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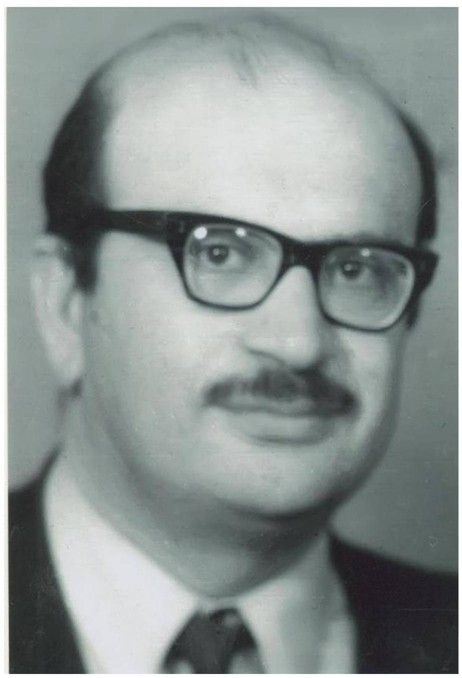
On the Occasion of Givi Khuskivadze's 80th Birthday Anniversary



Givi Khuskivadze, the well-known Georgian mathematician, doctor of sciences in physics and mathematics, has turned 80.

Givi Khuskivadze was born on March 28, 1932. In 1950 he finished with Gold medal the Tbilisi 8th Secondary school and in 1955 he graduated with honour from the Tbilisi State University, faculty of mechanics and mathematics. In 1955–1958 he went through a full post-graduate course at the same University. In 1963 he defended first his candidate's thesis and then, in 1999, he became doctor of phys.-math. sciences. Since 1958, all his life was connected with A. Razmadze Mathematical Institute, where he held posts of junior and then of senior and leading researcher. Just within the precincts of that Institute G. Khuskivadze carried out his remarkable investigations in fundamental problems of the real and complex analysis which earned him immense authority and respect among specialists engaged in this area.

In G. Khuskivadze's personality were harmoniously combined brilliant talent of a widely and well-educated person, on the one hand, and rare modesty, sensitiveness and honesty, on the other hand. For all these qualities G. Khuskivadze won sincere respect of all his colleagues and friends.

Givi Khuskivadze passed away on December 5, 2011, not having reached only three months to see his 80th anniversary.

Givi Khuskivadze's works deal with various actual problems of the theory of functions. The profound and fine results on the Cauchy type integrals and connected with them singular integrals greatly contributed to further development and elaboration of methods of the theory of the Cauchy type integrals. Of special interest are his works in such important parts of the function theory as the theory of integrals, boundary value problems for analytic and harmonic functions, conformal mappings, etc.

Here we cite an incomplete list of the results obtained by Givi Khuskivadze in a course of his scientific activity.

Since the functions defined by the Cauchy singular integral with density summable in the Lebesgue sense may turn out to be not summable, there naturally arose the question to generalize the notion of the Lebesgue integral in a way that the above-mentioned functions become integrable in that new sense. There were suggested and studied various generalizations of the notion of the Lebesgue integral, the so-called B and A integrals (see, e.g., A. Kolmogorov, E. Titchmarsh, P. Ul'yanov). These integrals, despite the fact that they provide us with a positive solution to the above-posed question, they have certain drawbacks. For example, there exists the A-integrable on the interval $[a,b]$ function which is A-integrable on neither interval $(\alpha,\beta)\subset[a,b]$.

G. Khuskivadze suggested a simple, free from any exotic drawbacks, easily observable generalization of the Lebesgue integral which he called an $\widetilde{L}$-integral. The basic properties inherent in the $\widetilde{L}$-integrability were revealed, having thus constructed such an extension of the Lebesgue integral which allowed one to get a complete picture of that part of the theory of integral which is connected with the notion of the singular integral in the Cauchy sense. In particular, the representation of the Cauchy type $\widetilde{L}$-integral taken with respect to a closed curve by means of the Cauchy $\widetilde{L}$-integral, is obtained. Besides, the cases are embraced in which the lines of integration are taken from a wide classes of curves, and equalities connecting generalized integrals with the Lebesgue integral are obtained.

The $\widetilde{L}$-integral finds its effective applications. For example, if the Riemann boundary value problem is treated in the assumption that an unknown function is representable by the Cauchy–Lebesgue type integral, then in the framework of the Lebesgue integral, it is impossible to get an acceptable picture of solvability. However, if the statement of the problem is replaced by the requirement for a solution of the problem to be representable by the Cauchy type $\widetilde{L}$-integral, then the Riemann problem can be solved in a standard way [23], [24].

The notion of the $\widetilde{L}$-integral has been successfully used for investigation of properties of conformal mappings of a unit circle onto a simply-connected domain. (In particular, the criterion for the representability of that integral as an exponential function of the Cauchy type $\widetilde{L}$-integral is established.) [24], [28].

Of great importance is Givi Khuskivadze's contribution to the investigation of problems dealt with the boundedness in Lebesgue spaces of the operator generated by the Cauchy singular integral, and also the properties of the Cauchy type integrals ([1]–[10], [37]). These problems of theoretical and applied interest always attracted his attention. A number of results in this direction have been obtained in collaboration with V. Paatashvili ([11], [12], [18], [19], [22], [31]).

They revealed the necessary and sufficient conditions for the boundedness of the Cauchy singular operator S_Γ from $L^p(\Gamma)$ to $L^q(\Gamma)$, $p \geq q$, $p > 1$, when Γ is a countable set of concentric circumferences whose sum of lengths is finite ([11], [18], [21]).

Next, it has been proved that for the boundedness of the operator S_Γ in the spaces $L^p(\Gamma)$ it is necessary that the condition $\sup\limits_{\zeta\in\Gamma} \ell(\zeta, r) = O(r)$ be fulfilled; $\ell(\zeta, r)$ here is a linear measure of that part of the curve Γ which finds itself in a circle with center $\zeta \in \Gamma$, of radius r. The fulfilment of this condition is sufficient for curves of a special class which contains the curves Γ_0 such that S_{Γ_0} is bounded in neither class $L^p(\Gamma_0)$, $p > 1$. These results were obtained long before the G. David's work in which he proved that the above-cited condition is necessary and sufficient for the boundedness of the operator S_Γ in the spaces $L^p(\Gamma)$, $p > 1$, for an arbitrary rectifiable curve Γ [12], [23], [27].

G. Khuskivadze constructed examples illustrating how the geometry of the curve Γ affects the boundedness of the operator S_Γ in the Lebesgue spaces. He pointed out: (i) a curve Γ_1 for which S_{Γ_1} is bounded from $L^p(\Gamma_1)$ to $L^{p-\varepsilon}(\Gamma)$ for any $p > 1$ and $\varepsilon \in (0, p - \varepsilon)$, but is unbounded in $L^p(\Gamma)$; (ii) a curve Γ_2 for which even the function $S_{\Gamma_2}(1)$ is summable in neither positive power. It should here be noted that G. Khuskivadze was very skillful in finding "examples in essence" which focused attention on different aspects of the problem under consideration ([10]).

One range of Givi Khuskivadze's works are devoted to the investigation of properties of conformal mappings of simply- connected domains (a portion of these works is carried out jointly with V. Paatashvili).

A new step in studying these properties is application of the methods and results of the theory of the Riemann boundary value problem to the case of a unit circle. This method allowed to obtain new, simple proofs of the well-known theorems on conformal mappings (Lindeloff, Cellogue, Smirnov, Warschawskii) and also to establish some important facts dealt with these mappings. Namely, depending on the geometry boundary of the domain D, it became possible to reveal those Hardy classes to which the derivative $z'(w)$ of the conformal mapping of a circle onto the domain D belongs, and also those values of numbers p for which the operator $Tf = z'S_\Gamma \frac{f}{z'}$ is bounded in the space $L^p(\Gamma)$ ([13], [14], [20]–[28], [32]).

Of the results mentioned above, a more significant particular case is the representation of $z'(w)$ in the case of arbitrary piecewise smooth curves which generalizes and complements Warschawski's theorem on the behaviour of $z'(w)$ in the case of piecewise Lyapunov curves.

The validity of these investigations is evident owing to a role that conformal mappings play in the analysis. Special attention earns one more of their dignities. They are especially useful in studying the Riemann and Riemann–Hilbert boundary value problems under new assumptions regarding the boundary D. (The Riemann's problem, well-known in the case of a circle, made it possible to establish properties of z' for a wide class of domains and then, using these properties, to investigate boundary value problems for these classes of domains.)

Owing to the developments in the investigation of properties of the Cauchy type integrals with density from $L^p(\Gamma;\omega)$, the process G. Khuskivadze took an active part in, it became possible to make progress in studying boundary value problems of the theory of analytic functions, when a solution is required to be represented by the Cauchy type integral with density from $L^p(\Gamma;\omega)$. It also became possible to move forward in investigating the problems under rather general assumptions with regard to Γ, ω and to the coefficients in the boundary conditions.

In G. Khuskivadze's works dealt with a simply-connected domain, the Riemann–Hilbert problem is reduced, by using the well-known N. Muskhelishvili method, to the Riemann problem in which the principal coefficient is a product of three functions which depend, respectively, on the curve Γ, weight ω and on the coefficients of the initial problem. This allows one to obtain conditions for the solvability under various assumptions regarding the data of the problem and to construct solutions themselves. In the case of doubly-connected domains, the problem is successfully investigated by means of its reduction to an equivalent system of singular integral equations ([15], [17], [23]).

Special attention is focused on the investigation of the Dirichlet problem. Here are revealed such interesting classes of harmonic functions in which the problem in domains with a piecewise smooth boundary (depending on the angle sizes at angular points) may turn out to be uniquely solvable, ambiguously solvable, or unsolvable at all ([16], [23], [33], [35], [36], [38], [40]).

A rather complete picture of the solvability is obtained for Zaremba's mixed boundary value problem, as well, i.e., for that in which we seek for a function, harmonic in the domain D, when the value of an unknown function is given on one part of the boundary and the value of its normal derivative is given on the remaining part of the boundary ([29], [30], [34], [35], [39]).

The results presented above show that Givi Khuskivadze was a person of brilliant talent and subtle mathematical mentality. G. Khuskivadze's remarkable personality will forever remain in the hearts of his friends and colleagues.

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V. Kokilashvili, V. Paatashvili, T. Oziashvili

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ON TWO-POINT SINGULAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF LINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. The two-point boundary value problem is considered for the system of linear generalized ordinary differential equations with singularities on a non-closed interval. The constant term of the system is a vector-function with bounded total variations components on the closure of the interval, and the components of the matrix-function have bounded total variations on every closed interval from this interval.

The general sufficient conditions are established for the unique solvability of this problem in the case where the system has singularities. Singularity is understand in a sense the components of the matrix-function corresponding to the system may have unbounded variations on the interval.

Relying on these results the effective conditions are established for the unique solvability of the problem.

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Key words and phrases. Systems of linear generalized ordinary differential equations, singularity, the Lebesgue–Stiltjes integral, two-point boundary value problem.

რეზიუმე. განზოგადებულ ჩვეულებრივ წრფივ დიფერენციალურ განტოლებათა სისტემისათვის სინგულარობებით არაჩაკეტილ ინტერვალზე განხილულია ორწერტილოვანი სასაზღვრო ამოცანა. ამ სისტემის თავისუფალი წევრი სასრული ვარიაციის კომპონენტებიანი ვექტორული ფუნქციაა აღნიშნული ინტერვალის ჩაკეტვაზე, ხოლო მატრიცული ფუნქციის კომპონენტებს კი აქვს სასრული ვარიაციები ყოველ ჩაკეტილ სეგმენტზე ამ ინტერვალიდან.

მიღებულია ამ ამოცანის ცალსახად ამოხსნადობის ზოგადი საკმარისი პირობები, როცა სისტემას გააჩნია სინგულარობები. სინგულარობა გაიგება იმ აზრით, რომ სისტემის შესაბამისი მატრიცული ფუნქციის კომპონენეტებს შეიძლება ჰქონდეს შემოუსაზღვრელი ვარიაციები განსახილველ შუალედზე.

ამ შედეგებზე დაყრდნობით დადგენილია ამ ამოცანის ცალსახად ამოხსნადობის ეფექტური პირობები.

1. Statement of the Problem and Basic Notation

In the present paper, for a system of linear generalized ordinary differential equations with singularities

$$dx(t) = dA(t) \cdot x(t) + df(t) \tag{1.1}$$

we consider the two-point boundary value problem

$$x_i(a+) = 0 \;\; (i = 1, \dots, n_0), \;\; x_i(b-) = 0 \;\; (i = n_0 + 1, \dots, n), \tag{1.2}$$

where $-\infty < a < b < +\infty$, $n_0 \in \{1, \dots, n\}$, $x_1, \dots, x_n$ are the components of the desired solution x, $n_0 \in \{1, \dots, n\}$, $f : [a, b] \to \mathbb{R}^n$ is a vector-function with bounded total variation components, and $A :]a, b[\to \mathbb{R}^{n\times n}$ is a matrix-function with bounded total variation components on every closed interval from the interval $]a, b[$.

We investigate the question of unique solvability of the problem (1.1), (1.2), when the system (1.1) has singularities. Singularity is understand in a sense that the components of the matrix-function A may have unbounded variation on the closed interval $[a, b]$, in general. On the basis of this theorem we obtain effective criteria for the solvability of this problem.

Analogous and related questions are investigated in [17–24] and [26] (see also references therein) for the singular two-point and multipoint boundary value problems for linear and nonlinear systems of ordinary differential equations, and in [1, 3, 6, 8, 10] (see also references therein) for regular two-point and multipoint boundary value problems for systems of linear and nonlinear generalized differential equations. As for the two-point and multipoint singular boundary value problems for generalized differential systems, they are little studied and, despite some results given in [12] and [13] for two-point singular boundary value problem, their theory is rather far from completion even in the linear case. Therefore, the problem under consideration is actual.

To a considerable extent, the interest in the theory of generalized ordinary differential equations has been motivated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see e.g. [1–13, 15, 16, 25, 27–29] and references therein).

Throughout the paper, the use will be made of the following notation and definitions.

$\mathbb{R} =] - \infty, +\infty[$; $R_+ = [0, +\infty[$; $[a, b]$, $]a, b[$ and $]a, b]$, $[a, b[$ are, respectively, closed, open and half-open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{il})_{i,l=1}^{n,m}$ with the norm

$$\|X\| = \sum_{i,l=1}^{n,m} |x_{il}|.$$

$\mathbb{R}_+^{n\times m} = \big\{(x_{il})_{i,l=1}^{n,m} : \; x_{il} \geq 0 \; (i = 1, \dots, n; \; l = 1, \dots, m)\big\}$.

$O_{n\times m}$ (or O) is the zero $n \times m$ matrix.

If $X=(x_{il})_{i,l=1}^{n,m}\in\mathbb{R}^{n\times m}$, then $|X|=(|x_{il}|)_{i,l=1}^{n,m}$.

$\mathbb{R}^n=R^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$; $\mathbb{R}_+^n=\mathbb{R}_+^{n\times 1}$.

If $X\in\mathbb{R}^{n\times m}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n\times n$-matrix; δ_{il} is the Kroneker symbol, i.e., $\delta_{ii}=1$ and $\delta_{il}=1$ for $i\neq l$ $(i,l=1,\dots,n)$.

$\bigvee\limits_c^d(X)$, where $a<c<d<b$, is the variation of the matrix-function $X:\,]a,b[\,\to\mathbb{R}^{n\times m}$ on the closed interval $[c,d]$, i.e., the sum of total variations of the latter components x_{il} $(i=1,\dots,n;\ l=1,\dots,m)$ on this interval; if $d<c$, then $\bigvee\limits_c^d(X)=-\bigvee\limits_d^c(X)$; $V(X)(t)=(v(x_{il})(t))_{i,l=1}^{n,m}$, where $v(x_{il})(c_0)=0$, $v(x_{il})(t)=\bigvee\limits_{c_0}^t(x_{il})$ for $a<t<b$, and $c_0=(a+b)/2$.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X:\,]a,b[\,\to\mathbb{R}^{n\times m}$ at the point $t\in\,]a,b[$ (we assume $X(t)=X(a+)$ for $t\le a$ and $X(t)=X(b-)$ for $t\ge b$, if necessary).

$d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all matrix-functions of bounded variation $X:[a,b]\to\mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X)<+\infty$);

$\|X\|_s=\sup\big\{\|X(t)\|:\ t\in[a,b]\big\}$, $\|X\|_v=\|X(a)\|+\bigvee\limits_a^b(X)$;

$\mathrm{BV}_s([a,b],\mathbb{R}^{n\times m})$ is the normed space $(\mathrm{BV}([a,b],\mathbb{R}^{n\times m}),\|\cdot\|_s)$;

$\mathrm{BV}_v([a,b],\mathbb{R}^{n\times m})$ is the Banach space $(\mathrm{BV}([a,b],\mathbb{R}^{n\times m}),\|\cdot\|_v)$.

$\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\,]a,b[\,\to\mathbb{R}^{n\times m}$ such that $\bigvee\limits_c^d(X)<+\infty$ for every $a<c<d<b$.

If $X\in\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$, $\det(I_n+(-1)^jd_jX(t))\neq 0$ for $t\in\,]a,b[$ $(j=1,2)$, and $Y\in\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times m})$, then $\mathcal{A}(X,Y)(t)\equiv\mathcal{B}(X,Y)(c_0,t)$, where $\mathcal{B}$ is the operator defined by

$$\mathcal{B}(X,Y)(t,t)=O_{n\times m}\ \text{ for }\ t\in\,]a,b[\,,$$

$$\mathcal{B}(X,Y)(s,t)=Y(t)-Y(s)+\sum_{s<\tau\le t}d_1X(\tau)\cdot\big(I_n-d_1X(\tau)\big)^{-1}d_1Y(\tau)-$$

$$-\sum_{s\le\tau<t}d_2X(\tau)\cdot\big(I_n+d_2X(\tau)\big)^{-1}d_2Y(\tau)\ \text{ for }\ a<s<t<b$$

and

$$\mathcal{B}(X,Y)(s,t)=-\mathcal{B}(X,Y)(t,s)\ \text{ for }\ a<t<s<b.$$

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If $\alpha \in \mathrm{BV}([a,b],\mathbb{R})$ has no more than a finite number of points of discontinuity, and $m \in \{1,2\}$, then $D_{\alpha m} = \{t_{\alpha m 1}, \dots, t_{\alpha m n_{\alpha m}}\}$ ($t_{\alpha m1} < \dots < t_{\alpha m n_{\alpha m}}$) is the set of all points from $[a,b]$ for which $d_m\alpha(t) \neq 0$, and $\mu_{\alpha m} = \max\{d_m\alpha(t) : t \in D_{\alpha m}\}$ ($m = 1,2$).

If $\beta \in \mathrm{BV}([a,b],\mathbb{R})$, then

$$\nu_{\alpha m \beta j} = \max\left\{ d_j\beta(t_{\alpha m l}) + \sum_{t_{\alpha m\, l+1-m} < \tau < t_{\alpha m\, l+2-m}} d_j\beta(\tau) : \ l = 1, \dots, n_{\alpha m} \right\}$$

($j,m = 1,2$); here $t_{\alpha 20} = a-1$, $t_{\alpha 1 n_{\alpha 1}+1} = b+1$.

$s_j : \mathrm{BV}([a,b],\mathbb{R}) \to \mathrm{BV}([a,b],\mathbb{R})$ ($j = 0,1,2$) are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0,$$
$$s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau) \ \text{ and } \ s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \ \text{ for } \ a < t \le b,$$

and

$$s_0(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \ \text{ for } \ t \in [a,b].$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\, dg(\tau) = \int_{]s,t[} x(\tau)\, ds_0(g)(\tau) + \sum_{s<\tau\le t} x(\tau) d_1g(\tau) + \sum_{s\le\tau<t} x(\tau) d_2g(\tau),$$

where $\int_{]s,t[} x(\tau)\, ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_0(g))$ corresponding to the function $s_0(g)$; if $a = b$, then we assume $\int_a^b x(t)\, dg(t) = 0$. Moreover, we put

$$\int_{s+}^t x(\tau)\, dg(\tau) = \lim_{\varepsilon\to 0,\ \varepsilon>0} \int_{s+\varepsilon}^t x(\tau)\, dg(\tau)$$

and

$$\int_s^{t-} x(\tau)\, dg(\tau) = \lim_{\varepsilon\to 0,\ \varepsilon>0} \int_s^{t-\varepsilon} x(\tau)\, dg(\tau).$$

$L([a,b],\mathbb{R};g)$ is the space of all functions $x : [a,b] \to \mathbb{R}$ measurable and integrable with respect to the measure $\mu(g)$ with the norm

$$\|x\|_{L,g} = \int_a^b |x(t)|\, dg(t).$$

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int\limits_s^t x(\tau)\,dg(\tau) = \int\limits_s^t x(\tau)\,dg_1(\tau) - \int\limits_s^t x(\tau)\,dg_2(\tau) \ \text{ for } \ s \le t.$$

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to R^{l\times n}$ is a nondecreasing matrix-function and $D \subset \mathbb{R}^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to D$ such that $x_{kj} \in L([a,b],R;g_{ik})$ $(i = 1,\dots,l;\ k = 1,\dots,n;\ j = 1,\dots,m)$;

$$\int\limits_s^t dG(\tau)\cdot X(\tau) = \bigg(\sum_{k=1}^n \int\limits_s^t x_{kj}(\tau)dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m} \ \text{ for } \ a \le s \le t \le b,$$

$$S_j(G)(t) \equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n} \ \ (j = 0,1,2).$$

If $G_j : [a,b] \to \mathbb{R}^{l\times n}$ $(j = 1,2)$ are nondecreasing matrix-functions, $G(t) \equiv G_1(t) - G_2(t)$ and $X : [a,b] \to \mathbb{R}^{n\times m}$, then

$$\int\limits_s^t dG(\tau)\cdot X(\tau) = \int\limits_s^t dG_1(\tau)\cdot X(\tau) - \int\limits_s^t dG_2(\tau)\cdot X(\tau) \ \text{ for } \ s \le t,$$

$$S_k(G) = S_k(G_1) - S_k(G_2) \quad (k = 0,1,2),$$

$$L([a,b],D;G) = \bigcap_{j=1}^2 L([a,b],D;G_j),$$

The inequalities between the vectors and between the matrices are understood componentwise.

We assume that the vector-function $f = (f_i)_{i=1}^n$ belongs to $\mathrm{BV}([a,b],\mathbb{R}^n)$, and the matrix-function $A = (a_{il})_{i,l=1}^n$ is such that $a_{il} \in \mathrm{BV}([a,b],\mathbb{R})$ $(i \ne l;\ i,l = 1,\dots,n)$, $a_{ii} \in \mathrm{BV}(]a,b],\mathbb{R})$ $(i = 1,\dots,n_0)$ and $a_{ii} \in \mathrm{BV}([a,b[\,,\mathbb{R})$ $(i = n_0+1,\dots,n)$.

A vector-function $x = (x_i)_{i=1}^n$ is said to be a solution of the system (1.1) if $x_i \in \mathrm{BV}_{loc}(]a,b],\mathbb{R})$ $(i = 1,\dots,n_0)$, $x_i \in \mathrm{BV}_{loc}([a,b[\,,\mathbb{R})$ $(i = n_0+1,\dots,n)$ and

$$x_i(t) = x_i(s) + \sum_{l=1}^n \int\limits_s^t x_l(\tau)\,da_{il}(\tau) + f_i(t) - f_i(s)$$

for $a<s\le t\le b$ $(i=1,\dots,n_0)$ and for $a\le s<t<b$ $(i=n_0+1,\dots,n)$.

Under the solution of the problem (1.1), (1.2) we mean a solution $x(t) = (x_i(t))_{i=1}^n$ of the system (1.1) such that the one-sided limits $x_i(a+)$ $(i = 1,\dots,n_0)$ and $x_i(b-)$ $(i = n_0+1,\dots,n)$ exist and the equalities (1.2) are fulfilled. We assume $x_i(a) = 0$ $(i = 1,\dots,n_0)$ and $x_i(b) = 0$ $(i = n_0+1,\dots,n)$, if necessary.

A vector-function $x \in \mathrm{BV}([a,b],\mathbb{R}^n)$ is said to be a solution of the system of generalized differential inequalities

$$dx(t) - dB(t)\cdot x(t) - dq(t) \le 0 \ (\ge 0) \ \text{ for } \ t \in [a,b],$$

where $B \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$, $q \in \mathrm{BV}([a,b],\mathbb{R}^n)$, if

$$x(t) - x(s) + \int\limits_s^t dB(\tau)\cdot x(\tau) - q(t) + q(s) \le 0 \ (\ge 0) \ \text{ for } \ a \le s \le t \le b.$$

Without loss of generality we assume that $A(a) = O_{n\times n}$, $f(a) = 0$. Moreover, we assume

$$\det(I_n + (-1)^j d_j A(t)) \neq 0 \ \text{ for } \ t \in]a,b[\ \ (j=1,2). \tag{1.3}$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding system (see [29, Theorem III.1.4]).

If $s \in]a,b[$ and $\alpha \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R})$ are such that

$$1 + (-1)^j d_j\beta(t) \neq 0 \ \text{ for } \ (-1)^j(t-s) < 0 \ \ (j=1,2),$$

then by $\gamma_\beta(\cdot\,,s)$ we denote the unique solution of the Cauchy problem

$$d\gamma(t) = \gamma(t)d\beta(t), \ \ \gamma(s) = 1.$$

It is known (see [15, 16]) that

$$\gamma_\alpha(t,s) = \begin{cases} \exp\big(s_0(\beta)(t) - s_0(\beta)(s)\big)\times \\ \qquad \times \prod\limits_{s<\tau\le t}(1-d_1\alpha(\tau))^{-1}\prod\limits_{s\le\tau<t}(1+d_2\beta(\tau)) & \text{for } \ t>s, \\ \exp\big(s_0(\beta(t) - s_0(\beta(s)\big)\times \\ \qquad \times \prod\limits_{t<\tau\le s}(1-d_1\beta(\tau))\prod\limits_{t\le\tau<s}(1+d_2\beta(\tau))^{-1} & \text{for } \ t<s, \\ 1 & \text{for } \ t=s. \end{cases} \tag{1.4}$$

It is evident that if the last inequalities are fulfilled on the whole interval $[a,b]$, then $\gamma_\alpha^{-1}(t)$ exists for every $t \in [a,b]$.

Definition 1.1. Let $n_0 \in \{1,\dots,n\}$. We say that a matrix-function $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ belongs to the set $\mathcal{U}(a+,b-;n_0)$ if the functions c_{il} $(i \neq l;\ i,l = 1,\dots,n)$ are nondecreasing on $[a,b]$ and the system

$$\operatorname{sgn}\Big(n_0+\frac{1}{2}-i\Big)\,dx_i(t) \le \sum_{l=1}^n x_l(t)\,dc_{il}(t) \ \text{ for } \ t \in [a,b] \ \ (i=1,\dots,n) \tag{1.5}$$

has no nontrivial nonnegative solution satisfying the condition (1.2).

The similar definition of the set $\mathcal{U}$ has been introduced by I. Kiguradze for ordinary differential equations (see [20, 21]).

Theorem 1.1. *Let the components of the matrix-function* $A=(a_{il})_{i,l=1}^n\in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$ *satisfy the conditions*

$$\begin{gathered}\big(s_0(a_{ii})(t)-s_0(a_{ii})(s)\big)\operatorname{sgn}\Big(n_0+\frac{1}{2}-i\Big)\leq\\ \leq s_0(c_{ii}-\alpha_i)(t)-s_0(c_{ii}-\alpha_i)(s)\ \ \textit{for}\ \ a<s<t<b\ \ (i=1,\dots,n),\end{gathered}\tag{1.6}$$

$$\begin{gathered}(-1)^j\big(\big|1+(-1)^jd_ja_{ii}(t)\big|-1\big)\operatorname{sgn}\Big(n_0+\frac{1}{2}-i\Big)\leq\\ \leq d_j\big(c_{ii}(t)-\alpha_i(t)\big)\ \ \textit{for}\ \ t\in\,]a,b]\ \ (j=1,2;\ i=1,\dots,n_0)\\ \textit{and for}\ \ t\in[a,b[\ \ (j=1,2;\ i=n_0+1,\dots,n),\end{gathered}\tag{1.7}$$

$$\begin{gathered}\big|s_0(a_{il})(t)-s_0(a_{il})(s)\big|\leq\\ \leq s_0(c_{il})(t)-s_0(c_{il})(s)\ \ \textit{for}\ \ a<s<t<b\ \ (i\neq l;\ \ i,l=1,\dots,n)\end{gathered}\tag{1.8}$$

and

$$|d_ja_{il}(t)|\leq d_jc_{il}(t)\ \ \textit{for}\ \ t\in[a,b]\ \ (i\neq l;\ i,l=1,\dots,n),\tag{1.9}$$

where

$$C=(c_{il})_{i,l=1}^n\in\mathcal{U}(a+,b-;\ n_0),\tag{1.10}$$

$\alpha_i:\,]a,b]\to\mathbb{R}$ $(i=1,\dots,n_0)$ *and* $\alpha_i:[a,b[\,\to\mathbb{R}$ $(i=n_0+1,\dots,n)$ *are nondecreasing functions such that*

$$\begin{gathered}\lim_{t\to a+}d_2\alpha_i(t)<1\ \ (i=1,\dots,n_0),\\ \lim_{t\to b-}d_1\alpha_i(t)<1\ \ (i=n_0+1,\dots,n)\end{gathered}\tag{1.11}$$

and

$$\begin{gathered}\lim_{t\to a+}\ \lim_{k\to\infty}\sup\gamma_{\beta_i}(t,a+1/k)=0\ \ (i=1,\dots,n_0),\\ \lim_{t\to b-}\ \lim_{k\to\infty}\sup\gamma_{\beta_i}(t,b-1/k)=0\ \ (i=n_0+1,\dots,n),\end{gathered}\tag{1.12}$$

where $\beta_i(t)\equiv\alpha_i(t)\operatorname{sgn}\big(n_0+\frac{1}{2}-i\big)$ $(i=1,\dots,n)$. *Then the problem* (1.1), (1.2) *has one and only one solution.*

Corollary 1.1. *Let the components of the matrix-function* $A=(a_{il})_{i,l=1}^n\in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$ *satisfy the conditions*

$$\begin{gathered}\big(s_0(a_{ii})(t)-s_0(a_{ii})(s)\big)\operatorname{sgn}\Big(n_0+\frac{1}{2}-i\Big)\leq-\big(s_0(\alpha_i)(t)-s_0(\alpha_i)(s)\big)\\ +\int\limits_s^t h_{ii}(\tau)\,ds_0(\beta_i)(\tau)\ \ \textit{for}\ \ a<s<t<b\ \ (i=1,\dots,n),\end{gathered}\tag{1.13}$$

$$\begin{gathered}(-1)^j\big(\big|1+(-1)^jd_ja_{ii}(t)\big|-1\big)\operatorname{sgn}\Big(n_0+\frac{1}{2}-i\Big)\leq\\ \leq h_{ii}(t)d_j\beta_i(t)-d_j\alpha_i(t)\big)\ \ \textit{for}\ \ t\in\,]a,b]\ \ (j=1,2;\ i=1,\dots,n_0)\\ \textit{and for}\ \ t\in[a,b[\ \ (j=1,2;\ i=n_0+1,\dots,n),\end{gathered}$$

$$\big|s_0(a_{il})(t)-s_0(a_{il})(s)\big|\leq$$
$$\leq\int\limits_s^t h_{il}(\tau)\,ds_0(\beta_l)(\tau)\ \ \textit{for}\ \ a<s<t<b\ \ (i\neq l;\ \ i,l=1,\dots,n) \tag{1.14}$$

and

$$|d_j a_{il}(t)|\leq h_{il}(t)d_j\beta_l(t)\ \ \textit{for}\ \ t\in[a,b]\ \ (i\neq l;\ i,l=1,\dots,n), \tag{1.15}$$

where $\alpha_i:]a,b]\to\mathbb{R}$ $(i=1,\dots,n_0)$ *and* $\alpha_i:[a,b[\to\mathbb{R}$ $(i=n_0+1,\dots,n)$ *are nondecreasing functions satisfying the conditions* (1.11) *and* (1.12), β_l $(l=1,\dots,n)$ *are functions nondecreasing on* $[a,b]$ *and having not more than a finite number of points of discontinuity,* $h_{ii}\in L^\mu([a,b],\mathbb{R};\beta_i)$, $h_{il}\in L^\mu([a,b],\mathbb{R}_+;\beta_l)$ $(i\neq l;\ i,l=1,\dots,n)$, $1\leq\mu\leq+\infty$. *Let, moreover,*

$$r(\mathcal{H})<1, \tag{1.16}$$

where the $3n\times 3n$*-matrix* $\mathcal{H}=(\mathcal{H}_{j+1\,m+1})_{j,m=0}^2$ *is defined by*

$$\mathcal{H}_{j+1\,m+1}=\big(\lambda_{kmij}\|h_{ik}\|_{\mu,s_m(\beta_i)}\big)_{i,k=1}^n\quad(j,m=0,1,2),$$
$$\xi_{ij}=\big(s_j(\beta_i)(b)-s_j(\beta_i)(a)\big)^{\frac{1}{\nu}}\quad(j=0,1,2,;\ \ i=1,\dots,n);$$
$$\lambda_{k0i0}=\begin{cases}\Big(\dfrac{4}{\pi^2}\Big)^{\frac{1}{\nu}}\xi_{k0}^2 & \textit{if}\ \ s_0(\beta_i)(t)\equiv s_0(\beta_k)(t),\\ \xi_{k0}\xi_{i0} & \textit{if}\ \ s_0(\beta_i)(t)\not\equiv s_0(\beta_k)(t)\ \ (i,k=1,\dots,n);\end{cases}$$
$$\lambda_{kmij}=\xi_{km}\xi_{ij}\ \ \textit{if}\ \ m^2+j^2>0,\ \ mj=0\ \ (j,m=0,1,2;\ \ i,k=1,\dots,n),$$
$$\lambda_{kmij}=\Big(\frac{1}{4}\,\mu_{\alpha_k m}\nu_{\alpha_k m\alpha_i j}\sin^{-2}\frac{\pi}{4n_{\alpha_k m}+2}\Big)^{\frac{1}{\nu}}\ \ (j,m=1,2;\ \ i,k=1,\dots,n),$$

and $\frac{1}{\mu}+\frac{2}{\nu}=1$. *Then the problem* (1.1), (1.2) *has one and only one solution.*

Remark 1.1. The $3n\times 3n$-matrix $\mathcal{H}$, appearing in Corollary 1.1 can be replaced by the $n\times n$-matrix

$$\bigg(\max\Big\{\sum_{j=0}^2\lambda_{kmij}\|h_{ik}\|_{\mu,S_m(\alpha_k)}:\ m=0,1,2\Big\}\bigg)_{i,k=1}^n.$$

By Remark 1.1, Corollary 1.1 has the following form for $h_{il}(t)\equiv h_{il}=const$ $(i,l=1,\dots,n)$, $\alpha_i(t)\equiv\alpha(t)$ $(i=1,\dots,n)$, $\beta_i(t)\equiv\beta(t)$ $(i=1,\dots,n)$ and $\mu=+\infty$.

Corollary 1.2. *Let the components of the matrix-function* $A=(a_{il})_{i,l=1}^n\in\mathrm{BV}_{loc}(]a,b[,\mathbb{R}^{n\times n})$ *satisfy the conditions*

$$\big(s_0(a_{ii})(t)-s_0(a_{ii})(s)\big)\operatorname{sgn}\Big(n_0+\frac{1}{2}-i\Big)\leq h_{ii}\big(s_0(\beta)(t)-s_0(\beta)(s)\big)-$$
$$-\big(s_0(\alpha)(t)-s_0(\alpha)(s)\big)\ \ \textit{for}\ \ a<s<t<b\ \ (i=1,\dots,n),$$

$$(-1)^j\big(\big|1+(-1)^j d_j a_{ii}(t)\big|-1\big)\operatorname{sgn}\Big(n_0+\frac{1}{2}-i\Big)\leq h_{ii}d_j\beta(t)-d_j\alpha(t)\big)$$
$$\text{for } t\in\,]a,b] \ \ (j=1,2;\ i=1,\dots,n_0)$$
$$\text{and for } t\in[a,b[\ \ (j=1,2;\ i=n_0+1,\dots,n),$$
$$\big|s_0(a_{il})(t)-s_0(a_{il})(s)\big|\leq h_{il}\big(s_0(\beta)(t)-s_0(\beta)(s)\big)$$
$$\text{for } a<s<t<b \ \ (i\neq l;\ \ i,l=1,\dots,n)$$

and

$$|d_j a_{il}(t)|\leq h_{il}d_j\beta(t) \ \ \text{for } t\in[a,b] \ \ (i\neq l;\ i,l=1,\dots,n)$$

hold, where $\alpha:[a,b]\to\mathbb{R}$ *is a nondecreasing function satisfying the conditions* (1.11) *and* (1.12), β *is a function nondecreasing on* $[a,b]$ *and having not more than a finite number of points of discontinuity,* $h_{ii}\in\mathbb{R}$, $h_{il}\in\mathbb{R}_+$ $(i\neq l;\, i,l=1,\dots,n)$. *Let, moreover,*

$$\rho_0\, r(\mathcal{H})<1,$$

where

$$\mathcal{H}=(h_{ik})_{i,k=1}^n,\quad \rho_0=\max\bigg\{\sum_{j=0}^{2}\lambda_{mj}: m=0,1,2\bigg\},$$
$$\lambda_{00}=\frac{2}{\pi}\,\big(s_0(\beta)(b)-s_0(\beta)(a)\big),$$
$$\lambda_{0j}=\lambda_{j0}=\big(s_0(\beta)(b)-s_0(\alpha)(a)\big)^{\frac12}\big(s_j(\beta)(b)-s_j(\beta)(a)\big)^{\frac12}\quad (j=1,2),$$
$$\lambda_{mj}=\frac{1}{2}\,\big(\mu_{\alpha m}\nu_{\alpha m\alpha j}\big)^{\frac12}\sin^{-1}\frac{\pi}{4n_{\alpha m}+2}\quad (m,j=1,2).$$

Then the problem (1.1), (1.2) *has one and only one solution.*

Theorem 1.2. *Let the components of the matrix-function* $A=(a_{il})_{i,l=1}^n\in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$ *satisfy the conditions* (1.6)–(1.9), *where* $c_{il}(t)\equiv h_{il}\beta_i(t) + \beta_{il}(t)$ $(i,l=1,\dots,n)$,

$$d_2\beta_i(a)\leq 0 \ \text{ and } \ 0\leq d_1\beta_i(t)<|\eta_i|^{-1} \ \text{ for } a<t\leq b \ \ (i=1,\dots,n_0),\quad (1.17)$$
$$d_1\beta_i(b)\leq 0 \ \text{ and } \ 0\leq d_2\beta_i(t)<|\eta_i|^{-1} \ \text{ for } a\leq t<b \ \ (i=n_0+1,\dots,n),\ (1.18)$$

where $\alpha_i:\,]a,b]\to\mathbb{R}$ $(i=1,\dots,n_0)$ *and* $\alpha_i:[a,b[\,\to\mathbb{R}$ $(i=n_0+1,\dots,n)$ *are nondecreasing functions satisfying the conditions* (1.11) *and* (1.12), $h_{ii}<0$, $h_{il}\geq 0$, $\eta_i<0$ $(i\neq l;\, i,l=1,\dots,n)$, β_{ii} $(i=1,\dots,n)$ *are the functions nondecreasing on* $[a,b]$; $\beta_{il},\beta_i\in\mathrm{BV}([a,b],\mathbb{R})$ $(i\neq l; i,l=1,\dots,n)$ *are the functions nondecreasing on the interval* $]a,b]$ *for* $i\in\{1,\dots,n_0\}$ *and on the interval* $[a,b[$ *for* $i\in\{n_0+1,\dots,n\}$. *Let, moreover, the condition* (1.16) *hold, where* $\mathcal{H}=(\xi_{il})_{i,l=1}^n$,

$$\xi_{ii}=\eta_i,\ \ \xi_{il}=\frac{h_{il}}{|h_{ii}|}\ \ (i\neq l;\ i,l=1,\dots,n),$$
$$\eta_i=V\big(\mathcal{A}(\zeta_i,a_i)\big)(b)-V\big(\mathcal{A}(\zeta_i,a_i)\big)(a+)\ \ \text{for } i\in\{1,\dots,n_0\},$$

$$\eta_i = V\big(\mathcal{A}(\zeta_i, a_i)\big)(b-) - V\big(\mathcal{A}(\zeta_i, a_i)\big)(a) \ \ \textit{for} \ \ i \in \{n_0+1, \dots, n\};$$

$$\zeta_i(t) \equiv \sum_{k=l}^{n} \beta_{il}(t) \ \ (i = 1, \dots, n),$$

$$a_i(t) \equiv \big(\beta_i(t) - \beta_i(a+)\big)h_{ii} \ \ \textit{for} \ \ a < t \leq b \ \ (i = 1, \dots, n_0),$$

$$a_i(t) \equiv \big(\beta_i(b-) - \beta_i(t)\big)h_{ii} \ \ \textit{for} \ \ a \leq t < b \ \ (i = n_0+1, \dots, n).$$

Then the problem (1.1), (1.2) *has one and only one solution.*

Remark 1.2. If

$$\eta_i < 1 \ \ (i = 1, \dots, n), \tag{1.19}$$

then, in Theorem 1.2, we can assume that

$$\xi_{ii} = 0, \ \ \xi_{il} = \frac{h_{il}}{(1-\eta_i)|h_{ii}|} \ \ (i \neq l; \ i, l = 1, \dots, n). \tag{1.20}$$

Theorem 1.3. *Let the matrix-function $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}([a,b], \mathbb{R}^{n\times n})$ be such that the functions c_{il} $(i \neq l;\ i, l = 1, \dots, n)$ are nondecreasing on $[a,b]$ and the problem* (1.5), (1.2) *has a nontrivial nonnegative solution, i.e., the condition* (1.10) *is violated. Let, moreover, $\alpha_i :]a,b] \to \mathbb{R}$ $(i = 1, \dots, n_0)$ and $\alpha_i : [a,b[\to \mathbb{R}$ $(i = n_0+1, \dots, n)$ be nondecreasing functions satisfying the conditions* (1.11), (1.12) *and*

$$1 + (-1)^j \operatorname{sgn}\Big(n_0 + \frac{1}{2} - i\Big) d_j\big(c_{ii}(t) - \alpha_i(t)\big) > 0$$
$$\textit{for} \ \ t \in]a,b] \ \ (j = 1,2; \ i = 1, \dots, n_0)$$
$$\textit{and for} \ \ t \in [a,b[\ \ (j = 1,2; \ i = n_0+1, \dots, n). \tag{1.21}$$

Then there exist a matrix-function $A = (a_{il})_{i,l=1}^n \in \mathrm{BV}([a,b], \mathbb{R}^{n\times n})$, a vector-function $f = (f_l)_{l=1}^n \in \mathrm{BV}([a,b], R^n)$ and nondecreasing functions $\widetilde{\alpha}_i :]a,b] \to \mathbb{R}$ $(i = 1, \dots, n_0)$ and $\widetilde{\alpha}_i : [a,b[\to \mathbb{R}$ $(i = n_0+1, \dots, n)$ such that the conditions (1.6)–(1.12) *and*

$$\widetilde{\alpha}_i(t) - \widetilde{\alpha}_i(s) \leq \alpha_i(t) - \alpha_i(s)$$
$$\textit{for} \ \ a < t < s \leq b \ \ \textit{and for} \ \ a \leq t < s < b \ \ (i = n_0+1, \dots, n) \tag{1.22}$$

are fulfilled, but the problem (1.1), (1.2) *is unsolvable. In addition, if the matrix-function $C = (c_{il})_{i,l=1}^n$ is such that*

$$\det\left((\delta_{il} + (-1)^j \varepsilon_l d_j c_{il}(t) \operatorname{sgn}\Big(n_0 + \frac{1}{2} - i\Big)_{i,l=1}^n \right) \neq 0$$
$$\textit{for} \ \ t \in [a,b]; \ \ \varepsilon_1, \dots, \varepsilon_n \in [a,b] \ \ (j = 1,2), \tag{1.23}$$

then the matrix-function $A = (a_{il})_{i,l=1}^n$ satisfies the condition (1.3).

Remark 1.3. The condition (1.23) holds, for example, if either

$$\sum_{l=1}^{n} |d_j c_{il}(t)| < 1 \ \ \text{for} \ \ t \in [a,b] \ \ (j = 1,2; \ \ i = 1, \dots, n), \tag{1.24}$$

$$\sum_{l=1,\, l\neq i}^{n} |d_j c_{il}(t)| < 1 + (-1)^j \operatorname{sgn}\Big(n_0 + \frac{1}{2} - i\Big) d_j c_{ii}(t)$$

$$\text{for } \ t \in [a,b] \ \ (j=1,2; \ \ i=1,\dots,n) \tag{1.25}$$

or

$$\sum_{l=1,\, l\neq i}^{n} |d_j c_{li}(t)| < 1 + (-1)^j \operatorname{sgn}\Big(n_0 + \frac{1}{2} - i\Big) d_j c_{ii}(t)$$

$$\text{for } \ \ t \in [a,b] \ \ (j=1,2; \ \ i=1,\dots,n). \tag{1.26}$$

2. Auxiliary Propositions

Lemma 2.1. *Let* $t_0 \in [a,b]$, α *and* $q \in \mathrm{BV}_{loc}([a,t_0[\,,\mathbb{R}^n) \cap \mathrm{BV}_{loc}(]t_0,b],\mathbb{R}^n)$ *be such that*

$$1 + (-1)^j \operatorname{sgn}(t-t_0) d_j \alpha(t) > 0 \ \ \textit{for} \ \ t \in [a,b] \ \ (j=1,2). \tag{2.1}$$

Let, moreover, $x \in \mathrm{BV}_{loc}([a,t_0[\,,\mathbb{R}^n) \cap \mathrm{BV}_{loc}(]t_0,b],\mathbb{R}^n)$ *be a solution of the linear generalized differential inequality*

$$\operatorname{sgn}(t-t_0) dx(t) \le x(t) d\alpha(t) + dq(t) \tag{2.2}$$

on the intervals $[a,t_0[$ *and* $]t_0,b]$, *satisfying the inequalities*

$$x(t_0+) \le y(t_0+) \ \ \textit{and} \ \ x(t_0-) \le y(t_0-), \tag{2.3}$$

where $y \in \mathrm{BV}_{loc}([a,t_0[\,,\mathbb{R}^n) \cap \mathrm{BV}_{loc}(]t_0,b],\mathbb{R}^n)$ *is a solution of the general differential equality*

$$\operatorname{sgn}(t-t_0) dy(t) = y(t) d\alpha(t) + dq(t). \tag{2.4}$$

Then

$$x(t) \le y(t) \ \ \textit{for} \ \ t \in [a,t_0[\cup]t_0,b]. \tag{2.5}$$

Proof of Lemma 2.1. Assume $t_0 < b$ and consider the closed interval $[t_0 + \varepsilon, b]$, where ε is an arbitrary sufficiently small positive number.

By (2.1), the Cauchy problem

$$d\gamma(t) = \gamma(t) d\alpha(t), \ \ \gamma(s) = 1$$

has the unique solution γ_s for every $s \in [t_0+\varepsilon, b]$ and, by (1.4), this is positive, i.e.,

$$\gamma_s(t) > 0 \ \ \text{for} \ \ t \in [t_0+\varepsilon, b]. \tag{2.6}$$

According to the variation-of-constant formula (see [29, Corollary III.2.14]), from (2.4) we have

$$y(t) = q(t) - q(s) + \\ + \gamma(t) \Big\{ \gamma^{-1}(s) y(s) - \int_s^t \big(q(\tau) - q(s)\big) d\gamma^{-1}(\tau) \Big\} \ \ \text{for} \ \ s,t \in [t_0+\varepsilon, b], \tag{2.7}$$

where $\gamma(t) \equiv \gamma_{t_0+\varepsilon}(t)$.

From (2.2), we have

$$dx(t) \le x(t)d\alpha(t) + d\big(q(t) - q_\varepsilon(t)\big) \ \text{ for } \ t \in [t_0+\varepsilon, b]$$

and, therefore,

$$x(t) = q(t) - q(t_0+\varepsilon) - q_\varepsilon(t) + q_\varepsilon(t_0+\varepsilon) + \gamma(t)\bigg\{\gamma^{-1}(t_0+\varepsilon)x(t_0+\varepsilon) - $$
$$- \int\limits_{t_0+\varepsilon}^{t} \big(q(\tau) - q(t_0+\varepsilon) - q_\varepsilon(\tau) + q_\varepsilon(t_0+\varepsilon)\big)\, d\gamma^{-1}(\tau)\bigg\} \ \text{ for } \ t \in [t_0+\varepsilon, b],$$

where

$$q_\varepsilon(t) = -x(t) + x(t_0+\varepsilon) + q(t) - q(t_0+\varepsilon) + \int\limits_{t_0+\varepsilon}^{t} x(\tau)d\alpha(\tau) \quad \text{for } \ t \in [t_0+\varepsilon, b].$$

Hence, by (2.7), we get

$$x(t) = y(t) + \gamma(t)\gamma^{-1}(t_0+\varepsilon)\big(x(t_0+\varepsilon) - y(t_0+\varepsilon)\big) + \\ + g_\varepsilon(t) \ \text{ for } \ t \in [t_0+\varepsilon, b], \tag{2.8}$$

where

$$g_\varepsilon(t) = -q_\varepsilon(t) + q_\varepsilon(t_0+\varepsilon) + \gamma(t) \int\limits_{t_0+\varepsilon}^{t} \big(q_\varepsilon(\tau) - q_\varepsilon(t_0+\varepsilon)\big)d\gamma^{-1}(\tau) \quad \text{for } \ t \in [t_0+\varepsilon, b].$$

Using the formula of integration-by-parts (see [29, Theorem I.4.33]), we find

$$g_\varepsilon(t) = -\gamma(t)\bigg(\int\limits_{t_0+\varepsilon}^{t} \gamma^{-1}(\tau)\, ds_0(q_\varepsilon)(\tau) + \\ + \sum_{t_0+\varepsilon<\tau\le t} \gamma^{-1}(\tau-)d_1q_\varepsilon(\tau) + \sum_{t_0+\varepsilon\le\tau<t} \gamma^{-1}(\tau+)d_2q_\varepsilon(\tau)\bigg) \ \text{ for } \ t \in [t_0+\varepsilon, b]. \tag{2.9}$$

According to (2.6) and (2.9), we have

$$g_\varepsilon(t) \le 0 \ \text{ for } \ t \in [t_0+\varepsilon, b],$$

since by the definition of a solution of the generalized differential inequality (2.2) the function q_ε is nondecreasing on the interval $]t_0, b]$. By the equality $\gamma(t_0+\varepsilon) = 1$, from this and (2.8) we get

$$x(t) \le y(t) + \gamma(t)\big(x(t_0+\varepsilon) - y(t_0+\varepsilon)\big) \ \text{ for } \ t \in [t_0+\varepsilon, b].$$

Passing to the limit as $\varepsilon \to 0$ in the latter inequality and taking into account (2.3) and (2.6), we conclude

$$x(t) \le y(t) \text{ for } t \in]t_0, b].$$

Analogously we can show the validity of the inequality (2.5) for $t \in [a, t_0[$. The lemma is proved. $\square$

The following lemma makes more precise the ones (see Lemma 6.5) in [10].

Lemma 2.2. *Let $t_1, \dots, t_n \in [a, b]$, $l_i : \mathrm{BV}_v([a,b], \mathbb{R}^n_+) \to \mathbb{R}_+$ $(i = 1, \dots, n)$ be linear bounded functionals, and $C_{kj} = (c_{kjil})_{i,l=1}^{n_k, n_j} \in \mathrm{BV}([a,b], \mathbb{R}^{n_k \times n_j})$ $(k, j = 1, 2)$ be such that the system*

$$\begin{gathered}
\operatorname{sgn}(t - t_i) dx_i(t) \le \sum_{l=1}^{n_1} x_l(t) dc_{11il}(t) + \sum_{l=1}^{n_2} x_{n_1+l}(t) dc_{12il}(t) \\
\text{for } t \in [a, b], \ \ t \ne t_i \ \ (i = 1, \dots, n_1), \\
(-1)^j d_j x_i(t_i) \le \sum_{l=1}^{n_1} x_{1l}(t_i) d_j c_{11il}(t_i) + \sum_{l=1}^{n_2} x_{n_1+l}(t_i) d_j c_{12il}(t_{1i}) \\
(j = 1, 2; \ \ i = 1, \dots, n_1), \\
dx_{n_1+i}(t) = \sum_{l=1}^{n_1} x_l(t) dc_{21il}(t) + \sum_{l=1}^{n_2} x_{n_1+l}(t) dc_{22il}(t) \\
\text{for } t \in [a, b] \ \ (i = 1, \dots, n_2),
\end{gathered} \tag{2.10}$$

has a nontrivial nonnegative solution under the condition

$$\begin{gathered}
x_i(t_i) \le l_i(x_1, \dots, x_n) \text{ for } i \in N_n, \\
x_i(t_i) = l_i(x_1, \dots, x_n) \text{ for } i \in \{1, \dots, n\} \setminus N_n,
\end{gathered} \tag{2.11}$$

where n_1 and n_2 $(n_1 + n_2 = n)$ are some nonnegative integers, and N_n is some subset of the set $\{1, \dots, n\}$. Let, moreover, the functions $\alpha_1, \dots, \alpha_{n_1} \in \mathrm{BV}([a,b], \mathbb{R}^n)$ be such that

$$d_j \alpha_i(t) \ge 0 \text{ for } t \in [a, b] \ \ (j = 1, 2; \ \ i = 1, \dots, n_1) \tag{2.12}$$

and

$$1 + (-1)^j \operatorname{sgn}(t - t_i) d_j \big(c_{11ii}(t) - \alpha_i(t)\big) > 0 \\
\text{for } t \in [a, b] \ \ (j = 1, 2; \ \ i = 1, \dots, n_1). \tag{2.13}$$

Then there exist matrix-functions $\widetilde{C}_{k1} = (\widetilde{c}_{k1il})_{i,l=1}^{n_k, n_1} \in \mathrm{BV}([a,b], \mathbb{R}^{n_k \times n_1})$ $(k = 1, 2)$, functions $\widetilde{\alpha}_i \in \mathrm{BV}([a,b], \mathbb{R}^n)$ $(i = 1, \dots, n_1)$, linear bounded functionals $\widetilde{l}_i : \mathrm{BV}_v([a,b], \mathbb{R}^n) \to \mathbb{R}$ $(i = 1, \dots, n)$ and numbers $c_{0i} \in \mathbb{R}$ $(i = 1, \dots, n)$ such that

$$\begin{gathered}
s_0(\widetilde{c}_{11ii})(t) - s_0(\widetilde{c}_{11ii})(s) \le \\
\le \big(s_0(c_{11ii} - \widetilde{\alpha}_i)(t) - s_0(c_{11ii} - \widetilde{\alpha}_i)(s)\big) \operatorname{sgn}(t - s) \\
\text{for } (t - s)(s - t_i) > 0, \ \ s, t \in [a, b] \ \ (i = 1, \dots, n_1),
\end{gathered} \tag{2.14}$$

$$(-1)^{j+m}\big(\big|1+(-1)^j d_j\widetilde{c}_{11ii}(t)\big|-1\big) \le d_j\big(c_{ii}(t)-\widetilde{\alpha}_i(t)\big)$$
$$\text{for } (-1)^m(t-t_i)>0 \ \ (j,m=1,2; i=1,\dots,n_1); \tag{2.15}$$
$$|s_0(\widetilde{c}_{21il})(t)-s_0(\widetilde{c}_{21il})(s)| \le$$
$$\le \bigvee_s^t (s_0(c_{21il})) \ \text{ for } a\le s\le t\le b \ \ (i=1,\dots,n_2,\ l=1,\dots,n_1), \tag{2.16}$$
$$\big|d_j\widetilde{c}_{21il}(t)\big| \le \big|d_j\widetilde{c}_{21il}(t)\big| \ \text{ for } t\in[a,b] \ \ (i=1,\dots,n_2,\ l=1,\dots,n_1), \tag{2.17}$$
$$0\le d_j\widetilde{\alpha}_i \le d_j\alpha_i(t) \ \text{ for } t\in[a,b] \ \ (j=1,2;\ \ i=1,\dots,n_1), \tag{2.18}$$

and the system

$$dx(t)=d\widetilde{A}(t)\cdot x(t) \tag{2.19}$$

under the n-condition

$$x_i(t_i)=\widetilde{l}_i(x_1,\dots,x_n)+c_{0i} \ \ (i=1,\dots,n) \tag{2.20}$$

is unsolvable, where

$$\widetilde{A}(t)\equiv\begin{pmatrix}\widetilde{C}_{11}(t), & C_{12}(t)\\ \widetilde{C}_{21}(t), & C_{22}(t)\end{pmatrix}. \tag{2.21}$$

Proof. Let $x=(x_i)_{i=1}^n$ be the nonnegative solution of the problem (2.10), (2.11). Let, moreover, $\varphi_i\in \mathrm{BV}([a,b],\mathbb{R})$ $(i=1,\dots,n_1)$ be the functions defined by

$$\varphi_i(t)\equiv\Bigg(\sum_{l=1}^{n_1}\int_{t_i}^t x_l(\tau)\,dc_{11\,il}(\tau)+$$
$$+\sum_{l=1}^{n_2}\int_{t_i}^t x_{n_1+l}(\tau)\,dc_{12\,il}(\tau)-\int_{t_i}^t x_i(\tau)db_i(\tau)\Bigg)\operatorname{sgn}(t-t_i) \ \ (i=1,\dots,n_1),$$

where $b_i(t)\equiv c_{11ii}-\alpha_i(t)$.

By the condition (2.13), it is evident that the Cauchy problem

$$dy(t)=y(t)\,d\widetilde{b}_i(t)+d\varphi_i(t), \tag{2.22}$$
$$y(t_i)=x_i(t_i), \tag{2.23}$$

where $\widetilde{b}_i(t)\equiv b_i(t)\operatorname{sgn}(t-t_i)$, has a unique solution y_i for every $i\in\{1,\dots,n_1\}$.

In addition, by (2.10) it is easy to verify that the function

$$z_i(t)\equiv x_i(t)-y_i(t)$$

satisfies the conditions of Lemma 2.1 and the problem

$$du(t)=u(t)\,d\widetilde{b}_i(t),\ \ u(t_i)=0$$

has only the trivial solution for every $i\in\{1,\dots,n_1\}$.

According to this lemma, we have

$$x_i(t) \le y_i(t) \ \text{ for } \ t \in [a,b] \ \ (i = 1, \dots, n_1)$$

and therefore

$$x_i(t) = \eta_i(t) y_i(t) \ \text{ for } \ t \in [a,b] \ \ (i = 1, \dots, n_1),$$

where for every $i \in \{1, \dots, n\}$, $\eta_i(t) = x_i(t)/y_i(t)$ if $t \in [a,b]$ is such that $y_i(t) \neq 0$, and $\eta_i(t) = 1$ if $t \in [a,b]$ is such that $y_i(t) = 0$.

It is evident that

$$0 \le \eta_i(t) \le 1 \ \text{ for } \ t \in [a,b] \ \text{ and } \ \eta_i(t_i) = 1 \ \ (i = 1, \dots, n). \tag{2.24}$$

Moreover, for every $i \in \{1, \dots, n\}$, $\eta_i : [a,b] \to [0,1]$ is the function bounded and measurable with respect to every measure along with x_i and y_i are integrable functions.

Hence there exist the integrals appearing in the notation

$$\begin{gathered}
\widetilde{c}_{11\,ii}(t) \equiv \big(c_{11ii}(t) - \widetilde{\alpha}_i(t)\big)\,\mathrm{sign}(t - t_i) \ \ (i = 1, \dots, n_1), \\
\widetilde{c}_{11\,il}(t) \equiv \mathrm{sgn}(t - t_i) \int\limits_{t_i}^{t} \eta_l(\tau)\, dc_{11\,il}(\tau) \ \ (i \neq l; \ \ i, l = 1, \dots, n_1)
\end{gathered} \tag{2.25}$$

and

$$\widetilde{c}_{21\,il}(t) \equiv \int\limits_{t_i}^{t} \eta_l(\tau)\, dc_{21\,il}(\tau) \ \ (i = 1, \dots, n_2; \ \ l = 1, \dots, n_1), \tag{2.26}$$

where

$$\widetilde{\alpha}_i(t) \equiv \int\limits_{t_i}^{t} \big(1 - \eta_i(\tau)\big)\, d\alpha_i(\tau) \ \ (i = 1, \dots, n_1). \tag{2.27}$$

Due to (2.11) and (2.22)–(2.24), the vector-function $z(t) = (z_i(t))_{i=1}^n$, $z_i(t) = y_i(t)$ $(i = 1, \dots, n_1)$, $z_{n_1+i}(t) = x_{n_1+i}(t)$ $(i = 1, \dots, n_2)$, is a nontrivial nonnegative solution of the problem

$$dz(t) = d\widetilde{A}(t) \cdot z(t), \tag{2.28}$$

$$z_i(t_i) = \widetilde{l}_i(z_1, \dots, z_n) \ \ (i = 1, \dots, n), \tag{2.29}$$

where the matrix-function $\widetilde{A}$ is defined by (2.21), (2.25)–(2.27); $\widetilde{l}_i$: $\mathrm{BV}_v([a,b], \mathbb{R}^n) \to \mathbb{R}$ $(i = 1, \dots, n)$ are linear bounded functionals defined by

$$\begin{gathered}
\widetilde{l}_i(z_1, \dots, z_{n_1}, z_{n_1+1}, \dots, z_n) = \\
= \delta_i l_i(\eta_1 z_1, \dots, \eta_{n_1} z_{n_1}, z_{n_1+1}, \dots, z_n) \ \text{ for } \ (z_l)_{l=1}^n \in \mathrm{BV}_v([a,b], \mathbb{R}^n),
\end{gathered} \tag{2.30}$$

and $\delta_i \in [0,1]$ $(i = 1, \dots, n)$, $\delta_i = 1$ for $i \in \{1, \dots, n\} \setminus N_n$, are some numbers.

On the other hand, by Remark 1.2 from [9], there exist numbers $c_{0i} \in \mathbb{R}$ $(i = 1, \dots, n)$ such that the problem (2.19), (2.20) is not solvable, where the matrix-function $\widetilde{A}(t)$ and the linear functionals $\widetilde{l}_i$ $(i=1,\dots,n)$ are defined as above.

Let us show the estimates (2.14)–(2.18). To this end, we use the following formulas obtained from Theorem I.4.12 and Lemma I.4.23 given in [29]. Let the functions $g \in \mathrm{BV}([a,b],\mathbb{R})$ and $f:[a,b]\to\mathbb{R}$ be such that the integral $\varphi(t)=\int\limits_a^t f(\tau)dg(\tau)$ exists for $t\in[a,b]$. Then the equalities

$$s_0(\varphi)(t) \equiv \int\limits_a^t f(\tau)ds_0(g)(\tau), \quad d_j\varphi(t) \equiv f(t)d_jg(t) \ \ (j=1,2) \tag{2.31}$$

hold.

Using (2.31), from (2.24)–(2.26) we get the estimates (2.14), (2.16) and (2.17). Moreover, by (2.12), (2.24) and (2.31), the estimate (2.18) holds. As for the estimate (2.15), it holds by general inequality $a-|b|\leq(a-b)\operatorname{sgn} a$ for the cases $t>t_i$, $j=1$ $(i=1,\dots,n_1)$ and $t<t_i$, $j=2$ $(i=1,\dots,n_1)$, and follows from (2.13) by using (2.18) for the cases $t>t_i$, $j=2$ $(i=1,\dots,n_1)$ and $t<t_i$, $j=1$ $(i=1,\dots,n_1)$. The lemma is proved. □

Remark 2.1. In Lemma 2.2, if the functions α_i and c_{21kl} are nondecreasing for some $i\in\{1,\dots,n_1\}$ and $k\in\{1,\dots,n_2\}$, $l\in\{1,\dots,n_1\}$, then the functions $\widetilde{\alpha}_i$ and $\widetilde{c}_{21kl}$, respectively, are nondecreasing as well, and

$$\widetilde{\alpha}_i(t)-\widetilde{\alpha}_i(s)\leq\alpha_i(t)-\alpha_i(s) \ \text{ and } \ \widetilde{c}_{21kl}(t)-\widetilde{c}_{21kl}(s)\leq c_{21kl}(t)-c_{21kl}(s)$$
$$\text{for } \ a\leq s<t\leq b.$$

The statement of Remark 2.1 follows from (2.26) and (2.27) with regard for (2.24).

3. Proofs of the Main Results

Proof of Theorem 1.1. Let us assume

$$t_{*k}=a+\frac{1}{k} \ \text{ and } \ t_k^*=b-\frac{1}{k} \ \ (k=1,2,\dots);$$

$$a_{ilk}(t)=\begin{cases} c_{il}(t)-c_{il}(t_{*k}-)+a_{il}(t_{*k}-) \ \text{ for } \ a\leq t<t_{*k},\\ a_{il}(t) \ \text{ for } \ t_{*k}\leq t\leq t_k^*,\\ c_{il}(t)-c_{il}(t_k^*+)+a_{il}(t_k^*+) \ \text{ for } \ t_k^*<t\leq b\\ \qquad (i,l=1,\dots,n;\ k=1,2,\dots) \end{cases} \tag{3.1}$$

and

$$A_k(t)\equiv(a_{ilk}(t))_{i,l=1}^n \ \ (k=1,2,\dots).$$

It is evident that $A_k\in\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$.

For every natural k, consider the system

$$dx(t)=dA_k(t)\cdot x(t)+df(t) \ \text{ for } \ t\in[a,b]. \tag{3.2}$$

We show that the problem $(3.2), (1.2)$ has the unique solution. By Theorem 1.1 from [9] (see also [28]), for this it suffices to verify that the corresponding homogeneous system

$$dx(t) = dA_k(t) \cdot x(t) \quad \text{for} \quad t \in [a, b] \tag{3.2_0}$$

has only the trivial solution under the condition (1.2).

Let us show that the problem $(3.2_0), (1.2)$ has only the trivial solution.

Indeed, if $x = (x_i)_{i=1}^n$ is an arbitrary solution of this problem, then due to Lemma 6.1 from [10], with regard for the conditions (1.6)–(1.9), the vector-function x satisfies the system (1.5) of generalized differential inequalities. But, by the condition (1.10), this system has only the trivial solution under the condition (1.2). Thus $x_i(t) \equiv 0$ $(i = 1, \dots, n)$.

We put

$$t_i = a \quad \text{for} \quad i \in \{1, \dots, n_0\} \quad \text{and} \quad t_i = b \quad \text{for} \quad i \in \{n_0 + 1, \dots, n\}. \tag{3.3}$$

Let now k be an arbitrary fixed natural number, and $x_k = (x_{ik})_{i=1}^n$ be the unique solution of the problem $(3.2), (1.2)$. Then by the conditions (1.6)–(1.9) and the equalities (3.1) and (3.2), using Lemma 2.2 from [8] (or Lemma 6.1 from [10]), we find that the vector-function $x_k = (x_{ik})_{i=1}^n$ satisfies the system

$$\operatorname{sgn}(t - t_i) d|x_{ik}(t)| \le \sum_{l=1}^n |x_{lk}(t)| \, dc_{il}(t) + \operatorname{sgn}[x_{ik}(t)(t - t_i)] df_i(t)$$
$$\text{for} \quad t \in [a, b], \quad t \ne t_i \quad (i = 1, \dots, n),$$
$$(-1)^j d_j |x_{ik}(t_i)| \le \sum_{l=1}^n |x_{lk}(t_i)| \, d_j c_{il}(t_i) + (-1)^j \operatorname{sgn}[x_{ik}(t_i)] df_i(t_i)$$
$$(j = 1, 2; \quad i = 1, \dots, n),$$

where $t_1, \dots, t_n$ are defined by (3.3). From this, we have

$$\operatorname{sgn}(t - t_i) d|x_{ik}(t)| \le \sum_{l=1}^n |x_{lk}(t)| \, dc_{il}(t) + dv(f_i)(t)$$
$$\text{for} \quad t \in [a, b], \quad t \ne t_i \quad (i = 1, \dots, n),$$
$$(-1)^j d_j |x_{ik}(t_i)| \le \sum_{l=1}^n |x_{lk}(t_i)| \, d_j c_{il}(t_i) + d_j v(f_i)(t_i) \quad (j = 1, 2; \ i = 1, \dots, n).$$

Therefore, due to Lemma 2.4 from [8], there exists a number $\rho_0 > 0$ independent of k such that

$$\|x_{ik}\|_s \le \rho_0 \quad (i = 1, \dots, n; \quad k = 1, 2, \dots). \tag{3.4}$$

Let for every natural k, $t_{ik} = a + \frac{1}{k}$ and $\Delta_{ik} =]t_{ik}, b]$ for $i \in \{1, \dots, n_0\}$, and $t_{ik} = b - \frac{1}{k}$ and $\Delta_{ik} = [a, t_{ik}[$ for $i \in \{n_0 + 1, \dots, n\}$. Then, as above, using Lemma 2.2 from [8] and the estimate (3.4), we conclude that there exists a sufficiently large natural number k_0 such that for every $k \in \{k_0 + 1, k_0 + 2, \dots\}$, the vector-function $x_k = (x_{ik})_{i=1}^n$ satisfies the inequalities

$$\begin{gathered}
\operatorname{sgn}(t - t_{ik}) d|x_{ik}(t)| \le -|x_{ik}(t)| d\alpha_i(t) + dq_i(t) \\
\text{for } t \in \Delta_{ik} \ \ (i = 1, \dots, n), \\
(-1)^j d_j |x_{ik}(t_{ik})| \le -|x_{ik}(t_{ik})| d_j \alpha_i(t_{ik}) + d_j q_i(t_{ik}) \\
(j = 1, 2; \ \ i = 1, \dots, n),
\end{gathered} \tag{3.5}$$

where

$$q_i(t) \equiv \rho_0 \Big(\bigvee_{t_{ik}}^{t} (c_{ii}) + \sum_{l=1, l \ne i}^{n} \big(c_{il}(t) - c_{il}(t_{ik}) \big) \Big) \operatorname{sgn}(t - t_{ik}) + \\
+ v(f_i)(t) - v(f_i)(t_{ik}) \ \ (i = 1, \dots, n).$$

Let $i \in \{1, \dots, n_0\}$ and $k \in \{k_0 + 1, k_0 + 2, \dots\}$. Consider the Cauchy problem

$$d\gamma(t) = -\gamma(t)\, d\alpha_i(t), \quad \gamma(t_{ik}) = 1.$$

Due to the condition (1.11), this problem has the unique solution γ_{ik} on the interval $\Delta_{ik\delta} = [t_{ik}, a + \delta]$ for sufficiently small $\delta > 0$. Then $\gamma_{ik}(t) = \gamma_{\beta_i}(t, t_{ik})$ for $t \in \Delta_{ik\delta}$, where the function γ_{α_i} is defined according to (1.4). Moreover, this function is positive and nonincreasing on the interval $t \in \Delta_{ik\delta}$. In addition, we can assume without loss of generality that the conditions of Lemma 2.1 are fulfilled on this interval. Therefore, according to this lemma, (3.5) and the variation-of-constant formula mentioned above, we have the estimate

$$|x_{ik}(t)| \le q_i(t) - q_i(t_{ik}) + \\
+ \gamma_{ik}(t) \Big\{ \rho_0 - \int_{t_{ik}}^{t} \big(q_i(\tau) - q_i(t_{ik}) d\gamma_{ik}^{-1}(\tau) \Big\} \ \text{ for } t \in \Delta_{ik\delta}. \tag{3.6}$$

Taking into account the first equality of the condition (1.12) and the fact that the function q_i is nondecreasing on $\Delta_{ik\delta}$, from (3.6) we get

$$\lim_{t \to a+} \sup \Big\{ |x_{ik}(t)| : \ k = k_0 + 1, k_0 + 2, \dots \Big\} = 0 \ \ (i = 1, \dots, n_0). \tag{3.7}$$

Analogously, using the second parts of the conditions (1.11) and (1.12), as above we show that

$$\lim_{t \to b-} \sup \Big\{ |x_{ik}(t)| : \ k = k_0 + 1, k_0 + 2, \dots \Big\} = 0 \ \ (i = n_0 + 1, \dots, n). \tag{3.8}$$

Without loss of generality, we can assume that the natural number k_0 is such that $a < t_{1k_0} < t_{2k_0} < b$. Consider the sequence x_k $(k = k_0 + 1, k_0 + 2, \dots)$. Then by (3.1), (3.4) and the definition of the solution of the system

(3.2), we have

$$\|x_k(t)-x_k(s)\| \leq \|f(t)-f(s)\| + \left\| \int\limits_s^t dA_k(\tau)\cdot(x_k(\tau)-x_k(s)) \right\| \leq$$

$$\leq \|f(t)-f(s)\| + \rho_0 \bigvee_s^t (A_{k_0}) \ \text{ for } \ t_{1k_0} \leq s < t \leq t_{2k_0},$$

since $A_k(t) = A_{k_0}(t) = A(t)$ for $t \in [t_{1k_0}, t_{2k_0}]$ $(k = k_0+1, k_0+2, \dots)$. Hence there exists a positive number ρ_{k_0} such that

$$\bigvee_{t_{1k_0}}^{t_{2k_0}} (x_k) \leq \rho_{k_0} \ \ (k = k_0+1, k_0+2, \dots).$$

Consequently, in view of Helly's choose theorem, without loss of generality we can assume that the sequence x_k $(k = k_0+1, k_0+2, \dots)$ converges to some function $x_0 = (x_{io})_{i=1}^n \in \mathrm{BV}([t_{1k_0}, t_{2k_0}b], \mathbb{R}^n)$. If we continue this process, then in a standard way we can assume without loss of generality that

$$\lim_{k\to\infty} x_k(t) = x_0(t) \ \text{ for } \ t \in]a,b[\,, \tag{3.9}$$

where $x_0 = (x_{io})_{i=1}^n \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$.

Let now $[a_0,b_0] \subset]a,b[$ be an arbitrary closed interval. Then

$$\|x_k(t)-x_k(s)\| \leq l_k + \|g_(t) - g_(s)\|$$
$$\text{for } \ a_0 \leq s < t \leq b_0 \ \ (k = k_0+1, k_0+2, \dots),$$

where

$$g(t) = f(t) + \int\limits_{a_0}^t dA_{k_0}(\tau)\cdot x_0(\tau), \quad l_k = \left\| \int\limits_{a_0}^{b_0} dV(A_{k_0})(\tau)\cdot|x_k(\tau)-x_0(\tau)| \right\|.$$

On the other hand, due to (3.9) and the Lebesgue theorem, we have $l_k \to 0$ as $k \to \infty$. Therefore, according to Lemma 2.3 from [7],

$$\lim_{k\to\infty} x_k(t) = x_0(t) \ \text{ uniformly on } \ [a_0,b_0].$$

Moreover, by (3.7), the sequences $\{x_{ik}\}_{k=1}^\infty (i = 1,\dots,n_0)$ converge uniformly on the interval $]a,t_0]$, and by (3.8), the sequences $\{x_{ik}\}_{k=1}^\infty$ $(i = n_0+1,\dots,n)$ converge uniformly on the interval $[t_0,b[$ for every $t_0 \in]a,b[$. Therefore, there exist one-sided limits $x_{i0}(a+)$ $(i = 1,\dots,n_0)$ and $x_{i0}(b-)$ $(i = n_0+1,\dots,n)$ and, in addition, they are equal to zero. Thus, due to (3.1) and (3.2), we have established that $x_0 \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$ is a solution of the problem (1.1), (1.2).

Let us show that the problem (1.1), (1.2) has only one solution. Let x and y be two arbitrary solutions of the problem. Then the function

$z(t) \equiv x(t) - y(t)$, $z(t) \equiv (z_i(t))_{i=1}^n$, will be a solution of the homogeneous problem

$$dz(t) = dA(t) \cdot z(t),$$
$$z_i(a+) = 0 \ \ (i = 1, \dots, n_0), \quad z_i(b-) = 0 \ \ (i = n_0 + 1, \dots, n).$$

From this, by (1.6)–(1.9), z is a solution of the system of differential inequalities (1.5) under the condition (1.2). Thus, due to the condition (1.10), we conclude that $z(t) \equiv 0$. The theorem is proved. $\square$

Proof of Corollary 1.1. The proof of this corollary slightly differs from that of Lemma 2.6 given in [3]. We give the main aspect of this proof for completeness.

It suffices to show that the problem (1.5), (1.2) has only the trivial nonnegative solution.

Let $(x_i)_{i=1}^n$ be an arbitrary nonnegative solution of the problem (1.5), (1.2). Let $i \in \{1, \dots, n_0\}$ be fixed, and ε be an arbitrary sufficiently small positive number. Then by (1.13)–(1.15) and Hölder's inequality, we have

$$|x_i(t)| \le |x_i(a+\varepsilon)| + \sum_{\sigma=0}^{2} \sum_{k=0}^{n} \left(\|h_{ik}\|_{\mu, s_\sigma(\beta_k)} \left| \int_{a+\varepsilon}^{t} |x_k(\tau)|^{\frac{\nu}{2}} \, ds_\sigma(\beta_k)(\tau) \right|^{\frac{2}{\nu}} \right) \quad \text{for } \ t \in \,]a, b].$$

This, in view of Minkowski's inequality, implies

$$\|x_i\|_{\nu, s_j(\beta_i)} \le |x_i(a+\varepsilon)| \big(s_j(\beta_i)(b) - s_j(\beta_i)(a) \big)^{\frac{1}{\nu}} + \sum_{\sigma=0}^{2} \sum_{k=0}^{n} \|h_{ik}\|_{\mu, s_\sigma(\beta_k)} \times$$
$$\times \left(\int_a^b \left| \int_{a+\varepsilon}^{t} |x_k(\tau)|^{\frac{\nu}{2}} ds_\sigma(\beta_k)(\tau) \right|^2 ds_j(\beta_i)(t) \right)^{\frac{1}{\nu}} \quad (j = 0, 1, 2). \quad (3.10)$$

On the other hand, by virtue of Hölder's inequality, in case $\sigma^2 + j^2 + (i-k)^2 > 0$, $j = 0$, and by the generalized Wirtinger's inequalities (see Lemma 2.5 from [3]), in the other case we have

$$\lim_{\varepsilon \to 0} \left(\int_a^b \left| \int_{a+\varepsilon}^{t} |x_k(\tau)|^{\frac{\nu}{2}} \, ds_\sigma(\beta_k)(\tau) \right|^2 ds_j(\beta_i)(t) \right)^{\frac{1}{\nu}} \le$$
$$\le \lambda_{k\sigma ij} \left(\int_{a+}^{b} |x_k(\tau)|^{\nu} \, ds_\sigma(\beta_k)(\tau) \right)^{\frac{1}{\nu}} \quad (j, \sigma = 0, 1, 2; \quad k = 1, \dots, n).$$

By this, (1.2) and (3.10), we get

$$\|x_i\|_{\nu,s_j(\beta_i)} \leq$$

$$\leq \sum_{\sigma=0}^{2}\sum_{k=0}^{n}\lambda_{k\sigma ij}\|h_{ik}\|_{\mu,s_\sigma(\beta_k)}\|x_k\|_{\nu,s_\sigma(\beta_k)} \ \ (j=0,1,2;\ i=1,\dots,n_0). \tag{3.11}$$

Analogously, we show that the estimate (3.11) is valid for $i \in \{n_0 + 1,\dots,n\}$, as well.

Therefore,

$$(I_{3n} - \mathcal{H})r \leq 0, \tag{3.12}$$

where $r \in \mathbb{R}^{3n}$ is the vector with the components

$$r_{i+nj} = \|x_i\|_{\nu,s_j(\beta_i)} \ \ (j = 0,1,2;\ \ i = 1,\dots,n).$$

From (3.12), due to (1.2) and (1.16), we find that $r = 0$ and $x_i(t) \equiv 0$ $(i = 1,\dots,n)$. Consequently, the problem (1.5), (1.2) has no nontrivial nonnegative solution. The corollary is proved. □

Proof of Theorem 1.2. It suffices to show that the problem (1.5), (1.2), where $c_{il}(t) = h_{il}\beta_i(t) + \beta_{il}(t)$ $(i,l = 1,\dots,n)$, has only the trivial nonnegative solution.

Let $(x_i)_{i=1}^n$ be an arbitrary nonnegative solution of the problem (1.5), (1.2). Let $i \in \{1,\dots,n_0\}$ be fixed. Then from (1.5), we have

$$dx_i(t) \leq x_i(t)da_i(t) + dg_i(t) \ \text{ for } \ t \in]a,b], \tag{3.13}$$

where

$$g_i(t) = g_{1i}(t) + g_{2i}(t),$$

$$g_{1i}(t) = \sum_{l=1,l\neq i}^{n} r_l h_{il}\big(\beta_i(t) - \beta_i(a+)\big) \ \text{ and } \ g_{2i}(t) = \sum_{l=1}^{n} r_l\big(\beta_{il}(t) - \beta_{il}(a+)\big)$$

and

$$r_l = \sup\big\{\|x_l(t)\| :\ t \in]a,b]\big\} \ \ (l = 1,\dots,n).$$

Hence the function x_i satisfies the inequality (2.2) for $t_0 = a$, $\alpha(t) \equiv a_i(t)$ and $q(t) \equiv g_i(t)$. Moreover, by (1.17), the condition (2.1) is fulfilled. Therefore, according to Lemma 2.1, we find

$$x_i(t) \leq y_i(t) \ \text{ for } \ a < t \leq b, \tag{3.14}$$

where y_i is the solution of the Cauchy problem of the equation

$$dy(t) = y(t)da_i(t) + dg_i(t), \ \ y(a+) = 0.$$

Due to the variation-of-constant formula mentioned above, we have

$$y_i(t) = g_i(t) - \lambda_i(t)\int_{a+}^{t} g_i(\tau)d\lambda_i^{-1}(\tau) \ \text{ for } \ t \in]a,b], \tag{3.15}$$

where λ_i is the solution of the Cauchy problem

$$d\lambda(t) = \lambda(t)da_i(t), \ \ \lambda(a+) = 1.$$

From (3.15), using the formula of integration-by-parts (see [29, Theorem I.4.33]), we conclude

$$y_i(t) = \lambda_i(t)\psi_i(t), \tag{3.16}$$

where

$$\psi_i(t) = \int\limits_{a+}^{t} \lambda_i^{-1}(\tau)dg_i(\tau) - \sum_{a<\tau<t} d_1 g_i(t) d_1\lambda_i(\tau) + \sum_{a<\tau<t} d_2 g_i(t) d_2\lambda_i(\tau) \quad \text{for } \ a < t \le b.$$

Moreover, by the equalities

$$d_j\lambda_i^{-1}(t) = -\lambda_i^{-1}(t)\cdot \big(1 + (-1)^j d_j a_i(t)\big)^{-1} d_j a_i(t) \ \ (j = 1, 2),$$

we have

$$\psi_i(t) = \psi_{1i}(t) + \psi_{2i}(t) \ \text{ for } \ a < t \le b,$$

where

$$\psi_{ji}(t) = \int\limits_{a+}^{t} \lambda_i^{-1}(\tau)\, d\mathcal{A}(g_{ji}, a_i)(\tau) \ \text{ for } \ a < t \le b \ \ (j = 1, 2).$$

Then by the equality $d\lambda_i^{-1}(t) = -\lambda_i^{-1}(t)d\mathcal{A}(a_i.a_i)(t)$ (see Lemma 2.1 from [11]) and the definition of the operator $\mathcal{A}$, we get

$$\psi_{1i}(t) = \sum_{l=1, l\neq i}^{n} r_l h_{il} \int\limits_{a+}^{t} \lambda_i^{-1}(\tau)\, d\mathcal{A}(a_i, a_i)(\tau) = \sum_{l=1, l\neq i}^{n} r_l \frac{h_{il}}{|h_{ii}|} \big(\lambda_i^{-1}(t) - 1\big)$$

and

$$\begin{aligned}
\psi_{2i}(t) &= r_i \int\limits_{a+}^{t} \lambda_i^{-1}(\tau) d\mathcal{A}(\zeta_i, a_i)(\tau) \le \\
&\le r_i\lambda_i^{-1}(t)\Big(V\big(\mathcal{A}(\zeta_i, a_i)\big)(t) - V\big(\mathcal{A}(\zeta_i, a_i)\big)(a+)\Big) \le \\
&\le r_i\eta_i\lambda_i^{-1}(t) \ \text{ for } \ a < t \le b.
\end{aligned}$$

Hence, in view of (3.14) and (3.16), we find

$$r_i \le \eta_i r_i + \sum_{l=1, l\neq i}^{n} r_l \frac{h_{il}}{|h_{ii}|} \tag{3.17}$$

for $i \in \{1, \dots, n_0\}$.

Analogously, we show the validity of the estimate (3.17) for $i \in \{n_0 + 1, \dots, n\}$, too.

Thus the constant vector $r=(r_i)_{i=1}^n$ satisfies the system of inequalities

$$(I-\mathcal{H})r\leq 0. \tag{3.18}$$

Therefore, according to the condition (1.16), we have $r=0$ and $x_i(t)\equiv 0$ $(i=1,\dots,n)$. The theorem is proved. □

Let us show Remark 1.2. Due to the condition (1.19), it is evident that (3.17) implies that the constant vector r, appearing in the proof of Theorem 1.2, satisfies the system (3.18), where the constant matrix $\mathcal{H}=(\xi_{il})_{i,l=1}^n$ is defined by (1.20). Therefore, by (1.16), we obtain $x_i(t)\equiv 0$ $(i=1,\dots,n)$ as in the proof of Theorem 1.2.

Proof of Theorem 1.3. Let the vector-function $x^*=(x_i^*)_{i=1}^n$ be the nontrivial nonnegative solution of the system (1.5) under the condition (1.2). Obviously, it will be a solution of the system (2.10), (2.11), where $C_{11}(t)\equiv C(T)$, $C_{12}(t)\equiv O_{n_1\times n_2}$, $C_{21}(t)\equiv O_{n_2\times n_1}$, $C_{22}(t)\equiv O_{n_2\times n_2}$, $t_i=a$ and $l_i(x_1,\dots,x_n)\equiv -d_2x_i(a)$ for $i\in\{1,\dots,n_0\}$, $t_i=b$ and $l_i(x_1,\dots,x_n)\equiv d_1x_i(b)$ for $i\in\{n_0+1,\dots,n\}$, and $N_n=\varnothing$. In addition, the condition (1.21) of Theorem 1.3 is equivalent to the condition (2.13) of Lemma 2.2. Therefore, according to Lemma 2.2 and Remark 2.1, there exist a matrix-function $\widetilde{A}\in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ and nondecreasing functions $\widetilde{\alpha}_i:[a,b]\to\mathbb{R}$ $(i=1,\dots,n)$ satisfying the conditions (2.14))–(2.18) of Lemma 2.2 and the condition (1.22), and a constant vector $c=(c_i)_{i=1}^n\in\mathbb{R}^n$ such that the system

$$dz(t)=d\widetilde{A}(t)\cdot z(t)$$

under the condition

$$z_i(t_i)=l_i(z_1,\dots,z_n)+c_i\ \ (i=1,\dots,n)$$

is unsolvable, where $z(t)=(z_i(t))_{i=1}^n$ and, due to the equalities (2.30), we have $\widetilde{l}_i(z_1,\dots,z_n)\equiv l_i(z_1,\dots,z_n)$. Consequently, using the mapping $x_i(t)=z_i(t)+c_i$ $(i=1,\dots,n)$ and definitions of the functionals l_i $(i=1,\dots,n)$, it is not difficult to see that the problem (1.1), (1.2) is not solvable as well, where $A(t)\equiv\widetilde{A}(t)$ and $f(t)\equiv\widetilde{A}(t)\cdot c$. Moreover, it is evident that in this case the conditions (1.6)–(1.9) coincide with the conditions (2.14)–(2.17), respectively. From the conditions (2.18) (or (1.22)) and (1.11) it follows that the functions $\widetilde{\alpha}_i$ $(i=1,\dots,n)$ satisfy the condition (1.22) as well. Therefore there exists the sufficiently small $\delta>0$ such that

$$\begin{aligned}1+(-1)^jd_j\widetilde{\beta}_i(t)>0\ \ \text{for}\ \ t\in\,]a,a+\delta[\ \ (i=1,\dots,n_0)&\\ \text{or}\ \ t\in\,]b-\delta,b[\ \ (i=n_0+1,\dots,n),&\end{aligned} \tag{3.19}$$

where $\widetilde{\beta}_i(t)\equiv\widetilde{\alpha}_i(t)\,\mathrm{sgn}(n_0+\frac{1}{2}-i)$.

Let us show that the condition (1.12) is valid. Let $i\in\{1,\dots,n_0\}$ be fixed and let a natural number k_0 be such that $a+\frac{1}{k}<a+\delta$ for $k>k_0$. Then, by the condition (3.19), there exists the nonnegative function $\gamma_{\widetilde{\beta}_i}(t)$ $(t\in\,]a,a+\delta[)$, since the corresponding Cauchy problem is uniquely solvable.

Let $t \in]a, a+\delta[$ and $k > k_0$ be such that $a + \frac{1}{k} < t$. Then, by definition of the solution, we have

$$\gamma_{\widetilde{\beta}_i}(t) = 1 + \int\limits_{a+\frac{1}{k}}^{t} \gamma_{\widetilde{\beta}_i}(\tau)\, d\widetilde{\beta}_i(\tau) \leq$$

$$\leq 1 + \int\limits_{a+\frac{1}{k}}^{t} \gamma_{\widetilde{\beta}_i}(\tau)\, d\alpha_i(\tau) + \int\limits_{a+\frac{1}{k}}^{t} \gamma_{\widetilde{\beta}_i}(\tau)\, d\big(\widetilde{\alpha}_i(\tau) - \alpha_i(\tau)\big).$$

Consequently, the function $\gamma_{\widetilde{\beta}_i}$ is a solution of the problem

$$\operatorname{sgn}(t - t_{ik}) d\gamma_(t) \leq \gamma_(t) d\widetilde{\beta}_i(t) \ \text{ for } \ t \in]t_{ik}, a+\delta[\,, \quad \gamma(t_{ik}) = 1,$$

where $t_{ik} = a + \frac{1}{k}$. On the other hand, the function $\gamma_{\widetilde{\beta}_i}$ is the unique solution of the problem

$$\operatorname{sgn}(t - t_{ik}) d\gamma_(t) = \gamma_(t) d\beta_i(t) \ \text{ for } \ t \in]t_{ik}, a+\delta[\,, \quad \gamma(t_{ik}) = 1.$$

Therefore, due to Lemma 2.1, we have

$$\gamma_{\widetilde{\beta}_i}(t) \leq \gamma_{\beta_i}(t) \ \text{ for } \ t \in]t_{ik}, a+\delta[\,.$$

From this, by (1.12) it follows that the function $\gamma_{\widetilde{\beta}_i}$ satisfies the first equality of the condition (1.12).

Analogously we show the second equality of the condition (1.12).

Let now the condition (1.23) hold. By definition of the matrix-function $A(t) \equiv \widetilde{A}(t)$ (see (2.21), (2.25)–(2.27)), we get

$$d_j A(t) = \left(\eta_i(t) d_j c_{il}(t) \operatorname{sgn}\left(n_0 + \frac{1}{2} - i\right)\right)_{i,l=1}^{n} \ \text{ for } \ t \in [a,b] \ \ (j = 1, 2).$$

From this, by (1.23), it follows that the condition (1.3) holds. Thus the theorem is proved. $\square$

Consider now Remark 1.3. The first case is evident. Indeed, by definition of the matrix-function $A = (a_{il})_{i,l=1}^{n}$, we have

$$d_j a_{il}(t) = \eta_l(t) d_j c_{il}(t) \operatorname{sgn}\left(n_0 + \frac{1}{2} - i\right) \ \text{ for } \ t \in [a,b] \ \ (j = 1, 2; \ i, l = 1, \dots, n)$$

and

$$|d_j a_{il}(t)| \leq |d_j c_{il}(t)| \ \text{ for } \ t \in [a,b] \ \ (j = 1, 2; \ \ i, l = 1, \dots, n).$$

Taking this into account, by (1.24), we have

$$\sum_{l=1}^{n} |d_j a_{il}(t)| < 1 \ \text{ for } \ t \in [a,b] \ \ (j = 1, 2; \ \ i = 1, \dots, n).$$

Hence the condition (1.23) holds.

Let now the condition (1.25) be valid. Then we have

$$\sum_{l=1,\, l\neq i}^{n} \left|\varepsilon_i d_j c_{il}(t)\,\mathrm{sgn}\left(n_0+\frac{1}{2}-i\right)\right| \leq$$

$$\leq \varepsilon_i+(-1)^j\varepsilon_i d_j c_{ii}(t)\,\mathrm{sgn}\left(n_0+\frac{1}{2}-i\right)\leq 1+(-1)^j\varepsilon_i d_j c_{ii}(t)\,\mathrm{sgn}\left(n_0+\frac{1}{2}-i\right)$$

$$\text{for } t\in[a,b] \ \ (j=1,2;\ \ i=1,\dots,n). \tag{3.20}$$

Therefore, by Hadamard's theorem (see [14, p. 382]), the condition (1.23) holds. Remark 1.3 is proved analogously to the conditions (1.26).

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NECESSARY CONDITIONS FOR THE EXISTENCE AND SUFFICIENT CONDITIONS FOR THE NONEXISTENCE OF SOLUTIONS TO A CERTAIN FRACTIONAL TELEGRAPH EQUATION

Abstract. We consider the Cauchy problem for the semi-linear fractional telegraph equation

$$\mathbf{D}_{0|t}^{2\gamma}u+\mathbf{D}_{0|t}^{\gamma}u+(-\Delta)^{\frac{\beta}{2}}u=h(x,t)|u|^p$$

with the given initial data, where $p>1$, $\frac{1}{2}\le\gamma<1$ and $0<\beta<2$. The Nonexistence results and the necessary conditions for global existence are established.

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რეზიუმე. განხილულია კოშის ამოცანა ნახევრადწრფივი ფრაქციონალური ტელეგრაფის განტოლებისათვის

$$\mathbf{D}_{0|t}^{2\gamma}u+\mathbf{D}_{0|t}^{\gamma}u+(-\Delta)^{\frac{\beta}{2}}u=h(x,t)|u|^p$$

მოცემული საწყისი პირობებით, სადაც $p>1$, $\frac{1}{2}\le\gamma<1$ და $0<\beta<2$. დადგენილია არარსებობის შედეგები და აუცილებელი პირობები გლობალური არსებობისათვის.

1. Introduction

The telegraph equation has recently been considered by many authors, see for instance [2, 3, 8, 12, 15] and references therein. Cascaval *et al.* [2] discussed the fractional telegraph equations

$$D^{2\beta}u + D^{\beta}u - \Delta u = 0$$

dealing with well-posedness and presenting a study involving asymptotic by using the Riemann–Liouville approach, it has been shown that as t tends to infinity, solutions of the telegraph equations can be approximated by solving the parabolic part. Beghin and Orsingher [15] discussed the time fractional telegraph equations and telegraph processes with Brownian time, showing that some processes are governed by time-fractional telegraph equations with well-posedness. Chen *et al.* [3] also discussed and derived the solution of the time-fractional telegraph equation with three kinds of nonhomogeneous boundary conditions.

To focus our motivation, we shall mention below only some results related to Todorova and Yordanov [20] for the Cauchy problem

$$u_{tt} - \Delta u + u_t = |u|^p, \quad u(0) = u_0, \quad u_t(0) = u_1. \tag{1}$$

It has been shown that the damped wave equation has the diffuse structure as $t \to \infty$ (see e.g. [20, 22]). This suggests that problem (1) should have $p_c(n) := 1 + \frac{2}{n}$ as critical exponent which is called the Fujita exponent [5, 7] named after Fujita, in general space dimension. Indeed, Todorova and Yordanov have showed that the critical exponent is exactly $p_c(n)$, that is, if $p > p_c(n)$, then all small initial data solutions of (1) are global, while if $1 < p < p_c(n)$, then all solutions of (1) with initial data having positive average value blow-up in finite time regardless of the smallness of the initial data.

In this paper, we consider the following nonlinear fractional telegraph equation:

$$\begin{cases} \mathbf{D}^{2\gamma}_{0|t}u + \mathbf{D}^{\gamma}_{0|t}u + (-\Delta)^{\frac{\beta}{2}}u = h(x,t)|u|^p & \text{in } \ Q = \mathbb{R}^n \times \mathbb{R}_+ \\ u(0,x) = u_0(x) \ \text{ and } \ u_t(0,x) = u_1(x), & \qquad x \in \mathbb{R}^n, \end{cases} \tag{2}$$

where $\mathbf{D}^{\gamma}_{0|t}$ (resp. $\mathbf{D}^{2\gamma}_{0|t}u$) denotes the so-called fractional time-derivative of power γ (resp. 2γ), $\gamma \in [1/2, 1]$ in the Caputo sense (see [11], [18]), $(-\Delta)^{\frac{\beta}{2}}$ ($\beta \in [0,2]$) is the ($\beta/2$)-fractional power of the Laplacian ($-\Delta$) defined by

$$(-\Delta)^{\frac{\beta}{2}}v(x,t) = \mathcal{F}^{-1}(|\xi|^{\beta}\mathcal{F}(v)(\xi))(x,t),$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ is its inverse, $h(x,t)$ is the positive function satisfying certain growth condition. We will generalize the results obtained in [20] to the problem (2). The nonexistence results as well as the necessary conditions for local and global existence are obtained.

The difficulties we encounter here arise mainly from the nonlocal nature of the fractional derivative operators; to overcome these difficulties, we present a brief and versatile proof of the equation (2) which is based on the method used by Mitidieri and Pohozaev [14], Pohozaev and Tesei [17], Hakem [6], Berbiche [1], Fino and Karch [4] and Zhang [22]. This method consists in a judicious choice of the test function in the weak formulation of the sought for solution of (2).

This paper is organized as follows: in Section 2, we present some definitions, properties concerning fractional derivative and prove results concerning positivity of solutions; Section 3 contains the proof of the blow-up result; in Section 4, we establish some necessary conditions for local and global existence.

2. Preliminaries

In this section we present some definitions of a fractional derivative and a result concerning the positivity of a solution.

The left-hand fractional derivative and the right-hand fractional derivative in the Riemann–Liouville sense for $\Psi \in L^1(0,T)$, $0<\alpha<1$, are defined as follows:

$$D^{\alpha}_{0|t}\Psi(t) = \frac{1}{\Gamma(1-\alpha)}\,\frac{d}{dt}\int_0^t \frac{\Psi(\sigma)}{(t-\sigma)^{\alpha}}\,d\sigma,$$

where the symbol Γ stands for the usual Euler gamma function, and

$$D^{\alpha}_{t|T}\Psi(t) = -\frac{1}{\Gamma(1-\alpha)}\,\frac{d}{dt}\int_t^T \frac{\Psi(\sigma)}{(\sigma-t)^{\alpha}}\,d\sigma,$$

respectively.

The Caputo derivative

$$\mathbf{D}^{\alpha}_{0|t}\Psi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\Psi^{'}(\sigma)}{(t-\sigma)^{\alpha}}\,d\sigma$$

requires $\Psi^{'} \in L^1(0,T)$. Clearly, we have

$$D^{\alpha}_{0|t}\Psi(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\Psi(0)}{t^{\alpha}} + \int_0^t \frac{\Psi^{'}(\sigma)}{(t-\sigma)^{\alpha}}\,d\sigma\right]$$

and

$$D^{\alpha}_{t|T}\Psi(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\Psi(T)}{(T-t)^{\alpha}} - \int_t^T \frac{\Psi^{'}(\sigma)}{(\sigma-t)^{\alpha}}\,d\sigma\right]. \tag{3}$$

Therefore, the Caputo derivative is related to the Riemann–Liouville derivative by

$$\mathbf{D}^{\alpha}_{0|t}\Psi(t) = D^{\alpha}_{0|t}[\Psi(t)-\Psi(0)] \tag{4}$$

and, in general,

$$\mathbf{D}^{\alpha}_{0|t}\Psi(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{\Psi^{(n)}(\sigma)}{(t-\sigma)^{2\gamma-n}}\, d\sigma, \quad n = [\alpha]+1, \quad \alpha > 0,$$

we have the formula of integration by parts (see [18, p. 26]),

$$\int\limits_0^T f(t) D^{\alpha}_{0|t} g(t)\, dt = \int\limits_0^T g(t) D^{\alpha}_{t|T} f(t)\, dt, \quad 0 < \alpha < 1.$$

We show the following result:

Proposition 1 (Positivity of solutions)**.** *If $u_0 \geq 0$, $u_1 = 0$, $f \geq 0$ and u is a solution of the nonhomogeneous problem*

$$\begin{cases} \mathbf{D}^{2\gamma}_{0|t} u + \mathbf{D}^{\gamma}_{0|t} u + (-\Delta)^{\frac{\beta}{2}} u = f(x,t), & (x,t) \in \mathbb{R}^n \times \mathbb{R}_+ \\ u(0,x) = u_0(x) \ \ \textit{and} \ \ u_t(0,x) = 0, & x \in \mathbb{R}^n, \end{cases} \tag{5}$$

then u is nonnegative.

Proof. Applying the temporal Laplace and spatial Fourier transforms to (5), we get

$$s^{2\gamma}\widetilde{u}(x,s) - s^{2\gamma-1}u_0(x) + s^{\gamma}\widetilde{u}(x,s) + (-\Delta)^{\beta/2}\widetilde{u}(x,s) = \widetilde{f}(x,s),$$

$$s^{2\gamma}\widehat{\widetilde{u}}(k,s) - s^{2\gamma-1}\widehat{u}_0(k) + s^{\gamma}\widehat{\widetilde{u}}(k,s) + |k|^{\beta}\widehat{\widetilde{u}}(k,s) = \widehat{\widetilde{f}}(k,s).$$

Then we derive

$$\widehat{\widetilde{u}}(k,s) = \frac{s^{2\gamma-1}+s^{\gamma-1}}{s^{2\gamma}+s^{\gamma}+|k|^{\beta}}\hat{u}_0(k) + \frac{1}{s^{2\gamma}+s^{\gamma}+|k|^{\beta}}\widehat{\widetilde{f}}(k,s) :=$$

$$:= \widehat{\widetilde{G}}_1(k,s)\hat{u}_0(k) + \widehat{\widetilde{G}}_2(k,s)\widehat{\widetilde{f}}(k,s), \tag{6}$$

where

$$\widehat{\widetilde{G}}_2(k,s) := \frac{1}{s^{2\gamma}+s^{\gamma}+|k|^{\beta}}\,, \tag{7}$$

$$\begin{gathered} \widehat{\widetilde{G}}_1(k,s) := \frac{s^{2\gamma-1}+s^{\gamma-1}}{s^{2\gamma}+s^{\gamma}+|k|^{\beta}} := \widehat{\widetilde{G}}_{1,1}(k,s) + \widehat{\widetilde{G}}_{1,2}, \\ \widehat{\widetilde{G}}_{1,1}(k,s) := \frac{s^{2\gamma-1}}{s^{2\gamma}+s^{\gamma}+|k|^{\beta}}\,, \quad \widehat{\widetilde{G}}_{1,2} := \frac{s^{\gamma-1}}{s^{2\gamma}+s^{\gamma}+|k|^{\beta}}\,. \end{gathered} \tag{8}$$

We invert the Fourier transform in (6) and obtain

$$u(x,t) = \int\limits_{\mathbb{R}^n} G_1(x-y)u_0(y)\, dy + \int\limits_{\mathbb{R}^n}\int\limits_0^t G_2(x-y,\tau)f(x,\tau)\, d\tau\, dy,$$

where $G_1(x,t)$, $G_2(x,t)$ is the corresponding Green's function or the fundamental solution obtained when $u_0(x) = \delta(x)$, $f = 0$ and $u_0(x) = 0$, $f(x,t) = \delta(x)\delta(t)$, respectively, which is characterized by (7), (8).

To express the Green's function, we recall two Laplace transform pairs and one Fourier transform pair,

$$F_1^{(\gamma)}(ct) := t^{-\gamma} M_\gamma(ct^{-\gamma}) \stackrel{\mathcal{L}}{\longleftrightarrow} s^{\gamma-1} e^{-cs^\gamma},$$
$$F_2^{(\gamma)}(ct) := c w_\gamma(ct) \stackrel{\mathcal{L}}{\longleftrightarrow} e^{-(s/c)^\gamma},$$

where M_γ denotes the so-called M function (of the Wright type) of order γ, which is defined by

$$M_\mu(z) = \sum_{i=0}^{\infty} \frac{(-z)^i}{i!\Gamma(-\mu i + (1-\mu))}, \quad 0 < \mu < 1.$$

Mainardi, see, for example, [12] has shown that $M_\mu(z)$ is positive for $z > 0$, the other general properties can be found in some references (see e.g. [12, 13, 16]).

w_μ $(0 < \mu < 1)$ denotes the one-sided stable (or Lévy) probability density which can be explicitly expressed by the Fox function [19]

$$w_\mu(t) = \mu^{-1} t^{-2} H_{11}^{10} \left(t^{-1} \middle| \begin{array}{c} (-1, 1) \\ (-1/\mu, 1/\mu) \end{array} \right).$$

It is well known that

$$e^{-\lambda|x|^\beta} \stackrel{\mathcal{F}}{\longrightarrow} p(x, \lambda), \quad 0 < \beta \le 2,$$

where $p(x, \lambda)$ is the probability density function.

From ([21, pp. 259–263]) we have

$$p(x, \lambda) := \int_0^{+\infty} f_{\lambda, \frac{\beta}{2}}(\tau) T(x, \tau)\, d\tau \quad \text{for} \quad 0 < \beta \le 2,$$

and

$$p(x, \lambda) = T(x, \lambda) \quad \text{if} \quad \beta = 2,$$

where

$$f_{\lambda, \frac{\beta}{2}}(s) = \int_{\tau - i\infty}^{\tau + i\infty} e^{zs - \lambda z^{\frac{\beta}{2}}}\, dz \ge 0, \quad T(x, \lambda) = \Big(\frac{1}{4\pi\lambda}\Big)^{\frac{n}{2}} e^{-\frac{|x|^2}{4\lambda}}, \quad \tau > 0, \quad \lambda > 0.$$

Then the Fourier–Laplace transform of Green's function G_1 can be rewritten in the integral form

$$\widehat{\widetilde{G}}_1(k, s) = (s^{2\gamma-1} + s^{\gamma-1}) \int_0^{+\infty} e^{-v(s^{2\gamma} + s^\gamma + |k|^\beta)}\, dv =$$
$$= \int_0^{+\infty} (s^{2\gamma-1} e^{-vs^{2\gamma}}) e^{-vs^\gamma} e^{-v|k|^\beta}\, dv + \int_0^{+\infty} (s^{\gamma-1} e^{-vs^\gamma}) e^{-vs^{2\gamma}} e^{-v|k|^\beta}\, dv =$$

$$= \int_0^{+\infty} \mathcal{L}\big\{F_1^{(2\gamma)}(vt)\big\}\mathcal{L}\big\{F_2^{(\gamma)}(v^{-1/\gamma}t)\big\}\mathcal{F}\{p(x,v)\}\,dv +$$

$$+ \int_0^{+\infty} \mathcal{L}\big\{F_1^{(\gamma)}(vt)\big\}\mathcal{L}\big\{F_2^{(2\gamma)}(v^{-1/2\gamma}t)\big\}\mathcal{F}\{p(x,v)\}\,dv =$$

$$= \int_0^{+\infty} \mathcal{L}\Big[F_1^{(2\gamma)}(vt) * F_2^{(\gamma)}(v^{-1/\gamma}t)\Big]\mathcal{F}\{p(x,v)\}\,dv +$$

$$+ \int_0^{+\infty} \mathcal{L}\Big[F_1^{(\gamma)}(vt) * F_2^{(2\gamma)}(v^{-1/2\gamma}t)\Big]\mathcal{F}\{p(x,v)\}\,dv.$$

Going back to the space-time domain, we obtain the relation

$$G_1(x,t) = \int_0^{+\infty} F_1^{(2\gamma)}(vt) * F_2^{(\gamma)}(v^{-1/\gamma}t)p(x,v)\,dv +$$

$$+ \int_0^{+\infty} F_1^{(\gamma)}(vt) * F_2^{(2\gamma)}(v^{-1/2\gamma}t)p(x,v)\,dv.$$

By the same technique, we obtain the expression of $G_2(x,t)$

$$\widehat{\widetilde{G}}_2(k,s) = \int_0^{+\infty} e^{-v(s^{2\gamma}+s^{\gamma}+|k|^{\beta})}\,dv = \int_0^{+\infty} e^{-vs^{2\gamma}}e^{-vs^{\gamma}}e^{-v|k|^{\beta}}\,dv =$$

$$= \int_0^{+\infty} \mathcal{L}\left[F_2^{(2\gamma)}(v^{-1/2\gamma}t) * F_2^{(\gamma)}(v^{-1/\gamma}t)\right]\mathcal{F}\{p(x,v)\}\,dv.$$

Going back to the space-time domain, we obtain the relation

$$G_2(x,t) = \int_0^{+\infty} \left[F_2^{(2\gamma)}(v^{-1/2\gamma}t) * F_2^{(\gamma)}(v^{-1/\gamma}t)\right]\{p(x,v)\}\,dv.$$

Thus, by the nonnegativity property of functions $F_1^{(\gamma)}$, $F_2^{(\gamma)}$, $p(x,v)$, we deduce that the solution u is nonnegative. $\square$

3. Blow-up of Solutions

This section is devoted to the blow-up of solutions of the problem (2), where we have assumed that the function h satisfies $h\left(Ry, T^{\beta/\gamma}\tau\right) = R^{\sigma}T^{\rho\beta/\gamma}h\left(y,\tau\right)$ for large R and T, where σ, ρ are some positive constants, under some restrictions on the initial data.

Definition 1. Let $u_0 \geq 0$, $u_0 \in L^1(\mathbb{R}^n)$, $u_1 = 0$. A function $u \in L^p_{loc}(Q_T)$ is a weak solution to (2) defined on $Q_T := \mathbb{R}^n \times [0,T]$, if

$$\int\limits_{Q_T} h\varphi|u|^p \, dx\, dt + \int\limits_{\mathbb{R}^n} u_0 D^{2\gamma-1}_{t|T}\varphi(0)\, dx + \int\limits_{Q_T} u_0 D^{\gamma}_{t|T}\varphi \, dx\, dt =$$
$$= \int\limits_{Q_T} uD^{2\gamma}_{t|T}\varphi \, dx\, dt + \int\limits_{Q_T} u(-\Delta)^{\frac{\beta}{2}}\varphi \, dx\, dt + \int\limits_{Q_T} uD^{\gamma}_{t|T}\varphi \, dx\, dt$$

for any test function $\varphi \in C^{2,1}_{x,t}(Q_T)$ such that

$$\varphi(x,T) = D^{2\gamma-1}_{t|T}\varphi(x,T) = 0.$$

If in the above definition $T = +\infty$, the solution is called global.

We now are in a position to announce our first result.

Theorem 1. *Let* $n \geq 1$, $1 < p < \min\left(\rho + 1, \frac{1}{1-\gamma}\right)$. *Assume that* $u_0 \in L^1(\mathbb{R}^n)$, $u_0(x) \geq 0$, *and* $u_1 = 0$. *If*

$$p \leq p_c = 1 + \frac{\gamma(\sigma + \frac{\beta}{\gamma}\rho) + \gamma\beta}{(1-\gamma)\beta + n\gamma}\,,$$

then the problem (2) *admits no global weak positive solutions other than the trivial one.*

Proof. The proof proceeds by contradiction. Suppose that u is a nontrivial nonnegative solution to problem (2) which exists globally in time. For later use, let Φ be a smooth nonincreasing function such that

$$\Phi(z) = \begin{cases} 1 & \text{if } z \leq 1, \\ 0 & \text{if } z \geq 2, \end{cases}$$

and $0 \leq \Phi \leq 1$. Let

$$\varphi(x,t) := \Phi^l\Big(\frac{t^{2\gamma}}{R^{2\beta}}\Big)\Phi^l\Big(\frac{|x|}{R}\Big) = \varphi_1^l(t)\varphi_2^l(x),$$

where R is a fixed positive number and l is a positive number to be chosen later. Multiplying the equation (2) by $\varphi(x,t)$ and integrating the result on $Q_{TR^{\beta/\gamma}}$, we obtain

$$\int\limits_{Q_{TR^{\beta/\gamma}}} h\varphi|u|^p \, dx\, dt + \int\limits_{\mathbb{R}^n} u_0 D^{2\gamma-1}_{t|TR^{\beta/\gamma}}\varphi(0)\, dx + \int\limits_{Q_{TR^{\beta/\gamma}}} u_0 D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi \, dx\, dt =$$
$$= \int\limits_{Q_{TR^{\beta/\gamma}}} uD^{2\gamma}_{t|TR^{\beta/\gamma}}\varphi \, dx\, dt + \int\limits_{Q_{TR^{\beta/\gamma}}} u(-\Delta)^{\frac{\beta}{2}}\varphi \, dx\, dt + \int\limits_{Q_{TR^{\beta/\gamma}}} uD^{\gamma}_{t|TR^{\beta/\gamma}}\varphi \, dx\, dt. \quad (9)$$

Now we estimate the right-hand side of (9). We have

$$\int\limits_{Q_{TR^{\beta/\gamma}}} u(-\Delta)^{\frac{\beta}{2}}\varphi\,dx\,dt = \int\limits_{Q_{TR^{\beta/\gamma}}} (h\Phi^l)^{\frac{1}{p}} u (h\Phi^l)^{-\frac{1}{p}} (-\Delta)^{\frac{\beta}{2}}\Phi^l\,dx\,dt \le$$
$$\le l \int\limits_{Q_{TR^{\beta/\gamma}}} (h\Phi^l)^{\frac{1}{p}} u (h\Phi^l)^{-\frac{1}{p}} \Phi^{l-1}(-\Delta)^{\frac{\beta}{2}}\Phi\,dx\,dt,$$

where we have used the Ju's inequality $(-\Delta)^{\beta/2}\xi^l(x) \le l\xi^{l-1}(x)(-\Delta)^{\beta/2}\xi(x)$ which is satisfied for every $\xi \in C_0^\infty(\mathbb{R}^n)$ (see [10]).

By the ε-Young's inequality, we can estimate

$$\int\limits_{Q_{TR^{\beta/\gamma}}} u(-\Delta)^{\frac{\beta}{2}}\varphi\,dx\,dt \le \varepsilon l \int\limits_{Q_{TR^{\beta/\gamma}}} h\Phi u^p\,dx\,dt +$$
$$+ C(\varepsilon) \int\limits_{Q_{TR^{\beta/\gamma}}} h^{\frac{-q}{p}} \Phi^{(l-1-\frac{l}{p})q} |(-\Delta)^{\frac{\beta}{2}}\Phi|^q\,dx\,dt =$$
$$= \varepsilon l \int\limits_{Q_{TR^{\beta/\gamma}}} h\Phi u^p\,dx\,dt + C(\varepsilon) \int\limits_{Q_{TR^{\beta/\gamma}}} h^{\frac{-q}{p}} \varphi^{(1-\frac{q}{l})} \left|(-\Delta)^{\frac{\beta}{2}}\varphi^{\frac{1}{l}}\right|^q\,dx\,dt < \infty, \tag{10}$$

so, we choose $l > q$ to ensure the convergence of the integral in (10).

$$\int\limits_{Q_{TR^{\beta/\gamma}}} u D^{2\gamma}_{t|TR^{\beta/\gamma}}\varphi\,dx\,dt \le$$
$$\le \varepsilon \int\limits_{Q_{TR^{\beta/\gamma}}} h\varphi u^p\,dx\,dt + C(\varepsilon) \int\limits_{Q_{TR^{\beta/\gamma}}} (h\varphi)^{1-q} |D^{2\gamma}_{t|TR^{\beta/\gamma}}\varphi|^q\,dx\,dt, \tag{11}$$

and

$$\int\limits_{Q_{TR^{\beta/\gamma}}} u D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi\,dx\,dt \le$$
$$\le \varepsilon \int\limits_{Q_{TR^{\beta/\gamma}}} h\varphi u^p\,dx\,dt + C(\varepsilon) \int\limits_{Q_{TR^{\beta/\gamma}}} (h\varphi)^{1-q} |D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi|^q\,dx\,dt, \tag{12}$$

where q is the conjugate of p. Gathering up (10), (11) and (12), with ε small enough, we infer that

$$\int\limits_{Q_{TR^{\beta/\gamma}}} h\varphi|u|^p\,dx\,dt + \int\limits_{Q_{TR^{\beta/\gamma}}} u_0 D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi\,dx\,dt \leq$$

$$\leq C \int\limits_{Q_{TR^{\beta/\gamma}}} h^{\frac{-q}{p}}\varphi^{(1-\frac{q}{l})}|(-\Delta)^{\frac{\beta}{2}}\varphi^{\frac{1}{l}}|^q\,dx\,dt +$$

$$+\,C \int\limits_{Q_{TR^{\beta/\gamma}}} (h\varphi)^{1-q}\left(|D^{2\gamma}_{t|TR^{\beta/\gamma}}\varphi|^q + |D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi|^q\right)\,dx\,dt, \quad (13)$$

for some positive constant C independent of R and T. At this stage, let us perform the change of variables $\tau = t/R^{\frac{\beta}{\gamma}}$, $y = \frac{x}{R}$, and $\varphi(x,t) = \psi(y,\tau)$, clearly

$$\tau = t/R^{\frac{\beta}{\gamma}}, \quad x = Ry, \quad dxdt = R^{n+\frac{\beta}{\gamma}}dyd\tau.$$

We have the estimates

$$\int\limits_{Q_{TR^{\beta/\gamma}}} h^{\frac{-q}{p}}\varphi^{(1-\frac{q}{l})}\left|(-\Delta)^{\frac{\beta}{2}}\varphi^{\frac{1}{l}}\right|^q\,dx\,dt =$$

$$= R^{-\beta q+n+\beta/\gamma+(1-q)(\sigma+\frac{\beta}{\gamma}\rho)}\int\limits_{Q_T} h^{1-q}\psi^{(1-\frac{q}{l})}\left|(-\Delta)^{\frac{\beta}{2}}\psi^{\frac{1}{l}}\right|^q\,dy\,d\tau,$$

$$\int\limits_{Q_{TR^{\beta/\gamma}}} (h\varphi)^{1-q}\left|D^{2\gamma}_{t|TR^{\beta/\gamma}}\varphi\right|^q\,dx\,dt =$$

$$= R^{-\frac{\beta}{\gamma}(2\gamma)q+n+\frac{\beta}{\gamma}+(1-q)(\sigma+\frac{\beta}{\gamma}\rho)}\int\limits_{Q_T} (h\psi)^{1-q}\left|D^{2\gamma}_{\tau|T}\psi\right|^q\,dy\,d\tau,$$

and

$$\int\limits_{Q_{TR^{\beta/\gamma}}} (h\varphi)^{1-q}\left|D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi\right|^q\,dx\,dt =$$

$$= R^{-\beta q+n+\frac{\beta}{\gamma}+(1-q)\left(\sigma+\frac{\beta}{\gamma}\rho\right)}\int\limits_{Q_T} (h\psi)^{1-q}\left|D^{\gamma}_{\tau|T}\psi\right|^q\,dy\,d\tau.$$

It is clear from (3) that $D^{2\gamma-1}_{t|TR^{\beta/\gamma}}\varphi \geq 0$, $D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi \geq 0$. Then we obtain

$$\int\limits_{Q_{TR^{\beta/\gamma}}} h\varphi|u|^p \, dx\, dt \leq$$

$$\leq C(\varepsilon)R^{-\beta q+n+\beta/\gamma+(1-q)(\sigma+\frac{\beta}{\gamma}\rho)}\Big[\int\limits_{Q_T} h^{1-q}\psi^{1-\frac{q}{l}}\big|(-\Delta)^{\frac{\beta}{2}}\psi^{\frac{1}{l}}\big|^q \, dy\, d\tau +$$

$$+\int\limits_{Q_T}(h\psi)^{1-q}\Big(\big|D^{\gamma}_{\tau|T}\psi\big|^q+\big|D^{2\gamma}_{\tau|T}\psi\big|^q\Big)\, dy\, d\tau\Big], \quad (14)$$

where C is positive constant independent of R. Now let $R \to +\infty$ in (14). We distinguish two cases. If $p < p_c$ (which is equivalent $-\beta q + n + \beta/\gamma + (1-q)(\sigma + \frac{\beta}{\gamma}\rho) < 0$), then we have

$$\int\limits_{\mathbb{R}^n\times\mathbb{R}^+} h|u|^p \, dx\, dt \leq 0.$$

This implies that $u \equiv 0$ a.e. on $\mathbb{R}^n \times \mathbb{R}^+$ since $h(x,t) > 0$ a.e. on $\mathbb{R}^n \times \mathbb{R}^+$. This is a contradiction.

In the case $p = p_c$ (i.e. critical case), from (14) we find that

$$\int\limits_{\mathbb{R}^n\times\mathbb{R}^+} h|u|^p \, dx\, dt \leq C. \quad (15)$$

Let us modify the test function φ by introducing a new fixed number S $(1 < S < R)$ such that

$$\varphi(x,t) := \Phi^l\Big(\frac{t^{2\gamma}}{(SR)^{2\beta}}\Big)\Phi^l\Big(\frac{|x|}{R}\Big),$$

we set $x = yR$, $t = (SR)^{\frac{\beta}{\gamma}}\tau$,

$$\Omega_{SR} = \Big\{(x,t) \in \mathbb{R}^n \times \mathbb{R}^+ : \ |x| \leq 2R, t^{2\gamma} \leq 2(SR)^{2\beta}\Big\},$$
$$\Omega = \Big\{(y,\tau) \in \mathbb{R}^n \times \mathbb{R}^+ : \ |y| \leq 2, \tau^{2\gamma} \leq 2\Big\}.$$

Then we have

$$\int\limits_{\Omega_{SR}} h^{\frac{-q}{p}}\varphi^{(1-\frac{q}{l})}\big|(-\Delta)^{\frac{\beta}{2}}\varphi^{\frac{1}{l}}\big|^q \, dx\, dt =$$

$$= S^{\beta/\gamma+(1-q)\frac{\beta}{\gamma}\rho}\int\limits_{\Omega} h^{1-q}\psi^{1-\frac{q}{l}}\big|(-\Delta)^{\frac{\beta}{2}}\psi^{\frac{1}{l}}\big|^q \, dy\, d\tau,$$

$$\int\limits_{\Omega_{SR}} (h\varphi)^{1-q} \big| D^{\gamma}_{t|TR^{\beta/\gamma}} \varphi \big|^q \, dx\, dt =$$

$$= S^{-\beta q + \beta/\gamma + (1-q)\frac{\beta}{\gamma}\rho} \int\limits_{\Omega} (h\psi)^{1-q} \big| D^{\gamma}_{\tau|T} \psi \big|^q \, dy\, d\tau,$$

and

$$\int\limits_{\Omega_{SR}} (h\varphi)^{1-q} \big| D^{2\gamma}_{t|TR^{\beta/\gamma}} \varphi \big|^q \, dx\, dt =$$

$$= S^{-2\beta q + \beta/\gamma + (1-q)\frac{\beta}{\gamma}\rho} \int\limits_{\Omega} (h\psi)^{1-q} \big| D^{2\gamma}_{t|T} \psi \big|^q \, dy\, d\tau.$$

Combining the above estimates we find

$$\begin{aligned}
&(1-3\varepsilon) \int\limits_{\Omega_{SR}} h\varphi u^{p_c} \, dx\, dt \le \\
&\le S^{\frac{\beta}{\gamma} + (1-q)\frac{\beta}{\gamma}\rho} \Bigg(\int\limits_{\Omega} h^{1-q} \psi^{1-\frac{q}{l}} |(-\Delta)^{\frac{\beta}{2}} \psi^{\frac{1}{l}}|^q \, dy\, d\tau \Bigg) + S^{-\beta q + \beta/\gamma + (1-q)\frac{\beta}{\gamma}\rho} \times \\
&\qquad \times \Bigg(\int\limits_{\Omega} (h\psi)^{1-q} |D^{\gamma}_{\tau|T} \psi|^q \, dy\, d\tau + \int\limits_{\Omega} (h\psi)^{1-q} |D^{2\gamma}_{\tau|T} \psi|^q \, dy\, d\tau \Bigg). \qquad (16)
\end{aligned}$$

Now, by taking $\varepsilon = \frac{1}{6}$ and using (15), we obtain via (16), after passing to the limit as $R \to \infty$,

$$\int\limits_{\mathbb{R}^n \times \mathbb{R}_+} h u^p \, dx\, dt \le C \Big(S^{-\beta q + \beta/\gamma + (1-q)\frac{\beta}{\gamma}\rho} + S^{\frac{\beta}{\gamma} + (1-q)\,\frac{\beta}{\gamma}\,\rho} \Big), \qquad (17)$$

we notice that the assumption $p < \min(\rho + 1, \frac{1}{1-\gamma})$ yields $-\beta q + \beta/\gamma + (1-q)\,\frac{\beta}{\gamma}\,\rho < 0$ and $\frac{\beta}{\gamma} + (1-q)\,\frac{\beta}{\gamma}\,\rho < 0$, and the left-hand side of (17) is independent of S. Passing to the limit $S \to \infty$, we get immediately

$$\int\limits_{\mathbb{R}^n \times \mathbb{R}^+} h|u|^p \, dx\, dt \le 0.$$

Thus $\int\limits_{\mathbb{R}^n \times \mathbb{R}^+} h|u|^p \, dx\, dt = 0$, which implies $u \equiv 0$ a.e. and completes the proof. $\square$

Remark 1. When $\beta = 2$, $\gamma = 1$ and $h = 1$, this agrees with Todorova–Yordanov [20].

4. The Necessary Conditions for the Local and Global Existence

In this section we assume that $\inf\limits_{t>0} h(x,t) > 0$, we see that the existence of solutions of the problem (2) depends on the behavior of initial data at infinity.

Theorem 2. *Let u be a local solution to* (2)*, where $T < +\infty$, and $1 < p < \frac{1}{1-\gamma}$. Assume that $u_0 \geq 0$ and $u_1 \geq 0$. Then the following two estimates*

$$\lim_{|x|\to+\infty} \inf \left(\inf_{t>0} h\right)^{q-1} u_0(x) \leq C\left(T^{\gamma(1-q)} + T^{\gamma-2\gamma q}\right),$$
$$\lim_{|x|\to+\infty} \inf \left(\inf_{t>0} h\right)^{q-1} u_1(x) \leq C'\left(T^{2\gamma-1-\gamma q} + T^{2\gamma(1-q)-1}\right)$$

hold for some positive constants C and C'.

Proof. Multiply the equation (2) by $\varphi(x,t)$ and integrating the result on $\Omega_R \times [0,T]$, we get

$$\int\limits_{\Omega_R\times[0,T]} h\varphi|u|^p \, dx\, dt + \int\limits_{\Omega_R} u_0 D_{t|T}^{2\gamma-1}\varphi(0)\, dx +$$
$$+ \int\limits_{\Omega_R\times[0,T]} u_0 D_{t|T}^{\gamma}\varphi \, dx\, dt + \int\limits_{\Omega_R\times[0,T]} u_1 D_{t|T}^{2\gamma-1}\varphi \, dx\, dt =$$
$$= \int\limits_{\Omega_R\times[0,T]} u D_{t|T}^{2\gamma}\varphi \, dx\, dt + \int\limits_{\Omega_R\times[0,T]} u(-\Delta)^{\frac{\beta}{2}}\varphi \, dx\, dt + \int\limits_{\Omega_R\times[0,T]} u D_{t|T}^{\gamma}\varphi \, dx\, dt. \quad (18)$$

where $\Omega_R := \{x \in \mathbb{R}^n;\ R \leq |x| \leq 2R\}$. Let us consider the function $\Phi \in H^\beta([1,2])$, $\Phi \geq 0$, such that $(-\Delta)^{\beta/2}\Phi = K\Phi$ for some positive constants K. We take

$$\varphi(x,t) := \Phi\Big(\frac{x}{R}\Big)\Big(1 - \frac{t^2}{T^2}\Big)^l, \ \ (x,t) \in \Omega_R \times [0,T], \ \ l > q.$$

Applying the ε-Young's inequality to the right-hand side of (18), one obtains

$$\int\limits_{\Omega_R} u_0 D_{t|T}^{2\gamma-1}\varphi(0)\, dx + \int\limits_{\Omega_R\times[0,T]} u_0 D_{t|T}^{\gamma}\varphi \, dx\, dt + \int\limits_{\Omega_R\times[0,T]} u_1 D_{t|T}^{2\gamma-1}\varphi \, dx\, dt \leq$$
$$\leq C \int\limits_{\Omega_R\times[0,T]} (h\varphi)^{\frac{-q}{p}} \left(\left|(-\Delta)^{\frac{\beta}{2}}\varphi\right|^q + \left|D_{t|T}^{2\gamma}\varphi\right|^q + \left|D_{t|T}^{\gamma}\varphi\right|^q\right) dx\, dt. \quad (19)$$

In order to estimate the right-hand side of (19) in terms of T and R, we have

$$\int\limits_{\Omega_R\times[0,T]} (h\varphi)^{1-q}\left|(-\Delta)^{\beta/2}\varphi\right|^q \, dx\, dt = CTR^{-\beta q} \int\limits_{\Omega_R} h^{1-q}\Phi\Big(\frac{x}{R}\Big)\, dx,$$

where we have used $(-\Delta)^{\beta/2}\Phi\big(\frac{x}{R}\big) = KR^{-\beta}\Phi(\frac{x}{R})$. An easy computation (using the Euler substitution $y = \frac{s-t}{T-t}$) yields

$$D^{\gamma}_{t|T}\Big(1-\frac{t^2}{T^2}\Big)^l = \frac{-T^{2l}}{\Gamma(1-\gamma)}\times$$
$$\times\sum_{k=0}^{l} 2^{l-k}C^l_k M_{lk} t^{l-k-1}(T-t)^{l-k-\gamma}\big[(l-k)T-(2l+1-\gamma)t\big], \quad (20)$$

where $M_{lk} := \Gamma(l+1)\sum\limits_{n=0}^{k} C^k_n \frac{\Gamma(n-\beta+1)}{\Gamma(l-\beta+n+2)}$ and $C^l_k = \frac{l!}{k!(l-k!)}$,

$$D^{2\gamma}_{t|T}(1-\frac{t^2}{T^2})^l = \frac{T^{2l}}{\Gamma(2-2\gamma)}\sum_{k=0}^{l} 2^{l-k}C^l_k M_{lk} t^{l-k-2}(T-t)^{l-k-2\gamma}\times$$
$$\times\Big[(l-k)(l-k-1)T^2-2tT(l-k)(2l-2\gamma+1)+(2l-2\gamma+1)(2l-2\gamma+2)t^2\Big], \quad (21)$$

and

$$\int_0^T D^{\gamma}_{t|T}\Big(1-\frac{t^2}{T^2}\Big)^l\,dt = \frac{T^{1-\gamma}}{\Gamma(1-\gamma)}\sum_{k=0}^{l} L_{\gamma k}C^l_k, \quad (22)$$

where

$$L_{\gamma k} := \frac{\Gamma(l+1)\Gamma(k+1-\gamma)}{\Gamma(l+k+2-\gamma)}.$$

By (20) and (21), we can see that

$$\Big|D^{\gamma}_{t|T}\Big(1-\frac{t^2}{T^2}\Big)^l\Big| \le \frac{T^{-\gamma}}{\Gamma(1-\gamma)}\sum_{k=0}^{l} 2^{(l-k)}(3l+1-\gamma-k)C^l_k M_{lk} \quad (23)$$

and

$$\Big|D^{2\gamma}_{t|T}\Big(1-\frac{t^2}{T^2}\Big)^l\Big| \le \frac{T^{-2\gamma}}{\Gamma(2-2\gamma)}\times$$
$$\times\sum_{k=0}^{l} 2^{(l-k)}C^l_k M_{lk}\Big[(l-k)(l-k-1)+(2l+1-2\gamma)(4l-2k+2-2\gamma)\Big]. \quad (24)$$

Passing to the new variable $t = T\tau$ and by the relations (22), (23) and (24), we obtain

$$\int_{\Omega_R\times[0,T]} u_1 D^{2\gamma-1}_{t|T}\varphi\,dx\,dt = \frac{C_3}{\Gamma(1-\alpha)}T^{-2\gamma+2}\int_{\Omega_R} u_1(x)\Phi\Big(\frac{x}{R}\Big)\,dx, \quad (25)$$

$$\int_{\Omega_R\times[0,T]} (h\varphi)^{1-q}\big|D^{\gamma}_{t|T}\varphi\big|^q\,dx\,dt \le CT^{1-\gamma q}\int_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx, \quad (26)$$

$$\int_{\Omega_R\times[0,T]} (h\varphi)^{1-q}\big|D^{2\gamma}_{t|T}\varphi\big|^q\,dx\,dt \le CT^{1-2\gamma q}\int_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx, \quad (27)$$

and

$$\int\limits_{\Omega_R\times[0,T]} (h\varphi)^{1-q}\big|(-\Delta)^{\frac{\beta}{2}}\varphi\big|^q\,dx\,dt \le$$
$$\le CTR^{-\beta q}\int\limits_{\Omega_R}\big[\inf_{t>0}h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx. \quad (28)$$

Gathering all the estimates (25)–(28) together with (19), we find

$$T^{1-\gamma}\int\limits_{\Omega_R}u_0(x)\Phi\Big(\frac{x}{R}\Big)\,dx + T^{2-2\gamma}\int\limits_{\Omega_R}u_1(x)\Phi\Big(\frac{x}{R}\Big)\,dx \le$$
$$\le C\big(T^{1-\gamma q}+T^{1-2\gamma q}+TR^{-\beta q}\big)\int\limits_{\Omega_R}\big[\inf_{t>0}h(x,t)\big]^{1-q}(x)\Phi\Big(\frac{x}{R}\Big)\,dx. \quad (29)$$

The estimate (29) and the following estimates

$$\int\limits_{\Omega_R}u_0(x)\Phi\Big(\frac{x}{R}\Big)\,dx \ge$$
$$\ge \inf_{|x|>R}\Big(u_0(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)\int\limits_{\Omega_R}\big[\inf_{t>0}h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx,$$
$$\int\limits_{\Omega_R}u_1(x)\Phi\Big(\frac{x}{R}\Big)\,dx \ge$$
$$\ge \inf_{|x|>R}\Big(u_1(xt)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)\int\limits_{\Omega_R}\big[\inf_{t>0}h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx,$$

yield

$$\Big(T^{-\gamma}\inf_{|x|>R}\Big(u_0(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)+T^{1-2\gamma}\inf_{|x|>R}\Big(u_1(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)\Big)\times$$
$$\times\int\limits_{\Omega_R}\big[\inf_{t>0}h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx \le$$
$$\le C\big[T^{-\gamma q}+T^{-2\gamma q}+R^{-\beta q}\big]\int\limits_{\Omega_R}\big[\inf_{t>0}h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx. \quad (30)$$

Dividing the both sides of (30) by $\int\limits_{\Omega_R}\big[\inf\limits_{t>0}h(x,t)\big]^{1-q}\Phi\big(\frac{x}{R}\big)\,dx > 0$, after passing to the limit $R\to+\infty$, we deduce

$$T^{-\gamma}\lim_{|x|\to+\infty}\inf\Big(u_0(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)+$$
$$+T^{1-2\gamma}\lim_{|x|\to+\infty}\inf\Big(u_1(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)\le C(T^{-\gamma q}+T^{-2\gamma q}).$$

Then we have

$$\lim_{|x|\to+\infty}\inf\Big(u_0(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)\le C(T^{\gamma-\gamma q}+T^{\gamma-2\gamma q})$$

and

$$\lim_{|x|\to+\infty}\inf\Big(u_1(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)\le C(T^{2\gamma-1-\gamma q}+T^{2\gamma(1-q)-1}). \qquad \square$$

Corollary 1. *Assume that the problem* (2) *has a nontrivial global solution. Then at least one of the following conditions is satisfied:*

$$\lim_{|x|\to+\infty}\inf\Big(u_0(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)=0,$$

$$\lim_{|x|\to+\infty}\inf\Big(u_1(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)=0.$$

Corollary 2. *If one of the conditions*

$$\lim_{|x|\to+\infty}\inf\Big(\big[\inf_{t>0}h(x,t)\big]^{q-1}u_0(x)\Big)=+\infty$$

or

$$\lim_{|x|\to+\infty}\inf\Big(\big[\inf_{t>0}h(x,t)\big]^{q-1}u_1(x)\Big)=+\infty$$

is fulfilled, then the problem (2) *cannot have any local weak solution.*

Theorem 3. *Suppose that the problem* (2) *has a global solution. Then there exist two positive constants K_1 and K_2 such that*

$$\lim_{|x|\to+\infty}\inf\Big(u_0(x)|x|^{\beta(q-1)}\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)\le K_1,$$

and

$$\lim_{|x|\to+\infty}\inf\Big(u_1(x)|x|^{\frac{\beta}{\gamma}(\gamma(q-1)+1-\gamma)}\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)\le K_2.$$

Proof. From the relation (30) we infer that

$$\inf_{|x|>R}\Big(\big[\inf_{t>0}h(x,t)\big]^{q-1}u_0(x)\Big)\int_{\Omega_R}\big[\inf_{t>0}h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx\le$$

$$\le C\big[T^{\gamma-\gamma q}+T^{\gamma-2\gamma q}+T^{\gamma}R^{-\beta q}\big]\int_{\Omega_R}\big[\inf_{t>0}h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx$$

Then, by taking $T>1$, we have

$$\inf_{|x|>R}\Big(u_0(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)\int_{\Omega_R}\big[\inf_{t>0}h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx\le$$

$$\le C\big[T^{\gamma-\gamma q}+T^{\gamma}R^{-\beta q}\big]\int_{\Omega_R}\big[\inf_{t>0}h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx. \qquad (31)$$

Now, taking in (31) $T = R^{\frac{\beta}{\gamma}}$, we find

$$\inf_{|x|>R}\Big(u_0(x)\big[\inf_{t>0} h(x,t)\big]^{q-1}\Big)\int\limits_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx \le$$
$$\le CR^{\beta(1-q)}\int\limits_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx.$$

The last inequality implies

$$\inf_{|x|>R}\Big(u_0(x)|x|^{\beta(q-1)}\big[\inf_{t>0} h(x,t)\big]^{q-1}\Big)\times$$
$$\times\int\limits_{\Omega_R}|x|^{\beta(1-q)}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx \le$$
$$\le C2^{2\beta(q-1)}\int\limits_{\Omega_R}|x|^{\beta(1-q)}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx. \quad (32)$$

After division of both sides of (32) by

$$\int\limits_{\Omega_R}|x|^{\beta(1-q)}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx > 0,$$

we deduce that

$$\inf_{|x|>R}\Big(u_0(x)|x|^{\beta(q-1)}\big[\inf_{t>0} h(x,t)\big]^{q-1}\Big) \le C2^{2\beta(q-1)}.$$

Finally, we pass to the limit $|x| \to +\infty$.

Similarly, we have

$$\inf_{|x|>R}\Big(u_1(x)\big[\inf_{t>0} h(x,t)\big]^{q-1}\Big)\int\limits_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx \le$$
$$\le C\big[T^{2\gamma-1-\gamma q} + T^{2\gamma-1-2\gamma q} + T^{2\gamma-1}R^{-\beta q}\big]\int\limits_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx,$$

and, by taking $T > 1$, we get

$$\inf_{|x|>R}\Big(\big[\inf_{t>0} h(x,t)\big]^{q-1}u_1(x)\Big)\int\limits_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx \le$$
$$\le C\big[T^{2\gamma-1-\gamma q} + T^{2\gamma-1}R^{-\beta q}\big]\int\limits_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx.$$

Likewise, $T = R^{\frac{\beta}{\gamma}}$. Therefore, by the substitution, we find

$$\inf_{|x|>R}\Big(\big[\inf_{t>0} h(x,t)\big]^{q-1}u_1(x)\Big)\int\limits_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx \le$$

$$\le CR^{\frac{\beta}{\gamma}(2\gamma-1)-\beta q}\int\limits_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx.$$

Hence

$$\inf_{|x|>R}\Big(|x|^{\beta q-\frac{\beta}{\gamma}(2\gamma-1)}\big[\inf_{t>0} h(x,t)\big]^{q-1}u_1(x)\Big)\times$$

$$\times\int\limits_{\Omega_R}|x|^{\frac{\beta}{\gamma}(2\gamma-1)-\beta q}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx \le$$

$$\le C2^{2\left(\frac{\beta}{\gamma}(2\gamma-1)-\beta q\right)}\int\limits_{\Omega_R}|x|^{\frac{\beta}{\gamma}(2\gamma-1)-\beta q}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx. \quad (33)$$

Finally, we divide both sides of the resulting relation by the expression

$$\int\limits_{\Omega_R}|x|^{\frac{\beta}{\gamma}(2\gamma-1)-\beta q}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx > 0,$$

and pass to the limit as $|x| \to +\infty$. □

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SOME PROPERTIES OF THE GENERALIZED EULER BETA FUNCTION

Abstract. A generalization of the Euler beta function for the case of multi-dimensional variable is proposed. In this context ordinary beta function is defined as a function of two-dimensional variable. An analogue of the Euler formula for this new function is proved for arbitrary dimension. There is found out the connection of defined function with multi-dimensional hypergeometric Laurichella's function and the theorem on cancelation of multi-dimensional hypergeometric functions singularities is proved. Such generalizations (among others) may be helpful to construct corresponding physical (string) models including different number of fields, as far the (bosonic) string theory reproduces the Euler beta function (Veneziano amplitude) and its multi-dimensional analogue.

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რეზიუმე. შემოთავაზებულია ეილერის ბეტა ფუნქციის განზოგადება მრავალგანზომილებიანი ცვლადისათვის. ამ მიდგომაში, ჩვეულებრივი ბეტა ფუნქცია განიმარტება, როგორც ორგანზომილებიანი ცვლადის ფუნქცია. დამტკიცებულია ეილერის ფორმულის ანალოგი ამ ახალი ფუნქციისათვის ნებისმიერი განზომილების შემთხვევაში. ნაპოვნია კავშირი შემოღებულ ფუნქციასა და ლაურიჩელას ჰიპერგეომეტრიულ ფუნქციას შორის და დამტკიცებულია, რომ მიღებულ გამოსახულებაში მრავალი ცვლადის ჰიპერგეომეტრიული ფუნქციის სინგულარობები კვეცავს ერთმანეთს. შესაძლოა მსგავსი განზოგადებები სასარგებლო აღმოჩნდეს ფიზიკური (სიმების) თეორიის ასაგებად, როცა პროცესებში მონაწილეობს რამდენიმე ველი, რამდენადაც (ბოზონური) სიმის თეორია (ვენეციანოს მოდელი) აღიწერება ეილერის ბეტა ფუნქციით.

In the articles [1], [2] there was proposed and investigated the function

$$B_n(r_0, r_1, \dots, r_n) = B_n(\mathbf{r}) =$$
$$= \begin{cases} 1 & \text{if } n = 0, \\ \det^{-1}[x_j^{i-1}]_{i,j=\overline{1,n}} \det\big[x_j^{i-1} b_{ij}\big]_{i,j=\overline{1,n}} & \text{if } n \geq 1 \end{cases} \tag{1}$$
$$(0 = x_0 < x_1, x_2 < \dots < x_n),$$

where

$$b_{ij} = \int\limits_{x_{j-1}/x_j}^{1} u^{i-1}(1-u)^{r_j-1} \prod_{\substack{k=0 \\ k\neq j}}^{n} \Big(\frac{x_j u - x_k}{x_j - x_k}^{r_k-1}\Big)\, du \ \ (i, j = 1, \dots, n).$$

The function (1) is a multidimensional generalization of the Euler beta function

$$B(r_0, r_1) = B_1(\mathbf{r}) = \int\limits_0^1 u^{r_0-1}(1-u)^{r_1-1}\, du.$$

A multidimensional analogues of Euler's beta function had been studied by mathematicians such as Selberg, Weil and Deligne among many others (see e.g. [3]–[7]). In [1] we have shown that for any $n \in \mathbb{N}$ and for $r_0 > 0$, $r_j \in \mathbb{N}$ $(j = 1, \dots, n)$ an analogue of the Euler formula is valid:

$$B_n(\mathbf{r}) = \frac{\prod\limits_{j=0}^{n} \Gamma(r_j)}{\Gamma\Big(\sum\limits_{j=0}^{n} r_j\Big)}\,, \tag{2}$$

where $\Gamma(t) = \int\limits_0^\infty e^{-u} u^{t-1}\, du$ is the Euler Gamma function.

In [2] there is investigated the case of the dimension $n = 2$. The analogue of the Euler formula has been proved for any complex parameters r_0, r_1, r_2 $(\operatorname{Re} r_j > 0,\ j = 0, 1, 2)$ and the complications arising when $n \geq 3$ are shown. The relations between $B_n(\mathbf{r})$ and hypergeometric functions of one and of many variables are shown too. Number of relations for the Gauss hypergeometric function is obtained. The analytic formulae for some new definite integrals of the special functions are obtained as well as for the elementary ones.

In present article we prove (2) for any $n \in \mathbb{N}$ and for $r_j \in \mathbb{C}$ $(\operatorname{Re} r_j > 0,$ $j = 0, 1, \dots, n)$. The key of the proof is the known

Carlson theorem ([8])**.** *If the function $f(z)$ of a complex variable z is regular in the semi-plane* $\operatorname{Re} z > A$, $A \in \mathbb{R}$, *and if the conditions*

(a) $\lim\limits_{|z|\to\infty} |f(z)| \exp(-k|z|) \leq const,\ 0 < k < \pi;$

(b) $f(z) = 0$ *for* $z = 0, 1, 2, \dots,$

are valid, then $f(z) = 0$ *for any* $z \in \mathbb{C}$.

For proving (2) we need the following

Lemma. *On the complex plane of the variable* $r_j \in \mathbb{C}$ $(j = 0, 1, \ldots, n)$ *the function* $B_0(r_0, r_1, \ldots, r_n) = B_n(\mathbf{r})$ *is bounded when* $|r_j| \to \infty$ *if other variables* $r_0, \ldots, r_{j-1}, r_{j+1}, \ldots, r_n$ *are fixed and* $\operatorname{Re} r_j \geq 1$ $(j = 0, 1 \ldots, n)$*:*

$$|B_n(\mathbf{r})| \leq M < \infty. \tag{3}$$

The bound M *does not depend on the variables* $r_0, r_1, \ldots, r_n$ $(\operatorname{Re} r_j \geq 1$, $j = 0, 1 \ldots, n)$.

Proof. The substitutions

$$u = \widetilde{u}, \quad j = 1; \quad u = 1 - \widetilde{u}\Big(1 - \frac{x_{j-1}}{x_j}\Big), \quad j = 2, \ldots, n \quad (n \geq 2),$$

give to the formula (1) the form

$$B_n(\mathbf{r}) = \frac{\det[x_j^{i-1}\widetilde{b}_{ij}]_{i,j=\overline{1,n}}}{\det[x_j^{i-1}]_{i,j=\overline{1,n}}}\,, \tag{4}$$

where

$$\widetilde{b}_{i1} = \prod_{k=2}^{n}\Big(1 - \frac{x_1}{x_k}\Big)^{1-r_k}\int\limits_0^1 \widetilde{u}^{r_0+i-2}(1-\widetilde{u})^{r_1-1}\prod_{k=2}^{n}\Big(1 - \frac{x_1}{x_k}\,\widetilde{u}\Big)^{r_k-1}\,d\widetilde{u},$$

$$\widetilde{b}_{ij} = \Big(1 - \frac{x_{j-1}}{x_j}\Big)^{r_j}\times$$

$$\times\int\limits_0^1\Big[1 - \Big(1 - \frac{x_{j-1}}{x_j}\Big)\widetilde{u}\Big]^{r_0+i-2}\widetilde{u}^{r_j-1}\prod_{\substack{k=0\\k\neq j}}^{n}\Big(1 - \frac{x_j - x_{j-1}}{x_j - x_k}\Big)^{r_k-1}\,d\widetilde{u},$$

$$i = 1, \ldots, n; \quad j = 2 \ldots, n.$$

All these integrals converge if the conditions

$$\operatorname{Re} r_j > 0, \quad j = 0 \ldots, n; \quad 0 = x_0 < x_1 < x_2 < \cdots < x_n$$

are fulfilled. Note that

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int\limits_0^1 u^{a-1}(1-u)^{c-a-1}\prod_{k=1}^{n}(1 - z_k u)^{b_k}\,du =$$

$$= F\Big({a \atop c}; b_1, z_1; \ldots; b_n, z_n\Big) \quad (\operatorname{Re} c > \operatorname{Re} a > 0),$$

where $F\Big({a \atop c}; b_1, z_1; \ldots; b_n, z_n\Big)$ denotes the multi-variable hypergeometric function – one of the four Lauricella's functions of the arguments $z_1, \ldots, z_n$ (see

[9]), which can be expressed as absolutely convergent power-sum

$$F\Big({a \atop c}; b_1, z_1; \ldots; b_n, z_n\Big) \equiv F_D\big(a; b_1, \ldots, b_n; c; z_1, \ldots, z_n\big) =$$
$$= \sum_{k_1,k_2,\ldots,k_n=0}^{\infty} \frac{(a)_{k_1+\cdots+k_n}}{(c)_{k_1+\cdots+k_n}} \prod_{j=1}^{n} \frac{(b_j)_{k_j} z_j^{k_j}}{k_j!},$$

if $|z_j| < 1$, $j = 1, \ldots, n$. Here $(a)_k$ denotes the Pochhammer symbol:

$$(a)_0 = 1, \quad (a)_k = a(a+1)\cdots(a+k-1), \quad k \in \mathbb{N}.$$

Thus the formula (4) expresses the above-defined function (1) via the determinant of Lauricella's multi-variable hypergeometric functions.

Let us rewrite the formula (4) as follows

$$B_n(\mathbf{r}) \det[z_j^{i-1}]_{i,j=\overline{1,n}} = M_1 \det[z_j^{i-1} \widetilde{\widetilde{b}}_{ij}]_{i,j=\overline{1,n}}, \tag{5}$$

where

$$z_k = \frac{x_k}{x_n}, \quad 0 < z_0 < z_1 < \cdots < z_{n-1} < z_n = 1, \tag{6}$$

$$M_1 = \prod_{k=2}^{n} \Big(1 - \frac{z_1}{z_k}\Big)^{1-r_k} \Big(1 - \frac{z_{k-1}}{z_k}\Big)^{r_k} =$$
$$= \prod_{k=2}^{n} \Big(1 - \frac{z_1}{z_k}\Big) \prod_{k=3}^{n} \Big(\frac{z_k - z_{k-1}}{z_k - z_1}\Big)^{r_k} \tag{7}$$

and

$$\widetilde{\widetilde{b}}_{i1} = \int_0^1 u^{r_0+i-2}(1-u)^{r_1-1} \prod_{k=2}^{n} \Big(1 - u\,\frac{z_1}{z_k}\Big)^{r_k-1} du,$$

$$\widetilde{\widetilde{b}}_{ij} = \int_0^1 \Big(1 - \frac{z_j - z_{j-1}}{z_j}\Big)^{r_0+i-2} u^{r_j-1} \prod_{\substack{k=1 \\ k\neq j}}^{n} \Big(1 - \frac{z_j - z_{j-1}}{z_j - z_k}\Big)^{r_k-1} du,$$
$$j = 2\ldots, n.$$

Due to the inequalities (6), the expression (5) can be estimated as follows:

$$\big|B_n(\mathbf{r})\big| \det[z_j^{i-1}]_{i,j=\overline{1,n}} =$$
$$= |M_1| \Big|\det[z_j^{i-1}\widetilde{\widetilde{b}}_{ij}]_{i,j=\overline{1,n}}\Big| \leq |M_1| \operatorname{per}\big[z_j^{i-1}|\widetilde{\widetilde{b}}_{ij}|\big]_{i,j=\overline{1,n}}, \tag{8}$$

where $\operatorname{per}[a_{ij}]_{i,j=\overline{1,n}}$ stands for the permanent of a matrix $[a_{ij}]_{i,j=\overline{1,n}}$ (see e.g. [10]):

$$\operatorname{per}[a_{ij}]_{i,j=\overline{1,n}} = \sum_{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)}, \quad \sigma : i \mapsto \sigma(i).$$

Further estimations give:

$$|\tilde{\tilde{b}}_{i1}| \le \int_0^1 \left| u^{r_0+i-2}(1-u)^{r_1-1} \prod_{k=2}^{n} \Big(1 - u\,\frac{z_1}{z_k}\Big)^{r_k-1} \right| du =$$

$$= \int_0^1 u^{\mathrm{Re}\, r_0+i-2}(1-u)^{\mathrm{Re}\, r_1-1} \prod_{k=2}^{n} \Big(1 - u\,\frac{z_1}{z_k}\Big)^{\mathrm{Re}\, r_k-1} \Big| du \equiv$$

$$\equiv \int_0^1 g_{i1}(u) f_1(u)\, du,$$

$$|\tilde{\tilde{b}}_{ij}| \le \tag{9}$$

$$\le \int_0^1 \left| \Big(1 - u\,\frac{z_j - z_{j-1}}{z_j}\Big)^{r_0+i-2} u^{r_j-1} \prod_{\substack{k=1\\k\ne j}}^{n} \Big(1 - u\,\frac{z_j - z_{j-1}}{z_j - z_k}\Big)^{r_k-1} \right| du =$$

$$= \int_0^1 \left| \Big(1 - u\,\frac{z_j - z_{j-1}}{z_j}\Big)^{\mathrm{Re}\, r_0+i-2} u^{\mathrm{Re}\, r_j-1} \prod_{\substack{k=1\\k\ne j}}^{n} \Big(1 - u\,\frac{z_j - z_{j-1}}{z_j - z_k}\Big)^{\mathrm{Re}\, r_k-1} \right| du \equiv$$

$$\equiv \int_0^1 f_{ij}(u) g_j(u)\, du, \quad j = 2, \ldots, n,$$

where we have denoted

$$f_1(u) = \prod_{k=1}^{n} \Big(1 - u\,\frac{z_1}{z_k}\Big)^{\mathrm{Re}\, r_k-1},$$

$$g_{i1}(u) = u^{\mathrm{Re}\, r_0+i-2},$$

$$f_{ij}(u) = \Big(1 - u\,\frac{z_j - z_{j-1}}{z_j}\Big)^{\mathrm{Re}\, r_0+i-2} \prod_{k=1}^{j-1} \Big(1 - u\,\frac{z_j - z_{j-1}}{z_j - z_k}\Big)^{\mathrm{Re}\, r_k-1},$$

$$g_j(u) = u^{\mathrm{Re}\, r_j-1} \prod_{k=j+1}^{n} \Big(1 + u\,\frac{z_j - z_{j-1}}{z_k - z_j}\Big)^{\mathrm{Re}\, r_k-1},$$

$$j = 2, \ldots, n; \quad i = 1, \ldots, n.$$

Let us use the mean value theorem in the integrals (9). As it is known, if the function $f(x)$ is monotonic and $f(x) \ge 0$ when $x \in [a, b]$, and if $g(x)$ is integrable, then the Bonnet formulae are valid (see e.g. [11, II, n°306]):

$$\int_a^b f(u)g(u)\, du = f(a) \int_a^\eta g(u)\, du, \quad a \le \eta \le b \ \text{ if } \ f(x) \ \text{ decreases},$$

$$\int\limits_a^b f(u)g(u)\,du = f(b)\int\limits_\xi^b g(u)\,du,\quad a\le\xi\le b \ \text{ if } \ f(x) \ \text{ increases,}$$
$$(x\in[a,b]).$$

It is obvious that if $\operatorname{Re} r_j \ge 1$, $j=0,1,\dots,n$, then the factors $f_1(u)$ and $f_{ij}(u)$ in (9) decrease for $u\in[0,1]$ and the factors $g_{i1}(u)$ and $g_j(u)$ increase $(j=2,\dots,n;\ i=1,\dots,n)$. Hence, according to the Bonnet formulae, for $\operatorname{Re} r_j\ge 1$, $j=0,1,\dots,n$ (due to this conditions all integrals converge) one gets

$$\begin{aligned}
&|\widetilde{\widetilde{b}}_{i1}| \le f_1(0)\int\limits_0^{\eta_{i1}} g_{i1}(u)\,du = f_1(0)g_{i1}(\eta_{i1})\int\limits_{\xi_{i1}}^{\eta_{i1}} du \le \\
&\qquad \le f_1(0)g_{i1}(1)(\eta_{i1}-\xi_{i1}) \le 1,\\
&|\widetilde{\widetilde{b}}_{ij}| \le g_j(1)\int\limits_{\xi_{ij}}^{1} f_{ij}(u)\,du = g_j(1)f_{ij}(\xi_{ij})\int\limits_{\xi_{ij}}^{\eta_{i1}} du \le \\
&\qquad \le g_j(1)f_{ij}(0)(\eta_{ij}-\xi_{ij}) \le \prod_{k=j+1}^{n}\Big(\frac{z_k-z_{j-1}}{z_k-z_j}\Big)^{\operatorname{Re} r_k-1},\\
&\qquad\qquad 0=z_0<z_1<\dots<z_n,\quad 0\le\xi_{ij}\le\eta_{ij}\le1,\quad i,j=1\dots,n.
\end{aligned}\tag{10}$$

According to (7), (8) and (10) one has

$$\begin{gathered}
\big|B_n(\mathbf{r})\big|\det[z_j^{i-1}]_{i,j=\overline{1,n}} \le \\
\le |M_1|\operatorname{per}\big[z_j^{i-1}|\widetilde{\widetilde{b}}_{ij}|\big]_{i,j=\overline{1,n}} \le M_2\operatorname{per}[z_j^{i-1}]_{i,j=\overline{1,n}},
\end{gathered}$$
$$M_2=\prod_{k=2}^{n}\Big(1-\frac{z_1}{z_k}\Big)\prod_{k=3}^{n}\Big(\frac{z_k-z_{k-1}}{z_k-z_1}\Big)^{\operatorname{Re} r_k}\prod_{j=2}^{n-1}\prod_{k=j+1}^{n}\Big(\frac{z_k-z_{j-1}}{z_k-z_j}\Big)^{\operatorname{Re} r_k-1}.$$

Because of obvious equalities

$$\begin{aligned}
&\prod_{j=2}^{n-1}\prod_{k=j+1}^{n}\Big(\frac{z_k-z_{j-1}}{z_k-z_j}\Big)^{\operatorname{Re} r_k-1} = \\
&\qquad = \prod_{k=3}^{n}\bigg[\prod_{j=2}^{k-1}\Big(\frac{z_k-z_{j-1}}{z_k-z_j}\Big)\bigg]^{\operatorname{Re} r_k-1} = \prod_{k=3}^{n}\Big(\frac{z_k-z_1}{z_k-z_{k-1}}\Big)^{\operatorname{Re} r_k-1}
\end{aligned}$$

one obtains

$$M_2=\prod_{k=2}^{n}\Big(1-\frac{z_1}{z_k}\Big)\prod_{k=3}^{n}\Big(\frac{z_k-z_{k-1}}{z_k-z_1}\Big)=\prod_{k=2}^{n}\Big(1-\frac{z_{k-1}}{z_k}\Big).$$

Inserting these results into the inequality (8), one obtains the estimation:

$$\left|B_n(\mathbf{r})\right| \leq \frac{\operatorname{per}[z_j^{i-1}]_{i,j=\overline{1,n}}}{\det[z_j^{i-1}]_{i,j=\overline{1,n}}} \prod_{k=2}^{n} \Big(1 - \frac{z_{k-1}}{z_k}\Big), \quad \operatorname{Re} r_j \geq 1, \quad j = 0, 1, \dots, n.$$

Hence, we have got the estimation (3) with M to be expressed as

$$M = \frac{\operatorname{per}[z_j^{i-1}]_{i,j=\overline{1,n}}}{\det[z_j^{i-1}]_{i,j=\overline{1,n}}} \prod_{k=1}^{n} \Big(1 - \frac{z_{k-1}}{z_k}\Big), \tag{11}$$

which, obviously, does not depend on the variables $r_0, r_1, \dots, r_n$ ($\operatorname{Re} r_j \geq 1$, $j = 0, 1, \dots, n$). $\square$

Note. The restriction $\operatorname{Re} r_j \geq 1$, $j = 0, 1, \dots, n$, is essential. Let, e.g., $n = 1$. In this case (11) gives $M = 1$ and one gets

$$\left|B_1(\mathbf{r})\right| = \Big|\int_0^1 u^{r_0-1}(1-u)^{r_1-1}\,du\Big| = \Big|\frac{\Gamma(r_0)\Gamma(r_1)}{\Gamma(r_0+r_1)}\Big| \leq 1 \ \text{ if } \ \operatorname{Re} r_0, r_1 \geq 1,$$

while in the opposite case when $\operatorname{Re} r_0, r_1 < 1$, e.g. for $r_0 = r_1 = 1/2$ one has $B_1(1/2, 1/2) = \pi > 1$, and the estimation (3) is not fulfilled.

Now we get the following

Statement. *The function*

$$f(r_0, r_1, \dots, r_n) = B_n(r_0, r_1, \dots, r_n) - \frac{\prod\limits_{j=0}^{n} \Gamma(r_j)}{\Gamma\big(\sum\limits_{j=0}^{n} r_j\big)} \tag{12}$$

satisfies all conditions of Carlson theorem on the complex plane of each variable r_j if other variables $r_0, r_1, \dots, r_{j-1}, r_{j+1}, \dots, r_n$ are fixed and $\operatorname{Re} r_j \geq 1$, $j = 0, 1, \dots, n$.

Proof of the Statement. Due to the fact that the function $\Gamma(z)$ is analytic everywhere on the (open) complex plane except the points $z = 0, -1, -2, \dots$, the function (12) is analytic if $\operatorname{Re} r_j > 0$, $j = 0, 1, \dots, n$ ($r_j = +\infty$ is a regular point of both summands of (12), $j = 0, 1, \dots, n$). In [1] we have proved that $f_0(r_0) = f(r_0, r_1, \dots, r_n) = 0$ if the variable r_0 is real and $r_0 > 0$, $r_j \in \mathbb{N}$, $j = 1, \dots, n$. Hence, according to the analytic function uniqueness theorem, if $r_j \in \mathbb{N}$, $j = 1, \dots, n$, then $f_0(r_0) = 0$ everywhere on the complex plane of the variable r_0 except, may be, the points $z = 0, -1, -2, \dots$. So, the function $f_0(r_0)$ satisfies the Statement.

Let us fix the numbers $r_j \in \mathbb{N}$, $j = 1, \dots, n$ and $r_0 \in \mathbb{C}$, $\operatorname{Re} r_0 > 0$. Under these conditions the function $f_1(r_1) = f(r_0, r_1, \dots, r_n)$ is analytic on the complex semi-plane $\operatorname{Re} r_1 > 0$ and $f_1(r_1) = 0$ for $r_1 = 1, 2, \dots$.

Let us show that if $\operatorname{Re} r_0 \geq 1$, $r_j \in \mathbb{N}$, $j = 2, \dots, n$, then

$$\lim_{|r_1| \to \infty} |f_1(r_1)| \exp(-k|r_1|) \leq \mathit{const}, \quad \operatorname{Re} r_1 \geq 1, \quad 0 < k < \pi.$$

Indeed, if $|z| \to \infty$ and $\operatorname{Re} z > 0$, in accordance with the asymptotic behavior of the Euler's Gamma function (see e.g. [12, Eq. 1.18(5)]) for any fixed number $\rho \in \mathbb{C}$ we have

$$\lim_{|z|\to\infty} \frac{\Gamma(z)}{\Gamma(z+\rho)} \exp\big(\rho \ln |z|\big) = \lim_{|z|\to\infty} \frac{\Gamma(z)}{\Gamma(z+\rho)} |z|^{\rho} = 1. \tag{13}$$

Hence, due to the estimation (3) we obtain:

$$\begin{aligned}
0 \le \lim_{|r_1|\to\infty} &|f_1(r_1)| \exp(-k|r_1|) = \\
&= \lim_{|r_1|\to\infty} \exp(-k|r_1|) \left| B_n(\mathbf{r}) - \frac{\Gamma(r_1)}{\Gamma(r_1+\rho_1)} \Gamma(r_0) \prod_{j=2}^{n} \Gamma(r_j) \right| \le \\
&\le \lim_{|r_1|\to\infty} \exp(-k|r_1|) \left\{ \big|B_n(\mathbf{r})\big| + \left| \frac{\Gamma(r_1)}{\Gamma(r_1+\rho_1)} \Gamma(r_0) \prod_{j=2}^{n} \Gamma(r_j) \right| \right\} \le \\
&\le M \lim_{|r_1|\to\infty} \exp(-k|r_1|) + \Gamma(r_0) \prod_{j=2}^{n} \Gamma(r_j) \times \\
\times \lim_{|r_1|\to\infty} &\left\{ \frac{\Gamma(r_1)}{\Gamma(r_1+\rho_1)} \exp\big(\rho_1 \ln |r_1|\big) \right\} \lim_{|r_1|\to\infty} \Big\{ |r_1|^{-\rho_1} \exp(-k|r_1|) \Big\} = 0,
\end{aligned}$$

where we denote $\rho_1 = r_0 + r_2 + \cdots + r_n$. Therefore we get

$$\lim_{|r_1|\to\infty} |f_1(r_1)| e^{-k|r_1|} = 0$$

for any fixed values of $k > 0$ and ρ_1. In particular, one can choose $0 < k < \pi$, $\operatorname{Re} r_0 \ge 1$, $r_j \in \mathbb{N}$, $j = 2, \ldots, n$.

Thus, if $\operatorname{Re} r_0 \ge 1$, $r_j \in \mathbb{N}$, $j = 2, \ldots, n$, then the function $f_1(r_1) = f(r_0, r_1, \ldots, r_n)$ satisfy all conditions of the Carlson Theorem and therefore $f_1(r_1) = 0$, $\operatorname{Re} r_1 \ge 1$.

Now let us suppose that the Statement is valid for the variables r_j with the indices $j = 0, 1, \ldots, m$, where $1 \le m \le n-1$, i.e. let the function

$$f(r_0, r_1, \ldots, r_{m-1}, r_m, \ldots, r_n) \equiv f_m(r_m)$$

satisfies the conditions:

$$\begin{gathered}
f_m(r_m) = f(r_0, r_1, \ldots, r_{m-1}, r_m, \ldots, r_n) = 0 \;\text{ if }\; r_m = 1, 2, \ldots, \\
\lim_{|r_m|\to\infty} |f_m(r_m)| \exp(-k|r_m|) \le const, \;\; 0 < k < \pi \;\text{ if }\; \operatorname{Re} r_m \ge 1 \\
\big(\operatorname{Re} r_j \ge 1, \;\; j = 0, 1, \ldots, m-1, \;\; r_{m+k} \in \mathbb{N}, \;\; k = 1, \ldots, n-m\big).
\end{gathered}$$

Therefore, according to the Carlson Theorem one gets

$$\begin{gathered}
f_m(r_m) = f(r_0, r_1, \ldots, r_{m-1}, r_m, \ldots, r_n) = 0 \;\text{ if }\; r_m \in \mathbb{C} \qquad (14) \\
\big(\operatorname{Re} r_j \ge 1, \;\; j = 0, 1, \ldots, m-1, \;\; r_{m+k} \in \mathbb{N}, \;\; k = 1, \ldots, n-m\big).
\end{gathered}$$

Thus we have shown that

$$f_{m+1}(r_{m+1}) = f(r_0, r_1, \ldots, r_m, r_{m+1}, \ldots, r_n) = 0$$

if $\operatorname{Re} r_j \geq 1$, $j = 0, 1, \ldots, m$, $r_{m+k} \in \mathbb{N}$, $k = 1, \ldots, n-m$. Besides, due to the estimates (3) and (13)

$$\lim_{|r_{m+1}|\to\infty} \left|f_{m+1}(r_{m+1})\right| \exp(-k|r_{m+1}|) \leq$$
$$\leq \lim_{|r_{m+1}|\to\infty} \left| \exp(-k|r_{m+1}|) \left\{ M + \frac{\Gamma(r_{m+1})}{\Gamma(r_{m+1}+\rho_{m+1})} \prod_{\substack{j=0 \\ j\neq m+1}}^{n} \Gamma(r_j) \right\} = 0\right.$$

if $\operatorname{Re} r_{m+1} \geq 1$, where k and ρ_{m+1} are fixed numbers such that

$$0 < k < \pi, \quad \rho_{m+1} = r_0 + r_1 + \cdots + r_m + r_{m+2} + \cdots + r_n$$
$$\big(\operatorname{Re} r_j \geq 1, \ \ j = 0, 1, \ldots, m, \ \ r_{m+k} \in \mathbb{N}, \ \ k = 2, \ldots, n-m\big).$$

So, the proposition of the Statement is fulfilled according to Full Mathematical Induction Principle. □

The Statement enables one to prove the following

Theorem 1. *For any number $n \in \mathbb{N}$ the function $B_n(\mathbf{r})$ satisfies the formula (2) – n-dimensional analogue of the Euler formula – if $\operatorname{Re} r_j > 0$, $j = 0, 1, \ldots, n$.*

Proof. According to the Statement the formula (2) is fulfilled if $\operatorname{Re} r_j \geq 1$, $j = 0, 1, \ldots, n$. Therefore, due to analyticity of the function (12) if $\operatorname{Re} r_j > 0$, $j = 0, 1, \ldots, n$, (2) is fulfilled on the open semi-plane $\operatorname{Re} r_j > 0$ of each variable $r_j \in \mathbb{C}$, $j = 0, 1, \ldots, n$. □

We have shown in [2] that the limits of the function $B_2(\mathbf{r}) = B_2(r_0, r_1, r_2)$ when $x_1 \to x_0 = 0$ ($x_1/x_2 = z \to 0$) and $x_2 \to x_1$ ($x_1/x_2 = z \to 1$) (see the definition (1)) exist and satisfy the relation

$$\left.\frac{\det[x_j^{i-1} b_{ij}]_{i,j=1,2}}{\det[x_j^{i-1}]_{i,j=1,2}}\right|_{x_1\to x_0=0} = \left.\frac{\det[x_j^{i-1} b_{ij}]_{i,j=1,2}}{\det[x_j^{i-1}]_{i,j=1,2}}\right|_{x_2\to x_1} =$$
$$= B_1(r_0, r_1) B_1(r_0 + r_1, r_2).$$

Therefore for $n = 2$ the formula (2) remains valid even if the only restrictions on the variables $x_0, x_1, \ldots, x_n$ are $x_j \geq x_0 = 0$, $j = 1, \ldots, n$ (instead of the restrictions $0 = x_0 < x_1 < \cdots < x_n$ which we have in (1)). It is easy to show that the same is valid for any $n \geq 3$. Namely, one has the following

Theorem 2. *For any $n, l, k \in \mathbb{N}$, $l \leq k \leq n - l$ and $x_j \geq x_0 = 0$, $\operatorname{Re} r_j > 0$, $j = 1, \cdots, n$, the function $B_n(\mathbf{r})$ satisfies the formula*

$$B_n(r_0, r_1, \ldots, r_n) =$$
$$= B_{n-k}(r_0, r_1, \ldots, r_{l-1}, r_l + \cdots + r_{l+k}, r_{l+k+1}, \ldots, r_n) B_k(r_l, \ldots, r_{l+k}) =$$
$$= \frac{\prod\limits_{j=0}^{n} \Gamma(r_j)}{\Gamma\big(\sum\limits_{j=0}^{n} r_j\big)} \,. \tag{15}$$

Proof. The theorem is trivial for $n = 0$ and $n = 1$; for $n = 2$, in fact, the formula (15) is proved in [2]. In the case $n \geq 3$ one has to consider separately the cases $x_1 \to x_0 = 0$ and $x_l \to x_{l-1}$, $l \geq 2$.

In the case $x_1 \to x_0 = 0$ from the definition (1) one gets:

$$x_1^{i-1} b_{i1} = x_1^{i-1} \int\limits_0^1 u^{r_0+i-2}(1-u)^{r_1-1} \prod_{k=2}^{n} \Big(\frac{x_1 u - x_k}{x_1 - x_k}\Big)^{r_k-1} du \underset{x_1\to 0}{\longrightarrow}$$

$$\underset{x_1\to 0}{\longrightarrow} \begin{cases} B_1(r_0, r_1), & i = 1, \\ 0 & i \geq 2, \end{cases}$$

$$x_j^{i-1} b_{ij} =$$

$$= x_j^{i-1} \int\limits_{x_{j-1}/x_j}^{1} u^{r_0+i-2}(1-u)^{r_j-1} \prod_{\substack{k=2\\k\neq j}}^{n} \Big(\frac{x_j u - x_k}{x_j - x_k}\Big)^{r_k-1} \Big(\frac{x_j u - x_1}{x_j - x_1}\Big)^{r_1-1} du \underset{x_1\to 0}{\longrightarrow}$$

$$\underset{x_1\to 0}{\longrightarrow} x_j^{i-1} \int\limits_{x_{j-1}/x_j}^{1} u^{r_0+r_1+i-3}(1-u)^{r_j-1} \prod_{\substack{k=2\\k\neq j}}^{n} \Big(\frac{x_j u - x_k}{x_j - x_k}\Big)^{r_k-1} du, \quad j \geq 2,$$

$$i = 1, \dots, n.$$

Hence, according to (1) and (2),

$$B_n(\mathbf{r})\big|_{x_1\to 0} = B_1(r_0, r_1) \prod_{2\leq k<j\leq n} (x_j - x_k)^{-1} \prod_{2\leq j\leq n} x_j^{-1} \times$$

$$\times \det \Bigg[x_j^{i-1} \int\limits_{x_{j-1}/x_j}^{1} u^{r_0+r_1+i-3}(1-u)^{r_j-1} \prod_{\substack{k=2\\k\neq j}}^{n} \Big(\frac{x_j u - x_k}{x_j - x_k}\Big)^{r_k-1} du \Bigg]_{i,j=2}^{n} =$$

$$(0 = x_1 < \dots < x_n)$$

$$= B_1(r_0, r_1) B_{n-1}(r_0 + r_1, r_2, \dots, r_n) =$$

$$= \frac{\Gamma(r_0)\Gamma(r_1)\Gamma(r_0+r_1)\Gamma(r_2)\cdots\Gamma(r_n)}{\Gamma(r_0+r_1)\Gamma(r_0+r_1+r_2+\dots+r_n)} = \frac{\Gamma(r_0)\Gamma(r_1)\cdots\Gamma(r_n)}{\Gamma(r_0+r_1+\dots+r_n)}. \quad (16)$$

Similarly, in the case when $x_l \to x_{l-1}$, $l \geq 2$, one obtains:

$$x_{l-1}^{i-1} b_{i\,l-1} = x_{l-1}^{i-1} \Big(1 - \frac{x_{l-1}}{x_l}\Big)^{1-r_l} \times$$

$$\times \int\limits_{x_{l-2}/x_{l-1}}^{1} u^{i-1}(1-u)^{r_{l-1}-1} \Big(1 - \frac{x_{l-1}}{x_l} u\Big)^{r_l-1} \prod_{\substack{k=0\\k\neq l-1,l}}^{n} \Big(\frac{x_{l-1} u - x_k}{x_{l-1} - x_k}\Big)^{r_k-1} du,$$

$$x_l^{i-1} b_{il} = x_l^{i-1} \int\limits_{x_{l-1}/x_l}^{1} u^{i-1} \Big(\frac{x_l u - x_{l-1}}{x_l - x_{l-1}}\Big)^{r_{l-1}-1} (1-u)^{r_l - 1} \prod_{\substack{k=0\\ k\neq l-1,l}}^{n} \Big(\frac{x_l u - x_k}{x_l - x_k}\Big)^{r_k - 1} du,$$

$$x_j^{i-1} b_{ij} = x_j^{i-1} \int\limits_{x_{j-1}/x_j}^{1} u^{i-1} (1-u)^{r_j - 1} \prod_{\substack{k=0\\ k\neq j}}^{n} \Big(\frac{x_j u - x_k}{x_j - x_k}\Big)^{r_k - 1} du,$$

$$j = 1, \ldots, n, \quad j \neq l-1, l$$

$$(i = 1, \ldots, n).$$

The substitution

$$\frac{x_l u - x_{l-1}}{x_l - x_{l-1}} = 1 - \widetilde{u}, \quad u = 1 - \Big(1 - \frac{x_{l-1}}{x_l}\Big)\widetilde{u},$$

$$x_l u - x_k = x_l - x_k - \widetilde{u}(x_l - x_{l-1})$$

gives to the second of these integrals the form

$$x_l^{i-1} b_{il} = x_l^{i-1} \Big(1 - \frac{x_{l-1}}{x_l}\Big)^{r_l} \times$$

$$\times \int\limits_0^1 \Big(1 - u\,\frac{x_l - x_{l-1}}{x_l}\Big)^{i-1} (1-u)^{r_{l-1}-1} u^{r_l - 1} \prod_{\substack{k=0\\ k\neq l-1,l}}^{n} \Big(1 - u\,\frac{x_l - x_{l-1}}{x_l - x_k}\Big)^{r_k - 1} du.$$

Inserting these results in (1), after obvious simplifications we find:

$$B_n(\mathbf{r})\big|_{x_l \to x_{l-1}} =$$

$$= \frac{B_1(r_{l-1}, r_l) \det B^{(l)}}{x_{l-1} \det[x_j^{i-1}]_{\substack{i=\overline{1,n-2}\\ j=\overline{1,n}\\ j\neq l-1,l}} \prod\limits_{1\le k\le l-2} (x_{l-1} - x_k)^2 \prod\limits_{l+1\le m\le n} (x_m - x_{l-1})^2}, \tag{17}$$

where $B^{(l)} = [B_{ik}^{(l)}]$ is the matrix whose i-th row, $i = 1, \ldots, n$, has the form

$$\Big[x_1^{i-1}\widetilde{b}_{i1}, \ldots, x_{l-2}^{i-1}\widetilde{b}_{i\,l-2}, x_{l-1}^{i-1}\widetilde{b}_{i\,l-1}, x_{l+1}^{i-1}\widetilde{b}_{i\,l+1}, \ldots, x_n^{i-1}\widetilde{b}_{in}\Big] \tag{18}$$

and

$$\widetilde{b}_{i\,l-1} = \int\limits_{x_{l-2}/x_{l-1}}^{1} u^{i-1} (1-u)^{r_{l-1}+r_l-2} \prod_{\substack{k=0\\ k\neq l-1,l}}^{n} \Big(\frac{x_{l-1} u - x_k}{x_{l-1} - x_k}\Big)^{r_k - 1} du,$$

$$\widetilde{b}_{ij} = \int\limits_{x_{j-1}/x_j}^{1} u^{i-1} (1-u)^{r_j - 1} \Big(\frac{x_j u - x_{l-1}}{x_j - x_{l-1}}\Big)^{r_{l-1}+r_l-2} \prod_{\substack{k=0\\ k\neq j,l-1,l}}^{n} \Big(\frac{x_j u - x_k}{x_j - x_k}\Big)^{r_k - 1} du,$$

$$i, j = 1, \ldots, n, \quad j \neq l-1, l.$$

The last step of our transformations is to multiply the row with number $i-1$ in the determinant of the matrix (18) on x_{l-1} and to extract it from the row with number i. Then in the l-th column of the determinant we obtain the Kroneker symbol δ_{1i},

$$\delta_{1i}=\begin{cases}1, & i=1\\ 0, & i\neq 1\end{cases}$$

and in the other ones we get:

$$x_{l-1}^{i-1}\widetilde{b}_{i\,l-1}\longrightarrow x_{l-1}^{i-1}\widetilde{b}'_{i\,l-1}=$$

$$=-x_{l-1}^{i-1}\int\limits_{x_{l-2}/x_{l-1}}^{1}u^{i-2}(1-u)^{r_{l-1}+r_l-1}\prod_{\substack{k=0\\k\neq l-1,l}}^{n}\Big(\frac{x_{l-1}u-x_k}{x_{l-1}-x_k}\Big)^{r_k-1}du,$$

$$x_j^{i-1}\widetilde{b}_{ij}\longrightarrow x_j^{i-1}\widetilde{b}'_{ij}=(x_j-x_{l-1})x_j^{i-2}\times \tag{19}$$

$$\times\int\limits_{x_{j-1}/x_j}^{1}u^{i-2}(1-u)^{r_j-1}\Big(\frac{x_ju-x_{l-1}}{x_j-x_{l-1}}\Big)^{r_{l-1}+r_l-1}\prod_{\substack{k=0\\k\neq j,l-1,l}}^{n}\Big(\frac{x_ju-x_k}{x_j-x_k}\Big)^{r_k-1}du,$$

$$i,j=1,\dots,n,\quad j\neq l-1,l.$$

Expanding the determinant obtained with respect to the l-th column elements and inserting the result in (17), one gets

$$B_n(\mathbf{r})\big|_{x_l\to x_{l-1}}=$$

$$=\frac{B_1(r_{l-1},r_l)(-1)^{l-1}}{\det[x_j^{i-1}]_{\substack{i=\overline{1,n-2}\\ j=\overline{1,n}\\ j\neq l-1,l}}\prod\limits_{1\le k\le l-2}(x_{l-1}-x_k)\prod\limits_{l+1\le m\le n}(x_m-x_{l-1})}(-1)^{l+1}\times$$

$$\times\det\Big[x_1^{i-1}\widetilde{b}'_{i1},\dots,x_{l-2}^{i-1}\widetilde{b}'_{i\,l-2},x_{l-1}^{i-1}\widetilde{b}'_{i\,l-1},x_{l+1}^{i-1}\widetilde{b}'_{i\,l+1},\dots,x_n^{i-1}\widetilde{b}'_{in}\Big]_{i=\overline{1,n-1}}=$$

$$=B_1(r_{l-1},r_l)\,\frac{\det\big[x_1^{i-1}\widetilde{b}'_{i1},\dots,x_{l-1}^{i-1}\widetilde{b}'_{i\,l-1},x_{l+1}^{i-1}\widetilde{b}'_{i\,l+1},\dots,x_n^{i-1}\widetilde{b}'_{in}\big]_{i=\overline{1,n-1}}}{\det\big[x_1^{i-1},\dots,x_{l-1}^{i-1},x_{l+1}^{i-1},\dots,x_n^{i-1}\big]_{i=\overline{1,n-1}}},$$

where the entries $\widetilde{b}_{ij}$, $i,j=1,\dots,n$, are defined in (19). So, we obtain the formula (16) in the case under consideration, too. Now the statement of Theorem 2 follows from (16) according to Full Mathematical Induction Principle. □

Theorem 3. *The integrals' singularities of the formula* (1) *determinant's entries, i.e. the singularities of Lauricella's hypergeometric functions on the complex plane of each variable $x_j\in\mathbb{C}$, $j=0,\dots,n$, cancel each other in the formula* (1).

Proof. According to Theorem 2, the function (1) does not depend on the variables $x_j\ge 0$ if $\operatorname{Re} r_j>0$, $j=0,\dots,n$. Hence, according to the analytic function uniqueness theorem, the function (1) does not depend on the variables $x_j\in\mathbb{C}$ if $\operatorname{Re} r_j>0$, $j=0,\dots,n$, while the integrals in the formula

(1) – Lauricella's hypergeometric functions – have singularities with respect to variables $x_j \in \mathbb{C}$. $\square$

As far as the (bosonic) string theory [13] reproduces the Euler beta function (Veneziano amplitude) and its multidimensional analogue, it seems to be helpful to take an advantage of the proposed generalization of the Euler beta function when one attempts to construct physical (string) models [14] including a number of fermionic fields, as far as the expression

$$\det\left[x_j^{i-1}\int\limits_{x_{j-1}/x_j}^{1} u^{i-1}(1-u)^{r_j-1}\prod_{\substack{k=0\\k\neq j}}\Big(\frac{x_j u - x_k}{x_j - x_k}\Big)^{r_k-1}du\right]_{i,j=\overline{1,n}} = \\ = \det[x_j^{i-1}]_{i,j=\overline{1,n}}\,\frac{\prod\limits_{j=0}^{n}\Gamma(r_j)}{\Gamma\Big(\sum\limits_{j=0}^{n} r_j\Big)}$$

is skew-symmetryc with respect to the variables $x_j \in \mathbb{C}$.

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ON SOME PROPERTIES OF ANALYTIC FUNCTIONS FROM SMIRNOV CLASS WITH A VARIABLE EXPONENT

Dedicated to the bless memory of my dear friend Givi Khuskivadze

Abstract. Let D be a simply connected domain bounded by a simple closed rectifiable curve Γ and $L^{p(t)}(D)$ denote the Lebesgue space with variable exponent.

The present work reveals different conditions regarding the functions $p(t)$ and the domain D under fulfilment of which the Cauchy type integrals with density from $L^{p(t)}(\Gamma)$ belong to the Smirnov class $E^{p(t)}(D)$.

When the domain D is bounded by the Lavrent'yev curve, the analogue of the well-known Smirnov's theorem is stated: if $\phi \in E^{p_1(\cdot)}(D)$, $\phi^+(t) \in L^{p_2(t)}(\Gamma)$, then $\phi \in E^{\widetilde{p}(t)}(D)$, where $\widetilde{p}(t) = \max(p_1(t), p_2(t))$.

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Key words and phrases. Smirnov classes of analytic functions, variable exponent, Cauchy type integral, regular curves, Lavrent'yev curves.

რეზიუმე. ვთქვათ, D მარტივი, შეკრული, გაწრფევადი Γ წირით შემოსაზღვრული სასრული არეა, ხოლო $L^{p(t)}(D)$ არის ლებეგის ცვლადმაჩვენებლიანი სივრცე.

გამოვლენილია $p(t)$ ფუნქციისა და D არის მიმართ ისეთი პირობები, რომელთა შესრულება იწვევს $L^{p(t)}(\Gamma)$ კლასის სიმკვრივის მქონე კოშის ტიპის ინტეგრალთა მიკუთვნებას სმირნოვის $E^{p(t)}(D)$ კლასისადმი.

როცა D შემოსაზღვრულია ლავრენტიევის Γ წირით, დადგენილია სმირნოვის ცნობილი თეორემის შემდეგი სახის ანალოგი: თუ $\phi \in E^{p_1(\cdot)}(D)$, ხოლო $\phi^+(t) \in L^{p_2(t)}(\Gamma)$, მაშინ $\phi \in E^{\widetilde{p}(t)}(D)$, სადაც $\widetilde{p}(t) = \max(p_1(t), p_2(t))$.

1. Introduction

Quite recently it became clear that for investigation of a number of questions dealing with analysis and in studying the problems of applied character, the Lebesgue spaces $L^{p(t)}$ with a variable exponent are very useful. In particular, in studying boundary value problems of the theory of analytic and harmonic functions it is advisable to consider them in classes of functions representable by the Cauchy type integral with density from $L^{p(t)}$ and their real parts as well as in classes of functions which reasonably generalize Smirnov classes $E^p(D)$ in the case of a variable exponent $p(t)$.

The works [1]–[3] suggest one (of the possible) such generalization under which all significant properties, inherent in these classes for a constant p, remain valid.

In the present paper we continue investigation of these classes. Special attention is attached to the problem of finding different conditions for the domains D and functions $p(t)$ under fulfilment of which the Cauchy type integrals with density from $L^{p(t)}(\Gamma)$ belong to the class $E^{p(t)}(D)$ (Γ is a simple closed curve bounding the domain D).

To achieve the purpose in view, for the domains bounded by piecewise smooth curves we establish one criterion in order for the analytic in D function ϕ to belong to the class $E^{p(t)}(D)$ (depending on the properties of conformal mapping of the unit circle onto D). However, when the domain D is bounded by the Lavrent'yev curve (i.e. the curves with the chord-arc condition), the analogue of the well-known Smirnov's theorem is fully justified; namely, the conditions are revealed under which: if $\phi(x) \in E^{p_1(t)}(D)$ and $\phi^+(x) \in L^{p_2(t)}(\Gamma)$, then $\phi(z) \in E^{\widetilde{p}(t)}(D)$, $\widetilde{p}(t) = \max(p_1(t), p_2(t))$.

2. Some Definitions and Auxiliary Statements

2.1. The Curves.

(i) Let D be a simply connected domain bounded by a simple finite rectifiable curve $\Gamma = \{t \in \mathbb{C} :\ t = t(s),\ 0 \le s \le l < \infty\}$ with arc-length measure $\nu(t) = s$. Let $\Gamma(t,r) = \Gamma \cap B(t,r)$, where $B(t,r) = \{\tau \in \mathbb{C} : |\tau - t| < r\}$, $t \in \Gamma$, $r > 0$.

A curve Γ is called Carleson one (or regular one), if

$$\sup_{t\in\Gamma, r>0} \frac{\nu[\Gamma(t,r)]}{r} < \infty.$$

(ii) By Λ we denote a set of all Lavrent'yev curves, i.e., the curves Γ for which

$$\sup_{t_1,t_2\in\Gamma} \frac{s(t_1,t_2)}{|t_1-t_2|} < \infty,$$

where $s(t_1,t_2)$ is length of the smallest of the two arcs lying on Γ and connecting the points t_1 and t_2.

(iii) If Γ is a piecewise smooth closed simple curve with angular points A_k, $k = 1,\dots,n$, and it is boundary of the domain D, and $\pi\nu_k$, $0 \le \nu_k \le 2$

are sizes of interior with respect to D angles at these points, we say that

$$\Gamma \in C^1_D(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n).$$

The set of piecewise Lyapunov curves with the same properties we denote by

$$C^{1,L}_D(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n).$$

(iv) Assume

$$S_\Gamma : f \to S_\Gamma f, \quad (S_\Gamma f)(t) = \frac{1}{\pi i} \int\limits_\Gamma \frac{f(\tau)}{\tau - t}\, d\tau, \quad t \in \Gamma.$$

We write $\Gamma \in R^p$, $p > 1$, if the operator is continuous in $L^p(\Gamma)$.

2.2. Conformal Mappings.

2.2.1. If $z = z(w)$ is a conformal mapping of the circle $U = \{w : \ |w| < 1\}$ onto the domain D with the boundary $\Gamma \in C^{1,L}_D(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 < \nu_k \le 2$, then

$$z'(w) \sim \prod_{k=1}^n (w - a_k)^{\nu_k - 1}, \quad A_k = z(a_k), \tag{1}$$

where $f \sim g$ denotes that $0 < \inf |\frac{f}{g}| \le \sup |\frac{f}{g}| < \infty$ [4].

2.2.2. If Γ is a simple closed curve bounding the domain D, and $\Gamma \in \Lambda$, then there exist positive numbers η and σ such that

$$z' \in H^{1+\eta}, \quad \frac{1}{z'} \in H^\sigma, \tag{2}$$

where H^σ is the Hardy class of analytic in U functions (see, e.g., [5, p. 170]).

2.2.3. If $\Gamma \in C^1_D(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 < \nu_k \le 2$, then

$$z'(w) \sim \prod_{k=1}^n (w - a_k)^{\nu_k - 1} \exp \int\limits_\gamma \frac{\psi(\zeta)}{\zeta - w}\, ds, \tag{3}$$

where $\psi(\zeta)$ is the real continuous function on γ, $\gamma = \{\zeta : \ |\zeta| = 1\}$ ([6], see also [7, p. 144]).

2.2.4. Let D be the bounded domain with a simple rectifiable boundary Γ, and let $z = z(w)$ be the conformal mapping of U onto D. D is said to be Smirnov's domain (and Γ is said to be Smirnov's curve), if the function $\ln |z'(w)|$ is representable by the Poisson integral, i.e.,

$$\ln |z'(re^{i\varphi})| = \frac{1}{2\pi} \int\limits_0^{2\pi} \ln |z'(e^{i\vartheta})| \, \frac{1 - r^2}{1 + r^2 - 2r\cos(\vartheta - \rho)}\, d\vartheta$$

(for these classes see, e.g., [8, pp. 250–252]).

2.3. Some Properties of the Operator S_Γ and of the Cauchy Type Integrals.

(i) If $p > 1$, then $\Gamma \in R^p$ if and only if Γ is a regular curve ([9]).

(ii) If Γ is a simple closed curve bounding the domain D and the operator S_Γ is continuous from $L^p(\Gamma)$ to $L^s(\Gamma)$, $p > 1$, $s \le p$, then:

(a) D is Smirnov's domain and

(b) the Cauchy type integral

$$(K_\Gamma f)(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{f(\tau)}{\tau - z}\, d\tau, \quad z \in D, \quad f \in L^p(\Gamma),$$

belongs to the Smirnov class $E^s(D)$.

In particular, if Γ is a regular curve, then $(K_\Gamma f)(z)$ belongs to the class $E^p(D)$ when $f \in L^p(\Gamma)$, $p > 1$ ([10], [11], see also [7, p. 29]).

(c) Smirnov's Theorem: if D is Smirnov's domain and $\phi \in E^{p_1}(D)$, while $\phi \in L^{p_2}(\Gamma)$, $p_2 > p_1$, then $\phi \in E^{p_2}(D)$ ([12], see also [8, p. 260]).

2.4. Spaces $L^{p(t)}(\Gamma;\omega)$. Classes of Exponents $\mathcal{P}[\Gamma]$ and $\widetilde{\mathcal{P}}(\Gamma)$. Let Γ be a simple rectifiable curve with the equation $t = t(s)$, $0 \le s \le l$, with arc-length measure, and let on Γ be assigned measurable functions $p(t)$ and $\omega(t)$, where $p(t)$ is positive and $\omega(t)$ is almost everywhere other than zero finite function.

Consider a set of measurable on Γ functions $f(t)$ for which

$$I_\Gamma^{p(\cdot)}(f\omega) = \int\limits_0^b \big|f(t(s))\omega(t(s))\big|^{p(t(s))}\, ds < \infty.$$

Denote

$$\|f\|_{L^{p(\cdot)}(\Gamma;\omega)} = \inf\Big\{\lambda > 0 :\ I^{p(\cdot)}\Big(\frac{f\omega}{\lambda}\Big) \le 1\Big\}.$$

By $L^{p(\cdot)}(\Gamma;\omega)$ we denote a space of measurable functions f such that $\|f\|_{L^{p(\cdot)}(\Gamma;\omega)} < \infty$. Assume $L^{p(\cdot)}(\Gamma) := L^{p(\cdot)}(\Gamma;1)$. (For detailed account on these spaces see, e.g., [13]).

2.4.1. *Classes of Functions* $\mathcal{P}(\Gamma)$ *and* $\widetilde{\mathcal{P}}(\Gamma)$. The spaces $L^{p(\cdot)}(\Gamma;\omega)$ in which the function $p(t)$ satisfies the conditions below are thoroughly studied and frequently used in applications:

(1) there is the constant A such that for any t_1, t_2 we have

$$|p(t_1) - p(t_2)| < \frac{A}{|\ln|t_1 - t_2||}\,; \tag{4}$$

(2)

$$\min_{t\in\Gamma} p(t) = \underline{p} > 1. \tag{5}$$

The set of all functions $p(t)$ satisfying the conditions (4), (5) we denote by $\mathcal{P}(\Gamma)$.

If $p \in \mathcal{P}(\Gamma)$, then the set $L^{p(\cdot)}(\Gamma;\omega)$ is the Banach space with the norm $\|\cdot\|_{L^{p(\cdot)}(\Gamma;\omega)}$.

Along with the class $\mathcal{P}(\Gamma)$, we introduce into consideration one more class of functions $\mathcal{P}_{1+\varepsilon}(\Gamma)$, $\varepsilon > 0$. This is a subset of those functions $p(t)$ from $\mathcal{P}(\Gamma)$ for which the condition (4) is replaced by the condition

$$|p(t_1) - p(t_2)| < \frac{A}{|\lambda|t_1 - t_2||^{1+\varepsilon}}\,. \tag{6}$$

Assume

$$\widetilde{\mathcal{P}}(\Gamma) = \bigcup_{\varepsilon>0} \mathcal{P}_{1+\varepsilon}. \tag{7}$$

2.5. The Hardy and Smirnov Classes with a Variable Exponent. Let D be the inner domain bounded by a simple closed curve Γ, and let $p = p(t)$ be the given on Γ measurable positive function. Moreover, let $z = z(w)$ be the conformal mapping of the circle U with boundary γ onto the domain D, and let $\omega = \omega(z)$ be the measurable on D function.

By $E^{p(t)}(D;\omega)$ we denote a set of all those analytic in D functions $\phi(z)$ for which

$$\sup_{0<z<1} \int\limits_0^{2\pi} \Big|\phi(z(re^{i\vartheta}))\omega(z(re^{i\vartheta}))\Big|^{p(z(e^{i\vartheta}))} |z'(re^{i\vartheta})|\,d\vartheta < \infty. \tag{8}$$

Assume

$$H^{p(\cdot)}(\omega) := E^{p(\cdot)}(U;\omega), \quad H^{p(\cdot)} := H^{p(\cdot)}(1).$$

For the constant p, these classes coincide with the well-known Smirnov and Hardy classes.

2.5.1. *On the Continuity of the Operator S_Γ in the Spaces $L^{p(\cdot)}(\Gamma;\omega)$.* In [14], the authors have proved theorems on the continuity of the operator S_Γ in the spaces $L^{p(\cdot)}(\Gamma;\omega)$. (More earlier works relating to this subject-matter can be found therein).

Combining the results of these theorems, we find that the theorem below is valid.

Theorem A. *For the operator S_Γ to be continuous in the space $L^{p(\cdot)}(\Gamma;\omega)$, where $p \in \mathcal{P}(\Gamma)$ and*

$$\omega(t) = \prod_{k=1}^{n} |t - t_k|^{\alpha_k}, \;\; t_k \in \Gamma, \;\; \alpha \in \mathbb{R},$$

it is necessary and sufficient that Γ is a regular curve and α_k satisfy the condition

$$-\frac{1}{p(t_k)} < \alpha_k < \frac{1}{p'(t_k)}\,, \;\; k = 1, \dots, n.$$

3. One Criterion for Belonging of the Analytic Function to the Class $E^{p(\cdot)}(D)$

If $p(t) = p = const$, then when studying the properties of functions from classes $E^p(D)$, the fact that the involution of the function $\phi \in E^p(D)$ is equivalent to the belonging of the function $\Psi(w) = \phi(z(w))[z'(w)]^{1/p}$ to the Hardy class H^p plays an important role. For variable p, the function $\Psi(w)$ is not even analytic.

It is desirable to have a certain analogue of the above-indicated result for a variable exponent, as well. It is particularly desirable to reveal those classes of domains D and functions $p(t)$ for which reasonable generalization of the above property would be possible.

In [2], such aim has been achieved under the assumption that $p \in \mathcal{P}(\Gamma)$ and the domain D is bounded by a piecewise Lyapunov curve, free from external cusps. Relying on the theorem from item 2.2.1, the following theorem is proved.

Theorem B. *If D is the bounded domain with the boundary $\Gamma \in C_D^{1,L}(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 < \nu_k \le 2$, and $p \in \mathcal{P}(\Gamma)$, then the analytic in D function $\phi(z)$ belongs to the class $E^{p(\cdot)}(D)$ if and only if*

$$\Psi(w) = \phi(z(w)) \prod_{k=1}^{n} (w - a_k)^{\frac{\nu_k - 1}{l(a_k)}} \in H^{l(\cdot)}, \ \ l(\tau) = p(z(\tau)). \tag{9}$$

3.1. In this section we will show that Theorem B can be generalized to a sufficiently wide class of functions $p(t)$ for arbitrary piecewise smooth curves.

Theorem 1. *Let $\Gamma \in C_D^1(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 \le \nu_k \le 2$, and $z = z(w)$ be conformal mapping of the circle U onto the domain. Next, let p be the function of the class*

$$Q(\Gamma) = \Big\{ p : \ p \in \widetilde{\mathcal{P}}(\Gamma), \ l(\tau) = p(z(\tau)) \in \widetilde{\mathcal{P}}(\gamma) \Big\}. \tag{10}$$

The analytic in D function $\phi(z)$ belongs to the class $E^{p(\cdot)}(D)$ if and only if

$$\Psi(w) = \Phi(z(w))\rho(w) \in H^{l(\cdot)}, \tag{11}$$

where

$$\rho(w) = \prod_{k=1}^{n} (w - a_k)^{\nu_k - 1} l(a_k) \exp \int_{\gamma} \frac{\psi(\zeta)}{l(\zeta)} \frac{d\zeta}{\zeta - w}, \ \ z(a_k) = A_k, \tag{12}$$

in which $\psi(\zeta)$ is the function from the representation (3) of the function $z'(w)$.

When $\Gamma \in C_D^{1,L}(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 < \nu_k \le 2$, and $p \in \mathcal{P}(\Gamma)$, the condition (11) is equivalent to the condition (9).

Proof. Let $\phi \in E^{p(\cdot)}(D)$. This is equivalent to the fact that

$$\Psi(w) = \phi(z(w)) \in H^{l(\cdot)}(m(w)), \tag{13}$$

where

$$m(w) = m(re^{i\vartheta}) = |z'(re^{i\vartheta})|^{\frac{1}{p(z(e^{i\vartheta}))}}. \tag{14}$$

Thus

$$\phi(z) \in E^{p(\cdot)}(D) \Longleftrightarrow \Psi(w) = \phi(z(w)) \in H^{l(\cdot)}(m(w)). \tag{15}$$

Let us now make use of the result given in [15]:
if $l \in \widetilde{\mathcal{P}}(\gamma)$, then

$$m(w) \sim m_0(w) = m_0(re^{i\vartheta}) = \\ = \prod_{k=1}^{n} (w - a_k)^{\frac{\nu_k - 1}{l(a_k)}} \exp\bigg(\frac{1}{l(e^{i\vartheta})} \int_\gamma \frac{\psi(\zeta)}{\zeta - re^{i\vartheta}}\, d\zeta\bigg), \tag{16}$$

and

$$m_0(w) \sim \rho(w) = \prod_{k=1}^{n} (w - a_k)^{\frac{\nu_k - 1}{l(a_k)}} \exp \int_\gamma \frac{\psi(\zeta)}{l(\zeta)}\, \frac{d\zeta}{\zeta - w}\,. \tag{17}$$

It follows from (16), (17) that $m(w) \sim \rho(w)$, and hence by virtue of (15), we conclude that

$$H^{l(\cdot)}(m(w)) = H^{l(\cdot)}(\rho(w)), \tag{18}$$

whence, in view of (13), it follows that the first statement of the theorem is valid.

Let now $\Gamma \in C_D^{1,L}(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 < \nu_k \le 2$, and $p \in \mathcal{P}(\Gamma)$. In this case, the function ψ in the representation (3) belongs to the Hölder class ([7, pp. 146] and [16]). Therefore the function $\int_\gamma \frac{\psi(\zeta)}{\zeta - w}\, d\zeta$ is bounded in U (see, e.g., [17, pp. 50, 71]). But then in U are bounded likewise the functions

$$\exp\bigg(\pm \frac{1}{l(e^{i\vartheta})} \int_\gamma \frac{\psi(\zeta)}{\zeta - re^{i\vartheta}}\, d\zeta\bigg).$$

Thus, on the basis of (16), we find that the second statement of the theorem is also valid. □

3.2. One Condition for Coincidence of the Classes $Q(\Gamma)$ and $\widetilde{\mathcal{P}}(\Gamma)$.

Theorem 2. *If the domain D is such that for conformal mapping $z = z(w)$ of the circle U onto D we have*

$$z'(w) \in \bigcup_{\delta > 0} H^{1+\delta}, \tag{19}$$

then

$$Q(\Gamma) = \widetilde{\mathcal{P}}(\Gamma). \tag{20}$$

Proof. By virtue of the definition of the class of functions $Q(\Gamma)$ (see (10)), it suffices to state that: if $p \in \widetilde{\mathcal{P}}(\Gamma)$, then $l \in \widetilde{\mathcal{P}}(\gamma)$. Towards this end, we shall use the following statement from [3]:

If $p \in \mathcal{P}(\Gamma)$, then under the condition (19), we have

$$|l(\tau_1 - l(\tau_2)| \leq \frac{A}{|\ln|z(\tau_1) - z(\tau_2)||} < \frac{A'}{|\ln|\tau_1 - \tau_2||}\,. \tag{21}$$

If $p \in \widetilde{\mathcal{P}}(\Gamma)$, then there exists the number $\varepsilon > 0$ for which the condition (6) is fulfilled. Then

$$|l(\tau_1 - l(\tau_2)| \leq \frac{A}{|\ln|z(\tau_1) - z(\tau_2)||^{1+\varepsilon}}$$

and (21) yields $|l(\tau_1 - l(\tau_2)| \leq A'|\ln|\tau_1 - \tau_2||^{-(1+\varepsilon)}$. Consequently, $l \in \mathcal{P}_{1+\varepsilon}(\gamma)$, and hence $l \in \widetilde{\mathcal{P}}(\gamma)$. □

Corollary 1. *If* $\Gamma \in \Lambda$*, then the equality* (20) *holds.*

This statement follows immediately from Theorem 2, if we take into account the fact that the inclusions (2) in the case under consideration are valid (see item 2.2.2).

Corollary 2. *If* $\Gamma \in C_D^1(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$*,* $0 < \nu_k \leq 2$*,* $k = 1, \dots, n$*, then* $Q(\Gamma) = \widetilde{\mathcal{P}}(\Gamma)$*.*

Indeed, since the function $\exp \int\limits_{\gamma} \frac{\psi(\zeta)}{\zeta - w}\, d\zeta$ for the continuous real ψ belongs to $\bigcap\limits_{\delta > 1} H^{\delta}$ (see [12] and [7, p. 96]), it is not difficult to state that $z' \in H^{1+\delta_0}$ for some $\delta_0 > 0$.

Corollary 3. *In the assumption of Corollary* 2*, the class* $Q(\Gamma)$ *in Theorem* 1 *can be replaced by the class* $\widetilde{\mathcal{P}}(\Gamma)$*.*

3.3. One Subset of the Class $\widetilde{\mathcal{P}}(\Gamma)$ Contained in $Q(\Gamma)$. Note first that according to Corollary 2, for $p \in \widetilde{\mathcal{P}}(\Gamma)$ the curves Γ of the class $\Gamma \in C_D^1(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, $0 < \nu_k \leq 2$, belong to $Q(\Gamma)$. However, if for some j we have $\nu_j = 0$, then this statement is, generally speaking, doubtful. Therefore for such curves it is desirable to indicate certain sets of functions $p(t)$ for which the equality (20) remains valid.

Let $p(t)$ be such a function from $\mathcal{P}(\Gamma)$ ($\widetilde{\mathcal{P}}(\Gamma)$) which is constant in some neighborhoods of the points A_{ν_j}. By virtue of the above-said, there exists the number $\sigma > 0$ such that as soon as $|t_1 - t_2| < \sigma$, the inequality (4) ((6)) will be fulfilled. Since the conformal mapping of the domains of above-mentioned type transfers the arcs of the boundary Γ into those of the circumference γ (see., e.g., [18, p. 46]), there exist neighborhoods of the points a_{ν_j} at which the condition (4) ((6)) is fulfilled. Consequently, there exists the number $\sigma_\gamma > 0$ such that for $|\tau_1 - \tau_2| < \sigma_\gamma$, $\tau_1, \tau_2 \in \gamma$, the inequality (4) ((6)) will be fulfilled. It is easy to verify that (4) ((6)) is valid for any pairs τ_1, τ_2 lying on γ. This implies that $l(\tau) \in \mathcal{P}(\gamma)$.

From the above, in particular, it follows that for the curves and functions $p(t)$ under consideration, we have $\widetilde{\mathcal{P}}(\Gamma) = Q(\Gamma)$. Moreover, in these assumptions, the set $Q(\Gamma)$ in Theorem 1 can be replaced by the set $\widetilde{\mathcal{P}}(\Gamma)$.

4. The Cauchy Type Integrals and Smirnov Classes

It is not difficult to state that if D is a simply connected domain bounded by a simple rectifiable curve Γ, and $p \in \mathcal{P}(\Gamma)$, then the functions of the class $E^{p(\cdot)}(D)$ are representable by the Cauchy type integral with density from $L^{p(\cdot)}(\Gamma)$ (see Theorem 3 below). However, one fails to inverse this statement to a full entent. It is shown in [2] that in the case of piecewise Lyapunov curves this way is quite possible.

In this section we prove that the integrals $(K_\Gamma\varphi)(z)$, $\varphi \in L^{p(\cdot)}(\Gamma)$, belong to $E^{p(\cdot)}(D)$ under some, very important for applications, assumptions regarding Γ and $p(t)$, including the case in which Γ is an arbitrary piecewise smooth curve, and $p(t) \in Q(\Gamma)$.

4.1. The Representability of Functions from $E^{p(\cdot)}(D)$ by the Cauchy Type Integral.

Theorem 3. *If D is the inner domain bounded by a simple rectifiable curve Γ, and $\phi \in E^{p(\cdot)}(D)$, where $p \in \mathcal{P}(\Gamma)$, then ϕ is representable by the Cauchy type integral with density from $L^{p(\cdot)}(\Gamma)$.*

Proof. It follows from the definition of the class $E^{p(\cdot)}(D)$ that $E^{p(\cdot)}(D) \subset E^{\underline{p}}(D)$, and since $p \in \mathcal{P}(\Gamma)$, hence $\underline{p} > 1$. Thus $E^{p(\cdot)}(D) \subset E^1(D)$. This implies that ϕ is representable by the Cauchy type integral, i.e.,

$$\phi(z) = (K_\Gamma\phi^+)(z), \quad z \in D, \tag{22}$$

(see, e.g., [8, p. 205]). Moreover, the function $F(w) = \phi(z(w))[z'(w)]^{1/\underline{p}}$ is of the Hardy class $H^{\underline{p}}$, and hence almost everywhere on γ there exists an angular boundary value $F^+(\tau)$. Since $z' \in H^1$ (see, e.g., [8, p. 405]), there likewise exists $[z'(w)]^+ = z'(\tau)$. Thus the boundary value of the function $\Phi(z(w))$ exists. Relying on this fact, we can conclude that

$$\lim_{r\to 1}\Big(|\Phi(re^{i\vartheta})|^{p(z(e^{i\vartheta}))}|z'(re^{i\vartheta})|\Big) = \big|\phi(z(e^{i\vartheta}))\big|^{p(z(e^{i\vartheta}))}|z'(e^{i\vartheta})|.$$

Using the Fatou lemma, by virtue of (8), we conclude that

$$\int_0^{2\pi} \big|\phi(z(e^{i\vartheta}))\big|^{p(z(e^{i\vartheta}))}|z'(e^{i\vartheta})|\,d\vartheta < \infty.$$

The above-said is equivalent to the fact that $\int_\Gamma |\phi^+(t)|^{p(t)}\,|dt| < \infty$, i.e., $\phi^+ \in L^{p(\cdot)}(\Gamma)$. But then the equality (22) implies that $\phi(z)$ is represented by the Cauchy type integral with density from $L^{p(\cdot)}(\Gamma)$. $\square$

4.2. On the Belonging of the Cauchy Type Integral to the Class $E^{p(\cdot)}(D)$ for Domains with Piecewise Smooth Boundaries.

Theorem 4. *Let D be the simply connected finite domain bounded by the curve Γ of the class $C^1_D(A_1, A_2, \dots, A_n; \nu_1, \nu_2, \dots, \nu_n)$, and $p \in Q(\Gamma)$. Then the Cauchy type integral $\phi(z) = (K_\Gamma\varphi)(z)$, where $\varphi \in L^{p(\cdot)}(\Gamma)$, belongs to the class $E^{p(\cdot)}(D)$.*

Proof. Since $L^{p(\cdot)}(\Gamma) \subset L^{\underline{p}}(\Gamma)$, therefore $\varphi \in L^{\underline{p}}(\Gamma)$, $\underline{p} > 1$. It is easy to verify that the piecewise smooth curve is regular, and according to statement (ii) of item 2.3, we can conclude that $\phi \in E^{\underline{p}}(D)$. Using Theorem 1, we find that the analytic in U function $\Psi(w) = \phi(z(w))\rho(w))$, where ρ, defined by the equality (12), belongs to the class $H^{\underline{p}}$. Let us now show that $\Psi^+ \in L^{l(\cdot)}(\gamma)$.

As far as Γ is the regular curve, and from the condition $p \in Q(\Gamma)$ follows $p \in \mathcal{P}(\Gamma)$, the operator S_Γ is continuous in $L^{p(\cdot)}(\Gamma)$ (see statement (i) of item 2.3). Thus the function $\phi^+(t_0) = \frac{1}{2}\varphi(t_0) + \frac{1}{2}(S_\Gamma\varphi)(t_0)$ belongs to $L^{p(\cdot)}(\Gamma)$, i.e., the function $\phi(z(\tau))[z'(\tau)]^{\frac{1}{p(z(\tau))}} \sim \phi(z(\tau))m^+(\tau)$ (see (14)) belongs to $L^{p(\cdot)}(\Gamma)$. This is the same thing as $\Psi^+ \in L^{l(\cdot)}(\gamma)$.

Thus $\Psi \in H^{\underline{p}}$ and $\Psi^+ \in L^{l(\cdot)}(\gamma)$, where $l \in \mathcal{P}(\gamma)$. We now apply the generalized Smirnov's theorem: if $\Psi(z) \in H^{l_1(\cdot)}$ and $\Psi^+(t) \in L^{l_2(\cdot)}(\gamma)$, $l_2 \in \mathcal{P}(\gamma)$, then $\Psi(z) \in H^{\widetilde{l}}(\cdot)$, where $\widetilde{l}(t) = \max(l_1(\tau), l_2(\tau))$ (under such a statement, this theorem has been proven in [2]). In our case, $\widetilde{l}(\tau) = \max(\underline{p}, l(\tau)) = l(\tau)$. Hence $\Psi(w) \in H^{l(\cdot)}$, i.e., $\phi(z(w)) \in H^{l(\cdot)}(\rho) = H^{l(\cdot)}(m(w))$ (see(18)), and this is the same thing as $\phi(z) \in E^{p(\cdot)}(D)$. $\square$

4.3. On the Belonging of the Cauchy Type Integrals with Density from $L^{p(\cdot)}(\Gamma)$ to the Class $E^{p(\cdot)}(D)$ when $p(t)$ is the Hölder Continuous Function.

If we assume that $p(t)$ is the Hölder class function, then the class of piecewise smooth curves in Theorem 4 can be replaced by another wide set of curves.

Upon our investigation we use Theorem 5 proven below. This theorem generalizes Smirnov's theorem (see 2.3.1) to the case of classes $E^{p(\cdot)}(D)$, when D belongs to a rather wide class of functions.

4.3.1. *Generalization of Smirnov's Theorem.*

Theorem 5. *Let Γ be the simple, rectifiable, closed, regular curve bounding the domain D such that*

$$z'(w) \in \bigcup_{\sigma>1} H^\sigma, \quad \frac{1}{z'(w)} \in \bigcup_{\eta>0} H^\eta, \tag{23}$$

where $z = z(w)$ is the conformal mapping of the circle U onto the domain D.

If $\phi(z) \in E^{\mu(\cdot)}(D)$, $\min\limits_{t\in\Gamma} \mu(t) = \delta > 0$ and $\phi^+(t) \in L^{p(t)}(\Gamma)$, where $p(t)$ is the Hölder class function on Γ, then $\phi(z) \in E^{\widetilde{p}(\cdot)}(D)$, $\widetilde{p}(t) = \max(\mu(t), p(t))$.

Proof. Assume $\Psi(w)=\phi(z(w))$ and show that the function $\Psi(w)$ in the adopted assumptions belongs to a certain Hardy class H^ε, $\varepsilon>0$.

Let ε be a number from the interval $(0,\delta)$. We have

$$I_r=\int\limits_0^{2\pi}|\Psi(re^{i\vartheta})|^\varepsilon\,d\vartheta=\int\limits_0^{2\pi}|\phi(re^{i\vartheta})|^\varepsilon|z'(re^{i\vartheta})|^{\frac{\varepsilon}{\delta}}|z'(re^{i\vartheta})|^{-\frac{\varepsilon}{\delta}}\,d\vartheta.$$

Using Hölder's inequality with the exponent $\delta/\varepsilon>1$, we obtain

$$I_r\le\bigg(\int\limits_0^{2\pi}|\phi(re^{i\vartheta})|^\delta|z'(re^{i\vartheta})|\,d\vartheta\bigg)^{\frac{\varepsilon}{\delta}}\bigg(\int\limits_0^{2\pi}|z'(re^{i\vartheta})|^{-\frac{\varepsilon}{\delta-\varepsilon}}\,d\vartheta\bigg)^{\frac{\delta-\varepsilon}{\delta}}\le$$
$$\le[M(r)]^{\frac{\varepsilon}{\delta}}\bigg(\int\limits_0^{2\pi}\frac{d\vartheta}{|z'(re^{i\vartheta})|^{\frac{\varepsilon}{\delta-\varepsilon}}}\bigg)^{\frac{\delta-\varepsilon}{\delta}},\quad M(r)=\int\limits_0^{2\pi}|\phi(re^{i\vartheta})|^\delta|z'(re^{i\vartheta})|\,d\vartheta. \tag{24}$$

It follows from the condition $\phi\in E^{\mu(\cdot)}(D)$ that $\phi\in E^\delta(D)$, and hence

$$\sup_{0<r<1}M(r)=C<\infty. \tag{25}$$

Further, the condition $\frac{1}{z'}\in\bigcup\limits_{\eta>0}H^\eta$ provides us with $\frac{1}{z'}\in H^{\eta_0}$ for some $\eta_0>0$. We choose ε such that $\frac{\varepsilon}{\delta-\varepsilon}=\eta_0$ (i.e., we take $\varepsilon=\varepsilon_0=\frac{\delta\eta_0}{1+\eta_0}$).

Since $\frac{1}{z'}\in H^{\eta_0}$, therefore

$$\sup_{0<r<1}\int\limits_0^{2\pi}\frac{d\vartheta}{|z'(re^{i\vartheta})|^{\eta_0}}<\infty.$$

In view of the above-said and the inequality (25), from (24) it follows that $\sup I_r<\infty$. Thus we have stated that $\Psi\in H^{\varepsilon_0}$, $\varepsilon_0=\frac{\delta\eta_0}{1+\eta_0}$.

Since $\Psi\in H^{\varepsilon_0}$, we have $\Psi(w)=e^{i\lambda}b(w)\sigma(w)D(w)$, where $b(w)$ is the Blaschke product, $\sigma(w)\ne0$, $|\sigma(w)|\le1$, $\lambda\in\mathbb{R}$, and

$$D(w)=\exp\frac{1}{2\pi}\int\limits_0^{2\pi}\ln|\Psi(e^{i\varphi})|\,\frac{e^{i\varphi}+w}{e^{i\varphi}-w}\,d\varphi,\quad |w|<1$$

(see [8, p. 110]).

Assume $l(\tau):=l(e^{i\vartheta})=p(z(e^{i\vartheta}))=p(z(\tau))$, $\tau=e^{i\vartheta}$. Then since $p(t)$ is the Hölder class function on Γ, there exist numbers M and $\alpha\in(0,1]$ such that $|p(t_1)-p(t_2)|<M|t_1-t_2|^\alpha$. Consequently,

$$|p(t_1)-p(t_2)|=\big|p(z(\tau_1))-p(z(\tau_2))\big|\le$$
$$\le AM|z(\tau_1-z(\tau_2)|^\alpha=AM\bigg|\int\limits_{\tau_1}^{\tau_2}z'(\tau)\,d\tau\bigg|^\alpha.$$

It follows from the inclusion $z' \in \bigcup\limits_{\sigma>1} H^{\sigma}$ (see (23)) that $z' \in H^{\sigma_0}$ for some $\sigma_0 > 1$. Then the last inequality (in view of the fact that on γ we have $s(\tau_1, \tau_2) \sim |\tau_1 - \tau_2|$) yields

$$|l(\tau_1) - l(\tau_2) \leq AM \bigg(\int\limits_{\tau_1}^{\tau_2} |z'(\tau)|^{\sigma_0} \, |d\tau| \bigg)^{\frac{\alpha}{\sigma_0}} |\tau_1 - \tau_2|^{\frac{\sigma_0 - 1}{\sigma_0} \alpha}.$$

Thus $l(\tau)$ is the function from the Hölder class on γ. In view of the above, we can apply the inequality proven in [2]:

$$|\Psi(re^{i\vartheta})|^{l(\vartheta)} \leq A(r, \vartheta) B(r, \vartheta), \tag{26}$$

where

$$A(r, \vartheta) = \exp \frac{1}{2\pi} \int\limits_0^{2\pi} l(\varphi) \ln |\widetilde{\Psi}(e^{i\varphi})| P(r, \vartheta - \varphi) \, d\varphi,$$

$$\widetilde{\Psi}(e^{i\varphi}) = \begin{cases} \Psi(e^{i\varphi}), & \text{if } |\Psi(e^{i\varphi})| \geq 1 \\ 1, & \text{if } |\Psi(e^{i\varphi})| < 1 \end{cases}, \quad P(r, x) = \frac{1 - r^2}{1 + r^2 - 2r \cos x} \,,$$

and for $B(r, \vartheta)$, the following estimate is valid:

$$|B(r, \vartheta)| \leq k_1 \exp k_2 \int\limits_0^{2\pi} |\Psi(e^{i\varphi})| \, d\varphi = k_3,$$

where k_1, k_2 does not depend on Ψ.

The inequality (26) results now in

$$\int\limits_0^{2\pi} |\Psi(re^{i\vartheta})|^{l(\vartheta)} |z'(re^{i\vartheta})| \, d\vartheta \leq$$

$$\leq k_3 \int\limits_0^{2\pi} \exp \bigg(\frac{1}{2\pi} \int\limits_0^{2\pi} \ln |\widetilde{\Psi}(e^{i\varphi})|^{l(\varphi)} P(r, \vartheta - \varphi) \, d\varphi \bigg) |z'(re^{i\vartheta})| \, d\vartheta. \tag{27}$$

Since Γ is the regular curve, therefore D is Smirnov's domain (see statement (ii) of item 2.3), and hence

$$|z'(re^{i\vartheta})| = |z'(w)| = \exp \ln |z'(w)| =$$

$$= \exp \frac{1}{2\pi} \int\limits_0^{2\pi} \ln |z'(re^{i\vartheta})| P(r, \vartheta - \varphi) \, d\varphi. \tag{28}$$

Moreover, we have

$$\ln|\widetilde{\Psi}(e^{i\varphi})|^{l(\varphi)} = \ln\Big|\frac{|\widetilde{\Psi}(e^{i\varphi})|^{l(\varphi)}z'(e^{i\varphi})}{z'(e^{i\varphi})}\Big| = \\ = \ln\Big[|\widetilde{\Psi}(e^{i\varphi})|^{l(\varphi)}|z'(e^{i\varphi})| - \ln|z'(e^{i\varphi})|\Big]. \quad (29)$$

From (27), by virtue of (28) and (29), we can conclude that

$$\int_0^{2\pi} |\Psi(re^{i\vartheta})|^{l(\vartheta)}|z'(re^{i\vartheta})|\,d\vartheta \leq \\ \leq k_3 \int_0^{2\pi} \exp\frac{1}{2\pi}\int_0^{2\pi} \ln|\widetilde{\Psi}(e^{i\varphi})|^{l(\varphi)}|z'(e^{i\varphi})|P(z,\vartheta-\varphi)\,d\varphi\,d\vartheta \leq \\ \leq k_3 \int_0^{2\pi} |\widetilde{\Psi}(re^{i\varphi})|^{l(\varphi)}|z'(e^{i\varphi})|\,d\varphi \leq \\ \leq k_3 \int_0^{2\pi} |\Psi(e^{i\varphi})|^{l(\varphi)}|z'(e^{i\varphi})|\,d\varphi + \int_0^{2\pi} |z'(e^{i\varphi})|\,d\varphi \leq \\ \leq k_3 \int_0^{2\pi} |\Psi(e^{i\varphi})|^{l(\varphi)}|z'(e^{i\varphi})|\,d\varphi + k_4. \quad (30)$$

By the assumption of the theorem, $\phi^+ \in L^{p(\cdot)}(\Gamma)$. But

$$\int_0^{2\pi} |\Psi(e^{i\varphi})|^{l(\varphi)}|z'(e^{i\varphi})|\,d\varphi = \int_\Gamma |\phi^+(t)|^{p(t)}\,|dt|$$

and from (30) follows

$$\sup_{r<1}\int_0^{2\pi} |\Psi(e^{i\varphi})|^{l(\varphi)}|z'(e^{i\varphi})|\,d\varphi < \infty.$$

Hence $\phi \in E^{p(\cdot)}(D)$; and since $\phi \in E^{\mu(\cdot)}(D)$, then $\phi \in E^{\widetilde{p}(\cdot)}(D)$, $\widetilde{p}(t) = \max(p(t),\mu(t))$. □

4.4. The Cauchy Type Integrals in the Domains with Lavrentiev Boundary.

Theorem 6. *If D is the inner domain bounded by a simple rectifiable curve of the class Λ, and p is the Hölder class function on Γ, then the Cauchy type integral $\phi(z) = (K_\Gamma\varphi)(z)$, where $\varphi \in L^{p(\cdot)}(\Gamma)$, belongs to the class $E^{p(\cdot)}(D)$.*

Proof. In the case under consideration, the both conditions in (23) are fulfilled. Moreover, it can be easily verified that any curve from Λ is regular one. Next, since $\varphi \in L^{\underline{p}}(\Gamma)$, $\underline{p} = \min_{t\in\Gamma} p(t)$, in view of property (ii) in item 2.3, we conclude that $\phi \in E^{\underline{p}}(D)$. Along with the above-said, $\phi^+ = \frac{1}{2}\,\varphi + \frac{1}{2}\,S_\Gamma\varphi$, $\varphi \in L^{\underline{p}(\cdot)}(\Gamma)$. Since $p \in \mathcal{P}(\Gamma)$, therefore $S_\Gamma\varphi \in L^{p(\cdot)}(\Gamma)$ (see Theorem A). Consequently, $\phi^+ \in L^{p(\cdot)}(\Gamma)$.

Thus $\phi \in E^{\underline{p}}(D)$ and $\phi^+ \in L^{p(\cdot)}(\Gamma)$, where $p(t)$ is the Hölder class function on Γ. This implies that all requirements of Theorem 5 are fulfilled and hence $\phi \in E^{p(\cdot)}(D)$. □

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LONG-TIME BEHAVIOR OF EVOLUTION INCLUSION WITH NON-DAMPED IMPULSIVE EFFECTS

Dedicated to Professor I.T. Kiguradze
on the occasion of his birthday

Abstract. In this paper we consider an evolution inclusion with impulse effects at fixed moments of time from the point of view of the theory of global attractors. For an upper semicontinuous multivalued term which does not provide the uniqueness of the Cauchy problem, we give sufficient conditions on non-damped multivalued impulse perturbations, which allow us to construct a multivalued non-autonomous dynamical system and prove for it the existence of a compact global attractor in the phase space.

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Key words and phrases. Evolution inclusion, impulse perturbation, multivalued dynamical system, global attractor.

რეზიუმე. გლობალურ ატრაქტორთა მეთოდით გამოკვლეულია ევოლუციური ჩართვა იმპულსური ზემოქმედებით დროის ფიქსირებულ მომენტებში. ზემოდან ნახევრადუწყვეტი მრავალსახა მარჯვენა მხარისათვის არამილევადი მრავალსახა იმპულსურ შეშფოთებაზე დადებულ გარკვეულ შეზღუდვებში აგებულია არაავტონომიური დინამიური სისტემა და დამტკიცებულია ფაზურ სივრცეში მისი კომპაქტური გლობალური ატრაქტორის არსებობა.

Introduction

One of the possible ways for the description of qualitative behavior of the solutions of evolution problem is the proving of the existence in a phase space of the problem of invariant attracting set, a global attractor. In contrast to finite-dimensional problems, in the case of infinite-dimensional situation of the dissipativity condition of a system does not ensure the existence of the compact attractor, and the resolving of this problem is based essentially on one-parameter semigroups apparatus. This approach was founded in the seventies of the past century by J. Hale and O. A. Ladyzhenskaya. It was then developed by J. Hale [7], [8] for autonomous infinite-dimensional systems generated by equations with delay, but his abstract results concerning the existence of global attractors of dynamical systems mostly coincided with the results due to O. A. Ladyzhenskaya [17], [18], which have been gained in studying the dynamics of solutions of a two-dimensional system of Navier–Stokes equations.

The essence of these results is based on the fact that for the given evolution problem

$$\begin{cases} \partial_t u(t) = F(u(t)), \\ u(0) = u_0 \in E, \end{cases} \tag{1}$$

for which, as is known, it is globally and uniquely solvable in some class W, and $u(t) \in E \ \forall\, t \in \Im_+$, where $\Im$ is a nontrivial semigroup of the additive group $\mathbb{R}$, $\Im_+ = \Im \cap [0, +\infty)$, the one-parametric family of mappings $\{V(t, \cdot\,) : E \mapsto E\}_{t \in \Im_+}$ is constructed, where

$$V(t, u_0) := \big\{u(t) |\ \ u(\,\cdot\,) \text{ is the solution of } (1)\big\}. \tag{2}$$

On the strength that the problem (1) is autonomous, the family of mappings (2) is a semidynamical system, for which the invariant, compact, attracting set in the phase space is found – a global attractor, which is minimal among closed attracting sets and maximal among invariant compact sets.

In the papers of J. Hale [7], [8], O. A. Ladyzhenskaya [17], [18], M. I. Vishik [1], R. Temam [26] and of other mathematicians the existence and properties of global attractors were established in many nonlinear equations of mathematical physics.

Owing to these works, the theory of global attractors of dynamical systems has became almost completed and for a wide class of autonomous well-posed evolution dissipative problems it gives response to the question about the existence of a global attractor, its connectedness, stability, robustness, regularity, structure and dimension.

At the same time, a large class of autonomous problems was left aside, for which there is a global solvability theorem in phase space and there is no uniqueness theorem or it hasn't proved yet. These are the three-dimensional Navier–Stokes system, the three-dimensional Benard system, the system of equations of chemical kinetics under general conditions on parameters, wave equations in the case of nonlinearity of general polynomial form, evolution nonlinear equations with non-Lipschitz function of interface, as well as an evolution inclusion that arises while investigating evolution equations with discontinuous coefficients. The problem of studying dynamics of systems with possible nonuniqueness of a solution was solved in two ways. G. R. Sell [25], M. I. Vishik [5] suggested the concept of a trajectory attractor, in the context of which the dynamical system is constructed in the space of trajectories on the basis of a shift operator. For that (already a single-valued) dynamical system one can find an attracting set, a trajectory attractor. But it is important to note that in the course of this approach the connection with the system‘s phase space has been lost. Another approach proposed in the papers due to J. M. Ball [2], V. S. Melnik [19], [20], assumed a possible nonuniqueness of the solution by introducing a multivalued analogue of the one-parameter semigroup (2).

Let us assume that the problem (1) is globally solved in the class W, $u(t) \in E \; \forall\, t \in \Im_+$. Then correctly defined (multivalued in the general case) is a family of mappings $\{G(t,\,\cdot\,) : E \mapsto 2^E\}_{t\in\Im_+}$, where

$$G(t,u_0) := \big\{u(t) |\;\; u(\,\cdot\,) \in W \text{ is the solution of (1)}\big\}. \tag{3}$$

The family of mappings (3) showing that the conditions

$$\begin{cases} G(0,x) = x & \forall\, x \in E, \\ G(t+s,x) \subset G(t,G(s,x)) & \forall\, x \in E, \;\; t,s \in \Im_+, \end{cases}$$

are fulfilled, is called an m-semiflow.

The global attractor of the m-semiflow in the phase space E is called a compact set Ξ which satisfies the following conditions:

1) $\forall\, t \in \Im_+ \Xi \subset G(t,\Xi)$ (semiinvariance),

2) for any bounded $B \subset E \, dist(G(t,B),\Xi) \to 0$, $t \to +\infty$ (attraction).

As it turned out, the mappings of type (3) occur naturally in the evolution equations without the uniqueness of a solution and also in evolution inclusions. For most of them, the existence of a global attractor was proved.

Eventually, the apparatus of global attractors of one-parameter semigroups turned out to be not an easy-to-use for research of the qualitative behavior of evolution systems, but it admits the generalization of nonautonomous systems. In [4] by V. V. Chepyzhov and M. I. Vishik, such type of generalization was realized by introducing an additional parameter, that was responsible for non-autonomous terms. Moreover, the application for equations with almost periodic in time right-hand part, as well as cascade systems were examined.

This scheme has been generalized in the case of ambiguous solvability by O. V. Kapustyan, V. S. Melnik, J. Valero[10]. The main idea of this approach consists in that for the problem

$$\begin{cases} \partial_t u(t) = F_{\sigma(t)}(u(t)), \\ u(\tau) = u_\tau \in E, \end{cases} \tag{4}$$

it is assumed that a non-autonomous term $\sigma(t)$ belongs to some space Σ, where $\{T(h) : \Sigma \mapsto \Sigma\}_{h\in\Im_+}$ is a semigroup, $\forall\,\sigma \in \Sigma$, $\tau \in \Im$, $u_\tau \in E$, the problem (4) is expected to be globally solvable in some class W_τ, $u(t) \in E$ $\forall\,t \geq \tau$. Thus we can correctly define the mapping (possibly multivalued):

$$U_\sigma(t,\tau,u_\tau) := \big\{u(t)|\ \ u(\,\cdot\,) \in W \text{ is the solution of (4)}\big\}. \tag{5}$$

It describes the dynamics of solutions of problems (4). If the following conditions are fulfilled for (5), $\forall\,\sigma \in \Sigma$

$$\begin{cases} U_\sigma(\tau,\tau,u_\tau) = u_\tau, & \\ U_\sigma(t,\tau,u_\tau) \subset U_\sigma(t,s,U_\sigma(s,\tau,u_\tau)) & \forall\,t \geq s \geq \tau, \\ U_\sigma(t+h,\tau+h,u_\tau) \subset U_{T(h)\sigma}(t,\tau,u_\tau) & \forall\,h \in \Im_+, \end{cases}$$

then the family of mappings (5) is called a family of m-processes, for which the global attractor is determined in the phase space E as a compact set Θ_Σ, for which the conditions below are fulfilled:

1) for any bounded $B \subset E$ $\forall\,\tau \in R$ $dist(U_\Sigma(t,\tau,B),\Theta_\Sigma) \to 0$, $t \to +\infty$,

2) Θ_Σ is minimal in a class of closed sets, which satisfies 1).

As it turned out, the dynamics of many classes of evolution problems can be described in terms of global attractors of m-processes. Random ambiguously solvable dynamical systems and evolution inclusions with non-autonomous right-hand part were investigated with the exception of the above-mentioned equations with almost periodic right-hand part and cascade systems. Consequently, such an essential non-autonomous object as evolution equations with impulses perturbations at fixed moments, can likewise be described in terms of non-autonomous dynamical processes. The existence of global attractors for evolution equations with impulsive effects was, for the first time, obtained in [11], [12], but only in the case of damped impulsive effects, that is, when values of impulsive perturbations tend to zero. This fact is essentially used in proving of the existence of global attractor, because in reality it is proved that every element of global attractor belongs to some trajectory of a non-perturbed evolution problem.

In the present article, relying on the theory of impulsive differential equations [24], the authors prove that the evolution inclusion with translation-compact perturbations at fixed moments [13] generates a multivalued dynamical system for which there exists the compact global attractor.

Global Attractors of Multivalued Processes

Let (X,ρ) be a metric space, $\Im_d = \{(t,\tau) \in \Im^2|\ \ t \geq \tau\}$, $P(X)$ be a set of all non-empty subsets of X, $\beta(X)$ be a set of all non-empty, bounded subsets of X, and Σ be some metric space, for which the semigroup $\{T(h) : \Sigma \mapsto \Sigma\}_{h\in\Im_+}$ is defined.

Definition 1. We say that the family of multivalued processes (MP) is defined, $\{U_\sigma : \Im_d \times X \mapsto P(X)\}_{\sigma\in\Sigma}$ $\forall\,\sigma \in \Sigma$, if the following conditions are fulfilled:

1) $U_\sigma(\tau,\tau,x) = x \quad \forall\, x \in X,\ \forall\,\tau \in \Im$,

2) $U_\sigma(t,\tau,x) \subseteq U_\sigma(t,s,U_\sigma(s,\tau,x)) \quad \forall\, t \geq s \geq \tau,\ \forall\, x \in X$,

3) $U_\sigma(t+h,\tau+h,x) \subseteq U_{T(h)\sigma}(t,\tau,x) \quad \forall\, t \geq \tau,\ \forall\, h \in \Im_+$,

where for $A \subset X$, $B \subset \Sigma$ $U_B(t,s,A) = \bigcup\limits_{\sigma\in B}\bigcup\limits_{x\in A} U_\sigma(t,s,x)$.

Definition 2. The compact set $\Theta_\Sigma \subset X$ is called a global attractor of the family of MP $\{U_\sigma\}_{\sigma\in\Sigma}$ if the following conditions are fulfilled:

1) Θ_Σ is a uniformly attracting set, i.e. $\forall\,\tau \in \mathbb{R}$, $\forall\, B \in \beta(X)$

$$dist(U_\Sigma(t,\tau,B),\Theta_\Sigma) \to 0, \quad t \to +\infty; \tag{6}$$

2) Θ_Σ is a minimal set in the class of all closed uniformly attracting sets.

Theorem 1. *Let the family* MP $\{U_\sigma\}_{\sigma\in\Sigma}$ *satisfy the following conditions:*

1) $\exists\, B_0 \in \beta(X)\ \forall\, B \in \beta(X)\ \forall\,\tau \in \Im\ \exists\, T = T(B,\tau)\ \forall\, t \geq T\ U_\Sigma(t,\tau,B) \subset B_0$*;*

2) $\forall\, B \in \beta(X)\ \forall\,\tau \in \Im\ \forall\, t_n \to +\infty$ *any* $\xi_n \in U_\Sigma(t_n,\tau,B)$ *is precompact in* X*.*

Then there exists Θ_Σ *which is the global attractor of* MP $\{U_\sigma\}_{\sigma\in\Sigma}$*. If, moreover,* $\forall\, h \in \Im_+$ $T(h)\Sigma = \Sigma$ *and in condition* 3) *from Definition* 1 *the equality is fulfilled, then it suffices to check only the conditions* 1), 2) *from the theorem for* $\tau = 0$*.*

Proof. For any $B \in \beta(X)$, $\tau \in \Im$, let us consider a set

$$\omega_\Sigma(\tau,B) = \bigcap_{s\geq 0}\overline{\bigcup_{t\geq s} U_\Sigma(t,\tau,B)}. \tag{7}$$

Under the condition 2) we find in a standard way that $\omega_\Sigma(\tau,B) \neq \varnothing$ is a compact, attracting set B, i.e.,

$$dist\big(U_\Sigma(t,\tau,B),\omega_\Sigma(\tau,B)\big) \to 0, \quad t \to +\infty,$$

and it is a minimal closed set possessing this property. Then the set

$$\Theta_\Sigma = cl_X\Big(\bigcup_{\tau\in\Im}\ \bigcup_{B\in\beta(X)} \omega_\Sigma(\tau,B)\Big) \tag{8}$$

satisfies the conditions 1), 2) from Definition 2.

Let us prove its compactness. Since $\forall\, B \in \beta(X)$ $\forall\,\tau \in \Im$ $\exists\, T = T(B,\tau)$ $\forall\, t \geq T$ $U_\Sigma(t,\tau,B) \subset B_0$, therefore $\forall\, p \in \Im_+$

$$\begin{aligned} U_\Sigma(t+p,\tau,B) &\subset U_\Sigma\big(t+p,t,U_\Sigma(t,\tau,B)\big) \subset \\ &\subset U_\Sigma(t+p,t,B_0) \subset U_{T(t)\Sigma}(p,0,B_0) \subset U_\Sigma(p,0,B_0). \end{aligned}$$

Thus $\forall\, s \geq T$, $\forall\, p \in \Im_+$

$$\bigcup_{t' \geq s+p} U_\Sigma(t', \tau, B) \subset U_\Sigma(p, 0, B_0).$$

Then $\forall\, s' \in \Im_+$

$$\bigcup_{p \geq s'} \bigcup_{t' \geq s+p} U_\Sigma(t', \tau, B) \subset \bigcup_{p \geq s'} U_\Sigma(p, 0, B_0),$$

$$cl_X\Big(\bigcup_{t' \geq s+s'} U_\Sigma(t', \tau, B)\Big) \subset cl_X\Big(\bigcup_{p \geq s'} U_\Sigma(p, 0, B_0)\Big),$$

$$\bigcap_{s' \geq 0} cl_X\Big(\bigcup_{t' \geq s+s'} U_\Sigma(t', \tau, B)\Big) \subset \bigcap_{s' \geq 0} cl_X\Big(\bigcup_{p \geq s'} U_\Sigma(p, 0, B_0)\Big),$$

$$\bigcap_{s'' \geq s} cl_X\Big(\bigcup_{t' \geq s''} U_\Sigma(t', 0, B)\Big) \subset \omega_\Sigma(0, B_0).$$

Thereby, $\omega_\Sigma(\tau, B) \subset \omega_\Sigma(0, B_0)$, hence $\Theta_\Sigma = \omega_\Sigma(0, B_0)$, and the desired compactness is proved. The second part of the theorem follows from the following inclusions: if $\tau \geq 0$

$$U_\Sigma(t, \tau, B) \subset U_{T(\tau)\Sigma}(t-\tau, 0, B) \subset U_\Sigma(t-\tau, 0, B);$$

if $\tau < 0$

$$U_\Sigma(t, \tau, B) = U_{T(-\tau)\Sigma}(t, \tau, B) = U_\Sigma(t-\tau, 0, B).$$

The theorem is proved. □

The Statement of the Impulsive Problem and the Properties of Solutions

Given a triplet $V \subset H \subset V^*$ of Hilbert spaces with a compact and dense embedding, $\langle \cdot\,, \cdot \rangle$ is a canonical duality between V and V^*. Let us denote by $\|\cdot\|$ and $(\cdot\,, \cdot)$ the norm and the scalar product in the space H, $\|\cdot\|_V$ is a norm in the space V. Assume that the inequality $\|u\|^2 \leq \alpha \|u\|_V^2$ is fulfilled.

We consider a linear continuous operator $A : V \to V^*$, which for the constants $\lambda_1 > 0, \lambda_2 > 0$ satisfies the following conditions:

$$\forall\, u \in V \quad \langle Au, u \rangle \geq \lambda_1 \|u\|_V^2, \tag{9}$$

$$\forall\, u, v \in V \quad |\langle Au, v \rangle| \leq \lambda_2^{\frac{1}{2}} \langle Au, u \rangle^{\frac{1}{2}} \|v\|_V. \tag{10}$$

From the condition (9), we obtain the estimate $|\langle Au, v\rangle| \leq \lambda_2 \|u\|_V \|v\|_V$. Then using Lax–Milgram's lemma, we have that $\exists\, A^{-1} \in L(V^*, V)$ and, moreover, $\|A^{-1}\| \leq \frac{1}{\lambda_1}$, $\|A\| \leq \lambda_2$.

Suppose that the multivalued perturbation $F : H \mapsto P(H)$ satisfies the conditions

$$\forall\, y \in H \quad F(y) \ \text{ is convex, closed, bounded subset of } \ H; \tag{11}$$

F is w-upper semicontinuous (w-u.s.), and has no more than linear growth, i.e.

$$\forall \varepsilon > 0 \ \forall y_0 \in H \ \exists \delta > 0 \quad y \in O_\delta(y_0), \ \ F(y) \subset O_\varepsilon(F(y_0)); \tag{12}$$

$$\exists C \geq 0 \ \forall y \in H \quad \|F(y)\|_+ \leq C(1 + \|y\|). \tag{13}$$

Here, for $B \subset H$, we denote $\|B\|_+ = \sup\limits_{b \in B} \|b\|$.

Consider the problem

$$\begin{cases} \dfrac{dy}{dt} + Ay \in F(y) + h(t), \quad t > \tau, \\ y(\tau) = y_\tau, \end{cases} \tag{14}$$

where $\tau \in \mathbb{R}$, $y_\tau \in H$, the operator A and the multivalued function F satisfy the conditions (9), (10), (11)–(13), $h \in L^2_{loc}(\mathbb{R}, H)$.

Definition 3. By the solution of the problem (14) on (τ, T) is meant the function $y \in L^2(\tau, T; V)$ with $\frac{dy}{dt} \in L^2(\tau, T; V^*)$ such that there exists $f \in L^2(\tau, T; H)$, $f(t) \in F(y(t))$ almost everywhere (a.e.), and

$$\begin{cases} \dfrac{dy}{dt} + Ay = f(t) + h(t), \\ y(\tau) = y_\tau. \end{cases} \tag{15}$$

It is known [6] that for all $\tau \in \mathbb{R}$, $T > \tau$, $y_\tau \in H$ under the conditions (9), (10), (11)–(13) the problem (14) has at least one solution and, moreover, any solution of problem (14) belongs to the space $C([\tau; T]; H)$. Thus, there is a reason to speak about global solvability of (14) on $(\tau, +\infty)$.

For the problem (14), we formulate the following impulsive problem: at fixed time moments $\{\tau_i\}_{i \in Z}$, $\tau_{i+1} - \tau_i \geq \gamma > 0$, every solution of the problem (14) in the phase space H undergoes impulsive perturbation of the form:

$$y(\tau_i + 0) - y(\tau_i) \in g(y(\tau_i)) + \Psi_i, \ \ i \in \mathbb{Z}, \tag{16}$$

where $g : H \mapsto H$ is the given function and $\Psi_i \subset H$ are the given sets.

Then $\forall \tau \in [\tau_i, \tau_{i+1})$, $\forall y_\tau \in H$, the Cauchy problem for (14), (16) is globally solvable in the sense that $\forall y_\tau \in H$ there exists the function $y(\cdot)$, which is the solution of (14) on (τ, τ_{i+1}), $(\tau_{i+1}, \tau_{i+2}), \ldots$, $y(\tau) = y_\tau$, and at the time moments $\{\tau_i, \tau_{i+1}, \ldots\}$, the function $y(\cdot)$ satisfies the relation (16) and is left-continuous.

Let us define some properties of the solution for the problem (14), (16). Towards this end, we consider an auxiliary problem

$$\begin{cases} \dfrac{dy}{dt} + Ay = f(t), \\ y(\tau) = y_\tau. \end{cases} \tag{17}$$

It is known [3], [26] that the problem (17) under the conditions (9), (10) for any $y_\tau \in H$, $T > \tau$, $f \in L^2(\tau, T; H)$ has a unique solution in the Hilbert space

$$W(\tau,T)=\Big\{y|\ \ y\in L^2(\tau,T;V),\ \frac{dy}{dt}\in L^2(\tau,T;V^*)\Big\},$$

which is denoted by $y=I(f,y_\tau)$. Moreover, the function $t\mapsto\|y(t)\|$ is absolutely continuous on $[\tau,T]$ and a.e. on (τ,T) the equality

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|^2+\langle Ay(t),y(t)\rangle=(f(t),y(t)) \tag{18}$$

is valid.

Lemma 1. *We have a sequence of problems* (17) *with right-hand parts* $f_n\in L^2(\tau,T;H)$ *and initial datas* $y^n_\tau\in H$. *Let* $f_n\xrightarrow{w} f$ *in* $L^2(\tau,T;H)$, $y^n_\tau\xrightarrow{w} y_\tau$ *in* H. *Then* $y_n=I(f_n,y^n_\tau)\to y=I(f,y_\tau)$ *in* $C([\delta,T];H)$ $\forall\,\delta\in(\tau,T)$. *If* $y^n_\tau\to y_\tau$ *in* H, *then* $y_n\to y$ *in* $C([\tau,T];H)$.

Proof. From (18), we have an estimation for $\tau\le s\le t\le T$,

$$\|y_n(t)\|^2+2\lambda_1\int_s^t\|y_n(p)\|_V^2\,dp\le\|y_n(s)\|^2+2\int_s^t(f_n(p),y_n(p))\,dp. \tag{19}$$

From (19), due to the boundedness of $\{f_n\}$ in $L^2(\tau,T;H)$, the boundedness of $\{y^n_\tau\}$ in H and (7), we have that $\exists\,M>0\ \forall\,n\ge1$,

$$\sup_{t\in[\tau,T]}\|y_n(t)\|+\int_\tau^T\|y_n(p)\|_V^2\,dp+\int_\tau^T\Big\|\frac{dy_n}{dt}\Big\|_{V^*}^2\,dp\le M. \tag{20}$$

Hence there exists $y\in W(\tau,T)$ such that $y_n\xrightarrow{w} y$ in $W(\tau,T)$. Then under the compactness of the embedding $W(\tau,T)\subset L^2(\tau,T;H)$, we obtain $y_n\to y$ in $L^2(\tau,T;H)$, and it means that $y_n(t)\to y(t)$ in H for almost all $t\in(\tau,T)$, and, besides, $y_n(t_n)\xrightarrow{w} y(t_0)$ in $H\ \ \forall\,t_n\to t_0\in[\tau,T]$. Hence, in particular, $y=I(f,y_\tau)$.

Let us now consider the functions

$$J_n(t)=\|y_n(t)\|^2-2\int_\tau^t(f_n(p),y_n(p))\,dp,\quad J(t)=\|y(t)\|^2-2\int_\tau^t(f(p),y(p))\,dp.$$

These functions under (19) are monotonous non-increasing, continuous, and $J_n(t)\to J(t)$ a.e. on (τ,T). Then $J_n(t)\to J(t)$ in $C([\delta,T])\ \ \forall\,\delta\in(\tau,T)$.

Let

$$\max_{t\in[\delta,T]}\|y_n(t)-y(t)\|=\|y_n(t_n)-y(t_n)\|$$

and on some subsequence $t_n\to t_0$.

Thus, under (20),

$$\int_{t_0}^{t_n}\big|(f_n(p),y_n(p))\big|\,dp\le M\int_{t_0}^{t_n}\|y_n(p)\|\,dp\to0,\ \ n\to+\infty.$$

Then

$$\int_{\tau}^{t_n} (f_n(p), y_n(p))\, dp \longrightarrow \int_{\tau}^{t_0} (f(p), y(p))\, dp.$$

Hence, under the weak convergence of $y_n(t_n)$ to $y(t_0)$, we have a system of inequalities

$$J(t_0) \le \underline{\lim} \|y_n(t_n)\|^2 - 2\int_{\tau}^{t_0} (f(p), y(p))\, dp \le$$

$$\le \overline{\lim} \|y_n(t_n)\|^2 - 2\int_{\tau}^{t_0} (f(p), y(p))\, dp \le \overline{\lim} J_n(t_n) = J(t_0).$$

It follows that there exists $\lim\limits_{n\to+\infty} \|y_n(t_n)\| = \|y(t_0)\|$ such that $y_n(t_n) \to y(t_0)$ in H. Hence, on some subsequence, $y_n \to y$ in $C([\delta, T]; H)$. Since (17) has a unique solution, the convergence goes along the whole sequence.

If $y_\tau^n \to y_\tau$, then $J_n(\tau) \to J(\tau)$, hence $J_n \to J$ in $C([\tau, T])$ and, similarly to the previous arguments, we obtain $y_n \to y$ in $C([\tau, T]; H)$. The lemma is proved. $\square$

The following lemma provides us with the sufficient conditions of dissipativity for the impulsive problem (14), (16).

Lemma 2. *Let the conditions*

$$\sup_{i\in Z} \|\Psi_i\|_+ < \infty, \tag{21}$$

$$\exists\, D > 0 \ \ \forall\, u \in H \quad \|g(u)\| \le D(1 + \|u\|), \tag{22}$$

$$\|h\|_+^2 := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|h(s)\|^2\, ds < \infty, \tag{23}$$

$$-\frac{2\lambda_1}{\alpha} + 2C + \frac{1}{\gamma} \ln(1 + (D+1)^2) < 0 \tag{24}$$

be fulfilled. Then

$$\begin{gathered} \exists\, R > 0 \ \text{ such that } \forall\, r \ge 0 \ \forall\, y_0 \in H, \ \ \|y_0\| \le r, \\ \text{and for any solution } y(\cdot) \text{ of the problem (14), (16) on } (0, +\infty) \\ \text{with } y(0) = y_0, \ \ \exists\, T = T(r) \ \text{ such that } \forall\, t \ge T, \ \ \|y(t)\| \le R. \end{gathered} \tag{25}$$

Proof. From the inequality

$$\frac{d}{dt} \|y(t)\|^2 + \frac{2\lambda_1}{\alpha} \|y(t)\|^2 \le 2C\|y(t)\|^2 + 2C\|y(t)\| + 2\|h(t)\|\, \|y(t)\| \tag{26}$$

and under the condition (24), for a.a. t we have the estimation

$$\frac{d}{dt} \|y(t)\|^2 + \delta \|y(t)\|^2 \le C_1\big(\|h(t)\|^2 + 1\big), \tag{27}$$

where the constants $\delta = \frac{2\lambda_1}{\alpha} - 2C > 0$, $C_1 > 0$ depend only on the constants of the problem (14), (16). Moreover, taking (16) into account, we have

$$\Big| \|y(\tau_i + 0)\|^2 - \|y(\tau_i)\|^2 \Big| \le (D+1)^2 \|y(\tau_i)\|^2 + C_2,$$

where the constant $C_2 > 0$ depends only on the constants of the problem (14), (16). It turns out that the function $t \mapsto \|y(t)\|^2$ is the solution of the impulsive problem

$$\begin{cases} \dfrac{d}{dt}\, \|y(t)\|^2 + \delta \|y(t)\|^2 \le C_1\big(\|h(t)\|^2 + 1\big), \\ \|y(\tau_i + 0)\|^2 - \|y(\tau_i)\|^2 \le (D+1)^2 \|y(\tau_i)\|^2 + C_2, \end{cases}$$

and the solutions of this problem at every moment cannot exceed the solutions of the problem

$$\begin{cases} \dfrac{d}{dt}\, x(t) + \delta x(t) = C_1\big(\|h(t)\|^2 + 1\big), \\ x(\tau_i + 0) - x(\tau_i) = (D+1)^2 x(\tau_i) + C_2. \end{cases} \tag{28}$$

For every $x_0 \in \mathbb{R}$, the solution $x(\cdot)$ of the problem (28) with $x(0) = x_0$ is defined by the formula [24]

$$x(t) = e^{-\delta t}\big(1 + (D+1)^2\big)^{i(t,0)} \cdot x_0 + \\ + \int_0^t C_1\big(\|h(p)\|^2 + 1\big) e^{-\delta(t-p)} \big(1 + (D+1)^2\big)^{i(t,p)}\, dp + \\ + C_2 \sum_{0 \le \tau_i < t} e^{-\delta(t-\tau_i)} \big(1 + (D+1)^2\big)^{i(t,\tau_i)},$$

where $i(t,s)$ is a number of points τ_i on $[s,t)$.

By the condition (24), $\exists\, \mu > 0$ such that

$$-\delta + \frac{1}{\gamma}\, \ln(1 + (D+1)^2) \le -\mu < 0,$$

and $\forall\, t > 0$, we have the inequality

$$\int_0^t \|h(s)\|^2 e^{-\mu(t-s)}\, ds \le \\ \le \int_{t-1}^t \|h(s)\|^2\, ds + e^{-\mu} \int_{t-2}^{t-1} \|h(s)\|^2\, ds + e^{-2\mu} \int_{t-3}^{t-2} \|h(s)\|^2\, ds + \cdots \le \\ \le \|h\|_+^2 (1 - e^{-\mu})^{-1},$$

then for $x_0 = \|y(0)\|^2$, it is easy to get an estimation for all $t \ge 0$

$$\|y(t)\|^2 \le x(t) \le e^{-\mu t} \|y(0)\|^2 + M,$$

from which follows the condition (25). The lemma is proved. □

The Construction of the Semigroup of Translations for Impulsive Systems with Nondamped Perturbations

Let us begin with the presentation of the concept of translation-compact functions [5]. Let $(\mathbb{M}, \rho_{\mathbb{M}})$ be a complete metric space. We consider the space $\mathbb{C}(\mathbb{R}; \mathbb{M})$ of continuous functions from $\mathbb{R}$ to $\mathbb{M}$ with topology of uniform convergence on the compacts, i.e.,

$$\sigma_n \to \sigma \text{ in } \mathbb{C}(\mathbb{R}; \mathbb{M}) \Longleftrightarrow$$
$$\Longleftrightarrow \forall [t_1, t_2] \subset \mathbb{R}, \quad \max_{t \in [t_1, t_2]} \rho_{\mathbb{M}}(\sigma_n(t), \sigma(t)) \to 0, \quad n \to \infty.$$

The defined topology can be described by using the metric, and with this metric $\mathbb{C}(\mathbb{R}; \mathbb{M})$ will be the complete metric space.

For the fixed $\sigma(\cdot) \in \mathbb{C}(\mathbb{R}; \mathbb{M})$, define the set

$$H(\sigma) := cl_{\mathbb{C}(\mathbb{R};\mathbb{M})}\big\{\sigma(t + \cdot) | \ t \in \mathbb{R}\big\}.$$

Definition 4. The function $\sigma(\cdot) \in \mathbb{C}(\mathbb{R}; \mathbb{M})$ is called a translation-compact function (tr.-c.) in $\mathbb{C}(\mathbb{R}; \mathbb{M})$ if $H(\sigma)$ is compact in $\mathbb{C}(\mathbb{R}; \mathbb{M})$.

The concept of the translation-compactness, as the form of generalization of almost periodicity, was presented in [5]. In this paper, an example of translation-compact but not almost periodic function is given.

Lemma 3 ([5])**.** *If $\sigma \in \mathbb{C}(\mathbb{R}; \mathbb{M})$ is tr.-c. function in $\mathbb{C}(\mathbb{R}; \mathbb{M})$, then*

1) *any $\sigma_1(\cdot) \in H(\sigma)$ is also tr.-c. in $\mathbb{C}(\mathbb{R}; \mathbb{M})$, $H(\sigma_1) \subseteq H(\sigma)$;*

2) $\exists R > 0 \ \forall \sigma_1(\cdot) \in H(\sigma) \ \sup\limits_{s \in \mathbb{R}} \rho_{\mathbb{M}}(\sigma_1(s), 0) \le R$*;*

3) *the translation group $\{T(t)\}_{t \in \mathbb{R}}$, $T(t)\sigma(s) = \sigma(t + s)$, for any $t \in \mathbb{R}$ is continuous in the topology $\mathbb{C}(\mathbb{R}; \mathbb{M})$, and $T(t)H(\sigma) = H(\sigma)$.*

Let us consider the space $L^{2,w}_{loc}(\mathbb{R}; H)$, that is, the space $L^2_{loc}(\mathbb{R}; H)$ with a local weak convergence topology, i.e.,

$$\sigma_n \to \sigma \text{ in } L^{2,w}_{loc}(\mathbb{R}; H) \Longleftrightarrow \forall [t_1, t_2] \subset \mathbb{R} \ \ \forall \eta \in L^2(t_1, t_2; H),$$
$$\int_{t_1}^{t_2} (\sigma_n(t) - \sigma(t), \eta(t))\, dt \to 0, \quad n \to \infty.$$

In the same way as above, for the function $\sigma \in L^{2,w}_{loc}(\mathbb{R}; H)$ we consider the set

$$H(\sigma) := cl_{L^{2,w}_{loc}(\mathbb{R};H)}\big\{\sigma(t + s) | \ t \in \mathbb{R}\big\}.$$

Definition 5. The function $\sigma(\cdot) \in L^{2,w}_{loc}(\mathbb{R}; H)$ is to be called translation-compact (tr.-c.) in $L^{2,w}_{loc}(\mathbb{R}; H)$, if $H(\sigma)$ is compact in $L^{2,w}_{loc}(\mathbb{R}; H)$.

Lemma 4 ([5])**.** *The function $\sigma \in L^{2,w}_{loc}(\mathbb{R}; H)$ is tr.-c. in $L^{2,w}_{loc}(\mathbb{R}; H) \Leftrightarrow$ $\|\sigma\|^2_+ < \infty$.*

Lemma 5 ([5]). *If $\sigma \in L^{2,w}_{loc}(\mathbb{R}; H)$ is tr.-c. in $L^{2,w}_{loc}(\mathbb{R}; H)$, then*

1) *any $\sigma_1(\cdot) \in H(\sigma)$ is also tr.-c. in $L^{2,w}_{loc}(\mathbb{R}; H)$, $H(\sigma_1) \subseteq H(\sigma)$;*

2) *$\forall\, \sigma_1(\cdot) \in H(\sigma)$ $\|\sigma_1\|_+^2 \le \|\sigma\|_+^2$;*

3) *the translation group $\{T(t)\}_{t\in\mathbb{R}}$, $T(t)\sigma(s) = \sigma(t+s)$, for any $t \in \mathbb{R}$ is continuous in the topology $L^{2,w}_{loc}(\mathbb{R}; H)$, and $T(t)H(\sigma) = H(\sigma)$.*

We proceed to the construction of translation-compact distribution as the generalization of almost periodic distribution [24].

Consider the separable Banach space

$$D = \big\{\varphi \in \mathbb{C}^1(\mathbb{R})|\ \ D^1\varphi \text{ is absolutely continuous on } \mathbb{R},\\ D^j\varphi \in L^1(\mathbb{R}),\ j = 0, 1, 2\big\}$$

with the norm

$$|\varphi|_D := \max_{j=0,1,2} \bigg\{ \int_{-\infty}^{+\infty} |\ \ D^j\varphi(t)|\, dt \bigg\}.$$

Let $(X, \|\cdot\|)$ be the Banach space. We consider a subset of the space $L(D, X)$ of all linear continuous operators from D into X for fixed $K > 0$:

$$W_K = \big\{h \in L(D, X)|\ \ \|h\|_{L(D,X)} \le K\big\}.$$

Lemma 6. *There exists the function ρ_{W_K} on W_K for which the following conditions are fulfilled:*

1) *(W_K, ρ_{W_K}) is the complete metric space;*

2) *$\rho_{W_K}(A_n, A) \to 0 \Longleftrightarrow \forall\, \varphi \in D$ $A_n\varphi \to A\varphi$;*

3) *$\rho_{W_K}(A_1, A_2) \le L\|A_1 - A_2\|_{L(D,X)}$.*

Proof. Let $\{x_i\}$ be a dense set in D. There is

$$\rho_{W_K}(A, B) = \sum_{i=1}^{\infty} \alpha_i \frac{\|Ax_i - Bx_i\|}{1 + \|Ax_i - Bx_i\|}$$

for $\alpha_i > 0$, $\sum\limits_{i=1}^{\infty} \alpha_i < \infty$, the metric is determined in W_K, and the condition 2) is fulfilled. Moreover, in this formula we always can choose numbers $\{\alpha_i\}$ such that the inequality $\sum\limits_{i=1}^{\infty} \alpha_i\|x_i\| < \infty$ holds. Let us now prove that (W_K, ρ_{W_K}) is a complete metric space.

Indeed, if $\rho_{W_K}(A_n, A_m) \to 0$, then $A_n\varphi - A_m\varphi \to 0$ for any $\varphi \in D$. We put $A\varphi := \lim A_n\varphi$, then A is linear. Thus, $\|A\varphi\| \le K|\varphi|_D$ under $\|A_n\varphi\| \le K|\varphi|_D$, so $A \in W_K$. Since $\sum\limits_{i=1}^{\infty} \alpha_i\|x_i\| < \infty$, it follows that $\rho_{W_K}(A, B) \le L\|A - B\|_{L(D,X)}$. The lemma is proved. □

Next, for any $s \in \mathbb{R}$, we consider the map $T(s) : W_K \mapsto W_K$ such that

$$(T(s)h)\varphi(\cdot) = h\varphi(\cdot - s) \;\; \forall\, h \in W_K, \;\; \forall\, \varphi \in D.$$

It is easy to find that $T(s)W_K = W_K \; \forall\, s \in \mathbb{R}$ and $\{T(s)\}$ is a continuous group in W_K.

Definition 6. The element $h \in W_K$ is called a translation-compact distribution if the function $T(\cdot)h : \mathbb{R} \mapsto W_K$ is translation-compact in $\mathbb{C}(\mathbb{R}; W_K)$.

Here, the set

$$\Sigma_K = cl_{W_K}\big\{T(s)h \mid \; s \in \mathbb{R}\big\} \tag{29}$$

is called a minimal flow which is generated by $h \in W_K$.

Lemma 7. *If $h \in W_K$ is the translation-compact distribution, then Σ_K is compact in W_K and $T(s)\Sigma_K = \Sigma_K$ for any $s \in \mathbb{R}$. If for $h \in W_K$, the mapping $T(\,\cdot\,)h : \mathbb{R} \mapsto W_K$ is uniformly continuous in $\mathbb{R}$ and Σ_K is compact in W_K, then h is the translation-compact distribution.*

Let the sequences $\{f_i\}_{i\in\mathbb{Z}} \subset X$, $\{t_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}$ be given, and the following conditions be fulfilled:

$$\begin{gathered} \sup_{i\in\mathbb{Z}} \|f_i\| \le K, \;\; \{f_i\}_{i\in\mathbb{Z}} \text{ is precompact in } X, \\ t_i = ai + c_i \text{ for } a > 0, \;\; \sup_{i\in\mathbb{Z}} |c_i| < \infty, \;\; t_{i+1} - t_i \ge \gamma > 0. \end{gathered} \tag{30}$$

Then $h \in L(D, X)$ is determined by $h = \sum\limits_i f_i \delta_{t_i}$, $h\varphi = \sum\limits_i f_i \varphi(t_i)$ and

$$\|h\varphi\| \le \Big\| \sum_i f_i \varphi(t_i) \Big\| \le K \sum_i |\varphi(t_i)| \, \frac{t_{i+1} - t_i}{t_{i+1} - t_i} \le \frac{2K}{\gamma} \, |\varphi|_D \,.$$

The last inequality is a consequence of the following lemma.

Lemma 8. *If $\varphi \in D$, then the inequality*

$$\sum_i \|\varphi(t_i)\| (t_{k+1} - t_k) \le \int\limits_{\mathbb{R}} \big(\|\varphi(t)\| + \|\varphi'(t)\|\big)\, dt$$

holds.

Proof. The lemma can be considered as already proven if the inequality

$$|\varphi(t)|(t_{k+1} - t_k) \le \int\limits_{t_k}^{t_{k+1}} \big(|\varphi(s)| + |\varphi'(s)|\big)\, ds$$

holds for $k \in \mathbb{Z}$, where $t \in [t_k, t_{k+1}]$.

Summing the following inequalities

$$\int_{t_k}^{t} \varphi'(s)(s-t_k)\,ds = \varphi(t)(t-t_k) - \int_{t_k}^{t} \varphi(s)\,ds,$$

$$\int_{t}^{t_{k+1}} \varphi'(s)(s-t_{k+1})\,ds = -\varphi(t)(t-t_k) + (t_{k+1}-t_k)\varphi(t) - \int_{t}^{t_{k+1}} \varphi(s)\,ds,$$

we get

$$\varphi(t)(t_{k+1}-t_k) = \int_{t_k}^{t_{k+1}} \varphi(s)\,ds + \int_{t_k}^{t} \varphi'(s)(s-t_k)\,ds + \int_{t}^{t_{k+1}} \varphi'(s)(s-t_k-1)\,ds.$$

So,

$$\begin{aligned}
&|\varphi(t)|(t_{k+1}-t_k) \le \\
&\le \int_{t_k}^{t_{k+1}} |\varphi(s)|ds + (t-t_k)\int_{t_k}^{t} |\varphi'(s)|\,ds + (t_k+1-t)\int_{t}^{t_{k+1}} |\varphi'(s)|\,ds \le \\
&\le \int_{t_k}^{t_{k+1}} |\varphi(s)|\,ds + (t-t_k)\int_{t_k}^{t_{k+1}} |\varphi'(s)|\,ds + (t_k+1-t)\int_{t_k}^{t_{k+1}} |\varphi'(s)|\,ds \le \\
&\qquad\qquad \le \int_{t_k}^{t_{k+1}} \big(|\varphi(s)| + |\varphi'(s)|\big)\,ds.
\end{aligned}$$

The lemma is proved. □

Denote $W = W_{\frac{2K}{\gamma}}$, $\Sigma = \Sigma_{\frac{2K}{\gamma}}$. Under the conditions that $\{f_i\}_{i\in\mathbb{Z}}$ is precompact, and $\{c_i\}_{i\in\mathbb{Z}}$ is bounded, we can use the following property: for any sequence of integers $\{m_n\}$ there exist sequences $\{m_k\}$ and $\{\widetilde{f}_i\}_{i\in Z} \subset X$, $\{\widetilde{c}_i\}_{i\in Z} \subset \mathbb{R}$ such that for all $i \in \mathbb{Z}$,

$$\|f_{i+m_k} - \widetilde{f}_i\| \to 0, \;\; |c_{i+m_k} - \widetilde{c}_i| \to 0, \;\; k \to \infty. \tag{31}$$

As is known [24], the uniform with respect to $i \in \mathbb{Z}$ convergence in (31) characterizes almost periodic sequences.

Theorem 2. *Let the conditions* (30) *be fulfilled. Then* $h = \sum\limits_i f_i\delta_{t_i}$ *is the translation-compact distribution, and for any* $g \in \Sigma$*, the representation* $g = \sum\limits_i l_i\delta_{\tau_i}$ *holds, and also the sequences* $\{l_i\} \subset X$*,* $\{\tau_i\} \subset \mathbb{R}$ *satisfy the condition* (30). *Moreover, if* $g^n = \sum\limits_i l_i^n\delta_{\tau_i^n} \longrightarrow g = \sum\limits_i l_i\delta_{\tau_i}$ *in* Σ*, then* $l_i^n \to l_i$ *in* X*,* $\tau_i^n \to \tau_i$ *in* $\mathbb{R}$ $\forall\, i \in \mathbb{Z}$.

Proof. At our first step, we prove that the mapping $h = \sum_i f_i \delta_{t_i}$ is the translation-compact distribution, if and only if the function $F(t) = \sum_i f_i \varphi(t - t_i)$ is translation-compact in $\mathbb{C}(\mathbb{R}; X)$ for any $\varphi \in D$.

$$\big\|(T(s)h)\varphi - (T(t)h)\varphi\big\| \le \Big\| \sum_i (f_i \varphi(t_i - s) - f_i \varphi(t_i - t)) \Big\| \le$$
$$\le \frac{C}{\gamma} |t - s| \sum_i \big\|\varphi'(t_i^*)(t_i - t_{i-1})\big\|$$

for $t > s$, where $t_i^* \in [t_i - t, t_i - s]$. Here, without loss of generality, we assume that $t - s < \gamma$. Then for an arbitrary number i, $t - s < t_i - t_{i-1}$ and $t_i^* \in [t_i - t, t_i - s] \subset [t_{i-1} - s, t_i - s]$. Relying on the proof of Lemma 8, we have

$$|\varphi'(t_i^*)|(t_i - t_{i-1}) \le \int\limits_{t_{i-1}-s}^{t_i - s} \big(|\varphi'(r)| + |\varphi''(r)|\big)\, dr.$$

Thus,

$$\big\|(T(s)h)\varphi - (T(t)h)\varphi\big\| \le \frac{2C}{\gamma} |t - s|\, |\varphi|_D,$$

so,

$$\|T(s)h - T(t)h\|_{L(D,X)} \le \frac{2C}{\gamma} |t - s|,$$
$$\|F(s) - F(t)\| \le \frac{2C}{\gamma} |t - s|\, |\varphi|_D,$$

and also, the functions $F(\cdot)$, $T(\cdot)h$ are uniformly continuous in $\mathbb{R}$. If h is the translation-compact distribution, then $\{T(s)h|\ \ s \in \mathbb{R}\}$ is precompact in W. Thus, on the basis of Lemma 7, we find that $\{F(s)|\ \ s \in \mathbb{R}\}$ is precompact in X for any $\varphi \in D$, and also, the mapping F is translation-compact in $\mathbb{C}(\mathbb{R}; X)$.

Inversely, let F be the translation-compact in $\mathbb{C}(\mathbb{R}; X)$. We choose $\{\varphi_j\}_{j \ge 1} \subset D$, *supp* $\varphi_j \subset [-\frac{1}{j}, \frac{1}{j}]$, $\varphi_j \ge 0$, $\int\limits_{-\infty}^{+\infty} \varphi_j(t)\, dt = 1$, and consider the mapping F_j which is defined as follows:

$$F_j \varphi = \int\limits_{-\infty}^{+\infty} \sum_i f_i \varphi_j(t - t_i) \varphi(t)\, dt \quad \forall\, \varphi \in D.$$

Then

$$\|F_j\varphi\| = \Big\| \int_{-\infty}^{+\infty} \varphi_j(t) \sum_i f_i \varphi(t+t_i)\, dt \Big\| \le$$
$$\le C \int_{-\infty}^{\infty} \varphi_j(t) \sum_i |\varphi(t+t_i)|\, dt \le C \sum_i \big|\varphi(\theta_i^j + t_i)\big|,$$

where $\theta_i^j \in [-\frac{1}{j}, \frac{1}{j}]$. Here, without loss of generality, we assume that $\frac{1}{j} < \gamma$. Hence, we have

$$\|F_j\varphi\| \le C \sum_i \big|\varphi(\theta_i^j + t_i)\big| \le \frac{C}{2\gamma} \sum_i |\varphi(\theta_i^j + t_i)|(t_{i+1} - t_{i-1}) \le$$
$$\le \frac{C}{2\gamma} \sum_i \int_{t_{i-1}}^{t_{i+1}} \big(|\varphi(s)| + |\varphi'(s)|\big)\, ds \le \frac{2C}{\gamma}\, |\varphi|_D,$$

by virtue of Lemma 8, i.e. $F_j \in W$. Let us show that F_j is the translation-compact distribution. We start with

$$\big\|(T(t')F_j)\varphi - (T(t'')F_j)\varphi\big\| =$$
$$= \Big\| \int_{-\infty}^{+\infty} \sum_i f_i \varphi_j(t-t_i)\big(\varphi(t-t') - \varphi(t-t'')\big)\, dt \Big\| \le$$
$$\le C \int_{-\infty}^{+\infty} \varphi_j(t) \sum_i \big|\varphi(t+t_i-t') - \varphi(t+t_i-t'')\big|\, dt \le$$
$$\le C \sum_i \big|\varphi(t^*_{i,j} - t') - \varphi(t^*_{i,j} - t'')\big|,$$

where $t^*_{i,j} \in [t_i - \frac{1}{j}, t_i + \frac{1}{j}]$. Then

$$\big\|(T(t')F_j)\varphi - (T(t'')F_j)\varphi\big\| \le C|t' - t''| \sum_i \big|\varphi'(\theta_i^j))\big|,$$

holds for $t'' < t'$, where

$$\theta_i^j \in [t^*_{i,j} - t', t^*_{i,j} - t''] \subset \Big[t_i - \frac{1}{j} - t', t_i + \frac{1}{j} - t''\Big] \subset$$
$$\subset \Big[t_i - \frac{1}{j} - t', t_i + \frac{1}{j} - t' + |t' - t''|\Big] \subset [t_{i-1} - t', t_{i+1} - t'].$$

Hence, if $\frac{1}{j} < \gamma/2$, $|t' - t''| < \gamma/2$, we have the estimation

$$\left\|(T(t')F_j)\varphi - (T(t'')F_j)\varphi\right\| \leq$$
$$\leq \frac{C}{2\gamma}\,|t' - t''| \sum_i \left|\varphi'(\theta_i^j))\right|(t_{i+1} - t_{i-1}) \leq \frac{2C}{\gamma}\,|t' - t''|\,|\varphi|_D$$

to be fulfilled. Thus, we have proved that $T(\cdot)F_j$ is uniformly continuous. It remains to prove that $\{T(s)F_j| \;\; s \in \mathbb{R}\}$ is a precompact set in W. Let $s_n \to \infty$ be an arbitrary sequence. Since the function $F_j(t) = \sum\limits_i f_i\varphi_j(t-t_i)$ is translation-compact in $\mathbb{C}(\mathbb{R}; X)$, there exists the subsequence (denoted as $\{s_n\}$), and when $R > 0$, the statement

$$\sup_{|t|\leq R} \left\|F_j(t - s_n) - F_j(t - s_m)\right\| \to 0, \;\; n, m \to \infty$$

holds. Note that on the basis of diagonal method we can use the general subsequence s_n for all φ_j Since for all $\varphi \in D$, $\varepsilon > 0$ there exists $R > 0$, and also $\int\limits_{|t|>R} |\varphi(t)|\, dt < \varepsilon$, hence

$$\left|\int\limits_{|t|>R} \sum_i f_i\varphi_j(t + s_n - t_i)\right|\varphi(t)\,|dt| \leq \frac{2C}{\gamma}\,|\varphi_j|_D \int\limits_{|t|>R} |\varphi(t)|\, dt < C(j)\varepsilon.$$

Then

$$\left\|(T(s_n)F_j)\varphi - (T(s_m)F_j)\varphi\right\| =$$
$$= \left\|\int\limits_{-\infty}^{\infty} \sum_i f_i\varphi_j(t - t_i)\varphi(t - s_n)\, dt - \int\limits_{-\infty}^{\infty} \sum_i f_i\varphi_j(t - t_i)\varphi(t - s_m)\, dt\right\| =$$
$$= \left\|\int\limits_{-\infty}^{\infty} \Big(\sum_i f_i\varphi_j(t + s_n - t_i) - \sum_i f_i\varphi_j(t + s_m - t_i)\Big)\varphi(t)\, dt\right\| \leq$$
$$\leq \left\|\int\limits_{-R}^{R} \sum_i f_i(\varphi_j(t + s_n - t_i) - \varphi_j(t + s_m - t_i))\varphi(t)\, dt\right\| + 2C(j)\varepsilon.$$

That's why for all $\varepsilon > 0$, $j \geq 1$, $\varphi \in D$ there exists $N = N(\varepsilon, j, \varphi)$ such that $\forall\, m, n \geq N$

$$\left\|(T(s_n)F_j)\varphi - (T(s_m)F_j)\varphi\right\| < \varepsilon.$$

Hence, the set $\{T(s)F_j|\ \ s \in \mathbb{R}\}$ is precompact in W. Relying on Lemma 8, for all $\varphi \in D$

$$\left\| \int_{-\infty}^{\infty} \sum_i f_i \varphi_j(t-t_i)\varphi(t)\,dt - \sum_i f_i \varphi(t_i) \right\| =$$

$$= \left\| \int_{-\infty}^{\infty} \sum_i f_i \varphi_j(t-t_i)(\varphi(t)-\varphi(t_i)\,dt \right\| \le$$

$$\le C \sum_i \int_{t_i-\frac{1}{j}}^{t_i+\frac{1}{j}} \frac{1}{j} \max_{\theta \in [t_i-\frac{1}{j}, t_i+\frac{1}{j}]} |\varphi'(\theta)| \varphi_j(t-t_i)\,dt \le \frac{2C}{\gamma}\,\frac{1}{j}\,|\varphi|_D,$$

i.e. $\|F_j - h\|_{L(D;X)} \le \frac{2C}{\gamma}\,\frac{1}{j}$. Then for all $\varphi \in D$ and $\varepsilon > 0$, there exist $j(\varepsilon,\varphi)$ and $N(j,\varepsilon,\varphi)$ such that for any $n, m > N$,

$$\begin{aligned}
&\big\|(T(s_n)h)\varphi - (T(s_m)h)\varphi\big\| \le \\
&\qquad \le \big\|T(s_n)h - T(s_n)F_j\big\|_{L(D;X)} |\varphi|_D + \\
&+ \big\|T(s_m)h - T(s_m)F_j\big\|_{L(D;X)} |\varphi|_D + \big\|(T(s_n)F_j)\varphi - (T(s_m)F_j)\varphi\big\| \le \\
&\qquad \le \frac{4C}{\gamma}\,\frac{1}{j}\,|\varphi|_D + \big\|(T(s_n)F_j)\varphi - (T(s_m)F_j)\varphi\big\| < \varepsilon.
\end{aligned}$$

Hence, the set Σ is compact in W. Thus the desired equivalence is proved.

Let us now prove the first statement. We show that the set $\{F(s)|\ \ s \in \mathbb{R}\}$ is precompact in X for all $\varphi \in D$. Let $s_n \to \infty$ be an arbitrary sequence. Then there exists the sequence $\{m_n\} \subset \mathbb{Z}$, such that $|s_n - am_n| \le a$, and on some subsequence $s_n - am_n \to b$, $n \to \infty$. On the basis of $\{m_n\}$, we choose $\{m_k\} \subset \{m_n\}$, $\widetilde{f}_i$, $\widetilde{c}_i$ from (31). Let $\widetilde{t}_i = ai - b + \widetilde{c}_i$. By (31), $\sup\limits_i \|\widetilde{f}_i\| \le C$, $\sup\limits_i |\widetilde{c}_i - b| < \infty$. Moreover, if $t_{i+1} - t_i = a + c_{i+1} - c_i \ge \gamma$, $\widetilde{t}_{i+1} - \widetilde{t}_i = a + \widetilde{c}_{i+1} - \widetilde{c}_i$, then from (31) it follows that $\widetilde{t}_{i+1} - \widetilde{t}_i \ge \gamma$. Thus the sequences $\{\widetilde{f}_i\} \subset X$ and $\{\widetilde{c}_i - b\} \subset \mathbb{R}$ satisfy the conditions (30). Therefore, for any $i \in \mathbb{Z}$, from the convergence $s_k - t_{i+m_k} \to -\widetilde{t}_i$, $k \to \infty$, we have

$$\begin{aligned}
&\Big\| \sum_i f_i \varphi(s_k - t_i) - \sum_i \widetilde{f}_i \varphi(-\widetilde{t}_i) \Big\| \le \\
&\qquad \le \Big\| \sum_{|i| \le N} \big(f_{i+m_k}\varphi(s_k - t_{i+m_k}) - \widetilde{f}_i \varphi(-\widetilde{t}_i)\big) \Big\| + \\
&+ C \Big\| \sum_{|i|>N} \Big(\big|\varphi(-ai + s_k - am_k - c_{i+m_k})\big| + \big|\varphi(-ai + b - \widetilde{c}_i)\big| \Big) \Big\|.
\end{aligned}$$

Then $\forall\, \varepsilon > 0$ there exist $N \ge 1$, $K(\varepsilon, N)$, such that $\forall\, k \ge K(\varepsilon, N)$

$$\Big\| F(s_k) - \sum_i \widetilde{f}_i \varphi(-\widetilde{t}_i) \Big\| < \varepsilon.$$

Thus, h is the translation-compact distribution.

Consider now an arbitrary element $g \in \Sigma$. Then there exists the sequence $\{s_n\}$, such that $T(s_n)h \to g$ in W, i.e.

$$\big\|(T(s_n)h)\varphi - g\varphi\big\| \to 0, \;\; n \to \infty, \;\; \forall\, \varphi \in D.$$

Similarly to the above-mentioned, for all $\varphi \in D$ we have

$$(T(s_n)h)\varphi = \sum_i f_i \varphi(t_i - s_n) \longrightarrow \sum_i \widetilde{f}_i \varphi(\widetilde{t}_i).$$

Then (31) yields $\widetilde{h} = \sum\limits_i \widetilde{f}_i \delta_{\widetilde{t}_i} \in W$. Hence, $g = \widetilde{h}$. We have proved the first part of the lemma.

Let now $g^n = \sum\limits_i l_i^n \delta_{\tau_i^n} \longrightarrow g = \sum\limits_i l_i \delta_{\tau_i}$ in Σ. From the previous considerations we have $\|l_i^n\| \le C$, $\{l_i^n\}_{i \in \mathbb{Z}} \subset K$ for any $n \ge 1$, where $K = cl_X\{f_i\}_{i\in\mathbb{Z}}$ is compact in X, and $\tau_i^n = ai + c_i^n$, $\{c_i^n\}_{i\in\mathbb{Z}}$ is uniformly bounded as $n \ge 1$. Then there exists $\{\widetilde{l}_i\}_{i\in\mathbb{Z}} \subset K$, $\{\widetilde{c}_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}$ such that $l_i^{n_k} \to \widetilde{l}_i$ in X, $c_i^{n_k} \to \widetilde{c}_i$ in $\mathbb{R}$ $\forall\, i \in \mathbb{Z}$. We put $\widetilde{\tau}_i = ai + \widetilde{c}_i$, $\widetilde{g} = \sum\limits_i \widetilde{l}_i \delta_{\widetilde{\tau}_i}$. Then $\forall\, \varphi \in D$

$$\begin{aligned} \|g^{n_k}\varphi - \widetilde{g}\varphi\| &= \Big\| \sum_i \big(l_i^{n_k} \varphi(\tau_i^{n_k}) - \widetilde{l}_i \varphi(\widetilde{\tau}_i)\big)\Big\| \le \\ &\le \sum_{i=-N}^{N} \big\| l_i^{n_k}\varphi(\tau_i^{n_k}) - \widetilde{l}_i \varphi(\widetilde{\tau}_i)\big\| + C \sum_{|i|>N} \big(|\varphi(\tau_i^{n_k})| + |\varphi(\widetilde{\tau}_i)|\big). \end{aligned}$$

Since $\{c_i^n\}_{i\in\mathbb{Z}}$ is uniformly bounded as $n \ge 1$, the estimation $\forall\, k \ge 1$ $C \sum\limits_{|i|>N} \big(|\varphi(\tau_i^{n_k})| + |\varphi(\widetilde{\tau}_i)|\big) < \frac{\varepsilon}{2}$ holds for all $\varepsilon > 0$, where $N \ge 1$. Then for all $\varepsilon > 0$, there exist $k(\varepsilon) \ge 1$ such that $\|g^{n_k}\varphi - \widetilde{g}\varphi\| < \varepsilon$ $\forall\, k \ge k(\varepsilon)$, i.e. $g^{n_k} \to \widetilde{g}$ in Σ. Hence, $\widetilde{g} = g$, and the theorem is proved. □

The Existence of a Global Attractor for a Nonautonomous Impulsive-Perturbed Evolutional Inclusion

Let $\forall\, i \in \mathbb{Z}$

$$\Psi_i = [f_i, g_i] = \big\{ \lambda f_i + (1-\lambda) g_i \big|\;\; \lambda \in [0,1] \big\}, \tag{32}$$

and the sequences $\{\tau_i\}$, $\{f_i\}$, $\{g_i\}$ satisfy the conditions (30), i.e.,

$$\begin{gathered} \{f_i\}_{i\in\mathbb{Z}} \subset H, \;\; \sup_{i\in\mathbb{Z}} \|f_i\| \le K, \;\; \{f_i\}_{i\in\mathbb{Z}} \text{ is precompact in } H, \\ \{g_i\}_{i\in\mathbb{Z}} \subset H, \;\; \sup_{i\in\mathbb{Z}} \|g_i\| \le K, \;\; \{g_i\}_{i\in\mathbb{Z}} \text{ is precompact in } H, \\ \{\tau_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}, \;\; \tau_i = ai + c_i, \;\; a > 0, \;\; \sup_{i\in\mathbb{Z}} |c_i| < \infty, \;\; \tau_{i+1} - \tau_i \ge \gamma > 0. \end{gathered} \tag{33}$$

Let us construct a non-autonomous multivalued dynamical system for (14), (16).

On the basis of Lemmas 4 and 5, the set

$$\Sigma^1 = cl_{L^{2,w}_{loc}(\mathbb{R};H)}\big\{h(t+\cdot\,) \mid \ t \in \mathbb{R}\big\}$$

is compact in $L^{2,w}_{loc}(\mathbb{R};H)$, with the action of continuous group of shifts on it $\{T^1(s) : \Sigma^1 \mapsto \Sigma^1\}_{s\in\mathbb{R}}$, and $\forall\, s \in \mathbb{R}\ T^1(s)\Sigma^1 = \Sigma^1$.

By virtue of Theorem 2, both of the mappings $h = \sum\limits_i f_i\delta_{\tau_i}$ and $p = \sum\limits_i g_i\delta_{\tau_i}$ are translation-compact distributions. Moreover, if the linear continuous mapping is defined by $(h,p) : D \mapsto H^2$, $(h,p)(\varphi) := (h\varphi, p\varphi)$, then it is easy to find that the set

$$\Sigma^2 = cl_{W^2}\big\{T(s)(h,p) \mid \ s \in \mathbb{R}\big\}$$

satisfies the following conditions: Σ^2 is compact in W^2, $T(s)\Sigma^2 = \Sigma^2\ \forall\, s \in \mathbb{R}$, and for all $\sigma^2 \in \Sigma^2$, we have $\sigma^2 = (\widetilde{h},\widetilde{p})$, where $\widetilde{h} = \sum\limits_i \widetilde{f}_i\delta_{\widetilde{\tau}_i}$, $\widetilde{p} = \sum\limits_i \widetilde{g}_i\delta_{\widetilde{\tau}_i}$ and $\{\widetilde{f}_i\}$, $\{\widetilde{g}_i\}$, $\{\widetilde{\tau}_i = ai + \widetilde{c}_i\}$ fulfill (33).

Consider the impulsive problem

$$\begin{cases} \dfrac{du}{dt} + Au \in F(u) + l(t), \quad t > \tau, \\ u(\tau) = u_\tau, \end{cases} \tag{34}$$

$$u(\widetilde{\tau}_i + 0) - u(\widetilde{\tau}_i) \in g(u(\widetilde{\tau}_i)) + \widetilde{\Psi}_i = g(u(\widetilde{\tau}_i)) + [\widetilde{f}_i, \widetilde{g}_i], \ \ i \in \mathbb{Z}, \tag{35}$$

for all $\sigma = (\sigma^1,\sigma^2) \in \Sigma := \Sigma^1 \times \Sigma^2$, where $\sigma^1 = l$, $\sigma^2 = (\widetilde{h},\widetilde{p})$.

The impulsive problem (34), (35) is globally solvable in the sense of solvability of the problem (14), (16). Then for all $\sigma = (\sigma^1,\sigma^2) \in \Sigma$, $\tau \in \mathbb{R}$, $u_\tau \in H$, one can correctly construct the multivalued mapping

$$U_\sigma : \mathbb{R}_d \times H \mapsto P(H),$$

$$U_\sigma(t,\tau,u_\tau) = \big\{u(t) \mid \ u(\cdot) \text{ is the solution of (34), (35)}, \ u(\tau) = u_\tau\big\}. \tag{36}$$

Theorem 3. *Let for the problem* (14), (16) *the conditions* (9)–(13), (21)–(24), (32), (33) *be fulfilled. Then the formula* (36) *defines the family of MP* $\{U_\sigma\}_{\sigma\in\Sigma}$*, for which there exists the compact global attractor in the phase space* H*.*

Proof. Let us prove that (36) defines the family of processes and $U_\sigma(t+h,\tau+h,x) = U_{T(h)\sigma}(t,\tau,x)$ holds for all $(t,\tau) \in \mathbb{R}_d$, $h \in \mathbb{R}_+$, $x \in H$. Let $\xi \in U_\sigma(t,\tau,x)$. Then $\xi = u(t)$, $u(\cdot)$ is the solution of (34), (35), $u(\tau) = x$. Thus, $\forall\, s \in (\tau,t)\ u(s) \in U_\sigma(s,\tau,x)$. Let $\omega(p) = u(p)$, if $p \geq s$. Then $\omega(\cdot)$ is the solution of (34), (35), $\omega(s) = u(s)$, i.e., $\xi = u(t) = \omega(t) \in U_\sigma(t,s,u(s)) \subset U_\sigma(t,s,U_\sigma(s,\tau,x))$. Let $\xi \in U_\sigma(t+s,\tau+s,x)$. Then $\xi = u(t+s)$, $u(\cdot)$ is the solution of (34), (35), $u(\tau+s) = x$. We put $v(p) = u(p+s)$, $p \geq \tau$. If $\tau+s \in (\widetilde{\tau}_{i-1},\widetilde{\tau}_i]$, then $u(\cdot)$ is the solution of (34) on $(\tau+s,\widetilde{\tau}_i), (\widetilde{\tau}_i,\widetilde{\tau}_{i+1}), \ldots$, such that

$$u(\widetilde{\tau}_j + 0) - u(\widetilde{\tau}_j) \in g(u(\widetilde{\tau}_j)) + \widetilde{\Psi}_j, \ \ j \geq i,$$

holds. Thus, $v(\cdot)$ is the solution of (34) on $(\tau, \widetilde{\tau}_i - s), (\widetilde{\tau}_i - s, \widetilde{\tau}_{i+1} - s), \ldots,$ such that

$$v(\widetilde{\tau}_j - s + 0) - v(\widetilde{\tau}_j - s) \in g(v(\widetilde{\tau}_j - s)) + \widetilde{\Psi}_j, \ \ j \geq i,$$

and $v(\tau) = u(\tau + s) = x$ hold. Hence, $\xi = u(t + s) = v(t) \in U_{T(s)\sigma}(t, \tau, x)$. Let $\xi \in U_{T(s)\sigma}(t, \tau, x)$. Then $\xi = u(t)$, $u(\cdot)$ is the solution of (34), (35) with parameter $T(s)\sigma$, $u(\tau) = x$. We put $v(p) := u(p - s)$, $p \geq \tau + s$. Then $v(\cdot)$ is the solution of (34), (35) with parameter σ, $v(\tau + s) = u(\tau) = x$. Thus, $\xi = u(t) = v(t + s) \in U_\sigma(t + s, \tau + s, x)$.

Let us check the conditions of Theorem 1, using both the equality proven above and the equality $T(s)\Sigma = \Sigma$. Since $\{\widetilde{f}_i\}$, $\{\widetilde{g}_i\}$, $\{\widetilde{\tau}_i = ai + \widetilde{c}_i\}$ satisfy (33), and basing on Lemma 5 $\|l\|_+^2 \leq \|h\|_+^2$, from Lemma 2 we can get

$$\exists\, R_0 > 0 \ \ \forall\, r \geq 0 \ \ \exists\, T = T(r) \ \ \forall\, t \geq T(r) \ \ U_\Sigma(t, 0, B_r) \subset B_{R_0} \tag{37}$$

and thus we obtain the uniform dissipativity condition 1) from Theorem 1. Let us show that condition 2) from Theorem 1 holds, that is, the sequence $\xi_n \in U_\Sigma(t_n, 0, B_r)$ is precompact for any $t_n \to \infty$ and $r > 0$. Since

$$\xi_n \in U_{\sigma_n}(t_n, 0, B_r) \subset U_{\sigma_n}(t_n, t_n - \widetilde{t}, U_{\sigma_n}(t_n - \widetilde{t}, 0, B_r) \subset U_{T(t_n)\sigma_n}(\widetilde{t}, 0, B_{R_0}),$$

it remains only to prove that $\xi_n \in U_{(\sigma_n^1, \sigma_n^2)}(\widetilde{t}, 0, u_n^0)$ is precompact in H, when $\widetilde{t} \in (0, \gamma)$, $u_n^0 \to u_0$ weakly in H, $\sigma_n^1 = l_n \to \sigma^1 = l$ in Σ^1, $\sigma_n^2 = (\widetilde{h}^n, \widetilde{p}^n) \to \sigma^2 = (\widetilde{h}, \widetilde{p})$ in Σ^2, $\widetilde{h}^n = \sum\limits_i \widetilde{f}_i^n \delta_{\widetilde{\tau}_i^n}$, $\widetilde{p}^n = \sum\limits_i \widetilde{g}_i^n \delta_{\widetilde{\tau}_i^n}$, $\widetilde{\tau}_{i+1}^n - \widetilde{\tau}_i^n \geq \gamma > 0$, $\widetilde{\tau}_i^n = ai + \widetilde{c}_i^n$. By Theorem 2, $\widetilde{h} = \sum\limits_i \widetilde{f}_i \delta_{\widetilde{\tau}_i}$, $\widetilde{p} = \sum\limits_i \widetilde{g}_i \delta_{\widetilde{\tau}_i}$, where $\widetilde{\tau}_i = ai + \widetilde{c}_i$ and $\widetilde{f}_i^n \to \widetilde{f}_i$, $\widetilde{g}_i^n \to \widetilde{g}_i$ in H, $\widetilde{\tau}_i^n \to \widetilde{\tau}_i$ in $\mathbb{R}$ $\forall\, i \in \mathbb{Z}$. Thus, $\xi_n = u_n(\widetilde{t})$, where $u_n(\cdot)$ is the solution of the problem

$$\begin{cases} \dfrac{du_n}{dt} + Au_n \in F(u_n) + l_n(t), \quad t > 0, \\ u_n(0) = u_n^0, \end{cases} \tag{38}$$

$$u_n(\widetilde{\tau}_i^n + 0) - u_n(\widetilde{\tau}_i^n) \in g(u_n(\widetilde{\tau}_i^n)) + \widetilde{\Psi}_i^n = g(u_n(\widetilde{\tau}_i^n)) + [\widetilde{f}_i^n, \widetilde{g}_i^n], \ \ i \in \mathbb{Z}. \tag{39}$$

There exists no more than one moment of impulsive perturbations $\widetilde{\tau}_i^n$ on $[0, \widetilde{t})$ for any $n \geq 1$, and the number i depends on n, i.e., $i = i(n)$.

If for infinitely many numbers $n \geq 1$ there are no moments of impulsive perturbations on $[0, \widetilde{t})$, then $\{\xi_n = u_n(\widetilde{t})\}$ is precompact by virtue of Lemma 1, estimate (13) and the dissipativity condition (37).

Let $i(n) \in \mathbb{Z}$ to be exist for any $n \geq 1$, such that $\widetilde{\tau}_{i(n)}^n \in [0, \widetilde{t})$. As $\widetilde{\tau}_{i(n)}^n = ai(n) + \widetilde{c}_{i(n)}^n$ and $\{\widetilde{c}_i^n\}$ is uniformly bounded on $n \geq 1$, then there exists $i_0 \in \mathbb{N}$, such that $i(n) \in [-i_0, i_0] \cap \mathbb{Z}$ $\forall\, n \geq 1$. Then $i(n) \equiv i \in [-i_0, i_0] \cap \mathbb{Z}$, for infinitely many $n \geq 1$, and for some subsequence $\{u_n(\cdot)\}$ we have the following impulsive problem:

$$u_n(\widetilde{\tau}_i^n + 0) - u_n(\widetilde{\tau}_i^n) \in g(u_n(\widetilde{\tau}_i^n)) + \widetilde{\Psi}_i^n = g(u_n(\widetilde{\tau}_i^n)) + [\widetilde{f}_i^n, \widetilde{g}_i^n],$$

for fixed $i \in \mathbb{Z}$. Since for any $y_n \in \widetilde{\Psi}_i^n$ $y_n = \lambda_n \widetilde{f}_i^n + (1-\lambda_n)\widetilde{g}_i^n$, $\lambda_n \in [0,1]$, therefore on the subsequence $y_n \to y \in \lambda \widetilde{f}_i + (1-\lambda)\widetilde{g}_i \in \widetilde{\Psi}_i = [\widetilde{f}_i, \widetilde{g}_i]$. Let us consider all possible situations.

If $\widetilde{\tau}_i^n \in (0, \widetilde{t})$ and $\widetilde{\tau}_i^n \to \widetilde{\tau}_i \in (0, \widetilde{t})$, then by Lemma 1 $u_n(\widetilde{\tau}_i^n) \to u(\widetilde{\tau}_i)$, where $u(\cdot)$ is the solution of (14), $u(0) = u_0$. Since

$$\xi_n = u_n(\widetilde{t}) \in U_{\widetilde{\sigma}_n}\big(\widetilde{t}, \widetilde{\tau}_i^n, u_n(\widetilde{\tau}_i^n)\big) \subset U_{T(\widetilde{\tau}_i^n)\widetilde{\sigma}_n}\big(\widetilde{t} - \widetilde{\tau}_i^n, 0, u_n(\widetilde{\tau}_i^n)\big), \tag{40}$$

therefore $\xi_n = v_n(\widetilde{t} - \widetilde{\tau}_i^n)$, where $v_n(\cdot)$ is the solution of the following problem:

$$\begin{cases} \dfrac{dv_n}{dt} + Av_n \in F(v_n) + l_n(t + \widetilde{\tau}_i^n), \\ v_n\big|_{t=0} = v_n(0) \in u_n(\widetilde{\tau}_i^n) + g(u_n(\widetilde{\tau}_i^n)) + \widetilde{\Psi}_i^n. \end{cases} \tag{41}$$

Denote $\widetilde{l}_n(t,x) = l_n(t + \widetilde{\tau}_i^n, x) = T^1(\widetilde{\tau}_i^n) l_n(t,x)$. Since $l_n \in \Sigma^1$ is compact in $L^{2,w}_{loc}(\mathbb{R}; H)$ and $T^1(p)\Sigma^1 = \Sigma^1$, $\forall p \in \mathbb{R}$, the subsequence $\widetilde{l}_n \to \widetilde{l}$ in Σ^1. Then $v_n(0) \to v_0$ weakly in H, hence, $\xi_n = v_n(\widetilde{t} - \widetilde{\tau}_i^n) \to v(\widetilde{t} - \widetilde{\tau}_i)$ and $\{\xi_n\}$ is precompact in H, by Lemma 1.

If $\widetilde{\tau}_i^n \in (0, \widetilde{t})$, $\widetilde{\tau}_i^n \searrow 0$ (or $\widetilde{\tau}_i^n = 0$ for infinitely many $n \geq 1$), then $u_n(\widetilde{\tau}_i^n) \to u_0$ weakly in H, by Lemma 1. In a similar way,

$$\xi_n = u_n(\widetilde{t}) \in U_{T(\widetilde{\tau}_i^n)\widetilde{\sigma}_n}\big(\widetilde{t} - \widetilde{\tau}_i^n, 0, u_n(\widetilde{\tau}_i^n)\big),$$

i.e., $\xi_n = v_n(\widetilde{t} - \widetilde{\tau}_i^n)$, $v_n(0) \in g\big(u_n(\widetilde{\tau}_i^n) + u_n(\widetilde{\tau}_i^n) + \widetilde{\Psi}_i^n\big)$.

Due to the weak convergence of $u_n(\widetilde{\tau}_i^n)$ to u_0, it is easy to find that the sequence $\{v_n(0)\}$ is bounded in H. Thus the sequence $v_n(0) \to v_0$ converges weakly in H. Then $\xi_n = v_n(\widetilde{t} - \widetilde{\tau}_i^n) \to v(\widetilde{t})$, by lemma 1, hence $\{\xi_n\}$ is precompact in H.

If $\widetilde{\tau}_i^n \in (0, \widetilde{t})$, $\widetilde{\tau}_i^n \nearrow \widetilde{t}$, then

$$\xi_n = v_n(\widetilde{t} - \widetilde{\tau}_i^n) \in U_{T(\widetilde{\tau}_i^n)\widetilde{\sigma}_n}\big(\widetilde{t} - \widetilde{\tau}_i^n, 0, u_n(\widetilde{\tau}_i^n)\big),$$

where $u_n(\widetilde{\tau}_i^n) \to u(\widetilde{t}) \in U_{(l,\sigma_{s_0})}(\widetilde{t}, 0, u_0)$, $v_n(0) \in g(u_n(\widetilde{\tau}_i^n)) + u_n(\widetilde{\tau}_i^n) + \widetilde{\Psi}_i^n$, $v_n(0) \to v_0$ weakly in H. Since $\widetilde{t} - \widetilde{\tau}_i^n \nearrow 0$, and by Lemma 1 $\xi_n = v_n(\widetilde{t} - \widetilde{\tau}_i^n) \to v_0$ and $\{\xi_n\}$ is precompact in H. Thus the theorem is proved. □

Remark. In many dissipative nonautonomous problems it is expected that for a global attractor $\Theta_\Sigma \subseteq U_\Sigma(t, \tau, \Theta_\Sigma)$. But a trivial example of periodic one-dimensional problem shows that in impulsive problems, in general, this is not true.

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MULTIPLE POSITIVE SOLUTIONS FOR A CLASS OF FRACTIONAL SINGULAR BOUNDARY VALUE PROBLEMS

Abstract. For $(n-1,n]$ order singular fractional differential equations, conditions are established guaranteeing, respectively the existence of multiple positive solutions and the nonexistence of a positive solution of a class of boundary value problems.

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Key words and phrases. Singular boundary value problem, cone, positive solution, fractional derivative, Caputo's fractional integral, fixed point.

რეზიუმე. $(n-1,n]$ რიგის სინგულარული ფრაქციონალური დიფერენციალური განტოლებებისათვის დადგენილია პირობები, რომლებიც სათანადოდ უზრუნველყოფენ სასაზღვრო ამოცანათა ერთი კლასის ჯერადი დადებითი ამონახსნების არსებობასა და დადებითი ამონახსნის არარსებობას.

1. Introduction

The boundary value problem (BVP, for short), singular boundary value problem, and fractional order boundary value problem arise in a variety of differential applied mathematics and physics and hence, they have received much attention (see [1,2,6–12] and references therein). For example, in [1], Qiu and Bai considered the existence of positive solutions to BVP in the nonlinear fractional differential equation

$$\begin{cases} {}^C D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \end{cases}$$

where $2 < \alpha \leq 3$, and $f : (0,1] \times [0, +\infty) \to [0, +\infty)$ with $\lim\limits_{t \to 0+} f(t, u) = +\infty$ is continuous, that is, $f(t, u)$ may be singular at $t = 0$. They obtained the existence of at least one positive solution by using Krasnoselskii's fixed point theorem and nonlinear alternative of Leray–Schauder type in a cone.

In [14], Kaufmann obtained the existence and nonexistence of positive solutions to the nonlinear fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \qquad t \in (0, \tau), \\ I^{\gamma} u(0^+) = 0, \quad I^{\beta} u(\tau) = 0, \end{cases}$$

where $\tau \in (0, T]$, $1 - \alpha < \gamma \leq 2 - \alpha$, $2 - \alpha < \beta < 0$, D_{0+}^{α} is the Riemann–Liouville differential operator of order α, $f \in C([0, T] \times \mathbb{R})$ is nonnegative.

In this paper, we consider the following singular fractional boundary value problem of the form

$$\begin{aligned} {}^C D_{0+}^{\alpha} u(t) + \lambda f(t, u(t)) &= 0, \;\; 0 < t < 1, \\ u^{(j)}(0) &= 0, \;\; 0 \leq j \leq n-1, \;\; j \neq 2, \\ u''(1) &= 0, \end{aligned} \tag{1.1}$$

where $n - 1 < \alpha \leq n$, $n \geq 4$, ${}^C D_{0+}^{\alpha}$ are the Caputo's fractional derivatives and $f : (0,1) \times (0, +\infty) \to [0, +\infty)$ is continuous, that is, $f(t, u)$ may be singular at $t = 0, 1$ and $u = 0$. When constructing a special cone and using approximation method and fixed point index theory, we have obtain the existence of multiple positive solutions and nonexistence for BVP (1.1).

The main features of the paper are as follows. Firstly, the degree of singularity in [1] is lower than that of the present paper (for details, please see our examples). Here, $f(t, u)$ may be singular not only at $t = 0, 1$, but also at $u = 0$. Secondly, the results we obtained are the existence of multiple positive solutions and nonexistence of positive solutions, while [1] just obtained the existence of at least one positive solution. Finally, BVP (1.1) is more general and extensive than that in [1].

The paper is organized as follows. Section 2 contains some definitions and lemmas. Moreover, the Green's function and its properties are derived. In Section 3, by constructing a special cone and using approximation method and fixed point index theory, the existence of multiple positive solutions and nonexistence result are established. Finally, in Section 4, two examples are worked out to demonstrate our main results.

2. Preliminaries

For convenience of the reader, we present some necessary definitions from fractional calculus theory (see [3, 5]).

Definition 2.1. The fractional (arbitrary) order integral of the function $h \in L^1([a,b])$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,\mathrm{d}s,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.2. For a function h given on the interval $[a,b]$, the αth Caputo fractional-order derivative of h, is defined by

$$({}^C D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)\,ds.$$

Here, n is the smallest integer greater than or equal to α.

Lemma 2.3. *Let $\alpha > 0$. Then the differential equation*

$${}^C D_{0+}^\alpha u(t) = 0$$

has solutions $u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$ for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, where n is the smallest integer greater than or equal to α.

Lemma 2.4. *Assume that $u \in C(0,1) \cap L^1[0,1]$ with a derivative of order n that belongs to $C(0,1) \cap L^1[0,1]$. Then*

$$I_{0+}^\alpha\, {}^C D_{0+}^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, where n is the smallest integer greater than or equal to α.

Lemma 2.5. *The relation*

$$I_{0+}^\alpha I_{0+}^\beta \varphi = I_{0+}^{\alpha+\beta} \varphi$$

is valid in the following case:

$$\operatorname{Re}\beta > 0, \quad \operatorname{Re}(\alpha+\beta) > 0, \quad \varphi \in L^1[a,b].$$

In the rest of this paper, we suppose $\alpha \in (n-1, n]$, $n \geq 4$.

Lemma 2.6. *Given $g \in C[0,1]$, the unique solution of*

$$\begin{aligned} {}^{C}D_{0+}^{\alpha} u(t) + g(t) = 0,\ \ 0 < t < 1, \\ u^{(j)}(0) = 0,\ \ 0 \le j \le n-1,\ \ j \ne 2, \\ u''(1) = 0 \end{aligned} \tag{2.1}$$

is

$$u(t) = \int_0^1 G(t,s) g(s)\,\mathrm{d}s, \tag{2.2}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(\alpha-1)(\alpha-2)}{2}\, t^2 (1-s)^{\alpha-3} - (t-s)^{\alpha-1}, & s \le t, \\ \dfrac{(\alpha-1)(\alpha-2)}{2}\, t^2 (1-s)^{\alpha-3}, & t \le s. \end{cases} \tag{2.3}$$

Proof. Let $u \in C[0,1]$ be a solution of (2.1). By Lemma 2.3,

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, g(s)\,\mathrm{d}s.$$

From $u^{(j)}(0) = 0, 0 \le j \le n-1, j \ne 2, u''(1) = 0$, we have $c_0 = c_1 = c_3 = \cdots = c_{n-1} = 0$ and

$$c_2 = \frac{(\alpha-1)(\alpha-2)}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-3} g(s)\,\mathrm{d}s.$$

Then

$$\begin{aligned} u(t) &= c_2 t^2 - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\,\mathrm{d}s = \\ &= \frac{1}{\Gamma(\alpha)} \Bigg(\int_0^t \Big(\frac{(\alpha-1)(\alpha-2)}{2}\, t^2 (1-s)^{\alpha-3} - (t-s)^{\alpha-1} \Big) g(s)\,\mathrm{d}s + \\ &\quad + \int_t^1 \frac{(\alpha-1)(\alpha-2)}{2}\, t^2 (1-s)^{\alpha-3} g(s)\,\mathrm{d}s \Bigg) = \\ &= \int_0^1 G(t,s) g(s)\,\mathrm{d}s. \end{aligned}$$

The proof is completed. □

Lemma 2.6 indicates that the solution of the BVP (1.1) coincides with the fixed point of the operator T defined as

$$Tu(t) = \int_0^1 G(t,s) f(s,u(s))\,\mathrm{d}s, \quad \forall\, u \in C[0,1]. \tag{2.4}$$

Lemma 2.7. *The function* $G(t,s)$ *defined by* (2.3) *has the following properties:*

(i) $G(t,s) > 0, \quad \forall\, t,s \in [0,1].$ (2.5)

(ii) $G(t,s) \le H(s) \le \dfrac{(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)}$, (2.6)

where

$$H(s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(\alpha-1)(\alpha-2)}{2}\, s^2(1-s)^{\alpha-3} - (1-s)^{\alpha-1}, & s \le t, \\ \dfrac{(\alpha-1)(\alpha-2)}{2}\, s^2(1-s)^{\alpha-3}, & t \le s, \end{cases} \tag{2.7}$$

(iii) $G(t,s) \ge t^2 G(\tau,s), \quad \forall\, t,s,\tau \in [0,1].$ (2.8)

Proof. First, since $\alpha \in (n-1,n]$ and $n \ge 4$, it is easy to see

$$\frac{(\alpha-1)(\alpha-2)}{2} > 1.$$

Furthermore, for $s,t \in [0,1]$,

$$\frac{(\alpha-1)(\alpha-2)}{2}\, t^2(1-s)^{\alpha-3} > t^2(1-s)^{\alpha-3} \ge$$
$$\ge (t-s)^2(t-s)^{\alpha-3} = (t-s)^{\alpha-1}.$$

Obviously, we can get (2.5).

Next, for the given $s \in (0,1)$, we can find that $G(t,s)$ is increasing with respect to t. For $t \le s$,

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\, \frac{(\alpha-1)(\alpha-2)}{2}\, t^2(1-s)^{\alpha-3} \le$$
$$\le \frac{1}{\Gamma(\alpha)}\, \frac{(\alpha-1)(\alpha-2)}{2}\, s^2(1-s)^{\alpha-3}$$

and for $t \ge s$,

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \Big(\frac{(\alpha-1)(\alpha-2)}{2}\, t^2(1-s)^{\alpha-3} - (t-s)^{\alpha-1} \Big),$$
$$G_t(t,s) = \frac{1}{\Gamma(\alpha)} \Big((\alpha-1)(\alpha-2)t(1-s)^{\alpha-3} - (\alpha-1)(t-s)^{\alpha-2} \Big) =$$
$$= \frac{1}{\Gamma(\alpha-1)} \Big((\alpha-2)t(1-s)^{\alpha-3} - (t-s)^{\alpha-2} \Big) \ge$$
$$\ge \frac{1}{\Gamma(\alpha-1)} \Big((t-s)(t-s)^{\alpha-3} - (t-s)^{\alpha-2} \Big) = 0.$$

Hence, we have

$$G(t,s) \le G(1,s) = \frac{1}{\Gamma(\alpha)}\Big(\frac{(\alpha-1)(\alpha-2)}{2}(1-s)^{\alpha-3} - (1-s)^{\alpha-1}\Big) = \\ = \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha)}\Big(\frac{(\alpha-1)(\alpha-2)}{2} - (1-s)^2\Big).$$

By the definition of $H(s)$, we know

$$H(s) \le \frac{1}{2\Gamma(\alpha)}(\alpha-1)(\alpha-2)(1-s)^{\alpha-3} = \frac{(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)},$$

which means that (2.6) holds.

Finally, for $t \le s$, we have

$$\frac{G(t,s)}{H(s)} = \frac{\frac{1}{\Gamma(\alpha)}\frac{(\alpha-1)(\alpha-2)}{2}t^2(1-s)^{\alpha-3}}{\frac{1}{\Gamma(\alpha)}\frac{(\alpha-1)(\alpha-2)}{2}s^2(1-s)^{\alpha-3}} = \frac{t^2}{s^2} \ge t^2;$$

for $t \ge s$,

$$\frac{G(t,s)}{H(s)} = \frac{\frac{1}{\Gamma(\alpha)}\big(\frac{(\alpha-1)(\alpha-2)}{2}t^2(1-s)^{\alpha-3} - (t-s)^{\alpha-1}\big)}{\frac{1}{\Gamma(\alpha)}\big(\frac{(\alpha-1)(\alpha-2)}{2}(1-s)^{\alpha-3} - (1-s)^{\alpha-1}\big)} = \\ = \frac{1}{\frac{(\alpha-1)(\alpha-2)}{2} - (1-s)^2}\Big(\frac{(\alpha-1)(\alpha-2)}{2}t^2 - \frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-3}}\Big).$$

Since $s \le t \le 1$, $s \ge ts$ and $t-s \le t-ts$, we can get $(t-s)^{\alpha-3} \le (1-s)^{\alpha-3}$, $(t-s)^2 \le (t-ts)^2$. Thus,

$$\frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-3}} = \frac{(t-s)^2(t-s)^{\alpha-3}}{(1-s)^{\alpha-3}} \le \frac{(t-ts)^2(1-s)^{\alpha-3}}{(1-s)^{\alpha-3}} = t^2(1-s)^2.$$

Therefore,

$$\frac{G(t,s)}{H(s)} \ge \frac{1}{\frac{(\alpha-1)(\alpha-2)}{2} - (1-s)^2}\Big(\frac{(\alpha-1)(\alpha-2)}{2}t^2 - t^2(1-s)^2\Big) = t^2,$$

which implies that (iii) holds. The proof is completed. □

Lemma 2.8. *Let P be a cone of the real Banach space E, Ω be a bounded open set of E, $\theta \in \Omega$, $A : P \cap \overline{\Omega} \to P$ be completely continuous.*

(i) *If $x \ne \mu Ax$ for $x \in P \cap \partial\Omega$ and $\mu \in [0,1]$, then $i(A, P\cap\Omega, P) = 1$.*
(ii) *If $\inf_{x \in P\cap\partial\Omega} \|Ax\| > 0$ and $Ax \ne \mu x$ for $x \in P \cap \partial\Omega$ and $\mu \in (0,1]$, then $i(A, P\cap\Omega, P) = 0$.*

Let $J = [0,1]$. The basic space used in this paper is $E = C[J,\mathbb{R}]$. It is well known that E is a Banach space with norm $\|u\| = \max_{t\in J}|u(t)|$ $(\forall\, u \in E)$. From Lemma 2.7, it is easy to see that

$$Q := \Big\{u \in C[J,\mathbb{R}^+] :\ u(t) \ge t^2 u(s),\ \forall\, t,\ s \in J\Big\} \tag{2.9}$$

is a cone of E. Moreover, by (2.9), we have for all $u \in Q$,

$$u(t) \geq t^2 \|u\|, \quad \forall\, t \in J. \tag{2.10}$$

A function u is said to be a solution of BVP (1.1) if u satisfies (1.1). In addition, if $u(t) > 0$ for $t \in (0,1)$, then u is said to be a positive solution of BVP (1.1). Obviously, if $u \in Q \setminus \{\theta\}$ is a solution of BVP (1.1), then u is a positive solution of BVP (1.1), where θ denotes the zero element of the Banach space E.

3. Main Results

For convenience, we list the following assumptions.

(H1) $f \in C[(0,1) \times (0,+\infty), \mathbb{R}^+]$ and for every pair of positive numbers R and r with $R > r > 0$,

$$\int_0^1 (1-s)^{\alpha-3} f_{r,R}(s)\, ds < +\infty,$$

where $f_{r,R}(s) := \max\{f(s,u) :\ u \in [rs^2, R]\}$ for all $s \in (0,1)$.

(H2) For every $R > 0$, there exists $\psi_R \in C[J, \mathbb{R}^+]$ $(\psi_R \neq \theta)$ such that $f(t,u) \geq \psi_R(t)$ for $t \in (0,1)$ and $u \in (0,R]$.

(H3) There exists an interval $[a,b] \subset (0,1)$ such that $\lim\limits_{u \to +\infty} f(s,u)/u = +\infty$ uniformly with respect to $s \in [a,b]$.

We remark that (H2) allows $f(t,u)$ being singular at $t = 0, 1$, and $u = 0$. Assumption (H3) shows that f is superlinear in u. The following theorem is our main results of this paper.

Theorem 3.1. *Assume* (H1)–(H3) *are satisfied. Then there exist positive numbers λ^* and λ^{**} with $\lambda^* < \lambda^{**}$ such that BVP* (1.1) *has at least two positive solutions for $\lambda \in (0, \lambda^*)$ and no solution for $\lambda > \lambda^{**}$.*

To overcome difficulties arising from singularity, we first consider the approximate problem

$$\begin{aligned} {}^C D_{0+}^{\alpha} u(t) + \lambda f_n(t, u(t)) &= 0, \quad 0 < t < 1, \\ u^{(j)}(0) &= 0, \quad 0 \leq j \leq n-1, \quad j \neq 2, \\ u''(1) &= 0, \end{aligned} \tag{3.1}$$

where $f_n(t,u) =: f\big(t, \max\{\frac{1}{n}, u\}\big)$, $n \in \mathbb{N}$. Define an operator A_n^λ on Q by

$$(A_n^\lambda u)(t) := \lambda \int_0^1 G(t,s) f_n(s, u(s))\, ds, \tag{3.2}$$

where $G(t,s)$ is defined by (2.3).

Obviously, $u = A_n^\lambda u$ is the corresponding integral equation of (3.1). Therefore, $u \in E$ is a solution of (3.1) if $u \in E$ is a fixed point of A_n^λ.

Furthermore, u is a positive solution of (3.1) if $u \in Q \setminus \{\theta\}$ is a fixed point of A_n^λ.

By (3.2), it is easy to see that A_n^λ is well defined on Q for each $n \in \mathbb{N}$ if the condition (H1) holds. For the sake of proving our main results we first prove some lemmas.

Lemma 3.2. *Under the condition* (H1)*,* $A_n^\lambda : Q \to Q$ *is completely continuous.*

Proof. First, we show that $A_n^\lambda Q \subset Q$ for each $n \in \mathbb{N}$ and $\lambda > 0$. From Lemma 2.7, it follows that

$$(A_n^\lambda u)(t) = \lambda \int_0^1 G(t,s) f_n(s,u(s))\,ds \geq$$

$$\geq t^2 \lambda \int_0^1 G(\tau,s) f_n(s,u(s))\,ds = t^2 (A_n^\lambda u)(\tau), \quad \forall\, t,\tau \in J, \quad u \in Q.$$

Therefore, $A_n^\lambda Q \subset Q$ for each $n \in \mathbb{N}$ and $\lambda > 0$.

Next, by standard methods and Ascoli–Arzela theorem one can prove that $A_n^\lambda : Q \to Q$ is completely continuous. So it is omitted. □

Lemma 3.3. *Suppose the conditions* (H1) *and* (H2) *hold. Then for each* $r > 0$ *there exists a positive number* $\lambda(r)$ *such that*

$$i(A_n^\lambda, Q_r, Q) = 1$$

for $\lambda \in (0, \lambda(r))$ *and* n *sufficiently large, where* $Q_r = \{u \in Q : \|u\| < r\}$.

Proof. For each $r > 0$ and $n > \frac{1}{r}$, let

$$\lambda(r) := r \left[\frac{1}{2\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} f_{r,r}(s)\,ds \right]^{-1}.$$

We assert $\|A_n^\lambda u\| < \|u\|$ for each $\lambda \in (0, \lambda(r))$ and $u \in \partial Q_r$. In fact, using (2.10) and

$$G(t,s) \leq \frac{1}{2\Gamma(\alpha-2)} (1-s)^{\alpha-3} \ \text{ for } \ t,s \in J,$$

one can obtain

$$\|A_n^\lambda u\| \leq \lambda \int_0^1 \frac{1}{2\Gamma(\alpha-2)} (1-s)^{\alpha-3} f_n(s,u(s))\,ds =$$

$$= \lambda \frac{1}{2\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} f_{r,r}(s)\,ds < r =$$

$$= \|u\| \ \text{ for } \lambda \in (0, \lambda(r)) \text{ and } u \in \partial Q_r.$$

Therefore, by Lemma 2.8, we have $i(A_n^\lambda, Q_r, Q) = 1$ for $\lambda \in (0, \lambda(r))$. □

Lemma 3.4. *Suppose the conditions* (H1) *and* (H2) *hold. Then for any given* $\lambda \in (0, \lambda(r))$ *there exists* $r' \in (0, r)$ *such that*

$$i(A_n^\lambda, Q_{r'}, Q) = 0$$

for n *sufficiently large, where* r *and* $\lambda(r)$ *are the same as in Lemma* 3.3.

Proof. Choose a positive number r' with

$$r' < \min\left\{r, \lambda \max_{t \in J} \int_0^1 G(t,s)\psi_r(s)\,ds\right\},$$

where $\psi_r(s)$ is defined as in (H2). Now, we claim that

$$A_n^\lambda u \neq \mu u, \quad \forall\, u \in \partial Q_{r'}, \quad \mu \in (0,1], \tag{3.3}$$

for $n > 1/r'$. Suppose, on the contrary, that there exist $u_0 \in \partial Q_{r'}$ and $\mu_0 \in (0,1]$ such that $A_n^\lambda u_0 = \mu_0 u_0$, namely,

$$u_0(t) \geq (A_n^\lambda u_0)(t) = \lambda \int_0^1 G(t,s) f_n(s, u_0(s))\,ds, \quad \forall t \in J.$$

Notice that $|u_0(s)| \leq r' < r$ and $n > \frac{1}{r'}$ imply $f_n(s, u_0(s)) \geq \psi_r(s)$ for $s \in (0,1)$. Therefore,

$$u_0(t) \geq (A_n^\lambda u_0)(t) \geq \lambda \int_0^1 G(t,s)\psi_r(s)\,ds,$$

that is,

$$r' \geq \lambda \max_{t \in J} \int_0^1 G(t,s)\psi_r(s)\,ds,$$

which is in contradiction with the selection of r'. This means that (3.3) holds. Thus, by Lemma 2.8, we have $i(A_n^\lambda, Q_{r'}, Q) = 0$ for $n > \frac{1}{r'}$. □

Lemma 3.5. *Suppose the condition* (H3) *holds. Then for every* $\lambda \in (0, \lambda(r))$*, there exists* $R > r$ *such that*

$$i(A_n^\lambda, Q_R, Q) = 0$$

for all $n \in \mathbb{N}$*, where* $\lambda(r)$ *is the same as in Lemma* 3.3.

Proof. By (H3) we know that there exists $R' > \max\{r, 1\}$ such that

$$\frac{f(t,u)}{u} > L := \left[a^2\left(\lambda \min_{t \in [a,b]} \int_a^b G(t,s)\,ds\right)\right]^{-1} \quad \text{for} \ \ u > R'. \tag{3.4}$$

Let $R := 1 + \frac{R'}{a^2}$. Then for $u \in \partial Q_R$, by (2.10) we have $u(t) \geq a^2\|u\| > R'$ as $t \in [a,b]$. Now we show that

$$A_n^\lambda u \neq \mu u \ \ \text{for} \ u \in \partial Q_R \ \text{and} \ \mu \in (0,1]. \tag{3.5}$$

Suppose, on the contrary, that there exist $u_0 \in \partial Q_R$ and $\mu_0 \in (0,1]$ such that $A_n^\lambda u_0 = \mu_0 u_0$, that is,

$$u_0(t) \geq (A_n^\lambda u_0)(t) = \lambda \int_0^1 G(t,s) f_n(s, u_0(s))\, ds, \quad \forall\, t \in J.$$

Furthermore,

$$u_0(t) \geq (A_n^\lambda u_0)(t) > \lambda \biggl(\int_a^b G(t,s) \cdot L u_0(s)\, ds \biggr) >$$

$$> \biggl(\lambda \min_{t \in [a,b]} \int_a^b G(t,s)\, ds \biggr) L a^2 R = R$$

for $t \in [a,b]$. That is in contradiction with $\|u_0\| = R$, which means that (3.5) holds. Therefore, by Lemma 2.8, we have $i(A_n^\lambda, Q_R, Q) = 0$ for $n \in \mathbb{N}$. $\square$

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. For each $r > 0$, by Lemmas 3.3–3.5, there exist three positive numbers $\lambda(r), r'$, and R with $r' < r < R$ such that

$$i(A_n^\lambda, Q_{r'}, Q) = 0, \quad i(A_n^\lambda, Q_r, Q) = 1, \quad i(A_n^\lambda, Q_R, Q) = 0 \tag{3.6}$$

for n sufficiently large. Without loss of generality, suppose (3.6) holds for $n \geq n_0$. By virtue of the excision property of the fixed point index, we get

$$i\bigl(A_n^\lambda, Q_r \setminus \overline{Q_{r'}}, Q\bigr) = 1, \quad i\bigl(A_n^\lambda, Q_R \setminus \overline{Q_r}, Q\bigr) = -1$$

for $n \geq n_0$. Therefore, using the solution property of the fixed point index, there exist $u_n \in Q_r \setminus \overline{Q_{r'}}$ and $v_n \in Q_R \setminus \overline{Q_r}$ satisfying $A_n^\lambda u_n = u_n$ and $A_n^\lambda v_n = v_n$ as $n \geq n_0$. By the proof of Lemma 3.3, we know that there is no positive fixed point on ∂Q_r. Thus, $u_n \neq v_n$. Moreover, from (2.10) it follows that

$$r' t^2 \leq u_n(t) < r \ \text{ and } \ r t^2 < v_n(t) \leq R \ \text{ for } \ t \in J. \tag{3.7}$$

Further, we show that $\{u_n(t)\}_{n \geq n_0}$ are equicontinuous on J. To see this, we need to prove only that $\lim\limits_{t \to 0+} u_n(t) = 0$ uniformly with respect to $n \in \{n_0, n_0+1, n_0+2, \dots\}$ and $\{u_n(t)\}_{n \geq n_0}$ are equicontinuous on any subinterval of $(0,1]$. We first claim that $\lim\limits_{t \to 0+} u_n(t) = 0$ uniformly with respect to $n \in \{n_0, n_0+1, n_0+2, \dots\}$.

For arbitrary $\varepsilon > 0$, by (H1), there exists $\overline{\delta} > 0$ such that

$$\lambda \int_0^{\overline{\delta}} \frac{1}{2\Gamma(\alpha-2)} (1-s)^{\alpha-3} f_{r',r}(s)\, ds \leq \frac{\varepsilon}{3}\,. \tag{3.8}$$

Choose $\delta \in (0, \overline{\delta})$ sufficiently small such that

$$\lambda\delta^2 \int\limits_0^1 \frac{1}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} f_{r',r}(s)\,ds < \frac{\varepsilon}{3}\,. \tag{3.9}$$

Therefore, by (2.6), (3.8) and (3.9), we know for $t \in (0, \delta)$ and $\forall n \geq n_0$ that

$$\begin{aligned}
u_n(t) = \lambda \int\limits_0^1 G(t,s) f_n(s, u_n(s))\,ds \leq \\
&\leq \lambda \int\limits_0^t \frac{1}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} f_{r',r}(s)\,ds + \\
&\quad + \lambda\Bigg(\int\limits_t^{\overline{\delta}} + \int\limits_{\overline{\delta}}^1 \Bigg) \frac{t^2}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} f_{r',r}(s)\,ds \leq \\
&\leq 2\lambda \int\limits_0^{\overline{\delta}} \frac{1}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} f_{r',r}(s)\,ds + \\
&\quad + \lambda t^2 \int\limits_{\overline{\delta}}^1 \frac{1}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} f_{r',r}(s)\,ds \leq \\
&\leq 2\lambda \int\limits_0^{\overline{\delta}} \frac{1}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} f_{r',r}(s)\,ds + \\
&\quad + \lambda\delta^2 \int\limits_0^1 \frac{1}{2\Gamma(\alpha-2)}\,(1-s)^{\alpha-3} f_{r',r}(s)\,ds \leq \\
&\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This implies that $\lim\limits_{t\to 0+} u_n(t) = 0$ uniformly with respect to $n \in \{n_0, n_0 + 1, n_0 + 2, \dots\}$.

Now we are in a position to show that $\{u_n(t)\}_{n \geq n_0}$ are equicontinuous on any subinterval $[a, b]$ of $(0, 1]$. Notice that

$$u_n(t) = \lambda \int\limits_0^1 G(t,s) f_n(s, u_n(s))\,ds, \quad \forall\, t \in (0,1].$$

Thus, for $t \in [a, b]$, we have

$$|u_n'(t)| = \lambda \Big| \int_0^1 G_t(t,s) f_n(s, u_n(s)) ds \Big| \le$$

$$\le \frac{\lambda}{\Gamma(\alpha)} \Big(\int_0^t \Big| (\alpha-1)(\alpha-2)t(1-s)^{\alpha-3} - (\alpha-1)(t-s)^{\alpha-2} \Big| f_{r',r}(s)\, ds +$$

$$+ \int_t^1 \Big| (\alpha-1)(\alpha-2)t(1-s)^{\alpha-3} \Big| f_{r',r}(s)\, ds \Big) \le$$

$$\le \frac{\lambda(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \int_0^1 t(1-s)^{\alpha-3} f_{r',r}(s)\, ds \le$$

$$\le \frac{\lambda}{\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} f_{r',r}(s)\, ds < +\infty,$$

which implies that $\{u_n(t)\}_{n\ge n_0}$ are equicontinuous on $[a,b]$. Similarly as above, we can get that $\{v_n(t)\}_{n\ge n_0}$ are equicontinuous on $[0,1]$.

Then, the Ascoli–Arzela theorem guarantees the existence of $u, v \in Q \setminus \{\theta\}$ and two subsequences $\{u_{n_i}\}$ of $\{u_n\}$ and $\{v_{n_i}\}$ of $\{v_n\}$ such that $\lim\limits_{i\to+\infty} u_{n_i}(t) = u(t)$ and $\lim\limits_{i\to+\infty} v_{n_i}(t) = v(t)$ both uniformly with respect to $t \in J$. Moreover, by (H1), (3.7), and Lebesgue dominated convergence theorem, we obtain

$$u(t) = \lambda \int_0^1 G(t,s) f(s,u(s))\, ds, \quad v(t) = \lambda \int_0^1 G(t,s) f(s,v(s))\, ds, \ \ \forall\, t \in J$$

with $r' \le \|u\| \le r \le \|v\| \le R$. On the other hand, similarly to the proof of Lemma 3.3, it is easy to see $\|u\| < r < \|v\|$.

Choose $r = 1$. From the above we know that there exists $\lambda(1) > 0$ such that for each $\lambda \in (0, \lambda(1))$, BVP (1.1) has at least two positive solutions u_λ and v_λ with $0 < \|u_\lambda\| < 1 < \|v_\lambda\|$. Let

$$\lambda^* := \sup\{\overline{\lambda} > 0 : \text{ (1.1) have at least two positive solutions as } \lambda \in (0, \overline{\lambda})\}.$$

So, we get the existence of λ^* satisfying that BVP (1.1) has multiple positive solutions as $\lambda \in (0, \lambda^*)$.

Now we are in a position to prove the existence of λ^{**}. As above, we still choose $r = 1$ and corresponding $\lambda(1)$, R, r'. Here we show that BVP (1.1) has no positive solution for λ sufficiently large.

First suppose $\lambda \ge \lambda^*$. If BVP (1.1) has a positive solution u for some $\lambda \ge \lambda^*$, then by the corresponding integral equation

$$u(t) = \lambda \int_0^1 G(t,s) f(s,u(s))\, ds \tag{3.10}$$

and a process similar to the proof of Lemmas 3.4 and 3.5 (replacing λ in (3.4) with $\lambda(1)$), we obtain $r' < \|u\| < R$. This together with the condition (H2) and (3.10) guarantees that $u(t) \geq \lambda \int_0^1 G(t,s)\psi_R(s)ds$, that is, $R > \|u\| \geq \lambda \cdot \max_{t\in J} \int_0^1 G(t,s)\psi_R(s)\,ds$, which implies $\lambda < \big(\max_{t\in J} \int_0^1 G(t,s)\psi_R(s)\,ds\big)^{-1} R$. Therefore, we have obtained the existence of λ^{**}. The proof of Theorem 3.1 is complete. □

If $f(t,u)$ is not singular at $u = 0$, we have the following result, under the hypothesis

(H4) $f \in C[(0,1) \times [0,+\infty), \mathbb{R}^+]$ is nondecreasing with respect to u and for every positive number R,

$$\int_0^1 (1-s)^{\alpha-3} f_{0,R}(s)\,ds < +\infty,$$

where $f_{0,R}(s) = \max\{f(s,u) : \ u \in [0,R]\}$ for all $s \in (0,1)$.

Theorem 3.6. *Assume that the conditions* (H2)–(H4) *hold. Then there exist two positive numbers λ^* and λ^{***} with $\lambda^* \leq \lambda^{***}$ such that*

(i) *BVP* (1.1) *has at least two positive solutions for $\lambda \in (0, \lambda^*)$;*
(ii) *BVP* (1.1) *has at least one positive solution for $\lambda \in (0, \lambda^{***}]$;*
(iii) *BVP* (1.1) *has no solutions for $\lambda > \lambda^{***}$.*

Proof. Notice that the condition (H4) implies (H1). Therefore, the existence of λ^* can be obtained just as in Theorem 3.1. Now we claim that

$$\lambda^{***} := \sup\Big\{\lambda \in \mathbb{R}^+ : (1.1) \text{ has at least one positive solution}\Big\} \tag{3.11}$$

is required. First, from the proof of Theorem 3.1, we know that $\lambda^{***} \leq \lambda^{**}$. In the following we prove that (1.1) with $\lambda = \lambda^{***}$ has a positive solution $u^* \in Q$.

By (3.11), there exist two sequences $\{\lambda_n\}$ and $\{u_n\} \subset Q \setminus \{\theta\}$ such that $\{u_n\}$ is a positive solution of BVP (1.1) with $\lambda = \lambda_n$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_n \to \lambda^{***}$. Without loss of generality, suppose $\lambda_n \geq \lambda^*/2$ for each $n \in \mathbb{N}$. Similarly to the proof of Lemmas 3.4, 3.5 and Theorem 3.1, we can find that there exist two positive numbers r_1 and R_1 satisfying $r_1 \leq \|u_n\| \leq R_1$ for each $n \in \mathbb{N}$, and $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ which convergences to a function $u^* \in \overline{Q}_{R_1} \setminus Q_{r_1}$ uniformly as $t \in J$. Notice that

$$u_{n_k}(t) = \lambda_{n_k} \int_0^1 G(t,s) f(s, u_{n_k}(s))\,ds, \quad \forall\, t \in J.$$

Letting $k \to +\infty$, by the condition (H4) and Lebesgue dominated convergence theorem, we get

$$u^*(t) = \lambda^{***} \int_0^1 G(t,s) f(s, u^*(s))\, ds, \quad \forall\, t \in J.$$

This implies that $u^*(t)$ is a positive solution of BVP (1.1) with $\lambda = \lambda^{***}$.

Now we are in a position to prove that BVP (1.1) has at least one positive solution $u_\lambda(t)$ for each $\lambda \in (0, \lambda^{***})$. Notice that for $\lambda \in (0, \lambda^{***})$,

$$\begin{aligned} {}^C D_{0+}^{\alpha} u^*(t) &= \lambda^{***} f(t, u^*(t)) \geq \lambda f(t, u^*(t)), \quad t \in (0,1), \\ u^{*(j)}(0) &= 0, \quad 0 \leq j \leq n-1, \quad j \neq 2, \\ (u^*)''(1) &= 0. \end{aligned} \tag{3.12}$$

This implies that $u^*(t)$ is an upper solution of BVP (1.1). On the other hand, $u(t) \equiv 0$ is a lower solution for BVP (1.1). Applying [4, p. 244, Theorem 2.1], one can obtain that BVP (1.1) has at least one positive solution $u_\lambda(t) \in [0, u^*(t)]$ $(t \in J)$ for each $\lambda \in (0, \lambda^{***})$. ☐

4. Examples

Example 4.1. Consider the fractional singular boundary value problem

$$\begin{gathered} {}^C D_{0+}^{7/2} u(t) + \lambda \left[\frac{1}{\sqrt{t(1-t)}} \left(u^{-1/6} + u^2 \sin^2 t \right) \right] = 0, \quad t \in (0,1), \\ u(0) = u'(0) = u''(1) = u'''(0) = 0. \end{gathered} \tag{4.1}$$

Then there exist positive numbers λ^* and λ^{**} with $\lambda^* < \lambda^{**}$ such that BVP (4.1) has at least two positive solutions for $\lambda \in (0, \lambda^*)$ and no solution for $\lambda > \lambda^{**}$.

Proof. BVP (4.1) can be regarded as a BVP of the form (1.1), where $\alpha = \frac{7}{2}$, and

$$f(t,u) = \frac{1}{\sqrt{t(1-t)}} \left(u^{-1/6} + u^2 \sin^2 t \right).$$

We prove that $f(t,u)$ satisfies the conditions (H1)–(H3). For each pair of positive numbers R and r with $R > r > 0$, we know

$$f_{r,R}(t) \leq \frac{1}{\sqrt{t(1-t)}} \left((rt^2)^{-1/6} + R^2 \right).$$

Then

$$\int_0^1 (1-t)^{1/2} f_{r,R}(t)\, dt \leq \int_0^1 \frac{1}{\sqrt{t}} \left((rt^2)^{-1/6} + R^2 \right) dt < +\infty.$$

This means that the condition (H1) is satisfied. To see that (H2) holds, we notice that for each $R > 0$, one can choose $\psi_R(t) = R^{-1/6} / \sqrt{t(1-t)}$, which satisfies $\psi_R \neq \theta$ and $f(t,u) \geq \psi_R(t)$ for $t \in (0,1)$ and $u \in (0, R]$. Finally, it is easy to see that (H3) is satisfied since we can choose any subinterval of $[a, b] \subset (0, 1)$ satisfying $\lim\limits_{u \to +\infty} f(s, u)/u = +\infty$ uniformly with respect to $s \in [a, b]$. By Theorem 3.1, the conclusion follows. □

Analogously, using Theorem 3.6, we can prove that the following statement holds.

Example 4.2. Consider the fractional singular boundary value problem

$$\begin{aligned} {}^{C}D_{0^+}^{\alpha} u(t) &= \lambda t^{-1/2}(1-t)^{3-\alpha}(1 + e^{u} + u^2 \sin t), \quad t \in (0,1), \\ u^{(j)}(0) &= 0, \quad 0 \le j \le n-1, \quad j \ne 2, \\ u''(1) &= 0. \end{aligned} \tag{4.2}$$

where $\alpha \in (n-1, n]$, $n \ge 4$. Then there exist two positive numbers λ^* and λ^{***} with $\lambda^* \le \lambda^{***}$ such that:

(i) BVP (4.2) has at least two positive solutions for $\lambda \in (0, \lambda^*)$;
(ii) BVP (4.2) has at least one positive solution for $\lambda \in (0, \lambda^{***}]$;
(iii) BVP (4.2) has no solution for $\lambda > \lambda^{***}$.

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Short Communications

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SOME MULTI–POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER SINGULAR DIFFERENTIAL EQUATIONS

Abstract. For second order nonlinear differential equations with non-integrable singularities with respect to the time variable, unimprovable sufficient conditions for solvability and unique solvability of multi-point boundary value problems are established.

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Let $-\infty < a < b < +\infty$, $f :]a,b[\,\times R \to R$ be the function satisfying the local Carathéodory conditions, and let $p :]a,b[\, \to [0,+\infty[$ be the measurable function such that

$$p(t) > 0 \quad \text{almost everywhere on } \ ]a,b[\,, \quad \int\limits_a^b \frac{dt}{p(t)} < +\infty.$$

In the interval $[a,b]$, we consider the differential equation

$$\big(p(t)u'\big)' = f(t,u) \tag{1}$$

with the multi-point boundary conditions

$$\sum_{i=1}^{m} \alpha_i u(a_i) = c_1, \quad \sum_{i=1}^{n} \beta_i u(b_i) = c_2. \tag{2}$$

Here m and n are natural numbers, $\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_n,c_1,c_2$ are real constants,

$$a \le a_i \le a_0 < b_0 \le b_j \le b \ \ (i = 1,\dots,m; \ j = 1,\dots,n).$$

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Moreover, if $m = 1$ ($n = 1$), it is assumed that $a = a_0 = a_1$ ($b = b_0 = b_1$), and if $m \geq 2$ ($n \geq 2$), then

$$a = a_1 < \cdots < a_m = a_0 \quad \big(b_0 = b_1 < \cdots < b_n = b\big).$$

We are interested, in general, in the cases where the function f with respect to the time variable has non-integrable singularities at the points a and b. In that sense the problem (1), (2) is singular.

For $m = n = 1$, the singular problem (1), (2) is investigated in detail (see [1]–[4], [9], [14]–[16] and the references therein).

The optimal conditions for the unique solvability of problems of the type (1), (2) in the case, when the equation (1) is linear, are contained in [7], [8], [11], [12].

Various particular cases of the nonlinear singular problem (1), (2) are studied in [6], [10], [13]. Nevertheless, in the general case that problem remains so far studied insufficiently. In the present paper, new and unimprovable in a certain sense sufficient conditions for solvability and unique solvability of the above-mentioned problem are given.

We will seek a solution of the problem (1), (2) in the space of continuous functions $u : [a, b] \to R$ which are absolutely continuous together with $t \to p(t)u'(t)$ on an arbitrary closed interval, contained in $]a, b[$.

We introduce the following functions:

$$f^*(t, y) = \max\big\{|f(t, x)| : |x| \leq y\big\} \quad \text{for } a < t < b, \ y \geq 0;$$

$$f_0(t, y) = \sup\Big\{\frac{1}{2}\big(|f(t, x)| - f(t, x)\operatorname{sgn} x\big) : |x| \leq y\Big\} \quad \text{for } a < t < b, \ y \geq 0;$$

$$\delta(t) = \int\limits_a^t \frac{ds}{p(s)} \quad \text{for } a \leq t \leq b.$$

In the statements of the main results of the present paper, besides the functions f^*, f_0, and δ, there are appearing also the functions ψ_1, ψ_2, and ψ_0, which are defined in the following manner:
if $m = 1$ ($n = 1$), then

$$\psi_1(t) = 0 \ \text{ for } a \leq t \leq b \quad \Big(\psi_2(t) = \beta_1(\delta(b) - \delta(t)) \ \text{ for } a \leq t \leq b\Big);$$

if $m > 2$, then

$$\psi_1(t) = 0 \ \text{ for } a \geq a_0, \quad \psi_1(t) = \psi_1(a_{k+1}) + \Big(\sum_{i=k+1}^{m} \alpha_i\Big)\big(\delta(a_{k+1}) - \delta(t)\big)$$
$$\text{for } a_k \leq t \leq a_{k+1} \ (k = 1, \dots, m-1);$$

and if $n > 2$, then

$$\psi_2(b) = 0, \quad \psi_2(t) = \psi_2(b_{k+1}) + \Big(\sum_{i=k+1}^{n} \beta_i\Big)\big(\delta(b_{k+1}) - \delta(t)\big)$$
$$\text{for } b_k \leq t < b_{k+1} \ (k = 1, \dots, n-1),$$

$$\psi_2(t) = \psi_2(b_0) + \Big(\sum_{i=1}^{n} \beta_i\Big)\big(\delta(b_0) - \delta(t)\big) \ \text{ for } \ a \le t < b_0,$$

and

$$\psi_0(b) = 0, \quad \psi_0(t) = \psi_0(b_{k+1}) +$$
$$+\Big(\sum_{i=1}^{k} \beta_i\Big)\big(\delta(b_{k+1}) - \delta(t)\big) \ \text{ for } \ b_k \le t < b_{k+1} \ \ (k = 1, \dots, n-1), \tag{3}$$
$$\psi_0(t) = \psi_0(b_0) \ \text{ for } \ a \le t < b_0.$$

It is clear that

$$\sum_{i=1}^{n} \beta_i = 0 \Longrightarrow \psi_0(t) \equiv -\psi_2(t).$$

Let

$$\chi(t,s) = \begin{cases} 1 & \text{for } \ s \le t, \\ 0 & \text{for } \ s > t. \end{cases}$$

The following simple lemma is valid.

Lemma 1. *The boundary value problem*

$$\big(p(t)u'\big)' = 0; \quad \sum_{i=1}^{m} \alpha_i u(a_i) = 0, \quad \sum_{i=1}^{n} \beta_i u(b_i) = 0 \tag{4}$$

has only the trivial solution if and only if

$$\Delta = \Big(\sum_{i=1}^{n} \beta_i\Big)\psi_1(a) - \Big(\sum_{i=1}^{m} \alpha_i\Big)\psi_2(a) \ne 0. \tag{5}$$

Moreover, if the condition (5) *is satisfied, then the Green function of the problem* (4) *admits the representation*

$$g(t,s) = \frac{1}{\Delta}\Big[\psi_1(s)\psi_2(a) - \psi_2(s)\psi_1(a) + \Big(\psi_2(s)\sum_{i=1}^{m}\alpha_i - \psi_1(s)\sum_{i=1}^{n}\beta_i\Big)\delta(t)\Big] +$$
$$+\chi(t,s)(\delta(t) - \delta(s))$$

and

$$r = \sup\Big\{\frac{|g(t,s)|}{\delta(s)(\delta(b) - \delta(s))} : \ a \le t \le b, \ \ a < s < b\Big\} < +\infty. \tag{6}$$

We study the problem (1), (2) in the case, where

$$\int_a^b \delta(t)(\delta(b) - \delta(t)) f^*(t,y)\,dt < +\infty \quad \text{for } \ y \ge 0. \tag{7}$$

Moreover, if $a_0 > a$, then it is assumed that

$$\limsup_{\tau\to t,\, y\to+\infty} \int_t^\tau \delta(s)\frac{f^*(s,y)}{y}\,ds < 1 \quad \text{for } \ a \le t < a_0, \tag{8}$$

and if $b_0 < b$, then

$$\limsup_{\tau\to t,\, y\to+\infty} \int\limits_{\tau}^{t} (\delta(b)-\delta(s))\frac{f^*(s,y)}{y}\,ds < 1 \quad \text{for} \;\; b_0 < t \le b. \tag{9}$$

Along with $(1), (2)$ we consider the problem

$$\big(p(t)u'\big)' = \lambda f(t,u); \tag{10}$$

$$\sum_{i=1}^{m} \alpha_i u(a_i) = \lambda c_1, \quad \sum_{i=1}^{n} \beta_i u(b_i) = \lambda c_2, \tag{11}$$

dependent on a parameter $\lambda \in]0,1[$.

On the basis of Corollary 1.2 from [5] and Lemma 1, the following statements are proved.

Theorem 1 (The principle of a priori boundedness)**.** *Let the conditions* (5)*,* (7) *be fulfilled and let there exist a positive constant* y_0 *such that for any* $\lambda \in]0,1[$ *every solution of the problem* $(10), (11)$ *admits the estimate*

$$|u(t)| \le y_0 \quad \textit{for} \;\; a \le t \le b.$$

Then the problem $(1), (2)$ *has at least one solution.*

Theorem 2. *Let the inequality* (5) *hold and let there exist a positive constant* y_0 *such that*

$$r\int\limits_{a}^{b} \delta(s)(\delta(b)-\delta(s))f^*(s,y_0)\,ds \le y_0, \tag{12}$$

where r *is a number given by the equality* (6)*. Then the problem* $(1), (2)$ *has at least one solution.*

Theorem 3. *Let the inequality* (5) *hold and let in the domain* $]a,b[\,\times R$ *the condition*

$$\big|f(t,x_1)-f(t,x_2)\big| \le h(t)|x_1-x_2|$$

be fulfilled, where $h :]a,b[\, \to [0,+\infty[$ *is a measurable function such that*

$$r\int\limits_{a}^{b} \delta(s)(\delta(b)-\delta(s))h(s)\,ds < 1. \tag{13}$$

If, moreover,

$$\int\limits_{a}^{b} \delta(s)(\delta(b)-\delta(s))|f(s,0)|\,ds < +\infty,$$

then the problem $(1), (2)$ *has one and only one solution.*

Consider now the case, where

$$\alpha_i > 0 \ \ (i = 1, \dots, m), \quad \beta_i > 0 \ \ (i = 1, \dots, n). \tag{14}$$

Then the condition (5) is satisfied since

$$\Delta < -\Big(\sum_{i=1}^{m} \alpha_i\Big) \sum_{k=1}^{n-1} \Big(\sum_{i=k+1}^{n} \beta_i\Big)\big(\delta(b_{k+1}) - \delta(b_k)\big) < 0.$$

Let g_0 be the Green function of the boundary value problem

$$\big(p(t)u'\big)' = 0; \quad u(a) = u(b) = 0,$$

i.e.,

$$g_0(t,s) = \Big(\frac{\delta(s)}{\delta(b)} - 1\Big)\delta(t) + \chi(t,s)(\delta(t) - \delta(s)).$$

The following theorem is valid.

Theorem 4. *Let the conditions* (7)–(9)*, *and* (14) *be fulfilled. Let, moreover, there exist a positive constant* y_0 *such that*

$$\int_a^b |g_0(t,s)| f_0(s,y)\, ds < y \quad \textit{for} \ \ a \le t \le b, \ \ y > y_0. \tag{15}$$

Then the problem (1), (2) *has at least one solution.*

Corollary 1. *Let the inequalities* (14) *hold. Let, moreover, in the domain* $]a,b[\times R$ *the inequality*

$$f(t,x) \operatorname{sgn} x \ge -h(t)|x| - h_0(t) \tag{16}$$

be fulfilled, and in the domain $\big(]a,a_0[\cup]b_0,b[\big) \times R$ *the inequality*

$$|f(t,x)| \le h_0(t)(1 + |x|) \tag{17}$$

hold, where $h :]a,b[\to [0,+\infty[$ *and* $h_0 :]a,b[\to [0,+\infty[$ *are measurable functions such that*

$$\int_a^b \delta(s)(\delta(b) - \delta(s))h(s)\, ds \le \delta(b), \tag{18}$$

$$\int_a^b \delta(s)(\delta(b) - \delta(s))h_0(s)\, ds < +\infty. \tag{19}$$

Then the problem (1), (2) *has at least one solution.*

* For $m = 1$ ($n = 1$), the condition (7) (the condition (8)) is dropped out.

Theorem 5. *Let in the domain* $]a_0, b_0[\times R$ *the condition*

$$\big[f(t,x_1) - f(t,x_2)\big]\operatorname{sgn}(x_1 - x_2) \geq -h(t)|x_1 - x_2| \tag{20}$$

be fulfilled, and in the domain $\big(]a, a_0[\cup]b_0, b[\big)\times R$ *the condition*

$$\big|f(t,x_1) - f(t,x_2)\big| \leq \bar h(t)|x_1 - x_2| \tag{21}$$

hold, where $h :]a, b[\to [0, +\infty[$ *and* $\bar h :]a, a_0[\cup]b_0, b[\to [0, +\infty[$ *are measurable functions. If, moreover, the inequalities* (14), (18), *and* (19) *are satisfied, where*

$$h_0(t) = \begin{cases} |f(t,0)| & \text{for } t \in]a_0, b_0[\,, \\ |f(t,0)| + \bar h(t) & \text{for } t \in]a, b[\setminus]a_0, b_0[\,, \end{cases} \tag{22}$$

then the problem (1), (2) *has one and only one solution.*

Remark 1. If we take into account Example 1.1 from [4], then it becomes evident that the conditions (12), (13), (15), and (18) in Theorems 2–5 are unimprovable in the sense that they cannot be replaced, respectively, by the conditions

$$r\int_a^b \delta(s)(\delta(b) - \delta(s))f^*(s, y_0)\,ds \leq (1+\varepsilon)y_0,$$

$$r\int_a^b \delta(s)(\delta(b) - \delta(s))h(s)\,ds \leq 1+\varepsilon,$$

$$\int_a^b |g_0(t,s)|f_0(s,y)\,ds \leq (1+\varepsilon)y \quad \text{for } a \leq t \leq b, \ y \geq y_0,$$

$$\int_a^b \delta(s)(\delta(b) - \delta(s))h(s)\,ds \leq (1+\varepsilon)\delta(b),$$

no matter how small $\varepsilon > 0$ would be.

Consider now the case, where

$$\alpha_i > 0 \ \ (i = 1,\dots,m), \quad n > 2, \quad \beta_i > 0 \ \ (i = 1,\dots,n-1), \quad \beta_n = \\ = -\sum_{i=1}^{n-1}\beta_i, \quad \sum_{k=1}^{n-1}\Big(\sum_{i=1}^{k}\beta_i\Big)\big(\delta(b_{k+1}) - \delta(b_k)\big) = 1. \tag{23}$$

In that case the inequality (5) is also satisfied since

$$\Delta = -\Big(\sum_{i=1}^m \alpha_i\Big)\psi_2(a) = \Big(\sum_{i=1}^m \alpha_i\Big)\psi_0(a) = \sum_{i=1}^m \alpha_i > 0.$$

Let g_1 be the Green function of the boundary value problem

$$\big(p(t)u'\big)' = 0; \quad u(a) = 0, \quad \sum_{i=1}^{n} \beta_i u(b_i) = 0.$$

Then in view of (3) and (23) we have

$$g_1(t,s) = -\psi_0(s)\delta(t) + \chi(t,s)(\delta(t) - \delta(s)).$$

Lemma 2. *If along with* (23) *the condition*

$$\sum_{k=j}^{n-1} \Big(\sum_{i=1}^{k} \beta_i\Big)\big(\delta(b_{k+1}) - \delta(b_k)\big) \geq \frac{\delta(b) - \delta(b_j)}{\delta(b)} \quad (j = 1, \dots, n) \tag{24}$$

holds, then

$$g_1(t,s) \leq g_0(t,s) < 0 \quad \text{for } a < t < b$$

and

$$|g_1(t,s)| \leq \delta^{\mu}(t)\delta^{1-\mu}(s)\psi_0(s) \quad \text{for } a \leq t,\ s \leq b, \ \ 0 \leq \mu \leq 1.$$

For any $x \in R$, we suppose

$$[x]_+ = \frac{1}{2}(|x| + x).$$

On the basis of Theorem 1 and Lemma 2, the following theorems are proved.

Theorem 6. *Let the conditions* (23) *and* (24) *hold. Let, moreover, in the domains* $]a,b[\times R$ *and* $\big(]a,a_0[\cup]b_0,b[\big) \times R$ *the inequalities* (16) *and* (17) *be satisfied, respectively, where* $h :]a,b[\to [0,+\infty[$ *and* $h_0 :]a,b[\to [0,+\infty[$ *are measurable functions satisfying the conditions*

$$\int_a^b \delta^{\mu}(s)(\delta(b) - \delta(s))h(s)\,ds < +\infty, \quad \int_a^b \delta^{\mu}(s)(\delta(b) - \delta(s))h_0(s)\,ds < +\infty, \tag{25}$$

$$\int_a^b \delta(s)\psi_0(s)\Big[h(s) - \frac{\mu(1-\mu)\ell}{p(s)\psi_0(s)\delta^2(s)}\Big]_+ ds \leq 1 \tag{26}$$

for some $\mu \in]0,1]$ *and* $\ell \in]0,1]$. *Then the problem* (1), (2) *has at least one solution.*

Theorem 7. *Let the conditions* (23) *and* (24) *hold, and let in the domains* $]a,b[\times R$ *and* $\big(]a,a_0[\cup]b_0,b[\big) \times R$ *the inequalities* (20) *and* (21) *be satisfied, respectively, where* $h :]a,b[\to [0,+\infty[$ *and* $\bar{h} :]a,a_0[\cup]b_0,b[\to [0,+\infty[$ *are measurable functions. If, moreover, for some* $\mu \in]0,1]$ *and* $\ell \in]0,1]$ *the conditions* (25) *and* (26) *are satisfied, where* h_0 *is a function given by the equality* (22), *then the problem* (1), (2) *has one and only one solution.*

Remark 2. The condition (26) in Theorems 6 and 7 is unimprovable and it cannot be replaced by the condition

$$\int_a^b \delta(s)\psi_0(s)\Big[h(s) - \frac{\mu(1-\mu)\ell}{p(s)\psi_0(s)\delta^2(s)}\Big]_+ ds \le 1 + \varepsilon - \ell,$$

no matter how small $\varepsilon > 0$ would be.

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ON THE CAUCHY–NICOLETTI WEIGHTED PROBLEM FOR HIGHER ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. The unimprovable in a certain sense conditions are established which, respectively, ensure the solvability and well-posedness of the weighted Cauchy–Nicoletti problem for higher order nonlinear singular differential equations.

რეზიუმე. დადგენილია გარკვეული აზრით არაგაუმჯობესებადი პირობები, რომლებიც, სათანადოდ, უზრუნველყოფენ კოში–ნიკოლეტის წონიანი ამოცანის ამოხსნადობასა და კორექტულობას მაღალი რიგის არაწრფივი სინგულარული ფუნქციონალურ-დიფერენციალური განტოლებებისათვის.

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Let $-\infty < a < b < +\infty$, $n \geq 2$ be a natural number and f be an operator defined on some set $D(f) \subset C^{n-1}([a,b])$ and mapping $D(f)$ onto $L([a,b])$. We consider the functional differential equation

$$u^{(n)}(t) = f(u)(t) \tag{1}$$

with the Cauchy–Nicoletti weighted conditions

$$\limsup_{t\to t_i} \Big(\frac{|u^{(i-1)}(t)|}{\rho_i(t)}\Big) < +\infty \quad (i = 1, \dots, n). \tag{2}$$

Here $t_i \in [a,b]$ $(i = 1,\dots,n)$ and $\rho_i : [a,b] \to [0;+\infty[$ $(i = 1,\dots,n)$ are continuous functions such that

$$\rho_n(t_n) = 0, \quad \rho_n(t) > 0 \ \text{ for } \ t \neq t_n, \quad \rho_i(t_i) = 0,$$
$$\Big|\int\limits_{t_i}^{t} \rho_{i+1}(s)\,ds\Big| \leq \rho_i(t) \ \text{ for } \ a \leq t \leq b \ \ (i = 1,\dots,n-1).$$

By $C^{n-1}_{\rho_1,\dots,\rho_n}([a,b])$ we denote a set of functions $u \in C^{n-1}([a,b])$ such that

$$\mu(u) = \max\big\{\mu_1(u),\dots,\mu_n(u)\big\} < +\infty,$$

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where

$$\mu_i(u) = \sup\left\{\frac{|u^{(i-1)}(t)|}{\rho_i(t)} \,:\, a \le t \le b,\ t \ne t_i\right\}.$$

For an arbitrary $x > 0$, assume

$$C^{n-1}_{\rho_1,\dots,\rho_n;x}([a,b]) = \Big\{u \in C_{\rho_1,\dots,\rho_n}([a,b]) :\ \mu(u) \le x\Big\},$$
$$f^*(\rho_1,\dots,\rho_n;x)(t) = \sup\Big\{|f(u)(t)| :\ u \in C^{n-1}_{\rho_1,\dots,\rho_n;x}([a,b])\Big\}.$$

We investigate the problem (1), (2) in the case, where

$$C^{n-1}_{\rho_1,\dots,\rho_n}([a,b]) \subset D(f) \tag{3}$$

and for any $x > 0$ the conditions

$$f : C^{n-1}_{\rho_1,\dots,\rho_n;x}([a,b]) \longrightarrow L([a,b]) \text{ is continuous} \tag{4}$$

and

$$\int\limits_a^b f^*(\rho_1,\dots,\rho_n;x)(t)\,dt < +\infty$$

are fulfilled.

Of special interest is the case, where

$$D(f) \ne C^{n-1}([a,b]).$$

In this sense the equation (1) is singular one.

In the case, where f is the Nemytski's operator, i.e., when

$$f(u)(t) \equiv f_0\big(t, u(t), \dots, u^{(n-1)}(t)\big),$$

where $f : (]a,b[\,\setminus\{t_1,\dots,t_n\}) \times \mathbb{R}^n \to \mathbb{R}$ is the function satisfying the local Carathéodory conditions, the problems of the type (1), (2) are investigated thoroughly (see [1]–[6] and references therein). The problem (1), (2) is also investigated in the case, where

$$f(u)(t) \equiv f_0\big(t, u(\tau_1(t)), \dots, u^{(n-1)}(\tau_n(t))\big);$$
$$t_1 = \dots = t_n \ \text{ and } \ \rho_{i+1}(t) = \rho_i'(t) \ \ (i = 1,\dots,n)$$

(see [7]–[9]).

However, the problem mentioned above remains still little studied in a general case. Just this case we consider in the present paper.

The function $u \in D(f)$ with an absolutely continuous $(n-1)$th derivative is said to be a solution of the equation (1) if it almost everywhere on $]a,b[$ satisfies this equation.

A solution of the equation (1) satisfying the boundary conditions (2) is called a solution of the problem (1), (2).

Theorem 1. *Let the conditions* (3) *and* (4) *be fulfilled, and there exist constants* $\alpha \in]0,1[$ *and* $x_0 > 0$ *such that*

$$\left| \int_{t_n}^{t} f^*(\rho_1,\dots,\rho_n;x)(s)\,ds \right| \le \alpha\rho_n(x) \ \ \text{for } a \le t \le b, \ \ x \ge x_0. \qquad (5)$$

Then the problem (1), (2) *has at least one solution.*

Corollary 1. *Let there exist integrable functions* p *and* $q : [a,b] \to [0;+\infty[$ *such that*

$$\sup\left\{ \Big| \int_{t_n}^{t} p(s)\,ds \Big| / \rho_n(t) : \ \ a \le t \le b, \ \ t \ne t_n \right\} < 1, \qquad (6)$$

$$\sup\left\{ \Big| \int_{t_n}^{t} q(s)\,ds \Big| / \rho_n(t) : \ \ a \le t \le b, \ \ t \ne t_n \right\} < +\infty \qquad (7)$$

and for any $u \in C^{n-1}_{\rho_1,\dots,\rho_n}([a,b])$ *almost everywhere on* $]a,b[$ *the condition*

$$\big|f(u)(t)\big| \le \rho(t)\mu(u) + q(t)$$

is fulfilled. Then the problem (1), (2) *has at least one solution.*

Along with the problem (1), (2) we consider the perturbed problem

$$v^{(n)}(t) = f(v)(t) + h(t), \qquad (8)$$

$$\limsup_{t\to t_i} \Big(\frac{|v^{(i-1)}(t)|}{\rho_i(t)} \Big) < +\infty \ \ (i = 1,\dots,n), \qquad (9)$$

where $h :]a,b[\to \mathbb{R}$ is the integrable function such that

$$\mu_0(h) = \sup\left\{ \Big| \int_{t_n}^{t} h(s)\,ds \Big| / \rho_n(t) : \ \ a \le t \le b, \ \ t \ne t_n \right\} < +\infty. \qquad (10)$$

Definition 1. The problem (1), (2) is said to be well-posed if for any integrable function $h :]a,b[\to \mathbb{R}$ satisfying the condition (10), the problem (8), (9) is uniquely solvable, and there exists an independent of h positive constant r such that

$$\mu(u - v) \le r\mu_0(h),$$

where u and v are, respectively, the solutions of the problems (1), (2) and (8), (9).

Theorem 2. *Let there exist an integrable function* $p : [a,b] \to [0,+\infty[$ *satisfying the inequality* (6) *such that for any* u *and* $v \in C^{n-1}_{\rho_1,\dots,\rho_n}([a,b])$ *almost everywhere on* $]a,b[$ *the condition*

$$\big|f(u)(t) - f(v)(t)\big| \le p(t)\mu(u - v)$$

is fulfilled. If, moreover, the inequality (7), *where* $q(t) \equiv |f(0)(t)|$, *is fulfilled, then the problem* (1), (2) *is well-posed.*

Note that the condition (5) in Theorem 1, where $\alpha \in]0,1[$, is unimprovable and it cannot be replaced by the condition

$$\bigg|\int\limits_{t_n}^{t} f^*(\rho_1,\dots,\rho_n;x)(s)\,ds\bigg| \leq \rho_n(t)x \ \text{ for } \ a \leq t \leq b, \ \ x \geq x_0.$$

Similarly, in Corollary 1 and in Theorem 2, the strict inequality (6) cannot be replaced by the nonstrict inequality

$$\sup\bigg\{\bigg|\int\limits_{t_n}^{t} p(s)\,ds\bigg|/\rho_n(t): \ \ a \leq t \leq b, \ \ t \neq t_n\bigg\} \leq 1.$$

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