2)}$ (see Lemma 3.1).

Lemma 3.3. Let the conditions of Theorem 3.1 hold. Then there exist the numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that

$$\max_{t \in [t_{00}, r_2 + \delta_2]} |\Delta y(t)| \leq O(\varepsilon\delta\mu) \tag{3.18}$$

for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times \Im_-^{(2)}$. Moreover,

$$\Delta y(t_{00}) = \varepsilon\big[\delta\varphi(t_{00}) + (\dot{\varphi}_0^- - f_0^-)\delta t_0\big] + o(\varepsilon\delta\mu). \tag{3.19}$$

Proof. Let $\varepsilon_2' \in (0, \varepsilon_1)$ be insomuch small that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2') \times \Im_{-}^{(2)}$ the following inequalities

$$t_0 + \tau_i > t_{00}, \quad i = \overline{1, s}, \tag{3.20}$$

hold, where $t_0 = t_{00} + \varepsilon\delta t_0$, $\tau_i = \tau_{i0} + \varepsilon\delta\tau_i$. On the interval $[t_{00}, r_2 + \delta_1]$, the function $\Delta y(t) = y(t) - y_0(t)$, where $y(t) = y(t; \mu + \varepsilon\delta\mu)$, satisfies the equation

$$\dot{\Delta} y(t) = a(t; \varepsilon\delta\mu) + \varepsilon b(t; \varepsilon\delta\mu), \tag{3.21}$$

where

$$
\begin{aligned}
a(t; \varepsilon\delta\mu) &= f_0(t_0, \tau_1, \ldots, \tau_s, \varphi, y_0 + \Delta y)(t) - f_0(t_{00}, \tau_{10}, \ldots, \tau_{s0}, \varphi_0, y_0)(t), \\
b(t; \varepsilon\delta\mu) &= \delta f(t_0, \tau_1, \ldots, \tau_s, \varphi, y_0 + \Delta y)(t).
\end{aligned} \tag{3.22}
$$

We rewrite the equation (3.21) in the integral form

$$\Delta y(t) = \Delta y(t_{00}) + \int_{t_{00}}^{t} \left[a(\xi; \varepsilon\delta\mu) + \varepsilon b(\xi; \varepsilon\delta\mu) \right] d\xi.$$

Hence it follows that

$$|\Delta y(t)| \leq |\Delta y(t_{00})| + a_1(t; t_{00}, \varepsilon\delta\mu) + \varepsilon b_1(t_{00}; \varepsilon\delta\mu), \tag{3.23}$$

where

$$a_1(t; t_{00}, \varepsilon\delta\mu) = \int_{t_{00}}^{t} |a(\xi; \varepsilon\delta\mu)| \, d\xi, \quad b_1(t_{00}; \varepsilon\delta\mu) = \int_{t_{00}}^{r_2 + \delta_1} |b(\xi; \varepsilon\delta\mu)| \, d\xi.$$

Let us prove the formula (3.19). We have

$$
\begin{aligned}
\Delta y(t_{00}) &= y(t_{00}; \mu_0 + \varepsilon\delta\mu) - y_0(t_{00}) = \varphi_0(t_0) + \varepsilon\delta\varphi(t_0) \\
&\quad + \int_{t_0}^{t_{00}} \left[f_0\big(t, y_0(t) + \Delta y(t), \varphi(t - \tau_1), \ldots, \varphi(t - \tau_s)\big) + \varepsilon b(t; \varepsilon\delta\mu) \right] dt - \varphi_0(t_{00}) \quad (3.24)
\end{aligned}
$$

(see (3.20)). Since

$$\int_{t_{00}}^{t_0} \dot{\varphi}_0(t) \, dt = \varepsilon\dot{\varphi}_0^- \delta t_0 + o(\varepsilon\delta\mu)$$

and

$$\lim_{\varepsilon \to 0} \delta\varphi(t_0) = \delta\varphi(t_{00}) \quad \text{uniformly with respect to} \quad \delta\mu \in \Im_{-}^{(2)}$$

(see (3.3)), we have

$$
\begin{aligned}
&\varphi_0(t_0) + \varepsilon\delta\varphi(t_0) - \varphi_0(t_{00}) \\
&\quad = \int_{t_{00}}^{t_0} \dot{\varphi}_0(t) \, dt + \varepsilon\delta\varphi(t_{00}) + \varepsilon\big[\delta\varphi(t_0) - \delta\varphi(t_{00})\big] = \varepsilon[\dot{\varphi}_0^- \delta t_0 + \delta\varphi(t_{00})] + o(\varepsilon\delta\mu). \quad (3.25)
\end{aligned}
$$

It is clear that if $t \in [t_0, t_{00}]$, then

$$\lim_{\varepsilon \to 0} \big(t, y_0(t) + \Delta y(t), \varphi(t - \tau_1), \ldots, \varphi(t - \tau_s)\big) = \lim_{t \to t_{00-}} \big(t, y_0(t), \varphi_0(t - \tau_{10}), \ldots, \varphi_0(t - \tau_{s0})\big) = w_0$$

(see (3.17)). Consequently,

$$\lim_{\varepsilon \to 0} \sup_{t \in [t_0, t_{00}]} \left| f_0\big(t, y_0(t) + \Delta y(t), \varphi(t - \tau_1), \ldots, \varphi(t - \tau_s)\big) - f_0^- \right| = 0.$$

This relation implies that

$$\int\limits_{t_0}^{t_{00}} f_0\big(t, y_0(t) + \Delta y(t), \varphi(t - \tau_1), \ldots, \varphi(t - \tau_s)\big)\, dt = -\varepsilon f_0^- \delta t_0$$

$$+ \int\limits_{t_0}^{t_{00}} \Big[f_0\big(t, y_0(t) + \Delta y(t), \varphi(t - \tau_1), \ldots, \varphi(t - \tau_s)\big) - f_0^- \Big]\, dt = -\varepsilon f_0^- \delta t_0 + o(\varepsilon\delta\mu). \quad (3.26)$$

Further, we have

$$\int\limits_{t_0}^{t_{00}} |b(t; \varepsilon\delta\mu)|\, dt \leq \int\limits_{t_0}^{t_{00}} \sum_{i=1}^{k} |\lambda_i|\, \big|\delta f_i(t_0, \tau_1, \ldots, \tau_s, \varphi, y_0 + \Delta y)(t)\big|\, dt \leq \gamma \sum_{i=1}^{k} \int\limits_{t_0}^{t_{00}} m_{\delta f_i, K_1}(t)\, dt \quad (3.27)$$

(see (3.16), (3.3) and (3.14)).

From (3.24), by virtue of (3.25)–(3.27), we obtain (3.19).

Let us now prove the inequality (3.18). Towards this end, we estimate $a_1(t; t_{00}, \varepsilon\delta\mu)$, $t \in [t_{00}, r_2 + \delta_1]$. Obviously,

$$a_1(t; t_{00}, \varepsilon\delta\mu) \leq \int\limits_{t_{00}}^{t} L_{f_0, K_1}(\xi) |\Delta y(\xi)|\, d\xi + \sum_{i=1}^{s} a_{2i}(t; t_{00}, \varepsilon\delta\mu), \quad (3.28)$$

where

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) = \int\limits_{t_{00}}^{t} L_{f_0, K_1}(\xi) \big| h(t_0, \varphi, y_0 + \Delta y)(\xi - \tau_i) - h(t_{00}, \varphi_0, y_0)(\xi - \tau_{i0}) \big|\, d\xi$$

(see (3.22)).

Let $t_{00} + \tau_{i0} \leq r_2$ and let ε_2' be insomuch small that $t_{00} + \tau_i < r_2 + \delta_1$.

Furthermore, let

$$\rho_{i1} = \min\{t_0 + \tau_i, t_{00} + \tau_{i0}\}, \quad \rho_{i2} = \max\{t_{00} + \tau_i, t_{00} + \tau_{i0}\}.$$

It is easy to see that

$$\rho_{i2} \geq \rho_{i1} > t_{00} \quad \text{and} \quad \rho_{i2} - \rho_{i1} = O(\varepsilon\delta\mu).$$

Let $t \in [t_{00}, \rho_{i1})$, then for $\xi \in [t_{00}, t]$, we have $\xi - \tau_i < t_0$ and $\xi - \tau_{i0} < t_{00}$, and hence,

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) = \int\limits_{t_{00}}^{t} L_{f_0, K_1}(\xi) \big| \varphi(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0}) \big|\, d\xi.$$

From the boundedness of the function $\dot{\varphi}_0(t)$, $t \in I_1$, it follows that

$$\big| \varphi(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0}) \big| = \big| \varphi_0(\xi - \tau_i) + \varepsilon\delta\varphi(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0}) \big|$$

$$= O(\varepsilon\delta\mu) + \bigg| \int\limits_{\xi - \tau_{i0}}^{\xi - \tau_i} \dot{\varphi}_0(t)\, dt \bigg| = O(\varepsilon\delta\mu). \quad (3.29)$$

Thus, for $t \in [t_{00}, \rho_{i1}]$, we have

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) \leq O(\varepsilon\delta\mu), \quad i = \overline{1, s}. \quad (3.30)$$

Let $t \in [\rho_{i1}, \rho_{i2}]$, then

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) \leq a_{2i}(\rho_{i1}; t_{00}, \varepsilon\delta\mu) + a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) \leq O(\varepsilon\delta\mu) + a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu).$$

Let $\rho_{i1} = t_0 + \tau_i$ and $\rho_{i2} = t_{00} + \tau_i$, i.e., $t_0 + \tau_i < t_{00} + \tau_{i0} < t_{00} + \tau_i$. We have

$$a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) \leq \int_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{f_0, K_1}(\xi)\big|y(\xi - \tau_i; \mu_0 + \varepsilon\delta\mu) - \varphi_0(\xi - \tau_{i0})\big|\, d\xi$$

$$+ \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{f_0, K_1}(\xi)\big|y(\xi - \tau_i; \mu_0 + \varepsilon\delta\mu) - y_0(\xi - \tau_{i0})\big|\, d\xi$$

$$\leq \int_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{f_0, K_1}(\xi)\big|y(\xi - \tau_i; \mu_0 + \varepsilon\delta\mu) - \varphi(\xi - \tau_i)\big|\, d\xi$$

$$+ \int_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{f_0, K_1}(\xi)\big|\varphi(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})\big|\, d\xi$$

$$+ \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{f_0, K_1}(\xi)\big|y(\xi - \tau_i; \mu_0 + \varepsilon\delta\mu) - \varphi(\xi - \tau_i)\big|\, d\xi$$

$$+ \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{f_0, K_1}(\xi)\big|\varphi(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})\big|\, d\xi$$

$$+ \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{f_0, K_1}(\xi)\big|\varphi_0(\xi - \tau_{i0}) - y_0(\xi - \tau_{i0})\big|\, d\xi$$

$$\leq o(\varepsilon\delta\mu) + \int_{t_0+\tau_i}^{t_{00}+\tau_i} L_{f_0, K_1}(\xi)\big|y(\xi - \tau_i; \mu_0 + \varepsilon\delta\mu) - \varphi(\xi - \tau_i)\big|\, d\xi$$

$$+ \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{f_0, K_1}(\xi)\big|\varphi_0(\xi - \tau_{i0}) - y_0(\xi - \tau_{i0})\big|\, d\xi$$

$$= o(\varepsilon\delta\mu) + \int_{t_0}^{t_{00}} L_{f_0, K_1}(\xi + \tau_i)\big|y(\xi; \mu_0 + \varepsilon\delta\mu) - \varphi(\xi)\big|\, d\xi$$

$$+ \int_{t_{00}}^{t_{00}+\tau_i-\tau_{i0}} L_{f_0, K_1}(\xi + \tau_{i0})|\varphi_0(\xi) - y_0(\xi)|\, d\xi$$

(see (3.29)). The functions $f_0(t, x, x_1, \ldots, x_s)$, $(t, x, x_1, \ldots, x_s) \in I \times O^{1+s}$ and $\dot{\varphi}_0(t)$, $t \in I_1$, are bounded; therefore,

$$\big|y(\xi; \mu_0 + \varepsilon\delta\mu) - \varphi(\xi)\big|$$

$$= \left|\varphi(t_0) + \int_{t_0}^{\xi} \big[f_0(t_0, \tau_1, \ldots, \tau_s, \varphi, y_0 + \Delta y)(t) + b(t; \varepsilon\delta\mu)\big]\, dt - \varphi(\xi)\right| \leq O(\varepsilon\delta\mu), \quad \xi \in [t_0, t_{00}], \quad (3.31)$$

and

$$\big|\varphi_0(\xi) - y_0(\xi)\big| = \left|\varphi_0(\xi) - \varphi_0(t_{00}) - \int_{t_{00}}^{\xi} f_0(t_{00}, \tau_{10}, \ldots, \tau_{s0}, \varphi_0, y_0)(t)\, dt\right| \leq O(\varepsilon\delta\mu),$$

$$\xi \in [t_{00}, t_{00} + \tau_i - \tau_{i0}] \quad (\tau_i > \tau_{i0}).$$

Thus,

$$a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) = o(\varepsilon\delta\mu).$$

Let $\rho_{i1} = t_0 + \tau_i$ and $\rho_{i2} = t_{00} + \tau_{i0}$, then

$$a_{i2}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) = \int\limits_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{f_0,K_1}(\xi)\big|y(\xi - \tau_i; \mu_0 + \varepsilon\delta\mu) - \varphi_0(\xi - \tau_{i0})\big|\, d\xi = o(\varepsilon\delta\mu).$$

Let $\rho_{i1} = t_{00} + \tau_{i0}$ and $\rho_{i2} = t_{00} + \tau_i$, i.e., $t_{00} + \tau_{i0} < t_0 + \tau_i < t_{00} + \tau_i$. We have

$$a_{i2}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) \leq \int\limits_{t_{00}+\tau_{i0}}^{t_0+\tau_i} L_{f_0,K_1}(\xi)\big|\varphi(\xi - \tau_i) - y_0(\xi - \tau_{i0})\big|\, d\xi$$

$$+ \int\limits_{t_0+\tau_i}^{t_{00}+\tau_i} L_{f_0,K_1}(\xi)\big|y(\xi - \tau_i; \mu_0 + \varepsilon\delta\mu) - y_0(\xi - \tau_{i0})\big|\, d\xi = o(\varepsilon\delta\mu).$$

Consequently, for $t \in [t_{00}, \rho_{i2}]$, the inequality (3.30) holds.

Let $t \in [\rho_{i2}, r_2 + \delta_1]$, then $t - \tau_i \geq t_0$ and $t - \tau_{i0} \geq t_{00}$, therefore,

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) = a_{2i}(\rho_{i2}; t_{00}, \varepsilon\delta\mu) + \int\limits_{\rho_{i2}}^{t} L_{f_0,K_1}(\xi)\big|y_0(\xi - \tau_i) + \Delta y(\xi - \tau_i) - y_0(\xi - \tau_{i0})\big|\, d\xi$$

$$\leq O(\varepsilon\delta\mu) + \int\limits_{\rho_{i2}-\tau_i}^{t-\tau_i} L_{f_0,K_1}(\xi + \tau_i)|\Delta y(\xi)|\, d\xi + \int\limits_{\rho_{i2}}^{t} L_{f_0,K_1}(\xi)\big|y_0(\xi - \tau_i) - y_0(\xi - \tau_{i0})\big|\, d\xi$$

$$\leq O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t} \chi(\xi + \tau_i) L_{f_0,K_1}(\xi + \tau_i)|\Delta y(\xi)|\, d\xi + \int\limits_{\rho_{i2}}^{r_2+\delta_1} L_{f_0,K_1}(\xi)\big|y_0(\xi - \tau_i) - y_0(\xi - \tau_{i0})\big|\, d\xi.$$

Further, for $\xi \in [\rho_{i2}, r_2 + \delta_1]$,

$$\big|y_0(\xi - \tau_i) - y_0(\xi - \tau_{i0})\big| \leq \int\limits_{\xi-\tau_{i0}}^{\xi-\tau_I} \big|f_0(t_{00}, \tau_{i1}, \ldots, \tau_{is}, y_0)(t)\big|\, dt \leq O(\varepsilon\delta\mu).$$

Thus, for $t \in [t_{00}, r_2 + \delta_1]$, we get

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) \leq O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t} \chi_1(\xi + \tau_i) L_{f_0,K_1}(\xi + \tau_i)|\Delta y(\xi)|\, d\xi. \tag{3.32}$$

We now consider the case where $t_{00} + \tau_{i0} > r_2$. Let $\delta_2 \in (0, \delta_1)$ and $\varepsilon_2'' \in (0, \varepsilon_1)$ be insomuch small numbers that $t_{00} + \tau_{i0} > r_2 + \delta_2$ and $t_0 + \tau_i > r_2 + \delta_2$ for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2'') \times \Im_{-}^{(2)}$.

It is easy to see that

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) \leq \int\limits_{t_{00}}^{t} L_{f_0,K_1}(\xi)\big|\varphi(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})\big|\, dt \leq O(\varepsilon\delta\mu).$$

Thus, for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{00}, r_2 + \delta_2] \times (0, \varepsilon_2) \times \Im_{-}^{(2)}$ and $i = \overline{1,s}$, where $\varepsilon_2 = \min(\varepsilon_2', \varepsilon_2'')$, the inequality (3.32) holds.

Consequently, we have

$$a_1(t; t_{00}, \varepsilon\delta\mu) \leq O(\varepsilon\delta\mu) + \int_{t_{00}}^{t} \left[L_{f,K_1}(\xi) + \sum_{i=1}^{s} \chi_1(\xi + \tau_i) L_{f,K_1}(\xi + \tau_i) \right] |\Delta y(\xi)| \, d\xi, \tag{3.33}$$

$$t \in [t_{00}, r_2 + \delta_1],$$

(see (3.28)). Obviously,

$$b_1(t_{00}, \varepsilon\delta\mu) \leq \gamma \int_{t_{00}}^{r_2+\delta_2} \sum_{i=1}^{k} \left| \delta f_i(t_0, \tau_1, \ldots, \tau_s, y_0 + \Delta y)(t) \right| dt \leq \gamma \sum_{i=1}^{k} \int_{I} m_{\delta f_i, K_1}(t) \, dt. \tag{3.34}$$

According to (3.19), (3.33) and (3.34), the inequality (3.23) directly implies

$$|\Delta y(t)| \leq O(\varepsilon\delta\mu) + \int_{t_{00}}^{t} \left[L_{f,K_1}(\xi) + \sum_{i=1}^{s} \chi_1(\xi + \tau_i) L_{f,K_1}(\xi + \tau_i) \right] |\Delta y(\xi)| \, d\xi, \ \ t \in [t_{00}, r_2 + \delta_2].$$

By the Gronwall–Bellman inequality lemma, from the above we obtain (3.18). $\qquad\square$

3.3   Proof of Theorem 3.1

Let $r_1 = t_{00}$ and $r_2 = t_{10}$ as in Lemma 3.1, then

$$x_0(t) = \begin{cases} \varphi_0(t), & t \in [\widehat{\tau}, t_{00}), \\ y_0(t), & t \in [t_{00}, t_{10}], \end{cases}$$

and for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im_{-}^{(2)}$,

$$x(t; \mu_0 + \varepsilon\delta\mu) = \begin{cases} \varphi(t), & t \in [\widehat{\tau}, t_0), \\ y(t; \mu_0 + \varepsilon\delta\mu), & t \in [t_0, t_{10} + \delta_1] \end{cases}$$

(see (3.13)).

We note that $\delta\mu \in \Im_{-}^{(2)}$, i.e., $t_0 < t_{00}$, therefore

$$\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t) & \text{for } t \in [\widehat{\tau}, t_0), \\ y(t; \mu_0 + \varepsilon\delta\mu) - \varphi_0(t) & \text{for } t \in [t_0, t_{00}), \\ \Delta y(t) & \text{for } t \in [t_{00}, t_{10} + \delta_1] \end{cases}$$

(see (3.4) and (3.16)).

By Lemma 3.3 and the relation

$$\left| y(t; \mu_0 + \varepsilon\delta\mu) - \varphi_0(t) \right| \leq O(\varepsilon\delta\mu), \ \ t \in [t_0, t_{00}],$$

we have

$$|\Delta x(t)| \leq O(\varepsilon\delta\mu) \ \ \forall (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im_{-}^{(2)}, \tag{3.35}$$

$$\Delta x(t_{00}) = \varepsilon \left[\delta\varphi(t_{00}) + (\dot{\varphi}_0^- - f_0^-)\delta t_0 \right] + o(\varepsilon\delta\mu). \tag{3.36}$$

The function $\Delta x(t)$ satisfies the equation

$$\dot{\Delta} x(t) = f_0[t, x_0 + \Delta x] + \varepsilon\delta f[t, x_0 + \Delta x] - f_0[t]$$

$$= f_{0x}[t]\Delta x(t) + \sum_{i=1}^{s} f_{0x_i}[t]\Delta x(t - \tau_{i0}) + \varepsilon\delta f[t] + \sum_{i=1}^{2} \vartheta_i(t; \varepsilon\delta\mu) \tag{3.37}$$

on the interval $[t_{00}, t_{10} + \delta_2]$, where

$$f_0[t, x_0 + \Delta x] = f_0\Big(t, x_0(t) + \Delta x(t), x_0(t - \tau_1) + \Delta x(t - \tau_1), \ldots, x_0(t - \tau_s) + \Delta x(t - \tau_s)\Big),$$

$$f_0[t] = f_0\big(t, x_0(t), x_0(t - \tau_{10}), \ldots, x_0(t - \tau_{s0})\big),$$

$$\delta f[t, x_0 + \Delta x] = \delta f\Big(t, x_0(t) + \Delta x(t), x_0(t - \tau_1) + \Delta x(t - \tau_1), \ldots, x_0(t - \tau_s) + \Delta x(t - \tau_s)\Big),$$

$$\delta f[t] = \delta f\big(t, x_0(t), x_0(t - \tau_1), \ldots, x_0(t - \tau_s)\big),$$

$$\vartheta_1(t; \varepsilon\delta\mu) = f_0[t, x_0 + \Delta x] - f_0[t] - f_{0x}[t]\Delta x(t) - \sum_{i=1}^{s} f_{0x_i}[t]\Delta x(t - \tau_{i0}), \tag{3.38}$$

$$\vartheta_2(t; \varepsilon\delta\mu) = \varepsilon\big[\delta f[t, x_0 + \Delta x] - \delta f[t]\big]. \tag{3.39}$$

By using the Cauchy formula, one can represent the solution of the equation (3.37) in the form

$$\Delta x(t) = Y(t_{00}; t)\Delta x(t_{00}) + \varepsilon \int_{t_{00}}^{t} Y(\xi; t)\delta f[\xi]\, d\xi + \sum_{p=0}^{2} R_p(t; t_{00}, \varepsilon\delta\mu), \quad t \in [t_{00}, t_{10} + \delta_2], \tag{3.40}$$

where

$$\begin{cases} R_0(t; t_{00}, \varepsilon\delta\mu) = \displaystyle\sum_{i=1}^{s} R_{i0}(t; t_{00}, \varepsilon\delta\mu), \\[2mm] R_{i0}(t; t_{00}, \varepsilon\delta\mu) = \displaystyle\int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}]\Delta x(\xi)\, d\xi, \\[2mm] R_p(t; t_{00}, \varepsilon\delta\mu) = \displaystyle\int_{t_{00}}^{t} Y(\xi; t)\vartheta_p(\xi; \varepsilon\delta\mu)\, d\xi, \quad p = 1, 2, \end{cases} \tag{3.41}$$

and $Y(\xi; t)$ is the matrix function satisfying the equation (3.8) and the condition (3.9). The function $Y(\xi; t)$ is continuous on the set

$$[t_{00} - \delta_2, t_{00}] \times [t_{00}, t_{10} + \delta_2] \subset \Pi.$$

Therefore,

$$Y(t_{00}; t)\Delta x(t_{00}) = \varepsilon Y(t_{00}; t)\big[\delta\varphi(t_{00}) + (\dot{\varphi}_0^- - f_0^-)\delta t_0\big] + o(t; \varepsilon\delta\mu) \tag{3.42}$$

(see (3.36)). One can readily see that

$$R_{i0}(t; t_{00}, \varepsilon\delta\mu) = \varepsilon \int_{t_{00}-\tau_{i0}}^{t_0} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}]\delta\varphi(\xi)\, d\xi + \int_{t_0}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}]\Delta x(\xi)\, d\xi$$

$$= \varepsilon \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}]\delta\varphi(\xi)\, d\xi + o(t; \varepsilon\delta\mu)$$

(see (3.35)). Thus

$$R_0(t; t_{00}, \varepsilon\delta\mu) = \varepsilon \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t) f_{0x_i}[\xi + \tau_{i0}]\delta\varphi(\xi)\, d\xi + o(t; \varepsilon\delta\mu). \tag{3.43}$$

We introduce the notations:

$$f_0[t; \theta, \varepsilon\delta\mu] = f_0\Big(t, x_0(t) + \theta\Delta x(t), x_0(t - \tau_{10}) + \theta(x_0(t - \tau_1) - x_0(t - \tau_{10}) + \Delta x(t - \tau_1)), \ldots,$$

$$x_0(t - \tau_{s0}) + \theta\big(x_0(t - \tau_s) - x_0(t - \tau_{s0}) + \Delta x(t - \tau_s)\big)\Big),$$

$$\sigma(t; \theta, \varepsilon\delta\mu) = f_{0x}[t; \theta, \varepsilon\delta\mu] - f_{0x}[t], \quad \varrho_i(t; \theta, \varepsilon\delta\mu) = f_{0x_i}[t; \theta, \varepsilon\delta\mu] - f_{0x_i}[t].$$

It is easy to see that

$$
f_0([t, x_0 + \Delta x] - f_0[t] = \int\limits_0^1 \frac{d}{d\theta} f_0[t; \theta, \varepsilon\delta\mu] \, d\theta
$$

$$
= \int\limits_0^1 \left\{ f_{0x}[t; \theta, \varepsilon\delta\mu]\Delta x(t) + \sum_{i=1}^s f_{0x_i}[t; \theta, \varepsilon\delta\mu]\big(x_0(t - \tau_i) - x_0(t - \tau_{i0}) + \Delta x(t - \tau_i)\big) \right\} d\theta
$$

$$
= \left[\int\limits_0^1 \sigma(t; \theta, \varepsilon\delta\mu) \, d\theta \right] \Delta x(t) + \sum_{i=1}^s \left[\int\limits_0^1 \varrho_i(t; \theta, \varepsilon\delta\mu) \, d\theta \right] \big(x_0(t - \tau_i) - x_0(t - \tau_{i0}) + \Delta x(t - \tau_i)\big)
$$

$$
+ f_{0x}[t]\Delta x(t) + \sum_{i=1}^s f_{0x_i}[t]\big(x_0(t - \tau_i) - x_0(t - \tau_{i0}) + \Delta x(t - \tau_i)\big).
$$

Taking into account the last relation for $t \in [t_{00}, t_{10} + \delta_2]$, we have

$$
R_1(t; t_{00}, \varepsilon\delta\mu) = \sum_{p=3}^6 R_p(t; t_{00}, \varepsilon\delta\mu),
$$

where

$$
R_3(t; t_{00}, \varepsilon\delta\mu) = \int\limits_{t_{00}}^t Y(\xi; t)\sigma_1(\xi; \varepsilon\delta\mu)\Delta x(\xi) \, d\xi, \quad \sigma_1(\xi; \varepsilon\delta\mu) = \int\limits_0^1 \sigma(\xi; s, \varepsilon\delta\mu) \, ds,
$$

$$
R_4(t; t_{00}, \varepsilon\delta\mu) = \sum_{i=1}^s \int\limits_{t_{00}}^t Y(\xi; t)\varrho_{i2}(\xi; \varepsilon\delta\mu)\big[x_0(\xi - \tau_i) - x_0(\xi - \tau_{i0}) + \Delta x(\xi - \tau_i)\big] \, d\xi,
$$

$$
\varrho_{i2}(\xi; \varepsilon\delta\mu) = \int\limits_0^1 \varrho_{i1}(\xi; \theta, \varepsilon\delta\mu) \, d\theta,
$$

$$
R_5(t; t_{00}, \varepsilon\delta\mu) = \sum_{i=1}^s \int\limits_{t_{00}}^t Y(\xi; t)f_{0x_i}[\xi] \big[x_0(\xi - \tau_i) - x_0(\xi - \tau_{i0})\big] \, d\xi,
$$

$$
R_6(t; t_{00}, \varepsilon\delta\mu) = \sum_{i=1}^s \int\limits_{t_{00}}^t Y(\xi; t)f_{0x_i}[\xi] \big[\Delta x(\xi - \tau_i) - \Delta x(\xi - \tau_{i0})\big] \, d\xi
$$

(see (3.38)). The function $x_0(t)$, $t \in [\widehat{\tau}, t_{10} + \delta_2]$, is absolutely continuous, and for each fixed Lebesgue point $\xi_i \in (t_{00}, t_{10} + \delta_2)$ of the function $\dot{x}_0(\xi - \tau_{i0})$ we get

$$
x_0(\xi_i - \tau_i) - x_0(\xi_i - \tau_{i0}) = \int\limits_{\xi_i}^{\xi_i - \varepsilon\delta\tau_i} \dot{x}_0(\varsigma - \tau_{i0}) \, d\varsigma = -\varepsilon\dot{x}_0(\xi_i - \tau_{i0})\delta\tau_i + \gamma_i(\xi_i; \varepsilon\delta\mu), \qquad (3.44)
$$

where
$$
\lim_{\varepsilon \to 0} \frac{\gamma_i(\xi_i; \varepsilon\delta\mu)}{\varepsilon} = 0 \quad \text{uniformly for} \quad \delta\mu \in \Im_-^{(2)}. \qquad (3.45)
$$

Thus (3.44) is valid for almost all points of the interval $(t_{00}, t_{10} + \delta_2)$. From (3.44), taking into account the boundedness of the function

$$
\dot{x}_0(t) = \begin{cases} \dot{\varphi}_0(t), & t \in [\widehat{\tau}, t_{00}], \\ f_0\big(t, x_0(t), x_0(t - \tau_{10}), \ldots, x_0(t - \tau_{s0})\big), & t \in (t_{00}, t_{10} + \delta_2], \end{cases}
$$

it follows that

$$\left|x_0(\xi_i - \tau_i) - x_0(\xi_i - \tau_{i0})\right| \le O(\varepsilon\delta\mu) \ \ \text{and} \ \left|\frac{\gamma_i(\xi_i; \varepsilon\delta\mu)}{\varepsilon}\right| \le const. \tag{3.46}$$

It is clear that

$$\left|\Delta x(\xi - \tau_i) - \Delta x(\xi - \tau_{i0})\right| = \begin{cases} o(\xi; \varepsilon\delta\mu) & \text{for } \xi \in [t_{00}, \rho_{i1}], \\ O(\xi; \varepsilon\delta\mu) & \text{for } \xi \in [\rho_{i1}, \rho_{i2}] \end{cases} \tag{3.47}$$

(see (3.35)).

Let $\xi \in [\rho_{i2}, t_{10} + \delta_1]$, then $\xi - \tau_i \ge t_{00}$, $\xi - \tau_{i0} \ge t_{00}$. Therefore,

$$\left|\Delta x(\xi - \tau_i) - \Delta x(\xi - \tau_{i0})\right| \le \int\limits_{\xi - \tau_{i0}}^{\xi - \tau_i} |\dot\Delta x(s)| \, ds$$

$$\le \int\limits_{\xi - \tau_{i0}}^{\xi - \tau_i} L_{f_0, K_1}(\varsigma) \Big[\left|\Delta x(\varsigma)\right| + \sum_{i=1}^{s} \left|x_0(\varsigma - \tau_i) - x_0(\varsigma - \tau_{i0})\right| + \left|\Delta x(\varsigma - \tau_i)\right|\Big] \, d\varsigma$$

$$+ \varepsilon\alpha \int\limits_{\xi - \tau_{i0}}^{\xi - \tau_i} \sum_{j=1}^{k} m_{\delta f_j, K_1}(\varsigma) \, d\varsigma = o(\xi; \varepsilon\delta\mu) \tag{3.48}$$

(see (3.37), (3.15), (3.46) and (3.35)).

According to (3.35), (3.44) and (3.46)–(3.48), for the expressions $R_p(t; t_{00}, \varepsilon\delta\mu)$, $p = 3, 4, 5$, we have

$$|R_3(t; t_{00}, \varepsilon\delta\mu)| \le \|Y\| O(\varepsilon\delta\mu)\sigma_2(\varepsilon\delta\mu), \ \ \sigma_2(\varepsilon\delta\mu) = \int\limits_{t_{00}}^{t_{10}+\delta_1} |\sigma_1(\xi; \varepsilon\delta\mu)| \, d\xi,$$

$$|R_4(t; t_{00}, \varepsilon\delta\mu)| \le \|Y\| O(\varepsilon\delta\mu) \sum_{i=1}^{s} \rho_{i2}(\varepsilon\delta\mu), \ \ \rho_{i2}(\varepsilon\delta\mu) = \int\limits_{t_{00}}^{t_{10}+\delta_1} |\rho_{i1}(\xi; \varepsilon\delta\mu)| \, d\xi,$$

$$R_5(t; t_{00}, \varepsilon\delta\mu) = -\varepsilon \sum_{i=1}^{s} \Big[\int\limits_{t_{00}}^{t} Y(\xi; t) f_{0x_i}[\xi]\dot x_0(\xi - \tau_{i0}) \, d\xi\Big]\delta\tau_i + \sum_{i=1}^{s} \gamma_{i1}(t; \varepsilon\delta\mu),$$

where

$$\|Y\| = \sup\{|Y(\xi; t)| : (\xi, t) \in \Pi\}, \ \ \gamma_{i1}(t; \varepsilon\delta\mu) = \int\limits_{t_{00}}^{t} Y(\xi; t) f_{0x_i}[\xi]\gamma_i(\xi; \varepsilon\delta\mu) \, d\xi.$$

Obviously,

$$\left|\frac{\gamma_{i1}(t; \varepsilon\delta\mu)}{\varepsilon}\right| \le \|Y\| \int\limits_{t_{00}}^{t_{10}+\delta_1} |f_{0x_i}[\xi]| \left|\frac{\gamma_i(\xi; \varepsilon\delta\mu)}{\varepsilon}\right| d\xi.$$

By the Lebesgue theorem on the passage to the limit under the integral sign, we have

$$\lim_{\varepsilon\to 0} \sigma_2(\varepsilon\delta\mu) = 0, \ \ \lim_{\varepsilon\to 0} \rho_{i2}(\varepsilon\delta\mu) = 0, \ \ \lim_{\varepsilon\to 0} \left|\frac{\gamma_{i1}(t; \varepsilon\delta\mu)}{\varepsilon}\right| = 0$$

uniformly for $(t, \delta\mu) \in [t_{00}, t_{10} + \delta_1] \times \Im_-^{(2)}$ (see (3.45)). Thus,

$$R_p(t; t_{00}, \varepsilon\delta\mu) = o(t; \varepsilon\delta\mu), \ \ p = 3, 4, \tag{3.49}$$

$$R_5(t; t_{00}, \varepsilon\delta\mu) = -\varepsilon \sum_{i=1}^{s} \Big[\int\limits_{t_{00}}^{t} Y(\xi; t) f_{0x_i}[\xi]\dot x_0(\xi - \tau_{i0}) \, d\xi\Big]\delta\tau_i + o(t; \varepsilon\delta\mu). \tag{3.50}$$

Further,

$$|R_6(t;t_{00},\varepsilon\delta\mu)| \le \|Y\| \int_{t_{00}}^{t_{10}+\delta_1} \sum_{i=1}^{s} |f_{0x_i}[\xi]| |\Delta x(\xi-\tau_i) - \Delta x(\xi-\tau_{i0})| \, d\xi = o(\varepsilon\delta\mu). \tag{3.51}$$

On the basis of (3.49)–(3.51), we obtain

$$R_1(t;t_{00},\varepsilon\delta\mu) = -\varepsilon \sum_{i}^{s} \left[\int_{t_{00}}^{t} Y(\xi;t) f_{0x_i}[\xi] \dot{x}_0(\xi-\tau_{i0}) \, d\xi \right] \delta\tau_i + o(t;\varepsilon\delta\mu). \tag{3.52}$$

Next,

$$|R_2(t;t_{00},\varepsilon\delta\mu)|$$

$$\le \varepsilon\gamma \int_{t_{00}}^{t_{10}+\delta_1} \sum_{j=1}^{k} L_{\delta f_j, K_1}(\xi) \left[|\Delta x(\xi)| + \sum_{i=1}^{s} \left(|x_0(\xi-\tau) - x_0(\xi-\tau_0)| + |\Delta x(\xi-\tau)| \right) \right] d\xi \le o(\varepsilon\delta\mu) \tag{3.53}$$

(see (3.39)).

From (3.40), by virtue of (3.42), (3.43), (3.52) and (3.53), we obtain (3.5), where $\delta x(t;\delta\mu)$ has the form (3.6).

3.4 Lemma on estimation of the increment of a solution with respect to the variation set $\Im_+^{(2)}$

Lemma 3.4. *Let the conditions of Theorem 3.2 hold. Then there exist the numbers $\varepsilon_2 \in (0,\varepsilon_1)$ and $\delta_2 \in (0,\delta_1)$ such that*

$$\max_{t \in [t_0, r_2+\delta_2]} |\Delta y(t)| \le O(\varepsilon\delta\mu) \tag{3.54}$$

for arbitrary $(\varepsilon,\delta\mu) \in [0,\varepsilon_2] \times \Im_+^{(2)}$. Moreover,

$$\Delta y(t_0) = \varepsilon \big[\delta\varphi(t_{00}) + (\dot{\varphi}_0^+ - f_0^+)\delta t_0 \big] + o(\varepsilon\delta\mu). \tag{3.55}$$

Proof. Let a number $\varepsilon_2' \in (0,\varepsilon_1)$ be insomuch small that for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_2') \times \Im_+^{(2)}$, the inequalities

$$t_{00} + \tau_i > t_0, \quad t_{00} + \tau_{i0} > t_0, \quad i = \overline{1,s}, \tag{3.56}$$

hold, where $t_0 = t_{00} + \varepsilon\delta t_0$. On the interval $[t_0, r_2+\delta_1]$, the function $\Delta y(t) = y(t) - y_0(t)$ satisfies the equation

$$\dot{\Delta}y(t) = a(t;\varepsilon\delta\mu) + \varepsilon b(t;\varepsilon\delta\mu) \tag{3.57}$$

(see (3.21)). We rewrite the equation (3.57) in the integral form

$$\Delta y(t) = \Delta y(t_0) + \int_{t_0}^{t} \big[a(\xi;\varepsilon\delta\mu) + \varepsilon b(\xi;\varepsilon\delta\mu) \big] \, d\xi.$$

Hence it follows that

$$|\Delta y(t)| \le |\Delta y(t_0)| + a_1(t;t_0,\varepsilon\delta\mu) + \varepsilon b(t_0,\varepsilon\delta\mu). \tag{3.58}$$

Let us prove the formula (3.55). We have

$$\Delta y(t_0) = y(t_0; \mu_0 + \varepsilon\delta\mu) - y_0(t_0)$$

$$= \varphi_0(t_0) + \varepsilon\delta\varphi(t_0) - \varphi_0(t_{00}) - \int_{t_{00}}^{t_0} \big[f_0\big(t, y_0(t), \varphi_0(t-\tau_{10}), \ldots, \varphi_0(t-\tau_{s0}) \big) \big] \, dt \tag{3.59}$$

(see (3.56)). Since

$$\int\limits_{t_{00}}^{t_0} \dot\varphi_0(t)\,dt = \varepsilon\dot\varphi_0^+\delta t_0 + o(\varepsilon\delta\mu)$$

and

$$\lim_{\varepsilon\to 0}\delta\varphi(t_0) = \delta\varphi(t_{00}) \ \text{ uniformly with respect to } \ \delta\mu\in\Im_+^{(2)}$$

(see (3.3)), we get

$$\varphi_0(t_0) + \varepsilon\delta\varphi(t_0) - \varphi_0(t_{00})$$
$$= \int\limits_{t_{00}}^{t_0}\dot\varphi_0(t)\,dt + \varepsilon\delta\varphi(t_{00}) + \varepsilon[\delta\varphi(t_0) - \delta\varphi(t_{00})] = \varepsilon[\dot\varphi_0^+\delta t_0 + \delta\varphi(t_{00})] + o(\varepsilon\delta\mu). \quad (3.60)$$

It is clear that if $t\in[t_{00}, t_0]$, then

$$\lim_{\varepsilon\to 0}\big(t, y_0(t)+\Delta y(t), \varphi(t-\tau_1), \ldots, \varphi(t-\tau_s)\big) = \lim_{t\to t_{00}+}\big(t, y_0(t), \varphi_0(t-\tau_{10}), \ldots, \varphi_0(t-\tau_{s0})\big) = w_0$$

(see (3.17)). Consequently,

$$\lim_{\varepsilon\to 0}\sup_{t\in[t_{00}, t_0]}\Big|f_0(t, y_0(t)+\Delta y(t), \varphi(t-\tau_1), \ldots, \varphi(t-\tau_s)) - f_0^+\Big| = 0.$$

This relation implies that

$$\int\limits_{t_0}^{t_{00}} f_0\big(t, y_0(t)+\Delta y(t), \varphi(t-\tau_1), \ldots, \varphi(t-\tau_s)\big)\,dt$$

$$= -\varepsilon f_0^+\delta t_0 + \int\limits_{t_0}^{t_{00}}\Big[f_0\big(t, y_0(t)+\Delta y(t), \varphi(t-\tau_1), \ldots, \varphi(t-\tau_s)\big) - f_0^+\Big]\,dt = -\varepsilon f_0^+\delta t_0 + o(\varepsilon\delta\mu). \quad (3.61)$$

From (3.59), by virtue of (3.60) and (3.61), we obtain (3.55).

In order to prove the inequality (3.54) we estimate $a_1(t; t_0, \varepsilon\delta\mu)$, $t\in[t_0, r_2+\delta_1]$. Obviously,

$$a_1(t; t_0, \varepsilon\delta\mu) \le \int\limits_{t_0}^{t} L_{f_0, K_1}(\xi)|\Delta y(\xi)|\,d\xi + \sum_{i=1}^{s} a_{2i}(t; t_0, \varepsilon\delta\mu) \qquad (3.62)$$

(see (3.28)).

Let there exist $t_{00}+\tau_{i0} \le r_2$ and let $\varepsilon_2' \in (0, \varepsilon_1)$ be insomuch small that $t_0+\tau_i < r_2+\delta_1$. Furthermore, let

$$\rho_{i1} = \min\{t_{00}+\tau_i, t_{00}+\tau_{i0}\}, \ \ \rho_{i2} = \max\{t_0+\tau_i, t_{00}+\tau_{i0}\}.$$

It is easy to see that

$$\rho_{i2} \ge \rho_{i1} > t_0, \ \ \rho_{i2} - \rho_{i1} = O(\varepsilon\delta\mu).$$

Let $t\in[t_0, \rho_{i1})$, then for $\xi\in[t_0, t]$ we have $\xi-\tau_i < t_0$ and $\xi-\tau_{i0} < t_{00}$. Therefore,

$$a_{2i}(t; t_0, \varepsilon\delta\mu) = \int\limits_{t_0}^{t} L_{f_0, K_1}(\xi)\big|\varphi(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0}\big|\,d\xi.$$

From the boundedness of the function $\dot\varphi_0(t)$, $t\in I_1$, follows

$$\left|\varphi(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})\right| = \left|\varphi_0(\xi - \tau_i) + \varepsilon\delta\varphi(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})\right|$$

$$= O(\varepsilon\delta\mu) + \left| \int_{\xi-\tau_{i0}}^{\xi-\tau_i} \dot{\varphi}_0(t)\, dt \right| = O(\varepsilon\delta\mu). \qquad (3.63)$$

Thus, for $t \in [t_0, \rho_{i1}]$, we have

$$a_{2i}(t; t_0, \varepsilon\delta\mu) \leq O(\varepsilon\delta\mu), \quad i = \overline{1, s}. \qquad (3.64)$$

Let $t \in [\rho_{i1}, \rho_{i2}]$, then

$$a_{2i}(t; t_0, \varepsilon\delta\mu) \leq a_{2i}(\rho_{i1}; t_0, \varepsilon\delta\mu) + a_{2i}(\rho_{21}; \rho_{i1}, \varepsilon\delta\mu) \leq O(\varepsilon\delta\mu) + a_{2i}(\rho_{21}; \rho_{i1}, \varepsilon\delta\mu)$$

(see (3.64)).

Let $\rho_{i1} = t_{00} + \tau_i$ and $\rho_{i2} = t_0 + \tau_i$, i.e., $t_{00} + \tau_i < t_{00} + \tau_{i0} < t_0 + \tau_i$. We have

$$a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) \leq \int_{t_{00}+\tau_i}^{t_{00}+\tau_{i0}} L_{f_0,K_1}(\xi)\left|\varphi(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})\right| d\xi$$

$$+ \int_{t_{00}+\tau_{i0}}^{t_0+\tau_i} L_{f_0,K_1}(\xi)\left|\varphi(\xi - \tau_i) - y_0(\xi - \tau_{i0})\right| d\xi \leq o(\varepsilon\delta\mu)$$

(see (3.63)). Consequently, for $t \in [t_0, \rho_{i2}]$, the inequality (3.64) holds.

Let $t \in [\rho_{i2}, r_2 + \delta_1]$, then $t - \tau_i \geq t_0$ and $t - \tau_{i0} \geq t_{00}$. Therefore

$$a_{2i}(t; t_0, \varepsilon\delta\mu) = a_{2i}(\rho_{i2}; t_0, \varepsilon\delta\mu) + \int_{\rho_{i2}}^{t} L_{f_0,K_1}(\xi)\left|y_0(\xi - \tau_i) + \Delta y(\xi - \tau_i) - y_0(\xi - \tau_{i0})\right| d\xi$$

$$\leq O(\varepsilon\delta\mu) + \int_{\rho_{i2}-\tau_i}^{t-\tau_i} L_{f_0,K_1}(\xi + \tau_i)|\Delta y(\xi)|\, d\xi + \int_{\rho_{i2}}^{t} L_{f_0,K_1}(\xi)\left|y_0(\xi - \tau_i) - y_0(\xi - \tau_{i0})\right| d\xi$$

$$\leq O(\varepsilon\delta\mu) + \int_{t_0}^{t} \chi_1(\xi + \tau_i) L_{f_0,K_1}(\xi + \tau_i)|\Delta y(\xi)|\, d\xi.$$

Consequently, in this case we have

$$a_1(t; t_0, \varepsilon\delta\mu) \leq O(\varepsilon\delta\mu) + \int_{t_0}^{t} \left[L_{f,K_1}(\xi) + \sum_{i=1}^{s} \chi_1(\xi + \tau_i) L_{f,K_1}(\xi + \tau_i)\right] |\Delta y(\xi)|\, d\xi, \quad t \in [t_0, r_2 + \delta_1], \quad (3.65)$$

(see (3.62)).

We now consider the case where $t_{00} + \tau_{i0} > r_2$. Let the numbers $\delta_2 \in (0, \delta_1)$ and $\varepsilon_2'' \in (0, \varepsilon_1)$ be insomuch small that $t_0 + \tau_i > r_2 + \delta_2$ for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2'') \times \Im_+^{(2)}$. It is easy to see that

$$a_{2i}(t; t_0, \varepsilon\delta\mu) \leq \int_{t_0}^{t} L_{f_0,K_1}(\xi)\left|\varphi(\xi - \tau_i) - \varphi_0(\xi - \tau_{i0})\right| dt \leq O(\varepsilon\delta\mu).$$

Thus, for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{00}, r_2 + \delta_2] \times (0, \varepsilon_2) \times \Im_+^{(2)}$, where $\varepsilon_2 = \min(\varepsilon_2', \varepsilon_2'')$, the inequality (3.65) holds.

Obviously,

$$b(t_0, \varepsilon\delta\mu) \leq \gamma \int_{t_0}^{r_2+\delta_2} \sum_{i=1}^{k} \left|\delta f_i(t_0, \tau_1, \ldots, \tau_s, y_0 + \Delta y)(t)\right| dt \leq \gamma \sum_{i=1}^{k} \int_{I} m_{\delta f_i, K_1}(t)\, dt. \qquad (3.66)$$

According to (3.55), (3.65) and (3.66), the inequality (3.58) directly implies

$$|\Delta y(t)| \leq O(\varepsilon\delta\mu) + \int\limits_{t_0}^{t} \left[L_{f,K_1}(\xi) + \sum_{i=1}^{s} \chi_1(\xi+\tau_i)L_{f,K_1}(\xi+\tau_i) \right] |\Delta y(\xi)|\, d\xi, \quad t \in [t_0, r_2+\delta_2].$$

By the Gronwall–Bellman inequality, from the above we obtain (3.54). □

3.5 Proof of Theorem 3.2

First of all, we note that $\delta\mu \in \Im_+^{(2)}$ i.e., $t_{00} < t_0$, therefore we have

$$\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t) & \text{for } t \in [\widehat{\tau}, t_{00}), \\ \varphi(t) - y_0(t) & \text{for } t \in [t_{00}, t_0), \\ \Delta y(t) & \text{for } t \in [t_0, t_{10}+\delta_1]. \end{cases}$$

In a similar way (see (3.31)), one can prove

$$|\varphi(t) - y_0(t)| = O(t; \varepsilon\delta\mu), \quad t \in [t_{00}, t_0].$$

According to the last relation and Lemma 3.4, we have

$$|\Delta x(t)| \leq O(\varepsilon\delta\mu) \ \ \forall (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10}+\delta_2] \times [0, \varepsilon_2] \times \Im_+^{(2)}$$

and

$$\Delta x(t_0) = \varepsilon\big[\delta\varphi(t_{00}) + (\dot{\varphi}_0^+ - f_0^+)\big]\delta t_0 + o(\varepsilon\delta\mu).$$

Let $\widehat{t} \in (t_{00}, t_{10})$ be a fixed point, and let $\varepsilon_2 \in (0, \varepsilon_1)$ be insomuch small that $t_0 < \widehat{t}$ for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im_+^{(2)}$. The function $\Delta x(t)$ satisfies the equation (3.37) on the interval $[t_0, t_{10}+\delta_2]$. Therefore, by using the Cauchy formula, we can represent it in the form

$$\Delta x(t) = Y(t_0; t)\Delta x(t_0) + \varepsilon \int\limits_{t_0}^{t} Y(\xi; t)\delta f[\xi]\, d\xi + \sum_{i=0}^{2} R_i(t; t_0, \varepsilon\delta\mu), \tag{3.67}$$

where $Y(\xi; t)$ is the matrix function satisfying the equation (3.8) and the condition (3.9). The matrix function $Y(\xi; t)$ is continuous on $[t_{00}, \widehat{t}] \times [\widehat{t}, t_{10}+\delta_2]$, therefore

$$Y(t_0; t)\Delta x(t_0) = \varepsilon Y(t_{00}; t)\big[\delta\varphi(t_{00}) + (\dot{\varphi}_0^+ - f_0^+)\big]\delta t_0 + o(\varepsilon\delta\mu). \tag{3.68}$$

Let us now transform

$$R_0(t; t_0, \varepsilon\delta\mu) = \sum_{i=1}^{s} R_{i0}(t; t_0, \varepsilon\delta\mu).$$

It is not difficult to see that

$$R_{i0}(t; t_0, \varepsilon\delta\mu) = \varepsilon \int\limits_{t_0-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0}; t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\, d\xi + \int\limits_{t_{00}}^{t_0} Y(\xi+\tau_0; t)f_{0x_i}[\xi+\tau_{i0}]\Delta x(\xi)\, d\xi$$

$$= \varepsilon \int\limits_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0}; t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\, d\xi + o(t; \varepsilon\delta\mu).$$

Thus,

$$R_0(t; t_0, \varepsilon\delta\mu) = \varepsilon \sum_{i=1}^{s} \int\limits_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0}; t)f_{0x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\, d\xi + o(t; \varepsilon\delta\mu). \tag{3.69}$$

In a similar way, with nonessential changes, for $t \in [\widehat{t}, t_{10} + \delta_2]$, one can prove

$$R_1(t; t_0, \varepsilon\delta\mu) = -\varepsilon \sum_{i=1}^{s} \int_{t_{00}}^{t} Y(\xi; t)\big[f_{0x_i}[\xi]\dot{x}_0(\xi - \tau_{i0})\delta\tau_i\big]\, d\xi + o(t; \varepsilon\delta\mu), \tag{3.70}$$

$$R_2(t; t_0, \varepsilon\delta\mu) = o(t; \varepsilon\delta\mu). \tag{3.71}$$

Finally, we note that for $t \in [\widehat{t}, t_{10} + \delta_2]$,

$$\varepsilon \int_{t_0}^{t} Y(\xi; t)\delta f[\xi]\, d\xi = \varepsilon \int_{t_{00}}^{t} Y(\xi; t)\delta f[\xi]\, d\xi + o(t; \varepsilon\delta\mu). \tag{3.72}$$

Taking into account (3.68)–(3.72), from (3.67) we obtain (3.5), where $\delta x(t; \varepsilon\delta\mu)$ has the form (3.10).

4 Optimal control problems and necessary conditions of optimality

4.1 Preliminaries and necessary criticality condition

In this subsection by E_z we will denote a vector space. The k-dimensional vector space E_z^k will be identified with the space $\mathbb{R}^k$. By the module of an element $z \in E_z^k$ we will mean the Euclidean module

$$|z|^2 = z^\top z = \sum_{i=1}^{k} (z^i)^2.$$

In what follows, finite-dimensional vector spaces will be endowed with the Euclidean topology. Let $z_i \in E_z$, $i = \overline{1, k}$. The set

$$L = \Big\{ z = \sum_{i=1}^{k} \lambda_i z_i : \ \lambda_i \in \mathbb{R}, \ \ i = \overline{1, k} \Big\}$$

is called the finite-dimensional linear manifold generated by the points z_i, $i = \overline{1, k}$. If $z_0 \in L$, then we say that the manifold L passes through the point z_0 and it will be denoted by L_{z_0}. In what follows, we will write the manifold L_{z_0} in the equivalent form

$$L_{z_0} = \Big\{ z = z_0 + \sum_{i=1}^{k} \lambda_i z_i : \ \lambda_i \in \mathbb{R}, \ \ i = \overline{1, k} \Big\}. \tag{4.1}$$

For each fixed $\alpha > 0$, the set

$$\Big\{ \sum_{i=1}^{k} \lambda_i z_i : \ |\lambda_i| \leq \alpha, \ \ i = \overline{1, k} \Big\} \tag{4.2}$$

is a convex bounded neighborhood of zero in the space $L_{z_0} - z_0$.

Definition 4.1. We say that points $z_i \in E_z$, $i = \overline{0, k}$, are in a general position if the vectors $z_i - z_0$, $i = \overline{1, k}$, are linearly independent.

From this definition it follows that for any z_i, $i = \overline{1, k}$, the system of vectors $z_0 - z_i, \ldots, z_{i-1} - z_i, z_{i+1} - z_i, \ldots, z_k - z_i$ is linearly independent, as well.

Definition 4.2. Let the points z_i, $i = \overline{0, k}$, be in a general position. The convex hull of points z_i, $i = \overline{0, k}$, i.e., $\text{co}(\{z_0, \ldots, z_k\})$, is called a k-dimensional simplex.

Clearly, a k-dimensional simplex is a convex compact set in the linear finite dimensional manifold generated by the points z_i, $i = \overline{0, k}$. It is easy to note that

$$\mathrm{co}(\{z_0, \dots, z_k\}) = z_0 + \mathrm{co}(\{0, z_1 - z_0, \dots, z_k - z_0\})$$

$$= z_0 + \Big\{ \sum_{i=1}^{k} \lambda_i (z_i - z_0) : \ \lambda_i \geq 0, \ \sum_{i=1}^{k} \lambda_i \leq 1 \Big\}. \tag{4.3}$$

From the relations (4.2), (4.3) and Definition 4.2 follow Lemmas 4.1 and 4.2.

Lemma 4.1. *The simplex* $\mathrm{co}(\{z_0, \dots, z_k\})$ *has a nonempty interior.*

Lemma 4.2. *Each point z of a simplex $\mathrm{co}(\{z_0, \dots, z_k\})$ can be uniquely represented in the form*

$$z = \sum_{i=0}^{k} \lambda_i z_i, \ \ \text{where} \ \lambda_i \geq 0, \ i = \overline{0, k}, \ \text{and} \ \sum_{i=0}^{k} \lambda_i = 1.$$

Lemma 4.3. *Let $M \subset E_z^k$ and, moreover, let $0 \in \mathrm{int}\, M$. Then in E_z^k there exists a k-dimensional simplex which is contained in M and contains $0 \in E_z^k$ as an interior point.*

Proof. Let $\mathrm{co}(\{z_0, \dots, z_k\}) \subset E_z^k$ be a k-dimensional simplex. By Lemma 4.1, there exists $\widehat{z} \in \mathrm{int}\, \mathrm{co}(\{z_0, \dots, z_k\})$ such that the k-dimensional simplex

$$-\widehat{z} + \mathrm{co}(\{z_0, \dots, z_k\}) = \mathrm{co}(\{z_0 - \widehat{z}, \dots, z_k - \widehat{z}\})$$

contains $0 \in E_z^k$ as an interior point. By assumption, there exists a convex neighborhood

$$V = \big\{ z \in E_z^k : \ |z| < \varepsilon_0 \big\}, \ \ \varepsilon_0 > 0,$$

of zero contained in M.

Let $\varepsilon > 0$ be a number such that $\varepsilon(z_i - \widehat{z}) \in V$, $i = \overline{0, k}$. Hence the k-dimensional simplex $\varepsilon\, \mathrm{co}(\{z_0 - \widehat{z}, \dots, z_k - \widehat{z}\})$ is contained in M and contains $0 \in E_z^k$ as an interior point. $\qquad\square$

Lemma 4.4. *Let a linear mapping*

$$g : E_z \to E_g^k \tag{4.4}$$

and a k-dimensional simplex $\mathrm{co}(\{g_0, \dots, g_k\}) \subset E_g^k$ be given. Let z_i, $i = \overline{0, k}$, be certain inverse images of the points g_i, $i = \overline{0, k}$, under the mapping (4.4), respectively. Then $\mathrm{co}(\{z_0, \dots, z_k\}) \subset E_z$ is a k-dimensional simplex, and the restriction of the mapping

$$g : \mathrm{co}(\{z_0, \dots, z_k\}) \longrightarrow \mathrm{co}(\{g_0, \dots, g_k\}) \tag{4.5}$$

is a homeomorphism.

Proof. Let there exist numbers λ_i, $i = \overline{1, k}$, such that

$$\sum_{i=1}^{k} \lambda_i (z_i - z_0) = 0, \ \ \ \sum_{i=1}^{k} |\lambda_i| \neq 0.$$

Obviously,

$$g\Big(\sum_{i=1}^{k} \lambda_i (z_i - z_0) \Big) = \sum_{i=1}^{k} \lambda_i (g_i - g_0) = 0;$$

in turn, this contradicts the linear independence of the elements $g_i - g_0$, $i = \overline{1, k}$.

Therefore, $\mathrm{co}(\{z_0, \dots, z_k\})$ is a k-dimensional simplex. By Lemma 4.2, the mapping (4.5) is a homeomorphism. $\qquad\square$

Let the Hausdorff vector topology be given in E_z, which transforms E_z into a topological vector space.

Lemma 4.5. *Let $W \subset E_z$ and let a mapping*

$$P : W \to E_p^k, \tag{4.6}$$

continuous in the topology induced from E_z, be given. Further, let $K \subset W$ be a compact set. Then for any $\varepsilon > 0$, there exists a neighborhood $V_\varepsilon \subset E_z$ of zero such that

$$|P(z') - P(z'')| \le \varepsilon \ \ \forall (z', z'') \in K \times W, \ \ z' - z'' \in V_\varepsilon.$$

Proof. For each point $z' \in K$, there exists a convex neighborhood $V(z') \subset E_z$ of zero such that

$$|P(z') - P(z)| \le \frac{\varepsilon}{3} \ \ \forall z \in (z' + V(z')) \cap W.$$

The system of sets $\{z' + V(z') : \ z' \in K\}$ composes an open covering of the compact set K. Hence there exists a finite subcovering $\{z_i' + V(z_i') : \ i = \overline{1, m}\}$ of the set K.

Clearly, for $z \in (z_i' + V(z_i')) \cap W$,

$$|P(z_i') - P(z)| \le \frac{\varepsilon}{3}. \tag{4.7}$$

By the continuity of the mapping (4.6), for $2\varepsilon/3$, there exist convex neighborhoods $V_i \supset V(z_i')$, $i = \overline{1, m}$, of zero such that

$$|P(z_i') - P(z)| \le 2\frac{\varepsilon}{3} \ \ \forall z \in (z_i' + V_i) \cap W. \tag{4.8}$$

Obviously, the sets

$$\widehat{V}_i = V_i - V(z_i') = V_i + (-1)V(z_i'), \ \ V_\varepsilon = \bigcap_{i=1}^{m} \widehat{V}_i$$

are the neighborhoods of zero in E_z, and for an arbitrary point $z \in (z_i' + V(z_i') + \widehat{V}_i) \cap W$, the inequality (4.8) holds.

Let $(z', z'') \in K \times W$, $z' - z'' \in V_\varepsilon$ and the point z' belong to some of the sets $z_k' + V(z_k')$, $1 \le k \le m$. Further,

$$z'' - z_k' = z'' - z' + z' - z_k' \in V_\varepsilon + V(z_k') \subset \widehat{V}_k + V(z_k') = V_k.$$

Taking into account the inequalities (4.7) and (4.8), we have

$$|P(z') - P(z'')| \le |P(z') - P(z_k')| + |P(z_k') - P(z'')| \le \frac{\varepsilon}{3} + \frac{2}{3}\varepsilon = \varepsilon. \qquad \square$$

Definition 4.3. The set Ψ of a subsets from E_z is called a filter if it satisfies the following conditions:

(a) if $A \in \Psi$ and $B \in \Psi$, then $A \cap B \in \Psi$;

(b) if $A \in \Psi$ and $B \supset A$, then $B \in \Psi$;

(c) $\varnothing \notin \Psi$.

The set of all neighborhoods of a fixed point of the space E_z serves as an example of a filter.

Definition 4.4. A set $\widehat{\Re}$ of a subset of E_z is called a basis of a filter if it has the following properties:

(a) for any $A \in \widehat{\Re}$ and $B \in \widehat{\Re}$, there exists $C \in \widehat{\Re}$ such that $C \subset A \cap B$;

(b) $\varnothing \notin \Re$.

The set Ψ of all subsets each of which contains a certain set from $\widehat{\Re}$ is the filter generated by the basis $\widehat{\Re}$.

Theorem 4.1 (Carathéodory). *Let $M \subset E_z^k$. Then any point $z \in \mathrm{co}(M)$ can be represented in the form*

$$z = \sum_{i=0}^{k} \lambda_i z_i,$$

where $z_i \in M$, $\lambda_i \geq 0$, $i = \overline{0,k}$, and $\sum_{i=0}^{k} \lambda_i = 1$.

Theorem 4.2 (Brouwer). *Let $\mathrm{co}(\{z_0, \ldots, z_k\}) \subset E_z$ be a k-dimensional simplex. Then each continuous mapping*

$$g : \mathrm{co}(\{z_0, \ldots, z_k\}) \longrightarrow \mathrm{co}(\{z_0, \ldots, z_k\})$$

has a fixed point, i.e., there exists a point $z \in \mathrm{co}(\{z_0, \ldots, z_k\})$ such that $g(z) = z$.

Theorem 4.3. *Let $M \subset E_z^k$ be a convex set and $0 \in \partial M$. Then there exists a nonzero k-dimensional vector $\pi = (\pi_1, \ldots, \pi_k)$ such that*

$$\pi z = \sum_{i=1}^{k} \pi_i z^i \leq 0 \ \ \forall z \in M.$$

Let $E_z = E_x^k \times E_\varsigma$ be a vector space of points $z = (x, \varsigma)$. Assume that $D \subset E_z$ is a certain set and a mapping

$$P : D \to E_p^m \tag{4.9}$$

is given. Let Ψ be an arbitrary filter in E_z.

Definition 4.5. We say that the mapping (4.9) is defined on the filter Ψ if there exists an element $W \in \Psi$ such that $W \subset D$.

Definition 4.6. Let the mapping (4.9) be defined on the filter Ψ. The mapping (4.9) is said to be critical on the filter Ψ if for any point z_0 belonging to all elements of the filter Ψ, there exists an element $W \subset \Psi$ such that $W \subset D$ and $P(z_0) \in \partial P(W)$.

Definition 4.7. We say that the mapping (4.9) defined on the filter Ψ is continuous on Ψ if there exists an element $W \in \Psi$ such that $W \subset D$ and the restriction

$$P : W \to E_p^m$$

of the mapping (4.9) is continuous in the topology induced from E_z.

Let $X \subset E_z^k$ be a locally convex topological subspace, i.e., for an arbitrary neighborhood $V_x \subset X$ of a point $x \in X$, there exists a convex neighborhood $\widehat{V}_x \subset X$ contained in V_x. The following lemma is easily proved.

Lemma 4.6. *Let $\widehat{x} \in X$ be a fixed point. Further, let $V_0 \subset X - \widehat{x}$ be a convex bounded neighborhood of zero, and let $V_1 \subset X - \widehat{x}$ be a certain neighborhood of zero. Then there exists a number $\varepsilon_0 > 0$ such that*

$$\varepsilon V_0 \subset V_1 \ \ \forall \varepsilon \in (0, \varepsilon_0).$$

Definition 4.8. A set $D \subset X \times E_\varsigma$ is said to be finitely locally convex if for an arbitrary point $z = (x, \varsigma) \in D$ and for arbitrary manifold $L_\varsigma \subset E_\varsigma$, there exist convex neighborhoods $V_x \subset X$ and $V_\varsigma \subset E_\varsigma$ of the points x and ς, respectively, such that

$$V_x \times V_\varsigma \subset D.$$

Lemma 4.6 and Definition 4.8 directly imply the following lemma.

Lemma 4.7. *Let D be a finitely locally convex set, and let $z_0 = (x_0, \varsigma_0) \in D$. Further, let $V_0 \subset X - x_0$ and $V \subset L_{\varsigma_0} - \varsigma_0$ be bounded convex neighborhoods of zero (see (4.2)). Then there exists a number $\varepsilon_0 > 0$ such that*

$$z_0 + \varepsilon \delta z \in D \ \ \forall (\varepsilon, \delta z) \in (0, \varepsilon_0) \times V_0 \times V, \ \ \delta z = (\delta x, \delta \varsigma). \tag{4.10}$$

Definition 4.9. We say that the mapping (4.9) has a differential at a point $z_0 = (x_0, \varsigma_0) \in D$ if there exists a linear mapping

$$dP_{z_0} : E_{\delta z} = E_z - z_0 \to E_{\delta p}^m \tag{4.11}$$

such that for any manifold

$$L_{\varsigma_0} = \left\{ \varsigma_0 + \sum_{i=1}^{k} \lambda_i \delta \varsigma_i : \ \lambda_i \in \mathbb{R}, \ i = \overline{1, k} \right\} \subset E_\varsigma$$

(see (4.1)) the representation

$$P(z_0 + \varepsilon \delta z) - P(z_0) = \varepsilon dP_{z_0}(\delta z) + o(\varepsilon \delta z) \ \ \forall (\varepsilon, \delta z) \in (0, \varepsilon_0) \times V_0 \times V$$

holds, where $V_0 \subset X - x_0$ and $V \subset L_{\varsigma_0} - \varsigma_0$ are bounded neighborhoods of zero; $\varepsilon_0 > 0$ is the number for which (4.10) holds;

$$\lim_{\varepsilon \to 0} \frac{o(\varepsilon \delta z)}{\varepsilon} = 0 \ \ \text{uniformly in} \ \ \delta z \in V_0 \times V.$$

The mapping (4.11) is called the differential of the mapping (4.9) at the point z_0.

Definition 4.10 (Gamkrelidze)**.** A filter Ψ in E_z is said to be quasiconvex if for any element $W \in \Psi$ and any natural number k, there exists an element $W_1 = W_1(W, k) \in \Psi$ such that for arbitrary points $z_i \in W_1$, $i = \overline{0, k}$, and an arbitrary neighborhood of zero $V \subset E_z$, there exists a continuous mapping

$$\phi : \mathrm{co}(\{z_0, \ldots, z_k\}) \longrightarrow W. \tag{4.12}$$

satisfying the condition

$$(z - \phi(z)) \in V \ \ \forall z \in \mathrm{co}(\{z_0, \ldots, z_k\}).$$

Obviously, in Definition 4.10, we can assume that $W_1 \subset W$, since any element $W_2 \subset W \cap W_1$ of the filter has the indicated property of the element W_1. Therefore, in what follows, we will assume that $W_1 \subset W$.

Definition 4.11. A filter Ψ is said to be convex if there exists a basis of the filter consisting of convex sets.

Lemma 4.8. *Every convex filter Ψ in E_z is quasiconvex.*

Proof. For any element $W \in \Psi$, there exists a convex element $W_2 \subset W$, which can be taken as W_1; as the mapping (4.12), it is necessary to take the identity mapping. $\qquad \square$

Let E_z be a topological vector space, $X \subset E_x^k$ be a locally convex topological space, and $D \subset X \times E_\varsigma$ be a finitely locally convex set.

Let a mapping

$$P : D \to E_p^m \tag{4.13}$$

be given, and let Ψ be a filter in E_z.

By $\mathrm{co}[\Psi]$ we denote the convex filter whose elements are the sets $\mathrm{co}(W)$, where W is an arbitrary element of the filter Ψ.

Theorem formulated below is an analogue of R. V. Gamkrelidze and G. L. Kharatishvili's Theorem on the necessary criticality condition to mappings defined on a finitely locally convex set. The proof of the following theorem is performed according to the scheme presented in [7–9] with only nonessential changes.

Theorem 4.4. *Let the mapping (4.13) be continuous on $\mathrm{co}[\Psi]$ and critical on Ψ. Further, let the filter Ψ be quasiconvex. Then for any point $z_0 = (x_0, \varsigma_0)$ belonging to all sets of the filter Ψ at which the mapping (4.13) has the differential (4.11), there exists an element $\widehat{W} \in \Psi$ such that zero of the space E_{dp}^m is a boundary point of the set*

$$dP_{z_0}(\mathrm{co}(\widehat{W}) - z_0) \subset E_{dp}^m. \tag{4.14}$$

Proof. By the assumption, there exist elements $W_i \in \Psi$, $i = 1, 2$, such that $\mathrm{co}(W_1) \subset D$, $W_2 \subset D$, and, moreover, the mapping

$$P : \mathrm{co}(W_1) \rightarrow E_p^m \tag{4.15}$$

is continuous and $P(z_0) \in \partial P(W_2)$. Clearly, $W_3 = W_1 \cap W_2 \in \Psi$ and $P(z_0) \in \partial P(W_3)$.

Let the conditions of the theorem hold, but for any $W \in \Psi$ lying in D, the point $0 \in E_{dp}^m$ is an interior point of the set

$$dP_{z_0}(\mathrm{co}(W) - z_0) \subset E_{dp}^m.$$

Let us show that this contradicts the choice of the element W_3. Precisely, we prove the solvability of the following equation

$$P(z) = P(z_0) + p, \quad z \in W_3, \tag{4.16}$$

with respect to z and for any vector $p \in E_p^m$ whose module is sufficiently small, and, therefore, we prove that $P(z_0)$ is an interior point of the set $P(W_3) \subset E_p^m$, which contradicts the choice of W_3. By $W_4 = W_4(W_3; (m+1)^2) \subset W_3$ we denote the element of the quasiconvex filter Ψ (see Definition 4.10) satisfying the following condition: for any neighborhood of zero $V \subset E_z$ and any $1 + (m+1)^2$ points $z_0, \ldots, z_{(m+1)^2}$ from W_4, there exists a continuous mapping

$$\phi : \mathrm{co}(\{z_0, \ldots, z_{(m+1)^2}\}) \longrightarrow W_3 \tag{4.17}$$

satisfying the condition

$$(z - \phi(z)) \in V \quad \forall z \in \mathrm{co}(\{z_0, \ldots, z_{(m+1)^2}\}). \tag{4.18}$$

According to the assumption made, $0 \in E_{dp}^m$ is an interior point of the convex set

$$dP_{z_0}(\mathrm{co}(W_4) - z_0) \subset E_{dp}^m. \tag{4.19}$$

Hence there exist $m+1$ points $dp_i \in dP_{z_0}(\mathrm{co}(W_4) - z_0)$ that are in general position, and, moreover, the m-dimensional simplex $\mathrm{co}(\{dp_0, \ldots, dp_m\})$ containing $0 \in E_{dp}^m$ as an interior point (see Lemma 4.3).

By the linearity of the mapping (4.11),

$$dP_{z_0}(\mathrm{co}(W_4) - z_0) = \mathrm{co}(dP_{z_0}(W_4 - z_0)).$$

Each of the points

$$dp_i \in \mathrm{co}(dP_{z_0}(W_4 - z_0)), \quad i = \overline{0, m},$$

is represented in the form

$$dp_i = \sum_{j=0}^{m} \mu_{ij} dp_{ij}, \quad dp_{ij} \in dP_{z_0}(W_4 - z_0), \quad \mu_{ij} \geq 0, \quad \sum_{j=0}^{m} \mu_{ij} = 1$$

(see Theorem 4.1). Let $\delta z_{ij} \in W_4 - z_0$ be some inverse images of the points dp_{ij} under the mapping

$$dP_{z_0} : W_4 - z_0 \longrightarrow E_{dp}^m,$$

and let

$$\delta z_i = \sum_{j=0}^{m} \mu_{ij} \delta z_{ij}, \quad i = \overline{0, m}. \tag{4.20}$$

Obviously,

$$dP_{z_0}(\delta z_i) = dp_i, \quad i = \overline{0, m}.$$

By Lemma 4.4, the points $\delta z_i = (\delta x_i, \delta \varsigma_i)$, $i = \overline{0, m}$, are in general position and the mapping

$$dP_{z_0} : \mathrm{co}(\{\delta z_0, \ldots, \delta z_m\}) \longrightarrow \mathrm{co}(\{dp_0, \ldots, dp_m\}) \tag{4.21}$$

is a homeomorphism.

Let $z \in \mathrm{co}(\{z_0, z_0 + \delta z_0, \ldots, z_0 + \delta z_m\})$. Then

$$z = z_0 + \sum_{i=0}^{m} \lambda_i \delta z_i = z_0 + \sum_{i=0}^{m} \sum_{j=0}^{k} \lambda_i \mu_{ij} \delta z_{ij} = \left(1 - \sum_{i=0}^{m} \sum_{j=0}^{m} \lambda_i \mu_{ij}\right) z_0 + \sum_{i=0}^{m} \sum_{j=0}^{m} \lambda_i \mu_{ij} (z_0 + \delta z_{ij}),$$

$$\lambda_i \geq 0, \quad \sum_{i=0}^{m} \lambda_i \leq 1$$

(see (4.3) and (4.20)). Hence

$$\mathrm{co}\left(\{z_0, z_0 + \delta z_0, \ldots, z_0 + \delta z_m\}\right) \subset \mathrm{co}\left(\{z_0, z_0 + \delta z_{00}, \ldots, z_0 + \delta z_{ij}, \ldots, z_0 + \delta z_{mm}\}\right). \tag{4.22}$$

Further, let us show that for $\varepsilon \in [0, 1]$, the inclusion

$$z_0 + \varepsilon \, \mathrm{co}(\{\delta z_0, \ldots, \delta z_m\}) \subset \mathrm{co}\left(\{z_0, z_0 + \delta z_0, \ldots, z_0 + \delta z_m\}\right) \tag{4.23}$$

holds. Indeed, it is clear that every point $z_0 + \varepsilon \, \mathrm{co}(\{\delta z_0, \ldots, \delta z_m\})$ is represented in the form

$$z = z_0 + \varepsilon \sum_{i=0}^{m} \lambda_i \delta z_i = (1 - \varepsilon) z_0 + \varepsilon \sum_{i=0}^{m} \lambda_i (z_0 + \delta z_i) \in \mathrm{co}\left(\{z_0, z_0 + \delta z_0, \ldots, z_0 + \delta z_m\}\right).$$

The inclusions (4.22) and (4.23) imply

$$z_0 + \varepsilon \, \mathrm{co}(\{\delta z_0, \ldots, \delta z_m\}) \subset \mathrm{co}\left(\{z_0, z_0 + \delta z_0, \ldots, z_0 + \delta z_m\}\right)$$
$$\subset \mathrm{co}(W_4) \subset \mathrm{co}(W_3) \subset D \; \forall \varepsilon \in [0, 1]. \tag{4.24}$$

Taking into account

$$D - z_0 \subset (X - x_0) \times (E_\varsigma - \varsigma_0),$$

we see that the latter relation directly implies the inclusion

$$\varepsilon \, \mathrm{co}(\{\delta x_0, \ldots, \delta x_m\}) \subset (X - x_0), \quad \varepsilon \in [0, 1]. \tag{4.25}$$

Let $L_{\varsigma_0} \subset E_\varsigma$ be the manifold generated by the points $\varsigma_0, \delta\varsigma_0, \ldots, \delta\varsigma_m$:

$$L_{\varsigma_0} = \left\{\varsigma_0 + \sum_{i=0}^{m+1} \lambda_i \delta\varsigma_i : \; \lambda_i \in \mathbb{R}, \; i = \overline{0, m+1}\right\}, \quad \delta\varsigma_{m+1} = \varsigma_0.$$

Obviously,

$$\varepsilon \, \mathrm{co}(\{\delta\varsigma_0, \ldots, \delta\varsigma_m\}) \subset L_{\varsigma_0} - \varsigma_0. \tag{4.26}$$

Let $V_0 \subset X - x_0$ and $V \subset L_{\varsigma_0} - \varsigma_0$ be convex bounded neighborhoods of zero. There exists a number $\varepsilon_1 \in (0, 1)$ such that

$$\varepsilon_1 \, \mathrm{co}(\{\delta x_0, \ldots, \delta x_m\}) \subset V_0, \quad \varepsilon_1 \, \mathrm{co}(\{\delta\varsigma_0, \ldots, \delta\varsigma_m\}) \subset V$$

(see (4.25) and (4.26)). Hence

$$\varepsilon_1 \, \mathrm{co}(\{\delta z_0, \ldots, \delta z_m\}) \subset V_0 \times V. \tag{4.27}$$

Let $w(\varepsilon) = \varepsilon \varepsilon_1$, $\varepsilon \in (0, 1)$. Lemma 4.7 implies the existence of a number $\varepsilon_2 \in (0, 1)$ such that

$$z_0 + w(\varepsilon) \delta z \in D \; \forall (\varepsilon, \delta z) \in (0, \varepsilon_2) \times V_0 \times V.$$

Denote by $d > 0$ the distance from the point $0 \in E_{dp}^m$ to the boundary of the simplex $\mathrm{co}(\{dp_0, \ldots, dp_m\})$.

The differentiability of the mapping (4.13) at the point z_0 implies the existence of a number $\varepsilon_3 \in (0, \varepsilon_2)$ such that

$$P(z_0 + w(\varepsilon) \delta z) = P(z_0) + w(\varepsilon) dP_{z_0}(\delta z) + o(w(\varepsilon) \delta z) \; \forall (\varepsilon, \delta z) \in (0, \varepsilon_3) \times V_0 \times V; \tag{4.28}$$

moreover,

$$\frac{|o(w(\varepsilon)\delta z)|}{w(\varepsilon)} \leq \frac{d}{3} \quad \forall (\varepsilon, \delta z) \in (0, \varepsilon_3) \times V_0 \times V. \tag{4.29}$$

Obviously, on account of (4.27), the relations (4.28) and (4.29) hold for $(\varepsilon, \delta z) \in (0, \varepsilon_3) \times \mathrm{co}(\{\delta z_0, \ldots, \delta z_m\})$.

The mapping (4.15) is continuous on $\mathrm{co}(W_3)$ in the topology of the space $X \times L_{\varsigma_0}$. Therefore, $P(z_0 + w(\varepsilon)\delta z)$ is continuous in $\delta z \in \mathrm{co}(\{\delta z_0, \ldots, \delta z_m\})$ (see (4.24)). Using (4.28), we conclude from the above-said that for each $\varepsilon \in (0, \varepsilon_3)$, the function $o(w(\varepsilon)\delta z)$ is continuous on $\mathrm{co}(\{\delta x_0, \ldots, \delta x_m\})$.

Further, the continuity of the mapping P on $\mathrm{co}(W_3)$ and the compactness of the set $z_0 + w(\varepsilon)\mathrm{co}(\{\delta z_0, \ldots, \delta z_k\}) \subset \mathrm{co}(W_3)$ imply that for each $\varepsilon \in (0, \varepsilon_3)$, there exists a neighborhood of zero $V_\varepsilon \subset E_z$ such that for

$$z' \in z_0 + w(\varepsilon)\mathrm{co}(\{\delta x_0, \ldots, \delta x_m\}), \quad z'' \in \mathrm{co}(W_3), \quad z' - z'' \in V_\varepsilon, \tag{4.30}$$

we have

$$|P(z') - P(z'')| \leq w(\varepsilon)\frac{d}{3} \tag{4.31}$$

(see Lemma 4.5).

The conditions (4.17), (4.18) and the relation (4.22) directly imply the existence of a family of continuous mappings

$$\phi_\varepsilon : \mathrm{co}\left(\{z_0, z_0 + \delta z_0, \ldots, z_0 + \delta z_m\}\right) \longrightarrow W_4,$$

depending on $\varepsilon \in (0, \varepsilon_3)$ and satisfying the condition

$$z - \phi_\varepsilon(z) \in V_\varepsilon \quad \forall z \in \mathrm{co}\left(\{z_0, z_0 + \delta z_0, \ldots, z_0 + \delta z_m\}\right).$$

For $\varepsilon \in (0, \varepsilon_3)$, the simplex $z_0 + w(\varepsilon)\mathrm{co}(\{\delta z_0, \ldots, \delta z_m\})$ is contained in $\mathrm{co}(\{z_0, z_0 + \delta z_0, \ldots, z_0 + \delta z_m\})$ (see (4.23), and, therefore,

$$z - \phi_\varepsilon(z) \in V_\varepsilon \quad \forall z \in z_0 + w(\varepsilon)\mathrm{co}(\{\delta z_0, \ldots, \delta z_m\}). \tag{4.32}$$

Let us now show that the equation

$$P(\phi_\varepsilon(z)) = P(z_0) + w(\varepsilon)p, \quad z \in z_0 + w(\varepsilon)\mathrm{co}(\{\delta z_0, \ldots, \delta z_m\}) \tag{4.33}$$

is solvable in z for a sufficiently small ε and an arbitrary $p \in E_p^m$ satisfying the condition

$$|p| \leq \frac{d}{3}. \tag{4.34}$$

Indeed, we rewrite this equation in the form

$$P(z) = P(z_0) + w(\varepsilon)p + P(z) - P(\phi_\varepsilon(z)), \quad z \in z_0 + w(\varepsilon)\mathrm{co}(\{\delta z_0, \ldots, \delta z_m\}),$$

or, using (4.28), in the form of the following equation in δz:

$$dP_{z_0}(\delta z) = p - \frac{o(w(\varepsilon)\delta z)}{w(\varepsilon)} + \frac{P(z_0 + w(\varepsilon)\delta z) - P(\phi_\varepsilon(z_0 + w(\varepsilon)\delta z)}{w(\varepsilon))}, \quad \delta z \in \mathrm{co}(\{\delta z_0, \ldots, \delta z_m\}). \tag{4.35}$$

The relations (4.29)–(4.32) and (4.34) imply

$$\left(p - \frac{o(w(\varepsilon)\delta z)}{w(\varepsilon)} + \frac{P(z_0 + w(\varepsilon)\delta z) - P(\phi_\varepsilon(z_0 + w(\varepsilon)\delta z)}{w(\varepsilon))}\right) \in \mathrm{co}(\{\delta p_0, \ldots, \delta p_m\})$$

and hence the equation (4.35) is equivalent to the equation

$$\delta z = dP_{z_0}^{-1}\left(p - \frac{o(w(\varepsilon)\delta z)}{w(\varepsilon)} + \frac{P(z_0 + w(\varepsilon)\delta z) - P(\phi_\varepsilon(z_0 + w(\varepsilon)\delta z)}{w(\varepsilon)}\right), \tag{4.36}$$

where

$$dP_{z_0}^{-1} : \mathrm{co}(\{dp_0, \ldots, dp_m\}) \longrightarrow \mathrm{co}(\{\delta z_0, \ldots, \delta z_m\})$$

is a continuous mapping, inverse to the mapping (4.21).

We can consider the right-hand side of the equation (4.36) as a continuous self-mapping of the simplex $\mathrm{co}(\{\delta z_0, \ldots, \delta z_m\})$, and hence each fixed point of this mapping is a solution of the equation (4.36) (see Theorem 4.2). Thus, we have proved the solvability of the equation (4.33) for an arbitrary p satisfying (4.34) and, therefore, the solvability of the equation (4.16) for p whose modules are sufficiently small. $\square$

Theorem 4.5. *Let the conditions of Theorem 4.4 hold. Then for any point z_0 belonging to all sets Ψ at which the differential (4.11) exists, there exist an element $\widehat{W} \in \Psi$ and a vector $\pi = (\pi_1, \ldots, \pi_m) \neq 0$ such that*

$$\pi dP_{z_0}(\delta z) = \sum_{i=1}^{m} \pi_i dP_{z_0}^i(\delta z) \leq 0 \ \ \forall \in \mathrm{cone}(\mathrm{co}(\widehat{W}) - z_0), \tag{4.37}$$

where $\mathrm{cone}(\widehat{W})$ is the cone generated by the set $\widehat{W}$.

Proof. Set (4.14), being the image of a convex set under a linear mapping, is also convex. Since $0 \in E_{dp}^m$ is a boundary point of the convex set (4.14), by Theorem 4.3, there exists a nonzero m-dimensional vector for which

$$\pi dP_{z_0}(\delta z) \leq 0 \ \ \forall \delta z \in \mathrm{co}(\widehat{W} - z_0).$$

This implies (4.37). $\square$

4.2 Gamkrelidze's approximation lemma

Let $U_0 \subset \mathbb{R}^r$ be an open set. Now let us consider the function $f(t, x, x_1, \ldots, x_s, u), (t, x, x_1, \ldots, x_s, u) \in I \times O^{s+1} \times U_0$, satisfying the following conditions: for almost all $t \in I$, the function $f : I \times O^{s+1} \times U_0 \to \mathbb{R}^n$ is continuous and continuously differentiable in $(x, x_1, \ldots, x_s) \in O^{s+1}$; for each $(x, x_1, \ldots, x_s, u) \in O^{s+1} \times U_0$, the function $f(t, x, x_1, \ldots, x_s, u)$ and the matrices $f_x(t, x, \cdot), f_{x_i}(t, x, \cdot), i = \overline{1, s}$, are measurable on I; for any compact sets $K \subset O$ and $M \subset U_0$, there exists a function $m_{K,M}(t) \in L_1(I, \mathbb{R}_+)$ such that for any $(x, x_1, \ldots, x_s, u) \in K^{s+1} \times M$ and almost all $t \in I$,

$$|f(t, x, x_1, \ldots, x_s, u)| + |f_x(t, x, \cdot)| + \sum_{i=1}^{s} |f_{x_i}(t, x, \cdot)| \leq m_{K,M}(t).$$

Introduce the set

$$F = \big\{ f(t, x, x_1, \ldots, x_s) = f(t, x, x_1, \ldots, x_s, u(t)) : \ u \in \Omega(I, U) \big\},$$

where $U \subset U_0$ is a given set. The set F can be identified with a subset of the space $E_f^{(1)}$. A family of subintervals

$$\sigma = \big\{ I_\beta = [t_\beta, t_{\beta+1}] : \ \beta = \overline{1, m} \big\},$$

where $a = t_1 < t_2 < \cdots < t_{m-1} < t_m = b$, is called a σ-partition of the interval I.

Let the points $f_i(t, x, x_1, \ldots, x_s) = f(t, x, x_1, \ldots, x_s, u_i(t)) \in F$, $i = \overline{1, k+1}$, and the σ-partition of the interval be given. Using these data, to each point λ of the k-dimensional simplex

$$\Sigma = \Big\{ \lambda = (\lambda_1, \ldots, \lambda_{k+1}) : \ \lambda_i \geq 0, \ \sum_{i=1}^{k+1} \lambda_i = 1 \Big\}$$

we can uniquely put in correspondence the subdivision of each intervals I_β into $k + 1$ subintervals $I_{\beta_i}(\lambda)$, $i = \overline{1, k+1}$, defined by the condition

$$\mathrm{mes}\, I_{\beta_i}(\lambda) = \lambda_i \,\mathrm{mes}\, I_\beta, \ \ i = \overline{1, k+1}, \tag{4.38}$$

if $\lambda_i = 0$, then the corresponding interval degenerates into a point. Define the mapping

$$\phi_\sigma : \Sigma \to F \tag{4.39}$$

by the formula

$$\phi_\sigma(\lambda) = f_\lambda(t, x, x_1, \ldots, x_s) = f\big(t, x, x_1, \ldots, x_s, u_\lambda(t)\big),$$

where

$$u_\lambda(t) = u_i(t), \ \ t \in I_{\beta_i}(\lambda), \ \ \beta = \overline{1, m}, \ \ i = \overline{1, k+1}.$$

It is clear that

$$f_\lambda(t, x, x_1, \ldots, x_s) = f_i(t, x, x_1, \ldots, x_s), \ \ t \in I_{\beta_i}(\lambda), \ \ (x, x_1, \ldots, x_s) \in O^{s+1}, \tag{4.40}$$
$$\beta = \overline{1, m}, \ \ i = \overline{1, k+1}.$$

The relations (4.38) and (4.40) play principal role in proving the following

Lemma 4.9 (Gamkrelidze's approximation lemma [6,7,10])**.** *For an arbitrary σ-partition, the mapping (4.39) is continuous, i.e., for an arbitrary point $\widehat{\lambda} \in \Sigma$ and an arbitrary neighborhood $V_{K,\varepsilon} \in \Re$, there exists a number $\delta > 0$ such that*

$$(f_\lambda - f_{\widehat{\lambda}}) \in V_{K,\varepsilon} \ \ \forall \lambda \in \big\{ \lambda \in \Sigma : \ |\lambda - \widehat{\lambda}| < \delta \big\}.$$

Moreover, for an arbitrary neighborhood $V_{K,\varepsilon} \in \Re$, there exists a σ-partition such that for $\forall \lambda \in \Sigma$, we have

$$\Big(\sum_{i=1}^{k+1} \lambda_i f_i - f_\lambda \Big) \in V_{K,\varepsilon},$$

i.e.,

$$\left| \int_{t'}^{t''} \Big[\sum_{i=0}^{s} \lambda_i f_i(t, x, x_1, \ldots, x_s) - f_\lambda(t, x, x_1, \ldots, x_s) \Big] \, dt \right| \le \varepsilon$$
$$\forall (t', t'', x, x_1, \ldots, x_s, \lambda) \in I^2 \times K^{s+1} \times \Sigma.$$

Let $\theta_\nu > \cdots > \theta_1 > 0$ be the given numbers with $\theta_i = m_i h$, where m_i, $i = \overline{1, \nu}$, are natural numbers and $h > 0$ is a real number. Let the function $f(t, x, x_1, \ldots, x_s, u, u_1, \ldots, u_\nu)$, $(t, x, x_1, \ldots, x_s, u_1, \ldots, u_\nu) \in I \times O^{s+1} \times U_0^{\nu+1}$, satisfy the following conditions: for almost all $t \in I$, the function $f : I \times O^{s+1} \times U_0^{\nu+1} \to \mathbb{R}^n$ is continuous and continuously differentiable in $(x, x_1, \ldots, x_s) \in O^{s+1}$; for each $(x, x_1, \ldots, x_s, u, u_1, \ldots, u_\nu) \in O^{1+s} \times U_0^{\nu+1}$, the function $f(t, x, x_1, \ldots, x_s, u, u_1, \ldots, u_\nu)$ and the matrices $f_x(t, x, \cdot)$, $f_{x_i}(t, x, \cdot)$, $i = \overline{1, s}$, are measurable on I; for any compact sets $K \subset O$ and $M \subset U_0$, there exists a function $m_{K,M}(t) \in L_1(I, \mathbb{R}_+)$ such that for any $(x, x_1, \ldots, x_s, u, u_1, \ldots, u_\nu) \in K^{s+1} \times M^{\nu+1}$ and almost all $t \in I$,

$$\big| f(t, x, x_1, \ldots, x_s, u, u_1, \ldots, u_\nu) \big| + |f_x(t, x, \cdot)| + \sum_{i=1}^{s} |f_{x_i}(t, x, \cdot)| \le m_{K,M}(t).$$

Introduce the set

$$F_1 = \Big\{ f(t, x, x_1, \ldots, x_s) = f\big(t, x, x_1, \ldots, x_s, u(t), u(t - \theta_1), \ldots, u(t - \theta_\nu)\big) : \ u \in \Omega(I_2, U) \Big\},$$

where $I_2 = [a - \theta_\nu, b], \Omega(I_2, U) \subset E_u(I_2)$.

Consider the functions $f_i(t, x, x_1, \ldots, x_s) = f(t, x, x_1, \ldots, x_s, u_i(t), u_i(t - \theta_1), \ldots, u_i(t - \theta_\nu)) \in F_1$, $i = \overline{0, s}$. In this case, we consider the $\widehat{\sigma}$-partition which means that we partition the interval $[a - \theta_\nu, b]$ in the following way. Let $\gamma > 0$ be the minimum number satisfying the condition $b + \gamma - a + \theta_{nu} = lh$, where l is a natural number and let $I^{(\alpha)}$, $\alpha = \overline{1, l}$, be a system of intervals of length h adjacent to each other such that the left endpoint of the interval $I^{(1)}$ coincides with the point $a - \theta_\nu$, the right endpoint

of it coincides with the left endpoint of the subsequent interval $I^{(2)}$, etc., and the right endpoint of $I^{(l)}$ coincides with the endpoint $b + \gamma$. Next, we divide each of the intervals $I^{(\alpha)}$ by a partial interval $I_\beta^{(\alpha)}$, $\beta = \overline{1, m}$, in a unified way so that the right endpoint of one of the partial intervals $I_\beta^{(l)}$ coincides with the point b. To an arbitrary point $\lambda \in \Sigma$, we put in correspondence a subdivision into partial intervals $I_{\beta_i}^{(\alpha)}(\lambda)$ common for all $I_\beta^{(\alpha)}$ and defined by the condition (4.38).

Let the points $f_i(t, x, x_1, \ldots, x_s) = f(t, x, x_1, \ldots, x_s, u_i(t), u_i(t - \theta_1), \ldots, u_i(t - \theta_\nu)) \in F_1$, $i = \overline{0, s}$, and $\widehat{\sigma}$-partition of the interval be given.

Let us define the mapping

$$\phi_{\widehat{\sigma}} : \Sigma \to F_1 \tag{4.41}$$

by the formula

$$\phi_{\widehat{\sigma}}(\lambda) = f_\lambda(t, x, x_1, \ldots, x_s) = f(t, x, x_1, \ldots, x_s, u_\lambda(t), u_\lambda(t - \theta_1), \ldots, u_\lambda(t - \theta_\nu)),$$

where

$$u_\lambda(t) = u_i(t), \ \ t \in I_{\beta_i}^{(\alpha)}(\lambda), \ \ \alpha = \overline{1, l}, \ \ \beta = \overline{1, m}, \ \ i = \overline{1, k+1}.$$

It is clear that

$$f_\lambda(t, x, x_1, \ldots, x_s) = f_i(t, x, x_1, \ldots, x_s), \ \ t \in I_{\beta_i}^{(\alpha)}(\lambda).$$

The latter relation allows one to prove generalization of Lemma 4.9.

Lemma 4.10. *For an arbitrary $\widehat{\sigma}$-partition, the mapping (4.41) is continuous. Moreover, for an arbitrary neighborhood $V_{K,\varepsilon} \in \Re$, there exists a $\widehat{\sigma}$-partition such that for $\forall \lambda \in \Sigma$, we have*

$$\left(\sum_{i=1}^{k+1} \lambda_i f_i - f_\lambda \right) \in V_{K,\varepsilon}. \tag{4.42}$$

Lemma 4.11 ([20, p. 66]). *Let $z_i \in E_z$, $i = \overline{1, k+1}$. There exist a subset $\Sigma_0 \subset \Sigma$ and a function $\phi(z)$, $z \in \mathrm{co}(\{z_1, \ldots, z_{k+1}\})$, such that the mapping*

$$\phi : \mathrm{co}(\{z_1, \ldots, z_{k+1}\}) \longrightarrow \Sigma_0 \ \ (z \longmapsto \lambda \in \Sigma_0) \tag{4.43}$$

is a homeomorphism.

Lemma 4.12. *Let $f_i(t, x, x_1, \ldots, x_s) \in F_1$, $i = \overline{1, k+1}$. Then for an arbitrary $V_{K,\varepsilon} \in \Re$, there exists a continuous mapping*

$$\phi_0 : \mathrm{co}(\{f_1, \ldots, f_{k+1}\}) \longrightarrow F_1 \tag{4.44}$$

satisfying the condition

$$(z - \phi_0(z) \in V_{K,\varepsilon} \ \ \forall z \in \mathrm{co}(\{f_1, \ldots, f_{k+1}\}). \tag{4.45}$$

Proof. On the set Σ_0 we define the mapping (4.41), i.e., $\phi_{\widehat{\sigma}}(\lambda) = f_\lambda \in F_1 \ \forall \lambda \in \Sigma_0$. By Lemma 4.10, the mapping (4.41) is continuous and (4.42) is valid. Define now the continuous mapping (4.44) by the formula $\phi_0(z) = \phi_{\widehat{\sigma}}(\phi(z))$, $z \in \mathrm{co}(\{f_1, \ldots, f_{k+1}\})$, where

$$z \longmapsto \phi(z) = \lambda \in \Sigma_0 \ \ \text{and} \ \ \phi_{\widehat{\sigma}}(\phi(z)) = f_{\phi(z)} = f_\lambda \in F_1$$

(see (4.43)). The relation (4.42) implies (4.45). □

4.3   Example of a quasiconvex filter

Let $f_0(t, x, x_1, \ldots, x_s) = f(t, x, x_1, \ldots, x_s, u_0(t), u_0(t - \theta_1), \ldots, u_0(t - \theta_\nu)) \in F_1$ be a fixed point. In F_1, let us define the filter Ψ using the basis

$$\Re_1 = \big\{ W_{K,\delta} : \ K \subset O \text{ is a compact set}, \ \delta > 0 \text{ is an arbitrary number} \big\},$$

where

$$W_{K,\delta} = \Big\{ f(t, x, x_1, \ldots, x_s) = f(t, x, x_1, \ldots, x_s, u(t), u(t - \theta_1), \ldots, u(t - \theta_\nu)) \in F_1 : H_1(f - f_0 : K) \leq \delta \Big\},$$

$$H_1(f:K) = \int\limits_I \left[\sup_{(x,x_1,\ldots,x_s)\in K^{s+1}} \left(|f(t,x,x_1,\ldots,x_s)| + |f_x(t,x,\cdot)| + \sum_{i=1}^{s} |f_{x_i}(t,x,\cdot)| \right) \right] dt, \quad f \in E_f^{(1)},$$

(see Lemma 2.1).

Lemma 4.13. *The filter Ψ is quasiconvex.*

Proof. Let an arbitrary element $W \in \Psi$ and an arbitrary natural number k be given. There exists an element $W_{K,\delta} \in \Re_1$ such that $W_{K,\delta} \subset W$. Let us show that as W_1 in the definition of a quasiconvex filter, we can take $W_{K,\frac{\delta}{k+1}}$.

Assume that the points

$$f_i(t,x,x_1,\ldots,x_s) = f\big(t,x,x_1,\ldots,x_s,u_i(t),u_i(t-\theta_1),\ldots,u_i(t-\theta_\nu)\big) \in W_{K,\frac{\delta}{k+1}}, \quad i = \overline{1,k+1},$$

are such that

$$H_1(f_i - f_0);K) \le \frac{\delta}{k+1}, \quad i = \overline{1,k+1}.$$

By Lemma 4.12, there exists a continuous mapping

$$\phi_0 : \mathrm{co}(\{f_1,\ldots,f_{k+1}\}) \longrightarrow F_1$$

defined by the formula

$$\phi_0 = \phi_{\widehat{\sigma}}(\phi(z)) = f_\lambda, \quad \lambda \in \Sigma_0,$$

and satisfying the condition

$$(z - \phi_0(z)) \in V_{K,\varepsilon} \quad \forall z \in \mathrm{co}(\{f_1,\ldots,f_{k+1}\}).$$

It remains to prove that $f_\lambda \in W_{K,\delta} \; \forall \lambda \in \Sigma_0$. For this purpose, let us estimate the quantity $H_1(f_\lambda - f_0; K)$. Owing to the specific character of the $\widehat{\sigma}$-partition, we have

$$f_\lambda(t,x,x_1,\ldots,x_s) = f_i(t,x,x_1,\ldots,x_s), \quad t \in I_{\beta_i}^{(\alpha)}(\lambda) \cap I.$$

Taking into account the latter assertion, we have

$$H_1(f_\lambda - f_0; K) = \sum_{\alpha=1}^{l} \int\limits_{I^{(\alpha)}\cap I} \left[\sup_{(x,x_1,\ldots,x_s)\in K^k} \left(\left| f_\lambda(t,x,\cdot) - f_0(t,x,\cdot) \right| \right. \right.$$

$$\left. \left. + \left| \frac{\partial}{\partial x} f_\lambda(t,x,\cdot) - \frac{\partial}{\partial x} f_0(t,x\cdot) \right| + \sum_{j=1}^{s} \left| \frac{\partial}{\partial x_j} f_\lambda(t,x,\cdot) - \frac{\partial}{\partial x_j} f_0(t,x,\cdot) \right| \right) \right] dt$$

$$\le \sum_{\alpha=1}^{l} \sum_{\beta=1}^{m} \sum_{i=1}^{k+1} \int\limits_{I_{\beta_i}^{(\alpha)}(\lambda)\cap I} \left[\sup_{(x,x_1,\ldots,x_s)\in K^k} \left(\left| f_i(t,x,\cdot) - f_0(t,x,\cdot) \right| \right. \right.$$

$$\left. \left. + \left| \frac{\partial}{\partial x} f_i(t,x,\cdot) - \frac{\partial}{\partial x} f_0(t,x,\cdot) \right| + \sum_{j=1}^{s} \left| \frac{\partial}{\partial x_j} f_\lambda(t,x,\cdot) \frac{\partial}{\partial x_j} f_0(t,x,\cdot) \right| \right) \right] dt$$

$$\le \sum_{i=1}^{k+1} H_1(f_i - f_0; K) \le \delta.$$

Hence $\phi_0(z) \in W_{K,\delta}$. $\qquad\square$

Lemma 4.14. *In the space $E_f^{(1)}$, let the set*

$$W^{(1)} = \big\{ f \in E_f^{(1)} : \; H_1(f - f_0; K_0) \le \delta_0 \big\},$$

where $\delta_0 > 0$ is a fixed number and $K_0 \subset O$ is a compact set, be given. Then for an arbitrary $W \in \Psi$, the inclusion

$$\text{cone}\left([W^{(1)}]_W - f_0\right) \supset F_1 - f_0 \tag{4.46}$$

holds. Here $[W]_{w^{(1)}}$ denotes the closure (with respect to $W^{(1)}$) of the set $W^{(1)} \cap W$ in the topology on $W^{(1)}$ induced by the topology on E_f.

Proof. Clearly, $W_{K_0,\delta_0} \subset W^{(1)}$ and there exists W_{K_1,δ_1} contained in W. Therefore,

$$W^{(1)} \cap W \supset W_{K_1,\delta_1} \cap W_{K_0,\delta_0} \supset W_{K_2,\delta_2}, \tag{4.47}$$

where $K_2 = K_0 \cup K_1$, $\delta_2 = \min\{\delta_1, \delta_0\}$. To prove the inclusion (4.46), it suffices to show that

$$\text{cone}\left([W^{(1)}]_{w_{K_2,\delta_2}} - f_0\right) \supset F_1 - f_0$$

(see (4.47)). Let $f - f_0 \in F_1 - f_0$ and $z_\lambda = (1-\lambda)f_0 + \lambda f$, $\lambda \in [0,1]$; let $\{\varepsilon_i\}$ be a sequence converging to zero. By Lemma 4.12, we can construct a sequence of continuous mappings

$$\phi_0^{(i)} : \text{co}(\{f_0, f\}) \longrightarrow F_1, \quad i = 1, 2, \ldots,$$

such that

$$z_\lambda - \phi_0^{(i)}(z_\lambda) \in V_{K_2,\varepsilon_i}, \quad \lambda \in [0,1], \quad i = 1, 2, \ldots, \tag{4.48}$$

where

$$\phi_0^{(i)}(z_\lambda) = f_\lambda(t, x, x_1, \ldots, x_s) = f\left(t, x, x_1, \ldots, x_s, u_\lambda(t), u_\lambda(t - \theta_1), \ldots, u_\lambda(t - \theta_\nu)\right),$$

$$u_\lambda(t) = \begin{cases} u_0(t), & t \in I_{\beta_1}^{(\alpha)} \cap I_2, \\ u(t), & t \in I_{\beta_2}^{(\alpha)} \cap I_2, \end{cases} \quad \alpha = \overline{1,l}, \quad \beta = \overline{1, m_i}, \quad m_i = m(\varepsilon_i).$$

Let us now prove the existence of $\lambda_0 \in (0,1)$ such that

$$\phi_0^{(i)}(z_\lambda) \in W_{K_2,\delta_2}, \quad i = 1, 2, \ldots, \quad \forall \lambda \in [0, \lambda_0]. \tag{4.49}$$

For the expression $H_1(f - f_0; K_2)$, taking into account the relation $f_\lambda(t, x, x_1, \ldots, x_s) = f_0(t, x, x_1, \ldots, x_s)$, $t \in t \in I_{\beta_1}^{(\alpha)}(\lambda) \cap I$, we have

$$H_1(f_\lambda - f_0; K_2) = \sum_{\alpha=1}^{l} \int_{I_{2i}^{(\alpha)}(\lambda)} \left[\sup_{(x,x_1,\ldots,x_s) \in K_2^{1+s}} \left(\left| f(t, x, \cdot) - f_0(t, x, \cdot) \right| \right. \right.$$
$$\left. \left. + \left| \frac{\partial}{\partial x} f(t, x, \cdot) - \frac{\partial}{\partial x} f_0(t, x, \cdot) \right| + \sum_{j=1}^{s} \left| \frac{\partial}{\partial x_j} f(t, x, \cdot) - \frac{\partial}{\partial x_j} f_0(t, x, \cdot) \right| \right) \right] dt,$$

where

$$I_{2i}^{(\alpha)}(\lambda) = \bigcup_{\beta=1}^{m_i} \left(I_{\beta_2}^{(\alpha)}(\lambda) \cap I \right).$$

The specific character of the $\widehat{\sigma}$-partition implies

$$\text{mes}\left(\sum_{\alpha=1}^{l} I_{2i}^{(\alpha)}(\lambda) \right) \longrightarrow \sum_{\alpha=1}^{l} \sum_{\beta=1}^{m_1} \text{mes} I_{\beta_2}^{\alpha}(\lambda) = \lambda \sum_{\alpha=1}^{l} \sum_{\beta=1}^{m_1} \text{mes} I_{\beta}^{\alpha} \leq \lambda \, \text{mes} I_2.$$

Therefore,

$$\text{mes}\left(\sum_{\alpha=1}^{l} I_{2i}^{(\alpha)}(\lambda) \right) \longrightarrow 0 \quad \text{as} \quad \lambda \to 0.$$

uniformly in $i = 1, 2, \ldots$. Hence there exists $\lambda_0 \in (0, 1)$ for which

$$H_1(f_\lambda - f_0; K_2) \leq \delta_2.$$

The inclusion (4.48) is proved. The condition (4.49) implies $\phi_0^{(i)}(z_\lambda) \to z_\lambda$ as $i \to \infty$. Therefore,

$$z_\lambda \in [W^{(1)}]_{W_{K_2,\delta_2}} \text{ for } \lambda \in [0, \lambda_0],$$

and hence

$$z_\lambda - f_0 \in \text{cone}\left([W^{(1)}]_{W_{K_2,\delta_2}}\right), \quad \lambda \in [0, \lambda_0],$$

but $z_\lambda - f_0 = \lambda(f - f_0)$. Thus $f - f_0 \in \text{cone}([W^{(1)}]_{W_{K_2,\delta_2}})$. $\qquad\qquad\square$

4.4   The optimal control problem with the discontinuous initial condition

Consider the optimal control problem

$$\dot{x}(t) = f\big(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_s), u(t), u(t - \theta_1), \ldots, u(t - \theta_\nu)\big), \tag{4.50}$$
$$t \in [t_0, t_1] \subset I, \quad u \in \Omega(I_2, U),$$
$$x(t) = \varphi(t), \quad t \in [\widehat{\tau}, t_0), \quad x(t_0) = x_0, \quad \varphi \in \Phi_1, \quad x_0 \in X_0, \tag{4.51}$$
$$q^i\big(t_0, t_1, \tau_1, \ldots, \tau_s, x_0, x(t_1)\big) = 0, \quad i = \overline{1, l}, \tag{4.52}$$
$$q^0\big(t_0, t_1, \tau_1, \ldots, \tau_s, x_0, x(t_1)\big) \longrightarrow \min, \tag{4.53}$$

where $\theta_\nu > \cdots > \theta_1 > 0$, $\Phi_2 = \{\varphi \in PC(I_2, \mathbb{R}^n) : \varphi(t) \in N\}$, $N \subset O$ is a convex set; $X_0 \subset O$ is a convex compact set; the scalar-valued functions $q^i(t_0, t_1, \tau_1, \ldots, \tau_s, x_0, x_1)$, $i = \overline{0, l}$, are continuously differentiable on $I^2 \times [\theta_{11}, \theta_{12}] \times \cdots \times [\theta_{s1}, \theta_{s2}] \times O^2$.

The problem (4.50)–(4.53) is called an optimal control problem with the discontinuous initial condition.

Definition 4.12. Let $v = (t_0, t_1, \tau_1, \ldots, \tau_s, x_0, \varphi, u) \in A = (a, b) \times (a, b) \times (\theta_{11}, \theta_{12}) \times \cdots \times (\theta_{s1}, \theta_{s2}) \times X_0 \times \Phi_2 \times \Omega(I, U)$. A function $x(t) = x(t; v) \in O$, $t \in [\widehat{\tau}, t_1]$, is called a solution of the equation (4.50) with the discontinuous initial condition (4.51), or a solution corresponding to the element v and defined on the interval $[\widehat{\tau}, t_1]$, if it satisfies the condition (4.51) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (4.50) a.e. on $[t_0, t_1]$.

Definition 4.13. An element $v = (t_0, t_1, \tau_1, \ldots, \tau_s, x_0, \varphi, u) \in A$ is said to be admissible if the corresponding solution $x(t) = x(t; v)$ satisfies the boundary conditions (4.52).

Denote by A_0 the set of admissible elements.

Definition 4.14. An element $v_0 = (t_{00}, t_{10}, \tau_{10}, \ldots, \tau_{s0}, x_{00}, \varphi_0, u_0) \in A_0$ is said to be optimal if there exist a number $\delta_0 > 0$ and a compact set $K_0 \subset O$ such that for an arbitrary element $v \in A_0$ satisfying the condition

$$|t_{00} - t_0| + |t_{10} - t_1| + \sum_{i=1}^{s} |\tau_{i0} - \tau_i| + |x_{00} - x_0| + \|\varphi_0 - \varphi\|_{I_1} + H_1(f_0 - f; K_0) \leq \delta_0,$$

the inequality

$$q^0\big(t_{00}, t_{10}, \tau_{10}, \ldots, \tau_{s0}, x_{00}, x_0(t_{10})\big) \leq q^0\big(t_0, t_1, \tau_1, \ldots, \tau_s, x_0, x(t_1)\big)$$

holds. Here

$$f_0 = f_0(t, x, x_1, \ldots, x_s) = f\big(t, x, x_1, \ldots, x_s, u_0(t), u_0(t - \theta_1), \ldots, u_0(t - \theta_k)\big)$$

and

$$f = f(t, x, x_1, \ldots, x_s) = f\big(t, x, x_1, \ldots, x_s, u(t), u(t - \theta_1), \ldots, u(t - \theta_k)\big).$$

Theorem 4.6. *Let v_0 be an optimal element and let the following conditions hold:*

4.1. $\tau_{s0} > \cdots > \tau_{10}$ *and* $t_{00} + \tau_{s0} < t_{10}$, *with* $\tau_{i0} \in (\theta_{i0}, \theta_{i+10})$, $i = \overline{1, s-1}$;

4.2. $\theta_i = m_i h$, $i = \overline{1, \nu}$, *where* m_i, $i = \overline{1, \nu}$, *are natural numbers,* $h > 0$ *is a real number;*

4.3. *the function* $\varphi_0(t)$ *is absolutely continuous and* $\dot{\varphi}_0(t)$ *is bounded;*

4.4. *the function* $f_0(w)$, $w = (t, x, x_1, \ldots, x_s) \in I \times O^{s+1}$, *is bounded;*

4.5. *there exists the finite limit*

$$\lim_{w \to w_0} f_0(w) = f_0^-, \quad w \in (a, t_{00}] \times O^{s+1},$$

where $w_0 = (t_{00}, x_{00}, \varphi_0(t_{00} - \tau_{10}), \ldots, \varphi_0(t_{00} - \tau_{s0}))$;

4.6. *there exist the finite limits*

$$\lim_{(w_{1i}, w_{2i}) \to (w_{1i}^0, w_{2i}^0)} [f_0(w_{1i}) - f_0(w_{2i})] = f_{0i},$$

where $w_{1i}, w_{2i} \in (a, b) \times O^{s+1}$, $i = \overline{1, s}$,

$$w_{1i}^0 = \Big(t_{00} + \tau_{i0}, x_0(t_{00} + \tau_{i0}), x_0(t_{00} + \tau_{i0} - \tau_{10}), \ldots, x_0(t_{00} + \tau_{i0} - \tau_{i-10}),$$
$$x_{00}, x_0(t_{00} + \tau_{i0} - \tau_{i+10}), \ldots, x_0(t_{00} + \tau_{i0} - \tau_{s0}) \Big),$$
$$w_{2i}^0 = \Big(t_{00} + \tau_{i0}, x_0(t_{00} + \tau_{i0}), x_0(t_{00} + \tau_{i0} - \tau_{10}), \ldots, x_0(t_{00} + \tau_{i0} - \tau_{i-10}),$$
$$\varphi_0(t_{00}), x_0(t_{00} + \tau_{i0} - \tau_{i+10}), \ldots, x_0(t_{00} + \tau_{i0} - \tau_{s0}) \Big);$$

4.7. *there exists the finite limit*

$$\lim_{w \to w_{s+1}} f_0(w) = f_{s+1}^-, \quad w \in (t_{00}, t_{10}] \times O^{s+1}, \quad w_{s+1} = \big(t_{10}, x_0(t_{10}), x_0(t_{10} - \tau_{10}), \ldots, x_s(t_{10} - \tau_{s0}) \big).$$

Then there exist a vector $\pi = (\pi_0, \ldots, \pi_l) \neq 0$, *with* $\pi_0 \leq 0$, *and a solution* $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ *of the equation*

$$\dot{\psi}(t) = -\psi(t) f_{0x}[t] - \sum_{i=1}^{s} \psi(t + \tau_{i0}) f_{0x_i}[t + \tau_{i0}], \quad t \in [t_{00}, t_{10}], \quad \psi(t) = 0, \ t > t_{10}, \qquad (4.54)$$

such that the following conditions hold:

4.8. *the conditions for the moments* t_{00} *and* t_{10}:

$$\pi Q_{0 t_0} \geq \psi(t_{00}) f_0^- + \sum_{i=1}^{s} \psi(t_{00} + \tau_{i0}) f_{0i}, \quad \pi Q_{0 t_1} \geq -\psi(t_{10}) f_{s+1}^-,$$

where

$$Q_0 = (q^0, \ldots, q^l)^\top, \quad Q_{0 t_0} = \frac{\partial}{\partial t_0} Q_0;$$

4.9. *the conditions for the delays* τ_{i0}, $i = \overline{1, s}$,

$$\pi Q_{0 \tau_i} = \psi(t_{00} + \tau_{i0}) f_{0i} + \int_{t_{00}}^{t_{10}} \psi(t) f_{0x_i}[t] \dot{x}_0(t - \tau_{i0}) \, dt = 0, \quad i = \overline{1, s};$$

4.10. *the condition of the vector* x_{00},

$$(\pi Q_{0x_0} + \psi(t_{00}))x_{00} = \max_{x_0 \in X_0} (\pi Q_{0x_0} + \psi(t_{00}))x_0;$$

4.11. *the integral maximum principle for the initial function* $\varphi_0(t)$,

$$\sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} \psi(t+\tau_{i0})f_{0x_i}[t+\tau_{i0}]\varphi_0(t)\,dt = \max_{\varphi(t) \in \Phi_1} \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} \psi(t+\tau_{i0})f_{0x_i}[t+\tau_{i0}]\varphi(t)\,dt;$$

4.12. *the integral maximum principle for the control function* $u_0(t)$,

$$\int_{t_{00}}^{t_{10}} \psi(t)f_0[t]\,dt$$

$$= \max_{u(t) \in \Omega(I_2, U)} \int_{t_{00}}^{t_{10}} \psi(t)f\Big(t, x_0(t), x_0(t-\tau_{10}), \ldots, x_0\big(t-\tau_{i0}, u(t), u(t-\theta_1), \ldots, u(t-\theta_\nu)\big)\Big)\,dt;$$

4.13. *the condition for the function* $\psi(t)$,

$$\psi(t_{10}) = \pi Q_{0x_1}.$$

Theorem 4.7. *Let v_0 be an optimal element and let the conditions* 4.1–4.4 *and* 4.6 *hold. Moreover, there exist the finite limits*

$$\lim_{w \to w_0} f_0(w) = f_0^+, \quad w \in [t_{00}, b) \times O^{s+1}, \qquad \lim_{w \to w_{s+1}} f_0(w) = f_{s+1}^+, \quad w \in [t_{10}, b) \times O^{s+1}. \qquad (4.55)$$

Then there exist a vector $\pi = (\pi_0, \ldots, \pi_l) \neq 0$, with $\pi_0 \leq 0$, and a solution of the equation (4.54) *such that the conditions* 4.9–4.13 *hold. Moreover,*

$$\pi Q_{0t_0} \leq \psi(t_{00})f_0^- + \sum_{i=1}^{s} \psi(t_{00}+\tau_{i0})f_{0i}, \quad \pi Q_{0t_1} \leq -\psi(t_{10})f_{s+1}^-.$$

Theorem 4.8. *Let v_0 be an optimal element and let the conditions of Theorem* 4.6 *hold. Moreover, there exist the finite limits f_0^+, f_{s+1}^+, with $f_0^- = f_0^+ := \widehat{f}_0$, $f_{s+1}^- = f_{s+1}^+ := \widehat{f}_{s+1}$. Then there exist a vector $\pi = (\pi_0, \ldots, \pi_l) \neq 0$, with $\pi_0 \leq 0$, and a solution of the equation* (4.54) *such that the conditions* 4.9–4.13 *hold. Moreover,*

$$\pi Q_{0t_0} = \psi(t_{00})\widehat{f}_0 + \sum_{i=1}^{s} \psi(t_{00}+\tau_{i0})f_{0i}, \quad \pi Q_{0t_1} = -\psi(t_{10})\widehat{f}_{s+1}.$$

Theorem 4.9. *Let v_0 be an optimal element and let the conditions* 4.1–4.5 *and* 4.7 *hold. Moreover, there exist the finite limits*

$$\lim_{(w_{1i}, w_{2i}) \to (w_{1i}^0, w_{2i}^0)} [f_0(w_{1i}) - f_0(w_{2i})] = f_{0i}^-,$$

where $w_{1i}, w_{2i} \in (a, t_{00}+\tau_{i0}) \times O^{s+1}$, $i = \overline{1, s}$. Then there exist a vector $\pi = (\pi_0, \ldots, \pi_l) \neq 0$, with $\pi_0 \leq 0$, and a solution of the equation (4.54) *such that the conditions* 4.8–4.13 *hold. Moreover,*

$$\pi Q_{0\tau_i} \geq \psi(t_{00}+\tau_{i0})f_{0i}^- + \int_{t_{00}}^{t_{10}} \psi(t)f_{0x_i}[t]\dot{x}_0(t-\tau_{i0})\,dt, \quad i = \overline{1, s}.$$

Theorem 4.10. *Let v_0 be an optimal element and let the conditions* 4.1–4.5 *and* (4.55) *hold. Moreover, there exist the finite limits*

$$\lim_{(w_{1i}, w_{2i}) \to (w_{1i}^0, w_{2i}^0)} \big[f_0(w_{1i}) - f_0(w_{2i})\big] = f_{0i}^+,$$

where $w_{1i}, w_{2i} \in [t_{00} + \tau_{i0}, b) \times O^{s+1}$, $i = \overline{1, s}$. Then there exist a vector $\pi = (\pi_0, \dots, \pi_l) \neq 0$, with $\pi_0 \leq 0$, and a solution of the equation (4.54) *such that the conditions* 4.8–4.13 *hold. Moreover,*

$$\pi Q_{0\tau_i} \leq \psi(t_{00} + \tau_{i0})f_{0i}^+ + \int_{t_{00}}^{t_{10}} \psi(t)f_{0x_i}[t]\dot{x}_0(t - \tau_{i0})\, dt, \quad i = \overline{1, s}.$$

4.5 Proof of Theorem 4.6

Auxiliary assertions.   Let $K \subset O$ be a compact set and let $\alpha > 0$ be a certain given number. In the spaces $E_f^{(1)}$ and E_f, we define, respectively, the sets

$$W_{K,\alpha} = \big\{\delta f \in E_f^{(1)} :\ H_1(\delta f; K) \leq \alpha\big\},$$

$$W(K; \alpha) = \bigg\{\delta f \in E_f :\ \exists\, m_{\delta f, K}(t), L_{\delta f, K}(t) \in L_1(I, R_+),\ \int_I \big[m_{\delta f, K}(t) + L_{\delta f, K}(t)\big]\, dt \leq \alpha\bigg\}.$$

Lemma 4.15. *Let $K_i \subset O$, $i = 1, 2$, be compact sets, and, moreover, let $K_1 \subset \text{int } K_2$ and $\alpha_1 > 0$ be a certain number. Then there exists a number $\alpha_2 > 0$ such that*

$$W_{K_2, \alpha_1} \subset W(K_1; \alpha_2). \tag{4.56}$$

Proof. Let $\delta f \in W_{K_2, \alpha_1}$. Hence

$$\int_I \sup\Big\{|\delta f(t, x, x_1, \dots, x_s)| + |\delta f_x(t, x, \cdot)| + \sum_{i=1}^s |\delta f_{x_i}(t, x, \cdot)| :\ (x, x_1, \dots, x_s) \in K_2^{s+1}\Big\}\, dt \leq \alpha_1.$$

For a.e. $t \in I$ and every $(x', x_1', \dots, x_s') \in K_1^{s+1}, (x', x_1', \dots, x_s') \in K_1^{s+1}$ the inequality

$$\big|\delta f(t, x', x_1', \dots, x_s') - \delta f(t, x'', x_1'', \dots, x_s'')\big| \leq L_{\delta f, K_1}(t)\Big[|x' - x''| + \sum_{i=1}^s |x_i' - x_i''|\Big]$$

holds, where

$$L_{\delta f, K_1}(t) = n^2 s(\alpha_0 + 1)$$

$$\times \sup\Big\{|\delta f(t, x, x_1, \dots, x_s)| + |\delta f_x(t, x, \cdot)| + \sum_{i=1}^s |\delta f_{x_i}(t, x, \cdot)| :\ (x, x_1, \dots, x_s) \in K_2\Big\}$$

(see Lemma 2.2).

On the other hand, it is obvious that for $(t, x, x_1, \dots, x_s) \in I \times K_1^{s+1}$, we have

$$|\delta f(t, x, x_1, \dots, x_s)| \leq m_{\delta f, K_1}(t) = \sup\Big\{|\delta f(t, x, x_1, \dots, x_s)| :\ (x, x_1, \dots, x_s) \in K_1^{s+1}\Big\}.$$

Using the relations obtained above, we get

$$\int_I \big[m_{\delta f, K_1}(t) + L_{\delta f, K_1}(t)\big]\, dt \leq \alpha_1\big[1 + n^2 s(\alpha_0 + 1)\big] := \alpha_2.$$

The inclusion (4.56) is proved.   □

To each element

$$\kappa = (t_0, t_1, \tau_1, \ldots, \tau_s, x_0, \varphi, f) \in (a, b) \times (a, b) \times (\theta_{11}, \theta_{12}) \times \cdots \times (\theta_{s1}, \theta_{s2}) \times X_0 \times \Phi_1 \times E_f^{(1)}$$

we put in correspondence the functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_s)\big), \quad t \in [t_0, t_1],$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [\widehat{\tau}, t_0), \quad x(t_0) = x_0.$$

Definition 4.15. The solution corresponding to an element $\kappa = (t_0, t_1, \tau_1, \ldots, \tau_s, x_0, \varphi, f)$ is called a solution $x(t; \mu)$, $\mu = (t_0, \tau_1, \ldots, \tau_s, x_0, \varphi, f)$, defined on $[\widehat{\tau}, t_1]$, and denoted by $x(t; \kappa)$.

Therefore,

$$x_0(t) = x(t; v_0) = x(t; \kappa_0) = x(t; \mu_0), \quad t \in [\widehat{\tau}, t_{10}], \tag{4.57}$$

where

$$\kappa_0 = (t_{00}, t_{10}, \tau_{10}, \ldots, \tau_{s0}, x_{00}, \varphi_0, f_0), \quad \mu_0 = (t_{00}, \tau_{10}, \ldots, \tau_{s0}, x_{00}, \varphi_0, f_0).$$

The following lemma is a direct consequence of Theorem 1.2.

Lemma 4.16. *Let $\alpha_1 > 0$ be a certain given number, and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $\mathrm{cl}\,\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then there exists a number $\delta_1 > 0$ such that to each element*

$$\kappa \in V(\kappa_0; K_1, \delta_1, \alpha_1) = (B(t_{00}; \delta_1) \cap I) \times (B(t_{10}; \delta_1) \cap I) \times (B(\tau_{10}; \delta_1) \cap (\theta_{11}, \theta_{12})) \times \cdots$$
$$\times (B(\tau_{s0}; \delta_1) \cap (\theta_{s1}, \theta_{s2})) \times (B(x_{00}, \delta_1) \cap O) \times (B(\varphi_0; \delta_1) \cap \Phi_2) \times \big[f_0 + (W_{K_1, \alpha_1} \cap V_{K_1, \delta_1}) \big]$$

there corresponds the solution $x(t; \kappa) \in K_1$, $t \in [\widehat{\tau}, t_1]$. Moreover, for each $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) \in (0, \delta_1)$ such that for an arbitrary $\kappa \in V(\kappa_0; K_1, \delta_1, \alpha_1)$, the inequality

$$|x(t_{10}; \kappa_0) - x(t_1; \kappa)| \leq \varepsilon$$

holds.

Remark 4.1. Lemma 4.16 remains valid if we replace the set $V(\kappa_0; K_1, \delta_1, \alpha_1)$ by the set

$$V(\kappa_0; K_1, \delta_1) = (B(t_{00}; \delta_1) \cap I) \times (B(t_{10}; \delta_1) \cap I) \times (B(\tau_{10}; \delta_1) \cap (\theta_{11}, \theta_{12})) \times \cdots$$
$$\times (B(\tau_{s0}; \delta_1) \cap (\theta_{s1}, \theta_{s2})) \times (B(x_{00}, \delta_1) \cap O) \times (B(\varphi_0; \delta_1) \cap \Phi_2) \times [f_0 + W_{K_1, \delta_1}].$$

Let us now consider the topological vector space

$$E_\kappa = \mathbb{R}^{2+s+n} \times \mathrm{PC}(I_1, \mathbb{R}^n) \times E_f^{(1)}$$

with the points $\kappa = (y, \varsigma)$, where $y = (t_0, t_1, \tau_1, \ldots, \tau_s, x_0)^\top$, $\varsigma = (\varphi, f)$.

The set

$$X = [a, t_{00}] \times [t_{00}, t_{10}] \times [\theta_{11}, \theta_{12}] \times \cdots \times [\theta_{s1}, \theta_{s2}] \times O \subset \mathbb{R}^{2+s+n}$$

is a locally convex subspace in the topology induced from $\mathbb{R}^{2+s+n}$.

By $D_0 \subset E_\kappa$ we denote the set of elements $\kappa \in X \times \Phi_2 \times E_f^{(1)}$ such that the solution $x(t; \kappa)$ corresponds to each of them. The set D_0 is nonempty, since $\kappa_0 \in D_0$.

Lemma 4.17. *The set D_0 is finitely convex.*

Proof. Let $\widehat{\kappa} = (\widehat{y}, \widehat{\varsigma}) \in D_0$ be an arbitrary fixed point, and $L_{\widehat{\varsigma}} \subset E_\varsigma$ be a linear manifold, i.e.,

$$L_{\widehat{\varsigma}} = \left\{ \widehat{\varsigma} + \delta\varsigma : \quad \delta\varsigma = \sum_{i=1}^k \lambda_i \delta\varsigma_i, \quad \lambda_i \in \mathbb{R}, \quad i = \overline{1, k} \right\},$$

where $\delta\varsigma_i \in E_\varsigma$, $i = \overline{1,k}$, are fixed points. There exists a number $\delta_1 > 0$ such that with each element $\kappa \in V(\widehat{\kappa}; K_1, \delta_1)$ we associate the solution $x(t; \widehat{\varsigma}) \in K_1$ (see Remark 4.1).

Let a number $\delta \in (0, \delta_1)$ be insomuch small that the neighborhood of the point $\widehat{\varsigma}$

$$V_{\widehat{\varsigma}} = \left\{ \widehat{\varsigma} + \sum_{i=1}^{k} \lambda_i \delta\varsigma_i : \ |\lambda_i| \leq \delta, \ i = \overline{1,k} \right\}$$

is contained in the set

$$(B(\widehat{\varphi}; \delta_1) \cap \Phi_2) \times [\widehat{f} + W_{K_1, \delta_1}].$$

Therefore, there exist convex neighborhoods

$$V_{\widehat{y}} = (B(\widehat{t_0}; \delta) \cap (a, \widehat{t_0})) \times (B(\widehat{t_1}; \delta) \cap (\widehat{t_0}, \widehat{t_1})) \times (B(\widehat{x_0}; \delta_1) \cap O) \subset X, \ V_{\widehat{\varsigma}} \subset L_{\widehat{\varsigma}}$$

such that

$$V_{\widehat{y}} \times V_{\widehat{\varsigma}} \subset D_0.$$

Hence the set D_0 is finitely locally convex with respect to the space $X \times E_\varsigma$. $\square$

On the set D_0, let us define the mapping

$$S : D_0 \to \mathbb{R}^n$$

by the formula

$$S(\kappa) = x(t_1; \kappa).$$

Lemma 4.18. *The mapping S is differentiable at the point κ_0 and*

$$dS_{\kappa_0}(\delta\kappa) = \delta x(t_{10}; \delta\kappa) + f_{s+1}^{-}\delta t_1 \ \ \forall \delta\kappa = (\delta t_0, \delta t_1, \delta\tau_1, \ldots, \delta\tau_s, \delta x_0, \delta\varphi, \delta f) \in E_\kappa - \kappa_0, \qquad (4.58)$$

where

$$\delta x(t_{10}; \delta\kappa) = \delta x(t_{10}; \delta\mu) = -\left[Y(t_{00}; t)f_0^{-} + \sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t)f_{0i} \right]\delta t_0$$

$$- \sum_{i=1}^{s} \left[Y(t_{00} + \tau_{i0}; t)f_{0i} + \int_{t_{00}}^{t} Y(\xi; t)f_{0x_i}[\xi]\dot{x}_0(\xi - \tau_{i0}) \, d\xi \right]\delta\tau_i + Y(t_{00}; t)\delta x_0$$

$$+ \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t)f_{0x_i}[\xi + \tau_{i0}]\delta\varphi(\xi) \, d\xi + \int_{t_{00}}^{t} Y(\xi; t)\delta f[\xi] \, d\xi \qquad (4.59)$$

and $\delta\mu = (\delta t_0, \delta\tau_1, \ldots, \delta\tau_s, \delta x_0, \delta\varphi, \delta f) \in E_\mu^{(1)} - \mu_0$.

Proof. Let $L_{\varsigma_0} \subset E_\varsigma$ be a linear manifold, and let

$$V_0 \subset X - y_0, \ V \subset L_{\varsigma_0} - \varsigma_0$$

be bounded convex neighborhoods of zero, where $y_0 = (t_{00}, t_{10}, \tau_{10}, \ldots, \tau_{s0}, x_{00})^\top$ and $\varsigma_0 = (\varphi_0, f_0)$.

The finite local convexity of the set D_0 implies the existence of a number $\varepsilon_0 > 0$ such that for an arbitrary $(\varepsilon, \delta\varsigma) \in (0, \varepsilon_0) \times V_0 \times V$, $\varsigma_0 + \varepsilon\delta\varsigma \in D_0$, and

$$x(t_{10} + \varepsilon\delta t_1; \kappa_0 + \varepsilon\delta\kappa) - x(t_{10} + \varepsilon\delta t_1; \kappa_0) = x(t_{10} + \varepsilon\delta t_1; \mu_0 + \varepsilon\delta\mu) - x(t_{10} + \varepsilon\delta t_1; \mu_0)$$

$$= \Delta x(t_{10} + \varepsilon\delta t_1; \varepsilon\delta\mu) = \varepsilon\delta x(t_{10} + \varepsilon\delta t_1; \delta\mu) + o(t_{10} + \varepsilon\delta t_1; \varepsilon\delta\mu),$$

where the variation $\delta x(t_{10} + \varepsilon\delta t_1; \delta\mu)$ is calculated by the formula (2.7).

We have

$$S(\kappa_0 + \varepsilon\delta\kappa) - S(\kappa_0) = x(t_{10} + \varepsilon\delta t_1; \kappa_0 + \varepsilon\delta\kappa) - x_0(t_{10})$$
$$= x(t_{10} + \varepsilon\delta t_1; \kappa_0 + \varepsilon\delta\kappa) - x_0(t_{10} + \varepsilon\delta t_1) + x_0(t_{10} + \varepsilon\delta t_1) - x_0(t_{10})$$
$$= \varepsilon\delta x(t_{10} + \varepsilon\delta t_1; \delta\mu) + o(t_{10} + \varepsilon\delta t_1; \varepsilon\delta\mu) + \int\limits_{t_{10}}^{t_{10}+\varepsilon\delta t_1} f_0[t]\, dt. \quad (4.60)$$

It is easy to note that
$$\lim_{i \to 0} \delta x(t_{10} + \varepsilon\delta t_1; \delta\mu) = \delta x(t_{10}; \delta\mu)$$

uniformly in $\delta\kappa \in V_0 \times V$, (i.e., uniformly for the corresponding $\delta\mu$) and

$$\int\limits_{t_{10}}^{t_{10}+\varepsilon\delta t_1} f_0[t]\, dt = \varepsilon f_{s+1}^- \delta t_1 + o(\varepsilon\delta\kappa).$$

Taking into account these relations and the variation formula (2.7), from (4.60) we obtain

$$S(\kappa_0 + \varepsilon\delta\kappa) - S(\kappa_0) = \varepsilon\big[\delta x(t_{10}; \delta\kappa) + f_{s+1}^- \delta t_1\big] + o(\varepsilon\delta\kappa) = \varepsilon dS_{\kappa_0}(\delta\kappa) + o(\varepsilon\delta\kappa), \quad (4.61)$$

where $\delta x(t_{10}; \delta\kappa)$ has the form (2.59). $\hfill\square$

Differentiability of the mapping at the point z_0**.** Consider the vector space

$$E_z = \mathbb{R} \times E_\kappa$$

of points $z = (\xi, \kappa)$.

Introduce the sets

$$X = \mathbb{R}_+ \times X_0, \quad D = \mathbb{R}_+ \times D_0.$$

The set is finitely locally convex in the subspace $X \times E_\varsigma \subset E_z$ (see Lemma 4.17).

On the set D, let us define the mapping

$$P : D \to \mathbb{R}^{l+1}$$

by the formula
$$P(z) = Q(t_0, t_1, \tau_1, \ldots, \tau_s, x_0, S(\kappa)) + (\xi, 0, \ldots, 0)^\top,$$

where $Q = q^0, \ldots, q^l$ and $S(\kappa) = x(t_1; \kappa)$.

Lemma 4.19. *The mapping P is differentiable at the point $z_0 = (0, \kappa_0)$ and*

$$dP_{z_0}(\delta z) = \Big\{Q_{0t_0} - Q_{0x_1} Y(t_{00}; t_{10}) f_0^- - \sum_{i=1}^s Q_{0x_1} Y(t_{00} + \tau_{i0}; t_{10}) f_{0i}\Big\}\delta t_0 + \Big\{Q_{0t_1} + Q_{0x_1} f_{s+1}^-\Big\}\delta t_1$$

$$+ \sum_{i=1}^s \bigg\{Q_{0\tau_i} - Q_{0x_1} Y(t_{00} + \tau_{i0}; t_{10}) f_{0i} - \int\limits_{t_{00}}^{t_{10}} Q_{0x_1} Y(t; t_{10}) f_{0x_i}[t] \dot{x}_0(t - \tau_{i0})\, dt\bigg\}\delta\tau_i$$

$$+ \Big\{Q_{0x_0} + Q_{0x_1} Y(t_{00}; t_{10})\Big\}\delta x_0 + \sum_{i=1}^s \int\limits_{t_{00}-\tau_{i0}}^{t_{00}} Q_{0x_1} Y(t + \tau_{i0}; t) f_{0x_i}[t + \tau_{i0}]\delta\varphi(t)\, dt$$

$$+ \int\limits_{t_{00}}^{t_{10}} Q_{0x_1} Y(t; t_{10})\delta f[t]\, dt + (\delta\xi, 0, \ldots, 0)^\top, \quad \delta z = (\delta\xi, \delta\kappa) \in E_z - z_0. \quad (4.62)$$

Proof. Let $L_{\varsigma_0} \subset E_\varsigma$ be an arbitrary linear manifold and let

$$V_0 \subset X - (0, y_0)^\top, \ \ V \subset L_{\varsigma_0} - \varsigma_0$$

be arbitrary bounded convex neighborhoods of zero. There exists a number $\varepsilon_0 > 0$ such that for arbitrary $\varepsilon \in (0, \varepsilon_0)$ and $\delta z \in V_0 \times V$,

$$z_0 + \varepsilon \delta z \in D,$$

and the formula (4.61) holds.

We have

$$P(z_0 + \varepsilon \delta z) - P(z_0) = Q\big(t_{00} + \varepsilon \delta t_0, t_{10} + \varepsilon \delta t_1, \tau_{10} + \varepsilon \delta \tau_1, \ldots, \tau_{s0} + \varepsilon \delta \tau_s, x_{00} + \varepsilon \delta x_0, S(\kappa_0 + \varepsilon \delta \kappa)\big)$$
$$- Q\big(t_{00}, t_{10}, \tau_{10}, \ldots, \tau_{s0}, x_{00}, S(\kappa_0)\big) + \varepsilon(\delta \xi, 0, \ldots, 0)^\top.$$

Let a number $\varepsilon_0 > 0$ be insomuch small that

$$S(\kappa_0) + t(S(\kappa_0 + \varepsilon \delta \kappa) - S(\kappa_0)) \in O \ \ \forall (t, \varepsilon) \in (0, 1) \times (0, \varepsilon_0), \ \ \forall \delta z \in V_0 \times V,$$

where $\delta z = (\delta \xi, \delta \kappa)$ (see Lemma 4.16).

Let us now transform the difference

$$Q\big(t_{00} + \varepsilon \delta t_0, t_{10} + \varepsilon \delta t_1, \tau_{10} + \varepsilon \delta \tau_1, \ldots, \tau_{s0} + \varepsilon \delta \tau_s, x_{00} + \varepsilon \delta x_0, S(\kappa_0 + \varepsilon \delta \kappa)\big)$$
$$- Q\big(t_{00}, t_{10}, \tau_{10}, \ldots, \tau_{s0}, x_{00}, S(\kappa_0)\big)$$

$$= \int_0^1 \frac{d}{dt} Q\big(t_{00} + \varepsilon t \delta t_0, t_{10} + \varepsilon t \delta t_1, \tau_{10} + \varepsilon t \delta \tau_1, \ldots,$$

$$\tau_{s0} + \varepsilon t \delta \tau_s, x_{00} + \varepsilon t \delta x_0, S(\kappa_0) + t\big(S(\kappa_0 + \varepsilon \delta \kappa) - S(\kappa_0)\big)\big) dt$$

$$= \varepsilon \Big[Q_{0t_0} \delta t_0 + Q_{0t_1} \delta t_1 + \sum_{i=1}^s Q_{0\tau_i} \delta \tau_i + Q_{0x_0} \delta x_0 + Q_{0x_1} dS_{\kappa_0}(\delta \kappa) \Big] + \alpha(\varepsilon \delta z),$$

where

$$\alpha(\varepsilon \delta z) = \varepsilon \int_0^1 \Big\{ \big[Q_{0t_0}[\varepsilon; t] - Q_{0t_0}\big] \delta t_0 + \big[Q_{0t_1}[\varepsilon; t] - Q_{0t_1}\big] \delta t_1 + \sum_{i=1}^s \big[Q_{0\tau_i}[\varepsilon; t] - Q_{0\tau_i}\big] \delta \tau_i$$

$$+ \big[Q_{0x_0}[\varepsilon; t] - Q_{0x_0}\big] \delta x_0 + \big[Q_{0x_1}[\varepsilon; t] - Q_{0x_1}\big] S_{\kappa_0}(\delta \kappa) + Q_{0x_1}[\varepsilon; t] o(\varepsilon \delta \kappa) \Big\} dt,$$

$$Q_{0t_0}[\varepsilon; t] = Q_{t_0}\big(t_{00} + \varepsilon t \delta t_0, t_{10} + \varepsilon t \delta t_1, \tau_{10} + \varepsilon t \delta \tau_1, \ldots,$$

$$\tau_{s0} + \varepsilon t \delta \tau_s, x_{00} + \varepsilon t \delta x_0, S(\kappa_0) + t\big(S(\kappa_0 + \varepsilon \delta \kappa) - S(\kappa_0)\big)\big).$$

It is easy to note that

$$\lim_{\varepsilon \to 0} \big[Q_{0t_i}[\varepsilon; t] - Q_{0t_i}\big] = 0, \ \ i = 1, 2, \ \ \lim_{\varepsilon \to 0} \big[Q_{0\tau_i}[\varepsilon; t] - Q_{0\tau_i}\big] = 0, \ \ i = \overline{1, s},$$

$$\lim_{\varepsilon \to 0} \big[Q_{0x_i}[\varepsilon; t] - Q_{0x_i}\big] = 0, \ \ i = 0, 1.$$

Therefore, $\alpha(\varepsilon \delta z) = o(\varepsilon \delta z)$. Thus,

$$P(z_0 + \varepsilon \delta z) - P(z_0) = \varepsilon \Big[Q_{0t_0} \delta t_0 + Q_{0t_1} \delta t_1 + \sum_{i=1}^s Q_{0\tau_i} \delta \tau_i + Q_{0x_1} dS_{\kappa_0}(\delta \kappa) + (\delta \xi, 0, \ldots, 0)^\top \Big] + o(\varepsilon \delta z).$$

Due to the relations (4.58) and (4.59) from the above equality we get (4.62). □

Quasiconvexity of the filter Ψ_{z_0}. Continuity of the mapping P on the filter $\mathrm{co}[\Psi_{z_0}]$. In the topological vector space E_z, let us define the filter Ψ_{z_0} as the direct product

$$\Psi_{z_0} = \Psi_{\widehat{y}_0} \times \Psi_{\varphi_0} \times \Psi$$

of two filters $\Psi_{\widehat{y}_0}$, $\widehat{y}_0 = (0, y_0)^\top$, and Ψ_{φ_0} which are defined, respectively, by the convex bases

$$\Big\{ (B_0 \cap \mathbb{R}_+) \times (B_{t_{00}} \cap (a, t_{00}]) \times (B_{t_{01}} \cap (a, t_{10}]) \times (B_{\tau_{10}} \cap (\theta_{11}, \theta_{12})) \times \cdots$$
$$\times (B_{\tau_{s0}} \cap (\theta_{s1}, \theta_{s2})) \times (B_{x_{00}} \cap O): \quad B_0, \ldots, B_{x_{00}} \text{ are convex neighborhoods} \Big\},$$
$$\Big\{ B_{\varphi_0} \cap \Phi_1: \quad B_{\varphi_0} \subset \mathrm{PC}(I_1, \mathbb{R}^n) \text{ is a convex neighborhood} \Big\}.$$

The filter Ψ has been introduced in Subsection 4.3.

There exists a number $\delta_1 > 0$ such that the set

$$W = \mathbb{R}_+ \times (B(t_{00}; \delta_1) \cap (a, t_{00}]) \times (B(t_{01}; \delta_1) \cap (a, t_{10}]) \times (B(\tau_{10}; \delta_1) \cap (\theta_{11}, \theta_{12})) \times \cdots$$
$$\times B(\tau_{s0}; \delta_1) \cap (\theta_{s1}, \theta_{s2})) \times (B(x_{00}; \delta_1) \cap O)) \times (B(\varphi_0; \delta_1) \cap \Phi_1)) \times W_{f_0}^{(1)}(K_1, \delta_1) \subset D$$

and, moreover, the mapping

$$P : W \to \mathbb{R}_p^{l+1}$$

is continuous in the topology induced from E_z. Here

$$W_{f_0}^{(1)}(K_1, \delta_1) = \big\{ f \in E_f^{(1)}: \quad H_1(f - f_0 : K_1) \leq \delta_1 \big\}.$$

The element W_{K_1, δ_1} of the filter Ψ is contained in the convex set $W_{f_0}^{(1)}(K_1, \delta_1)$. Therefore,

$$\mathrm{co}(W_{z_0}) \subset W \subset D,$$

where

$$W_{z_0} = \mathbb{R}_+ \times (B(t_{00}; \delta_1) \cap (a, t_{00}]) \times (B(t_{01}; \delta_1) \cap (a, t_{10}]) \times (B(\tau_{10}; \delta_1) \cap (\theta_{11}, \theta_{12})) \times \cdots$$
$$\times B(\tau_{s0}; \delta_1) \cap (\theta_{s1}, \theta_{s2})) \times (B(x_{00}; \delta_1) \cap O)) \times (B(\varphi_0; \delta_1) \cap \Phi_1)) \times W_{K_1, \delta_1} \in \Psi_{z_0}.$$

Hence there exists an element $W_{z_0} \in \Psi$ such that

$$P : \mathrm{co}(W_{z_0}) \to \mathbb{R}^{l+1}$$

is continuous. Therefore, the mapping P is defined and continuous on the filter $\mathrm{co}([\Psi_{z_0}])$.

Criticality of the mapping P on the filter Ψ_{z_0}. The point $z_0 = (0, \kappa_0)$ belongs to all elements of the filter Ψ_{z_0}, and, moreover,

$$P(z_0) = \big(q^0 \big(t_{00}, t_{10}, \tau_{10}, \ldots, \tau_{s0}, x_{00}, x_0(t_{10}) \big), 0, \ldots, 0 \big)^\top.$$

Introduce the set

$$\mho = \Big\{ \kappa = (t_0, t_1, \tau_1, \ldots, \tau_s, x_0, \varphi, f): \quad f = f \big(t, x, x_1, \ldots, x_s, u(t), u(t - \theta_1), \ldots, u(t - \theta_\nu) \big),$$
$$w = (t_0, t_1, \tau_1, \ldots, \tau_s, x_0, \varphi, u) \in W_0 \Big\}.$$

For an arbitrary element

$$z = (\xi, \kappa) \in W_{z_0} \cap (\mathbb{R}_+ \times \mho),$$

where $W_{z_0} \in \Psi_{z_0}$, we have

$$P(z) = \big(q^0(t_0, t_1, \tau_1, \ldots, \tau_s, x_0, x(t_1); \kappa), 0, \ldots, 0 \big)^\top.$$

The element $w_0 \in W_0$ is optimal; therefore, there exists an element $W_{z_0}(K_2; \delta_2) \in \Psi_{z_0}$, where $\delta_2 \in (0, \widehat{\delta})$ and $K_2 \subset O$ is a compact set containing $\widehat{K}$ such that for an arbitrary element

$$z \in W_{z_0}(K_2; \delta_2) \cap (\mathbb{R}_+ \times \mho)$$

the inequality

$$q^0\big(t_{00}, t_{10}, \tau_{10}, \ldots, \tau_{s0}, x_{00}, x_0(t_{10})\big) \leq q^0\big(t_0, t_1, \tau_1, \ldots, \tau_s, x_0, x(t_1; \kappa)\big) + \xi$$

holds. It is easy to see that

$$P(W_{z_0} \cap (\mathbb{R}_+ \times \mho)) \subset \mathbb{R}_0 = \big\{(p^1, 0 \ldots, 0)^\top \in \mathbb{R}^{p+1}\big\}$$

and the point $P(z_0)$ is a boundary point of the set $P(W_{z_0}(K_2; \delta_2) \cap (\mathbb{R}_+ \times \mho))$ with respect to the space $\mathbb{R}_0$.

Therefore, $P(z_0) \in \partial(P(W_{z_0}(K_2; \delta_2) \cap \mathbb{R}_0))$, and, the more so, $P(z_0) \in \partial(P(W_{z_0}(K_2; \delta_2)))$.

Deduction of the necessary optimality conditions. All the conditions of Theorem 4.5 hold. Therefore, there exist a nonzero vector $\pi = (\pi_0, \ldots, \pi_l)$ and an element $\widehat{W}_{z_0} \in \Psi_{z_0}$ such that the inequality

$$\pi dP_{z_0}(\delta z) \leq 0 \quad \forall \delta z \in \mathrm{cone}(\widehat{W}_{z_0} - z_0) \tag{4.63}$$

holds, where $dP_{z_0}(\delta z)$ has the form (4.62).

Introduce the function

$$\psi(t) = \pi Q_{0x_1} Y(t : t_{10}); \tag{4.64}$$

as is easily seen, it satisfies the equation (4.54) and the conditions

$$\psi(t_{10}) = \pi Q_{0x_1}, \quad \psi(t) = 0, t > t_{10}. \tag{4.65}$$

Taking into account (4.62), (4.64) and (4.65), from the inequality (4.63) we obtain

$$\left\{\pi Q_{0t_0} - \psi(t_{00})f_0^- - \sum_{i=1}^s \psi(t_{00} + \tau_{i0})f_{0i}\right\}\delta t_0 + \left\{\pi Q_{0t_1} + \psi(t_{10})f_{s+1}^-\right\}\delta t_1$$

$$+ \sum_{i=1}^s \left\{\pi Q_{0\tau_i} - \psi(t_{00} + \tau_{i0})f_{0i} - \int_{t_{00}}^{t_{10}} \psi(t)f_{0x_i}[t]\dot{x}_0(t - \tau_{i0})\, dt\right\}\delta\tau_i + \left\{\pi Q_{0x_0} + \psi(t_{00})\right\}\delta x_0$$

$$+ \sum_{i=1}^s \int_{t_{00}-\tau_{i0}}^{t_{00}} \psi(t + \tau_{i0})f_{0x_i}[t + \tau_{i0}]\delta\varphi(t)\, dt + \int_{t_{00}}^{t_{10}} \psi(t)\delta f[t]\, dt + \pi_0\delta\xi, \quad \delta z \in \mathrm{cone}(\widehat{W}_{z_0} - z_0). \tag{4.66}$$

The condition $\delta z \in \mathrm{cone}(\widehat{W}_{z_0} - z_0)$ is equivalent to the conditions

$$\delta\xi \in \mathbb{R}_+, \quad \delta t_0 \in (-\infty, 0], \quad \delta t_1 \in (-\infty, 0], \quad \delta\tau_i \in \mathbb{R}, \quad i = \overline{1, s},$$

$$\delta x_0 \in \mathrm{cone}(\widehat{W}_{x_{00}} - x_{00}), \quad \delta\varphi \in \mathrm{cone}(\widehat{W}_{\varphi_0} - \varphi_0), \quad \delta f \in \mathrm{cone}(\widehat{W}_{f_0} - f_0),$$

where

$$\widehat{W}_{x_{00}} = B_{x_{00}} \cap X_0, \quad \widehat{W}_{\varphi_0} = B_{\varphi_0} \cap \Phi_1 \in \Psi_{\varphi_0}, \quad \widehat{W}_{f_0} \in \Psi_{f_0}.$$

Let $\delta t_0 = \delta t_1 = \delta\tau_1 = \cdots = \delta\tau_i = 0$ and $\delta x_0 = \delta\varphi = \delta f = 0$ in (4.66), we obtain

$$\pi_0\delta\xi \leq 0 \quad \forall \delta\xi \in \mathbb{R}_+.$$

This implies

$$\pi_0 \leq 0.$$

Setting $\delta\xi = \delta t_0 = \delta\tau_1 = \cdots = \delta\tau_i = 0$, and $\delta x_0 = \delta\varphi = \delta f = 0$; then, taking into account the fact that $\delta t_0 \in (-\infty, 0]$, from (4.66) for the initial moment t_{00} we obtain the following condition:

$$\pi Q_{0t_0} \geq \psi(t_{00})f_0^- + \sum_{i=1}^{s} \psi(t_{00} + \tau_{i0})f_{0i}.$$

If $\delta\xi = \delta t_0 = \delta\tau_1 = \cdots = \delta\tau_i = 0$ and $\delta x_0 = \delta\varphi = \delta f = 0$ in the inequality (4.66), then for the final moment t_{10} we obtain the following condition:

$$\pi Q_{0t_1} \geq -\psi(t_{10})f_{s+1}^-.$$

If $\delta\xi = \delta t_0 = \delta t_1 = 0$ and $\delta x_0 = \delta\varphi = \delta f = 0$, we get

$$\sum_{i=1}^{s} \left\{ \pi Q_{0\tau_i} - \psi(t_{00} + \tau_{i0})f_{0i} - \int_{t_{00}}^{t_{10}} \psi(t)f_{0x_i}[t]\dot{x}_0(t - \tau_{i0})\,dt \right\}\delta\tau_i \leq 0 \ \ \forall\delta\tau_i \in \mathbb{R}, \ \ i = \overline{1,s}.$$

From the above follow the conditions for the delays τ_{i0}, $i = \overline{1,s}$:

$$\pi Q_{0\tau_i} = \psi(t_{00} + \tau_{i0})f_{0i} + \int_{t_{00}}^{t_{10}} \psi(t)f_{0x_i}[t]\dot{x}_0(t - \tau_{i0})\,dt, \ \ i = \overline{1,s}.$$

Let $\delta\xi = \delta t_0 = \delta t_1 = \delta\tau_1 = \cdots = \delta\tau_i = 0$ and $\delta\varphi = \delta f = 0$ in (4.66). Then

$$\{\pi Q_{0x_0} + \psi(t_{00})\}\delta x_0 \leq 0, \ \ \delta x_0 \in \mathrm{cone}((B_{x_{00}} \cap X_0) - x_{00}).$$

Let us prove the inclusion

$$\mathrm{cone}((B_{x_{00}} \cap X_0) - x_{00}) \supset X_0 - x_{00}.$$

Indeed, let $x_0 \in X_0$ be arbitrary point. The set x_0 is convex, therefore, for an arbitrary $\varepsilon \in [0,1]$, the point $x_\varepsilon = x_{00} + \varepsilon(x_0 - x_{00}) \in X_0$. On the other hand, for a sufficiently small $\varepsilon > 0$, $x_\varepsilon \in B_{x_{00}}$. Hence $x_\varepsilon - x_{00} = \varepsilon(x_0 - x_{00}) \in (B_{x_{00}} \cap X_0) - x_{00}$. This implies $x_0 - x_{00} \in \mathrm{cone}((B_{x_{00}} \cap X_0) - x_{00})$. Thus,

$$\{\pi Q_{0x_0} + \psi(t_{00})\}x_{00} = \max_{x_0 \in X_0} \{\pi Q_{0x_0} + \psi(t_{00})\}x_0.$$

Let $\delta\xi = \delta t_0 = \delta t_1 = \delta\tau_1 = \cdots = \delta\tau_i = 0$ and $\delta x_0 = \delta f = 0$. We have

$$\sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} \psi(t + \tau_{i0})f_{0x_i}[t + \tau_{i0}]\delta\varphi(t)\,dt \leq 0 \ \ \forall\delta\varphi \in \mathrm{cone}(\widehat{W}_{\varphi_0} - \varphi_0).$$

Analogously, we can prove

$$\mathrm{cone}(\widehat{W}_{\varphi_0} - \varphi_0) \supset \Phi_1 - \varphi_0.$$

Thus,

$$\sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} \psi(t + \tau_{i0})f_{0x_i}[t + \tau_{i0}]\varphi_0(t)\,dt = \max_{\varphi(t)\in\Phi_1} \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t_{00}} \psi(t + \tau_{i0})f_{0x_i}[t + \tau_{i0}]\varphi(t)\,dt.$$

We now consider the case where $\delta\xi = \delta t_0 = \delta t_1 = \delta\tau_1 = \cdots = \delta\tau_i = 0$ and $\delta x_0 = \delta\varphi = 0$. From (4.66) we obtain

$$\int_{t_{00}}^{t_{10}} \psi(t)\delta f[t]\,dt \leq 0, \ \ \delta f \in \mathrm{cone}(\widehat{W}_{f_0} - f_0).$$

Now, using the last inequality, let us prove the integral maximum principle. For this purpose, we have to prove the continuity of the mapping

$$\delta f \longrightarrow \int_{t_{00}}^{t_{10}} \delta f[t] dt, \quad \delta f[t] = \delta f\big(t, x_0(t), x_0(t - \tau_{10}), \dots, x_0(t - \tau_{s0})\big) \tag{4.67}$$

on the set $W^{(1)}(K_1; \alpha)$ in the topology induced from $E_f^{(1)}$. Here $K_1 \subset O$ is a compact set containing a certain neighborhood of the set $\varphi_0(I_2) \cup x_0([t_{00}, t_{10}])$ and $\alpha > 0$ is a certain number.

Let $\delta f_i \in W^{(1)}(K_1; \alpha)$, $i = 1, 2, \dots,$ and $\lim_{i \to \infty} H_0(\delta f_i; K_1) = 0$. The mapping (4.67) is continuous if

$$\lim_{i \to \infty} \int_{t_{00}}^{t_{10}} \psi(t) \delta f_i[t] \, dt = 0. \tag{4.68}$$

Integration by parts yields

$$\int_{t_{00}}^{t_{10}} \delta \psi(t) f_i[t] dt = \psi(t_{00}) \int_{t_{00}}^{t_{10}} \delta f_i[t] \, dt - \int_{t_{00}}^{t_{10}} \psi(t) \left(\int_{t_{00}}^{t} \delta f_i[\xi] \, d\xi \right) dt.$$

By Lemma 1.5, we have

$$\lim_{i \to \infty} \int_{t_{00}}^{t} \delta f_i[\xi] \, d\xi = 0$$

uniformly in $t \in [t_{00}, t_{10}]$.

Therefore, the relation (4.68) holds. The continuity of the mapping (4.67) allows us to strengthen inequality given above:

$$\int_{t_{00}}^{t_{10}} \psi(t) \delta f[t] dt \le 0, \quad \delta f \in \text{cone}([W^{(1)}(K_1; \alpha)]_{\widehat{W}_{f_0}} - f_0).$$

According to Lemma 4.14,

$$\text{cone}([W^{(1)}(K_1; \alpha)]_{\widehat{W}_{f_0}} - f_0) \supset F_1 - f_0. \tag{4.69}$$

From (4.69) it follows the integral maximum principle

$$\int_{t_{00}}^{t_{10}} \psi(t) f_0[t] \, dt = \max_{u(t) \in \Omega(I, U)} \int_{t_{00}}^{t_{10}} \psi(t) f\Big(t, x_0(t), x_0(t - \tau_{10}), \dots, x_0(t - \tau_{s0}), u(t), u\big(t - \theta_1, \dots, u(t - \theta_\nu)\big)\Big) \, dt.$$

Theorem 4.6 is proved.

To conclude this subsection, it should be noted that Theorems 4.7–4.10 are proved in a similar way by using the corresponding variation formulas of a solution.

4.6 The optimal control problem with the continuous initial condition

Consider the optimal control problem

$$\dot{x}(t) = f\big(t, x(t), x(t - \tau_1), \dots, x(t - \tau_s), u(t), u(t - \theta_1), \dots, u(t - \theta_\nu)\big), \tag{4.70}$$

$$t \in [t_0, t_1] \subset I, \quad u \in \Omega(I_2, U),$$

$$x(t) = \varphi(t), \quad t \in [\widehat{\tau}, t_0], \quad \varphi \in \Phi_3, \tag{4.71}$$

$$q^i\big(t_0, t_1, \tau_1, \dots, \tau_s, \varphi(t_0), x(t_1)\big) = 0, \quad i = \overline{1, l}, \tag{4.72}$$

$$q^0\big(t_0, t_1, \tau_1, \dots, \tau_s, \varphi(t_0), x(t_1)\big) \longrightarrow \min, \tag{4.73}$$

where $\Phi_3 = \{\varphi \in C(I_1, \mathbb{R}^n) : \varphi(t) \in N\}$, $N \subset O$ is a convex set.

The problem (4.70)–(4.73) is called an optimal control problem with the continuous initial condition.

Definition 4.16. Let $v = (t_0, t_1, \tau_1, \ldots, \tau_s, \varphi, u) \in A_1 = (a, b) \times (a, b) \times (\theta_{11}, \theta_{12}) \times \cdots \times (\theta_{s1}, \theta_{s2}) \times \Phi_3 \times \Omega(I, U))$. A function $x(t) = x(t; v) \in O$, $t \in [\widehat{\tau}, t_1]$, is called a solution of the equation (4.70) with the discontinuous initial condition (4.71), or a solution corresponding to the element v and defined on the interval $[\widehat{\tau}, t_1]$, if it satisfies the condition (4.71) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (4.70) a.e. on $[t_0.t_1]$.

Definition 4.17. An element $v = (t_0, t_1, \tau_1, \ldots, \tau_s, \varphi, u) \in A_1$ is said to be admissible if the corresponding solution $x(t) = x(t; v)$ satisfies the boundary conditions (4.72).

Denote by A_{10} the set of admissible elements.

Definition 4.18. An element $v_0 = (t_{00}, t_{10}, \tau_{10}, \ldots, \tau_{s0}, \varphi_0, u_0) \in V_{10}$ is said to be optimal if there exist a number $\delta_0 > 0$ and a compact set $K_0 \subset O$ such that for an arbitrary element $v \in A_{10}$ satisfying the condition

$$|t_{00} - t_0| + |t_{10} - t_1| + \sum_{i=1}^{s} |\tau_{i0} - \tau_i| + \|\varphi_0 - \varphi\|_{I_1} + H_1(f_0 - f; K_0) \leq \delta_0$$

the inequality

$$q^0\big(t_{00}, t_{10}, \tau_{10}, \ldots, \tau_{s0}, \varphi_0(t_{00}), x_0(t_{10})\big) \leq q^0\big(t_0, t_1, \tau_1, \ldots, \tau_s, \varphi(t_0), x(t_1)\big)$$

holds.

Theorem 4.11. *Let v_0 be an optimal element and let the following conditions hold:*

4.14. *$\tau_{s0} > \cdots > \tau_{10}$ and with $\tau_{i0} \in (\theta_{i0}, \theta_{i+10})$, $i = \overline{1, s-1}$;*

4.15. *$\theta_i = m_i h$, $i = \overline{1, \nu}$, where m_i, $i = \overline{1, \nu}$, are natural numbers, $h > 0$ is a real number;*

4.16. *the function $\varphi_0(t)$ is absolutely continuous and $\dot{\varphi}_0(t)$ is bounded;*

4.17. *the function $f_0(w)$, $w = (t, x, x_1, \ldots, x_s) \in I \times O^{1+s}$, is bounded;*

4.18. *there exist the finite limits*

$$\lim_{t \to t_{00}-} \dot{\varphi}_0(t) = \dot{\varphi}_0^-, \quad \lim_{w \to w_0} f_0(w) = f_0^-, \quad w \in (a, t_{00}] \times O^{1+s},$$

where $w_0 = (t_{00}, \varphi_0(t_{00}), \varphi_0(t_{00} - \tau_{10}), \ldots, \varphi_0(t_{00} - \tau_{s0}))$;

4.19. *there exists the finite limit*

$$\lim_{w \to w_{s+1}} f_0(w) = f_{s+1}^-, \quad w \in (t_{00}, t_{10}] \times O^{s+1}, \quad w_{s+1} = \big(t_{10}, x_0(t_{10}), x_0(t_{10} - \tau_{10}), \ldots, x_s(t_{10} - \tau_{s0})\big).$$

Then there exist a vector $\pi = (\pi_0, \ldots, \pi_l) \neq 0$ with $\pi_0 \leq 0$, and a solution $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ of the equation

$$\dot{\psi}(t) = -\psi(t) f_{0x}[t] - \sum_{i=1}^{s} \psi(t + \tau_{i0}) f_{0x_i}[t + \tau_{i0}], \quad t \in [t_{00}, t_{10}],$$

$$\psi(t) = 0, \quad t > t_{10}, \tag{4.74}$$

such that the following conditions hold:

4.20. *the conditions for the moments t_{00} and t_{10}:*

$$\pi Q_{0t_0} \geq \psi(t_{00})[\dot{\varphi}_0^- - f_0^-], \quad \pi Q_{0t_1} \geq -\psi(t_{10}) f_{s+1}^-;$$

4.21. *the conditions for the delays* τ_{i0}, $i = \overline{1,s}$:

$$\pi Q_{0\tau_i} = \int\limits_{t_{00}}^{t_{10}} \psi(t) f_{0x_i}[t] \dot{x}_0(t - \tau_{i0}) \, dt, \quad i = \overline{1,s};$$

4.22. *the maximum principle for the initial function* $\varphi_0(t)$:

$$[Q_{0x_0} + \psi(t_{00})]\varphi_0(t_{00}) + \sum_{i=1}^{s} \int\limits_{t_{00}-\tau_{i0}}^{t_{00}} \psi(t + \tau_{i0}) f_{0x_i}[t + \tau_{i0}]\varphi_0(t) \, dt,$$

$$= \max_{\varphi(t) \in \Phi_2} [Q_{0x_0} + \psi(t_{00})]\varphi(t_{00}) + \sum_{i=1}^{s} \int\limits_{t_{00}-\tau_{i0}}^{t_{00}} \psi(t + \tau_{i0}) f_{0x_i}[t + \tau_{i0}]\varphi(t) \, dt;$$

4.23. *the integral maximum principle for the control function* $u_0(t)$:

$$\int\limits_{t_{00}}^{t_{10}} \psi(t) f_0[t] \, dt$$

$$= \max_{u(t) \in \Omega(I_2, U)} \int\limits_{t_{00}}^{t_{10}} \psi(t) f\Big(t, x_0(t), x_0(t - \tau_{10}), \ldots, x_0\big(t - \tau_{i0}, u(t), u(t - \theta_1), \ldots, u(t - \theta_\nu)\big)\Big) \, dt;$$

4.24. *the condition for the function* $\psi(t)$
$$\psi(t_{10}) = \pi Q_{0x_1}.$$

Theorem 4.12. *Let v_0 be an optimal element and let the conditions* 4.14–4.17 *hold. Moreover, there exist the finite limits*

$$\lim_{t \to t_{00}+} \dot{\varphi}(t) = \dot{\varphi}_0^+, \quad \lim_{w \to w_0} f_0(w) = f_0^+, \quad w \in [t_{00}, b) \times O^{1+s},$$
$$\lim_{w \to w_{s+1}} f_0(w) = f_{s+1}^+, \quad w \in [t_{10}, b) \times O^{s+1}.$$

Then there exist a vector $\pi = (\pi_0, \ldots, \pi_l) \neq 0$ with $\pi_0 \leq 0$, and a solution of the equation (4.74) *such that the conditions* 4.21–4.24 *hold. Moreover,*

$$\pi Q_{0t_0} \leq \psi(t_{00})[\dot{\varphi}_0^+ - f_0^+], \quad \pi Q_{0t_1} \leq -\psi(t_{10})f_{s+1}^+.$$

Theorem 4.13. *Let v_0 be an optimal element and let the conditions of Theorems* 4.11 *and* 4.12 *hold. Moreover, $\dot{\varphi}_0^- - f_0^- = \dot{\varphi}_0^+ - f_0^+ := \widehat{f}_0$, $f_{s+1}^- = f_{s+1}^+ := \widehat{f}_{s+1}$. Then there exist a vector $\pi = (\pi_0, \ldots, \pi_l) \neq 0$, with $\pi_0 \leq 0$, and a solution of the equation* (4.74) *such that the conditions* 4.21–4.24 *hold. Moreover,*

$$\pi Q_{0t_0} = \psi(t_{00})\widehat{f}_0, \quad \pi Q_{0t_1} = -\psi(t_{10})\widehat{f}_{s+1}.$$

By variation formulas of a solution (see Section 3), Theorems 4.11–4.13 are proved by the analogous scheme.

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p-MOMENT EXPONENTIAL STABILITY OF DIFFERENTIAL EQUATIONS WITH RANDOM IMPULSES AND THE ERLANG DISTRIBUTION

Abstract. The investigation of differential equations with random impulses combines ideas in the qualitative theory of differential equations and probability theory. The p-moment exponential stability of the solutions is defined and studied when the waiting time between two consecutive impulses is Erlang distributed. The study is based on the application of Lyapunov functions. Some examples are given to illustrate the results.

რეზიუმე. შემთხვევით იმპულსებიანი დიფერენციალური განტოლებების გამოკვლევა აერთია-ნებს დიფერენციალურ განტოლებათა თვისებრივი თეორიისა და ალბათობის თეორიის იდეებს. ამონახსნთა p-მომენტური ექსპონენციალური მდგრადობა განსაზღვრულია და შესწავლილია იმ შემთხვევაში, როცა ორ მომდევნო იმპულსს შორის ლოდინის დრო არის ერლანგის წესით განაწილებული. გამოკვლევა ეფუძნება ლიაპუნოვის ფუნქციების გამოყენებას. შედეგების საილუსტრაციოდ მოყვანილია მაგალითები.

1   Introduction

In some real world phenomena the investigated process changes instantaneously at uncertain moments. In modeling such processes it is necessarily to combine deterministic differential equations with random variables presenting the moments of impulses. The presence of randomness in the jump condition changes the behavior of solutions of differential equations significantly. The study of properties of solutions combines methods of deterministic differential equations and probability theory. Impulsive differential equations with random impulsive moments differ from stochastic differential equations with jumps [5, 9–11]. Investigations concerning deterministic differential equations with random impulses were considered in [2, 3, 7, 8, 12], but there are some inaccuracies there concerning properties of deterministic variables and random variables.

In this paper we study nonlinear differential equations subject to impulses occurring at random moments. Inspired by queuing theory and the distribution for the waiting time, we study the case of Erlang distributed random variables between two consecutive moments of impulses. The p-moment exponential stability of the solution is investigated by employing Lyapunov's functions.

2   Random impulses in differential equations

Let the increasing sequence of nonnegative points $\{T_k\}_{k=0}^{\infty}$ be given with $\lim_{k\to\infty} \{T_k\} = \infty$. Consider the initial value problem for the system of *impulsive differential equations* (IDE) with fixed points of impulses

$$
\begin{aligned}
x' &= f(t, x(t)) \text{ for } t \in (T_k, T_{k+1}], \quad k = 0, 1, 2, \ldots, \\
x(T_k + 0) &= I_k(x(T_k - 0)) \text{ for } k = 1, 2, \ldots, \\
x(T_0) &= x_0,
\end{aligned}
\tag{2.1}
$$

where $x, x_0 \in \mathbb{R}^n$, $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, $I_k : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$.

We will assume the following condition is satisfied

H1. $f(t, 0) = 0$ *and* $I_k(t, 0) = 0$ *for* $t \geq 0$, $k = 1, 2, \ldots$.

Let the probability space $(\Omega, \mathcal{F}, P)$ be given. Let $\{\tau_k\}_{k=1}^{\infty}$ be a sequence of independent random variables that are defined on the sample space Ω. We will call the random variables τ_k waiting times. Assume $\sum_{k=1}^{\infty} \tau_k = \infty$ with probability 1.

We will assume the following condition is satisfied:

H2. *The random variables* $\{\tau_k\}_{k=1}^{\infty}$, $\tau_k \in Erlang(\alpha_k, \lambda)$ *are independent random variables.*

We will recall some properties of Erlang distribution:

(i) If $X \in Erlang(\alpha_1, \lambda)$ and $Y \in Erlang(\alpha_2, \lambda)$ are independent random variables, then $X + Y \in Erlang(\alpha_1 + \alpha_2, \lambda)$;

(ii) The cumulative distribution function (CDF) of $Erlang(\alpha, \lambda)$ is

$$
F(x; \alpha, \lambda) = 1 - \sum_{j=1}^{\alpha-1} \frac{(\lambda x)^j}{j!} \, e^{-\lambda x}, \quad x \geq 0,
$$

and the density function is

$$
f(x; \alpha, \lambda) = \lambda \frac{(\lambda x)^{\alpha-1}}{(\alpha-1)!} \, e^{-\lambda x}, \quad x > 0.
$$

Define the sequence of random variables $\{\xi_k\}_{k=0}^{\infty}$ such that

$$
\xi_k = T_0 + \sum_{i=1}^{k} \tau_i, \quad k = 1, 2, \ldots,
$$

where $T_0 \geq 0$ is a fixed point.

We note that $\{\xi_k\}_{k=0}^{\infty}$ is an increasing sequence of random variables defined by the recurrence formula $\xi_0 = T_0$, $\xi_k = \xi_{k-1} + \tau_k$, $k = 1, 2, \ldots$. The random variable ξ_n will be called the waiting time and it gives the arrival time of n-th impulses.

Remark 2.1. The random variables $\Xi_n = \xi_n - T_0 = \sum_{i=1}^{n} \tau_i \in Erlang\left(\sum_{i=1}^{n} \alpha_i, \lambda\right)$ are continuous with CDF

$$F\left(t; \sum_{i=1}^{n} \alpha_i, \lambda\right) = P(\Xi_n < t) = 1 - \sum_{j=1}^{\sum_{i=1}^{n} \alpha_i - 1} \frac{(\lambda t)^j}{j!} e^{-\lambda t} \text{ for } t \geq 0.$$

Let the points t_k be arbitrary values of the random variables τ_k, $k = 1, 2, \ldots$, correspondingly. Define the increasing sequence of points $T_k = T_0 + \sum_{i=1}^{k} t_i$, $k = 1, 2, 3 \ldots$, that are values of the random variables ξ_k.

Consider the initial value problem (IVP) for the system of impulsive fractional differential equations (IFrDE) with fixed points of impulses (2.1). The solution of IVP for IDE (2.1) depends not only on the initial condition (T_0, x_0) but on the moments of impulses T_k, $k = 1, 2, \ldots$, i.e. the solution depends on the initially chosen arbitrary values t_k of the random variables τ_k, $k = 1, 2, \ldots$. We denote the solution of the initial value problem (2.1) by $x(t; T_0, x_0, \{T_k\})$. We will assume that $x(T_k; T_0, x_0, \{t_k\}) = \lim_{t \to T_k - 0} x(t; T_0, x_0, \{t_k\})$ for any $k = 1, 2, \ldots$.

The set of all solutions $x(t; T_0, x_0, \{T_k\})$ of IVP for IDE (2.1) for any values t_k of the random variables τ_k, $k = 1, 2, \ldots$, generates a stochastic process with state space $\mathbb{R}^n$. We denote it by $x(t; T_0, x_0, \{\tau_k\})$ and we will say that it is a solution of the following initial value problem for impulsive differential equations with random moments of impulses (RIDE)

$$\begin{aligned} x'(t) &= f(t, x(t)) \text{ for } t \geq T_0, \ \xi_k < t < \xi_{k+1}, \ k = 0, 1, \ldots, \\ x(\xi_k + 0) &= I_k(x(\xi_k - 0)) \text{ for } k = 1, 2, \ldots, \\ x(T_0) &= x_0. \end{aligned} \qquad (2.2)$$

Definition 2.1. Let t_k be a value of the random variable τ_k, $k = 1, 2, 3, \ldots$, and $T_k = T_0 + \sum_{i=1}^{k} t_i$, $k = 1, 2, \ldots$. Then the solution $x(t; T_0, x_0, \{T_k\})$ of the IVP for the IDE with fixed points of impulses (2.1) is called *a sample path solution* of the IVP for the RIDE (2.2).

Any sample path solution $x(t; T_0, x_0, \{T_k\}) \in C^1((T_k, T_{k+1}], \mathbb{R}^n)$, $k = 0, 1, 2, \ldots$.

Definition 2.2. A stochastic process $x(t; T_0, x_0, \{\tau_k\})$ with an uncountable state space $\mathbb{R}^n$ is said to be a solution of the IVP for the system of RIDE (2.2) if for any values t_k of the random variable τ_k, $k = 1, 2, 3, \ldots$, and $T_k = T_0 + \sum_{i=1}^{k} t_i$, $k = 1, 2, \ldots$, the corresponding function $x(t; T_0, x_0, \{T_k\})$ is a sample path solution of the IVP for RIDE (2.2).

Example.

Case 1 (differential equation). Consider the following scalar ordinary differential equation (ODE) $x' = 0$, $x(0) = x_0 \neq 0$. Its solution $x(t) = x_0$, $t \geq 0$, is stable but does not approach 0.

Case 2 (impulsive differential equations with fixed points of impulses). Consider the following IVP for the scalar IDE (2.1)

$$\begin{aligned} x' &= 0 \text{ for } t \geq 0, \ t \neq T_k, \\ x(T_k + 0) &= ax(T_k - 0) \text{ for } k = 1, 2, \ldots, \\ x(0) &= x_0 \neq 0, \end{aligned} \qquad (2.3)$$

where a is a constant.

The solution of IVP (2.3) is the piecewise continuous function $x(t; x_0) = a^k x_0$ for $t \in (T_k, T_{k+1}]$. The behavior of $x(t; x_0)$ depends significantly on the amplitude a of the impulses.

If $|a| < 1$ then $|x(t; x_0)|$ approaches 0.

Case 3 (differential equation with random points of impulses). Consider the following partial case of the IVP for RIDE (2.2)

$$
\begin{aligned}
x' &= 0 \ \text{ for } \ t \geq 0, \ \ \xi_k < t < \xi_{k+1}, \\
x(\xi_k + 0) &= ax(\xi_k - 0) \ \text{ for } \ k = 1, 2, \dots, \\
x(0) &= x_0 \neq 0,
\end{aligned}
\tag{2.4}
$$

where $x \in \mathbb{R}$, a is a constant and the random variables ξ_k are defined above.

Let for any $k = 1, 2, \dots$ the point t_k be an arbitrary value of the random variable τ_k and $T_k = \sum_{i=0}^{k} t_i$, $k = 1, 2, 3 \dots$, i.e. T_k is a value of the random variable ξ_k. Consider the IVP for the corresponding IDE

$$
\begin{aligned}
x' &= 0 \ \text{ for } \ t \geq 0, \ \ t \neq T_k, \\
x(T_k + 0) &= ax(T_k - 0) \ \text{ for } \ k = 1, 2, \dots, \\
x(0) &= x_0.
\end{aligned}
\tag{2.5}
$$

The solution of (2.5) is $x(t; 0, x_0, \{T_k\}) = a^k x_0$ for $T_k < t \leq T_{k+1}$. It depends on both initial value x_0 and the moments of impulses T_k, i.e. on the initially chosen arbitrary values t_k of the random variables τ_k, $k = 1, 2, \dots$.

The set of all solutions of the IVP (2.5) for any values t_k of the random variables τ_k generates a stochastic process $x(t; x_0, \{\tau_k\}) = a^k x_0$ for $\xi_k < t \leq \xi_{k+1}$ which has an expected value

$$
E\big|x(t; x_0, \{\tau_k\})\big| = |x_0| P(T_0 < t < \xi_1) + \sum_{k=1}^{\infty} |a^k x_0| P(\xi_k < t < \xi_{k+1}),
$$

i.e. it depends significantly on the distribution of the random variables τ_k.

3 Preliminary probability results

Lemma 3.1. *Let the condition* (H2) *be satisfied. Then the probability that there will be exactly k impulses until time t, $t \geq T_0$, is given by*

$$
P(S_k(t)) = e^{-\lambda(t - T_0)} \sum_{j = \sum_{i=1}^{k-1} \alpha_i}^{\sum_{i=1}^{k} \alpha_i - 1} \frac{(\lambda(t - T_0))^j}{j!}, \quad t \geq T_0,
$$

where the events $S_k(t) = \{\omega \in \Omega : \ \xi_k(\omega) < t < \xi_{k+1}(\omega)\}$, $k = 1, 2, \dots$.

Proof. According to Remark 2.1 we get

$$
\begin{aligned}
P(S_k(t)) &= P(\xi_k < t < \xi_{k+1}) = P(\xi_k - T_0 < t - T_0 < \xi_{k+1} - T_0) \\
&= F_{\Xi_k}\left(t - T_0; \sum_{i=1}^{k} \alpha_i, \lambda\right) - F_{\Xi_{k+1}}\left(t - T_0; \sum_{i=1}^{k+1} \alpha_i, \lambda\right). \qquad \square
\end{aligned}
$$

Corollary. *Let the condition* (H2) *be satisfied with $\alpha_i = \alpha$, $i = 1, 2, \dots$. Then the probability that there will be exactly k impulses until time t, $t \geq T_0$, is given by*

$$
P(S_k(t)) = e^{-\lambda(t - T_0)} \sum_{j = (k-1)\alpha}^{k\alpha - 1} \frac{(\lambda(t - T_0))^j}{j!}.
$$

We now obtain a formula for the solution of the initial value problem for a scalar linear differential equation with random moments of impulses:

$$
\begin{aligned}
u' &= -m_k u \ \text{ for } \ t \geq T_0, \ \ \xi_k < t < \xi_{k+1}, \\
u(\xi_k + 0) &= b_k u(\xi_k - 0) \ \text{ for } \ k = 1, 2, \ldots, \\
u(T_0) &= u_0,
\end{aligned}
\tag{3.1}
$$

where $u_0 \in \mathbb{R}$, $m_k > 0$, $k = 0, 1, 2, \ldots$, and $b_k \neq 1$, $k = 1, 2, \ldots$, are real constants.

Lemma 3.2. *Let the condition* (H2) *be satisfied and the nonincreasing sequence of real positive numbers* $\{m_i\}_{i=0}^{\infty}$ *be such that*

$$
\sum_{k=0}^{\infty} e^{-m_k(t-T_0)} \prod_{i=1}^{k} |b_i| < \infty.
$$

Then the solution of the IVP for the linear RIDE (3.1) *is given by the formula*

$$
u(t; T_0, u_0, \{\tau_k\}) =
\begin{cases}
u_0 e^{-m_0(t-T_0)} & \text{for } T_0 < t < \tau_1 \\
u_0 \Big(\prod_{i=1}^{k} b_i\Big) e^{-\sum_{i=1}^{k} m_{i-1}\tau_i} e^{-m_k(t-\xi_k)} & \text{for } \xi_k < t < \xi_{k+1}, \ \ k = 1, 2, \ldots,
\end{cases}
\tag{3.2}
$$

and the expected value of the solution satisfies the inequality

$$
E\big(\big|u(t; T_0, u_0, \{\tau_k\})\big|\big) \leq |u_0| \sum_{k=0}^{\infty} e^{-m_k(t-T_0)} \prod_{i=1}^{k} |b_i| \ \ \text{for } t \geq T_0.
$$

Proof. The formula for the solution follows from the formula for the solution of the corresponding IVP for the linear IDE with fixed points of impulses and Definition 2.2.

According to Lemma 3.1, formula (3.2), the independence of the random variables τ_k and inequality $E(\eta) \leq E(\xi)$ for the random variables $\eta, \xi : \ 0 \leq \eta \leq \xi$ we have

$$
\begin{aligned}
E\big(\big|u(t; T_0, u_0, \{\tau_k\})\big|\big) &= |u_0| e^{-m_0(t-T_0)} P\big(\tau_1 > t - T_0\big) \\
&+ \sum_{k=1}^{\infty} |u_0| \Big(\prod_{i=1}^{k} |b_i|\Big) e^{-m_k(t-T_0)} \prod_{i=1}^{k} E\big(e^{-(m_{i-1}-m_k)\tau_i}\big) P(S_k(t)) \ \text{ for } t \geq T_0. \quad (3.3)
\end{aligned}
$$

Using the definition of the density function of the Erlang distribution and substituting $(m_i - m_k + \lambda)x = s$ we get

$$
\begin{aligned}
E e^{(m_k-m_i)\tau_i} &= \int_0^{\infty} e^{(m_k-m_i)x} \lambda \frac{(\lambda x)^{\alpha_i-1}}{(\alpha_i-1)!} e^{-\lambda x}\, dx = \frac{(\lambda)^{\alpha_i}}{(\alpha_i-1)!} \int_0^{\infty} e^{-(m_i-m_k+\lambda)x} x^{\alpha_i-1}\, dx \\
&= \frac{1}{(m_i-m_k+\lambda)^{\alpha_i}} \frac{(\lambda)^{\alpha_i}}{(\alpha_i-1)!} \int_0^{\infty} e^{-s} s^{\alpha_i-1}\, ds = \Big(\frac{\lambda}{m_i-m_k+\lambda}\Big)^{\alpha_i}.
\end{aligned}
\tag{3.4}
$$

Substitute (3.4) in (3.3), use Lemma 3.1 and obtain

$$
\begin{aligned}
E\big(\big|u(t; T_0, u_0, \{\tau_k\})\big|\big) &= |u_0| e^{-(m_0+\lambda)(t-T_0)} \sum_{j=1}^{\alpha_1-1} \frac{(\lambda(t-T_0))^j}{j!} \\
&+ \sum_{k=1}^{\infty} |u_0| e^{-(m_k+\lambda)(t-T_0)} \Big(\prod_{i=1}^{k} |b_i|\Big) \prod_{i=1}^{k} \Big(\frac{\lambda}{m_i-m_k+\lambda}\Big)^{\alpha_i} \sum_{j=\sum_{i=1}^{k-1}\alpha_i}^{\sum_{i=1}^{k}\alpha_i-1} \frac{(\lambda(t-T_0))^j}{j!}.
\end{aligned}
\tag{3.5}
$$

Inequalities (3.5) and $\frac{\lambda}{m_i-m_k+\lambda} \leq 1$ prove the lemma. $\qquad\square$

Remark 3.1. Note that the conditions of Lemma 3.2 are satisfied for $m_k = m$ and $|b_i| = \frac{1}{2}$.

4 p-exponential stability

In this paper we will use Lyapunov functions $V(t, x) : J \times \Delta \to \mathbb{R}_+$, which are continuous on $J \times \Delta$ and locally Lipschitzian with respect to its second argument, where $J \subset \mathbb{R}_+$ and $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$, and their Dini derivatives.

Definition 4.1. Let $p > 0$. Then the trivial solution $(x_0 = 0)$ of the RIDE (2.2) is said to be p-moment exponentially stable if for any $x_0 \in \mathbb{R}^n$ there exist constants α, $\mu > 0$ such that $E[\|x(t; T_0, x_0, \{\tau_k\})\|^p] < \alpha \|x_0\|^p e^{-\mu(t-T_0)}$ for all $t > T_0$, where $x(t; T_0, x_0, \{\tau_k\}$ is the solution of the IVP for the RIDE (2.2).

Theorem 4.1. *Let the following conditions be satisfied:*

1. *The conditions* (H1), (H2) *hold.*

2. *The function* $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ *and there exist positive constants* a, b *such that*

   (i) $a\|x\|^p \leq V(t, x) \leq b\|x\|^p$ *for* $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$;

   (ii) *there exists a constant* $m \geq 0$ *such that*
   $$D_{(2.1)}^+ V(t, x) \leq -mV(t, x) \ \text{for} \ t \in \mathbb{R}_+, \ \ x \in \mathbb{R}^n;$$

   (iii) *for any* $k = 1, 2, \ldots$ *there exist functions* $w_k \in C(\mathbb{R}_+, \mathbb{R}_+)$ *and constants* $C_k > 0$, $w_k(t) \leq C_k$ *for* $t \geq 0$ *such that* $\sum_{k=0}^{\infty} \prod_{i=1}^{k} C_i = C < \infty$ *and*
   $$V(t, I_k(x)) \leq w_k(t) V(t, x) \ \text{for} \ t \in \mathbb{R}_+, \ \ x \in \mathbb{R}^n. \tag{4.1}$$

Then the trivial solution of the RIDE (2.2) *is p-moment exponentially stable.*

Proof. Let $(T_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ be an arbitrary initial data and the stochastic process $x_\tau(t) = x(t; T_0, x_0, \{\tau_k\})$ be a solution of the IVP for the RIDE (2.2).

Now consider the IVP for the scalar linear RIDE (3.1) with $m_k = m$, $b_k = C_k$ and $u_0 = V(T_0, x_0)$ with a solution $u_\tau(t) = u(t; T_0, x_0, \{\tau_k\})$. According to Lemma 3.2 the inequality

$$E\big(|u_\tau(t)|\big) \leq |u_0|e^{-m(t-T_0)} \sum_{k=0}^{\infty} \prod_{i=1}^{k} |C_i| = C|u_0|e^{-m(t-T_0)}$$

holds.

Let t_k be arbitrary values of the random variables τ_k, $k = 1, 2, \ldots$, and $T_k = T_0 + \sum_{i=1}^{k} t_i$, $k = 1, 2, \ldots$. Consider the sample path solutions $x(t) = x(t; T_0, x_0, \{T_k\})$ and $u(t) = u(t; T_0, x_0, \{T_k\})$.

Let $v(t) = V(t, x(t))$ for $t \geq T_0$. The function $v(t)$ satisfies the linear impulsive differential inequalities with fixed points of impulses

$$\begin{aligned} D_+ v(t) &\leq -m\, v_\tau(t) \ \text{for} \ T_k < t < T_{k+1}, \\ v(T_k+) &\leq C_k v(T_k), \ \ k = 1, 2, \ldots, \\ v(T_0) &= V(T_0, x_0). \end{aligned} \tag{4.2}$$

The function $m(t) = v(t) - u(t)$, $t \geq T_0$, is a piecewise continuous function and satisfies IVP (4.2) with a zero initial condition. Therefore $m(t) \leq 0$ on $[T_0, \infty)$ (for details see the books [4, 6]). Therefore $v_\tau(t) \leq u_\tau(t)$ where the set of $v(t; T_0, x_0, \{T_k\})$ for any values t_k of the random variables τ_k, $k = 1, 2, \ldots$, generates a stochastic process $v_\tau(t)$ with state space $\mathbb{R}^n$.

From the condition 2 (i) of Theorem 4.1 we obtain

$$E\big(\|x_\tau(t)\|^p\big) = \frac{1}{a} E\big(a\|x_\tau(t)\|^p\big) \leq \frac{1}{a} E(V(t, x_\tau(t))) = \frac{1}{a} E(v_\tau(t)) \leq \frac{1}{a} E(u_\tau(t))$$

$$\leq \frac{C}{a} V(T_0, x_0)e^{-m(t-T_0)} \leq \frac{Cb}{a} \|x_0\|^p e^{-m(t-T_0)}, \ \ t \geq T_0. \qquad \square$$

Remark 4.1. If $\alpha_k = 1$ for all k, i.e. the random variables $\tau_k \in Exp(\lambda)$, the p-moment exponential stability is studied in [1].

Acknowledgments

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SOLVABILITY OF PERIODIC BOUNDARY VALUE PROBLEMS OF FRACTIONAL DIFFERENTIAL SYSTEMS WITH IMPULSE EFFECTS

Abstract. Two new classes of periodic boundary value problems of coupled impulsive fractional differential equations are proposed. Sufficient conditions are given for the existence of solutions of these problems. The analysis relies on the well known Schauder's fixed point theorem. The obtained results show that the Riemann–Liouville fractional derivative and the Caputo's fractional derivative have similar properties. Examples are given to illustrate the main theorems.

რეზიუმე. შეუღლებული იმპულსური წილადწარმოებულიანი დიფერენციალური განტოლებე-ბისთვის შემოტანილია პერიოდული სასაზღვრო ამოცანების ორი ახალი კლასი. მოცემულია ამ ამოცანების ამონახსნის არსებობის საკმარისი პირობები. ანალიზი ეყრდნობა შაუდერის ცნობილ თეორემას უძრავი წერტილის შესახებ. მიღებული შედეგები აჩვენებს, რომ რიმან-ლიუვილის და კაპუტოს წილად წარმოებულებს აქვს მსგავსი თვისებები. მთავარი თეორემების საილუსტრაციოდ მოყვანილია მაგალითები.

1   Introduction

The fractional derivatives serve an excellent tool for the description of hereditary properties of various materials and processes. Fractional differential equations arise naturally in many engineering and scientific disciplines such as physics, chemistry, biology, electrochemistry, electromagnetic, control theory, economics, signal and image processing, aerodynamics, and porous media. The boundary value problems for nonlinear fractional differential equations have been addressed by several researchers during the last decades. There have been many results obtained on the existence of solutions of boundary value problems for nonlinear fractional differential equations (see [7, 8, 33, 35, 36, 47, 55, 58]).

Applications of fractional order differential systems are in many fields, as for example, rheology, mechanics, chemistry, physics, bioengineering, robotics and many others (see [11]). Diethehm [12] proposed the model of the type (which is called a multi-order fractional differential system):

$$\begin{cases} {}^cD_{0^+}^{n_i}y_i(t) = f_i\big(t, y_1(t), \ldots, y_n(t)\big), & i = 1, 2, \ldots, n, \\ y_j(0) = y_{j,0}, & j = 1, 2, \ldots, n. \end{cases} \tag{1.1}$$

Here ${}^cD_{0^+}^*$ is the standard Caputo's fractional derivative. This system contains many models as special cases, see Chen's fractional order system [51, 52] with a double scroll attractor, Genesio-Tesi fractional-order system [19], Lu's fractional order system [13], Volta's fractional-order system [38, 39], Rossler's fractional-order system [27] and so on. Other applications of fractional differential systems may be seen in Chapter 10 in [40].

In [16, 37, 49], the fractional order nonlinear dynamical model of interpersonal relationships

$$\begin{cases} D^\alpha x_1(t) + \alpha_1 x_1(t) = A_1 + \beta_1 x_2(t)\big(1 - \varepsilon x_2^2(t)\big), \\ D^\alpha x_2(t) + \alpha_2 x_2(t) = A_2 + \beta_2 x_1(t)\big(1 - \varepsilon x_1^2(t)\big), \end{cases} \tag{1.2}$$

was proposed, where $0 < \alpha \le 1$, $\alpha_i > 0$, β_i, A_i $(i = 1, 2)$, ε are the real constants. These parameters are oblivion, reaction, and attraction constants. The variables x_1 and x_2 are the measures of love of individuals and for their respective partners, where positive and negative measures represent feelings. In the equations in (1.2), we assume that feelings decay exponentially fast in the absence of partners. The parameters specify the romantic style of individuals 1 and 2. For instance, α_i describes the extent to which individual i is encouraged by his/her own feeling. In other words, α_i indicates the degree to which an individual has internalized a sense of his/her self-worth. In addition, it can be used as the level of anxiety and dependency on other person's approval in romantic relationships. The parameter β_i represents the extent to which individual i is encouraged by his/her partner, and/or expects his/her partner to be supportive. It measures the tendency to seek or avoid closeness in a romantic relationship. Therefore, the term $-\alpha_i x_i$ says that the love measure of i, in the absence of the partner, decays exponentially and α_i is the time required for love to decay (see [37]).

From the viewpoint of the theoretics and practice, it is natural for mathematicians to investigate the impulsive fractional differential equations. In recent years, many authors [1, 9, 15, 17, 20, 22, 25, 26, 28, 29, 34, 41, 46, 47, 54] studied the existence or uniqueness of solutions of impulsive initial or boundary value problems for fractional differential equations. For examples, impulsive anti-periodic boundary value problems (see [2–4, 43]), impulsive periodic boundary value problems (see [44]), impulsive initial value problems (see [10, 14, 31, 50]), two-point, three-point or multi-point impulsive boundary value problems (see [5, 45, 57]), impulsive boundary value problems on infinite intervals (see [56]). However, there has been no papers concerned with the solvability of periodic boundary value problems of impulsive fractional differential systems.

In [9], the authors have studied the solvability of the following periodic boundary value problem:

$$\begin{cases} D_{t_i^+}^\alpha x(t) - \lambda x(t) = p(t)f(t, x(t)), & t \in (t_i, t_{i+1}), \ i = 0, 1, \ldots, m, \\ x(1) - \lim\limits_{t \to 0} t^{1-\alpha}x(t) = 0, \\ \lim\limits_{t \to t_i^+} (t - t_i)^{1-\alpha}\big[x(t) - x(t_i)\big] = I_i(x(t_i)), & i = 1, 2, \ldots, m, \end{cases}$$

where $\alpha \in (0,1)$, $0 = t_0 < t_1 < \cdots < t_{m+1} = 1$, $D_{t_i^+}^\alpha$ is the standard Riemann–Liouville fractional derivative, $I_i : R \to R$ is continuous, $\lambda \neq 0$, f is continuous at every points in $(t_i, t_{i+1}] \times R$ and for every function $v \in C^0(t_i, t_{i+1}]$ the limit $\lim_{t \to t_i^+} v(t)$ exists (finite), then $\lim_{t \to t_i^+} f(t, (t - t_i)^{\alpha-1} v(t))$ exists (finite).

In [24], Liu studied the existence of solutions of the following periodic type boundary value problem of nonlinear singular fractional differential equation

$$\begin{cases} D_{0+}^\beta \left[\Phi(\rho(t) D_{0+}^\alpha u(t)) \right] = q(t) f(t, u(t), D_{0+}^\alpha u(t)), \quad t \in (0,1), \\[2mm] \lim_{t \to 1} t^{1-\alpha} u(t) - \lim_{t \to 0} t^{1-\alpha} u(t) = \int_0^1 G(s, u(s), D_{0+}^\alpha u(s))\, ds, \\[2mm] \lim_{t \to 1} t^{1-\beta} \Phi(\rho(t) D_{0+}^\alpha u(t)) - \lim_{t \to 0} t^{1-\beta} \Phi(\rho(t) D_{0+}^\alpha u(t)) = \int_0^1 H(s, u(s), D_{0+}^\alpha u(s))\, ds, \\[2mm] \lim_{t \to t_1^+} u(t) = I(t_1, u(t_1), D_{0+}^\alpha u(t_1)), \\[2mm] \lim_{t \to t_1^+} \Phi(\rho(t) D_{0+}^\alpha u(t)) = J(t_1, u(t_1), D_{0+}^\alpha u(t_1)). \end{cases}$$

where $0 < \alpha, \beta \leq 1$, D_{0+}^α (or D_{0+}^β) is the Riemann–Liouville fractional derivative of order α (or β), $\Phi : R \to R$ is a sup-multiplicative-like function with supporting function ω, its inverse function is denoted by $\Phi^{-1} : R \to R$ with supporting function ν, $0 < t_1 < 1$, $I, J : (0,1) \times R^2 \to R$ are continuous functions, $\phi, \psi : (0,1) \to R$ with $\phi|_{(0,t_1]}, \rho|_{(0,t_1]} \in L^1(0, t_1)$ and $\phi|_{(t_1,1]}, \rho|_{(t_1,1]} \in L^1(t_1, 1)$, $\rho : (0,1) \to [0, +\infty)$ with $\rho|_{(0,t_1]} \in C^0(0, t_1]$ and $\rho|_{(t_1,1]} \in C^0(t_1, 1)$ satisfies that there exist numbers $L > 0$ and $k > -\alpha$ such that $\rho(t) \geq \frac{t^{-k} \nu(t^{\beta-1})}{L}$ for all $t \in (0,1)$, $t \neq t_1$, $q : (0,1) \to R$ with $q|_{(0,t_1]} \in C^0(0, t_1]$ and $q|_{(t_1,1]} \in C^0(t_1, 1)$ and there exist numbers $L_1 > 0$ and $k_1 > -\beta$ such that $|q(t)| \leq L_1 t^{k_1}$ for all $t \in (0,1)$, f, G, H defined on $(0, t_1) \bigcup (t_1, 1) \times R \times R$ are **impulsive Carathéodory functions** that may be singular at $t = 0$, t_1 and 1.

One knows that both of the fractional derivatives (the Riemann–Liouville fractional derivative and the Caputo's fractional derivative) are actually nonlocal operators because integrals are nonlocal operators. Moreover, calculating time fractional derivatives of a function at some time requires all the past history and hence fractional derivatives can be used for modeling systems with memory. In [9], the fractional derivative has a variable base points t_i $(i = 0, 1, 2, \ldots, m)$. This action may short the memory time. However, in applications, fractional differential equation involves fractional derivative that has a constant base point.

In this paper, we discuss the following impulsive periodic boundary value problems of singular fractional differential systems with a constant base point $t = 0$:

$$\begin{cases} D_{0+}^{\alpha_1} x(t) - \lambda_1 x(t) = p_1(t) f_1(t, x(t), y(t)), \quad t \in (t_i, t_{i+1}), \ i \in N_0, \\[1mm] D_{0+}^{\alpha_2} y(t) - \lambda_2 y(t) = p_2(t) f_2(t, x(t), y(t)), \quad t \in (t_i, t_{i+1}), \ i \in N_0, \\[1mm] x(1) - \lim_{t \to 0} t^{1-\alpha_1} x(t) = 0, \\[1mm] y(1) - \lim_{t \to 0} t^{1-\alpha_2} y(t) = 0, \\[1mm] \lim_{t \to t_i^+} (t - t_i)^{1-\alpha_1} x(t) = I(t_i, x(t_i), y(t_i)), \quad i \in N, \\[1mm] \lim_{t \to t_i^+} (t - t_i)^{1-\alpha_2} y(t) = J(t_i, x(t_i), y(t_i)), \quad i \in N, \end{cases} \tag{1.3}$$

and

$$\begin{cases} {}^{c}D_{0^+}^{\alpha_1}x(t) - \lambda_1 x(t) = p_3(t)f_3\big(t, x(t), y(t)\big), & t \in (t_i, t_{i+1}), \ i \in N_0, \\ {}^{c}D_{0^+}^{\alpha_2}y(t) - \lambda_2 y(t) = p_4(t)f_4\big(t, x(t), y(t)\big), & t \in (t_i, t_{i+1})), \ i \in N_0, \\ x(1) - \lim_{t \to 0} x(t) = 0, \\ y(1) - \lim_{t \to 0} y(t) = 0, \\ \lim_{t \to t_i^+} x(t) = I\big(t_i, x(t_i), y(t_i)\big), & i \in N, \\ \lim_{t \to t_i^+} y(t) = J\big(t_i, x(t_i), y(t_i)\big), & i \in N, \end{cases} \qquad (1.4)$$

where

(a) $0 < \alpha_1, \alpha_2 < 1$, $\lambda_1, \lambda_2 \in R$, D^* is the standard Riemann–Liouville fractional derivative of order $* > 0$, ${}^{c}D^*$ is the standard Caputo's fractional derivative of order $* > 0$;

(b) m is a positive integer, $0 = t_0 < t_1 < t_2 < \cdots < t_{m-1} < t_m < t_{m+1} = 1$, $N_0 = \{0, 1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, m\}$;

(c) p_1, p_2 are continuous on $(0, 1)$ and $p_1, p_2 \in L^1(0, 1)$ and there exist constants $k_j > -1$, $l_j \in (-\alpha_j, 0]$ with $1 + k_j + l_j > 0$ $(j = 1, 2)$ such that $|p_j(t)| \le t^{k_j}(1-t)^{l_j}$ for all $t \in (0, 1)$, $j = 1, 2$;

(c1) p_3, p_4 are continuous on $(0, 1)$ and $p_1, p_2 \in L^1(0, 1)$ and there exist constants $k_j > -1$, $l_j \in (-\alpha_j, 0]$ with $\alpha_j + k_j + l_j > 0$ $(j = 1, 2)$ such that $|p_j(t)| \le t^{k_j}(1-t)^{l_j}$ for all $t \in (0, 1)$, $j = 3, 4$;

(d) f_1, f_2 defined on $\bigcup_{i=0}^{m}(t_i, t_{i+1}) \times R^2$ are **I-Carathéodory functions** (see the definition in Section 2), $I, J : \{t_i : \ i \in N\} \times R^2 \to R$ are **discrete I-Carathéodory functions**;

(d1) f_3, f_4 defined on $\bigcup_{i=0}^{m}(t_i, t_{i+1}) \times R^2$ are **II-Carathéodory functions** (see the definition in Section 2), $I, J : \{t_i : \ i \in N\} \times R^2 \to R$ are **discrete II-Carathéodory functions**.

A pair of functions $x, y : (0, 1] \to R$ is called a solution of BVP (1.3) if

$$x\big|_{(t_i, t_{i+1}]} \in C^0(t_i, t_{i+1}], \quad y\big|_{(t_i, t_{i+1}]} \in C^0(t_i, t_{i+1}], \ i \in N_0, \qquad (1.5)$$

and the limits

$$\lim_{t \to t_i^+} (t - t_i)^{1-\alpha_1} x(t), \quad \lim_{t \to t_i^+} (t - t_i)^{1-\alpha_2} y(t), \ i \in N_0,$$

exist and x, y satisfy all equations in (1.3).

A pair of functions $x, y : (0, 1] \to R$ is called a solution of BVP (1.4) if the limits

$$\lim_{t \to t_i^+} x(t), \quad \lim_{t \to t_i^+} y(t), \ i \in N_0,$$

exist and x, y satisfy all equations in (1.4).

To the best of the authors knowledge, no one has studied the existence of solutions for BVPs (1.3) and (1.4). We obtain results on the existence of at least one solution for BVPs (1.3) and (1.4), respectively. Two examples are given to illustrate the efficiency of the main theorems.

The remainder of this paper is organized as follows: in Section 2, we present preliminary results. In Sections 3 and 4, the existence theorems and their proofs on BVPs (1.3) and (1.4) are given, respectively. Finally, we present examples to show the applications of the main theorems.

2   Preliminaries

For the convenience of the readers, we firstly present the necessary definitions from the fractional calculus theory. These definitions and results can be found in [23, 40].

Let the Gamma function, Beta function and two classical Mittag–Leffler special functions be defined by

$$\Gamma(\alpha) = \int\limits_0^{+\infty} x^{\alpha-1} e^{-x}\, dx, \quad \mathbf{B}(p,q) = \int\limits_0^1 x^{p-1} (1-x)^{q-1}\, dx,$$

$$E_{\delta,\delta}(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(\delta k + \delta)}, \quad E_{\delta,1}(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(\delta k + 1)},$$

respectively, for $\alpha > 0$, $p > 0$, $q > 0$, $\delta > 0$. We note that $E_{\delta,\delta}(x) > 0$ for all $x \in R$ and $E_{\delta,\delta}(x)$ is strictly increasing in x. Then for $x > 0$, we have

$$E_{\delta,\delta}(-x) < E_{\delta,\delta}(0) = \frac{1}{\Gamma(\delta)} < E_{\delta,\delta}(x).$$

Definition 2.1 ([23]). Let $c \in R$. The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $g : (c, \infty) \to R$ is given by

$$I_{c^+}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int\limits_c^t (t-s)^{\alpha-1} g(s)\, ds,$$

provided that the right-hand side exists.

Definition 2.2 ([23]). Let $c \in R$. The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g : (c, \infty) \to R$ is given by

$$D_{c^+}^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int\limits_c^t \frac{g(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where $\alpha < n \le \alpha + 1$, i.e., $n = \lceil \alpha \rceil$, provided that the right-hand side exists.

Lemma 2.1 ([23]). *Let $\alpha < n \le \alpha + 1$, $u \in C^0(c, \infty) \cap L^1(c, \infty)$. Then*

$$I_{c^+}^\alpha D_{c^+}^\alpha u(t) = u(t) + C_1 (t-c)^{\alpha-1} + C_2 (t-c)^{\alpha-2} + \cdots + C_n (t-c)^{\alpha-n},$$

where $C_i \in R$, $i = 1, 2, \ldots, n$.

We use the function space

$$X = \Big\{ x : \ (0,1] \to R : \ x\big|_{(t_i, t_{i+1}]} \in C^0(t_i, t_{i+1}], \ i \in N_0,$$

$$\text{there exist the limits } \lim_{t \to t_i^+} (t - t_i)^{1-\alpha_1} x(t), \ i \in N_0 \Big\}.$$

Define

$$\|x\| = \|x\|_X = \max \Big\{ \sup_{t \in (t_i, t_{i+1}]} (t - t_i)^{1-\alpha_1} |x(t)| : \ i \in N_0 \Big\}.$$

Lemma 2.2. *X is a Banach space with the norm $\|\cdot\|$ defined.*

Proof. In fact, it is easy to see that X is a normed linear space with the norm $\|\cdot\|$. Let $\{x_u\}$ be a Cauchy sequence in X. Then $\|x_u - x_v\| \to 0$, $u, v \to +\infty$. It follows that

$$\sup_{t \in (t_i, t_{i+1}]} (t - t_i)^{1-\alpha_1} \big| x_u(t) - x_v(t) \big| \longrightarrow 0, \ \ v, u \to +\infty, \ \ i \in N_0.$$

Define $x_i = x\big|_{(t_i, t_{i+1}]}$ and

$$(t - t_i)^{1-\alpha_1} \overline{x}_i(t) = \begin{cases} \lim\limits_{t \to t_i^+} (t - t_i)^{1-\alpha_1} x(t), & t = t_i, \\ (t - t_i)^{1-\alpha_1} x(t), & t \in (t_i, t_{i+1}]. \end{cases}$$

We know that $t \to (t - t_i)^{1-\alpha_1} \overline{x}(t)$ is continuous on $[t_i, t_{i+1}]$. Thus $t \to (t - t_i)^{1-\alpha_1} \overline{x}_{u,i}(t)$ is a Cauchy sequence in $C[t_i, t_{i+1}]$. Then $(t - t_i)^{1-\alpha_1} \overline{x}_{u,i}(t)$ uniformly converges to some $x_{0,i}$ in $C[t_i, t_{i+1}]$ as $u \to +\infty$. It follows that

$$\sup_{t \in [t_i, t_{i+1}]} \left| (t - t_i)^{1-\alpha_1} \overline{x}_{u,i}(t) - x_{0,i} \right| \longrightarrow 0, \quad u \to +\infty, \quad i \in N_0.$$

That is,

$$\sup_{t \in [t_i, t_{i+1}]} (t - t_i)^{1-\alpha_1} \left| \overline{x}_{u,i}(t) - (t - t_i)^{\alpha_1-1} x_{0,i} \right| \longrightarrow 0, \quad u \to +\infty, \quad i \in N_0.$$

Let $x_0(t) = (t - t_i)^{\alpha_1-1} x_{0,i}(t)$ for $t \in (t_i, t_{i+1}]$, $i \in N_0$. It is easy to see that $x_0 \in X$ and $x_u \to x_0$ as $u \to +\infty$ in X. It follows that X is a Banach space. The proof is complete. $\qquad\square$

Define

$$Y = \Big\{ y : \ (0,1] \to R : \ y\big|_{(t_i, t_{i+1}]} \in C^0(t_i, t_{i+1}], \ i \in N_0,$$

$$\text{there exist the limits } \lim_{t \to t_i^+} (t - t_i)^{1-\alpha_2} y(t), \ i \in N_0, \ \Big\}$$

with the norm

$$\|y\| = \|y\|_Y = \max \Big\{ \sup_{t \in (t_i, t_{i+1}]} (t - t_i)^{1-\alpha_2} |y(t)| : \ i \in N_0 \Big\}.$$

Then Y is a Banach space. Choose $E = X \times Y$ with the norm $\|(x,y)\| = \max\{\|x\|_X, \|y\|_Y\}$. Then E is a Banach space. We will seek for solutions of BVP (1.3) in E.

Definition 2.3. We call $F : \bigcup\limits_{i=0}^{m} (t_i, t_{i+1}) \times R^2 \to R$ an **I-Carathéodory function** if it satisfies the following conditions:

(i) $t \to F(t, (t - t_i)^{\alpha_1-1} u, (t - t_i)^{\alpha_2-1} v)$ are measurable on (t_i, t_{i+1}), $i \in N_0$ for any $(u,v) \in R^2$;

(ii) $(u,v) \to F\left(t, (t - t_i)^{\alpha_1-1} u, (t - t_i)^{\alpha_2-1} v\right)$ are continuous on R^2 for all $t \in (t_i, t_{i+1})$, $i \in N_0$;

(iii) for each $r > 0$, there exists $M_r \geq 0$ such that $|F(t, (t - t_i)^{\alpha_1-1} u, (t - t_i)^{\alpha_2-1} v)| \leq M_r$ for all $t \in (t_i, t_{i+1})$, $i \in N_0$ and $|u|, |v| \leq r$.

We call $G : \{t_i : \ i \in N\} \times R^2 \to R$ a **discrete I-Carathéodory function** if it satisfies the following conditions:

(i) $(u,v) \to G(t_i, (t_i - t_{i-1})^{\alpha_1-1} u, (t_i - t_{i-1})^{\alpha_2-1} v)$, $i \in N$ are continuous on R^2;

(ii) for each $r > 0$, there exists $M_r \geq 0$ such that $|G(t_i, (t_i - t_{i-1})^{\alpha_1-1} u, (t_i - t_{i-1})^{\alpha_2-1} v)| \leq M_r$ for almost all $i \in N$ and $|u|, |v| \leq r$.

Definition 2.4. We call $F : \bigcup\limits_{i=0}^{m} (t_i, t_{i+1}) \times R^2 \to R$ a **II-Carathéodory function** if it satisfies the following conditions:

(i) $t \to F(t, u, v)$ are measurable on (t_i, t_{i+1}), $i \in N_0$ for any $(u,v) \in R^2$;

(ii) $(u,v) \to F(t, u, v)$ are continuous on R^2 for all $t \in (t_i, t_{i+1})$, $i \in N_0$;

(iii) for each $r > 0$, there exists $M_r \geq 0$ such that $|F(t, u, v)| \leq M_r$ for all $t \in (t_i, t_{i+1})$, $i \in N_0$ and $|u|, |v| \leq r$.

We call $G : \{t_i : i \in N\} \times R^2 \to R$ a **discrete II-Carathéodory function** if it satisfies the following conditions:

(i) $(u, v) \to G(t_i, u, v)$, $i \in N$ are continuous on R^2;

(ii) for each $r > 0$, there exists $M_r \geq 0$ such that $|G(t_i, u, v)| \leq M_r$ for almost all $i \in N$ and $|u|, |v| \leq r$.

We also use the function space

$$X_1 = \Big\{ x : (0,1] \to R : x\big|_{(t_i, t_{i+1}]} \in C^0(t_i, t_{i+1}], \ i \in N_0, \text{ there exist the limits } \lim_{t \to t_i^+} x(t), \ i \in N_0 \Big\}.$$

Define

$$\|x\| = \|x\|_{X_1} = \max\Big\{ \sup_{t \in (t_i, t_{i+1}]} |x(t)| : \ i \in N_0 \Big\}.$$

Then X_1 is the Banach space with the norm $\| \cdot \|_{X_1}$ defined. Choose $E_1 = X_1 \times X_1$ with the norm $\|(x, y)\| = \max\{\|x\|_{X_1}, \|y\|_{X_1}\}$. Then E_1 is a Banach space. We will seek for solutions of BVP (1.4) in E_1.

To ease expression, we denote $\delta_{\alpha, \lambda}(t, t_i) = (t - t_i)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t - t_i)^\alpha)$ for $t \in (t_i, t_{i+1}]$ and $\alpha \in (0, 1]$ and $\lambda \in R$.

3   Solvability of BVP (1.3)

In this section, we study the solvability of BVP (1.3) by seeking solutions in the Banach space E.

Lemma 3.1. *Suppose that $\sigma \in L^1(0, 1)$ and there exist numbers $k_1 > -1$ and $\max\{-\alpha_1, -k_1 - 1\} < l_1 \leq 0$ such that $|\sigma(t)| \leq t^{k_1}(1 - t)^{l_1}$ for all $t \in (0, 1)$. Then $x \in X$ is a solution of*

$$D_{0+}^{\alpha_1} x(t) - \lambda_1 x(t) = \sigma(t), \quad t \in (t_i, t_{i+1}), \quad i \in N_0, \tag{3.1}$$

if and only if there exist constants A_i $(i \in N_0)$ such that

$$x(t) = \Gamma(\alpha_1) \sum_{j=0}^{i} A_j \delta_{\alpha_1, \lambda_1}(t, t_j) + \int_0^t \delta_{\alpha_1, \lambda_1}(t, s)\sigma(s)\, ds, \quad t \in (t_i, t_{i+1}], \quad i \in N_0. \tag{3.2}$$

Proof. We do two steps:

Step 1. Suppose that $x \in X$ is a solution of (3.1). By (3.26) in [8], we know that there exist numbers A_0 such that

$$x(t) = \Gamma(\alpha_1) A_0 \delta_{\alpha_1, \lambda_1}(t, 0) + \int_0^t \delta_{\alpha_1, \lambda_1}(t, s)\sigma(s)\, ds, \quad t \in (0, t_1]. \tag{3.3}$$

We know that (3.2) holds when $i = 0$. Now suppose that (3.2) holds for $i = 0, 1, 2, \ldots, n$ $(n \leq m - 1)$. We will prove that (3.2) holds for $i = n + 1$. Suppose that

$$x(t) = \Phi(t) + \Gamma(\alpha_1) \sum_{j=0}^{n} A_j \delta_{\alpha, \lambda_1}(t, t_j) + \int_0^t \delta_{\alpha_1, \lambda_1}(t, s)\sigma(s)\, ds, \quad t \in (t_{n+1}, t_{n+2}]. \tag{3.4}$$

It is easy to check that for $t \in (t_{n+1}, t_{n+2}]$

$$\int_0^t \frac{x(s)}{(t - s)^{\alpha_1}}\, ds = \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{x(s)}{(t - s)^{\alpha_1}}\, ds + \int_{t_{n+1}}^t \frac{x(s)}{(t - s)^{\alpha_1}}\, ds$$

$$= \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{\sum_{u=0}^{j} A_u \Gamma(\alpha_1)(s-t_u)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-t_u)^{\alpha_1}) + \int_0^s (s-v)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\,dv}{(t-s)^{\alpha_1}}\,ds$$

$$+ \int_{t_{n+1}}^{t} \frac{\Phi(s) + \sum_{j=0}^{n} A_j \Gamma(\alpha_1)(s-t_j)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-t_j)^{\alpha_1}) + \int_0^s (s-v)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\,dv}{(t-s)^{\alpha_1}}\,ds$$

$$= \int_{t_{n+1}}^{t} \frac{\Phi(s)}{(t-s)^{\alpha_1}}\,ds + \sum_{j=0}^{n}\sum_{u=0}^{j} A_u \Gamma(\alpha_1) \int_{t_j}^{t_{j+1}} \frac{(s-t_u)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-t_u)^{\alpha_1})}{(t-s)^{\alpha_1}}\,ds$$

$$+ \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \int_{0}^{s} \frac{(s-v)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)dv}{(t-s)^{\alpha_1}}\,ds$$

$$+ \sum_{j=0}^{n} A_j \Gamma(\alpha_1) \int_{t_{n+1}}^{t} \frac{(s-t_j)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-t_j)^{\alpha_1})}{(t-s)^{\alpha_1}}\,ds$$

$$+ \int_{t_{n+1}}^{t} \int_{0}^{s} \frac{(s-v)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\,dv}{(t-s)^{\alpha_1}}\,ds$$

$$= \int_{t_{n+1}}^{t} \frac{\Phi(s)}{(t-s)^{\alpha}}\,ds + \sum_{j=0}^{n}\sum_{u=0}^{j} A_u \Gamma(\alpha_1) \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1}(s-t_u)^{\alpha_1-1} \sum_{w=0}^{+\infty} \frac{\lambda_1^w(s-t_u)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\,ds$$

$$+ \sum_{j=0}^{n} A_j \Gamma(\alpha_1) \int_{t_{n+1}}^{t} (t-s)^{-\alpha_1}(s-t_j)^{\alpha_1-1} \sum_{w=0}^{+\infty} \frac{\lambda_1^w(s-t_j)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\,ds$$

$$+ \sum_{j=0}^{n} \int_{0}^{t_j} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1}(s-v)^{\alpha_1-1} \sum_{w=0}^{+\infty} \frac{\lambda_1^w(s-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\,ds\,\sigma(v)\,dv$$

$$+ \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \int_{v}^{t_{j+1}} (t-s)^{-\alpha_1}(s-v)^{\alpha_1-1} \sum_{w=0}^{+\infty} \frac{\lambda_1^w(s-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\,ds\sigma(v)\,dv$$

$$+ \int_{0}^{t_{n+1}} \int_{t_{n+1}}^{t} (t-s)^{-\alpha_1}(s-v)^{\alpha_1-1} \sum_{w=0}^{+\infty} \frac{\lambda_1^w(s-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\,ds\,\sigma(v)\,dv$$

$$+ \int_{t_{n+1}}^{t} \int_{v}^{t} (t-s)^{-\alpha_1}(s-v)^{\alpha_1-1} \sum_{w=0}^{+\infty} \frac{\lambda_1^w(s-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\,ds\,\sigma(v)\,dv$$

$$= \int_{t_{n+1}}^{t} \frac{\Phi(s)}{(t-s)^{\alpha}}\,ds + \sum_{j=0}^{n}\sum_{u=0}^{j} A_u \Gamma(\alpha_1) \sum_{w=0}^{+\infty} \frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1}(s-t_u)^{\alpha_1 w+\alpha_1-1}\,ds$$

$$+ \sum_{j=0}^{n} A_j \Gamma(\alpha_1) \sum_{w=0}^{+\infty} \frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))} \int_{t_{n+1}}^{t} (t-s)^{-\alpha_1}(s-t_j)^{\alpha_1 w+\alpha_1-1}\,ds$$

$$+ \sum_{j=0}^{n} \int_{0}^{t_j} \sum_{w=0}^{+\infty} \frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1}(s-v)^{\alpha_1 w+\alpha_1-1}\,ds\,\sigma(v)\,dv$$

$$+\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\sum_{w=0}^{+\infty}\frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))}\int_{v}^{t_{j+1}}(t-s)^{-\alpha_1}(s-v)^{\alpha_1 w+\alpha_1-1}\,ds\,\sigma(v)\,dv$$

$$+\int_{0}^{t_{n+1}}\sum_{w=0}^{+\infty}\frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))}\int_{t_{n+1}}^{t}(t-s)^{-\alpha_1}(s-v)^{\alpha_1 w+\alpha_1-1}\,ds\,\sigma(v)\,dv$$

$$+\int_{t_{n+1}}^{t}\sum_{w=0}^{+\infty}\frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))}\int_{v}^{t}(t-s)^{-\alpha_1}(s-v)^{\alpha_1 w+\alpha_1-1}\,ds\,\sigma(v)\,dv$$

$$=\int_{t_{n+1}}^{t}\frac{\Phi(s)}{(t-s)^{\alpha}}\,ds+\sum_{j=0}^{n}\sum_{u=0}^{j}A_u\Gamma(\alpha_1)\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-t_u)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{\frac{t_j-t_u}{t-t_u}}^{\frac{t_{j+1}-t_u}{t-t_u}}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega$$

$$+\sum_{j=0}^{n}A_j\Gamma(\alpha_1)\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-t_j)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{\frac{t_{n+1}-t_j}{t-t_j}}^{1}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega$$

$$+\sum_{j=0}^{n}\int_{0}^{t_j}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{\frac{t_j-v}{t-v}}^{\frac{t_{j+1}-v}{t-v}}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$+\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{0}^{\frac{t_{j+1}-v}{t-v}}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$+\int_{0}^{t_{n+1}}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{\frac{t_{n+1}-v}{t-v}}^{1}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$+\int_{t_{n+1}}^{t}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{0}^{1}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$=\int_{t_{n+1}}^{t}\frac{\Phi(s)}{(t-s)^{\alpha}}\,ds+\sum_{u=0}^{n}A_u\Gamma(\alpha_1)\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-t_u)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{0}^{\frac{t_{n+1}-t_u}{t-t_u}}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega$$

$$+\sum_{j=0}^{n}A_j\Gamma(\alpha_1)\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-t_j)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{\frac{t_{n+1}-t_j}{t-t_j}}^{1}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega$$

$$+\sum_{j=1}^{n}\int_{0}^{t_j}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{\frac{t_j-v}{t-v}}^{\frac{t_{j+1}-v}{t-v}}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$+\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\sum_{w=0}^{+\infty}\frac{\lambda_1^w(t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))}\int_{0}^{\frac{t_{j+1}-v}{t-v}}(1-\omega)^{-\alpha_1}\omega^{\alpha_1 w+\alpha_1-1}\,d\omega\,\sigma(v)\,dv$$

$$
+ \int_0^{t_{n+1}} \sum_{w=0}^{+\infty} \frac{\lambda_1^w (t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))} \int_{\frac{t_{n+1}-v}{t-v}}^1 (1-\omega)^{-\alpha_1} \omega^{\alpha_1 w + \alpha_1 - 1} \, d\omega \, \sigma(v) \, dv
$$

$$
+ \int_{t_{n+1}}^t \sum_{w=0}^{+\infty} \frac{\lambda_1^w (t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))} \int_0^1 (1-\omega)^{-\alpha_1} \omega^{\alpha_1 w + \alpha_1 - 1} \, d\omega \, \sigma(v) \, dv
$$

$$
= \int_{t_{n+1}}^t \frac{\Phi(s)}{(t-s)^\alpha} \, ds + \sum_{u=0}^n A_u \Gamma(\alpha_1) \sum_{w=0}^{+\infty} \frac{\lambda_1^w (t-t_u)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))} \int_0^1 (1-\omega)^{-\alpha_1} \omega^{\alpha_1 w + \alpha_1 - 1} \, d\omega
$$

$$
+ \sum_{j=1}^n \int_0^{t_j} \sum_{w=0}^{+\infty} \frac{\lambda_1^w (t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))} \int_{\frac{t_j - v}{t-v}}^{\frac{t_{j+1}-v}{t-v}} (1-\omega)^{-\alpha_1} \omega^{\alpha_1 w + \alpha_1 - 1} \, d\omega \, \sigma(v) \, dv
$$

$$
+ \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \sum_{w=0}^{+\infty} \frac{\lambda_1^w (t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))} \int_0^{\frac{t_{j+1}-v}{t-v}} (1-\omega)^{-\alpha_1} \omega^{\alpha_1 w + \alpha_1 - 1} \, d\omega \, \sigma(v) \, dv
$$

$$
+ \int_0^{t_{n+1}} \sum_{w=0}^{+\infty} \frac{\lambda_1^w (t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))} \int_{\frac{t_{n+1}-v}{t-v}}^1 (1-\omega)^{-\alpha_1} \omega^{\alpha_1 w + \alpha_1 - 1} \, d\omega \, \sigma(v) \, dv
$$

$$
+ \int_{t_{n+1}}^t \sum_{w=0}^{+\infty} \frac{\lambda_1^w (t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))} \int_0^1 (1-\omega)^{-\alpha_1} \omega^{\alpha_1 w + \alpha_1 - 1} \, d\omega \, \sigma(v) \, dv
$$

$$
= \int_{t_{n+1}}^t \frac{\Phi(s)}{(t-s)^\alpha} \, ds + \sum_{u=0}^n A_u \Gamma(\alpha_1) \sum_{w=0}^{+\infty} \frac{\lambda_1^w (t-t_u)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))} \int_0^1 (1-\omega)^{-\alpha_1} \omega^{\alpha_1 w + \alpha_1 - 1} \, d\omega
$$

$$
+ \int_0^t \sum_{w=0}^{+\infty} \frac{\lambda_1^w (t-v)^{\alpha_1 w}}{\Gamma(\alpha_1(w+1))} \int_0^1 (1-\omega)^{-\alpha_1} \omega^{\alpha_1 w + \alpha_1 - 1} \, d\omega \, \sigma(v) \, dv.
$$

One can see that for $t \in (t_{n+1}, t_{n+2}]$

$$
\frac{1}{\Gamma(1-\alpha_1)} \left(\int_0^t \frac{x(s)}{(t-s)^\alpha} \, ds \right)' = \frac{1}{\Gamma(1-\alpha_1)} \left(\int_{t_{n+1}}^t \frac{\Phi(s)}{(t-s)^\alpha} \, ds \right)'
$$

$$
+ \frac{1}{\Gamma(1-\alpha_1)} \sum_{u=0}^n A_u \Gamma(\alpha_1) \sum_{w=1}^{+\infty} \frac{\lambda_1^w \alpha_1 w (t-t_u)^{\alpha_1 w - 1}}{\Gamma(\alpha_1(w+1))} \int_0^1 (1-\omega)^{-\alpha_1} \omega^{\alpha_1 w + \alpha_1 - 1} \, d\omega
$$

$$
+ \frac{1}{\Gamma(1-\alpha_1)} \int_0^t \sum_{w=1}^{+\infty} \frac{\lambda_1^w \alpha_1 w (t-v)^{\alpha_1 w - 1}}{\Gamma(\alpha_1(w+1))} \int_0^1 (1-\omega)^{-\alpha_1} \omega^{\alpha_1 w + \alpha_1 - 1} \, d\omega \, \sigma(v) \, dv
$$

$$
+ \frac{1}{\Gamma(1-\alpha_1)} \frac{1}{\Gamma(\alpha_1)} \int_0^1 (1-\omega)^{-\alpha_1} \omega^{\alpha_1 - 1} \, d\omega \, \sigma(t)
$$

$$
= \sigma(t) + D_{t_{n+1}^+}^{\alpha_1} \Phi(t) + \frac{1}{\Gamma(1-\alpha_1)} \sum_{u=0}^n A_u \Gamma(\alpha_1) \sum_{w=1}^{+\infty} \frac{\lambda_1^w \alpha_1 w (t-t_u)^{\alpha_1 w - 1}}{\Gamma(\alpha_1(w+1))} \mathbf{B}(1-\alpha_1, \alpha_1 w + \alpha_1)
$$

$$+ \frac{1}{\Gamma(1-\alpha_1)} \int\limits_0^t \sum_{w=1}^{+\infty} \frac{\lambda_1^w \alpha_1 w(t-v)^{\alpha_1 w - 1}}{\Gamma(\alpha_1(w+1))} \mathbf{B}(1-\alpha_1, \alpha_1 w + \alpha_1) \sigma(v)\, dv$$

$$= \sigma(t) + D_{t_{n+1}^+}^{\alpha_1} \Phi(t) + \sum_{u=0}^n A_u \Gamma(\alpha_1) \sum_{w=1}^{+\infty} \frac{\lambda_1^w \alpha_1 w(t-t_u)^{\alpha_1 w - 1}}{\Gamma(\alpha_1 w + 1))} + \int\limits_0^t \sum_{w=1}^{+\infty} \frac{\lambda_1^w \alpha_1 w(t-v)^{\alpha_1 w - 1}}{\Gamma(\alpha_1 w + 1))} \sigma(v)\, dv$$

$$= \sigma(t) + D_{t_{n+1}^+}^{\alpha_1} \Phi(t) + \lambda_1 \sum_{j=0}^n A_j \Gamma(\alpha_1)(t-t_j)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(\lambda_1(t-t_j)^{\alpha_1})$$

$$+ \lambda_1 \int\limits_0^t (t-s)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\, ds.$$

So,

$$\sigma(t) = D_{0^+}^{\alpha_1} x(t) - \lambda_1 x(t) = \frac{1}{\Gamma(1-\alpha_1)} \left(\int\limits_0^t \frac{x(s)}{(t-s)^\alpha}\, ds \right)'$$

$$- \lambda_1 \left(\Phi(t) + \sum_{j=0}^n A_j \Gamma(\alpha_1)(t-t_j)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(\lambda_1(t-t_j)^{\alpha_1}) \right.$$

$$\left. + \int\limits_0^t (t-s)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\, ds \right)$$

$$= \sigma(t) + D_{t_{n+1}^+}^{\alpha_1} \Phi(t) - \lambda_1 \Phi(t).$$

It follows that $D_{t_{n+1}^+}^{\alpha_1} \Phi(t) - \lambda_1 \Phi(t) = 0$ for $t \in (t_{n+1}, t_{n+2}]$. Hence there exists a constant $A_{n+1} \in R$ such that $\Phi(t) = A_{n+1} \Gamma(\alpha_1)(t-t_{n+1})^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(\lambda_1(t-t_{n+1})^{\alpha_1})$. Substituting Φ into (9), we know that (3.2) holds for $i = n+1$. Thus, due to induction, (3.2) holds for all $i \in N_0$.

Step 2. Now suppose that x satisfies (3.2). We prove that $x \in X$ and x satisfies (3.1).
Firstly, we prove that $x \in X$. In fact, we have for $0 < t_1 < t_2 \le 1$ that

$$\left| \int\limits_0^{t_2} (t_2 - s)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(\lambda_1(t_2-s)^{\alpha_1})\sigma(s)\, ds - \int\limits_0^{t_1} (t_1 - s)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(\lambda_1(t_1-s)^{\alpha_1})\sigma(s)\, ds \right|$$

$$\le \int\limits_{t_1}^{t_2} (t_2 - s)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(\lambda_1(t_2-s)^{\alpha_1}) s^{k_1}(1-s)^{l_1}\, ds$$

$$+ \int\limits_0^{t_1} \left| (t_2 - s)^{\alpha_1 - 1} - (t_1 - s)^{\alpha_1 - 1} \right| E_{\alpha_1,\alpha_1}(\lambda_1(t_2-s)^{\alpha_1}) s^{k_1}(1-s)^{l_1}\, ds$$

$$+ \int\limits_0^{t_1} (t_1 - s)^{\alpha_1 - 1} \left| E_{\alpha_1,\alpha_1}(\lambda_1(t_2-s)^{\alpha_1}) - E_{\alpha_1,\alpha_1}(\lambda_1(t_1-s)^{\alpha_1}) \right| s^{k_1}(1-s)^{l_1}\, ds$$

$$\le \sum_{w=0}^{+\infty} \frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))} \int\limits_{t_1}^{t_2} (t_2 - s)^{\alpha w + \alpha_1 + l_1 - 1} s^{k_1}\, ds$$

$$+ \sum_{w=0}^{+\infty} \frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))} \int\limits_0^{t_1} \left[(t_1 - s)^{\alpha_1 - 1} - (t_2 - s)^{\alpha_1 - 1} \right] (t_2 - s)^{\alpha_1 w} s^{k_1}(1-s)^{l_1}\, ds$$

$$+ \int_0^{t_1} (t_1 - s)^{\alpha_1 - 1} \big| E_{\alpha_1,\alpha_1}(\lambda_1(t_2 - s)^{\alpha_1}) - E_{\alpha_1,\alpha_1}(\lambda_1(t_1 - s)^{\alpha_1}) \big| s^{k_1}(1 - s)^{l_1} \, ds$$

$$=: M_1 + M_2 + M_3.$$

We see that the first term M_1 satisfies

$$M_1 = \sum_{w=0}^{+\infty} \frac{\lambda_1^w}{\Gamma(\alpha_1(w + 1))} t_2^{\alpha_1 w + \alpha_1 + k_1 + l_1} \int_{\frac{t_1}{t_2}}^{1} (1 - \omega)^{\alpha w + \alpha_1 + l_1 - 1} \omega^{k_1} \, d\omega$$

$$\leq E_{\alpha_1,\alpha_1}(\lambda_1 t_2^{\alpha_1}) \int_{\frac{t_1}{t_2}}^{1} (1 - \omega)^{\alpha_1 + l_1 - 1} \omega^{k_1} \, d\omega \longrightarrow 0 \,' \text{ as } t_1 \to t_2.$$

The second term M_2 satisfies

$$M_2 \leq \sum_{w=0}^{+\infty} \frac{\lambda_1^w}{\Gamma(\alpha_1(w + 1))} \int_0^{t_1} \big[(t_1 - s)^{\alpha_1 - 1} - (t_2 - s)^{\alpha_1 - 1} \big] (1 - s)^{l_1} s^{k_1} \, ds$$

$$\leq \sum_{w=0}^{+\infty} \frac{\lambda_1^w}{\Gamma(\alpha_1(w+1))} \int_0^{t_1} (t_1-s)^{\alpha_1-1} (t_2-s)^{l_1} s^{k_1} \, ds - \sum_{w=0}^{+\infty} \frac{\lambda^w}{\Gamma(\alpha_1(w+1))} \int_0^{t_1} (t_2-s)^{\alpha_1+l_1-1} s^{k_1} \, ds$$

$$\leq \sum_{w=0}^{+\infty} \frac{\lambda_1^w}{\Gamma(\alpha_1(w + 1))} \int_0^{t_1} (t_1 - s)^{\alpha_1 + l_1 - 1} s^{k_1} \, ds - \sum_{w=0}^{+\infty} \frac{\lambda^w}{\Gamma(\alpha_1(w + 1))} \int_0^{t_1} (t_2 - s)^{\alpha_1 + l_1 - 1} s^{k_1} \, ds$$

$$= \sum_{w=0}^{+\infty} \frac{\lambda_1^w}{\Gamma(\alpha_1(w + 1))} t_1^{\alpha_1 + k_1 + l_1} \int_0^{1} (1 - \omega)^{\alpha_1 + l_1 - 1} \omega^{k_1} \, d\omega$$

$$- \sum_{w=0}^{+\infty} \frac{\lambda^w}{\Gamma(\alpha_1(w + 1))} t_2^{\alpha_1 + k_1 + l_1} \int_0^{\frac{t_1}{t_2}} (1 - \omega)^{\alpha_1 + l_1 - 1} \omega^{k_1} \, d\omega$$

$$= E_{\alpha_1,\alpha_1}(\lambda_1) \big[t_1^{\alpha_1 + k_1 + l_1} - t_2^{\alpha_1 + k_1 + l_1} \big] \mathbf{B}(\alpha_1 + l_1, k_1 + 1)$$

$$- E_{\alpha_1,\alpha_1}(\lambda_1) t_2^{\alpha_1 + k_1 + l_1} \int_{\frac{t_1}{t_2}}^{1} (1 - \omega)^{\alpha_1 + l_1 - 1} \omega^{k_1} \, d\omega \longrightarrow 0 \text{ as } t_1 \to t_2.$$

Now for the third term M_3, we know that $E_{\alpha_1,\alpha_1}(\lambda_1 x^\alpha)$ is uniformly continuous on $[0, t_2]$, thus for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|E_{\alpha_1,\alpha_1}(\lambda_1 x_1^\alpha) - E_{\alpha_1,\alpha_1}(\lambda_1 x_2^\alpha)| < \varepsilon$ for $|x_1 - x_2| < \delta$. Then for $s \in [0, t_1]$ and $|t_1 - t_2| < \delta$, we have $t_1 - s, t_2 - s \in [0, t_2]$ and $|(t_1 - s) - (t_2 - s)| < \delta$. So $|E_{\alpha_1,\alpha_1}(\lambda_1(t_2 - s)^{\alpha_1}) - E_{\alpha_1,\alpha_1}(\lambda_1(t_1 - s)^{\alpha_1})| < \varepsilon$. Hence,

$$M_3 \leq \int_0^{t_1} (t_1 - s)^{\alpha_1 + l_1 - 1} \big| E_{\alpha_1,\alpha_1}(\lambda_1(t_2 - s)^{\alpha_1}) - E_{\alpha_1,\alpha_1}(\lambda_1(t_1 - s)^{\alpha_1}) \big| s^{k_1} \, ds$$

$$\leq \varepsilon \int_0^{t_1} (t_1 - s)^{\alpha_1 + l_1 - 1} s^k \, ds \leq \varepsilon \mathbf{B}(\alpha_1 + l_1, k_1 + 1).$$

Thus $M_3 \to 0$ as $t_1 \to t_2$.

So, $t \longrightarrow \int_0^t (t - s)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(\lambda_1(t - s)^{\alpha_1}) \sigma(s) \, ds$ is continuous on $(0, 1]$. Then $x \in C^0(t_i, t_{i+1}]$, $i \in N_0$. For $t \in (0, t_1]$, we get

$$t^{1-\alpha_1}\left|\int\limits_0^t (t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\,ds\right| \le t^{1-\alpha_1}\int\limits_0^t (t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})s^{k_1}(1-s)^{l_1}\,ds$$

$$= t^{1-\alpha_1}\int\limits_0^t (t-s)^{\alpha_1-1}\sum_{i=0}^{+\infty}\frac{\lambda_1^i(t-s)^{\alpha_1 i}}{\Gamma(\alpha_1 i+\alpha_1)}s^{k_1}(1-s)^{l_1}\,ds \le t^{1-\alpha_1}\int\limits_0^t (t-s)^{\alpha_1+l_1-1}\sum_{i=0}^{+\infty}\frac{\lambda_1^i(t-s)^{\alpha_1 i}}{\Gamma(\alpha_1 i+\alpha_1)}s^{k_1}\,ds$$

$$= t^{1-\alpha_1}\sum_{i=0}^{+\infty}\frac{\lambda_1^i}{\Gamma(\alpha_1 i+\alpha_1)}\int\limits_0^t (t-s)^{\alpha_1+\alpha_1 i+l_1-1}s^{k_1}\,ds$$

$$= t^{1-\alpha_1}\sum_{i=0}^{+\infty}\frac{\lambda_1^i}{\Gamma(\alpha_1 i+\alpha_1)}t^{\alpha_1+\alpha_1 i+l_1+k_1}\int\limits_0^1 (1-w)^{\alpha_1+\alpha_1 i+l_1-1}w^{k_1}\,dw$$

$$\le t^{1-\alpha_1}\sum_{i=0}^{+\infty}\frac{\lambda_1^i}{\Gamma(\alpha_1 i+\alpha_1)}t^{\alpha_1+\alpha_1 i+l_1+k_1}\int\limits_0^1 (1-w)^{\alpha_1+l_1-1}w^{k_1}\,dw$$

$$= t^{1+l_1+k_1}\mathbf{B}(\alpha_1+l_1,k_1+1)\sum_{i=0}^{+\infty}\frac{\lambda_1^i t^{\alpha_1 i}}{\Gamma(\alpha_1 i+\alpha_1)} = t^{1+l_1+k_1}\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,\alpha_1}(\lambda_1 t^{\alpha_1}).$$

From $k_1+l_1+1>0$, we get

$$\lim_{t\to 0}t^{1-\alpha_1}\left|\int\limits_0^t (t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\,ds\right| = 0.$$

Then $\lim\limits_{t\to 0^+}t^{1-\alpha_1}x(t)$ exists. It is similar to prove that $\lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha_1}x(t)$, $i\in N$, exist. Hence $x\in X$.

By direct computation, similarly to Step 1, we can show for $t\in(t_i,t_{i+1}]$, $i\in N_0$, that

$$D_{0^+}^{\alpha_1}x(t)-\lambda_1 x(t) = \frac{1}{\Gamma(1-\alpha)}\left(\int\limits_0^t (t-s)^{-\alpha_1}x(s)\,ds\right)' - \lambda_1 x(t) = \sigma(t).$$

So, x is a solution of (3.1) in X. $\square$

Lemma 3.2. *Suppose that* $\Delta = 1-\Gamma(\alpha_1)E_{\alpha_1,\alpha_1}(\lambda_1)\neq 0$, $\sigma\in L^1(0,1)$ *and there exist numbers* $k_1>-1$ *and* $\max\{-\alpha_1,-k_1-1\}<l_1\le 0$ *such that* $|\sigma(t)|\le t^{k_1}(1-t)^{l_1}$ *for all* $t\in(0,1)$. *Then* $x\in X$ *is a solution of*

$$\begin{cases} D_{0^+}^{\alpha_1}x(t)-\lambda_1 x(t)=\sigma(t), & t\in(t_i,t_{i+1}), \ i\in N_0, \\ x(1)-\lim\limits_{t\to 0}t^{1-\alpha_1}x(t)=0, \\ \lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha_1}x(t)=I_i, & i\in N, \end{cases} \tag{3.5}$$

if and only if $x\in X$ *and*

$$x(t) = \frac{\Gamma(\alpha_1)^2\delta_{\alpha_1,\lambda_1}(t,0)}{\Delta}\sum_{j=1}^m I_j\delta_{\alpha_1,\lambda_1}(1,t_j) + \frac{\Gamma(\alpha_1)\delta_{\alpha_1,\lambda_1}(t,0)}{\Delta}\int\limits_0^1 \delta_{\alpha_1,\lambda_1}(1,s)\sigma(s)\,ds$$

$$+\Gamma(\alpha_1)\sum_{j=1}^i I_j\delta_{\alpha_1,\lambda_1}(t,t_j) + \int\limits_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds, \quad t\in(t_i,t_{i+1}], \ i\in N_0. \tag{3.6}$$

Proof. Let x be a solution of (3.5). Similarly to Step 2 in Lemma 3.1, since $l \leq 0$ for $t \in (0, t_1]$, one can see that

$$t^{1-\alpha_1} \left| \int_0^t \delta_{\alpha_1, \lambda_1}(t, s)\sigma(s)\, ds \right| = t^{1-\alpha_1} \left| \int_0^t (t-s)^{\alpha_1 - 1} E_{\alpha_1, \alpha_1}(\lambda_1 (t-s)^{\alpha_1})\sigma(s)\, ds \right|$$

$$\leq t^{1 + l_1 + k_1} \mathbf{B}(\alpha_1 + l_1, k_1 + 1) E_{\alpha_1, \alpha_1}(\lambda_1 t^{\alpha_1}).$$

From $k_1 + l_1 + 1 > 0$, we get

$$\lim_{t \to 0} t^{1-\alpha_1} \left| \int_0^t \delta_{\alpha_1, \lambda_1}(t, s)\sigma(s)\, ds \right| = 0. \tag{3.7}$$

From Lemma 3.1, there exist constants $A_i \in R \; (i \in N_0)$ such that

$$x(t) = \Gamma(\alpha_1) \sum_{j=0}^{i} A_j \delta_{\alpha_1, \lambda_1}(t, t_j) + \int_0^t \delta_{\alpha_1, \lambda_1}(t, s)\sigma(s)\, ds, \;\; t \in (t_i, t_{i+1}], \;\; i \in N_0. \tag{3.8}$$

Note that $E_{\alpha_1, \alpha_1}(0) = \frac{1}{\Gamma(\alpha_1)}$. It follows from the boundary conditions and the impulse assumption in (3.5) that

$$\Gamma(\alpha_1) \sum_{j=0}^{m} A_j \delta_{\alpha_1, \lambda_1}(1, t_j) + \int_0^1 \delta_{\alpha_1, \lambda_1}(1, s)\sigma(s)\, ds - A_0 = 0, \quad A_i = I_i, \;\; i \in N.$$

It follows that

$$A_0 = \frac{1}{\Delta} \left(\Gamma(\alpha_1) \sum_{j=1}^{m} I_j \delta_{\alpha_1, \lambda_1}(1, t_j) + \int_0^1 \delta_{\alpha_1, \lambda_1}(1, s)\sigma(s)\, ds \right).$$

Substituting $A_i \; (i \in N_0)$ into (3.8), we obviously get (3.6) by changing the term order.

On the other hand, if x satisfies (3.6), then it is easy to see that $x\big|_{(t_i, t_{i+1}]}$ is continuous and the limit $\lim_{t \to t_{i+}} (t - t_i)^{1-\alpha_1} x(t)$ exists for $i \in N_0$. Using Lemma 3.1, we can prove that x satisfies (3.5). $\quad\square$

Lemma 3.3. *Suppose that* $\nabla = 1 - \Gamma(\alpha_2) E_{\alpha_2, \alpha_2}(\lambda_2) \neq 0$, $\sigma \in L^1(0, 1)$ *and there exist numbers* $k_2 > -1$ *and* $\max\{-\alpha_2, -k_2 - 1\} < l_2 \leq 0$ *such that* $|\sigma(t)| \leq t^{k_2}(1-t)^{l_2}$ *for all* $t \in (0, 1)$. *Then* $x \in Y$ *is a solution of*

$$\begin{cases} D_{0^+}^{\alpha_2} x(t) - \lambda_2 x(t) = \sigma(t), & t \in (t_i, t_{i+1}), \;\; i \in N_0, \\ x(1) - \lim_{t \to 0} t^{1-\alpha_2} x(t)\, dt = 0, \\ \lim_{t \to t_i^+} (t - t_i)^{1-\alpha_2} x(t) = J_i, & i \in N \end{cases} \tag{3.9}$$

if and only if $x \in Y$ *and*

$$x(t) = \frac{\Gamma(\alpha_2)^2 \delta_{\alpha_2, \lambda_2}(t, 0)}{\nabla} \sum_{j=1}^{m} J(t_j, x(t_j), y(t_j))\delta_{\alpha_2, \lambda_2}(1, t_j) + \frac{\Gamma(\alpha_2)\delta_{\alpha_2, \lambda_2}(t, 0)}{\nabla} \int_0^1 \delta_{\alpha_2, \lambda_2}(1, s)\sigma(s)\, ds$$

$$+ \Gamma(\alpha_2) \sum_{j=1}^{i} J(t_j, x(t_j), y(t_j))\delta_{\alpha_2, \lambda_2}(t, t_j) + \int_0^t \delta_{\alpha_2, \lambda_2}(t, s)\sigma(s)\, ds, \;\; t \in (t_i, t_{i+1}], \;\; i \in N_0. \tag{3.10}$$

Proof. The proof is similar to that of Lemma 3.2, and hence we omit it. $\quad\square$

Let Δ, ∇ be defined in Lemmas 3.2 and 3.3. Define the nonlinear operator T on E by $T(x,y)(t) = ((T_1(x,y))(t), (T_2(x,y))(t))$ with

$$(T_1(x,y))(t) = \frac{\Gamma(\alpha_1)^2 \delta_{\alpha_1,\lambda_1}(t,0)}{\Delta} \sum_{j=1}^{m} \delta_{\alpha_1,\lambda_1}(1,t_j) I(t_j, x(t_j), y(t_j))$$

$$+ \frac{\Gamma(\alpha_1)\delta_{\alpha_1,\lambda_1}(t,0)}{\Delta} \int_0^1 \delta_{\alpha_1,\lambda_1}(1,s) p_1(s) f_1(s, x(s), y(s)) \, ds$$

$$+ \Gamma(\alpha_1) \sum_{j=1}^{i} I(t_j, x(t_j), y(t_j)) \delta_{\alpha_1,\lambda_1}(t,t_j)$$

$$+ \int_0^t \delta_{\alpha_1,\lambda_1}(t,s) p_1(s) f_1(s, x(s), y(s)) \, ds, \quad t \in (t_i, t_{i+1}], \ \ i \in N_0,$$

and

$$(T_2(x,y))(t) = \frac{\Gamma(\alpha_2)^2 \delta_{\alpha_2,\lambda_2}(t,0)}{\nabla} \sum_{j=1}^{m} \delta_{\alpha_2,\lambda_2}(1,t_j) J(t_j, x(t_j), y(t_j))$$

$$+ \frac{\Gamma(\alpha_2)\delta_{\alpha_2,\lambda_2}(t,0)}{\nabla} \int_0^1 \delta_{\alpha_2,\lambda_2}(1,s) p_2(s) f_2(s, x(s), y(s)) \, ds$$

$$+ \Gamma(\alpha_2) \sum_{j=1}^{i} J(t_j, x(t_j), y(t_j)) \delta_{\alpha_2,\lambda_2}(t,t_j)$$

$$+ \int_0^t \delta_{\alpha_2,\lambda_2}(t,s) p_2(s) f_2(s, x(s), y(s)) \, ds, \quad t \in (t_i, t_{i+1}], \ \ i \in N_0,$$

for $(x,y) \in E$.

Lemma 3.4. *Suppose that f_1, f_2 are* **I-Carathéodory functions***, I, J are discrete I-Carathéodory functions. Then $(x,y) \in E$ is a solution of BVP (1.3) if and only if (x,y) is a fixed point of T in E and $T : E \to E$ is well defined and is completely continuous.*

Proof. It is easy to see from Lemmas 3.2 and 3.3 that $(x,y) \in E$ is a solution of BVP (1.3) if and only if (x,y) is a fixed point of T in E. We divide the remainder of the proof into the following steps:

Step (i). We prove that $T : E \to E$ is well defined. Similarly to the proofs of Lemma 3.1, we can prove that $T_1(x,y) \in X$ and $T_2(x,y) \in Y$ for all $(x,y) \in E$. Thus $T : E \to E$ is well defined.

Step (ii). We prove that T is continuous.

Let $(x_n, y_n) \in E$ with $(x_n, y_n) \to (x_0, y_0)$ as $n \to \infty$. We can show that $T(x_n, y_n) \to T(x_0, y_0)$ as $n \to \infty$ by using the dominant convergence theorem. We refer the readers to the papers [42, 48, 53].

Step (iii). Prove that T is compact, i.e., $T(\overline{\Omega})$ is relatively compact for every bounded subset $\Omega \subset E$.

Let Ω be a bounded open nonempty subset of E. Then there exists $r > 0$ such that

$$\|(x,y)\| = \max \left\{ \sup_{t \in (t_i, t_{i+1}]} (t - t_i)^{1-\alpha_1} |x(t)|, \ \sup_{t \in (t_i, t_{i+1}]} (t - t_i)^{1-\alpha_2} |y(t)| : \ i \in N_0 \right\} \leq r < +\infty \quad (3.11)$$

holds for all $(x,y) \in \overline{\Omega}$. Since f_1, f_2 are **I-Carathéodory functions** and I, J are **discrete I-Cara-**

théodory functions, there exist constants $M_I, M_J, M_{f_1}, M_{f_2} \geq 0$ such that

$$
\begin{aligned}
\left| f_1(t, x(t), y(t)) \right| &= \left| f\Big(t, \frac{(t-t_i)^{1-\alpha_1} x(t)}{(t-t_i)^{1-\alpha_1}}, \frac{(t-t_i)^{1-\alpha_2} y(t)}{(t-t_i)^{1-\alpha_2}}\Big) \right| \leq M_{f_1}, \quad t \in (t_i, t_{i+1}), \ i \in N_0, \\
\left| f_2(t, x(t), y(t)) \right| &\leq M_{f_2}, \quad t \in (t_i, t_{i+1}), \ i \in N_0, \\
\left| I(t_i, x(t_i), y(t_i)) \right| &= \left| I\Big(t_i, \frac{(t_i-t_{i-1})^{1-\alpha_1} x(t_i)}{(t_i-t_{i-1})^{1-\alpha_1}}, \frac{(t_i-t_{i-1})^{1-\alpha_2} y(t_i)}{(t_i-t_{i-1})^{1-\alpha_2}}\Big) \right| \leq M_I, \quad i \in N, \\
\left| J(t_i, x(t_i), y(t_i)) \right| &\leq M_J, \quad i \in N.
\end{aligned}
\tag{3.12}
$$

This step is done by three sub-steps.

Sub-step (iii1). Prove that $T(\overline{\Omega})$ is uniformly bounded.

Using the definition of T_1, for $t \in (0, t_1]$ we have

$$
\begin{aligned}
t^{1-\alpha_1} |(T_1(x,y))(t)| &\leq \frac{\Gamma(\alpha_1)^2 E_{\alpha_1,\alpha_1}(|\lambda_1| t_1^{\alpha_1})}{|\Delta|} \sum_{j=1}^{m} (1-t_j)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(1-t_j)^{\alpha_1}) M_I \\
&\quad + \frac{\Gamma(\alpha_1) E_{\alpha_1,\alpha_1}(|\lambda_1| t_1^{\alpha_1})}{|\Delta|} \int_0^1 (1-s)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(1-s)^{\alpha_1}) s^{k_1}(1-s)^{l_1} \, ds \, M_{f_1} \\
&\quad + t^{1-\alpha_1} \int_0^t (t-s)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1}) s^{k_1}(1-s)^{l_1} \, ds \, M_{f_1} \\
&\leq \frac{\Gamma(\alpha_1) E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|} (1-t_m)^{\alpha_1-1} m \Gamma(\alpha_1) E_{\alpha_1,\alpha_1}(|\lambda_1|) M_I \\
&\quad + \Big(\frac{\Gamma(\alpha_1) E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|} + 1\Big) \mathbf{B}(\alpha_1 + l_1, k_1 + 1) E_{\alpha_1,\alpha_1}(|\lambda_1|) M_{f_1}.
\end{aligned}
$$

Similarly, we can prove for $t \in (t_i, t_{i+1}]$, $i \in N$, that

$$
\begin{aligned}
(t-t_i)^{1-\alpha_1} &|(T_1(x,y))(t)| \\
&\leq (t_{i+1}-t_i)^{1-\alpha_1} \frac{\Gamma(\alpha_1)^2 t_1^{\alpha_1-1} E_{\alpha_1,\alpha_1}(|\lambda_1| t_{i+1}^{\alpha_1})}{|\Delta|} \sum_{j=1}^{m}(1-t_j)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(1-t_j)^{\alpha_1}) M_I \\
&\quad + (t_{i+1}-t_i)^{1-\alpha_1} \frac{\Gamma(\alpha_1) t_1^{\alpha_1-1} E_{\alpha_1,\alpha_1}(|\lambda_1| t_{i+1}^{\alpha_1})}{|\Delta|} \mathbf{B}(\alpha_1 + l_1, k_1 + 1) E_{\alpha_1,\alpha_1}(\lambda_1) M_{f_1} \\
&\quad + \Gamma(\alpha_1) \sum_{j=1}^{i} E_{\alpha_1,\alpha_1}(|\lambda_1|(t_{i+1}-t_j)^{\alpha_1}) M_I \\
&\quad + \max\{1, t_1^{\alpha_1 + k_1 + l_1}\} \mathbf{B}(\alpha_1 + l_1, k_1 + 1) E_{\alpha_1,\alpha_1}(|\lambda_1| t_{i+1}^{\alpha_1}) M_{f_1} \\
&\leq \Big(\frac{\Gamma(\alpha_1) t_1^{\alpha_1-1} E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|} (1-t_m)^{\alpha_1-1} + 1\Big) m \Gamma(\alpha_1) E_{\alpha_1,\alpha_1}(|\lambda_1|) M_I \\
&\quad + \Big(\frac{\Gamma(\alpha_1) t_1^{\alpha_1-1} E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|} + \max\{1, t_1^{\alpha_1 + k_1 + l_1}\}\Big) \mathbf{B}(\alpha_1 + l_1, k_1 + 1) E_{\alpha_1,\alpha_1}(|\lambda_1|) M_{f_1}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|T_1(x,y)\|_X &\leq \Big(\frac{\Gamma(\alpha_1) t_1^{\alpha_1-1} E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|} (1-t_m)^{\alpha_1-1} + 1\Big) m \Gamma(\alpha_1) E_{\alpha_1,\alpha_1}(|\lambda_1|) M_I \\
&\quad + \Big(\frac{\Gamma(\alpha_1) t_1^{\alpha_1-1} E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|} + \max\{1, t_1^{\alpha_1 + k_1 + l_1}\}\Big) \mathbf{B}(\alpha_1 + l_1, k_1 + 1) E_{\alpha_1,\alpha_1}(|\lambda_1|) M_{f_1}.
\end{aligned}
$$

We can also find that

$$\|T_2(x,y)\| \leq \Big(\frac{\Gamma(\alpha_2)t_1^{\alpha_2-1}E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|}(1-t_m)^{\alpha_2-1}+1\Big)m\Gamma(\alpha_2)E_{\alpha_2,\alpha_2}(|\lambda_2|)M_J$$

$$+\Big(\frac{\Gamma(\alpha_2)t_1^{\alpha_2-1}E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|}+\max\{1,t_1^{\alpha_2+k_2+l_2}\}\Big)\mathbf{B}(\alpha_2+l_2,k_2+1)E_{\alpha_2,\alpha_2}(|\lambda_2|)M_{f_2}.$$

From the above discussion, $T(\overline{\Omega})$ is uniformly bounded.

Sub-step (iii2). Prove that both $\{(t-t_i)^{1-\alpha_1}(T_1(x,y))(t): (x,y) \in \overline{\Omega}\}$ and $\{(t-t_i)^{1-\alpha_2}(T_2(x,y))(t): (x,y) \in \overline{\Omega}\}$ are equi-continuous on each subinterval $[a,b] \subseteq (t_i, t_{i+1}], i \in N_0$, respectively.

Let $s_2 \leq s_1$ and $s_1, s_2 \in [a,b] \subseteq (0,t_1]$. We can prove

$$\left|s_1^{1-\alpha_1}(T_1(x,y))(s_1) - s_2^{1-\alpha_1}(T_1(x,y))(s_2)\right|$$

$$\leq \Gamma(\alpha_1)^2\left|\frac{E_{\alpha_1,\alpha_1}(\lambda_1 s_1^{\alpha_1}) - E_{\alpha_1,\alpha_1}(\lambda_1 s_2^{\alpha_1})}{\Delta}\right|\Big|\sum_{j=1}^{m}(1-t_j)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(1-t_j)^{\alpha_1})M_I$$

$$+\Gamma(\alpha_1)\left|\frac{E_{\alpha_1,\alpha_1}(\lambda_1 s_1^{\alpha_1}) - E_{\alpha_1,\alpha_1}(\lambda_1 s_2^{\alpha_1})}{\Delta}\right|\Big|\int_0^1(1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(1-s)^{\alpha_1})s^{k_1}(1-s)^{l_1}\,ds$$

$$+\left|s_1^{1-\alpha_1}\int_0^{s_1}(s_1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(s_1-s)^{\alpha_1})p_1(s)f_1(s,x(s),y(s))\,ds\right.$$

$$\left.-s_2^{1-\alpha_1}\int_0^{s_2}(s_2-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(s_2-s)^{\alpha_1})p_1(s)f_1(s,x(s),y(s))\,ds\right|.$$

(i) It is easy to show that $E_{\alpha_1,\alpha_1}(\lambda_1 x^{\alpha_1})$ is uniformly continuous on $[0,b]$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|E_{\alpha_1,\alpha_1}(\lambda_1 s_1^{\alpha_1}) - E_{\alpha_1,\alpha_1}(\lambda_1 s_2^{\alpha_1})| < \varepsilon$ for all $s_1, s_2 \in [0,b]$ with $|s_1-s_2| < \delta$.

(ii) Both $\sum_{j=1}^{m}(1-t_j)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(1-t_j)^{\alpha_1})M_I$ and $\int_0^1(1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(1-s)^{\alpha_1})s^{k_1}(1-s)^{l_1}\,ds$ are constants.

(iii) It is easy to prove that

$$x^\mu - y^\mu \leq (x-y)^\mu \text{ for all } \mu \in (0,1], \ x \geq y > 0,$$

$$x^\mu - y^\mu \leq \mu x^{\mu-1}(x-y) \text{ for all } \mu > 1, \ x \geq y > 0,$$

$$h_1(x) = xc^{x-1} \text{ is increasing on } \Big(0,-\frac{1}{\ln c}\Big) \text{ and decreasing on } \Big(-\frac{1}{\ln c},+\infty\Big), \ c \in (0,1),$$

$$h_2(x) = -x\ln x \geq \min\big\{-(b-a)\ln(b-a), -b\ln b\big\} \text{ for all } x \in [b-a,b].$$

It follows that

$$h_1(x) \leq \max\Big\{1,\frac{1}{e}\frac{1}{-c\ln c}\Big\} \leq \max\Big\{1,\frac{1}{e}\frac{1}{\min\{-(b-a)\ln(b-a),-b\ln b\}}\Big\}, \ x > 1, \ c \in [b-a,b].$$

Then we have for $s_2, s_1 \in [a,b]$ with $s_1 > s_2$ that

$$\left|s_1^{1-\alpha_1}\int_0^{s_1}(s_1-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(s_1-s)^{\alpha_1})p_1(s)f_1(s,x(s),y(s))\,ds\right.$$

$$\left.-s_2^{1-\alpha_1}\int_0^{s_2}(s_2-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(s_2-s)^{\alpha_1})p_1(s)f_1(s,x(s),y(s))\,ds\right|$$

$$=\left|s_1^{1-\alpha_1}\int_0^{s_1}(s_1-s)^{\alpha_1-1}\sum_{k=0}^{+\infty}\frac{\lambda_1^k(s_1-s)^{\alpha_1 k}}{\Gamma(\alpha_1(k+1))}p_1(s)f_1(s,x(s),y(s))\,ds\right.$$

$$- s_2^{1-\alpha_1} \int_0^{s_2} (s_2 - s)^{\alpha_1 - 1} \sum_{k=0}^{+\infty} \frac{\lambda_1^k (s_2 - s)^{\alpha_1 k}}{\Gamma(\alpha(k+1))} \, p_1(s) f_1(s, x(s), y(s)) \, ds \Bigg|$$

$$= \Bigg| \sum_{k=0}^{+\infty} \frac{|\lambda_1|^k}{\Gamma(\alpha_1(k+1))} s_1^{1-\alpha_1} \int_0^{s_1} (s_1 - s)^{\alpha_1(k+1)-1} p_1(s) f_1(s, x(s), y(s)) \, ds$$

$$- \sum_{k=0}^{+\infty} \frac{|\lambda_1|^k}{\Gamma(\alpha(k+1))} s_2^{1-\alpha_1} \int_0^{s_2} (s_2 - s)^{\alpha_1(k+1)-1} p_1(s) f_1(s, x(s), y(s)) \, ds \Bigg|$$

$$\leq \sum_{k=0}^{+\infty} \frac{|\lambda_1|^k}{\Gamma(\alpha(k+1))} \Bigg| s_1^{1-\alpha_1} \int_0^{s_1} (s_1 - s)^{\alpha_1(k+1)-1} p_1(s) f_1(s, x(s), y(s)) \, ds$$

$$- s_2^{1-\alpha_1} \int_0^{s_2} (s_2 - s)^{\alpha_1(k+1)-1} p_1(s) f_1(s, x(s), y(s)) \, ds \Bigg|$$

$$\leq \sum_{k=0}^{+\infty} \frac{|\lambda_1|^k}{\Gamma(\alpha_1(k+1))} \Bigg[\big| s_1^{1-\alpha_1} - s_2^{1-\alpha_1} \big| \int_0^{s_1} (s_1 - s)^{\alpha_1(k+1)-1} |p_1(s)| \, |f_1(s, x(s), y(s))| \, ds$$

$$+ s_2^{1-\alpha_1} \int_{s_2}^{s_1} (s_1 - s)^{\alpha_1(k+1)-1} |p_1(s)| \, |f_1(s, x(s), y(s))| \, ds$$

$$+ s_2^{1-\alpha_1} \int_0^{s_2} \big| (s_1 - s)^{\alpha_1(k+1)-1} - (s_2 - s)^{\alpha_1(k+1)-1} \big| |p_1(s)| \, |f_1(s, x(s), y(s))| \, ds \Bigg]$$

$$\leq M_{f_1} \sum_{k=0}^{+\infty} \frac{|\lambda_1|^k}{\Gamma(\alpha_1(k+1))} \Bigg[\big| s_1^{1-\alpha_1} - s_2^{1-\alpha_1} \big| \int_0^{s_1} (s_1 - s)^{\alpha_1(k+1)-1} s^{k_1} (1-s)^{l_1} \, ds$$

$$+ s_2^{1-\alpha_1} \int_{s_2}^{s_1} (s_1 - s)^{\alpha_1(k+1)-1} s^{k_1} (1-s)^{l_1} \, ds$$

$$+ s_2^{1-\alpha_1} \int_0^{s_2} \big| (s_1 - s)^{\alpha_1(k+1)-1} - (s_2 - s)^{\alpha_1(k+1)-1} \big| s^{k_1} (1-s)^{l_1} \, ds \Bigg]$$

$$\leq M_{f_1} \sum_{k=0}^{+\infty} \frac{|\lambda_1|^k}{\Gamma(\alpha_1(k+1))} \Bigg[\big| s_1^{1-\alpha_1} - s_2^{1-\alpha_1} \big| \int_0^{s_1} (s_1 - s)^{\alpha_1(k+1)+l_1-1} s^{k_1} \, ds$$

$$+ s_2^{1-\alpha_1} \int_{s_2}^{s_1} (s_1 - s)^{\alpha_1(k+1)+l_1-1} s^{k_1} \, ds$$

$$+ s_2^{1-\alpha_1} \int_0^{s_2} \big| (s_1 - s)^{\alpha_1(k+1)-1} - (s_2 - s)^{\alpha_1(k+1)-1} \big| s^{k_1} (1-s)^{l_1} \, ds \Bigg]$$

$$= M_{f_1} \sum_{k=0}^{+\infty} \frac{|\lambda_1|^k}{\Gamma(\alpha_1(k+1))} \Bigg[\big| s_1^{1-\alpha_1} - s_2^{1-\alpha_1} \big| s_1^{\alpha_1(k+1)+k_1+l_1} \int_0^1 (1-w)^{\alpha_1(k+1)+l_1-1} w^{k_1} \, dw$$

$$+ s_2^{1-\alpha_1} s_1^{\alpha_1(k+1)+k_1+l_1} \int_{\frac{s_2}{s_1}}^1 (1-w)^{\alpha_1(k+1)+l_1-1} w^{k_1} \, dw$$

$$+ s_2^{1-\alpha_1} \int_0^{s_2} \left| (s_1 - s)^{\alpha_1(k+1)-1} - (s_2 - s)^{\alpha_1(k+1)-1} \right| s^{k_1}(1-s)^{l_1} \, ds \Bigg]$$

$$\leq M_{f_1} \sum_{k=0}^{+\infty} \frac{|\lambda_1|^k}{\Gamma(\alpha_1(k+1))} \Bigg[\left| s_1^{1-\alpha_1} - s_2^{1-\alpha_1} \right| \max\left\{ a^{\alpha_1+k_1+l_1}, b^{\alpha_1+k_1+l_1} \right\} \int_0^1 (1-w)^{\alpha_1+l_1-1} w^{k_1} \, dw$$

$$+ \max\left\{ a^{\alpha_1+k_1+l_1}, b^{\alpha_1+k_1+l_1} \right\} \int_{\frac{s_2}{s_1}}^1 (1-w)^{\alpha_1+l_1-1} w^{k_1} \, dw$$

$$+ s_2^{1-\alpha_1} \int_0^{s_2} \left| (s_1 - s)^{\alpha_1(k+1)-1} - (s_2 - s)^{\alpha_1(k+1)-1} \right| s^{k_1}(1-s)^{l_1} \, ds \Bigg].$$

One can see

$$\left| s_1^{1-\alpha_1} - s_2^{1-\alpha_1} \right| \longrightarrow 0, \quad \int_{\frac{s_2}{s_1}}^1 (1-w)^{\alpha_1+l_1-1} w^{k_1} \, dw \longrightarrow 0 \ \text{ as } \ s_1 \to s_2.$$

On the other hand, for $\alpha_1(k+1) - 1 \in (0, 1]$ we have

$$s_2^{1-\alpha_1} \int_0^{s_2} \left| (s_1 - s)^{\alpha_1(k+1)-1} - (s_2 - s)^{\alpha_1(k+1)-1} \right| s^{k_1}(1-s)^{l_1} \, ds$$

$$\leq (s_1 - s_2)^{\alpha_1(k+1)-1} \int_0^{s_2} s^{k_1}(s_2 - s)^{l_1} \, ds = (s_1 - s_2)^{\alpha_1(k+1)-1} s_2^{k_1+l_1+1} \int_0^1 (1-w)^{l_1} w^{k_1} \, ds$$

$$\leq (s_1 - s_2)^{\alpha_1(k+1)-1} \int_0^1 (1-w)^{l_1} w^{k_1} \, dw \longrightarrow 0$$

$$\text{as } s_1 \to s_2 \text{ uniformly for } \ k = \left[\frac{1}{\alpha_1}\right] - 1, \left[\frac{1}{\alpha_1}\right], \ldots, \left[\frac{2}{\alpha_1}\right] - 1.$$

For $\alpha_1(k+1) - 1 > 1$,

$$s_2^{1-\alpha_1} \int_0^{s_2} \left| (s_1 - s)^{\alpha_1(k+1)-1} - (s_2 - s)^{\alpha_1(k+1)-1} \right| s^{k_1}(1-s)^{l_1} \, ds$$

$$\leq (s_1 - s_2) \int_0^{s_2} (\alpha_1(k+1) - 1)(s_1 - s)^{\alpha_1(k+1)-2} s^{k_1}(s_2 - s)^{l_1} \, ds$$

$$\leq (s_1 - s_2) \max\left\{ 1, \frac{1}{e} \frac{1}{\min\{-(b-a)\ln(b-a), -b\ln b\}} \right\} \int_0^{s_2} s^{k_1}(s_2 - s)^{l_1} \, ds$$

$$\leq (s_1 - s_2) \max\left\{ 1, \frac{1}{e} \frac{1}{\min\{-(b-a)\ln(b-a), -b\ln b\}} \right\} s_2^{k_1+l_1+1} \int_0^1 w^{k_1}(1-w)^{l_1} \, dw$$

$$\leq (s_1 - s_2) \max\left\{ 1, \frac{1}{e} \frac{1}{\min\{-(b-a)\ln(b-a), -b\ln b\}} \right\} \int_0^1 w^{k_1}(1-w)^{l_1} \, dw \longrightarrow 0 \ \text{ as } \ s_1 \to s_2.$$

For $\alpha_1(k+1) - 1 \le 0$,

$$s_2^{1-\alpha_1} \int_0^{s_2} \left| (s_1 - s)^{\alpha_1(k+1)-1} - (s_2 - s)^{\alpha_1(k+1)-1} \right| s^{k_1} (1-s)^{l_1} \, ds$$

$$\le \int_0^{s_2} \left[(s_2 - s)^{\alpha_1(k+1)-1} - (s_1 - s)^{\alpha_1(k+1)-1} \right] s^{k_1} (s_2 - s)^{l_1} \, ds$$

$$\le \int_0^{s_2} (s_2 - s)^{\alpha_1(k+1)+l_1-1} s^{k_1} ds - \int_0^{s_2} (s_1 - s)^{\alpha_1(k+1)-1} s^{k_1} (s_2 - s)^{l_1} \, ds$$

$$\le \int_0^{s_2} (s_2 - s)^{\alpha_1(k+1)+l_1-1} s^{k_1} ds - \int_0^{s_2} (s_1 - s)^{\alpha_1(k+1)+l_1-1} s^{k_1} \, ds$$

$$= s_2^{\alpha_1(k+1)+k_1+l_1} \int_0^1 (1-w)^{\alpha_1(k+1)+l_1-1} w^{k_1} \, dw - s_1^{\alpha_1(k+1)+k_1+l_1} \int_0^{\frac{s_2}{s_1}} (1-w)^{\alpha_1(k+1)+l_1-1} w^{k_1} \, dw$$

$$\le \left| s_2^{\alpha_1(k+1)+k_1+l_1} - s_1^{\alpha_1(k+1)+k_1+l_1} \right| \int_0^1 (1-w)^{\alpha_1(k+1)+l_1-1} w^{k_1} \, dw$$

$$+ s_1^{\alpha_1(k+1)+k_1+l_1} \int_{\frac{s_2}{s_1}}^1 (1-w)^{\alpha_1(k+1)+l_1-1} w^{k_1} \, dw$$

$$\le \left| s_2^{\alpha_1(k+1)+k_1+l_1} - s_1^{\alpha_1(k+1)+k_1+l_1} \right| \int_0^1 (1-w)^{\alpha_1+l_1-1} w^{k_1} \, dw$$

$$+ \max \left\{ a^{\alpha_1+k_1+l_1}, b^{\alpha_1+k_1+l_1} \right\} \int_{\frac{s_2}{s_1}}^1 (1-w)^{\alpha_1+l_1-1} w^{k_1} \, dw \longrightarrow 0$$

as $s_1 \to s_2$ uniformly for $k = 0, 1, 2, \ldots, \left[\dfrac{1}{\alpha_1} \right] - 1$.

From the above discussion, there exists $\delta > 0$ such that

$$\left| s_1^{1-\alpha_1} - s_2^{1-\alpha_1} \right| < \varepsilon, \quad \int_{\frac{s_2}{s_1}}^1 (1-w)^{\alpha_1+l_1-1} w^{k_1} \, dw < \varepsilon,$$

$$s_2^{1-\alpha_1} \int_0^{s_2} \left| (s_1 - s)^{\alpha_1(k+1)-1} - (s_2 - s)^{\alpha_1(k+1)-1} \right| s^{k_1} (1-s)^{l_1} \, ds < \varepsilon$$

hold for all $s_1, s_2 \in [a, b]$, $0 < s_1 - s_2 < \delta$, $k = 0, 1, 2, \ldots$. Then

$$\left| s_1^{1-\alpha_1} \int_0^{s_1} (s_1 - s)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s_1 - s)^{\alpha_1}) p_1(s) f_1(s, x(s), y(s)) \, ds \right.$$

$$\left. - s_2^{1-\alpha_1} \int_0^{s_2} (s_2 - s)^{\alpha_1-1} E_{\alpha_1,\alpha_1}(\lambda_1(s_2 - s)^{\alpha_1}) p_1(s) f_1(s, x(s), y(s)) \, ds \right|$$

$$< \varepsilon M_{f_1} \sum_{k=0}^{+\infty} \frac{|\lambda_1|^k}{\Gamma(\alpha_1(k+1))} \left[\max\left\{ a^{\alpha_1+k_1+l_1}, b^{\alpha_1+k_1+l_1} \right\} \int_0^1 (1-w)^{\alpha_1+l_1-1} w^{k_1} \, dw \right.$$

$$\left. + \max\left\{ a^{\alpha_1+k_1+l_1}, b^{\alpha_1+k_1+l_1} \right\} + 1 \right]$$

$$= \varepsilon M_{f_1} E_{\alpha_1,\alpha_1}(|\lambda_1|) \left[\max\left\{ a^{\alpha_1+k_1+l_1}, b^{\alpha_1+k_1+l_1} \right\} \int_0^1 (1-w)^{\alpha_1+l_1-1} w^{k_1} \, dw \right.$$

$$\left. + \max\left\{ a^{\alpha_1+k_1+l_1}, b^{\alpha_1+k_1+l_1} \right\} + 1 \right], \quad s_1, s_2 \in [a,b], \ 0 < s_1 - s_2 < \delta, \ k = 0,1,2,\dots.$$

From (i), (ii) and (iii), we have

$$\left| s_1^{1-\alpha_1} (T_1(x,y))(s_1) - s_2^{1-\alpha_1} (T_1(x,y))(s_2) \right| \longrightarrow 0 \text{ uniformly in } [a,b] \subset (0,t_1] \text{ as } s_1 \to s_2.$$

Similarly, we can show that for $s_1, s_2 \in [a,b] \subseteq (t_i, t_{i+1}]$, $i \in N$,

$$\left| (s_1 - t_i)^{1-\alpha_1} (T_1(x,y))(s_1) - (s_2 - t_i)^{1-\alpha_1} (T_1(x,y))(s_2) \right| \longrightarrow 0 \text{ uniformly as } s_1 \to s_2,$$

and for $s_1, s_2 \in [a,b] \subseteq (t_i, t_{i+1}]$, $i \in N_0$,

$$\left| (s_1 - t_i)^{1-\alpha_2} (T_2(x,y))(s_1) - (s_2 - t_i)^{1-\alpha_2} (T_2(x,y))(s_2) \right| \longrightarrow 0 \text{ uniformly as } s_1 \to s_2.$$

All of these expressions complete this step.

Sub-step (iii3). Prove that both $\{(t-t_i)^{1-\alpha_1}(T_1(x,y))(t) : (x,y) \in \overline{\Omega}\}$ and $\{(t-t_i)^{1-\alpha_2}(T_2(x,y))(t) : (x,y) \in \overline{\Omega}\}$ are equi-convergent at $t = t_i$, $i \in N_0$, respectively.

For $i \in N_0$, since $\lim_{t \to t_i^+} (t-t_i)^{1-\alpha_1}(T_1(x,y))(t)$ exists, we can easily see that $\{(t-t_i)^{1-\alpha_1}(T_1(x,y))(t) : (x,y) \in \overline{\Omega})\}$ is equi-convergent at $t = t_i$, $i \in N_0$. Similarly, we can show that $(t-t_i)^{1-\alpha_2}(T_2(x,y))(t) : (x,y) \in (\overline{\Omega})\}$ is equi-convergent at $t = t_i$, $i \in N_0$. So $T(\overline{\Omega})$ is relatively compact. Then T is completely continuous. $\qquad \square$

Now, we prove the first theorem of this paper by using the Schauder's fixed point theorem [30]. We need the following assumptions:

(D1) f_1, f_2 are **I-Carathéodory functions**, I, J are discrete I-Carathéodory functions and there exist non-decreasing functions $\phi_i, \psi_i : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, measurable functions $\phi_i, \overline{\phi}_i, \overline{\psi}_i : (0,1) \to R$ $(i = 1,2)$ and constants I_j, J_j such that

$$\left| f_1\left(t, \frac{x}{(t-t_i)^{1-\alpha_1}}, \frac{y}{(t-t_i)^{1-\alpha_2}}\right) - \phi_1(t) \right| \leq \overline{\phi}_1(|x|,|y|), \ t \in (t_i, t_{i+1}), \ x,y \in R, \ i \in N_0,$$

$$\left| f_2\left(t, \frac{x}{(t-t_i)^{1-\alpha_1}}, \frac{y}{(t-t_i)^{1-\alpha_2}}\right) - \phi_2(t) \right| \leq \overline{\phi}_2(|x|,|y|), t \in (t_i, t_{i+1}), x,y \in R, i \in N_0,$$

$$\left| I\left(t_j, \frac{x}{(t_j-t_{j-1})^{1-\alpha_1}}, \frac{y}{(t_j-t_{j-1})^{1-\alpha_2}}\right) - I_j \right| \leq \overline{\psi}_1(|x|,|y|), \ j \in N, \ x,y \in R,$$

$$\left| J\left(t_j, \frac{x}{(t_j-t_{j-1})^{1-\alpha_1}}, \frac{y}{(t_j-t_{j-1})^{1-\alpha_2}}\right) - J_j \right| \leq \overline{\psi}_2(|x|,|y|), \ j \in N, \ x,y \in R.$$

(D2) f_1, f_2 are **I-Carathéodory functions**, I, J are discrete I-Carathéodory functions and there exist nonnegative constants $I_i, J_i, b_i, a_i, B_i, A_i, \tau_j, \sigma_j$ $(j = 1,2,\dots,n)$ and measurable functions

$\phi_i : (0,1) \to R \ (i = 1, 2)$ such that

$$\left| f_1\left(t, \frac{x}{(t-t_i)^{1-\alpha_1}}, \frac{y}{(t-t_i)^{1-\alpha_2}}\right) - \phi_1(t)\right| \le \sum_{j=1}^{n} a_j |x|^{\tau_j} |y|^{\sigma_j}, \quad t \in (t_i, t_{i+1}), \quad x, y \in R, \quad i \in N_0,$$

$$\left| f_2\left(t, \frac{x}{(t-t_i)^{1-\alpha_1}}, \frac{y}{(t-t_i)^{1-\alpha_2}}\right) - \phi_2(t)\right| \le \sum_{j=1}^{n} b_j |x|^{\tau_j} |y|^{\sigma_j}, \quad t \in (t_i, t_{i+1}), \quad x, y \in R, \quad i \in N_0,$$

$$\left| I\left(t_j, \frac{x}{(t_j-t_{j-1})^{1-\alpha_1}}, \frac{y}{(t_j-t_{j-1})^{1-\alpha_2}}\right) - I_j\right| \le \sum_{j=1}^{n} A_j |x|^{\tau_j} |y|^{\sigma_j}, \quad j \in N, \quad x, y \in R,$$

$$\left| J\left(t_j, \frac{x}{(t_j-t_{j-1})^{1-\alpha_1}}, \frac{y}{(t_j-t_{j-1})^{1-\alpha_2}}\right) - J_j\right| \le \sum_{j=1}^{n} B_j |x|^{\tau_j} |y|^{\sigma_j}, \quad j \in N, \quad x, y \in R.$$

Define

$$\Phi_1(t) = \frac{\Gamma(\alpha_1)^2 \delta_{\alpha_1,\lambda_1}(t,0)}{\Delta} \sum_{j=1}^{m} \delta_{\alpha_1,\lambda_1}(1,t_j) I_j + \frac{\Gamma(\alpha_1)\delta_{\alpha_1,\lambda_1}(t,0)}{\Delta} \int_0^1 \delta_{\alpha_1,\lambda_1}(1,s) p_1(s)\phi_1(s)\,ds$$

$$+ \Gamma(\alpha_1) \sum_{j=1}^{i} \delta_{\alpha_1,\lambda_1}(t,t_j) I_j + \int_0^t \delta_{\alpha_1,\lambda_1}(t,s) p_1(s)\phi_1(s)\,ds, \quad t \in (t_i, t_{i+1}], \quad i \in N_0,$$

and

$$\Phi_2(t) = \frac{\Gamma(\alpha_2)^2 \delta_{\alpha_2,\lambda_2}(t,0)}{\nabla} \sum_{j=1}^{m} \delta_{\alpha_2,\lambda_2}(1,t_j) J_j + \frac{\Gamma(\alpha_2)\delta_{\alpha_2,\lambda_2}(t,0)}{\nabla} \int_0^1 \delta_{\alpha_2,\lambda_2}(1,s) p_2(s)\phi_2(s)\,ds$$

$$+ \Gamma(\alpha_2) \sum_{j=1}^{i} \delta_{\alpha_2,\lambda_2}(t,t_j) J_j + \int_0^t \delta_{\alpha_2,\lambda_2}(t,s) p_2(s)\phi_2(s)\,ds, \quad t \in (t_i, t_{i+1}], \quad i \in N_0.$$

Denote

$$P_1 = \left(\frac{\Gamma(\alpha_1) t_1^{\alpha_1-1} E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|} + \max\left\{1, t_1^{\alpha_1+k_1+l_1}\right\}\right) \mathbf{B}(\alpha_1 + l_1, k_1 + 1) E_{\alpha_1,\alpha_1}(|\lambda_1|),$$

$$Q_1 = \left(\frac{\Gamma(\alpha_1) t_1^{\alpha_1-1} E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|}(1-t_m)^{\alpha_1-1} + 1\right) m\Gamma(\alpha_1) E_{\alpha_1,\alpha_1}(|\lambda_1|),$$

$$P_2 = \left(\frac{\Gamma(\alpha_2) t_1^{\alpha_2-1} E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|} + \max\left\{1, t_1^{\alpha_2+k_2+l_2}\right\}\right) \mathbf{B}(\alpha_2 + l_2, k_2 + 1) E_{\alpha_2,\alpha_2}(|\lambda_2|),$$

$$Q_2 = \left(\frac{\Gamma(\alpha_2) t_1^{\alpha_2-1} E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|}(1-t_m)^{\alpha_2-1} + 1\right) m\Gamma(\alpha_2) E_{\alpha_2,\alpha_2}(|\lambda_2|).$$

Theorem 3.1. *Suppose that* (D1) *holds. Then BVP* (1.3) *has at least one solution if*

$$\begin{aligned} P_1\overline{\phi}_1\big(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\big) + Q_1\overline{\psi}_1\big(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\big) \le r_1, \\ P_2\overline{\phi}_2\big(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\big) + Q_2\overline{\psi}_2\big(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\big) \le r_2 \end{aligned} \tag{3.13}$$

has a couple of positive solutions (r_1, r_2).

Proof. To apply the Schauder's fixed point theorem, we have to define a closed convex bounded subset Ω of E such that $T(\Omega) \subseteq \Omega$.

Let $r_1 > 0$, $r_2 > 0$, denote $\Omega = \{(x, y) \in E : \ \|x - \Phi_1\| \le r_1, \ \|y - \Phi_2\| \le r_2\}$. For $(x, y) \in \Omega$, we get

$$\|x\| \le \|x - \Phi_1\| + \|\Phi_1\| \le r_1 + \|\Phi_1\|, \quad \|y\| \le \|y - \Phi_2\| + \|\Phi_2\| \le r_2 + \|\Phi_2\|.$$

Then (D1) implies that

$$\left|f_1(t,x(t),y(t)) - \phi_1(t)\right| = \left|f\left(t, \frac{(t-t_i)^{1-\alpha_1}x(t)}{(t-t_i)^{1-\alpha_1}}, \frac{(t-t_i)^{1-\alpha_2}y(t)}{(t-t_i)^{1-\alpha_1}}\right) - \phi_1(t)\right|$$

$$\leq \overline{\phi}_1\left(|(t-t_i)^{1-\alpha_1}x(t)|, |(t-t_i)^{1-\alpha_2}y(t)|\right) \leq \phi_1\left(\|x\|, \|y\|\right)$$

$$\leq \overline{\phi}_1\left(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\right), \;\; t \in (t_i, t_{i+1}), \;\; i \in N_0,$$

$$\left|f_2(t,x(t),y(t)) - \phi_2(t)\right| \leq \overline{\phi}_2\left(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\right), \;\; t \in (t_i, t_{i+1}), \;\; i \in N_0,$$

$$\left|I(t_j, x(t_j), y(t_j)) - I_j\right| \leq \overline{\psi}_1\left(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\right), \;\; j \in N,$$

$$\left|J(t_j, x(t_j), y(t_j)) - J_j\right| \leq \overline{\psi}_2\left(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\right), \;\; j \in N.$$

By the definition of T, using the above inequalities, similarly to Step (iii1) in the proof of Lemma 3.4, we get

$$\|T_1(x,y) - \Phi_1\| \leq \left(\frac{\Gamma(\alpha_1)t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|}(1-t_m)^{\alpha_1-1} + 1\right)$$

$$\times m\Gamma(\alpha_1)E_{\alpha_1,\alpha_1}(|\lambda_1|)\overline{\psi}_1\left(r_1 + \|\Phi_1\|, \; r_2 + \|\Phi_2\|\right)$$

$$+ \left(\frac{\Gamma(\alpha_1)t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|} + \max\left\{1, t_1^{\alpha_1+k_1+l_1}\right\}\right)$$

$$\times \mathbf{B}(\alpha_1+l_1, k_1+1)E_{\alpha_1,\alpha_1}(|\lambda_1|)\overline{\phi}_1\left(r_1 + \|\Phi_1\|, \; r_2 + \|\Phi_2\|\right)$$

$$= P_1\overline{\phi}_1\left(r_1 + \|\Phi_1\|, \; r_2 + \|\Phi_2\|\right) + Q_1\overline{\psi}_1\left(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\right).$$

Similarly, we have

$$\|T_2(x,y) - \Phi_2\| \leq \left(\frac{\Gamma(\alpha_2)t_1^{\alpha_2-1}E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|}(1-t_m)^{\alpha_2-1} + 1\right)$$

$$\times m\Gamma(\alpha_2)E_{\alpha_2,\alpha_2}(|\lambda_2|)\overline{\psi}_2\left(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\right)$$

$$+ \left(\frac{\Gamma(\alpha_2)t_1^{\alpha_2-1}E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|} + \max\left\{1, t_1^{\alpha_2+k_2+l_2}\right\}\right)$$

$$\times \mathbf{B}(\alpha_2+l_2, k_2+1)E_{\alpha_2,\alpha_2}(|\lambda_2|)\overline{\phi}_2\left(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\right)$$

$$= P_2\overline{\phi}_2\left(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\right) + Q_2\overline{\psi}_2\left(r_1 + \|\Phi_1\|, \; r_2 + \|\Phi_2\|\right).$$

From the assumption, the inequality of system (3.13) has positive solution (r_1, r_2). We choose $\Omega = \{(x,y) \in E : \|x - \Phi_1\| \leq r_1, \|y - \Phi_2\| \leq r_2\}$. Then we get $T(\Omega) \subset \Omega$. Hence the Schauder's fixed point theorem implies that T has a fixed point $(x,y) \in \Omega$. So (x,y) is a solution of BVP (1.3). $\qquad\square$

Denote

$$P_j = \left(\frac{\Gamma(\alpha_1)t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|}(1-t_m)^{\alpha_1-1} + 1\right)m\Gamma(\alpha_1)E_{\alpha_1,\alpha_1}(|\lambda_1|)A_j$$

$$+ \left(\frac{\Gamma(\alpha_1)t_1^{\alpha_1-1}E_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Delta|} + \max\left\{1, t_1^{\alpha_1+k_1+l_1}\right\}\right)\mathbf{B}(\alpha_1+l_1, k_1+1)E_{\alpha_1,\alpha_1}(|\lambda_1|)a_j, \; j=1,2,\ldots,n,$$

$$Q_j = \left(\frac{\Gamma(\alpha_2)t_1^{\alpha_2-1}E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|}(1-t_m)^{\alpha_2-1} + 1\right)m\Gamma(\alpha_2)E_{\alpha_2,\alpha_2}(|\lambda_2|)B_j$$

$$+ \left(\frac{\Gamma(\alpha_2)t_1^{\alpha_2-1}E_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\nabla|} + \max\left\{1, t_1^{\alpha_2+k_2+l_2}\right\}\right)\mathbf{B}(\alpha_2+l_2, k_2+1)E_{\alpha_2,\alpha_2}(|\lambda_2|)b_j, \; j=1,2,\ldots,n.$$

Theorem 3.2. *Suppose that* (D2) *holds. Then BVP* (1.3) *has at least one solution if*

$$\sum_{j=1}^n P_j\left[r_1 + \|\Phi_1\|\right]^{\tau_j}\left[r_2 + \|\Phi_2\|\right]^{\sigma_j} \leq r_1, \quad \sum_{j=1}^n Q_j\left[r_1 + \|\Phi_1\|\right]^{\tau_j}\left[r_2 + \|\Phi_2\|\right]^{\sigma_j} \leq r_2 \qquad (3.14)$$

has a couple of positive solutions (r_1, r_2).

Proof. To apply the Schauder's fixed point theorem, we have to define a closed convex bounded subset Ω of E such that $T(\Omega) \subseteq \Omega$.

Let $r_1 > 0, r_2 > 0$, denote $\Omega = \{(x, y) \in E : \|x - \Phi_1\| \le r_1, \|y - \Phi_2\| \le r_2\}$. For $(x, y) \in \Omega$, we get

$$\|x\| \le \|x - \Phi_1\| + \|\Phi_1\| \le r_1 + \|\Phi_1\|, \quad \|y\| \le \|y - \Phi_2\| + \|\Phi_2\| \le r_2 + \|\Phi_2\|.$$

Then (D2) implies that

$$\left| f_1(t, x(t), y(t)) - \phi_1(t) \right| \le \sum_{j=1}^n a_j \left[r_1 + \|\Phi_1\| \right]^{\tau_j} \left[r_2 + \|\Phi_2\| \right]^{\sigma_j}, \quad t \in (t_i, t_{i+1}), \quad i \in N_0,$$

$$\left| f_2(t, x(t), y(t)) - \phi_2(t) \right| \le \sum_{j=1}^n b_j \left[r_1 + \|\Phi_1\| \right]^{\tau_j} \left[r_2 + \|\Phi_2\| \right]^{\sigma_j}, \quad t \in (t_i, t_{i+1}), \quad i \in N_0,$$

$$\left| I(t_j, x(t_j), y(t_j)) - I_j \right| \le \sum_{j=1}^n A_j \left[r_1 + \|\Phi_1\| \right]^{\tau_j} \left[r_2 + \|\Phi_2\| \right]^{\sigma_j}, \quad j \in N,$$

$$\left| J(t_j, x(t_j), y(t_j)) - J_j \right| \le \sum_{j=1}^n B_j \left[r_1 + \|\Phi_1\| \right]^{\tau_j} \left[r_2 + \|\Phi_2\| \right]^{\sigma_j}, \quad j \in N.$$

By the definition of T, using the above inequalities, similarly to Step (iii1) in the proof of Lemma 3.4, we get

$$\|T_1(x, y) - \Phi_1\| \le \left(\frac{\Gamma(\alpha_1) t_1^{\alpha_1 - 1} E_{\alpha_1, \alpha_1}(|\lambda_1|)}{|\Delta|} (1 - t_m)^{\alpha_1 - 1} + 1 \right)$$

$$\times m\Gamma(\alpha_1) E_{\alpha_1, \alpha_1}(|\lambda_1|) \sum_{j=1}^n A_j \left[r_1 + \|\Phi_1\| \right]^{\tau_j} \left[r_2 + \|\Phi_2\| \right]^{\sigma_j}$$

$$+ \left(\frac{\Gamma(\alpha_1) t_1^{\alpha_1 - 1} E_{\alpha_1, \alpha_1}(|\lambda_1|)}{|\Delta|} + \max \left\{ 1, t_1^{\alpha_1 + k_1 + l_1} \right\} \right)$$

$$\times \mathbf{B}(\alpha_1 + l_1, k_1 + 1) E_{\alpha_1, \alpha_1}(|\lambda_1|) \sum_{j=1}^n a_j \left[r_1 + \|\Phi_1\| \right]^{\tau_j} \left[r_2 + \|\Phi_2\| \right]^{\sigma_j}$$

$$= \sum_{j=1}^n P_j \left[r_1 + \|\Phi_1\| \right]^{\tau_j} \left[r_2 + \|\Phi_2\| \right]^{\sigma_j}.$$

Similarly, we have

$$\|T_2(x, y) - \Phi_2\| \le \left(\frac{\Gamma(\alpha_2) t_1^{\alpha_2 - 1} E_{\alpha_2, \alpha_2}(|\lambda_2|)}{|\nabla|} (1 - t_m)^{\alpha_2 - 1} + 1 \right)$$

$$\times m\Gamma(\alpha_2) E_{\alpha_2, \alpha_2}(|\lambda_2|) \sum_{j=1}^n B_j \left[r_1 + \|\Phi_1\| \right]^{\tau_j} \left[r_2 + \|\Phi_2\| \right]^{\sigma_j}$$

$$+ \left(\frac{\Gamma(\alpha_2) t_1^{\alpha_2 - 1} E_{\alpha_2, \alpha_2}(|\lambda_2|)}{|\nabla|} + \max \left\{ 1, t_1^{\alpha_2 + k_2 + l_2} \right\} \right)$$

$$\times \mathbf{B}(\alpha_2 + l_2, k_2 + 1) E_{\alpha_2, \alpha_2}(|\lambda_2|) \sum_{j=1}^n b_j \left[r_1 + \|\Phi_1\| \right]^{\tau_j} \left[r_2 + \|\Phi_2\| \right]^{\sigma_j}$$

$$= \sum_{j=1}^n Q_j \left[r_1 + \|\Phi_1\| \right]^{\tau_j} \left[r_2 + \|\Phi_2\| \right]^{\sigma_j}.$$

From the assumption, the inequality of system (3.14) has positive solution (r_1, r_2). We choose $\Omega = \{(x, y) \in E : \|x - \Phi_1\| \le r_1, \|y - \Phi_2\| \le r_2\}$. Then we get $T(\Omega) \subset \Omega$. Hence the Schauder's fixed point theorem implies that T has a fixed point $(x, y) \in \Omega$. So (x, y) is a solution of BVP (1.3). $\qquad \square$

Remark 3.1. Suppose that (D2) holds. Fix $r_1 > 0$ and $r_2 > 0$. Since

$$\lim_{A_j,a_j \to 0} \sum_{j=1}^{n} Q_j \left[r_1 + \|\Phi_1\| \right]^{\tau_j} \left[r_2 + \|\Phi_2\| \right]^{\sigma_j} = \lim_{B_j,b_j \to 0} \sum_{j=1}^{n} P_j \left[r_1 + \|\Phi_1\| \right]^{\tau_j} \left[r_2 + \|\Phi_2\| \right]^{\sigma_j} = 0,$$

(3.14) holds for sufficiently small nonnegative constants b_i, a_i $(i = 1, 2, \ldots, n)$, B_i, A_i $(i = 1, 2, \ldots, n)$. So it is easy to see that BVP (1.3) has at least one solution if the nonnegative constants b_i, a_i $(i = 1, 2, \ldots, n)$, B_i, A_i $(i = 1, 2, \ldots, n)$ are sufficiently small.

4   Solvability of BVP (1.4)

In this section, we study the solvability of BVP (1.4). We will seek for solutions of BVP (1.4) in E_1.

Lemma 4.1. *Suppose that $\sigma \in L^1(0,1)$ and there exist numbers $k > -1$ and $\max\{-\alpha_1, -k-1\} < l \le 0$ such that $|\sigma(t)| \le t^k (1-t)^l$ for all $t \in (0,1)$. Then $x \in X_1$ is a solution of*

$$^c D_{0^+}^{\alpha_1} x(t) - \lambda_1 x(t) = \sigma(t), \quad t \in (t_i, t_{i+1}), \quad i \in N_0, \tag{4.1}$$

if and only if there exist constants c_i $(i \in N_0)$ such that

$$x(t) = \Gamma(\alpha_1) \sum_{j=0}^{i} c_j E_{\alpha_1,1}(\lambda_1 (t - t_j)^{\alpha_1}) + \int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds, \quad t \in (t_i, t_{i+1}], \quad i \in N_0. \tag{4.2}$$

Proof. Suppose that x satisfies (4.2). Firstly, we prove that $\int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds$ is convergent. In fact, we have for $t \in (t_i, t_{i+1}]$ that

$$\left| \int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\,ds \right| = \left| \int_0^t (t-s)^{\alpha_1-1} \mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\,ds \right|$$

$$\le \int_0^t (t-s)^{\alpha_1-1} \mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})|\sigma(s)|\,ds$$

$$\le \int_0^t (t-s)^{\alpha_1-1} \mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1}) s^k (1-s)^l\,ds$$

$$= \sum_{j=0}^{+\infty} \frac{\lambda_1^j}{((j+1)\alpha_1)} \int_0^t (t-s)^{\alpha_1-1}(t-s)^{\alpha_1 j} s^k (1-s)^l\,ds$$

$$\le \sum_{j=0}^{+\infty} \frac{\lambda_1^j}{((j+1)\alpha_1)} \int_0^t (t-s)^{\alpha_1+l-1}(t-s)^{\alpha_1 j} s^k\,ds$$

$$= \sum_{j=0}^{+\infty} \frac{\lambda_1^j}{((j+1)\alpha_1)} t^{\alpha_1+\alpha_1 j+k+l} \int_0^1 (1-w)^{\alpha_1+\alpha_1 j+l-1} w^k\,dw$$

$$\le \sum_{j=0}^{+\infty} \frac{\lambda_1^j t^{\alpha_1 j}}{((j+1)\alpha_1)} t^{\alpha_1+k+l} \int_0^1 (1-w)^{\alpha_1+l-1} w^k\,dw$$

$$= t^{\alpha_1+k+l} \mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1 t^{\alpha_1}) \mathbf{B}(\alpha_1+l, k+1).$$

One knows that $x\big|_{(t_i,t_{i+1}]}$ is continuous. Now, we prove that x satisfies (4.1). In fact, for $t \in (t_0, t_1]$, we have

$$
{}^{c}D_{0^+}^{\alpha_1} x(t) = \frac{1}{(1-\alpha_1)} \int_0^t (t-s)^{-\alpha_1} \Big(c_0 \mathbf{E}_{\alpha_1,1}(\lambda_1(s)^{\alpha_1})
$$

$$
+ \int_0^s (s-v)^{\alpha_1-1} \mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\, dv \Big)' ds
$$

$$
= \frac{1}{(1-\alpha_1)} \int_0^t (t-s)^{-\alpha_1} \Big(c_0 \sum_{j=0}^{+\infty} \frac{\lambda_1^j s^{\alpha_1 j}}{(\alpha_1 j+1)} + \sum_{j=0}^{+\infty} \frac{\lambda_1^j}{((j+1)\alpha_1)} \int_0^s (s-v)^{\alpha_1-1}(s-v)^{\alpha_1 j}\sigma(v)\, dv \Big)' ds
$$

$$
= \frac{c_0}{(1-\alpha_1)} \sum_{j=1}^{+\infty} \int_0^t (t-s)^{-\alpha_1} \frac{\alpha_1 j \lambda_1^j s^{\alpha_1 j-1}}{\Gamma(\alpha_1 j+1)}\, ds
$$

$$
+ \frac{1}{(1-\alpha_1)} \sum_{j=0}^{+\infty} \frac{\lambda_1^j}{((j+1)\alpha_1)} \int_0^t (t-s)^{-\alpha_1} \Big(\int_0^s (s-v)^{\alpha_1+\alpha_1 j-1}\sigma(v)\, dv \Big)' ds
$$

$$
= \frac{c_0}{(1-\alpha_1)} \sum_{j=1}^{+\infty} \frac{\alpha_1 j \lambda_1^j}{(\alpha_1 j+1)} t^{-\alpha_1+\alpha_1 j} \int_0^1 (1-w)^{-\alpha_1} w^{\alpha_1 j-1}\, dw + \sum_{j=0}^{+\infty} \lambda_1^j {}^{c}D_{0^+}^{\alpha_1} I_{0^+}^{\alpha_1+\alpha_1 j}\sigma(t)
$$

$$
= \frac{c_0}{(1-\alpha_1)} \sum_{j=1}^{+\infty} \frac{\alpha_1 j \lambda_1^j}{(\alpha_1 j+1)} t^{-\alpha_1+\alpha_1 j} \frac{(1-\alpha_1)(\alpha_1 j)}{(1+\alpha_1 j-\alpha_1)} + \sigma(t) + \sum_{j=1}^{+\infty} \lambda_1^j \int_0^t \frac{(t-s)^{\alpha_1 j-1}}{(\alpha_1 j)}\sigma(s)\, ds
$$

$$
= \lambda_1 c_0 \sum_{j=1}^{+\infty} \frac{\alpha_1 j \lambda_1^{j-1}}{(\alpha_1 j+1)} t^{\alpha_1(j-1)} \frac{(\alpha_1 j)}{(1+\alpha_1 j-\alpha_1)} + \sigma(t) + \lambda_1 \int_0^t (t-s)^{\alpha_1-1} \sum_{j=1}^{+\infty} \lambda_1^{j-1} \frac{(t-s)^{\alpha_1(j-1)}}{(\alpha_1 j)}\sigma(s)\, ds
$$

$$
= \lambda_1 c_0 \sum_{j=1}^{+\infty} \frac{\lambda_1^{j-1} t^{\alpha_1(j-1)}}{(\alpha_1 j+1)} \frac{(\alpha_1 j+1)}{(\alpha_1(j-1)+1)} + \sigma(t) + \lambda_1 \int_0^t (t-s)^{\alpha_1-1} \mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\, ds
$$

$$
= \lambda_1 x(t) + \sigma(t).
$$

Now, suppose that $t \in (t_i, t_{i+1}]$ $(i \geq 1)$. Similarly we have

$$
{}^{c}D_{0^+}^{\alpha_1} x(t) = \frac{1}{(1-\alpha_1)} \int_0^t (t-s)^{-\alpha_1} x'(s)\, ds
$$

$$
= \frac{1}{(1-\alpha_1)} \Big[\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1} x'(s)\, ds + \int_{t_i}^t (t-s)^{-\alpha_1} x'(s)\, ds \Big]
$$

$$
= \frac{1}{(1-\alpha_1)} \Big[\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1}
$$

$$
\times \Big(\sum_{\kappa=0}^{j} c_\kappa \mathbf{E}_{\alpha_1,1}(\lambda_1(s-t_\kappa)^{\alpha}) + \int_0^s (s-v)^{\alpha_1-1} \mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\, dv \Big)' ds
$$

$$
+ \int_{t_i}^t (t-s)^{-\alpha_1} \Big(\sum_{\kappa=0}^{i} c_\kappa \mathbf{E}_{\alpha_1,1}(\lambda_1(s-t_\kappa)^{\alpha_1}) + \int_0^s (s-v)^{\alpha_1-1} \mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\, dv \Big)' ds \Big]
$$

$$= \frac{1}{(1-\alpha_1)} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1} \Big[\sum_{\kappa=0}^{j} c_\kappa \mathbf{E}_{\alpha_1,1}(\lambda_1(s-t_\kappa)^{\alpha_1}) \Big]' ds$$

$$+ \frac{1}{(1-\alpha_1)} \int_{t_i}^{t} (t-s)^{-\alpha_1} \Big[\sum_{\kappa=0}^{i} c_\kappa \mathbf{E}_{\alpha_1,1}(\lambda_1(s-t_\kappa)^{\alpha_1}) \Big]' ds$$

$$+ \frac{1}{(1-\alpha_1)} \int_{t_0}^{t} (t-s)^{-\alpha_1} \Big[\int_0^s (s-v)^{\alpha_1-1} \mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(s-v)^{\alpha_1})\sigma(v)\,dv \Big]' ds$$

$$= \frac{1}{(1-\alpha_1)} \sum_{j=0}^{i-1} \sum_{\kappa=0}^{j} c_\kappa \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1} \Big[\sum_{m=0}^{+\infty} \frac{\lambda_1^m (s-t_\kappa)^{m\alpha_1}}{(m\alpha_1+1)} \Big]' ds$$

$$+ \frac{1}{(1-\alpha_1)} \sum_{\kappa=0}^{i} c_\kappa \int_{t_i}^{t} (t-s)^{-\alpha_1} \Big[\sum_{m=0}^{+\infty} \frac{\lambda_1^m (s-t_\kappa)^{m\alpha_1}}{(m\alpha_1+1)} \Big]' ds$$

$$+ \frac{1}{(1-\alpha_1)} \sum_{m=0}^{+\infty} \frac{\lambda_1^m}{(\alpha_1(m+1))} \int_0^t (t-s)^{-\alpha_1} \Big[\int_0^s (s-v)^{\alpha_1+m\alpha_1-1}\sigma(v)\,dv \Big]' ds$$

$$= \frac{1}{(1-\alpha_1)} \sum_{j=0}^{i-1} \sum_{\kappa=0}^{j} c_\kappa \sum_{m=1}^{+\infty} \frac{\lambda_1^m m\alpha_1}{(m\alpha_1+1)} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha_1}(s-t_\kappa)^{m\alpha_1-1}\,ds$$

$$+ \frac{1}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{m\alpha_1\lambda_1^m}{(m\alpha_1+1)} \sum_{\kappa=0}^{i} c_\kappa \int_{t_i}^{t} (t-s)^{-\alpha_1}(s-t_\kappa)^{m\alpha_1-1}\,ds + \sum_{m=0}^{+\infty} \lambda_1^{mc} D_{0^+}^{\alpha_1} I_{0^+}^{\alpha_1(m+1)}\sigma(t)$$

$$= \frac{1}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{\lambda_1^m m\alpha_1}{(m\alpha_1+1)} \sum_{j=0}^{i-1} \sum_{\kappa=0}^{j} c_\kappa (t-t_\kappa)^{m\alpha_1-\alpha_1} \int_{\frac{t_j-t_\kappa}{t-t_\kappa}}^{\frac{t_{j+1}-t_\kappa}{t-t_\kappa}} (1-w)^{-\alpha_1}w^{m\alpha_1-1}\,dw$$

$$+ \frac{1}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{m\alpha_1\lambda_1^m}{(m\alpha_1+1)} \sum_{\kappa=0}^{i} c_\kappa (t-t_\kappa)^{\alpha_1 m-\alpha_1} \int_{\frac{t_i-y_\kappa}{t-t_\kappa}}^{1} (1-w)^{-\alpha_1}w^{m\alpha_1-1}\,dw + \sum_{m=0}^{+\infty} \lambda_1^m I_{0^+}^{\alpha_1 m}\sigma(t)$$

$$= \frac{1}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{\lambda_1^m m\alpha_1}{(m\alpha_1+1)} \sum_{\kappa=0}^{i-1} c_\kappa (t-t_\kappa)^{m\alpha_1-\alpha_1} \sum_{j=\kappa}^{i-1} \int_{\frac{t_j-t_\kappa}{t-t_\kappa}}^{\frac{t_{j+1}-t_\kappa}{t-t_\kappa}} (1-w)^{-\alpha_1}w^{m\alpha_1-1}\,dw$$

$$+ \frac{1}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{m\alpha_1\lambda_1^m}{(m\alpha_1+1)} \sum_{\kappa=0}^{i-1} c_\kappa (t-t_\kappa)^{\alpha_1 m-\alpha_1} \int_{\frac{t_i-y_\kappa}{t-t_\kappa}}^{1} (1-w)^{-\alpha_1}w^{m\alpha_1-1}\,dw$$

$$+ \frac{c_i}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{m\alpha_1\lambda_1^m}{(m\alpha_1+1)} \int_0^1 (1-w)^{-\alpha_1}w^{m\alpha_1-1}\,dw + \sigma(t)$$

$$+ \lambda_1 \int_0^t \sum_{m=1}^{+\infty} (t-s)^{\alpha_1-1} \frac{\lambda_1^{m-1}(t-s)^{\alpha_1(m-1)}}{(\alpha_1 m)}\sigma(s)\,ds$$

$$= \frac{1}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{\lambda_1^m m\alpha_1}{(m\alpha_1+1)} \sum_{\kappa=0}^{i-1} c_\kappa (t-t_\kappa)^{m\alpha_1-\alpha_1} \int_0^1 (1-w)^{-\alpha_1}w^{m\alpha_1-1}\,dw$$

$$+ \frac{c_i}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{m\alpha_1 \lambda_1^m}{(m\alpha_1 + 1)} \int_0^1 (1-w)^{-\alpha_1} w^{m\alpha_1 - 1}\, dw + \sigma(t)$$

$$+ \lambda_1 \int_0^t \sum_{m=1}^{+\infty} (t-s)^{\alpha_1 - 1} \frac{\lambda_1^{m-1}(t-s)^{\alpha_1(m-1)}}{(\alpha_1 m)} \sigma(s)\, ds = \lambda_1 x(t) + \sigma(t).$$

Thus, due to induction, we conclude that x satisfies (4.1) if x satisfies (4.2).

We now suppose that x is a solution of (4.1). We will prove that x satisfies (4.2). Since x is continuous on $(t_0, t_1]$ and the limit $\lim_{t \to 0^+} x(t)$ exists, by Theorem 6.11 in [21], there exists $c_0 \in R$ such that

$$x(t) = c_0 \mathbf{E}_{\alpha_1, 1}(\lambda_1 t^{\alpha_1}) + \int_0^t \delta_{\alpha_1, \lambda_1}(t, s) \sigma(s)\, ds, \quad t \in (t_0, t_1].$$

This means that (4.2) holds for $i = 0$. Suppose

$$x(t) = \Phi(t) + c_0 \mathbf{E}_{\alpha_1, 1}(\lambda_1 t^{\alpha_1}) + \int_0^t \delta_{\alpha_1, \lambda_1}(t, s) \sigma(s)\, ds, \quad t \in (t_1, t_2]. \tag{4.3}$$

By ${}^c D_{0+}^{\alpha_1} x(t) - \lambda_1 x(t) = \sigma(t)$, $t \in (t_1, t_2]$, we get

$$\sigma(t) + \lambda_1 x(t) = {}^c D_{0+}^{\alpha_1} x(t) = \frac{1}{(1-\alpha_1)} \int_0^t (t-s)^{-\alpha_1} x'(s)\, ds$$

$$= \frac{1}{(1-\alpha_1)} \left(\int_0^{t_1} (t-s)^{-\alpha_1} x'(s)\, ds + \int_{t_1}^t (t-s)^{-\alpha_1} x'(s)\, ds \right)$$

$$= \frac{1}{(1-\alpha_1)} \left[\int_0^{t_1} (t-s)^{-\alpha_1} \left(c_0 \mathbf{E}_{\alpha_1, 1}(\lambda_1 s^{\alpha_1}) + \int_0^s (s-v)^{\alpha_1 - 1} \mathbf{E}_{\alpha_1, \alpha_1}(\lambda_1 (s-v)^{\alpha_1}) \sigma(v)\, dv \right)' ds \right.$$

$$+ \int_{t_1}^t (t-s)^{-\alpha_1} \left(\Phi(s) + c_0 \mathbf{E}_{\alpha_1, 1}(\lambda_1 s^{\alpha_1}) + \int_0^s (s-v)^{\alpha_1 - 1} \mathbf{E}_{\alpha_1, \alpha_1}(\lambda_1 (s-v)^{\alpha_1}) \sigma(v)\, dv \right)' ds$$

$$= \frac{1}{(1-\alpha_1)} \int_0^{t_1} (t-s)^{-\alpha_1} \left(c_0 \mathbf{E}_{\alpha_1, 1}(\lambda_1 s^{\alpha_1}) \right)' ds + \frac{1}{(1-\alpha_1)} \int_{t_1}^t (t-s)^{\alpha_1 - 1} \left(c_0 \mathbf{E}_{\alpha_1, 1}(\lambda_1 s^{\alpha_1}) \right)' ds$$

$$+ \frac{1}{(1-\alpha_1)} \int_0^t (t-s)^{-\alpha_1} \left(\int_0^s (s-v)^{\alpha_1 - 1} \mathbf{E}_{\alpha_1, \alpha_1}(\lambda_1 (s-v)^{\alpha_1}) \sigma(v)\, dv \right)' ds$$

$$+ \frac{1}{(1-\alpha_1)} \int_{t_1}^t (t-s)^{\alpha_1 - 1} \Phi'(s)\, ds$$

$$= \frac{c_0}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{m\alpha_1 \lambda_1^m}{(m\alpha_1 + 1)} \int_0^{t_1} (t-s)^{-\alpha_1} s^{m\alpha_1 - 1}\, ds$$

$$+ \frac{c_0}{(1-\alpha_1)} \sum_{m=1}^{+\infty} \frac{m\alpha_1 \lambda_1^m}{(m\alpha_1 + 1)} \int_{t_1}^t (t-s)^{-\alpha_1} s^{m\alpha_1 - 1}\, ds$$

$$+ \frac{1}{(1-\alpha_1)} \int_0^t (t-s)^{-\alpha_1} \left(\int_0^s (s-v)^{\alpha_1 - 1} \sum_{m=0}^{+\infty} \frac{\lambda_1^m (s-v)^{\alpha_1 m}}{(\alpha_1 (m+1))} \sigma(v)\, dv \right)' ds$$

$$+ \frac{1}{(1-\alpha_1)} \int_{t_1}^{t} (t-s)^{\alpha_1-1} \Phi'(s)\, ds$$

$$= \sigma(t) + \lambda_1 \int_0^t (t-s)^{\alpha_1-1} \mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\, ds + \lambda_1 c_0 \mathbf{E}_{\alpha_1,1}(\lambda_1 t^{\alpha_1}) + {}^c D_{t_1^+}^{\alpha_1}\Phi(t)$$

$$= \sigma(t) + \lambda_1 x(t) + {}^c D_{t_1^+}^{\alpha_1}\Phi(t) - \lambda_1\Phi(t).$$

It follows that ${}^c D_{t_1^+}^{\alpha_1}\Phi(t) - \lambda_1\Phi(t) = 0$ for all $t \in (t_1, t_2]$. By Theorem 6.11 in [21], there exists $c_1 \in R$ such that $\Phi(t) = c_1 \mathbf{E}_{\alpha,1}(\lambda(t-t_1)^{\alpha})$ for $t \in (t_1, t_2]$. Substituting Φ into (22), we find that (4.2) holds for $i = 2$. Now suppose that (4.2) holds for all $i = 0, 1, 2, \ldots, n \le m-1$. By a similar method used above, we can prove that (4.2) holds for $i = n+1$. Due to induction, x satisfies (4.2), $x|_{(t_i, t_{i+1}]}$ is continuous and $\lim_{t \to t_i^+} x(t)$ exists. □

Lemma 4.2. *Suppose that* $\Delta = 1 - \Gamma(\alpha_1)E_{\alpha_1,1}(\lambda_1) \ne 0$, $\sigma \in L^1(0,1)$ *and that there exist numbers* $k > -1$ *and* $\alpha_1 + k + l > 0$ *such that* $|\sigma(t)| \le t^k(1-t)^l$ *for all* $t \in (0,1)$. *Then* $x \in X$ *is a solution of*

$$\begin{cases} {}^c D_{0^+}^{\alpha_1} x(t) - \lambda_1 x(t) = \sigma(t), & t \in (t_i, t_{i+1}), \ i \in N_0, \\ x(1) - \lim_{t \to 0} x(t) = 0, \\ \lim_{t \to t_i^+} x(t) = I_i, & i \in N, \end{cases} \tag{4.4}$$

if and only if $x \in X$ *and*

$$x(t) = \frac{\Gamma(\alpha_1)^2 E_{\alpha_1,1}(\lambda_1 t^{\alpha_1})}{\Delta} \sum_{j=1}^m I_j E_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1}) + \frac{\Gamma(\alpha_1)E_{\alpha_1,1}(\lambda_1 t^{\alpha_1})}{\Delta} \int_0^1 \delta_{\alpha_1,\lambda_1}(1,s)\sigma(s)\, ds$$

$$+ \Gamma(\alpha_1) \sum_{j=1}^i I_j E_{\alpha_1,1}(\lambda_1(t-t_j)^{\alpha_1}) + \int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\, ds, \ t \in (t_i, t_{i+1}], \ i \in N_0. \tag{4.5}$$

Proof. Let x be a solution of (4.4). Similarly to Step 2 in Lemma 4.1 since $l \le 0$ for $t \in (0, t_1]$, one can see that

$$\left| \int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\, ds \right| \le t^{\alpha_1+l+k}\mathbf{B}(\alpha_1+l, k+1)E_{\alpha_1,\alpha_1}(\lambda_1 t^{\alpha_1}).$$

From $\alpha_1 + k + l > 0$, we get

$$\lim_{t \to 0} \left| \int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})\sigma(s)\, ds \right| = 0. \tag{4.6}$$

Due to Lemma 4.1, there exist constants $A_i \in R(i \in N_0)$ such that

$$x(t) = \Gamma(\alpha_1) \sum_{j=0}^i A_j E_{\alpha_1,1}(\lambda_1(t-t_j)^{\alpha_1}) + \int_0^t \delta_{\alpha_1,\lambda_1}(t,s)\sigma(s)\, ds, \ t \in (t_i, t_{i+1}], \ i \in N_0. \tag{4.7}$$

Boundary conditions and the impulse assumption in (4.4) yield that

$$\Gamma(\alpha_1) \sum_{j=0}^m A_j E_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1}) + \int_0^1 \delta_{\alpha_1,\lambda_1}(1,s)\sigma(s)\, ds - A_0 = 0, \quad A_i = I_i, \ i \in N.$$

It follows that

$$A_0 = \frac{1}{\Delta}\left(\Gamma(\alpha_1)\sum_{j=1}^{m} I_j E_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1}) + \int_0^1 \delta_{\alpha_1,\lambda_1}(1,s)\sigma(s)\,ds\right).$$

Substituting A_i ($i \in N_0$) into (4.7), we obviously get (4.5) by changing the term order.

On the other hand, if x satisfies (4.5), then it is easy to see that $x|_{(t_i,t_{i+1}]}$ is continuous and the limit $\lim_{t\to t_{i+}} x(t)$ exists for $i \in N_0$. Using Lemma 4.1, we can prove that x satisfies (4.4). $\qquad\square$

Lemma 4.3. *Suppose that $\nabla = 1 - \Gamma(\alpha_2)E_{\alpha_2,1}(\lambda_2) \neq 0$, $\sigma \in L^1(0,1)$ and there exist numbers $k > -1$ and $\max\{-\alpha_2, -k-1\} < l \leq 0$ such that $|\sigma(t)| \leq t^k(1-t)^l$ for all $t \in (0,1)$. Then $x \in Y$ is a solution of*

$$\begin{cases} {}^c D_{0^+}^{\alpha_2} x(t) - \lambda_2 x(t) = \sigma(t), \quad t \in (t_i, t_{i+1}), \ i \in N_0, \\ x(1) - \lim_{t\to 0} x(t) = 0, \\ \lim_{t\to t_i^+} x(t) = J_i, \quad i \in N, \end{cases} \tag{4.8}$$

if and only if $x \in Y$ and

$$x(t) = \frac{\Gamma(\alpha_2)^2 E_{\alpha_2,1}(\lambda_2 t^{\alpha_2})}{\nabla}\sum_{j=1}^{m} J_j E_{\alpha_2,1}(\lambda_2(1-t_j)^{\alpha_2}) + \frac{\Gamma(\alpha_2)E_{\alpha_2,1}(\lambda_2 t^{\alpha_2})}{\nabla}\int_0^1 \delta_{\alpha_2,\lambda_2}(1,s)\sigma(s)\,ds$$

$$+ \Gamma(\alpha_2)\sum_{j=1}^{i} J_j E_{\alpha_2,1}(\lambda_2(t-t_j)^{\alpha_2}) + \int_0^t \delta_{\alpha_2,\lambda_2}(t,s)\sigma(s)\,ds, \quad t \in (t_i, t_{i+1}], \ i \in N_0. \tag{4.9}$$

Proof. The proof is similar to that of Lemma 4.2 and we omit it here. $\qquad\square$

Let Δ, ∇ be defined as in Lemmas 4.2 and 4.3. Define the nonlinear operator T on E by $T(x,y)(t) = ((T_1(x,y))(t), (T_2(x,y))(t))$ for $(x,y) \in E$ with

$$(T_1(x,y))(t) = \frac{\Gamma(\alpha_1)^2 E_{\alpha_1,1}(\lambda_1 t^{\alpha_1})}{\Delta}\sum_{j=1}^{m} E_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1})I(t_j, x(t_j), y(t_j))$$

$$+ \frac{\Gamma(\alpha_1)E_{\alpha_1,1}(\lambda_1 t^{\alpha_1})}{\Delta}\int_0^1 \delta_{\alpha_1,\lambda_1}(1,s)p_3(s)f_3(s, x(s), y(s))\,ds$$

$$+ \Gamma(\alpha_1)\sum_{j=1}^{i} E_{\alpha_1,1}(\lambda_1(t-t_j)^{\alpha_1})I(t_j, x(t_j), y(t_j))$$

$$+ \int_0^t \delta_{\alpha_1,\lambda_1}(t,s)p_3(s)f_3(s, x(s), y(s))\,ds, \quad t \in (t_i, t_{i+1}], \ i \in N_0,$$

$$(T_2(x,y))(t) = \frac{\Gamma(\alpha_2)^2 E_{\alpha_2,1}(\lambda_2 t^{\alpha_2})}{\nabla}\sum_{j=1}^{m} E_{\alpha_2,1}(\lambda_2(1-t_j)^{\alpha_2})J(t_j, x(t_j), y(t_j))$$

$$+ \frac{\Gamma(\alpha_2)E_{\alpha_2,1}(\lambda_2 t^{\alpha_2})}{\nabla}\int_0^1 \delta_{\alpha_2,\lambda_2}(1,s)p_4(s)f_4(s, x(s), y(s))\,ds$$

$$+ \Gamma(\alpha_2)\sum_{j=1}^{i} E_{\alpha_2,1}(\lambda_2(t-t_j)^{\alpha_2})J(t_j, x(t_j), y(t_j))$$

$$+ \int_0^t \delta_{\alpha_2,\lambda_2}(t,s)p_4(s)f_4(s, x(s), y(s))\,ds, \quad t \in (t_i, t_{i+1}], \ i \in N_0.$$

Lemma 4.4. *Suppose that f_3, f_4 are* **II-Carathéodory functions***, I, J discrete* **II-Carathéodory** *functions, Then $(x, y) \in E$ is a solution of BVP (1.4) if and only if (x, y) is a fixed point of T in E_1 and $T : E_1 \to E_1$ is well defined and completely continuous.*

Proof. The proof is similar to that of of Lemma 3.4 and is omitted. $\qquad\qquad\qquad\qquad\square$

Now, we prove the main theorem in this section by using the Schauder's fixed point theorem [30]. We need the following assumptions:

(D3) f_3, f_4 are **II-Carathéodory functions**, I, J are discrete II-Carathéodory functions and there exist non-decreasing functions $\phi_i, \psi_i : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$, measurable functions $\phi_i : (0, 1) \to R$ $(i = 1, 2)$ and constants I_j, J_j such that

$$\left|f_3(t, x, y) - \phi_3(t)\right| \leq \overline{\phi}_3(x, y), \quad t \in (t_i, t_{i+1}), \quad x, y \in R, \quad i \in N_0,$$

$$\left|f_4(t, x, y) - \phi_4(t)\right| \leq \overline{\phi}_4(x, y), \quad t \in (t_i, t_{i+1}), \quad x, y \in R, \quad i \in N_0,$$

$$\left|I(t_j, x, y) - I_j\right| \leq \overline{\psi}_3(x, y), \quad j \in N, \quad x, y \in R,$$

$$\left|J(t_j, x, y) - J_j\right| \leq \overline{\psi}_4(x, y), \quad j \in N, \quad x, y \in R.$$

(D4) f_3, f_4 are **II-Carathéodory functions**, I, J are discrete II-Carathéodory functions and there exist nonnegative constants I_i, J_i, b_i, a_i, B_i, A_i, τ_j, σ_j $(j = 1, 2, \ldots, n)$ and measurable functions $\phi_i : (0, 1) \to R$ $(i = 1, 2)$ such that

$$\left|f_3(t, x, y) - \phi_1(t)\right| \leq \sum_{j=1}^{n} a_j |x|^{\tau_j} |y|^{\sigma_j}, \quad t \in (t_i, t_{i+1}), \quad x, y \in R, \quad i \in N_0,$$

$$\left|f_4(t, x, y) - \phi_2(t)\right| \leq \sum_{j=1}^{n} b_j |x|^{\tau_j} |y|^{\sigma_j}, \quad t \in (t_i, t_{i+1}), \quad x, y \in R, \quad i \in N_0,$$

$$\left|I(t_j, x, y) - I_j\right| \leq \sum_{j=1}^{n} A_j |x|^{\tau_j} |y|^{\sigma_j}, \quad j \in N, \quad x, y \in R,$$

$$\left|J(t_j, x, y) - J_j\right| \leq \sum_{j=1}^{n} A_j |x|^{\tau_j} |y|^{\sigma_j}, \quad j \in N, \quad x, y \in R.$$

Define

$$\Phi_1(t) = \frac{\Gamma(\alpha_1)^2 E_{\alpha_1,1}(\lambda_1 t^{\alpha_1})}{\Delta} \sum_{j=1}^{m} E_{\alpha_1,1}(\lambda_1 (1 - t_j)^{\alpha_1}) I_j$$

$$+ \frac{\Gamma(\alpha_1) E_{\alpha_1,1}(\lambda_1 t^{\alpha_1})}{\Delta} \int_0^1 \delta_{\alpha_1,\lambda_1}(1, s) p_1(s) \phi_1(s) \, ds + \Gamma(\alpha_1) \sum_{j=1}^{i} E_{\alpha_1,1}(\lambda_1 (t - t_j)^{\alpha_1}) I_j$$

$$+ \int_0^t \delta_{\alpha_1,\lambda_1}(t, s) p_1(s) \phi_1(s) \, ds, \quad t \in (t_i, t_{i+1}], \quad i \in N_0,$$

$$\Phi_2(t) = \frac{\Gamma(\alpha_2)^2 E_{\alpha_2,1}(\lambda_2 t^{\alpha_2})}{\nabla} \sum_{j=1}^{m} E_{\alpha_2,1}(\lambda_2 (1 - t_j)^{\alpha_2}) J_j$$

$$+ \frac{\Gamma(\alpha_2) E_{\alpha_2,1}(\lambda_2 t^{\alpha_2})}{\nabla} \int_0^1 \delta_{\alpha_2,\lambda_2}(1, s) p_2(s) \phi_2(s) \, ds + \Gamma(\alpha_2) \sum_{j=1}^{i} E_{\alpha_2,1}(\lambda_2 (t - t_j)^{\alpha_2}) J_j$$

$$+ \int_0^t \delta_{\alpha_2,\lambda_2}(t, s) p_2(s) \phi_2(s) \, ds, \quad t \in (t_i, t_{i+1}], \quad i \in N_0.$$

Denote

$$P_1 = \frac{\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)}{|\Delta|} \mathbf{B}(\alpha_1 + l_1, k_1 + 1)E_{\alpha_1,1}(|\lambda_1|) + \mathbf{B}(\alpha_1 + l_1, k_1 + 1)E_{\alpha_1,1}(|\lambda_1|),$$

$$Q_1 = \frac{\Gamma(\alpha_1)^2 E_{\alpha_1,1}(|\lambda_1|}{|\Delta|} mE_{\alpha_1,1}(|\lambda_1|) + m\Gamma(\alpha_1)E_{\alpha_1,\alpha_1}(|\lambda_1|),$$

$$P_2 = \frac{\Gamma(\alpha_2)E_{\alpha_2,1}(|\lambda_2|)}{|\nabla|} \mathbf{B}(\alpha_2 + l_2, k_2 + 1)E_{\alpha_2,1}(|\lambda_2|) + \mathbf{B}(\alpha_2 + l_2, k_2 + 1)E_{\alpha_2,1}(|\lambda_2|),$$

$$Q_2 = \frac{\Gamma(\alpha_2)^2 E_{\alpha_2,1}(|\lambda_2|}{|\nabla|} mE_{\alpha_2,1}(|\lambda_2|) + m\Gamma(\alpha_2)E_{\alpha_2,1}(|\lambda_2|).$$

Theorem 4.1. *Suppose that* (D3) *holds. Then BVP* (1.4) *has at least one solution if*

$$P_1\overline{\phi}_3\big(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\big) + Q_1\overline{\psi}_3\big(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\big) \leq r_1,$$
$$P_2\overline{\phi}_4\big(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\big) + Q_2\overline{\psi}_4\big(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\big) \leq r_2 \tag{4.10}$$

has a couple of positive solutions (r_1, r_2).

Proof. To apply the Schauder's fixed point theorem, we have to define a closed convex bounded subset Ω of E such that $T(\Omega) \subseteq \Omega$.

Let $r_1 > 0$, $r_2 > 0$, denote $\Omega = \{(x, y) \in E : \|x - \Phi_1\| \leq r_1, \|y - \Phi_2\| \leq r_2\}$. For $(x, y) \in \Omega$, we get

$$\|x\| \leq \|x - \Phi_1\| + \|\Phi_1\| \leq r_1 + \|\Phi_1\|, \quad \|y\| \leq \|y - \Phi_2\| + \|\Phi_2\| \leq r_2 + \|\Phi_2\|.$$

Then (D3) implies that

$$\big|f_3(t, x(t), y(t)) - \phi_3(t)\big| \leq \overline{\phi}_3\big(|x(t)|, |y(t)|\big) \leq \phi_1\big(\|x\|, \|y\|\big)$$
$$\leq \overline{\phi}_3\big(r_1 + \|\Phi_2\|, r_2 + \|\Phi_2\|\big), \quad t \in (t_i, t_{i+1}), \quad i \in N_0,$$
$$\big|f_4(t, x(t), y(t)) - \phi_4(t)\big| \leq \overline{\phi}_4\big(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\big), \quad t \in (t_i, t_{i+1}), \quad i \in N_0,$$
$$\big|I(t_j, x(t_j), y(t_j)) - I_j\big| \leq \overline{\psi}_3\big(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\big), \quad j \in N,$$
$$\big|J(t_j, x(t_j), y(t_j)) - J_j\big| \leq \overline{\psi}_4\big(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|\big), \quad j \in N.$$

By the definition of T, using the above inequalities and the first inequality in the proof of Lemma 4.2, we get

$$\|T_1(x, y) - \Phi_1\| \leq \frac{\Gamma(\alpha_1)^2 E_{\alpha_1,1}(\lambda_1 t^{\alpha_1})}{|\Delta|} \sum_{j=1}^{m} E_{\alpha_1,1}(\lambda_1(1 - t_j)^{\alpha_1})\big|I(t_j, x(t_j), y(t_j)) - I_j\big|$$

$$+ \frac{\Gamma(\alpha_1)E_{\alpha_1,1}(\lambda_1 t^{\alpha_1})}{|\Delta|} \int_0^1 \delta_{\alpha_1,\lambda_1}(1, s)|p_3(s)|\big|f_3(s, x(s), y(s)) - \phi_3(s)\big|\, ds$$

$$+ \Gamma(\alpha_1) \sum_{j=1}^{i} E_{\alpha_1,1}(\lambda_1(t - t_j)^{\alpha_1})\big|I(t_j, x(t_j), y(t_j)) - I_j\big|$$

$$+ \int_0^t \delta_{\alpha_1,\lambda_1}(t, s)|p_3(s)|\big|f_3(s, x(s), y(s)) - \phi_3(s)\big|\, ds$$

$$\leq \frac{\Gamma(\alpha_1)^2 E_{\alpha_1,1}(|\lambda_1|}{|\Delta|} mE_{\alpha_1,1}(|\lambda_1|)\big(\overline{\psi}_3(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|)\big)$$

$$+ \frac{\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)}{|\Delta|} \int_0^1 \delta_{\alpha_1,\lambda_1}(1, s)|p_3(s)|\, ds(v)$$

$$+ m\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)\big(\overline{\psi}_3(r_1 + \|\Phi_1\|, r_2 + \|\Phi_2\|)\big)$$

$$+ \int_0^t \delta_{\alpha_1,\lambda_1}(t,s)|p_3(s)|\,ds\,\big(\overline{\phi}_3(r_1+\|\Phi_2\|,r_2+\|\Phi_2\|)\big)$$

$$\leq \frac{\Gamma(\alpha_1)^2 E_{\alpha_1,1}(|\lambda_1|)}{|\Delta|}\,mE_{\alpha_1,1}(|\lambda_1|)\big(\overline{\psi}_3(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|)\big)$$

$$+ \frac{\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)}{|\Delta|}\,\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,1}(|\lambda_1|)\big(\overline{\phi}_3(r_1+\|\Phi_2\|,r_2+\|\Phi_2\|)\big)$$

$$+ m\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)\big(\overline{\psi}_3(r_1+\|\Phi_1\|,\ r_2+\|\Phi_2\|)\big)$$

$$+ t^{\alpha_1+l_1+k_1}\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,1}(\lambda_1 t^{\alpha_1})\big(\overline{\phi}_3(r_1+\|\Phi_2\|,r_2+\|\Phi_2\|)\big)$$

$$\leq P_1\overline{\phi}_3\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big)+Q_1\overline{\psi}_3\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big).$$

Similarly, we have

$$\|T_2(x,y)-\Phi_2\| \leq P_2\overline{\phi}_4\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big)+Q_2\overline{\psi}_4\big(r_1+\|\Phi_1\|,r_2+\|\Phi_2\|\big).$$

From the assumption, the inequality of system (4.10) has a positive solution (r_1,r_2). We choose $\Omega = \{(x,y) \in E : \|x-\Phi_1\| \leq r_1,\ \|y-\Phi_2\| \leq r_2\}$. Then we get $T(\Omega) \subset \Omega$. Hence the Schauder's fixed point theorem implies that T has a fixed point $(x,y) \in \Omega$. So (x,y) is a solution of BVP (1.4). $\qquad\square$

Denote

$$P_j = \Big(\frac{\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)}{|\Delta|}+1\Big)m\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)A_j$$

$$+ \Big(\frac{\Gamma(\alpha_1)E_{\alpha_1,1}(|\lambda_1|)}{|\Delta|}+1\Big)\mathbf{B}(\alpha_1+l_1,k_1+1)E_{\alpha_1,1}(|\lambda_1|)a_j,\ \ j=1,2,\ldots,n,$$

$$Q_j = \Big(\frac{\Gamma(\alpha_2)E_{\alpha_2,1}(|\lambda_2|)}{|\nabla|}+1\Big)m\Gamma(\alpha_2)E_{\alpha_2,1}(|\lambda_2|)B_j$$

$$+ \Big(\frac{\Gamma(\alpha_2)E_{\alpha_2,1}(|\lambda_2|)}{|\nabla|}+1\Big)\mathbf{B}(\alpha_2+l_2,k_2+1)E_{\alpha_2,1}(|\lambda_2|)b_j,\ \ j=1,2,\ldots,n.$$

Theorem 4.2. *Suppose that* (D4) *holds. Then BVP* (1.4) *has at least one solution if*

$$\sum_{j=1}^n P_j\big[r_1+\|\Phi_1\|\big]^{\tau_j}\big[r_2+\|\Phi_2\|\big]^{\sigma_j} \leq r_1,\quad \sum_{j=1}^n Q_j\big[r_1+\|\Phi_1\|\big]^{\tau_j}\big[r_2+\|\Phi_2\|\big]^{\sigma_j} \leq r_2 \qquad (4.11)$$

has a couple of positive solutions (r_1,r_2).

Proof. The proof is similar to that of Lemma 3.2 and is omitted. $\qquad\square$

Remark 4.1. Suppose that (D4) holds. Fix $r_1 > 0$ and $r_2 > 0$. Since

$$\lim_{A_j,a_j \to 0} \sum_{j=1}^n Q_j\big[r_1+\|\Phi_1\|\big]^{\tau_j}\big[r_2+\|\Phi_2\|\big]^{\sigma_j} = \lim_{B_j,b_j \to 0} \sum_{j=1}^n P_j\big[r_1+\|\Phi_1\|\big]^{\tau_j}\big[r_2+\|\Phi_2\|\big]^{\sigma_j} = 0,$$

(4.11) holds for sufficiently small nonnegative constants b_i, a_i $(i=1,2,\ldots,n)$, B_i, A_i $(i=1,2,\ldots,n)$, we know that BVP (1.4) has at least one solution if the nonnegative constants b_i, a_i $(i=1,2,\ldots,n)$, B_i, A_i $(i=1,2,\ldots,n)$ are sufficiently small.

5   Examples

To illustrate the usefulness of our main results, we present two examples to see that Theorems 3.1 and 4.1 can be readily applied.

Example 5.1. Consider the following impulsive boundary value problem

$$
\begin{cases}
D_{0+}^{\frac{2}{5}}u(t) - u(t) = q_1(t)\big[c_1 + a_1\big[(t - t_i)^{\frac{3}{5}}u(t)\big]^{\sigma} + a_2\big[(t - t_i)^{\frac{2}{5}}v(t)\big]^{\sigma}\big], & i = 0,1,2,3, \\
D_{0+}^{\frac{3}{5}}v(t) - v(t) = q_2(t)\big[c_2 + b_1\big[(t - t_i)^{\frac{3}{5}}u(t)\big]^{\sigma} + b_2\big[(t - t_i)^{\frac{2}{5}}v(t)\big]^{\sigma}\big], & i = 0,1,2,3, \\
u(1) - \lim\limits_{t\to 0} t^{\frac{3}{5}}u(t) = 0, \quad v(1) - \lim\limits_{t\to 0} t^{\frac{2}{5}}v(t) = 0, \\
\lim\limits_{t\to t_i^+}(t - t_i)^{\frac{3}{5}}u(t) = \lim\limits_{t\to t_i^+}(t - t_i)^{\frac{2}{5}}v(t) = 0, & i = 1,2,3,
\end{cases}
\tag{5.1}
$$

where $c_1, c_2 \in R$, $b_1, a_1, b_2, a_2 \geq 0$ are constants, $0 = t_0 < t_1 = \frac{1}{4} < t_2 = \frac{1}{3} < t_3 = \frac{1}{2} < t_4 = 1$ with $m = 3$, $q_1(t) = q_2(t) = t^{-\frac{1}{15}}(1 - t)^{-\frac{1}{15}}$, $t \in (0, 1)$, $\sigma > 0$.

We apply Theorem 3.2 to get solutions of BVP (5.1). Corresponding to BVP (1.3), we have $\alpha_1 = \frac{2}{5}$, $\alpha_2 = \frac{3}{5}$, $\lambda_1 = \lambda_2 = 1$, q_1, q_2 satisfy the condition $|q_i(t)| \leq t^{k_i}(1 - t)^{l_i}$ with $k_i = l_i = -\frac{1}{15}$ $(i = 1, 2)$, f_1, f_2, I, J satisfy the following items:

$$
f_1\big(t, (t - t_i)^{-\frac{3}{5}}x, (t - t_i)^{-\frac{2}{5}}y\big) = c_1 + a_1 x^{\sigma} + a_2 y^{\sigma}, \quad t \in (t_i, t_{i+1}], \quad i = 0,1,2,3,
$$
$$
f_2\big(t, (t - t_i)^{-\frac{3}{5}}x, (t - t_i)^{-\frac{2}{5}}y\big) = c_2 + b_1 x^{\sigma} + b_2 y^{\sigma}, \quad t \in (t_i, t_{i+1}], \quad i = 0,1,2,3,
$$
$$
I(t_i, x, y) = J(t_i, x, y) = 0, \quad i = 1,2,3.
$$

It is easy to show that f_1, f_2 are I-Carathéodory functions, I, J are discrete I-Carathéodory functions. Furthermore, choose $\phi_1(t) = c_1$ and $\phi_2(t) = c_2$, $I_i = J_i = 0$. It is easy to see

$$
\Big|f_1\big(t, (t - t_i)^{-\frac{3}{5}}x, (t - t_i)^{-\frac{2}{5}}y\big) - \phi_1(t)\Big| \leq a_1|x|^{\sigma} + a_2|y|^{\sigma}, \quad t \in (t_i, t_{i+1}], \quad i = 0,1,2,3,
$$
$$
\Big|f_2\big(t, (t - t_i)^{-\frac{3}{5}}x, (t - t_i)^{-\frac{2}{5}}y\big) - \phi_1(t)\Big| \leq b_1|x|^{\sigma} + b_2|y|^{\sigma}, \quad t \in (t_i, t_{i+1}], \quad i = 0,1,2,3,
$$
$$
\big|I(t_i, x, y) - I_i\big| = \big|J(t_i, x, y) - J_i\big| = 0, \quad i = 1,2,3.
$$

By the Mathlab tool we find that

$$
2 < E_{\frac{2}{5}, \frac{2}{5}}(1) = \sum_{j=0}^{+\infty} \frac{1}{\Gamma(\frac{2}{5}(j+1))} = \frac{1}{\Gamma(\frac{2}{5})} + \frac{1}{\Gamma(\frac{4}{5})} + \frac{1}{\Gamma(\frac{6}{5})} + \cdots + \frac{1}{\Gamma(\frac{30}{5})} + \cdots
$$
$$
\leq 7 + \frac{1}{\Gamma(\frac{32}{5})} + \frac{1}{\Gamma(\frac{34}{5})} + \frac{1}{\Gamma(\frac{36}{5})} + \frac{1}{\Gamma(\frac{38}{5})} + \cdots
$$
$$
\leq 7 + \frac{1}{\Gamma(\frac{32}{5})} + \frac{1}{\Gamma(\frac{34}{5})} + \frac{1}{\Gamma(\frac{36}{5})} + \frac{5}{33\Gamma(\frac{32}{5})} + \frac{5}{35\Gamma(\frac{34}{5})} + \frac{5}{37\Gamma(\frac{36}{5})} + \cdots
$$
$$
\leq 7 + \frac{1}{\Gamma(\frac{32}{5})}\frac{1}{1 - \frac{5}{33}} + \frac{1}{\Gamma(\frac{34}{5})}\frac{1}{1 - \frac{5}{35}} + \frac{1}{\Gamma(\frac{36}{5})}\frac{1}{1 - \frac{5}{37}} \leq 8,
$$
$$
2 < E_{\frac{3}{5}, \frac{3}{5}}(1) = \frac{1}{\Gamma(\frac{3}{5})} + \frac{1}{\Gamma(\frac{6}{5})} + \frac{1}{\Gamma(\frac{9}{5})} + \frac{1}{\Gamma(\frac{12}{5})} + \frac{1}{\Gamma(\frac{15}{5})} + \frac{1}{\Gamma(\frac{21}{5})} + \frac{1}{\Gamma(\frac{24}{5})} + \cdots
$$
$$
\leq 4 + \frac{5}{7}\frac{1}{\Gamma(\frac{7}{5})} + \frac{5}{10}\frac{5}{5}\frac{1}{\Gamma(\frac{5}{5})} + \frac{5}{13}\frac{5}{8}\frac{1}{\Gamma(\frac{8}{5})} + \frac{5}{16}\frac{5}{11}\frac{5}{6}\frac{1}{\Gamma(\frac{6}{5})}
$$
$$
+ \frac{5}{19}\frac{5}{14}\frac{5}{9}\frac{1}{\Gamma(\frac{9}{5})} + \frac{5}{22}\frac{5}{17}\frac{5}{12}\frac{5}{7}\frac{1}{\Gamma(\frac{7}{5})} + \cdots
$$
$$
\leq 4 + \frac{5}{7}\frac{1}{\Gamma(\frac{7}{5})} + \frac{5}{10}\frac{1}{\Gamma(\frac{5}{5})} + \frac{5}{7}\frac{5}{7}\frac{1}{\Gamma(\frac{7}{5})} + \frac{5}{10}\frac{5}{10}\frac{1}{\Gamma(\frac{5}{5})}
$$
$$
+ \frac{5}{7}\frac{5}{7}\frac{5}{7}\frac{1}{\Gamma(\frac{7}{5})} + \frac{5}{10}\frac{5}{10}\frac{5}{10}\frac{1}{\Gamma(\frac{5}{5})} + \cdots
$$
$$
= 4 + \frac{1}{\Gamma(\frac{7}{5})}\frac{5}{7 - 5} + \frac{1}{\Gamma(\frac{5}{5})}\frac{5}{10 - 5} \leq 8.
$$

By direct computation, using $\alpha_j + k_j + l_j > 0$, we get

$$P_j = \left(\frac{\Gamma(\frac{2}{5})\sqrt[5]{64}\,E_{\frac{2}{5},\frac{2}{5}}(1)}{|1-\Gamma(\frac{2}{5})E_{\frac{2}{5},\frac{2}{5}}(1)|} + 1\right)\mathbf{B}\left(\frac{1}{3},\frac{14}{15}\right)E_{\frac{2}{5},\frac{2}{5}}(1)a_j < 1035a_j, \quad j = 1, 2,$$

$$Q_j = \left(\frac{\Gamma(\frac{3}{5})\sqrt[5]{16}\,E_{\frac{3}{5},\frac{3}{5}}(1)}{|1-\Gamma(\frac{3}{5})E_{\frac{3}{5},\frac{3}{5}}(1)|} + 1\right)\mathbf{B}\left(\frac{8}{15},\frac{14}{15}\right)E_{\frac{3}{5},\frac{3}{5}}(1)b_j < 16b_j, \quad j = 1, 2.$$

One finds that

$$\Phi_1(t) = c_1 \frac{\Gamma(\alpha_1)\delta_{\alpha_1,\lambda_1}(t,0)}{\Delta}\int_0^1 \delta_{\alpha_1,\lambda_1}(1,s)p_1(s)\,ds + c_1\int_0^t \delta_{\alpha_1,\lambda_1}(t,s)p_1(s)\,ds$$

and

$$\Phi_2(t) = c_2 \frac{\Gamma(\alpha_2)\delta_{\alpha_2,\lambda_2}(t,0)}{\nabla}\int_0^1 \delta_{\alpha_2,\lambda_2}(1,s)p_2(s)\,ds + c_2\int_0^t \delta_{\alpha_2,\lambda_2}(t,s)p_2(s)\,ds.$$

Then

$$\|\Phi_1\| \le |c_1|\left(\frac{\Gamma(\frac{2}{5})\sqrt[5]{64}\,E_{\frac{2}{5},\frac{2}{5}}(1)}{|1-\Gamma(\frac{2}{5})E_{\frac{2}{5},\frac{2}{5}}(1)|} + 1\right)\mathbf{B}\left(\frac{1}{3},\frac{14}{15}\right)E_{\frac{2}{5},\frac{2}{5}}(1) \le 1035|c_1|.$$

We can also get

$$\|\Phi_2\| \le |c_2|\left(\frac{\Gamma(\frac{3}{5})\sqrt[5]{16}\,E_{\frac{3}{5},\frac{3}{5}}(1)}{|1-\Gamma(\frac{3}{5})E_{\frac{3}{5},\frac{3}{5}}(1)|} + 1\right)\mathbf{B}\left(\frac{8}{15},\frac{14}{15}\right)E_{\frac{3}{5},\frac{3}{5}}(1) \le 16|c_2|.$$

Then Theorem 3.2 implies that BVP (5.1) has at least one solution if

$$\begin{aligned}
1035a_1\big[r_1 + 1035|c_1|\big]^\sigma + 1035a_2\big[r_2 + 16|c_2|\big]^\sigma &\le r_1, \\
16b_1\big[r_1 + 1035|c_1|\big]^\sigma + 16b_2\big[r_2 + 16|c_2|\big]^\sigma &\le r_2
\end{aligned} \tag{5.2}$$

has a couple of positive solutions (r_1, r_2). So BVP (5.1) has at least one solution for every $c_1, c_2 \in R$ and sufficiently small a_i, b_i.

Example 5.2. Consider the following impulsive boundary value problem

$$\begin{cases}
{}^cD_{0^+}^{\frac{2}{5}}u(t) - u(t) = q_1(t)\big[c_1 + a_1[u(t)]^\sigma + a_2[v(t)]^\sigma\big], & t \in (t_i, t_{i+1}), \; i = 0,1,2,3, \\
{}^cD_{0^+}^{\frac{3}{5}}v(t) - v(t) = q_2(t)\big[c_2 + b_1[u(t)]^\sigma + a_2[v(t)]^\sigma\big], & t \in (t_i, t_{i+1}), \; i = 0,1,2,3, \\
u(1) - \lim\limits_{t\to 0}u(t) = 0, \quad v(1) - \lim\limits_{t\to 0}v(t) = 0, \\
\lim\limits_{t\to t_i^+}u(t) = \lim\limits_{t\to t_i^+}v(t) = t_i, & i \in N,
\end{cases} \tag{5.3}$$

where $c_1, c_2 \in R$, $b_1, a_1, b_2, a_2 \ge 0$ are constants, $0 = t_0 < t_1 = \frac{1}{4} < t_2 = \frac{1}{3} < t_3 = \frac{1}{2} < t_4 = 1$ with $m = 3$, $q_1(t) = q_2(t) = t^{-\frac{1}{15}}(1-t)^{-\frac{1}{15}}$, $t \in (0,1)$.

We apply Theorem 4.2 to get solutions of BVP (5.3). Corresponding to BVP (1.4), we have $\alpha_i = \frac{2}{5}$, $\beta_i = \frac{3}{5}$, $\lambda_1 = \lambda_2 = 1$, q_1, q_2 satisfy the condition $|q_i(t)| \le t^{k_i}(1-t)^{l_1}$ with $k_i = l_i = -\frac{1}{15}$, f_3, f_4, I, J are defined by $f_3(t,x,y) = c_1 + a_1x^\sigma + a_2y^\sigma$, $f_4(t,x,y) = c_2 + b_1x^\sigma + b_2y^\sigma$ and $I(t_i,x,y) = J(t_i,x,y) = t_i$, $i = 1,2,3$.

It is easy to show that f_3, f_4 are II-Carathéodory functions, I, J are discrete II-Carathéodory functions. Choose $\phi_1(t) = c_1$ and $\phi_2(t) = c_2$, $I_i = J_i = t_i$. It is easy to see

$$\begin{aligned}
\big|f_3(t,x,y) - \phi_1(t)\big| &\le a_1|x|^\sigma + a_2|y|^\sigma, \quad t \in (t_i, t_{i+1}), \; x,y \in R, \\
\big|f_4(t,x,y) - \phi_2(t)\big| &\le b_1|x|^\sigma + b_2|y|^\sigma, \quad t \in (t_i, t_{i+1}), \; x,y \in R,
\end{aligned}$$

$$\left|I(t_i, x, y) - I_i\right| = \left|J(t_i, x, y) - J_i\right| = 0, \quad i \in N, \quad x, y \in R.$$

By the direct computation, we get

$$P_j = \left(\frac{\Gamma(\frac{2}{5})E_{\frac{2}{5},1}(1)}{|1 - \Gamma(\frac{2}{5})E_{\frac{2}{5},1}(1)|} + 1\right)\mathbf{B}\left(\frac{1}{3}, \frac{14}{15}\right)E_{\frac{2}{5},1}(1)a_j < 465a_j, \quad j = 1, 2,$$

$$Q_j = \left(\frac{\Gamma(\frac{3}{5})E_{\frac{3}{5},1}(1)}{|1 - \Gamma(\frac{3}{5})E_{\frac{3}{5},1}(1)|} + 1\right)\mathbf{B}\left(\frac{8}{15}, \frac{14}{15}\right)E_{\frac{3}{5},1}(1)b_j < 203b_j, \quad j = 1, 2.$$

Then Theorem 4.2 implies that BVP (5.3) has at least one solution if (5.2) holds. So BVP (5.3) has at least one solution for every $c_1, c_2 \in R$ and sufficiently small a_i, b_i.

Acknowledgements

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Short Communication

Malkhaz Ashordia

ON THE ANTIPERIODIC PROBLEM FOR SYSTEMS
OF NONLINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. The general theorem (principle of a priori boundedness) on the solvability of the antiperiodic problem for systems of nonlinear generalized ordinary differential equations is given.

რეზიუმე. განზოგადებულ ჩვეულებრივ არაწრფივ დიფერენციალურ განტოლებათა სისტემებისთვის განხილულია ანტიპერიოდული ამოცანა. მოყვანილია ზოგადი თეორემა (აპრიორულ შეფასებათა პრინციპი) ამ ამოცანის ამოხსნადობის შესახებ.

Let n be a natural number, $\omega > 0$ be a real number, $A : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ be a matrix-function with bounded total variation components on every closed interval of the real axis, and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a vector-function belonging to the Carathéodory class corresponding to the matrix-function A on every closed interval of the real axis.

Consider the nonlinear system of generalized ordinary differential equations

$$dx = dA(t) \cdot f(t, x) \tag{1}$$

in the antiperiodic problem

$$x(t + \omega) = -x(t) \ \text{ for } \ t \in \mathbb{R}. \tag{2}$$

We will assume that

$$A(t + \omega) = A(t) + C \ \text{ and } \ f(t + \omega, x) = -f(t, -x) \ \text{ for } \ t \in \mathbb{R}, \ \ x \in \mathbb{R}^n, \tag{3}$$

or

$$A(t + \omega) = -A(t) + C \ \text{ and } \ f(t + \omega, x) = f(t, -x) \ \text{ for } \ t \in \mathbb{R}, \ \ x \in \mathbb{R}^n, \tag{4}$$

where $C \in \mathbb{R}^{n \times n}$ is a constant matrix.

The theorem on the existence of a solution of the problem $(1), (2)$, which will be given below and called the principle of a priori boundedness, generalizes the well known Conti–Opial type theorems (see [6, 7, 12] for the case of ordinary differential equations) and supplements the earlier known criteria of solvability of nonlinear and initial boundary value problems for systems of generalized ordinary differential equations (see [1–5, 11, 13, 14] and the references therein).

Analogous and related questions are investigated in [7–10] (see also the references therein) for the boundary value problems for linear and nonlinear systems of ordinary differential and functional differential equations.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see [1–5, 11, 13, 14] and the references therein).

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$, $[a, b]$ $(a, b \in \mathbb{R})$ is a closed interval.

$\mathbb{R}^{n \times m}$ is the space of all real $n \times m$-matrices $X = (x_{il})_{i,l=1}^{n,m}$ with the norm $\|X\| = \sum\limits_{i,l=1}^{n,m} |x_{il}|$;

$\mathbb{R}_+^{n \times m} = \left\{ (x_{il})_{i,l=1}^{n,m} : x_{il} \geq 0 \ (i = 1, \dots, n; \ l = 1, \dots, m) \right\}$.

$O_{n \times m}$ (or O) is the zero $n \times m$-matrix.

If $X = (x_{il})_{i,l=1}^{n,m} \in \mathbb{R}^{n \times m}$, then $|X| = (|x_{il}|)_{i,l=1}^{n,m}$.

$\mathbb{R}^n = \mathbb{R}^{n \times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}$.

If $X \in \mathbb{R}^{n \times n}$, then $\det X$ is the determinant of X; I_n is the identity $n \times n$-matrix; $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_n$.

$\bigvee\limits_a^b (X)$ is the total variation of the matrix-function $X : \mathbb{R} \to \mathbb{R}^{n \times m}$ on the closed interval $[a, b]$, i.e., the sum of total variations of its components x_{il} $(i = 1, \dots, n; l = 1, \dots, m)$; $V(X)(t) = (v(x_{il})(t))_{i,l=1}^{n,m}$, where $v(x_{il})(0) = 0$, $v(x_{il})(t) = \bigvee\limits_0^t (x_{il})$ for $t > 0$ and $v(x_{il})(t) = -\bigvee\limits_t^0 (x_{il})$ for $t < 0$;

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X : [a, b] \to \mathbb{R}^{n \times m}$ at the point t (we assume $X(t) = X(a)$ for $t \leq a$ and $X(t) = X(b)$ for $t \geq b$, if necessary); $\triangle^- X(t) = X(t) - X(t-)$, $\triangle^+ X(t) = X(t+) - X(t)$;

$\mathrm{BV}([a, b], \mathbb{R}^{n \times m})$ is the set of all matrix-functions of bounded variation $X : [a, b] \to \mathbb{R}^{n \times m}$ (i.e., such that $\bigvee\limits_a^b (X) < +\infty$);

$\mathrm{BV}_s([a, b], \mathbb{R}^{n \times m})$ is the normed space of all $X \in BV([a, b], \mathbb{R}^{n \times m})$ with the norm $\|X\|_s = \sup\{\|X(t)\| : t \in [a, b]\}$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

$I \subset \mathbb{R}$ is an interval.

$C(I, \mathbb{R}^{n \times m})$ is the set of all continuous matrix-functions $X : I \to \mathbb{R}^{n \times m}$.

If B_1 and B_2 are normed spaces, then an operator $g : B_1 \to B_2$ (nonlinear, in general) is positive homogeneous if $g(\lambda x) = \lambda g(x)$ for every $\lambda \in \mathbb{R}_+$ and $x \in B_1$.

An operator $\varphi : \mathrm{BV}([a, b], \mathbb{R}^n) \to \mathbb{R}^n$ is called nondecreasing if for every $x, y \in \mathrm{BV}([a, b], \mathbb{R}^n)$ such that $x(t) \leq y(t)$ for $t \in [a, b]$ the inequality $\varphi(x)(t) \leq \varphi(y)(t)$ holds for $t \in [a, b]$.

If $\alpha : I \to \mathbb{R}$ is a nondecreasing function, then $D_\alpha = \{t \in I : \alpha(t+) - \alpha(t-) \neq 0\}$.

$s_1, s_2, s_c : \mathrm{BV}([a, b], \mathbb{R}) \to \mathrm{BV}([a, b], \mathbb{R})$ are the operators defined by

$$s_1(x)(a) = s_2(x)(a) = 0,$$

$$s_1(x)(t) = \sum_{a < \tau \leq t} \triangle^- x(\tau) \ \text{ and } \ s_2(x)(t) = \sum_{a \leq \tau < t} \triangle^+ x(\tau) \ \text{ for } \ a < t \leq b,$$

and

$$s_c(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \ \text{ for } \ t \in [a, b].$$

If $g : [a, b] \to \mathbb{R}$ is a nondecreasing function, $x : [a, b] \to \mathbb{R}$, then

$$\int\limits_s^t x(\tau)\, dg(\tau) = \int\limits_{]s,t[} x(\tau)\, ds_c(g)(\tau) + \sum_{s < \tau \leq t} x(\tau) \triangle^- g(\tau) + \sum_{s \leq \tau < t} x(\tau) \triangle^+ g(\tau) \ \text{ for } \ a \leq s < t \leq b,$$

where $\int_{]s,t[} x(\tau)\, ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s, t[$ with respect to the measure $\mu(s_c(g))$ corresponding to the function $s_c(g)$; if $a = b$, then we assume $\int_a^b x(t)\, dg(t) = 0$; thus $\int_s^t x(\tau)\, dg(\tau)$ is the Kurzweil–Stieltjes integral (see [11, 13, 14]).

$L([a, b], \mathbb{R}; g)$ is the space of all functions $x : [a, b] \to \mathbb{R}$, measurable and integrable with respect to the measure $\mu(g_c(g))$ for which

$$\sum_{a < t \leq b} |x(t)| \triangle^- g(t) + \sum_{a \leq t < b} |x(t)| \triangle^+ g(t) < +\infty,$$

with the norm $\|x\|_{L,g} = \int_a^b |x(t)| \, dg(t)$.

If $g_j : [a, b] \to \mathbb{R}$ $(j = 1, 2)$ are nondecreasing functions, $g(t) \equiv g_1(t) - g_2(t)$, and $x : [a, b] \to \mathbb{R}$, then

$$\int_s^t x(\tau) \, dg(\tau) = \int_s^t x(\tau) \, dg_1(\tau) - \int_s^t x(\tau) \, dg_2(\tau) \quad \text{for} \ a \le s \le t \le b.$$

If $G = (g_{ik})_{i,k=1}^{l,n} : [a, b] \to \mathbb{R}^{l \times n}$ is a nondecreasing matrix-function and $D \subset \mathbb{R}^{n \times m}$, then $L([a, b], D; G)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a, b] \to D$ such that $x_{kj} \in L([a, b], R; g_{ik})$ $(i = 1, \ldots, l; \ k = 1, \ldots, n; \ j = 1, \ldots, m)$;

$$\int_s^t dG(\tau) \cdot X(\tau) = \left(\sum_{k=1}^n \int_s^t x_{kj}(\tau) dg_{ik}(\tau) \right)_{i,j=1}^{l,m} \quad \text{for} \ a \le s \le t \le b,$$

$$S_j(G)(t) \equiv \left(s_j(g_{ik})(t) \right)_{i,k=1}^{l,n} \ (j = 1, 2) \ \text{ and } \ S_c(G)(t) \equiv \left(s_c(g_{ik})(t) \right)_{i,k=1}^{l,n}.$$

If $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}^{n \times m}$, then $Car([a, b] \times D_1, D_2; G)$ is the Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a, b] \times D_1 \to D_2$ such that for each $i \in \{1, \ldots, l\}$, $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, n\}$:

(i) the function $f_{kj}(\,\cdot\,, x) : I \to D_2$ is $\mu(s_c(g_{ik}))$-measurable for every $x \in D_1$;

(ii) the function $f_{kj}(t, \,\cdot\,) : D_1 \to D_2$ is continuous for $\mu(s_c(g_{ik}))$-almost every $t \in I$ and for every $t \in D_{g_{ik}}$, and $\sup\{|f_{kj}(\,\cdot\,, x)| : \ x \in D_0\} \in L([a, b], \mathbb{R}; g_{ik})$ for every compact $D_0 \subset D_1$.

If $G_j : [a, b] \to \mathbb{R}^{l \times n}$ $(j = 1, 2)$ are nondecreasing matrix-functions, $G(t) \equiv G_1(t) - G_2(t)$, and $X : [a, b] \to \mathbb{R}^{n \times m}$, then

$$\int_s^t dG(\tau) \cdot X(\tau) = \int_s^t dG_1(\tau) \cdot X(\tau) - \int_s^t dG_2(\tau) \cdot X(\tau) \quad \text{for} \ a \le s \le t \le b,$$

$$S_k(G)(t) \equiv S_k(G_1)(t) - S_k(G_2)(t) \ (k = 1, 2), \ \ S_c(G)(t) \equiv S_c(G_1)(t) - S_c(G_2)(t).$$

If $G_1(t) \equiv V(G)(t)$ and $G_2(t) \equiv V(G)(t) - G(t)$, then

$$L([a, b], D; G) = \bigcap_{j=1}^2 L([a, b], D; G_j),$$

$$Car([a, b] \times D_1, D_2; G) = \bigcap_{j=1}^2 Car([a, b] \times D_1, D_2; G_j).$$

If $G(t) \equiv \text{diag}(t, \ldots, t)$, then we omit G in the notation containing G.

The inequalities between the vectors and matrices are understood componentwise.

Below we assume that

$$A_1(t) \equiv V(A)(t) \ \text{ and } \ A_2(t) \equiv V(A)(t) - A(t).$$

A vector-function $x : \mathbb{R} \to \mathbb{R}^n$ is said to be a solution of the system (1) if its restriction on every closed interval $[a, b] \subset \mathbb{R}$ belongs to $\text{BV}([a, b], \mathbb{R}^n)$ and

$$x(t) = x(s) + \int_s^t dA(\tau) \cdot f(\tau, x(\tau)) \quad \text{for} \ s \le t.$$

Under a solution of the problem $(1), (2)$ we mean solutions of the system (1) satisfying the condition (2).

Let $B \in \mathrm{BV}([a,b], \mathbb{R}^{n \times n})$, $\eta : [a,b] \to \mathbb{R}^n$ and $q : \mathrm{BV}([a,b], \mathbb{R}^n) \to \mathrm{BV}([a,b], \mathbb{R}^n)$ be a matrix-function, a vector-function and an operator, respectively. Then by a solution of the system of generalized ordinary differential inequalities

$$dx - dB(t) \cdot x \leq d\eta(t) + dq(x) \ (\geq) \ \text{for} \ t \in [a,b]$$

we mean a vector-function $x \in \mathrm{BV}([a,b], \mathbb{R}^n)$ such that

$$x(t) - x(s) - \int_s^t dB(\tau) \cdot x(\tau) \leq \eta(t) - \eta(s) + q(x)(t) - q(x)(s) \ (\geq) \quad \text{for} \ a \leq s \leq t \leq b.$$

In addition, if the vector-function $\eta : [a,b] \to \mathbb{R}^n$ is nondecreasing and $g : \mathrm{BV}([a,b], \mathbb{R}^n) \to \mathrm{BV}([a,b], \mathbb{R}_+^n)$ is a positive homogeneous nondecreasing operator, then by $\Omega_{B,\eta,g}$ we denote a set of all solutions of the system

$$|dx - dB(t) \cdot x| \leq d\eta(t) + dg(|x|).$$

If $\eta(t) \equiv 0$ and q is a trivial operator, then we omit η and q in the notations containing ones. Thus Ω_B is the set of all solutions of the homogeneous system of generalized differential equations

$$dx = dB(t) \cdot x.$$

We define

$$\alpha_l(t) = \sum_{i=1}^n v(a_{il})(t) \ (l = 1, \ldots, n) \ \text{and} \ \alpha(t) = \sum_{i=1}^n \alpha_i(t) \ \text{for} \ t \in \mathbb{R}.$$

Under the conditions (3) or (4), it is not difficult to verify that if a vector-function x is a solution of the system (1), then the vector-function $y(t) = -x(t + \omega)$ $(t \in \mathbb{R})$ will be a solution of the system (1), as well. Indeed, by the definition of a solution of the system, using (3) or (4), we have

$$y(t) - y(s) = -\big(x(t+\omega) - x(s+\omega)\big) = -\int_{s+\omega}^{t+\omega} dA(\tau) \cdot f(\tau, x(\tau))$$

$$= -\int_s^t dA(\tau + \omega) \cdot f\big(\tau + \omega, x(\tau + \omega)\big) = \int_s^t dA(\tau) \cdot f(\tau, y(\tau)) \ \text{for} \ s < t.$$

Therefore, if $x \in \mathrm{BV}([a,b], \mathbb{R}^n)$ is a solution of the system (1) on the closed interval $[0, \omega]$ satisfying the condition

$$x(\omega) = -x(0), \tag{5}$$

then its ω-antiperiodic continuation, i.e. the vector-function $y(t) = (-1)^k x(t - k\omega)$ for $k\omega \leq t < (k+1)\omega$ $(k = 0, \pm 1, \pm 2, \ldots)$, will be a solution of the ω-antiperiodic problem (1), (2).

In connection with this fact we consider the boundary value problem (1), (5) on the closed interval $[0, \omega]$. Below, we will give the sufficient conditions guaranteeing the solvability of the later and hence of the problem (1), (2), as well.

Definition 1. The pair (P, l) of the matrix-function $P \in Car([0, \omega] \times \mathbb{R}^n, \mathbb{R}^{n \times n}; A)$ and the continuous operator $l : \mathrm{BV}_s([0, \omega], \mathbb{R}^n) \times \mathrm{BV}_s([0, \omega], \mathbb{R}^n) \to \mathbb{R}^n$ is said to be consistent if:

(i) for any fixed $x \in \mathrm{BV}_s([0, \omega], \mathbb{R}^n)$, the operator $l(x, \cdot) : \mathrm{BV}_s([0, \omega], \mathbb{R}^n) \to \mathbb{R}^n$ is linear;

(ii) for any $z \in \mathbb{R}^n$, x and $y \in \mathrm{BV}_s([0, \omega], \mathbb{R}^n)$, the inequalities

$$\|P(t, z)\| \leq \xi(t, \|z\|), \ \ \|l(x, y)\| \leq \xi_0(\|x\|_s) \cdot \|y\|_s$$

are fulfilled for $\mu(g_c(\alpha))$-almost all $t \in [0, \omega]$ and for $t \in D_\alpha$, where $\xi_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function, and $\xi : [0, \omega] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing in the second variable function such that $\xi(\cdot, s) \in L([0, \omega], \mathbb{R}_+; \alpha)$ for every $s \in \mathbb{R}_+$;

(iii) there exists a positive number β such that for any $x \in \mathrm{BV}_s([0, \omega], \mathbb{R}^n)$, $q \in L([0, \omega], \mathbb{R}^n; A)$ and $c_0 \in \mathbb{R}^n$, for which the conditions

$$\det\left(I_n - \triangle^- A(t) \cdot P(t, x(t))\right) \neq 0 \ \text{ for } \ t \in [0, \omega],$$
$$\det\left(I_n + \triangle^+ A(t) \cdot P(t, x(t))\right) \neq 0 \ \text{ for } \ t \in [0, \omega]$$

hold, an arbitrary solution x of the boundary value problem

$$dy = dA(t) \cdot \left(P(t, x(t))y + q(t)\right), \ \ l(x, y) = c_0$$

admits the estimate

$$\|y\|_s \leq \beta\left(\|c_0\| + \|q\|_{L,\alpha}\right).$$

Theorem 1. *Let $A \in \mathrm{BV}([0, \omega], \mathbb{R}^{n \times n})$, $f \in Car([0, \omega] \times R^n, R^n; A)$ and let there exist a positive number ρ and a consistent pair (P, l) of the matrix-function $P \in Car([0, \omega] \times \mathbb{R}^n, \mathbb{R}^{n \times n}; A)$ and the continuous operator $l : \mathrm{BV}_s([0, \omega], \mathbb{R}^n) \times \mathrm{BV}_s([0, \omega], \mathbb{R}^n) \to \mathbb{R}^n$ such that an arbitrary solution of the problem*

$$dx = dA(t) \cdot \left(P(t, x)x + \lambda[f(t, x) - P(t, x)]x\right), \tag{6}$$
$$\lambda(x(0) + x(\omega)) + (1 - \lambda)l(x, x) = 0 \tag{7}$$

admits the estimate

$$\|x\|_s \leq \rho \tag{8}$$

for any $\lambda \in]0, 1[$. Then the problem $(1), (2)$ is solvable.

Definition 2. Let $\mathcal{S} \subset \mathrm{BV}_s([0, \omega], \mathbb{R}^{n \times n})$, $\mathcal{L}$ be a subset of the set of all bounded vector-functionals $l : \mathrm{BV}_s([0, \omega], \mathbb{R}^n) \to \mathbb{R}^n$, and $y \in \mathrm{BV}([0, \omega], \mathbb{R}^n)$. We say that:

(i) a matrix-function $B_0 \in \mathrm{BV}([0, \omega], \mathbb{R}^{n \times n})$ belongs to the set $\mathcal{E}_{\mathcal{S}}^n$ if the condition

$$\det\left(I_n - \triangle^- B_0(t)\right) \neq 0 \ \text{ and } \ \det\left(I_n + \triangle^+ B_0(t)\right) \neq 0 \ \text{ for } \ t \in [0, \omega] \tag{9}$$

holds and there exists a sequence $B_k \in \mathcal{S}$ $(k = 1, 2, \dots)$ such that $\lim\limits_{k \to +\infty} \|B_k - B_0\|_s = 0$;

(ii) a vector-functional $l_0 : \mathrm{BV}_s([0, \omega], \mathbb{R}^n) \to \mathbb{R}^n$ belongs to the set $\mathcal{E}_{\mathcal{L}}^n(y)$ if there exists a sequence $l_k \in \mathcal{L}$ $(k = 1, 2, \dots)$ such that $\lim\limits_{k \to +\infty} l_k(y) = l_0(y)$.

Definition 3. Let $g_0 : \mathrm{BV}([0, \omega], \mathbb{R}_+^n) \to \mathrm{BV}([0, \omega], \mathbb{R}^n)$ be a positive homogeneous nondecreasing operator, and $h_0 : \mathrm{BV}_s([0, \omega], \mathbb{R}_+^n) \to \mathbb{R}_+^n$ be a positive homogeneous operator. We say that the pair $(\mathcal{S}, \mathcal{L})$ of a set $\mathcal{S} \subset \mathrm{BV}_s([0, \omega], \mathbb{R}^{n \times n})$ and a set $\mathcal{L}$ of some vector-functionals $l : \mathrm{BV}_s([0, \omega], \mathbb{R}^n) \to \mathbb{R}^n$ belongs to the Opial class $\mathcal{O}_{g_0, h_0}^n$ if:

(i) every operator $l \in \mathcal{L}$ is linear and continuous with respect to the norm $\| \cdot \|_s$;

(ii) there exist the numbers $r_0, \xi_0 \in \mathbb{R}_+$ and a nondecreasing function $\varphi : [0, \omega] \to \mathbb{R}$ such that the inequalities

$$\|B(0)\| \leq r_0, \ \ \|B(t) - B(s)\| \leq \varphi(t) - \varphi(s) \ \text{ for } \ 0 \leq s < t \leq \omega,$$
$$\|l(y)\| \leq \xi_0 \|y\|_s$$

are fulfilled for any $B \in \mathcal{S}$, $l \in \mathcal{L}$ and $y \in \mathrm{BV}_s([0, \omega], \mathbb{R}^n)$;

(iii) for $B_0 \in \mathcal{E}_{\mathcal{S}}^n$, if the function $y \in \mathrm{BV}_s([0, \omega], \mathbb{R}^n)$ is a solution of the system

$$|dy - dB_0(t) \cdot y| \leq dg_0(|y|)$$

under the condition

$$|l_0(y)| \leq h_0(|y|),$$

where $l_0 \in \mathcal{E}_{\mathcal{L}}^n(y)$, then $y(t) \equiv 0$.

If $g_0(y)(t) \equiv \int\limits_0^t dG_0(\tau) \cdot q_0(y)(\tau)$ for $y \in \mathrm{BV}([0,\omega], \mathbb{R}_+^n)$, where $G_0 : [0,\omega] \to \mathbb{R}^n$ is a nondecreasing matrix-function, and $q_0 : \mathrm{BV}_s([0,\omega], \mathbb{R}_+^n) \to \mathrm{BV}_s([0,\omega], \mathbb{R}_+^n)$ is a positive homogeneous operator, then we write $\mathcal{O}_{G_0,q_0,h_0}^n$ instead of $\mathcal{O}_{g_0,h_0}^n$.

Definition 4. Let $P \in Car([0,\omega] \times \mathbb{R}^n, \mathbb{R}^{n \times n}; A)$ and let $l : \mathrm{BV}_s([0,\omega], \mathbb{R}^n) \times \mathrm{BV}_s([0,\omega], \mathbb{R}^n) \to \mathbb{R}^n$ be a continuous vector-functional. We say that the pair (B_0, l_0) of the matrix-function $B_0 \in \mathrm{BV}([0,\omega], \mathbb{R}^{n \times n})$ and the vector-functional $l_0 : \mathrm{BV}_s([0,\omega], \mathbb{R}^n) \to \mathbb{R}^n$ belongs to the set $\mathcal{E}_{A,P,l}^n$ if there exists a sequence $x_k \in \mathrm{BV}_s([0,\omega], \mathbb{R}^n)$ $(k = 1, 2, \dots)$ such that the conditions

$$\lim_{k \to +\infty} \int\limits_a^t dA(\tau) \cdot P(\tau, x_k(\tau)) = B_0(t) \text{ uniformly on } [0,\omega] \tag{10}$$

$$\lim_{k \to +\infty} l(x_k, y) = l_0(y) \text{ for } y \in \Omega_{B_0}$$

are valid.

Definition 5. We say that the pair (P, l) of the matrix-function $P \in Car([0,\omega] \times \mathbb{R}^n, \mathbb{R}^{n \times n}; A)$ and the continuous operator $l : \mathrm{BV}_s([0,\omega], \mathbb{R}^n) \times \mathrm{BV}_s([0,\omega], \mathbb{R}^n) \to \mathbb{R}^n$ belongs to the Opial class $\mathcal{O}_A^n$ with respect to the matrix-function A if:

(i) for any fixed $x \in \mathrm{BV}_s([0,\omega], \mathbb{R}^n)$, the operator $l(x, \cdot) : \mathrm{BV}_s([0,\omega], \mathbb{R}^n) \to \mathbb{R}^n$ is linear;

(ii) for any $z \in \mathbb{R}^n$, x and $y \in \mathrm{BV}_s([0,\omega], \mathbb{R}^n)$, the inequalities

$$\|P(t, z)\| \leq \xi(t), \quad \|l(x, y)\| \leq \xi_0 \|y\|_s \tag{11}$$

are fulfilled for $\mu(g_c(\alpha))$-almost all $t \in [0,\omega]$ and for $t \in D_\alpha$, where $\xi_0 \in R_+$, and $\xi \in L([0,\omega], \mathbb{R}_+; \alpha)$;

(iii) the problem

$$dy = dB_0(t) \cdot y, \quad l_0(y) = 0$$

has only a trivial solution for every pair $(B_0, l_0) \in \mathcal{E}_{A,P,l}^n$.

Remark 1. By (10) and (11), the condition $\|\triangle^- A(t)\| \cdot \xi(t) < 1$ and $\|\triangle^+ A(t)\| \cdot \xi(t) < 1$ for $t \in [0,\omega]$ guarantees the condition (9).

Corollary 1. *Let* $A \in \mathrm{BV}([0,\omega], \mathbb{R}^{n \times n})$, $f \in Car([0,\omega] \times \mathbb{R}^n, \mathbb{R}^n; A)$ *and let there exist a positive number* ρ *and a pair* $(P, l) \in \mathcal{O}_A^n$ *such that an arbitrary solution of problem* (6), (7) *admits the estimate* (8) *for any* $\lambda \in]0, 1[$. *Then the problem* (1), (2) *is solvable.*

Corollary 2. *Let* $A \in \mathrm{BV}([0,\omega], \mathbb{R}^{n \times n})$, $f \in Car([0,\omega] \times \mathbb{R}^n, \mathbb{R}^n; A), P \in L([0,\omega], \mathbb{R}^{n \times n}; A)$, *and let* $l : \mathrm{BV}_s([0,\omega], \mathbb{R}^n) \to \mathbb{R}^n$ *be a bounded linear operator such that*

$$\det \left(I_n - \triangle^- A(t) \cdot P(t) \right) \neq 0 \ \text{ and } \ \det \left(I_n + \triangle^+ A(t) \cdot P(t) \right) \neq 0 \ \text{ for } \ t \in [0,\omega]$$

and the problem

$$dy = dA(t) \cdot P(t)y, \ \ l(y) = 0$$

has only a trivial solution. Let, moreover, there exist a positive number ρ *such that an arbitrary solution of the problem*

$$dx = dA(t) \cdot \left(P(t)x + \lambda[f(t, x) - P(t)x] \right),$$
$$\lambda(x(0) + x(\omega)) + (1 - \lambda)l(x) = 0$$

admits the estimate (8) *for any* $\lambda \in]0, 1[$. *Then the problem* (1), (2) *is solvable.*

The following result is analogous of the well-known one belonging to R. Conti and Z. Opial for boundary value problems for ordinary nonlinear differential equations (see [6, 7, 12]).

Corollary 3. *Let $A \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f \in Car([0,\omega] \times \mathbb{R}^n, \mathbb{R}^n; A)$ and let the pair $(P,l) \in \mathcal{O}_A^n$ be such that*

$$|f(t,x) - P(t,x)x| \le \beta(t, \|x\|) \ \text{ for } \ t \in [0,\omega], \ \ x \in \mathbb{R}^n, \tag{12}$$

$$|x(0) + x(\omega) - l(x,x)| \le l_0(|x|) + l_1(\|x\|_s) \ \text{ for } \ x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n), \tag{13}$$

where $\beta \in Car([0,\omega] \times \mathbb{R}_+, \mathbb{R}_+^n; A)$ is a nondecreasing in the second variable vector-function, $l_0 : \mathrm{BV}_s([0,\omega],\mathbb{R}_+^n) \to \mathbb{R}_+^n$ is a positive homogeneous continuous operator, and $l_1 \in C(\mathbb{R}_+, \mathbb{R}_+^n)$. Let, moreover,

$$\lim_{k\to+\infty} \frac{1}{\rho} \int_a^b dV(A)(\tau) \cdot \beta(\tau, \rho) = 0_n, \quad \lim_{\rho\to+\infty} \frac{l_1(\rho)}{\rho} = 0_n. \tag{14}$$

Then the problem $(1),(2)$ is solvable.

By $Y_P(x)$ we denote the fundamental matrix of the system

$$dy = dA(t) \cdot P(t, x(t))\, y$$

for every $x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$ satisfying the condition $Y_P(x)(a) = I_n$.

Corollary 4. *Let $A \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f \in Car([0,\omega] \times \mathbb{R}^n, \mathbb{R}^n; A)$, $P \in Car([0,\omega] \times \mathbb{R}^n, \mathbb{R}^{n\times n}; A)$ and a continuous operator $l : \mathrm{BV}_s([0,\omega],\mathbb{R}^n) \times \mathrm{BV}_s([0,\omega],\mathbb{R}^n) \to \mathbb{R}^n$, satisfying conditions* (i) *and* (ii) *of Definition 5, be such that the conditions* (12)–(14) *hold, where $\beta \in Car([0,\omega] \times \mathbb{R}_+, \mathbb{R}_+^n; A)$ is a nondecreasing in the second variable vector-function, $l_0 : \mathrm{BV}_s([0,\omega],\mathbb{R}_+^n) \to \mathbb{R}_+^n$ is a positive homogeneous continuous operator, and $l_1 \in C(\mathbb{R}_+, \mathbb{R}_+^n)$. Let, moreover,*

$$\inf\big\{|\det(l(x, Y_P(x)))| : \ x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)\big\} > 0. \tag{15}$$

Then the problem $(1),(2)$ is solvable.

Remark 2. In Corollary 4, the condition (15) cannot be replaced by the condition

$$\det\big(l(x, Y_P(x))\big) \ne 0 \ \text{ for } \ x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n). \tag{16}$$

The corresponding example for ordinary differential systems, i.e., for the case $A(t) \equiv \mathrm{diag}(t,\ldots,t)$ has been constructed in [8]. Basing on this example, it is not difficult to construct analogous examples for the case $A(t) \not\equiv \mathrm{diag}(t,\ldots,t)$. Consider the scalar boundary value problem

$$dx = \Big(\frac{|x|x}{1+|x|} + 1\Big)\, d\alpha(t), \ \ x(0) = -x(\omega),$$

where $\alpha(t) = 0$ for $0 \le t \le c$ and $\alpha(t) = -2$ for $c < t \le \omega$, and $c = \omega/2$. Every solution of the system has the form

$$x(t) = \begin{cases} x(0) & \text{for } \ 0 \le t \le c, \\ x(0) - 2\Big(\dfrac{|x(0)|x(0)}{1+|x(0)|} + 1\Big) & \text{for } \ c < t \le \omega. \end{cases}$$

This problem cannot be solvable because the equation $x(0) + x(\omega) = 0$ is unsolvable with respect to $x(0)$. On the other hand, if we assume $P(t,x) = \frac{|x|}{1+|x|}$ and $l(x,y) = y(0) + y(\omega)$, then

$$Y(t) = \begin{cases} 1 & \text{for } \ 0 \le t \le c, \\ 1 - \dfrac{2|x(c)|}{1+|x(c)|} & \text{for } \ c < t \le \omega \end{cases}$$

for $x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$ and, therefore,

$$\det\big(l(x, Y_P(x))\big) = \frac{2}{1+|x(c)|} \ \text{ for } \ x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n).$$

Consequently, all the conditions of Corollary 4 are fulfilled except of the condition (15), instead of which we have the condition (16).

Remark 3. In particular, in the results obtained above we can assume $l(x, y) \equiv x(0) + x(\omega)$ and $l(x) = l(x, x) \equiv x(0) + x(\omega)$. Thus, for instance, the second estimate in condition (ii) of Definition 1 is fulfilled. The condition (7) in Theorem 1 and Corollary 1 as well as the analogous condition in Corollary 2 coincide with the condition (2). The condition (13) is valid for $l_0 \equiv 0$ and $l_1 \equiv 0$.

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C O N T E N T S

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Short Communication