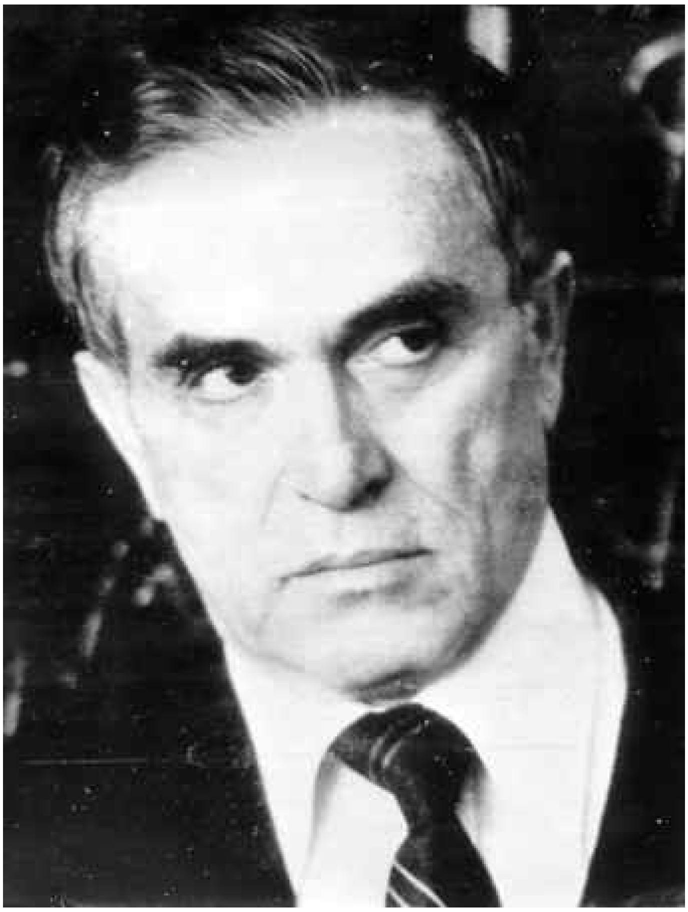
This issue is dedicated
to the memory of Academician Andria Bitsadze
on the occasion of his 100th birthday anniversary



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A Short Survey of Scientific Results of Academician Andria Bitsadze

> "It is too difficult to write about a scientist not only because of the great responsibility toward the history of science, but also because of the complexity of scientific creative process without which it is impossible to show his real personality".
>
> A. Bitsadze

Such an attitude of Andria Bitsadze to the problem cited in the epigraph is not accidental; a task to give an exhaustive description of his versatile activities seems to us insuperable. The true appraisal of human creativity and its crystallization occurs in the future generations. This point of view has been shared by A. Bitsadze. However, his creative work during his lifetime was properly evaluated by the mathematical community. This is confirmed at least by the fact that in the mathematical literature we are often encountered with the facts and terms associated with his name: Bitsadze's equation, Lavrent'ev–Bitsadze's equation, Bitsadze's general mixed problem, Bitsadze's extremum principle, Bitsadze's inversion formula, weakly and strongly connected Bitsadze's elliptic systems, Bitsadze–Samarski's problem, and others. We do not intend to touch upon his organizational, pedagogical or educational work with students, we will dwell only on his scientific results not pretending to present them in a perfect form.

We consider it appropriate to divide Andria Bitsadze's activity into several staged, keeping here chronology.

Elliptic equations and systems together with the problems posed for them take central place in Andria Bitsadze's creative work.

The fact that the condition of uniform ellipticity

$$k_0\Big(\sum_{i=1}^n \lambda_i^2\Big)^N \leq \det \sum_{i,j=1}^n A^{ij}(x)\lambda_i\lambda_j \leq k_1\Big(\sum_{i=1}^n \lambda_i^2\Big)^N, \quad k_0,k_1=const>0,$$

of the linear equation, or of the system

$$L(u):=\sum_{i,j=1}^n A^{ij}(x)\,\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^n B^i(x)\,\frac{\partial u}{\partial x_i}+C(x)u=F(x), \ \ u=(u_1,\dots,u_N)$$

ensures fredholmity of the boundary value problems in the domain D, in particular, of the first boundary value problem

$$u\big|_{\partial D}=f,$$

was assumed formerly indisputable.

Irregularity of this fact was illustrated by A. Bitsadze in a simple and clear for everyone example, called later on Bitsadze's system

$$\frac{\partial^2 u_1}{\partial x^2}-\frac{\partial^2 u_1}{\partial y^2}-2\,\frac{\partial^2 u_2}{\partial x\partial y}=0, \quad 2\,\frac{\partial^2 u_1}{\partial x\partial y}+\frac{\partial^2 u_2}{\partial x^2}-\frac{\partial^2 u_2}{\partial y^2}=0. \tag{1}$$

It turned out that the Dirichlet homogeneous problem for Bitsadze's system in a circular domain $D:\ (x-x_0)^2+(y-y_0)^2<R^2$ has an infinite set of linearly independent solutions, and all of them are representable explicitly by the formula

$$w:=u_1+iu_2=\big(R^2-|z-z_0|^2\big)\psi(z), \ \ z_0=x_0+iy_0$$

written in terms of an arbitrary analytic function $\psi(z)$ of the complex argument $z=x+iy$.

While this fact seemed unexpected and almost improbable, it became a subject of discussions for many mathematicians trying to explain this phenomenon. At his known seminar, I. Gelfand made an attempt to explain this fact by multiplicity of characteristic roots of system (1). In reply, A. Bitsadze has constructed another system

$$\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_1}{\partial y^2} + \sqrt{2}\,\frac{\partial^2 u_2}{\partial x \partial y} = 0, \quad \sqrt{2}\,\frac{\partial^2 u_1}{\partial x \partial y} - \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0 \tag{2}$$

with simple characteristic roots, the system for which the Dirichlet problem has likewise an infinite set of linearly independent solutions

$$w_k(z) = B_k\Big\{\big[(\mu\zeta + \overline{\zeta})^2 - 4\mu R^2\big]^k - (\mu\zeta - \overline{\zeta})^{2k}\Big\}, \;\; k = 1, 2, \dots,$$

where $\zeta = z - z_0$, $(1 + \sqrt{2})\mu = i$, and B_k are arbitrary complex constants. On the basis of those simple and refined examples, the theory of boundary value problems for elliptic systems has acquired a great deal of new trends. The widely known theory of nonfredholm boundary value problems is one of such them. These theories do not lose their importance even nowadays, and many of A. Bitsadze's followers and pupils devote them their researches.

Afterwards, there arose the natural question to single out classes of elliptic systems with solvable, in a certain sense, boundary value problems, in particular, solvable in the Fredholm, Noether or Hausdorff sense. In this direction, it is impossible to hold back about the question on weakly connected Bitsadze's systems for which the Dirichlet problem is always fredholmian one.

It was considered earlier that solvability of boundary value problems is determined only by the principal part of the system. A. Bitsadze has expressed somewhat different opinion that coefficients of the system with lower order derivatives may significantly affect the solvability of the problem. To justify this concept, he introduced the notion of strongly connected elliptic systems that cover systems (1) and (2) constructed earlier in the form of particular examples. As it has become clear, the solvable in one or another sense boundary value problems for elliptic systems with Bitsadze's operators in the principal part may turn out to be unsolvable on adding the lower order terms.

The above-mentioned fundamental effects were discovered by A. Bitsadze by using the apparatus of the theory of functions of a complex variable. This instrument is well suited for a homogeneous system consisting only of the principal part

$$A\,\frac{\partial^2 u}{\partial x^2} + 2B\,\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = 0 \tag{3}$$

with two independent variables. A. Bitsadze has constructed a general regular solution of system (1) in the form

$$u(x, y) = \operatorname{Re} \sum_{j=1}^{n} \sum_{l=1}^{k_j} \sum_{k=0}^{l-1} C_{lkj}\overline{z}_j^{\,k} \varphi_{jl}^{(k)}(z_j),$$

where $\varphi_{jl}(z_j)$ are analytic functions of the complex variable $z_j = x + \lambda_j y$, and λ_j are the roots of the corresponding to system (3) characteristic polynomial $Q(\lambda) = \det(A + 2B\lambda + C\lambda^2)$ with positive imaginary parts. As regards the N-component vectors C_{lkj}, they are the solutions of the fully defined system of linear algebraic equations.

The instruments of the theories of analytic functions and of one-dimensional singular integral equations make it possible to investigate many boundary value problems in the case of two independent variables. If there are more than two variables, then there arise considerable difficulties due to the lack of a complete theory of multidimensional singular integral equations. Using a multidimensional analogue of the Sokhotski–Plemelj theorem, A. Bitsadze has studied the first boundary value problem for the well-known Moisel–Theodorescu system, reduced it to a multidimensional system of singular integral equations with a special matrix kernel and constructed the inversion formula which in the literature is called "Bitsadze's inversion formula".

Among the problems formulated for elliptic equations and systems, even, in particular, for harmonic functions, the problem with an oblique derivative is regarded as one of the basic ones, when on the boundary of the n-dimensional domain D there is the condition

$$\sum_{i=1}^{n} \ell_i(x)\,\frac{\partial u}{\partial x_i} = f(x), \quad x \in \partial D.$$

As far back as in G. Giraud's works it has been shown that if the direction of the vector $\ell := (\ell_1, \dots, \ell_n)$ at none of the boundary points meets the tangent, the problem becomes solvable in Fredholm's sense. Otherwise, the situation changes insomuch that many scientists were inclined to consider this problem atypical for elliptic equations. Considering these nonstandard cases, A. Bitsadze has shown this problem not at all to exceed the bounds of typical problems and proved the theorems on a number and existence of solutions. As it has become clear, the problem with an oblique derivative may turn out to be simultaneously subdefinite and overdetermined. For the problem to be well-posed, it is necessary, proceeding from the structure of interconnection between the vector field ℓ and the domain, to release some set of boundary points from the conditions and impose additional conditions on some set of points. To illustrate this, we consider one simplest example when the vector field meets the boundary at k isolated points. In this case a number of linearly independent solutions of the problem under consideration does not exceed k.

The objects of A. Bitsadze's investigations are not always ordinary. He studied the problems which are, as a rule, not subjected to the standard conditions ensuring the existence and uniqueness of solutions. To such problems may belong those suggested by A. Bitsadze for elliptic equations with parabolic degeneration with weighted conditions on the boundary. These problems were dictated by their practical necessity. For such problems not only the conditions of uniform or strong ellipticity violate, but they degenerate parabolically either on the whole boundary, or on its certain part. In addition, a set of points of parabolic degeneration may turn out to be even a characteristic. For example, for the equation

$$\frac{\partial^2 u}{\partial x^2} + y^m\,\frac{\partial^2 u}{\partial y^2} + a\,\frac{\partial u}{\partial x} + b\,\frac{\partial u}{\partial y} + cu = 0, \quad y > 0, \quad m > 0,$$

the line of degeneration $y = 0$ is simultaneously its multiple characteristic. In such a case, the role of coefficients with the lower order derivatives extends, and depending on them, not all solutions may be bounded. M. Keldysh considered this problem in the class of bounded functions, and hence neglected unbounded solutions. A. Bitsadze replaced the requirement of the boundedness by the following weighted boundary conditions:

$$u\big|_{\sigma} = f, \quad \lim_{y\to 0} \psi(x,y)u(x,y) = \varphi(x), \ \ 0 \le x \le 1,$$

where $\sigma \cup \{y = 0,\ 0 \le x \le 1\}$ is the boundary of the domain, and the weighted function ψ on the line of degeneration vanishes. These problems have brought to light new practical and theoretical validity of weighted functional spaces that before and after formulation of those problems have become the subject of a great number of research works.

The hyperbolic equations and systems aren't less rich with the effects connected with parabolic degeneration. Many factors affect the solvability of the problems formulated here; they include an order of parabolic degeneration, orientation of a set of degeneration points with respect to characteristic manifolds, etc. As distinct from a separately taken equation, hyperbolic systems show a lot of unexpected properties even without parabolic degeneration. Thus, for example, the well-known Goursat problem for a scalar equation is quite well-posed. The constructed by A. Bitsadze hyperbolic system

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + 2\,\frac{\partial^2 u_2}{\partial x \partial y} = 0, \quad 2\,\frac{\partial^2 u_1}{\partial x \partial y} + \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

has shown that the corresponding homogeneous problem may have an infinite set of linearly independent solutions, and what is more, the lower order terms of the system may affect significantly the well-posedness of the problem. This fact has given a great impetus to many important researches and stimulated the development of a series of scientific trends.

In the middle of the past century, mathematics has found new significant applications that should, seemingly, be explained by an unprecedented rate of technical progress. The major achievements in transonic and supersonic velocities have drawn attention of scientists to many problems, including those of mixed type equations in which M. Lavrent'ev has shown spacial interest and awoken it in A. Bitsadze. Combining the methods of the theory of analytic functions, of partial differential equations and singular integral equations, A. Bitsadze created a powerful and, at the same time, elegant apparatus, convenient for solving the problems formulated for the mixed type equations. Effectiveness of the suggested method has been tested on the boundary value problems for the Lavrent'ev–Bitsadze's equation

$$\frac{\partial^2 u}{\partial x^2} + \operatorname{sgn} y \, \frac{\partial^2 u}{\partial y^2} = 0$$

being the model of the well-known Tricomi's equation for which A. Bitsadze posed a great number of actual problems and established a series of significant facts known in the literature as "Bitsadzian facts". Here we will mention only Bitsadze's extremum principle. For the Tricomi's equation

$$y \, \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

along with the Tricomi's problem has also been considered the Dirichlet problem expecting its solvability. This was needed, mainly, for practical, concrete purpose.

A. Bitsadze has shown that this problem was not always well-posed, and for it to be solvable, it was necessary to release a definite part of the boundary of hyperbolic subdomain from the conditions. To formulate the problems responding practical purposes in which the whole boundary is occupied with the conditions, A. Bitsadze suggested several versions. In one of his versions he linked the solution values at different boundary points with the functional law. This nonlocal problem is well-posed. It has prompted the ways of its natural generalization to a multidimensional case.

To every well-posed plane problem there may be assigned several spatial versions, of which we will dwell only on those which maximally approach practical problems. The spatial version of the above-mentioned problem of exactly such a nature is easily generalizable and provides us with a well-posed problem. As concerns the Tricomi's problem, it has several generalization versions that make it possible to demonstrate the structure of a set of type variation points. This set of points may turn out to be a surface, oriented to the space and time. This moment determines two essentially different trends in the theory of boundary value problems for multidimensional mixed type equations.

Equations refer to different types, depending on their characteristic roots. If the equation, along with its real characteristic roots, has complex ones, then it belongs to the composite type equations. Such equations include, for example, the Laplace differentiated equation. If instead of the Laplace operator is differentiated Tricomi's operator, we obtain the mixed-composite type operator. For the equation of such a complicated nature, A. Birsadze formulated a great number of actual problems and obtained important results.

We have mentioned above the nonlocal problem in which the values of an unknown solution are interconnected at different boundary points. Of practical and theoretical interest are the problems, in which the boundary values of solutions are connected by the specific law with their values on a set of interior points of the domain. Among the problems of such a kind the Bitsadze-Samarski's problem takes central place. Its simplest and visual version is formulated as follows: Find in a unit circle a harmonic function u satisfying the condition

$$u(x,y) - u(\delta x, \delta y) = f(x,y), \;\; x^2 + y^2 = 1,$$

where the constant $\delta \in (0,1)$.

Practical problems in modeling are reduced, mainly, to the nonlinear equations. This is, seemingly, the fact that explains special interest to the above formulated problems. The powerful methods used for linear equations, in the nonlinear case are not always effective. It is a great advantage to reveal even a separate class of their solutions. The constructed by A. Bitsadze exact solutions of special type nonlinear equations

$$\sum_{i,j=1}^{n} a^{ij}(x) \Big[\frac{\partial^2 u}{\partial x_i \partial x_j} - b(u) \, \frac{\partial u}{\partial x_i} \, \frac{\partial u}{\partial x_j} \Big] + \sum_{i=1}^{n} c^i(x) \, \frac{\partial u}{\partial x_i} + d(x,u) = 0 \tag{4}$$

have found versatile practical and theoretical applications. Equations of type (4) cover a large number of models corresponding to the well-known equations of gravitation field, ferromagnetism theory, Heisenberg equations and Lorentz-covariant equations.

A large number of A. Bitsadze's creative achievements, including those mentioned above, have become long ago a corner stone on which scientific trends in the modern theory of partial differential equations are constructed.

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Otar Jokhadze

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EXISTENCE AND STABILITY RESULTS FOR NONLINEAR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSES

Abstract. In this paper, we establish the existence and uniqueness of solutions for a class of boundary value problems for nonlinear implicit fractional differential equations with impulse and Caputo's fractional derivatives, the stability of this class of problems is considered, as well. The arguments are based upon the Banach contraction principle and the Schaefer's fixed point theorem. We present two examples to show the applicability of our results.

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1. Introduction

In this paper, we establish existence, uniqueness and stability results to the following boundary value problems (BVPs) for nonlinear implicit fractional differential equations with impulses

$$ {}^cD^\alpha_{t_k} y(t) = f\big(t, y, {}^cD^\alpha_{t_k} y(t)\big) \text{ for each } t \in (t_k, t_{k+1}],\ \ k = 0, \dots, m,\ \ 0 < \alpha \le 1, \tag{1} $$

$$ \Delta y\big|_{t=t_k} = I_k(y(t_k^-)),\ \ k = 1, \dots, m, \tag{2} $$

$$ ay(0) + by(T) = c, \tag{3} $$

where ${}^cD^\alpha_{t_k}$ is the Caputo's fractional derivative, $f : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $I_k : \mathbb{R} \to \mathbb{R}$ are given functions, and a, b, c are real constants with $a+b \neq 0$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta y\big|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim\limits_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim\limits_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

In recent years, there has been a significant development in the theory of fractional differential equations. It is caused by its applications in the modeling of many phenomena in various fields of science and engineering such as acoustic, control theory, signal processing, porous media, electrochemistry, viscoelasticity, rheology, polymer physics, proteins, optics, economics, astrophysics, chaotic dynamics, statistical physics, thermodynamics, biosciences, bioengineering, etc. See, for example, [1, 6, 7, 15, 20, 27], and the references therein. On the other hand, impulsive differential equations have received much attention, we refer the reader to the books [2, 10, 16, 22, 24, 26], and the papers [13, 19, 29], and the references therein. Very recently, boundary value problems of fractional differential equations have received a considerable attention because they occur in the mathematical modeling of a variety of physical processes; see, for example, [3, 4, 8, 9, 14, 28, 31]. In [11, 12], the authors give some existence and uniqueness results for some classes of implicit fractional order differential equations. In [23], the authors consider the existence of multiple positive solutions of systems of nonlinear Caputo's fractional differential equations with general separated boundary conditions.

Motivated by the works mentioned above, in this paper we present some existence and uniqueness results for a class of boundary value problems for implicit fractional differential equations. The present paper is organized as follows. In Section 2, some notations are introduced and we recall some preliminaries about fractional calculus and auxiliary results. In Section 3, two results for the problem (1)–(3) are presented: the first one is based on the Banach contraction principle, and the second one on Schaefer's fixed point theorem. In Section 4, we present Ulam–Hyers stability result for the problem (1)–(2). Finally, in the last Section, we give two examples to illustrate the applicability of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $T > 0$, $J = [0, T]$. By $C(J, \mathbb{R})$ we denote the Banach space of continuous functions from J into $\mathbb{R}$ with the norm

$$ \|y\|_\infty = \sup\big\{|y(t)| : \ t \in J\big\}; $$

$L^1(J, \mathbb{R})$ is the space of Lebesgue-integrable functions $w : J \to \mathbb{R}$ with the norm

$$ \|w\|_1 = \int\limits_0^T |w(s)|\, ds, $$

$$ AC^n(J) = \big\{h : J \to \mathbb{R} : h, h', \dots h^{(n-1)} \in C(J, \mathbb{R}) \text{ and } h^{(n-1)} \text{ is absolutely continuous}\big\}. $$

In what follows, $\alpha > 0$. Consider the set of functions

$$ PC(J, \mathbb{R}) = \Big\{y :\ J \to \mathbb{R} : y \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 0, \dots, m \\ \text{and there exist } \ y(t_k^-) \ \text{ and } \ y(t_k^+),\ k = 1, \dots, m \ \text{ with } \ y(t_k^-) = y(t_k)\Big\}. $$

$PC(J, \mathbb{R})$ is a Banach space with the norm

$$ \|y\|_{PC} = \sup_{t\in J} |y(t)|. $$

Let $J_0 = [t_0, t_1]$ and $J_k = (t_k, t_{k+1}]$ where $k = 1, \dots, m$.

Definition 2.1 ([21, 25])**.** The fractional (arbitrary) order integral of the function $h \in L^1([0,T], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds,$$

where Γ is the Euler's gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$, $\alpha > 0$.

Definition 2.2 ([21, 25])**.** For a function $h \in AC^n(J)$, the Caputo's fractional-order derivative of order α is defined by

$$({}^cD_0^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s)\, ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.3 ([21, 25])**.** *Let $\alpha \geq 0$ and $n = [\alpha] + 1$. Then*

$$I^\alpha({}^cD_0^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^k(0)}{k!}\, t^k.$$

Lemma 2.4 ([21])**.** *Let $\alpha > 0$. Then the differential equation*

$${}^cD_0^\alpha k(t) = 0$$

has solutions $k(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.

Lemma 2.5 ([21])**.** *Let $\alpha > 0$. Then*

$$I^{\alpha c}D_0^\alpha k(t) = k(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.

D. Bainov and S. Hristova [5] introduced the following integral inequality of Gronwall type for piecewise continuous functions which can be used in the sequel.

Lemma 2.6. *Let for $t \geq t_0 \geq 0$ the inequality*

$$x(t) \leq a(t) + \int_{t_0}^t g(t,s) x(s)\, ds + \sum_{t_0 < t_k < t} \beta_k(t) x(t_k)$$

holds, where $\beta_k(t)$ $(k \in \mathbb{N})$ are nondecreasing functions for $t \geq t_0$, $a \in PC([t_0, \infty), \mathbb{R}_+)$, a is nondecreasing and $g(t,s)$ is a continuous nonnegative function for $t, s \geq t_0$ and nondecreasing with respect to t for any fixed $s \geq t_0$. Then, for $t \geq t_0$, the following inequality is valid:

$$x(t) \leq a(t) \prod_{t_0 < t_k < t} (1 + \beta_k(t)) \exp\left(\int_{t_0}^t g(t,s)\, ds \right).$$

Definition 2.7. A function $y \in PC(J, \mathbb{R}) \cap AC(J_k)$ is said to be a solution of (1)–(3) if y satisfies the equation ${}^cD_{t_k}^\alpha y(t) = f(t, y(t), {}^cD_{t_k}^\alpha y(t))$ on J_k and the conditions

$$\begin{aligned} \Delta y\big|_{t=t_k} &= I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ ay(0) + by(T) &= c. \end{aligned}$$

Here, we adopt the concepts from Wang *et al.* [30] and introduce Ulam's type stability concepts for the problem (1)–(3). Let $z \in PC(J,\mathbb{R})$, $\varepsilon > 0$, $\psi > 0$, and $\varphi \in PC(J,\mathbb{R}_+)$ be nondecreasing. We consider the set of inequalities

$$\begin{cases} \left|{}^cD^\alpha z(t) - f\big(t, z(t), {}^cD^\alpha z(t)\big)\right| \le \varepsilon, & t \in (t_k, t_{k+1}], \ \ k = 1, \dots, m, \\ \left|\Delta y(t_k) - I_k\big(y(t_k^-)\big)\right| \le \varepsilon, & k = 1, \dots, m, \end{cases} \tag{4}$$

$$\begin{cases} \left|{}^cD^\alpha z(t) - f\big(t, z(t), {}^cD^\alpha z(t)\big)\right| \le \varphi(t), & t \in (t_k, t_{k+1}], \ \ k = 1, \dots, m, \\ \left|\Delta y(t_k) - I_k\big(y(t_k^-)\big)\right| \le \psi, & k = 1, \dots, m, \end{cases} \tag{5}$$

and

$$\begin{cases} \left|{}^cD^\alpha z(t) - f\big(t, z(t), {}^cD^\alpha z(t)\big)\right| \le \varepsilon\varphi(t), & t \in (t_k, t_{k+1}], \ \ k = 1, \dots, m, \\ \left|\Delta y(t_k) - I_k\big(y(t_k^-)\big)\right| \le \varepsilon\psi, & k = 1, \dots, m. \end{cases} \tag{6}$$

Definition 2.8. The problem (1)–(3) is Ulam–Hyers stable if there exists a real number $c_{f,m} > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in PC(J,\mathbb{R}) \cap AC(J_k)$ of (4) there exists a solution y of the problem (1)–(3) with

$$|z(t) - y(t)| \le c_{f,m}\varepsilon, \ \ t \in J.$$

Definition 2.9. The problem (1)–(3) is generalized Ulam–Hyers stable if there exists $\theta_{f,m} \in C(\mathbb{R}_+,\mathbb{R}_+)$, $\theta_{f,m}(0) = 0$ such that for each solution $z \in PC(J,\mathbb{R}) \cap AC(J_k)$ of (4) there exists a solution y of the problem (1)–(3) with

$$|z(t) - y(t)| \le \theta_{f,m}(\varepsilon), \ \ t \in J.$$

Definition 2.10. The problem (1)–(3) is Ulam–Hyers–Rassias stable with respect to (φ, ψ) if there exists $c_{f,m,\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in PC(J,\mathbb{R}) \cap AC(J_k)$ of (6) there exists a solution y of the problem (1)–(3) with

$$|z(t) - y(t)| \le c_{f,m,\varphi}\varepsilon(\varphi(t) + \psi), \ \ t \in J.$$

Definition 2.11. The problem (1)–(3) is generalized Ulam–Hyers–Rassias stable with respect to (φ, ψ) if there exists $c_{f,m,\varphi} > 0$ such that for each solution $z \in PC(J,\mathbb{R}) \cap AC(J_k)$ of (5) there exists a solution y of the problem (1)–(3) with

$$|z(t) - y(t)| \le c_{f,m,\varphi}(\varphi(t) + \psi), \ \ t \in J.$$

Remark 2.12. It is clear that:

(i) Definition 2.8 implies Definition 2.9;

(ii) Definition 2.10 implies Definition 2.11;

(iii) Definition 2.10 for $\varphi(t) = \psi = 1$ implies Definition 2.8.

Remark 2.13. A function $z \in PC(J,\mathbb{R}) \cap AC(J_k)$ is a solution of (6) if and only if there are $\sigma \in PC(J,\mathbb{R})$ and a sequence σ_k, $k = 1, \dots, m$ (which depend on z), such that

(i) $|\sigma(t)| \le \varepsilon\varphi(t)$, $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$, and $|\sigma_k| \le \varepsilon\psi$, $k = 1, \dots, m$;

(ii) ${}^cD^\alpha z(t) = f(t, z(t), {}^cD^\alpha z(t)) + \sigma(t)$, $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$;

(iii) $\Delta z(t_k) = I_k(z(t_k^-)) + \sigma_k$, $k = 1, \dots, m$.

One can have similar remarks for inequalities (4) and (5).

Theorem 2.14 ([18]) (Ascoli–Arzela theorem)**.** *Let $A \subset C(J,\mathbb{R})$. A is relatively compact (i.e., $\overline{A}$ is compact) if:*

1. *A is uniformly bounded, i.e., there exists $M > 0$ such that*

$$|f(x)| < M \ \text{ for every } \ f \in A \ \text{ and } \ x \in J.$$

2. *A is equicontinuous, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each x, $\overline{x} \in J$, $|x - \overline{x}| \le \delta$ implies $|f(x) - f(\overline{x})| \le \varepsilon$, for every $f \in A$.*

Theorem 2.15 ([17]) (The Banach fixed point theorem)**.** *Let C be a non-empty closed subset of a Banach space X. Then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 2.16 ([17]) (The Schaefer's fixed point theorem)**.** *Let X be a Banach space and $N : X \longrightarrow X$ be a completely continuous operator. If the set $\mathcal{E} = \{y \in X : \ y = \lambda N y \text{ for some } \lambda \in (0,1)\}$ is bounded, then N has fixed points.*

3. The Existence of Solutions

To prove the existence of solutions to (1)–(3), we need the following auxiliary Lemma.

Lemma 3.1. *Let $0 < \alpha \leq 1$ and let $\sigma : J \to \mathbb{R}$ be continuous. A function $y \in PC(J, \mathbb{R})$ is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \dfrac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \dfrac{b}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\sigma(s)\,ds + \dfrac{b}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}\sigma(s)\,ds - c\right] \\ \qquad + \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s)\,ds, \quad \text{if } t \in [0,t_1] \\ \dfrac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \dfrac{b}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\sigma(s)\,ds + \dfrac{b}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}\sigma(s)\,ds - c\right] \\ \qquad + \sum_{i=1}^{k} I_i(y(t_i^-)) + \dfrac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\sigma(s)\,ds \\ \qquad + \dfrac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}\sigma(s)\,ds, \quad \text{if } t \in (t_k, t_{k+1}], \end{cases} \tag{7}$$

where $k = 1, \dots, m$, if and only if $y \in PC(J, \mathbb{R}) \cap AC(J_k)$ is a solution of the fractional BVP

$$^cD^\alpha y(t) = \sigma(t), \quad t \in J_k, \tag{8}$$

$$\Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{9}$$

$$ay(0) + by(T) = c. \tag{10}$$

Proof. Assume that y satisfies (8)–(10). If $t \in [0, t_1]$, then

$$^cD^\alpha y(t) = \sigma(t).$$

By Lemma 2.5

$$y(t) = c_0 + I^\alpha \sigma(t) = c_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s)\,ds$$

for $c_0 \in \mathbb{R}$. If $t \in (t_1, t_2]$, then Lemma 2.5 implies

$$\begin{aligned} y(t) = y(t_1^+) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)\,ds &= \Delta y\big|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)\,ds \\ &= I_1(y(t_1^-)) + \left[c_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\sigma(s)\,ds\right] + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)\,ds \\ &= c_0 + I_1(y(t_1^-)) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)\,ds. \end{aligned}$$

If $t \in (t_2, t_3]$, then from Lemma 2.5 we get

$$y(t) = y(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)\,ds = \Delta y\big|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)\,ds$$
$$= I_2(y(t_2^-)) + \left[c_0 + I_1(y(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\sigma(s)\,ds\right]$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)\,ds$$
$$= c_0 + \left[I_1(y(t_1^-)) + I_2(y(t_2^-))\right] + \left[\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\sigma(s)\,ds\right]$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)\,ds.$$

Repeating the process in this way, the solution $y(t)$ for $t \in (t_k, t_{k+1}]$, where $k = 1, \dots, m$, can be written as

$$y(t) = c_0 + \sum_{i=1}^{k} I_i(y(t_i^-)) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\sigma(s)\,ds.$$

Applying the boundary condition $ay(0) + by(T) = c$ we get

$$c = c_0(a+b) + b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\sigma(s)\,ds + \frac{b}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1}\sigma(s)\,ds.$$

Then

$$c_0 = \frac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\sigma(s)\,ds + \frac{b}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1}\sigma(s)\,ds - c\right].$$

Thus, if $t \in (t_k, t_{k+1}]$, where $k = 1, \dots, m$, then

$$y(t) = \frac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\sigma(s)\,ds + \frac{b}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1}\sigma(s)\,ds - c\right]$$
$$+ \sum_{i=1}^{k} I_i(y(t_i^-)) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\sigma(s)\,ds.$$

Conversely, assume that y satisfies the impulsive fractional integral equation (7). If $t \in [0, t_1]$, then $ay(0) + by(T) = c$ and, using the fact that ${}^cD^\alpha$ is the left inverse of I^α, we get

$${}^cD^\alpha y(t) = \sigma(t) \ \text{ for each } \ t \in [0, t_1].$$

If $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$, using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we get

$${}^cD^\alpha y(t) = \sigma(t) \ \text{ for each } \ t \in (t_k, t_{k+1}].$$

Also, we can easily show that

$$\Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m.$$

We are now in a position to state and prove our existence result for the problem (1)–(3) based on the Banach fixed point theorem. □

Theorem 3.2. *Assume*

$(H1)$ *the function* $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous;*

$(H2)$ *there exist constants* $K > 0$ *and* $0 < L < 1$ *such that*

$$\big|f(t,u,v) - f(t,\overline{u},\overline{v})\big| \le K|u-\overline{u}| + L|v-\overline{v}|$$

for any $u, v, \overline{u}, \overline{v} \in \mathbb{R}$ *and* $t \in J$*;*

$(H3)$ *there exists a constant* $\widetilde{l} > 0$ *such that*

$$\big|I_k(u) - I_k(\overline{u})\big| \le \widetilde{l}\,|u-\overline{u}|$$

for each $u, \overline{u} \in \mathbb{R}$ *and* $k = 1, \dots, m$.

If

$$\Big(\frac{|b|}{|a+b|} + 1\Big)\Big[m\widetilde{l} + \frac{(m+1)KT^{\alpha}}{(1-L)\Gamma(\alpha+1)}\Big] < 1, \tag{11}$$

then there exists a unique solution for the BVP (1)–(3).

Proof. Transform the problem (1)–(3) into a fixed point problem. Consider the operator $N : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by

$$\begin{aligned} N(y)(t) = \frac{-1}{a+b}\Bigg[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}g(s)\,ds + \frac{b}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}g(s)\,ds - c\Bigg] \\ + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}g(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}g(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \end{aligned} \tag{12}$$

where $g \in C(J,\mathbb{R})$ is such that

$$g(t) = f\big(t, y(t), g(t)\big).$$

Clearly, the fixed points of operator N are solutions of problem (1)–(3).

Let $u, w \in PC(J,\mathbb{R})$. Then for $t \in J$ we have

$$\begin{aligned} \big|N(u)(t) - N(w)(t)\big| \le \frac{|b|}{|a+b|}\Bigg[\sum_{i=1}^{m}\big|I_i(u(t_i^-)) - I_i(w(t_i^-))\big| \\ + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|g(s)-h(s)|\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}|g(s)-h(s)|\,ds\Bigg] \\ + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|g(s)-h(s)|\,ds \\ + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|g(s)-h(s)|\,ds + \sum_{0<t_k<t}\big|I_k(u(t_k^-)) - I_k(w(t_k^-))\big|, \end{aligned}$$

where $g, h \in C(J,\mathbb{R})$ are such that

$$g(t) = f\big(t, u(t), g(t)\big)$$

and

$$h(t) = f\big(t, w(t), h(t)\big).$$

By $(H2)$, we have

$$|g(t)-h(t)| = \big|f(t,u(t),g(t)) - f(t,w(t),h(t))\big| \le K|u(t)-w(t)| + L|g(t)-h(t)|.$$

Then

$$|g(t)-h(t)| \le \frac{K}{1-L}\,|u(t)-w(t)|.$$

Therefore, for each $t \in J$,

$$\begin{aligned}
\big|N(u)(t)-N(w)(t)\big| &\le \frac{|b|}{|a+b|}\left[\sum_{k=1}^{m}\widetilde{l}\,\big|u(t_k^-)-w(t_k^-)\big|\right.\\
&\left.+\frac{K}{(1-L)\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|u(s)-w(s)|\,ds+\frac{K}{(1-L)\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}|u(s)-w(s)|\,ds\right]\\
&+\frac{K}{(1-L)\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|u(s)-w(s)|\,ds\\
&+\frac{K}{(1-L)\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|u(s)-w(s)|\,ds+\sum_{k=1}^{m}\widetilde{l}\,\big|u(t_k^-)-w(t_k^-)\big|\\
&\le\Big(\frac{|b|}{|a+b|}+1\Big)\Big[m\widetilde{l}+\frac{mKT^\alpha}{(1-L)\Gamma(\alpha+1)}+\frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\Big]\|u-w\|_{PC}.
\end{aligned}$$

Thus

$$\|N(u)-N(w)\|_{PC}\le\Big(\frac{|b|}{|a+b|}+1\Big)\Big[m\widetilde{l}+\frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)}\Big]\|u-w\|_{PC}.$$

By (11), the operator N is a contraction. Hence, by the Banach contraction principle, N has a unique fixed point which is a unique solution of the problem (1)–(3). □

Our second result is based on the Schaefer's fixed point theorem.

Theorem 3.3. *Assume that* $(H1)$, $(H2)$ *and the following conditions are fulfilled:*

$(H4)$ *there exist* $p,q,r\in C(J,\mathbb{R}_+)$ *with* $r^*=\sup_{t\in J}r(t)<1$ *such that*

$$|f(t,u,w)|\le p(t)+q(t)|u|+r(t)|w|\ \ \textit{for}\ \ t\in J\ \ \textit{and}\ \ u,w\in\mathbb{R};$$

$(H5)$ *the functions* $I_k:\mathbb{R}\to\mathbb{R}$ *are continuous and there exist constants* $M^*,N^*>0$ *such that*

$$|I_k(u)|\le M^*|u|+N^*\ \ \textit{for each}\ \ u\in\mathbb{R},\ \ k=1,\dots,m.$$

If

$$\Big(\frac{|b|}{|a+b|}+1\Big)\Big(mM^*+\frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\Big)<1, \tag{13}$$

then the BVP (1)–(3) *has at least one solution on* J.

Proof. Let the operator N be defined by (12). We shall use the Schaefer's fixed point theorem to prove that N has a fixed point. The proof will be given in several steps.

Step 1: *N is continuous.* Let $\{u_n\}$ be a sequence such that $u_n\to u$ in $PC(J,\mathbb{R})$. Then for each $t\in J$

$$\begin{aligned}
\big|N(u_n)(t)-N(u)(t)\big| &\le \frac{|b|}{|a+b|}\left[\sum_{i=1}^{m}\big|I_k(u_n(t_k^-))-I_k(u(t_k^-))\big|\right.\\
&\left.+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|g_n(s)-g(s)|\,ds+\frac{1}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}|g_n(s)-g(s)|\,ds\right]\\
&+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|g_n(s)-g(s)|\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|g_n(s)-g(s)|\,ds+\sum_{0<t_k<t}\big|I_k(u_n(t_k^-))-I_k(u(t_k^-))\big|,
\end{aligned}\tag{14}$$

where $g_n,g\in C(J,\mathbb{R})$ are such that

$$g_n(t)=f\big(t,u_n(t),g_n(t)\big)$$

and

$$g(t) = f\big(t, u(t), g(t)\big).$$

By $(H2)$, we have

$$|g_n(t) - g(t)| = \big|f(t, u_n(t), g_n(t)) - f(t, u(t), g(t))\big| \le K|u_n(t) - u(t)| + L|g_n(t) - g(t)|.$$

Then

$$|g_n(t) - g(t)| \le \frac{K}{1-L}\,|u_n(t) - u(t)|.$$

Since $u_n \to u$, we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. Let $\eta > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \le \eta$ and $|g(t)| \le \eta$. Then we have

$$(t-s)^{\alpha-1}|g_n(s) - g(s)| \le (t-s)^{\alpha-1}\big[|g_n(s)| + |g(s)|\big] \le 2\eta(t-s)^{\alpha-1}$$

and

$$(t_k-s)^{\alpha-1}|g_n(s) - g(s)| \le (t_k-s)^{\alpha-1}\big[|g_n(s)| + |g(s)|\big] \le 2\eta(t_k-s)^{\alpha-1}.$$

For each $t \in J$, the functions $s \to 2\eta(t-s)^{\alpha-1}$ and $s \to 2\eta(t_k-s)^{\alpha-1}$ are integrable on $[0,t]$, then the Lebesgue Dominated Convergence Theorem and (14) imply that

$$\big|N(u_n)(t) - N(u)(t)\big| \longrightarrow 0 \text{ as } n \to \infty$$

and hence

$$\big\|N(u_n) - N(u)\big\|_{PC} \longrightarrow 0 \text{ as } n \to \infty.$$

Consequently, N is continuous.

Step 2: *F maps bounded sets into bounded sets in $PC(J,\mathbb{R})$.* Indeed, it is enough to show that for any $\eta^* > 0$ there exists a positive constant ℓ such that for each $u \in B_{\eta^*} = \{u \in PC(J,\mathbb{R}) : \|u\|_{PC} \le \eta^*\}$, $\|N(u)\|_{PC} \le \ell$. For each $t \in J$ we have

$$N(u)(t) = \frac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(u(t_i^-)) + \frac{b}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}g(s)\,ds + \frac{b}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}g(s)\,ds - c\right]$$
$$+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}g(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}g(s)\,ds + \sum_{0<t_k<t} I_k(u(t_k^-)), \tag{15}$$

where $g \in C(J,\mathbb{R})$ is such that

$$g(t) = f\big(t, u(t), g(t)\big).$$

By $(H4)$, for each $t \in J$ we have

$$\begin{aligned}|g(t)| &= \big|f(t, u(t), g(t))\big| \le p(t) + q(t)|u(t)| + r(t)|g(t)|\\ &\le p(t) + q(t)\eta^* + r(t)|g(t)| \le p^* + q^*\eta^* + r^*|g(t)|,\end{aligned}$$

where $p^* = \sup_{t\in J} p(t)$ and $q^* = \sup_{t\in J} q(t)$. Then

$$|g(t)| \le \frac{p^* + q^*\eta^*}{1-r^*} := M.$$

Thus (15) implies

$$\begin{aligned}|N(u)(t)| \le{}& \frac{|b|}{|a+b|}\left[m(M^*|u| + N^*) + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)}\right]\\ &+ \frac{|c|}{|a+b|} + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} + m\big(M^*|u| + N^*\big)\\ \le{}& \left(\frac{|b|}{|a+b|} + 1\right)\left[m(M^*|u| + N^*) + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)}\right] + \frac{|c|}{|a+b|}.\end{aligned}$$

Therefore

$$\|N(u)\|_{PC} \le \left(\frac{|b|}{|a+b|} + 1\right)\left[m(M^*\eta^* + N^*) + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)}\right] + \frac{|c|}{|a+b|} := \ell.$$

Step 3: *F maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R})$.* Let $\tau_1,\tau_2\in J$, $\tau_1<\tau_2$, B_{η^*} be a bounded set of $PC(J,\mathbb{R})$ as in Step 2, and let $u\in B_{\eta^*}$. Then

$$\begin{aligned}\big|N(u)(\tau_2)-N(u)(\tau_1)\big| &\le \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\big|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\big|\,|g(s)|\,ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\big|(\tau_2-s)^{\alpha-1}\big|\,|g(s)|\,ds+\sum_{0<t_k<\tau_2-\tau_1}\big|I_k(u(t_k^-))\big|\\ &\le\frac{M}{\Gamma(\alpha+1)}\big[2(\tau_2-\tau_1)^\alpha+(\tau_2^\alpha-\tau_1^\alpha)\big]+(\tau_2-\tau_1)\big(M^*|u|+N^*\big)\\ &\le\frac{M}{\Gamma(\alpha+1)}\big[2(\tau_2-\tau_1)^\alpha+(\tau_2^\alpha-\tau_1^\alpha)\big]+(\tau_2-\tau_1)\big(M^*\eta^*+N^*\big).\end{aligned}$$

As $\tau_1\to\tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Ascoli–Arzela theorem, we can conclude that $N:PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ is completely continuous.

Step 4: *A priori bounds.* Now it remains to show that the set

$$E=\big\{u\in PC(J,\mathbb{R}):\ u=\lambda N(u)\text{ for some }0<\lambda<1\big\}$$

is bounded. Let $u\in E$, then $u=\lambda N(u)$ for some $0<\lambda<1$. Thus for each $t\in J$

$$\begin{aligned}u(t)&=\frac{-1}{a+b}\left[b\lambda\sum_{i=1}^m I_i(u(t_i^-))+\frac{b\lambda}{\Gamma(\alpha)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}g(s)\,ds+\frac{b\lambda}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}g(s)\,ds-c\lambda\right]\\ &\quad+\frac{\lambda}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}g(s)\,ds+\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}g(s)\,ds+\lambda\sum_{0<t_k<t}I_k(u(t_k^-)).\end{aligned}\tag{16}$$

By $(H4)$, for each $t\in J$ we have

$$|g(t)|=\big|f(t,u(t),g(t))\big|\le p(t)+q(t)|u(t)|+r(t)|g(t)|\le p^*+q^*|u(t)|+r^*|g(t)|.$$

Thus

$$|g(t)|\le\frac{1}{1-r^*}\big(p^*+q^*|u(t)|\big)\le\frac{1}{1-r^*}\big(p^*+q^*\|u\|_{PC}\big).$$

This implies, by (16) and $(H5)$, that for each $t\in J$

$$\begin{aligned}|u(t)|&\le\frac{|b|}{|a+b|}\Big[m\big(M^*\|u\|_{PC}+N^*\big)+\frac{mT^\alpha(p^*+q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)}+\frac{T^\alpha(p^*+q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)}\Big]\\ &\quad+\frac{|c|}{|a+b|}+\frac{mT^\alpha(p^*+q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)}+\frac{T^\alpha(p^*+q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)}+m\big(M^*\|u(t)\|_{PC}+N^*\big).\end{aligned}$$

Then

$$\begin{aligned}\|u\|_{PC}&\le\Big(\frac{|b|}{|a+b|}+1\Big)\Big[m\big(M^*\|u(t)\|_{PC}+N^*\big)+\frac{(m+1)(p^*+q^*\|u\|_{PC})T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\Big]+\frac{|c|}{|a+b|}\\ &\le\Big(\frac{|b|}{|a+b|}+1\Big)\Big(mN^*+\frac{(m+1)p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\Big)\\ &\quad+\frac{|c|}{|a+b|}+\Big(\frac{|b|}{|a+b|}+1\Big)\Big(mM^*+\frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\Big)\|u\|_{PC}.\end{aligned}$$

Thus

$$\begin{aligned}&\Big[1-\Big(\frac{|b|}{|a+b|}+1\Big)\Big(mM^*+\frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\Big)\Big]\|u\|_{PC}\\ &\qquad\le\Big(\frac{|b|}{|a+b|}+1\Big)\Big[\frac{|c|}{|a+b|}+mN^*+\frac{(m+1)p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\Big].\end{aligned}$$

Finally, by (13), we obtain

$$\|u\|_{PC} \leq \frac{(\frac{|b|}{|a+b|}+1)\big[mN^* + \frac{(m+1)p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)} + \frac{|c|}{|a+b|}\big]}{\big[1-(\frac{|b|}{|a+b|}+1)(mM^* + \frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)})\big]} := R.$$

This shows that the set E is bounded. As a consequence of the Schaefer's fixed point theorem, we deduce that N has a fixed point which is a solution of the problem (1)–(3). □

4. Ulam–Hyers Rassias Stability

Now, we state the following Ulam–Hyers–Rassias stable result.

Theorem 4.1. *Assume that* $(H1)$–$(H3)$, (11) *and the following condition are satisfied:*

$(H6)$ *there exists a nondecreasing function* $\varphi \in PC(J,\mathbb{R}_+)$ *and there exists* $\lambda_\varphi > 0$ *such that for any* $t \in J$:

$$I^\alpha \varphi(t) \leq \lambda_\varphi \varphi(t).$$

Then the problem (1)–(2) *is Ulam–Hyers–Rassias stable with respect to* (φ, ψ).

Proof. Let $z \in PC(J,\mathbb{R}) \cap AC(J_k)$ be a solution of (6). Denote by y the unique solution of the BVP

$$\begin{cases} {}^cD^\alpha_{t_k} y(t) = f\big(t, y(t), {}^cD^\alpha_{t_k} y(t)\big), & t \in (t_k, t_{k+1}],\ \ k = 1,\dots,m, \\ \Delta y(t_k) = I_k(y(t_k^-)), & k = 1,\dots,m, \\ ay(0) + by(T) = c, \\ y(0) = z(0). \end{cases}$$

Using Lemma 3.1, for each $t \in (t_k, t_{k+1}]$ we obtain

$$y(t) = y(0) + \sum_{i=1}^{k} I_i(y(t_i^-)) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} g(s)\, ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)\, ds, \ \ t \in (t_k, t_{k+1}],$$

where $g \in C(J,\mathbb{R})$ is such that

$$g(t) = f(t, y(t), g(t)).$$

Since z is a solution of (6), by Remark 2.13, we have

$$\begin{cases} {}^cD^\alpha_{t_k} z(t) = f\big(t, z(t), {}^cD^\alpha_{t_k} z(t)\big) + \sigma(t), & t \in (t_k, t_{k+1}],\ \ k = 1,\dots,m, \\ \Delta z(t_k) = I_k(z(t_k^-)) + \sigma_k, & k = 1,\dots,m. \end{cases} \tag{17}$$

Clearly, the solution of (17) is given by

$$z(t) = z(0) + \sum_{i=1}^{k} I_i(z(t_i^-)) + \sum_{i=1}^{k} \sigma_i + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s)\, ds$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \sigma(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} h(s)\, ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \sigma(s)\, ds, \ \ t \in (t_k, t_{k+1}],$$

where $h \in C(J,\mathbb{R})$ is such that

$$h(t) = f\big(t, z(t), h(t)\big).$$

Hence for each $t \in (t_k, t_{k+1}]$ it follows that

$$
\begin{aligned}
|z(t) - y(t)| \le & \sum_{i=1}^{k} |\sigma_i| + \sum_{i=1}^{k} \left| I_i(z(t_i^-)) - I_i(y(t_i^-)) \right| \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |h(s) - g(s)|\, ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |\sigma(s)|\, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |h(s) - g(s)|\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |\sigma(s)|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|z(t) - y(t)| \le & \, m\varepsilon\psi + (m+1)\varepsilon\lambda_\varphi \varphi(t) + \sum_{i=1}^{k} \widetilde{l} \left| z(t_i^-) - y(t_i^-) \right| \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |h(s) - g(s)|\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |h(s) - g(s)|\, ds.
\end{aligned}
$$

By $(H2)$, we get

$$
|h(t) - g(t)| = \left| f\big(t, z(t), h(t)\big) - f\big(t, y(t), g(t)\big) \right| \le K|z(t) - y(t)| + L|g(t) - h(t)|.
$$

Then

$$
|h(t) - g(t)| \le \frac{K}{1 - L} |z(t) - y(t)|.
$$

Therefore, for each $t \in J$,

$$
\begin{aligned}
|z(t) - y(t)| \le & \, m\varepsilon\psi + (m+1)\varepsilon\lambda_\varphi \varphi(t) + \sum_{i=1}^{k} \widetilde{l} \left| z(t_i^-) - y(t_i^-) \right| \\
& + \frac{K}{(1-L)\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |z(s) - y(s)|\, ds \\
& + \frac{K}{(1-L)\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |z(s) - y(s)|\, ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|z(t) - y(t)| \le & \sum_{i=1}^{k} \widetilde{l} \left| z(t_i^-) - y(t_i^-) \right| + \varepsilon(\psi + \varphi(t))\big(m + (m+1)\lambda_\varphi\big) \\
& + \frac{K(m+1)}{(1-L)\Gamma(\alpha)} \int_{0}^{t} (t - s)^{\alpha-1} |z(s) - y(s)|\, ds.
\end{aligned}
$$

Applying Lemma 2.6, we get

$$
\begin{aligned}
|z(t) - y(t)| \le & \, \varepsilon(\psi + \varphi(t))(m + (m+1)\lambda_\varphi) \\
& \times \left[\prod_{0 < t_k < t} (1 + \widetilde{l}) \exp\left(\int_{0}^{t} \frac{K(m+1)}{(1-L)\Gamma(\alpha)} (t - s)^{\alpha-1}\, ds \right) \right] \le c_\varphi \varepsilon(\psi + \varphi(t)),
\end{aligned}
$$

where

$$c_\varphi = (m + (m+1)\lambda_\varphi)\Big[\prod_{k=1}^{m}(1+\widetilde{l})\exp\Big(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\Big)\Big]$$
$$= \big(m + (m+1)\lambda_\varphi\big)\Big[(1+\widetilde{l})\exp\Big(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\Big)\Big]^m.$$

Thus, the problem (1)–(2) is Ulam–Hyers–Rassias stable with respect to (φ, ψ). □

Next, we present the following Ulam–Hyers stability result.

Theorem 4.2. *Assume that* $(H1)$*–*$(H3)$ *and* (11) *are satisfied. Then the problem* (1)*–*(2) *is Ulam–Hyers stable.*

Proof. Let $z \in PC(J,\mathbb{R}) \cap AC(J_k)$ be a solution of (4). Denote by y the unique solution of the BVP

$$\begin{cases} {}^cD^\alpha_{t_k} y(t) = f\big(t, y(t), {}^cD^\alpha_{t_k} y(t)\big), \quad t \in (t_k, t_{k+1}], \quad k = 1, \dots, m, \\ \Delta y(t_k) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ ay(0) + by(T) = c, \\ y(0) = z(0). \end{cases}$$

Similarly as in the proof of Theorem 4.1 we get the inequality

$$|z(t) - y(t)| \le \sum_{i=1}^{k} \widetilde{l}\,\big|(z(t_i^-)) - (y(t_i^-))\big|$$
$$+ m\varepsilon + \frac{T^\alpha \varepsilon (m+1)}{\Gamma(\alpha+1)} + \frac{K(m+1)}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - y(s)|\, ds.$$

Applying Lemma 2.6, we obtain

$$|z(t) - y(t)| \le \varepsilon\Big(\frac{m\Gamma(\alpha+1) + T^\alpha(m+1)}{\Gamma(\alpha+1)}\Big)$$
$$\times \Big[\prod_{0<t_k<t} (1+\widetilde{l})\exp\Big(\int_0^t \frac{K(m+1)}{(1-L)\Gamma(\alpha)}(t-s)^{\alpha-1}\, ds\Big)\Big] \le c_\varphi \varepsilon,$$

where

$$c_\varphi = \Big(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{\Gamma(\alpha+1)}\Big)\Big[\prod_{k=1}^{m}(1+\widetilde{l})\exp\Big(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\Big)\Big]$$
$$= \Big(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{\Gamma(\alpha+1)}\Big)\Big[(1+\widetilde{l})\exp\Big(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\Big)\Big]^m.$$

which completes the proof of the theorem. □

Moreover, if we set $\gamma(\varepsilon) = c\varepsilon$, $\gamma(0) = 0$, then the problem (1)–(2) is generalized Ulam–Hyers stable.

Remark 4.3. Our results for the boundary value problem (1)–(3) are appropriate for the following problems:

- Initial value problem: $a = 1$, $b = 0$, $c = 0$.
- Terminal value Problem: $a = 0$, $b = 1$, c is arbitrary.
- Anti-periodic problem: $a = 1$, $b = 1$, $c = 0$.

However, our results are not applicable for the periodic problem, i.e., for $a = 1$, $b = -1$, $c = 0$.

5. Examples

Example 1. Consider the following impulsive boundary value problem:

$$ {}^cD^{\frac{1}{2}}_{t_k} y(t) = \frac{1}{5e^{t+2}(1+|y(t)|+|{}^cD^{\frac{1}{2}}_{t_k} y(t)|)} \quad \text{for each } t \in J_0 \cup J_1, \tag{18} $$

$$ \Delta y\big|_{t=\frac{1}{2}} = \frac{|y(\frac{1}{2}^-)|}{10+|y(\frac{1}{2}^-)|}\,, \tag{19} $$

$$ 2y(0) - y(1) = 3, \tag{20} $$

where $J_0 = [0, \frac{1}{2}]$, $J_1 = (\frac{1}{2}, 1]$, $t_0 = 0$ and $t_1 = \frac{1}{2}$. Set

$$ f(t,u,v) = \frac{1}{5e^{t+2}(1+|u|+|v|)}\,, \quad t \in [0,1], \quad u, v \in \mathbb{R}. $$

Clearly, the function f is jointly continuous.

For each $u, v, \overline{u}, \overline{v} \in \mathbb{R}$ and $t \in [0,1]$

$$ \big|f(t,u,v) - f(t,\overline{u},\overline{v})\big| \le \frac{1}{5e^2}\big(|u-\overline{u}| + |v-\overline{v}|\big). $$

Hence the condition $(H2)$ is satisfied with $K = L = \frac{1}{5e^2}$.

Let

$$ I_1(u) = \frac{u}{10+u}\,, \quad u \in [0,\infty). $$

Let $u, v \in [0,\infty)$. Then we have

$$ \big|I_1(u) - I_1(v)\big| = \Big|\frac{u}{10+u} - \frac{v}{10+v}\Big| = \frac{10|u-v|}{(10+u)(10+v)} \le \frac{1}{10}\,|u-v|. $$

Thus the condition

$$ \Big(\frac{|b|}{|a+b|}+1\Big)\Big[m\widetilde{l} + \frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)}\Big] = 2\Big[\frac{1}{10} + \frac{\frac{2}{5e^2}}{(1-\frac{1}{5e^2})\Gamma(\frac{3}{2})}\Big] = 2\Big[\frac{4}{(5e^2-1)\sqrt{\pi}} + \frac{1}{10}\Big] < 1 $$

is satisfied with $T = 1$, $a = 2$, $b = -1$, $c = 3$, $m = 1$ and $\widetilde{l} = \frac{1}{10}$. From Theorem 3.2 it follows that the problem (18)–(20) has a unique solution on J.

Set for any $t \in [0,1]$, $\varphi(t) = t$, $\psi = 1$. Since

$$ I^{\frac{1}{2}}\varphi(t) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{\frac{1}{2}-1} s\,ds \le \frac{2t}{\sqrt{\pi}}\,, $$

the condition $(H6)$ is satisfied with $\lambda_\varphi = \frac{2}{\sqrt{\pi}}$. From this it follows that the problem (18)–(19) is Ulam–Hyers–Rassias stable with respect to (φ,ψ).

Example 2. Consider the following impulsive anti-periodic problem:

$$ {}^cD^{\frac{1}{2}}_{t_k} y(t) = \frac{2+|y(t)|+|{}^cD^{\frac{1}{2}}_{t_k} y(t)|}{108e^{t+3}(1+|y(t)|+|{}^cD^{\frac{1}{2}}_{t_k} y(t)|)} \quad \text{for each } t \in J_0 \cup J_1, \tag{21} $$

$$ \Delta y\big|_{t=\frac{1}{3}} = \frac{|y(\frac{1}{3}^-)|}{6+|y(\frac{1}{3}^-)|}\,, \tag{22} $$

$$ y(0) = -y(1), \tag{23} $$

where $J_0 = [0, \frac{1}{3}]$, $J_1 = (\frac{1}{3}, 1]$, $t_0 = 0$, and $t_1 = \frac{1}{3}$. Set

$$ f(t,u,v) = \frac{2+|u|+|v|}{108e^{t+3}(1+|u|+|v|)}\,, \quad t \in [0,1], \quad u, v \in \mathbb{R}. $$

Clearly, the function f is jointly continuous. For any $u, v, \overline{u}, \overline{v} \in \mathbb{R}$ and $t \in [0,1]$

$$ \big|f(t,u,v) - f(t,\overline{u},\overline{v})\big| \le \frac{1}{108e^3}\big(|u-\overline{u}| + |v-\overline{v}|\big). $$

Hence the condition $(H2)$ is satisfied with $K = L = \frac{1}{108e^3}$. For each $t \in [0,1]$ we have

$$|f(t,u,v)| \leq \frac{1}{108e^{t+3}}\big(2 + |u| + |v|\big).$$

Thus the condition $(H4)$ is satisfied with $p(t) = \frac{1}{54e^{t+3}}$ and $q(t) = r(t) = \frac{1}{108e^{t+3}}$.

Let

$$I_1(u) = \frac{u}{6+u}\,, \quad u \in [0,\infty).$$

For each $u \in [0,\infty)$ we have

$$|I_1(u)| \leq \frac{1}{6}\,u + 1.$$

Thus the condition $(H5)$ is satisfied with $M^* = \frac{1}{6}$ and $N^* = 1$. Therefore the condition

$$\Big(\frac{|b|}{|a+b|} + 1\Big)\Big(mM^* + \frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\Big) = \frac{3}{2}\Big(\frac{1}{6} + \frac{4}{(108e^3-1)\sqrt{\pi}}\Big) < 1,$$

is satisfied with $T = 1$, $a = 1$, $b = 1$, $c = 0$, $m = 1$ and $q^*(t) = r^*(t) = \frac{1}{108e^3}$. From Theorem 3.3 it follows that the problem (21)–(23) has at least one solution on J.

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ON THE CONVERGENCE RATE ANALYSIS OF ONE DIFFERENCE SCHEME FOR BURGERS' EQUATION

Abstract. We consider an initial boundary value problem for the 1D nonlinear Burgers' equation. A three-level finite difference scheme is studied. Two-level scheme is used to find the values of unknown function on the first level. The obtained algebraic equations are linear with respect to the values of the unknown function for each new level. It is proved that the scheme is convergent at rate $O(\tau^{k-1}+h^{k-1})$ in discrete L_2-norm when an exact solution belongs to the Sobolev space W_2^k, $2<k\leq 3$.

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1. Introduction

We will study the finite difference method for a numerical solution of initial boundary value problem for a forced Burgers' equation

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} - \nu\,\frac{\partial^2 u}{\partial x^2} = f, \quad (x,t) \in Q, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \quad t \in [0,T), \quad u(x,0) = \varphi(x), \quad x \in [0,1], \tag{1.2}$$

where $Q = (0,1) \times (0,T)$, and parameter $\nu = const > 0$ defines the kinematic viscosity.

Assume that a solution of this problem belongs to the fractional-order Sobolev space $W_2^k(Q)$, $k > 2$, whose norms and seminorms we denote by $\|\cdot\|_{W_2^k(Q)}$ and $|\cdot|_{W_2^k(Q)}$, respectively.

Certain numerical methods (Galerkin, least squares, collocation, method of lines, finite differences, etc.) are devoted to problems posed for Burgers' equation (see, e.g., [1, 2, 3, 7, 10, 11, 14, 15, 16, 19]). In some cases, the Hopf–Cole transformation [9, 13] is used before approximation in order to reduce Burgers' equation to a linear heat equation.

H. Sun and Z. Z. Sun [19] investigated a three-level difference scheme for the problem (1.1), (1.2) and ascertained a second-order convergence in the maximum-norm under the assumption that the exact solution belongs to $\mathcal{C}^{4,3}(\overline{Q})$.

In the present article, a three-level difference scheme is studied for the problem (1.1), (1.2). All the obtained algebraic equations are linear with respect to the values of an unknown function on the upper level. It is proved that the scheme is convergent at rate $O(\tau^{k-1} + h^{k-1})$ when an exact solution belongs to the Sobolev space $W_2^k(Q)$, $2 < k \le 3$. The error estimate is derived by using the certain well-known techniques (see, e.g., [18, 4]) that employ the generalized Bramble–Hilbert Lemma. For the upper layers, the difference equations are the same as in [19] and are obtained by using the well known approximations for derivatives. For the first layer, the difference equations are constructed with the help of approximation of $\partial(u)^2/\partial x$ by the way offered in [5, 6]. In the case of sufficiently smooth solutions, they represent the second order approximations for obtaining additional initial data. At the same time, they represent approximation of the equation (1.1) to within the accuracy $O(\tau + h^2)$.

Despite the last circumstance, the order of convergence by discrete L_2-norm does not decrease and remains still second order on sufficiently smooth solutions. "The study of the local approximation is insufficient for determination of the order of the difference approximation and proper evaluation of the quality of a difference operator" (Samarskii [17, Chapter 2, Section 1.3, Example 1]).

2. A Finite Difference Scheme and Main Results

The finite domain $[0,1] \times [0,T]$ is divided into rectangle grids by the points $(x_i, t_j) = (ih, j\tau)$, $i = 0,1,\dots,n$, $j = 0,1,2,\dots,J$, where $h = 1/n$ and $\tau = T/J$ denote the spatial and temporal mesh sizes, respectively.

Let $\overline{\omega} = \{x_i : \ i = 0,1,\dots,n\}$, $\omega = \{x_i : \ i = 1,2,\dots,n-1\}$, $\omega^+ = \{x_i : \ i = 1,2,\dots,n\}$.

The value of the mesh function U at the node (x_i, t_j) is denoted by U_i^j, that is, $U(ih, j\tau) = U_i^j$. For the sake of simplicity sometimes we will use notation without subscripts: $U_i^j = U$, $U_i^{j+1} = \widehat{U}$, $U_i^{j-1} = \check{U}$. Moreover, let

$$\overline{U}^{\,0} = \frac{U^1 + U^0}{2}\,, \quad \overline{U}^{\,j} = \frac{U^{j+1} + U^{j-1}}{2}\,, \quad j = 1,2,\dots\,.$$

We define the difference quotients in x and t directions as follows:

$$(U_i)_{\overline{x}} = \frac{U_i - U_{i-1}}{h}\,, \quad (U_i)_{\overset{\circ}{x}} = \frac{1}{2h}\,(U_{i+1} - U_{i-1}), \quad (U_i)_{\overline{x}\,x} = \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}\,,$$

$$(U^j)_t = \frac{U^{j+1} - U^j}{\tau}\,, \quad (U^j)_{\overset{\circ}{t}} = \frac{U^{j+1} - U^{j-1}}{2\tau}\,, \quad (U^j)_{\overline{t}\,t} = \frac{U^{j+1} - 2U^j + U^{j-1}}{\tau^2}\,.$$

Let H_0 be a set of functions defined on the mesh $\overline{\omega}$ and equal to zero at $x = 0$ and $x = 1$. On H_0 we define the following inner product and norm:

$$(U,V) = \sum_{x \in \omega} hU(x)V(x), \quad \|U\| = (U,U)^{1/2}.$$

Let, moreover,

$$(U,V] = \sum_{x\in\omega^+} hU(x)V(x), \quad \|U]| = (U,U]^{1/2}.$$

We need the following averaging operators for the functions defined on Q:

$$\widehat{\mathcal{S}}v := \frac{1}{\tau}\int\limits_t^{t+h} v(x,\xi)\,d\xi, \quad \overset{\circ}{\mathcal{S}}v := \frac{1}{2\tau}\int\limits_{t-h}^{t+h} v(x,\xi)\,d\xi,$$

$$\widehat{\mathcal{P}}v := \frac{1}{h}\int\limits_x^{x+h} v(\xi,t)\,d\xi, \quad \mathcal{P}v := \frac{1}{h^2}\int\limits_{x-h}^{x+h} \big(h-|x-\xi|\big)v(\xi,t)\,d\xi.$$

Note that

$$\overset{\circ}{\mathcal{S}}\,\frac{\partial v}{\partial t} = v_{\overset{\circ}{t}}, \quad \widehat{\mathcal{S}}\,\frac{\partial v}{\partial t} = v_t, \quad \mathcal{P}\,\frac{\partial^2 v}{\partial x^2} = v_{\overline{x}\,x}, \quad \mathcal{P}\,\frac{\partial v}{\partial x} = \widehat{\mathcal{P}}v_{\overline{x}}.$$

We approximate the problem (1.1), (1.2) by of the difference scheme:

$$\mathcal{L}U_i^j = F_i^j, \ \ i=1,2,\dots,n-1, \ \ j=0,1,\dots,J-1, \tag{2.1}$$

$$U_0^j = U_n^j = 0, \ \ j=0,1,\dots,J, \quad U_i^0 = \varphi(x_i), \ \ i=0,1,\dots,n. \tag{2.2}$$

where

$$\mathcal{L}U^0 := (U^0)_t + \frac{1}{3}\,\Lambda U^0 - \nu(\overline{U}^{\,0})_{\overline{x}\,x},$$

$$\Lambda U^0 := U^0(\overline{U}^{\,0})_{\overset{\circ}{x}} + (U^0\overline{U}^{\,0})_{\overset{\circ}{x}}, \quad F^0 := \mathcal{P}\overline{f}^{\,0},$$

$$\mathcal{L}U^j := (U^j)_{\overset{\circ}{t}} + \frac{1}{3}\Lambda U^j - \nu(\overline{U}^{\,j})_{\overline{x}\,x}, \ \ j=1,2,\dots\,,$$

$$\Lambda U^j = U^j(\overline{U}^{\,j})_{\overset{\circ}{x}} + (U^j\overline{U}^{\,j})_{\overset{\circ}{x}}, \quad F^j := \mathcal{P}\overline{f}^{\,j}.$$

Theorem 2.1. *The finite difference scheme* (2.1), (2.2) *is uniquely solvable.*

Proof. Note that

$$(YV_{\overset{\circ}{x}} + (YV)_{\overset{\circ}{x}}, V) = 0, \ \text{ if } \ V\in H_0. \tag{2.3}$$

Considering inner products $(\mathcal{L}U^j, \overline{U}^{\,j})$ and $(\mathcal{L}U^0, \overline{U}^{\,0})$, we obtain

$$\frac{1}{4\tau}\left(\|U^{j+1}\|^2 - \|U^{j-1}\|^2\right) + \nu\|\overline{U}_{\overline{x}}^{\,j}]|^2 = (F^j, \overline{U}^{\,j}), \ \ j=1,2,\dots\,, \tag{2.4}$$

$$\frac{1}{2\tau}\left(\|U^1\|^2 - \|U^0\|^2\right) + \nu\|\overline{U}_{\overline{x}}^{\,0}]|^2 = (F^0, \overline{U}^{\,0}). \tag{2.5}$$

Summing up the equalities (2.4) with respect to j from 1 to k, we get

$$\frac{1}{2\tau}\left(\|U^{k+1}\|^2 + \|U^k\|^2 - \|U^1\|^2 - \|U^0\|^2\right) + 2\nu\sum_{j=1}^k \|\overline{U}_{\overline{x}}^{\,j}]|^2 = 2\sum_{j=1}^k (F^j, \overline{U}^{\,j}). \tag{2.6}$$

Adding the equalities (2.5) and (2.6) gives

$$\frac{1}{2\tau}\left(\|U^{k+1}\|^2 + \|U^k\|^2\right) + 2\nu\sum_{j=0}^k \sigma_j\|\overline{U}_{\overline{x}}^{\,j}]|^2 = \frac{1}{\tau}\,\|U^0\|^2 + 2\sum_{j=0}^k \sigma_j(F^j, \overline{U}^{\,j}), \ \ k=1,2,\dots\,, \tag{2.7}$$

where $\sigma_j = 1$ for $j\ge 1$ and $\sigma_0 = 1/2$.

If we rewrite the equality (2.5) in the form

$$\frac{1}{2\tau}\left(\|U^1\|^2 + \|U^0\|^2\right) + \nu\|\overline{U}_{\overline{x}}^{\,0}]|^2 = \frac{1}{\tau}\,\|U^0\|^2 + (F^0, \overline{U}^{\,0}), \tag{2.8}$$

we will see that the equalities (2.7), (2.8) can be written all in the same key

$$\frac{1}{2}\left(\|U^{j+1}\|^2 + \|U^j\|^2\right) + 2\nu\tau\sum_{k=0}^j \sigma_k\|\overline{U}_{\overline{x}}^{\,k}]|^2 = \|\varphi\|^2 + 2\tau\sum_{k=0}^j \sigma_k(F^j, \overline{U}^{\,j}), \ \ j=0,1,2,\dots\,. \tag{2.9}$$

Since the difference scheme (2.1), (2.2) is linear on each new level with respect to the unknown values, its unique solvability follows directly from (2.9). □

Remark. Let the external source $f(x,t)$ be equal to 0. Then we rewrite (2.9) as

$$E(U^j)+\nu\sum_{k=0}^{j}\sigma_k\tau\|\overline{U}^k_{\overline{x}}\|^2=0.5\|\varphi\|^2,\ \ j=0,1,\dots .$$

The left-hand side of this equality is the energy of the system at time $t=t_j$. As we see, the difference scheme is energy conservative and, besides, kinetic energy

$$E(U^j):=\frac{\|U^{j+1}\|^2+\|U^j\|^2}{4}$$

is monotonically decreasing, i.e.,

$$E(U^{j+1})\le E(U^j)\ \text{ for }\ j\ge 0.$$

Theorem 2.2. *Let the exact solution of the initial boundary value problem* (1.1), (1.2) *belong to* $W_2^k(Q)$, $2<k\le 3$. *Then the convergence rate of the finite difference scheme* (2.1), (2.2) *is determined by the estimate*

$$\|U^j-u^j\|\le c(\tau^{k-1}+h^{k-1})\|u\|_{W_2^k(Q)},$$

where $c=c(u)$ *denotes the positive constant, independent of* h *and* τ.

The correctness of Theorem 2.2 follows from the consequence of Lemmas 3.1, 4.2 and 4.4, proved in the next sections.

3. A Priori Estimate of Discretization Error

Let $Z:=U-u$, where u is an exact solution of the problem (1.1), (1.2), and U is a solution of the finite difference scheme (2.1), (2.2). Substituting $U=Z+u$ into (2.1), (2.2), we obtain

$$Z^j_{\overset{\circ}{t}}-\nu\overline{Z}^j_{\overline{x}x}=-\frac{1}{3}\,(\Lambda U^j-\Lambda u^j)+\Psi^j, \tag{3.1}$$

$$Z^0_t-\nu\overline{Z}^0_{\overline{x}x}=-\frac{1}{3}\,(\Lambda U^0-\Lambda u^0)+\Psi^0, \tag{3.2}$$

$$Z^0=0,\quad Z^j_0=Z^j_n=0,\ \ j=0,1,2,\dots , \tag{3.3}$$

where $\Psi^j:=F^j-\mathcal{L}u^j$.

Denote

$$B_j:=\|Z^j\|^2+\|Z^{j-1}\|^2,\ \ j=1,2,\dots .$$

Lemma 3.1. *For a solution of the problem* (3.1)–(3.3), *the relations*

$$B_1\le\|\tau\Psi^0\|^2, \tag{3.4}$$

$$B_{j+1}\le c_1B_1+c_2\tau\sum_{k=1}^{j}\|\Psi^k\|^2,\ \ j=1,2,\dots , \tag{3.5}$$

are valid, where

$$c_1=\exp\left(\frac{Tc_*^2}{3\nu}\right),\ \ c_2=\frac{c_1}{2\nu}\,,\ \ c_*=\|u\|_{C^1(\overline{Q})}.$$

Proof. Multiplying (3.2) by $\overline{Z}^0$, we obtain

$$(Z^0_t,\overline{Z}^0)+\nu(\overline{Z}^0_{\overline{x}},\overline{Z}^0_{\overline{x}})=-\frac{1}{3}\,(\Lambda U^0-\Lambda u^0,\overline{Z}^0)+(\Psi^0,\overline{Z}^0).$$

Taking into account $U^0=u^0$ we have

$$\Lambda U^0-\Lambda u^0=u^0\overline{Z}^0_{\overset{\circ}{x}}+(u^0\overline{Z}^0)_{\overset{\circ}{x}},$$

therefore due to (2.3)

$$(\Lambda U^0-\Lambda u^0,\overline{Z}^0)=0$$

and we get

$$(Z_t^0,\overline{Z}^{\,0})+\nu(\overline{Z}^{\,0}_{\overline{x}},\overline{Z}^{\,0}_{\overline{x}})=(\Psi^0,\overline{Z}^{\,0}).$$

From this, via $Z^0=0$, we see that

$$\frac{1}{2\tau}\,\|Z^1\|^2+\frac{\nu}{4}\,\|Z^1_{\overline{x}}\|^2=\frac{1}{2}\,(\Psi^0,Z^1),$$

or

$$\|Z^1\|^2+\frac{\nu\tau}{2}\,\|Z^1_{\overline{x}}\|^2=(\tau\Psi^0,Z^1),$$

where

$$\|Z^1\|^2+\frac{\nu\tau}{2}\,\|Z^1_{\overline{x}}\|^2\le\frac{1}{4}\,\|\tau\Psi^0\|^2+\|Z^1\|^2$$

and

$$\|Z^1_{\overline{x}}\|^2\le\frac{\tau}{2\nu}\,\|\Psi^0\|^2,$$

and also

$$\|Z^1\|^2\le\|\tau\Psi^0\|\,\|Z^1\|$$

and

$$\|Z^1\|\le\|\tau\Psi^0\|.$$

On the basis of the above consideration, we come to the conclusion that (3.4) is true.

Now, let us multiply (3.1) by $\overline{Z}^{\,j}$ scalarly:

$$\frac{1}{4\tau}\,\big(\|Z^{j+1}\|^2-\|Z^{j-1}\|^2\big)+\nu\|\overline{Z}^{\,j}_{\overline{x}}]|^2=-\frac{1}{3}\,(\Lambda U^j-\Lambda u^j,\overline{Z}^{\,j})+(\Psi^j,\overline{Z}^{\,j}),\ \ j=1,2,\dots. \tag{3.6}$$

Noticing in the right-hand side of (3.6) that

$$\Lambda U^j-\Lambda u^j=(U^j\overline{Z}^{\,j}_{\overset{\circ}{x}}+(U^j\overline{Z}^{\,j})_{\overset{\circ}{x}})+(Z^j\overline{U}^{\,j}_{\overset{\circ}{x}}+(Z^j\overline{U}^{\,j})_{\overset{\circ}{x}}),$$

and taking into account (2.3), we obtain

$$(\Lambda U^j-\Lambda u^j,\overline{Z}^{\,j})=(Z^j\overline{u}^{\,j}_{\overset{\circ}{x}}+(Z^j\overline{u}^{\,j})_{\overset{\circ}{x}},\overline{Z}^{\,j})=(Z^j\overline{u}^{\,j}_{\overset{\circ}{x}},\overline{Z}^{\,j})-(Z^j\overline{u}^{\,j},\overline{Z}^{\,j}_{\overset{\circ}{x}})=(Z^j\overline{Z}^{\,j},\overline{u}^{\,j}_{\overset{\circ}{x}})-(Z^j\overline{Z}^{\,j}_{\overset{\circ}{x}},\overline{u}^{\,j}).$$

Applying here the Cauchy–Bunyakovsky inequality, the ε-inequality, and finally the Friedrichs' inequality

$$\|V\|^2\le\frac{1}{8}\,\|V_{\overline{x}}]|^2,$$

we obtain

$$\begin{aligned}\big|(\Lambda U^j-\Lambda u^j,\overline{Z}^{\,j})\big|&\le c_*\big(\|Z^j\|\,\|\overline{Z}^{\,j}\|+\|Z^j\|\,\|\overline{Z}^{\,j}_{\overline{x}}]|\big)\\&\le c_*\Big(\frac{\varepsilon}{2}\,\|Z^j\|^2+\frac{1}{2\varepsilon}\,\|\overline{Z}^{\,j}\|^2+\frac{\varepsilon}{2}\,\|Z^j\|^2+\frac{1}{2\varepsilon}\,\|\overline{Z}^{\,j}_{\overline{x}}]|^2\Big)\le c_*\Big(\varepsilon\|Z^j\|^2+\frac{9}{16\varepsilon}\,\|\overline{Z}^{\,j}_{\overline{x}}]|^2\Big).\end{aligned} \tag{3.7}$$

Now, let us estimate the second term in the right-hand side of (3.6)

$$|(\Psi^j,\overline{Z}^{\,j})|\le\|\Psi^j\|\,\|\overline{Z}^{\,j}\|\le\frac{\varepsilon}{2c_*}\|\Psi^j\|^2+\frac{c_*}{2\varepsilon}\|\overline{Z}^{\,j}\|^2\le\frac{\varepsilon}{2c_*}\,\|\Psi^j\|^2+\frac{c_*}{16\varepsilon}\,\|\overline{Z}^{\,j}_{\overline{x}}]|^2. \tag{3.8}$$

After substituting (3.7) and (3.8) in (3.6), we arrive at

$$\begin{aligned}\frac{1}{4\tau}\,\big(\|Z^{j+1}\|^2-\|Z^{j-1}\|^2\big)+\nu\|\overline{Z}^{\,j}_{\overline{x}}]|^2&\le c_*\Big(\frac{\varepsilon}{3}\,\|Z^j\|^2+\frac{3}{16\varepsilon}\,\|\overline{Z}^{\,j}_{\overline{x}}]|^2\Big)+\frac{\varepsilon}{2c_*}\,\|\Psi^j\|^2+\frac{c_*}{16\varepsilon}\,\|\overline{Z}^{\,j}_{\overline{x}}]|^2\\&\le\frac{\varepsilon c_*}{3}\,\|Z^j\|^2+\frac{\varepsilon}{2c_*}\,\|\Psi^j\|^2+\frac{c_*}{4\varepsilon}\,\|\overline{Z}^{\,j}_{\overline{x}}]|^2.\end{aligned}$$

Here choose $\varepsilon=\frac{c_*}{4\nu}$. Then we obtain

$$\frac{1}{4\tau}\,\big(\|Z^{j+1}\|^2-\|Z^{j-1}\|^2\big)\le\frac{1}{8\nu}\,\|\Psi^j\|^2+\frac{c_*^2}{12\nu}\,\|Z^j\|^2,$$

that is,

$$\|Z^{j+1}\|^2-\|Z^{j-1}\|^2\le\frac{\tau}{2\nu}\,\|\Psi^j\|^2+\frac{c_*^2\tau}{3\nu}\,\|Z^j\|^2,\ \ j=1,2,\dots. \tag{3.9}$$

Suppose

$$a:=\frac{c_*^2}{3\nu}\,,\quad b:=\frac{1}{2\nu}\,.$$

From (3.9) we find
$$B_{j+1} \le (1+a\tau)B_j + b\tau\|\Psi^j\|^2, \ \ j=1,2,\dots,$$
whence
$$B_{j+1} \le (1+a\tau)^j B_1 + b\tau(1+a\tau)^{j-1}\sum_{k=1}^{j}\|\Psi^k\|^2, \ \ j=1,2,\dots. \tag{3.10}$$
Since $j \le T/\tau$, we obtain
$$(1+a\tau)^j \le (1+a\tau)^{T/\tau} \le \exp(Ta),$$
and on the basis of (3.10), the validity of (3.5) follows directly. Thus Lemma 3.1 is proved. □

4. Estimation of the Truncation Error

In order to determine the rate of convergence of the finite difference scheme (2.1), (2.2) with the help of Lemma 3.1, it is sufficient to estimate a truncation error eventuated while replacing a differential equation by a difference scheme, Ψ. Towards this end, we will need the following result.

Lemma 4.1. *Assume that the linear functional $l(u)$ is bounded in $W_2^k(E)$, where $k=\overline{k}+\epsilon$, $\overline{k}$ is an integer, $0<\epsilon\le 1$, and $l(P)=0$ for every polynomial P of degree $\overline{k}$ in two variables. Then, there exists a constant c, independent of u, such that $|l(u)| \le c|u|_{W_2^k(E)}$.*

This lemma is a particular case of the Dupont–Scott approximation theorem [12] and represents a generalization of the Bramble–Hilbert lemma [8] (see, e.g., [18, p. 29]).

Let us introduce the elementary rectangles $e=e(x,t)=\{(x,t):\ |x-x_i|\le h,\ |t-t_j|\le\tau\}$, $e_0=(x_{i-1},x_{i+1})\times(0,\tau)$, $Q_\tau=(0,1)\times(0,\tau)$, $Q_j=(0,1)\times(t_{j-1},t_{j+1})$.

Lemma 4.2. *If a solution u of the problem* (1.1), (1.2) *belongs to the Sobolev space $W_2^k(Q)$, $2<k\le 3$, then for the truncation error $\Psi^j = F^j - \mathcal{L}u^j$ the estimate*
$$\|\Psi^j\|^2 \le c(\tau+h)^{2k-3}\|u\|^2_{W_2^k(Q_j)}, \ \ j\ge 1,$$
is true, where the constant $c>0$ does not depend on the mesh steps.

Proof. Apply operator $\mathcal{P}$ to the equation (1.1):
$$\frac{1}{2}\,\mathcal{P}\Big(\frac{\partial u^{j-1}}{\partial t}+\frac{\partial u^{j+1}}{\partial t}+\Big(u\,\frac{\partial u}{\partial x}\Big)^{j+1}+\Big(u\,\frac{\partial u}{\partial x}\Big)^{j-1}\Big)-\frac{\nu}{2}\,(u^{j+1}+u^{j-1})_{\overline{x}\,x}=F^j.$$
With the help of this equality, the expression Ψ can be written in the form
$$\Psi=\chi_1+\chi_3+\frac{1}{6}\,\chi_4,$$
where
$$\chi_1=\mathcal{P}\Big(\frac{\partial\overline{u}}{\partial t}\Big)+\overset{\circ}{\mathcal{S}}\Big(\frac{\partial u}{\partial t}\Big),$$
$$\chi_2:=\frac{1}{4}\,\mathcal{P}\Big(\frac{\partial(\widehat{u})^2}{\partial x}+\frac{\partial(\check{u})^2}{\partial x}\Big)-\frac{1}{2}(u)^2_{\overset{\circ}{x}},\quad \chi_4:=3(u)^2_{\overset{\circ}{x}}-2\Lambda u.$$
We assert that the following inequalities hold for $\alpha=1,2,3$:
$$|\chi_\alpha|\le c(\tau+h)^{k-2}\|u\|_{W_2^k(e)}, \ \ 2<k\le 3. \tag{4.1}$$
First of all, note that χ_1, as a linear functional with respect to $u(x,t)$, vanishes on the polynomials of second degree and is bounded in W_2^k, $k>1$. Consequently, using Lemma 4.1 and the well known techniques from [18], we see that the estimate (4.1) for $\alpha=1$ is true.

Now, let us note that
$$\chi_2=\chi_2(u)=\ell(v):=\frac{1}{2}\,(\widehat{\mathcal{P}}\overset{\circ}{\mathcal{S}}v_{\overline{x}}-v_{\overset{\circ}{x}}),\ \ v:=(u)^2.$$
The linear functional $\ell(v)$ is bounded for $v\in W_2^k$, $k>2$, and vanishes on polynomials of second degree. For this functional the estimate
$$|\ell(v)|\le c(\tau+h)^{k-2}\|v\|_{W_2^k(e)}, \ \ 2<k\le 3, \tag{4.2}$$
is obtained.

Since Sobolev space $W_2^k(Q)$, $k > 1$, is an algebra with respect to a pointwise multiplication, consequently, $\|uu\|_{W_2^k(e)} \le c\|u\|_{W_2^k(e)}$, $c = c(u)$. Therefore, (4.2) proves the validity of (4.1) in the case where $\alpha = 2$.

We will present estimates χ_3 in a more convenient form. We have

$$\begin{aligned}\chi_3 &= 3(u)^2_{\overset{\circ}{x}} - u(\widehat{u} + \check{u})_{\overset{\circ}{x}} - (u(\widehat{u} + \check{u}))_{\overset{\circ}{x}}\\ &= 3(u)^2_{\overset{\circ}{x}} - u(\widehat{u} - 2u + \check{u})_{\overset{\circ}{x}} - \big(u(\widehat{u} - 2u + \check{u})\big)_{\overset{\circ}{x}} - 2uu_{\overset{\circ}{x}} - 2(uu)_{\overset{\circ}{x}}\\ &= (u)^2_{\overset{\circ}{x}} - 2uu_{\overset{\circ}{x}} - \tau^2 uu_{\bar{t}t\overset{\circ}{x}} - \tau^2(uu_{\bar{t}t})_{\overset{\circ}{x}},\end{aligned}$$

whence

$$\chi_3 = h^2 u_{\overset{\circ}{x}} u_{\bar{x}x} - \tau^2 uu_{\bar{t}t\overset{\circ}{x}} - \tau^2(uu_{\bar{t}t})_{\overset{\circ}{x}} := \chi_3' + \chi_3'' + \chi_3''', \tag{4.3}$$

since

$$(u)^2_{\overset{\circ}{x}} - 2uu_{\overset{\circ}{x}} = u_{\overset{\circ}{x}}(u_{i+1} + u_{i-1}) - 2uu_{\overset{\circ}{x}} = h^2 u_{\overset{\circ}{x}} u_{\bar{x}x}.$$

When $u \in W_2^k(Q)$, $2 < k \le 3$, the terms in the right-hand side of (4.3) can be estimated as follows:

$$|\chi_3'| \le h^2\|u\|_{C^1(\overline{Q})}|u_{\bar{x}x}| \le c(\tau + h)^{k-2}\|u\|_{W_2^2(e)} \le c(\tau + h)^{k-2}\|u\|_{W_2^{k-2}(e)},$$

$$|\chi_3''| \le \tau^2\|u\|_{C^1(\overline{Q})}|u_{\bar{t}t\overset{\circ}{x}}| \le c(\tau + h)^{k-2}\|u\|_{W_2^{k-2}(e)},$$

$$|\chi_3'''| = \tau^2|u_{i+1}u_{\bar{t}t\overset{\circ}{x}} + u_{\overset{\circ}{x}}u_{\bar{t}t,i-1}| \le \|u\|_{C^1(\overline{Q})}(|u_{\bar{t}t\overset{\circ}{x}}| + |u_{\bar{t}t,i-1}|) \le c(\tau + h)^{k-2}\|u\|_{W_2^{k-2}(e)}$$

and therefore (4.1) is true for $\alpha = 3$ also.

Finally, (4.1) yields

$$\|\chi_\alpha\|^2 = \sum_{x\in\omega} h|\chi_\alpha|^2 \le c(\tau + h)^{2k-3}\|u\|^2_{W_2^k(Q_j)}, \quad \alpha = 1, 2, 3,$$

which completes the proof of Lemma 4.2. □

Lemma 4.3. *For any function $v \in W_2^k(Q)$, $1 < k \le 3$, the inequalities*

$$\|v^0_{\overset{\circ}{x}t}\| \le c(\tau + h)^{k-3}\|v\|_{W_2^k(Q)}, \tag{4.4}$$

$$\|v^0_{\bar{x}x}\| \le c(\tau + h)^{k-3}\|v\|_{W_2^k(Q)} \tag{4.5}$$

are true.

Proof. $v^0_{\overset{\circ}{x}t}$ is bounded when $v \in W_2^\lambda(Q)$, $\lambda > 1$, and vanishes on the first degree polynomials. Therefore for $1 < \lambda \le 2$ we have

$$|v^0_{\overset{\circ}{x}t}| \le c(\tau + h)^{\lambda-3}\|v\|_{W_2^\lambda(e_0)},$$

$$\|v^0_{\overset{\circ}{x}t}\|^2 = \sum_{\omega} h|v^0_{\overset{\circ}{x}t}|^2 \le c(\tau + h)^{2\lambda-5}\|v\|^2_{W_2^\lambda(Q_\tau)},$$

which confirms the validity of (4.4) in the case where $1 < k \le 2.5$. Further,

$$|v^0_{\overset{\circ}{x}t}| = \frac{1}{2\tau h}\bigg|\int_0^\tau\int_{x_{i-1}}^{x_{i+1}} \frac{\partial^2 v}{\partial x\partial t}\,dx\,dt\bigg| \le (2\tau h)^{-1/2}\Big\|\frac{\partial^2 v}{\partial x\partial t}\Big\|_{L_2(e_0)},$$

$$\|v^0_{\overset{\circ}{x}t}\| \le c\tau^{-1/2}\Big\|\frac{\partial^2 v}{\partial x\partial t}\Big\|_{L_2(Q_\tau)}. \tag{4.6}$$

In order to obtain the desired estimate, it is sufficient to use the inequality giving estimate of the L_2-norm of the function in the near-border stripe via its W_2^λ-norm in the domain (cf. [18, p. 161])

$$\|v\|_{L_2(Q_\tau)} \le c\tau^{1/2}\|v\|_{W_2^\lambda(Q)}, \quad 0.5 < \lambda \le 1.$$

This relation along with (4.6) confirms the validity of (4.4) for $2.5 < k \le 3$.

When $1 < k \le 2.5$, (4.5) can be proved similarly to the previous case. In the event of $2.5 < k \le 3$, we use the relation

$$|u_{\bar{x}x}| \le |\mathcal{P}\widehat{\mathcal{S}}\,\frac{\partial^2 u}{\partial x^2}| + |(u - \widehat{\mathcal{S}})_{\bar{x}x}|.$$

Here the first term in the right-hand side is estimated again analogously to the previous case, and for the second term Lemma 4.1 is used. $\square$

Lemma 4.4. *If a solution u of the problem* (1.1), (1.2) *belongs to the Sobolev space $W_2^k(Q)$, $k>2$, then for the truncation error $\Psi^0 = F^0 - \mathcal{L}u^0$ the estimate*

$$\|\Psi^0\| \le c(\tau+h)^{k-2}\|u\|^2_{W_2^k(Q)}, \;\; 2<k\le 3,$$

is true, where the constant $c>0$ does not depend on the mesh steps.

Proof. Apply operator $\mathcal{P}$ to the equation (1.1):

$$F^0 = \frac{1}{2}\,\mathcal{P}(f^0+f^1) = \frac{1}{2}\,\mathcal{P}\Big(\frac{\partial u^0}{\partial t} + \frac{\partial u^1}{\partial t}\Big) + \frac{1}{4}\,\mathcal{P}\bigg(\frac{\partial (u)^2}{\partial x}\Big|_{t=0} + \frac{\partial (u)^2}{\partial x}\Big|_{t=\tau}\bigg) - \nu \overline{u}_{\overline{x}\,x}.$$

Via this equality we rewrite Ψ^0 as

$$\Psi^0 = \zeta_1 - \frac{1}{6}\,\zeta_2 - \frac{1}{2}\,\zeta_3, \;\; t=0,$$

where

$$\begin{aligned} \zeta_1 &:= \mathcal{P}\,\frac{\partial \overline{u}}{\partial t} - u_t^0, \quad \zeta_2 := 2(u\overline{u}_{\overset{\circ}{x}} + (u\overline{u})_{\overset{\circ}{x}}) - \frac{3}{2}\,\big((\widehat{u})^2+(u)^2\big)_{\overset{\circ}{x}}, \\ \zeta_3 &:= \frac{1}{2}\,\big((\widehat{u})^2+(u)^2\big)_{\overset{\circ}{x}} - \frac{1}{2}\,\mathcal{P}\bigg(\frac{\partial (u)^2}{\partial x}\Big|_{t=0} + \frac{\partial (u)^2}{\partial x}\Big|_{t=\tau}\bigg). \end{aligned} \tag{4.7}$$

We assert that the inequalities

$$\|\zeta_\alpha\| \le c(\tau+h)^{k-2}\|u\|_{W_2^k(Q)}, \;\; 2<k\le 3, \tag{4.8}$$

hold for $\alpha = 1,2,3$.

Expression ζ_1 can be estimated similarly to χ_1.

Further, notice that

$$\zeta_3 = \zeta_3(u) = I(v) := \frac{1}{2}\,(\widehat{v}+v)_{\overset{\circ}{x}} - \frac{1}{2}\,\mathcal{P}\Big(\frac{\partial \widehat{v}}{\partial x} + \frac{\partial v}{\partial x}\Big), \;\; v := (u)^2.$$

It is easy to verify that $I(v)$, as a linear functional with respect to v, vanishes on the polynomials of second degree and is bounded when $v \in W_2^k(Q)$, $k>2$. For that functional we can derive the following estimate

$$\|I(v)\| \le c(\tau+h)^{k-2}\|v\|_{W_2^k(Q)}, \;\; 2<k\le 3.$$

The latter along with $\|uu\|_{W_2^k(Q)} \le c\|u\|^2_{W_2^k(Q)}$, $k>1$, states the validity of (4.8) in the case $\alpha=3$, as well.

Now, let us pass to the estimation of ζ_2. If we take into account that

$$2u\overline{u}_{\overset{\circ}{x}} = 2uu_{\overset{\circ}{x}} + \tau uu_{\overset{\circ}{x}t}, \quad 2uu_{\overset{\circ}{x}} = (u)^2{}_{\overset{\circ}{x}} - h^2 u_{\overset{\circ}{x}} u_{\overline{x}\,x},$$

(4.7) will give

$$\begin{aligned} \zeta_2 &= \tau uu_{\overset{\circ}{x}t} - h^2 u_{\overset{\circ}{x}} u_{\overline{x}\,x} + \frac{1}{2}\,\big(4u\overline{u} - 3(\widehat{u})^2 - (u)^2\big)_{\overset{\circ}{x}} \\ &= \tau uu_{\overset{\circ}{x}t} - h^2 u_{\overset{\circ}{x}} u_{\overline{x}\,x} - \frac{1}{2}\,\Big(2\big[(\widehat{u})^2 - (u)^2\big] + \big[(\widehat{u})^2 - 2\widehat{u}u + (u)^2\big]\Big)_{\overset{\circ}{x}} \end{aligned}$$

or

$$\zeta_2 = \tau uu_{\overset{\circ}{x}t} - h^2 u_{\overset{\circ}{x}} u_{\overline{x}\,x} - \tau (u)^2{}_{\overset{\circ}{x}t} - \frac{\tau^2}{2}\,(u_t)^2{}_{\overset{\circ}{x}} := \zeta_2' + \zeta_2'' + \zeta_2''' + \zeta_2''''. \tag{4.9}$$

In the right-hand side of (4.9), the first and the second terms can be estimated by using Lemma 4.3:

$$\begin{aligned} \|\zeta_2'\| &\le c\tau\|u\|_{C(\overline{Q})}\|u_{\overset{\circ}{x}t}\| \le c(\tau+h)^{k-2}\|u\|_{W_2^k(Q)}, \;\; 2<k\le 3, \\ \|\zeta_2''\| &\le ch\|u\|_{C(\overline{Q})}\|u_{\overline{x}\,x}\| \le c(\tau+h)^{k-2}\|u\|_{W_2^k(Q)}, \;\; 2<k\le 3. \end{aligned}$$

The term ζ_2''' can be estimated in a similar way, if we make replacement $(u)^2 := v$ in it.

Change the term ζ_2'''' as follows:

$$\frac{\tau^2}{2}\,(u_t)^2{}_{\overset{\circ}{x}} = \frac{\tau^2}{2}\,\frac{(u_t)^2_{i+1} - (u_t)^2_{i-1}}{2h} = \frac{\tau^2}{2}\,\frac{(u_{t,i+1} - u_{t,i-1})(u_{t,i+1} + u_{t,i-1})}{2h} = \tau^2 u_{\overset{\circ}{x}t}\,\frac{u_{t,i+1} + u_{t,i-1}}{2}\,,$$

from which again via Lemma 4.3 we get

$$\|\zeta_2''''\| \le c\tau |u_{\overset{\circ}{x}t}| \le c(\tau+h)^{k-2}\|u\|_{W_2^k(Q)}, \quad 2<k\le 3.$$

Finally, all of these estimates confirm the validity of (4.8) in the case $\alpha = 2$.

The inequalities (4.8) prove Lemma 4.4. □

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ON A CLASS OF NONLINEAR NONAUTONOMOUS ORDINARY DIFFERENTIAL EQUATIONS OF n-TH ORDER

Abstract. Asymptotic representations of solutions of nonautonomous nonlinear ordinary differential n-th order equations that are close, in a certain sense, to linear equations are established.

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რეზიუმე. n-ური რიგის არაავტონომიური არაწრფივი ჩვეულებრივი დიფერენციალური განტოლებებისთვის, რომლებიც გარკვეული აზრით ახლოს არიან წრფივ განტოლებებთან, დადგენილია ამონახსნების ასიმპტოტური წარმოდგენები.

1. Introduction and Preliminaries

Consider the differential equation

$$y^{(n)} = \alpha_0 p(t) y |\ln|y||^{\sigma}, \tag{1.1}$$

where $\alpha_0 \in \{-1, 1\}$, $\sigma \in \mathbb{R}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $-\infty < a < \omega \leq +\infty$[1].

A solution y of the equation (1.1) is called a $P_\omega(\lambda_0)$-solution, if it is defined on the interval $[t_y, \omega[\subset [a, \omega[$, and satisfies the conditions:

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{either} & 0, \\ \text{or} & \pm\infty \end{cases} \quad (k = 0, 1, \dots, n-1), \quad \lim_{t\uparrow\omega} \frac{(y^{(n-1)}(t))^2}{y^{(n)}(t) y^{(n-2)}(t)} = \lambda_0. \tag{1.2}$$

For each such solution, the representation $y(t)|\ln|y(t)|| = |y(t)|^{1+o(1)} \operatorname{sign} y(t)$ as $t \uparrow \omega$, holds. Therefore, when we study these solutions, the equation (1.1) is asymptotically close to linear differential equations

$$y^{(n)} = \alpha_0 p(t) y, \tag{1.3}$$

such asymptotic behavior of solutions has been studied extensively (see, e.g., [9, Chapter 1]).

For $n = 2$ and any $\sigma \in \mathbb{R}$, asymptotic behavior as $t \uparrow \omega$ of all possible types of $P_\omega(\lambda_0)$-solutions of the differential equation (1.1) was studied in [1, 2, 3, 5, 7].

We introduce the following auxiliary notation:

$$a_{0k} = (n-k)\lambda_0 - (n-k-1) \ \ (k = 1, \dots, n) \ \text{ for } \ \lambda_0 \in \mathbb{R},$$

$$\pi_\omega(t) = t - \omega, \ \text{ if } \ \omega < +\infty,$$

$$I_A(t) = \int_A^t [\pi_\omega(\tau)]^{n-1} p(\tau)\, d\tau,$$

$$A = \omega, \ \text{ if } \int_a^\omega |\pi_\omega(\tau)|^{n-1} p(\tau)\, d\tau < +\infty.$$

The following theorem concerning the differential equation (1.1) has been established in [4].

Theorem 1.1. *Let $\sigma \neq n$ and $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{n-2}{n-1}, 1\}$. Then for the existence of a $P_\omega(\lambda_0)$-solution of the equation* (1.1) *it is necessary, and if the inequality*

$$\sigma \neq a_{01}\Big(1 + \sum_{k=1}^{n-1} \frac{1}{a_{0k}}\Big) \tag{1.4}$$

holds and the algebraic equation

$$\prod_{j=1}^{n-1}(a_{0j} + \rho) + \sum_{k=1}^{n-1} \prod_{j=1}^{k-1}(a_{0j} + \rho) \prod_{j=k+1}^{n-1} a_{0j} = 0 \tag{1.5}$$

with respect to ρ has no roots with zero real part, then it is sufficient for the inequality

$$\alpha_0 \Big(\prod_{k=1}^{n-1} a_{0k}\Big) \big[(\lambda_0 - 1)\pi_\omega(t)\big]^n > 0 \ \textit{ for } \ t \in [a, \omega[\,, \tag{1.6}$$

and the conditions

$$\lim_{t\uparrow\omega} p^{\frac{1}{n}}(t) |\pi_\omega(t)| \Big|\frac{n-\sigma}{n} J_B(t)\Big|^{\frac{\sigma}{n-\sigma}} = \frac{|a_{01}|}{|\lambda_0 - 1|} \left(\frac{\prod_{k=1}^{n-1} |a_{0k}|^{\frac{1}{n}}}{|a_{01}|}\right)^{\frac{n}{n-\sigma}},$$

[1] We assume that $a > 1$ for $\omega = +\infty$, and $\omega - a < 1$ for $\omega < +\infty$.

to take place. Moreover, each of these solutions admits the following asymptotic representations as $t \uparrow \omega$*:*

$$\ln|y(t)| = \nu\Big(\frac{|a_{01}|}{\prod_{k=1}^{n-1}|a_{0k}|^{\frac{1}{n}}}\Big)^{\frac{n}{n-\sigma}}\Big|\frac{n-\sigma}{n}\,J_B(t)\Big|^{\frac{n}{n-\sigma}}[1+o(1)], \tag{1.7}$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{a_{0k}}{(\lambda_0-1)\pi_\omega(t)}\,[1+o(1)] \ \ (k=1,\dots,n-1), \tag{1.8}$$

where

$$\nu = \operatorname{sign}\big[a_{01}(\lambda_0-1)(n-\sigma)\pi_\omega(t)J_B(t)\big].$$

In addition to these conditions, if the algebraic equation (1.5) *has the* m*-roots* (*including multiples*)*, the real parts of which have a sign opposite to the sign of the function* $(\lambda_0-1)\pi_\omega(t)$ *on the interval* $[a,\omega[$ *, and the inequality*

$$\Big(\frac{\sigma}{a_{01}} - 1 - \sum_{k=1}^{n-1}\frac{1}{a_{0k}}\Big)\Big(1+\sum_{k=1}^{n-1}\frac{1}{a_{0k}}\Big) > 0$$

is satisfied, then the equation (1.1) *has* m*-parametric family of solutions with the representations* (1.7) *and* (1.8)*, and when the opposite inequality holds, it has* $m+1$*-parametric family of such solutions.*

From this theorem the following corollary for the linear differential equation (1.3) is obtained.

Corollary 1.1. *For the existence of* $P_\omega(\lambda_0)$*-solution of the equation* (1.3)*, where* $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{n-2}{n-1}, 1\}$*, it is necessary, and if the algebraic equation* (1.5) *with respect to* ρ *has no roots with zero real part, then it is sufficient that the inequality* (1.6) *and the condition*

$$\lim_{t\uparrow\omega} p(t)\pi_\omega^n(t) = \frac{\alpha_0\prod_{k=1}^{n-1}a_{0k}}{(\lambda_0-1)^n} \tag{1.9}$$

are satisfied. For each of these solutions the asymptotic representations

$$\ln|y(t)| = \frac{\alpha_0(\lambda_0-1)^{n-1}I_A(t)}{\prod_{k=2}^{n-1}a_{0k}}\,[1+o(1)], \tag{1.10}$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{a_{0k}}{(\lambda_0-1)\pi_\omega(t)}\,[1+o(1)] \ \ (k=1,\dots,n-1), \tag{1.11}$$

take place as $t\uparrow\omega$*. Moreover, if in addition to these conditions, the algebraic equation* (1.5) *has the* m*-roots* (*including multiples*)*, the real parts of which have a sign, opposite to that of the function* $(\lambda_0-1)\pi_\omega(t)$ *on the interval* $[a,\omega[$ *, then for the equation* (1.1) *there exists* $m+1$*-parametric family of solutions with the representations* (1.10) *and* (1.11)*.*

We note that this corollary refers to the case where the differential equation (1.3) is asymptotically close to the Euler equations.

If

$$\lim_{t\uparrow\omega} p(t)\pi_\omega^n(t) = c_0 \neq 0$$

and the next algebraic equation with respect to λ_0

$$c_0(\lambda_0-1)^n = \alpha_0\prod_{k=1}^{n-1}\big[(n-k)\lambda_0-(n-k-1)\big],$$

which we obtain from (1.9) by taking into account the inequality (1.6), has n distinct real roots λ_{0j} $(j = 1, \dots, n)$, then the fundamental system of solutions y_j $(j = 1, \dots, n)$ of the differential equation (1.3) admits as $t \uparrow \omega$ the following asymptotic representations:

$$\ln|y_j(t)| = \frac{\alpha_0(\lambda_{0j}-1)^{n-1} I_A(t)}{\prod_{k=2}^{n-1}[(n-j)\lambda_{0j}-(n-j-1)]}\,[1+o(1)],$$
$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{(n-j)\lambda_{0j}-(n-j-1)}{(\lambda_{0j}-1)\pi_\omega(t)}\,[1+o(1)] \ \ (k = 1, \dots, n-1;\ \ j = 1, \dots, n).$$

From the previous statements it is clear that the case for $\lambda_0 = 1$ is a special one in the study of $P_\omega(\lambda_0)$-solutions. This case is the subject of this work.

2. The Main Result and the Necessary Auxiliary Statements for its Establishment

We introduce the function $J_B(t)$, setting

$$J_B(t) = \int\limits_B^t p^{\frac{1}{n}}(\tau)\,d\tau, \quad B = \begin{cases} a, & \text{if } \int\limits_a^\omega p^{\frac{1}{n}}(\tau)\,d\tau = +\infty, \\ \omega, & \text{if } \int\limits_a^\omega p^{\frac{1}{n}}(\tau)\,d\tau < +\infty. \end{cases}$$

The main result of this paper is the following

Theorem 2.1. *Let $\sigma \neq n$. Then for the existence of $P_\omega(1)$-solution of the equation* (1.1) *it is necessary that for some $\mu \in \{-1, 1\}$ the inequality*

$$\alpha_0\mu^n > 0 \tag{2.1}$$

and the condition

$$\lim_{t\uparrow\omega} |\pi_\omega(t)| p^{\frac{1}{n}}(t) |J_B(t)|^{\frac{\sigma}{n-\sigma}} = +\infty \tag{2.2}$$

hold. Moreover, each of these solutions admits the following asymptotic representations as $t \uparrow \omega$

$$\ln|y(t)| = \nu \Big|\frac{n-\sigma}{n} J_B(t)\Big|^{\frac{n}{n-\sigma}}[1+o(1)], \tag{2.3}$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \mu p^{\frac{1}{n}}(t) \Big|\frac{n-\sigma}{n} J_B(t)\Big|^{\frac{\sigma}{n-\sigma}}[1+o(1)] \ \ (k = 1, \dots, -1), \tag{2.4}$$

where

$$\nu = \mu \operatorname{sign}\Big(\frac{n-\sigma}{n}\, J_B(t)\Big).$$

If the function $p : [a, \omega[\, \to\,]0, +\infty[$ is continuously differentiable, and there exists the limit (finite or equal to $\pm\infty$)

$$\lim_{t\uparrow\omega} \frac{(p^{\frac{1}{n}}(t)|J_B(t)|^{\frac{\sigma}{n-\sigma}})'}{p^{\frac{2}{n}}(t)|J_B(t)|^{\frac{2\sigma}{n-\sigma}}}, \tag{2.5}$$

and if (2.1) *and* (2.2) *hold, then the equation* (1.1) *has at least one $P_\omega(1)$-solution which admits the asymptotic representations* (2.3), (2.4) *as $t \uparrow \omega$. If $\mu = 1$ and $\sigma > n$, then there exists $(n-1)$-parametric family of solutions, if $\mu = 1$ and $\sigma < n$, then we get n-parametric family of solutions, if $\mu = -1$ and $\sigma < n$, then we obtain one parametric family of solutions.*

To prove Theorem 2.1, we will use the following lemma which can be deduced from Lemmas 10.1–10.6 in [6].

Lemma 2.1. *Let $y : [t_0, \omega[\to \mathbb{R} \setminus \{0\}$ be an arbitrary $P_\omega(1)$-solution of the equation* (1.1)*. Then we have the following asymptotic relations:*

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} \sim \frac{y'(t)}{y(t)} \quad (k = 1, \dots, n) \quad \text{as} \quad t \uparrow \omega \tag{2.6}$$

and

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) y^{(k)}(t)}{y^{(k-1)}(t)} = \pm\infty \quad (k = 1, \dots, n). \tag{2.7}$$

Along with this lemma, we will also need the next result on the existence of vanishing at infinity solutions of a system of quasi-linear differential equations

$$\begin{cases} v_k' = \beta_0 \Big[f_k(\tau, v_1, \dots, v_n) + \sum\limits_{i=1}^n c_{ki} v_i + V_k(v_1, \dots, v_n) \Big] & (k = 1, \dots, n-1), \\ v_n' = H(\tau) \Big[f_n(\tau, v_1, \dots, v_n) + \sum\limits_{i=1}^n c_{ni} v_i + V_n(v_1, \dots, v_n) \Big], \end{cases} \tag{2.8}$$

in which $\beta_0 \in \mathbb{R} \setminus \{0\}$, $c_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, n)$, $H : [\tau_0, +\infty[\to \mathbb{R} \setminus \{0\}$ is a continuous function, $f_k : [\tau_0, +\infty[\times \mathbb{R}^n_{\frac{1}{2}}$ $(k = 1, \dots, n)$ are continuous functions satisfying the conditions

$$\lim_{t \uparrow \omega} f_k(\tau, v_1, \dots, v_n) = 0 \ \text{ uniformly in } \ (v_1, \dots, v_n) \in \mathbb{R}^n_{\frac{1}{2}}, \tag{2.9}$$

where

$$\mathbb{R}^n_{\frac{1}{2}} = \Big\{ (v_1, \dots, v_n) \in \mathbb{R}^n : \ |v_i| \le \frac{1}{2} \ (i = 1, \dots, n) \Big\},$$

and $V_k : \mathbb{R}^n_{\frac{1}{2}} \to \mathbb{R}$ $(k = 1, \dots, n)$ are continuously differentiable functions such that

$$V_k(0, \dots, 0) = 0 \ \ (k = 1, \dots, n), \quad \frac{\partial V_k(0, \dots, 0)}{\partial v_i} = 0 \ \ (i, k = 1, \dots, n). \tag{2.10}$$

By Theorem 2.6 of [8], for a system of the differential equations (2.8), we have the following

Lemma 2.2. *Let the function $H : [\tau_0, +\infty[\, [\mathbb{R} \setminus \{0\}$ be continuously differentiable and satisfy the following conditions:*

$$\lim_{\tau \to +\infty} H(\tau) = 0, \quad \lim_{\tau \to +\infty} \frac{H'(\tau)}{H(\tau)} = 0, \quad \int\limits_{\tau_0}^{+\infty} H(\tau) \, d\tau = \pm\infty, \tag{2.11}$$

and the matrices $C_n = (c_{ki})_{k,i=1}^n$ and $C_{n-1} = (c_{ki})_{k,i=1}^{n-1}$ are such that $\det C_n \neq 0$, and C_{n-1} has no eigenvalues with a zero real part. Then the system of differential equations (2.8) *has at least one solution $(v_k)_{k=1}^n : [\tau_1, +\infty[\to \mathbb{R}^n_{\frac{1}{2}}$ $(\tau_1 \ge \tau_0)$, which tends to zero as $t \to +\infty$. Moreover, if among the eigenvalues of the matrix C_{n-1} there are m eigenvalues* (*taking into account multiplicity*)*, the real parts of which have opposite sign to β_0, then the system* (2.8) *has m-parametric family of solutions if $H(\tau)(\det C_n)(\det C_{n-1}) > 0$, and $m + 1$-parametric family of solutions if the inequality holds in opposite direction.*

3. Proof of the Main Theorem and the Corollary to a Linear Differential Equation

Proof of Theorem 2.1. *Necessity.* Let $y : [t_y, \omega[\to \mathbb{R} \setminus \{0\}$ be an arbitrary $P_\omega(1)$ solution of (1.1). Then, according to Lemma 2.1, the conditions (2.6) and (2.7) are satisfied. In view of (2.7), in a left neighborhood of ω,

$$\operatorname{sign} \Big(\frac{y'(t)}{y(t)} \Big) = \mu, \quad \text{where} \ \ \mu \in \{-1; 1\}. \tag{3.1}$$

Since from (1.1)

$$\frac{y^{(n)}(t)}{y(t)} = \alpha_0 p(t)|| \ln |y(t)||^{\sigma}$$

and by (2.6)

$$\frac{y^{(n)}(t)}{y(t)} = \frac{y^{(n)}(t)}{y^{(n-1)}(t)} \cdot \frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} \cdots \frac{y'(t)}{y(t)} \sim \Big(\frac{y'(t)}{y(t)}\Big)^n \text{ as } t \uparrow \omega,$$

then

$$\Big(\frac{y'(t)}{y(t)}\Big)^n = \alpha_0 p(t)| \ln |y(t)||^{\sigma}[1+o(1)] \text{ as } t \uparrow \omega.$$

Hence, in view of (3.1), it is clear that the inequality (2.1) holds, and so we have the asymptotic relation

$$\frac{y'(t)}{y(t)| \ln |y(t)||^{\frac{\sigma}{n}}} = \mu p^{\frac{1}{n}}(t)[1+o(1)] \text{ as } t \uparrow \omega. \tag{3.2}$$

Since $\sigma \neq n$, therefore, integrating this relation from t_y to t and taking into account the definition of $P_\omega(1)$-solution, we find that

$$| \ln |y(t)||^{\frac{n-\sigma}{n}} \operatorname{sign}\big(\ln |y(t)|\big) = \frac{\mu(n-\sigma)}{n}\, J_B(t)[1+o(1)] \text{ as } t \uparrow \omega.$$

Thus (2.3) holds. Taking into account (2.3), from (3.2) we obtain the representation

$$\frac{y'(t)}{y(t)} = \mu p^{\frac{1}{n}}(t)\Big|\frac{n-\sigma}{n}\, J_B(t)\Big|^{\frac{\sigma}{n-\sigma}}[1+o(1)] \text{ as } t \uparrow \omega,$$

from which, by (2.6) and (2.7), it follows that the condition (2.2) holds and we have the asymptotic representation (2.4).

Sufficiency. Let $p : [a, \omega[\to]0, +\infty[$ be continuously differentiable function for which there is a finite or equal to $\pm\infty$ limit (2.5). We show that in this case, if the conditions (2.1) and (2.2) are satisfied, then the equation (1.1) has solutions defined in the left neighborhood of ω and admits as $t \uparrow \omega$ the asymptotic representations (2.3) and (2.4).

We choose arbitrary $a_0 \in]a, \omega[$. By (2.2) we get

$$\int_{a_0}^{\omega} p^{\frac{1}{n}}(t)|J_B(t)|^{\frac{\sigma}{\sigma-n}}\, dt = +\infty,$$

hence, taking into account the form of the function J_B, it follows that

$$\lim_{t\uparrow\omega} |J_B(t)|^{\frac{n}{n-\sigma}} = +\infty. \tag{3.3}$$

Next, we establish that the limit (2.5) is equal to zero. Assume the contrary. Then, by virtue of its existence,

$$\lim_{t\uparrow\omega} Q(t) = \begin{cases} \text{either} & const \neq 0, \\ \text{or} & \pm\infty, \end{cases} \tag{3.4}$$

where

$$Q(t) = \frac{(p^{\frac{1}{n}}(t)|J_B(t)|^{\frac{\sigma}{n-\sigma}})'}{p^{\frac{2}{n}}(t)|J_B(t)|^{\frac{2\sigma}{n-\sigma}}}.$$

Integrating the function Q from a_0 to t, we obtain

$$\int_{a_0}^{t} Q(\tau)\, d\tau = -\frac{1}{p^{\frac{1}{n}}(t)|J_B(t)|^{\frac{\sigma}{n-\sigma}}} + C, \tag{3.5}$$

where C is a constant. If $\omega = +\infty$, then $\pi_\omega(t) = t$, and in this case, by (2.2), we have

$$\lim_{t\to+\infty} \frac{\int_{a_0}^t Q(\tau)\,d\tau}{t} = 0.$$

However, this is impossible since by the de L'Hospital's rule and (3.4),

$$\lim_{t\to+\infty} \frac{\int_{a_0}^t Q(\tau)\,d\tau}{t} = \lim_{t\to+\infty} Q(t) \neq 0.$$

If $\omega < \infty$, then $\pi_\omega(t) = t - \omega$ and by (2.2)

$$\lim_{t\uparrow\omega} p^{\frac{1}{n}}(t)|J_B(t)|^{\frac{\sigma}{n-\sigma}} = +\infty.$$

Therefore, from (3.5) it follows that

$$\lim_{t\uparrow\omega} \int\limits_{a_0}^{t} Q(\tau)\,d\tau = C.$$

Due to this condition, the equation (3.5) can be rewritten as

$$\int\limits_{\omega}^{t} Q(\tau)\,d\tau = -\frac{1}{p^{\frac{1}{n}}(t)|J_B(t)|^{\frac{\sigma}{n-\sigma}}}\,.$$

Dividing this relation by $\pi_\omega(t)$, taking then the limit as $t\uparrow\omega$ and using (2.2) we obtain

$$\lim_{t\uparrow\omega} \frac{\int_{\omega}^t Q(\tau)\,d\tau}{t-\omega} = 0.$$

However, the last equality is impossible because the limit owing to the de L'Hospital's rule and (3.4), is nonzero. Therefore, the assumption that the limit (2.5) is not equal to zero was incorrect.

Now, applying to the equation (1.1) the transformation

$$\begin{aligned}
&\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \mu p^{\frac{1}{n}}(t)\Big|\frac{n-\sigma}{n}\,J_B(t)\Big|^{\frac{\sigma}{n-\sigma}}[1+v_k(\tau)] \quad (k=1,\dots,n-1),\\
&\ln|y(t)| = \nu\Big|\frac{n-\sigma}{n}\,J_B(t)\Big|^{\frac{n}{n-\sigma}}[1+v_n(\tau)], \quad \tau = \Big|\frac{n-\sigma}{n}\,J_B(t)\Big|^{\frac{n}{n-\sigma}},
\end{aligned} \tag{3.6}$$

we obtain the following system of differential equations:

$$\begin{cases}
v_k' = \mu\,(1+v_k)[v_{k+1} - v_k - \mu h(\tau)] \quad (k=1,\dots,n-2),\\
v_{n-1}' = \mu\Big[\dfrac{|1+v_n|^\sigma}{(1+v_1)\cdots(1+v_{n-2})} - (1+v_{n-1})^2 - \mu h(\tau)(1+v_{n-1})\Big],\\
v_n' = g(\tau)(v_1 - v_n),
\end{cases} \tag{3.7}$$

in which

$$g(\tau(t)) = \Big|\frac{n-\sigma}{n}\,J_B(t)\Big|^{-\frac{n}{n-\sigma}}, \quad h(\tau(t)) = \frac{(p^{\frac{1}{n}}(t)|\frac{n-\sigma}{n}\,J_B(t)|^{\frac{\sigma}{n-\sigma}})'}{p^{\frac{2}{n}}(t)|\frac{n-\sigma}{n}\,J_B(t)|^{\frac{2\sigma}{n-\sigma}}}\,.$$

We will consider this system on the set $[\tau_0,+\infty[\,\times\mathbb{R}^n_{\frac{1}{2}}$, where

$$\tau_0 = \Big|\frac{n-\sigma}{n}\,J_B(a_0)\Big|^{\frac{n}{n-\sigma}}, \quad \mathbb{R}^n_{\frac{1}{2}} = \Big\{(v_1,\dots,v_n)\in\mathbb{R}^n:\ |v_k| \le \frac{1}{2}\ (k=1,\dots,n)\Big\}.$$

By (3.3) and the fact that the limit (2.5) is equal to zero as established above, we have

$$\lim_{\tau\to+\infty} g(\tau) = \lim_{t\uparrow\omega} g(\tau(t)) = 0, \qquad \lim_{\tau\to+\infty} h(\tau) = \lim_{t\uparrow\omega} h(\tau(t)) = 0. \tag{3.8}$$

Moreover,

$$\int\limits_{\tau_0}^{+\infty} g(\tau)\,d\tau = \frac{n}{n-\sigma}\int\limits_{a_0}^{\omega} \frac{p^{\frac{1}{n}}(s)\,ds}{J_B(s)} = \frac{n}{n-\sigma}\,\ln|J_B(s)|\Big|_{a_0}^{\omega} = \pm\infty. \tag{3.9}$$

By separating linear parts in the equations of the system (3.7), we obtain a system of differential equations (2.8) in which

$$\beta_0=\mu,\quad H(\tau)=g(\tau),\quad f_k(\tau,v_1,\dots,n)=-\mu(1+v_k)h(\tau)\ \ (k=1,\dots,n-1),$$
$$f_n(\tau,v_1,\dots,n)\equiv 0,\quad V_k(v_1,\dots,v_n)=v_kv_{k+1}-v_k^2\ \ (k=1,\dots,n-2),$$
$$V_{n-1}(v_1,\dots,v_n)=\frac{|1+v_n|^\sigma}{(1+v_1)\cdots(1+v_{n-2})}+\sum_{i=1}^{n-2}v_i-v_{n-1}^2-\sigma v_n,\quad V_n(v_1,\dots,v_n)\equiv 0,$$
$$c_{kk}=-1,\quad c_{kk+1}=1,\quad c_{ki}=0\ \text{ for }\ i\neq k,k+1\ \ (k=1,\dots,n-2),$$
$$c_{n-1i}=-1\ \ (i=1,\dots,n-2),\quad c_{n-1n-1}=-2,\quad c_{n-1n}=\sigma,$$
$$c_{n1}=1,\quad c_{ni}=0\ \ (i=2,\dots,n-1),\quad c_{nn}=-1.$$

Here the functions V_k $(k=1,\dots,n)$ satisfy (2.10) and by (3.8) and (3.9) the conditions (2.9) and (2.11) hold. Furthermore, for the matrices $C_{n-1}=(c_{ki})_{k,i=1}^{n-1}$ and $C_n=(c_{ki})_{k,i=1}^{n}$, we find

$$\det C_n=(-1)^{n+1}[\sigma-n],\quad \det[C_{n-1}-\rho E]=(-1)^{n+1}\sum_{k=1}^{n}(1+\rho)^{k-1}.$$

Therefore, $(\det C_n)(\det C_{n-1})=n(\sigma-n)$ and the characteristic equation of the matrix C_{n-1} has the form

$$\sum_{k=1}^{n}(1+\rho)^{k-1}=0.$$

The roots of this equation differ from the roots of $(1+\rho)^n=1$. Clearly, all such roots have negative real parts.

Hence, taking into account the condition $\sigma\neq n$, it is clear that the system of differential equations (3.7) satisfy all the conditions of Lemma 2.2. On the basis of this lemma, the given system of differential equations has at least one solution $(v_k)_{k=1}^n : [\tau_1,+\infty[\,\to\mathbb{R}^n$ $(\tau_1\geq\tau_0)$, which tends to zero as $\tau\to+\infty$. Moreover, if $\mu=1$ and $\sigma>n$, there exist $(n-1)$-parametric family of such solutions and n-parametric family in case $\mu=1$ and $\sigma<n$. If $\mu=-1$ and $\sigma<n$, there exists one-parametric family of solutions. Each such solution of the system (3.7) by virtue of the substitutions (3.6) corresponds to y-solution of the differential equation (1.1), which admits the asymptotic representations (2.3), (2.4) as $t\uparrow\omega$. It is not difficult to see that using the conditions (2.1) and (2.2) any of these solutions is a $P_\omega(1)$-solution. □

From this theorem we get the following corollary for the linear differential equation (1.3).

Corollary 3.1. *For the existence of $P_\omega(1)$-solution of the differential equation* (1.3)*, it is necessary, and if the function $p:[a,\omega[\,\to]0,+\infty[$ is continuously differentiable and $\lim_{t\uparrow\omega}p'(t)p^{-\frac{n+1}{n}}(t)$ is finite or equal to $\pm\infty$, then it is sufficient that for some $\mu\in\{-1;1\}$, the inequality* (2.1) *holds and the condition*

$$\lim_{t\uparrow\omega}p(t)|\pi_\omega(t)|^n=+\infty \tag{3.10}$$

is fulfilled.

Moreover, for each of these solutions there take place the following asymptotic representations as $t \uparrow \omega$:

$$\ln |y(t)| = \mu\, J_B(t)[1 + o(1)],$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \mu p^{\frac{1}{n}}(t)[1 + o(1)] \quad (k = 1, \dots, n-1),$$

whereas, for $\mu = 1$, there exists an n-parametric family of $P_\omega(1)$-solutions for this representation, and for $\mu = -1$, there exists one-parametric family of solutions.

This corollary complements the results given in [9, Chapter 1, § 6] on the asymptotic behavior of solutions of linear differential equations. In view of (3.10), it does not refer to the cases where the differential equation (1.3) is asymptotically close to the Euler equation and the equation with almost constant coefficients.

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THE CAUCHY–DARBOUX PROBLEM FOR WAVE EQUATIONS WITH A NONLINEAR DISSIPATIVE TERM

Abstract. The Cauchy–Darboux problem for wave equations with a nonlinear dissipative term is investigated. The questions on the existence, uniqueness and nonexistence of a global solution of the problem are considered. The question of local solvability of the problem is also discussed.

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რეზიუმე. გამოკვლეულია კოში-დარბუს ამოცანა ტალღის განტოლებისთვის არაწრფივი დისიპატიური წევრით. განხილულია არსებობის, ერთადერთობის და გლობალური ამონახსნის არარსებობის საკითხები. ასევე განიხილება დასმული ამოცანის ლოკალური ამოხსნადობის საკითხი.

1. Statement of the Problem

In a plane of independent variables x and t we consider a wave equation with a nonlinear dissipative term (see [16, p. 57], [17])

$$Lu := u_{tt} - u_{xx} + g(x,t,u)u_t = f(x,t), \tag{1.1}$$

where f, g are the given and u is an unknown real functions.

By $D_T := \{(x,t):\ 0 < x < \widetilde{k}t,\ 0 < t < T\}$ we denote a triangular domain lying inside of the characteristic angle $t > |x|$ and bounded by the segments $\widetilde{\gamma}_{1,T}:\ x = \widetilde{k}t$, $\widetilde{\gamma}_{2,T}:\ x = 0$, $0 \le t \le T$ and $\widetilde{\gamma}_{3,T}:\ t = T$, $0 \le x \le \widetilde{k}T$, $0 < \widetilde{k} < 1$. For $T = +\infty$, we assume that $D_\infty := \{(x,t) \in \mathbb{R}^2:\ 0 < x < \widetilde{k}t,\ 0 < t < +\infty\}$.

For the equation (1.1), we consider the Cauchy–Darboux problem on finding a solution $u(x,t)$ in the domain D_T by the conditions [2, p. 284]

$$u\big|_{\widetilde{\gamma}_{1,T}} = 0, \quad u_x\big|_{\widetilde{\gamma}_{2,T}} = 0. \tag{1.2}$$

Note that, the problem

$$u_{tt} - u_{xx} + a(x,t)u_x + b(x,t)u_t + c(x,t)u + d(x,t,u) = f(x,t),$$
$$(\alpha_i u_x + \beta_i u_t + \gamma_i u)\big|_{\gamma_{i,T}} = 0,\ \ i = 1,2;\ \ u(0,0) = 0$$

in a linear case has been investigated in [4,11,12,18,22,23] and in a nonlinear case in [1,6–8,10,13–15]. As is mentioned in [4,23], the problems of such a matter arise under mathematical simulation of small harmonic wedge oscillations in a supersonic flow and of string oscillations in a cylinder filled with a viscous liquid. It should also be noted that when passing from the nonlinearity $d(x,t,u)$ appearing in [1, 6–8, 10, 13–15] to the nonlinearity of type $g(x,t,u)u_t$ in the equation (1.1), as it will be seen below when studying the boundary value problem, there arise difficulties, and not only of technical character.

Below, we will show that under definite requirements imposed on the nonlinear function g the problem (1.1), (1.2) is locally solvable. The conditions of global solvability of the problem will be obtained, violation of these conditions may, generally speaking, give rise to a soluion destruction after a lapse of a finite time interval. The question on the uniqueness of a solution of the problem (1.1), (1.2) will also be considered in the present work.

Definition 1.1. Let $f \in C(\overline{D}_T)$, $g \in C(\overline{D}_T \times \mathbb{R})$. The function u is said to be a general solution of the problem (1.1), (1.2) of the class C^1 in the domain D_T if $u \in C^1(\overline{D}_T)$ and there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \widetilde{\Gamma}_T)$ such that $u_n \to u$ and $Lu_n \to f$, as $n \to \infty$, respectively, in the spaces $C^1(\overline{D}_T)$ and $C(\overline{D}_T)$, where $\overset{\circ}{C}{}^2(\overline{D}_T, \widetilde{\Gamma}_T) := \{v \in C^2(\overline{D}_T):\ v|_{\widetilde{\gamma}_{1,T}} = 0,\ v_x|_{\widetilde{\gamma}_{2,T}} = 0\}$, $\widetilde{\Gamma}_T := \widetilde{\gamma}_{1,T} \cup \widetilde{\gamma}_{2,T}$.

Remark 1.1. Below, for the sake of simplicity of our exposition, sometimes instead of a generalized solution of the problem (1.1), (1.2) of the class C^1 in the domain D_T we will speak about a generalized solution of that problem.

Remark 1.2. Obviously, a classical solution of the problem (1.1), (1.2) from the space $u \in \overset{\circ}{C}{}^2(\overline{D}_T, \widetilde{\Gamma}_T)$ is a generalized solution of that problem. In its turn, if a generalized solution of the problem (1.1), (1.2) belongs to the space $C^2(\overline{D}_T)$, it will also be a classical solution of the problem.

Definition 1.2. Let $f \in C(\overline{D}_T)$, $g \in C(\overline{D}_T \times \mathbb{R})$. We say that the problem (1.1), (1.2) is locally solvable in the class C^1, if there is a positive number $T_0 = T_0(f,g) \le T$ such that for any $T_1 < T_0$, this problem has a generalized solution of the class C^1 in the domain D_{T_1}.

Definition 1.3. Let $f \in C(\overline{D}_\infty)$, $g \in C(\overline{D}_\infty \times \mathbb{R})$. We say that the problem (1.1), (1.2) is globally solvable in the class C^1, if for any finite $T > 0$ this problem has a generalized solution of the class C^1 in the domain D_{T_1}.

When investigating the problem (1.1), (1.2), below, in Section 4, we will need to study the following mixed problem: in the domain $D_{t_1,t_2} := D_T \cap \{t_1 < t < t_2\}$, where $0 < t_1 < t_2 \le T$, find a solution $u(x,t)$ of the equation (1.1) by the initial

$$u\big|_{t=t_1} = \varphi, \quad u_t\big|_{t=t_1} = \psi \tag{1.3}$$

and boundary
$$u\big|_{\partial D_{t_1,t_2}\cap\widetilde{\gamma}_{1,T}} = 0, \quad u_x\big|_{\partial D_{t_1,t_2}\cap\widetilde{\gamma}_{2,T}} = 0 \tag{1.4}$$
conditions.

Remark 1.3. Analogously, just as in the case of the problem (1.1), (1.2), we introduce the notions of a generalized solution, local and global solvability of the problem (1.1), (1.3), (1.4).

2. Equivalent Reduction of the Problem (1.1), (1.2) to the Nonlinear Integro-Differential Equation of Volterra Type

In new independent variables $\xi = \frac{1}{2}(t+x)$, $\eta = \frac{1}{2}(t-x)$, the domain D_T will pass into a triangular domain E_T with vertices at the points $O(0,0)$, $Q_1(\frac{T}{1+k}, \frac{kT}{1+k})$, $Q_2(\frac{T}{2}, \frac{T}{2})$ of the plane of variables ξ, η, and the problem (1.1), (1.2) will pass into the problem (see Figure 2.1)
$$\widetilde{L}\widetilde{u} := \widetilde{u}_{\xi\eta} + \frac{1}{2}\, g(\xi-\eta, \xi+\eta, \widetilde{u})(\widetilde{u}_\xi + \widetilde{u}_\eta) = \widetilde{f}(\xi,\eta), \quad (\xi,\eta)\in E_T, \tag{2.1}$$
$$\widetilde{u}\big|_{\gamma_{1,T}} = 0, \quad (\widetilde{u}_\xi - \widetilde{u}_\eta)\big|_{\gamma_{2,T}} = 0, \tag{2.2}$$
with respect to a new unknown function $\widetilde{u}(\xi,\eta) := u(\xi-\eta, \xi+\eta)$ with the right-hand side $\widetilde{f}(\xi,\eta) := f(\xi-\eta,\xi+\eta)$. Here,
$$\gamma_{1,T} : \eta = k\xi, \ \ 0 \le \xi \le \xi_T := \frac{T}{1+k}, \quad \gamma_{2,T} : \xi = \eta, \ \ 0 \le \eta \le \eta_T := \frac{T}{2}, \tag{2.3}$$
$$0 < k := \frac{1-\widetilde{k}}{1+\widetilde{k}} < 1. \tag{2.4}$$

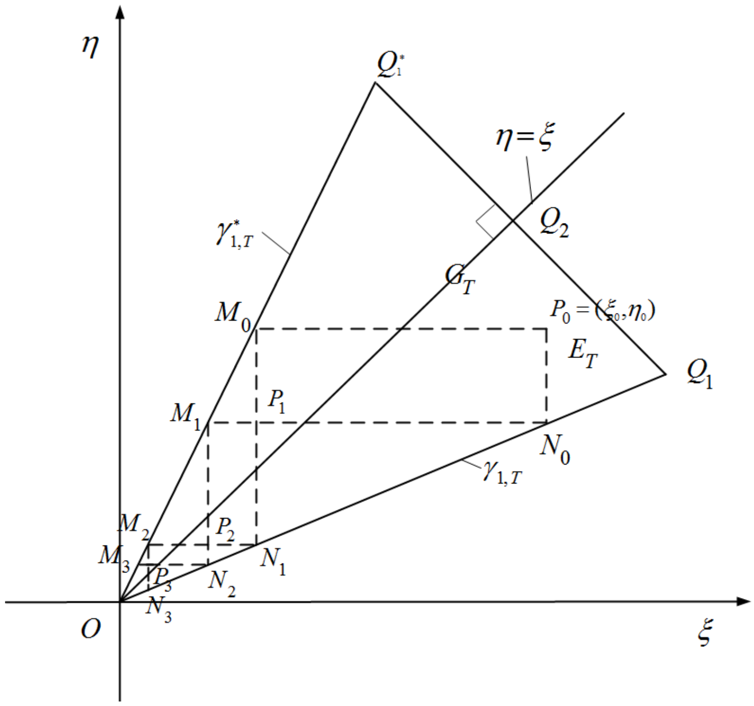

Figure 1

Remark 2.1. According to Definition 1.1, we introduce the notion of a generalized solution $\widetilde{u}$ of the problem (2.1), (2.2) of the class C^1 in the domain E_T, i.e., there exists a sequence of function $\widetilde{u}_n \in \overset{\circ}{C}{}^2(\overline{E}_T, \Gamma_T) := \{w \in C^2(\overline{E}_T) : \ w|_{\gamma_{1,T}} = 0, \ (w_\xi - w_\eta)|_{\gamma_{2,T}} = 0\}$ such that
$$\lim_{n\to\infty} \|\widetilde{u}_n - \widetilde{u}\|_{C(\overline{E}_T)} = 0, \quad \lim_{n\to\infty} \|\widetilde{L}\widetilde{u}_n - \widetilde{f}\|_{C(\overline{E}_T)} = 0, \tag{2.5}$$

where $\Gamma_T := \gamma_{1,T} \cup \gamma_{2,T}$.

Note that, if u is a generalized solution of the problem (1.1), (1.2) in a sense of Definition 1.1, then $\widetilde{u}$ will be a generalized solution of the problem (2.1), (2.2) in a sense of the given definition, and vice versa.

By G_T we denote a triangular domain with vertices at the points O, $Q_1, Q_1^*(\frac{kT}{1+k}, \frac{T}{1+k})$, symmetric with respect to the straight line $\xi = \eta$, and as is easily seen, $G_T \cap \{\eta < \xi\} = E_T$.

We continue the functions $\widetilde{u}_n$ and $\widetilde{f}$ evenly with respect to the straight line $\xi = \eta$ into the domain G_T retaining for them previous notation, i.e.,

$$\widetilde{u}_n(\xi,\eta) = \widetilde{u}_n(\eta,\xi), \quad \widetilde{f}(\xi,\eta) = \widetilde{f}(\eta,\xi), \ \ (\xi,\eta) \in G_T. \tag{2.6}$$

Remark 2.2. Since $\widetilde{f}|_{\overline{E}_T} \in C(\overline{E}_T)$ and $\widetilde{u}_n|_{\overline{E}_T} \in \overset{\circ}{C}{}^2(\overline{E}_T, \Gamma_T)$, taking into account (2.6), we have

$$\widetilde{f} \in C(\overline{G}_T), \quad \widetilde{u}_n \in C^2(\overline{G}_T), \tag{2.7}$$

$$\widetilde{u}_n\big|_{\gamma_{1,T}} = 0, \quad \widetilde{u}_n\big|_{\gamma_{1,T}^*} = 0, \tag{2.8}$$

where $\gamma_{1,T}^* := OQ_1^* \in \partial G_T$, i.e., $\gamma_{1,T}^* : \ \xi = k\eta$, $0 \le \eta \le \frac{T}{1+k}$.

Remark 2.3. Note that, for the functions $\widetilde{u}_n$, $\widetilde{f}$, continued to the domain G_T, the limiting equalities of type (2.5) remain valid, i.e.,

$$\lim_{n\to\infty} \|\widetilde{u}_n - \widetilde{u}\|_{C(\overline{G}_T)} = 0, \quad \lim_{n\to\infty} \|\widetilde{L}\widetilde{u}_n - \widetilde{f}\|_{C(\overline{G}_T)} = 0. \tag{2.9}$$

If $P_0 = (\xi_0, \eta_0) \in E_T$, we denote by $P_1M_0P_0N_0$ the characteristic with respect to the equation (2.1) rectangle whose vertices N_0 and M_0 lie, respectively, on the segments $\gamma_{1,T}$ and $\gamma_{1,T}^*$, i.e., by virtue of (2.3): $N_0 = (\xi_0, k\xi_0)$, $M_0 = (k\eta_0, \eta_0)$, $P_1 = (k\eta_0, k\xi_0)$. Since $P_1 \in G_T$, we construct analogously the characteristic rectangle $P_2M_1P_1N_1$ with vertices N_1 and M_1 lying, respectively, on the segments $\gamma_{1,T}$ and $\gamma_{1,T}^*$. Continuing this process, we get the characteristic rectangle $P_{i+1}M_iP_iN_i$ for which $N_i \in \gamma_{1,T}$, $M_i \in \gamma_{1,T}^*$, where $N_i = (\xi_i, k\xi_i)$, $M_i = (k\eta_i, \eta_i)$, $P_{i+1} = (k\eta_i, k\xi_i)$, if $P_i = (\xi_i, \eta_i)$, $i = 0, 1, \dots$.

It is easily seen that

$$\begin{aligned} P_{2m} &= (k^{2m}\xi_0, k^{2m}\eta_0), \quad P_{2m+1} = (k^{2m+1}\eta_0, k^{2m+1}\xi_0), \\ M_{2m} &= (k^{2m+1}\eta_0, k^{2m}\eta_0), \quad M_{2m+1} = (k^{2m+2}\xi_0, k^{2m+1}\xi_0), \quad m = 0, 1, 2, \dots . \\ N_{2m} &= (k^{2m}\xi_0, k^{2m+1}\xi_0), \quad N_{2m+1} = (k^{2m+1}\eta_0, k^{2m+2}\eta_0), \end{aligned} \tag{2.10}$$

As is known, for any function v of the class C^2 in the closed characteristic rectangle $P_{i+1}M_iP_iN_i$ the equality (see, e.g., [3, p. 173])

$$v(P_i) = v(M_i) + v(N_i) - v(P_{i+1}) + \int\limits_{P_{i+1}M_iP_iN_i} \widetilde{\Box}\, v \, d\xi_1 d\,\eta_1, \ \ i = 0, 1, \dots, \tag{2.11}$$

where $\widetilde{\Box} = \frac{\partial^2}{\partial\xi\partial\eta}$, is valid.

From (2.10), by virtue of (2.8), we have $\widetilde{u}_n(M_i) = \widetilde{u}_n(N_i) = 0$, $i = 0, 1, 2, \dots$. Therefore, (2.11) for $v = \widetilde{u}_n$ results in

$$\begin{aligned} \widetilde{u}_n(\xi_0,\eta_0) &= \widetilde{u}_n(P_0) = \widetilde{u}_n(M_0) + \widetilde{u}_n(N_0) - \widetilde{u}_n(P_1) + \int\limits_{P_1M_0P_0N_0} \widetilde{\Box}\, \widetilde{u}_n \, d\xi_1\, d\eta_1 \\ &= -\widetilde{u}_n(P_1) + \int\limits_{P_1M_0P_0N_0} \widetilde{\Box}\, \widetilde{u}_n \, d\xi_1\, d\eta_1 \\ &= -\widetilde{u}_n(M_1) - \widetilde{u}_n(N_1) + \widetilde{u}_n(P_2) - \int\limits_{P_2M_1P_1N_1} \widetilde{\Box}\, \widetilde{u}_n \, d\xi_1\, d\eta_1 + \int\limits_{P_1M_0P_0N_0} \widetilde{\Box}\, \widetilde{u}_n \, d\xi_1\, d\eta_1 \\ &= \widetilde{u}_n(P_2) - \int\limits_{P_2M_1P_1N_1} \widetilde{\Box}\, \widetilde{u}_n \, d\xi_1\, d\eta_1 + \int\limits_{P_1M_0P_0N_0} \widetilde{\Box}\, \widetilde{u}_n \, d\xi_1\, d\eta_1 = \cdots \end{aligned}$$

$$= (-1)^m \widetilde{u}_n(P_m) + \sum_{i=0}^{m-1} (-1)^i \int\limits_{P_{i+1}M_iP_iN_i} \widetilde{\square}\, \widetilde{u}_n \, d\xi_1 \, d\eta_1, \quad (\xi_0, \eta_0) \in E_T. \tag{2.12}$$

Since the point P_m from (2.12) tends to the point O, as $m \to \infty$, by virtue of (2.8), we have $\lim\limits_{m\to\infty} \widetilde{u}_n(P_m) = 0$. Hence, passing in the equality (2.12) to the limit, as $m \to \infty$, for the function $\widetilde{u}_n \in C^2(\overline{G}_T)$ in the domain E_T we obtain the following integral representation:

$$\widetilde{u}_n(\xi_0, \eta_0) = \sum_{i=0}^{\infty} (-1)^i \int\limits_{P_{i+1}M_iP_iN_i} \widetilde{\square}\, \widetilde{u}_n \, d\xi_1 \, d\eta_1, \quad (\xi_0, \eta_0) \in E_T. \tag{2.13}$$

Remark 2.4. Since $\widetilde{\square}\, \widetilde{u}_n \in C(\overline{E}_T)$ and there are the inequalities (2.4), and owing to (2.10),

$$\operatorname{mes} P_{i+1}M_iP_iN_i = k^{2i}(\xi_0 - k\eta_0)(\eta_0 - k\xi_0), \tag{2.14}$$

therefore the series in the right-hand side of the equality (2.13) is uniformly and absolutely convergent.

It can be easily seen that by virtue of (2.4) and (2.14),

$$\Big| \sum_{i=0}^{\infty} (-1)^i \int\limits_{P_{i+1}M_iP_iN_i} \widetilde{\square}\, \widetilde{u}_n \, d\xi_1 \, d\eta_1 - \sum_{i=0}^{\infty} (-1)^i \int\limits_{P_{i+1}M_iP_iN_i} \widetilde{f} \, d\xi_1 \, d\eta_1 \Big|$$

$$\le \sum_{i=0}^{\infty} \big\| \widetilde{\square}\, \widetilde{u}_n - \widetilde{f} \big\|_{C(\overline{G}_T)} \operatorname{mes} P_{i+1}M_iP_iN_i = \big\| \widetilde{\square}\, \widetilde{u}_n - \widetilde{f} \big\|_{C(\overline{G}_T)} \sum_{i=0}^{\infty} k^{2i}(\xi_0 - k\eta_0)(\eta_0 - k\xi_0)$$

$$\le \frac{\xi_0 \eta_0}{1-k^2} \big\| \widetilde{\square}\, \widetilde{u}_n - \widetilde{f} \big\|_{C(\overline{G}_T)}. \tag{2.15}$$

Remark 2.5. By (2.5) for $g = 0$ and (2.15), passing in the equality (2.13) to the limit, as $n \to \infty$, for a generalized solution $\widetilde{u}$ of the problem (2.1), (2.2) we obtain the following integral representation:

$$\widetilde{u}(\xi_0, \eta_0) = \sum_{i=0}^{\infty} (-1)^i \int\limits_{P_{i+1}M_iP_iN_i} \widetilde{f} \, d\xi_1 \, d\eta_1, \quad (\xi_0, \eta_0) \in E_T. \tag{2.16}$$

Remark 2.6. From the above reasonings it follows that for any $\widetilde{f} \in C(\overline{E}_T)$, the linear problem (2.1), (2.2) has a unique generalized solution $\widetilde{u}$ which is representable in the form of a uniformly and absolutely convergent series (2.16) and for $\widetilde{f} \in C^1(\overline{E}_T)$ is a classical solution of that problem, i.e., $\widetilde{u} \in \overset{\circ}{C}{}^2(\overline{E}_T, \Gamma_T)$.

According to (2.16), we introduce into consideration the operator $\widetilde{\square}^{-1} : C(\overline{E}_T) \to C(\overline{E}_T)$ acting by the formula

$$(\widetilde{\square}^{-1} \widetilde{f})(\xi, \eta) := \sum_{i=0}^{\infty} (-1)^i \int\limits_{P_{i+1}M_iP_iN_i} \widetilde{f} \, d\xi_1 \, d\eta_1, \quad (\xi, \eta) \in E_T. \tag{2.17}$$

In the integrand here, according to (2.6), under $\widetilde{f}$ we mean the right-hand side of the equation (2.1) which is continued evenly from the domain E_T to the domain G_T with respect to the straight line $\xi = \eta$, and due to (2.7), we have $\widetilde{f} \in C(\overline{E}_T)$.

Remark 2.7. By virtue of (2.17) and Remark 2.6, a unique generalized solution $\widetilde{u}$ of the problem (2.1), (2.2) is representable in the form $\widetilde{u} = \widetilde{\square}^{-1} \widetilde{f}$, and in view of (2.4), (2.14), the estimate

$$|\widetilde{u}(\xi, \eta)| \le \sum_{i=0}^{\infty} \int\limits_{P_{i+1}M_iP_iN_i} |\widetilde{f}| \, d\xi_1 \, d\eta_1 \le \xi\eta \|\widetilde{f}\|_{C(\overline{E}_T)} \sum_{i=0}^{\infty} k^{2i}$$

$$\le \frac{\xi^2 + \eta^2}{2(1-k^2)} \|\widetilde{f}\|_{C(\overline{E}_T)} \le \frac{T^2}{1-k^2} \|\widetilde{f}\|_{C(\overline{E}_T)}$$

holds which in its turn yields

$$\|\widetilde{\square}^{-1}\|_{C(\overline{E}_T) \longrightarrow C(\overline{E}_T)} \le \frac{T^2}{1-k^2}\,. \tag{2.18}$$

Remark 2.8. Standard reasonings (see, e.g., [9]) show that the function $\widetilde{u} \in C^1(\overline{E}_T)$ is the generalized solution of the problem (2.1), (2.2), if and only if it is a solution of the following nonlinear Volterra type integro-differential equation

$$\widetilde{u}(\xi,\eta) + \frac{1}{2}\,\widetilde{\square}^{-1}\big(g(\xi-\eta,\xi+\eta,\widetilde{u})(\widetilde{u}_\xi+\widetilde{u}_\eta)\big)(\xi,\eta) = (\widetilde{\square}^{-1}\widetilde{f})(\xi,\eta), \quad (\xi,\eta)\in E_T. \tag{2.19}$$

3. Local Solvability of the Problem (1.1), (1.2)

Lemma 3.1. *The operator $\widetilde{\square}^{-1}$ defined by the formula* (2.17) *is the linear continuous operator acting from the space $C(\overline{E}_T)$ to the space $C^1(\overline{E}_T)$.*

Proof. To this end, we first show that for $\widetilde{f} \in C(\overline{E}_T)$, the series from the right-hand side of (2.17), differentiated formally with respect to ξ and to η converges uniformly on the set $\overline{E}_T$. Indeed, as it can be easily verified, we have

$$\begin{aligned}(L_1\widetilde{f})(\xi,\eta) &:= \frac{\partial}{\partial\xi}\,\big[(\widetilde{\square}^{-1}\widetilde{f})(\xi,\eta)\big] \\ &= \sum_{n=0}^{\infty}\Bigg[k^{2n}\int\limits_{N_{2n}P_{2n}}\widetilde{f}\,d\eta_1 + k^{2n+2}\int\limits_{P_{2n+2}M_{2n+1}}\widetilde{f}\,d\eta_1 - k^{2n+1}\int\limits_{M_{2n+1}N_{2n}}\widetilde{f}\,d\xi_1\Bigg],\end{aligned} \tag{3.1}$$

$$\begin{aligned}(L_2\widetilde{f})(\xi,\eta) &:= \frac{\partial}{\partial\eta}\,\big[(\widetilde{\square}^{-1}\widetilde{f})(\xi,\eta)\big] \\ &= \sum_{n=0}^{\infty}\Bigg[k^{2n}\int\limits_{M_{2n}P_{2n}}\widetilde{f}\,d\xi_1 + k^{2n+2}\int\limits_{P_{2n+2}N_{2n+1}}\widetilde{f}\,d\xi_1 - k^{2n+1}\int\limits_{N_{2n+1}M_{2n}}\widetilde{f}\,d\eta_1\Bigg].\end{aligned} \tag{3.2}$$

By virtue of (2.10), we have the equalities

$$|N_{2m}P_{2m}| = k^{2m}(\eta-k\xi), \quad |P_{2m+2}M_{2m+1}| = k^{2m+1}(\xi-k\eta), \quad |M_{2m+1}N_{2m}| = k^{2m}(1-k^2)\xi,$$
$$|M_{2m}P_{2m}| = k^{2m}(\xi-k\eta), \quad |P_{2m+2}N_{2m+1}| = k^{2m+1}(\eta-k\xi), \quad |N_{2m+1}M_{2m}| = k^{2m}(1-k^2)\eta,$$

which in view of (2.4) imply that the series (3.1) and (3.2) are uniformly and absolutely convergent, and the estimate

$$\max\big\{\|L_1\widetilde{f}\|_{C(\overline{E}_T)}, \|L_2\widetilde{f}\|_{C(\overline{E}_T)}\big\} \le \frac{3T}{1-k^4}\,\|\widetilde{f}\|_{C(\overline{E}_T)} \tag{3.3}$$

holds.

From (3.3), in view of (2.18) and the fact that $\|v\|_{C^1} := \max\{\|v\|_C, \|v_\xi\|_C, \|v_\eta\|_C\}$, it follows that Lemma 3.1 is valid. □

Introducing the notation $v_1 := \widetilde{u}$, $v_2 := \widetilde{u}_\xi$, $v_3 := \widetilde{u}_\eta$ and differentiating formally the equality (2.19) with respect to ξ and η for $(\xi,\eta)\in E_T$, we obtain

$$\begin{cases} v_1(\xi,\eta) = -\dfrac{1}{2}\,\widetilde{\square}^{-1}\big(g(\xi-\eta,\xi+\eta,v_1)(v_2+v_3)\big) + (\widetilde{\square}^{-1}\widetilde{f})(\xi,\eta), \\ v_2(\xi,\eta) = -\dfrac{1}{2}L_1\big(g(\xi-\eta,\xi+\eta,v_1)(v_2+v_3)\big) + (L_1\widetilde{f})(\xi,\eta), \\ v_3(\xi,\eta) = -\dfrac{1}{2}L_2\big(g(\xi-\eta,\xi+\eta,v_1)(v_2+v_3)\big) + (L_2\widetilde{f})(\xi,\eta), \end{cases} \tag{3.4}$$

where the linear operators L_1 and L_2 are defined by the equalities (3.1) and (3.2).

Remark 3.1. It is not difficult to check that if $\widetilde{u} \in C^1(\overline{E}_T)$ is a solution of the nonlinear equation (2.19), then the functions $v_1 := \widetilde{u}$, $v_2 := \widetilde{u}_\xi$, $v_3 := \widetilde{u}_\eta$ of the class $C(\overline{E}_T)$ satisfy the system of nonlinear equations (3.4), and vice versa, if the functions v_1, v_2 and v_3 of the class $C(\overline{E}_T)$ satisfy the system of equations (3.4), then $v_1 \in C^1(\overline{E}_T)$ and $v_{1\xi} = v_2$, $v_{2\eta} = v_3$, and $\widetilde{u} = v_1$ will be a solution of the equation (2.19) of the class $C^1(\overline{E}_T)$.

We will now proceed to the proof of the local solvability of the system of nonlinear integral equations (3.4).

Let us consider the following conditions:

$$|g(x,t,s)| \le m(r), \quad \big|g(x,t,s_2) - g(x,t,s_1)\big| \le c(r)|s_2 - s_1|, \quad (x,t) \in \overline{D}_T, \quad |s|, |s_1|, |s_2| \le r, \tag{3.5}$$

where $m(r)$ and $c(r)$ are some nonnegative continuous functions of argument $r \ge 0$. Obviously, the conditions (3.5) will be fulfilled if $g, g_s \in C(\overline{D}_T \times \mathbb{R})$.

Theorem 3.1. *Let $f \in C(\overline{D}_T)$ and the function $g \in C(\overline{D}_T \times \mathbb{R})$ satisfy the conditions* (3.5)*. Then there exists a positive number $T_0 = T_0(f,g) \le T$ such that for any $T_1 < T_0$ the problem* (1.1), (1.2) *has at least one generalized solution in the domain D_{T_1}.*

Proof. By Remarks 2.1 and 2.8, the problem (1.1), (1.2) in the space $C^1(\overline{D}_T)$ is equivalent to the system of nonlinear integral equations (3.4) in the class $C(\overline{E}_T)$. Below, we will prove the solvability of the system (3.4) by using the principle of contracted mappings (see, e.g., [21, p. 390]).

Assume $V := (v_1, v_2, v_3)$ and introduce the vector operator $\Phi := (\Phi_1, \Phi_2, \Phi_3)$ acting by the formula

$$\begin{cases} (\Phi_1 V)(\xi,\eta) = -\dfrac{1}{2}\,\widetilde{\square}^{-1}\big(g(\xi-\eta,\xi+\eta,v_1)(v_2+v_3)\big) + (\widetilde{\square}^{-1}\widetilde{f})(\xi,\eta), \\ (\Phi_2 V)(\xi,\eta) = -\dfrac{1}{2}\,L_1\big(g(\xi-\eta,\xi+\eta,v_1)(v_2+v_3)\big) + (L_1\widetilde{f})(\xi,\eta), \\ (\Phi_3 V)(\xi,\eta) = -\dfrac{1}{2}\,L_2\big(g(\xi-\eta,\xi+\eta,v_1)(v_2+v_3)\big) + (L_2\widetilde{f})(\xi,\eta). \end{cases} \tag{3.6}$$

Taking into account (3.6), the system (3.4) can be rewritten in the vector form

$$V = \Phi V. \tag{3.7}$$

Let

$$\|V\|_{X_T} := \max_{1 \le i \le 3} \big\{\|v_i\|_{C(\overline{E}_T)}\big\}, \quad V \in X_T := C(\overline{E}_T; \mathbb{R}^3),$$

where $C(\overline{E}_T; \mathbb{R}^3)$ is a set of continuous vector functions $V : \overline{E}_T \to \mathbb{R}^3$.

We fix the number $R > 0$ and denote by $B_R(T) := \{V \in X_T : \ \|V\|_{X_T} \le R\}$ a closed ball of radius R in the Banach space X_T with center in a zero element.

Below, we will prove that there exists the positive number $T_0 = T_0(f,g) \le T$ such that for any $T_1 < T_0$:

(i) Φ maps the ball $B_R(T_1)$ into itself;

(ii) Φ is a contractive mapping on the set $B_R(T_1)$.

Indeed, by the estimates (2.18), (3.3) and the first inequality (3.5), from (3.6) for $V \in B_R(T_1)$, when $T_1 < T$, we have

$$\begin{aligned} \big|(\Phi_1 V)(\xi,\eta)\big| &\le \frac{T_1^2}{1-k^2}\,\big(Rm(R) + \|\widetilde{f}\|_{C(\overline{E}_T)}\big), \quad (\xi,\eta) \in E_{T_1}, \\ \big|(\Phi_i V)(\xi,\eta)\big| &\le \frac{3T_1}{1-k^4}\,\big(Rm(R) + \|\widetilde{f}\|_{C(\overline{E}_T)}\big), \quad (\xi,\eta) \in E_{T_1}, \quad i = 2,3. \end{aligned} \tag{3.8}$$

From these estimates, owing to the fact that $k^2 < 1$, it follows that

$$\|\Phi V\|_{X_{T_1}} \le \frac{T_1(T_1+3)}{1-k^2}\,\big(RM(R) + \|\widetilde{f}\|_{C(\overline{E}_T)}\big). \tag{3.9}$$

For the fixed $R > 0$, we require for the value T_1 to be so small that

$$\frac{T_1(T_1+3)}{1-k^2}\,\big(Rm(R) + \|\widetilde{f}\|_{C(\overline{E}_T)}\big) \le R. \tag{3.10}$$

Then from (3.9) and (3.10) it follows that $\Phi U \in B_R(T_1)$, and hence the condition (i) is fulfilled.

Next, by (2.18) and (3.5), from (3.6), for $V_i = (v_i^1, v_i^2, v_i^3) \in B_R(T_1)$, $i = 1,2$, we have

$$\begin{aligned} \big|(\Phi_1 V_2 - \Phi_1 V_1)(\xi,\eta)\big| &= \frac{1}{2}\,\Big|\widetilde{\square}^{-1}\Big[g(\xi-\eta,\xi+\eta,v_2^1)(v_2^2+v_2^3) - g(\xi-\eta,\xi+\eta,v_1^1)(v_1^2+v_1^3)\Big]\Big| \\ &= \frac{1}{2}\,\Big|\widetilde{\square}^{-1}\Big[\big(g(\xi-\eta,\xi+\eta,v_2^1) - g(\xi-\eta,\xi+\eta,v_1^1)\big)(v_2^2+v_2^3) + g(\xi-\eta,\xi+\eta,v_1^1)(v_2^2-v_1^2+v_2^3-v_1^3)\Big]\Big| \\ &\le \frac{T_1^2}{1-k^2}\,(Rc(R)+m(R))\|V_2 - V_1\|_{X_{T_1}}. \end{aligned}$$

Analogously, taking into account (3.3), we have

$$\big|(\Phi_i V_2-\Phi_i V_1)(\xi,\eta)\big| \le \frac{3T_1}{1-k^4}\,(Rc(R)+m(R))\|V_2-V_1\|_{X_{T_1}},\ \ i=2,3. \tag{3.11}$$

We now choose the number T_1 so small that

$$\frac{T_1(T_1+3)}{1-k^2}\,(Rc(R)+m(R)) \le q = const < 1, \tag{3.12}$$

and hence $\|\Phi V_2-\Phi V_1\|_{X_{T_1}} \le q\|V_2-V_1\|_{X_{T_1}}$. Thus the operator Φ is a contractive mapping on the set $B_R(T_1)$, i.e., the condition (ii) is fulfilled.

It follows from (3.11) and (3.12) that there exists the number $T_0=T_0(f,g)\le T$ such that for any $T_1<T_0$, both conditions (i) and (ii) are fulfilled for the mapping $\Phi: B_R(T_1)\to B_R(T_1)$. Therefore, by the principle of contracted mappings, there exists the solution V of the equation (3.7) in the space $C(\overline{E}_{T_1};\mathbb{R}^3)$. □

Remark 3.2. From the above reasonings as in proving Theorem 3.1 dealt with the contraction of the mapping Φ, it immediately follows that if u_1 and u_2 are two possible solutions of the problem (1.1), (1.2) of the class $C^1(\overline{D}_T)$, then there exists the positive number $T_1=T_1(\|u_1\|,\|u_2\|)\le T$ such that $u_1|_{D_{T_1}}=u_2|_{D_{T_1}}$.

4. A priori Estimates of a Solution of the Problem (1.1), (1.3), (1.4) in the Classes $C(\overline{D}_{t_1,t_2})$ and $C^1(\overline{D}_{t_1,t_2})$

Assume

$$\begin{aligned}
\omega_\tau &:= \overline{D}_{t_1,t_2}\cap\{t=\tau\},\ \ t_1\le\tau\le t_2,\\
\gamma_{i;t_1,t_2} &:= \overline{D}_{t_1,t_2}\cap\widetilde{\gamma}_{i,T},\ \ i=1,2,\\
\Gamma_{t_1,t_2} &:= \gamma_{1;t_1,t_2}\cup\gamma_{2;t_1,t_2},
\end{aligned}$$

and introduce into consideration the space

$$\overset{\circ}{C}{}^2(\overline{D}_{t_1,t_2},\Gamma_{t_1,t_2}) := \Big\{v\in C^2(\overline{D}_{t_1,t_2}):\ v\big|_{\gamma_{1;t_1,t_2}}=0,\ v_x\big|_{\gamma_{2;t_1,t_2}}=0\Big\}.$$

Let

$$f\in C(\overline{D}_T),\ \ g\in C(\overline{D}_T\times\mathbb{R}),\ \ \varphi\in C^1(\overline{\omega}_{t_1}),\ \ \psi\in C(\overline{\omega}_{t_1}). \tag{4.1}$$

Definition 4.1. The function $u\in C^1(\overline{D}_{t_1,t_2})$ is said to be a generalized solution of the problem (1.1), (1.3), (1.4) if there exists a sequence of functions $u_n\in \overset{\circ}{C}{}^2(\overline{D}_{t_1,t_2},\Gamma_{t_1,t_2})$ such that the limiting equalities

$$\lim_{n\to\infty}\|u_n-u\|_{C^1(\overline{D}_{t_1,t_2})}=0,\qquad \lim_{n\to\infty}\|Lu_n-f\|_{C(\overline{D}_{t_1,t_2})}=0 \tag{4.2}$$

and

$$\lim_{n\to\infty}\|u_n|_{\overline{\omega}_{t_1}}-\varphi\|_{C^1(\overline{\omega}_{t_1})}=0,\qquad \lim_{n\to\infty}\|u_{nt}|_{\overline{\omega}_{t_1}}-\psi\|_{C(\overline{\omega}_{t_1})}=0 \tag{4.3}$$

hold.

Lemma 4.1. *Let the conditions* (4.1) *and*

$$g(x,t,s)\ge -M_T,\ \ (x,t,s)\in\overline{D}_T\times\mathbb{R},\ \ M_T:=const>0, \tag{4.4}$$

be fulfilled. Then for a generalized solution $u\in C^1(\overline{D}_{t_1,t_2})$ of the problem (1.1), (1.3), (1.4) *an a priori estimate*

$$\|u\|_{C(\overline{D}_{t_1,t_2})}\le c_1\big(\|f\|_{C(\overline{D}_{t_1,t_2})}+\|\varphi\|_{C^1(\overline{\omega}_{t_1})}+\|\psi\|_{C(\overline{\omega}_{t_1})}\big) \tag{4.5}$$

with the positive constant $c_1=c_1(T)$, independent of u, f, φ, and ψ is valid.

Proof. Let u be a generalized solution of the problem (1.1), (1.3), (1.4). Then by Definition 4.1, there exists the sequence of functions $u_n\in\overset{\circ}{C}{}^2(\overline{D}_{t_1,t_2},\Gamma_{t_1,t_2})$ such that the limiting equalities (4.2), (4.3) are valid.

Consider the function $u_n\in\overset{\circ}{C}{}^2(\overline{D}_{t_1,t_2},\Gamma_{t_1,t_2})$ as a solution of the following mixed problem

$$Lu_n=f_n, \tag{4.6}$$

$$u_n\big|_{\overline{\omega}_{t_1}}=\varphi_n,\quad u_{nt}\big|_{\overline{\omega}_{t_1}}=\psi_n, \tag{4.7}$$

$$u_n\big|_{\gamma_{1;t_1,t_2}}=0,\quad u_{nx}\big|_{\gamma_{2;t_1,t_2}}=0. \tag{4.8}$$

Here,

$$\varphi_n:=u_n\big|_{\overline{\omega}_{t_1}},\quad \psi_n:=u_{nt}\big|_{\overline{\omega}_{t_1}},\quad f_n:=Lu_n. \tag{4.9}$$

Multiplying both parts of the equality (4.6) by u_{nt} and integrating the obtained equality with respect to the domain $D_{t_1,t_2;\tau}:=\{(x,t)\in D_{t_1,t_2}:\ t_1<t<\tau\}$, $t_1<\tau\le t_2$, we have

$$\frac{1}{2}\int\limits_{D_{t_1,t_2;\tau}}(u_{nt}^2)_t\,dx\,dt-\int\limits_{D_{t_1,t_2;\tau}}u_{nxx}u_{nt}\,dx\,dt+\int\limits_{D_{t_1,t_2;\tau}}g(x,t,u_n)u_{nt}^2\,dx\,dt=\int\limits_{D_{t_1,t_2;\tau}}f_nu_{nt}\,dx\,dt.$$

Taking into account (4.8) and applying Green's formula to the left-hand side of the last equality, we obtain

$$\int\limits_{D_{t_1,t_2;\tau}}f_nu_{nt}\,dx\,dt=\int\limits_{\gamma_{1;t_1,\tau}}\frac{1}{2\nu_t}\Big[(u_{nx}\nu_t-u_{nt}\nu_x)^2+u_{nt}^2(\nu_t^2-\nu_x^2)\Big]\,ds$$
$$+\frac{1}{2}\int\limits_{\omega_\tau}(u_{nx}^2+u_{nt}^2)\,dx-\frac{1}{2}\int\limits_{\omega_{t_1}}(u_{nx}^2+u_{nt}^2)\,dx+\int\limits_{D_{t_1,t_2;\tau}}g(x,t,u_n)u_{nt}^2\,dx\,dt, \tag{4.10}$$

where $\nu:=(\nu_x,\nu_t)$ is a unit vector of the outer normal to $D_{t_1,t_2;\tau}$.

Taking into account the fact that the operator $\nu_t\,\frac{\partial}{\partial x}-\nu_x\,\frac{\partial}{\partial t}$ is the directional derivative, tangent to $\gamma_{1;t_1,\tau}$, owing to the first condition (4.8), we have

$$(u_{nx}\nu_t-u_{nt}\nu_x)\big|_{\gamma_{1;t_1,\tau}}=0. \tag{4.11}$$

Since $\nu_x=\frac{1}{\sqrt{1+\widetilde{k}^2}}$, $\nu_t=\frac{-\widetilde{k}}{\sqrt{1+\widetilde{k}^2}}$ and $0<\widetilde{k}<1$, therefore

$$(\nu_t^2-\nu_x^2)\big|_{\gamma_{1;t_1,\tau}}<0. \tag{4.12}$$

Consequently, by (4.4), (4.11), (4.12), from (4.10), we have

$$w_n(\tau):=\int\limits_{\omega_\tau}(u_{nx}^2+u_{nt}^2)\,dx\le\int\limits_{\omega_{t_1}}(u_{nx}^2+u_{nt}^2)\,dx+2\int\limits_{D_{t_1,t_2;\tau}}f_nu_{nt}\,dx\,dt+2M_T\int\limits_{D_{t_1,t_2;\tau}}u_{nt}^2\,dx\,dt. \tag{4.13}$$

Bearing in mind the inequality $2f_nu_{nt}\le u_{nt}^2+f_n^2$, by (4.7) and (4.13), we get

$$w_n(\tau)\le(1+2M_T)\int\limits_{D_{t_1,t_2;\tau}}u_{nt}^2\,dx\,dt+\int\limits_{D_{t_1,t_2;\tau}}f_n^2\,dx\,dt+\int\limits_{\omega_{t_1}}(\varphi_n'^2+\psi_n^2)\,dx,$$

whence, in view of the expression for the function $w_n(\tau)$, it follows that

$$w_n(\tau)\le m_T\int\limits_0^\tau w_n(\sigma)\,d\sigma+\|f_n\|^2_{L_2(D_{t_1,t_2;\tau})}+\|\varphi_n'\|^2_{L_2(\omega_{t_1})}+\|\psi_n\|^2_{L_2(\omega_{t_1})},$$

where $m_T:=1{+}2M_T$. Hence, since the value $\|f_n\|^2_{L_2(D_{t_1,t_2;\tau})}$, being the function of τ, is nondecreasing, by the Gronwall's lemma (see, e.g., [5, p. 13]), we have

$$w_n(\tau)\le\exp(m_T\tau)\Big[\|f_n\|^2_{L_2(D_{t_1,t_2;\tau})}+\|\varphi_n'\|^2_{L_2(\omega_{t_1})}+\|\psi_n\|^2_{L_2(\omega_{t_1})}\Big]. \tag{4.14}$$

If $(x,t)\in\overline{D}_{t_1,t_2}$, then by virtue of the first condition (4.8), we obtain the equality

$$u_n(x,t)=u_n(x,t)-u_n(\widetilde{k}t,t)=\int\limits_{\widetilde{k}t}^{x}u_{nx}(\sigma,t)\,d\sigma,$$

which owing to the Schwartz inequality and (4.14) results in

$$
\begin{aligned}
|u_n(x,t)|^2 &\le \int\limits_x^{\widetilde{k}t} d\sigma \int\limits_x^{\widetilde{k}t} [u_{nx}(\sigma,t)]^2\,d\sigma \le (\widetilde{k}t - x) \int\limits_{\omega_t} [u_{nx}(\sigma,t)]^2\,d\sigma \le (\widetilde{k}t - x) w_n(t) \le \widetilde{k}t w_n(t) \\
&\le \widetilde{k}t_2 \exp(m_T t_2) \Big[\|f_n\|^2_{C(\overline{D}_T)} \operatorname{mes} D_{t_1,t_2;\tau} + \operatorname{mes}\omega_{t_1} \big(\|\varphi_n\|^2_{C^1(\overline{\omega}_{t_1})} + \|\psi_n\|^2_{C(\overline{\omega}_{t_1})} \big) \Big] \\
&= \frac{1}{2}\,\widetilde{k}^2 t_2 (t_2^2 - t_1^2) \exp(m_T t_2) \|f_n\|^2_{C(\overline{D}_T)} + \widetilde{k}^2 t_1 t_2 \exp(m_T t_2) \|\varphi_n\|^2_{C^1(\overline{\omega}_{t_1})} \\
&\qquad + \widetilde{k}^2 t_1 t_2 \exp(m_T t_2) \|\psi_n\|^2_{C(\overline{\omega}_{t_1})}. \qquad (4.15)
\end{aligned}
$$

Thus, using the obvious inequality

$$\Big(\sum_{i=1}^n a_i^2 \Big)^{\frac{1}{2}} \le \sum_{i=1}^n |a_i|,$$

we obtain

$$
\begin{aligned}
\|u_n\|_{C(\overline{D}_{t_1,t_2})} &\le T\widetilde{k} \sqrt{\frac{T}{2}}\, \exp\Big(\frac{Tm_T}{2}\Big) \|f_n\|_{C(\overline{D}_T)} \\
&\qquad + T\widetilde{k} \exp\Big(\frac{Tm_T}{2}\Big) \|\varphi_n\|_{C^1(\overline{\omega}_{t_1})} + T\widetilde{k} \exp\Big(\frac{Tm_T}{2}\Big) \|\psi_n\|_{C(\overline{\omega}_{t_1})}.
\end{aligned}
$$

Passing in the last inequality to the limit, as $n \to \infty$, by virtue of (4.2), (4.3), (4.9), we obtain the estimate (4.5) in which

$$c_1(T) = T\widetilde{k} \exp\Big(\frac{Tm_T}{2}\Big) \max\Big\{ \sqrt{\frac{T}{2}}\,, 1 \Big\}. \qquad \square$$

Remark 4.1. Repeating the same reasoning as in Lemma 4.1, for a generalized solution of the problem (1.1), (1.2) we obtain an a priori estimate

$$\|u\|_{C(\overline{D}_T)} \le c_0 \|f\|_{C(\overline{D}_T)},$$

where

$$c_0 = T\widetilde{k} \sqrt{\frac{T}{2}}\, \exp\Big(\frac{m_T T}{2}\Big).$$

Below, using the classical method of characteristics and taking into account (4.5), we obtain a priori estimate in the space $C^1(\overline{D}_{t_1,t_2})$ for a generalized solution of the problem (1.1), (1.3), (1.4).

We have the following

Lemma 4.2. *Under the conditions of Lemma* 4.1*, if*

$$t_2 - t_1 \le \frac{1}{2}\,\widetilde{k} t_1, \qquad (4.16)$$

for a generalized solution of the problem (1.1), (1.3), (1.4) *an a priori estimate*

$$\|u\|_{C^1(\overline{D}_{t_1,t_2})} \le \big(2T\|f\|_{C(\overline{D}_T)} + \|\varphi\|_{C^1(\overline{\omega}_{t_1})} + \|\psi\|_{C(\overline{\omega}_{t_1})} \big) \exp\big[2(K_{\varphi,\psi} + 1)T \big] \qquad (4.17)$$

holds. Here,

$$K_{\varphi,\psi} := K\big(\|f\|_{C(\overline{D}_T)} + \|\varphi\|_{C^1(\overline{\omega}_{t_1})} + \|\psi\|_{C(\overline{\omega}_{t_1})} \big), \qquad (4.18)$$

where

$$K(s) := \sup_{(x,t)\in D_T,\, |s_1| \le c_1 s} |g(x,t,s_1)| < +\infty,$$

c is the constant from the a priori estimate (4.5)*, and*

$$\|u\|_{C^1(\overline{D}_{t_1,t_2})} := \max\Big\{ \|u\|_{C(\overline{D}_{t_1,t_2})}, \|u_x\|_{C(\overline{D}_{t_1,t_2})}, \|u_t\|_{C(\overline{D}_{t_1,t_2})} \Big\}.$$

Proof. Let u be a generalized solution of the problem (1.1), (1.3), (1.4). The limiting equalities (4.2), (4.3) are valid, where u_n can be considered as a solution of the problem (4.6)–(4.8) with right-hand sides f_n, φ_n, ψ_n from (4.9). For the fixed natural n we introduce the functions

$$u_n^1 := u_{nt} - u_{nx}, \quad u_n^2 := u_{nt} + u_{nx}, \quad u_n^3 := u_n, \qquad (4.19)$$

which in view of (4.7), (4.8) for $t_1 \le t \le t_2$ satisfy the initial and boundary conditions

$$u_n^1\big|_{\omega_{t_1}} = \psi_n - \varphi_n', \quad u_n^2\big|_{\omega_{t_1}} = \psi_n + \varphi_n', \quad u_n^3\big|_{\omega_{t_1}} = \varphi_n, \tag{4.20}$$

$$\Big(u_n^2 + \frac{1-\widetilde{k}}{1+\widetilde{k}}\,u_n^1\Big)\Big|_{\gamma_{1;t_1,t_2}} = 0, \quad u_n^3\big|_{\gamma_{1;t_1,t_2}} = 0, \quad (u_n^1 - u_n^2)\big|_{\gamma_{2;t_1,t_2}} = 0. \tag{4.21}$$

By virtue of (1.1), and (4.19), the unknown functions u_n^1, u_n^2, u_n^3 satisfy the following system of partial differential equations of the first order

$$\begin{cases} \dfrac{\partial u_n^1}{\partial t} + \dfrac{\partial u_n^1}{\partial x} = f_n(x,t) - \dfrac{1}{2}\, g(x,t,u_n^3)(u_n^1 + u_n^2), \\ \dfrac{\partial u_n^2}{\partial t} - \dfrac{\partial u_n^2}{\partial x} = f_n(x,t) - \dfrac{1}{2}\, g(x,t,u_n^3)(u_n^1 + u_n^2), \\ \dfrac{\partial u_n^3}{\partial t} - \dfrac{\partial u_n^3}{\partial x} = u_n^1. \end{cases} \tag{4.22}$$

Taking into account (4.16), we divide the domain D_{t_1,t_2} into three subdomains

$$D_{1;t_1,t_2} := \big\{(x,t) \in D_{t_1,t_2} :\ t - t_1 < x < (1+\widetilde{k})t_1 - t\big\},$$

$$D_{2;t_1,t_2} := \big\{(x,t) \in D_{t_1,t_2} :\ 0 < x < t - t_1\big\},$$

$$D_{3;t_1,t_2} := \big\{(x,t) \in D_{t_1,t_2} :\ (1+\widetilde{k})t_1 - t < x < \widetilde{k}t\big\}.$$

For $(x,t) \in D_{1;t_1,t_2}$, integration equations of the system (4.22) along the corresponding characteristic curves and bearing in mind the initial conditions (4.20), we obtain

$$\begin{cases} u_n^1(x,t) = -\dfrac{1}{2}\displaystyle\int_{t_1}^{t} g(P_\tau, u_n^3(P_\tau))\big(u_n^1(P_\tau) + u_n^2(P_\tau)\big)\,d\tau + \int_{t_1}^{t} f_n(P_\tau)\,d\tau + \psi_n(x-t+t_1) - \varphi_n'(x-t+t_1), \\ u_n^2(x,t) = -\dfrac{1}{2}\displaystyle\int_{t_1}^{t} g(Q_\tau, u_n^3(Q_\tau))\big(u_n^1(Q_\tau) + u_n^2(Q_\tau)\big)\,d\tau + \int_{t_1}^{t} f_n(Q_\tau)\,d\tau + \psi_n(x+t-t_1) + \varphi_n'(x+t-t_1), \\ u_n^3(x,t) = \displaystyle\int_{t_1}^{t} u_n^1(Q_\tau)\,d\tau + \varphi_n(x+t-t_1), \end{cases}$$

where $P_\tau := (x - t + \tau, \tau)$, $Q_\tau := (x + t - \tau, \tau)$. Passing in this system to the limit, as $n \to \infty$, in the space $C(\overline{D}_{1;t_1,t_2})$ and taking into account (4.2), (4.3), (4.6), (4.7), (4.9) and (4.10), we have

$$\begin{cases} u^1(x,t) = -\dfrac{1}{2}\displaystyle\int_{t_1}^{t} g(P_\tau, u^3(P_\tau))\big(u^1(P_\tau) + u^2(P_\tau)\big)\,d\tau + \int_{t_1}^{t} f(P_\tau)\,d\tau + \psi(x-t+t_1) \\ \qquad\qquad -\varphi'(x-t+t_1), \\ u^2(x,t) = -\dfrac{1}{2}\displaystyle\int_{t_1}^{t} g(Q_\tau, u^3(Q_\tau))\big(u^1(Q_\tau) + u^2(Q_\tau)\big)\,d\tau + \int_{t_1}^{t} f(Q_\tau)\,d\tau + \psi(x+t-t_1) \\ \qquad\qquad +\varphi'(x+t-t_1), \\ u^3(x,t) = \displaystyle\int_{t_1}^{t} u^1(Q_\tau)\,d\tau + \varphi(x+t-t_1). \end{cases} \tag{4.23}$$

Here, by (4.2) and (4.19),

$$u^1 := u_t - u_x, \quad ; u^2 := u_t + u_x, \quad u^3 := u. \tag{4.24}$$

In case $(x,t) \in D_{2;t_1,t_2}$, integrating equations of the system (4.22) along the corresponding characteristic curves and taking into account the initial conditions (4.20), we obtain

$$\begin{cases} u_n^1(x,t) = u_n^1(0,t-x) - \dfrac{1}{2}\displaystyle\int_{t-x}^{t} g(P_\tau, u_n^3(P_\tau))\big(u_n^1(P_\tau) + u_n^2(P_\tau)\big)\,d\tau + \int_{t-x}^{t} f_n(P_\tau)\,d\tau, \\ u_n^2(x,t) = -\dfrac{1}{2}\displaystyle\int_{t_1}^{t} g(Q_\tau, u_n^3(Q_\tau))\big(u_n^1(Q_\tau) + u_n^2(Q_\tau)\big)\,d\tau + \int_{t_1}^{t} f_n(Q_\tau)\,d\tau + \psi_n(x+t-t_1) \\ \qquad\qquad +\varphi_n'(x+t-t_1), \\ u_n^3(x,t) = \displaystyle\int_{t_1}^{t} u_n^1(Q_\tau)d\tau + \varphi_n(x+t-t_1). \end{cases} \tag{4.25}$$

Since due to (4.21) the equality $u_n^1(0,t-x) = u_n^2(0,t-x)$ holds, bearing in mind the second equality of the obtained system and the notation $P_\tau^2 := (t-x-\tau,\tau)$, we can rewrite the system (4.25) in the form

$$\begin{cases} u_n^1(x,t) = -\dfrac{1}{2}\displaystyle\int_{t_1}^{t-x} g(P_\tau^2, u_n^3(P_\tau^2))\big(u_n^1(P_\tau^2) + u_n^2(P_\tau^2)\big)\,d\tau + \int_{t_1}^{t-x} f_n(P_\tau^2)\,d\tau + \psi_n(t-x-t_1) \\ \qquad\qquad +\varphi_n'(t-x-t_1) - \dfrac{1}{2}\displaystyle\int_{t-x}^{t} g(P_\tau, u_n^3(P_\tau))\big(u_n^1(P_\tau) + u_n^2(P_\tau)\big)\,d\tau + \int_{t-x}^{t} f_n(P_\tau)\,d\tau, \\ u_n^2(x,t) = -\dfrac{1}{2}\displaystyle\int_{t_1}^{t} g(Q_\tau, u_n^3(Q_\tau))\big(u_n^1(Q_\tau) + u_n^2(Q_\tau)\big)\,d\tau + \int_{t_1}^{t} f_n(Q_\tau)\,d\tau + \psi_n(x+t-t_1) + \varphi_n'(x+t-t_1), \\ u_n^3(x,t) = \displaystyle\int_{t_1}^{t} u_n^1(Q_\tau)d\tau + \varphi_n(x+t-t_1). \end{cases}$$

Passing here to the limit as $n \to \infty$ in the space $C(\overline{D}_{2;t_1,t_2})$ and taking into account (4.2), (4.3), (4.6), (4.7), (4.9) and (4.19), we have

$$\begin{cases} u^1(x,t) = -\dfrac{1}{2}\displaystyle\int_{t_1}^{t-x} g(P_\tau^2, u^3(P_\tau^2))\big(u^1(P_\tau^2) + u^2(P_\tau^2)\big)\,d\tau + \int_{t_1}^{t-x} f(P_\tau^2)\,d\tau + \psi(t-x-t_1) \\ \qquad\qquad +\varphi'(t-x-t_1) - \dfrac{1}{2}\displaystyle\int_{t-x}^{t} g(P_\tau, u^3(P_\tau))\big(u^1(P_\tau) + u^2(P_\tau)\big)\,d\tau + \int_{t-x}^{t} f(P_\tau)\,d\tau, \\ u^2(x,t) = -\dfrac{1}{2}\displaystyle\int_{t_1}^{t} g(Q_\tau, u^3(Q_\tau))\big(u^1(Q_\tau) + u^2(Q_\tau)\big)\,d\tau + \int_{t_1}^{t} f(Q_\tau)\,d\tau + \psi(x+t-t_1) \\ \qquad\qquad +\varphi'(x+t-t_1), \\ u^3(x,t) = \displaystyle\int_{t_1}^{t} u^1(Q_\tau)\,d\tau + \varphi(x+t-t_1). \end{cases} \tag{4.26}$$

For $(x,t)\in D_{3;t_1,t_2}$, integrating equations of the system (4.22) along the characteristic curves, in view of the initial and boundary conditions (4.20), (4.21), we obtain

$$\begin{cases} u_n^1(x,t)=-\dfrac{1}{2}\displaystyle\int_{t_1}^{t} g(P_\tau,u_n^3(P_\tau))\big(u_n^1(P_\tau)+u_n^2(P_\tau)\big)\,d\tau+\int_{t_1}^{t} f_n(P_\tau)\,d\tau+\psi_n(x-t+t_1)-\varphi_n'(x-t+t_1),\\ u_n^2(x,t)=u_n^2\Big(\dfrac{\widetilde{k}(x+t)}{\widetilde{k}+1}\,,\dfrac{x+t}{\widetilde{k}+1}\Big)-\dfrac{1}{2}\displaystyle\int_{\frac{x+t}{\widetilde{k}+1}}^{t} g(Q_\tau,u_n^3(Q_\tau))\big(u_n^1(Q_\tau)+u_n^2(Q_\tau)\big)\,d\tau+\int_{\frac{x+t}{\widetilde{k}+1}}^{t} f_n(Q_\tau)\,d\tau,\\ u_n^3(x,t)=\displaystyle\int_{\frac{x+t}{\widetilde{k}+1}}^{t} u_n^1(Q_\tau)\,d\tau. \end{cases} \tag{4.27}$$

Since by (4.21) there is on $\gamma_{1;t_1,t_2}$ the equality $u_n^2=\frac{\widetilde{k}-1}{\widetilde{k}+1}\,u_n^1$, due to the first equality of the obtained system and the notation $P_\tau^3:=(\frac{\widetilde{k}-1}{\widetilde{k}+1}\,(x+t)+\tau,\tau)$, the system (4.27) can be rewritten in the form

$$\begin{cases} u_n^1(x,t)=-\dfrac{1}{2}\displaystyle\int_{t_1}^{t} g(P_\tau,u_n^3(P_\tau))\big(u_n^1(P_\tau)+u_n^2(P_\tau)\big)\,d\tau+\int_{t_1}^{t} f_n(P_\tau)\,d\tau+\psi_n(x-t+t_1)-\varphi_n'(x-t+t_1),\\ u_n^2(x,t)=\dfrac{\widetilde{k}-1}{\widetilde{k}+1}\bigg[-\dfrac{1}{2}\displaystyle\int_{t_1}^{\frac{x+t}{\widetilde{k}+1}} g(P_\tau^3,u_n^3(P_\tau^3))\big(u_n^1(P_\tau^3)+u_n^2(P_\tau^3)\big)\,d\tau+\int_{t_1}^{\frac{x+t}{\widetilde{k}+1}} f_n(P_\tau^3)\,d\tau\\ \qquad+\psi_n\Big(\dfrac{\widetilde{k}-1}{\widetilde{k}+1}\,(x+t)+t_1\Big)-\varphi_n'\Big(\dfrac{\widetilde{k}-1}{\widetilde{k}+1}(x+t)+t_1\Big)\bigg]\\ \qquad-\dfrac{1}{2}\displaystyle\int_{\frac{x+t}{\widetilde{k}+1}}^{t} g(Q_\tau,u_n^3(Q_\tau))\big(u_n^1(Q_\tau)+u_n^2(Q_\tau)\big)\,d\tau+\int_{\frac{x+t}{\widetilde{k}+1}}^{t} f_n(Q_\tau)\,d\tau,\\ u_n^3(x,t)=\displaystyle\int_{\frac{x+t}{\widetilde{k}+1}}^{t} u_n^1(Q_\tau)\,d\tau. \end{cases}$$

Passing in this system to the limit, as $n\to\infty$, in the space $C(\overline{D}_{3;t_1,t_2})$ and taking into account (4.2), (4.3), (4.6), (4.7), (4.9) and (4.10), we have

$$\begin{cases} u^1(x,t)=-\dfrac{1}{2}\displaystyle\int_{t_1}^{t} g(P_\tau,u^3(P_\tau))\big(u^1(P_\tau)+u^2(P_\tau)\big)\,d\tau+\int_{t_1}^{t} f(P_\tau)\,d\tau+\psi(x-t+t_1)-\varphi'(x-t+t_1),\\ u^2(x,t)=\dfrac{\widetilde{k}-1}{\widetilde{k}+1}\bigg[-\dfrac{1}{2}\displaystyle\int_{t_1}^{\frac{x+t}{\widetilde{k}+1}} g(P_\tau^3,u^3(P_\tau^3))\big(u^1(P_\tau^3)+u^2(P_\tau^3)\big)\,d\tau+\int_{t_1}^{\frac{x+t}{\widetilde{k}+1}} f(P_\tau^3)\,d\tau\\ \qquad+\psi\Big(\dfrac{\widetilde{k}-1}{\widetilde{k}+1}(x+t)+t_1\Big)-\varphi'\Big(\dfrac{\widetilde{k}-1}{\widetilde{k}+1}(x+t)+t_1\Big)\bigg]\\ \qquad-\dfrac{1}{2}\displaystyle\int_{\frac{x+t}{\widetilde{k}+1}}^{t} g(Q_\tau,u^3(Q_\tau))\big(u^1(Q_\tau)+u^2(Q_\tau)\big)\,d\tau+\int_{\frac{x+t}{\widetilde{k}+1}}^{t} f(Q_\tau)\,d\tau,\\ u^3(x,t)=\displaystyle\int_{\frac{x+t}{\widetilde{k}+1}}^{t} u^1(Q_\tau)\,d\tau. \end{cases} \tag{4.28}$$

By the a priori estimate (4.5), for a generalized solution $u^3 = u$ of the problem (1.1), (1.3), (1.4) we get
$$\big|g(x,t,u^3(x,t))\big| \le K_{\varphi,\psi}, \quad (x,t) \in \overline{D}_{t_1,t_2}, \tag{4.29}$$
where $K_{\varphi,\psi}$ is defined in (4.18).

Let
$$v^i(t) := \sup_{(\xi,\tau)\in\overline{D}_{t_1,t}} |u^i(\xi,\tau)|, \quad i=1,2,3, \quad F(t) := \sup_{(\xi,\tau)\in\overline{D}_{t_1,t}} |f(\xi,\tau)|. \tag{4.30}$$
It follows from (4.23), (4.26) and (4.28) by virtue of (4.29) and (4.30) that
$$|u^i(x,t)| \le (K_{\varphi,\psi}+1)\int\limits_{t_1}^{t} \big[v^1(\tau)+v^2(\tau)\big]\,d\tau + 2t\|f\|_{C(\overline{D}_{t_1,t})} + \|\varphi\|_{C^1(\overline{\omega}_{t_1})} + \|\psi\|_{C(\overline{\omega}_{t_1})}, \quad i=1,2,3.$$
whence taking into account the fact that the right-hand sides of these inequalities are nondecreasing, by virtue of (4.30), we obtain
$$|v^i(t)| \le (K_{\varphi,\psi}+1)\int\limits_{t_1}^{t} \big[v^1(\tau)+v^2(\tau)\big]\,d\tau + 2t_2\|f\|_{C(\overline{D}_{t_1,t_2})} + \|\varphi\|_{C^1(\overline{\omega}_{t_1})} + \|\psi\|_{C(\overline{\omega}_{t_1})},$$
$$t_1 \le t \le t_2, \quad i=1,2,3.$$

Putting $v(t) := \max\limits_{1\le i\le 3} v^i(t)$, the obtained inequalities result in
$$v(t) \le 2(K_{\varphi,\psi}+1)\int\limits_{t_1}^{t} v(\tau)\,d\tau + 2t_2\|f\|_{C(\overline{D}_{t_1,t_2})} + \|\varphi\|_{C^1(\overline{\omega}_{t_1})} + \|\psi\|_{C(\overline{\omega}_{t_1})}, \quad t_1 \le t \le t_2. \tag{4.31}$$
From (4.31), applying Gronwall's lemma, we obtain
$$v(t) \le \Big[2t_2\|f\|_{C(\overline{D}_{t_1,t_2})} + \|\varphi\|_{C^1(\overline{\omega}_{t_1})} + \|\psi\|_{C(\overline{\omega}_{t_1})}\Big] \exp\big[2(K_{\varphi,\psi}+1)t\big], \quad t_1 \le t \le t_2.$$

From (4.24) and (4.30), it now easily follows that
$$\|u\|_{C^1(\overline{D}_{t_1,t_2})} \le \Big[2t_2\|f\|_{C(\overline{D}_{t_1,t_2})} + \|\varphi\|_{C^1(\overline{\omega}_{t_1})} + \|\psi\|_{C(\overline{\omega}_{t_1})}\big] \exp[2(K_{\varphi,\psi}+1)t_2\Big],$$
which proves Lemma 4.2. □

5. The Uniqueness of a Solution of the Problems (1.1), (1.2) and (1.1), (1.3), (1.4)

Lemma 5.1. *Let the conditions* (3.5), (4.1), (4.4), (4.16) *be fulfilled. Then the problem* (1.1), (1.3), (1.4) *may have no more than one generalized solution of the class* $C^1(\overline{D}_{t_1,t_2})$.

Proof. Indeed, assume that the problem (1.1), (1.3), (1.4) has two possible different generalized solutions u^1 and u^2 of the class C^1 in the domain D_{t_1,t_2}. According to Definition 1.1, there exists a sequence of functions $u_n^i \in \overset{\circ}{C}{}^2(\overline{D}_{t_1,t_2},\Gamma_{t_1,t_2})$ such that the limiting equalities
$$\lim_{n\to\infty}\|u_n^i - u^i\|_{C^1(\overline{D}_{t_1,t_2})} = 0, \quad \lim_{n\to\infty}\|Lu_n^i - f\|_{C(\overline{D}_{t_1,t_2})} = 0 \tag{5.1}$$
and
$$\lim_{n\to\infty}\|u_n^i|_{\overline{\omega}_{t_1}} - \varphi\|_{C^1(\overline{\omega}_{t_1})} = 0, \quad \lim_{n\to\infty}\|u_{nt}^i|_{\overline{\omega}_{t_1}} - \psi\|_{C(\overline{\omega}_{t_1})} = 0, \quad i=1,2, \tag{5.2}$$
hold.

We take advantage here the well-known notation $\square := \partial^2/\partial t^2 - \partial^2/\partial x^2$ and put $\omega_n := u_n^2 - u_n^1$. It can be easily seen that the function $\omega_n \in \overset{\circ}{C}{}^2(\overline{D}_{t_1,t_2},\Gamma_{t_1,t_2})$ satisfies the following equalities:
$$\square\omega_n + g_n = f_n,, \tag{5.3}$$
$$\omega_n\big|_{\overline{\omega}_{t_1}} = \widetilde{\varphi}_n, \quad \omega_{nt}\big|_{\overline{\omega}_{t_1}} = \widetilde{\psi}_n, \tag{5.4}$$
$$\omega_n\big|_{\gamma_{1;t_1,t_2}} = 0, \quad \omega_{nx}\big|_{\gamma_{2;t_1,t_2}} = 0, \tag{5.5}$$
where
$$g_n := g(x,t,u_n^2)u_{nt}^2 - g(x,t,u_n^1)u_{nt}^1, \quad f_n := Lu_n^2 - Lu_n^1, \tag{5.6}$$

$$\widetilde{\varphi}_n := \omega_n\big|_{\overline{\omega}_{t_1}}, \quad \widetilde{\psi}_n := \omega_{nt}\big|_{\overline{\omega}_{t_1}}, \tag{5.7}$$

and by virtue of (5.2) and (5.7), the equalities

$$\lim_{n\to\infty} \|\widetilde{\varphi}_n\|_{C^1(\overline{\omega}_{t_1})} = 0, \quad \lim_{n\to\infty} \|\widetilde{\psi}_n\|_{C(\overline{\omega}_{t_1})} = 0, \ \ i = 1, 2, \tag{5.8}$$

hold.

By the first equality of (5.1), there is the number $A = const > 0$, independent of the indices i and n, such that

$$\|u_n^i\|_{C^1(\overline{D}_{t_1,t_2})} \le A. \tag{5.9}$$

According to the second equalities of (5.1) and (5.6), we have

$$\lim_{n\to\infty} \|f_n\|_{C(\overline{D}_{t_1,t_2})} = 0. \tag{5.10}$$

By (3.5), (5.9) and the first equality of (5.6), it is not difficult to see that

$$g_n^2 = \Big(g(x,t,u_n^2)\omega_{nt} + \big(g(x,t,u_n^2) - g(x,t,u_n^1)\big)u_{nt}^1\Big)^2 \le 2m^2(A)\omega_{nt}^2 + 2A^2c^2(A)\omega_n^2. \tag{5.11}$$

Multiplying both parts of the equality (5.3) by ω_{nt} and integrating the obtained equality with respect to the domain D_{t_1,t_2}, by virtue of (5.4), (5.5), just in the same manner as when obtaining inequality (4.13), from (4.10)–(4.12), we have

$$w_n(\tau) := \int\limits_{\omega_\tau} (\omega_{nx}^2 + \omega_{nt}^2)\,dx \le \int\limits_{\omega_{t_1}} \big(\widetilde{\varphi}_n{}'^2 + \widetilde{\psi}_n^2\big)\,dx + 2\int\limits_{D_{t_1,t_2;\tau}} (f_n - g_n)\omega_{nt}\,dx\,dt. \tag{5.12}$$

By virtue of the estimate (5.11) and the Cauchy inequality, we obtain

$$\begin{aligned}
2\int\limits_{D_{t_1,t_2;\tau}} (f_n - g_n)\omega_{nt}\,dx\,dt &\le \int\limits_{D_{t_1,t_2;\tau}} (f_n - g_n)^2 dxdt + \int\limits_{D_{t_1,t_2;\tau}} \omega_{nt}^2\,dx\,dt \\
&\le 2\int\limits_{D_{t_1,t_2;\tau}} f_n^2\,dx\,dt + 2\int\limits_{D_{t_1,t_2;\tau}} g_n^2\,dx\,dt + \int\limits_{D_{t_1,t_2;\tau}} \omega_{nt}^2\,dx\,dt \\
&\le 2\int\limits_{D_{t_1,t_2;\tau}} f_n^2\,dx\,dt + 4A^2c^2(A)\int\limits_{D_{t_1,t_2;\tau}} \omega_n^2\,dx\,dt + (1+4m^2(A))\int\limits_{D_{t_1,t_2;\tau}} \omega_{nt}^2\,dx\,dt.
\end{aligned} \tag{5.13}$$

Next, in view of the equality

$$\omega_n(x,t) = \int\limits_{\widetilde{k}t}^{x} \omega_{nx}(\xi,t)\,d\xi, \quad (x,t) \in \overline{D}_{t_1,t_2;\tau}$$

which follows from the first equality of (5.5), reasoning in a standard manner, we obtain the following inequality:

$$\int\limits_{D_{t_1,t_2;\tau}} \omega_n^2\,dx\,dt \le (\widetilde{k}T)^2 \int\limits_{D_{t_1,t_2;\tau}} \omega_{nx}^2\,dx\,dt. \tag{5.14}$$

It follows from (5.12)–(5.14) that

$$\begin{aligned}
w_n(\tau) &\le \int\limits_{\omega_{t_1}} (\widetilde{\varphi}_n'^2 + \widetilde{\psi}_n^2)\,dx + 2\int\limits_{D_{t_1,t_2;\tau}} f_n^2\,dx\,dt \\
&\qquad + 4k^2T^2A^2c^2(A)\int\limits_{D_{t_1,t_2;\tau}} \omega_{nx}^2\,dx\,dt + (1+4m^2(A))\int\limits_{D_{t_1,t_2;\tau}} \omega_{nt}^2\,dx\,dt \\
&\le \int\limits_{\omega_{t_1}} (\widetilde{\varphi}_n'^2 + \widetilde{\psi}_n^2)\,dx + 2\int\limits_{D_{t_1,t_2;\tau}} f_n^2\,dx\,dt + \big(4k^2T^2A^2c^2(A) + 1 + 4m^2(A)\big)\int\limits_{D_{t_1,t_2;\tau}} (\omega_{nx}^2 + \omega_{nt}^2)\,dx\,dt
\end{aligned}$$

$$= \big(4k^2T^2A^2c^2(A)+1+4m^2(A)\big)\int\limits_{t_1}^{\tau} w_n(\sigma)\,d\sigma + \int\limits_{\omega_{t_1}} (\widetilde{\varphi}_n'^2+\widetilde{\psi}_n^2)\,dx + 2\int\limits_{D_{t_1,t_2;\tau}} f_n^2\,dx\,dt,$$

whence due to the Gronwall's lemma, we find that

$$w_n(\tau)\le c_2\big(\|\widetilde{\varphi}'_n\|^2_{L_2(\omega_{t_1})}+\|\widetilde{\psi}_n\|^2_{L_2(\omega_{t_1})}+2\|f_n\|^2_{L_2(D_{t_1,t_2})}\big),\ \ t_1<\tau\le t_2, \tag{5.15}$$

where

$$c_2:=\exp\big(4k^2T^2A^2c^2(A)+1+4m^2(A)\big)(t_2-t_1).$$

Reasoning analogously as in the obtaining estimate (4.15) and taking into account obvious inequalities

$$\|f_n\|^2_{L_2(D_{t_1,t_2})}\le\|f_n\|^2_{C(\overline{D}_{t_1,t_2})}\operatorname{mes}D_{t_1,t_2},\quad \|\widetilde{\varphi}'_n\|^2_{L_2(\omega_{t_1})}\le\|\widetilde{\varphi}'_n\|^2_{C(\overline{\omega}_{t_1})}\operatorname{mes}\omega_{t_1},$$
$$\|\widetilde{\psi}_n\|^2_{L_2(\omega_{t_1})}\le\|\widetilde{\psi}_n\|^2_{C(\overline{\omega}_{t_1})}\operatorname{mes}\omega_{t_1},$$

by virtue of (5.15), for $(x,t)\in\overline{D}_{t_1,t_2}$ we have

$$|\omega_n(x,t)|^2\le\widetilde{k}tw_n(t)\le\widetilde{k}Tc_2\Big(\operatorname{mes}\omega_{t_1}\|\widetilde{\varphi}'_n\|^2_{C(\overline{\omega}_{t_1})}+\operatorname{mes}\omega_{t_1}\|\widetilde{\psi}_n\|^2_{C(\overline{\omega}_{t_1})}+2\operatorname{mes}D_{t_1,t_2}\|f_n\|^2_{C(\overline{D}_{t_1,t_2})}\Big)$$
$$\le c_2(\widetilde{k}T)^2(1+T)\big(\|\widetilde{\varphi}'_n\|^2_{C(\overline{\omega}_{t_1})}+\|\widetilde{\psi}_n\|^2_{C(\overline{\omega}_{t_1})}+\|f_n\|^2_{C(\overline{D}_{t_1,t_2})}\big).$$

Hence it immediately follows that

$$\|\omega_n\|_{C(\overline{D}_{t_1,t_2})}\le\widetilde{k}T\sqrt{c_2(1+T)}\,\big(\|\widetilde{\varphi}'_n\|_{C(\overline{\omega}_{t_1})}+\|\widetilde{\psi}_n\|_{C(\overline{\omega}_{t_1})}+\|f_n\|_{C(\overline{D}_{t_1,t_2})}\big). \tag{5.16}$$

According to the definition of the function ω_n and the first equality of (5.1), we can easily see that

$$\lim_{n\to\infty}\|\omega_n\|_{C^1(\overline{D}_{t_1,t_2})}=\|u^2-u^1\|_{C^1(\overline{D}_{t_1,t_2})}$$

and all the more,

$$\lim_{n\to\infty}\|\omega_n\|_{C(\overline{D}_{t_1,t_2})}=\|u^2-u^1\|_{C(\overline{D}_{t_1,t_2})}.$$

Therefore, passing in the inequality (5.16) to the limit, as $n\to\infty$, and taking into account (5.8) and (5.10), we obtain $\|u^2-u^1\|_{C(\overline{D}_{t_1,t_2})}=0$, i.e. $u^1=u^2$. □

Theorem 5.1. *Let the conditions* (3.5), (4.1), (4.4) *be fulfilled. Then the problem* (1.1), (1.2) *may have no more than one generalized solution of the class* $C^1(\overline{D}_T)$.

Proof. We take a natural number n so large that $\Delta=\frac{T-T_1}{n}<\frac{1}{2}\widetilde{k}T_1$, where T_1 is the number appearing in Remark 3.2, and put $T_i:=T_1+(i-1)\Delta$, $i=2,\dots,n+1$. Then if u_1 and u_2 are the two possible solutions of the problem (1.1), (1.2) of the class $C^1(\overline{D}_T)$, then owing to Remark 3.2, we have $u_1|_{D_{T_1}}=u_2|_{D_{T_1}}$, whence by virtue of Lemma 5.1, we find that $u_1|_{D_{T_1,T_2}}=u_2|_{D_{T_1,T_2}}$. Further, continuing analogous reasoning step by step, in the domains $D_{T_2,T_3},D_{T_3,T_4},\dots,D_{T_n,T_{n+1}}$ we find that $u_1|_{D_{T_i,T_{i+1}}}=u_2|_{D_{T_i,T_{i+1}}}$, $i=2,\dots,n$, and hence $u_1|_{D_T}=u_2|_{D_T}$. Thus this proves the uniqueness of a solution of the problem (1.1), (1.2) in the class $C^1(\overline{D}_T)$. □

6. Solvability of the Problem (1.1), (1.2)

As is known, if a global a priori estimate of a solution is obtained and the existence of a local solution of the evolution problem is established, then reasoning in a standard manner, we obtain the existence of the global solution of that problem (see, e.g., [20]). In our case, the a priori estimate of a solution of the problem (1.1), (1.3), (1.4) is obtained under the assumption that the height $\Delta t:=t_2-t_1$ of the trapezoid D_{t_1,t_2} is less than the defined value (see (4.16)). Therefore, in this case, to prove the existence of the global solution, we have to modify the above-mentioned general approach, making it convenient for our case.

Remark 6.1. In the assumption that the condition (4.16) is fulfilled, we consider first the question on the solvability of the problem (1.1), (1.3), (1.4) of the class C^1 in the domain D_{t_1,t_2} taking into account that if u is a generalized solution of that problem of the class C^1 in the domain D_{t_1,t_2}, then $u^1 := u_t - u_x$, $u^2 := u_t + u_x$, $u^3 := u$ is a continuous solution of the system of nonlinear Volterra type integral equations (4.23), (4.26), (4.28), respectively, in the domains $D_{1;t_1,t_2}$, $D_{2;t_1,t_2}$, $D_{3;t_1,t_2}$, and vice versa, if u^1, u^2, u^3 is a continuous solution of the above-mentioned system, then $u := u^3$ is a generalized solution of the problem (1.1), (1.3), (1.4) of the class C^1 in the domain D_{t_1,t_2}, and the equalities $u^1 := u_t - u_x$, $u^2 := u_t + u_x$ are valid.

We rewrite the systems (4.23), (4.26) and (4.28) in the vector form

$$U(P) = (\Phi U)(P), \quad P \in D_{t_1,t_2}, \tag{6.1}$$

where $U := (u^1, u^2, u^3)$ and $\Phi := (\Phi^1, \Phi^2, \Phi^3)$, and the operators

$$\Phi^1(U) := \Phi(U)\big|_{D_{1;t_1,t_2}}, \quad \Phi^2(U) := \Phi(U)\big|_{D_{2;t_1,t_2}}, \quad \Phi^3(U) := \Phi(U)\big|_{D_{3;t_1,t_2}} \tag{6.2}$$

are defined by the right-hand sides of the systems (4.23), (4.26) and (4.28), respectively.

Let

$$\|U\|_{X_{t_1,t_2}} := \max_{1\le i\le 3} \big\{\|u^i\|_{C(\overline{D}_{t_1,t_2})}\big\}, \quad U \in X_{t_1,t_2} := C(\overline{D}_{t_1,t_2}; \mathbb{R}^3).$$

We fix the number $R > 0$ and denote by $B_R(t_1,t_2) := \{U \in X_{t_1,t_2} : \ \|U\|_{X_{t_1,t_2}} \le R\}$ a closed ball of radius R in the Banach space X_{t_1,t_2} with the center in a zero element.

Below, it will be shown that there exists the positive number $t_2^0 \in (t_1, T]$ such that for any $t_2 < t_2^0$:

(i) Φ maps the ball $B_R(t_1,t_2)$ into itself;

(ii) Φ is a contracting mapping on the set $B_R(t_1,t_2)$.

Assume

$$R = 2\big(2T\|f\|_{C(\overline{D}_T)} + \|\varphi\|_{C^1(\overline{\omega}_{t_1})} + \|\psi\|_{C(\overline{\omega}_{t_1})}\big).$$

For $\|U\|_{X_{t_1,t_2}} \le R$, by virtue of (6.1), from (4.31), we have

$$\begin{aligned}
\big|(\Phi U)(x,t)\big| &\le 2(K_{\varphi,\psi}+1)\int_{t_1}^{t} v(\tau)\,d\tau + 2t_2\|f\|_{C(\overline{D}_{t_1,t_2})} + \|\varphi\|_{C^1(\overline{\omega}_{t_1})} + \|\psi\|_{C(\overline{\omega}_{t_1})} \\
&\le 2(K_{\varphi,\psi}+1)R(t-t_1) + 2T\|f\|_{C(\overline{D}_T)} + \|\varphi\|_{C^1(\overline{\omega}_{t_1})} + \|\psi\|_{C(\overline{\omega}_{t_1})}, \quad t_1 \le t \le t_2,
\end{aligned}$$

whence for

$$\Delta t_1 := t_2 - t_1 \le \frac{1}{4(K_{\varphi,\psi}+1)}\,, \tag{6.3}$$

we obtain

$$|(\Phi U)(x,t)| \le R, \quad (x,t) \in D_{t_1,t_2}. \tag{6.4}$$

The value K here is defined in Lemma 4.2.

Thus, by (6.4), in the case (6.3), the operator Φ maps the ball $B_R(t_1,t_2)$ into itself, i.e., item (i) is fulfilled.

Let us now show that item (ii) is likewise fulfilled, that is, the operator Φ is a contracted mapping in that ball. Indeed, for $U_i := (u_i^1, u_i^2, u_i^3)$, $i = 1,2$, and $P \in D_{1;t_1,t_2}$, from (4.23), by virtue of (3.5) for

$$(\Phi_1^1 U)(P) := -\frac{1}{2}\int_{t_1}^{t} g(P_\tau, u^3(P_\tau))\big(u^1(P_\tau)+u^2(P_\tau)\big)\,d\tau + \int_{t_1}^{t} f(P_\tau)\,d\tau + \psi(x-t+t_1) - \varphi'(x-t+t_1),$$

we have

$$\begin{aligned}
\big|(\Phi_1^1 U_2 - \Phi_1^1 U_1)(x,t)\big| \le \frac{1}{2}\int_{t_1}^{t} \Big(&\big|g(P_\tau, u_2^3(P_\tau)) - g(P_\tau, u_1^3(P_\tau))\big|\,\big|u_2^1(P_\tau) + u_2^2(P_\tau)\big| \\
&+ \big|g(P_\tau, u_1^3(P_\tau))\big|\,\big|u_2^1(P_\tau) - u_1^1(P_\tau) + u_2^2(P_\tau) - u_1^2(P_\tau)\big|\Big)\,d\tau
\end{aligned}$$

$$\le c(R)R\Delta t_1\|u_2^3-u_1^3\|_{C(\overline{D}_{1;t_1,t_2})}+\frac{1}{2}\,m(R)\Delta t_1\big(\|u_2^1-u_1^1\|_{C(\overline{D}_{1;t_1,t_2})}+\|u_2^2-u_1^2\|_{C(\overline{D}_{1;t_1,t_2})}\big)$$
$$\le (c(R)R+m(R))\Delta t_1\|U_2-U_1\|_{C(\overline{D}_{1;t_1,t_2})},$$

whence in view of (4.23) and (6.3), for

$$\Delta t_1:=t_2-t_1=\min\Big\{\frac{1}{2}\,\widetilde{k}t_1,\frac{1}{4(K_{\varphi,\psi}+1)}\,,\frac{1}{2(c(R)R+m(R))}\Big\} \tag{6.5}$$

we obtain

$$\big|(\Phi_1^1U_2-\Phi_1^1U_1)(x,t)\big|\le\frac{1}{2}\,\|U_2-U_1\|_{C(\overline{D}_{1;t_1,t_2})},\ \ (x,t)\in D_{1;t_1,t_2}. \tag{6.6}$$

The estimates, analogous to (6.6) are likewise valid for the operators

$$(\Phi_2^1U)(P):=-\frac{1}{2}\int_{t_1}^{t}g(Q_\tau,u^3(Q_\tau))\big(u^1(Q_\tau)+u^2(Q_\tau)\big)\,d\tau+\int_{t_1}^{t}f(Q_\tau)\,d\tau+\psi(x+t-t_1)+\varphi'(x+t-t_1)$$

and

$$(\Phi_3^1U)(P):=\int_{t_1}^{t}u^1(Q_\tau)\,d\tau+\varphi(x+t-t_1)$$

from (6.2), namely,

$$\big|(\Phi_i^1U_2-\Phi_i^1U_1)(x,t)\big|\le\frac{1}{2}\,\|U_2-U_1\|_{C(\overline{D}_{1;t_1,t_2})},\ \ (x,t)\in D_{1;t_1,t_2},\ \ i=2,3. \tag{6.7}$$

The same reasonings in the case (6.5) result in the following estimates:

$$\big|(\Phi_j^iU_2-\Phi_j^iU_1)(x,t)\big|\le\frac{1}{2}\,\|U_2-U_1\|_{C(\overline{D}_{i;t_1,t_2})},\ \ (x,t)\in D_{i;t_1,t_2},\ \ i=2,3;\ \ j=1,2,3. \tag{6.8}$$

Bearing in mind (6.1), (6.2), (6.5)–(6.8), the estimate

$$\|\Phi U_2-\Phi U_1\|_{C(\overline{D}_{t_1,t_2})}\le\frac{1}{2}\,\|U_2-U_1\|_{C(\overline{D}_{t_1,t_2})},\ \ (x,t)\in D_{t_1,t_2} \tag{6.9}$$

holds.

Thus, in the case (6.5), by virtue of (6.4), (6.9) and theorem on the contracted mapping it follows that the system (6.1) in the class $C(\overline{D}_{t_1,t_2})$ is solvable, and hence the following lemma is valid.

Lemma 6.1. *The problem* (1.1), (1.3), (1.4) *has a unique solution of the class* C^1 *in the domain* D_{t_1,t_2} *if the condition* (6.5) *is fulfilled.*

Let $t_1=T_1<T$, where T_1 is taken from Theorem 3.1 when the problem (1.1), (1.2) has a unique generalized solution of the class C^1 in the triangular domain D_{T_1}.

We take a natural number n so large that the inequality

$$\frac{T-T_1}{n}<\frac{1}{2}\,\widetilde{k}T_1 \tag{6.10}$$

holds.

Accordingly, we divide the interval $[T_1,T]$ into n equal segments $[T_1,T_2],[T_2,T_3],\dots,[T_n,T_{n+1}]$ of the same length $\Delta:=\frac{T-T_1}{n}$.

In the domain D_{T_1,T_2}, consider the problem (1.1), (1.3), (1.4) in which as the initial functions φ and ψ we take traces of the solution u and its derivative u_t of the problem (1.1), (1.2) in the domain D_{T_1} on the interval ω_{T_1}. In view of (6.10), the condition (4.16) of Lemma 4.2 is fulfilled, and hence we have the following a priori estimate

$$\|u\|_{C^1(\overline{D}_{T_1,T_2})}\le L_1:=\big(2T\|f\|_{C(\overline{D}_T)}+\|\varphi\|_{C^1(\overline{\omega}_{T_1})}+\|\psi\|_{C(\overline{\omega}_{T_1})}\big)\exp\big[2(K_{\varphi,\psi}+1)T\big]. \tag{6.11}$$

Remark 6.2. From the definition of the value $K=K(s)$, $s\ge0$ it is easy to see that it is the nondecreasing function with respect to the variable s.

Remark 6.3. It is not difficult to see that by virtue of (6.11) and (4.17), if u is a solution of the problem (1.1), (1.3), (1.4) of the class C^1 in the domain D_{T_1,T_2}, then the estimate

$$\|u|_{t=\tau}\|_{C^1(\overline{\omega}_\tau)} + \|u_t|_{t=\tau}\|_{C(\overline{\omega}_\tau)} \le 2L_1 \quad \forall\,\tau \in [T_1,T_2] \tag{6.12}$$

is valid, and hence

$$K_{\varphi_\tau,\psi_\tau} = K\big(\|f\|_{C(\overline{D}_T)} + \|u|_{t=\tau}\|_{C^1(\overline{\omega}_\tau)} + \|u_t|_{t=\tau}\|_{C(\overline{\omega}_\tau)}\big) \le K\big(\|f\|_{C(\overline{D}_T)} + 2L_1\big) \quad \forall\,\tau \in [T_1,T_2]. \tag{6.13}$$

By Lemma 6.1, in view of (6.5) and (6.13), for the value Δt_1 for which there exists the unique solution of the problem (1.1), (1.3), (1.4) of the class C^1 in the domain D_{T_1,t_2}, where $t_2 = T_1 + \Delta t_1$, the following lower bound

$$\Delta t_1 \ge \min\Big\{\frac{1}{2}\,\widetilde{k}t_1,\ \frac{1}{4(K(\|f\|_{C(\overline{D}_T)} + 2L_1) + 1)},\ \frac{1}{2(c(R)R + m(R))}\Big\} \tag{6.14}$$

is valid.

Continuing this process of constructing a local solution of the problem (1.1), (1.3), (1.4) in the domains D_{t_{i-1},t_i}, by (6.14), for the length Δt_i of the interval $[t_{i-1},t_i]$, independently on the step number i, there exists the natural number i_0 such that $t_{i_0} \ge t_2$. This latter means that the problem (1.1), (1.3), (1.4) has the unique solution in the domain D_{T_1,T_2}. The same process, owing to the estimate (6.14), allows one to construct step by step a unique solution of the problem (1.1), (1.3), (1.4) in the domains $D_{T_2,T_3},\dots,D_{T_n,T_{n+1}}$, and since $T_{n+1} = T$, this proves the existence of a generalized solution of the problem (1.1), (1.2) in the domain D_T.

Thus the following theorem is valid.

Theorem 6.1. *Let $f \in C(\overline{D}_T)$, $g \in C(\overline{D}_T \times \mathbb{R})$ and the conditions* (3.5) *and* (4.4) *be fulfilled. Then the problem* (1.1), (1.2) *has a unique generalized solution of the class C^1 in the domain D_T.*

Remark 6.4. From Theorem 6.1 we arrive at the global solvability of the problem (1.1), (1.2) in the sense of Definition 1.3.

7. The Case of Nonexistence of a Global Solution of the Problem (1.1), (1.2)

Below, we will show that violation of the condition (4.4) may result in the nonexistence of global solvability of the problem (1.1), (1.2) in the sense of Definition 1.3. To simplify our exposition, we consider the case $\widetilde{k} = 1$, i.e., when $\widetilde{\gamma}_{1,T}$ is the characteristic of the equation (1.1). Indeed, let $g(x,t,s) = -|s|^\alpha s$, $s \in \mathbb{R}$ and the nonlinearity exponent $\alpha > -1$.

Lemma 7.1. *Let u be a strong generalized solution of the problem* (1.1), (1.2) *of the class C^1 in the domain D_T in the sense of Definition* 1.1. *Then the following integral equality*

$$\int\limits_{D_T} u\,\square\,\varphi\,dx\,dt = \int\limits_{D_T} |u|^\alpha u u_t \varphi\,dx\,dt + \int\limits_{D_T} f\varphi\,dx\,dt \tag{7.1}$$

is valid for any function φ such that

$$\varphi \in C^2(\overline{D}_T),\ \ \varphi\big|_{\widetilde{\gamma}_{3,T}} = 0,\ \ \varphi_t\big|_{\widetilde{\gamma}_{3,T}} = 0,\ \ \varphi_x\big|_{\widetilde{\gamma}_{2,T}} = 0. \tag{7.2}$$

Proof. According to the definition of a strong generalized solution u of the problem (1.1), (1.2) of the class C^1 in the domain D_T, the function $u \in C^1(\overline{D}_T)$ and there exists the sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T,\widetilde{\Gamma}_T)$ such that the equalities

$$\lim_{n\to\infty} \|u_n - u\|_{C^1(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|Lu_n - f\|_{C(\overline{D}_T)} = 0 \tag{7.3}$$

are valid.

Assume $f_n := Lu_n$. We multiply both parts of the equality $Lu_n = f_n$ by the function φ and integrate the obtained equality with respect to the domain D_T. As a result of integration by parts of the left part of that equality, in view of (7.2) and the conditions (1.2), we obtain

$$\int\limits_{D_T} u_n\,\square\,\varphi\,dx\,dt = \int\limits_{D_T} |u_n|^\alpha u_n u_{nt}\varphi\,dx\,dt + \int\limits_{D_T} f_n\varphi\,dx\,dt.$$

Passing in this equality to the limit, as $n \to \infty$, owing to (7.3), we obtain (7.1). $\square$

Below, the use will be made of the test functions method (see, e.g., [19, pp. 10–12]). We introduce into consideration the function $\varphi^0 := \varphi^0(x,t)$ such that

$$\varphi^0 \in C^2(\overline{D}_\infty),\ \ \varphi^0 + \varphi^0_t \leq 0,\ \ \varphi^0\big|_{D_{T=1}} > 0,\ \ \varphi^0_x\big|_{\widetilde{\gamma}_{2,\infty}} = 0,\ \ \varphi^0\big|_{t\geq 1} = 0 \tag{7.4}$$

and

$$\kappa_0 := \int\limits_{D_{T=1}} \frac{|\Box\varphi^0|^{p'}}{|\varphi^0|^{p'-1}}\,dx\,dt < +\infty,\ \ p' = \frac{\alpha+2}{\alpha+1}. \tag{7.5}$$

It can be easily verified that in the capacity of the function φ^0 satisfying the conditions (7.4) and (7.5), we can take the function

$$\varphi^0(x,t) = \begin{cases} [x(1-t)]^n, & (x,t)\in D_{T=1},\\ 0, & t\geq 1 \end{cases}$$

for a sufficiently large positive n.

Put $\varphi_T(x,t) := \varphi^0(\frac{x}{T}, \frac{t}{T})$, $T>0$. By virtue of (7.4), it can be easily seen that

$$\varphi_T \in C^2(\overline{D}_T),\ \ \varphi_T + T\,\frac{\partial\varphi_T}{\partial t} \leq 0,\ \ \varphi_T\big|_{D_T} > 0,\ \ \frac{\partial\varphi_T}{\partial x}\bigg|_{\widetilde{\gamma}_{2,T}} = 0,\ \ \varphi_T\big|_{\widetilde{\gamma}_{3,T}} = 0,\ \ \frac{\partial\varphi_T}{\partial t}\bigg|_{\widetilde{\gamma}_{3,T}} = 0. \tag{7.6}$$

Given f, we consider the function

$$\zeta(T) := \int\limits_{D_T} f\varphi_T\,dx\,dt,\ \ T>0. \tag{7.7}$$

The following theorem on the nonexistence of global solvability of the problem (1.1), (1.2) holds.

Theorem 7.1. *Let $g(x,t,s) = -|s|^\alpha s$, $s\in\mathbb{R}$, $\alpha > -1$, $f\in C(\overline{D}_\infty)$, and $f\geq 0$ in the domain D_∞. Then if*

$$\liminf_{T\to+\infty} \zeta(T) > 0, \tag{7.8}$$

there exists the positive number $T^* := T^*(f)$ such that for $T > T^*$ the problem (1.1), (1.2) *fails to have a strong generalized solution u of the class C^1 in the domain D_T.*

Proof. Suppose that in the conditions of this theorem there exists a strong generalized solution u of the problem (1.1), (1.2) of the class C^1 in the domain D_T. Then by Lemma 7.1, there is the equality (7.1) in which, due to (7.6), in the capacity of the function φ is taken the function $\varphi = \varphi_T$, i.e.,

$$\int\limits_{D_T} u\,\Box\varphi_T\,dx\,dt = \int\limits_{D_T} |u|^\alpha u u_t \varphi_T\,dx\,dt + \int\limits_{D_T} f\varphi_T\,dx\,dt. \tag{7.9}$$

Taking into account (1.2) and (7.6), we have

$$\begin{aligned}\int\limits_{D_T} |u|^\alpha u u_t\varphi_T\,dx\,dt &= \frac{1}{\alpha+2}\int\limits_{D_T} \varphi_T\,\frac{\partial}{\partial t}\,|u|^{\alpha+2}\,dx\,dt\\ &= -\frac{1}{\alpha+2}\int\limits_{D_T} |u|^{\alpha+2}\,\frac{\partial\varphi_T}{\partial t}\,dx\,dt \geq \frac{1}{(\alpha+2)T}\int\limits_{D_T} |u|^{\alpha+2}\varphi_T\,dx\,dt.\end{aligned}$$

Hence by (7.7), it follows from (7.9) that

$$\frac{1}{pT}\int\limits_{D_T} |u|^p\varphi_T\,dx\,dt \leq \int\limits_{D_T} u\,\Box\varphi_T\,dx\,dt - \zeta(T),\ \ p := \alpha+2 > 1. \tag{7.10}$$

If in the Young's inequality with parameter $\varepsilon > 0$

$$ab \leq \frac{\varepsilon}{p}\,a^p + \frac{1}{p'\varepsilon^{p'-1}}\,b^{p'};\ \ a,b\geq 0,\ \ \frac{1}{p}+\frac{1}{p'} = 1,\ \ p>1,$$

we take $a=|u|\varphi_T^{\frac{1}{p}}$, $b=\frac{|\Box\varphi_T|}{\varphi_T^{\frac{1}{p}}}$, $\varepsilon=\frac{1}{T}$, then in view of the fact that $\frac{p'}{p}=p'-1$, we obtain

$$|u\,\Box\,\varphi_T|=|u|\varphi_T^{\frac{1}{p}}\,\frac{|\Box\,\varphi_T|}{\varphi_T^{\frac{1}{p}}}\leq\frac{1}{pT}\,|u|^p\varphi_T+\frac{T^{p'-1}}{p'}\,\frac{|\Box\,\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,.$$

By virtue of (7.10) and the last inequality, we have

$$0\leq\frac{T^{p'-1}}{p'}\int\limits_{D_T}\frac{|\Box\,\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt-\zeta(T). \tag{7.11}$$

Since $\varphi_T(x,t):=\varphi^0(\frac{x}{T}\,,\frac{t}{T})$, in view of (7.4), (7.5), after the change of variables $x=Tx_1$, $t=Tt_1$, it can be easily verified that

$$\int\limits_{D_T}\frac{|\Box\,\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt=\frac{1}{T^{2(p'-1)}}\int\limits_{D_{T=1}}\frac{|\Box\,\varphi^0|^{p'}}{|\varphi^0|^{p'-1}}\,dx_1d\,t_1=\frac{\kappa_0}{T^{2(p'-1)}}\,.$$

Hence, bearing in mind (7.11), we obtain

$$0\leq\frac{\kappa_0}{p'T^{p'-1}}-\zeta(T). \tag{7.12}$$

Since $p'=\frac{p}{p-1}>1$, by virtue of (7.5), we have

$$\lim_{T\to+\infty}\frac{\kappa_0}{p'T^{p'-1}}=0.$$

Therefore, owing to (7.8), there exists the positive number $T^*:=T^*(f)$ such that for $T>T^*$, the right-hand side of the inequality (7.12) is negative, whereas the left-hand side equals zero. The obtained contradiction shows that if u is a strong generalized solution of the problem (1.1), (1.2) of the class C^1 in the domain D_T, then necessarily $T\leq T^*$, which proves Theorem 7.1. □

Remark 7.1. It is easy to check that if $f\in C(\overline{D}_\infty)$, $f\geq 0$, and $f(x,t)\geq ct^{-m}$ for $t\geq 1$, where $c=const>0$, $0\leq m=const\leq 2$, then the condition (7.8) is fulfilled and hence for $g=-|s|^\alpha s$, $s\in\mathbb{R}$, $\alpha>-1$ the problem (1.1), (1.2) for sufficiently large T fails to have a strong generalized solution u of the class C^1 in the domain D_T.

Indeed, introducing in (7.7)the transformation of independent variables x and t by formula $x=Tx_1$, $t=Tt_1$, after simple transformations we will have

$$\begin{aligned}\zeta(T)&=T^2\int\limits_{D_{T=1}}f(Tx_1,Tt_1)\varphi^0(x_1,t_1)\,dx_1\,dt_1\\&\geq cT^{2-m}\int\limits_{D_{T=1}\cap\{t_1\geq T^{-1}\}}t_1^{-m}\varphi^0(x_1,t_1)\,dx_1\,dt_1+T^2\int\limits_{D_{T=1}\cap\{t_1<T^{-1}\}}f(Tx_1,Tt_1)\varphi^0(x_1,t_1)\,dx_1\,dt_1\end{aligned}$$

in the assumption that $T>1$. Further, let $T_1>1$ be an arbitrary fixed number. Then from the last inequality, when $T\geq T_1>1$, for the function ζ we have

$$\zeta(T)\geq cT^{2-m}\int\limits_{D_{T=1}\cap\{t_1\geq T^{-1}\}}t_1^{-m}\varphi^0(x_1,t_1)\,dx_1\,dt_1\geq c\int\limits_{D_{T=1}\cap\{t_1\geq T_1^{-1}\}}t_1^{-m}\varphi^0(x_1,t_1)\,dx_1\,dt_1,$$

which immediately results in the validity of (7.8).

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CERTAIN PROPERTIES OF GENERALIZED ANALYTIC FUNCTIONS FROM SMIRNOV CLASS WITH A VARIABLE EXPONENT

Abstract. Let D be a simply connected domain bounded by a simple, closed, rectifiable curve Γ, $p = p(t)$ be the given on Γ positive measurable function, and $z = z(\zeta)$, $\zeta = re^{i\vartheta}$ be conformal mapping of the circle $U = \{\zeta : \ |\zeta| < 1\}$ onto the domain D.

The function $W(z)$, generalized-analytical in I. Vekua's sense, belongs to the Smirnov class $E^{p(t)}(A; B; D)$, if

(1) $W \in U^{s,2}(A; B; D)$;

(2) $\sup\limits_{0<r<1} \int\limits_0^{2\pi} |W(z(re^{i\vartheta}))|^{p(z(e^{i\vartheta}))} |z'(re^{i\vartheta})| \, d\vartheta < \infty$

(see [15]).

When $p(t)$ is Log–Hölder function continuous in Γ and $\min p(t) = \underline{p} > 1$, we considers the problems of representability of functions from $E^{p(t)}(A; B; D)$ by the generalized Cauchy integral, show the connection between the generalized Cauchy type integral and the generalized singular integral; of special interest is the question of extendability of functions from those classes, and the symmetry principle is proved.

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Key words and phrases. Generalized analytic functions, variable exponent, Smirnov classes of generalized analytic functions, generalized Cauchy and Cauchy type integrals.

რეზიუმე. ვთქვათ, D ცალადბმული არეა შემოსაზღვრული მარტივი, გაწრფევადი, შეკრული Γ წირით, $p = p(t)$ მასზე განსაზღვრული დადებითი ზომადი ფუნქციაა, ხოლო $z = z(\zeta)$, $\zeta = re^{i\vartheta}$ ერთეულოვანი $U = \{\zeta : \ |\zeta| < 1\}$ წრის D არეზე კონფორმულად ამსახველი ფუნქციაა.

D არეში ი. ვეკუას აზრით განზოგადებული ანალიზური $W(z)$ ფუნქცია ეკუთვნის სმირნოვის ცვლადმაჩვენებლიან $E^{p(t)}(A; B; D)$ კლასს, თუ:

(1) $W \in U^{s,2}(A; B; D)$;

(2) $\sup\limits_{0<r<1} \int\limits_0^{2\pi} |W(z(re^{i\vartheta}))|^{p(z(e^{i\vartheta}))} |z'(re^{i\vartheta})| \, d\vartheta < \infty$

(იხ. [15]).

ნაშრომში განიხილება $E^{p(t)}(A; B; D)$ კლასის ფუნქციათა კოშის ინტეგრალით წარმოდგენადობის საკითხი, კოშის ტიპის განზოგადებული ინტეგრალით წარმოდგენილი ფუნქციის სმირნოვის კლასისადმი მიკუთვნების საკითხი, შესწავლილია სმირნოვის კლასის ფუნქციათა გაგრძელებადობის საკითხი; დამტკიცებულია სიმეტრიის პრინციპი, როცა $p(t)$ Log–Hölder-ის აზრით უწყვეტი ფუნქციაა Γ-ზე და $\min p(t) = \underline{p} > 1$.

1. Introduction

The Hardy classes H^p of analytic in a unit circle U functions and their generalizations, i.e., Smirnov classes $E^p(D)$, $p > 0$, are the main objects of investigation of mathematical analysis (see [2, 3, 10, 16], etc.). They have a great number of applications in the boundary value problems of the theory of analytic functions.

Recently, the Lebesgue spaces with a variable exponent $L^{p(t)}$ and their applications attract attention of many mathematicians. This tendency has touched upon certain questions of the theory of analytic functions. The notions of Hardy and Smirnov classes (with a variable exponent) of analytic functions have been introduced in [5] and [6] and successfully applied to the boundary value problems; a part of those applications are reflected in [7].

For a constant p, the analogues of Smirnov classes for generalized analytic functions are presented in [4, 11–14] and some boundary value problems in these classes are studied therein.

The perspective to investigate the boundary value problems for generalized analytic functions more thoroughly made it necessary to introduce Smirnov classes with a variable exponent. But towards this end, one has, first of all, to know the properties of generalized Cauchy type integrals and generalized singular integrals with densities from the class $L^{p(t)}$. These questions have been studied in [9]. In particular, the validity of analogues of Sokhotski-Plemelj's formulas in the case of arbitrary, simple, rectifiable curves and summable densities has been proved, and the continuity in the space $L^{p(t)}(\Gamma)$ (with weight) of the operator $\widetilde{S}_\Gamma$ generated by a generalized singular integral when Γ is the Carleson curve has been proved, as well. All that made it possible to introduce the notion of Smirnov classes with a variable exponent for generalized analytic functions and to establish a series of their properties [15]; some of them we will frequently refer to in this work, are cited below, in Subsection 3.1. It should be noted here that in [15] the questions of extension and the symmetry principle for the introduced classes were left unconsidered; the case of unbounded domains was‘ considered superficially; the belonging of the generalized Cauchy type integrals with density from $L^{p(t)}$ to Smirnov classes was not considered in detail.

The present paper, being the continuation of our previous work [15], deals with the problems just mentioned and provides us with many new properties of the generalized Cauchy type integrals and Smirnov classes (with a variable exponent) of generalized analytic functions.

Relying mainly on the results obtained in [9, 15], we have succeeded in investigating the Riemann problem for generalized analytic functions from the introduced Smirnov classes with a variable exponent [8].

2. Preliminaries

2.1. **Generalized analytic functions in I. N. Vekua's sense.** Let D be a simply connected domain bounded by a simple, closed, rectifiable curve Γ and $A(z)$, $B(z)$ be the functions given on D. We extend them by zero on the set $E \setminus D$ when E is the complex plane, retaining the same notation for the obtained functions.

Let $s > 0$ and $L^s(D)$ be a set of functions f, summable on D, of degree s. If $D = E$, then we put $f_\nu(z) \equiv z^\nu f(\frac{1}{z})$, $\nu \in (-\infty, +\infty)$. The set of functions f for which

$$f \in L^s(U), \quad f_\nu(z) \in L^s(U), \ \ s \geq 1, \quad U = \{z : \ |z| < 1\},$$

we denote by $L^{s,\nu}(E)$.

A solution $W(z)$ of the equation

$$LW = \partial_{\overline{z}} W + A(z)W + B(z)\overline{W} = 0 \tag{2.1}$$

is said to be regular in the domain D, if every point $z_0 \in D$ possesses the neighborhood $D(z_0) \subset D$, where W has a generalized in Sobolev sense derivative $\partial_{\overline{z}} W \equiv \frac{1}{2}\left(\frac{\partial W}{\partial x} + i\,\frac{\partial W}{\partial y}\right)$.

If $A, B \in L^{s,2}(D)$, then we denote by $U^{s,2}(A; B; D)$ the set of all regular solutions of the equation (2.1). For $s > 2$, the equation (2.1) has regular solutions and each solution $W(z)$ is representable in the form

$$W(z) = \Phi_{W}(z) \exp \omega_{W}(z) \ \ (= \Phi \exp \omega), \tag{2.2}$$

where Φ_W is analytic in D function, and

$$\omega_W(z) = \frac{1}{2\pi i} \iint\limits_D \Big(A(\zeta) + B(\zeta) \, \frac{\overline{W(\zeta)}}{W(\zeta)} \Big) \, \frac{ds}{\zeta - z} \, .$$

The function ω_W belongs to the Hölder class $H_{\frac{s-2}{s}}(E)$ [17, pp. 156, 163]. The function $\Phi_W(z)$ is called a normal analytic divisor of the generalized analytic function $W(z)$ [17, p. 160].

2.2. **Principal kernels of the class $U^{s,2}(A;B;D)$.** Let

$$\phi_1(z) = \frac{1}{2(t-z)} \, , \quad \phi_2(z) = \frac{1}{2i(t-z)} \, ,$$

where t is a fixed point of the plane E. Then there exist the functions $X_j(z)$, $j = 1,2$ (solutions of the equation (2.1)), such that:

(1) $X_{j,0}(z) = \frac{X_j(z)}{\phi_j(z)} \in H_{\frac{s-2}{s}}(E)$;

(2) the functions $X_{j,0}(z)$ are continuous in D and continuously extendable on E;

(3) $X_{j,0}(z) \neq 0$;

(4) $X_{j,0}(t) = 1$.

The functions

$$\Omega_1(z,t) = X_1(z,t) + iX_2(z,t), \quad \Omega_2(z,t) = X_1(z,t) - iX_2(z,t)$$

are called principal normalized kernels of the class $U^{s,2}(A;B;D)$, $s > 2$ [17, p. 193]. There exist bounded functions $m_1(z,t)$, $m_2(z,t)$ such that

$$\Omega_1(z,t) = \frac{1}{t-z} + \frac{m_1(z,t)}{|t-z|^{\alpha}} \, , \quad \Omega_2(z,t) = \frac{m_2(z,t)}{|t-z|^{\alpha}} \, , \quad \alpha = \frac{2}{s} \tag{2.3}$$

(see [17, p. 179]).

2.3. **The generalized Cauchy type integral and generalized singular integral.** Let

$$\Gamma = \big\{ t \in E : \ t = t(\sigma), \ 0 \le \sigma \le \ell \big\},$$

where σ is the arc coordinate of the point t.

If Ω_1, Ω_2 are the principal normalized kernels of the class $U^{s,2}(A;B;D)$ and $f \in L(\Gamma)$, then the function

$$W(z) = (\widetilde{K}_\Gamma f)(z) = \frac{1}{2\pi i} \int\limits_\Gamma \Omega_1(z,\tau) f(\tau) \, d\tau - \Omega_2(z,\tau) \overline{f}(\tau) \, d\overline{\tau}$$

is a regular solution of the equation (2.1) of the class $U^{s,2}(A;B;D)$ [17, pp. 156, 168].

The function $(\widetilde{K}_\Gamma f)(z)$ is called the generalized Cauchy type integral. The corresponding singular integral is defined by the equality

$$(\widetilde{S}_\Gamma f)(t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int\limits_{\Gamma - \Gamma_\varepsilon(t)} \Omega_1(\tau,z) f(\tau) \, d\tau - \Omega_2(z,\tau) \overline{f}(\tau) \, d\overline{\tau},$$

where $\Gamma_\varepsilon(t)$ is a small in length arc lying on Γ with the ends $t(\sigma-\varepsilon)$ and $t(\sigma+\varepsilon)$.

Under different assumptions for Γ and f, the integrals $(\widetilde{K}_\Gamma f)(z)$ and $(\widetilde{S}_\Gamma f)(t)$ and their interconnections have been studied in [11–14] (for details see [9]). In particular, analogues of Sokhotski–Plemelj's formulas have been obtained. Here we cite the most general results stated in [9].

If Γ is a simple rectifiable curve and $f \in L(\Gamma)$, then the generalized Cauchy type integral $(\widetilde{K}_\Gamma f)(z)$ for almost all $t \in \Gamma$ has angular boundary values $(\widetilde{K}_\Gamma f)^+(t)$ and $(\widetilde{K}_\Gamma f)^-(t)$, and the equalities

$$(\widetilde{K}_\Gamma f)^{\pm}(t) = \pm \frac{1}{2} \, f(t) + \frac{1}{2} \, (\widetilde{S}_\Gamma f)(t) \tag{2.4}$$

are valid.

2.4. **The space $L^{p(t)}(\Gamma)$.** Let $p = p(t)$ be a measurable positive function on Γ. Assume

$$\|f\|_{p(t)} = \inf\Big\{\lambda > 0 : \int\limits_0^\ell \Big|\frac{f(t(\sigma))}{\lambda}\Big|^{p(t(\sigma))} d\sigma \le 1\Big\}$$

and

$$L^{p(t)}(\Gamma) = \big\{f : \|f\|_{p(t)} < \infty\big\}.$$

2.5. **The class of exponents $\mathcal{P}(\Gamma)$.** By $\mathcal{P}(\Gamma)$ we denote a union of those measurable on Γ positive functions $p(t)$ for which:

(1) there exists a constant $c(p)$ such that for any $t_1, t_2 \in \Gamma$ we have

$$|p(t_1) - p(t_2)| < c(p)\big|\ln|t_1 - t_2|\big|^{-1};$$

(2) $\underline{p} = \inf\limits_{t\in\Gamma} p(t) > 1$.

2.6. **On the continuity of the operator $\widetilde{S}_\Gamma$ in the space $L^{p(t)}(\Gamma)$.** Not touching upon the questions dealing with the investigation of that operator for constant p, we will cite here the most general result for the variable $p(t)$ [9].

If Γ is the Carleson curve (in the sequel, we will write $\Gamma \in \mathbb{R}$) and $p(t) \in \mathcal{P}(\Gamma)$, then the operator $\widetilde{S}_\Gamma : f(t) \to (\widetilde{S}_\Gamma f)(t)$ is continuous in $L^{p(t)}(\Gamma; \omega)$, where ω belongs to the definite class of weighted functions, inclusive all admissible power functions of the type

$$\omega = |t - a|^\alpha, \quad -\frac{1}{p(a)} < \alpha < \frac{1}{p'(a)} \quad a \in \Gamma, \quad p'(t) = \frac{p(t)}{p(t) - 1}\,.$$

3. The Variable Smirnov Classes of Generalized Analytic Functions

3.1. **The case of a bounded domain.** Let D be a finite domain bounded by a simple rectifiable curve Γ and μ be a measurable function different from zero almost everywhere on Γ.

We say that the generalized analytic function $W(z)$ belongs to the Smirnov class $E^{p(t)}(A; B; \mu; D)$ if:

(1) $W \in U^{s,2}(A; B; D)$, $s > 2$;

(2)

$$\sup_{0<r<1} \int\limits_0^{2\pi} \big|W(z(re^{i\vartheta}))\mu(z(re^{i\vartheta}))\big|^{p(z(e^{i\vartheta}))} |z'(re^{i\vartheta})|\, d\vartheta < \infty, \tag{3.1}$$

where $z = z(re^{i\vartheta})$ is conformal mapping of U onto D.

Assume $E^{p(t)}(A; B; D) = E^{p(t)}(A; B; 1; D)$.

This class of functions has been considered in [15]. Here we present the results from [15] which we will need in the sequel.

Statement 3.1. *The function $W \in U^{s,2}(A; B; D)$, $s > 2$, belongs to $E^{p(t)}(A; B; D)$ if and only if its normal analytic divisor Φ_W (see Subsection 2.1) belongs to $E^{p(t)}(D)$, i.e.,*

$$\sup_{0<r<1} \int\limits_0^{2\pi} \big|\Phi_W(z(re^{i\vartheta}))\big|^{p(z(e^{i\vartheta}))} |z'(re^{i\vartheta})|\, d\vartheta < \infty. \tag{3.2}$$

Statement 3.2. *The function $W(z) \in E^{p(t)}(A; B; D)$, $\underline{p} > 0$, has angular boundary values $W^+(t)$ for almost all $t \in \Gamma$ and, moreover, $W^+(t) \in L^{p(t)}(\Gamma)$. If $p \in \mathcal{P}(\Gamma)$, then*

$$(\widetilde{K}_\Gamma W^+)(z) = \begin{cases} W(z), & z \in D, \\ 0, & z \in E \setminus D. \end{cases} \tag{3.3}$$

Remark 3.1. It follows from Statement 3.1 that if $W \in E^{p(t)}(A; B; D)$, $\underline{p} > 0$, and $W^+(t) = 0$, $t \in \mathcal{E}$, $\mathcal{E} \subset \Gamma$, $\operatorname{mes} \mathcal{E} > 0$, then $W(z) \equiv 0$, $z \in D$.

Statement 3.3. *If $W \in U^{s,2}(A;B;D)$, $s>2$, and it belongs to $E^1(\widetilde{A};\widetilde{B};D)$, where*

$$\widetilde{A}(z)=\begin{cases} A(z), & z\in D,\\ 0, & z\in E\setminus D,\end{cases} \qquad \widetilde{B}(z)=\begin{cases} B(z), & z\in D,\\ 0, & z\in E\setminus D,\end{cases}$$

then it is representable by the formula

$$W(z)=\frac{1}{2\pi i}\int\limits_{\Gamma}\Omega_1(z,t)W^+(t)\,dt-\Omega_2(z,t)\overline{W^+}(t)\,d\overline{t},$$

when $\Omega_k(z,t)$, $k=1,2$, are the principal normalized kernels of the class $U^{s,2}(\widetilde{A};\widetilde{B};E)$.

Statement 3.4. *If $A,B\in L^{s,2}(D)$, $\Gamma\in\mathbb{R}$, $p\in\mathcal{P}(\Gamma)$, $\overline{p}\,'=\sup\limits_{t\in\Gamma}p'(t)$, $\frac{s}{2}>\overline{p}\,'$, $f\in L^{p(t)}(\Gamma)$, then $\widetilde{K}_\Gamma f$ belongs to $E^{p(t)}(A;B;D)$.*

Corollary 3.1. *If $A,B\in L^\infty(D)$, $\Gamma\in\mathbb{R}$, $p\in\mathcal{P}(\Gamma)$, $f\in L^{p(t)}(\Gamma)$, then $(\widetilde{K}_\Gamma f)(z)\in E^{p(t)}(A;B;D)$.*

3.2. **The case of an unbounded domain.** We will consider only those unbounded domains D whose boundary is a simple, closed, rectifiable curve. For the sake of simplicity, we consider only conformal mappings $z=z(s)$ of the circle U onto the domain D (which we denote by D^-) for which $z(0)=\infty$ and assume that $W\in E^{p(t)}(A;B;D^-)$ if the conditions (3.1) are fulfilled.

From the definition it follows that if $W\in E^1(A;B;D)$, then $W(\infty)=0$. If $p\in\mathcal{P}(\Gamma)$, then this is likewise valid when $W\in E^{p(t)}(A;B;D^-)$ (since $E^{p(t)}(A;B;D^-)\in E^1(A;B;D^-)$).

Theorem 3.1. *If D^- is an outer domain bounded by a simple, closed, rectifiable curve Γ, and $W\in E^1(A;B;D^-)$, then*

$$W(z)=(\widetilde{K}_\Gamma W^-)(z),\ \ z\in D^-, \tag{3.4}$$

where Γ denotes the curve oriented so that moving around it leaves D^- on the left.

Proof. Denote by Γ_ρ the image of the circumference $\{\zeta:\ |\zeta|=\rho<1\}$ under the conformal mapping of the circle U onto the domain D^-. Further, let Γ_R be the circumference $\{z:\ |z|=R>1\}$. Then for ρ, close to unity, and for sufficiently large R, the curve Γ_ρ lies inside of the circle $\{z:\ |z|<R\}$. The function $W(z)$ defined in a doubly-connected domain $\mathcal{E}$ with the boundary $\Gamma_\rho\cup\Gamma_R$ is representable by the Cauchy integral [17, p. 186], that is,

$$W(z)=(\widetilde{K}_{\Gamma_\rho}W)(z)+(\widetilde{K}_{\Gamma_R}W)(z). \tag{3.5}$$

We have

$$W(z(\rho e^{i\vartheta}))=\Phi_{W}(z(\rho e^{i\vartheta}))\exp\omega_{W}(z(\rho e^{i\vartheta})).$$

Assume

$$\varphi_\rho(\vartheta)=\Phi_{W}(z(\rho e^{i\vartheta}))z'(\rho e^{i\vartheta})i\rho e^{i\vartheta},$$

then

$$W(z(\rho e^{i\vartheta}))z'(\rho e^{i\vartheta})i\rho e^{i\vartheta}=\varphi_\rho(\vartheta)\exp\omega_{W}(z(\rho e^{i\vartheta})).$$

Therefore

$$\begin{aligned}(\widetilde{K}_{\Gamma_\rho}W)(z(re^{i\beta}))=\frac{1}{2\pi i}\int\limits_0^{2\pi}&\Omega_1\big(z(re^{i\beta}),z(\rho e^{i\vartheta})\big)\varphi_\rho(\vartheta)\exp\big(\omega_{W}(z(\rho e^{i\vartheta}))\big)\,d\vartheta\\ &-\Omega_2\big(z(re^{i\beta}),z(\rho e^{i\vartheta})\big)\overline{\varphi}_\rho(\vartheta)\exp\big(\omega_{W}(z(\rho e^{i\vartheta}))\big)\,d\vartheta.\end{aligned} \tag{3.6}$$

Since $W(z)\in E^1(A;B;D^-)$, Φ_{W} belongs to the class $E^1(D^-)$ (see Statement 3.1). Consequently, the sequence $\{\varphi_\rho(\vartheta)\}$ for $\vartheta\to1$ converges in the space $L([0,2\pi])$ to the function $\varphi_1(\vartheta)$ [16, p. 89].

Since $\exp(\omega_{W}(z(\zeta)))$ is continuous in $\overline{U}$, from the above-said it follows that the sequence $\{W(z(\rho e^{i\vartheta}))\}$ for $\rho\to1$ converges in $L([0,2\pi])$ to $W(z(e^{i\vartheta}))$.

Let $\rho_0\in(0,1)$ and $\varepsilon>0$ be a small number such that $\rho_0(1+\varepsilon)=\rho_1<1$. We take the point $z(re^{i\beta})$, $r\in(0,\rho_0)$. If $\rho\in(\rho_1,1)$, then

$$\big|z(re^{i\beta})-z(\rho e^{i\vartheta})\big|\geq\operatorname{dist}\big(z(re^{i\beta}),\Gamma_{\rho_0}\cup\Gamma_{\rho_1}\big)=m_0>0.$$

By the equality (2.3), there exists a number c such that

$$|\Omega_1(z,t)| < \frac{c}{|z-t|} = \frac{c}{|z(re^{i\beta}) - z(\rho e^{i\vartheta})|} \le \frac{c}{m_0}\,.$$

Owing to this fact, if we put

$$g_\rho(\vartheta) = \Omega_1\big(z(re^{i\beta}), z(\rho e^{i\vartheta})\big)\varphi_\rho(\vartheta),$$

then

$$|g_\rho(\vartheta)| < \frac{c}{m_0}\,|\varphi_\rho(\vartheta)|.$$

From the convergence of $\{\varphi_\rho\}$ to φ_1 in $L([0,2\pi])$ it follows that for any set $\mathcal{E} \subset [0,2\pi]$ the sequence $\{\varphi_\rho\}$ converges to φ_1 in $L(\mathcal{E})$ (see, e.g., [17]). According to the Hahn–Banach theorem [1, p. 255], we can conclude that the family $\{\varphi_\rho\}$ has absolutely continuous integrals of the same degree. Moreover, as $\rho \to 1$, the sequence $|g_\rho(\vartheta)|$ converges almost everywhere to $g_1(\vartheta)$.

Now, owing to the Vitali theorem [1, p. 255], we can conclude that in (3.6) the limiting passage under the integral sign is admissible and hence

$$\lim_{\rho\to 1}(\widetilde{K}_{\Gamma_\rho}W)(z(re^{i\beta})) = \frac{1}{2\pi i}\int\limits_0^{2\pi} \Omega_1\big(z(re^{i\beta}), z(e^{i\vartheta})\big)\Phi_W(z(re^{i\vartheta}))ie^{i\vartheta}\exp\omega_W(z(e^{i\vartheta}))\,d\vartheta$$
$$- \Omega_2\big(z(re^{i\beta}), z(e^{i\vartheta})\big)\overline{\Phi_W(z(re^{i\vartheta}))z'(e^{i\vartheta})ie^{i\vartheta}\exp\omega_W(z(e^{i\vartheta}))}\,d\vartheta = (\widetilde{K}_\Gamma W)(z(re^{i\beta})). \quad (3.7)$$

Let us prove that

$$\lim_{R\to\infty}(\widetilde{K}_{\Gamma_R}W)(z) = 0.$$

Let $|z| = R$ and $t \in \Gamma_R$. Then $|t| = R$ and it can be easily verified that $|\Omega_j(z,t)| < \frac{M}{R-|z|}$. Therefore

$$|(\widetilde{K}_{\Gamma_R}W)| < 2M\int\limits_0^{2\pi} \frac{|W(Re^{i\vartheta})|}{(R-|z|)^\alpha}\,d\vartheta, \quad \alpha = \frac{2}{s}\,.$$

Since $\lim\limits_{R\to\infty}|W(Re^{i\vartheta})| = 0$ for large R, we have $|W(Re^{i\vartheta})| \le M_0$ and hence

$$|(\widetilde{K}_{\Gamma_R}W)| \le \frac{2\pi M M_0}{(R-|z|)^\alpha} \longrightarrow 0.$$

This, together with (3.5) and (3.7), results in the equality (3.4). □

Remark 3.2. If orientation on Γ is chosen such that when moving around in this direction the domain D^+ leaves to the left, then the formula (3.4) takes the form

$$W(z) = -(\widetilde{K}_\Gamma W^-)(z), \quad z \in D^-.$$

3.3. **On the belonging of the function** $(\widetilde{K}_\Gamma f)(z))$ **to Smirnov class.** First, let us prove an analogue of Statement 3.4 for an unbounded domain. Towards this end, we will need the following

Lemma 3.1. *Let*

(1) Γ *be a simple, closed, rectifiable curve bounding the finite* D^+ *and the infinite* D^- *domains;*

(2) $p \in \mathcal{P}(\Gamma)$*;*

(3) $\zeta = \zeta(z)$ *be conformal mapping of* U^+ *onto* D^-*;*

(4) $\omega(\zeta) = \frac{k}{\zeta - a}$, $a \in D^+$, $\zeta \in D^-$, *and* k *be the constant such that* $k \le [\operatorname{dist}(a,\Gamma)]^2 = d^2$, *hence* $\widetilde{\Gamma} = \partial\widetilde{D}$, $\omega : D^- \to \widetilde{D}$, *where* $\widetilde{D}$ *is the bounded domain;*

(5) *the function* $\tau = \frac{k}{t-a}$ *map* Γ *onto* $\widetilde{\Gamma}$.

Assume $\widetilde{p}(\tau) = p(\frac{k}{\tau} + a)$. *Then*

$$\widetilde{p}(\tau) \in \mathcal{P}(\widetilde{\Gamma}). \quad (3.8)$$

Proof. Let $|\tau_1-\tau_2|<\frac{1}{2}$. We have

$$\big|\widetilde{p}(\tau_1)-\widetilde{p}(\tau_2)\big|=\Big|p\Big(\frac{k}{\tau_1}+a\Big)-p\Big(\frac{k}{\tau_2}+a\Big)\Big|\leq\frac{c(p)}{|\ln\frac{k|\tau_2-\tau_1|}{|\tau_1\tau_2|}|}\,. \tag{3.9}$$

Since $|\tau_1|\geq d$, $|\tau_2|\geq d$, owing to the condition (4), we obtain $\frac{k}{|\tau_1\tau_2|}\leq\frac{k}{d^2}\leq 1$. Therefore $\frac{k|\tau_1-\tau_2|}{|\tau_1\tau_2|}\leq|\tau_1-\tau_2|<\frac{1}{2}$, which implies that

$$\Big|\ln\frac{k|\tau_1-\tau_2|}{|\tau_1\tau_2|}\Big|>\big|\ln|\tau_1-\tau_2|\big|,$$

and from (3.9) we can conclude that $|\widetilde{p}(\tau_1)-\widetilde{p}(\tau_2)|<\frac{c(p)}{|\ln|\tau_1-\tau_2||}$. Moreover, it is obvious that $\min\widetilde{p}(\tau)=\min\limits_{t\in\gamma}p(t)=\underline{p}>1$. Thus the inclusion (3.8) is proved. □

Theorem 3.2. *Let Γ be the simple, closed, rectifiable curve bounding the domain D^-, and let the conditions*

$$A(z),B(z)\in L^\infty(D^-),\ \ \Gamma\in\mathbb{R},\ \ f\in L^{p(t)}(\Gamma),\ \ p\in\mathcal{P}(\Gamma), \tag{3.10}$$

be fulfilled. Then the function

$$W(z)=(\widetilde{K}_\Gamma f)(z),\ \ z\in D^-,$$

belongs to the class $E^{p(t)}(A;B;D^-)$.

Proof. We choose a point a from D^+ and assume $\zeta=\frac{k}{z-a}$, where k is chosen as in Lemma 3.1. Then $z=a+\frac{k}{\zeta}$ and

$$W\Big(\frac{k}{\zeta}+a\Big)=(\widetilde{K}_{\widetilde{\Gamma}}f)\Big(\frac{k}{\zeta}+a\Big). \tag{3.11}$$

We replace the integral variable in the right-hand side of (3.11) by the equality $t=\frac{k}{\tau}+a$. As a result, we obtain

$$\widetilde{W}(\zeta)=\frac{1}{2\pi i}\int\limits_{\Gamma}\widetilde{\Omega}_1(\zeta,\tau)F(\tau)\,d\tau-\widetilde{\Omega}_2(\zeta,\tau)\overline{F}(\tau)\,d\overline{\tau}, \tag{3.12}$$

where

$$\widetilde{W}(\zeta)=W\Big(\frac{k}{\zeta}+a\Big),\quad \widetilde{\Omega}_j(\zeta,\tau)=\Omega_j\Big(\frac{k}{\zeta}+a,\frac{k}{\tau}+a\Big),\ \ j=1,2,\quad F(\tau)=-\frac{f(\frac{k}{\tau}+a)}{\tau^2}\,k. \tag{3.13}$$

Since $f\in L^{p(t)}(\Gamma)$, we have $F\in L^{\widetilde{p}(\tau)}(\widetilde{\gamma})$, $\widetilde{p}(\tau)=p(\frac{k}{\tau}+a)$. In our assumptions Lemma 3.1 is applicable by virtue of which we have $\widetilde{p}(\tau)\in\mathcal{P}(\widetilde{\Gamma})$.

It can be easily verified that $\widetilde{\Omega}_k(\zeta,\tau)$, $k=1,2$, are the kernels of the type of principal normal kernels. Therefore following the proof of Statement 3.4 (see Theorem 3 of [15]), we find that $\widetilde{W}(\zeta)\in E^{\widetilde{p}(\tau)}(\widetilde{A};\widetilde{B};\widetilde{D})$. It is not difficult to show that $W\in E^{p(t)}(A;B;D)$. □

From Statement 3.4 and Theorem 3.2 follows one statement on the generalized Cauchy type integral which we formulate in the form of the following

Lemma 3.2. *Let Γ be the simple, closed, rectifiable curve dividing the plane E into the domains D^+ and D^-; next, let*

$$A(z),B(z)\in L^\infty(E),\ \ \Gamma\in\mathbb{R},\ \ f\in L^{p(t)}(\Gamma),\ \ p\in\mathcal{P}(\Gamma). \tag{3.14}$$

Then the narrowings on D^+ and D^- of the function $W(z)=(\widetilde{K}_\Gamma f)(z)$ belong to the classes $E^{p(t)}(A;B;D^+)$ and $E^{p(t)}(A;B;D^-)$, respectively, vice versa, if $W_1(z)\in E^{p(t)}(A;B;D^+)$ and $W_2(z)\in E^{p(t)}(A;B;D^-)$, then the function

$$W(z)=\begin{cases}W_1(z), & z\in D^+,\\ W_2(z), & z\in D^-\end{cases}$$

is representable by the generalized Cauchy type integral with density from $L^{p(t)}(\Gamma)$.

Proof. First, we note that if $W \in E^{p(t)}(A;B;D^+)$, then according to Statement 3.2 we have

$$(\widetilde{K}_\Gamma W^+)(z) = \begin{cases} W(z), & z \in D^+, \\ 0, & z \in D^- \end{cases} \tag{3.15}$$

(see (3.3)).

Relying on Remark 3.2, it is not difficult to establish that if $W \in E^{p(t)}(A;B;D^-)$, then

$$(\widetilde{K}_\Gamma W^-)(z) = \begin{cases} 0, & z \in D^+, \\ -W(z), & z \in D^-. \end{cases} \tag{3.16}$$

Let now $W(z) = (\widetilde{K}_\Gamma f)(z)$; if we consider it in the domain D^+, then according to Statement 3.4 we find that $W \in E^{p(t)}(A;B;D^+)$, but if we consider W in the domain D^-, then it belongs to $W \in E^{p(t)}(A;B;D^-)$, by Theorem 3.2.

The formulas (2.4) result in $W^+ - W^- = f$, hence $W = \widetilde{K}_\Gamma(W^+ - W^-)$.

Since for W_1 and W_2 respectively the relations (3.15) and (3.16) are valid, we have

$$\big[\widetilde{K}_\Gamma(W_1^+ - W_2^-)\big](z) = \begin{cases} W_1(z), & z \in D^+, \\ -W_2(z), & z \in D^-. \end{cases} \tag{3.17}$$

Obviously, $[W_1^+(t) - W_2^-(t)] \in L^{p(\cdot)}(\Gamma)$, hence $W(z) \in \widetilde{K}^{p(\cdot)}(\Gamma)$. □

4. Certain Properties of Integrals $\widetilde{K}_\Gamma f$ and $\widetilde{S}_\Gamma f$

Theorem 4.1. *In order for the function $W(z) \in U^{s,2}(A;B;D)$, $s > 2$, the equality*

$$W(z) = (\widetilde{K}_\Gamma W^+)(z) \tag{4.1}$$

to take place, it is necessary and sufficient that for almost all $t \in \Gamma$ the equality

$$(\widetilde{S}_\Gamma W^+)(t) = W^+(t) \tag{4.2}$$

to hold.

Proof. The necessity. It follows from the representation (4.1) that $W^+ \in L(\Gamma)$. By the equalities (2.4) we have

$$W^+(t) = \frac{1}{2}\, W^+(t) + \frac{1}{2}\, (\widetilde{S}_\Gamma W^+)(t),$$

and hence the equality (4.2) is valid.

Sufficiency. Let the equality (4.2) hold. Let us show that the equality (4.1) is likewise valid.

Consider the function

$$M(z) = W(z) - (\widetilde{K}_\Gamma W^+)(z), \ \ z \in D.$$

We have

$$M^+ = W^+ - \frac{1}{2}\,(W^+ + \widetilde{S}_\Gamma W^+) = \frac{1}{2}\,(W^+ - \widetilde{S}_\Gamma W^+). \tag{4.3}$$

By virtue of (4.2), we can conclude that $M^+(t) = 0$.

Since $W \in U^{s,2}(A;B;D)$, $s > 2$, we have $\widetilde{K}_\Gamma W^+ \in U^{s,2}(A;B;D)$ (see Subsection 2.3); consequently, $M(z) \in U^{s,2}(A;B;D)$. Therefore we have the representation

$$M(z) = \Phi_M(z)\omega_M(z), \ \ z \in D,$$

(see Subsection 2.1, the equality (2.2)). Here $\omega_M(z) \neq 0$ everywhere on $E \setminus \Gamma$.

Consequently, $\omega_M^+ \neq 0$, and from the equality $M^+ = 0$ we conclude that $\Phi_M^+(t) = 0$ almost everywhere on Γ. From the theorem on the uniqueness of analytic functions we find that $\Phi_M(z) = 0$; hence $M(z) = 0$, and from (4.3) follows (4.1). □

Remark 4.1. If D is an unbounded domain, then for the equality $W(z) = -(\widetilde{K}_\Gamma W^-)(z)$ it is necessary and sufficient that the equality

$$(\widetilde{S}_\Gamma W^-)(t) = -W^-(t)$$

to be fulfilled.

Theorem 4.2. *Let*

$$A, B \in L^\infty(D), \quad \Gamma \in \mathbb{R}, \quad p \in \mathcal{P}(\Gamma). \tag{4.4}$$

For the generalized analytic function $W(z)$ to have the boundary function $W^+(z)$ of the class $L^{p(t)}(\Gamma)$ and the equality

$$W(z) = (\widetilde{K}_\Gamma W^+)(\tau) \tag{4.5}$$

to hold, it is necessary and sufficient that $W(z)$ belong to the class $E^{p(t)}(A; B; D)$.

Proof. *The necessity.* Let the conditions (4.4) be fulfilled and there exist $W^+(t)$ and $W^+ \in L^{p(t)}(\Gamma)$, then by Corollary 3.1 we conclude that $(\widetilde{K}_\Gamma W^+)(z) \in E^{p(t)}(A; B; D)$.

Sufficiency. Let $W \in E^{p(t)}(A; B; D)$ and $p \in \mathcal{P}(\Gamma)$, then $W \in E^1(A; B; D)$. According to Statement 3.3 and Theorem 3.1, the equality (4.5) holds. This allows us to conclude that $W^+ \in L^{p(t)}(\Gamma)$, by virtue of Statement 3.2. □

Remark 4.2. Theorem 4.2 is a certain analogue of the Fichtenholz theorem [9, p. 97].

Theorem 4.3. *If the assumptions* (4.4) *holds and $f \in L^{p(t)}(\Gamma)$, then*

$$\widetilde{S}_\Gamma^2 f = f \tag{4.6}$$

holds.

Proof. By virtue of Corollary 3.1, the function $W(z) = (\widetilde{K}_\Gamma f)(z)$ belongs to $E^{p(t)}(A; B; D)$. Then by Statement 3.2 we have $(\widetilde{K}_\Gamma W^+)(z) = W(z)$. Now, by Theorem 4.1 we can conclude that $W^+(t) = (\widetilde{S}_\Gamma W^+(t)$. Using the first of the formulas (2.4), we write the last equality in the form

$$\frac{1}{2}\,(f + \widetilde{S}_\Gamma f) = \frac{1}{2}\,\widetilde{S}_\Gamma(f + \widetilde{S}_\Gamma f)$$

from which follows the equality (4.6). □

Tracing the proof of the theorem, we easily find that the following assertion is valid.

Lemma 4.1. *Let $W = \Phi_{_W} \exp \omega_{_W}$ be the function of the class $U^{s,2}(A; B; D)$, $s > 2$, and φ be analytic function in D, then*

$$\varphi W = \Phi_{_{\varphi W}} \exp \omega_{_{\varphi W}} \in U^{s,2}\Big(A; B\,\frac{\varphi}{\overline{\varphi}}\,; D\Big),$$

where

$$\Phi_{_{\varphi W}} = \varphi \Phi_{_W} \quad \textit{and} \quad \omega_{_{\varphi W}} = \omega_{_W}.$$

Proof. Since $\partial_{\overline{z}} \varphi = 0$, we have $\partial_{\overline{z}}(\varphi W) = \varphi \partial_{\overline{z}} W$. Moreover, $\partial_{\overline{z}} W + AW + B\overline{W} = 0$, hence

$$\partial_{\overline{z}} \varphi W + A\varphi W + B\,\frac{\varphi}{\overline{\varphi}}\,\overline{\varphi W} = 0.$$

This implies that $\varphi W = U^{s,2}(A; B\,\frac{\varphi}{\overline{\varphi}}\,; D)$.

Find the function $\omega_{_{\varphi W}}$. We have [17, p. 192]

$$\omega_{_{\varphi W}}(\zeta) = \frac{1}{\pi} \iint\limits_D \Big(A(t) + B(t)\,\frac{\varphi(t)}{\overline{\varphi}(t)}\,\frac{\overline{\varphi W}}{\varphi W}\Big)\,\frac{d\xi\, d\eta}{t - \zeta} = \frac{1}{\pi} \iint\limits_D \Big(A(t) + B(t)\,\frac{\overline{W}(t)}{W(t)}\Big)\,\frac{d\xi\, d\eta}{t - \zeta} = \varphi_{_W}(\zeta).$$

Next, taking into account the above equality, we obtain

$$\varphi W = \varphi \Phi_{_W} \exp \omega_{_W} = \{\varphi \Phi_{_W}\} \exp \omega_{_W},$$

from which we get both provable equalities. □

5. Extensions of Generalized Smirnov Class Analytic Functions

Theorem 5.1. *Let D_1 and D_2 be the domains lying outside of each other, bounded with simple rectifiable curves of the class $\mathbb{R}$, and:*

(1) *boundaries of the domains D_1 and D_2 have common arc Γ, so that $\partial D_1 = \Gamma_1 \cup \Gamma$, $\partial D_2 = \Gamma_2 \cup \Gamma$;*

(2) *$p_1(t) \in \mathcal{P}(\Gamma_1)$, $p_2(t) \in \mathcal{P}(\Gamma_2)$;*

(3) *$A_1, B_1 \in L^\infty(D_1)$, $A_2, B_2 \in L^\infty(D_2)$ and $W_1 \in E^{p_1(t)}(A_1; B_1; D_1)$, $W_2 \in E^{p_2(t)}(A_2; B_2; D_2)$;*

(4) *$p_1(a) = p_2(a)$, $p_1(b) = p_2(b)$, where a and b are the ends of the arc Γ;*

(5) *$W_1(t) = W_2(t)$, $t \in \Gamma$.*

Then the function

$$W(z) = \begin{cases} W_1(z), & z \in D_1, \\ W_2(z), & z \in D_2, \\ W_1(t) = W_2(t), & t \in \Gamma, \end{cases} \tag{5.1}$$

belongs to the Smirnov class $E^{p(t)}(A; B; D)$, where $D = D_1 \cup D_2 \cup \Gamma$,

$$p(t) = \begin{cases} p_1(t), & t \in \Gamma_1, \\ p_2(t), & t \in \Gamma_2, \end{cases},$$

and

$$A(z) = \begin{cases} A_1(z), & z \in D_1, \\ A_2(z), & z \in D_2, \end{cases} \qquad B(z) = \begin{cases} B_1(z), & z \in D_1, \\ B_2(z), & z \in D_2. \end{cases}$$

Proof. Assume

$$\widetilde{A}_k(z) = \begin{cases} A_k(z), & z \in D_k, \\ 0, & z \in E \setminus D_k, \end{cases} \qquad \widetilde{B}_k(z) = \begin{cases} B_k(z), & z \in D_k, \\ 0, & z \in E \setminus D_k, \end{cases} \qquad k = 1, 2.$$

Then $A = \widetilde{A}_1 + \widetilde{A}_2$, $\widetilde{B} = \widetilde{B}_1 + \widetilde{B}_2$. By virtue of the assumption (3), we have $A, B \in L^\infty(D)$. Further, owing to (3.3) and assumption (3),

$$(\widetilde{K}_{\Gamma_1 \cup \Gamma} W_1)(z) = 0, \ \ z \in D_2, \quad (\widetilde{K}_{\Gamma_2 \cup \Gamma} W_2)(z) = 0, \ \ z \in D_1. \tag{5.2}$$

In these integrals, the integration sets are $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$. In addition, the curve $\Gamma_1 \cup \Gamma$ is oriented so that moving in this direction, the domain D_1 leaves to the left, analogously, $\Gamma_2 \cup \Gamma$ is oriented so that moving in this direction, the domain D_2 leaves to the left. These orientations on Γ generate on Γ opposite directions. Therefore, if we denote the oriented arc of Γ on the boundary ∂D_1 of the domain D_1 by Γ^+, then on ∂D_2 it will be Γ^-.

In the domain D, let us consider the function

$$\begin{aligned} F(z) &= (\widetilde{K}_{\Gamma_1 \cup \Gamma^+} W_1)(z) + (\widetilde{K}_{\Gamma_1 \cup \Gamma^-} W_2)(z) = F_1(z) + F_2(z) = (\widetilde{K}_{\Gamma_1} W_1)(z) + (\widetilde{K}_{\Gamma_2} W_2)(z) \\ &= \frac{1}{2\pi i} \int\limits_{\Gamma^+} \Omega_1(z,t) W_1^+(t)\, dt - \Omega_2(z,t) \overline{W}_1(t)\, d\bar{t} + \frac{1}{2\pi i} \int\limits_{\Gamma^-} \Omega_1(z,t) W_2(t)\, dt - \Omega_2(z,t) \overline{W}_2(t)\, d\bar{t}, \end{aligned}$$

where Ω_1, Ω_2 are the principal kernels of the class $U^\infty(A; B; E)$.

We write $F(z)$ in the form

$$\begin{aligned} F(z) &= (\widetilde{K}_{\Gamma_1} W_1)(z) + (\widetilde{K}_{\Gamma_2} W_2)(z) \\ &\quad + \frac{1}{2\pi i} \int\limits_{\Gamma^+} \Omega_1(z,t)(W_1(t) - W_2(t))\, dt - \frac{1}{2\pi i} \int\limits_{\Gamma^-} \Omega_2(z,t)(\overline{W}_1(t) - \overline{W}_2(t))\, d\bar{t} \\ &= (K_{\Gamma_1 \cup \Gamma_2} W)(z) \end{aligned} \tag{5.3}$$

(we have taken into account that $W_1(t) = W_2(t)$, $t \in \Gamma$).

In view of the equality (5.2) we have

$$(\widetilde{K}_{\Gamma_1\cup\Gamma_2}W)(z)=\begin{cases}W_1(z), & z\in D_1,\\ W_2(z), & z\in D_2,\end{cases}$$

that is, $F(z)=W(z)$, $z\in D_1\cup D_2$. Moreover, for $t\in\Gamma$ we have

$$\lim_{z\widehat{\to}t,\, z\in D_k} F(z)=W_k(t),$$

that is, $F(t)=W_1(t)=W_2(t)$, $t\in\Gamma$. Consequently, almost everywhere on D, we get

$$F(z)=W(z). \tag{5.4}$$

The function $p(t)$ given on $\Gamma_1\cup\Gamma_2$ is, by assumption (4), of the class $\mathcal{P}(\Gamma_1\cup\Gamma_2)$. Therefore, it can be easily seen from (5.3) that $F(z)$ is the generalized Cauchy type integral with density from $L^{p(t)}(\Gamma_1\cup\Gamma_2)$. In view of Statement 3.4, we can conclude that $F(z)\in E^{p(t)}(A;B;D)$, and hence owing to (5.4), $W(z)\in E^{p(t)}(A;B;D)$, as well. □

6. The Symmetry Principle for Smirnov Class Functions

Before we proceed to formulating and proving the above-mentioned principle, we will prove below the following Lemmas 6.1 and 6.2. We denote $U^+=U$, $U^-=E\setminus\overline{U^+}$.

Lemma 6.1. *Let the domain D lie in U^+ and a part of its boundary lie on γ. Assume $D_*=\{\zeta:\zeta=\frac{1}{\overline{z}},\ z\in D\}$, and let $A(z),B(z)\in L^{s,2}(D)$, $s>2$. Then the functions*

$$A_0(\zeta)=\begin{cases}A(\zeta), & \zeta\in D,\\ -\dfrac{1}{\overline{\zeta}^{\,2}}\,\overline{A}\Big(\dfrac{1}{\overline{\zeta}}\Big), & \zeta\in D_*,\end{cases}\qquad B_0(\zeta)=\begin{cases}B(\zeta), & \zeta\in D,\\ -\dfrac{1}{\overline{\zeta}^{\,2}}\,\overline{B}\Big(\dfrac{1}{\overline{\zeta}}\Big), & \zeta\in D_*\end{cases} \tag{6.1}$$

belong to the class $L^{s,2}(D\cup D_*)$.

Proof. Show that $A_0\in L^{s,2}(D\cup D_*)$. Let $\zeta=x+iy$ and

$$J=\iint\limits_{D_*}|A_0(\zeta)|^s\,dx\,dy=\iint\limits_{D}\Big|-\frac{1}{\overline{\zeta}\,2}\,A\Big(\frac{1}{\overline{\zeta}}\Big)\Big|^s\,dx\,dy.$$

Assume $\tau=\alpha+i\beta$ and transform the variable ζ by the equality $\zeta=\frac{1}{\overline{\tau}}$, i.e., $x=\frac{\alpha}{\alpha^2+\beta^2}$, $y=\frac{\beta}{\alpha^2+\beta^2}$. Then

$$J=\iint\limits_{D}|\tau^2A(\tau)|^s|I|\,d\alpha\,d\beta,$$

where

$$I=\begin{vmatrix}x'_\alpha & x'_\beta\\ y'_\alpha & y'_\beta\end{vmatrix}=\begin{vmatrix}(\beta^2-\alpha^2)(\alpha^2+\beta^2)^{-2} & -2\alpha\beta(\alpha^2+\beta^2)^{-2}\\ -2\alpha\beta(\alpha^2+\beta^2)^{-2} & (\alpha^2-\beta^2)(\alpha^2+\beta^2)^{-2}\end{vmatrix}=-\frac{1}{(\alpha^2+\beta^2)^2}=-\frac{1}{|\tau|^4}\,.$$

Therefore

$$I=\iint\limits_{D}|\tau^2A(\tau)|^s\,\frac{d\alpha\,d\beta}{|\tau|^4}=\iint\limits_{D}|A(\tau)|^s|\tau|^{2(s-2)}\,d\alpha\,d\beta=\iint\limits_{D}|A(\tau)|^s\,d\alpha\,d\beta<\infty.$$

(We have taken into account that $s>2$, $|\tau|<1$ and $A,B\in L^\infty(D)$.) This implies that $A_0\in L^{s,2}(D\cup D_*)$.

In the same manner we can prove that $B_0\in L^{s,2}(D\cup D_*)$. □

Assume that the domain D is bounded by a simple, rectifiable, closed curve, $D\subset U^+$ and a part of the boundary D is the arc lying on γ.

Given $W(z)$ on D, we put

$$W_*(z)=\begin{cases}W(z), & z\in D,\\ -\overline{W}\Big(\dfrac{1}{\overline{z}}\Big), & z\in D_*.\end{cases}$$

Lemma 6.2. *Let $W(z) \in E^1(A;B;D)$ and either $z=0 \notin D$, or $z=0 \in D$, $W(0)=0$. Then $W_*(z) \in E^1(A_0;B_0;D_*)$, where A_0, B_0 are defined by the equality* (6.1).

Proof. According to the definition of the class $E^1(A_0,B_0,D_*)$, we have to establish that

$$W_*(z) \in U^{s,2}(A_0;B_0;D_*) \tag{6.2}$$

and

$$\sup_{0<r<1} \int\limits_0^{2\pi} \big|W_*(z(re^{i\vartheta}))\big|\,|z'(re^{i\vartheta}|^r\,d\vartheta < \infty, \tag{6.3}$$

where $z = z(re^{i\vartheta})$ is conformal mapping of U^+ onto D_*.

We start from the first one. By Lemma 6.1, $A_0, B_0 \in L^{s,2}(D \cup D_*)$. Therefore we have to prove that for $z \in D_*$ we have the equality

$$\partial_{\overline{z}} W_* + A_0(z)W_*(z) + B_0(z)\overline{W}_*(z) = 0. \tag{6.4}$$

Assuming $W(z) = u(z) + iv(z)$, we have

$$\partial_{\overline{z}} W_* = -\partial_{\overline{z}} \overline{W}\Big(\frac{1}{\overline{z}}\Big) = -\Big[\partial_{\overline{z}}\Big(u\Big(\frac{1}{\overline{z}}\Big) - iv\Big(\frac{1}{\overline{z}}\Big)\Big]\Big(-\frac{1}{\overline{z}^2}\Big). \tag{6.5}$$

But

$$u_{\overline{z}}\Big(\frac{1}{\overline{z}}\Big) + iv_{\overline{z}}\Big(\frac{1}{\overline{z}}\Big) = -\overline{A}\Big(\frac{1}{\overline{z}}\Big)\Big(u\Big(\frac{1}{\overline{z}}\Big) + iv\Big(\frac{1}{\overline{z}}\Big)\Big) - \overline{B}\Big(\frac{1}{\overline{z}}\Big)\Big(u\Big(\frac{1}{\overline{z}}\Big) + iv\Big(\frac{1}{\overline{z}}\Big)\Big),$$

and from (6.5), we get

$$\begin{aligned}
-\partial_{\overline{z}} W_*(z) &= \Big(-\frac{1}{\overline{z}^2}\Big)\Big[\overline{A}\Big(\frac{1}{\overline{z}}\Big)\overline{W}\Big(\frac{1}{\overline{z}}\Big) - \overline{B}\Big(\frac{1}{\overline{z}}\Big)W\Big(\frac{1}{\overline{z}}\Big)\Big] \\
&= -\frac{1}{\overline{z}^2}\,\overline{A}\Big(\frac{1}{\overline{z}}\Big)W_*(z) - \frac{1}{\overline{z}^2}\,\overline{B}\Big(\frac{1}{\overline{z}}\Big)\overline{W}_*(z) = A_0(z)W_*(z) + B_0(z)\overline{W}_*(z),
\end{aligned}$$

that is,

$$\partial_{\overline{z}} W_*(z) + A_0(z)W_*(z) + B_0(z)\overline{W}_*(z) = 0, \ \ z \in D_*.$$

Let now $W \in E^1(A;B;D)$. This implies that

$$\sup_{0<r<1} \int\limits_0^{2\pi} \big|W(\zeta(re^{i\vartheta}))\zeta'(re^{i\vartheta})\big|r\,d\vartheta = M < \infty, \tag{6.6}$$

where the function $\zeta = \zeta(re^{i\vartheta})$ is conformal mapping of U^+ onto D and if $0 \in D$, then $\zeta(0) = 0$.

The function $z = \frac{1}{\zeta(re^{i\vartheta})}$ is the conformal mapping of U^+ onto D_*.

We need to prove that

$$\sup_{0<r<1} \int\limits_0^{2\pi} \Big|W_*\Big(\frac{1}{\zeta(re^{i\vartheta})}\Big)\,\frac{\zeta'(re^{i\vartheta})}{\zeta^2(re^{i\vartheta})}\Big|r\,d\vartheta < \infty$$

We have

$$\begin{aligned}
J_r &= \int\limits_0^{2\pi} \Big|W_*\Big(\frac{1}{\zeta(re^{i\vartheta})}\Big)\,\frac{\zeta'(re^{i\vartheta})}{\zeta^2(re^{i\vartheta})}\Big|r\,d\vartheta \\
&= \int\limits_0^{2\pi} \Big|\overline{W}(\overline{\zeta})\,\frac{\zeta'}{\zeta^2}\Big|r\,d\vartheta = \int\limits_0^{2\pi} |W(\overline{\zeta})\,\frac{\overline{\zeta}\,'}{\overline{\zeta}^2}\Big|r\,d\vartheta \\
&= \int\limits_0^{2\pi} |W(\overline{\zeta})\,\frac{(\overline{\zeta})'}{\zeta^2}\,\frac{\overline{\zeta}\,'}{(\overline{\zeta})'}\Big|r\,d\vartheta = \int\limits_0^{2\pi} |W(\overline{\zeta})\,\frac{(\overline{\zeta})'}{\zeta^2}\Big|r\,d\vartheta.
\end{aligned} \tag{6.7}$$

If $0 \notin D$, then

$$J_r < \frac{M}{[\operatorname{dist}(0;D)]^2} = \frac{M}{m^2}\,. \tag{6.8}$$

If $0 \in D$, then $\zeta(0) = 0$, $W(0) = 0$, hence for small r (say, for $0 < r < r_0$) we have $|W(re^{-i\vartheta})| \sim r$, $|\zeta(re^{i\vartheta})| < cr$. Owing to that facts, there exists the constant c such that $|W(re^{i\vartheta})| < cr$, $|\zeta(r(e^{i\vartheta})| \sim cr$. Therefore, for small r we get

$$J_r < \int\limits_0^{2\pi} \frac{cr}{r^2}\,|\zeta'(re^{i\vartheta})|r\,d\vartheta = \frac{c}{c_1}\int\limits_0^{2\pi} |\zeta'(re^{i\vartheta})|\,d\vartheta = d < \infty. \tag{6.9}$$

Now, from (6.8), (6.9), when $r \in (0,1)$, we have $J_r < (\frac{M}{m^2} + d)$. This implies that the inequality (6.3) is valid, and since (6.2) is already proved, we have $W_* \in E^1(A_0; B_0; D_*)$. □

Corollary 6.1. *If $W \in E^{p(t)}(A; B; D)$, $p \in \mathcal{P}(\Gamma)$ and either $0 \notin D$, or $0 \in D$ and $W(0) = 0$, then $W \in E^{\ell(\tau)}(A; B; D_*)$, $\ell(\tau) = p(z(\frac{1}{\overline{\tau}}) \equiv p(z(\tau))$.*

Indeed, since $E^{p(t)}(A; B; D) \subset E^1(A; B; D)$, we have $W_* = \widetilde{K}_{\Gamma_*} W_*^+$, where Γ_* is the boundary of the domain D_*. In addition,

$$\int\limits_{\Gamma_*} |W_*(\zeta)|^{p(\zeta)} |z'(\zeta)|\,|d\zeta| = \int\limits_{\gamma} \Big|W\Big(\frac{1}{\overline{\tau}}\Big)\Big|^{p(z(\frac{1}{\overline{\tau}}))} \Big|\frac{1}{\tau^2}\Big|\,d\tau = \int\limits_{\gamma} |W(\tau)|^{p(z(\tau))}\,|d\tau| < \infty.$$

(We have taken into account that if $\tau \in \gamma$, then $\frac{1}{\overline{\tau}} = \tau$.)

Theorem 6.1 (The symmetry principle for the Smirnov class functions)**.** *Let:*

(1) *D be the simply connected domain bounded by a simple, closed, rectifiable curve $\gamma_2 \cup \gamma_1 \in \mathbb{R}$, lying inside of U^+, and the arc γ_1 lying on γ;*

(2) *$A, B \in L^\infty(D)$;*

(3) *D_* be a mirror image of D with respect to γ;*

(4) *$W \in E^{p(t)}(A; B; D)$;*

(5) *$W^+(t) + \overline{W^+}(t) = 0$, $t \in \gamma_1$;*

(6) *A_0 and B_0 are defined by the equalities* (6.1)*, $D_0 = D \cup D_* \cup \gamma_1$ and $p_0(t) = \begin{cases} p(t), & t \in \gamma_2, \\ p(\frac{1}{\overline{t}}), & t \in (\gamma_2)_*. \end{cases}$*

Then, if either $z = 0 \notin D$, or $0 \in D$ and $W(0) = 0$, then there exists a function $F \in E^{p_0(t)}(A_0; B_0; D_0)$ which for $z \in D$ coincides with $W(z)$, and for $z \in D_*$ with $W_*(z)$, but if $t \in \gamma_1$, then $F(t) = W^+(t) = -\overline{W^+}(t)$.

Proof. Assume $W_1(z) = W(z)$, $z \in D$, and $W_2(z) = W_*(z)$, $z \in D_*$. For the points t lying on γ_1, we have

$$W_1(t) = \lim_{z \widehat{\to} t,\, z \in D} W_1(z) = W(t), \quad W_2(t) = \lim_{z \widehat{\to} t,\, z \in D_*} \Big[-\overline{W}\Big(\frac{1}{\overline{z}}\Big)\Big] = -\overline{W}\Big(\frac{1}{\overline{t}}\Big) = -\overline{W}(t).$$

Due to the condition $W^+(t) + \overline{W^+}(t) = 0$, $t \in \gamma_1$, we have

$$W_1(t) = W_2(t), \ \ t \in \gamma_1.$$

We have the right to apply Theorem 5.1 due to which the function $F(z)$ given by the equality (5.3) coincides with the function W given by the equality (5.1). Thus the proof of theorem is complete. □

Corollary 6.2. *If $A(z), B(z) \in L^\infty(D)$, $W(z) \in E^{p(t)}(A; B; U^+)$, $W(0) = 0$, and $W^+(t) + \overline{W^+}(t) = 0$, $t \in \gamma$, then $W_*(z) \in E^{p_*(t)}(A_0, B_0; U^-)$, where $p_*(t) = p(\frac{1}{\overline{t}}) = p(t)$.*

Indeed, if we take $D = U^+$, $\gamma_1 = \gamma$, then $D_* = U^-$, and hence the validity of Corollary 6.2 follows from Theorem 6.1.

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THE DIRICHLET BOUNDARY VALUE PROBLEM OF THERMO-ELECTRO-MAGNENO ELASTICITY FOR HALF SPACE

Abstract. We prove the uniqueness theorem for the Dirichlet boundary value problem of statics of the thermo-electro-magneto-elasticity theory in the case of a half-space. The corresponding unique solution is represented explicitly by means of the inverse Fourier transform under some natural restrictions imposed on the boundary vector function.

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რეზიუმე. ნახევარსივრცის შემთვევაში დამტკიცებულია თერმო-ელექტრო-მაგნეტო დრეკადობის თეორიის დირიხლეს სასაზღვრო ამოცანისათვის ერთადერთობის თეორემა. გარკვეულ ბუნებრივ შეზღუდვებში, რომლებსაც ვადებთ სასაზღვრო ვექტორ-ფუნქციას, შესაბამისი დირიხლეს სასაზღვრო ამოცანის ერთადერთი ამონახსნი წარმოდგენილია ცხადი სახით შებრუნებული ფურიეს გარდაქმნის მეშვეობით.

1. Introduction

Although natural materials rarely show full coupling between elastic, electric, magnetic and thermal fields, some artificial materials do. In [14] it is reported that the fabrication of $BaTiO_3$-$CoFe_2O_4$ composite had the magnetoelectric effect not existing in either constituent. Other examples of similar complex coupling can be found in the references [1]–[6], [8]–[10], [13], [15].

The mathematical model of the thermo-electro-magneto-elasticity theory is described by the non-self-adjoint 6×6 system of second order partial differential equations with the appropriate boundary and initial conditions. The problem is to determine three components of the elastic displacement vector, the electric and magnetic scalar potential functions and the temperature distribution. Other field characteristics (e.g., mechanical stresses, electric and magnetic fields, electric displacement vector, magnetic induction vector, heat flux vector and entropy density) can be then determined by the gradient equations and the constitutive equations.

In the paper we prove the uniqueness theorem of solutions for Dirichlet boundary value problems of statics for half-space.

We show that under some natural restriction on the boundary vector functions the corresponding unique solution is represented by the inverse Fourier transform.

2. Basic Equations and Formulation of Boundary Value Problems

2.1. **Field equations.** Throughout the paper $u = (u_1, u_2, u_3)^\top$ denotes the displacement vector, σ_{ij} is the mechanical stress tensor, $\varepsilon_{kj} = 2^{-1}(\partial_k u_j + \partial_j u_k)$ is the strain tensor, $E = (E_1, E_2, E_3)^\top = -\operatorname{grad} \varphi$ and $H = (H_1, H_2, H_3) = -\operatorname{grad} \psi$ are electric and magnetic fields, respectively, $D = (D_1, D_2, D_3)^\top$ is the electric displacement vector and $B = (B_1, B_2, B_3)^\top$ is the magnetic induction vector, φ and ψ stand for the electric and magnetic potentials, ϑ is the temperature increment, $q = (q_1, q_2, q_3)^\top$ is the heat flux vector, and S is the entropy density. We employ the notation $\partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial_j$, $\partial_t = \partial/\partial_t$; the superscript $(\cdot)^\top$ denotes transposition operation; the summation over the repeated indices is meant from 1 to 3, unless stated otherwise.

In this subsection we collect the field equations of the linear theory of thermo-electro-magneto-elasticity for a general anisotropic case and introduce the corresponding matrix partial differential operators [11].

Constitutive relations:

$$
\begin{aligned}
\sigma_{rj} = \sigma_{jr} &= c_{rjkl}\varepsilon_{kl} - e_{lrj}E_l - q_{lrj}H_l - \lambda_{rj}\vartheta, \quad r, j = 1, 2, 3, \\
D_j &= e_{jkl}\varepsilon_{kl} + \varkappa_{jl}E_l + a_{jl}H_l + p_j\vartheta, \quad j = 1, 2, 3, \\
B_j &= q_{jkl}\varepsilon_{kl} + a_{jl}E_l + \mu_{jl}H_l + m_j\vartheta, \quad j = 1, 2, 3, \\
S &= \lambda_{kl}\varepsilon_{kl} + p_kE_k + m_kH_k + \gamma\vartheta.
\end{aligned}
$$

Fourier Law: $q_j = -\eta_{jl}\partial_l\vartheta, \;\; j = 1, 2, 3.$

Equations of motion: $\partial_j\sigma_{rj} + X_r = \varrho\partial_t^2 u_r, \;\; r = 1, 2, 3.$

Quasi-static equations for electro-magnetic fields where the rate of magnetic field is small (electric field is curl free) and there is no electric current (magnetic field is curl free): $\partial_j D_j = \varrho_e, \;\; \partial_j B_j = 0.$

Linearised equation of the entropy balance: $T_0\partial_t S - Q = -\partial_j q_j,$

Here ϱ is the mass density, ϱ_e is the electric density, c_{rjki} are the elastic constants, e_{jki} are the piezoelectric constants, q_{jki} are the piezomagnetic constants, $\varkappa_{jk}$ are the dielectric (permittivity) constants, μ_{jk} are the magnetic permeability constants, a_{jk} are the coupling coefficients connecting electric and magnetic fields, p_j and m_j are constants characterizing the relation between thermodynamic processes and electro-magnetic effects, λ_{jk} are the thermal strain constants, η_{jk} are the heat conductivity coefficients, $\gamma = \varrho c T_0^{-1}$ is the thermal constant, T_0 is the initial reference temperature, c is the specific heat per unit mass, $X = (X_1, X_2, X_3)^\top$ is a mass force density, Q is a heat source intensity. The constants involved in these equations satisfy the symmetry conditions

$$
\begin{gathered}
c_{rjkl} = c_{jrkl} = c_{klrj}, \quad e_{klj} = e_{kjl}, \quad q_{klj} = q_{kjl}, \quad \varkappa_{kj} = \varkappa_{jk}, \\
\lambda_{kj} = \lambda_{jk}, \quad \mu_{kj} = \mu_{jk}, \quad \eta_{kj} = \eta_{jk}, \quad a_{kj} = a_{jk}, \quad r, j, k, l = 1, 2, 3.
\end{gathered}
\tag{2.1}
$$

From physical considerations it follows (see, e.g., [7], [12])

$$c_{rjkl}\xi_{rj}\xi_{kl} \geq c_0\xi_{kl}\xi_{kl}, \quad \varkappa_{kj}\xi_k\xi_j \geq c_1|\xi|^2, \quad \mu_{kj}\xi_k\xi_j \geq c_2|\xi|^2, \quad \eta_{kj}\xi_k\xi_j \geq c_3|\xi|^2, \tag{2.2}$$

for all $\xi_{kj} = \xi_{jk} \in \mathbb{R}$ and for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, where c_0, c_1, c_2 and c_3 are positive constants. More careful analysis related to the positive definiteness of the potential energy and thermodynamical laws insure positive definiteness of the matrix

$$\Xi = \begin{bmatrix} [\varkappa_{kj}]_{3\times3} & [a_{kj}]_{3\times3} & [p_j]_{3\times1} \\ [a_{kj}]_{3\times3} & [\mu_{kj}]_{3\times3} & [m_j]_{3\times1} \\ [p_j]_{1\times3} & [m_j]_{1\times3} & \gamma \end{bmatrix}_{7\times7}. \tag{2.3}$$

Further we introduce the following generalised stress operator

$$\mathcal{T}(\partial, n) := \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times3} & [e_{lrj}n_j\partial_l]_{3\times3} & [q_{lrj}n_j\partial_l]_{3\times1} & [-\lambda_{rj}n_j]_{3\times1} \\ [-e_{jkl}n_j\partial_l]_{1\times3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & -p_jn_j \\ [-q_{jkl}n_j\partial_l]_{1\times3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & -m_jn_j \\ [0]_{1\times3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6\times6}.$$

Evidently, for a six vector $U := (u, \varphi, \psi, \vartheta)^\top$ we have

$$\mathcal{T}(\partial, n)U = (\sigma_{1j}n_j, \sigma_{2j}n_j, \sigma_{3j}n_j, -D_jn_j, -B_jn_j, -q_jn_j)^\top. \tag{2.4}$$

The components of the vector $\mathcal{T}U$ given by (2.4) have the physical sense: the first three components correspond to the mechanical stress vector in the theory of thermo-electro-magneto-elasticity, the forth, fifth and sixth ones are respectively the normal components of the electric displacement vector, magnetic induction vector and heat flux vector with opposite sign.

From the above equations we derive the following equations of statics

$$A(\partial)U(x) = \Phi(x),$$

where $U = (u_1, \dots, u_6)^\top := (u, \varphi, \psi, \vartheta)^\top$ is the sought for vector function and $\Phi = (\Phi_1, \dots, \Phi_6)^\top := (-X_1, -X_2, -X_3, -\varrho_e, 0, -Q)^\top$ is a given vector function; $A(\partial) = [A_{pq}(\partial)]_{6\times6}$ is the matrix differential operator

$$A(\partial) = \begin{bmatrix} [c_{rjkl}\partial_j\partial_l]_{3\times3} & [e_{lrj}\partial_j\partial_l]_{3\times3} & [q_{lrj}\partial_j\partial_l]_{3\times1} & [-\lambda_{rj}\partial_j]_{3\times1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -p_j\partial_j \\ [-q_{jkl}\partial_j\partial_l]_{1\times3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -m_j\partial_j \\ [0]_{1\times3} & 0 & 0 & \eta_{jl}\partial_j\partial_l \end{bmatrix}_{6\times6}.$$

From the symmetry conditions (2.1), inequalities (2.2) and positive definiteness of the matrix (2.3) it follows that $A(\partial)$ is a formally non-self adjoint strongly elliptic operator.

2.2. **Formulation of boundary value problems.** Let $\mathbb{R}^3$ be divided by some plane into two half-spaces. Without loss of generality we can assume that these half-spaces are

$$\mathbb{R}^3_1 := \{x \mid x = (x_1, x_2, x_3) \in \mathbb{R}^3 \text{ and } x_3 > 0\} \quad \text{and}$$
$$\mathbb{R}^3_2 := \{x \mid x = (x_1, x_2, x_3) \in \mathbb{R}^3 \text{ and } x_3 < 0\};$$

$n = (n_1, n_2, n_3) = (0, 0, -1)$ is the outward unit normal vector with respect to $\mathbb{R}^3_1$; $S := \partial\mathbb{R}^3_{1,2}$.

Now we formulate the basic boundary value problems of the thermo-electro-magneto-elasticity theory for a half-space.

Dirichlet problem $(D)^\pm$**.** Find a solution vector $U = (u, \varphi, \psi, \vartheta)^\top \in [C^1(\overline{\mathbb{R}^3_{1,2}})]^6 \cap [C^2(\mathbb{R}^3_{1,2})]^6$ to the system of equations

$$A(\partial)U = 0 \quad \text{in} \quad \mathbb{R}^3_{1,2} \tag{2.5}$$

satisfying the Dirichlet type boundary condition

$$\{U\}^\pm = f \quad \text{on} \quad S. \tag{2.6}$$

The symbols $\{\cdot\}^\pm$ denote the one-sided limits on S from $\mathbb{R}^3_1$ (sign "+") and $\mathbb{R}^3_2$ (sign "−").

We require that the boundary data involved in the above setting possess the following smoothness property: $f \in \overset{\circ}{C}{}^\infty(\mathbb{R}^2)$, where $\overset{\circ}{C}{}^\infty(\mathbb{R}^2)$ is the space of infinitely differentiable functions with compact support.

Let $\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}$ and $\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}$ denote the direct and inverse generalized Fourier transforms in the space of tempered distributions (the Schwartz space $\mathcal{S}'(\mathbb{R}^2)$) which for regular summable functions g and h read as follows

$$\begin{aligned}\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[g] &= \int\limits_{\mathbb{R}^2} g(\widetilde{x})e^{i\widetilde{x}\cdot\widetilde{\xi}}d\widetilde{x},\\ \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[h] &= \frac{1}{4\pi^2}\int\limits_{\mathbb{R}^2} h(\widetilde{\xi})e^{-i\widetilde{x}\cdot\widetilde{\xi}}d\widetilde{\xi},\end{aligned} \tag{2.7}$$

where $\widetilde{x}=(x_1,x_2)$, $\widetilde{\xi}=(\xi_1,\xi_2)$, $d\widetilde{x}=dx_1\,dx_2$, $\widetilde{x}\cdot\widetilde{\xi}=x_1\xi_1+x_2\xi_2$.

Note that if $g(x)=g(x_1,x_2,x_3)=g(\widetilde{x},x_3)$, then

$$\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[\partial_{x_j}g(x)] = -i\xi_j\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[g] = -i\xi_j\widehat{g}(\widetilde{\xi},x_3), \quad j=1,2,$$

and hence

$$\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[\nabla_x g(x)] = \begin{bmatrix}-i\xi_1\\ -i\xi_2\\ \partial_{x_3}\end{bmatrix}\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[g(x)] = P(-i\widetilde{\xi},\partial x_3)\widehat{g}(\widetilde{\xi},x_3), \tag{2.8}$$

here $\widehat{g}(\widetilde{\xi},x_3)=\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}[g]$ and

$$P=P(-i\widetilde{\xi},\partial_{x_3}) = (-i\xi_1,-i\xi_2,\partial_{x_3})^\top. \tag{2.9}$$

Applying Fourier transform (2.7) in (2.5)–(2.6) and taking into account (2.9) we arrive at the problem:

$$A(P)\widehat{U}(\widetilde{\xi},x_3)=0, \quad x_3\in(0;+\infty) \ \text{ or } \ x_3\in(-\infty;0), \tag{2.10}$$

$$\Big\{\widehat{U}(\widetilde{\xi},x_3)\Big\}^{\pm}_{(x_3\to 0\pm)} = \widehat{f}(\widetilde{\xi}). \tag{2.11}$$

We see that (2.10) is the system of ordinary differential equations of second order for each $\widetilde{\xi}\in\mathbb{R}^2$.

3. Uniqueness Theorems

We start with constructing a system of linear independent solutions to the system (2.10).

Let us denote by $k_j=k_j(\widetilde{\xi})$, $j=\overline{1,12}$, the roots of the equation

$$\det A(-i\xi)=0 \tag{3.1}$$

with respect to ξ_3, where $A(-i\xi)$ is the symbol matrix of the operator $A(\partial)$.

Note that $\det A(-i\xi)$ is a homogeneous polynomial of order 12 and the equation (3.1) has no real roots, $\operatorname{Im}k_j\neq 0$, $j=\overline{1,12}$. These roots are continuously dependent on the coefficients of (3.1) and the number of roots with positive and negative imaginary parts are equal. Denote by $k_1,k_2,\dots,k_6$ roots with positive imaginary parts and by $k_7,\dots,k_{12}$ with negative ones.

Let us construct the following matrices:

$$\Phi^{(+)}(\widetilde{\xi},x_3) = \int\limits_{\ell^+} A^{-1}(-i\xi)e^{-i\xi_3x_3}\,d\xi_3, \tag{3.2}$$

$$\Phi^{(-)}(\widetilde{\xi},x_3) = \int\limits_{\ell^-} A^{-1}(-i\xi)e^{-i\xi_3x_3}\,d\xi_3, \tag{3.3}$$

where ℓ^+ (respectively, ℓ^-) is a closed simple curve of positive counterclockwise orientation (respectively, negative clockwise orientation) in the upper (respectively, lower) complex half-plane $\operatorname{Re}\xi_3>0$ (respectively, $\operatorname{Re}\xi_3<0$) enclosing all the roots with respect to ξ_3 of the equation $\det A(-i\xi)=0$ with positive (respectively, negative) imaginary parts (see Fig. 1). Clearly, (3.2) and (3.3) do not depend on the shape of ℓ^+ (respectively, ℓ^-).

With the help of the Cauchy integral theorem for analytic functions, we conclude that the entries of the matrix $\Phi^{(+)}(\widetilde{\xi},x_3)=[\Phi^{(+)}_{kj}(\widetilde{\xi},x_3)]_{6\times 6}$ are increasing exponentially as $x_3\to+\infty$ and are decreasing exponentially as $x_3\to-\infty$ $(-i\xi_3x_3=-i(\xi_3'+i\xi_3'')x_3=-i\xi_3'x_3+\xi_3''x_3)$.

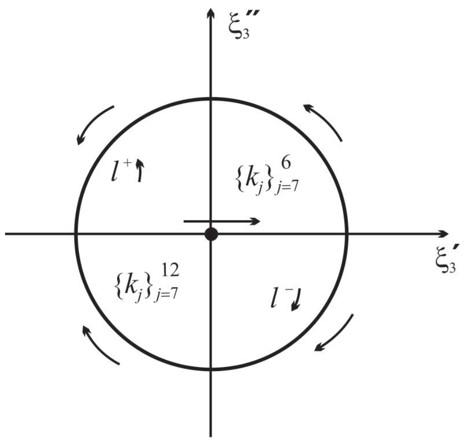

FIGURE 1

Analogously, the entries of the matrix $\Phi^{(-)}(\widetilde{\xi}, x_3) = [\Phi^{(-)}_{kj}(\widetilde{\xi}, x_3)]_{6\times 6}$ are increasing exponentially as $x_3 \to -\infty$ and vanish exponentially as $x_3 \to +\infty$.

Lemma 3.1. *The columns of* $\Phi^{(\pm)}(\widetilde{\xi}, x_3)$ *are linearly independent solutions to system* (2.10)*.*

Proof. Applying the Cauchy integral theorem we have

$$\begin{aligned} A(P)\Phi^{(\pm)}(\widetilde{\xi}, x_3) &= \int\limits_{\ell^\pm} A(-i\xi)A^{-1}(-i\xi)e^{-i\xi_3 x_3}d\xi_3 \\ &= \int\limits_{\ell^\pm} I_6 e^{-\xi_3 x_3} d\xi_3 = 0, \end{aligned}$$

where I_6 is the 6×6 unit matrix. Now we prove that the columns of the matric $\Phi^{(+)}(\widetilde{\xi}, x_3)$

$$\overset{(1)}{\Phi}{}^{(+)}, \overset{(2)}{\Phi}{}^{(+)}, \dots, \overset{(6)}{\Phi}{}^{(+)}$$

are linearly independent vector functions.

Assume that there exists a complex vector $(C_1, C_2, \dots, C_6) =: C \in \mathbb{C}^6$ $(C = C(\widetilde{\xi}))$ such that $\sum\limits_{j=1}^{6} C_j \overset{(j)}{\Phi}{}^{(+)}(\widetilde{\xi}, x_3) = 0$ or

$$\Phi^{(+)}(\widetilde{\xi}, x_3)C = 0. \tag{3.4}$$

If $x_3 = 0$, then from (3.2) and (3.4) we get

$$\Phi^{(+)}(\widetilde{\xi}, 0)C = \int\limits_{\ell^+} A^{-1}(-i\xi)\, d\xi_3\; C = 0. \tag{3.5}$$

Taking into account that (see (3.27) in [11])

$$\int\limits_{\ell^+} A^{-1}(-i\xi)\, d\xi_3 = \int\limits_{-\infty}^{\infty} A^{-1}(-i\xi)\, d\xi_3,$$

one can rewrite (3.5) as follows

$$\int\limits_{-\infty}^{\infty} A^{-1}(-i\xi)C\, d\xi_3 = \int\limits_{-\infty}^{\infty} A^{-1}_{kj}(-i\xi)C_j\, d\xi_3 = 0, \quad k = \overline{1,6},$$

or

$$\int\limits_{-\infty}^{+\infty}\begin{bmatrix}[c_{rjkl}\xi_j\xi_l]_{3\times3} & [e_{lrj}\xi_j\xi_l]_{3\times1} & [q_{lrj}\xi_j\xi_l]_{3\times1} & [-\lambda_{rj}\xi_j]_{3\times1}\\ [-e_{jkl}\xi_j\xi_l]_{1\times3} & \varkappa_{jl}\xi_j\xi_l & a_{jl}\xi_j\xi_l & -ip_j\xi_j\\ [-q_{jkl}\xi_j\xi_l]_{1\times3} & a_{jl}\xi_j\xi_l & \mu_{jl}\xi_j\xi_l & -im_j\xi_j\\ [0]_{1\times3} & 0 & 0 & \eta_{jl}\xi_j\xi_l\end{bmatrix}^{-1}_{6\times6} C\,d\xi_3=0. \tag{3.6}$$

The integrand in (3.6) is $\Psi := -A^{-1}(-i\xi)C$ and hence $C=-A(-i\xi)\Psi$. Using these notation we can write

$$\int\limits_{-\infty}^{+\infty}\Psi_k\,d\xi_3=0,\quad \int\limits_{-\infty}^{+\infty}\overline{\Psi}_k\,d\xi_3=0,\quad k=\overline{1,6},\quad\text{and}$$

$$\int\limits_{-\infty}^{+\infty}\sum_{r=1}^{3}C_r\overline{\Psi}_r\,d\xi_3=0,\quad \int\limits_{-\infty}^{+\infty}\overline{C}_4\Psi_4\,d\xi_3=0, \tag{3.7}$$

$$\int\limits_{-\infty}^{+\infty}\overline{C}_5\Psi_5\,d\xi_3=0,\quad \int\limits_{-\infty}^{+\infty}\overline{C}_6\Psi_6\,d\xi_3=0.$$

Using (2.2) and the last equality of (3.7) we conclude that $\Psi_6=0$.

Taking the sum of the first five equalities of (3.7) we obtain

$$\int\limits_{-\infty}^{+\infty}\Big\{\sum_{r=1}^{3}(-A(-i\xi)\Psi)_r\,\overline{\Psi}_r+\Big(-\overline{A(-i\xi)\Psi}\Big)_4\Psi_4+\Big(-\overline{A(-i\xi)\Psi}\Big)_5\Psi_5\Big\}d\xi_3$$

$$=\int\limits_{-\infty}^{+\infty}\big\{c_{rjkl}\xi_j\xi_l\Psi_k\overline{\Psi}_r+e_{lrj}\xi_j\xi_l\Psi_4\overline{\Psi}_r+q_{lrj}\xi_j\xi_l\Psi_5\overline{\Psi}_r-e_{jkl}\xi_j\xi_l\overline{\Psi}_r\Psi_4$$
$$+\varkappa_{jl}\xi_j\xi_l\overline{\Psi}_4\Psi_4+a_{jl}\xi_j\xi_l\overline{\Psi}_5\Psi_4-q_{jkl}\xi_j\xi_l\overline{\Psi}_r\Psi_5+a_{jl}\xi_j\xi_l\overline{\Psi}_4\Psi_5+\mu_{jl}\xi_j\xi_l\overline{\Psi}_5\Psi_5\big\}d\xi_3=0,$$

i.e.

$$\int\limits_{-\infty}^{+\infty}\big\{c_{rjkl}\xi_j\xi_l\Psi_k\overline{\Psi}_r+\varkappa_{jl}\xi_j\xi_l\overline{\Psi}_4\Psi_4+a_{jl}\xi_j\xi_l(\overline{\Psi}_5\Psi_4+\Psi_5\overline{\Psi}_4)+\mu_{jl}\xi_j\xi_l\overline{\Psi}_5\Psi_5\big\}d\xi_3=0.$$

De to (2.1), (2.2) and positive definiteness of the matrix (2.3) from the last equality we conclude that $\Psi_k=0$, $k=\overline{1,5}$, and therefore together with $\Psi_6=0$ we have $C_k=0$, $k=\overline{1,6}$.

Hence the columns of the matrix $\Phi^{(+)}(\widetilde{\xi},x_3)$ are linearly independent vector functions. Similarly, it can be proved that the columns of the matrix $\Phi^{(-)}(\widetilde{\xi},x_3)$ defined by (3.3) are linearly independent vector functions. □

Theorem 3.2. *The boundary value problems* (2.10)–(2.11) *have only one solution in the space of functions vanishing at infinity.*

Proof. If $x_3\in(0;+\infty)$, then we look for a solution of the Dirichlet problem in the following form

$$\widehat{U}(\widetilde{\xi},x_3)=\Phi^{(-)}(\widetilde{\xi},x_3)C,\quad x_3>0,$$

where $C=(C_1,\dots,C_6)$ is unknown vector depending only on $\widetilde{\xi}$.

From (2.11) we have

$$\Phi^{(-)}(\widetilde{\xi},0)C=\widehat{f}(\widetilde{\xi})$$

and since $\det\Phi^{(-)}(\widetilde{\xi},0)\neq0$, $|\widetilde{\xi}|\neq0$, due to Lemma 3.1 we obtain

$$C=[\Phi^{(-)}(\widetilde{\xi},0)]^{-1}\widehat{f}(\widetilde{\xi}).$$

Therefore the unique solution has the following form

$$\widehat{U}(\widetilde{\xi},x_3)=\Phi^{(-)}(\widetilde{\xi},x_3)[\Phi^{(-)}(\widetilde{\xi},0)]^{-1}\widehat{f}(\widetilde{\xi}),\quad x_3>0. \tag{3.8}$$

Similarly, if $x_3\in(-\infty;0)$, then the unique solution of the Dirichlet problem has the form

$$\widehat{U}(\widetilde{\xi},x_3)=\Phi^{(+)}(\widetilde{\xi},x_3)[\Phi^{(+)}(\widetilde{\xi},0)]^{-1}\widehat{f}(\widetilde{\xi}),\quad x_3<0. \tag{3.9}$$

The theorem is proved. $\square$

Lemma 3.3. *There hold the following relations*

$$[\Phi^{(-)}(\widetilde{\xi},0)]^{-1}=\begin{bmatrix}[O(|\widetilde{\xi}|)]_{5\times5} & [O(1)]_{5\times1}\\ [0]_{1\times5} & O(|\widetilde{\xi}|)\end{bmatrix}_{6\times6}. \tag{3.10}$$

Proof. It can be shown (see [11]) that the entries of the matrix $A^{-1}(-i\xi)$ are homogeneous functions in ξ and

$$A^{-1}(-i\xi)=\begin{bmatrix}[O(|\xi|^{-2})]_{5\times5} & [O(|\xi|^{-3})]_{5\times1}\\ [0]_{1\times5} & O(|\xi|^{-2})\end{bmatrix}_{6\times6}. \tag{3.11}$$

Assume that $\xi_1=t_1|\widetilde{\xi}|$, $\xi_2=t_2|\widetilde{\xi}|$, $\xi_3=t_3|\widetilde{\xi}|$, where $\xi=(\xi_1,\xi_2,\xi_3)=(\widetilde{\xi},\xi_3)$, $t_1^2+t_2^2=1$. If $|\widetilde{\xi}|\neq0$, from (3.11) we obtain

$$\Phi^{(-)}_{kj}(\widetilde{\xi},0)=\int\limits_{-\infty}^{+\infty}A^{-1}_{kj}(-i\xi)\,d\xi_3=\int\limits_{-\infty}^{+\infty}O(|\xi|^{-m})\,d\xi_3,\quad m=2\ \text{ or }\ m=3.$$

Hence

$$|\Phi^{(-)}_{kj}(\widetilde{\xi},0)|\leq\int\limits_{-\infty}^{+\infty}\frac{\widetilde{c}}{|\xi|^m}\,d\xi_3=\int\limits_{-\infty}^{+\infty}\frac{\widetilde{c}}{\Big(\sqrt{t_1^2|\widetilde{\xi}|^2+t_2^2|\widetilde{\xi}|^2+t_3^2|\widetilde{\xi}|^2}\Big)^m}|\widetilde{\xi}|\,dt_3$$
$$=\frac{\widetilde{c}}{|\widetilde{\xi}|^{m-1}}\int\limits_{-\infty}^{+\infty}\frac{dt_3}{(1+t_3^2)^{m/2}}=\frac{\widetilde{c}_1}{|\widetilde{\xi}|^{m-1}}\,;$$

here $\widetilde{c}>0$ and $\widetilde{c}_1>0$ are some constants.

We derive the following relations

$$\Phi^{(-)}(\widetilde{\xi},0)=\begin{bmatrix}[O(|\widetilde{\xi}|^{-1})]_{5\times5} & [O(|\widetilde{\xi}|^{-2})]_{5\times1}\\ [0]_{1\times5} & O(|\widetilde{\xi}|^{-1})\end{bmatrix}_{6\times6}. \tag{3.12}$$

It can easily be checked that $\det\Phi^{(-)}(\widetilde{\xi},0)=O(|\widetilde{\xi}|^{-6})$ and there exist constants $c_1^*>0$ and $c_2^*>0$ such that

$$c_1^*|\widetilde{\xi}|^{-6}\leq|\det\Phi^{(-)}(\widetilde{\xi},0)|\leq c_2^*|\widetilde{\xi}|^{-6}. \tag{3.13}$$

If $\Phi^{(-)}_c(\widetilde{\xi},0)$ is the corresponding matrix of cofactors, then

$$[\Phi^{(-)}(\widetilde{\xi},0)]^{-1}=\frac{1}{\det\Phi^{(-)}(\widetilde{\xi},0)}\,\Phi^{(-)}_c(\widetilde{\xi},0).$$

Taking into account (3.12) and (3.13) we arrive at the relation

$$[\Phi^{(-)}(\widetilde{\xi},0)]^{-1}=\frac{1}{\det\Phi^{(-)}(\widetilde{\xi},0)}\begin{bmatrix}[O(|\widetilde{\xi}|^{-5})]_{5\times5} & [O(|\widetilde{\xi}|^{-6})]_{5\times1}\\ [0]_{1\times5} & O(|\widetilde{\xi}|^{-5})\end{bmatrix}_{6\times6}$$
$$=\begin{bmatrix}[O(|\widetilde{\xi}|)]_{5\times5} & [O(1)]_{5\times1}\\ [0]_{1\times5} & O(|\widetilde{\xi}|)\end{bmatrix}_{6\times6}. \qquad \square$$

Remark 3.4. Note that $\Phi^{(-)}(\widetilde{\xi},x_3)$ has the same behaviour (3.12) as $\Phi^{(-)}(\widetilde{\xi},0)$ for arbitrary x_3 and due to (3.10)

$$\Phi^{(-)}(\widetilde{\xi},x_3)[\Phi^{(-)}(\widetilde{\xi},0)]^{-1}=\begin{bmatrix}[O(1)]_{5\times5} & [O(|\widetilde{\xi}|^{-1})]_{5\times1}\\ [0]_{1\times5} & O(1)\end{bmatrix}_{6\times6}. \tag{3.14}$$

Theorem 3.5. *The Dirichlet boundary value problems* (2.5)–(2.6) *have at most one solution* $U=(u,\varphi,\psi,\vartheta)^\top$ *in the space* $[C^1(\overline{\mathbb{R}^3_{1,2}})]^6\cap[C^2(\mathbb{R}^3_{1,2})]^6$ *provided*

$$\partial^\alpha\vartheta(x)=O(|x|^{-1-|\alpha|}), \tag{3.15}$$

$$\partial^\alpha\widetilde{U}(x)=O(|x|^{-1-|\alpha|}\ln|x|)\quad\textit{as}\quad|x|\to\infty \tag{3.16}$$

for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. *Here* $\widetilde{U} = (u, \varphi, \psi)^\top$.

Proof. Let $U^{(1)} = (u^{(1)}, \varphi^{(1)}, \psi^{(1)}, \vartheta^{(1)})^\top$ and $U^{(2)} = (u^{(2)}, \varphi^{(2)}, \psi^{(2)}, \vartheta^{(2)})$ be two solutions of the problem under consideration with properties indicated in the theorem for $\mathbb{R}^3_1$. It is evident that the difference

$$V = (u', \varphi', \psi', \vartheta') = U^{(1)} - U^{(2)}$$

solves the corresponding homogeneous problem.

Therefore for the temperature function we get the separated homogeneous Dirichlet problem

$$[A(\partial)V]_6 = \eta_{jl}\partial_j\partial_l\vartheta' = 0 \quad \text{in} \quad \mathbb{R}^3_1, \tag{3.17}$$

$$\{\vartheta'\}^+ = 0 \quad \text{on} \quad S. \tag{3.18}$$

By Green's formula (see (2.83) in [11]) for $B^+(0;R) := \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 \le R^2 \text{ and } x_3 > 0\}$ and (3.17)–(3.18) we have

$$\begin{aligned}\int\limits_{B^+(0;R)} \eta_{jl}\partial_l\vartheta'\partial_j\vartheta'\,dx &= \int\limits_{\partial B^+(0;R)} \{\eta_{jl}n_j\partial_l\vartheta'\}^+\{\vartheta'\}^+ dS \\ &= \int\limits_{\Sigma^+(0;R)} \{\eta_{jl}n_j\partial_l\vartheta'\}^+\{\vartheta'\}^+\,d\Sigma.\end{aligned} \tag{3.19}$$

Here $\Sigma^+(0;R)$ is the upper half sphere.

Taking the limit as $R \to \infty$ in (3.19) according to (3.15) we get

$$\int\limits_{\mathbb{R}^3_1} \eta_{jl}\partial_l\vartheta'\partial_j\vartheta'\,dx = 0.$$

Due to (2.2) $\vartheta' = \mathit{const}$ and from (3.15) we conclude that $\vartheta' = 0$.

Therefore the five dimensional vector $\widetilde{V} = (u', \varphi', \psi')^\top$ constructed by the first five components of the solution vector V, solves the following homogeneous boundary value problem

$$\begin{aligned}\widetilde{A}(\partial)\widetilde{V} = 0 \quad &\text{in} \quad \mathbb{R}^3_1, \\ \{\widetilde{V}\}^+ = 0 \quad &\text{on} \quad S,\end{aligned} \tag{3.20}$$

where $\widetilde{A}(\partial)$ is the 5×5 differential operator of statics of the electro-magneto-elasticity theory without taking into account thermal effects (see (2.85) in [11]).

Using the limiting procedure as above in the corresponding Green's identity for vectors satisfying decay conditions (3.16) we obtain

$$\int\limits_{\mathbb{R}^3_1} [\widetilde{A}(\partial)\widetilde{V}\cdot\widetilde{V} + \widetilde{\mathcal{E}}(\widetilde{V}, \widetilde{V})]\,dx = \lim_{R\to\infty} \int\limits_{\Sigma^+(0;R)} [\widetilde{\mathcal{T}}\widetilde{V}]^+ \cdot [\widetilde{V}]^+\,d\Sigma. \tag{3.21}$$

Here $\widetilde{\mathcal{T}}(\partial, n)$ is the corresponding 5×5 generalized stress operator (see (2.86) in [11]) and

$$\widetilde{\mathcal{E}}(\widetilde{V}, \widetilde{V}) = c_{rjkl}\partial_l u'_k\partial_j u'_r + \varkappa_{jl}\partial_l\varphi'\partial_j\varphi' + a_{jl}(\partial_l\varphi'\partial_j\psi' + \partial_j\psi'\partial_l\varphi') + \mu_{jl}\partial_l\psi'\partial_j\psi'. \tag{3.22}$$

If $\widetilde{V}$ is a solution of (3.20) satisfying (3.16), then from (3.21) we have

$$\int\limits_{\mathbb{R}^3_1} \widetilde{\mathcal{E}}(\widetilde{V}, \widetilde{V})\,dx = 0. \tag{3.23}$$

From (3.20), (3.22) and (3.23) along with (2.2) we get

$$u'(x) = a \times x + b, \quad \varphi'(x) = b_4, \quad \psi' = b_5,$$

where $a = (a_2, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are arbitrary constant vectors and b_4, b_5 are arbitrary constants. Now, in view of (3.16) we arrive at the equalities $u'(x) = 0$, $\varphi'(x) = 0$, $\psi'(x) = 0$ for all $x \in \mathbb{R}^3_1$, consequently, $U^{(1)} = U^{(2)}$ in $\mathbb{R}^3_1$.

The proof is similar for the domain $\mathbb{R}^3_2$. □

Theorem 3.6. *Let* $f \in \overset{\circ}{C}{}^{\infty}(\mathbb{R}^2)$ *and*

$$\int\limits_{\mathbb{R}^2} f(\widetilde{x})\,d\widetilde{x} = 0, \quad \int\limits_{\mathbb{R}^2} f(\widetilde{x})x_j\,d\widetilde{x} = 0, \quad j = 1, 2.$$

Then the Dirichlet boundary value problems (2.5)–(2.6) *possess unique solutions which can be represented in the following form*

$$U(x) = \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}\Big[\Phi^{(-)}(\widetilde{\xi}, x_3)[\Phi^{(-)}(\widetilde{\xi}, 0)]^{-1}\widehat{f}(\widetilde{\xi})\Big], \quad x_3 > 0, \tag{3.24}$$

or

$$U(x) = \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}\Big[\Phi^{(+)}(\widetilde{\xi}, x_3)[\Phi^{(+)}(\widetilde{\xi}, 0)]^{-1}\widehat{f}(\widetilde{\xi})\Big], \quad x_3 < 0. \tag{3.25}$$

Proof. It suffices to show that the vector functions (3.24) and (3.25) satisfy the conditions (3.15)–(3.16). This will be done if we prove that the following relations hold for all $x \in \mathbb{R}^3$:

$$x_j\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[\widehat{U}(\widetilde{\xi}, x_3)] \le O(1), \quad j = 1, 2, 3, \tag{3.26}$$

and

$$x_j^2\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[\widehat{U}(\widetilde{\xi}, x_3)] \le O(1), \quad j = 1, 2, 3, \tag{3.27}$$

where $\widehat{U}(\widetilde{\xi}, x_3)$ is defined by (3.8) or (3.9). For $j = 1$ or $j = 2$, we find

$$\begin{aligned}
x_j\int\limits_{\mathbb{R}^2}\widehat{U}(\widetilde{\xi}, x_3)e^{-i\widetilde{\xi}\cdot\widetilde{x}}\,d\widetilde{\xi} &= i\int\limits_{\mathbb{R}^2}\widehat{U}(\widetilde{\xi}, x_3)\,\frac{\partial e^{-i\widetilde{\xi}\cdot\widetilde{x}}}{\partial\xi_j}\,d\widetilde{\xi} = i\lim_{R\to\infty}\int\limits_{K(0;R)}\widehat{U}(\widetilde{\xi}, x_3)\,\frac{\partial e^{-i\widetilde{\xi}\cdot\widetilde{x}}}{\partial\xi_j}\,d\widetilde{\xi}\\
&= -i\lim_{R\to\infty}\bigg(\int\limits_{K(0;R)}\frac{\partial\widehat{U}(\widetilde{\xi}, x_3)}{\partial\xi_j}\,e^{-i\widetilde{\xi}\cdot\widetilde{x}}\,d\widetilde{\xi} - \int\limits_{\partial K(0;R)}\widehat{U}(\widetilde{\xi}, x_3)\,e^{-i\widetilde{\xi}\cdot\widetilde{x}}\,\frac{\xi_j}{R}\,ds\bigg)\\
&= -i\lim_{R\to\infty}\int\limits_{K(0;R)}\frac{\partial\widehat{U}(\widetilde{\xi}, x_3)}{\partial\xi_j}\,e^{-i\widetilde{\xi}\cdot\widetilde{x}}\,d\widetilde{\xi} = -i\int\limits_{\mathbb{R}^2}\frac{\partial\widehat{U}(\widetilde{\xi}, x_3)}{\partial\xi_j}\,e^{-i\widetilde{\xi}\cdot\widetilde{x}}\,d\widetilde{\xi},
\end{aligned} \tag{3.28}$$

where $K(0, R)$ is the circle of radius R centered at the origin.

Under the restriction on f we conclude that $\widehat{f} \in \mathcal{S}(\mathbb{R}^2)$ and $\widehat{f}(\widetilde{\xi}) = O(|\widetilde{\xi}|^2)$ as $|\widetilde{\xi}| \to 0$, where $\mathcal{S}$ is the space of rapidly decreasing functions. Therefore in view of (3.14) we have

$$\begin{aligned}
&\frac{\partial\widehat{U}(\widetilde{\xi}, x_3)}{\partial\xi_j} = O(1), \quad |\widetilde{\xi}| \to 0 \quad \text{and}\\
&\frac{\partial\widehat{U}(\widetilde{\xi}, x_3)}{\partial\xi_j} = O(|\widetilde{\xi}|^{-k}), \quad |\widetilde{\xi}| \to \infty, \quad k \ge 2,
\end{aligned} \tag{3.29}$$

uniformly for all $x \in \mathbb{R}^3$. Then the relations (3.28) and (3.29) imply (3.26). The condition (3.27) can be proved similarly if we note that

$$\begin{aligned}
&\frac{\partial^2\widehat{U}(\widetilde{\xi}, x_3)}{\partial\xi_j^2} = O\bigg(\frac{1}{|\widetilde{\xi}|}\bigg), \quad |\widetilde{\xi}| \to 0 \quad \text{and}\\
&\frac{\partial^2\widehat{U}(\widetilde{\xi}, x_3)}{\partial\xi_j^2} = O(|\widetilde{\xi}|^{-k-1}), \quad |\widetilde{\xi}| \to \infty, \quad k \ge 2,
\end{aligned}$$

uniformly for all $x \in \mathbb{R}^3$.

Note that

$$x_3\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[\widehat{U}(\widetilde{\xi}, x_3)] = x_3\int\limits_{\mathbb{R}^2}\bigg(\int\limits_{\ell^-}A^{-1}(-\xi)e^{-i\xi_3x_3}d\xi_3\bigg)[\Phi^{(-)}(\widetilde{\xi}, 0)]^{-1}\widehat{f}(\widetilde{\xi})e^{-i\widetilde{\xi}\cdot\widetilde{x}}d\widetilde{\xi}. \tag{3.30}$$

Using the Cauchy integral theorem for analytic functions and the relations (3.10), (3.11), from (3.30) we get

$$\begin{aligned} & x_3\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[\widehat{U}(\widetilde{\xi},x_3)] \\ &\quad = x_3\int\limits_{\mathbb{R}^2} e^{-|\widetilde{\xi}|x_3}\begin{bmatrix}[O(|\widetilde{\xi}|^{-1})]_{5\times 5} & [O(|\widetilde{\xi}|^{-2})]_{5\times 1}\\ [0]_{1\times 5} & O(|\widetilde{\xi}|^{-1})\end{bmatrix}\begin{bmatrix}[O(|\widetilde{\xi}|)]_{5\times 5} & [O(1)]_{5\times 1}\\ [0]_{1\times 5} & O(|\widetilde{\xi}|)\end{bmatrix}\widehat{f}(\widetilde{\xi})\,d\widetilde{\xi} \\ &\quad = x_3\int\limits_{\mathbb{R}^2} e^{-|\widetilde{\xi}|x_3}[O(1)]_{6\times 6}\widehat{f}(\widetilde{\xi})\,d\widetilde{\xi} = I_1+I_2, \end{aligned} \tag{3.31}$$

where

$$I_1 = x_3\int\limits_{|\xi|\le M} e^{-|\widetilde{\xi}|x_3}[O(1)]_{6\times 6}\widehat{f}(\widetilde{\xi})\,d\widetilde{\xi} \quad\text{and}\quad I_2 = x_3\int\limits_{|\xi|>M} e^{-|\widetilde{\xi}|x_3}[O(1)]_{6\times 6}\widehat{f}(\widetilde{\xi})\,d\widetilde{\xi}.$$

Since $\widehat{f}(\widetilde{\xi})\in S(\mathbb{R}^2)$, it is easy to check that $I_1=O(1)$ and $I_2=O(1)$ and hence (3.26) holds.

We can prove the boundedness of the vector function $x_3^2\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}[\widehat{U}(\widetilde{\xi},x_3)]$ quite similarly taking into account that $\widehat{f}(\widetilde{\xi})=O(|\widetilde{\xi}|^2)$ as $|\widetilde{\xi}|\to 0$. $\square$

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Short Communications

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ON THE SOLVABILITY OF THE ANTIPERIODIC PROBLEM FOR LINEAR SYSTEMS OF IMPULSIVE EQUATIONS

Abstract. The antiperiodic boundary value problem for systems of linear impulsive equations is considered. The Green type theorem on the unique solvability of the problem is established, and its solution is represented. The effective necessary and sufficient (among them spectral sufficient) conditions for the unique solvability of the problem are also given.

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In the present paper, we consider the system of linear impulsive equations on the real axis with a finite number of impulses points

$$\frac{dx}{dt} = P(t)x + p(t) \ \text{ for a.e. } \ t \in \mathbb{R}, \tag{1}$$

$$x(\tau_{kj}+) - x(\tau_{kj}-) = Q_{kj}x(\tau_{kj}-) + q_{kj} \ \ (j = 1, \dots, m_0; \ \ k = 0, \pm 1, \pm 2, \dots) \tag{2}$$

under the ω-antiperiodic condition

$$x(t + \omega) = -x(t) \ \text{ for } \ t \in \mathbb{R}, \tag{3}$$

where $k\omega \le \tau_{k1} < \dots < \tau_{km_0} < (k+1)\omega$, $\tau_{k+1\,j} = \tau_{k\,j} + \omega$ $(j = 1, \dots, m_0;\ k = 0, \pm 1, \pm 2, \dots)$, m_0 is a fixed natural number, ω is a fixed positive number, $P \in L_{loc}(\mathbb{R}; \mathbb{R}^{n\times n})$ is a ω-periodic matrix-function, $p \in L_{loc}(\mathbb{R}; \mathbb{R}^n)$ is a ω-antiperiodic vector-function, $Q_{kj} \in \mathbb{R}^{n\times n}$ $(j = 1, \dots, m;\ k = 0, \pm 1, \pm 2, \dots)$ and $q_{kj} \in \mathbb{R}^n$ $(j = 1, \dots, m;\ k = 0, \pm 1, \pm 2, \dots)$ are, respectively, constant $n \times n$-matrices and n-vectors.

Below we present the Green type theorem on the solvability of the problem (1), (2); (3) and give representation of its solution. In addition, we give effective necessary and sufficient (spectral type) conditions for the unique solvability of the problem. The general linear boundary value problem for the system (1), (2) and the nonlinear problems for impulsive systems are investigated sufficiently well in [1,5,6,8–11,16–18] (see also the references therein), where, in particular, the Green type theorems for the unique solvability have been obtained. Some questions of periodic problems for the system (1), (2) are investigated in [10, 11, 16–18]. Moreover, they are a particular case of the problems considered in [3, 4, 6, 19]. As to the antiperiodic problem, it is rather far from completeness. Thus the problem under consideration what follows, is actual.

In the paper we establish some spectral conditions for the unique solvability of the problem which follows from the analogous results for the generalized linear differential systems.

In the paper, the use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$; $[a, b]$ and $]a, b[$ $(a, b \in \mathbb{R})$ are, respectively, closed and open intervals. $\mathbb{Z}$ is a set of all integers.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\|=\max\limits_{j=1,\dots,m}\sum\limits_{i=1}^{n}|x_{ij}|$.

$O_{n\times m}$ (or O) is the zero $n\times m$ matrix.

If $X=(x_{ij})_{i,j=1}^{n,m}\in\mathbb{R}^{n\times m}$, then $|X|=(|x_{ij}|)_{i,j=1}^{n,m}$.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$; $\mathbb{R}^n_+=\mathbb{R}^{n\times 1}_+$.

If $X\in\mathbb{R}^{n\times n}$, then:

X^{-1} is the matrix inverse to X;

$\det X$ is the determinant of X;

$r(X)$ is the spectral radius of X;

I_n is the identity $n\times n$-matrix.

The inequalities between the real matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

If $X:[a,b]\to\mathbb{R}^{n\times m}$ is a matrix-function, then $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t ($X(a-)=X(a)$, $X(b+)=X(b)$).

$L([a,b];\mathbb{R}^{n\times m})$ is the set of all measurable and Lebesgue integrable on $[a,b]$ matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$;

$L_{loc}(\mathbb{R},\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{R}\to\mathbb{R}^{n\times m}$ whose restrictions on every closed interval $[a,b]$ from $\mathbb{R}$ belong to $L([a,b],\mathbb{R}^{n\times n})$.

$C([a,b];\mathbb{R}^{n\times l})$ is the set of all continuous on $[a,b]$ matrix-functions $X:[a,b]\to\mathbb{R}^{n\times l}$;

$C_{loc}(\mathbb{R},\mathbb{R}^{n\times l})$ is the set of all matrix-functions $X:\mathbb{R}\to\mathbb{R}^{n\times l}$ whose restrictions on every closed interval $[a,b]$ from $\mathbb{R}$ belong to $C([a,b],\mathbb{R}^{n\times l})$.

$\widetilde{C}([a,b];\mathbb{R}^{n\times l})$ is the set of all absolutely continuous on $[a,b]$ matrix-functions $X:[a,b]\to\mathbb{R}^{n\times l}$;

$\widetilde{C}([a,b];\mathbb{R}^{n\times l};\tau_1,\dots,\tau_m)$, where $\tau_1,\dots,\tau_m\in[a,b]$, is the set of all matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$, having the one-sided limits $X(\tau_k-)$ $(k=1,\dots,m)$ and $X(\tau_k+)$ $(k=1,\dots,m)$, whose restriction on an arbitrary closed interval $[c,d]$ from $[a,b]\setminus\{\tau_k\}_{k=1}^m$ belong to $\widetilde{C}([c,d];\mathbb{R}^{n\times l})$.

For the pair $\{X;\{Y_l\}_{l=1}^m\}$, consisting of the matrix-function $X\in L([0,\omega],\mathbb{R}^{n\times n})$ and a sequence of constant $n\times n$ matrices $Y_1,\dots,Y_m$, we put

$$\begin{aligned}
&\big[(X;\{Y_l\}_{l=1}^m)(t)\big]_0=I_n \ \text{ for } \ 0\le t\le\omega,\\
&\big[(X;\{Y_l\}_{l=1}^m)(0)\big]_i=O_{n\times n} \ \ (i=1,2,\dots),\\
&\big[(X;\{Y_l\}_{l=1}^m)(t)\big]_{i+1}=\int\limits_0^t X(\tau)\cdot\big[(X;\{Y_l\}_{l=1}^m)(\tau)\big]_i\,d\tau\\
&\qquad+\sum_{a\le\tau_l<t}Y_l\cdot\big[(X;\{Y_l\}_{l=1}^m)(\tau_l)\big]_i \ \text{ for } \ 0<t\le\omega \ \ (i=1,2,\dots).
\end{aligned}\tag{4}$$

We say that the pair $\{X;\{Y_l\}_{l=1}^m\}$ satisfies the Lappo–Danilevskiĭ condition, if the matrices $Y_1,\dots,Y_m$ are pairwise permutable and there exists $t_0\in[a,b]$ such that

$$\int\limits_{t_0}^t X(\tau)\,dX(\tau)=\int\limits_{t_0}^t dX(\tau)\cdot X(\tau) \ \text{ for } \ t\in[0,\omega]$$

and

$$X(t)Y_l=Y_lX(t) \ \text{ for } \ t\in[0,\omega] \ \ (l=1,\dots,m).$$

Under a solution of the system (1), (2) we understand a continuous from the left vector-function $x:\mathbb{R}\to\mathbb{R}^n$ whose restrictions on $[k\omega,(k+1)\omega]$ belong to $\widetilde{C}([k\omega,(k+1)\omega];\mathbb{R}^n;\tau_{k1},\dots,\tau_{km_0})$ for every $k\in\mathbb{Z}$ and satisfying both the system (1) for a.e. $t\in\mathbb{R}$ and the equality (2) for every $j\in\{1,\dots,m_0\}$.

In the sequel, we assume everywhere that $P(t+\omega)=P(t)$ and $q(t+\omega)=-q(t)$ for $t\in\mathbb{R}$, $\tau_{0j}=\tau_j$, $q_{0j}=q_j$, $Q_{kj}=Q_j$ and $q_{k+1j}=-q_{kj}$ $(j=1,\dots,m_0;\ k=0,\pm1,\pm2,\dots)$. Moreover, we assume that

$$\det(I_n+Q_j)\neq 0 \ \ (j=1,\dots,m_0).\tag{5}$$

Note that the condition (5) guarantees the unique solvability of the system (1), (2) under the Cauchy condition $x(t_0)=c_0$.

Alongside with the system (1), (2), we consider the corresponding homogeneous system

$$\frac{dx}{dt} = P(t)x \ \text{ for a.e. } \ t \in \mathbb{R}, \tag{1_0}$$

$$x(\tau_{kj}+) - x(\tau_{kj}-) = Q_{kj}x(\tau_{kj}-) \ \ (j = 1, \dots, m_0; \ k = 0, \pm 1, \pm 2, \dots). \tag{2_0}$$

Moreover, along with (3) we consider the condition

$$x(0) = -x(\omega). \tag{6}$$

Proposition 1. *The following statements are valid:*

(a) *if x is a solution of the system* (1), (2)*, then the function $y(t) = -x(t+\omega)$ $(t \in \mathbb{R})$ is a solution of the system* (1), (2)*, as well;*

(b) *the problem* (1), (2); (3) *is solvable if and only if the system* (1), (2) *on the closed interval $[0, \omega]$ has a solution satisfying the boundary condition* (6)*. Moreover, the set of restrictions of solutions of the problem* (1), (2); (3) *on $[0, \omega]$ coincides with the set of solutions of the problem* (1), (2); (6).

Based on this proposition we give the following definition.

Let

$$D = I_n + Y(\omega),$$

where Y is the fundamental matrix of the problem $(1_0), (2_0); (6)$ under the condition $Y(0) = I_n$.

Definition 1. Let $\det D \neq 0$. A matrix-function $\mathcal{G} : [0, \omega] \times [0, \omega] \to \mathbb{R}^{n\times n}$ is said to be the Green matrix of the problem $(1_0), (2_0); (6)$ if:

(a) for every $s \in]0, \omega[$, the matrix-function $\mathcal{G}(\,\cdot\,, s)$ satisfies the impulsive homogeneous matrix equation

$$\frac{dX}{dt} = P(t)X \ \text{ for a. e. } \ t \in \mathbb{R},$$
$$X(\tau_j+) - X(\tau_j-) = Q_jX(\tau_j-) \ \ (j = 1, \dots, m_0);$$

(b)

$$\mathcal{G}(t, t+) - \mathcal{G}(t, t-) = Y(t)D^{-1}Y(\omega)Y^{-1}(t) \ \text{ for } \ t \in]0, \omega[\, \setminus \{\tau_1, \dots, \tau_{m_0}\},$$
$$\mathcal{G}(\tau_j, \tau_j+) - \mathcal{G}(\tau_j, \tau_j-) = Y(\tau_j)D^{-1}Y(\omega)Y^{-1}(\tau_j)(I_n + Q_j)^{-1} \ \ (j = 1, \dots, m_0);$$

(c)

$$\mathcal{G}(t+, t) - \mathcal{G}(t-, t) = I_n \ \text{ for } \ t \in]0, \omega[\, \setminus \{\tau_1, \dots, \tau_{m_0}\},$$
$$\mathcal{G}(\tau_j+, \tau_j) - \mathcal{G}(\tau_j-, \tau_j) = I_n + Q_jY(\tau_j)D^{-1}\big(I_n + Y^{-1}(\tau_j)\big) \ \ (j = 1, \dots, m_0);$$

(d)

$$\mathcal{G}(t, \cdot) \in \widetilde{C}\big([0, \omega]; \mathbb{R}^{n\times n}; \tau_1, \dots, \tau_{m_0}\big) \ \text{ for } \ t \in [0, \omega];$$

(e) the equality

$$\int_0^\omega \big(\mathcal{G}(0, s) + \mathcal{G}(\omega, s)\big) \cdot p(s)\, ds + \sum_{j=1}^{m_0} \big(\mathcal{G}(0, \tau_j+) + \mathcal{G}(\omega, \tau_j+)\big) \cdot q_j = 0$$

holds for every $p \in L([0, \omega], \mathbb{R}^n)$ and $q_1, \dots, q_{m_0} \in \mathbb{R}^n$.

The Green matrix of the problem $(1_0), (2_0); (6)$ exists and is unique in the following sense. If $\mathcal{G}(t, s)$ and $\mathcal{G}_1(t, s)$ are two matrix-functions satisfying the conditions (a)–(e) of Definition 1, then

$$\mathcal{G}(t, s) - \mathcal{G}_1(t, s) \equiv Y(t)H_*(s),$$

where $H_* \in \widetilde{C}([0, \omega]; \mathbb{R}^{n\times n}; \tau_1, \dots, \tau_{m_0})$ is a matrix-function such that

$$H_*(s+) = H_*(s-) = C = const \ \text{ for } \ s \in [0, \omega],$$

and $C \in \mathbb{R}^{n\times n}$ is a constant matrix.

In particular, the matrix-function $\mathcal{G}$ defined by

$$\mathcal{G}(t,s)=\begin{cases} Y(t)D^{-1}(I_n+Y^{-1}(s)) & \text{for } 0\le s<t\le\omega,\\ Y(t)D^{-1}\big(I_n-Y(\omega)Y^{-1}(s)\big) & \text{for } 0\le t<s\le\omega,\\ \text{arbitrary} & \text{for } t=s \end{cases}$$

is the Green matrix of the problem $(1_0),(2_0);(6)$.

Theorem 1. *The problem* (1),(2) *has a unique ω-antiperiodic solution x if and only if the corresponding homogeneous system* $(1_0),(2_0)$ *has only the trivial solution satisfying the condition* (6), *i.e., when*

$$\det(I_n+Y(\omega))\neq 0. \tag{7}$$

If the last condition holds, then the solution x admits the notation

$$x(t)=\int\limits_0^{\omega}\mathcal{G}(t,s)\cdot p(s)\,ds+\sum_{j=1}^{m_0}\mathcal{G}(t,\tau_j+)\cdot q_j \quad \textit{for } t\in[0,\omega], \tag{8}$$

where $\mathcal{G}:[0,\omega]\times[0,\omega]\to\mathbb{R}^{n\times n}$ is the Green matrix $\mathcal{G}$ of the problem $(1_0),(2_0)$;(6) *on* $[0,\omega]$.

Corollary 1. *Let the pair $\{P,\{Q_j\}_{j=1}^{m_0}\}$ satisfy the Lappo–Danilevskiĭ condition. Then the problem* (1),(2) *has a unique ω-antiperiodic solution if and only if*

$$\det\bigg(I_n+\exp\bigg(\int\limits_0^{\omega}P(s)\,ds\bigg)\prod_{j=1}^{m_0}(I_n+Q_j)\bigg)\neq 0.$$

Note that if the pair $\{P,\{Q_j\}_{j=1}^{m_0}\}$ satisfies the Lappo–Danilevskiĭ condition, then

$$Y(t)\equiv\exp\bigg(\int\limits_0^{\omega}P(s)\,ds\bigg)\prod_{j=1}^{m_0}(I_n+Q_j)$$

and, therefore, the condition (7) is of the form given in the corollary.

Remark 1. If the system $(1_0),(2_0)$ has a nontrivial ω-antiperiodic solution, then there exist the vector-function $p\in L_{loc}(\mathbb{R},\mathbb{R}^n)$ and constant vectors q_{kj} ($j=1,\dots,m_0$; $k=0,\pm1,\pm2,\dots$) such that $q(t+\omega)=-q(t)$ for $t\in\mathbb{R}$, $q_{k+1j}=-q_{kj}$ ($j=1,\dots,m_0$; $k=0,\pm1,\pm2,\dots$), but the system (1), (2) has no ω-antiperiodic solution.

In general, it is quite difficult to verify the condition (7) directly even in the case where one is able to write out the fundamental matrix of the system $(1_0),(2_0)$ explicitly. Therefore it is important to find of effective conditions which would guarantee the absence of nontrivial ω-antiperiodic solutions of the homogeneous system $(1_0),(2_0)$. Below we give the results concerning the subset question. Analogous results have been obtained by T. Kiguradze for the ordinary differential equations (see [12,13]).

Theorem 2. *The system* (1), (2) *has a unique ω-antiperiodic solution if and only if there exist natural numbers k and m such that the matrix*

$$M_k=-\sum_{i=0}^{k-1}\big[(P;\{Q_l\}_{l=1}^{m_0})(\omega)\big]_i$$

is nonsingular and

$$r(M_{k,m})<1, \tag{9}$$

where

$$M_{k,m}=\big[(P;\{Q_l\}_{l=1}^{m_0})(\omega)\big]_m+\sum_{i=0}^{m-1}\big[(P;\{Q_l\}_{l=1}^{m_0})(\omega)\big]_i\cdot|M_k^{-1}|\big[(P;\{Q_l\}_{l=1}^{m_0})(\omega)\big]_k,$$

and $[(P;\{Q_l\}_{l=1}^{m_0})(\omega)]_i$ $(i=0,\dots,m-1)$ *are defined by* (4).

Corollary 2. *Let there exist a natural j such that*

$$\big[(P;\{Q_l\}_{l=1}^{m_0})(\omega)\big]_j = 0 \ \ (i=1,\dots,j)$$

and

$$\det\Big(\big[(P;\{Q_l\}_{l=1}^{m_0})(\omega)\big]_{j+1}\Big) \neq 0,$$

where $[(P;\{Q_l\}_{l=1}^{m_0})(\omega)]_i$ $(i=0,\dots,m-1)$ are defined by (4). Then there exists $\varepsilon_0>0$ such that the system

$$\frac{dx}{dt} = \varepsilon P(t)x + p(t) \ \text{ for a.e. } t\in\mathbb{R},$$
$$x(\tau_{kj}+) - x(\tau_{kj}-) = \varepsilon Q_j x(\tau_{kj}-) + q_{kj} \ \ (j=1,\dots,m_0;\ k=0,\pm1,\pm2,\dots)$$

have one and only one ω-antiperiodic solution for every $\varepsilon\in]0,\varepsilon_0[$.

Theorem 3. *Let the homogeneous system*

$$\frac{dx}{dt} = P_0(t)x \ \text{ for a. e. } t\in\mathbb{R},, \tag{10_0}$$
$$x(\tau_{kj}+) - x(\tau_{kj}-) = Q_{0kj}x(\tau_{kj}-) \ \ (j=1,\dots,m_0;\ \ k=0,\pm1,\pm2,\dots) \tag{11_0}$$

has only the trivial ω-antiperiodic solution, where $P_0\in L_{loc}(\mathbb{R};\mathbb{R}^{n\times n})$ is $\omega>0$-periodic matrix-function, $Q_{0kj}\in\mathbb{R}^{n\times n}$ $(j=1,\dots,m;\ k=0,\pm1,\pm2,\dots)$ are constant $n\times n$-matrices such that $Q_{0kj}=Q_{0j}$ $(j=1,\dots,m_0;\ k=0,\pm1,\pm2,\dots)$ and

$$\det(I_n+Q_{0j})\neq 0 \ \ (j=1,\dots,m_0).$$

Let, moreover, the matrix-function $P_0\in L_{loc}(\mathbb{R};\mathbb{R}^{n\times n})$ and constant matrices $Q_j(j=1,\dots,m_0)$ admit the estimate

$$\int\limits_0^\omega |\mathcal{G}_0(t,\tau)|\,|P(\tau)-P_0(\tau)|\,d\tau + \sum_{j=1}^{m_0}|\mathcal{G}_0(t,\tau_j+)(Q_j-Q_{0j})| \le M \ \text{ for } t\in[0,\omega],$$

where $\mathcal{G}_0(t,\tau)$ is the Green matrix of the problem $(10_0),(11_0);(6)$, and $M\in\mathbb{R}_+^{n\times n}$ is a constant matrix such that

$$r(M)<1.$$

Then the system $(1),(2)$ has one and only one ω-antiperiodic solution.

The representation (8) can be replaced by a more simple and suitable form by introducing the concept of the Green matrix for the problem $(1_0),(2_0);(3)$.

Definition 2. The matrix-function $\mathcal{G}_\omega:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^{n\times n}$ is said to be the Green matrix of the problem $(1_0),(2_0);(3)$ if:

(a) $\mathcal{G}_\omega(t+\omega,\tau+\omega)=\mathcal{G}_\omega(t,\tau)$, $\mathcal{G}_\omega(t,t+\omega)+\mathcal{G}_\omega(t,\tau)=-I_n$ for $t,\tau\in\mathbb{R}$;

(b) the matrix-function $\mathcal{G}_\omega(\,\cdot\,,\tau):\mathbb{R}\to\mathbb{R}^{n\times n}$ is a fundamental matrix of the system $(1_0),(2_0)$ for every $\tau\in\mathbb{R}$.

Proposition 2. *Let the problem $(1_0),(2_0)$ have only a trivial solution. Then there exists the unique Green matrix of the problem, which has the form*

$$\mathcal{G}_\omega(t,\tau) = -Y(t)(I_n+Y^{-1}(\omega))^{-1}Y^{-1}(\tau) \ \text{ for } t,\tau\in\mathbb{R}.$$

Theorem 4. *Let the condition*

$$\det(I_n\pm Q_j)\neq 0 \ \ (j=1,\dots,m_0)$$

hold and the boundary value problem $(1_0),(2_0);(3)$ have only a trivial solution. Then the ω-antiperiodic problem $(1),(2);(3)$ has a unique solution x admitting the representation

$$\begin{aligned} x(t) = \int\limits_t^{t+\omega} \mathcal{G}_\omega(t,\tau)p(\tau)\,d\tau &+ \sum_{t\le\tau_{kj}<(k+1)\omega} \mathcal{G}_\omega(t,\tau_{kj})(I_n-Q_j^2)^{-1}q_{kj} \\ &+ \sum_{(k+1)\omega\le\tau_{k+1\,j}<t+\omega} \mathcal{G}_\omega(t,\tau_{k+1\,j})(I_n-Q_j^2)^{-1}q_{k+1\,j} \ \text{ for } t\in(k\omega,(k+1)\omega] \ \ (k=0;\pm1;\pm;\dots), \end{aligned} \tag{12}$$

where $\mathcal{G}_\omega$ is the Green matrix of the problem $(1_0), (2_0); (3)$.

Using the properties of the Green matrix $\mathcal{G}_\omega(t,\tau)$ (see Definition 2 (a)), the representation (12) can be rewriten in the form

$$x(t) = \int\limits_t^{t+\omega} \mathcal{G}_\omega(t,\tau)p(\tau)\,d\tau + (-1)^{k+1} \sum_{0\le\tau_j<t-k\omega} \mathcal{G}_\omega(t-\omega,\tau_j)(I_n - Q_j^2)^{-1}q_j$$
$$+ (-1)^k \sum_{t-k\omega\le\tau_j<\omega} \mathcal{G}_\omega(t-k\omega,\tau_j)(I_n - Q_j^2)^{-1}q_j \ \text{ for } \ t\in(k\omega,(k+1)\omega] \ \ (k=0;\pm1;\pm;\dots).$$

Note that the results obtained in the paper, follow from the corresponding results given in [7] for the generalized differential system of the form

$$dx(t) = dA(t)\cdot x(t) + df(t)$$

since the impulsive system (1), (2) is the particular case of the last system under the assumptions that

$$A(0) = O_{n\times n}, \ \ A(t) = \int\limits_0^t P(\tau)\,d\tau + \sum_{0\le\tau_j<t} Q_j \ \text{ for } \ t\in(0,\omega],$$

$$f(0) = 0, \ \ f(t) = \int\limits_0^t p(\tau)\,d\tau + \sum_{0\le\tau_j<t} q_j \ \text{ for } \ t\in(0,\omega],$$

and

$$A(t+\omega) = A(t) \ \text{ and } \ f(t+\omega) = -f(t) \ \text{ for } \ t\in\mathbb{R}\setminus[0,\omega].$$

It is not difficult to verify that

$$A(t) = \int\limits_{k\omega}^t P(\tau)\,d\tau + \sum_{k\omega\le\tau_{kj}<t} Q_j + kA(\omega) \ \text{ for } \ t\in(k\omega,(k+1)\omega]$$

and

$$f(t) = \int\limits_{k\omega}^t p(\tau)\,d\tau + \sum_{k\omega\le\tau_{kj}<t} q_j + \varphi(k)f(\omega) \ \text{ for } \ t\in(k\omega,(k+1)\omega] \quad (k=0;\pm1,\pm2,\dots),$$

where $\varphi(k)=0$ if k is an even integer, and $\varphi(k)=1$ if k is an odd one.

The theory of generalized ordinary differential equations has been introduced by J. Kurzweil [14,15] in connection with the investigation of the well-posed problem for the Cauchy problem for ordinary differential equations.

Finally, we note that, to a considerable extent, the interest to the theory of generalized ordinary differential equations has been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see [1, 2, 5, 7, 19] and the references therein).

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ON THE WELL-POSEDNESS OF ANTIPERIODIC PROBLEM FOR SYSTEMS OF LINEAR GENERALIZED DIFFERENTIAL EQUATIONS

Abstract. The question of well-posedness of antiperiodic boundary value problem for systems of linear generalized differential equations is considered. The necessary and sufficient as well as the effective sufficient conditions are found for the well-posedness of the problem.

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We consider the question of well-posedness of the ω-antiperiodic problem for linear generalized ordinary differential equations of the form

$$dx(t) = dA(t) \cdot x(t) + df(t) \ \text{ for } \ t \in \mathbb{R}, \tag{1}$$

$$x(t+\omega) = -x(t) \ \text{ for } \ t \in \mathbb{R}, \tag{2}$$

where $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $f : \mathbb{R} \to \mathbb{R}^n$ are, respectively, the matrix- and vector-functions with bounded variation components on the every closed interval $[a,b]$ from $\mathbb{R}$, and ω is a fixed positive number.

Let the system (1) have a unique ω-antiperiodic solution x^0.

Along with the system (1), consider a sequence of systems

$$dx(t) = dA_k(t) \cdot x(t) + df_k(t) \ \ (k = 1, 2, \dots) \tag{1_k}$$

where $A_k : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $f_k : \mathbb{R} \to \mathbb{R}^n$ are, respectively, the matrix- and vector-functions with bounded variation components on every closed interval $[a,b]$ from $\mathbb{R}$.

In the present paper, the necessary and sufficient conditions are given for a sequence of ω-antiperiodic problems $(1_k), (2)$ $(k = 1, 2, \dots)$ to have a unique solution x_k for a sufficiently large k and

$$\lim_{k\to+\infty} x_k(t) = x^0(t) \ \text{ uniformly on } \ \mathbb{R}. \tag{3}$$

The analogous questions for the linear general boundary value problems are investigated in [2, 6, 10, 11, 19] (see also the references therein) for linear generalized differential systems, in [3–5, 14] (see also the references therein) for nonlinear generalized differential systems and equations, and in [1, 9, 12, 13, 16] (see also the references therein) for ordinary differential and impulsive systems.

The problem on the solvability of the ω-antiperiodic boundary value problem (1), (2) can be found in [8].

As to the well-posedness question concerning of the antiperiodic problem, it is sufficiently far from by completeness. Thus the problem considered in the present paper is actual.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see [3,7,14,15,17,18] and the references therein).

The theory of generalized ordinary differential equations has been introduced by J. Kurzweil [14,15] in connection with the investigation of the well-posed problem for the Cauchy problem for ordinary differential equations.

In the paper, the use will be made of the following notation and definitions:

$\mathbb{R} =]-\infty, +\infty[$ is the real axis;

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\|=\max_{j=1,\dots,m}\sum_{i=1}^{n}|x_{ij}|;$$

$O_{n\times m}$ (or O) is the zero $n\times m$ matrix; I_n is the identity $n\times n$-matrix.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such. The inequalities between the real matrices are understood componentwise.

If $X:[a,b]\to\mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_a^b(X)$ is the sum of total variations on $[a,b]$ of its components x_{ij} $(i=1,\dots,n;\ j=1,\dots,m)$; $V(X)(t)=(V(x_{ij})(t))_{i,j=1}^{n,m}$, where $V(x_{ij})(a)=0$, $V(x_{ij})(t)=\bigvee\limits_a^t(x_{ij})$ for $a<t\le b$; $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t $(X(a-)=X(a),\ X(b+)=X(b))$; $d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the normed space of all bounded variation matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ (i.e., $\bigvee\limits_a^b(X)<\infty$) with the norm $\|X\|_s=\sup\{\|X(t)\|:\ t\in[a,b]\}$.

$\mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ whose restrictions on every closed interval $[a,b]$ from $\mathbb{R}$ belong to $\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$.

$\mathrm{BV}_\omega^+(\mathbb{R},\mathbb{R}^{n\times m})$ and $\mathrm{BV}_\omega^-(\mathbb{R},\mathbb{R}^{n\times m})$ are the sets of all matrix-functions $G:\mathbb{R}\to\mathbb{R}^{n\times m}$ whose restrictions on $[0,\omega]$ belong to $\mathrm{BV}([0,\omega],\mathbb{R}^{n\times m})$, and there exist a constant matrix $C\in\mathbb{R}^{n\times m}$ such that, respectively,

$$G(t+\omega)=G(t)+C\ \text{ and }\ G(t+\omega)=-G(t)+C\ \text{ for }\ t\in\mathbb{R}.$$

$s_c, s_j:\mathrm{BV}([a,b],\mathbb{R})\to\mathrm{BV}([a,b],\mathbb{R})$ $(j=1,2)$ are the operators defined, respectively, by

$$s_1(x)(a)=s_2(x)(a)=0,$$

$$s_1(x)(t)=\sum_{a<\tau\le t}d_1x(\tau)\ \text{ and }\ s_2(x)(t)=\sum_{a\le\tau<t}d_2x(\tau)\ \text{ for }\ a<t\le b,$$

and

$$s_c(x)(t)=x(t)-s_1(x)(t)-s_2(x)(t)\ \text{ for }\ t\in[a,b].$$

If $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, $x:[a,b]\to\mathbb{R}$ and $a\le s<t\le b$, then

$$\int\limits_s^t x(\tau)\,dg(\tau)=\int\limits_{]s,t[}x(\tau)\,ds_c(g)(\tau)+\sum_{s<\tau\le t}x(\tau)d_1g(\tau)+\sum_{s\le\tau<t}x(\tau)d_2g(\tau),$$

where $\int\limits_{]s,t[}x(\tau)\,ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_c(g))$ corresponding to the function $s_c(g)$.

If $a=b$, then we assume

$$\int\limits_a^b x(t)\,dg(t)=0,$$

and if $a>b$, then we assume

$$\int\limits_a^b x(t)\,dg(t)=-\int\limits_b^a x(t)\,dg(t).$$

Thus $\int\limits_a^b x(\tau)\,dg(\tau)$ is the Kurzweil–Stieltjes integral (see [14–19]).

If $g(t)\equiv g_1(t)-g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int\limits_s^t x(\tau)\,dg(\tau)=\int\limits_s^t x(\tau)\,dg_1(\tau)-\int\limits_s^t x(\tau)\,dg_2(\tau)\ \text{ for }\ s\le t.$$

If $G=(g_{ik})_{i,k=1}^{l,n}\in \mathrm{BV}([a,b],\mathbb{R}^{l\times n})$ and $X=(x_{kj})_{k,j=1}^{n,m}:[a,b]\to\mathbb{R}^{n\times m}$, then

$$S_c(G)(t)\equiv \big(s_c(g_{ik})(t)\big)_{i,k=1}^{l,n},\quad S_j(G)(t)\equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n}\quad (j=1,2)$$

and

$$\int\limits_a^b dG(\tau)\cdot X(\tau)=\bigg(\sum_{k=1}^n\int\limits_a^b x_{kj}(\tau)\,dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}.$$

We introduce the operators. If $X\in \mathrm{BV}_{loc}(\mathbb{R},;\mathbb{R}^{n\times n})$ and $Y\in \mathrm{BV}_{loc}(\mathbb{R},;\mathbb{R}^{n\times m})$, then

$$\mathcal{B}(X,Y)(t)=X(t)Y(t)-X(0)Y(0)-\int\limits_0^t dX(\tau)\cdot Y(\tau);$$

if, in addition, $\det(X(t))\neq 0$ for $t\in\mathbb{R}$, then

$$\mathcal{I}(X,Y)(t)=\int\limits_0^t d\big(X(\tau)+\mathcal{B}(X,Y)(\tau)\big)\cdot X^{-1}(\tau);$$

and if, moreover, $\det(I_n+(-1)^jd_jX(t))\neq 0$ for $t\in\mathbb{R}$ $(j=1,2)$, then

$$\begin{aligned}
&\mathcal{A}(X,Y)(0)=O_{n\times m},\\
&\mathcal{A}(X,Y)(t)=Y(t)-Y(0)+\sum_{0<\tau\le t}d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)\\
&\qquad\qquad -\sum_{0\le\tau<t}d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau)\ \text{ for }\ t>0,\\
&\mathcal{A}(X,Y)(t)=-\mathcal{A}(X,Y)(t)\ \text{ for }\ t<0.
\end{aligned}$$

We say that the matrix-function $X\in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ satisfies the Lappo–Danilevskiĭ condition if the matrices $S_c(X)(t)$, $S_1(X)(t)$ and $S_2(X)(t)$ are pairwise permutable for every $t\in[a,b]$, and there exists $t_0\in[a,b]$ such that

$$\int\limits_{t_0}^t S_c(X)(\tau)\,dS_c(X)(\tau)=\int\limits_{t_0}^t dS_c(X)(\tau)\cdot S_c(X)(\tau)\ \text{ for }\ t\in[a,b].$$

A vector-function $\mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^{n\times m})$ is said to be a solution of the system (1) if

$$x(t)-x(s)=\int\limits_s^t dA(\tau)\cdot x(\tau)+f(t)-f(s)\ \text{ for }\ s<t;\ \ s,t\in\mathbb{R}.$$

We assume that

$$A,A_k\in \mathrm{BV}_\omega^+(\mathbb{R},\mathbb{R}^{n\times n})\ \text{ and }\ f,f_k\in \mathrm{BV}_\omega^-(\mathbb{R},\mathbb{R}^n)\ \ (k=1,2,\dots),$$

i.e.,

$$A(t+\omega)=A(t)+C,\quad A_k(t+\omega)=A_k(t)+C_k\ \text{ for }\ t\in\mathbb{R}\ \ (k=1,2,\dots)$$

and

$$f(t+\omega)=-f(t)+c,\quad f_k(t+\omega)=-f_k(t)+c_k\ \text{ for }\ t\in\mathbb{R}\ \ (k=1,2,\dots),$$

where $C,C_k\in\mathbb{R}^{n\times n}$ $(k=1,2,\dots)$ and $c,c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ are, respectively, some constant matrix and vector. In addition, without loss of generality, we assume that

$$A(0)=A_k(0)=O_{n\times n},\quad f(0)=f_k(0)=0\ \ (k=1,2,\dots)$$

(the last condition is assumed for every generalized linear systems, as well). Moreover, we assume

$$\det\big(I_n+(-1)^jd_jA(t)\big)\neq 0\ \text{ for }\ t\in\mathbb{R}\ \ (j=1,2).$$

Alongside with the system (1), we consider the corresponding homogeneous system

$$dx(t)=dA(t)\cdot x(t).\tag{4_0}$$

Moreover, along with the problem (2), we consider the problem

$$x(0)=-x(\omega). \tag{5}$$

If the matrix-function A satisfies the Lappo–Danilevskiĭ's condition, then the fundamental matrix Y, $Y(0)=I_n$, of the system (4_0) is defined by

$$Y(t)\equiv \exp(S_0(A)(t))\prod_{0\le\tau<t}(I_n+d_2A(\tau))\prod_{0<\tau\le t}(I_n-d_1A(\tau))^{-1} \ \text{ for } \ t\in[0,\omega].$$

Definition 1. We say that a sequence (A_k,f_k) $(k=1,2,\dots)$ belongs to the set $\mathcal{S}(A,f)$ if the ω-antiperiodic problem $(1_k),(2)$ has a unique solution x_k for any sufficiently large k, and the condition (3) holds.

Proposition 1. *The following statements are valid:*

(a) *if x is a solution of the system* (1)*, then the vector-function $y(t)=-x(t+\omega)$ $(t\in\mathbb{R})$ will be a solution of the system* (1)*, as well;*

(b) *the problem* $(1),(2)$ *is solvable if and only if the system* (1) *on the closed interval $[0,\omega]$ has a solution satisfying the boundary condition* (5)*. Moreover, the set of restrictions of solutions of the problem* $(1),(2)$ *on $[0,\omega]$ coincides with the set of solutions of the problem* $(1),(5)$*.*

Theorem 1. *The inclusion*

$$\big((A_k,f_k)\big)_{k=1}^{+\infty}\in\mathcal{S}(A,f) \tag{6}$$

is valid if and only if there exists a sequence of matrix-functions $H,H_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ such that

$$\lim_{k\to+\infty}\sup\bigvee_a^b(H_k+\mathcal{B}(H_k,A_k))<+\infty, \tag{7}$$

$$\inf\Big\{\big|\det(H(t))\big|:\ t\in[0,\omega]\Big\}>0, \tag{8}$$

and the conditions

$$\lim_{k\to+\infty}H_k(t)=H(t), \tag{9}$$

$$\lim_{k\to+\infty}\mathcal{B}(H_k,A_k)(t)=\mathcal{B}(H,A)(t), \tag{10}$$

$$\lim_{k\to+\infty}\mathcal{B}(H_k,f_k)(t)=\mathcal{B}(H,f)(t)$$

are fulfilled uniformly on $[0,\omega]$.

Theorem 2. *Let $A_*\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f_*\in\mathrm{BV}([0,\omega],\mathbb{R}^n)$ be such that*

$$\det\big(I_n+(-1)^jd_jA_*(t)\big)\neq 0 \ \textit{ for } \ t\in[0,\omega] \ \ (j=1,2) \tag{11}$$

and the system

$$dx(t)=dA_*(t)\cdot x(t)+df_*(t) \tag{12}$$

have a unique ω-antiperiodic solution x_*. Let, moreover, there exist sequences of matrix- and vector-functions $H_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ and $h_k\in\mathrm{BV}([0,\omega],\mathbb{R}^n)$ $(k=1,2\dots)$, respectively, such that $h_k(0)=-h_k(\omega)$ $(k=1,2,\dots)$,

$$\inf\Big\{\big|\det(H_k(t))\big|:\ t\in[0,\omega]\Big\}>0 \ \ (k=1,2,\dots), \tag{13}$$

and

$$\lim_{k\to+\infty}\sup\bigvee_a^b A_{*k}<+\infty, \tag{14}$$

and the conditions

$$\lim_{k\to+\infty}A_{*k}(t)=A_*(t), \tag{15}$$

$$\lim_{k\to+\infty}f_{*k}(t)=f_*(t)$$

are fulfilled uniformly on $[0,\omega]$, *where*

$$A_{*k}(t) \equiv \mathcal{I}_k(H_k, A_k)(t) \ \ (k=1,2,\dots),$$

$$f_{*k}(t) \equiv h_k(t) - h_k(0) + \mathcal{B}_k(H_k, f_k)(t) - \int\limits_0^t dA_{*k}(\tau)\cdot h_k(t) \ \ (k=1,2,\dots).$$

Then the system (1_k) *has a unique* ω*-antiperiodic solution* x_k *for any sufficiently large* k, *and*

$$\lim_{k\to+\infty} \|H_k x_k + h_k - x_*\|_s = 0.$$

Corollary 1. *Let the conditions* (7) *and* (8) *hold, and let the conditions* (9), (10) *and*

$$\lim_{k\to+\infty}\bigg(\mathcal{B}(H_k, f_k-\varphi_k)(t) + \int\limits_0^t d\mathcal{B}(H_k, A_k)(s)\cdot\varphi_k(s)\bigg) = \mathcal{B}(H,f)(t)$$

be fulfilled uniformly on $[0,\omega]$, *where* $H, H_k \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$. *Then the system* (1_k) *has a unique* ω*-antiperiodic solution* x_k *for any sufficiently large* k *and*

$$\lim_{k\to+\infty}\|x_k-\varphi_k-x_*\|_s = 0.$$

Corollary 2. *Let the conditions* (7) *and* (8) *hold, and let the conditions* (9),

$$\lim_{k\to+\infty}\int\limits_0^t H_k(s)\,dA_k(s) = \int\limits_0^t H(s)\,dA(s), \quad \lim_{k\to+\infty}\int\limits_0^t H_k(s)\,df_k(s) = \int\limits_0^t H(s)\,df(s),$$

$$\lim_{k\to+\infty} d_jA_k(t) = d_jA(t) \ \ (j=1,2), \ \ \text{and} \ \ \lim_{k\to+\infty} d_jf_k(t) = d_jf(t) \ \ (j=1,2)$$

be fulfilled uniformly on $[0,\omega]$, *where* $H, H_k \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$. *Let, moreover, either*

$$\lim_{k\to+\infty}\sup\sum_{a\le t\le b}\big(\|d_jA_k(t)\| + \|d_jf_k(t)\|\big) < +\infty \ \ (j=1,2)$$

or

$$\lim_{k\to+\infty}\sup\sum_{a\le t\le b}\|d_jH_k(t)\| < +\infty \ \ (j=1,2). \tag{16}$$

Then the inclusion (6) *holds.*

Corollary 3. *Let the conditions* (7) *and* (8) *hold, and let the conditions* (9),

$$\lim_{k\to+\infty} A_k(t) = A(t), \tag{17}$$

$$\lim_{k\to+\infty} f_k(t) = f(t)), \tag{18}$$

$$\lim_{k\to+\infty}\int\limits_0^t d\big(H^{-1}(s)H_k(s)\big)\cdot A_k(s) = A_*(t),$$

$$\lim_{k\to+\infty}\int\limits_0^t d\big(H^{-1}(s)H_k(s)\big)\cdot f_k(s) = f_*(t)$$

be fulfilled uniformly on $[0,\omega]$, *where* $H, H_k, A_* \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, *and* $f_* \in \mathrm{BV}([0,\omega],\mathbb{R}^n)$. *Let, moreover, the system*

$$dx(t) = d\big(A(t)-A_*(t)\big)\cdot x(t) + d\big(f(t)-f_*(t)\big)$$

have a unique ω*-antiperiodic solution. Then*

$$\big((A_k,f_k)\big)_{k=1}^{+\infty} \in \mathcal{S}(A-A_*, f-f_*).$$

Corollary 4. *Let there exist a natural number m and matrix-functions $B_j \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(j = 0,\dots,m-1$ such that*

$$\lim_{k\to+\infty} \sup \bigvee_a^b (A_{km}) < +\infty,$$

and the conditions

$$\lim_{k\to+\infty} \big(A_{km}(t) - A_{km}(0)\big) = A(t),$$
$$\lim_{k\to+\infty} \big(f_{km}(t) - f_{km}(0)\big) = f(t)$$

be fulfilled uniformly on $[0,\omega]$, where

$$H_{k0}(t) \equiv I_n, \quad H_{k\,j+1\,0}(t) \equiv \prod_{j+1}^{1} \big(I_n - A_{kl}(t) + A_{kl}(0) + B_l(t) - B_l(0)\big),$$
$$A_{k\,j+1} \equiv H_{kj}(t) + \mathcal{B}(H_{kj}, A_k)(t), \quad f_{k\,j+1} \equiv \mathcal{B}(H_{kj}, f_k)(t).$$

Then the inclusion (6) holds.

If $m = 1$, then Corollary 4 has the following form

Corollary 5. *Let*

$$\lim_{k\to+\infty} \sup \bigvee_a^b (A_k) < +\infty$$

and the conditions (17) and (18) be fulfilled uniformly on $[0,\omega]$. Then the inclusion (6) holds.

Theorem 1′. *Let $A_* \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f_* \in \mathrm{BV}([0,\omega],\mathbb{R}^n)$ be such that the condition (11) hold and the system (12) has a unique ω-antiperiodic solution x_*. Let, moreover, there exist sequences of matrix- and vector-functions $H_k \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$ and $B, B_k \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$, and a sequence of vector-functions $h_k \in \mathrm{BV}([0,\omega],\mathbb{R}^n)$ $(k = 1,2\dots)$, respectively, such that $h_k(0) = -h_k(\omega)$ $(k = 1,2,\dots)$, the conditions (13),*

$$\lim_{k\to+\infty} \sup \bigvee_a^b (A_{*k} - B_k) < +\infty, \tag{19}$$

$$\det\big(I_n+(-1)^j d_j B(t)\big)\neq 0, \ \ \det\big(I_n+(-1)^j d_j B_k(t)\big)\neq 0 \ \ \textit{for} \ \ t\in[0,\omega] \ \ (j=1,2; \ \ k=0,1,\dots) \tag{20}$$

hold, and the conditions

$$\lim_{k\to+\infty} Z_k(t) = Z(t), \tag{21}$$
$$\lim_{k\to+\infty} \mathcal{B}\big(Z_k^{-1}, A_{*k}(t)\big) = \mathcal{B}(Z^{-1}, A_*(t)), \tag{22}$$
$$\lim_{k\to+\infty} \mathcal{B}\big(Z_k^{-1}, f_{*k}(t)\big) = \mathcal{B}(Z^{-1}, f_*(t)) \tag{23}$$

are fulfilled uniformly on $[0,\omega]$, where A_{*k} and f_{*k} are the matrix- and vector-functions appearing in Theorem 2, and Z_k (Z) is the fundamental matrix of the system

$$dx(t) = dB_k(t)\cdot x(t) \ \ \big(dx(t) = dB(t)\cdot x(t)\big) \tag{24}$$

under the condition

$$Z_k(0) = I_n \ \ (Z(0) = I_m) \ \ (k = 1,2,\dots). \tag{25}$$

Then the conclusion of Theorem 2 is true.

Below, everywhere, just as in the above theorem, it will be assumed that Z_k (Z) is the fundamental matrix of the system (24) under the condition (25) for every $k \in \{1,2,\dots\}$, as well.

Corollary 6. *Let the conditions (8), (19),*

$$\lim_{k\to+\infty} \sup \sum_{0\le t\le\omega} \|d_j B_k(t)\| < +\infty \ \ (j = 1,2) \tag{26}$$

and

$$\det\big(I_n+(-1)^j d_j B(t)\big)\neq 0 \;\; for \;\; t\in[0,\omega] \;\; (j=1,2; \;\; k=0,1,\dots) \tag{27}$$

hold and let the conditions (9),

$$\lim_{k\to+\infty} B_k(t)=B(t), \tag{28}$$

$$\lim_{k\to+\infty}\int_0^t Z_k^{-1}(s)\,d\mathcal{A}(B_k,A_{*k})(s)=\int_0^t Z^{-1}(s)\,d\mathcal{A}(B,A_*)(s) \tag{29}$$

and

$$\lim_{k\to+\infty}\int_0^t Z_k^{-1}(s)\,d\mathcal{A}(B_k,f_{*k})(s)=\int_0^t Z^{-1}(s)\,d\mathcal{A}(B,f_*)(s) \tag{30}$$

be fulfilled uniformly on $[0,\omega]$, *where* H, $H_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, *and* B *and* $B_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ *satisfy the Lappo–Danilevskiĭ condition;* $A_{*k}(t)\equiv\mathcal{I}(H_k,A_k)(t)$ $(k=1,2,\dots)$,

$$f_{*k}(t)\equiv -H_k(t)\varphi_k(t)+H_k(0)\varphi_k(0)+\mathcal{B}(H_k,f_k)(t)+\int_0^t dA_{*k}(s)\cdot H_k(s)\varphi_k(s),$$
$$\varphi_k\in\mathrm{BV}([0,\omega],\mathbb{R}^n) \;\; (k=1,2,\dots),$$

and A_* *and* f_* *are the matrix- and vector-functions appearing in Theorem* $1'$. *Then the conclusion of Corollary* 1 *is true.*

In the Lappo–Danilevskiĭ case, for every $k\in\{1,2,\dots\}$, we have

$$Z_k(t)\equiv\exp(S_0(B_k)(t))\prod_{0\le\tau<t}\big(I_n+d_2B_k(\tau)\big)\prod_{0<\tau\le t}\big(I_n-d_1B_k(\tau)\big)^{-1}.$$

Corollary 7. *Let the conditions* (8), (19) *hold and let the conditions* (9), (15), (27) *and*

$$\lim_{k\to+\infty}\int_0^t \exp(-B_k(s))\,df_{*k}(s)=\int_0^t \exp(-B(s))\,df_*(s)$$

be fulfilled uniformly on $[0,\omega]$, *where* H, $H_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, *and* B *and* $B_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ *are the continuous matrix-functions satisfying the Lappo–Danilevskiĭ condition; and* A_*, A_{*k} *and* f_*, f_{*k}, φ_k $(k=1,2,\dots)$ *are, respectively, matrix- and vector-functions appearing in Corollary* 6. *Then the conclusion of Corollary* 1 *is true.*

Corollary 8. *Let there exist a sequence of matrix-functions* H *and* H_k $(k=0,1,\dots)$ *from* $\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ *such that the matrix-functions* $S_c(A)$ *and* $S_c(A_{*k})$ $(k=1,2,\dots)$ *satisfy the Lappo–Danilevskiĭ condition and the conditions* (8) *and*

$$\lim_{k\to+\infty}\sup\sum_{0\le t\le\omega}\|d_jA_{*k}(t)\|<+\infty \;\; (j=1,2)$$

hold, let the conditions (9),

$$\lim_{k\to+\infty}S_c(A_{*k})(t)=S_c(A_*)(t), \quad \lim_{k\to+\infty}S_j(A_{*k})=S_j(A_*)(t) \;\; (j=1,2)$$

and

$$\lim_{k\to+\infty}\int_0^t \exp\big(-S_c(A_{*k})(s)\big)\,df_{*k})(s)=\int_0^t \exp\big(-S_c(A_{*k})(s)\big)\,df_*)(s)$$

be fulfilled uniformly on $[0,\omega]$, *where* A_*, A_{*k} *and* f_*, f_{*k}, φ_k $(k=1,2,\dots)$ *are, respectively, the matrix-and vector-functions appearing in Corollary* 6. *Then the conclusion of Corollary* 1 *is true.*

Theorem 2′. *The inclusion* (6) *is valid if and only if there exist the sequences of matrix-functions* H, H_k *and* B, $B_k \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ *such that the conditions* (8), (20) *and*

$$\lim_{k\to+\infty}\sup\bigvee_a^b(\mathcal{I}(H_k,A_k)-B_k)<+\infty$$

hold, and the conditions (9), (21),

$$\lim_{k\to+\infty}\mathcal{B}\big(Z_k^{-1},\mathcal{I}(H_k,A_k)\big)(t)=\mathcal{B}\big(Z^{-1},\mathcal{I}(H,A)\big)(t)$$

and

$$\lim_{k\to+\infty}\mathcal{B}\big(Z_k^{-1},\mathcal{I}(H_k,f_k)\big)(t)=\mathcal{B}\big(Z^{-1},\mathcal{I}(H,f)\big)(t)$$

are fulfilled uniformly on $[0,\omega]$.

Corollary 9. *Let the conditions* (20) *and*

$$\lim_{k\to+\infty}\sup\bigvee_a^b(A_k-B_k)<+\infty \tag{31}$$

hold and the conditions (21),

$$\lim_{k\to+\infty}\mathcal{B}(Z_k^{-1},A_k)(t)=\mathcal{B}(Z^{-1},A)(t) \tag{32}$$

and

$$\lim_{k\to+\infty}\mathcal{B}(Z_k^{-1},f_k)(t)=\mathcal{B}(Z^{-1},f)(t) \tag{33}$$

be fulfilled uniformly on $[0,\omega]$*, where* B *and* $B_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$*. Then the inclusion* (6) *holds.*

Corollary 10. *Let the conditions* (26), (27) *and* (31) *hold and the conditions* (29),

$$\lim_{k\to+\infty}\int_0^t Z_k^{-1}(s)\,d\mathcal{A}(B_k,A_k)(s)=\int_0^t Z^{-1}(s)\,d\mathcal{A}(B,A)(s)$$

and

$$\lim_{k\to+\infty}\int_0^t Z_k^{-1}(s)\,d\mathcal{A}(B_k,f_k)(s)=\int_0^t Z^{-1}(s)\,d\mathcal{A}(B,f)(s)$$

be fulfilled uniformly on $[0,\omega]$*, where* B *and* $B_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ *satisfy the Lappo–Danilevskiĭ condition. Then the inclusion* (6) *holds.*

Corollary 11. *Let the condition* (31) *hold and the conditions* (17), (29) *and*

$$\lim_{k\to+\infty}\int_0^t \exp(-B_k(s))\,df_k(s)=\int_0^t \exp(-B(s))\,df(s)$$

be fulfilled uniformly on $[0,\omega]$*, where* B *and* $B_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ *are the continuous matrix-function satisfying the Lappo–Danilevskiĭ condition. Then the inclusion* (6) *holds.*

Corollary 12. *Let the matrix-functions* $S_c(A)$ *and* $S_c(A_k)$ $(k=0,1,\dots)$*,* $A(t)\equiv A_0(t)$*, satisfy the Lappo–Danilevskiĭ condition and the condition*

$$\lim_{k\to+\infty}\sup\sum_{0\le t\le\omega}\|d_jA_k(t)\|<+\infty\ \ (j=1,2)$$

hold. Let, moreover, the conditions

$$\lim_{k\to+\infty}S_c(A_k)(t)=S_c(A)(t),\quad \lim_{k\to+\infty}S_j(A_k)=S_j(A)(t)\ \ (j=1,2)$$

and

$$\lim_{k\to+\infty}\int_0^t \exp\big(-S_c(A_k)(s)\big)\,df_k(s)=\int_0^t \exp\big(-S_c(A)(s)\big)\,df(s)$$

be fulfilled uniformly on $[0,\omega]$. *Then the inclusion* (6) *holds.*

Remark 1. The condition (8) is equivalent to the condition

$$\det\big(H(t-)\cdot H(t+)\big)\neq 0 \ \text{ for } \ t\in[0,\omega].$$

Remark 2. Let $A_*(t)\equiv\mathcal{I}(H,A)(t)$ and (9) be fulfilled uniformly on $[0,\omega]$. Then the condition (14) holds and (15) is fulfilled uniformly on $[0,\omega]$ if and only if the condition (7) holds and (10) is fulfilled uniformly on $[0,\omega]$, respectively.

Remark 3. Without loss of generality we can assume that $H(t)\equiv I_n$ in Theorems 1 and 1′ and in the above corollaries.

Remark 4. In designations of Theorem 1′:

(a) if (19) holds and the conditions (21),

$$\lim_{k\to+\infty}\int\limits_0^t Z_k^{-1}(s)\,d\big(A_{*k}(s)-B_k(s)\big)=\int\limits_0^t Z_k^{-1}(s)\,d\big(A_*(s)-B(s)\big) \tag{34}$$

and

$$\lim_{k\to+\infty} d_j\big(A_{*k}(t)-B_k(t)\big)=d_j\big(A_*(t)-B(t)\big) \ \ (j=1,2) \tag{35}$$

are fulfilled uniformly on $[0,\omega]$, then (22) is fulfilled uniformly on $[0,\omega]$, as well. On the other hand, if the condition (19) holds and the conditions (21) and

$$\lim_{k\to+\infty}\big(A_{*k}(t)-B_k(t)\big)=A_*(t)-B(t)$$

are fulfilled uniformly on $[0,\omega]$, then the conditions (34) and (35) are fulfilled uniformly on $[0,\omega]$, as well;

(b) if

$$\lim_{k\to+\infty}\sup\sum_{0\le t\le\omega}\|d_jf_{*k}(t)\|<+\infty \ \ (j=1,2)$$

and the conditions (21),

$$\lim_{k\to+\infty}\int\limits_0^t Z_k^{-1}(s)\,df_{*k}(s)=\int\limits_0^t Z_k^{-1}(s)\,df_*(s) \tag{36}$$

and

$$\lim_{k\to+\infty} d_jf_{*k}(t)=d_jf_*(t) \ \ (j=1,2) \tag{37}$$

are fulfilled uniformly on $[0,\omega]$, then the condition (24) is fulfilled uniformly on $[0,\omega]$, as well;

(c) if $B(t)\equiv A_*(t)$ and $B_k(t)\equiv A_{*k}(t)$ $(k=1,2,\dots)$, then (19) vanishes and (22) follows from (21).

Remark 5. In designations of Corollary 6:

(a) if (19) holds and (15) and (28) are fulfilled uniformly on $[0,\omega]$, then (29) is fulfilled uniformly on $[0,\omega]$, as well;

(b) if (26) and (27) holds and (28), (36) and (37) are fulfilled uniformly on $[0,\omega]$, then (30) is fulfilled uniformly on $[0,\omega]$, as well.

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OSCILLATORY SOLUTIONS OF HIGHER ORDER NONLINEAR NONAUTONOMOUS DIFFERENTIAL SYSTEMS

Abstract. Oscillatory properties of solutions of higher order nonlinear nonautonomous differential systems are considered. In particular, unimprovable in a certain sense conditions are found under which all proper solutions of those systems are oscillatory.

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On an infinite interval $[a,+\infty[$, we consider the differential system

$$u_i^{(n_i)} = g_i\big(t, u_1, \dots, u_1^{(n_1-1)}, u_2, \dots, u_2^{(n_2-1)}\big) \ \ (i=1,2), \tag{1}$$

where $n_1 \geq 1$, $n_2 \geq 2$, $a > 0$, $g_i : [a,+\infty[\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \to \mathbb{R}$ $(i=1,2)$ are continuous functions, satisfying on $[a,+\infty[\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ one of the following two conditions

$$\begin{aligned} &g_1(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(y_1) \geq f_1(t,y_1)\operatorname{sgn}(y_1),\\ &g_2(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(x_1) \leq -f_2(t,x_1)\operatorname{sgn}(x_1), \end{aligned} \tag{2}$$

or

$$\begin{aligned} &g_1(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(y_1) \geq f_1(t,y_1)\operatorname{sgn}(y_1),\\ &g_2(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(x_1) \geq f_2(t,x_1)\operatorname{sgn}(x_1). \end{aligned} \tag{3}$$

Here $f_i[a,+\infty[\times\mathbb{R} \to \mathbb{R}$ $(i=1,2)$ are nondecreasing in the second argument continuous functions such that

$$f_i(t,x)\operatorname{sgn}(x) \geq 0 \ \ (i=1,2).$$

The present paper is devoted to the investigation of oscillatory properties of solutions of system (1). Previously, such properties have been investigated only in the cases when system (1) can be reduced to one differential equation of order $n = n_1 + n_2$ (see, [1–13, 15] and the references therein), or when $n_1 = n_2 = 1$ (see, [14]).

A solution of system (1) defined on some interval $[a_0,+\infty[\,\subset[a,+\infty[$ is said to be **proper** if it does not identically equal to zero in any neighbourhood of $+\infty$.

A proper solution (u_1,u_2) of system (1) is said to be **oscillatory** if at least one of its components changes sign in any neighbourhood of $+\infty$, and is said to be **Kneser** solution if in the interval $[a_0,+\infty[$ it satisfies the inequalities

$$\begin{aligned} &(-1)^i u_1^{(i)}(t)u_1(t) \geq 0 \quad (i=1,\dots,n_1),\\ &(-1)^k u_2^{(k)}(t)u_2(t) \geq 0 \ \ (k=1,\dots,n_2). \end{aligned}$$

Assume

$$n = n_1 + n_2,$$

and introduce the definitions.

Definition 1. System (1) has the **property** A_0 if every its proper solution for even n is oscillatory, and for odd n either is oscillatory or is a Kneser solution.

Definition 2. System (1) has the **property** B_0 if every its proper solution for even n is either oscillatory, or is a Kneser solution, or satisfies the condition

$$\lim_{t\to+\infty} |u^{(n_i-1)}(t)| > 0 \ \ (i=1,2), \tag{4}$$

and for n odd either is oscillatory or satisfies condition (4).

If m is a natural number, then by $\mathcal{N}_m^0$ we denote the set of those $k \in \{1,\dots,m\}$ for which $m+k$ is even.

For an arbitrary natural k, we put

$$I_k(t,x) = x\Big[t^{n_1-1} + \int\limits_a^t (t-s)^{n_1-1}\big|f_1(s,xs^{k-1})\big|\,ds\Big].$$

Theorem 1. *Let condition* (2) *be satisfied and for any $x \neq 0$ and $k \in \mathcal{N}_{n_2-1}^0$ the equalities*

$$\int\limits_a^{+\infty} |f_1(t,x)|\,dt = +\infty, \qquad \int\limits_a^{+\infty} t^{n_2-1}|f_2(t,x)|\,dt = +\infty, \tag{5}$$

$$\int\limits_a^{+\infty} t^{n_2-k-1}\big|f_2(t,I_k(t,x))\big|\,dt = +\infty \tag{6}$$

be fulfilled. Then system (1) *has the property A_0.*

Theorem 2. *Let condition* (3) *be satisfied. If, moreover, $n_2 > 2$ ($n_2 = 2$) and for any $x \neq 0$ and $k \in \mathcal{N}_{n_2-2}^0$ equalities* (5) *and* (6) *hold* (*for any $x \neq 0$ equalities* (5) *is fulfilled*), *then system* (1) *has the property B_0.*

If $n_1 = 1$, $n_2 = n-1$,

$$g_1(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2}) = y_1, \quad g_2(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2}) = f(t,x_1),$$

then system (1) is equivalent to the differential equation

$$u^{(n)} = f(t,u). \tag{7}$$

We consider the last equation in the case where $f : [a,+\infty[\,\times\mathbb{R} \to \mathbb{R}$ is a continuous function satisfying either the condition

$$f(t,0)=0, \quad f(t,x) \le f(t,y) \ \text{ for } \ t>a, \ \ x<y, \tag{8}$$

or the condition

$$f(t,0)=0, \quad f(t,x) \ge f(t,y) \ \text{ for } \ t>a, \ \ x<y. \tag{9}$$

A solution u of the equation (1), defined on some interval $[a_0,+\infty[\,\subset [a,+\infty[$, is said to be **proper** if is not identically zero in any neighborhood of $+\infty$.

A proper solution $u : [a_0+\infty[\,\to \mathbb{R}$ is said to be oscillatory if it changes the sign in any neighborhood of $+\infty$ and side to be **Kneser solution**

$$(-1)^i u^{(i)}(t)u(t) \ge 0 \ \text{ for } \ t \ge a_0 \ \ (i=1,\dots,n).$$

For equation (6), Definitions 1, 2 and Theorems 1 and 2 have the following forms.

Definition 3. Equation (7) has the property A_0 if any proper solution of this equation in case n even is oscillatory and in case n odd either is oscillatory or is a Kneser solution.

Definition 4. Equation (7) has the property B_0 if any proper solution of this equation in case n even either is oscillatory, or is a Kneser solution, or satisfies the condition

$$\lim_{t\to+\infty} |u^{(n-2)}(t)| = +\infty, \tag{10}$$

and in case n odd either is oscillatory or satisfies condition (10).

Theorem 3. *If along with* (8) *the condition*

$$\int\limits_a^{+\infty} t^{n-k-1}\big|f(t,xt^{k-1})\big|\,dt=+\infty \ \ for \ \ x\neq 0, \ \ k\in\mathcal{N}_{n-1}^0 \tag{11}$$

holds, then equation (7) *has the property* A_0.

Theorem 4. *If* $n\geq 3$ *and along with* (9) *the condition*

$$\int\limits_a^{+\infty} t^{n-k-1}\big|f(t,xt^{k-1})\big|\,dt=+\infty \ \ for \ \ x\neq 0, \ \ k\in\mathcal{N}_{n-2}^0 \tag{12}$$

holds, then equation (7) *has the property* B_0.

The conditions of Theorems 1–4 are in a certain sense unimprovable. Moreover, the following statements are valid.

Theorem 5. *Let condition* (8) *be satisfied and for any* $x\neq 0$ *there exist numbers* $t_x\geq a$ *and* $\delta(x)>0$ *such that*

$$t^{n-k-1}\big|f(t,xt^{k-1})\big|\geq\delta(x)\big|f(t,xt^{n-1})\big| \ \ for \ \ t\geq t_x, \ \ k\in\mathcal{N}_{n-1}^0.$$

Then for the differential equation (6) *to have the property* A_0 *it is necessary and sufficient equalities* (11) *to be fulfilled.*

Theorem 6. *Let conditions* (9) *be fulfilled,* $n\geq 3$ *and for any* $x\neq 0$ *there exist numbers* $t_x\geq a$ *and* $\delta(x)>0$ *such that*

$$t^{n-k-2}\big|f(t,xt^{k-1})\big|\geq\delta(x)\big|f(t,xt^{n-2})\big| \ \ for \ \ t\geq t_x, \ \ k\in\mathcal{N}_{n-2}^0.$$

Then for the differential equation (2) *to have the property* B_0 *it is necessary and sufficient equalities* (12) *to be fulfilled.*

An essential difference between the above formulated theorems and the results obtained earlier (see, e.g., [1–15]) is that they cover the case, where the right-hand sides of system (1) and of equation (7) are slowly increasing with respect to the phase variable functions.

As an example, let us consider the differential equation

$$u^{(n)}=g_0(t)f_0(u)+g_1(t)\ln\big(1+|u|\big)\operatorname{sign}(u), \tag{13}$$

$g_i:[a,+\infty[\,\to\mathbb{R}$ $(i=0,1)$ are continuous functions, $f_0:\mathbb{R}\to\mathbb{R}$ is a continuous, nondecreasing function such that

$$f_0(x)x>0 \ \text{ for } \ x\neq 0, \quad \sup\big\{|f_0(x)|:\ x\in\mathbb{R}\big\}<+\infty.$$

Theorems 5 and 6 result in the following corollaries.

Corollary 1. *If* $n\geq 3$ *and* $g_0(t)\leq 0$, $g_1(t)\leq 0$ *for* $t\geq a$, *then for equation* (13) *to have property* A_0 *it is necessary and sufficient the equality*

$$\int\limits_a^{+\infty}\big[g_0(t)+g_1(t)\ln t\big]\,dt=-\infty$$

to be fulfilled.

Corollary 2. *If* $n\geq 4$ *and* $g_0(t)\geq 0$, $g_1(t)\geq 0$ *for* $t\geq a$, *then for differential equation* (13) *to have property* B_0 *it is necessary and sufficient the equality*

$$\int\limits_a^{+\infty} t\big[g_0(t)+g_1(t)\ln t\big]\,dt=+\infty$$

to be satisfied.

Consider now the case where the right-hand sides of system (1) on the set $[a,+\infty[\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ satisfy either the inequalities

$$\begin{gathered} g_1(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(y_1)\geq p_1(t)|y_1|^{\lambda_1},\\ g_2(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(x_1)\leq -p_2(t)|x_1|^{\lambda_2}, \end{gathered}\tag{14}$$

or the inequalities

$$\begin{gathered} g_1(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(y_1)\geq p_1(t)|y_1|^{\lambda_1},\\ g_2(t,x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\operatorname{sgn}(x_1)\geq p_2(t)|x_1|^{\lambda_2}, \end{gathered}\tag{15}$$

where

$$\lambda_1>0,\ \ \lambda_2>0,\ \ \lambda_1\lambda_2>1,$$

and $p_i:[a,+\infty[\to[0,+\infty[$ are continuous functions.

Along with system (1), let us consider its particular cases

$$u_1^{(n_1)}=p_1(t)|u_2|^{\lambda_1}\operatorname{sgn}(u_2),\quad u_2^{(n_2)}=-p_2(t)|u_1|^{\lambda_2}\operatorname{sgn}(u_1),\tag{16}$$

and

$$u_1^{(n_1)}=p_1(t)|u_2|^{\lambda_1}\operatorname{sgn}(u_2),\quad u_2^{(n_2)}=p_2(t)|u_1|^{\lambda_2}\operatorname{sgn}(u_1).\tag{17}$$

Theorem 7. *If along with* (14) (*along with* (15)) *the conditions*

$$\int\limits_a^{+\infty}p_1(t)\,dt=+\infty,\tag{18}$$

$$\int\limits_a^{+\infty}t^{n_2-1}\bigg[\int\limits_a^t(t-s)^{n_1-1}\Big(\frac{s}{t}\Big)^{(n_2-1)\lambda_1}p_1(s)\,ds\bigg]^{\lambda_2}p_2(t)\,dt=+\infty,\tag{19}$$

$$\lim_{x\to+\infty}\int\limits_a^x t^{n_1-1}\bigg[\int\limits_t^x(s-t)^{n_2-1}p_2(s)\,ds\bigg]^{\lambda_1}p_1(t)\,dt=+\infty\tag{20}$$

are fulfilled, then system (1) *has the property* A_0 (*the property* B_0).

Note that if

$$\liminf_{t\to+\infty}\frac{\int_a^t(t-s)^{n_1-1}s^{(n_2-1)\lambda_1}p_1(s)\,ds}{t^{(n_2-1)\lambda_1}\int_a^t(t-s)^{n_1-1}p_1(s)\,ds}>0,\tag{21}$$

then condition (19) takes the form

$$\int\limits_a^{+\infty}t^{n_2-1}\bigg[\int\limits_a^t(t-s)^{n_2-1}p_1(s)\,ds\bigg]^{\lambda_2}p_2(t)\,dt=+\infty.\tag{22}$$

For system (16), from Theorem 5 it follows

Corollary 3. *If conditions* (18) *and* (21) *are fulfilled, then for system* (16) (*system* (17)) *to have the property* A_0 (*the property* B_0), *it is necessary and sufficient the equalities* (20) *and* (22) *to be satisfied.*

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Mem. Differential Equations Math. Phys. 69 (2016), 129–133

Nino Partsvania

ON OSCILLATORY AND MONOTONE SOLUTIONS OF NONLINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

Abstract. The nonlinear functional differential system with deviating arguments

$$u_1^{(n_1)}(t) = f_1\big(t, u_2(\tau_1(t))\big), \quad u_2^{(n_2)}(t) = f_2\big(t, u_1(\tau_2(t))\big)$$

is considered, where $f_i : [a, +\infty[\, \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ and $\tau_i : [a, +\infty[\to \mathbb{R}$ $(i = 1, 2)$ are continuous functions, and $\tau_i(t) \to +\infty$ as $t \to +\infty$ $(i = 1, 2)$. Conditions are found under which any proper solution of that system is, respectively: a) oscillatory, b) either oscillatory or Kneser solution, c) either oscillatory or rapidly increasing.

რეზიუმე. განხილულია გადახრილარგუმენტებიანი არაწრფივი ფუნქციონალურ-დიფერენციალური სისტემა

$$u_1^{(n_1)}(t) = f_1\big(t, u_2(\tau_1(t))\big), \quad u_2^{(n_2)}(t) = f_2\big(t, u_1(\tau_2(t))\big),$$

სადაც $f_i : [a, +\infty[\, \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ და $\tau_i : [a, +\infty[\to \mathbb{R}$ $(i = 1, 2)$ უწყვეტი ფუნქციებია და $\tau_i(t) \to +\infty$, როცა $t \to +\infty$ $(i = 1, 2)$. ნაპოვნია პირობები, რომელთა შესრულებისას ამ სისტემის ნებისმიერი წესიერი ამონახსნი სათანადოდ არის: ა) რხევადი, ბ) ან რხევადი, ან კნეზერული, გ) ან რხევადი, ან სწრაფად ზრდადი.

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The present paper is devoted to the investigation of asymptotic properties of solutions of the nonlinear functional differential system

$$u_1^{(n_1)}(t) = f_1\big(t, u_2(\tau_1(t))\big), \quad u_2^{(n_2)}(t) = f_2\big(t, u_1(\tau_2(t))\big). \tag{1}$$

Here, $n_1 \geq 1$, $n_2 \geq 2$, $a > 0$, while $f_i : [a, +\infty[\, \times \mathbb{R} \to \mathbb{R}$ and $\tau_i : [a, +\infty[\to \mathbb{R}$ $(i = 1, 2)$ are continuous functions. Moreover,

$$\lim_{t \to +\infty} \tau_i(t) = +\infty \quad (i = 1, 2),$$

and one of the following two conditions

$$f_i(t, 0) = 0, \quad (-1)^{i-1} f_i(t, x) \leq (-1)^{i-1} f_i(t, y) \;\text{ for }\; t > a, \;\; x < y \;\; (i = 1, 2); \tag{2}$$

$$f_i(t, 0) = 0, \quad f_i(t, x) \leq f_i(t, y) \;\text{ for }\; t \geq a, \;\; x < y \;\; (i = 1, 2) \tag{3}$$

is satisfied.

Asymptotic (including oscillatory) properties of solutions of the system (1) previously have been investigated mainly in the cases where this system can be reduced to one $n_1 + n_2$-order functional differential equation, or in the cases where $n_1 = n_2 = 1$ (see [1–7, 11, 12, 15–19] and the references therein). The case, where $n_1 + n_2 > 2$, $\tau_i(t) \not\equiv t$ $(i = 1, 2)$, and the system (1) cannot be reduced to one equation, still remains practically unstudied. The results of the present paper concern namely this case.

Let $a_0 \geq a$. A vector function $(u_1, u_2) : [a_0, +\infty[\to \mathbb{R}^2$ is said to be a **solution of the system** (1) if u_1 and u_2 are, respectively, n_1-times and n_2-times continuously differentiable functions, and there exist continuous functions $v_i :]-\infty, a_0] \to \mathbb{R}$ $(i = 1, 2)$ such that on $[a_0, +\infty[$ the equalities (1) are fulfilled, where

$$u_i(t) = v_i(t) \;\text{ for }\; t \leq a_0 \;\; (i = 1, 2).$$

A solution (u_1, u_2) of the system (1), defined on some interval $[a_0, +\infty[\, \subset [a, +\infty[$, is said to be **proper** if it is not identically zero in any neighborhood of $+\infty$.

A proper solution of the system (1) is said to be **oscillatory** if at least one of its components changes the sign in any neighborhood of $+\infty$.

Note that if one of the conditions (2) and (3) is satisfied, then both components of every oscillatory solution of the system (1) change the sign in any neighborhood of $+\infty$.

A nontrivial solution $(u_1,u_2):[a_0,+\infty[\,\to\mathbb{R}$ of the system (1) is said to be a **Kneser solution** if on $[a_0,+\infty[$ it satisfies the inequalities

$$(-1)^i u_1^{(i)}(t)u_1(t)\geq 0 \ \ (i=1,\dots,n_1),$$
$$(-1)^k u_2^{(k)}(t)u_2(t)\geq 0 \ \ (k=1,\dots,n_2),$$

and it is said to be **rapidly increasing** if

$$\lim_{t\to+\infty}|u_i^{(n_i-1)}(t)|>0 \ \ (i=1,2).$$

Let

$$n=n_1+n_2,$$

and following I. Kiguradze [8, 9] introduce the definitions.

Definition 1. The system (1) has the **property** A_0 if every its proper solution for n even is oscillatory, and for n odd either is oscillatory or is a Kneser solution.

Definition 2. The system (1) has the **property** B_0 if every its proper solution for n even either is oscillatory, or is a Kneser solution, or is rapidly increasing, and for n odd either is oscillatory or is rapidly increasing.

I. T. Kiguradze [8, 9] has established unimprovable in a certain sense conditions under which the differential system

$$u_1^{(n_1)}(t)=f_1(t,u_2(t)), \quad u_2^{(n_2)}(t)=f_2(t,u_1(t))$$

has the property A_0 (the property B_0). The theorems below are the generalizations of those results for the system (1).

If m is a natural number, then by $\mathcal{N}_m^0$ we denote the set of those $k\in\{1,\dots,m\}$ for which $m+k$ is even.

For any natural k, we put

$$\varphi_k(t,x)=x\bigg[|\tau_2(t)|^{n_1-1}+\int_a^{\tau_2(t)}(\tau_2(t)-s)^{n_1-1}\big|f_1\big(t,x|\tau_1(s)|^{k-1}\big)\big|\,ds\bigg].$$

Theorem 1. *Let the condition* (2) *hold and let for any $x\neq 0$ and $k\in\mathcal{N}_{n_2-1}^0$ the equalities*

$$\int_a^{+\infty}|f_1(t,x)|\,dt=+\infty, \quad \int_a^{+\infty}t^{n_2-1}|f_2(t,x)|\,dt=+\infty, \tag{4}$$
$$\int_a^{+\infty}t^{n_2-k-1}\big|f_2(t,\varphi_k(t,x))\big|\,dt=+\infty \tag{5}$$

be satisfied. Then the system (1) *has the property A_0.*

Theorem 2. *Let $n_2>2$ ($n_2=2$) and the condition* (3) *hold. If, moreover, for any $x\neq 0$ and $k\in\mathcal{N}_{n_2-2}^0$ the equalities* (4) *and* (5) *are satisfied* (*for any $x\neq 0$ the equalities* (4) *are satisfied*)*, then the system* (1) *has the property B_0.*

Remark 1. For the equality (5) to be satisfied for any $x\neq 0$ and $k\in\mathcal{N}_{n_2-1}^0$ it is sufficient that the equality

$$\int_a^{+\infty}\big|f_2\big(t,x|\tau_2(t)|^{n_1-1}\big)\big|\,dt=+\infty$$

be satisfied for any $x\neq 0$.

The conditions of Theorems 1 and 2 do not guarantee the existence of proper solutions appearing in the definitions of the properties A_0 and B_0. The problem on the existence of such solutions needs additional investigation. In particular, for the system (1) we have to study the initial problem

$$u_i^{(k-1)}(a)=c_{ik} \ \ (k=1,\dots,n_i; \ \ i=1,2), \tag{6}$$

the Kneser problem

$$\sum_{i=1}^{2}\sum_{k=1}^{n_i}|u_i^{(k-1)}(a)|=c_0,\ \ (-1)^{k-1}u_i^{(k-1)}(t)u_i(t)>0 \ \text{ for } \ t\geq a \ \ (k=1,\dots,n_i;\ \ i=1,2), \tag{7}$$

and the Kiguradze problem [10]

$$\begin{gathered} u_1^{(k-1)}(a)=\alpha_{1k}u_2^{(n_2-1)}(a)+c_{1k} \ \ (k=1,\dots,n_1),\\ u_2^{(k-1)}(a)=\alpha_{2k}u_1^{(n_2-1)}(a)+c_{2k} \ \ (k=1,\dots,n_2-1), \quad \liminf_{t\to+\infty}|u_2^{(n_2-1)}(t)|<+\infty. \end{gathered} \tag{8}$$

The following lemma is valid.

Lemma 1. *If the conditions*

$$a\leq\tau_i(t)<t,\ \ f_i(t,x)\neq 0 \ \ \textit{for} \ \ t>a,\ \ x\neq 0 \ \ (i=1,2),$$

and

$$\sum_{i=1}^{2}\sum_{k=1}^{n_i}|c_{ik}|>0$$

are fulfilled, then the problem (1), (6) *is solvable and every its solution is proper.*

On the basis of the methods proposed in [13] and [14], the following lemmas can be proved.

Lemma 2. *If* $c_0>0$,

$$\tau_i(t)>t \ \ \textit{for} \ \ t>a \ \ (i=1,2),$$

and

$$f_1(t,x)x>0,\ \ (-1)^{n_1+n_2}f_2(t,x)x>0 \ \ \textit{for} \ \ t>a,\ \ x\neq 0,$$

then the problem (1), (7) *is solvable.*

Lemma 3. *Let the conditions*

$$a\leq\tau_i(t)<t,\ \ f_i(t,x)x>0 \ \ \textit{for} \ \ t\geq a,\ \ x\neq 0 \ \ (i=1,2),$$

$$f_1(t,x)\leq f_1(t,y) \ \ \textit{for} \ \ t\geq a,\ \ x\leq y,$$

and

$$\int_a^{+\infty}\big|f_1\big(t,x|\tau_1(t)|^{n_2-1}\big)\big|\,dt=+\infty \ \ \textit{for} \ \ x\neq 0$$

hold. If, moreover,

$$\alpha_{1j}>0,\ \ \alpha_{2k}>0 \ \ (j=1,\dots,n_1;\ \ k=1,\dots,n_2-1),\quad \sum_{j=1}^{n_1}|c_{1j}|+\sum_{k=1}^{n_2-1}|c_{2k}|>0,$$

then the problem (1), (8) *is solvable and every its solution is proper.*

Theorem 1 and Lemmas 1 and 2 yield the following propositions.

Theorem 3. *Let* n_1+n_2 *be even and along with* (2) *the condition*

$$\tau_i(t)<t,\ \ f_i(t,x)\neq 0 \ \ \textit{for} \ \ t\geq a,\ \ x\neq 0 \ \ (i=1,2) \tag{9}$$

be satisfied. If, moreover, for any $x\neq 0$ *and* $k\in\mathcal{N}_{n_2-1}^0$ *the equalities* (4) *and* (5) *are fulfilled, then the system* (1) *has an infinite set of proper solutions and every such solution is oscillatory.*

Theorem 3′. *Let* n_1+n_2 *be odd and along with* (2) *the condition*

$$\tau_i(t)>t,\ \ f_i(t,x)\neq 0 \ \ \textit{for} \ \ t>a,\ \ x\neq 0 \ \ (i=1,2) \tag{10}$$

hold. If, moreover, for any $x\neq 0$ *and* $k\in\mathcal{N}_{n_2-1}^0$ *the equalities* (4) *and* (5) *are satisfied, then:*

(i) *the system* (1) *has an infinite set of proper Kneser solutions and every such solution is vanishing at infinity;*

(ii) *an arbitrary nontrivial solution* (u_1, u_2) *of the system* (1), *defined on some interval* $[a_0, +\infty[\subset [a, +\infty[$ *and satisfying the inequality*

$$\min\left\{(-1)^k u_i^{(k)}(a_0)u_i(a_0) : \ k = 1, \dots, n_i - 1; \ i = 1, 2\right\} \le 0,$$

is oscillatory.

On the basis of Theorem 2 and Lemma 3 the following theorem can be proved.

Theorem 4. *Let* $n_1 + n_2$ *be odd and the conditions* (3) *and* (9) *hold. If, moreover,* $n_2 > 2$ $(n_2 = 2)$ *and for any* $x \neq 0$ *and* $k \in \mathcal{N}^0_{n_2-2}$ *the equalities* (4) *and* (5) *are satisfied* (*for any* $x \neq 0$ *the equalities* (4) *are satisfied*), *then the system* (1) *has infinite sets of oscillatory and rapidly increasing solutions.*

Remark 2. If $n_1 + n_2$ is even and the conditions (3) and (10) hold, then by Lemma 3 the system (1) has an infinite set of proper Kneser solutions. However, in this case the problem on the existence of oscillatory and rapidly increasing solutions of that system remains open.

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Mem. Differential Equations Math. Phys. 69 (2016), 135–138

Jitka Vacková

BOUNDED SOLUTIONS OF NONLINEAR DIFFERENTIAL SYSTEMS WITH DEVIATING ARGUMENTS

Abstract. For systems of nonlinear differential equations with deviating arguments, sufficient conditions for the existence and uniqueness of bounded on $(-\infty, +\infty)$ solutions are established.

რეზიუმე. გადახრილ არგუმენტებიან არაწრფივ დიფერენციალურ განტოლებათა სისტემებისთვის დადგენილია $(-\infty, +\infty)$ შუალედში შემოსაზღვრული ამონახსნების არსებობისა და ერთადერთობის საკმარისი პირობები.

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Consider the system of nonlinear differential equations with deviating arguments

$$x_i'(t) = g_i(t)x_i(t) + f_i(t, x_1(\tau_{i1}(t)), \dots, x_n(\tau_{in}(t))) \ \ (i = 1, \dots, n), \tag{1}$$

where $\tau_{ij} : \mathbb{R} \to \mathbb{R}$ $(i, j = 1, \dots, n)$ are measurable in any finite interval functions, $g_i \in L_{loc}(\mathbb{R}, \mathbb{R})$ $(i = 1, \dots, n)$ and $f_i : \mathbb{R}^{n+1} \to \mathbb{R}$ $(i = 1, \dots, n)$ are functions satisfying the local Carathéodory conditions.

A vector function $(x_i)_{i=1}^n : \mathbb{R} \to \mathbb{R}^n$ is said to be a **bounded solution of the system** (1) if it is absolutely continuous in any finite interval, satisfies the system (1) almost everywhere on $\mathbb{R}$ and

$$\sup \Big\{ \sum_{i=1}^n |x_i(t)| : \ t \in \mathbb{R} \Big\} < +\infty.$$

For systems of ordinary differential equations, the problem on the existence of bounded solutions is investigated in detail (see, [4–7] and the references therein). In particular, for both linear [5] and essentially nonlinear differential systems [4, 6], I. Kiguradze has established unimprovable in a certain sense conditions guaranteeing, respectively, the existence and uniqueness of a bounded solution.

By R. Hakl [1, 2] effective sufficient conditions are established for the existence of a unique solution of a linear differential system with deviating arguments

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^n p_{ij}(t)x_j(\tau_{ij}(t)) + q_i(t) \ \ (i = 1, \dots, n).$$

In the present paper, based on the method of a priori estimates elaborated in [3, 4, 8–10], the Kiguradze type theorems on the existence and uniqueness of a bounded solution of the system (1) are established.

Throughout the paper the following notation is used.

$\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, \infty)$.

$\mathbb{R}^n$ is the space of n-dimensional vectors $x = (x_i)_{i=1}^n$ with the components $x_i \in \mathbb{R}$ $(i = 1, \dots, n)$.

$\mathbb{R}^{n\times n}$ is the space of $n \times n$ matrices $X = (x_{ij})_{i,j=1}^n$ with the components $x_{ij} \in \mathbb{R}$ $(i, j = 1, \dots, n)$.

$\mathbb{R}_+^{n\times n} = \{X = (x_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n} : x_{ij} \in \mathbb{R}_+ \ (i, j = 1, \dots, n)\}$.

$r(X)$ is the spectral radius of the matrix $X \in \mathbb{R}^{n\times n}$.

$L_{loc}(\mathbb{R}, \mathbb{R})$ is the space of summable in any finite interval functions $u : \mathbb{R} \to \mathbb{R}$.

Theorem 1. *Let there exist a constant matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}_+^{n\times n}$, a nonnegative number b, and nonnegative functions p_{ij}, $q_i \in L_{loc}(\mathbb{R},\mathbb{R})$ $(i,j = 1,\dots,n)$ such that*

$$r(A) < 1, \tag{2}$$

$$\big|f_i(t,x_1,\dots,x_n)\big| \le \sum_{j=1}^n p_{ij}(t)|x_j| + q_i(t) \quad \textit{for } t\in\mathbb{R},\ (x_j)_{j=1}^n \in \mathbb{R}^n \ (i=1,\dots,n),$$

$$\Big|\int_{t_i}^t \exp\Big(\int_s^t g_i(\xi)\,d\xi\Big)p_{ij}(s)\,ds\Big| \le a_{ij} \quad \textit{for } t\in\mathbb{R} \ (i,j=1,\dots,n), \tag{3}$$

$$\sum_{i=1}^n \Big|\int_{t_i}^t \exp\Big(\int_s^t g_i(\xi)\,d\xi\Big)q_i(s)\,ds\Big| \le b \quad \textit{for } t\in\mathbb{R}, \tag{4}$$

where $t_i \in \{-\infty,+\infty\}$ $(i=1,\dots,n)$. Then the system (1) *has at least one bounded solution.*

Theorem 2. *Let there exist a constant matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}_+^{n\times n}$, a nonnegative number b, and nonnegative functions $p_{ij} \in L_{loc}(\mathbb{R},\mathbb{R})$ $(i,j = 1,\dots,n)$ such that along with* (2), (3) *the conditions*

$$\begin{aligned}&\big|f_i(t,x_1,\dots,x_n) - f_i(t,y_1,\dots,y_n)\big| \\ &\quad\le \sum_{j=1}^n p_{ij}(t)|x_j - y_j| \quad \textit{for } t\in\mathbb{R},\ (x_j)_{j=1}^n\in\mathbb{R}^n,\ (y_j)_{j=1}^n\in\mathbb{R}^n \ (i=1,\dots,n),\end{aligned} \tag{5}$$

$$\sum_{i=1}^n \Big|\int_{t_i}^t \exp\Big(\int_s^t g_i(\xi)\,d\xi\Big)|f_i(s,0\dots,0)|\,ds\Big| \le b \quad \textit{for } t\in\mathbb{R} \tag{6}$$

and

$$\limsup_{t\to t_i} \int_0^t g_i(s)\,ds = +\infty \ \ (i=1,\dots,n) \tag{7}$$

be fulfilled, where $t_i \in \{-\infty,+\infty\}$ $(i=1,\dots,n)$. Then the system (1) *has one and only one bounded solution.*

Let us describe a scheme of proving the above-formulated theorems.

For an arbitrary natural number m, we consider the system of differential equations

$$x_i'(t) = g_i(t)x_i(t) + \lambda f_i\big(t, x_1(\tau_{i\,1m}(t)),\dots,x_n(\tau_{i\,nm}(t))\big) \ \ (i=1,\dots,n) \tag{8}$$

and the system of differential equations

$$\big|x_i'(t) - g_i(t)x_i(t)\big| \le \sum_{j=1}^n p_{ij}(t)\big|x_j(\tau_{i\,jm}(t))\big| + q_i(t) \ \ (i=1,\dots,n) \tag{9}$$

with the boundary conditions

$$x_i(\sigma_i m) = 0 \ \ (i=1,\dots,n). \tag{10}$$

Here $\lambda\in[0,1]$, $\sigma_i \in \{-1,1\}$ $(i=1,\dots,n)$,

$$\tau_{i\,jm}(t) = \begin{cases}\tau_{ij}(t) & \text{for } |\tau_{ij}(t)| \le m,\\ m & \text{for } \tau_{ij}(t) > m,\\ -m & \text{for } \tau_{ij}(t) < -m\end{cases}$$

and $p_{ij}\in L_{loc}(\mathbb{R},\mathbb{R})$, $q_i \in L_{loc}(\mathbb{R},\mathbb{R})$ $(i,j=1,\dots,n)$ are nonnegative functions.

An absolutely continuous vector function $(x_i)_{i=1}^n : [-m,m]\to\mathbb{R}^n$ is said to be a **solution of the system** (8) (of **the system** (9)) if it almost everywhere on $[-m,m]$ satisfies this system. A solution of the system (8) (of the system (9)), satisfying the boundary conditions (10), is called a **solution of the problem** (8)**,** (10) (of **the problem** (9)**,** (10)).

The following lemmas are valid.

Lemma 1. *Let there exist a positive constant ρ such that for an arbitrary natural number m and arbitrary $\lambda \in [0,1]$ every solution of the problem* (8), (10) *admits the estimate*

$$\max\Big\{\sum_{i=1}^{n}|x_i(t)|:\ -m\le t\le m\Big\}\le \rho. \tag{11}$$

Then the system (1) *has at least one bounded solution.*

Lemma 2. *Let inequalities* (2)–(4)*, where $t_i \in \{-\infty,+\infty\}$ $(i=1,\dots,n)$, $A=(a_{ij})_{i,j=1}^n \in \mathbb{R}_+^{n\times n}$ and $b\in\mathbb{R}_+$, be fulfilled. Moreover, let the condition*

$$\sigma_i=\begin{cases}1 & \text{if } t_i=+\infty,\\ -1 & \text{if } t_i=-\infty\end{cases}$$

for any $i\in\{1,\dots,n\}$ be fulfilled. Then there exists a positive constant ρ such that for an arbitrary natural m every solution of the problem (9), (10) *admits the estimate* (11)*.*

Theorem 1 follows from Lemmas 1 and 2.

Assume now that the conditions of Theorem 2 are fulfilled. Then by Theorem 1, the system (1) has at least one bounded solution $(x_i)_{i=1}^n$. Our aim is to show that an arbitrary bounded solution $(\overline{x}_i)_{i=1}^n$ of that system coincides with $(x_i)_{i=1}^n$. Assume

$$u_i(t)=\overline{x}_i(t)-x_i(t)\ \ (i=1,\dots,n)$$

and

$$\rho_i=\sup\big\{|u_i(t)|:\ t\in\mathbb{R}\big\}\ \ (i=1,\dots,n).$$

Then according to the condition (5), the vector function $(u_i)_{i=1}^n$ is a bounded solution of the system of differential inequalities

$$\big|u_i'(t)-g_i(t)u_i(t)\big|\le\sum_{j=1}^n p_{ij}(t)\rho_j\ \ (i=1,\dots,n).$$

If we now take the conditions (3) and (7) into account, then it becomes clear that

$$|u_i(t)|\le\sum_{j=1}^n\bigg|\int_{t_i}^{t}\exp\Big(\int_s^t g_i(\xi)\,d\xi\Big)p_{ij}(s)\,ds\bigg|\rho_j\le\sum_{j=1}^n a_{ij}\rho_j\ \text{ for } t\in\mathbb{R}\ \ (i=1,\dots,n)$$

and

$$\rho_i\le\sum_{j=1}^n a_{ij}\rho_j\ \ (i=1,\dots,n).$$

Hence, in view of (2), it follows that

$$\rho_i=0\ \ (i=1,\dots,n),$$

and, consequently,

$$\overline{x}_i(t)\equiv x_i(t)\ \ (i=1,\dots,n).$$

Example. Consider the differential equation

$$x'(t)=g(t)\big[x(t)+a|x(\tau(t))|+b\big], \tag{12}$$

where $a\in\mathbb{R}_+$, $b>0$, $\tau:\mathbb{R}\to\mathbb{R}$ is a measurable in any infinite interval function and $g\in L_{loc}(\mathbb{R},\mathbb{R})$ is a nonnegative function such that

$$\int_0^{+\infty} g(s)\,ds=+\infty. \tag{13}$$

The equation (12) follows from the system (1) in case

$$n=1,\ \ \tau_1(t)=\tau(t),\ \ g_1(t)=g(t),\ \ f_1(t,x_1)=g_1(t)(a|x_1|+b). \tag{14}$$

On the other hand, the equalities (13) and (14) guarantee the fulfilment of the conditions (3), (5)–(7), where

$$t_1=+\infty,\ \ a_{11}=a,\ \ p_{11}(t)=a_{11}g_1(t),$$

whence by Theorem 2, it follows that if

$$a < 1, \tag{15}$$

then the equation (12) has a unique bounded solution.

Let us now show that the condition (15) is also necessary for the existence of a bounded solution of the equation (1). Indeed, let the equation (12) have a bounded solution x. If we put

$$\delta = \inf\big\{|x(t)| : \ t \in \mathbb{R}\big\},$$

then with regard for (13), we find

$$\begin{aligned} -x(t) &= \int\limits_t^{+\infty} \exp\bigg(\int\limits_s^t g(\xi)\,d\xi\bigg) g(s)\big[a|x(\tau(s))| + b\big]\,ds \\ &\geq (a\delta + b) \int\limits_t^{+\infty} \exp\bigg(\int\limits_s^t g(\xi)\,d\xi\bigg) g(s)\,ds = a\delta + b > 0 \ \text{ for } \ t \in \mathbb{R} \end{aligned}$$

and

$$\delta \geq a\delta + b.$$

Consequently, the inequality (15) is fulfilled.

The above-constructed example shows that the condition (2) in Theorems 1 and 2 is unimprovable and it cannot be replaced by the condition

$$r(A) \leq 1.$$

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