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Academician Boris Khvedelidze
(November 7, 1915 – March 27, 1993)

This year we mark the 100-th birthday anniversary of the outstanding Georgian mathematician, distinguished scientist, Member of the Georgian National Academy of Sciences, Professor Boris Khvedelidze.

Boris Khvedelidze was born on 7 November 1915 in Chiatura (West Georgia). In 1918 his family resettled in Tbilisi. Father Vladimir Khvedelidze and mother Olgha Berishvili-Khvedelidze were doctors.

In 1931 Boris Khvedelidze graduated from the pedagogical technicum (formerly the 9-th labor school). During the school years Boris Khvedelidze was highly interested in history, philosophy and less in mathematics. After graduating from school he wanted to enroll in an engineering faculty, following the strong advice of his father. But, since he was not from working-class family, a necessary condition for this was a two years labor experience. Therefore he worked as a librarian in 1931–1933. Since members of working-class families had priority to be accepted for a course in engineering, Boris Khvedelidze enrolled in the Faculty of Physics and Mathematics (with specialization in mathematics) of the State University, hoping to change later to the engineering studies. During the first year at the university the professors Levan Gokieli and Archil Kharadze influenced him to give up the idea of becoming an engineer and he made his final choice towards mathematic. Later he decided to study intensively topics of complex analysis, differential and integral equations, inspired by the lectures of Ilya Vekua.

After graduating from the Tbilisi State University (TSU) in 1938 with honor, Boris Khvedelidze enrolled in a PhD course at the Mathematical Institute of the Georgian Branch of the Academy of Sciences of the Soviet Union (later Andrea Razmadze Mathematical Institute of the Georgian National Academy of Sciences). His mentor was the famous Georgian mathematician Ilya Vekua. He was lucky to witness the emergence of the famous seminars of Niko Muskhelishvili on Singular Integral Equations, where he participated very actively during many years. Boris Khvedelidze received his PhD degree in 1942 with a thesis entitled "The Poincare boundary value problem for a linear second order elliptic differential equation".

During the PhD studies, in 1938, Boris Khvedelidze started to teach mathematics at the Georgian Agricultural Institute as an assistant. From 1939 on he was teaching mathematics at the Tbilisi State University (TSU). After getting his PhD degree, Boris Khvedelidze was elected as a docent (associated professor) of TSU until 1951. In 1943-1944 he was Vice Dean of the Faculty of Physics and Mathematics of TSU.

From 1942 until 1953 he was working as a junior researcher and from 1943 as a senior researcher at the Mathematical Institute. In 1945–1948 he was the Scientific Secretary of the Institute.

On 26 December 1951, Boris Khvedelidze and his family (spouse and son) became victims of Stalin's repression. The family was deported to South Kazakhstan "for a rough violation of Soviet legality" (the reason behind was his uncles decision, after participation in the World War II, to stay in France and not to return to Soviet Union after the war!). All members of families of close relatives of "traitors" became subject to deportation from Georgia. Since Boris Khvedelidze was living with his mother in the apartment left by his father, they fall under this "human" rule of Stalinlaws.

From September 1952 until February 1954 Boris Khvedelidze was teaching mathematics in a zoo-veterinary professional school in a remote village of South Kazachstan. On December 9, 1953, the Supreme Court of Soviet Union in Moscow denounced the decision of deportation of the Khvedelidze family and they were allowed to repatriate. On February 22, 1954, Boris Khvedelidze returned to Tbilisi and was restored as a senior researcher at the Mathematical Institute.

In his hand-written autobiography Boris Khvedelidze recalls one episode of his deportation. The properties of all deported families were subject to obligatory confiscation. To prevent the worst B. Khvedelidze donated his father's rich library to the TSU (the rector at that time was Niko Ketskhoveli). Boris Khvedelidze was very thankfulto TSU and, in particular, to Niko Ketskhoveli that he got back his entire library after repatriation in 1954.

In 1956–1958 and from 1967 on until his last year Boris Khvedelidze was a professor of TSU. In 1958–1967 he hold one of three chairs in mathematics at the State Polytechnical Institute (now Technical University of Georgia).

In 1980–1993 Boris Khvedelidze held the Chair of Algebra and Geometry of Abkhazian State University (Sukhumi).

In 1957 Boris Khvedelidze defended his habilitation thesis "Linear discontinuous boundary value problems of function theory, singular integral equations and some of their applications", which was published in the same year in the journal "Trudy Tbilisskogo Matematicheskogo Instituta" (Proceedings of A. Razmadze Mathematical Institute), Vol. 23 (1956), pp. 3–158.

From 1954 until his last days Boris Khvedelidze worked at A. Razmadze Mathematical Institute first as a senior researcher and was elected in 1957 as Head of the Department of Function Theory and Functional Analysis.

The most important part of the scientific heritage of Boris Khvedelidze is, in our opinion, the theory of singular integral equations (SIEs) in Lebesgue spaces with exponential weight, where he obtained results similar to those in the theory of SIEs developed by Niko Muskhelishvili and his disciples in Hoelder classes with and without weight. This work, which was the core of his habilitation thesis in 1956, was one of the first, along with papers of S. Mikhlin, Israel Gohberg and Harold Widon, where methods of functional analysis were widely used in SIEs and its applications.The theorem on the boundedness of the Cauchy Singular Integral Operator in the Lebesgue spaces with exponential weight is until nowadays known as the "Khvedelidze Theorem".

In 1967 Boris Khvedelidze was elected a corresponding member of the Georgian Academy of Sciences and in 1983 he became a full member of the Academy. In the same year he got the distinction of "Honored Scientist".

From 1962 on, when the Georgian Mathematical Union was refunded, he became a vice-president of this institution for many years.

His disciples are Givi Khuskivadze (PhD in 1963), Vakhtang Paatashvili (PhD in 1964), Stefan Toklikishvili (PhD in 1968), Eteri Gordadze (PhD in 1969), Zoia Denisova (PhD in 1973), Elizaveta Ischenko (PhD in 1989).

I consider myself as a disciple of Boris Khvedelidze as well. He was my mentor during last years at the university, supervised my diploma work, send me to my PhD mentor Israel Gohberg to Chisineu and after my PhD in 1968 I worked in his department at A. Razmadze Mathematical Institute until he passed away in March 1993.

In conclusion it is proper to mention that the present short biography is based on the extended autobiography written by Boris Khvedelidze himself.

Roland Duduchava

Professor,
Head of the Department of Mathematical Physics of
A. Razmadze Mathematical Institute,
President of the Georgian Mathematical Union

List of Publications of B. Khvedelidze

(i) Monographs and Memoirs

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L. P. Castro, R. C. Guerra, and N. M. Tuan

ON INTEGRAL OPERATORS GENERATED BY THE FOURIER TRANSFORM AND A REFLECTION

Dedicated to the memory of Professor Boris Khvedelidze (1915–1993)
on the 100th anniversary of his birthday

Abstract. We present a detailed study of structural properties for certain algebraic operators generated by the Fourier transform and a reflection. First, we focus on the determination of the characteristic polynomials of such algebraic operators, which, e.g., exhibit structural differences when compared with those of the Fourier transform. Then, this leads us to the conditions that allow one to identify the spectrum, eigenfunctions, and the invertibility of this class of operators. A Parseval type identity is also obtained, as well as the solvability of integral equations generated by those operators. Moreover, new convolutions are generated and introduced for the operators under consideration.

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Key words and phrases. Characteristic polynomials, Fourier transform, reflection, algebraic integral operators, invertibility, spectrum, integral equation, Parseval identity, convolution.

რეზიუმე. ჩვენ წარმოგიდგენთ ფურიეს გარდაქმნითა და არეკვლის ოპერატორებით წარმოქნილი ზოგიერთი ალგებრული ოპერატორის სტრუქტურული თვისებების დეტალურ კვლევას. პირველ რიგში, ყურადღება გამახვილებულია ასეთი ალგებრული ოპერატორებისთვის მახასიათებელი პოლინომების განსაზღვრაზე, რომელიც დაგვანახებს სტრუქტურულ განსხვავებებს ფურიეს გარდაქმნასთან შედარებით. ამას მივყავართ პირობებთან, რომლებიც საშუალებას მოგვცემს მოვახდინოთ სპექტრის, საკუთრივი ფუნციებისა და ამ კლასში შებრუნებადი ოპერატორების იდენტიფიცირება. მიღებულია პარსევალის ტიპის იგივეობა და შესწავლილია ამ ოპერატორებით წარმოქმნილი ინტეგრალური განტოლებების ამოხსნადობა. განხილული ოპერატორებისთვის შემოტანილია ახალი ნახვევის ოპერატორის ცნება.

1. Introduction

In several types of mathematical applications it is useful to apply more than once the Fourier transformation (or its inverse) to the same object, as well as to use algebraic combinations of the Fourier transform. This is the case e.g. in wave diffraction problems which – although being initially modeled as boundary value problems – can be translated into single equations by applying operator theoretical methods and convenient operators upon the use of algebraic combinations of the Fourier transform (cf. [8–10]). Additionally, in such processes it is also useful to construct relations between convolution type operators [7], generated by the Fourier transform, and some simpler operators like the reflection operator; cf. [5, 6, 11, 21]. Some of the most known and studied classes of this type of operators are the Wiener–Hopf plus Hankel and Toeplitz plus Hankel operators.

It is also well-known that several of the most important integral transforms are involutions when considered in appropriate spaces. For instance, the Hankel transform J, the Cauchy singular integral operator S on a closed curve, and the Hartley transforms (typically denoted by H_1 and H_2, see [2–4, 17]) are involutions of order 2. Moreover, the Fourier transform F and the Hilbert transform $\mathcal{H}$ are involutions of order 4 (i.e. $\mathcal{H}^4 = I$, in this case simply because $\mathcal{H}$ is an anti-involution in the sense that $\mathcal{H}^2 = -I$).

Those involution operators possess several significant properties that are useful for solving problems which are somehow characterized by those operators, as well as several kinds of integral equations, and ordinary and partial differential equations with transformed argument (see [1, 15, 16, 18, 20, 22–26]).

Let $W : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the reflection operator defined by

$$(W\varphi)(x) := \varphi(-x),$$

and let now $\langle \cdot , \cdot \rangle_{L^2(\mathbb{R}^n)}$ denote the usual inner product in $L^2(\mathbb{R}^n)$. Moreover, let F denote the Fourier integral operator given by

$$(Ff)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} e^{-i\langle x,y\rangle} f(y)\, dy.$$

In view of the above-mentioned interest, in the present work we propose a detailed study of some of the fundamental properties of the following operator, generated by the operators I (identity operator), F and W:

$$T := aI + bF + cW : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \tag{1.1}$$

where $a, b, c \in \mathbb{C}$. In very general terms, we can consider the operator T as a Fourier integral operator with reflection which allows to consider similar operators to the Cauchy integral operator with reflection (see [12–14, 19] and the references therein). Anyway, it is also well-known that $F^2 = W$. In this paper, the operator T, together with its properties, can be seen as a starting point to further studies of the Fourier integral operators with more general shifts that will be addressed in the forthcoming papers.

The paper is organized as follows. In the next section, we will justify that T is an algebraic operator and we will deduce their characteristic polynomials for distinct cases of the parameters a, b and c. Then, the conditions that allow to identify the spectrum, eigenfunctions, and the invertibility of the operator are obtained. Moreover, Parseval type identities are derived, and the solvability of integral equations generated by those operators is described. In addition, new operations for the operators under consideration are introduced such that they satisfy the corresponding property of the classical convolution.

2. Characteristic Polynomials

In order to have some global view on corresponding linear operators, we start by recalling the concept of algebraic operators.

An operator L defined on the linear space X is said to be algebraic if there exists a non-zero polynomial $P(t)$, with variable t and coefficients in the complex field $\mathbb{C}$, such that $P(L) = 0$. Moreover, the algebraic operator L is said be of order N if $P(L) = 0$ for a polynomial $P(t)$ of degree N, and $Q(L) \neq 0$ for any polynomial Q of degree less than N. In such a case, P is said to be the characteristic polynomial of L (and its roots are called the characteristic roots of L). As an example, for the operators J, S, H_1, H_2 and $\mathcal{H}$, mentioned in the previous section, we may directly identify their characteristic polynomials in the following corresponding way:

$$P_J(t) = t^2 - 1; \quad P_S(t) = t^2 - 1;$$
$$P_{H_1}(t) = t^2 - 1; \quad P_{H_2}(t) = t^2 - 1; \quad P_{\mathcal{H}}(t) = t^2 + 1.$$

As above mentioned, it is well-known that the operator F is an involution of order 4 (thus $F^4 = I$, where I is the identity operator in $L^2(\mathbb{R}^n)$). In other words, F is an algebraic operator which has a characteristic polynomial given by $P_F(t) = t^4 - 1$. Such polynomial has obviously the following four characteristic roots: 1, $-i$, -1, i.

We will consider the following four projectors correspondingly generated with the help of F:

$$P_0 = \frac{1}{4}\,(I + F + F^2 + F^3),$$
$$P_1 = \frac{1}{4}\,(I + iF - F^2 - iF^3),$$
$$P_2 = \frac{1}{4}\,(I - F + F^2 - F^3),$$
$$P_3 = \frac{1}{4}\,(I - iF - F^2 + iF^3),$$

and that satisfy the identities

$$\begin{cases} P_j P_k = \delta_{jk} P_k & \text{for } \ j,k = 0,1,2,3, \\ P_0 + P_1 + P_2 + P_3 = I, \\ F = P_0 - iP_1 - P_2 + iP_3, \end{cases} \tag{2.1}$$

where

$$\delta_{jk} = \begin{cases} 0, & \text{if } \ j \neq k, \\ 1, & \text{if } \ j = k. \end{cases}$$

Moreover, we have

$$F^2 = P_0 - P_1 + P_2 - P_3, \tag{2.2}$$

$$F^4 = P_0 + P_1 + P_2 + P_3 = I. \tag{2.3}$$

It is also clear that

$$\alpha P_0 + \beta P_1 + \gamma P_2 + \delta P_3 = 0$$

if and only if

$$\alpha = \beta = \gamma = \delta = 0.$$

Having in mind this property, in the sequel, for denoting the operator

$$A = \alpha P_0 + \beta P_1 + \gamma P_2 + \delta P_3,$$

we will use the notation $(\alpha;\beta;\gamma;\delta) = A$.

Obviously, $A^n = (\alpha^n;\beta^n;\gamma^n;\delta^n)$, for every $n \in \mathbb{N}$, where we admit that $A^0 = I$.

Theorem 2.1. *Let us consider the operator*

$$T = aI + bF + cW, \ \ a,b,c \in \mathbb{C}. \tag{2.4}$$

The characteristic polynomial of this T is:

(i)

$$P_T(t) = t^2 - 2at + (a^2 - c^2) \tag{2.5}$$

if and only if

$$b = 0 \ \ \textit{and} \ \ c \neq 0; \tag{2.6}$$

(ii)

$$\begin{aligned} P_T(t) = t^3 - \big[(3a+c) + ib\big]t^2 + \big[3(a^2-c^2) + 2a(c+ib)\big]t \\ + \big[-a^3 - ia^2b - a^2c - 3b^2c + 3ac^2 + ibc^2 + c^3\big] \end{aligned} \tag{2.7}$$

if and only if

$$bc \neq 0 \ \ \textit{and} \ \ \left(c = \frac{b}{2}\,(1-i) \ \ \textit{or} \ \ c = -\frac{b}{2}\,(1+i) \right); \tag{2.8}$$

(iii)

$$P_T(t) = t^3 + \big[-(3a+c) + ib\big]t^2 + \big[3(a^2-c^2) + 2a(c-ib)\big]t + \big[-a^3 + ia^2b - a^2c - 3b^2c + 3ac^2 - ibc^2 + c^3\big] \tag{2.9}$$

if and only if

$$bc \neq 0 \ \ and \ \ \left(c = \frac{b}{2}(1+i) \ \ or \ \ c = -\frac{b}{2}(1-i)\right); \tag{2.10}$$

(iv)

$$P_T(t) = t^4 - 4at^3 + (6a^2 - 2c^2)t^2 + (-4a^3 - 4b^2c + 4ac^2)t + (a^2-c^2)^2 + b^2(4ac - b^2) \tag{2.11}$$

if and only if

$$\begin{cases} c \neq \frac{b}{2}(1-i), \\ c \neq -\frac{b}{2}(1+i), \\ c \neq \frac{b}{2}(1+i), \\ c \neq -\frac{b}{2}(1-i) \end{cases} \tag{2.12}$$

and $b \neq 0$.

Proof. We can write the operator T in the following form:

$$\begin{aligned} T &= a(P_0 + P_1 + P_2 + P_3) + b(P_0 - iP_1 - P_2 + iP_3) \\ &\qquad + c(P_0 - P1 + P_2 - P_3) \\ &= (a+c+b)P_0 + (a-c-ib)P_1 \\ &\qquad + (a+c-b)P_2 + (a-c+ib)P_3 \\ &= (a+c+b; a-c-ib; a+c-b; a-c+ib). \end{aligned} \tag{2.13}$$

In order to determine the characteristic polynomial of the operator T, for each one of the cases, we may begin by considering a polynomial of order 2, that is, $P_T(t) = t^2 + mt + n$. In fact, a polynomial of order 1 is the characteristic polynomial of the operator T if and only if $b = 0$ and $c = 0$, but in this case, we obtain the trivial operator $T = aI$. That $P_T(t)$ is the characteristic polynomial of T if and only if $P_T(T) = 0$ and if there does not exist any polynomial Q with $\deg(Q) < 2$ such that $Q(T) = 0$.

Moreover, the condition $P_T(T) = 0$ is equivalent to

$$\begin{cases} (a+c+b)^2 + m(a+c+b) + n = 0, \\ (a-c-ib)^2 + m(a-c-ib) + n = 0, \\ (a+c-b)^2 + m(a+c-b) + n = 0, \\ (a-c+ib)^2 + m(a-c+ib) + n = 0. \end{cases}$$

The solution of this system is $b = 0$ and $c = 0$ (but in this case, we obtain the trivial operator $T = aI$) or that

$$\begin{cases} b = 0, \\ c \neq 0, \\ m = -2a, \\ n = a^2 - c^2. \end{cases}$$

So, if $b = 0$ and $c \neq 0$, then $P_T(t) = t^2 - 2at + a^2 - c^2$. Indeed, by using the operator T written in the above form (2.13), it is possible to verify that $P_T(T) = 0$:

$$\begin{aligned} &T^2 - 2aT + (a^2 - c^2)I \\ &= \big((a+c)^2; (a-c)^2; (a+c)^2; (a-c)^2\big) - 2a(a+c; a-c; a+c; a-c) \\ &\qquad + (a^2 - c^2)(1;1;1;1) = (0;0;0;0). \end{aligned}$$

Now, we will prove that there does not exist any polynomial Q with $\deg(Q) < 2$ such that $Q(T) = 0$.

Suppose that there exists a polynomial Q, defined by $Q(t) = t + m$, that satisfies $Q(T) = 0$. In this case, we would have the following system of equations:

$$\begin{cases} (a+c) + m = 0, \\ (a-c) + m = 0, \end{cases}$$

which is equivalent to $c = 0$, but this is not the case under the conditions imposed before.

Conversely, assume that $P_T(t) = t^2 - 2at + (a^2 - c^2)$ is the characteristic polynomial of T. Thus, $P_T(T) = 0$, which is equivalent to

$$\begin{aligned} 0 = T^2 - 2aT + (a^2 - c^2)I \\ &= \big((a+c)^2; (a-c)^2; (a+c)^2; (a-c)^2\big) \\ &\quad - 2a(a+c; a-c; a+c; a-c) + (a^2 - c^2)(1;1;1;1). \end{aligned}$$

This implies that $b = 0$ and $c = 0$ (which is the case of the trivial operator) or that $b = 0$. So, case (i) is proved.

To obtain the characteristic polynomial for the other cases, we have to consider polynomials with degree greater than 2. So, let us consider a polynomial $P_T(t) = t^3 + mt^2 + nt + p$ and repeat the same procedure. Thus, $P_T(T) = 0$ is equivalent to

$$\begin{cases} (a+c+b)^3 + m(a+c+b)^2 + n(a+c+b) + p = 0, \\ (a-c-ib)^3 + m(a-c-ib)^2 + n(a-c-ib) + p = 0, \\ (a+c-b)^3 + m(a+c-b)^2 + n(a+c-b) + p = 0, \\ (a-c+ib)^3 + m(a-c+ib)^2 + n(a-c+ib) + p = 0. \end{cases}$$

This system has as solutions $b = 0$ and $c = 0$ (in this case, we obtain the operator $T = aI$) or $b = 0$ and $c \neq 0$ (but for this case, the characteristic polynomial is of order 2 – case (i)) or

$$\begin{cases} b \neq 0, \\ c = \dfrac{b}{2}(1-i) \ \text{ or } \ c = -\dfrac{b}{2}(1+i), \\ m = -\big[(3a+c)+ib\big], \\ n = 3(a^2-c^2)+2a(c+ib), \\ p = -a^3 - ia^2b - a^2c - 3b^2c + 3ac^2 + ibc^2 + c^3 \end{cases}$$

or

$$\begin{cases} b \neq 0, \\ c = \dfrac{b}{2}(1+i) \ \text{ or } \ c = -\dfrac{b}{2}(1-i), \\ m = \big[-(3a+c)+ib\big], \\ n = 3(a^2-c^2)+2a(c-ib), \\ p = -a^3 + ia^2b - a^2c - 3b^2c + 3ac^2 - ibc^2 + c^3. \end{cases}$$

So,

- if $c = \frac{b}{2}(1-i)$ or $c = -\frac{b}{2}(1+i)$, then
$$\begin{aligned} P_T(t) = t^3 &- \big[(3a+c)+ib\big]t^2 + \big[3(a^2-c^2)+2a(c+ib)\big]t \\ &+ \big[-a^3 - ia^2b - a^2c - 3b^2c + 3ac^2 + ibc^2 + c^3\big]; \end{aligned}$$
- If $c = \frac{b}{2}(1+i)$ or $c = -\frac{b}{2}(1-i)$, then
$$\begin{aligned} P_T(t) = t^3&\big[-(3a+c)+ib\big]t^2 + \big[3(a^2-c^2)+2a(c-ib)\big]t \\ &+ \big[-a^3 + ia^2b - a^2c - 3b^2c + 3ac^2 - ibc^2 + c^3\big]. \end{aligned}$$

If we consider the case $c = \frac{b}{2}(1-i)$, by using the operator T written in the above form (2.13), we can prove that $P_T(T) = 0$. Indeed,

$$\begin{aligned} T^3 &- \big[(3a+c)+ib\big]T^2 + \big[3(a^2-c^2)+2a(c+ib)\big]T \\ &+ \big[-a^3 - ia^2b - a^2c - 3b^2c + 3ac^2 + ibc^2 + c^3\big]I \\ &= \big([a+c+b]^3; [a-c-ib]^3; [a+c-b]^3; [a-c+ib]^3\big) \\ &- \big[(3a+c)+ib\big]\big([a+c+b]^2; [a-c-ib]^2; [a+c-b]^2; [a-c+ib]^2\big) \\ &+ \big[3(a^2-c^2)+2a(c+ib)\big](a+c+b; a-c-ib; a+c-b; a-c+ib) \\ &+ \big[-a^3 - ia^2b - a^2c - 3b^2c + 3ac^2 + ibc^2 + c^3\big](1;1;1;1) \\ &= (0;0;0;0). \end{aligned}$$

Now we will prove that there does not exist any polynomial G with $\deg(G) < 3$ such that $G(T) = 0$.

Suppose that there exists a polynomial G, defined by $G(t) = t^2 + mt + n$, that satisfies $G(T) = 0$. In this case, we would have the following system of equations:

$$\begin{cases}(a+c+b)^2+m(a+c+b)+n=0,\\(a-c-ib)^2+m(a-c-ib)+n=0,\\(a+c-b)^2+m(a+c-b)+n=0,\\(a-c+ib)^2+m(a-c+ib)+n=0.\end{cases}$$

For $c=\frac{b}{2}\,(1-i)$, we find that the second and third equations are equivalent. So, the last system is equivalent to

$$\begin{cases}(a+c+b)^2+m(a+c+b)+n=0,\\(a-c-ib)^2+m(a-c-ib)+n=0,\\(a-c+ib)^2+m(a-c+ib)+n=0,\end{cases}$$

which is equivalent to $b=0$. This is a contradiction under the initial conditions of the theorem. In this way, we can say that there does not exist a polynomial G such that $\deg(G)<3$ and this fulfills $G(T)=0$.

So, we can conclude that under these conditions,

$$\begin{aligned}P_T(t)=t^3-\big[(3a+c)+ib\big]t^2+\big[3(a^2-c^2)+2a(c+ib)\big]t\\+\big[-a^3-ia^2b-a^2c-3b^2c+3ac^2+ibc^2+c^3\big].\end{aligned}$$

Conversely, suppose that $P_T(t)$ is the characteristic polynomial of T. In this case, we have $P_T(T)=0$, which is equivalent to

$$\begin{aligned}0&=T^3-\big[(3a+c)+ib\big]T^2+\big[3(a^2-c^2)+2a(c+ib)\big]T\\&\quad+\big[-a^3-ia^2b-a^2c-3b^2c+3ac^2+ibc^2+c^3\big]\\&=\big([a+c+b]^3;[a-c-ib]^3;[a+c-b]^3;[a-c+ib]^3\big)\\&\quad-\big[(3a+c)+ib\big]\big([a+c+b]^2;[a-c-ib]^2;[a+c-b]^2;[a-c+ib]^2\big)\\&\quad+\big[3(a^2-c^2)+2a(c+ib)\big](a+c+b;a-c-ib;a+c-b;a-c+ib)\\&\quad+\big[-a^3-ia^2b-a^2c-3b^2c+3ac^2+ibc^2+c^3\big](1;1;1;1).\end{aligned}$$

This implies that $b=0$ (which is the case (i)), $c=\frac{b}{2}\,(1-i)$ or $c=-\frac{b}{2}\,(1+i)$.

The remaining conditions in (2.8) and (2.10) can be proved in a similar way.

If

$$\begin{cases}c\neq\dfrac{b}{2}\,(1-i),\\c\neq-\dfrac{b}{2}\,(1+i),\\c\neq\dfrac{b}{2}\,(1+i),\\c\neq-\dfrac{b}{2}\,(1-i),\end{cases}$$

then (2.7) and (2.9) are not anymore characteristic polynomials of T.

Additionally, if we consider a polynomial $P_T(t) = t^4 + mt^3 + nt^2 + pt + q$, such that $P_T(T) = 0$, we obtain the following system of equations:

$$\begin{cases} (a+c+b)^4 + m(a+c+b)^3 + n(a+c+b)^2 + p(a+c+b) + q = 0, \\ (a-c-ib)^4 + m(a-c-ib)^3 + n(a-c-ib)^2 + p(a+c+b) + q = 0, \\ (a+c-b)^4 + m(a+c-b)^3 + n(a+c-b)^2 + p(a+c+b) + q = 0, \\ (a-c+ib)^4 + m(a-c+ib)^3 + n(a-c+ib)^2 + p(a+c+b) + q = 0. \end{cases}$$

This is equivalent to $b = c = 0$ (which is the trivial case $T = aI$) or to $b = 0$ and $c \neq 0$ (which is the case (i)) or to the cases (ii) and (iii) or

$$\begin{cases} b \neq 0, \\ m = -4a, \\ n = 6a^2 - 2c^2, \\ p = -4a^3 - 4b^2c + 4ac^2, \\ q = (a^2 - c^2) + b^2(4ac - b^2). \end{cases}$$

In this case, we can say that if $b \neq 0$ and if (2.12) holds, then

$$P_T(t) = t^4 - 4at^3 + (6a^2 - 2c^2)t^2 + (-4a^3 - 4b^2c + 4ac^2)t \\ + (a^2 - c^2)^2 + b^2(4ac - b^2).$$

On the other hand, with the use of operator T (written as in (2.13)), we can directly prove that $P_T(T) = 0$. Indeed,

$$\begin{aligned} T^4 - 4aT^3 + (6a^2 - 2c^2)T^2 + (-4a^3 - 4b^2c + 4ac^2)T \\ + \big[(a^2 - c^2)^2 + b^2(4ac - b^2)\big]I \\ = \big([a+c+b]^4; [a-c-ib]^4; [a+c-b]^4; [a-c+ib]^4\big) \\ - 4a\big([a+c+b]^3; [a-c-ib]^3; [a+c-b]^3; [a-c+ib]^3\big) \\ + (6a^2 - 2c^2)\big([a+c+b]^2; [a-c-ib]^2; [a+c-b]^2; [a-c+ib]^2\big) \\ + (-4a^3 - 4b^2c + 4ac^2)(a+c+b; a-c-ib; a+c-b; a-c+ib) \\ + \big[(a^2 - c^2)^2 + b^2(4ac - b^2)\big](1;1;1;1) = (0;0;0;0). \end{aligned}$$

Now, we will prove that there does not exist any polynomial G with $\deg(G) < 4$ that satisfies $G(T) = 0$ under these conditions. Towards this end, suppose that there exists a polynomial G, defined by $G(t) = t^3 + mt^2 + nt + p$, that satisfies $G(T) = 0$. In this case, we would have the following system of equations:

$$\begin{cases} (a+c+b)^3 + m(a+c+b)^2 + n(a+c+b) + p = 0, \\ (a-c-ib)^3 + m(a-c-ib)^2 + n(a-c-ib) + p = 0, \\ (a+c-b)^3 + m(a+c-b)^2 + n(a+c-b) + p = 0, \\ (a-c+ib)^3 + m(a-c+ib)^2 + n(a-c+ib) + p = 0, \end{cases}$$

which is equivalent to $b = 0$ or $c = \frac{b}{2}(1-i)$ or $c = -\frac{b}{2}(1+i)$ or $c = \frac{b}{2}(1+i)$ or $c = -\frac{b}{2}(1-i)$.

This is a contradiction under the conditions of part (iii) of the Theorem. In this way, we can say that there does not exist a polynomial G with $\deg(G) < 4$ that satisfies $G(T) = 0$.

So, we can conclude that under these conditions

$$\begin{aligned} P_T(t) = t^4 - 4at^3 + (6a^2 - 2c^2)t^2 + (-4a^3 - 4b^2c + 4ac^2)t \\ + (a^2 - c^2)^2 + b^2(4ac - b^2). \end{aligned}$$

Conversely, suppose that $P_T(t)$ is the characteristic polynomial of T. Consequently, we have $P_T(T) = 0$, which is equivalent to

$$\begin{aligned} 0 &= T^4 - 4aT^3 + (6a^2 - 2c^2)T^2 + (-4a^3 - 4b^2c + 4ac^2)T \\ &\quad + (a^2 - c^2)^2 + b^2(4ac - b^2) \\ &= \big([a+c+b]^4; [a-c-ib]^4; [a+c-b]^4; [a-c+ib]^4\big) \\ &\quad - 4a\big([a+c+b]^3; [a-c-ib]^3; [a+c-b]^3; [a-c+ib]^3\big) \\ &\quad + (6a^2 - 2c^2)\big([a+c+b]^2; [a-c-ib]^2; [a+c-b]^2; [a-c+ib]^2\big) \\ &\quad + (-4a^3 - 4b^2c + 4ac^2)(a+c+b; a-c-ib; a+c-b; a-c+ib) \\ &\quad + \big[(a^2 - c^2)^2 + b^2(4ac - b^2)\big](1; 1; 1; 1). \end{aligned}$$

This condition is universal, and hence this case is proved. □

3. Invertibility, Spectrum and Integral Equations

We will now investigate the operator T in view of invertibility, spectrum, convolutions and associated integral equations. This will be done in the next subsections, by separating different cases of the parameters a, b and c, due to their corresponding different nature. The case of $b = 0$ and $c \neq 0$ is here omitted simply because this is the easiest case (in the sense that for this case we even do not have an integral structure: T is just a combination of the reflection and the identity operators).

3.1. Case $b \neq 0$ and $c = \frac{b}{2}(1-i)$.

In this subsection we will concentrate on the properties of the operator $T = aI + bF + cW$, $a, b, c \in \mathbb{C}$, $b, c \neq 0$, in the special case of $c = \frac{b}{2}(1-i)$ (whose importance is justified by the results of Section 2).

If we consider the following characteristic polynomial:

$$\begin{aligned} P_T(t) = t^3 - \big[(3a+c) + ib\big]t^2 + \big[3a^2 + 2ib(a+c) + 2ac - (b^2 + c^2)\big]t \\ + \big[-a^3 - ia^2b + ab^2 + ib^3 - a^2c - 2iabc - b^2c + ac^2 - ibc^2 + c^3\big] \end{aligned}$$

and if $c := \frac{b}{2}\,(1-i)$, we obtain that this polynomial is equivalent to

$$P_T(t) = t^3 - \Big[3a + \frac{b}{2}\,(1+i)\Big]t^2 + \Big[3a^2 + ab(1+i) + \frac{3}{2}\,ib^2\Big]t + \Big[-a^3 - \frac{1}{2}\,a^2b(1+i) - \frac{3}{2}\,iab^2 - \frac{5}{4}\,b^3(1-i)\Big].$$

3.1.1. *Invertibility and spectrum.* We will now present a characterization for the invertibility and the spectrum of the present T.

Theorem 3.1. *The operator T (with $c = \frac{b}{2}\,(1-i)$) is an invertible operator if and only if*

$$a + \Big(\frac{3}{2} - \frac{i}{2}\Big)b \neq 0, \quad a - \Big(\frac{1}{2} + \frac{i}{2}\Big)b \neq 0 \ \text{ and } \ a - \Big(\frac{1}{2} - \frac{3i}{2}\Big)b \neq 0. \tag{3.1}$$

In this case, the inverse operator is defined by

$$T^{-1} = \frac{1}{a^3 + \frac{1}{2}\,a^2b(1+i) + \frac{3}{2}\,iab^2 + \frac{5}{4}\,b^3(1-i)} \times \Big[T^2 - \Big(3a + \frac{b}{2}\,(1+i)\Big)T + \Big(3a^2 + ab(1+i) + \frac{3}{2}\,ib^2\Big)I\Big]. \tag{3.2}$$

Proof. Suppose that the operator T is invertible. Choosing the Hermite functions φ_k, we have:

- for $|k| \equiv 0 \pmod 4$, $(T\varphi_k)(x) = (a + \frac{3}{2}\,b - \frac{i}{2}\,b)\varphi_k(x)$, which implies that $a + (\frac{3}{2} - \frac{i}{2})b \neq 0$;
- for $|k| \equiv 1, 2 \pmod 4$, $(T\varphi_k)(x) = (a - \frac{b}{2} - \frac{i}{2}b)\varphi_k(x)$. So, $a - (\frac{1}{2} + \frac{i}{2})b \neq 0$;
- for $|k| \equiv 3 \pmod 4$, $(T\varphi_k)(x) = (a - \frac{b}{2} + \frac{3i}{2}b)\varphi_k(x)$, which implies that $a - (\frac{1}{2} - \frac{3i}{2})b \neq 0$.

Summarizing, we have:

$$\big(T\varphi_k\big)(x) = \begin{cases} \Big(a + \Big(\frac{3}{2} - \frac{i}{2}\Big)b\Big)\varphi_k(x) & \text{if } |k| \equiv 0 \pmod 4, \\ \Big(a - \Big(\frac{1}{2} + \frac{i}{2}\Big)b\Big)\varphi_k(x) & \text{if } |k| \equiv 1, 2 \pmod 4, \\ \Big(a - \Big(\frac{1}{2} - \frac{3i}{2}\Big)b\Big)\varphi_k(x) & \text{if } |k| \equiv 3 \pmod 4. \end{cases} \tag{3.3}$$

Conversely, suppose that we have (3.1). This implies that

$$a^3 + \frac{1}{2}\,a^2b(1+i) + \frac{3}{2}\,iab^2 + \frac{5}{4}\,b^3(1-i) \neq 0.$$

Hence, it is possible to consider the operator defined in (3.2) and, by a straightforward computation, verify that this is, indeed, the inverse of T. □

Remark 3.2.

(1) It is not difficult to see that
$$t_1 := a + \Big(\frac{3}{2} - \frac{i}{2}\Big)b, \quad t_2 := a - \Big(\frac{1}{2} + \frac{i}{2}\Big)b, \quad t_3 := a - \Big(\frac{1}{2} - \frac{3i}{2}\Big)b$$
are the roots of the polynomial $P_T(t)$. Consequently, t_1, t_2, t_3 are the characteristic roots of $P_T(t)$.

(2) T is not a unitary operator, unless $b = 0$ and $a = e^{i\alpha}$, $\alpha \in \mathbb{R}$, which is a somehow trivial case and is not under the conditions we have here imposed to this operator.

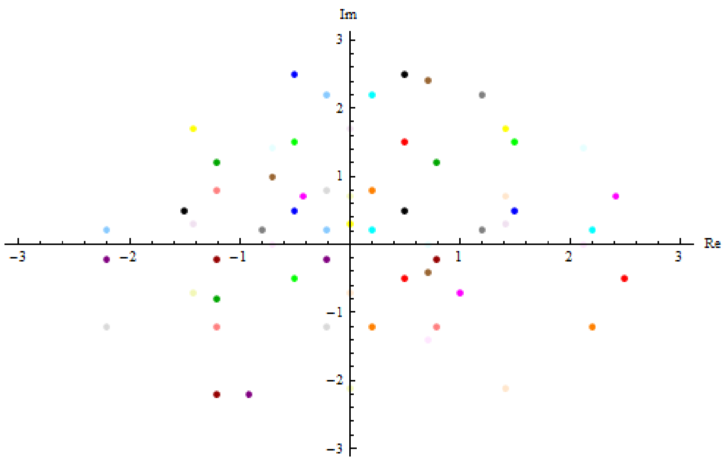

FIGURE 1. The spectrum of the operator T for different values of the parameters a and b.

Theorem 3.3. *The spectrum of the operator T is given by*
$$\sigma(T) = \Big\{a + \Big(\frac{3}{2} - \frac{i}{2}\Big)b, a - \Big(\frac{1}{2} + \frac{i}{2}\Big)b, a - \Big(\frac{1}{2} - \frac{3i}{2}\Big)b\Big\}$$
(*see Figure* 1).

Proof. For any $\lambda \in \mathbb{C}$, we have
$$\begin{aligned} t^3 - \Big[3a + \frac{b}{2}(1+i)\Big]t^2 + \Big[3a^2 + ab(1+i) + \frac{3}{2}ib^2\Big]t \\ + \Big[-a^3 - \frac{1}{2}a^2b(1+i) - \frac{3}{2}iab^2 - \frac{5}{4}b^3(1-i)\Big] \\ = (t-\lambda)\Big[t^2 + \Big(\lambda - 3a - \frac{b}{2}(1+i)\Big)t \\ + \Big(\lambda^2 - 3a\lambda - \frac{b}{2}(1+i) + 3a^2 + ab(1+i) + \frac{3}{2}ib^2\Big)\Big] + P_T(\lambda). \end{aligned}$$

Suppose that

$$\lambda \notin \left\{ a + \Big(\frac{3}{2} - \frac{i}{2}\Big)\, b, a - \Big(\frac{1}{2} + \frac{i}{2}\Big) b, a - \Big(\frac{1}{2} - \frac{3i}{2}\Big) b \right\}.$$

This implies that

$$\begin{aligned} P_T(\lambda) = \lambda^3 - \Big[3a + \frac{b}{2}\,(1+i)\Big]\lambda^2 + \Big[3a^2 + ab(1+i) + \frac{3}{2}\, ib^2\Big]\lambda \\ + \Big[- a^3 - \frac{1}{2}\, a^2 b(1+i) - \frac{3}{2}\, iab^2 - \frac{5}{4}\, b^3(1-i)\Big] \neq 0. \end{aligned}$$

Then the operator $T - \lambda I$ is invertible, and its inverse operator is defined by

$$\begin{aligned} (T - \lambda I)^{-1} = -\frac{1}{P_T(\lambda)} \Big[T^2 + \Big(\lambda - 3a - \frac{b}{2}\,(1+i)\Big) T \\ + \Big(\lambda^2 - 3a\lambda - \frac{b}{2}\,(1+i) + 3a^2 + ab(1+i) + \frac{3}{2}\, ib^2\Big) I \Big]. \end{aligned}$$

So, we have proved that if $T - \lambda I$ is not invertible, then $\lambda \in \sigma(T)$.

Conversely, if we choose $\lambda = t_1$, we obtain:

$$\begin{aligned} \Big[T - \Big(a + \Big(\frac{3}{2} - \frac{i}{2}\Big) b\Big) I\Big] \Big[T^2 + (-2a + b(1-i))T \\ + \Big(a^2 - \frac{ab}{2}\,(1-3i) + 2b^2 - \frac{b}{2}\,(1+i)\Big) I\Big] = -P_T(\lambda) I. \end{aligned}$$

As $\lambda = a + (\frac{3}{2} - \frac{i}{2})b$, then $P_T(\lambda) = 0$. So, if $T - (a + (\frac{3}{2} - \frac{i}{2})b)I$ is invertible, then

$$T^2 + \big(- 2a + b(1-i)\big)T + \Big(a^2 - \frac{ab}{2}\,(1-3i) + 2b^2 - \frac{b}{2}\,(1+i)\Big) I = 0,$$

which implies that $b = 0$ and this is a contradiction. So, $T - (a + (\frac{3}{2} - \frac{i}{2})b)I$ is not invertible.

The same procedure can be repeated for $\lambda = t_2, t_3$, in which cases we obtain the same desired conclusion. □

Thanks to the identity (3.3), we obtain three types of eigenfunctions of T, represented as follows:

$$\Phi_I(x) = \sum_{|k|=0 \pmod 4}^{K} \alpha_k \varphi_k(x), \quad k \in \mathbb{C}, \tag{3.4}$$

$$\Phi_{II}(x) = \sum_{|k|=1,2 \pmod 4}^{K} \alpha_k \varphi_k(x), \quad k \in \mathbb{C}, \tag{3.5}$$

$$\Phi_{III}(x) = \sum_{|k|=3 \pmod 4}^{K} \alpha_k \varphi_k(x), \quad k \in \mathbb{C}. \tag{3.6}$$

3.1.2. *Parseval type identity.*

Theorem 3.4. *A Parseval type identity for T is given by*

$$\langle Tf, Tg\rangle_{L^2(\mathbb{R}^n)} = \Big[\,|a|^2 + \frac{3}{2}\,|b|^2\Big]\langle f, g\rangle_{L^2(\mathbb{R}^n)} + 2\Re\{a\bar{b}\}\langle f, Fg\rangle_{L^2(\mathbb{R}^n)} + \Re\big\{b(1-i)\overline{a}\big\}\langle f, Wg\rangle_{L^2(\mathbb{R}^n)} + |b|^2\langle f, F^{-1}g\rangle_{L^2(\mathbb{R}^n)}, \quad (3.7)$$

for any $f, g \in L^2(\mathbb{R}^n)$.

Proof. For any $f, g \in L^2(\mathbb{R}^n)$, it is straightforward to verify the following identities:

$$\langle Wf, Wg\rangle_{L^2(\mathbb{R}^n)} = \langle f, g\rangle_{L^2(\mathbb{R}^n)}, \quad (3.8)$$

$$\langle f, Wg\rangle_{L^2(\mathbb{R}^n)} = \langle Wf, g\rangle_{L^2(\mathbb{R}^n)}. \quad (3.9)$$

If we have in mind (3.8)–(3.9) and as well that for any $f, g \in L^2(\mathbb{R}^n)$:

$$\begin{aligned} \langle Wf, Fg\rangle_{L^2(\mathbb{R}^n)} &= \langle f, F^{-1}g\rangle_{L^2(\mathbb{R}^n)}, \\ \langle Ff, Wg\rangle_{L^2(\mathbb{R}^n)} &= \langle f, F^{-1}g\rangle_{L^2(\mathbb{R}^n)}, \\ \langle Ff, Fg\rangle_{L^2(\mathbb{R}^n)} &= \langle f, g\rangle_{L^2(\mathbb{R}^n)}, \\ \langle Ff, g\rangle_{L^2(\mathbb{R}^n)} &= \langle f, Fg\rangle_{L^2(\mathbb{R}^n)}, \end{aligned} \quad (3.10)$$

then (3.7) directly appears by using (1.1). □

3.1.3. *Integral equations generated by T.* Now we will consider the operator equation, generated by the operator T (on $L^2(\mathbb{R}^n)$), of the following form

$$m\varphi + nT\varphi + pT^2\varphi = f, \quad (3.11)$$

where $m, n, p \in \mathbb{C}$ are given, $|m| + |n| + |p| \neq 0$, and f is predetermined.

As we proved previously, the polynomial $P_T(t)$ has the single roots $t_1 = a + (\frac{3}{2} - \frac{i}{2})b$, $t_2 = a - (\frac{1}{2} + \frac{i}{2})b$ and $t_3 = a - (\frac{1}{2} - \frac{3i}{2})b$. The projectors induced by T, in the sense of the Lagrange interpolation formula, are given by

$$P_1 = \frac{(T - t_2I)(T - t_3I)}{(t_1 - t_2)(t_1 - t_3)} = \frac{T^2 - (t_2 + t_3)T + t_2t_3}{(t_1 - t_2)(t_1 - t_3)}\,, \quad (3.12)$$

$$P_2 = \frac{(T - t_1I)(T - t_3I)}{(t_2 - t_1)(t_2 - t_3)} = \frac{T^2 - (t_1 + t_3)T + t_1t_3}{(t_2 - t_1)(t_2 - t_3)}\,, \quad (3.13)$$

$$P_3 = \frac{(T - t_1I)(T - t_2I)}{(t_3 - t_1)(t_3 - t_2)} = \frac{T^2 - (t_1 + t_2)T + t_1t_2}{(t_3 - t_1)(t_3 - t_2)}\,. \quad (3.14)$$

Then we have

$$P_jP_k = \delta_{jk}P_k, \quad T^\ell = t_1^\ell P_1 + t_2^\ell P_2 + t_3^\ell P_3, \quad (3.15)$$

for any $j, k = 1, 2, 3$, and $\ell = 0, 1, 2$. The equation (3.11) is equivalent to the equation

$$a_1P_1\varphi + a_2P_2\varphi + a_3P_3\varphi = f, \quad (3.16)$$

where $a_j = m + nt_j + pt_j^2$, $j = 1, 2, 3$.

Theorem 3.5.

(i) *The equation* (3.11) *has a unique solution for every f if and only if $a_1a_2a_3 \neq 0$. In this case, the solution of* (3.11) *is given by*

$$\varphi = a_1^{-1}P_1f + a_2^{-1}P_2f + a_3^{-1}P_3f. \tag{3.17}$$

(ii) *If $a_j = 0$, for some $j = 1, 2, 3$, then the equation* (3.11) *has a solution if and only if $P_jf = 0$. If this condition is satisfied, then the equation* (3.11) *has an infinite number of solutions given by*

$$\varphi = \sum_{\substack{j\leq 3\\ a_j\neq 0}} a_j^{-1}P_jf + z, \quad \textit{where} \quad z \in \ker\Big(\sum_{\substack{j\leq 3\\ a_j\neq 0}} P_j\Big). \tag{3.18}$$

Proof. Suppose that the equation (3.11) has a solution $\varphi \in L^2(\mathbb{R}^n)$. Applying P_j to both sides of the equation (3.16), we obtain a system of three equations:

$$a_jP_j\varphi = P_jf, \quad j = 1, 2, 3.$$

In this way, if $a_1a_2a_3 \neq 0$, then we have the following system of equations:

$$\begin{cases} P_1\varphi = a_1^{-1}P_1f, \\ P_2\varphi = a_2^{-1}P_2f, \\ P_3\varphi = a_3^{-1}P_3f. \end{cases} \tag{3.19}$$

Using the identity

$$P_1 + P_2 + P_3 = I,$$

we obtain (3.17). Conversely, we can verify that φ fulfills (3.16).

If $a_1a_2a_3 = 0$, then $a_j = 0$, for some $j \in \{1, 2, 3\}$. Therefore, it follows that $P_jf = 0$. Then, we have

$$\sum_{\substack{j\leq 3\\ a_j\neq 0}} P_j\varphi = \sum_{\substack{j\leq 3\\ a_j\neq 0}} a_j^{-1}P_jf.$$

Using the fact that $P_jP_k = \delta_{jk}P_k$, we get

$$\Big(\sum_{\substack{j\leq 3\\ a_j\neq 0}} P_j\Big)\varphi = \Big(\sum_{\substack{j\leq 3\\ a_j\neq 0}} P_j\Big)\Big[\sum_{\substack{j\leq 3\\ a_j\neq 0}} a_j^{-1}P_jf\Big]$$

or, equivalently,

$$\Big(\sum_{\substack{j\leq 3\\ a_j\neq 0}} P_j\Big)\Big[\varphi - \sum_{\substack{j\leq 3\\ a_j\neq 0}} a_j^{-1}P_jf\Big] = 0.$$

Therefore, we can obtain the solution (3.18).

Conversely, we can verify that φ fulfills (3.16). As the Hermite functions are the eigenfunctions of T, we can say that the cardinality of all functions φ in (3.18) is infinite. □

3.1.4. *Convolution.* In this subsection we will focus on obtaining a new convolution $\overset{T}{*}$ for the operator T. We will perform it for the case $b \neq 0$ and $c = \frac{b}{2}(1-i)$, although the same procedure can be implemented for other cases of the parameters.

This means that we are identifying the operations that have a correspondent multiplication property for the operator T as the usual convolution has for the Fourier transform $(Tf)(Tg) = T(f \overset{T}{*} g)$.

Theorem 3.6. *For the operator $T = aI + bF + cW$, with $a, b, c \in \mathbb{C}$, $b \neq 0$ and $c = \frac{b}{2}(1-i)$, and $f, g \in L^2(\mathbb{R}^n)$, we have the following convolution:*

$$\begin{aligned} f \overset{T}{*} g = C\Big[&A_1 fg + A_2 (Wf)(Wg) + A_3 (f\, Wg + g\, Wf) \\ &+ A_4 (f\, Fg + g\, Ff) + A_5 \big((Wf)(F^{-1}g) + (F^{-1}f)(Wg)\big) \\ &+ A_6 \big((Wf)(Fg) + (Ff)(Wg)\big) + A_7 (g\, F^{-1}f + f\, F^{-1}g) \\ &+ A_8 ((Ff)(Fg)) + A_9 \big((F^{-1}f)(F^{-1}g)\big) + A_{10} (F(fg)) \\ &+ A_{11} \big(F(f\, Wg) + F(g\, Wf)\big) + A_{12} (F^{-1}(fg)) \\ &+ A_{13} \big(F(f\, Fg) + F(g\, Ff)\big) + A_{14} \big(F^{-1}(f\, Fg) + F^{-1}(g\, Ff)\big) \\ &+ A_{15} \big(F((Ff)(Wg)) + F((Wf)(Fg))\big) \\ &+ A_{16} \big(F^{-1}((Ff)(Wg)) + F^{-1}((Wf)(Fg))\big) \\ &+ A_{17} F((Ff)(Fg)) + A_{18} F^{-1}((Ff)(Fg))\Big], \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} C &= \frac{1}{a^3 + \frac{1}{2}\, a^2 b(1+i) + \frac{3}{2}\, iab^2 + \frac{5}{4}\, b^3 (1-i)}, \\ A_1 &= a^4 + \frac{a^3 b}{2}\,(1+i) + ia^2 b^2 + \frac{ab^3}{4}\,(1+i) + \frac{ib^4}{4}\,, \\ A_2 &= -\frac{a^2 b^2}{2}\,(1+i) - \frac{ab^3}{2}\,(1-i) + \frac{b^4}{2} - \frac{a^3 b}{2}\,(1-i), \\ A_3 &= \frac{a^3 b}{2}\,(1+i) + \frac{a^2 b^2}{2}\,(1+i) + \frac{ab^3}{2}\,(1+i) - \frac{ab^3}{4}\,(1-i), \\ A_4 &= a^3 b + \frac{a^2 b^2}{2}\,(1+i) + iab^3, \quad A_5 = -\frac{a^2 b^2}{2}\,(1-i) - \frac{ab^3}{2}\,, \\ A_6 &= \frac{a^2 b^2}{2}\,(1-i) + \frac{ab^3}{2} + \frac{b^4}{2}\,(1+i), \quad A_7 = i\,\frac{ab^3}{2} - \frac{b^4}{4}\,(1-i), \\ A_8 &= a^2 b^2 + \frac{ab^3}{2}\,(1+i) + ib^4, \quad A_9 = -\frac{ab^3}{2}\,(1-i) - \frac{b^4}{2}\,, \\ A_{10} &= -a^3 b - \frac{a^2 b^2}{2}\,(1+i) - \frac{b^4}{2}\,(1+i), \\ A_{11} &= -\frac{a^2 b^2}{2}\,(1-i) - \frac{ab^3}{2} - iab^3, \end{aligned}$$

$$A_{12} = i\,\frac{ab^3}{2} - \frac{b^4}{4}\,(1-i) + a^2b^2(1-i), \quad A_{13} = -a^2b^2 - \frac{ab^3}{2}\,(1+i),$$
$$A_{14} = ab^3(1-i), \quad A_{15} = -\frac{ab^3}{2}\,(1-i) - \frac{b^4}{2}\,,$$
$$A_{16} = -ib^4, \quad A_{17} = -ab^3 - \frac{b^4}{2}\,(1+i), \quad A_{18} = b^4(1-i).$$

Proof. Using the definition of T and a direct (but long) computation, we obtain the equivalence between (3.20) and

$$f \overset{T}{*} g = \frac{1}{a^3 + \frac{1}{2}\,a^2b(1+i) + \frac{3}{2}\,iab^2 + \frac{5}{4}\,b^3(1-i)}$$
$$\times \left[T^2 - \Big(3a + \frac{b}{2}\,(1+i)\Big)T + \Big(3a^2 + ab(1+i) + \frac{3}{2}\,ib^2\Big)I\right]\big[(Tf)(Tg)\big].$$

Thus, having in mind (3.2), we identify the last identity with

$$f \overset{T}{*} g = T^{-1}\big[(Tf)(Tg)\big],$$

which is equivalent to

$$(Tf)(Tg) = T\big(f \overset{T}{*} g\big),$$

as desired. □

3.2. **Case** $b \neq 0$ **and** $c \neq \pm\frac{b}{2}(1 \pm i)$**.** In the case of the operator $T := aI + bF + cW$, $a, b, c \in \mathbb{C}$, $b \neq 0$ and $c \neq \pm\frac{b}{2}\,(1 \pm i)$, whose characteristic polynomial is

$$P_T(t) = t^4 - 4at^3 + (6a^2 - 2c^2)t^2 + (-4a^3 - 4b^2c + 4ac^2)t$$
$$+ (a^2 - c^2)^2 + b^2(4ac - b^2),$$

we have the following properties.

3.2.1. *Invertibility and spectrum.*

Theorem 3.7. *T is an invertible operator if and only if*

$$a + c + b \neq 0, \quad a - c - ib \neq 0, \quad a + c - b \neq 0, \quad a - c + ib \neq 0. \tag{3.21}$$

In this case, the inverse operator is defined by the formula

$$T^{-1} = -\frac{1}{(a^2 - c^2)^2 + b^2(4ac - b^2)}$$
$$\times \left[T^3 - 4aT^2 + (6a^2 - 2c^2)T - (-4a^3 - 4b^2c + 4ac^2)I\right]. \tag{3.22}$$

Proof. Suppose that the operator T is invertible. Using the Hermite functions φ_k, we have:

$$(T\varphi_k)(x) = \begin{cases} (a+c+b)\varphi_k(x) & \text{if } |k| \equiv 0 \pmod 4, \\ (a-c-ib)\varphi_k(x) & \text{if } |k| \equiv 1 \pmod 4, \\ (a+c-b)\varphi_k(x) & \text{if } |k| \equiv 2 \pmod 4, \\ (a-c+ib)\varphi_k(x) & \text{if } |k| \equiv 3 \pmod 4. \end{cases} \tag{3.23}$$

Therefore,

- for $|k| \equiv 0 \pmod 4$, $(T\varphi_k)(x) = (a+b+c)\varphi_k(x)$, which implies that $a+c+b \neq 0$;
- for $|k| \equiv 1 \pmod 4$, $(T\varphi_k)(x) = (a-ib-c)\varphi_k(x)$, which implies that $a-c-ib \neq 0$;
- for $|k| \equiv 2 \pmod 4$, $(T\varphi_k)(x) = (a-b+c)\varphi_k(x)$, which implies that $a+c-b \neq 0$;
- for $|k| \equiv 3 \pmod 4$, $(T\varphi_k)(x) = (a+ib-c)\varphi_k(x)$, which implies that $a-c+ib \neq 0$.

Conversely, suppose that (3.21) holds. So,

$$(a^2-c^2)^2 + b^2(4ac-b^2) \neq 0.$$

Hence, it is easy to verify that the operator defined in (3.22) is the inverse of the operator T. □

Remark 3.8.

(1) The characteristic roots of the polynomial $P_T(t)$ are

$$t_1 = a+c+b, \quad t_2 = a-c-ib, \quad t_3 = a+c-b, \quad t_4 = a-c+ib.$$

(2) T is not a unitary operator, unless $a=0$, $b=e^{i\beta}$, $c=0$, $\beta \in \mathbb{R}$, (which is the operator $T = bF$, with $b \in \mathbb{C} \setminus \{0\}$) or $a = e^{i\alpha}$, $b=0$, $c=0$ or $a=0$, $b=0$, $c=e^{i\gamma}$, $\alpha, \varphi \in \mathbb{R}$, which are not under the conditions here considered for this operator.

Theorem 3.9. *The spectrum of the operator T is defined by*

$$\sigma(T) = \big\{a+c+b, a-c-ib, a+c-b, a-c+ib\big\}.$$

Proof. For any $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} t^4 - 4at^3 + (6a^2-2c^2)t^2 + \big[-4a^3-4b^2c+4ac^2\big]t + (a^2-c^2)^2 + b^2(4ac-b^2) \\ = (t-\lambda)\Big[t^3 + (\lambda-4a)t^2 + (\lambda^2 - 4a\lambda + 6a^2 - 2c^2)t \\ + \big(\lambda^3 - 4a\lambda^2 + (6a^2-2c^2)\lambda - 4a^3 - 4b^2c + 4ac^2\big)\Big] + P_T(\lambda). \end{aligned}$$

If $\lambda \notin \{a+c+b, a-c-ib, a+c-b, a-c+ib\}$, then

$$\begin{aligned} P_T(\lambda) = \lambda^4 - 4a\lambda^3 + (6a^2-2c^2)\lambda^2 \\ + [-4a^3 - 4b^2c + 4ac^2]\lambda + (a^2-c^2)^2 + b^2(4ac-b^2) \neq 0. \end{aligned}$$

In this way, the operator $T - \lambda I$ is invertible, and its inverse operator is defined by the following formula:

$$\begin{aligned} (T-\lambda I)^{-1} = -\frac{1}{P_T(\lambda)}\Big[T^3 + (\lambda-4a)T^2 + (\lambda^2 - 4a\lambda + 6a^2 - 2c^2)T \\ + \big(\lambda^3 - 4a\lambda^2 + (6a^2-2c^2)\lambda - 4a^3 - 4b^2c + 4ac^2\big)I\Big]. \end{aligned}$$

In this way, we have proved that if $T - \lambda I$ is not invertible, then $\lambda \in \sigma(T)$.

Conversely, if we choose $\lambda = t_1$, we obtain:

$$\begin{aligned}&\big(T - (a+c+b)I\big)\Big[T^3 + (-3a+b+c)T^2\\ &+ (3a^2 - 2ab + b^2 - 2ac + 2bc - c^2)T + (-a^3 + a^2b - ab^2 + b^3 + 4ac\\ &+ a^2c - 2abc - b^2c - 3ac^2 + bc^2 - c^3)I\Big] = -P_T(\lambda)I.\end{aligned}$$

As $\lambda = a+c+b$, $P_T(\lambda) = 0$. So, if $T - (a+c+b)I$ is invertible, then

$$\begin{aligned}&T^3 + (-3a+b+c)T^2 + (3a^2 - 2ab + b^2 - 2ac + 2bc - c^2)T\\ &+ (-a^3 + a^2b - ab^2 + b^3 + 4ac + a^2c - 2abc - b^2c - 3ac^2 + bc^2 - c^3)I = 0,\end{aligned}$$

which implies that $a = 0$ and $b = 0$ or that $b = 0$ and $c = 0$, which is not under the conditions imposed for this operator. So, we reach to a contradiction. Hence, $T - (a - c - b(1+i))I$ is not invertible.

Arguing in the same way for $\lambda = t_2, t_3, t_4$, we obtain a very similar conclusion. □

Thanks to the identity (3.23), we obtain four types of eigenfunctions of T, represented as follows:

$$\Phi_I(x) = \sum_{|k|\equiv 0 \pmod 4}^{K} \alpha_k \varphi_k(x), \quad k \in \mathbb{C}, \tag{3.24}$$

$$\Phi_{II}(x) = \sum_{|k|\equiv 1 \pmod 4}^{K} \alpha_k \varphi_k(x), \quad k \in \mathbb{C}, \tag{3.25}$$

$$\Phi_{III}(x) = \sum_{|k|\equiv 2 \pmod 4}^{K} \alpha_k \varphi_k(x), \quad k \in \mathbb{C}, \tag{3.26}$$

$$\Phi_{IV}(x) = \sum_{|k|\equiv 3 \pmod 4}^{K} \alpha_k \varphi_k(x), \quad k \in \mathbb{C}. \tag{3.27}$$

3.2.2. *Parseval type identity.* In the present case, a Parseval type identity takes the following form.

Theorem 3.10. *In the present case, a Parseval type identity for T is given by*

$$\begin{aligned}\langle Tf, Tg\rangle_{L^2(\mathbb{R}^n)} &= \big[\,|a|^2 + |b|^2 + |c|^2\big]\langle f, g\rangle_{L^2(\mathbb{R}^n)} + 2\Re\{a\bar{b}\}\langle f, Fg\rangle_{L^2(\mathbb{R}^n)}\\ &+ 2\Re\{a\bar{c}\}\langle f, Wg\rangle_{L^2(\mathbb{R}^n)} + 2\Re\{b\bar{c}\}\langle f, F^{-1}g\rangle_{L^2(\mathbb{R}^n)}\end{aligned} \tag{3.28}$$

for any $f, g \in L^2(\mathbb{R}^n)$.

Proof. The formula (3.28) is a direct consequence of (1.1), (3.8), (3.9) and (3.10). □

3.2.3. *Integral equations generated by T.* As before, we will now consider in the present case the following operator equation generated by the operator T, on $L^2(\mathbb{R}^n)$,

$$m\varphi + nT\varphi + pT^2\varphi = f, \tag{3.29}$$

where $m, n, p \in \mathbb{C}$ are given, $|m| + |n| + |p| \neq 0$, and f is predetermined.

The polynomial $P_T(t)$ has the single roots: $t_1 = a+c+b$, $t_2 = a-c-ib$, $t_3 = a+c-b$, $t_4 = a-c+ib$. Using the Lagrange interpolation structure, we construct the projectors induced by T:

$$\begin{aligned} P_1 &= \frac{(T-t_2I)(T-t_3I)(T-t_4I)}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} \\ &= \frac{T^3-(t_2+t_3+t_4)T^2+(t_2t_3+t_2t_4+t_3t_4)T-t_2t_3t_4I}{(t_1-t_2)(t_1-t_3)(t_1-t_4)}, \end{aligned} \tag{3.30}$$

$$\begin{aligned} P_2 &= \frac{(T-t_1I)(T-t_3I)(T-t_4I)}{(t_2-t_1)(t_2-t_3)(t_2-t_4)} \\ &= \frac{T^3-(t_1+t_3+t_4)T^2+(t_1t_3+t_1t_4+t_3t_4)T-t_1t_3t_4I}{(t_2-t_1)(t_2-t_3)(t_1-t_4)}, \end{aligned} \tag{3.31}$$

$$\begin{aligned} P_3 &= \frac{(T-t_1I)(T-t_2I)(T-t_4I)}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} \\ &= \frac{T^3-(t_1+t_2+t_4)T^2+(t_1t_2+t_1t_4+t_2t_4)T-t_1t_2t_4I}{(t_3-t_1)(t_3-t_2)(t_3-t_4)}, \end{aligned} \tag{3.32}$$

$$\begin{aligned} P_4 &= \frac{(T-t_1I)(T-t_2I)(T-t_3I)}{(t_4-t_1)(t_4-t_2)(t_4-t_3)} \\ &= \frac{T^3-(t_1+t_2+t_3)T^2+(t_1t_2+t_1t_3+t_2t_3)T-t_1t_2t_3I}{(t_4-t_1)(t_4-t_2)(t_4-t_3)}. \end{aligned} \tag{3.33}$$

Then, we have

$$P_jP_k = \delta_{jk}P_k; \;\; T^\ell = t_1^\ell P_1 + t_2^\ell P_2 + t_3^\ell P_3 + t_4^\ell P_4, \tag{3.34}$$

for any $j, k = 1, 2, 3, 4$, and $\ell = 0, 1, 2$. The equation (3.29) is equivalent to the equation

$$a_1P_1\varphi + a_2P_2\varphi + a_3P_3\varphi + a_4P_4\varphi = f, \tag{3.35}$$

where $a_j = m + nt_j + pt_j^2$, $j = 1, 2, 3, 4$.

Theorem 3.11.

(i) *Equation* (3.29) *has a unique solution for every f if and only if $a_1a_2a_3a_4 \neq 0$. In this case, the solution is given by*

$$\varphi = a_1^{-1}P_1f + a_2^{-1}P_2f + a_3^{-1}P_3f + a_4^{-1}P_4f. \tag{3.36}$$

(ii) *If $a_j = 0$, for some $j = 1, 2, 3, 4$, then the equation* (3.11) *has a solution if and only if $P_jf = 0$. If we have this, then the equation*

(3.11) *has an infinite number of solutions given by*

$$\varphi = \sum_{\substack{j\le 4\\ a_j\neq 0}} a_j^{-1} P_j f + z, \quad \text{where } \ z \in \ker\Big(\sum_{\substack{j\le 4\\ a_j\neq 0}} P_j\Big). \tag{3.37}$$

Proof. Suppose that the equation (3.29) has a solution $\varphi \in L^2(\mathbb{R}^n)$. Applying P_j to both sides of the equation (3.35), we obtain the system of four equations: $a_j P_j \varphi = P_j f,\ j = 1, 2, 3, 4$.

If $a_1 a_2 a_3 a_4 \neq 0$, then we have the following system of equations:

$$\begin{cases} P_1\varphi = a_1^{-1} P_1 f, \\ P_2\varphi = a_2^{-1} P_2 f, \\ P_3\varphi = a_3^{-1} P_3 f, \\ P_4\varphi = a_4^{-1} P_4 f. \end{cases} \tag{3.38}$$

Using the identity

$$P_1 + P_2 + P_3 + P_4 = I,$$

we obtain (3.36). Conversely, we can verify that φ fulfills (3.35).

If $a_1 a_2 a_3 a_4 = 0$, then $a_j = 0$ for some $j \in \{1, 2, 3, 4\}$. It follows that $P_j f = 0$. Then, we have

$$\sum_{\substack{j\le 4\\ a_j\neq 0}} P_j \varphi = \sum_{\substack{j\le 4\\ a_j\neq 0}} a_j^{-1} P_j f.$$

Using $P_j P_k = \delta_{jk} P_k$, we obtain

$$\Big(\sum_{\substack{j\le 4\\ a_j\neq 0}} P_j\Big)\varphi = \Big(\sum_{\substack{j\le 4\\ a_j\neq 0}} P_j\Big)\Big[\sum_{\substack{j\le 4\\ a_j\neq 0}} a_j^{-1} P_j f\Big].$$

Equivalently,

$$\Big(\sum_{\substack{j\le 4\\ a_j\neq 0}} P_j\Big)\Big[\varphi - \sum_{\substack{j\le 4\\ a_j\neq 0}} a_j^{-1} P_j f\Big] = 0.$$

So, we obtain the solution (3.37).

Conversely, we can verify that φ fulfills (3.35). As the Hermite functions are the eigenfunctions of T, we can say that the cardinality of all functions φ in (3.37) is infinite. □

3.2.4. *Convolution.* In this subsection we will present a new convolution $\overset{T}{*}$ for the operator T. We will perform it for the case $b \neq 0$ and $c \neq \pm\frac{b}{2}(1 \pm i)$. This means that we are identifying the operations that have a correspondent multiplication property for the operator T as the usual convolution has for the Fourier transform $(Tf)(Tg) = T(f \overset{T}{*} g)$.

Theorem 3.12. *For the operator $T = aI + bF + cW$, with $a, b, c \in \mathbb{C}$, $b \neq 0$ and $c \neq \pm\frac{b}{2}(1 \pm i)$, and $f, g \in L^2(\mathbb{R}^n)$, we have the following convolution:*

$$\begin{aligned} f \overset{T}{*} g = C\Big[& A_1 fg + A_2(Wf)(Wg) + A_3(f\,Wg + g\,Wf) \\ & + A_4(f\,Fg + g\,Ff) + A_5\big((Wf)(F^{-1}g) + (F^{-1}f)(Wg)\big) \\ & + A_6\big((Wf)(Fg) + (Ff)(Wg)\big) + A_7(g\,F^{-1}f + f\,F^{-1}g) \\ & + A_8((Ff)(Fg)) + A_9\big((F^{-1}f)(F^{-1}g)\big) \\ & + A_{10}(F(fg)) + A_{11}\big(F(f\,Wg) + F(g\,Wf)\big) + A_{12}(F^{-1}(fg)) \\ & + A_{13}\big(F(f\,Fg) + F(g\,Ff)\big) + A_{14}\big(F^{-1}(f\,Fg) + F^{-1}(g\,Ff)\big) \\ & + A_{15}\big(F\big((Ff)(Wg)\big) + F\big((Wf)(Fg)\big)\big) \\ & + A_{16}\big(F^{-1}\big((Ff)(Wg)\big) + F^{-1}\big((Wf)(Fg)\big)\big) \\ & + A_{17}F\big((Ff)(Fg)\big) + A_{18}F^{-1}\big((Ff)(Fg)\big)\Big], \end{aligned} \tag{3.39}$$

where

$$C = -\frac{1}{(a^2 - c^2)^2 + b^2(4ac - b^2)},$$
$$A_1 = 7a^5 - 7a^3c^2 + 7a^2b^2c - ab^2c^2 + a^2c^3 - c^5,$$
$$A_2 = 7a^3c^2 - 7ac^4 + 7b^2c^3 - a^3b^2 + a^4c - a^2c^3,$$
$$A_3 = 7a^4c - 7a^2c^3 + 7ab^2c^2 - a^2b^2c + a^3c^2 - ac^4,$$
$$A_4 = 7a^4b - 7a^2bc^2 + 7ab^3c, \quad A_5 = -a^2b^3 + a^3bc - abc^3,$$
$$A_6 = 7a^3bc - 7abc^3 + 7b^3c^2, \quad A_7 = -ab^3c + a^2bc^2 - bc^4,$$
$$A_8 = 7a^3b^2 - 7ab^2c^2 + 7b^4c, \quad A_9 = -ab^4 + a^2b^2c - b^2c^3,$$
$$A_{10} = a^4b + a^2bc^2 + b^3c - 2abc^3,$$
$$A_{11} = a^3bc + abc^3 + ab^3c - 2a^2bc^2,$$
$$A_{12} = a^2bc^2 + bc^4 + a^2b^3 - 2a^3bc, \quad A_{13} = a^3b^2 + ab^2c^2,$$
$$A_{14} = ab^4 - 2a^2b^2c, \quad A_{15} = a^2b^2c + b^2c^3,$$
$$A_{16} = b^4c - 2ab^2c^2, \quad A_{17} = a^2b^3 + b^3c^2, \quad A_{18} = b^5 - 2ab^3c.$$

Proof. Using the definition of T, by computation we obtain the equivalence between (3.39) and

$$\begin{aligned} f \overset{T}{*} g = & -\frac{1}{(a^2 - c^2)^2 + b^2(4ac - b^2)} \\ & \times \Big[T^3 - 4aT^2 + (6a^2 - 2c^2)T - (-4a^3 - 4b^2c + 4ac^2)I\Big]\big[(Tf)(Tg)\big]. \end{aligned}$$

Consequently, having in mind (3.22), we identify the last identity with

$$f \overset{T}{*} g = T^{-1}\big[(Tf)(Tg)\big],$$

which is equivalent to

$$(Tf)(Tg) = T\big(f \overset{T}{*} g\big),$$

as desired. □

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Authors' addresses:

L. P. Castro, R. C. Guerra

CIDMA – Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal.

E-mail: castro@ua.pt; ritaguerra@ua.pt

N. M. Tuan

Department of Mathematics, College of Education, Viet Nam National University, G7 Build., 144 Xuan Thuy Rd., Cau Giay Dist., Hanoi, Vietnam.

E-mail: tuannm@hus.edu.vn

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O. Chkadua, R. Duduchava, and D. Kapanadze

THE SCREEN TYPE DIRICHLET BOUNDARY VALUE PROBLEMS FOR ANISOTROPIC PSEUDO-MAXWELL'S EQUATIONS

Dedicated to Professor Boris Khvedelidze
on the occasion of his 100th birthday anniversary

Abstract. We investigate the Dirichlet type boundary value problems for anisotropic pseudo-Maxwell's equations in screen type problems. It is shown that the problems with tangent Dirichlet traces are well-posed in tangent Sobolev spaces and they can equivalently be reduced to the Dirichlet boundary value problems in usual Sobolev spaces. Using the potential method and theory if pseudeodifferential equations the uniqieness and existence theorems are proved. Asymptotic expansions of solutions near the screen edge are derived and used to establish the best Hölder smoothness for solutions.

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რეზიუმე. გამოკვლეულია ეკრანის ტიპის დირიხლეს სასაზღვრო ამოცანები ანიზოტროპული ფსევდო-მაქსველის განტოლებებისათვის. ნაჩვენებია, რომ ამოცანები დირიხლეს მხები კვალებით კორექტულად არიან დასმული მხებ სობოლევის სივრცეებში და ისინი ეკვივალენტურად დაიყვანება სობოლევის სივრცეებში დასმულ დირიხლეს ამოცანებზე. პოტენციალთა მეთოდისა და ფსევდო-დიფერენციალურ განტოლებათა მეთოდის გამოყენებით დამტკიცებულია არსებობისა და ერთადერთობის თეორემები. მიღებულია ამონახსნის ასიმპტოტური დაშლა ეკრანის საზღვრის მახლობლობაში, რომლის გამოყენებით დადგენილია ამონახსნის ჰელდერული უწყვეტობის საუკეთესო მაჩვენებელი.

1. Introduction

The study of boundary value problems in electromagnetism naturally leads us to the pseudo-Maxwell's equations with inherited tangent boundary conditions, which are in some sense non-standard for the system of elliptic equations, cf. the works of Buffa, Costabel, Christiansen, Dauge, Hazard, Lenoir, Mitrea, Nicaise and others. Due to the presence of tangent boundary conditions the usage of the potential methods for the investigation is complicated and the case of tangent Dirichlet type boundary condition is mostly studied by variational methods. Our goal is to investigate well-posedness of the screen type Dirichlet boundary value problems for pseudo-Maxwell's equations

$$A(D)\boldsymbol{U} := \operatorname{curl}\, \mu^{-1} \operatorname{curl}\, \boldsymbol{U} - s\varepsilon \operatorname{grad}\operatorname{div}(\varepsilon\, \boldsymbol{U}) - \omega^2 \varepsilon\, \boldsymbol{U} = 0 \ \text{ in } \ \mathbb{R}^3 \setminus \overline{\mathscr{C}} \quad (1.1)$$

with the help of the potential method and tools of pseudodifferential equations; here, $\mathscr{C} \subset \mathbb{R}^3$ denotes a screen which is a compact, orientable and non self-intersecting surface with the boundary.

The present investigation covers the anisotropic case when the coefficients in (1.1) are real-valued and constant matrices

$$\varepsilon = [\varepsilon_{jk}]_{3\times 3}, \quad \mu = [\mu_{jk}]_{3\times 3}, \tag{1.2}$$

which are symmetric and positive definite,

$$\langle \varepsilon\xi, \xi\rangle \geq c|\xi|^2, \quad \langle \mu\xi, \xi\rangle \geq d|\xi|^2, \ \ \forall\, \xi \in \mathbb{R}^3\,,$$

for some positive constants $c > 0$, $d > 0$, where

$$\langle \eta, \xi\rangle := \sum_{j=1}^{3} \eta_j \overline{\xi}_j, \ \ \eta, \xi \in \mathbb{C}^3,$$

s in (1.2) is a positive real number and the frequency parameter ω is assumed to be non-zero and complex valued, i.e., $\operatorname{Im}\omega \neq 0$.

2. Formulation of the Problems

From now on throughout the paper, unless stated otherwise, Ω denotes either a bounded $\Omega^+ \subset \mathbb{R}^3$ or an unbounded $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega}^+$ domain with the smooth, non-self-intersecting boundary $\mathscr{S} := \partial\Omega^+$ and $\boldsymbol{\nu}$ is the outer unit normal vector field to $\mathscr{S}$. Whenever necessary, we will specify the case.

By $\mathscr{C}$ we denote a subsurface of $\mathscr{S}$ (a screen) with a boundary $\partial\mathscr{C}$, which has two faces $\mathscr{C}^-$ and $\mathscr{C}^+$ and inherits the orientation from $\mathscr{S}$: $\mathscr{C}^+$ borders the inner domain Ω^+ and $\mathscr{C}^-$ borders the outer domain Ω^-. The unbounded domain with a screen configuration is denoted by

$$\mathbb{R}^3_{\mathscr{C}} := \mathbb{R}^3 \setminus \overline{\mathscr{C}}.$$

The space $\widetilde{\mathbb{H}}^r(\mathscr{C})$ comprises those functions $\varphi \in \mathbb{H}^r(\mathscr{S})$ which are supported in $\overline{\mathscr{C}}$ (functions with the "vanishing traces on the boundary"). For the detailed definitions and properties of these spaces we refer, e.g., to [13, 14, 16, 17]).

It is well-known that $\mathbb{H}^{r-1/2}(\mathscr{S})$ is a trace space for $\mathbb{H}^r(\Omega)$, provided that $r > 1/2$ and the corresponding trace operator is denoted by $\gamma_{\mathscr{S}}$. For the detailed definitions and properties of these spaces we refer, e.g., to [17].

Let us note that since $\mathscr{S}$ is smooth, the Dirichlet trace $\gamma_{\mathscr{S}}\boldsymbol{U}$, the tangential (Dirichlet) traces $\gamma_\tau\boldsymbol{U} = \gamma_{\mathscr{S}}(\boldsymbol{\nu}\times\boldsymbol{U})$ and $\gamma_\pi\boldsymbol{U} = \gamma_{\mathscr{C}}[(\boldsymbol{\nu}\times\boldsymbol{U})\times\boldsymbol{\nu}]$, the normal (Dirichlet) traces $\gamma_n\boldsymbol{U} = \langle\boldsymbol{\nu},\gamma_{\mathscr{S}}\boldsymbol{U}\rangle$ (i.e., $\gamma_n\boldsymbol{U} = \boldsymbol{\nu}\cdot\gamma_{\mathscr{S}}\boldsymbol{U}$) are well defined for the elements of $\mathbb{H}^1(\Omega)$ and $\gamma_\tau\boldsymbol{U},\gamma_\pi\boldsymbol{U}$ belong to the Sobolev space

$$\mathbb{H}^{\frac12}_t(\mathscr{S}) := \big\{\boldsymbol{U}\in(H^{\frac12}(\Gamma))^3:\ \boldsymbol{\nu}\cdot\boldsymbol{U}=0 \text{ on } \mathscr{S}\big\}$$

of tangential vector fields of order $1/2$ on the surface $\mathscr{S}$, while $\gamma_n\boldsymbol{U}\in H^{\frac12}(\mathscr{S})$ and $\gamma_{\mathscr{S}}\boldsymbol{U}\in\mathbb{H}^{\frac12}(\mathscr{S})$.

First, for the smooth functions, using the Gauß formula (integration by parts), we obtain the following Green's formulae:

$$\begin{aligned}(\boldsymbol{A}(D)\boldsymbol{U},\boldsymbol{V})_{\Omega^+} &= (\boldsymbol{\nu}\times\mu^{-1}\operatorname{curl}\boldsymbol{U},\boldsymbol{V}_\pi)_{\mathscr{S}} - (s\operatorname{div}(\varepsilon\boldsymbol{U}),\varepsilon\boldsymbol{\nu}\cdot\boldsymbol{V})_{\mathscr{S}}\\ &\quad+\boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U},\boldsymbol{V})_{\Omega^+} - \omega^2(\varepsilon\,\boldsymbol{U},\boldsymbol{V})_{\Omega^+},\end{aligned}\tag{2.1}$$

where $\boldsymbol{a}_{\varepsilon,\mu}$ is the natural bilinear differential form associated with Green's formulae (2.1)

$$\boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U},\boldsymbol{V})_\Omega := (\mu^{-1}\operatorname{curl}\boldsymbol{U},\operatorname{curl}\boldsymbol{V})_\Omega + s\,(\operatorname{div}(\varepsilon\boldsymbol{U}),\operatorname{div}(\varepsilon\boldsymbol{V}))_\Omega.\tag{2.2}$$

and $\boldsymbol{V}_\pi := \boldsymbol{V} - \langle\boldsymbol{\nu},\boldsymbol{V}\rangle\boldsymbol{\nu}$.

Note that Green's formula (2.1) allows us to define the Neumann's trace

$$\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U} := s\operatorname{div}(\varepsilon\boldsymbol{U})\varepsilon\boldsymbol{\nu} - \boldsymbol{\nu}\times\mu^{-1}\operatorname{curl}\boldsymbol{U},\tag{2.3}$$

for an arbitrary vector $\boldsymbol{U}\in\mathbb{H}^1(\Omega^+)$ provided that $\boldsymbol{A}(D)\boldsymbol{U}\in\mathbb{L}_2(\Omega^+)$ by the duality as follows

$$(\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U},\boldsymbol{V})_{\mathscr{S}} = \boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U},\boldsymbol{V})_{\Omega^+} - (\boldsymbol{A}(D)\boldsymbol{U},\boldsymbol{V})_{\Omega^+} - \omega^2(\varepsilon\,\boldsymbol{U},\boldsymbol{V})_{\Omega^+},\tag{2.4}$$

for all $\boldsymbol{V}\in\mathbb{H}^1(\Omega^+)$.

Theorem 2.1 (cf. [6])**.** *In* (1.1)*, the operator*

$$\boldsymbol{A}(D)\boldsymbol{U} := \operatorname{curl}\mu^{-1}\operatorname{curl}\boldsymbol{U} - s\,\varepsilon\operatorname{grad}\operatorname{div}(\varepsilon\boldsymbol{U}) - \omega^2\varepsilon\boldsymbol{U}$$

is elliptic, has a positive definite principal symbol and is self-adjoint.

Now we are ready to formulate the screen type Dirichlet boundary value problems (BVPs) for anisotropic pseudo-Maxwell's equations:

The Dirichlet boundary value problem D:

Find $\boldsymbol{U}\in\mathbb{H}^1(\mathbb{R}^3_{\mathscr{C}})$ such that

$$\begin{cases}\boldsymbol{A}(D)\boldsymbol{U}=0 & \text{in } \mathbb{R}^3_{\mathscr{C}},\\ \gamma^\pm(\boldsymbol{U}) = \mathbf{g}^\pm & \text{on } \mathscr{C},\end{cases}\tag{2.5}$$

where the given data $\mathbf{g}^\pm$ satisfy the conditions

$$\mathbf{g}^\pm\in\mathbb{H}^{1/2}(\mathscr{C}),\quad \mathbf{g}^+-\mathbf{g}^-\in r_{\mathscr{C}}\widetilde{\mathbb{H}}^{1/2}(\mathscr{C}).\tag{2.6}$$

The Dirichlet boundary value problem D_τ:

Find $\boldsymbol{U} \in \mathbb{H}^1_{\varepsilon\boldsymbol{\nu},0}(\mathbb{R}^3_{\mathscr{C}}) := \{\boldsymbol{U} \in \mathbb{H}^1(\mathbb{R}^3_{\mathscr{C}}) : \ \langle \varepsilon\boldsymbol{\nu}, \gamma_{\mathscr{C}^\pm}\boldsymbol{U}\rangle = 0 \text{ on } \mathscr{C}\}$ such that

$$\begin{cases} \boldsymbol{A}(D)\boldsymbol{U} = 0 & \text{in } \mathbb{R}^3_{\mathscr{C}}, \\ \gamma^\pm_\tau(\boldsymbol{U}) = \mathbf{f}^\pm & \text{on } \mathscr{C}, \end{cases} \tag{2.7}$$

where the given data $\mathbf{f}^\pm$ satisfy the conditions

$$\mathbf{f}^\pm \in \mathbb{H}^{1/2}_t(\mathscr{C}), \quad \mathbf{f}^+ - \mathbf{f}^- \in r_{\mathscr{C}}\widetilde{\mathbb{H}}^{1/2}_t(\mathscr{C}). \tag{2.8}$$

The Dirichlet boundary value problem D_π:

Find $\boldsymbol{U} \in \mathbb{H}^1_{\varepsilon\boldsymbol{\nu},0}(\mathbb{R}^3_{\mathscr{C}})$ such that

$$\begin{cases} \boldsymbol{A}(D)\boldsymbol{U} = 0 & \text{in } \mathbb{R}^3_{\mathscr{C}}, \\ \gamma^\pm_\pi(\boldsymbol{U}) = \mathbf{f}^\pm & \text{on } \mathscr{C}, \end{cases} \tag{2.9}$$

where the given data $\mathbf{f}^\pm$ satisfy the conditions

$$\mathbf{f}^\pm \in \mathbb{H}^{1/2}_t(\mathscr{C}), \quad \mathbf{f}^+ - \mathbf{f}^- \in r_{\mathscr{C}}\widetilde{\mathbb{H}}^{1/2}_t(\mathscr{C}). \tag{2.10}$$

Before we proceed it is worth to note that tangent boundary conditions in *Problems* D_τ and D_π are motivated by tight connections between boundary value problems for pseudo-Maxwell's equation and Maxwell's equation, where the boundary operators γ_τ and γ_π are natural, cf. [1–3,7] and others. However, since we consider smooth screens there is a connection between the traces γ_τ and γ_π established by the geometric operation $\boldsymbol{\nu} \times \cdot$ which is in fact a rotation operator and therefore from the uniqueness, existence and regularity results for the *Problem* D_τ we get the same results for the *Problem* D_π, and vice versa. Moreover, the uniqueness, existence and regularity results for these problems are an easy consequence of the results obtained for the *Problem D* below due to the following formula:

$$\mathbf{g} = (\boldsymbol{\nu} \times \mathbf{g}) \times \boldsymbol{\nu} + \frac{\langle \varepsilon\boldsymbol{\nu}, \mathbf{g}\rangle - \langle \varepsilon\boldsymbol{\nu}, (\boldsymbol{\nu} \times \mathbf{g}) \times \boldsymbol{\nu}\rangle}{\langle \varepsilon\boldsymbol{\nu}, \boldsymbol{\nu}\rangle}\,\boldsymbol{\nu}, \tag{2.11}$$

which holds true for the smooth vector field $\boldsymbol{\nu}$ and any $\mathbf{g} \in \mathbb{H}^{\frac{1}{2}}(\mathscr{S})$. Indeed, first, from the decomposition

$$\mathbf{g} = \boldsymbol{\nu} \times (\mathbf{g} \times \boldsymbol{\nu}) + \langle \boldsymbol{\nu}, \mathbf{g}\rangle\boldsymbol{\nu} \tag{2.12}$$

we have

$$\langle \varepsilon\boldsymbol{\nu}, \mathbf{g}\rangle = \langle \varepsilon\boldsymbol{\nu}, \boldsymbol{\nu} \times (\mathbf{g} \times \boldsymbol{\nu})\rangle + \langle \boldsymbol{\nu}, \mathbf{g}\rangle\langle \varepsilon\boldsymbol{\nu}, \boldsymbol{\nu}\rangle. \tag{2.13}$$

Now, by expressing $\langle \boldsymbol{\nu}, \mathbf{g}\rangle$ from (2.13) and inserting it into (2.12), we get (2.11). Further, if $\boldsymbol{U}$ is a unique solution of the *Problem D* with the boundary data

$$\mathbf{g}^\pm = \mathbf{f}^\pm \times \boldsymbol{\nu} - \frac{\langle \varepsilon\boldsymbol{\nu}, \mathbf{f}^\pm \times \boldsymbol{\nu}\rangle}{\langle \varepsilon\boldsymbol{\nu}, \boldsymbol{\nu}\rangle}\,\boldsymbol{\nu},$$

where $\mathbf{f}^{\pm}$ satisfy the conditions (2.8) (therefore $\mathbf{g}^{\pm}$ satisfy the conditions (2.6)), we need to show that $\boldsymbol{U}\in\mathbb{H}^1_{\varepsilon\boldsymbol{\nu},0}(\mathbb{R}^3_{\mathscr{C}})$ and $\gamma^{\pm}_{\tau}(\boldsymbol{U})=\mathbf{f}^{\pm}$. Clearly, we have

$$\langle\varepsilon\boldsymbol{\nu},\gamma_{\mathscr{C}^{\pm}}\boldsymbol{U}\rangle=\langle\varepsilon\boldsymbol{\nu},\mathbf{g}^{\pm}\rangle=\langle\varepsilon\boldsymbol{\nu},\mathbf{f}^{\pm}\times\boldsymbol{\nu}\rangle-\frac{\langle\varepsilon\boldsymbol{\nu},\mathbf{f}^{\pm}\times\boldsymbol{\nu}\rangle}{\langle\varepsilon\boldsymbol{\nu},\boldsymbol{\nu}\rangle}\langle\varepsilon\boldsymbol{\nu},\boldsymbol{\nu}\rangle=0$$

and

$$\gamma^{\pm}_{\tau}(\boldsymbol{U})=\boldsymbol{\nu}\times(\mathbf{f}^{\pm}\times\boldsymbol{\nu})-\frac{\langle\varepsilon\boldsymbol{\nu},\mathbf{f}^{\pm}\times\boldsymbol{\nu}\rangle}{\langle\varepsilon\boldsymbol{\nu},\boldsymbol{\nu}\rangle}(\boldsymbol{\nu}\times\boldsymbol{\nu})=\boldsymbol{\nu}\times(\mathbf{f}^{\pm}\times\boldsymbol{\nu})=\mathbf{f}^{\pm},$$

since $\mathbf{f}^{\pm}\in\mathbb{H}^{1/2}_t(\mathscr{C})$. Thus it is sufficient to study the *Problem D*.

3. Vector Potentials

The elliptic operator $\boldsymbol{A}(D)$ in (1.1) has the fundamental solution (cf. [13])

$$\mathbf{F}_{\boldsymbol{A}}(x):=\mathscr{F}^{-1}_{\xi\to x}\big[\mathscr{A}^{-1}(\xi)\big]=\mathscr{F}^{-1}_{\xi'\to x'}\bigg[\pm\frac{1}{2\pi}\int\limits_{\mathscr{L}}e^{-i\tau x_3}\mathscr{A}^{-1}(\xi',\tau)\,d\tau\bigg],$$
$$\xi'=(\xi_1,\xi_2)^{\top}\in\mathbb{R}^2,\ \ x=(x',x_3)\in\mathbb{R}^3,$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform and $\mathscr{A}(\xi)$ is the full symbol of the operator $\boldsymbol{A}(D)$:

$$\mathscr{A}(\xi):=\sigma_{\rm curl}(\xi)\mu^{-1}\sigma_{\rm curl}(\xi)+s\,\varepsilon\big[\xi_j\xi_k\big]_{3\times3}\varepsilon-\omega^2\varepsilon,\ \ \xi=(\xi_1,\xi_2,\xi_3)^{\top}\in\mathbb{R}^3,$$

where

$$\sigma_{\rm curl}(\xi):=\begin{bmatrix}0 & i\xi_3 & -i\xi_2\\ -i\xi_3 & 0 & i\xi_1\\ i\xi_2 & -i\xi_1 & 0\end{bmatrix}.$$

If $x_3<0$ (if, respectively, $x_3>0$), we fix the sign "+" (the sign "−") and a contour $\mathscr{L}$ in the upper (in the lower) complex half-plane, which encloses all roots of the polynomial equation $\det\mathscr{A}(\xi)=0$ in the corresponding half-planes.

Let us consider, respectively, the *single-layer* and *double-layer* potential operators

$$\mathbf{V}\boldsymbol{U}(x):=\oint\limits_{\mathscr{S}}\mathbf{F}_{\boldsymbol{A}}(x-\tau)\boldsymbol{U}(\tau)\,dS, \tag{3.1}$$

$$\mathbf{W}\boldsymbol{U}(x):=\oint\limits_{\mathscr{S}}\Big[\big(\boldsymbol{T}(D,\boldsymbol{\nu}(\tau))\mathbf{F}_{\boldsymbol{A}}\big)(x-\tau)\Big]^{\top}\boldsymbol{U}(\tau)\,dS,\ \ x\in\Omega, \tag{3.2}$$

related to pseudo-Maxwell's equations in (1.1). Obviously,

$$\boldsymbol{A}(D)\mathbf{V}\boldsymbol{U}(x)=\boldsymbol{A}(D)\mathbf{W}\boldsymbol{U}(x)=0,\ \ \forall\,\boldsymbol{U}\in\mathbb{L}_1(\mathscr{S}),\ \ \forall\,x\in\Omega. \tag{3.3}$$

For the next Propositions 3.1–3.4 and for their proofs we refer, e.g., to [9, 11, 15].

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a domain with the smooth boundary $\mathscr{S} = \partial\Omega$.*

The potential operators above map continuously the spaces

$$\begin{aligned} \mathbf{V} &: \mathbb{H}^r(\mathscr{S}) \to \mathbb{H}^{r+3/2}(\Omega), \\ \mathbf{W} &: \mathbb{H}^r(\mathscr{S}) \to \mathbb{H}^{r+1/2}(\Omega), \quad \forall\, r \in \mathbb{R}. \end{aligned} \tag{3.4}$$

The direct values $\mathbf{V_{-1}}$, $\mathbf{W_0}$ and $\mathbf{V_{+1}}$ of the potential operators $\mathbf{V}$, $\mathbf{W}$ and $\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{W}$ are pseudodifferential operators of order -1, 0 and 1, respectively, and map continuously the spaces

$$\begin{aligned} \mathbf{V_{-1}} &: \mathbb{H}^r(\mathscr{S}) \to \mathbb{H}^{r+1}(\mathscr{S}), \\ \mathbf{W_0} &: \mathbb{H}^r(\mathscr{S}) \to \mathbb{H}^{r}(\mathscr{S}), \\ \mathbf{V_{+1}} &: \mathbb{H}^r(\mathscr{S}) \to \mathbb{H}^{r-1}(\mathscr{S}), \quad \forall\, r \in \mathbb{R}. \end{aligned} \tag{3.5}$$

Proposition 3.2. *The potential operators on an open, compact, smooth surface $\mathscr{C} \subset \mathbb{R}^3$ have the following mapping properties:*

$$\begin{aligned} \mathbf{V} &: \widetilde{\mathbb{H}}^r(\mathscr{C}) \to \mathbb{H}^{r+3/2}(\mathbb{R}^3_{\mathscr{C}}), \\ \mathbf{W} &: \widetilde{\mathbb{H}}^r(\mathscr{C}) \to \mathbb{H}^{r+1/2}(\mathbb{R}^3_{\mathscr{C}}), \quad \forall\, r \in \mathbb{R}. \end{aligned} \tag{3.6}$$

The direct values $\mathbf{V_{-1}}$, $\mathbf{W_0}$ and $\mathbf{V_{+1}}$ of the potential operators $\mathbf{V}$, $\mathbf{W}$ and $\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{W}$ are pseudodifferential operators of order -1, 0 and 1, respectively, and have the following mapping properties:

$$\begin{aligned} \mathbf{V_{-1}} &: \widetilde{\mathbb{H}}^r(\mathscr{C}) \to \mathbb{H}^{r+1}(\mathscr{C}), \\ \mathbf{W_0} &: \widetilde{\mathbb{H}}^r(\mathscr{C}) \to \mathbb{H}^{r}(\mathscr{C}), \\ \mathbf{V_{+1}} &: \widetilde{\mathbb{H}}^r(\mathscr{C}) \to \mathbb{H}^{r-1}(\mathscr{C}), \quad \forall\, r \in \mathbb{R}. \end{aligned} \tag{3.7}$$

Proposition 3.3. *For the traces of potential operators we have the following Plemelji formulae:*

$$(\gamma_{\mathscr{S}^-}\mathbf{V}\boldsymbol{U})(x) = (\gamma_{\mathscr{S}^+}\mathbf{V}\boldsymbol{U})(x) = \mathbf{V_{-1}}\boldsymbol{U}(x), \tag{3.8}$$

$$(\gamma_{\mathscr{S}^\pm}\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{V}\boldsymbol{U})(x) = \mp\frac{1}{2}\boldsymbol{U}(x) + (\mathbf{W_0})^*(x,D)\boldsymbol{U}(x), \tag{3.9}$$

$$(\gamma_{\mathscr{S}^\pm}\mathbf{W}\boldsymbol{U})(x) = \pm\frac{1}{2}\boldsymbol{U}(x) + \mathbf{W_0}(x,D)\boldsymbol{U}(x), \tag{3.10}$$

$$(\gamma_{\mathscr{S}^-}\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{W}\boldsymbol{U})(x) = (\gamma_{\mathscr{S}^+}\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{W}\boldsymbol{U})(x) = \mathbf{V_{+1}}\boldsymbol{U}(x), \tag{3.11}$$

$$x \in \mathscr{S}, \quad \boldsymbol{U} \in \mathbb{H}^s_p(\mathscr{S}),$$

where $(\mathbf{W_0})^*(x,D)$ is the adjoint to the pseudodifferential operator $\mathbf{W_0}(x,D)$, the direct value of the potential operator $\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{V}$ on the boundary $\mathscr{S}$.

Proposition 3.4. *Let the boundary $\mathscr{S} = \partial\Omega^\pm$ be a compact smooth surface. Solutions to pseudo-Maxwell's equations with anisotropic coefficients ε and μ are represented as*

$$\boldsymbol{U}(x) = \pm\mathbf{W}(\gamma_{\mathscr{S}^\pm}\boldsymbol{U})(x) \mp \mathbf{V}(\gamma_{\mathscr{S}^\pm}\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U})(x), \quad x \in \Omega^\pm, \tag{3.12}$$

where $\gamma_{\mathscr{S}^\pm}\boldsymbol{T}(D,\boldsymbol{\nu})\Psi$ is Neumann's trace operator (see (2.3)*) and $\gamma_{\mathscr{S}^\pm}\Psi$ is Dirichlet's trace operator.*

If $\mathscr{C}\subset\mathbb{R}^3$ is an open compact smooth surface, then a solution to pseudo-Maxwell's equations with anisotropic coefficients ε and μ is represented as

$$\boldsymbol{U}(x)=\mathbf{W}([\boldsymbol{U}])(x)-\mathbf{V}([\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}])(x),\quad x\in\mathbb{R}^3_{\mathscr{C}},$$

$$[\boldsymbol{U}]:=\gamma_{\mathscr{C}^+}\boldsymbol{U}-\gamma_{\mathscr{C}^-}\boldsymbol{U},\quad \big[\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}\big]:=\gamma_{\mathscr{C}^+}\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}-\gamma_{\mathscr{C}^-}\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}.$$

As a consequence of the representation formula (3.12) we derive the following

Corollary 3.5. *For a complex valued frequency, a solution to the screen type boundary value problems for pseudo-Maxwell's equations decays at infinity exponentially, i.e.,*

$$\boldsymbol{U}(x)=\mathscr{O}\big(e^{-\alpha|x|}\big)\ \text{ as }\ |x|\to\infty\ \text{ provided that } \operatorname{Im}\omega\neq 0 \tag{3.13}$$

for some $\alpha>0$.

Theorem 3.6. *The Problem D has at most one solution.*

Proof. The proof is standard and uses Green's formula (cf. (2.1)–(2.4)). Let R be a sufficiently large positive number and $B(R)$ be the ball centered at the origin with radius R. Set $\Omega_R:=\mathbb{R}^3_{\mathscr{C}}\cap B(R)$. Note that the domain Ω_R has a piecewise smooth boundary S_R including both sides of $\mathscr{C}$.

Let $\boldsymbol{U}$ be a solution of the homogeneous problem. Then applying Green's formula for $\boldsymbol{V}=\boldsymbol{U}$ in Ω_R and passing to the limit $R\to\infty$, taking into account the estimate

$$\boldsymbol{U}(x)=\mathscr{O}\big(e^{-\alpha|x|}\big)\ \text{ as }\ |x|\to\infty\ \text{ for }\ \alpha>0,$$

we get

$$\boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U},\boldsymbol{U})_{\mathbb{R}^3}-\omega^2(\varepsilon\,\boldsymbol{U},\boldsymbol{U})_{\mathbb{R}^3}=0.$$

Since ε and μ^{-1} are positive definite constant matrices, $s>0$, and $\operatorname{Im}\omega\neq 0$, it follows that

$$(\varepsilon\,\boldsymbol{U},\boldsymbol{U})_{\mathbb{R}^3}=0,$$

and therefore $\boldsymbol{U}\equiv 0$ in $\mathbb{R}^3$. □

4. The Screen Type Dirichlet Problem

Let $\ell\mathbf{f}^+\in\mathbb{H}^{-1/2}(\mathscr{S})$ be a fixed extension of the function $\mathbf{f}^+\in\mathbb{H}^{-1/2}(\mathscr{C})$ up to the entire closed surface $\mathscr{S}$ and let $\ell_0(\mathbf{f}^+-\mathbf{f}^-)\in\mathbb{H}^{-1/2}_{\varepsilon\boldsymbol{\nu},0}(\mathscr{S})$ be an extension by zero of the function $\mathbf{f}^+-\mathbf{f}^-\in r_{\mathscr{C}}\widetilde{\mathbb{H}}^{-1/2}(\mathscr{C})$, cf. (2.6). Then any extension of the function $\mathbf{f}^+\in\mathbb{H}^{-1/2}(\mathscr{C})$ onto $\mathscr{S}$ is given as

$$\ell^+\mathbf{f}^+=\ell\mathbf{f}^+ +\ ,$$

where   is an arbitrary element of the space $\widetilde{\mathbb{H}}^{1/2}(\mathscr{C}^c)$, $\mathscr{C}^c:=\mathscr{S}\setminus\overline{\mathscr{C}}$. Therefore, any extension of the function $\mathbf{f}^-\in\mathbb{H}^{1/2}(\mathscr{C})$ onto $\mathscr{S}$ is defined as

$\ell^-\mathbf{f}^- := \ell^+\mathbf{f}^+ - \ell_0(\mathbf{f}^+ - \mathbf{f}^-) \in \mathbb{H}^{1/2}(\mathscr{S})$ and we have

$$\begin{aligned} r_{\mathscr{C}}\ell^-\mathbf{f}^- &= \mathbf{f}^+ - (\mathbf{f}^+ - \mathbf{f}^-) = \mathbf{f}^-, \\ r_{\mathscr{C}^c}\ell^+\mathbf{f}^+ &= r_{\mathscr{C}^c}\ell^-\mathbf{f}^-. \end{aligned} \tag{4.1}$$

We look for a solution of the screen type Dirichlet problem (2.5)-(2.6) in the form of single-layer potentials:

$$\boldsymbol{U}(x) = \begin{cases} \mathbf{V}(\mathbf{V_{-1}})^{-1}\ell^+\mathbf{f}^+(x), & x \in \Omega^+, \\ \mathbf{V}(\mathbf{V_{-1}})^{-1}\ell^-\mathbf{f}^-(x), & x \in \Omega^-. \end{cases} \tag{4.2}$$

Then $\boldsymbol{U}$ satisfies the basic differential equation (1.1) in the domains $\Omega^\pm$, as well as the boundary conditions on $\mathscr{C}$. From the ellipticity of the differential operator $\boldsymbol{A}(D)$ it follows that a generalized solution of the equation $\boldsymbol{A}(D)\boldsymbol{U} = 0$ is analytic in $\mathbb{R}^3_{\mathscr{C}}$ and following continuity conditions

$$\begin{cases} r_{\mathscr{C}^c}\gamma_{\mathscr{S}^+}\boldsymbol{U} - r_{\mathscr{C}^c}\gamma_{\mathscr{S}^-}\boldsymbol{U} = 0, \\ r_{\mathscr{C}^c}\gamma_{\mathscr{S}^+}(\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}) - r_{\mathscr{C}^c}\gamma_{\mathscr{S}^-}(\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}) = 0 \end{cases} \tag{4.3}$$

hold across the complementary surface $\mathscr{C}^c$. It is clear that by our construction the first equation in (4.3) is satisfied, cf. (3.8) and (4.1). From the second equation, by applying (3.9) and (4.1) we derive the equation

$$r_{\mathscr{C}^c}\Big(-\frac{1}{2}\mathbf{I} + (\mathbf{W_0})^*\Big)(\mathbf{V_{-1}})^{-1}\ell^+\mathbf{f}^+ - r_{\mathscr{C}^c}\Big(\frac{1}{2}\mathbf{I} + (\mathbf{W_0})^*\Big)(\mathbf{V_{-1}})^{-1}\ell^-\mathbf{f}^- = 0,$$

which is a strongly elliptic pseudo-differential equation on the surface $\mathscr{C}$

$$-r_{\mathscr{C}^c}(\mathbf{V_{-1}})^{-1} \quad = \mathbf{F}, \tag{4.4}$$

with the known right-hand side

$$\mathbf{F} := r_{\mathscr{C}^c}(\mathbf{V_{-1}})^{-1}\ell\mathbf{f}^+ - r_{\mathscr{C}^c}\Big(\frac{1}{2}\mathbf{I} + (\mathbf{W_0})^*\Big)(\mathbf{V_{-1}})^{-1}\ell_0(\mathbf{f}^+ - \mathbf{f}^-) \in \mathbb{H}^{\frac{1}{2}}(\mathscr{C}^c).$$

The principal homogeneous symbol $\sigma_{-(\mathbf{V_{-1}})^{-1}}(x,\xi)$ of the operator $-(\mathbf{V_{-1}})^{-1}$ is even with respect to ξ for all $x \in \overline{\mathscr{C}}$. This implies that the matrix

$$\big(\sigma_{-(\mathbf{V_{-1}})^{-1}}(x',0,0,-1)\big)^{-1}\sigma_{-(\mathbf{V_{-1}})^{-1}}(x',0,0,+1) = I, \;\; x' \in \partial\mathscr{C}, \tag{4.5}$$

has trivial eigenvalues. Using the equality (4.5) analogously to Lemma 3.12 from [6] we can prove the following theorem.

Theorem 4.1. *The operator*

$$-r_{\mathscr{C}^c}(\mathbf{V_{-1}})^{-1} : \widetilde{\mathbb{H}}^s(\mathscr{C}^c) \to \mathbb{H}^{s-1}(\mathscr{C}^c)$$

is invertible for all $0 < s < 1$.

From Theorem 4.1 the following existence result follows immediately.

Theorem 4.2. *The Problem D possesses a unique solution $\boldsymbol{U} \in \mathbb{H}^1(\mathbb{R}^3_{\mathscr{C}})$ which can be represented by single- layer potentials*

$$\boldsymbol{U} = \begin{cases} \mathbf{V}(\mathbf{V_{-1}})^{-1}(\ell\mathbf{f}^+ + \ ) & in \ \Omega^+, \\ \mathbf{V}(\mathbf{V_{-1}})^{-1}\big(\ell\mathbf{f}^+ + \quad - \ell_0(\mathbf{f}^+ - \mathbf{f}^-)\big) & in \ \Omega^-, \end{cases}$$

where is a solution of the uniquely solvable pseudo-differential equation (4.4).

Moreover, if the conditions

$$\mathbf{f}^{\pm} \in \mathbb{H}^{\frac{1}{2}+s}(\mathscr{C}), \quad \mathbf{f}^+ - \mathbf{f}^- \in r_{\mathscr{C}}\widetilde{\mathbb{H}}^{\frac{1}{2}+s}(\mathscr{C}).$$

for the data in (2.6) *hold, a solution $\boldsymbol{U}$ of the screen type Dirichlet problem belongs to the space $\mathbb{H}^{1+s}(\mathbb{R}^3_{\mathscr{C}})$ for all $s \in [0, 1/2)$.*

Finally, we characterize the asymptotic behaviour of solutions of the problem D-I near the screen edge $\partial\mathscr{C}$.

Let $x' \in \partial\mathscr{C}$ and $\Pi_{x'}$ be the plane passing through the point x' and orthogonal to the curve $\partial\mathscr{C}$. We introduce the polar coordinates (r, α), $z \geq 0$, $-\pi \leq \alpha \leq \pi$, on the plane $\Pi_{x'}$, with pole at the point x', such that the points $(r, \pm\pi)$ describe the faces of the screen $\mathscr{C}$ in the vicinity of the boundary $\partial\mathscr{C}$. We assume that the boundary data $\mathbf{f}^{\pm}$ are infinitely smooth. Applying the results obtained in [4, 5, 8, 12], near the screen edge we obtain the following asymptotic expansion:

$$\boldsymbol{U}(x', r, \alpha) = \mathbf{d}_0(x', \alpha) r^{\frac{1}{2}} + \sum_{k=1}^{M} \mathbf{d}_k(x', \alpha) r^{\frac{1}{2}+k} + \boldsymbol{U}_{M+1}(x', r, \alpha), \qquad (4.6)$$

where $\mathbf{d}_k \in (C^\infty(\partial\mathscr{C} \times [-\pi, \pi]))^3$, $k = 0, \dots, M$, $\boldsymbol{U}_{M+1} \in C^{M+1}(\overline{\Omega}^{\pm})$.

Note that from asymptotic expansion (4.6) it follows that $\boldsymbol{U}$ has $C^{\frac{1}{2}}$-smoothness in the tubular neighbourhood of the screen edge $\partial\mathscr{C}$.

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Authors' addresses:

O. Chkadua

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia.

2. Sokhumi State University, 9 Politkovskaya Str., Tbilisi 0186, Georgia.

E-mail: chkadua@rmi.ge

R. Duduchava, D. Kapanadze

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia.

E-mail: roldud@gmail.com; daka@rmi.ge

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R. Duduchava and T. Tsutsunava

INTEGRO-DIFFERENTIAL EQUATIONS OF PRANDTL TYPE IN THE BESSEL POTENTIAL SPACES

Dedicated to Professor Boris Khvedelidze, a mathematician, teacher and mentor, on the occasion of his 100th birthday anniversary

Abstract. The purpose of the present research is to investigate the Fredholm criteria for the Prandtl-type integro-differential equation with piecewise-continuous coefficients in the Bessel potential spaces $\mathbb{H}^s_p(\mathbb{R})$.

We reduce the integro-differential equations to an equivalent system of Mellin type convolution equation. Applying the recent results to Mellin convolution equations with meromorphic kernels in Bessel potential spaces obtained by V. Didenko & R. Duduchava [3] and R. Duduchava [9], the Fredholm criteria (and in some cases, the unique solvability criteria) of the above-mentioned integro-differential equations are obtained.

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რეზიუმე. წინამდებარე სტატიის მიზანია გამოიკვლიოს ფრედჰოლმურობის კრიტერიუმი პრანდტლის ტიპის ინტეგრო-დიფერენციალური განტოლებისათვის უბან-უბან უწყვეტი კოეფიციენტებით ბესელის პოტენციალთა სივრცეში $\mathbb{H}^s_p(\mathbb{R})$. გამოსაკვლევი ინტეგრო-დიფერენციალური განტოლებების სისტემა დაიყვანება ექვივალენტურ მელინის კონვოლუციის ტიპის სისტემაზე, რომლისთვისაც გამოყენება ვ. დიდენკოს, რ. დუდუჩავას [3] და რ. დუდუჩავას [9] მიერ ბოლო დროს მიღებული შედეგები მელინის კონვოლუციის ტიპის განტოლებებისათვის მერომორფული ბირთვებით ბესელის პოტენციათა სივრცეებში, სადაც დადგენილია ფრედჰოლმურობის (და, რიგ შემთხვევებში, ერთადერთი ამოხსნადობის) კრიტერიუმები ზემოთ ხსენებული ინტეგრო-დიფერენციალური განტოლებებისათვის.

Introduction and the Formulation of the Main Theorem

We study the following integro-differential equation in the Bessel potential space setting

$$\varphi(t) - \frac{a(t)}{\pi} \int_{\mathbb{R}} \frac{\varphi'(\tau)}{\tau - t}\, d\tau = f(t), \tag{1}$$

$$\varphi \in \mathbb{H}^s_p(\mathbb{R}), \ \ \varphi(0) = 0, \ \ f \in \mathbb{H}^{s-1}_p(\mathbb{R}), \ \ \frac{1}{p} < s < 1 + \frac{1}{p}, \ \ 1 < p < \infty,$$

where $a(t)$ is a piecewise-constant coefficient: $a(t) = a_-$ for $t < 0$ and $a(t) = a_+$ for $t > 0$. Such boundary integral equations occur as an equivalent reformulation of many problems in the classical two-dimensional elasticity (stringers attached to plates, rigid inclusions in elastic plates, stamps applied to elastic plates etc., see [16]) in aerodynamics (airfoil equation) and in many other problems. In Section 1 we expose an example from Section 6, [18], where the model initial stringer problem was considered and solved in a spaceless setting by a somewhat different method, namely by the method of complex analysis. We endow the example with the non-classical setting when the displacement vector $u + iv$ is sought in the Bessel potential space $\mathbb{H}^{s+1/p}_p$ and the stresses σ_x, σ_y, τ_{xy} belong to the Bessel potential space $\mathbb{H}^{s+1/p-1}_p$.

Based on the investigations from [3,9], in Section 4 is defined the symbol $\mathcal{A}^s_p(\omega)$ of the equation (1), which is a continuous 2×2 matrix-function on the infinite rectangle $\mathfrak{R}$. For an elliptic symbol $\inf_{\omega \in \mathfrak{R}} |\det \mathcal{A}^s_p(\omega)| \neq 0$, the increment of the argument $\frac{1}{2\pi} \arg \det \mathcal{A}^s_p(\omega)$ is an integer and called the index $\operatorname{ind} \det \mathcal{A}^s_p$. The following theorem is the main result for the equation (1) in the present paper.

Theorem 0.1. *Let,* $1 < p < \infty$, $-1 \leqslant s \leqslant 1$, $a_\pm \in \mathbb{C}$.

The equation (1) *is Fredholm if and only if the following two conditions hold:*

- (i) *The coefficients* $a_\pm$ *are not negative reals:* $a_\pm \in \mathbb{C} \setminus \overline{\mathbb{R}^-}$, $\overline{\mathbb{R}^-} := (-\infty, 0]$*;*
- (ii) *The parameters* p *and* s *are not the solutions to the following transcendental equation:*

$$\cos^2 \frac{\pi}{p} \sin^2 \pi \left(\frac{1}{p} + s \right) - \sin^2 \frac{\pi}{p} = 0. \tag{2}$$

If the conditions i and ii hold and $1 < p < 4$, *then the equation* (1) *has a unique solution for all* $1 < p < 4$ *and arbitrary* $-1 \leqslant s \leqslant 1$.

If the conditions i and ii hold and $4 \leqslant p < \infty$, *then the transcendental equation* (2) *has two pairs of solutions* $\{p, s_p\}$ *and* $\{p, s_p - 1\}$, *where* $s_p > 0$, $s_p - 1 < 0$. *Then the equation* (1) *has*

- (i) *a unique solution for all* $s_p - 1 < s < s_p$*;*

(ii) *a unique solution for all right-hand sides which are orthogonal to the solution of the dual homogeneous equation for all* $s_p < s \leqslant 1$ (*the equation has index* -1)*;*

(iii) *a non-unique solution for all right-hand sides provided* $-1 \leqslant s < s_p - 1$*; the homogeneous equation has one linearly independent solution* (*the equation has index* 1)*.*

The same method which we use in the present paper, applies also to the equations with complex conjugated unknown functions

$$\begin{gathered} a_1(x)\varphi(x) + a_2(x)\varphi'(x) + a_3(x)\overline{\varphi(x)} + a_4(x)\overline{\varphi'(x)} \\ + \frac{a_5(x)}{\pi i}\int\limits_\Gamma \frac{\varphi(t)}{t-x}\,dt + \frac{a_6(x)}{\pi}\int\limits_\Gamma \frac{\varphi'(t)}{t-x}\,dt + \frac{a_7(x)}{\pi i}\int\limits_\Gamma \frac{\overline{\varphi(t)}}{t-x}\,dt \\ + \frac{a_8(x)}{\pi}\int\limits_\Gamma \frac{\overline{\varphi'(t)}}{t-x}\,dt + \frac{a_9(x)}{\pi i}\overline{\int\limits_\Gamma \frac{\varphi(t)}{t-x}\,dt} \\ + \frac{a_{10}(x)}{\pi}\overline{\int\limits_\Gamma \frac{\varphi'(t)}{t-x}\,dt} = f(x), \quad x\in\Gamma, \qquad (3) \\ \varphi\in\mathbb{H}^1_p(\Gamma), \quad f\in\mathbb{L}_p(\Gamma), \quad a_j\in PC(\Gamma), \quad j=1,\dots,10, \end{gathered}$$

where Γ is a union of smooth curves, open or closed, including infinite beams (e.g. $\mathbb{R}$). Such equations occur in many problems of elasticity (see e.g. [6–8, 17]).

For the investigation of equation (1) on $\mathbb{R}$ we first convert it into a system of Mellin convolution equations with constant coefficients on the semi-axes $\mathbb{R}^+$. Then the results on Mellin convolution equations in the Bessel potential spaces (see [3,9]) are applied and provide the criteria for the initial equation to have the Fredholm property and write formula for the index.

For the investigation of equation (3) first a quasi-localization is applied, which assigns to it at each point $t\in\Gamma$ the same equation, but either on the axes $\mathbb{R}$ with piecewise constant coefficients, which have jumps only at 0, or on the beam $\mathbb{R}^+$ with constant coefficients ("freezing coefficients" at the localization points; see details in [1, 2, 4, 15]). The obtained equations are investigated just as in the case of equation (1). It is proved that equation (3) is Fredholm one, if and only if all local equations are Fredholm (the local and global Fredholmness for the localized equations coincide).

The details of this investigation will be available in a forthcoming publication.

The present paper is organized as follows: in Section 1 we describe the stringer problem which leads to the integro-differential equation (1) we are going to investigate. In Section 2 we observe Fourier convolution operators in the Bessel potential spaces. The key result on commutants of the Mellin convolution operators and Bessel potentials is represented in Section 3. In the Section 4 we investigate integro-differential equation (1) in the Bessel potential space $\mathbb{B}^s_p(\mathbb{R})$ and prove the key results, including Theorem 0.1.

1. The Integro-Differential Equation of the Stringer Problem

In the present section we expose some details how the Prandtl-type equation (1) is derived as an equivalent boundary integral equation for a model stringer problem. The procedure is very well described in the literature and we only expose some details to show in which space is it correct to look for a solution of a boundary integral equation. In foregoing papers the space where solution belongs was either ignored (see e.g. [16, 18]), or a solution was sought in the Lebesgue space $\mathbb{L}_p$ (see e.g. [6–8]). It should be noted here that the Fredholm property of equation (1) might be essentially different in Lebesgue and Bessel potential spaces (see [3, 9] and Section 3 below).

Suppose a piecewise homogenous thin elastic plate, consisting of two semi-infinite parts occupy the upper $\operatorname{Im} z > 0$ and the lower $\operatorname{Im} z < 0$ complex half-planes of the variable $z = x + iy$. It is reinforced along the junction line $y = 0$. A piecewise homogenous infinite elastic stringer consists of two semi-infinite bars $x > 0$ and $x < 0$, joined to one another and having elastic moduli E_- and E_+ and small cross sections S_- and S_+, respectively. The plates have thicknesses h_-, h_+, Poisson's ratios ν_-, ν_+ and share moduli μ_-, μ_+. Here and below the subscript $+$ corresponds to the plate occupying the upper half-plane $\operatorname{Im} z > 0$ and the subscript $-$ corresponds to the plate occupying the lower half-plane $\operatorname{Im} z < 0$. The plates are joined so that their middle surfaces are identical. The stringer is attached ideally rigidly to the plates and symmetrically both with respect to the junction line of the plates and with respect to their middle surfaces.

Problem S: *Find complex potentials that describe the stress state of the plates and the contact stresses under the stringer.*

To write the corresponding boundary integral equation we follow [18] and apply the complex potentials.

First, we write the equilibrium equations in the interval $[x, x + \Delta x]$:

$$N(x + \Delta x) - N(x) + \big[h_-\tau^+_{xy}(x) - h_- + \tau^-_{xy}(x)\big]\Delta x = 0, \qquad (4)$$
$$\big[h_-\sigma^+_{xy}(x) - h_- + \sigma^-_{xy}(x)\big]\Delta x = 0.$$

After dividing both sides by Δx and taking the limit as $\Delta x \to 0$, we obtain

$$N'(x) + h_-\tau^+_{xy}(x) - h_- + \tau^-_{xy}(x) = 0, \quad h_-\sigma^+_y(x) - h_- + \sigma^-_y(x) = 0, \qquad (5)$$

where N is the normal stress in the stringer calculated for the entire thickness of the stringer, τ^-_{xy} and σ^+_{xy} are the share and normal stresses in the plates calculated per unit thickness of the plates. $N(x) = E_-S_-\varepsilon(x)$ at $x > 0$ and $N(x) = E_+S_+\varepsilon(x)$ at $x < 0$. The stringer is rigidly attached to the plates. Within the model adopted, this is taken into account by equating the displacement vector $u + iv$ of points in the stringer and the displacement vectors $(u+iv)^+$ and $(u+iv)^-$ of the corresponding points in the upper and lower plates on the line $y = 0$. Thus, we obtain the following system of boundary conditions:

$$\begin{gathered} A(x)u''(x)+h_-\tau^+_{xy}(x)-h_-+\tau^-_{xy}(x)=0, \quad h_-\sigma^+_y(x)-h_-+\sigma^-_y(x)=0, \\ (u+iv)(x) = (u+iv)^-(x) = (u+iv)^+(x), \quad x \in \mathbb{R}\setminus\{0\}, \end{gathered} \tag{6}$$

where $A(x) = E_-S_-$ for $x < 0$ and $A(x) = E_+S_+$ for $x > 0$. Conditions (6) must be supplemented with the equilibrium condition of the stringer

$$\int\limits_{-\infty}^{\infty} \big[h_-\tau^+_{xy}(x) - h_- + \tau^-_{xy}(x)\big]\, dx + P_\infty = 0.$$

It is natural to look for a weak solution. Namely, in the classical setting the displacement vector $u + iv$ belongs to the Sobolev (energy) space $\mathbb{H}^1$ and the stresses σ_x, σ_y, τ_{xy}, which are compiled of the partial derivatives of the displacement vector $u + iv$ in the plates with respect to the variable x in the Hilbert space $\mathbb{L}_2$:

$$u + iv \in \mathbb{H}^1(\mathbb{C}^- \cup \mathbb{C}^+), \quad \sigma_x, \sigma_y, \tau_{x,y} \in \mathbb{L}_2(\mathbb{C}), \tag{7}$$

where $\mathbb{C}^-$ denotes the lower and $\mathbb{C}^+$ the upper complex half-planes.

The displacement vector $u+iv$ and the stresses σ_x, σ_y, τ_{xy} are found by means of the Kolosov–Muskhelishvili's formulae

$$\begin{gathered} \sigma_x(z) + \sigma_y(z) = 4\,\mathrm{Re}\,\Phi_\pm(z), \\ \sigma_y(z) - i\tau_{xy}(z) = \Phi_\pm(z) - \Phi_\pm(\overline{z}) + (z-\overline{z})\overline{\Phi_\pm(z)}, \\ 2\mu_\pm \frac{d}{dx}\big[u(z) + iv(z)\big] = \Phi_\pm(z) - \Phi_\pm(\overline{z}) + (z-\overline{z})\overline{\Phi_\pm(z)}, \quad \pm\,\mathrm{Im}\, z>0, \\ \Phi_\pm(z) = \begin{cases} \Phi^+_\pm(z), & \mathrm{Im}\, z > 0, \\ \Phi^-_\pm(z), & \mathrm{Im}\, z < 0, \end{cases} \qquad \mu_\pm = \frac{3-\nu_\pm}{1+\nu_\pm}\,, \end{gathered} \tag{8}$$

where $\Phi_\pm(z)$ are piecewise holomorphic functions (complex potentials) with a line of discontinuity along the real axis and they vanish at infinity. Based on the representation of the potentials as the Cauchy integrals,

$$\Phi^+_-(z) = \frac{1}{2\pi(1+\delta\kappa_-)} \int\limits_{-\infty}^{\infty} \frac{g(t)}{t-z}\, dt, \quad \Phi^-_-(z) = \frac{\kappa_+}{2\pi(\kappa_+ + \delta)} \int\limits_{-\infty}^{\infty} \frac{g(t)}{t-z}\, dt, \tag{9}$$

for the unknown density we derive the following equation from (8):

$$\begin{gathered} g(x) - \frac{a(x)}{\pi}\frac{d}{dx} \int\limits_{-\infty}^{\infty} \frac{g(t)}{t-x}\, dt = g(x) - \frac{a(x)}{\pi} \int\limits_{-\infty}^{\infty} \frac{g'(t)}{t-x}\, dt = 0, \\ g \in \mathbb{H}^{-\frac{1}{2}}(\mathbb{R}), \quad x \in \mathbb{R} \end{gathered} \tag{10}$$

(see [18, Section 6] for details), where

$$\begin{aligned} &\delta = \frac{h_+\mu_+}{h_-\mu_-}, \quad g(x) = \frac{A(x)}{2h_-\mu_-}\frac{d}{dx}\,\mathrm{Re}\left[\kappa_-\Phi^+_-(x) + \Phi^-_-(x)\right], \\ &a(x) = a_\pm \ \text{ for } \ \pm x > 0, \ \ a_\pm := \frac{E_\pm S_\pm}{4h_+\mu_+}\,\frac{\kappa_+(\kappa_- + \delta) + \kappa_-(1+\kappa_+\delta)}{(\kappa_+ + \delta)(1+\kappa_+\delta)} > 0. \end{aligned} \tag{11}$$

Equation (10) coincides with (1) and in the classical setting (7) due to the Kolosov–Muskhelishvili's formulae (8), we have $\Phi_\pm \in \mathbb{L}_2(\mathbb{C}^\pm)$. Then, due to the representation formulae (9), the unknown function g in equation (11) has to be found in the trace space

$$g \in \mathbb{H}^{-1/2}(\mathbb{R}). \tag{12}$$

In the non-classical setting,

$$u + iv \in \mathbb{H}^s_p(\mathbb{C}^- \cup \mathbb{C}^+), \ \ \sigma_x, \sigma_y, \tau_{x,y} \in \mathbb{H}^{s-1}_p(\mathbb{C}), \ \ 1 < p < \infty, \ \ s > \frac{1}{p} \tag{13}$$

(we should impose the constraint $s > 1/p$ to ensure the existence of the trace $(u+iv)^+$ on the boundary), the integral equation (11) has to be solved in the trace space

$$g \in \mathbb{H}^{s-1/p-1}_p(\mathbb{R}). \tag{14}$$

2. Fourier Convolution Operators in the Bessel Potential Spaces $\mathbb{H}^s_p(\mathbb{R}^+)$

To formulate the next theorem we need to introduce the Fourier convolution and Bessel potential operators.

Let $a \in \mathbb{L}_{\infty,loc}(\mathbb{R})$ be a locally bounded $m \times m$ matrix function. The Fourier convolution operator (FCO) with the symbol a is defined by

$$W^0_a := \mathcal{F}^{-1} a \mathcal{F}. \tag{15}$$

Here

$$\mathcal{F}u(\xi) := \int_{\mathbb{R}^n} e^{i\xi x} u(x)\, dx, \ \ \xi \in \mathbb{R}^n. \tag{16}$$

is the Fourier transformation and

$$\mathcal{F}^{-1}v(\xi) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\xi x} v(\xi)\, d\xi, \ \ x \in \mathbb{R}^n. \tag{17}$$

is its inverse transformation. If the operator

$$W^0_a : \mathbb{H}^s_p(\mathbb{R}) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}) \tag{18}$$

is bounded, we say that a is an $\mathbb{L}_p$-multiplier of order r and use "$\mathbb{L}_p$-multiplier" if the order is 0. The set of all $\mathbb{L}_p$-multipliers of order r (of order 0) is denoted by $\mathfrak{M}^r_p(\mathbb{R})$ (by $\mathfrak{M}_p(\mathbb{R})$, respectively). Let

$$\widetilde{\mathfrak{M}}^r_p(\mathbb{R}) := \bigcap_{p-\varepsilon<q<p+\varepsilon} \mathfrak{M}^r_q(\mathbb{R}), \quad \widetilde{\mathfrak{M}}_p(\mathbb{R}) := \bigcap_{p-\varepsilon<q<p+\varepsilon} \mathfrak{M}_q(\mathbb{R}).$$

For an $\mathbb{L}_p$-multiplier of order r, $a \in \mathfrak{M}_p^r(\mathbb{R})$, the Fourier convolution operator (FCO) on the semi-axis $\mathbb{R}^+$ is defined by the equality

$$W_a = r_+ W_a^0 : \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^{s-r}(\mathbb{R}^+), \tag{19}$$

where $r_+ := r_{\mathbb{R}^+} : \mathbb{H}_p^s(\mathbb{R}) \longrightarrow \mathbb{H}_p^s(\mathbb{R}^+)$ is the restriction operator to the semi-axes $\mathbb{R}^+$.

We did not use in the definition of the class of multipliers $\mathfrak{M}_p^r(\mathbb{R})$ the parameter $s \in \mathbb{R}$. This is due to the fact that $\mathfrak{M}_p^r(\mathbb{R})$ is independent of s: if the operator W_a in (19) is bounded for some $s \in \mathbb{R}$, it is bounded for all other values of s.

Another definition of the multiplier class $\mathfrak{M}_p^r(\mathbb{R})$ is written as follows: $a \in \mathfrak{M}_p^r(\mathbb{R})$ if and only if $\lambda^{-r} a \in \mathfrak{M}_p(\overline{\mathbb{R}}) = \mathfrak{M}_p^0(\overline{\mathbb{R}})$, where $\lambda^r(\xi) := (1+|\xi|^2)^{r/2}$. This assertion is one of the consequences of Theorem 2.1 below.

The Bessel potential operators are defined as follows:

$$\begin{gathered} \mathbf{\Lambda}_\gamma^r = W_{\lambda_\gamma^r}^0 : \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{H}}_p^{s-r}(\mathbb{R}^+), \\ \mathbf{\Lambda}_{-\gamma}^r = r_+ W_{\lambda_{-\gamma}^r}^0 \ell : \mathbb{H}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^{s-r}(\mathbb{R}^+), \\ \lambda_{\pm\gamma}^r(\xi) := (\xi \pm \gamma)^r, \quad \xi \in \mathbb{R}, \quad \operatorname{Im}\gamma > 0, \end{gathered} \tag{20}$$

and they arrange isomorphisms of the corresponding spaces (see [6,9]). Here, $\ell : \mathbb{H}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^s(\mathbb{R})$ is some extension operator and the final result is independent of the choice of an extension ℓ (we did not needed the extension operator in (19), since the space $\widetilde{\mathbb{H}}_p^s(\mathbb{R}^+)$ is automatically embedded in $\mathbb{H}_p^s(\mathbb{R})$ by extending the functions with 0).

Theorem 2.1. *Let $1 < p < \infty$. Then*

1. *For any $r, s \in \mathbb{R}$, $\gamma \in \mathbb{C}$, $\operatorname{Im}\gamma > 0$ the convolution operators*

$$\begin{gathered} \mathbf{\Lambda}_\gamma^r = W_{\lambda_\gamma^r} : \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{H}}_p^{s-r}(\mathbb{R}^+), \\ \mathbf{\Lambda}_{-\gamma}^r = r_+ W_{\lambda_{-\gamma}^r}^0 \ell : \mathbb{H}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^{s-r}(\mathbb{R}^+), \\ \lambda_{\pm\gamma}^r(\xi) := (\xi \pm \gamma)^r, \quad \xi \in \mathbb{R}, \quad \operatorname{Im}\gamma > 0, \end{gathered} \tag{21}$$

arrange isomorphisms of the corresponding spaces (*see* [6,14]). *Here, $\ell : \mathbb{H}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^s(\mathbb{R})$ is some extension operator and the final result is independent of the choice of an extension ℓ. r_+ is the restriction from the axes $\mathbb{R}$ to the semi-axes $\mathbb{R}^+$.*

2. *For any operator $\mathbf{A} : \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^{s-r}(\mathbb{R}^+)$ of order r, the following diagram is commutative*

$$\begin{array}{ccc} \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) & \xrightarrow{\ \mathbf{A}\ } & \mathbb{H}_p^{s-r}(\mathbb{R}^+) \\ \mathbf{\Lambda}_\gamma^{-s} \big\downarrow & & \big\downarrow \mathbf{L}_{-\gamma}^{s-r} \\ \mathbb{L}_p(\mathbb{R}^+) & \xrightarrow[\mathbf{\Lambda}_{-\gamma}^{s-r} \mathbf{A} \mathbf{\Lambda}_\gamma^{-s}]{} & \mathbb{L}_p(\mathbb{R}^+) \end{array} . \tag{22}$$

Diagram (22) provides an equivalent lifting of the operator $\mathbf{A}$ *of order* r *to the operator* $\mathbf{\Lambda}^{s-r}_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *of order* 0.

3. *For any bounded convolution operator* $W_a : \mathbb{H}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+)$ *of order* r *and for any pair of complex numbers* γ_1, γ_2 *such that* $\operatorname{Im}\gamma_j > 0$, $j = 1, 2$, *the lifted operator*

$$\begin{gathered}\mathbf{\Lambda}^{\mu}_{-\gamma_1} W_a \mathbf{\Lambda}^{\nu}_{\gamma_2} = W_{a_{\mu,\nu}} : \mathbb{H}^{s+\nu}_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r-\mu}_p(\mathbb{R}^+),\\ a_{\mu,\nu}(\xi) := (\xi-\gamma_1)^{\mu} a(\xi)(\xi+\gamma_2)^{\nu}\end{gathered} \tag{23}$$

is again a Fourier convolution.

In particular, the lifted operator $W_{a_{s-r,-s}} = \mathbf{\Lambda}^{s-r}_{-\gamma} W_a \mathbf{\Lambda}^{-s}_{\gamma} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *has the symbol*

$$a_{s-r,-s}(\xi) = \lambda^{s-r}_{-\gamma}(\xi) a(\xi) \lambda^{-s}_{\gamma}(\xi) = \Big(\frac{\xi-\gamma}{\xi+\gamma}\Big)^{s-r} \frac{a(\xi)}{(\xi+i)^r}\,.$$

Remark 2.2. For any pair of multipliers $a \in \mathfrak{M}^r_p(\mathbb{R})$, $b \in \mathfrak{M}^s_p(\mathbb{R})$ the corresponding convolution operators on the axes W^0_a and W^0_b have the property $W^0_a W^0_b = W^0_b W^0_a = W^0_{ab}$.

For the corresponding Wiener–Hopf operators on the half-axes a similar equality

$$W_a W_b = W_{ab} \tag{24}$$

holds if and only if either the function $a(\xi)$ has an analytic extension in the lower half-plane, or the function $b(\xi)$ has an analytic extension in the upper half-plane (see [6]).

Note that actually (23) is a consequence of (24).

Let $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ denote the one point compactification of the real axes $\mathbb{R}$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ denote the two point compactification of $\mathbb{R}$. By $C(\dot{\mathbb{R}})$ (by $C(\overline{\mathbb{R}})$, respectively) we denote the space of continuous functions $g(x)$ on $\mathbb{R}$ which have the equal limits at the infinity $g(-\infty) = g(+\infty)$ (limits at the infinity may be different $g(-\infty) \neq g(+\infty)$. By $PC(\dot{\mathbb{R}})$ is denoted the space of piecewise-continuous functions on $\dot{\mathbb{R}}$, having the limits $a\ (t \pm 0)$ at all points $t \in \dot{\mathbb{R}}$, including the infinity.

Proposition 2.3 ([6, Lemma 7.1] and [10, Proposition 1.2])**.** *Let* $1<p<\infty$, $a \in C(\dot{\mathbb{R}}^+)$, $b \in C(\dot{\mathbb{R}}) \cap \widetilde{\mathfrak{M}}_p(\dot{\mathbb{R}})$ *and* $a(\infty) = b(\infty) = 0$. *Then the operators* $aW_b, W_b\, aI : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *are compact.*

Moreover, these operators are bounded in all Bessel potential space, and, due to Krasnoselskij interpolation theorem for compact operators, are compact in these spaces.

Proposition 2.4 ([6, Lemma 7.4] and [10, Lemma 1.2])**.** *Let* $1 < p < \infty$ *and let* a *and* b *satisfy at least one of the conditions*

(i) $a \in C(\overline{\mathbb{R}}^+)$, $b \in \widetilde{\mathfrak{M}}_p(\mathbb{R}) \cap PC(\overline{\mathbb{R}})$;

(ii) $a \in PC(\overline{\mathbb{R}}^+)$, $b \in C\widetilde{\mathfrak{M}}_p(\overline{\mathbb{R}})$.

Then the commutants $[aI, W_b]$ *and* $[aI, \mathfrak{M}_b^0]$, *where* $\mathfrak{M}_b^0$ *is a Mellin convolution operator* (*see the next Section* 3), *are compact operators in the space* $\mathbb{L}_p(\mathbb{R}^+)$.

Moreover, these operators are compact in all Bessel potential and Besov spaces, where they are bounded, due to Krasnoselskij interpolation theorem for compact operators.

The differentiation is a Fourier convolution operator with the symbol $-i\xi$:

$$r_+\partial_t\psi = r_+\partial_t\mathcal{F}^{-1}\mathcal{F}\psi = r_+\mathcal{F}^{-1}(-i\xi)\mathcal{F}\psi = W_{-i\xi}\psi, \quad \psi \in C_0^\infty(\mathbb{R}^+). \tag{25}$$

Using (25) and (20), we get

$$r_+(\partial_t\mathbf{\Lambda}_{\pm\gamma}^{-1} - I) = r_+(\mathbf{\Lambda}_{\pm\gamma}^{-1}\partial_t - I) = W_g, \quad g(\xi) := \frac{\xi}{\xi \pm \gamma} - 1, \quad \xi \in \mathbb{R}.$$

The symbol $g(\xi)$ is infinitely smooth and vanishes at infinity: $g(\pm\infty) = 0$. Then, due to Proposition 2.3, the operators

$$v_0\big[r_+(\partial_t\mathbf{\Lambda}_{\pm\gamma}^{-1} - I)\big], \qquad \big[r_+(\partial_t\mathbf{\Lambda}_{\pm\gamma}^{-1} - I)\big]v_0I \tag{26}$$

are compact for all $v_0 \in C_0^\infty(\mathbb{R}^+)$ (and even for all sufficiently smooth $v_0 \in C^m(\mathbb{R}^+)$) which vanish at infinity $v_0(\infty) = 0$. The compactness of the operators in (26) imply the local invertibility of ∂_t (with the local inverse $\mathbf{\Lambda}_{\pm\gamma}^{-1}$) even at all finite points $t \in \mathbb{R}^+$.

3. Mellin Convolution Operators in the Bessel Potential Spaces $\mathbb{H}_p^s(\mathbb{R}^+)$

In the present section we expose auxiliary results from [9] (also see [3, 6, 10]), which are essential for the investigation of boundary integral equation (1).

Consider a Mellin convolution operator $\mathfrak{M}_a^0$ in the Bessel potential spaces

$$\mathfrak{M}_a^0 := \mathcal{M}_\beta^{-1}a\mathcal{M}_\beta : \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^s(\mathbb{R}^+), \tag{27}$$

where

$$\mathcal{M}_\beta v(\xi) := \int_0^\infty \tau^{\beta - i\xi}v(\tau)\frac{d\tau}{\tau}\,, \quad \xi \in \mathbb{R},$$

$$\mathcal{M}_\beta^{-1}u(t) := \frac{1}{2\pi}\int_{-\infty}^\infty t^{i\xi-\beta}u(\xi)\,d\xi, \quad t \in \mathbb{R}^+.$$

are the Mellin transformation and the inverse to it.

The symbol $a(\xi)$ of this operator is an $n\times n$ matrix function $a \in C\mathfrak{M}_p^0(\overline{\mathbb{R}})$ continuous on the real axis $\mathbb{R}$ with the only possible jump at infinity.

The most important example of a Mellin convolution operator is an integral operator of the form

$$\mathfrak{M}^0_a\mathbf{u}(t) := c_0\mathbf{u}(t) + \frac{c_1}{\pi i}\int_0^\infty \frac{\mathbf{u}(\tau)}{\tau - t}\,dt + \int_0^\infty \mathcal{K}\Big(\frac{t}{\tau}\Big)\mathbf{u}(\tau)\,\frac{d\tau}{\tau} \tag{28}$$

with $n\times n$ matrix coefficients and $n\times n$ matrix kernel. $\mathfrak{M}^0_a$ is a bounded operator in the weighted Lebesgue space of vector-functions

$$\mathfrak{M}^0_a : \mathbb{L}_p(t^\gamma,\mathbb{R}^+) \longrightarrow \mathbb{L}_p(t^\gamma,\ \mathbb{R}^+),\ \ 1<p<\infty,\ \ -1<\gamma<p-1, \tag{29}$$

endowed with the norm

$$\big\|u \mid \ \mathbb{L}_p(t^\gamma,\mathbb{R}^+)\big\| := \Bigg[\int_0^\infty t^\gamma|u(t)|^p\,dt\Bigg]^{1/p}$$

under the following constraint on the kernel (on each entry of the matrix kernel)

$$\int_0^\infty t^{\beta-1}\mathcal{K}(t)\,dt < \infty,\ \ \beta := \frac{1+\gamma}{p}\,,\ \ 0<\beta<1 \tag{30}$$

(cf. [6]). The symbol of the operator (28) is the Mellin transform of the kernel

$$\begin{aligned} a(\xi) &:= c_0 + c_1\coth\pi(i\beta+\xi) + \mathcal{M}_\beta\mathcal{K}(\xi)\\ &:= c_0 + c_1\coth\pi(i\beta+\xi) + \int_0^\infty t^{\beta-i\xi}\mathcal{K}(t)\frac{dt}{t}\,,\ \ \xi\in\mathbb{R}, \end{aligned} \tag{31}$$

and the symbol is responsible for the Fredholm properties of the operator.

Obviously,

$$\mathfrak{M}^0_a\mathfrak{M}^0_b\varphi = \mathfrak{M}^0_{ab}\varphi = \mathfrak{M}^0_b\mathfrak{M}^0_a\varphi,\ \ \varphi\in C_0^\infty(\mathbb{R}^+),$$

for arbitrary $a\in\mathfrak{M}^r_p(\mathbb{R})$ and $b\in\mathfrak{M}^s_p(\mathbb{R})$.

Theorem 3.1. *Let $1<p<\infty$ and $-1<\gamma<p-1$ (or $1\leqslant p\leqslant\infty$ provided $c_1=0$ in (28)). The operator $\mathfrak{M}^0_a$ in (28)–(29) with a symbol $a\in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$, is a Fredholm operator if and only if its symbol is invertible (is elliptic)*

$$\inf_{\xi\in\mathbb{R}}|\det a(\xi)|>0. \tag{32}$$

If the symbol is elliptic, the operator is invertible and $\mathfrak{M}^0_{a^{-1}}$ is the inverse.

Things are different in the Bessel potential spaces. Let us recall some results from [9, Section 2].

Consider the kernels which are meromorphic functions on the complex plane $\mathbb{C}$, vanishing at infinity,

$$\mathcal{K}(t) := \sum_{j=0}^N \frac{d_j}{(t-c_j)^{m_j}}\,, \tag{33}$$

having poles at $c_0, c_1, \ldots, c_N \in \mathbb{C} \setminus \{0\}$ and complex coefficients $d_j \in \mathbb{C}$.

Definition 3.2 (see [9]). We call a kernel $\mathcal{K}(t)$ in (33) admissible if for those poles $c_0, \ldots, c_\ell$ which belong to the positive semi-axes $\arg c_0 = \cdots = \arg c_\ell = 0$, the corresponding multiplicities is one $m_0 = \cdots = m_\ell = 1$.

For example: The Mellin convolution operator

$$\mathbf{K}_c^m v(t) := \frac{1}{\pi} \int_0^\infty \frac{\tau^{m-1} v(\tau)}{(t - c\tau)^m}\, d\tau, \quad -\pi < \arg c < \pi, \quad v \in \mathbb{L}_p(\mathbb{R}^+) \tag{34}$$

has an admissible kernel for arbitrary $m = 1, 2, \ldots$ if the following constraint holds: for a real $\arg c = 0$ and positive $c > 0$ necessarily $m = 1$.

Proposition 3.3 (see [9, Corollary 2.3, Theorem 2.4]). *Let $1 < p < \infty$ and $-1 < \gamma < p - 1$ (or $1 \leqslant p \leqslant \infty$ provided $c_1 = 0$ in (28)), $s \in \mathbb{R}$ and $\mathcal{K}(t)$ in (33) be an admissible kernel. Then the Mellin convolution operator*

$$\mathfrak{M}_a^0 \mathbf{u}(t) := c_0 \mathbf{u}(t) + \int_0^\infty \mathcal{K}\Big(\frac{t}{\tau}\Big) \mathbf{u}(\tau) \frac{d\tau}{\tau}$$

is bounded in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ and, also, in the Bessel potential space in the following setting:

$$\mathfrak{M}_a^0 : r\widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^s(\mathbb{R}^+). \tag{35}$$

Theorem 3.4 ([3, Theorem 5.1] and [9]). *Let $s \in \mathbb{R}$ and $1 < p < \infty$.*

If $-\pi \leqslant \arg c < \pi$, $\arg c \neq 0$, $0 < \arg \gamma < \pi$ and $0 < \arg(-c\,\gamma) < \pi$, the Mellin convolution operator between the Bessel potential spaces

$$\mathbf{K}_c^1 : \widetilde{\mathbb{H}}_p^r(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^r(\mathbb{R}^+) \tag{36}$$

is lifted to the equivalent operator

$$\mathbf{\Lambda}_{-\gamma}^s \mathbf{K}_c^1 \mathbf{\Lambda}_\gamma^{-s} = c^{-s} \mathbf{K}_c^1 W_{g^s_{-c\gamma,\gamma}} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+), \tag{37}$$

where $c^{-s} = |c|^{-s} e^{-is \arg c}$ and

$$g^s_{\delta,\mu}(\xi) := \Big(\frac{\xi + \delta}{\xi + \mu}\Big)^s. \tag{38}$$

If $-\pi \leqslant \arg c < \pi$, $\arg c \neq 0$, $0 < \arg \gamma < \pi$ and $-\pi < \arg(-c\,\gamma) < 0$, the Mellin convolution operator between the Bessel potential spaces (36) is lifted to the equivalent operator

$$\begin{aligned} \mathbf{\Lambda}_{-\gamma}^s \mathbf{K}_c^1 \mathbf{\Lambda}_\gamma^{-s} &= c^{-s} W_{g^s_{-\gamma,-\gamma_0}} \mathbf{K}_c^1 W_{g^s_{-c\gamma_0,\gamma}} \\ &= c^{-s} \mathbf{K}_c^1 W_{g^s_{-\gamma,-\gamma_0} g^s_{-c\gamma_0,\gamma}} + \mathbf{T} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+), \end{aligned} \tag{39}$$

where $\mathbf{T} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ is a compact operator.

Let us consider the Banach algebra $\mathfrak{A}_p(\mathbb{R}^+)$ generated by Mellin convolution and Fourier convolution operators, i.e. by the operators

$$\mathbf{A} := \sum_{j=1}^{m} W_{d_j} \mathfrak{M}^0_{a_j} W_{b_j} \tag{40}$$

and their compositions in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$. Here, $\mathfrak{M}^0_{a_j}$ are the Mellin convolution operators with continuous $N \times N$ matrix symbols $a_j \in C\mathfrak{M}_p(\overline{R})$, W_{b_j}, W_{d_j} are Fourier convolution operators with $N \times N$ matrix symbols $b_j, d_j \in C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\}) := C\mathfrak{M}_p(\overline{\mathbb{R}}^- \cup \overline{\mathbb{R}}^+)$. The algebra of $N \times N$ matrix $\mathbb{L}_p$-multipliers $C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\})$ consists of those piecewise-continuous $N \times N$ matrix multipliers $b \in \mathfrak{M}_p(\mathbb{R}) \cap PC(\overline{\mathbb{R}})$ which are continuous on the semi-axis $\mathbb{R}^-$ and $\mathbb{R}^+$, but might have finite jump discontinuities at 0 and at infinity.

To define the symbol of the operator $\mathbf{A}$ in (40) which governs the Fredholm property and the index of $\mathbf{A}$ (see Theorem 3.5, below) we consider the infinite clockwise oriented "rectangle" $\mathfrak{R} := \Gamma_1 \cup \Gamma_2^- \cup \Gamma_2^+ \cup \Gamma_3$, where (cf. Figure 1)

$$\Gamma_1 := \overline{\mathbb{R}} \times \{+\infty\}, \;\; \Gamma_2^{\pm} := \{\pm\infty\} \times \overline{\mathbb{R}}^+, \;\; \Gamma_3 := \overline{\mathbb{R}} \times \{0\}.$$

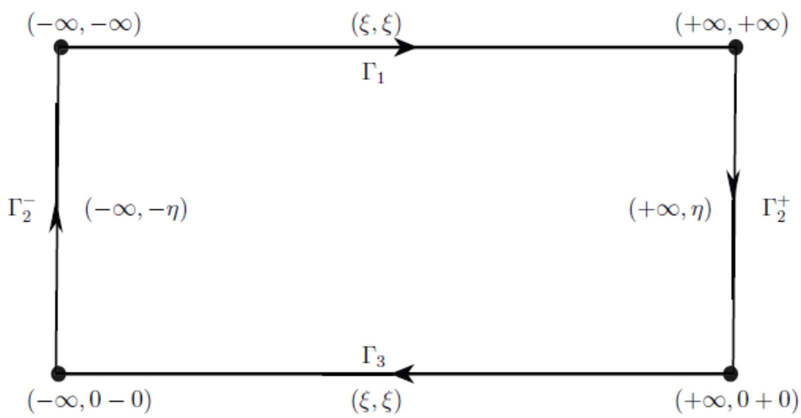

FIGURE 1. The domain $\mathfrak{R}$ of definition of the symbol $\mathcal{A}^s_p(\omega)$.

The symbol $\mathcal{A}_p(\omega)$ of the operator $\boldsymbol{A}$ in (40) is a function on the set $\mathfrak{R}$, viz.

$$\mathcal{A}_p(\omega) := \begin{cases} \sum_{j=1}^{m} (d_j)_p(\infty,\xi) a_j(\xi)(b_j)_p(\infty,\xi), & \omega = (\xi,\xi) \in \overline{\Gamma_1}, \\ \sum_{j=1}^{m} d_j(\eta) a_j(+\infty) b_j(\eta), & \omega = (+\infty,\eta) \in \Gamma_2^+, \\ \sum_{j=1}^{m} d_j(-\eta) a_j(-\infty) b_j(-\eta), & \omega = (-\infty,\eta) \in \Gamma_2^-, \\ \sum_{j=1}^{m} (d_j)_p(0,\xi) a_j(\xi)(b_j)_p(0,\xi), & \omega = (\xi,0) \in \overline{\Gamma_3}. \end{cases} \tag{41}$$

The connecting function $g_p(\infty,\xi)$ in (41) for a piecewise continuous function $g \in PC(\overline{\mathbb{R}})$ is defined as follows:

$$\begin{aligned} g_p(x,\xi) &:= \frac{1}{2}\big[g(x+0)+g(x-0)\big] - \frac{i}{2}\big[g(x+0)-g(x-0)\big]\cot\pi\Big(\frac{1}{p}-i\xi\Big) \\ &= e^{i\pi\,\frac{g_x^+ + g_x^-}{2}}\,\frac{\cos\pi(\frac{1}{p}+\frac{g_x^+ - g_x^-}{2}-i\xi)}{\sin\pi(\frac{1}{p}-i\xi)}\,, \quad \xi\in\mathbb{R}, \end{aligned} \tag{42}$$

$$g_x^\pm := \frac{1}{\pi i}\ln\, g(x\pm 0), \quad \operatorname{Re} g_x^\pm = \frac{1}{\pi}\arg\, g(x\pm 0), \quad x \in \dot{\mathbb{R}} := \mathbb{R}\cup\{\infty\}.$$

The function $g_p(\infty,\xi)$ fills up the discontinuity (the jump) of $g(\xi)$ at ∞ between $g(-\infty)$ and $g(+\infty)$ with an oriented arc of the circle (see [9, Section 4] for further details).

The symbol $\mathcal{A}_p(\omega)$ is continuous on the rectangle $\mathfrak{R}$ and if it is elliptic

$$\inf_{\omega\in\mathfrak{R}} \big|\det\mathcal{A}_p(\omega)\big| > 0, \tag{43}$$

the increment of the argument $(1/2\pi)\arg\mathcal{A}_p(\omega)$ when ω ranges through $\mathfrak{R}$ in the positive direction, is an integer. This integer is called the winding number or the index of the symbol and is denoted by $\operatorname{ind}\det\mathcal{A}_p$.

Theorem 3.5 ([9, Theorem 4.13])**.** *Let* $1<p<\infty$ *and let* $\mathbf{A}$ *be defined by* (40)*. Then* $\mathbf{A} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *is a Fredholm operator if and only if its symbol* $\mathcal{A}_p(\omega)$ *is elliptic. If* $\mathbf{A}$ *is Fredholm, the index of the operator is*

$$\operatorname{Ind}\mathbf{A} = -\operatorname{ind}\det\mathcal{A}_p. \tag{44}$$

4. Investigation of the Integro-Differential Equation (1)

For the investigation of equation (1) we apply the approach developed in [11] and the localization technique.

Proof of Theorem 0.1. Let us introduce the notation

$$\varphi_1(t) := \varphi(-t), \quad f_1(t) = f(-t),$$
$$\varphi_2(t) := \varphi(t), \quad f_2(t) = f(t) \ \text{ for } \ t>0.$$

Then $\varphi_1'(t) := -\varphi'(-t)$, $\varphi_2'(t) := \varphi'(t)$ and the integral equation (1) is then written in the following form:

$$\begin{cases} \varphi_1(t) + \dfrac{a_-}{\pi}\displaystyle\int_0^\infty \frac{\varphi_1'(\tau)}{t-\tau}\,d\tau - \frac{a_-}{\pi}\int_0^\infty \frac{\varphi_2'(\tau)}{t+\tau}\,d\tau = f_1(t), \\ \varphi_2(t) - \dfrac{a_+}{\pi}\displaystyle\int_0^\infty \frac{\varphi_1'(\tau)}{t+\tau}\,d\tau + \frac{a_+}{\pi}\int_0^\infty \frac{\varphi_2'(\tau)}{t-\tau}\,d\tau = f_2(t), \end{cases} \quad t \in \mathbb{R}^+, \tag{45}$$

$$\varphi_1, \varphi_2 \in \mathbb{H}^s_p(\mathbb{R}^+), \quad f_1, f_2 \in \mathbb{H}^{s-1}_p(\mathbb{R}^+).$$

Moreover, by physical arguments (the system (45) is an equivalent reformulation of the Problem S (see (11), (13)) and we can assume that:

(i) a solution to the system (45) vanishes at 0

$$\varphi_1, \varphi_2 \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+). \tag{46}$$

(ii) the system (45) has a unique solution φ_1, φ_2 in the classical setting $s = 1/2$, $p = 2$:

$$\varphi_1, \varphi_2 \in \widetilde{\mathbb{H}}^{1/2}(\mathbb{R}^+), \quad f_1, f_2 \in \mathbb{H}^{-1/2}(\mathbb{R}^+). \tag{47}$$

The system of integral equations (45) is of Mellin type,

$$\begin{cases} \varphi_1(t) + a_-\Big[\boldsymbol{K}^1_1\,\varphi_1'(t) - \boldsymbol{K}^1_{-1}\,\varphi_2'(t)\Big] = f_1(t), \\ \varphi_2(t) - a_+\Big[\boldsymbol{K}^1_{-1}\,\varphi_1'(t) - \boldsymbol{K}^1_1\,\varphi_2'(t)\Big] = f_2(t), \end{cases} \tag{48}$$

$$\varphi_1, \varphi_2 \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}+), \quad f_1, f_2 \in \mathbb{H}^{s-1}_p(\mathbb{R}^+),$$

where

$$\mathbf{K}^1_c\varphi(t) := \frac{1}{\pi}\int_0^\infty \frac{\varphi(\tau)}{t - c\,\tau}\,d\tau, \quad 0 < |\arg c| \leqslant \pi, \quad \varphi \in \mathbb{L}_p(\mathbb{R}^+),$$

is a Mellin convolutions operator with a meromorphic kernel (see Definition 3.2).

Since $\varphi_j \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$, $f_j \in \mathbb{H}^{s-1}_p(\mathbb{R}^+)$, $j = 1, 2$, we introduce new functions

$$\varphi_1 = \Lambda^{-s}_\gamma\psi_1, \quad \varphi_2 = \Lambda^{-s}_\gamma\psi_2, \quad f_1 = \Lambda^{-s+1}_{-\gamma}g_1 \quad f_2 = \Lambda^{-s+1}_{-\gamma}g_2,$$
$$\operatorname{Im}\gamma > 0, \quad \psi_1, \psi_2, g_1, g_2 \in \mathbb{L}_p(\mathbb{R}^+),$$

use the equality

$$\frac{d\varphi(t)}{dt} = \varphi'(t) = (\boldsymbol{W}_{-i\xi}\varphi)(t)$$

and get

$$\begin{cases} \Lambda^{-s}_\gamma\psi_1 + a_-\Big[\boldsymbol{K}^1_1\boldsymbol{W}_{-i\xi}\Lambda^{-s}_\gamma\psi_1 - \boldsymbol{K}^1_{-1}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_\gamma\psi_2\Big] = \Lambda^{-s+1}_{-\gamma}g_1, \\ \Lambda^{-s}_\gamma\psi_2 - a_+\Big[\boldsymbol{K}^1_{-1}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_\gamma\psi_1 - \boldsymbol{K}^1_1\boldsymbol{W}_{-i\xi}\Lambda^{-s}_\gamma\psi_2\Big] = \Lambda^{-s+1}_{-\gamma}g_2. \end{cases}$$

Here, the pair of functions ψ_1, ψ_2 is unknown while the pair g_1, g_2 is known (prescribed).

The system is already lifted to the $\mathbb{L}_p$-space setting, and we will write it in a convenient form by applying the Bessel potential operator Λ^{s-1}_{γ} to the both parts of the equations:

$$\begin{cases} \Lambda^{s-1}_{-\gamma}\Lambda^{-s}_{\gamma}\psi_1 + a_-\Big[\Lambda^{s-1}_{-\gamma}\boldsymbol{K}^1_1\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_1 - \Lambda^{s-1}_{-\gamma}\boldsymbol{K}^1_{-1}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_2\Big] \\ \qquad\qquad = \Lambda^{s-1}_{-\gamma}\Lambda^{-s+1}_{-\gamma}g_1 = g_1, \\ \Lambda^{s-1}_{-\gamma}\Lambda^{-s}_{\gamma}\psi_2 - a_+\Big[\Lambda^{s-1}_{-\gamma}\boldsymbol{K}^1_{-1}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_1 - \Lambda^{s-1}_{-\gamma}\boldsymbol{K}^1_1\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_2\Big] \\ \qquad\qquad = \Lambda^{s-1}_{-\gamma}\Lambda^{-s+1}_{-\gamma}g_2 = g_2, \end{cases}$$
$$\psi_1, \psi_2, g_1, g_2 \in \mathbb{L}_p(\mathbb{R}^+),$$

since $\Lambda^{s-1}_{-\gamma}\Lambda^{-s+1}_{-\gamma}u = u$ (see [6]). By using the equality

$$\boldsymbol{\Lambda}^r_{-\gamma}\mathbf{K}^1_c\boldsymbol{\Lambda}^{-r}_{\gamma} = c^{-r}\mathbf{K}^1_c\boldsymbol{\Lambda}^r_{-c\gamma}\boldsymbol{\Lambda}^{-r}_{\gamma}$$

proved in [3, Theorem 5.1] for arbitrary $c \in \mathbb{C}$ and again the equality $\Lambda^{s-1}_{-\gamma}\Lambda^{-s+1}_{\gamma} = I$, we rewrite the system in the following form:

$$\begin{cases} \Lambda^{s-1}_{-\gamma}\Lambda^{-s}_{\gamma}\psi_1 \\ \quad +a_-\Big[\boldsymbol{K}^1_1\Lambda^{s-1}_{-\gamma}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_1 - (-1)^{-s+1}\boldsymbol{K}^1_{-1}\Lambda^{s-1}_{\gamma}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_2\Big] = g_1, \\ \Lambda^{s-1}_{-\gamma}\Lambda^{-s}_{\gamma}\psi_2 \\ \quad -a_+\Big[(-1)^{-s+1}\boldsymbol{K}^1_{-1}\Lambda^{s-1}_{\gamma}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_1 - \boldsymbol{K}^1_1\Lambda^{s-1}_{-\gamma}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma}\psi_2\Big] = g_2, \end{cases}$$
$$\psi_1, \psi_2, g_1, g_2 \in \mathbb{L}_p(\mathbb{R}^+).$$

Next, we apply the equalities

$$\Lambda^r_{\mu} = \boldsymbol{W}_{(\xi+\mu)^r}, \quad \boldsymbol{W}_a\boldsymbol{W}_b = \boldsymbol{W}_{ab},$$

where the second one holds if $a(\xi)$ has an analytic extension in the lower half-plane or $b(\xi)$ has an analytic extension in the upper half-plane (see [3,6] for details). The above equalities imply, in particular, that

$$\Lambda^{s-1}_{-\gamma}\Lambda^{-s}_{\gamma} = \boldsymbol{W}_{(\xi-\gamma)^{s-1}}\boldsymbol{W}_{(\xi+\gamma)^{-s}} = \boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s\,\frac{1}{\xi-\gamma}},$$
$$\Lambda^{s-1}_{\gamma}\boldsymbol{W}_{-i\xi}\Lambda^{-s}_{\gamma} = \boldsymbol{W}_{\frac{-i\xi}{\xi+\gamma}}.$$

Finally, we arrive to the following system of convolution equations, which is an equivalent reformulation of the system (48) in the $\mathbb{L}_p$-space setting:

$$\begin{cases} \boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s\,\frac{1}{\xi-\gamma}}\psi_1 + a_-\boldsymbol{K}^1_1\boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s\,\frac{-i\xi}{\xi-\gamma}}\psi_1 \\ \qquad\qquad +(-1)^s a_-\boldsymbol{K}^1_{-1}\boldsymbol{W}_{\frac{-i\xi}{\xi+\gamma}}\psi_2 = g_1, \\ \boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s\,\frac{1}{\xi-\gamma}}\psi_2 + (-1)^s a_+\boldsymbol{K}^1_{-1}\boldsymbol{W}_{\frac{-i\xi}{\xi+\gamma}}\psi_1 \\ \qquad\qquad +a_+\boldsymbol{K}^1_1\boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s\,\frac{-i\xi}{\xi-\gamma}}\psi_2 = g_2, \end{cases} \tag{49}$$

$$\psi_1, \psi_2, g_1, g_2 \in \mathbb{L}_p(\mathbb{R}^+).$$

Let us rewrite the system (49) as follows

$$\boldsymbol{A}\Psi = \mathbf{G}, \quad \Psi := \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \in \mathbb{L}_p(\mathbb{R}^+), \quad \mathbf{G} := \begin{pmatrix} g_1 \\ g_2\end{pmatrix} \in \mathbb{L}_p(\mathbb{R}^+)$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s\,\frac{1}{\xi-\gamma}} + a_-\boldsymbol{K}^1_1\boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s\,\frac{-i\xi}{\xi-\gamma}} & e^{\pi si}a_-\boldsymbol{K}^1_{-1}\boldsymbol{W}_{\frac{-i\xi}{\xi+\gamma}} \\ e^{\pi si}a_+\boldsymbol{K}^1_{-1}\boldsymbol{W}_{\frac{-i\xi}{\xi+\gamma}} & \boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s\,\frac{1}{\xi-\gamma}} + a_+\boldsymbol{K}^1_1\boldsymbol{W}_{(\frac{\xi-\gamma}{\xi+\gamma})^s\,\frac{-i\xi}{\xi-\gamma}} \end{bmatrix}$$

According to [9, Formulae (41), (81)], the symbols of operators $\boldsymbol{K}^1_1 = \mathfrak{M}^0_{\mathcal{K}^1_1}$ and $\boldsymbol{K}^1_{-1} = \mathfrak{M}^0_{\mathcal{K}^1_{-1}}$ are, respectively,

$$\mathcal{K}^1_1(\xi) = -i\coth\pi(i\beta+\xi) = -\cot\pi(\beta - i\xi), \;\; \mathcal{K}^1_{-1}(\xi) = \frac{1}{\sin\pi(\beta-i\xi)}.$$

With the shorthand notation,

$$b_1(\xi) = \Big(\frac{\xi-\gamma}{\xi+\gamma}\Big)^s\frac{1}{\xi-\gamma}, \;\; b_2(\xi) = \Big(\frac{\xi-\gamma}{\xi+\gamma}\Big)^s\frac{-i\xi}{\xi-\gamma}, \;\; b_3(\xi) = \frac{-i\xi}{\xi+\gamma},$$

we rewrite the operator $\boldsymbol{A}$ as follows

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}^-_1 & \boldsymbol{A}^-_2 \\ \boldsymbol{A}^+_2 & \boldsymbol{A}^+_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{W}_{b_1} + a_-\mathfrak{M}^0_{\mathcal{K}^1_1}\boldsymbol{W}_{b_2} & e^{\pi si}a_-\mathfrak{M}^0_{\mathcal{K}^1_{-1}}\boldsymbol{W}_{b_3} \\ e^{\pi si}a_+\mathfrak{M}^0_{\mathcal{K}^1_{-1}}\boldsymbol{W}_{b_3} & \boldsymbol{W}_{b_1} + a_+\mathfrak{M}^0_{\mathcal{K}^1_1}\boldsymbol{W}_{b_2} \end{bmatrix} \tag{50}$$

and investigate the operator $\boldsymbol{A} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$.

It is easy to see that the functions $b_1(\xi)$, $b_2(\xi)$ $b_3(\xi)$ and $\mathcal{K}^1_{\pm 1}$ have the following limits:

$$b_1(\pm\infty) = 0, \quad b_1(0) = -\frac{e^{\pi si}}{\gamma},$$
$$b_2(-\infty) = -i, \quad b_2(+\infty) = -ie^{2\pi si}, \quad b_2(0) = 0,$$
$$b_3(\pm\infty) = -i, \quad b_3(0) = 0,$$
$$\mathcal{K}^1_1(\pm\infty) = \pm i, \quad \mathcal{K}^1_{-1}(\pm\infty) = 0.$$

Then, according to [9, Formulae (85), (86)] (also see the earlier paper [10]), the symbol of the operators $\boldsymbol{A}_1^{\pm}$ and $\boldsymbol{A}_2^{\pm}$ in (50) are written as follows:

$$\mathcal{A}_p^s(\omega)=\begin{bmatrix}(\mathcal{A}_1^-)_p^s(\omega) & (\mathcal{A}_2^-)_p^s(\omega)\\ (\mathcal{A}_2^+)_p^s(\omega) & (\mathcal{A}_1^+)_p^s(\omega)\end{bmatrix},$$

where

$$\begin{aligned}
(\mathcal{A}_1^{\pm})_p^s(\omega)&=ia_{\pm}\cot\pi(\beta-i\xi)\Big(\frac{e^{2\pi si}+1}{2}+\frac{e^{2\pi si}-1}{2i}\,\cot\pi(\beta-i\xi)\Big)\\
&=ia_{\pm}e^{\pi si}\,\frac{\cot\pi(\beta-i\xi)\sin\pi(\beta-i\xi+s)}{\sin\pi(\beta-i\xi)}\\
&=ia_{\pm}e^{\pi si}\,\frac{\cos\pi(\beta-i\xi)\sin\pi(\beta-i\xi+s)}{\sin^2\pi(\beta-i\xi)}\,,\\
(\mathcal{A}_2^{\pm})_p^s(\omega)&=\frac{-ia_{\pm}e^{\pi si}}{\sin\pi(\beta-i\xi)}\quad\text{if }\ \omega=(\xi,\infty)\in\overline{\Gamma_1},\\
(\mathcal{A}_1^{\pm})_p^s(\omega)&=\Big(\frac{-\eta-\gamma}{-\eta+\gamma}\Big)^s\,\frac{1}{-\eta-\gamma}-ia_{\pm}\Big(\frac{-\eta-\gamma}{-\eta+\gamma}\Big)^s\,\frac{i\eta}{-\eta-\gamma}\\
&=-\Big(\frac{\eta-\gamma}{\eta+\gamma}\Big)^{-s}\,\frac{1+a_{\pm}\eta}{\eta+\gamma}\,,\\
(\mathcal{A}_2^{\pm})_p^s(\omega)&=0\ \ \text{if }\ \omega=(+\infty,\eta)\in\overline{\Gamma_2^-}\,,\\
(\mathcal{A}_1^{\pm})_p^s(\omega)&=\Big(\frac{\eta-\gamma}{\eta+\gamma}\Big)^s\,\frac{1}{\eta-\gamma}+ia_{\pm}\Big(\frac{\eta-\gamma}{\eta+\gamma}\Big)^s\,\frac{-i\eta}{\eta-\gamma}\\
&=\Big(\frac{\eta-\gamma}{\eta+\gamma}\Big)^s\,\frac{1+a_{\pm}\eta}{\eta-\gamma},\\
(\mathcal{A}_2^{\pm})_p^s(\omega)&=0\ \ \text{if }\ \omega=(-\infty,\eta)\in\overline{\Gamma_2^+}\,,\\
(\mathcal{A}_1^{\pm})_p^s(\omega)&=-\frac{e^{\pi si}}{\gamma}\\
(\mathcal{A}_2^{\pm})_p^s(\omega)&=0\ \ \text{if }\ \omega=(\xi,\infty)\in\overline{\Gamma_3},
\end{aligned}$$

and $\beta=\frac{1}{p}$, $\xi\in\mathbb{R}$, $\eta\in\mathbb{R}^+$. Then

$$\det\mathcal{A}_p^s(\omega)$$
$$=\begin{cases}-a_-a_+e^{2\pi si}\,\dfrac{\cos^2\pi(\beta-i\xi)\sin^2\pi(\beta-i\xi+s)-\sin^2\pi(\beta-i\xi)}{\sin^4\pi(\beta-i\xi)}\,,\\ \qquad\qquad\omega=(\xi,\infty)\in\overline{\Gamma_1},\\ \mp\Big(\dfrac{\eta-\gamma}{\eta+\gamma}\Big)^{\mp 2s}\,\dfrac{(1+a_-\eta)(1+a_+\eta)}{\eta^2-\gamma^2}\,,\quad\omega=(\pm\infty,\eta)\in\Gamma_2^{\pm},\\ \dfrac{e^{2\pi si}}{\gamma^2}\,,\quad\omega=(\xi,0)\in\overline{\Gamma_3}.\end{cases}\tag{51}$$

The symbol $\mathcal{A}_p^s(\omega)$ is non-elliptic (i.e., $\det\mathcal{A}_p^s(\omega)=0$) if and only if

(i) $(1+a_-\eta)(1+a_+\eta)\neq 0$ for all $0<\eta<\infty$. This condition holds if and only if coefficients $a_\pm$ are not negative reals: $a_\pm\in\mathbb{C}\setminus\overline{\mathbb{R}^-}$;

(ii) The parameters p and s are solutions to the equation

$$\cos^2\pi\Big(\frac{1}{p}-i\xi\Big)\sin^2\pi\Big(\frac{1}{p}+s-i\xi\Big)-\sin^2\pi\Big(\frac{1}{p}-i\xi\Big)=0. \tag{52}$$

By analyzing the transcendental equation (52), we come to the following conclusions.

(52) have a solution only for $\xi=0$ and (52) transforms into an equivalent transcendental equation (2).

For $1<p<4$, (2) has no solution for any $-1\leqslant s\leqslant 1$, because if we write it in an equivalent form

$$\sin^2\pi\Big(\frac{1}{p}+s\Big)=\tan^2\frac{\pi}{p} \tag{53}$$

the right-hand side is more than 1, while the left-hand side is less than or is equal to 1. On the other hand, in the classical setting $p=2$, $s=-\frac{1}{2}$ equation (1) has a unique solution (see (47)). Since this pair belongs to the quadrat $1<p<4$, $-1\leqslant s\leqslant 1$, where equation (1) is Fredholm, it has the same kernel and co-kernel in all these cases, i.e., is uniquely solvable for all $1<p<4$ and all $-1\leqslant s\leqslant 1$ (see [5] and [12] for the proof of the assertion).

For $4\leqslant p<\infty$, (53) has, due to the periodicity, two pairs of solutions $\{p,s_p\}$ and $\{p,s_p-1\}$, where $s_p>0$, $s_p-1<0$. It can be shown that for $s_p-1<s<s_p$, for $-1\leqslant s<s_p-1$, and for $s_p<s\leqslant 1$ the symbol $\mathcal{A}^s_p$ has index 0, +1 and −1, respectively. Manipulating with the properties of kernels and co-kernels in embedded spaces, we can prove easily that equation (1) has, respectively, no kernel and co-kernel (is uniquely solvable), has no kernel, but 1-dimensional co-kernel (has a unique solution for all right-hand sides which are orthogonal to the solution of the dual homogeneous equation) and has the 1-dimensional kernel, but no co-kernel (has a non-unique solution for all right-hand sides), respectively. □

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Authors' address:

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia.

E-mail: `RolDud@gmail.com`; `tamta.wuwunava@mail.ru`

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Lasha Ephremidze and Ilya Spitkovsky

MATRIX SPECTRAL FACTORIZATION WITH PERTURBED DATA

Dedicated to Boris Khvedelidze's 100-th birthday anniversary

Abstract. A necessary condition for the existence of spectral factorization is positive definiteness a.e. on the unit circle of a matrix function which is being factorized. Correspondingly, the existing methods of approximate computation of the spectral factor can be applied only in the case where the matrix function is positive definite. However, in many practical situations an empirically constructed matrix spectral densities may lose this property. In the present paper we consider possibilities of approximate spectral factorization of matrix functions by their known perturbation which might not be positive definite on the unit circle.

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რეზიუმე. სპექტრალური ფაქტორიზაციის არსებობისათვის აუცილებელია გასაფაქტორიზირებელი მატრიც ფუნქცია იყოს დადებითად განსაზღვრული თ.ყ. ერთეულოვან წრეწირზე. შესაბამისად, არსებული მეთოდები სპექტრალური თანამამრავლის მიახლოებით პოვნისა შეიძლება გამოყენებულ იქნას მხოლოდ დადებითად განსაზღვრული მატრიცებისათვის, მაშინ როცა მრავალ პრაქტიკულ სიტუაციაში ემპირიულად შედგენილი სპექტრალური სიმკვრივის მატრიცი შეიძლება უკვე აღარ იყოს ამ თვისების მატარებელი. წინამდებარე ნაშრომში განხილულია შესაძლებლობები მატრიც ფუნქციების მიახლოებითი ფაქტორიზაციისა მათი "შეშფოთებული" ვერსიის მიხედვით, რომელნიც შეიძლება აღარ იყვნენ დადებითად განსაზღვრულნი.

1. Introduction

Matrix Spectral Factorization Theorem [9], [5] asserts that if

$$S(t) = \begin{pmatrix} s_{11}(t) & s_{12}(t) & \cdots & s_{1r}(t) \\ s_{21}(t) & s_{22}(t) & \cdots & s_{2r}(t) \\ \vdots & \vdots & \vdots & \vdots \\ s_{r1}(t) & s_{r2}(t) & \cdots & s_{rr}(t) \end{pmatrix}, \tag{1}$$

$|t| = 1$, is a positive definite (a.e.) matrix function with integrable entries, $s_{ij} \in L^1(\mathbb{T})$, and if the Paley–Wiener condition

$$\log \det S \in L^1(\mathbb{T}) \tag{2}$$

is satisfied, then (1) admits a (left) spectral factorization

$$S(t) = S^+(t)S^-(t) = S^+(t)(S^+(t))^*, \tag{3}$$

where S^+ is an $r \times r$ outer analytic matrix function with entries from the Hardy space $H^2(\mathbb{D})$, $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$, and $S^-(z) = (S^+(1/\overline{z}))^*$, $|z| > 1$. It is assumed that (3) holds for boundary values a.e. on $\mathbb{T}$. A spectral factor S^+ is unique up to a constant right unitary multiplier (see e.g. [3]).

In the scalar case, $r = 1$, a spectral factor of a positive function f can be explicitly written by the formula

$$f^+(z) = \exp\left(\frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log f(e^{i\theta})\, d\theta \right) \tag{4}$$

and it is well-known that if (1) is a Laurent polynomial matrix

$$S(t) = \sum_{k=-N}^{N} C_k t^k, \ \ C_k \in \mathbb{C}^{r \times r}, \tag{5}$$

then the spectral factor $S^+(t) = \sum_{k=0}^{N} A_k t^k$ is a polynomial matrix of the same order (see e.g. [2]).

A challenging practical problem is actual approximate computation of matrix coefficients of analytic function S^+ for a given matrix function (1). Starting with Wiener's original efforts [10], various methods have been developed to approach this problem (see the survey papers [7], [8] and references therein). Recently, a new algorithm of matrix spectral factorization has been proposed in [6]. This algorithm can be applied to any matrix function which satisfies the necessary and sufficient condition (2) for the existence of factorization. (Most of other algorithms impose additional restrictions on (1), such as S to be rational or strictly positive definite on the boundary.) In the present paper we would like to demonstrate that (at least in the polynomial case) the proposed algorithm can be also applied to the so-called "perturbed" data which looses the property of positive definiteness. Namely, we consider and solve the following problem.

In most practical applications of spectral factorization, a power spectral density S is constructed from empirical observations which are always subject to small numerical errors. Thus, instead of theoretically existing matrix spectral density (1), which is always positive definite (a.e.) on $\mathbb{T}$, we have to deal with $\widehat{S}$ which may no longer be even positive semi-definite on $\mathbb{T}$. The classical illustrative example is when $S(t)=\sum\limits_{k=-n}^{n} C_k t^k$ is a Laurent matrix polynomial with $\det S(t_0)=0$ for some $t_0\in\mathbb{T}$ and we disturb the coefficients C_k. The question arises if the above mentioned spectral factorization algorithm can treat $\widehat{S}$ as positive definite in order to correct this "small error" in data and find S^+ approximately. (Most of existing matrix spectral factorization algorithms do not make sense for non positive definite data.) Below we provide a positive answer to this question. To be specific, for polynomial matrix functions, depending on algorithm proposed in [6], we explicitly describe a computational procedure which can be applied to any polynomial data (say, maps $\mathfrak{C}_n:\mathcal{P}_N(m\times m)\to\mathcal{P}_N^+(m\times m)$, $n=1,2,\ldots$, see Section 2 for the notation) in such a manner that the following statement is true.

Theorem 1. *Let S be a polynomial matrix function* (5) *which is positive semi-definite on $\mathbb{T}$ and has a spectral factor S^+, and let S_n, $n=1,2,\ldots$, be a sequence of arbitrary* (*not necessarily positive semi-definite on $\mathbb{T}$*) *polynomial matrix functions of the same degree N such that*

$$\|S_n-S\|_{L^1}\to 0. \tag{6}$$

Then

$$\|\mathfrak{C}_n(S_n)-S^+\|_{L^2}\to 0. \tag{7}$$

The paper is organized as follows. In the next section, we introduce the notation that will be used throughout the paper. In Section 3, we review the matrix spectral factorization algorithm proposed in [6] and in Section 4 we describe the strategy dealing with non positive definite matrices. In Section 5, we consider the above formulated problem in the scalar case and solve it for polynomial functions. A partial solution of the problem is provided for general spectral densities. The main Theorem 1 along with some auxiliary lemmas are proved in Section 6.

2. Notation

Let $\mathbb{T}=\{z\in\mathbb{C}:\ |z|=1\}$ with the standard Lebesgue measure $d\mu$ on it and $\overline{\mathbb{D}}=\{z\in\mathbb{C}:\ |z|\leq 1\}$. As usual, $L^p=L^p(\mathbb{T})$, $0<p<\infty$, denotes the Lebesgue space of p-integrable complex functions defined on $\mathbb{T}$, and $\mathbb{C}^{m\times m}$, $L^p(\mathbb{T})^{m\times m}$, etc., denote the set of $m\times m$ matrices with entries from $\mathbb{C}$, $L^p(\mathbb{T})$, etc. If $S\in\mathbb{C}^{r\times r}$ is a matrix (function) and $m\leq r$, then $S_{[m]}$ stands for the upper-left $m\times m$ submatrix of S ($S_{[0]}$ is assumed to be 1). For a matrix (function) M, its Hermitian conjugate matrix (function) is denoted by $M^* = \overline{M}^T$. Finally, I_m is the $m \times m$ unit matrix.

The kth Fourier coefficient of an integrable (matrix) function $f \in L^1(\mathbb{T})$ ($f \in L^1(\mathbb{T})^{m\times m}$) is denoted by $c_k\{f\}$ ($C_k\{f\} \in \mathbb{C}^{m\times m}$). For $p \geq 1$, $L^p_+(\mathbb{T}) := \{f \in L_p(\mathbb{T}) : \ c_k\{f\} = 0$ whenever $k < 0\}$, and, for $n \geq 0$, $L^p_{n-}(\mathbb{T}) := \{f \in L_p(\mathbb{T}) : \ c_k\{f\} = 0$ whenever $k < -n\}$. Moreover, $\mathcal{P}_N := \{\sum_{k=-N}^{N} c_k z^k, \ c_k \in \mathbb{C}\}$ is the set of trigonometric polynomials of degree at most N and $\mathcal{P}_N^+ := \{\sum_{k=0}^{N} c_k z^k, \ c_k \in \mathbb{C}\}$. Also, $\mathcal{P} = \cup \mathcal{P}_N$ and $\mathcal{P}^+ = \cup \mathcal{P}_N^+$, while $\mathbb{Q}[z] := \{p/q : \ p, q \in \mathcal{P}^+\}$ stands for the set of rational functions.

The Hardy space of analytic functions in $\mathbb{D}$, $H^p = H^p(\mathbb{D})$ is identified with $L^p_+(\mathbb{T})$ for $p \geq 1$, and $H^p_O = H^p_O(\mathbb{D})$ is the set of outer analytic functions from H_p. A square matrix function is called outer if its determinant is an outer function.

For a real function f, let ${}_\delta f$ be the truncated function

$$ {}_\delta f(t) = \begin{cases} f(t) & \text{if } f(t) > \delta, \\ \delta & \text{if } f(t) \leq \delta \end{cases} \tag{8} $$

(we usually use the argument "t" for functions defined on $\mathbb{T}$). Also, let $f^{(+)} = \max(0, f)$ and $f^{(-)} = \max(0, -f)$.

The notation $f_n \rightrightarrows f$ means that f_n converges to f in measure. Observe that

$$ f_n \rightrightarrows f \Longrightarrow f_n^{(+)} \rightrightarrows f^{(+)}. \tag{9} $$

We will also use the following implication (see, e.g. [4, Corollary 1]):

$$ \|f_n - f\|_{L^2} \to 0, \ \ |u_n| \leq 1, \ \ u_n \rightrightarrows u \Longrightarrow \|f_n u_n - f u\|_{L^2} \to 0. \tag{10} $$

3. Overview of the Matrix Spectral Factorization Algorithm

The first step of the matrix spectral factorization algorithm proposed in [6] is the triangular factorization

$$ S(t) = M_S(t) M_S^*(t), \tag{11} $$

where $M_S(t)$ is the lower triangular matrix

$$ M_S(t) = \begin{pmatrix} f_1^+(t) & 0 & \cdots & 0 & 0 \\ \xi_{21}(t) & f_2^+(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_{r-1,1}(t) & \xi_{r-1,2}(t) & \cdots & f_{r-1}^+(t) & 0 \\ \xi_{r1}(t) & \xi_{r2}(t) & \cdots & \xi_{r,r-1}(t) & f_r^+(t) \end{pmatrix}, \tag{12} $$

$\xi_{ij} \in L^2(\mathbb{T})$, $f_i^+ \in H^2_O$. Then M_S is post multiplied by the unitary matrix functions of the special form $\mathbf{U}_2, \mathbf{U}_3, \ldots, \mathbf{U}_r$, so that to make the left-upper

$m \times m$ submatrices of M_S analytic step-by-step, $m = 2, 3, \ldots, r$. As a result, we get (see [4, formula (47)])

$$S^+(t) = M_S(t)\mathbf{U}_2(t)\mathbf{U}_3(t)\cdots\mathbf{U}_r(t), \tag{13}$$

where each $\mathbf{U}_m$ has a block matrix form

$$\mathbf{U}_m(t) = \begin{pmatrix} U_m(t) & 0 \\ 0 & I_{r-m} \end{pmatrix},$$

and $U_m(t)$ is the special unitary matrix function

$$U(t) = \begin{pmatrix} u_{11}(t) & u_{12}(t) & \cdots & u_{1,m-1}(t) & u_{1m}(t) \\ u_{21}(t) & u_{22}(t) & \cdots & u_{2,m-1}(t) & u_{2m}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m-1,1}(t) & u_{m-1,2}(t) & \cdots & u_{m-1,m-1}(t) & u_{m-1,m}(t) \\ \overline{u_{m1}(t)} & \overline{u_{m2}(t)} & \cdots & \overline{u_{m,m-1}(t)} & \overline{u_{mm}(t)} \end{pmatrix}, \tag{14}$$

$$u_{ij} \in L_+^\infty, \quad \det U(t) = 1 \text{ a.e.}, \tag{15}$$

while, for each $m \le r$, the left-upper $m \times m$ submatrix of $M_S\mathbf{U}_2\mathbf{U}_3\ldots\mathbf{U}_m$ is a spectral factor of the left-upper $m \times m$ submatrix of S, i.e.,

$$\big(M_S(t)\mathbf{U}_2(t)\mathbf{U}_3(t)\cdots\mathbf{U}_m(t)\big)_{[m]} = S^+_{[m]}. \tag{16}$$

An explicit description of the representation (13) and its approximate computation are discussed in [6], [4]. In particular, when the left-upper $(m-1) \times (m-1)$ submatrix of (12) has already been made analytic, the matrix function $\big(M_S\mathbf{U}_2\mathbf{U}_3\ldots\mathbf{U}_{m-1}\big)_{[m]}$ has the form

$$\big(M_S\mathbf{U}_2\cdots\mathbf{U}_{m-1}\big)_{[m]}$$
$$= \begin{pmatrix} & & & & 0 \\ & & S^+_{[m-1]} & & \vdots \\ & & & & 0 \\ \zeta_1 & \zeta_2 & \ldots & \zeta_{m-1} & f_m^+ \end{pmatrix} = \begin{pmatrix} & & & & 0 \\ & & S^+_{[m-1]} & & \vdots \\ & & & & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} F, \tag{17}$$

where F is the matrix function

$$F(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \zeta_1(t) & \zeta_2(t) & \zeta_3(t) & \cdots & \zeta_{m-1}(t) & f_m^+(t) \end{pmatrix}. \tag{18}$$

Remark 1. Note that matrix function (17) multiplied by its Hermitian conjugate gives $S_{[m]}$. Therefore, the following equation

$$S^+_{[m-1]} \begin{pmatrix} \overline{\zeta_1} \\ \overline{\zeta_2} \\ \vdots \\ \overline{\zeta_{m-1}} \end{pmatrix} = \begin{pmatrix} s_{1m} \\ s_{2m} \\ \vdots \\ s_{m-1,m} \end{pmatrix} \tag{19}$$

holds.

The analyticity of the m-th row in (17) is achieved by application of the following

Theorem 2 (see [4, Lemma 4])**.** *For any matrix function F of the form* (18)*, where*

$$\zeta_i \in L^2(\mathbb{T}), \quad i = 1, 2, \dots, m-1, \quad f^+ \in H^2_O, \tag{20}$$

there exists a unitary matrix function U of the form (14), (15)*, such that*

$$FU \in L^2_+(\mathbb{T})^{m\times m}.$$

Remark 2. Note that under the above circumstances,

$$S^+_{[m]} = \begin{pmatrix} & & & & 0 \\ & & S^+_{[m-1]} & & \vdots \\ & & & & 0 \\ \zeta_1 & \zeta_2 & \dots & \zeta_{m-1} & f^+_m \end{pmatrix} U. \tag{21}$$

In order to compute (14) approximately, (18) is approximated by its Fourier series. More specifically, let F_n be the matrix function (18) in which the last row is replaced by

$$\big(\zeta_1^n(t), \zeta_2^n(t), \dots, \zeta_{m-1}^n(t), f_n^+(t)\big), \quad \zeta_i^n \in L^2(\mathbb{T}), \tag{22}$$

where $\zeta_i^n(t) = \sum\limits_{k=-n}^{\infty} c_k\{\zeta_i\} t^k$, $f_n^+ = f_m^+$. Then the following result is invoked:

Theorem 3 (see [6, Theorem 1])**.** *Let F_n be a matrix function of the form* (18), (22)*, where*

$$\zeta_i^n \in L^2_{n-}(\mathbb{T}), \quad i = 1, 2, \dots, \quad \textit{and} \quad f_n^+(0) \neq 0. \tag{23}$$

Then there exists a unique unitary matrix function U_n of the form (14)*, where $u_{ij} \in \mathcal{P}_n^+$,* $\det U_n(t) = 1$ *(on $\mathbb{T}$), and $U_n(1) = I_m$, such that*

$$F_n U_n \in L^2_+(\mathbb{T})^{m\times m}.$$

Note that [6] in fact provides an explicit construction of U_n.

In order to justify the approximating properties of the algorithm, the following convergence theorem is proved in [4].

Theorem 4 (cf. [4, Theorem 2]). *Let F be a matrix function of the form* (18), (20)*, and let F_n, $n = 1, 2, \dots$, be a sequence of matrix functions of the form* (18) *with the last row replaced by* (22)*. Let also*

$$\zeta_i^n \to \zeta_i, \;\; f_n^+ \to f^+ \;\; \text{in } L^2 \;\; \text{and } \; f_n^+ \in H^2_O. \tag{24}$$

If U_n, $n = 1, 2, \dots$, are the corresponding unitary matrix functions defined according to Theorem 2, then U_n converges in measure:

$$U_n \rightrightarrows U,$$

and $F_n U_n$ converges in L^2 to the spectral factor of FF^*.

4. Treatment of Nonpositive Definite Matrix Functions

The main argument which helps to deal with the matrix functions $\widehat{S}$ which are close to S, but are not necessarily positive semi-definite (a.e. on $\mathbb{T}$) is the observation that Theorem 4 remains valid if in (24) we replace the condition $f_n^+ \in H^2_O$ by a weaker requirement $f_n^+(0) \neq 0$, as in (23). Theorem 3 guarantees that the corresponding U_n exists in this case as well. Because of the importance of this fact for our goals, we formulate this result separately.

Theorem 5. *Let F be a matrix function of the form* (18), (20)*, and let F_n, $n = 1, 2, \dots$, be a sequence of matrix function of the form* (18), (22), (23) *such that*

$$\zeta_i^n \to \zeta_i \;\; \text{and} \;\; f_n^+ \to f^+ \;\; \text{in } L^2.$$

If U_n, $n = 1, 2, \dots$, are the corresponding unitary matrix functions defined according to Theorem 3, then U_n converges in measure:

$$U_n \rightrightarrows U,$$

and $F_n U_n$ converges in L^2 to the spectral factor of FF^*.

Remark 3. It should be observed that under the above circumstances $F_n U_n$ might not be the canonical spectral factor of $F_n F_n^*$ (which was the case in the situation of Theorem 4), since $\det(F_n U_n) = \det(F_n) = f_n^+$ might have zeros inside the unit circle. Thus the phrase in the end of the first column on page 2320 in [6] contains a small inaccuracy.

Although Theorems 4 and 5 look alike, there is a significant difference between their meaning, as it is explained in the above remark. Nevertheless, the proof of Theorem 5 does not require any additional efforts, and the proof of Theorem 4 given in [4] goes through without any changes. Therefore we do not provide the proof of Theorem 5 here.

For a positive definite (a.e. on $\mathbb{T}$) matrix function (1) the triangular factorization (11) can be performed by the recurrent formulas which are similar to Cholesky factorization formulas for constant positive definite matrices.

Namely, for the entries of matrix function (12), we can write (see [6], formulas (56)–(58)):

$$f_m^+ = \sqrt{\det S_{[m]} / \det S_{[m-1]}} = \det S^+_{[m]} / \det S^+_{[m-1]}\,, \tag{25}$$

where $\sqrt{f}$ is the scalar spectral factor of f defined by (4),

$$\xi_{i1} = s_{i1} / \overline{f_1^+},\ \ i = 2, 3, \dots, r, \tag{26}$$

$$\xi_{ij} = \Big(s_{ij} - \sum_{k=1}^{j-1} \xi_{ik} \overline{\xi_{jk}}\Big) \Big/ \overline{f_j^+},\ \ j=2,3,\dots,r-1,\ \ i=j+1, j+2, \dots, r. \tag{27}$$

If now $\widehat{S}$ is not necessarily positive definite, then $\det \widehat{S}_{[m]}$ might become negative on a set of positive measure and we would not be able to define the scalar spectral factor of $\det S_{[m]} / \det S_{[m-1]}$. However, we could still define $M_{\widehat{S}}$ using formulas (25)–(27) if we were able to determine the $\sqrt{f}$ for not necessarily positive function f. In the following section, we define a "scalar spectral factor" of not necessarily positive function for specific cases. If we continue the computational procedures described in Section 3 for $M_{\widehat{S}}$ in place of M_S, we get $M_{\widehat{S}} \mathbf{U}_2 \mathbf{U}_3 \cdots \mathbf{U}_r$ which is similar to expression (13) and therefore we would expect its closeness to S^+. For polynomial matrix functions, we perform these procedures in an explicit way.

5. The Scalar Case

If $0 \le f \in L^1(\mathbb{T})$ and $\log f \in L^1(\mathbb{T})$, then the spectral factor f^+ can be written by the formula (4). However, if we only know that $\widehat{f}$ is close to f in L^1 norm, then $\widehat{f}$ might even not be non-negative a.e. (we discard the imaginary part of $\widehat{f}$ if it exists, so we assume here that $\widehat{f}$ is a real function). Even if $\widehat{f}$ were positive, $\log \widehat{f}$ should also be close to $\log f$ in order to claim the closeness of $\widehat{f}^+$ to f^+ (see [4], [1]). Therefore, we consider

$$_\delta f^+(z) = \exp\left(\frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log {_\delta f}(e^{i\theta})\, d\theta\right) \tag{28}$$

(see (8)) and prove the following

Lemma 1. *Let* $0 \le f \in L^1(\mathbb{T})$ *and* $\log f \in L^1(\mathbb{T})$. *Suppose* $f_n \in L^1(\mathbb{T})$ *and*

$$\|f_n - f\|_{L^1} \to 0. \tag{29}$$

Then

$$\lim_{\delta \to 0+} \lim_{n \to \infty} \|{_\delta f_n^+} - f^+\|_{L^2} = 0. \tag{30}$$

Proof. It has been proved in [4] that if $0 \le f_n \in L^1(\mathbb{T})$, (29) holds and $\int_0^{2\pi} \log f_n(e^{i\theta})\, d\theta \to \int_0^{2\pi} \log f(e^{i\theta})\, d\theta$, then $\|f_n^+ - f^+\|_{H^2} \to 0$. Hence, we can show first that

a) $\lim\limits_{n\to\infty} \|{}_\delta f_n - {}_\delta f\|_{L^1} = 0$ for each $\delta > 0$ and

$$\lim_{n\to\infty} \int\limits_{\mathbb{T}} \log {}_\delta f_n(e^{i\theta})\, d\theta = \int\limits_{\mathbb{T}} \log {}_\delta f(e^{i\theta})\, d\theta, \tag{31}$$

which implies that $\lim\limits_{n\to\infty} \|{}_\delta f_n^+ - {}_\delta f^+\|_{L^2} = 0$ and, consequently,

$$\lim_{n\to\infty} \|{}_\delta f_n^+ - f^+\|_{L^2} = \|{}_\delta f^+ - f^+\|_{L^2}, \tag{32}$$

and then

b) $\lim\limits_{\delta\to 0+} \|{}_\delta f - f\|_{L^1} \to 0$ and $\lim\limits_{\delta\to 0+} \int\limits_{\mathbb{T}} \log {}_\delta f(e^{i\theta})\, d\theta = \int\limits_{\mathbb{T}} \log f(e^{i\theta})\, d\theta$, which implies that

$$\lim_{\delta\to 0+} \|{}_\delta f^+ - f^+\|_{L^2} = 0. \tag{33}$$

The relation (30) will then follow from (32) and (33).

Part b) is an easy exercise in Lebesgue integration theory, and we will thus concentrate on a). It is easy to realize that $|{}_\delta f_n - {}_\delta f| \le |f_n - f|$ and therefore the first part of a) follows from (29), which also implies that $\log {}_\delta f_n \rightrightarrows \log {}_\delta f$ as $n \to \infty$. In addition, $[\log {}_\delta f_n]^{(\pm)} \rightrightarrows [\log {}_\delta f]^{(\pm)}$ (see (9)).

The necessary and sufficient condition for (29) is that

$$f_n \rightrightarrows f \quad \text{and} \quad \sup_{n>k, \mu(E)<\varepsilon} \int\limits_E f_n\, dm \to 0 \quad \text{as} \quad k \to \infty \quad \text{and} \quad \varepsilon \to 0.$$

Therefore, $\|[\log {}_\delta f_n]^{(+)} - [\log {}_\delta f]^{(+)}\|_{L^1} \to 0$ and, in addition, $\|[\log {}_\delta f_n]^{(-)} - [\log {}_\delta f]^{(-)}\|_{L^1} \to 0$ due to the bounded convergence theorem. Thus (31) follows. □

The relation (30) shows that for any sequence f_n, $n = 1, 2, \ldots$, satisfying (29) there exist $\delta_n \to 0+$ such that

$$\lim_{n\to\infty} \|{}_{\delta_n} f_n^+ - f^+\|_{L^2} = 0. \tag{34}$$

However, (34) does not hold for every sequence $\delta_n \to 0+$ and, in general, it is hard to determine conditions on δ_n which would guarantee (34).

An explicit computational procedure is proposed for polynomial case. Namely, for any polynomial $p(z) = \sum\limits_{k=-N}^{N} c_k z^k$ (which might not be positive or even real on $\mathbb{T}$), let

$$\check{p}(t) = \begin{cases} \Re\mathrm{e}\{p(t)\} & \text{when } \Re\mathrm{e}\{p(t)\} > 0, \\ 1 & \text{when } \Re\mathrm{e}\{p(t)\} \le 0, \end{cases} \qquad t \in \mathbb{T}, \tag{35}$$

let $\check{p}^+$ be the spectral factor of $\check{p}$:

$$\check{p}^+ = \exp\left(\frac{1}{4\pi} \int\limits_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \check{p}(e^{i\theta})\, d\theta \right), \tag{36}$$

and let $\widetilde{p}^+$ be its Fourier approximation up to degree N:

$$\widetilde{p}_n^+(z) = \sum_{k=0}^{N} c_k\{\widehat{p}_n^+\} z^k. \tag{37}$$

We prove the following

Lemma 2. *Let*

$$f(t) = \sum_{k=-N}^{N} c_k t^k \geq 0 \ \ for \ \ t \in \mathbb{T}, \tag{38}$$

and let

$$f_n(t) = \sum_{k=-N}^{N} c_k^{\{n\}} t^k$$

be such a sequence that

$$f_n \to f. \tag{39}$$

Then

$$\widetilde{f}_n^+ \to f^+. \tag{40}$$

Proof. We will show that

$$\|\check{f}_n^+ - f^+\|_{H^2} \to 0 \tag{41}$$

which implies (40), by virtue of the definition (37).

In order to prove (41), it is sufficient to show that (see [4])

$$\|\check{f}_n - f\|_{L^1} \to 0 \tag{42}$$

and

$$\int_{\mathbb{T}} \log \check{f}_n(t)\, dt \to \int_{\mathbb{T}} \log f(t)\, dt. \tag{43}$$

Let $E_n := \{t \in \mathbb{T} : \Re\mathrm{e}\{f_n(t)\} > 0\}$. Then

$$\check{f}_n = \mathbf{1}_{E_n} \Re\mathrm{e}\{f_n\} + \mathbf{1}_{\mathbb{T}\setminus E_n} = \Re\mathrm{e}\{f_n\}^{(+)} + \mathbf{1}_{\mathbb{T}\setminus E_n}. \tag{44}$$

Since (39) holds, we have $\|\Re\mathrm{e}\{f_n\} - f\|_{L^1} \to 0$, which implies that

$$\left\|\Re\mathrm{e}\{f_n\}^{(+)} - f^{(+)}\right\|_{L^1} = \left\|\Re\mathrm{e}\{f_n\}^{(+)} - f\right\|_{L^1} \to 0. \tag{45}$$

Since $\Re\mathrm{e}\{f_n\} \rightrightarrows f$, $\mu\{f \leq 0\} = 0$, and 0 is the continuity point of the distribution function $t \mapsto \mu\{f \leq t\}$, we have

$$\mu(\mathbb{T} \setminus E_n) \to 0, \tag{46}$$

which implies that $\|\mathbf{1}_{\mathbb{T}\setminus E_n}\|_{L^1} \to 0$ and (42) follows from (44) and (45).

In order to prove (43), we need the following

Lemma 3. *Let $\mathcal{P}'_N \subset \mathcal{P}^+_N$ be the set of monic polynomials with the degrees not exceeding N. Then*

$$\lim_{\delta \to 0+} \sup_{f \in \mathcal{P}'_N} \left| \int_{\{t\in\mathbb{T}:\ |f(t)|<\delta\}} \log|f(t)|\, d\mu \right| = 0. \tag{47}$$

Proof. Let

$$f(z)=\prod_{k=1}^{N}(z-z_k).$$

Then $\{t\in\mathbb{T}:\ |f(t)|<\delta\}\subset\bigcup_{k=1}^{N}\{t\in\mathbb{T}:\ |t-z_k|<\delta^{1/N}\}$ and

$$\Bigg|\int\limits_{\{|f|<\delta\}}\log|f|\,d\mu\Bigg|\le\int\limits_{\bigcup_{k=1}^{N}\{|t-z_k|<\delta^{1/N}\}}\big|\log|f|\big|\,d\mu$$

$$\le\sum_{j=1}^{N}\int\limits_{\bigcup_{k=1}^{N}\{|t-z_k|<\delta^{1/N}\}}\big|\log|t-z_j|\big|\le\sum_{j=1}^{N}\sum_{k=1}^{N}\int\limits_{\{|t-z_k|<\delta^{1/N}\}}\big|\log|t-z_j|\big|$$

$$\le N^2\int\limits_{\{|t-1|<\delta^{1/N}\}}\big|\log|t-1|\big|\,d\mu\to 0\ \text{ as }\ \delta\to 0+\,.$$

Consequently, (47) holds. □

We continue with the proof of (43) as follows.

Since $\Re\mathrm{e}\{f_n\}\to f$ and $\check{f}_n-\Re\mathrm{e}\{f_n\}\rightrightarrows 0$ by virtue of (35) and (46), the convergence in measure

$$\check{f}_n\rightrightarrows f \tag{48}$$

holds, which implies that ${}_\delta\check{f}_n\rightrightarrows{}_\delta f$ for each $\delta>0$, and since $\check{f}_n$ are uniformly bounded as well, from the above by virtue of (39), we have

$$\int\limits_{\mathbb{T}}\log{}_\delta\check{f}_n\,d\mu\to\int\limits_{\mathbb{T}}\log{}_\delta f\,d\mu. \tag{49}$$

On the other hand,

$$\lim_{\delta\to0+}\Bigg|\int\limits_{\mathbb{T}}\log{}_\delta f\,d\mu-\int\limits_{\mathbb{T}}\log f\,d\mu\Bigg|\le\lim_{\delta\to0+}\int\limits_{\{f\le\delta\}}\log f\,d\mu=0 \tag{50}$$

as $\log f\in L^1(\mathbb{T})$, and

$$\Bigg|\int\limits_{\mathbb{T}}\log{}_\delta\check{f}_n\,d\mu-\int\limits_{\mathbb{T}}\log\check{f}_n\,d\mu\Bigg|\le\Bigg|\int\limits_{\{0<\check{f}_n\le\delta\}}\log\check{f}_n\,d\mu\Bigg|$$

$$\le\Bigg|\int\limits_{\{0<\Re\mathrm{e}\{f_n\}\le\delta\}}\log\Re\mathrm{e}\{f_n\}\,d\mu\Bigg|\to 0\ \text{ as }\ \delta\to0+, \tag{51}$$

by virtue of Lemma 3, since $\Re\mathrm{e}\{f_n\}(t)=\Re\mathrm{e}\{\sum_{k=-N}^{N}c_k^{\{n\}}t^k\}=\sum_{k=-N}^{N}\check{c}_k^{\{n\}}t^k$ are trigonometric polynomials and $\big(\check{c}_N^{\{n\}}\big)^{-1}t^N\Re\mathrm{e}\{f_n\}(t)\in\mathcal{P}'_{2N}$, while $\check{c}_N^{\{n\}}$

are uniformly bounded. Now (43) follows from (49), (50) and (51) since

$$\lim_{n\to\infty}\Big|\int_{\mathbb{T}}\log\check{f}_n\,d\mu-\int_{\mathbb{T}}\log f\,d\mu\Big|$$
$$\leq \lim_{n\to\infty}\limsup_{\delta\to 0+}\Big|\int_{\mathbb{T}}\log\check{f}_n\,d\mu-\int_{\mathbb{T}}\log{}_\delta\check{f}_n\,d\mu\Big|$$
$$+\lim_{n\to\infty}\limsup_{\delta\to 0+}\Big|\int_{\mathbb{T}}\log{}_\delta\check{f}_n\,d\mu-\int_{\mathbb{T}}\log{}_\delta f\,d\mu\Big|$$
$$+\limsup_{\delta\to 0+}\Big|\int_{\mathbb{T}}\log{}_\delta f\,d\mu-\int_{\mathbb{T}}\log f\,d\mu\Big|=0. \quad \square$$

6. The Matrix Case

In this section we introduce the computational procedures $\mathfrak{C}_n:\mathcal{P}_N(m\times m)\to\mathcal{P}_N^+(m\times m)$, $n=1,2,\ldots$, and prove Theorem 1. First, we need two auxiliary lemmas.

Lemma 4. *Let*

$$f(z)=\frac{p(z)}{q(z)}\in\mathbb{Q}[z]\cap L^\infty(\mathbb{T}) \tag{52}$$

be a rational function without poles in $\overline{\mathbb{D}}$, satisfying

$$|f(z)|<C \ \ for\ \ z\in\mathbb{T}, \tag{53}$$

and let

$$p_n\to p \ \ and\ \ q_n\to q, \tag{54}$$

where $\deg(p_n)=\deg(p)$ *and* $\deg(q_n)=\deg(q)$, $n=1,2,\ldots.$

Let $\omega_{k,n}=\exp(\frac{2\pi k}{n}i)$, $n=1,2,\ldots$, $k=0,1,\ldots,n-1$, *be the Discrete Fourier Transform nodes. Then*

$$V_n:=\frac{2\pi}{n}\sum_{k=0}^{n-1}|h_n(\omega_{k,n})|^2\to\|f\|_2^2 \ \ as\ \ n\to 0, \tag{55}$$

where

$$h_n(\omega)=\begin{cases}p_n(\omega)/q_n(\omega) & if\ \ |p_n(\omega)/q_n(\omega)|\leq C,\\ 0 & if\ \ |p_n(\omega)/q_n(\omega)|>C.\end{cases}$$

Proof. By virtue of (54), for each $R<\infty$, the set of polynomials p_n, $n=1,2,\ldots$, is uniformly bounded on $\overline{\mathbb{D}(0,R)}=\{z\in\mathbb{C}:\ |z|\leq R\}$, i.e.,

$$\sup_n\sup_{z\in\overline{\mathbb{D}(0,R)}}|p_n(z)|<\infty.$$

Let $q(z)=b\prod\limits_{k=1}^{N}(z-z_k)$, $b\neq 0$. If $q_n(z)=b_n\prod_{k=1}^N(z-z_{k,n})$, $n=1,2,\ldots$, and we label the zeroes of q_n accordingly, then we get

$$b_n\to b \ \text{ and } \ z_{k,n}\to z_k,\ \ k=1,2,\ldots,N,\ \text{ as } \ n\to\infty.$$

Since the zeros of q_n are concentrated around the points z_k, $k = 1, 2, \dots, N$, for each $\varepsilon > 0$, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\mu\Big\{\mathbb{T} \setminus \bigcup_{k=1}^{N} \overline{D(z_k, \delta)}\Big\} < \varepsilon \tag{56}$$

and the functions

$$f_n := p_n/q_n, \quad n \geq n_0, \tag{57}$$

are uniformly bounded in

$$D_\varepsilon := D(0, 1+\delta) \setminus \bigcup_{k=1}^{N} \overline{D(z_k, \delta)}. \tag{58}$$

Consequently, the set of functions (57) is a normal family and converges uniformly on every compact in (58) which implies, by virtue of (53), that there exists $n_1 \geq n_0$ such that

$$|f_n(z)| < C \quad \text{for} \quad n \geq n_1 \quad \text{and} \quad z \in \mathbb{T} \cap D_\varepsilon.$$

Consequently, $h_n = f_n$ on $\mathbb{T} \cap D_\varepsilon$ for $n \geq n_1$ and h_n converges uniformly to f in $\mathbb{T} \cap D_\varepsilon$.

Since the derivatives of a normal family of functions form a normal family as well, we have that h_n together with h_n' converge uniformly on $\mathbb{T} \cap D_\varepsilon$. Consequently,

$$\frac{2\pi}{n} \sum_{\{k:\ \omega_{kn} \in D_\varepsilon\}} |h_n(\omega_{k,n})|^2 \to \int_{\mathbb{T} \cap D_\varepsilon} |f|^2 \, d\mu$$

as $n \to \infty$, while

$$\Bigg| \int_{\mathbb{T} \cap D_\varepsilon} |f|^2 \, d\mu - \int_{\mathbb{T}} |f|^2 \, d\mu \Bigg| \leq \sup_{\mu(E) < \varepsilon} \int_E |f|^2 \, d\mu \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and

$$\limsup_{n \to \infty} \Bigg| \frac{2\pi}{n} \sum_{\{k:\ \omega_{kn} \in D_\varepsilon\}} |h_n(\omega_{k,n})|^2 - \frac{2\pi}{n} \sum_{k=0}^{n-1} |h_n(\omega_{k,n})|^2 \Bigg|$$

$$= \limsup_{n \to \infty} \frac{2\pi}{n} \sum_{\{k:\ \omega_{kn} \notin D_\varepsilon\}} |h_n(\omega_{k,n})|^2 \leq C\mu\{\mathbb{T} \setminus D_\varepsilon\} \leq C\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Hence (55) holds. □

Lemma 5. *Let f, p_n, q_n and V_n be the same as in Lemma 4. Assume that $q_n(0) \neq 0$, $n = 1, 2, \dots$, and let*

$$\sum_{k=0}^{\infty} \alpha_k z^k \sim \frac{p_n(z)}{q_n(z)}$$

be the Tailor expansion of p_n/q_n *in the neighborhood of zero. Define* $\mathcal{L}_n[p_n,q_n]\in\mathcal{P}_n^+$ *by*

$$\mathcal{L}_n[p_n,q_n](z):=\begin{cases}\sum_{k=0}^{l}\alpha_k z^k & \text{if } \sum_{k=0}^{l}|\alpha_k|^2\le V_n<\sum_{k=0}^{l+1}|\alpha_k|^2 \text{ and } l<n,\\ \sum_{k=0}^{n}\alpha_k z^k & \text{if } \sum_{k=0}^{n}|\alpha_k|^2\le V_n.\end{cases} \tag{59}$$

Then

$$\big\|\mathcal{L}_n[p_n,q_n]-f\big\|_{L^2(\mathbb{T})}\to 0 \ \text{ as } n\to\infty. \tag{60}$$

Proof. The Tailor coefficients of $f=p/q$ can be expressed recurrently in terms of coefficients of p and q. Thus, because of (54), we have

$$\text{for each } k\ge 0,\quad c_k\big\{\mathcal{L}_n[p_n,q_n]\big\}\to c_k\{f\} \ \text{ as } n\to\infty. \tag{61}$$

By virtue of Lemma 4 and definition (59), taking into account (61), we also have

$$\big\|\mathcal{L}_n[p_n,q_n]\big\|_{L^2(\mathbb{T})}\to\|f\|_{L^2(\mathbb{T})} \ \text{ as } n\to\infty.$$

The convergence in (60) now follows from the general fact that in a Hilbert space the weak convergence, when combined with the convergence of norms, implies strong convergence. □

We are ready now to introduce the computational procedure $\mathfrak{C}=\mathfrak{C}_n$ described in the introduction, which can be applied to any $S_n\in\mathcal{P}^{r\times r}$, such that Theorem 1 holds.

Note that if S is a polynomial matrix function (5), then for each m, $1<m\le r$, the first $m-1$ entries $\zeta_1,\zeta_2,\dots,\zeta_{m-1}$ of the m-th row of $M_S\mathbf{U}_2\mathbf{U}_3\dots\mathbf{U}_{m-1}$ in (17) are rational functions, since they can be determined by Cramer's rule from equation (19) as

$$\zeta_i(t)=\overline{p_i(t)/q(t)}=t^N\overline{(t^Np_i(t)/q(t))}, \tag{62}$$

where $q=\det S^+_{[m-1]}\in\mathcal{P}^+_{N(m-1)}$ (it is free of zeros in $\mathbb{D}$) and p_i is the determinant of the matrix $S^+_{[m-1]}$, the i-th column of which is replaced by $[s_{2m},\dots,s_{m-1,m}]^T$, implying $z^Np_i\in\mathcal{P}^+_{Nm}$.

We compute the diagonal entries $\widehat{f}^+_{1,n},\widehat{f}^+_{2,n},\dots,\widehat{f}^+_{r,n}$ of the "triangular factor" of S_n by the formulas: $\widehat{f}^+_{1,n}=\widetilde{(S_n)_{[1]}}^{\,+}$ (see Section 2 for notation $S_{[m]}$ and definitions (35)–(37)) and

$$\widehat{f}^+_{m,n}=\mathcal{L}_n\Big[\det\big(\widetilde{(S_n)_{[m]}}\big)^+,\det\big(\widetilde{(S_n)_{[m-1]}}\big)^+\Big] \tag{63}$$

(see Lemma 5 for definitions). Set

$$(\widehat{S}_n)^+_{[1]}=\widehat{f}^+_{1,n}=\widetilde{(S_n)_{[1]}}^{\,+} \tag{64}$$

and for each $m=2,3,\dots,r$ we recurrently construct

$$(\widehat{S}_n)^+_{[m]}(t)=\sum_{k=0}^N \widehat{A}_{k,n}t^k,\ \ \widehat{A}_{k,n}\in\mathbb{C}^{m\times m},$$

an approximate "spectral factor" of $(S_n)_{[m]}$, making an assumption that $(\widehat{S}_n)^+_{[m-1]}$ has already been constructed and performing the following operations. Let

$$\widehat{\zeta}_{i,n}=t^N\overline{\mathcal{L}_n[t^N\widehat{p}_{i,n},\widehat{q}_n(t)]}\,,\ \ i=1,2,\dots,m-1, \tag{65}$$

where $\widehat{p}_{i,n}$ and $\widehat{q}_i$ are defined similar to (62), namely, $\widehat{q}_n=\det\big((\widehat{S}_n)^+_{[m-1]}\big)$ and $\widehat{p}_{i,n}$ is the determinant of the matrix $(\widehat{S}_n)^+_{[m-1]}$ with its i-th column replaced by $[\widehat{s}_{2m},\dots,\widehat{s}_{m-1,m}]^T$. For the matrix $\widehat{F}_{n,m}$ of the form (18) with the last row

$$(\widehat{\zeta}_{1,n},\widehat{\zeta}_{2,n},\dots,\widehat{\zeta}_{m-1,n},\widehat{f}^+_{m,n}), \tag{66}$$

using Theorem 3, we construct the unitary matrix function $U_{m,n}$ such that $\widehat{F}_{n,m}U_{m,n}\in(\mathcal{P}^+)^{m\times m}$. By virtue of formula (21), the matrix function

$$\widehat{S}\cdot U:=\begin{pmatrix} & & & & 0\\ & & \det\big((\widehat{S}_n)^+_{[m-1]}\big) & & \vdots\\ & & & & 0\\ \widehat{\zeta}_{1,n} & \widehat{\zeta}_{2,n} & \dots & \widehat{\zeta}_{m-1,n} & \widehat{f}^+_{m,n}\end{pmatrix}U_{m,n} \tag{67}$$

is a candidate for $(\widehat{S}_n)^+_{[m]}$. Since we know that $S^+_{[m]}\in(\mathcal{P}^+_N)^{m\times m}$, we discard coefficients of the entries in (67) with indices outside the range $[0,N]$ and let

$$(\widehat{S}_n)^+_{[m]}(z):=\sum_{k=0}^N C_k\{\widehat{S}\cdot U\}z^k,\ \ m=2,3,\dots,r. \tag{68}$$

We define

$$\mathfrak{C}_n(S_n)=(\widehat{S}_n)^+_{[r]}. \tag{69}$$

Let us prove now the convergence (7).

Consider the equation (63). Since, because of (6), $\det\big((S_n)_{[m]}\big)\to\det S_{[m]}$ as $n\to\infty$, we have $\det\widetilde{((S_n)_{[m]})}^+\to\det S^+_{[m]}$, $m=1,2,\dots,r$, by virtue of Lemma 2 (in particular,

$$\widetilde{(S_n)_{[1]}}^+=(\widehat{S}_n)^+_{[1]}\to S^+_{[1]}, \tag{70}$$

see (64)), while the limiting functions $\det S^+_{[m]}$ are free of zeros in $\mathbb{D}$ and $f^+_m=\det S^+_{[m]}/\det S^+_{[m-1]}\in L_2(\mathbb{T})$ (see (25)) do not have poles on $\mathbb{T}$. Consequently, the hypotheses of Lemma 5 hold and therefore

$$\widehat{f}^+_{m,n}\to f^+_m \text{ in } L^2 \ \text{ as } \ n\to\infty,\ \ m=2,3,\dots,r. \tag{71}$$

Since (70) holds, we assume invoking induction that

$$(\widehat{S}_n)^+_{[m-1]} \to S^+_{[m-1]} \text{ in } L^2 \text{ as } n \to \infty, \tag{72}$$

and prove (72) for $m-1$ replaced by m.

Consider now the equation (65). The sequences of polynomials $p_{i,n}$ and q_n also satisfy the hypothesis of Lemma 5 and therefore

$$\widehat{\zeta}_{i,n} \to \zeta_i \text{ in } L^2 \text{ as } n \to \infty. \tag{73}$$

Thus, taking into account the relation (71) also, we have that the sequence of matrix functions $\widehat{F}_{n,m}$ of the form (18), (66) converges in L^2. Consequently, we can apply Theorem 5 to conclude that the sequence of unitary matrix functions $U_{n,m}$ in the equation (67) is convergent in measure which, by virtue of (10), implies that the product in (67) and, consequently, (68) are convergent. Namely,

$$(\widehat{S}_n)^+_{[m]} \to S^+_{[m]} \text{ in } L^2 \text{ as } n \to \infty. \tag{74}$$

We get (7) if we substitute $m = r$ in (74), and thus the proof of Theorem 1 is completed.

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Authors' addresses:

Lasha Ephremidze

1. A. Razmadze Mathematical Institute of Iv. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia.

2. Division of Science and Mathematics New York University Abu Dhabi (NYUAD), Saadiyat Island, P.O. Box 129188, Abu Dhabi, United Arab Emirates.

E-mail: `lasha@rmi.ge`; `le23@nyu.edy`

Ilya Spitkovsky

1. Division of Science and Mathematics New York University Abu Dhabi (NYUAD), Saadiyat Island, P.O. Box 129188, Abu Dhabi, United Arab Emirates.

2. Department of Mathematics, College of William and Mary, Williamsburg, VA 23187, USA.

E-mail: `ims2@nyu.edy`; `ilya@math.wm.edu`

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Sergo Kharibegashvili

THE EXISTENCE OF SOLUTIONS OF ONE NONLOCAL IN TIME PROBLEM FOR MULTIDIMENSIONAL WAVE EQUATIONS WITH POWER NONLINEARITY

Abstract. For multidimensional wave equations with power nonlinearity we investigate the question on the existence of solutions in a nonlocal in time problem whose particular case is a periodic case.

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Key words and phrases. Nonlocal in time conditions, multidimensional wave equations, power nonlinearity.

რეზიუმე. მრავალგანზომილებიანი ტალღის განტოლებისათვის ხარისხოვანი არაწრფივობით გამოკვლეულია დროით არალოკალური ამოცანის ამონახსნების არსებობის საკითხი. ამ ამოცანის კერძო შემთხვევაა პერიოდული ამოცანა.

1. Statement of the Problem

In the space $\mathbb{R}^{n+1}$ of variables $x = (x_1, \dots, x_n)$ and t, in the cylindrical domain $D = \Omega \times (0, T)$, where Ω is some open Lipschitz domain in $\mathbb{R}^n$, we consider a nonlocal problem of finding a solution $u(x, t)$ of the equation

$$L_\lambda u := u_{tt} - \sum_{i=1}^{n} u_{x_i x_i} + 2au_t + cu + \lambda |u|^\alpha u = F(x, t), \quad (x, t) \in D_T, \quad (1.1)$$

satisfying the homogeneous boundary condition

$$\Big(\frac{\partial u}{\partial \nu} + \sigma u\Big)\Big|_\Gamma = 0 \quad (1.2)$$

on the lateral boundary $\Gamma : \partial\Omega \times (0, T)$ of the cylinder D_T and the homogeneous nonlocal conditions

$$\mathcal{K}_\mu u := u(x, 0) - \mu u(x, T) = 0, \quad x \in \Omega, \quad (1.3)$$

$$\mathcal{K}_\mu u_t := u_t(x, 0) - \mu u_t(x, T) = 0, \quad x \in \Omega, \quad (1.4)$$

where F is the given function; α, λ, μ, a, c and σ are the given constants and $\alpha > 0$, $\lambda\mu \neq 0$; $\frac{\partial}{\partial \nu}$ is the derivative with respect to the outer normal to ∂D_T, $n \geq 2$.

Remark 1.1. A great number of works are devoted to the investigation of nonlocal problems. In the case of abstract evolution equations and partial differential equations of hyperbolic type, the nonlocal problems are studied in [1–13, 17, 21]. Note that for $|\mu| \neq 1$ it suffices to restrict ourselves to the case $|\mu| < 1$, since the case $|\mu| > 1$ reduces to the previous one if we pass from the variable t to the variable $t' = T - t$. The case $|\mu| = 1$ will be treated in the final Section 4. In particular, the problem (1.1)–(1.4) for $\mu = 1$ can be treated as a periodic problem.

We introduce into consideration the following spaces of functions:

$$\overset{\circ}{C}{}^2_\mu(D_T) := \Big\{ v \in C^2(D_T) : \Big(\frac{\partial v}{\partial \nu} + \sigma v\Big)\Big|_\Gamma = 0, \ \mathcal{K}_\mu v = 0, \ \mathcal{K}_\mu v_t = 0 \Big\}, \quad (1.5)$$

$$\overset{\circ}{W}{}^1_{2,\mu}(D_T) := \Big\{ v \in W_2^1(D_T) : \ \mathcal{K}_\mu v = 0 \Big\}, \quad (1.6)$$

where $W_2^1(D_T)$ is the well-known Sobolev space consisting of functions of the class $L_2(D_T)$ whose all generalized first order derivatives belong likewise to $L_2(D_T)$, and the equality $\mathcal{K}_\mu v = 0$ is understood in a sense of the trace theory [16, p. 71].

Definition 1.1. Let $F \in L_2(\Omega_T)$. The function u will be said to be a strong generalized solution of the problem (1.1)–(1.4) of the class W_2^1 in the domain D_T if $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ and there exists a sequence of functions

$u_m \in \overset{\circ}{C}{}^2_\mu(\overline{D}_T)$ such that $u_m \to u$ in the space $\overset{\circ}{W}{}^1_{2,\mu}(D_T)$, and $L_\lambda u_m \to F$ in the space $L_2(D_T)$.

Note that the above definition of a solution of the problem (1.1)–(1.4) remains valid in a linear case, as well, that is for $\lambda = 0$.

Remark 1.2. Obviously, a classical solution of the problem (1.1)–(1.4) from the space $C^2(\overline{D}_T)$ is a strong generalized solution of that problem of the class W_2^1 in the domain D_T in a sense of Definition 1.1.

Remark 1.3. It should be noted that even in a linear case, that is for $\lambda = 0$, the problem (1.1)–(1.4) is not always well-posed. For example, for $\lambda = a = c = 0$ and $|\mu| = 1$, the homogeneous problem corresponding to (1.1)–(1.4) may have infinite set of linearly independent solutions, whereas in order for the inhomogeneous problem to be solvable, it is necessary that a finite or an infinite set of conditions in the form of functional equalities imposed on the right-hand side F of equation (1.1) be fulfilled (see Remark 4.1 below).

The present paper is organized as follows. In Section 2, for some conditions on the coefficients of the problem (1.1)–(1.4) an a priori estimate for a strong generalized solution of the class W_2^1 is proved. In Section 3, on the basis of the obtained a priori estimate it is proved that the problem (1.1)–(1.4) is solvable. In the last Section 4, as an application of the results obtained in the previous sections, we consider the case $|\mu| = 1$.

2. An a Priori Estimate of Solution of the Problem (1.1)–(1.4)

Consider the conditions

$$a \geq 0, \quad c \geq 0, \quad \sigma \geq 0. \tag{2.1}$$

Lemma 2.1. *Let $\lambda > 0$, $|\mu| < 1$, and let $F \in L_2(D_T)$ and conditions* (2.1) *be fulfilled. Then for a strong generalized solution u of the problem* (1.1)–(1.4) *of the class W_2^1 in the domain D_T in a sense of Definition* 1.1 *the a priori estimate*

$$\|u\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} \leq c_1 \|F\|_{L_2(D_T)} + c_2 \tag{2.2}$$

with nonnegative constants $c_i = c_i(\lambda, \mu, \Omega, T)$, independent of u and F, and $c_1 > 0$, is valid, whereas in a linear case, that is for $\lambda = 0$, if $\sigma > 0$, the constant $c_2 = 0$, and by virtue of (2.2), *a solution of the problem* (1.1)–(1.4) *is unique in a sense of Definition* 1.1.

Proof. Let u be a strong generalized solution of the problem (1.1)–(1.4) of the class W_2^1 in the domain D_T. By Definition 1.1, there exists the sequence of functions $u_m \in \overset{\circ}{C}{}^2_\mu(\overline{D}_T)$ (see (1.5)) such that

$$\lim_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} = 0, \quad \lim_{m\to\infty} \|L_\lambda u_m - F\|_{L_2(D_T)} = 0. \tag{2.3}$$

Let us consider the function $u_m \in \overset{\circ}{C}{}^2_\mu(\overline{D}_T)$ as a solution of the problem

$$L_\lambda u_m = F_m, \quad (x,t) \in D_T, \tag{2.4}$$

$$\Big(\frac{\partial u_m}{\partial \nu} + \sigma u_m\Big)\Big|_\Gamma = 0, \quad \mathcal{K}_\mu u_m = 0, \quad \mathcal{K}_\mu u_{mt} = 0. \tag{2.5}$$

Here

$$F_m := L_\lambda u_m. \tag{2.6}$$

Multiplying both parts of equality (2.4) by $2u_{mt}$ and integrating with respect to the domain $D_\tau := D_T \cap \{t < \tau\}$, $0 < \tau \le T$, we obtain

$$\begin{aligned} &\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u_m}{\partial t}\Big)^2 dx\,dt - 2\int\limits_{D_\tau} \sum_{i=1}^n \frac{\partial^2 u_m}{\partial x_i^2}\,\frac{\partial u_m}{\partial t}\,dx\,dt \\ &+4a\int\limits_{D_\tau} u_{mt}^2\,dx\,dt + c\int\limits_{D_\tau} (u_m^2)_t\,dx\,dt + \frac{2\lambda}{\alpha+2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\,|u_m|^{\alpha+2}\,dx\,dt \\ &= 2\int\limits_{D_\tau} F_m u_{mt}\,dx\,dt. \end{aligned} \tag{2.7}$$

Assume $\omega_\tau := \{(x,t) \in \overline{D}_T :\ x \in \Omega,\ t = \tau\}$, $0 \le \tau \le T$, where ω_0 and Ω_T are, respectively, the lower and upper bases of the cylindrical domain D_T. Let $\nu := (\nu_{x_1}, \nu_{x_2}, \dots, \nu_{x_n}, \nu_t)$ be the unit vector of the outer normal to ∂D_τ. Since

$$\nu_{x_i}\big|_{\omega_\tau \cup \omega_0} = 0, \ \ i = 1, \dots, n,$$
$$\nu_t\big|_{\Gamma_\tau := \Gamma \cap \{t \le \tau\}} = 0, \quad \nu_t\big|_{\omega_\tau} = 1, \quad \nu_t\big|_{\omega_0} = -1,$$

therefore, taking into account (2.5) and integrating by parts, we have

$$\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u_m}{\partial t}\Big)^2 dx\,dt = \int\limits_{D_\tau} \Big(\frac{\partial u_m}{\partial t}\Big)^2 \nu_t\,ds = \int\limits_{\omega_\tau} u_{mt}^2\,dx - \int\limits_{\omega_0} u_{mt}^2\,dx, \tag{2.8}$$

$$\begin{aligned} -2\int\limits_{D_\tau} \sum_{i=1}^n \frac{\partial^2 u_m}{\partial x_i^2}\,\frac{\partial u_m}{\partial t}\,dx\,dt &= \int\limits_{D_\tau} \sum_{i=1}^n \Big[(u_{mx_i}^2)_t - 2(u_{mx_i}u_{mt})_{x_i}\Big]\,dx\,dt \\ &= \int\limits_{\omega_\tau} \sum_{i=1}^n u_{mx_i}^2\,dx - \int\limits_{\omega_0} \sum_{i=1}^n u_{mx_i}^2\,dx - 2\int\limits_{\Gamma_\tau} \Big[\sum_{i=1}^n u_{mx_i}\nu_i\Big] u_{mt}\,ds \\ &= \int\limits_{\omega_\tau} \sum_{i=1}^n u_{mx_i}^2\,dx - \int\limits_{\omega_0} \sum_{i=1}^n u_{mx_i}^2\,dx + 2\int\limits_{\Gamma_\tau} \sigma u_m u_{mt}\,ds \\ &= \int\limits_{\omega_\tau} \sum_{i=1}^n u_{mx_i}^2\,dx - \int\limits_{\omega_0} \sum_{i=1}^n u_{mx_i}^2\,dx + \sigma\int\limits_{\Gamma_\tau} (u_m^2)_t\,ds \end{aligned}$$

$$= \int_{\omega_\tau} \sum_{i=1}^{n} u_{mx_i}^2 \, dx - \int_{\omega_0} \sum_{i=1}^{n} u_{mx_i}^2 \, dx + \sigma \int_{\partial\omega_\tau} u_m^2 \, ds - \sigma \int_{\partial\omega_0} u_m^2 \, ds, \tag{2.9}$$

$$\frac{2\lambda}{\alpha+2} \int_{D_\tau} \frac{\partial}{\partial t} |u_m|^{\alpha+2} \, dx \, dt$$

$$= \frac{2\lambda}{\alpha+2} \int_{\omega_\tau} |u_m|^{\alpha+2} \, dx - \frac{2\lambda}{\alpha+2} \int_{\omega_0} |u_m|^{\alpha+2} \, dx, \tag{2.10}$$

$$\int_{D_\tau} (u_m^2)_t \, dx \, dt = \int_{\omega_\tau} u_m^2 \, dx - \int_{\omega_0} u_m^2 \, dx.$$

Assuming

$$w_m(\tau) = \int_{\omega_\tau} \Big[c u_m^2 + u_{mt}^2 + \sum_{i=1}^{n} u_{mx_i}^2 + \frac{2\lambda}{\alpha+2} |u_m|^{\alpha+2} \Big] \, dx$$

$$+ \sigma \int_{\partial\omega_\tau} u_m^2 \, ds \tag{2.11}$$

by virtue of (2.8), (2.9), (2.10) and (2.7), we obtain

$$w_m(\tau) + 4a \int_{D_\tau} u_{mt}^2 \, dx \, dt = w_m(0) + 2 \int_{D_\tau} F_m \frac{\partial u_m}{\partial t} \, dx \, dt. \tag{2.12}$$

Since $2F_m u_{mt} \le \varepsilon^{-1} F_m^2 + \varepsilon u_{mt}^2$ for every $\varepsilon = const > 0$, it follows from (2.12), owing to $a \ge 0$, that

$$w_m(\tau) \le w_m(0) + \varepsilon \int_{D_\tau} u_{mt}^2 \, dx \, dt + \varepsilon^{-1} \int_{D_\tau} F_m^2 \, dx \, dt. \tag{2.13}$$

Next, by virtue of (2.11), $\lambda > 0$ and $\sigma \ge 0$, we have

$$\int_{D_\tau} u_{mt}^2 \, dx \, dt = \int_0^\tau \Big[\int_{\omega_s} u_{mt}^2 \, dx \Big] \, ds \le \int_0^\tau w_m(s) \, ds,$$

whence, with regard for (2.13), we obtain

$$w_m(\tau) \le \varepsilon \int_0^\tau w_m(\xi) \, d\xi + w_m(0) + \varepsilon^{-1} \int_{D_\tau} F_m^2 \, dx \, dt, \quad 0 < \tau \le T. \tag{2.14}$$

Since $D_\tau \subset D_T$, $0 < \tau \le T$, the right-hand side of inequality (2.14) is a nondecreasing function of the variable τ, and by Gronwall's lemma, it follows from (2.14) that

$$w_m(\tau) \le \Big[w_m(0) + \varepsilon^{-1} \int_{D_\tau} F_m^2 \, dx \, dt \Big] e^{\varepsilon\tau}, \quad 0 < \tau \le T. \tag{2.15}$$

By virtue of (2.5), $\lambda > 0$, $\sigma \geq 0$, $|\mu| < 1$, $\alpha > 0$, from (2.12) we get

$$w_m(0) = \int\limits_{\Omega} \Big[c u_m^2(x,0) + u_{mt}^2(x,0) + \sum_{i=1}^{n} u_{mx_i}^2(x,0) + \frac{2\lambda}{\alpha+2} \, |u_m^2(x,0)|^{\alpha+2} \Big] \, dx + \sigma \int\limits_{\partial\Omega} u_m^2(x,0) \, ds$$
$$= \int\limits_{\Omega} \Big[\mu^2 c u_m^2(x,T) + \mu^2 u_{mt}^2(x,T) + \mu^2 \sum_{i=1}^{n} u_{mx_i}^2(x,T) + \frac{2\lambda|\mu|^{\alpha+2}}{\alpha+2} \, |u_m(x,T)|^{\alpha+2} \Big] \, dx + \sigma \int\limits_{\partial\Omega} \mu^2 u_m^2(x,T) \, ds \leq \mu^2 w_m(T). \tag{2.16}$$

Using inequality (2.15) for $\tau = T$, by virtue of (2.16), we find that

$$w_m(0) \leq \mu^2 w_m(T) \leq \mu^2 \Big[w_m(0) + \varepsilon^{-1} \int\limits_{D_\tau} F_m^2 \, dx \, dt \Big] e^{\varepsilon T}. \tag{2.17}$$

Since $|\mu| < 1$, we can choose a positive constant $\varepsilon = \varepsilon(\mu, T)$ so small that

$$\mu_1 = \mu^2 e^{\varepsilon T} < 1. \tag{2.18}$$

For example, in the capacity of ε from (2.18) we can take the number $\varepsilon = \frac{1}{T} \ln(\frac{1}{|\mu|})$.

Owing to (2.18), from (2.17) we obtain

$$w(0) \leq (1-\mu_1)^{-1} \mu^2 \varepsilon^{-1} e^{\varepsilon T} \|F_m\|^2_{L_2(D_T)}. \tag{2.19}$$

Taking into account (2.19), from (2.15) we find that

$$w_m(\tau) \leq \gamma \|F_m\|^2_{L_2(D_T)}, \quad 0 < \tau \leq T. \tag{2.20}$$

Here

$$\gamma = \Big[(1-\mu_1)^{-1} \mu^2 \varepsilon^{-1} e^{\varepsilon T} + \varepsilon^{-1} \Big] e^{\varepsilon T}, \quad \varepsilon = \frac{1}{T} \ln\Big(\frac{1}{|\mu|}\Big). \tag{2.21}$$

By virtue of $\lambda > 0$, $\alpha > 0$, $c \geq 0$, $\sigma \geq 0$ and (2.11), we have

$$\int\limits_{\omega_\tau} u_m^2 \, dx = \int\limits_{\omega_\tau, \, |u_m| \leq 1} u_m^2 \, dx + \int\limits_{\omega_\tau, \, |u_m| > 1} u_m^2 \, dx \leq \operatorname{mes} \Omega + \int\limits_{\omega_\tau, \, |u_m| > 1} |u_m|^{\alpha+2} \, dx \leq \operatorname{mes} \Omega + \frac{\alpha+2}{2\lambda} \, w_m(\tau). \tag{2.22}$$

It follows from (2.11), (2.20) and (2.22) that

$$\int\limits_{\omega_\tau}\Big[u_m^2+u_{mt}^2+\sum_{i=1}^n u_{mx_i}^2\Big]\,dx\le \operatorname{mes}\Omega+\frac{\alpha+2}{2\lambda}\,w_m(\tau)+w_m(\tau)$$
$$=\operatorname{mes}\Omega+\Big(1+\frac{\alpha+2}{2\lambda}\Big)\gamma\|F_m\|^2_{L_2(D_T)},\quad 0<\tau\le T. \tag{2.23}$$

By (2.23), we obtain

$$\|u_m\|^2_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)}=\int\limits_0^T\Big[\int\limits_{\omega_\tau}\Big(u_m^2+u_{mt}^2+\sum_{i=1}^n u_{mx_i}^2\Big)\,dx\Big]\,d\tau$$
$$\le T\operatorname{mes}\Omega+T\Big(1+\frac{\alpha+2}{2\lambda}\Big)\gamma\|F_m\|^2_{L_2(D_T)},\quad 0<\tau\le T. \tag{2.24}$$

Taking from both parts of inequality (2.24) the square root and using the obvious inequality $(a^2+b^2)^{1/2}\le|a|+|b|$, we have

$$\|u_m\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)}\le c_1\|F_m\|_{L_2(D_T)}+c_2. \tag{2.25}$$

Here, due to (2.21), for $\lambda>0$, we get

$$\begin{cases} c_1=\Big(T\Big(1+\dfrac{\alpha+2}{2\lambda}\Big)\Big[(1-\mu_1)^{-1}\mu^2\varepsilon^{-1}e^{\varepsilon T}+\varepsilon^{-1}\Big]e^{\varepsilon T}\Big)^{\frac12}, \\ \qquad\qquad \varepsilon=\dfrac1T\,\ln\Big(\dfrac{1}{|\mu|}\Big), \\ c_2=(T\operatorname{mes}\Omega)^{\frac12}. \end{cases} \tag{2.26}$$

Bearing in mind limiting equalities (2.3) and passing in inequality (2.25) to the limit, as $m\to\infty$, we obtain (2.2). Thus Lemma 2.1 is proved for $\lambda>0$.

In a linear case, that is for $\lambda=0$, but $\sigma>0$, the proof of a priori estimate (2.2) with $c_2=0$ follows from the following reasoning. As is known, the norm of the space $W_2^1(\Omega)$ for $\sigma>0$ is equivalent to the norm

$$\|v\|^2=\int\limits_\Omega\sum_{i=1}^n v_{x_i}^2\,dx+\sigma\int\limits_{\partial\Omega}v^2\,ds$$

[18, p. 147] that is, in particular, there exists the positive constant $c_0=c_0(\Omega,\sigma)$ such that

$$\|v\|^2_{W_2^1(\Omega)}=\int\limits_\Omega\Big[v^2+\sum_{i=1}^n v_{x_i}^2\Big]\,dx$$
$$\le c_0\Big[\int\limits_\Omega\sum_{i=1}^n v_{x_i}^2\,dx+\sigma\int\limits_{\partial\Omega}v^2\,ds\Big]\quad\forall\,v\in W_2^1(\Omega). \tag{2.27}$$

By (2.1),(2.27), instead of (2.22) and (2.23), with regard for (2.11), we will have

$$\int\limits_{\omega_\tau} \Big[u_m^2 + u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2\Big]\,dx \\ \le \int\limits_{\omega_\tau} u_m^2\,dx + w_m(\tau) \le (c_0+1)w_m(\tau). \quad (2.28)$$

From (2.20) and (2.28), analogously to how we have obtained (2.24), it follows that

$$\|u_m\|^2_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} \le \int\limits_0^T (c_0+1)w_m(\tau)\,d\tau \le T(c_0+1)\gamma\|F_m\|^2_{L_2(D_T)}. \quad (2.29)$$

Passing in inequality (2.29) to the limit, as $m \to \infty$, and taking into account (2.3), we obtain estimate (2.2) in which

$$\begin{cases} c_1 = \Big(T(c_0+1)\Big[(1-\mu_1)^{-1}\mu^2\varepsilon^{-1}e^{\varepsilon T} + \varepsilon^{-1}\Big]e^{\varepsilon T}\Big)^{\frac{1}{2}}, \\ c_2 = 0, \end{cases} \quad (2.30)$$

what proves Lemma 2.1 in case $\lambda = 0$ and $\sigma > 0$. □

Remark 2.1. In Section 3, the question on the solvability of the problem (1.1)–(1.4) is reduced to that of finding a uniform with respect to the parameter $s \in [0,1]$ a priori estimate for a strong generalized solution of the equation

$$v_{tt} - \sum_{i=1}^n v_{x_i x_i} + s(c-a^2)v + s\lambda\exp(-\alpha a t)|v|^\alpha v \\ = s\exp(at)F(x,t), \quad (x,t) \in D_T, \quad (2.31)$$

satisfying both the boundary condition

$$\Big(\frac{\partial v}{\partial \nu} + \sigma v\Big)\Big|_\Gamma = 0 \quad (2.32)$$

and the nonlocal conditions

$$(\mathcal{K}_{\mu_0}v)(x) = 0, \quad (\mathcal{K}_{\mu_0}v_t)(x) = 0, \quad x \in \Omega, \quad (2.33)$$

where $\mu_0 = \mu\exp(-aT)$, $|\mu| < 1$, and the operator $\mathcal{K}_{\mu_0}$ for $\mu = \mu_0$ is defined in (1.3). To obtain a uniform with respect to τ a priori estimate for the solution of the problem (2.31)–(2.33) it is sufficient that instead of (2.1) be fulfilled the more bounded conditions

$$a \ge 0, \ \ c \ge a^2, \ \ \sigma > 0. \quad (2.34)$$

For this case, we present in the proof of Lemma 2.1 certain changes. Assuming

$$\widetilde{w}_m(\tau)=\int\limits_{\omega_\tau}\bigg[s(c-a^2)v_m^2+v_{mt}^2+\sum_{i=1}^n v_{mx_i}^2+\frac{2s\lambda}{\alpha+2}\,\exp(-\alpha a\tau)|v_m|^{\alpha+2}\bigg]\,dx + \sigma\int\limits_{\partial\omega_\tau} v_m^2\,ds,$$

instead of equality (2.12) for u_m, in regard to the function v_m we get

$$\widetilde{w}_m(\tau)+\frac{2s\lambda a}{\alpha+2}\int\limits_{D_\tau}\exp(-\alpha at)|v_m|^{\alpha+2}\,dx\,dt = \widetilde{w}_m(0)+2s\int\limits_{D_\tau}\exp(at)F_m v_{mt}\,dx\,dt,$$

whence by virtue of $s\lambda a\geq 0$, $s\in[0,1]$, analogously to (2.13)–(2.15), we, respectively, obtain

$$\widetilde{w}_m(\tau)\leq \widetilde{w}_m(0)+\varepsilon\int\limits_{D_T} v_{mt}^2\,dx\,dt+\varepsilon^{-1}\exp(2aT)\int\limits_{D_T}F_m^2\,dx\,dt,$$

$$\widetilde{w}_m(\tau)\leq \varepsilon\int\limits_0^T w_m(\xi)\,d\xi+\widetilde{w}_m(0)+\varepsilon^{-1}\exp(2aT)\int\limits_{D_T}F_m^2\,dx\,dt,$$

$$\widetilde{w}_m(\tau)\leq \bigg[\widetilde{w}_m(0)+\varepsilon^{-1}\exp(2aT)\int\limits_{D_T}F_m^2\,dx\,dt\bigg]e^{\varepsilon\tau},\quad 0<\tau\leq T.$$

Further, by (2.33), (2.34) and $\mu_0=\mu\exp(-at)$, $|\mu|<1$, taking into account the fact that

$$|\mu_0|^{\alpha+2}=|\mu_0|^2\exp(-\alpha aT)|\mu_0|^\alpha\exp(\alpha aT) = |\mu_0|^2\exp(-\alpha aT)|\mu|^\alpha\leq|\mu_0|^2\exp(-\alpha aT),$$

we instead of (2.16) have

$$\widetilde{w}_m(0)=\int\limits_\Omega\bigg[s(c-a^2)v_m^2(x,0)+v_{mt}^2(x,0)+\sum_{i=1}^n v_{mx_i}^2(x,0) + \frac{2s\lambda}{\alpha+2}\,|v_m(x,0)|^{\alpha+2}\bigg]\,dx+\sigma\int\limits_{\partial\Omega}v_m^2(x,0)\,ds$$

$$=\int\limits_\Omega\Big[\mu_0^2 s(c-a^2)v_m^2(x,T)+\mu_0^2v_{mt}^2(x,T)$$

$$+\mu_0^2\sum_{i=1}^{n}v_{mx_i}^2(x,T)+\frac{2s\lambda|\mu_0|^{\alpha+2}}{\alpha+2}\,|v_m(x,T)|^{\alpha+2}\Big]\,dx$$

$$+\sigma\int\limits_{\partial\Omega}\mu_0^2v_m^2(x,T)\,ds\le\mu_0^2\widetilde{w}_m(T).$$

Analogously, instead of (2.17)–(2.21) we, respectively, obtain

$$\widetilde{w}_m(0)\le\mu_0^2\widetilde{w}_m(T)\le\mu_0^2\bigg[\widetilde{w}_m(0)+\varepsilon^{-1}\exp(2aT)\int\limits_{D_T}F_m^2\,dx\,dt\bigg]e^{\varepsilon T},$$

$$\mu_2=\mu_0^2e^{\varepsilon T}<1,$$

$$\widetilde{w}_m(0)\le(1-\mu_2)^{-1}\mu_0^2\varepsilon^{-1}e^{\varepsilon T}\exp(2aT)\|F_m\|_{L_2(D_T)}^2,$$

$$\widetilde{w}_m(\tau)\le\widetilde{\gamma}\|F_m\|_{L_2(D_T}^2,\ \ 0<\tau\le T,$$

$$\widetilde{\gamma}=\Big[(1-\mu_2)^{-1}\mu_0^2\varepsilon^{-1}e^{\varepsilon T}+\varepsilon^{-1}\Big]\exp(2a+\varepsilon)T,$$

where by virtue of $|\mu_0|\le|\mu|$, we can take in the capacity of ε the same number $\varepsilon=\frac{1}{T}\ln(\frac{1}{|\mu|})$ as in (2.21). Next, analogously to how from (2.20) and (2.28) we have got a priori estimate (2.2) with the constants c_1 and c_2, from (2.30) we will have

$$\|v\|_{\overset{\circ}{W}{}^1_{2,\mu_0}(D_T)}\le c_3\|F\|_{L_2(D_T)},\tag{2.35}$$

where the positive constant

$$c_3=\Big\{T(c_0+1)\Big[(1-\mu_2)^{-1}\mu_0^2\varepsilon^{-1}e^{\varepsilon T}+\varepsilon^{-1}\Big]\exp(2a+\varepsilon)T\Big\}^{\frac{1}{2}}\tag{2.36}$$

does not depend on v, F and on the parameter $s\in[0,1]$.

3. The Existence of a Solution of the Problem (1.1)–(1.4)

To prove that the problem (1.1)–(1.4) has a solution in case $|\mu|<1$, we will use the well-known facts dealing with the solvability of the following mixed problem

$$u_{tt}-\sum_{i=1}^{n}u_{x_ix_i}=F(x,t),\ \ (x,t)\in D_T,\tag{3.1}$$

$$\Big(\frac{\partial u}{\partial\nu}+\sigma u\Big)\Big|_{\Gamma}=0,\quad u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x),\ \ x\in\Omega,\tag{3.2}$$

where F, φ and ψ are the given functions, $\sigma=const>0$.

For $F\in L_2(D_T)$, $\varphi\in W_2^1(\Omega)$, $\psi\in L_2(\Omega)$ a unique generalized solution u of the problem (3.1), (3.2) from the space $E_{2,1}(D_T)$ with the norm

$$\|v\|_{E_{2,1}(D_T)}^2=\sup_{0\le t\le T}\int\limits_{\omega}\Big[v^2+v_t^2+\sum_{i=1}^{n}v_{x_i}^2\Big]\,dx$$

is given by the formula [16, pp. 214, 226], [19, pp. 292, 294]

$$u = \sum_{k=1}^{\infty} \biggl(a_k \cos \mu_k t + b_k \sin \mu_k t + \frac{1}{\mu_k} \int\limits_0^t F_k(\tau) \sin \mu_k (t-\tau)\, d\tau \biggr) \varphi_k(x), \quad (3.3)$$

where $\lambda_k = -\mu_k^2$, $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \leq \cdots$ are eigen-functions and $\lim\limits_{k\to\infty} \mu_k = 0$, while $\varphi_k \in W_2^1(\Omega)$ are the corresponding eigen-functions of the spectral problem $\Delta w = \lambda w$, $(\frac{\partial w}{\partial \nu} + \sigma w)\big|_{\partial\Omega} = 0$ in the domain Ω ($\Delta := \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$) which form simultaneously an orthonormalized basis in $L_2(\Omega)$ and orthogonal basis in $W_2^1(\Omega)$ in a sense of the scalar product

$$(v, w)_{W_2^1(\Omega)} = \int\limits_{\Omega} \sum_{i=1}^{n} v_{x_i} w_{x_i}\, dx + \int\limits_{\partial\Omega} \sigma v w\, ds$$

[16, p. 237], that is,

$$(\varphi_k, \varphi_l)_{L_2(\Omega)} = \delta_k^l, \quad (\varphi_k, \varphi_l)_{W_2^1(\Omega)} = -\lambda_k \delta_k^l, \ \ \delta_k^l = \begin{cases} 1, & l = k, \\ 0, & l \neq k. \end{cases} \quad (3.4)$$

Here

$$a_k = (\varphi, \varphi_k)_{L_2(\Omega)}, \quad b_k = \mu_k^{-1} (\psi, \varphi_k), \ \ k = 1, 2, \dots, \quad (3.5)$$

$$F(x,t) = \sum_{k=1}^{\infty} F_k(t) \varphi_k(x), \qquad F_k(t) = (F, \varphi_k)_{L_2(\omega_t)}, \ \ \omega_\tau := D_T \cap \{t = \tau\}, \quad (3.6)$$

and for the solution u from (3.3) the estimate

$$\|u\|_{E_{2,1}(D_T)} \leq \gamma \Bigl(\|F\|_{L_2(D_T)} + \|\varphi\|_{W_2^1(\Omega)} + \|\psi\|_{L_2(\Omega)} \Bigr) \quad (3.7)$$

with the positive constant γ, independent of F, φ and ψ, is valid [16, pp. 214, 226].

Let us consider now the linear problem

$$L_0 u := \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = F(x,t), \ \ (x,t) \in D_T, \quad (3.8)$$

$$\Bigl(\frac{\partial u}{\partial \nu} + \sigma u \Bigr) \Big|_{\Gamma} = 0, \quad (3.9)$$

$$u(x,0) - \mu u(x,T) = 0, \quad u_t(x,0) - \mu u_t(x,T) = 0, \ \ x \in \Omega, \quad (3.10)$$

corresponding to (1.1)–(1.4) in case $a = c = \lambda = 0$.

Show that for $|\mu| < 1$, for any $F \in L_2(D_T)$, there exists a unique strong generalized solution of the problem (3.8)–(3.10). Indeed, since the space of finite infinitely differentiable functions $C_0^\infty(D_T)$ is dense in $L_2(D_T)$, therefore for $F \in L_2(D_T)$ and for any natural number m there exists the function $F_m \in C_0^\infty(D_T)$ such that

$$\|F_m - F\|_{L_2(D_T)} < \frac{1}{m}\,. \tag{3.11}$$

On the other hand, for the function F_m in the space $L_2(D_T)$ the decomposition [16]

$$F_m(x,t) = \sum_{k=1}^{\infty} F_{m,k}(t)\varphi_k(x), \quad F_{m,k}(t) = (F_m, \varphi_k)_{L_2(\Omega)} \tag{3.12}$$

is valid.

Therefore there exists the natural number ℓ_m, $\lim\limits_{m\to\infty} \ell_m = \infty$, such that for

$$\widetilde{F}_m(x,t) = \sum_{k=1}^{\ell_m} F_{m,k}(t)\varphi_k(x) \tag{3.13}$$

the inequality

$$\|\widetilde{F}_m - F_m\|_{L_2(D_T)} < \frac{1}{m} \tag{3.14}$$

holds.

It follows from (3.11) and (3.14) that

$$\lim_{m\to\infty} \|\widetilde{F}_m - F\|_{L_2(D_T)} = 0. \tag{3.15}$$

The solution $u = u_m$ of the problem (3.1), (3.2) for

$$\varphi = \sum_{k=1}^{\ell_m} \widetilde{a}_k \varphi_k, \quad \psi = \sum_{k=1}^{\ell_m} \mu_k \widetilde{b}_k \varphi_k, \quad F = \widetilde{F}_m$$

is given by formula (3.3) which with regard for (3.4)–(3.6) and (3.13) takes the form

$$u_m = \sum_{k=1}^{\ell_m} \biggl(\widetilde{a}_k \cos \mu_k t + \widetilde{b}_k \sin \mu_k t + \frac{1}{\mu_k} \int_0^t F_{m,k}(\tau) \sin \mu_k (t-\tau)\, d\tau \biggr) \varphi_k(x). \tag{3.16}$$

By the construction, the function u_m from (3.16) satisfies equation (3.8) and the boundary condition (3.9) for $F = \widetilde{F}_m$ from (3.13).

Define now unknown coefficients $\widetilde{a}_k$ and $\widetilde{b}_k$ in such a way that the function u_m from (3.16) likewise satisfy the nonlocal conditions (3.10). Towards this end, we substitute the right-hand side of (3.16) into equalities (3.10). As a result, taking into account that the system of functions $\{\varphi_k(x)\}$ forms the basis in $L_2(\Omega)$, to find coefficients $\widetilde{a}_k$ and $\widetilde{b}_k$, we obtain the following system of linear algebraic equations

$$\begin{cases} (1-\mu\cos\mu_k T)\widetilde{a}_k - (\mu\sin\mu_k T)\widetilde{b}_k \\ \qquad\qquad = \dfrac{\mu}{\mu_k}\displaystyle\int\limits_0^T F_{m,k}(\tau)\sin\mu_k(T-\tau)\,d\tau, \\ (\mu\mu_k\sin\mu_k T)\widetilde{a}_k + \mu_k(1-\mu\cos\mu_k T)\widetilde{b}_k \\ \qquad\qquad = \mu\displaystyle\int\limits_0^T F_{m,k}(\tau)\cos\mu_k(T-\tau)\,d\tau, \end{cases} \tag{3.17}$$

$k=1,2,\dots,\ell_m$, whose solution is

$$\widetilde{a}_k = \Big[d_{1k}\mu\mu_k\sin\mu_k T - d_{2k}(1-\mu\cos\mu_k T)\Big]\Delta_k^{-1}, \quad k=1,2,\dots,\ell_m, \tag{3.18}$$

$$\widetilde{b}_k = \Big[d_{2k}(1-\mu\cos\mu_k T) - d_{1k}\mu\mu_k\sin\mu_k T\Big]\Delta_k^{-1}, \quad k=1,2,\dots,\ell_m. \tag{3.19}$$

Here

$$d_{1k} = \frac{\mu}{\mu_k}\int\limits_0^T F_{m,k}(\tau)\sin\mu_k(T-\tau)\,d\tau,$$

$$d_{2k} = \mu\int\limits_0^T F_{m,k}(\tau)\cos\mu_k(T-\tau)\,d\tau$$

and since $|\mu|<1$, for the determinant Δ_k of system (3.17), we have

$$\Delta_k = \mu_k\Big[(1-\mu\cos\mu_k T)^2 + \mu^2\sin^2\mu_k T\Big] \geq \mu_k\big(1-|\mu|\big)^2 > 0. \tag{3.20}$$

Below, the Lipschitz domain Ω will be assumed to be such that the eigen-functions $\varphi_k \in C^2(\overline{\Omega})$, $k\geq 1$. For example, this fact will hold if $\partial\Omega \in C^{[\frac{n}{2}]+3}$ [18, p. 227]. This may take place also in the case of piecewise smooth Lipschitz domain, for example, for the parallelepiped $\Omega = \{x\in\mathbb{R}^n : |x_i| < a_i,\ i=1,\dots,n\}$, the corresponding eigen-functions $\varphi_k \in C^\infty(\overline{\Omega})$ [19] (see also Remark 4.1). Thus, since $F_m \in C_0^\infty(D_T)$, by virtue of (3.12), the function $F_{m,k}\in C^2([0,T])$, and hence the function u_m from (3.16) belongs to the space $C^2(\overline{D}_T)$. Next, by the construction, the function u_m from (3.16) will belong to the space $\overset{\circ}{C}{}^2_\mu(D_T)$ which has been defined in (1.5), and

$$L_0 u_m = \widetilde{F}_m, \quad L_0(u_m - u_k) = \widetilde{F}_m - \widetilde{F}_k. \tag{3.21}$$

From (3.21) and a priori estimate (2.2) for $a=c=\lambda=0$ in which by Lemma 2.1 the constant $c_2=0$, we have

$$\|u_m - u_k\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} \leq c_1\|\widetilde{F}_m - \widetilde{F}_k\|_{L_2(D_T)}. \tag{3.22}$$

By virtue of (3.15), it follows from (3.22) that the sequence $u_m \in \overset{\circ}{C}{}^2_\mu(D_T)$ is fundamental in the whole space $\overset{\circ}{W}{}^1_{2,\mu}(D_T)$. Therefore there exists the function $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ such that by (3.15) and (3.21) the limiting equalities (2.3) are valid for $\lambda = 0$. The latter means that the function u is a strong generalized solution of the problem (3.8)–(3.10). The uniqueness of that solution follows from a priori estimate (2.2) in which $\lambda = 0$ and the constant $c_2 = 0$, that is,

$$\|u\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} \leq c_1 \|f\|_{L_2(D_T)}. \tag{3.23}$$

Remark 3.1. Thus the linear problem (3.8)–(3.10) has a unique strong generalized solution $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ for which we can write $u = \Box_\mu^{-1}(F)$, where $\Box_\mu^{-1} : L_2(D_T) \to \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ is the linear continuous operator whose norm by virtue of (3.23) admits the estimate

$$\|\Box_\mu^{-1}\|_{L_2(D_T)\to \overset{\circ}{W}{}^1_{2,\mu}(D_T)} \leq c_1. \tag{3.24}$$

Remark 3.2. Regarding a new unknown function $v := u\exp(at)$, the problem (1.1)–(1.4) can be written in the form

$$\begin{aligned} \widetilde{L}_\lambda v := v_{tt} - \sum_{i=1}^n v_{x_i x_i} + (c-a^2)v + \lambda \exp(-\alpha a t)|v|^\alpha v \\ = \exp(at)F(x,t), \quad (x,t) \in D_T, \end{aligned} \tag{3.25}$$

$$\Big(\frac{\partial v}{\partial \nu} + \sigma v\Big)\Big|_\Gamma = 0, \tag{3.26}$$

$$(\mathcal{K}_{\mu_0} v)(x) = 0, \quad (\mathcal{K}_{\mu_0} v_t)(x) = 0, \quad x \in \Omega, \tag{3.27}$$

where $\mu_0 = \mu\exp(-aT)$. Note that the problems (1.1)–(1.4) and (3.25)–(3.27) are equivalent in a sense that u is a strong generalized solution of the problem (1.1)–(1.4), if and only if v is a strong generalized solution of the problem (3.25)–(3.27), that is $v \in \overset{\circ}{W}{}^1_{2,\mu_0}(D_T)$, and there exists the sequence of functions $v_m \in \overset{\circ}{C}{}^2_{\mu_0}(D_T)$ such that $v_m \to v$ in the space $\overset{\circ}{W}{}^1_{2,\mu_0}(D_T)$, and $\widetilde{L}_\lambda v_m \to \exp(at)F(x,t)$ in the space $L_2(D_T)$.

Remark 3.3. The embedding operator $I : W_2^1(D_T) \to L_q(D_T)$ is the linear, continuous, compact operator for $1 < q < \frac{2(n+1)}{n-1}$, when $n > 1$ [16, p. 81]. At the same time, the Nemytski's operator $\mathcal{N} : L_q(D_T) \to L_2(D_T)$ acting by the formula $\mathcal{N} v = (c-a^2)v + \lambda\exp(-\alpha a t)|v|^\alpha v$ is continuous and bounded if $q \geq 2(\alpha+1)$ [14, p. 349], [15, pp. 66, 67]. Thus, if $\alpha < \frac{2}{n-1}$, that is $2(\alpha+1) < \frac{2(n+1)}{n-1}$, then there exists the number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \geq 2(\alpha+1)$. Therefore, in this case the operator

$$\mathcal{N}_0 = \mathcal{N} I : \overset{\circ}{W}{}^1_{2,\mu_0}(D_T) \to L_2(D_T) \tag{3.28}$$

will be continuous and compact. Moreover, from $w \in \overset{\circ}{W}{}^1_{2,\mu_0}(D_T)$ it all the more follows that $\exp(-\alpha at)|v|^\alpha v \in L_2(D_T)$, and if $v_m \to v$ in the space $\overset{\circ}{W}{}^1_{2,\mu_0}(D_T)$, then $\exp(-\alpha at)|v_m|^\alpha v_m \to \exp(-\alpha at)|v|^\alpha v$ in the space $L_2(D_T)$.

Remark 3.4. Under the assumption that $a \geq 0$ and $|\mu| < 1$, we have $|\mu_0| < 1$, and taking into account Remarks 3.1 and 3.2, the function $v \in \overset{\circ}{W}{}^1_{2,\mu_0}(D_T)$ is a strong generalized solution of the problem (3.25)–(3.27), if and only if v is a solution of the following functional equation

$$v = \square^{-1}_{\mu_0}\Big((a^2 - c)v - \lambda \exp(-\alpha at)|v|^\alpha v\Big) + \square^{-1}_{\mu_0}\big(\exp(at)F\big) \tag{3.29}$$

in the space $\overset{\circ}{W}{}^1_{2,\mu_0}(D_T)$.

We rewrite equation (3.29) in the form

$$v = A_0 v := -\square^{-1}_{\mu_0}(\mathcal{N}_0 v) + \square^{-1}_{\mu_0}\big(\exp(at)F\big), \tag{3.30}$$

where the operator $\mathcal{N}_0 : \overset{\circ}{W}{}^1_{2,\mu_0}(D_T) \to L_2(D_T)$ from (3.28) is, by Remark 3.3, continuous and compact one. Consequently, owing to (3.24), the operator $A_0 : \overset{\circ}{W}{}^1_{2,\mu_0}(D_T) \to \overset{\circ}{W}{}^1_{2,\mu_0}(D_T)$ from (3.30) is likewise continuous and compact for $0 < \alpha < \frac{2}{n-1}$. At the same time, by Remarks 2.1, 3.2 and 3.4, if conditions (2.34) are fulfilled for every value of parameter $s \in [0,1]$ and for every solution v of equation $v = sA_0 v$ with the parameter $s \in [01,]$, then a priori estimate (2.35) with nonnegative constant c_3 from (2.36), independent of v, F and s, is valid. Therefore, by the Lerè–Schauder theorem [20, p. 375], equation (3.30), and hence by Remarks 3.2 and 3.4, the problem (1.1)–(1.4) has at least one solution $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$. Thus we have proved the following

Theorem 3.1. *Let* $0 < \alpha < \frac{2}{n-1}$, $\lambda > 0$, $|\mu| < 1$ *and conditions* (2.34) *be fulfilled. Then for any* $F \in L_2(D_T)$ *the problem* (1.1)*–*(1.4) *has at least one strong generalized solution of the class* W^1_2 *in the domain* D_T *in a sense of Definition* 1.1.

4. The Case $|\mu| = 1$

Instead of conditions (2.1) we consider now the conditions

$$a > 0, \quad c \geq a^2, \quad \sigma > 0. \tag{4.1}$$

Theorem 4.1. *Let* $0 < \alpha < \frac{2}{n-1}$, $\lambda > 0$, $|\mu| = 1$ *and conditions* (4.1) *be fulfilled. Then for any* $F \in L_2(D_T)$ *the problem* (1.1)*–*(1.4) *has at least one strong generalized solution of the class* W^1_2 *in the domain* D_T *in a sense of Definition* 1.1.

Proof. Regarding a new unknown function $v := u\exp(at)$, the problem (1.1)–(1.4) by Remark 3.2 reduces equivalently to the nonlocal problem (3.25)–(3.27), where by virtue of $a > 0$, for the number $\mu_0 = \mu\exp(-aT)$ we have $|\mu_0| < 1$. Therefore if the conditions of Theorem 4.1 are fulfilled, then repeating reasoning mentioned in proving Theorem 3.1 we can conclude that the problem (3.25)–(3.27) and hence the problem (1.1)–(1.4) has at least one strong generalized solution of the class W_2^1 in the domain D_T. □

Remark 4.1. It should be noted that for $|\mu| = 1$ the homogeneous problem corresponding to (1.1)–(1.4) may have even in a linear case, i.e., for $\lambda = 0$, a finite or even an infinite set of linearly independent solutions, if conditions (4.1) are violated, whereas for the solvability of that problem the function $F \in L_2(D_T)$ must satisfy, respectively, a finite or an ininite number of conditions of solvability of type $\ell(F) = 0$, where ℓ is the linear continuous functional in $L_2(D_T)$. Indeed, let us consider the case $\lambda = a = c = 0$, $\sigma = 1$. When $\mu = 1$, we denote by $\Lambda(1)$ a set of those μ_k from (3.3) for which the ratio $\frac{\mu_k T}{2\pi}$ is a natural number, i.e., $\Lambda(1) = \{\mu_k : \frac{\mu_k T}{2\pi} \in \mathbb{N}\}$. Formulas (3.18) and (3.19) for finding unknown coefficients $\widetilde{a}_k$ and $\widetilde{b}_k$ in the representation (3.16) have been obtained from the system of linear algebraic equations (3.17). In case $\lambda(1) \neq \varnothing$ and $\mu_k \in \Lambda(1)$, $\mu = 1$, the determinant of system (3.17) given by formula (3.20) is equal to zero. Moreover, in this case all coefficients $\widetilde{a}_k$ and $\widetilde{b}_k$ in the left-hand side of system (3.17) are equal to zero. Therefore, in accordance with (3.3), the homogeneous problem corresponding to (3.8), (3.9) and (3.10) is satisfied with the function

$$u_k(x,t) = (C_1\cos\mu_k t + C_2\sin\mu_k t)\varphi_k(x), \tag{4.2}$$

where C_1 and C_2 are arbitrary constant numbers, and in this case the necessary conditions for the solvability of the inhomogeneous problem (3.8)–(3.10) corresponding to $\mu_k \in \Lambda(1)$ are

$$\begin{aligned} \ell_{k,1}(F) &= \int\limits_{D_T} F(x,t)\varphi_k(x)\sin\mu_k(T-t)\,dx\,dt = 0, \\ \ell_{k,2}(F) &= \int\limits_{D_T} F(x,t)\varphi_k(x)\cos\mu_k(T-t)\,dx\,dt = 0. \end{aligned} \tag{4.3}$$

Analogously, in case $\mu = -1$, we denote by $\Lambda(-1)$ a set of those μ_k from (3.3) for which the ratio $\frac{\mu_k T}{\pi}$ is an odd natural number. For $\mu_k \in \Lambda(-1)$, $\mu = -1$, the function u_k from (4.2) is, likewise, a solution of the homogeneous problem corresponding to (3.8)–(3.10), and conditions (4.3) are the necessary ones for solvability of that problem. For example, for $n = 2$, the eigen-numbers and eigen-functions of the spectral problem $\Delta w = \lambda w$, $(\frac{\partial w}{\partial \nu} + w)\big|_{\partial\Omega} = 0$ are

$$\lambda_k = -\frac{1}{4}\left[(2k_1-1)^2 + (2k_2-1)^2\right], \quad k = (k_1,k_2) \in \mathbb{N}^2,$$

$$\varphi_k(x_1,x_2)=d_k\big(\sin\widetilde{\mu}_{k_1}x_1+\widetilde{\mu}_{k_1}\cos\widetilde{\mu}_{k_1}x_1\big)\big(\sin\widetilde{\mu}_{k_2}x_2+\widetilde{\mu}_{k_2}\cos\widetilde{\mu}_{k_2}x_2\big),$$

where $\widetilde{\mu}_{k_i}=\frac{1}{2}\,(2k_i-1)$, $\mu_k=\frac{1}{2}\,\sqrt{(2k_1-1)^2+(2k_2-1)^2}$, and d_k is the normalizing factor defined from the condition $\|\varphi_k\|_{L_2(\Omega)}=1$. It can be easily seen that if the number T is such that $\frac{T}{2\sqrt{2}\,\pi}\in\mathbb{N}$, then for any $k=(k_1,k_2)$ such that $k_1=k_2$ we have $\mu_k\in\Lambda(1)$. In this case, i.e., for $\mu=1$ and $\frac{T}{2\sqrt{2}\,\pi}\in\mathbb{N}$, the homogeneous problem corresponding to (3.8)–(3.10) will have an infinite set of linearly independent solutions of type (4.2), and for the solvability of that problem it is necessary that an infinite number of conditions of type (4.3) for $k=(k_1,k_2)$ such that $k_1=k_2\in\mathbb{N}$ are fulfilled. The case $\mu=-1$ is considered analogously.

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Author's address:

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia.

E-mail: kharibegashvili@yahoo.com

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Nahum Krupnik

INFLUENCE OF SOME B. V. KHVEDELIDZE'S RESULTS ON THE DEVELOPMENT OF FREDHOLM THEORY FOR SIOs WITH PC COEFFICIENTS IN $L_p^n(\Gamma, \rho)$

Dedicated to the 100th Anniversary of Boris Vladimirovich Khvedelidze

Abstract. A concise survey on the construction of the spectra, symbols and index-formulas for singular integral operators with piecewise continuous coefficients in the spaces $L_p^n(\Gamma, \rho)$ is given. Influence of some results by B. V. Khvedelidze on this research is shown. Several interesting associated results, obtained during this research, and their applications are discussed in appendix. An open question is stated. Some historical information, related to this paper is presented in the introduction.

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რეზიუმე. ჩვენ წარმოგიდგენთ მოკლე მიმოხილვას სპექტრის, სიმბოლოს და ინდექსის აგების შესახებ უბან-უბან უწყვეტ კოეფიციენტებიანი სინგულარული ინტეგრალური ოპერატორებისათვის $L_p(\Gamma, \rho)$ სივრცეებში. ნაჩვენებია ბ. ხვედელიძის ზოგიერთი შედეგის გავლენა დასახელებულ კვლევებზე. დანართში განხილულია ზოგიერთი საინტერესო ასოცირებული შედეგი და მათი გამოყენება, რომლებიც მიღებულია ასეთი კვლევების დროს. დასმულია ზოგიერთი ამოცანა, რომელიც საჭიროებს გადაწყვეტას. შესავალში ჩართულია ისტორიული ინფორმაცია, რომელიც ეხება მოცემულ სტატიას.

1. Introduction

About eighty years ago S. G. Mikhlin [24] in solving the regularization problem for two-dimensional singular integral operators (SIOs) assigned to each such an operator A a function $\sigma(A)(x)$, which he called a *symbol*, and he showed that the regularization is possible if $\inf_x |\sigma(A)(x)| > 0$. Thereafter (as widely known) the notion of the symbol was extended to multi-dimensional and one-dimensional SIOs by many authors. In particular, for one-dimensional singular operator $A = aI + bS + T$, where $a(t)$, $b(t)$ are continuous functions on a simple closed contour Γ, T is a compact operator and

$$Sf(t) := \frac{1}{\pi i} \int\limits_{\Gamma} \frac{f(\tau)}{\tau - t}\, d\tau \quad (t \in \Gamma), \tag{1.1}$$

the symbol in the space $L_p(\Gamma, \rho)$ $(1 < p < \infty)$ was defined by the equality

$$\sigma(aI + bS + T)(t, z) = a(t) + z\, b(t) \quad \big((t, z) \in \Gamma \times \{\pm 1\}\big). \tag{1.2}$$

For a long period of time, symbols of SIOs were used for the following (sufficient) conditions:

$$\text{If } \inf_x |\sigma(A)(x)| > 0, \text{ then } A \text{ is a Fredholm operator.}$$

An important role in raising the status of the symbols (for many classes of operators) was played by Gelfand's theory of maximal ideals in Banach algebras. Using this theory, I. Gohberg obtained the following important results.

Theorem 1.1 ([3]). *Let $A := aI+bS+T$ and $\sigma(A)(t, z)$ denote, respectively, the singular integral operator and its symbol, defined in* (1.2). *Then*

$$A \in F(L_2(\Gamma)) \Longleftrightarrow \sigma(A)(t, z) \neq 0, \quad \forall\, (t, z) \in \Gamma \times \{\pm 1\}, \tag{1.3}$$

where $F(\mathcal{B})$ denote the set of all Fredholm operators on Banach space $\mathcal{B}$.

To formulate a next theorem, we need the following notations. Let Ω denote the unit sphere in an n-dimensional space $\mathbb{R}^n$; $Y_n(\theta)$ $(\theta \in \Omega$, $n = 1, 2, \ldots)$ the sequence of all n-dimensional spherical functions, numbered in some order; Y_n the simplest singular integral operator (see [24] or [5])

$$(Y_n f)(x) = \frac{1}{\gamma_n} \int\limits_{\mathbb{R}^n} \frac{y_n(\nu)}{|x - y|^n}\, f(y)\, dy$$

with the symbol $Y_n(\theta)$; $\mathcal{T}$ the ideal of all compact operators in the algebra L $(L_p(\mathbb{R}^n))$ $(1 < p < \infty)$; $\mathcal{A}_p$ the Banach subalgebra of L $(L_p(\mathbb{R}^n))$, generated by the operators

$$Af(x) := a_0(x) f(x) + \sum_{n=1}^{r} a_n(x)(Y_n f)(x) + T \quad (T \in \mathcal{T})$$

with continuous coefficients $a_n(x)$ and with the symbols

$$A(x,\theta) = a_0(x) + \sum_{n=1}^{r} a_n(x) Y_n(\theta).$$

Theorem 1.2 ([5]). *The quotient algebra $\widehat{\mathcal{A}}_2 = \mathcal{A}_2/\mathcal{T}$ is a commutative Banach algebra; the symbols $A(x,\theta)$ coincides with the functions of element $\widehat{A} \in \widehat{\mathcal{A}}_2$ on the compact space of maximal ideals of the algebra $\widehat{\mathcal{A}}_2$ and*

$$A \in F(L_2(\mathbb{R}^n)) \Longleftrightarrow A(x,\theta) \neq 0, \ \ \forall\,(x,\theta) \in \mathbb{R}^n \times \Omega. \tag{1.4}$$

Theorems 1.1, 1.2 were extended in [4,6] to systems of the corresponding SIOs.

With the appearance of the (revolutionary) results [3–6] the concept of the symbols of SIOs achieved a higher status: responsibility for the necessary and sufficient conditions of Fredholmness (see (1.3), (1.4)). This inspired many mathematicians, interested in the theory of symbol of SIOs, to generalize these results, obtained by Gohberg, to other Banach spaces[1].

The author of this survey was inspired, too. And in the papers [19, 20] the main results from [3–6] were extended to spaces L_p and L_p^n $(1 < p < \infty)$.

Shortly thereafter, I. Gohberg invited me to join him for studying the Fredholm theory of one-dimensional SIOs *with piecewise continuous coefficients* on $L_p^n(\Gamma)$: to obtain the spectrum, symbols and formulas for computation the index. I gladly accepted this invitation.

The Fredholm theory for SIOs with PC coefficients, obtained in [7–10], is briefly described in Sections 2, 3. The influence of some results of B. V. Khvedelidze on this cycle of researches is described in Section 4. In Section 5, we construct a counterexample, related to a scalar symbol in algebra generated by SIOs with PC coefficients. In appendix (Section 6), some associated results and their applications, obtained in [19, 20] and [8], are shown. An open question is stated.

It is my pleasure to thank my friend Prof. Roland Duduchava[2] for useful remarks and comments.

2. On the Spectrum and Index of SIOs with PC Coefficients

Recall (for convenience) several notations and definitions.

Let $L_p(\Gamma,\rho)$, $1 < p < \infty$ denote a weighted Banach space with

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}, \ \ -1 < \beta_k < p - 1,$$

$$\|f\|^p_{L_p(\Gamma,\rho)} = \int_\Gamma |f(t)|^p \rho(t)\,|dt|,$$

[1]Many results, obtained in this research, are described in the (encyclopedic) book [25].

[2]Note, that Roland Duduchava had a privilege to be the student of both: B. V. Khvedelidze and I. C. Gohberg!

PC(Γ) is the set of all piecewise continuous functions[3] on Γ; $A = aP + bQ$; $C = cP + Q$, where $a, b, c \in$ PC, $P := (I + S)/2$, $Q = (I - S)/2$, and the operator S is defined by (1.1).

In this section we assume, for simplicity, that Γ is a simple closed oriented Lyapunov contour, $0 \in D^+$ and the function $c(t)\,(\in \mathrm{PC}(\Gamma))$ has only one point t_0 of discontinuity:

$$c(t_0 - 0) = z_1, \quad c(t_0 + 0) = z_2. \tag{2.1}$$

Definition 2.1. We denote by $\nu(z_1, z_2, \delta)$ $(0 < \delta < \pi)$ the circular arc joining the points z_1 to z_2 and having the following properties:

1^0. Let $\delta \in (0, \pi)$. Then from any interior point $z \in \nu(z_1, z_2, \delta)$ one sees the straight line $[z_1, z_2]$ under the angle δ, and running through the arc from z_1 to z_2, this straight line is located in the left-hand side.

2^0. Let $\delta \in (\pi, 2\pi)$, then we define $\nu(z_1, z_2, \delta) := \nu(z_2, z_1, 2\pi - \delta)$.

3^0. Finally, $\nu(z_1, z_2, \pi)$ denotes the straight line $[z_1, z_2]$.

Next, we denote by $W_{p,\rho}(c)$ the plane curve which results from the range of the function $c(t)$ by adding the arc $\nu(c(t_0 - 0), c(t_0 + 0), \frac{2\pi(1+\beta)}{p})$. We orient the curve $W_{p,\rho}(c)$ in the natural manner. Also, we write $W_p(c)$ if $\rho(t) \equiv 1$.

Definition 2.2. The function $c(t)\,(\in \mathrm{PC}(\Gamma))$ is called $\{p, \rho\}$-non-singular, if the curve $W_{p,\rho}(c)$ does not contain the origin.

Definition 2.3. Let the function $c(t)$ be a $\{p, \rho\}$-non-singular. Then the winding number of the curve $W_{p,\rho}(c)$ around the point $z = 0$ is called $\{p, \rho\}$-*index* of the function $c(t)$. This index is abbreviated by $\mathrm{ind} c^{p,\rho}$.

Theorem 2.4. *The operator $C = cP + Q$ is at least one-side invertible on $L_p(\Gamma, \rho)$ if and only if the function $c(t)$ is $\{p, \rho\}$- non-singular. Let the function $c(t)$ be $\{p, \rho\}$-non-singular. Then the operator C is invertible, invertible only from the left or invertible only from the right, depending on whether the number $k := \mathrm{ind}\, c^{p,\rho}$ is equal to zero, positive or negative, respectively. If $k > 0$, then dim* $\mathrm{coker}(C) = k$, *and if $k < 0$, then dim* $\ker(C) = -k$.

Remark 2.5. If the function $c(t)$ has several points t_k of discontinuity, then $W_{p,\rho}(c)$ results from the range of the function $c(t)$ by adding several arcs $\nu(c(t_k - 0), c(t_k + 0), \delta)$.

Theorem 2.6. *The operator $A = aP + bQ$ is Fredholm on $L_p(\Gamma, \rho)$ if and only if $b(t \pm 0) \neq 0$ $(t \in \Gamma)$, and the function $c(t) = a(t)/b(t)$ is $\{p, \rho\}$-non-singular.*

Remark 2.7. A theorem similar to Theorem 2.6, was obtained by H. Widom [28] for the case where Γ is a measurable subset of $\mathbb{R}$.

[3]See (for details) the definition of PC(Γ) in [13, p. 62].

Remark 2.8. For the space $L_2(\Gamma)$, the results of Theorems 2.4 and 2.6 were obtained in [7]. For the spaces $L_p(\Gamma)$ and $L_p(\Gamma,\rho)$ in [8, 9].

After the papers [8, 9] were published, we (the authors) were periodically asked (at seminars and conferences) various questions related to these papers. Most often we were asked the following

Question 2.9. *How did you guess* (*or, how did you come*) *to adding these special circular arcs, depending on* p,ρ *and joining the points* $c(t_k \pm 0)$*?*

In Sections 4, we show a way, paved by B. V. Khvedelidze, on which we came to the idea of these circular arcs.

3. SIOs with Matrix PC-Coefficients in $L_p^n(\Gamma,\rho)$

Let $R := AP + BQ$ denote a singular integral operator with piecewise continuous matrix coefficients $A := [a_{ik}]_{i,k=1}^n$ and $B := [b_{ik}]_{i,k=1}^n$. Sufficient conditions for the operator R to be Fredholm in $L_p^n(\Gamma,\rho)$ was first obtained by B. V. Khvedelidze (see [17, Chapter 2]). Then the Fredholm criterion was obtained in our work [10]. See also [25, Chapter 5, Section 6] for some additional historical details.

Let $C := [c_{ik}]_{i,k=1}^n$ be a piecewise continuous matrix function, and let $t_1,\dots,t_r$ be the points of discontinuity of the matrix C. To each point t_s $(s = 1,\dots,r)$ we attach a matrix-valued arc

$$\nu(t_s,\mu) := \frac{e^{i\mu\theta_s}\sin(1-\mu)\theta_s}{\sin\theta_s}\,G(t_s-0) + \frac{e^{i(\mu-1)\theta_s}\sin\mu\theta_s}{\sin\theta_s}\,G(t_s+0), \quad (3.1)$$

where $\theta_s = \pi - \frac{2\pi(1+\beta_s)}{p}$, and we assume that

$$\rho(t) = \prod_{k=1}^{m} |t-t_k|^{\beta} \ \ (m \geq r). \quad (3.2)$$

We associate with the matrix C a continuous matrix curve $C^{p,\rho}(t,\mu)$, obtained by adding r arcs $\nu_(t_s,\mu)$ to the range of the matrix C.

Definition 3.1. The matrix function $C := [c_{ik}]_{i,k=1}^n$ is called $\{p,\rho\}$-nonsingular if $0 \notin \det C^{p,\rho}(t,\mu)$. Let $C(t)$ be $\{p,\rho\}$-nonsingular matrix function, then its $\{p,\rho\}$ index is defined by the equality $\operatorname{ind} C^{p,\rho} := \operatorname{ind}\det C^{p,\rho}(t,\mu)$.

Theorem 3.2. *The operator* $R = AP + BQ$ *is a Fredholm operator on* $L_p^n(\Gamma,\rho)$ *if and only if* $\det B(t\pm 0) \neq 0$ *for all* $t\in\Gamma$ *and the matrix function* $C(t) := B(t)^{-1}A(t)$ *is* $\{p,\rho\}$*-nonsingular. If these conditions are fulfilled, then the index of operator* R *in the space* $L_p^n(\Gamma,\rho)$ *is defined by the equality* $\operatorname{ind} R = -\operatorname{ind} C^{p,\rho}$.

4. Influence of Some Results by B. V. Khvedelidze

In this section we assume, for simplicity, that Γ is a simple closed oriented Lyapunov contour, $0 \in D^+$ and $1 \in \Gamma$.

By the time we (Gohberg–Krupnik) started to work on the Fredholm theory of SIOs with piecewise continuous coefficients on $L_p(\Gamma)$ $(1 < p < \infty)$, the following statement was well known:

Proposition 4.1. *The spectrum and Fredholm spectrum for one-dimensional SIOs with continuous coefficients in the spaces $L_p(\Gamma)$ do not depend on $p \in (1, \infty)$.*

Naturally, there arose the following

Question 4.2. *Is Proposition* 4.1 *true in the case of piecewise continuous coefficients?*

In order to get the answer to this question (as well as to some other questions), we referred to Khvedelidze's works [16–18]. First we turned our attention to the following important statements.

Theorem 4.3 ([16])**.** *Let $1 < p < \infty$ and $\rho = |t - t_0|^\beta$ $(t_0 \in \Gamma)$. If $-1 < \beta < p - 1$, then the singular operator S is bounded in $L_p(\Gamma, \rho)$.*

Corollary 4.4. *The operator $(t - t_0)^\delta S(t - t_0)^{-\delta}$ $(t_0 \in \Gamma)$ is bounded in $L_p(\Gamma)$ if and only if*

$$-\frac{1}{p} < \operatorname{Re}\delta < 1 - \frac{1}{p}\,.$$

Next, using suitable ideas and results from [16–18], we have proved the following

Theorem 4.5. *The operator $A = t^\gamma P + Q$ with $\operatorname{Re}\gamma \in (0, 1)$ is a Fredholm operator in $L_p(\Gamma)$ for all $p \neq 1/\operatorname{Re}\gamma$.*

Proof. Following [17, 18], we considered the following two factorizations of the function $\psi(t) = t^\gamma$ $(\operatorname{Re}\gamma \in (0, 1))$:

$$\psi(t) = (t - 1)^\gamma \Big(\frac{t-1}{t}\Big)^{-\gamma} = \psi_+(t)\psi_-(t)$$

and

$$\psi(t) = (t - 1)^{\gamma-1} t \Big(\frac{t-1}{t}\Big)^{1-\gamma} = \xi_+(t)\, t\, \xi_-(t). \tag{4.1}$$

We assumed that Γ satisfies the conditions, formulated above (before (2.1)). Without loss of generality, we also assumed that $t_0 = 1$ and $0 \in D^+$.

Let $A = \psi(t)P + Q = \psi_-(\psi_+ P + \psi_-^{-1} Q)$ and $B = (\psi_+^{-1} P + \psi_- Q)\psi_-^{-1}$. It is not difficult to check that $AB = BA = I$. Therefore, the operator A is invertible in some space $L_p(\Gamma)$, if and only if the operator B is bounded in L_p. Using the representation

$$B = \frac{1}{2}\left[(\psi^{-1} + 1)I + (\psi^{-1} - 1)\psi_- S \psi_-^{-1} I\right]$$

and Corollary 4.4, it was obtained in [17] that the operator A is invertible in L_p for all p:

$$\frac{1-p}{p} < \operatorname{Re}\gamma < \frac{1}{p}\,, \ \text{ i.e., for } \ p < \frac{1}{\operatorname{Re}\gamma}\,.$$

Next, we used factorization (4.1) and represented the operator A in the form $A = A_1T$, where $A_1 = \xi_+(t)\xi_-(t)P + Q$ and $T = tP + Q$. The operator T is Fredholm in L_p for all $p \in (1, \infty)$. Like in the first factorization, one can obtain here that the operator A_1 is invertible in $L_p(\Gamma)$ for all p such that

$$\frac{1-p}{p} < \operatorname{Re}\gamma - 1 < \frac{1}{p}\,, \text{ i.e., for } p > \frac{1}{\operatorname{Re}\gamma}\,.$$

Thus, for all $p > 1/\operatorname{Re}\gamma$, the operator A is a Fredholm one with $\operatorname{ind}^p A = 1$. □

Example 4.6. Let $\gamma = 1/2$. The operator $A = t^{1/2}P + Q$ is invertible in L_p for all $p < 2$ and it is a Fredholm with $\operatorname{ind}^p = 1$ for all $p > 2$. This follows from Theorem 4.5. For $p = 2$, the operator A is not Fredholm. This does not follow from Theorem 4.5, but it follows from the paper [7] in which the Fredholm theory for SIOs with PC coefficients in $L_2(\Gamma)$ was developed.

Remark 4.7. Example 4.6 shows the spectral behavior of the point $\lambda = 0$ of the operator $A - \lambda I$ and, in particular, gives the negative answer to Question 4.2.

In order to analyze the spectral behavior of other points λ, we consider $\psi(t) = t^{1/2}$, $A = \psi P + Q$, and $\lambda \notin \{\psi(t) : t \in \Gamma\}$. We represent operator $A - \lambda I$ in the form

$$A - \lambda I = (1-\lambda)R, \text{ where } R := \left(\frac{\psi(t)-\lambda}{1-\lambda}P + Q\right) := g(t)P + Q. \tag{4.2}$$

It follows from (4.2) that

$$\frac{g(1-0)}{g(1+0)} = \frac{\lambda+1}{\lambda-1} := z = re^{i\theta} = e^{i\theta+\ln r}. \tag{4.3}$$

Following [17], we consider such a function $h(t) = t^\gamma$, that

$$\frac{h(1-0)}{h(1+0)} = e^{2\pi i\gamma} = e^{i\theta+\ln r} \Longrightarrow \operatorname{Re}\gamma = \frac{\theta}{2\pi}. \tag{4.4}$$

Now we can prove the following

Theorem 4.8. *Let $\psi(t) = t^{1/2}$, $A = \psi P + Q$,*

$$\lambda \notin \big\{\psi(t) : t \in \Gamma\big\}, \text{ and } \frac{\lambda+1}{\lambda-1} \neq r\exp\frac{2\pi i}{p} \quad (0 \le r < \infty). \tag{4.5}$$

Then the operator $A - \lambda I$ is a Fredholm operator in $L_p(\Gamma)$.

Proof. It follows from (4.5) and (4.3) that $\theta \neq 2\pi/p$ and from (4.4) that $\operatorname{Re}\gamma \neq 1/p$. Thus (see Theorem 4.5), operator $H = hP + Q$ is Fredholm in $L_p(\Gamma)$. Equalities (4.3), (4.4) provide that the function $h(t)/g(t)$ is continuous on Γ, and hence operators R (as well as operator $A - \lambda I$) under the condition $\theta \neq 2\pi/p$ is a Fredholm operator in L_p, too. This proves the theorem. □

Remark 4.9. It remains to describe the set ℓ of the points $\lambda \in \mathbb{C} \setminus \{\psi(t) : t \in \Gamma\}$ (candidates for "non-Fredholm points"), for which the second condition in (4.5) is not satisfied. This is not difficult.

Let $z = (\lambda + 1)/(\lambda - 1) = r \exp(2\pi i/p)$ $(0 \le r < \infty)$. If $r = 0$, then $\lambda = -1$, if $r = \infty$, then $\lambda = 1$. If $r = 1$, then, $\lambda = -i \cot \frac{\pi}{p}$. Thus, ℓ is a circular arc with the chord $[-1, 1]$. The point $-i \cot \frac{\pi}{p}$ is located on the circular arc ℓ, and from this point one sees the segment $[-1, 1]$ under the angle $\delta = \frac{2\pi}{p}$.

Conclusion 4.10. *Let $\psi(t) = t^{1/2}$ and $A = \psi P + Q$. Then the set of the points $\lambda \in \mathbb{C} \setminus \{\psi(t) : t \in \Gamma\}$, which are candidates for "non-Fredholm points" of operator $A - \lambda I$ in $L_p(\Gamma)$, coincides with the circular arc $\nu(-1, 1, \frac{2\pi}{p})$.*

This is the way on which we came to the idea of circular arc, and it gives the answer to Question 2.9.

5. Symbols for Algebras of SIOs with PC Coefficients

Let $\mathcal{E}$ denote a subalgebra of the algebra $\mathcal{A} := L(\mathcal{B})$, where B is a Banach space. We say that algebra $\mathcal{E}$ is with a (scalar) Fredholm symbol if there exists a collection $\{h_y\}_{y \in Y}$, of multiplicative functionals $h_y : \mathcal{E} \to \mathbb{C}$ such that

$$A \in \mathcal{E} \cap F(\mathcal{B}) \Longleftrightarrow h_y(A) \neq 0, \quad \forall\, y \in Y. \tag{5.1}$$

Compare (5.1) with scalar symbols in (1.2) and (1.4), where the sets Y_1, Y_2 are defined, respectively, by the equalities:

$$Y_1 = \Gamma \times \{\pm 1\} \quad \text{and} \quad Y_2 = \mathbb{R}^n \times \Omega.$$

After the results in [7–10] were obtained a natural question arose:

Question 5.1. *Is algebra $\mathcal{E}$, generated by SIOs with piecewise continuous coefficients on $L_p(\Gamma, \rho)$, with a scalar symbol?*

We (I. Gohberg and N. Krupnik) tried to get a positive answer to this question. But (instead), we constructed a counterexample (see below). After some thought, we decided to construct a *matrix symbol* for algebras, generated by (scalar) SIOs with PC coefficients. This idea opened a next cycle of our common research, review of which is beyond the scope of this article.

We conclude this section with a counterexample, mentioned above.[4]

Lemma 5.2. *Let $\mathcal{E}$ denote the algebra generated by SIOs with PC coefficients on $L_p(\Gamma)$, where Γ is a unite circle, and let $G = \lambda I + CP - PC$, where $C := c(t)I$. If algebra $\mathcal{E}$ is with a scalar symbol, then G is a Fredholm operator for each $\lambda \neq 0$.*

[4] To my knowledge, such a counterexample has never been published.

Proof. Let algebra $\mathcal{A}$ be with a scalar symbol. Then

$$h_x(G) = \lambda + h_x(C)h_x(P) - h_x(P)h_x(C) = \lambda \neq 0, \ \ \forall \lambda \neq 0.$$

From the definition of scalar symbol it follows that $G \in F(L_p(\Gamma))$ for all $\lambda \neq 0$. □

Lemma 5.3. *Let $p = 2$, $c(t) = t^{1/2}$ and $c(1 \pm 0) = \pm 1$. Then there exists $\lambda \neq 0$ such that the operator G, defined in Lemma* 5.2*, is not Fredholm.*

Proof. It follows from Shur's representation

$$R := \begin{bmatrix} I & C \\ P & \lambda I + CP \end{bmatrix} = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} I & C \\ 0 & I \end{bmatrix}$$

that the operator $G \in F(L_2(\Gamma))$ if and only if the operator $R \in F(L_2^2(\Gamma))$. The operator R can be represented in the form

$$R = \begin{bmatrix} 1 & c(t) \\ 1 & \lambda + c(t) \end{bmatrix} P + \begin{bmatrix} 1 & c(t) \\ 0 & \lambda \end{bmatrix} Q = AP + BQ.$$

Since $\det B(t) \neq 0$, the operator R is Fredholm if and only if the matrix $M_\lambda := B^{-1}A$ is 2-nonsingular. In particular (see Theorem 3.2 and equalities (3.1), (3.2)), this means that

$$0 \notin \det \nu_\lambda(1, \mu), \ \ \text{where} \ \ \nu_\lambda(t, \mu) := (1 - \mu)M_\lambda(1 - 0) + \mu M_\lambda(t + 0).$$

But for $\mu = 1/2$, we have the equality

$$\nu_\lambda\Big(1, \frac{1}{2}\Big) = \frac{1}{2\lambda}\left(\begin{bmatrix} \lambda - 1 & -1 \\ 1 & \lambda + 1 \end{bmatrix} + \begin{bmatrix} \lambda + 1 & -1 \\ 1 & \lambda - 1 \end{bmatrix} \right) = \frac{1}{\lambda} \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}$$

and for $\lambda_0 = i$, we receive $\det \nu_{\lambda_0}(1, 1/2) = 0$. This proves that the operator $G = iI + t^{1/2}P - Pt^{1/2}I$ is not an F-operator in $L_2(\Gamma)$. □

Corollary 5.4. *Combining these two lemmas, we obtain the negative answer to Question* 5.1*.*

6. Appendix: Several (Side) Results Associated with the Main Results in [19, 20] and [8]

6.1. **Banach spaces versus Hilbert spaces.** Let $L(\mathcal{B})$ $(L(\mathcal{H}))$ denote the algebra of all linear bounded operators in the Banach (Hilbert) space $\mathcal{B}$ $(\mathcal{H})$ and $GL(B)$ be the group of invertible operators. By $\mathcal{T}(\mathcal{B})$ we denote the ideal of all compact operators on $\mathcal{B}$ and by $F(\mathcal{B})$ the set of all Fedholm operators on $\mathcal{B}$.

Analyzing the proofs of Theorems 1.1 and 1.2 for the purpose of transferring them to Banach spaces, an idea appeared to find a replacement of the following well known Proposition 6.1 (so that it would work in Banach spaces):

Proposition 6.1. *For any operator $A \in L(\mathcal{H})$, there exist two operators $A_1, A_2 \in L(\mathcal{H})$ such that $A = A_1 + iA_2$,* $\operatorname{spec}(A_i) \subset \mathbb{R}$ $(i = 1, 2)$ *and the relation*

$$A = A_1 + iA_2 \in GL(\mathcal{H}) \Longleftrightarrow \overline{A} := A_1 - iA_2 \in GL(\mathcal{H})$$

holds.

Indeed, one can take $A_1 = (A + A^*)/2$, and $A_2 = (A - A^*)/2i$.

In the paper [19], the following version of substitution was proposed.

Theorem 6.2. *Let operators $A_1, A_2 \in L(\mathcal{B})$, $A_1A_2 = A_2A_1$ and* $\operatorname{spec}(A_i) \subset \mathbb{R}$ $(i = 1, 2)$*. Then*

$$A := A_1 + iA \in GL(\mathcal{B}) \Longleftrightarrow \overline{A} := A_1 - iA_2 \in GL(\mathcal{B}).$$

Corollary 6.3. *Let operators $A_1, A_2 \in L(\mathcal{B})$, $A_1A_2 - A_2A_1 \in \mathcal{T}(\mathcal{B})$ and* $\operatorname{spec}(A_i) \subset \mathbb{R}$ $(i = 1, 2)$*. Then*

$$A := A_1 + iA \in F(\mathcal{B}) \Longleftrightarrow \overline{A} := A_1 - iA_2 \in F(\mathcal{B}).$$

Remark 6.4. These (side) results were first used in [19] for extending Gohberg's Theorem 1.2 from L_2 to L_p. Thereafter, Theorem 6.2 and Corollary 6.3 were used for different purposes by many authors. For illustration we consider two examples.

In 1962 Kharazov and Khvedelidze proved the following statement [15]:

Theorem 6.5. *Let $A = a(t)I + b(t)S$ be a SIO with continuous coefficients on a closed contour in $L_p(\Gamma)$. If A is a Fredholm operator in both $L_p(\Gamma)$ and $L_q(\Gamma)$, $(p^{-1} + q^{-1} = 1)$, then $a(t)^2 - b(t)^2 \neq 0$ on Γ.*

Let us show (for illustration) how Theorem 6.5 and Corollary 6.3 could be combined for a simple extension of Gohberg's Theorem 1.1 from $L_2(\Gamma)$ to $L_p(\Gamma)$.

Theorem 6.6. *The operator $A = aI + bS$ with continuous coefficients on a closed contour Γ is Fredholm in $L_p(\Gamma)$ if and only if $a(t)^2 - b(t)^2 \neq 0$ on Γ.*

Proof. The sufficiency of this condition was proved earlier by B. V. Khvedelidze [17]. Now, let $A \in F(L_p)$. It follows from Corollary 6.3 that $\overline{A} = \overline{a}I + \overline{b}S \in F(L_p)$, too. Therefore, the operator $\overline{A}^* = aI + bS + T$, $T \in \mathcal{T}(L_p(\Gamma))$ is Fredholm in L_p^*. Thus, the operator A is a Fredholm operator in both $L_p(\Gamma)$ and $L_q(\Gamma)$. Using Theorem 6.5, we obtain $a(t)^2 - b(t)^2 \neq 0$. □

For a second illustration, consider the following theorem which is proved by using Theorem 6.2.

Theorem 6.7. *Let $\mathcal{K}$ be a Banach algebra and let $\mathcal{K}_0$ be commutative subalgebra of $\mathcal{K}$, which possesses a symmetric sufficient family of multiplicative functionals. Then $\mathcal{K}_0$ is inverse closed in $\mathcal{K}$. See* [21, *Theorem* 13.3] *for details.*

We conclude this subsection with an **open**

Question 6.8. *Can we replace in Theorem* 6.2 *the Banach space* $\mathcal{B}$ *with a normed or topological* (*or even with non-topological*) *space?*

6.2. **The circular arc $\nu_p(c)$ and exact values of the norms of operators S, P, Q on $L_p(\Gamma)$.** It is well known (especially now) that the norms of SIOs play an important role in various applications. But, by the time we were working on the paper [8], almost nothing was known about these norms. We decided to illustrate the results (we just received) for this paper with possible estimation of the norms of operators S, P, Q. We started with the following experiment:

It is evident that for any operator R on the Banach space $\mathcal{B}$ the relation $I+R \notin GL(B) \Longrightarrow \|R\| \geq 1$ holds. We considered the operator $A := cP+Q$, where the function $c(t)$ ($|t| = 1$) takes only two values : $r\exp(\pm\pi i/p)$, $r > 0$, $p \geq 2$. In this case, $\nu_p(c)$ is a circular arc which connects these two points, and from the point $0 \in \nu_p(c)$ the segment $[r\exp(-\pi i/p), r\exp(\pi i/p)]$ is seen at the angle $2\pi/p$.

It follows from Theorem 2.4 that the operator A is not invertible. But $A = I + (c-1)P$, $|c(t)-1| = r^2+1-2r\cos\frac{\pi}{p}$ does not depend on t, and its minimal value (for a fixed number p) equals $\sin\frac{\pi}{p}$ (when $r = \cos\frac{\pi}{p}$). Taking $r = cos\frac{\pi}{p}$, we obtain

$$1 \leq \left\| \sin\frac{\pi}{p}\, P \right\|, \;\text{ therefore }\; \|P\| \geq \left(\sin\frac{\pi}{p}\right)^{-1}.$$

This was the best estimation we could extract from our experiment. Using same approach, we obtained the following estimates:

$$\|Q\|_p \geq |Q|_p \geq \frac{1}{\sin(\pi/p)}\,, \quad \|P\|_p \geq |P|_p \geq \frac{1}{\sin(\pi/p)}\,, \tag{6.1}$$

$$\|S\|_p \geq |S|_p \geq \cot\frac{\pi}{2p^*}\,, \tag{6.2}$$

where $|A| := \inf\limits_{T} \|A+T\|$, T are compact operators, and $p^* = \max(p, p/(p-1))$.

These estimates acquired greater significance (for us) when we were able to prove the accuracy of some estimates. For example,

$$\|S\|_p = \begin{cases} \cot\dfrac{\pi}{2p} & \text{if } \; p = 2^n, \\[2ex] \tan\dfrac{\pi}{2p} & \text{if } \; p = \dfrac{2^n}{2^n-1}\,, \end{cases} \qquad n = 1, 2, \ldots \tag{6.3}$$

(see [8, Section 3] for details). And we formulated the following

Conjecture 6.9. *Inequalities* (6.1), (6.2) *can be replaced by equalities.*

These results (associated with the main part of results in [8]) and Conjecture 6.9 gave rise to a large number of publications dedicated to the best constants, and such publications continue to appear. Almost all new result related to best constants required new ideas and methods for their proofs.

Some problems turned out to be very complicated. For example, it took more than 30 years of attempts of many authors to confirm Conjecture 6.9 for analytical projections P and Q. This was done by B. Hollenbeck and I. Verbitsky (see [14] and the list of references in this paper). The operator S was more lucky. Conjecture 6.9 was confirmed by S. K. Pichorides [26] in 1972. Some addendum to his paper was obtained in [23]. A survey related to best constant in the theory of one-dimensional SIO is written in the paper [22].

6.3. **One more associated result.** Denote by $\mathcal{E}$ a subalgebra of the Banach algebra $\mathcal{A} = L(\mathcal{B})$, where $\mathcal{B}$ is a Banach space, and by $M_n(\mathcal{E})$ the algebra of all $n \times n$-matrices with the entries from $\mathcal{E}$. Comparing the results in articles [3, 5] and [4, 6] related, respectively, to the symbols of SIOs with scalar and matrix coefficients, the following statement was predicted:

Theorem 6.10. *Let the algebra $\mathcal{E}$ be commutative modulo compact operators, and let $R \in M_n(\mathcal{E})$. Then*

$$R \in F(L(\mathcal{B}^n)) \Longleftrightarrow \det(R) \in F(L(\mathcal{B})). \tag{6.4}$$

Remark 6.11. When one writes the determinant $\det(R)$, the order of the factors is irrelevant, since the possible determinants differ from one another by a compact term.

In [20], a following statement, associated with Theorem 6.10, was obtained:

Theorem 6.12. *Let $\mathcal{K}$ be an associative and, generally speaking, noncommutative ring with identity e. Assume that $a_{mk} \in \mathcal{K}$ $(m, k \le n)$ for some $n \in \mathbb{N}$, and $a_{mk}a_{pq} = a_{pq}a_{mk}$, $\forall\, m, k, p, q = 1, \dots, n$. Then the matrix $A := [a_{mk}]_{m,k=1}^n$ is invertible in $M_n(\mathcal{K})$ if and only if the element $\Delta := \det A$ is invertible in $\mathcal{K}$.*

The proof of Theorem 6.10 was represented in [20], as a corollary from the general Theorem 6.12.

These two theorems (6.10 and 6.12) proved to be useful for many classes of equations and they were included in many publications, even in the publications of the current millennium (see, for example, Lemma 1.2.34 and related statements in [27]). Theorem 6.10 was first used in the proof of Theorem 6 from [19].

Consider one more example of application of Theorem 6.10. Let $T_a := [a_{i-k}]_{i,k=1}^{\infty}$ denote the Toeplitz operator, generated by a function $a(t) = \sum\limits_{j=-\infty}^{\infty} a_j t^j \in L_\infty(S^1)$. The following statement is proved in [11, Section 3].

Theorem 6.13. *Algebra $\mathcal{E} \subset L(\ell_2)$, generated by Toeplitz operators $T_a := [a_{i-k}]$, where $a(t)$ are piecewise continuous functions on the unite circle , is with a scalar Fredholm symbol. In particular, the symbol of operator Ta is defined by the equality*

$$a(t, \mu) = \mu a(t+0) + (1-\mu)a(t-0) \quad \big(|t| = 1,\ \ 0 \le \mu \le 1\big). \tag{6.5}$$

The following corollary follows directly from Theorems 6.13 and 6.10:

Corollary 6.14. *Let* $A := [A_{i,k}]_{i,k=1}^n$ $(A_{i,k} \in \mathcal{E})$. *Then*

$$A \in F(L(\ell_2^n)) \Longleftrightarrow \det A \in F(L(l_2)).$$

Remark 6.15. In order to get the analogue of Theorem 6.13 and Corollary 6.14 for ℓ_p spaces with $p \neq 2$, it was necessary to obtain some additional results, related to Toeplitz operators on ℓ_p. In contrast with the space ℓ_2, here the Khvedelidze and Gohberg–Krupnik approaches did not work. But, Rolland Duduchava proposed a new approach and succeeded in solving the necessary problems (see [1, 2]). This made it possible to obtain in [12] the analogues of Theorem 6.13 for ℓ_p $(1 < p < \infty)$ and to use (automatically) Theorem 6.10 in ℓ_p^n.

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Author's address:

208–7460, Bathurst St. Thornhill, L4J 7K9, Ontario, Canada.
E-mail: krupnik13@rogers.com

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Vladimir Rabinovich

ON SPECTRAL PROPERTIES AND INVERTIBILITY OF SOME OPERATORS OF MATHEMATICAL PHYSICS

Dedicated to Boris Khvedelidze's 100-th birthday anniversary

Abstract. The main aim of the paper is to study the Fredholm property, essential spectrum, and invertibility of some operators of the Mathematical Physics, such that the Schrödinger and Dirac operators with complex electric potentials, and Maxwell operators in absorbing at infinity media. This investigation is based on the limit operators method, and the uniqueness continuation property for the operators under consideration.

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რეზიუმე. სტატიის ძირითადი მიზანია მათემატიკური ფიზიკის ზოგიერთი ოპერატორის ფრედჰოლმის თვისებების შესწავლა, როგორიცაა შროდინგერის და დირაკის ოპერატორები კომპლექსური ელექტრული პოტენციალებით და მაქსველის ოპერატორები არეში, რომელსაც გააჩნია შთანთქმის თვისება უსასრულობის მიდამოში. ეს გამოკვლევა ეფუძნება ზღვრული ოპერატორების მეთოდს და განხილვის ქვეშ მყოფი ოპერატორების ერთადერთობის უწყვეტობის თვისებას.

1. Introduction

The main aim of the paper is the study of the Fredholm property, essential spectrum, and invertibility of some operators of the Mathematical Physics, such that the Schrödinger and Dirac operators with complex electric potentials, and Maxwell operators in absorbing at infinity media. This investigation is based on the limit operators method [23]. Earlier this method was applied to the investigation of the location of essential spectra of perturbed pseudodifferential operators with applications to electromagnetic Schrödinger operators, square-root Klein–Gordon, and Dirac operators under general assumptions with respect to the behavior of real valued magnetic and electric potentials at infinity. By means of this method a very simple and transparent proof of the well known Hunziker, van Winter, Zjislin theorem (HWZ-Theorem) for multi-particle Hamiltonians has been obtained [14, 15]. In the papers [19, 20, 22] the limit operators method was applied to the study of the location of the essential spectrum of discrete Schrödinger operators on $\mathbb{Z}^n$, and on periodic combinatorial graphs. We also note the recent papers [16–18] devoted to applications of the limit operators method to the investigation of the Fredholm properties of boundary and transmission problems, and the boundary equations for unbounded domains.

The paper is organized as follows. In Section 2 we give some notations and an auxiliary material. In Section 3 we consider the Fredholm property of strongly elliptic second order systems of differential operators of the form

$$Au(x) = \sum_{k,l=1}^{n} (i\partial_{x_k} - a_k(x))b^{kl}(x)(i\partial_{x_l} - a_l(x))u(x) + W(x)u(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where a_k are real-valued functions on $\mathbb{R}^n$ and b^{kl} are $N \times N$ Hermitian matrices, W is a complex-valued $N \times N$ matrix. We suppose that a_k, and the coefficients of the matrix b^{kl} belong to $C^1_{b,u}(\mathbb{R}^n)$, and the coefficients of the matrix W belong to $C_{b,u}(\mathbb{R}^n)$, where $C_{b,u}(\mathbb{R}^n)$ is the class of bounded uniformly continuous functions on $\mathbb{R}^n$, and $C^1_{b,u}(\mathbb{R}^n)$ is the class of functions a on $\mathbb{R}^n$ such that $\partial_{x_j} a \in C_{b,u}(\mathbb{R}^n)$, $j = 1, \ldots, n$. In this section we prove that if

$$\liminf_{x\to\infty} \inf_{\|h\|_{\mathbb{C}^N}=1} \Im(W(x)h, h) > 0,$$

then $A : H^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N)$ is a Fredholm operator of the index 0. In Section 4, applying the results of Section 3, we study the spectra of electromagnetic Schrödinger operators on $\mathbb{R}^n$ with real magnetic and complex electric potentials Φ. We prove that if

$$\liminf_{x\to\infty} \Im(\Phi(x)) > 0, \tag{1.2}$$

where Φ is the electric potential, then the *essential spectrum* of the Schrödinger operator does not have intersections with the real line $\mathbb{R}$. If, in addition to (1.2),

$$\Im(\Phi(x)) \geq 0, \quad x \in \mathbb{R}^n, \tag{1.3}$$

then the *spectrum* of the Schrödinger operator does not intersect the real line $\mathbb{R}$. Under the proof of the last result we have used *the uniqueness of the continuation* for elliptic operators (see e.g. [4, 9, 10]). Note that there is an extensive literature devoted to the spectral properties of the Schrödinger operators (see e.g. [1, 5, 24–26]).

Section 5 is devoted to the investigation of spectra of the Dirac operators with real-valued magnetic and complex-valued electric potentials. We suppose here that the magnetic and electric potentials are slowly oscillating at infinity. We prove here that the conditions (1.2), (1.3) provide us with the spectrum of the Dirac operator which does not contain the real values. For the proof we use the results of Section 3 and the uniqueness of the continuation for some almost diagonal strongly elliptic systems of second order.

In Section 6, we consider the harmonic Maxwell system on $\mathbb{R}^3$ for isotropic nonhomogeneous media. We suppose that the electric and magnetic permittivities ε and μ are the slowly oscillating at infinity complex valued functions. We prove that the operator of Maxwell's system is invertible in admissible functional spaces if the electromagnetic medium is absorbing at infinity, that is,

$$\liminf_{x \to \infty} \Im(\varepsilon(x)\mu(x)) > 0.$$

The proof of this result is based on the realization of the Maxwell system in a quaternionic form (see e.g. [8, 11, 12]), applications of results of Section 3, and the uniqueness of the continuation for almost diagonal strongly elliptic systems of second order.

2. Auxiliary Material

2.1. **Notation.** We will use the following standard notation.

- Given Banach spaces X, Y, $\mathcal{L}(X, Y)$ is the space of all bounded linear operators from X into Y. We abbreviate $\mathcal{L}(X, X)$ to $\mathcal{L}(X)$. If X is a Hilbert spaces, then $(x, y)_X$ is a scalar product in X of x, y.
- $L^2(\mathbb{R}^n, \mathbb{C}^N)$ is the Hilbert space of all measurable functions on $\mathbb{R}^n$ with values in $\mathbb{C}^N$ provided with the norm

$$\|u\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} := \left(\int_{\mathbb{R}^n} \|u(x)\|^2_{\mathbb{C}^N} \, dx \right)^{1/2}.$$

- $H^s(\mathbb{R}^n, \mathbb{C}^N)$ is a Sobolev space of distributions with norm

$$\|u\|_{H^s(\mathbb{R}^n,\mathbb{C}^N)} := \left(\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s \|\widehat{u}(\xi)\|^2_{\mathbb{C}^N} \, d\xi \right)^{1/2},$$

where $\widehat{u}$ is the Fourier transform of u.

- We also use the standard multi-index notations. Thus, $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j \in \mathbb{N} \cup \{0\}$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is its length, and

$$\partial^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}; \quad D^\alpha := (-i\partial_{x_1})^{\alpha_1} \cdots (-i\partial_{x_n})^{\alpha_n}.$$

  Finally, $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ for $\xi \in \mathbb{R}^n$.
- $C_b(\mathbb{R}^n)$ is the C^*-algebra of all bounded continuous functions on $\mathbb{R}^n$.
- $C_{b,u}(\mathbb{R}^n)$ is the C^*-subalgebra of $C_b(\mathbb{R}^n)$ of all uniformly continuous functions.
- $C_b^k(\mathbb{R}^n)$ is the C^*-subalgebra of $C_b(\mathbb{R}^n)$ of k-times differentiable functions such that $\partial_x^\alpha a \in C_b(\mathbb{R}^n)$ for $|\alpha| \le k$, and $a \in C_{b,u}^k(\mathbb{R}^n)$ if $a \in C_b^k(\mathbb{R}^n)$ and $\partial_x^\alpha a \in C_{b,u}(\mathbb{R}^n)$ for $|\alpha| = k$.
- We say that $a \in C_0^k(\mathbb{R}^n)$ if $a \in C_b^k(\mathbb{R}^n)$ and $\lim_{x\to\infty} a(x) = 0$.
- We denote by $SO(\mathbb{R}^n)$ a C^*-subalgebra of $C_b(\mathbb{R}^n)$ which consists of all functions a, slowly oscillating at infinity in the sense that

$$\lim_{x\to\infty} \sup_{y\in K} |a(x+y) - a(x)| = 0$$

  for every compact subset K of $\mathbb{R}^n$.
- We denote by $SO^k(\mathbb{R}^n)$ the set of functions $a \in C_b^k(\mathbb{R}^n)$ such that

$$\lim_{x\to\infty} \frac{\partial a(x)}{\partial x_j} = 0, \quad j = 1, \dots, n.$$

  Evidently, $SO^k(\mathbb{R}^n) \subset SO(\mathbb{R}^n)$.
- If $\mathcal{A}(\mathbb{R}^n)$ is an algebra of functions on $\mathbb{R}^n$, then we set

$$\mathcal{A}(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^N)) = \mathcal{A}(\mathbb{R}^n) \otimes \mathcal{L}(\mathbb{C}^N).$$

- $B_R = \{x \in \mathbb{R}^n : |x| < R\}$, and $B'_R = \{x \in \mathbb{R}^n : |x| > R\}$.

2.2. **Fredholm properties of matrix partial differential operators and limit operators.** We consider matrix partial differential operators of order m of the form

$$(Au)(x) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u(x), \quad x \in \mathbb{R}^n, \tag{2.1}$$

under the assumption that the coefficients a_α belong to $C_{b,u}(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^N))$. One can see that $A : H^m(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N)$ is a bounded operator.

The operator A is said to be *elliptic* at the point $x \in \mathbb{R}^n$ if

$$\det a_0(x, \xi) \neq 0$$

for every point $\xi \neq 0$, where

$$a_0(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$$

is the main symbol of A, and A is called *uniformly elliptic* if

$$\inf_{x\in\mathbb{R}^n,\, \omega\in S^{n-1}} \left| \det \sum_{|\alpha|=m} a_\alpha(x)\omega^\alpha \right| > 0,$$

where S^{n-1} refers to the unit sphere in $\mathbb{R}^n$.

The Fredholm properties of the operator $A : H^s(\mathbb{R}^n, \mathbb{C}^N) \to H^{s-m}(\mathbb{R}^n, \mathbb{C}^N)$ can be expressed in terms of its limit operators which are defined as follows (see e.g. [21]). Let $h : \mathbb{N} \to \mathbb{R}^n$ be a sequence tending to infinity. Since $a_\alpha \in C_{b,u}(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^N))$, the Arcelà–Ascoli's theorem combined with a Cantor diagonal argument implies that there exists a subsequence g of h such that the sequences of the functions $x \mapsto a_\alpha(x + g(k))$ converge as $k \to \infty$ to a limit function a_α^g uniformly on every compact set $K \subset \mathbb{R}^n$ for every multi-index α. The operator

$$A^g := \sum_{|\alpha|\leq m} a_\alpha^g D^\alpha$$

is called the *limit operator of A defined by the sequence g*. We denote by $\mathrm{Lim}(A)$ the set of all limit operators of the differential operator A.

Theorem 2.1 ([21])**.** *Let A be a differential operator of the form* (2.1). *Then $A : H^m(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N)$ is a Fredholm operator if and only if:*

(i) *A is a uniformly elliptic operator on $\mathbb{R}^n$;*

(ii) *all limit operators of A are invertible as operators from $H^m(\mathbb{R}^n, \mathbb{C}^N)$ to $L^2(\mathbb{R}^n, \mathbb{C}^N)$.*

Note that the uniform ellipticity of the operator A implies the a priori estimate

$$\|u\|_{H^2(\mathbb{R}^n,\mathbb{C}^N)} \leq C \left(\|Au\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} + \|u\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \right). \tag{2.2}$$

This estimate allows one to consider the uniformly elliptic differential operator A as a closed unbounded operator on $L^2(\mathbb{R}^n, \mathbb{C}^N)$ with a dense domain $H^m(\mathbb{R}^n, \mathbb{C}^N)$. It turns out (see [2, p. 27–32]) that A, considered as an unbounded operator in this way, is an (unbounded) Fredholm operator if and only if A, considered as a bounded operator from $H^m(\mathbb{R}^n, \mathbb{C}^N)$ to $L^2(\mathbb{R}^n, \mathbb{C}^N)$, is a Fredholm operator.

We say that $\lambda \in \mathbb{C}$ belongs to the *essential spectrum* of A if the operator $A - \lambda I$ is not Fredholm as an unbounded differential operator. As above, we denote the essential spectrum of A by $\mathrm{sp}_{ess} A$ and the common spectrum of A (considered as an unbounded operator) by $\mathrm{sp}\, A$. Then the assertion of Theorem 2.1 can be stated as follows.

Theorem 2.2 ([21])**.** *Let A be a uniformly elliptic differential operator of the form* (2.1)*. Then*

$$\mathrm{sp}_{ess}\, A = \bigcup_{A^g \in \mathrm{Lim}(A)} \mathrm{sp}\, A^g. \tag{2.3}$$

3. Fredholm Property of Systems of Strongly Elliptic Partial Differential Operators on $\mathbb{R}^n$

We consider the system of partial differential equations of second order on $\mathbb{R}^n$ in the divergent form

$$Au(x) = \sum_{k,l=1}^{n} (i\partial_{x_k} - a_k(x)) b^{kl}(x)(i\partial_{x_l} - a_l(x))u(x) \\ + W(x)u(x), \quad x \in \mathbb{R}^n, \tag{3.1}$$

where

$$a_k \in C^1_{b,u}(\mathbb{R}^n), \quad b^{kl} \in C^1_{b,u}(\mathbb{R}^n), \quad \mathcal{L}(\mathbb{R}^n)), \quad W \in C_{b,u}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n)), \tag{3.2}$$

a_k are real-valued functions, b^{kl} are Hermitian matrices, that is, $b^{kl}(x)^* = b^{kl}(x)$, and W is a complex-valued matrix. The conditions (3.2) provide the boundedness of $A : H^2(\mathbb{R}^n, \mathbb{C}^n) \to L^2(\mathbb{R}^n, \mathbb{C}^n)$. We suppose that the operator A is strongly elliptic, that is there exists a constant $\gamma > 0$ such that for every $h \in \mathbb{C}^N$ and $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n$,

$$\sum_{k,l=1}^{n} (b^{kl}(x)h, h)_{\mathbb{C}^N} \nu_k \nu_l \geq \gamma \|h\|^2_{\mathbb{C}^N} \|\nu\|^2_{\mathbb{R}^n}. \tag{3.3}$$

Theorem 3.1. *Let the conditions* (3.2)*,* (3.3) *and*

$$\liminf_{x \to \infty} \inf_{\|h\|_{\mathbb{C}^N}=1} \Im \langle W(x)h, h\rangle_{\mathbb{C}^N} > 0 \tag{3.4}$$

hold. Then $A : H^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^n)$ is a Fredholm operator of the index 0*.*

Proof. Since A is a uniformly elliptic operator, by the condition (3.3) we have to prove that all limit operators A^g of the operator A are invertible from $H^2(\mathbb{R}^n, \mathbb{C}^n) \to L^2(\mathbb{R}^n, \mathbb{C}^n)$. The limit operators A^g are of the form

$$A^g u(x) = \sum_{k,l=1}^{n} (i\partial_{x_k} - a^g_k(x))(b^{kl})^g(x)(i\partial_{x_l} - a^g_l(x))u(x) \\ + W^g(x)u(x), \quad x \in \mathbb{R}^n. \tag{3.5}$$

The condition (3.4) implies that there exists $\epsilon > 0$ such that for every $x \in \mathbb{R}^n$,

$$\Im (W^g(x)h, h)_{\mathbb{C}^N} \geq \epsilon \|h\|^2_{\mathbb{C}^N}. \tag{3.6}$$

Then for every $u \in H^2(\mathbb{R}^n, \mathbb{C}^N)$,

$$\begin{aligned}\left|(A^g u, u)_{L^2(\mathbb{R}^n,\mathbb{C})}\right| &\geq \Im(A^g u, u)_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \\ &= \int\limits_{\mathbb{R}^n} \Im(W^g u, u)_{\mathbb{C}^N}\, dx \geq \epsilon \|u\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)}. \end{aligned} \tag{3.7}$$

This estimate yields that there exists an inverse in the algebraic sense operator $(A^g)^{-1}$, bounded in $L^2(\mathbb{R}^n, \mathbb{C}^N)$. Since A is a uniformly elliptic operator on $\mathbb{R}^n$, the following a priory estimate

$$\|u\|_{H^2(\mathbb{R}^n,\mathbb{C}^N)} \leq C\left(\|Au\|_{L^2(\mathbb{R}^n,\mathbb{C}^n)} + \|u\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\right) \tag{3.8}$$

holds. The last estimate implies that all limit operators $A^g : H^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^n)$ are invertible. Then by Theorem 2.1, $A : H^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^n)$ is a Fredholm operator. Let us prove that $\operatorname{index} A = 0$. We consider the family of differential operators $A_\mu = A + \mu^2 I$, $\mu \geq 0$. As above, one can prove that $A_\mu : H^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^n)$ are Fredholm operators. Note that A_μ is an elliptic family depending on the parameter $\mu \geq 0$ (see e.g. [3]). Hence there exists $\mu_0 > 0$ such that A_μ is an invertible operator for $\mu > \mu_0$. Hence $\operatorname{index} A = 0$ because the family $A_\mu : H^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N)$ is continuously depending on the parameter μ. □

4. Schrödinger Operators with a Complex Potential

We consider the Schrödinger operator

$$\begin{aligned}\mathcal{H}u(x) := \frac{1}{2m}\Big(D_j + \frac{e}{c}\, a_j(x)\Big)\rho^{jk}(x)\Big(D_j + \frac{e}{c}\, a_k(x)\Big)u(x) \\ + e\Phi(x)u(x), \quad x \in \mathbb{R}^n,\end{aligned}$$

where $D_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j}$, $\hbar$ is a Planck constant, m is the electron mass, c is the light speed in the vacuum, $\mathbf{a} = (a_1, \ldots, a_n)$ is a magnetic potential, and Φ is an electrical potential on $\mathbb{R}^n$, the latter equipped with a Riemannian metric $\rho = (\rho_{jk})^n_{j,k=1}$ which is subject to the positivity condition

$$\inf_{x\in\mathbb{R}^n,\, \omega\in S^{n-1}} \rho_{jk}(x)\omega^j\omega^k > 0, \tag{4.1}$$

where $\rho_{jk}(x)$ refers to the matrix, inverse to $\rho^{jk}(x)$. Here and in what follows, we make use of Einstein's summation convention.

We suppose that ρ^{jk}, a_j are real-valued functions in $C^1_{b,u}(\mathbb{R}^n)$ and a complex valued electric potential $\Phi \in C_{b,u}(\mathbb{R}^n)$. Under these conditions, $\mathcal{H}$ can be considered as a closed unbounded operator on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$. If Φ is a real-valued function, then $\mathcal{H}$ is a self-adjoint operator and $\mathcal{H}$ has a real spectrum.

Theorem 4.1. (i) *Let*

$$\liminf_{x\to\infty} \Im\Phi(x) > 0. \tag{4.2}$$

Then the essential spectrum of the operator $\mathcal{H}$ does not contain real values.

(ii) *Let the condition* (4.2) *hold and*

$$\Im\Phi(x) \geq 0 \tag{4.3}$$

for every $x \in \mathbb{R}^n$. Then the spectrum of the operator $\mathcal{H}$ does not contain real values.

Proof. (i) According to formula (2.3),

$$\mathrm{sp}_{ess}\,\mathcal{H} = \bigcup_{A^g \in \mathrm{Lim}\,(\mathcal{H})} \mathrm{sp}\,\mathcal{H}^g, \tag{4.4}$$

where

$$\mathcal{H}^g u(x) := \frac{1}{2m}\Big(D_j + \frac{e}{c}\,a_j^g(x)\Big)(\rho^{jk})^g(x)\Big(D_k + \frac{e}{c}\,a_k^g(x)\Big)u(x) \\ + e\Phi^g(x)u(x), \quad x \in \mathbb{R}^n.$$

We set $\Phi_\lambda = \Phi - \lambda I$, $\lambda \in \mathbb{R}$. The condition (4.2) implies that

$$\inf_{x\in\mathbb{R}^n} \Im\Phi_\lambda^g(x) > 0. \tag{4.5}$$

This condition implies that the operator $\mathcal{H}^g - \lambda I$, $\lambda \in \mathbb{R}$ is invertible with a bounded in $L^2(\mathbb{R}^n)$ inverse operator $(\mathcal{H}^g - \lambda I)^{-1}$. Hence $\mathbb{R} \ni \lambda \notin \mathrm{sp}\,\mathcal{H}^g$. Formula (4.4) implies that $(\mathrm{sp}_{ess}\,\mathcal{H}) \cap \mathbb{R} = \varnothing$.

(ii) As in the proof of Theorem 3.1, we obtain that $\mathcal{H}_\lambda = \mathcal{H} + \lambda I : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are Fredholm operators of the index zero. Let us prove that $\ker \mathcal{H}_\lambda = \{0\}$. Let $u \in \ker \mathcal{H}_\lambda$. Estimates (4.1), (4.2), and (4.3) imply that there exists ϵ and $R > 0$ such that

$$\begin{aligned} 0 = \Im(\mathcal{H}_\lambda u, u)_{L^2(\mathbb{R}^n,\mathbb{C}^N)} &= \Im \int_{\mathbb{R}^n} (e\Phi(x)u(x), u(x))_{\mathbb{C}^N}\,dx \\ &= \Im \int_{|x|<R} (e\Phi(x)u(x), u(x))_{\mathbb{C}^N}\,dx + \Im \int_{|x|\geq R} (e\Phi(x)u(x), u(x))_{\mathbb{C}^N}\,dx \\ &\geq \epsilon \|u\|^2_{L^2(B'_R,\mathbb{C}^N)}. \end{aligned} \tag{4.6}$$

Since $\ker \mathcal{H}_\lambda \subset H^2(\mathbb{R}^n)$, the estimate (4.6) implies that

$$u\Big|_{\partial B_R} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial B_R} = 0, \tag{4.7}$$

where $\frac{\partial u}{\partial \nu}$ is a normal derivative to the sphere ∂B_R. By the uniqueness of a solution of the Cauchy problem, for elliptic equations with the oldest Lipschitz coefficients (see e.g. [4,7,9,10]), we obtain that the Cauchy problem

$$Au(x) = 0, \quad x \in B_R, \\ u\Big|_{\partial B_R} = 0, \quad \frac{\partial u}{\partial \nu}\,|_{\partial B_R} = 0$$

has the trivial solution only. Hence $u = 0$ on $\mathbb{R}^n$. That is, $\ker \mathcal{H}_\lambda = \{0\}$ and $\mathcal{H}_\lambda : H^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N)$ is an invertible operator. This implies that $\operatorname{sp} \mathcal{H} \cap \mathbb{R} = \varnothing$. □

5. Dirac Operators with Complex Electric Potentials

In this section we consider the Dirac operator on $\mathbb{R}^3$, equipped with the Riemannian metric tensor (ρ_{jk}) depending on $x \in \mathbb{R}^3$ (for a general account on Dirac operators see, for example, [28]). We suppose that there is a constant $C > 0$ such that

$$\rho_{jk}(x)\xi^j\xi^k \geq C|\xi|^2, \quad x \in \mathbb{R}^3, \tag{5.1}$$

where we use as above Einstein's summation convention. Let ρ^{jk} be the tensor, inverse to ρ_{jk}, and let $\phi^{jk}(x) = \sqrt{\rho^{jk}(x)}$ be the positive square root. The Dirac operator on $\mathbb{R}^3$ is the matrix operator defined as

$$\mathcal{D} := \frac{c}{2}\gamma_k(\phi^{jk}P_j + P_j\phi^{jk}) + c^2 m\gamma_0 + e\Phi E_4 \tag{5.2}$$

acting on vector functions on $\mathbb{R}^3$ with values in $\mathbb{C}^4$. In (5.2), the γ_k, $k = 0, 1, 2, 3$, are the 4×4 Dirac matrices, i.e., they satisfy

$$\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk}E_4 \tag{5.3}$$

for all choices of $j, k = 0, 1, 2, 3$, E_4 is the 4×4 unit matrix,

$$P_j = D_j + \frac{e}{c}a_j, \quad D_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j}, \quad j = 1, 2, 3,$$

where $\hbar$ is the Planck constant, $\mathbf{a} = (a_1, a_2, a_3)$ is the vector potential of the magnetic field $\mathbf{H}$, that is, $\mathbf{H} = \nabla \times a$, Φ is the scalar potential of the electric field $\mathbf{E}$, that is, $\mathbf{E} = \nabla\Phi$, and m and e are the mass and the charge of the electron, c is a light speed in the vacuum.

We suppose that

$$\rho^{jk}, a_j \in SO^2(\mathbb{R}^3), \quad j, k = 1, 2, 3, \quad \Phi \in SO^1(\mathbb{R}^3), \tag{5.4}$$

and ρ^{jk}, A_j are real-valued functions, and electrical potential Φ can be a complex function. We consider the operator $\mathcal{D}$ as an unbounded operator on the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $H^1(\mathbb{R}^3, \mathbb{C}^4)$.

Note that the main symbol of $\mathcal{D}$ is $\sigma_{\mathcal{D}}(x, \xi) = c\phi^{jk}(x)\xi_j\gamma_k$. Using (5.3) and the identity $\phi^{jk}\phi^{rt}\delta_{kt} = \rho^{jr}$, we obtain that

$$\begin{aligned}\sigma_{\mathcal{D}}(x, \xi)^2 &= c^2\hbar^2\phi^{jk}(x)\phi^{rt}(x)\xi_j\xi_r\gamma_k\gamma_t \\ &= c^2\hbar^2\phi^{jk}(x)\phi^{rt}(x)\delta_{kt}\xi_j\xi_r = (c^2\hbar^2\rho^{jr}(x)\xi_j\xi_r)E_4.\end{aligned}$$

Together with (5.1), this equality shows that $\mathcal{D}$ is a uniformly elliptic matrix differential operator on $\mathbb{R}^3$. Hence the following a priory estimate

$$\|u\|_{H^1(\mathbb{R}^3, \mathbb{C}^4)} \leq C\left(\|\mathcal{D}u\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)} + \|u\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)}\right)$$

holds which implies that $\mathcal{D}$ is a closed operator in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $H^1(\mathbb{R}^3, \mathbb{C}^4)$. It follows from the conditions (5.4) that the limit operators $\mathcal{D}^g$

of $\mathcal{D}$ defined by sequences $g : \mathbb{Z} \to \mathbb{R}^3$ tending to infinity are the operators with the constant coefficients of the form

$$\mathcal{D}^g = c\gamma_k(\phi^{jk})^g\Big(D_j + \frac{e}{c}\, a^g_j\Big) + mc^2\gamma_0 - e\Phi^g E_4,$$

where

$$\begin{gathered}(\phi^{jk})^g := \lim_{m\to\infty}\phi^{jk}(g(m)),\\ a^g_j := \lim_{m\to\infty} a_j(g(m)), \quad \Phi^g := \lim_{m\to\infty}\Phi(g(m)).\end{gathered} \tag{5.5}$$

The operator $\mathcal{D}$ is unitarily equivalent to the operator

$$\mathcal{D}^g_1 = c\gamma_k(\phi^{jk})^g D_j + \gamma_0 mc^2 + e\Phi^g,$$

and the equivalence is realized by the unitary operator $T_{a^g} : f \mapsto e^{i\,\frac{e}{c}\,a^g\cdot x} f$, $a^g := (a^g_1, a^g_2, a^g_3)$. Let $\Phi \in SO(\mathbb{R}^3)$, and $\Phi_\infty \subset \mathbb{C}$ be the set of all particular limits $\Phi^g = \lim\limits_{m\to\infty}\Phi(g(m))$ defined by sequences $\mathbb{R}^3 \ni g(m) \to \infty$.

Theorem 5.1. *Let the conditions* (5.1) *be fulfilled. Then the Dirac operator*

$$\mathcal{D} : H^1(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$$

is a Fredholm operator if and only if

$$\Phi_\infty \cap (-\infty, -mc^2] = \varnothing, \quad \Phi_\infty \cap [mc^2, +\infty) = \varnothing. \tag{5.6}$$

Proof. Set

$$\widehat{\mathcal{D}}^g_0(\xi) := c\hbar\gamma_k(\phi^{jk})^g\xi_j + mc^2\gamma_0 \ \text{ and } \ (\rho^{jk})^g := \lim_{m\to\infty}\rho^{jk}(g(m)).$$

Then

$$\begin{aligned}(\widehat{\mathcal{D}}^g_0(\xi) - e\Phi^g E_4)&(\widehat{\mathcal{D}}^g_0(\xi) + e\Phi^g E_4)\\ &= \big(c^2\hbar^2(\rho^{jk})^g\xi_j\xi_k + m^2c^4 - (e\Phi^g)^2\big)E_4.\end{aligned} \tag{5.7}$$

The condition (5.6) and the identity (5.7) imply that

$$\det\big((\widehat{\mathcal{D}}^g_0(\xi) + e\Phi^g)E_4\big) \neq 0$$

for every $\xi \in \mathbb{R}^3$. Hence, the operator $\mathcal{D}^g_1 : H^1(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$ is invertible and, consequently, so is $\mathcal{D}^g$. By Theorem 2.1, $\mathcal{D}$ is a Fredholm operator. For the reverse implication, assume that the condition (5.6) is not fulfilled. Then there exist $\Phi^g \in \mathbb{C}$ and a vector $\xi^0 \in \mathbb{R}^3 \setminus \{0\}$ such that

$$c^2(\rho^{jk})^g\xi^0_j\xi^0_k + m^2c^4 - (e\Phi^g)^2 = 0.$$

Given ξ^0, we find a vector $u \in \mathbb{C}^4$ such that $v := (\widehat{\mathcal{D}}^g_0(\xi^0) - (e\Phi^g)E)u \neq 0$. Then (5.7) implies that $(\mathcal{D}^g_0(\xi^0) + e\Phi^g E_4)v = 0$, whence $\det(\widehat{\mathcal{D}}^g_0(\xi^0) + e\Phi^g E_4) = 0$. Thus, the operator $\mathcal{D}^g$ is not invertible. By Theorem 2.1, $\mathcal{D}$ cannot be a Fredholm operator. □

Theorem 5.2. *If the condition* (5.1) *is satisfied, then*

$$\operatorname{sp}_{ess}\mathcal{D} = e\Phi_\infty + (-\infty, -mc^2] + [mc^2 + \infty),$$

where + denotes the algebraic sum of sets on the complex plane, and $e\Phi_\infty$ *is the set of particular limits of the function* $e\Phi$ *at infinity.*

Proof. Let $\lambda \in \mathbb{C}$. The symbol of the operator $\mathcal{D}^g - \lambda I$ is the function $\xi \mapsto \widehat{\mathcal{D}}_0^g(\xi) + (e\Phi^g - \lambda)E_4$. Invoking (5.7)), we obtain

$$(\widehat{\mathcal{D}}_0^g(\xi) - (e\Phi^g - \lambda)E_4)(\widehat{\mathcal{D}}_0^g(\xi) + (e\Phi^g - \lambda)E_4) \\ = \left[c^2\hbar^2(\rho^{jk})^g\xi_j\xi_k + m^2c^4 - (e\Phi^g - \lambda^2)\right]E_4. \quad (5.8)$$

Then eigenvalues $\lambda_\pm^g(\xi)$ of the matrix $\mathcal{D}_0^g(\xi) - e\Phi^g E_4$ are given by

$$\lambda_\pm^g(\xi) := e\Phi^g \pm (c^2\rho_g^{jk}\xi_j\xi_k + m^2c^4)^{1/2}. \quad (5.9)$$

This implies that

$$\operatorname{sp}\mathcal{D}^g = [e\Phi^g + mc^2, +\infty) \cup (-\infty, e\Phi^g - mc^2].$$

Hence,

$$\operatorname{sp}_{ess}\mathcal{D} = \cup_g \operatorname{sp}\mathcal{D}^g = e\Phi_\infty + [mc^2, +\infty) + [-\infty, -mc^2). \qquad \square$$

Theorem 5.3. *Let the condition* (5.3) *be satisfied and*

$$\inf \Im\Phi^2(x) \geq 0, \quad \liminf \Im\Phi^2(x) > 0. \quad (5.10)$$

Then the Dirac operator

$$\mathcal{D} : H^1(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$$

is invertible.

Proof. Let $\mathcal{D}_\mu = \mathcal{D} + \mu I : H^1(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$, $\mu \geq 0$. Then according to Theorem 5.1, $\mathcal{D}_\mu$ is the continuous family of Fredholm operators. Moreover, $\mathcal{D}_\mu$ is an elliptic family. This implies that there exists $\mu_0 > 0$ large enough such that $\mathcal{D}_\mu$ are invertible operators for $\mu \geq \mu_0$. Hence $\operatorname{index}\mathcal{D} = 0$. Let us prove that $\ker\mathcal{D} = \{0\}$. Note if $u \in \ker\mathcal{D}$, then $u \in \ker A$, where

$$A = (\mathcal{D}_0 - e\Phi E_4)(\mathcal{D}_0 + e\Phi E_4),$$

and

$$\mathcal{D}_0 = \frac{c}{2}\gamma_k(\phi^{jk}P_j + P_j\phi^{jk}) + c^2 m\gamma_0.$$

Since $\rho^{jk} \in SO^2(\mathbb{R}^3)$ and $\Phi \in SO^1(\mathbb{R}^3)$, we obtain that

$$A = (\mathcal{D}_0 - e\Phi E_4)(\mathcal{D}_0 + e\Phi E_4) = L + \mathcal{R}, \quad (5.11)$$

where

$$L = \left[(c^2\hbar^2 P_j\rho^{jk}P_k) + m^2c^4 - (e\Phi)^2\right]E_4$$

is the diagonal 4×4 matrix operator with strongly elliptic differential operators of second order on the main diagonal, and

$$\mathcal{R} = \sum_{j=1}^{3} r^j\partial_{x_j} + r^0$$

is a 4×4 matrix differential operator of the first order with coefficients $r^j \in C_0(\mathbb{R}^3, \mathcal{L}(\mathbb{C}^4))$, $j = 0, 1, 2, 3$. Let $u \in \ker A$. Then we obtain

$$0 = (Au, u)_{L^2(\mathbb{R}^3, \mathbb{C}^4)} = c^2\hbar^2 \int\limits_{\mathbb{R}^3} \rho^{jk}(P_j u, P_k u)_{\mathbb{C}^4}\, dx$$

$$+ \int\limits_{\mathbb{R}^3} (m^2c^4 - (e\Phi(x))^2)\|u(x)\|^2_{\mathbb{C}^4}\, dx + \int\limits_{\mathbb{R}^3} (\mathcal{R}u(x), u(x))_{\mathbb{R}^4}\, dx. \tag{5.12}$$

Since $r^j \in C_0(\mathbb{R}^3, \mathcal{L}(\mathbb{C}^4))$ for every $\varepsilon > 0$, there exists $R_0 > 0$ such that

$$\|\mathcal{R}u\|_{L^2(B'_R, \mathbb{C}^4)} \le \varepsilon \|u\|_{L^2(B'_R, \mathbb{C}^4)} \tag{5.13}$$

for $R \ge R_0$. Let $R \ge R_0$ be such that

$$\inf_{\mathbb{B}_R} \Im(e\Phi(x))^2 \ge \epsilon - \varepsilon > 0.$$

The condition (5.10) and formulas (5.12)), (5.13) yield

$$0 = \Im(Au, u)_{L^2(\mathbb{R}^3, \mathbb{C}^4)} \ge (\epsilon - \varepsilon) \int\limits_{B'_R} \|u(x)\|^2_{\mathbb{C}^4}\, dx. \tag{5.14}$$

Note that the operator of second order A is uniformly elliptic. This implies that $\ker A \subset H^2(\mathbb{R}^3, \mathbb{C}^4)$. Hence $u|_{B'_R} = 0$ implies that u is a solution of the homogeneous Cauchy problem

$$Au = 0, \quad x \in B_R, \tag{5.15}$$

$$u\big|_{\partial B_R} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial B_R} = 0.$$

The matrix operator $A = L + \mathcal{R}$ is a perturbation of the diagonal elliptic operator L of second order by the first order operator $\mathcal{R}$ with bounded coefficients, conserving the Carleman estimates (see e.g. [27, Chapter 14], [6], [7]). Hence the Cauchy problem (5.15) has the trivial solution only, and $\ker \mathcal{D} = \{0\}$. Hence $\mathcal{D}$ is an invertible operator. □

Corollary 5.4. *Let the conditions* (5.3), (5.10) *be satisfied. Then the spectrum of $\mathcal{D}$ does not have real values.*

6. Maxwell's Equation with Complex Electric and Magnetic Permittivity

6.1. **Maxwell's system.** We consider the Maxwell's system describing the harmonic electromagnetic fields

$$\nabla \times \mathbf{H} = i\omega \mathbf{D} + \mathbf{j}, \tag{6.1}$$

$$\nabla \times \mathbf{E} = -i\omega \mathbf{B}, \tag{6.2}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{6.3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{6.4}$$

where $\omega > 0$ is a frequency of harmonic vibrations of the electromagnetic field,

$\rho = \rho(x)$ is the volume charge density,

$\mathbf{j} = \mathbf{j}(x)$ is the current density,

$\mathbf{E} = \mathbf{E}(x)$ is the electric field intensivity,

$\mathbf{H} = \mathbf{H}(x)$ is the magnetic field intensivity,

$\mathbf{D} = \mathbf{D}(x)$ is the electric induction vector,

$\mathbf{B} = \mathbf{B}(x)$ is the electric induction vector.

The Maxwell equations are provided by the constitutive relations connecting the vectors $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{D}$, $\mathbf{B}$. We consider relations corresponding to isotropic nonhomogeneous media:

$$\mathbf{D}(x) = \varepsilon(x)\mathbf{E}(x), \tag{6.5}$$

$$\mathbf{B}(x) = \mu(x)\mathbf{H}(x), \tag{6.6}$$

where $\varepsilon = \varepsilon(x), \mu(x)$ are electric and magnetic permittivity given by complex-valued functions on $\mathbb{R}^3$ depending on the frequency ω, such that

$$\inf|\varepsilon(x)| > 0, \quad \inf|\mu(x)| > 0.$$

(In what follows, we will omit the dependence of these functions on ω).

The system (6.1)–(6.6) can be written as

$$\begin{aligned} \nabla \times \mathbf{H} &= i\omega\varepsilon\mathbf{H} + \mathbf{j}, \\ \nabla \times \mathbf{E} &= -i\omega\mu\mathbf{H}, \\ \nabla \cdot \varepsilon\,\mathbf{E} &= \rho, \\ \nabla \cdot \mu\,\mathbf{H} &= 0. \end{aligned} \tag{6.7}$$

We associate with the system (6.7) the operator $M : H^1(\mathbb{R}^3, \mathbb{C}^6) \to L^2(\mathbb{R}^3, \mathbb{C}^8)$.

6.2. **Quaternionic representation of Maxwell's system.** To study the Fredholm property and invertibility of the Maxwell's operators, it is convenient to consider their quaternionic realizations (see the book [11]). We let $\mathbb{H}$ denote the complex quaternionic algebra, which is the associative algebra over the field $\mathbb{C}$ generated by four elements 1, $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ satisfying the conditions

$$\mathbf{e}^1\mathbf{e}^2 = \mathbf{e}^3, \quad \mathbf{e}^2\mathbf{e}^3 = \mathbf{e}^1, \quad \mathbf{e}^3\mathbf{e}^1 = \mathbf{e}^2$$

and

$$1^2 = 1, \quad (\mathbf{e}^k)^2 = -1, \quad 1\mathbf{e}^k = \mathbf{e}^k 1 = \mathbf{e}^k, \quad \mathbf{e}^k\mathbf{e}^j = -\mathbf{e}^j\mathbf{e}^k$$

for $j, k = 1, 2, 3$. Each of the elements $1, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ commutes with the imaginary unit i. Hence, every element $\check{q} \in \mathbb{H}$ has a unique decomposition

$$\check{q} = q_0 + q_1\mathbf{e}^1 + q_2\mathbf{e}^2 + q_3\mathbf{e}^3 =: q_0 + \mathbf{q}$$

with $q_j \in \mathbb{C}$. The number q_0 is called the scalar part of the quaternion q, and $\mathbf{q}$ is its vector part. One can also think of $\mathbb{H}$ as a complex linear space of dimension 4 with usual linear operations.

With respect to the base $\{1, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ of this space, the operator of multiplication from the left and from the right by 1 has the unit 4×4 matrix E^0 as its matrix representation, whereas the matrix representations E_l^j and E_r^j of the operators of multiplication from the left and from the right by $\mathbf{e}_j$, $j = 1, 2, 3$, are real and skew-symmetric matrices. In what follows, if $\check{a}$ is a quaternion, we denote in a usual way the operator multiplication by $\check{a}$ from the left as $\mathbb{H} \ni \check{u} \to \check{a}\check{u} \in \mathbb{H}$, and we denote the operator multiplication by $\check{a}$ from the right as $\mathbb{H} \ni \check{u} \to \check{a}^r\check{u} = \check{u}\check{a} \in \mathbb{H}$. Let $\check{a} = a_0 + a_1\mathbf{e}^1 + a_2\mathbf{e}^2 + a_3\mathbf{e}^3$. Then the operators $\check{u} \to \check{a}\check{u}$ and $\check{u} \to \check{a}^r\check{u}$ have in the base $\{1, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$

$$\text{the matrices } \mathfrak{M}_{\check{a}} \text{ and } \mathfrak{M}_{\check{a}^r}\text{: } \mathfrak{M}_{\check{a}} = \sum_{j=0}^{3} a_j E_l^j, \mathfrak{M}_{\check{a}^r} = \sum_{j=0}^{3} a_j E_r^j.$$

The space $\mathbb{H}$ carries also the structure of a complex Hilbert space via the scalar product

$$(\check{q}, \check{r})_{\mathbb{H}} := q_0\overline{r_0} + q_1\overline{r_1} + q_2\overline{r_2} + q_3\overline{r_3}.$$

By $L^2(\mathbb{R}^3, \mathbb{H})$ we denote the Hilbert space of all measurable and squared integrable quaternion valued functions $\check{u}(x) = u(x) + \mathbf{u}(x)$ on $\mathbb{R}^3$ which is provided with the scalar product

$$(\check{u}, \check{v})_{L^2(\mathbb{R}^3,\mathbb{H})} = \int_{\mathbb{R}^3} (\check{u}(x), \check{v}(x))_{\mathbb{H}}\, dx,$$

and by $H^s(\mathbb{R}^3, \mathbb{H})$ the Sobolev space of order $s \in \mathbb{R}$ with the norm

$$\|\check{u}\|_{H^s(\mathbb{R}^3,\mathbb{H})} = \left(\int_{\mathbb{R}^3} \|(1-\Delta)^{s/2}\check{u}(x)\|^2_{L^2(\mathbb{R}^3,\mathbb{H})}\, dx \right)^{1/2}.$$

It is clear that $L^2(\mathbb{R}^3, \mathbb{H})$ and $H^s(\mathbb{R}^3, \mathbb{H})$ are isometrically isomorphic to $L^2(\mathbb{R}^3, \mathbb{C}^4)$ and $H^s(\mathbb{R}^3, \mathbb{C}^4)$. Let

$$D\check{u}(x) = \mathbf{e}^j \partial_{x_j}\check{u}(x), \quad x \in \mathbb{R}^3,$$

be the Moisil–Teodorescu differential operator of the first order acting from $H^s(\mathbb{R}^3, \mathbb{H})$ into $H^{s-1}(\mathbb{R}^3, \mathbb{H})$. The operator D has remarkable properties:

$$D\check{u}(x) = Du_0(x) + D\mathbf{u}(x) = -\nabla \cdot u(x) + \nabla u_0(x) + \nabla \times \mathbf{u}(x) \qquad (6.8)$$

for the quaternionic function $\check{u} = u_0 + \mathbf{u}$ and

$$D^2\check{u} = -\Delta\check{u}, \quad \check{u} \in H^2(\mathbb{R}^3, \mathbb{H})), \qquad (6.9)$$

where $\Delta = \sum_{j=1}^{3} \partial_{x_j^2}$ is the Laplacian. In what follows, we need the formula of differentiation of the product of a quaternion function $\check{f} \in C^1(\mathbb{R}^3, \mathbb{H})$ by a scalar function $a \in C^1(\mathbb{R}^3)$,

$$D(a\check{f}) = a(D\check{f}) + (\nabla a)\check{f}. \qquad (6.10)$$

Properties (6.8), (6.10) allow us to write Maxwell's system (6.1)–(6.6) in the quaternionic form (see [11, p. 88]),

$$D\mathbf{E}(x) = \varepsilon^{-1}(x)\nabla\varepsilon(x)\cdot\mathbf{E} - i\omega\mu(x)\mathbf{H}(x) - \frac{\rho(x)}{\varepsilon(x)}, \tag{6.11}$$

$$D\mathbf{H}(x) = \mu^{-1}(x)\nabla\mu(x)\cdot\mathbf{H} + i\omega\varepsilon(x)\mathbf{E}(x) + \mathbf{j}(x), \tag{6.12}$$

where $\mathbf{a}\cdot\mathbf{b} = \sum_{j=1}^{3} a_j b_j$. Applying formula

$$\mathbf{a}\cdot\mathbf{b} = -\frac{1}{2}(\mathbf{ab}+\mathbf{ba}),$$

where **ab** and **ba** denote the product of the vectors as quaternions, we obtain

$$D\mathbf{E}(x) = -\frac{1}{2}\varepsilon^{-1}(x)(\nabla\varepsilon(x)) + (\nabla\varepsilon(x))^r\mathbf{E}(x) - i\omega\mu(x)\mathbf{H}(x) - \frac{\rho(x)}{\varepsilon(x)},$$

$$D\mathbf{H}(x) = -\frac{1}{2}\mu^{-1}(x)(\nabla\mu(x) + (\nabla\mu(x))^r)\mathbf{H}(x) + i\omega\varepsilon(x)\mathbf{E}(x) + \mathbf{j}(x).$$

We associate with the system (6.11), (6.12) the quaternionic matrix operator

$$\mathcal{M}\begin{pmatrix}\mathbf{E}(x)\\ \mathbf{H}(x)\end{pmatrix} = \begin{pmatrix} D\mathbf{E}(x) + \frac{1}{2}(\varepsilon^{-1}(x)\nabla\varepsilon(x) + \nabla\varepsilon(x)^r\mathbf{E}(x)) + i\omega\mu(x)\mathbf{H}(x) \\ D\mathbf{H}(x) + \frac{1}{2}\left(\mu^{-1}(x)(\nabla\mu(x) + \nabla\mu(x)^r\mathbf{H}(x)) - i\omega\varepsilon(x)\mathbf{E}(x)\right)\end{pmatrix} \tag{6.13}$$

acting from $H^1(\mathbb{R}^3,\mathbb{H}^2)$ into $L^2(\mathbb{R}^3,\mathbb{H}^2), \mathbb{H}^2 = \mathbb{H}\times\mathbb{H}$.

Remark 6.1. Since a quaternionic system of equations can be written in the matrix-vectorial form, we can apply the limit operators approach for investigation of the Fredholm property of the operator $\mathcal{M}$.

6.3. **Fredholm property and invertibility.**

Theorem 6.2. *Let*

$$\liminf_{x\to\infty} \Im k^2(x) > 0, \tag{6.14}$$

where $k^2(x) = \omega^2\varepsilon(x)\mu(x)$ *is square of the wave number of Maxwell's system. Then* $\mathcal{M} : H^1(\mathbb{R}^3,\mathbb{H}^2) \to L^2(\mathbb{R}^3,\mathbb{H}^2)$ *is a Fredholm operator of the index* 0.

Proof. We follow to the above given scheme of the proof of the Fredholm properties. The main symbol of $\mathcal{M}$ is a quaternionic matrix function

$$\sigma_{\mathcal{M}}(\xi) = \begin{pmatrix} i\mathbf{e}^j\xi_j & 0 \\ 0 & i\mathbf{e}^j\xi_j\end{pmatrix},$$

and

$$\sigma^2_{\mathcal{M}}(\xi) = \begin{pmatrix} |\xi|^2E_4 & 0 \\ 0 & |\xi|^2E_4\end{pmatrix}, \quad |\xi|^2 = |\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2. \tag{6.15}$$

Let

$$\widetilde{\sigma}_{\mathcal{M}}(\xi) = \begin{pmatrix} iE_l^j \xi_j & 0 \\ 0 & iE_l^j \xi_j \end{pmatrix}$$

be the main symbol of $\mathcal{M}$ in the matrix representation. Then (6.15) implies that $\mathcal{M}$ is a uniformly elliptic operator. The limit operators $\mathcal{M}^g$ are those with constant coefficients

$$\mathcal{M}^g \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix} = \begin{pmatrix} D\mathbf{E}(x) + i\omega\mu^g \mathbf{H}(x) \\ D\mathbf{H}(x) - i\omega\varepsilon^g \mathbf{E}(x) \end{pmatrix}, \tag{6.16}$$

where

$$\mu^g = \lim_{m\to\infty} \mu(g(m)), \quad \varepsilon^g = \lim_{m\to\infty} \varepsilon(g(m))$$

and

$$\lim_{x\to\infty} \nabla\mu(x) = \lim_{x\to\infty} \nabla\varepsilon(x) = 0,$$

since $\varepsilon, \mu \in SO^2(\mathbb{R}^n)$. We will prove that the condition (6.14) provides the invertibility of the operators $\mathcal{M}^g : H^1(\mathbb{R}^3, \mathbb{H}^2) \to L^2(\mathbb{R}^3, \mathbb{H}^2)$. Indeed, let

$$\mathcal{M}^g \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} D\mathbf{E} + i\omega\mu^g \mathbf{H} \\ D\mathbf{H} - i\omega\varepsilon^g \mathbf{E} \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ \mathbf{\Phi} \end{pmatrix}. \tag{6.17}$$

The system (6.17) is reduced to two independent equations

$$-(\Delta + (k^g)^2)\mathbf{E} = D\mathbf{F} - i\omega\mu^g \mathbf{\Phi}, \tag{6.18}$$

$$-(\Delta + (k^g)^2)\mathbf{H} = D\mathbf{\Phi} + i\omega\varepsilon^g \mathbf{F}, \tag{6.19}$$

where $(k^g)^2 = \omega^2 \varepsilon^g \mu^g$ is a square of the wave number of the limit operator. Since $\Im(k^g)^2 > 0$ the operators $(\Delta^2 + (k^g)^2) : H^2(\mathbb{R}^3, \mathbb{H}(\mathbb{C})) \to L^2(\mathbb{R}^3, \mathbb{H}(\mathbb{C}))$ are invertible, and we obtain

$$(\mathcal{M}^g)^{-1} \begin{pmatrix} \mathbf{F} \\ \mathbf{\Phi} \end{pmatrix} = \begin{pmatrix} -D(\Delta + (k^g)^2)^{-1}\mathbf{F} - i\omega\mu^g(\Delta + (k^g)^2)^{-1}\mathbf{\Phi} \\ -D(\Delta + (k^g)^2)^{-1}\mathbf{\Phi} + i\omega\varepsilon^g(\Delta + (k^g)^2)^{-1}\mathbf{F} \end{pmatrix}. \tag{6.20}$$

It follow from (6.20) that $(\mathcal{M}^g)^{-1}$ is a bounded operator from $L^2(\mathbb{R}^3, \mathbb{H}^2)$ into $H^1(\mathbb{R}^3, \mathbb{H}^2)$. Hence the limit operators

$$\mathcal{M}^g : H^1(\mathbb{R}^3, \mathbb{H}^2) \to L^2(\mathbb{R}^3, \mathbb{H}^2)$$

are invertible. Thus Theorem 2.1 implies that

$$\mathcal{M} : H^1(\mathbb{R}^3, \mathbb{H}^2) \to L^2(\mathbb{R}^3, \mathbb{H}^2)$$

is a Fredholm operator.

Let us consider the family of operators $\mathcal{M}_\lambda = \mathcal{M} + \lambda I$, $\lambda \geq 0$. It is easy to see that $\mathcal{M}_\lambda$ is the family of elliptic systems with a parameter. Moreover, as above, $\mathcal{M}_\lambda$ is a Fredholm family, continuously depending on the parameter $\lambda \geq 0$. Hence index $\mathcal{M} = 0$. □

Theorem 6.3. *Let $\varepsilon, \mu \in SO^2(\mathbb{R}^3)$, and*

$$\Im k^2(x) \geq 0, \tag{6.21}$$

and the condition (6.14) *be satisfied. Then the operator*

$$\mathcal{M} : H^1(\mathbb{R}^3, \mathbb{H}^2) \to L^2(\mathbb{R}^3, \mathbb{H}^2)$$

is invertible.

Proof. It remains to prove that $\ker \mathcal{M} = \{0\}$. Suppose that $\mathbf{u} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in \ker \mathcal{M}$. Then $\mathbf{u}$ satisfies the homogeneous system of equations

$$D\mathbf{E}(x) = \varepsilon^{-1}(x)\nabla\varepsilon(x) \cdot \mathbf{E} - i\omega\mu(x)\mathbf{H}(x), \tag{6.22}$$

$$D\mathbf{H}(x) = \mu^{-1}(x)\nabla\mu(x) \cdot \mathbf{H} + i\omega\varepsilon(x)\mathbf{E}(x). \tag{6.23}$$

Applying differentiation formula (6.10)), we reduce this system to the following ones:

$$\begin{aligned} (D^2 - k^2(x))\mathbf{E}(x) - D(\varepsilon^{-1}(x)\nabla\varepsilon(x) \cdot \mathbf{E}) + i\omega\nabla\mu(x) \cdot \mathbf{H}(x) = 0, \\ (D^2 - k^2(x))\mathbf{H}(x) - D(\mu^{-1}(x)\nabla\mu(x) \cdot \mathbf{H}) - i\omega\nabla\varepsilon(x) \cdot \mathbf{E}(x) = 0. \end{aligned} \tag{6.24}$$

Hence $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$ satisfies the homogeneous system of quaternionic equations

$$\mathcal{B}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} := -(\Delta + k^2(x))\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} + \mathcal{T}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

where

$$\mathcal{T}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} := \begin{pmatrix} -D(\varepsilon^{-1}(x)\nabla\varepsilon(x) \cdot \mathbf{E}) + i\omega\nabla\mu(x) \cdot \mathbf{H}(x) \\ -i\omega\nabla\mu(x) \cdot \mathbf{E}(x) - D(\mu^{-1}(x)\nabla\mu(x) \cdot \mathbf{H}) \end{pmatrix}.$$

Note that $\mathcal{T}$ is a matrix quaternionic differential operator of the first order with coefficients in the class $C_0^1(\mathbb{R}^n)$. This implies that

$$\begin{aligned} &\lim_{R\to\infty} \|\varphi_{B'_R}\mathcal{T}\|_{\mathcal{L}(H^2(\mathbb{R}^3,\mathbb{H}^2),L^2(\mathbb{R}^3,\mathbb{H}^2))} \\ &= \lim_{R\to\infty} \|\mathcal{T}\varphi_{B'_R}\|_{\mathcal{L}(H^2(\mathbb{R}^3,\mathbb{H}^2),L^2(\mathbb{R}^3,\mathbb{H}^2))} = 0, \end{aligned}$$

where $\varphi_{B'_R} \in C^\infty(\mathbb{R}^3)$, $0 \le \varphi_{B'_R} \le 1$, $\operatorname{supp}\varphi_{B'_R} \subset B'_R$, $\varphi_{B'_R}(x) = 1$ if $x \in B'_{2R}$. Note that $\ker\mathcal{B} \in H^2(\mathbb{R}^3, \mathbb{C}^6)$ because the operator $\mathcal{B}$ is uniformly elliptic on $\mathbb{R}^3$. Repeating the proof of triviality of the kernel of the Dirac operator, we obtain $\ker\mathcal{M} = \{0\}$. □

Theorems 6.2 and 6.3 imply the following result.

Theorem 6.4. *Let* $\varepsilon, \mu \in SO^2(\mathbb{R}^3)$. *Then:*

(i) *If the condition* (6.14) *is satisfied, then the operator*

$$M : H^1(\mathbb{R}^3, \mathbb{C}^6) \to L^2(\mathbb{R}^3, \mathbb{C}^8)$$

of the Maxwell system is a Fredholm one;

(ii) *If the conditions* (6.14) *and* (6.21) *are satisfied, then*

$$M : H^1(\mathbb{R}^3, \mathbb{C}^6) \to L^2(\mathbb{R}^3, \mathbb{C}^8)$$

is an invertible operator.

Note that the electric and magnetic permittivity in the *dispersive electromagnetic media* are complex-valued functions of the form (see e.g. [13]):

$$\begin{aligned} \varepsilon(x) &= \varepsilon_0\Big(1 + \frac{i\sigma_\varepsilon(x)}{\omega}\Big), \\ \mu(x) &= \mu_0\Big(1 + \frac{i\sigma_\mu(x)}{\omega}\Big), \end{aligned} \tag{6.25}$$

where ε_0, μ_0 are electric and magnetic permittivity in the vacuum, $\sigma_\varepsilon(x)$, $\sigma_\mu(x)$ are absorption coefficients for the electric and magnetic permittivity satisfying the conditions:

$$\sigma_\varepsilon(x) \geq 0, \quad \sigma_\mu(x) \geq 0. \tag{6.26}$$

This implies that

$$k^2(x) = \frac{\omega^2}{c_0^2}\Big(1 + \frac{i\sigma_\varepsilon(x)}{\omega}\Big)\Big(1 + \frac{i\sigma_\mu(x)}{\omega}\Big),$$

where c_0 is the light speed in the vacuum.

Thus Theorem 6.4 provides us with the following result.

Theorem 6.5. *Let* $\sigma_\varepsilon, \sigma_\mu \in SO^2(\mathbb{R}^3)$. *Then Maxwell's operator* $M : H^1(\mathbb{R}^3, \mathbb{C}^6) \to L^2(\mathbb{R}^3, \mathbb{C}^8)$ *is invertible if at least one of the conditions*

$$\liminf_{x\to\infty} \sigma_\varepsilon(x) > 0, \quad \liminf_{x\to\infty} \sigma_\mu(x) > 0$$

in (6.25) *holds.*

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Author's address:

National Polytechnic Institute of Mexico, ESIME Zacatenco, edif. 1, Av. Luis Enrique Erro S/N, C.P. 07738, Mexico, DF

E-mail: vladimir.rabinovich@gmail.com

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Short Communications

Malkhaz Ashordia

ANTIPERIODIC BOUNDARY VALUE PROBLEM FOR SYSTEMS OF LINEAR GENERALIZED DIFFERENTIAL EQUATIONS

Abstract. The antiperiodic boundary value problem for systems of linear generalized differential equations is considered. The Green type theorem on the unique solvability of the problem is established and representation of its solution is given. The effective necessary and sufficient (among them spectral) conditions for the unique solvability of the problem are also given.

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In the present paper we study the question of the solvability for the system of linear generalized ordinary differential equations

$$dx(t) = dA(t) \cdot x(t) + df(t) \tag{1}$$

under the $\omega > 0$-antiperiodic condition

$$x(t + \omega) = -x(t) \quad \text{for} \quad t \in \mathbb{R}, \tag{2}$$

where $A = (a_{ik})_{i,k=1}^n : \mathbb{R} \to \mathbb{R}^{n \times n}$ and $f = (f_i)_{i=1}^n : \mathbb{R} \to \mathbb{R}^n$ are, respectively, the matrix- and vector-functions with bounded variation components on the every closed interval $[a, b]$ from $\mathbb{R}$, and ω is a fixed positive number.

We establish the Green type theorem on the solvability of the problem (1), (2) and represent the solution of the problem. In addition, we give the effective necessary and sufficient conditions (spectral type) for unique solvability of the problem.

The general linear boundary value problem for the system (1) is investigated sufficiently well (see e.g. [6, 8, 15] and the references therein), and the Green type theorems for the unique solvability are obtained. Certain questions dealt with the periodic problem for the system (1) have been investigated in [2–5, 7, 14] (see also the references therein), but the specifical properties analogous to those established for the ordinary differential case (see e.g. [11]) are not available. As for the antiperiodic problem, it is sufficient far from a full value. Thus the problem under considered in the paper, is very actual.

In the paper we establish some special conditions for the unique solvability of the problem (1), (2).

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see [1, 8–10, 13, 14] and the references therein).

The theory of generalized ordinary differential equations has been introduced by J. Kurzweil [13] in connection with investigation of the well-possed problem for the Cauchy problem for ordinary differential equations.

Throughout the paper, the use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $[a, b]$ and $]a, b[$ $(a, b \in \mathbb{R})$ are, respectively, closed and open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$\mathbb{R}_+^{n\times m} = \big\{(x_{ij})_{i,j=1}^{n,m} : x_{ij} \ge 0 \ \ (i = 1, \dots, n; \ \ j = 1, \dots, m)\big\}$.

$O_{n\times m}$ (or O) is the zero $n \times m$ matrix.

If $X = (x_{ij})_{i,j=1}^{n,m} \in \mathbb{R}^{n\times m}$, then

$$|X| = (|x_{ij}|)_{i,j=1}^{n,m}.$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^{n}$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

If $X \in \mathbb{R}^{n\times n}$, then:

X^{-1} is the matrix inverse to X;
$\det X$ is the determinant of X;
$r(X)$ is spectral radius of X;
X^T is the matrix transposed to X;
$\lambda_0(X)$ and $\lambda^0(X)$ are, respectively, the minimal and maximal eigenvalues of the symmetric X matrix.

I_n is the identity $n \times n$-matrix.

The inequalities between the real matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

If $X : [a,b] \to \mathbb{R}^{n\times m}$ is a matrix-function, then $\overset{b}{\underset{a}{\mathrm{V}}}(X)$ is the sum of total variations on $[a,b]$ of its components x_{ij} $(i = 1,\dots,n;\ j = 1,\dots,m)$; $V(X)(t) = (V(x_{ij})(t))_{i,j=1}^{n,m}$, where $V(x_{ij})(a) = 0$, $V(x_{ij})(t) = \overset{t}{\underset{a}{\mathrm{V}}}(x_{ij})$ for $a < t \le b$; $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t $(X(a-) = X(a),\ X(b+) = X(b))$.

$d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\|X\|_s = \sup\{\|X(t)\| : t \in [a,b]\}$, $|X|_s = (\|x_{ij}\|_s)_{i,j=1}^{n,m}$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the normed space of all bounded variation matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$ (i.e., such that $\overset{b}{\underset{a}{\mathrm{V}}}(X) < \infty$) with the norm $\|X\|_s$.

$\mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$ whose restrictions on every closed interval $[a,b]$ from $\mathbb{R}$ belong to $\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$.

$\mathrm{BV}_\omega^+(\mathbb{R},\mathbb{R}^{n\times m})$ and $\mathrm{BV}_\omega^-(\mathbb{R},\mathbb{R}^{n\times m})$ are the sets of all matrix-functions $G : \mathbb{R} \to \mathbb{R}^{n\times m}$ whose restrictions on $[0,\omega]$ belong to $\mathrm{BV}([0,\omega],\mathbb{R}^{n\times m})$ and there exists a constant matrix $C \in \mathbb{R}^{n\times m}$ such that, respectively,

$$G(t+\omega) = G(t) + C \quad \text{for} \quad t \in \mathbb{R}$$

and

$$G(t+\omega) = -G(t) + C \quad \text{for} \quad t \in \mathbb{R}.$$

$\mathrm{BV}([a,b],\mathbb{R}_+^{n\times m}) = \big\{X \in \mathrm{BV}([a,b],\mathbb{R}^{n\times m}) : X(t) \ge O_{n\times m} \text{ for } t \in [a,b]\big\}$.

$s_c, s_j : \mathrm{BV}([a,b],\mathbb{R}) \to \mathrm{BV}([a,b],\mathbb{R})$ $(j = 1,2)$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0,$$
$$s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau) \quad \text{and} \quad s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \quad \text{for} \ \ a < t \le b,$$

and

$$s_c(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \quad \text{for} \quad t \in [a,b].$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,ds_c(g)(\tau) + \sum_{s<\tau\le t} x(\tau)d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)d_2g(\tau),$$

where $\int_{]s,t[} x(\tau)\,ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_c(g))$ corresponding to the function $s_c(g)$.

If $a = b$, then we assume

$$\int_a^b x(t)\,dg(t) = 0,$$

and if $a > b$, then we assume

$$\int_a^b x(t)\,dg(t) = -\int_b^a x(t)\,dg(t).$$

Hence $\int_a^b x(\tau)\,dg(\tau)$ is the Kurzweil–Stieltjes integral (see [12,13]).

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_s^t x(\tau)\,dg_1(\tau) - \int_s^t x(\tau)\,dg_2(\tau) \quad \text{for} \quad s \le t.$$

If $G = (g_{ik})_{i,k=1}^{l,n} \in \mathrm{BV}([a,b], \mathbb{R}^{l\times n})$ and $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to \mathbb{R}^{n\times m}$, then

$$S_c(G)(t) \equiv \big(s_c(g_{ik})(t)\big)_{i,k=1}^{l,n}, \quad S_j(G)(t) \equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n} \quad (j = 1, 2)$$

and

$$\int_a^b dG(\tau)\cdot X(\tau) = \bigg(\sum_{k=1}^n \int_a^b x_{kj}(\tau)\,dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}.$$

We introduce the operator. If $X \in \mathrm{BV}_{loc}(\mathbb{R},;\mathbb{R}^{n\times n})$,

$$\det(I_n + (-1)^j d_j X(t)) \neq 0 \ \text{ for } t \in \mathbb{R} \ \ (j = 1, 2),$$

and $Y \in \mathrm{BV}_{loc}(\mathbb{R},;\mathbb{R}^{n\times m})$, then

$$\mathcal{A}(X,Y)(0) = O_{n\times m},$$

$$\mathcal{A}(X,Y)(t) = Y(t) - Y(0) + \sum_{0<\tau\le t} d_1X(\tau)\cdot (I_n - d_1X(\tau))^{-1}\, d_1Y(\tau)$$

$$- \sum_{0\le\tau<t} d_2X(\tau)\cdot (I_n + d_2X(\tau))^{-1}\, d_2Y(\tau) \quad \text{for} \quad t > 0,$$

$$\mathcal{A}(X,Y)(t) = -\mathcal{A}(X,Y)(t) \quad \text{for} \quad t < 0.$$

We say that the matrix-function $X \in \mathrm{BV}([a,b], \mathbb{R}^{n\times n})$ satisfies the Lappo–Danilevskiĭ condition if the matrices $S_c(X)(t)$, $S_1(X)(t)$ and $S_2(X)(t)$ are pairwise permutable for every $t \in [a,b]$ and there exists $t_0 \in [a,b]$ such that

$$\int_{t_0}^t S_c(X)(\tau)\,dS_c(X)(\tau) = \int_{t_0}^t dS_c(X)(\tau)\cdot S_c(X)(\tau) \quad \text{for} \ \ t \in [a,b].$$

A vector-function $\mathrm{BV}_{loc}(\mathbb{R}, \mathbb{R}^{n\times m})$ is said to be a solution of the system (1) if

$$x(t) - x(s) = \int\limits_s^t dA(\tau)\cdot x(\tau) + f(t) - f(s) \ \text{ for } \ s < t; \quad s, t \in \mathbb{R}.$$

Under a solution of the problem (1), (2) we understand a solution x of the system (1), satisfying the condition (2).

We assume that

$$A \in \mathrm{BV}^{+}_{\omega}(\mathbb{R}, \mathbb{R}^{n\times n}) \quad \text{and} \quad f \in \mathrm{BV}^{-}_{\omega}(\mathbb{R}, \mathbb{R}^{n}),$$

i.e.,

$$A(t+\omega) = A(t) + C \quad \text{and} \quad f(t+\omega) = -f(t) + c \quad \text{for} \quad t \in \mathbb{R}, \tag{3}$$

where $C \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}^n$ are, respectively, some constant matrix and a vector; and

$$\det\big(I_n + (-1)^j d_j A(t)\big) \neq 0 \quad \text{for} \quad t \in \mathbb{R} \ \ (j = 1, 2). \tag{4}$$

If a matrix-function $X \in \mathrm{BV}([0, \omega], \mathbb{R}^{n\times n})$ is such that $\det(I_n - d_1 X(t)) \neq 0$ for $t \in [0, \omega]$, then we put

$$\begin{aligned} &[X(t)]_0 = (I_n - d_1 X(t))^{-1}, \\ &\qquad [X(t)]_i = (I_n - d_1 X(t))^{-1} \int\limits_0^t dX_-(\tau)\cdot [X(\tau)]_{i-1} \\ &\qquad\qquad \text{for} \quad t \in [0, \omega] \quad (i = 1, 2, \dots), \end{aligned} \tag{5_1}$$

$$\begin{aligned} &(X(t))_0 = O_{n\times n}, \quad (X(t))_1 = X(t), \quad (X(t))_{i+1} = \int\limits_0^t dX_-(\tau)\cdot (X(\tau))_i \\ &\qquad\qquad \text{for} \quad t \in [0, \omega] \quad (i = 1, 2, \dots), \end{aligned} \tag{6_1}$$

and

$$\begin{aligned} &V_1(X)(t) = |(I_n - d_1 X(t))^{-1}| V(X_-)(t), \\ &V_{i+1}(X)(t) = |(I_n - d_1 X(t))^{-1}| \int\limits_0^t dV(X_-)(\tau)\cdot V_i(X)(\tau) \\ &\qquad\qquad \text{for } \ t \in [0, \omega] \ \ (i = 1, 2, \dots), \end{aligned} \tag{7_1}$$

where $X_-(t) \equiv X(t-)$; and if $X \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ is such that $\det(I_n + d_2X(t)) \neq 0$ for $t \in [0,\omega]$, then we put

$$[X(t)]_0 = (I_n + d_2X(t))^{-1},$$

$$[X(t)]_i = (I_n + d_2X(t))^{-1} \int_\omega^t dX_+(\tau)\cdot[X(\tau)]_{i-1}$$

$$\text{for} \quad t \in [0,\omega] \quad (i = 1,2,\dots), \tag{5_2}$$

$$(X(t))_0 = O_{n\times n}, \quad (X(t))_1 = X(t), \quad (X(t))_{i+1} = \int_\omega^t dX_+(\tau)\cdot(X(\tau))_i$$

$$\text{for} \quad t \in [0,\omega] \quad (i = 1,2,\dots) \tag{6_2}$$

and

$$V_1(X)(t) = |(I_n + d_2X(t))^{-1}|(V(X_+)(t)(b) - V(X_+)(t)|,$$

$$V_{i+1}(X)(t) = |(I_n + d_2X(t))^{-1}|\left|\int_\omega^t dV(X_+)(\tau)\cdot V_i(X)(\tau)\right|$$

$$\text{for } t \in [0,\omega] \ (i = 1,2,\dots), \tag{7_2}$$

where $X_+(t) \equiv X(t+)$.

Alongside with the system (1), we consider the corresponding homogeneous system

$$dx(t) = dA(t)\cdot x(t). \tag{1_0}$$

Moreover, along with the condition (2), we consider the condition

$$x(0) = -x(\omega). \tag{8}$$

Definition 1. Let the condition (3) hold. A matrix-function $\mathcal{G} : [0,\omega] \times [0,\omega] \to \mathbb{R}^{n\times n}$ is said to be the Green matrix of the problem $(1_0), (8)$ if:

(a) for every $s \in]0,\omega[$, the matrix-function $\mathcal{G}(\cdot,s)$ satisfies the matrix equation

$$dX(t) = dA(t)\cdot X(t)$$

both on $[0,s[$ and $]s,\omega]$;

(b)

$$\mathcal{G}(t,t+) - \mathcal{G}(t,t-) = Y(t)D^{-1}\Big\{Y^{-1}(t)(I_n - d_1A(t))^{-1} + Y(\omega)Y^{-1}(t)(I_n + d_2A(t))^{-1}\Big\} \text{ for } t \in]a,b[;$$

(c) $\mathcal{G}(t,\cdot) \in BV([0,\omega],\mathbb{R}^{n\times n})$ for every $t \in [0,\omega]$;

(d) the equality

$$\int_0^\omega d_s\,(\mathcal{G}(0,s) + \mathcal{G}(\omega,s))\cdot f(s) = 0$$

holds for every $f \in BV([0,\omega],\mathbb{R}^n)$.

The Green matrix of the problem (1_0) exists and it is unique in the following sense. If $\mathcal{G}(t,s)$ and $\mathcal{G}_1(t,s)$ are two matrix-functions satisfying the conditions (a)–(d) of Definition 1, then

$$\mathcal{G}(t,s) - \mathcal{G}_1(t,s) \equiv Y(t)H_*(s),$$

where $H_* \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ is a matrix-function such that

$$H_*(s+) = H_*(s-) = C = const \quad \text{for} \quad s \in [0,\omega],$$

and $C \in \mathbb{R}^{n\times n}$ is a constant matrix.

In particular,

$$\mathcal{G}(t,s) = \begin{cases} -Y(t)\,(I_n + Y(\omega))^{-1}\,Y^{-1}(s) & \text{for } 0 \le s < t \le \omega, \\ Y(t)\,(I_n + Y(\omega))^{-1}\,Y(\omega)Y^{-1}(s) & \text{for } 0 \le t < s \le \omega, \\ \text{an arbitrary} & \text{for } \ t = s. \end{cases}$$

Theorem 1. *Let the conditions* (3) *and* (4) *hold. Then the problem* (1), (2) *has the unique solution x if and only if the corresponding homogeneous system* (1_0) *has only the trivial solution satisfying the condition* (8)*, i.e., when*

$$\det(Y(0) + Y(\omega)) \neq 0, \tag{9}$$

where Y is a fundamental matrix of the system (1_0)*. If the last condition holds, then the solution x admits the notation*

$$x(t) = \int_0^\omega d_s\mathcal{G}(t,s)\cdot f(s) \quad \textit{for} \quad t \in [0,\omega], \tag{10}$$

where $\mathcal{G} : [a,b]\times[a,b] \to \mathbb{R}^{n\times n}$ is the Green matrix $\mathcal{G}$ of the problem (1_0), (8)*.*

Corollary 1. *Let the conditions* (3) *and* (4) *hold, and the matrix-function A satisfy the Lappo–Danilevskiĭ condition. Then the problem* (1), (8) *has the unique solution if and only if*

$$\det\Big(I_n + \exp(S_0(A)(\omega)) \prod_{0\le\tau<\omega}(I_n + d_2A(\tau)) \prod_{a<\tau\le\omega}(I_n - d_1A(\tau))^{-1}\Big) \neq 0.$$

Note that if the matrix-function A satisfies the Lappo–Danilevskiĭ condition, then the matrix-function Y is defined by $Y(a) = I_n$ and

$$Y(t) \equiv \exp(S_0(A)(t)) \prod_{0\le\tau<t}(I_n + d_2A(\tau)) \prod_{0<\tau\le t}(I_n - d_1A(\tau))^{-1}$$

is the fundamental matrix of the system (1_0).

Remark 1. Let the system (1_0) have a nontrivial ω-antiperiodic solution. Then there exist $f \in \mathrm{BV}^-_\omega(\mathbb{R},\mathbb{R}^n)$ such that the system (1) has no ω-antiperiodic solution.

In general, it is quite difficult to verify the condition (9) directly even in the case where one is able to write out the fundamental matrix of the system (1_0) explicitly. Therefore it is important to seek for effective conditions which would guarantee the absence of nontrivial ω-antiperiodic solutions of the homogeneous system (1_0). Below we present the results concerning our question. Analogous results have been obtained by T. Kiguradze for the ordinary differential equations (see [11, 12]).

Theorem 2. *Let the conditions* (3) *and* (4) *hold. Then the system* (1) *has the unique ω-antiperiodic solution if and only if there exist natural numbers k and m such that the matrix*

$$M_k = -\sum_{i=0}^{k-1} \left([A(0)]_i + [A(\omega)]_i\right)$$

is nonsingular and

$$r(M_{k,m}) < 1, \tag{11}$$

where

$$M_{k,m} = V_m(A)(c) + \left(\sum_{i=0}^{m-1} |[A(\cdot)]_i|_s\right) \cdot |M_k^{-1}| [V_k(A)(0) + V_k(A)(\omega)],$$

$[A(t)]_i$ $(i = 0, \dots, m-1)$ *and* $V_i(A)(t)$ $(i = 0, \dots, m-1)$ *are defined, respectively, by* (5_l) *and* (7_l) *for some* $l \in \{1, 2\}$*, and* $c = (2-l)\omega$.

Corollary 2. *Let the conditions* (3) *and* (4) *hold. Then the system* (1) *has the unique ω-antiperiodic solution if and only if there exist natural numbers k and m such that the matrix*

$$M_k = -\sum_{i=0}^{k-1} \left[(A(0))_i + (A(\omega))_i\right]$$

is nonsingular and the inequality (11) *holds, where*

$$M_{k,m} = (V(A)(c))_m$$
$$+ \left(I_n + \sum_{i=0}^{m-1} |(A(\cdot))_i|_s\right) \cdot |M_k^{-1}| \left[(V(A)(0))_k + (V(A)(\omega))_k\right],$$

$(A(t))_i$ $(i = 0, \dots, m-1)$ *and* $(V(A)(t))_i$ $(i = 0, \dots, m-1)$ *are defined by* (6_l) *for some* $l \in \{1, 2\}$*, and* $c = (2-l)\omega$.

Corollary 3. *Let the conditions* (3) *and* (4) *hold. Let, moreover, there exist a natural j such that*

$$(A(0))_i = -(A(\omega))_i \quad (i = 1, \dots, j-1)$$

and

$$\det\left((A(0))_j + (A(\omega))_j\right) \neq 0,$$

where $(A(t))_i$ $(i = 0, \dots, j)$ *are defined by* (6_l) *for some* $l \in \{1, 2\}$. *Then there exists* $\varepsilon_0 > 0$ *such that the system*

$$dx(t) = \varepsilon dA(t) \cdot x(t) + df(t)$$

has one and only one ω*-antiperiodic solution for every* $\varepsilon \in]0, \varepsilon_0[$.

Theorem 3. *Let the conditions* (3) *and* (4) *hold, and let a matrix-function* $A_0 \in \mathrm{BV}^+_\omega(\mathbb{R}, \mathbb{R}^{n\times n})$ *be such that*

$$\det\big(I_n + (-1)^j d_j A_0(t)\big) \neq 0 \quad for \quad t \in [0, \omega] \quad (j = 1, 2)$$

and the homogeneous system

$$dx(t) = dA_0(t) \cdot x(t)$$

has only the trivial ω*-antiperiodic solution. Let, moreover, the matrix-function* $A \in \mathrm{BV}^+_\omega(\mathbb{R}, \mathbb{R}^{n\times n})$ *admit the estimate*

$$\int\limits_0^\omega |\mathcal{G}_0(t,\tau)| dV(S_0(A - A_0))(\tau) + \sum_{0<\tau\leq\omega} |\mathcal{G}_0(t,\tau-) \cdot d_1(A(\tau) - A_0(\tau))| + \sum_{0\leq\tau<\omega} |\mathcal{G}_0(t,\tau+) \cdot d_2(A(\tau) - A_0(\tau))| \leq M,$$

where $\mathcal{G}_0(t,\tau)$ *is the Green matrix of the problem* (1_0), (8), *and* $M \in \mathbb{R}^{n\times n}_+$ *is a constant matrix such that*

$$r(M) < 1.$$

Then the system (1) *has one and only one* ω*-antiperiodic solution.*

The presentation (10) can be replaced by a more simple and suitable form if we introduce the concept of the Green matrix for the problem $(1_0), (2)$.

Definition 2. The matrix function $\mathcal{G}_\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ is said to be the Green matrix of the problem $(1_0), (2)$ if:

(a) $\mathcal{G}_\omega(t+\omega, \tau+\omega) = \mathcal{G}_\omega(t,\tau)$, $\mathcal{G}_\omega(t, t+\omega) + \mathcal{G}_\omega(t,\tau) = -I_n$ for $t, \tau \in \mathbb{R}$;

(b) the matrix function $\mathcal{G}_\omega(.,\tau) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a fundamental matrix of the system (1_0) for every $\tau \in \mathbb{R}$.

Theorem 4. *Let the conditions* (3) *and*

$$\det\big(I_n \pm d_j A(t)\big) \neq 0 \quad for \quad t \in \mathbb{R}$$

hold and the boundary value problem (1_0), (2) *have only the trivial solution. Then the system* (1) *has the unique* ω*-antiperiodic solution* x *and it admits the notation*

$$x(t) = \int\limits_t^{t+\omega} \mathcal{G}_\omega(t,\tau) d\mathcal{A}\left(A, \mathcal{A}(-A, f)\right)(\tau) \quad for \quad t \in \mathbb{R},$$

where $\mathcal{G}_\omega$ *is the Green matrix of the problem* $(1_0), (2)$.

If $s \in \mathbb{R}$ and $\beta \in \mathrm{BV}[0,\omega],\mathbb{R})$ are such that

$$1+(-1)^j d_j\beta(t) \neq 0 \quad \text{for} \quad (-1)^j(t-s)<0 \quad (j=1,2),$$

then by $\gamma_s(\beta)$ we denote the unique solution of the Cauchy problem

$$d\gamma(t)=\gamma(t)\,d\beta(t), \quad \gamma(s)=1.$$

It is known (see [9, 10]) that

$$\gamma_s(\beta)(t)=\begin{cases} \exp(s_0(\beta)(t)-s_0(\beta)(s)) \prod\limits_{s<\tau\le t}(1-d_1\beta(\tau))^{-1} \\ \quad \times \prod\limits_{s\le\tau<t}(1+d_2\beta(\tau)) \ \text{ for } \ s<t\le\omega, \\ \exp(s_0(\beta)(t)-s_0(\beta)(s)) \prod\limits_{t<\tau\le s}(1-d_1\beta(\tau)) \\ \quad \times \prod\limits_{t\le\tau<s}(1+d_2\beta(\tau))^{-1} \ \text{ for } \ 0\le t<s, \\ \qquad 1 \qquad\qquad \text{for } \ t=s. \end{cases} \tag{12}$$

Let $g:[0,\omega]\to\mathbb{R}$ be a nondecreasing function, and $P=(p_{ik})_{i,k=1}^n$, where $p_{ik}\in L([0,\omega],\mathbb{R};g)$ $(i,k=1,\dots,n)$. Then by $Q_\omega(P;g)$ we denote the set of all matrix-functions $A=(a_{ik})_{i,k=1}^n \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ such that

$$b_{ik}(t)=\int\limits_0^t p_{ik}(\tau)dg(\tau) \quad \text{for} \quad t\in[0,\omega] \quad (i,k=1,\dots,n),$$

where

$$b_{ik}(t)\equiv a_{ik}(t)-\frac{1}{2}\bigg(\sum_{l=1}^n\sum_{0<\tau\le t} d_1a_{li}(\tau)\cdot d_1a_{lk}(\tau) \\ -\sum_{0\le\tau<t} d_2a_{li}(\tau)\cdot d_2a_{lk}(\tau)\bigg) \quad (i,k=1,\dots,n).$$

Theorem 5. *Let the conditions* (3) *and* $A\in Q_\omega(P;g)$ *hold. Let, moreover, either*

$$\sum_{i,k=1}^n p_{ik}(t)x_ix_k \ge \alpha(t)\sum_{i=1}^n x_i^2 \ \textit{ for } \ \mu(g)\text{-a.a. } t\in[0,\omega], \ (x_i)_{i=1}^n\in\mathbb{R}^n, \\ 1-2\alpha(t)d_1g(t)>0, \quad 1+2\alpha(t)d_2g(t)\neq 0 \ \textit{ for } \ 0\le t<\omega \tag{13}$$

and

$$\gamma_\omega(g_\alpha)(0)>1 \tag{14}$$

or

$$\sum_{i,k=1}^n p_{ik}(t)x_ix_k \le \beta(t)\sum_{i=1}^n x_i^2 \ \textit{ for } \ \mu(g)\text{-a.a. } t\in[0,\omega], \ (x_i)_{i=1}^n\in\mathbb{R}^n, \\ 1+2\beta(t)d_2g(t)>0, \quad 1-2\beta(t)d_1g(t)\neq 0 \ \textit{ for } \ 0<t\le\omega \tag{15}$$

and

$$\gamma_0(g_\beta)(\omega) < 1, \tag{16}$$

where $\alpha, \beta \in L([0,\omega], \mathbb{R}; g)$, *the functions* $\gamma_\omega(g_\alpha), \gamma_0(g_\beta)$ *are defined by* (12), *and*

$$g_\alpha(t) \equiv 2\int_0^t \alpha(\tau)dg(\tau) \quad \text{and} \quad g_\beta(t) \equiv 2\int_0^t \beta(\tau)dg(\tau) \tag{17}$$

Then the system (1) *has the unique* ω*-antiperiodic solution.*

Corollary 4. *Let the conditions* (3) *and* $A \in Q_\omega(P; g)$ *hold. Let, moreover, either the conditions* (13) *and* (14) *or the conditions* (15) *and* (16) *hold, where* $\alpha(t) \equiv \lambda_0(P^*(t))$, $\beta(t) \equiv \lambda^0(P^*(t))$, $P^*(t) \equiv P(t) + P^T(t)$, *the functions* $\gamma_\omega(g_\alpha), \gamma_0(g_\beta)$ *are defined by* (12), *and the functions* g_α *and* g_β *are defined by* (17). *Then the system* (1) *has the unique* ω*-antiperiodic solution.*

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Author's addresses:

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia.

2. Sokhumi State University, 12 Politkovskaia Str., Tbilisi 0186, Georgia.

E-mail: ashord@rmi.ge

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IVAN KIGURADZE

PERIODIC TYPE BOUNDARY VALUE PROBLEMS FOR SINGULAR IN PHASE VARIABLES NONLINEAR NONAUTONOMOUS DIFFERENTIAL SYSTEMS

Dedicated to the Blessed Memory of Professor B. Khvedelidze

Abstract. The unimprovable in a certain sense conditions guaranteeing the existence and uniqueness of positive solutions of periodic type boundary value problems for singular in phase variables nonlinear nonautonomous differential systems are established.

რეზიუმე. დადგენილია გარკვეული აზრით არაგაუმჯობესებადი პირობები, რომლებიც უზრუნველყოფენ პერიოდულის ტიპის სასაზღვრო ამოცანების დადებითი ამონახსნების არსებობასა და ერთადერთობას ფაზური ცვლადების მიმართ სინგულარული არაწრფივი არაავტონომიური დიფერენციალური სისტემებისათვის.

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Let $-\infty < a < b < +\infty$, $\mathbb{R}_{0+} =]0, +\infty[$,

$$\mathbb{R}^n_{0+} = \Big\{(x_i)_{i=1}^n \in \mathbb{R}^n : \; x_1 > 0, \dots, x_n > 0\Big\}$$

and $f_i : [a,b] \times \mathbb{R}^n_{0+} \to \mathbb{R}$ $(i = 1, \dots, n)$ are functions satisfying the local Carathéodory conditions, i.e. $f_i(\,\cdot\,, x_1, \dots, x_n) : [a,b] \to \mathbb{R}$ $(i = 1, \dots, n)$ are measurable for all $(x_i)_{i=1}^n \in \mathbb{R}^n_{0+}$, $f_i(t, \,\cdot\,, \dots, \,\cdot\,) : \mathbb{R}^n_{0+} \to \mathbb{R}$ $(i = 1, \dots, n)$ are continuous for almost all $t \in [a,b]$ and for any $\rho > 0$ and $\rho_0 \in]0, \rho[$ the function

$$f^*_{\rho_0,\rho}(t) = \max\bigg\{\sum_{i=1}^n \big|f_i(t, x_1, \dots, x_n)\big| : \; \rho_0 \le x_1 \le \rho, \dots, \rho_0 \le x_n \le \rho\bigg\}$$

is integrable on $[a,b]$.

Consider the differential system

$$\frac{du_i}{dt} = f_i(t, u_1, \dots, u_n) \;\; (i = 1, \dots, n) \tag{1}$$

with the boundary conditions

$$u_i(a) = \varphi_i(u_i(b)) \;\; (i = 1, \dots, n), \tag{2}$$

where $\varphi_i : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ $(i = 1, \dots, n)$ are continuous functions.

A particular case of (2) are the periodic boundary conditions

$$u_i(a) = u_i(b) \ \ (i = 1, \dots, n).$$

Thus the conditions (2) we call the periodic type boundary conditions.

A solution $(u_i)_{i=1}^n : [a, b] \to \mathbb{R}_{0+}^n$ of the system (1) satisfying the boundary conditions (2) is called a positive solution of the problem (1), (2).

For singular in phase variables first and second order differential equations, periodic type boundary value problems are studied in detail (see, e.g., [1, 2, 3, 5, 7]). As for the system (1), for it problems of the type (2) are investigated mainly only in the regular case, i.e., in the case where the functions f_i $(i = 1, \dots, n)$ are continuous, or satisfy the local Carathéodory conditions on the set $[a, b] \times \mathbb{R}_+^n$ and $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are continuous functions, where

$$\mathbb{R}_+ = [0, +\infty[\quad \mathbb{R}_+^n = \Big\{ (x_i)_{i=1}^n : \ x_1 > 0, \dots, x_n > 0 \Big\}$$

(see [1, 2] and the references therein).

Theorems below on the existence of a positive solution of the problem (1), (2) cover the cases in which the system under consideration has singularities in phase variables, in particular, the case where for arbitrary $i, k \in \{1, \dots, n\}$ and $x_j > 0$ $(j = 1, \dots, n;\ j \neq k)$ the equality

$$\lim_{x_k \to 0} \big| f_i(t, x_1, \dots, x_n) \big| = +\infty$$

is fulfilled.

In Theorems 1 and 2 it is assumed, respectively, that the functions f_i $(i = 1, \dots, n)$ and φ_i $(i = 1, \dots, n)$ on the sets $[a, b] \times \mathbb{R}_{0+}^n$ and $\mathbb{R}_{0+}$ satisfy the inequalities

$$\sigma_i \big(f_i(t, x_1, \dots, x_n) - p_i(t) x_i \big) \geq q_i(t, x_i) \ \ (i = 1, \dots, n), \tag{3}$$

$$\sigma_i \big(\varphi_i(x) - \alpha_i x \big) \geq 0 \ \ (i = 1, \dots, n), \tag{4}$$

$$q_i(t, x_i) \leq \sigma_i \big(f_i(t, x_1, \dots, x_n) - p_i(t) x_i \big) \leq$$

$$\leq \sum_{k=1}^n p_{ik}(t, x_1 + \dots + x_n) x_k + q_0(t, x_1, \dots, x_n) \ \ (i = 1, \dots, n), \tag{5}$$

$$\sigma_i(\varphi_i(x) - \alpha_i x) \geq 0, \quad \sigma_i(\varphi_i(x) - \beta_i x) \leq \beta_0 \ \ (i = 1, \dots, n). \tag{6}$$

Here,

$$\sigma_i \in \{-1, 1\}, \ \ \alpha_i > 0, \ \ \beta_i > 0, \ \ \sigma_i(\beta_i - \alpha_i) \geq 0 \ \ (i = 1, \dots, n), \ \ \beta_0 \geq 0,$$

$p_i : [a, b] \to \mathbb{R}$ $(i = 1, \dots, n)$ are integrable functions, $p_{ik} : [a, b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ and $q_i : [a, b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i, k = 1, \dots, n)$ are integrable in the first argument and nonincreasing and continuous in the second argument functions, and $q_0 : [a, b] \times \mathbb{R}_{0+}^n \to \mathbb{R}_+$ is an integrable in the first argument and nonincreasing and continuous in the last n arguments function. Moreover, p_i

and q_i $(i = 1, \dots, n)$ satisfy the conditions

$$\sigma_i\left(\alpha_i \exp\left(\int_a^b p_i(s)\,ds\right) - 1\right) < 0 \quad (i = 1, \dots, n), \tag{7}$$

$$\sigma_i\left(\beta_i \exp\left(\int_a^b p_i(s)\,ds\right) - 1\right) < 0 \quad (i = 1, \dots, n), \tag{8}$$

$$\int_a^b q_i(t, x)\,dt > 0 \quad \text{for} \quad x > 0 \quad (i = 1, \dots, n). \tag{9}$$

Along with $(1), (2)$ we consider the auxiliary problem

$$\frac{du_i}{dt} = (1 - \lambda)\big(p_i(t)u_i + \sigma_i q_i(t, u_i)\big) + \\ + \lambda f_i(t, u_1, \dots, u_n) + \sigma_i\varepsilon \quad (i = 1, \dots, n), \tag{10}$$

$$u_i(a) = (1 - \lambda)\alpha_i u_i(b) + \lambda\varphi_i(u_i(b)) \quad (i = 1, \dots, n), \tag{11}$$

depending on the parameters $\lambda > 0$ and $\varepsilon > 0$.

Theorem 1 (**Principle of a priori boundedness**). *Let the inequalities* (3)*,* (4)*,* (7)*, and* (9) *be fulfilled and let there exist positive constants* ε_0 *and* ρ *such that for arbitrary* $\lambda \in [0, 1]$ *and* $\varepsilon \in]0, \varepsilon_0]$ *every positive solution* $(u_i)_{i=1}^n$ *of the problem* $(10), (11)$ *admits the estimates*

$$u_i(t) < \rho \quad (i = 1, \dots, n).$$

Then the problem $(1), (2)$ *has at least one positive solution.*

By $X = (x_{ik})_{i,k=1}^n$ and $r(X)$ we denote the $n \times n$ matrix with components $x_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, n)$ and the spectral radius of the matrix X, respectively. For any integrable function $p : [a, b] \to \mathbb{R}$ and positive number β satisfying the condition

$$\Delta(p, \beta) = 1 - \beta \exp\left(\int_a^b p(s)\,ds\right) \neq 0,$$

we put

$$g(p, \beta)(t, s) = \\ = \begin{cases} \dfrac{1}{\Delta(p, \beta)} \exp\left(\displaystyle\int_s^t p(\tau)\,d\tau\right) & \text{for} \quad a \le s \le t \le b, \\ \dfrac{\beta}{\Delta(p, \beta)} \exp\left(\displaystyle\int_a^b p(\tau)\,d\tau + \int_s^t p(\tau)\,d\tau\right) & \text{for} \quad a \le t < s \le b \end{cases}$$

and

$$w(p,\beta)(t) = \frac{1}{\Delta(p,\beta)}\left[(1-\beta)\exp\left(\int\limits_a^t p(s)\,ds\right) + \beta\exp\left(\int\limits_a^b p(s)\,ds\right) - 1\right].$$

Theorem 2. *Let the inequalities* (5), (6), (8), *and* (9) *be fulfilled and let there exist continuous functions* $\ell_i : [a,b] \to \mathbb{R}_{0+}$ $(i = 1,\dots,n)$ *such that*

$$\lim_{x\to+\infty} r(H(x)) < 1, \tag{12}$$

where $H(x) = (h_{ik}(x))_{i,k=1}^n$ *and*

$$h_{ik}(x) =$$

$$= \max\left\{\frac{1}{\ell_i(t)}\int\limits_a^b \big|g(p_i,\beta_i)(t,s)\big| p_{ik}(s,x)\ell_k(s)\,ds :\ a \le t \le b\right\}\ \ (i,k=1,\dots,n).$$

Then the problem (1), (2) *has at least one positive solution.*

This theorem can be proved on the basis of Theorem 1 and Theorems 2.1, 2.2 and 3.1 of [3].

Now we pass to the case, where

$$\sigma_i p_i(t) \le 0 \ \text{ for } \ a \le t \le b, \quad \sigma_i \int\limits_a^b p_i(t)\,dt < 0 \ \ (i = 1,\dots,n)$$

and the inequalities (5) have the form

$$q_i(t,x_i) \le \sigma_i\big(f_i(t,x_1,\dots,x_n) - p_i(t)x_i\big) \le$$

$$\le |p_i(t)| \sum_{k=1}^n \frac{h_{ik}(x_1+\cdots+x_n)}{|w(p_k,\beta_k)(t)|}\, x_k + q_0(t,x_1,\dots,x_n) \ \ (i = 1,\dots,n), \tag{13}$$

where $h_{ik} : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ $(i,k = 1,\dots,n)$ are continuous nonincreasing functions, and σ_i, q_i $(i = 1,\dots,n)$ and q_0 are the numbers and functions satisfying the above conditions.

>From Theorem 2 it follows the following corollary.

Corollary 1. *If along with* (6), (8) *and* (13) *the inequality* (12) *is fulfilled, where* $H(x) = (h_{ik}(x))_{i,k=1}^n$, *then the problem* (1), (2) *has at least one positive solution.*

As an example, we consider the problems

$$\frac{du_i}{dt} = \sigma_i\Big(\sum_{k=1}^n p_{ik}u_k + f_{0i}(t,u_1,\dots,u_n)\Big) \ \ (i = 1,\dots,n), \tag{14}$$

$$u_i(a) = u_i(b) \ \ (i = 1,\dots,n), \tag{15}$$

and

$$\frac{du_i}{dt} = \sigma_i \sum_{k=1}^{n} \frac{|1-\beta_k| h_{ik}}{(1-\beta_k)(t-a) + \beta_k(b-a)}\, u_k + \\ + \sigma_i f_{0i}(t, u_1, \dots, u_n) \ \ (i = 1, \dots, n), \tag{16}$$

$$u_i(a) = \beta_i u_i(b) \ \ (i = 1, \dots, n), \tag{17}$$

where $\sigma_i \in \{-1, 1\}$ $(i = 1, \dots, n)$, p_{ik} $(i, k = 1, \dots, n)$ and β_i $(i = 1, \dots, n)$ are the constants satisfying the inequalities

$$p_{ii} < 0, \ \ p_{ik} \geq 0 \ \ (i \neq k; \ \ i, k = 1, \dots, n), \tag{18}$$

$$\beta_i > 0, \ \ \sigma_i(\beta_i - 1) < 0 \ \ (i = 1, \dots, n), \tag{19}$$

h_{ik} $(i, k = 1, \dots, n)$ are nonnegative constants and $f_{0i} : [a, b] \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are functions satisfying the local Carathédoty conditions. Moreover, on the set $[a, b] \times \mathbb{R}^n_{0+}$ the inequalities

$$q_i(t, x_i) \leq f_{0i}(t, x_1, \dots, x_n) \leq q_0(t, x_1, \dots, x_n) \ \ (i = 1, \dots, n)$$

are fulfilled, where $q_0 : [a, b] \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ is an integrable in the first argument and nonincreasing and continuous in the last n arguments function, and $q_i : [a, b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are integrable in the first argument and nonincreasing in the second argument functions satisfying the conditions (9).

Corollary 2. *For the existence of at least one positive solution of the problem* (14), (15) *it is necessary and sufficient that real parts of the eigenvalues of the matrix* $(p_{ik})^n_{i,k=1}$ *be negative.*

Corollary 3. *For the existence of at least one positive solution of the problem* (16), (17) *it is necessary and sufficient that the matrix* $H = (h_{ik})^n_{i,k=1}$ *satisfy the inequality*

$$r(H) < 1. \tag{20}$$

Remark 1. In the conditions of Corollaries 2 and 3 the functions f_{0i} $(i = 1, \dots, n)$ may have singularities of arbitrary order in the least n arguments. For example, in (14) and (16) we may assume that

$$f_{0i}(t, x_1, \dots, x_n) = \sum_{k=1}^{n} \big(q_{1ik}(t) x_k^{-\mu_{1ik}} + q_{2ik} \exp(x_k^{-\mu_{2ik}})\big) \ \ (i = 1, \dots, n),$$

where μ_{1ik}, μ_{2ik} $(i, k = 1, \dots, n)$ are positive constants and $q_{1ik} : [a, b] \to \mathbb{R}_+$, $q_{2ik} : [a, b] \to \mathbb{R}_+$ $(i, k = 1, \dots, n)$ are integrable functions such that

$$\int_a^b \big(q_{1ii}(t) + q_{2ii}(t)\big)\, dt > 0 \ \ (i = 1, \dots, n).$$

The uniqueness of a positive solution of the problem (1), (2) can be proved only in the case where each function f_i has the singularity in the i-th phase variable only. More precisely, we consider the case when the system (1) has the following form

$$\frac{du_i}{dt} = p_i(t)u_i + \sigma_i\big(f_{0i}(t,u_1,\dots,u_n) + q_i(t,u_i)\big) \quad (i=1,\dots,n). \tag{21}$$

The particular cases of (21) are the differential systems

$$\frac{du_i}{dt} = \sigma_i\bigg(\sum_{k=1}^{n} p_{ik}u_k + q_i(t,u_i)\bigg) \quad (i=1,\dots,n) \tag{22}$$

and

$$\frac{du_i}{dt} = \sigma_i\bigg(\sum_{k=1}^{n} \frac{|1-\beta_k|h_{ik}}{(1-\beta_k)(t-a)+\beta_k(b-a)}\, u_k + q_i(t,u_i)\bigg) \quad (i=1,\dots,n). \tag{23}$$

Here $\sigma_i \in \{-1,1\}$ $(i=1,\dots,n)$, $p_i : [a,b] \to \mathbb{R}$ $(i=1,\dots,n)$ are integrable functions, $f_{0i} : [a,b] \times \mathbb{R}_+^n \to \mathbb{R}_+$ $(i=1,\dots,n)$ are functions satisfying the local Carathéodoty conditions, and $q_i : [a,b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i=1,\dots,n)$ are integrable in the first argument and nonincreasing and continuous in the second argument functions. Moreover, p_i and q_i $(i=1,\dots,n)$ satisfy the conditions (8) and (9). As for p_{ik} and β_i $(i,k=1,\dots,n)$, they are the constants satisfying the inequalities (18) and (19), and h_{ik} $(i,k=1,\dots,n)$ are nonnegative constants.

Theorem 3. *Let on the sets $[a,b] \times \mathbb{R}_+^n$ and $\mathbb{R}_+$ the conditions*

$$\sigma_i\big(f_{0i}(t,x_1,\dots,x_n) - f_{0i}(t,y_1,\dots,y_n)\big)\operatorname{sgn}(x_i-y_i) \le$$
$$\le \sum_{k=1}^{n} p_{ik}(t)|x_k-y_k| \quad (i=1,\dots,n)$$

and

$$\sigma_i(\varphi_i(x)-\alpha_i x) \ge 0, \quad \sigma_i\Big[(\varphi_i(x)-\varphi_i(y))\operatorname{sgn}(x-y) - \beta_i|x-y|\Big] \le 0$$
$$(i=1,\dots,n)$$

hold, where $p_{ik} : [a,b] \to \mathbb{R}_+$ $(i,k=1,\cdots,n)$ are integrable functions. Let, moreover, there exist continuous functions $\ell_i : [a,b] \to \mathbb{R}_{0+}$ $(i=1,\dots,n)$ such that the matrix $H = (h_{ik})_{i,k=1}^n$ with the components

$$h_{ik} = \max\bigg\{\frac{1}{\ell_i(t)}\int_a^b \big|g(p_i,\beta_i)(t,s)\big|p_{ik}(s)\ell_k(s)\,ds : \ a \le t \le b\bigg\} \ (i,k=1,\dots,n)$$

satisfies the inequality (20). *Then the problem* (21), (2) *has a unique positive solution.*

Theorem 3 results in the following corollaries.

Corollary 4. *For the existence of a unique positive solution of the problem* (22), (15) *it is necessary and sufficient that real parts of eigenvalues of the matrix $(p_{ik})_{i,k=1}^n$ be negative.*

Corollary 5. *For the existence of a unique positive solution of the problem* (23), (17) *it is necessary and sufficient that the matrix* $H = (h_{ik})_{i,k=1}^n$ *satisfy the inequality* (20).

Remark 2. In the conditions of Theorem 3 and its corollaries, the functions q_i $(i = 1, \dots, n)$ may have singularities of arbitrary order in the second argument. For example, in (21), (22) and (23) we may assume that

$$q_i(t, x) = q_{i1}(t)x^{-\mu_{i1}} + q_{i2}(t)\exp(x^{-\mu_{i2}}) \quad (i = 1, \dots, n),$$

where $\mu_{i1} > 0$, $\mu_{i2} > 0$ $(i = 1, \dots, n)$, and $q_{ik} : [a, b] \to \mathbb{R}_+$ $(i = 1, \dots, n;$ $k = 1, 2)$ are integrable functions such that

$$\int\limits_a^b \big(q_{i1}(t) + q_{i2}(t)\big)\,dt > 0 \quad (i = 1, \dots, n).$$

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Author's address:

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia.

E-mail: kig@rmi.ge

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