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CONTACT PROBLEMS OF PLANE ELASTICITY THEORY AND RELATED BOUNDARY VALUE PROBLEMS OF FUNCTION THEORY

Abstract. In the work the boundary value problems of the theory of analytic functions with displacement are considered, namely: Carleman type problems with continuous and unbounded coefficients for strip and circular ring, the Riemann–Hilbert problems for doubly connected domains and discontinuous coefficients for ring. The contact problems of the elasticity theory for unbounded (isotropic, anisotropic and piecewise-homogeneous) domains with rectilinear boundaries with elastic fastening are investigated. The boundary value problems of plane theory of elasticity for anisotropic domains with cracks and inclusions are studied as well as the third basic and mixed boundary value problems for doubly-connected domains. The methods of analytic functions, integral transformations and theory of integral equations are applied. The solvability conditions of problems are formulated and proved. New methods of factorization are developed and the solutions of problems are represented in explicit form.

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რეზიუმე. ნაშრომში განხილულია ანალიზურ ფუნქციათა თეორიის გადაადგილებიანი სასაზღვრო ამოცანები, კერძოდ, კარლემანის ტიპის ამოცანები უწყვეტი და შემოუსაზღვრელი კოეფიციენტებით ზოლისა და რგოლისთვის, რიმან-ჰილბერტის ამოცანები ორადბმული არისთვის და წყვეტილი კოეფიციენტებით რგოლისთვის. გამოკვლეულია დრეკადობის თეორიის საკონტაქტო ამოცანები წრფივი საზღვრის მქონე შემოუსაზღვრელი იზოტროპული, ანიზოტროპული და უბან-უბან ერთგვაროვანი სხეულებისთვის დრეკადი გამაგრებებით. შესწავლილია დრეკადობის ბრტყელი თეორიის სასაზღვრო ამოცანები ჭრილებისა დ ჩართვების მქონე ანიზოტროპული სხეულებისთვის, აგრეთვე დრეკადობის თეორიის მესამე ძირითადი და შერეული სასაზღვრო ამოცანები ორადბმული არეებისთვის. გამოყენებულია ანალიზურ ფუნქციათა თეორიის, ინტეგრალური გარდაქმნებისა და ინტეგრალურ განტოლებათა თეორიის მეთოდები, ჩამოყალიბებულია და დამტკიცებულია ამოცანათა ამოხსნადობის პირობები, დამუშავებულია ფაქტორიზაციის ახალი მეთოდები და ამონახსნები წარმოდგენილია ცხადი სახით.

Introduction

One of the important areas of the elasticity theory, which studies contact problems of the interaction of thin-walled elements such as stringers with massive elastic bodies of various shapes, cracks propagating onto the body surface, and also problems with partly unknown boundaries, has been steadily developing since the 60s of the last century. The interest shown in these problems is due to their use for the solution of many serious problems related to engineering structures and machine-building. A fundamental work in this area belongs to E. Meland [**70**] who obtained an exact solution for a half-plane and the whole plane stiffened with an infinite stringer to which concentrated force is applied along its axis.

In the subsequent works [**35**], [**33**], [**46**], [**60**], [**123**], the problem was studied in the case where a semi-infinite stringer is fixed to an elastic plane or to the edge of an elastic half-plane.

Various problems for a half-plane stiffened with one or several stringers of finite length are considered in the works of many authors. Among them special mention should be made of E. Reissner [**97**], E. V. Benscoter [**32**], H. Bufler [**36**], N. Arutunyan [**7**], N. Arutunyan and S. Mkhitaryan [**8**], [**9**], B. Abramyan [**1**], where the problems are reduced to singular integro-differential equations and approximate solutions are obtained by different methods.

Detailed results for stringers and the bibliography are presented in F. Muki and E. Sternberg [**72**], [**73**], E. Sternberg [**115**] and in the survey paper by B. Abramyan [**2**].

Various contact problems are solved by the Wiener–Hopf method in the works of B. Lebedev and B. Nuller [**64**], B. Nuller [**80**], [**82**].

Problems of cracks propagating onto the body surface and the problem of a crack propagating to the interface of a piecewise-homogenous plane were also investigated by the Wiener–Hopf method (R. Bantsuri [**12**], H. Bueckner [**34**], G. R. Irwin [**45**], W. T. Koiter [**59**], A. Khrapkov [**54**]–[**57**], B. Smetanin [**111**], [**112**], R. Srivastavnazian Prom. [**114**], V. A. Wigglesworth [**125**], V. S. Tonoyan, S. A. Melkumyan [**117**]–[**119**]).

There exist a lot of contact problems of important applied character that cannot be solved effectively by the commonly used methods. Among these problems are the problem for a wedge with elastic stiffener, the third basic problem for a doubly connected domain bounded by broken lines and problems with a partly unknown boundary.

The contact problems considered in the present monograph can be attributed to three types depending on a mathematical method used to solve them. Problems of the first type are reduced by means of the Fourier transform to a Carleman type problem for a strip. Problems of the second type are reduced by the Fourier transform to the Riemann problem (Wiener–Hopf problem) (see N. Wiener and E. Hopf [**124**], N. I. Muskhelishvili [**76**], F. D. Gakhov [**42**]). Problems of the third type are reduced by the conformal mapping to the Riemann–Hilbert problem for a circular ring.

Examples of problems of the first and third types are contact problems for a wedge and for a doubly connected domain bounded by broken lines.

Carleman type problems for a strip and a circular ring are studied in the monograph in the most comprehensive way. Their effective solutions and some of their applications are the subject of Chapter 1.

In our opinion, Carleman type problems, which arose naturally when studying the contact problems, are of independent mathematical interest and their application area is much wider than that indicated in the works D. Lebedev and I. Skalskaya [**65**], B. Nuller [**81**], [**83**], B. Nuller and L. Stsepneva [**84**], G. Vasilyev [**120**], A. Krasnov and L. Tikhonenko [**58**].

The monograph consists of four chapters.

In Chapter 1, the Carleman type boundary value problems are solved for a strip and a circular ring. Their solutions are obtained in effective form and the Noether theorems as to their solvability are proved. The Riemann–Hilbert problem for a circular ring is also solved in effective form.

We introduce the class of functions $A_0^\beta(\mu)$ that are analytic in a strip $0 < \operatorname{Im} z < \beta$, continuously extendable on the boundary and satisfy the condition $\Phi(z)e^{-\mu|z|} \to 0$ for $|z| \to \infty$, $\mu \geq 0$.

For functions of the class $A_0^\beta(\mu)$ we obtain formulas analogous to the Cauchy integral formula, where instead of the usual Cauchy kernel $(t-z)^{-1}$ we introduce the kernel $[\mathbf{sh}\, p(t-z)]^{-1}$. The properties of functions represented by integrals analogous to Cauchy type integrals are studied. These formulas and integrals used to solve the boundary value problems of the analytic function theory considered in Chapter 1 play the same role as the Cauchy formula and a Cauchy type integral used in solving the Riemann linear conjugation problem.

In the first chapter, we consider the following Carleman problem for a strip: Find a function $\varphi(x) \in A_0^\beta(\mu)$ by the boundary condition

$$\varphi(x) = \lambda G(x)\varphi(x+a) + F(x), \quad -\infty < x < \infty, \tag{1}$$

where $G(x)$, $F(x)$ are given functions. Also,

$$a = \alpha + i\beta, \quad G(x) \neq 0, \quad -\infty < x < \infty,$$

and

$$G(\infty) = G(-\infty) = 1, \quad \lambda = \pm 1.$$

The problem is solved by the factorization method.

The solvability conditions and solution of problem (1) are obtained in explicit form.

Problem (1) is considered for $\alpha = 0$ when $G(x) = G_0(x)P_n(x)$, where $G_0(x) \neq 0$, $-\infty < x < \infty$, is a nonzero function of the class H (Hölder) including the point $x = \pm\infty$, and $P_n(x)$ is a polynomial having no real roots.

As different from our previous approach, here we also use factorization of a function of the form $ix \pm 2\beta$. This factorization is carried out using the Fourier integral transform. In that case, too, the solution of the problem is derived in explicit form and Noether type theorems are formulated. For $\lambda = -1$, $G(x) \in R$ (R is the Wiener class), $F(x) \in L_2$, problem (1) is reduced by the conformal mapping to the Riemann problem. The same technique is used to solve the problem in Yu. Cherski [**40**].

It should be said that in the above setting the solution of problem (1) by the method of reduction is less effective because of a difficulty associated with canonical factorization of functions of the form

$$\begin{cases} P_n[\ln t] & \text{if } \ t > 0, \\ 1 & \text{if } \ t < 0. \end{cases}$$

The homogeneous problem

$$\varphi(t+1) = G(t)\varphi(t), \quad t = \alpha + iy, \quad -\infty < y < \infty,$$

was considered under the assumption that $G(t)$ is a meromorphic function by E. Barnes [**31**] in 1904.

The Carleman type boundary value problem of the analytic function theory for a circular ring which we investigate here is formulated as follows: Find a function $\varphi(z)$, that is holomorphic in the ring $D = \{1 < |z| < R\}$ and continuously extendable on the boundary, using the boundary condition

$$\varphi(at) = G(t)\varphi(t) + f(t), \quad t \in \gamma = \{t : |t| = 1\}, \tag{2}$$

where a is a fixed point of the circumference $|t| = R$, $G(t)$ and $f(t)$ are functions of the class H given on γ, and $G(t) \neq 0$ almost everywhere on γ. The problem is solved in effective form and the Noether theorem is proved.

Furthermore, problem (2) is studied under the assumption that the functions $G(t)$ and $f(t)$ have first kind discontinuities at a finite number of points of γ. A solution is found in the effective form.

The Riemann–Hilbert problem considered in Chapter 1 is formulated as follows: Find a holomorphic function $\varphi(z)$ in the ring $D = \{z : 1 < |z| < R\}$ by the boundary condition

$$\mathrm{Re}[a(t)\varphi(t)] = C(t), \quad t \in \gamma_0 \cup \gamma_1,$$

where γ_0 and γ_1 are respectively the circumferences $|t| = R$ and $|t| = 1$, $a(t) \neq 0$ almost everywhere. $a(t)$ is a given complex function, and $C(t)$ is also a given real-valued function. It is assumed that the functions $a(t)$ and

$C(t)$ satisfy the Hölder condition. The problem is solved effectively and the Noether theorem is proved.

The Riemann–Hilbert problem is also investigated in the case where $a(t)$ and $C(t)$ have a finite number of points of first kind discontinuities. The problem is reduced to the Carleman type problem for a circular ring. The solution is constructed in the effective form and the Noether theorem is proved.

We give one more application of the solution of the Carleman type problem for a circular ring in solving an infinite system of linear equations

$$a^n \varphi_n = \sum_{m=-\infty}^{\infty} K_{n-m} \varphi_m = f_n,$$

where $\{K_n\}_{-\infty}^{\infty}$, $\{f_n\}_{-\infty}^{\infty}$ are given vectors and $\{\varphi_n\}_{-\infty}^{\infty}$ the sought vectors from ℓ_1, $|a| \neq 1$ is the known constant.

By means of the discrete Fourier transform, this system is reduced to the Carleman type problem for a circular ring where it is assumed that its coefficient and free term belong to the Wiener ring and a solution is sought also in this same ring. Using the Wiener–Levy theorem (see [**44**]) and the well-known theorem on conjugate functions (see [**126**]) we prove that the boundary values of the solution of the Carleman type problem for a circular ring are functions of the Wiener ring.

Chapter 2 of the monograph is dedicated to the investigation of contact problems of the plane elasticity theory of isotropic and anisotropic bodies when the problems are reduced to a Carleman type problem for a strip.

Contact problems are investigated for an elastic wedge-shaped plate when one of the wedge faces is stifferened with a semi-infinite stringer, and the concentrated force acting along the stringer is applied to its tip.

Using the Kolosov–Muskhelishvili formulas and the Fourier transform, the formulated problem is reduced to the Carleman type problem for a strip which is studied in Chapter 1. We construct the exact solution and study the behavior of tangential contact stresses at the wedge vertex and at infinity. For $0 \leq \alpha \leq \pi$, the problem is considered in J. Alblas and W. Kuypers [**3**].

The problem is also considered in the case of an anisotropic plate when the stringer stiffness is constant or variable. Using S. Lekhnitski's formulas and the Fourier transform, the problem is also reduced to the Carleman type problem for a strip and its exact solution is constructed. The behavior of tangential contact stresses at the wedge vertex and at infinity is studied.

The contact problem is studied for an anisotropic elastic wedge when one of the faces is supported by a semi-infinite beam and the beam stiffness is assumed to be constant or variable; the other wedge face is free. The tangential contact stress between the beam and the wedge is assumed to be equal to zero. It is required to find a distribution of stresses in the wedge and beam deflections when the beam is under the action of normally distributed or concentrated forces. The problem is reduced to the Carleman type problem for a strip. The exact solution of the contact problem is obtained by solving the problem by means of the inverse Fourier transform. The behavior of tangential contact stress at the wedge vertex and at infinity is studied.

A great number of works are dedicated to the investigation of static contact problems for various domains stiffened with elastic supports or inclusions in the form of plates of small thickness – see e.g. V. Aleksandrov and S. Mkhitaryan [**5**], V. Aleksandrov and Ye. Kovalenko [**4**], G. Popov and L. Tikhonenko [**95**], [**96**], G. Popov [**94**], V. Reut and L. Tikhonenko [**98**], N. Shavlakadze [**101**]–[**106**], V. Sitnik and L. Tikhonenko [**110**], V. Sitnik [**109**].

Chapter 3 deals with problems for an anisotropic wedge with a finite cut running from the wedge vertex along the bisectrix. It is assumed that the cut is under the action of stresses.

We also study the problem of an orthotropic wedge having a cut of finite length along the bisectrix that runs from the wedge vertex. It is assumed that the wedge faces are free from external stresses, while arbitrary stresses are applied to the cut banks. Using S. Lekhnitski's formulas [**66**], [**67**] and applying the Fourier transform , the problem reduces to three linear conjugation problems for a half-plane. The behavior at the cut end is studied. The stress intensity coefficient is defined in terms of an integral.

We consider the problem for a piecewise-homogeneous plane consisting of two orthotropic half-planes with, generally speaking, different elastic constants when one of the half-planes has a cut perpendicular to the interface straight line, and symmetric normal stresses are applied to the cut banks. The problem is solved using methods of the analytic function theory. Integral representations are obtained for unknown complex potentials, where the derivative of normal displacement of points of the cut edge serves as density. Using these integral representations, from the boundary conditions at the cut edges we obtain a singular integral equation having a fixed singularity at the cut edge lying on the interface line. The equation is solved by the Wiener–Hopf method. The exact solution of the equation is constructed, by means of which complex potentials are written in explicit form. The behavior of stresses near the cut ends is studied. It is established that near the end of the cut located on the interface line the stress may have – depending on a material – a singularity of any order less than one, whereas near the other cut end the order of a stress singularity is equal to $\frac{1}{2}$ independently of a material. Moreover, the intensity coefficient value is defined explicitly by means of integrals.

We consider a piecewise-homogeneous elastic plate stiffened with a semi-infinite inclusion intersecting the interface at the straight angle and loaded by tangential forces. The problem consists in defining contact stresses in the neighborhood of singular points. Applying the analytic function theory, the problem is reduced to a system of integro-differential equations on the semi-axis. The solution is obtained in explicit form.

Problems for a plane weakened by a finite system of rectilinear cuts located along one straight line are studied in N. Muskhelishvili [**75**], D. Sherman [**107**], G. Cherepanov [**37**], [**39**] and other works. All of these problems are reduced to problems of linear conjugation with respect to complex potentials.

In Chapter 4 we investigate problems for doubly-connected domains. These problems are solved using the results obtained in Chapter 1. In this chapter, we give an effective solution of the third basic problem of the elasticity theory for an isotropic body occupying a doubly-connected domain bounded by convex closed broken lines.

For the sake of definiteness, we consider the case of a finite domain. For the case of an infinite domain the problem is solved in G. A. Kapanadze [**50**].

Using the Kolosov–Muskhelishvili formulas and the conformal mapping, the considered problem reduces to a successive solution of two Riemann–Hilbert problems for a circular ring with piecewise-constant coefficients. For a simply connected domain bounded by the closed broken line, the third basic boundary value problem of the elasticity theory is solved in G. N. Polozhii [**89**]–[**93**]. By a technique different from ours, G. N. Polozhii reduces the problem to successive solutions of the Dirichlet and Riemann–Hilbert problems for a circle.

We use the results obtained in Chapter 1 to solve the following contact problem of new type: Given an elastic isotropic homogeneous plate shaped as a polygon weakened by some curvilinear hole, it is required to define the shape and location of the hole and also the stressed state of the plate assuming that on the external boundary of the plate the tangential stress is equal to zero, the normal displacement takes a constant value on every side of the polygon, and on the boundary of the hole free from external stresses the tangential normal stress takes the constant value $\sigma_\theta = K$. Using the methods of the analytic function theory and the Kolosov–Muskhelishvili formulas, the finding of the hole boundary reduces to the solution of the Riemann–Hilbert problem for a circular ring $1 < |\zeta| < R$ with piecewise-constant coefficients with respect to a function conformally mapping the domain occupied by the plate onto the circular ring. We seek the coefficient discontinuity points which under the conformal mapping are the images of the polygon vertices. The necessary and sufficient conditions for the problem to be solvable are obtained. Using these conditions we define the discontinuity points of coefficients. The discontinuity points of the coefficients are defined when the polygon is regular and the principal vectors of external forces applied to every point of the polygon side have one and the same value; the solvability condition reduces to one equation with respect to R and K. It is shown that then the problem is always solvable and the formula is obtained by means of which K is expressed through R. According to this formula, to various values of K there correspond various holes and the hole narrows as K decreases.

In the fourth chapter we also study the plate bending problem for the square weakened by five unknown equistable holes, of which four are identical, equidistant from the center of the square, symmetric with respect to the segments connecting the midpoints of the opposite sides of the square, and intersecting them. The fifth hole, symmetric with respect to the diagonals, contains the center of the square. The neighborhoods of the square vertices are cut out by regular unknown equistable arcs, symmetric with respect to the diagonals. Rigid strips are glued to the linear parts of the boundary. The plate is bent by concentrated moments applied to the midpoints of the strips.

We investigate the axially symmetric problem for a rectangle weakened by a finite number of unknown equistable holes. On the boundary of the rectangle, normal displacements have constant values, the tangential stress is equal to zero.

Problems for an infinite homogeneous isotropic plate weakened by curvilinear holes are studied in N. B. Banichuk [**10**], [**11**], O. G. Kosmodamianski and G. M. Ivanov [**61**], S. B. Vigdergauz [**122**], G.P. Cherepanov [**39**] when the stresses σ_x^∞, σ_y^∞ and σ_{xy}^∞ are given at infinity and it is required to find equistable holes; axially symmetric problems of the plane elasticity theory and bending problems for a plate with a partly unknown boundary were studied in G. A. Kapanadze [**48**]–[**50**], N. Odishelidze, F. Criado-Aldeanueva [**86**]–[**88**], R. D. Bantsuri [**23**], [**26**].

Some of our results obtained in the monograph are announced for the first time. The works [**22**]–[**30**] of the author were published in complete form, while other works of the author in the abridged form.

CHAPTER 1

Boundary Value Problems of the Theory of Analytic Functions with Displacements

1.1. Integral Representations of Holomorphic Functions in a Strip

Let the function $\Phi(z)$, $z = x + iy$, be holomorphic in a strip $\{a<y<b, -\infty < x < \infty\}$, continuous in a closed strip $\{a \le y \le b, -\infty < x < \infty\}$ and satisfy the condition $\Phi(z)e^{\mu|z|} \to 0$ for $|z| \to \infty$, $\mu \ge 0$. The class of functions satisfying these conditions will be denoted by $A_a^b(\mu)$.

Let

$$\Phi_k(z) \in A_0^\beta(\mu_k), \quad \mu_k < \frac{\pi\beta[3 + (-1)^k]}{2(\alpha^2 + \beta^2)}, \quad k = 1, 2, \tag{1.1.1}$$

where α and β are real numbers, $\beta > 0$. Then the following formulas are valid:

$$\Phi_1(z) = \frac{1}{2a} \int_{-\infty}^{+\infty} \frac{\Phi_1(t) + \Phi_1(t + a)}{\sinh p(t - z)}\, dt, \quad 0 < \mathcal{I}_m z < \beta, \tag{1.1.2}$$

$$\Phi_2(z) = \frac{\cosh pz}{2a} \int_{-\infty}^{+\infty} \frac{\Phi_2(t) - \Phi_2(t+a)}{\cosh pt \sinh p(t-z)}\, dt + \Phi_2\left(\frac{a}{2}\right), \quad 0 < \mathcal{I}_m z < \beta, \tag{1.1.3}$$

where $p = \frac{\pi i}{a}$, $a = \alpha + i\beta$.

The above formulas are obtained using the theorem on residues.

If $\Phi_k(z)$ has the form

$$\Phi_k(z) = \Psi_k(z) + \sum_{j=1}^{n} A_j \left(z - \frac{a}{2}\right)^{-j}, \quad \Psi_k(z) \in A_0^\beta(\mu_k), \quad k = 1, 2,$$

then we have

$$\Phi_1(z) = \frac{1}{2a} \int_{-\infty}^{+\infty} \frac{\Phi_1(t) + \Phi_1(t+a)}{\sinh p(t-z)}\, dt - \sum_{j=1}^{n} \frac{(-p)^j A_j}{j!} \left(\frac{1}{\cosh pz}\right)^{(j-1)}, \quad 0 < \mathcal{I}_m z < \beta, \tag{1.1.4}$$

$$\Phi_2(z) = \frac{\cosh pz}{2a} \int\limits_{-\infty}^{+\infty} \frac{\Phi_2(t) - \Phi_2(t+a)}{\cosh pt \sinh p(t-z)}\, dt$$
$$- \sum_{j=1}^{n} \frac{A_j(-p)^j}{j!} (\tanh pz)^{(j-1)} + \Phi_2\left(\frac{a}{2}\right), \quad 0 < \mathcal{I}_m z < \beta. \quad (1.1.5)$$

Let further $F_k(t)$, $k = 1, 2$, be the functions given on the real axis L and having the form $F_k(x) = f_k(x)e^{\mu_k |x|}$, $f_k(\pm\infty) = 0$, where $f_k(t)$ are the functions satisfying the Hölder condition everywhere on L, μ_k are the numbers satisfying inequality (1.1.1).

Consider the integrals

$$\Phi_1(z) = \frac{1}{2a} \int\limits_{-\infty}^{+\infty} \frac{F_1(t)}{\sinh p(t-z)}\, dt, \quad 0 < \mathcal{I}_m z < \beta, \quad (1.1.6)$$

$$\Phi_2(z) = \frac{\cosh pz}{2a} \int\limits_{-\infty}^{+\infty} \frac{F_2(t)}{\cosh pt \sinh p(t-z)}\, dt, \quad 0<\mathcal{I}_m z<\beta. \quad (1.1.7)$$

It is obvious that these functions are holomorphic in a strip $0 < y < \beta$.

Using the Sohotski–Plemelj formulas we can show that the boundary values of Φ_1 and Φ_2 are expressed by the formulas

$$\begin{aligned} \Phi_1(t_0) &= \frac{F_1(t_0)}{2} + \frac{1}{2a} \int\limits_{-\infty}^{+\infty} \frac{F_1(t)}{\sinh p(t-t_0)}\, dt, \\ \Phi_1(t_0+a) &= \frac{F_1(t_0)}{2} - \frac{1}{2a} \int\limits_{-\infty}^{+\infty} \frac{F_1(t)}{\sinh p(t-t_0)}\, dt; \end{aligned} \quad (1.1.8)$$

$$\begin{aligned} \Phi_2(t_0) &= \frac{F_2(t_0)}{2} + \frac{\cosh pz}{2a} \int\limits_{-\infty}^{+\infty} \frac{F_2(t)}{\cosh pt \sinh p(t-t_0)}\, dt, \\ \Phi_2(t_0+a) &= -\frac{F_2(t_0)}{2} + \frac{\cosh pt_0}{2a} \int\limits_{-\infty}^{+\infty} \frac{F_2(t)}{\cosh pt \sinh p(t-t_0)}\, dt. \end{aligned} \quad (1.1.9)$$

From Plemelj–Privalov's theorem it follows that that the boundary values of Φ_1 and Φ_2 satisfy the Hölder condition on the finite part of the boundary.

Let us investigate the behavior of these functions in the neighborhood of a point at infinity. First we consider the case with $\mu_k = 0$, $k = 1, 2$.

Rewrite formula (1.1.6) as

$$\Phi_1(z) = \frac{1}{2a}\int\limits_{-\infty}^{+\infty}\left[\frac{1}{\sinh p(t-z)} - \frac{a}{p}\frac{1}{(t-z)(t+a-z)}\right]F_1(t)\,dt$$
$$+\frac{1}{2ap}\int\limits_{-\infty}^{+\infty}\frac{F_1(t)}{t-z}\,dt - \frac{1}{2ap}\int\limits_{-\infty}^{+\infty}\frac{F_1(t)}{t+a-z}\,dt,\quad 0<\mathcal{I}_m z<\beta.$$

Here the first term is holomorphic in the closed strip $0 \le \mathcal{I}_m z \le \beta$ and tends to zero at infinity. The second and the third term are analytic in the strip $0 < \mathcal{I}_m z < \beta$, vanish at infinity and their boundary values satisfy the Hölder condition, including points at infinity [**76**].

Therefore $\Phi_1(z) \in A_0^\beta(0)$.

Now let us consider the function $\Phi_2(z)$. Rewrite formula (1.1.7) as

$$\Phi_2(z) = \frac{1}{2a}\int\limits_{-\infty}^{+\infty}\frac{(\cosh pz - \cosh pt)F_2(t)}{\cosh pt \sinh p(t-z)}\,dt + \frac{1}{2a}\int\limits_{-\infty}^{+\infty}\frac{F_2(t)}{\sinh p(t-z)}\,dt.$$

As we have shown, the second term here belongs to the class $A_0^\beta(0)$.

Denote the first term by $\mathcal{I}$ and rewrite it as

$$\mathcal{I} = -\frac{1}{2a}\int\limits_{-\infty}^{+\infty}\frac{\sinh\frac{p}{2}(t+z)F_2(t)}{\cosh pt \sinh\frac{p}{2}(t-z)}\,dt$$
$$= -\frac{1}{2a}\int\limits_{-\infty}^{+\infty}\frac{\sinh\frac{p}{2}(2x-\tau+iy)F_2(x-\tau)}{\cosh p(x-\tau)\cosh\frac{p}{2}(\tau+iy)}\,d\tau$$
$$= -\frac{1}{2a}\left(\int\limits_{-\infty}^{0} + \int\limits_{0}^{+\infty}\right)\frac{\sinh\frac{p}{2}(2x-\tau+iy)F_2(x-\tau)}{\cosh p(x-\tau)\cosh\frac{p}{2}(\tau+iy)}\,d\tau.$$

Let $x > 0$. Then the first integral will be bounded in the strip $0 \le \mathcal{I}_m z < \beta$, since $2x-\tau < 2(x-\tau)$.

Rewrite the second integral as

$$\frac{1}{2a}\int\limits_{0}^{+\infty}\frac{\sinh\frac{p}{2}(2x-\tau+iy)F_2(x-\tau)}{\cosh p(x-\tau)\cosh\frac{p}{2}(\tau+iy)}\,d\tau$$
$$= \left(\frac{1}{2a}\int\limits_{-\infty}^{0} + \frac{1}{2a}\int\limits_{0}^{x}\right)\frac{\sinh\frac{p}{2}(x+t+iy)F_2(t)}{\cosh pt \cosh\frac{p}{2}(x-t+iy)}\,dt.$$

The first term is bounded since $x+t<x-t$. The second term can be written in the form

$$\frac{1}{2a}\int\limits_0^x \frac{\sinh\frac{p}{2}([(x-t+iy)+2t]F_2(t)}{\cosh pt\cosh\frac{p}{2}(x-t+iy)}\,dt$$

$$=\frac{1}{2a}\int\limits_0^x \tanh\frac{p}{2}\,(x-t+iy)F_2(t)\,dt+\frac{1}{2a}\int\limits_0^x \tanh ptF_2(t)\,dt. \quad (1.1.10)$$

Since the function $\tanh\frac{p}{2}z=\tanh\frac{|p|^2}{2\pi}(\beta+\alpha i)z$ is holomorphic in the strip $0\le \mathcal{I}_m z\le\delta<\beta$ and $|\tanh\frac{p}{2}z|\to 1$, the estimate

$$|\Phi_2(z)|<|\Phi_0(x)|+\varepsilon|x| \quad (1.1.11)$$

holds for the function Φ_2 when x are large in the closed strip $0\le\mathcal{I}_m z\le\delta$, $\Phi_0(x)$ is bounded for $x>0$ and $\varepsilon<0$ is an arbitrarily small number. A similar estimate is also true for the case $x<0$. In the same manner we can obtain an estimate of form (1.1.11) in the strip $0<\delta\le\mathcal{I}_m z\le\beta$ provided that the function $\Phi_2(z)$ is represented as

$$\Phi_2(z)=\frac{1}{2a}\int\limits_{-\infty}^{+\infty}\frac{\cosh pz+\cosh pt}{\cosh pt\sinh p(t-z)}\,F_2(t)\,dt-\frac{1}{2a}\int\limits_{-\infty}^{+\infty}\frac{F_2(t)}{\sinh p(t-z)}\,dt.$$

Now let us consider the case with $\mu_k>0$, $k=1,2$. Rewrite (1.1.6) as follows:

$$\Phi_1(z)=\frac{1}{2a}\int\limits_{-\infty}^{+\infty}\frac{\cosh\mu_1 t\varphi_1(t)}{\sinh p(t-z)}\,dt,\quad \varphi_1(t)\equiv f_1(t)e^{\mu_1|t|}/\cosh\mu_1 t.$$

It is obvious that $\varphi_1(t)$ satisfies the Hölder condition in the neighbourhood of a point at infinity.

We write the function $\Phi_1(z)$ in the form

$$\Phi_1(z)=\frac{1}{2a}\int\limits_{-\infty}^{+\infty}\frac{\varphi_1(t)\cosh[\mu_1(t-z)+\mu_1 z]}{\sinh p(t-z)}\,dt$$

$$=\frac{\cosh\mu_1 z}{2a}\int\limits_{-\infty}^{+\infty}\frac{\cosh\mu_1(t-z)}{\sinh p(t-z)}\,\varphi_1(t)\,dt+\frac{\sinh\mu_1 z}{2a}\int\limits_{-\infty}^{+\infty}\frac{\sinh(t-z)\mu_1}{\sinh p(t-z)}\,\varphi_1(t)\,dt.$$

Since $\mu_1<\pi\beta/(\alpha^2+\beta^2)=\operatorname{Re}p$, we have $\Phi_1(z)\in A_0^\beta(\mu_1)$. Taking this into account and applying the arguments used when investigating the behavior of the function $\Phi_2(z)$ in the case with $F_2(\pm\infty)=0$, we show that

$$\Phi_2(z)\in A_0^\beta(\beta_2).$$

Let us formulate the results obtained above as the following statement.

Theorem 1. *If the functions $F_k(x)e^{-\mu_k|x|}$ $(k=1,2)$ satisfy the Hölder condition everywhere on L and $F_k(x)e^{-\mu_k|x|} \to 0$ for $|x| \to +\infty$, where μ_k are some numbers satisfying inequality* (1.1.1)*, then $\Phi_k \in A_0^\beta(\mu_k)$ for $\mu_1 \geq 0$, $\mu_2 > 0$, $\exp \Phi_2 \in A_0^\beta(\varepsilon)$ for $\mu_2 = 0$, where ε is an arbitrarily small positive number.*

Formulas (1.1.8) and (1.1.9) imply

$$\Phi_1(t) + \Phi_1(t+a) = F_1(t), \quad t \in (-\infty, \infty), \tag{1.1.12}$$

$$\Phi_2(t) - \Phi_2(t+a) = F_2(t), \quad t \in (-\infty, \infty), \tag{1.1.13}$$

i.e., $\Phi_1(z)$ and $\Phi_2(z)$ defined by (1.1.6) and (1.1.7) are solutions of boundary value problems (1.1.12) and (1.1.13) of the class $A_0^\beta(\mu_k)$, $k=1,2$.

Clearly, if the function $\Phi_2(z)$ is a solution of problem (1.1.13), then the function $W(z) = c + \Phi_2(z)$ will also be a solution. We will show that problems (1.1.12) and (1.1.13) do not have other solutions of the class $A_0^\beta(\mu_k)$, $k=1,2$. For this we should prove

Theorem 2. *If $F_2(t) \in L(-\infty,\infty)$, then for a solution of problem* (1.1.13) *of the class $A_0^\beta(0)$ to exist it is necessary and sufficient that the condition*

$$\int_{-\infty}^{\infty} F_2(t)\,dt = 0$$

be fulfilled.

Proof. We can rewrite formula (1.1.7) as

$$\Phi_2(z) = \frac{1}{2a}\int_{-\infty}^{\infty} \coth p(t-z) F_2(t)\,dt - \frac{1}{2a}\int_{-\infty}^{\infty} \tanh pt F_2(t)\,dt. \tag{1.1.14}$$

It is obvious that the limits of $\Phi_2(z)$ exist for $x \to \pm\infty$, $0 \leq y \leq \beta$, and

$$C + \Phi_2(\pm\infty + iy) = \pm\frac{1}{2a}\int_{-\infty}^{\infty} F_2(t)\,dt - \frac{1}{2a}\int_{-\infty}^{\infty} \tanh pt F_2(t)\,dt + C. \tag{1.1.15}$$

Taking

$$C = \frac{1}{2a}\int_{-\infty}^{\infty} F_2(t) \tanh pt\; dt$$

and setting

$$\int_{-\infty}^{\infty} F_2(t)\,dt = 0, \tag{1.1.16}$$

we find by virtue of (1.1.15) and (1.1.16) that a solution of problem (1.1.13) has the form

$$\Phi_2(z) = \frac{1}{2a} \int\limits_{-\infty}^{\infty} \coth p(t-z) F_2(t)\, dt \tag{1.1.17}$$

and belongs to the class $A_0^\beta(0)$.

The necessity is proved by integrating equality (1.1.13) and applying the Cauchy theorem. □

It remains to prove

Theorem 3. *If the function* $\varphi \in A_0^\beta\big(\frac{\pi\beta(3+\lambda)}{2(\alpha^2+\beta^2)}\big)$, $\lambda = \pm 1$, *and satisfies the condition* $\varphi(z) = \lambda\varphi(x+a)$, *then it is constant and, for* $\lambda = -1$, *is equal to zero.*

Proof. Let $\lambda = -1$ and

$$\Psi(z) = \frac{\varphi(z)}{\cosh pz} + \varphi\left(\frac{a}{2}\right) \frac{a}{\pi(z-\frac{a}{2})}\,. \tag{1.1.18}$$

The function $\Psi(z) \in A_0^\beta(0)$ and satisfies the condition

$$\Psi(x) - \Psi(x+a) = \frac{2a^2}{\pi}\varphi\left(\frac{a}{2}\right) \frac{1}{x^2 - a^2/4}\,. \tag{1.1.19}$$

Since $\Psi(z)$ is a solution of problem (1.1.19) of the class $A_0^\beta(0)$, the condition

$$\frac{2a^2}{\pi}\varphi\left(\frac{a}{2}\right) \int\limits_{-\infty}^{\infty} \frac{dx}{x^2 - a^2/4} = 4ai\varphi\left(\frac{a}{2}\right) = 0$$

is fulfilled on account of Theorem 2. Thus $\Psi(z)$ is a solution of the homogeneous problem

$$\Psi(x) - \Psi(x+a) = 0, \quad -\infty < x < +\infty.$$

If we introduce the function

$$\Psi_1(z) = \frac{\Psi(z) - \Psi\big(\frac{a}{2}\big)}{\cosh pz}\,,$$

then we have

$$\Psi_1(x) + \Psi_1(x+a) = 0, \quad -\infty < x < +\infty.$$

By applying the Fourier transform to the latter equality we obtain

$$\widehat{\Psi}_1(1 + e^{i\alpha t}) \equiv 0.$$

Hence we have $\widehat{\Psi}_1(t) \equiv 0$, $\Psi_1(z) = 0$. Therefore by (1.1.18) $\varphi(z) = 0$. We have thereby proved the theorem for $\lambda = -1$.

Let $\lambda = 1$. Then $\varphi(x) - \varphi(x+a) = 0$.

The function
$$\Psi(z)=\varphi(z)-\varphi\left(\frac{3}{4}a\right) \tag{1.1.20}$$
also satisfies this condition and $\varphi\big(\frac{3}{4}\,a\big)=0$.

We introduce the notation
$$\Psi_0(z)=\frac{\Psi(z)}{\cosh 2pz}+\frac{a}{2\pi}\,\frac{\Psi\big(\frac{a}{4}\big)}{z-\frac{a}{4}}.$$
Now, repeating the above arguments, we find that $\Psi\big(\frac{a}{4}\big)=0$, i.e., $\Psi_0(z)\in A_0^\beta(0)$ and satisfies the condition
$$\Psi_0(x)-\Psi_0(x+a)=0.$$
But, as shown above, in that case $\Psi_0(z)=\Psi(z)=0$ and therefore equality (1.1.20) implies
$$\varphi(z)=\varphi\left(\frac{3}{4}\,a\right),$$
which proves the theorem. □

1.2. A Carleman Type Problem with a Continuous Coefficient for a Strip

Let us consider the following problem: find a function Φ of the class $A_0^\beta(\mu)$ by the boundary condition
$$\Phi(x)=\lambda G(x)\Phi(x+a)+F(x),\quad -\infty<x<+\infty, \tag{1.2.1}$$
where $a=\alpha+i\beta$, $\beta>0$, $\mu<\pi\beta(3+\lambda)/2(\alpha^2+\beta^2)$, F and G are the given functions satisfying the Hölder condition including a point at infinity, $G\neq 0$ and $F(\pm\infty)=0$, $G(-\infty)=G(\infty)=1$, the constant λ takes the value 1 or -1.

The integer number $\varkappa=\frac{1}{2\pi}\,[\arg G(x)]_{-\infty}^{+\infty}$, where $[\arg G(x)]_{-\infty}^{+\infty}$ denotes an increment of the function $\arg G(x)$ when x runs over the entire real axis from $-\infty$ to ∞, is called the index of the function $G(x)$. The index of $G_0(x)=G(x)[(x-a/2)/(x+a/2)]^\varkappa$ is equal to zero and therefore any branch of the function $\ln G_0(x)$ is continuous all over the real axis. We choose a branch that vanishes at infinity. By formulas (1.1.7) and (1.1.9), $G(x)$ can be represented as
$$G(x)=\frac{X(x)}{X(x+a)}\,, \tag{1.2.2}$$
where
$$\begin{aligned}
X(z)&=\Big(z-\frac{a}{2}\Big)^{\varkappa}X_0(z),\\
X_0(z)&=\exp\left(\frac{\cosh pz}{2a}\int\limits_{-\infty}^{+\infty}\frac{\ln G_0(t)}{\cosh pt\sinh p(t-z)}\,dt\right).
\end{aligned} \tag{1.2.3}$$

By virtue of Theorem 1, $X_0(z)$ and $[X_0(z)]^{-1} \in A_0^\beta(\varepsilon)$, where ε is an arbitrarily small positive number.

Using (1.2.2), we rewrite condition (1.2.1) as

$$\frac{\Phi(x)}{X(x)} = \lambda \frac{\Phi(x+a)}{X(x+a)} + \frac{F(x)}{X(x)}\,, \quad -\infty < x < \infty. \tag{1.2.4}$$

The function $\Phi(z)/X(z)$ is holomorphic in the strip $0 < \mathcal{I}_m z < \beta$ except perhaps for the point $z = \frac{a}{2}$ at which it may have a pole of order $\varkappa$, for $\varkappa > 0$ and satisfies the condition

$$(\Phi(z)/X(z))e^{-\mu|z|} \to 0 \quad \text{for} \quad |x| \to \infty \quad \text{and} \quad 0 \le y \le \beta,$$

where $0 < \mu < \pi\beta(3+\lambda)/2(\alpha^2+\beta^2)$. By (1.1.4) and (1.1.5), condition (1.2.4) implies

$$\Phi(z) = \frac{X(z)}{2a} \int\limits_{-\infty}^{+\infty} \frac{F(t)}{X(t)\sinh p(t-z)}\, dt + X(z)\varphi_1(z) \ \text{ for } \lambda = -1, \tag{1.2.5}$$

$$\Phi(z) = \frac{X(z)\cosh pz}{2a} \int\limits_{-\infty}^{+\infty} \frac{F(t)}{X(t)\cosh pt \sinh p(t-z)}\, dt \\ + X(z)\varphi_2(z) \quad \text{for} \quad \lambda = 1, \tag{1.2.6}$$

where

$$\varphi_1(z) = \begin{cases} 0, & \varkappa \le 0, \\ \sum\limits_{k=1}^{\varkappa-1} C_k (1/\cosh pz)^{(k)}, & \varkappa > 0, \end{cases} \tag{1.2.7}$$

$$\varphi_2(z) = \begin{cases} 0, & \varkappa < 0, \\ \sum\limits_{k=1}^{\varkappa} C_k (\tanh pz)^{(k)}, & \varkappa \ge 0, \end{cases} \tag{1.2.8}$$

C_k are arbitrary constants.

Let us investigate the behavior of the function

$$\varphi(z) = \frac{X(z)}{2a} \int\limits_{-\infty}^{+\infty} \frac{F(t)}{X(t)\sinh p(t-z)}\, dt, \quad 0 \le \mathcal{I}_m z \le \beta, \tag{1.2.9}$$

in the neighborhood of a point at infinity. The function $X(z)$ can be represented as

$$X(z) = \Big(z - \frac{a}{2}\Big)^{\varkappa} \exp\Gamma_1(z) \cdot \exp\Gamma_2(z),$$

where

$$\Gamma_1(z) = -\frac{1}{2a} \int\limits_{-\infty}^{\infty} \frac{\sinh \frac{p}{2}(z+t) \ln G_0(t)}{\cosh pt \cosh \frac{p}{2}(t-z)}\, dt,$$

$$\Gamma_2(z) = \frac{1}{2a} \int\limits_{-\infty}^{\infty} \frac{\ln G_0(t)}{\sinh p(t-z)}\, dt.$$

As has been shown above, $\Gamma_2(z) \in A_0^\beta(0)$, i.e., $(\exp \Gamma_2(z) - 1) \in A_0^\beta(0)$.

By differentiating the function $\Gamma_1(z)$ we obtain

$$\Gamma_1'(z) = \frac{1}{2a} \int\limits_{-\infty}^{\infty} \frac{\ln G_0(t)}{\cosh \frac{p}{2}\,(t-z)}\, dt, \quad 0 \le \mathcal{I}_m z \le \beta_0 < \beta.$$

It is easy to verify that $\Gamma_1'(z) \to 0$ for $|z| \to \infty$ and therefore for any $\varphi(z)$ there is a number N such that

$$|\Gamma_1'(x+iy)| < \varepsilon \quad \text{for} \quad |x| > N, \quad 0 \le y \le \beta_0 < \beta. \tag{1.2.10}$$

We represent $\varphi(z)$ as

$$\varphi(z) = \frac{X(z)}{2a} \int\limits_{-N}^{N} \frac{F(t)}{X(t)} \cdot \frac{dt}{\sinh p(t-z)}$$

$$+ \frac{X(z)}{2a} \Bigg(\int\limits_{-\infty}^{N} + \int\limits_{N}^{\infty} \Bigg) \frac{F(t)}{X(t) \sinh p(t-z)}\, dt, \;\; 0 \le \mathcal{I}_m z \le \beta_0.$$

It is easy to show that the first and the second term vanish as $x \to +\infty$. We will show that the third term also tends to zero as $x \to +\infty$, $0 \le y \le \beta_0 < \beta$. This term will be denoted by $\mathcal{I}$.

$$\mathcal{I} = \int\limits_{N}^{\infty} \frac{\exp(\Gamma_2(z) - \Gamma_2(t)) \exp(\Gamma_1(z) - \Gamma_1(t)) (z - \frac{a}{2})^{\varkappa} F(t)}{(t - \frac{a}{2})^{\varkappa} \sinh p(t-z)}\, dt.$$

Assume that $\varkappa \ge 0$ and represent the function $\left(z - \frac{a}{2}\right)^{\varkappa}$ as

$$\Big(z - \frac{a}{2}\Big)^{\varkappa} = \varkappa! \sum_{n=1}^{\varkappa} \frac{(t - \frac{a}{2})^{\varkappa - n} (z-t)^n}{(\varkappa - n)!\; n!} + \Big(t - \frac{a}{2}\Big)^{\varkappa}. \tag{1.2.11}$$

Inequality (1.2.10) implies that

$$|\Gamma_1(z) - \Gamma_1(t)| \le \left| \int\limits_{t}^{z} \Gamma'(s)\, ds \right| \le \varepsilon |t-z|, \;\; t < N, \;\; \varkappa > N, \;\; 0 \le y \le \beta_0,$$

i.e., $\mathrm{Re}(\Gamma_1(t) - \Gamma_1(z)) - \varepsilon|t-z| < 0$. Thus we have

$$\big| \exp[\Gamma_1(t) - \Gamma_1(z)) - \varepsilon|t-z| - 1 \big| < A|t-z|, \quad t > N, \;\; x > N.$$

The latter inequality and formula (1.2.11) imply

$$
\begin{aligned}
|\mathcal{I}| \leq c \sum_{n=1}^{\varkappa} \int_N^\infty \frac{e^{\varepsilon|x-t|}|x-t+iy|^n |F(t)|}{(\varkappa-n)!n!|\sinh p(x-t+iy)||t-\frac{a}{2}|^n}\, dt \\
+ c_1 \int_N^\infty \frac{e^{\varepsilon|x-t|}[A|x-t| + (1-e^{-\varepsilon|x-t|})]|F(t)|}{|\sinh p(x-t+iy)|}\, dt \\
+ \left| \int_N^\infty \frac{F(t)}{2|a| \sinh p(t-z)}\, dt \right|.
\end{aligned}
$$

where, as shown above, the third term is the modulus of a function of the class $A_0^\beta(0)$. Since ε is an arbitrarily small number and $F(\infty) = F(-\infty) = 0$, the first two terms are the convolutions of functions summable with functions tending to zero for $x < 0$. Therefore they tend to zero for $x \to \infty$, $0 \leq y \leq \beta_0 < \beta$.

It can be shown in a similar manner that $\varphi(z) \to 0$ for $x \to -\infty$, $0 \leq y \leq \beta_0$, as well. It is not difficult to prove that the function $\varphi(z)$ tends to zero for $|x| \to \infty$, $\beta_0 \leq \mathcal{I}_m z \leq \beta$. When $\varkappa < 0$, one can use the same reasoning to show that $\varphi(z) \to 0$ for $|x| \to \infty$, $0 \leq y \leq \beta$, provided that z and t are exchanged in equality (1.2.11). Thus the function Φ represented by (1.2.5) tends to zero as $|x| \to +\infty$, $0 \leq y \leq \beta$. Quite similarly, it is proved that for the function Φ defined by (1.2.6) we have $\Phi(z)e^{-\varepsilon|z|} \to 0$ as $|x| \to \infty$, $0 \leq y \leq \beta$.

For $\varkappa < 0$ the function $X(z)$ has a pole of order $-\varkappa$ at the point $z = \frac{a}{2}$. In that case the solution exists only if the following conditions are fulfilled:

$$
\int_{-\infty}^{\infty} \frac{F(t)}{X(t)} \left(\frac{1}{\cosh pt} \right)^{(k)} dt = 0, \quad k = 0, \dots, (-\varkappa - 1) \ \text{ for } \ \lambda = -1, \tag{1.2.12}
$$

$$
\int_{-\infty}^{\infty} \frac{F(t)}{X(t)} \left(\frac{e^{pt}}{\cosh pt} \right)^{(k)} dt = 0, \quad k = 1, \dots, (-\varkappa - 1) \ \text{ for } \ \lambda = 1. \tag{1.2.13}
$$

The results obtained can be formulated as

Theorem 4. *For $\lambda = -1$ and $\varkappa \geq 0$, problem* (1.2.1) *is solvable in the class $A_0^\beta(0)$ and a general solution is given by* (1.2.5) *with formula* (1.2.7) *taken into account. If $\varkappa < 0$, then the problem is solvable if condition* (1.2.12) *is fulfilled. In these conditions problem* (1.2.1) *has a unique solution in the class $A_0^\beta(0)$ which is given by formula* (1.2.5) *for $\varphi_1 = 0$.*

Theorem 5. *if $\lambda = 1$ and $\varkappa \geq -1$, then problem* (1.2.1) *is solvable in the class $A_0^\beta(\varepsilon)$ and its solution is given by* (1.2.6) *with* (1.2.8) *taken into account; for $\varkappa < -1$, the solution exists provided that condition* (1.2.13) *is* *fulfilled. If these conditions are fulfilled, then problem* (1.2.1) *has a unique solution in the class* $A_0^\beta(\varepsilon)$. *This solution is given by* (1.2.6) *where* $\varphi_2 = 0$.

1.3. A Carleman Type Problem with Unbounded Coefficients for a Strip

Problems of the elasticity theory can often be reduced to a Carleman type problem with coefficients polynomially increasing or decreasing at infinity. We will consider such a case below.

We write the boundary condition of the problem in the form

$$\Phi(x) = P_n(x)G(x)\Phi(x+i\beta) + F(x), \quad -\infty < x < \infty, \tag{1.3.1}$$

where $G(x)$ and $F(x)$ satisfy the conditions discussed in Section 1.2, and $P_n(x)$ is a polynomial without real zeros. Condition (1.3.1) can be rewritten as

$$\Phi(x) = q[x^2+4\beta^2]^{[\frac{n}{2}]}(2\beta - ix)^{\delta(n)}G_0(x)\Phi(x+i\beta) + F(x), \tag{1.3.2}$$

where $\delta(n) = 0$ for even n and $\delta(n) = 1$ for odd n; q is a complex number; $G_0(x)$ is a Hölder class function including a point at infinity $G_0(-\infty) = G_0(\infty) = 1$.

As shown above, the function $G_0(x)$ can be represented as

$$G_0(x) = \frac{X_0(x)}{X_0(x+i\beta)}\,, \quad -\infty < x < \infty, \tag{1.3.3}$$

where

$$X_0(z) = \left(z - \frac{i\beta}{2}\right)^{\varkappa} \exp\left(\frac{\cosh pz}{2i\beta}\int\limits_{-\infty}^{\infty} \frac{\ln\left[G_0(t)(\frac{t+i\beta/2}{t-i\beta/2})^{\varkappa}\right]}{\cosh pt \sinh p(t-z)}\,dt\right). \tag{1.3.4}$$

Write the function $[x^2+4\beta^2]^{[\frac{n}{2}]}(2\beta-ix)^{\delta(n)}$ in form (1.3.3). Let us find solutions of the problems

$$X_1(x) = (2\beta+ix)X_1(x+i\beta), \quad -\infty < x < +\infty, \tag{1.3.5}$$
$$X_2(x+i\beta) = (2\beta-ix)X_2(x), \quad -\infty < x < +\infty. \tag{1.3.6}$$

Applying the Fourier transform to conditions (1.3.5) and (1.3.6), we obtain the differential equations

$$(f_1(t)e^{\beta t})' = (1-2\beta e^{\beta t})f_1(t), \quad -\infty < t < +\infty,$$
$$f_2'(t) = (2\beta - e^{-\beta t})f_2(t), \quad -\infty < t < +\infty,$$

where $f_1(t)$ and $f_2(t)$ denote the Fourier transforms of the functions $X_1(x)$ and $X_2(x)$.

By performing the reverse Fourier transformation of the solutions of these equations we obtain the solutions of problems (1.3.5) and (1.3.6):

$$X_1(z) = \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{\beta}e^{\beta t} + 3\beta z + itz\right) dt, \ \ 0 < \mathcal{I}_m z < \beta, \tag{1.3.7}$$

$$X_2(z) = \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{\beta}e^{-\beta t} - 2\beta t + itz\right) dt, \ \ 0 < \mathcal{I}_m z < \beta. \tag{1.3.8}$$

On substituting $e^{\beta t} = \beta\tau$, we have

$$\begin{aligned} X_1(z) &= \beta^2 \beta^{\frac{iz}{\beta}} \int_0^\infty e^{-\tau}\tau^{2+\frac{iz}{\beta}}\, d\tau = \beta^2\beta^{\frac{iz}{\beta}}\,\Gamma\left(3 + \frac{iz}{\beta}\right), \\ X_2(z) &= \beta^{-\frac{iz}{\beta}} \int_0^\infty e^{-\tau}\tau^{1-\frac{iz}{\beta}}\, d\tau = \beta\beta^{-\frac{iz}{\beta}}\,\Gamma\left(2 - \frac{iz}{\beta}\right). \end{aligned} \tag{1.3.9}$$

We introduce the notation

$$X_3(z) = \left[\frac{X_1(z)}{X_2(z)}\right]^{\left[\frac{n}{2}\right]} (X_2(z))^{-\delta(n)}, \quad 0 < \mathcal{I}_m z < \beta. \tag{1.3.10}$$

Using Stirling's formulas [**113**], we obtain from (1.3.9) and (1.3.10) the following representations of the functions $X_1(z)$ and $X_2(z)$ in the neighbourhood of a point at infinity:

$$|X_1(z)| = C_1(y)e^{-\frac{\pi}{2\beta}|x|}|x|^{\frac{5}{2}-\frac{y}{\beta}}\left(1 + O\left(\frac{1}{x}\right)\right), \quad 0 \le y \le \beta,$$

$$|X_2(z)| = C_2(y)e^{-\frac{\pi}{2\beta}|x|}|x|^{\frac{3}{2}+\frac{y}{\beta}}\left(1 + O\left(\frac{1}{x}\right)\right), \quad 0 \le y \le \beta,$$

where $C_1(y)$, $C_2(y)$ are the non-vanishing bounded functions.

By virtue of these formulas, for sufficiently large values of $|z|$ (1.3.10)) implies

$$|X_3(z)| = C(y)\big(|x|^{\frac{\beta-2y}{\beta}}\big)^{\left[\frac{n}{2}\right]}\big(e^{-\frac{\pi}{2\beta}|x|}|x|^{\frac{3}{2}+\frac{y}{\beta}}\big)^{-\delta(n)}\left(1 + O\left(\frac{1}{x}\right)\right). \tag{1.3.11}$$

By equalities (1.3.3) and (1.3.11), condition (1.3.2) can be rewritten as

$$\frac{\Phi(x)}{X(x)} - q\frac{\Phi(x+i\beta)}{X(x+i\beta)} = \frac{F(x)}{X(x)}, \quad -\infty < x < \infty, \tag{1.3.12}$$

where $X(z) = X_0(z)X_3(z)$.

The function $\Phi(z)/X(z)$ is holomorphic in the strip $0 < \mathcal{I}_m z < \beta$ except perhaps for the point $z = i\beta/2$, where for $\varkappa > 0$ it may have a pole of order not higher than $\varkappa$, and satisfies the condition

$$\big(\Phi(z)/X(z)\big)e^{-\mu|z|} \to 0 \quad \text{for} \quad |z| \to \infty, \quad \mu < \frac{\pi}{2\beta} + \varepsilon.$$

Write q in the form

$$q = \frac{X_4(x)}{X_4(x+i\beta)}\,, \quad X_4(z) = \exp\left(\frac{iz}{\beta}\ln q\right).$$

From (1.2.7) and (1.2.5) it follows that if q is not a real positive number, then a general solution of problem (1.3.1) is given by the formula

$$\Phi(z) = \frac{X(z)}{2i\beta}\int\limits_{-\infty}^{\infty}\frac{\exp\left(\frac{\pi-\delta+i\gamma}{\beta}(z-t)\right)}{X(t)\sinh p(t-z)}F(t)\,dt + X(z)\varphi(z), \tag{1.3.13}$$

where $\gamma = \ln|q|$, $\delta = \arg q$, $0 < \delta < 2\pi$.

$$\varphi(z) = \sum_{j=0}^{\varkappa-1} C_j\frac{d^j}{dz^j}\left(\exp\frac{(\pi-\delta+i\gamma)z}{\beta}/\cosh pz\right). \tag{1.3.14}$$

For $\varkappa \geq 0$, the solution of problem (1.3.1) is given by formulas (1.3.13) and (1.3.14). Note that for $\varkappa \leq 0$ it is assumed that $\varphi(z) \equiv 0$. For $\varkappa < 0$, the function $X(z)$ has, at the point $z = \frac{i\beta}{2}$, a pole of order $-\varkappa$ and therefore the bounded solution exists in the finite part of the strip only if the conditions $\varphi(z) = 0$;

$$\int\limits_{-\infty}^{\infty} F(t)\Psi_j(t)\,dt = 0, \quad \Psi_j(t) = \frac{d^j}{dt^j}\left(\frac{\exp(\frac{\delta-\pi-ij}{\beta})t}{\cosh pt}\right), \tag{1.3.15}$$
$$j = 0,\dots,(-1-\varkappa),$$

are fulfilled. Thus, like in Section 1.2, it can be easily proved that in the case of even n problem (1.3.1) has a solution $\Phi(z) \in A_0^\beta(0)$ for any $\delta \in (0,2\pi)$, while in the case of odd n it has a solution $\Phi(z) \in A_0^\beta\big(\frac{\pi-2\delta}{2\beta}+\varepsilon\big)$ for $\delta \in \big(0,\frac{\pi}{2}\big]$; $\Phi(z) \in A_0^\beta(0)$ for $\delta \in \big(\frac{\pi}{2}\,,\frac{3}{2}\pi\big)$; $\Phi(z) \in A_0^\beta\big(\frac{2\delta-3\pi}{2\beta}+\varepsilon\big)$ for $\delta \in \big[\frac{3}{2}\pi, 2\pi\big)$, where $\varepsilon > 0$ is an arbitrarily small number.

When $q > 0$, by substituting

$$\Phi(z) = X_4(x)\Psi(t)$$

condition (1.3.12) can be reduced to the form

$$\frac{\Psi(x)}{X(x)} - \frac{\Psi(x+i\beta)}{X(x+i\beta)} = \frac{F(x)X_4(x)}{X(x)}\,, \quad -\infty < x < \infty. \tag{1.3.16}$$

By virtue of formula (1.3.15) a general solution of problem (1.3.1) has the form

$$\Phi(z) = \frac{X^*(z)}{2i\beta}\int\limits_{-\infty}^{\infty}\frac{F(t)}{X^*(t)\sinh p(t-z)}\,dt + X^*(z)\varphi_2(z), \tag{1.3.17}$$

where $X^*(z) = X(z)\cosh pz X_4(z)$,

$$\varphi_2(z) = \begin{cases} \sum\limits_{j=0}^{\varkappa-1} C_j \dfrac{d^j}{dz^j}(\tanh pz) + Cx, & \text{for} \quad \varkappa > 0, \\ C, & \text{for} \quad \varkappa = 0, \\ 0, & \text{for} \quad \varkappa \le -1, \end{cases} \tag{1.3.18}$$

C, C_j, $j = 0, \dots, (\varkappa - 1)$, are arbitrary constants. If $\varkappa < -1$, then the solution exists only provided that the condition

$$\int\limits_{-\infty}^{\infty} \frac{F(t)}{X^*(z)} \cdot \frac{d^j}{dt^j}\left(\frac{1}{\cosh pt}\right) dt = 0, \quad j = 0, \dots, (-\varkappa - 2),$$

is fulfilled.

One can prove that $\Phi(z) \in A_0^\beta(\varepsilon)$ for even n and $\Phi(z) \in A_0^\beta(\pi/(2\beta)+\varepsilon)$ for odd n; here ε is a small positive integer.

Remark. Formulas (1.3.8) and (1.3.9) can be obtained by applying formulas (1.3.3) and (1.3.4).

Indeed, if in formula (1.3.4) $G_0(t)$ is replaced by the function $(2\beta - ix)^{-1}$, then we have

$$X_2(z) = \exp\left(\frac{\cosh pz}{2i\beta} \int\limits_{-\infty}^{\infty} \frac{\ln i - \ln(x + 2i\beta)}{\cosh px \sinh p(x - z)}\, dx\right). \tag{1.3.19}$$

Under the function $\ln z$ we understand $\ln z = \ln|z| + \arg z$, $-\pi < \arg z < \pi$. After rewriting $\ln(x + 2i\beta)$ as

$$\ln(x + 2i\beta) = \sum_{k=0}^{n}\Big[\ln(x + i\beta(k+2)) - \ln(x + i\beta(k+3))\Big] + \ln(x + i\beta(3+n))$$

and substituting this expression into (1.3.19), by virtue of (1.1.3) we obtain

$$\begin{aligned} \omega(z) &= \frac{\cosh pz}{2i\beta} \int\limits_{-\infty}^{\infty} \frac{\ln i - \ln(x + 2i\beta)}{\cosh px \sinh p(x - z)}\, dx \\ &= \sum_{k=0}^{n}\left[\ln(x + i\beta(k+2)) - \ln\left(\frac{5i\beta}{2} + ki\beta\right)\right] \\ &\qquad + \frac{\cosh pz}{2i\beta} \int\limits_{-\infty}^{\infty} \frac{\ln(1+n)\beta)}{\cosh pt \sinh p(t - z)}\, dt + O\left(\frac{1}{n}\right). \end{aligned}$$

If we perform some simple transformations and calculate the latter integral by the formula

$$\frac{\cosh pt}{2i\beta} \int\limits_{-\infty}^{\infty} \frac{\ln[(n+1)\beta]}{\cosh px \sinh p(x - z)}\, dx = \ln[(1+n)\beta]\left(\frac{iz}{\beta} + \frac{1}{2}\right),$$

then we have

$$\omega(z)=\sum_{k=1}^{n}\ln\left[\left(1+\frac{\zeta}{k}\right)e^{-\frac{\zeta}{k}}\right]-\zeta\bigg(\ln(n+1)-\sum_{k=1}^{n}\frac{1}{k}\bigg)-\ln\beta^{\zeta}$$
$$-\frac{5}{2}\bigg(\ln(n+1)-\sum_{k=1}^{n}\frac{1}{k}\bigg)+\ln\zeta+C_n,\quad \zeta=\frac{z+2i\beta}{i\beta}.$$

Passing to the limit as $n\to+\infty$, by virtue of (1.3.19) we obtain

$$X_2(z)=A\zeta\prod_{1}^{\infty}\left(1+\frac{\zeta}{k}\right)e^{-\frac{\zeta}{k}}e^{-c\zeta}\beta^{\zeta}=A\Gamma\left(2-\frac{iz}{\beta}\right)\beta^{2-\frac{iz}{\beta}}.$$

1.4. On a Conjugation Boundary Value Problem with Displacements

As an application of the results obtained in Section 1.2, we will consider one kind of a conjugation problem with displacements, when the boundary is a real axis. Denote by S^+ and S^- the upper and the lower half-plane, respectively.

Let us consider the following problem:

Find a piecewise-holomorphic function bounded all over the plane using the boundary condition

$$\Phi^+(x)=G(x)\Phi^-[\alpha(x)]+f(x),\quad -\infty<x<+\infty, \tag{1.4.1}$$

where $G(x)$ and $f(x)$ are the given functions satisfying the Hölder condition, $G(x)\neq 0$, $G(\infty)=G(-\infty)=1$, $f(+\infty)=f(-\infty)=0$,

$$\alpha(x)=\begin{cases}x, & x<0,\\ bx, & x\geq 0,\end{cases}$$

b is a constant.

If we denote by $\varkappa$ the index of the function $G(x)$, then $G(x)$ can be represented as [**76**]

$$G(x)=\frac{X^+(x)}{X^-(x)}\,,\quad X(z)=\begin{cases}\exp\omega(z), & z\in S^+,\\ \left(\dfrac{z+i}{z-i}\right)^{\varkappa}\exp\omega(z), & z\in S^-,\end{cases} \tag{1.4.2}$$

$$\omega(z)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln G_0(t)}{t-z}\,dt,\quad G_0(x)=G(x)\left(\frac{x+i}{x-i}\right)^{\varkappa}.$$

On putting the value of $G(x)$ into (1.4.1), we obtain

$$\frac{\Phi^+(x)}{X^+(x)}-\frac{\Phi^-(\alpha(x))}{X^-(x)}=\frac{f(x)}{X^+(x)}\,,\quad -\infty<x<+\infty. \tag{1.4.3}$$

For $x < 0$, condition (1.4.3) takes the form

$$\frac{\Phi^+(x)}{X^+(x)} - \frac{\Phi^-(x)}{X^-(x)} = \frac{f(x)}{X^+(x)}\,. \tag{1.4.4}$$

A general solution of problem (1.4.4) can be written as

$$\Phi(z) = \frac{X(z)}{2\pi i}\int\limits_{-\infty}^{0} \frac{f(t)}{X^+(t)(t-z)}\,dt + X(z)\Phi_0(z). \tag{1.4.5}$$

The function $\Phi(z)$ is holomorphic on the plane cut along the positive semi-axis except perhaps for the neighborhood of the point $z = -i$ at which it has a pole of order $\varkappa$ for $\varkappa > 0$.

For $\varkappa < 0$ the function $X(z)$ has a pole of order $-\varkappa$ at the point $z = -i$. Therefore for a bounded solution to exist it is necessary that the condition

$$\Phi_0^{(k)}(-i) + \frac{k!}{2\pi i}\int\limits_{-\infty}^{0} \frac{f(t)}{X^+(t)(t+i)^{k+1}}\,dt = 0, \quad k = 0,1,\ldots,(-\varkappa-1), \tag{1.4.6}$$

be fulfilled.

If we put the value of $\Phi(z)$ into (1.4.3), then we have

$$\Phi_0^+(x) = G_1(x)\Phi_0^-(bx) + f_0(x), \quad 0 < x < \infty, \tag{1.4.7}$$

where $G_1(x) = \frac{X^-(bx)}{X^-(x)}$, $f_0(x) = \frac{f(x)}{X^+(x)} - A^+(x) + G_1(x)A^-(bx)$,

$$A(z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{0} \frac{f(t)}{X^+(t)(t-z)}\,dt.$$

The function $z = e^{\zeta}$, $\zeta = \xi + i\eta$, maps **the** strip $0 < \eta < 2\pi$ onto the plane having a cut along the axis $x > 0$.

On introducing the notation $\Phi_0(e^{\zeta}) = \Psi_0(\zeta)$, $0 < \eta < 2\pi$, we obtain

$$\Phi_0^+(x) = \Psi_0(\xi), \quad \Phi_0^-(bx) = \Psi_0(\xi + \ln b + 2\pi i), \quad -\infty < \xi < +\infty. \tag{1.4.8}$$

Thus problem (1.4.7) is reduced to the problem considered in Chapter 1, Section 2

$$\Psi_0(\xi) = G^+(\xi)\Psi_0(\xi + \ln b + 2\pi i) + F_0(\xi), \quad -\infty < \xi < +\infty, \tag{1.4.9}$$

where $G^+(\xi) = G_1(e^{\xi})$, $F_0(\xi) = f_0(e^{\xi})$, $G^*(-\infty) = G^*(\infty) = 1$,

$$\mathcal{J}_n dG^* = 0, \quad F_0(+\infty) = 0, \quad F_0(-\infty) = \frac{f(0)}{X^+(0)}.$$

Since for $\varkappa > 0$ the function $\Phi_0(z)$ can have a pole of order $\varkappa$ at the point $z = -i$, we seek for a solution Ψ_0 of problem (1.4.9) in the class of functions satisfying the condition

$$\Psi_0(\zeta)\left(\frac{\zeta - \frac{3}{2}\pi i}{\zeta + \frac{3}{2}\pi i}\right)^{\varkappa} \in A_0^{\beta}(\mu), \quad \mu < \frac{4\pi^2}{4\pi^2 + \ln b}. \tag{1.4.10}$$

By virtue of formula (1.2.6) it is easy to show that a general solution of problem (1.4.9) is given by the formula

$$\Psi_0(\zeta)=\frac{X^*(\zeta)\cosh p\zeta}{2a}\int\limits_{-\infty}^{+\infty}\frac{F_0(t)}{X^+(t)\cosh pt\sinh p(t-\zeta)}\,dt + X^*(\zeta)\Psi(\zeta), \tag{1.4.11}$$

where $a=\ln b+2\pi i$, $p=\frac{\pi i}{a}$,

$$\psi(\zeta)=\begin{cases}\sum\limits_{k=0}^{\varkappa}c_k\coth^k p\left(\zeta-\dfrac{3}{2}\pi i\right), & \varkappa\geq 0,\\ c_{-1}, & \varkappa=-1,\\ 0, & \varkappa<-1,\end{cases}$$

$$X^*(\zeta)=\exp\left(\frac{\cosh p\zeta}{2a}\int\limits_{-\infty}^{+\infty}\frac{\ln G^*(t)}{\cosh pt\sinh p(t-\zeta)}\,dt\right).$$

Returning to the variable z, we obtain

$$\Psi_0(\zeta)=\frac{X_0(z)}{a}\int\limits_0^{+\infty}\frac{t^{2p-1}f_0(t)}{(t^{2p}-z^{2p})X_0^+(t)}\,dt+X_0(z)(\varphi_0(z)-A), \tag{1.4.12}$$

$$X_0(z)=\exp\left(\frac{1}{a}\int\limits_0^{\infty}\frac{\ln G_1(t)t^{2p-1}}{t^{2p}-z^{2p}}\,dt\right),\quad A=\frac{1}{a}\int\limits_0^{\infty}\frac{t^{2p-1}f_0(t)}{(t^{2p}+1)X_0^*(t)}\,dt.$$

With (1.4.5) and (1.4.12) taken into account we conclude that a general solution of problem (1.4.1) has the form

$$\Phi(z)=X(z)\Bigg[\frac{1}{2\pi i}\int\limits_{-\infty}^{0}\frac{f(t)}{X^+(t)(t-z)}\,dt + \frac{X_0(z)}{a}\int\limits_0^{\infty}\frac{t^{2p-1}f_0(t)}{X_0^+(t)(t^{2p}-z^{2p})}\,dt+X_0(z)(\varphi_0(z)-A)\Bigg], \tag{1.4.13}$$

$$\varphi_0(z)=\begin{cases}\sum\limits_{k=0}^{\varkappa}c_k\left(\dfrac{z^{2p}+(-i)^{2p}}{z^{2p}-(-i)^{2p}}\right)^k, & \varkappa\geq 0,\\ c_{-1}, & \varkappa=-1,\\ 0, & \varkappa<-1.\end{cases} \tag{1.4.14}$$

The function z^{2p} is holomorphic on the plane cut along the positive axis if under this function we mean the branch for which, assuming that $z\to 1$, the limit from the upper half-plane is equal to 1, and t^{2p} denotes the function value of the upper edge of the cut at the point t.

For $\varkappa = -1$, the function $X_0(z)$ has a pole of first order at the point $z = -i$. In that case $\varphi_0(z) = C_{-1}$ and $X_0(-i) \neq 0$ and therefore the constant c_1 can be chosen so that for $z = -i$ the expression enclosed in the square brackets on the right-hand side of (1.4.13) will vanish. Hence when $\varkappa \geq -1$, problem (1.4.1) has a bounded solution for an arbitrary right-hand side. When $\varkappa < -1$, for a bounded solution to exist it is necessary and sufficient that the conditions

$$\frac{d^k}{dz^k}\left[\frac{1}{2\pi i}\int\limits_{-\infty}^{0}\frac{f(t)}{X^+(t)(t-z)}\,dt+\frac{X_0(z)}{a}\int\limits_{0}^{\infty}\frac{t^{2p-1}f_0(t)}{X_0^+(t)(t^{2p}-z^{2p})}\,dt-AX_0(z)\right]=0,$$
$$z=-i,\quad k=1,\dots,-\varkappa,$$

be fulfilled. Then the solution is given by formula (1.4.13).

For $b = 1$ we have $p = \frac{1}{2}$, $X_0(z) \equiv 1$, $f_0(t) \equiv f(t)$ and formulas (1.4.13) and (1.4.14) give a solution of the conjugation problem.

Conjugation problems with displacements are investigated in [**62**], [**63**], [**69**] in the case with $\alpha'(t)$ belonging to the Hölder class.

1.5. A Carleman Type Problem with Continuous Coefficients for the Circular Ring

In this paragraph we consider the following boundary value problem of the analytic function theory.

Find a function $\varphi(z)$, analytic in the ring $D = \{1 < |z| < R\}$ and continuous in the closed ring $\overline{D}$, by the boundary condition

$$\varphi(at) = G(t)\varphi(t) + f(t), \quad t \in \gamma, \tag{1.5.1}$$

where γ denotes the circumference of unit radius and center at the origin, $a = Re^{i\alpha}$, $G(t)$ and $f(t)$ belong to the class H on γ, $G(t) \neq 0$ everywhere on γ. We will call $G(t)$ a coefficient, and $f(t)$ a free term of problem (1.5.1).

The integer number $\varkappa = \frac{1}{2\pi}\left[\arg G(t)\right]_\gamma$ called the index of the function $G(t)$ will be called in our case the index of problem (1.5.1).

As we will see in the sequel, the solution of problem (1.5.1) is reduced to the solution of the problem with constant coefficients

$$\varphi_0(at) = g\varphi_0(t) + f_0(t), \quad t \in \gamma. \tag{1.5.2}$$

After multiplying equality (1.5.2) by $t^{-(n+1)}$, integrating over γ and applying the Cauchy theorem, we obtain

$$(a^n - g)\int\limits_\gamma \varphi(t)t^{-(n+1)}\,dt = \int\limits_\gamma f(t)t^{-(n+1)}\,dt, \quad n = 0, \pm 1, \pm 2, \dots\ .$$

Hence it follows that if $a^n - g \neq 0$, $n = 0, \pm 1, \pm 2, \dots$, then problem (1.5.2) has a unique solution given by the formula

$$\varphi(z) = \sum_{n=-\infty}^{\infty}\frac{f_n}{a^n - g}\,z^n, \quad f_n = \int\limits_\gamma f(t)e^{-(n+1)}\,dt. \tag{1.5.3}$$

If $a^m = g$ for some integer number m, then for problem (1.5.2) to be solvable it is necessary and sufficient that the condition

$$\int\limits_{\gamma} f(t)t^{-(m+1)}\,dt = 0 \tag{1.5.4}$$

be fulfilled.

When this condition is fulfilled, we have

$$\varphi(z) = \sum_{m\neq n=-\infty}^{\infty} \frac{f_n}{a^n - g}\, z^n + Cz^m, \tag{1.5.5}$$

where C is an arbitrary constant.

To continue our investigation of the problem it is convenient to rewrite formulas (1.5.3) and (1.5.5) in the form

$$\varphi(z) = \frac{1}{2\pi i}\int\limits_{\gamma} K_g\left(\frac{z}{t}\right)\frac{f(t)}{t}\,dt, \quad a^n - g \neq 0; \tag{1.5.6}$$

$$\varphi(z) = \frac{1}{2\pi i}\int\limits_{\gamma} K_g^*\left(\frac{z}{t}\right)\frac{f(t)}{t}\,dt + Cz^m, \quad a^m = g, \tag{1.5.7}$$

where

$$K_g(z) = \frac{-a}{z-a} + \frac{1}{g(1-z)} + g\sum_{n=0}^{\infty}\frac{1}{a^n - g}\left(\frac{z}{a}\right)^n + \frac{1}{g}\sum_{n=-\infty}^{-1}\frac{a^n z^n}{a^n - g}\,. \tag{1.5.8}$$

and $K_g^*(z)$ is obtained from (1.5.8) by removing the term the denominator of which vanishes, i.e. the term with denominator $n = m$.

By virtue of the Plemelj–Privalov formula, formulas (1.5.6), (1.5.7), (1.5.8) imply that a solution of problem (1.5.1) belongs to the class H.

Let us now prove

Proposition. *The number of linearly independent solutions of the boundary value problem*

$$\varphi(at) = gt^k\varphi(t), \quad k > 0, \tag{1.5.9}$$

is equal to k (k is a natural number).

Proof. Multiplying both parts of equality (1.5.9) by t^n, integrating and applying the Cauchy theorem, we obtain

$$C_{n+k} = \frac{C_n}{ga^{n+k}}\,, \tag{1.5.10}$$

where

$$C_j = \int\limits_{\gamma}\varphi(t)t^j\,dt, \quad j = 0, \pm 1, \pm 2, \dots\ .$$

In formula (1.5.10), $C_0; C_1, \dots, C_k; \dots$ can be chosen arbitrarily constant and used for other definitions. Hence problem (1.5.9) has k linearly independent solutions.

For $k < 0$, if system (1.5.10) has a nontrivial solution, then $|C_n| \to \infty$ as $n \to \infty$, but this contradicts the property of Fourier coefficients. Therefore for $k<0$, $\varphi(z)=0$. $\square$

A more general proposition is true: *If $IndG(t) = \varkappa > 0$, then the homogeneous problem*

$$\varphi(at) = G(t)\varphi(t) \tag{1.5.11}$$

has exactly $\varkappa$ linearly independent solutions.

Let us first show that there exist a function $X_1(z)$, which is analytic in the ring D and is everywhere different from zero, and a number μ_1 such that

$$G(t) = \mu_1 t^{\varkappa} \frac{X_1(at)}{X_1(t)}, \quad t \in \gamma. \tag{1.5.12}$$

Indeed, from condition (1.5.12) we have

$$\ln X_1(at) - \ln X_1(t) = \ln[\bar{t}^{\,-\varkappa} G(t)] - \ln \mu_1.$$

This is a boundary value problem of form (1.5.1). From the solvability condition we define the value for μ_1,

$$\mu_1 = \exp\left(\frac{1}{2\pi i}\int\limits_{\gamma} \ln\left[G(t)t^{-\varkappa}\right]\frac{dt}{t}\right),$$

while, by virtue of (1.5.7), a solution has the form

$$X_1(z) = \exp\left(\frac{1}{2\pi i}\int\limits_{\gamma} K_1^*\left(\frac{z}{t}\right)\ln\left[t^{-\varkappa}G(t)\right]\frac{dt}{t}\right).$$

Note that the fulfillment of the condition $\varkappa > 0$ has not been assumed here.

Now, with the aid of representation (1.5.12), the boundary value problem (1.5.11) is reduced to a problem of form (1.5.9), whence the validity of the above proposition follows.

We will call the boundary value problems

$$\varphi(at) = G(t)\varphi(t) + f(t), \quad t \in \gamma, \tag{1.5.13}$$

and

$$\varphi(\overline{a}\,t) = \overline{G(t)}\,\psi(t) + g(t), \quad t \in \gamma, \tag{1.5.14}$$

adjoint.

Let us prove the following lemma.

LEMMA. *If the functions $\varphi(z)$ and $\psi(z)$ are holomorphic in $D = \{1 < |z| < R\}$ and continuous in $\overline{D}$, then the equality*

$$\int\limits_{\gamma} \varphi(at)\overline{\psi(t)}\,\frac{dt}{t} = \int\limits_{\gamma} \overline{\psi(\overline{a}t)}\varphi(t)\,\frac{dt}{t}$$

is valid.

Proof. Expand the functions $\varphi(z)$ and $\psi(z)$ into Laurent series

$$\varphi(z)=\sum_{n=-\infty}^{\infty}\varphi_n z^n,\quad \psi(z)=\sum_{n=-\infty}^{\infty}\psi_n z^n,\quad 1<|z|<|a|.$$

Hence we obtain

$$\varphi\left(\frac{a}{p}\,t\right)=\sum_{n=-\infty}^{\infty}\varphi_n\left(\frac{a}{p}\right)^n t^n,\quad \psi(pt)=\sum_{m=-\infty}^{\infty}\psi_m p^m t^m,\quad 1<p<|a|.$$

Since these series converge modulo, the following equality is valid:

$$\varphi\left(\frac{a}{p}\,t\right)\cdot\overline{\psi(pt)}=\sum_{n=-\infty}^{\infty}t^n\sum_{m=-\infty}^{\infty}\varphi_{n-m}a^{n-m}p^{m-n}\overline{\psi}_{-m}p^{-m}\,.$$

This series converges modulo because so do the series

$$\sum_{n=-\infty}^{\infty}|a|^n|\varphi_n|p^{-n},\qquad \sum_{m=-\infty}^{\infty}|\overline{\psi}_m|p^m.$$

The integration of the preceding equality gives

$$\int\limits_{\gamma}\varphi\left(\frac{a}{p}\,t\right)\overline{\psi(pt)}\,\frac{dt}{t}=2\pi i\sum_{n=-\infty}^{\infty}\varphi_n\overline{\psi}_n a^n\,.$$

Analogously, we obtain

$$\int\limits_{\gamma}\varphi(pt)\overline{\psi\left(\frac{\overline{a}}{p}\,t\right)}\,\frac{dt}{t}=2\pi i\sum_{n=-\infty}^{\infty}\varphi_n\overline{\psi}_n a^n\,.$$

Thus the lemma is true. □

After multiplying equality (1.5.1) by $\overline{\psi(t)t}$ and integrating, by virtue of the above lemma we obtain

$$\int\limits_{\gamma}\varphi(t)\left[\overline{\psi(\overline{a}t)}-G(t)\,\overline{\psi(t)}\right]\frac{dt}{t}=\int\limits_{\gamma}f(t)\overline{\psi(t)}\,\frac{dt}{t}\,.$$

If $\psi(z)$ is a solution of the homogeneous problem corresponding to problem (1.5.14), then the integrand in the left-hand part of the last equality is equal to zero. Hence the necessary condition for problem (1.5.1) to be solvable is the equality

$$\int\limits_{\gamma}f(t)\overline{\psi(t)}\,\frac{dt}{t}=0. \tag{1.5.15}$$

In an analogous manner we establish that the necessary condition for problem (1.5.14) to be solvable is the equality

$$\int\limits_{\gamma}g(t)\overline{\varphi(t)}\,\frac{dt}{t}=0, \tag{1.5.16}$$

where $\varphi(z)$ is a solution of the homogeneous problem corresponding to problem (1.5.1).

Let z_0 be an arbitrary fixed point of the ring D, then it is obvious that the index of the function

$$G_0(t) = G(t)\left(\frac{t-z_0}{at-z_0}\right)^{\varkappa}$$

is equal to zero.

As has been shown above, in (1.5.12) the function $G_0(t)$ can be represented as

$$G_0(t) = \mu\,\frac{X_0(t)}{X_0(at)}\,, \quad t \in \gamma, \tag{1.5.17}$$

where

$$X_0(z) = \exp\left(\frac{1}{2\pi i}\int\limits_{\gamma} K_1^*\left(\frac{z}{t}\right)\ln\frac{G_0(t)}{\mu}\,\frac{dt}{t}\right), \quad z \in D, \tag{1.5.18}$$

$$\mu = \exp\left(\frac{1}{2\pi i}\int\limits_{\gamma} \ln G_0(t)\,\frac{dt}{t}\right). \tag{1.5.19}$$

Therefore the function $G(t)$ can be written in the form

$$\begin{aligned} G(t) &= \mu\,\frac{X(at)}{X(t)}\,, \\ X(z) &= (z-z_0)^{\varkappa} X_0(z), \quad z \in D. \end{aligned} \tag{1.5.20}$$

We will call the function $X(z)$ a canonical function of problem (1.5.1). It can be easily shown that it depends on a choice of the point z_0. Thus problem (1.5.1) has an uncountable number of canonical functions.

Inserting the value of $G(t)$ defined by (1.5.20) into condition (1.5.1), we obtain

$$\frac{\varphi(at)}{X(at)} - \mu\,\frac{\varphi(t)}{X(t)} = \frac{f(t)}{X(at)}\,, \quad t \in \gamma. \tag{1.5.21}$$

For $\varkappa \le 0$, the function $\varphi(z)/X(z)$ is holomorphic in the ring D, while for $\varkappa > 0$ it may have, at the point z_0, a pole of order not higher than $\varkappa$.

Thus problem (1.5.1) reduces to problem (1.5.21) with constant coefficients, whose particular solutions are given by formulas (1.5.6) or (1.5.7) depending on the fact whether the equation $a^n - \mu = 0$ has or does not have an integer solution. A general solution of the problem is obtained by adding a general solution of the corresponding homogeneous problem.

Let us now prove that if $\varkappa \ne 0$, then the point z_0 can be chosen so that $a^n - \mu \ne 0$ for any integer n.

Indeed, assume that z_1 and z_2 are arbitrary points of D, μ_1 and μ_2 are their corresponding numbers defined by formula (1.5.19)

$$\mu_k = \exp\left(\frac{1}{2\pi i}\int\limits_{\gamma} \ln\left[G(t)\left(\frac{t-z_k}{at-z_k}\right)^{\varkappa}\right]\frac{dt}{t}\right), \quad k = 1, 2.$$

Hence

$$\frac{\mu_1}{\mu_2} = \exp\left[\frac{1}{2\pi i}\int\limits_\gamma \ln\left(\frac{t-z_1}{t-z_2}\right)^{\varkappa}\frac{dt}{t} + \frac{1}{2\pi i}\int\limits_\gamma \ln\left(\frac{at-z_2}{at-z_1}\right)^{\varkappa}\frac{dt}{t}\right].$$

Since the function $\ln\left(\frac{at-z_2}{at-z_1}\right)^{\varkappa}$ is holomorphic outside the circumference γ and vanishes at infinity, by the Cauchy formula

$$\frac{1}{2\pi i}\int\limits_\gamma \ln\left(\frac{at-z_2}{at-z_1}\right)^{\varkappa}\frac{dt}{t} = 0.$$

The function $\ln\left(\frac{t-z_1}{t-z_2}\right)^{\varkappa}$ is holomorphic inside the circumference γ and therefore by the Cauchy formula

$$\frac{1}{2\pi i}\int\limits_\gamma \ln\left(\frac{t-z_1}{t-z_2}\right)^{\varkappa}\frac{dt}{t} = \ln\left(\frac{z_1}{z_2}\right)^{\varkappa}.$$

Assume that $z_1 = \rho e^{i\alpha_1}$, $z_2 = \rho e^{i\alpha_2}$, $1 < \rho < |a|$. Then we have

$$\frac{\mu_1}{\mu_2} = e^{i(\alpha_1-\alpha_2)}.$$

If α_1 and α_2 are chosen so that $(\alpha_1-\alpha_2)\varkappa \neq 2\pi$, then we obtain $\mu_1 \neq \mu_2$ and $\mu_1 = \mu_2$. In view of the fact that the equation $a^n - \mu = 0$ may have only one integer solution, if it turns out that $a^m - \mu_1 = 0$ for some integer m, then $a^n - \mu_2 \neq 0$ for any integer number n. We have thus shown that for $\varkappa \neq 0$ the point z_0 can be chosen so that $a^n - \mu \neq 0$ for any integer n. In the sequel, it will be assumed that the latter condition is fulfilled for $\varkappa \neq 0$.

To construct solutions of the homogeneous problem corresponding to problem (1.5.21) which are of order x at the point z_0, we have to consider the adjoint problem (1.5.14).

Rewrite the condition of problem (1.5.14) in the form

$$\psi(\overline{a}t) = \left(\frac{\overline{z}_0 t - \overline{a}}{z_0 t - 1}\right)^{\varkappa}\overline{G_1(t)}\,\psi(t) + g(t). \tag{1.5.22}$$

Since the index of the coefficient $G_1(t)$ is equal to zero, it can be represented as

$$\overline{G_1(t)} = \overline{\mu}\,\frac{X_0'(\overline{a}t)}{X_0'(t)}, \tag{1.5.23}$$

where

$$X_0'(z) = \exp\left(\frac{1}{2\pi i}\,K_1'\left(\frac{z}{t}\right)\ln\left[\overline{G_1(t)\mu^{-1}}\right]\frac{dt}{t}\right). \tag{1.5.24}$$

The function $K_1'(z)$ is obtained from $K_1^*(z)$ if we replace a by $\overline{a}$. By means of (1.5.23) condition (1.5.22) can be rewritten in the form

$$\frac{\psi(\overline{a}t)}{X'(\overline{a}t)} - \overline{\mu}\,\frac{\psi(t)}{X'(t)} = \frac{g(t)}{X'(\overline{a}t)}, \quad t \in \gamma, \tag{1.5.25}$$

where

$$X'(z) = X_0'(z)\,(\overline{z}_0 z - \overline{a})^{-\varkappa}\, z^{\varkappa}, \quad z \in D. \tag{1.5.26}$$

Equalities (1.5.17), (1.5.20), (1.5.24) and (1.5.26) yield

$$\overline{X_1'\left(\frac{1}{\overline{z}}\right)} = \frac{1}{X(az)}\,, \quad \frac{1}{|a|} < |z| < 1. \tag{1.5.27}$$

For $\varkappa \geq 0$, a solution of problem (1.5.25) is given by the formula

$$\psi(z) = \frac{X'(z)}{2\pi i} \int\limits_{\gamma} K_{\overline{\mu}}'\left(\frac{z}{t}\right) \frac{g(t)}{X'(\overline{a}t)}\,\frac{dt}{t}\,,$$

but since the function $X'(z)$ has a pole of order $\varkappa$ at the point $z = \overline{a}/\overline{z}_0$, to obtain a bounded solution of problem (1.5.14) it is necessary and sufficient that the conditions

$$\int\limits_{\gamma} \frac{d^j}{dz^j}\, K_{\overline{\mu}}'\left(\frac{z}{t}\right) \frac{g(t)}{X'(\overline{a}t)}\,\frac{dt}{t} = 0, \quad j = 0, 1, \ldots, \varkappa - 1, \;\; z = \overline{a}/\overline{z}_0,$$

be fulfilled, which by virtue of equality (1.5.27) can be written in the form

$$\int\limits_{\gamma} \overline{\varphi_j(t)}\,\overline{X(t)} g(t)\,\frac{dt}{t} = 0, \quad j = 0, 1, \ldots, \varkappa - 1, \tag{1.5.28}$$

where

$$\overline{\varphi_j(t)} = \frac{d^j}{dt^j}\, K_{\overline{\mu}}'\left(\frac{z}{t}\right), \quad z = \overline{a}/\overline{z}_0, \quad j = 0, 1, \ldots, \varkappa - 1.$$

Condition (1.5.16) is the necessary one, while condition (1.5.28) is the necessary and sufficient one for problem (1.5.14) to be solvable. Hence it can be expected that the functions $X(z)\varphi_j(z)$ are solutions of the homogeneous problem corresponding to problem (1.5.1).

If this is so, then $\varphi_j(z)$ must be solutions of the equation

$$\varphi(at) - \mu\varphi(t) = 0, \quad t \in \gamma, \tag{1.5.29}$$

having poles of order $j+1$ at the point $z = z_0$.

It is easy to verify that

$$\overline{K_{\overline{\mu}}'\left(\frac{\overline{a}}{\overline{z}_0 t}\right)} = -\varphi(z_0; t),$$

where

$$\begin{aligned}\varphi(\lambda; z) = {} & \frac{z}{z-\lambda} + \frac{\mu z}{z - a\lambda} + \sum_{n=0}^{\infty} \frac{\lambda^n}{z^n(a^n\mu - 1)} \\ & + \mu^2 \sum_{n=-\infty}^{-1} \frac{a^{2n}\lambda^n}{z^n(a^n\mu - 1)}\end{aligned} \tag{1.5.30}$$

is such a solution of problem (1.5.29) that has a pole of first order at the point $x = \lambda$.

Thus a general solution of problem (1.5.29) having, at the point $z = z_0$, a pole of order not higher than $\varkappa$ is given by the formula

$$\varphi_\varkappa(z) = \sum_{j=0}^{\varkappa-1} C_j \left. \frac{d^j \varphi(\lambda; z)}{d\lambda^j} \right|_{\lambda=z_0}, \tag{1.5.31}$$

where C_j are arbitrary complex constants. When $\varkappa > 0$, the homogeneous problem corresponding to problem (1.5.1) has $\varkappa$ linearly independent solutions

$$\varphi_j(z) = X(z) \left. \frac{d^j \varphi(\lambda; z)}{d\lambda^j} \right|_{\lambda=z_0'}, \quad j = 0, 1, \dots, \varkappa - 1.$$

Therefore, by virtue of (1.5.6) and (1.5.21), for $\varkappa > 0$ a general solution of problem (1.5.2) is given by the formula

$$\varphi(z) = \frac{X(z)}{2\pi i} \int\limits_\gamma K_\mu \left(\frac{z}{t}\right) \frac{f(t)}{X(at)} \frac{dt}{t} + \varphi_\varkappa(z) X(z), \quad z \in D. \tag{1.5.32}$$

For $\varkappa < 0$, a solution of problem (1.5.1) exists if and only if the conditions

$$\int\limits_\gamma \frac{d^j}{dz^j} \frac{K_\mu\left(\frac{z}{t}\right)}{X(at)} \frac{f(t)}{t} \, dt = 0, \quad j = 0, 1, \dots, -\varkappa - 1. \tag{1.5.33}$$

are fulfilled. In that case, a solution of problem (1.5.1) is given by formula (1.5.32) where we should put $\varphi_\varkappa(z) \equiv 0$.

For $\varkappa = 0$ and $a^n - \mu \neq 0$, $n = 0, \pm 1, \pm 2, \dots$, a solution of problem (1.5.1) is given by formula (1.5.32) where we should put $\varphi_\varkappa(z) \equiv 0$.

For $\varkappa = 0$ and $a^m = \mu$, problem (1.5.2) is solvable if and only if the condition

$$\int\limits_\gamma \frac{f(t)}{X(at) t^{m+1}} \, dt = 0 \tag{1.5.34}$$

is fulfilled.

In that case, by virtue of (1.5.7) and (1.5.21) a solution of the problem has the form

$$\varphi(z) = \frac{X(z)}{2\pi i} \int\limits_\gamma K_\mu^* \left(\frac{z}{t}\right) \frac{f(t)}{X(at) t} \, dt + C X(z) z^m, \tag{1.5.35}$$

while we have the function

$$\psi_0(z) = C X'(z) z^m \tag{1.5.36}$$

as a solution of the adjoint homogeneous problem and condition (1.5.34) for problem (1.5.1) takes the form

$$\int\limits_\gamma f(t) \, \overline{\psi_0(t)} \, \frac{dt}{t} = 0. \tag{1.5.37}$$

For $\varkappa < 0$, the adjoint homogeneous problem has $-\varkappa$ linearly independent solutions

$$\psi_{j+1}(z) = \frac{d^j}{dz^j}\,\psi(\lambda; z)X'(z), \quad \lambda = \frac{\overline{a}}{\overline{z}_0}\,, \tag{1.5.38}$$

where $\psi(\lambda; z)$ is given in the form of series (1.5.30) where the numbers a and μ should be replaced by their complex-conjugate values.

It is not difficult either to show that

$$\overline{\psi_{j+1}(t)} = \frac{d^j K_\mu\left(\frac{z}{t}\right)}{dz^j}\bigg|_{z=z_0} \cdot \frac{1}{X(at)}\,. \tag{1.5.39}$$

Taking the last equality into account, the solvability conditions and (1.5.33) take the form

$$\int\limits_\gamma \overline{\psi_j(t)}\,f(t)\,\frac{dt}{t} = 0, \quad j = 1, 2, \dots, -\varkappa. \tag{1.5.40}$$

Thus, for problem (1.5.1) the following Noether type theorems are valid:

1. If $\varkappa > 0$ or $\varkappa = 0$ and $a^m - \mu \neq 0$, then the adjoint homogeneous problem has no solution different from zero, and problem (1.5.1) is always solvable.

2. If $\varkappa < 0$ or $\varkappa = 0$ and $a^n - \mu = 0$, then the adjoint homogeneous problem has a nontrivial solution of form (1.5.38) and (1.5.36), and for problem (1.5.1) to be solvable it is necessary and sufficient that conditions (1.5.40) and (1.5.37) be fulfilled.

1.6. A Carleman Type Problem with Discontinuous Coefficients

Let us consider the following Carleman type problem

$$\varphi(at) = G(t)\varphi(t) + f(t), \quad t \in \gamma, \tag{1.6.1}$$

when the functions $G(t)$ and $f(t)$ have discontinuities of first kind at a finite number of points of the boundary, and satisfy the Hölder condition on each closed arc whose ends are discontinuity points. It is assumed that $G(t) \neq 0$ everywhere on γ. Of the unknown function $\varphi(z)$ it is required that it be continuously extendable on the boundary of the domain D except for the discontinuity points c of the functions $G(t)$ and $f(t)$ and their corresponding points on the boundary of the ring D near which $\varphi(z)$ must satisfy the condition

$$|\varphi(z)| < const\,|z - c|^{-\alpha}, \quad |\varphi(z)| < const\,|z - ac|^{-\alpha}, \quad 0 \le \alpha < 1.$$

Following [**76**], the discontinuity points, at which the condition

$$\arg G(c - 0) = \arg G(c + 0)$$

is fulfilled, are called singular and all other points nonsingular.

Take a point $t_1 \in \gamma$ at which $G(t)$ is continuous and choose any value $\ln G(t_1 + 0)$. Moving t away from t_1 in the positive direction, we can change the function $\ln G(t)$ continuously until t reaches the first nonsingular point c.

Having reached this point, $\arg G(t)$ obtains the well-defined value $\arg G(c-0)$.

When passing through the point c, we choose the value $\arg G(c+0)$ so that one of the conditions

$$0 < \frac{1}{2\pi}\Big(\arg G(c-0) - \arg G(c+0)\Big) < 1 \tag{1.6.2}$$

or

$$-1 < \frac{1}{2\pi}\Big(\arg G(c-0) - \arg G(c+0)\Big) < 0 \tag{1.6.3}$$

is fulfilled.

Continuing the movement of the point t in the positive direction on γ and choosing the value $\arg G(t)$ so that one of conditions (1.6.2) or (1.6.3) is fulfilled at each nonsingular point, we obtain, on returning to the initial point t_1, a well-defined value for the function $\ln G(t)$ on each of the arcs into which the contour γ is divided by discontinuity points of $G(t)$ and the point t_1.

Assume

$$\varkappa = \frac{1}{2\pi i}\Big[\ln G(t_1-0) - \ln G(t_1+0)\Big] = \frac{1}{2\pi}\,[\arg G(t)]_\gamma\ ,$$

where the symbol [ ] denotes an increment of the bracketed expression that takes place when t moves around the contour γ in the positive direction and condition (1.6.2) or (1.6.3) is fulfilled. It is obvious that $\varkappa$ is an integer number.

Consider the function

$$G_0(t) = \left(\frac{t-z_0}{at-z_0}\right)^{\varkappa} G(t),$$

where z_0 is some fixed point of the ring D. It is obvious for the above-indicated choice of $\arg G(t)$ we have

$$[\ln G_0(t)]_\gamma = 0.$$

Absolutely in the same manner as in Section 1.5, the function $G(t)$ can be written in the form

$$G(t) = \mu\,\frac{X(at)}{X(t)}\,, \quad t \in \gamma, \tag{1.6.4}$$

where

$$\begin{aligned} X(z) &= (z-z_0)^{\varkappa} \exp\left(\frac{1}{2\pi i}\int\limits_\gamma K_1^*\left(\frac{z}{t}\right)\,\ln\frac{G_0(t)}{\mu}\,\frac{dt}{t}\right),\\ \mu &= \exp\left(\frac{1}{2\pi i}\int\limits_\gamma \ln G_0(t)\,\frac{dt}{t}\right). \end{aligned} \tag{1.6.5}$$

Since

$$K_1^*\left(\frac{z}{t}\right) = \left(\frac{at}{at-z}\right) + \frac{t}{t-z} + K^0\left(\frac{z}{t}\right),$$

where

$$K^0\left(\frac{z}{t}\right)=\sum_{n=1}^{\infty}\frac{1}{a^n-1}\left(\frac{z}{at}\right)^n+\sum_{n=-\infty}^{-1}\frac{a^n}{a^n-1}\left(\frac{z}{t}\right)^n,$$

is holomorphic in the ring $\frac{1}{|a|}<\left|\frac{z}{t}\right|<|a|^2$, the function $X(z)$ is continuously extendable at all points of the boundary of the ring D except for the discontinuity points c of the function $G(t)$ and their corresponding points ac. Near these points, $X(z)$ is representable as

$$X(z)=\big[(z-c)(z-ac)\big]^{\alpha+i\beta}\Omega(t) \tag{1.6.6}$$

(see [**76**, Chapter 3, Section 26]), where

$$\alpha=\frac{1}{2\pi}\arg\frac{G(c-0)}{G(c+0)}, \quad \beta=\frac{1}{2\pi}\ln\left|\frac{G(c-0)}{G(c+0)}\right|.$$

The function $\Omega(z)$ is holomorphic near the points c and ac and tends to the well-defined nonzero limit as $z\to c$ or $z\to ac$.

As we see from formula (1.6.6), the function $X(z)$ is bounded near all singular points and those nonsingular points, for which condition (1.6.2) is fulfilled, and also near the points corresponding to them. Moreover, for the nonsingular points c $\lim X(z)\to 0$ as $z\to c$ or $z\to ac$.

When solving various applied problems it is sometimes required to find solutions of the Carleman type problem (1.6.1) which are bounded near certain prescribed nonsingular points $c_1,c_2,\dots,c_p$ and the points $ac_1,ac_2,\dots,ac_p$ corresponding to them. Following [**76**, Section 77], we call the solutions of problem (1.6.1) satisfying this condition the solutions of the class $h(c_1,c_2,\dots,c_p)$[1].

We will denote the class corresponding to $p=0$ by $h(0)$ or h_0. If m is the number of all nonsingular points and $c_1,c_2,\dots,c_p$ are all these points, then the class $h(c_1,c_2,\dots,c_m)$ is sometimes denoted by h_m. The class h_0 contains all other classes and the class h_m is contained in all others.

If it is assumed that the function $\ln G(t)$ in (1.6.5) is chosen so that condition (1.6.2) is fulfilled at the nonsingular points $c_1,c_2,\dots,c_p$, while condition (1.6.3) is fulfilled at all other nonsingular points $c_{p+1},c_{p+2},\dots,c_m$, then, by virtue of the above reasoning, the function $X(z)$ defined by formula (1.6.5) is bounded at the points $c_1,c_2,\dots,c_p$. We call this function $X(z)$ a canonical function of problem (1.6.1) from the class $h(c_1,c_2,\dots,c_p)$, while the number $\varkappa$ corresponding to it is called an index of the problem from the class $h(c_1,c_2,\dots,c_p)$. It is obvious that the function $X(z)$ is holomorphic in the ring D and different from zero everywhere except for the point z_0 where it has zero of order $\varkappa$ for $\varkappa>0$ and has a pole of order $-\varkappa$ for $\varkappa<0$.

[1]If in the indicated conditions the solution of problem (1.6.1) is bounded at the points $c_1,c_2,\dots,c_p$, it will also be bounded at the points $ac_1,ac_2,\dots,ac_p$ corresponding to them.

Let $c_{m+1},\dots,c_n$ be singular points. If all discontinuity points of the function $G(t)$ are nonsingular, then $n=m$.If all discontinuity points are singular, then $m=0$.

A canonical function $X(z)$ of the class $h(c_1,c_2,\dots,c_p)$ is continuously extendable on the boundary of the ring D except for the points $c_1,c_2,\dots,c_m$ and the points corresponding to them, is bounded near the points $c_{m+1},c_{m+2},\dots,c_n,ac_{m+1},\dots,ac_n$ and near the nonsingular points c_k and ac_k, $k=p+1,\dots,m$, it admits an estimate

$$|X(z)|<\frac{const}{|z-c_k|^\alpha}\,,\quad |X(z)|<\frac{const}{|z-ac_k|^\alpha}\,,\quad 0<\alpha<1.$$

Substituting the value of the function $G(t)$ defined by (1.6.4) into condition (1.6.1), we obtain

$$\frac{\varphi(at)}{X(at)}-\mu\,\frac{\varphi(t)}{X(t)}=\frac{f(t)}{X(at)}\,,\quad t\in\gamma. \tag{1.6.7}$$

For $\varkappa\neq 0$, the point z_0 can be chosen so that $a^n-\mu\neq 0$, $n=0,\pm1,\pm2,\dots$. For $\varkappa>0$, by formula (1.5.5) we obtain

$$\varphi(z)=\frac{X(z)}{2\pi i}\int\limits_\gamma K_\mu\left(\frac{z}{t}\right)\frac{f(t)\,dt}{tX(at)}+X(z)\varphi_\varkappa(z), \tag{1.6.8}$$

where $K_\mu(z)$ and $\varphi_\varkappa(z)$ are the same functions as in Section 1.5.

Using the corollaries from [**76**, Section 26], we conclude that the solution belongs to the class $h(c_1,\dots,c_p)$ and, near the singular points, is almost bounded.

For $\varkappa<0$, a solution of the class exists if and only if the following conditions are fulfilled:

$$\int\limits_\gamma \frac{d^jK_\mu\left(\frac{z}{t}\right)}{dz^j}\,\frac{f(t)}{X(at)}\,\frac{dt}{t}=0,\quad z=z_0,\quad j=0,1,\dots,-\varkappa-1. \tag{1.6.9}$$

When these conditions are fulfilled, the problem has a unique solution given by formula (1.6.7) where we should put $\varphi_\varkappa(z)=0$.

For $\varkappa=0$ and $a^n\neq\mu$, a solution from the class $h(c_1,\dots,c_p)$ always exists and is given by the same formula (1.6.7) for $\varphi_\varkappa(z)=0$.

When $\varkappa=0$ and $a^m=\mu$ for some integer number m, the problem is solvable if the condition

$$\int\limits_\gamma \frac{f(t)\,ft}{t^{m+1}X(at)}\,dt=0 \tag{1.6.10}$$

is fulfilled and the solution is given by the formula

$$\varphi(z)=\frac{X(z)}{2\pi i}\int\limits_\gamma K_\mu^*\left(\frac{z}{t}\right)\frac{f(t)}{X(at)}\,\frac{dt}{t}+bz^m, \tag{1.6.11}$$

where b is a complex constant.

If now condition (1.6.2) is replaced by condition (1.6.3). then we obtain the most general solution of problem (1.6.1) on which no restriction is imposed at the nonsingular points. Such a solution is not attributed to the class h_0. It is obvious that the index $\varkappa_0$ of this class is greater than the indices of all other classes. The index $\varkappa$ is the class $h(c_1,\dots,c_p)$ is related to $\varkappa$ by

$$\varkappa = \varkappa_0 - p.$$

The class h_m is a subclass of all other classes, its index $\varkappa_m$ is less than other indices and

$$\varkappa_m = \varkappa_0 - m.$$

We call the problem

$$\psi(\overline{a}t) = \overline{G(t)}\psi(t), \quad t \in \gamma, \tag{1.6.12}$$

the adjoint problem to (1.6.1). It is obvious that the singular (nonsingular) points of problem (1.6.1) are the singular (nonsingular) points of problem (1.6.12).

Accordingly, the class $h = h(c_1,\dots,c_p)$ of solutions of problem (1.6.1) and the class $h' = h(c_{p+1},\dots,c_m)$ of solutions of problem (1.6.12) are called the adjoint classes.

The canonical functions of the adjoint problems (1.6.12) and (1.6.1) of the adjoint classes are related by (see Section 1.5)

$$X'(z) = \frac{1}{\overline{X(a/\overline{z})}}, \quad z \in D, \tag{1.6.13}$$

while the corresponding indices by

$$\varkappa' = \varkappa.$$

For $\varkappa > 0$, the homogeneous problem (1.6.1) has, in the class $h(c_1,\dots,c_p)$, $\varkappa$ linearly independent solutions of the form

$$\varphi_j(z) = X(z)\,\frac{d^j}{dz^j}\,\varphi(\lambda;z)\bigg|_{\lambda=z_0}, \quad j = 0,1,\dots,\varkappa-1. \tag{1.6.14}$$

For $\varkappa < 0$, an adjoint homogeneous problem of the class $h(c_m,\dots,c_n)$ has $-\varkappa$ linearly independent solutions of the form

$$\psi_j(z) = X'(z)\,\frac{d^j}{dz^j}\,\psi(\lambda;z)\bigg|_{\lambda=\frac{\overline{a}}{\overline{z}_0}}, \quad j = 0,1,\dots,-\varkappa-1. \tag{1.6.15}$$

For $\varkappa = 0$ and $a^n \neq \mu$, adjoint homogeneous problems in the respective adjoint classes have no solutions.

For $\varkappa = 0$ and $a^m = \mu$, each of the adjoint homogeneous problems has one solution in the adjoint classes of the form

$$\varphi(z) = CX(z)z^m, \quad \psi(z) = X'(z)z^m. \tag{1.6.16}$$

In view of (1.6.13) the second formula (1.6.16) gives

$$\overline{\psi(t)} = \frac{1}{X(t)t^m}\,.$$

It is likewise easy to show that

$$\overline{\psi_j(t)} = \frac{1}{X(at)}\,\frac{d^j}{dz^j}\,K_\mu\left(\frac{z}{t}\right), \quad z = z_0, \quad j = 0, 1, \dots, -\varkappa - 1. \qquad (1.6.17)$$

Thus, for problem (1.6.1) the following Noether type theorems are valid:

1. For $\varkappa > 0$ or $\varkappa = 0$ and $a^n \neq \mu$, the homogeneous problem (1.6.12) has no nonzero solutions in the class $h(c_{p+1}, \dots, c_m)$, while problem (1.6.1) is always solvable in the class $h(c_1, \dots, c_p)$ and its solution is given by formula (1.6.11).

2. For $\varkappa < 0$, the homogeneous problem (1.6.12) has $-\varkappa$ linearly independent solutions of form (1.6.15) and, by virtue of equalities (1.6.9) and (1.6.17), for problem (1.6.1) to be solvable it is necessary and sufficient that the conditions

$$\int\limits_\gamma f(t)\overline{\psi_j(t)}\,\frac{dt}{t} = 0, \;\; j = 0, 1, \dots, -\varkappa - 1$$

be fulfilled. When these conditions are fulfilled, the solution of problem (1.6.1) is given by formula (1.6.8) where it is assumed that $\varphi_\varkappa(z) = 0$.

3. If $\varkappa = 0$ and $a^m = \mu$ for some integer number m, then each of the adjoint homogeneous problems has one solution φ_0, ψ_0, each, in the adjoint classes, which are given by formula (1.6.16), and for the nonhomogeneous problem (1.6.1) to be solvable it is necessary and sufficient that the condition

$$\int\limits_\gamma f(t)\overline{\psi_0(t)}\,\frac{dt}{t} = 0$$

be fulfilled. When this condition is fulfilled, a solution of problem (1.6.1) looks like (1.6.11).

1.7. The Riemann–Hilbert Problem for Doubly Connected Domains

Let D be a finite or infinite domain bounded by the smooth closed contours $L_0, L_1, \dots, L_n$, of which the first one covers all others. L will be understood as the set of all these contours. The Riemann–Hilbert problem is formulated as follows.

Find a function $\varphi(z)$, holomorphic in D and continuous in $D + L$, by the condition

$$\operatorname{Re}\big[a(t)\varphi(t)\big] = c(t), \;\; t \in L, \qquad (1.7.1)$$

where $a(t)$ and $c(t)$ are functions of the class H given on L.

This problem is a particular case of quite a general problem posed by Riemann. The problem was considered by Hilbert.

In subsequent years, the problem was the subject of investigation by many authors. For the detailed references see the monographs by N. Muskhelishvili [**76**], I. N. Vekua [**121**], F. D. Gakhov [**42**], where various methods are presented for its solution.

In the case of a simply connected domain, N. I. Muskhelishvili [**76**] proposed an effective technique of solution of the Riemann–Hilbert problem by reducing it to a linear conjugation problem. This technique is based on the application of conformal mapping of the domain onto the circle. Therefore the assumption that the domain is simply connected is quite essential.

Below we give a method of effective solution of the Riemann–Hilbert problem for a doubly connected domain.

Since an arbitrary doubly connected domain can always be conformally mapped onto the circular ring [**52**], we assume that the domain $D = \{1 < |z| < R\}$. We denote by L_0 the external boundary of the ring D, and by L_1 the internal boundary.

The boundary condition is written as follows:

$$\overline{a_0(t)}\varphi(t) + a_0(t)\overline{\varphi(t)} = 2c_0(t), \quad t \in L_0, \tag{1.7.2}$$

$$\overline{a_1(t)}\varphi(t) + a_1(t)\overline{\varphi(t)} = 2c_1(t), \quad t \in L_1, \tag{1.7.3}$$

where a_0, a_1, c_0, c_1 are given function of the class H; $a_0(t) \neq 0$, $a_1(t) \neq 0$ everywhere.

We introduce the notation: $\varkappa_j = (-1)^j \operatorname{Ind} a_j(t)$, $j = 0, 1$, $\varkappa = \varkappa_0 + \varkappa_1$. The number $\varkappa$ is called the index of the Riemann–Hilbert problem.

We rewrite condition (1.7.3) in the form

$$\varphi(t) + \frac{a_1(t)}{\overline{a_1(t)}}\,\overline{\varphi(t)} = \frac{2c_1(t)}{\overline{a_1(t)}}\,, \quad t \in L_1 \tag{1.7.4}$$

and represent the function $a_1(t)/\overline{a_1(t)}$ as (see [**76**])

$$\frac{a_1(t)}{\overline{a_1(t)}} = \frac{a^-(t)}{\overline{a^-(t)}}\, t^{-2\varkappa_1}, \tag{1.7.5}$$

where

$$\begin{aligned} a^-(z) &= e^{i\beta} \exp\bigg(-\frac{1}{2\pi i}\int\limits_{L_1} \ln\bigg(\frac{a_1(t)}{\overline{a_1(t)}}\, t^{2\varkappa_1}\bigg)\frac{dt}{t-z}\bigg), \quad |z| > 1, \\ \beta &= \frac{1}{2\pi i}\int\limits_{L_1} \arg\big(t^{\varkappa_1} a_1(t)\big)\,\frac{dt}{t}\,. \end{aligned} \tag{1.7.6}$$

Substituting (1.7.5) into condition (1.7.4) and introducing the notation

$$\psi(z) = \begin{cases} \varphi(z) z^{\varkappa_1} [a^-(z)]^{-1}, & 1 < |z| < R, \\ -\overline{\varphi\Big(\dfrac{1}{\overline{z}}\Big)} z^{-\varkappa_1} \bigg[\overline{a^-\Big(\dfrac{1}{\overline{z}}\Big)}\bigg]^{-1}, & \dfrac{1}{R} < |z| < 1, \end{cases} \tag{1.7.7}$$

we obtain

$$\psi^+(t) - \psi^-(t) = -\frac{2c_1(t)t^{\varkappa_1}}{\overline{a_1(t)}a^-(t)}\,, \quad t \in L_1. \tag{1.7.8}$$

Hence follows

$$\psi(z) = A(z) - \frac{A(0)}{2} + \omega_1(z), \tag{1.7.9}$$

where

$$A(z) = -\frac{1}{\pi i}\int\limits_{L_1} \frac{c_1(t)t^{\varkappa_1}}{\overline{a_1(t)}a^-(t)(t-z)}\,dt, \quad |z| \neq 1 \tag{1.7.10}$$

and $\omega_1(z)$ is a holomorphic function in the ring $1/R < |z| < R$.

Formula (1.7.7) implies that the function $\psi(z)$ satisfies the condition $\psi(z) + \overline{\psi(1/\overline{z})} = 0$ for $1 < |z| < R$ and therefore the function $\omega_1(z)$ must satisfy the condition

$$\omega_1(z) = -\overline{\omega_1\left(\frac{1}{\overline{z}}\right)}, \quad \frac{1}{R} < |z| < R. \tag{1.7.11}$$

Formulas (1.7.7) and (1.7.9) yield

$$\varphi(z) = a^-(z)z^{-\varkappa_1}\left[A(z) - \frac{A(0)}{2} + \omega_1(z)\right], \quad 1 < |z| < R, \tag{1.7.12}$$

$$\overline{\varphi\left(\frac{1}{\overline{z}}\right)} = -\overline{a\left(\frac{1}{\overline{z}}\right)}z^{\varkappa_1}\left[A(z) - \frac{A(0)}{2} + \omega_1(z)\right], \quad \frac{1}{R} < |z| < R, \tag{1.7.13}$$

or

$$\overline{\varphi(z)} = -\overline{a^-(z)}(\overline{z})^{-\varkappa_1}\left[A\left(\frac{1}{\overline{z}}\right) - \frac{A(0)}{2} + \omega_1(\overline{z})\right], \quad 1<|z|<R. \tag{1.7.14}$$

Substituting values (1.7.12) and (1.7.14) into formula (1.7.2) and introducing the notation

$$\zeta = Rz, \quad \omega(\zeta) = \omega_1\left(\frac{\zeta}{R}\right) = \omega_1(z), \quad 1 < |\zeta| < R^2. \tag{1.7.15}$$

for finding the function $\omega(\zeta)$ we obtain the boundary condition

$$\omega(R^2\sigma) = G(\sigma)\omega(\sigma) + F(\sigma), \quad |\sigma| = 1, \tag{1.7.16}$$

where

$$G(\sigma) = \frac{a_0(R\sigma)\overline{a^-(R\sigma)}}{\overline{a_0(R\sigma)}a^-(R\sigma)}\,\sigma^{2\varkappa_1}, \quad |\sigma| = 1, \tag{1.7.17}$$

$$\begin{aligned} F(\sigma) = {} & G(\sigma)A\left(\frac{\sigma}{R}\right) - A(\sigma R) \\ & - \frac{A(0)}{2}\,[G(\sigma) - 1] + \frac{2c_0(R\sigma)(R\sigma)^{\varkappa_1}}{\overline{a_0(R\sigma)}a^-(R\sigma)}\,, \end{aligned} \tag{1.7.18}$$

The function $D_1 = \{1 < |z| < R^2\}$ which we want to define in the ring $\omega(\zeta)$ must satisfy the additional condition

$$\omega(\zeta) = -\overline{\omega\left(\frac{R^2}{\bar{\zeta}}\right)}, \quad \zeta \in D_1, \tag{1.7.19}$$

(1.7.16) is the problem which we studied in 1.5 The solution of this problem obtained in that paragraph may not satisfy the additional condition (1.7.19), but using this solution we may construct the solution satisfying condition (1.7.19). Indeed, passing in (1.7.16) to the conjugate values, we see that if the function $\omega_0(\zeta)$ satisfies condition (1.7.16), then the function $\omega_*(\zeta) = -\overline{\omega_0(R^2/\bar{\zeta})}$ satisfies the condition

$$\omega_*(R^2\sigma) = G(\sigma)\omega_*(\sigma) - \overline{F(\sigma)}G(\sigma), \quad |\sigma| = 1.$$

It can be shown that

$$\overline{F(\sigma)}G(\sigma) = -F(\sigma).$$

Thus the functions $\omega_0(\zeta)$ and $\omega_*(\zeta)$ satisfy one and the same boundary condition (1.7.16) and therefore the function

$$\omega(\zeta) = \frac{1}{2}\left[\omega_0(\zeta) + \omega_*(\zeta)\right] = \frac{1}{2}\left[\omega_0(\zeta) - \overline{\omega_0\left(\frac{R^2}{\bar{\zeta}}\right)}\,\right]$$

also satisfies conditions (1.7.16) and (1.7.9). Thus $\omega(\zeta)$ is a solution of the problem posed.

The solution of the considered problem can written in simpler terms. For this, in formula (1.5.20), the function $\zeta - \zeta_0$ should be replaced by such a function $W_\varkappa(\zeta)$ that satisfies the conditions

$$W_\varkappa(\zeta) = \overline{W_\varkappa\left(\frac{R^2}{\bar{\zeta}}\right)}, \quad W_\varkappa(\zeta_0) = 0 \ \text{ for } \ \varkappa > 0, \quad R < |\zeta| < R^2.$$

An example of such a function is

$$W_\varkappa(\zeta) = (\zeta - \zeta_0)^\varkappa (R^2 - \zeta\bar{\zeta}_0)^\varkappa \zeta^{-\varkappa}, \quad R < |\zeta| < R^2.$$

The index of the function $W_\varkappa(\sigma)/W(R^2\sigma)$ is equal to $-2\varkappa$, and it module is equal to one.

Thus, for $\varkappa > 0$ a solution of the problem posed has the form (see 1.5)

$$\omega(\zeta) = \frac{X(\zeta)}{2\pi i}\int\limits_\gamma K_\mu\left(\frac{\zeta}{\sigma}\right)\frac{F(\sigma)}{\sigma X(R^2\sigma)}\,d\sigma + X(\zeta)Q(\zeta), \tag{1.7.20}$$

where

$$X(\zeta) = e^{i\alpha} W_\varkappa(\zeta) \exp\left(\frac{1}{2\pi i}\int_\gamma K_1^*\left(\frac{\zeta}{\sigma}\right)\ln\frac{G(\sigma)W_\varkappa(\sigma)}{\mu W_\varkappa(R^2\sigma)}\,\frac{d\sigma}{\sigma}\right),$$

$$\mu = \exp\left(\frac{1}{2\pi i}\int_\gamma \ln\frac{G(\sigma)W_\varkappa(\sigma)}{W_\varkappa(R^2\sigma)}\,\frac{d\sigma}{\sigma}\right), \tag{1.7.21}$$

$$\alpha = -\frac{1}{2\pi i}\int_\gamma \ln\frac{G(\sigma)W_\varkappa(\sigma)}{\mu W_\varkappa(R^2\sigma)}\,\frac{d\sigma}{\sigma}\,.$$

It is obvious that $|\mu| = 1$. The point ζ_0 can always be chosen so that $\mu \neq 1$ for $\varkappa \neq 0$.

For $\varkappa > 0$,

$$Q(\zeta) = \sum_{j=0}^{\varkappa-1}\left[c_j\varphi_j(\zeta) - \overline{c}_j\overline{\varphi_j\left(\frac{R^2}{\overline{\zeta}}\right)}\right],$$

$$\varphi_j(\zeta) = \frac{d^j}{d\lambda^j}\,\varphi(\lambda,\zeta)\bigg|_{\lambda=\zeta_0}\,,$$

where $c_0, c_1, \dots, c_{\varkappa-1}$ are complex constants.

It is easy to show that

$$K_\mu\left(\frac{\zeta}{\sigma}\right) = -\overline{\mu}\,\overline{K_\mu\left(\frac{R^2}{\overline{\zeta}\sigma}\right)},\ \ \zeta \in D_1,\ \ \mu \neq 1$$

and

$$K_1^*\left(\frac{\zeta}{\sigma}\right) = -\overline{K_1^*\left(\frac{R^2}{\sigma\overline{\zeta}}\right)} + 2,\ \ \zeta \in D.$$

By these equalities and the condition

$$\frac{\overline{F(\sigma)}}{\overline{X(\sigma R^2)}} = \frac{F(\sigma)}{\mu X(R^2\sigma)}\,,$$

we conclude that the function represented by (1.7.20) is a solution of the problem posed, i.e. it satisfies condition (1.7.19), too.

If $\varkappa < 0$, then a solution of problem (1.7.16) exists provided that the necessary and sufficient conditions

$$\int_\gamma \frac{d^j}{d\zeta^j}\,K_\mu\left(\frac{\zeta}{\sigma}\right)\,\frac{F(\sigma)}{X(\sigma R^2)}\,\frac{d\sigma}{\sigma} = 0,\ \ j = 0, 1, \dots, -\varkappa - 1, \tag{1.7.22}$$

are fulfilled for $\zeta = \zeta_0$ and $\zeta = R^2/\overline{\zeta}_0$ and is represented by formula (1.7.20) where it is assumed that $Q(\zeta) \equiv 0$.

If conditions (1.7.22) are fulfilled at the point $\zeta = \zeta_0$, they will be fulfilled at the point $\zeta = R^2/\overline{\zeta}_0$, too, since

$$\omega(\zeta) = -\overline{\omega\left(\frac{R^2}{\overline{\zeta}}\right)}.$$

For $\varkappa = 0$ and $\mu \neq 1$, problem (1.7.16) has a unique solution given by formula (1.7.20) where $Q(\zeta) \equiv 0$.

If $\varkappa = 0$ and $\mu = 1$, then provided that the following necessary and sufficient condition

$$\int\limits_{|\sigma|=1} \frac{F(\sigma)}{\sigma X(R^2\sigma)}\, d\sigma = 0$$

for the existence of a solution is fulfilled, a solution of the problem is given by the formula

$$\omega(\zeta) = \frac{X(\zeta)}{2\pi i} \int\limits_{\gamma} \frac{K_1^*\big(\frac{\zeta}{\sigma}\big)F(\sigma)}{\sigma X(R^2\sigma)}\, d\sigma + c_i X(\zeta), \tag{1.7.23}$$

where c is an arbitrary real constant.

The first three terms in formula (1.7.18) of the expression of the function $F(\sigma)$ are transformed as follows

$$\begin{aligned} G(\sigma)\left[A\left(\frac{\sigma}{R}\right) - \frac{A(0)}{2}\right] - A(R\sigma) + \frac{A(0)}{2} \\ = \frac{\mu X(R^2\sigma)}{X(\sigma)}\left[\frac{1}{\pi i}\int\limits_{|t|=1} \frac{f_1(t)}{t-\sigma/R}\, dt - \frac{A(0)}{2}\right] \\ -\frac{1}{\pi i}\int\limits_{|t|=1} \frac{f_1(t)}{t-R\sigma}\, dt + \frac{A(0)}{2}\,, \quad |\sigma| = 1, \end{aligned}$$

and substituted into formula (1.7.20). Inverting the order of integration in the double integrals and applying the Cauchy residue theorem, we obtain

$$\begin{aligned} \omega(\zeta) = \frac{X(\sigma)}{\pi i}\left[\int\limits_{\gamma} \frac{K_\mu\big(\frac{\zeta}{\sigma}\big)f_0(R\sigma)}{X(R^2\sigma)\sigma}\, d\sigma - \mu\int\limits_{\gamma} \frac{K_\mu\big(\frac{\zeta}{\sigma}\big)f_1(\sigma)}{X(R\sigma)\sigma}\, d\sigma\right] \\ + X(\zeta)Q(\zeta) - A\left(\frac{\zeta}{R}\right) + \frac{A(0)}{2}\,. \end{aligned}$$

Replacing now in this formula ζ/R by z, $\omega(\zeta/R)$ by $\omega_1(\zeta)$ and substituting the obtained values into (1.7.12), we have

$$\begin{aligned} \varphi(z) = \frac{a^-(z)z^{-\varkappa_1}X(Rz)}{\pi i}\left[\int\limits_{L_0} \frac{K_\mu\big(\frac{R^2z}{\sigma}\big)f_0(\sigma)}{X(R\sigma)\sigma}\, d\sigma - \mu\int\limits_{L_1} \frac{K_\mu\big(\frac{z}{\sigma}\big)f_1(\sigma)}{X(R\sigma)\sigma}\, d\sigma\right] \\ + X(Rz)Q(Rz)a^-(z)z^{-\varkappa_1}, \end{aligned} \tag{1.7.24}$$

where

$$f_j(t) = \frac{c_j(t)\sigma^{\varkappa_1}}{a_j(t)a^-(t)}\,, \quad t \in L_j, \quad j = 0, 1.$$

So, we come to the conclusions:

1. For $\varkappa > 0$, a solution of the Riemann–Hilbert problem is given by formula (1.7.24).

2. For $\varkappa < 0$, for the problem to be fulfilled it is necessary and sufficient that the conditions

$$\int_{L_0} \frac{d^j K_\mu\big(R^2\,\frac{z}{t}\big) f_0(t)}{dz^j X(Rt)t}\,dt - \mu \int_{L_1} \frac{d^j K\big(\frac{z}{t}\big) f_0(t)}{dz^j} \cdot \frac{f_1(t)}{X(Rt)t}\,\frac{dt}{t} = 0, \tag{1.7.25}$$

$$j = 0, 1, \dots, -\varkappa - 1, \quad z = z_0,$$

be fulfilled. Then the problem has a unique solution given by (1.7.24) where $Q(Rz) \equiv 0$.

3. For $\varkappa = 0$ and $\mu \neq 1$, the solution (unique) is given by the same formula (1.7.24) if $Q(Rz) = 0$.

4. For $\varkappa = 0$ and $\mu = 1$, the Riemann–Hilbert problem has a solution if and only if the condition

$$\int_{L_0} \frac{f_0(t)}{tX(Rt)}\,dt - \int_{L_1} \frac{f_1(t)}{tX(Rt)}\,dt = 0, \tag{1.7.26}$$

is fulfilled. Then a solution has the form

$$\varphi(z) = \frac{z^{-\varkappa_1} a^-(z) X(Rz)}{\pi i} \times \bigg[\int_{L_0} \frac{K_\mu^*\big(R^2\,\frac{z}{t}\big) f_0(t)}{tX(Rt)}\,dt - \mu \int_{L_1} \frac{K_\mu^*\big(\frac{z}{t}\big) f_1(t)}{tX(Rt)}\,dt + C\bigg], \tag{1.7.27}$$

where c is a real constant.

It can be easily shown that for $\varkappa = 0$ the number μ can be represented as follows

$$\mu = \exp\bigg(\frac{1}{\pi} \int_0^{2\pi} \arg \frac{a_0(Re^{i\theta})}{a_1(e^{i\theta})}\,d\theta\bigg).$$

Hence it follows that if the integral

$$\int_0^{2\pi} \arg \frac{a_0(Re^{i\theta})}{a_1(e^{i\theta})}\,d\theta \neq 0,$$

then the Riemann–Hilbert problem is solvable for any right-hand part, but if this integral is equal to zero, then the Riemann–Hilbert problem is solvable only provided that condition (1.7.26) is fulfilled.

1.8. The Riemann–Hilbert Problem with Discontinuous Coefficients for a Ring

Let us now consider the case where the functions $a_0(t)$, $a_1(t)$, $c_0(t)$, $c_1(t)$ have discontinuities of first kind at a finite number of points of the boundary and, on each closed arc whose ends are discontinuity points, satisfy the

Hölder condition and it is assumed that $a_j(t) \neq 0$ everywhere on L_j, $j = 0, 1$. Denote the number of discontinuity points on the internal contour L_1 by n_1, and that on the external contour L_0 by n_0.

The Riemann–Hilbert problem is formulated as follows: Find in the ring $D = \{1 < |z| < R\}$ a holomorphic function $\varphi(z)$, continuously extendable to all points of the boundary except perhaps for the discontinuity points of the functions $a_0(t)$, $a_1(t)$ and $c_0(t)$, $c_1(t)$ near which

$$\varphi(z) < \frac{const}{|z-c|^\alpha}\,, \quad 0 \leq \alpha < 1,$$

by the boundary condition

$$\varphi(t) + \frac{a_0(t)}{\overline{a_0(t)}}\,\overline{\varphi(t)} = \frac{2c_0(t)}{\overline{a_0(t)}}\,, \ \ t \in L_0, \tag{1.8.1}$$

$$\varphi(t) + \frac{a_1(t)}{\overline{a_1(t)}}\,\overline{\varphi(t)} = \frac{2c_1(t)}{\overline{a_1(t)}}\,, \ \ t \in L_1. \tag{1.8.2}$$

Like in 1.6, the discontinuity points, at which the argument of the relation $a_j/\overline{a}_j$, $j = 0, 1$ can be changed continuously when passing through them, are called singular, and all other points nonsingular.

Analogously to what has been done in 1.6, a solution will be sought in the class $h(c_1, \dots, c_q, c_{n+1}, \dots, c_{n+p})$, i.e. in the class of functions bounded near the nonsingular points $c_1, \dots, c_q, c_{n+1}, \dots, c_{n+p}$, where the points $c_1, c_2, \dots, c_q$ and $c_{n+1}, c_{n+2}, \dots, c_{n+p}$, $q < n$, $p < n$ lie on the contours L_1 and L_0, respectively.

The index $\varkappa$ of this class is defined by the formula

$$\varkappa = \varkappa_0 + \varkappa_1, \tag{1.8.3}$$

where

$$\varkappa_0 = \frac{1}{2\pi}\left[\arg\frac{a_0(t)}{\overline{a_0(t)}}\right]_{L_0}, \quad \varkappa_1 = \frac{1}{2\pi}\left[\arg\frac{a_1(t)}{\overline{a_1(t)}}\right]_{L_1},$$

$[\ ]_{L_j}$, $j = 0, 1$, denotes an increment of the bracketed function that takes place when the point t passes over the contour L_j in the positive direction, i.e. in the direction leaving the point of the ring D on the left. The values of $\arg\big(a_0(t)/\overline{a_0(t)}\big)$ and $\arg\big(a_1(t)/\overline{a_1(t)}\big)$ are chosen so that condition (1.6.2) is fulfilled at the nonsingular points $c_1, c_2, \dots, c_q$ and $c_{n+1}, c_{n+2}, \dots, c_{n+p}$, and condition (1.6.3) at the nonsingular points $c_{q+1}, \dots, c_{q+l}, c_{n+p+1}, \dots, c_{n+r}$, $l < n$, $r < m$.

As different from the case of a continuous coefficient, $\varkappa_0$ and $\varkappa_1$ may be both even and odd.

The coefficient of the boundary condition (1.8.2) can be represented as

$$\frac{a_1(t)}{\overline{a_1(t)}} = t^{-\varkappa_1}\,\frac{a^-(t)}{\overline{a^-(t)}}\,, \tag{1.8.4}$$

where

$$
\begin{aligned}
a^-(z) &= e^{i\beta_0} \exp\left(-\frac{1}{2\pi i} \int\limits_{L_1} \ln\left[\frac{a_1(t)}{\overline{a_1(t)}}\, t^{\varkappa_1}\right] \frac{dt}{t-z} \right), \quad |z|>1, \\
\beta_0 &= \frac{1}{2\pi i} \int\limits_{L_1} \ln\left[\frac{a_1(t)}{\overline{a_1(t)}}\, t^{\varkappa_1}\right] \frac{dt}{t}\,.
\end{aligned} \tag{1.8.5}
$$

Let us first consider the case where $\varkappa_1$ is an even number. Like in 1.7 we obtain

$$
\varphi(z) = z^{-\frac{\varkappa_1}{2}} a^-(z) \left[A_1(z) - \frac{A_1(0)}{2} + \omega_1(z)\right], \quad 1<|z|<R, \tag{1.8.6}
$$

$$
\overline{\varphi(z)} = -(\overline{z})^{-\frac{\varkappa_1}{2}} \overline{a_1(z)} \left[A_1\left(\frac{1}{\overline{z}}\right) - \frac{A_1(0)}{2} + \omega_1\left(\frac{1}{\overline{z}}\right)\right], \quad 1<|z|<R, \tag{1.8.7}
$$

where

$$
A_1(z) = -\frac{1}{\pi i} \int\limits_{L_1} \frac{c_1(t) t^{\frac{\varkappa_1}{2}}}{\overline{a_1(t)} a^-(t)} \frac{dt}{t-z}\,, \quad z \neq 1, \tag{1.8.8}
$$

and $\omega(\zeta)$ is a holomorphic function in the ring $1/R<|z|<R$ that satisfies the condition

$$
\overline{\omega_1\left(\frac{1}{\overline{z}}\right)} = -\omega_1(z). \tag{1.8.9}
$$

Substituting the boundary values of $\varphi(z)$ and $\overline{\varphi(z)}$ defined by (1.8.6) and (1.8.7) into formula (1.8.1) and using the otation

$$
\zeta = Rz, \;\; \omega(\zeta) = \omega_1(z) = \omega_1\left(\frac{\zeta}{R}\right), \quad 1<|\zeta|<R^2, \tag{1.8.10}
$$

we obtain

$$
\omega(R^2\sigma) = G(\sigma)\omega(\sigma) + F(\sigma), \;\; \sigma \in \gamma, \tag{1.8.11}
$$

where

$$
\begin{aligned}
G(\sigma) &= \frac{a_0(R\sigma)\overline{a^-(R\sigma)}}{\overline{a(R\sigma)}a^-(R\sigma)}\, \sigma^{\varkappa_1}, \\
F(\sigma) &= G(\sigma)\left[A_1\left(\frac{\sigma}{R}\right) - \frac{A_1(0)}{2}\right] - A_1(R\sigma) + \frac{A_1(0)}{2} + \frac{2c_0(R\sigma)(R\sigma)^{\varkappa_1/2}}{\overline{a_0(R\sigma)}a^-(R\sigma)}\,.
\end{aligned}
$$

Formulas (1.8.9) and (1.8.10) imply that $\omega(\zeta)$ must satisfy the condition

$$
\overline{\omega\left(\frac{R^2}{\overline{\zeta}}\right)} = -\omega(\zeta), \;\; 1<|\zeta|<R^2. \tag{1.8.12}
$$

Thus the finding of a solution of the Riemann–Hilbert problem in the class $h(c_1,\dots,c_q,c_{n+1},\dots,c_{n+p})$ reduces to finding a solution of the Carleman type problem studied in 1.5 in the class $h\big(\frac{c_{n+1}}{R},\dots,\frac{c_{n+p}}{R}\big)$ with the additional condition (1.8.11). Note that the index of problem (1.8.10) of the class $h\big(\frac{c_{n+1}}{R},\dots,\frac{c_{n+p}}{R}\big)$ coincides with the index of the Riemann–Hilbert problem of the class $h(c_1,\dots,c_q,c_{n+1},\dots,c_{n+p})$.

Let us introduce the function

$$W_\varkappa(\zeta) = (\zeta - Re^{i\alpha_0})^{-\varkappa}\zeta^{\delta(n)}e^{i\,\frac{\alpha_0\varkappa}{2}}, \tag{1.8.13}$$

where

$$\delta(n) = \begin{cases} \dfrac{\varkappa}{2} & \text{for even } \varkappa, \\ \dfrac{\varkappa-1}{2} & \text{for odd } \varkappa, \end{cases}$$

α_0 is a fixed number, $\alpha_0 \in [0, 2\pi]$.

It is obvious that

$$\overline{W_\varkappa\left(\frac{R^2}{\overline{\zeta}}\right)} = \begin{cases} W_\varkappa(\zeta) & \text{for even } \varkappa, \\ -\zeta\,\dfrac{W_\varkappa(\zeta)}{R} & \text{for odd } \varkappa. \end{cases}$$

As we have done in 1.6, we find a general solution of problem (1.8.10) for $\varkappa > 0$

$$\omega(\zeta) = \frac{X(\zeta)}{2\pi i}\int\limits_\gamma \frac{K_\mu\left(\frac{\zeta}{\sigma}\right)F(\sigma)}{X(R^2\sigma)\sigma}\,d\sigma + Q(\zeta)X(\zeta), \tag{1.8.14}$$

where

$$X(\zeta) = W_\varkappa(\zeta)e^{i\beta_0}\exp\left(\frac{1}{2\pi i}\int\limits_\gamma K_1\left(\frac{\zeta}{\sigma}\right)\ln\left[\frac{G(\sigma)W_\varkappa(\sigma)}{\mu W_\varkappa(R^2\sigma)}\right]\frac{d\sigma}{\sigma}\right), \tag{1.8.15}$$

$$\mu = \exp\left(\frac{1}{2\pi i}\int\limits_\gamma \ln\frac{G(\sigma)W_\varkappa(\sigma)}{W_\varkappa(R^2\sigma)}\,\frac{d\sigma}{\sigma}\right),$$

$|\mu| = 1$ when $\varkappa$ is even, and $|\mu| = R$ when $\varkappa$ is odd.

$$\begin{aligned} Q(\zeta) &= \sum_{j=0}^{\varkappa-1}\left(d_j\varphi_j(\zeta) + \overline{d}_j\overline{\varphi_j\left(\frac{R^2}{\overline{\zeta}}\right)}\right) \quad \text{for odd } \varkappa, \\ Q(\zeta) &= \sum_{j=0}^{\varkappa-1}\left(d_j\varphi_j(z) - \overline{d}_j\overline{\varphi_j\left(\frac{R^2}{\overline{\zeta}}\right)}\right) \quad \text{for even } \varkappa, \end{aligned} \tag{1.8.16}$$

$\varphi_j(\zeta)$ is the same as in 1.7.

If we take into account that

$$\overline{K_\mu\left(\frac{R^2}{\sigma\overline{\zeta}}\right)} = \begin{cases} -\mu K_\mu\left(\dfrac{\zeta}{\sigma}\right) & \text{for even } \varkappa, \\ -\mu\,\dfrac{\zeta}{\sigma}\,K_\mu\left(\dfrac{\zeta}{\sigma}\right) & \text{for odd } \varkappa \end{cases}$$

and

$$\overline{X\left(\frac{R^2}{\overline{\zeta}}\right)} = \begin{cases} X(\zeta) & \text{for even } \varkappa, \\ -\dfrac{\zeta}{R}\,X(\zeta) & \text{for odd } \varkappa \end{cases}$$

and

$$\overline{F(\sigma)} = \frac{F(\sigma)}{G(\sigma)} + \frac{A_1(0)}{2}\left[1 - \frac{1}{G(\sigma)}\right],$$

then it can be easily shown that $\omega(\zeta)$ satisfies condition (1.8.11).

If $\varkappa < 0$, then the solution of problem (1.8.11) exists provided that the necessary conditions

$$\int\limits_{\gamma} \frac{d^j}{d\zeta^j} K_\mu\left(\frac{\zeta}{\sigma}\right) \frac{F(\sigma)}{X(R^2\sigma)} \frac{d\sigma}{\sigma} = 0, \quad j = 0, 1, \dots, -\varkappa - 1, \tag{1.8.17}$$

are fulfilled for $\zeta = Re^{i\alpha_0}$ and is represented by formula (1.8.14) where it should be assumed that $Q_\varkappa(\zeta) \equiv 0$.

When $\varkappa = 0$ and $\mu \neq 1$, problem (1.8.10) has a unique solution given by (1.8.14) where $Q_\varkappa(\zeta) \equiv 0$.

For $\varkappa = 0$ and $\mu = 1$ the solution exists only provided that

$$\int\limits_{\gamma} \frac{F(\sigma)}{X(R^2\sigma)\sigma} \, d\sigma = 0.$$

If this condition is fulfilled, then the solution is given by the formula

$$\omega(\zeta) = \frac{X(\zeta)}{2\pi i} \int\limits_{\gamma} \frac{K_\mu(\frac{\zeta}{\sigma}) F(\sigma)}{X(R^2\sigma)\sigma} \, d\sigma + ciX(\zeta),$$

where c is an arbitrary real constant.

Now we let us consider the case where $\varkappa_1$ is odd. The functions $\varphi(z)$ and $\overline{\varphi(\frac{1}{\bar z})}$ can be written in the form

$$\varphi(z) = z^{-\frac{\varkappa_1 - 1}{2}} a^-(z)\big[A_2(z) + \omega_2(z)\big], \quad 1 < |z| < R, \tag{1.8.18}$$

$$\overline{\varphi\left(\frac{1}{\bar z}\right)} = -z^{\frac{\varkappa_1 + 1}{2}} \overline{a^-\left(\frac{1}{\bar z}\right)}\big[A_2(z) + \omega_2(z)\big], \quad \frac{1}{R} < |z| < 1, \tag{1.8.19}$$

where

$$A_2(z) = -\frac{1}{\pi i} \int\limits_{L_1} \frac{c_1(t) t^{\frac{\varkappa_1 - 1}{2}}}{\overline{a_1(t)} a^-(t)} \frac{dt}{t - z}, \quad |z| \neq 1,$$

the function $\omega_2(z)$ is holomorphic in the ring $1/R < |z| < R$ and satisfies the condition

$$\overline{\omega_2\left(\frac{1}{\bar z}\right)} = -z\omega_2(z). \tag{1.8.20}$$

If we substitute the boundary values of $\varphi(z)$ and $\overline{\varphi(z)}$ defined by (1.8.18) and (1.8.19) into the boundary condition (1.8.1) and use the notation

$$\zeta = Rz, \quad \omega(\zeta) = \omega_2\left(\frac{\zeta}{R}\right) = \omega_2(z),$$

then for $\omega(\zeta)$ we obtain the Carleman type problem

$$\omega(R^2\sigma) = \frac{G(\sigma)}{R}\,\omega(\sigma) + F_1(\sigma), \quad \sigma \in \gamma,$$

where

$$G(\sigma) = \frac{a_0(R\sigma)\overline{a^-(R\sigma)}}{\overline{a_0(R\sigma)}a^-(R\sigma)}\,\sigma^{\varkappa_1},$$

$$F_1(\sigma) = G(\sigma)\left[A_2\left(\frac{\sigma}{R}\right) - A(R\sigma)\right] + \frac{c_0(R\sigma)(R\sigma)^{\frac{\varkappa_1-1}{2}}}{\overline{a_0(R\sigma)}a^-(R\sigma)}\,,$$

with the additional condition

$$\overline{\omega_*\left(\frac{R^2}{\overline{\zeta}}\right)} = \frac{\zeta}{R}\,\omega(\zeta). \tag{1.8.21}$$

If we introduce the notation $\mu/R = \mu^*$, then for $\varkappa > 0$ a solution of problem (1.8.1) can be represented as

$$\omega(\zeta) = \frac{X(\zeta)}{2\pi i}\int\limits_{\gamma} K_{\mu^*}\left(\frac{\zeta}{\sigma}\right)\frac{F_1(\sigma)}{\sigma X(R^2\sigma)}\,d\sigma + X(\zeta)Q_1(\zeta), \tag{1.8.22}$$

where

$$Q_1(\zeta) = \begin{cases} \displaystyle\sum_{j=0}^{\varkappa-1}\left[d_j\varphi_j(\zeta) - \overline{d}_j\,\overline{\frac{R}{\zeta}\,\varphi_j\left(\frac{R^2}{\overline{\zeta}}\right)}\right] & \text{for even } \varkappa, \\ \displaystyle\sum_{j=0}^{\varkappa-1}\left[d_j\varphi_j(\zeta) + \overline{d}_j\overline{\varphi_j\left(\frac{R^2}{\overline{\zeta}}\right)}\right] & \text{for odd } \varkappa, \\ 0 & \text{for } \varkappa = 0, \end{cases}$$

d_j are complex constants.

For $\varkappa < 0$, a solution is given by (1.8.22), where it is assumed that $Q_1(\zeta) \equiv 0$ provided that the following necessary and sufficient solvability conditions are fulfilled:

$$\int\limits_{\gamma}\frac{d^j}{d\zeta^j}\,K_{\mu^*}\left(\frac{\zeta}{\sigma}\right)\frac{F_1(\sigma)}{\sigma X(R^2\sigma)}\,d\sigma = 0, \quad \zeta = Re^{i\alpha_0}, \quad j = 0, 1, \ldots, -\varkappa - 1.$$

1.9. Solution of an Infinite System of Algebraic Equations

In this paragraph, we will show one more application of the results obtained in 1.5. Let us consider the following infinite system of algebraic equations

$$a^n\varphi_n - \sum_{m=-\infty}^{\infty} K_{n-m}\varphi_m = f_n, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{1.9.1}$$

where $|a| \neq 1$ is a complex constant; $f = \{f_n\}_{-\infty}^{\infty}$, $K = \{k_n\}_{-\infty}^{\infty}$ are given vectors and $\varphi = \{\varphi_n\}$ is an unknown vector in the space l_1. It can be assumed without loss of generality that $|a| > 1$.

The space l_1 is the commutative normed ring where the operation of multiplication is defined by the convolution

$$K\varphi = K * \varphi = \left\{ \sum_{m=-\infty}^{\infty} K_{n-m}\varphi_m \right\}_{-\infty}^{\infty}$$
$$= \left\{ \sum_{m=-\infty}^{\infty} \varphi_{n-m}K_m \right\}_{-\infty}^{\infty} = \varphi * K = \varphi K. \quad (1.9.2)$$

Assume

$$\psi = \{\psi_n\}_{-\infty}^{\infty} \in l_1$$

and consider the function

$$\Psi(t) = \sum_{n=-\infty}^{\infty} \psi_n t^n, \quad t = e^{i\theta}, \quad 0 \le \theta \le 2\pi. \quad (1.9.3)$$

Thus to each vector of the space l_1 there corresponds a function which is the sum of an absolutely summable Fourier series and, vice versa, to each function defined on the circumference $|t| = 1$ which can be expanded into an absolutely summable Fourier series there corresponds one vector of the ring l_1 of the form

$$\psi_n = \left\{ \frac{1}{2\pi i} \int\limits_{|t|=1} \Psi(t)t^{-(n+1)}\, dt \right\}_{-\infty}^{\infty}. \quad (1.9.4)$$

The class of functions defined on the circumference $|t| = 1$ and expandable into an absolutely converging Fourier series is called the Wiener class and denoted by W.

Equality (1.9.3), which to each vector $\psi \in l_1$ puts into correspondence a function $\Psi \in W$, is called the discrete Fourier transform, and equality (1.9.4) which provides the reciprocal correspondence is called the reciprocal discrete Fourier transform.

Since the discrete Fourier transform of a convolution is the product of functions, the class W is a commutative normed ring with the ordinary operation of multiplication where the norm is defined as follows

$$\|\Psi\|_W = \|\psi\|_{l_1} = \sum_{n=-\infty}^{\infty} |\psi_n|.$$

The system of equations

$$\overline{a}^{\,n}\psi_n - \sum_{m=-\infty}^{\infty} \overline{K}_{m-n}\psi_m = c_n, \quad n = 0, \pm 1, \pm 2, \dots, \quad (1.9.5)$$

where $c = \{c_n\}_{-\infty}^{\infty}$ is a given vector and $\psi = \{\psi_n\}_{-\infty}^{\infty}$ is the unknown vector of the ring l_1, will be called the adjoint system to (1.9.1).

After carrying out discrete Fourier transformation of systems (1.9.1) and (1.9.5) and applying the convolution transformation property, we obtain

$$\Phi(at) = K(t)\Phi(t) + F(t), \quad t \in \gamma, \tag{1.9.6}$$

$$\Psi(\overline{a}t) = \overline{K(t)}\Psi(t) + C(t), \quad t \in \gamma, \tag{1.9.7}$$

where

$$K(t) = \sum_{n=-\infty}^{\infty} K_n t^n, \quad F(t) = \sum_{n=-\infty}^{\infty} f_n t^n, \quad C(t) = \sum_{n=-\infty}^{\infty} c_n t^n, \quad t \in \gamma,$$

while

$$\Phi(z) = \sum_{n=-\infty}^{\infty} \varphi_n z^n, \quad \Psi(t) = \sum_{n=-\infty}^{\infty} \psi_n z^n, \quad z \in D, \quad D = \{1 < |z| < R\},$$

are the sought analytic functions with boundary values belonging to the ring W.

We have studied these problems in 1.5 for the case where the coefficient and free term belong to the class H (Hölder). But, as is well known, a function of the class H cannot belong to the class W and conversely. Therefore, in that case additional investigation of the problem is needed.

The next two theorems play an essential role in the study of problem (1.9.6).

Wiener–Levy Theorem. *If the function $\Omega(z)$ is analytic in the domain S, $F(t) \in W$ and its pre-image $F(\gamma) \in S$, then, together with F, also $\Omega(F) \in W$.*

Theorem. *If $F(t) \in W$, then there exists a Cauchy principal value of the singular integral*

$$\frac{1}{\pi i} \int\limits_{\gamma} \frac{F(t)}{t - t_0}\, dt, \quad t \in \gamma,$$

belonging to the class W.

With the aid of these theorem the solution of problem (1.9.6) can be constructed in the same manner as in Section 1.5, i.e. for $\varkappa > 0$ or $\varkappa = 0$ and $a^n - \mu \neq 0$ the solution can given by the formula

$$\Phi(z) = \frac{X(z)}{2\pi i} \int\limits_{\gamma} K_\mu\left(\frac{z}{t}\right) \frac{F(t)}{tX(at)}\, dt + X(z)\varphi_\varkappa(z), \quad z \in D, \tag{1.9.8}$$

where X, K_μ, $\varphi_\varkappa$ and μ are the same as in 1.5. If $\varkappa < 0$, then for problem (1.9.1) to be solvable it is necessary and sufficient that

$$\int\limits_{\gamma} F(t)\overline{\Psi_j(t)}\, \frac{dt}{t} = 0, \quad j = 1, 2, \ldots, -\varkappa, \tag{1.9.9}$$

where the functions $\Psi_j(t)$ are the boundary values of the functions $\Psi_j(z)$, $j = 1, 2, \dots, -\varkappa$, which make up a complete system of linearly independent solutions of the homogeneous problem adjoint to the considered one.

If $\varkappa = 0$ and $a^m - \mu = 0$ for some integer number m, then the homogeneous problems corresponding to problems (1.9.6) and (1.9.7) have each a solution, while for the nonhomogeneous problem (1.9.6) to be solvable it is necessary and sufficient that the condition

$$\int_\gamma F(t)\overline{\Psi(t)}\,\frac{dt}{t} = 0 \tag{1.9.10}$$

be fulfilled.

Since between the rings W and l_1 there arises an isomorphism, the above propositions hold also for system (1.9.1) whose solution is given by the inverse transform

$$\varphi = \left\{ \frac{1}{2\pi i} \int_\gamma \frac{\Phi(t)}{t^{n+1}}\,dt \right\}_{-\infty}^{\infty}. \tag{1.9.11}$$

Thus, if $\varkappa > 0$ or $\varkappa = 0$ and $a^n - \mu \neq 0$, $n = 0, \pm 1, \pm 2, \dots$, then the homogeneous system corresponding to system (1.9.1) has $\varkappa$ linearly independent solutions, its adjoint system has only a trivial solution, and the nonhomogeneous system (1.9.1) is always solvable.

If $\varkappa = 0$ and $a^m = \mu$ for some integer number m, then the homogeneous system corresponding to system (1.9.1) and its adjoint homogeneous system have one solution each. For the nonhomogeneous system (1.9.1) to be solvable it is necessary and sufficient that the condition

$$\sum_{n=-\infty}^{\infty} f_n \overline{\psi_n^{(0)}} = 0 \tag{1.9.12}$$

be fulfilled. This condition arises from condition (1.9.10) where $\psi = \{\psi_n\}_{-\infty}^{\infty}$ is a solution of the adjoint homogeneous system to (1.9.1).

If $\varkappa < 0$, then the homogeneous system corresponding to (1.9.1) has no solution, its adjoint system has $\varkappa$ linearly independent solutions, while for the nonhomogeneous system (1.9.1) to be solvable it is necessary and sufficient that the conditions

$$\sum_{n=-\infty}^{\infty} f_n \overline{\psi_n^{(j)}} = 0, \quad j = 1, 2, \dots, -\varkappa, \tag{1.9.13}$$

be fulfilled, where

$$\psi^{(j)} = \{\psi_n^j\}_{-\infty}^{\infty}, \quad j = 1, 2, \dots, -\varkappa,$$

are solutions of the homogeneous system corresponding to system (1.9.5).

In [**53**], system (1.9.1) is investigated for $|a| = 1$ by reducing it to the functional equation

$$\Phi(te^{i\theta}) - K(t)\Phi(t) = F(t), \quad t \in \gamma. \tag{1.9.14}$$

An important particular case of equation (1.9.14) is investigated for $K(t) \equiv 1$ in [**6**], [**7**]. It is obvious that the case investigated by us differs essentially from the case considered in these works since for equation (1.9.14) to be solvable it is necessary and sufficient that the condition $\operatorname{Ind} K(t) = 0$ be fulfilled.

CHAPTER 2

The Contact Problems for Unbounded Domains with Rectilinear Boundaries

2.1. Some Basic Formulas of the Elasticity Theory

In the sequel, we will use some basic formulas of the plane static elasticity theory. These formulas establish a relation of the stress $(\sigma_x, \sigma_y, \tau_{xy})$ and displacement (u, v) component to analytic functions of a complex variable.

For an isotropic body, stress and displacement components are expressed through two analytic functions by the well known Kolosov–Muskhelishvili formulas [**77**]

$$i\int_{z_0}^{z}(X_n + iY_n)\,ds = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + const, \tag{2.1.1}$$

$$2\mu(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{2.1.2}$$

where $\varphi(z)$ and $\Psi(z)$ are analytic functions in the domain S occupied by the body; μ is the shear modulus; $\varkappa = 3 - 4\nu$ for plane deformation; $\varkappa = (3-\nu)/(1+\nu)$ for plane stressed state; ν is Poisson's ratio; the integral is taken over any smooth arc l lying within the domain S and connecting the fixed point z_0 with the variable point z of the domain S; X_n and Y_n are the components of stress acting on the arc from the side of the positive normal, i.e.the normal directed to the right if one looks along the positive direction l. As is known,

$$\begin{aligned} X_n &= \sigma_x \cos(n,x) + \tau_{xy}\cos(n,y), \\ Y_n &= \tau_{xy}\cos(n,x) + \sigma_y\cos(n,y). \end{aligned} \tag{2.1.3}$$

For the case of an anisotropic body, S. G. Lekhnitski showed in [**67**] that if the equation

$$a_{11}s^4 - 2a_{16}s^3 + (2a_{12} + a_{66})s^2 - 2a_{26}s + a_{22} = 0, \tag{2.1.4}$$

where a_{11}, a_{12}, a_{22}, a_{16}, a_{26}, a_{66} are real constants depending on the elastic properties of the considered body, has no multiple roots, i.e. has four different pairwise conjugate roots[1]

$$s_1 = \alpha_1 + i\beta_1, \quad s_2 = \alpha_2 + i\beta_2, \quad s_3 = \alpha_1 - i\beta_1, \quad s_4 = \alpha_2 - i\beta_2;$$

[1] In [**67**] it was established that equation (2.1.4) has no real roots.

then stresses and displacements are expressed through two analytic functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ of the variables

$$z_1 = x_1 + iy_1 = (x + \alpha_1 y) + i(\beta_1 y),$$
$$z_2 = x_2 + iy_2 = (x + \alpha_2 y) + i(\beta_2 y)$$

as follows:

$$\begin{aligned} \sigma_x &= 2\,\mathrm{Re}\left[s_1^2\Phi_1(z_1) + s_2^2\Phi_2(z_2)\right], \\ \sigma_y &= 2\,\mathrm{Re}\left[\Phi_1(z_1) + \Phi_2(z_2)\right], \\ \tau_{xy} &= -2\,\mathrm{Re}\left[s_1\Phi_1(z_1) + s_2\Phi_2(z_2)\right], \end{aligned} \tag{2.1.5}$$

$$\begin{aligned} u &= 2\,\mathrm{Re}\left[p_1\varphi_1(z_1) + p_2\varphi_2(z_2)\right], \\ v &= 2\,\mathrm{Re}\left[q_1\varphi_1(z_1) + q_2\varphi_2(z_2)\right]. \end{aligned} \tag{2.1.6}$$

Here

$$\begin{aligned} p_1 &= a_{11}s_1^2 + a_{12} - a_{16}s_1, \\ p_2 &= a_{11}s_2^2 + a_{12} - a_{16}s_2^2, \\ q_1 &= a_{12}s_1 + a_{22}/s_1 - a_{26}, \\ q_2 &= a_{12}s_2 + a_{22}/s_2 - a_{26}, \\ \Phi_1(z_1) &= \frac{d\varphi_1(z_1)}{dz_1}\,, \\ \Phi_2(z_2) &= \frac{d\varphi_2(z_2)}{dz_2}\,. \end{aligned} \tag{2.1.7}$$

We also present the formula ([**66**, Section 8]) which can be used instead of (2.1.5)

$$(1 + is_1)\varphi_1(z_1) + (1 + i\overline{s}_1)\overline{\varphi_1(z_1)} + (1 + is_2)\varphi_2(z_2) + (1 + i\overline{s}_2)\overline{\varphi_2(z_2)}$$
$$= i\int_{z_0}^{z} (X_n + iY_n)\,ds + const. \tag{2.1.8}$$

If equation (2.1.4) has multiple roots, then stresses and displacements are expressed by formulas analogous to the formulas for isotropic body.

In the particular case, if the body is orthotropic and the direction of the x- and y-axes coincides with the principal directions of elasticity, then equation (2.1.4) takes the form

$$\frac{1}{E_1}\,s^4 + \left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right)s^2 + \frac{1}{E_2} = 0. \tag{2.1.9}$$

Here E_1, E_2 are Young's modulus with respect to the principal $x-$ and $y-$axes; G is the shear modulus, ν_1 is Poisson's ratio characterizing contraction along the y-axis for extension (compression) along the x-axis.

The roots of equation (2.1.4) are purely imaginary; $s_1 = i\beta_1$, $s_2 = i\beta_2$. It is assumed that $\beta_1 > \beta_2$.

Then

$$a_{16} = a_{26} = 0, \quad a_{11} = \frac{1}{E_1}\,, \quad a_{12} = -\frac{\nu_1}{E_1} = -\frac{\nu_2}{E_2}\,, \quad a_{66} = \frac{1}{G}\,,$$
$$p_1 = -\frac{\beta_1^2 + \nu_1}{E_1}\,, \quad p_2 = -\frac{\beta_2^2 + \nu_1}{E_1}\,, \tag{2.1.10}$$
$$q_1 = -i\left(\frac{\nu_1\beta_1}{E_1} + \frac{1}{E_2\beta_1}\right), \quad q_2 = -i\left(\frac{\nu_1\beta_2}{E_1} + \frac{1}{E_2\beta_2}\right).$$

Taking into account that $\beta_1^2\beta_2^2 = E_1/E_2$, we obtain

$$q_1 = -\frac{i\beta_1(\beta_2^2 + \nu_1)}{E_1}\,, \quad q_2 = -\frac{i\beta_2(\beta_1^2 + \nu_1)}{E_1}\,.$$

2.2. A Contact Problem for a Wedge with an Elastic Fastening

Let us assume that a thin isotropic wedge-shaped plate on the plane $z = x + iy$ occupies an angle $-\alpha < \arg z < 0$, $0 < \alpha \le 2\pi$.

Let one side $\arg z = -\alpha$ of the angle be free or fastened and a rectilinear rod be pasted to the other side $\arg z = 0$. We are to define the law of distribution of contact forces along the fastening line and the elastic equilibrium of the plate when the concentrated force P directed along the x-axis is applied to the rod end. The rod rigidity in bending is assumed to be negligibly small, i.e. $\sigma_y = 0$.

From the condition of equilibrium of any part $(0, x)$ of the rod we have

$$P + S_0\sigma_x^{(0)} - h\int\limits_0^x \tau_{xy}^{(0)}(s)\,ds = 0, \quad x > 0. \tag{2.2.1}$$

Here $\sigma_x^{(0)}$ is the normal stress acting in an arbitrary cross-section of the rod, $\tau_{xy}^{(0)}$ is the tangent stress acting on the rod along the contact line, S_0 is the rod cross-section, h is the plate thickness.

A condition of a complete contact of the elastic rod with the wedge has the form

$$\frac{du_0(x)}{dx} = \frac{du(x,0)}{dx}\,, \quad \tau_{xy}^{(0)} = \tau_{xy}(x;0) = \tau(x), \quad x > 0. \tag{2.2.2}$$

Furthermore, taking into account that $\sigma_y^{(0)} = \sigma_y = 0$, by Hooke's law we obtain

$$\frac{du_0(x)}{dx} = \frac{\sigma_x^{(0)}}{E_0}\,, \quad \frac{du(x,0)}{dx} = \frac{\sigma_x(x,0)}{E}\,, \tag{2.2.3}$$

where E and E_0 are respectively the elasticity moduli of the place and the rod.

By virtue of (2.2.2) and (2.2.3), condition (2.2.1) takes the form

$$P + K\sigma_x - h\int\limits_0^x \tau(s)\,ds = 0, \quad x > 0, \tag{2.2.4}$$

where

$$K = \frac{S_0 E_0}{E} \,.$$

Using the Kolosov–Muskhelishvili formulas the problem posed reduces to finding two holomorphic functions $\Phi(z)$ and $\Psi(z)$ in the angle by the boundary conditions

$$\Phi(t) + \overline{\Phi(t)} + t\overline{\Phi'(t)} + \overline{\Psi(t)} = -\tau(t), \quad t = x > 0, \tag{2.2.5}$$

$$K_1\Phi(t) - \overline{\Phi(t)} - e^{2i\alpha}\big[t\overline{\Phi'(t)} + \overline{\Psi(t)}\,\big] = 0, \quad \arg t = -\alpha, \tag{2.2.6}$$

$$2K\big[\Phi(t) + \overline{\Phi(t)}\,\big] = h\int\limits_0^t \tau(s)\,ds - P, \quad t = x > 0, \tag{2.2.7}$$

where

$K_1 = -1$ if stresses are given on the boundary;

$K_1 = \varkappa$ if the boundary is fastened.

Let us introduce the notation

$$\Psi_1(z) = \Phi(z) + z\Phi'(z) + \Psi(z), \quad -\alpha \le \arg z \le 0.$$

Then formulas (2.2.5) and (2.2.6) can be rewritten in the form

$$\Phi(t) + \overline{\Psi_1(t)} = -i\tau(t), \quad t = x > 0, \tag{2.2.8}$$

$$K_1\Phi(t) - (1 - e^{2i\alpha})(t\overline{\Phi(t)})' - e^{2i\alpha}\overline{\Psi_1(t)} = 0, \quad \arg t = -\alpha. \tag{2.2.9}$$

Of the functions $\Phi(z)$ and $\Psi_1(z)$ it is required that for large $|z|$ they have the form

$$\Phi(z) = O\left(\frac{1}{z}\right), \quad \Psi_1(z) = O\left(\frac{1}{z}\right)$$

and, near the angle vertex, satisfy the condition

$$z\Phi(z) \to 0, \quad z\Psi_1(z) \to 0 \;\text{ as }\; z \to 0.$$

We will seek these functions in the form

$$\Phi(z) = \frac{1}{\sqrt{2\pi}\,z}\int\limits_{-\infty}^{\infty} \frac{A_1(t)}{t}\,e^{-it\ln z}\,dt - \frac{c_1}{z}\,, \quad -\alpha < \arg z < 0, \tag{2.2.10}$$

$$\Psi_1(z) = \frac{1}{\sqrt{2\pi}\,z}\int\limits_{-\infty}^{\infty} \frac{A_2(t)}{t}\,e^{-it\ln z}\,dt - \frac{c_2}{z}\,, \quad -\alpha < \arg z < 0, \tag{2.2.11}$$

where

$$c_k = \lim_{z\to 0}\int\limits_{-\infty}^{\infty} \frac{A_k(t)}{t}\,e^{-it\ln z}\,dt = i\sqrt{\frac{\pi}{2}}\,A_k(0), \quad k = 1, 2. \tag{2.2.12}$$

In formulas (2.2.10) and (2.2.11), the integrals at the point $t = 0$are understood in the sense of Cauchy principal value.

It is not difficult to show that

$$\lim_{z\to\infty} z\Phi(z) = -2c_1, \quad \lim_{z\to\infty} z\Psi_1(z) = -2c_2.$$

Let us require of the function $\tau(x)$ that $x\tau(x) \to 0$ as $x \to 0$ or $x \to \infty$.

Then from condition (2.2.9) we obtain $x\Phi(x) + x\overline{\Psi_1(x)} \to 0$ as $x \to \infty$. Hence it should be required that $c_1 + \bar{c}_2 = 0$, i.e. $c_2 = -\bar{c}_1$ or, which is the same, $A_1(0) = A_2(0)$.

The substitution of values (2.2.10) and (2.2.11) into formulas (2.2.8) and (2.2.9) and the Fourier transform yield

$$\begin{cases} A_1(t) - \overline{A_2(-t)} = -itT(t), \\ K_1 e^{-\alpha t} A_1(t) + i(1-e^{-2i\alpha})te^{\alpha t}\overline{A_1(-t)} + e^{\alpha t}\overline{A_2(-t)} = 0, \end{cases} \tag{2.2.13}$$

where

$$T(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \tau(e^s)e^s e^{its}\, ds.$$

Since $\overline{T(-t)} = T(t)$, a solution of system (2.2.13) has the form

$$A_1(t) = -\frac{(K_1 e^{2\alpha t} + 1 + 2te^{i\alpha}\sin\alpha)itT(t)}{2K_1\,\mathbf{ch}\,2\alpha t + K_1^2 + 1 + 4t^2\sin^2\alpha}\,, \tag{2.2.14}$$

$$A_2(t) = \overline{A_1(-t)} + itT(t). \tag{2.2.15}$$

Using formulas (2.2.14) and (2.2.10) , from condition (2.2.7) we obtain

$$\frac{1}{2i\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{K_1\,\mathbf{sh}\,2\alpha t + t\sin 2\alpha}{\left(\frac{K_1+1}{2}\right)^2 + K_1\,\mathbf{sh}^2\,\alpha t + t^2\sin^2\alpha}\, T(t)e^{-it\ln x}\, dt$$
$$= Hx\bigg(\int\limits_0^x \tau(s)\, ds - \frac{P}{h}\bigg) + 2\operatorname{Re} c_1, \tag{2.2.16}$$

where $H = h/2K$.

Hence, in the case $K_1 = -1$, i.e. when one side $\arg z = -\alpha$ of the angle is free from external stresses, we obtain

$$\frac{1}{2i\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{\mathbf{sh}\,2\alpha t - t\sin 2\alpha}{\mathbf{sh}^2\,\alpha t - t^2\sin^2\alpha}\, T(t)e^{-it\ln x}\, dt$$
$$= Hx\bigg(\int\limits_0^x \tau(s)\, ds - \frac{P}{h}\bigg) + 2\operatorname{Re} c_1, \tag{2.2.17}$$

and, in the case $K_1 = \varkappa$, i.e. when the side $\arg z = -\alpha$ of the angle is rigidly fixed, we have

$$\frac{1}{2i\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{\varkappa \,\mathbf{sh}\, 2\alpha t + t \sin 2\alpha}{\left(\frac{\varkappa+1}{2}\right)^2 + \varkappa \,\mathbf{sh}^2\, \alpha t + t^2 \sin^2 \alpha} T(t) e^{-it \ln x}\, dt$$
$$= Hx\bigg(\int\limits_0^x \tau(s)\, ds - \frac{P}{h}\bigg) + 2 \operatorname{Re} c_1, \quad (2.2.18)$$

Though equations (2.2.17) and (2.2.18) look superficially alike, they essentially differ from each other. Indeed, the point $t = 0$ is a pole of first order for the coefficients of the unknown function $T(t)$ in equation (2.2.17), and a zero of first order in equation (2.2.18). Therefore these equations should be considered separately.

Thus the problems posed reduce to equations (2.2.17) and (2.2.18).

We will first consider the case where the side $\arg z = -\alpha$ of the angle is free, i.e. $K_1 = -1$. Then from equalities (2.2.12) and (2.2.14) we obtain

$$c_1 = \frac{\pi}{2\sqrt{2\pi}} \, \frac{\alpha - e^{-i\alpha} \sin \alpha}{\alpha^2 - \sin^2 \alpha} \, T(0),$$

and since

$$\int\limits_0^\infty \tau(s)\, ds = \frac{P}{h}\,,$$

we have

$$T(0) = \frac{P}{\sqrt{2\pi}\, h} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^s \tau(e^s)\, ds = \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty \tau(s)\, ds$$

and

$$c_1 = \frac{\alpha - e^{-i\alpha} \sin \alpha}{\alpha^2 - \sin^2 \alpha} \, \frac{P}{4h}\,.$$

After substituting

$$\ln x = \xi$$

and introducing the notation

$$G(t) = \frac{\mathbf{sh}^2\, 2\alpha t - t \sin 2\alpha}{\mathbf{sh}^2\, \alpha t - t^2 \sin^2 \alpha}\, t,$$

equation (2.2.17) takes the form

$$\frac{1}{2i\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\frac{G(t)}{t}\,T(t)e^{-it\xi}\,dt = He^{\xi}\bigg(\int\limits_{-\infty}^{\xi}e^{s}\tau(e^{s})\,ds-\frac{P}{h}\bigg)+2\operatorname{Re}c_1. \tag{2.2.19}$$

Since the function in the right-hand part is not integrable (it does not vanish as $x\to\infty$), this equation cannot be solved by the standard Fourier transform.

Let us introduce a new function

$$\varphi_1(\xi)=\int\limits_{-\infty}^{\xi}e^{s}\tau(e^{s})\,ds-\frac{e^{\pi\xi}(P-\lambda e^{-\xi})}{h(1+e^{\pi\xi})}\,, \tag{2.2.20}$$

where

$$\lambda=8K\operatorname{Re}c_1.$$

After carrying out Fourier transformation in equality (2.2.20) we obtain

$$T(t)=-it\Phi_1(t)+\frac{P}{\sqrt{2\pi}\,h}\,\frac{t}{\mathbf{sh}\,t}-\frac{\lambda}{\sqrt{2\pi}\,h}\,\frac{t}{\mathbf{sh}(t+i)}\,, \tag{2.2.21}$$

where $\Phi_1(t)$ is the Fourier transform of the function $\varphi_1(\xi)$.

Taking into account formulas (2.2.20) and (2.2.21) and making some elementary transformations, from equation (2.2.19) we obtain for the function $\varphi_1(\xi)$ the equation

$$\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}G(t)\Phi_1(t)e^{-it\xi}\,dt+He^{\xi}\varphi_1(\xi)=f(\xi),\quad -\infty<\xi<\infty, \tag{2.2.22}$$

where

$$f(\xi)=\frac{Pe^{\xi}}{2K(1+e^{\pi\xi})}+\frac{P}{2\pi ih}\int\limits_{-\infty}^{\infty}\frac{G(t)-G(0)}{\mathbf{sh}\,t}\,e^{-it\xi}\,dt -\frac{\lambda}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{G(t)}{\mathbf{sh}(t+i)}\,e^{-it\xi}\,dt.$$

It is easy to see that the function $f(\xi)$ is integrable all over the axis, while the integrand $G(t)\Phi_1(t)$ in (2.2.22) is continuous at the point $t=0$.

Carrying out Fourier transformation in equation (2.2.22) we obtain

$$G(t)\Phi_1(t)+H\Phi_1(t-i)=F(t), \tag{2.2.23}$$

where

$$F(t)=\frac{P}{2\sqrt{2\pi}\,Ki\,\mathbf{sh}(t-i)}-P\,\frac{G(t)-G(0)}{\sqrt{2\pi}\,ih\,\mathbf{sh}\,t}-\frac{\lambda G(t)}{\sqrt{2\pi}\,i\,\mathbf{sh}(t+i)}\,. \tag{2.2.24}$$

Thus we reduce the considered problem to the problem studied in Section 1.3 in the case where the coefficient and the free term are meromorphic functions.

To meet our purposes, the problem with the boundary condition (2.2.23) is formulated as follows:

Using condition (2.2.23),find a function $\Phi_1(w)$, which is analytic in the strip $-1 < \operatorname{Im} w < 1$ except for a finite number of points lying in the strip $0 < \operatorname{Im} w < 1$ where it may have poles, and which vanishes at infinity.

It is obvious that if we manage to find a function $\Phi_1^-(w)$ which is holomorphic in the strip $-1 < \operatorname{Im} w < 0$, continuous on the boundary and satisfying condition (2.2.23), the solution of the formulated problem is

$$\Phi_1(w) = \begin{cases} \Phi_1^-(w), & -1 < \operatorname{Im} w < 0, \\ \Phi_1^+(w), & 0 < \operatorname{Im} w < 1, \end{cases} \tag{2.2.25}$$

where

$$\Phi_1^+(w) = \frac{F(w) - H\Phi(w-i)}{G(w)}\,, \quad 0 < \operatorname{Im} w < 1. \tag{2.2.26}$$

But since the function $F(w)$ is holomorphic in the strip $0 < \operatorname{Im}(w) < 1$, the function $\Phi_1^+(w)$ will have poles at the points which are zeros of the function $G(w)$.

Let us write the function $G(t)$ in the form

$$G(t) = tG_0(t) = itG_0(t)\,\mathbf{th}\,\frac{\pi}{2}\,t \cdot \frac{\mathbf{sh}\,\frac{\pi}{2}\,(t-i)}{\mathbf{sh}\,\frac{\pi}{2}\,t}\,.$$

Since the index of the function $G_0(t)\,\mathbf{th}\,\frac{\pi}{2}\,t$ is equal to zero and $\ln\big[G_0(t)\,\mathbf{th}\,\frac{\pi}{2}\,t\big] \in L_1(-\infty,\infty)$, this function can be represented as

$$G_0(t)\,\mathbf{th}\,\frac{\pi}{2}\,t = \frac{X_0(t-i)}{X_0(t)}\,, \tag{2.2.27}$$

where

$$X_0(w) = \exp\bigg(-\frac{1}{2i}\int\limits_{-\infty}^{\infty} \ln\Big[G_0(t)\,\mathbf{th}\,\frac{\pi}{2}\,t\Big]\,\mathbf{cth}\,\pi(t-w)\,dt\bigg).$$

The function $X_0(z)$ is holomorphic in the strip, continuous on the boundary and bounded in the closed strip $-1 \le \operatorname{Im} w \le 0$ and at infinity.

Therefore for the function $G(t)$ we have

$$G(t) = it\,\frac{X_0(t-i)\,\mathbf{sh}\,\frac{\pi}{2}\,(t-i)}{X_0(t)\,\mathbf{sh}\,\frac{\pi}{2}\,t}\,. \tag{2.2.28}$$

Substituting (2.2.28) into condition (2.2.23) and introducing the notation

$$\Psi_2(w) = \frac{iw\Phi_1^-(w)}{X_0(w)\,\mathbf{sh}\,\frac{\pi}{2}\,w}\,, \tag{2.2.29}$$

we obtain

$$(1+it)\Psi_2(t)+H\Psi_2(t-i)=\frac{F(t)(1+it)}{X_0(t-i)\,\mathbf{sh}\,\frac{\pi}{2}\,(t-i)}\,.$$

As follows from Section 1.3, the solution of the problem is given by the formula

$$\Psi_2(w)=-\frac{X_1(w)}{2iH}\int\limits_{-\infty}^{\infty}\frac{F(t)(1+it)}{X(t)\,\mathbf{sh}\,\pi(t-w)}\,dt\,,\quad -1<\operatorname{Im}w<0, \tag{2.2.30}$$

where

$$X(w)=X_0(w)\,\mathbf{sh}\,\frac{\pi}{2}\,wX_1(w),$$
$$X_1(w)=\exp(-iw\ln H)\Gamma(1+iw).$$

Now (2.2.29) implies

$$\Phi_1^-(w)=-\frac{X(w)}{2Hw}\int\limits_{-\infty}^{\infty}\frac{F(t)(1+it)}{X(t)\,\mathbf{sh}\,\pi(t-w)}\,dt. \tag{2.2.31}$$

As shown in Section 1.3, the function $X_1(w)$ in the strip $-1<\operatorname{Im}w<0$ satisfies the condition

$$D_1|t|^{\frac{1}{2}}e^{-\frac{\pi}{2}\,|t|}<\big|X_1(t+i\tau)\big|<D_2|t|^{\frac{3}{2}}e^{-\frac{\pi}{2}\,|t|},$$

and therefore $X(w)$ in this strip admits an estimate

$$D_3|t|^{\frac{1}{2}}<\big|X(t+i\tau)\big|<D_4|t|^{\frac{3}{2}}.$$

Since the function $F(w)$ exponentially vanishes at infinity, it is easy to prove that the function $\Phi_1^-(w)$, too, possesses this property. Therefore the function $\Phi_1(w)$ defined by (2.2.25), (2.2.26) and (2.2.31) will be holomorphic in the strip $-1<\operatorname{Im}<1$, exponentially vanishing at infinity, bounded throughout the strip except for the points of the upper half of the strip which are zeros of the function $G(w)$ at which it has poles of first order.

Now due to formula (2.2.21) we conclude that the function $T(t)$ is analytically extendable in the strip $-1<\operatorname{Im}w<1$ except for the points of the strip $0<\operatorname{Im}w<1$ which are the roots of the function $G(w)$. Furthermore, the function $T(w)$ exponentially vanishes at infinity for $-1\le\operatorname{Im}w<1$, is continuously extendable all over the boundary except for the point $w=i$ and the points of the boundary $\operatorname{Im}w<1$ which are the roots of the function $G(w)$.

Since $T(t)$ is an integrable function, the contact tangent stress is defined by the inversion formula

$$\tau(x)=\frac{x^{-1}}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}T(t)e^{-it\ln x}\,dt,\quad x>0. \tag{2.2.32}$$

Let us investigate the behavior of this function at infinity and near the angular point $x=0$.

Recalling the character of the function $\Phi_1(w)$, by the Cauchy formula we can write

$$\varphi_1'(\ln x) = \frac{i}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} t\Phi_1(t)e^{-it\ln x}\,dt = -\frac{ix^{-1}}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} (t-i)\Phi_1(t-i)e^{-it\ln x}\,dt.$$

Therefore, for sufficiently large values of x we have

$$\varphi_1'(\ln x) = O\left(\frac{1}{x}\right).$$

Differentiating formula (2.2.20) and passing to the variable x, by the latter relation we obtain

$$\tau(x) = \frac{\lambda}{hx^2} + O\left(\frac{1}{x^2}\right). \tag{2.2.33}$$

If $\alpha < \beta$, where β is the smallest positive root of the equation $\operatorname{tg} 2\beta = 2\beta$ ($\beta \approx 2,247$), then it can be proved that the function $G(w)$ has no roots in the strip $0 \leq \operatorname{Im} w \leq 1$.

If $\alpha > \beta$, then the function $G(w)$ has roots in this strip, these roots being purely imaginary. The closest root to the real axis is denoted by $i\tau_0$.

If $\alpha = \beta$, then $G(w)$ has only one root $w = i$ in the strip.

Using formulas (2.2.21), (2.2.24) and (2.2.26), the function $T(t)$ can be represented as

$$T(t) = -\frac{Pt}{2\sqrt{2\pi}\,KG(t)\,\mathbf{sh}(t-i)} + T_0(t), \tag{2.2.34}$$

where $T_0(t)$ is analytically extendable in the strip $-\varepsilon < \operatorname{Im} w < 1+\varepsilon$ for $\alpha < \beta$ and has a root at the point $w = i$ for $\alpha = \beta$.

Carrying out the inverse Fourier transformation of equality (2.2.34) for $\alpha < \beta$ and applying the Cauchy formula, in the neighborhood of a zero we obtain the representation

$$\tau(x) = P\,\frac{\alpha\sin 2\alpha - 2\sin\alpha}{2\alpha\cos 2\alpha - \sin 2\alpha} + \varepsilon(x), \tag{2.2.35}$$

where

$$\varepsilon(x) \to 0 \ \text{ as } \ x \to 0.$$

When $\alpha = \beta$, equality (2.2.34) can be rewritten in the form

$$T(t) = -P\,\frac{\alpha\cos\alpha - \sin\alpha}{2\sqrt{2\pi}\,K\alpha^2\cos^2\alpha}\,\frac{1}{(t-i)^2} + T_1(t), \tag{2.2.36}$$

where the function $T_1(t)$ is holomorphic throughout the strip $-\varepsilon < \operatorname{Im} w < 1+\varepsilon$ except for the point $w = i$ where it has a pole of first order and for sufficiently large values of $|t|$ is represented as

$$T_1(t) = O\left(\frac{1}{t^2}\right).$$

Carrying out the inverse Fourier transformation of equality (2.2.36) and applying the Cauchy formula, we obtain

$$\tau(x) = -\frac{P(\alpha - \operatorname{tg}\alpha)}{2\alpha^2 K}\ln x + \varphi_0(x), \quad 0 < x < 1,$$

where $\varphi_0(x)$ is a bounded function.

If $\alpha > \beta$, then $T(w)$ has, at the point $w = i\tau_0$, a pole of first order and, using the generalized Cauchy theorem, from (2.2.32) we obtain

$$\tau(x) = mx^{\tau_0 - 1}(1 + \varphi_0(x)),$$

where

$$m = \sqrt{2\pi}\lim_{\tau\to\tau_0} T(t)(\tau - \tau_0),$$

and $\varphi_0(x)$ is a continuous function that vanishes at the point $x = 0$. It is obvious that if $\alpha = \pi; \frac{3}{2}\pi; 2\pi$, then, correspondingly, $\tau_0 = \frac{1}{2}; \frac{1}{3}; \frac{1}{4}$.

Let us consider the case where the side of the angle is rigidly fastened. If it is assumed that the function $\tau(x)$ is integrable, then by the Fourier transformation of equality (2.2.18) we obtain the homogeneous problem

$$\frac{\varkappa\,\mathbf{sh}\,2\alpha t + t\sin 2\alpha}{2\big[\big(\frac{\varkappa+1}{2}\big)^2 + \varkappa\,\mathbf{sh}^2\,\alpha t + t^2\sin^2\alpha\big]}\,T(t) + HT(t - i) = 0.$$

The coefficient of this problem vanishes at the point $t = 0$. It can be proved that this problem has no solution which is the Fourier transform of a summable function , i.e. if it is assumed that $\tau(x)$ is summable, then $x\tau(x) = 0$. Therefore $\tau(x) = 0$ except for the point $x = 0$ where it may turn into the Dirac function $\delta(x)$.

Thus we obtain that if one side of the angle is rigidly fastened and to the other side a stringer is pasted, whose end is subjected to the action of the tensile force P, then this force does not spread all over the body.

2.3. A Contact Problem for an Anisotropic Wedge with an Elastic Fastening

Let us now investigate the problem considered in Section 2.2 for the case where a thin plate occupying an angle $-\theta < \arg z < 0$, $0 < \theta < 2\pi$, on the plane is anisotropic. It is obvious that then the following formulas are valid:

$$P + S_0\sigma_x^{(0)} - h\int_0^x \tau_{xy}^{(0)}\,ds = 0, \quad x > 0, \tag{2.3.1}$$

$$\frac{du_0(x)}{dx} = \frac{du(x,0)}{dx}, \quad \tau_{xy}^{(0)}(x) = \tau_{xy}(x,0) = \tau(x), \quad x > 0. \tag{2.3.2}$$

Furthermore, since, by condition, $\sigma_y^{(0)}(x) = \sigma_y(x) = 0$, on the boundary $x > 0$, $y = 0$ Hooke's law takes the form

$$\frac{du^0(x)}{dx} = \frac{\sigma_x^0(x)}{E_0}, \quad \frac{du(x,0)}{dx} = a_{16}\tau_{xy}(x,0) + a_{11}\sigma_x(x,0), \tag{2.3.3}$$

where E_0 is the shear modulus of the rod; a_{11}, a_{16} are the elastic constants of the plate. By virtue of (2.3.2) and (2.3.3) condition (2.3.1) takes the form

$$P + K_1\sigma_x + K_2\tau(x) - h\int_0^x \tau(s)\,ds = 0, \quad x > 0, \tag{2.3.4}$$

where

$$K_1 = S_0E_0a_{11}, \quad K_2 = S_0E_0a_{16}.$$

Consider two planes of complex variables $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$ obtained respectively from the plane $z = x + iy$ by the affine transforms

$$x_k = x + \alpha_k y, \quad y_k = \beta_k y, \quad \beta_k > 0, \quad k = 1, 2,$$

where $s_k = \alpha_k + i\beta_k$ $(k = 1, 2)$ are the roots of equation (2.1.4) and, besides, $s_1 \neq s_2$.

By means of these transforms, the given domain S $(-\theta < \arg z < 0)$ transforms, on the plane of a complex variable z, to domains S_k $(-\theta_k < \arg z_k < 0)$ on the plane z_k $(k = 1, 2)$, where

$$\operatorname{tg}\theta_k = \frac{\beta_k\sin\theta}{\cos\theta - \alpha_k\sin\theta}\,, \quad 0 < \theta_k < 2\pi.$$

Due to formula (2.1.8), the above-formulated problem reduces to the solution of the following boundary value problem of the theory of functions of a complex variable: find two analytic functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ in the domains S_1 and S_2, respectively, by the boundary conditions

$$(s_1 - \overline{s}_2)t_1\Phi_1(t_1) + (\overline{s}_1 - \overline{s}_2)\overline{t}_1\overline{\Phi_1(t_1)} + (s_2 - \overline{s}_2)t_2\Phi_2(t_2) = 0, \tag{2.3.5}$$

$$t_k = \rho(\cos\theta - s_k\sin\theta), \quad \rho = |t| \geq 0,$$

$$(s_1 - \overline{s}_2)\Phi_1(t_1) + (\overline{s}_1 - \overline{s}_2)\overline{\Phi_1(t_1)} + (s_2 - \overline{s}_2)\Phi_2(t_2) = -\tau(x), \tag{2.3.6}$$

$$t_1 = t_2 = x > 0,$$

$$2\operatorname{Re}\Big[K_1a\Phi_1(x) + (K_2 - 2\alpha_2K_1)\tau_{xy}\Big] = h\int_0^x \tau_{xy}(s)\,ds - P, \quad x > 0, \tag{2.3.7}$$

where

$$a = (s_1 - s_2)(s_1 - \overline{s}_2).$$

Let us assume that stresses and rotations vanish at infinity. Then for large $|z_k|$ we write

$$\Phi_k(z_k) = \frac{\gamma_k}{z_k} + o\left(\frac{1}{z_k}\right), \quad k = 1, 2. \tag{2.3.8}$$

Let us also assume that the functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ are continuously extendable to all points of the boundary except perhaps for the points $z_k = 0$ at which they satisfy the conditions

$$\lim_{z_k\to 0} z_k\Phi_k(z_k) = 0.$$

Therefore we will seek analytic functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ in the form

$$\Phi_k(z_k) = \frac{1}{\sqrt{2\pi}\, z_k} \int\limits_{-\infty}^{\infty} \frac{A_k(t)}{t}\, e^{it\ln z_k}\, dt - \frac{c_k}{z_k}\,, \quad z_k \in S_k, \tag{2.3.9}$$

where

$$c_k = \lim_{z_k\to 0} \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{A_k(t)}{t}\, e^{it\ln z_k}\, dt, \quad k=1,2.$$

At the point $t=0$, the integrals are understood in the sense of the Cauchy principal value.

It can be easily shown that

$$c_k = i\sqrt{\frac{\pi}{2}}\, A_k(0) = -\lim_{z_k\to\infty} \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{A_k(t)}{t}\, e^{it\ln z_k}\, dt, \tag{2.3.10}$$

whence

$$\gamma_k = -2c_k = -i\sqrt{2\pi}\, A_k(0). \tag{2.3.11}$$

Formulas (2.3.9) and (2.3.5) imply that c_1 and c_2 satisfy the condition

$$(s_2-\overline{s}_2)c_2 = (\overline{s}_2 - s_1)c_1 + (\overline{s}_2 - \overline{s}_1)\overline{c}_1.$$

Substituting expressions (2.3.9) into the boundary conditions (2.3.5) and (2.3.6) and after that carrying out Fourier transformation, we obtain

$$\left.\begin{aligned} &(s_1-\overline{s}_2)A_1(t)e^{\delta t} - (\overline{s}_1-\overline{s}_2)\overline{A_1(-t)}e^{-\gamma t} + (s_2-\overline{s}_2)A_2(t)e^{i\mu t} = 0,\\ &(s_1-\overline{s}_2)A_1(t) - (\overline{s}_1-\overline{s}_2)\overline{A_1(-t)} + (s_2-\overline{s}_2)A_2(t) = -tT(t), \end{aligned}\right\} \tag{2.3.12}$$

where

$$\mu = \ln|\cos\theta - s_1\sin\theta| - \ln|\cos\theta - s_2\sin\theta|,$$
$$\gamma = \theta_1+\theta_2, \quad \delta = \theta_1-\theta_2,$$
$$T(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^s\tau(e^s)e^{-its}\, ds. \tag{2.3.13}$$

It is obvious that

$$\overline{T(-t)} = T(t).$$

Solving system (2.3.12) with respect to the unknown function $A_1(t)$, we have

$$A_1(t) = \frac{(\overline{s}_1-s_2)e^{-\delta t} + (\overline{s}_2-\overline{s}_1)e^{-\gamma t} + (s_2-\overline{s}_2)e^{-i\mu t}}{2\big[|s_1-s_2|^2\,\mathbf{ch}\,\gamma t - |s_1-\overline{s}_2|^2\,\mathbf{ch}\,\delta t + 4\beta_1\beta_2\cos\mu t\big]}\, tT(t). \tag{2.3.14}$$

The function A_2 is obtained from (2.3.14) by permutation of s_1 and s_2, θ_1 and θ_2.

We can prove that for real t the denominator of expression (2.3.14) does not vanish anywhere except for the point $t=0$. At this point it has a double root. Therefore the functions $A_1(t)$ and $A_2(t)$ are continuous throughout the axis if the function $\tau(x)$ is absolutely integrable. Further we will see that these functions are not only continuous but are also analytically extendable in some strip and analytically vanish at infinity. Therefore in equality (2.3.14), the integral at the point $t = 0$ exists in the sense of Cauchy principal value.

Since, by condition, stresses vanish at infinity, passing in (2.3.4) to infinity we obtain

$$T(0) = \frac{P}{\sqrt{2\pi}\, h}\,.$$

Therefore from (2.3.14) it follows that

$$A_1(0) = \frac{(\overline{s}_1 - \overline{s}_2)\gamma - (\overline{s}_1 - s_2)\delta - i\mu(s_2 - \overline{s}_2)}{|s_1 - s_2|^2\gamma^2 - |s_1 - \overline{s}_2|^2\delta^2 - 4\beta_1\beta_2\mu^2}\,\frac{P}{\sqrt{2\pi}\, h}\,. \tag{2.3.15}$$

Thus the constants c_1, c_2, γ_1, γ_2 can be defined by equalities (2.3.10), (2.3.11) and (2.3.15).

Substituting the values of $\Phi_1(z_1)$ defined by (2.3.14) and (2.3.9) into the boundary condition (2.3.7) and carrying out some transformations, we obtain

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}\, i}\int\limits_{-\infty}^{\infty}\left[\frac{\Delta_1(t)}{\Delta(t)} - i(\alpha_1 - \alpha_2)\right] T(t) e^{it\ln x}\, dt \\ &+ \left[\frac{K_2}{K_1} - (\alpha_1 + \alpha_2)\right] x\tau(x) - Hx\bigg(\int\limits_0^x \tau(s)\, ds - \frac{P}{h}\bigg) = 2\operatorname{Re} ac_1, \end{aligned} \tag{2.3.16}$$

where

$$\begin{aligned} \Delta(t) &= |s_1 - s_2|^2\, \mathbf{ch}\,\gamma t - |s_1 - \overline{s}_2|^2\, \mathbf{ch}\,\delta t + 4\beta_1\beta_2\cos\mu t, \\ \Delta_1(t) &= -(\beta_1 + \beta_2)|s_1 - s_2|^2\, \mathbf{sh}\,\gamma t \\ &\qquad + (\beta_1 - \beta_2)|s_1 - \overline{s}_2|^2\, \mathbf{sh}\,\delta t + 4|\alpha_1 - \alpha_2|\beta_1\beta_2\sin\mu t, \\ H &= \frac{h}{K_1}\,. \end{aligned}$$

Using the inverse Fourier transform, we have

$$\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} T(t) e^{it\ln x}\, dt = x\tau(x).$$

By means of the latter equality, equation (2.3.16) can be rewritten in the form

$$\frac{1}{i\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\frac{\Delta_1(t)}{\Delta(t)}\,T(t)e^{it\ln x}\,dt$$
$$+\left[\frac{K_2}{K_1}-(\alpha_1+\alpha_2)\right]x\tau(x)-Hx\bigg(\int\limits_0^x\tau(s)\,ds-\frac{P}{h}\bigg)=2\operatorname{Re}ac_1,$$

where $K_2/K_1 = a_{16}/a_{11}$ is a half of the second coefficient of the characteristic equation with the opposite sign. Hence by the Viéte formulas we have

$$\frac{K_2}{K_1}=\frac{1}{2}\left(s_1+\overline{s}_1+s_2+\overline{s}_2\right)=\alpha_1+\alpha_2$$

or, which is the same,

$$\frac{K_2}{K_1}-\alpha_1-\alpha_2=0.$$

Thus equation (2.3.16) can be rewritten as

$$\frac{1}{i\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\frac{\Delta_1(t)}{\Delta(t)}\,T(t)e^{it\ln x}\,dt-Hx\bigg(\int\limits_0^x\tau(s)\,ds-\frac{P}{h}\bigg)=2\operatorname{Re}ac_1. \quad (2.3.17)$$

Passing to the limit in equation (2.3.17) as $x\to 0$ we obtain

$$2\operatorname{Re}ac_1=-G(0)\,\frac{P}{2h}\,,$$

where

$$G(t)=\frac{\Delta_1(t)}{\Delta(t)}\,t.$$

The substitution of $\ln x=\xi$ makes equation (2.3.17) take the form

$$\frac{1}{i\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\frac{G(t)T(t)}{t}\,e^{it\xi}\,dt-He^{\xi}\bigg(\int\limits_{-\infty}^{\xi}\tau(e^s)e^s\,ds-\frac{P}{h}\bigg)$$
$$=-\frac{P}{2h}\,G(0). \quad (2.3.17_0)$$

Thus the considered problem reduces to an equation like (2.3.18) in the preceding paragraph.

Let us introduce the notation

$$\Psi_1(\xi)=\int\limits_{-\infty}^{\xi}e^s\tau(e^s)\,ds-e^{\pi\xi}\,\frac{P+\lambda e^{-\xi}}{h(1+e^{\pi\xi})}\,,\quad -\infty<\xi<\infty, \quad (2.3.18)$$

where

$$\lambda=\frac{K_1P}{h}\,G(0).$$

The inverse Fourier transformation of equality (2.3.18) gives

$$it\Psi_1(t) = T(t) + \frac{Pt}{\sqrt{2\pi}\,h\,\mathbf{sh}\,t} + \frac{\lambda t}{\sqrt{2\pi}\ \mathbf{sh}(t-i)}\,, \tag{2.3.19}$$

where $\Psi_1(t)$ is the inverse Fourier transform of the function $\Psi_1(\xi)$. Using (2.3.18) and (2.3.19), equation (2.3.17$_0$) can be rewritten in the form

$$\frac{1}{i\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} G(t)\Psi_1(t)e^{it\xi}\,dt - He^{\xi}\Psi_1(\xi) = f(\xi), \quad -\infty < \xi < \infty, \tag{2.3.20}$$

where

$$\begin{aligned} f(\xi) = &-\frac{P}{K_1}\,\frac{e^{\xi}}{1+e^{\pi\xi}} \\ &+ \frac{P}{2\pi ih}\int\limits_{-\infty}^{\infty} \frac{G(t)-G(0)}{\mathbf{sh}\,t}\,e^{it\xi}\,dt + \frac{\lambda}{2\pi ih}\int\limits_{-\infty}^{\infty} \frac{G(t)}{\mathbf{sh}(t-i)}\,e^{it\xi}\,dt. \end{aligned}$$

The inverse Fourier transformation of (2.3.20) yields

$$G(t)\Psi_1(t) - H\Psi_1(t+i) = F(t), \quad -\infty < t < \infty, \tag{2.3.21}$$

where

$$F(t) = \frac{P}{i\sqrt{2\pi}\,K_1\,\mathbf{sh}(t+i)} + \frac{P}{2\pi ih}\,\frac{G(t)-G(0)}{\mathbf{sh}\,t} + \frac{\lambda}{2\pi ih}\,\frac{G(t)}{\mathbf{sh}(t-i)}\,.$$

So, for isotropic and anisotropic plates the problems reduce to one and the same boundary value problem which is a particular case of the problem considered in Chapter 1. The free term and the coefficient of problem (2.3.21) are analytic in the strip except for the points which are poles of the function $G(t)$. Rewrite (2.3.21) as follows

$$\Psi_1(t) - H[G(t)]^{-1}\Psi_1(t+i) = F(t)G^{-1}(t). \tag{2.3.22}$$

The coefficient and the free term of problem (2.3.22) are analytic in the strip $-1 < \operatorname{Im} w < 1$ except for the points where $G(w) = 0$. At these points they have poles of first order.

The considered problem reduces to the following problem: find a function $\Psi_1(w)$ which is holomorphic in the strip $-1 < \operatorname{Im} w < 1$, vanishes at infinity, is bounded throughout the strip except for the points w_k ($k = 1, 2, \dots, n$) that are zeros of the function $G(w)$ in the lower half-plane and satisfies condition (2.3.22).

We introduce the notation

$$\Psi_1(w) = \begin{cases} \Psi_1^+(w), & 0 < \operatorname{Im} w < 1, \\ \Psi_1^-(w), & -1 < \operatorname{Im} w < 0, \end{cases}$$

where $\Psi_1^+(w)$ denotes the solution of the following problem: by the boundary condition (2.3.22) find a function which is holomorphic in the strip

$0 < \operatorname{Im} w < 1$, vanishes at infinity, is continuously extendable on the strip boundary.

By solving this problem the solution of the preceding problem can be constructed as follows

$$\Psi_1(w) = \begin{cases} \Psi_1^+(w), & 0 < \operatorname{Im} w < 1, \\ H\Psi_1(w+i)[G(w)]^{-1} + F(w)[G(w)]^{-1}, & -1 < \operatorname{Im} w < 0. \end{cases} \tag{2.3.23}$$

It is obvious that $\Psi_1(w)$ is holomorphic in the strip $-1 < \operatorname{Im} w < 1$ except for the points $w_k = t_k + i\tau_k$ $(k = 1, 2, \dots, n)$, $-1 < \tau_k < 0$. If $G(w)$ has no roots in the strip $-1 < \operatorname{Im} w < 0$, the function $\Psi_1(w)$ will be analytic throughout the strip $-1 < \operatorname{Im} w < 1$.

Let us introduce the notation

$$G_0(t) = -i\,\frac{\Delta_1(t)}{\Delta(t)(\beta_1+\beta_2)}\,\frac{\mathbf{sh}\,\frac{\pi}{2}\,t}{\mathbf{sh}\,\frac{\pi}{2}\,(t+i)}\,. \tag{2.3.24}$$

The function $G_0(t)$ is positive all over the axis, $G_0(-\infty) = G(\infty) = 1$, $\operatorname{Ind} G_0(t) = 0$ and $\ln G_0(t) \in L_1(-\infty;\infty)$. Therefore it can be represented as

$$G_0(t) = \frac{X_0(t+i)}{X_0(t)}\,, \quad -\infty < t < \infty, \tag{2.3.25}$$

where

$$X_0(w) = \exp\left(\frac{1}{2i}\int\limits_{-\infty}^{\infty} \ln G_0(t)\,\mathbf{cth}\,\pi(w-t)\,dt\right), \quad 0 < \operatorname{Im} w < 1. \tag{2.3.26}$$

Following (2.3.24) and (2.3.25), the function $G(t)$ can be written in the form

$$G(t) = it\,\frac{X_0(t+i)}{X_0(t)}\,\frac{\mathbf{sh}\,\frac{\pi}{2}\,(t+i)}{\mathbf{sh}\,\frac{\pi}{2}\,t}\,(\beta_1+\beta_2).$$

Inserting this value into condition (2.3.22) and introducing the notation

$$\Psi_0(w) = \frac{w\Psi_1^+(w)}{X(w)\,\mathbf{sh}\,\frac{\pi}{2}\,w}\,, \quad 0 < \operatorname{Im} w < 1, \tag{2.3.27}$$

we obtain

$$H_0(1-it)\Psi_0(t) + \Psi_0(t+i) = \frac{F(t)(it-1)}{HX_0(t+i)\,\mathbf{ch}\,\frac{\pi}{2}\,t}\,, \tag{2.3.28}$$

where

$$H_0 = \frac{\beta_1+\beta_2}{H}\,.$$

As has been shown above, $H_0(1-it)$ is representable in the form

$$H_0(1-it) = \frac{X_1(t+i)}{X_1(t)}\,, \tag{2.3.29}$$

where

$$X_1(t) = \Gamma(1-it)e^{-it\ln H_0}.$$

Substituting this value into the boundary condition (2.3.28) and taking into account equality (2.3.27) we obtain

$$\frac{t\Psi_1^+(t)}{X_0(t)X_1(t)\,\mathbf{sh}\,\frac{\pi}{2}\,t} + \frac{(t+i)\Psi_1^+(t+i)}{X_0(t+i)X_1(t+i)\,\mathbf{sh}\,\frac{\pi}{2}\,(t+i)}$$
$$= \frac{F(t)(it-1)}{HX_0(t+i)X_1(t+i)\,\mathbf{ch}\,\frac{\pi}{2}\,t}\,.$$

By virtue of (1.2.7) a solution of this problem has the form

$$\Psi_1^+(w) = \frac{X(w)}{2iHw}\int_{-\infty}^{\infty}\frac{F(t)(t+i)}{X(t+i)\,\mathbf{sh}\,\pi(t-w)}\,dt,\quad 0<\operatorname{Im}w<1, \tag{2.3.30}$$

where

$$X(w) = X_0(w)\Gamma(1-iw)\exp(-iw\ln H_0)\,\mathbf{sh}\,\frac{\pi}{2}\,w.$$

Since $F(t)$ exponentially vanishes at infinity and $X(w)$ satisfies the condition

$$c_3|t|^{\frac{1}{2}} < |X(w)| < c_4|t|^{\frac{3}{2}},\quad w=t+i\tau,\quad 0\le\tau\le 1$$

the integral in (2.3.30) exponentially decreases, i.e. $\Psi_1^+(w)$ is continuous in a closed strip $0<\operatorname{Im}w<1$ and exponentially vanishes at infinity.

Thus the solution of the problem posed is provided by (2.3.23). When $\tau_0<-1$, the solution is analytic throughout the strip $-1<\operatorname{Im}w<1$ and exponentially vanishes at infinity; when $\tau_0>-1$, the solution has poles of first order at the points $w_k = t_k + i\tau_k$ $(k=0,1,\dots,n)$. The function $T(t)$ defined by (2.3.19) is of the same nature and for it, as can be easily verified, the equality $\overline{T(-t)} = T(t)$ holds.

The stress $\tau(x)$ can be calculated by the formula

$$\tau(x) = \frac{x^{-1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}T(t)e^{it\ln x}\,dt.$$

Just like in the case of an isotropic body, for sufficiently large values of x, $\tau(x)$ has the form

$$\tau(x) = \frac{\lambda}{hx^2} + o\left(\frac{1}{x^2}\right).$$

For $\tau_0<-1$, $\tau(x)$ is bounded at the point $x=0$, while for $\tau_0>-1$ it has near this point the form

$$\tau(x) = \varphi_1(x)x^{|\tau_0|-1},$$

where $\varphi_1(x)$ is bounded at the point $x=0$.

As different from the case of an isotropic body, in the case of general anisotropy, as follows from the boundary condition, the stresses $\sigma_x(x)$ and

$\tau_{xy}(x)$ are simultaneously bounded or infinitely increase but so that the sum $K_1\sigma_x(x) + K_2\tau_{xy}(x)$ remains bounded.

Let us consider two particular cases:

1. When $\theta = \pi$, i.e. the body is a half-plane, we have

$$\Delta_1(t) = -(\beta_1 + \beta_2)|s_1 - s_2|^2 \,\mathbf{sh}\, 2\pi t,$$
$$\Delta(t) = 2|s_1 - s_2|^2 \,\mathbf{sh}^2\, \pi t$$

and

$$G(t) = -(\beta_1 + \beta_2)\,\frac{\mathbf{ch}\,\pi t}{\mathbf{sh}\,\pi t}\, t.$$

The function $G(w)$ has the only one, frequently imaginary, root $w_0 = -\frac{i}{2}$ in the strip $-1 < \operatorname{Im} w < 0$, while the contact stress near the point $x = 0$ has the form

$$\tau(x) = \frac{c}{\sqrt{x}} + \varphi_0(x),$$

where $\varphi_0(x)$ is a bounded function.

2. When $\theta = 2\pi$, i.e. when the body occupies the entire plane cut along the positive part of the real axis, we have

$$\Delta_1(t) = -(\beta_1 + \beta_2)|s_1 - s_2|^2 \,\mathbf{sh}\, 4\pi t,$$
$$\Delta(t) = 2|s_1 - s_2|^2 \,\mathbf{sh}^2\, \pi t$$

and

$$G(t) = -(\beta_1 + \beta_2)\,\frac{\mathbf{ch}\,2\pi t}{\mathbf{sh}\,2\pi t}\, t.$$

The function $G(w)$ has purely imaginary roots $w_0 = -\frac{1}{4}\,i$, $w_1 = -\frac{3}{4}\,i$ in the strip $-1 < \operatorname{Im} w < 0$, while the contact stress near the point has the form

$$\tau(x) = c_1 x^{-\frac{3}{4}} + c_2 x^{-\frac{1}{4}} + \varphi_0(x), \quad x > 0,$$

where $\varphi_0(x)$ is a bounded function.

Thus for an anisotropic body the stress $\tau(x)$ has the same features as for an isotropic case in analogous situations.

Let us now consider the case in which the body is orthotropic and the principal axes of anisotropy coincide with the coordinate axes. Then $a_{16} = \alpha_1 = \alpha_2 = 0$ and since $K_2 = a_{16}E_0S_0 = 0$, condition (2.3.4) takes the form

$$K_1\sigma_x + P - h\int\limits_0^x \tau(s)\, ds = 0,$$

i.e. the form it has in the case of an isotropic body. The stress $\sigma_x(x)$ is always bounded for $x = 0$. In this setting,

$$\Delta_1(t) = -(\beta_1 + \beta_2)(\beta_1 - \beta_2)^2 \,\mathbf{sh}\, \gamma t + (\beta_1 + \beta_2)^2(\beta_1 - \beta_2)\,\mathbf{sh}\, \delta t,$$
$$\Delta(t) = (\beta_1 - \beta_2)^2 \,\mathbf{ch}\, \gamma t - (\beta_1 + \beta_2)^2 \,\mathbf{ch}\, \delta t + 4\beta_1\beta_2 \cos \mu t,$$

and since $\beta_1 > \beta_2$, we have $|\operatorname{tg}\theta_1| \geq |\operatorname{tg}|\theta_2|$.

Assuming $\theta < \pi/2$, we have $\theta_2 < \theta_1 < \pi/2$.

Let us consider the function

$$\Delta_1(w) = (\beta_2^2 - \beta_1^2)\,[(\beta_1 - \beta_2)\,\mathbf{sh}\,\gamma w - (\beta_1 + \beta_2)\,\mathbf{sh}\,\delta w]$$

and prove that the equation

$$(\beta_1 - \beta_2)\,\mathbf{sh}\,\gamma w - (\beta_1 + \beta_2)\sin\delta w = 0 \tag{2.3.31}$$

has no roots in the strip $-1 < \operatorname{Im} w < 0$.

Dividing this equation by β_1 and using the equality $\frac{\beta_2}{\beta_1} = \frac{\operatorname{tg}\theta_2}{\operatorname{tg}\theta_1}$, we obtain

$$1 \pm \frac{\beta_2}{\beta_1} = 1 \pm \frac{\operatorname{tg}\theta_2}{\operatorname{tg}\theta_1} = \frac{\sin(\theta_1 \pm \theta_2)}{\cos\theta_1 \cdot \cos\theta_2}\ \operatorname{ctg}\theta_1.$$

For $\theta \neq \pi/2$, equation (2.3.31) takes the form

$$\sin\delta\,\mathbf{sh}\,\gamma t - \sin\gamma\,\mathbf{sh}\,\delta t = 0.$$

For $0 < \theta < \pi$ it can be proved that in the strip the latter equation may have only the imaginary root $w = i\tau$. In that case, it is equivalent to the equation

$$\sin\delta\sin\gamma\tau - \sin\gamma\sin\delta\tau = 0, \quad -1 \le \tau \le 0. \tag{2.3.32}$$

If τ_0 is the root of this equation, then $-\tau_0$ is also the root and therefore it can be assumed that $0 < \tau < 1$.

Let $\theta < \pi/2$, then $\theta_2 < \theta_1 < \pi/2$. Consider the function

$$f(\tau) = \sin\delta\,\frac{\sin\gamma\tau}{\sin\delta\tau} - \sin\gamma.$$

We have

$$\begin{aligned}
f(0) &= \frac{\gamma}{\delta}\sin\delta - \sin\gamma = \gamma\left(\frac{\sin\delta}{\delta} - \frac{\sin\gamma}{\gamma}\right) > 0,\\
f(1) &= 0,\\
f_1'(\tau) &= \gamma\tau(\gamma-\tau)\left[\frac{\sin\gamma\tau\cdot\cos\delta\tau}{\gamma\tau} - \frac{\sin(\gamma-\tau)\tau}{(\gamma-\tau)\tau}\right]\\
&\le \gamma\tau(\gamma-\tau)\left[\frac{\sin\gamma\tau}{\gamma\tau} - \frac{\sin(\gamma-\delta)\tau}{(\gamma-\delta)\tau}\right] \le 0.
\end{aligned}$$

Since $f(\tau)$ is a decreasing function on the interval $(0;1)$ and $f(1) = 0$, we have $f(\tau) > 0$ for $\tau \in (0,1)$. Therefore the equation has no roots for $\theta < \pi/2$.

When $\theta = \pi/2$, we have $\delta = 0$ and $\gamma = \pi$,

$$\Delta(w) = (\beta_2 - \beta_1)(\beta_1^2 - \beta_2^2)\,\mathbf{sh}\,\pi w.$$

This function has no zeros in the strip $-1 < \operatorname{Im} w < 0$ and therefore the stress $\tau_{xy}(x)$ is bounded.

When $\theta = 3\pi/2$, we have $\theta_1 = \theta_2 = 3\pi/2$ and

$$\Delta_1(t) = (\beta_1^2 - \beta_2^2)(\beta_2 - \beta_1)\,\mathbf{sh}\,3\pi t.$$

This function has zeros at the points $w = -\frac{1}{3}\,i; -\frac{2}{3}\,i$ and near the point $x = 0$ the stress $\tau_{xy}(x)$ is written as

$$\tau_{xy} = c_0 x^{-\frac{2}{3}} + c_1 x^{-\frac{1}{3}} + \varphi_0(x), \ \ x > 0,$$

where $\varphi_0(x)$ is continuous in an interval $0 \le x < \infty$.

Thus, for an orthotropic body the stress τ_{xy} has, for $\theta = \frac{\pi}{2}\,; \pi; \frac{3\pi}{2}\,; 2\pi$, the same character as for an isotropic body.

When $\pi/2 < \theta < \pi$, by choosing numbers γ and δ or, which is the same, β_1 and β_2 we can make equation (2.3.32) have a root in the interval $(-1; 0)$. This means that for $\theta \in (\pi/2; \pi)$ the stress τ_{xy} can be both bounded and unbounded.

Since $\operatorname{tg}\theta_k = \beta_k \operatorname{tg}\theta$, $\pi/2 < \theta < \pi$, by choosing a number $\beta_k > 0$, $|\operatorname{tg}\theta_k|$ can be made both arbitrarily large andr arbitrarily small, therefore θ_k can be arbitrarily approximated both to $\pi/2$ and to π.

Let us now show that if $\pi/2 < \theta < \pi$, then the material and therefore the parameters β_1 and β_2 can always be chosen so that the stress τ_{xy} at the point $x = 0$ be bounded or unbounded.

It has been shown above that for $0 < \theta < \pi/2$ the stress is bounded in the neighborhood of the point $x = 0$ for any orthotropic material.

We will show below that for any material there exists a number $\pi/2 < \theta_0 < \pi$ such that the stress $\theta < \theta_0$ is bounded for τ_{xy} and unbounded for $\theta > \theta_0$.

Let us consider the situation

$$f(\tau) = \sin\gamma\tau\sin\delta - \sin\gamma\sin\delta\tau.$$

It is obvious that $f(1) = 0$. For $\tau = 1$ to be the double root of equation (2.3.31) the following condition

$$f'(1) = \gamma\cos\gamma\sin\delta - \delta\sin\gamma\cos\delta = 0$$

must be fulfilled. Hence we obtain

$$\gamma - \operatorname{tg}\gamma \cdot \frac{\delta\cos\delta}{\sin\delta} = 0$$

Denote by the angle θ_0 the respective angles θ_1 and θ_2 ($\operatorname{tg}\theta_k = \beta_k \operatorname{tg}\theta_0$, $k = 1, 2$) which satisfy the equation

$$(\theta_1 + \theta_2) - (\theta_1 - \theta_2)\operatorname{tg}(\theta_1 + \theta_2)\operatorname{ctg}(\theta_1 - \theta_2) = 0. \tag{2.3.33}$$

For $\beta_1 \to \beta_2 = 1$, i.e. for $\theta_1 \to \theta_2 = \theta_0$ it follows that $\operatorname{tg} 2\theta_0 = 2\theta_0$. This is the necessary and sufficient condition for the stress τ_{xy} to be bounded which has been obtained in Section 2.2.

When $\theta = \theta_0$, equation (2.3.31) has the double root at the point $w = -i$, while at the point $x = 0$ the stress τ_{xy} has a logarithmic singularity. When $\theta < \theta_0$, the function τ_{xy} is bounded. When $\theta > \theta_0$, equation (2.3.31) has the root $-1 < \tau_0 < 0$ and the stress τ_{xy} can be written in the form

$$\tau_{xy}(x) = x^{|\tau_0|-1}\varphi_0(x), \ \ x > 0,$$

where $\varphi_0(x)$ is bounded for $x \ge 0$.

2.4. The Bending Problem of a Beam Resting on the Elastic Foundation

Let an elastic anisotropic body occupy an angle $-\theta < \arg z < 0$ ($0 < \theta < 2\pi$) on the plane $z = x+iy$. Assume that one boundary $\arg z = 0$ of the body supports a beam with rigidity D to which a distributed normal load with intensity $p(x)$ is applied. Also assume $p(x)$ is a bounded summable functions equal to zero outside some interval. There is no friction between the beam and the wedge. The other side of the boundary is free of external stresses.

As is known, the vertical displacement of points of the beam midline satisfies the equation

$$D\,\frac{d^4V_n}{dx^4} = p(x) + \sigma_y(x), \quad x > 0, \tag{2.4.1}$$

where

$$D = \frac{E_0h^3}{12(1-\nu^2)}$$

is the beam rigidity and $\sigma_y(x)$ is the sought contact stress satisfying the equilibrium condition

$$\begin{cases} \int\limits_0^\infty \sigma_y(x)\,dx = -\int\limits_0^\infty p(x)\,dx = P, \\ \int\limits_0^\infty x\sigma_y(x)\,dx = -\int\limits_0^\infty xp(x)\,dx = M. \end{cases} \tag{2.4.2}$$

Thus the posed problem reduces to the problem of equilibrium of an elastic body with the following boundary conditions

$$\begin{cases} D\,\dfrac{d^3v}{dx^3} = -\int\limits_0^x p(s)\,ds - \int\limits_0^x \sigma_y(s)\,ds, \\ \tau_{xy}(x;0) = 0, \quad x > 0, \end{cases} \tag{2.4.3}$$

$$X_n(t) = Y_n(t) = 0, \quad \arg t = -\theta, \tag{2.4.4}$$

with $\sigma_y(x)$ satisfying condition (2.4.2).

Let us consider two planes of complex variables $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ which are respectively obtained from the plane $z = x + iy$ by the affine transforms

$$x_1 = x + \alpha_1 y; \quad y_1 = \beta_1 y; \quad x_2 = x + \alpha_2 y; \quad y_2 = \beta_2 y; \quad \beta_1 > \beta_2 > 0.$$

By means of these transforms, the domain $S = \{-\theta < \arg z < 0\}$ of the plane of the variable z becomes the domain $S_k = \{-\theta_k < \arg z_k < 0\}$ of the plane of the variable z_k.

If the roots of the characteristic equation (2.1.4) $s_1 \neq s_2$, then the posed problem reduces, by virtue of formulas (2.1.6), (2.1.8), to finding holomorphic functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ in the domains S_1 and S_2, respectively, using the following boundary conditions

$$\begin{gathered}(s_1-\overline{s}_2)t_1\Phi_1(t_1)+(\overline{s}_1-\overline{s}_2)\overline{t_1\Phi_1(t_1)}+(s_2-\overline{s}_2)t_2\Phi_2(t_2)=0,\\ t_k=\rho(\cos\theta-s_k\sin\theta),\quad \rho=|t|>0,\end{gathered} \tag{2.4.5}$$

$$\begin{aligned}(s_1-\overline{s}_2)\Phi_1(x)+(\overline{s}_1-\overline{s}_2)\overline{\Phi_1(x)}+(s_2-\overline{s}_2)\Phi_2(x)&\\ =-\overline{s}_2\sigma_y(x),\quad x>0,&\end{aligned} \tag{2.4.6}$$

$$2\operatorname{Re}\big[q_1\Phi_1''(x)+q_2\Phi_2''(x)\big]=-\frac{1}{D}\int\limits_0^x\big[p(s)+\sigma_y(s)\big]\,ds,\quad x>0. \tag{2.4.7}$$

It is required of the functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ that they satisfy the conditions

$$\lim z_k\Phi_k(z_k)\to 0 \ \text{ as } \ z_k\to 0 \ \ (k=1,2)$$

and for sufficiently large $|z_k|$ have the form

$$\Phi_k(z_k)=\frac{\gamma_k}{z_k}+o\left(\frac{1}{z_k}\right)\quad (k=1,2). \tag{2.4.8}$$

A solution of the problem will be sought in the form

$$\Phi_k(z_k)=\frac{1}{z_k\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\frac{A_k(t)}{t}\,e^{it\ln z_k}\,dt-i\sqrt{\frac{\pi}{2}}\,\frac{A_k(0)}{z_k}\,, \tag{2.4.9}$$

where $A_1(0)$ and $A_2(0)$ satisfy the condition

$$(s_2-\overline{s}_2)A_2(0)=(\overline{s}_2-s_1)A_1(0)+(\overline{s}_1-\overline{s}_2)\overline{A_1(0)}.$$

Substituting value (2.4.9) into the boundary conditions (2.4.5) and (2.4.6) and arguing as in Section 2.3, we obtain

$$\begin{aligned}A_1(t)&=\frac{\overline{s}_1(s_2-\overline{s}_2)e^{i\mu t}+\overline{s}_2(\overline{s}_1-s_2)e^{-\delta t}+s_2(\overline{s}_2-\overline{s}_1)e^{-\gamma t}}{2\Delta(t)}\,tN(t),\\ A_2(t)&=\frac{\overline{s}_2(s_2-\overline{s}_2)e^{-i\mu t}+\overline{s}_1(\overline{s}_2-s_1)e^{\delta t}+s_1(\overline{s}_2-\overline{s}_1)e^{-\gamma t}}{2\Delta(t)}\,tN(t),\end{aligned} \tag{2.4.10}$$

where

$$\begin{gathered}\Delta(t)=|s_1-s_2|^2\,\mathbf{ch}\,\gamma t-|s_1-\overline{s}_2|^2\,\mathbf{ch}\,\delta t+4\beta_1\beta_2\cos\mu t,\\ \gamma=\theta_1+\theta_2,\quad \delta=\theta_1-\theta_2,\\ \mu=\ln\left|\frac{\cos\theta-s_1\sin\theta}{\cos\theta-s_2\sin\theta}\right|,\quad N(t)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\sigma_y(e^s)e^{s-its}\,ds.\end{gathered}$$

The first equality (2.4.2) yields

$$N(0)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\sigma_y(e^s)e^s\,ds=\int\limits_{0}^{\infty}\sigma_y(x)\,dx=\frac{P}{\sqrt{2\pi}}\,.$$

Passing to the limit in equalities (2.4.10) as $t\to 0$ and taking the value $N(0)=\frac{P}{\sqrt{2\pi}}$ into account, we obtain

$$A_1(0)=\frac{-2\beta_2\mu\overline{s}_1-\delta\overline{s}_2(\overline{s}_1-s_2)+\gamma\overline{s}_2(\overline{s}_1-\overline{s}_2)}{|s_1-s_2|^2\gamma^2-|s_1-\overline{s}_2|^2\delta^2-4\beta_1\beta_2\mu^2}\cdot\frac{P}{\sqrt{2\pi}}\,.$$

Putting the values of $\Phi_1(z_1)$ and $\Phi_2(z_2)$ represented by formula (2.4.9) into the boundary condition (2.4.7) and using equality (2.4.10), we have

$$\begin{aligned}&2\,\mathrm{Re}\big[q_1\Phi_1''(x)+q_2\Phi_2''(x)\big]\\&\quad=\frac{1}{x^3\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\frac{(it-1)(it-2)(\Delta_2+i\Delta_1)}{\Delta(t)}\,N(t)e^{it\ln x}\,dt+\frac{c}{x^3}\,,\end{aligned}\qquad(2.4.11)$$

where

$$\begin{aligned}c&=-2\sqrt{\pi}\ \mathrm{Im}\big[q_1A_1(0)+q_2A_2(0)\big],\\\Delta_1(t)&=a_1\,\mathbf{sh}\,\gamma t+b_1\,\mathbf{sh}\,\delta t+c_1\sin\mu t,\\\Delta_2(t)&=a_2\,\mathbf{ch}\,\gamma t+b_2\,\mathbf{ch}\,\delta t+c_2\cos\mu t,\\c_1&=2\beta_2\,\mathrm{Im}[\overline{q}_1s_1]-2\beta_1\,\mathrm{Im}[\overline{q}_2s_2],\\c_2&=2\beta_2\,\mathrm{Im}[\overline{q}_1s_1]+2\beta_1\,\mathrm{Im}[\overline{q}_2s_2],\\a_2+ia_1&=\big(\overline{s_2q_1}-\overline{q_2s_1}\big)(s_2-s_1),\\b_2+ib_1&=\big(\overline{q_1}s_2-q_2\overline{s_1}\big)(s_1-s_2).\end{aligned}$$

The substitution of the values of q_1 and q_2 defined by equality (2.1.7) into these formulas and some simple transformations give

$$\begin{aligned}a_1&=a_{22}|s_1-s_2|^2\,\mathrm{Im}\left(\frac{1}{\overline{s}_1}+\frac{1}{\overline{s}_2}\right),\\b_1&=a_{22}|s_1-\overline{s}_2|^2\,\mathrm{Im}\left(\frac{1}{\overline{s}_2}-\frac{1}{\overline{s}_1}\right),\\a_2&=|s_2-s_1|^2K_2,\quad b_2=-|s_1-\overline{s}_2|^2K_2,\\c_1&=4\beta_1\beta_2\,\mathrm{Re}\left(\frac{1}{s_1}-\frac{1}{s_2}\right),\quad c_2=4\beta_1\beta_2K_2,\\K_2&=a_{22}\,\mathrm{Re}\left[\frac{1}{s_1}+\frac{1}{s_2}\right]-a_{26}.\end{aligned}$$

By Viète’s theorem we have

$$\begin{aligned}a_{22}&=s_1\cdot\overline{s}_1\cdot s_2\cdot\overline{s}_2\cdot a_{11},\\\frac{2a_{26}}{a_{11}}&=s_1s_2\overline{s}_1+s_2\overline{s}_1\cdot\overline{s}_2+s_2\cdot s_1\overline{s}_2+s_1\cdot\overline{s}_1\cdot\overline{s}_2,\end{aligned}$$

whence it follows that

$$K_2 = 0$$

and therefore

$$a_2 = b_2 = c_2 = \Delta_2(t) = 0.$$

Thus formula (2.4.11) takes the form

$$\begin{aligned} 2\operatorname{Re}\left[q_1\Phi_1''(x) + q_2\Phi_2''(x)\right] \\ = \frac{i}{x^3\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{(it-1)(it-2)\Delta_1(t)N(t)}{\Delta(t)}\, e^{it\ln x}\, dt + \frac{c}{x^3}\,. \end{aligned} \tag{2.4.12}$$

Comparing equalities (2.4.7) and (2.4.12) we obtain

$$\begin{aligned} \frac{i}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{(it-1)(it-2)\Delta_1(t)N(t)}{\Delta(t)}\, e^{it\ln x}\, dt + \frac{x^3}{D} \int\limits_0^x \sigma_y(s)\, ds \\ = -\frac{x^3}{D} \int\limits_0^x p(s)\, ds + c. \end{aligned} \tag{2.4.13}$$

The functions $\Delta(t)$ and $\Delta_1(t)$ do not vanish anywhere except for the point $t = 0$. The point $t = 0$ is a zero of second order for the function $\Delta(t)$, and a zero of second order for the function $\Delta_1(t)$.

Making a substitution in formula (2.4.12)

$$\xi = \ln x,$$

we obtain

$$\begin{aligned} \frac{i}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{(it-1)(it-2)\Delta_1(t)N(t)}{\Delta(t)}\, e^{it\xi}\, dt + \frac{e^{3\xi}}{D} \int\limits_{-\infty}^{\xi} \sigma_y(e^s)e^s\, ds \\ = -\frac{e^{3\xi}}{D} \int\limits_{-\infty}^{\xi} p(e^s)e^s\, ds + c. \end{aligned} \tag{2.4.14}$$

Since the function in the right-hand part of equation (2.4.14) is not integrable, we have to introduce, as we have done in Section 2.3, a new unknown function

$$\Psi_1(\xi) = \int\limits_{-\infty}^{\infty} e^s\sigma_y(e^s)\, ds - \frac{e^{\pi\xi}(p - \lambda e^{-3\xi})}{1 + e^{\pi}\xi}\,. \tag{2.4.15}$$

After differentiating this equality and performing the inverse Fourier transformation, we obtain

$$N(t) = it\Psi_1(t) + \frac{pt}{\sqrt{2\pi}\,\mathbf{sh}\, t} - \frac{\lambda t}{\sqrt{2\pi}\,\mathbf{sh}(t - 3i)}\,, \tag{2.4.16}$$

where $\Psi_1(t)$ is the inverse Fourier transform of the function $\Psi_1(\xi)$.

Substituting values (2.4.16) and (2.4.15) into formula (2.4.14), making some transformations and choosing λ by the equality

$$\lambda = 8\,\mathrm{Im}\big[q_1A_1(0)+q_2A_2(0)\big],$$

we obtain

$$\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\frac{(t+i)(t+2i)t\Delta_1(t)\Psi_1(t)}{\Delta(t)}\,e^{it\xi}\,dt+\frac{1}{D}\,e^{3\xi}\Psi_1(\xi)=f(\xi), \quad (2.4.17)$$

$$-\infty<\xi<\infty,$$

where

$$f(\xi)=-\frac{P}{\sqrt{2\pi}\,i}\int\limits_{-\infty}^{\infty}\frac{G(t)-G(0)}{\mathbf{sh}\,t}\,e^{it\xi}\,dt$$

$$+\frac{\lambda}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{G(t)e^{it\xi}}{\mathbf{sh}(t-3i)}\,dt-\frac{e^{3\xi}}{D}\bigg(\int\limits_{-\infty}^{\xi}p(e^s)\,ds-\frac{Pe^{\pi\xi}}{1+e^{\pi\xi}}\bigg),$$

$$G(t)=-\frac{(t+i)(t+2i)\Delta_1(t)}{\Delta(t)}\,.$$

Since outside some interval, $p(x)=0$ for a sufficiently large value of n, if $\xi>n$, we have $p(e^{\xi})=0$ and therefore

$$e^{3\xi}\bigg(\int\limits_{-\infty}^{\xi}e^sp(e^s)\,ds-\frac{Pe^{\pi\xi}}{1+e^{\pi\xi}}\bigg)=\frac{Pe^{3\xi}}{1+e^{\pi\xi}}\quad\text{for}\ \ \xi>n,$$

i.e. the function $f(\xi)$ is integrable along the whole axis.

By the inverse Fourier transformation of equation (2.4.17) we obtain

$$t(t+i)(t+2i)\,\frac{\Delta_1(t)}{\Delta(t)}\,\Psi_1(t)+\frac{1}{D}\,\Psi_1(t+3i)=F(t), \quad -\infty<t<\infty, \quad (2.4.18)$$

where $F(t)$ is the inverse Fourier transform of the function $f(\xi)$. The function $F(t)$ is analytically extendable in the strip $-3<\mathrm{Im}\,w<3$ except for the point $w=(3-\pi)i$ and points w which are the roots of the function $\Delta(w)$, where it has poles and vanishes at infinity.

We have thus come to the following problem of the analytic function theory: by the boundary condition (2.3.22), find a function, which is analytic in the strip $-3<\mathrm{Im}\,w<0$ except for the points $t=i,2i$ and w_k, where it may have poles, and vanishing at infinity.

The coefficient of the problem can be written in the form

$$\frac{t(t+i)(t+2i)\Delta_1(t)}{\Delta(t)}$$

$$=-\frac{ait(t+i)(2t-3i)}{(t-2i)(2t+3i)}\,\frac{\Delta_1(t)}{a\Delta(t)}\,(t^2+4)\,\mathbf{th}\,\frac{\pi}{6}\,t\,\frac{\mathbf{sh}\,\frac{\pi}{6}\,(t+3i)}{\mathbf{sh}\,\frac{\pi}{6}\,t}\,\frac{2t+3i}{2t-3i}\,,$$

where

$$a = a_{22} \operatorname{Im}\left(\frac{1}{\bar{s}_1} + \frac{1}{\bar{s}_2}\right) = \lim_{t\to\infty} \frac{\Delta(t)}{\Delta_1(t)}\,.$$

We introduce the notation

$$G_0(t) = \frac{\Delta_1(t)(t+i)(2t-3i)}{a\Delta(t)(t-2i)(2t+3i)}\ \mathbf{th}\,\frac{\pi}{6}\,t.$$

The function $G_0(t)$ is continuous all over the axis and $G_0(-\infty) = G_0(\infty) = 1$. Substituting the function $G_0(t)$ as a product of two functions

$$\frac{\Delta_1(t)}{a\Delta(t)}\ \mathbf{th}\,\frac{\pi}{6}\,t \ \text{ and } \ \frac{t+i}{t-2i}\cdot\frac{2t-3i}{2t+3i}$$

of which one takes positive values, while the other has one zero in the upper half-plane, we see that $\operatorname{Ind} G_0(t) = 0$.

It is easy to verify that the branch of the function $G_0(t)$ which vanishes at infinity is integrable all over the axis.

As has been shown in Section 1.2, the function $G_0(t)$ can be written in the form

$$G_0(t) = \frac{X_0(t+3i)}{X_0(t)}\,, \quad -\infty < t < \infty, \tag{2.4.19}$$

where

$$X_0(w) = \exp\left(\frac{1}{6i}\int\limits_{-\infty}^{\infty} \ln G_0(t) \coth\frac{\pi}{3}\,(t-w)\,dt\right), \quad 0 < \operatorname{Im} w < 3. \tag{2.4.20}$$

In Section 1.3 we have shown that the function $G_0(x)$ is representable in the form

$$t^2 + 4 = \frac{X_1(t+3i)}{X_1(t)}\,, \quad -\infty < t < \infty, \tag{2.4.21}$$

where

$$X_1(w) = \frac{3^{-2iw}\Gamma\big(\frac{2-iw}{3}\big)}{\Gamma\big(\frac{5+iw}{3}\big)}\,, \quad 0 \le \operatorname{Im} w \le 3.$$

The function $X_1(w)$ satisfies, in the strip $0 \le \operatorname{Im} w \le 3$, the condition

$$\frac{D_1}{|2+it|} < |X_1(w)| < D_2|2+it|.$$

We represent the number D/a as

$$\frac{D}{a} = \frac{X_2(t+3i)}{X_2(t)}\,, \quad -\infty < t < \infty, \tag{2.4.22}$$

where

$$X_2(w) = \exp\left(-\,iw\ln\sqrt[3]{\frac{D}{a}}\,\right).$$

If we substitute expressions (2.4.19), (2.4.21) and (2.4.22) into formula (2.4.18) and introduce the notation

$$X_3(w) = \frac{X_0(w)X_2(w)X_1(w)\big(w - \frac{3}{2}\,i\big)\,\mathbf{sh}\,\frac{\pi}{6}\,w}{w}\,,$$

then we obtain

$$\frac{(3-it)\Psi_1(t)}{X_3(t)} + \frac{\Psi_1(t+3i)}{X_3(t+3i)} = \frac{DF(t)}{X_3(t+3i)}\,, \quad -\infty < t < \infty. \tag{2.4.23}$$

By virtue of (1.3.9),

$$3 - it = \frac{X_4(t+3i)}{X_4(t)}\,,$$

where

$$X_4(w) = 3^{iw}\Gamma\left(\frac{3-iw}{3}\right).$$

Introducing one more notation

$$X(w) = X_3(w)\cdot X_4(w)$$

condition (2.4.23) can be given the form

$$\frac{\Psi_1(t)}{X(t)} + \frac{\Psi_1(t+3i)}{X(t+3i)} = \frac{DF(t)}{X(t+3i)}\,, \quad -\infty < t < \infty. \tag{2.4.24}$$

The function $[X(z)]^{-1}$ is holomorphic in the strip $0 < \operatorname{Im} w < 3$ except for the point $w = \frac{3}{2}\,i$ where it has a pole of second order. Let us investigate the behavior of this function for large $|w|$.

The functions $X_0(w)$ and $X_2(w)$ are bounded throughout the strip, while $X_1(w)$ and $X_4(w)$ admit the following estimate for sufficiently large $|w|$

$$|X_1(w)| = O\big(|t|^{\frac{2\tau}{3}-1}\big), \quad |X_4(w)| = O\big(|t|^{\frac{1}{2}+\frac{\tau}{3}}\big)e^{-\frac{\pi}{6}|t|}, \quad w = t+i\tau, \quad 0<\tau\le 3.$$

Hence it follows that for sufficiently large $|w|$, $X(w)$ admits the estimate

$$|X(w)| = O(t^{\tau-\frac{1}{2}}), \quad 0 \le \tau \le 3. \tag{2.4.25}$$

Thus the function $\Psi_1(w)/X(w)$ is holomorphic in the strip $0 < \operatorname{Im} w < 3$ except for the point $w = 3i/2$ where it may have a pole of second order. According to condition (2.4.16), the function $w\Psi_1(w)$ vanishes at infinity and therefore $\Psi_1(w)/X(w)$, too, vanishes at infinity.

By virtue of (1.1.4), the solution of problem (2.4.24) is given by the formula

$$\Psi_1(w) = \frac{DX(w)}{6i}\int_{-\infty}^{\infty}\frac{F(t)}{X(t+3i)\,\mathbf{sh}\,\frac{\pi}{3}\,(t-w)}\,dt + \frac{A_0X(w)}{\mathbf{ch}\,\frac{\pi}{3}\,w}\,, \tag{2.4.26}$$

$$0 < \operatorname{Im} w < 3.$$

Since the function $F(t)$ is analytically extendable in the strip $-3 < \operatorname{Im} w < 0$, a solution of this problem in the strip $-3 < \operatorname{Im} w < 3$ will have the form

$\Psi_1(w)$

$$= \begin{cases} \dfrac{DX(w)}{6i} \displaystyle\int_{-\infty}^{\infty} \dfrac{F(t)}{X(t+3i)\,\mathbf{sh}(t-w)}\,dt + \dfrac{A_0X(w)}{\mathbf{ch}\,\frac{\pi}{3}\,w}\,, & 0<\operatorname{Im} w<3, \\ \dfrac{DF(w)-\Psi^+(w+3i)}{(t+i)(t+3i)w}\;\dfrac{\Delta(w)}{\Delta_1(w)}\,, & -3<\operatorname{Im}<0. \end{cases} \tag{2.4.27}$$

The function represented by formula (2.4.27) is holomorphic in the strip $-3 < \operatorname{Im} w < 3$ except for the points $w = -i$, $w = -2i$, $w = (3-\pi)i$, $w = t_k + i\tau_k$ $(k = 0, \dots, n)$, where $t_k + i\tau_k$ are zeros of the function $\Delta_1(w)$ in the lower half-plane, $\operatorname{Im} w < 0$, and $|\tau_0| < |\tau_1| < \cdots < |\tau_n|$.

For sufficiently large $|t|$, the function $F(t)$ has the form $F(t) = O(1/|t|^{2+\varepsilon})$ because we have required of the function $p(x)$ that it be bounded and integrable. Taking now estimate (2.4.25) into account, we conclude that for sufficiently large $|t|$, the function $F(t)/X(t+3i)$ admits the estimate

$$\frac{F(t)}{X(t+3i)} = O(t^{-k}),$$

where $k > 4$.

The integral in the right-hand part of formula (2.4.26) will decrease in the same manner in the closed strip $0 \le \operatorname{Im} w \le 3$.

By virtue of formulas (1.1.8) which are analogous to the Sokhotski–Plemelj formula, from (2.4.26) we obtain

$$\begin{aligned} \Psi_1^+(t_0) &= \frac{X(t_0)F(t_0)D}{2X(t_0+3i)} + \frac{X(t_0)}{6i} \int_{-\infty}^{\infty} \frac{F(t)}{X(t+3i)\,\mathbf{sh}\,\frac{\pi}{3}\,(t-t_0)}\,dt \\ &\quad + \frac{A_0X(t_0)}{\mathbf{ch}\,\frac{\pi}{3}\,t_0}\,, \\ \Psi_1^-(t_0) &= \frac{DF(t_0)}{2} + \frac{X_0(t+3i)}{6i} \int_{-\infty}^{\infty} \frac{F(t)}{X(t+3i)\,\mathbf{sh}\,\frac{\pi}{3}\,(t-t_0)}\,dt \\ &\quad - \frac{A_0X(t_0+3i)}{\mathbf{ch}\,\frac{\pi}{3}\,t_0}\,. \end{aligned} \tag{2.4.28}$$

Since $\Psi_1(t)$ vanishes at infinity being of order more than four, the integral in formula (2.4.17) exists in the ordinary sense, while the integrals in formulas (2.4.9) and (2.4.13) exist at infinity in the ordinary sense and, at the point $t = 0$, in the sense of Cauchy principal value.

From formula (2.4.16) it follows that the function $W(t)$ is analytically extendable in the strip $-3 < \operatorname{Im} < 3$ except perhaps for the points $w = i; 2i$ and $w = t_k + i\tau_k$. Therefore it can be written in the form

$$N(w) = iw\Psi_1(w) + \frac{Pw}{\sqrt{2\pi}\;\mathbf{sh}\,w} - \frac{\lambda w}{\sqrt{2\pi}\;\mathbf{sh}(w-3i)}\,. \tag{2.4.29}$$

Using the condition

$$M = \int_0^\infty x\sigma_y(x)\,dx = \int_{-\infty}^\infty e^{2s}\sigma_y(e^s)\,ds = \sqrt{2\pi}\,Ni,$$

from expression (2.4.29) we obtain

$$\Psi_1(i) = \frac{P}{\sqrt{2\pi}\,\sin 1} + \frac{\lambda}{\sqrt{2\pi}\,\sin 2} - \frac{M}{\sqrt{2\pi}}\,.$$

Substituting into this equality the value of $\Psi_1(i)$ defined from (2.4.26), we obtain the value of the constant $A(0)$.

The contact stress $\sigma_y(x)$ is obtained from the function $N(t)$ by means of the Fourier transform

$$\sigma_y(x) = x^{-1}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty N(t)e^{it\ln x}\,dt, \quad x > 0. \tag{2.4.30}$$

By arguments analogous to those used in Section 2.2 we prove that for sufficiently large $|x|$

$$\sigma_y(x) = \frac{\lambda}{x^4} + o\left(\frac{1}{x^4}\right).$$

Note that for $N(w)$ the function $|\tau_0| > 1$ is holomorphic in the strip $-1 < \operatorname{Im} w < 3$, while for $|\tau_0| < 1$ it has a pole of first order at the point $w = t_0 + i\tau_0$. In the same manner as in the preceding paragraphs it can be proved that for $|\tau_0| > 1$ the function $\sigma_y(x)$ is bounded in the neighborhood of the point $x = 0$ and is representable in the form

$$\sigma_y(x) = x^{|\tau_0|-1}\varphi(x), \quad x > 0,$$

for $|\tau_0| < 1$; $\varphi(x)$ is continuous on a semi-axis $x \geq 0$.

Let us now consider the particular cases.

1. Assume that the domain S is a half-plane, then

$$\theta_1 = \theta_2 = \theta = \pi, \quad \delta = 0, \quad \gamma = 2\pi, \quad \mu = 0,$$
$$\Delta(t) = 2|s_1 - s_2|^2\,\mathbf{sh}^2\,\pi t,$$
$$\Delta_1(t) = 2a|s_1 - s_2|\,\mathbf{sh}\,\pi t\cdot\mathbf{ch}\,\pi t,$$

i.e.

$$\frac{\Delta_1(t)}{\Delta(t)} = \frac{a\,\mathbf{ch}\,\pi t}{\mathbf{sh}\,\pi t}\,.$$

Hence it follows that

$$\tau_0 = -\frac{1}{2}\,, \quad \tau_1 = -\frac{3}{2}\,,$$

and the function $\sigma_y(x)$ can be represented as

$$\sigma_y(x) = \frac{c}{\sqrt{x}} + \varphi_0(x), \quad x > 0,$$

where $\varphi_0(x)$ is continuous on a semi-axis $x \geq 0$.

When $\theta = 2\pi$, which means that the plane is cut in the positive direction of the real axis, we obtain

$$\theta_1 = \theta_2 = \theta = 2\pi, \quad \delta = 0, \quad \gamma = 4\pi, \quad \mu = 0,$$
$$\Delta(t) = 2|s_1 - s_2|^2 \,\mathbf{sh}^2\, 2\pi t,$$
$$\Delta_1(t) = 2|s_1 - s_2|^2 a\,\mathbf{sh}\, 2\pi t \cdot \mathbf{ch}\, 2\pi t,$$

and

$$\tau_0 = -\frac{1}{4}, \quad \tau_1 = -\frac{3}{4},$$

and the function $\sigma_y(x)$ can be represented as

$$\sigma_y(x) = c_0 x^{-\frac{3}{4}} + c_1 x^{-\frac{1}{4}} + \varphi_0(x).$$

2. Let the body be orthotropic and the elastic anisotropy axes be parallel to the coordinate axes. Then $a_{16} = a_{26} = 0$, the characteristic equation will be biquadratic and its roots be purely imaginary: $s_1 = i\beta_1$, $s_2 = i\beta_2$.

Furthermore we have

$$\Delta_1(t) = \frac{a_{22}(\beta_1^2 - \beta_2^2)(\beta_1 - \beta_2)\,\mathbf{sh}\,\gamma t + a_{22}(\beta_1 + \beta_2)^2(\beta_1 - \beta_2)\,\mathbf{sh}\,\delta t}{\beta_1\beta_2},$$
$$\Delta(t) = (\beta_1 - \beta_2)^2\,\mathbf{ch}\,\gamma t - (\beta_1 + \beta_2)^2\,\mathbf{ch}\,\delta t + 4\beta_2\beta_2 \cos\mu t.$$

The boundary condition (2.4.18) will take the form

$$t(t+i)(t+2i)a_1 \frac{(\beta_1 + \beta_2)\big[(\beta_1 + \beta_2)\,\mathbf{sh}\,\gamma t + (\beta_1 - \beta_2)\,\mathbf{sh}\,\delta t\big]}{(\beta_1 - \beta_2)^2\,\mathbf{ch}\,\gamma t - (\beta_1 + \beta_2)^2\,\mathbf{ch}\,\delta(t) + 4\beta_1\beta_2\cos\mu t}\,\Psi_1(t) + \Psi_1(t+3i) = DF(t), \tag{2.4.31}$$

where

$$a_1 = a_{22}\,\frac{\beta_1 + \beta_2}{\beta_1\beta_2}\,D.$$

We can show that the equation

$$(\beta_1 + \beta_2)\,\mathbf{sh}\,\gamma w + (\beta_1 - \beta_2)\,\mathbf{sh}\,\delta w = 0 \ \text{ for } \ \theta < \pi/2 \tag{2.4.32}$$

has no roots in the strip $-1 < \operatorname{Im} w < 0$.

Indeed, since $\operatorname{tg}\theta_k = \beta_k \operatorname{tg}\theta$ $(k = 1, 2)$, we rewrite equation (2.4.32) as

$$\sin\delta \sin\gamma w + \sin\gamma \sin\delta w = 0.$$

We can prove that the equation has only imaginary roots for $\theta < \pi/2$, i.e. it is equivalent to the equation

$$\sin\delta \sin\gamma\tau + \sin\gamma \sin\delta\tau = 0 \tag{2.4.33}$$

for $-1 < \tau < 0$.

Both summands in the left-hand part of equation (2.4.33) are negative and therefore it has no solution.

If $\theta = \pi/2$, then $\gamma = \pi$, $\delta = 0$ and equation (2.4.33) takes the form

$$\sin\pi w = 0.$$

This equation has, in the strip $-1 < \operatorname{Im} w < 0$, a unique root $w = -i$. This number is at the same time a simple root of the function $\Delta(w)$. The coefficient of problem (2.4.31) does not have a zero in the strip $-1 \le \operatorname{Im} w < 0$. Therefore the contact stress $\sigma_y(x)$ is bounded near the point $x = 0$.

If $\theta > \pi/2$, then $\theta_1 > \pi/2$, $\theta_2 > \pi/2$ and equation (2.4.33) always has a solution in the strip $-1 < \operatorname{Im} w < 0$, therefore the function is representable in the form

$$\sigma_y(x) = |x|^{|\tau_0|-1}\varphi_0(x),$$

where $\varphi_0(x)$ is bounded on the half-axis $x \ge 0$.

In particular, if $\theta = \frac{3\pi}{2}$, then $\theta_1 = \theta_2 = \frac{3\pi}{2}$, $\delta = 0$, $\gamma = 3\pi$. Therefore in that case $\sigma_y(x)$ is representable in the form

$$\sigma_y(x) = c_0 x^{-\frac{2}{3}} + c_1 x^{-\frac{1}{3}} + \varphi_0(x), \quad x > 0,$$

where $\varphi_0(x)$ continuous near the point $x = 0$.

3. Assume now that the body is isotropic. In that case, the boundary conditions cannot be represented as (2.4.3), (2.4.6) and (2.4.7) because we have obtained them under the assumption that the characteristic equation has no multiple roots. For an isotropic body, the characteristic equation has the multiple roots

$$s_1 = s_2 = i, \quad s_3 = s_4 = -i.$$

But by the expressions obtained using the above-mentioned conditions, which do not contain complex potentials, we can obtain the results for an isotropic body if we take the limit as $s_1 \to s_2 = i$. Namely, assuming $\beta_2 = 1$, taking into account that $\theta_2 = 0$, and $\gamma \cdot \delta \cdot \mu$ depend on β_1, and passing to the limit as $\beta_1 \to 1$ in (2.4.31), we obtain the problem of an isotropic body.

Let us represent the coefficient of problem (2.4.31) in the form

$$a_1 \, \frac{\mathbf{sh}\,\gamma t + (1+\beta_1)\,\frac{\mathbf{sh}\,\delta t}{\beta_1 - 1}}{\mathbf{ch}\,\gamma t - \frac{(\beta_1+1)^2\,\mathbf{ch}\,\gamma t - 4\beta_1 \cos \mu t}{(\beta_1-1)^2}} \, .$$

Evaluating the indeterminate form as $\beta_1 \to 1$, we obtain

$$2a_{22}D\,\frac{\mathbf{sh}\,2\theta t + 2t\sin\theta\cos\theta}{\mathbf{sh}^2\,\theta t - t^2\sin^2\theta}\, . \tag{2.4.34}$$

If now the relation $\Delta_1(t)/\Delta(t)$ in solution (2.4.28) is replaced by (2.4.34), we obtain the contact stress $\sigma_y(x)$ for an isotropic body.

4. In the case where the concentrated force P is applied at some point $x_0 > 0$ of the beam, the function $f(\xi)$ is analytic in the intervals $(-\infty; \ln x_0)$ and $(\ln x_0; \infty)$, and has, at the point $\ln x_0$, a discontinuity of first kind. It can be proved that the function $F(t)$ vanishes at infinity when being of first order, $\Psi(t)$ vanishes when being of second order, and $N(t)$ vanishes when being of third order. Therefore the integrand in (2.4.17) vanishes at infinity when being of first order, and the integral itself exists in the Plancheral sense [**116**] and for it the inverse Fourier transform is valid.

When $x_0 = 0$, which means that the concentrated force is applied to the beam end, the function $f(\xi)$ is analytic all over the axis and vanishes exponentially at infinity. The functions $F(t)$ and $N(t)$ possess the same property.

2.5. The Contact Problem for an Anisotropic Wedge-Shaped Plate with an Elastic Fastening of Variable Stiffness

Contact problems of the interaction between elastic bodies of various shapes (including wedge-shaped bodies) and thin elastic elements in the form of stringers or inclusions were considered in [**5**], [**4**], [**94**]. Problems for an elastic isotropic or anisotropic wedge, supported by a rod of constant stiffness [**13**], [**19**], [**81**], [**95**], as well as the problem for an elastic isotropic wedge, supported along the bisector by an elastic rod of variable stiffness [**83**] were studied by means of boundary-value problems of the theory of analytical functions.

In this section, we consider the elastic anisotropic thin wedge-shaped plate occupying an angle $-\theta < \arg z < \theta$, $0 < \theta < 2\pi$ in the plane. One side of the angle $\arg z = -\theta$ is free of stresses and the rod of variable tensile stiffness is glued to the other side $\arg z = 0$. We will determine the distribution of contact forces along the fastening line as well as the elastic equilibrium of the plate under tangential load of intensity $\tau_0(x)$ applied along the rod. It is assumed that the bending stiffness of the rod is negligibly small, i.e. $\sigma_y^0 = 0$.

From the equilibrium condition for any part $(0, x)$ of the rod we have

$$S_0(x)\sigma_x^0(x) - h\int\limits_0^x \left[\tau_{xy}^0(s) - \tau_0(s)\right]\,dx = 0, \quad x > 0. \tag{2.5.1}$$

A condition for a full contact between the elastic rod and the wedge has the form (the prime denotes differentiation with respect to x)

$$u_0'(x) = u'(x,0), \quad \tau_{xy}^0(x) = \tau_{xy}(x,0) \equiv \tau(x), \quad x > 0. \tag{2.5.2}$$

By Hooke's law, taking into account that $\sigma_y^0 = \sigma_y = 0$, we have

$$u_0'(x) = \sigma_x^0(x)/E_0(x), \quad u'(x,0) = a_{16}\tau_{xy}(x,0) + a_{11}\sigma_x(x,0). \tag{2.5.3}$$

Here $E_0(x)$ is the modulus of elasticity of the rod, a_{11} and a_{16} are the elasticity constants of the plate, $\sigma_x^0(x)$, $\tau_{xy}^0(x)$ and $\sigma_x(x,y)$, $\tau_{xy}(x,y)$ are the normal and shear stresses of the rod and the wedge, respectively, $u_0(x)$ and $u(x,y)$ are the horizontal displacements of the rod and elastic wedge, respectively; $s_0(x)$ is the cross-section area of the rod and h is the plate thickness.

Taking equations (2.5.2) and (2.5.3) into account, we can rewrite condition (2.5.1) in the form

$$\begin{gathered}
k_1(x)\sigma_x(x) + k_2(x)\tau(x) - hJ(x) = 0, \quad x > 0, \\
k_1(x) = s_0(x)E_0(x)a_{11}, \quad k_2(x) = s_0(x)E_0(x)a_{16}, \\
J(x) = \int_0^x [\tau(s) - \tau_0(s)]\, ds.
\end{gathered} \tag{2.5.4}$$

An equilibrium condition for the rod has the form

$$J(\infty) = 0. \tag{2.5.5}$$

Consider two planes of complex variables: $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, which are obtained from the plane $z = x + iy$ by the affine transforms $x_n = x + \alpha_n y$ and $y_n = \beta_n y$, $\beta_n > 0$, respectively, where $s_n = \alpha_n + i\beta_n$ $(n = 1.2)$ are the roots of the characteristic equation, where $s_1 \neq s_x$ [**66**].

The domain $S(-\theta < \arg z < 0)$ in the plane of the complex variable z is mapped by means of these transforms into the domains $S_n(-\theta_n < \arg z_n < 0)$, respectively, in the plane z_n $(n = 1, 2)$ where

$$\operatorname{tg}\theta_n = \beta_n \sin\theta(\cos\theta - \alpha_n\sin\theta)^{-1}, \quad 0 < \theta_n < 2\pi.$$

The problem thus reduces, by means of the well-known relations from [**66**] defining the stress vector components in terms of two analytical functions, to the solution of the following boundary value problem of the theory of functions of a complex-variable: find two functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ that are analytic in the domains S_1 and S_2, respectively, using the boundary conditions

$$(s_1 - \overline{s}_2)t_1\Phi_1(t_1) + (\overline{s}_1 - \overline{s}_2)\overline{t}_1\overline{\Phi_1(t_1)} + (s_2 - \overline{s}_2)t_2\Phi_2(t_2) = 0, \tag{2.5.6}$$
$$t_n = \rho(\cos\theta - s_n\sin\theta), \quad \rho = |t| \geq 0,$$

$$(s_1 - \overline{s}_2)\Phi_1(t_1) + (\overline{s}_1 - \overline{s}_2)\overline{\Phi_1(t_1)} + (s_2 - \overline{s}_2)\Phi_2(t_2) = -\tau(x), \tag{2.5.7}$$
$$t_1 = t_2 = x > 0,$$

$$2\operatorname{Re}[k_1(x)a\Phi_1(x)] + [k_2(x) - 2\alpha_2 k_1(x)]\tau(x) = hJ(x), \quad x > 0, \tag{2.5.8}$$
$$a = (s_1 - s_2)(s_1 - \overline{s}_2).$$

Assume that stresses and rotations vanish at infinity, for large $|z_n|$ we obtain

$$\Phi_n(z_n) = \gamma_n/z_n + O(1/z_n), \quad n = 1, 2.$$

Assume further that the functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ are continuously extendable to all boundary points, except perhaps for the points $z_n = 0$, at which they satisfy the conditions

$$\lim z_n\Phi_n(t_n) = 0 \quad \text{when} \quad z_n \to 0.$$

So, we will look for functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ in the form

$$\Phi_n(z_n) = \frac{1}{\sqrt{2\pi}\, z_n} \int\limits_{-\infty}^{\infty} \frac{A_n(t)}{t}\, e^{it\ln z_n}\, dt - \frac{a_n}{z_n}\,, \quad z_n \in S_n, \tag{2.5.9}$$

where

$$a_n = \lim_{z_n \to 0} \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{A_n(t)}{t}\, e^{it\ln z_n}\, dt, \quad n = 1, 2. \tag{2.5.10}$$

At the point $t = 0$ the integrals are considered in the sense of the principal Cauchy value. It can be shown that $a_n = -i\sqrt{\pi/2}\, A_n(0)$ from which it follows that $\gamma_n = -2a_n = i\sqrt{2\pi}\, A_n(0)$. We can also conclude from Eqs (2.5.6) and (2.5.9) that a_1 and a_2 satisfy the condition

$$(s_2 - \overline{s}_2)a_2 = (\overline{s}_2 - s_1)a_1 + (\overline{s}_2 - \overline{s}_1)\overline{a}_1.$$

Substituting (2.5.9) into conditions (2.5.6) and (2.5.7), carrying out Fourier transformation and solving the latter system for $A_n(t)$ $(n = 1, 2)$, we obtain

$$A_1(t) = \frac{1}{2\Delta(t)} \left[(\overline{s}_1 - s_2)e^{-\delta t} + (\overline{s}_2 - \overline{s}_1)e^{-\gamma t} + (s_2 - \overline{s}_2)e^{-i\mu t}\right] tT(t), \tag{2.5.11}$$

$$\Delta(t) = |s_1 - s_2|^2\, \mathbf{ch}\, \gamma t - |s_1 - \overline{s}_2|^2\, \mathbf{ch}\, \delta t + 4\beta_1\beta_2 \cos \mu t,$$

$$T(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^s \tau(e^s) e^{-its}\, dt,$$

$$\gamma = \theta_1 + \theta_2, \quad \delta = \theta_1 - \theta_2, \quad \mu = \ln|\cos\theta - s_1 \sin\theta| - \ln|\cos\theta - s_2 \sin\theta|.$$

The function $A_2(t)$ is obtained from the expression for $A_1(t)$ by interchanging s and s_2 and θ_1 and θ_2. It is obvious that $\overline{T(-t)} = T(t)$. Since the stress vanishes at infinity, taking the limit in the relation for $T(t)$ we obtain

$$T(0) = T_0/\sqrt{2\pi}\,, \quad T_0 = \int\limits_0^{\infty} \tau(t)\, dt = \int\limits_0^{\infty} \tau_0(t)\, dt.$$

It can be proved that the function $\Delta(t)$ vanishes nowhere for real t except for the point $t = 0$ where it has a double zero root. The function in square brackets in the equation for $A_1(t)$ behaves similarly. Consequently, if the function $\tau(x)$ is absolutely integrable, then the functions $A_1(t)$ and $A_2(t)$ is continuous over the axis. Therefore equation (2.5.11) implies

$$A_1(0) = \frac{(\overline{s}_1 - \overline{s}_2)\gamma - (\overline{s}_1 - s_2)\delta - i\mu(s_2 - \overline{s}_2)}{|s_1 - s_2|^2\gamma^2 - |s_1 - \overline{s}_2|^2\delta^2 - 4\beta_1\beta_2\mu^2}\, \frac{T_0}{\sqrt{2\pi}}\,. \tag{2.5.12}$$

Hence the constants a_1, a_2, γ_1 and γ_2 are well-defined.

Substituting the value of the function $\Phi_1(z_1)$ defined by equations (2.5.9) and (2.5.11) into the boundary condition (2.5.8), by Vieta's formula for characteristic equations we get

$$\frac{1}{\sqrt{2\pi}\,i}\int\limits_{-\infty}^{\infty}\frac{\Delta_1(t)}{\Delta(t)}\,T(t)e^{it\ln x}\,dt-\frac{hx}{k_1(x)}\,J(x)=2\operatorname{Re}aa_1, \tag{2.5.13}$$

$$\Delta_1(t)=-(\beta_1+\beta_2)|s_1-s_2|^2\,\mathbf{sh}\,\gamma t+(\beta_1-\beta_2)|s_1-\overline{s}_2|^2\,\mathbf{sh}\,\delta t$$
$$+4|\alpha_1-\alpha_2|\beta_1\beta_2\sin\mu t.$$

Let $k_1(x)=d_0x^\alpha$, $d_0>0$ and α be any real number. After substituting $\ln x=\xi$, equation (2.5.13) takes the form

$$\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\frac{G(t)}{t}\,T(t)e^{it\xi}\,dt$$
$$-He^{-k\xi}\bigg(\int\limits_{-\infty}^{\xi}[\tau(e^s)-\tau_0(e^s)]e^s\,ds\bigg)=2\operatorname{Re}aa_1, \tag{2.5.14}$$
$$G(t)=\frac{\Delta_1(t)}{\Delta(t)}\,t,\quad k=\alpha-1,\quad H=\frac{h}{d_0}\,.$$

Differentiating both sides of (2.5.14) and applying the inverse Fourier transformation to the resulting relation with the complex variable $t=t_0-i\varepsilon$ as a parameter (ε is an arbitrarily small positive number), we obtain

$$G(t)\Psi(t)-H\Psi(t-ik)=F(t),\quad -\infty-i\varepsilon<t<+\infty-i\varepsilon, \tag{2.5.15}$$
$$t\Psi(t)=T(t)-T_0(t),\quad F(t)=-\frac{G(t)T_0(t)}{t}\,,$$
$$T_0(t)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}e^s\tau_0(e^s)e^{-its}\,dx.$$

Assume that $k>0$. The problem under consideration reduces to the following Carleman type problem for the strip: find a function $\Psi(z)$ which is holomorphic in the strip $-k-\varepsilon<\operatorname{Im}z<-\varepsilon$, vanishes at infinity, is continuously extendable on the strip boundary and satisfies condition (2.5.15).

Using the results obtained earlier [**15**], the function $\Psi(z)$ can be written in the form

$$\Psi(z)=\frac{\chi(z)}{2ikH}\int\limits_{-\infty-i\varepsilon}^{+\infty-i\varepsilon}\frac{F(t)}{\chi(t-ik)}\,\Big(\mathbf{sh}\,\frac{\pi}{k}(t-z)\Big)^{-1}\,dt, \tag{2.5.16}$$
$$-k-\varepsilon<\operatorname{Im}z<-\varepsilon,$$
$$\chi(z)=\frac{1}{z}\,\chi_k(z)\varkappa(z)\,\mathbf{sh}\,\frac{\pi}{2k}\,z,\quad \varkappa(z)=k^{iz/k}\Gamma\left(\frac{k+iz}{k}\right)\exp(iz\ln H_0^{1/k}),$$

$$\chi_k(z) = \exp\left\{\frac{1}{2ik}\int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} \ln G_k(t)\coth\frac{\pi}{k}(t-z)\,dt\right\},$$

$$G_k(t) = -\frac{\Delta_1(t)}{(\beta_1+\beta_2)\Delta(t)}\,\mathbf{th}\,\frac{\pi}{2k}\,t, \quad H_0 = \frac{\beta_1+\beta_2}{H}\,.$$

Now assume that $k \geq 1$. If the function $T_0(z)$ is analytically extendable in the strip $-1 < \operatorname{Im} z < 1$ and vanishes exponentially at infinity, then condition (2.5.15) and equation (2.5.16) imply that the function

$$\Psi_1(z) = \begin{cases} \Psi(z), & -k-\varepsilon < \operatorname{Im} z < -\varepsilon, \\ [F(z) + H\Psi(z-ik)]/G(z), & -\varepsilon < \operatorname{Im} z < k-\varepsilon, \end{cases}$$

is holomorphic in the strip $-k-\varepsilon < \operatorname{Im} z < k-\varepsilon$, vanishes exponentially at infinity, and is bounded all over the strip except for the points $z_j^+ = t_j^+ + i\tau_j^+$ $(j = 0, 1, \dots, p)$ which are the zeroes of the function $G(z)$ in the upper strip.

Thus, according to the Cauchy formula, the required contact stress can be represented as

$$\tau(x)-\tau_0(x) = \frac{x^{-1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t\Psi(t)e^{it\ln x}\,dt = \frac{x^{-1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(t-ik)\Psi(t-ik)e^{i(t-ik)\ln x}\,dt.$$

Consequently, in the neighborhood of the angle vertex we obtain $\tau(x) - \tau_0(x) = x^{k-1}\varphi_0(x)$ (as $x \to 0$), where $\varphi_0(x)$ is a bounded function near the point $x = 0$. For large x we get

$$\tau(x) - \tau_0(x) = O(1/x^{1+\tau_0^+}).$$

If $0 < k < 1$, the function $\Psi(z)$ given by (2.5.16) is analytically continuous in the strip $-1 < \operatorname{Im} z < 1$ except for the points $\omega_j^- = \lambda_j^- + i\mu_j^-$ $(j = 0, 1, l)$ which are the poles of the function $G(z)$ in the same strip. Then shear stress near the point $x = 0$ is represented as follows:

$$\tau(x) - \tau_0(x) = \frac{x^{-1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(t-i)\Psi(t-i)e^{i(t-i)\ln x}\,dt$$

$$+\frac{x^{-1}}{\sqrt{2\pi}}\operatorname{res}\left[z\Psi(z)e^{iz\ln x}\right]_{\omega_0^-=\lambda_0^-+i\mu_0^-} = c_1x^{-(\mu_0^-+1)} + \varphi_1(x), \quad c_1 = const,$$

where $\varphi_1(x)$ is a bounded function for $x \geq 0$.

Let us consider the case where $k < 0$ $(\alpha < 1)$, i.c. where the stiffness of the rod grows near the angle vertex and vanishes at infinity, and the principal vector of external load is shifted to the wedge. Putting $m = -k$, we can write condition (2.5.15) in the form

$$G(t)\Psi_0(t) - H\Psi_0(t+im) = F(t), \quad -\infty - i\varepsilon < t < +\infty - i\varepsilon. \quad (2.5.17)$$

Now we will consider the following problem: find a function $\Psi_2(z)$ which is holomorphic in the strip $-m-\varepsilon < \operatorname{Im} z < m-\varepsilon$, vanishes at infinity and is bounded all over the strip, except for the points $z_j^- = t_j^- + i\tau_j^-$

$(j = 0, 1, \ldots, q)$ which are the zeroes of the function $G(z)$ in the lower half-space.

If we first solve the problem of finding a function $\Psi_0(z)$ which is holomorphic in the strip $-\varepsilon < \operatorname{Im} z < m - \varepsilon$, vanishes at infinity and is continuously extendable on the strip boundary by the boundary condition (2.5.17), then the solution of the preceding problem will be the function

$$\Psi_2(z) = \begin{cases} \Psi_0(z), & -\varepsilon < \operatorname{Im} z < m - \varepsilon, \\ [F(z) + H\Psi_0(z + im)]/G(z), & -m - \varepsilon < \operatorname{Im} z < -\varepsilon, \end{cases}$$

Using the results obtained in [**15**], the function Ψ_0 can be written in the form

$$\Psi_0(z) = -\frac{\widetilde{\chi}(z)}{2imH} \int\limits_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} \frac{F(t)}{\widetilde{\chi}(t + im)} \left(\mathbf{sh}\, \frac{\pi}{m}\,(t - z)\right)^{-1} dt, \tag{2.5.18}$$

$$\widetilde{\chi}(z) = \frac{1}{z}\, \chi_m(z) \widetilde{\varkappa}(z)\, \mathbf{sh}\, \frac{\pi}{2m}\, z,$$

$$\widetilde{\varkappa}(z) = m^{-iz/m} \Gamma\left(\frac{m - iz}{m}\right) \exp(-iz \ln H_0^{1/m}).$$

If $\tau_0^- < 1$, then the function $\Psi_2(z)$ is analytically extendable in the strip $-1 < \operatorname{Im} z < m - \varepsilon$ and the shear stress $\tau(x) - \tau_0(x)$ is bounded at the point $x = 0$. If $\tau_0^- > -1$, then the function $\Psi_2(z)$ has the pole very near to the real axis at the point $z_0^- = t_0^- + i\tau_0^-$, the function $T(t) - T_0(t)$ has a similar property and the unknown contact stress near the point $x = 0$ can be represented as

$$\tau(x) - \tau_0(x) = c_2 x^{-(\tau_0^- + 1)} + \varphi_2(x).$$

For large x we have

$$\tau(x) - \tau_0(x) = O(1/x^{1+m}).$$

For $\alpha = 1$ $(k = m = 0)$, condition (2.5.17) gives

$$\Psi(z) = F(z)/(G(z) - H)$$

and shear stress has the form

$$\tau(x) - \tau_0(x) = O(x^{\lambda - 1}) \quad \text{as} \quad x \to 0, \quad \lambda = \operatorname{Im} \mu,$$

where μ is chosen from the zeros nearest to the real axis of the functions $\Delta(z)$ and $G(z) - H$ in the lower half-space.

For $\alpha < 1$, when $\theta = \pi$, i.e. the anisotropic body is a half-plane, the function

$$G(z) = -(\beta_1 + \beta_2) z \coth \pi z$$

has a unique purely imaginary root $z_0 = -i/2$ in the strip $-1 < \operatorname{Im} z < 0$, and shear stress near the point $x = 0$ has the form

$$\tau(x) - \tau_0(x) = c_2 x^{-1/2} + \varphi_2(x).$$

When $\theta = 2\pi$, i.e. the body occupies the entire plane cut along the positive part of the real axis, then

$$G(z) - (\beta_1 + \beta_2) z \coth 2\pi z.$$

This function has pure imaginary roots $z_0 = -i/4$, $z_1 = -3i/4$ in the strip $-1 < \operatorname{Im} z < 0$, and the shear stress as $x \to 0$ has following form

$$\tau(x) - \tau_0(x) = c_3 x^{-3/4} + c_4 x^{-1/4} + \varphi_3(x).$$

Here $\varphi_2(x)$ and $\varphi_3(x)$ arc bounded functions for $x \geq 0$, and c_2, c_3 and c_4 are constants.

For $1 < \alpha \leq 2$, when $\theta = \pi$ the function $G(z)$ has a pole at the point $\omega_0^- = -i$, the shear stress is bounded in the neighborhood of the angle vertex. When $\theta = 2\pi$ as $x \to 0$, shear stress has a singularity of square root order.

Analogous results are obtained for an isotropic body in [**9**].

Now consider the case with an orthotropic body. Then

$$\Delta_1(t) = -(\beta_1 + \beta_2)(\beta_1 - \beta_2)^2 \,\mathbf{sh}\, \gamma t + (\beta_1 + \beta_2)^2 (\beta_1 - \beta_2) \,\mathbf{sh}\, \delta t,$$
$$\Delta(t) = (\beta_1 - \beta_2)^2 \,\mathbf{ch}\, \gamma t - (\beta_1 + \beta_2)^2 \,\mathbf{ch}\, \delta t + 4\beta_1\beta_2 \cos \mu t.$$

One can prove that, for $0 < \theta < \pi$, the equation $\Delta_1(t) = 0$ can have only the imaginary root in the strip $-1 < \operatorname{Im} z < 0$, while the equation $\Delta(z) = 0$ does not have any roots in this strip. Moreover, for $\theta < \pi/2$ $(\theta_2 < \theta_1 < \pi/2)$, the equation $\Delta_1(z) = 0$ does not have any roots in the strip $-1 < \operatorname{Im} z < 0$.

For $\alpha < 1$, if $\theta = 2\pi/3$, the function $\Delta_1(z)$ has zeroes at the points $z_0^- = -i/3$, $z_1^- = -2i/3$ and the stress at the point $x = 0$ has the estimate

$$\tau(x) - \tau_0(x) = \widetilde{c}_1 x^{-2/3} + \widetilde{c}_2 x^{-1/3} + \widetilde{\varphi}_3(x),$$

where $\widetilde{\varphi}_3(x)$ is a bounded function for $x \geq 0$, and $\widetilde{c}_1$ and $\widetilde{c}_2$ are constants.

When $\pi/2 < \theta < \pi$, by an appropriate choice of numbers δ and γ or numbers β_1 and β_2, we can make the equation $\Delta_1(z) = 0$ have a root in the strip $-1 \leq \operatorname{Im} z < 0$. This means that the stress $\tau(x) - \tau_0(x)$ can be both bounded and unbounded at the point $x = 0$.

2.6. The Bending Problem of a Beam Resting on the Elastic Foundation

Contact problems of the interaction of differently shaped elastic bodies with thin elastic elements in the form of stringers, beams or inclusions were considered in [**5**], [**4**], [**94**]. Problems for an elastic isotropic or anisotropic wedge, reinforced with elastic elements of constant stiffness [**13**], [**17**], [**19**], [**81**], [**95**], and, also, the problem for an elastic isotropic wedge reinforced along the bisectrix by an elastic rod of variable stiffness [**83**] were investigated using boundary value problems of the analytic functions theory. The contact problem for an anisotropic wedge-shaped plate with an elastic support of variable stiffness was considered in [**27**].

Let us assume that a beam with stiffness $D(x)$ lies on one boundary ($\arg z = 0$) of an elastic anisotropic body which occupies an angle $-\theta \le \arg z \le 0$ in the plane $z = x + iy$ and that the distributed normal load $P_0(x)$ is applied to the beam. $P_0(x)$ is assumed to be a bounded summable function, equal to zero outside some interval. There is no friction between the beam and the wedge. The other boundary of the wedge ($\arg z = -\theta$) is stress-free, $0 < \theta < 2\pi$.

The problem reduces to the following problem of elastic angle equilibrium

$$\frac{d^2}{dx^2}\, D(x)\, \frac{d^2 v}{dx^2} = P_0(x) - P(x), \quad \tau_{xy}(x,0) = 0, \quad x > 0;$$
$$D(x) = \frac{E_0(x) h^3(x)}{12(1-\nu_0^2)}\,, \tag{2.6.1}$$

$$X_n(t) = Y_n(t) = 0, \quad \arg t = -\theta, \tag{2.6.2}$$

where $P(x)$ is the required contact stress satisfying the equilibrium conditions

$$\int\limits_0^\infty P(t)\,dt = \int\limits_0^\infty P_0(t)\,dt = P_0, \quad \int\limits_0^\infty tP(t)\,dt = \int\limits_0^\infty tP_0(t)\,dt = M_0, \tag{2.6.3}$$

$E_0(x)$ is the elasticity modulus of the beam, $h(x)$ is its thickness, ν_0 is Poisson's ratio and $v(x)$ is the vertical displacement of the points of the beam.

We will consider two planes of complex variables: $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ obtained from the plane $z = x + iy$ by the affine transforms

$$x_1 = x + \alpha_1 y, \quad y_1 = \beta_1 y, \quad x_2 = x + \alpha_2 y, \quad y_2 = \beta_2 y; \quad \beta_1 > \beta_2 > 0.$$

By these transformats, the domain $S(-\theta \le \arg z \le 0)$ of the plane of the variable z transforms to the domain $S_k(-\theta_k \le \arg z_k \le 0)$ of the plane of the variable z_k ($k = 1,2$), $\operatorname{tg}\theta_k = \beta_k \sin\theta(\cos\theta - \alpha_k \sin\theta)^{-1}$.

If the roots of the characteristic equation $s_1 \ne s_2$, then, by virtue of the well-known formulas [**66**], the problem reduces to finding holomorphic functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ in the domains S_1 and S_2, respectively, by the following boundary conditions

$$(s_1 - \overline{s}_2) t_1 \Phi_1(t_1) + (\overline{s}_1 - \overline{s}_2)\overline{t}_1 \overline{\Phi_1(t_1)} + (s_2 - \overline{s}_2) t_2 \Phi_2(t_2) = 0,$$
$$t_k = \rho(\cos\theta - s_k \sin\theta), \quad \rho = |t| > 0, \tag{2.6.4}$$

$$(s_1 - \overline{s}_2)\Phi_1(t) + (\overline{s}_1 - \overline{s}_2)\overline{\Phi_1(t)} + (s_2 - \overline{s}_2)\Phi_2(t) = -\overline{s}_2 P(t), \quad t > 0, \tag{2.6.5}$$

$$2\operatorname{Re}\left[q_1 \Phi_1'(x) + q_2 \Phi_2'(x)\right] = \frac{1}{D(x)} \int\limits_0^x dt \int\limits_0^t [P_0(s) - P(s)]\,ds, \quad x > 0. \tag{2.6.6}$$

It is required of the functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ to satisfy the conditions

$$\lim z_k \Phi_k(z_k) \to 0, \quad z_k \to 0, \quad k = 1, 2,$$

and, for sufficiently large $|z_k|$, to have the form

$$\Phi_k(z_k) = \gamma_k/z_k + O(1/z_k), \quad k = 1, 2, \tag{2.6.7}$$

$$\Phi_k(z_k) = \frac{1}{\sqrt{2\pi}\, z_k} \int_{-\infty}^{\infty} \frac{A_k(t)}{t}\, e^{it\ln z_k} - i\sqrt{\frac{\pi}{2}}\, \frac{A_k(0)}{z_k}\,, \quad z_k \in S_k. \tag{2.6.8}$$

Furthermore, we assume that $A_k(0)$ satisfies the condition

$$(s_2 - \overline{s}_2)A_2(0) = (\overline{s}_2 - s_1)A_1(0) + (\overline{s}_1 - \overline{s}_2)\overline{A_1(0)}\,.$$

The substitution of (2.6.8) into the boundary conditions (2.6.4) and (2.6.5) yields

$$A_k(t) = \Big[\overline{s}_k(s_2 - \overline{s}_2)e^{i(3-2k)\mu t} + \overline{s}_{3-k}(\overline{s}_k - s_{3-k})e^{-(3-2k)\delta t} + s_{3-k}(\overline{s}_2 - \overline{s}_1)e^{-\gamma t}\Big]\, \frac{tN(t)}{2\Delta(t)}\,, \tag{2.6.9}$$

where

$$\mu = \ln\left|\frac{\cos\theta - s_1\sin\theta}{\cos\theta - s_2\sin\theta}\right|, \quad \delta = \theta_1 - \theta_2, \quad \gamma = \theta_1 + \theta_2,$$

$$\Delta(t) = |s_1 - s_2|^2\, \mathbf{ch}\,\gamma t - |s_1 - \overline{s}_2|^2\, \mathbf{ch}\,\delta t + 4\beta_1\beta_2\cos\mu t,$$

$$N(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} P(e^s)e^s e^{-its}\, ds.$$

The first equality of (2.6.3) gives

$$\sqrt{2\pi}\, N(0) = \int_{-\infty}^{\infty} P(e^s)e^s\, ds = \int_{0}^{\infty} P(t)\, dt = P_0.$$

Taking the limit in equalities (2.6.9) as $t \to 0$, we obtain

$$A_k(0) = \frac{2(-1)^k\mu\beta_2\overline{s}_k + (-1)^k\delta\overline{s}_{3-k}(\overline{s}_k - s_{3-k}) + \gamma s_{3-k}(\overline{s}_1 - \overline{s}_2)}{|s_1 - s_2|^2\gamma^2 - |s_1 - \overline{s}_1|^2\delta^2 - 4\beta_1\beta_2\mu^2}\, \frac{P_0}{\sqrt{2\pi}}\,.$$

Substituting the functions $\Phi_k(z_k)$ defined by (2.6.8) into the boundary condition (2.6.6) and keeping in mind equality (2.6.9), we have

$$2\,\mathrm{Re}\left[q_1\Phi_1'(x) + q_2\Phi_2'(x)\right] = \frac{1}{\sqrt{2\pi}\, x^2} \int_{-\infty}^{\infty} \frac{(it-1)(\Delta_2 + i\Delta_1)N(t)e^{it\ln x}}{\Delta(t)}\, dt + \frac{c}{x^2}\,, \tag{2.6.10}$$

where

$$c = \sqrt{\pi}\ \mathrm{Im}[q_1A_1(0) + q_2A_2(0)],$$

$$\Delta_1(t) = a_1^+\,\mathbf{sh}\,\gamma t + a_1^-\,\mathbf{sh}\,\delta t + c_1^-\sin\mu t,$$

$$\Delta_2(t) = a_2^+\,\mathbf{ch}\,\gamma t + a_2^-\,\mathbf{ch}\,\delta t + c_1^+\cos\mu t,$$

$$
\begin{aligned}
c_1^{\pm} &= 2\beta_2 \operatorname{Im}[\overline{q}_1 s_1] \pm 2\beta_1 \operatorname{Im}[\overline{q}_2 s_2], \\
a_2^- + i a_1^- &= (\overline{q}_1 s_2 - q_2 \overline{s}_1)(s_1 - s_2), \\
a_2^+ + i a_1^+ &= (\overline{q_1 s_2} - \overline{q_2 s_1})(s_2 - s_1).
\end{aligned}
$$

Substituting the values of q_1, and q_2 [**66**] into the above formulas, performing the operation of reduction and applying Vieta's theorem, we obtain

$$
a_1^+ = a_{22}|s_1 - s_2|^2 \operatorname{Im}\left(\frac{1}{\overline{s}_1} + \frac{1}{\overline{s}_2}\right), \quad a_1^- = a_{22}|s_1 - \overline{s}_2|^2 \operatorname{Im}\left(\frac{1}{\overline{s}_2} - \frac{1}{\overline{s}_1}\right),
$$

$$
a_2^+ = a_2^- = c_1^+ = 0, \quad c_1^- = 4\beta_1\beta_2 \operatorname{Re}\left(\frac{1}{s_1} - \frac{1}{s_2}\right), \quad \Delta_2(t) = 0,
$$

where a_{22} is one of the constants of elasticity of the plate.

Thus, by (2.6.10), condition (2.6.6) takes the form

$$
\begin{gathered}
-\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} G(t)N(t)e^{it\ln x}\,dt + \frac{x^2}{D(x)} \int\limits_0^x dt \int\limits_0^t [P(s) - P_0(s)]\,ds = c; \\
G(t) = \frac{(1-it)\Delta_1(t)}{\Delta(t)}\,.
\end{gathered}
\tag{2.6.11}
$$

The functions $\Delta(t)$ and $\Delta_1(t)$ do not vanish anywhere except for the point $t = 0$. The point $t = 0$ is a second order zero for the function $\Delta(t)$ and a first order zero for the function $\Delta_1(t)$.

We put $D(x) = d_0 x^{p+2}$, $d_0 > 0$, where p is any real number. After substituting $\xi_0 = \ln x$ into formula (2.6.11), differentiating both sides of the resulting equality and carrying out the inverse Fourier transformation, we obtain

$$
d_0 t(p+it)G(t)\Psi(t) + \Psi(t - ip) = F(t), \quad -\infty - i\varepsilon < t < \infty - i\varepsilon, \tag{2.6.12}
$$

where

$$
\Psi(t) = \frac{N(t) - N_0(t)}{t}\,, \quad F(t) = -d_0 G(t)(p+it)N_0(t),
$$

$$
N_0(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^s P_0(e^s) e^{-its}\,ds
$$

and ε is a positive number which can be arbitrarily small.

There arises the following problem: find a function which is homomorphic in the strip $-p - \varepsilon < \operatorname{Im} z < -\varepsilon$, vanishes at infinity, is continuable on the strip boundary and satisfies condition (2.6.12).

The function $F(t)$ is analytically extendable in the strip $0 < \operatorname{Im} z < p$ except for the points which are the roots of the function $\Delta(t)$, where $F(t)$ has poles and vanishes at infinity.

Assume that $p>0$. Then the coefficients of the problem can be given the form

$$\frac{t(t-ip)(t+i)\Delta_1(t)}{\Delta(t)}=it(t^2+p^2)T_p(t)\,\frac{\Delta_1(t)}{\Delta(t)}\ \mathbf{th}\,\frac{\pi}{2p}\,t\,\frac{\mathbf{sh}\,\frac{\pi}{2p}(t-ip)}{\mathbf{sh}\,\frac{\pi}{2p}t}\,;$$

$$T_p(t)=\frac{t+i}{t+ip}\,.$$

Consider the function

$$G_p(t)=T_p(t)U_p(t),$$

where

$$U_p(t)=\frac{\Delta_1(t)}{a\Delta(t)}\ \mathbf{th}\,\frac{\pi}{2p}\,t,\quad a=\lim_{t\to\infty}\frac{\Delta_1(t)}{\Delta(t)}=a_{22}\,\mathrm{Im}\left(\frac{1}{\overline{s}_1}+\frac{1}{\overline{s}_2}\right).$$

The function $G_p(t)$ is continuous all over the axis and $G_p(-\infty)=G_p(+\infty)=0$. The function $U_p(t)$ takes positive values,the function $T_p(t)$ has a unique zero and one pole in the lower half-plane and therefore $\mathrm{Ind}\,G_p(t)=0$. The branch of the function In $G_p(t)$ which vanishes at infinity is integrable all over the axis.

By virtue of the results obtained in [**15**], the functions $G_p(t)$, t^2+p^2 and the number ad_0 can be represented as

$$G_p(t)=\frac{X_p(t-ip)}{X_p(t)}\,,\quad t^2+p^2=\frac{X_1(t-ip)}{X_1(t)}\,,\quad ad_0=\frac{X_2(t-ip)}{X_2(t)}\,,\qquad (2.6.13)$$
$$-\infty-i\varepsilon<t<(\infty-i\varepsilon),$$

where

$$X_p(z)=\exp\left\{\frac{1}{2i\rho}\int\limits_{-\infty-i\varepsilon}^{\infty-i\varepsilon}\ln G_p(t)\coth\pi(t-z)\,dt\right\},$$
$$X_1(z)=p^{2tz/p}\Gamma(1+iz/p)/\Gamma(2-iz/p),$$
$$X_2(z)=\exp(i(z/p)\ln(ad_0)),\quad -p-\varepsilon<\mathrm{Im}\,z<-\varepsilon.$$

The substitution of expressions (2.6.13) into (2.6.12) yields

$$\begin{gathered}\frac{\Psi(t)}{X(t)}+\frac{\Psi(t-ip)}{X(t-ip)}=\frac{F(t)}{X(t-ip)}\,,\quad -\infty-i\varepsilon<t<\infty-i\varepsilon,\\ X(z)=\frac{1}{z}\,X_p(z)X_1(z)X_2(z)\,\mathbf{sh}\,\frac{\pi}{2p}\,zp^{iz/p}\Gamma(1+iz/p).\end{gathered}\qquad (2.6.14)$$

The functions $X_p(z)$ and $X_2(z)$ are bounded all over the strip and, for sufficiently large $|z|$, the function $X_1(z)$ admits the estimate

$$|X_1(z)|=O(|t|^{-2\tau/p-1}),\quad z=t+i\tau,\quad -p<\tau<0.$$

Hence it follows that

$$X(z)=O(|t|^{-3\tau/p-1/2}),\quad -p<\tau<0.$$

Now the solution of problem (2.6.14) can be written in the form

$$\Psi(z) = \frac{X(z)}{2ip} \int\limits_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{F(t)}{X(t-ip)\,\mathbf{sh}\,\frac{\pi}{p}(t-z)}\,dt, \tag{2.6.15}$$
$$-p-\varepsilon < \operatorname{Im} z < -\varepsilon.$$

Assume that $p \geq 1$. If the function $N_0(t)$ is analytically continuable in the strip $-1 < \operatorname{Im} z < 1$ and vanishes exponentially at infinity, it follows from condition (2.6.12) and formula (2.6.15) that the function

$$\Psi_1(z) = \begin{cases} \Psi(z), & -p-\varepsilon < \operatorname{Im} z < -\varepsilon, \\ \dfrac{F(z) - \Psi(z-ip)}{d_0 z(p+iz)G(z)}\,, & -\varepsilon < \operatorname{Im} < p-\varepsilon, \end{cases}$$

is holomorphic in the strip $-p-\varepsilon < \operatorname{Im} z < p-\varepsilon$, vanishes exponentially at infinity and is bounded all over the strip, except for the points $z_j^+ = t_j^+ + i\tau_j^+$ $(j = 1, 2, \ldots, l)$ which are the zeros of the function $G(z)$ in the strip $-\varepsilon < \operatorname{Im} z < p-\varepsilon$.

Applying the Cauchy formula, the required contact stress can be represented as

$$\Delta P(x) = P(x) - P_0(x) = \frac{1}{\sqrt{2\pi}\,x} \int\limits_{-\infty}^{\infty} t\Psi(t)e^{it\ln x}\,dt$$
$$= \frac{1}{\sqrt{2\pi}\,x} \int\limits_{-\infty}^{\infty} (t-ip)\Psi(t-ip)e^{i(t-ip)\ln x}\,dt.$$

Thus near the angle vertex $(x \to 0)$ we have $\Delta P(x) = x^{p-1}g(x)$, where $g(x)$ is a bounded function when $x \geq 0$. For large x, we have $\Delta P(x) = O(x^{-(1+\tau_1^+)})$.

If $0 < p < 1$, then the function $\Psi(z)$ given by formula (2.6.15) is analytically continuable throughout the strip $-1 < \operatorname{Im} z < -\varepsilon$ except for the points $w_j^- = \lambda_j^- + i\mu_j^-$ $(j = 1, 2, \ldots, q)$ which are the poles of the function $G(z)$ in the same strip. The normal contact stress can then be represented near the point $x = 0$ as follows: $\Delta P(x) = \widetilde{c}x^{-(1+\mu_1^-)} + \widetilde{g}(x)$, where $\widetilde{g}(x)$ is a bounded function when $x \geq 0$, $\widetilde{c} = \mathit{const}$.

We will now consider the case where $p < 2$, i.e. the rod stiffness increases at the angle vertex and decreases at infinity. Introducing the notation $m = -p$ $(m > 0)$ and arguing as above, we can write condition (2.6.12) in the form

$$\frac{\Psi_0(t)}{\widetilde{X}(t)}+\frac{\Psi_0(t+im)}{\widetilde{X}(t+im)}=\frac{F_0(t)}{\widetilde{X}(t)}\,,\quad -\infty+i\varepsilon<t<\infty+i\varepsilon,$$

$$\widetilde{X}(z)=\frac{1}{z}\,X_m(z)k(z)(z-im/2)\,\mathbf{sh}\,\frac{\pi}{2m}\,z,\quad \varepsilon<\operatorname{Im}z<m+\varepsilon,$$

$$X_m(z)=\exp\left\{\frac{1}{2im}\int\limits_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\ln G_m(t)\coth\frac{\pi}{m}(t-z)\,dt\right\}, \tag{2.6.16}$$

$$k(z)=\exp(-iz/m\ \ln(ad_0))m^{-3tz/m}\Gamma^2(1+iz/m)/\Gamma(2+iz/m),$$

$$G_m(t)=\frac{t+i}{a(t-im)}\,\frac{2t-im\Delta_1(t)}{2t+im\Delta(t)}\ \mathbf{th}\,\frac{\pi}{2m}\,t\,.$$

For sufficiently large $|z|$, the function $\widetilde{X}(z)$ admits the estimate

$$|\widetilde{X}(z)|=O(|t|^{3\tau/m-5/2}),\quad 0<\tau<m.$$

The function $\Psi_0(z)/\widetilde{X}(z)$ is holomorphic in the strip $\varepsilon<\operatorname{Im}z<m+\varepsilon$ except for the point $z=im/2$ where it can have a first-order pole. Therefore the solution of problem (2.6.16) is given by

$$\Psi_0(z)=\frac{\widetilde{X}(z)}{2im}\int\limits_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\frac{F_0(t)}{\widetilde{X}(t+im)\,\mathbf{sh}\,\frac{\pi}{m}(t-z)\,dt}+\frac{A_0\widetilde{X}(z)}{\mathbf{ch}\,\frac{\pi}{m}\,z}\,,$$

$$F_0(z)=d_0(iz-1)(m-iz)(\Delta_1(z)/\Delta(z))N_0(z),$$

$$A_0=const,\quad \varepsilon<\operatorname{Im}z<m+\varepsilon.$$

Using the equality

$$\int\limits_0^\infty t(P(t)-P_0(t))\,dt=0$$

we obtain $\Psi_0(i)=0$, from which we define the constant A_0.

The function

$$\Psi_2(z)=\begin{cases}\Psi_0(z), & \varepsilon<\operatorname{Im}z<m+\varepsilon,\\[2mm] \dfrac{F_0(z)+\Psi_0(z+im)}{d_0z(m-iz)G(z)}\,, & -m-\varepsilon<\operatorname{Im}<\varepsilon,\end{cases}$$

is holomorphic in the strip $-m+\varepsilon<\operatorname{Im}z<m+\varepsilon$, vanishes at infinity and is continuable on the strip boundary, except for the points $z_j^-=t_j^-+i\tau_j^-$ $(j=1,2,\dots,n)$ which are the zeros of the function $G(z)$ in the strip $-m+\varepsilon<\operatorname{Im}z<\varepsilon$.

If $\tau_1^-<-1$, then the function $\Psi_2(z)$ is analytically continuable in the strip $-1<\operatorname{Im}z<m+\varepsilon$ and the normal contact stress $\Delta P(x)$ is bounded in the neighborhood of the point $x=0$.

If $\tau_1^- > -1$, the function $\Psi_2(z)$ has a pole very close to the real axis at the point $z_1^- = t_1^- + i\tau_1^-$ and, consequently, the contact stress in the neighborhood of the point $x = 0$ can be represented as

$$\Delta P(x) = \widetilde{c}_1 x^{-(1+\tau_1^-)} + \widetilde{g}_1(x),$$

where $\widetilde{g}_1(x)$ is a bounded function when $x \geq 0$, $\widetilde{c}_1 = const$. For large x, we have

$$\Delta P(x) = O(x^{-1-m}), \quad x \to \infty.$$

Let us consider some special cases. As will be clear from the discussion below, in these cases we have

$$\Delta P(x) = \begin{cases} O(x^{p-1}), & p \geq 1, \\ O(x^{\xi}), & 0 < p < 1, \quad x \to 0. \\ O(x^{\eta}), & p < 0, \end{cases} \tag{2.6.17}$$

Assume that the domain S is a half-plane. Then

$$\theta_1 = \theta_2 = \theta = \pi, \quad \delta = 0, \quad \gamma = 2\pi, \quad \mu = 0,$$

$$\Delta(t) = 2|s_1 - s_2|^2 \,\mathbf{sh}\,\pi t, \quad \Delta_1(t) = 2|s_1 - s_2|^2 a\,\mathbf{sh}\,\pi t\,\mathbf{ch}\,\pi t.$$

Hence we obtain that $\tau_1^- = -1/2$, $\mu_1^- = -1$ and the function $\Delta P(x)$ satisfies relations (2.6.17) when $\xi = 0$, $\eta = -1/2$.

When $\theta = 2\pi$, i.e. the plane is cut along the real positive axis, we obtain

$$\theta_1 = \theta_2 = 2\pi, \quad \delta = 0, \quad \gamma = 4\pi, \quad \mu = 0,$$

$$\Delta(t) = 2|s_1 - s_2|^2 \,\mathbf{sh}\,2\pi t, \quad \Delta_1(t) = 2|s_1 - s_2|^2 a\,\mathbf{sh}\,2\pi t\,\mathbf{ch}\,2\pi t.$$

Therefore $\tau_1^- = -1/4$, $\mu_1^- = -1/2$ and the function $\Delta P(x)$ satisfies relations (2.6.17) when $\xi = -1/2$, $\eta = -3/4$. Note that for $p = m = 0$, condition (2.6.12) gives

$$\Psi(z) = F(z)/(id_0 z^2 G(z) + 1)$$

and the estimate

$$\Delta P(x) = O(x^{\lambda-1}) \quad \text{when} \quad x \to 0$$

holds for the normal stress, where $\lambda = -\operatorname{Im}\mu$ and μ is the zero of the function $id_0 z^2 G(z) + 1$ in the lower half-plane which is very close to the real axis.

In a special case where the body is orthotropic and one of its axes of anisotropy is parallel to the edge of the wedge which supports the beam, we prove that for $p < 0$, the normal contact stress near the beam end is bounded when $\theta \leq \pi/2$ and has the form $\Delta P(x) = O(x^{-\tau_0})$, $x \to 0$, for $\theta > \pi/2$ where $0 < \tau_0 \leq 3/4$. In particular, we have $\Delta P(x) = O(x^{-2/3})$, $x \to 0$, when $\theta = 3\pi/2$.

CHAPTER 3

The Problems of Plane Theory of Elasticity for an Anizotropic Body with Cracks and Inclusions

3.1. Solution of the First Basic Boundary Value Problem of the Elasticity Theory for an Orthotropic Wedge with a Finite Cut

Let on the plane of a complex variable $z = x + iy$ an elastic orthotropic body occupy an angle $-\alpha < \arg z < \alpha$, $0 < \alpha < 2\pi$, which is cut from the angle vertex along the bisectrix segment. Assume that the length of the cut is equal to one.

Let the boundary of the body $\arg z = \pm\alpha$ be free of external stresses (this can be assumed without loss of generality) and let the following stress components be given on the cut:

$$\sigma_y = p_1(x), \quad \tau_{xy} = q_1(x) \quad \text{on the upper edge of the cut,}$$
$$\sigma_y = p_2(x), \quad \tau_{xy} = q_2(x) \quad \text{on the lower edge of the cut,}$$

where $p_1(x)$, $p_2(x)$, $q_1(x)$, $q_2(x)$ are absolutely continuous functions. Assume further that the principal of elasticity coincide with the coordinate axes.

Let S be the domain occupied by the body. The domains S_1 and S_2 are respectively obtained from the domain S by the affine transforms

$$x_k = x, \quad y_k = \beta_k y \quad (k = 1, 2),$$

$\beta_1 > \beta_2 > 0$ are the angles of D_k cut along a segment of the real axis $\mathcal{J} = [0, 1]$, i.e. $S_k = D_k - \mathcal{J}$, where

$$D_k = \big\{ -\alpha_k < \arg z_k < \alpha_k \big\},$$
$$\alpha_k = \arg(\cos\alpha + i\beta_k \sin\alpha).$$

Write D_k in the form $D_k = D_{k1} \cup D_{k2}$ where

$$D_{kn} = \Big\{ \alpha_k(1-n) < \arg z_k < \alpha_k(2-n) \Big\}, \quad k, n = 1, 2.$$

Introduce the notation

$$\Phi_k(z_k) = \Phi_{kn}(z_k) \quad \text{for} \quad z_k \in D_k.$$

Since the functions $\Phi_k(z_k)$ are analytic in the domains S_k, they must satisfy the conditions

$$\Phi_{k1}(x_k) = \Phi_{k2}(x_k) \quad \text{for} \quad x_k > 1, \quad k = 1, 2.$$

According to formula (2.1.8), the problem we want to consider can be formulated as follows: find functions $\Phi_{kn}(z_k)$, $n, k = 1, 2$, in the domains D_{kn} by the boundary conditions

$$\begin{gathered}(\beta_1 + \beta_2) t_1 \Phi_{1n}(t_1) + (\beta_2 - \beta_1) t_1 \overline{\Phi_1(t_1)} + 2\beta_2 t_2 \Phi_2(t_2) = 0, \\ t_k = \rho \left[\cos\alpha + (3 - n) i\beta_k \sin\alpha\right], \quad n = 1, 2, \quad k = 1, 2,\end{gathered} \tag{3.1.1}$$

$$\begin{gathered}(\beta_1 + \beta_2)\Phi_{1n}(x) + (\beta_2 - \beta_1)\overline{\Phi_{1n}(x)} + 2\beta_2 \Phi_1(x) = \beta_2 \sigma_y^{(n)} + i\tau_{xy}^{(n)}, \\ \sigma_y^{(n)} = p_n(x), \quad \tau_{xy}^{(n)} = q_n(x), \quad x \in \mathcal{J},\end{gathered} \tag{3.1.2}$$

$$\sigma_y^{(1)} = \sigma_y^{(2)} = \sigma_y, \quad \tau_{xy}^{(1)} = \tau_{xy}^{(2)} = \tau_{xy}, \quad x > 1, \tag{3.1.3}$$

$$\Phi_{11}(x) - \Phi_{12}(x) = \Phi_{21}(x) - \Phi_{22}(x) = 0, \quad x > 1. \tag{3.1.4}$$

Analytic functions $\Phi_{kn}(z_k)$, $k, n = 1, 2$, will be sought in the form

$$\begin{aligned}\Phi_{kn}(z_k) = \left(\frac{1}{z_k} - \frac{d}{dz_k}\right) \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{A_{kn}(t)}{t}\, e^{-it\ln z_k}\, dt - \\ - i\sqrt{\frac{\pi}{2}}\, \frac{A_{kn}(0)}{z_k}\,.\end{aligned} \tag{3.1.5}$$

The integrals at the point $t = 0$ are understood in the sense of the Cauchy principal value.

Like in the preceding paragraphs, it is assumed that $A_{kn}(0)$, $k, n = 1, 2$, satisfy the conditions

$$2\beta_2 A_{2n}(0) = -(\beta_1 - \beta_2)\overline{A_{1n}(0)} - (\beta_1 + \beta_2) A_{1n}(0). \tag{3.1.6}$$

I will be shown below that the sought functions $A_{kn}(t)$, $k, n = 1, 2$, are the Fourier transforms of the summable functions $a_{kn}(\xi)$ which are continuous all over the whole axis except perhaps for the point $\xi = 0$.

The class of such functions is denoted by R_0'. If $A_{kn}(t) \in R_0'$, then it is easy to show that in representations (3.1.5), the integrals exist in the sense of the Cauchy principal value and it is possible to pass to the limit both under the sign of the differential and under the sign of the integral for z_k tending to a point of the boundary D_{kn}.

Let us introduce some other notation and definitions.

Denote by R_0 a set of all functions

$$F(t) = \int\limits_{-\infty}^{\infty} f(\xi) e^{it\xi}\, d\xi \ \text{ where } \ f(\xi) \in L_1(-\infty; \infty).$$

R_0 is a ring of continuous functions on the closed straight line [**116**]. Denote, further, by R_0^+ (R_0^-) a subring of R_0 composed of the functions

$$F^+(t) = \int_0^\infty f(\xi)e^{it\xi}\,d\xi \quad \left(F^-(t) = \int_{-\infty}^0 f(\xi)e^{it\xi}\,d\xi\right).$$

The ring obtained by expansion of the ring R_0 (R_0^+; R_0^-) by adding 1 to it is denoted by R (R^+; R^-). It is obvious that a function $\Phi^+ \in R_0^+$ ($\Phi^- \in R_0^-$) is the limiting value of a function, analytic in the upper (lower) half-plane and vanishing at infinity.

Denote by $N_n(t)$, $T_n(t)$, $P_n(t)$, $Q_n(t)$ the Fourier transforms of the functions $e^\xi\sigma_y^{(n)}(e^\xi)$, $e^\xi\tau_{xy}^{(n)}(e^\xi)$, $e^\xi p_n(e^\xi)$, $e^\xi q_n(e^\xi)$, respectively.

Substituting expression (3.1.5) into the boundary conditions and arguing as in the preceding paragraphs, we obtain

$$\begin{aligned}A_{kn}(t) = &-\frac{\delta_{kn}(t)+\gamma_{kn}(t)-2\beta_k e^{-i\mu_k t}}{2(1+it)\Delta(t)}\,(\beta_1+\beta_2-\beta_k)tN_n(t)\\ &+\frac{\gamma_{kn}(t)-\delta_{kn}(t)+2\beta_k e^{-i\mu_k t}}{2(1+it)\Delta(t)}\,itT_n(t) \quad (k,n=1,2),\end{aligned}$$

where

$$\begin{gathered}\Delta(t) = (\beta_1-\beta_2)^2\,\mathbf{ch}\,\gamma t - (\beta_1+\beta_2)^2\,\mathbf{ch}\,\delta t + 4\beta_1\beta_2\cos\mu t,\\ \mu_k = (-1)^k \ln\left|\frac{\cos\alpha + i\beta_2\sin\alpha}{\cos\alpha+i\beta_1\sin\alpha}\right|,\\ \delta_{1n} = (\beta_1+\beta_2)\exp\left[(2n-3)\delta t\right], \quad \delta_{2n} = (\beta_1+\beta_2)\exp\left[(3-2n)\delta t\right],\\ \gamma_{1n} = (\beta_1-\beta_2)\exp\left[(2n-3)\gamma t\right], \quad \gamma_{2n} = (\beta_2-\beta_1)\exp\left[(2n-3)\gamma t\right],\\ \gamma = \gamma_1+\gamma_2.\end{gathered}$$

By virtue of condition (3.1.4) we have

$$\begin{aligned}\left(\frac{1}{x}-\frac{d}{dx}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{A_{k1}(t)-A_{k2}(t)}{t}\,e^{-it\ln x}\,dt\\ = i\sqrt{\frac{\pi}{2}}\,\frac{[A_{k1}(0)-A_{k2}(0)}{x}\,, \quad x>0. \quad (3.1.7)\end{aligned}$$

Using the notation $\ln x = \xi$, we obtain

$$\begin{aligned}\left(1-\frac{d}{d\xi}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{A_{k1}(t)-A_{k2}(t)}{t}\,e^{-it\xi}\,dt\\ = i\sqrt{\frac{\pi}{2}}\,\big[A_{k1}(0)-A_{k2}(0)\big], \quad -\infty<\xi<\infty. \quad (3.1.8)\end{aligned}$$

If we assume that $A_{kn}(t) \in R_0$ and pass to the limit, then we obtain that the limit in the right-hand part of equality (3.1.8) is equal to

$$-\sqrt{\frac{\pi}{2}}\, i\big(A_{k1}(0) - A_{k2}(0)\big)$$

as $\xi \to \infty$. Hence it follows that

$$A_{k1}(0) - A_{k2}(0) = 0, \quad k = 1, 2. \tag{3.1.9}$$

Now it can be shown that if $A_{kn}(t) \in R_0$, then

$$A_{k1}(t) - A_{k2}(t) \in R_0^-.$$

The function

$$\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{A_{k1}(t) - A_{k2}(t)}{t}\, e^{-it\xi}\, dt \quad \text{for} \quad \xi > 0$$

vanishes at infinity and satisfies condition (3.1.8). Therefore

$$\Psi_k(\xi) = \frac{i}{\sqrt{2\pi}} \frac{d}{d\xi} \int\limits_{-\infty}^{\infty} \frac{A_{k1}(t) - A_{k2}(t)}{t}\, e^{-it\xi}\, dt = 0 \quad \text{for} \quad \xi > 0.$$

Since $A_{k1}(t) - A_{k2}(t) \in R_0$, the function $\Psi_k(\xi) \in L_1$ and the Fourier inversion formula

$$A_{k1}(t) - A_{k2}(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{0} \Psi_k(\xi) e^{it\xi}\, d\xi \in R_0^-$$

is valid.

By conditions (3.1.4) we have

$$N_n(t) = P_n(t) + N^+(t), \quad T_n(t) = Q_n(t) + T^+(t), \tag{3.1.10}$$

where

$$N^+(t) = \frac{1}{\sqrt{2\pi}} \int\limits_0^{\infty} \sigma_y(e^\xi) e^\xi e^{it\xi}\, d\xi, \quad T^+(t) = \frac{1}{\sqrt{2\pi}} \int\limits_0^{\infty} e^\xi \tau_{xy}(e^\xi) e^{it\xi}\, d\xi$$

are the sought functions of the class R_0^+.

From conditions (3.1.7)–(3.1.9) we obtain

$$A_{11}(t) - A_{12}(t) = \frac{\beta_2 t \Delta_1(t)}{(1 + it)\Delta(t)}\, N^+(t) + \frac{t\Delta_2(t)}{(1 + it)\Delta(t)}\, iT^+(t) + f_1(t), \tag{3.1.11}$$

$$A_{21}(t) - A_{22}(t) = -\frac{\beta_1 t \Delta_1(t)}{(1 + it)\Delta(t)}\, N^+(t) - \frac{t\Delta_2(t)}{(1 + it)\Delta(t)}\, iT^+(t) + f_2(t), \tag{3.1.12}$$

where

$$\Delta_n(t) = (\beta_1 + \beta_2)\, \mathbf{sh}\, \delta t + (-1)^n (\beta_1 - \beta_2) \sin \gamma t, \quad n = 1, 2,$$

$f_1(t)$, $f_2(t)$ are given functions of the class R_0.

Since the functions $A_{k1}(t)-A_{k2}(t)\in R_0$, $k=1,2$, (3.1.11) and (3.1.12) are the conditions of Riemann boundary value problems for two pairs of functions.

Our problem reduces to the following two Riemann problems:

$$\begin{aligned} &A_{11}(t)-A_{12}(t)+A_{21}(t)-A_{22}(t)\\ &\qquad=\frac{(\beta_2-\beta_1)t\Delta_1(t)}{(1+it)\Delta(t)}\,N^+(t)+f_1(t)+f_2(t), \qquad (3.1.13)\\ &\big[A_{11}(t)-A_{12}(t)\big]\beta_1+\big[A_{21}(t)-A_{22}(t)\big]\beta_2\\ &\qquad=\frac{(\beta_1-\beta_2)t\Delta_1(t)}{(1+it)\Delta(t)}\,iT^+(t)+\beta_1f_1(t)+\beta_2f_2(t). \qquad (3.1.14)\end{aligned}$$

Since (3.1.13) and (3.1.14) are problems of the same type, we will solve only problem (3.1.13).

Let us introduce the notation

$$\begin{cases} \Phi^-(t)=\sqrt{1+it}\,\Big(A_{11}(t)-A_{12}(t)+A_{21}(t)-A_{22}(t)\Big),\\ \Phi^+(t)=-\sqrt{1-it}\,N^+(t), \end{cases} \qquad (3.1.15)$$

where under the radicals $\sqrt{1+iw}$ and $\sqrt{1-iw}$ we understand respectively the branches, holomorphic on the plane cut along the lines $(i;i\infty)$ and $(-i;-i\infty)$, and the branches taking positive values on the uncut part of the imaginary axis.

Substituting (3.1.15) into expression (3.1.13), we obtain

$$\Phi^+(t)=\frac{\Delta(t)\sqrt{1+it}}{(\beta_1-\beta_2)t\Delta_(t)}\,\Phi^-(t)+g(t), \quad -\infty<t<\infty. \qquad (3.1.16)$$

It is easy to show that by the conditions we have made as to the given stresses, $g(t)\in R_0'$, and since the coefficient of the problem belongs to the class R and is positive all over the axis $-\infty<t<\infty$, the index of problem (3.1.15) is equal to zero.

From equalities (3.1.14) it follows that if $N^+(t)\in R_0^+$, then $\Phi^+(t)$ may not belong to the class R_0^+, it may increase at infinity by order less than half. But we know that if the homogeneous problem has such a solution, then it is bounded. Therefore we will seek for a solution of problem (3.1.15) in the class R.

Due to [**25**], a solution of the boundary value problem (3.1.15) in the class of functions $\Phi^\pm(t)\in R_0^\pm$ is given by the formula

$$\Phi^\pm(t_0)=\frac{X^\pm(t_0)}{2}\left[\pm\frac{g(t_0)}{X^+(t_0)}+\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{g(t)}{X^+(t)(t-t_0)}\,dt+c\right], \qquad (3.1.17)$$

where

$$X(z)=\exp\left[\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\ln\frac{\Delta(t)\sqrt{1+t^2}}{t(\beta_1-\beta_2)\Delta_1(t)}\,\frac{dt}{t-z}\right]. \tag{3.1.18}$$

By the Wiener–Levy [**44**] and Wiener [**44**] theorems we have

$$X^{\pm}(t)\in R^{\pm},\quad \left[X^{\pm}(t)\right]^{-1}\in R^{\pm}.$$

It can be easily shown that $X^{\pm}(t)-1$ and $[X^{\pm}(t)]^{-1}-1$ are the Fourier transforms of summable and bounded functions on the whole axis.

Taking into account that by virtue of (3.1.9) and (3.1.15) $\Phi^{-}(0)=0$, from (3.1.17) we obtain

$$c=\frac{g(0)}{X^{+}(0)}-\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{g(t)}{tX^{+}(t)}\,dt. \tag{3.1.19}$$

From equalities (3.1.15) and (3.1.17) we have

$$N^{+}(t_0)=-\frac{g(t_0)}{2\sqrt{1-it_0}}-\frac{X^{+}(t_0)}{2\pi i\sqrt{1-it_0}}\int\limits_{-\infty}^{\infty}\frac{g(t)}{X^{+}(t)(t-t_0)}\,dt-c\,\frac{X^{+}(t_0)}{\sqrt{1-it_0}}\,.$$

This formula can be rewritten as

$$\begin{aligned}N^{+}(t_0)=&-\frac{g(t_0)}{2\sqrt{1-it_0}}-\frac{X^{+}(t_0)-1}{2\pi i\sqrt{1-it_0}}\int\limits_{-\infty}^{\infty}\frac{g(t)}{X^{+}(t)(t-t_0)}\,dt\\&-\frac{1}{2\pi i\sqrt{1-it_0}}\int\limits_{-\infty}^{\infty}\left(\frac{1}{X^{+}(t)}-1\right)\frac{g(t)}{t-t_0}\,dt\\&-\frac{1}{2\pi i\sqrt{1+it}}\int\limits_{-\infty}^{\infty}\frac{g(t)}{t-t_0}\,dt-\frac{c(X^{+}(t_0)-1)}{\sqrt{1-it_0}}-\frac{c}{\sqrt{1-it_0}}\,.\end{aligned} \tag{3.1.20}$$

Since $X^{+}(t)-1$ and $1/X^{+}(t)-1$ are the Fourier transforms of bounded functions, the second, the third and the fifth summand in the right-hand part of (3.1.20) are the Fourier transforms of continuous functions on the closed right-hand semi-axis.

It is proved that the first and the fourth summand are the Fourier transforms of functions, continuous on the whole semi-axis except perhaps for the point $\xi=0$ where they have logarithmic singularities, i.e.

$$N^{+}(t)=N_0^{+}(t)-\frac{c}{\sqrt{1-it}}\,.$$

Hence it follows that the function $\sigma_y(e^{\xi};\theta)$ can be represented in the form

$$\sigma_y(e^{\xi};0)=\varphi_0(\xi)e^{-\xi}-\frac{ce^{-\xi}}{\sqrt{\xi}}\,,\quad \xi>0,$$

or, if we return to the variable x, in the form

$$\sigma_y(x;0) = \varphi_1(x) - \frac{c_1}{x^2\sqrt{x-1}}\,, \quad x > 1, \tag{3.1.21}$$

where $\varphi_1(x)$ is a continuous function for $x > 0$ that may have a logarithmic singularity near the point $x = 0$.

In the particular case, where symmetric normal stresses are applied to the cut edges, i.e. $p_1(x) = p_2(x) = p(x)$, $q_1(x) = q_2(x) = 0$, conditions (3.1.11) and (3.1.12) take the form

$$\begin{aligned} &A_{11}(t) - A_{12}(t) \\ &\quad = \frac{\beta_2\Delta_1(t)t}{(1+it)\Delta t}\,\big(N^+(t) + p^-(t)\big) + \frac{\Delta_2(t)t}{\Delta(t)(1+it)}\,iT^+(t), \end{aligned} \tag{3.1.22}$$

$$\begin{aligned} &A_{21}(t) - A_{22}(t) \\ &\quad = -\frac{\beta_1\Delta_1(t)t}{(1+it)\Delta t}\,\big(N^+(t) + p^-(t)\big) - \frac{\Delta_2(t)t}{\Delta(t)(1+it)}\,iT^+(t). \end{aligned} \tag{3.1.23}$$

Hence it follows that

$$(A_{11}(t) - A_{12}(t))\,\beta_1 + (A_{21}(t) - A_{22}(t))\,\beta_2 = \frac{(\beta_1-\beta_2)\Delta_2(t)}{(1+it)\Delta t}\,itT^+(t).$$

It is easy to see that the problem has only a trivial solution in the class $R_0^\pm$, i.e.

$$T^+(t) = 0, \quad \beta_1\,(A_{11}(t) - A_{12}(t)) = -\beta_2\,(A_{21}(t) - A_{22}(t))\,.$$

In that case, equalities (3.1.22) and (3.1.23) are equivalent.

We introduce the notation

$$\begin{aligned} &(A_{11}(t) - A_{12}(t))\,\frac{\beta_1-\beta_2}{\beta_1}\,\sqrt{1+it} = \Phi^-(t); \\ &\sqrt{1-it}\,N^+(t) = \Phi^+(t). \end{aligned}$$

Now formula (3.1.22) takes the form

$$\Phi^+(t) = \frac{\Delta(t)\sqrt{1+t^2}}{t(\beta_2-\beta_1)\Delta_1(t)}\,\Phi^-(t) - P(t)\sqrt{1-it}\,, \quad -\infty < t < \infty. \tag{3.1.24}$$

The solution of this problem is given by formulas (3.1.17) and (3.1.18), where it is assumed that

$$g(t) = -P(t)\sqrt{1-it}\,.$$

If $p(x) = \sum\limits_{k=0}^{n} a_k x^k$, where a_k are constant values, we have

$$P(t) = \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{n}\frac{a_k}{k+1+it}\,.$$

The function $P(t)$ is a homomorphism all over the plane except perhaps for the point $w = (k+1)i$, $k = 0, 1, \ldots, n$.

We rewrite equality (3.1.24) in the form

$$\sqrt{1-it}\left(N^+(t)+\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{n}\frac{a_k}{1+k+it}\right)=\frac{\Delta(t)\sqrt{1+t^2}}{(\beta_1-\beta_2)\Delta_1(t)t}\,.$$

Taking into account that

$$\frac{\Delta(t)\sqrt{1+t^2}}{t(\beta_1-\beta_2)\Delta(t)}=\frac{X^+(t)}{X^-(t)}\,,$$

we have

$$\frac{\sqrt{1-it}}{X^+(t)}\left(N^+(t)+\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{n}\frac{a_k}{1+k+it}\right)=\frac{\Phi^-(t)}{X^-(t)}\,.$$

Applying the generalized Liouville theorem, we obtain

$$N^+(w)=\frac{X^+(w)}{\sqrt{1-iw}}\left(\sum_{k=0}^{n}\frac{c_k}{1+k+iw}+c\right)-\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{n}\frac{a_k}{1+k+iw}\,, \quad (3.1.26)$$

$$\Phi^-(w)=X^-(w)\left(\sum_{k=0}^{n}\frac{c_k}{1+k+iw}+c\right). \quad (3.1.27)$$

Since $\Phi^-(0)=0$, from (3.1.27) we obtain

$$c=-\sum_{k=0}^{n}\frac{c_k}{k+1}\,.$$

Multiplying expression (3.1.26) by $\prod\limits_{k=0}^{n}(k+1+iw)$ and replacing w by $(k+1)i$, we have

$$c_k=\frac{a_k}{\sqrt{2\pi}}\,\frac{\sqrt{(k+2)i}}{X^+((k+1)i)}\,,\quad k=0,1,\dots,n.$$

Thus the constants $c_1,c_2,\dots,c_n$ are well defined.

Formula (3.1.26) implies

$$N^+(w)=\frac{X^+(w)}{\sqrt{2\pi}\sqrt{1-iw}}\sum_{k=0}^{n}\frac{a_k\sqrt{(k+2)i}}{X^+(i+ki)(k+1+iw)}$$
$$-\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{n}\frac{a_k}{k+1+iw}+\frac{cX^+(w)}{\sqrt{1-iw}}\,. \quad (3.1.28)$$

The function $N^+(t)$ can be represented as

$$N^+(t)=N_0^+(t)+\frac{c}{\sqrt{1-it}}\,, \quad (3.1.29)$$

where $N_0^+(t)$ is the Fourier transform of a continuous function on the closed semi-axis $\xi\ge 0$.

From equality (3.1.29) we obtain

$$\sigma_y(x;0) = \varphi_0(x) + \frac{c}{x^2\sqrt{x-1}}\,, \quad x > 1,$$
$$c = -\frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n} a_k\sqrt{k+2}\, \exp\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(k+1)\ln G(t)}{t^2+(k+1)^2}\, dt\right), \tag{3.1.30}$$

where

$$G(t) = \frac{\Delta(t)\sqrt{1+t^2}}{(\beta_1-\beta_2)t\Delta_1(t)}\,.$$

Until now we have assumed that the stresses given on the cut are absolutely continuous. As will be seen below, this condition is not necessary. The solution of the problem can be constructed not only in the case of absolutely continuousF boundary conditions, but also even in the case of concentrated force.

Assume that the concentrated force P is applied to the point x_0 of the cut, i.e.

$$p_1(x) = p_2(x) = P\delta(x-x_0),$$
$$q_1(x) = q_2(x) = 0.$$

Then

$$P(t) = \frac{Px_0\exp(it\ln x_0)}{\sqrt{2\pi}}\,.$$

The substitution of this value into the boundary condition (3.1.24) gives

$$N^+(t)\sqrt{1-it} = G(t)\Phi^-(t) - \frac{Px_0\sqrt{1-it}\,\exp(it\ln x_0)}{\sqrt{2\pi}}\,. \tag{3.1.31}$$

Since the free term of problem (3.1.31) increases at infinity, to solve the problem we cannot apply formulas (3.1.17) in a straightforward manner.

Dividing equality (3.1.31) by $1-it$, we obtain

$$\frac{N^+(t)}{\sqrt{1-it}} = G(t)\,\frac{\Phi^-(t)}{1-it} - \frac{Px_0\exp(it\ln x_0)}{\sqrt{2\pi}\,\sqrt{1-it}}\,. \tag{3.1.32}$$

Since the function $\Phi^-(t)/(1-it)$ is holomorphic in the lower half-plane except for the point $w=-i$ where it has a pole of first order, the solution of problem (3.1.32) is given by the formulas

$$N^+(w) = -\frac{X(w)}{2\pi i\sqrt{2\pi}}\,\sqrt{1-iw}\,Px_0\int_{-\infty}^{\infty}\frac{\exp(it\ln x_0)}{X^+(t)\sqrt{1-it}\,(t-w)}\,dt$$
$$+\frac{cX^+(w)}{\sqrt{1-iw}}\,, \quad \operatorname{Im} w > 0,$$

$$\Phi^-(w) = -\frac{PX(w)(1-iw)x_0}{2\pi i\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{\exp(it\ln x_0)}{X^+(t)\sqrt{1-it}\,(t-w)}\,dt$$
$$+\,cX^-(w)\,,\quad \operatorname{Im} w < 0.$$

Hence we have

$$N^+(t_0) = \frac{Px_0\exp(it\ln x_0)}{2\sqrt{2\pi}}$$
$$-\frac{\sqrt{1-it_0}\,X^+(t_0)Px_0}{2\pi i\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{e^{it\ln x_0}}{X^+(t)\sqrt{1-it}\,(t-t_0)}\,dt + \frac{cX^+(t_0)}{\sqrt{1-it}}\,; \quad (3.1.33)$$

$$\Phi^-(t_0) = \frac{Px_0e^{it\ln x_0}\sqrt{1-it_0}}{2\sqrt{2\pi}G(t_0)}$$
$$-\frac{Px_0(1-it_0)}{2\pi i\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{e^{it\ln x_0}}{X^+(t)\sqrt{1-it}\,(t-t_0)}\,dt + cX^-(t_0). \qquad (3.1.34)$$

Since $\Phi^-(0)=0$, from (3.1.34) we define the values

$$c = \frac{Px_0}{2\pi i\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{\exp(it\ln x_0)}{X^+(t)\sqrt{1-it}\,t}\,dt - \frac{Px_0}{2\sqrt{2\pi}\,X^+(t_0)}\,.$$

Using the equality

$$\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{\exp(it\ln x_0)}{X^+(t)\sqrt{1-it}\,t}\,dt = -i\sqrt{\frac{\pi}{2}} \int\limits_{0}^{\infty} K(y)\operatorname{sign}(-\ln x_0 - y)\,dy,$$

where $K(y)$ is the function whose Fourier transform is the function $(X^+(t)\sqrt{1-it})^{-1}$, we obtain

$$c = \frac{Px_0}{2\sqrt{2\pi}} \int\limits_{0}^{\infty} K(y)\operatorname{sign}(-\ln x_0 - y)\,dy - \frac{Px_0}{2\sqrt{2\pi}X^+(0)}\,.$$

Since $\ln x_0 < 0$, we have

$$\int\limits_{0}^{\infty} K(y)\operatorname{sign}(-\ln x_0 - y)\,dy = \int\limits_{0}^{-\ln x_0} K(y)\,dy - \int\limits_{-\ln x_0}^{\infty} K(y)\,dy$$
$$= 2\int\limits_{0}^{-\ln x_0} K(y)\,dy - \int\limits_{0}^{\infty} K(y)\,dy = 2\int\limits_{0}^{-\ln x_0} K(y)\,dy - \frac{1}{X^+(0)}\,,$$

and finally we obtain

$$c = -\frac{Px_0}{\sqrt{2\pi}} \int\limits_0^{-\ln x_0} K(y)\,dy.$$

Let us introduce the notation

$$X^+(t) = X_0(t) + 1, \quad \big[X^+(t)\big]^{-1} = X_1(t) + 1.$$

If now we take into account that for sufficiently large values of $|t|$ $X_0(t)$ and $X_1(t)$ have order $O(\frac{1}{|t|})$, then, after elementary calculations, from (3.1.33) we obtain

$$N^+(t_0) = -\frac{Px_0\sqrt{1-it_0}}{2\sqrt{2\pi}}\left(\frac{e^{it\ln x_0}}{\sqrt{1-it_0}} + \frac{1}{\pi i}\int\limits_{-\infty}^{\infty} \frac{e^{it\ln x_0}}{\sqrt{1-it}\,(t-t_0)}\,dt\right.$$
$$\left. - \frac{1}{\pi i(1+it_0)}\int\limits_{-\infty}^{\infty} \frac{X_1(t)e^{it\ln x_0}}{\sqrt{1-it}}\,dt\right) + \frac{c}{\sqrt{1-it_0}} + \Phi_1(t_0),$$

where $\Phi_1(t_0)$ is the Fourier transform of a function $\varphi_1(\xi)$ that is continuous all over the whole axis except for the point $\xi = 0$ where it may have a logarithmic singularity.

Since the function $1/\sqrt{1-it}$ is the Fourier transform of the function

$$\varphi(\xi) = \begin{cases} \dfrac{e^{-\xi}}{\sqrt{\pi\zeta}} & \text{for } \xi > 0, \\ 0 & \text{for } \xi < 0, \end{cases}$$

the expression $e^{it\ln x_0}/\sqrt{1-it}$ is the Fourier transform of the function $\varphi(\xi - \ln x_0)$, and the expression

$$\frac{1}{\pi i}\int\limits_{-\infty}^{\infty} \frac{\exp(it\ln x_0)}{\sqrt{1-it}\,(t-t_0)}\,dt$$

is the Fourier transform of the function $\varphi(\xi - \ln x_0)\operatorname{sign}\xi$. Hence it follows that $N^+(t)$ can be written in the form

$$N^+(t_0) = \left(\frac{Px_0^2}{2\pi\sqrt{-\ln x_0}} - \frac{Px_0}{2\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} \frac{X_1^+(t)e^{it\ln x_0}}{\sqrt{1-it}}\,dt + c\right)\frac{1}{\sqrt{1-it_0}} + \Phi_2(t_0),$$

i.e.

$$N^+(t) = \frac{K}{\sqrt{1-it}} + \Phi_2(t).$$

Hence we obtain

$$\sigma_y(x;0) = \frac{K}{x^2\sqrt{x-1}} + \varphi(x).$$

3.2. First Basic Problem of a Piecewise-Homogeneous Orthotroipic Half-Plane with a Cut Perpendicular to the Boundary Line

Let the domain S occupied by a piecewise-homogeneous orthotropic elastic body be the whole plane of the complex variable $z = x + iy$ cut along the segment $[0, 1]$ of the Ox-axis. It is assumed that the left-hand ($\operatorname{Re} z < 0$) and right-hand ($\operatorname{Re} z > 0$) half-planes are homogeneous and the principal directions of elasticity coincide with the coordinate axes.

We denote by S_1 and S_2 the right-hand and the left-hand half-plane, respectively. The stress and displacement components as well as the elastic constants and other values related to S_1 and S_2 are denoted by the symbols 1 and 2, respectively.

Let the symmetric normal stresses

$$(\sigma_y^{(1)})^+ = (\sigma_y^{(1)})^- = p(x), \quad (\tau_{xy}^{(1)})^+ = (\tau_{xy}^{(1)})^- = 0$$

be applied the edges of the cut $0 \leq x \leq 1$. Here $p(x)$ is an absolutely continuous functions; the signs $(+)$ and $(-)$ denote respectively the boundary values on the upper and the lower edge of the cut.

As is known, the stresses and displacements are written in the form

$$\left.\begin{aligned}
\sigma_x^{(k)} &= -2\operatorname{Re}\big[\beta_k^2\Phi_k(z_k) + \gamma_k^2\Psi_k(\zeta_k)\big],\\
\sigma_y^{(k)} &= 2\operatorname{Re}\big[\Phi_k(z_k) + \Psi_k(\zeta_k)\big],\\
\tau_{xy}^{(k)} &= 2\operatorname{Im}\operatorname{Re}\big[\beta_k\Phi_k(z_k) + \gamma_k\Psi_k(\zeta_k)\big],
\end{aligned}\right\} \tag{3.2.1}$$

$$\left.\begin{aligned}
u_k &= 2\operatorname{Re}\big[\rho_k\varphi_k(z_k) + r_k\psi_k(\zeta_k)\big],\\
v_k &= -2\operatorname{Im}\big[\beta_k r_k\varphi_k(z_k) + \gamma_k p_k\psi_k(\zeta_k)\big],
\end{aligned}\right\} \tag{3.2.2}$$

$$z_k = x + i\beta_k y, \;\; \zeta_k = x + i\gamma_k y, \;\; (x, y) \in S_k,$$

where

$$\Phi_k(z_k) = \varphi_k'(z_k), \quad \Psi_k(\zeta_k) = \psi_k'(\zeta_k),$$

$$\mu^4 + \Big(\frac{E_k}{G_k} - 2\nu_k\Big)\mu^2 + \frac{E_k}{E_k^*} = 0, \tag{3.2.3}$$

$$p_k = -\frac{\beta_k^2 + \nu_k}{E_k}\,, \quad r_k = -\frac{\gamma_k^2 + \nu_k}{E_k}\,, \;\; k = 1, 2.$$

$\pm\beta_k i$, $\pm\gamma_k i$ are the roots of equation (3.2.3).

Using formulas (3.2.1), (3.2.2), we reduce the problem posed to finding holomorphic functions $\Phi_k(z_k)$, $\Psi_k(\zeta_k)$, $k = 1, 2$, in the domains S_1 and S_2, respectively, by the boundary conditions on the cut $0 \leq x \leq 1$ and the boundary line $x = 0$:

$$\begin{aligned} 2\,\mathrm{Re}\left[\Phi_1^{\pm}(x) + \Psi_1^{\pm}(x)\right] &= p(x), \quad 0 < x < 1, \\ \mathrm{Im}\left[\beta_1\Phi_1^{\pm}(x) + \gamma_1\Psi_1^{\pm}(x)\right] &= 0, \end{aligned} \tag{3.2.4}$$

$$\begin{aligned} \mathrm{Re}\left[\beta_1^2\Phi_1(t_1) + \gamma_1^2\Psi_1(\sigma_1)\right] &= \mathrm{Re}\left[\beta_2^2\Phi_2(t_2) + \gamma_2^2\Psi_2(\sigma_2)\right], \\ \mathrm{Im}\left[\beta_1\Phi_1(t_1) + \gamma_1\Psi_1(\sigma_1)\right] &= \mathrm{Im}\left[\beta_2\Phi_2(t_2) + \gamma_2\Psi_2(\sigma_2)\right], \end{aligned} \tag{3.2.5}$$

$$\begin{aligned} \mathrm{Im}\left[p_1\beta_1\Phi_1(t_1) + r_1\gamma_1\Psi_1(\sigma_1)\right] &= \mathrm{Im}\left[p_2\beta_2\Phi_2(t_2) + r_2\gamma_2\Psi_2(\sigma_2)\right], \\ \mathrm{Re}\left[\beta_1^2 r_1\Phi_1(t_1) + \gamma_1^2 p_1\Psi_1(\sigma_1)\right] &= \mathrm{Re}\left[\beta_2^2 r_2\Phi_2(t_2) + \gamma_2^2 p_2\Psi_2(\sigma_2)\right], \end{aligned} \tag{3.2.6}$$

where

$$t_k = i\beta_k y, \quad \sigma_k = i\gamma_k y, \quad k = 1, 2.$$

Due to the symmetry

$$\left.\begin{aligned} &\tau_{xy}^{(1)} = v_1(x;0) = 0 \ \text{ for } \ x > 1, \\ &\frac{\partial u_1^+}{\partial x} - \frac{\partial u_1^-}{\partial x} = 0, \quad (\tau_{xy}^{(1)})^+ - (\tau_{xy}^{(1)})^- = 0, \\ &\frac{\partial v_1^+}{\partial x} - \frac{\partial u_1^-}{\partial x} = 2\,\frac{\partial v_1^+}{\partial x}\,, \quad (\sigma_y^{(1)})^+ - (\sigma_y^{(1)})^- = 0, \end{aligned}\right\} \quad x > 0. \tag{3.2.7}$$

After substituting into formulas (3.2.7) the boundary values of the stress and displacement components defined by equalities (3.2.1) and (3.2.2) we have

$$\begin{aligned} \mathrm{Re}\Big[\Phi_1^+(x) - \Phi_1^-(x) + \Psi_1^+(x) - \Psi_1^-(x)\Big] &= 0, \\ \mathrm{Im}\Big[\beta_1\big(\Phi_1^+(x) - \Phi_1^-(x)\big) + \gamma_1\big(\Psi_1^+(x) - \Psi_1^-(x)\big)\Big] &= 0, \\ \mathrm{Re}\Big[p_1\big(\Phi_1^+(x) - \Phi_1^-(x)\big) + r_1\big(\Psi_1^+(x) - \Psi_1^-(x)\big)\Big] &= 0, \\ \mathrm{Im}\Big[\beta_1 r_1\big(\Phi_1^+(x) - \Phi_1^-(x)\big) + \gamma_1 p_1\big(\Psi_1^+(x) - \Psi_1^-(x)\big)\Big] &= -i\,\frac{\partial v_1^+(x,0)}{\partial x}\,. \end{aligned}$$

This system has the unique solution

$$\left.\begin{aligned} \Phi_1^+(x) - \Phi_1^-(x) &= i\,\frac{f(x)}{\beta_1(p_1 - r_1)}\,, \\ \Psi_1^+(x) - \Psi_1^-(x) &= i\,\frac{f(x)}{\gamma_1(r_1 - p_1)}\,, \end{aligned}\right\} \quad x > 0, \tag{3.2.8}$$

where

$$f(x) \equiv \frac{\partial v_1^+(x;0)}{\partial x}\,.$$

Since $f(x)=0$ for $x>1$, a general solution of problem (3.2.8) is written in the form

$$\begin{aligned}\Phi_1(z_1)&=\frac{1}{2\pi i\beta(p_1-r_1)}\int\limits_0^1\frac{f(t)}{t-z_1}\,dt+W_1(z_1),\\ \Psi_1(\zeta_1)&=\frac{1}{2\pi i\gamma(r_1-p_1)}\int\limits_0^1\frac{f(t)}{t-\zeta_1}\,dt+W_2(\zeta_1),\end{aligned}\tag{3.2.9}$$

where $W_1(z_1)$ and $W_2(z_2)$ are analytic functions in the half-planes $\operatorname{Re} z_1>0$ and $\operatorname{Re}\zeta_1>0$, respectively.

Let us rewrite formulas (3.2.9) as

$$\begin{aligned}\Phi_1(z_1)&=\frac{W_0(z_1)}{\beta_1}+W_1(z_1),\\ \Psi_1(\zeta_1)&=\frac{W_0(\zeta_1)}{\gamma_1}+W_2(\zeta_1),\end{aligned}\tag{3.2.10}$$

where

$$W_0(z)=\frac{1}{2\pi(p_1-r_1)}\int\limits_0^1\frac{f(t)}{t-z}\,dt.$$

Now substituting the boundary values of formulas (3.2.10) into equalities (3.2.6), multiplying the resulting expressions by $\frac{1}{2\pi i}\frac{dt}{t-z}$, $t=iy$, $z=x+iy$, $x>0$, integrating along the imaginary axis and using the fact that if $\Phi(z)$ is holomorphic in the half-plane $\operatorname{Re} z>0$ ($\operatorname{Re} z<0$), then $\overline{\Phi(iy)}$ is the boundary value of the holomorphic function $\overline{\Phi(-\overline{z})}$ in the half-plane $\operatorname{Re} z<0$ ($\operatorname{Re} z>0$), we obtain by means of the Cauchy theorem and formula the system

$$\begin{gathered}\beta_1^2W_1(\beta_1z)+\gamma_1^2W_2(\gamma_1z)-\beta_2^2\overline{\Phi_2(\beta_2\overline{z})}-\gamma_2^2\overline{\Psi_2(-\gamma_2\overline{z})}\\ =-\beta_1\overline{W_0(-\beta_1\overline{z})}+\gamma_1\overline{W_0(-\gamma_1\overline{z})},\\ \beta_1W_1(\beta_1z)+\gamma_1W_2(\gamma_1z)+\beta_2\overline{\Phi_2(\beta_2\overline{z})}+\gamma_2\overline{\Psi_2(-\gamma_2\overline{z})}\\ =\overline{W_0(-\beta_1\overline{z})}-\overline{W_0(-\gamma_1\overline{z})},\\ p_1\beta_1W_1(\beta_1z)+r_1\gamma_1W_2(\gamma_1z)+p_2\beta_2\overline{\Phi_2(-\beta_2\overline{z})}+\gamma_2r_2\overline{\Psi_2(-\gamma_2\overline{z})}\\ =p_1\overline{W_0(-\beta_1\overline{z})}-r_1\overline{W_0(-\gamma_1\overline{z})},\\ \beta_1^2r_1W_1(\beta_1z)+\gamma_1^2p_1W_2(\gamma_1z)-\beta_1^2r_2\overline{\Phi_2(-\beta_2\overline{z})}-\gamma_2^2p_2\overline{\Psi_2(\gamma_2\overline{z})}\\ =-\beta_1r_1\overline{W_0(-\beta_1\overline{z})}+\gamma_1p_1\overline{W_0(-\gamma_1\overline{z})}.\end{gathered}$$

Having solved this system for the functions $W_1(\beta_1 z)$ and $W_2(\gamma_1 z)$, we have

$$W_1(\beta_1 z) = \frac{\Delta_{12}}{\Delta\beta_1}\,\overline{W_0(-\beta_1\overline{z})} - \frac{\Delta_{22}}{\Delta\gamma_1}\,\overline{W_0(-\gamma_1\overline{z})}, \tag{3.2.11}$$

$$W_2(\gamma_1 z) = \frac{\Delta_{11}}{\Delta\beta_1}\,\overline{W_0(-\beta_1\overline{z})} + \frac{\Delta_{21}}{\Delta\gamma_1}\,\overline{W_0(-\gamma_1\overline{z})}, \tag{3.2.12}$$
$$\operatorname{Re} z > 0,$$

where

$$\Delta = \begin{vmatrix} \beta_1 & \gamma_1 & -\beta_2 & -\gamma_2 \\ 1 & 1 & 1 & 1 \\ p_1 & r_1 & p_2 & r_2 \\ r_1\beta_1 & p_1\gamma_1 & -\beta_2 r_2 & -\gamma_2 p_2 \end{vmatrix} \beta_1\beta_2\gamma_1\gamma_2,$$

$$\Delta_{ij} = \begin{vmatrix} -a_i & a_j & -\beta_1 & -\gamma_2 \\ 1 & 1 & 1 & 1 \\ b_i & b_j & p_2 & r_2 \\ -c_j & c_j & -r_2\beta_2 & -p_2\gamma_2 \end{vmatrix} a_i a_j \beta_2\gamma_2,$$

$$a_1 = \beta_1,\ \ b_1 = p_1,\ \ c_1 = r_1\beta_1,\ \ a_2 = \gamma_1,\ \ b_2 = r_1,\ \ c_2 = p_1\gamma_1.$$

Let us replace z in equality (3.2.11) by z_1/β_1, and z in equality (3.2.12) by ζ_1/γ_1, and insert the value of W_0, we obtain

$$\begin{aligned} W_1(z_1) &= \frac{\Delta_{12}}{\Delta\beta_1}\,\overline{W_0(-\overline{z}_1)} - \frac{\Delta_{22}}{\Delta\gamma_1}\,\overline{W_0\Big(-\frac{\gamma_1}{\beta_1}\,\overline{z}_1\Big)}, \\ W_2(z_1) &= \frac{\Delta_{11}}{\Delta\beta_1}\,\overline{W_0\Big(-\frac{\beta_1}{\gamma_1}\,\overline{\zeta}_1\Big)} - \frac{\Delta_{21}}{\Delta\gamma_1}\,\overline{W_0(-\overline{\zeta}_1)}. \end{aligned} \tag{3.2.13}$$

It is easy to verify that $\gamma_1^3\Delta_{11} = \beta_1^3\Delta_{22}$.

For $0 < x < 1$, the boundary condition (3.2.4) is equivalent to the condition

$$\begin{aligned} &\operatorname{Re}\Big[\Phi_1^+(x) + \Phi_1^-(x) + \Psi_1^+(x) + \Psi_1^-(x)\Big] = \sigma_y^{(1)},\ \ x > 0, \\ &\operatorname{Re}\Big[\Phi_1^+(x) - \Phi_1^-(x) + \Psi_1^+(x) - \Psi_1^-(x)\Big] = 0,\ \ x > 0. \end{aligned} \tag{3.2.14}$$

We have already used the second condition (3.2.14) and therefore the functions $\Phi_1(z_1)$ and $\Psi_1(\zeta_1)$ represented by formulas (3.2.9) and (3.2.13) satisfy this condition for any $f(x)$. It is also obvious that these functions satisfy condition (3.2.5). Thus, to find $f(x)$ it remains to use only the first condition (3.2.14).

If we introduce the boundary values of $\Phi_1(z_1)$ and $\Psi_1(\zeta_1)$ into the first equality (3.2.14) and take into account the relation

$$E_1(r_1 - p_1) = (\beta_1 - \gamma_1)(\beta_1 + \gamma_1),$$

then we obtain

$$\int_0^1 \frac{f(t)}{t-x}\,dt - K_1\int_0^1 \frac{f(t)}{t+x}\,dt - K_2\gamma_1\int_0^1 \frac{f(t)}{\beta_1 t+\gamma_1 x}\,dt$$
$$-K_2\beta_1\int_0^1 \frac{f(t)}{\gamma_1 t+\beta_1 x}\,dt = K_3\pi\sigma_y^{(1)}(x), \quad x>0, \tag{3.2.15}$$

where

$$K_1 = \frac{\Delta_{12}\gamma_1 + \Delta_{21}\beta_1}{\Delta(\beta_1-\gamma_1)}\,,$$
$$K_2 = \frac{\gamma_1^2\Delta_{11}}{\beta_1\Delta(\gamma_1-\beta_1)}\,,$$
$$K_3 = \frac{(\gamma_1+\beta_1)\gamma_1\beta_1}{E_1}\,.$$

If $0<x<1$, then $\sigma_y^{(1)}(x)=p(x)$ and (3.2.15) is a singular integral equation which also has a fixed singularity at the point $x=0$.

Below it will be shown that at the point $x=0$ the singularity order of the obtained equation can be any number less than 1.

Since the displacement must be bounded at the point $x=0$, it is necessary to require of the sought function $f(x)$ to satisfy the condition

$$xf(x)\to 0 \ \text{ as } \ x\to 0.$$

Multiplying equations (3.2.15) by x and using the equality

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{\partial v_1}{\partial x}\,dx = v_1^+(1)-v_1^+(0)=0, \tag{3.2.16}$$

we obtain

$$\int_0^1 \frac{tf(t)}{t-x}\,dt + \int_0^1 Q\Big(\frac{x}{t}\Big)f(t)\,dt = K_3\pi x\sigma_y^{(1)}(x), \tag{3.2.17}$$

where

$$Q(x) = K_1(1+x)^{-1} + K_2\beta_1(\beta_1+\gamma_1 x)^{-1} + K_2\gamma_1(\gamma_1+\beta_1 x)^{-1}.$$

Let us substitute $x=e^{\xi_0}$, $y^t=e^{\xi}$ into formula (3.2.17). Then we have

$$\int_{-\infty}^0 \frac{f(e^\xi)e^\xi}{1-\exp(\xi_0-\xi)}\,d\xi + \int_{-\infty}^\infty Q(e^{\xi_0-\xi})f(e^\xi)e^\xi\,d\xi = K_3\pi e^{\xi_0}\sigma_y^{(1)}(e^{\xi_0}). \tag{3.2.18}$$

By the Fourier transformation of this equation we obtain

$$\Phi^+(t) = G(t)\Phi^-(t) - K_3 iP(t), \quad -\infty<t<\infty, \tag{3.2.19}$$

where

$$G(t) = \frac{\mathbf{ch}\,\pi t + K_1 + 2K_2 \cos \mu t}{\mathbf{sh}\,\pi t}\,, \quad \mu = \ln \frac{\beta_1}{\gamma_1}\,,$$

$$\Phi^+(t) = -\frac{K_3 i}{\sqrt{2\pi}} \int\limits_0^\infty \sigma_y(e^\xi) e^{\xi(1+it)}\, d\xi,$$

$$\Phi^-(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^\infty f(e^\xi) e^{\xi(1+it)}\, d\xi,$$

$$P(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^0 p(e^\xi) e^{\xi(1+it)}\, d\xi.$$

Since the function $p(e^\xi)e^\xi$ exponentially vanishes as $\xi \to -\infty$, the function $P(w)$, where $w = t + i\tau$, will be analytic in the half-plane $\operatorname{Im} w < 1$.

Also note that

$$\Phi^-(0) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^0 f(e^\xi) e^\xi\, d\xi = \frac{1}{\sqrt{2\pi}} \int\limits_0^1 f(t)\, dt = 0.$$

Let us now consider the function

$$G_1(t) = \mathbf{ch}\,\pi t + K_1 + 2K_2\, \mathbf{ch}\, i\mu t.$$

We prove that if the condition

$$\nu_k < \sqrt{\frac{E_k}{E_k^*}}$$

is fulfilled, then

$$G_1(0) > 0, \quad G_1'' > 0.$$

It is obvious that if $K_2 < 0$, then $G_1(t) > G_1(0)$ and $G_1(t) > 0$ on the whole axis, and since $G_1'' > 0$, $G_1(0) > 0$, for $K_2 > 0$ we have

$$G_1''(t) = \pi^2\, \mathbf{ch}\,\pi t - 2K_2\mu^2\, \mathbf{ch}\, i\mu t > G_1''(0),$$

i.e. in this case the function $G'(t)$ increases and, at the point $t = 0$, attains its minimum. Hence it follows that the function $G_1(t)$ also increases and attains its minimum at the point $t = 0$. Since $G_1(0) > 0$, we have $G_1(t) > 0$.

The function $G(t)$ has a first order pole at the point $t = 0$, and a first kind discontinuity at infinity because $G(\infty) = -G(-\infty) = 1$. The boundary condition (3.2.19) can be rewritten as follows

$$\frac{\Phi^+(t)}{\sqrt{t+i}} = \frac{G(t)t}{\sqrt{1+t^2}}\, \frac{\Phi^-(t)}{t}\, \sqrt{t-i} - \frac{K_{3i} P(t)}{\sqrt{t+i}}\,, \tag{3.2.20}$$

where $\sqrt{w+i}$ and $\sqrt{w-i}$ denote the branches which are analytic in the half-planes cut along the rays drawn from the points $w = -i$ and $w = i$, respectively, along the direction x and which take respectively a positive and a negative value on the upper side of the cut. For such a choice of branches the function $\sqrt{1+w^2}$ is analytic in the strip $-1 < \operatorname{Im} w < 1$ and takes a positive value on the real axis.

Since the relation $w/\sqrt{w+i}$ is holomorphic in the half-plane $\operatorname{Im} w < 1$, the relation $\Phi^+(w)/\sqrt{w+i}$ is holomorphic in the half-plane $\operatorname{Im} w > 0$, $G(t) \neq 0$ and $\Phi^-(t) = 0$, the function $\Phi^-(w)\sqrt{w-i}/w$ will be holomorphic everywhere in the half-plane $\operatorname{Im} w < 1$, except those points which are zeros of the function $G(w)$ and lie in the upper half-plane.

Thus the considered problem can be formulated as follows: Using condition (3.2.20), find a function $\Phi^+(w)$, which is holomorphic in the upper half-plane $\operatorname{Im} w > 0$ and vanishes at infinity, and a function $\Phi^-(w)$, which is holomorphic in the half-plane $\operatorname{Im} w < 1$, except the points w_n which are the roots of the function $G(w)$, vanishes at infinity and is continuous on the real axis $w = t$.

The function $G_0(t) = G(t)t(1+t^2)^{-\frac{1}{2}}$ is positive and continuous on the whole real axis and $G_0(\infty) = G_0(-\infty) = 1$ and therefore $\operatorname{Ind} G_0(t) = 0$.

The solution of problem (3.2.19) is given by the formulas

$$\Phi^+(w) = -\frac{X(w)K_3\sqrt{w+i}}{2\pi} \int_{-\infty}^{\infty} \frac{P(t)}{X^+(t)\sqrt{i+t}\,(t-w)}\,, \quad \operatorname{Im} w > 0, \quad (3.2.21)$$

$$\Phi^-(w) = -\frac{X(w)K_3 w}{2\pi\sqrt{w-i}} \int_{-\infty}^{\infty} \frac{P(t)}{X^+(t)(t-w)\sqrt{t+i}}\,, \quad \operatorname{Im} w \leq 0, \qquad (3.2.22)$$

$$\Phi^-(w) = \frac{\Phi^+(w) + K_3 i P(w)}{G(w)}\,, \quad 0 < \operatorname{Im} w < 1, \qquad (3.2.23)$$

$$X(w) = \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln[t(t^2+1)^{-\frac{1}{2}}G(t)]}{t-w}\,dt\right), \quad \operatorname{Im} w \neq 0. \qquad (3.2.24)$$

Using the Sokhotski–Plemelj formulas it can be verified that

$$\Phi^-(t-i0) = \Phi^-(t+i0), \quad \operatorname{Im} w < 0,$$

and therefore the function $\Phi^-(w)$ is holomorphic in the half-plane except the points w_k, $k = 0, 1, \dots, n$, which lie in the upper half-plane and are zeros of the function $G(w)$.

One can prove that $G(i) < 0$, and since $G(0) > 0$, $G(w)$ has at least one purely imaginary zero $w_0 = i\tau_0$, $0 < \tau_0 < 1$.

Let us rewrite the function $\Phi^+(w)$ as

$$\Phi^+(w) = -\frac{X^+(w)K_3}{2\pi\sqrt{w+i}} \left[\int_{-\infty}^{\infty} \frac{(t+i)P(t)}{X^+(t)\sqrt{t+i}\,(t-w)}\,dt - \int_{-\infty}^{\infty} \frac{P(t)}{X^+(t)\sqrt{t+i}}\,dt\right]$$

or as

$$\Phi^+(w) = \Phi_0^+(w) + \frac{X^+(w)K_3}{2\pi\sqrt{w+i}} \int\limits_{-\infty}^{\infty} \frac{P(t)}{X^+(t)\sqrt{t+i}}\, dt.$$

The boundary value $\Phi_0^+(t)$ of the function $\Phi_0^-(w)$ is the Fourier transform of the bounded function, i.e.

$$\Phi^+(t_0) = \Phi_0^+(t_0) + \frac{cK_3}{\sqrt{t+i}}\,, \tag{3.2.25}$$

where

$$c = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{P(t)}{X^+(t)\sqrt{t+i}}\, dt. \tag{3.2.26}$$

The function $P(t)$ is the Fourier transform of a real function and therefore $\overline{P(-t)} = P(t)$. Moreover, $\overline{X^{\pm}(-t)} = X^{\pm}(t)$, $\overline{X_0^{\pm}(-t)} = \pm i X_0^{\pm}(t)$, where $X_0^{\pm}(t) = \sqrt{t \pm i}$. Based on the above reasoning, we easily conclude that

$$\overline{\Phi^+(-t)} = -\Phi^+(t) \;\text{ and }\; \overline{\Phi^-(t)} = \Phi^-(t),$$

i.e. $\Phi^+(t)$ is the Fourier transform of a purely imaginary function, while $\Phi^-(t)$ is that of a real function. Therefore the solution of the considered problem can be obtained by the inverse Fourier transformations of the functions $\Phi^+(t)$, $\Phi^-(t)$.

Let us perform the inverse Fourier transformation of equality (3.2.25) and go back to the variable x. By elementary calculations we obtain

$$\sigma_y^{(1)}(x,0) = -\frac{c\exp(\frac{\pi}{4}\, i)}{\pi x^2\sqrt{x-1}} + \varphi_0(x), \quad x > 1, \tag{3.2.27}$$

where $\varphi_0(x)$ is bounded for $x \geq 0$.

It is easy to show that $ce^{i\pi/4}$ is a real number. In that case, if the force applied to the boundaries of the cut is constant, i.e. $p(x) = p = const$, we have

$$P(t) = \frac{1}{\sqrt{2\pi}}\, \frac{p}{1+it}\,.$$

The substitution of this value into formula (3.2.26) yields

$$c = \frac{p}{2\pi\sqrt{2\pi}\, i} \int\limits_{-\infty}^{\infty} \frac{dt}{X^+(t)\sqrt{t+i}\,(t-i)} = \frac{1}{2\sqrt{\pi}\, i}\, \frac{p}{X^+(i)}\,.$$

Using formula (3.2.24) and taking into account that the integral density is an even value, we obtain

$$c = \frac{p}{2\sqrt{\pi}\, i}\, \exp\bigg(-\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\ln[t(t^2+1)^{-\frac{1}{2}}G(t)]}{t^2+1}\, dt \bigg). \tag{3.2.28}$$

Thus we have obtained that the normal stress has a singularity of order 1/2 in the neighborhood of the cut end $x = 1$, as should have been expected.

Let us now proceed to investigating the behavior of the function $f(x)$ in the neighborhood of the cut ends. Applying the same reasoning as above, we see that in the neighborhood of the point $x = 1$ the function $f(x)$ is represented in the form

$$f(x) = \frac{c_1}{\sqrt{1-x}} + \varphi_1(x), \tag{3.2.29}$$

where the function $\varphi_1(x)$ may have a logarithmic singularity in the neighborhood of the point $x = 1$.

For $0 \le \operatorname{Im} w < 1$ we easily obtain

$$\Phi^-(w) = \frac{c_2}{\sqrt{w-i}} + \Phi_0^-(w), \tag{3.2.30}$$

where the function $\Phi_0^-(w)$ is holomorphic throughout the strip $0 < \operatorname{Im} w < 1$, except perhaps the points $w_0 = i\tau_0$, $\tau_0 < \beta < 1$, at which it has a first order pole, and for sufficiently large values of $|w|$ it can be written in the form

$$\Phi_0(w) = O\Big(\frac{1}{|w|}\Big).$$

Multiply the function $\Phi_0^-(w)$ by $e^{-i\xi w}$, $\xi < 0$, and integrate the obtained expression along the rectangle with vertices at the points $(-N;0)$, $(N;0)$, (N,β), $(-N,\beta)$. Applying the Cauchy theorem for a multiply connected domain we obtain

$$\int\limits_{-N}^{N} \Phi_0^-(t)e^{-it\xi}\,dt = e^{\beta\xi}\int\limits_{-N}^{N} \Phi_0^-(t+i\beta)e^{-it\xi}\,dt + c_1e^{\tau_0\xi} + \varepsilon(N,\xi),$$

where $\varepsilon(N,\xi) \to 0$ as $N \to \infty$. Thus we have established that for $N \to \infty$ the integrals exist in the sense of Plancherel [**116**]. By the Fourier transformation, from (3.2.30) we obtain

$$e^{\xi}f(e^{\xi}) =, \frac{Me^{\xi}}{\sqrt{-\xi}} + \frac{e^{\beta\xi}}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} \Phi_0(t+i\beta)e^{-t\xi}\,dt + \frac{c_1e^{\tau_0\xi}}{\sqrt{2\pi}}\,, \ \ \xi < 0,$$

where M is real, and

$$c_1 = 2\pi i\lim_{\tau\to\tau_0}(\tau - i\tau_0)\Phi^-(\tau).$$

By formula (3.2.23) we can write

$$\lim_{\tau\to\tau_0}\Phi^-(\tau)(\tau - i\tau_0) = \frac{\Phi^+(i\tau_0) + K_3iP(i\tau_0)}{\pi\sin\pi\tau_0 - 2K_2\mu\,\mathbf{sh}\,\mu\tau_0}\,\sin\pi\tau_0,$$

and therefore

$$c_1 = -\frac{1}{\sqrt{2\pi}}\,\frac{K_3\int\limits_0^\infty \sigma_y(x)x^{-\tau_0}\,dx}{\pi\sin\pi\tau_0 - 2K_2\mu\,\mathbf{sh}\,\mu\tau_0}\,.$$

Thus we have

$$f(x)=\frac{N}{\sqrt{\ln\frac{1}{x}}}+x^{\beta-1}\varphi_0(x)+\frac{c_1}{\sqrt{2\pi}}\,x^{\tau_0-1}=O(x^{\tau_0-1}).$$

Applying the properties of Cauchy type integrals in the neighborhood of the ends of the open contour [**76**], it can be shown that near the points $z=0$ and $x=1$ the functions $\Phi_1(z_1)$ and $\Psi_1(\zeta_1)$ have the same character as the function $f(x)$. It can further be shown that the functions $\Phi_2(z_2)$ and $\Psi_2(\zeta_2)$ and the stress components $\sigma_y^{(2)}$, $\sigma_x^{(2)}$ and $\tau_{xy}^{(2)}$ have the analytic character in the neighborhood of the point $x=0$.

In the particular case of the problem to be considered below we see that τ_0 can take any value from the interval $(0;1)$.

1. Assume that the domain S_2 is obtained from the domain S_1 by the rotation of the elastic axis by an angle of $90°$. Then we shall have

$$E_2=E_1^*,\ \ E_2^*=E_1,\ \ \nu_2=\nu_1^*=\frac{E_1^*}{E_1}\,\nu_1,\ \ G_2=G_1.$$

The characteristic equation for the body S_2 will take the form

$$\mu^4+\Big(\frac{E_1^*}{G}-2\,\frac{E_1^*}{E_1}\,\nu_1\Big)\mu^2+\frac{E_1^*}{E_1}=0.$$

or

$$\Big(\frac{1}{\mu}\Big)^4+\big(\frac{E_1}{G_1}-2\nu_1\Big)\Big(\frac{1}{\mu}\Big)^2+\frac{E_1}{E_1^*}=0.$$

The roots of this equation are $\pm i/\gamma_1$, $\pm i/\beta_1$, i.e. $\beta_2=1/\gamma_1$, $\gamma_2=1/\beta_1$. Furthermore

$$p_2=-\Big(\frac{\beta_2^2+\nu_2}{E_2}\Big)=-\Big(\frac{1}{\gamma_1^2}+\frac{E_1^*}{E_1}\,\nu_1\Big)\,\frac{1}{E_1^*}=-\frac{1}{E_1}\,\Big(\frac{E_1}{\gamma_1^2E_1^*}+\nu_1\Big)=-\frac{\beta_1^2+\nu_1}{E_1}\,,$$

i.e. $p_2=p_1$. Analogously, we obtain

$$\Delta=-(p_1-r_1)^2\,\frac{(\alpha_1\beta_1+1)^2}{\gamma_1\beta_1}\,,\ \ \Delta_{11}=0,$$

$$\Delta_{12}=-\Delta_{21}=(p_1-r_1)^2\,\frac{\gamma_1^2\beta_1^2-1}{\gamma_1\beta_1}\,,$$

$$K_1=(\Delta_{12}\gamma_1+\Delta_{21}\beta_1)(\beta_1-\gamma_1)\Delta=\frac{\beta_1^2\gamma_1^2-1}{(\gamma_1\beta_1+1)^2}$$

$$=\frac{\beta_1\gamma_1-1}{\beta_1\gamma_1+1}=\frac{\sqrt{E_1}-\sqrt{E_1^*}}{\sqrt{E_1}+\sqrt{E_1^*}}\,,$$

$$G(t)=\Big(\mathbf{ch}\,\pi t+\frac{\sqrt{E_1}-\sqrt{E_1^*}}{\sqrt{E_1}+\sqrt{E_1^*}}\Big)\,\frac{1}{\mathbf{sh}\,\pi t}\,.$$

Hence it follows that if $E_1<E_1^*$, then

$$\tau_0=\frac{1}{\pi}\,\arccos\Big(\frac{\sqrt{E_1}-\sqrt{E_1^*}}{\sqrt{E_1}+\sqrt{E_1^*}}\Big)<\frac{1}{2}\,.$$

If $E_1^* < E_1$, then

$$\tau_0 = 1 - \frac{1}{\pi}\,\arccos\Big(\frac{\sqrt{E_1} - \sqrt{E_1^*}}{\sqrt{E_1} + \sqrt{E_1^*}}\Big) > \frac{1}{2}\,.$$

If $E_1^* = E_1$, then

$$\tau_0 = \frac{1}{2}\,.$$

The latter fact corresponds to the case in which the cut plane is homogeneous.

The considered example shows that the more rigid the left-hand half-plane is, the lower the concentration degree is near the end which is on the boundary line.

In the case in which the cut edges are under the action of the constant load p, the stress σ_y near the right-hand end of the cut be represented in the form

$$\sigma_y(x,0) = -\frac{c\sqrt{2i}}{\pi x^2\sqrt{x-1}} + \varphi_0(x), \quad x > 1,$$

where $\varphi_0(x)$ is continuous on the closed semi-axis $x \geq 0$ and

$$c = \frac{1}{\sqrt{2i}}\,\exp\bigg(-\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\ln\big|\frac{(\mathbf{ch}\,\pi t+K)t}{\mathbf{sh}\,\pi t\sqrt{t^2+1}}\big|}{t^2+1}\,dt\bigg)$$

or

$$c = \frac{c_0}{\sqrt{2\pi}}\,\exp\bigg(-\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\ln(1+\frac{K}{\mathbf{ch}\,\pi t})}{t^2+1}\,dt\bigg),$$

$$c_0 = \exp\bigg(-\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\Big[\ln\frac{t\,\mathbf{ch}\,\pi t}{(1+t^2)^{1/2}\,\mathbf{sh}\,\pi t}\Big]\frac{dt}{t^2+1}\bigg),$$

$$K = \frac{\sqrt{E_1^*} - \sqrt{E_1}}{\sqrt{E_1^*} + \sqrt{E_1}}\,.$$

For the homogeneous body, $K = 0$ and $c_0 = c$, i.e. c_0 is the concentration coefficient corresponding to the homogeneous plane.

When $E_1^* > E_1$, $K > 0$ and $\ln(1+\frac{K}{\mathbf{ch}\,\pi t}) > 0$, while when $E_1^* < E_1$, $K < 0$ and $\ln(1+\frac{K}{\mathbf{ch}\,\pi t}) < 0$. In the former case $c < c_0$ and in the latter case $c > c_0$.

This example shows that the concentration degree near the left-hand end of the cut may be an arbitrary number from the interval $(0,1)$. This number takes a value greater than a half when, as compared with the right-hand half-plane, the left-hand half-plane is less rigid along the y-axis, and takes a value smaller than a half when, as compared with the left-hand half-lane, the right-hand half-plane is less rigid along the y-axis. In the former case, the coefficient is greater than that of the homogeneous body, while in the former case the coefficient is smaller.

2. Assume that the bodies S_1 and S_2 are isotropic, then $\beta_1 = \gamma_1 = \gamma_2 = \beta_2 = 1$. The characteristic equation will have the multiple roots and therefore formulas (3.2.1), (3.2.2) will not be valid, but the expressions not containing complex potentials will remain in force if we assume that $\gamma_1 = \gamma_2 = 1$ and pass to the limit as $\beta_1 \to 1$ and $\beta_2 \to 1$.

In particular, if we pass to the limit in equation (3.2.17) as $\beta_k \to \gamma_k \to 1$, then, after evaluating the indeterminacy, we obtain

$$Q(x) = \frac{1}{2}\Big(\frac{1-\alpha}{\alpha+\varkappa_1} + \frac{\varkappa_1 - \alpha\varkappa_2}{1+\alpha+\varkappa_2}\Big)(x+1)^{-1} + \frac{2(\alpha-1)}{\alpha+\varkappa_1}\,\frac{x^2-x}{(1+x)^3}$$

where

$$\alpha = \frac{\mu_1}{\mu_2}\,, \quad \varkappa_k = \frac{3-\nu_k}{1+\nu_k}\,, \quad k=1,2.$$

The coefficient of the boundary value problem (3.2.19) takes the form

$$G(t) = \frac{\mathbf{ch}\,\pi t + \frac{1}{2}\big(\frac{1-\alpha}{\alpha+\varkappa_1} + \frac{\varkappa_1-\alpha\varkappa_2}{1+\alpha\varkappa_2}\big) + \frac{2(\alpha-1)}{\alpha+\varkappa_1}\,t^2}{\mathbf{sh}\,\pi t}\,.$$

For isotropic bodies, this problem is studied in [**54**].

3.3. The Contact Problem for Piecewise-Homogeneous Plane with a Semi-Infinite Inclusion

We consider a piecewise-homogeneous elastic plate stiffened with a semi-infinite inclusion under the action of tangential stresses with intensity $\tau_k^0(x)$. The problem consists in defining contact tangential stresses $\tau_k(x)$ along the contact line and in establishing their behavior at singular points.

In mathematical terms the problem reads as follows: let the elastic body S occupy the plane of a complex variable $z = x+iy$, which, along the line $L = (-\infty, 1)$, contains an elastic inclusion with elasticity modulus $E_0(x)$, thickness $h_0(x)$, the Poisson ratio ν_0, and consists of two half-planes $S_1 = \{z \mid \operatorname{Re} z > 0,\ z \notin \overline{\ell}_1 = [0,1]\}$ and $S_2 = \{z \mid \operatorname{Re} z < 0,\ z \notin \overline{\ell}_2 =]-\infty, 0]\}$ that are sealed together along the axis $x = 0$. The values and functions related to S_k will be marked by the index k ($k = 1,2$), and the boundary values of other functions on the upper and lower edges of the inclusion will be marked by the signs $(+)$ and $(-)$, respectively.

On the interface boundary we have the following conditions of continuity

$$\sigma_x^{(1)} = \sigma_x^{(2)}, \quad \tau_{xy}^{(1)} = \tau_{xy}^{(2)}, \quad u_1 = u_2, \quad v_1 = v_2\,. \tag{3.3.1}$$

On the segment ℓ_k we have the conditions

$$\frac{du_k^{(0)}(x)}{dx} = \frac{1}{E(x)}\bigg\{P_0\delta_{1k} - (-1)^{k+1}\int\limits_{a_k}^{x}\big[\tau_k(t) - \tau_k^0(t)\big]\,dt\bigg\}, \tag{3.3.2}$$

$$x \in \ell_k, \quad k = 1,2,$$

where $u_k^{(0)}(x)$ are horizontal displacements of the inclusion points $a_1 = 0$, $a_2 = -\infty$, $E(x) = \frac{E_0(x)h_0(x)}{1-\nu_0^2}$, while the equilibrium conditions for separate parts of the inclusion have the form

$$\int_{-\infty}^{0} (\tau_2(t) - \tau_2^{(0)}(t))\, dt = P_0, \quad P_0 - \int_{0}^{1} (\tau_1(t) - \tau_1^{(0)}(t))\, dt = P, \tag{3.3.3}$$

where P_0 and P are the unknown axial stresses at the points $x = 0$ and $x = 1$, respectively. From the Kolosov–Muskhelishvili [**77**] formulas

$$\begin{aligned} &\varphi_k(z) + z\overline{\varphi_k'(z)} + \overline{\psi(z)} = i\int_{z_k}^{z} (X_n^{(k)} + iY_n^{(k)})\, ds \equiv R_k(z), \\ &\aleph_k\varphi_k(z) - x\overline{\varphi_k'(z)} - \overline{\psi(z)} = 2\mu_k(u_k + i\nu_k), \quad \aleph_k = 3 - 4\nu_k, \end{aligned} \tag{3.3.4}$$

we obtain

$$\begin{aligned} &\varphi_k^+(x) - \varphi_k^-(x) = \frac{i}{1+\aleph_k}\int_{x_k}^{x} \tau_k(t)\, dt \equiv if_k(t), \\ &\psi_k^+(x) - \psi_k^-(x) = -i(\aleph_k f_k(x) + xf_k'(x)), \quad x \in \ell_k, \quad k = 1, 2. \end{aligned}$$

For $z \in S_k$ a solution of these problems has the form

$$\begin{aligned} \varphi_k(z) &= \frac{1}{2\pi}\int_{\ell_k} \frac{f_k(t)}{t-z}\, dt + W_k(z) = w_k(z) + W_k(z), \\ \psi_k(z) &= -\frac{1}{2\pi}\int_{\ell_k} \frac{(\aleph_k f_k(t) + tf_k'(t))}{t-z}\, dt + Q_k(z) \\ &= q_k(z) + Q_k(z), \quad z \in S_k, \end{aligned} \tag{3.3.5}$$

where $W_k(z)$ and $Q_k(z)$ are analytic functions in S_k.

By the introduction of the function

$$\omega_k(z) = -z\varphi_k'(z) + \psi_k(z) = \eta_k(z) + \Omega_k(z), \tag{3.3.6}$$

where

$$\eta_k(z) = -zw_k'(z) + q_k(z), \quad \Omega_k(z) = -zW_k(z) + Q_k(z),$$

equalities (3.3.4) take the form

$$\begin{aligned} \varphi_k(z) + (z+\overline{z})\,\overline{\varphi_k'(z)} + \overline{\omega_k(z)} &= R_k(z), \\ \aleph_k\varphi_k(z) - (z+\overline{z})\,\overline{\varphi_k'(z)} - \overline{\omega_k(z)} &= 2\mu_k(u_k + iv_k). \end{aligned}$$

After writing conditions (3.3.1) in terms of these functions and applying the singular operator

$$S(\cdot) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{(\cdot)}{t-z}\, dt, \quad z \in S_k,$$

to the resulting equalities, for the functions $W_1(z)$, $\Omega_1(z)$, $\overline{W_2(-\overline{z})}$, $\overline{\Omega_2(-\overline{z})}$ we obtain a system of four equations whose solution has the form

$$\begin{aligned}
W_1(z) &= e_1\overline{\eta_1(-\overline{z})} + r_2 w_2(z),\\
\Omega_1(z) &= h_1\overline{w_1(-\overline{z})} + m_2\eta_2(z),\\
\overline{W_2(-\overline{z})} &= e_2\eta_2(z) + r_1\overline{w_1(-\overline{z})},\\
\overline{\Omega_2(-\overline{z})} &= h_2 w_2(z) + m_1\overline{\eta_1(-\overline{z})}.
\end{aligned}$$

Using these relations, from formulas (3.3.5), (3.3.6) we find the following expressions for $\varphi_k(z)$ and $\psi_k(z)$

$$\begin{aligned}
\varphi_k(z) &= \frac{1}{2\pi}\int\limits_{\ell_k}\left[\frac{1}{t-z} - \frac{e_k\aleph_k}{t+z} + \frac{e_k z}{(t+z)^2}\right] f_k(t)\,dt\\
&\quad - \frac{e_k}{2\pi}\int\limits_{\ell_k}\frac{tf_k'(t)}{t+z}\,dt + \frac{r_{3-k}}{2\pi}\int\limits_{\ell_{3-k}}\frac{f_{3-k}(t)}{t-z}\,dt,\\
\psi_k(z) &= \frac{1}{2\pi}\int\limits_{\ell_k}\left[\frac{-\aleph_k}{t-z} + \frac{h_k}{t+z} + \frac{e_k(1+\aleph_k)z}{(t+z)^2} - \frac{2e_k z^2}{(t+z)^3}\right] f_k(t)\,dt\\
&\quad + \frac{1}{2\pi}\int\limits_{\ell_k}\left[\frac{-1}{t-z} + \frac{e_k z}{(t+z)^2}\right] tf_k'(t)\,dt\\
&\quad + \frac{1}{2\pi}\int\limits_{\ell_{3-k}}\left[\frac{-m_3\aleph_{3-k}}{t-z} + \frac{(r_{3-k} - m_{3-k})z}{(t-z)^2}\right] f_{3-k}(t)\,dt\\
&\quad - \frac{m_{3-k}}{2\pi}\int\limits_{\ell_{3-k}}\frac{tf_{3-k}'(t)}{t-z}\,dt, \quad k = 1,2,
\end{aligned} \tag{3.3.7}$$

where

$$\begin{gathered}
e_1 = \frac{\mu_1 - \mu_2}{\Delta_1}\,,\ \ r_2 = \frac{\mu_1(\aleph_2+1)}{\Delta_1}\,,\ \ m_2 = \frac{\mu_2(\aleph_1+1)}{\Delta_1}\,,\ \ h_2 = \frac{\aleph_2\mu_1 - \aleph_1\mu_2}{\Delta_1}\,,\\
e_2 = \frac{\mu_1 - \mu_2}{\Delta_2}\,,\ \ r_1 = \frac{\mu_2(\aleph_1+1)}{\Delta_2}\,,\ \ m_1 = \frac{\mu_1(\aleph_2+1)}{\Delta_2}\,,\ \ h_1 = \frac{\aleph_1\mu_2 - \aleph_2\mu_1}{\Delta_2}\,,\\
\Delta_1 = \aleph_1\mu_2 + \mu_1\,,\ \ \Delta_2 = \aleph_2\mu_1 + \mu_2.
\end{gathered}$$

If we substitute relations (3.3.7) into the equality

$$u_k(z) = \frac{1}{2\mu_k}\,\mathrm{Re}\left[\aleph_k\varphi_k(z) - z\overline{\varphi_k'(z)} - \overline{\psi_k(z)}\right]$$

take the limit as $z \to x \pm i0$, then we come to a system of singular integro-differential equations

$$\frac{E(x)}{4\pi\mu_1}\int\limits_0^\infty \left[\frac{2\aleph_1}{t-x}+\frac{\aleph_1^2 e_1+h_1}{t+x}-\frac{4e_1t^2}{(t+x)^3}\right]\widetilde{f}_1'(t)\,dt$$

$$-\frac{E(x)}{4\pi\mu_1}\int\limits_0^\infty \left[\frac{r_2\aleph_1+m_2\aleph_2}{t+x}+\frac{2(m_2-r_2)t}{(t+x)^2}\right]\widetilde{f}_2'(t)\,dt$$

$$=\begin{cases}-(1+\aleph_1)\widetilde{f}_1(x)+T_1(x), & x\in(0,1),\\ E(x)u_1'(x), & x\in(1,\infty),\end{cases}$$

$$-\frac{E(x)}{4\pi\mu_2}\int\limits_0^\infty \left[\frac{2\aleph_2}{t-x}+\frac{\aleph_2^2 e_2+h_2}{t+x}-\frac{4e_2t^2}{(t+x)^3}\right]\widetilde{f}_2'(t)\,dt$$

$$+\frac{E(x)}{4\pi\mu_2}\int\limits_0^\infty \left[\frac{r_1\aleph_2+m_1\aleph_1}{t+x}+\frac{2(m_1-r_1)t}{(t+x)^2}\right]\widetilde{f}_1'(t)\,dt$$

$$=(1+\aleph_2)\widetilde{f}_2(x)+T_2(x),\quad x\in(0,\infty),\qquad(3.3.8)$$

where

$$\widetilde{f}_1(x)=\begin{cases}f_1(x), & x\in(0,1),\\ 0, & x\in(1,\infty),\end{cases}\qquad \widetilde{f}_2(x)=f_2(-x).$$

The functions $T_1(x)$ and $T_2(x)$ depend on the known value $\tau_k^{(0)}(x)$ $(k=1,2)$ and on the unknown constants P_0 and P, i.e.

$$T_1(x)=P_0+\int\limits_0^x\tau_1^0(t)\,dt-g_1(x),\quad T_2(x)=\int\limits_{-x}^\infty\tau_2^0(-t)\,dt-g_2(x),$$

$$g_1(x)=\frac{E(x)}{2\mu_1}\left((\aleph_1-1)\alpha(x)-x\alpha'(x)-\delta(x)\right),$$

$$g_2(x)=\frac{E(x)}{2\mu_2}\left((\aleph_2-1)\beta(-x)-x\beta'(-x)-\gamma(-x)\right),$$

where

$$\alpha(x)=C_1\left\{\frac{e_1}{(1+x)^2}+\frac{1}{1-x}+\frac{e_1\aleph_1}{1+x}\right\}+C_2\,\frac{r_2}{x},$$

$$\beta(x)=C_1\,\frac{r_2}{1-x}+C_2\,\frac{(1-e_2\aleph_2)}{x},$$

$$\gamma(x)=C_1\left\{\frac{m_1\aleph_1}{1-x}+\frac{m_1-r_1}{(1-x)^2}\right\}+C_2\,\frac{h_2-\aleph_2}{x},$$

$$\delta(x)=C_1\left\{\frac{\aleph_1}{1-x}+\frac{h_1}{1+x}-\frac{e_1(1+\aleph_1)}{(1+x)^2}-\frac{2e_1x}{(1+x)^3}\right\}-C_2\,\frac{m_2\aleph_2}{x},$$

$$C_1 = \frac{T_1^0 + P_0 - P}{2\pi(1+\aleph_1)}, \quad C_2 = \frac{P_0 + T_2^0}{2\pi(1+\aleph_2)},$$

$$T_1^0 = \int\limits_0^1 \tau_1^0(t)\,dt, \quad T_2^0 = \int\limits_{-\infty}^0 \tau_2^0(t)\,dt.$$

To solve system (3.3.8) when the inclusion rigidity changes by a linear law, i.e. $E(x) = h|x|$, $x \in (-\infty, 1)$, after substituting $t = e^\zeta$ and $x = e^\xi$ into (3.3.8) and making Fourier transformation [**42**], we obtain a system

$$\begin{aligned} &G_1(s)F^-(s) + G_2(s)\Phi(s) = -(1+\aleph_1)F^-(s) + \Psi^+(s) + P_1(s), \\ &G_3(s)\Phi(s) + G_4(s)F^-(s) = -(1+\aleph_2)\Phi(s) + P_2(s), \\ &\qquad\qquad s = s_0 - i\varepsilon, \quad \varepsilon > 0, \end{aligned} \tag{3.3.9}$$

where

$$F^-(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^0 \widetilde{f}_1(e^\xi)e^{i\xi z}\,d\xi, \quad \Phi(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \widetilde{f}_2(e^\xi)e^{i\xi z}\,d\xi,$$

$$\Psi^+(z) = \frac{1}{\sqrt{2\pi}} \int\limits_0^{+\infty} u'(e^\xi)e^{i\xi z}\,d\xi, \quad P_1(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^0 T_1(e^\xi)e^{i\xi z}\,d\xi,$$

$$P_2(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} T_2(e^\xi)e^{i\xi z}\,d\xi,$$

$$G_1(z) = \frac{1}{\mathbf{sh}(\pi z)} \Big[2\aleph_1\,\mathbf{ch}(\pi z) + \aleph_1^2 e_1 + h_1 - 2e_1 i z^2(z+i)\Big],$$

$$G_2(z) = \frac{1}{\mathbf{sh}(\pi z)} \Big[(r_2\aleph_1 + m_2\aleph_2) + 2(m_2 - r_2)z^2\Big],$$

$$G_3(z) = \frac{1}{\mathbf{sh}(\pi z)} \Big[2\aleph_2\,\mathbf{ch}(\pi z) + \aleph_2^2 e_2 + h_2 - 2e_2 i z^2(z+i)\Big],$$

$$G_4(z) = \frac{1}{\mathbf{sh}(\pi z)} \Big[(r_1\aleph_2 + m_1\aleph_1) + 2(m_1 - r_1)z^2\Big].$$

If from system (3.3.9) we eliminate the function

$$\Phi(s) = \frac{P_2(s) - G_4(s)F^-(s)}{G_3(s) - (1+\aleph_2)}, \tag{3.3.9$'$}$$

then we obtain

$$G(s)F^-(s) = \Psi^+(s) + H(s), \tag{3.3.10}$$

where

$$G(s) = G_1(s) + (1+\aleph_1) - \frac{G_2(s)G_4(s)}{G_3(s) - (1+\aleph_2)},$$

$$H(s) = P_1(s) - \frac{P_2(s)G_2(s)}{G_3(s) - (1+\aleph_2)}.$$

It is easy to show that the function $G(s) \to 3\aleph_1 + 1 \equiv \alpha$ as $t \to +\infty$, $G(s) \to 1 - \aleph_1 \equiv \beta$ $(\beta < 0)$ as $t \to -\infty$, $G(t) = \frac{t^2 g_0(t)}{\mathbf{sh}\,\pi t}$, $g_0(t) > 0$.

Condition (3.3.10) can be rewritten in the form

$$\frac{\sqrt{1+t^2}}{t}\, G(t)\, \frac{tF^-(t)}{\sqrt{t-i}} = \Psi^+(t)\sqrt{t+i} + H(t)\sqrt{t+i}\,, \tag{3.3.11}$$

where under $\sqrt{z+i}$ and $\sqrt{z-i}$ we understand the branches which are analytic in the planes cut along the rays radiating from the points $z = -i$ and $z = i$ in the direction x and which take respectively positive and negative values on the upper edge of the cut. With such a choice of branches, the function $\sqrt{1+z^2}$ is analytic in the strip $-1 < \operatorname{Im} z < 1$ and takes a positive value on the real axis.

Thus the posed problem can be formulated as follows: using condition (3.3.11), find a function $\Psi^+(z)$, holomorphic in the half-plane $\operatorname{Im} z > 0$ and vanishing at infinity, and a function $F^-(z)$, holomorphic in a half-plane $\operatorname{Im} z < 1$ except for the points which are the roots of the function $G(z)$ and the poles of the function $H(z)$, vanishing at infinity and continuous on the real axis.

A solution of problem (3.3.11) has the form

$$\begin{aligned}
F^-(z) &= \frac{\sqrt{z-i}\,X(z)}{z}\left(\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\sqrt{t+i}\,H(t)}{X^+(t)(t-z)}\,dt + \frac{c}{z-i}\right), \quad \operatorname{Im} z < 0,\\
\Psi^+(z) &= \frac{X(z)}{\sqrt{z+i}}\left(\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\sqrt{t+i}\,H(t)}{X^+(t)(t-z)}\,dt + \frac{c}{z-i}\right), \quad \operatorname{Im} z > 0,\\
F^-(z) &= \left\{\Psi^+(z) + H(z)\right\} G^{-1}(z), \quad 0 < \operatorname{Im} z < 1,
\end{aligned} \tag{3.3.12}$$

where

$$X(z) = \exp\left\{\frac{z+i}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln G_0(t)}{(t+i)(t-z)}\,dt\right\}, \quad G_0(t) = \frac{\sqrt{1+t^2}}{t}\,G(t).$$

The constant c is defined from the condition $F^-(0) = O(1)$,

$$c = \frac{H(0)}{2\sqrt{i}\,X^+(t)} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H(t)\sqrt{t+i}}{X^+(t)t}\,dt.$$

It is easy to show that $F^-(x+i0) = F^-(x-i0)$ and therefore the function $F^-(z)$ is holomorphic in the half-plane except for the points which are the zeros of the function $G(z)$ and the poles of the function $H(z)$ in the upper half-plane. Our aim is to investigate the behavior of contact stresses near the singular points $z = 0$ and $z = 1$. It can be shown that in (3.3.12) $F^-(x) = \frac{c_0}{\sqrt{x-i}} + F_0^-(x)$, where $F_0^-(x)$ is the Fourier transform of a continuous function $f_0(x)$ on the semi-axis except perhaps for a point $x \le 0$

where it may have a logarithmic singularity. By the inverse transformation we obtain $\tau_1(x) = O((1-x)^{-1/2})$, $x \to 1-$.

Let us now study the behavior of the function $\tau_1(x)$ near the point $z = 0$. The poles of the function $F^-(z)$ in the domain $D_0 = \{z : 0 < \operatorname{Im} z < 1\}$ are the zeros of the functions $g(z) = (G_1(z) + (1+\aleph_1))(G_3(z) - (1+\aleph_2)) - G_2(z)G_4(z)$ and $g_1(z) = G_3(z) - (1+\aleph_2)$.

Assume that $i\tau_0$ is the smallest modulo a simple zero of the functions $g(z)$ and $g_1(z)$ in the domain D_0. Then, applying to the function $e^{i\xi z}F^-(z)$ the Cauchy theorem on residues for the rectangle $D(N)$ with the boundary $L(N)$ consisting of segments $[-N, N]$, $[N+i0, N+i\beta_0]$, $[N+i\beta_0, -N+i\beta_0]$, $[-N+i\beta_0, -N+i0]$, $\tau_0 < \beta_0 < \tau_0^1$, $(g(i\tau_0^1) = 0$ or $g_1(i\tau_0^1) = 0$, we obtain

$$\int\limits_{L(N)} F^-(t)e^{-i\xi t}dt = \int\limits_{-N}^{N} F^-(t)e^{-i\xi t}\,dt$$
$$- e^{\beta_0\xi}\int\limits_{-N}^{N} F^-(t+i\beta_0)e^{-i\xi t}\,dt + \rho(N,\xi) = K_1 e^{\xi\tau_0}, \quad (3.3.13)$$

where $\rho(N,\xi) \to 0$ as $N \to \infty$. Passing to the limit in (3.3.13) and returning to the old variables, we obtain

$$\tau_1(x) = (1+\aleph_1)f_1'(x) = (1+\aleph_1)K_1x^{\tau_0-1} + O(x^{\beta_0-1}), \quad x \to 0+.$$

Analogously, defining the function $\Phi(t)$ by (3.3.9′) and making the inverse Fourier transformation, after some calculations we obtain

$$\tau_2(x) = (1+\aleph_2)f_2'(x) = O(x^{\mu_0-1}), \quad x \to 0-,$$

here $i\mu_0$ $(\mu_0 \geq \tau_0)$ is the smallest modulo a simple zero of the function $g_1(z)$ in the domain D_0.

Suppose the functions $g(x)$ and $g_1(x)$ do not have simple zeros in the domain D_0. In that case, contact stresses may have a singularity of logarithmic type at the origin (if for instance the point $z = i$ is a double zero of the function $g(x)$ or $g_1(x)$).

It should be noted that the obtained system of integro-differential equations (3.3.8) reduces to one equation in the following quite interesting cases:

1. When a semi-infinite inclusion has constant rigidity or rigidity changes according to a qualitative (nonlinear) law and reaches the interface between two materials, for the Fourier transform of the sought functions we obtain a boundary value problem of the theory of analytic functions with shear for a strip (this is a Carleman type problem).
2. When the stiffness of a semi-infinite or finite inclusion changes by a linear law and reaches the interface of two materials, for the Fourier transform of the sought function we obtain an algebraic equation or a boundary value problem of linear conjugation.

The application of the theory of analytic functions and integral transformations makes it possible to obtain effective solutions of the above-stated concrete problems.

CHAPTER 4

The Problem for Doubly-Connected Domains

4.1. Solution of the Third Basic Problem of the Elasticity Theory for Doubly-Connected Domains Bounded by Broken Lines

Let on the domain S an elastic body occupy a finite doubly-connected domain $z = x + iy$ bounded by two mutually disjoint closed convex broken lines L_0 and L_1. Also assume that L_0 is the external and L_1 the internal boundary of the domain S. Let the origin lie within the contour L_1.

Denote by $A_1, A_2, \dots, A_p$ and $A_{p+1}, \dots, A_{p+q}$ the vertices of the broken lines L_1 and L_0, respectively, and by $\Gamma_k = A_k A_{k+1}$ the their sides for $k = 1, 2, \dots, p+q$, $k \neq p, p+q$, $\Gamma_p = A_p A_1$, $\Gamma_{p+q} = A_{p+q} A_{p+1}$.

We will consider the following problem.

Given on the boundary L $(L = L_0 \cup L_1)$ the tangent component T of acting external forces and the normal component v_n of the displacement vector, find an elastic equilibrium of the domain S.

Assume that $v_n'(t)$ and $\Gamma(t)$ belong to the class H_0 for the nodes A_k, $k = 1, 2, \dots, p+q$.

The third basic problem for multiply connected domains bounded by smooth contours is investigated in [**108**].

The third basic problem for domains mapped conformally on the circle by means of rational functions is solved in [**74**], and for the polygon - in [**89**]–[**93**].

Using the Kolosov–Muskhelishvili method [**77**] we can reduce the problem to finding two analytic functions $\varphi(z)$ and $\psi(z)$ of the complex variable $z = x + iy$ in the domain S by the following conditions on L:

$$\operatorname{Re}\Big[\big(\varkappa\varphi_1(t) - t\overline{\varphi_1'(t)} - \overline{\psi_1(t)}\,\big)e^{-i\alpha(t)}\Big] = 2\mu v_n, \quad t \in L, \tag{4.1.1}$$

$$\begin{gathered}\operatorname{Re}\Big[\big(\varphi_1(t) + t\overline{\varphi_1'(t)} + \overline{\psi_1(t)}\big)e^{-i\alpha(t)}\Big] \\ = \operatorname{Re} e^{-i\alpha(t)}\Big[i\int\limits_{t_j}^{t}\big(N(t_0) + iT(t_0)\big)e^{i\alpha(t_0)}\,ds_0\Big] + c_j, \\ t \in L_j, \quad j = 0, 1,\end{gathered} \tag{4.1.2}$$

where $\varkappa$ and μ are elastic constants, s_0 is the arc abscissa of the point t_0 counted from the point t_j, $j = 0, 1$; c_0, c_1 are some constants, $\alpha(t)$ is the angle between the external normal n and the positive direction of the x-axis.

It is obvious that $\alpha(t)$ is a piecewise-constant function, i.e. $\alpha(t) = \alpha_k$ on Γ_k, $k = 1, 2, \ldots, p+q$. Without loss of generality it can be assumed that $\pi > \alpha_1 > \cdots > \alpha_p > -\pi$ and $0 \le \alpha_{p+1} < \alpha_{p+2} < \cdots < \alpha_{p+q} < 2\pi$.

The constants c_0, c_1 are a priori unknown; it is assumed that $c_1 = 0$, $c_0 = A + iB$, where A and B are the sought real constants.

Since $\alpha(t)$ is piecewise-constant, the right-hand part of equality (4.1.2) can be rewritten in the form

$$\operatorname{Re} e^{-i\alpha(t)} i \int_{t_j}^{t} (N + iT) e^{i\alpha(t)}\, ds_0 = -\int_{t_j}^{t} T(t_0) \cos\big[\alpha(t) - \alpha(t_0)\big]\, ds_0 + C(t),$$

where

$$\begin{aligned}
C(t) &= \int_{t_j}^{t} N(t_0) \sin\big[\alpha(t) - \alpha(t_0)\big]\, ds_0 \\
&= \sum_{r=1}^{k} \int_{\Gamma_r} N(t) \sin[\alpha_k - \alpha_r]\, ds = C_k, \quad k = 1, 2, \ldots, p, \\
C(t) &= \int_{A_{p+1}}^{t} N(t_0) \sin\big[\alpha(t) - \alpha(t_0)\big]\, ds_0 \\
&= \sum_{r=p+1}^{k} \int_{\Gamma_r} N(t) \sin[\alpha_k - \alpha_r]\, ds = C_k, \quad k = p+1. \ldots, p+q.
\end{aligned}$$

From this we see that $c_1 = c_{p+1} = 0$.

As is known [**77**], the functions $\varphi_1(z)$ and $\psi_1(z)$ are written in the form

$$\varphi_1(z) = \varphi_2(z) - \frac{X + iY}{2\pi(1 + \varkappa)} \ln z, \tag{4.1.3}$$

$$\psi_1(z) = \psi_2(z) + \frac{(X - iY)\varkappa}{2\pi(1 + \varkappa)} \ln z, \tag{4.1.4}$$

where $\varphi_2(z)$ and $\psi_2(z)$ are holomorphic functions in the domain S, and (X, Y) are the projections of the principal vector of external forces applied to L_1. These constants are a priori unknown and are to be defined together with the functions $\varphi_2(z)$, $\psi_2(z)$.

Thus the problem reduces to defining the holomorphic functions $\varphi_2(z)$ and $\psi_2(z)$ in the domain S, and $p + q - 2$ real constants $c_2, \ldots, c_p, c_{p+r}, \ldots, c_{p+q}$, A, B, X, Y. As we will see below, the constants c_p and c_{p+q} can be expressed through the constants X and Y.

Indeed, if we multiply the equality

$$X + iY = \int_{L_1} (N + iT) e^{i\alpha(t)}\, ds$$

by $e^{-i\alpha p}$ and equate the real parts to each other, then we obtain

$$C_p = X\sin\alpha_p - Y\cos\alpha_p + \int\limits_{L_1} T(t)\cos\big(\alpha_p - \alpha(t)\big)\,ds. \qquad (*)$$

From the equilibrium condition we have

$$\int\limits_{L_0}(N+iT)e^{i\alpha(t)}\,ds = -\int\limits_{L_1}(N+iT)e^{i\alpha(t)}\,ds = -(X+iY).$$

By transformations analogous to those above we obtain

$$C_{p+q} = -X\sin\alpha_{p+q} + Y\cos\alpha_{p+q} + \int\limits_{L_0} T(t)\cos(\alpha_{p+q} - \alpha(t))\,ds. \qquad (**)$$

Thus it remains for us to define only $p+q$ constants and the functions $\varphi_2(z)$ and $\psi_2(z)$.

Summing the boundary conditions (4.1.1) and (4.1.2) we have

$$\operatorname{Re}\big[\varphi_1(t)e^{-i\alpha(t)}\big] = f_j(t) + \frac{C(t)}{\varkappa+1} + \operatorname{Re}\frac{e^{-i\alpha}c_j}{\varkappa+1}\,, \quad t\in L_j, \qquad (4.1.5)$$

where

$$f_j(t) = \frac{2\mu}{\varkappa+1}\,v_n(t) - \frac{1}{\varkappa+1}\int\limits_{t_j}^{t} T(t_0)\cos\big[\alpha(t)-\alpha(t_0)\big]\,ds_0,$$

$$j=0,1, \quad t_1 = A_1, \quad t_0 = A_{p+1}.$$

As is known [**52**], any doubly-connected domain is conformally mapped on a circular ring $r<|\zeta|<R$, where r can be chosen arbitrarily and R is defined uniquely for this domain.

Assume that the function $z=z(\zeta)$ maps conformally the domain S on the circular ring $D=\{1<|\zeta|<R\}$, the contour L_1 transforms to the circumference $|\zeta|=1$ and the contour L_0 to the circumference $|\zeta|=R$.

Denote by a_k the points of the boundary of the circle D which correspond to the points A_k of the boundary of the domain S, and by γ_k the arcs corresponding to the segments Γ_k, $k=1,2,\dots,p+q$.

Remark. The problem of finding the functions reduces to the homogeneous Riemann–Hilbert problem for the ring with piecewise-constant coefficients. The index of the problem $z=z(\zeta)$ is equal to zero and $\mu=1$.

As has been shown above, this problem has a unique nontrivial solution given by formula (1.7.27).

Let us introduce the notation

$$\varphi_1(z) = \varphi_1(z(\zeta)) = \varphi(\zeta), \quad \psi_1(z) = \psi_1(z(\psi)) = \psi(\zeta), \qquad (4.1.6)$$

Then, using (4.1.3) and (4.1.4), we obtain

$$\varphi(\zeta) = \varphi_0(\zeta) - \frac{X+iY}{2\pi(1+\varkappa)}\,\ln\zeta, \qquad (4.1.7)$$

$$\psi(\zeta)=\psi_0(\zeta)-\frac{(X-iY)\varkappa}{2\pi(1+\varkappa)}\,\ln\zeta, \tag{4.1.8}$$

where the functions $\varphi_0(\zeta)$ and $\psi_0(\zeta)$ are holomorphic in the ring D.

By the latter equalities the boundary condition (4.1.5) can be rewritten in the form

$$\begin{aligned}\varphi_0(\sigma)+e^{i\beta(\sigma)}\overline{\varphi_0(\sigma)}&=2e^{i\alpha(\sigma)}f_2(\sigma), \quad |\sigma|=R,\\ \varphi_0(\sigma)+e^{2i\alpha(\sigma)}\overline{\varphi_0(\sigma)}&=2e^{i\alpha(\sigma)}f_3(\sigma), \quad |\sigma|=1,\end{aligned} \tag{4.1.9}$$

where

$$\beta(\sigma)=2\alpha_{p+k}-2\pi(k-1), \quad \sigma\in\gamma_{p+k}, \quad k=1,2,\dots,q,$$

$$\begin{cases} f_2(\sigma)=f_0(t(\sigma))+\dfrac{c(\sigma)}{\varkappa+1} \\ \qquad +\operatorname{Re}\left[e^{-\alpha(\sigma)}\,\dfrac{X+iY}{2\pi(1+\varkappa)}\,\ln\sigma\right], & |\sigma|=R, \\ f_3(\sigma)=f_1(t(\sigma))+\dfrac{c(\sigma)}{\varkappa+1} \\ \qquad +\operatorname{Re}\left[e^{-\alpha(\sigma)}\left(\dfrac{X+iY}{2\pi(1+\varkappa)}\,\ln\sigma+A+iB\right)\right], & |\sigma|=1. \end{cases} \tag{4.1.10}$$

Since it is assumed that the displacement vector projection on the x- and y-axes is continuous and that $N+iT$ is integrable, by the Kolosov–Muskhelishvili formulas we obtain that the functions $\varphi_2(z)$ and $\overline{z}\varphi_2'(z)+\psi_2(z)$ are continuous in the closed domain $\overline{S}=S+L$. Hence it follows that a solution of problem (4.1.9) can be sought in the class of bounded functions, i.e. in the class $h(a_1;\dots,a_{p+q})$.

It can be easily shown that the index of problem (4.1.9) of the class $h(a_1;\dots,a_{p+q})$ is equal to $\varkappa=-q$, while by virtue of formulas (1.7.24) a solution of the class h_{p+q} has the form

$$\begin{aligned}\varphi_0(\zeta)&=\frac{g(\zeta)X(R\zeta)}{\pi i}\\ &\times\left[\int\limits_{\ell_0}\frac{K_\lambda\big(\frac{R^2\zeta}{\sigma}\big)f_2(\sigma)}{X(R\sigma)\sigma}\,d\sigma+\lambda\int\limits_{\ell_1}\frac{K_\lambda\big(\frac{\zeta}{\sigma}\big)f_3(\sigma)}{X(R\sigma)\sigma}\,d\sigma\right], \quad \zeta\in D,\end{aligned} \tag{4.1.11}$$

where $l=l_0\cup l_1$; l_0 and l_1 denote respectively the circumferences $|\sigma|=R$ and $|\sigma|=1$;

$$\begin{aligned}K_\lambda(z)&=\frac{R^2}{R^2-z}+\frac{1}{\lambda}\,\frac{1}{1-z}+\lambda\sum_{n\geq 1}\frac{1}{R^{2n}-\lambda}\,\Big(\frac{z}{R^2}\Big)^n\\ &+\frac{1}{\lambda}\sum_{n\leq -1}\frac{R^{2n}z^n}{R^{2n}-\lambda}+\begin{cases}\dfrac{\lambda}{1-\lambda} & \text{for } \lambda\neq 1,\\ 0 & \text{for } \lambda=1.\end{cases}\end{aligned} \tag{4.1.12}$$

The function $g(\zeta)$ defined by (1.8.5) can now be represented as

$$g(\zeta)=e^{-i\beta_0}\,\frac{(\zeta-a_1)(\zeta-a_2)}{a_1\zeta}\left(\frac{\zeta-a_1}{\zeta-a_p}\right)^{\frac{\alpha_p}{\pi}}\prod_{k=2}^{p-1}\left(\frac{\zeta-a_{k+1}}{\zeta-a_k}\right)^{\frac{\alpha_k}{\pi}},\quad |\zeta|>1.$$

Under the expression $\big[(\zeta-a_{k+1})/(\zeta-a_k)\big]^{\alpha_k/\pi}$ we mean a holomorphic branch on the plane cut along the arcs γ_k, for which

$$\lim_{\zeta\to\infty}\frac{\zeta-a_{k+1}}{\zeta-a_k}\to 1.$$

It is known that λ can be given in the following manner

$$\lambda=\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\ln G(e^{i\theta})\,d\theta\right),$$

where

$$G(\sigma)=\frac{e^{2i\alpha(\sigma)}\overline{g(R\sigma)}W_q^*(\sigma)}{g(R\sigma)W_q^*(R^2\sigma)}\,.$$

. In that case, instead of the function

$$W_q(\zeta)=(\zeta-Re^{i\alpha_0})\zeta^{[\frac{q}{2}]}e^{\frac{i\theta_0 q}{2}}$$

we can take the function

$$W_q^*(\zeta)=\prod_{k=1}^{q}(\zeta-\zeta_kR)^{-1}\zeta^{[\frac{q}{2}]}e^{\frac{i\theta_0 q}{2}},$$

where

$$\zeta_k=\exp i\left(\alpha_0+\frac{2\pi(k-1)}{q}\right),\quad k=1,2,\dots,q,\quad 0\le\theta_0<2\pi,$$

$$X(R\zeta)=e^{i\gamma_0}W_q^*(R\zeta)\exp\left[\frac{1}{2\pi i}\int\limits_{|\sigma|=1}K_1\left(\frac{R\zeta}{\sigma}\right)\ln\left(\frac{G(\sigma)}{\lambda}\right)\frac{d\sigma}{\sigma}\right],\tag{4.1.13}$$

$$\gamma_0=-\frac{1}{2\pi}\int_0^{2\pi}\arg\frac{G(\sigma)}{\lambda}\,d\theta,\quad \sigma=e^{i\theta}.$$

It can be easily verified that $W_q^*(\zeta)$ satisfies the condition

$$\overline{W_q^*\left(\frac{R^2}{\overline{\zeta}}\right)}=\begin{cases}W_q^*(\zeta) & \text{for even } q,\\ -\dfrac{\zeta}{R}\,W_q^*(\zeta) & \text{for odd } q.\end{cases}\tag{4.1.14}$$

As has been shown in § 1.8, the function $X(\tau)$ satisfies the condition

$$\overline{X\left(\frac{R^2}{\overline{\zeta}}\right)}=\begin{cases}X(\zeta) & \text{for even } q,\\ -\dfrac{\zeta}{R}\,X(\zeta) & \text{for odd } q.\end{cases}\tag{4.1.15}$$

With (4.1.14) taken into account, it can be easily shown that $|\lambda| = R$ for odd q and $|\lambda| = 1$ for even q and that, in the latter case, ζ_0 can be chosen so that $|\lambda| \neq 1$.

Since the function $W_q^*(R\zeta)$ has first order poles at the points $\zeta = \zeta_k$, $k = 1, 2, \dots, q$, for the function $\varphi_0(\zeta)$ to be bounded it is necessary and sufficient that the condition

$$\lim_{\zeta\to\zeta_k} \int\limits_{\ell_0} K_\lambda\left(\frac{R^2\zeta}{\sigma}\right) \frac{f_2(\sigma)}{X(R\sigma)g(\sigma)} \frac{d\sigma}{\sigma}$$

$$- \lim_{\zeta\to\zeta_k} \int\limits_{\ell_1} \frac{K_\lambda\left(\frac{\zeta}{\sigma}\right) f_3(\sigma)}{X(R\sigma)g(\sigma)\sigma}\, d\sigma = 0, \quad k = 1, \dots, q \qquad (4.1.16)$$

be fulfilled.

Since the expression $f^*(\sigma)/X(R\sigma)g(\sigma)$ vanishes at the points ζ_k, we can pass to the limit under the integral sign in formula (4.1.16) and thus write the solvability condition of problem (4.1.15) in the form

$$\int\limits_{\ell_0} K_\lambda\left(\frac{R^2\zeta_k}{\sigma}\right) \frac{f_2(\sigma)}{X(R\sigma)g(\sigma)} \frac{d\sigma}{\sigma} + \int\limits_{\ell_1} \frac{K_\lambda\left(\frac{\zeta_k}{\sigma}\right) f_3(\sigma)}{X(R\sigma)g(\sigma)\sigma}\, d\sigma = 0, \qquad (4.1.17)$$

$$k = 0, 1, \dots, q.$$

Substituting here the value $f^*(\sigma)$ defined by formula (4.1.10), we obtain a system of linear algebraic equations

$$\sum_{j=1}^{q+p} a_{kj} X_j = d_k, \quad k = 1, 2, \dots, q, \qquad (4.1.18)$$

where a_{kj} are well-defined constants independent of the functions $f_0(z(\sigma))$ and $f_1(z(\sigma))$, and d_k are constants depending on these functions:

$$d_k = \frac{1}{2\pi i}\left[- \int\limits_{|\sigma|=1} K_\lambda\left(\frac{R^2\zeta_k}{\sigma}\right) \frac{f_1(t(\sigma))e^{i\alpha(\sigma)}}{X(R^2(\sigma))g(\sigma)} \frac{d\sigma}{\sigma}\right.$$

$$\left. + \int\limits_{|\sigma|=R} K_\lambda\left(\frac{R^2\zeta_k}{\sigma}\right) \frac{f_0(t(\sigma))e^{i\alpha(\sigma)}}{X(R^2\sigma)g(\sigma)} \frac{d\sigma}{\sigma}\right],$$

$$X_j = C_j \text{ for } j = 2, 3, \dots, p+q-1 \text{ and } j \neq p, p+1,$$

$$X_1 = X, \quad X_p = Y, \quad X_{p+1} = A, \quad X_{p+q} = B.$$

Let us now show that a_{kj} and d_k are real numbers. For this it suffices to show that for real $f^*(\sigma)$ the left-hand part of equality (4.1.17) takes a real value or it becomes real if multiplied by some complex numbers.

We rewrite the left-hand part of equality (4.1.17) as follows

$$\frac{1}{2\pi i}\int\limits_{|\sigma|=1} K_\lambda\left(\frac{R\zeta_k}{\sigma}\right)\frac{f_2(\sigma)e^{i\alpha(R\sigma)}}{X(R^2\sigma)g(\sigma)}\,\frac{d\sigma}{\sigma}$$
$$-\frac{1}{2\pi i}\int\limits_{|\sigma|=1} K_\lambda\left(\frac{R^2\zeta_k}{\sigma}\right)\frac{f_3(\sigma)}{X(R\sigma)g(\sigma)}\,\frac{d\sigma}{\sigma}=D(\zeta_k).$$

Passing in this expression to the conjugate values and then using the following properties of the functions $K_\lambda(\zeta)$, $X(\zeta)$ and $g(\zeta)$, we obtain

$$\overline{K_\lambda\left(\frac{R^2\zeta_k}{\sigma}\right)}=-\begin{cases}K_\lambda\left(\dfrac{R^2\zeta_k}{\sigma}\right) & \text{for even } q,\\[2mm] \dfrac{\sigma}{\zeta_k}\,K_\lambda\left(\dfrac{R^2\zeta_k}{\sigma}\right) & \text{for odd } q,\end{cases}$$

$$\overline{K_\lambda\left(\frac{R\zeta_k}{\sigma}\right)}=-\begin{cases}\lambda K_\lambda\left(\dfrac{R\zeta_k}{\sigma}\right) & \text{for even } q,\\[2mm] \lambda\,\dfrac{\sigma}{R\zeta_k}\,K_\lambda\left(\dfrac{R\zeta_k}{\sigma}\right) & \text{for odd } q,\end{cases}$$

$$\overline{X(R\sigma)}=-\begin{cases}X(R\sigma) & \text{for even } q,\\ \sigma X(R\sigma) & \text{for odd } q,\end{cases}$$

$$\overline{X(R^2\sigma)}=-\begin{cases}X(\sigma) & \text{for even } q,\\ -\dfrac{\sigma}{R}\,X(\sigma) & \text{for odd } q,\end{cases}$$

$$X(R^2\sigma)=X(\sigma)\lambda e^{2i\alpha}\,\frac{\overline{g(R\sigma)}}{g(\sigma)}\quad\text{and}\quad g(\sigma)=e^{2i\alpha}\overline{g(\sigma)},$$

we have

$$\overline{D(\zeta_k)}=\begin{cases}D(\zeta_k) & \text{for even } q,\\ \zeta_k D(\zeta_k) & \text{for odd } q.\end{cases}$$

Thus, when q is even, (4.1.18) is a system with real coefficients, and when q is odd, system (4.1.18) can be made such if we multiply it by $\zeta_k^{1/2}$.

In the sequel, we will show that there exists a unique value of the constants X_k, $k=1,2,\dots,p+q$, that satisfies system (4.1.18) or conditions (4.1.17).

The function $K_1(\zeta/\sigma)$ can be written in the form

$$K_1\left(\frac{R\zeta}{\sigma}\right)=\frac{R\sigma}{R\sigma-\zeta}+\frac{\sigma}{\sigma-\zeta}+K_1^0\left(\frac{\zeta}{\sigma}\right),\ \ 1<|\zeta|<R,$$

where $K_1^0(\zeta/\sigma)$ is analytic in the ring $1/R<|\zeta|<R^2$. Therefore $X(R\zeta)$ is analytic in the ring $1/R<|\zeta|<R$ except for the points $\zeta=\zeta_k$, continuously extendable on the boundary $|\zeta|=R$, the boundary value $X(R_\sigma)$ satisfies

the Hölder condition [**76**] on the circumference $|\sigma| = R$ and vanishes at the points a_k by order less than one.

Let us write the function $K_\lambda(\zeta/\sigma)$ in the form

$$K_\lambda\left(\frac{\zeta}{\sigma}\right) = \frac{\sigma}{\sigma-\zeta} + \frac{R^2\sigma}{R^2\sigma-\zeta} + K_\lambda^0\left(\frac{\zeta}{\sigma}\right), \quad 1 < |\zeta| < R,$$

where $K_\lambda^0(\zeta/\sigma)$ is analytic in the ring $1/R_\lambda < |\zeta| < R^2$ for $\sigma \in l$. Then $\varphi_0(\zeta)$ can be represented as

$$\varphi_0(\zeta) = \frac{2g(\zeta)X(R\zeta)}{2\pi i}\left[\int\limits_l \frac{f^*(\sigma)e^{i\alpha(\sigma)}}{X(R\sigma)g(\sigma)(\sigma-\zeta)}\,d\sigma + \int\limits_l \frac{f^*(\sigma)K_\lambda^0\big(\frac{R^2\zeta}{\sigma}\big)e^{i\alpha(\sigma)}}{X(R\sigma)g(\sigma)\sigma}\,d\sigma\right]. \tag{4.1.19}$$

The second summand in the right-hand part of equality (4.1.19) is holomorphic in the ring $1 < |\zeta| < R$ and continuous in the closed ring $\overline{D}$. The first summand is a Cauchy type integral whose density satisfied the Hölder condition on every open arc γ_k. Therefore by the Plemelj–Privalov theorem [**76**] the function $\varphi_0(\zeta)$ is continuously extendable on the open arcs γ_k and its boundary value satisfies on these arcs the Hölder condition. If we now use the results of [**76**, § 26], then we will satisfy ourselves that $\varphi_0(\zeta)$ is continuously extendable on the whole boundary and its boundary value satisfies on it the Hölder condition.

Since the function $z = z(\zeta)$ is continuous in the closed ring $\overline{D}$ and its boundary value satisfies the Hölder condition, the function $\varphi_2(z)$, too, satisfies this condition in the closed domain $\overline{D}$.

Let us now study the behavior of a derivative of the function $\varphi_0(\zeta)$ near the ends a_k. It suffices to consider only the end $\zeta = a_1$.

We rewrite equality (4.1.19) as follows

$$\varphi_0(\zeta) = X(R\zeta)\varphi_3(\zeta) + g(\zeta)\varphi_4(\zeta), \tag{4.1.20}$$

where

$$\varphi_3(\zeta) = \frac{g(\zeta)}{2\pi i}\int\limits_l \frac{f^*(\sigma)e^{i\alpha(\sigma)}}{X(R\sigma)g(\sigma)(\sigma-\zeta)}\,d\sigma,$$

$$\varphi_4(\zeta) = \frac{X(R\sigma)}{2\pi i}\int\limits_l \frac{f^*(\sigma)K_\lambda^0\big(\frac{R^2\zeta}{\sigma}\big)e^{i\alpha(\sigma)}}{X(R\sigma)g(\sigma)}\,\frac{d\sigma}{\sigma}\,.$$

Near $\zeta = a_1$, the derivative of the function $g(\zeta)$ is representable as

$$g(\zeta) = \frac{g(\zeta)}{\zeta - a_1}\left[1 + \frac{\alpha_p}{\pi}\,\frac{a_p - a_1}{\zeta - a_p}\right.$$
$$\left. + (\zeta - a_1)\sum_{r=2}^{\rho-1}\frac{\alpha_r(\alpha_r - \alpha_{r+1})}{(\zeta - a_r)(\zeta - a_{r+1})} - \frac{\zeta - a_1}{\zeta}\right].$$

Since the function $\varphi_k(\zeta)$ is analytic near the point a_1, after differentiating equality (4.1.20) we satisfy ourselves that near the point $\varphi_0'(\zeta)$ the derivative a_1 is representable as

$$\varphi_0'(\zeta) = X(R\zeta)\varphi_3'(\zeta) + M(\zeta)\left(\frac{\zeta - a_p}{\zeta - a_1}\right)^{-\frac{\alpha_p}{\pi}}, \tag{4.1.21}$$

where $M(\zeta)$ is an analytic function in the neighborhood of the point $\zeta = a_1$ and bounded at the point a_1. In the sequel, the function possessing the above-mentioned properties will be denoted by $M(\zeta)$.

Let us introduce the notation

$$\phi(\zeta) = (\zeta - a_1)\varphi_3(\zeta). \tag{4.1.22}$$

We can rewrite $\phi(\zeta)$ as follows

$$\phi(\zeta) = \frac{g(\zeta)}{2\pi i}\int\limits_l \frac{F(\sigma)(\sigma - a_1)}{(\sigma - \zeta)g(\sigma)}\,d\sigma - \frac{g(\zeta)}{2\pi i}\int\limits_l \frac{F(\sigma)}{g(\sigma)}\,d\sigma. \tag{4.1.23}$$

where

$$F(\sigma) = \frac{f^*(\sigma)e^{i\alpha(\sigma)}}{X(R\sigma)}\,.$$

Differentiating equalities (4.1.22) and applying the results of [**68**], we obtain

$$\phi'(\zeta) - \varphi_3(\zeta) = M(\zeta)g(\zeta),$$
$$\varphi_3'(\zeta) = \frac{\phi'(\zeta) - \varphi_3(\zeta)}{\zeta - a_1} = \frac{M(\zeta)g(\zeta)}{\zeta - a_1}\,.$$

Therefore in the neighborhood of the point $\varphi_0'(\zeta)$ the derivative $\zeta = a_1$ is representable as

$$\varphi_0'(\zeta) = M(\zeta)\left(\frac{\zeta - a_p}{\zeta - a_1}\right)^{-\frac{\alpha_p}{\pi}}.$$

If we now use the equality

$$\varphi_1'(\zeta) = \frac{\varphi'(\zeta)}{\omega'(\zeta)} = \frac{\varphi_0'(\zeta)}{\omega'(\zeta)} - \frac{X + iY}{2\pi(1 + \varkappa)\omega'(\zeta)}$$

take into consideration the fact that in the neighborhood of the point a_1, $\omega'(\zeta)$ can be represented as

$$\omega'(\zeta) = \omega_0(\zeta)(\zeta - a_1)^{1 + \frac{\alpha_p}{\pi}},$$

where $\omega_0(\zeta)$ is bounded in the neighborhood of a_1 and $\omega_0(a_1) \neq 0$, then we have

$$\varphi_1'(\zeta) = \frac{\Omega(\zeta)}{\zeta - a_1}\,. \tag{4.1.24}$$

As is known, near the point A_1 the inverse function $z = z(\zeta)$ of $\zeta = \zeta(z)$ is written in the form

$$\zeta - a_1 = \Omega(z)(z - A_1)^{\frac{\pi}{\delta_1}},$$

where $\Omega(z)$ is a nonzero function bounded near the point $z = A_1$, δ_1 is the value of the angle with the vertex at the point A_1.

From the above reasoning it follows that in the neighborhood of the point A_1, $\varphi_1'(z)$ satisfies the condition

$$|\varphi_1'(z)| < \frac{M}{|z - A_1|^{\pi/\delta_1}}\,.$$

It is obvious that this representation holds for any point A_k, $k = 1, 2, \dots, p$. Therefore, in the neighborhood of the point we have the inequality

$$|\varphi_1'(z)| < \frac{M}{|z - A_k|^{\pi/\delta_k}}\,,$$

where $\delta_k > \pi$ is the value of the angle with vertex at the point A_k.

By an analogous reasoning it can be shown that the representation

$$|\varphi_1'(z)| < M \ln |z - A_k|$$

holds near the points A_k, $k = p + 1, \dots, p + q$.

Using the Plemelj–Privalov theorem we prove that the boundary values of the function $\varphi_1'(z)$ belong to the class H^* on L_1, and to the class H_ε on L_0.

If we assume that the second derivative of the function $f_j(t)$, $j = 0, 1$, satisfies the condition H_0, then by a reasoning analogous to that above it can be shown that the boundary values of the function $\varphi_1''(z)$ belong to the class H on the interior segments Γ_k, whereas near the ends of the segments they satisfy the condition

$$|\varphi_1''(z)| < \frac{M}{|z - A_k|^{\delta}}\,, \quad 1 \le \delta \le 2.$$

Let us now define the function $\psi_1(z)$. Substituting (4.1.11) into condition (4.1.1), we obtain the Riemann–Hilbert problem for the function $\psi_1(z)$ whose right-hand part is unbounded near the points A_k, $k = 1, \dots, p + q$. Hence the solution of the obtained problem should be sought in the class of unbounded functions. Using formulas (1.8.21) by means of which we considered the Riemann–Hilbert problem in the case of the bounded right-hand part, we can investigate the problem in the considered case, too, but this will put us into a difficult position, especially when proving the continuity of the expression $\overline{z}\varphi_1'(z) + \psi(z)$ in a closed domain. This difficulty can be overcome by reducing the considered problem to a problem with the bounded right-hand part.

Let us rewrite the boundary condition (4.1.1) as follows

$$\operatorname{Re} e^{i\alpha}\big[P(t)\varphi_1'(t)+\psi_1(t)\big] = -2\mu v_n + \operatorname{Re} e^{i\alpha}\Big[(\bar t - P(t))\varphi_1'(t) - \varkappa\varphi_1(t)\Big], \quad (4.1.25)$$

where $P(t)$ is the interpolation polynomial satisfying the conditions

$$P(A_k)=\overline{A}_k, \quad k=1,2,\dots,p+q.$$

We have thus reduced the problem to the case we have studied above, i.e. when the right-hand part in the boundary condition (4.1.25) is bounded, and a solution is sought in the class of bounded functions. The index of problem (4.1.25) of the class h_{p+q} is equal to $-p$, and the solution can be constructed analogously to the preceding one, the solvability condition having form (4.1.17). This condition is a system of linear algebraic equations with real coefficients of form (4.1.18):

$$\sum_{j=1}^{p+q} b_{kj}X_j = d_k^{(2)}, \quad k=1,2,\dots,p, \qquad (4.1.18')$$

where b_{kj}, $k=1,2,\dots,p$, $j=1,\dots,p+q$, are the known constants not depending on the functions v_n and T, $d_k^{(2)}$ are also the known constants which can be expressed through the functions v_n and T and which vanish for $v_n=T=0$. X_j has the same meaning as in equation (4.1.18).

Thus (4.1.18)–(4.1.18′) is a system consisting of $p+q$ equations with respect to the unknowns X_j. This system can be represented in the form

$$\sum_{j=1}^{p+q} K_{sj}X_j = d_s, \quad s=1,2,\dots,p+q, \qquad (4.1.26)$$

where

$$K_{sj}=\begin{cases} a_{sj} & \text{for } s=1,2,\dots,q;\ \ j=1,2,\dots,p+q,\\ b_{s-q} & \text{for } s=q+1,\dots,p+q;\ \ j=1,2,\dots,p+q,\end{cases}$$

$$d_s=\begin{cases} d_s^{(1)} & \text{for } s=1,2,\dots,p,\\ d_{s-q}^{(2)} & \text{for } s=q+1,\dots,q+p.\end{cases}$$

Let us now show that the determinant of system (4.1.26) differs from zero. Indeed, assume $T=v_n=0$, then all $d_s=0$ and system (4.1.26) becomes homogeneous. If the determinant of the latter system is equal to zero, then the homogeneous system will have nontrivial solutions. Assume that X_j^0, $j=1,\dots,p+q$, is one of the solutions, then the problem will have a solution which we denote by $\varphi_0(z)$ and $\psi_0(z)$. These functions satisfy the conditions

$$\begin{gathered}\operatorname{Re}\big[e^{-i\alpha}\varphi_0(t)\big]=\frac{C^0(t)}{\varkappa+1}+\operatorname{Re}\big[e^{-i\alpha_j}C_j\big],\ \ t\in L_j,\ \ j=1,2,\\ \operatorname{Re}\Big[\big(\varkappa\varphi_0(t)-t\overline{\varphi_0'(t)}-\overline{\psi_0(t)}\big)e^{-i\alpha}\Big]=0,\ \ t\in L.\end{gathered}\tag{4.1.27}$$

Let S_ε be a doubly connected domain bounded by the broken lines $L_0^{(\varepsilon)}$ and $L_1^{(\varepsilon)}$ and lying in the domain S. Assume that the sides of the broken lines $L_0^{(\varepsilon)}$ and $L_1^{(\varepsilon)}$ are parallel to the sides of L_0 and L_1, respectively, and lie from them at a sufficiently small distance ε.

Consider the integral

$$\begin{aligned}J_\varepsilon=\frac{1}{2\mu}\int\limits_{L_\varepsilon}&\operatorname{Re}\Big\{2\varphi_0'(t)-e^{2i\alpha}\big[\bar t\varphi_0''(t)+\psi_0'(t)\big]\Big\}\\ &\times\operatorname{Re}\Big\{\Big[\varkappa\varphi_0(t)-t\overline{\varphi_0'(t)}-\overline{\psi_0(t)}\Big]e^{-i\alpha}\Big\}\,ds\\ +\frac{1}{2\mu}\int\limits_{L_\varepsilon}&\operatorname{Im}\Big\{-e^{2i\alpha}\big[\bar t\varphi_0''(t)+\psi_0'(t)\big]\Big\}\\ &\times\operatorname{Im}\Big\{\Big[\varkappa\varphi_0(t)-t\overline{\varphi_0'(t)}-\overline{\psi(t)}\Big]e^{-i\alpha}\Big\}\,ds,\end{aligned}\tag{4.1.28}$$

or

$$\begin{aligned}J_\varepsilon&=\frac{1}{2\mu}\operatorname{Re}\int\limits_{L_\varepsilon}\Big[\big(\varphi_0'(t)+\overline{\varphi_0'(t)}\big)e^{-i\alpha}-e^{i\alpha}\big(\bar t\varphi_0''(t)+\psi_0'(t)\big)\Big]\\ &\qquad\times\Big[\varkappa\varphi_0(t)-t\overline{\varphi_0'(t)}-\overline{\psi_0(t)}\Big]\,ds\\ &=\frac{1}{2\mu}\operatorname{Re}\Big\{\int\limits_{L_\varepsilon}\Big[\varphi_0'(t)+\overline{\varphi_0'(t)}-\bar t\varphi_0''(t)-\psi_0'(t)\Big]\\ &\qquad\times\Big[\varkappa\varphi_0(t)-t\overline{\varphi_0'(t)}-\overline{\psi(t)}\Big]\cos\alpha\,ds\\ &\quad-i\int\limits_{L_\varepsilon}\Big[\varphi_0'(t)+\overline{\varphi_0'(t)}+\bar t\varphi_0''(t)+\psi_0'(t)\Big]\\ &\qquad\times\Big[\varkappa\varphi_0(t)-t\overline{\varphi_0'(t)}-\overline{\psi_0(t)}\Big]\sin\alpha\,ds\Big\}.\end{aligned}$$

Using Green's formula, we obtain

$$J_\varepsilon=2(\varkappa-1)\iint\limits_{S_\varepsilon}\big[\operatorname{Re}\varphi_0'(z)\big]^2\,dx\,dy+\iint\limits_{S_\varepsilon}\big|\overline{z}\varphi_0''(z)+\psi_0'(t)\big|^2\,dx\,dy.\tag{4.1.29}$$

Since $\varkappa\varphi_0(z)-z\overline{\varphi_0'(z)}-\overline{\psi_0(z)}$ is continuously extendable in the closed domain $\overline{S}$, and $\varphi_0'(z)$ and $\overline{z}\varphi_0''(z)+\psi_0'(t)$ are continuously extendable all over the boundary L except perhaps for the points A_k at which they may reduce to infinity of order less than one, we may pass in expression (4.1.29) to the limit for $\varepsilon \to 0$. Since for $\varepsilon \to 0$ we have $L_\varepsilon \to L$, $S_\varepsilon \to S$

$$\operatorname{Re} e^{-i\alpha}\Big[\varkappa\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)}\Big] \longrightarrow 0,$$

$$\operatorname{Im} e^{2i\alpha}\Big[\overline{t}\varphi_0''(t) + \psi'(t)\Big] \longrightarrow T \longrightarrow 0,$$

by passing to the limit as $\varepsilon \to 0$ we obtain from (4.1.29) that

$$2(\varkappa-1)\iint\limits_D \big[\operatorname{Re}\varphi_0'(z)\big]^2\,dx\,dy + \iint\limits_D \big[\overline{z}\varphi_0''(z) + \psi_0'(z)\big|^2\,dx\,dy = 0.$$

This implies

$$\varphi_0'(t) = ic,\ \ \varphi_0(z) = icz + D_1,\ \ \psi_0(z) = D_2, \tag{4.1.30}$$

where c is a real constant, and D_1 anD_2 are complex constants. The substitution of values (4.1.30) into the second condition (4.1.27) gives

$$\operatorname{Re} e^{-i\alpha}\Big[(\varkappa+1)ict + \varkappa D_1 - \overline{D}_2\Big] = 0. \tag{4.1.31}$$

When $t \in \Gamma_1$, we have $\alpha(t) = \pi$, $t \in x_1 + iy = \operatorname{Re} A_1 + iy$, $\operatorname{Im} A_1 < y < \operatorname{Im} A_2$ and condition (4.1.31) takes the form

$$\operatorname{Re}\Big[(\varkappa+1)cy - \varkappa D_1 + \overline{D}_2\Big] = (\varkappa+1)yc - \operatorname{Re}[\varkappa D_1 - D_2] = 0.$$

Hence it follows that $c = 0$, $\operatorname{Re}[\varkappa D_1 - D_2] = 0$.

When $t \in \Gamma_2$, condition (4.1.31) takes the form

$$\operatorname{Re} e^{-i\alpha_2}[\varkappa D_1 - \overline{D}_2] = \cos\alpha_2 \operatorname{Re}(\varkappa D_1 - \overline{D}_2) + \sin\alpha_2 \operatorname{Im}(\varkappa D_1 - \overline{D}_2) = 0.$$

Since $\alpha_2 \neq \pi, 2\pi$, we obtain

$$\operatorname{Im}(\varkappa \cdot D_1 - \overline{D}_2) = 0,\ \ \varkappa D_1 - \overline{D}_2 = 0.$$

Thus we have

$$\varphi_0(z) = 0,\ \ \psi_0(z) = \varkappa\overline{D}_1,$$

and from representation (4.1.3) it follows that $X = Y = 0$, which by virtue of $(*)$ and $(**)$ implies $c_p = c_{p+q} = 0$.

From the first condition (4.1.27) we obtain

$$\operatorname{Re} e^{-i\alpha(t)} D_1 = \frac{C(t)}{\varkappa+1}\,,\ \ t \in L_0 \tag{4.1.32}$$

whence it follows that

$$\begin{cases}\operatorname{Re} e^{-i\alpha_{p+1}} D_1 = 0,\\ \operatorname{Re} e^{-i\alpha_{p+q}} D_1 = 0.\end{cases}$$

The determinant of this system $\sin(\alpha_{p+q} - \alpha_{p+1}) \neq 0$; therefore $D_1 = 0$ and from equality (4.1.32) we obtain

$$C_{p+1} = C_{p+2} = C_{p+q} = 0.$$

Since $C_1 = C_{p+1} = 0$, from the first equality (4.1.27) for $t \in L_0$ we analogously obtain

$$A = B = C_1 = C_2 = \cdots = C_{p+1} = 0.$$

Thus the determinant of system (4.1.26) is different from zero and therefore the problem has a unique solution.

4.2. Defining a Hole of Uniform Strength in a Polygonal Plate

In this paragraph we investigate the problem of finding a hole with a uniformly strong boundary in a finite plate.

Let us consider an isotropic and homogeneous plate shaped as a convex polygon weakened by a curvilinear hole. Assume that the normal displacement u_n on each side of the polygon has a constant value, the tangent stress on the external boundary of the plate is equal to zero, while the internal boundary is under the action of the constant normal force and the tangent stress is equal to zero. We can consider two cases where 1) the values of the constant u_n are given, and 2) the values of the principal vector are given on either side of the external boundary of the plate.

The mechanical meaning of the first case consists in the following: an elastic washer is inserted into the hole of polygonal configuration made in a fixed rigid body. Prior to deformation the shape of the washer contour differs but little from the shape of the hole. In the second case it is assumed that the dies with rectilinear bases adjoin the sides of the plate.

We pose the following problem: find a stressed state of the body and the boundary of the hole assuming that the boundary of the hole is uniformly strong. Let on the plane of the complex variable $z = x + iy$ the plate occupy the domain S bounded by the closed convex broken line $A_1 A_2 \cdots A_n$ which we denote by L_1 and by the smooth closed contour L_2 lying inside L_1. To simplify the notation, the affixes of the points A_k, $k = 1, 2, \dots, n$, which are the vertices of the broken line are denoted by the same symbols.

It is also assumed that the point $z = 0$ lies within the sought contour L_2.

We make use of the following formulas [**77**]

$$\varkappa\varphi(z) - \overline{z\varphi'(z)} - \overline{\psi(z)} = 2\mu(u + iv), \tag{4.2.1}$$

$$\varphi(z) + \overline{z\varphi'(z)} + \overline{\psi(z)} = i \int\limits_{z_0}^{z} (X_n + iY_n)\, ds + const, \tag{4.2.2}$$

$$X_x + Y_y = 4 \operatorname{Re} \varphi'(z), \tag{4.2.3}$$

where $\varphi(z)$, $\psi(z)$ are holomorphic functions in the domain S occupied by the body; u, v are the displacement components on the coordinate axes; X_x, Y_y are stress components. The integral in formula (4.2.2) is taken over any smooth arc l that lies within S and connects an arbitrarily fixed point z_0 with a variable point z of the domain S; X_n and Y_n denote the component of stress acting on the arc l from the side of the normal directed to the right relative to the direction on l leading from z_0 to z.

By the Kolosov–Muskhelishvili formulas (4.2.1), (4.2.2) for two sought holomorphic functions φ and ψ in the domain S we obtain the boundary condition

$$\operatorname{Re}\Big[e^{-i\alpha(t)}\big(\varkappa\varphi(t)-\overline{t\varphi'(t)}-\overline{\psi(t)}\big)\Big]=2\mu u_n \ \text{ on } L_1, \tag{4.2.4}$$

$$\operatorname{Re}\Big[e^{-i\alpha(t)}\big(\varphi(t)+\overline{t\varphi'(t)}+\overline{\psi(t)}\big)\Big]=C(t) \ \text{ on } L_1, \tag{4.2.5}$$

where $\alpha(t)$ is the angle formed by the normal to L_1 at the point t with the ox-axis,

$$C(t)=\operatorname{Re}\left[i\int\limits_0^s N(t_0)e^{i(\alpha(t_0)-\alpha(t))}\,ds_0\right],$$

$N(t)$ is the normal stress to L_1 at the point t, s is the arc abscissa at the point t counted from the point A_1 in the positive direction.

Taking into account that $\alpha(t)$ is a piecewise-constant function, we obtain

$$C(t)=\sum_{j=1}^{k}\sin(\alpha_k-\alpha_j)\int\limits_{s_j}^{s_{j+1}}N(t_0)\,ds_0$$

$$\text{for } \ t\in A_lA_{k+1} \ \ k=1,2,\dots,n; \ \ A_{n+1}=A_1,$$

where α_k's are the values of the function $\alpha(t)$ on A_kA_{k+1}, $k=1,2,\dots,n$; s_j is the arc abscissa of the point A_j, i.e. the length of the broken line $A_1A_2\cdots A_j$. It is obvious that $C(t)$ is also a piecewise-constant function.

For the functions φ and ψ, from formulas (4.2.2), (4.2.3) we obtain the condition on L_2

$$\varphi(t)+\overline{t\varphi'(t)}+\overline{\psi(t)}=B_1+iB_2 \ \text{ on } L_2, \tag{4.2.6}$$

$$4\operatorname{Re}\varphi'(t)=\sigma_t=K \ \text{ on } L_2, \tag{4.2.7}$$

where B_1, B_2, K are real constants.

Since in the first case u_n is a given piecewise-constant function, and in the second case $C(t)$ is also a given piecewise-constant function, by virtue of formulas (4.2.4) and (4.2.5) both cases reduce to identical problems of the analytic function theory. We will consider the second case where the values of the principal vector of external stress are given on the segments A_kA_{k+1}

$$P_k=\int\limits_{s_k}^{s_{k+1}}N(s)\,ds, \ \ k=1,2,\dots,n.$$

From the equilibrium condition we have

$$\sum_{k=1}^{n}P_ke^{i\alpha_k}=0. \tag{4.2.8}$$

Using formulas (4.2.1)–(4.2.3) and applying physical argumentation we conclude that the function $\varphi(z)$ is continuous in the closed domain S, whereas $\varphi'(z)$ and $\psi(z)$ are continuously extendable on the domain boundary except perhaps for the points A_k, $k = 1, 2, \dots, n$, near which they admit an estimate of the form

$$|\varphi'(z)|, \ \ |\psi(z)| < M|z - A_k|^{-\delta}, \ \ 0 \le \delta < 1.$$

Taking into account the fact that L_1 is a broken line, by the summation of formulas (4.2.4) and (4.2.5) and the next differentiation we obtain $\operatorname{Im} \varphi'(t) = 0$ on L_1. The latter equality and condition define uniquely $\varphi'(z) = \frac{K}{4}$, whence, neglecting the constant summand which does not influence the stressed state of the body, we obtain

$$\varphi(z) = \frac{1}{4}\, Kz. \tag{4.2.9}$$

Thus the boundary conditions (4.2.5), (4.2.6) take the form

$$\operatorname{Re}\left[e^{-i\alpha(t)}\left(\frac{K}{2}\, t + \overline{\psi(t)}\right)\right] = C(t) \ \text{ on } L_1, \tag{4.2.10}$$

$$\overline{\psi(t)} + \frac{K}{2}\, t = B \ \text{ on } L_2, \quad B = B_1 + iB_2. \tag{4.2.11}$$

If $t \in A_k A_{k+1}$, then

$$(t - A_k) = i\rho e^{i\alpha_k}, \ \ \rho = |t - A_k|,$$

whence

$$\operatorname{Re}(te^{-i\alpha(t)}) = \operatorname{Re}\left(A(t)e^{-i\alpha(t)}\right), \ \ t \in L_1, \tag{4.2.12}$$

where $A(t) = A_k$ for $t \in A_k A_{k+1}$, $k = 1, 2, \dots, n$.

Let the function $z = \omega(\zeta)$ conformally map the circular ring $1 < |\zeta| < R$ onto the domain S, where R is the unknown number to be determined. Assume that the circumference $|\zeta| = R$ is mapped onto L_1. Assume that to the vertices $A_1, A_2, \dots, A_n$ there correspond the points $a_1, a_2, \dots, a_n$ from the circumference $|\zeta| = R$. Let $a_k = Re^{i\delta_k}$, $k = 1, 2, \dots, n$, where δ_k are unknown numbers. Assume that $0 = \delta_1 < \delta_2 < \dots < \delta_n < 2\pi$. From conditions (4.2.10)–(4.2.12) we have

$$\operatorname{Re}\left[e^{-i\alpha(\sigma)}\left(\frac{1}{2}\,\omega(\sigma) + \overline{\psi_0(\sigma)}\right)\right] = C(\sigma), \ \ |\sigma| = R, \tag{4.2.13}$$

$$\overline{\psi_0(\sigma)} + \frac{1}{2}\, K\omega(\sigma) = B, \ \ |\sigma| = 1, \tag{4.2.14}$$

$$\operatorname{Re}\left[e^{-i\alpha(\sigma)}\omega(\sigma)\right] = \operatorname{Re}\left[e^{-i\alpha(\sigma)}A(\sigma)\right], \ \ |\sigma| = R, \tag{4.2.15}$$

where $\psi_0(\zeta) = \psi[\omega(\zeta)]$, $1 < |\zeta| < R$. For the sake of simplicity we rite $\alpha(\sigma)$, $A(\sigma)$, $C(\sigma)$ instead of $\alpha[\omega(\sigma)]$, $A[\omega(\sigma)]$, $C[\omega(\sigma)]$, respectively. These functions are defined all over the plane by the equalities

$$\alpha(r\sigma) = \alpha(\sigma), \quad A(r\sigma) = A(\sigma), \quad C(r\sigma) = C(\sigma), \ \ 0 < r < \infty, \ \ |\sigma| = 1.$$

Let $W(\zeta)$ be the function defined by the equalities

$$W(\zeta)=\begin{cases}\dfrac{1}{2}\,K\omega\left(\dfrac{\zeta}{R}\right) & \text{for } R<|\zeta|<R^2,\\[2ex] B-\overline{\psi_0\left(\dfrac{R}{\zeta}\right)} & \text{for } 1<|\zeta|<R.\end{cases}\tag{4.2.16}$$

It is obvious that $W(\zeta)$ is a holomorphic function in domains $1<|\zeta|<R$ and $R<|\zeta|<R^2$. By virtue of condition (4.2.14), on the circumference $W(\zeta)$ the boundary values of $|\zeta|=R$, are equal to each other from the inside and outside. Therefore $W(\zeta)$ is holomorphic in the ring $1<|\zeta|<R^2$.

From (4.2.16) we have

$$\begin{aligned}\frac{1}{2}\,K\omega(R\sigma)&=W(R^2\sigma) \ \text{ for } \ |\sigma|=1,\\ \overline{\psi_0(R\sigma)}&=B-W(\sigma) \ \text{ for } \ |\sigma|=1.\end{aligned}$$

The substitution of the values into conditions (4.2.13), (4.2.15) gives

$$\operatorname{Re}\left[e^{-i\alpha(\sigma)}W(\sigma)\right]=f(\sigma), \quad \sigma\in\Gamma,\tag{4.2.17}$$

where

$$\Gamma=\Gamma_1\cup\Gamma_2, \ \ \Gamma_1=\{\sigma: \ |\sigma|=R^2\}, \ \ \Gamma_2=\{\sigma:|\sigma|=1\},$$

$$f(\sigma)=\begin{cases}\dfrac{1}{2}\,K\operatorname{Re}\left[e^{-i\alpha(\sigma)}A(\sigma)\right], & \sigma\in\Gamma_1,\\[2ex] \operatorname{Re}Be^{-i\alpha(\sigma)}-C(\sigma)+\dfrac{1}{2}\,K\operatorname{Re}\left[e^{-i\alpha(\sigma)}A(\sigma)\right], & \sigma\in\Gamma_2.\end{cases}\tag{4.2.18}$$

We have thus reduced the posed problem to the Riemann–Hilbert problem for the circular ring with piecewise-constant coefficients. All discontinuity points are nonsingular (see [**76**, p. 256]).

Since the function $W(\zeta)$ must be bounded on the domain boundary, a solution of problem (4.2.17) should be sought in the class of functions bounded on the boundary, i.e. in the class h_{2n} (see [**76**, p. 256]).

The coefficient index of problem (4.2.17) corresponding to this class is equal to $-n+2$ on Γ_1, and to -2 on Γ_2. Therefore the index of the Riemann–Hilbert problem (4.2.17) corresponding to the class h_{2n} is equal to $-n$.

Let us represent the boundary condition (4.2.17) in the form

$$\begin{aligned}W(\sigma)+e^{2i\alpha(\sigma)}\overline{W(\sigma)}&=2f_1(\sigma)e^{i\alpha(\sigma)} \ \text{ on } \Gamma_1,\\ W(\sigma)+e^{2i\alpha(\sigma)}\overline{W(\sigma)}&=2f_2(\sigma)e^{i\alpha(\sigma)} \ \text{ on } \Gamma_2,\end{aligned}\tag{4.2.19}$$

where $f_1(\sigma)$ and $f_2(\sigma)$ are the values of the function $f(\sigma)$ on Γ_1 and Γ_2, respectively.

Consider the function

$$\chi(z) = z \exp\left(i\beta + \int\limits_{\Gamma_2} \frac{\ln(e^{-2i\alpha(\sigma)}\sigma^2)}{\sigma - z}\, d\sigma\right), \quad |z| > 1,$$

where

$$\beta = -\frac{1}{4\pi} \int\limits_0^{2\pi} \arg\left(\sigma^2 e^{2i\alpha(\sigma)}\right) d\vartheta, \quad \vartheta = \arg\sigma.$$

$\chi(z)$ is holomorphic in the domain of its definition and satisfies the condition

$$\chi(\sigma) = e^{2i\alpha(\sigma)}\overline{\chi(\sigma)} \ \text{ for } \ |\sigma| = 1. \tag{4.2.20}$$

By virtue of (4.2.20) the boundary condition on Γ_2 in (4.2.19) can be written as follows

$$\frac{W(\sigma)}{\chi(\sigma)} + \frac{\overline{W(\sigma)}}{\overline{\chi(\sigma)}} = \frac{2f_2(\sigma)e^{i\alpha(\sigma)}}{\chi(\sigma)} \ \text{ for } \ |\sigma| = 1. \tag{4.2.21}$$

Consider the holomorphic function $\Psi(z)$

$$\Psi(z) = \begin{cases} \dfrac{W(z)}{\chi(z)} & \text{for } \ 1 < |z| < R^2, \\ -\dfrac{\overline{W(1/\overline{z})}}{\overline{\chi(1/\overline{z})}} & \text{for } \ \frac{1}{R^2} < |z| < 1, \end{cases} \tag{4.2.22}$$

on the set $(1 < |z| < R^2) \cup (1/R^2 < |z| < 1)$.

By (4.2.21)

$$\Psi^+(\sigma) - \Psi^-(\sigma) = -f_0(\sigma) \ \text{ for } \ |\sigma| = 1, \tag{4.2.23}$$

where

$$f_0(\sigma) = \frac{2f_2(\sigma)e^{i\alpha(\sigma)}}{\chi(\sigma)}\,. \tag{4.2.24}$$

Since the piecewise-holomorphic function

$$F(z) = -\frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{f_0(t)}{t - z}\, dt \tag{4.2.25}$$

also satisfies condition (4.2.23), we obtain

$$\Psi(z) = F(z) + W_1(z), \tag{4.2.26}$$

where $W_1(z)$ a holomorphic function in the ring $1/R^2 < |z| < R^2$. For the function Ψ to be representable as (4.2.22) we should subject the function W_1 to the condition

$$W_1(z) + \overline{W_1\left(\frac{1}{\overline{z}}\right)} = -F(z) - \overline{F\left(\frac{1}{\overline{z}}\right)}.$$

Using (4.2.20) and (4.2.25) we obtain

$$F(z)+\overline{F\left(\frac{1}{\overline{z}}\right)}=F(0). \tag{4.2.27}$$

Therefore the function W_1 in (4.2.26) should be subjected to the condition

$$W_1(z)+\overline{W_1\left(\frac{1}{\overline{z}}\right)}=-F(0). \tag{4.2.28}$$

Using formulas (4.2.19), (4.2.22), (4.2.28), for the function $W_1(z)$ we obtain the boundary condition

$$W_1(\sigma)-e^{2i\alpha(\sigma)}\,\frac{\overline{\chi(\sigma)}}{\chi(\sigma)}\,W_1\left(\frac{\sigma}{R^4}\right)=Q(\sigma),\ \ |\sigma|=R^2, \tag{4.2.29}$$

where

$$Q(\sigma)=\frac{1}{\chi(\sigma)}\left(2f_1(\sigma)e^{i\alpha(\sigma)}-\chi(\sigma)F(\sigma)+e^{2i\alpha(\sigma)}\overline{\chi(\sigma)}F\left(\frac{1}{\overline{\sigma}}\right)\right).$$

Let us introduce a new sought function

$$W_2(z)=W_1\left(\frac{z}{R^2}\right).$$

Then, by virtue of (4.2.29), we obtain

$$W_2(R^4\sigma)=G(\sigma)W_2(\sigma)+Q(R^2\sigma)\ \text{ for }\ |\sigma|=1, \tag{4.2.30}$$

where

$$G(\sigma)=e^{2i\alpha(\sigma)}\,\frac{\overline{\chi(R^2\sigma)}}{\chi(R^2\sigma)}\,,$$

$$Q(R^2\sigma)=\frac{2f_1(R^2\sigma)}{\chi(R^2\sigma)}-F(R^2\sigma)+G(\sigma)F\left(\frac{\sigma}{R^2}\right),\ \ |\sigma|=1, \tag{4.2.31}$$

and condition (4.2.28) takes the form

$$W_2(z)+\overline{W_2\left(\frac{R^4}{\overline{z}}\right)}=-F(0). \tag{4.2.32}$$

Thus we come to the problem of finding a holomorphic function W_2 in the ring $1<|z|<R^4$ by the boundary condition (4.2.30) and the additional condition (4.2.32). We have to find bounded solutions of this problem.

To solve problem (4.2.30), we make use of the following result [**18**].

Consider a problem with the boundary condition

$$\Phi(R^4\sigma)-\lambda\Phi(\sigma)=g(\sigma),\ \ |\sigma|=1, \tag{4.2.33}$$

where Φ is the sought holomorphic function in the ring $1<|z|<R^4$, g is a given function from the class H^*, λ is some number.

If $\lambda\neq R^{4n}$, $n=0,\pm1,\pm2,\dots$, then problem (4.2.33) has the unique solution

$$\Phi(z)=\frac{1}{2\pi i}\int\limits_{\Gamma_2}K_\lambda\left(\frac{z}{t}\right)\frac{g(t)}{t}\,dt \tag{4.2.34}$$

but if $\lambda = 1$, then for problem (4.2.33) to be solvable it is necessary and sufficient that the condition

$$\int\limits_{\Gamma_2} \frac{g(t)}{t}\, dt = 0$$

be fulfilled. In that case, for $\lambda = 1$ the solution of problem (4.2.32) is written in the form

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_2} K_1\left(\frac{z}{t}\right) \frac{g(t)}{t}\, dt + C, \tag{4.2.35}$$

where C is an arbitrary number.

In the above formulas, the function $K_\lambda(z)$ has the form

$$\begin{aligned} K_\lambda(z) = {} & \frac{R^4}{R^4 - z} + \frac{1}{\lambda} \cdot \frac{1}{1 - z} + \lambda \sum_{n \geq 1} \frac{1}{R^{4n} - \lambda} \left(\frac{z}{R^4}\right)^n \\ & + \frac{1}{\lambda} \sum_{n \leq -1} \frac{R^{4n} z^n}{R^{4n} - \lambda} + \begin{cases} \dfrac{\lambda}{1 - \lambda} & \text{for } \lambda \neq 1, \\ 0 & \text{for } \lambda = 1. \end{cases} \end{aligned} \tag{4.2.36}$$

Consider the function

$$T_n(z) = \prod_{k=1}^{n} (z - R^2 z_k)^{-1} z^{[\frac{n}{2}]} e^{i\,\frac{\vartheta_0 n}{2}}, \tag{4.2.37}$$

where

$$z_k = \exp\left(i\vartheta_0 + \frac{2\pi(k-1)}{n}\, i\right), \quad k = 1, 2, \ldots, n,$$

is a fixed number, ϑ_0, such that $0 \leq \vartheta_0 < 2\pi$ does not coincide with the points $a_1, a_2, \ldots, a_n$.

By direct calculations we find that

$$T_n\left(\frac{R^4}{\overline{z}}\right) = \begin{cases} -T_n(z) & \text{if } n \text{ is even}, \\ -\dfrac{z}{R^2}\, T_n(z), & \text{if } n \text{ is odd}. \end{cases} \tag{4.2.38}$$

Further it is not difficult to verify that

$$\operatorname{Ind}_{\Gamma_2} \frac{T_n(\sigma)}{T_n(R^4 c)} = n.$$

By the results of [**18**], we write the coefficient in the boundary condition (4.2.30) as follows

$$G(\sigma) = \lambda\, \frac{X(R^4\sigma)}{X(\sigma)}, \quad |\sigma| = 1, \tag{4.2.39}$$

where

$$X(z) = T_n(z) \exp\left(\frac{1}{2\pi i} \int\limits_{\Gamma_2} K\left(1; \frac{z}{\sigma}\right) \ln \frac{G(\sigma) T_n(\sigma)}{\lambda T_n(R^4\sigma)}\, \frac{d\sigma}{\sigma}\right), \tag{4.2.40}$$

$$\lambda=\exp\left(\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{1}{\sigma}\ln\frac{G(\sigma)T_n(\sigma)}{T_n(R^4\sigma)}\,d\sigma\right). \tag{4.2.41}$$

It is obvious that $X(z)$ is a function holomorphic in the ring $1<|z|<R^4$ except for the points R^2z_k, $k=1,2,\dots,n$, where it has poles of first order.

From (4.2.38) and (4.2.41) we derive

$$|\lambda|=\begin{cases}1 & \text{for even } n,\\ R^2 & \text{for odd } n.\end{cases}$$

Therefore for odd n we have $\lambda\neq R^{4n}$ $(n=0,\pm1,\pm2,\dots)$. The number ϑ_0 in formula (4.2.37) can be chosen so that $\lambda\neq 1$ for even n. Then $\lambda\neq R^{4n}$ $(n=0,\pm1,\pm2,\dots)$ for even n too.

Using (4.2.39), from the boundary condition (4.2.30) we obtain

$$\frac{W_2(R^4\sigma)}{X(R^4\sigma)}=\lambda\,\frac{W_2(\sigma)}{X(\sigma)}+\frac{Q(R^2\sigma)}{X(R^4\sigma)}\,.$$

From this, by virtue of (4.2.34), we find the solution of problem (4.2.39)

$$W_2(z)=\frac{X(z)}{2\pi i}\int\limits_{\Gamma_2}\frac{K_\lambda\big(\frac{z}{\sigma}\big)Q(R^2\sigma)}{\sigma X(R^4\sigma)}\,d\sigma. \tag{4.2.42}$$

Let us prove that this function satisfies condition (4.2.32), too.

Note the following properties of the function $K_\lambda(z)$ which can be verified by direct calculations:

$$\overline{K_1\left(\frac{R^4}{\overline{z}^{\,\sigma}}\right)}=2-K_1\left(\frac{z}{\sigma}\right),\quad |\sigma|=1, \tag{4.2.43}$$

if $\lambda\neq1$, $|\sigma|=1$,

$$\overline{K_1\left(\frac{R^4}{\overline{z}^{\,\sigma}}\right)}=\begin{cases}-\lambda K_\lambda\left(\dfrac{z}{\sigma}\right) & \text{for even } n,\\[2mm] -\lambda\,\dfrac{\sigma}{z}\,K_\lambda\left(\dfrac{z}{\sigma}\right) & \text{for odd } n.\end{cases} \tag{4.2.44}$$

Taking into account that

$$\left|\frac{G(\sigma)T_n(\sigma)}{\lambda T_n(R^4\sigma)}\right|=1$$

and using formulas (4.2.41) and (4.2.42), from formula (4.2.40) we obtain

$$\begin{aligned}\overline{X\left(\frac{R^4}{\overline{z}}\right)}&=-X(z)\ \text{ for even } n,\\ \overline{X\left(\frac{R^4}{\overline{z}}\right)}&=-\frac{z}{R^2}\,X(z)\ \text{ for odd } n.\end{aligned} \tag{4.2.45}$$

From formula (4.2.31) with (4.2.27) taken into account we derive

$$\overline{Q(R^2\sigma)}=\frac{1}{G(\sigma)}\,Q(R^2\sigma)+\frac{F(o)}{G(\sigma)}\,(1-G(\sigma)),\quad |\sigma|=1. \tag{4.2.46}$$

Note that $\Phi(z) \equiv 1$ is a solution of the problem with the boundary condition

$$\Phi(R^4\sigma) = G(\sigma)\Phi(\sigma) + (1 - G(\sigma)), \quad |\sigma| = 1,$$

and by virtue of (4.2.42) we obtain the identity

$$\frac{X(z)}{2\pi i} \int\limits_{\Gamma_2} \frac{K_\lambda\left(\frac{z}{\sigma}\right)(1 - G(\sigma))}{X(R^4\sigma)}\, d\sigma = 1. \tag{4.2.47}$$

By virtue of (4.2.39), (4.2.44)–(4.2.47) we make sure that the function $W_2(z)$ defined by (4.2.42) satisfies condition (4.2.32).

By (4.2.25), (4.2.31) and (4.2.39) we obtain

$$\begin{aligned} W_2(z) = \frac{X(z)}{\pi i} \int\limits_{\Gamma_2} \frac{K_\lambda\left(\frac{z}{\sigma}\right) f_1(R^2\sigma) e^{i\alpha(\sigma)}}{\sigma X(R^4\sigma)\chi(R^2\sigma)}\, d\sigma \\ - \frac{X(z)}{2\pi i} \int\limits_{\Gamma_2} f_0(t) \Bigg(\frac{1}{2\pi i} \int\limits_{\Gamma_2} K_\lambda\left(\frac{z}{\sigma}\right) \\ \times \left(-\frac{1}{X(R^4\sigma)(t - R^2\sigma)} + \frac{\lambda}{X(\sigma)(t - \sigma/R^2)} \right) \frac{d\sigma}{\sigma} \Bigg)\, dt. \end{aligned} \tag{4.2.48}$$

Consider the integral

$$I_1(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_2} K_\lambda\left(\frac{z}{\sigma}\right) \frac{\lambda R^2}{\sigma X(\sigma)(R^2 t - \sigma)}\, d\sigma.$$

Since

$$K_\lambda(z) = \frac{R^4}{R^4 - z} + \frac{1}{\lambda(1 - z)} + K_\lambda^0(z),$$

where $K_\lambda^0(z)$ is a holomorphic function in the ring $1 < |z| < R^4$, the integrand function of σ in the expression for $I_1(z)$ in the ring $1 < |\sigma| < R^4$ has poles at the points $\sigma = z$ and $\sigma = R^2 t$. Therefore by virtue of the Cauchy theorem

$$I_1(z) = \frac{1}{2\pi i} \left(\int\limits_{|\sigma| = R^4} + \int\limits_{\gamma_1^-} + \int\limits_{\gamma_2^-} \right) K_\lambda\left(\frac{z}{\sigma}\right) \frac{\lambda R^2}{\sigma X(\sigma)(R^2 t - \sigma)}\, d\sigma,$$

where γ_1 and γ_2 are circumferences in the ring $1 < |\sigma| < R^4$ with centers at the points $R^2 t$ and z.

By the Cauchy theorem,

$$\frac{1}{2\pi i} \int\limits_{\gamma_1^-} K_\lambda\left(\frac{z}{\sigma}\right) \frac{\lambda R^2}{\sigma X(\sigma)(R^2 t - \sigma)}\, d\sigma = \frac{\lambda}{t} K_\lambda\left(\frac{z}{R^2 t}\right) \frac{1}{X(R^2 t)}\,,$$

$$\frac{1}{2\pi i} \int\limits_{\gamma_2^-} K_\lambda\left(\frac{z}{\sigma}\right) \frac{\lambda R^2}{\sigma X(\sigma)(R^2 t - \sigma)}\, d\sigma = -\frac{\lambda R^2}{X(z)(R^2 t - \sigma)}\,.$$

Hence (4.2.48) yields

$$W_2(z)=\frac{X(z)}{\pi i}\int\limits_{\Gamma_2}\frac{K_\lambda\big(\frac{z}{\sigma}\big)f_1(R^2\sigma)e^{i\alpha(\sigma)}}{\sigma X(R^4\sigma)\chi(R^2\sigma)}\,d\sigma$$
$$+\frac{X(z)}{2\pi i}\int\limits_{\Gamma_2}\frac{f_0(t)}{2\pi i}\,dt\int\limits_{\Gamma_2}\frac{K_\lambda\big(\frac{z}{\sigma}\big)}{\sigma X(R^4\sigma)\chi(t-R^2\sigma)}\,d\sigma$$
$$-\frac{X(z)}{2\pi i}\int\limits_{\Gamma_2}\frac{\lambda f_0(t)K_\lambda\big(\frac{z}{R^2t}\big)}{tX(R^2t)}\,dt$$
$$+\frac{R^2}{2\pi i}\int\limits_{\Gamma_2}\frac{f_0(t)}{R^2t-z}\,dt$$
$$-\frac{X(z)}{2\pi i}\int\limits_{\Gamma_2}\frac{f_0(t)}{2\pi i}\left(\int\limits_{\Gamma_2}K_\lambda\left(\frac{z}{R^4\sigma}\right)\frac{\lambda}{\sigma X(R^4\sigma)(t-R^2\sigma)}\,d\sigma\right)dt.$$

By direct calculations we verify that

$$\lambda K_\lambda\left(\frac{z}{R^4}\right)=K_\lambda(z).$$

Then we obtain

$$W_1(z)=W_2(R^2z)=\frac{X(R^2z)}{\pi i}\int\limits_{\Gamma_2}\frac{K_\lambda\big(\frac{R^2z}{\sigma}\big)f_1(R^2\sigma)e^{i\alpha(\sigma)}}{\sigma X(R^4\sigma)\chi(R^2\sigma)}\,d\sigma$$
$$-\frac{X(R^2z)}{2\pi i}\int\limits_{\Gamma_2}\frac{\lambda f_0(\sigma)}{\sigma X(R^2\sigma)}\,K_\lambda\left(\frac{z}{\sigma}\right)\,d\sigma-F(z),\quad \frac{1}{R^2}<|z|<R^2.$$

By virtue of (4.2.22), (4.2.26) we have

$$W(z)=\chi(z)\big(W_1(z)+F(z)\big)=$$
$$=\frac{\chi(z)X(R^2z)}{2\pi i}\left(\int\limits_{\Gamma_2}\frac{2K_\lambda\big(\frac{R^2z}{\sigma}\big)f_1(R^2\sigma)}{\sigma X(R^4\sigma)\chi(R^2\sigma)}\,d\sigma\right.$$
$$\left.-\int\limits_{\Gamma_2}\frac{\lambda f_0(\sigma)}{\sigma X(R^2\sigma)}\,K_\lambda\left(\frac{z}{\sigma}\right)\,d\sigma\right).\quad (4.2.49)$$

Hence, using (4.2.24) and (4.2.49), we finally come to

$$W(z)=\frac{\chi(z)X(R^2z)}{\pi i}\int\limits_{\Gamma}\frac{K_\lambda\big(\frac{R^4z}{\sigma}\big)f(\sigma)e^{i\alpha(\sigma)}}{\sigma\chi(\sigma)X(R^2\sigma)}\,d\sigma,\quad 1<|z|<R^2,\quad (4.2.50)$$

where f is the given function defined by (4.2.18).

Since $X(R^2 z)$ has simple poles at the points $z = z_k$, for the function $W(z)$ to be bounded it is necessary and sufficient that the conditions

$$\int_\Gamma K_\lambda\left(\frac{R^4 z}{\sigma}\right) \frac{f(\sigma)e^{i\alpha(\sigma)}}{\sigma\chi(\sigma)X(R^2\sigma)}\, d\sigma = 0, \quad k = 1, 2, \dots, n \tag{4.2.51}$$

be fulfilled.

Let us write the function $K_\lambda(R^4 z/\sigma)$ in the form

$$K_\lambda\left(\frac{R^4 z}{\sigma}\right) = \frac{\sigma}{\sigma - z} + K_\lambda^0\left(\frac{R^4 z}{\sigma}\right), \quad 1 < |z| < R^2,$$

then by virtue of (4.2.50) we have

$$W(z) = \frac{\chi(z)X(R^2 z)}{\pi i}\left[\int_\Gamma \frac{f(\sigma)e^{i\alpha(\sigma)}}{\sigma X(R^2\sigma)\chi(\sigma)(\sigma - z)}\, d\sigma \right.$$
$$\left. + \int_\Gamma \frac{f(\sigma)K_\lambda^0\left(\frac{R^4 z}{\sigma}\right)e^{i\alpha(\sigma)}}{\sigma X(R^2\sigma)\chi(\sigma)}\, d\sigma\right]. \tag{4.2.52}$$

The second summand in the right-hand part of equality (4.2.52) is a holomorphic function in the ring D $(1 < |z| < R^2)$ and a continuous one in the closed ring $\overline{D}$. The first summand is a Cauchy type integral whose density is a Hölder-continuous function on each open arc (Ra_k, Ra_{k+1}), $(R^{-1}a_k, R^{-1}a_{k+1})$, $k = 1, 2, \dots, n$. Therefore, according to the Plemelj–Privalov theorem (see e.g. [**76**]), the function $W(z)$ is continuously extendable on these open arcs and its boundary value satisfies the Hölder condition on them. Applying now the results of N. I. Muskhelishvili's monograph [**76**, § 26], we see that $W(z)$ is continuously extendable on Γ and its boundary value is a Hölder-continuous function on Γ.

(4.2.51) is a system of n equations with respect to $n+3$ real unknowns K, B_1, B_2, R, δ_k, $k = 2, 3, \dots, n$, $0 < \delta_k < 2\pi$. To each solution of system (4.2.51), if it is solvable, we can assign, by formula (4.2.50), the unique solution of the Riemann–Hilbert problem (4.2.17). Hence solutions by formula (4.2.16) are defined by the functions ω and ψ_0:

$$\omega(\zeta) = \frac{2}{K} W(R\zeta), \quad 1 < |\zeta| < R, \tag{4.2.53}$$

$$\psi_0(\zeta) = \overline{B} - \overline{W\left(\frac{R}{\overline{\zeta}}\right)}, \quad 1 < |\zeta| < R. \tag{4.2.54}$$

Since $\omega'(\zeta)$ is shown to be different from zero in the domain of its definition, $z = \omega(\zeta)$ conformally maps a circular ring $1 < |\zeta| < R$ onto the domain S, and $t = \omega(\sigma)$, whereas $\omega(\sigma) = W(R\sigma)/K$ is the equation of the sought contour.

To show one important application case, we will prove that the system of algebraic equations (4.2.51) is always solvable and find its solution in explicit form.

Let L_1 be the boundary of a regular polygon. Assume that the die with rectilinear base adjoins each side of the polygon. Assume that a normal pressing concentrated force P is applied to the middle of each die. The origin is supposed to lie at the centre of the polygon $A_1A_2\cdots A_n$ and the ox-axis to be directed normally to the side A_1A_2. Then

$$A_k=\rho\exp\left(\frac{\pi i}{n}\,(2k-3)\right),\;\;\alpha_k=\frac{2\pi}{n}\,(k-1),\;\;k=1,2,\dots,n.$$

By the symmetry property it can be assumed that

$$a_k=Re^{\frac{2\pi}{n}\,(k-1)i},\;\;k=1,2,\dots,n.$$

This assumption is justified if system (4.2.51) is solvable with respect to the unknowns K, B_1, B_2, R.

Let us show that if one of conditions (4.2.51) is fulfilled, then all other conditions are fulfilled too.

First we give some equalities whose validity is easy to verify:

$$T_n(ze^{\frac{2\pi i}{n}})=\begin{cases}-T_n(z) & \text{if } n \text{ is even},\\ -e^{-\frac{\pi i}{n}}T_n(z), & \text{if } n \text{ is odd},\end{cases}$$

$$\alpha(\sigma e^{\frac{2\pi i}{n}})=\begin{cases}\alpha(\sigma)+\dfrac{2\pi}{n}\,, & \text{if }\;\sigma\in a_ka_{k+1},\;\;1\le k\le n-1,\\ \alpha(\sigma)-\dfrac{2\pi(n-1)}{n}\,, & \text{if }\;\sigma\in a_na_1,\end{cases}$$

$$\ln\left(e^{-2i\alpha(\sigma_0)}\sigma_0^2\right)\Big|_{\sigma_0=\sigma e^{\frac{2\pi i}{n}}}=\ln\left(e^{-2i\alpha(\sigma)}\sigma^2\right),$$

$$X(ze^{\frac{2\pi i}{n}})=e^{\frac{2\pi i}{n}}\chi(z),\quad G(\sigma e^{\frac{2\pi i}{n}})=G(\sigma),$$

$$A(\sigma e^{\frac{2\pi i}{n}})=e^{\frac{2\pi i}{n}}A(\sigma).$$

By means of these equalities we easily conclude that the function $X(z)$ satisfies the condition

$$X(ze^{\frac{2\pi i}{n}})=\begin{cases}-X(z) & \text{if } n \text{ is even},\\ -e^{-\frac{2\pi i}{n}}X(z), & \text{if } n \text{ is odd}.\end{cases}$$

In that case $f_1(\sigma)$ is a constant,

$$f_1(\sigma)=\frac{1}{2}\,K_\rho\cos\frac{\pi}{n}\,.$$

Let us now show that the constants B_1 and B_2 can be chosen so that the function f_2 would also be a constant.

In the considered case

$$C(\sigma)=-P\sum_{r=1}^{k-1}\sin\frac{2\pi}{n}\,r=-\frac{P}{2\sin\frac{\pi}{n}}\,\left(\cos\frac{\pi}{n}-\cos(2k-1)\,\frac{\pi}{n}\right),$$
$$\sigma\in a_ka_{k+1},\;\;k=1,2,\dots,n.$$

By virtue of (4.2.18),

$$f_2(\sigma) = B_1 \cos\alpha + B_2 \sin\alpha - C(\sigma) + f_1(\sigma).$$

Therefore if $\sigma \in a_k a_{k+1}$, then

$$\begin{aligned} f_2(\sigma) &= B_1 \cos\frac{2\pi}{n}(k-1) + B_2 \sin\frac{2\pi}{n}(k-1) + \frac{P}{2}\operatorname{ctg}\frac{\pi}{n} \\ &\quad - \frac{P}{2}\cos\frac{2\pi}{n}(k-1)\operatorname{ctg}\frac{\pi}{n} + \frac{P}{2}\sin\frac{2\pi}{n}(k-1) + f_1(\sigma). \end{aligned}$$

If we now take

$$B_1 = \frac{P}{2}\operatorname{ctg}\frac{\pi}{n}, \quad B_2 = -\frac{P}{2},$$

then we obtain

$$f_2(\sigma) = \frac{1}{2}\left(K_\rho \cos\frac{\pi}{n} + P\operatorname{ctg}\frac{\pi}{n}\right).$$

Thus $f_2(\sigma)$ is a constant.

If we introduce the notation

$$\begin{aligned} D(\zeta) = &\int\limits_{|\sigma|=1} K_\lambda\left(\frac{R^2\zeta}{\sigma}\right)\frac{f_1(\sigma)e^{i\alpha(\sigma)}}{\sigma X(R^4\sigma)\chi(R^2\sigma)}\,d\sigma \\ &- \int\limits_{|\sigma|=1} K_\lambda\left(\frac{R^4\zeta}{\sigma}\right)\frac{f_2(\sigma)e^{i\alpha(\sigma)}}{\sigma X(R^2\sigma)\chi(\sigma)}\,d\sigma, \end{aligned}$$

then conditions (4.2.51) take the form

$$D(\zeta_k) = 0, \quad k = 1, 2, \dots, n. \tag{4.2.55}$$

By virtue of the above equalities we readily obtain

$$D(\zeta e^{\frac{2\pi i}{n}}) = \begin{cases} -D(\zeta) & \text{if } n \text{ is even}, \\ -e^{\frac{\pi i}{n}} D(\zeta), & \text{if } n \text{ is odd}. \end{cases}$$

Hence it follows that if $D(\zeta_1) = 0$, then $D(\zeta_k) = 0$, $k = 2, 3, \dots, n$.

Therefore system (4.2.55) reduces to one equation with two unknowns

$$\begin{aligned} K_\rho &\int\limits_{|\sigma|=1} K_\lambda\left(\frac{R^2\zeta_1}{\sigma}\right)\frac{e^{i\alpha(\sigma)}}{\sigma X(R^4\sigma)\chi(R^2\sigma)}\,d\sigma \\ &= \left(K_\rho + \frac{P}{\sin\frac{\pi}{n}}\right)\int\limits_{|\sigma|=1} K_\lambda\left(\frac{R^4\zeta_1}{\sigma}\right)\frac{e^{i\alpha(\sigma)}}{\sigma X(R^2\sigma)\chi(\sigma)}\,d\sigma. \end{aligned}$$

Hence we obtain

$$K = \frac{P\gamma(R)}{\rho(\delta(R) - \gamma(R))\sin\frac{\pi}{n}},$$

where

$$\delta(R) = \int\limits_{|\sigma|=1} K_\lambda\left(\frac{R^2 e^{i\vartheta_0}}{\sigma}\right) \frac{e^{i\alpha(\sigma)}}{\sigma X(R^4\sigma)X(R^2\sigma)}\,d\sigma,$$
$$\gamma(R) = \int\limits_{|\sigma|=1} K_\lambda\left(\frac{R^4 e^{i\vartheta_0}}{\sigma}\right) \frac{e^{i\alpha(\sigma)}}{\sigma X(R^2\sigma)\chi(\sigma)}\,d\sigma. \tag{4.2.56}$$

Using formula (4.2.56) and assuming R to be given, we define the tangential normal stress value on the sought contour. Giving R various values, we obtain a table of relationship between K and R, i.e. the position of a uniformly strong contour can be defined by the given values of K.

4.3. Defining the Shapes of a Hole in Bent Plates

Given an isotropic homogeneous plate shaped as a polygon weakened by a curvilinear hole, we assume that a rigid strip is attached to each side of the polygon and the plate is bent by moments of force applied to the strips. The contour of the hole is assumed to be free from external forces. The tangential normal moment on the hole contour depends on the shape and position of the hole. We will consider the following problem: find a deflection of the plate and a hole contour such that the tangential normal moment would take a constant value on the sought contour.

A problem of finding a hole contour within an isotropic infinite plate was solved in the monograph by N. V. Banichuk [**11**] under the assumption that the plate is bent by moments of force applied at a point at infinity, the hole contour is free from load and the tangential moment on it is constant.

Let us assume that on the plane of a complex variable the midsurface of the plate occupies the doubly connected domain S bounded by the convex closed broken line $z = x + iy$ and the sought contour $A_1A_2\cdots A_n(L_1)$. Like in the preceding paragraph, the affixes of the points A_k are denoted by the same symbols. The plate deflection at the point $M(x,y)$ is denoted by $W(x,y)$. According to the approximate plate bending theory, the considered case W must satisfy the biharmonic equation

$$\Delta^2 W = 0, \quad z \in S,$$

and the boundary conditions

$$\frac{\partial W(t)}{\partial n} = d_k, \quad d_k = \operatorname{tg}\beta_k,$$
$$N(t) = 0 \;\text{ on }\; A_kA_{k+1}, \quad k = 1,2,\dots,n \;\; (A_{n+1} = A_1), \tag{4.3.1}$$

$$M_n(t) = 0, \quad M_{ns}(t) = 0, \quad M_s(t) = const = K \;\text{ on } L_2, \tag{4.3.2}$$

where n is the external normal, β_k's are constants (angles of rotation), $N(t)$ is the intersecting force, $M_n(t)$ is the normally bending moment, $M_{ns}(t)$ is the torque, $M_s(t)$ is the tangential normal moment, t is a point of the contour.

We can consider two cases:

1) the rotation angles β_k, $k = 1, 2, \dots, n$ are known;
2) the values of the principal bending moment M_k are given on each side $A_k A_{k+1}$ of the external plate boundary.

As is known, a solution of a biharmonic equation is written in the form

$$W(x, y) = \operatorname{Re}\left[\overline{z}\varphi(z) + \chi(z)\right], \quad z \in S, \tag{4.3.3}$$

where φ and χ are analytic functions in the domain S.

By (4.3.3) we obtain

$$\frac{\partial W}{\partial n} = \operatorname{Re}\left[i\,\frac{\partial \overline{t}}{\partial s}\left(\varphi(t) + \overline{t\varphi'(t)} + \overline{\psi(t)}\,\right)\right] \quad \text{on } L_1,$$

where $\psi(z) = \chi'(z)$.

Hence, by virtue of (4.3.1) we have

$$\operatorname{Re}\left[e^{-i\alpha(t)}\left(\varphi(t) + \overline{t\varphi'(t)} + \overline{\psi(t)}\,\right)\right] = d(t) \quad \text{on } L_1, \tag{4.3.4}$$

where $\alpha(t)$ is the angle formed by the normal to L_1 at a point t and the ox-axis, $d(t) = d_k$ for $t \in A_k A_{k+1}$, $k = 1, 2, \dots, n$.

Using (4.3.1) and the formula for intersecting force $N(t)$ [**76**] we obtain

$$\operatorname{Re}\left[e^{-i\alpha(t)}\left(\varkappa\varphi(t) - t\overline{t\varphi'(t)} - \overline{\psi(t)}\,\right)\right] = C(t) \quad \text{on } L_1, \tag{4.3.5}$$

where $C(t)$ is the value of a piecewise-constant function at a point t: $C(t) = C_k$ for $t \in A_k A_{k+1}$, $k = 1, 2, \dots, n$,

$$C_k = \sum_{j=1}^{k} \sin(\alpha_k - \alpha_j) M_j, \quad j = 1, 2, \dots, n,$$

$$M_j = \int\limits_{s_j}^{s_{j+1}} M_n(t(s))\, ds, \quad j = 1, 2, \dots, n,$$

is the principal bending moment acting on the side $A_j A_{j+1}$, $j = 1, 2, \dots, n$, $\varkappa = \frac{\sigma+3}{\sigma-1}$, σ is the Poisson ratio.

Let us now establish the boundary conditions for the functions φ and ψ on the sought contour L_2.

We make use of the formulas [**76**]

$$M_x + M_y = -2D(1+\sigma)\left[\varphi'(z) + \overline{\varphi'(z)}\,\right], \tag{4.3.6}$$

$$M_y - M_x + 2iM_{xy} = 2D(1-\sigma)\left(\overline{z}\varphi''(z) + \psi(z)\right), \tag{4.3.7}$$

where M_x, M_y, M_{xy} are bending moments, D is the cylindrical rigidity of the plate.

Since $M_x + M_y$ is invariant with respect to a choice of axes, from (4.3.6) we obtain

$$\operatorname{Re}\varphi'(t) = -\frac{K}{4D(1+\sigma)}\,, \quad t \in L_2. \tag{4.3.8}$$

By virtue of (4.3.2) and (4.3.7) we have

$$2D(1-\sigma)\big(\bar{t}\varphi''(t)+\psi(t)\big)e^{2i\theta}=K,\ \ t\in L_2, \tag{4.3.9}$$

where $\theta(t)$ is the angle formed by the tangent to L_2 at a point t with the ox-axis.

As has been said above, we can consider two cases where either the rotation angles of the links of the broken line L_1 or the values of the principal bending moment acting on each side of the broken line L_1 are given. From (4.3.4) and (4.3.5) we see that in both cases we obtain the identical problems of the analytic function theory. We will consider the case with given values of principal bending moments M_j, $j=1,2,\dots,n$.

Using formulas (4.3.4), (4.3.5), (4.3.8), we obtain like in the preceding paragraph

$$\varphi(z)=-\frac{K}{4D(1+\alpha)}\,z.$$

Hence formula (4.3.5) take the form

$$\operatorname{Re}\Big[e^{-i\alpha(t)}\big((\varkappa-1)pt-\overline{\psi(t)}\,\big)\Big]=C(t)\ \text{ on } L_1, \tag{4.3.10}$$

whereas formula (4.3.9) implies

$$\overline{\psi(t)}=qt+B\ \text{ on } L_2, \tag{4.3.11}$$

where

$$p=-\frac{K}{4D(1+\sigma)}\,,\quad q=\frac{K}{2D(1-\sigma)}\,,\ \ B=B_1+iB_2,$$

B_1 and B_2 are unknown real constants.

The boundary conditions (4.3.10), (4.3.11) have the form of the boundary conditions (4.1.9), (4.1.10) obtained for the problem considered in the preceding paragraph. Hence it is clear that the problem posed in this paragraph is solved in the same manner as the preceding problem.

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Contents