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Tamaz Tadumadze and Nika Gorgodze

VARIATION FORMULAS OF A SOLUTION AND INITIAL DATA OPTIMIZATION PROBLEMS FOR QUASI-LINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS INITIAL CONDITION

Dedicated to the 125th birthday anniversary of Professor A. Razmadze

Abstract. For the quasi-linear neutral functional differential equation the continuous dependence of a solution of the Cauchy problem on the initial data and on the nonlinear term in the right-hand side of that equation is investigated, where the perturbation nonlinear term in the right-hand side and initial data are small in the integral and standard sense, respectively. Variation formulas of a solution are derived, in which the effect of perturbations of the initial moment and the delay function, and also that of the discontinuous initial condition are detected. For initial data optimization problems the necessary conditions of optimality are obtained. The existence theorem for optimal initial data is proved.

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Key words and phrases. Quasi-linear neutral functional differential equation, continuous dependence of solution, variation formula of solution, effect of initial moment perturbation, effect of a discontinuous initial condition, effect of delay function perturbation, initial data optimization problem, necessary conditions of optimality, existence of optimal initial data.

რეზიუმე. კვაზიწრფივი ნეიტრალური ფუნქციონალურ-დიფერენციალური განტოლებებისათვის გამოკვლეულია კოშის ამოცანის ამონახსნის უწყვეტობა საწყისი მონაცემების და განტოლების მარჯვენა მხარის არაწრფივი შესაკრების შეშფოთებების მიმართ, სადაც მარჯვენა მხარის არაწრფივი შესაკრების და საწყისი მონაცემების შეშფოთებები, შესაბამისად, მცირეა ინტეგრალური და სტანდარტული აზრით. დამტკიცებულია ამონახსნის ვარიაციის ფორმულები, რომლებშიც გამოვლენილია საწყისი მომენტის და დაგვიანების ფუნქციის შეშფოთებების ეფექტები, წყვეტილი საწყისი პირობის ეფექტი. საწყისი მონაცემების ოპტიმიზაციის ამოცანებისთვის მიღებულია ოპტიმალურობის აუცილებელი პირობები. დამტკიცებულია ოპტიმალური საწყისი მონაცემების არსებობის თეორემა.

Introduction

Neutral functional differential equation (briefly-neutral equation) is a mathematical model of such dynamical system whose behavior depends on the prehistory of the state of the system and on its velocity (derivative of trajectory) at a given moment of time. Such mathematical models arise in different areas of natural sciences as electrodynamics, economics, etc. (see e.g. [1, 2, 4–6, 12, 13, 16]). To illustrate this, we consider a simple model of economic growth. Let $N(t)$ be a quantity of a product produced at the moment t which is expressed in money units. The fundamental principle of the economic growth has the form

$$N(t) = C(t) + I(t), \tag{0.1}$$

where $C(t)$ is the so-called an apply function and $I(t)$ is a quantity of induced investment. We consider the case where the functions $C(t)$ and $I(t)$ are of the form

$$C(t) = \alpha N(t), \ \ \alpha \in (0, 1), \tag{0.2}$$

and

$$I(t) = \alpha_1 N(t-\theta) + \alpha_2 \dot{N}(t) + \alpha_3 \dot{N}(t-\theta) + \alpha_0 \ddot{N}(t) + \alpha_4 \ddot{N}(t-\theta), \ \ \theta > 0. \tag{0.3}$$

From formulas (0.1)–(0.3) we get the equation

$$\ddot{N}(t) = \frac{1-\alpha}{\alpha_0}\, N(t) - \frac{\alpha_1}{\alpha_0}\, N(t-\theta) - \frac{\alpha_2}{\alpha_0}\, \dot{N}(t) - \frac{\alpha_3}{\alpha_0}\, \dot{N}(t-\theta) - \frac{\alpha_4}{\alpha_0}\, \ddot{N}(t-\theta)$$

which is equivalent to the following neutral equation:

$$\begin{cases} \dot{x}^1(t) = x^2(t), \\ \dot{x}^2(t) = \dfrac{1-\alpha}{\alpha_0}\, x^1(t) - \dfrac{\alpha_1}{\alpha_0}\, x^1(t-\theta) - \dfrac{\alpha_2}{\alpha_0}\, x^2(t) - \\ \qquad - \dfrac{\alpha_3}{\alpha_0}\, x^2(t-\theta) - \dfrac{\alpha_4}{\alpha_0}\, \dot{x}^2(t-\theta), \end{cases}$$

here $x^1(t) = N(t)$.

Many works are devoted to the investigation of neutral equations, including [1–7, 12–14, 17, 19, 25, 28].

We note that the Cauchy problem for the nonlinear with respect to the prehistory of velocity neutral equations is, in general, ill-posed when perturbation of the right-hand side of equation is small in the integral sense. Indeed, on the interval $[0, 2]$ we consider the system

$$\begin{cases} \dot{x}^1(t) = 0, \\ \dot{x}^2(t) = \big[x^1(t-1)\big]^2 \end{cases} \tag{0.4}$$

with the initial condition

$$\dot{x}^1(t) = 0, \ \ t \in [-1, 0), \quad x^1(0) = x^2(0) = 0. \tag{0.5}$$

The solution of the system (0.4) is

$$x_0^1(t) = x_0^2(t) \equiv 0.$$

We now consider the perturbed system

$$\begin{cases} \dot{x}_k^1(t) = p_k(t), \\ \dot{x}_k^2(t) = \big[x_k^1(t-1)\big]^2 \end{cases}$$

with the initial condition (0.5). Here,

$$p_k(t) = \begin{cases} \varsigma_k(t), & t \in [0,1], \\ 0, & t \in (1,2]. \end{cases}$$

The function $\varsigma_k(t)$ is defined as follows: for the given $k = 2, 3, \dots$, we divide the interval $[0,1]$ into the subintervals l_i, $i = 1, \dots, k$, of the length $1/k$; then we define $\varsigma_k(t) = 1$, $t \in l_1$, $\varsigma_k(t) = -1$, $t \in l_2$ and so on. It is easy to see that

$$\lim_{k\to\infty} \max_{s_1,s_2\in[0,1]} \left| \int\limits_{s_1}^{s_2} \varsigma_k(t)\,dt \right| = 0.$$

Taking into consideration the initial condition (0.5) and the structure of the function $\varsigma_k(t)$, we get

$$x_k^1(t) = \int\limits_0^t \varsigma_k(s)\,ds \ \text{ for } \ t \in [0,1], \quad x_k^1(t) = x_k^1(1) \ \text{ for } \ t \in (1,2]$$

and

$$x_k^2(t) = \int\limits_0^t \big[\dot{x}_k^1(s-1)\big]^2\,ds = 0 \ \text{ for } \ t \in [0,1],$$

$$x_k^2(t) = \int\limits_1^t \big[\dot{x}_k^1(s-1)\big]^2\,ds = \int\limits_1^t \varsigma_k^2(s-1)\,ds =$$

$$= \int\limits_1^t 1\,ds = t-1 \ \text{ for } \ t \in (1,2].$$

It is clear that

$$\lim_{k\to\infty} \max_{t\in[0,2]} \big|x_k^1(t) - x_0^1(t)\big| = 0, \quad \lim_{k\to\infty} \max_{t\in[0,2]} \big|x_k^2(t) - x_0^2(t)\big| \neq 0.$$

Thus, the Cauchy problem (0.4)–(0.5) is ill-posed.

The present work consists of two parts, naturally interconnected in their meaning.

Part I concerns the following quasi-linear neutral equation:

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f\big(t, x(t), x(\tau(t))\big) \tag{0.6}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \ \ \dot{x}(t) = v(t), \ \ t < t_0, \ \ x(t_0) = x_0. \tag{0.7}$$

We note that the symbol $\dot{x}(t)$ for $t < t_0$ is not connected with the derivative of the function $\varphi(t)$. The condition (0.7) is called the discontinuous initial condition, since, in general, $x(t_0) \neq \varphi(t_0)$.

In the same part we study the continuous dependence of a solution of the problem (0.6)–(0.7) on the initial data and on the nonlinear term in the right-hand side of the equation (0.6). Here, under initial data we mean the collection of an initial moment, delay function appearing in the phase coordinates, initial vector and initial functions. Moreover, we derive variation formulas of a solution.

In Part II we consider the control neutral equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f\big(t, x(t), x\big(\tau(t), u(t)\big)\big)$$

with the initial condition (0.7). Here under initial data we understand the collection of the initial moment t_0, delay function $\tau(t)$, initial vector x_0, initial functions $\varphi(t)$ and $v(t)$, and the control function $u(t)$. In the same part, the continuous dependence of a solution and variation formulas are used in proving both the necessary optimality conditions for the initial data optimization problem and the existence of optimal initial data.

In Section 1 we prove the theorem on the continuous dependence of a solution in the case where the perturbation of f is small in the integral sense and initial data are small in the standard sense. Analogous theorems without perturbation of a delay function are given [17, 28] for quasi-linear neutral equations. Theorems on the continuous dependence of a solution of the Cauchy and boundary value problems for various classes of ordinary differential equations and delay functional differential equations when perturbations of the right-hand side are small in the integral sense are given in [10, 11, 15, 18, 20, 21, 23, 26].

In Section 2 we prove derive variation formulas which show the effect of perturbations of the initial moment and the delay function appearing in the phase coordinates and also that of the discontinuous initial condition. Variation formulas for various classes of neutral equations without perturbation of delay can be found in [16, 24]. The variation formula of a solution plays the basic role in proving the necessary conditions of optimality [11, 15] and in sensitivity analysis of mathematical models [1,2,22]. Moreover, the variation formula allows one to obtain an approximate solution of the perturbed equation.

In Section 3 we consider initial data optimization problem with a general functional and under the boundary conditions. The necessary conditions are obtained for: the initial moment in the form of inequalities and equalities, the initial vector in the form of equality, and the initial functions and control function in the form of linearized integral maximum principle.

Finally, in Section 4 the existence theorem for an optimal initial data is proved.

1. Continuous Dependence of a Solution

1.1. **Formulation of main results.** Let $I=[a,b]$ be a finite interval and $\mathbb{R}^n$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$, where T is the sign of transposition. Suppose that $O\subset\mathbb{R}^n$ is an open set and let E_f be the set of functions $f:I\times O^2\to\mathbb{R}^n$ satisfying the following conditions: for each fixed $(x_1,x_2)\in O^2$ the function $f(\,\cdot\,,x_1,x_2):I\to\mathbb{R}^n$ is measurable; for each $f\in E_f$ and compact set $K\subset O$ there exist the functions $m_{f,K}(t),L_{f,K}(t)\in L(I,\mathbb{R}_+)$, where $\mathbb{R}_+=[0,\infty)$, such that for almost all $t\in I$

$$|f(t,x_1,x_2)|\le m_{f,K}(t),\ \ \forall\,(x_1,x_2)\in K^2,$$
$$\big|f(t,x_1,x_2)-f(t,y_1,y_2)\big|\le$$
$$\le L_{f,K}(t)\sum_{i=1}^{2}|x_i-y_i|,\ \ \forall\,(x_1,x_2)\in K^2,\ \ \forall\,(y_1,y_2)\in K^2.$$

We introduce the topology in E_f by the following basis of neighborhoods of zero:

$$\Big\{V_{K,\delta}:\ K\subset O \text{ is a compact set and } \delta>0 \text{ is an arbitrary number}\Big\},$$

where

$$V_{K,\delta}=\big\{\delta f\in E_f:\ \Delta(\delta f;K)\le\delta\big\},$$
$$\Delta(\delta f;K)=\sup\bigg\{\bigg|\int\limits_{t'}^{t''}\delta f(t,x_1,x_2)\,dt\bigg|:\ t',t''\in I,\ x_i\in K,\ i=1,2\bigg\}.$$

Let D be the set of continuously differentiable scalar functions (delay functions) $\tau(t)$, $t\in\mathbb{R}$, satisfying the conditions

$$\tau(t)<t,\ \ \dot\tau(t)>0,\ \ t\in\mathbb{R};\quad \inf\big\{\tau(a):\ \tau\in D\big\}:=\widehat\tau>-\infty,$$
$$\sup\big\{\tau^{-1}(b):\ \tau\in D\big\}:=\widehat\gamma<+\infty,$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$.

Let E_φ be the space of bounded piecewise-continuous functions $\varphi(t)\in\mathbb{R}^n$, $t\in I_1=[\widehat\tau,b]$, with finitely many discontinuities, equipped with the norm $\|\varphi\|_{I_1}=\sup\{|\varphi(t)|:\ t\in I_1\}$. By $\Phi_1=\{\varphi\in E_\varphi:\ \operatorname{cl}\varphi(I_1)\subset O\}$ we denote the set of initial functions of trajectories, where $\varphi(I_1)=\{\varphi(t):\ t\in I_1\}$; by E_v we denote the set of bounded measurable functions $v:I_1\to\mathbb{R}^n$, $v(t)$ is called the initial function of trajectory derivative.

By μ we denote the collection of initial data $(t_0,\tau,x_0,\varphi,v)\in[a,b)\times D\times O\times\Phi_1\times E_v$ and the function $f\in E_f$.

To each element $\mu=(t_0,\tau,x_0,\varphi,v,f)\in\Lambda=[a,b)\times D\times O\times\Phi_1\times E_v\times E_f$ we assign the quasi-linear neutral equation

$$\dot x(t)=A(t)\dot x(\sigma(t))+f\big(t,x(t),x(\tau(t))\big) \tag{1.1}$$

with the initial condition

$$x(t) = \varphi(t), \ \ \dot{x}(t) = v(t), \ \ t \in [\widehat{\tau}, t_0), \ \ x(t_0) = x_0. \tag{1.2}$$

Here $A(t)$ is a given continuous $n \times n$ matrix function and $\sigma \in D$ is a fixed delay function in the phase velocity. We note that the symbol $\dot{x}(t)$ for $t < t_0$ is not connected with a derivative of the function $\varphi(t)$. The condition (1.2) is called the discontinuous initial condition, since $x(t_0) \neq \varphi(t_0)$, in general.

Definition 1.1. Let $\mu = (t_0, \tau, x_0, \varphi, v, f) \in \Lambda$. A function $x(t) = x(t; \mu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is called a solution of the equation (1.1) with the initial condition (1.2) or a solution corresponding to the element μ and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies the condition (1.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (1.1) almost everywhere (a.e.) on $[t_0, t_1]$.

To formulate the main results, we introduce the following sets:

$$W(K;\alpha) = \Big\{ \delta f \in E_f : \ \exists\, m_{\delta f,K}(t), L_{\delta f,K}(t) \in L(I, \mathbb{R}_+), \int\limits_I \big[m_{\delta f,K}(t) + L_{\delta f,K}(t)\big]\, dt \le \alpha \Big\},$$

where $K \subset O$ is a compact set and $\alpha > 0$ is a fixed number independent of δf;

$$\begin{aligned} B(t_{00};\delta) &= \{t_0 \in I : |t_0 - t_{00}| < \delta\}, \\ B_1(x_{00};\delta) &= \{x_0 \in O : |x_0 - x_{00}| < \delta\}, \\ V(\tau_0;\delta) &= \{\tau \in D : ||\tau - \tau_0||_{I_2} < \delta\}, \\ V_1(\varphi_0;\delta) &= \{\varphi \in \Phi_1 : ||\varphi - \varphi_0||_{I_1} < \delta\}, \\ V_2(v_0;\delta) &= \{v \in E_v : ||v - v_0||_{I_1} < \delta\}, \end{aligned}$$

where $t_{00} \in [a, b)$ and $x_{00} \in O$ are the fixed points; $\tau_0 \in D$, $\varphi_0 \in \Phi_1$, $v_0 \in E_v$ are the fixed functions, $\delta > 0$ is the fixed number, $I_2 = [a, \widehat{\gamma}]$.

Theorem 1.1. *Let $x_0(t)$ be a solution corresponding to $\mu_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, v_0, f_0) \in \Lambda$, $t_{10} < b$, and defined on $[\widehat{\tau}, t_{10}]$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $\operatorname{cl}\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following assertions hold:*

1.1. *there exist numbers $\delta_i > 0$, $i = 0, 1$ such that to each element*

$$\begin{aligned} \mu = (t_0, \tau, x_0, \varphi, v, f_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha) = \\ = B(t_{00};\delta_0) \times V(\tau_0;\delta_0) \times B_1(x_{00};\delta_0) \times V_1(\varphi_0;\delta_0) \times V_2(v_0;\delta_0) \times \\ \times \big[f_0 + W(K_1;\alpha) \cap V_{K_1,\delta_0}\big] \end{aligned}$$

there corresponds the solution $x(t;\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ and satisfying the condition $x(t;\mu) \in K_1$;

1.2. *for an arbitrary* $\varepsilon > 0$ *there exists a number* $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0]$ *such that the following inequality holds for any* $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$:

$$\big|x(t;\mu) - x(t;\mu_0)\big| \le \varepsilon, \ \ \forall\, t \in [\widehat{t}, t_{10} + \delta_1], \ \ \widehat{t} = \max\{t_{00}, t_0\};$$

1.3. *for an arbitrary* $\varepsilon > 0$ *there exists a number* $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0]$ *such that the following inequality holds for any* $\mu \in V(\mu_0; K_1, \delta_3, \alpha)$:

$$\int\limits_{\widehat{\tau}}^{t_{10}+\delta_1} |x(t;\mu) - x(t;\mu_0)|\, dt \le \varepsilon.$$

Due to the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$.

In the space $E_\mu - \mu_0$, where $E_\mu = \mathbb{R} \times D \times \mathbb{R}^n \times E_\varphi \times E_v \times E_f$, we introduce the set of variations:

$$\Im = \Big\{\delta\mu = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi, \delta v, \delta f) \in E_\mu - \mu_0 : \ |\delta t_0| \le \beta, \ \|\delta\tau\|_{I_2} \le \beta,$$
$$|\delta x_0| \le \beta, \ \|\delta\varphi\|_{I_1} \le \beta, \ \|\delta v\|_{I_1} \le \beta, \ \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i,$$
$$|\lambda_i| \le \beta, \ i = 1, \dots, k\Big\},$$

where $\beta > 0$ is a fixed number and $\delta f_i \in E_f - f_0$, $i = i = 1, \dots, k$, are fixed functions.

Theorem 1.2. *Let* $x_0(t)$ *be a solution corresponding to* $\mu_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, v_0, f_0) \in \Lambda$ *and defined on* $[\widehat{\tau}, t_{10}]$, $t_{i0} \in (a, b)$, $i = 0, 1$. *Let* $K_1 \subset O$ *be a compact set containing a certain neighborhood of the set* $\operatorname{cl}\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. *Then the following conditions hold:*

1.4. *there exist the numbers* $\varepsilon_1 > 0$, $\delta_1 > 0$ *such that for an arbitrary* $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times \Im$ *we have* $\mu_0 + \varepsilon\delta\mu \in \Lambda$ *and the solution* $x(t; \mu_0 + \varepsilon\delta\mu)$ *defined on the interval* $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ *corresponds to that element. Moreover,* $x(t; \mu_0 + \varepsilon\delta\mu) \in K_1$*;*

1.5. $\lim\limits_{\varepsilon\to 0} \sup \Big\{|x(t; \mu_0 + \varepsilon\delta\mu) - x(t;\mu_0)| : \ t \in [\widehat{t}, t_{10} + \delta_1]\Big\} = 0,$

$$\lim_{\varepsilon\to 0} \int\limits_{\widehat{\tau}}^{t_{10}+\delta_1} \big|x(t; \mu_0 + \varepsilon\delta\mu) - x(t;\mu_0)\big|\, dt = 0$$

uniformly for $\delta\mu \in \Im$, *where* $\widehat{t} = \max\{t_0, t_0 + \varepsilon\delta t_0\}$.

Theorem 1.2 is the corollary of Theorem 1.1.

Let E_u be the space of bounded measurable functions $u(t) \in \mathbb{R}^r$, $t \in I$. Let $U_0 \subset \mathbb{R}^r$ be an open set and $\Omega = \{u \in E_u : \ \operatorname{cl} u(I) \subset U_0\}$. Let Φ_{11} be the set of bounded measurable functions $\varphi(t) \in O$, $t \in I_1$, with $\operatorname{cl}\varphi(I_1) \subset O$.

To each element $w = (t_0, \tau, x_0, \varphi, v, u) \in \Lambda_1 = [a, b) \times D \times O \times \Phi_{11} \times E_v \times \Omega$ we assign the controlled neutral equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f\big(t, x(t), x(\tau(t)), u(t)\big) \tag{1.3}$$

with the initial condition (1.2). Here, the function $f(t, x_1, x_2, u)$ is defined on $I \times O^2 \times U_0$ and satisfies the following conditions: for each fixed $(x_1, x_2, u) \in O^2 \times U_0$, the function $f(\,\cdot\,, x_1, x_2, u) : I \to \mathbb{R}^n$ is measurable; for each compact sets $K \subset O$ and $U \subset U_0$ there exist the functions $m_{K,U}(t), L_{K,U}(t) \in L(I, R_+)$ such that for almost all $t \in I$,

$$\big|f(t, x_1, x_2, u)\big| \le m_{K,U}(t), \ \ \forall\,(x_1, x_2, u) \in K^2 \times U,$$

$$\big|f(t, x_1, x_2, u_1) - f(t, y_1, y_2, u_2)\big| \le L_{f,K}(t)\Big[\sum_{i=1}^{2} |x_i - y_i| + |u_1 - u_2|\Big],$$
$$\forall\,(x_1, x_2) \in K^2, \ \ \forall\,(y_1, y_2) \in K^2, \ \ (u_1, u_2) \in U^2.$$

Definition 1.2. Let $w = (t_0, \tau, x_0, \varphi, v, u) \in \Lambda_1$. A function $x(t) = x(t; w) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is called a solution of the equation (1.3) with the initial condition (1.2), or a solution corresponding to the element w and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies the condition (1.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (1.3) a. e. on $[t_0, t_1]$.

Theorem 1.3. *Let $x_0(t)$ be a solution corresponding to $w_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, v_0, u_0) \in \Lambda_1$ and defined on $[\widehat{\tau}, t_{10}]$, $t_{10} < b$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $\operatorname{cl}\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following conditions hold:*

1.6. *there exist the numbers $\delta_i > 0, i = 0, 1$ such that to each element*

$$w = (t_0, \tau, x_0, \varphi, v, u) \in \widehat{V}(w_0; \delta_0) =$$
$$= B(t_{00}; \delta_0) \times V(\tau_0; \delta_0) \times B_1(x_{00}; \delta_0) \times V_1(\varphi_0; \delta_0) \times V_2(v_0; \delta_0) \times V_3(u_0; \delta_0)$$

there corresponds the solution $x(t; w)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ and satisfying the condition $x(t; w) \in K_1$, where $V_3(u_0; \delta_0) = \{u \in \Omega : \ \|u - u_0\|_I < \delta_0\}$;

1.7. *for an arbitrary $\varepsilon > 0$, there exists the number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $w \in \widehat{V}(w_0; \delta_2)$:*

$$\big|x(t; w) - x(t; w_0)\big| \le \varepsilon, \ \ \forall\, t \in [\widehat{t}, t_{10} + \delta_1], \quad \widehat{t} = \max\{t_0, t_{00}\};$$

1.8. *for an arbitrary $\varepsilon > 0$, there exists the number $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $w \in \widehat{V}(w_0; \delta_3)$:*

$$\int_{\widehat{\tau}}^{t_{10}+\delta_1} \big|x(t; w) - x(t; w_0)\big|\,dt \le \varepsilon.$$

In the space $E_w - w_0$, where $E_w = \mathbb{R} \times D \times \mathbb{R}^n \times \Phi_{11} \times E_v \times E_u$, we introduce the set of variations

$$\Im_1 = \Big\{ \delta w = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi, \delta v, \delta u) \in E_w - w_0 : \ |\delta t_0| \le \beta, \ \|\delta\tau\|_{I_2} \le \beta, \\ |\delta x_0| \le \beta, \ \|\delta\varphi\|_{I_1} \le \beta, \ \|\delta v\|_{I_1} \le \beta, \ \|\delta u\|_{I} \le \beta \Big\}.$$

Theorem 1.4. *Let $x_0(t)$ be a solution corresponding to $w_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, v_0, u_0) \in \Lambda_1$ and defined on $[\widehat{\tau}, t_{10}]$, $t_{i0} \in (a, b)$, $i = 0, 1$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $\operatorname{cl}\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following conditions hold:*

1.9. *there exist numbers $\varepsilon_1 > 0$, $\delta_1 > 0$ such that for an arbitrary $(\varepsilon, \delta w) \in [0, \varepsilon_1] \times \Im_1$ we have $w_0 + \varepsilon\delta w \in \Lambda_1$, and the solution $x(t; w_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to that element. Moreover, $x(t; w_0 + \varepsilon\delta w) \in K_1$;*

1.10. $\lim\limits_{\varepsilon\to 0} \sup \Big\{ |x(t; w_0 + \varepsilon\delta w) - x(t; w_0)| : \ t \in [\widehat{t}, t_{10} + \delta_1] \Big\} = 0,$

$$\lim_{\varepsilon\to 0} \int\limits_{\widehat{\tau}}^{t_{10}+\delta_1} \big| x(t; w_0 + \varepsilon\delta w) - x(t; w_0) \big| \, dt = 0$$

uniformly for $\delta w \in \Im_1$.

Theorem 1.4 is the corollary of Theorem 1.3.

1.2. **Preliminaries.** Consider the linear neutral equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + B(t)x(t) + C(t)x(\tau(t)) + g(t), \ \ t \in [t_0, b], \tag{1.4}$$

with the initial condition

$$x(t) = \varphi(t), \ \ \dot{x}(t) = v(t), \ \ t \in [\widehat{\tau}, t_0), \ \ x(t_0) = x_0, \tag{1.5}$$

where $B(t)$, $C(t)$ and $g(t)$ are the integrable on I matrix- and vector-functions.

Theorem 1.5 (Cauchy formula)**.** *The solution of the problem* (1.4)–(1.5) *can be represented on the interval $[t_0, b]$ in the following form:*

$$x(t) = \Psi(t_0; t)x_0 + \int\limits_{\sigma(t_0)}^{t_0} Y(\nu(\xi); t) A(\nu(\xi)) \dot{\nu}(\xi) v(\xi) \, d\xi + \\ + \int\limits_{\tau(t_0)}^{t_0} Y(\gamma(\xi); t) C(\gamma(\xi)) \dot{\gamma}(\xi) \varphi(\xi) \, d\xi + \int\limits_{t_0}^{t} Y(\xi; t) g(\xi) \, d\xi, \tag{1.6}$$

where $\nu(t) = \sigma^{-1}(t)$, $\gamma(t) = \tau^{-1}(t)$; $\Psi(\xi; t)$ and $Y(\xi; t)$ are the matrix-functions satisfying the system

$$\begin{cases} \Psi_\xi(\xi; t) = -Y(\xi; t)B(\xi) - Y(\gamma(\xi); t)C(\gamma(\xi))\dot{\gamma}(\xi), \\ Y(\xi; t) = \Psi(\xi; t) + Y(\nu(\xi); t)A(\nu(\xi))\dot{\nu}(\xi) \end{cases} \tag{1.7}$$

on (a,t) for any fixed $t \in (a,b]$ and the condition

$$\Psi(\xi;t) = Y(\xi;t) = \begin{cases} H, & \xi = t, \\ \Theta, & \xi > t. \end{cases} \tag{1.8}$$

Here, H is the identity matrix and Θ is the zero matrix.

This theorem is proved in a standard way [3, 9, 15]. The existence of a unique solution of the system (1.7) with the initial condition (1.8) can be easily proved by using the step method from right to left.

Theorem 1.6. *Let q be the minimal natural number for which the inequality*

$$\sigma^{q+1}(b) = \sigma^q(\sigma(b)) < a$$

holds. Then for each fixed instant $t \in (t_0,b]$, the matrix function $Y(\xi;t)$ on the interval $[t_0,t]$ can be represented in the form

$$Y(\xi;t) = \Psi(\xi;t) + \sum_{i=1}^{q} \Psi(\nu^i(\xi);t) \prod_{m=i}^{1} A(\nu^m(\xi)) \frac{d}{d\xi}\nu^m(\xi). \tag{1.9}$$

Proof. It is easy to see that as a result of a multiple substitution of the corresponding expression for the matrix functions $Y(\xi;t)$, using the second equation of the system (1.7), we obtain

$$\begin{aligned} Y(\xi;t) &= \Psi(\xi;t) + \Big[\Psi(\nu(\xi);t) + Y(\nu^2(\xi);t)A(\nu^2(\xi))\dot\nu(\nu(\xi))\Big]A(\nu(\xi))\dot\nu(\xi) = \\ &= \Psi(\xi;t) + \Psi(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi) + Y(\nu^2(\xi);t)A(\nu^2(\xi))A(\nu(\xi))\,\frac{d}{d\xi}\,\nu^2(\xi) = \\ &= \Psi(\xi;t) + \Psi(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi) + \\ &+ \Big[\Psi(\nu^2(\xi);t) + Y(\nu^3(\xi);t)A(\nu^3(\xi))\dot\nu(\nu^2(\xi))\Big]A(\nu^2(\xi))A(\nu(\xi))\,\frac{d}{d\xi}\,\nu^2(\xi) = \\ &= \Psi(\xi;t) + \Psi(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi) + \Psi(\nu^2(\xi);t)A(\nu^2(\xi))A(\nu(\xi))\,\frac{d}{d\xi}\nu^2(\xi) + \\ &+ Y(\nu^3(\xi);t)A(\nu^3(\xi))A(\nu^2(\xi))A(\nu(\xi))\,\frac{d}{d\xi}\,\nu^3(\xi). \end{aligned}$$

Continuing this process and taking into account (1.8), we obtain (1.9). □

Theorem 1.7. *The solution $x(t)$ of the equation*

$$\dot x(t) = A(t)\dot x(\sigma(t)) + g(t), \quad t \in [t_0,b]$$

with the initial condition

$$\dot x(t) = v(t), \quad t \in [\widehat\tau, t_0), \quad x(t_0) = x_0,$$

on the interval $[t_0,b]$ can be represented in the form

$$x(t) = x_0 + \int\limits_{\sigma(t_0)}^{t_0} Y(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi)v(\xi)\,d\xi + \int\limits_{t_0}^{t} Y(\xi;t)g(\xi)\,d\xi, \tag{1.10}$$

where

$$Y(\xi;t)=\alpha(\xi;t)H+\sum_{i=1}^{q}\alpha(\nu^i(\xi);t)\prod_{m=i}^{1}A(\nu^m(\xi))\frac{d}{d\xi}\nu^m(\xi), \tag{1.11}$$

$$\alpha(\xi;t)=\begin{cases}1, & \xi<t,\\ 0, & \xi>t.\end{cases}$$

Proof. In the above-considered case, $B(t)=C(t)=\Theta$, therefore the first equation of the system (1.7) is of the form

$$\Psi_\xi(\xi;t)=0,\ \ \xi\in[t_0,t].$$

Hence, taking into account (1.8), we have $\Psi(\xi;t)=\alpha(\xi;t)H$. From (1.6) and (1.9), we obtain (1.10) and (1.11), respectively. □

Theorem 1.8. *Let the function $g:I\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ satisfy the following conditions: for each fixed $(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n$, the function $g(\,\cdot\,,x_1,x_2):I\to\mathbb{R}^n$ is measurable; there exist the functions $m(t),L(t)\in L(I,\mathbb{R}_+)$ such that for almost all $t\in I$,*

$$\big|g(t,x_1,x_2)\big|\le m(t),\ \ \forall\,(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n,$$

$$\big|g(t,x_1,x_2)-g(t,y_1,y_2)\big|\le$$

$$\le L(t)\sum_{i=1}^{2}|x_i-y_i|,\ \ \forall\,(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n,\ \ \forall\,(y_1,y_2)\in\mathbb{R}^n\times\mathbb{R}^n.$$

Then the equation

$$\dot x(t)=A(t)\dot x(\sigma(t))+g\big(t,x(t),x(\tau(t))\big) \tag{1.12}$$

with the initial condition

$$x(t)=\varphi(t),\ \ \dot x(t)=v(t),\ \ t\in[\widehat\tau,t_0),\ \ x(t_0)=x_0. \tag{1.13}$$

has the unique solution $x(t)\in\mathbb{R}^n$ defined on the interval $[\widehat\tau,b]$ (see Definition 1.1).

Proof. The existence of a global solution will be proved by the step method with respect to the function $\nu(t)$. We divide the interval $[t_0,b]$ into the subintervals $[\xi_i,\xi_{i+1}]$, $i=0,\dots,l$, where $\xi_0=t_0,\xi_i=\nu^i(t_0)$, $i=1,\dots,l$, $\xi_{l+1}=b$, $\nu^1(t_0)=\nu(t_0)$, $\nu^2(t_0)=\nu(\nu(t_0)),\dots$.

It is clear that on the interval $[\xi_0,\xi_1]$ we have the delay differential equation

$$\dot x(t)=g\big(t,x(t),x(\tau(t))\big)+A(t)v(\sigma(t)) \tag{1.14}$$

with the initial condition

$$x(t)=\varphi(t),\ \ t\in[\widehat\tau,\xi_0),\ \ x(\xi_0)=x_0. \tag{1.15}$$

The problem (1.14)–(1.15) has the unique solution $z_1(t)$ defined on the interval $[\widehat\tau,\xi_1]$, i.e. the function $z_1(t)$ satisfies the condition (1.13) and on the interval $[\xi_0,\xi_1)]$ is absolutely continuous and satisfies the equation (1.12) a.e. on $[\xi_0,\xi_1]$. Thus, $x(t)=z_1(t)$ is the solution of the problem (1.12)–(1.13) defined on the interval $[\widehat{\tau},\xi_1]$.

Further, on the interval $[\xi_1,\xi_2]$ we have the equation

$$\dot{x}(t)=g\big(t,x(t),x(\tau(t))\big)+A(t)\dot{z}(\sigma(t)) \tag{1.16}$$

with the initial condition

$$x(t)=z_1(t),\ \ t\in[\widehat{\tau},\xi_1]. \tag{1.17}$$

Here,

$$\dot{z}(t)=\begin{cases} v(t), & t\in[\widehat{\tau},\xi_0),\\ \dot{z}_1(t), & t\in[\xi_0,\xi_1].\end{cases}$$

The problem (1.16)–(1.17) has the unique solution $z_2(t)$ defined on the interval $[\widehat{\tau},\xi_2]$. Thus, the function $x(t)=z_2(t)$ is the solution of the problem (1.12)–(1.13) defined on the interval $[\widehat{\tau},\xi_2]$.

Continuing this process, we can construct a solution of the problem (1.12)–(1.13) defined on the interval $[\widehat{\tau},b]$. □

Theorem 1.9. *Let* $x(t)$, $t\in[\widehat{\tau},b]$, *be a solution of the problem* (1.12)–(1.13), *then it is a solution of the integral equation*

$$\begin{aligned} x(t)=x_0+\int\limits_{\sigma(t_0)}^{t_0} Y(\nu(\xi);t)A(\nu(\xi))\dot{\nu}(\xi)v(\xi)\,d\xi+\\ +\int\limits_{t_0}^{t} Y(\xi;t)g(t,x(\xi),x(\tau(\xi)))\,d\xi,\ \ t\in[t_0,b], \end{aligned} \tag{1.18}$$

with the initial condition

$$x(t)=\varphi(t),\ \ t\in[\widehat{\tau},t_0), \tag{1.19}$$

where $Y(\xi;t)$ *has the form* (1.11).

This theorem is a simple corollary of Theorem 1.5.

Theorem 1.10. *If the integral equation* (1.18) *with the initial condition* (1.19) *has a solution, then it is unique.*

Proof. Let $x_1(t)$ and $x_2(t)$ be two solutions of the problem (1.18)–(1.19). We have

$$\begin{aligned} &\big|x_1(t)-x_2(t)\big|\le\\ &\le\|Y\|\int\limits_{t_0}^{t} L(\xi)\Big\{\big|x_1(\xi)-x_2(\xi)\big|+\big|x_1(\tau(\xi))-x_2(\tau(\xi))\big|\Big\}\,d\xi\le\\ &\le\|Y\|\Big\{\int\limits_{t_0}^{t}\big[L(\xi)+L(\gamma(\xi))\dot{\gamma}(\xi)\big]\,\big|x_1(\xi)-x_2(\xi)\big|\,d\xi\Big\}, \end{aligned}$$

where

$$\|Y\| = \sup\big\{|Y(\xi;t)|:\ (\xi,t)\in I\times I\big\}.$$

By virtue of Gronwall's inequality, we have $x_1(t)=x_2(t)$, $t\in[t_0,b]$. □

Theorem 1.11. *The solution of the problem* (1.18)–(1.19) *is the solution of the problem* (1.12)–(1.13)*.*

This theorem is a simple corollary of Theorems 1.7–1.9.

Theorem 1.12 ([24])**.** *Let $x(t)\in K_1$, $t\in I_1$, be a piecewise-continuous function, where $K_1\subset O$ is a compact set, and let a sequence $\delta f_i\in W(K_1;\alpha)$, $i=1,2,\dots$, satisfy the condition*

$$\lim_{i\to\infty}\Delta(\delta f_i;K_1)=0.$$

Then

$$\lim_{i\to\infty}\sup\bigg\{\bigg|\int\limits_{s_1}^{s_2} Y(\xi;t)\delta f_i\big(\xi,x(\xi),x(\tau(\xi))\big)\,d\xi\bigg|:\ s_1,s_2\in I\bigg\}=0$$

uniformly in $t\in I$.

Theorem 1.13 ([24])**.** *The matrix functions $\Psi(\xi;t)$ and $Y(\xi;t)$ have the following properties:*

1.11. *$\Psi(\xi;t)$ is continuous on the set $\Pi=\{(\xi,t):\ a\le\xi\le t\le b\}$;*
1.12. *for any fixed $t\in(a,b)$, the function $Y(\xi;t)$, $\xi\in[a,t]$, has first order discontinuity at the points of the set*

$$I(t_0;t)=\Big\{\sigma^i(t)=\sigma(\sigma^{i-1}(t))\in[a,t],\ i=1,2,\dots,\sigma^0(t)=t\Big\};$$

1.13. *$\lim\limits_{\theta\to\xi-}Y(\theta;t)=Y(\xi-;t)$, $\lim\limits_{\theta\to\xi+}Y(\theta;t)=Y(\xi+;t)$ uniformly with respect to $(\xi,t)\in\Pi$;*
1.14. *Let $\xi_i\in(a,b)$, $i=0,1$, $\xi_0<\xi_1$ and $\xi_0\ne I(\xi_0;\xi_1)$. Then there exist numbers δ_i, $i=0,1$, such that the function $Y(\xi;t)$ is continuous on the set $[\xi_0-\delta_0,\xi_0+\delta_0]\times[\xi_1-\delta_1,\xi_1-\delta_1]\subset\Pi$.*

1.3. **Proof of Theorem 1.1.** *On the continuous dependence of a solution for a class of neutral equation.* To each element $\mu=(t_0,\tau,x_0,\varphi,v,f)\in\Lambda$ we assign the functional differential equation

$$\dot y(t)=A(t)h(t_0,v,\dot y)(\sigma(t))+f(t_0,\tau,\varphi,y)(t) \tag{1.20}$$

with the initial condition

$$y(t_0)=x_0, \tag{1.21}$$

where $f(t_0,\tau,\varphi,y)(t)=f(t,y(t),h(t_0,\varphi,y)(\tau(t)))$ and $h(\,\cdot\,)$ is the operator given by the formula

$$h(t_0,\varphi,y)(t)=\begin{cases}\varphi(t) & \text{for } t\in[\widehat\tau,t_0),\\ y(t) & \text{for } t\in[t_0,b].\end{cases} \tag{1.22}$$

Definition 1.3. An absolutely continuous function $y(t) = y(t;\mu) \in O$, $t \in [r_1, r_2] \subset I$, is called a solution of the equation (1.20) with the initial condition (1.21), or a solution corresponding to the element $\mu \in \Lambda$ and defined on $[r_1, r_2]$ if $t_0 \in [r_1, r_2]$, $y(t_0) = x_0$ and satisfies the equation (1.20) a.e. on the interval $[r_1, r_2]$.

Remark 1.1. Let $y(t;\mu), t \in [r_1, r_2]$ be the solution of the problem (1.20)–(1.21). Then the function

$$x(t;\mu) = h\big(t_0, \varphi, y(\,\cdot\,;\mu)\big)(t), \ \ t \in [\widehat{\tau}, r_2]$$

is the solution of the equation (1.1) with the initial condition (1.2).

Theorem 1.14. *Let $y_0(t)$ be a solution corresponding to $\mu_0 \in \Lambda$ defined on $[r_1, r_2] \subset (a, b)$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \operatorname{cl}\varphi_0(I_1) \cup y_0([r_1, r_2])$. Then the following conditions hold:*

1.15. *there exist numbers $\delta_i > 0, i = 0, 1$ such that a solution $y(t;\mu)$ defined on $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ corresponds to each element*

$$\mu = (t_0, \tau, x_0, \varphi, v, f_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha).$$

Moreover,

$$\varphi(t) \in K_1, t \in I_1; \ \ y(t;\mu) \in K_1, \ \ t \in [r_1 - \delta_1, r_2 + \delta_1],$$

for arbitrary $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$;

1.16. *for an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0]$ such that the following inequality holds for any $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$:*

$$\big|y(t;\mu) - y(t;\mu_0)\big| \le \varepsilon, \ \ \forall\, t \in [r_1 - \delta_1, r_2 + \delta_1]. \tag{1.23}$$

Proof. Let $\varepsilon_0 > 0$ be so small that a closed ε_0-neighborhood of the set K_0:

$$K(\varepsilon_0) = \Big\{x \in \mathbb{R}^n : \ \exists\, \widehat{x} \in K_0 \ |x - \widehat{x}| \le \varepsilon_0\Big\}$$

lies in $intK_1$. There exist a compact set Q: $K_0^2(\varepsilon_0) \subset Q \subset K_1^2$ and a continuously differentiable function $\chi : \mathbb{R}^{2n} \to [0, 1]$ such that

$$\chi(x_1, x_2) = \begin{cases} 1 & \text{for } \ (x_1, x_2) \in Q, \\ 0 & \text{for } \ (x_1, x_2) \notin K_1^2 \end{cases} \tag{1.24}$$

(see Assertion 3.2 in [11, p. 60]).

To each element $\mu \in \Lambda$, we assign the functional differential equation

$$\dot{z}(t) = A(t)h(t_0, v, \dot{z})(\sigma(t)) + g(t_0, \tau, \varphi, z)(t) \tag{1.25}$$

with the initial condition

$$z(t_0) = x_0, \tag{1.26}$$

where $g(t_0, \tau, \varphi, z)(t) = g(t, z(t), h(t_0, \varphi, z)(\tau(t)))$ and $g = \chi f$. The function $g(t, x_1, x_2)$ satisfies the conditions

$$|g(t, x_1, x_2)| \le m_{f,K_1}(t), \ \ \forall\, x_i \in \mathbb{R}^n, \ \ i = 1, 2, \tag{1.27}$$

for $\forall\, x_i', x_i'' \in \mathbb{R}^n$, $i = 1, 2$, and for almost all $t \in I$

$$\big|g(t, x_1', x_2') - g(t, x_1'', x_2'')\big| \leq L_f(t) \sum_{i=1}^{2} |x_i' - x_i''|, \tag{1.28}$$

where

$$\begin{gathered} L_f(t) = L_{f,K_1}(t) + \alpha_1 m_{f,K_1}(t), \\ \alpha_1 = \sup\bigg\{ \sum_{i=1}^{2} |\chi_{x_i}(x_1, x_2)| :\ x_i \in \mathbb{R}^n,\ i = 1, 2 \bigg\} \end{gathered} \tag{1.29}$$

(see [15]).

By the definition of the operator $h(\,\cdot\,)$, the equation (1.25) for $t \in [a, t_0]$ can be considered as the ordinary differential equation

$$\dot z_1(t) = A(t) v(\sigma(t)) + g\big(t, z_1(t), \varphi(\tau(t))\big) \tag{1.30}$$

with the initial condition

$$z_1(t_0) = x_0, \tag{1.31}$$

and for $t \in [t_0, b]$, it can be considered as the neutral equation

$$\dot z_2(t) = A(t) \dot z_2(\sigma(t)) + g\big(t, z_2(t), z_2(\tau(t))\big) \tag{1.32}$$

with the initial condition

$$z_2(t) = \varphi(t), \ \ \dot z_2(t) = v(t), \ \ t \in [\widehat\tau, t_0), \ \ z_2(t_0) = x_0. \tag{1.33}$$

Obviously, if $z_1(t)$, $t \in [a, t_0]$, is a solution of problem (1.30)–(1.31) and $z_2(t), t \in [t_0, b]$, is a solution of problem (1.32)–(1.33), then the function

$$z(t) = \begin{cases} z_1(t), & t \in [a, t_0), \\ z_2(t), & t \in [t_0, b] \end{cases}$$

is a solution of the equation (1.25) with the initial condition (1.26) defined on the interval I.

We rewrite the equation (1.30) with the initial condition (1.31) in the integral form

$$z_1(t) = x_0 + \int_{t_0}^{t} \Big[A(\xi) v(\sigma(\xi)) + g\big(\xi, z_1(\xi), \varphi(\tau(\xi))\big) \Big]\, d\xi, \ \ t \in [a, t_0], \tag{1.34}$$

and the equation (1.32) with the initial condition (1.33) we write in the equivalent form

$$\begin{aligned} z_2(t) = x_0 &+ \int_{t_0}^{\nu(t_0)} Y(\xi; t) A(\xi) v(\sigma(\xi))\, d\xi + \\ &+ \int_{t_0}^{t} Y(\xi; t) g(\xi, z_2(\xi), z_2(\tau(\xi)))\, d\xi, \ \ t \in [t_0, b], \end{aligned} \tag{1.35}$$

where

$$z_2(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0)$$

(see Theorem 1.9 and (1.11)).

Introduce the following notation:

$$Y_0(\xi; t, t_0) = \begin{cases} H, & t \in [a, t_0), \\ Y(\xi; t), & t \in [t_0, b], \end{cases} \tag{1.36}$$

$$Y(\xi; t, t_0) = \begin{cases} H, & t \in [a, t_0), \\ Y(\xi; t), & t_0 \le t \le \min\{\nu(t_0), b\}, \\ \Theta, & \min\{\nu(t_0), b\} < t \le b. \end{cases} \tag{1.37}$$

Using this notation and taking into account (1.34) and (1.35), we can rewrite the equation (1.25) in the form of the equivalent integral equation

$$\begin{aligned} z(t) = x_0 + \int_{t_0}^{t} Y(\xi; t, t_0) A(\xi) v(\sigma(\xi))\, d\xi + \\ + \int_{t_0}^{t} Y_0(\xi; t, t_0) g(t_0, \tau, \varphi, z)(\xi)\, d\xi, \ \ t \in I. \end{aligned} \tag{1.38}$$

A solution of the equation (1.38) depends on the parameter

$$\mu \in \Lambda_0 = I \times D \times O \times \Phi_1 \times E_v \times \big(f_0 + W(K_1; \alpha)\big) \subset E_\mu$$

The topology in Λ_0 is induced by the topology of the vector space E_μ. Denote by $C(I, \mathbb{R}^n)$ the space of continuous functions $y : I \to \mathbb{R}^n$ with the distance $d(y_1, y_2) = \|y_1 - y_2\|_I$.

On the complete metric space $C(I, \mathbb{R}^n)$, we define a family of mappings

$$F(\,\cdot\,; \mu) : C(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n) \tag{1.39}$$

depending on the parameter μ by the formula

$$\zeta(t) = \zeta(t; z, \mu) =$$

$$= x_0 + \int_{t_0}^{t} Y(\xi; t, t_0) A(\xi) v(\sigma(\xi))\, d\xi + \int_{t_0}^{t} Y_0(\xi; t, t_0) g(t_0, \tau, \varphi, z)(\xi)\, d\xi.$$

Clearly, every fixed point $z(t; \mu), t \in I$, of the mapping (1.39) is a solution of the equation (1.25) with the initial condition (1.26).

Define the kth iteration $F^k(z;\mu)$ by

$$\zeta_k(t) = x_0 + \int\limits_{t_0}^{t} Y(\xi;t,t_0)A(\xi)v(\sigma(\xi))\,d\xi + \\ + \int\limits_{t_0}^{t} Y_0(\xi;t,t_0)g(t_0,\tau,\varphi,\zeta_{k-1})(\xi)\,d\xi, \quad k=1,2,\dots,$$

$$\zeta_0(t) = z(t).$$

Let us now prove that for a sufficiently large k, the family of mappings $F^k(z;\mu)$ is uniformly contractive. Towards this end, we estimate the difference

$$\big|\zeta_k'(t)-\zeta_k''(t)\big| = \big|\zeta_k(t;z',\mu)-\zeta_k(t;z'',\mu)\big| \le \\ \le \int\limits_a^t \big|Y_0(\xi;t,t_0)\big|\,\Big|g(t_0,\tau,\varphi,\zeta_{k-1}')(\xi)-g(t_0,\tau,\varphi,\zeta_{k-1}'')(\xi)\Big|\,d\xi \le \\ \le \int\limits_a^t L_f(\xi)\Big[\big|\zeta_{k-1}'(\xi)-\zeta_{k-1}''(\xi)\big| + \\ + \big|h(t_0,\varphi,\zeta_{k-1}')(\tau(\xi))-h(t_0,\varphi,\zeta_{k-1}'')(\tau(\xi))\big|\Big]\,d\xi, \quad k=1,2,\dots, \tag{1.40}$$

(see (1.28)), where the function $L_f(\xi)$ is of the form (1.29). Here, it is assumed that $\zeta_0'(\xi)=z'(\xi)$ and $\zeta_0''(\xi)=z''(\xi)$.

It follows from the definition of the operator $h(\,\cdot\,)$ that

$$h(t_0,\varphi,\zeta_{k-1}')(\tau(\xi))-h(t_0,\varphi,\zeta_{k-1}'')(\tau(\xi)) = h(t_0,0,\zeta_{k-1}'-\zeta_{k-1}'')(\tau(\xi)).$$

Therefore, for $\xi\in[a,\gamma(t_0)]$, we have

$$h(t_0,0,\zeta_{k-1}'-\zeta_{k-1}'')(\tau(\xi)) = 0. \tag{1.41}$$

Let $\gamma(t_0)<b$; then for $\xi\in[\gamma(t_0),b]$, we obtain

$$\big|h(t_0,0,\zeta_{k-1}'-\zeta_{k-1}'')(\tau(\xi))\big| = \big|\zeta_{k-1}'(\tau(\xi))-\zeta_{k-1}'')(\tau(\xi))\big|, \\ \sup\Big\{\big|\zeta_{k-1}'(\tau(t))-\zeta_{k-1}''(\tau(t))\big| : \ t\in[\gamma(t_0),\xi]\Big\} \le \\ \le \sup\Big\{\big|\zeta_{k-1}'(t)-\zeta_{k-1}''(t)\big| : \ t\in[a,\xi]\Big\}. \tag{1.42}$$

If $\gamma(t_0)>b$, then (1.41) holds on the whole interval I. The relation (1.40), together with (1.41) and (1.42), imply that

$$\big|\zeta_k'(t)-\zeta_k''(t)\big| \le \sup\Big\{\big|\zeta_k'(\xi)-\zeta_k''(\xi)\big| : \ \xi\in[a,t]\Big\} \le \\ \le 2\|Y_0\|\int\limits_a^t L_f(\xi_1)\sup\Big\{\big|\zeta_{k-1}'(\xi)-\zeta_{k-1}''(\xi)\big| : \ \xi\in[a,\xi_1]\Big\}\,d\xi_1, \quad k=1,2,\dots.$$

Therefore,

$$\left|\zeta_k'(t)-\zeta_k''(t)\right|\le$$
$$\le 2^2\|Y_0\|^2\int\limits_a^t L_f(\xi_1)\,d\xi_1\int\limits_a^{\xi_1}L_f(\xi_2)\sup\Big\{\left|\zeta_{k-2}'(\xi)-\zeta_{k-2}''(\xi)\right|:\ \xi\in[a,\xi_2]\Big\}\,d\xi_2.$$

By continuing this procedure, we obtain

$$\left|\zeta_k'(t)-\zeta_k''(t)\right|\le\big(2\|Y_0\|\big)^k\alpha_k(t)\|z'-z''\|_I,$$

where

$$\alpha_k(t)=\int\limits_a^t L_f(\xi_1)\,d\xi_1\int\limits_a^{\xi_1}L_f(\xi_2)\,d\xi_2\cdots\int\limits_a^{\xi_{k-1}}L_f(\xi_k)\,d\xi_k.$$

By the induction, one can readily show that

$$\alpha_k(t)=\frac{1}{k!}\bigg(\int\limits_a^t L_f(\xi)\,d\xi\bigg)^k.$$

Thus,

$$d\big(F^k(z';\mu),F^k(z'';\mu)\big)=$$
$$=\|\zeta_k'-\zeta_k''\|_I\le\big(2\|Y_0\|\big)^k\alpha_k(b)\|z'-z''\|_I=\widehat{\alpha}_k\|z'-z''\|_I.$$

Let us prove the existence of the number $\alpha_2>0$ such that

$$\int\limits_I L_f(t)\,dt\le\alpha_2,\ \ \forall\,f\in f_0+W(K_1;\alpha).$$

Indeed, let $(x_1,x_2)\in K_1^2$ and let $f\in f_0+W(K_1;\alpha)$, then

$$\left|f(t,x_1,x_2)\right|\le m_{f_0,K_1}(t)+m_{\delta f,K_1}(t):=m_{f,K_1}(t),\ \ t\in I.$$

Further, let $x_i',x_i''\in K_1$, $i=1,2$ then

$$\left|f(t,x_1',x_2')-f(t,x_1'',x_2'')\right|\le$$
$$\le\left|f_0(t,x_1',x_2')-f_0(t,x_1'',x_2'')\right|+\left|\delta f(t,x_1',x_2')-\delta f(t,x_1'',x_2'')\right|\le$$
$$\le\big(L_{f_0,K_1}(t)+L_{\delta f,K_1}(t)\big)\sum_{i=1}^2|x_i'-x_i''|=L_{f,K_1}(t)\sum_{i=1}^2|x_i'-x_i''|,$$

where $L_{f,K_1}(t)=L_{f_0,K_1}(t)+L_{\delta f,K_1}(t)$. By (1.29),

$$\int\limits_I L_f(t)\,dt=\int\limits_I \big(L_{f,K_1}(t)+\alpha_1 m_{f,K_1}(t)\big)\,dt=$$
$$=\int\limits_I \Big[L_{f_0,K_1}(t)+L_{\delta f,K_1}(t)+\alpha_1\big(m_{f_0,K_1}(t)+m_{\delta f,K_1}(t)\big)\Big]\,dt\le$$
$$\le\alpha(\alpha_1+1)+\int\limits_I \big[L_{f_0,K_1}(t)+\alpha_1 m_{f_0,K_1}(t)\big]\,dt=\alpha_2.$$

Taking into account this estimate, we obtain $\widehat{\alpha}_k\le(2\|Y_0\|\alpha_2)^k/k!$. Consequently, there exists a positive integer k_1 such that $\widehat{\alpha}_{k_1}<1$. Therefore, the k_1st iteration of the family (1.39) is contracting. By the fixed point theorem for contraction mappings (see [11, p. 90], [27, p. 110]), the mapping (1.39) has a unique fixed point for each μ. Hence it follows that the equation (1.25) with the initial condition (1.26) has a unique solution $z(t;\mu)$, $t\in I$.

Let us prove that the mapping $F^k(z(\,\cdot\,;\mu_0);\,\cdot\,):\Lambda_0\to C(I,\mathbb{R}^n)$ is continuous at the point $\mu=\mu_0$ for an arbitrary $k=1,2,\dots$. To his end, it suffices to show that if the sequence $\mu_i=(t_{0i},\tau_i,x_{0i},\varphi_i,v_i,f_i)\in\Lambda_0$, $i=1,2,\dots$, where $f_i=f_0+\delta f_i$, converges to $\mu_0=(t_{00},\tau_0,x_{00},\varphi_0,v_0,f_0)$, i.e. if

$$\lim_{i\to\infty}\Big(|t_{0i}-t_{00}|+\|\tau_i-\tau_0\|_{I_2}+$$
$$+|x_{0i}-x_{00}|+\|\varphi_i-\varphi_0\|_{I_1}+\|v_i-v_0\|_{I_1}+\Delta(\delta f_i;K_1)\Big)=0,$$

then

$$\lim_{i\to\infty}F^k\big(z(\,\cdot\,;\mu_0);\mu_i\big)=F^k\big(z(\,\cdot\,;\mu_0);\mu_0\big)=z(\,\cdot\,;\mu_0).\qquad(1.43)$$

We prove the relation (1.43) by induction. Let $k=1$, then we have

$$\big|\zeta_1^i(t)-z_0(t)\big|\le|x_{0i}-x_{00}|+$$
$$+\bigg|\int\limits_{t_{0i}}^t Y(\xi;t,t_{0i})A(\xi)v_i(\sigma(\xi))\,d\xi-\int\limits_{t_{00}}^t Y(\xi;t,t_{00})A(\xi)v_0(\sigma(\xi))\,d\xi\bigg|+$$
$$+\bigg|\int\limits_{t_{0i}}^t Y_0(\xi;t,t_{0i})g_i(t_{0i},\tau_i,\varphi_i,z_0)(\xi)\,d\xi-$$
$$-\int\limits_{t_{00}}^t Y_0(\xi;t,t_{00})g_0(t_{00},\tau_0,\varphi_0,z_0)(\xi)\,d\xi\bigg|=$$
$$=|x_{0i}-x_{00}|+a_i(t)+b_i(t),\quad(1.44)$$

where

$$\zeta_1^i(t) = \zeta_1(t; z_0, \mu_i), \quad z_0(t) = z(t; \mu_0),$$
$$g_i = \chi f_i = g_0 + \delta g_i, \quad g_0 = \chi f_0, \quad \delta g_i = \chi \delta f_i;$$

$$a_i(t) = \Big| \int_{t_{0i}}^{t} Y(\xi; t, t_{0i}) A(\xi) v_i(\sigma(\xi))\, d\xi - \int_{t_{00}}^{t} Y(\xi; t, t_{00}) A(\xi) v_0(\sigma(\xi))\, d\xi \Big|;$$

$$b_i(t) = \Big| \int_{t_{0i}}^{t} Y_0(\xi; t, t_{0i}) g_i(t_{0i}, \tau_{1i}, \varphi_i, z_0)(\xi)\, d\xi -$$
$$- \int_{t_{00}}^{t} Y_0(\xi; t, t_{00}) g_0(t_{00}, \tau_{10}, \varphi_0, z_0)(\xi)\, d\xi \Big|.$$

First of all, let us estimate $a_i(t)$. We have

$$a_i(t) \le \Big| \int_{t_{0i}}^{t_{00}} Y(\xi; t, t_{00}) A(\xi) v_0(\sigma(\xi))\, d\xi \Big| +$$
$$+ \int_I \Big| Y(\xi; t, t_{0i}) A(\xi) v_i(\sigma(\xi)) - Y(\xi; t, t_{00}) A(\xi) v_0(\sigma(\xi)) \Big|\, d\xi =$$
$$= a_{i1}(t) + a_{i2}(t). \tag{1.45}$$

Obviously,

$$\lim_{i\to\infty} a_{i1}(t) = 0 \ \text{ uniformly in } \ t \in I. \tag{1.46}$$

Furthermore,

$$a_{i2}(t) \le \int_I \big| Y(\xi; t, t_{0i}) - Y(\xi; t, t_{00}) \big| \, \big| A(\xi) v_i(\sigma(\xi)) \big|\, d\xi +$$
$$+ \int_I \big| Y(\xi; t, t_0) A(\xi) \big| \, \big| v_i(\sigma(\xi)) - v_0(\sigma(\xi)) \big|\, d\xi \le$$
$$\le \|A\| \, \|v_i\|_{I_1} a_{i3}(t) + \|YA\| \, \|v_i - v_0\|_{I_1}, \tag{1.47}$$

where

$$a_{i3}(t) = \int_I \big| Y(\xi; t, t_{0i}) - Y(\xi; t, t_{00}) \big|\, d\xi.$$

Let $t_{0i} < t_{00}$, and let a number i_0 be so large that $\nu(t_{0i}) > t_{00}$ for $i \ge i_0$. Then taking into account (1.37), we have

$$a_{i3}(t) = \int_{t_{0i}}^{t_{00}} |Y(\xi; t) - H|\, d\xi + \int_{\nu(t_{0i})}^{\nu(t_{00})} |Y(\xi; t)|\, d\xi \le$$
$$\le \|Y - H\| (t_{00} - t_{0i}) + \|Y\| \big(\nu(t_{00}) - \nu(t_{0i})\big),$$

therefore,

$$\lim_{i\to\infty} a_{i3}(t)=0 \ \text{ uniformly in } \ I. \tag{1.48}$$

Let $t_{0i}>t_{00}$. Choose a number i_0 so large that $\nu(t_{00})>t_{0i}$ for $i\geq i_0$. Then

$$a_{i3}(t)=\int\limits_{t_{00}}^{t_{0i}}|H-Y(\xi;t)|\,d\xi+\int\limits_{\nu(t_{00})}^{\nu(t_{0i})}|Y(\xi;t)|\,d\xi.$$

This implies (1.48). Taking into account (1.46)–(1.48), we obtain from (1.45) that

$$\lim_{i\to\infty} a_i(t)=0 \ \text{ uniformly in } \ I. \tag{1.49}$$

Now, let us estimate the summand $b_i(t)$. We have

$$\begin{aligned}b_i(t)\leq\bigg|\int\limits_{t_{0i}}^{t_{00}}Y_0(\xi;t,t_{0i})g_0(t_{00},\tau_0,\varphi_0,z_0)(\xi)\,d\xi\bigg|+\\+\bigg|\int\limits_{t_{0i}}^{t}\Big[Y_0(\xi;t,t_{0i})g_i(t_{0i},\tau_i,\varphi_i,z_0)(\xi)-\\-Y_0(\xi;t,t_{00})g_0(t_{00},\tau_0,\varphi_0,z_0)(\xi)\Big]\,d\xi=b_{i1}(t)+b_{i2}(t).\end{aligned} \tag{1.50}$$

Obviously,

$$\lim_{i\to\infty} b_{i1}(t)=0 \ \text{ uniformly in } \ I. \tag{1.51}$$

Furthermore,

$$\begin{aligned}b_{i2}(t)=\bigg|\int\limits_{t_{0i}}^{t}Y_0(\xi;t,t_{0i})\Big[g_i(t_{0i},\tau_i,\varphi_i,z_0)(\xi)-g_0(t_{0i},\tau_i,\varphi_i,z_0)(\xi)\Big]\,d\xi+\\+\int\limits_{t_{0i}}^{t}Y_0(\xi;t,t_{0i})\Big[g_0(t_{0i},\tau_i,\varphi_i,z_0)(\xi)-g_0(t_{00},\tau_0,\varphi_0,z_0)(\xi)\Big]\,d\xi+\\+\int\limits_{t_{0i}}^{t}\Big[Y_0(\xi;t,t_{0i})-Y_0(\xi;t,t_{00})\Big]g_0(t_{00},\tau_0,\varphi_0,z_0)(\xi)\,d\xi\bigg|\leq\\\leq\sum_{j=1}^{3}b_{i2}^{j}(t),\end{aligned} \tag{1.52}$$

where

$$b_{i2}^1(t) = \Big| \int_{t_{0i}}^{t} Y_0(\xi; t, t_{0i}) \delta g_i(t_{0i}, \tau_i, \varphi_i, z_0)(\xi)\, d\xi \Big|,$$

$$b_{i2}^2(t) = \Big| \int_{t_{0i}}^{t} Y_0(\xi; t, t_{0i}) \Big[g_0(t_{0i}, \tau_i, \varphi_i, z_0)(\xi) - g_0(t_{00}, \tau_0, \varphi_0, z_0)(\xi) \Big]\, d\xi \Big|,$$

$$b_{i2}^3(t) = \Big| \int_{t_{0i}}^{t} \Big[Y_0(\xi; t, t_{0i}) - Y_0(\xi; t, t_{00}) \Big] g_0(t_{00}, \tau_0, \varphi_0, z_0)(\xi)\, d\xi \Big|.$$

Now, let us estimate the expressions $b_{i2}^1(t)$. We have

$$\begin{aligned} b_{i2}^1(t) = \Big| \int_{t_{0i}}^{t} Y_0(\xi; t, t_{0i}) \Big[\delta g_i(t_{0i}, \tau_i, \varphi_i, z_0)(\xi) - \delta g_i(t_{0i}, \tau_i, \varphi_0, z_0)(\xi) \Big]\, d\xi + \\ + \int_{t_{0i}}^{t} Y_0(\xi; t, t_{0i}) \delta g_i(t_{0i}, \tau_i, \varphi_0, z_0)(\xi)\, d\xi \Big| \le \\ \le \|Y_0\| \int_I L_{\delta g_i, K_1}(\xi) \Big| h(t_{0i}, \varphi_i, z_0)(\tau_i(\xi)) - h(t_{0i}, \varphi_0, z_0)(\tau_i(\xi)) \Big|\, d\xi + \\ + \max_{t', t'' \in I} \Big| \int_{t'}^{t''} Y_0(\xi; t, t_{0i}) \delta g_i(t_{0i}, \tau_i, \varphi_0, z_0)(\xi)\, d\xi \Big| = \\ = b_{i2}^4 + b_{i2}^5(t). \end{aligned} \tag{1.53}$$

It is easy to see that

$$\begin{aligned} b_{i2}^4 \le \|Y_0\| \int_I L_{\delta g_i, K_1}(\xi) \big| \varphi_i(\tau_i(\xi)) - \varphi_0(\tau_i(\xi)) \big|\, d\xi \le \\ \le \|\varphi_i - \varphi_0\|_{I_1} \int_I L_{\delta g_i, K_1}(\xi)\, d\xi. \end{aligned}$$

The sequence

$$\int_I L_{\delta g_i, K_1}(\xi)\, d\xi, \quad i = 1, 2, \dots,$$

is bounded, therefore

$$\lim_{i \to \infty} b_{i2}^4 = 0.$$

Furthermore,

$$b_{i2}^5(t) \le \max_{t',t''\in I}\left|\int\limits_{t'}^{t''}\delta g_i\big(\xi, z_0(\xi), \varphi_0(\tau_i(\xi))\big)\,d\xi\right| +$$

$$+\max_{t',t''\in I}\left|\int\limits_{t'}^{t''} Y(\xi;t)\delta g_i(t_{0i},\tau_i,\varphi_0,z_0)(\xi)\,d\xi\right| = b_{i2}^6 + b_{i2}^7(t).$$

The function $\varphi_0(\xi)$, $\xi\in I_1$, is piecewise-continuous with a finite number of discontinuity points of the first kind, i.e. there exist subintervals (θ_q,θ_{q+1}), $q=1,\dots,m$, where the function $\varphi_0(t)$ is continuous, with

$$\theta_1=\widehat{\tau},\ \ \theta_{m+1}=b,\ \ I_1=\bigcup_{q=1}^{m-1}[\theta_q,\theta_{q+1})\cup[\theta_m,\theta_{m+1}].$$

We define on the interval I_1 the continuous functions $z_i(t)$, $i=1,\dots,m+1$, as follows:

$$z_1(t)=\varphi_{01}(t),\dots,z_m(t)=\varphi_{0m}(t),$$

$$z_{m+1}(t)=\begin{cases} z_0(a), & t\in[\widehat{\tau},a),\\ z_0(t), & t\in I,\end{cases}$$

where

$$\varphi_{0q}(t)=\begin{cases}\varphi_0(\theta_q+), & t\in[\widehat{\tau},\theta_q],\\ \varphi_0(t), & t\in(\theta_q,\theta_{q+1}),\\ \varphi_0(\theta_{q+1}-), & t\in[\theta_{q+1},b]\end{cases}\qquad q=1,\dots,m.$$

One can readily see that b_{i2}^6 satisfies the estimation

$$b_{i2}^6 \le \sum_{m_1=1}^m \max_{t',t''\in I}\left|\int\limits_{t'}^{t''}\delta g_i\big(t,z_0(t),z_{m_1}(\tau_i(t))\big)\,dt\right| \le$$

$$\le \sum_{m_1=1}^m \max_{t',t''\in I}\left|\int\limits_{t'}^{t''}\delta g_i\big(t,z_0(t),z_{m_1}(\tau_0(t))\big)\,dt\right| +$$

$$+\sum_{m_1=1}^m \max_{t',t''\in I}\left|\int\limits_{t'}^{t''}\Big|\delta g_i\big(t,z_0(t),z_{m_1}(\tau_i(t))\big)-\delta g_i\big(t,z_0(t),z_{m_1}(\tau_0(t))\big)\Big|\,dt\right| \le$$

$$\le \sum_{m_1=1}^m \max_{t',t''\in I}\left|\int\limits_{t'}^{t''}\delta g_i\big(t,z_0(t),z_{m_1}(\tau_0(t))\big)\,dt\right| +$$

$$+\sum_{m_1=1}^m \int\limits_I L_{\delta f_i,K_1}(t)\big|z_{m_1}(\tau_i(t))-z_{m_1}(\tau_0(t))\big|\,dt \le$$

$$\leq \sum_{m_1=1}^{m} \max_{t',t''\in I} \bigg| \int\limits_{t'}^{t''} \delta g_i\big(t, z_0(t), z_{m_1}(\tau_0(t))\big)\, dt \bigg| +$$

$$+ \sum_{m_1=1}^{m} \max_{t\in I} \big| z_{m_1}(\tau_i(t)) - z_{m_1}(\tau_0(t)) \big| \int\limits_I L_{\delta g_i, K_1}(t)\, dt. \quad (1.54)$$

Obviously,

$$\Delta(\delta g_i; K_1) = \Delta(\chi\delta f_i; K_1) \leq \Delta(\delta f_i; K_1)$$

(see (1.24)). Since $\Delta(\delta f_i; K_1) \to 0$ as $i \to \infty$, we have

$$\lim_{i\to\infty} \Delta(\delta g_i, K_1) = 0.$$

This allows us to use Theorem 1.12, which in turn, implies that

$$\lim_{i\to\infty} \max_{t',t''\in I} \bigg| \int\limits_{t'}^{t''} \delta g_i\big(t, z_0(t), z_{m_1}(\tau_0(t))\big)\, dt \bigg| = 0, \ \ \forall\, m_1 = 1, \dots, m.$$

Moreover, it is clear that

$$\lim_{i\to\infty} \max_{t\in I} \big| z_{m_1}(\tau_i(t)) - z_{m_1}(\tau_0(t)) \big| = 0.$$

The right-hand side of inequality (1.54) consists of finitely many summands, and therefore

$$\lim_{i\to\infty} b_{i2}^6 = 0.$$

For $b_{12}^7(t)$, in the analogous manner, we get

$$b_{12}^7(t) \leq \sum_{m_1=1}^{m+1} \max_{t',t''\in I} \bigg| \int\limits_{t'}^{t''} Y(\xi;t)\delta g_i\big(\xi, z_0(\xi), z_{m_1}(\tau_0(\xi))\big)\, d\xi \bigg| +$$

$$+ \|Y\| \sum_{m_1=1}^{m+1} \max_{t\in I} \big| z_{m_1}(\tau_i(t)) - z_{m_1}(\tau_0(t)) \big| \int\limits_I L_{\delta g_i, K_1}(t)\, dt,$$

from which we have

$$\lim_{i\to\infty} b_{i2}^7(t) = 0 \ \text{ uniformly in } \ I$$

(see Theorem 1.12).

Thus,

$$\lim_{i\to\infty} b_{i2}^5(t) = 0 \ \text{ uniformly in } \ I.$$

Consequently,

$$\lim_{i\to\infty} b_{i2}^1(t) = 0 \ \text{ uniformly in } \ I. \quad (1.55)$$

Next,

$$b_{i2}^2(t) \leq \|Y\| \int\limits_I L_{f_0}(t) \Big| h(t_{0i}, \varphi_i, z_0)(\tau_i(t)) - h(t_{00}, \varphi_0, z_0)(\tau_0(t)) \Big|\, dt \leq$$

$$\leq \|Y\|\bigg\{\int\limits_I L_{f_0}(t)\Big|h(t_{0i},\varphi_i,z_0)(\tau_i(t))-h(t_{0i},\varphi_0,z_0)(\tau_i(t))\Big|\,dt+$$

$$+\int\limits_I L_{f_0}(t)\Big|h(t_{0i},\varphi_0,z_0)(\tau_i(t))-h(t_{00},\varphi_0,z_0)(\tau_0(t))\Big|\,dt\bigg\}\leq$$

$$\leq \|Y\|\bigg\{\int\limits_I L_{f_0}(t)\Big|h(t_{0i},\varphi_i-\varphi_0)(\tau_i(t))\Big|\,dt+$$

$$+\int\limits_I L_{f_0}(t)\Big|h(t_{0i},\varphi_0,z_0)(\tau_i(t))-h(t_{00},\varphi_0,z_0)(\tau_i(t))\Big|+$$

$$+\int\limits_I L_{f_0}(t)\Big|h(t_{00},\varphi_0,z_0)(\tau_i(t))-h(t_{00},\varphi_0,z_0)(\tau_0(t))\Big|\bigg\}\times$$

$$\times\|Y\|\bigg\{\|\varphi_i-\varphi_0\|_{I_1}\int\limits_I L_{f_0}(t)\,dt+b_{i21}^2+b_{i22}^2\bigg\}$$

(see (1.22) and (1.36)). Introduce the notation

$$\rho_{0i}=\min\big\{\gamma_i(t_{00}),\gamma_i(t_{0i})\big\},\quad \theta_{0i}=\max\big\{\gamma_i(t_{00}),\gamma_i(t_{0i})\big\}.$$

We prove that

$$\lim_{i\to\infty}\gamma_i(t_{00})=\lim_{i\to\infty}\gamma_i(t_{0i})=\gamma_0(t_{00}).$$

The sequences $\{\gamma_i(t_{00})\}$ and $\gamma_i(t_{0i})$ are bounded. Without less of generality, we assume that

$$\lim_{i\to\infty}\gamma_i(t_{00})=\gamma_0,\quad \lim_{i\to\infty}\gamma_i(t_{0i})=\gamma_1.$$

We have

$$t_{00}=\tau_i(\gamma_i(t_{00}))=\tau_i(\gamma_i(t_{00}))-\tau_0(\gamma_i(t_{00}))+\tau_0(\gamma_i(t_{00})).$$

Clearly,

$$\lim_{i\to\infty}\Big|\tau_i(\gamma_i(t_{00}))-\tau_0(\gamma_i(t_{00}))\Big|\leq\lim_{i\to\infty}\|\tau_i-\tau_0\|_{I_2}=0.$$

Passing to the limit, we obtain $t_{00}=\tau_0(\gamma_0)$. The equation $\tau_0(t)=t_{00}$ has a unique solution $\gamma_0(t_{00})$, i.e. $\gamma_0=\gamma_0(t_{00})$.

Further,

$$t_{0i}=\tau_i(\gamma_i(t_{0i}))=\tau_i(\gamma_i(t_{0i}))-\tau_0(\gamma_i(t_{0i}))+\tau_0(\gamma_i(t_{0i})).$$

Hence we obtain $t_{00}=\tau_0(\gamma_1)$, i.e. $\gamma_1=\gamma_0(t_{00})$.

Thus,

$$\lim_{i\to\infty}(\rho_{0i}-\theta_{0i})=0.$$

Consequently,

$$b_{i21}^2=\int\limits_{\rho_{0i}}^{\theta_{0i}}L_{f_0}(t)\Big|h(t_{0i},\varphi_0,z_0)(\tau_i(t))-h(t_{00},\varphi_0,z_0)(\tau_i(t))\Big|\,dt\longrightarrow 0.$$

Introduce the notation

$$\rho_{1i} = \min\big\{\gamma_i(t_{00}), \gamma_0(t_{00})\big\}, \quad \theta_{1i} = \max\big\{\gamma_i(t_{00}), \gamma_0(t_{00})\big\}.$$

For b_{i22}^2, we have

$$b_{i22}^2 = \int\limits_{\rho_{1i}}^{\theta_{1i}} L_{f_0}(t)\Big|h(t_{00},\varphi_0,z_0)(\tau_i(t)) - h(t_{00},\varphi_0,z_0)(\tau_0(t))\Big|\,dt.$$

Analogously, it can be proved that

$$\lim_{i\to\infty}(\rho_{1i} - \theta_{1i}) = 0.$$

Thus, $b_{i22}^2 \to 0$. Consequently,

$$b_{i2}^2(t) \to 0. \tag{1.56}$$

Finally, we have

$$b_{12}^3(t) \le \Bigg|\int\limits_{t_{0i}}^{t_{00}} |Y(\xi;t) - H| m_{f_0,K_1}(\xi)\,d\xi\Bigg| \le \|Y - H\|\Bigg|\int\limits_{t_{0i}}^{t_{00}} m_{f_0,K_1}(\xi)\,d\xi\Bigg|$$

i.e.

$$\lim_{i\to\infty} b_{i2}^3(t) = 0 \ \text{ uniformly in } \ I.$$

Therefore,

$$\lim_{i\to\infty} |\zeta_1^i(t) - z_0(t)| = 0 \ \text{ uniformly in } \ I$$

(see (1.44), (1.45), (1.49)–(1.52), (1.55), (1.56)). Assume that the relation (1.43) holds for a certain $k > 1$. Let us prove its fulfilment for $k+1$. Elementary transformations yield

$$\big|\zeta_{k+1}^i(t) - z_0(t)\big| \le |x_{0i} - x_{00}| + a_i(t) + b_{ik}(t), \tag{1.57}$$

where

$$b_{ik}(t) = \Bigg|\int\limits_{t_{0i}}^{t} Y_0(\xi;t,t_{0i}) g_i(t_{0i},\tau_i,\varphi_i,\zeta_k^i)(\xi)\,d\xi - \\ - \int\limits_{t_{00}}^{t} Y_0(\xi;t,t_{00}) g_i(t_{00},\tau_0,\varphi_0,z_0)(\xi)\,d\xi\Bigg|$$

(see (1.44)). The quantity $a_i(t)$ has been estimated above, it remains to estimate $b_{ik}(t)$. We have

$$b_{ik}(t) \le \|Y_0\| \int\limits_I \Big|g_i(t_{0i},\tau_i,\varphi_i,\zeta_k^i)(\xi) - g_i(t_{0i},\tau_i,\varphi_i,z_0)(\xi)\Big|\,d\xi + \\ + \Bigg|\int\limits_{t_{0i}}^{t} Y_0(\xi;t,t_{0i}) g_i(t_{0i},\tau_i,\varphi_i,z_0)(\xi)\,d\xi -$$

$$-\int\limits_{t_{0i}}^{t} Y_0(\xi;t,t_{00})g_i(t_{00},\tau_0,\varphi_0,z_0)(\xi)\,d\xi\bigg| = b_{ik}^1(t)+b_i(t).$$

The function $b_i(t)$ has been estimated above. It is not difficult to see that the following inequality holds for $b_{ik}(t)$:

$$b_{ik}(t)\le 2\|Y_0\|\,\|\zeta_k^i-z_0\|\int\limits_I L_{f_i}(t)\,dt.$$

By the assumptions,

$$\lim_{i\to\infty}\|\zeta_k^i-z_0\|=0.$$

Therefore,

$$\lim_{i\to\infty} b_{ik}(t)=0 \ \text{ uniformly in } \ I.$$

Thus, we obtain from (1.57) that

$$\lim_{i\to\infty}\|\zeta_{k+1}^i-z_0\|=0.$$

We have proved (1.43) for every $k=1,2,\dots$. Let the number $\delta_1>0$ be so small that $[r_1-\delta_1,r_2+\delta_1]\subset I$ and $|z(t;\mu_0)-z(r_1;\mu_0)|\le\varepsilon_0/2$ for $t\in[r_1-\delta_1,r_1]$ and $|z(t;\mu_0)-z(r_2;\mu_0)|\le\varepsilon_0/2$ for $t\in[r_2,r_2+\delta_1]$.

From the uniqueness of the solution $z(t;\mu_0)$, we can conclude that $z(t;\mu_0)=y_0(t)$ for $t\in[r_1,r_2]$. Taking into account the above inequalities, we have

$$\Big(z(t;\mu_0),h\big(t_{00},\varphi_0,z(\,\cdot\,;\mu_0)(\tau_0(t))\big)\Big)\in K^2(\varepsilon_0/2)\subset Q,\ \ t\in[r_1-\delta_1,r_2+\delta_1].$$

Hence,

$$\chi\Big(z(t;\mu_0),h\big(t_{00},\varphi_0,z(\,\cdot\,;\mu_0)(\tau_0(t))\big)\Big)=1,\ \ t\in[r_1-\delta_1,r_2+\delta_1],$$

and the function $z(t;\mu_0)$ satisfies the equation (1.20) and the condition (1.21).

Therefore,

$$y(t;\mu_0)=z(t;\mu_0),\ \ t\in[r_1-\delta_1,r_2+\delta_1].$$

According to the fixed point theorem, for $\varepsilon_0/2$ there exists a number $\delta_0\in(0,\varepsilon_0)$ such that a solution $z(t;\mu)$ satisfying the condition

$$\big|z(t;\mu)-z(t;\mu_0)\big|\le\frac{\varepsilon_0}{2}\,,\ \ t\in I,$$

corresponds to each element $\mu\in V(\mu_0;K_1,\delta_0,\alpha)$.

Therefore, for $t\in[r_1-\delta_1,r_2+\delta_1]$

$$z(t;\mu)\in K(\varepsilon_0),\ \ \forall\,\mu\in V(\mu_0;K_1,\delta_0,\alpha).$$

Taking into account the fact that $\varphi(t)\in K(\varepsilon_0)$, we can see that for $t\in[r_1-\delta_1,r_2+\delta_1]$, this implies

$$\chi\Big(z(t;\mu),h\big(t_0,\varphi,z(\,\cdot\,;\mu)(\tau(t))\big)\Big)=1,\ \ \forall\,\mu\in V(\mu_0;K_1,\delta_0,\alpha).$$

Hence the function $z(t;\mu)$ satisfies the equation (1.20) and the condition (1.21), i.e.

$$y(t;\mu)=z(t;\mu)\in \operatorname{int} K_1,\ \ t\in[r_1-\delta_1,r_2+\delta_1],\ \ \mu\in V(\mu_0;K_1,\delta_0,\alpha). \quad (1.58)$$

The first part of Theorem 1.14 is proved. By the fixed point theorem, for an arbitrary $\varepsilon>0$, there exists a number $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0)$ such that for each $\mu\in V(\mu_0;K_1,\delta_2,\alpha)$,

$$\big|z(t;\mu)-z(t;\mu_0)\big|\leq\varepsilon,\ \ t\in I,$$

whence using (1.58), we obtain (1.23). □

Proof of Theorem 1.1. In Theorem 1.14, let $r_1=t_{00}$ and $r_2=t_{00}$. Obviously, the solution $x_0(t)$ satisfies on the interval $[t_{00},t_{10}]$ the following equation:

$$\dot y(t)=A(t)h(t_{00},v_0,\dot y)(\sigma(t))+f_0(t_{00},\tau_0,\varphi_0,y)(t).$$

Therefore, in Theorem 1.14, as the solution $y_0(t)$ we can take the function $x_0(t), t\in[t_{00},t_{10}]$.

By Theorem 1.14, there exist numbers $\delta_i>0, i=0,1$, and for an arbitrary $\varepsilon>0$, there exists a number $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0]$ such that the solution $y(t;\mu)$, $t\in[t_{00}-\delta_1,t_{10}+\delta_1]$, corresponds to each $\mu\in V(\mu_0;K_1,\delta_0,\alpha)$. Moreover, the following conditions hold:

$$\begin{cases}\varphi(t)\in K_1, & t\in I_1;\ \ y(t;\mu)\in K_1,\\ \big|y(t;\mu)-y(t;\mu_0)\big|\leq\varepsilon, & t\in[t_{00}-\delta_1,t_{10}+\delta_1],\\ & \mu\in V(\mu_0;K_1,\delta_2,\alpha).\end{cases} \quad (1.59)$$

For an arbitrary $\mu\in V(\mu_0;K_1,\delta_0,\alpha)$, the function

$$x(t;\mu)=\begin{cases}\varphi(t), & t\in[\widehat\tau,t_0),\\ y(t;\mu), & t\in[t_0,t_1+\delta_1].\end{cases}$$

is the solution corresponding to μ. Moreover, if $t\in[\widehat t,t_{10}+\delta_1]$, then $x(t;\mu_0)=y(t;\mu_0)$ and $x(t;\mu)=y(t;\mu)$. Taking into account (1.59), we can see that this implies 1.1 and 1.2. It is easy to notice that for an arbitrary $\mu\in V(\mu_0;K_1,\delta_0,\alpha)$, we have

$$\int\limits_{\widehat\tau}^{t_{10}+\delta_1}\big|x(t;\mu)-x(t;\mu_0)\big|\,dt=\int\limits_{\widehat\tau}^{\bar t}\big|\varphi(t)-\varphi_0(t)\big|\,dt+$$

$$+\int\limits_{\bar t}^{\widehat t}\big|x(t;\mu)-x(t;\mu_0)\big|\,dt+\int\limits_{\widehat t}^{t_{10}+\delta_1}|x(t;\mu)-x(t;\mu_0)|\,dt\leq$$

$$\leq\|\varphi-\varphi_0\|_{I_1}(b-\widehat\tau)+N|t_0-t_{00}|+\max_{t\in[\widehat t,t_{10}+\delta_1]}\big|x(t;\mu)-x(t;\mu_0\big|(b-\widehat\tau),$$

where

$$\bar t=\min\{t_0,t_{00}\},\quad N=\sup\big\{|x'-x''|:\ x',x''\in K_1\big\}.$$

By 1.1 and 1.2, this inequality implies 1.3. □

1.4. **Proof of Theorem 1.3.** To each element $w \in \Lambda_1$ we correspond the equation

$$\dot{y}(t) = A(t)h(t_0, v, \dot{y})(\sigma(t)) + f(t_0, \tau, \varphi, y, u)(t)$$

with the initial condition (1.21).

Theorem 1.15. *Let $y_0(t)$ be a solution corresponding to $w_0 = (t_0, \tau_0, x_{00}, \varphi_0, v_0, u_0) \in \Lambda_1$ and defined on $[r_1, r_2] \subset (a, b)$. Let $K_2 \subset O$ be a compact set containing a certain neighborhood of the set $\mathrm{cl}\,\varphi_0(I_1) \cup y_0([r_1, r_2])$. Then the following conditions hold:*

1.17. *there exist numbers $\delta_i > 0$, $i = 0, 1$ such that to each element*

$$w = (t_0, \tau, x_0, \varphi, v, u) \in \widehat{V}(w_0; \delta_0)$$

there corresponds the solution $y(t; w)$ defined on the interval $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ and satisfying the condition $y(t; w) \in K_2$;

1.18. *for an arbitrary $\varepsilon > 0$ there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0]$ such that the following inequality holds for any $w \in \widehat{V}(w_0; \delta_0$*

$$\big|y(t; w) - y(t; w_0)\big| \le \varepsilon, \ \ \forall\, t \in [r_1 - \delta_1, r_2 + \delta_1].$$

Theorem 1.15 is proved analogously to Theorem 1.14.

Proof of Theorem 1.3. In Theorem 1.15, let $r_1 = t_{00}$ and $r_2 = t_{10}$. Obviously, the solution $x_0(t)$ satisfies on the interval $[t_{00}, t_{10}]$ the following equation:

$$\dot{y}(t) = A(t)h(t_{00}, v_0, \dot{y})(\sigma(t)) + f(t_{00}, \tau_0, \varphi_0, y, u_0)(t).$$

Therefore, in Theorem 1.15, as the solution $y_0(t)$ we can take the function $x_0(t), t \in [t_{00}, t_{10}]$. Then the proof of the theorem completely coincides with that of Theorem 1.1; for this purpose, it suffices to replace the element μ by the element w and the set $V(\mu_0; K_1, \delta_0, \alpha)$ by the set $\widehat{V}(w_0; \delta_0)$ everywhere. □

2. Variation Formulas of a Solution

Let $D_1 = \{\tau \in D : \ \dot{\tau}(t) \ge e = const > 0, \ t \in \mathbb{R}\}$ and let $E_f^{(1)}$ be the set of functions $f : I \times O^2 \to \mathbb{R}^n$ satisfying the following conditions: the function $f(t, \cdot\,) : O^2 \to \mathbb{R}^n$ is continuously differentiable for almost all $t \in I$; the functions $f(t, x_1, x_2)$, $f_{x_1}(t, x_1, x_2)$ and $f_{x_2}(t, x_1, x_2)$ are measurable on I for any $(x_1, x_2) \in O^2$; for each $f \in E_f^{(1)}$ and compact set $K \subset O$, there exists a function $m_{f,K}(t) \in L(I, \mathbb{R}_+)$, such that

$$\big|f(t, x_1, x_2)\big| + \big|f_{x_1}(t, x_1, x_2)\big| + \big|f_{x_2}(t, x_1, x_2)\big| \le m_{f,K}(t)$$

for all $(x_1, x_2) \in K^2$ and almost all $t \in I$.

To each element

$$\mu = (t_0, \tau, x_0, \varphi, v, f) \in \Lambda_2 = [a, b) \times D_1 \times O \times \Phi_1 \times E_v \times E_f^{(1)}$$

we assign the neutral equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f\big(t, x(t), x(\tau(t))\big)$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \ \ \dot{x}(t) = v(t), \ \ t \in [\widehat{\tau}, t_0), \ \ x(t_0) = x_0.$$

Let $\mu_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, v_0, f_0) \in \Lambda_2$ be a given element and $x_0(t)$ be the solution corresponding to μ_0 and defined on $[\widehat{\tau}, t_{10}]$, with $a < t_{00} < t_{10} < b$.

In the space $E_\mu^{(1)} - \mu_0$, where $E_\mu^{(1)} = \mathbb{R} \times D_1 \times \mathbb{R}^n \times E_\varphi \times E_v \times E_f^{(1)}$, we introduce the set of variations:

$$\Im_2 = \Big\{\delta\mu = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi, \delta v, \delta f) \in E_\mu^{(1)} - \mu_0 :$$
$$|\delta t_0| \le \beta, \ \|\delta\tau\|_{I_2} \le \beta, \ |\delta x_0| \le \beta, \ \delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i,$$
$$\|\delta v\|_{I_1} \le \beta, \ \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, \ |\lambda_i| \le \beta, \ i = 1, \dots, k\Big\},$$

where $\delta\varphi_i \in E_\varphi - \varphi_0$, $\delta f_i \in E_f^{(1)} - f_0$, $i = 1, \dots, k$, are fixed functions.

The inclusion $E_f^{(1)} \subset E_f$ holds (see [15, Lemma 2.1.2]), therefore, according to Theorem 1.2, there exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im_2$ the element $\mu_0 + \varepsilon\delta\mu \in \Lambda_2$, and there corresponds the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$.

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$. Therefore, the solution $x_0(t)$ is assumed to be defined on the interval $[\widehat{\tau}, t_{10} + \delta_1]$.

Let us define the increment of the solution $x_0(t) = x(t; \mu_0)$:

$$\Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), \ \ \forall\, (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times \Im_2.$$

Theorem 2.1. *Let the following conditions hold:*

2.1. $\gamma_0(t_{00}) < t_{10}$, *where* $\gamma_0(t)$ *is the inverse function to* $\tau_0(t)$*;*

2.2. *the functions* $v_0(\sigma(t))$ *and* $v_0(t)$ *are continuous at the point* t_{00}*; the function* $\varphi_0(t)$ *is absolutely continuous and the function* $\dot{\varphi}_0(t)$ *is bounded;*

2.3. *for each compact set* $K \subset O$ *there exists a number* $m_K > 0$ *such that*

$$|f_0(z)| \le m_K, \ \ \forall\, z = (t, x, y) \in I \times K^2;$$

2.4. *there exist the limits*

$$\lim_{z \to z_0} f_0(z) = f_0^-, \ \ z \in (a, t_{00}] \times O^2,$$

$$\lim_{(z_1, z_2) \to (z_{10}, z_{20})} \big[f_0(z_1) - f_0(z_2)\big] = f_{01}^-, \ \ z_i \in (t_{00}, \gamma_0(t_{00})] \times O^2, \ \ i = 1, 2,$$

where

$$z_0 = (t_{00}, x_{00}, \varphi_0(\tau_0(t_{00}))), \quad z_{10} = \big(\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), x_{00}\big),$$
$$z_{20} = \big(\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), \varphi_0(t_{00})\big).$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *such that for arbitrary*

$$(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im_2^-,$$

where

$$\Im_2^- = \big\{\delta\mu \in \Im_2 : \ \delta t_0 \le 0, \ \delta\tau(\gamma_0(t_{00})) > 0\big\}$$

we have

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu), \tag{2.1}$$

where

$$\begin{aligned} \delta x(t; \delta\mu) = \Big\{ & Y(t_{00}-; t)\Big[v_0(t_{00}) - A(t_{00})v_0(\sigma(t_{00})) - f_0^-\Big] - \\ & - Y(\gamma_0(t_{00})-; t) f_{01}^- \dot\gamma_0(t_{00})\Big\}\delta t_0 + \\ & + Y(\gamma_0(t_{00})-; t) f_{01}^- \dot\gamma_0(t_{00})\delta\tau(\gamma_0(t_{00})) + \\ & + \int_{t_{00}}^{t} Y(s; t)\delta f[s]\, ds + \beta(t; \delta\mu), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \beta(t; \delta\mu) = {} & \Psi(t_{00}; t)\big[\delta x_0 - v_0(t_{00})\delta t_0\big] + \\ & + \int_{t_{00}}^{\gamma_0(t_{00})} Y(s; t) f_{0x_2}[s]\dot\varphi_0(\tau_0(s))\delta\tau(s)\, ds + \\ & + \int_{\gamma(t_{00})}^{t} Y(s; t) f_{0x_2}[s]\dot x_0(\tau_0(s))\delta\tau(s)\, ds + \\ & + \int_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s); t) f_{0x_2}[\gamma_0(s)]\dot\gamma_0(s)\delta\varphi(s)\, ds + \\ & + \int_{\sigma(t_{00})}^{t_{00}} Y(\nu(s); t) A(\nu(s))\dot\nu(s)\delta v(s)\, ds \end{aligned} \tag{2.3}$$

Here, $\Psi(s;t)$ *and* $Y(s;t)$ *are* $n\times n$ *matrix functions satisfying the system*

$$\begin{cases}\Psi_s(s;t)=-Y(s;t)f_{0x_1}[t]-Y(\gamma_0(s);t)f_{0x_2}[\gamma_0(s)]\dot\gamma_0(s),\\ Y(s;t)=\Psi(s;t)+Y(\nu(s);t)A(\nu(s))\dot\nu(s),\\ \qquad\qquad s\in[t_{00}-\delta_2,t],\ \ t\in[t_{00},t_{10}+\delta_2]\end{cases}$$

and the condition

$$\Psi(s;t)=Y(s;t)=\begin{cases}H, & s=t,\\ \Theta, & s>t;\end{cases}$$

H *is the identity matrix and* Θ *is the zero matrix,* $\nu(s)$ *is the inverse function to* $\sigma(s)$*,*

$$f_{0x_1}[s]=f_{0x_1}\big(s,x_0(s),x_0(\tau_0(s))\big),\ \ \delta f[s]=\delta f\big(s,x_0(s),x_0(\tau_0(s))\big).$$

Some Comments. The function $\delta x(t;\delta\mu)$ is called the variation of the solution $x_0(t)$, $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$, and the expression (2.2) is called the variation formula.

Theorem 2.1 corresponds to the case where the variation at the point t_{00} is performed on the left.

The expression

$$-Y(\gamma_0(t_{00})-;t)f_{01}^-\dot\gamma_0(t_{00})\delta t_0$$

is the effect of the discontinuous initial condition and perturbation of the initial moment t_{00}.

The expression

$$Y(\gamma_0(t_{00})-;t)f_{01}^-\dot\gamma_0(t_{00})\delta\tau(\gamma_0(t_{00}))+$$
$$+\int\limits_{t_{00}}^{\gamma_0(t_{00})}Y(s;t)f_{0x_2}[s]\dot\varphi_0(\tau_0(s))\delta\tau(s)\,ds+\int\limits_{\gamma_0(t_{00})}^{t}Y(s;t)f_{0x_2}[s]\dot x_0(\tau_0(s))\delta\tau(s)\,ds$$

is the effect of perturbation of the delay function $\tau_0(t)$ (see (2.2) and (2.3)).

The addend

$$Y(t_{00}-;t)\Big[v_0(t_{00})-A(t_{00})v_0(\sigma(t_{00}))-f_0^-\Big]\delta t_0+\Psi(t_{00};t)\big[\delta x_0-v_0(t_{00})\delta t_0\big]$$

is the effect of perturbations of the initial moment t_{00} and the initial vector x_{00}.

The expression

$$\int\limits_{\tau_0(t_{00})}^{t_{00}}Y(\gamma_0(s);t)f_{0x_2}[\gamma_0(s)]\dot\gamma_0(s)\delta\varphi(s)\,ds+$$
$$+\int\limits_{\sigma(t_{00})}^{t_{00}}Y(\nu(s);t)A(\nu(s))\dot\nu(s)v(s)\,ds+\int\limits_{t_{00}}^{t}Y(s;t)\delta f[s]\,ds$$

is the effect of perturbations of the initial functions $\varphi_0(t)$ and $v_0(s)$ and the function $f_0(t,x,y)$.

If $\varphi_0(t_{00})=x_{00}$, then $f_{01}^-=0$. If $\gamma_0(t_{00})=t_{10}$, then Theorem 2.1 is valid on the interval $[t_{10},t_{10}+\delta_2]$. If $\gamma_0(t_{00})>t_{10}$, then Theorem 2.1 is valid, with $\delta_2\in(0,\delta_1)$ such that $t_{10}+\delta_2<\gamma_0(t_{00})$; in this case $Y(\gamma_0(t_{00})-;t)=\Theta$.

Finally, we note that the variation formula allows us to obtain an approximate solution of the perturbed equation

$$\begin{aligned}\dot x(t)=A(t)\dot x(\sigma(t))+f_0\big(t,x(t),x(\tau_0(t)+\varepsilon\delta\tau(t))\big)+\\+\varepsilon\delta f\big(t,x(t),x(\tau_0(t)+\varepsilon\delta\tau(t))\big)\end{aligned}$$

with the perturbed initial condition

$$\begin{gathered}x(t)=\varphi_0(t)+\varepsilon\delta\varphi(t),\ \ \dot x(t)=v_0(t)+\varepsilon\delta v(t),\ \ t\in[\widehat\tau,t_{00}+\varepsilon\delta t_0),\\ x(t_{00}+\varepsilon\delta t_0)=x_{00}+\varepsilon\delta x_0.\end{gathered}$$

In fact, for a sufficiently small $\varepsilon\in(0,\varepsilon_2)$ it follows from (2.1) that

$$x(t;\mu_0+\varepsilon\delta\mu)\approx x_0(t)+\varepsilon\delta x(t;\delta\mu).$$

The matrix function $Y(\xi;t)$ for any fixed $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$ has first order discontinuity at the points of the set

$$\big\{\sigma(t),\sigma^2(t),\dots,\sigma^i(t),\dots\big\},$$

where $\sigma^i(t)=\sigma(\sigma^{i-1}(t))$, $i=1,2,\dots$; $\sigma^0(t)=t,\sigma^1(t)=\sigma(t)$ (see Theorem 1.13).

Theorem 2.2. *Let the conditions* 2.1–2.3 *of Theorem* 2.1 *hold. Moreover, there exist the limits*

$$\begin{gathered}\lim_{z\to z_0}f_0(z)=f_0^+,\ \ z\in[t_{00},\gamma_0(t_{00}))\times O^2,\\ \lim_{(z_1,z_2)\to(z_{10},z_{20})}\big[f_0(z_1)-f_0(z_2)\big]=f_{01}^+,\ \ z_i\in[\gamma_0(t_{00}),b)\times O^2,\ \ i=1,2.\end{gathered}$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary

$$(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times\Im_2^+,$$

where

$$\Im_2^+=\big\{\delta\mu\in\Im_2:\ \delta t_0\geq0,\ \delta\tau(\gamma_0(t_{00}))<0\big\},$$

formula (2.1) is valid, where

$$\begin{aligned}\delta x(t;\delta\mu)=\Big\{Y(t_{00}+;t)\Big[v_0(t_{00})-A(t_{00})v_0(\sigma(t_{00}))-f_0^+\Big]-\\-Y(\gamma_0(t_{00})+;t)f_{01}^+\dot\gamma_0(t_{00})\Big\}\delta t_0+\\+Y(\gamma_0(t_{00})+;t)f_{01}^+\dot\gamma_0(t_{00})\delta\tau(\gamma_0(t_{00}))+\\+\int_{t_{00}}^{t}Y(s;t)\delta f[s]\,ds+\beta(t;\delta\mu).\end{aligned}$$

Theorem 2.2 corresponds to the case where the variation at the point t_{00} is performed on the right.

Theorem 2.3. *Let the assumptions of Theorems* 2.1 *and* 2.2 *be fulfilled. Moreover,*

$$f_0^- = f_0^+ := \widehat{f}_0, \;\; f_{01}^- = f_{01}^+ := \widehat{f}_{01}$$

and

$$t_{00}, \gamma_0(t_{00}) \notin \big\{\sigma(t_{10}), \sigma^2(t_{10}), \dots\big\}.$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2] \times \Im_2$ formula (2.1) *holds, where*

$$\begin{aligned}\delta x(t;\delta\mu) = \Big\{ & Y(t_{00};t)\Big[v_0(t_{00}) - A(t_{00})v_0(\sigma(t_{00})) - \widehat{f}_0\Big] - \\ & - Y(\gamma_0(t_{00});t)\widehat{f}_{01}\dot{\gamma}_0(t_{00})\Big\}\delta t_0 + \\ & + Y(\gamma_0(t_{00});t) f_{01}\dot{\gamma}_0(t_{00})\delta\tau(\gamma_0(t_{00})) + \\ & + \int\limits_{t_{00}}^{t} Y(s;t)\delta f[s]\, ds + \beta(t;\delta\mu).\end{aligned}$$

Theorem 2.3 corresponds to the case where the variation at the point t_{00} two-sided is performed. If the function $f_0(t, x, y)$ is continuous, then

$$\widehat{f}_0 = f_0\big(t_{00}, \varphi_0(t_{00}), \varphi_0(\tau_0(t_{00}))\big)$$

and

$$\widehat{f}_{01} = f_0\big(\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), x_{00}\big) - f_0\big(\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), \varphi_0(t_{00})\big).$$

Let the function $f(t, x_1, x_2, u)$ be defined on $I \times O^2 \times U_0$ and satisfy the conditions: for almost all $t \in I$ the function $f(t, x, y, u)$ is continuously differentiable with respect to $(x_1, x_2, u) \in O^2 \times U_0$; for any fixed $(x_1, x_2, u) \in O_2 \times U_0$ the functions $f(t, x_1, x_2, u)$, $f_{x_1}(t, x_1, x_2, u)$, $f_{x_2}(t, x_1, x_2, u)$, $f_u(t, x_1, x_2, u)$ are measurable, for any compacts $K \subset O$ and $U \subset U_0$ there exists $m_{K,U}(t) \in L(I, R_+)$ such that

$$\big|f(t, x_1, x_2, u)\big| + \big|f_{x_1}(t, x_1, x_2, u)\big| + \big|f_{x_2}(t, x_1, x_2, u)\big| + \big|f_u(t, x_1, x_2, u)\big| \leq \\ \leq m_f(t)$$

for all $(x_1, x_2, u) \in K^2 \times U$ and almost all $t \in I$.

Let $w_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, v_0, u_0) \in \Lambda_1$ be the given element and $x_0(t)$ be the solution corresponding to w_0 and defined on $[\widehat{\tau}, t_{10}]$, with $a < t_{00} < t_{10} < b$.

In the space $E_w - w_0$ we introduce the set of variations

$$\Im_3 = \Big\{\delta w = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi, \delta v, \delta u) \in E_w - w_0 :$$
$$|\delta t_0| \le \beta,\ \|\delta\tau\|_{I_2} \le \beta,\ |\delta x_0| \le \beta,\ \delta\varphi = \sum_1^k \lambda_i \delta\varphi_i,$$
$$|\lambda_i| \le \beta,\ i = 1, \dots, k,\ \|\delta v\|_{I_1} \le \beta,\ \|\delta u\|_I \le \beta\Big\}.$$

There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta w) \in (0, \varepsilon_1) \times \Im_3$ the element $w_0 + \varepsilon\delta w \in \Lambda_1$ and there corresponds the solution $x(t; w_0 + \varepsilon\delta w)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$.

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$. Therefore, the solution $x_0(t)$ is assumed to be defined on the interval $[\widehat{\tau}, t_{10} + \delta_1]$.

Let us define the increment of the solution $x_0(t) = x(t; w_0)$:

$$\Delta x(t; \varepsilon\delta w) = x(t; w_0 + \varepsilon\delta w) - x_0(t),\quad \forall\, (t, \varepsilon, \delta w) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times \Im_3.$$

Theorem 2.4. *Let the following conditions hold:*

2.5. $\gamma_0(t_{00}) < t_{10}$, *where* $\gamma_0(t)$ *is the inverse function to* $\tau_0(t)$*;*

2.6. *the functions* $v_0(\sigma(t))$ *and* $v_0(t)$ *are continuous at the point* t_{00}*; the function* $\varphi_0(t)$ *is absolutely continuous and the function* $\dot\varphi_0(t)$ *is bounded;*

2.7. *for each compact sets* $K \subset O$ *and* $U \subset U_0$ *there exists a number* $m_{K,U} > 0$ *such that*
$$|f_0(z)| \le m_{K,U},\quad \forall\, z = (t, x, y, u) \in I \times K^2 \times U;$$

2.8. *there exist the limits*
$$\lim_{z \to z_0} f_0(z) = f_0^-,\quad z \in (a, t_{00}] \times O^2,$$
$$\lim_{(z_1, z_2) \to (z_{10}, z_{20})} \big[f_0(z_1) - f_0(z_2)\big] = f_{01}^-,\quad z_i \in (t_{00}, \gamma_0(t_{00})] \times O^2,\quad i = 1, 2,$$
where
$$z_0 = \big(t_{00}, x_{00}, \varphi_0(\tau_0(t_{00}))\big),\quad z_{10} = \big(\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), x_{00}\big),$$
$$z_{20} = \big(\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), \varphi_0(t_{00})\big),\quad f_0(z) = f(z, u_0(t)).$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *such that for arbitrary* $(t, \varepsilon, \delta w) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im_3^-$, *where*
$$\Im_3^- = \big\{\delta w \in \Im_3 :\ \delta t_0 \le 0,\ \delta\tau(\gamma_0(t_{00})) > 0\big\}$$
we have
$$\Delta x(t; \varepsilon\delta w) = \varepsilon\delta x(t; \delta w) + o(t; \varepsilon\delta w), \tag{2.4}$$

where

$$\delta x(t;\delta w) = \Big\{Y(t_{00}-;t)\Big[v_0(t_{00}) - A(t_{00})v_0(\sigma(t_{00})) - f_0^-\Big] - \\ - Y(\gamma_0(t_{00})-;t)f_{01}^-\dot{\gamma}_0(t_{00})\Big\}\delta t_0 + \\ + Y(\gamma_0(t_{00})-;t)f_{01}^-\dot{\gamma}_0(t_{00})\delta\tau(\gamma_0(t_{00})) + \\ + \int\limits_{t_{00}}^{t} Y(s;t)f_{0u}[s]\delta u(s)\,ds + \beta(t;\delta w),$$

and

$$\beta(t;\delta w) = \beta(t;\delta\mu).$$

Theorem 2.5. *Let the conditions* 2.5–2.7 *of Theorem* 2.4 *hold. Moreover, there exist the limits*

$$\lim_{z\to z_0} f_0(z) = f_0^+,\ \ z \in [t_{00}, \gamma_0(t_{00})) \times O^2,$$

$$\lim_{(z_1,z_2)\to(z_{10},z_{20})} \big[f_0(z_1) - f_0(z_2)\big] = f_{01}^+,\ \ z_i \in [\gamma_0(t_{00}), b) \times O^2,\ \ i = 1,2,$$

where $f_0(z) = f(z, u_0(t))$. *Then there exist numbers* $\varepsilon_2 \in (0,\varepsilon_1)$ *and* $\delta_2 \in (0,\delta_1)$ *such that for arbitrary*

$$(t,\varepsilon,\delta w) \in [t_{10}-\delta_2, t_{10}+\delta_2] \times (0,\varepsilon_2) \times \Im_3^+,$$

where

$$\Im_3^+ = \big\{\delta w \in \Im_3:\ \delta t_0 \geq 0,\ \delta\tau(\gamma_0(t_{00})) < 0\big\},$$

formula (2.4) *is valid, where*

$$\delta x(t;\delta w) = \Big\{Y(t_{00}+;t)\Big[v_0(t_{00}) - A(t_{00})v_0(\sigma(t_{00})) - f_0^+\Big] - \\ - Y(\gamma_0(t_{00})+;t)f_{01}^+\dot{\gamma}_0(t_{00})\Big\}\delta t_0 + \\ + Y(\gamma_0(t_{00})+;t)f_{01}^+\dot{\gamma}_0(t_{00})\delta\tau(\gamma_0(t_{00})) + \\ + \int\limits_{t_{00}}^{t} Y(s;t)f_{0u}[s]\,ds + \beta(t;\delta\mu).$$

Theorem 2.6. *Let the assumptions of Theorems* 2.4 *and* 2.5 *be fulfilled. Moreover,*

$$f_0^- = f_0^+ := \widehat{f}_0,\quad f_{01}^- = f_{01}^+ := \widehat{f}_{01}$$

and

$$t_{00}, \gamma_0(t_{00}) \notin \big\{\sigma(t_{10}), \sigma^2(t_{10}), \dots\big\}.$$

Then there exist numbers $\varepsilon_2 \in (0,\varepsilon_1)$ *and* $\delta_2 \in (0,\delta_1)$ *such that for arbitrary*

$$(t,\varepsilon,\delta w) \in [t_{10}-\delta_2, t_{10}+\delta_2] \times (0,\varepsilon_2] \times \Im_3$$

formula (2.4) *holds, where*

$$\delta x(t;\delta w)=\Big\{Y(t_{00};t)\Big[v_0(t_{00})-A(t_{00})v_0(\sigma(t_{00}))-\widehat{f}_0\Big]-\\ -Y(\gamma_0(t_{00});t)\widehat{f}_{01}\dot{\gamma}_0(t_{00})\Big\}\delta t_0+\\ +Y(\gamma_0(t_{00});t)f_{01}\dot{\gamma}_0(t_{00})\delta\tau(\gamma_0(t_{00}))+\\ +\int\limits_{t_{00}}^{t}Y(s;t)f_{0u}[s]\delta u(s)\,ds+\beta(t;\delta w).$$

2.1. **Proof of Theorem 2.1.** First of all, we note that Lemma 2.1 formulated below is a consequence of Theorem 1.14.

Lemma 2.1. *Let $y_0(t)$ be a solution corresponding to $\mu_0\in\Lambda$ and defined on $[r_1,r_2]\subset(a,b)$. Let $K_1\subset O$ be a compact set containing a certain neighborhood of the set $\operatorname{cl}\varphi_0(I_1)\cup y_0([r_1,r_2])$. Then there exist numbers $\varepsilon_1>0$ and $\delta_1>0$ such that for an arbitrary $(t,\delta\mu)\in(0,\varepsilon_1)\times\Im_2$, we have $\mu_0+\varepsilon\delta\mu\in\Lambda$, and the solution $y(t;\mu_0+\varepsilon\delta\mu)$ defined on $[r_1-\delta_1,r_2+\delta_1]\subset I$ corresponds to this element. Moreover,*

$$\varphi(t)\in K_1,\ \ t\in I_1;\quad y(t;\mu_0+\varepsilon\delta\mu)\in K_1,\ \ t\in[r_1-\delta_1,r_2+\delta_1];\\ \lim_{\varepsilon\to 0}y(t;\mu_0+\varepsilon\delta\mu)=y(t;\mu_0),$$

uniformly in $(t,\delta\mu)\in[r_1-\delta_1,r_2+\delta_1]\times\Im_2$.

The solution $y(t;\mu_0)$ on the interval $[r_1-\delta_1,r_2+\delta_1]$ is a continuation of the solution $y_0(t)$. Therefore, in what follows, we can assume that the solution $y_0(t)$ is defined on the whole interval $[r_1-\delta_1,r_2+\delta_1]$.

Let us define the increment of the solution $y_0(t)=y(t;\mu_0)$:

$$\Delta y(t)=\Delta y(t;\varepsilon\delta\mu)=y(t;\mu_0+\varepsilon\delta\mu)-y_0(t),\\ \forall\,(t,\varepsilon,\delta\mu)\in[r_1-\delta_1,r_2+\delta_1]\times(0,\varepsilon_1)\times\Im_2.$$

Obviously,

$$\lim_{\varepsilon\to\infty}\Delta y(t;\varepsilon\delta\mu)=0,\tag{2.5}$$

uniformly in $(t,\delta\mu)\in[r_1-\delta_1,r_2+\delta_1]\times\Im_2$.

Lemma 2.2. *Let $\gamma_0(t_{00})<r_2$ and let the conditions of Theorem 2.1 be fulfilled. Then there exists a number $\varepsilon_2\in(0,\varepsilon_1)$ such that for any $(t,\delta\mu)\in(0,\varepsilon_2)\times\Im_2^-$ the inequality*

$$\max_{t\in[t_{00},r_2+\delta_1]}|\Delta y(t)|\le O(\varepsilon\delta\mu)\tag{2.6}$$

is valid. Moreover,

$$\Delta y(t_{00})=\varepsilon\Big\{\delta x_0-\big[A(t_{00})v_0(\sigma(t_{00}))+f_0^-\big]\delta t_0\Big\}+o(\varepsilon\delta\mu).\tag{2.7}$$

Proof. Let $\varepsilon_2 \in (0, \varepsilon_1)$ be so small that for any $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times \Im_2^-$ the following relations are fulfilled:

$$\tau(t) := \tau_0(t) + \varepsilon\delta\tau(t) < t_0 := t_{00} + \varepsilon\delta t_0, \quad \forall\, t \in [t_0, t_{00}]. \tag{2.8}$$

The function $\Delta y(t)$ on the interval $[t_{00}, r_2 + \delta_1]$ satisfies the equation

$$\dot{\Delta} y(t) = A(t)h(t_{00}, \varepsilon\delta v, \dot{\Delta} y)(\sigma(t)) + \sum_{i=1}^{3} W_i(t; \varepsilon\delta\mu),$$

where

$$\begin{aligned} W_1(t;\varepsilon\delta\mu) &= A(t)\Big[h(t_0, v, \dot{y}_0 + \dot{\Delta} y)(\sigma(t)) - h(t_{00}, v, \dot{y}_0 + \dot{\Delta} y)(\sigma(t))\Big], \\ W_2(t;\varepsilon\delta\mu) &= f_0(t_0, \tau, \varphi, y_0 + \Delta y)(t) - f_0(t_{00}, \tau_0, \varphi_0, y_0)(t), \\ W_3(t;\varepsilon\delta\mu) &= \varepsilon\delta f(t_0, \tau, \varphi, y_0 + \Delta y)(t), \\ & v := v_0 + \varepsilon\delta v, \quad \varphi := \varphi_0 + \varepsilon\delta\varphi. \end{aligned}$$

We now consider the linear nonhomogeneous neutral equation

$$\dot{z}(t) = A(t)\dot{z}(\sigma(t)) + \sum_{i=1}^{3} W_i(t;\varepsilon\delta\mu) \tag{2.9}$$

with the initial condition

$$\dot{z}(t) = \varepsilon\delta v(t), \ \ t \in [\widehat{\tau}, t_0), \quad z(t_0) = \Delta y(t_0).$$

Due to the uniqueness it is easily seen that $z(t) = \Delta y(t)$, $t \in [t_{00}, r_2 + \delta_1]$. According to Theorem 1.7, the solution of the equation (2.9) can be written in the form

$$\begin{aligned} \Delta y(t) = \Delta y(t_{00}) + \varepsilon \int\limits_{\sigma(t_{00})}^{t_{00}} Y(\nu(\xi);t)A(\nu(\xi))\dot{\nu}(\xi)\delta v(\xi)\,d\xi + \\ + \sum_{i=1}^{3}\int\limits_{t_{00}}^{t} Y(\xi;t)W_i(\xi;\varepsilon\delta\mu)\,d\xi, \end{aligned}$$

where $Y(\xi;t)$ has the form (1.11). Hence

$$|\Delta y(t)| \le |\Delta y(t_{00})| + \varepsilon\|Y\|\,\|A\|\alpha\big[\nu(t_{00}) - t_{00}\big] + \|Y\|\sum_{i=1}^{3} W_i(\varepsilon\delta\mu), \tag{2.10}$$

where

$$W_1(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{r_2+\delta_2}|W_1(t;\varepsilon\delta\mu)|\,dt,\quad W_2(t;t_{00},\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t}|W_2(\xi;\varepsilon\delta\mu)|\,d\xi,$$

$$W_3(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{r_2+\delta_2}|W_1(t;\varepsilon\delta\mu)|\,dt,\quad \|A\|=\sup\big\{|A(t)|:\ t\in I\big\},$$

$$\|Y\|=\sup\Big\{|Y(\xi;t)|:\ (\xi,t)\in[t_{00},r_2+\delta_1]\times[t_{00},r_2+\delta_1]\Big\}.$$

Let us prove equality (2.7). We have

$$\begin{aligned}\Delta y(t_{00})&=y(t_{00};\mu_0+\varepsilon\delta\mu)-x_{00}=\\&=x_{00}+\varepsilon\delta x_0+\int\limits_{t_0}^{t_{00}}A(t)[v_0(\sigma(t))+\varepsilon\delta v(\sigma(t))]\,dt+\\&\qquad+\int\limits_{t_0}^{t_{00}}f_0(t,y(t;\mu_0+\varepsilon\delta\mu),\varphi(\tau(t)))\,dt+\\&\qquad+\varepsilon\int\limits_{t_0}^{t_{00}}\delta f(t,y(t;\mu_0+\varepsilon\delta\mu),\varphi(\tau(t)))\,dt-x_{00}=\\&=\varepsilon\big[\delta x_0-A(t_{00})v_0(\sigma(t_{00}))\delta t_0\big]+o(\varepsilon\delta\mu)+\\&\qquad+\int\limits_{t_0}^{t_{00}}f_0(t,y_0(t)+\Delta y(t),\varphi(\tau(t)))\,dt+\\&\qquad+\varepsilon\sum_{i=1}^{k}\lambda_i\int\limits_{t_0}^{t_{00}}\delta f_i(t,y_0(t)+\Delta y(t),\varphi(\tau(t)))\,dt.\end{aligned}\tag{2.11}$$

It is clear that if $t\in[t_0,t_{00}]$, then

$$\lim_{\varepsilon\to 0}(t,y_0(t)+\Delta y(t),\varphi(t))=z_0$$

(see (2.5)). Consequently,

$$\lim_{\varepsilon\to 0}\sup_{t\in[t_0,t_{00}]}\Big|f_0\big(t,y_0(t)+\Delta y(t),\varphi(\tau(t))\big)-f_0^-\Big|=0.$$

This relation implies that

$$\int\limits_{t_0}^{t_{00}} f_0\big(t, y_0(t) + \Delta y(t), \varphi(\tau(t))\big)\,dt =$$

$$= -\varepsilon f_0^- \delta t_0 + \int\limits_{t_0}^{t_{00}} [f_0(t, y_0(t) + \Delta y(t), \varphi(\tau(t))) - f_0^-]\,dt =$$

$$= -\varepsilon f_0^- \delta t_0 + o(\varepsilon\delta\mu). \tag{2.12}$$

Further, we have

$$|\lambda_i| \int\limits_{t_0}^{t_{00}} \Big|\delta f_i\big(t, y_0(t) + \Delta y(t), \varphi(\tau(t))\big)\Big|\,dt \le \alpha \int\limits_{t_0}^{t_{00}} m_{\delta f_i, K_1}(t)\,dt. \tag{2.13}$$

From (2.11), by virtue of (2.12) and (2.13), we obtain (2.7).

Now, let us prove inequality (2.6). To this end, we have to estimate the expressions $W_1(\varepsilon\delta\mu), W_2(t; t_{00}, \varepsilon\delta\mu)$ and $W_3(\varepsilon\delta\mu)$. We have

$$W_1(\varepsilon\delta\mu) \le \|A\| \int\limits_{\nu(t_0)}^{\nu(t_{00})} \Big|\dot y(\sigma(t); \mu_0 + \varepsilon\delta\mu) - v_0(\sigma(t)) - \varepsilon\delta v(\sigma(t))\Big|\,dt.$$

Using the step method, we can prove the boundedness of $|\dot y(t; \mu_0 + \varepsilon\delta\mu)\|, t \in [r_1 - \delta_1, r_2 + \delta_1]$ uniformly in $\delta\mu \in \Im_2^-$ i.e. there exist $M > 0$ such that

$$\Big|\dot y\big(\sigma(t); \mu_0 + \varepsilon\delta\mu\big) - v_0(\sigma(t)) - \varepsilon\delta v(\sigma(t))\Big| \le M,$$
$$t \in [\nu(t_0), \nu(t_{00})], \ \ \forall\, \delta\mu \in \Im_2.$$

Moreover,

$$\nu(t_{00}) - \nu(t_0) = \int\limits_{t_0}^{t_{00}} \dot\nu(t)\,dt = O(\varepsilon\delta\mu).$$

Thus,

$$W_1(\varepsilon\delta\mu) = O(\varepsilon\delta\mu). \tag{2.14}$$

Let us estimate $W_2(t; t_{00}, \varepsilon\delta\mu)$. It is clear that

$$\gamma_0(t_0) - \gamma(t_0) = \int\limits_{\tau_0(\gamma(t_0))}^{t_0} \dot\gamma_0(\xi)\,d\xi = \int\limits_{t_0 - \varepsilon\delta\tau(\gamma(t_0))}^{t_0} \dot\gamma_0(\xi)\,d\xi > 0$$

and $\gamma(t_0) > t_{00}$ (see (2.8)). For $t \in [t_{00}, \gamma(t_0)]$, we have $\tau(t) < t_0$ and $\tau_0(t) < t_{00}$, therefore we get

$$W_2(t;t_{00},\varepsilon\delta\mu) \le \int\limits_{t_{00}}^{t} L_{f_0,K_1}(\xi)\Big[|\Delta y(\xi)| + |\varphi(\tau(\xi)) - \varphi_0(\tau_0(\xi))|\Big]\,d\xi \le$$
$$\le \int\limits_{t_{00}}^{t} L_{f_0,K_1}(\xi)\Delta y(\xi)\,d\xi + \bigg|\int\limits_{\tau_0(\xi)}^{\tau(\xi)} |\dot\varphi_0(s)|\,ds\bigg| + O(\varepsilon\delta\mu) =$$
$$= \int\limits_{t_{00}}^{t} L_{f_0,K_1}(\xi)\Delta y(\xi)\,d\xi + O(\varepsilon\delta\mu). \tag{2.15}$$

For $t \in [\gamma(t_0), \gamma_0(t_{00})]$, we have

$$W_2(t;t_{00},\varepsilon\delta\mu) = W_2(\gamma(t_0);t_{00},\varepsilon\delta\mu) + \int\limits_{\gamma(t_0)}^{t} W_2(\xi;\varepsilon\delta\mu)\,d\xi \le$$
$$\le O(\varepsilon\delta\mu) + \int\limits_{\gamma(t_0)}^{\gamma_0(t_{00})} W_2(\xi;\varepsilon\delta\mu)\,d\xi \le O(\varepsilon\delta\mu) + 2m_{K_1}|\gamma_0(t_{00}) - \gamma(t_0)|.$$

Next,

$$\big|\gamma_0(t_{00}) - \gamma(t_0)\big| = \int\limits_{\tau_0(\gamma(t_0))}^{t_{00}} \dot\gamma_0(\xi)\,d\xi = \int\limits_{\tau_0(\gamma(t_0))+\varepsilon\delta\tau(\gamma(t_0))-\varepsilon\delta\tau(\gamma(t_0))}^{t_{00}} \dot\gamma_0(\xi)\,d\xi =$$
$$= \int\limits_{t_0-\varepsilon\delta\tau(\gamma(t_0))}^{t_{00}} \dot\gamma_0(\xi)\,d\xi = O(\varepsilon\delta\mu,)$$

Consequently,

$$W_2(t;t_{00},\varepsilon\delta\mu) = O(\varepsilon\delta\mu), \ \ t \in [\gamma(t_0), \gamma_0(t_{00})]. \tag{2.16}$$

For $t \in (\gamma_0(t_{00}), r_1 + \delta_1]$, we have

$$W_2(t;t_0,\varepsilon\delta\mu) = W_2\big(\gamma_0(t_{00});t_0,\varepsilon\delta\mu\big) + \int\limits_{\gamma_0(t_{00})}^{t} W_2(\xi;\varepsilon\delta\mu)\,d\xi \le$$
$$\le O(\varepsilon\delta\mu) + \bigg|\int\limits_{\gamma_0(t_{00})}^{\gamma(t_{00})} W_2(\xi;\varepsilon\delta\mu)\,d\xi\bigg| +$$

$$+\Bigg|\int\limits_{\gamma(t_{00})}^{t}\chi(\xi)L_{f_0,K_1}(\xi)|\Delta y(\tau(\xi))|\,d\xi\Bigg|+\Bigg|\int\limits_{\gamma(t_{00})}^{t}\big|y_0(\tau(\xi))-y_0(\tau_0(\xi))\big|\,d\xi\Bigg|\le$$

$$\le O(\varepsilon\delta\mu)+2m_{K_1}|\gamma_0(t_{00})-\gamma(t_{00})|+$$

$$+\int\limits_{t_{00}}^{t}\chi(\gamma(\xi))L_{f_0,K_1}(\gamma(\xi))\dot\gamma(\xi)|\Delta y(\xi)|\,d\xi+\int\limits_{t_{00}}^{r_1+\delta_1}\Bigg|\int\limits_{\tau_0(\xi)}^{\tau(\xi)}|\dot y_0(s)|\,ds\Bigg|\,d\xi=$$

$$=O(\varepsilon\delta\mu)+2m_{K_1}\Big[\big|\gamma(t_{00})-\gamma(t_0)\big|+\big|\gamma(t_{00})-\gamma_0(t_{00})\big|\Big]+$$

$$+\int\limits_{t_{00}}^{t}\chi(\gamma(\xi))L_{f_0,K_1}(\gamma(\xi))\dot\gamma(\xi)|\Delta y(\xi)|\,d\xi$$

where $\chi(\xi)$ is the characteristic function of I. Next,

$$\gamma(t_{00})-\gamma(t_0)=\int\limits_{t_0}^{t_{00}}\dot\gamma(\xi)\,d\xi\le\frac{1}{e}(t_{00}-t_0)=O(\varepsilon\delta\mu)$$

and

$$\big|\gamma(t_{00})-\gamma_0(t_{00})\big|=\Bigg|\int\limits_{t_{00}}^{\tau_0(\gamma(t_{00}))}\dot\gamma_0(t)\,dt\Bigg|=$$

$$=\Bigg|\int\limits_{t_{00}}^{\tau(\gamma(t_{00}))-\varepsilon\delta(\gamma(t_{00}))}\dot\gamma_0(t)\,dt\Bigg|=O(\varepsilon\delta\mu).$$

Thus,

$$W_2(t;t_0,\varepsilon\delta\mu)=O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{t}\chi(\gamma(\xi))L_{f_0,K_1}(\gamma(\xi))\dot\gamma(\xi)|\Delta y(\xi)|\,d\xi. \quad (2.17)$$

Finally, we note that

$$W_3(t;\varepsilon\delta\mu)=O(\varepsilon\delta\mu),\ \ t\in[t_{00},r_2+\delta_1] \quad (2.18)$$

(see (2.12)).

According to (2.7), (2.14)–(2.18), inequality (2.10) directly implies that

$$|\Delta y(t)|\le O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{t}\Big[L_{f_0,K_1}(\xi)+\chi(\gamma(\xi))L_{f_0,K_1}(\gamma(\xi))\dot\gamma(\xi)\Big]\Delta y(\xi)|\,d\xi$$

By virtue of Grounwall's lemma, we obtain

$$|\Delta y(t)| \leq$$

$$\leq O(\varepsilon\delta\mu)\exp\bigg\{\int_{t_{00}}^{t} L_{f_0,K_1}(\xi)\,d\xi + \int_{t_{00}}^{t} \chi(\gamma(\xi))L_{f_0,K_1}(\gamma(\xi))\dot{\gamma}(\xi)\,d\xi\bigg\} \leq$$

$$\leq \exp\bigg\{2\int_{I} L_{f_0,K_1}(\xi)\,d\xi\bigg\}.$$

The following assertion can be proved by analogy with Lemma 2.2. □

Lemma 2.3. *Let $\gamma_0(t_{00}) < r_2$ and let the conditions of Theorem* 2.2 *be fulfilled. Then there exists the number $\varepsilon_2 \in (0, \varepsilon_1)$ such that for any $(t, \delta\mu) \in (0, \varepsilon_2) \times \Im_2^+$ the inequality*

$$\max_{t\in[t_0,r_2+\delta_1]} |\Delta y(t)| \leq O(\varepsilon\delta\mu)$$

is valid. Moreover,

$$\Delta y(t_{00}) = \varepsilon\Big\{\delta x_0 - \big[A(t_{00})v_0(\sigma(t_{00})) + f_0^+\big]\delta t_0\Big\} + o(\varepsilon\delta\mu).$$

Proof of Theorem 2.1. Let $r_1 = t_{00}$ and $r_2 = t_{10}$ in Lemma 2.1. Then

$$x_0(t) = \begin{cases} \varphi_0(t), & t \in [\widehat{\tau}, t_{00}), \\ y_0(t), & t \in [t_{00}, t_{10}], \end{cases}$$

and for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im_2^-$

$$x(t;\mu_0 + \varepsilon\delta\mu) = \begin{cases} \varphi(t) = \varphi_0(t) + \varepsilon\delta\varphi(t), & t \in [\widehat{\tau}, t_0), \\ y(t;\mu_0 + \varepsilon\delta\mu), & t \in [t_0, t_{10} + \delta_1] \end{cases}$$

(see Remark 1.1). We note that $\delta\mu \in \Im_2^-$, i.e. $t_0 < t_{00}$, therefore

$$\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t), & t \in [\widehat{\tau}, t_0), \\ y(t;\mu_0 + \varepsilon\delta\mu) - \varphi_0(t), & t \in [t_0, t_{00}), \\ \Delta y(t), & t \in [t_{00}, t_{10} + \delta_1]. \end{cases}$$

$$\dot{\Delta} x(t) = \begin{cases} \varepsilon\delta v(t), & t \in [\widehat{\tau}, t_0), \\ \dot{y}(t;\mu_0 + \varepsilon\delta\mu) - v_0(t), & t \in [t_0, t_{00}), \\ \dot{\Delta} y(t), & t \in [t_{00}, t_{10} + \delta_1]. \end{cases}$$

By Lemma 2.2, we have

$$|\Delta x(t)| \leq O(\varepsilon\delta\mu), \ \ \forall\,(t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times \Im_2^-, \tag{2.19}$$

$$\Delta x(t_{00}) = \varepsilon\Big\{\delta x_0 - \big[A(t_{00})v_0(\sigma(t_{00})) + f_0^-\big]\delta t_0\Big\} + o(\varepsilon\delta\mu). \tag{2.20}$$

The function $\Delta x(t)$ satisfies the equation

$$\dot{\Delta}x(t) = A(t)\dot{\Delta}x(\sigma(t)) + \\ + f_{0x}[t]\Delta x(t) + f_{0y}[t]\Delta x(\tau_0(t)) + \varepsilon\delta f[t] + R_1[t] + R_2[t], \quad (2.21)$$

where

$$R_1[t] = f_0\big(t, x_0(t) + \Delta x(t), x_0(\tau(t))\big) + \Delta x(\tau(t)) - \\ - f_0[t] - f_{0x_1}[t]\Delta x(t) - f_{0x_2}[t]\Delta x(\tau_0(t)),$$

$$R_2[t] = \varepsilon\Big[\delta f\Big(t, x_0(t) + \Delta x(t), x_0(\tau(t)) + \Delta x(\tau(t))\Big) - \delta f[t]\Big].$$

By using the Cauchy formula, one can represent the solution of the equation (2.21) in the form

$$\Delta x(t) = \Psi(t_{00}; t)\Delta x(t_{00}) + \\ + \varepsilon \int_{t_{00}}^{t} Y(\xi; t)\delta f[\xi]\, d\xi + \sum_{i=-1}^{2} R_i[t; t_{00}], t \in [t_{00}, t_{10} + \delta_1], \quad (2.22)$$

where

$$R_{-1}[t; t_{00}] = \int_{\sigma(t_{00})}^{t_{00}} Y(\nu(\xi); t)A(\nu(\xi))\dot{\nu}(\xi)\dot{\Delta}x(\xi)\, d\xi,$$

$$R_0[t; t_{00}] = \int_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(\xi); t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\, d\xi,$$

$$R_i[t; t_{00}] = \int_{t_{00}}^{t} Y(\xi; t)R_i[\xi]\, d\xi, \quad i = 1, 2,$$

By Theorem 1.13, we get

$$\Phi(t_{00}; t)\Delta x(t_{00}) = \\ = \varepsilon\Phi(t_{00}; t)\Big\{\delta x_0 - \big[A(t_{00})v_0(\sigma(t_{00})) + f_0^-\big]\delta t_0\Big\} + o(t; \delta\mu) \quad (2.23)$$

(see (2.20)).

Now, let us transform $R_{-1}[t; t_{00}]$. We have

$$R_{-1}[t; t_{00}] = \varepsilon \int_{\sigma(t_{00})}^{t_0} Y(\nu(\xi); t)A(\nu(\xi))\dot{\nu}(\xi)\delta v(\xi)\, d\xi + \\ + \int_{t_0}^{t_{00}} Y(\nu(\xi); t)A(\nu(\xi))\dot{\nu}(\xi)\dot{\Delta}x(\xi)\, d\xi =$$

$$= \varepsilon \int\limits_{\sigma(t_{00})}^{t_{00}} Y(\nu(\xi);t)A(\nu(\xi))\dot{\nu}(\xi)\delta v(\xi)\,d\xi + o(t;\varepsilon\delta\mu)+$$

$$+ \int\limits_{t_0}^{t_{00}} Y(\nu(\xi);t)A(\nu(\xi))\dot{\nu}(\xi)\times$$

$$\times \Big[A(\xi)\big(v_0(\sigma(\xi)) + \varepsilon\delta v(\sigma(\xi))\big) + f_0(t_0,\tau,\varphi,y_0+\Delta y)(\xi)+$$

$$+ \varepsilon\delta f(t_0,\tau,\varphi,y_0+\Delta y)(\xi) - v_0(\xi)\Big]\,d\xi =$$

$$+ \varepsilon \int\limits_{\sigma(t_{00})}^{t_{00}} Y(\nu(\xi);t)A(\nu(\xi))\dot{\nu}(\xi)\delta v(\xi)\,d\xi - \varepsilon Y(\nu(t_{00})-;t)A(\nu(t_{00}))\times$$

$$\times \dot{\nu}(t_{00})\big[A(t_{00})v_0(\sigma(t_{00})) + f_0^- - v_0(t_{00})\big]\delta t_0 + o(t;\varepsilon\delta\mu) =$$

$$= \varepsilon \int\limits_{\sigma(t_{00})}^{t_{00}} Y(\nu(\xi);t)A(\nu(\xi))\dot{\nu}(\xi)\delta v(\xi)\,d\xi + \varepsilon\big[Y(t_{00}-;t) - \Phi(t_{00};t)\big]\times$$

$$\times \big[v_0(t_{00}) - A(t_{00})v_0(\sigma(t_{00})) - f_0^-\big]\delta t_0 + o(\varepsilon\delta\mu) \quad (2.24)$$

(see (1.7)).

For $R_0[t;t_{00}]$, we have

$$R_0[t;t_{00}] = \varepsilon \int\limits_{\tau_0(t_{00})}^{t_0} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\delta\varphi(\xi)\,d\xi+$$

$$+ \int\limits_{t_0}^{t_{00}} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\,d\xi =$$

$$= \varepsilon \int\limits_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\delta\varphi(\xi)\,d\xi + o(t;\varepsilon\delta\mu)+$$

$$+ \int\limits_{t_0}^{t_{00}} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\,d\xi. \quad (2.25)$$

Let a number $\delta_2 \in (0,\delta_1)$ be so small that $\gamma_0(t_{00}) < t_{10} - \delta_2$. Since $\gamma_0(t_{00}) > \gamma(t_0)$, therefore for $t \in [t_{10}-\delta_2, t_{10}+\delta_2]$, we have

$$R_1[t;t_{00}] = \sum_{i=1}^{3} \alpha_i[t],$$

where

$$\alpha_1[t]=\int\limits_{t_{00}}^{\gamma(t_0)} r[\xi;t]\,d\xi,\quad \alpha_2[t]=\int\limits_{\gamma(t_0)}^{\gamma_0(t_{00})} r[\xi;t]\,d\xi,$$

$$\alpha_3[t]=\int\limits_{\gamma_0(t_{00})}^{t} r[\xi;t]\,d\xi,\quad r[\xi;t]=Y(\xi;t)R_1[\xi].$$

Introducing the notation,

$$f_0[\xi;s]=$$
$$=f_0\big(\xi,x_0(\xi)+s\Delta x(\xi),x_0(\tau_0(\xi))+s\big(x_0(\tau(\xi))-x_0(\tau_0(\xi))+\Delta x(\tau(\xi))\big)\big),$$
$$\theta[\xi;s]=f_{0x_1}[\xi;s]-f_{0x_1}[\xi],\quad \rho[\xi;s]=f_{0x_2}[\xi;s]-f_{0x_2}[\xi],$$

Then we have

$$R_1[\xi]=\int\limits_0^1\frac{d}{ds}\,f_0[\xi;s]\,ds=$$
$$=\int\limits_0^1\Big\{f_{0x_1}[\xi;s]\Delta x(\xi)+f_{0x_2}[\xi;s]\Big(x_0(\tau(\xi))-x_0(\tau_0(\xi))+\Delta x(\tau(\xi))\Big)\Big\}\,ds-$$
$$-f_{0x_1}[\xi]\Delta x(\xi)-f_{0x_2}[\xi]\Delta x(\tau_0(\xi))=\bigg[\int\limits_0^1\theta[\xi;s]\,ds\bigg]\Delta x(\xi)+$$
$$+\bigg[\int\limits_0^1\rho[\xi;s]\,ds\bigg]\big(x_0(\tau(\xi))-x_0(\tau_0(\xi))+\Delta x(\tau(\xi))\big)+$$
$$+f_{0x_2}[\xi]\Big\{\big[x_0(\tau(\xi))-x_0(\tau_0(\xi))\big]+\big[\Delta x(\tau(\xi))-\Delta x(\tau_0(\xi))\big]\Big\}.$$

Taking into account the latter relation, we have

$$\alpha_1[t]=\sum_{i=1}^4\alpha_{1i}[t],$$

where

$$\alpha_{11}[t]=\int\limits_{t_{00}}^{\gamma(t_0)}Y(\xi;t)\theta_1[\xi]\Delta x(\xi)\,d\xi,\ \ \theta_1[\xi]=\int\limits_0^1\theta[\xi;s]\,ds,$$

$$\alpha_{12}[t]=\int\limits_{t_{00}}^{\gamma(t_0)}Y(\xi;t)\rho_1[\xi]\Big[x_0(\tau(\xi))-x_0(\tau_0(\xi))+\Delta x(\tau(\xi))\Big]\,d\xi,$$

$$\rho_1[\xi]=\int_0^1\rho[\xi;s]\,ds,$$

$$\alpha_{13}[t]=\int_{t_{00}}^{\gamma(t_0)}Y(\xi;t)f_{0x_2}[\xi]\big[\Delta x(\tau(\xi))-\Delta x(\tau_0(\xi))\big]\,d\xi,$$

$$\alpha_{14}[t]=\int_{t_{00}}^{\gamma(t_0)}Y(\xi;t)f_{0x_2}[\xi]\big[x_0(\tau(\xi))-x_0(\tau_0(\xi))\big]\,d\xi.$$

Further,

$$\gamma_0(t_0)-\gamma(t_0)=\int_{\tau_0(\gamma(t_0))}^{t_0}\dot\gamma_0(\xi)\,d\xi=\int_{t_0-\varepsilon\delta\tau(\gamma(t_0))}^{t_0}\dot\gamma_0(\xi)\,d\xi>0.$$

Therefore. for $\xi\in(t_{00},\gamma(t_0))$, we have $\tau(\xi)<t_0$, $\tau_0(\xi)<t_0$. Thus,

$$x_0(\tau(\xi))-x_0(\tau_0(\xi))=\varphi_0(\tau(\xi))-\varphi_0(\tau_0(\xi))$$

and

$$\Delta x(\tau(\xi))-\Delta x(\tau_0(\xi))=\varepsilon\big[\delta\varphi(\tau(\xi))-\delta\varphi(\tau_0(\xi))\big].$$

The function $\varphi_0(t), t\in I_1$ is absolutely continuous, therefore for each fixed Lebesgue point $\tau_0(\xi)\in I_1$ we get

$$\varphi_0(\tau(\xi))-\varphi_0(\tau_0(\xi))=\int_{\tau_0(\xi)}^{\tau(\xi)}\dot\varphi_0(s)\,ds=\varepsilon\dot\varphi_0(\tau_0(\xi))\delta\tau(\xi)+\gamma(\xi;\varepsilon\delta\mu), \tag{2.26}$$

where

$$\lim_{\varepsilon\to0}\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}=0 \ \text{ uniformly for } \ \delta\mu\in\Im_2^-. \tag{2.27}$$

Thus, (2.26) is valid for almost all points of the interval $(t_{00},\gamma(t_0))$. From (2.26), taking into account the boundedness of the function $\dot\varphi_0(t)$, we have

$$\big|\varphi_0(\tau(\xi))-\varphi_0(\tau_0(\xi))\big|\le O(\varepsilon\delta\mu) \ \text{ and } \ \Big|\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big|\le const. \tag{2.28}$$

According to (2.19) and (2.26)–(2.28). for the expressions $\alpha_{1i}[t]$, $i=1,\dots,4$, we have

$$\big|\alpha_{11}[t]\big|\le\|Y\|O(\varepsilon\delta\mu)\theta_2(\varepsilon\delta\mu),\ \ |\alpha_{12}[t]|\le\|Y\|O(\varepsilon\delta\mu)\rho_2(\varepsilon\delta\mu),$$

$$\big|\alpha_{13}[t]\big|\le o(\varepsilon\delta\mu),\ \ \alpha_{14}[t]=\varepsilon\int_{t_{00}}^{\rho_\varepsilon}Y(\xi;t)f_{0x_2}[\xi]\dot\varphi_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi,$$

where

$$\theta_2(\varepsilon\delta\mu)=\int_{t_{00}}^{b}\int_0^1\Big|f_{0x_1}\big(\xi,x_0(\xi)+s\Delta x(\xi),\varphi_0(\tau_0(\xi))\big)+$$

$$+s\Big(\varphi_0(\tau(\xi))-\varphi_0(\tau_0(\xi))-\varepsilon\delta\varphi(\tau_0(\xi))\Big)-f_{0x_1}\big(\xi,x_0(\xi),\varphi_0(\tau_0(\xi))\big)\Big|\,ds\,d\xi,$$

$$\rho_2(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{b}\int\limits_{0}^{1}\Big|f_{0x_2}\big(\xi,x_0(\xi)+s\Delta x(\xi),\varphi_0(\tau_0(\xi))\big)+$$
$$+\,s\Big(\varphi_0(\tau(\xi))-\varphi_0(\tau_0(\xi))-\varepsilon\delta\varphi(\tau_0(\xi))\Big)-$$
$$-\,f_{0x_2}\big(\xi,x_0(\xi),\varphi_0(\tau_0(\xi))\big)\Big|\,ds\,d\xi,$$

$$\gamma_1(t;\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t}Y(\xi;t)f_{0x_2}[\xi]\gamma(\xi;\varepsilon\delta\mu)\,d\xi.$$

Obviously,

$$\Big|\frac{\gamma(t;\varepsilon\delta\mu)}{\varepsilon}\Big|\le\|Y\|\int\limits_{t_{00}}^{\gamma_0(t_{00})}\big|f_{0x_2}[\xi]\big|\,\Big|\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big|\,d\xi.$$

By the Lebesgue theorem on the passage under the integral sign, we have

$$\lim_{\varepsilon\to0}\theta(\varepsilon\delta\mu)=\lim_{\varepsilon\to0}\rho(\varepsilon\delta\mu)=\Big|\frac{\gamma_1(t;\varepsilon\delta\mu)}{\varepsilon}\Big|=0$$

uniformly for $(t,\delta\mu)\in[t_{00},\gamma_0(t_{00})]\times\Im_2^-$ (see (2.26)).

Thus,

$$\alpha_{1i}[t]=o(\varepsilon\delta\mu),\quad i=1,2,3; \tag{2.29}$$

$$\alpha_{14}[t]=\varepsilon\int\limits_{t_{00}}^{\gamma(t_0)}Y(\xi;t)f_{0x_2}[\xi]\dot\varphi_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi+o(t;\varepsilon\delta\mu).$$

It is clear that

$$\varepsilon\int\limits_{\gamma(t_0)}^{\gamma_0(t_{00})}Y(\xi;t)f_{0x_2}[\xi]\dot\varphi_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi=o(t;\varepsilon\delta\mu),$$

i.e.

$$\alpha_{14}[t]=\varepsilon\int\limits_{t_{00}}^{\gamma_0(t_{00})}Y(\xi;t)f_{0x_2}[\xi]\dot\varphi_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi+o(t;\varepsilon\delta\mu). \tag{2.30}$$

On the basis of (2.28) and (2.29), we obtain

$$\alpha_1[t]=\varepsilon\int\limits_{t_{00}}^{\gamma_0(t_{00})}Y(\xi;t)f_{0x_2}[\xi]\dot\varphi_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi+o(t;\varepsilon\delta\mu). \tag{2.31}$$

Let us now transform $\alpha_2[t]$. We have

$$\alpha_2[t]=\sum_{i=1}^{3}\alpha_{2i}(t;\varepsilon\delta\mu),$$

where

$$\alpha_{21}[t]=\int\limits_{\gamma(t_0)}^{\gamma_0(t_{00})}Y(\xi;t)\Big[f_0\Big(\xi,x_0(\xi)+\Delta x(\xi),x_0(\tau(\xi))+\Delta x(\tau(\xi))\Big)-f_0[\xi]\Big]\,d\xi,$$

$$\alpha_{22}[t]=-\int\limits_{\gamma(t_0)}^{\gamma_0(t_{00})}Y(\xi;t)f_{0x_1}[\xi]\Delta x(\xi)\,d\xi,$$

$$\alpha_{23}[t]=-\int\limits_{\gamma(t_0)}^{\gamma_0(t_{00})}Y(\xi;t)f_{0x_2}[\xi]\Delta x(\tau_0(\xi))\,d\xi.$$

If $\xi\in(\gamma(t_0),\gamma_0(t_{00}))$, then

$$|\Delta x(\xi)|\leq O(\varepsilon\delta\mu),$$
$$x_0(\tau(\xi))+\Delta x(\tau(\xi))=y(\tau(\xi);\varepsilon\delta\mu)=y_0(\tau(\xi))+\Delta y(\tau(\xi);\varepsilon\delta\mu),$$
$$x_0(\tau_0(\xi))=\varphi_0(\tau_0(\xi)),$$

therefore,

$$\alpha_{22}[t]=o(t;\varepsilon\delta\mu),$$
$$\lim_{\varepsilon\to 0}\Big(\xi,x_0(\xi)+\Delta x(\xi),x_0(\tau(\xi))+\Delta x(\tau(\xi))\Big)=$$
$$=\lim_{\xi\to\gamma_0(t_{00})-}\big(\xi,x_0(\xi),y_0(\tau_0(\xi))\big)=z_{10},$$
$$\lim_{\varepsilon\to 0}\big(\xi,x_0(\xi),x_0(\tau_0(\xi))\big)=\lim_{\xi\to\gamma_0(t_{00})-}\big(\xi,x_0(\xi),\varphi_0(\tau_0(\xi))\big)=z_{20},$$

i.e.

$$\lim_{\varepsilon\to 0}\sup_{\xi\in[\gamma(t_0),\gamma_0(t_{00})]}\Big[f_0\Big(\xi,x_0(\xi)+\Delta x(\xi),x_0(\tau(\xi))+\Delta x(\tau(\xi))\Big)-$$
$$-f_0\big(\xi,x_0(\xi),x_0(\tau_0(\xi))\big)\Big]=f_{01}^{-}.$$

It is clear that

$$\gamma_0(t_{00})-\gamma(t_0)=\int\limits_{\tau_0(\gamma(t_0))}^{t_{00}}\dot\gamma_0(\xi)\,d\xi=$$
$$=\int\limits_{\tau(\gamma(t_0))-\varepsilon\delta\tau(\gamma(t_0))}^{t_{00}}\dot\gamma_0(\xi)\,d\xi=\int\limits_{t_0-\varepsilon\delta\tau(\gamma(t_0))}^{t_{00}}\dot\gamma_0(\xi)\,d\xi=O(\varepsilon\delta\mu)>0.$$

It is not difficult to see that

$$\alpha_{21}[t]=\int\limits_{\gamma(t_0)}^{\gamma_0(t_{00})} Y(\xi;t)f_{01}^{-}\,d\xi+o(t;\varepsilon\delta\mu)=$$

$$=\int\limits_{\tau_0(\gamma(t_0))}^{t_{00}} Y(\gamma_0(\xi);t)f_{01}^{-}\dot\gamma_0(\xi)\,d\xi+o(t;\varepsilon\delta\mu)=$$

$$=\int\limits_{t_{00}-\varepsilon(\delta\tau(\gamma_0(t_{00}))-\delta t_0)+o(\varepsilon\delta\mu)}^{t_{00}} Y(\gamma_0(\xi);t)f_{01}^{-}\dot\gamma_0(\xi)\,d\xi+o(t;\varepsilon\delta\mu)=$$

$$=\varepsilon Y(\gamma(t_{00})-;t)f_{01}^{-}\dot\gamma_0(t_{00})\big(\delta\tau(\gamma_0(t_{00}))-\delta t_0\big)+o(t;\varepsilon\delta\mu).$$

For $\xi\in[\gamma(t_0),\gamma_0(t_{00})]$, we have $\Delta x(\tau_0(\xi))=\varepsilon\delta\varphi(\tau_0(\xi))$, therefore

$$\alpha_{23}[t]=-\varepsilon\int\limits_{\gamma(t_0)}^{\gamma_0(t_0)} Y(\xi;t)f_{0x_2}[\xi]\delta\varphi(\tau_0(\xi))\,d\xi-$$

$$-\int\limits_{\gamma_0(t_0)}^{\gamma_0(t_{00})} Y(\xi;t)f_{0x_2}[\xi]\Delta x(\tau_0(\xi))\,d\xi=$$

$$=-\int\limits_{t_0}^{t_{00}} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot\gamma_0(\xi)\Delta x(\xi)\,d\xi+o(t;\varepsilon\delta\mu).$$

Consequently,

$$\alpha_2[t]=\varepsilon Y(\gamma_0(t_{00})-;t)f_{01}^{-}\dot\gamma_0(t_{00})\big(\delta\tau(\gamma_0(t_{00}))-\delta t_0\big)=$$

$$-\int\limits_{t_0}^{t_{00}} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot\gamma_0(\xi)\Delta x(\xi)\,d\xi+o(t;\varepsilon\delta\mu). \tag{2.32}$$

Transforming the expression $\alpha_3[t]$ for $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$, we have

$$\alpha_3[t]=\sum_{i=1}^{4}\alpha_{3i}[t],$$

where

$$\alpha_{31}[t]=\int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t)\theta_1[\xi]\Delta x(\xi)d\xi,$$

$$\alpha_{32}[t]=\int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t)\rho_1[\xi]\Big[x_0(\tau(\xi))-x_0(\tau_0(\xi))+\Delta x(\tau(\xi))\Big]\,d\xi,$$

$$\alpha_{33}[t]=\int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t)f_{0x_2}[\xi]\big[\Delta x(\tau(\xi))-\Delta x(\tau_0(\xi))\big]\,d\xi,$$

$$\alpha_{34}[t]=\int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t)f_{0x_2}[\xi]\big[x_0(\tau(\xi))-x_0(\tau_0(\xi))\big]\,d\xi.$$

For each Lebesgue point $\tau_0(\xi)$ of the function $\dot x_0(t), t\in[t_{00},t_{10}+\delta_2]$, we get

$$x_0(\tau(\xi))-x_0(\tau_0(\xi))=\int\limits_{\tau_0(\xi)}^{\tau(\xi)}\dot x_0(\xi)\,d\xi=\varepsilon\dot x_0(\tau_0(\xi))\delta\tau(\xi)+\widehat{\gamma}(\xi;\varepsilon\delta\mu), \tag{2.33}$$

where

$$\lim_{\varepsilon\to 0}\frac{\widehat{\gamma}(\xi;\varepsilon\delta\mu)}{\varepsilon}=0 \ \text{ uniformly for } \ \delta\mu\in\Im_2^-. \tag{2.34}$$

From (2.32), taking into account the boundedness of the function $\dot x_0(t)$, we have

$$\big|x_0(\tau(\xi))-x_0(\tau_0(\xi))\big|\le O(\varepsilon\delta\mu) \ \text{ and } \ \Big|\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big|\le const. \tag{2.35}$$

Further,

$$\big|\Delta x(\tau(\xi))-\Delta x(\tau_0(\xi))\big|\le\int\limits_{\tau_0(\xi)}^{\tau(\xi)}|\dot\Delta(x(s))|\,ds\le$$

$$\le\int\limits_{\tau_0(\xi)}^{\tau(\xi)}|A(s)|\,\big|\dot\Delta x(\sigma(s))\big|\,ds\le$$

$$\le\int\limits_{\tau_0(\xi)}^{\tau(\xi)}L_{f_0,K_1}(s)\Big(|\Delta x(s)|+\big|x_0(\tau(s))-x_0(\tau_0(s))\big|+\big|\Delta x(\tau(s))\big|\Big)\,ds\le$$

$$\le\|A\|\int\limits_{\tau(\xi)}^{\tau(\xi)}\big|\dot\Delta x(\sigma(s))\big|\,ds+o(\xi;\varepsilon\delta\mu).$$

If $[\sigma(\tau_0(\xi)),\sigma(\tau(\xi))]\subset[t_0,\nu(t_0)]$, then

$$\dot\Delta x(\sigma(s))=\varepsilon\delta v(\sigma(s)).$$

Thus, in this case we have

$$\big|\Delta x(\tau(\xi))-\Delta x(\tau_0(\xi))\big|=o(\xi;\varepsilon\delta\mu).$$

If $[\sigma(\tau_0(\xi)),\sigma(\tau(\xi))]\subset[\nu(t_0),\nu(t_{00})]$, then

$$\big|\dot\Delta x(\sigma(s))\big|=\big|\dot x(\sigma(s);\mu_0+\varepsilon\delta\mu)-v_0(\sigma(s))\big|$$

and

$$\big|\Delta x(\tau(\xi)) - \Delta x(\tau_0(\xi))\big| = O(\xi;\varepsilon\delta\mu).$$

It is clear that if $\xi \in [\gamma_0(\nu(t_0)),\gamma(\nu(t_{00}))]$, then

$$\big[\sigma(\tau_0(\xi)),\sigma(\tau(\xi))\big] \subset \big[\nu(t_0),\nu(t_{00})\big]$$

with

$$\lim_{\varepsilon\to 0}\big[\gamma_0(\nu(t_0)) - \gamma(\nu(t_{00}))\big] = 0,$$

therefore

$$\int\limits_{\gamma_0(\nu(t_0))}^{\gamma(\nu(t_{00}))} Y(\xi;t)f_{0y}\big[\Delta x(\tau(\xi)) - \Delta x(\tau_0)(\xi)\big]\,d\xi = o(\varepsilon\delta\mu).$$

Continuing this process analogously for $a_{33}[t]$, we get

$$\alpha_{33}[t] = o(t;\varepsilon\delta\mu).$$

According to (2.32) and (2.34), for the above expressions we have

$$\big|\alpha_{31}[t]\big| \le \|Y\|O(\varepsilon\delta\mu)\theta_3(\varepsilon\delta\mu), \quad \big|\alpha_{32}[t]\big| \le \|Y\|O(\varepsilon\delta\mu)\rho_3(\varepsilon\delta\mu),$$

$$\alpha_{34}[t] = \widehat{\gamma}_1(t;\varepsilon\delta\mu) + \varepsilon\int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t)f_{0x_2}[\xi]\dot{x}_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi,$$

where

$$\theta_3(\varepsilon\delta\mu) = \int\limits_{\gamma_0(t_{00})}^{t_{10}+\delta_2}\Big|f_{0x_1}(\xi,x_0(\xi)+s\Delta x(\xi),x_0(\tau_0(\xi)))+ \\ + s\Big(x_0(\tau(\xi)) - x_0(\tau_0(\xi)) + \Delta x(\tau(\xi))\Big) - \\ - f_{0x_1}\big(\xi,x_0(\xi),x_0(\xi)\big)\Big|\,d\xi,$$

$$\rho_3(\varepsilon\delta\mu) = \int\limits_{\gamma_0(t_{00})}^{t_{10}+\delta_2}\Big|f_{0x_2}(\xi,x_0(\xi)+s\Delta x(\xi),x_0(\tau_0(\xi)))+ \\ + s\Big(x_0(\tau(\xi)) - x_0(\tau_0(\xi)) + \Delta x(\tau(\xi))\Big) - \\ - f_{0x_2}\big(\xi,x_0(\xi),x_0(\xi)\big)\Big|\,d\xi,$$

$$\widehat{\gamma}_1(t;\varepsilon\delta\mu) = \int\limits_{\gamma_0(t_{00})}^{t_{10}+\delta_2} Y(\xi;t)f_{0x_2}[\xi]\widehat{\gamma}(\xi;\varepsilon\delta\mu)\,d\xi.$$

Obviously,

$$\Big|\frac{\widehat{\gamma}(t;\varepsilon\delta\mu)}{\varepsilon}\Big| \leq \|Y\| \int\limits_{\gamma_0(t_{00}}^{t_{10}+\delta_2} \big|f_{0x_2}[\xi]\big| \Big|\frac{\widehat{\gamma}(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big|\,d\xi.$$

By the Lebesgue theorem on the passage under the integral sign, we have

$$\lim_{\varepsilon\to 0}\theta_3(\varepsilon\delta\mu) = \lim_{\varepsilon\to 0}\rho_{\varepsilon\delta\mu} = 0$$

and

$$\lim_{\varepsilon\to 0}\Big|\frac{\widehat{\gamma}(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big| = 0$$

uniformly for $(t,\delta\mu)\in[\gamma_0(t_{00}),t_{10}+\delta_2]$ (see (2.33)).

Thus,

$$\alpha_{3i}[t] = o(t;\varepsilon\delta\mu), \quad i=1,2,$$

$$\alpha_{34}[t] = \varepsilon \int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t) f_{0x_2}[\xi]\dot{x}_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi + o(t;\varepsilon\delta\mu).$$

Consequently,

$$\alpha_3[t] = \varepsilon \int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t) f_{0x_2}[\xi]\dot{x}_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi + o(t;\varepsilon\delta\mu). \tag{2.36}$$

On the basis of (2.31), (2.32) and (2.36),

$$\begin{aligned} R_1[t;t_{00}] = \varepsilon &\int\limits_{t_{00}}^{\gamma_0(t_{00})} Y(\xi;t) f_{0x_2}[\xi]\dot{\varphi}_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi + \\ &+ \varepsilon Y(\gamma_0(t_{00})-;t) f_{01}^{-}\dot{\gamma}_0(t_{00})\big(\delta\tau(\gamma_0(t_{00})) - \delta t_0\big) - \\ &- \int\limits_{t_0}^{t_{00}} Y(\gamma_0(\xi);t) f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\,d\xi \times \\ &\times \varepsilon \int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t) f_{0x_2}[\xi]\dot{x}_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi + o(t;\varepsilon\delta\mu). \end{aligned} \tag{2.37}$$

Finally, let us estimate $R_2[t;t_{00}]$. We have

$$\big|R_2[t;t_{00}]\big| \leq \varepsilon\alpha\|Y\| \sum_{i=1}^{k} \beta_i(\varepsilon\delta\mu),$$

where

$$\beta_i(\varepsilon\delta\mu) =$$

$$= \int\limits_{t_{00}}^{t_{10}+\delta_2} L_{\delta f_i,K_1}(\xi)\Big[\,\big|\Delta x(\xi)\big| + \big|x_0(\tau(\xi)) - x_(\tau_0(\xi))\big| + \big|\Delta x(\tau(\xi))\big|\,\Big]\,d\xi.$$

It is clear that

$$\beta_i(\varepsilon\delta\mu) \le$$

$$\le \int\limits_{t_{00}}^{\gamma(t_0)} L_{\delta f_i,K_1}(\xi)\Big[O(\varepsilon\delta\mu) + \big|\varphi_0(\tau(\xi)) - \varphi_0(\tau_0(\xi))\big| + \varepsilon\big|\varphi(\tau(\xi))\big|\,\Big]\,d\xi +$$

$$+ \int\limits_{\gamma(t_0)}^{\gamma_0(t_{00})} L_{\delta f_i,K_1}(\xi)\Big[O(\varepsilon\delta\mu) + \big|x_0(\tau(\xi)) - x_0(\tau_0(\xi))\big| + \big|\Delta x(\tau(\xi))\big|\,\Big]\,d\xi +$$

$$+ \int\limits_{\gamma_0(t_{00})}^{t_{10}+\delta_2} L_{\delta f_i,K_1}(\xi)\Big[O(\varepsilon\delta\mu) + \big|x_0(\tau(\xi)) - x_0(\tau_0(\xi))\big| + O(\varepsilon\delta\mu)\Big]\,d\xi.$$

Obviously,

$$\lim_{\varepsilon\to 0}\beta(\varepsilon\delta\mu) = 0.$$

Thus,

$$R_2[t;t_{00}] = o(t;\varepsilon\delta\mu) \tag{2.38}$$

From (2.22), by virtue of (2.23)–(2.25), (2.37) and (2.38), we obtain (2.1), where $\delta x(t;\delta\mu)$ has the form (2.2). □

2.2. **Proof of Theorem 2.2.** Let $r_1 = t_{00}$ and $r_2 = t_{10}$ in Lemma 2.3. Then

$$x_0(t) = \begin{cases} \varphi_0(t), & t \in [\widehat{\tau}, t_{00}), \\ y_0(t), & t \in [t_{00}, t_{10}], \end{cases}$$

and for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_1) \times \Im_2^+$,

$$x(t;\mu_0+\varepsilon\delta\mu) = \begin{cases} \varphi(t) = \varphi_0(t) + \varepsilon\delta\varphi(t), & t \in [\widehat{\tau}, t_0), \\ y(t;\mu_0+\varepsilon\delta\mu), & t \in [t_0, t_{10}+\delta_1]. \end{cases}$$

We note that $\delta\mu \in \Im_2^+$, i.e. $t_0 > t_{00}$, therefore

$$\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t), & t \in [\widehat{\tau}, t_{00}), \\ \varphi(t) - x_0(t), & t \in [t_{00}, t_0), \\ \Delta y(t), & t \in [t_0, t_{10}+\delta_1], \end{cases}$$

$$\dot{\Delta} x(t) = \begin{cases} \varepsilon\delta v(t), & t \in [\widehat{\tau}, t_{00}), \\ v_0(t) + \varepsilon\delta v(t) - \dot{x}_0(t), & t \in [t_{00}, t_0), \\ \dot{\Delta} y(t), & t \in [t_0, t_{10}+\delta_1]. \end{cases}$$

By Lemma 2.3, we have

$$\big|\Delta x(t)\big| \le O(\varepsilon\delta\mu), \;\; \forall\,(t,\varepsilon,\delta\mu)\in[t_0,t_{10}+\delta_1]\times(0,\varepsilon_1)\times\Im_2^+, \tag{2.39}$$

$$\Delta x(t_0)=\varepsilon\Big\{\delta x_0-\big[A(t_{00})v_0(\sigma(t_{00}))+f_0^+\big]\delta t_0\Big\}+o(\varepsilon\delta\mu). \tag{2.40}$$

The function $\Delta x(t)$ satisfies the equation (2.21) on the interval $[t_0,t_{10}+\delta_1;]$ therefore, by using the Cauchy formula, we can represent it in the form

$$\Delta x(t)=\Psi(t_0;t)\Delta x(t_0)+\varepsilon\int\limits_{t_0}^{t}Y(\xi;t)\delta f[\xi]\,d\xi+\sum_{i=-1}^{2}R_i[t;t_0], \tag{2.41}$$
$$t\in[t_0,t_{10}+\delta_1].$$

Let $\delta_2\in(0,\delta_2)$ be so small that $\gamma_0(t_{00})<t_{10}-\delta_2$. The matrix function is continuous on $[t_{00},\gamma_0(t_{00})]\times[t_{10}-\delta_2,t_{10}+\delta_2]$ (see Theorem 1.13), therefore

$$\Phi(t_0;t)\Delta x(t_0)=$$
$$=\varepsilon\Phi(t_{00};t)\Big\{\delta x_0-\Big[A(t_{00})v_0(\sigma(t_{00}))+f_0^+\Big]\delta t_0\Big\}+o(t;\delta\mu) \tag{2.42}$$

(see (2.40)).

Let us now transform $R_{-1}[t;t_0]$. We have

$$R_{-1}[t;t_0]=\varepsilon\int\limits_{\sigma(t_0)}^{t_{00}}Y(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi)\delta v(\xi)\,d\xi+$$
$$+\int\limits_{t_{00}}^{t_0}Y(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi)\dot\Delta x(\xi)\,d\xi=$$
$$=\varepsilon\int\limits_{\sigma(t_{00})}^{t_{00}}Y(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi)\delta v(\xi)\,d\xi+o(t;\varepsilon\delta\mu)+$$
$$+\int\limits_{t_{00}}^{t_0}Y(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi)\times$$
$$\times\Big[A(\xi)\big(v_0(\sigma(\xi))+\varepsilon\delta v(\sigma(\xi))\big)+f_0\big(\xi,x_0(\xi),x_0(\tau_0(\xi))\big)\Big]\,d\xi=$$
$$=\varepsilon\int\limits_{\sigma(t_{00})}^{t_{00}}Y(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi)\delta v(\xi)\,d\xi+\varepsilon\big[Y(t_{00}+;t)-\Phi(t_{00};t)\big]\times$$
$$\times\Big[v_0(t_{00})-A(t_{00})v_0(\sigma(t_{00}))-f_0^+\Big]\delta t_0+o(\varepsilon\delta\mu). \tag{2.43}$$

For $R_0[t;t_0]$, we have

$$R_0[t;t_0]=\varepsilon\int\limits_{\tau_0(t_0)}^{t_{00}} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\delta\varphi(\xi)\,d\xi+ \\ +\int\limits_{t_{00}}^{t_0} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\,d\xi= \\ =\varepsilon\int\limits_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\delta\varphi(\xi)\,d\xi+o(t;\varepsilon\delta\mu)+ \\ +\int\limits_{t_{00}}^{t_0} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\,d\xi. \tag{2.44}$$

In a similar way, with inessential changes one can prove

$$R_1[t;t_0]=\varepsilon\int\limits_{t_{00}}^{\gamma_0(t_{00})} Y(\xi;t)f_{0x_2}[\xi]\dot{\varphi}_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi+ \\ +\varepsilon Y(\gamma_0(t_{00})-;t)f_{01}^{+}\dot{\gamma}_0(t_{00})\big(\delta\tau(\gamma_0(t_{00}))-\delta t_0\big)- \\ -\int\limits_{t_{00}}^{t_0} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\,d\xi\times \\ \times\varepsilon\int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t)f_{0x_2}[\xi]\dot{x}_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi+o(t;\varepsilon\delta\mu) \tag{2.45}$$

and

$$R_2(t;t_0)=o(t;\varepsilon\delta\mu). \tag{2.46}$$

Obviously,

$$\varepsilon\int\limits_{t_0}^{t} Y(\xi;t)\delta f[\xi]\,d\xi=\varepsilon\int\limits_{t_{00}}^{t} Y(\xi;t)\delta f[\xi]\,d\xi+o(t;\varepsilon\delta\mu). \tag{2.47}$$

Bearing in mind (2.42)–(2.47), from (2.41), we obtain (2.1) and the variation formula.

In the conclusion we note that the Theorems 2.3–2.6 can be proved by the scheme using in the proof of Theorems 2.1 and 2.2.

3. Initial Data Optimization Problem

3.1. **The Necessary conditions of optimality.** Let $t_{01},t_{02},t_1\in(a,b)$ be the given numbers with $t_{01}<t_{02}<t_1$ and let $X_0\subset O$, $K_0\subset O$, $K_1\subset O$,

$U\subset U_0$ be compact and convex sets. Then

$$D_2=\big\{\tau\in D:\ e_2>\dot\tau(t)>e_1>0\big\},$$
$$\Phi_1=\big\{\varphi\in E_\varphi:\ \varphi(t)\in K_0,\ t\in I_1\big\},\quad \Phi_2=\big\{v\in E_v:\ v(t)\in K_1,\ t\in I_1\big\},$$
$$\Omega_1=\big\{u\in\Omega:\ u(t)\in U,\ t\in I\big\}.$$

Consider the initial data optimization problem

$$\dot x(t)=A(t)\dot x(\sigma(t))+f\big(t,x(t),x(\tau(t)),u(t)\big),\ \ t\in[t_0,t_1],$$
$$x(t)=\varphi(t),\ \ \dot x(t)=v(t),\ \ t\in[\widehat\tau,t_0),\ \ x(t_0)=x_0,$$
$$q^i\big(t_0,x_0,x(t_1)\big)=0,\ \ i=1,\dots,l,$$
$$q^0\big(t_0,x_0,x(t_1)\big)\longrightarrow\min,$$

where

$$w=(t_0,\tau,x_0,\varphi,v,u)\in W_1=[t_{01},t_{02})\times D_2\times X_0\times\Phi_1\times\Phi_2\times\Omega_1$$

and $x(t)=x(t;w)$; $q^i(t_0,x_0,x)$, $i=0,\dots,l$, are the continuously differentiable functions on the set $I\times O^2$.

Definition 3.1. The initial data $w=(t_0,\tau,x_0,\varphi,v,u)\in W_1$ are said to be admissible, if the corresponding solution $x(t)=x(t;w)$ is defined on the interval $[\widehat\tau,t_1]$ and the conditions hold

$$q^i\big(t_0,x_0,x(t_1)\big)=0,\ \ i=1,\dots,l,$$

hold.

The set of admissible initial data will be denoted by W_{10}.

Definition 3.2. The initial data $w_0=(t_{00},\tau_0,x_{00},\varphi_0,v_0,u_0)\in W_{10}$ are said to be optimal, if for any $w=(t_0,\tau,x_0,\varphi,v,u)\in W_{10}$ we have

$$q^0\big(t_{00},x_{00},x_0(t_1)\big)\le q^0\big(t_0,x_0,x(t_1)\big),$$

where $x_0(t)=x(t;w_0),x(t)=x(t;w)$.

The initial data optimization problem consists in finding optimal initial data w_0.

Theorem 3.1. *Let $w_0\in W_{10}$ be optimal initial data and $t_{00}\in[t_{01},t_{02})$. Let the following conditions hold:*

(a) $\gamma_0(t_{00})<t_1$;
(b) *the functions $v_0(\sigma(t))$ and $v_0(t)$ are continuous at the point t_{00}; the function $\varphi_0(t)$ is absolutely continuous and the function $\dot\varphi_0(t)$ is bounded;*
(c) *for each compact sets $K\subset O$ and $U\subset U_0$ there exists a number $m_{K,U}>0$ such that*

$$|f_0(z)|\le m_{K,U},\ \ \forall\,z=(t,x_1,x_2,u)\in I\times K^2\times U;$$

(d) *there exist the limits*

$$\lim_{z\to z_0} f_0(z) = f_0^+, \;\; z\in[t_{00},t_{02})\times O^2,$$

$$\lim_{(z_1,z_2)\to(z_{10},z_{20})} \big[f_0(z_1)-f_0(z_2)\big] = f_{01}^+, \;\; z_i\in[\gamma_0(t_{00}),t_1)\times O^2, \;\; i=1,2,$$

where

$$z_0=\big(t_{00},x_{00},\varphi_0(\tau_0(t_{00}))\big), \quad z_{10}=\big(\gamma_0(t_{00}),x_0(\gamma_0(t_{00})),x_{00}\big),$$

$$z_{20}=\big(\gamma_0(t_{00}),x_0(\gamma_0(t_{00})),\varphi_0(t_{00})\big), \quad f_0(z)=f(z,u_0(t)).$$

Then there exist a vector $\pi=(\pi_0,\dots,\pi_l)\neq 0$, $\pi_0\le 0$, *and a solution* $(\chi(t),\psi(t))$ *of the system*

$$\begin{cases} \dot\chi(t)=-\psi(t)f_{0x_1}[t]-\psi(\gamma_0(t))f_{0x_2}[\gamma_0(t)]\dot\gamma_0(t), & \\ \psi(t)=\chi(t)+\psi(\nu(t))A(\nu(t))\dot\nu(t), & t\in[t_{00},t_1], \\ \chi(t)=\psi(t)=0, & t>t_1 \end{cases} \tag{3.1}$$

such that the conditions listed below hold:

3.1. *the condition for* $\chi(t)$ *and* $\psi(t)$

$$\chi(t_1)=\psi(t_1)=\pi Q_{0x},$$

where

$$Q=(q^0,\dots,q^l)^T, \quad Q_{0x}=Q_x\big(t_{00},x_{00},x_0(t_1)\big);$$

3.2. *the condition for the optimal initial moment* t_{00}

$$\pi Q_{0t_0}+\big(\psi(t_{00}+)-\chi(t_{00})\big)v_0(t_{00})-$$
$$-\psi(t_{00}+)\big(A(t_{00})v_0(\sigma(t_{00}))+f_0^+\big)-\psi(\gamma_0(t_{00}+))f_{01}^+\dot\gamma(t_{00})\le 0;$$

3.3. *the condition for the optimal initial vector* x_{00}

$$\big(\pi Q_{0x_0}+\psi(t_{00})\big)x_{00}\ge\big(\pi Q_{0x_0}+\psi(t_{00})\big)x_0, \;\; \forall\, x_0\in X_0;$$

3.4. *the condition for the optimal delay function* $\tau_0(t)$

$$\psi\big(\gamma_0(t_{00}+)\big)f_{01}^+t_{00}+\int_{t_{00}}^{\gamma_0(t_{00})}\psi(t)f_{0x_2}[t]\dot\varphi_0(\tau_0(t))\tau_0(t)\,dt+$$
$$+\int_{\gamma_0(t_{00})}^{t_1}\psi(t)f_{0x_2}[t]\dot x_0(\tau_0(t))\tau_0(t)\,dt\ge$$
$$\ge\psi(\gamma_0(t_{00}+))f_{01}^+\tau(\gamma_0(t_{00}))+\int_{t_{00}}^{\gamma_0(t_{00})}\psi(t)f_{0x_2}[t]\dot\varphi_0(\tau_0(t))\tau(t)\,dt+$$
$$+\int_{\gamma_0(t_{00})}^{t_1}\psi(t)f_{0x_2}[t]\dot x_0(\tau_0(t))\tau(t)\,dt, \;\; \forall\,\tau\in D_{21}=\big\{\tau\in D_2:\ \tau(\gamma_0(t_{00}))<t_{00}\big\};$$

3.5. *the condition for the optimal initial function* $\varphi_0(t)$

$$\int_{\tau_0(t_{00})}^{t_{00}} \psi(\gamma_0(t))f_{0x_2}[\gamma_0(t)]\dot\gamma_0(t)\varphi_0(t)\,dt \geq$$

$$\geq \int_{\tau_0(t_{00})}^{t_{00}} \psi(\gamma_0(t))f_{0x_2}[\gamma_0(t)]\dot\gamma_0(t)\varphi(t)\,dt, \ \ \forall\,\varphi\in\Phi_1;$$

3.6. *the condition for the optimal initial function* $v_0(t)$

$$\int_{\sigma(t_{00})}^{t_{00}} \psi(\nu(t))A(\nu(t))\dot\nu(t)v_0(t)\,dt \geq$$

$$\geq \int_{\sigma(t_{00})}^{t_{00}} \psi(\nu(t))A(\nu(t))\dot\nu(t)v(t)\,dt, \ \ \forall\,v\in\Phi_2;$$

3.7. *the condition for the optimal control function* $u_0(t)$

$$\int_{t_0}^{t_1} \psi(t)f_{0u}[t]u_0(t)\,dt \geq \int_{t_0}^{t_1} \psi(t)f_{0u}[t]u(t)\,dt, \ \ \forall\,u\in\Omega_1.$$

Here

$$f_{0x}[t]=f_x\big(t,x_0(t),x_0(\tau_0(t)),u_0(t)\big),$$

Theorem 3.2. *Let* $w_0\in W_{10}$ *be optimal initial data and* $t_{00}\in(t_{01},t_{02})$. *Let the conditions* (a), (b), (c) *hold. Moreover, there exist the limits*

$$\lim_{z\to z_0} f_0(z)=f_0^-, \ \ z\in(t_{01},t_{00}]\times O^2,$$

$$\lim_{(z_1,z_2)\to(z_{10},z_{20})}\big[f_0(z_1)-f_0(z_2)\big]=f_{01}^-, \ \ z_i\in(t_{00},\gamma_0(t_{00})]\times O^2, \ \ i=1,2,$$

Then there exist a vector $\pi=(\pi_0,\dots,\pi_l)\neq 0$, $\pi_0\leq 0$, *and a solution* $(\chi(t),\psi(t))$ *of the system* (3.1) *such that the conditions* 3.1, 3.3 *and* 3.5–3.7 *are fulfilled. Moreover,*

$$\pi Q_{0t_0}+\big(\psi(t_{00}-)-\chi(t_{00})\big)v_0(t_{00})-\psi(t_{00}-)\big(A(t_{00})v_0(\sigma(t_{00})+f_0^-\big)-$$
$$-\psi(\gamma_0(t_{00}-))f_{01}^-\dot\gamma(t_{00})\geq 0,$$

$$\psi(\gamma_0(t_{00}-))f_{01}^-t_{00}+\int_{t_{00}}^{\gamma_0(t_{00})} \psi(t)f_{0x_2}[t]\dot\varphi_0(\tau_0(t))\tau_0(t)\,dt+$$

$$+\int_{\gamma_0(t_{00})}^{t_1} \psi(t)f_{0x_2}[t]\dot x_0(\tau_0(t))\tau_0(t)\,dt\geq$$

$$\geq \psi(\gamma_0(t_{00}-))f_{01}^{-}\tau(\gamma_0(t_{00})) + \int\limits_{t_{00}}^{\gamma_0(t_{00})} \psi(t)f_{0x_2}[t]\dot{\varphi}_0(\tau_0(t))\tau(t)\,dt +$$

$$+ \int\limits_{\gamma_0(t_{00})}^{t_1} \psi(t)f_{0x_2}[t]\dot{x}_0(\tau_0(t))\tau(t)\,dt, \;\; \forall\,\tau\in D_{22}=\big\{\tau\in D_2:\; \tau(\gamma_0(t_{00}))>t_{00}\big\}.$$

Theorem 3.3. *Let $w_0 \in W_{10}$ be optimal initial data and $t_{00} \in (t_{01}, t_{02})$. Let the conditions of Theorems* 3.1 *and* 3.2 *hold. Moreover,*

$$f_0^- = f_0^+ := \widehat{f}_0, \quad f_{01}^- = f_{01}^+ := \widehat{f}_{01}$$

and

$$t_{00}, \gamma_0(t_{00}) \notin \big\{\sigma(t_1), \sigma^2(t_1), \dots\big\}.$$

Then there exist a vector $\pi = (\pi_0, \dots, \pi_l) \neq 0$, $\pi_0 \leq 0$, and a solution $(\chi(t), \psi(t))$ of the system (3.1) *such that the conditions* 3.1*,* 3.3 *and* 3.5–3.7 *are fulfilled, Moreover,*

$$\pi Q_{0t_0} + \big(\psi(t_{00}) - \chi(t_{00})\big)v_0(t_{00}) - \psi(t_{00})\big(A(t_{00})v_0(\sigma(t_{00})) + \widehat{f}_0\big) -$$
$$-\psi(\gamma_0(t_{00}))\widehat{f}_{01}\dot{\gamma}(t_{00}) = 0,$$

$$\psi(\gamma_0(t_{00}))\widehat{f}_{01}t_{00} + \int\limits_{t_{00}}^{\gamma_0(t_{00})} \psi(t)f_{0x_2}[t]\dot{\varphi}_0(\tau_0(t))\tau_0(t)\,dt +$$

$$+ \int\limits_{\gamma_0(t_{00})}^{t_1} \psi(t)f_{0x_2}[t]\dot{x}_0(\tau_0(t))\tau_0(t)\,dt \geq$$

$$\geq \psi(\gamma_0(t_{00}))\widehat{f}_{01}\tau(\gamma_0(t_{00})) + \int\limits_{t_{00}}^{\gamma_0(t_{00})} \psi(t)f_{0x_2}[t]\dot{\varphi}_0(\tau_0(t))\tau(t)\,dt +$$

$$+ \int\limits_{\gamma_0(t_{00})}^{t_1} \psi(t)f_{0x_2}[t]\dot{x}_0(\tau_0(t))\tau(t)\,dt, \;\; \forall\,\tau \in D_2.$$

3.2. **Proof of Theorem 3.1.** Denote by G_0 the set of such elements $w \in W_1^+ = [t_{00}, t_{02}) \times D_{21} \times X_0 \times \Phi_1 \times \Phi_2 \times \Omega_1$ to which there corresponds the solution $x(t; w)$, $t \in [\widehat{\tau}, t_1]$. On the basis of Theorem 3.3, there exist $\widehat{V}(w_0; \delta_0)$ such that

$$\widehat{V}_0(w_0; \delta_0) = \widehat{V}(w_0; \delta_0) \cap W_1^+ \subset G_0.$$

On the set $\widehat{V}_{01}(z_0; \delta_0) = [0, \delta_0) \times \widehat{V}_0(w_0; \delta_0)$, where $z_0 = (0, w_0))$, we define the mapping

$$P : \widehat{V}_{01}(z_0; \delta_0) \longrightarrow R_p^{1+l} \tag{3.2}$$

by the formula

$$P(z) = Q(t_0, x_0, x(t_1; w)) + (s, 0 \dots, 0)^T =$$
$$= \Big(q^0\big(t_1, x_0, x(t_1; w)\big) + s, q^1\big(t_1, x_0, x(t_1; w)\big), \dots, q^l\big(t_1, x_0, x(t_1; w)\big)\Big)^T,$$
$$z = (s, w).$$

Lemma 3.1. *The mapping P is differentiable at the point $z_0 = (0, w_0)$ and*

$$\begin{gathered}
dP_{z_0}(\delta z) = \Big\{Q_{0t_0} + Q_{0x}\big[Y(t_{00}+; t_1) - \Psi(t_{00}; t_1)\big]v_0(t_{00}) - \\
- Q_{0x}Y(t_{00}+; t_1)\Big[A(t_{00})v_0(\sigma(t_{00})) + f_0^+\Big] - \\
- Q_{0x}Y(\gamma_0(t_{00})+; t_1)f_{01}^+\dot\gamma_0(t_{00})\Big\}\delta t_0 + \Big\{Q_{0x_0} + Q_{0x}\Psi(t_{00}; t_1)\Big\}\delta x_0 + \\
+ Q_{0x}\Big\{Y(\gamma_0(t_{00})+; t_1)f_{01}^+\dot\gamma_0(t_{00})\delta\tau(\gamma_0(t_{00})) + \\
+ \int\limits_{t_{00}}^{\gamma_0(t_{00})} Y(t; t_1)f_{0x_2}[t]\dot\varphi_0(\tau(t))\delta\tau(t)\,dt + \int\limits_{\gamma_0(t_{00})}^{t_{00}} Y(t; t_1)f_{0x_2}[t]\dot x_0(\tau(t))\delta\tau(t)\,dt\Big\} + \\
+ Q_{0x}\Big\{\int\limits_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(t); t_1)f_{0x_2}[\gamma_0(t)]\dot\gamma_0(t)\delta\varphi(t)\,dt + \\
+ \int\limits_{\sigma_0(t_{00})}^{t_{00}} Y(\nu(t); t_1)f_{0x_2}[\nu(t)]\dot\nu_0(t)\delta v(t)\,dt\Big\} + \\
+ Q_{0x}\int\limits_{t_{00}}^{t_1} Y(t; t_1)f_{0u}[t]\delta u(t)\,dt + (\delta s, 0, \dots, 0). \qquad (3.3)
\end{gathered}$$

Proof. Obviously, for arbitrary $(\varepsilon, \delta z) \in (0, \delta_0) \times [\widehat V_{01}(z_0; \delta_0) - z_0]$, we have

$$z_0 + \varepsilon\delta z \in \widehat V_{01}(z_0; \delta_0).$$

Now we transform the difference

$$P(z_0 + \varepsilon\delta z) - P(z_0) =$$
$$= Q\Big(t_{00} + \varepsilon\delta t_0, x_{00} + \varepsilon\delta x_0, x(t_1; w_0 + \varepsilon\delta w)\Big) - Q_0 + \varepsilon(\delta s, 0, \dots, 0)^T.$$

It is easy to see that

$$Q(t_{00}+\varepsilon\delta t_{00},x_{00}+\varepsilon\delta x_0,x(t_1;w_0+\varepsilon\delta w))-Q_0$$
$$=\int\limits_0^1\frac{d}{d\xi}\,Q\Big(t_0+\varepsilon\xi\delta t_0,x_{00}+\varepsilon\xi\delta x_0,x_0(t_1)+\xi\big(x(t_1;w_0+\varepsilon\delta w)-x_0(t_1)\big)\Big)\,d\xi=$$
$$=\varepsilon\Big[Q_{0t_0}\delta t_0+Q_{0x_0}\delta x_0+Q_{0x}\delta x(t_1;\delta w)\Big]+\alpha(\varepsilon\delta w),$$

where

$$\alpha(\varepsilon\delta w)=\varepsilon\int\limits_0^1\big[Q_{t_0}(\varepsilon;\xi)-Q_{0t_0}\big]\delta t_0\,d\xi+\varepsilon\int\limits_0^1\big[Q_{x_0}(\varepsilon;\xi)-Q_{0x_0}\big]\delta x_0\,d\xi+$$
$$+\varepsilon\int\limits_0^1\big[Q_x(\varepsilon;t)-Q_{0x}\big]\delta x(t_1;\delta w)\,d\xi+o(\varepsilon\delta w)\int\limits_0^1 Q_{0x}(\varepsilon;\xi)\,d\xi,$$
$$Q_{t_0}(\varepsilon;\xi)=$$
$$=Q_{t_0}\Big(t_{00}+\varepsilon\xi\delta t_0,x_{00}+\varepsilon\xi\delta x_0,x_0(t_1)+\xi\big(x(t_1;w_0+\varepsilon\delta w)-x_0(t_1)\big)\Big).$$

Clearly, $\alpha(\varepsilon\delta w)=o(\varepsilon\delta w)$. Thus

$$P(z_0+\varepsilon\delta z)-P(z_0)=$$
$$=\varepsilon\Big[Q_{0t_0}\delta t_0+Q_{0x_0}\delta x_0+Q_{0x}\delta x(t_1;\delta w)+(\delta s,0,\dots,0)^\top\Big]+o(\varepsilon\delta w).$$

On the basis of Theorem 2.5, we have (3.3).

The set $\widehat{V}_{01}(z_0;\delta_0)$ is convex and the mapping (3.2) is continuous and differentiable. In a standard way we can prove the criticality of point z_0 with respect to the mapping (3.2), i.e. $P(z_0)\in\partial P(\widehat{V}_{01}(z_0;\delta_0))$ [10, 15]. These conditions guarantee fulfilment of the necessary condition of criticality [10, 15]. Thus, there exists the vector $\pi=(\pi_0,\dots,\pi_l)\neq 0$ such that the inequality

$$\pi dP_{z_0}(\delta z)\le 0,\quad \delta z\in\text{Cone}\,\big(\widehat{V}_{01}(z_0;\delta_0)-z_0\big),\tag{3.4}$$

is valid, where $dP_{z_0}(\delta z)$ has the form (3.3).

Let us introduce the functions

$$\chi(t)=\pi Q_{0x}\Psi(t;t_1),\quad \psi(t)=\pi Q_{0x}Y(t;t_1).\tag{3.5}$$

It is clear that the functions $\chi(t)$ and $\psi(t)$ satisfy the system (3.1) and the conditions

$$\chi(t_1)=\psi(t_1)=\pi Q_{0x},\quad \chi(t_1)=\psi(t_1)=0,\quad t>t_1.\tag{3.6}$$

Taking into consideration (3.3))–(3.5) and (3.6), from (3.4) we have

$$\Big\{\pi Q_{0t_0}+\big[\psi(t_{00}+)-\chi(t_{00})\big]v_0(t_{00})-$$
$$-\psi(t_{00}+)\big[A(t_{00})v_{00}(\sigma(t_{00}))+f_0^+\big]-\psi(\gamma_0(t_{00}+))f_{01}^+\dot\gamma_0(t_{00})\Big\}\delta t_0+$$
$$+\big\{\pi Q_{0x_0}+\chi(t_{00})\big\}\delta x_0+\psi(\gamma_0(t_{00})+)f_{01}^+\dot\gamma_0(t_{00})\delta\tau(\gamma_0(t_{00}))+$$
$$+\int\limits_{t_{00}}^{\gamma_0(t_{00})}\psi(t)f_{0x_2}[t]\dot\varphi_0(\tau(t))\delta\tau(t)\,dt+\int\limits_{\gamma_0(t_{00})}^{t_{00}}\psi(t)f_{0x_2}[t]\dot x_0(\tau(t))\delta\tau(t)\,dt+$$
$$+\int\limits_{\tau_0(t_{00})}^{t_{00}}\psi(\gamma_0(t))f_{0x_2}[\gamma(t)]\dot\gamma_0(t)\delta\varphi(t)\,dt+\int\limits_{\sigma(t_{00})}^{t_{00}}\psi(\nu(t))f_{0x_2}[\nu(t)]\dot\nu(t)\delta v(t)\,dt+$$
$$+\int\limits_{t_{00}}^{t_1}\psi(t)f_{0u}[t]\delta u(t)\,dt+\pi_0\delta s,\ \ \forall\,\delta z\in\operatorname{Cone}\big(\widehat V_{01}(z_0;\delta_0)-z_0\big). \tag{3.7}$$

The condition $\delta z\in Cone(\widehat V_{01}(z_0;\delta_0)-z_0)$ is equivalent to $\delta s\in[0,\infty)$, $\delta t_0\in[0,\infty)$,

$$\delta x_0\in\operatorname{Cone}\big(B(x_{00};\delta_0)\cap X_0-x_{00}\big)\supset X_0-x_{00},$$
$$\delta\tau\in\operatorname{Cone}\big(V(\tau_0:\delta_0)\cap D_{21}-\tau_0\big)\supset D_{21}-\tau_0,$$
$$\delta\varphi\in\operatorname{Cone}\Big(V_1(\varphi_0;\delta_0)\cap\Phi_1-\varphi_0\Big)\supset\Phi_1-\varphi_0,$$
$$\delta v\in\operatorname{Cone}\big(V_2(v_0;\delta_0)\cap\Phi_2-v_0\big)\supset\Phi_2-v_0,$$
$$\delta u\in\operatorname{Cone}\big(V_3(u_0;\delta_0)\cap\Omega_1-u_0\big)\supset\Omega_1-u_0.$$

Let $\delta t_0=0,\delta\tau=0,\delta x_0=0,\delta\varphi=\delta v=0,\delta u=0$, then from (3.7) we have $\pi\delta s\le 0$, $\forall\,\delta s\in[0,\infty)$, thus $\pi_0\le 0$.

Let $\delta s=0,\delta\tau=0,\delta x_0=0,\delta\varphi=\delta v=0,\delta u=0$, then we have

$$\Big\{\pi Q_{0t_0}+\big[\psi(t_{00}+)-\chi(t_{00})\big]v_0(t_{00})-\psi(t_{00}+)\big[A(t_{00})v_{00}(\sigma(t_{00}))+f_0^+\big]-$$
$$-\psi(\gamma_0(t_{00}+))f_{01}^+\dot\gamma_0(t_{00})\Big\}\delta t_0\le 0,\ \ \forall\,\delta t_0\in[0,\infty).$$

From this we obtain the condition for t_{00}.

If $\delta s=0$, $\delta t_0=0$, $\delta\tau=0$, $\delta\varphi=\delta v=0$, $\delta u=0$, then we obtain the condition for x_{00}. Let $\delta s=0$, $\delta t_0=0$, $\delta x_0=0$, $\delta\varphi=\delta v=0$, $\delta u=0$, then we have the condition for the optimal delay function $\tau_0(t)$ (see 3.4). Let $\delta s=0$, $\delta t_0=0$, $\delta\tau=0$, $\delta x_0=0$, $\delta v=0$, $\delta u=0$, then from (3.7) follows the condition for the initial function $\varphi_0(t)$. If $\delta s=0$, $\delta t_0=0$, $\delta\tau=0$, $\delta x_0=0$, $\delta\varphi=0$, $\delta u=0$, then we obtain the condition for $v_0(t)$. Finally, we consider the case, where $\delta s=0$, $\delta t_0=0$, $\delta\tau=0$, $\delta x_0=0$, $\delta\varphi=0$, $\delta v=0$, then we have the condition for the optimal control $u_0(t)$. Theorem 3.1 is proved completely. □

In the conclusion we note that the Theorems 3.2 and 3.3 are proved analogously by using the corresponding variation formulas.

4. The Existence Theorem of Optimal Initial Data

4.1. **Formulation of the main result.** Let $t_{01}, t_{02}, t_1 \in (a,b)$ be the given numbers with $t_{01} < t_{02} < t_1$ and let $X_0 \subset O$, $K_0 \subset O$, $U \subset U_0$ be compact sets. Then Φ_{11} is the set of measurable initial functions $\varphi(t) \in K_0$, $t \in I_1$, $\Omega_2 = \{u \in \Omega :\ u(t) \in U,\ t \in I\}$.

Consider the initial data optimization problem

$$\begin{gathered}\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f\big(t, x(t), x(\tau(t)), u(t)\big),\ \ t \in [t_0, t_1],\\ x(t) = \varphi(t),\ \ \dot{x}(t) = v(t),\ \ t \in [\widehat{\tau}, t_0),\ \ x(t_0) = x_0,\\ q^i\big(t_0, x_0, x(t_1)\big) = 0,\ \ i = 1, \dots, l,\\ J(w) = q^0\big(t_0, x_0, x(t_1)\big) \longrightarrow \min,\end{gathered}$$

where

$$w = (t_0, \tau, x_0, \varphi, v, u) \in W_2 = [t_{01}, t_{02}] \times D_2 \times X_0 \times \Phi_{12} \times \Phi_2 \times \Omega_2$$

and $x(t) = x(t; w)$. The set of admissible elements we denote by W_{20}.

Theorem 4.1. *There exists an optimal element w_0 if the following conditions hold:*

4.1. $W_{20} \neq \varnothing$;

4.2. *there exists a compact set $K_2 \subset O$ such that for an arbitrary $w \in W_{20}$,*

$$x(t; w) \in K_2, t \in [\widehat{\tau}, t_1];$$

4.3. *the sets*

$$P(t, x_1) = \big\{f(t, x_1, x_2, u) :\ (x_2, u) \in K_0 \times U\big\},\ \ (t, x_1) \in I \times O$$

and

$$P_1(t, x_1, x_2) = \big\{f(t, x_1, x_2, u) :\ u \in U\big\},\ \ (t, x_1, x_2) \in I \times O^2$$

are convex.

Remark 4.1. Let K_0 and U be convex sets, and

$$f(t, x_1, x_2, u) = B(t, x_1)x_2 + C(t, x_1)u.$$

Then the condition 4.3 of Theorem 4.1 holds.

4.2. **Auxiliary assertions.** To each element $w = (t_0, \tau, x_0, \varphi, v, u) \in W_2$ we correspond the functional differential equation

$$\dot{q}(t) = A(t)h(t_0, v, \dot{q})(\sigma(t)) + f\big(t, q(t), h(t_0, \varphi, q)(\tau(t)), u(t)\big) \tag{4.1}$$

with the initial condition

$$q(t_0) = x_0. \tag{4.2}$$

Let $K_i \subset O$, $i = 3, 4$ be compact sets and let K_4 contain a certain neighborhood of the set K_3.

Theorem 4.2. *Let $q_i(t) \in K_3, i = 1, 2, \dots,$ be a solution corresponding to the element $w_i = (t_{0i}, \tau_i, x_{0i}, \varphi_i, v_i, u_i) \in W_2$, $i = 1, 2, \dots,$ defined on the interval $[t_{0i}, t_1]$. Moreover,*

$$\lim_{i\to\infty} t_{0i} = t_{00}, \quad \lim_{i\to\infty} \|\tau_i - \tau_0\|_{I_2} = 0, \quad \lim_{i\to\infty} x_{0i} = x_{00}. \tag{4.3}$$

Then there exist numbers $\delta > 0$ and $M > 0$ such that for a sufficiently large i_0 the solution $\psi_i(t)$, $i \geq i_0$, corresponding to the element w_i, $i \geq i_0$, is defined on the interval $[t_{00} - \delta, t_1] \subset I$. Moreover,

$$\psi_i(t) \in K_4, \ \ \big|\dot{\psi}_i(t)\big| \leq M, \ \ t \in [t_{00} - \delta, t_1]$$

and

$$\psi_i(t) = q_i(t), \ \ t \in [t_{0i}, t_1] \subset [t_{00} - \delta, t_1].$$

Proof. Let $\varepsilon > 0$ be so small that a closed ε-neighborhood of the set K_3 : $K_3(\varepsilon) = \{x \in O : \ \exists \widehat{x} \in K_3, \ |x - \widehat{x}| \leq \varepsilon\}$ is contained $\operatorname{int} K_4$. There exist a compact set $Q \subset \mathbb{R}^n \times \mathbb{R}^n$ and a continuously differentiable function $\chi : \mathbb{R}^n \times \mathbb{R}^n \to [0, 1]$ such that

$$\chi(x_1, x_2) = \begin{cases} 1, & (x_1, x_2) \in Q, \\ 0, & (x_1, x_2) \notin K_4 \times [K_0 \cup K_4] \end{cases} \tag{4.4}$$

and

$$K_3(\varepsilon) \times [K_0 \cup K_3(\varepsilon)] \subset Q \subset K_4 \times [K_0 \cup K_4].$$

For each $i = 1, 2, \dots,$ the differential equation

$$\dot{\psi}(t) = A(t)h(t_{0i}, v_i, \dot{\psi})(\sigma(t)) + \phi\big(t, \psi(t), h(t_{0i}, \varphi_i, \psi)(\tau_i(t)), u_i(t)\big),$$

where

$$\phi(t, x_1, x_2, u) = \chi(x_1, x_2) f(t, x_1, x_2, u),$$

with the initial condition

$$\psi(t_{0i}) = x_{0i},$$

has the solution $\psi_i(t)$ defined on the interval I (see Theorem 1.15). Since

$$\big(q_i(t), h(t_{0i}, \varphi_i, q_i)(\tau_i(t))\big) \in K_3 \times [K_0 \cup K_3] \subset Q, t \in [t_{0i}, t_1],$$

therefore

$$\chi\big(q_i(t), h(t_{0i}, \varphi_i, q_i)(\tau_i(t))\big) = 1, \ \ t \in [t_{0i}, t_{1i}],$$

(see (4.6)), i.e.

$$\begin{aligned} &\phi\big(t, q_i(t), h(t_{0i}, \varphi_i, q_i)(\tau_i(t)), u_i(t)\big) = \\ &\qquad = f\big(t, q_i(t), h(t_{0i}, \varphi_i, q_i)(\tau_i(t)), u_i(t)\big), \ \ t \in [t_{0i}, t_1]. \end{aligned}$$

By the uniqueness,

$$\psi_i(t) = q_i(t), \ \ t \in [t_{0i}, t_1]. \tag{4.5}$$

There exists a number $M > 0$ such that

$$\big|\dot{\psi}_i(t)\big| \leq M, \ \ t \in I, \ \ i = 1, 2, \dots . \tag{4.6}$$

Indeed, first of all we note that

$$\Big|\phi\big(t,\psi_i(t),h(t_{0i},\varphi_i,\psi_i)(\tau_i(t)),u_i(t)\big)\Big| \leq$$
$$\leq \sup\Big\{\big|\phi(t,x_1,x_2,u)\big| : \ t\in I,\ x_1\in K_4,\ x_2\in K_4\cup K_0,\ u\in U\Big\} = N_1,$$
$$i=1,2,\dots .$$

It is not difficult to see that if $t\in[a,\nu(t_{0i}))$, then

$$\big|\dot\psi_i(t)\big| = \Big|A(t)v_i(\sigma_i(t)) + \phi\big(t,\psi_i(t),h(t_{0i},\varphi_i,\psi_i)(\tau_i(t)),u_i(t)\big)\Big| \leq$$
$$\leq \|A\|N_2 + N_1 = M_1,$$

where

$$N_2 = \sup\big\{|x| : \ x\in K_1\big\}.$$

Let $t\in[\sigma(t_{0i}),\sigma^2(t_{0i}))$, then

$$\big|\dot\psi_i(t)\big| \leq \|A\|\,\big|\dot\psi_i(\sigma(t))\big| + N_1 \leq \|A\|M_1 + N_1 = M_2.$$

Continuing this process, we obtain (4.6). Further, there exists a number $\delta_0>0$ such that for an arbitrary $i=1,2,\dots$, $[t_{0i}-\delta_0,t_1]\subset I$, and the following conditions hold:

$$\big|\psi_i(t_{0i})-\psi_i(t)\big| \leq \int\limits_t^{t_{0i}} \Big[\,\big|A(s)h(t_{0i},v_i,\dot\psi_i)(\sigma(s))\big| +$$
$$+\Big|\phi\big(s,\psi_i(s),h(t_{0i},\varphi_i,\psi_i)(\tau_i(\xi)),u_i(s)\big)\Big|\Big]\,ds \leq \varepsilon,\ \ t\in[t_{0i}-\delta_0,t_{0i}],$$

This inequality, with regard for $\psi_i(t_{0i})\in K_3$ (see (4.5)), yields

$$\big(\psi_i(t),h(t_{0i},\varphi_i,\psi_i)(\tau_i(t))\big)\in K_3(\varepsilon)\times[K_0\cup K_3(\varepsilon)],\ \ t\in[t_{0i}-\delta_0,t_1],$$

i.e.

$$\chi\big(\psi_i(t),h(t_{0i},\varphi_i,\psi_i)(\tau_i(t))\big)=1,\ \ t\in[t_{0i}-\delta_0,t_1],\ \ i=1,2,\dots,$$

Thus, $\psi_i(t)$ satisfies the equation (4.1) and the conditions $\psi_i(t_{0i})=x_{0i}$, $\psi_i(t)\in K_4$, $t\in[t_{0i}-\delta_0,t_1]$, i.e. $\psi_i(t)$ is the solution corresponding to the element w_i and defined on the interval $[t_{0i}-\delta_0,t_1]\subset I$. Let $\delta\in(0,\delta_0)$, according to (4.3), for a sufficiently large i_0, we have

$$[t_{0i}-\delta_0,t_1]\supset[t_{00}-\delta,t_1]\supset[t_{0i},t_1],\ \ i\geq i_0.$$

Consequently, $\psi_i(t), i\geq i_0$ are the solutions defined on the interval $[t_{00}-\delta,t_1]$ and satisfy the conditions $\psi_i(t)\in K_4$,

$$\big|\dot\psi_i(t)\big|\leq M,\ \ t\in[t_{00}-\delta,t_1],$$
$$\psi_i(t)=q_i(t),\ \ t\in[t_{0i},t_1].$$

□

Theorem 4.3 ([8]). *Let $p(t,u)\in\mathbb{R}^n$ be a continuous function on the set $I\times U$ and let*

$$P(t)=\big\{p(t,u):\ u\in U\big\}$$

be the convex set and

$$p_i\in L(I,\mathbb{R}^n),\ \ p_i(t)\in P(t)\ \ a.e.\ \ on\ I,\ \ i=1,2,\dots .$$

Moreover,

$$\lim_{i\to\infty}p_i(t)=p(t)\ \ weakly\ on\ \ I.$$

Then

$$p(t)\in P(t)\ \ a.e.\ \ on\ \ I$$

and there exists a measurable function $u(t)\in U$, $t\in I$ such that

$$p(t,u(t))=p(t)\ \ a.e.\ \ on\ \ I.$$

4.3. **Proof of Theorem 4.1.** Let $w_i=(t_{0i},\tau_i,x_{0i},\varphi_i,v_i,u_i)\in W_{20}$, $i=1,2,\dots$, be a minimizing sequence, i.e.

$$\lim_{i\to\infty}J(w_i)=\widehat{J}=\inf_{w\in W_{20}}J(w).$$

Without loss of generality, we assume that

$$\lim_{i\to\infty}t_{0i}=t_{00},\quad \lim_{i\to\infty}x_{0i}=x_{00}.$$

The set $D_2\subset C(I_2,\mathbb{R}^n)$ is compact and the set $\Phi_2\subset L(I_1,\mathbb{R}^n)$ is weakly compact (see Theorem 4.3), therefore we assume that

$$\lim_{i\to\infty}\tau_i(t)=\tau_0(t)\ \text{ uniformly in }\ t\in I_2=[a,\widehat{\gamma}],$$

and

$$\lim_{i\to\infty}v_i(t)=v_0(t)\ \text{ weakly in }\ t\in I_1,$$

the solution $x_i(t)=x(t;w_i)\in K_3$ is defined on the interval $[t_{0i},t_1]$. In a similar way (see proof of Theorem 4.2) we prove that $|\dot{x}_i(t)|\le N_3$, $t\in[t_{0i},t_1]$, $i=1,2,\dots$, $N_3>0$. By Theorem 4.2, there exists a number $\delta>0$ such that for a sufficiently large i_0 the solutions $x_i(t)$, $i\ge i_0$, are defined on the interval $[t_{00}-\delta,t_1]\subset I$. The sequence $x_i(t)$, $t\in[t_{00}-\delta,t_1]$, $i\ge i_0$, is uniformly bounded and equicontinuous. By the Arzèla–Ascoli lemma, from this sequence we can extract a subsequence which will again be denoted by $x_i(t)$, $i\ge i_0$, such that

$$\lim_{i\to\infty}x_i(t)=x_0(t)\ \text{ uniformly in }\ [t_{00}-\delta,t_1].$$

Further, from the sequence $\dot{x}_i(t)$, $i\ge i_0$, we can extract a subsequence which will again be denoted by $\dot{x}_i(t)$, $i\ge i_0$, such that

$$\lim_{i\to\infty}\dot{x}_i(t)=\varrho(t)\ \text{ weakly in }\ [t_{00}-\delta,t_1].$$

Obviously,

$$x_0(t) = \lim_{i\to\infty} x_i(t) =$$
$$= \lim_{i\to\infty} \Big[x_i(t_{00}-\delta) + \int\limits_{t_{00}-\delta}^{t} \dot{x}_i(s)\,ds\Big] = x_0(t_{00}-\delta) + \int\limits_{t_{00}-\delta}^{t} \varrho(s)\,ds.$$

Thus, $\dot{x}_0(t) = \varrho(t)$, i.e.

$$\lim_{i\to\infty} \dot{x}_i(t) = \dot{x}_0(t) \ \text{ weakly in } \ [t_{00}-\delta, t_1].$$

Further, we have

$$x_i(t) = x_{0i} +$$
$$+\int\limits_{t_{0i}}^{t}\Big[A(s)h(t_{0i}, v_i, \dot{x}_i)(\sigma(s)) + f\big(s, x_i(s), h(t_{0i}, \varphi_i, x_i)(\tau_i(s)), u_i(s)\big)\Big]\,ds =$$
$$= x_{0i} + \vartheta_{1i}(t) + \vartheta_{2i} + \theta_{1i}(t) + \theta_{2i}, \ \ t \in [t_{00}, t_1], \ \ i \geq i_0,$$

where

$$\vartheta_{1i}(t) = \int\limits_{t_{00}}^{t} A(s)h(t_{0i}, v_i, \dot{x}_i)(\sigma(s))\,ds,$$
$$\theta_{1i}(t) = \int\limits_{t_{00}}^{t} f\big(s, x_i(s), h(t_{0i}, \varphi_i, x_i)(\tau_i(s)), u_i(s)\big)\,ds,$$
$$\vartheta_{2i} = \int\limits_{t_{0i}}^{t_{00}} A(s)h(t_{0i}, v_i, \dot{x}_i)(\sigma(s))\,ds,$$
$$\theta_{2i} = \int\limits_{t_{0i}}^{t_{00}} f\big(s, x_i(s), h(t_{0i}, \varphi_i, x_i)(\tau_i(s)), u_i(s)\big)\,ds.$$

Obviously, $\vartheta_{2i} \to 0$ and $\theta_{2i} \to 0$ as $i \to \infty$.

First of all, we transform the expression $\vartheta_{1i}(t)$ for $t \in [t_{00}, t_1]$. For this purpose, we consider two cases. Let $t \in [t_{00}, \nu(t_{00})]$, we have

$$\vartheta_{1i}(t) = \vartheta_{1i}^{(1)}(t) + \vartheta_{1i}^{(2)}(t),$$

where

$$\vartheta_{1i}^{(1)}(t) = \int\limits_{t_{00}}^{t} A(s)h(t_{00}, v_i, \dot{x}_i)(\sigma(s))\,ds,$$
$$\vartheta_{1i}^{(2)}(t) = \int\limits_{t_{00}}^{t} \vartheta_{1i}^{(3)}(s)\,ds,$$

$$\vartheta_{1i}^{(3)}(s) = A(s)\Big[h(t_{0i}, v_i, \dot{x}_i)(\sigma(s)) - h(t_{00}, v_i, \dot{x}_i)(\sigma(s))\Big].$$

It is clear that

$$\big|\vartheta_{1i}^{(2)}(t)\big| \le \int\limits_{t_{00}}^{t_1} \big|\vartheta_{1i}^{(3)}(s)\big|\, ds, \ \ t \in [t_{00}, t_1]. \tag{4.7}$$

Suppose that $\nu(t_{0i}) > t_{00}$ for $i \ge i_0$, then

$$\vartheta_{1i}^{(3)}(s) = 0, \ \ s \in [t_{00}, t_{0i}^{(1)}) \cup (t_{0i}^{(2)}, t_1],$$

where

$$t_{0i}^{(1)} = \min\big\{\nu(t_{0i}), \nu(t_{00})\big\}, \quad t_{0i}^{(2)} = \max\big\{\nu(t_{0i}), \nu(t_{00})\big\}.$$

Since

$$\lim_{i\to\infty}(t_{0i}^{(2)} - t_{0i}^{(1)}) = 0,$$

therefore

$$\lim_{i\to\infty} \vartheta_{1i}^{(2)}(t) = 0 \ \text{ uniformly in } \ t \in [t_{00}, t_1] \tag{4.8}$$

(see (4.7)).

For $\vartheta_{1i}^{(1)}(t)$, $t \in [t_{00}, \nu(t_{00})]$, we get

$$\vartheta_{1i}^{(1)}(t) = \int\limits_{\sigma(t_{00})}^{\sigma(t)} A(\nu(s))h(t_{00}, v_i, \dot{v}_i)(s)\dot{\nu}(s)\, ds = \\ = \int\limits_{\sigma(t_{00})}^{\sigma(t)} A(\nu(s))\dot{\nu}(s)v_i(s)\, ds.$$

Obviously,

$$\lim_{i\to\infty} \vartheta_{1i}(t) = \int\limits_{\sigma(t_{00})}^{\sigma(t)} A(\nu(s)\dot{\nu}(s)v_0(s)\, ds = \int\limits_{t_{00}}^{t} A(s)v_0(\sigma(s))\, ds, \\ t \in [t_{00}, \sigma(t_{00})] \tag{4.9}$$

(see (4.8)).

Let $t \in [\nu(t_{00}), t_1]$, then

$$\vartheta_{1i}^{(1)}(t) = \vartheta_{1i}^{(1)}(\nu(t_{00})) + \vartheta_{1i}^{(4)}(t),$$

where

$$\vartheta_{1i}^{(4)}(t) = \int\limits_{\nu(t_{00})}^{t} A(s)h(t_{0i}, v_i, \dot{x}_i)(\sigma(s))\, ds.$$

Further,

$$\vartheta_{1i}^{(4)}(t) = \int\limits_{\nu(t_{00})}^{t} A(s)h(t_{00}, v_i, \dot{x}_i)(\sigma(s))\,ds = \int\limits_{t_{00}}^{\sigma(t)} A(\nu(s))\dot{\nu}(t)\dot{x}_i(s)\,ds.$$

Thus, for $t \in [\nu(t_{00}), t_1]$, we have

$$\lim_{i\to\infty} \vartheta_{1i}^{(1)}(t) = \int\limits_{t_{00}}^{\nu(t_{00})} A(t)v_0(\sigma(t))\,dt + \int\limits_{\nu(t_{00})}^{t} A(s)\dot{x}_0(\sigma(s))\,ds. \tag{4.10}$$

Now we transform the expression $\theta_{1i}(t)$ for $t \in [t_{00}, t_1]$. We consider two cases again. Letting $t \in [t_{00}, \gamma_0(t_{00})]$, we have

$$\begin{aligned}
\theta_{1i}(t) &= \theta_{1i}^{(1)}(t) + \theta_{1i}^{(2)}(t),\\
\theta_{1i}^{(1)}(t) &= \int\limits_{t_{00}}^{t} f\big(s, x_i(s), h(t_{00}, \varphi_i, x_i)(\tau_i(s)), u_i(s)\big)\,ds,\\
\theta_{1i}^{(2)}(t) &= \int\limits_{t_{00}}^{t} \theta_{1i}^{(3)}(s)\,ds,\\
\theta_{1i}^{(3)}(s) &= f\big(s, x_i(s), h(t_{0i}, \varphi_i, x_i)(\tau_i(s)), u_i(s)\big) -\\
&\quad - f\big(s, x_i(s), h(t_{00}, \varphi_i, x_i)(\tau_i(s)), u_i(s)\big).
\end{aligned}$$

It is clear that

$$\big|\theta_{1i}^{(2)}(t)\big| \le \int\limits_{t_{00}}^{t_{10}} \big|\theta_{1i}^{(3)}(s)\big|\,ds, \quad t \in [t_{00}, t_1].$$

Suppose that $\gamma_i(t_{0i}) > t_{00}$ for $i \ge i_0$, then

$$\theta_{1i}^{(3)}(s) = 0, \quad s \in [t_{00}, t_{0i}^{(3)}) \cup (t_{0i}^{(4)}, t_1],$$

where

$$t_{1i}^{(3)} = \min\big\{\gamma_i(t_{0i}), \gamma_i(t_{00})\big\}, \quad t_{1i}^{(4)} = \max\big\{\gamma_i(t_{0i}), \gamma_i(t_{00})\big\}.$$

Since

$$\lim_{i\to\infty} (t_{0i}^{(4)} - t_{0i}^{(3)}) = 0$$

therefore

$$\lim_{i\to\infty} \theta_{1i}^{(2)}(t) = 0 \ \text{ uniformly in } \ t \in [t_{00}, t_{10}].$$

For $\theta^{(1)}_{1i}(t)$, $t\in[t_{00},\gamma_0(t_{00})]$, we have

$$\theta^{(1)}_{1i}(t)=\int\limits_{\tau_i(t_{00})}^{\tau_i(t)} f\big(\gamma_i(s),x_i(\gamma_i(s)),h(t_{00},\varphi_i,x_i)(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)\,ds= \\ =\theta^{(4)}_{1i}(t)+\theta^{(5)}_{1i}(t),\ \ i\geq i_0,$$

where

$$\theta^{(4)}_{1i}(t)=\int\limits_{\tau_0(t_{00})}^{\tau_0(t)} f\big(\gamma_0(s),x_0(\gamma_0(s)),\varphi_i(s),u_i(\gamma_i(s))\big)\dot\gamma_0(s)\,ds,$$

$$\theta^{(5)}_{1i}(t)=\int\limits_{\tau_i(t_{00})}^{\tau_i(t)} f\big(\gamma_i(s),x_i(\gamma_i(s)),h(t_{00},\varphi_i,x_i)(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)\,ds- \\ -\int\limits_{\tau_0(t_{00})}^{\tau_0(t)} f\big(\gamma_0(s),x_0(\gamma_0(s)),\varphi_i(s),u_i(\gamma_i(s))\big)\dot\gamma_0(s)\,ds.$$

For $t\in[t_{00},\gamma_0(t_{00})]$, we obtain

$$\theta^{(5)}_{1i}(t)=\int\limits_{\tau_i(t_{00})}^{\tau_0(t_{00})} f\big(\gamma_i(s),x_i(\gamma_i(s)),h(t_{00},\varphi_i,x_i)(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)\,ds+ \\ +\int\limits_{\tau_0(t_{00})}^{\tau_0(t)} \Big[f\big(\gamma_i(s),x_i(\gamma_i(s)),\varphi_i(s),u_i(\gamma_i(s))\big)- \\ -f\big(\gamma_0(s),x_0(\gamma_0(s)),\varphi_i(s),u_i(\gamma_i(s))\big)\Big]\dot\gamma_i(s)\,ds+ \\ +\int\limits_{\tau_0(t)}^{\tau_i(t)} f\big(\gamma_i(s),x_i(\gamma_i(s)),h(t_{00},\varphi_i,x_i)(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)\,ds.$$

Suppose that $\|\tau_i-\tau_0\|\leq\delta$ as $i\geq i_0$, then

$$\lim_{i\to\infty} f\big(\gamma_i(s),x_i(\gamma_i(s)),x_2,u\big)=f\big(\gamma_0(s),x_0(\gamma_0(s)),x_2,u\big)$$

uniformly in $(s,x_2,u)\in[\tau_0(t_{00}),t_{00}]\times K_0\times U$, we have

$$\lim_{i\to\infty}\theta^{(5)}_{1i}(t)=0\ \text{ uniformly in }\ t\in[t_{00},\gamma_0(t_{00})].$$

From the sequence

$$f_i(s)=f\big(\gamma_0(s),x_0(\gamma_0(s)),\varphi_i(s),u_i(\gamma_i(s))\big),\ \ i\geq i_0,\ \ t\in[\tau_0(t_{00}),t_{00}],$$

we extract a subsequence which will again be denoted by $f_i(s), i \geq i_0$, such that

$$\lim_{i\to\infty} f_i(s) = f_0(s) \text{ weakly in the space } L\big([\tau_0(t_{00}), t_{00}], \mathbb{R}^n\big).$$

It is not difficult to see that

$$f_i(s) \in P(\gamma_0(s), x_0(\gamma_0(s))), \quad s \in [\tau_0(t_{00}), t_{00}].$$

By Theorem 4.3,

$$f_0(s) \in P(\gamma_0(s), x_0(\gamma_0(s))) \text{ a.e. } s \in [\tau_0(t_{00}), t_{00}]$$

and on the interval $[\tau_0(t_{00}), t_{00}]$ there exist measurable functions $\varphi_{01}(s) \in K_0, u_{01}(s) \in U$ such that

$$f_0(s) = f\big(\gamma_0(s), x_0(\gamma_0(s)), \varphi_{01}(s), u_{01}(s)\big) \text{ a.e. } s \in [\tau_0(t_{00}), t_{00}].$$

Thus,

$$\lim_{i\to\infty} \theta_{1i}^{(1)} = \lim_{i\to\infty} \theta_{1i}^{(4)}(t) = \int\limits_{\tau_0(t_{00})}^{\tau_0(t)} f_0(s)\dot{\gamma}_0(s)\, ds =$$

$$= \int\limits_{\tau_0(t_{00})}^{\tau_0(t)} f\big(\gamma_0(s), x_0(\gamma_0(s)), \varphi_{01}(s), u_{01}(s)\big)\dot{\gamma}_0(s)\, ds =$$

$$= \int\limits_{t_{00}}^{t} f\big(s, x_0(s), \varphi_{01}(\tau_0(s)), u_{01}(\tau_0(s))\big)\, ds, \quad t \in [t_{00}, \gamma_0(t_{00})]. \quad (4.11)$$

Let $t \in [\gamma_0(t_{00}), t_1]$, then

$$\theta_{1i}^{(1)}(t) = \theta_{1i}^{(1)}(\gamma_0(t_{00})) + \theta_{1i}^{(6)}(t),$$

where

$$\theta_{1i}^{(6)}(t) = \int\limits_{\gamma_0(t_{00})}^{t} f\big(s, x_i(s), h(t_{00}, \varphi_i, x_i)(\tau_i(s)), u_i(s)\big)\, ds.$$

It is clear that

$$\theta_{1i}^{(6)}(t) = \int\limits_{\tau_i(\gamma_0(t_{00}))}^{\tau_i(t)} f\big(\gamma_i(s), x_i(\gamma_i(s)), h(t_{00}, \varphi_i, x_i)(s), u_i(\gamma_i(s))\big)\dot{\gamma}_i(s)\, ds =$$

$$= \theta_{1i}^{(7)}(t) + \theta_{1i}^{(8)}(t), \quad i \geq i_0,$$

where

$$\theta_{1i}^{(7)}(t)=\int\limits_{t_{00}}^{\tau_0(t)} f\big(\gamma_0(t),x_0(\gamma_0(s)),x_0(s),u_i(\gamma_i(s))\big)\dot\gamma_0(s)\,ds,$$

$$\theta_{1i}^{(8)}(t)=\int\limits_{\tau_i(\gamma_0(t_{00}))}^{\tau_i(t)} f\big(\gamma_i(t),x_i(\gamma_i(s)),h(t_{00},\varphi_i,x_i)(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)\,ds-$$

$$-\int\limits_{t_{00}}^{\tau_0(t)} f\big(\gamma_0(s),x_0(\gamma_0(s)),x_0(s),u_i(\gamma_i(s))\big)\dot\gamma_0(s)\,ds.$$

For $t\in[\gamma_0(t_{00}),t_1]$, we have

$$\theta_{1i}^{(8)}(t)=\int\limits_{\tau_i(\gamma_0(t_{00}))}^{t_{00}} f\big(\gamma_i(s),x_i(\gamma_i(s)),h(t_{00},\varphi_i,x_i)(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)\,ds+$$

$$+\int\limits_{t_{00}}^{\tau_0(t)}\Big[f\big(\gamma_i(s),x_i(\gamma_i(s)),x_i(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)-$$

$$-f\big(\gamma_0(s),x_0(s+\gamma_0(s)),v_0(s),u_i(\gamma_i(s))\big)\dot\gamma_0(s)\Big]\,ds+$$

$$+\int\limits_{\tau_0(t)}^{\tau_i(t)} f\big(\gamma_i(s),x_i(\gamma_i(s)),h(t_{00},\varphi_i,x_i)(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)\,ds.$$

Thus,

$$\theta_{1i}^{(8)}(t)=0\ \text{ uniformly in }\ t\in[\gamma_0(t_{00}),t_1].$$

From the sequence

$$f_i(s)=f\big(\gamma_0(s),x_0(\gamma_0(s)),x_0(s),u_i(\tau_i(s))\big),\ \ i\ge i_0,\ \ t\in[t_{00},\tau_0(t_1)],$$

we extract a subsequence which will again be denoted by $F_i(s)$, $i\ge i_0$, such that

$$\lim_{i\to\infty} f_i(s)=f_0(s)\ \text{ weakly in the space }\ L\big([t_{00},\tau_0(t_1)],\mathbb{R}^n\big).$$

It is not difficult to see that

$$f_i(s)\in P_1\big(\gamma_0(s),x_0(\gamma_0(s)),x_0(s)\big),\ \ s\in[t_{00},\tau_0(t_1)].$$

By Theorem 4.3,

$$f_0(s)\in P_1\big(\gamma_0(s),x_0(\gamma_0(s)),x_0(s)\big)\ \text{ a.e. }\ s\in[t_{00},\tau_0(t_1)]$$

and on the interval $[t_{00},\tau_0(t_1)]$ there exists a measurable function $u_{02}(s)\in U$ such that

$$f_0(s)=f\big(\gamma_0(s),x_0(\gamma_0(s)),x_0(s),u_{02}(s)\big)\ \text{ a.e. }\ s\in[t_{00},\tau_0(t_1)].$$

Thus, for $t \in [\gamma_0(t_{00}), t_1]$, we have

$$\lim_{i\to\infty} \theta_{1i}^{(1)}(t) = \lim_{i\to\infty} \theta_{1i}^{(1)}(\gamma_0(t_{00})) + \lim_{i\to\infty} \theta_{1i}^{(7)}(t) =$$

$$= \int\limits_{t_{00}}^{\gamma_0(t_{00})} f\big(s, x_0(s), x_0(\tau_0(s)), u_{02}(s)\big)\, ds +$$

$$+ \int\limits_{\gamma_0(t_{00})}^{t} f\big(s, x_0(s), x_0(\tau_0(s)), u_{02}(\tau_0(s))\big)\, ds, \quad t \in [\gamma_0(t_{00}), t_1]. \quad (4.12)$$

We introduce the following notation:

$$\varphi_0(s) = \begin{cases} \widehat{\varphi}, & s \in [\widehat{\tau}, \tau_0(t_{00})) \cup (t_{00}, t_{02}], \\ \varphi_{01}(s), & s \in [\tau_0(t_{00}), t_{00}], \end{cases}$$

$$u_0(s) = \begin{cases} \widehat{u}, & s \in [a, t_{00}) \cup (t_1, b], \\ u_{01}(\tau_0(s)), & s \in [t_{00}, \tau_0(t_{00})], \\ u_{02}(\tau_0(s)), & s \in (\gamma_0(t_{00}), t_1], \end{cases}$$

where $\widehat{\varphi} \in K_0$ and $\widehat{u} \in U$ are the fixed points

$$x_0(t) = \begin{cases} \varphi_0(t), & t \in [\widehat{\tau}, t_{00}), \\ v_0(t), & t \in [t_{00}, t_1]; \end{cases}$$

$$\dot{x}_0(t) = v_0(t), \quad t \in [\widehat{\tau}, t_{00}),$$

Clearly, $w_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, v_0, u_0) \in W_2$. Taking into account (4.9)–(4.12), we obtain

$$x_0(t) = x_{00} + \int\limits_{t_{00}}^{t} \Big[A(s)\dot{x}_0(\sigma_0(t)) + f\big(s, x_0(s), x_0(\tau_0(s)), u_0(s)\big)\Big]\, ds,$$

$$t \in [t_{00}, t_{10}],$$

and

$$0 = \lim_{i\to\infty} q^i\big(t_{0i}, x_{0i}, x_i(t_1)\big) = q^i\big(t_{00}, x_{00}, x_0(t_1)\big), \quad i = 1, \dots, l,$$

i.e. the element w_0 is admissible and $x_0(t) = x(t; w_0)$, $t \in [\widehat{\tau}, t_1]$.

Further, we have

$$\widehat{J} = \lim_{i\to\infty} q^0\big(t_{0i}, x_{0i}, x_i(t_1)\big) = q^0\big(t_{00}, x_{00}, x_0(t_1)\big) = J(w_0).$$

Thus, w_0 is an optimal element.

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Authors' addresses:

Tamaz Tadumadze

1. Department of Mathematics, Faculty Exact and Natural Sciences of I. Javakhishvili Tbilisi State University, 13 University St., Tbilisi 0186, Georgia.

2. I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University, 2 University St., Tbilisi 0186, Georgia.

E-mail: tamaz.tadumadze@tsu.ge

Nika Gorgodze

A. Tsereteli Kutaisi State University, Department of Mathematics, 59 King Tamari St., Kutaisi 4600, Georgia.

E-mail: E-mail: nika_gorgodze@yahoo.com

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Tatiyana Barinova and Alexander Kostin

ON ASYMPTOTIC STABILITY OF SOLUTIONS OF SECOND ORDER LINEAR NONAUTONOMOUS DIFFERENTIAL EQUATIONS

Abstract. The sufficient conditions for asymptotic stability of solutions of second order linear differential equation

$$y'' + p(t)y' + q(t)y = 0$$

with continuously differentiable coefficients $p : [0, +\infty) \to \mathbb{R}$ and $q : [0, +\infty) \to \mathbb{R}$ are established in the case where the roots of the characteristic equation

$$\lambda^2 + p(t)\lambda + q(t) = 0$$

satisfy conditions

$$\operatorname{Re} \lambda_i(t) < 0 \ \text{ for } \ t \geq 0, \qquad \int\limits_{t_0}^{+\infty} \operatorname{Re} \lambda_i(t)\, dt = -\infty \ \ (i = 1, 2).$$

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რეზიუმე. მეორე რიგის წრფივი დიფერენციალური განტოლებისათვის

$$y'' + p(t)y' + q(t)y = 0$$

უწყვეტად წარმოებადი $p : [0, +\infty) \to \mathbb{R}$ და $q : [0, +\infty) \to \mathbb{R}$ კოეფიციენტებით დადგენილია ამონახსნების ასიმპტოტური მდგრადობის საკმარისი პირობები იმ შემთხვევაში, როცა მახასიათებელი

$$\lambda^2 + p(t)\lambda + q(t) = 0$$

განტოლების ფესვები აკმაყოფილებენ პირობებს

$$\operatorname{Re} \lambda_i(t) < 0, \ \text{ როცა } \ t \geq 0, \qquad \int\limits_{t_0}^{+\infty} \operatorname{Re} \lambda_i(t)\, dt = -\infty \ \ (i = 1, 2).$$

1. Introduction

This present paper is a continuation of the article *Sufficiency conditions for asymptotic stability of solutions of a linear homogeneous nonautonomous differential equation of second-order.*

In the theory of stability of linear homogeneous on-line systems (LHS) of ordinary differential equations

$$\frac{dY}{dt} = P(t)Y, \quad t \in [t_0; +\infty) = I,$$

where the matrix $P(t)$ is, in a general case, complex, of great importance is the study of the LHS stability depending on the roots $\lambda_i(t)$ $(i = \overline{1,n})$ of the characteristic equation

$$\det(P(t) - \lambda E) = 0.$$

L. Cesáro [1] has considered the system of differential equations of n-th order

$$\frac{dY}{dt} = \big[A + B(t) + C(t)\big]Y,$$

where A is a constant matrix, whose roots of the characteristic equation λ_i $(i = \overline{1,n})$ are distinct and satisfy the condition $\operatorname{Re}\lambda_i \leq 0$ $(i = \overline{1,n})$; $B(t) \to 0$ as $t \to +\infty$,

$$\int\limits_{t_0}^{+\infty} \Big\|\frac{dB(t)}{dt}\Big\|\,dt < +\infty, \quad \int\limits_{t_0}^{+\infty} \|C(t)\|\,dt < +\infty,$$

the roots of the characteristic equation of the matrix $A + B(t)$ have non-positive material parts.

In his work, C. P. Persidsky [2] considers the case in which elements of the matrix $P(t)$ are the functions of weak variation, that is, every function can be represented in the form

$$f(t) = f_1(t) + f_2(t),$$

where $f_1(t) \in C_I$, and there exists $\lim\limits_{t\to+\infty} f_1(t) \in \mathbb{R}$, and $f_2(t)$ is such that

$$\sup_{t\in I} |f_2(t)| < +\infty, \quad \lim_{t\to+\infty} f_2'(t) = 0,$$

and the condition $\operatorname{Re}\lambda_i(t) \leq a \in \mathbb{R}_-$ $(i = \overline{1,n})$ is fulfilled.

N. Y. Lyaschenko [3] has considered the case $\operatorname{Re}\lambda_i(t) < a \in \mathbb{R}_-$ $(i = \overline{1,n})$, $t \in I$,

$$\sup_{t\in I} \|A'(t)\| \leq \varepsilon.$$

The case $n = 2$ is thoroughly studied by N. I. Izobov.

I. K. Hale [4] investigated asymptotic behavior of LHS by comparing the roots of the characteristic equation with exponential functions

$$\operatorname{Re}\lambda_i(t) \leq -gt^\beta, \quad g > 0, \ \ \beta > -1 \ \ (i = \overline{1,n}).$$

Then there are the constants $K > 0$ and $0 < \rho < 1$ such that for solving the system

$$\frac{dy}{dt} = A(t)y$$

the estimate

$$\|y(t)\| \le K e^{-\frac{\rho g}{1+\beta} t^{1+\beta}} \|y(0)\|$$

is fulfilled.

In this paper we consider the problem of stability of a real linear homogeneous differential equation (LHDE) of second order

$$y'' + p(t)y' + q(t)y = 0 \quad t \in I \tag{1}$$

provided the roots $\lambda_i(t)$ $(i = 1, 2)$ of the characteristic equation

$$\lambda^2 + p(t)\lambda + q(t) = 0$$

are such that

$$\operatorname{Re}\lambda_i(t) < 0, \;\; t \in I, \quad \int\limits_{t_0}^{+\infty} \operatorname{Re}\lambda_i(t)\, dt = -\infty \;\; (i = 1, 2) \tag{2}$$

and there exist finite or infinite limits $\lim\limits_{t\to+\infty} \lambda_i(t)$ $(i = 1, 2)$. We have not yet encountered with the problems in such a formulation. The case where at least one of the roots satisfies the condition

$$0 < \int\limits_{t_0}^{+\infty} \big|\operatorname{Re}\lambda_i(t)\big|\, dt < +\infty \;\; (i = 1, 2)$$

should be considered separately.

Under the term "almost triangular LHS" we agree to understand each LHS

$$\frac{dy_i(t)}{dt} = \sum_{k=1}^{n} p_{ik}(t) y_k \;\; (i = \overline{1, n}) \tag{3}$$

with $p_{ik}(t) \in C_I$ $(i, k = \overline{1, n})$, which differs little from a linear triangular system

$$\frac{dy_i^*(t)}{dt} = \sum_{k=1}^{n} p_{ik}(t) y_k^* \;\; (i = \overline{1, n}), \tag{4}$$

and the conditions either of Theorem 0.1 or of Theorem 0.2 due to A. V. Kostin [5] are fulfilled.

Theorem 1. *Let the conditions*

1) *LHS* (4) *is stable when* $t \in I$*;*
2) *for a partial solution* $\sigma_i(t)$ $(i = \overline{1, n})$ *of a linear inhomogeneous triangular system*

$$\frac{d\sigma_i(t)}{dt} = \sum_{k=1}^{i-1} |p_{ik}(t)| + \operatorname{Re} p_{ii}(t)\sigma_i(t) + \sum_{k=i+1}^{n} |p_{ik}(t)|\sigma_k(t) \;\; (i = \overline{1, n}) \tag{5}$$

with the initial conditions $\sigma_i(t_0) = 0$ $(i = \overline{1,n})$ *the estimate of the form* $0 < \sigma_i(t) < 1 - \gamma$ $(i = \overline{1,n})$, $\gamma = const$, $\gamma \in (0,1)$ *holds for all* $t \in I$.

Then the zero solution of the system (3) *is a fortiori stable for* $t \in I$.

Theorem 2. *Let the system* (3) *satisfy all the conditions of Theorem* 1 *and, moreover,*

1) *triangular linear system* (4) *is asymptotically stable for* $t \in I$;
2) $\lim\limits_{t\to+\infty} \sigma_i(t) = 0$ $(i = \overline{1,n})$.

Then the zero solution of the system (3) *is asymptotically stable for* $t \in I$.

Theorem 3. *Let the system* (3) *satisfy all the conditions of Theorem* 1 *and, moreover,*

1) *none of the functions*

$$\psi_i(t) = \sum_{k=1}^{i-1} |p_{ik}(t)| \quad (i = \overline{2,n}) \not\equiv 0 \ \textit{ for } \ t \in I;$$

2) $\lim\limits_{t\to+\infty} \sigma_i(t) = 0$ $(i = \overline{1,n})$.

Then the zero solution of the system (3) *is stable for* $t \in I$.

We will also use the following lemma [5]:

Lemma 1. *If the functions* $p(t), q(t) \in C_I$, $\operatorname{Re} p(t) < 0$, $t \in I$,

$$\int\limits_{t_0}^{+\infty} \operatorname{Re} p(\tau)\, d\tau = -\infty, \quad \lim_{t\to+\infty} \frac{q(t)}{\operatorname{Re} p(t)} = 0,$$

then

$$e^{\int\limits_{t_0}^{t} \operatorname{Re} p(\tau)\, d\tau} \int\limits_{t_0}^{t} q(\tau) e^{-\int\limits_{\tau_0}^{\tau} \operatorname{Re} p(\tau_1)\, d\tau_1} d\tau = o(1), \quad t \to +\infty.$$

Further, it will be assumed that all limits and characters o, O are considered as $t \to +\infty$.

In case equation (1) has the form

$$y'' + p(t)y = 0, \tag{6}$$

where $p(t) \in C_I^2$, $p(t) > 0$ in I, $\lambda_1(t) = -i\sqrt{p}$, $\lambda_2(t) = i\sqrt{p}$, $p = p(t)$, there is the well-known I. T. Kiguradze's theorem [6]:

Theorem 4. *Let equation* (6) *be such that*

$$p(+\infty) = +\infty, \ \ p'p^{-\frac{3}{2}} = o(1), \quad (\ln p)^{-1} \int\limits_{a}^{t} \left|(p'p^{-\frac{3}{2}})'\right| d\tau = o(1), \ \ t \to t_0.$$

Then there take place the property of asymptotic stability.

2. The Main Results

2.1. **Reduction of equation** (1) **to the system of the form** (5)**.** Consider the real second order LHDE (1):

$$y'' + p(t)y' + q(t)y = 0, \;\; t \in I,$$

where $p(t), q(t) \in C_I^1$. Let $y = y_1$, $y' = y_2$. We reduce the equation to an equivalent system

$$\begin{cases} y_1' = 0 \cdot y_1 + 1 \cdot y_2, \\ y_2' = -q \cdot y_1 - p \cdot y_2. \end{cases} \tag{7}$$

Consider the characteristic equation of LHS (6):

$$\begin{vmatrix} 0-\lambda & 1 \\ -q & -p-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 + p\lambda + q = 0, \tag{8}$$

and assume that $\frac{p^2}{2} - q < 0$ at I. Then this equation has two complex-conjugate roots:

$$\lambda_1 = \alpha - i\beta, \quad \lambda_2 = \alpha + i\beta,$$

where $\lambda_i = \lambda_i(t)$ $(i = 1, 2)$, $\alpha = \alpha(t) \in C_I^1$, $\beta = \beta(t) \in C_I^1$. Given (2), we will consider the case

$$\alpha(t) < 0, \quad \int\limits_{t_0}^{+\infty} \alpha(t)\,dt = -\infty. \tag{9}$$

There is the question on the sufficient conditions for stability of the trivial solution of the system (7). Consider the following transformation for the system (7):

$$Y = C(t)Z, \quad C(t) = \begin{pmatrix} 1 & 1 \\ \lambda_1(t) & \lambda_2(t) \end{pmatrix}, \quad Z = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix},$$

where $z_i(t)$ are new unknown functions $(i = 1, 2)$.

$$Z' = (C^{-1}AC - C^{-1}C')Z,$$
$$\det C(t) = \lambda_2(t) - \lambda_1(1),$$
$$C^{-1}(t) = \frac{1}{\lambda_2(t) - \lambda_1(1)} \begin{pmatrix} \lambda_2(t) & -1 \\ -\lambda_1(t) & 1 \end{pmatrix},$$
$$C'(t) = \begin{pmatrix} 0 & 0 \\ \lambda_1'(t) & \lambda_2'(t) \end{pmatrix}, \quad C^{-1}C' = \frac{1}{\lambda_2(t) - \lambda_1(1)} \begin{pmatrix} -\lambda_1'(t) & -\lambda_2'(t) \\ \lambda_1'(t) & \lambda_2'(t) \end{pmatrix},$$
$$C^{-1}AC = \begin{pmatrix} \lambda_1(t) & 0 \\ 0 & \lambda_2(t) \end{pmatrix}.$$

The system withe respect to new unknowns $z_i(t)$ $(i = 1, 2)$ in a scalar form is

$$\begin{cases} z_1'(t) = \Big(\lambda_1(t) + \dfrac{\lambda_1'(t)}{\lambda_2(t)-\lambda_1(t)}\Big) z_1(t) + \dfrac{\lambda_2'(t)}{\lambda_2(t)-\lambda_1(t)} z_2(t), \\ z_2'(t) = -\dfrac{\lambda_1'(t)}{\lambda_2(t)-\lambda_1(t)} z_1(t) + \Big(\lambda_2(t) - \dfrac{\lambda_2'(t)}{\lambda_2(t)-\lambda_1(t)}\Big) z_2(t). \end{cases} \tag{10}$$

It is not difficult to see that

$$\operatorname{Re} \frac{\lambda_1'(t)}{\lambda_2(t)-\lambda_1(t)} = -\frac{1}{2}\,\frac{\beta'}{\beta}, \quad \operatorname{Re} \frac{\lambda_2'(t)}{\lambda_2(t)-\lambda_1(t)} = \frac{1}{2}\,\frac{\beta'}{\beta},$$

$$h(t) = \Big|\frac{\lambda_1'(t)}{\lambda_2(t)-\lambda_1(t)}\Big| = \Big|\frac{\lambda_2'(t)}{\lambda_2(t)-\lambda_1(t)}\Big| = \frac{1}{2}\sqrt{\Big(\frac{\beta'}{\beta}\Big)^2 + \Big(\frac{\alpha'}{\beta}\Big)^2}\,.$$

In accordance with Theorem 1 we write an auxiliary system of differential equations:

$$\begin{cases} \sigma_1'(t) = \Big(\alpha - \dfrac{1}{2}\,\dfrac{\beta'}{\beta}\Big)\sigma_1(t) + h(t)\sigma_2(t), \\ \sigma_2'(t) = h(t) + \Big(\alpha - \dfrac{1}{2}\,\dfrac{\beta'}{\beta}\Big)\sigma_2(t). \end{cases} \tag{11}$$

Consider a particular solution with initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$. This solution has the form

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int_{t_0}^{t} (\alpha - \frac{1}{2}\frac{\beta'}{\beta})\,d\tau} \displaystyle\int_{t_0}^{t} h(\tau) e^{-\int_{\tau_0}^{\tau} (\alpha - \frac{1}{2}\frac{\beta'}{\beta})\,d\tau_1}\,d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int_{t_0}^{t} (\alpha - \frac{1}{2}\frac{\beta'}{\beta})\,d\tau} \displaystyle\int_{t_0}^{t} h(\tau)\widetilde{\sigma}_2(\tau) e^{-\int_{\tau_0}^{\tau} (\alpha - \frac{1}{2}\frac{\beta'}{\beta})\,d\tau_1}\,d\tau. \end{cases} \tag{12}$$

Assume also that there exists a finite or an infinite limit

$$\lim_{t\to+\infty} \frac{\alpha}{\beta}\,.$$

2.2. **Various cases of behavior of the roots $\lambda_i(t)$ $(i = 1, 2)$.** We consider the following cases of behavior of the roots of the characteristic equation, assuming that the condition (9) is fulfilled:

1) $\alpha(+\infty) \in \mathbb{R}_-$, $\beta(+\infty) \in \mathbb{R}$;
2) $\alpha = o(1)$, $\beta = o(1)$, $\frac{\alpha}{\beta} \to const \neq 0$;
3) $\alpha = o(1)$, $\beta = o(1)$, $\frac{\alpha}{\beta} \to \infty$;
4) $\alpha = o(1)$, $\beta(+\infty) \in \mathbb{R} \setminus \{0\}$;
5) $\alpha = o(1)$, $\beta = o(1)$, $\frac{\alpha}{\beta} \to 0$;
6) $\alpha(+\infty) = -\infty$, $\beta(+\infty) = \infty$, $\frac{\alpha}{\beta} \to \infty$;
7) $\alpha(+\infty) = -\infty$, $\beta(+\infty) \in \mathbb{R} \setminus \{0\}$;

8) $\alpha(+\infty) = -\infty$, $\beta = o(1)$;

9) $\alpha(+\infty) = -\infty$, $\beta(+\infty) = \infty$, $\frac{\alpha}{\beta} \to const \neq 0$;

10) $\alpha = o(1)$, $\beta(+\infty) = \infty$;

11) $\alpha(+\infty) \in \mathbb{R}_-$, $\beta(+\infty) = \infty$;

12) $\alpha(+\infty) = -\infty$, $\beta(+\infty) = \infty$, $\frac{\alpha}{\beta} \to 0$.

Theorems 5–16 correspond to the above cases 1)–12).

Theorem 5. *Let the condition* (9) *be fulfilled and*

$$\alpha(+\infty) \in \mathbb{R}_-, \quad \beta(+\infty) \in \mathbb{R}.$$

Then the trivial solution of equation (1) *is asymptotically stable.*

This case is well known.

Theorem 6. *Let the condition* (9) *and the following conditions*

$$\alpha = o(1), \quad \beta = o(1), \quad \frac{\alpha}{\beta} \to const \neq 0,$$
$$\frac{\alpha'}{\alpha^2} = o(1), \quad \frac{\beta'}{\beta^2} = o(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. We consider the system (10), auxiliary system of differential equations (11) and its particular solution (12).

In this case

$$\lim_{t\to+\infty} \frac{h(t)}{\alpha - \frac{1}{2}\frac{\beta'}{\beta}} = \frac{1}{2}\lim_{t\to+\infty} \frac{\sqrt{(\frac{\beta'}{\beta^2})^2 + (\frac{\alpha'}{\beta^2})^2}}{\frac{\alpha}{\beta} - \frac{1}{2}\frac{\beta'}{\beta^2}} =$$
$$= \frac{1}{2}\lim_{t\to+\infty} \left|\frac{\alpha'}{\beta^2}\right| \frac{\beta}{\alpha} = \frac{1}{2}\lim_{t\to+\infty} \frac{|\alpha'|}{\alpha^2}\,\frac{\alpha}{\beta} = 0.$$

Consequently, $\widetilde{\sigma}_2(t) = o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty} \frac{h(t)}{\alpha - \frac{1}{2}\frac{\beta'}{\beta}}\,\widetilde{\sigma}_2(t) = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. Obviously, $\psi(t) = h(t) \not\equiv 0$ for $t \in I$. All the conditions of Theorem 3 are fulfilled and thus Theorem 6 is complete. To obtain the estimate of solutions $y_i(t)$ $(i = 1, 2)$ we make in the system (10) the following substitution:

$$z_i(t) = e^{\delta \int_{t_0}^{t} \alpha\, d\tau}\, \eta_i(t) \quad (i = 1, 2), \quad \delta \in (0, 1). \tag{13}$$

Then the system (10) takes the form

$$\begin{cases} \eta_1'(t) = \Big(\lambda_1(t) + \dfrac{\lambda_1'(t)}{\lambda_2(t)-\lambda_1(t)} - \delta\alpha\Big)\eta_1(t) + \dfrac{\lambda_2'(t)}{\lambda_2(t)-\lambda_1(t)}\,\eta_2(t), \\ \eta_2'(t) = -\dfrac{\lambda_1'(t)}{\lambda_2(t)-\lambda_1(t)}\,\eta_1(t) + \Big(\lambda_2(t) - \dfrac{\lambda_2'(t)}{\lambda_2(t)-\lambda_1(t)} - \delta\alpha\Big)\eta_2(t). \end{cases} \tag{14}$$

In accordance with Theorem 1, we write an auxiliary system of differential equations:

$$\begin{cases} \sigma_1'(t) = \Big((1-\delta)\alpha - \dfrac{1}{2}\,\dfrac{\beta'}{\beta}\Big)\sigma_1(t) + h(t)\sigma_2(t), \\ \sigma_2'(t) = h(t) + \Big((1-\delta)\alpha - \dfrac{1}{2}\,\dfrac{\beta'}{\beta}\Big)\sigma_2(t). \end{cases} \tag{15}$$

It's particular solution with the initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$ has the form

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int\limits_{t_0}^{t}\left((1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta}\right)d\tau} \displaystyle\int\limits_{t_0}^{t} h(\tau)e^{-\int\limits_{\tau_0}^{\tau}\left((1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta}\right)d\tau_1}\,d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int\limits_{t_0}^{t}\left((1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta}\right)d\tau} \displaystyle\int\limits_{t_0}^{t} h(\tau)\widetilde{\sigma}_2(\tau)e^{-\int\limits_{\tau_0}^{\tau}\left((1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta}\right)d\tau_1}\,d\tau. \end{cases} \tag{16}$$

It is not difficult to see that the replacement (13) does not affect the asymptotic stability. Taking into account the transformation $C(t)$,

$$\begin{cases} y_1(t) = z_1(t) + z_2(t), \\ y_2(t) = \lambda_1(t)z_1(t) + \lambda_2(t)z_2(t). \end{cases} \Longrightarrow$$

$$\Longrightarrow \begin{cases} y_1(t) = o\Big(e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big), \\ y_2(t) = o\Big(\lambda_1(t)e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau} + \lambda_2(t)e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big). \end{cases}$$

$$y_2(t) = o\Big(e^{\int\limits_{t_0}^{t}\left(\delta\alpha + \frac{\lambda_1'(t)}{\lambda_1(t)}\right)d\tau} + e^{\int\limits_{t_0}^{t}\left(\delta\alpha + \frac{\lambda_2'(t)}{\lambda_2(t)}\right)d\tau}\Big),$$

$$y_2(t) = o\Big(e^{\int\limits_{t_0}^{t}\alpha\left(\delta + \frac{1}{\alpha}\frac{\lambda_1'(t)}{\lambda_1(t)}\right)d\tau} + e^{\int\limits_{t_0}^{t}\alpha\left(\delta + \frac{1}{\alpha}\frac{\lambda_2'(t)}{\lambda_2(t)}\right)d\tau}\Big).$$

It is easy to see that

$$R(t) = \operatorname{Re}\frac{\lambda_1'(t)}{\lambda_1(t)} = \operatorname{Re}\frac{\lambda_2'(t)}{\lambda_2(t)} = \frac{\alpha'\alpha + \beta'\beta}{\alpha^2 + \beta^2}\,,$$

$$I(t) = \operatorname{Im}\frac{\lambda_1'(t)}{\lambda_1(t)} = -\operatorname{Im}\frac{\lambda_2'(t)}{\lambda_2(t)} = \frac{\alpha'\beta - \alpha\beta'}{\alpha^2 + \beta^2}\,.$$

Then

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,\mathrm{Re}\,\frac{\lambda_1'(t)}{\lambda_1(t)}=\lim_{t\to+\infty}\frac{\alpha'\alpha+\beta'\beta}{\alpha(\alpha^2+\beta^2)}=$$

$$=\lim_{t\to+\infty}\left(\frac{\frac{\alpha'}{\alpha^2}}{1+(\frac{\beta}{\alpha})^2}+\frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^3+\frac{\alpha}{\beta}}\right)=0,$$

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,\mathrm{Im}\,\frac{\lambda_1'(t)}{\lambda_1(t)}=\lim_{t\to+\infty}\frac{\alpha'\beta-\alpha\beta'}{\alpha(\alpha^2+\beta^2)}=$$

$$=\lim_{t\to+\infty}\left(\frac{\frac{\alpha'}{\alpha^2}}{\frac{\alpha}{\beta}+\frac{\beta}{\alpha}}-\frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2+1}\right)=0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)}=o(\alpha)\ \ (i=1,2).$$

Therefore,

$$y_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$ □

Theorem 7. *Let the condition* (9) *and the following conditions*

$$\alpha=o(1),\quad \beta=o(1),\quad \frac{\alpha}{\beta}\to\infty,$$

$$\frac{\alpha'}{\alpha}=o(\beta),\quad \frac{\beta'}{\beta^2}=O(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replace (13). We obtain the system (14), an auxiliary system of differential equations (15) and its particular solution (16).

In this case,

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\frac{\beta'}{\beta}}=\frac{1}{2}\lim_{t\to+\infty}\frac{\sqrt{(\frac{\beta'}{\beta})^2+(\frac{\alpha'}{\beta})^2}}{\alpha(1-\delta-\frac{1}{2}\frac{\beta'}{\alpha\beta})}=$$

$$=\frac{1}{2}\lim_{t\to+\infty}\sqrt{\left(\frac{\frac{\beta'}{\alpha\beta}}{1-\delta-\frac{1}{2}\frac{\beta'}{\alpha\beta}}\right)^2+\left(\frac{\frac{\alpha'}{\alpha\beta}}{1-\delta-\frac{1}{2}\frac{\beta'}{\alpha\beta}}\right)^2}=$$

$$=\frac{1}{2}\lim_{t\to+\infty}\sqrt{\left(\frac{\frac{\beta'}{\beta^2}\frac{\beta}{\alpha}}{1-\delta-\frac{1}{2}\frac{\beta'}{\beta^2}\frac{\beta}{\alpha}}\right)^2+\left(\frac{\frac{\alpha'}{\alpha\beta}}{1-\delta-\frac{1}{2}\frac{\beta'}{\beta^2}\frac{\beta}{\alpha}}\right)^2}=0.$$

Consequently, $\widetilde{\sigma}_2(t)=o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\frac{\beta'}{\beta}}\,\widetilde{\sigma}_2(t)=0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 7 is valid. Moreover,

$$z_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Next,

$$\lim_{t\to+\infty} \frac{1}{\alpha} R(t) = \lim_{t\to+\infty} \left(\frac{\frac{\alpha'}{\alpha\beta}}{\frac{\alpha}{\beta} + \frac{\beta}{\alpha}} + \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^3 + \frac{\alpha}{\beta}} \right) = 0,$$

$$\lim_{t\to+\infty} \frac{1}{\alpha} I(t) = \lim_{t\to+\infty} \left(\frac{\frac{\alpha'}{\alpha\beta}}{(\frac{\alpha}{\beta})^2 + 1} - \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2 + 1} \right) = 0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)} = o(\alpha) \ \ (i = 1, 2).$$

Therefore, just as in Theorem 6:

$$y_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1). \qquad \square$$

Theorem 8. *Let the condition* (9) *and the following conditions*

$$\alpha = o(1), \quad \beta(+\infty) \in \mathbb{R} \setminus \{0\},$$
$$\frac{\alpha'}{\alpha} = o(1), \quad \frac{\beta'}{\beta} = o(\alpha)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (13). We obtain the system (14), an auxiliary system of differential equations (15) and its particular solution (16).

In this case,

$$\lim_{t\to+\infty} \frac{h(t)}{\alpha - \frac{1}{2}\frac{\beta'}{\beta}} = \frac{1}{2} \lim_{t\to+\infty} \frac{\sqrt{(\frac{\beta'}{\beta})^2 + (\frac{\alpha'}{\beta})^2}}{\alpha(1 - \delta - \frac{1}{2}\frac{\beta'}{\alpha\beta})} =$$
$$= \frac{1}{2} \lim_{t\to+\infty} \frac{\sqrt{(\frac{\beta'}{\alpha\beta})^2 + (\frac{\alpha'}{\alpha\beta})^2}}{1 - \delta - \frac{1}{2}\frac{\beta'}{\alpha\beta}} = \frac{1}{2(1-\delta)} \lim_{t\to+\infty} \left| \frac{\alpha'}{\alpha\beta} \right| = 0.$$

Therefore, $\widetilde{\sigma}_2(t) = o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty} \frac{h(t)}{(1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta}} \widetilde{\sigma}_2(t) = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 8 is valid. Moreover,

$$z_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, \tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then

$$\lim_{t\to+\infty}\frac{1}{\alpha}R(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha}}{\alpha+\frac{\beta}{\alpha}\beta}+\frac{\frac{\beta'}{\alpha\beta}}{(\frac{\alpha}{\beta})^2+1}\Big)=0,$$

$$\lim_{t\to+\infty}\frac{1}{\alpha}I(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha}}{\frac{\alpha}{\beta}\alpha+\beta}-\frac{\frac{\beta'}{\alpha\beta}}{\frac{\alpha}{\beta}+\frac{\beta}{\alpha}}\Big)=0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)}=o(\alpha)\ \ (i=1,2).$$

Therefore, just as in Theorem 6,

$$y_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).\qquad\square$$

Theorem 9. *Let the condition* (9) *and the following conditions*

$$\alpha=o(1),\quad \beta=o(1),\quad \frac{\alpha}{\beta}\to 0,$$

$$\frac{\alpha'}{\alpha^2}=O(1),\quad \frac{\beta'}{\beta}=o(\alpha)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (13). We obtain the system (14), an auxiliary system of differential equations (15) and its particular solution (16).

In this case,

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\frac{\beta'}{\beta}}=\frac{1}{2}\lim_{t\to+\infty}\frac{\sqrt{(\frac{\beta'}{\alpha\beta})^2+(\frac{\alpha'}{\alpha\beta})^2}}{1-\delta-\frac{1}{2}\frac{\beta'}{\alpha\beta}}=$$

$$=\frac{1}{2(1-\delta)}\lim_{t\to+\infty}\Big|\frac{\alpha'}{\alpha\beta}\Big|=\frac{1}{2(1-\delta)}\lim_{t\to+\infty}\Big|\frac{\alpha'}{\alpha^2}\,\frac{\alpha}{\beta}\Big|=0.$$

Consequently, $\widetilde{\sigma}_2(t)=o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\frac{\beta'}{\beta}}\widetilde{\sigma}_2(t)=0.$$

Then $\widetilde{\sigma}_1(t)=o(1)$. This implies that Theorem 9 is valid. Moreover,

$$z_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

Then

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,R(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha^2}}{1+(\frac{\beta}{\alpha})^2}+\frac{\frac{\beta'}{\alpha\beta}}{(\frac{\alpha}{\beta})^2+1}\Big)=0,$$

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,I(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha^2}}{\frac{\alpha}{\beta}+\frac{\beta}{\alpha}}-\frac{\frac{\beta'}{\alpha\beta}}{\frac{\alpha}{\beta}+\frac{\beta}{\alpha}}\Big)=0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)}=o(\alpha)\ \ (i=1,2).$$

Therefore, just as in Theorem 6,

$$y_i(t)=o\Big(e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1). \qquad \square$$

Theorem 10. *Let the condition* (9) *and the following conditions*

$$\alpha(+\infty)=-\infty,\quad \beta(+\infty)=\infty,\quad \frac{\alpha}{\beta}\to\infty,$$
$$\frac{\alpha'}{\alpha}=O(1),\quad \frac{\beta'}{\beta^2}=O(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the following replacement:

$$z_1(t)\lambda_1(t)=\xi_1(t),\quad z_2(t)\lambda_2(t)=\xi_2(t). \tag{17}$$

Then the system (10) takes the form

$$\begin{cases}\xi_1'(t)=\Big(\lambda_1(t)+\dfrac{\lambda_1'(t)}{\lambda_2(t)-\lambda_1(t)}-\dfrac{\lambda_1'(t)}{\lambda_1(t)}\Big)\xi_1(t)+\\ \qquad+\dfrac{\lambda_2'(t)}{\lambda_2(t)-\lambda_1(t)}\,\dfrac{\lambda_1(t)}{\lambda_2(t)}\xi_2(t),\\ \xi_2'(t)=-\dfrac{\lambda_1'(t)}{\lambda_2(t)-\lambda_1(t)}\,\dfrac{\lambda_2(t)}{\lambda_1(t)}\xi_1(t)+\\ \qquad+\Big(\lambda_2(t)-\dfrac{\lambda_2'(t)}{\lambda_2(t)-\lambda_1(t)}-\dfrac{\lambda_2'(t)}{\lambda_2(t)}\Big)\xi_2(t).\end{cases} \tag{18}$$

In accordance with Theorem 1, we write an auxiliary system of differential equations:

$$\begin{cases}\sigma_1'(t)=\Big(\alpha-\dfrac{1}{2}\,\dfrac{\beta'}{\beta}-R(t)\Big)\sigma_1(t)+h(t)\sigma_2(t),\\ \sigma_2'(t)=h(t)+\Big(\alpha-\dfrac{1}{2}\,\dfrac{\beta'}{\beta}-R(t)\Big)\sigma_2(t).\end{cases}$$

Consider a particular solution with the initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$:

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int\limits_{t_0}^{t} \left(\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)\right) d\tau} \displaystyle\int_{t_0}^{t} h(\tau) e^{-\int\limits_{\tau_0}^{\tau} \left(\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)\right) d\tau_1} d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int\limits_{t_0}^{t} \left(\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)\right) d\tau} \displaystyle\int_{t_0}^{t} h(\tau)\widetilde{\sigma}_2(\tau) e^{-\int\limits_{\tau_0}^{\tau} \left(\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)\right) d\tau_1} d\tau. \end{cases}$$

In this case,

$$\lim_{t\to+\infty} \frac{1}{\beta} R(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha}}{\beta(1 + (\frac{\beta}{\alpha})^2)} + \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2 + 1} \Big) = 0.$$

Then

$$\lim_{t\to+\infty} \frac{h(t)}{\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)} = \frac{1}{2} \lim_{t\to+\infty} \frac{\sqrt{(\frac{\beta'}{\beta^2})^2 + (\frac{\alpha'}{\beta^2})^2}}{\frac{\alpha}{\beta} - \frac{1}{2}\frac{\beta'}{\beta^2} - \frac{1}{\beta} R(t)} =$$

$$= \frac{1}{2} \lim_{t\to+\infty} \sqrt{\Big(\frac{\beta'}{\beta^2}\frac{\beta}{\alpha}\Big)^2 + \Big(\frac{\alpha'}{\alpha\beta}\Big)^2} = \frac{1}{2} \lim_{t\to+\infty} \Big|\frac{\alpha'}{\alpha\beta}\Big| = 0.$$

Therefore, $\widetilde{\sigma}_2(t) = o(1)$, by Lemma 1. Next,

$$\lim_{t\to+\infty} \frac{h(t)}{\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)} \widetilde{\sigma}_2(t) = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 10 is valid. To obtain the estimate of solutions $y_i(t)$ $(i = 1, 2)$, we make in the system (18) the following replacement:

$$\xi_i(t) = e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau} \eta_i(t) \ \ (i = 1, 2), \ \ \delta \in (0, 1). \tag{19}$$

Then system (18) takes the form

$$\begin{cases} \eta_1'(t) = \Big(\lambda_1(t) + \dfrac{\lambda_1'(t)}{\lambda_2(t) - \lambda_1(t)} - \dfrac{\lambda_1'(t)}{\lambda_1(t)} - \delta\alpha\Big)\eta_1(t) + \\ \qquad + \dfrac{\lambda_2'(t)}{\lambda_2(t) - \lambda_1(t)} \dfrac{\lambda_1(t)}{\lambda_2(t)} \eta_2(t), \\ \eta_2'(t) = -\dfrac{\lambda_1'(t)}{\lambda_2(t) - \lambda_1(t)} \dfrac{\lambda_2(t)}{\lambda_1(t)} \eta_1(t) + \\ \qquad + \Big(\lambda_2(t) - \dfrac{\lambda_2'(t)}{\lambda_2(t) - \lambda_1(t)} - \dfrac{\lambda_2'(t)}{\lambda_2(t)} - \delta\alpha\Big)\eta_2(t). \end{cases} \tag{20}$$

In accordance with Theorem 1, we write an auxiliary system of differential equations:

$$\begin{cases} \sigma_1'(t) = \Big((1-\delta)\alpha - \dfrac{1}{2}\,\dfrac{\beta'}{\beta} - R(t)\Big)\sigma_1(t) + h(t)\sigma_2(t), \\ \sigma_2'(t) = h(t) + \Big((1-\delta)\alpha - \dfrac{1}{2}\,\dfrac{\beta'}{\beta} - R(t)\Big)\sigma_2(t). \end{cases} \tag{21}$$

Let us consider a particular solution with the initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$:

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int\limits_{t_0}^{t} \left((1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)\right) d\tau} \times \\ \qquad \times \int\limits_{t_0}^{t} h(\tau) e^{-\int\limits_{\tau_0}^{\tau} \left((1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)\right) d\tau_1} d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int\limits_{t_0}^{t} \left((1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)\right) d\tau} \times \\ \qquad \times \int\limits_{t_0}^{t} h(\tau)\widetilde{\sigma}_2(\tau) e^{-\int\limits_{\tau_0}^{\tau} \left((1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)\right) d\tau_1} d\tau. \end{cases} \tag{22}$$

It is not difficult to see that the replacement (19) does not affect the stability. At the same time,

$$\xi_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then, according to (17),

$$z_i(t) = o\Big(e^{\int\limits_{t_0}^{t} (\delta\alpha - \frac{\lambda_1'(t)}{\lambda_1(t)})\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Further,

$$\lim_{t\to+\infty} \frac{1}{\alpha} R(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha}}{\alpha(1 + (\frac{\beta}{\alpha})^2)} + \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^3 + \frac{\alpha}{\beta}} \Big) = 0,$$

$$\lim_{t\to+\infty} \frac{1}{\alpha} I(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha}}{\alpha(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})} - \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2 + 1} \Big) = 0.$$

Consequently,

$$\frac{\lambda_i'(t)}{\lambda_i(t)} = o(\alpha) \ \ (i = 1, 2)$$

and

$$z_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then

$$\begin{cases} y_1(t) = z_1(t) + z_2(t), \\ y_2(t) = \lambda_1(t)z_1(t) + \lambda_2(t)z_2(t) \end{cases} \Longrightarrow \begin{cases} y_1(t) = z_1(t) + z_2(t), \\ y_2(t) = \xi_1(t) + \xi_2(t) \end{cases} \Longrightarrow$$

$$\Longrightarrow y_i(t) = o\Big(e^{\delta \int_{t_0}^{t} \alpha\, d\tau}\Big) \quad (i = 1, 2), \quad \delta \in (0, 1). \qquad \square$$

Theorem 11. *Let the condition* (9) *and the following conditions*

$$\alpha(+\infty) = -\infty, \quad \beta(+\infty) \in \mathbb{R} \setminus \{0\},$$
$$\frac{\alpha'}{\alpha} = o(1), \quad \frac{\beta'}{\beta^2} = O(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (17). We get the system (18). In the system (18) we make the replacement (19). We obtain the system (20), an auxiliary system of differential equations (21) and its particular solution (22).

In this case,

$$\lim_{t\to+\infty} \frac{1}{\beta} R(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha}}{\beta(1 + (\frac{\beta}{\alpha})^2)} + \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2 + 1} \Big) = 0.$$

Then

$$\lim_{t\to+\infty} \frac{h(t)}{(1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)} = \frac{1}{2} \lim_{t\to+\infty} \frac{\sqrt{(\frac{\beta'}{\beta^2})^2 + (\frac{\alpha'}{\beta^2})^2}}{(1-\delta)\frac{\alpha}{\beta} - \frac{1}{2}\frac{\beta'}{\beta^2} - \frac{1}{\beta}R(t)} =$$

$$= \frac{1}{2(1-\delta)} \lim_{t\to+\infty} \sqrt{\Big(\frac{\beta'}{\beta^2}\,\frac{\beta}{\alpha}\Big)^2 + \Big(\frac{\alpha'}{\alpha\beta}\Big)^2} = \frac{1}{2(1-\delta)} \lim_{t\to+\infty} \Big|\frac{\alpha'}{\alpha\beta}\Big| = 0.$$

Therefore, $\widetilde{\sigma}_2(t) = o(1)$, by Lemma 1. Next,

$$\lim_{t\to+\infty} \frac{h(t)}{(1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)} \widetilde{\sigma}_2(t) = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 11 is valid. Thus

$$\xi_i(t) = o\Big(e^{\delta \int_{t_0}^{t} \alpha\, d\tau}\Big) \quad (i = 1, 2), \quad \delta \in (0, 1).$$

Then, according to (17),

$$z_i(t) = o\Big(e^{\int_{t_0}^{t} (\delta\alpha - \frac{\lambda_1'(t)}{\lambda_1(t)})\, d\tau}\Big) \quad (i = 1, 2), \quad \delta \in (0, 1).$$

Further,

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,R(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha}}{\alpha(1+(\frac{\beta}{\alpha})^2)}+\frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^3+\frac{\alpha}{\beta}}\Big)=0,$$

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,I(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha}}{\alpha(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})}-\frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2+1}\Big)=0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)}=o(\alpha)\ \ (i=1,2)$$

and

$$z_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

Then, just as in Theorem 10,

$$y_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).\qquad\square$$

Theorem 12. *Let the condition* (9) *and the following conditions*

$$\alpha(+\infty)=-\infty,\quad \beta=o(1),$$
$$\frac{\alpha'}{\alpha}=o(\beta),\quad \frac{\beta'}{\beta^2}=O(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (17). We get the system (18). In the system (18) we make the replacement (19). We obtain the system (20), an auxiliary system of differential equations (21) and its particular solution (22).

In this case,

$$\lim_{t\to+\infty}\frac{1}{\beta}\,R(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha\beta}}{1+(\frac{\beta}{\alpha})^2}+\frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2+1}\Big)=0.$$

Then

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\frac{\beta'}{\beta}-R(t)}=\frac{1}{2}\lim_{t\to+\infty}\frac{\sqrt{(\frac{\beta'}{\beta^2})^2+(\frac{\alpha'}{\beta^2})^2}}{(1-\delta)\,\frac{\alpha}{\beta}-\frac{1}{2}\,\frac{\beta'}{\beta^2}-\frac{1}{\beta}\,R(t)}=$$
$$=\frac{1}{2(1-\delta)}\lim_{t\to+\infty}\sqrt{\Big(\frac{\beta'}{\beta^2}\,\frac{\beta}{\alpha}\Big)^2+\Big(\frac{\alpha'}{\alpha\beta}\Big)^2}=\frac{1}{2(1-\delta)}\lim_{t\to+\infty}\Big|\frac{\alpha'}{\alpha\beta}\Big|=0.$$

Therefore, $\widetilde{\sigma}_2(t)=o(1)$, by Lemma 1. Next,

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\frac{\beta'}{\beta}-R(t)}\,\widetilde{\sigma}_2(t)=0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 12 is valid. Thus

$$\xi_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then, according to (17),

$$z_i(t) = o\Big(e^{\int\limits_{t_0}^{t} (\delta\alpha - \frac{\lambda_1'(t)}{\lambda_1(t)})\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then

$$\lim_{t\to+\infty} \frac{1}{\alpha} R(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha\beta}}{\frac{\alpha}{\beta} + \frac{\beta}{\alpha}} + \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^3 + \frac{\alpha}{\beta}} \Big) = 0,$$

$$\lim_{t\to+\infty} \frac{1}{\alpha} I(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha\beta}}{(\frac{\alpha}{\beta})^2 + 1} - \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2 + 1} \Big) = 0.$$

Hence

$$\frac{\lambda_i'(t)}{\lambda_i(t)} = o(\alpha) \ \ (i = 1, 2)$$

and

$$z_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then, just as in Theorem 10,

$$y_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \alpha\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1). \qquad \square$$

Theorem 13. *Let the condition* (9) *and thr following conditions*

$$\alpha(+\infty) = -\infty, \quad \beta(+\infty) = \infty, \quad \frac{\alpha}{\beta} \to const \neq 0,$$

$$\frac{\alpha'}{\alpha} = O(1), \quad \frac{\beta'}{\beta^2} = o(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (17). We get the system (18). In the system (18) we make the replacement (19). We obtain the system (20), an auxiliary system of differential equations (21) and its particular solution (22).

In this case,

$$\lim_{t\to+\infty} \frac{1}{\alpha} R(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha}}{\beta(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})} + \frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^3 + \frac{\alpha}{\beta}} \Big) = 0.$$

Then

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\frac{\beta'}{\beta}-R(t)}=\lim_{t\to+\infty}\frac{h(t)}{\alpha(1-\delta-\frac{1}{2}\frac{\beta'}{\alpha\beta}-\frac{1}{\alpha}R(t))}=$$

$$=\frac{1}{2}\lim_{t\to+\infty}\sqrt{\Big(\frac{\frac{\beta'}{\beta^2}\frac{\beta}{\alpha}}{1-\delta-\frac{1}{2}\frac{\beta'}{\beta^2}\frac{\beta}{\alpha}}\Big)^2+\Big(\frac{\frac{\alpha'}{\alpha\beta}}{1-\delta-\frac{1}{2}\frac{\beta'}{\beta^2}\frac{\beta}{\alpha}}\Big)^2}=$$

$$=\frac{1}{2(1-\delta)}\lim_{t\to+\infty}\Big|\frac{\alpha'}{\alpha\beta}\Big|=0.$$

Therefore, $\widetilde{\sigma}_2(t)=o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\frac{\beta'}{\beta}-R(t)}\,\widetilde{\sigma}_2(t)=0.$$

Then $\widetilde{\sigma}_1(t)=o(1)$. This implies that Theorem 13 is valid. Thus

$$\xi_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

Then, according to (17),

$$z_i(t)=o\Big(e^{\int_{t_0}^{t}(\delta\alpha-\frac{\lambda_1'(t)}{\lambda_1(t)})\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

As is shown above,

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,R(t)=0.$$

Then

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,I(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha}}{\alpha(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})}-\frac{\frac{\beta'}{\beta^2}}{(\frac{\alpha}{\beta})^2+1}\Big)=0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)}=o(\alpha)\ \ (i=1,2)$$

and

$$z_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

Then, just as in Theorem 10,

$$y_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

□

Theorem 14. *Let the condition* (9) *and the following conditions*

$$\alpha=o(1),\quad \beta(+\infty)=\infty,$$

$$\frac{\alpha'}{\alpha}=O(1),\quad \frac{\beta'}{\beta}=o(\alpha)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (17). We get the system (18). In the system (18) we make the replacement (19). We obtain the system (20), an auxiliary system of differential equations (21) and its particular solution (22).

In this case,

$$\lim_{t\to+\infty} \frac{1}{\alpha} R(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha}}{\beta(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})} + \frac{\frac{\beta'}{\alpha\beta}}{(\frac{\alpha}{\beta})^2 + 1} \Big) = 0.$$

Then

$$\lim_{t\to+\infty} \frac{h(t)}{(1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)} = \frac{1}{2} \lim_{t\to+\infty} \frac{\sqrt{(\frac{\beta'}{\alpha\beta})^2 + (\frac{\alpha'}{\alpha\beta})^2}}{(1 - \delta - \frac{1}{2}\frac{\beta'}{\alpha\beta} - \frac{1}{\alpha} R(t))} = 0.$$

Therefore, $\widetilde{\sigma}_2(t) = o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty} \frac{h(t)}{(1-\delta)\alpha - \frac{1}{2}\frac{\beta'}{\beta} - R(t)} \widetilde{\sigma}_2(t) = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 14 is valid. Moreover,

$$\xi_i(t) = o\Big(e^{\delta \int_{t_0}^{t} \alpha \, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then, according to (17),

$$z_i(t) = o\Big(e^{\int_{t_0}^{t} (\delta\alpha - \frac{\lambda_1'(t)}{\lambda_1(t)}) \, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

As is shown above,

$$\lim_{t\to+\infty} \frac{1}{\alpha} R(t) = 0.$$

Then

$$\lim_{t\to+\infty} \frac{1}{\alpha} I(t) = \lim_{t\to+\infty} \Big(\frac{\frac{\alpha'}{\alpha}}{\frac{\alpha}{\beta}\alpha + \beta} - \frac{\frac{\beta'}{\alpha\beta}}{\frac{\alpha}{\beta} + \frac{\beta}{\alpha}} \Big) = 0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)} = o(\alpha) \ \ (i = 1, 2)$$

and

$$z_i(t) = o\Big(e^{\delta \int_{t_0}^{t} \alpha \, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then, just as in Theorem 10,

$$y_i(t) = o\Big(e^{\delta \int_{t_0}^{t} \alpha \, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1). \qquad \square$$

Theorem 15. *Let the condition* (9) *and the following conditions*

$$\alpha(+\infty)\in\mathbb{R}_-,\quad \beta(+\infty)=\infty,$$
$$\frac{\alpha'}{\alpha}=O(1),\quad \frac{\beta'}{\beta}=o(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (17). We get the system (18). In the system (18) we make the replacement (19). We obtain the system (20), an auxiliary system of differential equations (21) and its particular solution (22).

In this case,

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,R(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha}}{\beta(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})}+\frac{\frac{\beta'}{\beta}}{\alpha((\frac{\alpha}{\beta})^2+1)}\Big)=0.$$

Then

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\frac{\beta'}{\beta}-R(t)}=\frac{1}{2}\lim_{t\to+\infty}\frac{\sqrt{(\frac{\beta'}{\alpha\beta})^2+(\frac{\alpha'}{\alpha\beta})^2}}{(1-\delta-\frac{1}{2}\frac{\beta'}{\alpha\beta}-\frac{1}{\alpha}\,R(t))}=0.$$

Therefore, $\widetilde{\sigma}_2(t)=o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha-\frac{1}{2}\frac{\beta'}{\beta}-R(t)}\,\widetilde{\sigma}_2(t)=0.$$

Then $\widetilde{\sigma}_1(t)=o(1)$. This implies that Theorem 15 is valid. Hence

$$\xi_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

Then, according to (17),

$$z_i(t)=o\Big(e^{\int_{t_0}^{t}(\delta\alpha-\frac{\lambda_1'(t)}{\lambda_1(t)})\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

As is shown above,

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,R(t)=0.$$

Then

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,I(t)=\lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha}}{\alpha(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})}-\frac{\frac{\beta'}{\beta}}{\alpha(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})}\Big)=0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)}=o(\alpha)\ \ (i=1,2)$$

and

$$z_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

Then, just as in Theorem 10,

$$y_i(t) = o\Big(e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1). \qquad \square$$

Theorem 16. *Let the condition* (9) *and the following conditions*

$$\alpha(+\infty) = -\infty, \quad \beta(+\infty) = \infty, \quad \frac{\alpha}{\beta}\to 0,$$

$$\frac{\alpha'}{\alpha^2} = O(1), \quad \frac{\beta'}{\beta} = O(1)$$

be fulfilled. Then the trivial solution of equation (1) *is asymptotically stable.*

Proof. In the system (10) we make the replacement (17). We get the system (18). In the system (18) we make the replacement (19). We obtain the system (20), an auxiliary system of differential equations (21) and its particular solution (22).

In this case,

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,R(t) = \lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha^2}}{1+(\frac{\beta}{\alpha})^2} + \frac{\frac{\beta'}{\beta}}{\alpha((\frac{\alpha}{\beta})^2+1)}\Big) = 0.$$

Then

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha - \frac{1}{2}\,\frac{\beta'}{\beta} - R(t)} = \frac{1}{2}\lim_{t\to+\infty}\frac{\sqrt{(\frac{\beta'}{\alpha\beta})^2 + (\frac{\alpha'}{\alpha\beta})^2}}{(1-\delta-\frac{1}{2}\,\frac{\beta'}{\alpha\beta} - \frac{1}{\alpha}\,R(t))} = 0.$$

Consequently, $\widetilde{\sigma}_2(t) = o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty}\frac{h(t)}{(1-\delta)\alpha - \frac{1}{2}\,\frac{\beta'}{\beta} - R(t)}\,\widetilde{\sigma}_2(t) = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 16 is valid. Moreover,

$$\xi_i(t) = o\Big(e^{\delta\int\limits_{t_0}^{t}\alpha\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

Then, according to (17),

$$z_i(t) = o\Big(e^{\int\limits_{t_0}^{t}(\delta\alpha - \frac{\lambda_1'(t)}{\lambda_1(t)})\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

As is shown above,

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,R(t) = 0.$$

Then

$$\lim_{t\to+\infty}\frac{1}{\alpha}\,I(t) = \lim_{t\to+\infty}\Big(\frac{\frac{\alpha'}{\alpha^2}}{\frac{\alpha}{\beta}+\frac{\beta}{\alpha}} - \frac{\frac{\beta'}{\beta}}{\alpha(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})}\Big) = 0.$$

Thus

$$\frac{\lambda_i'(t)}{\lambda_i(t)} = o(\alpha)\ \ (i=1,2)$$

and

$$z_i(t) = o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big) \quad (i=1,2), \quad \delta\in(0,1).$$

Then, just as in Theorem 10,

$$y_i(t) = o\Big(e^{\delta\int_{t_0}^{t}\alpha\,d\tau}\Big) \quad (i=1,2), \quad \delta\in(0,1). \qquad \square$$

2.3. **The case of purely imaginary roots $\lambda_i(t)$ $(i=1,2)$.** Let us analyze equation (6):

$$y'' + p(t)y = 0$$

where $p(t)\in C_I^2$, $p(t)>0$ in I. Then $\lambda_1(t) = -i\sqrt{p}$, $\lambda_2(t) = i\sqrt{p}$, $p=p(t)$.

Theorem 17. *Let the conditions*

$$\beta(+\infty) = +\infty, \quad \frac{\beta'}{\beta^2} = o(1), \quad \frac{(\frac{\beta'}{\beta^2})'}{\frac{\beta'}{\beta}} = o(1)$$

be fulfilled. Then the trivial solution of equation (6) *is asymptotically stable.*

Proof. In this case the system (10) takes the form

$$\begin{cases} z_1'(t) = \Big(-\dfrac{1}{2}\dfrac{\beta'}{\beta} - i\beta\Big)z_1(t) + \dfrac{1}{2}\dfrac{\beta'}{\beta}\,z_2(t),\\ z_2'(t) = \dfrac{1}{2}\dfrac{\beta'}{\beta}\,z_1(t) + \Big(-\dfrac{1}{2}\dfrac{\beta'}{\beta} + i\beta\Big)z_2(t). \end{cases}$$

In this system we make the following replacement:

$$z_i(t) = e^{\delta\int_{t_0}^{t}(-\frac{1}{2}\frac{\beta'}{\beta})\,d\tau}\varphi_i(t) \quad (i=1,2), \quad \delta\in(0,1).$$

As a result, we obtain the following system:

$$\begin{cases} \varphi_1'(t) = \Big(-\dfrac{1-\delta}{2}\dfrac{\beta'}{\beta} - i\beta\Big)\varphi_1(t) + \dfrac{1}{2}\dfrac{\beta'}{\beta}\,\varphi_2(t),\\ \varphi_2'(t) = \dfrac{1}{2}\dfrac{\beta'}{\beta}\,\varphi_1(t) + \Big(-\dfrac{1-\delta}{2}\dfrac{\beta'}{\beta} + i\beta\Big)\varphi_2(t). \end{cases} \tag{23}$$

Then in the system (23) we make the following replacement:

$$\begin{pmatrix}\varphi_1(t)\\ \varphi_2(t)\end{pmatrix} = \begin{pmatrix}1 & 0\\ r(t) & 1\end{pmatrix}\begin{pmatrix}\eta_1(t)\\ \eta_2(t)\end{pmatrix},$$

where $\eta_i(t)$ are the new unknown functions $(i=1,2)$. Then the system (23) takes the form

$$\begin{cases} \eta_1'(t) = \Big(-\dfrac{1-\delta}{2}\dfrac{\beta'}{\beta} + \dfrac{1}{2}\dfrac{\beta'}{\beta}\,r(t) - i\beta\Big)\eta_1(t) + \dfrac{1}{2}\dfrac{\beta'}{\beta}\,\eta_2(t),\\ \eta_2'(t) = \Big(\dfrac{1}{2}\dfrac{\beta'}{\beta} + 2i\beta r(t) - \dfrac{1}{2}\dfrac{\beta'}{\beta}\,r^2(t) - r'(t)\Big)\eta_1(t) +\\ \qquad + \Big(-\dfrac{1-\delta}{2}\dfrac{\beta'}{\beta} - \dfrac{1}{2}\dfrac{\beta'}{\beta}\,r(t) + i\beta\Big)\eta_2(t). \end{cases} \tag{24}$$

Suppose

$$\frac{1}{2}\,\frac{\beta'}{\beta}+2i\beta r(t)=0.$$

Then

$$r(t)=\frac{1}{4}\,\frac{\beta'}{\beta^2}\,i=o(1).$$

Then the system (24) takes the form

$$\begin{cases}\eta_1'(t)=\Big(-\dfrac{1-\delta}{2}\,\dfrac{\beta'}{\beta}+\dfrac{1}{8}\,\dfrac{\beta'}{\beta}\,\dfrac{\beta'}{\beta^2}\,i-i\beta\Big)\eta_1(t)+\dfrac{1}{2}\,\dfrac{\beta'}{\beta}\,\eta_2(t),\\ \eta_2'(t)=\Big(-\dfrac{1}{4}\,\Big(\dfrac{\beta'}{\beta^2}\Big)'i+\dfrac{1}{8}\,\dfrac{\beta'}{\beta}\,\Big(\dfrac{\beta'}{\beta^2}\Big)^2\Big)\eta_1(t)+\\ \qquad +\Big(-\dfrac{1-\delta}{2}\,\dfrac{\beta'}{\beta}-\dfrac{1}{8}\,\dfrac{\beta'}{\beta}\,\dfrac{\beta'}{\beta^2}\,i+i\beta\Big)\eta_2(t).\end{cases}\tag{25}$$

In accordance with Theorem 1, for the system (25) we write an auxiliary system of differential equations:

$$\begin{cases}\sigma_1'(t)=-\dfrac{1-\delta}{2}\,\dfrac{\beta'}{\beta}\,\sigma_1(t)+\dfrac{1}{2}\,\Big|\dfrac{\beta'}{\beta}\Big|\,\sigma_2(t),\\ \sigma_2'(t)=\dfrac{1}{8}\,\sqrt{4\Big(\Big(\dfrac{\beta'}{\beta^2}\Big)'\Big)^2+\Big(\dfrac{\beta'}{\beta}\Big)^2\Big(\dfrac{\beta'}{\beta^2}\Big)^4}-\dfrac{1-\delta}{2}\,\dfrac{\beta'}{\beta}\,\sigma_2(t).\end{cases}\tag{26}$$

We denote

$$g(t)=\frac{1}{8}\,\sqrt{4\Big(\Big(\frac{\beta'}{\beta^2}\Big)'\Big)^2+\Big(\frac{\beta'}{\beta}\Big)^2\Big(\frac{\beta'}{\beta^2}\Big)^4}.$$

Consider a particular solution of the system (26) with the initial conditions $\sigma_i(t_0)=0$ $(i=1,2)$:

$$\begin{cases}\widetilde{\sigma}_2(t)=e^{\int_{t_0}^{t}(-\frac{1-\delta}{2}\,\frac{\beta'}{\beta})\,d\tau}\displaystyle\int_{t_0}^{t} g(\tau)e^{-\int_{\tau_0}^{\tau}(-\frac{1-\delta}{2}\,\frac{\beta'}{\beta})\,d\tau_1}\,d\tau,\\ \widetilde{\sigma}_1(t)=e^{\int_{t_0}^{t}(-\frac{1-\delta}{2}\,\frac{\beta'}{\beta})\,d\tau}\displaystyle\int_{t_0}^{t}\frac{1}{2}\,\Big|\frac{\beta'}{\beta}\Big|\widetilde{\sigma}_2(\tau)e^{-\int_{\tau_0}^{\tau}(-\frac{1-\delta}{2}\,\frac{\beta'}{\beta})\,d\tau_1}\,d\tau.\end{cases}$$

In this case,

$$\lim_{t\to+\infty}\frac{g(t)}{-\frac{1-\delta}{2}\,\frac{\beta'}{\beta}}=-\frac{1}{4(1-\delta)}\lim_{t\to+\infty}\sqrt{4\Big(\frac{(\frac{\beta'}{\beta^2})'}{\frac{\beta'}{\beta}}\Big)^2+\Big(\frac{\beta'}{\beta^2}\Big)^4}=$$

$$=-\frac{1}{1-\delta}\lim_{t\to+\infty}\Big|\frac{(\frac{\beta'}{\beta^2})'}{\frac{\beta'}{\beta}}\Big|=0.$$

Therefore, $\widetilde{\sigma}_2(t) = o(1)$, by Lemma 1. Further, we have

$$\lim_{t\to+\infty} \frac{\frac{1}{2}\,|\frac{\beta'}{\beta}|}{-\frac{1-\delta}{2}\,\frac{\beta'}{\beta}}\,\widetilde{\sigma}_2(t) = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 17 is valid. Moreover, $\eta_i(t) = o(1)$ $(i = 1, 2)$. Then $\varphi_i(t) = o(1)$ $(i = 1, 2)$. Then we have obtained

$$z_i(t) = o\Big(e^{\delta \int_{t_0}^{t} (-\frac{1}{2}\,\frac{\beta'}{\beta})\,d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then

$$\begin{cases} y_1(t) = z_1(t) + z_2(t), \\ y_2(t) = -i\beta z_1(t) + i\beta z_2(t) \end{cases} \Longrightarrow \begin{cases} y_1(t) = o\Big(e^{\delta \int_{t_0}^{t} (-\frac{1}{2}\,\frac{\beta'}{\beta})\,d\tau}\Big), \\ y_2(t) = -i\beta z_1(t) + i\beta z_2(t) \end{cases} \Longrightarrow$$

$$\Longrightarrow |y_2(t)| = o\Big(e^{\int_{t_0}^{t} (-\frac{1}{2}\,\delta\,\frac{\beta'}{\beta} + \frac{\beta'}{\beta^2})\,d\tau}\Big) \Longrightarrow$$

$$\Longrightarrow \begin{cases} |y_1(t)| = o\Big(e^{\delta \int_{t_0}^{t} (-\frac{1}{2}\,\frac{\beta'}{\beta})\,d\tau}\Big), \\ |y_2(t)| = o\Big(e^{\int_{t_0}^{t} \delta(-\frac{1}{2}\,\frac{\beta'}{\beta} + o(1))\,d\tau}\Big), \end{cases} \qquad \delta \in (0, 1). \qquad \square$$

Remark 1. The condition

$$\frac{(\frac{\beta'}{\beta^2})'}{\frac{\beta'}{\beta}} = o(1)$$

is satisfied if there exists the corresponding limit.

Proof.

$$\lim_{t\to+\infty} \frac{(\frac{\beta'}{\beta^2})'}{\frac{\beta'}{\beta}} = \Big[\text{we use the inverse de L'Hospital's rule}\Big] =$$

$$= -\frac{1}{2} \lim_{t\to+\infty} \frac{\beta'}{\beta^2 \ln\beta} = 0. \qquad \square$$

Remark 2. The conditions of I. T. Kiguradze's Theorem 4 are equivalent to those of Theorem 17. But, in addition to Theorem 17, we have obtained the estimate of solutions of equation (6).

Proof.

$$\lim_{t\to+\infty} p'p^{-\frac{3}{2}} = \lim_{t\to+\infty} (\beta^2)'\beta^{-3} = \lim_{t\to+\infty} \frac{2\beta\beta'}{\beta^3} = 2\lim_{t\to+\infty} \frac{\beta'}{\beta^2} = 0,$$

$$\lim_{t\to+\infty} (\ln p)^{-1} \int\limits_a^t \left|(p'p^{-\frac{3}{2}})'\right| d\tau = \Big[\text{we use de L'Hospital's rule}\Big] =$$

$$= \lim_{t\to+\infty} \frac{|(p'p^{-\frac{3}{2}})'|}{\frac{p'}{p}} = \lim_{t\to+\infty} \frac{|(2\beta\beta'\beta^{-3})'|}{\frac{2\beta\beta'}{\beta^2}} = \lim_{t\to+\infty} \frac{|(\frac{\beta'}{\beta^2})'|}{\frac{\beta'}{\beta}} = 0. \quad \square$$

Conclusion

In the present paper we have revealed the sufficient conditions for asymptotic stability, as well as the estimate of solutions of the homogeneous linear non-autonomous second order differential equation in terms of the behavior of roots of the characteristic equation in the case of complex roots. The results of the work allow one to proceed both to investigating equations of higher order and to considering the problems on a simple stability and instability.

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Authors' address:

Odessa I. I. Mechnikov National University, 2 Dvoryanska St., Odessa 65082, Ukraine.

E-mail: Tani112358@mail.ru; kostin_a@ukr.net

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Ivan Kiguradze

A PRIORI ESTIMATES OF SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS FOR SINGULAR IN PHASE VARIABLES HIGHER ORDER DIFFERENTIAL INEQUALITIES AND SYSTEMS OF DIFFERENTIAL INEQUALITIES

Dedicated to the blessed memory of professor A. Razmadze

Abstract. For singular in phase variables higher order nonlinear differential inequalities and systems of nonlinear differential inequalities a priori estimates of solutions satisfying nonlinear boundary conditions of a certain type are established.

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რეზიუმე. ფაზური ცვლადების მიმართ სინგულარული მაღალი რიგის არაწრფივი დიფერენციალური უტოლობებისა და არაწრფივ დიფერენციალურ უტოლობათა სისტემებისათვის დადგენილია იმ ამონახსნების აპრიორული შეფასებები, რომლებიც გარკვეული სახის არაწრფივ სასაზღვრო პირობებს აკმაყოფილებენ.

Introduction

The boundary value problems for singular in phase variables second order differential equations attract attention of many mathematicians and are the subject of various investigations (see, e.g., [1–4, 6, 10, 12–14, 16, 17] and references therein). As for the singular in phase variables higher order differential equations and differential systems, for them only the initial and two-point problems [7,9], the Nikoletti perturbed problem [8] and the Kneser type problem [15] are studied.

The construction of the theory of boundary value problems for singular in phase variables differential equations and systems requires a priori estimates of solutions of singular in phase variables higher order differential inequalities and systems of differential inequalities, satisfying different nonlinear boundary conditions. The present paper contains such estimates.

We have used the following notation.

$x = (x_i)_{i=1}^n$ and $X = (x_{ik})_{i,k=1}^n$ are the n-dimensional vector column and the $n \times n$-matrix with the components x_i and x_{ik} $(i,k = 1,\dots,n)$ and the norms

$$\|x\| = \sum_{i=1}^n |x_i|, \quad \|X\| = \sum_{i,k=1}^n |x_{ik}|;$$

$r(X)$ is the spectral radius of the matrix X;

$\mathbb{R}_+ = [0,+\infty[\,$, $\mathbb{R}_{0+} =]0,+\infty[\,$;

$\mathbb{R}^n$ is the n-dimensional real Euclidean space;

$\mathbb{R}^n_{0+} = \big\{(x_i)_{i=1}^n \in \mathbb{R}^n : \ x_1 > 0,\dots,x_n > 0\big\}$;

$\widetilde{C}([a,b];\mathbb{R})$ is the space of absolutely continuous functions $u : [a,b] \to \mathbb{R}$;

$\widetilde{C}^m([a,b];\mathbb{R})$ is the space of m-times continuously differentiable functions $u : [a,b] \to \mathbb{R}$ whose derivative of m-th order is absolutely continuous;

$\widetilde{C}^m([a,b];\mathbb{R}^n_{0+})$ is the set of vector functions $(u_i)_{i=1}^n : [a,b] \to \mathbb{R}^n_{0+}$ with absolutely continuous components $u_i : [a,b] \to \mathbb{R}_{0+}$ $(i = 1,\dots,n)$.

1. Higher Order Differential Inequalities

In a finite interval $[a,b]$ we consider the n-th order differential inequality

$$g_0\big(t,u(t),\dots,u^{(n-1)}(t)\big) \le u^{(n)}(t) \le \le \sum_{k=1}^n g_k\big(t,u(t),\dots,u^{(n-1)}(t)\big)u^{(k-1)}(t) \quad (1.1)$$

with the boundary conditions

$$\alpha_i u^{(i-1)}(b) \le u^{(i-1)}(a) \le \beta_i u^{(i-1)}(b) + \beta_0 \ \ (i = 1,\dots,n). \quad (1.2)$$

Here $g_k : [a,b] \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ $(k = 0,\dots,n)$ are integrable in the first argument and continuous and nonincreasing in the last n arguments functions, α_i $(i = 1,\dots,n)$ and β_i $(i = 0,\dots,n)$ are constants such that

$$0 < \alpha_i \le \beta_i < 1 \ \ (i = 1,\dots,n), \ \ \beta_0 > 0. \quad (1.3)$$

We are mainly interested in the case where the differential inequality (1.1) is singular in phase variables, i.e., in the case when there exists a set of positive measure $I \subset [a,b]$ such that

$$\lim_{x_1+\cdots+x_n\to 0} g_k(t,x_1,\dots,x_n) = +\infty \ \text{ for } \ t\in I \ \ (k=0,\dots,n).$$

A function $u \in \widetilde{C}^{n-1}([a,b];\mathbb{R})$ is said to be a **solution of the differential inequality** (1.1) if

$$u^{(i-1)}(t) > 0 \ \text{ for } \ a \le t \le b \ \ (i=1,\dots,n)$$

and almost everywhere on $[a,b]$ the inequality (1.1) is fulfilled.

A solution of the differential inequality (1.1) satisfying the boundary conditions (1.2) is called a **solution of the problem** (1.1)**,** (1.2).

Before we give a theorem containing a priori estimates of solutions of the above-mentioned problem, we prove a simple lemma dealing with estimates of solutions of the differential inequality

$$u^{(n)}(t) \ge 0, \tag{1.4}$$

satisfying the boundary conditions (1.2).

Lemma 1.1. *An arbitrary solution u of the problem* (1.4)*,* (1.2) *admits the estimates*

$$\gamma_{0k}\ell \le u^{(k-1)}(t) \le \gamma_k(\ell+\beta_0) \ \text{ for } \ a\le t\le b \ \ (k=1,\dots,n), \tag{1.5}$$

where

$$\gamma_k = (b-a)^{n-k}\prod_{i=k}^{n}(1-\beta_i)^{-1} \ \ (k=1,\dots,n), \tag{1.6}$$

$$\gamma_{0k} = (b-a)^{n-k}\prod_{i=k}^{n}\frac{\alpha_i}{1-\alpha_i} \ \ (k=1,\dots,n), \tag{1.7}$$

and

$$\ell = \int_a^b u^{(n)}(s)\,ds. \tag{1.8}$$

Proof. In view of (1.2), (1.8), we have

$$u^{(n-1)}(b) = u^{(n-1)}(a) + \ell \ge \alpha_n u^{(n-1)}(b) + \ell,$$
$$u^{(n-1)}(b) \le \beta_n u^{(n-1)}(b) + \beta_0 + \ell,$$

and hence

$$u^{(n-1)}(b) \ge \frac{1}{1-\alpha_n}\,\ell, \quad u^{(n-1)}(b) \le \frac{1}{1-\beta_n}\,(\beta_0+\ell).$$

If along with this we take into account the inequality (1.4), it becomes obvious that

$$u^{(n-1)}(t) \geq u^{(n-1)}(a) \geq \alpha_n u^{(n-1)}(b) \geq \\ \geq \gamma_{0n}\ell, \quad u^{(n-1)}(t) \leq u^{(n-1)}(b) \leq \gamma_n(\beta_0+\ell) \text{ for } a \leq t \leq b.$$

This, according to the induction law and notations (1.6) and (1.7), results in the estimate (1.5). ☐

Theorem 1.1. *If along with* (1.3) *the conditions*

$$\int_a^b g_0(s,x,\dots,x)\,ds > 0 \ \textit{ for } \ x > 0, \tag{1.9}$$

$$\lim_{x\to+\infty} \sum_{k=1}^n \gamma_k \int_a^b g_k(s,x,\dots,x)\,ds < 1 \tag{1.10}$$

are fulfilled, then there exist positive constants δ and ρ such that an arbitrary solution of the problem (1.1), (1.2) *admits the estimates*

$$\delta \leq u^{(k-1)}(t) \leq \rho \ \textit{ for } \ a \leq t \leq b \ \ (k=1,\dots,n). \tag{1.11}$$

Proof. By the inequality (1.10), there exists a positive number x_0 such that

$$\Big(1+\frac{\beta_0}{x_0}\Big)\sum_{k=1}^n \gamma_k \int_a^b g_k(s,x_0,\dots,x_0)\,ds < 1. \tag{1.12}$$

Suppose

$$\gamma_0 = \min\big\{1,\gamma_{01},\dots,\gamma_{0n}\big\}, \quad \gamma = \max\big\{\gamma_1,\dots,\gamma_n\big\},$$
$$\rho = \Big(\frac{x_0}{\gamma_0}+\beta_0\Big)\gamma,$$

and

$$\delta = \gamma_0 \int_a^b g_0(s,\rho,\dots,\rho)\,ds.$$

Owing to (1.9), it is clear that $\delta > 0$.

Let u be an arbitrary solution of the problem (1.1), (1.2), and let ℓ be the number given by the equality (1.8). Then by Lemma 1.1, the inequalities (1.5) are valid. On the other hand, it follows from (1.1) and (1.5) that

$$\ell \leq (\ell+\beta_0)\sum_{k=1}^n \gamma_k \int_a^b g_k\big(s,\ell\gamma_0,\dots,\ell\gamma_0\big)\,ds \tag{1.13}$$

and

$$\ell \geq \int_a^b g_0\big(s,(\ell+\beta_0)\gamma,\dots,(\ell+\beta_0)\gamma\big)\,ds, \tag{1.14}$$

since g_k $(k = 0, \dots, n)$ are nonincreasing in the last n arguments functions.

Our aim is to prove that u admits the estimates (1.11). Let us first show that

$$\ell < \frac{x_0}{\gamma_0}\,. \tag{1.15}$$

Assume the contrary that

$$\ell \geq \frac{x_0}{\gamma_0}\,.$$

Then $\ell \geq x_0$. Thus taking into account the inequality (1.12), from the inequality (1.13) we find

$$\ell \leq \ell\Big(1 + \frac{\beta_0}{x_0}\Big) \sum_{k=1}^{n} \gamma_k \int\limits_a^b g_k(s, x_0, \dots, x_0)\, ds < \ell.$$

The obtained contradiction proves the validity of the estimate (1.15).

According to (1.5), (1.14) and (1.15), we have

$$u^{(k-1)}(t) < \Big(\frac{x_0}{\gamma_0} + \beta_0\Big)\gamma = \rho \ \text{ for } \ a \leq t \leq b \ \ (k = 1, \dots, n)$$

and

$$u^{(k-1)}(t) \geq \ell\gamma_0 \geq \gamma_0 \int\limits_a^b g_0(s, \rho, \dots, \rho)\, ds = \delta \ \text{ for } \ a \leq t \leq b.$$

Consequently, the estimates (1.11) are valid. □

As an example, we consider the differential inequality

$$p_0(t)q_0\big(u(t), \dots, u^{(n-1)}(t)\big) \leq u^{(n)}(t) \leq \\ \leq p(t)q\big(u(t), \dots, u^{(n-1)}(t)\big) + \sum_{k=1}^{n} p_k(t)u^{(k-1)}(t), \tag{1.16}$$

where $p_k : [a, b] \to \mathbb{R}_+$ $(k = 0, \dots, n)$, $p : [a, b] \to \mathbb{R}_+$ are integrable functions, and $q_0 : \mathbb{R}^n_{0+} \to \mathbb{R}_{0+}$, $q : \mathbb{R}^n_{0+} \to \mathbb{R}_{0+}$ are continuous and nonincreasing in all variables functions.

Corollary 1.1. *If*

$$\int\limits_a^b p_0(s)\, ds > 0, \quad \sum_{k=1}^{n} \gamma_k \int\limits_a^b p_k(s)\, ds < 1, \tag{1.17}$$

then there exist positive constants δ and ρ such that an arbitrary solution of the problem (1.16), (1.2) *admits the estimates* (1.11)*.*

Proof. Let

$$g_0(t, x_1, \dots, x_n) = p_0(t)q_0(x_1, \dots, x_n),$$

$$g_k(t, x_1, \dots, x_n) = \frac{p(t)}{nx_k}\, q(x_1, \dots, x_n) + p_k(t) \ \ (k = 1, \dots, n).$$

Then the differential inequality (1.16) takes the form (1.1). On the other hand, by virtue of (1.17), the functions $g_k : [a,b] \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ $(k = 0, \dots, n)$ satisfy the conditions (1.9) and (1.10). If now we apply Theorem 1.1, then validity of Corollary 1.1 becomes evident. □

Note that in the conditions of Theorem 1.1 or Corollary 1.1, the differential inequality under consideration may have singularities of arbitrary orders in phase variables. For example, In Corollary 1.1 as q_0 and q we can take the functions

$$q_0(x_1, \dots, x_n) = \ell_{01} \prod_{i=1}^{n} x_i^{-\lambda_{0i}} \exp\Big(\ell_{02} \prod_{j=1}^{n} x_j^{-\mu_{0j}}\Big),$$

$$q(x_1, \dots, x_n) = q_0(x_1, \dots, x_n) + \ell_1 \prod_{i=1}^{n} x_i^{-\lambda_i} \exp\Big(\ell_2 \prod_{j=1}^{n} x_j^{-\mu_j}\Big),$$

where λ_{0i}, λ_i, μ_{0i}, μ_i $(i = 1, \dots, n)$, ℓ_{0k}, ℓ_k $(k = 1, 2)$ are positive constants.

2. First Order Differential Inequalities

Let us consider the differential inequality

$$\sigma\big(u'(t) - p(t)u(t) - q\big(t, u(t)\big)\big) \geq 0 \tag{2.1}$$

with the boundary condition

$$\sigma\big(u(a) - \alpha u(b) - \alpha_0\big) \geq 0, \tag{2.2}$$

where $p : [a,b] \to \mathbb{R}$ is an integrable function, $q : [a,b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ is an integrable in the first argument and continuous and nonincreasing in the second argument function, $\sigma \in \{-1, 1\}$, $\alpha > 0$ and $\alpha_0 \geq 0$ are constants.

An absolutely continuous function $u : [a,b] \to \mathbb{R}_{0+}$ is said to be a **solution of the problem** (2.1)**,** (2.2) if it satisfies the condition (2.2) and almost everywhere on $[a,b]$ satisfies the differential inequality (2.1).

Along with (2.1), (2.2), we consider the boundary value problem of periodic type:

$$v'(t) = p(t)v(t) + q\big(t, v(t)\big), \tag{2.3}$$

$$v(a) = \alpha v(b) + \alpha_0. \tag{2.4}$$

The following theorem holds.

Theorem 2.1. *If*

$$\alpha \exp\bigg(\int_a^b p(s)\, ds\bigg) < 1 \tag{2.5}$$

and

$$\int_a^b q(s,x)\, ds > 0 \ \ \textit{for} \ \ x > 0, \tag{2.6}$$

then the problem (2.3), (2.4) *has a unique solution* v*, and an arbitrary solution* u *of the problem* (2.1), (2.2) *admits the estimate*

$$\sigma\big(u(t)-v(t)\big)\geq 0 \ \ \textit{for} \ \ a\leq t\leq b. \tag{2.7}$$

To prove the theorem, we need the following simple lemma.

Lemma 2.1. *Let* $t_0\in[a,b[$ *and* $c>0$*. Then the differential equation* (2.1) *under the initial condition*

$$v(t_0)=c \tag{2.8}$$

has a unique solution v *in the interval* $[t_0,b]$*, and an arbitrary solution* u *of the differential inequality* (2.1)*, satisfying the condition*

$$\sigma\big(u(t_0)-c\big)\geq 0,$$

admits the estimate

$$\sigma\big(u(t)-v(t)\big)\geq 0 \ \ \textit{for} \ \ t_0\leq t\leq b. \tag{2.9}$$

Proof. The unique solvability of the problem (2.1), (2.8) in the interval $[t_0,b]$ follows from the fact that $c>0$ and the function $q:[a,b]\times\mathbb{R}_{0+}\to\mathbb{R}_+$ is nonincreasing in the second argument.

Applying now Lemma 4.3 from [5], the validity of the estimate (2.9) becomes evident. □

Proof of Theorem 2.1. For the sake of definiteness we assume that $\sigma=1$ since the case where $\sigma=-1$ is considered analogously.

If $q:[a,b]\times\mathbb{R}_{0+}\to\mathbb{R}_+$ is a continuous and nonincreasing in the second argument function, then by Theorem 7 of [11], the conditions (2.5) and (2.6) guarantee the unique solvability of the problem (2.3), (2.4). If, however, q is integrable in the first and continuous and nonincreasing in the second argument, then using the method of proving of the above-mentioned theorem, we can show that the conditions (2.5) and (2.6) again guarantee the existence of a unique solution v of the problem (2.3), (2.4).

Let u be an arbitrary solution of the problem (2.1), (2.2). If

$$u(a)\geq v(a),$$

then by Lemma 2.1, the estimate (2.7) is valid.

To prove the theorem, it remains to show that the inequality

$$u(a)<v(a) \tag{2.10}$$

cannot take place.

Assume the contrary that the inequality (2.10) is valid. Then either

$$u(t)<v(t) \ \ \text{for} \ \ a<t<b, \tag{2.11}$$

or there exists $t_0\in\,]a,b[$ such that

$$u(t_0)\geq v(t_0). \tag{2.12}$$

Let the inequality (2.11) be fulfilled. Then in view of (2.1), almost everywhere on $[a, b]$ the inequality

$$u'(t) \geq p(t)u(t) + q\big(t, v(t)\big) \tag{2.13}$$

is fulfilled since q is the nonincreasing in the second argument function.

Put

$$w(t) = v(t) - u(t).$$

Then in view of the conditions (2.2), (2.4), (2.10) and (2.13),we have

$$0 < w(a) \leq \alpha w(b)$$

and

$$w'(t) \leq p(t)w(t) \ \text{ for almost all } \ t \in [a, b].$$

From these inequalities with regard for the condition (2.5) we find

$$w(b) \leq \exp\bigg(\int_a^b p(s)\,ds\bigg) w(a) \leq \alpha \exp\bigg(\int_a^b p(s)\,ds\bigg) w(b) < w(b).$$

The obtained contradiction proves that the inequality (2.11) cannot take place. Consequently, for some $t_0 \in]a, b[$ the inequality (2.12) is fulfilled.

By Lemma 2.1, the function u admits the estimate (2.9). From (2.4), (2.9) and (2.10), we find

$$u(a) < v(a) = \alpha v(b) + \alpha_0 \leq \alpha u(b) + \alpha_0,$$

which contradicts the inequality (2.2). The obtained contradiction proves that the inequality (2.10) cannot take place. Thus the theorem is proved. □

In conclusion of this section we consider the problem

$$\sigma\big(u'(t) - p(t)u(t) + q\big(t, u(t)\big)\big) \leq 0, \tag{2.14}$$

$$\sigma\big(u(a) - \alpha u(b) + \alpha_0\big) \leq 0, \tag{2.15}$$

and the differential equation

$$v'(t) = p(t)v(t) - q\big(t, v(t)\big) \tag{2.16}$$

with the boundary condition

$$v(a) = \alpha v(b) - \alpha_0. \tag{2.17}$$

As above we assume that $p : [a, b] \to \mathbb{R}$ is an integrable function, and $q : [a, b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ is an integrable in the first and continuous and nonincreasing in the second argument function, $\sigma \in \{-1, 1\}$, $\alpha > 0$ and $\alpha_0 \geq 0$.

On the basis of Theorem 2.1, the following statement can be proved.

Theorem 2.2. *If along with* (2.6) *the inequality*

$$\alpha\exp\bigg(\int\limits_a^b p(s)\,ds\bigg) > 1 \tag{2.18}$$

is fulfilled, then the problem (2.16), (2.17) *has a unique solution* v, *and an arbitrary solution* u *of the problem* (2.14), (2.15) *admits the estimate* (2.7).

If $q(t,x)\equiv q(t)$, then the differential inequalities (2.1), (2.14) and the differential equations (2.3) and (2.16) have the following forms

$$\sigma\big(u'(t)-p(t)u(t)-q(t)\big)\geq 0, \tag{2.19}$$

$$\sigma\big(u'(t)-p(t)u(t)+q(t)\big)\leq 0, \tag{2.20}$$

$$v'(t)=p(t)v(t)+q(t), \tag{2.21}$$

$$v'(t)=p(t)v(t)-q(t). \tag{2.22}$$

It is easy to see that for the unique solvability of the problem (2.21), (2.4) (of the problem (2.22), (2.17)) it is necessary and sufficient the inequality

$$1-\alpha\exp\bigg(\int\limits_a^b p(s)\,ds\bigg)\neq 0 \tag{2.23}$$

to be fulfilled.

Let the inequality (2.23) hold. Put

$$\Delta(p,\alpha)=1-\alpha\exp\bigg(\int\limits_a^b p(s)\,ds\bigg), \tag{2.24}$$

$$g(p,\alpha)(t,s)= \\ =\begin{cases}\dfrac{1}{\Delta(p,\alpha)}\exp\bigg(\displaystyle\int\limits_s^t p(\tau)\,d\tau\bigg) & \text{for } a\leq s\leq t\leq b,\\[2mm] \dfrac{\alpha}{\Delta(p,\alpha)}\exp\bigg(\displaystyle\int\limits_a^b p(\tau)\,d\tau+\int\limits_s^t p(\tau)\,d\tau\bigg) & \text{for } a\leq t< s\leq b.\end{cases} \tag{2.25}$$

Then the solution of the problem (2.21), (2.22) admits the representation

$$v(t)=\frac{\alpha_0}{\Delta(p,\alpha)}\exp\bigg(\int\limits_a^t p(\tau)\,d\tau\bigg)+\int\limits_a^b g(p,\alpha)(t,s)q(s)\,ds,$$

and the solution of the problem (2.22), (2.17) admits the representation

$$v(t)=-\frac{\alpha_0}{\Delta(p,\alpha)}\exp\bigg(\int\limits_a^t p(s)\,ds\bigg)-\int\limits_a^b g(p,\alpha)(t,s)q(s)\,ds.$$

On the other hand, in view of the fact that the number α is positive, (2.24) and (2.25) imply

$$\Delta(p,\alpha)g(p,\alpha)(t,s) > 0 \quad \text{for} \quad a \le s \le t \le b. \tag{2.26}$$

If along with this we take into account the fact that the function q is nonnegative, then it becomes evident that Theorems 2.1 and 2.2 yield the following propositions.

Corollary 2.1. *If the inequality* (2.5) (*the inequality* (2.18)) *is fulfilled, then an arbitrary solution of the problem* (2.19), (2.2) (*of the problem* (2.20), (2.15)) *admits the estimate* (2.7), *where*

$$v(t) = \frac{\alpha_0}{|\Delta(p,\alpha)|} \exp\bigg(\int_a^t p(s)\,ds\bigg) + \int_a^b \big|g(p,\alpha)(t,s)\big| q(s)\,ds \quad \textit{for} \quad a \le t \le b.$$

Lemma 2.2. *Let p be a constant sign function, satisfying the condition* (2.23). *Then*

$$\int_a^b \big|g(p,\alpha)(t,s)p(s)\big|\,ds \le \frac{\alpha+1+|\alpha-1|}{2}\,\bigg|\frac{\Delta(p,1)}{\Delta(p,\alpha)}\bigg| \quad \textit{for} \quad a \le t \le b, \tag{2.27}$$

$$\int_a^b \big|g(p,\alpha)(t,s)p(s)\big|\,ds \ge \frac{\alpha+1-|\alpha-1|}{2}\,\bigg|\frac{\Delta(p,1)}{\Delta(p,\alpha)}\bigg| \quad \textit{for} \quad a \le t \le b. \tag{2.28}$$

Proof. Due to the fact that p is of constant sign and the condition (2.26), there exists a number $\sigma_0 \in \{-1,1\}$ such that

$$\int_a^b \big|g(p,\alpha)(t,s)p(s)\big|\,ds = \sigma_0 w(t) \quad \text{for} \quad a \le t \le b, \tag{2.29}$$

where

$$w(t) = \int_a^b g(p,\alpha)(t,s)p(s)\,ds.$$

On the other hand, in view of the equalities (2.24) and (2.25), we find

$$w(t) = \frac{1-\alpha}{\Delta(p,\alpha)} \exp\bigg(\int_a^t p(s)\,ds\bigg) - 1.$$

Hence it is clear that

$$\min\big\{|w(a)|,|w(b)|\big\} \le |w(t)| \le \max\big\{|w(a)|,|w(b)|\big\}.$$

However,

$$w(a) = -\frac{\alpha\Delta(p,1)}{\Delta(p,\alpha)}\,, \quad w(b) = -\frac{\Delta(p,1)}{\Delta(p,\alpha)}\,.$$

Thus,

$$\min\{\alpha,1\}\left|\frac{\Delta(p,1)}{\Delta(p,\alpha)}\right| \leq |w(t)| \leq \max\{\alpha,1\}\left|\frac{\Delta(p,1)}{\Delta(p,\alpha)}\right| \text{ for } a\leq t\leq b,$$

according to which from the equality (2.29) it follows the estimates (2.27) and (2.28). ☐

3. Systems of Differential Inequalities

In this section, we establish a priori estimates of solutions of the system of differential inequalities

$$\begin{aligned} q_i\big(t,u_i(t)\big) \leq \sigma_i\big(u_i'(t)-p_i(t)u_i(t)\big) \leq \\ \leq \sum_{k=1}^n p_{ik}\big(t,u_1(t)+\cdots+u_n(t)\big)u_k(t)+ \\ + q_0\big(t,u_1(t),\dots,u_n(t)\big) \;\; (i=1,\dots,n), \end{aligned} \tag{3.1}$$

satisfying the boundary conditions

$$\sigma_i\big(u_i(a)-\alpha_i u_i(b)\big)\geq 0, \quad \sigma_i\big(u_i(a)-\beta_i u_i(b)\big)\leq \beta_0 \;\; (i=1,\dots,n). \tag{3.2}$$

Here

$$\begin{gathered} \sigma_i\in\{-1,1\}, \;\; \alpha_i>0, \;\; \beta_i>0, \\ \sigma_i(\beta_i-\alpha_i)>0 \;\; (i=1,\dots,n), \;\; \beta_0>0, \end{gathered} \tag{3.3}$$

$p_i:[a,b]\to\mathbb{R}$ $(i=1,\dots,n)$ are integrable functions, $q_i:[a,b]\times\mathbb{R}_{0+}\to\mathbb{R}_+$ and $p_{ik}:[a,b]\times\mathbb{R}_{0+}\to\mathbb{R}_+$ $(i,k=1,\dots,n)$ are integrable in the first and continuous and nonincreasing in the second argument functions, and $q_0:[a,b]\times\mathbb{R}^n_{0+}\to\mathbb{R}_+$ is an integrable in the first and continuous and nonincreasing in the last n arguments function.

A vector function $(u_i)_{i=1}^n:[a,b]\to\mathbb{R}^n_{0+}$ with absolutely continuous components $u_i:[a,b]\to\mathbb{R}_{0+}$ $(i=1,\dots,n)$ **is said to be a solution of the system** (3.1) if it satisfies that system almost everywhere on $[a,b]$.

A solution of the system (3.1), satisfying the boundary conditions (3.2), **is said to be a solution of the problem** (3.1), (3.2).

We investigate the problem (3.1), (3.2) in the case, where

$$\int_a^b q_i(s,x)\,ds>0 \;\text{ for }\; x>0 \;\; (i=1,\dots,n) \tag{3.4}$$

and

$$\sigma_i\bigg(\beta_i\exp\bigg(\int_a^b p_i(s)\,ds\bigg)-1\bigg)<0 \;\; (i=1,\dots,n). \tag{3.5}$$

Let g be the operator given by the equalities (2.24) and (2.25). Suppose

$$h_{ik}(x) =$$

$$= \max\left\{ \int\limits_a^b \big|g(p_i,\beta_i)(t,s)\big| p_{ik}(s,x)\,ds : \ a \le t \le b \right\} \ (i,k = 1,\dots,n) \quad (3.6)$$

and

$$H(x) = \big(h_{ik}(x)\big)_{i,k=1}^n \ \text{ for } \ x > 0. \quad (3.7)$$

Theorem 3.1. *Let along with* (3.3)–(3.5) *the condition*

$$\lim_{x\to+\infty} r(H(x)) < 1 \quad (3.8)$$

be fulfilled. Then there exist positive constants δ and ρ such that an arbitrary solution $(u_i)_{i=1}^n$ of the problem (3.1), (3.2) *admits the estimates*

$$\delta \le u_i(t) \le \rho \ \text{ for } \ a \le t \le b \ \ (i = 1,\dots,n). \quad (3.9)$$

To prove this theorem, along with the results from Section 2 we need the following lemma.

Lemma 3.1. *Let $h_{ik} : \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i,k = 1,\dots,n)$ be nonincreasing functions, and h_i $(i = 1,\dots,n)$ be nonnegative constants. Let, moreover, there exist a positive number x_0 such that*

$$r(H(x_0)) < 1, \quad (3.10)$$

where H is a matrix function given by the equality (3.7)*. Then arbitrary positive numbers $x_1,\dots,x_n$, satisfying the system of inequalities*

$$x_i \le \sum_{k=1}^n h_{ik}(x_1 + \dots + x_n)x_k + h_i \ \ (i = 1,\dots,n), \quad (3.11)$$

satisfy the inequality

$$\sum_{i=1}^n x_i \le x_0 + \big\|(E - H(x_0))^{-1}\big\| \sum_{i=1}^n h_i \quad (3.12)$$

as well, where E is a unit $n \times n$-matrix, and $(E - H(x_0))^{-1}$ is a matrix, inverse to the matrix $E - H(x_0)$.

Proof. Assume the contrary that

$$\sum_{i=1}^n x_i > x_0 + \big\|(E - H(x_0))^{-1}\big\| \sum_{i=1}^n h_i. \quad (3.13)$$

Then from (3.10) we have

$$x_i \le \sum_{k=1}^n h_{ik}(x_0)x_k + h_i \ \ (i = 1,\dots,n)$$

since h_{ik} $(i,k = 1,\dots,n)$ are nonincreasing functions. Consequently,

$$(E - H(x_0))\overline{x} \le \overline{h}, \quad (3.14)$$

where

$$\overline{x} = (x_i)_{i=1}^n, \quad \overline{h} = (h_i)_{i=1}^n.$$

The nonnegativeness of the matrix $H(x_0)$ and the condition (3.10) guarantee the nondegeneracy of the matrix $E - H(x_0)$ and the nonnegativeness of the matrix $(E - H(x_0))^{-1}$.

If we multiply both sides of the inequality (3.14) by $(E - H(x_0))^{-1}$, we obtain

$$\overline{x} \le (E - H(x_0))^{-1}\overline{h}.$$

Thus

$$\sum_{i=1}^n x_i \le (E - H(x_0))^{-1} \sum_{i=1}^n h_i,$$

which contradicts the inequality (3.13). The obtained contradiction proves the validity of the estimate (3.12). ☐

Proof of Theorem 3.1. According to the condition (3.8), there exists a positive number x_0 such that the inequality (3.10) holds.

(3.3) and (3.5) imply

$$\sigma_i\bigg(\alpha_i \exp\bigg(\int_a^b p_i(s)\,ds\bigg) - 1\bigg) < 0 \quad (i = 1, \dots, n). \tag{3.15}$$

On the other hand, by virtue of Theorems 2.1, 2.2 and the conditions (3.4) and (3.15) for any $i \in \{1, \dots, n\}$ the problem

$$\begin{aligned} v_i'(t) &= p_i(t)v(t) + \sigma_i q_i(t, v_i(t)), \\ v_i(a) &= \alpha_i v_i(b) \end{aligned}$$

has a unique solution v_i.

Put

$$\delta_i = \min\big\{v_i(t) : \ a \le t \le b\big\} \quad (i = 1, \dots, n),$$

$$h_i = \frac{\beta_0}{|\Delta(p_i, \beta_i)|} \exp\bigg(\int_a^b |p_i(s)|\,ds\bigg) + \tag{3.16}$$

$$+ \max\bigg\{\int_a^b \big|g(p_i, \beta_i)(t, s)\big| q_0(s, \delta_1, \dots, \delta_n)\,ds : \ a \le t \le b\bigg\} \quad (i = 1, \dots, n),$$

$$\delta = \min\{\delta_1, \dots, \delta_n\}, \quad \rho = x_0 + \big\|(E - H(x_0))^{-1}\big\| \sum_{i=1}^n h_i. \tag{3.17}$$

Let $(u_i)_{i=1}^n$ be a solution of the problem (3.1), (3.2). Our aim is to prove that this solution admits the estimates (3.9).

For each $i \in \{1, \dots, n\}$ the function u_i is a solution of the problem

$$\sigma_i\big(u_i'(t) - p_i(t)u_i(t)\big) \geq q_i(t, u_i(t)),$$
$$\sigma_i\big(u_i(a) - \alpha_i u_i(b)\big) \geq 0.$$

Hence by virtue of the conditions (3.4), (3.15) and Theorems 2.1 and 2.2 it follows that

$$u_i(t) \geq v_i(t) \ \text{ for } \ a \leq t \leq b$$

and, consequently,

$$u_i(t) \geq \delta_i \ \text{ for } \ a \leq t \leq b \ \ (i = 1, \dots, n). \tag{3.18}$$

According to (3.1), (3.2), and (3.18), for each $i \in \{1, \dots, n\}$ the function u_i is a solution of the problem

$$\sigma_i\big(u_i'(t) - p_i(t)u_i(t)\big) \leq \sum_{k=1}^{n} p_{ik}(t, x_1 + \dots + x_n)x_k + q_0(t, \delta_1, \dots, \delta_n),$$
$$\sigma_i\big(u_i(a) - \beta_i(t)u_i(b)\big) \leq \beta_0,$$

where

$$x_k = \max\big\{u_k(t) : a \leq t \leq b\big\} \ \ (k = 1, \dots, n). \tag{3.19}$$

Hence by virtue of the condition (3.5) and Corollary 2.1 it follows that

$$u_i(t) \leq \sum_{k=1}^{n} \left(\int_a^b \big|g(p_i, \beta_i)(t, s)\big| p_{ik}(s, x_1 + \dots + x_n)\, ds \right) x_k +$$
$$+ \frac{\beta_0}{|\Delta(p_i, \beta_i)|} \exp\left(\int_a^t p_i(s)\, ds \right) +$$
$$+ \int_a^b \big|g(p_i, \beta_i)(t, s)\big| q_0(s, \delta_1, \dots, \delta_n)\, ds \ \text{ for } \ a \leq t \leq b.$$

If along with this estimate we take into account the notations (3.6) and (3.16), then it becomes clear that the numbers $x_1, \dots, x_n$ satisfy the system of inequalities (3.11). By Lemma 3.1 these numbers satisfy the inequality (3.12) as well.

Due to (3.17) and (3.19), the estimates (3.12) and (3.18) result in the estimates (3.9). □

Corollary 3.1. *Let the functions p_i $(i = 1, \dots, n)$ are of constant sign,*

$$p_{ik}(t, x) \equiv |p_i(t)| p_{0ik}(x) \ \ (i, k = 1, \dots, n), \tag{3.20}$$

and let along with (3.3)–(3.5) *the condition*

$$\lim_{x \to +\infty} r(H_0(x)) < 1 \tag{3.21}$$

be fulfilled, where $p_{0ik} : \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i,k = 1,\dots,n)$ *are nonincreasing functions and*

$$H_0(x) = \left(\frac{\beta_i + 1 + |\beta_i - 1|}{2}\,\Big|\frac{\Delta(p_i,1)}{\Delta(p_i,\beta_i)}\Big| p_{0ik}(x)\right)_{i,k=1}^{n}, \tag{3.22}$$

and Δ *is a functional, given by the equality* (2.24). *Then there exist positive constants* δ *and* ρ *such that an arbitrary solution* $(u_i)_{i=1}^n$ *of the problem* (3.1), (3.2) *admits the estimates* (3.9).

Proof. By Lemma 2.2, the estimates

$$\int\limits_a^b \big|g(p_i,\beta_i)(t,s)p_i(s)\big|\,ds \leq$$
$$\leq \frac{\beta_i + 1 + |\beta_i - 1|}{2}\,\Big|\frac{\Delta(p_i,1)}{\Delta(p_i,\beta_i)}\Big| \ \text{ for } \ a \leq t \leq b \ \ (i = 1,\dots,n)$$

are valid, according to which (3.6) and (3.20) result in the inequalities

$$h_{ik}(x) \leq \frac{\beta_i + 1 + |\beta_i - 1|}{2}\,\Big|\frac{\Delta(p_i,1)}{\Delta(p_i,\beta_i)}\Big| p_{0ik}(x) \ \text{ for } \ x > 0 \ \ (i,k = 1,\dots,n).$$

Hence in view of (3.7) and (3.22) it is obvious that

$$H(x) \leq H_0(x) \ \text{ for } \ x > 0$$

and, consequently,

$$r(H(x)) \leq r(H_0(x)) \ \text{ for } \ x > 0.$$

Thus the inequalities (3.21) yield the inequality (3.8).

If now we apply Theorem 3.1, then the validity of Corollary 3.1 becomes evident. □

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Author's address:

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia.

E-mail: kig@rmi.ge

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Sergey Shchogolev

ON THE BLOCK SEPARATION OF THE LINEAR HOMOGENEOUS DIFFERENTIAL SYSTEM WITH OSCILLATING COEFFICIENTS IN THE RESONANCE CASE

Abstract. For the linear homogeneous differential system with oscillating coefficients the sufficient conditions of existence of linear transformation reducing this system to a block-diagonal form in a resonance case are obtained.

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რეზიუმე. წრფივი ერთგვაროვანი დიფერენციალური სისტემისათვის რხევადი კოეფიციენტებით დადგენილია ისეთი წრფივი გარდაქმნის არსებობის საკმარისი პირობები, რომელიც მას მიიყვანს უჯრულ-დიაგონალურ სახეზე რეზონანსულ შემთხვევაში.

1. Introduction

In the theory of differential equations of great importance is the problem of separation of a linear homogeneous n-th order differential system into k independent systems of orders $n_1, n_2, \ldots, n_k$ $(n_1 + n_2 + \cdots + n_k = n)$, in particular, separation of this system into n independent first-order differential equations (full separation). This problem has been considered, for example, in [1–8]. Obviously, it is impossible in a general case to construct transformations explicitly, leading to a separated system. Such a construction assumes for the initial system to be integrable. Therefore, in these studies there was no attempt to construct such a transformation explicitly; these works established only the conditions of its existence, investigated its properties and possibility for its approximate construction, particularly, in the form of asymptotic series. Of importance is also the question on the belonging of elements of a transforming matrix to the same classes as elements of the matrix of the original system.

In his articles [9–12], the author considers the problem of full separation of the system of the kind

$$\frac{dx}{dt} = (\Lambda(t,\varepsilon) + \mu B(t,\varepsilon,\theta))x, \tag{1}$$

where $\Lambda(t,\varepsilon) = \operatorname{diag}(\lambda_1(t,\varepsilon), \ldots, \lambda_n(t,\varepsilon))$, and the functions $\lambda_j(t,\varepsilon)$ $(j = \overline{1,n})$ are, in a definite sense, slowly varying, μ is a small positive parameter, elements of the matrix $B(t,\varepsilon,\theta)$ are represented by absolutely and uniformly convergent Fourier series with slowly varying coefficients and frequency $\varphi(t,\varepsilon) = \frac{d\theta}{dt}$. At the same time, the cases of resonance absence and presence of resonance, including the special case, have been investigated. For each of these cases the conditions were obtained under which the transforming matrix elements have a structure similar to that of the matrix $B(t,\varepsilon,\theta)$. In this article we study the possibility of block separation of the system (1) into two independent systems of smaller dimensions in a resonance case. Such a statement of the problem has some features as compared with the problems considered in [9–12].

2. Basic Notation and Definitions

Let $G = \{t, \varepsilon :\ t \in \mathbf{R},\ \varepsilon \in [0, \varepsilon_0],\ \varepsilon_0 \in \mathbf{R}^+\}$.

Definition 1. We say that the function $p(t,\varepsilon)$ is in general complex-valued, belongs to the class $S(m;\varepsilon_0)$, $m \in \mathbf{N} \cup \{0\}$, if $t, \varepsilon \in G$ and

(1) $p(t,\varepsilon) \in C^m(G)$ with respect to t;

(2) $d^k p(t,\varepsilon)/dt^k = \varepsilon^k p_k^*(t,\varepsilon)$, $\sup\limits_G |p_k^*(t,\varepsilon)| < +\infty$ $(0 \le k \le m)$.

Slow variation of a function is understood here in a sense of its belonging to the class $S(m;\varepsilon_0)$. As examples of this class of functions may serve in a general case complex-valued bounded together with their derivatives up to the m-th order, inclusive, functions depending on the "slow time" $\tau = \varepsilon t$: $\sin\tau$, $\operatorname{arctg}\tau$, etc.

Definition 2. We say that the function $f(t,\varepsilon,\theta(t,\varepsilon))$ belongs to the class $F(m;\varepsilon_0;\theta)$, $m \in \mathbf{N}\cup\{0\}$, if this function can be represented as

$$f(t,\varepsilon,\theta(t,\varepsilon)) = \sum_{n=-\infty}^{\infty} f_n(t,\varepsilon)\exp\big(in\theta(t,\varepsilon)\big),$$

where

(1) $f_n(t,\varepsilon) \in S(m;\varepsilon_0)$, $d^k f_n(t,\varepsilon)/dt^k = \varepsilon^k f_{nk}(t,\varepsilon)$ ($n \in \mathbf{Z}$, $0 \le k \le m$),

(2) $\|f\|_{F(m;\varepsilon_0;\theta)} \overset{def}{=} \sum\limits_{k=0}^{m}\sum\limits_{n=-\infty}^{\infty}\sup\limits_{G}|f_{nk}(t,\varepsilon)| < +\infty$,

(3) $\theta(t,\varepsilon) = \int\limits_0^t \varphi(\tau,\varepsilon)\,d\tau$, $\varphi(t,\varepsilon) \in \mathbf{R}^+$, $\varphi(t,\varepsilon) \in S(m;\varepsilon_0)$, $\inf\limits_G \varphi(t,\varepsilon) > 0$.

In particular, if $\varepsilon = 0$: $\varphi = const$, $\theta = \varphi t$, $f_n = const$, then functions of the class $F(m;\varepsilon_0;\theta)$ are transformed into $2\pi/\varphi$-periodic functions of variable t,

$$f(t) = \sum_{n=-\infty}^{\infty} f_n e^{in\varphi t},$$

such that

$$\sum_{n=-\infty}^{\infty} |f_n| < +\infty.$$

A set of functions of the class $F(m;\varepsilon_0;\theta)$ forms a linear space which transforms into a full normed space by means of the norm $\|\cdot\|_{F(m;\varepsilon_0,\theta)}$. The following chain of inclusions

$$F(0;\varepsilon_0;\theta) \supset F(1;\varepsilon_0;\theta) \supset \cdots \supset F(m;\varepsilon_0;\theta)$$

is valid.

Let there be given two functions of the class $F(m;\varepsilon_0;\theta)$:

$$u(t,\varepsilon,\theta(t,\varepsilon)) = \sum_{n=-\infty}^{\infty} u_n(t,\varepsilon)\exp(in\theta(t,\varepsilon)),$$

$$v(t,\varepsilon,\theta(t,\varepsilon)) = \sum_{n=-\infty}^{\infty} v_n(t,\varepsilon)\exp(in\theta(t,\varepsilon)).$$

We define product of those functions by the formula [13]:

$$(uv)(t,\varepsilon,\theta(t,\varepsilon)) = \sum_{n=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} u_{n-s}(t,\varepsilon)v_s(t,\varepsilon)\exp(in\theta(t,\varepsilon)).$$

Obviously, $uv \in F(m;\varepsilon_0;\theta)$. We state some properties of the norm $\|\cdot\|_{F(m;\varepsilon_0;\theta)}$. Let $u, v \in F(m;\varepsilon_0;\theta)$, $k = const$. Then

(1) $\|ku\|_{F(m;\varepsilon_0;\theta)} = |k|\|u\|_{F(m;\varepsilon_0;\theta)}$;

(2) $\|u+v\|_{F(m;\varepsilon_0;\theta)} \le \|u\|_{F(m;\varepsilon_0;\theta)} + \|v\|_{F(m;\varepsilon_0;\theta)}$;

(3) $\|uv\|_{F(m;\varepsilon_0;\theta)} \leq 2^m \|u\|_{F(m;\varepsilon_0;\theta)} \|v\|_{F(m;\varepsilon_0;\theta)}$.

For any $f(t,\varepsilon,\theta) \in F(m;\varepsilon_0;\theta)$, we denote:

$$\Gamma_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t,\varepsilon,u) \exp(-inu)\, du.$$

Let $A(t,\varepsilon,\theta) = (a_{jk}(t,\varepsilon,\theta))$ – $(M \times K)$ be the matrix with elements of the class $F(m;\varepsilon_0;\theta)$. We denote:

$$(A)_{jk} = a_{jk} \;\; (j = \overline{1,M}, \;\; k = \overline{1,K}),$$

$$\|A\|^*_{F(m;\varepsilon_0;\theta)} = \max_{1 \leq j \leq M} \sum_{k=1}^{K} \|(A)_{jk}\|_{F(m;\varepsilon_0;\theta)}.$$

3. Statement of the Problem

Consider the following system of differential equations:

$$\begin{aligned} \frac{dx_1}{dt} &= H_1(\varphi)x_1 + \mu(B_{11}(t,\varepsilon,\theta)x_1 + B_{12}(t,\varepsilon,\theta)x_2), \\ \frac{dx_2}{dt} &= H_2(\varphi)x_2 + \mu(B_{21}(t,\varepsilon,\theta)x_1 + B_{22}(t,\varepsilon,\theta)x_2), \end{aligned} \tag{2}$$

where $x_1 = \operatorname{colon}(x_{11}, \ldots, x_{1N_1})$, $x_2 = \operatorname{colon}(x_{21}, \ldots, x_{2N_2})$,

$$H_1(\varphi) = \begin{pmatrix} ip\varphi & 0 & \ldots & 0 & 0 \\ 1 & ip\varphi & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 1 & ip\varphi \end{pmatrix},$$

$$H_2(\varphi) = \begin{pmatrix} ir\varphi & 0 & \ldots & 0 & 0 \\ 1 & ir\varphi & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 1 & ir\varphi \end{pmatrix}$$

are the Jordan blocks of dimensions N_1 and N_2, respectively ($N_1 + N_2 = N$); $p, r \in \mathbf{Z}$; $B_{jk}(t,\varepsilon,\theta)$ are the $(N_j \times N_k)$ matrices with elements of the class $F(m;\varepsilon;\theta)$; $\varphi(t,\varepsilon)$ is the function appearing in the definition of class $F(m;\varepsilon;\theta)$; $\mu \in (0,1)$. In this sense, we deal with a resonance case.

We study the problem of existence and properties of the transformation of kind:

$$x_j = L_{j1}(t,\varepsilon,\theta,\mu)\widetilde{x}_1 + L_{j2}(t,\varepsilon,\theta,\mu)\widetilde{x}_2, \;\; j = 1,2, \tag{3}$$

where the elements of $(N_j \times N_k)$-matrices L_{jk} $(j,k = 1,2)$ belong to the class $F(m-1;\varepsilon_1;\theta)$ $(0 < \varepsilon_1 \leq \varepsilon_0)$, reducing the system (2) to the form:

$$\frac{d\widetilde{x}_1}{dt} = D_{N_1}(t,\varepsilon,\theta,\mu)\widetilde{x}_1, \quad \frac{d\widetilde{x}_2}{dt} = D_{N_2}(t,\varepsilon,\theta,\mu)\widetilde{x}_2, \tag{4}$$

where the elements of $(N_j \times N_j)$-matrices D_{N_j} $(j = 1, 2)$ likewise belong to the class $F(m-1; \varepsilon_1; \theta)$.

4. Auxiliary Results

Lemma 1. *Let there be given a matrix differential equation*

$$\frac{dX}{dt} = \Big(J_M + \sum_{l=1}^{q} P_l(t,\varepsilon,\theta)\mu^l\Big)X - X\Big(J_K + \sum_{l=1}^{q} Q_l(t,\varepsilon,\theta)\mu^l\Big), \tag{5}$$

where X is $(M \times K)$-matrix, $P_l(t,\varepsilon,\theta)$, $Q_l(t,\varepsilon,\theta)$ $(l = \overline{1,q})$ are matrices of dimensions $(M \times M)$ and $(K \times K)$ respectively with elements from the class $F(m; \varepsilon; \theta)$,

$$J_M = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad J_K = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

are Jordan blocks of dimensions M and K, respectively, whose diagonal elements are equal to zero, $\mu \in (0,1)$.

Then there exists $\mu_0 \in (0,1)$ such that for any $\mu \in (0, \mu_0)$ there exists transformation of the kind

$$X = \Big(E_M + \sum_{l=1}^{q} \Phi_l(t,\varepsilon,\theta)\mu^l\Big)Y\Big(E_K + \sum_{l=1}^{q} \Psi_l(t,\varepsilon,\theta)\mu^l\Big), \tag{6}$$

where Y is the $(M\times K)$-matrix, E_M, E_K are identity matrices of dimensions M and K respectively, the elements of $(M \times M)$-matrices Φ_l and those of $(K \times K)$-matrices Ψ_l $(l = \overline{1,q})$ belong to the class $F(m; \varepsilon; \theta)$ reducing equation (5) to the form:

$$\frac{dY}{dt} = \Big(J_M + \sum_{l=1}^{q} U_l(t,\varepsilon)\mu^l + \varepsilon \sum_{l=1}^{q} \widetilde{U}_l(t,\varepsilon,\theta)\mu^l + \mu^{q+1} W_1(t,\varepsilon,\theta,\mu)\Big)Y -$$
$$- Y\Big(J_K + \sum_{l=1}^{q} V_l(t,\varepsilon)\mu^l + \varepsilon \sum_{l=1}^{q} \widetilde{V}_l(t,\varepsilon,\theta)\mu^l + \mu^{q+1} W_2(t,\varepsilon,\theta,\mu)\Big), \tag{7}$$

where $U_l(t,\varepsilon)$, $V_l(t,\varepsilon)$ $(l = \overline{1,q})$ are the matrices of dimensions $(M \times M)$ and $(K \times K)$, respectively, with elements from the class $S(m; \varepsilon_0)$, $\widetilde{U}_l(t,\varepsilon)$ and $\widetilde{V}_l(t,\varepsilon)$ $(l = \overline{1,q})$ are the matrices of dimensions $(M \times M)$ and $(K \times K)$, respectively, with elements from the class $F(m-1; \varepsilon_0; \theta)$, W_1, W_2 are the matrices of dimensions $(M \times M)$ and $(K \times K)$, respectively, with elements from the class $F(m-1; \varepsilon_0; \theta)$.

Proof. We substitute (6) into the system (5) and require for the transformed system to have the form (7). Then for the matrices Φ_l, Ψ_l $(l = \overline{1,q})$ we obtain the following differential equations:

$$\frac{d\Phi_1}{dt} = J_M\Phi_1 - \Phi_1 J_M + P_1(t,\varepsilon,\theta) - U_1(t,\varepsilon) - \varepsilon\widetilde{U}_1(t,\varepsilon,\theta), \tag{8}$$

$$\frac{d\Psi_1}{dt} = J_K\Psi_1 - \Psi_1 J_K - Q_1(t,\varepsilon,\theta) + V_1(t,\varepsilon) + \varepsilon\widetilde{V}_1(t,\varepsilon,\theta), \tag{9}$$

$$\begin{aligned}\frac{d\Phi_l}{dt} &= J_M\Phi_l - \Phi_l J_M + P_l(t,\varepsilon,\theta) + \sum_{\nu=1}^{l-1} P_\nu(t,\varepsilon,\theta)\Phi_{l-\nu} - \\ &\quad - \sum_{\nu=1}^{l-1}\Phi_\nu U_{l-\nu}(t,\varepsilon) - \varepsilon\sum_{\nu=1}^{l-1}\Phi_\nu\widetilde{U}_{l-\nu}(t,\varepsilon,\theta) - \\ &\quad - U_l(t,\varepsilon) - \varepsilon\widetilde{U}_l(t,\varepsilon,\theta), \quad l=\overline{2,q},\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{d\Psi_l}{dt} &= J_K\Psi_l - \Psi_l J_K - Q_l(t,\varepsilon,\theta) - \sum_{\nu=1}^{l-1}\Psi_\nu Q_{l-\nu}(t,\varepsilon,\theta) + \\ &\quad + \sum_{\nu=1}^{l-1} V_\nu(t,\varepsilon)\Psi_{l-\nu} + \varepsilon\sum_{\nu=1}^{l-1}\widetilde{V}_\nu(t,\varepsilon,\theta)\Psi_{l-\nu} + \\ &\quad + V_l(t,\varepsilon) + \varepsilon\widetilde{V}_l(t,\varepsilon,\theta), \quad l=\overline{2,q}.\end{aligned} \tag{11}$$

The matrices W_1, W_2 are defined from the equations

$$\begin{gathered}\Big(E_M + \sum_{l=1}^{q}\Phi_l(t,\varepsilon,\theta)\mu^l\Big)W_1 = \\ = \sum_{s=0}^{q-1}\Big[\sum_{\sigma+\delta=s+q+1}(P_\sigma\Phi_\delta - \Phi_\delta U_\sigma)\Big]\mu^s - \varepsilon\sum_{s=0}^{q-1}\Big(\sum_{\sigma+\delta=s+q+1}\Phi_\sigma\widetilde{U}_\delta\Big)\mu^s,\end{gathered} \tag{12}$$

$$\begin{gathered}W_2\Big(E_K + \sum_{l=1}^{q}\Psi_l(t,\varepsilon,\theta)\mu^l\Big) = \\ = \sum_{s=0}^{q-1}\Big[\sum_{\sigma+\delta=s+q+1}(-\Psi_\sigma Q_\delta + V_\sigma\Psi_\delta)\Big]\mu^s + \varepsilon\sum_{s=0}^{q-1}\Big(\sum_{\sigma+\delta=s+q+1}\widetilde{V}_\sigma\Psi_\delta\Big)\mu^s,\end{gathered} \tag{13}$$

Based on the equations (8)–(11), we set

$$(U_l)_{sM} = \Gamma_0((T_l)_{sM}),$$

$$(\Phi_l)_{sM} = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty}\frac{\Gamma_n((\Phi_l)_{s-1,M} + (T_l)_{sM})}{in\varphi}\, e^{in\theta},$$

$$(\widetilde{U}_l)_{sM} = -\frac{1}{\varepsilon}\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty}\frac{d}{dt}\Big(\frac{\Gamma_n((\Phi_l)_{s-1,M} + (T_l)_{sM})}{in\varphi}\Big)e^{in\theta} - \Big(\sum_{\nu=1}^{l-1}\Phi_\nu\widetilde{U}_{l-\nu}\Big)_{sM},$$

$$(U_l)_{s,M-j} = \Gamma_0((T_l)_{s,M-j}),$$

$$(\Phi_l)_{s,M-j} = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{\Gamma_n((\Phi_l)_{s-1,M-j} - (\Phi_l)_{s,M-j+1} + (T_l)_{s,M-j})}{in\varphi}\, e^{in\theta},$$

$$(\widetilde{U}_l)_{s,M-j} = -\frac{1}{\varepsilon}\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{d}{dt}\Big(\frac{\Gamma_n((\Phi_l)_{s-1,M-j} - (\Phi_l)_{s,M-j+1} + (T_l)_{s,M-j})}{in\varphi}\Big)e^{in\theta} -$$

$$-\Big(\sum_{\nu=1}^{l-1}\Phi_\nu \widetilde{U}_{l-\nu}\Big)_{s,M-j} \quad (s=\overline{1,M};\ \ j=\overline{1,M-1}),$$

where

$$T_l = P_l + \sum_{\nu=1}^{l-1} P_\nu \Phi_{l-\nu} - \sum_{\nu=1}^{l-1}\Phi_\nu U_{l-\nu} \ \ (l=\overline{1,q}).$$

(if $l=1$, then we assume $\sum\limits_{\nu=1}^{l-1}$ to be equal to zero; if $s=1$, then we assume $(\Phi)_{s-1,j}$ to be equal to zero),

$$(V_l)_{sK} = \Gamma_0((R_l)_{sK}),$$

$$(\Psi_l)_{sK} = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{\Gamma_n((\Psi_l)_{s-1,K} + (R_l)_{sK})}{in\varphi}\, e^{in\theta},$$

$$(\widetilde{V}_l)_{sK} = -\frac{1}{\varepsilon}\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{d}{dt}\Big(\frac{\Gamma_n((\Psi_l)_{s-1,K} + (R_l)_{sK})}{in\varphi}\Big)e^{in\theta} -$$

$$-\Big(\sum_{\nu=1}^{l-1}\widetilde{V}_\nu \Psi_{l-\nu}\Big)_{sK},$$

$$(V_l)_{s,K-j} = \Gamma_0((R_l)_{s,K-j}),$$

$$(\Psi_l)_{s,K-j} = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{\Gamma_n((\Psi_l)_{s-1,K-j} - (\Psi_l)_{s,K-j+1} + (R_l)_{s,K-j})}{in\varphi} e^{in\theta},$$

$$(\widetilde{V}_l)_{s,K-j} = -\frac{1}{\varepsilon}\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{d}{dt}\Big(\frac{\Gamma_n((\Psi_l)_{s-1,K-j} - (\Psi_l)_{s,K-j+1} + (R_l)_{s,K-j})}{in\varphi}\Big)e^{in\theta} -$$

$$-\Big(\sum_{\nu=1}^{l-1}\widetilde{V}_\nu \Psi_{l-\nu}\Big)_{s,K-j} \quad (s=\overline{1,K};\ \ j=\overline{1,K-1}),$$

where

$$R_l = -Q_l - \sum_{\nu=1}^{l-1}\Psi_\nu Q_{l-\nu} + \sum_{\nu=1}^{l-1} V_\nu \Psi_{l-\nu} \ \ (l=\overline{1,q}).$$

(if $l=1$, then we set $\sum\limits_{\nu=1}^{l-1}$ to be equal to zero; if $s=1$, then we set $(\Psi)_{s-1,j}$ to be equal to zero).

Then for sufficiently small values μ, the matrices W_1, W_2 are uniquely defined from equations (12), (13). $\square$

Consider now the matrix differential equation

$$\frac{dX}{dt} = J_M X - X J_K + F(t,\varepsilon,\theta) + \mu(A(t,\varepsilon,\theta)X - \\ - XB(t,\varepsilon,\theta)) - \mu^2 XR(t,\varepsilon,\theta)X, \tag{14}$$

where X is the $(M \times K)$-matrix, F, A, B, R are matrices of dimensions $(M \times K)$, $(M \times M)$, $(K \times K)$, $(K \times M)$, respectively, whose all elements belong to the class $F(m;\varepsilon;\theta)$.

Lemma 2. *Let equation* (14) *satisfy one of the sets of conditions* I*,* II*,* III*:*

I. (1) $M < K$,

(2) $\sum\limits_{s=1}^{j} \Gamma_0((F)_{s,K-j+s}) \equiv 0$, $j = \overline{1,M}$,

(3) $\inf\limits_{G} |\Gamma_0((B)_{1K})| > 0$;

II. (1) $M = K$,

(2) $\sum\limits_{s=1}^{j} \Gamma_0((F)_{s,K-j+s}) \equiv 0$, $j = \overline{1,M}$,

(3) $\inf\limits_{G} |\Gamma_0((A)_{1M} - (B)_{1M})| > 0$;

III. (1) $M > K$,

(2) $\sum\limits_{s=1}^{j} \Gamma_0((F)_{s,K-j+s}) \equiv 0$, $j = \overline{1,K}$,

(3) $\inf\limits_{G} |\Gamma_0((A)_{1M})| > 0$.

Then there exists $\mu_1 \in]0,1[$ *such that for any* $\mu \in]0,\mu_1[$ *there exists the transformation of the kind*

$$X = \sum_{s=0}^{2q-1} \Xi_s(t,\varepsilon,\theta)\mu^s + \Phi(t,\varepsilon,\theta,\mu) Y \Psi(t,\varepsilon,\theta,\mu), \tag{15}$$

where the elements of $(M \times K)$*-matrices* Ξ_s $(s = \overline{0,2q-1})$*, of* $(M \times M)$*-matrix* Φ *and of* $(K \times K)$*-matrix* Ψ *belong to the class* $F(m;\varepsilon_0,\theta)$ $\forall\,\mu \in (0,\mu_1)$*, reducing the equation* (14) *to the form*

$$\frac{dY}{dt} = J_M Y - Y J_K + \Big(\sum_{l=1}^{q} U_l(t,\varepsilon)\mu^l\Big) Y - Y\Big(\sum_{l=1}^{q} V_l(t,\varepsilon)\mu^l\Big) + \\ + \varepsilon\big(\widetilde{U}(t,\varepsilon,\theta,\mu)Y - Y\widetilde{V}(t,\varepsilon,\theta,\mu)\big) + \\ + \mu^{q+1}\big(\widetilde{W}_1(t,\varepsilon,\theta,\mu)Y - Y\widetilde{W}_2(t,\varepsilon,\theta,\mu)\big) + \\ + \varepsilon G(t,\varepsilon,\theta,\mu) + \mu^{2q} H(t,\varepsilon,\theta,\mu) + \mu Y R_1(t,\varepsilon,\theta,\mu) Y, \tag{16}$$

where the elements of matrices U_l, V_l $(l = \overline{1, q})$ belong to the class $S(m; \varepsilon_0)$, and the elements of matrices $\widetilde{U}$, $\widetilde{V}$, $\widetilde{W}_1$, $\widetilde{W}_2$, G, H, R_1 of the corresponding dimensions belong to the class $F(m-1; \varepsilon_0; \theta)$.

Proof. Along with the equation (14), we consider an auxiliary matrix equation

$$\varphi(t,\varepsilon)\frac{d\Xi}{d\theta} = J_M\Xi - \Xi J_K + F(t,\varepsilon,\theta) + \\ + \mu(A(t,\varepsilon,\theta)\Xi - \Xi B(t,\varepsilon,\theta)) - \mu^2\Xi R(t,\varepsilon,\theta)\Xi, \quad (17)$$

where t, φ are considered as constants. The matrices-functions $F(t,\varepsilon,\theta)$, $A(t,\varepsilon,\theta)$, $B(t,\varepsilon,\theta)$, $R(t,\varepsilon,\theta)$ are 2π-periodic with respect to θ. We construct, according to the Poincare method of small parameter [14], an approximate 2π-periodic with respect to θ solution of the equation (17) in the form of a sum:

$$\Xi = \sum_{s=o}^{2q-1} \Xi_s(t,\varepsilon,\theta)\mu^s. \quad (18)$$

The coefficients Ξ_s are determined from the following chain of linear nonhomogeneous matrix differential equations:

$$\varphi(t,\varepsilon)\frac{d\Xi_0}{d\theta} = J_M\Xi_0 - \Xi_0 J_K + F(t,\varepsilon,\theta), \quad (19)$$

$$\varphi(t,\varepsilon)\frac{d\Xi_1}{d\theta} = J_M\Xi_1 - \Xi_1 J_K + A(t,\varepsilon,\theta)\Xi_0 - \Xi_0 B(t,\varepsilon,\theta), \quad (20)$$

$$\varphi(t,\varepsilon)\frac{d\Xi_2}{d\theta} = J_M\Xi_2 - \Xi_2 J_K + A(t,\varepsilon,\theta)\Xi_1 - \Xi_1 B(t,\varepsilon,\theta) - \\ - \Xi_0 R(t,\varepsilon,\theta)\Xi_0, \quad (21)$$

$$\varphi(t,\varepsilon)\frac{d\Xi_s}{d\theta} = J_M\Xi_s - \Xi_s J_K + A(t,\varepsilon,\theta)\Xi_{s-1} - \Xi_{s-1} B(t,\varepsilon,\theta) - \\ - \sum_{l=0}^{s-2} \Xi_l R(t,\varepsilon,\theta)\Xi_{s-2-l}, \quad s = \overline{3, 2q-1}. \quad (22)$$

First consider the case $M < K$. The condition I.(2) ensures the existence of a 2π-periodic with respect to θ solution $\Xi_0(t,\varepsilon,\theta)$ of the equation (19) having the form

$$\Xi_0(t,\varepsilon,\theta) = C_0(t,\varepsilon) + \widetilde{\Xi}_0(t,\varepsilon,\theta), \quad (23)$$

where $\widetilde{\Xi}_0(t,\varepsilon,\theta)$ is the known matrix whose elements belong to the class $F(m; \varepsilon_0; \theta)$, and $(M \times K)$-matrix $C_0(t,\varepsilon)$ has the form

$$C_0(t,\varepsilon) = \begin{pmatrix} c_{01}(t,\varepsilon) & 0 & \ldots & 0 & 0 & \ldots & 0 \\ c_{02}(t,\varepsilon) & c_{01}(t,\varepsilon) & \ldots & 0 & 0 & \ldots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ c_{0M}(t,\varepsilon) & c_{0,M-1}(t,\varepsilon) & \ldots & c_{01}(t,\varepsilon) & 0 & \ldots & 0 \end{pmatrix},$$

where the scalar functions $c_{01}, \dots, c_{0M}$ are determined from the following system of equations:

$$\sum_{s=1}^{j} \Gamma_0\big((A\Xi_0 - \Xi_0 B)_{s,K-j+s}\big) = 0, \quad j = \overline{1, M}. \tag{24}$$

We represent the matrices A and B in the form

$$A(t,\varepsilon,\theta) = A_0(t,\varepsilon) + \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} A_n(t,\varepsilon)e^{in\theta},$$

$$B(t,\varepsilon,\theta) = B_0(t,\varepsilon) + \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} B_n(t,\varepsilon)e^{in\theta}.$$

Then it is easy to verify that the system (24) is a system of M linear algebraic equations with respect to the functions $c_{01}, \dots, c_{0M}$:

$$\sum_{s=1}^{j} \big(A_0(t,\varepsilon)C_0 - C_0 B_0(t,\varepsilon)\big)_{s,K-j+s} = h_{0j}^*(t,\varepsilon), \quad j = \overline{1, M}, \tag{25}$$

where $h_{01}^*, \dots, h_{0M}^*$ are the known functions of the class $S(m;\varepsilon_0)$. Determinant of this system has a triangular form, and absolute values of all its diagonal elements are equal to $|(B_0(t,\varepsilon))_{1K}|$. Therefore, the condition I.(3) ensures the existence of a unique solution $c_{01}^*(t,\varepsilon), \dots, c_{0M}^*(t,\varepsilon)$ of the system (25), and this solution belongs to the class $S(m;\varepsilon_0)$.

Using the above found 2π-periodic with respect to θ solution (23) of the equation (19), we construct a 2π-periodic with respect to θ solution of the equation (20) of the form

$$\Xi_1(t,\varepsilon,\theta) = C_1(t,\varepsilon) + \widetilde{\Xi}_1(t,\varepsilon,\theta), \tag{26}$$

where $\widetilde{\Xi}_1(t,\varepsilon,\theta)$ is the known matrix, whose elements belong to the class $F(m;\varepsilon_0;\theta)$, and the $(M \times K)$-matrix $C_1(t,\varepsilon)$ has the form

$$C_1(t,\varepsilon) = \begin{pmatrix} c_{11}(t,\varepsilon) & 0 & \dots & 0 & 0 & \dots & 0 \\ c_{12}(t,\varepsilon) & c_{11}(t,\varepsilon) & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{1M}(t,\varepsilon) & c_{1,M-1}(t,\varepsilon) & \dots & c_{11}(t,\varepsilon) & 0 & \dots & 0 \end{pmatrix}.$$

The scalar functions $c_{11}, \dots, c_{1M}$ are determined from the system of linear algebraic equations

$$\sum_{s=1}^{j} \big(A_0(t,\varepsilon)C_1 - C_1 B_0(t,\varepsilon)\big)_{s,K-j+s} = h_{1j}^*(t,\varepsilon), \quad j = \overline{1, M}, \tag{27}$$

where $h_{11}^*, \dots, h_{1M}^*$ are the known functions of the class $S(m;\varepsilon_0)$. Therefore the condition I.(3) ensures the existence of a unique solution of the system (27), as well. Proceeding just as above, we find a 2π-periodic with respect to θ solutions of the equations (21), (22). The elements of all these solutions belong to the class $F(m;\varepsilon_0,\theta)$.

Consider now the case $M=K$. The condition II.(2) ensures the existence of a 2π-periodic with respect to θ solution of the equation (19) having the form (23), where the $(M\times M)$-matrix $C_0(t,\varepsilon)$ takes the form

$$C_0(t,\varepsilon)=\begin{pmatrix} c_{01}(t,\varepsilon) & 0 & \dots & 0 \\ c_{02}(t,\varepsilon) & c_{01}(t,\varepsilon) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ c_{0M}(t,\varepsilon) & c_{0,M-1}(t,\varepsilon) & \dots & c_{01}(t,\varepsilon) \end{pmatrix}.$$

The scalar functions $c_{01}(t,\varepsilon),\dots,c_{0M}(t,\varepsilon)$ are determined from the following system of linear algebraic equations:

$$\sum_{s=1}^{j}\big(A_0(t,\varepsilon)C_0-C_0B_0(t,\varepsilon)\big)_{s,K-j+s}=g^*_{0j}(t,\varepsilon),\ \ j=\overline{1,M}, \tag{28}$$

where $g^*_{01},\dots,g^*_{0M}$ are the known functions of the class $S(m;\varepsilon_0)$. Determinant of this system has a triangular form, and absolute values of all its diagonal elements are equal to $|(A_0(t,\varepsilon)_{1M}-(B_0(t,\varepsilon))_{1M}|$. Therefore the condition II.(3) ensures the existence of a unique solution $c^*_{01}(t,\varepsilon),\dots,c^*_{0M}(t,\varepsilon)$ of the system (28), and this solution belongs to the class $S(m;\varepsilon_0)$.

Thus we have fully determined the 2π-periodic with respect to θ solution of the equation (19). Next, in a full analogy with the case $M<K$, we determined 2π-periodic with respect to θ solutions of the equations (20), (21), (22).

In case $M>K$, the matrix $C_0(t,\varepsilon)$ in (23) is of the form

$$C_0(t,\varepsilon)=\begin{pmatrix} 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ c_{01}(t,\varepsilon) & 0 & \dots & 0 \\ c_{02}(t,\varepsilon) & c_{01}(t,\varepsilon) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ c_{0K}(t,\varepsilon) & c_{0,K-1}(t,\varepsilon) & \dots & c_{01}(t,\varepsilon) \end{pmatrix}.$$

The scalar functions $c_{01}(t,\varepsilon),\dots,c_{0K}(t,\varepsilon)$ are determined from the following system of linear algebraic equations:

$$\sum_{s=1}^{j}\big(A_0(t,\varepsilon)C_0-C_0B_0(t,\varepsilon)\big)_{s,K-j+s}=f^*_{0j}(t,\varepsilon),\ \ j=\overline{1,K}, \tag{29}$$

where $f^*_{01},\dots,f^*_{0M}$ are the known functions of the class $S(m;\varepsilon_0)$. Determinant of this system has a triangular form, and absolute values of all its diagonal elements are equal to $|(A_0(t,\varepsilon)_{1M}|$. Therefore the condition III.(3) ensures the existence of a unique solution $c^*_{01}(t,\varepsilon),\dots,c^*_{0K}(t,\varepsilon)$ of the system (29), and this solution belongs to the class $S(m;\varepsilon_0)$. Next, analogously to the case $M<K$, we determine a 2π-periodic with respect to θ solutions

of the equations (20), (21), (22). The elements of all these solutions belong to the class $F(m;\varepsilon_0;\theta)$.

Substituting in (14)

$$X = \sum_{s=0}^{2q-1} \Xi_s(t,\varepsilon,\theta)\mu^s + \widetilde{X}, \qquad (30)$$

where $\widetilde{X}$ is a new unknown matrix, we obtain

$$\frac{d\widetilde{X}}{dt} = J_M\widetilde{X} - \widetilde{X}J_K + \varepsilon G_1(t,\varepsilon,\theta,\mu) + \mu^{2q}H_1(t,\varepsilon,\theta,\mu)+ \\ + \Big(\sum_{l=1}^{q} P_l(t,\varepsilon,\theta)\mu^l\Big)\widetilde{X} - \widetilde{X}\Big(\sum_{l=1}^{q} Q_l(t,\varepsilon,\theta)\mu^l\Big)+ \\ + \mu^{q+1}\big(W_1^*(t,\varepsilon,\theta,\mu)\widetilde{X} - \widetilde{X}W_2^*(t,\varepsilon,\theta,\mu)\big) + \mu^2\widetilde{X}R(t,\varepsilon,\theta)\widetilde{X}. \qquad (31)$$

By Lemma 1, using the substitution of the kind

$$\widetilde{X} = \Big(E_M + \sum_{l=1}^{q} \Phi_l(t,\varepsilon,\theta)\mu^l\Big)Y\Big(E_K + \sum_{l=1}^{q} \Psi_l(t,\varepsilon,\theta)\mu^l\Big),$$

we reduce the equation (31) to the form (16). □

We introduce the matrices

$$U(t,\varepsilon,\mu) = \sum_{l=1}^{q} U_l(t,\varepsilon)\mu^l, \quad V(t,\varepsilon,\mu) = \sum_{l=1}^{q} V_l(t,\varepsilon)\mu^l,$$

where U_l and V_l ($l=\overline{1,q}$) are defined in Lemma 2.

Lemma 3. *Let the equation* (16) *satisfy the following conditions:*

(1) *eigenvalues* $\lambda_{1j}(t,\varepsilon,\mu)$ ($j=\overline{1,M}$) *of the matrix* $J_M+U(t,\varepsilon,\mu)$ *and* $\lambda_{2s}(t,\varepsilon,\mu)$ ($s=\overline{1,K}$) *of the matrix* $J_K+V(t,\varepsilon,\mu)$ *are such that*

$$\inf_G \Big|\mathrm{Re}\,\big(\lambda_{1j}(t,\varepsilon,\mu) - \lambda_{2s}(t,\varepsilon,\mu)\big)\Big| \ge \gamma_0\mu^{q_0}$$
$$(\gamma_0>0,\ \ 0<q_0\le q;\ \ j=\overline{1,M},\ \ s=\overline{1,K});$$

(2) *there exist the* $(M\times M)$*-matrix* $L_1(t,\varepsilon,\mu)$ *and the* $(K\times K)$*-matrix* $L_2(t,\varepsilon,\mu)$ *such that*

   (a) *all elements of these matrices belong to the class* $S(m;\varepsilon_0)\subset F(m;\varepsilon_0;\theta)$*;*

   (b) $\|L_j^{-1}(t,\varepsilon,\mu)\|^*_{F(m\varepsilon_0,\theta)} \le M_1\mu^{-\alpha}$*,* $M_1\in(0,+\infty)$*,* $\alpha\in[0,q]$*,* $j=1,2$*;*

   (c) $L_1^{-1}(J_M+U)L_1 = \Lambda_1(t,\varepsilon,\mu)$*,* $L_2(J_K+V)L_2^{-1} = \Lambda_2(t,\varepsilon,\mu)$*, where* $\Lambda_1=\mathrm{diag}\,(\lambda_{11},\ldots,\lambda_{1M})$*,* $\Lambda_2=\mathrm{diag}\,(\lambda_{21},\ldots,\lambda_{2K})$*;*

(3) $q>q_0+\alpha-1/2$.

Then there exist $\mu_2 \in (0,1)$ *and* $K^* \in (0,+\infty)$ *such that for any* $\mu \in (0,\mu_2)$ *the matrix differential equation* (16) *has a particular solution* $Y(t,\varepsilon,\theta,\mu)$ *such that all its elements belong to the class* $F(m-1;\varepsilon_1(\mu);\theta)$, *where* $\varepsilon_1(\mu) = \min(\varepsilon_0, K^*\mu^{2q_0+2\alpha-1})$.

Proof. In the equation (16), we perform the substitution

$$Y = \frac{\varepsilon+\mu^{2q}}{\mu^{q_0+2\alpha}}\, L_1(t,\varepsilon,\mu) Z L_2(t,\varepsilon,\mu), \tag{32}$$

where Z is a new unknown $(M \times K)$-matrix. We obtain

$$\begin{aligned}\frac{dZ}{dt} &= \Lambda_1(t,\varepsilon,\mu)Z - Z\Lambda_2(t,\varepsilon,\mu) + \varepsilon\big(\widetilde{U}_1(t,\varepsilon,\theta,\mu)Z - Z\widetilde{V}_1(t,\varepsilon,\theta,\mu)\big)+\\ &+ \mu^{q+1}\big(\widetilde{W}_3(t,\varepsilon,\theta,\mu)Z - Z\widetilde{W}_4(t,\varepsilon,\theta,\mu)\big)+\\ &+ \frac{\varepsilon\mu^{q_0+2\alpha}}{\varepsilon+\mu^{2q}}\, G_2(t,\varepsilon,\theta,\mu) + \frac{\mu^{2q+2\alpha+q_0}}{\varepsilon+\mu^{2q}}\, H_2(t,\varepsilon,\theta,\mu)+\\ &+ \frac{\varepsilon+\mu^{2q}}{\mu^{q_0+2\alpha-1}}\, ZR_2(t,\varepsilon,\theta,\mu)Z,\end{aligned} \tag{33}$$

where

$$G_2 = L_1^{-1}G_1L_2^{-1},\ \ H_2 = L_1^{-1}H_1L_2^{-1},$$
$$\widetilde{U}_1 = L_1^{-1}\widetilde{U}L_1 - \varepsilon^{-1}L_1^{-1}(dL_1/dt),\ \ \widetilde{V}_1 = L_2\widetilde{U}L_2^{-1} + \varepsilon^{-1}(dL_2/dt)L_2^{-1},$$
$$\widetilde{W}_3 = L_1^{-1}\widetilde{W}_1L_1,\ \ \widetilde{W}_4 = L_2\widetilde{W}_2L_2^{-1},\ \ R_2 = L_2R_1L_1.$$

All elements of these matrices belong to the class $F(m-1;\varepsilon_0;\theta)$.

Owing to the formulas for matrices G_2, H_2, $\widetilde{U}_1$, $\widetilde{V}_1$, $\widetilde{W}_3$, $\widetilde{W}_4$ and the condition 2(b) of the lemma, there exists $K_2 \in (0,+\infty)$ such that

$$\|G_2\|_{F(m-1;\varepsilon;\theta)} \le \frac{K_2}{\mu^{2\alpha}}\,, \quad \|H_2\|_{F(m-1;\varepsilon;\theta)} \le \frac{K_2}{\mu^{2\alpha}}\,,$$
$$\|\widetilde{U}_1\|_{F(m-1;\varepsilon;\theta)} \le \frac{K_2}{\mu^{\alpha}}\,, \quad \|\widetilde{V}_1\|_{F(m-1;\varepsilon;\theta)} \le \frac{K_2}{\mu^{\alpha}}\,,$$
$$\|\widetilde{W}_3\|_{F(m-1;\varepsilon;\theta)} \le \frac{K_2}{\mu^{\alpha}}\,, \quad \|\widetilde{W}_4\|_{F(m-1;\varepsilon;\theta)} \le \frac{K_2}{\mu^{\alpha}}\,, \quad \|R_2\|_{F(m-1;\varepsilon;\theta)} \le K_2.$$

Along with the equation (33), we consider the matrix linear differential equation

$$\begin{aligned}\frac{dZ_0}{dt} &= \Lambda_1(t,\varepsilon,\mu)Z_0 - Z_0\Lambda_2(t,\varepsilon,\mu)+\\ &+ \frac{\varepsilon\mu^{q_0+2\alpha}}{\varepsilon+\mu^{2q}}\, G_2(t,\varepsilon,\theta,\mu) + \frac{\mu^{2q+2\alpha+q_0}}{\varepsilon+\mu^{2q}}\, H_2(t,\varepsilon,\theta,\mu).\end{aligned} \tag{34}$$

It is easy to see that this equation is a system of MK independent scalar first-order differential equations

$$\frac{d(Z_0)_{js}}{dt} = \big(\lambda_{1j}(t,\varepsilon,\mu) - \lambda_{2s}(t,\varepsilon,\mu)\big)d(Z_0)_{js} + \\ + \frac{\varepsilon\mu^{q_0+2\alpha}}{\varepsilon+\mu^{2q}}\big(G_2(t,\varepsilon,\theta,\mu)\big)_{js} + \frac{\mu^{2q+2\alpha+q_0}}{\varepsilon+\mu^{2q}}\big(H_2(t,\varepsilon,\theta,\mu)\big)_{js}, \quad (35) \\ j=\overline{1,M}, \quad s=\overline{1,K}.$$

In [13], it has been shown that the conditions of the lemma provide us with the existence of a unique particular solution $(Z_0(t,\varepsilon,\theta,\mu))_{js}$ $(j=\overline{1,M}$, $s=\overline{1,K})$ of the system (35), which belongs to the class $F(m-1;\varepsilon_0;\theta)$, and in addition, there exists $K_0 \in (0,+\infty)$ such that

$$\|(Z_0)_{js}\|_{F(m-1;\varepsilon_0;\theta)} \le \\ \le \frac{K_0}{\mu^{q_0}}\Big(\frac{\varepsilon\mu^{q_0+2\alpha}}{\varepsilon+\mu^{2q}}\|(G_2)_{js}\|_{F(m-1;\varepsilon_0;\theta)} + \frac{\mu^{2q+2\alpha+q_0}}{\varepsilon+\mu^{2q}}\|(H_2)_{js}\|_{F(m-1;\varepsilon_0;\theta)}\Big).$$

Hence the equation (34) has a particular solution $Z_0(t,\varepsilon,\theta,\mu)$ all elements of which belong to the class $F(m-1;\varepsilon_0;\theta)$ and, in addition, there exists $K_1 \in (0,+\infty)$ such that

$$\|Z_0\|^*_{F(m-1;\varepsilon_0;\theta)} \le \\ \le \frac{K_1}{\mu^{q_0}}\Big(\frac{\varepsilon\mu^{q_0+2\alpha}}{\varepsilon+\mu^{2q}}\|G_2\|^*_{F(m-1;\varepsilon_0;\theta)} + \frac{\mu^{2q+2\alpha+q_0}}{\varepsilon+\mu^{2q}}\|H_2\|^*_{F(m-1;\varepsilon_0;\theta)}\Big). \quad (36)$$

We seek for a solution of the equation (33) all elements of which belong to the class $F(m-1;\varepsilon_1;\theta)$, by using the iterative method, identifying $Z_0(t,\varepsilon,\theta,\mu)$ as an initial approximation, and subsequent iterations are defined as a solutions all elements of which belong to the class $F(m-1;\varepsilon_1;\theta)$ of linear inhomogeneous matrix differential equations

$$\frac{dZ_{\nu+1}}{dt} = \Lambda_1(t,\varepsilon,\mu)Z_{\nu+1} - Z_{\nu+1}\Lambda_2(t,\varepsilon,\mu) + \\ + \frac{\varepsilon\mu^{q_0+2\alpha}}{\varepsilon+\mu^{2q}}G_2(t,\varepsilon,\theta,\mu) + \frac{\mu^{2q+2\alpha+q_0}}{\varepsilon+\mu^{2q}}H_2(t,\varepsilon,\theta,\mu) + \\ + \varepsilon\big(\widetilde{U}_1(t,\varepsilon,\theta,\mu)Z_\nu - Z_\nu\widetilde{V}_1(t,\varepsilon,\theta,\mu)\big) + \\ + \mu^{q+1}\big(\widetilde{W}_3(t,\varepsilon,\theta,\mu)Z_\nu - Z_\nu\widetilde{W}_4(t,\varepsilon,\theta,\mu)\big) + \\ + \frac{\varepsilon+\mu^{2q}}{\mu^{q_0+2\alpha-1}}Z_\nu R_2(t,\varepsilon,\theta,\mu)Z_\nu, \quad \nu=0,1,2,\ldots. \quad (37)$$

Denote

$$\Omega = \Big\{Z \in F(m-1;\varepsilon_0;\theta): \ \|Z-Z_0\|^*_{F(m-1;\varepsilon_0;\theta)} \le d\Big\}.$$

Using a technique known as contraction mapping principle [15], it is not difficult to show that if

$$K_1K_2\Big(\frac{\varepsilon+\mu^{q+1}}{\mu^{q_0+\alpha}}\,2^m\big(\|Z_0\|^*_{F(m-1;\varepsilon_0;\theta)}+d\big)+ \\ +\frac{\varepsilon+\mu^{2q}}{\mu^{2q_0+2\alpha-1}}\,2^{2m-2}\big(\|Z_0\|^*_{F(m-1;\varepsilon_0;\theta)}+d\big)^2\Big)\le d_0<d, \quad (38)$$

all iterations (37) belong to Ω. And if

$$K_1K_2\Big(\frac{\varepsilon+\mu^{q+1}}{\mu^{q_0+\alpha}}\,2^m+\frac{\varepsilon+\mu^{2q}}{\mu^{2q_0+2\alpha-1}}\,2^{2m-1}\big(\|Z_0\|^*_{F(m-1;\varepsilon_0;\theta)}+d\big)\Big)<1, \quad (39)$$

then the process (37) converges to a solution of the equation (33) all elements of which belong to class the $F(m-1;\varepsilon_1;\theta)$. The inequalities (38), (39) hold due to the conditions (3) of lemma for sufficiently small μ and $\varepsilon/\mu^{2q_0+2\alpha-1}$. Therefore $\varepsilon_1(\mu)=K^*\mu^{2q_0+2\alpha-1}$, where K^* is a sufficiently small constant. □

The following lemma is an immediate consequence of the above one.

Lemma 4. *Let the equation* (14) *satisfy all conditions of Lemma* 2*, and the equation* (16) *obtained from* (14) *by means of the transformation* (15) *satisfy the conditions of Lemma* 3*. Then there exists* $\mu_3\in(0,1)$*,* $K_3\in(0,+\infty)$ *such that for any* $\mu\in(0,\mu_3)$ *the equation* (14) *has a particular solution which belongs to the class* $F(m-1;\varepsilon_2(\mu);\theta)$*, where* $\varepsilon_2(\mu)=K_2\mu^{2q_0+2\alpha-1}$*, and* q_0*,* α *are defined in Lemma* 2*.*

5. The Basic Results

Getting back to the system (2), we make transformation

$$x_1=e^{ip\theta}y_1,\quad x_2=e^{ir\theta}y_2. \quad (39)$$

We obtain

$$\begin{aligned}\frac{dy_1}{dt}&=J_{N_1}y_1+\mu\big(\widetilde{B}_{11}(t,\varepsilon,\theta)y_1+\widetilde{B}_{12}(t,\varepsilon,\theta)y_2\big),\\ \frac{dy_2}{dt}&=J_{N_2}y_2+\mu\big(\widetilde{B}_{21}(t,\varepsilon,\theta)y_1+\widetilde{B}_{22}(t,\varepsilon,\theta)y_2\big),\end{aligned} \quad (40)$$

where

$$J_{N_1}=\begin{pmatrix}0&0&\dots&0&0\\1&0&\dots&0&0\\ \dots&\dots&\dots&\dots&\dots\\0&0&\dots&0&0\\0&0&\dots&1&0\end{pmatrix},\quad J_{N_2}=\begin{pmatrix}0&0&\dots&0&0\\1&0&\dots&0&0\\ \dots&\dots&\dots&\dots&\dots\\0&0&\dots&0&0\\0&0&\dots&1&0\end{pmatrix}$$

are the Jordan blocks of dimensions N_1 and N_2, respectively, whose diagonal elements are equal to zero, and all elements of matrices $\widetilde{B}_{jk}(t,\varepsilon,\theta)$ belong to the class $F(m;\varepsilon_0;\theta)$.

In the system (40) we make the transformation

$$y_1 = z_1 + \mu Q_{12}(t,\varepsilon,\theta,\mu)z_2, \quad y_2 = \mu Q_{21}(t,\varepsilon,\theta,\mu)z_1 + z_2. \tag{41}$$

Having required for the conditions of block diagonality for the above transformed system, we obtain for $(N_j \times N_k)$-matrices Q_{jk} the following system of the form

$$\begin{aligned} \frac{dQ_{jk}}{dt} &= J_{N_j}Q_{jk} - Q_{jk}J_{N_k} + \widetilde{B}_{jk}(t,\varepsilon,\theta) + \\ &+ \mu\big(\widetilde{B}_{jj}(t,\varepsilon,\theta)Q_{jk} - Q_{jk}\widetilde{B}_{kk}(t,\varepsilon,\theta)\big) - \mu^2 Q_{jk}\widetilde{B}_{kj}Q_{jk}, \\ &\qquad j,k = 1,2 \ \ (j \neq k). \end{aligned} \tag{42}$$

Then for the N_1-vector z_1 and N_2-vector z_2 we obtain the system

$$\frac{dz_1}{dt} = D_{N_1}(t,\varepsilon,\theta,\mu)z_1, \quad \frac{dz_2}{dt} = D_{N_2}(t,\varepsilon,\theta,\mu)z_2, \tag{43}$$

where

$$\begin{aligned} D_{N_1} &= J_{N_1} + \mu\widetilde{B}_{11}(t,\varepsilon,\theta) + \mu^2\widetilde{B}_{12}(t,\varepsilon,\theta)Q_{21}(t,\varepsilon,\theta,\mu), \\ D_{N_2} &= J_{N_1} + \mu\widetilde{B}_{22}(t,\varepsilon,\theta) + \mu^2\widetilde{B}_{21}(t,\varepsilon,\theta)Q_{12}(t,\varepsilon,\theta,\mu) \end{aligned} \tag{44}$$

are matrices of dimensions $(N_1 \times N_1)$ and $(N_2 \times N_2)$, respectively.

It is easy to see that the system (42) is divided into two independent equations, each of which has the form (14). Therefore, by Lemma 4, the following theorem is true.

Theorem. *Let each of the equations* (42) *satisfy all conditions of Lemma* 4. *Then there exists* $\mu_4 \in (0,1)$, $K_4 \in (0,+\infty)$ *such that for any* $\mu \in (0,\mu_4)$ *there exists the transformation of kind* (3) *with coefficients from the class* $F(m-1;\varepsilon_4(\mu);\theta)$, *where* $\varepsilon_4(\mu) = K_4\mu^{2q_0+2\alpha-1}$ (q_0 *and* α *are defined in Lemma* 2), *reducing the system* (2) *to a block-diagonal form* (4). *The matrices* D_{N_1}, D_{N_2} *are defined in terms of the expressions* (44).

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Author's address:

Odessa I. I. Mechnikov National University, 2 Dvoryanskaya St., Odessa 65082, Ukraine.

E-mail: sergas1959@gmail.com

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Short Communications

Malkhaz Ashordia

ON THE NONLOCAL NONLINEAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF GENERALIZED DIFFERENTIAL EQUATIONS WITH SINGULARITIES

Abstract. The general nonlocal boundary value problem is considered for systems of nonlinear generalized differential equations with singularities on a non-closed interval. Singularity is understood in a sense that the vector-function corresponding to the system may have unbounded variation with respect to the time variable on the whole interval. The sufficient conditions for the solvability of this problem are given.

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1. Statement of the Problem and Basic Notations

In the paper we investigate the question on the solvability of the system of generalized nonlinear differential equations

$$dx = dA(t) \cdot f(t,x) \tag{1.1}$$

under the general nonlinear boundary value problem

$$h(Hx) = 0, \tag{1.2}$$

where A and $H :]a,b[\to \mathbb{R}^{n\times n}$ are the matrix-functions with components of bounded variation on every closed interval from $]a,b[$, in addition, $\det H(t) \neq 0$ for $t \in]a,b[$; $f \in \mathrm{Car}_{loc}(]a,b[\times \mathbb{R}^n, \mathbb{R}^n; A)$, and $h : \mathrm{BV}_s([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ is a continuous operator.

The same question for the linear general and two-point boundary value problems for systems of generalized linear differential equations are investigated in [5]–[7].

The question on the existence of a solution of the problem (1.1), (1.2) when the matrix A and vector-function f are regular, i.e. $A\in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ and $f \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n;A)$, is investigated in [1]–[3], where the Conti–Opial type theorems for the solvability of the problem (1.1), (1.2) are obtained.

Analogous and related questions are investigated in [11] (see also the references therein) for the singular boundary value problems for ordinary differential systems, and in [8], [12]–[14], [16] (see also the references therein) for the regular boundary value problems for ordinary differential systems and for functional differential equations.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see e.g. [4], [9], [10], [15], [17], [18] and the references therein).

Throughout the paper the following notation and definitions will be used.

$\mathbb{R}=]-\infty,+\infty[$, $R_+=[0,+\infty[$; $[a,b]$ and $]a,b[$ $(a,b\in\mathbb{R})$ are the closed and open intervals, respectively.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X=(x_{il})_{i,l=1}^{n,m}$ with the norm

$$\|X\|=\sum_{i,l=1}^{n,m}|x_{il}|;$$

$$\mathbb{R}_+^{n\times m}=\Big\{(x_{il})_{i,l=1}^{n,m}:\ x_{il}\geq 0\ \ (i=1,\dots,n;\ l=1,\dots,m)\Big\}.$$

$O_{n\times m}$ (or O) is the zero $n\times m$-matrix.

If $X=(x_{il})_{i,l=1}^{n,m}\in\mathbb{R}^{n\times m}$, then $|X|=\big(|x_{il}|\big)_{i,l=1}^{n,m}$.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$; $\mathbb{R}_+^n=\mathbb{R}_+^{n\times 1}$.

If $X\in\mathbb{R}^{n\times n}$, then $\det X$ and X^{-1} are, respectively, the determinant of X and the matrix inverse to X; I_n is the identity $n\times n$-matrix.

$\bigvee\limits_c^d(X)$, where $a<c<d<b$, is the variation of the matrix-function $X:]a,b[\to\mathbb{R}^{n\times m}$ on the closed interval $[c,d]$, i.e., the sum of total variations of the latter components x_{il} $(i=1,\dots,n;\ l=1,\dots,m)$ on this interval; if $d<c$, then $\bigvee\limits_c^d(X)=-\bigvee\limits_d^c(X)$; $V(X)(t)=(v(x_{il})(t))_{i,l=1}^{n,m}$, where $v(x_{il})(t_0)=0$, $v(x_{il})(t)=\bigvee\limits_{t_0}^t(x_{il})$ for $a<t<b$, and $t_0=(a+b)/2$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X:]a,b[\to\mathbb{R}^{n\times m}$ at the point $t\in]a,b[$ (we assume $X(t)=X(a+)$ for $t\leq a$ and $X(t)=X(b-)$ for $t\geq b$, if necessary).

$d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all matrix-functions of the bounded variation $X:[a,b]\to\mathbb{R}^{n\times m}$ (i.e., $\overset{b}{\underset{a}{\vee}}(X)<+\infty$).

$\|X\|_s=\sup\big\{\|X(t)\|:\ t\in[a,b]\big\}$.

$\mathrm{BV}_s([a,b],\mathbb{R}^{n\times m})$ is the normed space $(\mathrm{BV}([a,b],\mathbb{R}^{n\times m}),\|\cdot\|_s)$.

$\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:]a,b[\to\mathbb{R}^{n\times m}$ such that $\overset{d}{\underset{c}{\vee}}(X)<+\infty$ for every $a<c<d<b$.

If $a<\alpha<\beta<b$ and $X\in\mathrm{BV}([\alpha,\beta],\mathbb{R}^{n\times m})$, then $X_{\alpha,\beta}\in\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is a matrix-function defined by

$$X_{\alpha,\beta}(t)=\begin{cases}X(\alpha-) & \text{for } a\le t<\alpha,\\ X(t) & \text{for } \alpha\le t\le\beta,\\ X(\beta+) & \text{for } \beta<t\le b.\end{cases}$$

Let $G\in\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$. By $\mathrm{BV}_G([a,b],\mathbb{R}^n)$ we denote the set of all vector-functions $x\in\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$ for which there exist the finite limits $\lim\limits_{t\to a+}G(t)x(t)$ and $\lim\limits_{t\to b-}G(t)x(t)$. It is evident that $x_G\in\mathrm{BV}([a,b],\mathbb{R}^n)$ for every $x\in\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$, where the vector-function $x_G:[a,b]\to\mathbb{R}^n$ is defined by

$$x_G(t)=\begin{cases}G(t)x(t) & \text{for } a<t<b,\\ \lim\limits_{t\to a+}G(t)x(t) & \text{for } t=a,\\ \lim\limits_{t\to b-}G(t)x(t) & \text{for } t=b.\end{cases}$$

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If $I\subset\mathbb{R}$ is an interval, then $C(I,\mathbb{R}^{n\times m})$ is the set of all continuous matrix-functions $X:I\to\mathbb{R}^{n\times m}$.

If B_1 and B_2 are normed spaces, then an operator $g:B_1\to B_2$ (nonlinear, in general) is positive homogeneous if

$$g(\lambda x)=\lambda g(x)$$

for every $\lambda\in\mathbb{R}_+$ and $x\in B_1$.

$s_1,s_2,s_c:\mathrm{BV}([a,b],\mathbb{R})\to\mathrm{BV}([a,b],\mathbb{R})$ are the operators defined, respectively, by

$$s_1(x)(a)=s_2(x)(a)=0,$$
$$s_1(x)(t)=\sum_{a<\tau\le t}d_1x(\tau)\quad\text{and}\quad s_2(x)(t)=\sum_{a\le\tau<t}d_2x(\tau)\ \text{ for } a<t\le b,$$

and

$$s_c(x)(t)=x(t)-s_1(x)(t)-s_2(x)(t)\ \text{ for } t\in[a,b].$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int\limits_s^t x(\tau)\, dg(\tau) = \int\limits_{]s,t[} x(\tau)\, ds_c(g)(\tau) + \sum_{s<\tau\le t} x(\tau) d_1 g(\tau) + \sum_{s\le\tau<t} x(\tau) d_2 g(\tau),$$

where $\int\limits_{]s,t[} x(\tau)\, ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu(s_c(g))$ corresponding to the function $s_c(g)$. If $a = b$, then we assume $\int\limits_a^b x(t)\, dg(t) = 0$; so that $\int\limits_s^t x(\tau)\, dg(\tau)$ is the Kurzweil–Stieltjes integral (see [20], [22], [24]). Moreover, we put

$$\int\limits_{s+}^t x(\tau)\, dg(\tau) = \lim_{\varepsilon\to 0,\ \varepsilon>0} \int\limits_{s+\varepsilon}^t x(\tau)\, dg(\tau)$$

and

$$\int\limits_s^{t-} x(\tau)\, dg(\tau) = \lim_{\varepsilon\to 0,\ \varepsilon>0} \int\limits_s^{t-\varepsilon} x(\tau)\, dg(\tau).$$

$L([a,b],\mathbb{R};g)$ is the space of all functions $x : [a,b] \to \mathbb{R}$, measurable and integrable with respect to the measure $\mu(g_c(g))$ for which

$$\sum_{a<\tau\le b} |x(t)| d_1 g(\tau) + \sum_{a\le\tau<b} |x(t)| d_2 g(t) < +\infty,$$

with the norm

$$\|x\|_{L,g} = \int\limits_a^b |x(t)|\, dg(t).$$

If $g_j : [a,b] \to \mathbb{R}$ $(j = 1,2)$ are nondecreasing functions, $g(t) \equiv g_1(t) - g_2(t)$, and $x : [a,b] \to \mathbb{R}$, then

$$\int\limits_s^t x(\tau)\, dg(\tau) = \int\limits_s^t x(\tau)\, dg_1(\tau) - \int\limits_s^t x(\tau)\, dg_2(\tau) \ \text{ for } \ a \le s \le t \le b.$$

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to \mathbb{R}^{l\times n}$ is a nondecreasing matrix-function and $D \subset \mathbb{R}^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to D$ such that $x_{kj} \in L([a,b],R;g_{ik})$ $(i = 1,\dots,l;\ k = 1,\dots,n;\ j = 1,\dots,m)$;

$$\int\limits_s^t dG(\tau)\cdot X(\tau) = \bigg(\sum_{k=1}^n \int\limits_s^t x_{kj}(\tau) dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m} \quad \text{for} \quad a \le s \le t \le b,$$

$$S_j(G)(t) \equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n} \ \ (j = 1,2) \ \text{ and } \ S_c(G)(t) \equiv \big(s_c(g_{ik})(t)\big)_{i,k=1}^{l,n}.$$

If $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}^{n\times m}$, then $\mathrm{Car}([a,b]\times D_1, D_2; G)$ is the Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that for each $i \in \{1,\dots,l\}$, $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$:

(i) the function $f_{kj}(\,\cdot\,,x) : [a,b] \to D_2$ is $\mu(s_c(g_{ik}))$-measurable for every $x \in D_1$;

(ii) the function $f_{kj}(t,\,\cdot\,) : D_1 \to D_2$ is continuous for $\mu(s_c(g_{ik}))$-almost every $t \in [a,b]$ and for every $t \in D_{g_{ik}}$, and

$$\sup\big\{|f_{kj}(\,\cdot\,,x)| :\ x \in D_0\big\} \in L([a,b],R;g_{ik})$$

for every compact $D_0 \subset D_1$;

$\mathrm{Car}_{loc}(]a,b[\times D_1, D_2; G)$ is the local Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} :]a,b[\times D_1 \to D_2$ the restriction of which on every closed interval $[\alpha,\beta]$ belongs to $\mathrm{Car}([\alpha,\beta]\times D_1, D_2; G)$ for every $a < \alpha < \beta < b$.

If $G_j : [a,b] \to \mathbb{R}^{l\times n}$ $(j = 1,2)$ are nondecreasing matrix-functions, $G(t) \equiv G_1(t) - G_2(t)$, and $X : [a,b] \to \mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau) = \int_s^t dG_1(\tau)\cdot X(\tau) - \int_s^t dG_2(\tau)\cdot X(\tau) \ \text{ for } \ a \le s \le t \le b,$$

$$S_k(G)(t) \equiv S_k(G_1)(t) - S_k(G_2)(t) \ \ (k = 1,2),$$
$$S_c(G)(t) \equiv S_c(G_1)(t) - S_c(G_2)(t).$$

If $G_1(t) \equiv V(G)(t)$ and $G_2(t) \equiv V(G)(t) - G(t)$, then

$$L([a,b],D;G) = \bigcap_{j=1}^{2} L([a,b],D;G_j),$$

$$\mathrm{Car}([a,b]\times D_1, D_2; G) = \bigcap_{j=1}^{2} \mathrm{Car}([a,b]\times D_1, D_2; G_j),$$

$$\mathrm{Car}_{loc}(]a,b[\times D_1, D_2; G) = \bigcap_{j=1}^{2} \mathrm{Car}_{loc}(]a,b[\times D_1, D_2; G_j).$$

If $G \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and $X : [a,b] \to \mathbb{R}^{n\times m})$, then

$$\mathcal{B}(G,X)(t) \equiv G(t)X(t) - G(a)X(a) - \int_{t_0}^t dG(\tau)\cdot X(\tau).$$

The inequalities between the matrices are understood componentwise.
Below we assume that

$$A_1(t) \equiv V(A)(t) \ \text{ and } \ A_2(t) \equiv V(A)(t) - A(t).$$

A vector-function $x \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$ is said to be a solution of the system (1.1) if

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot f(\tau, x(\tau)) \ \text{ for } \ a < s \le t < b.$$

Under a solution of the problem (1.1), (1.2) we mean solutions x of the system (1.1) such that $x \in \mathrm{BV}_H([a,b],\mathbb{R}^n)$ and the equality $h(x_H) = 0$ holds.

We say that the operator $g : \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n) \to \mathbb{R}^n)$ has some property in the set $\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$ if the operator $g_{\alpha,\beta} : \mathrm{BV}([\alpha,\beta],\mathbb{R}^n) \to \mathbb{R}^n$, defined by $g_{\alpha,\beta}(x) = g(x_{\alpha,\beta})$, has the same property for every $\alpha,\beta \in]a,b[$ $(\alpha < \beta)$; If, moreover, $B \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$, then we say that the problem

$$dx = dB(t)\cdot x \ \text{ for } \ t \in]a,b[\,, \ \ g(x) \le 0$$

has some property in $\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$, if the problem

$$dx = dB_{\alpha,\beta}(t)\cdot x \ \text{ for } \ t \in [\alpha,\beta], \ \ g_{\alpha,\beta}(x) \le 0$$

has the same property for every $\alpha,\beta \in]a,b[$ $(\alpha < \beta)$.

In particular, we say that the operator $g : \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n) \to \mathbb{R}^n)$ is continuous in the set $\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$ if

$$\lim_{k\to+\infty} g(x_{k;\alpha,\beta}) = g(x_{0;\alpha,\beta}) \ \text{ for every } \ a < \alpha < \beta < b,$$

where $x_0 \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$ and $x_k \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$ $(k = 1,2,\dots)$ is an arbitrary sequence such that

$$\lim_{k\to+\infty} x_{k;\alpha,\beta}(t) = x_0(t) \ \text{ uniformly on } \ [\alpha,\beta] \ \text{ for } \ a < \alpha < \beta < b.$$

Definition 1.1. Let a matrix-function $H \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$ be such that $\det H(t) \ne 0$ for $t \in]a,b[$. Let, moreover, $l : \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n) \to \mathbb{R}^n$ and $l_0 : \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n) \to \mathbb{R}^n_+$ be, respectively, linear continuous and positive homogeneous continuous operators in the set $\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$. Then by $\mathcal{O}(]a,b[\,,l,l_0;A,H)$ we denote the set of all matrix-functions $P \in \mathrm{Car}_{loc}(]a,b[\,\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ satisfying the Opial condition with respect to the set of four $(l,l_0;A;H)$, i.e.,

(i) there exists $\Phi \in L_{loc}(]a,b[\,,\mathbb{R}^{n\times n}_+;A)$ such that

$$|P(t,x)| \le \Phi(t) \ \text{ on the set } \ ]a,b[\,\times\mathbb{R}^n;$$

(ii)
$$\det\Big(I_n + (-1)^j\big(d_jB(t) + d_jH(t)\cdot H^{-1}(t)\big)\Big) \ne 0 \qquad (1.3)$$
$$\text{for } \ a < t < b \ \ (j = 1,2)$$

and the problem

$$dx = \big(dB(t) + dH(t)\cdot H^{-1}(t)\big)\cdot x, \ \ |l(x)| \le l_0(x)$$

has only the trivial solution in $]a,b[$ for every $B \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$ for which there exists a sequence $z_k \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n$ $(k=1,2,\dots)$ such that

$$\lim_{k\to+\infty}\int\limits_c^t d\mathcal{B}(H,A)(\tau)\cdot P(\tau,z_k(\tau)) = B(t) \ \text{ uniformly into } \ ]a,b[\,.$$

Remark 1.1. In particular, the condition (1.4) holds if

$$\big\|d_j\mathcal{B}(H,A)(t)\cdot\Phi(t)\big\| < 1 \ \text{ for } \ t\in\,]a,b[\ \ (j=1,2).$$

guarantees the condition (1.3).

Remark 1.2. If $H(t)\equiv I_n$, then Definition 1.1 coincides with the Opial class definition for the regular case on every closed interval $[\alpha,\beta]$ (see [2]).

We will assume that $H \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$ is a matrix-function such that $\det H(t)\neq 0$ for $t\in\,]a,b[$. Note that we can consider the case in which the matrix function H is regular only in the right and left neighborhood of the points a and b, respectively. In this case we assume that $H(t)=I_n$ if the point t does not belong to these neighborhoods.

2. Formulation of the Main Results

Theorem 2.1. *Let* $f=(f_l)_{l=1}^n$ *and* $f_k=(f_{kl})_{l=1}^n \in \mathrm{Car}_{loc}(]a,b[\,\times\mathbb{R}^n,\mathbb{R}^n;A)$ $(k=1,2,\dots)$,

$$\big|f_{kl}(t,x)\big| \le f_{0l}(t,x) \ \textit{ for } \ \mu(v(a_{il})) - \textit{for almost all } \ t\in\,]a,b[\,, \ \ x\in\mathbb{R}^n$$
$$(i,l=1,\dots,n; \ k=1,2,\dots)$$

and

$$\lim_{k\to+\infty} f_{kl}(t,x) = f_l(t,x) \ \textit{ for } \ \mu(v(a_{jil})) \ \textit{ for almost all } \ t\in\,]a,b[\,, \ \ x\in\mathbb{R}^n$$
$$(j=1,2; \ \ i,l=1,\dots,n; \ \ k=1,2,\dots),$$

where $f_l \in \mathrm{Car}_{loc}(]a,b[\,\times\mathbb{R}^n,\mathbb{R}^n;a_{il})$ $(i,l=1,\dots,n)$. *Let, moreover, for every natural* k*, the system*

$$dx = dA(t)\cdot f_k(t,x)$$

under the condition (1.2) *has a solution* x_k *such that*

$$\lim_{t\to a+}\sup\Big\{\big\|x_{k,H}(a+)-x_{k,H}(t)\big\| : \ k=1,2,\dots\Big\} = 0,$$
$$\lim_{t\to b-}\sup\Big\{\big\|x_{k,H}(b-)-x_{k,H}(t)\big\| : \ k=1,2,\dots\Big\} = 0$$

and

$$\sup\big\{\|x_k(t)\| : \ k=1,2,\dots\big\} \le \psi(t) \ \textit{ for } \ a<t<b,$$

where $\psi\in\mathrm{BV}_G([a,b],\mathbb{R}^n)$*. Then the sequence* x_k $(k=1,2,\dots)$ *contains a subsequence, convergent in the open interval* $]a,b[$*, and its limit is a solution of the problem* (1.1), (1.2).

Theorem 2.2. *Let the conditions*

$$\big|f(t, H^{-1}(t)x) - P(t, x)x\big| \le \alpha\big(t, \|x\|\big) \ \ \textit{for} \ \ t \in\,]a, b[\,, \ x \in \mathbb{R}^n,$$

and

$$\big|h(x) - l(x)\big| \le l_0(x) + l_1\big(\|x\|_v\big) \ \ \textit{in} \ \ \mathrm{BV}_{loc}(]a, b[\,, \mathbb{R}^n)$$

be fulfilled, where $l : \mathrm{BV}_{loc}(]a, b[\,, \mathbb{R}^n) \to \mathbb{R}^n$ *and* $l_0 : \mathrm{BV}_{loc}(]a, b[\,, \mathbb{R}^n) \to \mathbb{R}^n_+$ *are, respectively, linear continuous and positive homogeneous continuous operators in* $\mathrm{BV}_{loc}(]a, b[\,, \mathbb{R}^n)$; $P \in \mathcal{O}(]a, b[\,, l, l_0; A, H)$ *and a nondecreasing in the second variable matrix– and vector-functions, respectively,* $\alpha \in \mathrm{Car}_{loc}(]a, b[\,\times\mathbb{R}_+, \mathbb{R}^n_+; A)$ *and* $l_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ *are such that*

$$\lim_{\rho\to+\infty} \frac{1}{\rho} \int\limits_{a+}^{b-} dV(A)(t) \cdot \alpha(t, \rho) < 1 \ \ \textit{for} \ \ a < \alpha < \beta < b,$$

and

$$\lim_{\rho\to+\infty} \frac{l_1(\rho)}{\rho} < 1.$$

Then the problem (1.1), (1.2) *is solvable.*

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Author's addresses:

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia.

2. Sukhumi State University, 12 Politkovskaia St., Tbilisi 0186, Georgia.

E-mail: ashord@rmi.ge

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Nino Partsvania and Bedřich Půža

ON POSITIVE SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS FOR SINGULAR IN PHASE VARIABLES TWO-DIMENSIONAL DIFFERENTIAL SYSTEMS

Abstract. For the singular in phase variables differential system

$$u_i = f_i(t, u_1, u_2) \quad (i = 1, 2),$$

sufficient conditions are found for the existence of a positive on $]0, a[$ solution satisfying the nonlinear boundary conditions

$$\varphi(u_1) = 0, \quad u_2(a) = \psi(u_1(a)),$$

where $\varphi : C([0, a]; \mathbb{R}_+) \to \mathbb{R}$ is a continuous functional, while $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function.

რეზიუმე. ფაზური ცვლადების მიმართ სინგულარული დიფერენციალური სისტემისათვის

$$u_i = f_i(t, u_1, u_2) \quad (i = 1, 2)$$

ნაპოვნია $]0, a[$ შუალედში ისეთი დადებითი ამონახსნის არსებობის საკმარისი პირობები, რომელიც აკმაყოფილებს არაწრფივ სასაზღვრო პირობებს

$$\varphi(u_1) = 0, \quad u_2(a) = \psi(u_1(a)),$$

სადაც $\varphi : C([0, a]; \mathbb{R}_+) \to \mathbb{R}$ არის უწყვეტი ფუნქციონალი, ხოლო $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ არის უწყვეტი ფუნქცია.

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Let $a > 0$, $\mathbb{R}_- =]-\infty, 0]$, $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{R}_{0+} =]0, +\infty[$, $C([0, a]; \mathbb{R})$ be the Banach space of continuous functions $u : [0, a] \to \mathbb{R}$ with the norm

$$\|u\| = \max\big\{\|u(t)\| : \ a \le t \le b\big\},$$

and $C([0, a]; \mathbb{R}_+)$ be the set of all non-negative functions from $C([0, a]; \mathbb{R})$. Consider the two-dimensional differential system

$$\frac{du_i}{dt} = f_i(t, u_1, u_2) \ \ (i = 1, 2) \tag{1}$$

with the nonlinear boundary conditions

$$\varphi(u_1) = 0, \quad u_2(a) = \psi(u_1(a)), \tag{2}$$

where $f_i :]0,a[\times\mathbb{R}^2_{0+} \to \mathbb{R}_-$ $(i=1,2)$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions, while $\varphi : C([0,a];\mathbb{R}_+) \to \mathbb{R}_+$ is a continuous functional.

A continuous vector function $(u_1,u_2) : [0,a] \to \mathbb{R}^2_+$ is said to be **a positive solution of the differential system (1)** if it is continuously differentiable on an open interval $]0,a[$ and in this interval along with the inequalities

$$u_i(t) > 0 \quad (i=1,2) \tag{3}$$

satisfies the system (1).

A positive solution of the system (1) satisfying the conditions (2) is said to be **a positive solution of the problem (1), (2).**

We investigate the problem (1), (2) in the case where the functions f_i $(i=1,2)$ on the set $]0,a[\times\mathbb{R}^2_{0+}$ admit the estimates

$$\begin{aligned} g_{10}(t) \le -x^{\lambda_1} y^{-\mu_1} f_1(t,x,y) \le g_1(t),\\ g_{20}(t) \le -x^{\lambda_2} y^{\mu_2} f_2(t,x,y) \le g_2(t), \end{aligned} \tag{4}$$

where λ_i and μ_i $(i=1,2)$ are non-negative constants, and $g_{i0} :]0,a[\to \mathbb{R}_{0+}$ $(i=1,2)$, $g_i :]0,a[\to \mathbb{R}_{0+}$ $(i=1,2)$ are continuous functions such that

$$\int_0^a g_{i0}(t)\,dt < +\infty, \qquad \int_0^a g_i(t)\,dt < +\infty \quad (i=1,2).$$

If $\lambda_i > 0$ for some $i \in \{1,2\}$, then in view of (4) we have

$$\lim_{x\to 0} f_i(t,x,y) = +\infty \quad \text{for} \quad x>0, \quad 0<t<a.$$

And if $\mu_2 > 0$, then

$$\lim_{y\to 0} f_2(t,x,y) = +\infty.$$

Consequently, in both cases the system (1) has the singularity in at least one phase variable.

Boundary value problems for singular in phase variables second order nonlinear differential equations arise in different fields of natural science and are the subject of numerous studies (see e.g. [1–4, 7–14] and the references therein). In the recent paper by I. Kiguradze [5], optimal conditions are obtained for the solvability of the Cauchy–Nicoletti type nonlinear problems for singular in phase variables differential systems. As for the problems of the type (1), (2), they still remain unstudied in the above-mentioned singular cases. In the present paper, the attempt is made to fill this gap.

Along with the system (1) we consider the systems of differential inequalities

$$\begin{aligned} -u_1^{\lambda_1}(t) u_2^{-\mu_1}(t) u_1'(t) \ge g_{10}(t),\\ -u_1^{\lambda_2}(t) u_2^{\mu_2}(t) u_2'(t) \ge g_{20}(t), \end{aligned} \tag{5}$$

and

$$\begin{aligned} g_{10}(t) \le -u_1^{\lambda_1}(t) u_2^{-\mu_1}(t) u_1'(t) \le g_1(t),\\ g_{20}(t) \le -u_1^{\lambda_2}(t) u_2^{\mu_2}(t) u_2'(t) \le g_2(t). \end{aligned} \tag{6}$$

Let

$$\nu_0 = \frac{\mu_1}{1+\mu_2}\,, \quad \nu = 1 + \lambda_1 + \lambda_2\nu_0.$$

On the set $\{(t,x,y):\ 0 \le t \le a,\ x \ge 0,\ y \ge 0\}$ we introduce the functions

$$w_{10}(t,x,y) = \left[x^\nu + \nu \int_t^a g_{10}(s)\left(x^{\lambda_2} y^{1+\mu_2} + (1+\mu_2)\int_s^a g_{20}(\tau)\,d\tau\right)^{\nu_0} ds\right]^{\frac{1}{\nu}},$$

$$w_2(t,x,y) = \left[y^{1+\mu_2} + (1+\mu_2)\int_t^a w_{10}^{-\lambda_2}(s,x,y)g_2(s)\,ds\right]^{\frac{1}{1+\mu_2}},$$

$$w_1(t,x,y) = \left[x^{1+\lambda_1} + (1+\lambda_1)\int_t^a w_2^{\mu_1}(s,x,y)g_1(s)\,ds\right]^{\frac{1}{1+\lambda_1}},$$

$$w_{20}(t,x,y) = \left[y^{1+\mu_2} + (1+\mu_2)\int_t^a w_1^{-\lambda_2}(s,x,y)g_{20}(s)\,ds\right]^{\frac{1}{1+\lambda_2}}.$$

Note that the functions w_1, w_2, and w_{20} are defined on the set

$$\{(t,0,y):\ 0 \le t \le a,\ y \ge 0\}$$

only in the case, where

$$\int_0^a w_{10}^{-\lambda_2}(s,0,0)g_2(s)\,ds < +\infty. \tag{7}$$

A continuous vector function $(u_1,u_2) : [0,a] \to \mathbb{R}_+^2$ is said to be **a positive solution of the system of differential inequalities (5) (of the system of differential inequalities (6))** if it is continuously differentiable on an open interval $]0,a[$ and in this interval along with the inequalities (3) satisfies the system (5) (the system (6)).

The following statements are valid.

Lemma 1. *If the system of differential inequalities* (5) *has a positive solution* (u_1,u_2)*, then*

$$u_1(t) > w_{10}(t,x,y) \ \ \textit{for}\ \ 0 \le t \le a,$$

where

$$x = u_1(a), \quad y = u_2(a). \tag{8}$$

Lemma 2. *If the system of differential inequalities* (6) *has a positive solution* (u_1,u_2)*, then*

$$w_{i0}(t,x,y) < u_i(t) < w_i(t,x,y) \ \ \textit{for}\ \ 0 \le t \le a \ \ (i=1,2),$$

where x and y are numbers given by the equalities (8)*.*

On the basis of these lemmas we establish conditions guaranteeing, respectively, the existence or non-existence of at least one positive solution of problem (1), (2).

As this has already been said above, the theorems proven by us concern the case where the functions f_i $(i=1,2)$ admit the estimates (4). Moreover, everywhere below it is assumed that the functional φ is non-decreasing, i.e. for any $u \in C([0,a];\mathbb{R}_+)$ and $u_0 \in C([0,a];\mathbb{R}_+)$, it satisfies the inequality

$$\varphi(u+u_0) \geq \varphi(u).$$

For any non-negative constant x, we put $\varphi(x) = \varphi(u)$, where $u(t) \equiv x$.

Theorem 1. *Let*

$$\lim_{x\to+\infty} \varphi(x) = +\infty,$$

and let for some $\delta > 0$ the inequality

$$\varphi\big(w_1(\,\cdot\,,\delta,\psi(\delta))\big) \leq 0$$

hold. Then the problem (1), (2) *has at least one positive solution.*

Theorem 2. *If*

$$\varphi\big(w_{10}(\,\cdot\,,0,0)\big) > 0,$$

then the problem (1), (2) *has no positive solution.*

The particular cases of (2) are the nonlocal boundary conditions

$$\int_0^a \psi_0(u(s))\,d\sigma(s) = c, \quad u_2(a) = \psi(u_1(a)), \tag{9}$$

where $c \in \mathbb{R}$, $\psi_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, nondecreasing function, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, and $\sigma : [0,a] \to \mathbb{R}$ is a nondecreasing function such that

$$\sigma(a) - \sigma(0) = 1. \tag{10}$$

Theorems 1 and 2 imply the following corollary.

Corollary 1. *If*

$$\lim_{x\to+\infty} \psi_0(x) = +\infty$$

and for some $\delta > 0$ the inequality

$$c \geq \int_0^a \psi_0\big(w_1(s,\delta,\psi(\delta))\big)\,d\sigma(s) \tag{11}$$

holds, then the problem (1), (9) *has at least one positive solution. And if*

$$c < \int_0^a \psi_0\big(w_{10}(s,0,0)\big)\,d\sigma(s),$$

then the problem (1), (9) *has no positive solution.*

Note that due to the condition (10), for the inequality (11) to be fulfilled it is sufficient that

$$c \geq \psi_0\big(w_1(0,\delta,\psi(\delta))\big).$$

Corollary 2. *For an arbitrary $c > 0$, the differential system* (1) *has at least one positive solution satisfying the conditions*

$$u_1(a) = c, \quad u_2(0) = 0. \tag{12}$$

For $c = 0$, the problem (1), (12) becomes much more complicated, and to guarantee its solvability we have to impose additional restrictions of functions g_{i0} and g_i. More precisely, the following theorem is valid.

Theorem 3. *If*

$$\int_0^a w_{10}^{-\lambda_2}(s,0,0)g_2(s)\,ds < +\infty, \tag{13}$$

then the differential system (1) *has at least one positive solution satisfying the conditions*

$$u_1(a) = 0, \quad u_2(a) = 0. \tag{14}$$

The condition (13) in Theorem 3 is unimprovable in a certain sense. Moreover, the following theorem is true.

Theorem 4. *If*

$$\sup\Big\{g_i(t)/g_{i0}(t): \ \ 0 < t < a\Big\} < +\infty \ \ (i = 1,2),$$

then for the existence of at least one positive solution of the problem (1), (14) *it is necessary and sufficient the condition* (13) *to be fulfilled.*

Corollary 3. *Let*

$$\inf\Big\{t^{-\alpha_i}(a-t)^{-\beta_i}g_{i0}(t): \ \ 0 < t < a\Big\} > 0 \ \ (i = 1,2)$$

and

$$\sup\Big\{t^{-\alpha_i}(a-t)^{-\beta_i}g_i(t): \ \ 0 < t < a\Big\} < +\infty \ \ (i = 1,2).$$

Then for the existence of at least one positive solution of the problem (1), (14) *it is necessary and sufficient the inequalities*

$$\alpha_i > -1, \ \ \beta_i > -1 \ \ (i = 1,2), \quad (\alpha_2 + 1)(1 + \lambda_1) > (\alpha_1 + 1)\lambda_2$$

to be satisfied.

Theorems 3, 4 and Corollary 2 are analogs of the theorems by I. Kiguradze [6] for two-dimensional differential systems.

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Authors' addresses:

Nino Partsvania

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia;

2. International Black Sea University, 2 David Agmashenebeli Alley 13km, Tbilisi 0131, Georgia.

E-mail: `ninopa@rmi.ge`

Bedřich Půža

Faculty of Business and Management, Brno University of Technology, Kolejní 2906/4, Brno 612 00, Czech Republic.

E-mail: `puza@fbm.vutbr.cz`

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