Memoirs on Differential Equations and Mathematical Physics
Volume 57, 2012, 1–15

Eightieth Birthday Anniversary of Kusano Takaŝi

On December 30, 2012 Kusano Takaŝi, member of the Editorial Board of our journal, Professor Emeritus at Hiroshima University, Doctor of Science, became 80 years of age.

T. Kusano was born in a small town Shimabara, Nagasaki Prefecture, in Kyushu which is one of the four main islands forming the archipelago of Japan. In 1936 his family moved to Manchuria which was then a puppet nation made up by the Japanese Government in 1932 as his father took a job (assistant professor of mathematics) with Changchun Technical University.

His family lived in Changchun for ten years until they were repatriated to Japan in 1946 after World War II apart from his father who in the end of 1945 was arrested (by mistake) by the Soviet Army as a war criminal and was taken to Kazakhstan in USSR for forced labor.

In 1951 he left the Shimabara high school with honors and in 1955 he graduated from the Faculty of Science of the University of Tokyo, majoring in pure mathematics. Then he was admitted to its Graduate School by recommendation and began to study differential equations under the supervision of Professor Masuo Hukuhara. He remembers quite well that at the first meeting with his supervisor Professor Hukuhara said to him "We are now at the dawn of the era of nonlinear differential equations. Don't forget that fixed point theorems are one of the most important and useful tools in the analysis of nonlinear problems". His research subject was the qualitative theory of second order nonlinear partial differential equations of elliptic and parabolic types. After having obtained the Master's degree in 1957 he went to the Doctor's course, but in 1958 he had to leave the course halfway as he was offered the job of lecturer at Ibaraki University. It was in 1965 when he defended his doctoral dissertation submitted to the University of Tokyo.

After having taught at Ibaraki University (1958–1960), Nagasaki University (1960–1962), Chuo University (1962–1967) and Waseda University (1967–1969), he was appointed to be Full Professor of Hiroshima University in 1969 and served in Department of Mathematics, Faculty of Science, for twenty five years since then. In 1970 he changed his research subject from partial differential equations to ordinary differential equations under the influence of the oscillation theory created by Professor Ivan Kiguradze, and organized the seminar on oscillation of nonlinear ordinary differential equations with or without functional arguments. The seminar encouraged a number of graduate students to be active specialists in oscillation theory of differential equations, ordinary or partial. In 1994 he transferred to Fukuoka University to work for Department of Applied Mathematics, Faculty of Science for the last nine years of his career as a university professor. Since 2000, motivated by the work of Professor Vojislav Marić, he has been enthusiastic about the asymptotic analysis of positive solutions of differential equations by means of Karamata functions (or regular variation).

His scientific activities during his academic life include invited speeches at many international conferences on differential equations held in Europe and the Unites States, services as a member of the editorial board of the journals: Memoirs of Differential Equations and Mathematical Physics, Funkcialaj Ekvacioj, Hiroshima Mathematical Journal, Applied Mathematics E-Notes, and Studies of the University of Žilina (Mathematical Series), and supervision of nineteen students who successfully defended their Ph. D theses. Besides he is a permanent visiting professor of Northeast Normal University.

The main areas of T. Kusano's scientific investigations are broadly classified into the following four categories.

(I) Qualitative study of second order elliptic and parabolic partial differential equations:

(i) The construction of entire solutions (solutions defined in the entire space $\boldsymbol{R}^N$) and the solvability of exterior boundary value problems for a class of second order quasilinear elliptic equations. See e.g. [5], [6], [8].

(ii) The study of time change of solutions as functions of the space variable of the Cauchy problem for a class of second order linear parabolic equations and systems with unbounded coefficients. See e.g. [14], [17], [21].

(II) Oscillation theory of differential equations:

Since 1970 he has studied oscillation properties of differential equations in hopes of proceeding in the mainstream of oscillation theory created by F. V. Atkinson and I. T. Kiguradze. The equations considered by him include both ordinary and partial differential equations with or without functional arguments which can be regarded as generalizations of the Emden–Fowler equation. Various kinds of equations have been the objects of his investigations in this direction as listed below.

(i) Ordinary differential equations of generalized Emden–Fowler type in which the principal parts involve higher order linear differential operators such as $(p(t)x^{(n-m)})^{(m)}$ and $(p_{n-1}(t)(\cdots(p_1(t)(p_0(t)x)')'\cdots)')'$. See e.g. [39], [61], [68].

(ii) Ordinary differential equations of generalized Emden–Fowler type in which the principal parts involve second order nonlinear differential operators such as $(p(t)|x'|^{\alpha-1}x')'$.
   - (a) Half-linear equations [160], [165], [171].
   - (b) Non-half-linear equations [140], [151], [164].

(iii) Nonoscillatory ordinary differential equations which can be turned into oscillating systems as a result of introduction of functional arguments. See e.g. [72], [85], [159].

(iv) Ordinary differential equations of neutral type.
   - (a) First order equations having difference operators of degree 1 [132], [137], [142];
   - (b) Higher order equations having difference operators of degree 1 [129], [131], [143];
   - (c) Higher order equations having difference operators of higher degree [148], [163], [166].

(v) Partial differential equations.
   - (a) Nonlinear harmonic and metaharmonic equations [56], [63], [120];
   - (b) Non-elliptic equations [87], [100], [212].

(III) Existence and asymptotic behavior of positive solutions of nonlinear differential equations:

The following is a record of what he has done in his attempts at acquiring detailed and precise information about the asymptotic behavior of positive solutions of differential equations in mathematical physics.

(i) Positive solutions of ordinary differential equations of generalized Emden–Fowler type. See e.g. [83], [99], [104], [124], [125].
(ii) Positive solutions of second order semilinear elliptic equations in exterior domains. See e.g. [81], [98], [102].
(iii) Positive entire solutions of second order semilinear and quasilinear elliptic equations. See e.g. [107], [111], [157].
(iv) Positive entire solutions of higher order semilinear and quasilinear elliptic equations. See e.g. [117], [156], [172].
(v) Positive entire solutions of Monge-Ampère equations. See e.g. [135], [139], [141].

(IV) Asymptotic analysis of positive solutions in the framework of regular variation:

Inspired by the book of V. Marić entitled "Regular Variation and Differential Equations" published in 2000, he started studying theory of regular variation in the sense of Karamata and came before long to find a number of problems on differential equations that could be solved by means of regularly varying functions. What he has done in this regard is as follows.

(i) The construction of regularly varying solutions for various types of linear and nonlinear ordinary differential equations with or without functional arguments. See e.g. [218], [229], [235].
(ii) The introduction of the concept of generalized regularly varying functions and its application to the analysis of asymptotic behavior of positive solutions of more complicated differential equations than those considered in (i). See e.g. [223], [225].
(iii) The detection of the fact that if one's attention is restricted to generalized Emden–Fowler equations with regularly varying coefficients, then thorough and complete information can be obtained about the existence and asymptotic behavior of all possible regularly varying solutions of the equations under consideration. See e.g. [237]– [239].

We cordially wish Professor Kusano Takaŝi good health, long life and new successes in his scientific activities.

R. P. Agarwal, N. Izobov, I. Kiguradze,
T. Kiguradze, A. Lomtatidze, N. Partsvania,
M. Perestyuk, N. Rozov, M. Tvrdy, N. Yoshida

References

1. On a maximum principle for quasi-linear elliptic equations. *Proc. Japan Acad.* **38** (1962), 78–82.
2. On the maximum principle for quasi-linear parabolic equations of the second order. *Proc. Japan Acad.* **39** (1963), 211–216.
3. Remarks on some properties of solutions of some boundary value problems for quasi-linear parabolic and elliptic equations of the second order. *Proc. Japan Acad.* **39** (1963), 217–222.
4. On the Cauchy problem for a class of multicomponent diffusion systems. *Proc. Japan Acad.* **39** (1963), 634–638.
5. On bounded solutions of second order elliptic differential equations (with K. Akô). *J. Fac. Sci. Univ. Tokyo Sect. I* **11** (1964), 29–37.
6. On bounded solutions of elliptic partial differential equations of the second order. *Funkcial. Ekvac.* **7** (1965), 1–13.
7. On the first boundary problem for quasilinear systems of parabolic differential equations in non-cylindrical domains. *Funkcial. Ekvac.* **7** (1965), 103–118.
8. On bounded solutions of exterior boundary value problems for linear and quasilinear elliptic differential equations. *Japan. J. Math.* **35** (1965) 31–59.
9. A boundary value problem for nonlinear parabolic equations of the second order (with M. Kono). *Comment. Math. Univ. St. Paul.* **14** (1966) 85–96.
10. Periodic solutions of the first boundary problem for quasilinear parabolic equations of second order. *Funkcial. Ekvac* **9** (1966), 129–137.
11. A remark on a periodic boundary problem of parabolic type. *Proc. Japan Acad.* **42** (1966), 10–12.
12. Periodic solution of parabolic partial differential equations. (Japanese) *Sûgaku* **18** (1966), 104–106.
13. On the decay for large x of solutions of parabolic equations with unbounded coefficients. *Publ. Res. Inst. Math. Sci. Ser. A* **3** (1967/1968), 203–210.
14. On the behavior for large x of solutions of parabolic equations with unbounded coefficients. *Funkcial. Ekvac.* **11** (1968), 169–174 (1969).
15. Remarks on the behavior of solutions of second order parabolic equations with unbounded coefficients. *Funkcial. Ekvac.* **11** (1968) 197–205 (1969).
16. Asymptotic behavior of solutions of parabolic differential equations with bounded coefficients. *J. Sci. Hiroshima Univ. Ser. A-I Math.* **33** (1969), 151–159.
17. On the growth of solutions of parabolic differential inequalities with unbounded coefficients. *Funkcial. Ekvac.* **13** (1970), 45–50.
18. Oscillation of solutions of nonlinear differential delay equations of arbitrary order (with H. Onose). *Hiroshima Math. J.* **2** (1972), 1–13.
19. Oscillation theorems for delay equations of arbitrary order (with H. Onose). *Hiroshima Math. J.* **2** (1972), 263–270.
20. Nonlinear oscillation of a sublinear delay equation of arbitrary order (with H. Onose). *Proc. Amer. Math. Soc.* **40** (1973), 219–224.
21. Some parabolic equations with unbounded coefficients (with Lu-San Chen and T. Kuroda). *Funkcial. Ekvac.* **16** (1973), 1–28.
22. Weakly coupled parabolic systems with unbounded coefficients (with T. Kuroda and Lu-San Chen). *Hiroshima Math. J.* **3** (1973), 1–14.
23. Oscillations of functional differential equations with retarded argument (with H. Onose). *J. Differential Equations* **15** (1974), 269–277.
24. Oscillatory and asymptotic behavior of sublinear retarded differential equations (with H. Onose). *Hiroshima Math. J.* **4** (1974), 343–355.
25. On the oscillation of second order nonlinear ordinary differential equations (with H. Onose and H. Tobe). *Hiroshima Math. J.* **4** (1974), 491–499.

26. Oscillation theorems for second order differential equations with retarded argument (with H. Onose). *Proc. Japan Acad.* **50** (1974), 342–346.
27. An oscillation theorem for differential equations with deviating argument (with H. Onose). *Proc. Japan Acad.* **50** (1974), 809–811.
28. Oscillatory behavior of solutions of higher-order retarded differential equations. *Proceedings of the C. Caratheodory International Symposium (Athens, 1973)*, pp. 370–389. *Greek Math. Soc., Athens*, 1974.
29. Nonoscillatory solutions of differential equations with retarded arguments (with H. Onose). *Bull. Fac. Sci. Ibaraki Univ. Ser. A*, No. 7 (1975), 1–11.
30. Asymptotic behavior of nonoscillatory solutions of second order functional differential equations (with H. Onose). *Bull. Austral. Math. Soc.* **13** (1975), No. 2, 291–299.
31. Nonoscillation criteria for strongly elliptic systems (with N. Yoshida). *Boll. Un. Mat. Ital. (4)* **11** (1975), No. 1, 166–173.
32. Nonlinear oscillation of second order differential equations with retarded argument (with M. Naito). *Ann. Mat. Pura Appl. (4)* **106** (1975), 171–185.
33. Comparison and nonoscillation theorems for fourth order elliptic systems (with N. Yoshida). *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **58** (1975), No. 3, 328–337.
34. Picone's identity for ordinary differential operators of even order (with N. Yoshida). *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **58** (1975), No. 4, 524–530.
35. Remarks on the oscillatory behavior of solutions of functional differential equations with deviating argument (with H. Onose). *Hiroshima Math. J.* **6** (1976), No. 1, 183–189.
36. Nonoscillation theorems for differential equations with deviating argument (with H. Onose). *Pacific J. Math.* **63** (1976), No. 1, 185–192.
37. Asymptotic behaviour of non-oscillatory solutions of functional differential equations of arbitrary order (with H. Onose). *J. London Math. Soc. (2)* **14** (1976), No. 1, 106–112.
38. On the oscillation of solutions of nonlinear functional differential equations (with H. Onose). *Hiroshima Math. J.* **6** (1976), No. 3, 635–645.
39. Nonlinear oscillation of fourth order differential equations (with M. Naito). *Canad. J. Math.* **28** (1976), No. 4, 840–852.
40. A nonoscillation theorem for a second order sublinear retarded differential equation (with H. Onose). *Bull. Austral. Math. Soc.* **15** (1976), No. 3, 401–406.
41. On fourth-order non-linear oscillations (with M. Naito). *J. London Math. Soc. (2)* **14** (1976), No. 1, 91–105.
42. Unboundedness of solutions of parabolic differential inequalities (with M. Narita). *J. Math. Anal. Appl.* **57** (1977), No. 1, 68–75.
43. Asymptotic behavior of nonoscillatory solutions of nonlinear differential equations with forcing term (with H. Onose). *Ann. Mat. Pura Appl. (4)* **112** (1977), 231–239.
44. Nonlinear oscillation of second order functional differential equations with advanced argument (with H. Onose). *J. Math. Soc. Japan* **29** (1977), No. 3, 541–559.
45. Nonoscillation theorems for second order differential equations with forcing term (with H. Onose). *Quart. J. Math. Oxford Ser. (2)* **28** (1977), No. 110, 191–200.
46. Asymptotic decay of oscillatory solutions of second order differential equations with forcing term (with H. Onose). *Proc. Amer. Math. Soc.* **66** (1977), No. 2, 251–257.
47. On the oscillation of fourth-order nonlinear differential equations with deviating argument (with M. Naito). *Qualitative research on stability of solutions of functional differential equations (Proc. Sympos. Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976). Sûrikaisekikenkyûsho Kôkyûroku* No. 288 (1977), 71–79.
48. Vanishing oscillations of solutions of a class of differential systems with retarded argument (with Y. Kitamura). *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **62** (1977), No. 3, 325–334.

49. Asymptotic properties of solutions of n-th order differential equations with deviating argument (with Y. Kitamura and M. Naito). *Proc. Japan Acad. Ser. A Math. Sci.* **54** (1978), No. 1, 13–16.
50. Asymptotic properties of solutions of two-dimensional differential systems with deviating argument (with Y. Kitamura). *Hiroshima Math. J.* **8** (1978), No. 2, 305–326.
51. On the oscillation of a class of nonlinear differential systems with deviating argument (with Y. Kitamura). *J. Math. Anal. Appl.* **66** (1978), No. 1, 20–36.
52. An oscillation theorem for a superlinear functional differential equation with general deviating arguments (with Y. Kitamura). *Bull. Austral. Math. Soc.* **18** (1978), No. 3, 395–402.
53. Oscillation and a class of nonlinear differential systems with general deviating arguments (with Y. Kitamura). *Nonlinear Anal.* **2** (1978), No. 5, 537–551.
54. Nonlinear oscillation of a fourth-order elliptic equation (with Y. Kitamura). *J. Differential Equations* **30** (1978), No. 2, 280–286.
55. Nonlinear oscillation of fourth order differential equations with deviating argument (with M. Naito). *Rev. Roumaine Math. Pures Appl.* **23** (1978), No. 9, 1351–1365.
56. An oscillation theorem for a sublinear Schrödinger equation (with Y. Kitamura). *Utilitas Math.* **14** (1978), 171–175.
57. On the nonoscillation of perturbed functional-differential equations (with J. R. Graef, Y. Kitamura, H. Onose, and P. W. Spikes). *Pacific J. Math.* **83** (1979), No. 2, 365–373.
58. Oscillation of first-order nonlinear differential equations with deviating arguments (with Y. Kitamura). *Proc. Amer. Math. Soc.* **78** (1980), No. 1, 64–68.
59. Extremal solutions of general nonlinear differential equations (with K. Kreith). *Hiroshima Math. J.* **10** (1980), No. 1, 141–152.
60. Oscillation criteria for weakly superlinear differential equations of even order (with K. Kreith and M. Naito). *J. Differential Equations* **38** (1980), No. 1, 32–40.
61. Nonlinear oscillation of higher-order functional-differential equations with deviating arguments (with Y. Kitamura). *J. Math. Anal. Appl.* **77** (1980), No. 1, 100–119.
62. On asymptotic limits of nonoscillations in functional equations with retarded arguments (with B. Singh). *Hiroshima Math. J.* **10** (1980), No. 3, 557–565.
63. Oscillation criteria for semilinear metaharmonic equations in exterior domains (with Y. Kitamura). *Arch. Rational Mech. Anal.* **75** (1980/81), No. 1, 79–90.
64. Nonlinear oscillation criteria for singular elliptic differential operators (with N. Yoshida). *Funkcial. Ekvac.* **23** (1980), No. 2, 135–142.
65. Oscillation criteria for general linear ordinary differential equations (with M. Naito). *Pacific J. Math.* **92** (1981), No. 2, 345–355.
66. Oscillation criteria for a class of nonlinear partial differential equations (with N. Yoshida). *J. Math. Anal. Appl.* **79** (1981), No. 1, 236–243.
67. Comparison theorems for functional-differential equations with deviating arguments (with M. Naito). *J. Math. Soc. Japan* **33** (1981), No. 3, 509–532.
68. Oscillatory and asymptotic behaviour of solutions of a class of linear ordinary differential equations (with M. Naito and K. Tanaka). *Proc. Roy. Soc. Edinburgh Sect. A* **90** (1981), No. 1-2, 25–40.
69. On strong oscillation of even order differential equations with advanced arguments. *Hiroshima Math. J.* **11** (1981), No. 3, 617–620.
70. Asymptotic behavior of oscillatory solutions of a differential equation with deviating arguments (with B. Singh). *J. Math. Anal. Appl.* **83** (1981), No. 2, 395–407.
71. Oscillation theorems for nonlinear ordinary differential equations of even order (with K. Kreith). *Canad. Math. Bull.* **24** (1981), No. 4, 409–413.
72. On even-order functional-differential equations with advanced and retarded arguments. *J. Differential Equations* **45** (1982), No. 1, 75–84.
73. Oscillation criteria for a class of perturbed Schrödinger equations (with M. Naito). *Canad. Math. Bull.* **25** (1982), No. 1, 71–77.

74. Oscillatory solutions of functional-differential equations generated by deviation of arguments of mixed type (with A. F. Ivanov, Y. Kitamura, and V. N. Shevelo). *Hiroshima Math. J.* **12** (1982), No. 3, 645–655.
75. Boundedness of solutions of a class of higher order ordinary differential equations (with M. Naito). *J. Differential Equations* **46** (1982), No. 1, 32–45.
76. Nonoscillatory solutions of forced differential equations of the second order (with N. Kawano and M. Naito). *J. Math. Anal. Appl.* **90** (1982), No. 2, 323–342.
77. Asymptotic relationships between two higher order ordinary differential equations. *Internat. J. Math. Math. Sci.* **6** (1983), No. 3, 559–566.
78. On second order functional-differential equations and inequalities with deviating arguments (with N. Fukagai). *Monatsh. Math.* **96** (1983), No. 2, 107–118.
79. Forced oscillations in functional-differential equations with deviating arguments (with B. Singh). *Arch. Math. (Brno)* **19** (1983), No. 1, 9–17.
80. On the asymptotic behavior of oscillatory solutions of functional-differential equations (with J. R. Graef, H. Onose, and P. W. Spikes). *Funkcial. Ekvac.* **26** (1983), No. 1, 11–16.
81. Asymptotic properties of semilinear elliptic equations (with C. A. Swanson). *Funkcial. Ekvac.* **26** (1983), No. 2, 115–129.
82. Nonoscillation theorems for differential equations with general deviating arguments (with A. M. Fink). *Ordinary differential equations and operators (Dundee, 1982)*, 224–239, *Lecture Notes in Math.*, 1032, *Springer, Berlin*, 1983.
83. Nonoscillation theorems for a class of perturbed disconjugate differential equations (with A. M. Fink). *Japan. J. Math. (N.S.)* **9** (1983), No. 2, 277–291.
84. On almost oscillation of functional-differential equations with deviating arguments (with R. S. Dahiya and M. Naito). *J. Math. Anal. Appl.* **98** (1984), No. 2, 332–340.
85. Oscillation of even order linear functional differential equations with deviating arguments of mixed type. *J. Math. Anal. Appl.* **98** (1984), No. 2, 341–347.
86. Asymptotic theory of singular semilinear elliptic equations (with C. A. Swanson). *Canad. Math. Bull.* **27** (1984), No. 2, 223–232.
87. Oscillation properties of nonlinear hyperbolic equations (with K. Kreith and N. Yoshida). *SIAM J. Math. Anal.* **15** (1984), No. 3, 570–578.
88. Positive solutions of a class of second order semilinear elliptic equations in the plane (with H. Usami). *Math. Ann.* **268** (1984), No. 2, 255–264.
89. On positive entire solutions of a class of second order semilinear elliptic systems (with N. Kawano). *Math. Z.* **186** (1984), No. 3, 287–297.
90. Nonoscillation theorems for functional-differential equations of arbitrary order (with J. R. Graef, M. K. Grammatikopoulos, Y. Kitamura, H. Onose, and P. W. Spikes). *Internat. J. Math. Math. Sci.* **7** (1984), No. 2, 249–256.
91. Positive solutions of functional-differential equations with singular nonlinear terms (with B. Singh). *Nonlinear Anal.* **8** (1984), No. 9, 1081–1090.
92. Oscillation and nonoscillation criteria for a class of forced differential equations with deviating arguments (with J. R. Graef). *Bull. Inst. Math. Acad. Sinica* **12** (1984), No. 3, 211–217.
93. Oscillation theory of first order functional-differential equations with deviating arguments (with N. Fukagai). *Ann. Mat. Pura Appl. (4)* **136** (1984), 95–117.
94. Comparison theorems for functional-differential equations (with A. F. Ivanov and V. N. Shevelo). (Russian) *Akad. Nauk Ukrain. SSR Inst. Mat. Preprint* **1984**, No. 26, 19 pp.
95. Oscillation theorems of comparison type for nonlinear functional-differential equations with deviating arguments. *Ninth international conference on nonlinear oscillations, Vol. 2 (Kiev, 1981)*, 211–215, 476, *"Naukova Dumka", Kiev*, 1984.
96. On the elliptic equation $\Delta u = \varphi(x)u^\gamma$ in $\mathbf{R}^2$ (with N. Kawano and M. Naito). *Proc. Amer. Math. Soc.* **93** (1985), No. 1, 73–78.

97. Bounded entire solutions of second order semilinear elliptic equations with application to a parabolic initial value problem (with S. Oharu). *Indiana Univ. Math. J.* **34** (1985), No. 1, 85–95.
98. Growth properties of stationary Klein–Gordon equations (with C. A. Swanson). *SIAM J. Math. Anal.* **16** (1985), No. 3, 440–446.
99. Asymptotic behavior of solutions of a class of higher order ordinary differential equations (with B. Singh). *SIAM J. Math. Anal.* **16** (1985), No. 4, 770–784.
100. Forced oscillations of Timoshenko beams (with N. Yoshida). *Quart. Appl. Math.* **43** (1985), No. 2, 167–177.
101. Positive solutions of a class of second order semilinear elliptic systems in exterior domains (with N. Kawano). *Math. Nachr.* **121** (1985), 11–23.
102. Pairs of positive solutions of quasilinear elliptic equations in exterior domains (with C. A. Swanson and H. Usami). *Pacific J. Math.* **120** (1985), No. 2, 385–399.
103. Global existence theorems for solutions of nonlinear differential equations with prescribed asymptotic behaviour (with W. F. Trench). *J. London Math. Soc. (2)* **31** (1985), No. 3, 478–486.
104. Existence of global solutions with prescribed asymptotic behavior for nonlinear ordinary differential equations (with W. F. Trench). *Ann. Mat. Pura Appl. (4)* **142** (1985), 381–392 (1986).
105. Entire positive solutions of singular semilinear elliptic equations (with C. A. Swanson). *Japan. J. Math. (N.S.)* **11** (1985), No. 1, 145–155.
106. On entire solutions of second order semilinear elliptic equations (with S. Oharu). *J. Math. Anal. Appl.* **113** (1986), No. 1, 123–135.
107. Decaying entire positive solutions of quasilinear elliptic equations (with C. A. Swanson). *Monatsh. Math.* **101** (1986), No. 1, 39–51.
108. Asymptotic behavior of solutions of a class of second order nonlinear differential equations (with M. Naito and H. Usami). *Hiroshima Math. J.* **16** (1986), No. 1, 149–159.
109. Development of an aspect of the comparison method for functional-differential equations (with A. F. Ivanov, M. Naito, and V. N. Shevelo). (Russian) *Akad. Nauk Ukrain. SSR Inst. Mat. Preprint* **1986**, No. 4, 15 pp.
110. Positive entire solutions of superlinear elliptic equations (with M. Naito). *Hiroshima Math. J.* **16** (1986), No. 2, 361–366.
111. Unbounded positive entire solutions of semilinear elliptic equations (with C. A. Swanson). *Nonlinear Anal.* **10** (1986), No. 9, 853–861.
112. Global existence of solutions of mixed sublinear-superlinear differential equations (with W. F. Trench). *Hiroshima Math. J.* **16** (1986), No. 3, 597–606.
113. On the asymptotic behavior of solutions of nonlinear ordinary differential equations *Equadiff 6 (Brno, 1985)*, 465–468, *Univ. J. E. Purkyně, Brno*, 1986.
114. Some remarks on the supersolution-subsolution method for superlinear elliptic equations (with N. Fukagai and K. Yoshida). *J. Math. Anal. Appl.* **123** (1987), No. 1, 131–141.
115. Positive entire solutions of semilinear biharmonic equations (with C. A. Swanson). *Hiroshima Math. J.* **17** (1987), No. 1, 13–28.
116. Radial entire solutions to even order semilinear elliptic equations in the plane (with M. Naito and C. A. Swanson). *Proc. Roy. Soc. Edinburgh Sect. A* **105** (1987), 275–287.
117. Entire solutions of a class of even order quasilinear elliptic equations (with M. Naito and C. A. Swanson). *Math. Z.* **195** (1987), No. 2, 151–163.
118. Global existence of nonoscillatory solutions of perturbed general disconjugate equations (with W. F. Trench). *Hiroshima Math. J.* **17** (1987), No. 2, 415–431.
119. Oscillation of solutions of a class of functional-differential equations (with A. F. Ivanov). (Russian) *Ukrain. Mat. Zh.* **39** (1987), No. 6, 717–721, 814.

120. Oscillation theory of entire solutions of second order superlinear elliptic equations (with M. Naito). *Funkcial. Ekvac.* **30** (1987), No. 2–3, 269–282.
121. Systems of functional-differential equations with asymptotically constant solutions (with W. F. Trench). *Proc. Amer. Math. Soc.* **104** (1988), No. 4, 1091–1097.
122. Existence of decaying entire solutions of a class of semilinear elliptic equations (with E. S. Noussair and C. A. Swanson). *Proc. Amer. Math. Soc.* **104** (1988), No. 4, 1141–1147.
123. On the asymptotic behavior of solutions of nonlinear ordinary differential equations (with M. Naito and C. A. Swanson). *Czechoslovak Math. J.* **38(113)** (1988), No. 3, 498–519.
124. Unbounded nonoscillatory solutions of nonlinear ordinary differential equations of arbitrary order (with M. Naito). *Hiroshima Math. J.* **18** (1988), No. 2, 361–372.
125. Positive solutions of a class of nonlinear ordinary differential equations (with M. Naito). *Nonlinear Anal.* **12** (1988), No. 9, 935–942.
126. On the existence of positive decaying entire solutions for a class of sublinear elliptic equations (with Y. Furusho). *Canad. J. Math.* **40** (1988), No. 5, 1156–1173.
127. Asymptotic properties of entire solutions of even order quasilinear elliptic equations (with M. Naito and C. A. Swanson). *Japan. J. Math. (N.S.)* **14** (1988), No. 2, 275–308.
128. Radial entire solutions of even order semilinear elliptic equations (with M. Naito and C. A. Swanson). *Canad. J. Math.* **40** (1988), No. 6, 1281–1300.
129. Oscillation theory of higher order linear functional-differential equations of neutral type (with J. Jaroš). *Hiroshima Math. J.* **18** (1988), No. 3, 509–531.
130. On unbounded positive solutions of nonlinear differential equations with oscillating coefficients (with M. Švec). *Czechoslovak Math. J.* **39(114)** (1989), No. 1, 133–141.
131. Asymptotic behavior of nonoscillatory solutions of nonlinear functional-differential equations of neutral type (with J. Jaroš). *Funkcial. Ekvac.* **32** (1989), No. 2, 251–263.
132. The oscillatory nature of solutions of a class of first-order functional-differential equations of neutral type (with A. F. Ivanov). (Russian) *Ukrain. Mat. Zh.* **41** (1989), No. 10, 1370–1375, 1438; English transl.: *Ukrainian Math. J.* **41** (1989), No. 10, 1179–1183 (1990).
133. Sufficient conditions for oscillations in higher order linear functional-differential equations of neutral type (with J. Jaroš). *Japan. J. Math. (N.S.)* **15** (1989), No. 2, 415–432.
134. Radial entire solutions of a class of quasilinear elliptic equations (with C. A. Swanson). *J. Differential Equations* **83** (1990), No. 2, 379–399.
135. Existence theorems for elliptic Monge–Ampère equations in the plane (with C. A. Swanson). *Differential Integral Equations* **3** (1990), No. 3, 487–493.
136. On oscillation of linear neutral differential equations of higher order (with J. Jaroš). *Hiroshima Math. J.* **20** (1990), No. 2, 407–419.
137. On a class of first order nonlinear functional-differential equations of neutral type (with J. Jaroš). *Czechoslovak Math. J.* **40(115)** (1990), No. 3, 475–490.
138. Oscillation and asymptotic behavior of solutions of first-order functional-differential equations of neutral type (with Y. Kitamura). *Funkcial. Ekvac.* **33** (1990), No. 2, 325–343.
139. Existence theorems for Monge-Ampère equations in $\mathbf{R}^N$ (with C. A. Swanson). *Hiroshima Math. J.* **20** (1990), No. 3, 643–650.
140. Oscillation and nonoscillation theorems for a class of second order quasilinear differential equations (with Á. Elbert). *Acta Math. Hungar.* **56** (1990), No. 3-4, 325–336.
141. Entire solutions of real and complex Monge-Ampère equations (with C. A. Swanson). *SIAM J. Math. Anal.* **22** (1991), No. 1, 157–168.

142. Oscillation properties of first order nonlinear functional-differential equations of neutral type (with J. Jaroš). *Differential Integral Equations* **4** (1991), No. 2, 425–436.
143. Existence of oscillatory solutions for functional-differential equations of neutral type (with J. Jaroš). *Acta Math. Univ. Comenian. (N.S.)* **60** (1991), No. 2, 185–194.
144. Oscillation of solutions of second order nonlinear functional-differential equations of neutral type (with A. F. Ivanov). *Ukrain. Mat. Zh.* **43** (1991), No. 12, 1671–1683; English transl.: *Ukrainian Math. J.* **43** (1991), No. 12, 1556–1568 (1992).
145. On differential equation $y'' + p(t)|y'| \operatorname{sgn} y + q(t)y = 0$ (with Á. Elbert). *Hiroshima Math. J.* **22** (1992), No. 1, 203–218.
146. A general method for quasilinear elliptic problems in $\mathbf{R}^N$ (with C. A. Swanson). *J. Math. Anal. Appl.* **167** (1992), No. 2, 414–428.
147. Kiguradze classes for radial entire solutions of higher order quasilinear elliptic equations (with M. Naito). *Hiroshima Math. J.* **22** (1992), No. 2, 301–363.
148. On a class of functional-differential equations of neutral type (with J. Jaroš and Y. Kitamura). *Recent trends in differential equations*, 317–333, World Sci. Ser. Appl. Anal., 1, *World Sci. Publ., River Edge, NJ*, 1992.
149. Asymptotic analysis of solutions of systems of neutral functional-differential equations (with Y. Kitamura). Asymptotic methods in analysis and combinatorics. *J. Comput. Appl. Math.* **41** (1992), No. 1-2, 23–33.
150. Semilinear evolution equations with singularities in ordered Banach spaces (with S. Oharu). *Differential Integral Equations* **5** (1992), No. 6, 1383–1405.
151. Oscillation theory for a class of second order quasilinear ordinary differential equations with application to partial differential equations (with A. Ogata and H. Usami). *Japan. J. Math. (N.S.)* **19** (1993), No. 1, 131–147.
152. On the oscillation of solutions of second order quasilinear ordinary differential equations (with A. Ogata and H. Usami). *Hiroshima Math. J.* **23** (1993), No. 3, 645–667.
153. p-Laplace equations in unbounded domains. (Japanese) *Variational problems; some problems on nonlinear elliptic equations (Japanese) (Kyoto, 1993). Sūrikaisekikenkyūsho Kōkyūroku* No. 834, (1993), 64–69.
154. Kiguradze classes for radial entire solutions of higher order semilinear elliptic equations (with M. Naito). *Continuum mechanics and related problems of analysis (Tbilisi, 1991)*, 254–259, *"Metsniereba", Tbilisi*, 1993.
155. Oscillation of parabolic equations with oscillating coefficients (with N. Yoshida). *Hiroshima Math. J.* **24** (1994), No. 1, 123–133.
156. Existence of positive entire solutions for higher order quasilinear elliptic equations (with Y. Furusho). *J. Math. Soc. Japan* **46** (1994), No. 3, 449–465.
157. Symmetric positive entire solutions of second-order quasilinear degenerate elliptic equations (with Y. Furusho and A. Ogata). *Arch. Rational Mech. Anal.* **127** (1994), No. 3, 231–254.
158. Existence and asymptotic behavior of positive solutions of second order quasilinear differential equations (with A. Ogata). *Funkcial. Ekvac.* **37** (1994), No. 2, 345–361.
159. On oscillation of half-linear functional-differential equations with deviating arguments (with B. S. Lalli). *Hiroshima Math. J.* **24** (1994), No. 3, 549–563.
160. Strong oscillation and nonoscillation of quasilinear differential equations of second order (with Y. Naito and A. Ogata). *Differential Equations Dynam. Systems* **2** (1994), No. 1, 1–10.
161. Existence theorems for nonlinear functional-differential equations of neutral type (with Y. Kitamura and B. S. Lalli). *Georgian Math. J.* **2** (1995), No. 1, 79–92.
162. Nonoscillation theorems for a class of quasilinear differential equations of second order (with N. Yoshida). *J. Math. Anal. Appl.* **189** (1995), No. 1, 115–127.
163. Existence theorems for a neutral functional-differential equation whose leading part contains a difference operator of higher degree (with Y. Kitamura). *Hiroshima Math. J.* **25** (1995), No. 1, 53–82.

164. On a class of second order quasilinear ordinary differential equations (with M. Kitano). *Hiroshima Math. J.* **25** (1995), No. 2, 321–355.
165. Oscillation properties of half-linear functional-differential equations of the second order (with J. Wang). *Hiroshima Math. J.* **25** (1995), No. 2, 371–385.
166. Existence of oscillatory and nonoscillatory solutions for a class of neutral functional-differential equations (with Y. Kitamura and B. S. Lalli). *Math. Bohem.* **120** (1995), No. 1, 57–69.
167. Oscillation properties for a class of second order nonlinear functional-differential equations with deviating arguments (with M. P. Ch'ên). *Proceedings of the 2nd International Conference on Differential Equations (Fuzhou, 1994). Ann. Differential Equations* **10** (1995), No. 5, 9–13.
168. Symmetric positive entire solutions of nonlinear biharmonic equations (with Y. Furusho). (Russian) Translated from the English by A. I. Kirillov. *Differentsial'nye Uravneniya* **31** (1995), No. 2, 296–311, 367; English transl.: *Differential Equations* **31** (1995), No. 2, 273–287.
169. Positive solutions of second-order quasilinear differential equations with singular nonlinearities (with M. Kurokiba and D. Vang). (Russian) *Differ. Uravn.* **32** (1996), No. 12, 1630–1637, 1725; English transl.: *Differential Equations* **32** (1996), No. 12, 1623–1629 (1997).
170. Second-order half-linear eigenvalue problems (with M. Naito and T. Tanigawa). *Fukuoka Univ. Sci. Rep.* **27** (1997), No. 1, 1–7.
171. Oscillation and nonoscillation criteria for second order quasilinear differential equations (with Y. Naito). *Acta Math. Hungar.* **76** (1997), No. 1–2, 81–99.
172. A supersolution-subsolution method for nonlinear biharmonic equations in $\mathbf{R}^N$ (with Y. Furusho). *Czechoslovak Math. J.* **47(122)** (1997), No. 4, 749–768.
173. An oscillatory half-linear differential equation (with Á. Elbert and T. Tanigawa). *Arch. Math. (Brno)* **33** (1997), No. 4, 355–361.
174. Second-order semilinear differential equations with external forcing terms (with J. Jaroš). (Japanese) *Structure of functional equations and mathematical methods (Japanese) (Kyoto, 1996). Sūrikaisekikenkyūsho Kōkyūroku* No. 984 (1997), 191–197.
175. A singular eigenvalue problem for second order linear ordinary differential equations (with M. Naito). *International Symposium on Differential Equations and Mathematical Physics (Tbilisi, 1997). Mem. Differential Equations Math. Phys.* **12** (1997), 122–130.
176. Positive entire solutions to nonlinear biharmonic equations in the plane. Positive solutions of nonlinear problems (with Y. Furusho). *J. Comput. Appl. Math.* **88** (1998), No. 1, 161–173.
177. Studies on mathematical analysis of nonlinear structures. (Japanese) *Fukuoka Univ. Sci. Rep.* **28** (1998), No. 1, 21–25.
178. On second-order half-linear oscillations (with H. Hoshino, R. Imabayashi, and T. Tanigawa). *Adv. Math. Sci. Appl.* **8** (1998), No. 1, 199–216.
179. Singular eigenvalue problems for second order linear ordinary differential equations (with Á. Elbert and M. Naito). *Equadiff 9 (Brno, 1997). Arch. Math. (Brno)* **34** (1998), No. 1, 59–72.
180. Principal solutions of non-oscillatory half-linear differential equations (with Á. Elbert). *Adv. Math. Sci. Appl.* **8** (1998), No. 2, 745–759.
181. A singular eigenvalue problem for the Sturm–Liouville equation (with M. Naito). (Russian) *Differ. Uravn.* **34** (1998), No. 3, 303–312, 428; English transl.: *Differential Equations* **34** (1998), No. 3, 302–311.
182. Positive solutions to a class of second order differential equations with singular nonlinearities (with T. Tanigawa). *Appl. Anal.* **69** (1998), No. 3–4, 315–331.

183. Oscillation criteria for a class of functional parabolic equations (with N. Yoshida). *J. Appl. Anal.* **5** (1999), No. 1, 1–16.
184. Positive solutions of second order quasilinear ordinary differential equations with general nonlinearities (with J. Kiyomura and M. Naito). *Studia Sci. Math. Hungar.* **35** (1999), No. 1–2, 39–51.
185. Oscillation and nonoscillation theorems for second order quasilinear functional-differential equations of neutral type (with J. Jaroš and P. Marušiak). *Adv. Math. Sci. Appl.* **9** (1999), No. 1, 333–346.
186. On a class of doubly singular differential equations of second order (with J. Jaroš). *Fukuoka Univ. Sci. Rep.* **29** (1999), No. 1, 7–12.
187. Existence of nonoscillatory solutions for nonlinear integro-differential equations of second order (with B. Singh). *Fukuoka Univ. Sci. Rep.* **29** (1999), No. 1, 13–20.
188. A Picone type identity for second order half-linear differential equations (with J. Jaroš). *Acta Math. Univ. Comenian. (N.S.)* **68** (1999), No. 1, 137–151.
189. On oscillation of functional-differential equations with forcing terms (with H. Onose). *Fukuoka Univ. Sci. Rep.* **29** (1999), No. 2, 225–229.
190. Existence of positive decreasing solutions to a class of second order nonlinear differential systems (with T. Tanigawa). *Fukuoka Univ. Sci. Rep.* **29** (1999), No. 2, 231–242.
191. Singular differential equations without singular solutions (with M. Naito). *Fukuoka Univ. Sci. Rep.* **29** (1999), No. 2, 243–252.
192. On periodic solutions of higher-order nonautonomous ordinary differential equations (with I. T. Kiguradze). (Russian) *Differ. Uravn.* **35** (1999), No. 1, 72–78, 142; English transl.: *Differential Equations* **35** (1999), No. 1, 71–77.
193. Sturm–Liouville eigenvalue problems for half-linear ordinary differential equations (with M. Naito). *Methods and applications for functional equations (Japanese) (Kyoto, 1998). Sūrikaisekikenkyūsho Kōkyūroku* No. 1083 (1999), 32–43.
194. On the number of zeros of nonoscillatory solutions to second order half-linear differential equations (with Á. Elbert and M. Naito). *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **42** (1999), 101–131 (2000).
195. Asymptotic properties of solutions of second order quasilinear functional differential equations of neutral type (with P. Marušiak). *Math. Bohem.* **125** (2000), No. 2, 199–213.
196. A Picone-type identity and Sturmian comparison and oscillation theorems for a class of half-linear partial differential equations of second order (with J. Jaroš and N. Yoshida). *Lakshmikantham's legacy: a tribute on his 75th birthday. Nonlinear Anal.* **40** (2000), No. 1-8, Ser. A: Theory Methods, 381–395.
197. On the number of zeros of principal solutions to second-order half-linear ordinary differential equations (with M. Naito). *Mathematical models in functional equations* (*Japanese*) (*Kyoto,* 1999). *Sūrikaisekikenkyūsho Kōkyūroku* No. 1128 (2000), 122–126.
198. Forced superlinear oscillations via Picone's identity (with J. Jaroš and N. Yoshida). *Acta Math. Univ. Comenian. (N.S.)* **69** (2000), No. 1, 107–113.
199. Singular solutions of a singular differential equation (with M. Naito). *J. Inequal. Appl.* **5** (2000), No. 5, 487–496.
200. Positive decreasing solutions of systems of second order singular differential equations (with T. Tanigawa). *J. Inequal. Appl.* **5** (2000), No. 6, 581–602.
201. Existence of singular solutions for a class of systems of singular differential equations (with J. Jaroš and T. Tanigawa). *Fukuoka Univ. Sci. Rep.* **30** (2000), No. 2, 169–177.
202. On black hole solutions of second order differential equations with a singular nonlinearity in the differential operator (with J. Jaroš). *Funkcial. Ekvac.* **43** (2000), No. 3, 491–509.
203. On conditions for the existence and uniqueness of a periodic solution of nonautonomous differential equations (with I. T. Kiguradze). (Russian) *Differ. Uravn.* **36**

(2000), No. 10, 1301–1306, 1436; English transl.: *Differ. Equ.* **36** (2000), No. 10, 1436–1442.

204. On periodic solutions of even-order ordinary differential equations (with I. Kiguradze). *Ann. Mat. Pura Appl. (4)* **180** (2001), No. 3, 285–301.
205. Teimuraz Chanturia (with N. V. Azbelev, N. A. Izobov, I. Kiguradze, V. A. Kondrat'ev, V. M. Millionshchikov, N. Kh. Rozov, and V. Šeda). *Mem. Differential Equations Math. Phys.* **24** (2001), 1–4.
206. On the oscillation of solutions of 4-dimensional Emden–Fowler differential systems (with M. Naito and F. Wu). *Qualitative theory of functional equations and its application to mathematical science (Japanese) (Kyoto, 2000). Sūrikaisekikenkyūsho Kōkyūroku* No. 1216 (2001), 266–273.
207. Sturm–Liouville eigenvalue problems from half-linear ordinary differential equations (with M. Naito). *Rocky Mountain J. Math.* **31** (2001), No. 3, 1039–1054.
208. Picone-type inequalities for nonlinear elliptic equations and their applications (with J. Jaroš and N. Yoshida). *J. Inequal. Appl.* **6** (2001), No. 4, 387–404.
209. Oscillatory properties of solutions of superlinear-sublinear parabolic equations via Picone-type inequalities (with J. Jaroš and N. Yoshida). *Math. J. Toyama Univ.* **24** (2001), 83–91.
210. On the oscillation of solutions of 4-dimensional Emden–Fowler differential systems (with M. Naito and F. Wu). *Adv. Math. Sci. Appl.* **11** (2001), No. 2, 685–719.
211. Generalized Picone's formula and forced oscillations in quasilinear differential equations of the second order (with J. Jaroš and N. Yoshida). *Arch. Math. (Brno)* **38** (2002), No. 1, 53–59.
212. Oscillation properties of solutions of a class of nonlinear parabolic equations (with J. Jaroš and N. Yoshida). *J. Comput. Appl. Math.* **146** (2002), No. 2, 277–284.
213. On the number of zeros of nonoscillatory solutions to half-linear ordinary differential equations involving a parameter (with M. Naito). *Trans. Amer. Math. Soc.* **354** (2002), No. 12, 4751–4767 (electronic).
214. Oscillation criteria for a class of partial functional-differential equations of higher order (with T. Kiguradze and N. Yoshida). *J. Appl. Math. Stochastic Anal.* **15** (2002), No. 3, 271–282.
215. Picone-type inequalities for half-linear elliptic equations and their applications (with J. Jaroš and N. Yoshida). *Adv. Math. Sci. Appl.* **12** (2002), No. 2, 709–724.
216. On white hole solutions of a class of nonlinear ordinary differential equations of the second order (with J. Jaroš). *Funkcial. Ekvac.* **45** (2002), No. 3, 319–339.
217. Remarks on the existence of regularly varying solutions for second order linear differential equations (with J. Jaroš). *Publ. Inst. Math. (Beograd) (N.S.)* **72(86)** (2002), No. 3, 113–118.
218. Nonoscillation theory for second order half-linear differential equations in the framework of regular variation (with J. Jaroš and T. Tanigawa). *Results Math.* **43** (2003), No. 1-2, 129–149.
219. Asymptotics of some classes of nonoscillatory solutions of second-order half-linear differential equations (with V. Marić and T. Tanigawa). *Bull. Cl. Sci. Math. Nat. Sci. Math.* No. 28 (2003), 61–74.
220. On the well-posedness of initial-boundary value problems for higher-order linear hyperbolic equations with two independent variables (with T. I. Kiguradze). (Russian) *Differ. Uravn.* **39** (2003), No. 4, 516–526, 575; English transl.: *Differ. Equ.* **39** (2003), No. 4, 553–563.
221. On ill-posed initial-boundary value problems for higher-order linear hyperbolic equations with two independent variables (with T. I. Kiguradze). (Russian) *Differ. Uravn.* **39** (2003), No. 10, 1379–1394, 1438–1439; English transl.: *Differ. Equ.* **39** (2003), No. 10, 1454–1470.
222. Nonoscillatory linear and half-linear differential equations having regularly varying solutions (with V. Marić). *Adv. Math. Sci. Appl.* **14** (2004), No. 1, 351–357.

223. Self-adjoint differential equations and generalized Karamata functions (with J. Jaroš). *Bull. Cl. Sci. Math. Nat. Sci. Math.* No. 29 (2004), 25–60.
224. Bounded and periodic in a strip solutions of nonlinear hyperbolic systems with two independent variables (with T. Kiguradze). *Comput. Math. Appl.* **49** (2005), No. 2-3, 335–364.
225. Nonoscillatory half-linear differential equations and generalized Karamata functions (with J. Jaroš and T. Tanigawa). *Nonlinear Anal.* **64** (2006), No. 4, 762–787.
226. Picone-type inequalities for nonlinear elliptic equations with first-order terms and their applications (with J. Jaroš and N. Yoshida). *J. Inequal. Appl.* **2006**, Art. ID 52378, 17 pp.
227. On the structure of positive solutions of a class of fourth order nonlinear differential equations (with T. Tanigawa). *Ann. Mat. Pura Appl. (4)* **185** (2006), No. 4, 521–536.
228. Existence of regularly and rapidly varying solutions for a class of third order nonlinear ordinary differential equations (with J. Jaroš and V. Marić). *Publ. Inst. Math. (Beograd) (N.S.)* **79(93)** (2006), 51–64.
229. On a class of functional differential equations having slowly varying solutions (with V. Marić). *Publ. Inst. Math. (Beograd) (N.S.)* **80(94)** (2006), 207–217.
230. The asymptotic behaviour of solutions of a class of nonlinear second-order differential equations (with V. Marić and T. Tanigawa). *Bull. Lond. Math. Soc.* **39** (2007), No. 3, 413–418.
231. Comparison theorems for perturbed half-linear Euler differential equations (with J. Manojlović and T. Tanigawa). *Int. J. Appl. Math. Stat.* **9** (2007), No. J07, 77–94.
232. Slowly varying solutions of functional differential equations with retarded and advanced arguments (with V. Marić). *Georgian Math. J.* **14** (2007), No. 2, 301–314.
233. Regularly varying solutions to functional differential equations with deviating argument (with V. Marić). *Bull. Cl. Sci. Math. Nat. Sci. Math.* No. 32 (2007), 105–128.
234. Regularly varying solutions of generalized Thomas-Fermi equations (with V. Marić and T. Tanigawa). *Bull. Cl. Sci. Math. Nat. Sci. Math.* No. 34 (2009), 43–73.
235. Existence of regularly varying solutions with nonzero indices of half-linear differential equations with retarded arguments (with J. Manojlović and T. Tanigawa). *Comput. Math. Appl.* **59** (2010), No. 1, 411–425.
236. Regularly varying solutions of perturbed Euler differential equations and related functional differential equations (with V. Marić). *Publ. Inst. Math. (Beograd) (N.S.)* **88(102)** (2010), 1–20.
237. Precise asymptotic behavior of solutions of the sublinear Emden–Fowler differential equation (with J. Manojlović). *Appl. Math. Comput.* **217** (2011), No. 9, 4382–4396.
238. An asymptotic analysis of positive solutions of generalized Thomas-Fermi differential equations-the sub-half-linear case (with V. Marić and T. Tanigawa). *Nonlinear Anal.* **75** (2012), No. 4, 2474–2485.
239. Positive solutions of fourth order Emden–Fowler type differential equations in the framework of regular variation (with J. V. Manojlović). *Appl. Math. Comput.* **218** (2012), No. 12, 6684–6701.
240. Positive solutions of fourth order Thomas–Fermi type differential equations in the framework of regular variation (with J. Manojlović). *Acta. Appl. Math.* **121** (2012), 81–103; doi 10.1007/s10440-012-9691-5.

Memoirs on Differential Equations and Mathematical Physics
Volume 57, 2012, 17–40

Ravi P. Agarwal, Donal O'Regan, and Mohamed-Aziz Taoudi

FIXED POINT THEORY FOR MULTIVALUED WEAKLY CONVEX-POWER CONDENSING MAPPINGS WITH APPLICATION TO INTEGRAL INCLUSIONS

Abstract. In this paper we present new fixed point theorems for multivalued maps which are convex-power condensing relative to a measure of weak noncompactness and have weakly sequentially closed graph. These results are then used to investigate the existence of weak solutions to a Volterra integral inclusion with lack of weak compactness. In the last section we discuss convex-power condensing multivalued maps with respect to a measure of noncompactness.

2010 Mathematics Subject Classification. 47H10, 47H30.

Key words and phrases. Convex-power condensing multivalued maps, fixed point theorems, measure of weak noncompactness, weak solutions, Volterra integral inclusions.

რეზიუმე. ნაშრომში მოყვანილია ახალი თეორემები უძრავი წერტილის შესახებ მრავალსახა ასახვებისათვის, რომლებიც არიან ამოზნექილ-ხარისხოვანი კუმშვები სუსტი არაკომპაქტურობის ზომის მიმართ და გააჩნიათ სუსტად სეკვენციალურად ჩაკეტილი გრაფიკი. ეს შედეგები შემდეგ გამოყენებულია ვოლტერას ტიპის ინტეგრალური ჩართვების სუსტი ამონახსნების არსებობის საკითხის შესასწავლად სუსტი კომპაქტურობის არარსებობის შემთხვევაში. ბოლო პარაგრაფში განხილულია ამოზნექილ-ხარისხოვანი მრავალსახა ასახვები, რომლებიც არიან კუმშვები არაკომპაქტურობის ზომის მიმართ.

1. Introduction

Since the paper by Szep [32], the theory on the existence of weak solutions to differential equations in Banach spaces has become popular. We quote the contributions of Cramer, Lakshmikantham and Mitchell [6] in 1978 and more recently by Bugajewski [5], Cichon [9], [11], Cichon and Kubiaczyk [10], Mitchell and Smith [23], and O'Regan [24], [25], [26]. Motivated by the paper of Cichon [9], D. O'Regan [30] investigated the existence of weak solutions to the following inclusion which was modelled off a first order differential inclusion [7], [8], [9]

$$x(t) \in x_0 + \int_0^t G(s, x(s))\, ds, \quad t \in [0,T]; \tag{1.1}$$

here $G\colon [0,T] \times E \to 2^E$ and $x_0 \in E$ with E a real reflexive Banach space. The proofs involve a Arino–Gautier–Penot type fixed point theorem for multivalued mappings and the applications depend heavily upon the reflexiveness of the space E. In this paper, we establish existence results for the Volterra integral equation (1.1) in the case where E is nonreflexive. Our approach relies on the concept of convex-power condensing operators with respect to a measure of weak noncompactness. We note that Sun and Zhang [31] introduced the definition of a convex-power condensing operator with respect to the Kuratowski measure of noncompactness for single valued mappings and proved a fixed point theorem which extended the well-known Sadovskii's fixed point theorem and a fixed point theorem in Liu et al. [22]. [35], G. Zhang et al. established some fixed point theorems of Rothe and Altman types about convex-power condensing single valued operators with respect to the Kuratowski measure of noncompactness. These results were applied to a differential equation of first order with integral boundary conditions. In this paper we introduce the concept of a convex-power condensing multivalued operator with respect to a measure of weak noncompactness. We also prove some fixed point principles for this type of operator. Our fixed point results are not only of theoretical interest, but we discuss new applications, namely the existence of solutions to integral inclusions with lack of weak compactness. We illustrate this fact by deriving an existence theory for (1.1) in the case where E is nonreflexive.

For the remainder of this section we gather some notations and preliminary facts. Let X be a Banach space, let $\mathcal{B}(X)$ denote the collection of all nonempty bounded subsets of X and $\mathcal{W}(X)$ the subset of $\mathcal{B}(X)$ consisting of all weakly compact subsets of X. Also, let B_r denote the closed ball centered at 0 with radius r.

In our considerations the following definition will play an important role.

Definition 1.1 ([2])**.** A function $\psi\colon \mathcal{B}(X) \to \mathbb{R}_+$ is said to be a measure of weak noncompactness if it satisfies the following conditions:

(1) The family $\ker(\psi) = \{M \in \mathcal{B}(X) : \ \psi(M) = 0\}$ is nonempty and $\ker(\psi)$ is contained in the set of relatively weakly compact sets of X.

(2) $M_1 \subseteq M_2 \Longrightarrow \psi(M_1) \leq \psi(M_2)$.

(3) $\psi(\overline{co}(M)) = \psi(M)$, where $\overline{co}(M)$ is the closed convex hull of M.

(4) $\psi(\lambda M_1 + (1-\lambda)M_2) \leq \lambda\psi(M_1) + (1-\lambda)\psi(M_2)$ for $\lambda \in [0,1]$.

(5) If $(M_n)_{n\geq 1}$ is a sequence of nonempty weakly closed subsets of X with M_1 bounded and $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ such that $\lim_{n\to\infty} \psi(M_n) = 0$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty.

The family $\ker\psi$ described in (1) is said to be the kernel of the measure of weak noncompactness ψ. Note that the intersection set M_∞ from (5) belongs to $\ker\psi$ since $\psi(M_\infty) \leq \psi(M_n)$ for every n and $\lim_{n\to\infty} \psi(M_n) = 0$. Also, it can be easily verified that the measure ψ satisfies

$$\psi(\overline{M^w}) = \psi(M),$$

where $\overline{M^w}$ is the weak closure of M.

A measure of weak noncompactness ψ is said to be *regular* if

$$\psi(M) = 0 \ \text{ if and only if } M \text{ is relatively weakly compact.}$$

subadditive if

$$\psi(M_1 + M_2) \leq \psi(M_1) + \psi(M_2), \tag{1.2}$$

homogeneous if

$$\psi(\lambda M) = |\lambda|\psi(M), \quad \lambda \in \mathbb{R}, \tag{1.3}$$

set additive (or *have the maximum property*) if

$$\psi(M_1 \cup M_2) = \max(\psi(M_1), \psi(M_2)). \tag{1.4}$$

The first important example of a measure of weak noncompactness has been defined by De Blasi [13] as follows:

$$w(M) = \inf\Big\{r > 0 : \ \text{there exists } W \in \mathcal{W}(X) \text{ with } M \subseteq W + B_r\Big\},$$

for each $M \in \mathcal{B}(X)$.

Notice that $w(.)$ is regular, homogeneous, subadditive and set additive (see [13]).

The following results are crucial for our purposes. We first state a theorem of Ambrosetti type (see [23, 20] for a proof).

Theorem 1.1. *Let E be a Banach space and let $H \subseteq C([0,T],E)$ be bounded and equicontinuous. Then the map $t \to w(H(t))$ is continuous on $[0,T]$ and*

$$w(H) = \sup_{t\in[0,T]} w(H(t)) = w(H[0,T]),$$

where $H(t) = \{h(t)\colon h \in H\}$ and $H[0,T] = \bigcup_{t\in[0,T]} \{h(t)\colon h \in H\}$.

The following auxiliary result will also be needed.

Lemma 1.1 ([31]). *If $H \subseteq C([0,T],E)$ is equicontinuous and $x_0 \in C([0,T],E)$, then $\overline{co}(H \cup \{x_0\})$ is likewise equicontinuous in $C([0,T],E)$.*

In what follows, we shall recall some classical definitions and results regarding multivalued mappings. Let X and Y be topological spaces. A multivalued map $F\colon X \to 2^Y$ is a point to a set function if for each $x \in X$, $F(x)$ is a nonempty subset of Y. For a subset M of X we write $F(M) = \bigcup_{x\in M} F(x)$ and $F^{-1}(M) = \{x \in X : \ F(x) \cap M \neq \varnothing\}$. The *graph* of F is the set $Gr(F) = \{(x,y) \in X \times Y : \ y \in F(x)\}$. We say that F is *upper semicontinuous* (u.s.c. for short) at $x \in X$ if for every neighborhood V of $F(x)$ there exists a neighborhood U of x with $F(U) \subseteq V$ (equivalently, $F : X \to 2^Y$ is u.s.c. if for any net $\{x_\alpha\}$ in X and any closed set B in Y with $x_\alpha \to x_0 \in X$ and $F(x_\alpha) \cap B \neq \varnothing$ for all α, we have $F(x_0) \cap B \neq \varnothing$). We say that $F\colon X \to 2^Y$ is upper semicontinuous if it is upper semicontinuous at every $x \in X$. The function F is lower semicontinuous (l.s.c.) if the set $F^{-1}(B)$ is open for any open set B in Y . If F is l.s.c. and u.s.c., then F is continuous.
If Y is compact, and the images $F(x)$ are closed, then F is upper semicontinuous if and only if F has a closed graph. In this case, if Y is compact, we find that F is upper semicontinuous if $x_n \to x$, $y_n \to y$, and $y_n \in F(x_n)$, together imply that $y \in F(x)$. When X is a Banach space we say that $F\colon X \to 2^X$ is weakly upper semicontinuous if F is upper semicontinuous in X endowed with the weak topology. Also, $F\colon X \to 2^X$ is said to have weakly sequentially closed graph if the graph of F is sequentially closed w.r.t. the weak topology of X. In Section 4 we present fixed point theorems for multivalued convex-power maps with respect to a measure of noncompactness.
Now, we recall the following extension of the Arino–Gautier–Penot fixed point theorem for multivalued mappings. For a proof we refer the reader to [30, Theorem 2.2.].

Theorem 1.2. *Let X be a metrizable locally convex linear topological space and let C be a weakly compact, convex subset of X. Suppose $F\colon C \to \mathcal{C}(C)$ has a weakly sequentially closed graph. Then F has a fixed point; here $\mathcal{C}(C)$ denotes the family of nonempty, closed, convex subsets of C.*

In what follows, let X be a Banach space, C a nonempty closed convex subset of X, $F\colon C \to 2^C$ a multivalued mapping and $x_0 \in C$. For any $M \subseteq C$ we set

$$F^{(1,x_0)}(M)=F(M), \quad F^{(n,x_0)}(M)=F\left(\overline{co}\left(F^{(n-1,x_0)}(M) \cup \{x_0\}\right)\right)$$

for $n = 2, 3, \ldots$.

Definition 1.2. Let X be a Banach space, C a nonempty closed convex subset of X and ψ a measure of weak noncompactness on X. Let $F\colon C \to 2^C$ be a bounded multivalued mapping (that is it takes bounded sets into bounded ones) and $x_0 \in C$. We say that F is a ψ- convex-power condensing operator about x_0 and n_0 if for any bounded set $M \subseteq C$ with $\psi(M) > 0$ we have

$$\psi(F^{(n_0,x_0)}(M)) < \psi(M). \tag{1.5}$$

Obviously, $F\colon C \to 2^C$ is ψ-condensing if and only if it is ψ- convex-power condensing operator about x_0 and 1.

2. Fixed Point Theorems for Multivalued Mappings Relative to the Weak Topology

Theorem 2.1. *Let X be a Banach space and ψ be a regular and set additive measure of weak noncompactness on X. Let C be a nonempty closed convex subset of X, $x_0 \in C$ and n_0 be a positive integer. Suppose $F\colon C \to C(C)$ is ψ-convex-power condensing about x_0 and n_0. If F has weakly sequentially closed graph and $F(C)$ is bounded, then F has a fixed point in C.*

Proof. Let

$$\mathcal{F} = \big\{A \subseteq C,\ \overline{co}(A) = A,\ x_0 \in A \text{ and } F(x) \in C(A) \text{ for all } x \in A\big\}.$$

The set $\mathcal{F}$ is nonempty since $C \in \mathcal{F}$. Set $M = \bigcap_{A \in \mathcal{F}} A$. Now we show that for any positive integer n we have

$$\mathcal{P}(n) \qquad M = \overline{co}\left(F^{(n,x_0)}(M) \cup \{x_0\}\right).$$

To see this, we proceed by induction. Clearly, M is a closed convex subset of C and $F(M) \subseteq M$. Thus $M \in \mathcal{F}$. This implies $\overline{co}(F(M) \cup \{x_0\}) \subseteq M$. Hence $F(\overline{co}(F(M) \cup \{x_0\})) \subseteq F(M) \subseteq \overline{co}(F(M) \cup \{x_0\})$. Consequently, $\overline{co}(F(M) \cup \{x_0\}) \in \mathcal{F}$. Hence $M \subseteq \overline{co}(F(M) \cup \{x_0\})$. As a result $\overline{co}(F(M) \cup \{x_0\}) = M$. This shows that $\mathcal{P}(1)$ holds. Let n be fixed and suppose $\mathcal{P}(n)$ holds. This implies $F^{(n+1,x_0)}(M) = F(\overline{co}\left(F^{(n,x_0)}(M) \cup \{x_0\}\right) = F(M)$. Consequently,

$$\overline{co}\left(F^{(n+1,x_0)}(M) \cup \{x_0\}\right) = \overline{co}(F(M) \cup \{x_0\}) = M. \tag{2.1}$$

As a result

$$\overline{co}\left(F^{(n_0,x_0)}(M) \cup \{x_0\}\right) = M. \tag{2.2}$$

Using the properties of the measure of weak noncompactness, we get

$$\psi(M) = \psi\left(\overline{co}\left(F^{(n_0,x_0)}(M) \cup \{x_0\}\right)\right) = \psi(F^{(n_0,x_0)}(M)),$$

which yields that M is weakly compact. Since $F\colon M \to 2^M$ has weakly sequentially closed graph, the result follows from Theorem 1.2. $\square$

As an easy consequence of Theorem 2.1 we obtain the following sharpening of [30, Theorem 2.3].

Corollary 2.1. *Let X be a Banach space and ψ be a regular and set additive measure of weak noncompactness on X. Let C be a nonempty closed convex subset of X. Assume that $F\colon C \to C(C)$ has weakly sequentially closed graph with $F(C)$ bounded. If F is ψ-condensing, i.e. $\psi(F(M)) <$*

$\psi(M)$, whenever M is a bounded non-weakly compact subset of C, the F has a fixed point.

Remark 2.1. Theorem 2.1 is also an extension of its corresponding results in [28], [29].

Lemma 2.1. *Let $F\colon X \to 2^X$ be convex-power condensing about x_0 and n_0 (n_0 is a positive integer) with respect to a regular and set additive measure of weak noncompactness ψ. Let $\widetilde{F}\colon X \to 2^X$ be the operator defined on X by $\widetilde{F}(x) = F(x+x_0) - x_0$. Then, $\widetilde{F}$ is convex-power condensing about 0 and n_0 with respect to ψ. Moreover, F has a fixed point if $\widetilde{F}$ does.*

Proof. Let M be a bounded subset of X with $\psi(M) > 0$. We claim that for all integer $n \geq 1$, we have

$$\widetilde{F}^{(n,0)}(M) \subseteq F^{(n,x_0)}(M + x_0) - x_0. \tag{2.3}$$

To see this, we shall proceed by induction. Clearly,

$$\widetilde{F}^{(1,0)}(M) = \widetilde{F}(M) = F(M + x_0) - x_0 = F^{(1,x_0)}(M + x_0) - x_0. \tag{2.4}$$

Fix an integer $n \geq 1$ and suppose (2.3) holds. Then

$$\widetilde{F}^{(n,0)}(M) \cup \{0\} \subseteq \overline{co}\left(F^{(n,x_0)}(M + x_0) \cup \{x_0\}\right) - x_0. \tag{2.5}$$

Hence

$$\overline{co}\left(\widetilde{F}^{(n,0)}(M) \cup \{0\}\right) \subseteq \overline{co}\left(F^{(n,x_0)}(M + x_0) \cup \{x_0\}\right) - x_0. \tag{2.6}$$

As a result

$$\begin{aligned}\widetilde{F}^{(n+1,0)}(M) &= \widetilde{F}\left(\overline{co}\left(\widetilde{F}^{(n,0)}(M) \cup \{0\}\right)\right) \subseteq \\ &\subseteq \widetilde{F}\left(\overline{co}\left(F^{(n,x_0)}(M + x_0) \cup \{x_0\}\right) - x_0\right) = \\ &= F\left(\overline{co}\left(F^{(n,x_0)}(M + x_0) \cup \{x_0\}\right) - x_0\right) = \\ &= F^{(n+1,x_0)}(M + x_0) - x_0.\end{aligned}$$

This proves our claim. Consequently,

$$\begin{aligned}\psi(\widetilde{F}^{(n_0,0)}(M))) &\leq \psi(F^{(n_0,x_0)}(M + x_0) - x_0) \leq \\ &\leq \psi((F^{(n_0,x_0)}(M + x_0) < \psi(M + x_0) \leq \psi(M).\end{aligned}$$

This proves the first statement. The second statement is straightforward. $\square$

Theorem 2.2. *Let X be a Banach space and let ψ be a regular and set additive measure of weak noncompactness on X. Let Q and C be closed, convex subsets of X with $Q \subseteq C$. In addition, let U be a weakly open subset of Q with $F(\overline{U^w})$ bounded and $x_0 \in U$. Suppose $F\colon X \to 2^X$ is ψ-power-convex condensing map about x_0 and n_0 (n_0 is a positive integer). If F has* *a weakly sequentially closed graph and* $F(x) \in C(C)$ *for all* $x \in \overline{U^w}$, *then either*

$$F \text{ has a fixed point,} \tag{2.7}$$

or

$$\text{there is a point } u \in \partial_Q U \text{ and } \lambda \in (0,1) \text{ with } u \in \lambda F u; \tag{2.8}$$

here $\partial_Q U$ *is the weak boundary of* U *in* Q.

Proof. By replacing F, Q, C and U by $\widetilde{F}, Q - x_0, C - x_0$ and $U - x_0$ respectively and using Lemma 2.1, we may assume that $0 \in U$ and F is ψ-power-convex condensing about 0 and n_0. Now suppose (2.8) does not occur and F does not have a fixed point on $\partial_Q U$ (otherwise we are finished since (2.7) occurs). Let

$$M = \Big\{ x \in \overline{U^w} : \ x \in \lambda F x \text{ for some } \lambda \in [0,1] \Big\}.$$

The set M is nonempty since $0 \in U$. Also, M is weakly sequentially closed. Indeed, let (x_n) be the sequence of M which converges weakly to some $x \in \overline{U^w}$ and let (λ_n) be a sequence of $[0,1]$ satisfying $x_n \in \lambda_n F x_n$. Then for each n there is a $z_n \in F x_n$ with $x_n = \lambda_n z_n$. By passing to a subsequence if necessary, we may assume that (λ_n) converges to some $\lambda \in [0,1]$ and $\lambda_n \neq 0$ for all n. This implies that the sequence (z_n) converges to some $z \in \overline{U^w}$ with $x = \lambda z$. Since F has a weakly sequentially closed graph, then $z \in F(x)$. Hence $x \in \lambda F x$ and therefore $x \in M$. Thus M is weakly sequentially closed. We now claim that M is relatively weakly compact. Suppose $\psi(M) > 0$. Clearly,

$$M \subseteq co(F(M) \cup \{0\}). \tag{2.9}$$

By induction, note for all positive integers n we have

$$M \subseteq co\left(F^{(n,0)}(M) \cup \{0\}\right). \tag{2.10}$$

Indeed, fix an integer $n \geq 1$ and suppose (2.10) holds. Then

$$F(M) \subseteq F\left(\overline{co}\left(F^{(n,0)}(M) \cup \{0\}\right)\right) = F^{(n+1,0)}(M). \tag{2.11}$$

Hence

$$co(F(M) \cup \{0\}) \subseteq co\left(F^{(n+1,0)}(M) \cup \{0\}\right). \tag{2.12}$$

Combining (2.9) and (2.12), we arrive at

$$M \subseteq co\left(F^{(n+1,0)}(M) \cup \{0\}\right).$$

This proves (2.10). In particular, we have

$$M \subseteq co\left(F^{(n_0,0)}(M) \cup \{0\}\right).$$

Thus,

$$\psi(M) \leq \psi\left(co\left(F^{(n_0,0)}(M) \cup \{0\}\right)\right) = \psi(F(M)) < \psi(M), \tag{2.13}$$

which is a contradiction. Hence $\psi(M) = 0$ and therefore $\overline{M^w}$ is weakly compact. This proves our claim. Let now $x \in \overline{M^w}$. Since $\overline{M^w}$ is weakly compact, then there is a sequence (x_n) in M which converges weakly to x. Since M is weakly sequentially closed, we have $x \in M$. Thus $\overline{M^w} = M$. Hence M is weakly closed and therefore weakly compact. From our assumptions we have $M \cap \partial_Q U = \varnothing$. Since X endowed with the weak topology is a locally convex space, then there exists a weakly continuous mapping $\rho\colon \overline{U^w} \to [0,1]$ with $\rho(M) = 1$ and $\rho(\partial_Q U) = 0$ (see [15]). Let

$$T(x) = \begin{cases} \rho(x)F(x), & x \in \overline{U^w}, \\ 0, & x \in X \setminus \overline{U^w}. \end{cases}$$

Clearly, $T\colon X \to 2^X$ has a weakly sequentially closed graph since F does. Moreover, for any $S \subseteq C$ we have

$$T(S) \subseteq co(F(S) \cup \{0\}).$$

This implies that

$$\begin{aligned} T^{(2,0)}(S) &= T\Big(\overline{co}(T(S) \cup \{0\})\Big) \subseteq T\Big(\overline{co}(F(S) \cup \{0\})\Big) \subseteq \\ &\subseteq \overline{co}\Big(F\Big(\overline{co}(F(S) \cup \{0\}) \cup \{0\}\Big)\Big) = \overline{co}(F^{(2,0)}(S) \cup \{0\}). \end{aligned}$$

By induction,

$$\begin{aligned} T^{(n,0)}(S) &= T\left(\overline{co}(T^{(n-1,0)}(S) \cup \{0\})\right) \subseteq T\left(\overline{co}(F^{(n-1,0)}(S) \cup \{0\})\right) \subseteq \\ &\subseteq \overline{co}\left(F\left(\overline{co}(F^{(n-1,0)}(S) \cup \{0\}) \cup \{0\}\right)\right) = \overline{co}(F^{(n,0)}(S) \cup \{0\}), \end{aligned}$$

for each integer $n \geq 1$. Using the properties of the measure of weak noncompactness, we get

$$\psi(T^{(n_0,0)}(S)) \leq \psi(\overline{co}(F^{(n_0,0)}(S) \cup \{0\})) = \psi(F^{(n_0,0)}(S)) < \psi(S), \quad (2.14)$$

if $\psi(S) > 0$. Thus $T\colon X \to 2^X$ has a weakly sequentially closed graph and $T(x) \subseteq C(C)$ for all $x \in C$. Moreover, T is ψ-power-convex condensing about 0 and n_0. By Theorem 2.1 there exists $x \in C$ such that $w \in Tx$. Now $x \in U$ since $0 \in U$. Consequently, $x \in \rho(x)F(x)$ and so $x \in M$. This implies $\rho(x) = 1$ and so $x \in F(x)$. □

Now we present a fixed point theorem of Furi–Pera type for power-convex condensing multivalued mappings with weakly sequentially closed graph.

Theorem 2.3. *Let X be a Banach space and let ψ be a regular and set additive measure of weak noncompactness on X. Let C be a closed convex subset of X and Q a closed convex subset of C with $F(Q)$ bounded and $0 \in Q$. Also, assume $F\colon X \to 2^X$ has a weakly sequentially closed graph and is ψ-power-convex condensing about* 0 *and n_0 (n_0 is a positive integer) and $F(x) \in C(C)$ for all $x \in Q$. In addition, assume that the following conditions are satisfied:*

(i) *there exists a weakly continuous retraction* $r \colon X \to Q$, *with* $r(D) \subseteq \overline{co}(D \cup \{0\})$ *for any bounded subset* D *of* X *and* $r(x) = x$ *for all* $x \in Q$;

(ii) *there exists a* $\delta > 0$ *and a weakly compact set* Q_δ *with* $\Omega_\delta = \{x \in X : d(x, Q) \leq \delta\} \subseteq Q_\delta$; *here* $d(x, y) = \|x - y\|$;

(iii) *for any* $\Omega_\epsilon = \{x \in X : d(x, Q) \leq \epsilon, 0 < \epsilon \leq \delta\}$, *if* $\{(x_j, \lambda_j)\}_{j=1}^{\infty}$ *is a sequence in* $Q \times [0, 1]$ *with* $x_j \rightharpoonup x \in \partial_{\Omega_\epsilon} Q$, $\lambda_j \to \lambda$ *and* $x \in \lambda F(x), 0 \leq \lambda < 1$, *then* $\lambda_j F(x_j) \subseteq Q$ *for* j *sufficiently large; here* $\partial_{\Omega_\epsilon} Q$ *is the weak boundary of* Q *relative to* Ω_ϵ.

Then F *has a fixed point in* Q.

Proof. Consider $B = \{x \in X : x \in Fr(x)\}$. We first show that $B \neq \varnothing$. To see this, consider $Fr \colon C \to C(C)$. Clearly Fr has a weakly sequentially closed graph, since F has a weakly sequentially closed graph and r is weakly continuous. Now we show that Fr is ψ-power-convex condensing map about 0 and n_0. To see this, let A be a bounded subset of C and set $A' = \overline{co}(A \cup \{0\})$. Then, using assumption (i) we obtain

$$
\begin{aligned}
&(Fr)^{(1,0)}(A) \subseteq F(A'),\\
&(Fr)^{(2,0)}(A) = Fr\left(\overline{co}\left((Fr)^{(1,0)}(A) \cup \{0\}\right)\right) \subseteq\\
&\qquad\qquad \subseteq Fr\left(\overline{co}\left(F(A') \cup \{0\}\right)\right) \subseteq F\left(\overline{co}\left(F(A') \cup \{0\}\right)\right) =\\
&\qquad\qquad = F^{(2,0)}(A'),
\end{aligned}
$$

and by induction,

$$
\begin{aligned}
(Fr)^{(n_0,0)}(A) &= Fr\left(\overline{co}\left((Fr)^{(n_0-1,0)}(A) \cup \{0\}\right)\right) \subseteq\\
&\subseteq Fr\left(\overline{co}\left(F^{(n_0-1,0)}(A') \cup \{0\}\right)\right) \subseteq\\
&\subseteq F\left(\overline{co}\left(F^{(n_0-1,0)}(A') \cup \{0\}\right)\right) =\\
&= F^{(n_0,0)}(A').
\end{aligned}
$$

Thus

$$\psi\left((Fr)^{(n_0,0)}(A)\right) \leq \psi\left(F^{(n_0,0)}(A')\right) < \psi(A') = \psi(A),$$

whenever $\psi(A) \neq 0$. Invoking Theorem 2.1 we infer that there exists $y \in C$ with $y \in Fr(y)$. Thus $y \in B$ and $B \neq \varnothing$. In addition B is weakly sequentially closed, since Fr has a weakly sequentially closed graph. Moreover, we claim that B is weakly compact. To see this, first notice

$$B \subseteq Fr(B) \subseteq F(B') = F^{(1,0)}(B'),$$

where $B' = \overline{co}(B \cup \{0\})$. Thus

$$B \subseteq Fr(B) \subseteq Fr\left(F(B')\right) \subseteq F\left(\overline{co}\left(F(B') \cup \{0\}\right)\right) = F^{(2,0)}(B'),$$

and by induction

$$B \subseteq Fr(B) \subseteq Fr\left(F^{(n_0-1,0)}(B')\right) \subseteq$$
$$\subseteq F\left(\overline{co}\left(F^{(n_0-1,0)}(B') \cup \{0\}\right)\right) = F^{(n_0,0)}(B'),$$

Now if $\psi(B) \neq 0$, then

$$\psi(B) \leq \psi(F^{(n_0,0)}(B')) < \psi(B') = \psi(B),$$

which is a contradiction. Thus, $\psi(B) = 0$ and so B is relatively weakly compact and therefore $Fr(B)$ is relatively weakly compact, since r is weakly continuous and F has a sequentially closed graph. Now let $x \in \overline{B^w}$. Since $\overline{B^w}$ is weakly compact then there is a sequence (x_n) of elements of B which converges weakly to some x. Since B is weakly sequentially closed then $x \in B$. Thus, $\overline{B^w} = B$. This implies that B is weakly compact. We now show that $B \cap Q \neq \varnothing$. Suppose $B \cap Q = \varnothing$. Then, since B is weakly compact and Q is weakly closed we have from [16] that $d(B, Q) > 0$. Thus there exists ϵ, $0 < \epsilon < \delta$, with $\Omega_\epsilon \cap B = \varnothing$; here $\Omega_\epsilon = \{x \in X : \ d(x, Q) \leq \epsilon\}$. Now Ω_ϵ is closed convex and $\Omega_\epsilon \subseteq Q_\delta$. From our assumptions it follows that Ω_ϵ is weakly compact. Also since X is separable then the weak topology on Ω_ϵ is metrizable [14], [34], let d^* denote the metric. For $i \in \{0, 1 \ldots\}$, let

$$U_i = \left\{x \in \Omega_\epsilon : \ d^*(x, Q) < \frac{\epsilon}{i}\right\}.$$

For each $i \in \{0, 1 \ldots\}$ fixed, U_i is open with respect to d and so U_i is weakly open in Ω_ϵ. Also, $\overline{U_i^w} = \overline{U_i^d} = \left\{x \in \Omega_\epsilon : \ d^*(x, Q) \leq \epsilon/i\right\}$ and $\partial_{\Omega_\epsilon} U_i = \left\{x \in \Omega_\epsilon : d^*(x, Q) = \epsilon/i\right\}$. Keeping in mind that $\Omega_\epsilon \cap B = \varnothing$, Theorem 2.2 guarantees that there exists $y_i \in \partial_{\Omega_\epsilon} U_i$ and $\lambda_i \in (0, 1)$ with $y_i \in \lambda_i Fr(y_i)$. We now consider $D = \left\{x \in X : \ x \in \lambda Fr(x) \text{ for some } \lambda \in [0, 1]\right\}$.

First notice

$$D \subseteq Fr(D) \cup \{0\}.$$

Thus

$$D \subseteq Fr(D) \cup \{0\} \subseteq Fr\Big(\overline{co}\left(Fr(D) \cup \{0\}\right)\Big) \cup \{0\} = (Fr)^{(2,0)} \cup \{0\},$$

and by induction

$$D \subseteq Fr(D) \cup \{0\} \subseteq$$
$$\subseteq Fr\Big(\overline{co}\left((Fr)^{(n_0-1,0)}(D) \cup \{0\}\right)\Big) \cup \{0\} = (Fr)^{(n_0,0)} \cup \{0\},$$

Consequently,

$$\psi(D) \leq \psi\left((Fr)^{(n_0,0)} \cup \{0\}\right) \leq \psi\left((Fr)^{(n_0,0)}\right).$$

Since Fr is ψ-convex-power condensing about 0 and n_0 then $\psi(D) = 0$ and so D is relatively weakly compact.

The same reasoning as above implies that D is weakly compact. Then, up to a subsequence, we may assume that $\lambda_i \to \lambda^* \in [0, 1]$ and $y_i \rightharpoonup y^* \in \partial_{\Omega_\epsilon} U_i$.

Since F has a weakly sequentially closed graph then $y^* \in \lambda^* Fr(y^*)$. Notice $\lambda^* Fr(y^*) \nsubseteq Q$ since $y^* \in \partial_{\Omega_\epsilon} U_i$. Thus $\lambda^* \neq 1$ since $B \cap Q = \varnothing$. From assumption (iii) it follows that $\lambda_i Fr(y_i) \subseteq Q$ for j sufficiently large, which is a contradiction. Thus $B \cap Q \neq \varnothing$, so there exists $x \in Q$ with $x \in Fr(x)$, i.e. $x \in Fx$. □

Remark 2.2. In Theorem 2.3, we need $F\colon X \to 2^X$ ψ-convex-power condensing about 0 and n_0. However, the condition $F\colon X \to 2^X$ has weakly sequentially closed graph can be replaced by $F\colon Q \to 2^X$ has weakly sequentially closed graph.

3. Existence Results

In this section we shall discuss the existence of weak solutions to the Volterra integral inclusion

$$x(t) \in x_0 + \int_0^t G(s, x(s))\, ds, \quad t \in [0, T]; \tag{3.1}$$

here $G\colon [0,T] \times E \to C(E)$ and $x_0 \in E$ with E is a real Banach space. The integral in (3.1) is understood to be the Pettis integral and solutions to (3.1) will be sought in $C([0,T], E)$.

This equation will be studied under the following assumptions:

(i) for each continuous function $x\colon [0,T] \to E$ there exists a scalarly measurable function $v\colon [0,T] \to E$ with $v(t) \in G(t, x(t))$ a.e. on $[0,T]$ and v is Pettis integrable on $[0,T]$;

(ii) for any $r > 0$ there exists $\theta_r \in L^1[0,T]$ with $|G(t,u)| \leq \theta_r(t)$ for a.e. $t \in [0,T]$ and all $u \in E$ with $|z| \leq r$; here $|G(t,u)| = \sup\{|w| : w \in G(t,u)\}$;

(iii) there exists $\alpha \in L^1[0,T]$ and $\theta\colon [0,+\infty) \to (0,+\infty)$ a nondecreasing continuous function such that $|G(s,u)| \leq \alpha(s)\theta(|u|)$ for a.e. $s \in [0,t]$, and all $u \in E$, with
$$\int_0^T \alpha(s)\, ds < \int_{|x_0|}^{\infty} \frac{dx}{\theta(x)}\, ;$$

(iv) there is a constant $\tau \geq 0$ such that for any bounded subset S of E and for any $t \in [0,T]$ we have
$$w(G([0,t] \times S)) \leq \tau w(S);$$

(v) if (x_n) is a sequence of continuous functions from $[0,T]$ into E which converges weakly to x and if (v_n) is a sequence of Pettis integrable functions from $[0,T]$ into E such that $v_n(s)$ converges weakly to $v(s)$ and $v_n(s) \in G(s, x_n(s))$ for a.e. $s \in [0,T]$, then v is Pettis integrable with $v(s) \in G(s, x(s))$ for a.e. $s \in [0,T]$.

Theorem 3.1. *Let E be a Banach space and suppose* (i)–(iv) *hold. Then* (3.1) *has a solution in* $C([0,T],E)$.

Proof. Define a multivalued operator

$$F\colon C([0,T],E) \to C(C([0,T],E)). \tag{3.2}$$

by letting

$$Fx(t) = \left\{ x_0 + \int_0^t v(s)\,ds : \ v\colon [0,T] \to E \text{ Pettis integrable with } v(t) \in G(t,x(t)) \text{ a.e. } t \in [0,T] \right\}. \tag{3.3}$$

We first show that (3.2)–(3.3) make sense. To see this, let $x \in C([0,T],E)$. In view of our assumptions there exists a Pettis integrable $v\colon [0,T] \to E$ with $v(t) \in G(t,x(t))$ for a.e. $t \in [0,T]$. Thus F is well defined. Let $u(t) = x_0 + \int_0^t v(s)\,ds$. To see that $u \in C([0,T],E)$ first notice that there exists $r > 0$ with $|y| = \sup_{[0,T]} |x(t)| \le r$. From assumption (iii) it readily follows that there exists $\theta_r \in L^1[0,T]$ with

$$|G(t,x(t))| \le \theta_r(t) \ \text{ for a.e. } \ t \in [0,T]. \tag{3.4}$$

Let $t, t' \in [0,T]$ with $t < t'$. Without loss of generality assume $u(t) - u(t') \ne 0$. Invoking the Hahn–Banach theorem we deduce that there exists $\phi \in E^*$ (the topological dual of E) with $|\phi| = 1$ and $|u(t) - u(t')| = \phi(u(t) - u(t'))$. Thus

$$|u(t) - u(t')| = \phi\left(\int_t^{t'} v(s)\,ds \right) \le \int_t^{t'} \theta_r(s)\,ds.$$

Consequently, $u \in C([0,T],E)$. Our next task is to show that F has closed (in $C([0,T],E)$) values (note F has automatically convex values). Let $x \in C([0,T],E)$. Suppose $w_n \in Fx$, $n = 1,2,\dots$. Then there exists Pettis integrable $v_n\colon [0,T] \to E$, $n = 1,2,\dots$ with $v_n(s) \in G(s,x(s))$ a.e. $s \in [0,T]$. Suppose

$$w_n(t) \to x_0 + \int_0^t v(s)\,ds = w(t) \ \text{ in } \ C([0,T],E). \tag{3.5}$$

Fix $t \in (0,T]$ and $\phi \in E^*$. Then $\phi(v_n) \to \phi(v)$ in $L^1[0,t]$ so $\phi(v_n) \to \phi(v)$ in measure. Thus there exists a subsequence S of integers with

$$\phi(v_n(s)) \to \phi(v(s)) \ \text{ for a.e. } \ s \in [0,t] \quad (\text{as } \ n \to \infty \ \text{ in } \ S). \tag{3.6}$$

Now since $v_n(s) \in G(s, x(s))$ for a.e. $s \in [0,t]$ and since the values of G are closed and convex (so weakly closed) we have $v(s) \in G(s, x(s))$ for a.e. $s \in [0,t]$. Thus $w \in Fx$ and so F has closed (in $C([0,T],E)$) values. Now let

$$C = \Big\{ x \in C([0,T],E) : \ |x(t)| \le b(t) \text{ for } t \in [0,T] \text{ and } \\ |x(t) - x(s)| \le |b(t) - b(s)| \text{ for } t, s \in [0,T] \Big\},$$

where

$$b(t) = I^{-1}\Big(\int_0^t \alpha(s)\, ds \Big) \ \text{ and } \ I(z) = \int_{|x_0|}^{z} \frac{dx}{\theta(x)}\,.$$

Notice C is a closed, convex, bounded, equicontinuous subset of $C([0,T],E)$ with $0 \in C$. Let F be as defined in (3.2)–(3.3). We claim that $F(C) \subseteq C$. To see this take $u \in F(C)$. Then there exists $y \in C$ with $u \in Fy$ and there exists a Pettis integrable $v \colon [0,T] \to E$ with $u(t) = x_0 + \int_0^t v(s)\, ds$ and $v(t) \in G(t, y(t))$ for a.e. $t \in [0,T]$. Without loss of generality, assume $u(s) \neq 0$ for all $s \in [0,T]$. Then there exists $\phi_s \in E^*$ with $|\phi_s| = 1$ and $\phi_s(u(s)) = |u(s)|$. Consequently, for each fixed $t \in [0,T]$, we have

$$|u(t)| = \phi_t(u(t)) \le |x_0| + \int_0^t \alpha(s)\theta(|y(s)|)\, ds \le \\ \le |x_0| + \int_0^t \alpha(s)\theta(b(s))\, ds = |x_0| + \int_0^t b'(s)\, ds = b(t),$$

since

$$\int_{|x_0|}^{b(s)} \frac{dx}{\theta(x)} = \int_0^s \alpha(x)\, dx.$$

Next suppose t, $t' \in [0,T]$ with $t > t'$. Without loss of generality, assume $u(t) - u(t') \neq 0$. Then there exists $\phi \in E^*$ with $|\phi| = 1$ and $\phi(u(t) - u(t')) = |u(t) - u(t')|$. Consequently,

$$|u(t) - u(t')| \le \int_{t'}^{t} \alpha(s)\theta(|y(s)|)\, ds \le \\ \le \int_{t'}^{t} \alpha(s)\theta(|b(s)|)\, ds = \int_{t'}^{t} b'(s)\, ds = b(t) - b(t').$$

Thus, $u \in C$. This proves our claim. Our next task is to show that F has a weakly sequentially closed graph. To see this, let (x_n, y_n) be a sequence in $C \times C$ with $x_n \rightharpoonup x$, $y_n \rightharpoonup y$ and $y_n \in Fx_n$. Then for each $t \in [0,T]$ we have

$$y_n(t) = x_0 + \int_0^t v_n(s)\, ds \tag{3.7}$$

with $v_n\colon [0,T] \to E$, $n = 1, 2, \ldots$ Pettis integrable and $v_n(s) \in G(s, x_n(s))$ a.e. $s \in [0,T]$. Recall [23], since C is equicontinuous, that $x_n \rightharpoonup x$ if and only if $x_n(t) \rightharpoonup x(t)$ for each $t \in [0,T]$ and $y_n \rightharpoonup y$ if and only if $y_n(t) \rightharpoonup y(t)$ for each $t \in [0,T]$. Fix $t \in [0,T]$. Since $x_n(s) \rightharpoonup x(s)$ for each $s \in [0,t]$, then $S := \{x_n(s) :\ n \in \mathbb{N}\}$ is a relatively weakly compact subset of E for each $s \in [0,t]$. Using the fact that the De Blasi measure of weak noncompactness is regular we get $w(S) = 0$. From assumption (iv) it follows that $w(G([0,t] \times S) = 0$. Keeping in mind that $v_n(s) \in G(s, x_n(s))$ for a.e. $s \in [0,t]$ we obtain

$$\{v_n(s) :\ n \in \mathbb{N}\} \subseteq G([0,t] \times S)$$

for a.e. $s \in [0,t]$. Hence $w(\{v_n(s) : n \in \mathbb{N}\}) = 0$ for a.e. $s \in [0,t]$. This implies that the set $\{v_n(s) :\ n \in \mathbb{N}\}$ is relatively weakly compact for a.e. $s \in [0,t]$. Hence, by passing to a subsequence if necessary, we may assume that the sequence $v_n(s)$ is weakly convergent in E for a.e. $s \in [0,t]$. Let $v(s)$ be its weak limit. From our assumptions it follows that $v\colon [0,T] \to E$ is Pettis integrable and $v(s) \in G(s, x(s))$ for a. e. $s \in [0,t]$. The Lebesguev Dominated Convergence Theorem for the Pettis integral [18, Corollary 4] implies for each $\phi \in E^*$ that $\phi(y_n(t)) \to \phi\big(x_0 + \int_0^t v(s)\, ds\big)$ i.e. $y_n(t) \rightharpoonup x_0 + \int_0^t v(s)ds$. We can do this for each $t \in [0,T]$. Consequently, $y(t) = x_0 + \int_0^t v(s)ds \in Fx(t)$ for each $t \in [0,T]$, i.e. $y \in Fx$. Now we show that there is an integer n_0 such that F is w-power-convex condensing about 0 and n_0. To see this notice, for each bounded set $H \subseteq C$ and for each $t \in [0,T]$, that

$$F(H)(t) \subseteq x_0 + t\overline{co}\left(G\left([0,t] \times H[0,t]\right)\right). \tag{3.8}$$

Using the properties of the weak measure of noncompactness we get

$$\begin{gathered} w(F^{(1,0)}(H)(t)) = w(F(H)(t)) \leq \\ \leq tw\left(\overline{co}\left(G\left([0,t] \times H[0,t]\right)\right)\right) \leq tw(G([0,t] \times H[0,t]) \leq t\tau w(H[0,t]). \end{gathered}$$

Theorem 1.1 implies (since H is equicontinuous) that

$$w(F^{(1,0)}(H)(t)) \leq t\tau w(H). \tag{3.9}$$

Since $F^{(1,0)}(H)$ is equicontinuous, it follows from Lemma 1.1 that $F^{(2,0)}(H)$ is equicontinuous. Using (3.9) we get

$$w(F^{(2,0)}(H)(t)) =$$

$$= w\Big(\Big\{x_0 + \int_0^t v(s)\,ds\colon v(s) \in G(s, x(s)),\ x \in \overline{co}(F^{(1,0)}(H) \cup \{0\})\Big\}\Big) \le$$

$$\le w\Big(\Big\{\int_0^t v(s)\,ds\colon v(s) \in G(s, x(s)),\ x \in \overline{co}(F^{(1,0)}(H) \cup \{0\})\Big\}\Big) =$$

$$= w\Big(\Big\{\int_0^t v(s)\,ds\colon v(s) \in G(s, x(s)),\ x \in V\Big\}\Big),$$

where $V = \overline{co}(F^{(1,0)}(H) \cup \{0\})$. Fix $t \in [0, T]$. We divide the interval $[0, t]$ into m parts $0 = t_0 < t_1 < \cdots < t_m = t$ in such a way that $\Delta t_i = t_i - t_{i-1} = \frac{t}{m}$, $i = 1, \dots, m$. For each $x \in V$ and for each $v(s) \in G(s, x(s))$ we have

$$\int_0^t v(s)\,ds = \sum_{i=1}^m \int_{t_{i-1}}^{t_i} v(s)\,ds \in \sum_{i=1}^m \Delta t_i \overline{co}\big\{v(s)\colon\ s \in [t_{i-1}, t_i]\big\} \subseteq$$

$$\subseteq \sum_{i=1}^m \Delta t_i \overline{co}\Big(G\big([t_{i-1}, t_i] \times V([t_{i-1}, t_i])\big)\Big).$$

Using again Theorem 1.1 we infer that for each $i = 2, \dots, m$ there is a $s_i \in [t_{i-1}, t_i]$ such that

$$\sup_{s \in [t_{i-1}, t_i]} w(V(s)) = w(V[t_{i-1}, t_i]) = w(V(s_i)). \tag{3.10}$$

Consequently,

$$w\Big\{\int_0^t v(s)ds\colon x \in V\Big\} \le \sum_{i=1}^m \Delta t_i w(\overline{co}\Big(G\big([t_{i-1}, t_i] \times V([t_{i-1}, t_i])\big)\Big) \le$$

$$\le \tau \sum_{i=1}^m \Delta t_i w\big(\overline{co}(V([t_{i-1}, t_i])\big) \le \tau \sum_{i=1}^m \Delta t_i w(V((s_i)).$$

On the other hand, if $m \to \infty$ then

$$\sum_{i=1}^m \Delta t_i w(V((s_i)) \longrightarrow \int_0^t w(V(s))\,ds. \tag{3.11}$$

Using the regularity, the set additivity, the convex closure invariance of the De Blasi measure of weak noncompactness together with (3.9) we obtain

$$w(V(s)) = w(F^{(1,0)}(H)(s)) \le s\tau w(H) \tag{3.12}$$

and therefore

$$\int_0^t w(V(s))\,ds \le s\tau \frac{t^2}{2}\, w(H). \tag{3.13}$$

As a result

$$w(F^{(2,0)}(H)(t)) \le \frac{(\tau t)^2}{2}\, w(H). \tag{3.14}$$

By induction we get

$$w(F^{(n,0)}(H)(t)) \le \frac{(\tau t)^n}{n!}\, w(H). \tag{3.15}$$

Invoking Theorem 1.1 we obtain

$$w(F^{(n,0)}(H)) \le \frac{(\tau T)^n}{n!}\, w(H). \tag{3.16}$$

Since $\lim\limits_{n\to\infty} \frac{(\tau T)^n}{n!} = 0$, then there is a n_0 with $\frac{(\tau T)^{n_0}}{n_0!} < 1$. This implies

$$w(F^{(n_0,0)}(H)) < w(H). \tag{3.17}$$

Consequently, F is w-power-convex condensing about 0 and n_0. The result follows from Theorem 2.1. □

4. Multivalued Convex-Power Maps with Respect to a Measure of Noncompactness

In this section we shall prove some fixed point theorems for multivalued mappings relative to the strong topology on a Banach space. By a measure of noncompactness on a Banach space X we mean a map $\alpha\colon \mathcal{B}(X) \to \mathbb{R}_+$ which satisfies conditions (1)–(5) in Definition 1.1 relative to the strong topology instead of the weak topology. The concept of a measure of noncompactness was initiated by the fundamental papers of Kuratowski [21] and Darbo [12]. Measures of noncompactness play a very important role in nonlinear analysis, namely in the theories of differential and integral equations. Specifically, the so-called Kuratowski measure of noncompactness [21] and Hausdorff (or ball) measure of noncompactness [3] are frequently used. We say that a bounded multivalued mapping $F\colon C \to 2^C$, defined on a nonempty closed convex subset C of X, is a α-convex-power condensing operator about x_0 and n_0 if for any bounded set $M \subseteq C$ with $\alpha(M) > 0$ we have

$$\alpha(F^{(n_0,x_0)}(M)) < \alpha(M). \tag{4.1}$$

Clearly, $F\colon C \to 2^C$ is α-condensing if and only if it is α- convex-power condensing operator about x_0 and 1. We first state the following result:

Theorem 4.1. *Let X be a Banach space and α be a regular and set additive measure of noncompactness on X. Let C be a nonempty closed convex subset of X, $x_0 \in C$ and n_0 be a positive integer. Suppose $F\colon C \to C(C)$ is α-convex-power condensing about x_0 and n_0. If F has a closed graph with $F(C)$ bounded then F has a fixed point in C.*

Proof. Let

$$\mathcal{F} = \Big\{ A \subseteq C,\ \overline{co}(A) = A,\ x_0 \in A \text{ and } F(x) \in C(A) \text{ for all } x \in A \Big\}.$$

The set $\mathcal{F}$ is nonempty since $C \in \mathcal{F}$. Set $M = \bigcap\limits_{A \in \mathcal{F}} A$. The reasoning in Theorem 2.1 shows that for all integer $n \geq 1$ we have:

$$M = \overline{co}\big(F^{(n,x_0)}(M) \cup \{x_0\}\big) \tag{4.2}$$

Using the properties of the measure of noncompactness we get

$$\alpha(M) = \alpha\Big(\overline{co}\big(F^{(n_0,x_0)}(M) \cup \{x_0\}\big)\Big) = \alpha\left(F^{(n_0,x_0)}(M)\right),$$

which yields that M is compact. Since $F\colon M \to 2^M$ has a closed graph then F is upper semi-continuous. The result follows from the Bohnenblust–Karlin fixed point theorem [4]. □

As an easy consequence of Theorem 4.1 we obtain the following result.

Corollary 4.1. *Let X be a Banach space and α be a regular and set additive measure of noncompactness on X. Let C be a nonempty closed convex subset of X. Assume that $F\colon C \to C(C)$ has a closed graph with $F(C)$ bounded. If F is α-condensing, i.e. $\alpha(F(M)) < \alpha(M)$, whenever M is a bounded non-compact subset of C, then F has a fixed point.*

Lemma 4.1. *Let $F\colon X \to 2^X$ be α-convex-power condensing about x_0 and n_0 (n_0 is a positive integer), where α is a regular and set additive measure of noncompactness. Let $\widetilde{F}\colon X \to 2^X$ be the operator defined on X by $\widetilde{F}(x) = F(x + x_0) - x_0$. Then, $\widetilde{F}$ is α-convex-power condensing about 0 and n_0. Moreover, F has a fixed point if $\widetilde{F}$ does.*

Proof. Let M be a bounded subset of X with $\alpha(M) > 0$. The reasoning in Lemma 2.1 yields that for all integer $n \geq 1$, we have

$$\widetilde{F}^{(n,0)}(M) \subseteq F^{(n,x_0)}(M + x_0) - x_0.$$

Hence

$$\begin{aligned}\alpha\left(\widetilde{F}^{(n_0,0)}(M))\right) &\leq \alpha\left(F^{(n_0,x_0)}(M + x_0) - x_0\right) \leq \\ &\leq \alpha\left((F^{(n_0,x_0)}(M + x_0)\right) < \alpha(M + x_0) \leq \alpha(M).\end{aligned}$$

This proves the first statement. The second statement is straightforward. □

Theorem 4.2. *Let X be a Banach space and let α be a regular and set additive measure of noncompactness on X. Let Q and C be closed, convex subsets of X with $Q \subseteq C$. In addition, let U be an open subset of Q with $F(\overline{U})$ bounded and $x_0 \in U$. Suppose $F\colon X \to 2^X$ is α-power-convex condensing map about x_0 and n_0 (n_0 is a positive integer). If F has a closed graph and $F(x) \in C(C)$ for all $x \in \overline{U}$, then either*

$$F \text{ has a fixed point,} \tag{4.3}$$

or

$$\text{there is a point } u \in \partial_Q U \text{ and } \lambda \in (0,1) \text{ with } u \in \lambda F u; \tag{4.4}$$

here $\partial_Q U$ is the boundary of U in Q.

Proof. By replacing F, Q, C and U by $\widetilde{F}, Q - x_0, C - x_0$ and $U - x_0$ respectively and using Lemma 4.1 we may assume that $0 \in U$ and F is α-power-convex condensing about 0 and n_0. Now suppose (4.4) does not occur and F does not have a fixed point on $\partial_Q U$ (otherwise we are finished since (4.3) occurs). Let $M = \{x \in \overline{U} : \ x \in \lambda F x \text{ for some } \lambda \in [0,1]\}$. The set M is nonempty since $0 \in U$. Also M is closed. Indeed let (x_n) be sequence of M which converges to some $x \in \overline{U}$ and let (λ_n) be a sequence of $[0,1]$ satisfying $x_n \in \lambda_n F x_n$. Then for each n there is a $z_n \in F x_n$ with $x_n = \lambda_n z_n$. By passing to a subsequence if necessary, we may assume that (λ_n) converges to some $\lambda \in [0,1]$ and $\lambda_n \neq 0$ for all n. This implies that the sequence (z_n) converges to some $z \in \overline{U}$ with $x = \lambda z$. Since F has a closed graph then $z \in F(x)$. Hence $x \in \lambda F x$ and therefore $x \in M$. Thus M is closed. We now claim that M is relatively compact. Suppose $\alpha(M) > 0$. Clearly,

$$M \subseteq co(F(M) \cup \{0\}).$$

Arguing by induction as in the proof of Theorem 2.2, we can prove that for all integer $n \geq 1$ we have

$$M \subseteq co(F^{(n,0)}(M) \cup \{0\}).$$

This implies

$$\alpha(M) \leq \alpha\Big(co\big(F^{(n_0,0)}(M) \cup \{0\}\big)\Big) = \alpha(F(M)) < \alpha(M), \tag{4.5}$$

which is a contradiction. Hence $\alpha(M) = 0$ and therefore M is compact, since M is closed. From our assumptions we have $M \cap \partial_Q U = \varnothing$. By Urysohn Lemma [15] there exists a continuous mapping $\rho \colon \overline{U} \to [0,1]$ with $\rho(M) = 1$ and $\rho(\partial_Q U) = 0$. Let

$$T(x) = \begin{cases} \rho(x) F(x), & x \in \overline{U}, \\ 0, & x \in X \setminus \overline{U}. \end{cases}$$

Clearly, $T \colon X \to 2^X$ has a closed graph since F does. Moreover, for any $S \subseteq C$ we have

$$T(S) \subseteq co(F(S) \cup \{0\}).$$

This implies that

$$\begin{aligned} T^{(2,0)}(S) &= T\left(\overline{co}(T(S) \cup \{0\})\right) \subseteq T\left(\overline{co}(F(S) \cup \{0\})\right) \subseteq \\ &\subseteq \overline{co}\Big(F(\overline{co}(F(S) \cup \{0\}) \cup \{0\})\Big) = \overline{co}(F^{(2,0)}(S) \cup \{0\}). \end{aligned}$$

By induction

$$T^{(n,0)}(S) = T\left(\overline{co}(T^{(n-1,0)}(S) \cup \{0\})\right) \subseteq T\left(\overline{co}(F^{(n-1,0)}(S) \cup \{0\})\right) \subseteq$$
$$\subseteq \overline{co}\left(F(\overline{co}(F^{(n-1,0)}(S) \cup \{0\}) \cup \{0\})\right) = \overline{co}(F^{(n,0)}(S) \cup \{0\}),$$

for each integer $n \geq 1$. Using the properties of the measure of noncompactness we get

$$\alpha(T^{(n_0,0)}(S)) \leq \alpha\left(\overline{co}(F^{(n_0,0)}(S) \cup \{0\})\right) = \alpha(F^{(n_0,0)}(S)) < \alpha(S), \quad (4.6)$$

if $\alpha(S) > 0$. Thus $T\colon X \to 2^X$ has a closed graph and $T(x) \subseteq C(C)$ for all $x \in C$. Moreover, T is α-power-convex condensing about 0 and n_0. By Theorem 4.1 there exists $x \in C$ such that $x \in Tx$. Now $x \in U$ since $0 \in U$. Consequently, $x \in \rho(x)F(x)$ and so $x \in M$. This implies $\rho(x) = 1$ and so $x \in F(x)$. □

Theorem 4.3. *Let X be a Banach space and α a regular set additive measure of noncompactness on X. Let Q be a closed convex subset of X with $0 \in Q$ and n_0 a positive integer. Assume $F\colon X \to 2^X$ has a sequentially closed graph with $F(Q)$ bounded and $F(x) \in C(X)$ for all $x \in Q$. Also assume F is α-convex-power condensing about 0 and n_0 and*

$$\begin{cases} \text{if } \{(x_j, \lambda_j)\} \text{ is a sequence in } \partial Q \times [0,1] \\ \text{converging to } (x,\lambda) \text{ with } x \in \lambda F(x) \text{ and } 0 < \lambda < 1, \\ \text{then } \lambda_j F(x_j) \subseteq Q \text{ for } j \text{ sufficiently large} \end{cases} \quad (4.7)$$

holding. Also suppose the following condition holds:

$$\begin{cases} \text{there exists a continuous retraction } r\colon X \to Q \\ \text{with } r(z) \in \partial Q \text{ for } z \in X \setminus Q \text{ and } r(D) \subseteq co(D \cup \{0\}) \\ \text{for any bounded subset } D \text{ of } X. \end{cases} \quad (4.8)$$

Then, F has a fixed point.

Proof. Let $r\colon X \to Q$ be as described in (4.8). Consider $B = \{x \in X : x = Fr(x)\}$.

We first show that $B \neq \varnothing$. To see this, consider $Fr\colon X \to C(X)$. Clearly Fr has a sequentially closed graph, since F has a sequentially closed graph and r is continuous. Now we show that Fr is α-power-convex condensing map about 0 and n_0. To see this, let A be a bounded subset of X and set $A' = \overline{co}(A \cup \{0\})$. Then, using (4.8) we obtain

$$\begin{aligned}
(Fr)^{(1,0)}(A) &\subseteq F(A'),\\
(Fr)^{(2,0)}(A) &= Fr\left(\overline{co}\left((Fr)^{(1,0)}(A) \cup \{0\}\right)\right) \subseteq\\
&\subseteq Fr\left(\overline{co}\left(F(A') \cup \{0\}\right)\right) \subseteq F\left(\overline{co}\left(F(A') \cup \{0\}\right)\right) =\\
&= F^{(2,0)}(A'),
\end{aligned}$$

and by induction,

$$\begin{aligned}(Fr)^{(n_0,0)}(A) &= Fr\left(\overline{co}\left((Fr)^{(n_0-1,0)}(A)\cup\{0\}\right)\right)\subseteq\\ &\subseteq Fr\left(\overline{co}\left(F^{(n_0-1,0)}(A')\cup\{0\}\right)\right)\subseteq\\ &\subseteq F\left(\overline{co}\left(F^{(n_0-1,0)}(A')\cup\{0\}\right)\right)=\\ &= F^{(n_0,0)}(A').\end{aligned}$$

Thus

$$\alpha\left((Fr)^{(n_0,0)}(A)\right)\le\alpha\left(F^{(n_0,0)}(A')\right)<\alpha(A')=\alpha(A),$$

whenever $\alpha(A)\neq 0$. Invoking Theorem 4.1 we infer that there exists $y\in X$ with $y\in Fr(y)$. Thus $y\in B$ and $B\neq\varnothing$. In addition B is closed, since Fr has a sequentially closed graph. Moreover, we claim that B is compact. To see this, first notice

$$B\subseteq Fr(B)\subseteq F(B')=F^{(1,0)}(B'),$$

where $B'=\overline{co}(B\cup\{0\})$. Thus

$$B\subseteq Fr(B)\subseteq Fr\left(F(B')\right)\subseteq F\left(\overline{co}\left(F(B')\cup\{0\}\right)\right)=F^{(2,0)}(B'),$$

and by induction

$$\begin{aligned}B\subseteq Fr(B)\subseteq Fr\left(F^{(n_0-1,0)}(B')\right)\subseteq&\\ \subseteq F\left(\overline{co}\left(F^{(n_0-1,0)}(B')\cup\{0\}\right)\right)&=F^{(n_0,0)}(B'),\end{aligned}$$

Now if $\alpha(B)\neq 0$, then

$$\alpha(B)\le\alpha(F^{(n_0,0)}(B'))<\alpha(B')=\alpha(B),$$

which is a contradiction. Thus, $\alpha(B)=0$ and so B is relatively compact. Consequently, $B=\overline{B}$ is compact. We now show that $B\cap Q\neq\varnothing$. To do this, we argue by contradiction. Suppose $B\cap Q=\varnothing$. Then since B is compact and Q is closed there exists $\delta>0$ with $dist(B,Q)>\delta$. Choose $N\in\{1,2,\ldots\}$ such that $N\delta>1$. Define

$$U_i=\left\{x\in X:\ d(x,Q)<1/i\right\}\ \text{ for }\ i\in\{N,N+1,\ldots\};$$

here $d(x,Q)=\inf\{\|x-y\|:\ y\in Q\}$. Fix $i\in\{N,N+1,\ldots\}$. Since $dist(B,Q)>\delta$ then $B\cap\overline{U_i}=\varnothing$. Applying Theorem 4.2 to $Fr\colon\overline{U_i}\to C(X)$ we may deduce that there exists $(y_i,\lambda_i)\in\partial U_i\times(0,1)$ with $y_i=\lambda_i Fr(y_i)$. Notice in particular since $y_i\in\partial U_i\times(0,1)$ that

$$\lambda_i Fr(y_i)\notin Q\ \text{ for }\ i\in\{N,N+1,\ldots\}. \tag{4.9}$$

We now consider

$$D=\Big\{x\in X:\ x=\lambda Fr(x)\text{ for some }\lambda\in[0,1]\Big\}.$$

Clearly D is closed since F has a sequentially closed graph and r is continuous. Now we claim that D is compact. To see this, first notice

$$D \subseteq Fr(D) \cup \{0\}.$$

Thus

$$D \subseteq Fr(D) \cup \{0\} \subseteq Fr\Big(\overline{co}\,(Fr(D) \cup \{0\})\Big) \cup \{0\} = (Fr)^{(2,0)} \cup \{0\},$$

and by induction

$$D \subseteq Fr(D) \cup \{0\} \subseteq Fr\left(\overline{co}\left((Fr)^{(n_0-1,0)}(D) \cup \{0\}\right)\right) \cup \{0\} = (Fr)^{(n_0,0)} \cup \{0\},$$

Consequently,

$$\alpha(D) \leq \alpha\left((Fr)^{(n_0,0)} \cup \{0\}\right) \leq \alpha\left((Fr)^{(n_0,0)}\right).$$

Since Fr is α-convex-power condensing about 0 and n_0 then $\alpha(D) = 0$ and so D is relatively weakly compact. Consequently, $D = \overline{D}$ is compact. Then, up to a subsequence, we may assume that $\lambda_i \to \lambda^* \in [0,1]$ and $y_i \to y^* \in \partial U_i$. Hence $\lambda_i Fr(y_i) \to \lambda^* Fr(y^*)$ and therefore $y^* = \lambda^* Fr(y^*)$. Notice $\lambda^* Fr(y^*) \notin Q$ since $y^* \in \partial U_i$. Thus $\lambda^* \neq 1$ since $B \cap Q = \varnothing$. From assumption (4.7) it follows that $\lambda_i Fr(y_i) \in Q$ for j sufficiently large, which is a contradiction. Thus $B \cap Q \neq \varnothing$, so there exists $x \in Q$ with $x = Fr(x)$, i.e. $x = Fx$. □

Remark 4.1. If $0 \in int(Q)$ then we can choose $r\colon X \to Q$ in the statement of Theorem 4.3 as

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \quad \text{for} \quad x \in X;$$

here μ is the Minkowski functional [33] defined by

$$\mu(x) = \inf\big\{\lambda > 0 : \; x \in \lambda Q\big\},$$

for all $x \in X$. Clearly r is continuous, $r(X) \subseteq Q$ and $r(x) = x$ for all $x \in Q$. Also, for any subset A of X we have $r(A) \subseteq co(A \cup \{0\})$.

Remark 4.2. In Theorem 4.3, we need $F\colon X \to 2^X$ α-convex-power condensing about 0 and n_0. However, the condition $F\colon X \to 2^X$ has sequentially closed graph can be replaced by $F\colon Q \to 2^X$ has sequentially closed graph.

References

1. O. Arino, S. Gautier, and J.-P. Penot, A fixed point theorem for sequentially continuous mappings with application to ordinary differential equations. *Funkcial. Ekvac.* **27** (1984), No. 3, 273–279.
2. J. Banaś and J. Rivero, On measures of weak noncompactness. *Ann. Mat. Pura Appl. (4)* **151** (1988), 213–224.
3. J. Banaś and K. Goebel, Measures of noncompactness in Banach spaces. *Lecture Notes in Pure and Applied Mathematics*, 60. *Marcel Dekker, Inc., New York*, 1980.

4. H. F. Bohnenblust and S. Karlin, On a theorem of Ville. C*ontributions to the Theory of Games*, 155–160. *Annals of Mathematics Studies*, no. 24. *Princeton University Press, Princeton, N. J.*, 1950.
5. D. Bugajewski, On the existence of weak solutions of integral equations in Banach spaces. *Comment. Math. Univ. Carolin.* **35** (1994), No. 1, 35–41.
6. E. Cramer, V. Lakshmikantham, and A. R. Mitchell, On the existence of weak solutions of differential equations in nonreflexive Banach spaces. *Nonlinear Anal.* **2** (1978), No. 2, 169–177.
7. S. N. Chow and J. D. Schuur, An existence theorem for ordinary differential equations in Banach spaces. *Bull. Amer. Math. Soc.* **77** (1971), 1018–1020.
8. S. N. Chow and J. D. Schuur, Fundamental theory of contingent differential equations in Banach space. *Trans. Amer. Math. Soc.* **179** (1973), 133–144.
9. M. Cichoń, Weak solutions of differential equations in Banach spaces. *Discuss. Math. Differential Incl.* **15** (1995), No. 1, 5–14.
10. M. Cichoń and I. Kubiaczyk, Existence theorem for the Hammerstein integral equation. *Discuss. Math. Differential Incl.* **16** (1996), No. 2, 171–177.
11. M. Cichoń, On solutions of differential equations in Banach spaces. *Nonlinear Anal.* **60** (2005), No. 4, 651–667.
12. G. Darbo, Punti uniti in trasformazioni a codominio non compatto. (Italian) *Rend. Sem. Mat. Univ. Padova* **24** (1955), 84–92.
13. F. S. De Blasi, On a property of the unit sphere in a Banach space. *Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.)* **21(69)** (1977), No. 3-4, 259–262.
14. N. Dunford and J. T. Schwartz, Linear Operators. I. General Theory. With the assistance of W. G. Bade and R. G. Bartle. *Pure and Applied Mathematics*, Vol. 7 *Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London*, 1958.
15. R. Engelking, General topology. Translated from the Polish by the author. Second edition. *Sigma Series in Pure Mathematics*, 6. *Heldermann Verlag, Berlin*, 1989.
16. K. Floret,Weakly compact sets. *Lectures held at S.U.N.Y., Buffalo, in Spring* 1978. *Lecture Notes in Mathematics*, 801. *Springer, Berlin*, 1980.
17. J. García-Falset, Existence of fixed points and measures of weak noncompactness. *Nonlinear Anal.* **71** (2009), No. 7-8, 2625–2633.
18. R. F. Geitz, Pettis integration. *Proc. Amer. Math. Soc.* **82** (1981), No. 1, 81–86.
19. I. Kubiaczyk, On a fixed point theorem for weakly sequentially continuous mappings. *Discuss. Math. Differential Incl.* **15** (1995), No. 1, 15–20.
20. I. Kubiaczyk and S. Szufla, Kneser's theorem for weak solutions of ordinary differential equations in Banach spaces. *Publ. Inst. Math. (Beograd) (N.S.)* **32(46)** (1982), 99–103.
21. C. Kuratowski, Sur les espaces complets. *Fundamenta* **15** (1930), 301–309.
22. L. Liu, F. Guo, C. Wu, and Y. Wu, Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces. *J. Math. Anal. Appl.* **309** (2005), No. 2, 638–649.
23. A. R. Mitchell and C. Smith, An existence theorem for weak solutions of differential equations in Banach spaces. *Nonlinear equations in abstract spaces (Proc. Internat. Sympos., Univ. Texas, Arlington, Tex., 1977)*, pp. 387–403, *Academic Press, New York*, 1978.
24. D. O'Regan, Weak solutions of ordinary differential equations in Banach spaces. *Appl. Math. Lett.* **12** (1999), No. 1, 101–105.
25. D. O'Regan, Integral equations in reflexive Banach spaces and weak topologies. *Proc. Amer. Math. Soc.* **124** (1996), No. 2, 607–614.
26. D. O'Regan, Operator equations in Banach spaces relative to the weak topology. *Arch. Math. (Basel)* **71** (1998), No. 2, 123–136.
27. D. O'Regan, Fixed-point theory for weakly sequentially continuous mappings. *Math. Comput. Modelling* **27** (1998), No. 5, 1–14.

28. D. O'Regan, Fixed point theory for weakly contractive maps with applications to operator inclusions in Banach spaces relative to the weak topology. *Z. Anal. Anwendungen* **17** (1998), No. 2, 281–296.
29. R. P. Agarwal and D. O'Regan, Fixed point theory for weakly sequentially upper semicontinuous maps with applications to differential inclusions. *Nelīnīīnī Koliv.* **5** (2002), No. 3, 287–296; translation in *Nonlinear Oscil. (N. Y.)* **5** (2002), No. 3, 277–286.
30. D. O'Regan, Fixed point theorems for weakly sequentially closed maps. *Arch. Math. (Brno)* **36** (2000), No. 1, 61–70.
31. J. X. Sun and X. Y. Zhang, The fixed point theorem of convex-power condensing operator and applications to abstract semilinear evolution equations. (Chinese) *Acta Math. Sin.* **48** (2005), No. 3, 439–446.
32. A. Szép, Existence theorem for weak solutions of ordinary differential equations in reflexive Banach spaces. *Studia Sci. Math. Hungar.* **6** (1971), 197–203.
33. A. E. Taylor and D. C. Lay, Introduction to functional analysis. Second edition. *John Wiley & Sons, New York–Chichester–Brisbane*, 1980.
34. E. Zeidler, Nonlinear functional analysis and its applications. I. Fixed-point theorems. Translated from the German by Peter R. Wadsack. *Springer-Verlag, New York*, 1986.
35. G. Zhang, T. Zhang, and T. Zhang, Fixed point theorems of Rothe and Altman types about convex-power condensing operator and application. *Appl. Math. Comput.* **214** (2009), No. 2, 618–623.

(Received 28.04.2011)

Authors' addresses:

Ravi P. Agarwal

Department of Mathematics, Texas A&M University - Kingsville, 700 University Blvd., Kingsville, TX 78363-8202, USA.

e-mail: `Ravi.Agarwal@tamuk.edu`

Donal O'Regan

Department of Mathematics, National University of Ireland, Galway, Ireland.

e-mail: `donal.oregan@nuigalway.ie`

Mohamed-Aziz Taoudi

Université Cadi Ayyad, Laboratoire de Mathématiques et de Dynamique, de Populations, Marrakech, Maroc.

e-mail: `taoudi@ucam.ac.ma`

Memoirs on Differential Equations and Mathematical Physics
Volume 57, 2012, 41–50

Jaroslav Jaroš

GENERALIZED PICONE IDENTITY AND COMPARISON OF HALF-LINEAR DIFFERENTIAL EQUATIONS OF ORDER $4m$

Dedicated to Professor Kusano Takaŝi on the occasion of his 80th birthday anniversary

Abstract. A Picone-type identity and the Sturm-type comparison theorems are established for ordinary differential equations of the form

$$\big(p(t)\varphi(u^{(2m)})\big)^{(2m)} + q(t)\varphi(u) = 0$$

and

$$\big(P(t)\varphi(v^{(2m)})\big)^{(2m)} + Q(t)\varphi(v) = 0,$$

where $m \geq 1$, $p, P \in C^{2m}([a,b],(0,\infty))$, $q, Q \in C([a,b],\mathbf{R})$, $\varphi(s) := |s|^\alpha \operatorname{sgn} s$ and $\alpha > 0$.

2010 Mathematics Subject Classification. 34C10.

Key words and phrases. Picone's identity, comparison, half-linear differential operator.

რეზიუმე. ჩვეულებრივი დიფერენციალური განტოლებებისათვის

$$\big(p(t)\varphi(u^{(2m)})\big)^{(2m)} + q(t)\varphi(u) = 0$$

და

$$\big(P(t)\varphi(v^{(2m)})\big)^{(2m)} + Q(t)\varphi(v) = 0,$$

სადაც $m \geq 1$, $p, P \in C^{2m}([a,b],(0,\infty))$, $q, Q \in C([a,b],\mathbf{R})$, $\varphi(s) := |s|^\alpha \operatorname{sgn} s$ და $\alpha > 0$, დადგენილია პიკონეს ტიპის იგივობა და შტურმის შედარების თეორემა.

1. Introduction

In the classical Sturm comparison theory for linear self-adjoint differential equations of the second order a fundamental role plays by the so-called Picone's formula (see [14]). It states that if x, px', y and Py' are continuously differentiable functions on an interval I with $y(t) \neq 0$, then

$$\frac{d}{dt}\left[\frac{x}{y}\left(px'y - Pxy'\right)\right] =$$
$$= -\frac{x^2}{y}\,(Py')' + x(px')' + (p-P)x'^2 + P\left(x' - \frac{x}{y}\,y'\right)^2. \tag{1.1}$$

If, in addition, x and y solve in I the equations

$$-\big(p(t)u'\big)' + q(t)u = 0 \tag{1.2}$$

and

$$-\big(P(t)v'\big)' + Q(t)v = 0, \tag{1.3}$$

respectively, where $0 < P(t) \leq p(t)$ and $Q(t) \leq q(t)$ in I, and x have consecutive zeros at a and b ($a < b$), then integrating (1.1) between a and b, we obtain

$$0 = \int_a^b \left[\big(q(t) - Q(t)\big)x^2 + \big(p(t) - P(t)\big)x'^2 + P(t)\Big(x' - \frac{x}{y}\,y'\Big)^2\right] dt \tag{1.4}$$

and the Sturmian conclusion about the existence of a zero in $[a, b]$ for any solution y of the majorant equation (1.3) readily follows from (1.4).

Generalizations and extensions of the Sturm's comparison principle and underlying Picone-type identities to nonlinear equations and higher-order (ordinary and partial) differential operators have been obtained by various authors. We refer, in particular, to the papers [1]–[17] and the references cited therein.

The purpose of the present paper is to extend (1.1) to half-linear ordinary differential operators of the form

$$l_\alpha[x] \equiv \big(p\varphi(x^{(2m)})\big)^{(2m)} + q\varphi(x) \tag{1.5}$$

and

$$L_\alpha[y] \equiv \big(P\varphi(y^{(2m)})\big)^{(2m)} + Q\varphi(y), \tag{1.6}$$

where $m \geq 1$, $p, P \in C^{2m}([a, b], (0, \infty))$, $q, Q \in C([a, b], \mathbf{R})$ and $\varphi(s) := |s|^{\alpha-1}s$ for $s \neq 0$, $\alpha > 0$, and $\varphi(0) = 0$. Next, in Section 3, we illustrate the usefulness of the obtained identity by deriving Sturm's comparison theorems and other qualitative results concerning half-linear differential equations of the order $4m$.

In the linear case, i.e. if (1.5) and (1.6) reduce to a pair of $4m$th-order self-adjoint operators of the form $l_1[x] \equiv (px^{(2m)})^{(2m)} + qx$ and $L_1[y] \equiv (Py^{(2m)})^{(2m)} + Qy$, respectively, two different kinds of Picone-type identities are known in the literature. The first one which can be found in Kusano et al. [12] says (when specialized to (1.5) and (1.6)), that if $x \in D_{l_1}(I)$, $y \in D_{L_1}(I)$, and none of $y, y', \dots, y^{(2m-1)}$ vanishes in I, then

$$\frac{d}{dt}\left\{\sum_{k=0}^{2m-1}(-1)^k \frac{x^{(k)}}{y^{(k)}}\left[x^{(k)}(Py^{(2m)})^{(2m-k-1)} - y^{(k)}(px^{(2m)})^{(2m-k-1)}\right]\right\} =$$
$$= \frac{x^2}{y}L_1[y] - xl_1[x] + (q-Q)x^2 + (p-P)\big[x^{(2m)}\big]^2 +$$
$$+P\left[x^{(2m)} - \frac{x^{(2m-1)}}{y^{(2m-1)}}\,y^{(2m)}\right]^2 - y^{(2m-1)}\big(Py^{(2m)}\big)'\left[\frac{x^{(2m-1)}}{y^{(2m-1)}} - \frac{x^{(2m-2)}}{y^{(2m-2)}}\right]^2. \quad (1.7)$$

A typical comparison result based on the above formula is the following theorem (see [12]).

Theorem A. *Suppose there exists a nontrivial real-valued function* $u \in \mathcal{D}_{l_1}([a,b])$ *which satisfies*

$$\int_a^b ul_1[u]\,dt \le 0,$$
$$u(a) = u'(a) = \dots = u^{(2m-1)}(a) = u(b) = \dots = u^{(2m-1)}(b) = 0$$

and

$$\int_a^b \Big[\big(p(t) - P(t)\big)\big(u^{(2m)}\big)^2 + \big(q(t) - Q(t)\big)u^2\Big]\,dt \ge 0.$$

If $v \in \mathcal{D}_{L_1}([a,b])$ *satisfies*

$$vL_1[v] \ge 0 \ \ in \ \ (a,b), \quad where \ \ P(t) \ge 0,$$
$$v^{(k)}\big[P(t)v^{(2m)}\big]^{(2m-k)} \ge 0 \ \ in \ \ (a,b), \quad 1 \le k \le 2m-1,$$

and

$$\big[P(t)v^{(2m)}\big]^{(2m-\nu)} \ne 0 \ \ in \ \ (a,b) \ \ for\ some \ \ \nu, \quad 1 \le \nu \le 2m-1,$$

then at least one of $v, v', \dots, v^{(2m-1)}$ *has a zero in* (a,b).

Recently, Kusano–Yoshida's formula (1.7) was generalized to half-linear ordinary differential operators of an arbitrary even order (see [5]).

The second Picone type identity applied to (1.5) and (1.6) has been obtained by N. Yoshida [16]. The specialization to the one-dimensional case studied here says that if $x \in D_{l_1}(I)$, $y \in D_{L_1}(I)$ and none of $y, y', \dots, y^{(2m-2)}$ vanishes in I, then

$$\frac{d}{dt}\left\{\sum_{k=0}^{m-1}\frac{x^{(2m-2k-2)}}{y^{(2m-2k-2)}}\left[x^{(2m-2k-2)}\big(Py^{(2m)}\big)^{(2k+1)} - \right.\right.$$
$$\left.- \, y^{(2m-2k-2)}\big(px^{(2m)}\big)^{(2k+1)}\right] +$$

$$+\sum_{k=0}^{m-1}\left[\big(px^{(2m)}\big)^{(2m-2k-2)}x^{(2k+1)}-\big(Py^{(2m)}\big)^{(2k)}\left(\frac{\big(x^{(2m-2k-2)}\big)^2}{y^{(2m-2k-2)}}\right)'\right]\Bigg\}=$$

$$=\frac{x^2}{y}\,L_1[y]-xl_1[x]+(p-P)\big[x^{(2m)}\big]^2+(q-Q)x^2+$$

$$+P\Big[x^{(2m)}-\frac{x^{(2m-2)}}{y^{(2m-2)}}\,y^{(2m)}\Big]^2+$$

$$+\sum_{k=1}^{m-1}\frac{\big(Py^{(2m)}\big)^{(2k)}}{y^{(2m-2k)}}\left[x^{(2m-2k)}-\frac{x^{(2m-2k-2)}}{y^{(2m-2k-2)}}y^{(2m-2k)}\right]^2-$$

$$-2\sum_{k=0}^{m-1}\frac{\big(Py^{(2m)}\big)^{(2k)}}{y^{(2m-2k-2)}}\left[x^{(2m-2k-1)}-\frac{x^{(2m-2k-2)}}{y^{(2m-2k-2)}}\,y^{(2m-2k-1)}\right]^2. \quad (1.8)$$

The following comparison theorem can be easily obtained with the help of the identity (1.8) (see [16]).

Theorem B. *Assume that there exists a nontrivial function* $u\in D_{l_1}([a,b])$ *which satisfies*

$$\int_a^b ul_1[u]\,dt\le 0,$$

$$u(a)=u'(a)=\cdots=u^{(2m-1)}(a)=u(b)=u'(b)=\cdots=u^{(2m-1)}(b)=0$$

and

$$V[u]\equiv\int_a^b\left[(p(t)-P(t))\big(u^{(2m)}\big)^2+(q(t)-Q(t))u^2\right]dt\ge 0.$$

If $v\in D_{L_1}([a,b])$ *satisfies*

$$L_1[v]\ge 0 \quad \textit{in } (a,b),$$
$$(-1)^k v^{(2k)}(t)>0 \ \textit{ at some point } \ t\in(a,b),\ \ 0\le k\le m-1,$$
$$(-1)^{m+k}\big(Pv^{(2m)}\big)^{(2k)}\ge 0 \ \textit{ in } \ (a,b),\ \ 0\le k\le m-2,$$
$$\big(Pv^{(2m)}\big)^{(2m-2)}<0 \ \textit{ in } \ (a,b),$$

then at least one of the functions $v,v',\dots,v^{(2m-2)}$ *must vanish at some point of* $[a,b]$.

2. The Generalized Picone's Identity

Let $p,P\in C^{2m}([a,b],(0,\infty))$, $m\ge 1$ and $q,Q\in C([a,b],\mathbf{R})$. For a fixed $\alpha>0$ we define the function $\varphi:\mathbf{R}\to\mathbf{R}$ by $\varphi(s)=|s|^{\alpha-1}s$ for $s\ne 0$ and $\varphi(0)=0$, and consider ordinary differential operators of the form

$$l_\alpha[x]=\big(p(t)\varphi(x^{(2m)})\big)^{(2m)}+q(t)\varphi(x)$$

and

$$L_\alpha[y] = \big(P(t)\varphi(y^{(2m)})\big)^{(2m)} + Q(t)\varphi(y)$$

with the domains $D_{l_\alpha}(a,b)$ (resp., $D_{L_\alpha}(a,b)$) defined to be the sets of all functions x (resp., y) of the class $C^{2m}([a,b],\mathbf{R})$ such that $p\varphi(x^{(2m)})$ (resp., $P\varphi(y^{(2m)})$) are in $C^{2m}((a,b),\mathbf{R})\bigcap C([a,b],\mathbf{R})$.

Also, by Φ_α we denote the form defined for $X,Y\in\mathbf{R}$ and $\alpha>0$ by

$$\Phi_\alpha(X,Y) := |X|^{\alpha+1} + \alpha|Y|^{\alpha+1} - (\alpha+1)X\varphi(Y).$$

According to the Young inequality, it follows that $\Phi_\alpha(X,Y)\ge 0$ for all $X,Y\in\mathbf{R}$ and the equality holds if and only if $X=Y$.

We begin with the following lemma which can be verified by a routine computation.

Lemma 2.1. *If $x\in C^{2m}([a,b],\mathbf{R})$, $y\in D_{L_\alpha}((a,b))$ and none of $y,y',\dots,$ $y^{(2m-2)}$ vanishes in (a,b), then*

$$\begin{aligned}
&\frac{d}{dt}\Bigg\{\sum_{k=0}^{m-1}\Bigg[-\frac{|x^{(2m-2k-2)}|^{\alpha+1}}{\varphi(y^{(2m-2k-2)})}\big(P\varphi(y^{(2m)})\big)^{(2k+1)}+\\
&\qquad+\bigg(\frac{|x^{(2m-2k-2)}|^{\alpha+1}}{\varphi(y^{(2m-2k-2)})}\bigg)'\big(P\varphi(y^{(2m)})\big)^{(2k)}\Bigg]\Bigg\}=\\
&=-\frac{|x|^{\alpha+1}}{\varphi(y)}L_\alpha[y]+Q|x|^{\alpha+1}+P|x^{(2m)}|^{\alpha+1}-P\Phi_\alpha\Big(x^{(2m)},\frac{x^{(2m-2)}}{y^{(2m-2)}}\,y^{(2m)}\Big)-\\
&\quad-\sum_{k=1}^{m-1}\frac{\big(P\varphi(y^{(2m)})\big)^{(2k)}}{\varphi\big(y^{(2m-2k)}\big)}\Phi_\alpha\Big(x^{(2m-2k)},\frac{x^{(2m-2k-2)}}{y^{(2m-2k-2)}}\,y^{(2m-2k)}\Big)+\\
&\qquad+\alpha(\alpha+1)\sum_{k=0}^{m-1}\frac{\big(P\varphi(y^{(2m)})\big)^{(2k)}}{\varphi\big(y^{(2m-2k-2)}\big)}\big|x^{(2m-2k-2)}\big|^{\alpha-1}\times\\
&\qquad\qquad\times\Big[x^{(2m-2k-1)}-\frac{x^{(2m-2k-2)}}{y^{(2m-2k-2)}}\,y^{(2m-2k-1)}\Big]^2. \qquad (2.1)
\end{aligned}$$

We now establish a *stronger form* of Picone's identity in which the relatively weak hypothesis from Lemma 2.1 that x is any $2m$-times continuously differentiable function is replaced by the assumption that x is from the domain $\mathcal{D}_{l_\alpha}$ of the operator l_α.

Lemma 2.2. *If $x\in D_{l_\alpha}((a,b))$, $y\in D_{L_\alpha}((a,b))$ and none of $y,y',\dots,$ $y^{(2m-2)}$ vanishes in (a,b), then*

$$\begin{aligned}
&\frac{d}{dt}\Bigg\{\sum_{k=0}^{m-1}\Bigg[\frac{|x^{(2m-2k-2)}|^{\alpha+1}}{\varphi(y^{(2m-2k-2)})}\big(P\varphi(y^{(2m)})\big)^{(2k+1)}-\\
&\qquad-\big(P\varphi(y^{(2m)})\big)^{(2k)}\bigg(\frac{|x^{(2m-2k-2)}|^{\alpha+1}}{\varphi(y^{(2m-2k-2)})}\bigg)'+
\end{aligned}$$

$$+\big(p\varphi(x^{(2m)})\big)^{(2m-2k-2)}x^{(2k+1)}-x^{(2m-2k-2)}\big(p\varphi(x^{(2m)})\big)^{(2k+1)}\Big]\Bigg\}=$$

$$=\frac{|x|^{\alpha+1}}{\varphi(y)}\,L_\alpha[y]-xl_\alpha[x]+(p-P)|x^{(2m)}|^{\alpha+1}+(q-Q)|u|^{\alpha+1}+$$

$$+P\Phi_\alpha\Big(x^{(2m)},\frac{x^{(2m-2)}}{y^{(2m-2)}}\,y^{(2m)}\Big)+$$

$$+\sum_{k=1}^{m-1}\frac{\Big(P\varphi(y^{(2m)})\Big)^{(2k)}}{\varphi(y^{(2m-2k)})}\,\Phi_\alpha\Big(x^{(2m-2k)},\frac{x^{(2m-2k-2)}}{y^{(2m-2k-2)}}\,y^{(2m-2k)}\Big)-$$

$$-\alpha(\alpha+1)\sum_{k=0}^{m-1}\frac{\big(P\varphi(y^{(2m)})\big)^{(2k)}}{\varphi(y^{(2m-2k-2)})}\,\big|x^{(2m-2k-2)}\big|^{\alpha-1}\times$$

$$\times\Big[x^{(2m-2k-1)}-\frac{x^{(2m-2k-2)}}{y^{(2m-2k-2)}}y^{(2m-2k-1)}\Big]^2. \quad (2.2)$$

3. Applications

As the first application of the identity (2.1) we obtain the following result.

Theorem 3.1. *If there exists a nontrivial function* $u\in C^{2m}([a,b],\mathbf{R})$ *such that*

$$u(a)=u'(a)=\cdots=u^{(2m-1)}(a)=u(b)=\cdots=u^{(2m-1)}(b)=0 \quad (3.1)$$

and

$$M_\alpha[u]\equiv\int_a^b\Big[P(t)|u^{(2m)}|^{\alpha+1}+Q(t)|u|^{\alpha+1}\Big]\,dt\le 0, \quad (3.2)$$

then there does not exist a $v\in\mathcal{D}_{L_\alpha}([a,b])$ *satisfying*

$$L_\alpha[v]\ge 0 \;\; in \;\; (a,b), \quad (3.3)$$

$$v(a)>0,\quad v(b)>0, \quad (3.4)$$

$$(-1)^k v^{(2k)}>0 \;\; in \;\; [a,b],\;\; 1\le k\le m-1, \quad (3.5)$$

$$(-1)^{m+k}\big(P\varphi(v^{(2m)})\big)^{(2k)}\ge 0 \;\; in \;\; (a,b),\;\; 0\le k\le m-2, \quad (3.6)$$

and

$$\big(P\varphi(v^{(2m)})\big)^{(2m-2)}<0 \;\; in \;\; (a,b). \quad (3.7)$$

Proof. Suppose to the contrary that there exists a $v\in\mathcal{D}_{L_\alpha}([a,b])$ satisfying (3.3)–(3.7). Since $v(a)>0$, $v(b)>0$ and $v''(t)<0$ in (a,b), it follows that $v(t)>0$ on $[a,b]$. Integrating the identity (2.1) on $[a,b]$, we obtain

$$0\ge M_\alpha[u]-\int_a^b\frac{|u|^{\alpha+1}}{v^\alpha}\,L_\alpha[v]\,dt\ge$$

$$\geq -\alpha(\alpha+1)\int_a^b \frac{(P\varphi(v^{(2m)}))^{(2m-2)}}{v^\alpha}|u|^{\alpha-1}\Big(u'-\frac{u}{v}v'\Big)^2\,dt \geq 0.$$

It follows that $u' - uv'/v = 0$ in (a,b) and therefore $u/v = k$ in $[a,b]$ for some nonzero constant k. Since $u(a) = u(b) = 0$ and $v(a) > 0, v(b) > 0$, we have a contradiction. Hence there can exist no v satisfying (3.3)–(3.7). □

Theorem 3.2. *If there exists a nontrivial* $u \in C^{2m}([a,b],\mathbf{R})$ *satisfying* (3.1) *and* (3.2)*, then every solution* $v \in \mathcal{D}_{L_\alpha}((a,b))$ *of the inequality* (3.3) *satisfying* (3.5)–(3.7) *and*

$$v(t_0) > 0 \ \textit{ for some } \ t_0 \in (a,b) \tag{3.8}$$

has zero in $[a,b]$.

Proof. If the function v satisfies (3.3), (3.5)–(3.7) and (3.8), then either $v(a) < 0$, and hence v, must vanish somewhere in (a,b), or $v(a) \geq 0$. In the latter case, however, Theorem 3.1 implies that $v(a) = 0$ or $v(b) = 0$, and thus the proof is complete. □

As an application of the identity (2.2), we derive the Sturm-type comparison theorem. It belongs to weak comparison results in the sense that the conclusion regarding to v applies to $[a,b]$ rather than (a,b).

Theorem 3.3. *If there exists a nontrivial* $u \in \mathcal{D}_{l_\alpha}((a,b))$ *such that*

$$\int_a^b u l_\alpha[u]\,dt \leq 0, \tag{3.9}$$

$$u(a) = u'(a) = \cdots = u^{(2m-1)}(a) = u(b) = \cdots = u^{(2m-1)}(b) = 0, \tag{3.10}$$

$$V_\alpha[u] \equiv \int_a^b \Big[\big(p(t)-P(t)\big)|u^{(2m)}|^{\alpha+1} + \big(q(t)-Q(t)\big)|u|^{\alpha+1}\Big]\,dt \geq 0, \tag{3.11}$$

and if $v \in \mathcal{D}_{L_\alpha}((a,b))$ *satisfies*

$$L_\alpha[v] \geq 0 \ \textit{ in } \ (a,b), \tag{3.12}$$

$$(-1)^k v^{(2k)}(t_k) > 0 \ \textit{ at some point } \ t_k \in (a,b), \ \ 0 \leq k \leq m-1, \tag{3.13}$$

$$(-1)^{m+k}\big(P\varphi(v^{(2m)})\big)^{(2k)} \geq 0 \ \textit{ in } \ (a,b), \ \ 0 \leq k \leq m-2, \tag{3.14}$$

and

$$\big(P\varphi(v^{(2m)})\big)^{(2m-2)} < 0 \ \textit{ in } \ (a,b), \tag{3.15}$$

then at least one of $v, v'', \ldots, v^{(2m-2)}$ *vanishes somewhere in* $[a,b]$.

Proof. Suppose that none of $v, v', \ldots, v^{(2m-2)}$ vanishes in $[a,b]$. From the identity (2.2) integrated on $[a,b]$ we obtain, in view of the the conditions of the theorem, that

$$0=V_\alpha[u]+\int_a^b \frac{|u|^{\alpha+1}}{v^\alpha}\,L_\alpha[v]\,dt-\int_a^b u l_\alpha[u]\,dt+\int_a^b P\Phi_\alpha\Big(u^{(2m)},\frac{u^{(2m-2)}}{v^{(2m-2)}}\,v^{(2m)}\Big)\,dt+$$

$$+\int_a^b\bigg\{\sum_{k=1}^{m-1}\frac{\big(P\varphi(v^{(2m)})\big)^{(2k)}}{\varphi(v^{(2m-2k)})}\,\Phi_\alpha\Big(u^{(2m-2k)},\frac{u^{(2m-2k-2)}}{v^{(2m-2k-2)}}\,v^{(2m-2k)}\Big)\bigg\}\,dt-$$

$$-\alpha(\alpha+1)\int_a^b\bigg\{\sum_{k=0}^{m-1}\frac{\big(P\varphi(v^{(2m)})\big)^{(2k)}}{\varphi(v^{(2m-2k-2)})}\,|u^{(2m-2k-2)}|^{\alpha-1}\times$$

$$\times\Big[u^{2m-2k-1)}-\frac{u^{(2m-2k-2)}}{v^{(2m-2k-2)}}\,v^{(2m-2k-1)}\Big]^2\bigg\}\,dt\geq$$

$$\geq-\alpha(\alpha+1)\int_a^b\frac{\big(P\varphi(v^{(2m)})\big)^{(2m-2)}}{v^\alpha}\,|u|^{\alpha-1}\Big(u'-\frac{u}{v}\,v'\Big)^2\,dt\geq 0.$$

Consequently, $u'-uv'/v=0$ in (a,b), that is, $u/v=k$ in (a,b), and hence on $[a,b]$ by continuity, for some nonzero constant k. However, this is not the case since $u(a)=u(b)=0$, whereas $v(t)>0$ on $[a,b]$. This contradiction shows that at least one of $v,v',\dots,v^{(2m-2)}$ must vanish in $[a,b]$. □

Finally, we use the identity (2.2) to obtain a lower bound for the first eigenvalue of the nonlinear eigenvalue problem

$$l_\alpha[u]=\lambda\varphi(u)\ \text{ in }\ (a,b), \tag{3.16}$$

$$u(a)=u'(a)=\cdots=u^{(2m-1)}(a)=u(b)=\cdots=u^{(2m-1)}(b)=0. \tag{3.17}$$

Theorem 3.4. *Let λ_1 be the first eigenvalue of the problem* (3.16)–(3.17) *and $u_1\in\mathcal{D}_{l_\alpha}((a,b))$ be the corresponding eigenfunction. If there exists a function $v\in\mathcal{D}_{L_\alpha}((a,b))$ such that*

$$(-1)^k v^{(2k)}>0\ \text{ in }\ [a,b],\ \ 0\leq k\leq m-1,$$

$$(-1)^{m+k}\big(P\varphi(v^{(2m)})\big)^{(2k)}\geq 0\ \text{ in }\ (a,b),\ \ 0\leq k\leq m-1,$$

and if $V_\alpha[u_1]\geq 0$, then $\lambda_1\geq\inf\limits_{t\in(a,b)}\big[\frac{L_\alpha[v]}{v^\alpha}\big]$.

Proof. The identity (2.2) in view of the above hypotheses implies that

$$\lambda_1\int_a^b|u_1|^{\alpha+1}\,dt-\int_a^b|u_1|^{\alpha+1}\frac{L_\alpha[v]}{v^\alpha}\,dt\geq 0,$$

from which the conclusion follows readily. □

References

1. D. R. Dunninger, A Picone integral identity for a class of fourth order elliptic differential inequalities. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **50** (1971), 630–641.

2. M. S. P. Eastham, The Picone identity for self-adjoint differential equations of even order. *Mathematika* **20** (1973), 197–200.
3. Á. Elbert, A half-linear second order differential equation. *Qualitative theory of differential equations, Vol. I, II* (*Szeged,* 1979), pp. 153–180, *Colloq. Math. Soc. János Bolyai,* 30, *North-Holland, Amsterdam–New York*, 1981.
4. J. Jaroš, Comparison theorems for half-linear equations of fourth order. *Equadiff 9 – Conference on Differential Equations and Their Applications* (*Brno, Czech Republic, August* 25–29, 1997). Abstracts and Enlarged Abstracts (Ed. by Z. Došlá, J. Kalas, J. Vosmanský), *Masaryk University, Brno*, p. 233.
5. J. Jaroš, The higher-order Picone identity and comparison of half-linear differential equations of even order. *Nonlinear Anal.* **74** (2011), No. 18, 7513–7518.
6. J. Jaroš and T. Kusano, Second-order semilinear differential equations with external forcing terms. (Japanese) *Structure of functional equations and mathematical methods* (Japanese) (*Kyoto*, 1996). *Sūrikaisekikenkyūsho Kōkyūroku* No. 984 (1997), 191–197.
7. J. Jaroš and T. Kusano, A Picone type identity for second order half-linear differential equations. *Acta Math. Univ. Comenian. (N.S.)* **68** (1999), No. 1, 137–151.
8. K. Kreith, A comparison theorem for fourth order differential equations. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **46** (1969), 664–666.
9. K. Kreith, A Picone identity for fourth order differential equations. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **52** (1972), 455–456.
10. K. Kreith, Oscillation theory. *Lecture Notes in Mathematics.* 324. *Springer-Verlag, Berlin–Heidelberg–New York*, 1973.
11. K. Kreith, Picone's identity and generalizations. Collection of articles dedicated to Mauro Picone on the occasion of his ninetieth birthday. *Rend. Mat. (6)* **8** (1975), 251–262.
12. T. Kusano and N. Yoshida, Picone's identity for ordinary differential operators of even order. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **58** (1975), No. 4, 524–530.
13. W. Leighton and Z. Nehari, On the oscillation of solutions of self-adjoint linear differential equations of the fourth order. *Trans. Amer. Math. Soc.* **89** (1958), 325–377.
14. M. Picone, Sui valori eccezionali di un parametro da cui dipende un'equazione differenziale lineare ordinaria del second'ordine. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **11** (1910), 1–144.
15. T. Tanigawa and N. Yoshida, Picone identities for ordinary differential equations of fourth order. *Math. J. Toyama Univ.* **27** (2004), 91–99.
16. N. Yoshida, A Picone identity for elliptic differential operators of order $4m$ with applications. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **58** (1975), No. 3, 306–317.
17. N. Yoshida, Nonoscillation criteria for fourth order elliptic equations. *Hiroshima Math. J.* **5** (1975), 23–31.

(Received 03.06.2012)

Author's address:

Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University, 842 48 Bratislava, Slovakia.

e-mail: `jaros@fmph.uniba.sk`

Memoirs on Differential Equations and Mathematical Physics
Volume 57, 2012, 51–74

Liliya Koltsova and Alexander Kostin

THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF MONOTONE TYPE OF FIRST-ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS, UNRESOLVED FOR THE DERIVATIVE

Abstract. For the first-order nonlinear ordinary differential equation

$$F(t,y,y') = \sum_{k=1}^{n} p_k(t) y^{\alpha_k} (y')^{\beta_k} = 0,$$

unresolved for the derivative, asymptotic behavior of solutions of monotone type is established for $t \to +\infty$.

2010 Mathematics Subject Classification. 34D05, 34E10.

Key words and phrases. Nonlinear differential equations, monotone solutions, asymptotic properties.

რეზიუმე. გამოკვლეულია წარმოებულის მიმართ ამოუხსნელი პირველი რიგის არაწრფივი ჩვეულებრივი დიფერენციალური განტოლებების მონოტონური ტიპის ამონახსნების ასიმპტოტური თვისებები.

This article describes a first-order real ordinary differential equation:

$$F(t,y,y') = \sum_{k=1}^{n} p_k(t) y^{\alpha_k} (y')^{\beta_k} = 0, \tag{1}$$

$(t,y,y') \in D$, $D = \Delta(a) \times \mathbb{R}_1 \times \mathbb{R}_2$, $\Delta(a) = [a;+\infty[$, $a > 0$, $\mathbb{R}_1 = \mathbb{R}_+$, $\mathbb{R}_2 = \mathbb{R}_- \vee \mathbb{R}_+$; $p_k(t) \in \mathrm{C}_{\Delta(a)}$ $(k = \overline{1,n},\ n \geq 2)$; $\alpha_k, \beta_k \geq 0$ $(k = \overline{1,n})$, $\sum\limits_{k=1}^{n} \beta_k \neq 0$.

Further, we assume that all the expressions, appearing in the equation, make sense; and all functions we consider in the present paper are real.

We investigate the question on the existence and on the asymptotic behavior (as $t \to +\infty$) of unboundedly continuable to the right solutions (R-solutions) $y(t)$ of equation (1) and derivatives $y'(t)$ of these solutions which possess the following properties:

A) $0 < y(t) \in \mathrm{C}^1_{\Delta(t_1)}$, $\Delta(t_1) \subset \Delta(a)$, where t_1 is defined in the course of proving each theorem;

B) among the summands $p_k(t)(y(t))^{\alpha_k}(y'(t))^{\beta_k}$ $(k = \overline{1,n})$, the terms with numbers $i = \overline{1,s}$ $(2 \leq s \leq n)$ are asymptotically principal for the given R-solution $y(t)$, i.e., there exist:

$$\lim_{t\to+\infty} \frac{p_i(t)(y(t))^{\alpha_i}(y'(t))^{\beta_i}}{p_1(t)(y(t))^{\alpha_1}(y'(t))^{\beta_1}} \neq 0,\ \pm\infty \quad (i = \overline{1,s}),$$

$$\lim_{t\to+\infty} \frac{p_j(t)(y(t))^{\alpha_j}(y'(t))^{\beta_j}}{p_1(t)(y(t))^{\alpha_1}(y'(t))^{\beta_1}} = 0 \quad (j = \overline{s+1,n}).$$

It is obvious that $p_i(t) \neq 0$ $(i = \overline{1,s})$.

Lemma 1. *Let the equation*

$$\widetilde{F}(t,\xi,\eta) = 0, \tag{2}$$

$(t,\xi,\eta) \in D_1$, $D_1 = \Delta(a) \times [-h_1;h_1] \times [-h_2;h_2]$, $h_k \in \mathbb{R}_+$ $(k = 1,2)$, *satisfy the conditions:*

1) $\widetilde{F}(t,\xi,\eta) \in \mathrm{C}^{s_1 s_2 s_3}_{t\ \xi\ \eta}(D_1)$, $s_1, s_2, s_3 \in \{0,1,2,\ldots\}$, $s_2 \geq 1$, $s_3 \geq 2$;

2) $\exists\, \widetilde{F}(+\infty,0,0) = 0$;

3) $\exists\, \widetilde{F}'_\eta(+\infty,0,0) = A_1 \in \mathbb{R} \setminus \{0\}$;

4) $\sup\limits_{D_1} |\widetilde{F}''_{\eta\eta}(t,\xi,\eta)| = A_2 \in \mathbb{R}_+$.

Then in some domain $D_2 = \Delta(t_0) \times [-\widetilde{h}_1;\widetilde{h}_1] \times [-\widetilde{h}_2;\widetilde{h}_2]$, *where* $t_0 \geq a$, $0 < \widetilde{h}_1 \leq h_1$, $0 < \widetilde{h}_2 < \min\{h_2; \frac{|A_1|}{4A_2}\}$, *the equation* (2) *defines a unique function* $\eta = \widetilde{\eta}(t,\xi)$, *such that* $\widetilde{\eta}(t,\xi) \in \mathrm{C}^{s_1 s_2}_{t\ \xi}(D_3)$, $D_3 = \Delta(t_0) \times [-\widetilde{h}_1;\widetilde{h}_1]$, $\exists\, \widetilde{\eta}(+\infty,0) = 0$, $\widetilde{F}(t,\xi,\widetilde{\eta}(t,\xi)) \equiv 0$. *Moreover, for* $\xi = 0$, *the function* $\widetilde{\eta}(t,\xi)$

has the property

$$\widetilde{\eta}(t,0) \sim -\frac{\widetilde{F}(t,0,0)}{\widetilde{F}'_{\eta}(t,0,0)}\,. \tag{3}$$

Proof. Let us expand the function $\widetilde{F}(t,\xi,\eta)$ with respect to the variable η for $t \in \Delta(a)$, $\xi \in [-h_1; h_1]$ by using the Maclaurin's formula. Then the equation (2) can be written as:

$$\widetilde{F}(t,\xi,\eta) = \widetilde{F}(t,\xi,0) + \widetilde{F}'_{\eta}(t,\xi,0)\eta + R(t,\xi,\eta) = 0. \tag{4}$$

Obviously,

$$R(t,\xi,0) \equiv 0.$$

The equation (4) is equivalent to the implicit equation

$$\eta(t,\xi) = \frac{-\widetilde{F}(t,\xi,0) - R(t,\xi,\eta(t,\xi))}{\widetilde{F}'_{\eta}(t,\xi,0)}\,, \tag{5}$$

where

$$R(t,\xi,\eta) = \widetilde{F}(t,\xi,\eta) - \widetilde{F}(t,\xi,0) - \widetilde{F}'_{\eta}(t,\xi,0)\eta,$$

and, therefore,

$$R'_{\eta}(t,\xi,\eta) = \widetilde{F}'_{\eta}(t,\xi,\eta) - \widetilde{F}'_{\eta}(t,\xi,0).$$

Applying the Lagrange's theorem with respect to the variable η to the right-hand side of the above equation, we get:

$$\widetilde{F}'_{\eta}(t,\xi,\eta_2) - \widetilde{F}'_{\eta}(t,\xi,\eta_1) = \widetilde{F}''_{\eta\eta}(t,\xi,\eta^*)(\eta_2 - \eta_1), \quad \eta^* \in]\eta_1;\eta_2[\,,$$

$$\sup_{D_1} \left|\widetilde{F}'_{\eta}(t,\xi,\eta_2) - \widetilde{F}'_{\eta}(t,\xi,\eta_1)\right| \le$$

$$\le \sup_{D_1} \left|\widetilde{F}''_{\eta\eta}(t,\xi,\eta)\right| |\eta_2 - \eta_1| = A_2|\eta_2 - \eta_1|.$$

Assuming $\eta_1 = 0$, $\eta_2 = \eta$, we obtain:

$$\sup_{D_1} \left|R'_{\eta}(t,\xi,\eta)\right| \le A_2|\eta|.$$

We consider and evaluate also the difference $R(t,\xi,\eta_2) - R(t,\xi,\eta_1)$, $(t,\xi,\eta_i) \in D_1$ $(i = 1,2)$, applying the Lagrange's theorem with respect to the variable η:

$$R(t,\xi,\eta_2) - R(t,\xi,\eta_1) = R'_{\eta}(t,\xi,\eta^{**})(\eta_2 - \eta_1), \quad \eta^{**} \in]\eta_1;\eta_2[\,,$$

$$\sup_{D_1} \left|R(t,\xi,\eta_2) - R(t,\xi,\eta_1)\right| \le \sup_{D_1} \left|R'_{\eta}(t,\xi,\eta)\right| |\eta_2 - \eta_1| \le A_2|\eta_2 - \eta_1|^2.$$

Assuming $\eta_1 = 0$, $\eta_2 = \eta$, we get

$$\sup_{D_1} |R(t,\xi,\eta)| \le A_2|\eta|^2.$$

Consider the domain $D_2 \subset D_1$ in which

1) $\sup\limits_{D_2} |\widetilde{F}(t,\xi,0)| \le \frac{\widetilde{h}_2|A_1|}{4}$;

2) $\inf\limits_{D_2} |\widetilde{F}'_\eta(t,\xi,0)| > \frac{|A_1|}{2}$;

3) $\sup\limits_{D_2} |R(t,\xi,\eta)| \le A_2|\eta|^2 \le A_2\widetilde{h}_2^2$.

The fulfilment of conditions 1), 2) can be achieved by increasing t_0 and reducing $\widetilde{h}_1$ (by virtue of the conditions of the Lemma). The fulfilment of condition 3) is obvious.

To the equation (5) we put into the correspondence the operator

$$\eta(t,\xi) = T(t,\xi,\widetilde{\eta}(t,\xi)) \equiv \frac{-\widetilde{F}(t,\xi,0) - R(t,\xi,\widetilde{\eta}(t,\xi))}{\widetilde{F}'_\eta(t,\xi,0)},$$

where $\widetilde{\eta}(t,\xi) \in B_1 \subset B$, $B = \big\{\widetilde{\eta}(t,\xi) : \ \widetilde{\eta}(t,\xi) \in \mathrm{C}^{s_1 s_2}_{t\ \xi}(D_3),\ \widetilde{\eta}(+\infty,0) = 0,$ $\|\widetilde{\eta}(t,\xi)\| = \sup\limits_{D_3} |\widetilde{\eta}(t,\xi)|\big\}$ is the Banach space, $B_1 = \big\{\widetilde{\eta}(t,\xi) : \widetilde{\eta}(t,\xi) \in B,$ $\|\widetilde{\eta}(t,\xi)\| \le \widetilde{h}_2\big\}$ is a closed subset of the Banach space B.

We apply here the principle of contractive mappings.

1) Let us prove that if $\widetilde{\eta}(t,\xi) \in B_1$, then $\eta(t,\xi) = T(t,\xi,\widetilde{\eta}(t,\xi)) \in B_1$: $\widetilde{\eta}(t,\xi) \in \mathrm{C}^{s_1 s_2}_{t\ \xi}(D_3)$ and $\widetilde{\eta}(+\infty,0) = 0$, then by virtue of the structure of the operator, we get

$$\eta(t,\xi) \in \mathrm{C}^{s_1 s_2}_{t\ \xi}(D_3), \quad \eta(+\infty,0) = 0;$$

$$\|\widetilde{\eta}(t,\xi)\| \le \widetilde{h}_2 \Longrightarrow \|\eta(t,\xi)\| = \big\|T(t,\xi,\widetilde{\eta}(t,\xi))\big\| =$$
$$= \Big\|\frac{-\widetilde{F}(t,\xi,0) - R(t,\xi,\widetilde{\eta}(t,\xi))}{\widetilde{F}'_\eta(t,\xi,0)}\Big\| \le$$
$$\le \frac{1}{\inf\limits_{D_2} |\widetilde{F}'_\eta(t,\xi,\eta)|} \Big(\sup_{D_2} |\widetilde{F}(t,\xi,0)| + \sup_{D_2} |R(t,\xi,\widetilde{\eta}(t,\xi))|\Big) \le$$
$$\le \frac{2}{|A_1 t|} \Big(\sup_{D_2} |\widetilde{F}(t,\xi,0)| + A_2\widetilde{h}_2^2\Big) \le \frac{\widetilde{h}_2}{2} + \frac{\widetilde{h}_2}{2} \le \widetilde{h}_2.$$

2) Let us check the condition of contraction:

$$\widetilde{\eta}_1(t,\xi), \widetilde{\eta}_2(t,\xi) \in B_1 \Longrightarrow \big\|\eta_2(t,\xi) - \eta_1(t,\xi)\big\| =$$
$$= \Big\|\frac{R(t,\xi,\widetilde{\eta}_2(t,\xi)) - R(t,\xi,\widetilde{\eta}_1(t,\xi))}{\widetilde{F}'_\eta(t,\xi,0)}\Big\| \le$$
$$\le \frac{A_2}{\inf\limits_{D_2} |\widetilde{F}'_\eta(t,\xi,\eta)|} \big\|\widetilde{\eta}_2(t,\xi) - \widetilde{\eta}_1(t,\xi)\big\|^2 \le$$
$$\le \frac{2A_2}{|A_1|} \Big(\|\widetilde{\eta}_2(t,\xi)\| + \|\widetilde{\eta}_1(t,\xi)\|\Big) \big\|\widetilde{\eta}_2(t,\xi) - \widetilde{\eta}_1(t,\xi)\big\| \le$$
$$\le \frac{4A_2\widetilde{h}_2}{|A_1|} \big\|\widetilde{\eta}_2(t,\xi) - \widetilde{\eta}_1(t,\xi)\big\| = \gamma \big\|\widetilde{\eta}_2(t,\xi) - \widetilde{\eta}_1(t,\xi)\big\|,$$

where $\gamma = \frac{4A_2\widetilde{h}_2}{|A_1|} < 1$.

As a result, we have found that by the contractive mapping principle the equation (5) admits a unique solution $\eta = \widetilde{\eta}(t,\xi) \in B_1$.

Since $\widetilde{F}(t,\xi,\eta) \in \mathrm{C}^{s_1 s_2 s_3}_{t\ \xi\ \eta}(D_1)$, then by a local theorem on the differentiability of an implicit function, it can be stated that $\widetilde{\eta}(t,\xi) \in \mathrm{C}^{s_1 s_2}_{t\ \xi}(D_3)$.

Let us prove that $\widetilde{\eta}(t,\xi)$ has the property (3) for $\xi = 0$.

The function $\widetilde{\eta}(t,\xi) \in D_3$ satisfies the equation (4), which can be written as

$$\widetilde{F}(t,0,0) + \widetilde{F}'_\eta(t,0,0)\widetilde{\eta}(t,0) + O(\widetilde{\eta}^2) \equiv 0, \tag{6}$$

assuming $\xi = 0$.

As $O(\widetilde{\eta}^2) = O(1)\widetilde{\eta}^2 = o(1)\widetilde{\eta}$, then the equation (6) is equivalent to the equation

$$\widetilde{F}(t,0,0) + \widetilde{F}'_\eta(t,0,0)\widetilde{\eta}(t,0) + o(1)\widetilde{\eta}(t,0) \equiv 0.$$

Hence, taking into account that $\widetilde{F}'_\eta(+\infty,0,0) = A_1 \in \mathbb{R} \setminus \{0\}$, we can write

$$\widetilde{\eta}(t,0)\Big(1 + \frac{o(1)}{\widetilde{F}'_\eta(t,0,0)}\Big) = -\frac{\widetilde{F}(t,0,0)}{\widetilde{F}'_\eta(t,0,0)}\,. \tag{7}$$

The property (3) follows from the equality (7). $\square$

Lemma 2 ([2]). *Let the differential equation*

$$\xi' = \alpha(t) f(t,\xi), \tag{8}$$

$(t,\xi) \in D_3$, $D_3 = \Delta(t_0) \times [-\widetilde{h}_1; \widetilde{h}_1]$ $(\widetilde{h}_1 \in \mathbb{R}_+)$, *satisfy the conditions:*

1) $0 \neq \alpha(t) \in \mathrm{C}(\Delta(t_0))$, $\int\limits_{t_0}^{+\infty} \alpha(t)\,dt = \pm\infty$;
2) $f(t,\xi) \in \mathrm{C}^{01}_{t\xi}(D_3)$, $\exists\, f(+\infty,0) = 0$, $\exists\, f'_\xi(+\infty,0) \neq 0$;
3) $f'_\xi(t,\xi) \rightrightarrows f'_\xi(t,0)$ *under* $\xi \to 0$ *uniformly with respect to* $t \in \Delta(t_0)$.

Then there exists $t_1 \geq t_0$, *such that the equation* (8) *has a non-empty set of o-solutions*

$$\Omega = \big\{\xi(t) \in \mathrm{C}^1_{\Delta(t_1)} :\ \xi(+\infty) = 0\big\},$$

where

a) *if* $\operatorname{sign}(\alpha f'_\xi(+\infty,0)) = -1$, *then* Ω *is a one-parametric family of o-solutions of the equation* (8)*;*
b) *if* $\operatorname{sign}(\alpha f'_\xi(+\infty,0)) = 1$, *then* Ω *contains a unique element.*

The Existence and Asymptotics of R-Solutions of the Equation (1) with the Condition $y(+\infty) = 0 \vee +\infty$

The supposed asymptotics (to within a constant factor) of R-solution $y(t)$ with the condition $y(+\infty) = 0 \vee +\infty$ can be found from the ratio of the first two summands (we consider all possible cases with respect to the values of parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$). Taking into account that $p_1(t), p_2(t) \neq 0$

($t \in \Delta(a)$), we find that $y(t) \dot\sim v(t) > 0^*$ ($v \in \{v_i\}$, $i = \overline{1,4}$) under the condition that $v(+\infty) = 0 \vee +\infty$:

1) $v_1 = \left|\frac{p_1(t)}{p_2(t)}\right|^{\frac{1}{\alpha_2-\alpha_1}}$ ($\alpha_1 \neq \alpha_2$, $\beta_1 = \beta_2$), moreover, $p_1(t), p_2(t) \in \mathrm{C}^1_{\Delta(a)}$.

In all the rest asymptotics is used the function

$$I(A,t) = \int\limits_A^t \left|\frac{p_1(t)}{p_2(t)}\right|^{\frac{1}{\beta_2-\beta_1}} dt, \quad A = \begin{cases} a & (I(a,+\infty) = +\infty), \\ +\infty & (I(a,+\infty) \in \mathbb{R}_+ \cup \{0\}). \end{cases}$$

2) $v_2 = |I(A,t)|$ ($\alpha_1 = \alpha_2$, $\beta_1 \neq \beta_2$).

3) $v_3 = |I(A,t)|^{(\frac{\alpha_2-\alpha_1}{\beta_2-\beta_1}+1)^{-1}}$ ($\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$, $\alpha_1+\beta_1 \neq \alpha_2+\beta_2$).

4) $v_4 = e^{\ell_0|I(a,t)|}$ ($\ell_0 \in \mathbb{R}\setminus\{0\}$ and satisfies the conditions (13), (14), (16); $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$, $\alpha_1+\beta_1 = \alpha_2+\beta_2 \neq 0$; $I(a,+\infty) = +\infty$).

A solution is sought in the form

$$y(t) = v(t)(\ell + \xi(t)), \tag{9}$$

where $\ell \in \mathbb{R}_+$; $\xi(t) \in \mathrm{C}^1_{\Delta(a)}$, $\xi(+\infty) = 0$; $v(t) = v_k(t) \in \mathrm{C}^1_{\Delta(a)}$ (k is fixed, $k = \overline{1,4}$).

Differentiating the equation (9), we obtain:

$$y'(t) = v'(t)(\ell + \xi(t)) + v(t)\xi'(t) = v'(t)\Big(\ell + \xi(t) + \frac{v(t)}{v'(t)}\xi'(t)\Big).$$

Having denoted

$$\xi(t) + \frac{v(t)}{v'(t)}\xi'(t) = \eta(t), \tag{10}$$

$\eta(t) \in \mathrm{C}_{\Delta(a)}$, we get

$$y'(t) = v'(t)(\ell + \eta(t)). \tag{11}$$

The condition $y'(t) \sim \ell v'(t)$ requires the assumption that $\eta(+\infty) = 0$.

Substituting (9) and (11) into the equation (1), we obtain the equality

$$F(t, v(\ell+\xi), v'(\ell+\eta)) = \\ = \sum_{k=1}^n p_k(t)(v)^{\alpha_k}(\ell+\xi)^{\alpha_k}(v')^{\beta_k}(\ell+\eta)^{\beta_k} = 0, \tag{12}$$

which is satisfied by the functions $\xi(t)$, $\eta(t)$ and $(v'(t))^{\beta_k} : \Delta(a) \to \mathbb{R}_2$ ($k = \overline{1,n}$).

$^*f_i \dot\sim f_j$ ($i \neq j$) means that $\exists \lim\limits_{t\to+\infty} \frac{f_i}{f_j} \neq 0, \pm\infty$.

According to the condition B), indicated in the statement of the problem, we assume that

$$\frac{p_i(t)(v(t))^{\alpha_i}(v'(t))^{\beta_i}}{p_1(t)(v(t))^{\alpha_1}(v'(t))^{\beta_1}} =$$

$$= c_i^* + \varepsilon_i(t), \quad c_i^* \in \mathbb{R} \setminus \{0\},\ \varepsilon_i(+\infty) = 0 \ \ (i = \overline{1,s}); \tag{13}$$

$$\frac{p_j(t)(v(t))^{\alpha_j}(v'(t))^{\beta_j}}{p_1(t)(v(t))^{\alpha_1}(v'(t))^{\beta_1}} = \varepsilon_j(t), \quad \varepsilon_j(+\infty) = 0 \ \ (j = \overline{s+1,n}). \tag{14}$$

Then, after the division by $p_1(t)(v(t))^{\alpha_1}(v'(t))^{\beta_1}$, the equation (12) takes the form

$$\widetilde{F}(t,\xi,\eta) = \sum_{i=1}^{s}(c_i^* + \varepsilon_i(t))(\ell+\xi)^{\alpha_i}(\ell+\eta)^{\beta_i} +$$

$$+ \sum_{j=s+1}^{n} \varepsilon_j(t)(\ell+\xi)^{\alpha_j}(\ell+\eta)^{\beta_j} = 0. \tag{15}$$

Obviously, the condition $\widetilde{F}(+\infty,0,0) = 0$ is necessary for the existence of a solution and of its derivative of the form (9), (11), respectively.

Thus, for $v = v_k(t)$ $(k = \overline{1,4})$ it takes the form

$$\sum_{i=1}^{s} c_i^* \ell^{\alpha_i+\beta_i} = 0. \tag{16}$$

For $v = v_4(t) : \mathrm{sign}(v') = \mathrm{sign}(\ell_0)$, $c_i^* = c_i^*(\ell_0)$, ℓ_0, $\ell_0^{\beta_i} \in \mathbb{R}\setminus\{0\}$ $(i = \overline{1,s})$.

By virtue of its structure, the functions $\widetilde{F}(t,\xi,\eta) \in \mathrm{C}^{0\,\infty\,\infty}_{t\,\xi\,\eta}(D_1)$, $\frac{\partial^n \widetilde{F}}{\partial \xi^n}$, $\frac{\partial^m \widetilde{F}}{\partial \eta^m}$, $\frac{\partial^{n+m} \widetilde{F}}{\partial \xi^n \partial \eta^m}$ $(n = \overline{1,\infty},\quad m = \overline{1,\infty})$ are bounded in D_1, where $D_1 = \Delta(a) \times [-h_1; h_1] \times [-h_2; h_2]$, $0 < h_k < \ell$ $(k = 1,2)$.

Next, we will need expressions for the first and second order derivatives of the function $\widetilde{F}(t,\xi,\eta)$ with respect to the variables ξ and η:

$$\widetilde{F}'_{\xi}(t,\xi,\eta) = \sum_{i=1}^{s} \alpha_i c_i^* (\ell+\xi)^{\alpha_i-1}(\ell+\eta)^{\beta_i} +$$

$$+ \sum_{k=1}^{n} \alpha_k \varepsilon_k(t)(\ell+\xi)^{\alpha_k-1}(\ell+\eta)^{\beta_k},$$

$$\widetilde{F}'_{\eta}(t,\xi,\eta) = \sum_{i=1}^{s} \beta_i c_i^* (\ell+\xi)^{\alpha_i}(\ell+\eta)^{\beta_i-1} +$$

$$+ \sum_{k=1}^{n} \beta_k \varepsilon_k(t)(\ell+\xi)^{\alpha_k}(\ell+\eta)^{\beta_k-1},$$

$$\widetilde{F}''_{\xi\xi}(t,\xi,\eta) = \sum_{i=1}^{s} \alpha_i(\alpha_i - 1) c_i^* (\ell+\xi)^{\alpha_i-2}(\ell+\eta)^{\beta_i} +$$

$$+\sum_{k=1}^{n}\alpha_k(\alpha_k-1)\varepsilon_k(t)(\ell+\xi)^{\alpha_k-2}(\ell+\eta)^{\beta_k},$$

$$\widetilde{F}''_{\xi\eta}(t,\xi,\eta)=\widetilde{F}''_{\eta\xi}(t,\xi,\eta)=\sum_{i=1}^{s}\alpha_i\beta_i c_i^*(\ell+\xi)^{\alpha_i-1}(\ell+\eta)^{\beta_i-1}+$$

$$+\sum_{k=1}^{n}\alpha_k\beta_k\varepsilon_k(t)(\ell+\xi)^{\alpha_k-1}(\ell+\eta)^{\beta_k-1},$$

$$\widetilde{F}''_{\eta\eta}(t,\xi,\eta)=\sum_{i=1}^{s}\beta_i(\beta_i-1)c_i^*(\ell+\xi)^{\alpha_i}(\ell+\eta)^{\beta_i-2}+$$

$$+\sum_{k=1}^{n}\beta_k(\beta_k-1)\varepsilon_k(t)(\ell+\xi)^{\alpha_k}(\ell+\eta)^{\beta_k-2},$$

as well as the following notation:

$$\psi_{00}(t)=\sum_{k=1}^{n}\ell^{\alpha_k+\beta_k}\varepsilon_k(t),$$

$$\psi_{l0}(t)=\sum_{k=1}^{n}\alpha_k(\alpha_k-1)\cdots(\alpha_k-l+1)\varepsilon_k(t)\ell^{\alpha_k+\beta_k},$$

$$\psi_{0m}(t)=\sum_{k=1}^{n}\beta_k(\beta_k-1)\cdots(\beta_k-m+1)\varepsilon_k(t)\ell^{\alpha_k+\beta_k},$$

$$\psi_{lm}(t)=\sum_{k=1}^{n}\alpha_k(\alpha_k-1)\cdots(\alpha_k-l+1)\times$$

$$\times\,\beta_k(\beta_k-1)\cdots(\beta_k-m+1)\varepsilon_k(t)\ell^{\alpha_k+\beta_k},$$

$$S_{l0}=\sum_{i=1}^{s}\alpha_i(\alpha_i-1)\cdots(\alpha_i-l+1)c_i^*\ell^{\alpha_i+\beta_i},$$

$$S_{0m}=\sum_{i=1}^{s}\beta_i(\beta_i-1)\cdots(\beta_i-m+1)c_i^*\ell^{\alpha_i+\beta_i},$$

$$S_{lm}=\sum_{i=1}^{s}\alpha_i(\alpha_i-1)\cdots(\alpha_i-l+1)\times$$

$$\times\,\beta_i(\beta_i-1)\cdots(\beta_i-m+1)c_i^*\ell^{\alpha_i+\beta_i},$$

$$S_{l0},S_{0m},S_{lm}\in\mathbb{R}\ \ (l,m\in\mathbb{N}),\quad S=S_{10}^2S_{02}-2S_{10}S_{01}S_{11}+S_{01}^2S_{20},$$

$$\lambda_1=\frac{2S_{01}^3}{S}\in\mathbb{R},\quad \lambda_2=-\frac{2S_{01}^2\ell^2}{S}\in\mathbb{R}.$$

Theorem 1. *Let a function* $v(t)=v_k(t)$ $(k=\overline{1,4})$ *be a possible asymptotics of an R-solution of the equation* (1)*, which satisfies the conditions* $v(+\infty)=0\vee+\infty$, (13), *and* (14). *Let, moreover, there exist* $\ell\in\mathbb{R}_+$, *satisfying the condition* (16).

Then in order for the R-solution $y(t) \in \mathrm{C}^1_{\Delta(t_1)}$ *of the differential equation* (1) *with the asymptotic properties*

$$y(t) \sim \ell v(t), \quad y'(t) \sim \ell v'(t), \tag{17}$$

to exist, it is sufficient that the two following conditions

$$S_{01} \neq 0, \tag{18}$$

$$S_{10} + S_{01} \neq 0. \tag{19}$$

be fulfilled. Moreover, if $\operatorname{sign}\left(\frac{v'(S_{10}+S_{01})}{S_{01}}\right) = 1$*, then there exists a one-parameter set of R-solutions with the asymptotic properties* (17)*; if* $\operatorname{sign}\left(\frac{v'(S_{10}+S_{01})}{S_{01}}\right) = -1$*, then R-solution with the asymptotic* (17) *is unique.*

Proof. For the proof we will need the following properties of the function $\widetilde{F}(t, \xi, \eta)$:

$$\widetilde{F}'_{\xi}(+\infty, 0, 0) = \frac{S_{10}}{\ell};$$
$$\widetilde{F}'_{\eta}(+\infty, 0, 0) = \frac{S_{01}}{\ell} \neq 0$$

by virtue of the condition (18).

Owing to the conditions (16), (18) and to the properties of the function $\widetilde{F}(t, \xi, \eta)$, in some domain $D_2 \subset D_1$, $D_2 = \Delta(t_0) \times [-\widetilde{h}_1; \widetilde{h}_1] \times [-\widetilde{h}_2; \widetilde{h}_2]$, $t_0 \geq a$, $0 < \widetilde{h}_1 \leq h_1$, $0 < \widetilde{h}_2 < \min\left\{h_2; \frac{|S_{01}|}{4\ell \sup\limits_{D_1} |\widetilde{F}''_{\eta\eta}(t,\xi,\eta)|}\right\}$, for the equation (15) the conditions of Lemma 1 are satisfied. Consequently, there exists a unique function $\eta = \widetilde{\eta}(t, \xi) \in \mathrm{C}^{0\,\infty}_{t\,\xi}(D_3)$, $D_3 = \Delta(t_0) \times [-\widetilde{h}_1; \widetilde{h}_1]$, $\sup\limits_{D_3} \left|\frac{\partial^n \widetilde{\eta}}{\partial \xi^n}\right| < +\infty$ $(n = \overline{1, \infty})$, such that $\widetilde{F}(t, \xi, \widetilde{\eta}(t, \xi)) \equiv 0$, $\widetilde{\eta}(+\infty, 0) = 0$, $\|\widetilde{\eta}(t, \xi)\| \leq \widetilde{h}_2$. Moreover, we can write

$$\frac{\partial \widetilde{\eta}(t, \xi)}{\partial \xi} = -\frac{\widetilde{F}'_{\xi}(t, \xi, \widetilde{\eta})}{\widetilde{F}'_{\eta}(t, \xi, \widetilde{\eta})}.$$

Thus, in view of the replacement (10), we obtain the differential equation with respect to ξ:

$$\xi' = \frac{v'}{v}\big(-\xi + \widetilde{\eta}(t, \xi)\big). \tag{20}$$

The question on the existence of solutions of the form (9) reduces to the study of the differential equation (20).

Let us show that the conditions 1)–3) of Lemma 2 are satisfied for the equation (20). In this case we have: $\alpha(t) = \frac{v'(t)}{v(t)}$, $f(t, \xi) = -\xi + \widetilde{\eta}(t, \xi)$.

Obviously, the conditions 1) and 2) are satisfied.

1) Since $0 < v(t) \in \mathrm{C}^1(\Delta(a))$, therefore

$$0 \neq \alpha(t) \in \mathrm{C}(\Delta(t_0)), \quad \int\limits_{t_0}^{+\infty} \alpha(t)\, dt = \int\limits_{t_0}^{+\infty} \frac{v'(t)}{v(t)}\, dt = \pm\infty.$$

2) Since $\widetilde{\eta}(t,\xi)\in \mathrm{C}^{0\infty}_{t\,\xi}(D_3)$, then

$$f(t,\xi)\in \mathrm{C}^{0\infty}_{t\,\xi}(D_3),\quad \exists\, f(+\infty,0)=\widetilde{\eta}(+\infty,0)=0,$$

$$f'_\xi(t,\xi)=-1+\widetilde{\eta}'_\xi(t,\xi)=-1-\frac{\widetilde{F}'_\xi(t,\xi,\widetilde{\eta})}{\widetilde{F}'_\eta(t,\xi,\widetilde{\eta})}\,,$$

$$f'_\xi(+\infty,0)=-1-\frac{\widetilde{F}'_\xi(+\infty,0,\widetilde{\eta}(+\infty,0))}{\widetilde{F}'_\eta(+\infty,0,\widetilde{\eta}(+\infty,0))}=-\frac{S_{10}+S_{01}}{S_{01}}\neq 0$$

by virtue of the condition (19).

Let us check that the condition 3) is satisfied, that is,

$$\big\|f'_\xi(t,\xi)-f'_\xi(t,0)\big\|=\left\|\frac{\widetilde{F}'_\xi(t,\xi,\widetilde{\eta}(t,\xi))}{\widetilde{F}'_\eta(t,\xi,\widetilde{\eta}(t,\xi))}-\frac{\widetilde{F}'_\xi(t,0,\widetilde{\eta}(t,0))}{\widetilde{F}'_\eta(t,0,\widetilde{\eta}(t,0))}\right\|\rightrightarrows 0$$

as $\xi\to 0$ uniformly with respect to $t\in\Delta(t_0)$.

Towards this end, it suffices to verify that the following properties are satisfied:

3_1) $\widetilde{\eta}(t,\xi)\rightrightarrows\widetilde{\eta}(t,0)$ if $\xi\to 0$ uniformly with respect to $t\in\Delta(t_0)$,

3_2) $\widetilde{F}'_\xi(t,\xi,\widetilde{\eta}(t,\xi))\rightrightarrows\widetilde{F}'_\xi(t,0,\widetilde{\eta}(t,0))$ as $\xi\to 0$ uniformly with respect to $t\in\Delta(t_0)$,

3_3) $\widetilde{F}'_\eta(t,\xi,\widetilde{\eta}(t,\xi))\rightrightarrows\widetilde{F}'_\eta(t,0,\widetilde{\eta}(t,0))$, as $\xi\to 0$ uniformly with respect to $t\in\Delta(t_0)$ with regard for the fact that $F'_\eta(+\infty,0,\eta(+\infty,0))=S_{01}\neq 0$.

Let us estimate the differences $\widetilde{\eta}(t,\xi)-\widetilde{\eta}(t,0)$, $\widetilde{F}'_\xi(t,\xi,\widetilde{\eta}(t,\xi))-\widetilde{F}'_\xi(t,0,\widetilde{\eta}(t,0))$, $\widetilde{F}'_\eta(t,\xi,\widetilde{\eta}(t,\xi))-\widetilde{F}'_\eta(t,0,\widetilde{\eta}(t,0))$, applying the Lagrange's theorem to the first difference with respect to the variable ξ:

$$\widetilde{\eta}(t,\xi)-\widetilde{\eta}(t,0)=\widetilde{\eta}'_\xi(t,\xi^*)\xi,\quad \xi^*\in\,]0;\xi[\,.$$

As the functions $\varepsilon_k(t)$ $(k=\overline{1,n})$ are bounded in $\Delta(a)$ and $\|\widetilde{\eta}(t,\xi)\|\le\widetilde{h}_2$ in D_3, then we get the estimates in the form:

3_1) $\quad\big|\widetilde{\eta}(t,\xi)-\widetilde{\eta}(t,0)\big|=\big|\widetilde{\eta}'_\xi(t,\xi^*)\big|\,|\xi|=$

$$=\left|-\frac{\widetilde{F}'_\xi(t,\xi^*,\widetilde{\eta}(t,\xi^*))}{\widetilde{F}'_\eta(t,\xi^*,\widetilde{\eta}(t,\xi^*))}\right||\xi|\le O(1)|\xi|=O(\xi)\longrightarrow 0$$

as $\xi\to 0$ uniformly with respect to $t\in\Delta(t_0)$;

3_2) taking into account that $(\ell+\xi)^{\alpha_i-1}\to\ell^{\alpha_i-1}$ as $\xi\to 0$, $(\ell+\widetilde{\eta}(t,\xi))^{\beta_i}\to(\ell+\widetilde{\eta}(t,0))^{\beta_i}$ as $\xi\to 0$ uniformly with respect to $t\in\Delta(t_0)$ $(i=\overline{1,s})$, we get

$$\big|\widetilde{F}'_\xi(t,\xi,\widetilde{\eta}(t,\xi))-\widetilde{F}'_\xi(t,0,\widetilde{\eta}(t,0))\big|=$$

$$=\bigg|\sum_{i=1}^{s}\alpha_i c_i^*\Big[(\ell+\xi)^{\alpha_i-1}(\ell+\widetilde{\eta}(t,\xi))^{\beta_i}-\ell^{\alpha_i-1}(\ell+\widetilde{\eta}(t,0))^{\beta_i}\Big]+$$

$$+\sum_{k=1}^{n}\alpha_k\varepsilon_k(t)\Big[(\ell+\xi)^{\alpha_k-1}(\ell+\widetilde{\eta}(t,\xi))^{\beta_k}-\ell^{\alpha_k-1}(\ell+\widetilde{\eta}(t,0))^{\beta_k}\Big]\bigg|\longrightarrow 0$$

as $\xi \to 0$ uniformly with respect to $t \in \Delta(t_0)$;

3_3) analogously to 3_2), we get:

$$\left|\widetilde{F}'_\eta(t,\xi,\widetilde{\eta}(t,\xi)) - \widetilde{F}'_\eta(t,0,\widetilde{\eta}(t,0))\right| =$$

$$= \left|\sum_{i=1}^{s} \beta_i c_i^* \Big[(\ell+\xi)^{\alpha_i}(\ell+\widetilde{\eta}(t,\xi))^{\beta_i-1} - \ell^{\alpha_i}(\ell+\widetilde{\eta}(t,0))^{\beta_i-1}\Big] + \right.$$

$$\left. + \sum_{k=1}^{n} \beta_k \varepsilon_k(t) \Big[(\ell+\xi)^{\alpha_k}(\ell+\widetilde{\eta}(t,\xi))^{\beta_k-1} - \ell^{\alpha_k}(\ell+\widetilde{\eta}(t,0))^{\beta_k-1}\Big]\right| \longrightarrow 0$$

as $\xi \to 0$ uniformly with respect to $t \in \Delta(t_0)$.

Since $\widetilde{\eta}(+\infty,0) = 0$, therefore $F'_\eta(+\infty,0,\widetilde{\eta}(+\infty,0)) = S_{01} \neq 0$ by virtue of the condition (18).

Consequently, condition 3) is satisfied.

Then if sign $\left(\frac{v'(S_{10}+S_{01})}{S_{01}}\right) = 1$, then there exists a one-parameter set of o-solutions of the equation (20) in $\Delta(t_1) \subseteq \Delta(t_0)$.

If sign $\left(\frac{v'(S_{10}+S_{01})}{S_{01}}\right) = -1$, then a set of o-solutions of the equation (20) in $\Delta(t_1)$ contains the unique element.

Finally, having the dimension of a set of o-solutions of the equation (20), we have obtained the dimension of a set of R-solutions of the equation (1) with the asymptotic properties (17) in $\Delta(t_1)$. □

Theorem 2. *Let the conditions of Theorem* 1, *except for* (19), *be satisfied, and*

$$S \neq 0, \tag{21}$$

$$\psi_{00}(t)\ln^2 v(t) = o(1), \tag{22}$$

$$(\psi_{10}(t)+\psi_{01}(t))\ln v(t) = o(1). \tag{23}$$

Then there exists a one-parameter set of R-solutions $y(t) \in \mathrm{C}^1_{\Delta(t_1)}$ of the differential equation (1) *with the asymptotic properties*

$$y(t) = v(t)(\ell+\xi(t)), \quad y'(t) \sim \ell v'(t), \tag{24}$$

where $\xi(t) \sim \frac{\lambda_1 \ell}{\ln v(t)}$.

Proof. To prove the theorem, we will need the following properties and expressions of the function $\widetilde{F}(t,\xi,\eta)$:

$$\widetilde{F}(t,0,0) = \psi_{00}(t),$$

$$\widetilde{F}'_\xi(t,0,0) = \frac{1}{\ell}\sum_{i=1}^{s} \alpha_i c_i^* \ell^{\alpha_i+\beta_i} + \frac{1}{\ell}\sum_{k=1}^{n} \alpha_k \ell^{\alpha_k+\beta_k}\varepsilon_k(t),$$

$$\widetilde{F}'_\xi(+\infty,0,0) = \frac{S_{10}}{\ell};$$

$$\widetilde{F}'_\eta(t,0,0) = \frac{1}{\ell}\sum_{i=1}^{s} \beta_i c_i^* \ell^{\alpha_i+\beta_i} + \frac{1}{\ell}\sum_{k=1}^{n} \beta_k \ell^{\alpha_k+\beta_k}\varepsilon_k(t),$$

$$\widetilde{F}'_{\eta}(+\infty,0,0)=\frac{S_{01}}{\ell}\neq 0 \ \text{ by virtue of condition (18);}$$

$$\widetilde{F}''_{\xi\xi}(t,0,0)=\frac{1}{\ell^2}\sum_{i=1}^{s}\alpha_i(\alpha_i-1)c_i^*\ell^{\alpha_i+\beta_i}+ \\ +\frac{1}{\ell^2}\sum_{k=1}^{n}\ \alpha_k(\alpha_k-1)\ell^{\alpha_k+\beta_k}\varepsilon_k(t),$$

$$\widetilde{F}''_{\xi\xi}(+\infty,0,0)=\frac{S_{20}}{\ell^2}\,;$$

$$\widetilde{F}''_{\xi\eta}(t,0,0)=\widetilde{F}''_{\eta\xi}(t,0,0)= \\ =\frac{1}{\ell^2}\sum_{i=1}^{s}\alpha_i\beta_i c_i^*\ell^{\alpha_i+\beta_i}+\frac{1}{\ell^2}\sum_{k=1}^{n}\ \alpha_k\beta_k\ell^{\alpha_k+\beta_k}\varepsilon_k(t),$$

$$\widetilde{F}''_{\xi\eta}(+\infty,0,0)=\widetilde{F}''_{\eta\xi}(+\infty,0,0)=\frac{S_{11}}{\ell^2}\,;$$

$$\widetilde{F}''_{\eta\eta}(t,0,0)=\frac{1}{\ell^2}\sum_{i=1}^{s}\beta_i(\beta_i-1)c_i^*\ell^{\alpha_i+\beta_i}+ \\ +\frac{1}{\ell^2}\sum_{k=1}^{n}\ \beta_k(\beta_k-1)\ell^{\alpha_k+\beta_k}\varepsilon_k(t),$$

$$\widetilde{F}''_{\eta\eta}(+\infty,0,0)=\frac{S_{02}}{\ell^2}\,.$$

By virtue of the condition (18) and owing to the properties of the function $\widetilde{F}(t,\xi,\eta)$, in some domain $D_2\subset D_1$, $D_2=\Delta(t_0)\times[-\widetilde{h}_1;\widetilde{h}_1]\times[-\widetilde{h}_2;\widetilde{h}_2]$, $t_0\geq a$, $0<\widetilde{h}_1\leq h_1$, $0<\widetilde{h}_2<\min\Big\{h_2;\frac{|S_{01}|}{4\ell\sup\limits_{D_1}|\widetilde{F}''_{\eta\eta}(t,\xi,\eta)|}\Big\}$, for the equation (15) the conditions of Lemma 1 are fulfilled. Consequently, there exists a unique function $\eta=\widetilde{\eta}(t,\xi)$, $\widetilde{\eta}(t,\xi)\in \mathrm{C}^{0\,\infty}_{t\,\xi}(D_3)$, $D_3=\Delta(t_0)\times[-\widetilde{h}_1;\widetilde{h}_1]$, $\sup\limits_{D_3}\big|\frac{\partial^n\widetilde{\eta}}{\partial\xi^n}\big|<+\infty$ $(n=\overline{1,\infty})$, such that $\widetilde{F}(t,\xi,\widetilde{\eta}(t,\xi))\equiv 0$, $\widetilde{\eta}(+\infty,0)=0$, $\|\widetilde{\eta}(t,\xi)\|\leq\widetilde{h}_2$. Moreover, we can write:

$$\widetilde{\eta}(t,0)\sim-\frac{\widetilde{F}(t,0,0)}{\widetilde{F}'_{\eta}(t,0,0)}\,,$$

$$\widetilde{\eta}'_{\xi}(t,\xi)=-\frac{\widetilde{F}'_{\xi}(t,\xi,\widetilde{\eta})}{\widetilde{F}'_{\eta}(t,\xi,\widetilde{\eta})}\,,$$

$$\frac{\partial^2\widetilde{\eta}(t,\xi)}{\partial\xi^2}=-\frac{(\widetilde{F}'_{\xi})^2\widetilde{F}''_{\eta\eta}-2\widetilde{F}'_{\xi}\widetilde{F}'_{\eta}\widetilde{F}''_{\xi\eta}+(\widetilde{F}'_{\eta})^2\widetilde{F}''_{\xi\xi}}{(\widetilde{F}'_{\eta})^3}\,.$$

Thus, taking into account the replacement (10), we obtain the differential equation with respect to ξ:

$$\xi' = \frac{v'}{v}\,(-\xi + \widetilde{\eta}(t,\xi)). \tag{20}$$

The question of the existence of solutions of the type (9) reduces to the study of the differential equation (20).

Let us show that the conditions 1)–3) of Lemma 2 are satisfied for the equation (20). In this case we have: $\alpha(t) = \frac{v'(t)}{v(t)}$, $f(t,\xi) = -\xi + \widetilde{\eta}(t,\xi)$.

1) Since $0 < v(t) \in \mathrm{C}^1(\Delta(a))$, therefore

$$0 \neq \alpha(t) \in \mathrm{C}(\Delta(t_0)), \quad \int\limits_{t_0}^{+\infty} \alpha(t)\,dt = \int\limits_{t_0}^{+\infty} \frac{v'(t)}{v(t)}\,dt = \pm\infty.$$

2) Since $\widetilde{\eta}(t,\xi) \in \mathrm{C}^{0\,\infty}_{t\;\xi}(D_3)$, therefore

$$f(t,\xi) \in \mathrm{C}^{0\,\infty}_{t\;\xi}(D_3), \quad \exists\, f(+\infty,0) = \widetilde{\eta}(+\infty,0) = 0,$$

$$f'_\xi(t,\xi) = -1 + \widetilde{\eta}\,'_\xi(t,\xi) = -1 - \frac{\widetilde{F}\,'_\xi(t,\xi,\widetilde{\eta})}{\widetilde{F}\,'_\eta(t,\xi,\widetilde{\eta})}\,.$$

Taking into account the properties of the functions $\varepsilon_k(t)$ $(k = \overline{1,n})$ and also the conditions of the theorem, we obtain:

$$f'_\xi(+\infty,0) = -1 - \frac{\widetilde{F}\,'_\xi(+\infty,0,\widetilde{\eta}(+\infty,0))}{\widetilde{F}\,'_\eta(+\infty,0,\widetilde{\eta}(+\infty,0))} = -\frac{S_{10}+S_{01}}{S_{01}} = 0.$$

Thus, condition 2) is not satisfied, and we cannot apply Lemma 2 to the equation (20).

Since $f''_{\xi\xi}(t,\xi) = \widetilde{\eta}\,''_{\xi\xi}(t,\xi)$, therefore

$$f''_{\xi\xi}(+\infty,0) = \widetilde{\eta}\,''_{\xi\xi}(+\infty,0) = -\frac{S}{\ell S_{01}^3} = -\frac{2}{\lambda_1\ell}\,.$$

Consider the auxiliary differential equation with respect to ξ_1:

$$\xi_1' = -\frac{v'(t)}{\lambda_1 \ell v(t)}\,\xi_1^2.$$

and find one of its non-trivial solutions:

$$\xi_1 = \frac{\lambda_1\ell}{\ln v(t)}\,, \quad 0 \neq \xi(t)_1 \in \mathrm{C}^1_{\Delta(t_1)} \ \ (t_1 \geq t_0), \ \ \xi_1(+\infty) = 0.$$

We consider the question on the existence in the equation (20) of solutions of the form $\xi = \xi_1(1+\widetilde{\xi})$, where $\widetilde{\xi}(t) \in \mathrm{C}^1_{\Delta(t_1)}$, $\widetilde{\xi}(+\infty) = 0$. For the unknown function $\widetilde{\xi}$ we obtain the following differential equation:

$$\widetilde{\xi}\,' = \frac{v'\xi_1}{v}\bigg(-\frac{1}{\xi_1} - \frac{v\xi_1'}{v'\xi_1^2} + \Big(-\frac{1}{\xi_1} - \frac{v\xi_1'}{v'\xi_1^2}\Big)\widetilde{\xi} + \frac{\widetilde{\eta}(t,\xi_1(1+\widetilde{\xi}))}{\xi_1^2}\bigg), \tag{25}$$

$(t,\widetilde{\xi}) \in D_4$, $D_4 = \Delta(t_1) \times [-h_4; h_4]$ $(0 < h_4 \leq \widetilde{h}_1)$, $\frac{v(t)\xi_1'(t)}{v'(t)\xi_1^2(t)} \equiv -\frac{1}{\lambda_1\ell}\,.$

Let us show that the conditions 1)–3) of Lemma 2 are satisfied for the equation (25). In this case we have:

$$\alpha(t) = \frac{v'(t)\xi_1}{v(t)} = \frac{\lambda_1 \ell v'(t)}{v(t)\ln v(t)}\,,$$
$$f(t,\widetilde{\xi}) = -\frac{1}{\xi_1} + \frac{1}{\lambda_1\ell} + \Big(-\frac{1}{\xi_1} + \frac{1}{\lambda_1\ell}\Big)\widetilde{\xi} + \frac{\widetilde{\eta}(t,\xi_1(1+\widetilde{\xi}))}{\xi_1^2}\,.$$

Using the properties of functions $v(t)$, $\widetilde{\eta}(t,\xi)$, $\xi_1(t)$, we obtain:

1) $0 \neq \alpha(t) \in \mathrm{C}(\Delta(t_1))$, $\int\limits_{t_1}^{+\infty} \alpha(t)\,dt = \lambda_1\ell \int\limits_{t_1}^{+\infty} \frac{v'(t)}{v(t)\ln v(t)}\,dt = \infty$;

2) $f(t,\widetilde{\xi}) \in \mathrm{C}^{0\,\infty}_{t\,\widetilde{\xi}}(D_4)$;

$$f(t,0) = -\frac{1}{\xi_1} + \frac{1}{\lambda_1\ell} + \frac{\widetilde{\eta}(t,\xi_1)}{\xi_1^2}\,,$$
$$f'_{\widetilde{\xi}}(t,\widetilde{\xi}) = -\frac{1}{\xi_1} + \frac{1}{\lambda_1\ell} + \frac{\widetilde{\eta}'_{\xi}(t,\xi_1(1+\widetilde{\xi}))}{\xi_1}\,,$$
$$f'_{\widetilde{\xi}}(t,0) = -\frac{1}{\xi_1} + \frac{1}{\lambda_1\ell} + \frac{\widetilde{\eta}'_{\xi}(t,\xi_1)}{\xi_1}\,.$$

Let us expand the functions $\widetilde{\eta}(t,\xi_1)$ and $\widetilde{\eta}'_{\xi}(t,\xi_1)$ with respect to the variable ξ_1 in D_4 using the Maclaurin's formula:

$$\widetilde{\eta}(t,\xi_1) = \widetilde{\eta}(t,0) + \widetilde{\eta}'_{\xi_1}(t,0)\xi_1 + \frac{1}{2}\,\widetilde{\eta}''_{\xi_1^2}(t,0)\xi_1^2 + O(\xi_1^3),$$
$$\widetilde{\eta}'_{\xi}(t,\xi_1) = \widetilde{\eta}'_{\xi}(t,0) + \widetilde{\eta}''_{\xi\xi_1}(t,0)\xi_1 + O(\xi_1^2).$$

Using Lemma 1, we obtain:

$$\widetilde{\eta}(t,0) \sim -\frac{\ell\psi_{00}(t)}{S_{01} + o(1)}\,,$$
$$\widetilde{\eta}'_{\xi_1}(t,0) = \widetilde{\eta}'_{\xi}(t,0) =$$
$$= -\frac{\sum\limits_{i=1}^{s} \alpha_i c_i^* \ell^{\alpha_i - 1}(\ell + \widetilde{\eta}(t,0))^{\beta_i} + \sum\limits_{k=1}^{n} \alpha_k \varepsilon_k(t) \ell^{\alpha_k - 1}(\ell + \widetilde{\eta}(t,0))^{\beta_k}}{\sum\limits_{i=1}^{s} \beta_i c_i^* \ell^{\alpha_i}(\ell + \widetilde{\eta}(t,0))^{\beta_i - 1} + \sum\limits_{k=1}^{n} \beta_k \varepsilon_k(t) \ell^{\alpha_k}(\ell + \widetilde{\eta}(t,0))^{\beta_k - 1}}\,,$$
$$\widetilde{\eta}'_{\xi_1}(+\infty,0) = \widetilde{\eta}'_{\xi}(+\infty,0) = -\frac{S_{10}}{S_{01}}\,,$$
$$\widetilde{\eta}''_{\xi_1^2}(+\infty,0) = \widetilde{\eta}''_{\xi\xi_1}(+\infty,0) = \widetilde{\eta}''_{\xi^2}(+\infty,0) = -\frac{2}{\lambda_1\ell}\,.$$

Then

$$f(t,0) = \frac{\widetilde{\eta}(t,0)}{\xi_1^2} + \frac{\widetilde{\eta}'_{\xi_1}(t,0) - 1}{\xi_1} + \frac{1}{2}\,\widetilde{\eta}''_{\xi_1^2}(t,0) + \frac{1}{\lambda_1\ell} + O(\xi_1),$$
$$f'_{\widetilde{\xi}}(t,0) = \frac{\widetilde{\eta}'_{\xi}(t,0) - 1}{\xi_1} + \widetilde{\eta}''_{\xi\xi_1}(t,0) + \frac{1}{\lambda_1\ell} + O(\xi_1).$$

From the conditions (22), (23) and $S_{10}+S_{01}=0$ it follows that

$$\lim_{t\to+\infty}\frac{\widetilde{\eta}(t,0)}{\xi_1^2}=-\lim_{t\to+\infty}\frac{\psi_{00}(t)\ln^2 v(t)}{\ell S_{01}\lambda_1^2}=0,$$

$$\lim_{t\to+\infty}\frac{\widetilde{\eta}'_{\xi_1}(t,0)-1}{\xi_1}=\lim_{t\to+\infty}\frac{\widetilde{\eta}'_{\xi}(t,0)-1}{\xi_1}=$$

$$=-\lim_{t\to+\infty}\frac{\ln v(t)}{\lambda_1 S_{01}}\bigg(\sum_{k=0}^{\infty}\frac{S_{1k}+S_{0k+1}}{k!\ell^{k+1}}\,\widetilde{\eta}^{\,k}(t,0)+ +\sum_{k=0}^{\infty}\frac{\psi_{1k}+\psi_{0k+1}}{k!\ell^{k+1}}\,\widetilde{\eta}^{\,k}(t,0)\bigg)=0,$$

$$\lim_{t\to+\infty}\Big(\frac{1}{2}\,\widetilde{\eta}''_{\xi_1^2}(t,0)+\frac{1}{\lambda_1\ell}\Big)=0,$$

$$\lim_{t\to+\infty}\Big(\frac{1}{2}\,\widetilde{\eta}''_{\xi\xi_1}(t,0)+\frac{1}{\lambda_1\ell}\Big)=-\frac{1}{\lambda_1\ell}\,.$$

As a result, we have found that $f(+\infty,0)=0$, $f'_{\widetilde{\xi}}(+\infty,0)=-\frac{1}{\lambda_1\ell}\neq 0$.

3) Since

$$f''_{\widetilde{\xi}^2}(t,\widetilde{\xi})=\widetilde{\eta}''_{\xi^2}(t,\xi_1(1+\widetilde{\xi})),\quad f''_{\widetilde{\xi}^2}(t,0)=\widetilde{\eta}''_{\xi^2}(t,\xi_1)=\widetilde{\eta}''_{\xi^2}(t,0)+O(\xi_1),$$

$$f''_{\widetilde{\xi}^2}(+\infty,0)=\widetilde{\eta}''_{\xi^2}(+\infty,0)=-\frac{2}{\lambda_1\ell}\neq 0,$$

the condition 3) of Lemma 2 is automatically satisfied.

Then the differential equation (25) satisfies the conditions of Lemma 2, where since $\operatorname{sign}\big(\frac{v'\xi_1}{\lambda_1\ell v}\big)=1$, there exists for the fixed ℓ a one-parameter set of o-solutions of the equation (25) in $\Delta(t_1)$.

Finally, having the dimension of the set of o-solutions of the equation (25), we have likewise obtained the dimension of a set of R-solutions of the equation (1) with the asymptotic properties (24) in $\Delta(t_1)$. □

Consider now separately the exponential asymptotics $v_4=e^{\ell_0|I(a,t)|}$ (the values of the constants and functions we used, have been identified previously). We proceed from the assumption that of principal importance remain the first s terms, and also the fact that

1) $\alpha_k+\beta_k=\alpha_1+\beta_1\neq 0$ $(k=\overline{2,s})$;
2) $\alpha_k+\beta_k=\alpha_1+\beta_1\neq 0$ $(k=\overline{s+1,s_1})$;
3) $\alpha_k+\beta_k\neq\alpha_1+\beta_1$ $(k=\overline{s_1+1,n})$.

The possibility that the summands with powers of type 2) or 3) are absent is not excluded.

The assumptions 1)–3) and the condition (18) imply that the condition (19) is not satisfied, as

$$S_{10}+S_{01}=\sum_{i=1}^{s}\alpha_i c_i^* \ell^{\alpha_i+\beta_i}+\sum_{i=1}^{s}\beta_i c_i^* \ell^{\alpha_i+\beta_i}=$$
$$=\sum_{i=1}^{s}(\alpha_i+\beta_i)c_i^*\ell^{\alpha_i+\beta_i}=(\alpha_1+\beta_1)\sum_{i=1}^{s}c_i^*\ell^{\alpha_i+\beta_i}=0.$$

Therefore, Theorem 1 cannot be applied to the given asymptotics. If Theorem 2 is likewise not satisfied, then under certain conditions we can achieve fulfilment of the conditions of Theorem 2 by defining the asymptotics $v_4(t)$ more exactly.

Consider the more precise asymptotics

$$v_{41}(t)=e^{\ell_0\int\limits_a^t I_t'(a,t)(1+z(t))\,dt}, \tag{26}$$

where

$$I_t'(a,t)=\Big|\frac{p_1(t)}{p_2(t)}\Big|^{\frac{1}{\beta_2-\beta_1}},$$
$$z(t)\in \mathrm{C}_{\Delta(a)},\quad z(+\infty)=0\Longrightarrow v_{41}(+\infty)=v_4(+\infty)=0\vee+\infty.$$

A solution will be sought in the form

$$y(t)=v_{41}(t)(\ell+\xi(t)), \tag{27}$$

where $\xi(t)\in \mathrm{C}^1_{\Delta(a)}$, $\xi(+\infty)=0$.

Differentiating the equation (27), we obtain:

$$y'(t)=v_{41}'(t)(\ell+\eta(t)), \tag{28}$$
$$\eta(t)=\xi(t)+\frac{v_{41}(t)}{v_{41}'(t)}\,\xi'(t),\quad \eta(t)\in \mathrm{C}_{\Delta(a)}.$$

The condition $y'(t)\sim \ell v_{41}'(t)$ requires the assumption that $\eta(+\infty)=0$.

Substituting (27) and (28) into the equation (1), we obtain the equality:

$$\sum_{k=1}^{n}p_k(t)(v_{41}(t))^{\alpha_k}(v_{41}'(t))^{\beta_k}(\ell+\xi)^{\alpha_k}(\ell+\eta)^{\beta_k}=0. \tag{29}$$

In the equation (29) we put $\xi=0$, $\eta=0$ and get

$$\sum_{k=1}^{n}\ell^{\alpha_k+\beta_k}p_k(t)(v_{41}(t))^{\alpha_k}(v_{41}'(t))^{\beta_k}=0. \tag{30}$$

In accordance with the condition B), indicated in the statement of the problem, we consider the relations of the functions:

$$\frac{p_i(t)(v_{41}(t))^{\alpha_i}(v_{41}'(t))^{\beta_i}}{p_1(t)(v_{41}(t))^{\alpha_1}(v_{41}'(t))^{\beta_1}}=(c_i^*+\varepsilon_i(t))(1+z(t))^{\beta_i-\beta_1}=c_i^*+\varepsilon_{i1}(t), \tag{31}$$
$$\varepsilon_{i1}(+\infty)=0\ \ (i=\overline{1,s});$$

$$\frac{p_j(t)(v_{41}(t))^{\alpha_j}(v'_{41}(t))^{\beta_j}}{p_1(t)(v_{41}(t))^{\alpha_1}(v'_{41}(t))^{\beta_1}} = \varepsilon_j(t)(1+z(t))^{\beta_j-\beta_1} = \varepsilon_{j1}(t), \tag{32}$$

$$\varepsilon_{j1}(+\infty) = 0 \ \ (j = \overline{s+1, s_1});$$

$$\frac{p_k(t)(v_{41}(t))^{\alpha_k}(v'_{41}(t))^{\beta_k}}{p_1(t)(v_{41}(t))^{\alpha_1}(v'_{41}(t))^{\beta_1}} = \frac{e^{\ell_0(\alpha_k+\beta_k)\int_a^t I'_t(a,t)(1+z(t))\,dt}}{e^{\ell_0(\alpha_1+\beta_1)\int_a^t I'_t(a,t)(1+z(t))\,dt}}\times$$

$$\times(1+z(t))^{\beta_k-\beta_1} = \varepsilon_{k1}(t) \ \ (k = \overline{s_1+1, n}), \tag{33}$$

where

$$\lim_{t\to+\infty} \frac{e^{\ell_0(\alpha_k+\beta_k)\int_a^t I'_t(a,t)\,dt}}{e^{\ell_0(\alpha_1+\beta_1)\int_a^t I'_t(a,t)\,dt}} = 0 \Longrightarrow \varepsilon_{k1}(+\infty) = 0 \ \ (k = \overline{s_1+1, n}).$$

Then, after the division by $p_1(t)(v_{41}(t))^{\alpha_1}(v'_{41}(t))^{\beta_1}$, the equation (30) takes the form:

$$\ell^{\alpha_1+\beta_1}\left(\sum_{i=1}^{s} c_i^*(1+z(t))^{\beta_i-\beta_1} + \sum_{j=1}^{s_1}\varepsilon_j(t)(1+z(t))^{\beta_j-\beta_1}\right)+$$

$$+\sum_{k=s_1+1}^{n}\frac{e^{\ell_0(\alpha_k+\beta_k)\int_a^t I'_t(a,t)(1+z(t))\,dt}}{e^{\ell_0(\alpha_1+\beta_1)\int_a^t I'_t(a,t)(1+z(t))\,dt}}(1+z(t))^{\beta_k-\beta_1}\ell^{\alpha_k+\beta_k} = 0$$

or

$$F(t,z) = \ell^{\alpha_1+\beta_1}\left(\sum_{i=1}^{s} c_i^*(1+z)^{\beta_i} + \sum_{j=1}^{s_1}\varepsilon_j(t)(1+z)^{\beta_j}\right)+$$

$$+\sum_{k=s_1+1}^{n}\frac{e^{\ell_0(\alpha_k+\beta_k)\int_a^t I'_t(a,t)(1+z(t))\,dt}}{e^{\ell_0(\alpha_1+\beta_1)\int_a^t I'_t(a,t)(1+z(t))\,dt}}\,(1+z)^{\beta_k}\ell^{\alpha_k+\beta_k} = 0. \tag{34}$$

We introduce into consideration the domain $\widetilde{D} = \Delta(a)\times[-h;h]$. The function $F(t,z) \in \mathrm{C}^{0\infty}_{tz}(\widetilde{D})$.

We consider in $\widetilde{D}$ a part of the function $F(t,z)$:

$$\widetilde{F}(t,z) = \ell^{\alpha_1+\beta_1}\left(\sum_{i=1}^{s} c_i^*(1+z)^{\beta_i} + \sum_{j=1}^{s_1}\varepsilon_j(t)(1+z)^{\beta_j}\right). \tag{35}$$

Taking into account the conditions (16), (18), we get:

$$\widetilde{F}(+\infty,0) = 0;$$

$$\widetilde{F}'_z(+\infty,0) = S_{01} \neq 0;$$

$$\widetilde{F}''_{z^2}(+\infty,0) = S_{02}.$$

Then, by Lemma 1, the equation (35) determines a unique function $z = \widetilde{z}(t,\xi)$, such that $\widetilde{z}(t) \in \mathrm{C}(\Delta(a_1))$ $(a_1 \geq a)$, $\widetilde{z}(+\infty) = 0$.

As $\widetilde{z}(t)$ we take an approximate solution of the equation (35):

$$\widetilde{z}(t) = -\frac{\ell^{\alpha_1+\beta_1} \sum\limits_{j=1}^{s_1} \varepsilon_j(t)}{S_{01} + \ell^{\alpha_1+\beta_1} \sum\limits_{j=1}^{s_1} \beta_j \varepsilon_j(t)}\,. \tag{36}$$

Next, we will need the following functions:

$$\widetilde{\psi}_{00}(t) = \sum_{k=1}^{n} \ell^{\alpha_k+\beta_k} \varepsilon_{k1}(t),$$

$$\widetilde{\psi}_{10}(t) = \sum_{k=1}^{n} \alpha_k \varepsilon_{k1}(t) \ell^{\alpha_k+\beta_k},$$

$$\widetilde{\psi}_{01}(t) = \sum_{k=1}^{n} \beta_k \varepsilon_{k1}(t) \ell^{\alpha_k+\beta_k}.$$

We express $\widetilde{\psi}_{00}(t)$, $\widetilde{\psi}_{10}(t) + \widetilde{\psi}_{01}(t)$ through the previously introduced functions:

$$\begin{aligned}
\widetilde{\psi}_{00}(t) &= \sum_{k=1}^{n} \ell^{\alpha_k+\beta_k} \varepsilon_{k1}(t) = \\
&= \frac{\widetilde{z}^2(t)}{(1+\widetilde{z}(t))^{\beta_1}} \big[S_{02} + \psi_{02}(t) + O(\widetilde{z})\big] = O(\psi_{00}^2(t)); \\
\widetilde{\psi}_{10}(t) + \widetilde{\psi}_{01}(t) &= \sum_{k=1}^{n} (\alpha_k + \beta_k) \varepsilon_{k1}(t) \ell^{\alpha_k+\beta_k} = \\
&= \frac{(\alpha_1+\beta_1)\widetilde{z}^2(t)}{(1+\widetilde{z}(t))^{\beta_1}} \big[S_{02}+\psi_{02}(t)+O(\widetilde{z})\big] = O(\psi_{00}^2(t)).
\end{aligned}$$

Thus, using Theorem 2, we formulate a theorem for the more precise asymptotics

$$v_{41} = e^{\ell_0 \int\limits_a^t I_t'(a,t)(1+\widetilde{z}(t))\,dt}. \tag{37}$$

Theorem 3. *Let for the function $v = v_{41}(t)$ of the form* (37) *the conditions of Theorem* 1, *except for* (19), *be fulfilled, and*

$$S \neq 0, \tag{21}$$

$$S_{02} \neq 0, \tag{38}$$

$$\psi_{00}(t) \ln v_{41}(t) = o(1). \tag{39}$$

Then there exists a one-parameter set of R-solutions $y(t) \in \mathrm{C}^1_{\Delta(t_1)}$ of the differential equation (1) *with the asymptotic properties*

$$y(t) = v_{41}(t)(\ell + \xi(t)), \quad y'(t) \sim \ell v_{41}'(t), \tag{40}$$

where $\xi(t) \sim \frac{\lambda_1 \ell}{\ln v_{41}(t)}$.

The Existence and Asymptotics of R-Solutions of the Equation (1) with the Condition $y(+\infty) = \gamma \in \mathbb{R}_+$

Since $y(+\infty) = \gamma \in \mathbb{R}_+$, a supposed asymptotics will be sought for the derivative of n-solutions $y'(t)$ to within a constant factor of the ratio of the first two summands. Taking into account $p_1(t),\ p_2(t) \neq 0$ $(t \in \Delta(a))$, we get:

$$y'(t) \dot{\sim} w(t) = \left|\frac{p_1(t)}{p_2(t)}\right|^{\frac{1}{\beta_2 - \beta_1}} \quad (\beta_1 \neq \beta_2),$$

where $0 < w(t) \in \mathrm{C}_{\Delta(a)}$.

In the sequel, we will need the assumption that

$$\int\limits_a^{+\infty} w(t)\, dt < +\infty. \tag{41}$$

Let

$$y'(t) = w(t)(\ell + \eta(t)), \tag{42}$$

where $\ell, \ell^{\beta_k} \in \mathbb{R} \setminus \{0\}$ $(k = \overline{1,n})$; $\eta(t) \in \mathrm{C}_{\Delta(a)}$, $\eta(+\infty) = 0$.

Integrating (42), we obtain:

$$y(t) = \gamma - \int\limits_t^{+\infty} w(\tau)(\ell + \eta(\tau))\, d\tau,$$

where $\gamma \in \mathbb{R}_+$. Next, we show that the constants ℓ and γ are related to each other by the equation (49).

Denoting

$$-\int\limits_t^{+\infty} w(\tau)(\ell + \eta(\tau))\, d\tau = \xi(t), \tag{43}$$

$\xi(t) \in \mathrm{C}^1_{\Delta(a)}$, $\xi(+\infty) = 0$, we obtain:

$$y(t) = \gamma + \xi(t). \tag{44}$$

We substitute (42) and (44) into the equation (1) and obtain the equality:

$$F\big(t, \gamma + \xi, w(\ell + \eta)\big) = \sum_{k=1}^{n} p_k(t)(\gamma + \xi)^{\alpha_k} w^{\beta_k} (\ell + \eta)^{\beta_k} = 0, \tag{45}$$

which is satisfied by the functions $\xi(t)$ and $\eta(t)$.

In accordance with the condition B), indicated in the statement of the problem, we assume that:

$$\frac{p_i(t)(w(t))^{\beta_i}}{p_1(t)(w(t))^{\beta_1}} = \widetilde{c}_i + \varepsilon_i(t), \quad \varepsilon_i(+\infty) = 0, \ \ \widetilde{c}_i \in \mathbb{R} \setminus \{0\} \ \ (i = \overline{1,s}); \tag{46}$$

$$\frac{p_j(t)(w(t))^{\beta_j}}{p_1(t)(w(t))^{\beta_1}} = \varepsilon_j(t), \quad \varepsilon_j(+\infty) = 0 \ \ (j = \overline{s+1, n}). \tag{47}$$

Then, after the division by $p_1(t)(w(t))^{\beta_1}$, the equation (45) takes the form:

$$\widetilde{F}(t, \xi, \eta) = \sum_{i=1}^{s} (\widetilde{c}_i + \varepsilon_i(t))(\gamma + \xi)^{\alpha_i} (\ell + \eta)^{\beta_i} + \\ + \sum_{j=s+1}^{n} \varepsilon_j(t)(\gamma + \xi)^{\alpha_j} (\ell + \eta)^{\beta_j} = 0. \tag{48}$$

Obviously, the condition

$$\widetilde{F}(+\infty, 0, 0) = \sum_{i=1}^{s} \widetilde{c}_i \gamma^{\alpha_i} \ell^{\beta_i} = 0 \tag{49}$$

is necessary for the existence of a solution of the form (44) and of its derivative of the form (42).

Theorem 4. *Let a function $w(t)$ be a possible asymptotics of the derivative of R-solution of the equation* (1)*, which satisfies the conditions* (41), (46), (47). *Moreover, let there exist $\gamma \in \mathbb{R}_+$, $\ell \in \mathbb{R} \setminus \{0\}$, satisfying the condition* (49).

Then for the existence of R-solution $y(t) \in \mathrm{C}^1_{\Delta(t_1)}$ of the differential equation (1) *with the asymptotic properties*

$$y(t) \sim \gamma, \quad y'(t) \sim \ell w(t), \tag{50}$$

it is sufficient that the condition

$$\sum_{i=1}^{s} \beta_i \widetilde{c}_i \gamma^{\alpha_i} \ell^{\beta_i} \neq 0 \tag{51}$$

be satisfied.

In this connection, for each pair (γ, ℓ) the differential equation (1) *admits a unique R-solution $y(t)$ with the asymptotic properties* (50).

Proof. Owing to its structure, the functions $\widetilde{F}(t, \xi, \eta) \in \mathrm{C}^{0\infty\infty}_{t\,\xi\,\eta}(D_1)$, $\frac{\partial^n \widetilde{F}}{\partial \xi^n}$, $\frac{\partial^m \widetilde{F}}{\partial \eta^m}$, $\frac{\partial^{n+m} \widetilde{F}}{\partial \xi^n \partial \eta^m}$ $(n = \overline{1, \infty},\ m = \overline{1, \infty})$ are bounded in D_1, where $D_1 = \Delta(a) \times [-h_1; h_1] \times [-h_2; h_2]$, $0 < h_1 < \gamma$, $0 < h_2 < |\ell|$.

To prove the above theorem, we will need expressions of the derivatives of the function $\widetilde{F}(t, \xi, \eta)$ of first and order with respect to the variables ξ, η and also some of their properties:

$$\widetilde{F}'_{\xi}(t, \xi, \eta) = \sum_{i=1}^{s} \alpha_i \widetilde{c}_i (\gamma + \xi)^{\alpha_i - 1} (\ell + \eta)^{\beta_i} + \\ + \sum_{k=1}^{n} \alpha_k \varepsilon_k(t) (\gamma + \xi)^{\alpha_k - 1} (\ell + \eta)^{\beta_k},$$

$$\widetilde{F}'_{\xi}(+\infty,0,0)=\sum_{i=1}^{s}\alpha_i\widetilde{c}_i\gamma^{\alpha_i-1}\ell^{\beta_i}=\frac{1}{\gamma}\sum_{i=1}^{s}\alpha_i\widetilde{c}_i\gamma^{\alpha_i}\ell^{\beta_i};$$

$$\widetilde{F}'_{\eta}(t,\xi,\eta)=\sum_{i=1}^{s}\beta_i\widetilde{c}_i(\gamma+\xi)^{\alpha_i}(\ell+\eta)^{\beta_i-1}+ \\ +\sum_{k=1}^{n}\beta_k\varepsilon_k(t)(\gamma+\xi)^{\alpha_k}(\ell+\eta)^{\beta_k-1},$$

$$\widetilde{F}'_{\eta}(+\infty,0,0)=\sum_{i=1}^{s}\beta_i\widetilde{c}_i\gamma^{\alpha_i}\ell^{\beta_i-1}=\frac{1}{\ell}\sum_{i=1}^{s}\beta_i\widetilde{c}_i\gamma^{\alpha_i}\ell^{\beta_i}\neq 0$$

by virtue of condition (51);

$$\widetilde{F}''_{\eta\eta}(t,\xi,\eta)=\sum_{i=1}^{s}\beta_i(\beta_i-1)\widetilde{c}_i(\gamma+\xi)^{\alpha_i}(\ell+\eta)^{\beta_i-2}+ \\ +\sum_{k=1}^{n}\beta_k(\beta_k-1)\varepsilon_k(t)(\gamma+\xi)^{\alpha_k}(\ell+\eta)^{\beta_k-2}.$$

Owing to the conditions (49), (51) and the properties of the function $\widetilde{F}(t,\xi,\eta)$, in some domain $D_2\subset D_1$, $D_2=\Delta(t_0)\times[-\widetilde{h}_1;\widetilde{h}_1]\times[-\widetilde{h}_2;\widetilde{h}_2]$, $t_0\geq a$, $0<\widetilde{h}_1\leq h_1$, $0<\widetilde{h}_2<\min\Big\{h_2;\frac{\big|\sum_{i=1}^{s}\beta_i\widetilde{c}_i\gamma^{\alpha_i}\ell^{\beta_i}\big|}{4\ell\sup\limits_{D_1}\big|\widetilde{F}''_{\eta\eta}(t,\xi,\eta)\big|}\Big\}$, the equation (48) satisfies the conditions of Lemma 1. Consequently, there exists a unique function $\eta=\widetilde{\eta}(t,\xi)$, $\widetilde{\eta}(t,\xi)\in \mathrm{C}^{0\,\infty}_{t\,\xi}(D_3)$, $\sup\limits_{D_3}\big|\frac{\partial^n\widetilde{\eta}}{\partial\xi^n}\big|<+\infty$ $(n=\overline{1,\infty})$, such that $\widetilde{F}(t,\xi,\widetilde{\eta}(t,\xi))\equiv 0$, $\widetilde{\eta}(+\infty,0)=0$, $\|\widetilde{\eta}(t,\xi)\|\leq\widetilde{h}_2$. Moreover, we can write $\frac{\partial\widetilde{\eta}(t,\xi)}{\partial\xi}=-\frac{\widetilde{F}'_{\xi}(t,\xi,\widetilde{\eta})}{\widetilde{F}'_{\eta}(t,\xi,\widetilde{\eta})}$, $\sup\limits_{D_3}\big|\frac{\partial\widetilde{\eta}}{\partial\xi}\big|=M>0$.

In view of the replacement (43), we obtain the integral equation:

$$-\int_{t}^{+\infty}w(\tau)\big[\ell+\widetilde{\eta}(\tau,\xi(\tau))\big]\,d\tau=\xi(t). \tag{52}$$

The solution of the equation (52) will be sought in the class $\xi(t)\in \mathrm{C}^1_{\Delta(t_1)}$ $(t_1\geq t_0)$.

Next, we consider and estimate the difference $\widetilde{\eta}(t,\xi_2)-\widetilde{\eta}(t,\xi_1)$, $(t,\xi_i)\in D_3$ $(i=1,2)$, applying the Lagrange's theorem with respect to the variable ξ:

$$\widetilde{\eta}(t,\xi_2)-\widetilde{\eta}(t,\xi_1)=\widetilde{\eta}'_{\xi}(t,\xi^*)(\xi_2-\xi_1),\quad \xi^*\in]\xi_1;\xi_2[\,;$$
$$\big|\widetilde{\eta}(t,\xi_2)-\widetilde{\eta}(t,\xi_1)\big|\leq\sup_{D_3}|\widetilde{\eta}'_{\xi}(t,\xi)|\,|\xi_2-\xi_1|=M|\xi_2-\xi_1|.$$

Assuming $\xi_1=0$, $\xi_2=\xi$, we get:

$$|\widetilde{\eta}(t,\xi)|\leq M|\xi|.$$

To the equation (49) we out into the correspondence the operator

$$\xi(t) = T(t, \widetilde{\xi}(t)) \equiv -\int\limits_{t}^{+\infty} w(\tau)\big[\ell + \widetilde{\eta}(\tau, \widetilde{\xi}(\tau))\big]\, d\tau,$$

where $\widetilde{\xi}(t) \in B_1 \subset B$, $B = \big\{\widetilde{\xi}(t) :\ \widetilde{\xi}(t) \in \mathrm{C}^1_{\Delta(t_1)},\ \widetilde{\xi}(+\infty) = 0,\ \|\widetilde{\xi}(t)\| = \sup\limits_{\Delta(t_1)} |\widetilde{\xi}(t)|\big\}$ is the Banach space, $B_1 = \big\{\widetilde{\xi}(t) : \widetilde{\xi}(t) \in B,\ \|\widetilde{\xi}(t)\| \le \widetilde{h}_1\big\}$ is a closed subset of the Banach space B.

Using the contraction mapping principle, we:

1) prove that if $\widetilde{\xi}(t) \in B_1$, then $\xi(t) = T(t, \widetilde{\xi}(t)) \in B_1$: $\widetilde{\xi}(t) \in \mathrm{C}^1_{\Delta(t_1)}$ and $\widetilde{\xi}(+\infty) = 0$, and by virtue of the structure of the operator, we get $\xi(t) \in \mathrm{C}^1_{\Delta(t_1)}$, $\xi(+\infty) = 0$;

$$\begin{aligned}
&\|\widetilde{\xi}(t)\| \le \widetilde{h}_1 \Longrightarrow \|\xi(t)\| = \big\|T(t, \widetilde{\xi}(t))\big\| = \\
&\qquad = \bigg\| \int\limits_{t}^{+\infty} w(\tau)\big[\ell + \widetilde{\eta}(\tau, \widetilde{\xi}(\tau))\big]\, d\tau \bigg\| \le \int\limits_{t_1}^{+\infty} w(\tau)\big(|\ell| + \widetilde{h}_2\big)\, d\tau \le \widetilde{h}_1,
\end{aligned}$$

if t_1 is sufficiently large.

2) check the condition of contraction:

$$\begin{aligned}
&\widetilde{\xi}_1(t), \widetilde{\xi}_2(t) \in B_1 \Longrightarrow \|\xi_2(t) - \xi_1(t)\| = \\
&\qquad = \bigg\| \int\limits_{t}^{+\infty} w(\tau)\big[\widetilde{\eta}(\tau, \widetilde{\xi}_2(\tau)) - \widetilde{\eta}(\tau, \widetilde{\xi}_1(\tau))\big]\, d\tau \bigg\| \le \\
&\qquad \le M \int\limits_{t_1}^{+\infty} w(\tau)\, d\tau\, \big\|\widetilde{\xi}_2(t) - \widetilde{\xi}_1(t)\big\| = \gamma \big\|\widetilde{\xi}_2(\tau) - \widetilde{\xi}_1(\tau)\big\|,
\end{aligned}$$

where $\gamma = M \int\limits_{t_1}^{+\infty} w(\tau)\, d\tau < 1$, if t_1 is sufficiently large.

Thus, t_1 should necessarily be such that

$$\int\limits_{t_1}^{+\infty} w(\tau)\, d\tau < \min\left\{ \frac{\widetilde{h}_1}{|\ell| + \widetilde{h}_2}, \frac{1}{M} \right\}.$$

As a result, we have found that by the contractive mapping principle the equation (52) admits a unique solution $\xi = \widetilde{\xi}(t) \in B_1$.

Thus, we have obtained that for each pair of constants (γ, ℓ), satisfying the condition (49), the differential equation (1) admits a unique R-solution $y(t)$ with the asymptotic properties (50) in $\Delta(t_1)$. Thus the Theorem is complete. $\square$

References

1. A. V. Kostin, The asymptotic properties of solutions of first order ordinary differential equations. (Russian). *Differencial'nye Uravnenija* **4** (1968), 1184–1195.
2. A. V. Kostin, On the question of the existence of bounded particular solutions and of particular solutions tending to zero as $t \to +\infty$ for a system of ordinary differential equations. (Russian) *Differencial'nye Uravnenija* **1** (1965), 585–604.
3. A. V. Kostin, Asymptotics of the regular solutions of nonlinear ordinary differential equations. (Russian) *Differentsial'nye Uravneniya* **23** (1987), No. 3, 524–526.
4. A. V. Kostin, Stability and asymptotics of quasilinear nonautonomous differential systems. (Russian) *Uchebnoe posobie, Izd-vo OGU, Odessa*, 1984.

(Received 13.07.2011)

Authors' address:

Odessa I. I. Mechnikov National University, 2 Dvoryanskaya St., Odessa 65082, Ukraine.

E-mail: a_kostin@ukr.net; koltsova.liliya@gmail.com

Memoirs on Differential Equations and Mathematical Physics
Volume 57, 2012, 75–94

J. V. Manojlović and V. Marić

AN ASYMPTOTIC ANALYSIS OF POSITIVE SOLUTIONS OF THOMAS–FERMI TYPE SUBLINEAR DIFFERENTIAL EQUATIONS

Dedicated to Professor Kusano Takaŝi
on the occasion of his 80th birthday

Abstract. The set of positive solutions of Thomas–Fermi type differential equation

$$x'' = q(t)\phi(x)$$

is studied under the assumptions that q, ϕ are regularly varying functions in the sense of Karamata. It is shown that such solutions exist and their accurate asymptotic behavior at infinity is determined.

2010 Mathematics Subject Classification. Primary 34A34; Secondary 26A12.

Key words and phrases. Thomas–Fermi differential equation, sublinear case, existence, asymptotic behavior of solutions, positive solutions, regular variation.

რეზიუმე. გამოკვლეულია თომას–ფერმის ტიპის

$$x'' = q(t)\phi(x)$$

დიფერენციალური განტოლების დადებით ამონახსნთა სიმრავლე იმ შემთხვევაში, როცა q და ϕ კარამატას აზრით რეგულარულად ცვალებადი ფუნქციებია. სახელდობრ, ნაჩვენებია ასეთი ამონახსნების არსებობა და დადგენილია უსასრულობაში მათი ასიმპტოტური ყოფაქცევა.

1. Introduction

The present paper is devoted to the existence and the asymptotic analysis of positive solutions of nonlinear ordinary differential equations of *Thomas–Fermi type*

$$x'' = q(t)\phi(x) \tag{A}$$

assuming that $q : [a,\infty) \to (0,\infty)$, $a > 0$, is a continuous function which is regularly varying at infinity of index $\sigma \in \mathbb{R}$ and $\phi(x)$ is a positive, continuous function which is regularly varying at zero or at ∞ of index $\gamma \in (0,1)$.

We begin by stating some obvious but important facts valid for all positive solutions of equation (A): Let $x(t)$ be a positive solution of (A) on $[a,\infty)$, $a \geq 0$. Since all positive solutions are convex, it follows that $x'(t)$ is increasing, and hence either $x'(t) < 0$ on $[a,\infty)$ or $x'(t) > 0$ on $[t_0,\infty)$ for some $t_0 > a$. In the former case, $x'(t)$ tends to 0 as $t \to \infty$. In fact, if $x'(t)$ tends to some negative constant w_1, we have $x(t) \leq w_1\, t$, for $t \geq t_1 \geq t_0$, which contradicts positivity of $x(t)$. Moreover, $x(t)$ is positive and decreasing, so that it tends either to a positive constant or to 0 as $t \to \infty$. In the latter case, $x'(t)$ is positive and increasing, so it tends either to ∞ or to some positive constant as $t \to \infty$. Thus, $x'(t) \geq k$ for some positive constant k and for $t \geq t_1 \geq t_0$. Accordingly, by integration we get $x(t) \geq x(t_1) + k(t - t_1)$ which implies that $x(t) \to \infty$ as $t \to \infty$.

On the basis of the above observations all possible *positive decreasing solutions* of (A) fall into the following two types:

$$\lim_{t\to\infty} x(t) = \mathit{const} > 0, \quad \lim_{t\to\infty} x'(t) = 0, \tag{1.1}$$

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} x'(t) = 0, \tag{1.2}$$

while all possible *positive increasing solutions* of (A) fall into the following two types:

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} \frac{x(t)}{t} = \mathit{const} > 0, \tag{1.3}$$

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} x'(t) = \infty. \tag{1.4}$$

In our analysis we shall extensively use the class of regularly varying functions introduced by J. Karamata in 1930 by the following

Definition 1.1. A measurable function $f : [a,\infty) \to (0,\infty)$, $a > 0$, is said to be *regularly varying at infinity of index $\rho \in \mathbb{R}$* if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for all } \lambda > 0.$$

A measurable function $f : (0,a) \to (0,\infty)$ is said to be *regularly varying at zero of index $\rho \in \mathbb{R}$* if $f(\frac{1}{t})$ is regularly varying at ∞ i.e. if

$$\lim_{t\to 0+} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for all } \lambda > 0. \tag{1.5}$$

By $\mathrm{RV}(\rho)$ and $\mathcal{RV}(\rho)$ we denote, respectively, the set of regularly varying functions of index ρ at infinity and at zero. If, in particular, $\rho = 0$, the function f is called *slowly varying* at infinity or at zero. By SV and $\mathcal{SV}$ we denote, respectively, the set of slowly varying functions at infinity and at zero. Saying only regularly or slowly varying function, we mean regularity at infinity.

It follows from Definition 1.1 that any function $f(t) \in \mathrm{RV}(\rho)$ is written as

$$f(t) = t^\rho g(t) \quad \text{with} \quad g(t) \in \mathrm{SV}. \tag{1.6}$$

If, in particular, the function $g(t) \to k > 0$ as $t \to \infty$, it is called a *trivial slowly varying* one denoted by $g(t) \in tr$-SV, the function $f(t) \in \mathrm{RV}(\rho)$ is called a *trivial regularly varying of index* ρ, denoted by $f(t) \in tr$-$\mathrm{RV}(\rho)$. Otherwise $g(t)$ is called a *nontrivial slowly varying* function denoted by $g(t) \in ntr$-SV and $f(t)$ is called a *nontrivial* $\mathrm{RV}(\rho)$ *function*, denoted by $f(t) \in ntr$-$\mathrm{RV}(\rho)$. Similarly for the set $\mathcal{RV}(\rho)$.

Comprehensive treatises on regular variation are given in N. H. Bingham et al. [2] and by E. Seneta [15]. To help the reader, we present here a fundamental result which will be used throughout the paper.

Proposition 1.1 (Karamata's integration theorem). *Let $L(t) \in \mathrm{SV}$. Then*

(i) *if $\alpha > -1$,*

$$\int_a^t s^\alpha L(s)\,ds \sim \frac{1}{\alpha+1}\, t^{\alpha+1} L(t), \quad t \to \infty;$$

(ii) *if $\alpha < -1$,*

$$\int_t^\infty s^\alpha L(s)\,ds \sim -\frac{1}{\alpha+1}\, t^{\alpha+1} L(t), \quad t \to \infty;$$

(iii) *if $\alpha = -1$,*

$$m_1(t) = \int_a^t \frac{L(s)}{s}\,ds \in \mathrm{SV}, \quad m_2(t) = \int_t^\infty \frac{L(s)}{s}\,ds$$

and

$$\lim_{t\to\infty} \frac{L(t)}{m_i(t)} = 0, \quad i = 1, 2.$$

The symbol $\sim$ denotes the asymptotic equivalence

$$f(t) \sim g(t), \quad t \to \infty \Longleftrightarrow \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

Also, $f(t) \asymp g(t)$ means that there exist constants $0 < m < M$ such that

$$m g(t) \le f(t) \le M g(t), \quad t \ge t_0.$$

Throughout the text, "$t \geq t_0$" means that t is sufficiently large, so that t_0 need not to be the same at each occurrence.

We shall also use the following results:

Proposition 1.2. *Let* $q_1(t) \in \mathrm{RV}(\sigma_1)$, $q_2(t) \in \mathrm{RV}(\sigma_1)$, $q_3(t) \in \mathcal{RV}(\sigma_3)$. *Then*

(i) $g_1(t) + g_2(t) \in \mathrm{RV}(\sigma)$, $\sigma = \max(\sigma_1, \sigma_2)$;

(ii) $g_1(t)g_2(t) \in \mathrm{RV}(\sigma_1 + \sigma_2)$, $(g_1(t))^\alpha \in \mathrm{RV}(\alpha\sigma_1)$ *for any* $\alpha \in \mathbb{R}$;

(iii) $q_1(q_2(t)) \in \mathrm{RV}(\sigma_1\sigma_2)$ *if* $q_2(t) \to \infty$, *as* $t \to \infty$;
$q_3(q_2(t)) \in \mathrm{RV}(\sigma_3\sigma_2)$ *if* $q_2(t) \to 0$, *as* $t \to \infty$;

(iv) *for any* $\varepsilon > 0$ *and* $L(t) \in \mathrm{SV}$, *one has* $t^\varepsilon L(t) \to \infty$, $t^{-\varepsilon} L(t) \to 0$, *as* $t \to \infty$.

Proposition 1.3. *If* $f(t) \sim t^\alpha l(t)$ *as* $t \to \infty$ *with* $l(t) \in \mathrm{SV}$, *then* $f(t)$ *is a regularly varying function of index* α *i.e.* $f(t) = t^\alpha l^\star(t)$, $l^\star(t) \in \mathrm{SV}$, *where, in general,* $l^\star(t) \neq l(t)$, *but* $l^\star(t) \sim l(t)$ *as* $t \to \infty$.

Proposition 1.4. *A positive measurable function* $f(t)$ *belongs to* SV *if and only if for every* $\alpha > 0$, *there exist a non-decreasing function* Ψ *and a non-increasing function* ψ *with*

$$t^\alpha f(t) \sim \Psi(t), \quad \text{and} \quad t^{-\alpha} f(t) \sim \psi(t), \quad t \to \infty.$$

Proposition 1.5. *For the function* $f(t) \in \mathrm{RV}(\alpha)$, $\alpha > 0$, *there exists* $g(t) \in \mathrm{RV}(1/\alpha)$ *such that*

$$f(g(t)) \sim g(f(t)) \sim t \quad \text{as} \quad t \to \infty.$$

Here, g is an *asymptotic inverse of* f (and it is determined uniquely to within asymptotic equivalence).

Note, the same result holds for $t \to 0$ i.e. when $f(t) \in \mathcal{RV}(\alpha)$, $\alpha > 0$:

Proposition 1.6. *For the function* $f(t) \in \mathcal{RV}(\alpha)$, $\alpha > 0$, *there exists* $f(t) \in \mathcal{RV}(1/\alpha)$ *such that*

$$f(g(t)) \sim g(f(t)) \sim t \quad \text{as} \quad t \to 0.$$

This follows from Proposition 1.5, since by Definition 1.1 the assumption is equivalent to the saying that $f(1/t) \in \mathrm{RV}(-\alpha)$. Thus, one applies Proposition 1.5 to the function $1/f(1/t) \in \mathrm{RV}(\alpha)$.

The assumptions on q and ϕ, using notation (1.6), imply that equation (A) can be written in the form

$$x''(t) = t^\sigma l(t)\, x^\gamma L(x), \quad l(t) \in \mathrm{SV}, \quad L(x) \in \mathrm{SV} \ \text{or} \ L(x) \in \mathcal{SV}. \tag{1.7}$$

If in (1.7), $\gamma \in (0,1)$ or $\gamma > 1$, equation is called *sublinear* or *superlinear*, respectively.

The study of nonlinear differential equations of the form (A) in the framework of regular variation was initiated by Avakumović [1] (as the very first attempt of the kind in the theory of differential equations), followed by

Marić and Tomić [12]–[14] and some more recent results [4], [5], [7], [8], [10]. See also Marić [11, Chapter3]. These papers and some closely related ones [16], [17] are concerned exclusively with decreasing positive solutions of *superlinear* Thomas–Fermi type equations. No analysis from the viewpoint of regular variation, until recently in [9], seems to have been made of positive solutions of *sublinear* type of equations. There positive increasing solutions of the both types (1.3), (1.4) of the equation (A) (or (1.7)) with $\gamma \in (0,1)$ were analyzed. Very recently a paper [6] by Evtukhov and Samoilenko appeared. A more general equation $x^{(n)} = \alpha q(t)x(t)$ is studied and the existence and the asymptotics of solutions is obtained covering a subclass of regularly varying solutions. Here α may be $+1$ (Thomas–Fermi type), or -1 (Emden–Fowler one).

Our purpose here is to proceed further in studying positive solutions of sublinear equation (A) by establishing the sharp conditions for the existence and constructing the precise asymptotic forms of these. Besides regular variation, the main tools employed in the proof of our main results are the Schauder–Tychonoff fixed point theorem in locally convex spaces and the following generalized L'Hospital's rule (see [3]):

Lemma 1.1. *Let $f, g \in C^1[T,\infty)$ and*

$$\lim_{t\to\infty} g(t) = \infty \ \textit{ and } \ g'(t) > 0 \quad \textit{for all large } t$$

or

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \ \textit{ and } \ g'(t) < 0 \quad \textit{for all large } t.$$

Then

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \le \liminf_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f'(t)}{g'(t)} .$$

2. Results

To avoid repetitions, we state here basic conditions imposed on the functions q and ϕ in all theorems which follows:

$$q(t) \in \mathrm{RV}(\sigma), \quad \sigma\in\mathbb{R}, \tag{2.1}$$

$$\begin{aligned} &\text{a)} \quad \phi(x) \in \mathcal{RV}(\gamma), \quad \gamma \in (0,1);\\ &\text{b)} \quad \phi(x) \in \mathrm{RV}(\gamma), \quad \gamma \in (0,1). \end{aligned} \tag{2.2}$$

First, observe that in either of two cases a) or b) in (2.2), by Propositions 1.5 and 1.6 there exists an asymptotic inverse $\varphi(x)$ of the function $x/\phi(x)$.

In addition, in some of the theorems it is required that either

$$\begin{gathered} \phi(x) \in \mathcal{RV}(\gamma) \ \text{ satisfies } \ \phi(t^\lambda u(t)) \sim \phi(t^\lambda)u(t)^\gamma, \ \text{ as } \ t \to \infty,\\ \text{for each } \ \lambda \in \mathbb{R}^- \ \text{ and } \ u(t) \in \mathcal{SV} \cap C^1(\mathbb{R}), \end{gathered} \tag{2.3}$$

or

$$\begin{gathered} \phi(x) \in \mathrm{RV}(\gamma) \ \text{ satisfies } \ \phi(t^\lambda u(t)) \sim \phi(t^\lambda)u(t)^\gamma, \ \ t \to \infty,\\ \text{for each } \ \lambda \in \mathbb{R}^+ \ \text{ and } \ u(t) \in \mathrm{SV} \cap C^1(\mathbb{R}); \end{gathered} \tag{2.4}$$

In other words, the slowly varying part $L(x)$ of $\phi(x)$ must satisfy $L(t^\lambda u(t)) \sim L(t^\lambda)$, $t \to \infty$, for each slowly varying $u(t) \in C^1(\mathbb{R})$. It is easy to check that this is satisfied by for e.g.

$$L(t) = \prod_{k=1}^{N} (\log_k t)^{\alpha_k}, \quad \alpha_k \in \mathbb{R},$$

but not by

$$L(t) = \exp\Big(\prod_{k=1}^{N} (\log_k t)^{\beta_k}\Big), \quad \beta_k \in (0,1),$$

where $\log_k t = \log \log_{k-1} t$.

For the future analysis we need the following preparatory

Lemma 2.1. *Put*

$$Y_0(t) = \varphi\Big(\frac{t^2 q(t)}{\rho(\rho-1)}\Big), \tag{2.5}$$

and

$$I(t) = \int_t^\infty \int_s^\infty q(r)\phi(Y_0(r))\, dr\, ds, \tag{2.6}$$

where $\varphi(x)$ is an asymptotic inverse of the function $x/\phi(x)$ and ρ is given by

$$\rho = \frac{\sigma+2}{1-\gamma}\,. \tag{2.7}$$

If (2.2) a) *and* (2.1) *with $\sigma < -2$ hold, then as $t \to \infty$*

(i) $Y_0(t) \in \mathrm{RV}(\frac{\sigma+2}{1-\gamma})$ *and* $Y_0(t) \to 0$*;*

(ii) $I(t) \sim Y_0(t)$.

Proof. Since $t^2 q(t) \to 0$, $t \to \infty$, by Proposition 1.2-(iii), we conclude that $Y_0(t) \in \mathrm{RV}\,(\rho)$, with ρ given by (2.7). Thus, $Y_0(t)$ is expressed as $Y_0(t) = t^\rho \eta(t)$, $\eta(t) \in \mathrm{SV}$ and $Y_0(t) \to 0$, $t \to \infty$, because $\rho < 0$. Moreover, in view of (2.5), there follows

$$\frac{Y_0(t)}{\phi(Y_0(t))} \sim \frac{t^2 q(t)}{\rho(\rho-1)}, \quad t \to \infty\,. \tag{2.8}$$

Hence, by writing $I(t)$ in the form

$$I(t) = \int_t^\infty \int_s^\infty q(r)\, \frac{\phi(Y_0(r))}{Y_0(r)}\, Y_0(r)\, dr\, ds \sim$$

$$\sim \rho(\rho-1) \int_t^\infty \int_s^\infty r^{\rho-2}\eta(r)\, dr\, ds, \quad t \to \infty,$$

and applying Karamata's theorem twice on the last integral (Propositi-on 1.1-(ii)), one obtains the desired result. $\square$

To prove the existence and determine the exact asymptotic behavior of solutions $x(t) \in \mathrm{RV}(\rho)$, $\rho \in \mathbb{R}$ we shall consider the following three cases separately:

(i) $\rho < 0$ or $\rho > 1$,

(ii) $\rho = 0$,

(iii) $\rho = 1$.

Note, the case $\rho \in (0, 1)$ does not exist due to (1.1)–(1.4).

(i) Regularly varying solution of index $\rho < 0$ or $\rho > 1$.

Theorem 2.1. *Suppose that* (2.1)*,* (2.2) a) *and* (2.3) *hold. Then equation* (A) *possesses a decreasing regularly varying solution $x(t)$ of index $\rho < 0$ if and only if*

$$\sigma < -2. \tag{2.9}$$

Also, $x(t)$ satisfies (1.2)*.*

If, on the other hand, (2.1)*,* (2.2) b) *and* (2.4) *hold, then equation* (A) *possesses an increasing regularly varying solution $x(t)$ of index $\rho > 1$ if and only if*

$$\sigma > -\gamma - 1. \tag{2.10}$$

Also, $x(t)$ satisfies (1.4)*.*

In either case any such solution $x(t)$ has for $t \to \infty$ the exact asymptotic behavior

$$x(t) \sim \varphi\Big(\frac{t^2 q(t)}{\rho(\rho-1)}\Big), \tag{2.11}$$

where φ and ρ are as in Lemma 2.1*.*

Proof. We begin with the proof of the first part of Theorem 2.1, where $\rho < 0$. Let (2.1), (2.2) a) and (2.3) hold.

The "only if" part: Let $x(t) \in \mathrm{RV}(\rho)$, $\rho < 0$, be a decreasing solution of (A) on $[t_0, \infty)$. We express it as $x(t) = t^\rho \xi(t)$, $\xi(t) \in \mathrm{SV}$. To avoid ambiguity, notice that $\rho \in \mathbb{R}$ and has to be determined. Due to Proposition 1.2-(iv) $x(t) \to 0$ as $t \to \infty$, and as is pointed out in the Introduction, $x'(t) \to 0$ as $t \to \infty$. Integrating (A) over (t, ∞) and using (1.7), we get for $t \geq t_0$

$$-x'(t) = \int_t^\infty q(s)\phi(x(s))\,ds = \int_t^\infty s^{\sigma+\rho\gamma} l(s)\xi(s)^\gamma L(s^\rho \xi(s))\,ds. \tag{2.12}$$

The convergence of the last integral implies that $\sigma + \rho\gamma \leq -1$. However, the possibility $\sigma + \rho\gamma = -1$ is excluded. In fact, if this were the case, then (2.12) reduces to

$$-x'(t) = \int_t^\infty s^{-1} l(s)\xi(s)^\gamma L(s^\rho \xi(s))\,ds,$$

and since due to Proposition 1.1-(iii) the last integral is slowly varying, an integration over $[t,\infty)$ gives

$$x(t) \sim t \int_t^\infty s^{-1} l(s)\xi(s)^\gamma L(s^\rho \xi(s))\, ds \in \mathrm{RV}(1), \quad t \to \infty,$$

contradicting $\rho < 0$. Thus, we have $\sigma + \rho\gamma < -1$. Then, by Karamata's integration theorem from (2.12), we obtain

$$-x'(t) \sim \frac{t^{\sigma+\rho\gamma+1} l(t)\xi(t)^\gamma L(t^\rho \xi(t))}{-(\sigma+\rho\gamma+1)}\,, \quad t \to \infty. \tag{2.13}$$

Since $x(t) \to 0$ as $t \to \infty$, by integration we further get

$$\int^\infty \frac{t^{\sigma+\rho\gamma+1} l(t)\xi(t)^\gamma L(t^\rho \xi(t))}{-(\sigma+\rho\gamma+1)}\, dt < \infty,$$

and hence $\sigma + \rho\gamma + 1 \le -1$ i.e. $\sigma + \rho\gamma \le -2$. If $\sigma + \rho\gamma = -2$, then (2.13) reduces to

$$x'(t) \sim -t^{-1} l(t)\xi(t)^\gamma L(t^\rho \xi(t)), \quad t \to \infty,$$

and integration over $[t,\infty)$ yields

$$x(t) \sim \int_t^\infty s^{-1} l(s)\xi(s)^\gamma L(s^\rho \xi(s))\, ds \in \mathrm{SV}, \quad t \to \infty,$$

which leads to an impossibility that $\rho = 0$. Therefore, we must have $\sigma+\rho\gamma < -2$, in which case, integrating (2.13) over $[t,\infty)$, we get for $t \to \infty$

$$\begin{aligned} x(t) \sim \frac{t^{\sigma+\rho\gamma+2} l(t)\xi(t)^\gamma L(t^\rho \xi(t))}{[-(\sigma+\rho\gamma+1)]\,[-(\sigma+\rho\gamma+2)]} &= \\ &= \frac{t^2 q(t)\phi(t^\rho \xi(t))}{[-(\sigma+\rho\gamma+1)]\,[-(\sigma+\rho\gamma+2)]} \end{aligned} \tag{2.14}$$

implying, in view of Proposition 1.3, that the regularity index of $x(t)$ is $\rho = \sigma + \rho\gamma + 2$, i.e. $\rho = \frac{\sigma+2}{1-\gamma}$. Then, since $\rho < 0$, we conclude that $\sigma < -2$. Since, $(\sigma+\rho\gamma+1)(\sigma+\rho\gamma+2) = \rho(\rho-1)$, (2.14), due to (2.8), becomes

$$\frac{x(t)}{\phi(x(t))} \sim \frac{t^2 q(t)}{\rho(\rho-1)} \sim \frac{Y_0(t)}{\phi(Y_0(t))}\,, \quad t \to \infty. \tag{2.15}$$

Because $Y_0(t) \to 0$ and $x(t) \to 0$ as $t \to \infty$, (2.15) is, in view of Proposition 1.6, equivalent to (2.11).

The "if" part: Note that any solution $x(t)$ of the integral equation

$$x(t) = \int_t^\infty \int_s^\infty q(r)\phi(x(r))\, dr\, ds, \tag{2.16}$$

(if it exists) satisfies (A) and is obviously positive, decreasing and (1.2) holds. We shall prove that it indeed exists and possesses the properties stated in the Theorem.

Applying Proposition 1.4 to the function $\phi(x) \in \mathcal{RV}(\gamma)$ with $\gamma > 0$, we see that there exists a constant $A > 1$ such that

$$\phi(x) \leq A\,\phi(y) \quad \text{for each} \quad a > y \geq x > 0. \tag{2.17}$$

Due to Lemma 2.1, there exists $t_0 > a$ so that

$$\frac{Y_0(t)}{2} \leq I(t) \leq 2Y_0(t), \quad t \geq t_0. \tag{2.18}$$

In addition, since $Y_0(t) \to 0$ as $t \to \infty$ and (1.5) holds uniformly on each compact λ-set on $(0, \infty)$ ([2, Theorem 1.2.1]) there exists $t_0 > a$ such that

$$\frac{\lambda^\gamma}{2}\,\phi(Y_0(t)) \leq \phi(\lambda Y_0(t)) \leq 2\lambda^\gamma \phi(Y_0(t)) \quad \text{for} \quad t \geq t_0. \tag{2.19}$$

Choose $0 < k < 1$ and $K > 1$ such that

$$k^{1-\gamma} \leq \frac{1}{4A} \quad \text{and} \quad K^{1-\gamma} \geq 4A, \tag{2.20}$$

which is possible due to $0 < \gamma < 1$.

Now we choose t_0 such that (2.18) and (2.19) both hold and define the set $\mathcal{X}$ to be the set of continuous functions $x(t)$ on $[t_0, \infty)$ satisfying

$$kY_0(t) \leq x(t) \leq KY_0(t) \quad \text{for} \quad t \geq t_0. \tag{2.21}$$

It is clear that $\mathcal{X}$ is a closed convex subset of the locally convex space $C[t_0, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$. We shall show that the integral operator $\mathcal{F}$ defined by

$$\mathcal{F}x(t) = \int_t^\infty \int_s^\infty q(r)\phi(x(r))\,dr\,ds, \quad t \geq t_0,$$

is a continuous self-map on $\mathcal{X}$ and that $\mathcal{F}(\mathcal{X})$ is a relatively compact subset of $C[t_0, \infty)$ and then apply the Schauder–Tychonoff fixed point theorem. Notice that, in view of Lemma 2.1, the above integral converges on the set $\mathcal{X}$ under consideration.

Let $x(t) \in \mathcal{X}$. By using successively (2.17), (2.19) with $\lambda = K$ and $\lambda = k$, (2.20) and (2.18), one obtains

$$\begin{aligned}
\mathcal{F}x(t) &\leq A \int_t^\infty \int_s^\infty q(r)\phi(KY_0(r))\,dr\,ds \leq \\
&\leq 2A\,K^\gamma \int_t^\infty \int_s^\infty q(r)\phi(Y_0(r))\,dr\,ds \leq \\
&\leq 4A\,K^\gamma Y_0(t) \leq K\,Y_0(t), \quad t \geq t_0,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}x(t) &\geq \frac{1}{A}\int_t^\infty \int_s^\infty q(r)\phi(kY_0(r))\,dr\,ds \geq \\
&\geq \frac{k^\gamma}{2A}\int_t^\infty \int_s^\infty q(r)\phi(Y_0(r))\,dr\,ds \geq \\
&\geq \frac{k^\gamma}{4A}\,Y_0(t) \geq kY_0(t), \quad t \geq t_0.
\end{aligned}$$

Therefore, $\mathcal{F}x(t) \in \mathcal{X}$, that is, $\mathcal{F}$ maps $\mathcal{X}$ into itself.

Furthermore, it can be verified that $\mathcal{F}$ is a continuous map and $\mathcal{F}(\mathcal{X})$ is relatively compact in $C[t_0,\infty)$. Therefore, by the Schauder–Tychonoff fixed point theorem, there exists a fixed point $x(t)$ of $\mathcal{F}$ which satisfies the integral equation (2.16) and hence equation (A).

Now we prove that any such solution $x(t)$ has the asymptotic behavior (2.11). Because of (2.21), $x(t)$ satisfies

$$0 < \liminf_{t\to\infty} \frac{x(t)}{Y_0(t)} \leq \limsup_{t\to\infty} \frac{x(t)}{Y_0(t)} < \infty,$$

or in view of Lemma 2.1, we have

$$0 < \liminf_{t\to\infty} \frac{x(t)}{I(t)} \leq \limsup_{t\to\infty} \frac{x(t)}{I(t)} < \infty.$$

Put $Y_0(t) = t^\rho \eta(t)$, $\eta(t) \in \mathrm{SV}$. An application of Lemma 1.1, in view of assumption (2.3), yields

$$\begin{aligned}
L &= \limsup_{t\to\infty} \frac{x(t)}{I(t)} \leq \limsup_{t\to\infty} \frac{x''(t)}{I''(t)} = \limsup_{t\to\infty} \frac{q(t)\phi(x(t))}{q(t)\phi(Y_0(t))} = \\
&= \limsup_{t\to\infty} \frac{\phi(t^\rho\xi(t))}{\phi(t^\rho\eta(t))} = \limsup_{t\to\infty} \frac{\xi(t)^\gamma\phi(t^\rho)}{\eta(t)^\gamma\phi(t^\rho)} = \limsup_{t\to\infty} \frac{(x(t)/t^\rho)^\gamma}{(Y_0(t)/t^\rho)^\gamma} = \\
&= \Big(\limsup_{t\to\infty} \frac{x(t)}{Y_0(t)}\Big)^\gamma = \Big(\limsup_{t\to\infty} \frac{x(t)}{I(t)}\Big)^\gamma = L^\gamma.
\end{aligned}$$

Since $\gamma < 1$, from the above we conclude that

$$0 < L \leq 1. \tag{2.22}$$

Similarly, we can see that $l = \liminf\limits_{t\to\infty} \dfrac{x(t)}{I(t)}$ satisfies

$$1 \leq l < \infty. \tag{2.23}$$

From (2.22) and (2.23) we obtain that $l = L = 1$, which means that $x(t) \sim I(t) \sim Y_0(t)$, $t \to \infty$, i.e. (2.11) holds. This also shows, due to Proposition 1.3, that $x(t)$ is a regularly varying solution of (A) with the requested regularity index.

We now turn our attention to the second part of Theorem 2.1, where $\rho > 1$. Let (2.1), (2.2) b) and (2.4) hold.

The "only if" part: Suppose that (A) has solution of the form $x(t) = t^\rho \xi(t)$ on $[t_0, \infty)$ with $\rho > 1$ and $\xi(t) \in \mathrm{SV}$. Note that $x'(t) \to \infty$ and $x(t) \to \infty$ as $t \to \infty$. Integrating (A) on $[t_0, t]$, we have

$$x'(t) \sim \int_{t_0}^{t} q(s)\phi(x(s))\, ds = \int_{t_0}^{t} s^{\sigma+\rho\gamma} l(s)\xi(s)^\gamma L(s^\rho \xi(s))\, ds, \quad t \to \infty. \tag{2.24}$$

The divergence of the last integral as $t \to \infty$ means that $\sigma + \rho\gamma \geq -1$. But the possibility $\sigma + \rho\gamma = -1$ is precluded, because if this was the case, then

$$\int_{t_0}^{t} s^{-1} l(s)\xi(s)^\gamma L(s^\rho \xi(s))\, ds \in \mathrm{SV},$$

and hence integration of (2.24) on $[t_0, t]$ shows that

$$x(t) \sim t \int_{t_0}^{t} s^{-1} l(s)\xi(s)^\gamma L(s^\rho \xi(s))\, ds \in \mathrm{RV}(1),$$

which contradicts the condition $\rho > 1$. Thus, $\sigma + \rho\gamma > -1$. In this case, applying Karamata's integration theorem to the last integral in (2.24), we have

$$x'(t) \sim \frac{t^{\sigma+\rho\gamma+1} l(t)\xi(t)^\gamma L(t^\rho \xi(t))}{\sigma + \rho\gamma + 1}, \quad t \to \infty,$$

and integrating the above relation on $[t_0, t]$, we obtain

$$x(t) \sim \frac{t^{\sigma+\rho\gamma+2} l(t)\xi(t)^\gamma L(t^\rho \xi(t))}{(\sigma + \rho\gamma + 1)(\sigma + \rho\gamma + 2)} \in \mathrm{RV}(\sigma + \rho\gamma + 2), \quad t \to \infty, \tag{2.25}$$

which, in view of Proposition 1.3, shows that the regularity index of $x(t)$ is $\rho = \frac{\sigma+2}{1-\gamma}$. From the requirement $\rho > 1$ it follows that $\sigma > -\gamma - 1$. Exactly as when $\rho < 0$, (2.25) leads to the asymptotic formula (2.11).

The "if" part: It is proved in [9, Lemma 2.1, Theorem 2.1] that if the regularity index σ of $q(t)$ satisfies $\sigma > -\gamma - 1$, then the function $Y_0(t) \in \mathrm{RV}(\rho)$ satisfies the relation

$$Y_0(t) \sim \int_a^t \int_a^s q(r)\phi(Y_0(r))\, dr\, ds, \quad t \to \infty,$$

and there exists a positive increasing solution $x(t)$ of equation (A) which satisfies (1.4) and (2.21). Then, proceeding exactly as when $\rho < 0$, with application of Lemma 1.1 and using (2.4), we conclude that $x(t) \sim Y_0(t)$ as $t \to \infty$. This implies $x(t) \in \mathrm{RV}(\rho)$, with ρ given by (2.7), as before. $\square$

(ii) Regularly varying solutions of index $\rho = 0$.

We distinguish two subcases: $x(t) \in tr$-SV and $x(t) \in ntr$-SV.

Observe that slowly varying solutions must decrease. For otherwise (1.3) and (1.4) would hold contradicting Proposition 1.2-(iv).

Theorem 2.2. *Suppose that* (2.1) *and* (2.2) a) *hold. Equation* (A) *possesses a* (*decreasing*) *trivial slowly varying solution if and only if*

$$\int_{t_0}^{\infty} sq(s)\,ds < \infty. \tag{2.26}$$

Proof. The "only if" part: Suppose that (A) has a decreasing tr-SV-solution $x(t)$ on $[t_0,\infty)$ i.e. satisfying $x(t)\to c$, $t\to\infty$, $c>0$. Integrating (A) over $[t,\infty)$ and observing (1.1), one gets

$$-x'(t) = \int_{t}^{\infty} s^{\sigma} l(s)\phi(x(s))\,ds, \quad t \geq t_0, \tag{2.27}$$

implying $\sigma \leq -1$. But the case $\sigma = -1$ is impossible since then, by Proposition 1.1-(iii), the integral in (2.27) is an SV function, and another integration on $[t,\infty)$ would give $\rho = 1$. Thus $\rho < -1$ and by Karamata's theorem, (2.27) leads to

$$-x'(t) \sim \frac{t^{\sigma+1} l(t)\phi(x(t))}{-(\sigma+1)}\,, \quad t\to\infty, \tag{2.28}$$

which together with $x(t)\to c$, $t\to\infty$ yields

$$\int_{t_0}^{\infty} \frac{t^{\sigma+1} l(t)\phi(x(t))}{-(\sigma+1)} < \infty,$$

implying (2.26).

The "if" part: Suppose that (2.26) holds. Then there exists $t_0 \geq a$ such that

$$\int_{t_0}^{\infty} tq(t)\,dt \leq \frac{c}{2A\phi(c)}\,, \quad t\geq t_0, \tag{2.29}$$

where $A>1$ is a constant such that (2.17) holds. Let us now define the integral operator

$$\mathcal{F}x(t) = \frac{c}{2} + \int_{t}^{\infty}\int_{s}^{\infty} q(r)\phi(x(r))\,dr\,ds, \quad t\geq t_0,$$

and the set

$$\mathcal{X} = \left\{ x(t) \in C[t_0,\infty) : \ \frac{c}{2} \leq x(t) \leq c, \ \ t \geq t_0 \right\}.$$

If $x(t) \in \mathcal{X}$, then clearly, $\mathcal{F}x(t) \geq c/2$. Also, due to (2.29), we obtain

$$\int_t^\infty \int_s^\infty q(r)\phi(x(r))\,dr\,ds \leq A\phi(c) \int_t^\infty \int_s^\infty q(r)\,dr\,ds =$$

$$= A\phi(c) \int_t^\infty (r-t)q(r)dr \leq \frac{c}{2}\,, \quad t \geq t_0,$$

and hence $\mathcal{F}x(t) \leq c$ for $t \geq t_0$. This shows that $\mathcal{F}x(t) \in \mathcal{X}$, and hence $\mathcal{F}$ is a self-map of the closed convex set $\mathcal{X}$. Moreover, we can verify that $\mathcal{F}$ is continuous and $\mathcal{F}(\mathcal{X})$ is relatively compact in the topology of the locally convex space $C[t_0, \infty)$. Therefore, by the Schauder–Tychonoff fixed point theorem, $\mathcal{F}$ has a fixed point $x_0(t) \in \mathcal{X}$, which gives birth to a solution of equation (A) tending to a positive constant as $t \to \infty$. $\square$

Remark 2.1. It is clear that (2.26) implies $\sigma < -2$, or $\sigma = -2$ and $\int_t^\infty \frac{l(s)}{s}\,ds < \infty$.

Theorem 2.3. *Suppose that* (2.1) *and* (2.2) a) *hold. Equation* (A) *possesses a* (*decreasing*) *nontrivial slowly varying solution if and only if*

$$\sigma = -2 \;\; \text{and} \;\; \int_t^\infty tq(t)\,dt < \infty, \tag{2.30}$$

and any such solution $x(t)$ has the exact asymptotic behavior

$$x(t) \sim \Phi^{-1}(Q(t)), \;\; t \to \infty, \tag{2.31}$$

where

$$Q(t) = \int_t^\infty sq(s)\,ds, \;\; t \geq a, \;\; \text{and} \;\; \Phi(x) = \int_0^x \frac{dv}{\phi(v)}\,, \;\; x > 0. \tag{2.32}$$

Proof. The "only if" part: Suppose that (A) has a nontrivial SV-solution $x(t)$ on $[t_0, \infty)$, so it has to satisfy (1.2). Then, as in the proof of Theorem 2.2, we get (2.28) and conclude that σ must satisfy $\sigma + 1 \leq -1$. If $\sigma < -2$, integrating (2.28) over $[t, \infty)$ and applying Karamata's integration theorem, we obtain

$$x(t) \sim \frac{t^{\sigma+2} l(t)\phi(x(t))}{(\sigma+1)(\sigma+2)} \in \mathrm{RV}(\sigma+2), \;\; t \to \infty,$$

which is impossible because for the regularity index of $x(t)$ we would get $\rho = \sigma + 2 < 0$. Thus, one has $\sigma = -2$ and so, integration of (2.28) over $[t, \infty)$ gives

$$x(t) \sim \int_t^\infty s^{-1} l(s)\phi(x(s))\,ds, \;\; t \to \infty. \tag{2.33}$$

Let the integral in (2.33) be denoted by $\chi(t)$. Then, $\chi(t) \to 0$, $t \to \infty$ and satisfies

$$\chi'(t) = -t^{-1}l(t)\phi(x(t)) \sim -t^{-1}l(t)\phi(\chi(t)), \ \ t \to \infty,$$

that is

$$\frac{\chi'(t)}{\phi(\chi(t))} \sim -tq(t), \ \ t \to \infty.$$

An integration of the last relation over $[t, \infty)$ results in

$$\int_0^{\chi(t)} \frac{du}{\phi(u)} = \Phi(\chi(t)) \sim \int_t^\infty sq(s)\,ds = Q(t), \ \ t \to \infty, \tag{2.34}$$

or

$$\chi(t) \sim \Phi^{-1}(Q(t)), \ \ t \to \infty,$$

which is equivalent to (2.31) since by (2.33), $x(t) \sim \chi(t)$ as $t \to \infty$.

Observe that because of (2.2) a) and Proposition 1.2-(iv), the left-hand side integral in (2.34) converges at 0 and the same holds for the right-hand side one at ∞. Thus, the second condition in (2.30) also holds. In addition, since Φ is continuous and increasing and $\phi(x) \in \mathcal{RV}(1-\gamma)$, its inverse function exists and

$$\Phi^{-1}(x) \in \mathrm{RV}\Big(\frac{1}{1-\gamma}\Big). \tag{2.35}$$

The "if" part: Suppose that (2.30) holds, so that $q(t) = t^{-2}l(t)$, $l(t) \in \mathrm{SV}$. We show that $Y_1(t)$ defined by

$$Y_1(t) = \Phi^{-1}\Big(\int_t^\infty sq(s)\,ds\Big), \ \ t \geq a,$$

satisfies the integral asymptotic relation

$$\int_t^\infty \int_s^\infty q(r)\phi(Y_1(r))\,dr\,ds \sim Y_1(t), \ \ t \to \infty.$$

Notice that, in view of (2.30), $Q(t) \in \mathrm{SV}$ and $Q(t) \to 0$, $t \to \infty$, so that Proposition 1.2-(iii) and (2.35) show that $Y_1(t) \in \mathrm{SV}$. Also, $Y_1(t) \to 0$ as $t \to \infty$, so that $\phi(Y_1(t)) \in \mathrm{SV}$. Since $\Phi(Y_1(t)) = Q(t)$, we get

$$tq(t) = -\Phi'(Y_1(t))Y_1'(t) = -\frac{Y_1'(t)}{\phi(Y_1(t))}\,,$$

implying that $Y_1(t)$ is a solution of the differential equation

$$Y_1'(t) + tq(t)\phi(Y_1(t)) = 0.$$

Thus, applying Karamata's integration theorem, we have, due to the preceding differential equation,

$$\int_t^\infty \int_s^\infty q(r)\phi(Y_1(r))\,dr\,ds =$$

$$= \int_t^\infty \int_s^\infty r^{-2} l(r)\phi(Y_1(r))\,dr\,ds \sim \int_t^\infty s^{-1} l(s)\phi(Y_1(s))\,ds =$$

$$= \int_t^\infty sq(s)\phi(Y_1(s))\,ds = -\int_t^\infty Y_1'(s)\,ds = Y_1(t), \quad t \to \infty.$$

Then, by replacing in the proof of Theorem 2.1 the function $Y_0(t)$ by $Y_1(t)$, an application of the Schauder–Tychonoff fixed point theorem provides the existence of a decreasing solution $x(t)$ of equation (A) satisfying

$$x(t) \asymp Y_1(t). \tag{2.36}$$

We show that the obtained solution $x(t)$ of (A) is slowly varying and hence satisfies (2.31). Using (2.36) and (2.17), from equation (A) we get

$$x''(t) \asymp q(t)\phi(Y_1(t)) = t^{-2} l(t)\phi(Y_1(t)).$$

Integrating over $[t,\infty)$, we get

$$x'(t) \asymp t^{-1} l(t)\phi(Y_1(t)), \quad x(t) \asymp \int_t^\infty s^{-1} l(s)\phi(Y_1(s))\,ds.$$

Then

$$t\,\frac{x'(t)}{x(t)} \asymp l(t)\phi(Y_1(t)) \Big[\int_t^\infty s^{-1} l(s)\phi(Y_1(s))\,ds \Big]^{-1}. \tag{2.37}$$

Application of Karamata's integration theorem gives

$$\lim_{t\to\infty} l(t)\phi(Y_1(t)) \Big[\int_t^\infty s^{-1} l(s)\phi(Y_1(s))\,ds \Big]^{-1} = 0,$$

which implies with (2.37) that $tx'(t)/x(t) \to 0$ as $t \to \infty$. Therefore, by [11, Proposition 10], $x(t)$ is slowly varying and so enjoys the precise asymptotic behavior (2.31). This completes the proof of Theorem 2.3. □

Remark 2.2. If specially $\phi(x) = x^\gamma$, then formulas (2.11) and (2.31) read, respectively,

$$x(t) \sim \Big(\frac{t^2 q(t)}{\rho(\rho-1)} \Big)^{\frac{1}{1-\gamma}}, \quad x(t) \sim \Big(\int_t^\infty sq(s)\,ds \Big)^{\frac{1}{1-\gamma}}, \quad t \to \infty.$$

(iii) Regularly varying solutions of index $\rho = 1$.

This case is completely resolved by Theorems 3.2 and 3.3 in [9] and we present it here for the sake of completeness.

Theorem 2.4. *Suppose that* (2.1) *and* (2.2) b) *hold. Equation* (A) *possesses a trivial* RV(1) *solution if and only if*

$$\sigma < \gamma - 1, \quad or \ \sigma = -\gamma - 1 \ \ and \ \int_{t_0}^{\infty} q(t)\phi(t)\,dt < \infty.$$

If, in addition, (2.4) *holds for* $\lambda = 1$, *equation* (A) *possesses a nontrivial* RV(1) *solution if and only if*

$$\sigma = -\gamma - 1 \ \ and \ \int_{t_0}^{\infty} q(t)\phi(t)\,dt = \infty,$$

and any such solution has the exact asymptotic behavior

$$x(t) \sim t\left[(1-\gamma)\int_{a}^{t} q(s)\phi(s)\,ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty.$$

Remark 2.3. It is worthwhile mentioning that, due to Proposition 1.3, our results apply to a very wide class of equations (see Examples 2.1, 2.2).

Example 2.1. Consider differential equation (A) with

$$\phi(x) \sim x^{\gamma}\log(x+1) \ \text{ and } \ q(t) \sim \frac{3r(t)t^{\frac{\gamma-5}{2}}(\log t)^{\frac{1-\gamma}{2}}}{4\log(t^{-1/2}(\log t)^{1/2}+1)}, \quad t \to \infty, \tag{2.38}$$

where $0 < \gamma < 1$ and $r(t)$ is a continuous function on $[e, \infty)$ such that $\lim\limits_{t\to\infty} r(t) = 1$.

The function $q(t)$ is a regularly varying function of index $\sigma = \frac{\gamma-5}{2}$, which satisfies $\sigma < -2$, while $\phi(x) \in \mathcal{RV}(\gamma)$ fulfills the condition (2.3). Then $\rho = -1/2$ and it is easy to check that

$$\frac{t^2 q(t)}{\rho(\rho-1)} \sim \frac{t^{\frac{\gamma-1}{2}}(\log t)^{\frac{1-\gamma}{2}}}{\log(t^{-1/2}(\log t)^{1/2}+1)}, \quad t \to \infty.$$

Therefore, it follows from Theorem 2.1 that the equation possesses decreasing regularly varying solutions $x(t)$ of index $\rho = -1/2$, satisfying $x(t) \sim Y_0(t)$, $t \to \infty$ i.e.

$$\frac{x(t)^{1-\gamma}}{\log(x(t)+1)} = \frac{x(t)}{\phi(x(t))} \sim \frac{Y_0(t)}{\phi(Y_0(t))}, \quad t \to \infty.$$

In view of (2.8), we have

$$\frac{Y_0(t)}{\phi(Y_0(t))} \sim \Big(\frac{\log t}{t}\Big)^{\frac{1-\gamma}{2}}\left[\log\Big(\Big(\frac{\log t}{t}\Big)^{\frac{1}{2}}+1\Big)\right]^{-1}, \quad t \to \infty,$$

implying that

$$x(t) \sim \sqrt{\frac{\log t}{t}}, \quad t \to \infty.$$

Observe that if in (2.38) instead of "$\sim$" one has "$=$" and

$$r(t) = 1 - \frac{4}{3\log t} - \frac{1}{3(\log t)^2},$$

then, $x(t) = (\frac{\log t}{t})^{\frac{1}{2}} \in \mathrm{RV}(-1/2)$ is an exact solution.

Example 2.2. Consider equation (A) with

$$\phi(x) \sim x^\gamma \log(x^\delta + 1) \text{ and}$$
$$q(t) \sim \frac{f(t)}{2t^2(\log t)^{\frac{3-\gamma}{2}} \log((\log t)^{-\delta/2} + 1)}, \quad t \to \infty, \tag{2.39}$$

where $\gamma \in (0,1)$, $\delta > 0$ and $f(t)$ is a continuous function on $[e, \infty)$ such that $\lim_{t\to\infty} f(t) = 1$. Clearly, $q(t)$ is a regularly varying function of index $\sigma = -2$ and satisfies

$$Q(t) = \int_t^\infty sq(s)\,ds \sim \frac{1}{\delta(1-\gamma)(\log t)^{\frac{1-\gamma}{2}} \log(\log t)^{-1/2}} \to 0, \quad t \to \infty. \tag{2.40}$$

Also, $\phi(x) \in \mathcal{RV}(\gamma)$ and

$$\Phi(x) = \int_0^x \frac{dv}{\phi(v)} \sim \frac{1}{\delta(1-\gamma)x^{\gamma-1}\log x}, \quad x \to 0. \tag{2.41}$$

By Theorems 2.2 and 2.3, equation (A) has, along with a trivial slowly varying solution, a nontrivial SV-solution $x(t)$ whose asymptotic behavior is given by (2.31) or equivalently

$$\Phi(x(t)) \sim Q(t) = \int_t^\infty sq(s)\,ds, \quad t \to \infty. \tag{2.42}$$

Using (2.40) and (2.41), (2.42) is reduced to

$$\delta(1-\gamma)x(t)^{\gamma-1} \log x(t) \sim \delta(1-\gamma)\big((\log t)^{-1/2}\big)^{\gamma-1} \log(\log t)^{-1/2}, \quad t \to \infty,$$

implying that $x(t) \sim (\log t)^{-1/2}$ as $t \to \infty$. If in (2.39) instead of "$\sim$" one has "$=$" and, in particular, $f(t) = 1 + 3/2\log t$, then (A) possesses an exact nontrivial SV-solution $x(t) = (\log t)$.

Example 2.3. Consider equation (A) with

$$\phi(x) = x^\gamma \log(x+1), \quad q(t) = \big(t^{\gamma+1}(\log t)^\gamma \log(t\log t + 1)\big)^{-1}$$

with $\gamma \in (0,1)$. Note that ϕ fulfills the condition (2.4) with $\lambda = 1$. Also, $q(t) \in \mathrm{RV}(-\gamma - 1)$ and satisfies

$$q(t)\phi(t) \sim t(\log t)^{\gamma}, \quad t \to \infty$$

which for $t \to \infty$ gives

$$\int_{t_0}^{t} q(s)\phi(s)\, ds \sim \frac{(\log t)^{1-\gamma}}{1-\gamma} \to \infty.$$

Thus, by Theorem 2.4, the above-considered equation possesses nontrivial RV(1) solutions all of which have the same asymptotic behavior $x(t) \sim t \log t$, $t \to \infty$. In fact, an exact solution is $x(t) = t \log t$.

Acknowledgement

The first author is supported by the Research project OI-174007 of the Ministry of Education and Science of Republic of Serbia.

References

1. V. G. Avakumović, Sur l'équation différentielle de Thomas-Fermi. (French) *Acad. Serbe Sci. Publ. Inst. Math.* **1** (1947), 101–113.
2. N. H. Bingham, C. M. Goldie, and J. L. Teugels, Regular variation. *Encyclopedia of Mathematics and its Applications*, 27. *Cambridge University Press, Cambridge*, 1987.
3. O. Haupt and G. Aumann, Differential- und Integralrechnung. Unter bes. Berücksichtigung neuer Ergebnisse. Bd. 1: Einführung in die reelle Analysis. Bd. 2: Differentialrechnung. Bd. 3: Integralrechnung. (German) *Walter de Gruyter & Co., Berlin*, 1938.
4. J. Jaroš, T. Kusano, and T. Tanigawa, Nonoscillation theory for second order half-linear differential equations in the framework of regular variation. *Results Math.* **43** (2003), No. 1-2, 129–149.
5. J. Jaroš, T. Kusano, and T. Tanigawa, Nonoscillatory half-linear differential equations and generalized Karamata functions. *Nonlinear Anal.* **64** (2006), No. 4, 762–787.
6. V. M. Evtukhov and A. M. Samoĭlenko, Asymptotic representations of solutions of nonautonomous ordinary differential equations with regularly varying nonlinearities. (Russian) *Differ. Uravn.* **47** (2011), No. 5, 628–650; English transl.: *Differ. Equ.* **47** (2011), No. 5, 627–649.
7. T. Kusano and J. Manojlović, Asymptotic analysis of Emden–Fowler differential equations in the framework of regular variation. *Ann. Mat. Pura Appl. (4)* **190** (2011), No. 4, 619–644; doi: 10.1007/sl0231-010-0166-x.
8. T. Kusano and J. Manojlović, Asymptotic behavior of positive solutions of sublinear differential equations of Emden–Fowler type. *Comput. Math. Appl.* **62** (2011), No. 2, 551–565.
9. T. Kusano, J. Manojlović, and V. Marić, Increasing solutions of Thomas–Fermi type differential equations – the sublinear case. *Bull. T. de Acad. Serbe Sci. Arts, Classe Sci. Mat. Nat., Sci. Math.* **CXLIII** (2011), No. 36, 21–36.
10. T. Kusano, V. Marić, and T. Tanigawa, Regularly varying solutions of generalized Thomas-Fermi equations. *Bull. Cl. Sci. Math. Nat. Sci. Math.* No. 34 (2009), 43–73.
11. V. Marić, Regular variation and differential equations. *Lecture Notes in Mathematics*, 1726. *Springer-Verlag, Berlin*, 2000.

12. V. Marić and M. Tomić, Asymptotic properties of solutions of the equation $y'' = f(x)\phi(y)$. *Math. Z.* **149** (1976), No. 3, 261–266.
13. V. Marić and M. Tomić, Regular variation and asymptotic properties of solutions of nonlinear differential equations. *Publ. Inst. Math. (Beograd) (N.S.)* **21(35)** (1977), 119–129.
14. V. Marić and M. Tomić, Asymptotics of solutions of a generalized Thomas–Fermi equation. *J. Differential Equations* **35** (1980), No. 1, 36–44.
15. E. Seneta, Regularly varying functions. *Lecture Notes in Mathematics*, Vol. 508. *Springer-Verlag, Berlin–New York*, 1976.
16. S. D. Taliaferro, Asymptotic behavior of solutions of $y'' = \varphi(t)f(y)$. *SIAM J. Math. Anal.* **12** (1981), No. 6, 853–865.
17. S. D. Taliaferro, Asymptotic behavior of positive decreasing solutions of $y'' = F(t, y, y')$. *Geometric analysis and nonlinear partial differential equations* (*Denton, TX,* 1990), 105–127, *Lecture Notes in Pure and Appl. Math.*, 144, *Dekker, New York*, 1993.

(Received 05.09.2012)

Author's address:

Jelena V. Manojlović

University of Niš, Faculty of Science and Mathematics, Department of Mathematics, Višegradska 33, 18000 Niš, Serbia.

E-mail: jelenam@pmf.ni.ac.rs

Vojislav Marić

Serbian Academy of Science and Arts, Kneza Mihaila 35, 11000 Beograd, Serbia.

E-mail: vojam@uns.ac.rs

Memoirs on Differential Equations and Mathematical Physics
Volume 57, 2012, 95–108

Manabu Naito

A NOTE ON THE EXISTENCE OF SLOWLY GROWING POSITIVE SOLUTIONS TO SECOND ORDER QUASILINEAR ORDINARY DIFFERENTIAL EQUATIONS

Dedicated to Professor Takaŝi Kusano on his 80th birthday

Abstract. In this paper the second order quasilinear ordinary differential equations are considered, and a sufficient condition for the existence of a slowly growing positive solution is given.

2010 Mathematics Subject Classification. 34C10, 34C11.
Key words and phrases. Quasilinear ordinary differential equations, slowly growing positive solutions.

რეზიუმე. მეორე რიგის კვაზიწრფივი ჩვეულებრივი დიფერენციალური განტოლებებისათვის დადგენილია ნელად ზრდადი დადებითი ამონახსნის არსებობის საკმარისი პირობები.

1. Introduction

In this paper we consider the second order quasilinear ordinary differential equation

$$\big(|x'|^{\alpha}\mathrm{sgn}\, x'\big)' + p(t)|x|^{\beta}\mathrm{sgn}\, x = 0, \tag{1.1}$$

where α and β are positive constants and $p(t)$ is a positive and continuous function on an interval $[t_0, \infty)$. By a solution of (1.1) we mean a real-valued function $x = x(t)$ such that $x \in C^1[T, \infty)$, $T \geq t_0$, and $|x'|^{\alpha}\mathrm{sgn}\, x' \in C^1[T, \infty)$ and $x(t)$ satisfies (1.1) at every point of $[T, \infty)$, where T may depend on $x(t)$. A solution $x(t)$ of (1.1) is said to be *oscillatory* if there is a sequence $\{t_i\}_{i=1}^{\infty}$ such that $\lim\limits_{i\to\infty} t_i = \infty$ and $x(t_i) = 0$ $(i = 1, 2, \dots)$. If a solution $x(t)$ of (1.1) is not oscillatory, then it is said to be *nonoscillatory*. In other words, a solution $x(t)$ of (1.1) is called nonoscillatory if $x(t)$ is eventually positive or eventually negative. If $x(t)$ is a solution of (1.1), then so is $-x(t)$. Therefore there is no loss of generality in assuming that a nonoscillatory solution of (1.1) is eventually positive.

It is easily shown (Elbert [2], Elbert and Kusano [3]) that an eventually positive solution $x(t)$ of (1.1) satisfies one and only one of the following three conditions:

$$\lim_{t\to\infty} x(t) \ \text{ exists and is a positive finite number;} \tag{1.2}$$

$$\lim_{t\to\infty} x(t) = \infty \ \text{ and } \ \lim_{t\to\infty} \frac{x(t)}{t} = 0; \tag{1.3}$$

$$\lim_{t\to\infty} \frac{x(t)}{t} \ \text{ exists and is a positive finite number.} \tag{1.4}$$

A solution $x(t)$ of (1.1) which satisfies (1.2) [*resp.* (1.4)] is asymptotically equal to a positive constant function c [*resp.* a linear function ct] as $t \to \infty$ for some constant $c > 0$. The asymptotic growth of a solution $x(t)$ of (1.1) which satisfies (1.3) is asymptotically bigger than positive constant functions, and is asymptotically smaller than positive unbounded linear functions. In this paper we refer to eventually positive solutions $x(t)$ satisfying (1.3) as *slowly growing positive solutions*. Eventually positive solutions $x(t)$ satisfying (1.2), (1.3) and (1.4) are sometimes called *subdominant solutions*, *intermediate solutions* and *dominant solutions*, respectively ([1]).

It is well known that the following results hold ([2], [3], [7], [8]).

(A) Equation (1.1) has an eventually positive solution $x(t)$ satisfying (1.2) if and only if

$$\int_{t_0}^{\infty} \left[\int_{t}^{\infty} p(s)\, ds \right]^{1/\alpha} dt < \infty. \tag{1.5}$$

(B) Equation (1.1) has an eventually positive solution $x(t)$ satisfying (1.4) if and only if

$$\int_{t_0}^{\infty} t^{\beta} p(t)\, dt < \infty. \tag{1.6}$$

(C) Let $\alpha < \beta$. Equation (1.1) has an eventually positive solution if and only if (1.5) is satisfied.

(D) Let $\alpha > \beta$. Equation (1.1) has an eventually positive solution if and only if (1.6) is satisfied.

Now consider the problem of the existence of an eventually positive solution $x(t)$ satisfying (1.3), namely, a slowly growing positive solution. For the case $\alpha > \beta$ this problem has been solved finally by Naito [9]. The following statement is true:

(E) Let $\alpha > \beta$. Equation (1.1) has a slowly growing positive solution if and only if

$$\int_{t_0}^{\infty} t^{\beta} p(t)\, dt < \infty \ \text{ and } \ \int_{t_0}^{\infty} \left[\int_{t}^{\infty} p(s)\, ds \right]^{1/\alpha} dt = \infty. \tag{1.7}$$

More precisely, the statement (E) was proved by Kusano and Naito [6] for the case $\alpha = 1 > \beta$. The "if" part of (E) for the general case $\alpha > \beta$ was proved by Elbert and Kusano [3]. Very recently, the "only if" part of (E) for the general case $\alpha > \beta$ has been proved by Naito [9].

A characterization of the existence of slowly growing positive solutions of (1.1) for the case $\alpha < \beta$ seems to be a more difficult problem. For some results related to this case, see Cecchi, Došlá and Marini [1] and the references therein.

In this paper we attempt to discuss the existence of slowly growing positive solutions of (1.1) for the case $\alpha < \beta$. For this purpose, let us first consider the particular equation

$$\left(|x'|^{\alpha} \operatorname{sgn} x'\right)' + \kappa t^{-\mu} |x|^{\beta} \operatorname{sgn} x = 0 \ \ (\alpha < \beta), \tag{1.8}$$

where κ is a positive constant and μ is a real constant. It is easy to see that (1.8) has a slowly growing positive solution of the form ct^{ν} ($c > 0$, $0 < \nu < 1$) if and only if $\alpha + 1 < \mu < \beta + 1$, and that this solution is uniquely determined by

$$x_0(t) = c_0 t^{\nu_0} \tag{1.9}$$

with

$$\nu_0 = \frac{\mu - 1 - \alpha}{\beta - \alpha} \ \text{ and } \ c_0 = \left[\frac{\alpha(1-\nu_0)\nu_0^{\alpha}}{\kappa} \right]^{1/(\beta-\alpha)}. \tag{1.10}$$

Observe here that $0 < \nu_0 < 1$ under the conditions $\alpha < \beta$ and $\alpha + 1 < \mu < \beta + 1$. Then we may conjecture that if $p(t)$ is close to the function $\kappa t^{-\mu}$

($\kappa > 0$, $\alpha + 1 < \mu < \beta + 1$) in some sense, then (1.1) has a slowly growing positive solution $x(t)$ satisfying

$$\begin{cases} x(t) = x_0(t)(1 + o(1)) & (t \to \infty), \\ x'(t) = x_0'(t)(1 + o(1)) & (t \to \infty), \end{cases} \tag{1.11}$$

where $x_0(t)$ is defined by (1.9) and (1.10). This conjecture is true to a certain extent. In fact, the following theorem can be proved. For convenience, we write the equation (1.1) in the form

$$\big(|x'|^{\alpha}\mathrm{sgn}\, x'\big)' + \kappa t^{-\mu}(1 + \varepsilon(t))|x|^{\beta}\mathrm{sgn}\, x = 0, \tag{1.12}$$

where $\varepsilon(t)$ is a continuous function on $[t_0, \infty)$, $t_0 > 0$, such that $1+\varepsilon(t) > 0$ for $t \geq t_0$.

Theorem 1.1. *Consider the equation* (1.12) *under the condition*

$$0 < \alpha < \beta, \quad \alpha + 1 < \mu < \beta + 1, \quad \kappa > 0. \tag{1.13}$$

Set $x_0(t) = c_0 t^{\nu_0}$*, where* c_0 *and* ν_0 *are constants given by* (1.10)*. Suppose that there exists* $\ell > 0$ *such that*

$$\ell(\ell - 2\nu_0 + 1) - |1 - \alpha|(1 - \nu_0)\ell - (\beta - \alpha)(1 - \nu_0)\nu_0 > 0 \tag{1.14}$$

and

$$\lim_{t\to\infty} t^{\ell - 2\nu_0 + 1} \int_t^{\infty} s^{2(\nu_0 - 1)}|\varepsilon(s)|\, ds = 0. \tag{1.15}$$

Then the equation (1.12) *has a slowly growing positive solution* $x(t)$ *with the asymptotic property*

$$\begin{cases} x(t) = x_0(t)\big(1 + O(t^{-\ell})\big) & (t \to \infty), \\ x'(t) = x_0'(t)\big(1 + O(t^{-\ell})\big) & (t \to \infty). \end{cases}$$

The condition (1.14) is satisfied if $\ell > 0$ is taken sufficiently large. Therefore, if

$$\lim_{t\to\infty} t^{m} \int_t^{\infty} s^{2(\nu_0 - 1)}|\varepsilon(s)|\, ds = 0 \quad \text{for all } \ m > 0, \tag{1.16}$$

then there is $\ell_0 > 0$ such that for all $\ell \geq \ell_0$, both of the conditions (1.14) and (1.15) are satisfied. On the other hand, it is easy to see that (1.16) is equivalent to

$$\int_{t_0}^{\infty} s^{n}|\varepsilon(s)|\, ds < \infty \quad \text{for all } \ n > 0. \tag{1.17}$$

Thus we can conclude the following result as a corollary of Theorem 1.1.

Corollary 1.1. *Consider the equation* (1.12) *under the condition* (1.13)*. If* (1.17) *holds, then the equation* (1.12) *has a slowly growing positive solution* $x(t)$ *with the asymptotic property* (1.11)*.*

We give a simple example illustrating our theorem in the case $\alpha = 1$.

Example 1.1. Consider the equation

$$x'' + \kappa t^{-3}(1+\varepsilon(t))|x|^3 \operatorname{sgn} x = 0, \quad \kappa > 0, \tag{1.18}$$

where $\varepsilon(t)$ is a continuous function on $[1,\infty)$ such that $1+\varepsilon(t) > 0$ for $t \ge 1$. For this equation, $\alpha = 1$, $\beta = 3$, $\mu = 3$, $\kappa > 0$; and hence $\nu_0 = 1/2$ and $c_0 = 1/\big[2\sqrt{\kappa}\,\big]$. Consequently, the conditions (1.14) and (1.15) reduce to

$$\ell > \frac{\sqrt{2}}{2} \quad \text{and} \quad \lim_{t\to\infty} t^{\ell} \int_t^{\infty} \frac{|\varepsilon(s)|}{s}\, ds = 0. \tag{1.19}$$

Therefore, by Theorem 1.1, we can conclude that if (1.19) is satisfied for some ℓ, then (1.18) has a slowly growing positive solution $x(t)$ such that

$$\begin{cases} x(t) = \dfrac{1}{2\sqrt{\kappa}}\, t^{1/2}\big(1 + O(t^{-\ell})\big) & (t \to \infty), \\[2mm] x'(t) = \dfrac{1}{4\sqrt{\kappa}}\, t^{-1/2}\big(1 + O(t^{-\ell})\big) & (t \to \infty). \end{cases}$$

In the case $0 < \alpha < \beta$, assuming the existence of slowly growing positive solutions of (1.1), Kamo and Usami [4] have obtained the asymptotic forms as $t \to \infty$ of such solutions under a certain condition. Note, however, that the existence of slowly growing positive solutions of (1.1) is not proved.

In the case $0 < \beta < \alpha$, the asymptotic forms as $t \to \infty$ of slowly growing positive solutions of (1.1) has been discussed by Naito [9]. See also [4], [5].

2. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. First notice that if $x(t)$ is a positive solution of (1.1) on an interval $[T,\infty)$, $T \ge t_0$, then $x'(t) > 0$ for $t \ge T$. This fact is easily checked. For the proof of Theorem 1.1, we make use of the following lemma. In this lemma we consider the equations (1.1) and the auxiliary equation

$$\big(|x'|^{\alpha} \operatorname{sgn} x'\big)' + p_0(t)|x|^{\beta} \operatorname{sgn} x = 0, \tag{2.1}$$

where $p_0(t)$ is a positive continuous function on $[t_0,\infty)$, $t_0 > 0$.

Lemma 2.1. *Let $x_0(t)$ be an eventually positive solution of the auxiliary equation* (2.1). *If $x(t)$ is an eventually positive solution of* (1.1), *then*

$$u(t) = \frac{x(t)}{x_0(t)} \quad \textit{and} \quad v(t) = x_0(t)^2 \Big(\frac{x(t)}{x_0(t)}\Big)' \tag{2.2}$$

satisfy

$$u(t) > 0 \quad \textit{and} \quad \frac{1}{x_0(t)}\, v(t) + x_0'(t)u(t) > 0 \tag{2.3}$$

for all large t, and $(u(t), v(t))$ is a solution of the binary nonlinear system

$$\begin{cases} u' = \dfrac{1}{x_0(t)^2}\, v, \\ v' = \dfrac{1}{\alpha} \Big\{ p_0(t) x_0(t)^{\beta+1} x_0'(t)^{-\alpha+1} u - \\ \qquad\qquad -p(t) x_0(t)^{\beta+1} \Big[\dfrac{1}{x_0(t)}\, v + x_0'(t) u \Big]^{-\alpha+1} u^\beta \Big\} \end{cases} \tag{2.4}$$

for all large t.

Conversely, if $(u(t), v(t))$ is a solution of (2.4) *satisfying* (2.3)*, then $x(t) = x_0(t)u(t)$ is an eventually positive solution of* (1.1)*.*

Proof. Let $x(t)$ be an eventually positive solution of (1.1). By (2.2), we have

$$x'(t) = \frac{1}{x_0(t)}\, v(t) + x_0'(t) u(t).$$

Since $x'(t) > 0$ for all large t, it is obvious that $(u(t), v(t))$ satisfies (2.3) for all large t. Moreover, $x(t)$ satisfies

$$x''(t) + \frac{1}{\alpha}\, p(t) x(t)^\beta x'(t)^{-\alpha+1} = 0$$

for all large t. An analogous equality also holds for $x_0(t)$. Then we easily see that $(u(t), v(t))$ satisfies (2.4) for all large t. This proves the first half of the lemma.

To prove the second half, let $(u(t), v(t))$ be a solution of (2.4) satisfying (2.3). Then, a straightforward computation shows that $x(t) = x_0(t)u(t)$ is an eventually positive solution of (1.1). The details are left to the reader. The proof of Lemma 2.1 is complete. $\square$

Proof of Theorem 1.1. We apply Lemma 2.1 to the case $p_0(t) = \kappa t^{-\mu}$ and $x_0(t) = c_0 t^{\nu_0}$, where c_0 and ν_0 are constants given by (1.10). Then the existence of a solution $x(t)$ of (1.1) which satisfies $\lim\limits_{t\to\infty} [x(t)/x_0(t)] = 1$ is equivalent to the existence of a solution $(u(t), v(t))$ of (2.4) which satisfies

$$\lim_{t\to\infty} u(t) = 1 \tag{2.5}$$

and

$$\frac{1}{x_0(t)}\, v(t) + x_0'(t) u(t) > 0 \tag{2.6}$$

for all large t. Thus it is natural to consider the integral equation of the form

$$\begin{cases} u(t) = 1 - \displaystyle\int_t^\infty \frac{1}{x_0(s)^2}\, v(s)\, ds, \\ v(t) = -\dfrac{1}{\alpha} \displaystyle\int_t^\infty \Big\{ p_0(s) x_0(s)^{\beta+1} x_0'(s)^{-\alpha+1} u(s) - \\ \qquad -p(s) x_0(s)^{\beta+1} \Big[\dfrac{1}{x_0(s)}\, v(s) + x_0'(s) u(s) \Big]^{-\alpha+1} u(s)^\beta \Big\}\, ds, \end{cases} \tag{2.7}$$

where $p(t) = p_0(t)(1+\varepsilon(t)) = \kappa t^{-\mu}(1+\varepsilon(t))$.

Denote by X the set of all vector functions $(u(t), v(t)) \in C[T,\infty) \times C[T,\infty)$ such that

$$|u(t) - 1| \le L t^{-\ell} \quad \text{and} \quad |v(t)| \le M t^{-\ell+2\nu_0-1} \quad \text{for} \quad t \ge T, \tag{2.8}$$

where ℓ is a positive constant satisfying (1.14) and (1.15), and L, M, T are positive constants to be determined later. Note that, because of $\ell > 0$, the condition (1.14) implies $\ell - 2\nu_0 + 1 > 0$. We seek for a solution $(u(t), v(t))$ of (2.7) in the set X.

On account of (1.14), we can take a sufficiently small positive number d such that $0 < d < 1/2$ and

$$\ell(\ell - 2\nu_0 + 1) - |1-\alpha|(1-\nu_0)\ell(1-2d)^{-\alpha}(1+d)^\beta - \\ - (\beta - \alpha)(1-\nu_0)\nu_0(1+d) > 0. \tag{2.9}$$

Let M be an arbitrary positive number, and set $L = M/(\ell c_0^2)$ (> 0). Then, by (2.9),

$$\frac{L}{\ell - 2\nu_0 + 1}\, c_0^2 (\beta-\alpha)(1-\nu_0)\nu_0(1+d) + \\ + \frac{M}{\ell - 2\nu_0 + 1}\, |1-\alpha|(1-\nu_0)(1-2d)^{-\alpha}(1+d)^\beta < M.$$

For simplicity, let us use the letters C_1 and C_2 to denote, respectively, the first and the second terms in the left-hand side of the above inequality:

$$C_1 = \frac{L}{\ell - 2\nu_0 + 1}\, c_0^2 (\beta-\alpha)(1-\nu_0)\nu_0(1+d) \quad (> 0) \tag{2.10}$$

and

$$C_2 = \frac{M}{\ell - 2\nu_0 + 1}\, |1-\alpha|(1-\nu_0)(1-2d)^{-\alpha}(1+d)^\beta \quad (\ge 0). \tag{2.11}$$

We have $C_1 + C_2 < M$. Further, let

$$C_3 = D c_0^2 (1-\nu_0)\nu_0 (1+d)^\beta \quad (> 0), \tag{2.12}$$

where D is the positive constant defined by

$$D = \begin{cases} (1+2d)^{-\alpha+1} & \text{for } 0 < \alpha \le 1, \\ (1-2d)^{-\alpha+1} & \text{for } \alpha > 1. \end{cases} \tag{2.13}$$

Since

$$\lim_{u \to 1} \frac{u - u^{-\alpha+\beta+1}}{u-1} = \alpha - \beta,$$

there is $\delta > 0$ such that

$$|u - u^{-\alpha+\beta+1}| \le (1+d)(\beta-\alpha)|u-1| \ \text{ for } \ |u-1| \le \delta. \tag{2.14}$$

We take a number T sufficiently large so that the following inequalities hold for $t \ge T$:

$$Lt^{-\ell} \le d, \quad \frac{M}{c_0^2 \nu_0} t^{-\ell} \le d, \ \ Lt^{-\ell} \le \delta, \tag{2.15}$$

and

$$C_1 + C_2 + C_3 t^{\ell - 2\nu_0 + 1} \int_t^\infty s^{2(\nu_0 - 1)} |\varepsilon(s)| \, ds \le M. \tag{2.16}$$

Note that the inequality $C_1 + C_2 < M$ and the assumption (1.15) ensure the inequality (2.16).

Let X be the set of all vector functions $(u(t), v(t)) \in C[T,\infty) \times C[T,\infty)$ such that (2.8) holds. Define the operator $\Phi : X \to C[T,\infty) \times C[T,\infty)$ by $\Phi(u,v)(t) = (\Phi_1(u,v)(t), \Phi_2(u,v)(t))$ with

$$\Phi_1(u,v)(t) = 1 - \int_t^\infty \frac{1}{x_0(s)^2} \, v(s) \, ds, \quad t \ge T,$$

and

$$\Phi_2(u,v)(t) = -\frac{1}{\alpha} \int_t^\infty \Bigg\{ p_0(s) x_0(s)^{\beta+1} x_0'(s)^{-\alpha+1} u(s) - \\ - p(s) x_0(s)^{\beta+1} \left[\frac{1}{x_0(s)} \, v(s) + x_0'(s) u(s) \right]^{-\alpha+1} u(s)^\beta \Bigg\} ds, \quad t \ge T.$$

It will be shown with the aid of the Schauder–Tychonoff theorem that Φ has a fixed point $(u(t), v(t))$ in X $(\subset C[T,\infty) \times C[T,\infty))$. Here, the space $C[T,\infty) \times C[T,\infty)$ is regarded as the Fréchet space consisting of all continuous vector functions $(u(t), v(t))$ on $[T,\infty)$ with the topology of uniform convergence on compact subintervals of $[T,\infty)$.

(i) *The operator Φ is well defined on X and maps X into X.*

Let $(u(t), v(t)) \in X$. Then, by the first inequality in (2.15), we obtain $|u(t) - 1| \le Lt^{-\ell} \le d$ for $t \ge T$. Therefore,

$$(0 <) \ \ 1 - d \le u(t) \le 1 + d, \quad t \ge T. \tag{2.17}$$

We can show that

$$-\frac{1}{x_0(t)}|v(t)| + x_0'(t)u(t) \geq (1-2d)c_0\nu_0 t^{\nu_0-1}, \quad t \geq T, \tag{2.18}$$

and

$$\frac{1}{x_0(t)}|v(t)| + x_0'(t)u(t) \leq (1+2d)c_0\nu_0 t^{\nu_0-1}, \quad t \geq T. \tag{2.19}$$

In fact, it follows from (2.17) and the second inequality in (2.15) that

$$\begin{aligned}-\frac{1}{x_0(t)}|v(t)| + x_0'(t)u(t) &\geq -\frac{1}{c_0 t^{\nu_0}} M t^{-\ell+2\nu_0-1} + c_0\nu_0 t^{\nu_0-1}(1-d) = \\ &= (1-d)c_0\nu_0 t^{\nu_0-1}\Big\{1 - \frac{M}{(1-d)c_0^2\nu_0}\, t^{-\ell}\Big\} \geq \\ &\geq (1-d)c_0\nu_0 t^{\nu_0-1}\Big(1 - \frac{d}{1-d}\Big) = \\ &= (1-2d)c_0\nu_0 t^{\nu_0-1}, \quad t \geq T,\end{aligned}$$

which shows that (2.18) holds. The inequality (2.19) can be shown in a similar way.

Now let us define $y(t)$ by

$$y(t) = \frac{1}{x_0(t)}\, v(t) + x_0'(t)u(t), \quad t \geq T.$$

Then it follows from (2.18) and (2.19) that

$$(1-2d)c_0\nu_0 t^{\nu_0-1} \leq y(t) \leq (1+2d)c_0\nu_0 t^{\nu_0-1}, \quad t \geq T.$$

In particular, we have $y(t) > 0$ for $t \geq T$ and

$$y(t)^{-\alpha+1} \leq D c_0^{-\alpha+1}\nu_0^{-\alpha+1} t^{(\nu_0-1)(-\alpha+1)}, \quad t \geq T, \tag{2.20}$$

where D is the positive constant defined by (2.13).

For brevity, we define $\varphi_1(u,v)(t)$ and $\varphi_2(u,v)(t)$ by

$$\begin{aligned}\varphi_1(u,v)(t) &= \frac{1}{x_0(t)^2}\, v(t), \\ \varphi_2(u,v)(t) &= p_0(t)x_0(t)^{\beta+1}x_0'(t)^{-\alpha+1}u(t) - \\ &\quad - p(t)x_0(t)^{\beta+1}\left[\frac{1}{x_0(t)}\, v(t) + x_0'(t)u(t)\right]^{-\alpha+1} u(t)^{\beta},\end{aligned}$$

so that

$$\Phi_1(u,v)(t) = 1 - \int_t^\infty \varphi_1(u,v)(s)\, ds, \quad t \geq T,$$

$$\Phi_2(u,v)(t) = -\frac{1}{\alpha}\int_t^\infty \varphi_2(u,v)(s)\, ds, \quad t \geq T.$$

By (2.8), we obtain

$$|\varphi_1(u,v)(t)| \le \frac{1}{x_0(t)^2}\,|v(t)| \le M(c_0t^{\nu_0})^{-2}\,t^{-\ell+2\nu_0-1} = L\ell t^{-\ell-1} \tag{2.21}$$

for $t \ge T$. Thus, $\Phi_1(u,v)(t)$ is well defined on X and

$$|\Phi_1(u,v)(t) - 1| \le L\ell \int_t^\infty s^{-\ell-1}\,ds = Lt^{-\ell}, \quad t \ge T. \tag{2.22}$$

Since $p(t) = p_0(t)(1+\varepsilon(t))$, the function $\varphi_2(u,v)(t)$ can be estimated as follows:

$$\begin{aligned}
&|\varphi_2(u,v)(t)| \le \\
&\le \Big|p_0(t)x_0(t)^{\beta+1}x_0'(t)^{-\alpha+1}u(t) - p_0(t)x_0(t)^{\beta+1}x_0'(t)^{-\alpha+1}u(t)^{-\alpha+\beta+1}\Big| + \\
&\quad + \Big|p_0(t)x_0(t)^{\beta+1}x_0'(t)^{-\alpha+1}u(t)^{-\alpha+\beta+1} - \\
&\qquad - p_0(t)(1+\varepsilon(t))x_0(t)^{\beta+1}y(t)^{-\alpha+1}u(t)^{\beta}\Big| \le \\
&\le p_0(t)x_0(t)^{\beta+1}x_0'(t)^{-\alpha+1}\big|u(t) - u(t)^{-\alpha+\beta+1}\big| + \\
&\quad + p_0(t)x_0(t)^{\beta+1}\Big|\big[x_0'(t)u(t)\big]^{-\alpha+1} - y(t)^{-\alpha+1}\Big|u(t)^{\beta} + \\
&\quad + p_0(t)|\varepsilon(t)|x_0(t)^{\beta+1}y(t)^{-\alpha+1}u(t)^{\beta}.
\end{aligned}$$

Denote the first, second and third term of the last side in the above inequality by $\psi_1(u,v)(t)$, $\psi_2(u,v)(t)$ and $\psi_3(u,v)(t)$, respectively. Then

$$|\varphi_2(u,v)(t)| \le \psi_1(u,v)(t) + \psi_2(u,v)(t) + \psi_3(u,v)(t), \quad t \ge T. \tag{2.23}$$

In view of (2.8) and (2.15), we get $|u(t) - 1| \le Lt^{-\ell} \le \delta$ for $t \ge T$. Therefore, it follows from (2.14) that

$$\big|u(t) - u(t)^{-\alpha+\beta+1}\big| \le L(1+d)(\beta-\alpha)t^{-\ell}, \quad t \ge T.$$

Then it is easy to see that

$$\begin{aligned}
\psi_1(u,v)(t) &= p_0(t)x_0(t)^{\beta+1}x_0'(t)^{-\alpha+1}|u(t) - u(t)^{-\alpha+\beta+1}| \le \\
&\le \kappa t^{-\mu}(c_0t^{\nu_0})^{\beta+1}\left(c_0\nu_0t^{\nu_0-1}\right)^{-\alpha+1}L(1+d)(\beta-\alpha)t^{-\ell} = \\
&= \alpha(\ell - 2\nu_0 + 1)C_1t^{-\ell+2\nu_0-2}, \quad t \ge T,
\end{aligned}$$

where C_1 is the constant given by (2.10).

The mean value theorem implies that if $A > 0$ and $A + B > 0$, then the equality

$$A^{-\alpha+1} - (A+B)^{-\alpha+1} = (\alpha-1)(A+\theta B)^{-\alpha}B$$

holds for some θ, $0 < \theta < 1$. Applying the above equality to the cases $A = x_0'(t)u(t) > 0$ and $B = x_0(t)^{-1}v(t)$, and noting that $A + B = y(t) > 0$, we obtain

$$\begin{aligned}\Big|\big[x_0'(t)u(t)\big]^{-\alpha+1} - y(t)^{-\alpha+1}\Big| &= \\ &= |\alpha-1|\big[x_0'(t)u(t)+\theta x_0(t)^{-1}v(t)\big]^{-\alpha} x_0(t)^{-1}|v(t)| \le \\ &\le |\alpha-1|\big[x_0'(t)u(t)-x_0(t)^{-1}|v(t)|\big]^{-\alpha} x_0(t)^{-1}|v(t)|\end{aligned}$$

for $t \ge T$. Then, by (2.18) and (2.8), we get

$$\begin{aligned}\Big|\big[x_0'(t)u(t)\big]^{-\alpha+1} - y(t)^{-\alpha+1}\Big| &\le \\ &\le |\alpha-1|\big[(1-2d)c_0\nu_0 t^{\nu_0-1}\big]^{-\alpha}(c_0t^{\nu_0})^{-1}Mt^{-\ell+2\nu_0-1} = \\ &= |\alpha-1|(1-2d)^{-\alpha}c_0^{-\alpha-1}\nu_0^{-\alpha}Mt^{-\alpha(\nu_0-1)-\ell+\nu_0-1}\end{aligned}$$

for $t \ge T$. Then it is easy to see that

$$\begin{aligned}\psi_2(u,v)(t) &= p_0(t)x_0(t)^{\beta+1}\big|\big[x_0'(t)u(t)\big]^{-\alpha+1} - y(t)^{-\alpha+1}\big|u(t)^\beta \le \\ &\le \kappa t^{-\mu}(c_0t^{\nu_0})^{\beta+1}\,|\alpha-1|(1-2d)^{-\alpha}c_0^{-\alpha-1}\nu_0^{-\alpha}\times \\ &\qquad \times Mt^{-\alpha(\nu_0-1)-\ell+\nu_0-1}(1+d)^\beta = \\ &= \alpha(\ell-2\nu_0+1)C_2t^{-\ell+2\nu_0-2}, \quad t \ge T,\end{aligned}$$

where C_2 is the constant given by (2.11).

By virtue of (2.20) and (2.17), we find that

$$\begin{aligned}\psi_3(u,v)(t) &= p_0(t)|\varepsilon(t)|x_0(t)^{\beta+1}y(t)^{-\alpha+1}u(t)^\beta \le \\ &\le \kappa t^{-\mu}|\varepsilon(t)|(c_0t^{\nu_0})^{\beta+1}Dc_0^{-\alpha+1}\nu_0^{-\alpha+1}t^{(\nu_0-1)(-\alpha+1)}(1+d)^\beta = \\ &= \alpha C_3t^{2(\nu_0-1)}|\varepsilon(t)|, \quad t \ge T,\end{aligned}$$

where C_3 is the constant given by (2.12). Therefore, by the above estimates for $\psi_1(u,v)(t)$, $\psi_2(u,v)(t)$ and $\psi_3(u,v)(t)$, and by (2.23), we conclude that

$$|\varphi_2(u,v)(t)| \le \alpha(C_1+C_2)(\ell-2\nu_0+1)t^{-\ell+2\nu_0-2} + \alpha C_3t^{2(\nu_0-1)}|\varepsilon(t)| \quad (2.24)$$

for $t \ge T$. Therefore, $\Phi_2(u,v)(t)$ is well defined on X. Moreover, on account of (2.16), we can conclude that

$$\begin{aligned}|\Phi_2(u,v)(t)| &\le \left(C_1+C_2+C_3t^{\ell-2\nu_0+1}\int_t^\infty s^{2(\nu_0-1)}|\varepsilon(s)|\,ds\right)t^{-\ell+2\nu_0-1} \le \\ &\le Mt^{-\ell+2\nu_0-1}, \quad t \ge T.\end{aligned}$$

Thus, the operator $\Phi = (\Phi_1,\Phi_2)$ is well defined on X and maps X into itself. This proves the claim (i).

(ii) *The operator* $\Phi = (\Phi_1,\Phi_2)$ *is continuous on* X.

Assume that $(u_n,v_n) \in X$ $(n = 1,2,3,\dots)$, $(u_\infty,v_\infty) \in X$, and that $(u_n,v_n) \to (u_\infty,v_\infty)$ as $n \to \infty$ uniformly on any compact subinterval $[T,S]$ of $[T,\infty)$. The inequality (2.21) implies that, for every $(u_n,v_n) \in X$, the function $|\varphi_1(u_n, v_n)(t)|$ is bounded by the integrable function $L\ell t^{-\ell-1}$ on $[T, \infty)$. Therefore, by the Lebesgue dominated convergence theorem,

$$\Phi_1(u_n, v_n)(t) \to \Phi_1(u_\infty, v_\infty)(t) \text{ as } n \to \infty$$

uniformly on any compact subinterval $[T, S]$ of $[T, \infty)$. Similarly, using (2.24) and the Lebesgue dominated convergence theorem, we see that

$$\Phi_2(u_n, v_n)(t) \to \Phi_2(u_\infty, v_\infty)(t) \text{ as } n \to \infty$$

uniformly on any compact subinterval $[T, S]$ of $[T, \infty)$. This proves the claim (ii).

(iii) $\Phi(X)$ *is relatively compact.*

To prove the relative compactness of $\Phi(X)$, it is enough to show that $\Phi(X)$ is uniformly bounded and equicontinuous on any compact subinterval $[T, S]$ of $[T, \infty)$. The former follows from the fact that the inequalities $|\Phi_1(u, v)(t)| \le 1 + Lt^{-\ell}$ $(t \ge T)$, which is a consequence of (2.22), and $|\Phi_2(u, v)(t)| \le Mt^{-\ell+2\nu_0-1}$ $(t \ge T)$ hold for all $(u, v) \in X$. The latter follows from the fact that the inequalities (2.21) and (2.24) hold for all $(u, v) \in X$.

In view of (i)–(iii), the Schauder–Tychonoff theorem shows that Φ has a fixed point (u, v) in X. This fixed point $(u, v) = (u(t), v(t))$ $(\in X)$ is a solution of (2.7) on $[T, \infty)$, and satisfies (2.5) and (2.6). Consequently, $(u(t), v(t))$ $(\in X)$ is a solution of (2.4) which satisfies (2.3). Therefore, by Lemma 2.1, $x(t) = x_0(t)u(t)$ is an eventually positive solution of (1.12). By the previous arguments it is easy to see that

$$\frac{x(t)}{x_0(t)} = u(t) = 1 + O(t^{-\ell}) \text{ as } t \to \infty$$

and

$$\begin{aligned} \frac{x'(t)}{x_0'(t)} &= u(t) + \frac{1}{x_0(t)x_0'(t)}\, v(t) = \\ &= u(t) + \frac{1}{c_0^2\nu_0}\, t^{-2\nu_0+1} v(t) = 1 + O(t^{-\ell}) \text{ as } t \to \infty. \end{aligned}$$

This completes the proof of Theorem 1.1. □

References

1. M. Cecchi, Z. Došlá and M. Marini, Intermediate solutions for Emden-Fowler type equations: continuous versus discrete. *Adv. Dyn. Syst. Appl.* **3** (2008), No. 1, 161–176.
2. Á. Elbert, Oscillation and nonoscillation theorems for some nonlinear ordinary differential equations. *Ordinary and partial differential equations (Dundee, 1982)*, pp. 187–212, *Lecture Notes in Math.*, 964, *Springer, Berlin–New York*, 1982
3. Á. Elbert and T. Kusano, Oscillation and nonoscillation theorems for a class of second order quasilinear differential equations. *Acta Math. Hungar.* **56** (1990), No. 3-4, 325–336.

4. K. Kamo and H. Usami, Oscillation theorems for 4th-order quasilinear ordinary differential equations. (Japanese) *Dynamics of functional equations and related topics (Japanese) (Kyoto, 2001). Sūrikaisekikenkyūsho Kōkyūroku* No. 1254 (2002), 1–7.
5. K. Kamo and H. Usami, Characterization of slowly decaying positive solutions of second-order quasilinear ordinary differential equations with sub-homogeneity. *Bull. Lond. Math. Soc.* **42** (2010), No. 3, 420–428.
6. T. Kusano and M. Naito, Unbounded nonoscillatory solutions of nonlinear ordinary differential equations of arbitrary order. *Hiroshima Math. J.* **18** (1988), No. 2, 361–372.
7. D. D. Mirzov, Oscillatory properties of solutions of a system of nonlinear differential equations. *Differ. Equations* **9** (1973), 447–449 (1975).
8. D. D. Mirzov, Ability of the solutions of a system of nonlinear differential equations to oscillate. (Russian) *Mat. Zametki* **16** (1974), 571–576; English transl.: *Math. Notes* **16** (1974), 932–935 (1975).
9. M. Naito, On the asymptotic behavior of nonoscillatory solutions of second order quasilinear ordinary differential equations. *J. Math. Anal. Appl.* **381** (2011), No. 1, 315–327.

(Received 02.05.2012)

Author's address:

Department of Mathematics, Faculty of Science, Ehime University, Matsuyama 790-8577, Japan.

E-mail: mnaito@math.sci.ehime-u.ac.jp

Memoirs on Differential Equations and Mathematical Physics
Volume 57, 2012, 109–122

Yūki Naito

REMARKS ON SINGULAR STURM COMPARISON THEOREMS

Cordially dedicated to Professor Takaŝi Kusano on his 80th birthday

Abstract. In a finite or infinite open interval, the linear differential equations of second order with singularities at endpoints are considered. By making use of principal solutions at endpoints of the interval, we obtain sharper forms of the Strum comparison theorem.

2010 Mathematics Subject Classification. 34B24, 34C10.

Key words and phrases. Singular second order linear differential equations, Sturm comparison theorem, principal solutions

რეზიუმე. სასრულ ან უსასრულო შუალედში განხილულია მეორე რიგის წრფივი დიფერენციალური განტოლებები სინგულარობებით საზღვრით წერტილებში. აღნიშნული წერტილების მიმართ მთავარი ამონახსნების გამოყენებით დადგენილია არაგაუმჯობესებადი შტურმის შედარების თეორემა.

1. Introduction

We consider two differential equations

$$(p(t)u')' + q(t)u = 0, \tag{1.1}$$

$$(P(t)v')' + Q(t)v = 0 \tag{1.2}$$

on the intervals (α, ω) with $-\infty \le \alpha < \omega \le \infty$ and $[a, \omega)$ with $a \in (\alpha, \omega)$. Throughout the paper we assume, in (1.1) and (1.2), that $p(t)$, $q(t)$, $P(t)$, and $Q(t)$ are continuous functions on (α, ω), and satisfy

$$p(t) \ge P(t) > 0 \ \text{ and } \ Q(t) \ge q(t) \ \text{ on } \ (\alpha, \omega). \tag{1.3}$$

We consider the Sturm comparison theorems in the case where the continuity of the coefficients of equations is assumed only on (α, ω). The possibility that the interval is unbounded is not excluded. Concerning the Sturm comparison theorems for such singular equations, several results are summarized in Reid [13] and Swanson [14]. In this paper, motivated by the recent works by Chuaqui et. al. [2] and Aharonov and Elias [1], we will show sharper forms of the Strum's comparison theorem by making use of the principal solutions at endpoints of the interval.

Let us recall the definitions of principal and nonprincipal solutions to (1.1). Assume that (1.1) is nonoscillatory at $t = \omega$. It is well known [5, Ch. XI, Theorem 6.4] that (1.1) has a unique (neglecting a constant factor) solution $u_0(t)$ satisfying

$$\int^{\omega} \frac{ds}{p(s)u_0(s)^2} = \infty, \tag{1.4}$$

and any solution $u_1(t)$, linearly independent of $u_0(t)$, satisfies

$$\int^{\omega} \frac{ds}{p(s)u_1(s)^2} < \infty \tag{1.5}$$

and $u_0(t)/u_1(t) \to 0$ as $t \to \omega$. A solution $u_0(t)$ satisfying (1.4) is called a principal solution at $t = \omega$, and a solution $u_1(t)$ satisfying (1.5) is called a nonprincipal solution at $t = \omega$. The principal and nonprincipal solutions of (1.1) at $t = \alpha$ are defined similarly. For further information about the properties of principal and nonprincipal solutions, we refer to Hartman [5, Ch. XI] and Elbert and Kusano [3].

First we consider (1.1) and (1.2) on a half-open interval $[a, \omega)$ with $a \in (\alpha, \omega)$. The Sturm's comparison theorem can be stated usually as follows: (See, e.g., [5, Ch. XI, Theorem 3.1].)

Theorem A. *Let* $u(t) \not\equiv 0$ *be a solution of* (1.1) *on* $[a, \omega)$*, and let* $v(t)$ *be a solution of* (1.2) *on* $[a, \omega)$*. Assume that, for some* $n \in \mathbf{N} = \{1, 2, \dots\}$*, the solution* $u(t)$ *has exactly* n *zeros* $t = t_1 < t_2 < \cdots < t_n$ *in* (a, ω)*. If either* $u(a) = 0$ *or*

$$u(a) \ne 0, \ v(a) \ne 0, \ \text{and} \ \frac{p(a)u'(a)}{u(a)} \ge \frac{P(a)v'(a)}{v(a)},$$

then $v(t)$ has one of the following properties:

(i) *$v(t)$ has at least n zeros in (a, t_n);*

(ii) *$v(t)$ is a constant multiple of $u(t)$ on $[a, t_n]$ and*

$$p(t) \equiv P(t), \quad q(t) \equiv Q(t) \ \ on \ \ [a, t_n].$$

In the case where $u(t) \neq 0$ on (t_n, ω) in Theorem A, it seems interesting to put a question whether a solution $v(t)$ of (1.2) has at least one zero in (t_n, ω). Our results are the following.

Theorem 1. *Assume that (1.1) is nonoscillatory at $t = \omega$. Let $u_0(t)$ be a principal solution of (1.1) at $t = \omega$, and let $v(t)$ be a solution of (1.2) on $[a, \omega)$. Assume that $u_0(t) > 0$ on (a, ω). If either $u_0(a) = 0$ or*

$$u_0(a) \neq 0,\ v(a) \neq 0, \ \ and \ \frac{p(a)u_0'(a)}{u_0(a)} \geq \frac{P(a)v'(a)}{v(a)}\,, \tag{1.6}$$

then $v(t)$ has one of the following properties:

(i) *$v(t)$ has at least one zero in (a, ω);*

(ii) *$v(t)$ is a constant multiple of $u_0(t)$ on $[a, \omega)$, and*

$$p(t) \equiv P(t), \ \ q(t) \equiv Q(t) \ \ on \ \ [a, \omega).$$

Combining Theorems A and 1, we obtain the following

Theorem 2. *Assume that (1.1) is nonoscillatory at $t = \omega$. Let $u_0(t)$ be a principal solution of (1.1) at $t = \omega$, and let $v(t)$ be a solution of (1.2) on $[a, \omega)$. Assume that $u(t)$ has exactly n zeros in (a, ω) for some $n \in \mathbf{N}$. If either $u_0(a) = 0$ or (1.6) holds, then $v(t)$ has one of the following properties:*

(i) *$v(t)$ has at least $n + 1$ zeros in (a, ω);*

(ii) *$v(t)$ is a constant multiple of $u_0(t)$ on $[a, \omega)$ and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on $[a, \omega)$.*

Next, motivated by [1,2,11,12], we consider (1.1) and (1.2) on the interval (α, ω) with $-\infty \leq \alpha < \omega \leq \infty$.

Theorem 3. *Assume that there exists a solution $u_0(t)$ of (1.1) such that $u_0(t)$ has exactly $n - 1$ zeros in (α, ω) for some $n \in \mathbf{N}$ and is principal at both points $t = \alpha$ and $t = \omega$, that is,*

$$\int_{\alpha} \frac{1}{p(t)u_0(t)^2}\, dt = \infty \ \ and \ \ \int^{\omega} \frac{1}{p(t)u_0(t)^2}\, dt = \infty. \tag{1.7}$$

If $v(t)$ is a solution of (1.2) on (α, ω), then $v(t)$ has one of the following properties:

(i) *$v(t)$ has at least n zeros in (α, ω);*

(ii) *$v(t)$ is a constant multiple of $u_0(t)$ on (α, ω), and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on (α, ω).*

Let us consider some corollaries of Theorem 3. For the case where $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ on (α, ω) in Theorem 3, we will obtain the uniqueness of solution of (1.1) with prescribed numbers of zeros in (α, ω).

Corollary 1. *Assume that there exists a solution $u_0(t)$ of* (1.1) *such that $u_0(t)$ has exactly $n-1$ zeros in (α, ω) for some $n \in \mathbf{N}$ and satisfies* (1.7)*. Then any solution, linearly independent of u_0, has exactly n zeros in (α, ω), that is, the solution of* (1.1) *with $n-1$ zeros in (α, ω) is unique up to a constant factor.*

In the case where

$$p(t) \not\equiv P(t) \;\text{ or }\; q(t) \not\equiv Q(t) \;\text{ on }\; (\alpha, \omega), \tag{1.8}$$

as a corollary of Theorem 3, we obtain the following

Corollary 2. *Assume that* (1.8) *holds. If there exists a solution $u_0(t)$ of* (1.1) *such that $u_0(t)$ has exactly $n-1$ zeros in (α, ω) for some $n \in \mathbf{N}$ and satisfies* (1.7)*, then every solution v of* (1.2) *has at least n zeros in (α, ω).*

Remark 1.

(i) In the case where $u_0(t) > 0$ and $p(t) \equiv P(t) \equiv 1$ on (α, ω), the result in Corollary 2 was shown in [1, Theorem 1 (i)] by a different argument.

(ii) Let us consider the equation with a parameter $\lambda > 0$:

$$(p(t)u')' + \lambda q(t)u = 0 \tag{1.9}$$

on the interval (α, ω). In (1.9) we assume that $q \geq 0$, $q \not\equiv 0$ on (α, ω). For each $n \in \mathbf{N}$, let us denote by λ_n the parameter λ such that (1.9) has a solution u_0 which has exactly $n-1$ zeros in (α, ω) and satisfies (1.7). Corollary 2 implies that λ_n is unique for each $n \in \mathbf{N}$ if it exists. The existence of a sequence $\{\lambda_n\}_{n=1}^{\infty}$ was shown by Kusano and M. Naito [7,8] for the equation (1.9) on (a, ∞) under suitable conditions on p and q. (See also [10].) The extension of the results to the half-linear differential equations was done by [4, 9].

We will show that the condition (1.7) is likewise necessary for the uniqueness of a solution with prescribed numbers of zeros.

Theorem 4. *Assume that* (1.1) *has a solution $u(t)$ which has exactly $n-1$ zeros in (α, ω) with some $n \in \mathbf{N}$, and that any solution, linearly independent of u, has n zeros in (α, ω). Then $u(t)$ is principal at both points $t = \alpha$ and $t = \omega$, that is,* (1.7) *holds with $u_0 = u$.*

Finally, we consider comparison results on the existence of positive solutions of (1.1) and (1.2). Note that, by Corollary 2, if (1.8) holds, and if (1.1) has a positive solution u_0 satisfying (1.7), then (1.2) has no positive solution.

Theorem 5.

(i) *Assume that* (1.8) *holds. If* (1.2) *has a positive solution on* (α,ω), *then* (1.1) *has positive solutions* $u(t)$, $u_0(t)$, $\widetilde{u}_0(t)$ *on* (α,ω) *satisfying*

$$\int_\alpha \frac{1}{p(t)u(t)^2}\,dt < \infty, \quad \int^\omega \frac{1}{p(t)u(t)^2}\,dt < \infty, \tag{1.10}$$

$$\int_\alpha \frac{1}{p(t)u_0(t)^2}\,dt < \infty, \quad \int^\omega \frac{1}{p(t)u_0(t)^2}\,dt = \infty, \tag{1.11}$$

and

$$\int_\alpha \frac{1}{p(t)\widetilde{u}_0(t)^2}\,dt = \infty, \quad \int^\omega \frac{1}{p(t)\widetilde{u}_0(t)^2}\,dt < \infty, \tag{1.12}$$

respectively.

(ii) *Assume that* (1.1) *has a positive solution* $u(t)$ *on* (α,ω) *satisfying*

$$\int_\alpha \frac{1}{p(t)u(t)^2}\,dt < \infty \quad \text{or} \quad \int^\omega \frac{1}{p(t)u(t)^2}\,dt < \infty. \tag{1.13}$$

Then there exist continuous functions $P(t)$ *and* $Q(t)$ *satisfying* (1.3) *with* (1.8) *such that* (1.2) *has a positive solution on* (α,ω).

Remark 2. Some concrete examples of Theorem 5 (ii) were constructed by [1].

Theorem 1 is proved by employing Piconne's identity [6] together with some properties of principal solutions. We prove Theorem 3 by combining comparison results for the half-open intervals $(\alpha, a]$ and $[a,\omega)$. Making use of two principal solutions at $t=\alpha$ and $t=\omega$, we obtain Theorems 4 and 5.

2. Proofs of Theorems

To prove Theorem 1, we need the following lemmas.

Lemma 1. *Assume that* $q(t) \le 0$ *on* $[a,\omega)$ *in* (1.1). *Then* (1.1) *is nonoscillatory at* $t=\omega$ *and a principal solution* $u_0(t)$ *of* (1.1) *satisfies* $u_0(t) > 0$ *and* $u_0'(t) \le 0$ *on* $[a,\omega)$.

Lemma 2. *Assume that* (1.1) *is nonoscillatory at* $t=\omega$. *Let* $u_0(t)$ *be a principal solution of* (1.1), *and let* $v(t)$ *be a solution of* (1.2) *satisfying* $v(t) > 0$ *on* $[T,\omega)$ *with some* $T \ge a$. *Then* $u_0(t) > 0$ *on* $[T,\omega)$ *and*

$$\frac{p(t)u_0'(t)}{u_0(t)} \le \frac{P(t)v'(t)}{v(t)} \quad \text{on } [T,\omega).$$

Lemmas 1 and 2 are shown in [5, Ch. XI, Corollaries 6.4 and 6.5]. However, for reader's convenience, we give slightly simpler proofs of them.

Proof of Lemma 1. Let $u_i(t)$, $i = 1, 2$, be solutions of (1.1) determined by $u_i(a) = 1$ and $u_i'(a) = i$. It is easy to see that $(p(t)u_i'(t))' \geq 0$ and $u_i(t) > 0$ on $[a, \omega)$, $i = 1, 2$. Since $u_1(t)$ and $u_2(t)$ are linearly independent, either $u_1(t)$ or $u_2(t)$ is a nonprincipal solution. Without loss of generality, we may assume that $u_1(t)$ is a nonprincipal solution. By [5, Ch. XI, Corollary 6.3],

$$u_0(t) = u_1(t) \int_t^\infty \frac{ds}{p(s)u_1(s)^2} \quad \text{for } a \leq t < \omega,$$

is well defined and a principal solution of (1.1). Then we have $u_0(t) > 0$ on $[a, \omega)$. We obtain

$$p(t)u_0'(t) = p(t)u_1'(t) \int_t^\infty \frac{ds}{p(s)u_1(s)^2} - \frac{1}{u_1(t)} \quad \text{for } a \leq t < \omega.$$

Since $p(t)u_1'(t)$ is nondecreasing, we have

$$p(t)u_0'(t) \leq \int_t^\infty \frac{u_1'(s)}{u_1(s)^2}\, ds - \frac{1}{u_1(t)} \quad \text{for } a \leq t < \omega. \tag{2.1}$$

Note here that

$$\int_t^\infty \frac{u_1'(s)}{u_1(s)^2}\, ds - \frac{1}{u_1(t)} =$$

$$= \lim_{\tau\to\infty} \left(\int_t^\tau \frac{u_1'(s)}{u_1(s)^2}\, ds - \frac{1}{u_1(t)} \right) = \lim_{\tau\to\infty} \left(- \frac{1}{u_1(\tau)} \right) \leq 0.$$

Thus, from (2.1), we obtain $u_0'(t) \leq 0$ on $[a, \omega)$. □

Proof of Lemma 2. Let

$$w(t) = \exp\left(\int_T^t \frac{P(s)v'(s)}{p(s)v(s)}\, ds \right) \quad \text{for } T \leq t < \omega.$$

Then $w(t) > 0$ on $[T, \omega)$ and satisfies

$$p(t)w'(t) = \frac{P(t)v'(t)w(t)}{v(t)} \quad \text{for } T \leq t < \omega. \tag{2.2}$$

It follows that

$$(p(t)w')' = (P(t)v')' \frac{w}{v} + P(t)v' \left(\frac{w}{v}\right)'.$$

From (2.2) we note that

$$\left(\frac{w}{v}\right)' = \frac{vw' - v'w}{v^2} = \frac{w'}{v} - \frac{v'w}{v^2} = \left(\frac{1}{p(t)} - \frac{1}{P(t)}\right) \frac{P(t)v'w}{v^2}\,.$$

Thus, w satisfies

$$(p(t)w')' + Q_0(t)w = 0 \ \text{ for } \ T \le t < \omega,$$

where

$$Q_0(t) = Q(t) + \Big(\frac{1}{P(t)} - \frac{1}{p(t)}\Big)\Big(\frac{P(t)v'(t)}{v(t)}\Big)^2 \ \text{ for } \ T \le t < \omega.$$

Let

$$z(t) = \frac{u_0(t)}{w(t)} \ \text{ on } [T, \omega).$$

Since $z(t)$ satisfies

$$p(t)w(t)^2 z'(t) = p(t)u_0'(t)w(t) - p(t)u_0(t)w'(t),$$

we see that

$$(p(t)w(t)^2 z')' + w(t)^2(q(t) - Q_0(t))z = 0 \ \text{ for } \ T \le t < \omega. \tag{2.3}$$

Since $u_0(t)$ is a principal solution, by [5, Ch. XI, Lemma 2.1], we have

$$\int^{\infty} \frac{ds}{p(s)w(s)^2 z(s)^2} = \int^{\infty} \frac{ds}{p(s)u_0(s)^2} = \infty.$$

Thus $z(t)$ is a principal solution of (2.3). Note here that $Q_0(t) \ge Q(t) \ge q(t)$ on $[T, \omega)$. Then, by Lemma 1, we have $z(t) > 0$ and $z'(t) \le 0$ on $[T, \omega)$, which implies $u_0(t) > 0$ on $[T, \omega)$. Then it follows that

$$\frac{u_0'(t)}{u_0(t)} = \frac{w'(t)}{w(t)} + \frac{z'(t)}{z(t)} \le \frac{w'(t)}{w(t)} \ \text{ for } \ T \le t < \omega.$$

From (2.2) we conclude that

$$\frac{p(t)u_0'(t)}{u_0(t)} \le \frac{p(t)w'(t)}{w(t)} = \frac{P(t)v'(t)}{v(t)} \ \text{ for } \ T \le t < \omega. \qquad \square$$

Proof of Theorem 1. Assume that $v(t) > 0$ on (a, ω). By Picone's identity [6], we have

$$\frac{d}{dt}\Big(\frac{u_0}{v}\,(pu_0'v - Pu_0v')\Big) = (Q-q)u_0^2 + (p-P){u_0'}^2 + \frac{P(u_0'v - u_0v')^2}{v^2}. \tag{2.4}$$

Note that if $u_0(a) = v(a) = 0$, we obtain $\lim_{t\to a} u_0(t)^2/v(t) = 0$ by l'Hospital's rule. Then we have, if $u_0(a) = 0$,

$$\lim_{t\to a} \frac{u_0(t)}{v(t)}\big(p(t)u_0'(t)v(t) - P(t)u_0(t)v'(t)\big) = 0.$$

If (1.6) holds, then

$$\lim_{t\to a} \frac{u_0(t)}{v(t)}\big(p(t)u_0'(t)v(t) - P(t)u_0(t)v(t)'\big) =$$

$$= u_0(a)^2\Big(\frac{p(a)u_0'(a)}{u_0(a)} - \frac{P(a)v'(a)}{v(a)}\Big) \ge 0.$$

Therefore, integrating (2.4) over $[\tau, t]$ and letting $\tau \to a$, we have

$$u_0(t)^2\Big(\frac{p(t)u_0'(t)}{u_0(t)} - \frac{P(t)v'(t)}{v(t)}\Big) \geq$$
$$\geq \int_a^t \Big((Q-q)u_0^2 + (p-P){u_0'}^2 + \frac{P(u_0'v - u_0v')^2}{v^2}\Big)\, ds$$

for $a < t < \omega$. From Lemma 2, we have

$$\int_a^t \Big((Q-q)u_0^2 + (p-P){u_0'}^2 + \frac{P(u_0'v - u_0v')^2}{v^2}\Big)\, ds \leq 0 \ \text{ for } \ a < t < \omega,$$

which implies that

$$q(t) \equiv Q(t), \quad p(t) \equiv P(t), \ \text{ and } \ u_0(t)v'(t) \equiv u_0'(t)v(t) \ \text{ on } [a, \omega).$$

Hence, $v(t)$ is a constant multiple of $u_0(t)$ on $[a, \omega)$. This completes the proof of Theorem 1. □

Proof of Theorem 2. Let $t = t_1 < t_2 < \cdots < t_n$ be zeros of $u_0(t)$ in (a, ω). We note that $v(t)$ satisfies either (i) or (ii) in Theorem A on $[a, t_n]$.

By applying Theorem 1 on $[t_n, \omega)$, we find that either $v(t)$ has at least one zero in (t_n, ω) or $v(t)$ is a multiple constant of $u_0(t)$ on $[t_n, \omega)$ and $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ on $[t_n, \omega)$. In the former case, $v(t)$ has at least $n+1$ zeros in (a, ω). In the latter case, since $v(t_n) = 0$, we have either $v(t)$ has at least $n+1$ zeros in (a, ω) or $v(t)$ is a multiple constant of $u_0(t)$, $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ on $[a, \omega)$. This completes the proof of Theorem 2. □

In order to prove Theorem 3, we consider (1.1) and (1.2) on the half-open interval of the form $(\alpha, a]$ with $\alpha \geq -\infty$.

Lemma 3. *Assume that* (1.1) *is nonoscillatory at* $t = \alpha$. *Let* $u_0(t)$ *be a principal solution of* (1.1) *at* $t = \alpha$, *and let* $v(t)$ *be a solution of* (1.2) *on* $(\alpha, a]$. *Assume that* $u_0(t) > 0$ *on* (α, a). *If either* $u_0(a) = 0$ *or*

$$u_0(a) \neq 0, \ v(a) \neq 0, \ \text{ and } \ \frac{p(a)u_0'(a)}{u_0(a)} \leq \frac{P(a)v'(a)}{v(a)}\,,$$

then $v(t)$ *has one of the following properties:*

(i) $v(t)$ *has at least one zero in* (α, a)*;*

(ii) $v(t)$ *is a constant multiple of* $u_0(t)$ *on* $(\alpha, a]$, *and* $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ *on* $(\alpha, a]$.

Proof. Put

$$\widetilde{u}_0(t) = u_0(a-t) \ \text{ and } \ \widetilde{v}(t) = v(a-t).$$

Then $\widetilde{u}_0$ and $\widetilde{v}$ satisfy, respectively,

$$\big(\widetilde{p}(t)\widetilde{u}_0{}'\big)' + \widetilde{q}(t)\widetilde{u}_0 = 0 \ \text{ and } \ \big(\widetilde{P}(t)\widetilde{v}\,'\big)' + \widetilde{Q}(t)\widetilde{v} = 0 \ \text{ on } \ [0, \widetilde{\omega}),$$

where

$$\widetilde{p}(t) = p(a-t), \;\; \widetilde{q}(t) = q(a-t), \;\; \widetilde{P}(t) = P(a-t),$$
$$\widetilde{Q}(t) = Q(a-t), \;\text{ and }\; \widetilde{\omega} = a - \alpha.$$

Furthermore, we have

$$\int^{\widetilde{\omega}} \frac{1}{\widetilde{p}(t)\widetilde{u}_0(t)^2}\,dt = \infty, \quad \frac{\widetilde{p}(0)\widetilde{u}_0{}'(0)}{\widetilde{u}_0(0)} = -\frac{p(a)u_0'(a)}{u_0(a)}, \;\text{ and}$$
$$\frac{\widetilde{P}(0)\widetilde{v}\,'(0)}{\widetilde{v}(0)} = -\frac{P(a)v'(a)}{v(a)}.$$

By applying Theorem 1 to $\widetilde{u}_0$ and $\widetilde{v}$ on $[0, \widetilde{\omega})$, we obtain Lemma 3. □

Proof of Theorem 3. First we consider the case $n = 1$. We may assume that $u_0(t) > 0$ on (α, ω). We show that $v(t)$ is a constant multiple of $u_0(t)$ on (α, ω), if $v(t) > 0$ on (α, ω). Assume that $v(t) > 0$ on (α, ω). Take any $t_0 \in (\alpha, \omega)$. First we will verify that

$$\frac{p(t_0)u_0'(t_0)}{u_0(t_0)} = \frac{P(t_0)v'(t_0)}{v(t_0)}. \tag{2.5}$$

Assume to the contrary that (2.5) does not hold. If

$$\frac{p(t_0)u_0'(t_0)}{u_0(t_0)} > \frac{P(t_0)v'(t_0)}{v(t_0)}, \tag{2.6}$$

then $v(t)$ has at least one zero in (t_0, ω) by applying Theorem 1 with $a = t_0$. This is a contradiction. On the other hand, if the opposite inequality holds in (2.6), then $v(t)$ has at least one zero in (α, t_0) by Lemma 3. This is a contradiction. Thus we obtain (2.5).

By applying Theorem 1 and Lemma 3 with $a = t_0$ again, we conclude that $v(t)$ is a constant multiple of $u_0(t)$ on (α, ω), and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on (α, ω).

Next, we consider the case $n \geq 2$. Let $t = t_1 < t_2 < \cdots < t_{n-1}$ be zeros of $u_0(t)$ in (α, ω). By applying Theorem 2 with $a = t_1$, we have either $v(t)$ has at least $n-1$ zeros in (t_1, ω) or $v(t)$ is a multiple constant of $u_0(t)$ on $[t_1, \omega)$. Thus, $v(t)$ has at least $n-1$ zeros in (α, ω). Therefore, it suffices to show that if $v(t)$ has exactly $n-1$ zeros, then $v(t)$ is a multiple constant of $u_0(t)$ on (α, ω). Assume that $v(t)$ has exactly $n-1$ zeros. First we verify that $v(t_1) = 0$. (Recall that $t = t_1$ is the first zero of $u_0(t)$.) Assume to the contrary that $v(t_1) \neq 0$. By applying Theorem 2 and Lemma 3 with $a = t_1$, we see that $v(t)$ has at least $n-1$ zeros in (t_1, ω) and at least one zero in (α, t_1), respectively. Thus, $v(t)$ has at least n zeros in (α, ω). This is a contradiction. Thus we obtain $v(t_1) = 0$.

By applying Theorem 2 and Lemma 3 with $a = t_1$ again, we conclude that $v(t)$ is a constant multiple of $u_0(t)$ on (α, ω), and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on (α, ω). □

For the proof of Theorem 4, we need the following

Lemma 4. *Assume that* (1.1) *has a solution* $u(t)$ *which has exactly* $n-1$ *zeros in* (α, ω)*. Let* $u_0(t)$ *and* $\widetilde{u}_0(t)$ *be principal solutions of* (1.1) *at* $t = \omega$ *and* $t = \alpha$*, respectively. Then* $u_0(t)$ *and* $\widetilde{u}_0(t)$ *have at most* $n-1$ *zeros in* (α, ω)*.*

Proof. First we consider the case where $n = 1$, that is, $u(t)$ has no zero in (α, ω). Assume to the contrary that $u_0(t)$ has at least one zero in (α, ω). Let $t_0 \in (\alpha, \omega)$ be the largest zero of $u_0(t)$. We may assume that $u_0(t) > 0$ on (t_0, ω). By applying Theorem 1 with $a = t_0$, $p(t) \equiv P(t)$, and $q(t) \equiv Q(t)$, we see that $u(t)$ has at least one zero in $[t_0, \omega)$. This is a contradiction. Thus u_0 has no zero on (α, ω). By the similar argument as above, we see that $\widetilde{u}_0$ has no zero on (α, ω). Next, we consider the case where $n \geq 2$, that is $u(t)$ has exactly $n-1$ zeros in (α, ω). Assume to the contrary that $u_0(t)$ has at least n zeros in (α, ω). Let t_{n-1} be the $(n-1)$-th zero of $u(t)$. Note here that zeros of $u(t)$ and $u_0(t)$ do not coincide, since $u(t)$ and $u_0(t)$ are linearly independent. By the Sturm separation theorem, $u_0(t)$ has a zero $t_0 \in (t_{n-1}, \omega)$. By applying Theorem 1 with $a = t_0$, $p(t) \equiv P(t)$, and $q(t) \equiv Q(t)$, we see that $u(t)$ has at least one zero in (t_{n-1}, ω). This is a contradiction. Thus u_0 has at most $n-1$ zeros in (α, ω). By the similar argument as above, we see that $\widetilde{u}_0$ has at most $n-1$ zeros in (α, ω). $\square$

Proof of Theorem 4. Let u_0 and $\widetilde{u}_0$ be principal solutions of (1.1) at $t = \omega$ and $t = \alpha$, respectively. We show that the solution u is a multiple constant of u_0 on (α, ω), and also of $\widetilde{u}_0$ on (α, ω). Assume to the contrary that $u(t)$ and $u_0(t)$ are linearly independent. Then $u_0(t)$ has n zeros in (α, ω). This contradicts Lemma 4. Thus u is a multiple constant of u_0 on (α, ω). Similarly, we see that u is a multiple constant of $\widetilde{u}_0$ on (α, ω). Thus, the solution u is principal at both points $t = \alpha$ and $t = \omega$, and hence (1.7) holds with $u_0 = u$. $\square$

To prove Theorem 5, we have the following

Lemma 5. *Assume that there exists a positive solution* $v(t)$ *of* (1.2) *on* (α, ω)*.* (*Then* (1.1) *is nonoscillatory at* $t = \alpha$ *and* $t = \omega$*.*) *Let* $u_0(t)$ *and* $\widetilde{u}_0(t)$ *be principal solutions of* (1.1) *at* $t = \omega$ *and* $t = \alpha$*, respectively. Then* $u_0(t)$ *and* $\widetilde{u}_0(t)$ *have no zero on* (α, ω)*. Furthermore, if* $p(t) \not\equiv P(t)$ *or* $q(t) \not\equiv Q(t)$ *on* (α, ω)*, then* $u_0(t)$ *and* $\widetilde{u}_0(t)$ *are linearly independent on* (α, ω)*.*

Proof. Assume to the contrary that $u_0(t)$ has at least one zero in (α, ω). Let $t_0 \in (\alpha, \omega)$ be the largest zero of $u_0(t)$. We may assume that $u_0(t) > 0$ on (t_0, ω). By applying Theorem 1 with $a = t_0$, we find that any solution of (1.2) has at least one zero in $[t_0, \omega)$. This is a contradiction. Thus u_0 has no zero on (α, ω). By the similar argument, we see that $\widetilde{u}_0$ has no zero on (α, ω).

Assume that $p(t) \not\equiv P(t)$ or $q(t) \not\equiv Q(t)$ on (α, ω). In this case, we will show that $u_0(t)$ and $\widetilde{u}_0(t)$ are linearly independent on (α, ω). Assume to the contrary that $u_0(t)$ is a constant multiple of $\widetilde{u}_0(t)$ on (α, ω). Then $u_0(t)$ is also principal at $t = \alpha$, and hence (1.7) holds. Theorem 3 implies that $v(t)$ is a constant multiple of $u_0(t)$ on (α, ω), and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on (α, ω). This is a contradiction. Thus $u_0(t)$ and $\widetilde{u}_0(t)$ are linearly independent on (α, ω). □

Proof of Theorem 5. (i) Let u_0 and $\widetilde{u}_0$ be principal solutions of (1.1) at $t = \omega$ and $t = \alpha$, respectively. Lemma 5 implies that $u_0(t) > 0$ and $\widetilde{u}_0(t) > 0$ on (α, ω), and that $u_0(t)$ and $\widetilde{u}_0(t)$ are linear independent on (α, ω). Since a principal solution at $t = \alpha$ ($t = \omega$) is unique up to a constant factor, $u_0(t)$ and $\widetilde{u}_0(t)$ are nonprincipal at $t = \alpha$ and $t = \omega$, respectively. Thus we obtain (1.11) and (1.12). Put $u(t) = u_0(t) + \widetilde{u}_0(t)$. Then u is a positive solution of (1.1) on (α, ω), and nonprincipal at both points $t = \alpha$ and $t = \omega$. Thus (1.10) holds.

(ii) Let u_0 and $\widetilde{u}_0$ be principal solutions of (1.1) at $t = \omega$ and $t = \alpha$, respectively. Applying Lemma 5 with $P(t) \equiv p(t)$ and $Q(t) \equiv q(t)$ on (α, ω), we have $u_0(t) > 0$ and $\widetilde{u}_0(t) > 0$ on (α, ω). We show that $u_0(t)$ and $\widetilde{u}_0(t)$ are linearly independent on (α, ω). Assume to the contrary that $u_0(t)$ is a constant multiple of $\widetilde{u}_0(t)$ on (α, ω). Then $u_0(t)$ is also principal at $t = \alpha$, and hence (1.7) holds. Corollary 1 with $n = 1$ implies that any positive solution of (1.1) is a constant multiple of $u_0(t)$ on (α, ω). Since (1.1) has a positive solution u satisfying (1.13), this is a contradiction. Thus $u_0(t)$ and $\widetilde{u}_0(t)$ are linearly independent on (α, ω).

We note here that for any $t \in (\alpha, \omega)$,

$$\frac{p(t)u_0'(t)}{u_0(t)} < \frac{p(t)\widetilde{u}_0{}'(t)}{\widetilde{u}_0(t)}\,. \tag{2.7}$$

In fact, if (2.7) does not hold for some $t = t_0 \in (\alpha, \omega)$, then $\widetilde{u}_0$ has at least one zero in (t_0, ω) by Theorem 1. This is a contradiction. Thus (2.7) holds for any $t \in (\alpha, \omega)$.

For $\lambda \geq 0$, define $P_\lambda(t)$ and $Q_\lambda(t)$ by

$$P_\lambda(t) = \frac{p(t)}{1 + \lambda r(t)} \;\text{ and }\; Q_\lambda(t) = q(t) + \lambda r(t) \;\text{ on }\; (\alpha, \omega),$$

where $r(t)$ is a continuous function on (α, ω) satisfying $r(t) \geq 0$, $r(t) \not\equiv 0$ on (α, ω), and $r(t) \equiv 0$ on $(\alpha, t_1] \cup [t_2, \omega)$ with some $t_1 < t_2$. Let us consider the differential equation

$$(P_\lambda(t)v')' + Q_\lambda(t)v = 0 \;\text{ on }\; (\alpha, \omega). \tag{2.8}$$

Note that

$$P_\lambda(t) \equiv p(t) \;\text{ and }\; Q_\lambda(t) \equiv q(t) \;\text{ on } (\alpha, t_1] \cup [t_2, \omega) \text{ for all } \lambda \geq 0.$$

Then the solutions $u_0(t)$ and $\widetilde{u}_0(t)$ solve (2.8) on each interval $(\alpha, t_1]$ and $[t_2, \omega)$. For $\lambda \geq 0$, define $u_0(t;\lambda)$ and $\widetilde{u}_0(t;\lambda)$ by solutions of (2.8) satisfying

$$u_0(t;\lambda) \equiv u_0(t) \ \text{ on } \ [t_2,\omega) \ \text{ and } \ \widetilde{u}_0(t;\lambda) \equiv \widetilde{u}_0(t) \ \text{ on } \ (\alpha, t_1],$$

respectively. Then $u_0(t;\lambda)$ and $u_0'(t;\lambda)$ depend continuously on $\lambda \geq 0$ uniformly on any compact subinterval of (α,ω). In, particularly, $u_0(t;\lambda) \to u_0(t)$ and $\widetilde{u}_0(t;\lambda) \to \widetilde{u}_0(t)$ as $\lambda \to 0$ uniformly on $[t_1, t_2]$. Since (2.7) holds with $t = t_1$, for $\lambda > 0$ sufficiently small, we have

$$\frac{p(t_1)u_0'(t_1;\lambda)}{u_0(t_1;\lambda)} < \frac{p(t_1)\widetilde{u}_0{}'(t_1;\lambda)}{\widetilde{u}_0(t_1;\lambda)} \ \text{ and } \ u_0(t;\lambda) > 0 \ \text{ on } \ [t_1,\omega). \tag{2.9}$$

For $\lambda > 0$ satisfying (2.9), we will show that $\widetilde{u}_0(t;\lambda) > 0$ on (t_1,ω). Assume to the contrary that $\widetilde{u}_0(t;\lambda)$ has at least one zero $t_0 \in (t_1,\omega)$. Applying Theorem A with $a = t_0$, $u(t) \equiv \widetilde{u}_0(t;\lambda)$ and $v(t) \equiv u_0(t;\lambda)$, we see that $u_0(t;\lambda)$ has at least one zero in (t_1,ω). This is a contradiction. Thus $\widetilde{u}_0(t;\lambda) > 0$ on (t_1,ω), and hence $\widetilde{u}_0(t;\lambda) > 0$ on (α,ω). Then (1.2) with $P(t) \equiv P_\lambda(t)$ and $Q(t) \equiv Q_\lambda(t)$ has a positive solution on (α,ω). □

References

1. D. Aharonov and U. Elias, Singular Sturm comparison theorems. *J. Math. Anal. Appl.* **371** (2010), No. 2, 759–763.
2. M. Chuaqui, P. Duren, B. Osgood, and D. Stowe, Oscillation of solutions of linear differential equations. *Bull. Aust. Math. Soc.* **79** (2009), No. 1, 161–169.
3. Á. Elbert and T. Kusano, Principal solutions of non-oscillatory half-linear differential equations. *Adv. Math. Sci. Appl.* **8** (1998), No. 2, 745–759.
4. Á. Elbert, T. Kusano, and M. Naito, On the number of zeros of nonoscillatory solutions to second order half-linear differential equations. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **42** (1999), 101–131 (2000).
5. P. Hartman, Ordinary differential equations. *John Wiley & Sons, Inc., New York–London–Sydney*, 1964.
6. E. L. Ince, Ordinary differential equations. *Longmans, Green & Co, London*, 1927.
7. T. Kusano and M. Naito, A singular eigenvalue problem for second order linear ordinary differential equations. *International Symposium on Differential Equations and Mathematical Physics (Tbilisi, 1997). Mem. Differential Equations Math. Phys.* **12** (1997), 122–130.
8. T. Kusano and M. Naito, A singular eigenvalue problem for the Sturm-Liouville equation. (Russian) *Differ. Uravn.* **34** (1998), No. 3, 303–312, 428; English transl.: *Differential Equations* **34** (1998), No. 3, 302–311.
9. T. Kusano and M. Naito, On the number of zeros of nonoscillatory solutions to half-linear ordinary differential equations involving a parameter. *Trans. Amer. Math. Soc.* **354** (2002), No. 12, 4751–4767 (electronic).
10. Y. Naito, Classification of second order linear differential equations and an application to singular elliptic eigenvalue problems. *Bull. Lond. Math. Soc.* (to appear).
11. Z. Nehari, The Schwarzian derivative and schlicht functions. *Bull. Amer. Math. Soc.* **55** (1949), 545–551.
12. Z. Nehari, Some criteria of univalence. *Proc. Amer. Math. Soc.* **5** (1954), 700–704.
13. W. T. Reid, Sturmian theory for ordinary differential equations. With a preface by John Burns. *Applied Mathematical Sciences*, 31. *Springer-Verlag, New York–Berlin*, 1980.

14. C. A. Swanson, Comparison and oscillation theory of linear differential equations. *Mathematics in Science and Engineering*, Vol. 48. *Academic Press, New York–London*, 1968.

(Received 22.05.2012)

Author's address:

Department of Mathematics, Ehime University, Matsuyama 790-8577, Japan.

E-mail: ynaito@ehime-u.ac.jp

Memoirs on Differential Equations and Mathematical Physics
Volume 57, 2012, 123–162

Tomoyuki Tanigawa

GENERALIZED REGULARLY VARYING SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

Dedicated to 80th birthday anniversary of Professor Kusano Takaŝi

Abstract. The sharp sufficient conditions of the existence of generalized regularly varying solutions (in the sense of Karamata) of differential equations of the type

$$\big(p(t)\varphi(x'(t))\big)' \pm \sum_{i=1}^{n} \Big[q_i(t)\varphi\big(x(g_i(t))\big) + r_i(t)\varphi\big(x(h_i(t))\big)\Big] = 0$$

are established. Here, $p, q_i, r_i : [a, \infty) \to (0, \infty)$ are continuous functions, $g_i, h_i : [a, +\infty) \to R$ are continuous and increasing functions such that $g_i(t) < t$, $h_i(t) > t$ for $t \geq a$, $\lim\limits_{t\to\infty} g_i(t) = \infty$ and $\varphi(\xi) \equiv |\xi|^\alpha \operatorname{sgn} \xi$, $\alpha > 0$.

2010 Mathematics Subject Classification. 34C11, 26A12.

Key words and phrases. Second order nonlinear differential equation with deviating arguments, generalized regularly varying solution, asymptotic behavior of a solution.

რეზიუმე. დადგენილია კარამატას განზოგადოებული აზრით რეგულარულად ცვალებადი ამონახსნების არსებობის არაგაუმჯობესებადი საკმარისი პირობები

$$\big(p(t)\varphi(x'(t))\big)' \pm \sum_{i=1}^{n} \Big[q_i(t)\varphi\big(x(g_i(t))\big) + r_i(t)\varphi\big(x(h_i(t))\big)\Big] = 0$$

სახის დიფერენციალური განტოლებებისათვის, სადაც $p, q_i, r_i : [a, \infty) \to (0, \infty)$ უწყვეტი ფუნქციებია, $g_i, h_i : [a, +\infty) \to R$ ისეთი უწყვეტი არაზრდადი ფუნქციებია, რომ $g_i(t) < t$, $h_i(t) > t$, როცა $t \geq a$, $\lim\limits_{t\to\infty} g_i(t) = \infty$, ხოლო $\varphi(\xi) \equiv |\xi|^\alpha \operatorname{sgn} \xi$, $\alpha > 0$.

1. Introduction

The equation to be studied in this paper is

$$\big(p(t)\varphi(x'(t))\big)' \pm \sum_{i=1}^{n}\Big[q_i(t)\varphi\big(x(g_i(t))\big) + r_i(t)\varphi\big(x(h_i(t))\big)\Big] = 0 \qquad (\mathrm{A}_\pm)$$

$$\big(\,\varphi(\xi) = |\xi|^\alpha \operatorname{sgn}\xi, \ \ \alpha > 0, \ \ \xi \in \mathbb{R}\,\big),$$

where $p, q_i, r_i : [a,\infty) \to (0,\infty)$ are continuous functions, g_i, h_i are continuous and increasing functions with $g_i(t) < t$, $h_i(t) > t$ and $\lim\limits_{t\to\infty} g_i(t) = \infty$ for $i = 1, 2, \dots, n$. In what follows we always assume that the function $p(t)$ satisfies

$$\int_a^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} = \infty. \tag{1.1}$$

It is shown in the monograph [8] that the class of regularly varying functions in the sense of Karamata is a well-suited framework for the asymptotic analysis of nonoscillatory solutions of the second order linear differential equation of the form

$$x''(t) = q(t)x(t), \ \ q(t) > 0.$$

The study of asymptotic analysis of nonoscillatory solutions of functional differential equations with deviating arguments in the framework of regularly varying functions (called Karamata functions) was first attempted by Kusano and Marić [5], [6]. They established a sharp condition for the existence of a slowly varying solution of the second order functional differential equation with retarded argument of the form

$$x''(t) = q(t)x(g(t)), \tag{1.2}$$

and the following functional differential equation of the form

$$x''(t) \pm \big[q(t)x(g(t)) + r(t)x(h(t))\big] = 0, \tag{1.3}$$

where $q, r : [a,\infty) \to (0,\infty)$ are continuous functions, g, h are continuous and increasing with $g(t) < t$, $h(t) > t$ for $t \geqq a$, $\lim\limits_{t\to\infty} g(t) = \infty$.

It is well known that there is the qualitative similarity between linear differential equations and half-linear differential equations (see the book Došlý and Řehák [2]). Therefore, in our previous papers [4], [7] we proved how useful the regularly varying functions were for the study of nonoscillation and asymptotic analysis of the half-linear differential equation involving nonlinear Sturm–Liouville type differential operator of the form

$$\big(p(t)\varphi(x'(t))\big)' \pm f(t)\varphi(x(t)) = 0, \ \ p(t) > 0, \qquad (\mathrm{B}_\pm)$$

and the half-linear functional differential equation with both retarded and advanced arguments of the form

$$\big(\varphi(x'(t))\big)' \pm \Big[q(t)\varphi\big(x(g(t))\big) + r(t)\varphi\big(x(h(t))\big)\Big] = 0, \tag{1.4}$$

where $f : [a,\infty) \to (0,\infty)$ is a continuous function, p, g, h are just as in the above equations.

Theorem A (J. Jaroš, T. Kusano and T. Tanigawa [4]). *Suppose that* (1.1) *holds. The equations* ($B_\pm$) *have a normalized slowly varying solution with respect to* $P(t)$ *and a normalized regularly varying solution of index* 1 *with respect to* $P(t)$ *if and only if*

$$\lim_{t\to\infty} P(t)^{\alpha} \int_t^{\infty} f(s)\,ds = 0, \tag{1.5}$$

where the function $P(t)$ *is defined by*

$$P(t) = \int_a^t \frac{ds}{p(s)^{\frac{1}{\alpha}}}\,. \tag{1.6}$$

Theorem B (J. Manojlović and T. Tanigawa [7]). *Suppose that*

$$\lim_{t\to\infty} \frac{g(t)}{t} = 1 \ \text{ and } \ \lim_{t\to\infty} \frac{h(t)}{t} = 1$$

hold. Then the equations (1.4) *have a slowly varying solution and a regularly varying solution of index* 1 *if and only if*

$$\lim_{t\to\infty} t^{\alpha} \int_t^{\infty} q(s)\,ds = \lim_{t\to\infty} t^{\alpha} \int_t^{\infty} r(s)\,ds = 0.$$

The objective of this paper is to establish a sharp condition of the existence of a normalized slowly varying solution with respect to $P(t)$ and a normalized regularly varying solution of index 1 with respect to $P(t)$ of the equation ($A_\pm$). Our main result is the following

Theorem 1.1. *Suppose that*

$$\lim_{t\to\infty} \frac{P(g_i(t))}{P(t)} = 1 \ \text{ for } \ i = 1,2,\dots,n \tag{1.7}$$

and

$$\lim_{t\to\infty} \frac{P(h_i(t))}{P(t)} = 1 \ \text{ for } \ i = 1,2,\dots,n \tag{1.8}$$

hold. The equation ($A_\pm$) *possesses a normalized slowly varying solution with respect to* $P(t)$ *and a normalized regularly varying solution of index* 1 *with respect to* $P(t)$ *if and only if*

$$\lim_{t\to\infty} P(t)^{\alpha} \int_t^{\infty} q_i(s)\,ds = \lim_{t\to\infty} P(t)^{\alpha} \int_t^{\infty} r_i(s)\,ds = 0 \ \text{ for } \ i = 1,2,\dots,n. \tag{1.9}$$

This paper is organized as follows. In Section 2 we briefly recall the definitions and properties of the slowly varying and regularly varying functions of index ρ with respect to $P(t)$ which are called the generalized regularly varying functions introduced by Jaroš and Kusano [3]. Explicit expressions for the normalized slowly varying solution with respect to $P(t)$ and the normalized regularly varying solution of index 1 with respect to $P(t)$ of the equations ($\mathrm{B}_\pm$) obtained in [4] do not meet our need for application to the functional differential equations ($\mathrm{A}_\pm$), and thus we present a modified proof of Theorem A in Section 3. The proof of Theorem 1.1 which is based on Theorems A and B will be presented in Section 4. Some examples illustrating our result will also be presented in Section 5.

2. Definitions and Properties of the Generalized Regularly Varying Functions

For the reader's convenience we first state the definitions and some basic properties of the regularly varying functions and then refer to the generalized regularly varying functions. The generalized regularly varying functions are introduced for the first time by Jaroš and Kusano [3] in order to gain useful information about an asymptotic behavior of nonoscillatory solutions for the self-adjoint differential equations of the form

$$\big(p(t)x'(t)\big)' + f(t)x(t) = 0.$$

The definitions and properties of regularly varying functions:

Definition 2.1. A measurable function $f : [a,\infty) \to (0,\infty)$ is said to be a regularly varying of index ρ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \ \text{ for any } \ \lambda > 0, \ \ \rho \in \mathbb{R}.$$

Proposition 2.1 (Representation Theorem)**.** *A measurable function f : $[a,\infty) \to (0,\infty)$ is regularly varying of index ρ if and only if it can be written in the form*

$$f(t) = c(t) \exp\bigg\{ \int_{t_0}^{t} \frac{\delta(s)}{s}\, ds \bigg\}, \ \ t \geqq t_0,$$

for some $t_0 > a$, where $c(t)$ and $\delta(t)$ are measurable functions such that

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \ \ \textit{and} \ \ \lim_{t\to\infty} \delta(t) = \rho.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. The symbol SV is used to denote $\mathrm{RV}(0)$ and a member of $\mathrm{SV} = \mathrm{RV}(0)$ is referred to as a slowly varying function. If $f(t) \in \mathrm{RV}(\rho)$, then $f(t) = t^\rho L(t)$ for some $L(t) \in \mathrm{SV}$. Therefore, the class of slowly varying functions is of fundamental importance in the theory of regular variation. In addition to the functions tending to positive constants as $t \to \infty$, the following functions

$$\prod_{i=1}^{N} (\log_i t)^{m_i} \ (m_i \in \mathbb{R}), \quad \exp\left\{ \prod_{i=1}^{N} (\log_i t)^{n_i} \right\} \ (0 < n_i < 1), \quad \exp\left\{ \frac{\log t}{\log_2 t} \right\},$$

where $\log_1 t = \log t$ and $\log_k t = \log \log_{k-1} t$ for $k = 2, 3, \dots, N$, also belong to the set of slowly varying functions.

Proposition 2.2. *Let $L(t)$ be any slowly varying function. Then, for any $\gamma > 0$,*

$$\lim_{t\to\infty} t^{\gamma} L(t) = \infty \quad \textit{and} \quad \lim_{t\to\infty} t^{-\gamma} L(t) = 0.$$

Proposition 2.3 (Karamata's integration theorem)**.** *Let $L(t) \in \mathrm{SV}$. Then*

(i) *if $\gamma > -1$,*

$$\int_a^t s^{\gamma} L(s)\, ds \ \sim \ \frac{t^{\gamma+1}}{\gamma+1} L(t), \quad \textit{as} \ \ t \to \infty;$$

(ii) *if $\gamma < -1$,*

$$\int_t^{\infty} s^{\gamma} L(s)\, ds \ \sim \ -\frac{t^{\gamma+1}}{\gamma+1} L(t), \quad \textit{as} \ \ t \to \infty.$$

Here and hereafter the notation $\varphi(t) \sim \psi(t)$ as $t \to \infty$ is used to mean the asymptotic equivalence of $\varphi(t)$ and $\psi(t)$: $\lim_{t\to\infty} \psi(t)/\varphi(t) = 1$.

For an excellent explanation of the theory of regularly varying functions the reader is referred to the book [1].

The definitions and properties of generalized regularly varying functions:

Definition 2.2. A measurable function $f : [a, \infty) \to (0, \infty)$ is said to be slowly varying with respect to $P(t)$ if the function $f \circ P(t)^{-1}$ is slowly varying in the sense of Karamata, where the function $P(t)$ is defined by (1.6) and $P(t)^{-1}$ denotes the inverse function of $P(t)$. The totality of slowly varying functions with respect to $P(t)$ is denoted by SV_P.

Definition 2.3. A measurable function $g : [a, \infty) \to (0, \infty)$ is said to be regularly varying function of index ρ with respect to $P(t)$ if the function $g \circ P(t)^{-1}$ is regularly varying of index ρ in the sense of Karamata. The set of all regularly varying functions of index ρ with respect to $P(t)$ is denoted by $\mathrm{RV}_P(\rho)$.

Of fundamental importance is the following representation theorem for the generalized slowly and regularly varying functions, which is an immediate consequence of Proposition 2.1.

Proposition 2.4.

(i) *A function $f(t)$ is slowly varying with respect to $P(t)$ if and only if it can be expressed in the form*

$$f(t) = c(t)\exp\left\{\int_{t_0}^{t} \frac{\delta(s)}{p(s)^{\frac{1}{\alpha}}P(s)}\,ds\right\}, \quad t \geqq t_0 \tag{2.1}$$

for some $t_0 > a$, where $c(t)$ and $\delta(t)$ are measurable functions such that

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \quad \textit{and} \quad \lim_{t\to\infty} \delta(t) = 0.$$

(ii) *A function $g(t)$ is regularly varying of index ρ with respect to $P(t)$ if and only if it has the representation*

$$g(t) = c(t)\exp\left\{\int_{t_0}^{t} \frac{\delta(s)}{p(s)^{\frac{1}{\alpha}}P(s)}\,ds\right\}, \quad t \geqq t_0 \tag{2.2}$$

for some $t_0 > a$, where $c(t)$ and $\delta(t)$ are measurable functions such that

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \quad \textit{and} \quad \lim_{t\to\infty} \delta(t) = \rho.$$

If the function $c(t)$ in (2.1) (or (2.2)) is identically a constant on $[t_0,\infty)$, then the function $f(t)$ (or $g(t)$) is called normalized slowly varying (or normalized regularly varying of index ρ) with respect to $P(t)$. The totality of such functions is denoted by n-SV_P (or n-RV_P).

It is easy to see that if $g(t) \in \mathrm{RV}_P(\rho)$ (n-$\mathrm{RV}_P(\rho)$), then $g(t) = P(t)^\rho f(t)$ for some $f(t) \in \mathrm{SV}_P$ (or n-SV_P).

Proposition 2.5. *Let $f(t) \in \mathrm{SV}_P$. Then, for any $\gamma > 0$,*

$$\lim_{t\to\infty} P(t)^\gamma f(t) = \infty \quad \textit{and} \quad \lim_{t\to\infty} P(t)^{-\gamma} f(t) = 0. \tag{2.3}$$

The Karamata's integration theorem is generalized in the following manner.

Proposition 2.6 (The generalized Karamata's integration theorem)**.** *Let $f(t) \in$ n-SV_P. Then*

(i) *If $\gamma > -1$,*

$$\int_{t_0}^{t} \frac{P(s)^\gamma}{p(s)^{\frac{1}{\alpha}}} f(s)\,ds \sim \frac{P(t)^{\gamma+1}}{\gamma+1} f(t) \quad \textit{as} \quad t\to\infty; \tag{2.4}$$

(ii) *If $\gamma < -1$, $\int_{t_0}^{\infty} P(t)^\gamma f(t)/p(t)^{\frac{1}{\alpha}}\,dt < \infty$ and*

$$\int_{t}^{\infty} \frac{P(s)^\gamma}{p(s)^{\frac{1}{\alpha}}} f(s)\,ds \sim -\frac{P(t)^{\gamma+1}}{\gamma+1} f(t) \quad \textit{as} \quad t\to\infty. \tag{2.5}$$

3. The Existence of Generalized Regularly Varying Solution of Self-Adjoint Differential Equation without Deviating Arguments

Theorem 3.1. *Put* $F(t) = P(t)^\alpha \int_t^\infty f(s)\,ds$, $\widehat{F}(t) = \sup_{s \geqq t} F(s)$,

$$F_+(t,w) = |1 + F(t) - w|^{1+\frac{1}{\alpha}} + \Big(1 + \frac{1}{\alpha}\Big)w - 1, \tag{3.1}$$

and

$$F_-(t,w) = 1 + \Big(1 + \frac{1}{\alpha}\Big)w - |1 + F(t) - w|^{1+\frac{1}{\alpha}}. \tag{3.2}$$

(i) *The equation* (B_+) *possesses a* n-SV_P *solution* $x(t)$ *having the expression*

$$x(t) = \exp\bigg\{\int_{t_0}^{t} \Big(\frac{v(s) + F(s)}{p(s)P(s)^\alpha}\Big)^{\frac{1}{\alpha}}\,ds\bigg\}, \quad t \geqq t_0 \tag{3.3}$$

for some $t_0 > a$, *in which* $v(t)$ *satisfies*

$$v(t) = \alpha P(t)^\alpha \int_t^\infty \frac{(v(s) + F(s))^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}}\,ds, \quad t \geqq t_0 \tag{3.4}$$

and

$$0 \leqq v(t) \leqq \widehat{F}(t_0) \quad \text{for } t \geqq t_0 \tag{3.5}$$

if and only if (1.5) *holds.*

(ii) *The equation* (B_+) *possesses a* n-$RV_P(1)$ *solution* $x(t)$ *having the expression*

$$x(t) = \exp\bigg\{\int_{t_1}^{t} \Big(\frac{1 + F(s) - w(s)}{p(s)P(s)^\alpha}\Big)^{\frac{1}{\alpha}}\,ds\bigg\}, \quad t \geqq t_1 \tag{3.6}$$

for some $t_1 > a$, *in which* $w(t)$ *satisfies*

$$w(t) = \frac{\alpha}{P(t)} \int_t^\infty F_+(s, w(s))\,ds, \quad t \geqq t_1 \tag{3.7}$$

and

$$0 \leqq w(t) \leqq \sqrt{\widehat{F}(t_1)} \quad \text{for } t \geqq t_1 \tag{3.8}$$

if and only if (1.5) *holds.*

(iii) *The equation* (B_-) *possesses a* n-SV_P *solution* $x(t)$ *having the expression*

$$x(t) = \exp\bigg\{\int_{t_0}^{t} \Big(\frac{v(s) - F(s)}{p(s)P(s)^\alpha}\Big)^{\frac{1}{\alpha}*}\,ds\bigg\}, \quad t \geqq t_0 \tag{3.9}$$

for some $t_0 > a$, in which $v(t)$ satisfies

$$v(t) = \alpha P(t)^\alpha \int_t^\infty \frac{|v(s) - F(s)|^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}}\, ds, \quad t \geqq t_0 \tag{3.10}$$

and (3.5) *if and only if* (1.5) *holds. Here, the meaning of the asterisk notation is defined by $\xi^{\gamma *} = |\xi|^\gamma \operatorname{sgn} \xi$, $\gamma > 0$, $\xi \in \mathbb{R}$.*

(iv) *The equation* (B_-) *possesses a* n-$\mathrm{RV}_P(1)$ *solution $x(t)$ having the expression*

$$x(t) = \exp\left\{ \int_{t_1}^t \Big(\frac{1 - F(s) + w(s)}{p(s)P(s)^\alpha}\Big)^{\frac{1}{\alpha}} ds \right\}, \quad t \geqq t_1 \tag{3.11}$$

for some $t_1 > a$, in which $w(t)$ satisfies

$$w(t) = \frac{\alpha}{P(t)} \int_t^\infty F_-(s, w(s))\, ds, \quad t \geqq t_1 \tag{3.12}$$

and (3.8) *if and only if* (1.5) *holds.*

Our purpose in this section is to give a proof of the above Theorem 3.1. The following lemma will be needed for our purpose.

Lemma 3.1.

(i) *If $x(t)$, a nonoscillatory solution of* ($\mathrm{B}_\pm$)*, is not zero on $[a, \infty)$, then the function $u(t) = p(t)\varphi(x'(t)/x(t))$ satisfies the generalized Riccati equation*

$$u'(t) + \alpha \frac{|u(t)|^{1+\frac{1}{\alpha}}}{p(t)^{\frac{1}{\alpha}}} \pm f(t) = 0, \quad t \geqq a. \tag{$\mathrm{C}_\pm$}$$

(ii) *If $u(t)$ is a solution of* ($\mathrm{C}_\pm$)*, then the function*

$$x(t) = \exp\left\{ \int_a^t \Big(\frac{u(s)}{p(s)}\Big)^{\frac{1}{\alpha}*} ds \right\}$$

is a nonoscillatory solution of ($\mathrm{B}_\pm$) *on $[a, \infty)$.*

Proof of Theorem 3.1. Since the idea of the proof of Theorem 3.1 for the equation (B_-) is similar to the way of proving the equation (B_+), we restrict our attention to the proof for equation (B_+).

(The "only if" part): Let $x(t)$ be a positive solution of (B_+) belonging to n-SV_P or n-$\mathrm{RV}_P(1)$, respectively. Then, by the representation theorem,

$$x(t) = \exp\left\{ \int_{t_0}^t \frac{\delta(s)}{p(s)^{\frac{1}{\alpha}} P(s)}\, ds \right\}, \quad t \geqq t_0,$$

for some $t_0 > a$, where $\lim\limits_{t\to\infty} \delta(t) = 0$ or 1 according as $x(t) \in$ n-SV$_P$ or $x(t) \in$ n-RV$_P(1)$. Since the function

$$u(t) = p(t)\varphi\Big(\frac{x'(t)}{x(t)}\Big) = \varphi\Big(\frac{\delta(t)}{P(t)}\Big)$$

satisfies the generalized Riccati equation (C$_+$) and $u(t) \to 0$ as $t \to \infty$, we obtain

$$u(t) = \alpha \int_t^\infty \frac{|P(s)^\alpha u(s)|^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}}\, ds + \int_t^\infty f(s)\, ds$$

or

$$P(t)^\alpha u(t) = \alpha P(t)^\alpha \int_t^\infty \frac{|P(s)^\alpha u(s)|^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}}\, ds + \\ + P(t)^\alpha \int_t^\infty f(s)\, ds, \quad t \geqq t_0. \tag{3.13}$$

Letting $t \to \infty$ in (3.13), we easily conclude that (1.5) holds in either case of $P(t)^\alpha u(t) \to 0$ or $P(t)^\alpha u(t) \to 1$ as $t \to \infty$.

(The "if" part) Suppose that (1.5) holds.

(The existence of a n-SV$_P$ solution of (B$_+$)): Choose $t_0 > \max\{a, 1\}$ so large that

$$\phi = (2\widehat{F}(t_0))^{\frac{1}{\alpha}} \max\Big\{2, 1 + \frac{1}{\alpha}\Big\} < 1, \tag{3.14}$$

and define the set of continuous functions V and the integral operators $\mathcal{F}$ by

$$V = \Big\{v \in C_0[t_0, \infty) : \ \ 0 \leqq v(t) \leqq \widehat{F}(t_0), \ \ t \geqq t_0\Big\} \tag{3.15}$$

and

$$\mathcal{F}v(t) = \alpha P(t)^\alpha \int_t^\infty \frac{(v(s) + F(s))^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}}\, ds, \ \ t \geqq t_0, \tag{3.16}$$

where $C_0[t_0, \infty)$ denotes the Banach space consisting of all continuous functions on $[t_0, \infty)$ and tend to 0 as $t \to \infty$ and equipped with the norm $\|v\|_0 = \sup\limits_{t \geqq t_0} |v(t)|$. It can be verified that $\mathcal{F}$ is a contraction mapping on V. In fact, using (3.14), we see that $v \in V$ implies $\lim\limits_{t\to\infty} \mathcal{F}v(t) = 0$ and

$$\mathcal{F}v(t) \leqq \alpha(2\widehat{F}(t_0))^{1+\frac{1}{\alpha}} P(t)^\alpha \int_t^\infty \frac{ds}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}} = (2\widehat{F}(t_0))^{1+\frac{1}{\alpha}} \leqq \widehat{F}(t_0),$$

and that $v_1, v_2 \in V$ implies

$$\Big| |v_1(t) + F(t)|^{1+\frac{1}{\alpha}} - \big|v_2(t) + F(t)\big|^{1+\frac{1}{\alpha}} \Big| \leqq$$
$$\leqq \Big(1 + \frac{1}{\alpha}\Big)(2\widehat{F}(t_0))^{\frac{1}{\alpha}} |v_1(t) - v_2(t)| \leqq \phi |v_1(t) - v_2(t)|, \quad t \geqq t_0,$$

which ensures that $\mathcal{F}$ is a contraction mapping. Therefore, there exists a unique element $v_0 \in V$ such that $v_0 = \mathcal{F} v_0$, that is,

$$v_0(t) = \alpha P(t)^\alpha \int_t^\infty \frac{(v_0(s) + F(s))^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}}\, ds, \quad t \geqq t_0.$$

Obviously, $v_0(t)$ satisfies the integral equation

$$\Big(\frac{v_0(t)}{P(t)^\alpha}\Big)' + \frac{(v(t) + F(t))^{1+\frac{1}{\alpha}}}{p(t)^{\frac{1}{\alpha}} P(t)^{\alpha+1}} = 0, \quad t \geqq t_0. \tag{3.17}$$

By virtue of the function $v_0(t)$ we define the function

$$x_0(t) = \exp\Big\{ \int_{t_0}^t \Big(\frac{v_0(s) + F(s)}{p(s) P(s)^\alpha} \Big)^{\frac{1}{\alpha}} ds \Big\}, \quad t \geqq t_0.$$

Since the function $u(t) = v_0(t) + F(t)/P(t)^\alpha$ satisfies the generalized Riccati equation (C_+) associated with (B_+) which is easily seen to be equivalent to (3.17), $x_0(t)$ is a solution of the differential equation (B_+).

(The existence of a n-$RV_P(1)$ solution of (B_+)): We will construct a n-$RV_P(1)$ solution of (B_+). Let us consider the function

$$x(t) = \exp\Big\{ \int_{t_1}^t \Big(\frac{1 + F(s) - w(s)}{p(s) P(s)^\alpha} \Big)^{\frac{1}{\alpha}*} ds \Big\}, \quad t \geqq t_1 \tag{3.18}$$

for some $t_1 > a$ to be determined later. According to (ii) of Lemma 3.1, the function $x(t)$ is a solution of (B_+) on $[t_1, \infty)$ if $w(t)$ is chosen in such way that $u(t) = 1 + F(t) - w(t)/P(t)^\alpha$ satisfies the generalized Riccati equation (C_+) on $[t_1, \infty)$. Then the differential equation for $w(t)$ is derived:

$$w'(t) - \frac{\alpha}{p(t)^{\frac{1}{\alpha}} P(t)} w(t) + \frac{\alpha}{p(t)^{\frac{1}{\alpha}} P(t)} \big[1 - |1 + F(t) - w(t)|^{1+\frac{1}{\alpha}}\big] = 0. \tag{3.19}$$

We rewrite (3.19) as

$$(P(t) w(t))' - \frac{\alpha}{p(t)^{\frac{1}{\alpha}}} F_+(t, w(t)) = 0, \tag{3.20}$$

where $F_+(t, w(t))$ is defined with (3.1). It is convenient to express $F_+(t, w)$ as

$$F_+(t, w) = G(t, w) + H(t, w) + k(t), \tag{3.21}$$

with $G(t, w)$, $H(t, w)$ and $k(t)$ defined, respectively, by

$$G(t, w) = |1 + F(t) - w|^{1+\frac{1}{\alpha}} + \Big(1 + \frac{1}{\alpha}\Big)(1 + F(t))^{\frac{1}{\alpha}} w - (1 + F(t))^{1+\frac{1}{\alpha}}, \tag{3.22}$$

$$H(t,w)=\Big(1+\frac{1}{\alpha}\Big)\Big\{1-(1+F(t))^{\frac{1}{\alpha}}\Big\}\,w, \tag{3.23}$$

and

$$k(t)=(1+F(t))^{1+\frac{1}{\alpha}}-1. \tag{3.24}$$

Since $F(t)\to 0$ as $t\to\infty$ by hypothesis, we can choose $t_1>\max\{a,1\}$ such that

$$\Big(1+\frac{1}{\alpha}\Big)[K+L+\alpha]\sqrt{\widehat{F}(t_1)}\leqq 1, \tag{3.25}$$

where K and L are positive constants such that

$$\begin{aligned} &K=\Big(\frac{4}{3}\Big)^{1-\frac{1}{\alpha}}\ \text{ and }\ L=1\ \text{ if }\ \alpha>1;\\ &K=\Big(\frac{3}{2}\Big)^{\frac{1}{\alpha}-1}\ \text{ and }\ L=\Big(\frac{5}{4}\Big)^{\frac{1}{\alpha}-1}\ \text{ if }\ \alpha\leqq 1. \end{aligned} \tag{3.26}$$

Noting that since $1+1/\alpha>1$ and $K+L+\alpha\geqq 2$, we have in view of (3.25) that $\sqrt{\widehat{F}(t_1)}\leqq 1/2$ and $F(t)\leqq 1/4$ for all $t\geqq t_1$. It is easily shown that, using the mean value theorem and L'Hospital rule, the following inequalities hold for (3.22), (3.23) and (3.24):

$$\Big|\frac{\partial G(t,w)}{\partial w}\Big|\leqq\frac{1}{\alpha}\Big(1+\frac{1}{\alpha}\Big)K|w|, \tag{3.27}$$

$$\Big|\frac{\partial H(t,w)}{\partial w}\Big|\leqq\frac{1}{\alpha}\Big(1+\frac{1}{\alpha}\Big)L\,F(t), \tag{3.28}$$

$$|G(t,w)|\leqq\frac{1}{\alpha}\Big(1+\frac{1}{\alpha}\Big)L\,w^2, \tag{3.29}$$

$$|H(t,w)|\leqq\frac{1}{\alpha}\Big(1+\frac{1}{\alpha}\Big)L\,F(t)|w|, \tag{3.30}$$

and

$$|k(t)|\leqq\Big(1+\frac{1}{\alpha}\Big)F(t) \tag{3.31}$$

for $t\geqq t_1$ and for $|w|\leqq 1/4$.

Consider the set $W\subset C_0[t_1,\infty)$ defined by

$$W=\Big\{w\in C_0[t_1,\infty):\ |w(t)|\leqq\sqrt{\widehat{F}(t_1)}\,,\ \ t\geqq t_1\Big\} \tag{3.32}$$

and define the integral operator $\mathcal{G}:W\to C_0[t_1,\infty)$ by

$$\mathcal{G}w(t)=\frac{\alpha}{P(t)}\int_{t_1}^{t}\frac{F_+(s,w(s))}{p(s)^{\frac{1}{\alpha}}}\,ds,\ \ t\geqq t_1, \tag{3.33}$$

where $F_+(t,w)$ is given by (3.1). Then, it can be shown that $\mathcal{G}$ is a contraction mapping on W. In fact, if $w\in W$, then, by means of (3.29)–(3.31) and (3.25), we can see that

$$|\mathcal{G}w(t)| \leqq \frac{\alpha}{P(t)} \int_{t_1}^{t} \frac{1}{p(s)^{\frac{1}{\alpha}}} \Big[|G(s,w)| + |H(s,w)| + |k(s)|\Big]\, ds \leqq$$
$$\leqq \Big(1+\frac{1}{\alpha}\Big) \frac{1}{P(t)} \int_{t_1}^{t} \frac{1}{p(s)^{\frac{1}{\alpha}}} \Big[Lw(s)^2 + LF(s)|w(s)| + \alpha F(s)\Big]\, ds \leqq$$
$$\leqq \Big(1+\frac{1}{\alpha}\Big)\Big[L\widehat{F}(t_1) + L\widehat{F}(t_1)^{\frac{3}{2}} + \alpha\widehat{F}(t_1)\Big] =$$
$$= \Big(1+\frac{1}{\alpha}\Big)\widehat{F}(t_1)\Big[L + L\sqrt{\widehat{F}(t_1)} + \alpha\Big] \leqq$$
$$\leqq \sqrt{\widehat{F}(t_1)}\,\Big(1+\frac{1}{\alpha}\Big)[K+L+\alpha]\sqrt{\widehat{F}(t_1)} \leqq \sqrt{\widehat{F}(t_1)}\,, \quad t \geqq t_1.$$

Since $F_+(t,w(t)) \to 0$ as $t \to \infty$, we obtain $\lim\limits_{t\to\infty} \mathcal{G}w(t) = 0$. Thus, it follows that $\mathcal{G}w \in W$, and hence $\mathcal{G}$ maps W into itself. Moreover, if $w_1, w_2 \in W$, then, using (3.27) and (3.28), we obtain

$$\big|\mathcal{G}w_1(t) - \mathcal{G}w_2(t)\big| \leqq \frac{\alpha}{P(t)} \times$$
$$\times \int_{t_1}^{t} \frac{1}{p(s)^{\frac{1}{\alpha}}} \Big[\,\big|G(s,w_1(s)) - G(s,w_2(s))\big| + \big|H(s,w_1(s)) - H(s,w_2(s))\big|\,\Big]\, ds \leqq$$
$$\leqq \Big(1+\frac{1}{\alpha}\Big)\Big[K\sqrt{\widehat{F}(t_1)} + L\,\widehat{F}(t_1)\Big]\|w_1 - w_2\|_0 \leqq$$
$$\leqq \Big(1+\frac{1}{\alpha}\Big)[K+L]\sqrt{\widehat{F}(t_1)}\,\|w_1 - w_2\|_0,$$

which implies that

$$\big\|\mathcal{G}w_1 - \mathcal{G}w_2\big\|_0 \leqq \Big(1+\frac{1}{\alpha}\Big)[K+L]\sqrt{\widehat{F}(t_1)}\,\|w_1 - w_2\|_0.$$

In view of (3.25) this shows that $\mathcal{G}$ is a contraction mapping on W. Therefore, the contraction mapping principle ensures the existence of a unique fixed element $w_1 \in W$ such that $w_1 = \mathcal{G}w_1$, which is equivalent to the integral equation

$$w_1(t) = \frac{\alpha}{P(t)} \int_{t_1}^{t} \frac{F_+(s,w_1(s))}{p(s)^{\frac{1}{\alpha}}}\, ds, \quad t \geqq t_1. \tag{3.34}$$

Differentiation of (3.34) shows that $w_1(t)$ satisfies the differential equation (3.20), and substitution of this $w_1(t)$ into (3.6) gives rise to a solution $x(t)$ of the half-linear differential equation (B_+) defined on $[t_1,\infty)$. Furthermore, since $\lim\limits_{t\to\infty} w_1(t) = 0$, it follows from the representation theorem that

$x(t) \in \text{n-RV}_P(1)$. This completes the proof of Theorem 3.1 for the equation (B_+). $\square$

Remark 3.1. Consider another half-linear differential equation

$$\big(p(t)\varphi(x'(t))\big)' + \widetilde{f}(t)\varphi(x(t)) = 0, \tag{$\widetilde{\mathrm{B}}_+$}$$

where $\widetilde{f}(t)$ is a positive continuous function such that

$$\widetilde{f}(t) \geqq f(t), \quad t \geqq a$$

and

$$\lim_{t\to\infty} P(t)^\alpha \int_t^\infty \widetilde{f}(s)\, ds = 0.$$

We take $t_0 > \max\{a, 1\}$ so large that

$$(2\widetilde{F}(t_0))^{\frac{1}{\alpha}} \max\Big\{2, 1 + \frac{1}{\alpha}\Big\} < 1 \;\text{ where }\; \widetilde{F}(t) = P(t)^\alpha \int_t^\infty \widetilde{f}(s)\, ds.$$

Then, by means of Theorem 3.1, both $x_0(t)$ and $\widetilde{x}_0(t)$ are given, respectively, by (3.3) and

$$\widetilde{x}_0(t) = \exp\bigg\{ \int_{t_0}^t \Big(\frac{\widetilde{v}_0(s) + \widetilde{F}(s)}{p(s)P(s)^\alpha}\Big)^{\frac{1}{\alpha}} ds \bigg\}, \quad t \geqq t_0,$$

where $\widetilde{v}_0(t)$ is a solution of the integral equation

$$\widetilde{v}_0(t) = \alpha\, P(t)^\alpha \int_t^\infty \frac{(\widetilde{v}_0(s) + \widetilde{F}(s))^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}}\, ds, \quad t \geqq t_0.$$

We here compare $x_0(t)$ with $\widetilde{x}_0(t)$. From the proof of Theorem 3.1, $v_0(t)$ and $\widetilde{v}_0(t)$ are the fixed points of the contraction mapping $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ given, respectively, by (3.16) and

$$\widetilde{\mathcal{F}}\widetilde{v}(t) = \alpha\, P(t)^\alpha \int_t^\infty \frac{(\widetilde{v}(s) + \widetilde{F}(s))^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} p(s)^{\alpha+1}}\, ds, \quad t \geqq t_0.$$

Noting that $v_0(t)$ and $\widetilde{v}_0(t)$ are the limit points of uniform convergence on $[t_0, \infty)$ of the sequences defined by

$$v_{n+1}(t) = \mathcal{F}v_n(t), \quad t \geqq t_0, \quad n = 1, 2, \ldots, \quad v_1(t) = 0$$

and

$$\widetilde{v}_{n+1}(t) = \widetilde{\mathcal{F}}\widetilde{v}_n(t), \quad t \geqq t_0, \quad n = 1, 2, \ldots, \quad \widetilde{v}_1(t) = 0.$$

We conclude that $\widetilde{v}_0(t) \geqq v_0(t)$, $t \geqq t_0$, which implies that $\widetilde{x}_0(t) \geqq x_0(t)$ for $t \geqq t_0$.

4. The Existence of Generalized Regularly Varying Solution of Self-Adjoint Functional Differential Equation with Deviating Arguments

In this section we first present the proof of Theorem 1.1 for equation (A_+) and then give the proof for the equation (A_-).

4.1. **The proof of Theorem 1.1 for the equation $(\mathbf{A}_+)$.** (The "only if" part) Suppose that there exists a positive solution $x_1(t) \in$ n-SV_P or $x_2(t) \in$ n-$\mathrm{RV}_P(1)$ of (A_+). The equation (A_+) can be written as the half-linear differential equation without retarded and advanced arguments

$$\big(p(t)\varphi(x'(t))\big)' + \sum_{i=1}^{n} \big[q_{x,g_i}(t) + r_{x,h_i}(t)\big]\varphi(x(t)) = 0, \tag{4.1}$$

where

$$q_{x,g_i}(t) = q_i(t)\varphi\Big(\frac{x(g_i(t))}{x(t)}\Big) \text{ and } r_{x,h_i}(t) = r_i(t)\varphi\Big(\frac{x(h_i(t))}{x(t)}\Big), \quad i = 1, 2, \ldots, n. \tag{4.2}$$

Here, applying Theorem 3.1, we see that

$$\lim_{t\to\infty} P(t)^{\alpha} \int_t^{\infty} \sum_{i=1}^{n} \big[q_{x,g_i}(s) + r_{x,h_i}(s)\big]\, ds = 0$$

or

$$\lim_{t\to\infty} P(t)^{\alpha} \int_t^{\infty} \sum_{i=1}^{n} q_{x,g_i}(s)\, ds = \lim_{t\to\infty} P(t)^{\alpha} \int_t^{\infty} \sum_{i=1}^{n} r_{x,h_i}(s)\, ds = 0.$$

By the representation theorem, $x_j(t)$, $j = 1, 2$ can be expressed as

$$x_j(t) = \exp\bigg\{ \int_{t_0}^{t} \frac{\delta_j(s)}{p(s)^{\frac{1}{\alpha}} P(s)}\, ds \bigg\}, \quad j = 1,\ 2$$

for some $t_0 > a$, where $\delta_j(t)$ satisfies

$$\lim_{t\to\infty} \delta_j(t) = \begin{cases} 0 & (j = 1) \\ 1 & (j = 2). \end{cases}$$

The solutions $x_j(t)$, $j = 1, 2$ satisfy

$$\frac{x_j(g_i(t))}{x_j(t)} = \exp\bigg\{ -\int_{g_i(t)}^{t} \frac{\delta_j(s)}{p(s)^{\frac{1}{\alpha}} P(s)}\, ds \bigg\}, \quad t \geqq t_1,$$

and

$$\frac{x_j(h_i(t))}{x_j(t)} = \exp\bigg\{ \int_{t}^{h_i(t)} \frac{\delta_j(s)}{p(s)^{\frac{1}{\alpha}} P(s)}\, ds \bigg\}, \quad t \geqq t_1,$$

respectively, where t_1 is such that $g_i(t_1) \geqq t_0$, $i = 1, 2, \dots, n$. Then, using the properties of $\delta_j(t)$, (1.7) and (1.8), we see that

$$\int_{g_i(t)}^{t} \frac{|\delta_j(s)|}{p(s)^{\frac{1}{\alpha}} P(s)} \, ds \leqq \sup_{s \geqq g_i(t)} |\delta_j(s)| \cdot \log \frac{P(t)}{P(g_i(t))} \to 0 \ \text{ as } \ t \to \infty$$

and

$$\int_{t}^{h_i(t)} \frac{|\delta_j(s)|}{p(s)^{\frac{1}{\alpha}} P(s)} \, ds \leqq \sup_{s \geqq t} |\delta_j(s)| \cdot \log \frac{P(h_i(t))}{P(t)} \to 0 \ \text{ as } \ t \to \infty.$$

Thus, it follows that

$$\lim_{t \to \infty} \frac{x_j(g_i(t))}{x_j(t)} = \lim_{t \to \infty} \frac{x_j(h_i(t))}{x_j(t)} = 1, \quad i = 1, 2, \dots, n, \quad j = 1, 2. \tag{4.3}$$

Consequently, from (4.3) we find that (1.9) holds.

(The "if" part)

(The existence of a n-SV$_P$ solution of (A$_+$)): Suppose that (1.9) is satisfied. Choose $t_0 > a$ so large that $t_* = \min_{i=1,2,\dots,n} \left\{ \inf_{t \geqq t_0} g_i(t) \right\} > \max\{a, 1\}$,

$$\left\{ 2 \sum_{i=1}^{n} \left[\widehat{Q}_i(t_0) + 2^\alpha \widehat{R}_i(t_0) \right] \right\}^{\frac{1}{\alpha}} \max \left\{ 2, 1 + \frac{1}{\alpha} \right\} < 1 \tag{4.4}$$

and

$$\left(2 \sum_{i=1}^{n} \left[Q_i(t_0) + 2^\alpha R_i(t_0) \right] \right)^{\frac{1}{\alpha}} \log \frac{P(h_i(t))}{P(t)} \leqq \log 2, \quad t \geqq t_0, \tag{4.5}$$

where $Q_i(t)$, $R_i(t)$, $\widehat{Q}_i(t)$ and $\widehat{R}_i(t)$ for $i = 1, 2 \dots, n$ are defined by

$$Q_i(t) = P(t)^\alpha \int_t^\infty q_i(s) \, ds, \quad \widehat{Q}_i(t) = \sup_{s \geqq t} Q_i(s) \tag{4.6}$$

and

$$R_i(t) = P(t)^\alpha \int_t^\infty r_i(s) \, ds, \quad \widehat{R}_i(t) = \sup_{s \geqq t} R_i(s). \tag{4.7}$$

Let Ξ denote the set of all positive continuous nondecreasing functions $\xi(t)$ on $[t_*, \infty)$ satisfying

$$\xi(t) = 1 \ \text{ for } \ t_* \leqq t \leqq t_0; \tag{4.8}$$

$$\xi(t) \leqq \exp \left\{ \int_{t_0}^{\infty} \left(\frac{v_0(s) + \sum_{i=1}^{n} [Q_i(s) + 2^\alpha R_i(s)]}{p(s) P(s)^\alpha} \right)^{\frac{1}{\alpha}} ds \right\} \ \text{ for } \ t \geqq t_0; \tag{4.9}$$

$$\frac{\xi(h_i(t))}{\xi(t)} \leqq 2 \ \text{ for } \ t \geqq t_0 \tag{4.10}$$

for $i = 1, 2, \ldots, n$, where $v_0(t)$ satisfies the following integral equation:

$$v_0(t) = \alpha P(t)^\alpha \int_t^\infty \frac{(v_0(s) + \sum_{i=1}^n [Q_i(s) + 2^\alpha R_i(s)])^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}} \, ds, \quad t \geqq t_0. \tag{4.11}$$

We note that the function

$$X_0(t) = \exp\left\{ \int_{t_0}^t \left(\frac{v_0(s) + \sum_{i=1}^n [Q_i(s) + 2^\alpha R_i(s)]}{p(s)P(s)^\alpha} \right)^{\frac{1}{\alpha}} ds \right\}, \quad t \geqq t_0 \tag{4.12}$$

is a solution of the half-linear differential equation

$$\big(p(t)\varphi(x'(t))\big)' + \sum_{i=1}^n \big[q_i(t) + 2^\alpha r_i(t)\big]\varphi(x(t)) = 0, \tag{4.13}$$

since the function

$$u(t) = \frac{v_0(t) + \sum_{i=1}^n [Q_i(t) + 2^\alpha R_i(t)]}{P(t)^\alpha} \tag{4.14}$$

satisfies the generalized Riccati equation

$$u'(t) + \alpha \frac{|u(t)|^{1+\frac{1}{\alpha}}}{p(t)^{\frac{1}{\alpha}}} + \sum_{i=1}^n \big[q_i(t) + 2^\alpha r_i(t)\big] = 0. \tag{4.15}$$

Since $v_0(t) + \sum_{i=1}^n [Q_i(t) + 2^\alpha R_i(t)] \to 0$ as $t \to \infty$, $X_0(t)$ is a normalized slowly varying function with respect to $P(t)$ by the representation theorem. It is obvious that Ξ is a nonvoid closed and convex subset of the locally convex space $C[t_0, \infty)$ of all continuous functions on $[t_0, \infty)$ equipped with the metric topology of uniform convergence on compact subintervals of $[t_0, \infty)$.

For any $\xi \in \Xi$, we define $q_{\xi,g_i}(t)$ and $r_{\xi,h_i}(t)$ by

$$q_{\xi,g_i}(t) = q_i(t)\varphi\Big(\frac{\xi(g_i(t))}{\xi(t)}\Big) \ \text{ and } \ r_{\xi,h_i}(t) = r_i(t)\varphi\Big(\frac{\xi(h_i(t))}{\xi(t)}\Big), \tag{4.16}$$

respectively. Taking into account (4.10), we have

$$\sum_{i=1}^n q_{\xi,g_i}(t) \leqq \sum_{i=1}^n q_i(t), \quad \sum_{i=1}^n r_{\xi.h_i}(t) \leqq 2^\alpha \sum_{i=1}^n r_i(t), \tag{4.17}$$

and accordingly,

$$\sum_{i=1}^n Q_{\xi,g_i}(t) \leqq \sum_{i=1}^n Q_i(t), \quad \sum_{i=1}^n R_{\xi,h_i}(t) \leqq 2^\alpha \sum_{i=1}^n R_i(t) \tag{4.18}$$

where $Q_{\xi,g_i}(t)$ and $R_{\xi,h_i}(t)$ are defined by

$$Q_{\xi,g_i}(t) = P(t)^\alpha \int_t^\infty q_{\xi,g_i}(s)\, ds, \quad R_{\xi,h_i}(t) = P(t)^\alpha \int_t^\infty r_{\xi,h_i}(s)\, ds. \tag{4.19}$$

Consequently, it follows from (4.4) that

$$\left\{2\sum_{i=1}^{n}\left[\widehat{Q}_{\xi,g_i}(t_0)+2^{\alpha}\widehat{R}_{\xi,h_i}(t_0)\right]\right\}^{\frac{1}{\alpha}}\max\left\{2,1+\frac{1}{\alpha}\right\}<1, \tag{4.20}$$

where $\widehat{Q}_{\xi,g_i}(t) = \sup\limits_{s\geqq t} Q_{\xi,g_i}(t)$ and $\widehat{R}_{\xi,h_i}(t) = \sup\limits_{s\geqq t} R_{\xi,h_i}(s)$. Thus, Theorem 3.1 implies that for any $\xi \in \Xi$ the half-linear differential equation

$$\big(p(t)\varphi(x'(t))\big)'+\sum_{i=1}^{n}\big[q_{x,g_i}(t)+r_{x,h_i}(t)\big]\varphi(x(t))=0 \tag{4.21}$$

has a n-SV$_P$ solution

$$X_\xi(t)=\exp\left\{\int\limits_{t_0}^{t}\left(\frac{v_\xi(s)+\sum\limits_{i=1}^{n}[Q_{\xi,g_i}(s)+R_{\xi,h_i}(s)]}{p(s)P(s)^{\alpha}}\right)^{\frac{1}{\alpha}}ds\right\},\quad t\geqq t_0, \tag{4.22}$$

where $v_\xi(t)$ is a solution of the integral equation

$$v_\xi(t)=\alpha P(t)^{\alpha}\int\limits_{t}^{\infty}\frac{\big(v_\xi(s)+\sum\limits_{i=1}^{n}[Q_{\xi,g_i}(s)+R_{\xi,h_i}(s)]\big)^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}}P(s)^{\alpha+1}}\,ds,\quad t\geqq t_0, \tag{4.23}$$

and satisfies

$$0\leqq v_\xi(t)\leqq\sum_{i=1}^{n}\big[\widehat{Q}_{\xi,g_i}(t_0)+\widehat{R}_{\xi,h_i}(t_0)\big]\leqq\sum_{i=1}^{n}\big[\widehat{Q}_i(t_0)+2^{\alpha}\widehat{R}_i(t_0)\big]\ \text{ for }\ t\geqq t_0.$$

Let us now define the mapping Φ which assigns to each $\xi \in \Xi$ the function given by

$$\Phi\xi(t)=1\ \text{ for }\ t_*\leqq t\leqq t_0,\quad \Phi\xi(t)=X_\xi(t)\ \text{ for }\ t\geqq t_0. \tag{4.24}$$

To apply the Schauder–Tychonoff fixed point theorem to Φ we will show that Φ is a continuous mapping which sends Ξ into a relatively compact subset of Ξ.

(i) Φ maps Ξ into itself. Let $\xi \in \Xi$. Then

$$\begin{aligned}\Phi\xi(t)=X_\xi(t)&=\exp\left\{\int\limits_{t_0}^{t}\left(\frac{v_\xi(s)+\sum\limits_{i=1}^{n}[Q_{\xi,g_i}(s)+R_{\xi,h_i}(s)]}{p(s)P(s)^{\alpha}}\right)^{\frac{1}{\alpha}}ds\right\}\leqq\\&\leqq\exp\left\{\int\limits_{t_0}^{t}\left(\frac{v_\xi(s)+\sum\limits_{i=1}^{n}[Q_i(s)+2^{\alpha}R_i(s)]}{p(s)P(s)^{\alpha}}\right)^{\frac{1}{\alpha}}ds\right\}\leqq\\&\leqq\exp\left\{\int\limits_{t_0}^{t}\left(\frac{v_0(s)+\sum\limits_{i=1}^{n}[Q_i(s)+2^{\alpha}R_i(s)]}{p(s)P(s)^{\alpha}}\right)^{\frac{1}{\alpha}}ds\right\},\ t\geqq t_0,\end{aligned}$$

where we make use of the fact that $v_\xi(t) \leqq v_0(t)$, $t \geqq t_0$ for all $\xi \in \Xi$ (cf. Remark 3.1). Furthermore, since $v_\xi(t) \leqq \sum_{i=1}^{n} [\widehat{Q}_i(t_0) + 2^\alpha \widehat{R}_i(t_0)]$, using (4.5), we see that

$$\frac{\Phi(\xi(h_i(t)))}{\Phi(\xi(t))} = \exp\left\{ \int_t^{h_i(t)} \left(\frac{v_\xi(s) + \sum_{i=1}^{n} [Q_{\xi,g_i}(s) + R_{\xi,h_i}(s)]}{p(s)P(s)^\alpha} \right)^{\frac{1}{\alpha}} ds \right\} \leqq$$

$$\leqq \exp\left\{ \left(2 \sum_{i=1}^{n} \left[\widehat{Q}_i(t_0) + 2^\alpha \widehat{R}_i(t_0) \right] \right)^{\frac{1}{\alpha}} \int_t^{h_i(t)} \frac{ds}{p(s)^{\frac{1}{\alpha}} P(s)} \right\} =$$

$$= \exp\left\{ \left(2 \sum_{i=1}^{n} \left[\widehat{Q}_i(t_0) + 2^\alpha \widehat{R}_i(t_0) \right] \right)^{\frac{1}{\alpha}} \log \frac{P(h_i(t))}{P(t)} \right\} \leqq 2, \ t \geqq t_0.$$

This shows that $\Phi\xi \in \Xi$, that is, Φ is a self-map on Ξ.

(ii) $\Phi(\Xi)$ is relatively compact in $C[t_*, \infty)$. Since Φ maps Ξ into itself, that is, $\Phi(\Xi) \subset \Xi$, $\Phi(\Xi)$ is locally uniformly bounded on $[t_*, \infty)$, and since $\xi \in \Xi$ implies

$$0 \leqq \frac{d}{dt} \Phi\xi(t) = \frac{d}{dt} X_\xi(t) =$$

$$= \exp\left\{ \int_{t_0}^{t} \left(\frac{v_\xi(s) + \sum_{i=1}^{n} [Q_{\xi,g_i}(s) + R_{\xi,h_i}(s)]}{p(s)P(s)^\alpha} \right)^{\frac{1}{\alpha}} ds \right\} \times$$

$$\times \left(\frac{v_\xi(t) + \sum_{i=1}^{n} [Q_{\xi,g_i}(t) + R_{\xi,h_i}(t)]}{p(t)P(t)^\alpha} \right)^{\frac{1}{\alpha}} \leqq$$

$$\leqq \exp\left\{ \left(\sum_{i=1}^{n} [\widehat{Q}_i(t_0) + 2^\alpha \widehat{R}_i(t_0)] \right)^{\frac{1}{\alpha}} \log \frac{P(t)}{P(t_0)} \right\} \times$$

$$\times \left(\frac{2 \sum_{i=1}^{n} [\widehat{Q}_i(t_0) + 2^\alpha \widehat{R}_i(t_0)]}{p(t)P(t)^\alpha} \right)^{\frac{1}{\alpha}},$$

$\Phi(\Xi)$ is locally equi-continuous on $[t_*, \infty)$. From the Arzela–Ascoli lemma it then follows that $\Phi(\Xi)$ is relatively compact in $C[t_*, \infty)$.

(iii) Φ is a continuous mapping. Let $\{\xi_m(t)\}$ be a sequence of functions in Ξ converging to $\delta(t)$ uniformly on the compact subintervals of $[t_*, \infty)$. To prove the continuity of Φ, we have to prove that $\{\Phi\xi_m(t)\}$ converges to $\Phi\delta(t)$ uniformly on compact subintervals in $[t_*, \infty)$. Applying the mean value theorem, for $t \geqq t_*$ we obtain

$$\left|\Phi\xi_m(t) - \Phi\delta(t)\right| = |X_{\xi_m}(t) - X_\delta(t)| =$$

$$= \left|\exp\left\{\int\limits_{t_0}^{t}\left(\frac{v_{\xi_m}(s) + \sum\limits_{i=1}^{n}[Q_{\xi_m,g_i}(s) + R_{\xi_m,h_i}(s)]}{p(s)P(s)^\alpha}\right)^{\frac{1}{\alpha}} ds\right\} -\right.$$

$$\left. - \exp\left\{\int\limits_{t_0}^{t}\left(\frac{v_\delta(s) + \sum\limits_{i=1}^{n}[Q_{\delta,g_i}(s) + R_{\delta,h_i}(s)]}{p(s)P(s)^\alpha}\right)^{\frac{1}{\alpha}} ds\right\}\right|$$

$$\leqq \exp\left\{\int\limits_{t_0}^{t}\left(\frac{v_0(s) + \sum\limits_{i=1}^{n}[\widehat{Q}_i(s) + 2^\alpha \widehat{R}_i(s)]}{p(s)P(s)^\alpha}\right)^{\frac{1}{\alpha}} ds\right\} \times$$

$$\times \int\limits_{t_0}^{t} \frac{1}{p(s)^{\frac{1}{\alpha}}}\left|\left(\frac{v_{\xi_m}(s) + \sum\limits_{i=1}^{n}[Q_{\xi_m,g_i}(s) + R_{\xi_m,h_i}(s)]}{P(s)^\alpha}\right)^{\frac{1}{\alpha}} -\right.$$

$$\left. - \left(\frac{v_\delta(s) + \sum\limits_{i=1}^{n}[Q_{\delta,g_i}(s) + R_{\delta,h_i}(s)]}{P(s)^\alpha}\right)^{\frac{1}{\alpha}}\right| ds.$$

By means of the inequality $|x^\lambda - y^\lambda| \leqq |x-y|^\lambda$ for $x, y \in \mathbb{R}^+$ and $0 < \lambda < 1$, we find that the integrand of the last integral in the previous inequality is bounded from above by the function

$$\left|\left(\frac{v_{\xi_m}(t) + \sum\limits_{i=1}^{n}[Q_{\xi_m,g_i}(t) + R_{\xi_m,h_i}(t)]}{P(t)^\alpha}\right)^{\frac{1}{\alpha}} -\right.$$

$$\left. - \left(\frac{v_\delta(t) + \sum\limits_{i=1}^{n}[Q_{\delta,g_i}(t) + R_{\delta,h_i}(t)]}{P(t)^\alpha}\right)^{\frac{1}{\alpha}}\right| \leqq$$

$$\leqq \left(\frac{|v_{\xi_m}(t) - v_\delta(t)| + \sum\limits_{i=1}^{n}|Q_{\xi_m,g_i}(t) - Q_{\delta,g_i}(t)| + \sum\limits_{i=1}^{n}|R_{\xi_m,h_i}(t) - R_{\delta,h_i}(t)|}{P(t)^\alpha}\right)^{\frac{1}{\alpha}}$$

$$\text{if } \alpha > 1.$$

Similarity, using the mean value theorem, we find that

$$\left|\left(\frac{v_{\xi_m}(t) + \sum\limits_{i=1}^{n}[Q_{\xi_m,g_i}(t) + R_{\xi_m,h_i}(t)]}{P(t)^\alpha}\right)^{\frac{1}{\alpha}} -\right.$$

$$\left. - \left(\frac{v_\delta(t) + \sum\limits_{i=1}^{n}[Q_{\delta,g_i}(t) + R_{\delta,h_i}(t)]}{P(t)^\alpha}\right)^{\frac{1}{\alpha}}\right| \leqq$$

$$\leqq C_1 \frac{|v_{\xi_m}(t)-v_\delta(t)|+\sum\limits_{i=1}^{n}|Q_{\xi_m,g_i}(t)-Q_{\delta,g_i}(t)|+\sum\limits_{i=1}^{n}|R_{\xi_m,h_i}(t)-R_{\delta,h_i}(t)|}{P(t)^\alpha} \qquad \text{if } \alpha \leqq 1,$$

where C_1 is a constant depending only on α, $\widehat{Q}_i(t_0)$ and $\widehat{R}_i(t_0)$. Accordingly, the continuity of Φ is guaranteed if we prove that the two sequences

$$\frac{|v_{\xi_m}(t)-v_\delta(t)|}{P(t)^\alpha}, \quad \frac{\sum\limits_{i=1}^{n}|Q_{\xi_m,g_i}(t)-Q_{\delta,g_i}(t)|+\sum\limits_{i=1}^{n}|R_{\xi_m,h_i}(t)-R_{\delta,h_i}(t)|}{P(t)^\alpha} \tag{4.25}$$

converge to 0 on any compact subinterval of $[t_*,\infty)$. In fact, it can be shown more strongly that they converge to 0 uniformly on $[t_*,\infty)$. The uniform convergence of the second sequence in (4.25) follows from the Lebesgue dominated convergence theorem applied to the inequality

$$\frac{\sum\limits_{i=1}^{n}|Q_{\xi_m,g_i}(t)-Q_{\delta,g_i}(t)|+\sum\limits_{i=1}^{n}|R_{\xi_m,h_i}(t)-R_{\delta,h_i}(t)|}{P(t)^\alpha} \leqq$$

$$\leqq \int\limits_t^\infty \Bigg[\sum_{i=1}^{n} q_i(s)\bigg|\varphi\Big(\frac{\xi_m(g_i(s))}{\xi_m(s)}\Big)-\varphi\Big(\frac{\delta(g_i(s))}{\delta(s)}\Big)\bigg|+$$

$$+\sum_{i=1}^{n} r_i(s)\bigg|\varphi\Big(\frac{\xi_m(h_i(s))}{\xi_m(s)}\Big)-\varphi\Big(\frac{\delta(h_i(s))}{\delta(s)}\Big)\bigg|\Bigg]\,ds$$

for $t \geqq t_0$. To examine the first sequence in (4.25) we proceed as follows. Using (4.23) and the mean value theorem, we obtain

$$\frac{|v_{\xi_m}(t)-v_\delta(t)|}{P(t)^\alpha} \leqq \alpha \int\limits_t^\infty \frac{1}{p(s)^{\frac{1}{\alpha}}P(s)^{\alpha+1}} \times$$

$$\times \bigg|\Big(v_{\xi_m}(s)+\sum_{i=1}^{n}[Q_{\xi_m,g_i}(s)+R_{\xi_m,h_i}(s)]\Big)^{1+\frac{1}{\alpha}} -$$

$$-\Big(v_\delta(s)+\sum_{i=1}^{n}[Q_{\delta,g_i}(s)+R_{\delta,h_i}(s)]\Big)^{1+\frac{1}{\alpha}}\bigg|\,ds.$$

Therefore, we have

$$\frac{|v_{\xi_m}(t)-v_\delta(t)|}{P(t)^\alpha} \leqq$$

$$\leqq \alpha\tau_1\Bigg[\int\limits_t^\infty \frac{1}{p(s)^{\frac{1}{\alpha}}}\,\frac{|v_{\xi_m}(s)-v_\delta(s)|}{P(s)^{\alpha+1}}\,ds+\int\limits_t^\infty \frac{1}{p(s)^{\frac{1}{\alpha}}}\,\frac{S_{m,n}(s)}{P(s)^{\alpha+1}}\,ds\Bigg], \tag{4.26}$$

where τ_1 is a positive constant defined by

$$\tau_1 = \Big(1+\frac{1}{\alpha}\Big)\bigg\{2\sum_{i=1}^{n}[\widehat{Q}_i(t_0)+2^{\alpha}\widehat{R}_i(t_0)]\bigg\}^{\frac{1}{\alpha}} \tag{4.27}$$

and $S_{m,n}(t)$ is defined by

$$S_{m,n}(t) = \sum_{i=1}^{n}\Big[\big|Q_{\xi_m,g_i}(t)-Q_{\delta,g_i}(t)\big| + \big|R_{\xi_m,h_i}(t)-R_{\delta,h_i}(t)\big|\Big].$$

Note that $\tau_1 < 1$ by (4.27). Letting

$$Z_m(t) = \int_t^{\infty}\frac{|v_{\xi_m}(s)-v_\delta(s)|}{p(s)^{\frac{1}{\alpha}}P(s)^{\alpha+1}}\,ds, \tag{4.28}$$

we derive from (4.26) the following differential inequality for $z_m(t)$:

$$\big(P(t)^{\alpha\tau_1}Z_m(t)\big)' \geqq -\alpha\tau_1\frac{P(t)^{\alpha\tau_1-1}}{p(t)^{\frac{1}{\alpha}}}\int_t^{\infty}\frac{S_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}P(s)^{\alpha+1}}\,ds. \tag{4.29}$$

Noting that $P(t)^{\alpha\tau_1}Z_m(t)\to\infty$ and that the right-hand side of (4.29) is integrated over $[t,\infty)$, we obtain

$$Z_m(t) \leqq \frac{1}{P(t)^{\alpha\tau_1}}\int_t^{\infty}\frac{S_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}P(s)^{1+\alpha-\alpha\tau_1}}\,ds, \quad t\geqq t_0. \tag{4.30}$$

Combining (4.26) with (4.30), we have

$$\frac{|v_{\xi_m}(t)-v_\delta(t)|}{P(t)^{\alpha}} \leqq$$

$$\leqq \alpha\tau_1\left[\frac{1}{P(t)^{\alpha\tau_1}}\int_t^{\infty}\frac{S_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}P(s)^{1+\alpha-\alpha\tau_1}}\,ds + \int_t^{\infty}\frac{S_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}P(s)^{\alpha+1}}\,ds\right] \leqq$$

$$\leqq \frac{\alpha\tau_1}{P(t)^{\alpha\tau_1}}\int_t^{\infty}\frac{S_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}P(s)^{1+\alpha-\alpha\tau_1}}\,ds, \quad t\geqq t_0.$$

This shows that $|u_{\xi_m}(t)-v_\delta(t)|/P(t)^{\alpha}$ converges to 0 uniformly on $[t_*,\infty)$. We therefore conclude that the mapping Φ defined by (4.24) is continuous in the topology of $C[t_*,\infty)$. Thus, all the hypotheses of the Schauder–Tychonoff fixed point theorem are fulfilled, and hence there exists $\xi_0(t)\in\Xi$ satisfying the half-linear functional differential equation

$$\big(p(t)\varphi(\xi_0'(t))\big)' + \sum_{i=1}^{n}\big[q_{\xi_0,g_i}(t)+r_{\xi_0,h_i}(t)\big]\varphi(\xi_0(t)) = 0, \quad t\geqq t_0,$$

which is rewritten as

$$\big(p(t)\varphi(\xi_0'(t))\big)' + \sum_{i=1}^{n}\Big[q_i(t)\varphi\big(\xi_0(g_i(t))\big)+r_i(t)\varphi\big(\xi_0(h_i(t))\big)\Big] = 0, \quad t\geqq t_0.$$

This implies that the equation (A_+) has a n-SV_P solution $\xi_0(t)$ existing on $[t_0,\infty)$.

(The existence of a n-$RV_P(1)$ solution of (A_+)): Next, we will be concerned with the construction of a n-$RV_P(1)$ solution of equation (A_+) under the condition (1.9). Choose $t_1 > a$ so large that $t_* = \min\limits_{i=1,2,\dots,n}\big\{\inf\limits_{t\geqq t_1} g_i(t)\big\} > \max\{a,1\}$,

$$\Big(1+\frac{1}{\alpha}\Big)[K+L+\alpha]\sqrt{\sum_{i=1}^{n}\big[\widehat{Q}_i(t_1)+2^{\alpha}\widehat{R}_i(t_1)\big]} \leqq 1, \quad t \geqq t_1 \tag{4.31}$$

and

$$\bigg(\frac{3}{2}+\sum_{i=1}^{n}\big[\widehat{Q}_i(t_1)+2^{\alpha}\widehat{R}_i(t_1)\big]\bigg)^{\frac{1}{\alpha}}\log\frac{P(h_i(t))}{P(t)} \leqq \log 2, \quad t \geqq t_1. \tag{4.32}$$

Let $\mathbb{H}$ denote the set of all continuous nondecreasing functions $\eta(t)$ on $[t_*,\infty)$ satisfying

$$\eta(t) = 1 \ \text{ for } \ t_* \leqq t \leqq t_1; \tag{4.33}$$

$$1 \leqq \eta(t) \leqq \exp\Bigg\{\int_{t_1}^{t}\Bigg(\frac{\frac{3}{2}+\sum\limits_{i=1}^{n}[Q_i(s)+2^{\alpha}R_i(s)]}{p(s)P(s)^{\alpha}}\Bigg)^{\frac{1}{\alpha}} ds\Bigg\} \ \text{ for } \ t \geqq t_1; \tag{4.34}$$

$$\frac{\eta(h_i(t))}{\eta(t)} \leqq 2 \ \text{ for } \ t \geqq t_1, \quad i = 1,2,\dots,n. \tag{4.35}$$

For any $\eta \in \mathbb{H}$ we consider the differential equation

$$\big(p(t)\varphi(x'(t))\big)' + \sum_{i=1}^{n}[q_{\eta,g_i}(t)+r_{\eta,h_i}(t)]\varphi(x(t)) = 0, \quad t \geqq t_1, \tag{4.36}$$

where

$$q_{\eta,g_i}(t) = q_i(t)\varphi\Big(\frac{\eta(g_i(t))}{\eta(t)}\Big) \ \text{ and } \ r_{\eta,h_i}(t) = r_i(t)\varphi\Big(\frac{\eta(h_i(t))}{\eta(t)}\Big), \ i=1,2,\dots,n.$$

Since $\eta(g_i(t))/\eta(t) \leqq 1$ and $\eta(h_i(t))/\eta(t) \leqq 2$, we have

$$q_{\eta,g_i}(t) \leqq q_i(t) \ \text{ and } \ r_{\eta,h_i}(t) \leqq 2^{\alpha}r_i(t), \quad t \geqq t_1 \ \text{ for } \ i = 1,2,\dots,n,$$

so that

$$Q_{\eta,g_i}(t) := P(t)^{\alpha}\int_t^{\infty} q_{\eta,g_i}(s)\,ds \leqq P(t)^{\alpha}\int_t^{\infty} q_i(s)\,ds = Q_i(t), \quad t \geqq t_1, \tag{4.37}$$

$$R_{\eta,h_i}(t) := P(t)^{\alpha}\int_t^{\infty} r_{\eta,h_i}(s)\,ds \leqq 2^{\alpha}P(t)^{\alpha}\int_t^{\infty} r_i(s)\,ds = 2^{\alpha}R_i(t), \quad t \geqq t_1. \tag{4.38}$$

Accordingly, from (4.31) we have

$$\Big(1+\frac{1}{\alpha}\Big)[K+L+\alpha]\sqrt{\sum_{i=1}^{n}\big[\widehat{Q}_{\eta,g_i}(t_1)+\widehat{R}_{\eta,h_i}(t_1)\big]} \leqq 1, \tag{4.39}$$

where $\widehat{Q}_{\eta,g_i}(t) = \sup_{s\geqq t} Q_{\eta,g_i}(s)$ and $\widehat{R}_{\eta,h_i}(t) = \sup_{s\geqq t} R_{\eta,h_i}(s)$. Moreover, we notice that from $K+L+\alpha>2$ and $1+\frac{1}{\alpha}>1$ follows

$$\sqrt{\sum_{i=1}^{n}\big[\widehat{Q}_{\eta,g_i}(t_1)+\widehat{R}_{\eta,h_i}(t_1)\big]} \leqq \frac{1}{2}\,. \tag{4.40}$$

This enables us to apply Theorem 3.1 and thus we conclude that half-linear differential equation (4.36) has a n-RV$_P(1)$ solution of the form

$$X_\eta(t) = \exp\left\{\int_{t_1}^{t}\left(\frac{1+\sum\limits_{i=1}^{n}[Q_{\eta,g_i}(s)+R_{\eta,h_i}(s)]-w_\eta(s)}{p(s)P(s)^\alpha}\right)^{\frac{1}{\alpha}} ds\right\}, \quad t\geqq t_1, \tag{4.41}$$

where $w_\eta(t)$ is a solution of the integral equation

$$w_\eta(t) = \frac{\alpha}{P(t)}\int_{t}^{\infty}\frac{F_\eta(s,w_\eta(s))}{p(s)^{\frac{1}{\alpha}}}\,ds, \quad t\geqq t_1, \tag{4.42}$$

satisfying $|w_\eta(t)| \leqq \sqrt{\sum\limits_{i=1}^{n}[\widehat{Q}_{\eta,g_i}(t_1)+\widehat{R}_{\eta,h_i}(t_1)]}$ for $t\geqq t_1$. Furthermore, it follows from (4.40) that $|w_\eta(t)|\leqq 1/2$ for $t\geqq t_1$. Here $F_\eta(t,w_\eta(t))$ is

$$F_\eta(t,w_\eta)=\Big|1+\sum_{i=1}^{n}\big[Q_{\eta,g_i}(t)+R_{\eta,h_i}(t)\big]-w_\eta\Big|^{1+\frac{1}{\alpha}}+\Big(1+\frac{1}{\alpha}\Big)w_\eta-1, \quad t\geqq t_1.$$

Denote by Ψ the mapping which assigns to each $\eta\in\mathbb{H}$ the function $\Psi\eta(t)$ defined by

$$\Psi\eta(t)=1 \ \text{ for } \ t_*\leqq t\leqq t_1, \quad \Psi\eta(t)=X_\eta(t) \ \text{ for } \ t\geqq t_1. \tag{4.43}$$

(i) Ψ is a self-map on $\mathbb{H}$. For any $\eta\in\mathbb{H}$ from (4.37) and (4.38), for $t\geqq t_1$ we find that

$$\begin{aligned}\Big|1+\sum_{i=1}^{n}\big[Q_{\xi,g_i}(t)+R_{\xi,h_i}(t)\big]-w_\eta(t)\Big| &\leqq \\ &\leqq 1+\sum_{i=1}^{n}\big[Q_i(t)+2^\alpha R_i(t)\big]+|w_\eta(t)|\leqq \\ &\leqq \frac{3}{2}+\sum_{i=1}^{n}\big[Q_i(t)+2^\alpha R_i(t)\big],\end{aligned}$$

or accordingly,

$$X_\eta(t) \leqq \exp\left\{\int_{t_1}^{t}\left(\frac{\frac{3}{2}+\sum_{i=1}^{n}[Q_i(s)+2^\alpha R_i(s)]}{p(s)P(s)^\alpha}\right)^{\frac{1}{\alpha}} ds\right\}, \quad t \geqq t_1.$$

Moreover, we have

$$\begin{aligned}
\frac{\Psi\eta(h_i(t))}{\Psi\eta(t)} &= \exp\left\{\int_{t}^{h_i(t)} \frac{(1+\sum_{i=1}^{n}[Q_{\eta,g_i}(s)+R_{\eta,h_i}(s)]-w_\eta(s))^{\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}}P(s)} ds\right\} \leqq \\
&\leqq \exp\left\{\int_{t}^{h_i(t)} \frac{\left(\frac{3}{2}+\sum_{i=1}^{n}[Q_i(s)+2^\alpha R_i(s)]\right)^{\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}}P(s)} ds\right\} \leqq \\
&\leqq \exp\left\{\left(\frac{3}{2}+\sum_{i=1}^{n}\left[\widehat{Q}_i(s)+2^\alpha \widehat{R}_i(s)\right]\right)^{\frac{1}{\alpha}} \int_{t}^{h_i(t)} \frac{ds}{p(s)^{\frac{1}{\alpha}}P(s)}\right\} \leqq \\
&\leqq \exp\left\{\left(\frac{3}{2}+\sum_{i=1}^{n}\left[\widehat{Q}_i(s)+2^\alpha \widehat{R}_i(s)\right]\right)^{\frac{1}{\alpha}} \log\frac{P(h_i(t))}{P(t)}\right\} \leqq 2, \quad t \geqq t_1.
\end{aligned}$$

(ii) $\Psi(\mathbb{H})$ is relatively compact in $C[t_*, \infty)$. This is a consequence of the inclusion $\Psi(\mathbb{H}) \subset \mathbb{H}$ and the following inequality holding for any $\eta \in \mathbb{H}$:

$$\begin{aligned}
0 \leqq \frac{d}{dt}\Psi\eta(t) &= \frac{d}{dt}X_\eta(t) = \\
&\left(\frac{1+\sum_{i=1}^{n}[Q_{\eta,g_i}(t)+R_{\eta,h_i}(t)]-w_\eta(t)}{p(t)P(t)^\alpha}\right)^{\frac{1}{\alpha}} X_\eta(t) \leqq \\
&\leqq \left(\frac{\frac{3}{2}+\sum_{i=1}^{n}[\widehat{Q}_i(t_1)+2^\alpha \widehat{R}_i(t_1)]}{p(t)P(t)^\alpha}\right)^{\frac{1}{\alpha}} \times \\
&\times \exp\left\{\left(\frac{3}{2}+\sum_{i=1}^{n}\left[\widehat{Q}_i(t_1)+2^\alpha \widehat{R}_i(t_1)\right]\right)^{\frac{1}{\alpha}} \log\frac{P(t)}{P(t_1)}\right\}.
\end{aligned}$$

(iii) Ψ is continuous in the topology of $C[t_*, \infty)$. Let $\{\eta_n\}$ be a sequence in $\mathbb{H}$ converging to $\theta \in \mathbb{H}$, which amounts to supposing that the sequence $\{\eta_n(t)\}$ converges to $\theta(t)$ uniformly on the compact subintervals of $[t_*, \infty)$. We will show that $\{\Psi\eta_n(t)\}$ converges to $\Psi\theta(t)$ uniformly on the compact subintervals $[t_*, \infty)$. In order to simplify notation, for arbitrary $\eta \in \mathbb{H}$ we denote

$$V_\eta(t) = \frac{1+\sum_{i=1}^{n}[Q_{\eta,g_i}(t)+R_{\eta,h_i}(t)]-w_\eta(t)}{P(t)^\alpha}, \quad t \geqq t_1. \tag{4.44}$$

In view of (4.41), we have

$$\begin{aligned}
&\left|\Psi\eta_m(t)-\Psi\theta(t)\right| = \left|X_{\eta_m}(t)-X_\theta(t)\right| = \\
&\quad = \left|\exp\left\{\int\limits_{t_1}^{t}\Big(\frac{V_{\eta_m}(s)}{p(s)}\Big)^{\frac{1}{\alpha}}\,ds\right\} - \exp\left\{\int\limits_{t_1}^{t}\Big(\frac{V_\theta(s)}{p(s)}\Big)^{\frac{1}{\alpha}}\,ds\right\}\right| \leqq \\
&\quad \leqq \exp\left\{\int\limits_{t_1}^{t}\left(\frac{\frac{3}{2}+\sum\limits_{i=1}^{n}[Q_i(s)+2^\alpha R_i(s)]}{p(s)P(s)^\alpha}\right)^{\frac{1}{\alpha}} ds\right\}\times \\
&\qquad \times \int\limits_{t_1}^{t}\frac{1}{p(s)^{\frac{1}{\alpha}}}\left|(V_{\eta_m}(s))^{\frac{1}{\alpha}}-(V_\theta(s))^{\frac{1}{\alpha}}\right|\,ds.
\end{aligned}$$

As in the previous part of the proof, we can verify that the integrand of the last integral is bounded from the above by

$$\begin{aligned}
&\left(\frac{\sum\limits_{i=1}^{n}|Q_{\eta_m,g_i}(t)-Q_{\theta,g_i}(t)|+\sum\limits_{i=1}^{n}|R_{\eta_m,h_i}(t)-R_{\theta,h_i}(t)|+|w_{\eta_m}(t)-w_\theta(t)|}{P(t)^\alpha}\right)^{\frac{1}{\alpha}} \\
&\qquad\qquad\qquad\qquad \text{if } \alpha>1, \\
&C_2\,\frac{\sum\limits_{i=1}^{n}|Q_{\eta_m,g_i}(t)-Q_{\theta,g_i}(t)|+\sum\limits_{i=1}^{n}|R_{\eta_m,h_i}(t)-R_{\theta,h_i}(t)|+|w_{\eta_m}(t)-w_\theta(t)|}{P(t)^\alpha} \\
&\qquad\qquad\qquad\qquad \text{if } \alpha\leqq 1,
\end{aligned}$$

where C_2 is a constant depending only on α, $\widehat{Q}_i(t_1)$ and $\widehat{R}_i(t_1)$. Accordingly, it suffices to prove the uniform convergence to 0 on the compact subintervals of the two sequences

$$\frac{|w_{\eta_m}(t)-w_\theta(t)|}{P(t)^\alpha} \quad\text{and}\quad \frac{\pi_{m,n}(t)}{P(t)^\alpha}\,,$$

where

$$\pi_{m,n}(t)=\sum_{i=1}^{n}\left|Q_{\eta_m,g_i}(t)-Q_{\theta,g_i}(t)\right|+\sum_{i=1}^{n}\left|R_{\eta_m,h_i}(t)-R_{\theta,h_i}(t)\right|.$$

The uniform convergence of the sequence $\pi_{m,n}(t)/P(t)^\alpha$ is an immediate consequence of the Lebesgue dominated convergence theorem. Therefore, let us examine the sequence $|w_{\eta_m}(t)-w_\theta(t)|/P(t)^\alpha$. Applying the mean value theorem to $F_{\eta_m}(t,w_{\eta_m}(t))$ and $F_\theta(t,w_\theta(t))$ in $|w_{\eta_m}(t)-w_\theta(t)|/P(t)^\alpha$, we obtain for $t \geqq t_1$

$$\begin{aligned}
&\left|F_{\eta_m}(t, w_{\eta_m}(t)) - F_\theta(t, w_\theta(t))\right| \leqq \Big(1+\frac{1}{\alpha}\Big)|w_{\eta_m}(t) - w_\theta(t)| + \\
&+\left|\left|1+\sum_{i=1}^{n}\big[Q_{\eta_m,g_i}(t)+R_{\eta_m,h_i}(t)\big]-w_{\eta_m}(t)\right|^{1+\frac{1}{\alpha}} - \right. \\
&\qquad\left. - \left|1+\sum_{i=1}^{n}\big[Q_{\theta,g_i}(t)+R_{\theta,h_i}(t)\big]-w_\theta(t)\right|^{1+\frac{1}{\alpha}}\right| \leqq \\
&\qquad\leqq \Big(1+\frac{1}{\alpha}\Big)(1+\tau_2)|w_{\eta_m}(t)-w_\theta(t)| + \Big(1+\frac{1}{\alpha}\Big)\tau_2\pi_{n,m}(t),
\end{aligned}$$

where τ_2 is a positive constant depending only on α, $\widehat{Q}_i(t_1)$ and $\widehat{R}_i(t_1)$. Consequently, the sequence $|w_{\eta_m}(t) - w_\theta(t)|/P(t)^\alpha$ implies

$$\begin{aligned}
\frac{|w_{\eta_m}(t)-w_\theta(t)|}{P(t)^\alpha} &\leqq \frac{(\alpha+1)(1+\tau_2)}{P(t)^{\alpha+1}}\int_{t_1}^{t}\frac{|w_{\eta_m}(s)-w_\theta(s)|}{P(s)^\alpha}\,ds + \\
&+ \frac{(\alpha+1)\tau_2}{P(t)^{\alpha+1}}\int_{t_1}^{t}\frac{\pi_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}}\,ds, \quad t \geqq t_1. \qquad (4.45)
\end{aligned}$$

Putting for simplicity

$$W_m(t) = \int_{t_1}^{t}\frac{|w_{\eta_m}(s)-w_\theta(s)|}{P(s)^\alpha}\,ds, \qquad (4.46)$$

we transform (4.45) into

$$\big(P(t)^{-(\alpha+1)(1+\tau_2)}W_m(t)\big)' \leqq \frac{(\alpha+1)\tau_2}{p(t)^{\frac{1}{\alpha}}P(t)^{(\alpha+1)(1+\tau_2)+1}}\int_{t_1}^{t}\pi_{m,n}(s)\,ds, \quad t \geqq t_1,$$

which, after integration over $[t_1, t]$, yields

$$W_m(t) \leqq \frac{\tau_2}{1+\tau_2}\,P(t)^{(\alpha+1)(1+\tau_2)}\int_{t_1}^{t}\frac{\pi_{m,n}(s)}{P(s)^{(\alpha+1)(1+\tau_2)}}\,ds, \quad t \geqq t_1. \qquad (4.47)$$

Combining (4.45) with (4.47), we have

$$\begin{aligned}
\frac{|w_{\eta_m}(t)-w_\theta(t)|}{P(t)^\alpha} &\leqq \frac{(\alpha+1)\tau_2}{P(t)^{-(\alpha+1)\tau_2}}\int_{t_1}^{t}\frac{\pi_{m,n}(s)}{P(s)^{(\alpha+1)(1+\tau_2)}}\,ds + \\
&+ \frac{(\alpha+1)\tau_2}{P(t)^{\alpha+1}}\int_{t_1}^{t}\frac{\pi_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}}\,ds, \quad t \geqq t_1.
\end{aligned}$$

This ensures the desired convergence of the sequence $|w_{\eta_m}(t)-w_\theta(t)|/P(t)^\alpha$, whence the continuity of the mapping Ψ has been assured. Thus, all the hypotheses of the Schauder–Tychonoff fixed point theorem are fulfilled, and so there exists $\eta_0 \in \mathbb{H}$ such that $\eta_0 = \Psi\eta_0$. Since $\eta_0(t) = X_{\eta_0}(t)$ for $t \geqq t_1$, $\eta_0(t)$ satisfies the differential equation

$$\big(p(t)\varphi(\eta_0'(t))\big)' + \sum_{i=1}^{n} \big[q_{\eta_0,g_i}(t) + r_{\eta_0,h_i}(t)\big]\varphi(\eta_0(t)) = 0, \quad t \geqq t_1$$

or

$$\big(p(t)\varphi(\eta_0'(t))\big)' + \sum_{i=1}^{n} \Big[q_i(t)\varphi\big(\eta_0(g_i(t))\big) + r_i(t)\varphi\big(\eta_0(h_i(t))\big)\Big] = 0, \quad t \geqq t_1.$$

Therefore, $\eta_0(t)$ is a desired n-RV$_P(1)$ solution of the functional differential equation (A$_+$) on $[t_1, \infty)$.

The proof of Theorem 1.1 for the equation (A$_-$).

(The existence of a n-SV$_P$ solution of (A$_-$)): Suppose that (1.9) holds. Choose $t_0 > a$ so large that $t_* = \min\limits_{i=1,2,\ldots,n} \big\{\inf\limits_{t \geqq t_0} g_i(t)\big\} > \max\{a, 1\}$ and such that

$$\Big(2\sum_{i=1}^{n}\big[2^\alpha \widehat{Q}_i(t_0) + \widehat{R}_i(t_0)\big]\Big)^{\frac{1}{\alpha}} \max\Big\{2, 1 + \frac{1}{\alpha}\Big\} < 1, \tag{4.48}$$

$$\Big(2\sum_{i=1}^{n}\big[2^\alpha \widehat{Q}_i(t_0) + \widehat{R}_i(t_0)\big]\Big)^{\frac{1}{\alpha}} \log\frac{P(t)}{P(g_i(t))} < \log 2, \tag{4.49}$$

are satisfied for all $t \geqq t_0$, where $\widehat{Q}_i(t)$ and $\widehat{R}_i(t)$ are defined by (4.6) and (4.7).

Let $\mathbb{M}$ denote the set of all positive continuous nonincreasing functions $\mu(t)$ on $[t_*, \infty)$ with the properties

$$\mu(t) = 1 \ \text{ for } \ t_* \leqq t \leqq t_0; \tag{4.50}$$

$$\mu(t) \geqq \exp\left\{-\int_{t_0}^{t} \frac{\big(2\sum_{i=1}^{n}[2^\alpha\widehat{Q}_i(s) + \widehat{R}_i(s)]\big)^{\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}}P(s)}\,ds\right\} \ \text{ for } \ t \geqq t_0; \tag{4.51}$$

$$\frac{\mu(g_i(t))}{\mu(t)} \leqq 2 \ \text{ for } \ t \geqq t_0, \quad i = 1, 2, \ldots, n. \tag{4.52}$$

We here consider the following differential equation:

$$\big(p(t)\varphi(x'(t))\big)' = \sum_{i=1}^{n}\big[q_{\mu,g_i}(t) + r_{\mu,h_i}(t)\big]\varphi(x(t)) \tag{4.53}$$

where, for arbitrary $\mu \in \mathbb{M}$, the functions $q_{\mu,g_i}(t)$ and $r_{\mu,h_i}(t)$ are defined by (4.2). In view of Theorem 3.1, for each $\mu \in \mathbb{M}$, the equation (4.53) has a n-SV_P solution $X_\mu(t)$ having the representation

$$X_\mu(t)=\exp\left\{\int_{t_0}^{t}\left(\frac{r_\mu(s)-\sum_{i=1}^{n}[Q_{\mu,g_i}(s)+R_{\mu,h_i}(s)]}{p(s)P(s)^\alpha}\right)^{\frac{1}{\alpha}*}ds\right\},\quad t\geqq t_0,\quad (4.54)$$

where $r_\mu(t)$ is a solution of the integral equation

$$r_\mu(t)=\alpha P(t)^\alpha\int_t^\infty\frac{|r_\mu(s)-\sum_{i=1}^{n}[Q_{\mu,g_i}(s)+R_{\mu,h_i}(s)]|^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}}P(s)^{\alpha+1}}\,ds,\quad t\geqq t_0,\quad (4.55)$$

satisfying the inequality

$$0\leqq r_\mu(t)\leqq\sum_{i=1}^{n}\left[\widehat{Q}_{\mu,g_i}(t_0)+\widehat{R}_{\mu,h_i}(t_0)\right]\leqq$$

$$\leqq\sum_{i=1}^{n}\left[2^\alpha\widehat{Q}_i(t_0)+\widehat{R}_i(t_0)\right],\quad t\geq t_0.\quad (4.56)$$

Here $Q_{\mu,g_i}(t)$ and $R_{\mu,h_i}(t)$ are defined by (4.19) and $\widehat{Q}_{\mu,g_i}(t)=\sup\limits_{s\geqq t}Q_{\mu,g_i}(s)$ and $\widehat{R}_{\mu,h_i}(t)=\sup\limits_{s\geqq t}R_{\mu,h_i}(s)$. Furthermore, using the decreasing nature of $\mu(t)$, we have

$$q_{\mu,g_i}(t)\leqq 2^\alpha q_i(t)\ \text{ and }\ r_{\mu,h_i}(t)\leqq r_i(t),\quad t\geqq t_0,\quad i=1,2,\dots,n,$$

accordingly,

$$\sum_{i=1}^{n}\left[Q_{\mu,g_i}(t)+R_{\mu,h_i}(t)\right]\leqq\sum_{i=1}^{n}\left[2^\alpha Q_i(t)+R_i(t)\right]\leqq$$

$$\leqq\sum_{i=1}^{n}\left[2^\alpha\widehat{Q}_i(t_0)+\widehat{R}_i(t_0)\right],\quad t\geqq t_0.\quad (4.57)$$

Let us now define $\mathcal{H}$ to be the mapping which assigns to each $\mu\in\mathbb{M}$ the function $\mathcal{H}\mu$ given by

$$\mathcal{H}\mu(t)=1\ \text{ for }\ t_*\leqq t\leqq t_0,\quad \mathcal{H}\mu(t)=X_\mu(t)\ \text{ for }\ t\geqq t_0.\quad (4.58)$$

Proceeding as in the proof of the existence of n-SV_P solution of (A_+), it can be proved that $\mathcal{H}$ maps $\mathbb{M}$ into a relatively compact subset of $\mathbb{M}$ with the help of the Schauder–Tychonoff fixed point theorem, so that there exists a $\mu_0\in\mathbb{M}$ such that

$$\mu_0(t)=X_{\mu_0}(t)=$$

$$=\exp\left\{\int_{t_0}^{t}\left(\frac{r_{\mu_0}(s)-\sum_{i=1}^{n}[Q_{\mu_0,g_i}(s)+R_{\mu_0,h_i}(s)]}{p(s)P(s)^\alpha}\right)^{\frac{1}{\alpha}*}ds\right\},\quad t\geqq t_0.$$

This means that $\mu_0(t)$ is a solution satisfying the functional differential equation

$$\big(p(t)\varphi(\mu_0'(t))\big)' = \sum_{i=1}^{n} \big[q_{\mu_0,g_i}(t) + r_{\mu_0,h_i}(t)\big]\varphi(\mu_0(t)), \quad t \geqq t_0$$

or consequently,

$$\big(p(t)\varphi(\mu_0'(t))\big)' = \sum_{i=1}^{n} \Big[q_i(t)\varphi\big(\mu_0(g_i(t))\big) + r_i(t)\varphi\big(\mu_0(h_i(t))\big)\Big], \quad t \geqq t_0.$$

Therefore, we conclude that the equation (A_-) has a n-SV$_P$ solution.

(The existence of a n-RV$_P(1)$ solution of (A_-)): Suppose that (1.9) is satisfied. Choose $t_1 > a$ so large that $t_* = \min\limits_{i=1,2,\dots,n} \big\{\inf\limits_{t \geqq t_1} g_i(t)\big\} > \max\{a, 1\}$

$$\Big(1 + \frac{1}{\alpha}\Big)[\widetilde{K} + \widetilde{L} + \alpha]\sqrt{\sum_{i=1}^{n} \big[\widehat{Q}_i(t_1) + 2^{\alpha}\widehat{R}_i(t_1)\big]} \leqq 1 \tag{4.59}$$

and

$$\Bigg\{1 + \sqrt{\sum_{i=1}^{n} [\widehat{Q}_i(t_1) + 2^{\alpha} R_i(t_1)]}\Bigg\}^{\frac{1}{\alpha}} \log \frac{P(h_i(t))}{P(t)} \leqq \log 2, \tag{4.60}$$

where the functions $Q_i(t)$, $R_i(t)$, $\widehat{Q}_i(t)$ and $\widehat{R}_i(t)$ are defined by (4.6) and (4.7), while

$$\widetilde{K} = \begin{cases} \Big(\dfrac{4}{3}\Big)^{1-\frac{1}{\alpha}} & \text{if } \alpha > 1 \\ \Big(\dfrac{3}{2}\Big)^{\frac{1}{\alpha}-1} & \text{if } \alpha \leqq 1 \end{cases} \quad \text{and} \quad \widetilde{L} = \begin{cases} \Big(\dfrac{4}{3}\Big)^{1-\frac{1}{\alpha}} & \text{if } \alpha > 1 \\ 1 & \text{if } \alpha \leqq 1 \end{cases}. \tag{4.61}$$

Let $\mathbb{K}$ define the set of all positive continuous nondecreasing functions $\nu(t)$ on $[t_*, \infty)$ satisfying

$$\nu(t) = 1 \ \text{ for } \ t_* \leqq t \leqq t_1; \tag{4.62}$$

$$1 \leqq \nu(t) \leqq \exp\Bigg\{\int_{t_1}^{t} \Big(\frac{1 + \rho(s)}{p(s)P(s)^{\alpha}}\Big)^{\frac{1}{\alpha}} ds\Bigg\} \ \text{ for } \ t \geqq t_1; \tag{4.63}$$

$$\frac{\nu(h_i(t))}{\nu(t)} \leqq 2 \ \text{ for } \ t \geqq t_1, \quad i = 1, 2, \dots, n, \tag{4.64}$$

where $\rho(t)$ is a solution of the integral equation

$$\rho(t) = \Big(1 + \frac{1}{\alpha}\Big)\sum_{i=1}^{n} \big[\widehat{Q}_i(t_1) + 2^{\alpha}\widehat{R}_i(t_1)\big]\frac{1}{P(t)}\int_{t_1}^{t} \frac{[\widetilde{L}\rho(s) + \widetilde{L} + \alpha]}{p(s)^{\frac{1}{\alpha}}} ds. \tag{4.65}$$

In order to verify that $\rho(t)$ is a solution of (4.65), we now consider the integral operator $\mathcal{R}$ defined by

$$\mathcal{R}\rho(t) = \Big(1 + \frac{1}{\alpha}\Big) \sum_{i=1}^{n} \big[\widehat{Q}_i(t_1) + 2^{\alpha}\widehat{R}_i(t_1)\big] \times \\ \times \frac{1}{P(t)} \int_{t_1}^{t} \frac{1}{p(s)^{\frac{1}{\alpha}}} \big[\widetilde{L}\rho(s) + \widetilde{L} + \alpha\big]\, ds \tag{4.66}$$

on the set

$$\mathbb{P} = \Bigg\{ \rho \in C_0[t_1, \infty) : \ 0 \leqq \rho(t) \leqq \sqrt{\sum_{i=1}^{n} \big[\widehat{Q}_i(t_1) + 2^{\alpha}\widehat{R}_i(t_1)\big]}\,, \ \ t \geqq t_1 \Bigg\}.$$

It is easy to see that $\mathcal{R}$ sends $\mathbb{P}$ into itself and satisfies

$$\big\|\mathcal{R}\rho_1 - \mathcal{R}\rho_2\big\| \leqq \sum_{i=1}^{n} \big[\widehat{Q}_i(t_1) + 2^{\alpha}\widehat{R}_i(t_1)\big] \Big(1 + \frac{1}{\alpha}\Big) \widetilde{L} \|\rho_1 - \rho_2\|_0, \ \ \rho_1, \rho_2 \in \mathbb{R}.$$

Therefore, there exists a unique fixed point of $\mathcal{R}$ which solves the integral equation (4.65).

Consider a family of half-linear differential equations

$$\big(p(t)\varphi(x'(t))\big)' = \sum_{i=1}^{n} \big[q_{\nu,g_i}(t) + r_{\nu,h_i}(t)\big]\varphi(x(t)), \ \ t \geqq t_1, \tag{4.67}$$

where, for any $\nu \in \mathbb{K}$, the functions $q_{\nu,g_i}(t)$ and $r_{\nu,h_i}(t)$ are defined by

$$q_{\nu,g_i}(t) = q_i(t)\varphi\Big(\frac{\nu(g_i(t))}{\nu(t)}\Big) \ \text{ and } \ r_{\nu,h_i}(t) = r_i(t)\varphi\Big(\frac{\nu(h_i(t))}{\nu(t)}\Big).$$

Then, we define $Q_{\nu,g_i}(t)$, $R_{\nu,h_i}(t)$ for every $\nu \in \mathbb{K}$ by (4.19) and $\widehat{Q}_{\nu,g_i}(t) = \sup_{s \geqq t} Q_{\nu,g_i}(t)$, $\widehat{R}_{\nu,h_i}(t) = \sup_{s \geqq t} R_{\nu,h_i}(t)$. It follows from Theorem 3.1 that for each $\nu \in \mathbb{K}$, the equation (4.67) has a n-$\mathrm{RV}_P(1)$ solution $X_\nu(t)$ expressed in the form

$$X_\nu(t) = \exp\Bigg\{ \int_{t_1}^{t} \Bigg(\frac{1 - \sum_{i=1}^{n} [Q_{\nu,g_i}(s) + R_{\nu,h_i}(s)] + w_\nu(s)}{p(s)P(s)^{\alpha}} \Bigg)^{\frac{1}{\alpha}} ds \Bigg\}, \tag{4.68}$$
$$t \geqq t_1,$$

where $w_\nu(t)$ is a solution of the integral equation

$$w_\nu(t) = \frac{\alpha}{P(t)} \int_{t_1}^{t} \frac{\widetilde{F}_\nu(s, w_\nu(s))}{p(s)^{\frac{1}{\alpha}}}\, ds, \ \ t \geqq t_1 \tag{4.69}$$

and

$$\widetilde{F}_\nu(t,w_\nu) = 1+\Big(1+\frac{1}{\alpha}\Big)w_\nu - \Big|1-\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big]+w_\nu\Big|^{1+\frac{1}{\alpha}}. \quad (4.70)$$

We notice that for some fixed $\nu \in \mathbb{K}$, $w_\nu(t)$ is a fixed point of the contraction mapping $\mathcal{F}_\nu$ defined by

$$\mathcal{F}_\nu w_\nu(t) = \frac{\alpha}{P(t)}\int_{t_1}^{t}\frac{\widetilde{F}_\nu(s,w_\nu(s))}{p(s)^{\frac{1}{\alpha}}}\,ds, \quad t \geqq t_1, \quad (4.71)$$

which satisfies

$$|w_\nu(t)| \leqq \sqrt{\sum_{i=1}^{n}\big[\widehat{Q}_{\nu,g_i}(t_1)+\widehat{R}_{\nu,h_i}(t_1)\big]}, \quad t \geqq t_1. \quad (4.72)$$

Furthermore, using the increasing nature of $\nu(t)$, we obtain

$$q_{\nu,g_i}(t) \leqq q_i(t), \quad r_{\nu,h_i}(t) \leqq 2^\alpha r_i(t) \ \text{ for } \ t \geqq t_1, \quad \nu \in \mathbb{K},$$

or consequently,

$$\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big] \leqq$$
$$\leqq \sum_{i=1}^{n}\big[Q_i(t)+2^\alpha R_i(t)\big] \leqq \sum_{i=1}^{n}\big[\widehat{Q}_i(t_1)+2^\alpha \widehat{R}_i(t_1)\big], \quad t \geqq t_1. \quad (4.73)$$

We will show that for every $\nu \in \mathbb{K}$,

$$|w_\nu(t)| \leqq \rho(t), \quad t \geqq t_1. \quad (4.74)$$

To this end, it is convenient to express $\widetilde{F}_\nu(t,w_\nu)$ as

$$\widetilde{F}_\nu(t,w_\nu) = \widetilde{G}_\nu(t,w_\nu)+\widetilde{H}_\nu(t,w_\nu)+\widetilde{k}_\nu(t),$$

where $\widetilde{G}_\nu(t,w_\nu)$, $\widetilde{H}_\nu(t,w_\nu)$ and $\widetilde{k}_\nu(t)$ are defined, respectively, by

$$\widetilde{G}_\nu(t,w_\nu) = \Big(1-\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big]\Big)^{1+\frac{1}{\alpha}} +$$
$$+\Big(1+\frac{1}{\alpha}\Big)\Big(1-\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big]\Big)^{1+\frac{1}{\alpha}}w_\nu -$$
$$-\Big|1-\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big]+w_\nu\Big|^{1+\frac{1}{\alpha}},$$

$$\widetilde{H}_\nu(t,w_\nu) = \Big(1+\frac{1}{\alpha}\Big)\Big\{1-\Big(1-\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big]\Big)^{\frac{1}{\alpha}}\Big\}w_\nu,$$

and
$$\widetilde{k}_\nu(t) = 1 - \Big(1 - \sum_{i=1}^{n} \big[Q_{\nu,g_i}(t) + R_{\nu,h_i}(t)\big]\Big)^{1+\frac{1}{\alpha}}.$$
Using the mean value theorem, we find that for some $\theta \in (0,1)$ the inequalities hold:
$$\begin{aligned}
\big|\widetilde{H}_\nu(t, w_\nu(t))\big| &\leqq \frac{1}{\alpha}\Big(1+\frac{1}{\alpha}\Big)\bigg|1-(1-\theta)\Big(\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big]\Big)\bigg|^{\frac{1}{\alpha}-1} \times \\
&\qquad \times \sum_{i=1}^{n}\big[Q_{\nu,g_i}(t) + R_{\nu,h_i}(t)\big]|w_\nu(t)| \\
&\leqq \frac{1}{\alpha}\Big(1+\frac{1}{\alpha}\Big)\widetilde{L}\sum_{i=1}^{n}\big[\widehat{Q}_i(t_1) + 2^\alpha \widehat{R}_i(t_1)\big]|w_\nu(t)| \qquad (4.75)
\end{aligned}$$
and
$$\begin{aligned}
&\big|\widetilde{k}_\nu(t)\big| \leqq \\
&\leqq \Big(1+\frac{1}{\alpha}\Big)\bigg|1-(1-\theta)\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big]\bigg|^{\frac{1}{\alpha}}\sum_{i=1}^{n}\big[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)\big] \leqq \\
&\leqq \Big(1+\frac{1}{\alpha}\Big)\sum_{i=1}^{n}\big[\widehat{Q}_i(t_1) + 2^\alpha\widehat{R}_i(t_1)\big], \quad t \geqq t_1. \qquad (4.76)
\end{aligned}$$
Moreover, by means of the mean value theorem and L'Hospital rule, it follows that
$$\big|\widetilde{G}_\nu(t, w_\nu(t))\big| \leqq \frac{1}{\alpha}\Big(1+\frac{1}{\alpha}\Big)\widetilde{L}w_\nu^2(t), \quad t \geqq t_1. \qquad (4.77)$$
Let $\nu \in \mathbb{K}$ be fixed. Recalling that ρ and w_ν are the fixed point of the contraction mappings $\mathcal{R}$ and $\mathcal{F}_\nu$ defined by (4.66) and (4.71), we see that ρ and w_ν are constructed, respectively, as the limits as $n \to \infty$ of the sequences $\{\rho_n = \mathcal{R}\rho_{n-1},\ n = 1,2,\dots,$ with $\rho_0 = 0\}$ and $\{w_n = \mathcal{F}_\nu w_{n-1},\ n = 1,2,\dots,n,$ with $w_0 = 0\}$. First we note that for $t \geqq t_1$,
$$\begin{aligned}
&|w_1(t)| = \mathcal{F}_\nu w_0(t) = \\
&\qquad = \frac{\alpha}{P(t)}\int_{t_1}^{t}\frac{1}{p(s)^{\frac{1}{\alpha}}}\bigg[1 - \Big|1 + \sum_{i=1}^{n}\big[Q_{\nu,g_i}(s) + R_{\nu,h_i}(s)\big]\Big|^{1+\frac{1}{\alpha}}\bigg]\,ds \leqq \\
&\qquad \leqq \frac{\alpha+1}{P(t)}\int_{t_1}^{t}\frac{1}{p(s)^{\frac{1}{\alpha}}}\sum_{i=1}^{n}\big[Q_{\nu,g_i}(s) + R_{\nu,h_i}(s)\big]\,ds \leqq \\
&\qquad \leqq (\alpha+1)\sum_{i=1}^{n}\big[\widehat{Q}_i(t_1)(t) + 2^\alpha\widehat{R}_i(t_1)\big] \leqq \\
&\qquad \leqq \big[\widehat{Q}_i(t_1)(t) + 2^\alpha\widehat{R}_i(t_1)\big]\Big(1+\frac{1}{\alpha}\Big)[\widetilde{L} + \alpha] = \rho_1(t), \quad t \geqq t_1.
\end{aligned}$$

Then, assuming that $|w_n(t)| \leqq \rho_n(t)$, $t \geqq t_1$, for some $n \in \mathbb{N}$ and using (4.75), (4.76) and (4.77), we have

$$|w_{n+1}(t)| = \mathcal{F}_\nu w_n(t) =$$
$$= \frac{\alpha}{P(t)} \int_{t_1}^{t} \frac{1}{p(s)^{\frac{1}{\alpha}}} \Big[\big|\widetilde{G}_\nu(s, w_n(s))\big| + \big|\widetilde{H}_\nu(s, w_n(s))\big| + \big|\widetilde{k}_\nu(s)\big| \Big] \, ds \leqq$$
$$\leqq \frac{\alpha+1}{P(t)} \int_{t_1}^{t} \frac{1}{p(s)^{\frac{1}{\alpha}}} \bigg[\frac{\widetilde{L}}{\alpha} \, w_\nu^2(s) + \frac{\widetilde{L}}{\alpha} \sum_{i=1}^{n} \big[\widehat{Q}_i(t_1) + 2^\alpha \widehat{R}_i(t_1)\big] |w_n(s)| +$$
$$+ \sum_{i=1}^{n} \big[\widehat{Q}_i(t_1) + 2^\alpha \widehat{R}_i(t_1)\big] \bigg] \, ds \leqq$$
$$\leqq \Big(1+\frac{1}{\alpha}\Big) \sum_{i=1}^{n} \big[\widehat{Q}_i(t_1)+2^\alpha \widehat{R}_i(t_1)\big] \, \frac{1}{P(t)} \int_{t_1}^{t} \frac{[\widetilde{L} + \widetilde{L}\rho_n(s) + \alpha]}{p(s)^{\frac{1}{\alpha}}} \, ds = \rho_{n+1}(t).$$

Therefore, inductive arguments ensure the validity of (4.74).

We define by $\mathcal{M}$ the mapping which assigns to each $\nu \in \mathbb{K}$ the function $\mathcal{H}\nu(t)$ defined by

$$\mathcal{M}\nu(t) = 1 \ \text{ for } \ t_* \leqq t \leqq t_1, \ \ \mathcal{M}\nu(t) = X_\nu(t) \ \text{ for } \ t \geqq t_1.$$

(i) $\mathcal{M}$ is a self-map on $\mathbb{K}$, since it readily follows from (4.60) and $0 \leqq \rho(t) \leqq \sqrt{\sum_{i=1}^{n} [\widehat{Q}_i(t) + 2^\alpha \widehat{R}_i(t_1)]}$, $t \geqq t_1$ that

$$1 \leqq X_\nu(t) \leqq \exp\bigg\{ \int_{t_1}^{t} \Big(\frac{1+\rho(s)}{p(s)P(s)^\alpha}\Big)^{\frac{1}{\alpha}} \bigg\}, \ \ t \geqq t_1 \ \text{ for any } \ \nu \in \mathbb{K}$$

and

$$\frac{\mathcal{M}\nu(h_i(t))}{\mathcal{M}\nu(t)} =$$
$$= \exp\Bigg\{ \int_{t}^{h_i(t)} \Bigg(\frac{1 - \sum_{i=1}^{n} [Q_{\nu,g_i}(s) + R_{\nu,h_i}(s)] + w_\nu(s)}{p(s)P(s)^\alpha}\Bigg)^{\frac{1}{\alpha}*} ds \Bigg\} \leqq$$
$$\leqq \exp\bigg\{ \int_{t}^{h_i(t)} \Big(\frac{1+\rho(s)}{p(s)P(s)^\alpha}\Big)^{\frac{1}{\alpha}} ds \bigg\} \leqq$$
$$\leqq \exp\Bigg\{ \bigg(1 + \sqrt{\sum_{i=1}^{n} [\widehat{Q}_i(t_1) + 2^\alpha \widehat{R}_i(t_1)]}\bigg)^{\frac{1}{\alpha}} \log \frac{P(h_i(t))}{P(t)} \Bigg\} \leqq 2, \ \ t \geqq t_1.$$

(ii) $\mathcal{M}(\mathbb{K})$ is relatively compact in $C[t_*,\infty)$. The inclusion $\mathcal{M}(\mathbb{K}) \subset \mathbb{K}$ shows that $\mathcal{M}(\mathbb{K})$ is locally uniformly bounded on $[t_*,\infty)$. Since

$$\begin{aligned}
0 \leqq \frac{d}{dt}\,\mathcal{M}\nu(t) &= \frac{d}{dt}\,X_\nu(t) = \\
&= \left(\frac{1-\sum\limits_{i=1}^{n}[Q_{\nu,g_i}(t)+R_{\nu,h_i}(t)]+w_\nu(t)}{p(t)P(t)^\alpha}\right)^{\frac{1}{\alpha}} X_\nu(t) \leqq \\
&\leqq \frac{\left\{1+\sqrt{\sum\limits_{i=1}^{n}[\widehat{Q}_i(t_1)+2^\alpha\widehat{R}_i(t_1)]}\right\}^{\frac{1}{\alpha}}}{p(t)^{\frac{1}{\alpha}}P(t)} \times \\
&\times \exp\left\{\left(1+\sqrt{\sum_{i=1}^{n}[\widehat{Q}_i(t_1)+2^\alpha\widehat{R}_i(t_1)]}\right)^{\frac{1}{\alpha}}\log\frac{P(t)}{P(t_1)}\right\},
\end{aligned}$$

we conclude that $\mathcal{M}(\mathbb{K})$ is locally equi-continuous on $[t_*,\infty)$.

(iii) $\mathcal{M}$ is continuous in the topology of $C[t_*,\infty)$. Let $\{\nu_m(t)\}$ be a sequence in $\mathbb{K}$ converging to $\delta(t)$ uniformly on compact subintervals of $[t_*,\infty)$. We have to prove that $\{\mathcal{M}\nu_m(t)\}$ converges to $\mathcal{M}\delta(t)$ uniformly on any compact subintervals of $[t_*,\infty)$. In order to simplify the notation, for arbitrary $\nu\in\mathbb{K}$ we define

$$Z_\nu(t) = \frac{1-\sum\limits_{i=1}^{n}[q_{\nu,g_i}(t)+R_{\nu,h_i}(t)]+w_\nu(t)}{P(t)^\alpha}\,,\quad t\geqq t_1. \tag{4.78}$$

Then, using (4.68) and the mean value theorem, we get

$$\begin{aligned}
\big|\mathcal{M}\nu_m(t)-\mathcal{M}\delta(t)\big| &= \big|X_{\nu_m}(t)-X_\delta(t)\big| = \\
&= \left|\exp\left\{\int_{t_1}^{t}\Big(\frac{Z_{\nu_m}(s)}{p(s)}\Big)^{\frac{1}{\alpha}}ds\right\}-\exp\left\{\int_{t_1}^{t}\Big(\frac{Z_\delta(s)}{p(s)}\Big)^{\frac{1}{\alpha}}ds\right\}\right| \leqq \\
&\leqq \exp\left\{\int_{t_1}^{t}\Big(\frac{1+\rho(s)}{p(s)P(s)^\alpha}\Big)^{\frac{1}{\alpha}}ds\right\}\int_{t_1}^{t}\frac{|(Z_{\nu_m}(s))^{\frac{1}{\alpha}}-(Z_\delta(s))^{\frac{1}{\alpha}}|}{p(s)^{\frac{1}{\alpha}}}\,ds.
\end{aligned}$$

As in the proof of the existence of a n-RV$_P(1)$ solution of the equation (A_+), we can show that the integrand of the last integral is bounded from above by the functions

$$\left(\frac{\sum\limits_{i=1}^{n}|Q_{\nu_m,g_i}(t)-Q_{\delta,g_i}(t)|+\sum\limits_{i=1}^{n}|R_{\nu_m,h_i}(t)-R_{\delta,h_i}(t)|+|w_{\nu_m}(t)-w_\delta(t)|}{P(t)^\alpha}\right)^{\frac{1}{\alpha}}$$

$$\text{if } \alpha>1,$$

$$C_3 \frac{\sum\limits_{i=1}^{n} |Q_{\nu_m,g_i}(t) - Q_{\delta,g_i}(t)| + \sum\limits_{i=1}^{n} |R_{\nu_m,h_i}(t) - R_{\delta,h_i}(t)| + |w_{\nu_m}(t) - w_\delta(t)|}{P(t)^\alpha} \qquad \text{if } \alpha \leqq 1,$$

where C_3 is a positive constant depending only on α and $\rho(t_1)$. Therefore, it suffices to prove the uniform convergence to 0 on the compact subintervals of the two sequences

$$\frac{|w_{\nu_m}(t) - w_\delta(t)|}{P(t)^\alpha} \text{ and}$$

$$\frac{\sum\limits_{i=1}^{n} |Q_{\nu_m,g_i}(t) - Q_{\delta,g_i}(t)| + \sum\limits_{i=1}^{n} |R_{\nu_m,h_i}(t) - R_{\delta,h_i}(t)|}{P(t)^\alpha} = \frac{\widetilde{S}_{n,m}(t)}{P(t)^\alpha}. \tag{4.79}$$

The second sequence in (4.79) can be dealt with exactly as in the case of n-RV$_P(1)$ solution of the equation (A$_+$). In order to prove the uniform convergence of the first sequence in (4.79), we consider $\widetilde{F}_{\nu_m}(t, w_{\nu_m})$ and $\widetilde{F}_\delta(t, w_\delta)$ defined by (4.70). Applying the mean value theorem, for $t \geqq t_1$ we get

$$\begin{aligned}
\big|\widetilde{F}_{\nu_m}(t, w_{\nu_m}(t)) - \widetilde{F}_\delta(t, w_\delta(t))\big| &\leqq \Big(1 + \frac{1}{\alpha}\Big)\big|w_{\nu_m}(t) - w_\delta(t)\big| + \\
&+ \bigg|\Big|1 - \sum_{i=1}^{n} \big[Q_{\nu_m,g_i}(t) + R_{\nu_m,h_i}(t)\big] + w_{\nu_m}(t)\Big|^{1+\frac{1}{\alpha}} - \\
&- \Big|1 - \sum_{i=1}^{n} \big[Q_{\delta,g_i}(t) + R_{\delta,h_i}(t)\big] + w_\delta(t)\Big|^{1+\frac{1}{\alpha}}\bigg| \leqq \\
&\leqq \Big(1 + \frac{1}{\alpha}\Big)\big|w_{\nu_m}(t) - w_\delta(t)\big| + \Big(1 + \frac{1}{\alpha}\Big)\tau_3 \Big\{\widetilde{S}_{m,n}(t) + \big|w_{\nu_m}(t) - w_\delta(t)\big|\Big\} = \\
&= \Big(1 + \frac{1}{\alpha}\Big)(1 + \tau_3)\big|w_{\nu_m}(t) - w_\delta(t)\big| + \Big(1 + \frac{1}{\alpha}\Big)\tau_3 \widetilde{S}_{m,n}(t),
\end{aligned}$$

where τ_3 is a positive constant depending only on α, $\widehat{Q}_i(t_1)$ and $\widehat{R}_i(t_1)$. Consequently, the first sequence in (4.79) implies that

$$\begin{aligned}
\frac{|w_{\nu_m}(t) - w_\delta(t)|}{P(t)^\alpha} &\leqq \frac{(\alpha+1)(1+\tau_3)}{P(t)^{(\alpha+1)}} \int_{t_1}^{t} \frac{|w_{\nu_m}(s) - w_\delta(s)|}{p(s)^{\frac{1}{\alpha}}}\, ds + \\
&+ \frac{(\alpha+1)\tau_3}{P(t)^{\alpha+1}} \int_{t_1}^{t} \frac{\widetilde{S}_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}}\, ds, \quad t \geqq t_1.
\end{aligned} \tag{4.80}$$

Putting for brevity

$$\widetilde{W}_m(t) = \int_{t_1}^{t} \frac{|w_{\nu_m}(s) - w_\delta(s)|}{p(s)^{\frac{1}{\alpha}}}\, ds,$$

we derive the following differential inequality for $\widetilde{W}_m(t)$:

$$\big(P(t)^{-(\alpha+1)(1+\tau_3)}\widetilde{W}_m(t)\big)' \leqq$$
$$\leqq \frac{(\alpha+1)\tau_3}{p(t)^{\frac{1}{\alpha}}P(t)^{(\alpha+1)(\tau_3+1)+1}}\int_{t_1}^{t}\frac{\widetilde{S}_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}}\,ds,\quad t\geqq t_1. \quad (4.81)$$

Integrating (4.81) from t_1 to t, we obtain

$$\widetilde{W}_m(t)\leqq\frac{\tau_3}{\tau_3+1}\,\frac{1}{P(t)^{(\alpha+1)(\tau_3+1)}}\int_{t_1}^{t}\frac{\widetilde{S}_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}P(s)^{(\alpha+1)(\tau_3+1)}}\,ds,\quad t\geqq t_1. \quad (4.82)$$

Using (4.80) and (4.82), we conclude that

$$\frac{|w_{\nu_m}(t)-w_\delta(t)|}{P(t)^\alpha}\leqq\frac{(\alpha+1)\tau_3}{P(t)^{(\alpha+1)(\tau_3+2)}}\int_{t_1}^{t}\frac{\widetilde{S}_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}P(s)^{(\alpha+1)(\tau_3+1)}}\,ds+$$
$$+\frac{(\alpha+1)\tau_3}{P(t)^{\alpha+1}}\int_{t_1}^{t}\frac{\widetilde{S}_{m,n}(s)}{p(s)^{\frac{1}{\alpha}}}\,ds,\quad t\geqq t_1,$$

whence it follows that the sequence $|w_{\nu_m}(t)-w_\delta(t)|/P(t)^\alpha$ converges to 0 uniformly on $[t_1,\infty)$. Therefore, we have proved that the mapping $\mathcal{M}$ is continuous in the topology of $C[t_*,\infty)$. Thus, applying the Schauder–Tychonoff fixed point theorem, $\mathcal{M}$ has a fixed point ν_0 in $\mathbb{K}$. Since $\nu_0 = X_{\nu_0}(t)$ for $t\geqq t_1$, $\nu_0(t)$ satisfies the functional differential equation

$$\big(p(t)\varphi(\nu_0'(t))\big)'=\sum_{i=1}^{n}\Big[q_i(t)\varphi\big(\nu_0(g_i(t))\big)+r_i(t)\varphi\big(\nu_0(h_i(t))\big)\Big],\quad t\geqq t_1. \quad (4.83)$$

It is obvious that $\nu_0(t)$ is a n-RV$_P(1)$ solution of the equation (A$_-$). This completes the proof of Theorem 1.1.

5. Examples

We here present four examples illustrating application of Theorem 1.1 to the functional differential equations of the type (A$_+$) and (A$_-$), respectively. We begin with two examples of the existence of n-SV$_P$ and n-RV$_P(1)$ solutions of the type (A$_+$) with the case $i=1$.

Example 5.1. Consider the following functional differential equation with both retarded and advanced arguments

$$\big(e^{-\alpha t}\varphi(x'(t))\big)'+q(t)\varphi\Big(x\Big(t-\frac{1}{\log t}\Big)\Big)+r(t)\varphi\Big(x\Big(t+\frac{1}{\log t}\Big)\Big)=0,\quad (5.1)$$
$$t\geqq e,$$

where the functions $q(t)$ and $r(t)$ are given by

$$q(t) = \frac{\alpha}{2e^{\alpha t}\,t^{\alpha}}\Big(1+\frac{\lambda}{\log t}\Big)^{\alpha-1}\Big[1-\frac{\lambda}{t\log t}+\frac{\lambda(\lambda-1)}{t(\log t)^2}+\frac{\lambda}{\log t}\Big]\times \\ \times\Big(1-\frac{1}{t\log t}\Big)^{-\alpha}\Big\{1+\frac{\log(1-\frac{1}{t\log t})}{\log t}\Big\}^{-\alpha\lambda}$$

and

$$r(t) = \frac{\alpha}{2e^{\alpha t}t^{\alpha}}\Big(1+\frac{\lambda}{\log t}\Big)^{\alpha-1}\Big[1-\frac{\lambda}{t\log t}+\frac{\lambda(\lambda-1)}{t(\log t)^2}+\frac{\lambda}{\log t}\Big]\times \\ \times\Big(1+\frac{1}{t\log t}\Big)^{-\alpha}\Big\{1+\frac{\log(1+\frac{1}{t\log t})}{\log t}\Big\}^{-\alpha\lambda}$$

for λ being a positive constant. The function $p(t) = e^{-\alpha t}$ satisfies (1.1) and the function $P(t)$ given by (1.6) is $P(t) \sim e^t$. Moreover, the functions

$$g(t) = t - \frac{1}{\log t} \quad\text{and}\quad h(t) = t + \frac{1}{\log t} \tag{5.2}$$

satisfy the conditions (1.7) and (1.8). The condition (1.9) is satisfied for this equation, since

$$\int_t^\infty q(s)\,ds \sim \frac{\alpha}{2t^{\alpha}e^{\alpha t}} \quad\text{and}\quad \int_t^\infty h(s)\,ds \sim \frac{\alpha}{2t^{\alpha}e^{\alpha t}} \quad\text{as}\quad t\to\infty.$$

Therefore, equation (5.1) has a n-SV$_{e^t}$ solution $x(t)$ by Theorem 1.1. One such solution is $x(t) = t(\log t)^{\lambda}$.

Example 5.2. Consider the following functional differential equation:

$$\big(t^{\alpha}\varphi(x'(t))\big)' + q(t)\varphi\big(x(te^{-\frac{1}{t}})\big) + r(t)\varphi\big(x(te^{\frac{1}{t}})\big) = 0,\quad t \geqq e^e, \tag{5.3}$$

where the functions $q(t)$ and $r(t)$ are given by

$$q(t) = \frac{\alpha\mu}{2t(\log t)^{\alpha+1}\log_2 t}\Big(1-\frac{\mu}{\log_2 t}\Big)^{\alpha-1}\Big(1-\frac{\mu+1}{\log_2 t}\Big)\times \\ \times\Big(1-\frac{1}{t\log t}\Big)^{-\alpha}\Big\{1+\frac{\log(1-\frac{1}{t\log t})}{\log_2 t}\Big\}^{\alpha\mu}$$

and

$$r(t) = \frac{\alpha\mu}{2t(\log t)^{\alpha+1}\log_2 t}\Big(1-\frac{\mu}{\log_2 t}\Big)^{\alpha-1}\Big(1-\frac{\mu+1}{\log_2 t}\Big)\times \\ \times\Big(1+\frac{1}{t\log t}\Big)^{-\alpha}\Big\{1+\frac{\log(1+\frac{1}{t\log t})}{\log_2 t}\Big\}^{\alpha\mu},$$

respectively, and μ is a positive constant. The function $p(t) = t^{\alpha}$ satisfies (1.1) and the function $P(t)$ reduces to $P(t) \sim \log t$, while the functions

$g(t) = te^{-\frac{1}{t}}$ and $h(t) = te^{\frac{1}{t}}$ satisfy the conditions (1.7) and (1.8). Moreover, since

$$\int_t^\infty q(s)\,ds \sim \frac{\alpha\mu}{2t(\log t)^{\alpha+1}\log_2 t} \quad \text{and} \quad \int_t^\infty h(s)\,ds \sim \frac{\alpha\mu}{2t(\log t)^{\alpha+1}\log_2 t} \quad \text{as } t \to \infty,$$

the condition (1.9) is satisfied and thus, the equation (5.3) possesses a n-$\mathrm{RV}_{\log t}$ solution by Theorem 1.1. One such solution is $\log t/(\log_2 t)^\mu$.

Next, two examples illustrating application of Theorem 1.1 to the functional differential equation of the type (A_-) with the case $i = 1$ will be presented below.

Example 5.3. We consider the functional differential equation with both retarded and advanced arguments

$$\big(e^{-\alpha t}\varphi(x'(t))\big)' = q(t)\varphi\Big(x\Big(t-\frac{1}{\log t}\Big)\Big)+r(t)\varphi\Big(x\Big(t+\frac{1}{\log t}\Big)\Big), \quad t \geqq e, \tag{5.4}$$

where the functions $q(t)$ and $r(t)$ are given by

$$\begin{aligned} q(t) = {} & \frac{\alpha}{2t^\alpha e^{\alpha t}}\Big(1-\frac{\lambda}{\log t}\Big)^{\alpha-1}\times \\ & \times\Big[\Big(1+\frac{2}{t}\Big)\Big(1-\frac{\lambda}{\log t}\Big)+\frac{\lambda}{t\log t}\Big(1-\frac{\lambda}{\log t}\Big)+\frac{\lambda}{t(\log t)^2}\Big]\times \\ & \times\Big(1-\frac{1}{t\log t}\Big)^{\alpha}\Big\{1+\frac{\log(1-\frac{1}{t\log t})}{\log t}\Big\}^{-\alpha\lambda}, \\ r(t) = {} & \frac{\alpha}{2t^\alpha e^{\alpha t}}\Big(1-\frac{\lambda}{\log t}\Big)^{\alpha-1}\times \\ & \times\Big[\Big(1+\frac{2}{t}\Big)\Big(1-\frac{\lambda}{\log t}\Big)+\frac{\lambda}{t\log t}\Big(1-\frac{\lambda}{\log t}\Big)+\frac{\lambda}{t(\log t)^2}\Big]\times \\ & \times\Big(1+\frac{1}{t\log t}\Big)^{\alpha}\Big\{1+\frac{\log(1+\frac{1}{t\log t})}{\log t}\Big\}^{-\alpha\lambda} \end{aligned}$$

for λ being a positive constant. As in Example 5.1, it could be shown without difficulty that all conditions of Theorem 1.1 are satisfied, so that the equation (5.4) has a n-SV_{e^t} solution $x(t)$ by Theorem 1.1. One such solution is $(\log t)^\lambda/\mathrm{t}$.

Example 5.4. Consider the following functional differential equation:

$$\big(t^\alpha\varphi(x'(t))\big)' = q(t)\varphi\big(x(te^{-\frac{1}{t}})\big)+r(t)\varphi\big(x(te^{\frac{1}{t}})\big), \quad t \geqq e^e, \tag{5.5}$$

where the functions $q(t)$ and $r(t)$ are given by

$$q(t) = \frac{\alpha\mu}{2t(\log t)^{\alpha+1}\log_2 t}\Big(1+\frac{\mu}{\log_2 t}\Big)^{\alpha-1}\Big(1+\frac{\mu-1}{\log_2 t}\Big)\times$$

$$\times \Big(1 - \frac{1}{t\log t}\Big)^{-\alpha} \Big\{ 1 + \frac{\log(1 - \frac{1}{t\log t})}{\log_2 t} \Big\}^{-\alpha\mu}$$

and

$$r(t) = \frac{\alpha\mu}{2t(\log t)^{\alpha+1}\log_2 t}\Big(1 + \frac{\mu}{\log_2 t}\Big)^{\alpha-1}\Big(1 + \frac{\mu-1}{\log_2 t}\Big)\times$$
$$\times \Big(1 + \frac{1}{t\log t}\Big)^{-\alpha} \Big\{ 1 + \frac{\log(1 + \frac{1}{t\log t})}{\log_2 t} \Big\}^{-\alpha\mu},$$

respectively, and μ is a positive constant. As in Example 5.2, it can be verified that all conditions of Theorem 1.1 are satisfied. Therefore, the equation (5.5) possesses a n-$\mathrm{RV}_{\log t}$ solution $x(t)$. One such solution is $x(t) = \log t(\log_2 t)^{\mu}$.

Acknowledgement

This research was supported by Grant-in-Aid for Scientific Research (C) (No. 23540218), the Ministry of Education, Culture, Sports, Science and Technology, Japan.

References

1. N. H. Bingham, C. M. Goldie, and J. L. Teugels, Regular variation. *Encyclopedia of Mathematics and its Applications*, 27. *Cambridge University Press, Cambridge*, 1987.
2. O. Došlý and P. Řehák, Half-linear differential equations. *North-Holland Mathematics Studies*, 202. *Elsevier Science B.V., Amsterdam*, 2005.
3. J. Jaroš and T. Kusano, Self-adjoint differential equations and generalized Karamata functions. *Bull. Cl. Sci. Math. Nat. Sci. Math.* No. 29 (2004), 25–60.
4. J. Jaroš, T. Kusano, and T. Tanigawa, Nonoscillatory half-linear differential equations and generalized Karamata functions. *Nonlinear Anal.* **64** (2006), No. 4, 762–787.
5. T. Kusano and V. Marić, On a class of functional differential equations having slowly varying solutions. *Publ. Inst. Math. (Beograd) (N.S.)* **80(94)** (2006), 207–217.
6. T. Kusano and V. Marić, Slowly varying solutions of functional differential equations with retarded and advanced arguments. *Georgian Math. J.* **14** (2007), No. 2, 301–314.
7. J. Manojlović and T. Tanigawa, Regularly varying solutions of half-linear differential equations with retarded and advanced arguments. (Submitted for publication)
8. V. Marić, Regular variation and differential equations. *Lecture Notes in Mathematics*, 1726. *Springer-Verlag, Berlin*, 2000.

(Received 16.06.2012)

Author's address:

Kumamoto University, Department of Mathematics, Faculty of Education, 2 – 40 – 1, Kurokami, Chuo-ku, Kumamoto, 860 – 8555, Japan.

e-mail: `tanigawa@educ.kumamoto-u.ac.jp`

Memoirs on Differential Equations and Mathematical Physics
Volume 57, 2012, 163–176

Hiroyuki Usami and Takuro Yoshimi

THE EXISTENCE OF SOLUTIONS OF INTEGRAL EQUATIONS RELATED TO INVERSE PROBLEMS OF QUASILINEAR ORDINARY DIFFERENTIAL EQUATIONS

Dedicated to Professor T. Kusano
on the occasion of his 80-th birthday anniversary

Abstract. We consider nonlinear integral equations related to inverse problems for quasilinear ordinary differential equations. We establish a global existence of solutions for them by means of the method of successive approximations and fractional calculus. Our results give a generalization of the recent result in [3].

2010 Mathematics Subject Classification. 34A55, 45G05.

Key words and phrases. Inverse problem, nonlinear integral equation, fractional calculus.

რეზიუმე. განხილულია კვაზიწრფივ ჩვეულებრივ დიფერენციალურ განტოლებათა შებრუნებულ ამოცანებთან დაკავშირებული არაწრფივი ინტეგრალური განტოლებები. მიმდევრობითი მიახლოების მეთოდისა და წილადური აღრიცხვის გამოყენებით დამტკიცებულია მათი გლობალური ამოხსნადობა.

1. Introduction

In the present paper we will establish a global existence theorem of solutions to integral equations of the form

$$T(p) = 2\Big(\frac{m}{m+1}\Big)^{1/(m+1)} \int_0^p \frac{dv}{\big(\int_v^p f(u)\,du\big)^{1/(m+1)}}, \quad 0 \le p \le R, \qquad (1.1)$$

where $m, R > 0$ are constants, $T(p)$ is a given positive function. We seek for a solution f, that is of the class $C[0,R]$ and $f(u) > 0$ on $(0,R]$. If we set $F(u) = \int_0^u f(\xi)d\xi$, then (1.1) is rewritten as

$$T(p) = 2\Big(\frac{m}{m+1}\Big)^{1/(m+1)} \int_0^p \big(F(p) - F(v)\big)^{-1/(m+1)}\,dv, \quad 0 \le p \le R. \qquad (1.2)$$

Though equation (1.1) has a complicated appearance, it arises naturally from the following inverse problem for quasilinear ordinary differential equations:

Problem 1.1. Let $T(p)$ be a given positive function on $[0,R]$. Determine a nonlinearity $f(u)$ of an ordinary differential equation

$$\big(|u'|^{m-1}u'\big)' + f(u) = 0 \qquad (1.3)$$

so that, for each $p \in (0,R]$, the solution $u(t) = u(t;p)$ of the equation with the stationary (maximal) value p has a half-period $T(p)$. (Note that when $f(0) = 0$ and $f(u)$ is extended to the interval $[-R,R]$ as an odd function, every solution of (1.3) oscillates and is periodic.)

In fact, we will explain how Problem 1.1 relates to equation (1.1). Let $p \in [0,R]$, and $u = u(t;p)$ be the solution of (1.3) satisfying the constraints in Problem 1.1, that is,

$$\begin{gathered}\big(|u'|^{m-1}u'\big)' + f(u) = 0 \ \text{ on } \ [0,T(p)],\\ u(0) = u(T(p)) = 0, \ \text{ and } \ u(t) > 0 \ \text{ in } \ (0,T(p)),\end{gathered}$$

and

$$\max_{[0,T(p)]} u = u(T(p)/2) = p \ \text{ and } \ u'(T(p)/2) = 0.$$

Here, the symmetry of u on $[0,T(p)]$ has been employed. It is easy to see that

$$T(p) = 2\int_0^p \Big(u'(0)^{m+1} - \frac{m+1}{m}F(v)\Big)^{-1/(m+1)}\,dv.$$

Since $u'(0)^{m+1} = (m+1)F(p)/m$, we can get

$$T(p) = B_0 \int_0^p \big(F(p) - F(v)\big)^{-1/(m+1)}\, dv, \tag{1.4}$$

where

$$B_0 = 2\Big(\frac{m}{m+1}\Big)^{1/(m+1)}.$$

Accordingly, (1.2) has been obtained.

We transform equation (1.4) further to the form which is easy to analyze. By the change of variables $s = F(v)$, $t = F(p)$, that is, $p = p(t) = F^{-1}(t)$, this equation is transformed to

$$T(p(t)) = B_0 \int_0^t \frac{p'(s)}{(t-s)^{1/(m+1)}}\, ds, \quad 0 \le t \le F(R).$$

By using the Riemann–Liouville integral operator, which will be defined later in the next section, this is rewritten as

$$T(p(t)) = B_0 \Gamma\Big(\frac{m}{m+1}\Big) I^{m/(m+1)} p'(t).$$

Here, Γ denotes the Gamma function. Applying the Riemann–Liouville integral operator $I^{1/(m+1)}$ to the both sides, we have

$$I^{1/(m+1)} T(p)(t) = B_0 \Gamma\Big(\frac{m}{m+1}\Big) p(t),$$

that is,

$$p(t) = \frac{1}{B_0 \Gamma(\frac{m}{m+1})}\, I^{1/(m+1)} T(p)(t), \tag{1.5}$$

or equivalently,

$$p(t) = \frac{\sin\big(\frac{\pi}{m+1}\big)}{\pi B_0} \int_0^t \frac{T(p(s))}{(t-s)^{1-1/(m+1)}}\, ds. \tag{1.6}$$

(Here we have employed the property (2.2) appearing in the next section.)

When $m = 1$ and T is Lipschitzian, it is shown conversely [1], [3] that a solution $p(t)$ of (1.5) (with $m = 1$) is necessarily differentiable and satisfies (1.1) (with $m = 1$). Thus solving of equation (1.1) (as well as of Problem 1.1) is equivalent to finding a solution of (1.5) if $m = 1$.

In the paper we will show that such a result still holds for equation (1.5) with $m > 0$. This is the main objective of the paper. In fact, we can establish the following result:

Theorem 1.2. *Let $T(r)$ be a Lipschitz continuous positive function defined on* $[0, R]$. *Then there exists a* (*unique*) *solution f of* (1.1) *that is continuous on* $[0, R]$ *and positive on* $(0, R]$.

When $m = 1$, this theorem reduces to [3, Theorem 1.2].

The paper is organized as follows. In Section 2 we construct a solution of equation (1.5) by the method of successive approximations as a preliminary result. The proof of Theorem 1.2 is given in Section 3. Other related results can be found in [2], [4], [6].

Though the arguments in the paper are based essentially on those in [3], the fact that $m \neq 1$ causes some difficulties, in particular, in the proof of Proposition 3.2.

2. Preliminary Results

As a first step, we must introduce the Riemann–Liouville integral operators. Let $\delta > 0$ be a constant. We define the integral operator I^δ by

$$I^\delta \phi(t) = \frac{1}{\Gamma(\delta)} \int_0^t \frac{\phi(s)}{(t-s)^{1-\delta}}\, ds \tag{2.1}$$

for $\phi \in C[0,R]$, where Γ is the Gamma function. We can show by interchange of the order of integration that

$$I^{\delta_1} I^{\delta_2} = I^{\delta_1+\delta_2} \ \text{ on } \ C[0,R] \tag{2.2}$$

for $\delta_1, \delta_2 > 0$. See, for example, [2], [5]. Note that this property has been already used in the Introduction.

Let us construct a continuous solution of integral equation (1.5), namely (1.6), by successive approximation.

Proposition 2.1. *Suppose that $T(r)$ is Lipschitz continuous on $[0,R]$, and $T(r) > 0$ there. Then there exists a positive number q and a continuous function $p(t)$ such that*

(i) *$p(t)$ satisfies equation (1.5) on $[0,q]$;*

(ii) *$p(0) = 0$ and $p(q) = R$;*

(iii) *$0 < p(t) < R$ for $t \in (0,q)$.*

Proof. Let L be a constant satisfying

$$\big|T(r_1) - T(r_2)\big| \le L|r_1 - r_2| \tag{2.3}$$

for $r_1, r_2 \in [0,R]$. Put

$$T^* = \max_{[0,R]} T(r), \quad T_* = \min_{[0,R]} T(r), \quad \text{and} \quad \widetilde{R} = T^* R / T_*.$$

We extend $T(r)$ (defined on $[0,R]$) to the continuous function on $[0,\widetilde{R}\,]$ so that $T(r) \equiv T(\widetilde{R})$ on $[R,\widetilde{R}\,]$. (In what follows, we may denote the extension by the same symbol T for simplicity.) Then T still satisfies (2.3) for $r_1, r_2 \in [0,\widetilde{R}\,]$, and $T_* \le T(r) \le T^*$ on $[0,\widetilde{R}\,]$. Furthermore, we set

$$A = \frac{(m+1)\sin\big(\frac{\pi}{m+1}\big)}{\pi B_0}\,, \quad \widetilde{t} = \Big(\frac{R}{AT_*}\Big)^{m+1},$$

and

$$\underline{p}(t) = AT_* t^{1/(m+1)}, \quad \overline{p}(t) = AT^* t^{1/(m+1)} \quad \text{on} \quad [0, \widetilde{t}\,].$$

Let us define the sequence $\{p_n(t)\}_{n=0}^{\infty}$ inductively by $p_0(t) = \underline{p}(t)$ and

$$p_n(t) = \frac{1}{B_0 \Gamma(\frac{m}{m+1})} I^{1/(m+1)} T(p_{n-1})(t), \quad n = 1, 2, \dots . \tag{2.4}$$

We will show that $p_n(t)$, $n = 1, 2, \dots$, are well-defined, and

$$\underline{p}(t) \le p_n(t) \le \overline{p}(t) \quad \text{on} \quad [0, \widetilde{t}\,], \tag{2.5}$$

for $n = 0, 1, 2, \dots$, and hence $0 \le p_n(t) \le \widetilde{R}$.

For $p_0(t)$, inequalities (2.5) are obviously true. Let $p_{n-1}(t)$ satisfy them. Since $T(p_{n-1}(t)) \le T^*$, we have

$$\begin{aligned}
p_n(t) &\le \frac{T^*}{B_0 \Gamma(\frac{m}{m+1})} I^{1/(m+1)}(1) = \\
&= \frac{T^*}{B_0 \Gamma(\frac{m}{m+1}) \Gamma(\frac{1}{m+1})} \int_0^t \frac{ds}{(t-s)^{1-1/(m-1)}} = \\
&= \frac{(m+1)T^*}{B_0 \frac{\pi}{\sin(\pi/(m+1))}} t^{1/(m+1)} = AT^* t^{1/(m+1)} = \\
&= \overline{p}(t) \le \widetilde{R}.
\end{aligned}$$

Thus $p_n(t)$ is well-defined and satisfies $p_n(t) \le \overline{p}(t)$. Similarly, we can show that $p_n(t) \ge \underline{p}(t)$. We therefore find that (2.5) is true for all $n = 0, 1, 2, \dots$.

It follows from (2.4) that

$$\begin{aligned}
\big|p_{k+1}(t) - p_k(t)\big| &\le \frac{1}{B_0 \Gamma(\frac{m}{m+1})} I^{1/(m+1)} \big|T(p_k) - T(p_{k-1})\big|(t) \le \\
&\le \frac{L}{B_0 \Gamma(\frac{m}{m+1})} I^{1/(m+1)} |p_k - p_{k-1}|(t) \le \\
&\le \Big(\frac{L}{B_0 \Gamma(\frac{m}{m+1})}\Big)^2 I^{2/(m+1)} |p_{k-1} - p_{k-2}|(t)
\end{aligned}$$

for $k = 2, 3, \dots$. Repeating this procedure, we can get

$$\big|p_{k+1}(t) - p_k(t)\big| \le \Big(\frac{L}{B_0 \Gamma(\frac{m}{m+1})}\Big)^k I^{k/(m+1)} |p_1 - p_0|(t).$$

Putting $M = \max\limits_{[0,\widetilde{t}\,]} |p_1 - p_0|$, we find that

$$\max_{[0,\widetilde{t}\,]} |p_{k+1} - p_k| \le \frac{(m+1)M}{k\Gamma(\frac{k}{m+1})} \Big(\frac{L}{B_0 \Gamma(\frac{m}{m+1})}\Big)^k \widetilde{t}^{\,k/(m+1)} \equiv c_k.$$

By the Stirling's formula $\Gamma(z) = \sqrt{2\pi}\, e^{-z} z^{z-1/2}(1 + O(1/z))$, as $|z| \to \infty$, we find that $c_{k+1}/c_k \to 0$ as $k \to \infty$.

Consequently, the sequence $\{p_n(t)\}$ converges to a limit function $\widetilde{p}(t) \in C[0, \widetilde{t}\,]$ uniformly on $[0, \widetilde{t}\,]$. Moreover, by (2.5) we know that

$$\underline{p}(t) \le \widetilde{p}(t) \le \overline{p}(t) \ \text{ on } \ [0, \widetilde{t}\,].$$

In particular, $\widetilde{p}(\widetilde{t}) \ge \underline{p}(\widetilde{t}) = R$. So, there is a $q \in (0, \widetilde{t})$ such that $\widetilde{p}(t) < R$ on $[0, q)$ and $\widetilde{p}(q) = R$. We define a function $p(t)$ by the restriction of $\widetilde{p}(t)$ on $[0, q] : p(t) = \widetilde{p}\big|_{[0,q]}(t)$. Then $p(t)$ satisfies the desired properties (i)–(iii). This completes the proof. □

3. Proof of Theorem 1.2

To see Theorem 1.2, we first prove that the solution $p(t)$ constructed in Proposition 2.1 is differentiable and $p'(t) > 0$ on $(0, q]$. The discussion is based on the fractional calculus associated with the Riemann–Liouville integral operators introduced in the Introduction by (2.1) and corresponding differential operators D^δ defined by $D^\delta = (d/dt)I^{1-\delta} = DI^{1-\delta}$, $D = d/dt$.

Below, we introduce the weighted Hölder spaces. Let $0 < b < \infty$, $0 \le \alpha \le 1$, and $\eta \in \mathbf{R}$. We put for $\phi \in C(0, b]$

$$|\phi|_\eta = \sup_{t \in (0,b]} t^{-\eta} |\phi(t)|$$

and

$$|\phi|_{\alpha,\eta} = \sup_{t,s \in (0,b],\, t \ne s} \frac{|t^{\alpha-\eta}\phi(t) - s^{\alpha-\eta}\phi(s)|}{|t-s|^\alpha},$$

and define the Banach space $(C^\alpha(0, b]_\eta, \|\cdot\|_{\alpha,\eta})$ by

$$C^\alpha(0, b]_\eta = \Big\{ \phi \in C(0, b] \ \Big| \ \|\phi\|_{\alpha,\eta} = |\phi|_\eta + |\phi|_{\alpha,\eta} < \infty \Big\}.$$

It is easy to prove that $C^{\alpha_1}[0, b)_{\eta_1} \supset C^{\alpha_2}[0, b)_{\eta_2}$ if $\alpha_1 \le \alpha_2$ and $\eta_1 \le \eta_2$. Note that if $\eta > 0$, then $\phi \in C^\alpha(0, b]_\eta$ is a continuous function and $\phi(0) = 0$.

Lemma 3.1. *Let $\eta > -1$.*

(i) *Let $0 \le \alpha < \alpha + \delta < 1$. Then $I^\delta : C^\alpha(0, b]_\eta \to C^{\alpha+\delta}(0, b]_{\eta+\delta}$ is a bounded operator.*

(ii) *Let $0 < \alpha < \alpha + \delta \le 1$. Then $D^\delta : C^{\alpha+\delta}(0, b]_{\eta+\delta} \to C^\alpha(0, b]_\eta$ is a bounded operator. For $\phi \in C^{\alpha+\delta}(0, b]_{\eta+\delta}$, the derivative $D^\delta\phi$ is expressed as*

$$D^\delta\phi(t) = \frac{1}{\Gamma(1-\delta)} \left(\frac{\phi(t)}{t^\delta} + \delta \int_0^t \frac{\phi(t) - \phi(s)}{(t-s)^{\delta+1}}\, ds \right).$$

The proof of this lemma can be found in [3]; and related results in [2].

Since equation (1.5) has somewhat complicated appearance, we will consider equation (3.1) below instead of equation (1.5) without loss of generality.

Proposition 3.2. *Let τ be a Lipschitz continuous function defined on an interval containing* 0 *and assume that* $\tau(0) > 0$. *Suppose, furthermore, that a continuous function $x(t)$ defined on $[0, b]$, $0 < b < \infty$, satisfies* $x(t) = I^{1/(m+1)}(\tau \circ x)(t)$, $0 \le t \le b$, *that is,*

$$x(t) = \frac{1}{\Gamma(\frac{1}{m+1})} \int_0^t \frac{\tau(x(s))}{(t-s)^{1-1/(m+1)}}\, ds, \quad 0 \le t \le b. \tag{3.1}$$

Then $x(t)$ is differentiable and $x'(t) > 0$ on $(0, b]$.

The following simple lemma is employed in proving Proposition 3.2:

Lemma 3.3. Let $k, l > 0$ be constants satisfying $k + l \le 1$. Then,

$$s^k(t^l - s^l) \le (t-s)^{k+l}, \quad t \ge s \ge 0.$$

Proof of Proposition 3.2. In the sequel, we denote a Lipschitz constant of τ by L. We may assume that $m > 1$, because the case where $0 < m \le 1$ can be treated similarly. The proof is divided into several steps.

Step 1. We show that

$$x \in C^{\beta+1/(m+1)}(0, b]_{1/(m+1)} \;\text{ for any } \beta, \;\; 0 \le \beta < 1/(m+1). \tag{3.2}$$

To see this we first note that $\tau \circ x \in C^0(0, b]_0$. So the fact that $x = I^{1/(m+1)}(\tau \circ x)$ and Lemma 3.1-(i) imply that $x \in C^{1/(m+1)}(0, b]_{1/(m+1)}$. Since the Lipschitz continuity implies that

$$\begin{aligned}\bigl|\tau(x(t)) - \tau(x(s))\bigr| &\le L|x(t) - x(s)| \le \\ &\le L|x|_{1/(m+1),1/(m+1)}|t-s|^{1/(m+1)} \le C_1|t-s|^{1/(m+1)}\end{aligned}$$

for some constant $C_1 > 0$, it follows that

$$\begin{aligned}&\Bigl|t^{1/(m+1)}\tau(x(t)) - s^{1/(m+1)}\tau(x(s))\Bigr| \le \\ &\quad\le t^{1/(m+1)}\bigl|\tau(x(t)) - \tau(x(s))\bigr| + |\tau(x(s))| \cdot \bigl|t^{1/(m+1)} - s^{1/(m+1)}\bigr| \le \\ &\le b^{1/(m+1)}C_1|t-s|^{1/(m+1)} + \bigl(\max_{[0,b]}|\tau \circ x|\bigr)|t-s|^{1/(m+1)} \le C_2|t-s|^{1/(m+1)}\end{aligned}$$

for some constant $C_2 > 0$. Thus, $\tau \circ x \in C^{1/(m+1)}(0, b]_0$, and hence, $\tau \circ x \in C^\beta(0, b]_0$ for any β, $0 \le \beta < 1/(m+1)$. Noting $x(t) = I^{1/(m+1)}(\tau \circ x)(t)$, we can show (3.2) by Lemma 3.1-(i).

Step 2. We show that

$$\tau(x(t)) - \tau(x(0)) \in C^{\beta+1/(m+1)}(0, b]_{1/(m+1)} \;\text{ for any } \beta, \;\; 0 \le \beta < 1/(m+1).$$

In fact, by Step 1, we know that for some $C_3 > 0$,

$$|x(t)| \le C_3 t^{1/(m+1)}, \tag{3.3}$$

and

$$\bigl|t^\beta x(t) - s^\beta x(s)\bigr| \le C_3|t-s|^{\beta+1/(m+1)} \tag{3.4}$$

for any β, $0 \le \beta < 1/(m+1)$. By the Lipschitz continuity of τ and (3.3), we find that

$$\big|\tau(x(t)) - \tau(x(0))\big| \le L|x(t) - x(0)| = L|x(t)| \le C_4 t^{1/(m+1)} \tag{3.5}$$

for some $C_4 > 0$. On the other hand, by the Lipschitz continuity of τ, (3.4), and (3.5), we find that

$$\begin{gathered}
\Big|t^\beta\big\{\tau(x(t)) - \tau(x(0))\big\} - s^\beta\big\{\tau(x(s)) - \tau(x(0))\big\}\Big| = \\
= \Big|t^\beta\big\{\tau(x(t)) - \tau(x(s))\big\} - (t^\beta - s^\beta)\big\{\tau(x(s)) - \tau(x(0))\big\}\Big| \le \\
\le Lt^\beta|x(t) - x(s)| + LC_3 s^{1/(m+1)}|t^\beta - s^\beta| = \\
= L\Big|\big\{t^\beta x(t) - s^\beta x(s)\big\} - (t^\beta - s^\beta)x(s)\Big| + LC_3 s^{1/(m+1)}|t^\beta - s^\beta| \le \\
\le L\Big(C_3|t-s|^{\beta+1/(m+1)} + C_3|t-s|^\beta s^{1/(m+1)}\Big) + LC_3 s^{1/(m+1)}|t^\beta - s^\beta| = \\
= 2LC_3|t-s|^{\beta+1/(m+1)} + LC_3 s^{1/(m+1)}|t^\beta - s^\beta|.
\end{gathered}$$

Employing Lemma 3.3, we can get

$$\Big|t^\beta\big\{\tau(x(t)) - \tau(x(0))\big\} - s^\beta\big\{\tau(x(s)) - \tau(x(0))\big\}\Big| \le 3LC_3|t-s|^{\beta+1/(m+1)}.$$

Step 3. We show that

$$x \in C^{\beta+1/(m+1)}(0,b]_{1/(m+1)} \quad \text{for any} \quad \beta, \quad 0 \le \beta < 1 - 1/(m+1). \tag{3.6}$$

Since the constant $\tau(x(0))$ is of the class $C^{\beta+1/(m+1)}(0,b]_0$ and $C^{\beta+1/(m+1)}(0,b]_{1/(m+1)} \subset C^{\beta+1/(m+1)}(0,b]_0$, we find by Step 2 that $\tau \circ x \in C^{\beta+1/(m+1)}(0,b]_0, 0 \le \beta < 1/(m+1)$. Thus, by Lemma 3.1-(i) again,

$$\begin{gathered}
x = I^{1/(m+1)}(\tau \circ x) \in C^{\beta_1+2/(m+1)}(0,b]_{1/(m+1)}, \\
0 \le \beta_1 < \min\Big\{\frac{m-1}{m+1}, \frac{1}{m+1}\Big\}.
\end{gathered} \tag{3.7}$$

So, if $1 < m \le 2$, then we have established (3.6).

Below, we suppose that $m > 2$. Then from (3.7), we get $x \in C^{\beta_2+1/(m+1)}(0,b]_{1/(m+1)}$, $0 \le \beta_2 < 2/(m+1)$. By the argument developed in Step 2, we find that $\tau(x(t)) - \tau(x(0)) \in C^{\beta_2+1/(m+1)}(0,b]_{1/(m+1)}$, $0 \le \beta_2 < 2/(m+1)$; and hence $\tau(x(t)) \in C^{\beta_2+1/(m+1)}(0,b]_0$, $0 \le \beta_2 < 2/(m+1)$. Again, applying Lemma 3-(i), we have

$$\begin{gathered}
x = I^{1/(m+1)}(\tau \circ x) \in C^{\beta_2+2/(m+1)}(0,b]_{1/(m+1)}, \\
0 \le \beta_2 < \min\Big\{\frac{m-1}{m+1}, \frac{2}{m+1}\Big\}.
\end{gathered}$$

So, if $2 < m \le 3$, then we have established (3.6). If $m > 3$, then $x \in C^{\beta_3+1/(m+1)}(0,b]_{1/(m+1)}$, $0 \le \beta_3 < 3/(m+1)$.

Continuing this procedure, we finally reach to the relation

$$x \in C^{\widetilde{\beta}+1/(m+1)}(0,b]_{1/(m+1)}, \quad 0 \le \widetilde{\beta} < \frac{[m]}{m+1},$$

that is,

$$x \in C^{\widetilde{\beta}+1/(m+1)}(0,b]_{1/(m+1)}, \quad 0 \le \widetilde{\beta} < \frac{m-1}{m+1},$$

where $[m]$ denotes the largest integer, not exceeding m, as usual. Then, one more application of the argument in Step 2 and Lemma 3.1-(i) show that (3.6) is valid.

Step 4. We show that $x(t)$ is differentiable on $(0,b]$, and

$$t^{m/(m+1)}x'(t) = \frac{\tau(0)}{\Gamma(\frac{1}{m+1})} + O\big(t^{m/(m+1)}\big) \ \text{ as } \ t \to +0. \tag{3.8}$$

Therefore, $x'(t) > 0$ near $+0$.

To see this, we notice by Step 3 and the observation in Step 2 that

$$\tau \circ x(t) - \tau \circ x(0) \in C^{\beta+1/(m+1)}(0,b]_{1/(m+1)}, \quad 0 \le \beta < 1 - 1/(m+1), \tag{3.9}$$

and accordingly, $\tau \circ x \in C^{\beta+1/(m+1)}(0,b]_0$. Then, by Lemma 3.1-(ii), $D^{m/(m+1)}(\tau \circ x) \equiv DI^{1-m/(m+1)}(\tau \circ x)$ is well-defined; and so, $x' = DI^{1/(m+1)}(\tau \circ x)$ is well-defined, and

$$\begin{aligned} x' &= DI^{1-m/(m+1)}(\tau \circ x) \equiv \\ &\equiv D^{m/(m+1)}(\tau \circ x) \in C^{\beta-(m-1)/(m+1)}(0,b]_{-m/(m+1)}. \end{aligned}$$

Therefore, we obtain

$$I^{m/(m+1)}x' = \tau \circ x \in C^{\beta+1/(m+1)}(0,b]_0, \tag{3.10}$$

and

$$\begin{aligned} x'(t) &= D^{m/(m+1)}\big((\tau \circ x)(t) - (\tau \circ x)(0)\big) + D^{m/(m+1)}((\tau \circ x)(0)) = \\ &= D^{m/(m+1)}\big((\tau \circ x)(t) - (\tau \circ x)(0)\big) + \frac{\tau(0)}{\Gamma(\frac{1}{m+1})}\, t^{-m/(m+1)}, \end{aligned}$$

by Lemma 3.1-(ii). Since

$$D^{m/(m+1)}\big((\tau \circ x)(t) - (\tau \circ x)(0)\big) \in C^{\beta-(m-1)/(m+1)}(0,b]_0$$

by (3.9), we have

$$D^{m/(m+1)}\big((\tau \circ x)(t) - (\tau \circ x)(0)\big) = O(1) \ \text{ as } \ t \to +0.$$

This implies the validity of (3.8).

Step 5. Finally, we show that $x'(t) > 0$ on $(0,b]$.

The proof of this step is essentially the same as that of [3, Step 2 of the proof of Proposition 3.2]. To see this, let $0 < \varepsilon < 1/(m+1)$ and choose

$\beta \in (0, 1 - 1/(m+1))$, so that $1 - 1/(m+1) - \beta < \varepsilon$. (For example, $\beta = 1 - 1/(m+1) - \varepsilon/2$.) We get from (3.10) that

$$D^{1/(m+1)-\varepsilon} x' = D^{1-\varepsilon}(\tau \circ x). \tag{3.11}$$

By Lemma 3.1-(ii), the left-hand side of (3.11) can be rewritten as

$$D^{1/(m+1)-\varepsilon} x'(t) =$$

$$= \frac{1}{\Gamma(1 - \frac{1}{m+1} + \varepsilon)} \left(\frac{x'(t)}{t^{1/(m+1)-\varepsilon}} + \Big(\frac{1}{m+1} - \varepsilon\Big) \int_0^t \frac{x'(t) - x'(s)}{(t-s)^{1/(m+1)+1-\varepsilon}} \, ds \right).$$

To see $x'(t) > 0$ on $(0, b]$ by contradiction, we assume the contrary. Since $x'(t) > 0$ near the origin, there is an $a \in (0, b]$ such that $x'(t) > 0$ on $(0, a)$ and $x'(a) = 0$. Noting that

$$\int_0^a \frac{x'(s)}{(a-s)^{1+1/(m+1)-\varepsilon}} \, ds >$$

$$> a^{-1-1/(m+1)+\varepsilon} \int_0^a x'(s) \, ds = a^{-1-1/(m+1)} a^{\varepsilon} x(a),$$

we can find a constant $\rho > 0$ independent of ε such that

$$D^{1/(m+1)-\varepsilon} x'(t) \big|_{t=a} =$$

$$= -\frac{1/(m+1) - \varepsilon}{\Gamma(1 - \frac{1}{m+1} + \varepsilon)} \int_0^a \frac{x'(s)}{(a-s)^{1+1/(m+1)-\varepsilon}} \, ds \le -\rho. \tag{3.12}$$

On the other hand, the right-hand side of (3.11) with $t = a$ can be rewritten as

$$D^{1-\varepsilon}(\tau \circ x)(a) = \frac{1}{\Gamma(\varepsilon)} \left\{ \frac{\tau(x(a))}{a^{1-\varepsilon}} + (1-\varepsilon) \int_0^a \frac{\tau(x(a)) - \tau(x(s))}{(a-s)^{2-\varepsilon}} \, ds \right\} \equiv$$

$$\equiv \frac{1}{\Gamma(\varepsilon)} \left\{ \frac{\tau(x(a))}{a^{1-\varepsilon}} + (1-\varepsilon) \int_0^{a-\varepsilon} + (1-\varepsilon) \int_{a-\varepsilon}^a \right\}.$$

We observe that

$$(1-\varepsilon) \int_0^{a-\varepsilon} = (1-\varepsilon)\tau(x(a)) \int_0^{a-\varepsilon} \frac{ds}{(a-s)^{2-\varepsilon}} - (1-\varepsilon) \int_0^{a-\varepsilon} \frac{\tau(x(s))}{(a-s)^{2-\varepsilon}} \, ds =$$

$$= \tau(x(a))(\varepsilon^{\varepsilon-1} - a^{\varepsilon-1}) - (1-\varepsilon) \int_0^{a-\varepsilon} \frac{\tau(x(s))}{(a-s)^{2-\varepsilon}} \, ds,$$

so we get

$$D^{1-\varepsilon}(\tau\circ x)(a)=\frac{\varepsilon^{\varepsilon-1}\tau(x(a))}{\Gamma(\varepsilon)}-\frac{1-\varepsilon}{\Gamma(\varepsilon)}\int\limits_0^{a-\varepsilon}\frac{\tau(x(s))}{(a-s)^{2-\varepsilon}}\,ds+$$
$$+\frac{1-\varepsilon}{\Gamma(\varepsilon)}\int\limits_{a-\varepsilon}^{a}\frac{\tau(x(a))-\tau(x(s))}{(a-s)^{2-\varepsilon}}\,ds\equiv$$
$$\equiv J_1(\varepsilon)-J_2(\varepsilon)+J_3(\varepsilon),$$

where $J_i(\varepsilon)$, $i=1,2,3$, are defined naturally by the last equality. Below we will estimate each $J_i(\varepsilon)$ separately.

It is easy to see that

$$J_1(\varepsilon)=\frac{\varepsilon^{\varepsilon}}{\Gamma(\varepsilon+1)}\,\tau(x(a))\longrightarrow\tau(x(a))\ \text{ as }\ \varepsilon\to+0.$$

By the change of variables, the term $J_2(\varepsilon)$ is expressed as

$$J_2(\varepsilon)=\frac{(1-\varepsilon)\varepsilon^{\varepsilon}}{\Gamma(\varepsilon+1)}\int\limits_1^{a/\varepsilon}\frac{\tau(x(a-\varepsilon v))}{v^{2-\varepsilon}}\,dv=\frac{(1-\varepsilon)\varepsilon^{\varepsilon}}{\Gamma(\varepsilon+1)}\int\limits_1^{\infty}h_\varepsilon(v)\,dv,$$

where

$$h_\varepsilon(v)=\begin{cases}\tau(x(a-\varepsilon v))/v^{2-\varepsilon} & \text{if }\ 1\le v\le a/\varepsilon,\\ 0 & \text{if }\ v\ge a/\varepsilon.\end{cases}$$

Since $|h_\varepsilon(v)|\le Cv^{-2}$ on $[1,\infty)$ for some constant $C>0$, and $\lim\limits_{\varepsilon\to+0}h_\varepsilon(v)=\tau(x(a))/v^2$, the dominated convergence theorem implies that

$$J_2(\varepsilon)\longrightarrow\int\limits_1^{\infty}\frac{\tau(x(a))}{v^2}\,dv=\tau(x(a))\ \text{ as }\ \varepsilon\to+0.$$

Finally, let us examine $J_3(\varepsilon)$. Recall that $x'\in C^{\beta-(m-1)/(m+1)}(0,b]_{-m/(m+1)}$ for any β, $0\le\beta<1-1/(m+1)$. Hence

$$t^{m/(m+1)}|x'(t)|\le C_4$$

and

$$\left|t^{\beta+1/(m+1)}x'(t)-s^{\beta+1/(m+1)}x'(s)\right|\le C_4|t-s|^{\beta-(m-1)/(m+1)}$$

for some constant $C_4>0$. Therefore, for $t_0>0$, we have

$$|x'(t)-x'(s)|\le t^{-\beta-1/(m+1)}\left|t^{\beta+1/(m+1)}x'(t)-s^{\beta+1/(m+1)}x'(s)\right|+$$
$$+\,t^{-\beta-1/(m+1)}|x'(s)|\cdot|t-s|^{\beta+1/(m+1)}\le C(t_0)|t-s|^{\beta},\ \ t,s\in[t_0,b],$$

where $C(t_0) > 0$ is a constant depending on t_0. Thus for $s \le a$ near a, we have

$$\begin{aligned} \big|\tau(x(a)) - \tau(x(s))\big| &\le L|x(a) - x(s)| \le \\ &\le L \int_s^a |x'(v) - x'(a)|\, dv \le LC_5 \int_s^a (a-v)^\beta\, dv = \frac{LC_5}{\beta+1}\,(a-s)^{\beta+1} \end{aligned}$$

for some $C_5 > 0$. Consequently,

$$\begin{aligned} |J_3(\varepsilon)| &\le \frac{(1-\varepsilon)LC_5}{(\beta+1)\Gamma(\varepsilon)} \int_{a-\varepsilon}^a (a-s)^{\beta+\varepsilon-1}\, ds = \\ &= \frac{(1-\varepsilon)LC_5}{(\beta+1)(\beta+\varepsilon)\Gamma(\varepsilon+1)}\, \varepsilon^{\beta+1+\varepsilon} \longrightarrow 0 \ \text{ as } \ \varepsilon \to +0. \end{aligned}$$

Hence $\lim\limits_{\varepsilon\to+0} D^{1-\varepsilon}(\tau\circ x)(t)\big|_{t=a} = 0$. By (3.11) this contradicts (3.12). So, $x'(t) > 0$ on $(0, b]$.

The proof of Proposition 3.2 is complete. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $p(t)$ be the solution of equation (1.5) constructed in Proposition 2.1. By Proposition 3.2 we know that $p(t)$ is differentiable, $p'(t) > 0$ on $(0, q]$, $p' \in C(0, q]$, and $p(t)$ satisfies the asymptotic formula

$$t^{m/(m+1)} p'(t) = C_0 + O(t^{m/(m+1)}) \ \text{ as } \ t \to +0$$

for some constant $C_0 > 0$. Applying $I^{m/(m+1)}$ to the both sides of (1.5), we get

$$I^1 T(p)(t) = B_0 \Gamma\Big(\frac{m}{m+1}\Big) I^{m/(m+1)} p(t) = B_0 \int_0^t (t-s)^{-1/(m+1)} p(s)\, ds.$$

By the integration by parts, we have

$$I^1 T(p)(t) = \frac{(m+1)B_0}{m} \int_0^t (t-s)^{m/(m+1)} p'(s)\, ds$$

Differentiating this, we conclude that

$$T(p(t)) = B_0 \int_0^t \frac{p'(s)}{(t-s)^{1/(m+1)}}\, ds, \quad 0 \le t \le q \tag{3.13}$$

holds. Since $p'(t) > 0$ on $(0, q]$, $p = p(t)$ has the inverse function defined on $[0, R]$, which we denote by $t = F(p)$. Then F is differentiable on $[0, R]$ and satisfies $F'(u) = 1/p'(F(u))$. By putting $t = F(p)$ and $s = F(v)$ in (3.13), we have

$$T(p) = B_0 \int_0^p (F(p) - F(v))^{-1/(m+1)}\, dv, \quad 0 \le p \le R.$$

This means that F satisfies (1.2). So, the function f given by $f(u) = 1/p'(F(u))$ on $[0, R]$ gives the solution of integral equation (1.1). This completes the proof. □

Acknowledgement

The authors would like to express their sincere thanks to Professor Yutaka Kamimura of Tokyo University of Marine Science and Technology.

References

1. B. Alfawicka, Inverse problems connected with periods of oscillations described $\ddot{x} + g(x) = 0$. *Ann. Polon. Math.* **44** (1984), No. 3, 297–308.
2. Y. Kamimura, Sekibun hōteishiki: gyakumondai no shiten kara. (Japanese) *Kyōritsushuppan, Tokyo*, 2001.
3. Y. Kamimura, Global existence of a restoring force realizing a prescribed half-period. *J. Differential Equations* **248** (2010), No. 10, 2562–2584.
4. T. Ohsawa and T. Takiguchi, Abel-type integral transforms and the exterior problem for the Radon transform. *Inverse Probl. Sci. Eng.* **17** (2009), No. 4, 461–471.
5. S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional integrals and derivatives. Theory and applications. Edited and with a foreword by S. M. Nikol'skii. Translated from the 1987 Russian original. Revised by the authors. *Gordon and Breach Science Publishers, Yverdon*, 1993.
6. R. Schaaf, Global solution branches of two-point boundary value problems. *Lecture Notes in Mathematics*, 1458. *Springer-Verlag, Berlin*, 1990.

(Received 21.06.2012)

Authors' addresses:

Hiroyuki Usami

Department of Mathematical and Design Engineering, Faculty of Engineering, Gifu University, Gifu, 501-1193 Japan.

e-mail: `husami@gifu-u.ac.jp`

Takuro Yoshimi

Current affiliation is: Nippon Kouatsu Electric Co., Ltd., 8-288 Hiragiyama, Obu, 474-0053, Japan.

Department of Mathematical and Design Engineering, Graduate School of Engineering, Gifu University, Gifu, 501-1193 Japan.

Mem. Differential Equations Math. Phys. 57 (2012)

CONTENTS

მემუარები დიფერენციალურ განტოლებებსა და მათემატიკურ ფიზიკაში 57 (2012)

შ ი ნ ა ა რ ს ი