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INTERFACE CRACK PROBLEMS FOR METALLIC-PIEZOELECTRIC COMPOSITE STRUCTURES

Abstract. In the monograph we investigate three–dimensional interface crack problems for metallic-piezoelectric composite bodies with regard to thermal effects. We give a mathematical formulation of the physical problems when the metallic and piezoelectric bodies are bonded along some proper parts of their boundaries where interface cracks occur. By the potential method the interface crack problems are reduced to equivalent strongly elliptic systems of pseudodifferential equations on manifolds with boundary. We study the solvability of these systems in appropriate function spaces and prove uniqueness and existence theorems for the original interface crack problems. We analyse the regularity properties of the corresponding thermo-mechanical and electric fields near the crack edges and near the curves where the different boundary conditions collide. In particular, we characterize the stress singularity exponents and show that they can be explicitly calculated with the help of the principal homogeneous symbol matrices of the corresponding pseudodifferential operators. We expose some numerical calculations which demonstrate that the stress singularity exponents depend on the material parameters essentially.

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რეზიუმე. მონოგრაფიაში გამოკვლეულია მათემატიკური ამოცანები მეტალურ-პიეზოელექტრული კომპოზიტური სტრუქტურებისათვის თერმული ეფექტის გათვალისწინებით, როდესაც ბზარი მდებარეობს შედგენილი უბნობრივ ერთგვაროვანი სხეულის საკონტაქტო ზედაპირზე. ჩამოყალიბებულია ფიზიკურ-მექანიკური მოდელის შესაბამისი მათემატიკური სასაზღვრო-საკონტაქტო ამოცანების ფართო კლასი. პოტენციალთა მეთოდის გამოყენებით ეს არსებითად შერეული ბზარის ტიპის სასაზღვრო-საკონტაქტო ამოცანები დაყვანილია ეკვივალენტურ ინტეგრალურ (ფსევდოდიფერენციალურ) განტოლებათა სისტემაზე. დეტალურადაა შესწავლილი შესაბამისი მატრიცული ფსევდოდიფერენციალური ოპერატორების ფრედჰოლმურობის საკითხი და დადგენილია მათი შებრუნებადობა შესაბამის ფუნქციურ სივრცეებში. მიღებული შედეგების ბაზაზე დამტკიცებულია ამონახსნთა არსებობისა და ერთადერთობის თეორემები და დადგენილია ამონახსნთა რეგულარობა სინგულარობის წირების მიდამოში. შესწავლილია აგრეთვე ამონახსნთა ასიმპტოტიკა სინგულარობის წირების მიდამოში და დამუშავებულია ეფექტური მეთოდი ე. წ. ძაბვების სინგულარობის მაჩვენებლის მოსაძებნად. კონკრეტული მაგალითების განხილვით ნაჩვენებია, რომ ეს სინგულარობის მაჩვენებლები არსებითადაა დამოკიდებული კომპოზიტური სხეულის შემადგენელი მასალების მატერიალურ პარამეტრებზე.

Introduction

The monograph is dedicated to investigation of a mathematical model describing the interaction of the elastic, thermal, and electric fields in a three-dimensional composite structure consisting of a piezoelectric (ceramic) matrix and metallic inclusions (electrodes) bonded along some proper parts of their boundaries where interface cracks occur.

In spite of the fact that the piezoelectric phenomena were discovered long ago (see, e.g., [74]), the practical use of piezoelectric effects became possible only when piezoceramics and other materials (*metamaterials*) with pronounced piezoelectric properties were constructed. Nowadays, sensors and actuators made of such materials are widely used in medicine, aerospace, various industrial and domestic appliances, measuring and controlling devices. Therefore investigation of the mathematical models for such composite materials and analysis of the corresponding thermo-mechanical and electric fields became very actual and important for fundamental research and practical applications (for details see [31, 32, 46–48, 52, 59, 61, 70, 71] and the references therein).

Due to great theoretical and practical importance, problems of thermopiezoelectricity became very popular among mathematicians and engineers. Due to the references [34–42], during recent years more then 1000 scientific papers have been published annually! Most of them are engineering-technical papers dealing with the two-dimensional case.

Here we consider a general three-dimensional interface crack problem (ICP) for an anisotropic piezoelectric-metallic composite with regard to thermal effects and perform a rigorous mathematical analysis by the potential method. Similar problems for different type of metallic-piezoelastic composites without cracks and with interior cracks have been considered in [6–8].

In our analysis, we apply the Voigt's linear model in the piezoelectric part and the usual classical model of thermoelasticity in the metallic part to write the corresponding coupled systems of governing partial differential equations (see, e.g., [33, 56–58, 74]). As a result, in the piezoceramic part the unknown field is represented by a 5-component vector (three components of the displacement vector, the electric potential function and the temperature), while in the metallic part the unknown field is described by a 4-component vector (three components of the displacement vector and the temperature). Therefore, the mathematical modeling becomes complicated since we have to find reasonable efficient boundary, transmission and crack conditions for the physical fields possessing different dimensions in adjacent domains.

Since the crystal structures with central symmetry, in particular isotropic structures, do not reveal the piezoelectric properties in Voight's model, we have to consider anisotropic piezoelectric media. This also complicates the investigation. Thus, we have to take into account the composed anisotropic structure and the diversity of the fields in the ceramic and metallic parts.

The essential motivation for the choice of the interface crack problems treated in the monograph is that in a piezoceramic material, due to its brittleness, cracks arise often, especially when a piezoelectric device works at high temperature regime or under an intensive mechanical loading. The influence of the electric field on the crack growth has a very complex character. Experiments revealed that the electric field can either promote or retard the crack growth, depending on the direction of polarization and can even close an open crack [59].

As it is well known from the classical mathematical physics and the classical elasticity theory, in general, solutions to crack type and mixed boundary value problems have singularities near the crack edges and near the lines where different boundary conditions collide, regardless of the smoothness of given boundary data. The same effect can be observed in the case of our interface crack problems; namely, singularities of electric, thermal and stress fields appear near the crack edges and near the lines, where the boundary conditions collide and where the interfaces intersect the exterior boundary. Throughout the monograph we shall refer to such lines as *exceptional curves.*

In this monograph, we apply the potential method and reduce the ICPs to the equivalent system of pseudodifferential equations (ΨDEs) on a proper part of the boundary of the composed body. We analyse the solvability of the resulting boundary-integral equations in Sobolev–Slobodetskii (W_p^s), Bessel potential (H_p^s), and Besov ($B_{p,t}^s$) spaces and prove the corresponding uniqueness and existence theorems for the original ICPs. Moreover, our main goal is a detailed theoretical investigation of regularity properties of thermo-mechanical and electric fields near the exceptional curves and qualitative description of their singularities.

The monograph is organized as follows. In Section 1, we collect the field equations of the linear theory of thermoelasticity and thermopiezoelasticity, introduce the corresponding matrix partial differential operators and the generalized matrix boundary stress operators generated by the field equations, and formulate the boundary-transmission problems for a composed body consisting of metallic and piezoelectric parts with interface cracks. Depending on the physical properties of the metallic and piezoelectric materials and on surrounding media, one can consider different boundary, transmission and crack conditions for the thermal and electric fields. In particular, depending on the thermal insulation and dielectric properties of the crack gap, we present and discuss four possible mathematical models in Subsection 1.5, which are formulated as the interface crack problems:

- (ICP-A) - the crack gap is thermally insulated dielectric,
- (ICP-B) - the crack gap is thermally and electrically conductive,
- (ICP-C) - the crack gap is thermally insulated and electrically conductive, and
- (ICP-D) - the crack gap is heat-conducting dielectric.

Using Green's formulas, for these problems we prove the uniqueness theorems in appropriate function spaces.

Section 2 is devoted to the theory of pseudodifferential equations on manifolds with and without boundary which plays a crucial role in our analysis.

In Sections 3 and 4, we investigate properties of potential operators and prove some auxiliary assertions needed in our analysis. In particular, we study mapping properties of layer potentials and the corresponding boundary integral (pseudodifferential) operators in Sobolev–Slobodetskii, Bessel potential and Besov function spaces and establish Plemelji's type jump relations. We derive special representation formulas of solutions in terms of generalized layer potentials.

Sections 5 and 6 are the main parts of the present monograph. In Section 5, the interface crack problem (ICP-A) is reduced equivalently to the system of ΨDEs on manifolds with boundary and full analysis of solvability of these equations is given. Properties of the principal homogeneous symbol matrices are studied in detail and the existence, regularity and asymptotic properties of the solution fields are established. In particular, the global C^{α}-regularity results are shown with some $\alpha \in (0, \frac{1}{2})$. The exponent α is defined by the eigenvalues of a matrix which is explicitly constructed by the homogeneous symbol matrix of the corresponding pseudodifferential operator. In turn, these eigenvalues depend on the material parameters, in general. The exponent α actually defines the singularity exponents for the first order derivatives of solutions. In particular, they define stress singularity exponents. These questions are discussed in detail in Subsections 5.3 and 5.4. We calculate these exponents for particular cases explicitly, demonstrate their dependence on the material parameters and discuss problems related to the oscillating stress singularities. In Subsection 5.5, we present some numerical results and compare stress singularities at different type exceptional curves. As computations have shown, the stress singularities at the exceptional curves are different from -0.5 and essentially depend on the material parameters. We recall that for interior cracks the stress singularities do not depend on the material parameters and equal to -0.5 (see, e.g., [5, 13, 19, 30, 61]).

In Section 6, we consider the interface crack problem (ICP-B) which is reduced equivalently to a nonclassical system of boundary pseudodifferential equations which essentially differs from the system of pseudodifferential equations which appears in the study of the problem (ICP-A). This system is very involved, contains different dimensional matrix operators defined on overlapping submanifolds. Here we apply a different approach to carry out our analysis in order to prove the existence and regularity results for solutions of the problem (ICP-B). We study the asymptotic properties of solutions near the exceptional curves and characterize the corresponding stress singularity exponents. It is shown that these exponents again essentially depend on material parameters. The same approach can be applied to the problems (ICP-C) and (ICP-D).

1. Formulation of the Basic Problems and Uniqueness Results

1.1. **Geometrical description of the composite configuration.** Let $\Omega^{(m)}$ and Ω be bounded disjoint domains of the three-dimensional Euclidean space $\mathbb{R}^3$ with boundaries $\partial\Omega^{(m)}$ and $\partial\Omega$, respectively. Moreover, let $\partial\Omega$ and $\partial\Omega^{(m)}$ have a nonempty, simply connected intersection $\overline{\Gamma^{(m)}}$ of a positive measure, i.e., $\partial\Omega\cap\partial\Omega^{(m)}=\overline{\Gamma^{(m)}}$, $\operatorname{mes}\Gamma^{(m)}>0$. From now on $\Gamma^{(m)}$ will be referred to as an *interface surface*. Throughout the paper n and $\nu=n^{(m)}$ stand for the outward unit normal vectors to $\partial\Omega$ and to $\partial\Omega^{(m)}$, respectively. Clearly, $n(x)=-\nu(x)$ for $x\in\Gamma^{(m)}$.

Further, let $\overline{\Gamma^{(m)}}=\overline{\Gamma_T^{(m)}}\cup\overline{\Gamma_C^{(m)}}$, where $\Gamma_C^{(m)}$ is an open, simply connected proper part of $\Gamma^{(m)}$. Moreover, $\Gamma_T^{(m)}\cap\Gamma_C^{(m)}=\varnothing$ and $\partial\Gamma^{(m)}\cap\overline{\Gamma_C^{(m)}}=\varnothing$.

We set $S_N^{(m)}:=\partial\Omega^{(m)}\setminus\overline{\Gamma^{(m)}}$ and $S^*:=\partial\Omega\setminus\overline{\Gamma^{(m)}}$. Further, we denote by S_D some open, nonempty, proper sub-manifold of S^* and put $S_N:=S^*\setminus\overline{S_D}$. Thus, we have the following dissections of the boundary surfaces (see Figure 1)

$$\partial\Omega=\overline{\Gamma_T^{(m)}}\cup\overline{\Gamma_C^{(m)}}\cup\overline{S_N}\cup\overline{S_D},\quad \partial\Omega^{(m)}=\overline{\Gamma_T^{(m)}}\cup\overline{\Gamma_C^{(m)}}\cup\overline{S_N^{(m)}}.$$

Throughout the paper, for simplicity, we assume that $\partial\Omega^{(m)}$, $\partial\Omega$, $\partial S_N^{(m)}$, $\partial\Gamma_T^{(m)}$, $\partial\Gamma_C^{(m)}$, ∂S_D, ∂S_N are C^∞-smooth and $\partial\Omega^{(m)}\cap\overline{S_D}=\varnothing$, if not otherwise stated. Some results, obtained in the paper, also hold true when these manifolds and their boundaries are Lipschitz and we formulate them separately.

Let Ω be filled by an anisotropic homogeneous piezoelectric medium (ceramic matrix) and $\Omega^{(m)}$ be occupied by an isotropic or anisotropic homogeneous elastic medium (metallic inclusion). These two bodies interact along the interface $\Gamma^{(m)}$, where the interface crack $\Gamma_C^{(m)}$ occurs. Moreover, it is assumed that the composed body is fixed along the sub-surface S_D (the Dirichlet part of the boundary), while the sub-manifolds $S_N^{(m)}$ and S_N are the Neumann parts of the boundary (where the Neumann type boundary conditions are prescribed). In the metallic domain $\Omega^{(m)}$ we have a classical four-dimensional thermoelastic field represented by the displacement vector $u^{(m)}=(u_1^{(m)},u_2^{(m)},u_3^{(m)})^\top$ and temperature distribution function $u_4^{(m)}=\vartheta^{(m)}$, while in the piezoelectric domain Ω we have a five-dimensional physical field described by the displacement vector $u=(u_1,u_2,u_3)^\top$, temperature distribution function $u_4=\vartheta$ and the electric potential $u_5=\varphi$.

1.2. **Thermoelastic field equations.** Here we collect the field equations of the linear theory of thermoelasticity and introduce the corresponding

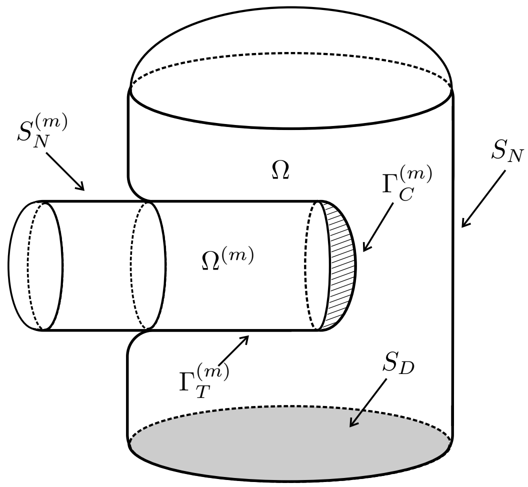

FIGURE 1. Metallic-piezoelectric composite

matrix partial differential operators (see [33, 58]). We will treat the general anisotropic case.

The basic governing equations of the classical thermoelasticity read as follows:

Constitutive relations:

$$s_{lk}^{(m)} = 2^{-1}(\partial_l\, u_k^{(m)} + \partial_k\, u_l^{(m)}), \tag{1.1}$$

$$\sigma_{ij}^{(m)} = \sigma_{ji}^{(m)} = c_{ijlk}^{(m)}\, s_{lk}^{(m)} - \gamma_{ij}^{(m)}\, \vartheta^{(m)} = c_{ijlk}^{(m)}\, \partial_l u_k^{(m)} - \gamma_{ij}^{(m)}\, \vartheta^{(m)}, \tag{1.2}$$

$$\mathcal{S}^{(m)} = \gamma_{ij}^{(m)}\, s_{ij}^{(m)} + \alpha^{(m)}\, [T_0^{(m)}]^{-1}\, \vartheta^{(m)}. \tag{1.3}$$

Fourier Law:

$$q_j^{(m)} = -\varkappa_{jl}^{(m)}\, \partial_l T^{(m)}. \tag{1.4}$$

Equations of motion:

$$\partial_i \sigma_{ij}^{(m)} + X_j^{(m)} = \varrho^{(m)}\, \partial_t^2 u_j^{(m)}. \tag{1.5}$$

Equation of the entropy balance:

$$T_0^{(m)}\, \partial_t \mathcal{S}^{(m)} = -\partial_j q_j^{(m)} + X_4^{(m)}. \tag{1.6}$$

Here $u^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)})^\top$ is the displacement vector, $\vartheta^{(m)} = T^{(m)} - T_0^{(m)}$ is the relative temperature (temperature increment); $\sigma_{kj}^{(m)}$ is the stress tensor in the theory of thermoelasticity, $s_{kj}^{(m)}$ is the strain tensor, $q^{(m)} = (q_1^{(m)}, q_2^{(m)}, q_3^{(m)})^\top$ is the heat flux vector; $\mathcal{S}^{(m)}$ is the entropy density, $\varrho^{(m)}$ is the mass density, $c_{ijkl}^{(m)}$ are the elastic constants, $\varkappa_{kj}^{(m)}$ are the thermal conductivity constants; $T_0^{(m)}>0$ is the initial temperature, that is the temperature in the natural state in the absence of deformation and electromagnetic fields; $\gamma_{kj}^{(m)}$ are the thermal strain constants; $\alpha^{(m)}=\varrho^{(m)}\,\widetilde{c}^{(m)}$ are the thermal material constants; $\widetilde{c}^{(m)}$ is the specific heat per unit mass; $X^{(m)}=(X_1^{(m)},X_2^{(m)},X_3^{(m)})^\top$ is a mass force density; $X_4^{(m)}$ is the heat source density; we employ the notation

$$\partial=\partial_x=(\partial_1,\partial_2,\partial_3),\quad \partial_j=\partial/\partial x_j,\quad \partial_t=\partial/\partial t.$$

The superscript $(\cdot)^\top$ denotes transposition operation.

Throughout the paper the Einstein convention about the summation over the repeated indices is meant from 1 to 3, unless stated otherwise.

Constants involved in the above equations satisfy the symmetry conditions:

$$c_{ijkl}^{(m)}=c_{jikl}^{(m)}=c_{klij}^{(m)},\ \ \gamma_{ij}^{(m)}=\gamma_{ji}^{(m)},\ \ \varkappa_{ij}^{(m)}=\varkappa_{ji}^{(m)},\ \ i,j,k,l=1,2,3. \tag{1.7}$$

Note that for an isotropic medium the thermomechanical coefficients are

$$c_{ijlk}^{(m)}=\lambda^{(m)}\,\delta_{ij}\,\delta_{lk}+\mu^{(m)}\,(\delta_{il}\,\delta_{jk}+\delta_{ik}\,\delta_{jl}),$$
$$\gamma_{ij}^{(m)}=\gamma^{(m)}\,\delta_{ij},\ \ \varkappa_{ij}^{(m)}=\varkappa^{(m)}\,\delta_{ij},$$

where $\lambda^{(m)}$ and $\mu^{(m)}$ are the Lamé constants and δ_{ij} is Kronecker's delta.

We assume that there are positive constants c_0 and c_1 such that

$$\begin{gathered} c_{ijkl}^{(m)}\,\xi_{ij}\,\xi_{kl}\geqslant c_0\,\xi_{ij}\,\xi_{ij},\ \ \varkappa_{ij}^{(m)}\xi_i\xi_j\geqslant c_1\,\xi_i\,\xi_i \\ \text{for all }\ \xi_{ij}=\xi_{ji}\in\mathbb{R},\ \ \xi_j\in\mathbb{R}. \end{gathered} \tag{1.8}$$

In particular, the first inequality implies that the density of potential energy

$$E^{(m)}\big(u^{(m)},u^{(m)}\big)=c_{ijlk}^{(m)}\ s_{ij}^{(m)}\ s_{lk}^{(m)},$$

corresponding to the displacement vector $u^{(m)}$, is positive definite with respect to the symmetric components of the strain tensor $s_{lk}^{(m)}=s_{kl}^{(m)}$.

Substituting (1.2) into (1.5) leads to the system of equations:

$$c_{ijlk}^{(m)}\,\partial_i\,\partial_l\,u_k^{(m)}-\gamma_{ij}^{(m)}\partial_i\vartheta^{(m)}+X_j^{(m)}=\varrho^{(m)}\,\partial_t^2\,u_j^{(m)},\ \ j=1,2,3. \tag{1.9}$$

Taking into account the Fourier law (1.4) and relation (1.3) from the equation of the entropy balance (1.6) we obtain the heat transfer equation

$$\varkappa_{il}^{(m)}\partial_i\,\partial_l\vartheta^{(m)}-\alpha^{(m)}\,\partial_t\vartheta^{(m)}-T_0^{(m)}\,\gamma_{il}^{(m)}\,\partial_t\partial_l u_i^{(m)}+X_4^{(m)}=0. \tag{1.10}$$

The simultaneous equations (1.9) and (1.10) represent the basic system of dynamics of the theory of thermoelasticity. If all the functions involved in these equations are harmonic time dependent, that is they represent a product of a function of the spatial variables (x_1,x_2,x_3) and the multiplier $\exp\{\tau\,t\}$, where $\tau=\sigma+i\omega$ is a complex parameter, we have the *pseudo-oscillation equations* of the theory of thermoelasticity. Note that the pseudo-oscillation equations can be obtained from the corresponding dynamical equations by the Laplace transform. If $\tau=i\,\omega$ is a pure imaginary number, with the so called frequency parameter $\omega \in \mathbb{R}$, we obtain the *steady state oscillation equations*. Finally, if $\tau = 0$, we get the *equations of statics*.

In this paper, we analyse the system of pseudo-oscillations

$$\begin{aligned} &c^{(m)}_{ijlk}\,\partial_i\partial_l\,u^{(m)}_k - \varrho^{(m)}\,\tau^2\,u^{(m)}_j - \gamma^{(m)}_{ij}\,\partial_i\vartheta^{(m)} + X^{(m)}_j = 0,\ j=\overline{1,3},\\ &-\tau\,T^{(m)}_0\,\gamma^{(m)}_{il}\,\partial_l u^{(m)}_i + \varkappa^{(m)}_{il}\,\partial_i\partial_l\vartheta^{(m)} - \tau\,\alpha^{(m)}\,\vartheta^{(m)} + X^{(m)}_4 = 0. \end{aligned} \tag{1.11}$$

In matrix form these equations can be rewritten as

$$A^{(m)}(\partial,\tau)\,U^{(m)}(x) + \widetilde{X}^{(m)}(x) = 0,$$

where $U^{(m)} = (u^{(m)},\vartheta^{(m)})^\top$ is the unknown vector function, while $\widetilde{X}^{(m)} = (X^{(m)}_1, X^{(m)}_2, X^{(m)}_3, X^{(m)}_4)^\top$ is a given vector, $A^{(m)}(\partial,\tau)$ is a formally non-selfadjoint matrix differential operator generated by equations (1.11),

$$A^{(m)}(\partial,\tau) = \big[\,A^{(m)}_{pq}(\partial,\tau)\,\big]_{4\times4}, \tag{1.12}$$

$$\begin{gathered} A^{(m)}_{jk}(\partial,\tau) = c^{(m)}_{ijlk}\,\partial_i\,\partial_l - \varrho^{(m)}\,\tau^2\,\delta_{jk},\\ A^{(m)}_{j4}(\partial,\tau) = -\gamma^{(m)}_{ij}\,\partial_i,\quad A^{(m)}_{4k}(\partial,\tau) = -\tau\,T^{(m)}_0\,\gamma^{(m)}_{kl}\partial_l,\\ A^{(m)}_{44}(\partial,\tau) = \varkappa^{(m)}_{il}\,\partial_i\,\partial_l - \alpha^{(m)}\tau,\quad j,k=1,2,3. \end{gathered}$$

By $A^{(m)*}(\partial,\tau)$ we denote the 4×4 matrix differential operator formally adjoint to $A^{(m)}(\partial,\tau)$, that is $A^{(m)*}(\partial,\tau) := [\overline{A^{(m)}(-\partial,\tau)}]^\top$, where the over-bar denotes the complex conjugation.

Denote by $A^{(m,0)}(\partial)$ the principal homogeneous part of the operator (1.12),

$$A^{(m,0)}(\partial) = \begin{bmatrix} [c^{(m)}_{ijlk}\,\partial_i\,\partial_l]_{3\times3} & [0]_{3\times1}\\ [0]_{1\times3} & \varkappa^{(m)}_{il}\,\partial_i\,\partial_l \end{bmatrix}_{4\times4}. \tag{1.13}$$

With the help of the symmetry conditions (1.7) and inequalities (1.8) it can easily be shown that $A^{(m,0)}(\partial)$ is a selfadjoint elliptic operator with a positive definite principal homogeneous symbol matrix, that is,

$$A^{(m,0)}(\xi)\,\eta\cdot\eta \geqslant c^{(m)}\,|\xi|^2\,|\eta|^2 \ \text{ for all }\ \xi\in\mathbb{R}^3 \ \text{ and for all }\ \eta\in\mathbb{C}^4$$

with some positive constant $c^{(m)} > 0$ depending on the material parameters.

Here and in what follows the central dot denotes the scalar product in $\mathbb{C}^N$, i.e., for $a = (a_1,\cdots,a_N)\in\mathbb{C}^N$ and $b=(b_1,\cdots,b_N)\in\mathbb{C}^N$ we set $a\cdot b := \sum\limits_{k=1}^{N} a_k\,\overline{b_k}$.

Components of the mechanical thermostress vector acting on a surface element with a normal $\nu = (\nu_1,\nu_2,\nu_3)$ read as follows

$$\sigma^{(m)}_{ij}\,\nu_i = c^{(m)}_{ijlk}\,\nu_i\,\partial_l u^{(m)}_k - \gamma^{(m)}_{ij}\,\nu_i\,\vartheta^{(m)},\quad j=1,2,3,$$

while the normal component of the heat flux vector (with opposite sign) has the form

$$-q^{(m)}_i\,\nu_i = \varkappa^{(m)}_{il}\,\nu_i\,\partial_l\vartheta^{(m)}.$$

We introduce the following generalized thermostress operator

$$\mathcal{T}^{(m)}(\partial,\nu)=\big[\,\mathcal{T}^{(m)}_{pq}(\partial,\nu)\,\big]_{4\times 4}, \tag{1.14}$$

$$\mathcal{T}^{(m)}_{jk}(\partial,\nu)=c^{(m)}_{ijlk}\,\nu_i\,\partial_l,\quad \mathcal{T}^{(m)}_{j4}(\partial,\nu)=-\gamma^{(m)}_{ij}\,\nu_i,$$

$$\mathcal{T}^{(m)}_{4k}(\partial,\nu)=0,\quad \mathcal{T}^{(m)}_{44}(\partial,\nu)=\varkappa^{(m)}_{il}\,\nu_i\,\partial_l,\ \ j,k=1,2,3.$$

For a four–vector $U^{(m)}=(u^{(m)},\vartheta^{(m)})^\top$ we have

$$\mathcal{T}^{(m)}\,U^{(m)}=(\,\sigma^{(m)}_{i1}\,\nu_i,\ \sigma^{(m)}_{i2}\,\nu_i,\ \sigma^{(m)}_{i3}\,\nu_i,\ -q^{(m)}_i\,\nu_i\,)^\top. \tag{1.15}$$

Clearly, the components of the vector $\mathcal{T}^{(m)}\,U^{(m)}$ given by (1.15) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of thermoelasticity, while the forth one is the normal component of the heat flux vector (with opposite sign).

We also introduce the boundary operator associated with the adjoint operator $A^{(m)*}(\partial,\tau)$ which appears in Green's formulae,

$$\widetilde{\mathcal{T}}^{(m)}(\partial,\nu,\tau)=\big[\,\widetilde{\mathcal{T}}^{(m)}_{pq}(\partial,\nu,\tau)\,\big]_{4\times 4},$$

$$\widetilde{\mathcal{T}}^{(m)}_{jk}(\partial,\nu,\tau)=c^{(m)}_{ijlk}\,\nu_i\,\partial_l,\quad \widetilde{\mathcal{T}}^{(m)}_{j4}(\partial,\nu,\tau)=\overline{\tau}\,T^{(m)}_0\,\gamma^{(m)}_{ij}\,\nu_i,$$

$$\widetilde{\mathcal{T}}^{(m)}_{4k}(\partial,\nu,\tau)=0,\quad \widetilde{\mathcal{T}}^{(m)}_{44}(\partial,\nu,\tau)=\varkappa^{(m)}_{il}\,\nu_i\,\partial_l,\ \ j,k=1,2,3.$$

The principal parts of the operators $\mathcal{T}^{(m)}$ and $\widetilde{\mathcal{T}}^{(m)}$ read as

$$\mathcal{T}^{(m,0)}(\partial,\nu)=\widetilde{\mathcal{T}}^{(m,0)}(\partial,\nu):=\begin{bmatrix}[c^{(m)}_{ijlk}\,\nu_i\,\partial_l]_{3\times 3} & [0]_{3\times 1}\\ [0]_{1\times 3} & \varkappa^{(m)}_{il}\,\nu_i\,\partial_l\end{bmatrix}_{4\times 4}. \tag{1.16}$$

1.3. **Thermopiezoelastic field equations.** In this subsection we collect the field equations of the linear theory of thermopiezoelasticity for a general anisotropic case and introduce the corresponding matrix partial differential operators (cf. [56, 61]). In the thermopiezoelasticity we have the following governing equations:

Constitutive relations:

$$s_{ij}=2^{-1}(\partial_i\,u_j+\partial_j\,u_i), \tag{1.17}$$

$$\sigma_{ij}=\sigma_{ji}=c_{ijkl}\,s_{kl}-e_{lij}E_l-\gamma_{ij}\,\vartheta=c_{ijkl}\,\partial_l u_k+e_{lij}\,\partial_l\varphi-\gamma_{ij}\,\vartheta, \tag{1.18}$$

$$\mathcal{S}=\gamma_{kl}\,s_{kl}+g_lE_l+\alpha\,T_0^{-1}\,\vartheta, \tag{1.19}$$

$$\begin{aligned}D_j&=e_{jkl}\,s_{kl}+\varepsilon_{jl}\,E_l+g_j\,\vartheta=\\ &=e_{jkl}\,\partial_l u_k-\varepsilon_{jl}\,\partial_l\varphi+g_j\,\vartheta,\ \ i,j=1,2,3.\end{aligned} \tag{1.20}$$

Fourier Law:

$$q_i=-\varkappa_{il}\,\partial_l T,\ \ i=1,2,3. \tag{1.21}$$

Equations of motion:

$$\partial_i\sigma_{ij}+X_j=\varrho\,\partial_t^2u_j,\ \ j=1,2,3. \tag{1.22}$$

Equation of the entropy balance:

$$T_0\,\partial_t \mathcal{S} = -\partial_j q_j + X_4. \tag{1.23}$$

Equation of static electric field:

$$\partial_i D_i - X_5 = 0. \tag{1.24}$$

Here $u = (u_1, u_2, u_3)^\top$ is the displacement vector, φ is the electric potential, ϑ is the temperature increment, σ_{kj} is the stress tensor in the theory of thermoelectroelasticity, s_{kj} is the strain tensor, D is the electric displacement vector, $E = (E_1, E_2, E_3) := -\operatorname{grad}\varphi$ is the electric field vector, $q = (q_1, q_2, q_3)$ is the heat flux vector, $\mathcal{S}$ is the entropy density, ϱ is the mass density, c_{ijkl} are the elastic constants, e_{kij} are the piezoelectric constants, ε_{kj} are the dielectric (permittivity) constants, γ_{kj} are thermal strain constants, $\varkappa_{kj}$ are thermal conductivity constants, T_0 is the initial reference temperature, that is the temperature in the natural state in the absence of deformation and electromagnetic fields, $\alpha := \varrho\,\widetilde{c}$ with $\widetilde{c}$ being the specific heat per unit mass, g_i are pyroelectric constants characterizing the relation between thermodynamic processes and piezoelectric effects, $X = (X_1, X_2, X_3)^\top$ is a mass force density, X_4 is a heat source density, X_5 is a charge density.

From the relations (1.18)–(1.24) we derive the linear system of the corresponding pseudo-oscillation equations of the theory of thermopiezoelasticity:

$$\begin{gathered}
c_{ijlk}\partial_i\partial_l u_k - \varrho\,\tau^2 u_j - \gamma_{ij}\partial_i\vartheta + e_{lij}\,\partial_l\partial_i\varphi + X_j = 0, \quad j = \overline{1,3}, \\
-\tau\,T_0\,\gamma_{il}\,\partial_l u_i + \varkappa_{il}\,\partial_i\,\partial_l\vartheta - \tau\,\alpha\,\vartheta + \tau\,T_0\,g_i\,\partial_i\varphi + X_4 = 0, \\
-e_{ikl}\,\partial_i\partial_l u_k - g_i\,\partial_i\vartheta + \varepsilon_{il}\,\partial_i\partial_l\varphi + X_5 = 0,
\end{gathered} \tag{1.25}$$

or in matrix form

$$A(\partial, \tau)\,U(x) + \widetilde{X}(x) = 0 \ \text{ in } \ \Omega, \tag{1.26}$$

where $U = (u, \vartheta, \varphi)^\top$, $\widetilde{X} = (X_1, X_2, X_3, X_4, X_5)^\top$, $A(\partial, \tau)$ is a formally nonselfadjoint matrix differential operator generated by equations (1.25)

$$A(\partial, \tau) = \big[\, A_{pq}(\partial, \tau) \,\big]_{5\times 5}, \tag{1.27}$$

$$\begin{gathered}
A_{jk}(\partial, \tau) = c_{ijlk}\,\partial_i\,\partial_l - \varrho\,\tau^2\,\delta_{jk}, \quad A_{j4}(\partial, \tau) = -\gamma_{ij}\,\partial_i, \\
A_{j5}(\partial, \tau) = e_{lij}\partial_l\partial_i, \quad A_{4k}(\partial, \tau) = -\tau\,T_0\,\gamma_{kl}\,\partial_l, \\
A_{44}(\partial, \tau) = \varkappa_{il}\,\partial_i\,\partial_l - \alpha\,\tau, \quad A_{45}(\partial, \tau) = \tau\,T_0\,g_i\,\partial_i, \\
A_{5k}(\partial, \tau) = -e_{ikl}\,\partial_i\partial_l, \quad A_{54}(\partial, \tau) = -g_i\partial_i, \\
A_{55}(\partial, \tau) = \varepsilon_{il}\,\partial_i\partial_l, \ \ j, k = \overline{1,3}.
\end{gathered}$$

By $A^*(\partial, \tau) := [\overline{A(-\partial, \tau)}]^\top$ we denote the operator formally adjoint to $A(\partial, \tau)$. Clearly, from (1.25)–(1.27) we obtain the equations and the operators of statics if $\tau = 0$. Denote by $A^{(0)}(\partial)$ the principal homogeneous part of the operator (1.27),

$$A^{(0)}(\partial)=\begin{bmatrix}[c_{ijlk}\,\partial_i\,\partial_l]_{3\times 3} & [0]_{3\times 1} & [e_{lij}\partial_l\partial_i]_{3\times 1}\\ [0]_{1\times 3} & \varkappa_{il}\,\partial_i\,\partial_l & 0\\ [-e_{ikl}\,\partial_i\partial_l]_{1\times 3} & 0 & \varepsilon_{il}\,\partial_i\partial_l\end{bmatrix}_{5\times 5}. \tag{1.28}$$

Evidently, the operator $A^{(0)}(\partial)$ is formally nonselfadjoint.
The constants involved in the above equations satisfy the symmetry conditions:

$$c_{ijkl}=c_{jikl}=c_{klij},\ \ e_{ijk}=e_{ikj},\ \ \varepsilon_{ij}=\varepsilon_{ji},$$
$$\gamma_{ij}=\gamma_{ji},\ \ \varkappa_{ij}=\varkappa_{ji},\ \ i,j,k,l=1,2,3.$$

Moreover, from physical considerations it follows that (see, e.g., [56]):

$$c_{ijkl}\,\xi_{ij}\,\xi_{kl}\geqslant c_0\,\xi_{ij}\,\xi_{ij}\ \text{ for all }\ \xi_{ij}=\xi_{ji}\in\mathbb{R}, \tag{1.29}$$

$$\varepsilon_{ij}\,\eta_i\,\eta_j\geqslant c_1\,\eta_i\,\eta_i\,,\ \ \varkappa_{ij}\,\eta_i\,\eta_j\geqslant c_2\,\eta_i\,\eta_i\ \text{ for all }\ \eta=(\eta_1,\eta_2,\eta_3)\in\mathbb{R}^3, \tag{1.30}$$

where c_0, c_1, and c_2 are positive constants. In addition, we require that (see, e.g., [56])

$$\varepsilon_{ij}\,\eta_i\,\overline{\eta}_j+\frac{\alpha}{T_0}\,|\,\zeta|^2-2\,\mathrm{Re}\,\big(\zeta\,g_l\,\overline{\eta}_l\big)\geqslant$$
$$\geqslant c_3\big(|\,\zeta|^2+|\,\eta|^2\big)\ \text{ for all }\ \zeta\in\mathbb{C},\ \ \eta\in\mathbb{C}^3 \tag{1.31}$$

with a positive constant c_3. A sufficient condition for the inequality (1.31) to be satisfied reads as

$$\frac{\alpha\,c_1}{3\,T_0}-g^2>0,$$

where $g=\max\{|g_1|,|g_2|,|g_3|\}$ and c_1 is the constant involved in (1.30).

With the help of the inequalities (1.29) and (1.30) it can easily be shown that the principal part of the operator $A(\partial,\tau)$ is strongly elliptic, that is,

$$\mathrm{Re}\,A^{(0)}(\xi)\,\eta\cdot\eta\geqslant c\,|\xi|^2\,|\eta|^2\ \text{ for all }\ \xi\in\mathbb{R}^3\ \text{ and for all }\ \eta\in\mathbb{C}^4$$

with some positive constant $c>0$ depending on the material parameters.

In the theory of thermopiezoelasticity the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n=(n_1,n_2,n_3)$ have the form

$$\sigma_{ij}\,n_i=c_{ijlk}\,n_i\,\partial_l u_k+e_{lij}\,n_i\,\partial_l\varphi-\gamma_{ij}\,n_i\,\vartheta\ \text{ for }\ j=1,2,3,$$

while the normal components of the electric displacement vector and the heat flux vector (with opposite sign) read as

$$-D_i\,n_i=-e_{ikl}\,n_i\,\partial_l u_k+\varepsilon_{il}\,n_i\,\partial_l\varphi-g_i\,n_i\,\vartheta,\quad -q_i\,n_i\ =\varkappa_{il}\,n_i\,\partial_l\vartheta.$$

Let us introduce the following matrix differential operator

$$\mathcal{T}(\partial,n) = \big[\,\mathcal{T}_{pq}(\partial,n)\,\big]_{5\times 5}, \tag{1.32}$$
$$\mathcal{T}_{jk}(\partial,n) = c_{ijlk}\,n_i\,\partial_l, \quad \mathcal{T}_{j4}(\partial,n) = -\gamma_{ij}\,n_i,$$
$$\mathcal{T}_{j5}(\partial,n) = e_{lij}\,n_i\,\partial_l, \quad \mathcal{T}_{4k}(\partial,n) = 0,$$
$$\mathcal{T}_{44}(\partial,n) = \varkappa_{il}\,n_i\,\partial_l, \quad \mathcal{T}_{45}(\partial,n) = 0,$$
$$\mathcal{T}_{5k}(\partial,n) = -e_{ikl}\,n_i\,\partial_l, \quad \mathcal{T}_{54}(\partial,n) = -g_i\,n_i,$$
$$\mathcal{T}_{55}(\partial,n) = \varepsilon_{il}\,n_i\,\partial_l, \quad j,k=1,2,3.$$

For a five–vector $U=(u,\vartheta,\varphi)^\top$ we have

$$\mathcal{T}(\partial,n)\,U = \big(\,\sigma_{i1}\,n_i,\ \sigma_{i2}\,n_i,\ \sigma_{i3}\,n_i,\ -q_i\,n_i,\ -D_i\,n_i\,\big)^\top. \tag{1.33}$$

Clearly, the components of the vector $\mathcal{T}\,U$ given by (1.33) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of thermoelectroelasticity, the forth and fifth ones are the normal components of the heat flux vector and the electric displacement vector (with opposite sign), respectively.

In Green's formulae there appear also the following boundary operator associated with the adjoint differential operator $A^*(\partial,\tau)$,

$$\widetilde{\mathcal{T}}(\partial,n,\tau) = \big[\,\widetilde{\mathcal{T}}_{pq}(\partial,n,\tau)\,\big]_{5\times 5},$$
$$\widetilde{\mathcal{T}}_{jk}(\partial,n,\tau) = c_{ijlk}\,n_i\,\partial_l, \quad \widetilde{\mathcal{T}}_{j4}(\partial,n,\tau) = \overline{\tau}\,T_0\,\gamma_{ij}\,n_i,$$
$$\widetilde{\mathcal{T}}_{j5}(\partial,n,\tau) = -e_{lij}\,n_i\,\partial_l, \quad \widetilde{\mathcal{T}}_{4k}(\partial,n,\tau) = 0,$$
$$\widetilde{\mathcal{T}}_{44}(\partial,n,\tau) = \varkappa_{il}\,n_i\,\partial_l, \quad \widetilde{\mathcal{T}}_{45}(\partial,n,\tau) = 0,$$
$$\widetilde{\mathcal{T}}_{5k}(\partial,n,\tau) = e_{ikl}\,n_i\,\partial_l, \quad \widetilde{\mathcal{T}}_{54}(\partial,n,\tau) = -\,\overline{\tau}\,T_0\,g_i\,n_i,$$
$$\widetilde{\mathcal{T}}_{55}(\partial,n,\tau) = \varepsilon_{il}\,n_i\,\partial_l.$$

The principal parts of the operators $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ read as

$$\mathcal{T}^{(0)}(\partial,n) := \begin{bmatrix} [c_{ijlk}\,n_i\,\partial_l]_{3\times 3} & [0]_{3\times 1} & [e_{lij}\,n_i\,\partial_l]_{3\times 1} \\ [0]_{1\times 3} & \varkappa_{il}\,n_i\,\partial_l & 0 \\ [-e_{ikl}\,n_i\,\partial_l]_{1\times 3} & 0 & \varepsilon_{il}\,n_i\,\partial_l \end{bmatrix}_{5\times 5}, \tag{1.34}$$

$$\widetilde{\mathcal{T}}^{(0)}(\partial,n) := \begin{bmatrix} [c_{ijlk}\,n_i\,\partial_l]_{3\times 3} & [0]_{3\times 1} & [-e_{lij}\,n_i\,\partial_l]_{3\times 1} \\ [0]_{1\times 3} & \varkappa_{il}\,n_i\,\partial_l & 0 \\ [e_{ikl}\,n_i\,\partial_l]_{1\times 3} & 0 & \varepsilon_{il}\,n_i\,\partial_l \end{bmatrix}_{5\times 5}. \tag{1.35}$$

1.4. **Green's formulae.** As it has been mentioned above, to avoid some misunderstanding related to the directions of normal vectors on the contact surface $\Gamma^{(m)}$, we denote by ν and n the unit outward normal vectors to $\partial\Omega^{(m)}$ and $\partial\Omega$ respectively. Here we recall Green's formulae for the differential operators $A^{(m)}(\partial,\tau)$ and $A(\partial,\tau)$ in $\Omega^{(m)}$ and Ω, respectively (see, e.g., [4, 6, 7, 9, 29, 30, 53]).

Let $\Omega^{(m)}$ and Ω be domains with smooth boundaries and

$$U^{(m)}=\big(u_1^{(m)},u_2^{(m)},u_3^{(m)},u_4^{(m)}\big)^\top\in[\,C^2(\overline{\Omega^{(m)}})\,]^4,$$
$$u^{(m)}=\big(u_1^{(m)},u_2^{(m)},u_3^{(m)}\big)^\top,$$
$$V^{(m)}=\big(v_1^{(m)},v_2^{(m)},v_3^{(m)},v_4^{(m)}\big)^\top\in[\,C^2(\overline{\Omega^{(m)}})\,]^4,$$
$$v^{(m)}=\big(v_1^{(m)},v_2^{(m)},v_3^{(m)}\big)^\top.$$

Then the following Green's formulae hold:

$$\int\limits_{\Omega^{(m)}}\Big[A^{(m)}(\partial,\tau)U^{(m)}\cdot V^{(m)}-U^{(m)}\cdot A^{(m)*}(\partial,\tau)\,V^{(m)}\Big]\,dx=$$
$$=\int\limits_{\partial\Omega^{(m)}}\Big[\{\mathcal{T}^{(m)}\,U^{(m)}\}^+\cdot\{V^{(m)}\}^+-\{U^{(m)}\}^+\cdot\{\widetilde{\mathcal{T}}^{(m)}V^{(m)}\}^+\Big]\,dS,\quad(1.36)$$

$$\int\limits_{\Omega^{(m)}}A^{(m)}(\partial,\tau)U^{(m)}\cdot V^{(m)}\,dx=\int\limits_{\partial\Omega^{(m)}}\{\mathcal{T}^{(m)}U^{(m)}\}^+\cdot\{V^{(m)}\}^+\,dS-$$
$$-\int\limits_{\Omega^{(m)}}\Big[E^{(m)}(u^{(m)},\overline{v^{(m)}})+\varrho^{(m)}\tau^2u^{(m)}\cdot v^{(m)}+\varkappa_{jl}^{(m)}\partial_ju_4^{(m)}\,\overline{\partial_lv_4^{(m)}}+$$
$$+\tau\alpha^{(m)}u_4^{(m)}\overline{v_4^{(m)}}+\gamma_{jl}^{(m)}\big(\tau T_0^{(m)}\partial_ju_l^{(m)}\overline{v_4^{(m)}}-u_4^{(m)}\overline{\partial_jv_l^{(m)}}\big)\Big]\,dx,\qquad(1.37)$$

$$\int\limits_{\Omega^{(m)}}\Big[\sum_{j=1}^3\big[A^{(m)}(\partial,\tau)U^{(m)}\big]_j\overline{u_j^{(m)}}+\frac{\tau}{|\tau|^2T_0^{(m)}}\,\big[\overline{A^{(m)}(\partial,\tau)U^{(m)}}\big]_4u_4^{(m)}\Big]\,dx=$$
$$=-\int\limits_{\Omega^{(m)}}\Big[\,E^{(m)}(u^{(m)},\overline{u^{(m)}})+\varrho^{(m)}\,\tau^2\,|u^{(m)}|^2+\frac{\alpha^{(m)}}{T_0^{(m)}}\,|u_4^{(m)}|^2+$$
$$+\frac{\tau}{|\tau|^2\,T_0^{(m)}}\;\varkappa_{lj}^{(m)}\,\partial_lu_4^{(m)}\,\overline{\partial_ju_4^{(m)}}\,\Big]\,dx+$$
$$+\int\limits_{\partial\Omega^{(m)}}\Big[\sum_{j=1}^3\big\{\mathcal{T}^{(m)}U^{(m)}\big\}_j^+\big\{\overline{u_j^{(m)}}\big\}^++$$
$$+\frac{\tau}{|\tau|^2\,T_0^{(m)}}\big\{\overline{\mathcal{T}^{(m)}U^{(m)}}\big\}_4^+\,\big\{u_4^{(m)}\big\}^+\Big]\,dS.\qquad(1.38)$$

Here $E^{(m)}(u^{(m)},\overline{v^{(m)}})=c_{ijlk}^{(m)}\,\partial_iu_j^{(m)}\,\overline{\partial_lv_k^{(m)}}$ and the differential operators $A^{(m)}(\partial,\tau)$, $A^{(m)*}(\partial,\tau)$,$\mathcal{T}^{(m)}=\mathcal{T}^{(m)}(\partial,\nu)$ and $\widetilde{\mathcal{T}}^{(m)}=\widetilde{\mathcal{T}}^{(m)}(\partial,\nu,\tau)$ are defined in Subsection 1.2.

Similarly, for arbitrary vector–functions

$$U=(u_1,u_2,u_3,u_4,u_5)^\top\in[\,C^2(\overline{\Omega})\,]^5,\;\;u=(u_1,u_2,u_3)^\top,$$
$$V=(v_1,v_2,v_3,v_4,v_5)^\top\in[\,C^2(\overline{\Omega})\,]^5,\;\;v\;=(v_1,v_2,v_3)^\top,$$

we have Green's formulae involving the differential operators of the thermo-electroelasticity theory:

$$\int\limits_{\Omega}\Big[A(\partial,\tau)\,U\cdot V-U\cdot A^{*}(\partial,\tau)\,V\Big]\,dx=$$
$$=\int\limits_{\partial\Omega}\Big[\{\,\mathcal{T}\,U\,\}^{+}\cdot\{\,V\,\}^{+}-\{\,U\,\}^{+}\cdot\{\,\widetilde{\mathcal{T}}\,V\,\}^{+}\Big]\,dS, \tag{1.39}$$

$$\int\limits_{\Omega}A(\partial,\tau)\,U\cdot V\,dx=\int\limits_{\partial\Omega}\{\,\mathcal{T}\,U\,\}^{+}\,\cdot\,\{\,V\,\}^{+}\,dS-$$
$$-\int\limits_{\Omega}\Big[E(u,\overline{v})+\varrho\,\tau^{2}\,u\cdot v+\gamma_{jl}\,(\,\tau\,T_0\,\partial_j u_l\,\overline{v_4}-u_4\,\overline{\partial_j v_l}\,)+$$
$$+\varkappa_{jl}\,\partial_j u_4\,\overline{\partial_l v_4}+\tau\,\alpha\,u_4\,\overline{v_4}+e_{lij}\,(\partial_l u_5\,\overline{\partial_i v_j}-\partial_i u_j\,\overline{\partial_l v_5})-$$
$$-g_l\,(\,\tau\,T_0\,\partial_l u_5\,\overline{v_4}+u_4\,\overline{\partial_l v_5})+\varepsilon_{jl}\,\partial_j u_5\,\overline{\partial_l v_5}\Big]\,dx, \tag{1.40}$$

$$\int\limits_{\Omega}\Big[\sum_{j=1}^{3}[A(\partial,\tau)\,U]_j\,\overline{u_j}+\frac{\tau}{|\tau|^2T_0}\,\big[\overline{A(\partial,\tau)U}\big]_4\,u_4+\big[\overline{A(\partial,\tau)U}\big]_5\,u_5\Big]\,dx=$$
$$=-\int\limits_{\Omega}\Big[E(u,\overline{u})+\varrho\tau^2|u|^2+\frac{\alpha}{T_0}|u_4|^2+\frac{\tau}{|\tau|^2T_0}\,\varkappa_{jl}\,\partial_l u_4\overline{\partial_j u_4}-$$
$$-2\mathrm{Re}\,\{g_l\,u_4\,\overline{\partial_l u_5}\}+\varepsilon_{jl}\,\partial_l u_5\,\overline{\partial_j u_5}\Big]\,dx+$$
$$+\int\limits_{\partial\Omega}\Big[\sum_{j=1}^{3}\{\mathcal{T}U\}_j^{+}\{\overline{u_j}\}^{+}+\frac{\tau}{|\tau|^2T_0}\{\overline{\mathcal{T}U}\}_4^{+}\{u_4\}^{+}+\{\overline{\mathcal{T}U}\}_5^{+}\{u_5\}^{+}\Big]\,dS. \tag{1.41}$$

Here $E(u,\overline{v})=c_{ijlk}\,\partial_i u_j\,\overline{\partial_l v_k}$ and the differential operators $A(\partial,\tau)$, $A^{*}(\partial,\tau)$, $\mathcal{T}=\mathcal{T}(\partial,n)$, and $\widetilde{\mathcal{T}}=\widetilde{\mathcal{T}}(\partial,n,\tau)$ are defined in Subsection 1.3.

For $\tau=0$ Green's formulae (1.36), (1.37), (1.40), and (1.39) remain valid and, in addition, there hold the following identities

$$\int\limits_{\Omega^{(m)}}\Big[\sum_{j=1}^{3}\big[\,A^{(m)}(\partial)\,U^{(m)}\big]_j\,\overline{u_j^{(m)}}+c_1\,\big[\,A^{(m)}(\partial)\,U^{(m)}\,\big]_4\,\overline{u_4^{(m)}}\Big]\,dx=$$
$$=-\int\limits_{\Omega^{(m)}}\Big[E^{(m)}(u^{(m)},\overline{u^{(m)}})+c_1\,\varkappa_{lj}^{(m)}\,\partial_l u_4^{(m)}\,\overline{\partial_j u_4^{(m)}}-\gamma_{jl}^{(m)}\,u_4^{(m)}\,\overline{\partial_j u_l^{(m)}}\Big]\,dx+$$
$$+\int\limits_{\partial\Omega^{(m)}}\Big[\sum_{j=1}^{3}\{\mathcal{T}^{(m)}U^{(m)}\}_j^{+}\{\,\overline{u_j^{(m)}}\}^{+}+c_1\{\mathcal{T}^{(m)}U^{(m)}\}_4^{+}\{\overline{u_4^{(m)}}\}^{+}\Big]\,dS, \tag{1.42}$$

$$\int\limits_{\Omega}\Big[\sum_{j=1}^{3}[\,A(\partial)\,U]_j\,\overline{u_j}+c\,[\,A(\partial)\,U\,]_4\,\overline{u_4}\,+[\,\overline{A(\partial)\,U}\,]_5\,u_5\Big]\,dx=$$

$$= -\int\limits_{\Omega} \Big[E(u,\overline{u}) + c\,\varkappa_{jl}\partial_l u_4 \overline{\partial_j u_4} - \gamma_{jl} u_4 \overline{\partial_l u_j} - g_l \overline{u_4} \partial_l u_5 + \varepsilon_{jl} \partial_l u_5 \overline{\partial_j u_5} \Big]\, dx +$$
$$+ \int\limits_{\partial\Omega} \Big[\sum_{j=1}^{3} \{\mathcal{T}\,U\}_j^+ \{\overline{u_j}\}^+ + c\,\{\mathcal{T}U\}_4^+\,\{\overline{u_4}\}^+ + \{\overline{\mathcal{T}U}\}_5^+\,\{u_5\}^+ \Big]\, dS, \quad (1.43)$$

where $A^{(m)}(\partial) := A^{(m)}(\partial, 0)$ and $A(\partial) := A(\partial, 0)$, and c_1 and c are arbitrary constants.

Remark that by a standard limiting procedure the above Green's formulae (1.37), (1.38), (1.40), and (1.41) can be generalized to Lipschitz domains and to vector–functions from the Sobolev spaces (see formulae (1.58), (1.59))

$$U^{(m)} \in \Big[W_p^1(\Omega^{(m)})\Big]^4, \ \ V^{(m)} \in \Big[W_{p'}^1(\Omega^{(m)})\Big]^4,$$
$$U \in \big[W_p^1(\Omega)\big]^5, \ \ V \in \big[W_{p'}^1(\Omega)\big]^5$$

with

$$A^{(m)}(\partial,\tau) U^{(m)} \in \Big[L_p(\Omega^{(m)})\Big]^4, \ \ A(\partial,\tau) U \in \left[L_p(\Omega)\right]^5,$$
$$1 < p < \infty, \ \ \frac{1}{p} + \frac{1}{p'} = 1.$$

In addition, if

$$A^{(m)*}(\partial,\tau) V^{(m)} \in [L_{p'}(\Omega^{(m)})]^4, \ \ A^*(\partial,\tau) V \in [L_{p'}(\Omega)]^5,$$

then formulae (1.36) and (1.39) hold true as well (see [25, 44, 50, 55]).

1.5. **Formulation of the interface crack problems.** Let us consider the metallic-piezoelectric composite structure described in Subsection 1.1 (see Figure 1). We assume that

(1) the composed body is fixed along the sub-surface S_D, i.e, there are given homogeneous Dirichlet data for the vector $U = (u, \vartheta, \varphi)^\top$;

(2) the sub-surface $S_N^{(m)}$ is either traction free or there is applied some surface force, i.e., the components of the mechanical stress vector $\sigma_{ij}^{(m)}\,\nu_i$, $j = 1, 2, 3$, are given on $S_N^{(m)}$;

(3) the sub-surface S_N is either traction free or there is applied some surface force, i.e., the components of the mechanical stress vector $\sigma_{ij}\,n_i$, $j = 1, 2, 3$, are given on S_N;

(4) along the transmission interface submanifold $\Gamma_T^{(m)}$ the piezoelectric and metallic solids are bonded, i.e., the rigid contact conditions are fulfilled which means that the displacement and mechanical stress vectors are continuous across $\Gamma_T^{(m)}$;

(5) the faces of the interface crack $\Gamma_C^{(m)}$ are traction free, i.e., the components of the mechanical stress vectors $\sigma_{ij}^{(m)}\,\nu_i$ and $\sigma_{ij}\,n_i$, $j = 1, 2, 3$, vanish on $\Gamma_C^{(m)}$.

Depending on the physical properties of the metallic and piezoelectric materials and also surrounding media, one can consider different boundary, transmission and crack conditions for the thermal and electric fields. For example,

(6) if some part of the boundary of the composed body is covered by a thermally insulated material then the normal components of the heat flux vectors $-q_i^{(m)}\,\nu_i$ and $-q_i\,n_i$ should be zero on the corresponding submanifold; in particular, these conditions hold on the crack faces if the crack gap is a thermal isolator;

(7) if some part of the boundary of the composed body is charge free, then the normal component of the electric displacement vector $-D_i\,n_i$ should be zero on the corresponding submanifold;

(8) if some part of the boundary of the composed body is covered by a metallic layer with applied charge, then the electric potential function φ should be given on the corresponding submanifold;

(9) if the crack gap is thermally conductive, the temperature and normal heat flux functions should satisfy continuity condition on the crack surface $\Gamma_C^{(m)}$;

(10) if the crack gap can be treated as a dielectric medium, the normal component of the electric displacement vector $-D_i\,n_i$ should be zero on $\Gamma_C^{(m)}$;

(11) due to the rigid contact conditions on $\Gamma_T^{(m)}$, for the electric potential function φ the Dirichlet condition should be given on $\Gamma_T^{(m)}$.

From the above arguments it follows that the physical problem under consideration is described by essentially mixed boundary, transmission and crack type conditions. Solutions to this kind crack and mixed boundary value problems and related mechanical, thermal and electrical characteristics usually have singularities in a neighbourhood of exceptional curves, $\partial\Gamma_C^{(m)}$, ∂S_D, $\partial\Gamma^{(m)}$.

Our goal is to formulate the above described problems mathematically, study their solvability in appropriate function spaces and analyse regularity properties of solutions. In particular, we describe dependence of the stress singularity exponents on the material parameters. As we will see below this dependence is quite nontrivial.

Let us introduce some notation.

Throughout the paper the symbol $\{\,\cdot\,\}^+$ denotes the interior one-sided trace operator on $\partial\Omega$ from Ω (respectively on $\partial\Omega^{(m)}$ from $\Omega^{(m)}$). Similarly, $\{\,\cdot\,\}^-$ denotes the exterior one-sided trace operator on $\partial\Omega$ from the exterior of Ω (respectively on $\partial\Omega^{(m)}$ from the exterior of $\Omega^{(m)}$).

By L_p, W_p^r, H_p^s, and $B_{p,q}^s$ with $r \geqslant 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, we denote the well-known Lebesgue, Sobolev–Slobodetskii, Bessel potential, and Besov function spaces, respectively (see, e.g., [43, 72]). Recall that $H_2^r = W_2^r = B_{2,2}^r$, $H_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$, and $H_p^k = W_p^k$, for any $r \geqslant 0$, for any $s\in\mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k. By $C_0^k(\mathbb{R}^n)$ we denote the set of functions with compact support possessing continuous derivatives up to order $k\geqslant 0$, $C_0^\infty(\mathbb{R}^n)=\bigcap\limits_{k=0}^{\infty}C_0^k(\mathbb{R}^n)$.

Let $\mathcal{M}_0$ be a smooth surface without boundary. For a smooth submanifold $\mathcal{M}\subset\mathcal{M}_0$ we denote by $\widetilde{H}_p^s(\mathcal{M})$ and $\widetilde{B}_{p,q}^s(\mathcal{M})$ the subspaces of $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$\widetilde{H}_p^s(\mathcal{M})=\big\{g:\ g\in H_p^s(\mathcal{M}_0),\ \operatorname{supp} g\subset\overline{\mathcal{M}}\big\},$$
$$\widetilde{B}_{p,q}^s(\mathcal{M})=\big\{g:\ g\in B_{p,q}^s(\mathcal{M}_0),\ \operatorname{supp} g\subset\overline{\mathcal{M}}\big\},$$

while $H_p^s(\mathcal{M})$ and $B_{p,q}^s(\mathcal{M})$ denote the spaces of restrictions on $\mathcal{M}$ of functions from $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$H_p^s(\mathcal{M})=\big\{r_{\mathcal{M}}f:\ f\in H_p^s(\mathcal{M}_0)\big\},$$
$$B_{p,q}^s(\mathcal{M})=\big\{r_{\mathcal{M}}f:\ f\in B_{p,q}^s(\mathcal{M}_0)\big\},$$

where $r_{\mathcal{M}}$ is the restriction operator onto $\mathcal{M}$.

From now on without loss of generality we assume that the mass force density, heat source density and charge density vanish in the corresponding regions, that is, $X_k^{(m)}=0$ in $\Omega^{(m)}$ for $k=\overline{1,4}$, $X_j=0$ in Ω for $j=\overline{1,5}$. Otherwise, we can write particular solutions to the nonhomogeneous differential equations (1.11) and (1.25) explicitly, in the form of volume Newtonian potentials,

$$U_0^{(m)}(x)=-N_\tau^{(m)}(X^{(m)})(x)\ \text{ and }\ U_0(x)=-N_\tau(X)(x),$$

where

$$N_\tau^{(m)}(X^{(m)})(x):=\int\limits_{\Omega^{(m)}}\Psi^{(m)}(x-y,\tau)X^{(m)}(y)\,dy,\ \ x\in\Omega^{(m)},$$
$$N_\tau(X)(x):=\int\limits_{\Omega}\Psi(x-y,\tau)\,X(y)\,dy,\ \ x\in\Omega,$$

with $\Psi^{(m)}(x-y,\tau)$ and $\Psi(x-y,\tau)$ being the fundamental solution matrices of the operators $A^{(m)}(\partial,\tau)$ and $A(\partial,\tau)$ respectively (see Subsection 4.1). Note that for $X^{(m)}\in[L_p(\Omega^{(m)})]^4$ and $X\in[L_p(\Omega)]^5$, we have $U_0^{(m)}\in[W_p^2(\Omega^{(m)})]^4$ and $U_0\in[W_p^2(\Omega)]^5$, and

$$A^{(m)}(\partial,\tau)\,N_\tau^{(m)}(X^{(m)})(x)=X^{(m)}(x),\ \ x\in\Omega^{(m)},\tag{1.44}$$
$$A(\partial,\tau)\,N_\tau(X)(x)=X(x),\ \ x\in\Omega,\tag{1.45}$$

for almost all $x\in\Omega^{(m)}$ and for almost all $x\in\Omega$ respectively. In addition, if $X^{(m)}\in[C^{0,\beta'}(\overline{\Omega^{(m)}})]^4$ and $X\in[C^{0,\beta'}(\overline{\Omega})]^5$ with some $\beta'>0$, then the relations (1.44) and (1.45) hold for all $x\in\Omega^{(m)}$ and for all $x\in\Omega$ respectively (see Section 4, Theorem 4.1).

Therefore, without loss of generality in what follows we will consider the homogeneous versions of the differential equations (1.11) and (1.25).

However, we have to take into consideration that the homogeneous boundary and transmission conditions described in the items (1)-(11) become then nonhomogeneous, in general.

Further, without loss of generality and for simplicity, throughout the paper we assume that the initial reference temperatures T_0 and $T_0^{(m)}$ in the adjacent domains Ω and $\Omega^{(m)}$ are the same: $T_0 = T_0^{(m)}$.

Now we are in a position to formulate mathematically the above described physical mixed interface crack problems. For illustration we formulate four typical problems: (ICP-A), (ICP-B), (ICP-C), and (ICP-D).

Problem (ICP-A) - the crack gap is thermally insulated dielectric:
Find vector-functions

$$U^{(m)} = (u_1^{(m)}, \dots, u_4^{(m)})^\top \;:\; \Omega^{(m)} \to \mathbb{C}^4,$$
$$U = (u_1, \dots, u_5)^\top \;:\; \Omega \to \mathbb{C}^5$$

belonging respectively to the spaces $[W_p^1(\Omega^{(m)})]^4$ and $[W_p^1(\Omega)]^5$ with $1 < p < \infty$ and satisfying

(i) *the systems of partial differential equations*:

$$\big[\, A^{(m)}(\partial_x, \tau)\, U^{(m)} \,\big]_j = 0 \;\text{ in }\; \Omega^{(m)},\;\; j = \overline{1,4}, \tag{1.46}$$
$$[\, A(\partial_x, \tau)\, U \,]_k = 0 \;\text{ in }\; \Omega,\;\; k = \overline{1,5}, \tag{1.47}$$

(ii) *the boundary conditions*:

$$r_{S_N^{(m)}} \big\{ \big[\, \mathcal{T}^{(m)}(\partial, \nu)\, U^{(m)} \,\big]_j \big\}^+ = Q_j^{(m)} \;\text{ on }\; S_N^{(m)},\;\; j = \overline{1,4}, \tag{1.48}$$
$$r_{S_N} \big\{ \big[\, \mathcal{T}(\partial, n)\, U \,\big]_k \big\}^+ = Q_k \;\text{ on }\; S_N,\;\; k = \overline{1,5}, \tag{1.49}$$
$$r_{S_D} \{u_k\}^+ = f_k \;\text{ on }\; S_D,\;\; k = \overline{1,5}, \tag{1.50}$$
$$r_{\Gamma_T^{(m)}} \{u_5\}^+ = f_5^{(m)} \;\text{ on }\; \Gamma_T^{(m)}, \tag{1.51}$$

(iii) *the transmission conditions on* $\Gamma_T^{(m)}$ *for* $j = \overline{1,4}$:

$$r_{\Gamma_T^{(m)}} \{u_j\}^+ - r_{\Gamma_T^{(m)}} \{u_j^{(m)}\}^+ = f_j^{(m)} \;\text{ on }\; \Gamma_T^{(m)}, \tag{1.52}$$
$$\begin{gathered} r_{\Gamma_T^{(m)}} \big\{ \big[\mathcal{T}(\partial, n) U\big]_j \big\}^+ + \\ + r_{\Gamma_T^{(m)}} \big\{ [\mathcal{T}^{(m)}(\partial, \nu) U^{(m)}]_j \big\}^+ = F_j^{(m)} \;\text{ on }\; \Gamma_T^{(m)}, \end{gathered} \tag{1.53}$$

(iv) *the interface crack conditions on* $\Gamma_C^{(m)}$:

$$r_{\Gamma_C^{(m)}} \big\{ [\, \mathcal{T}^{(m)}(\partial, \nu)\, U^{(m)} \,]_j \big\}^+ = \widetilde{Q}_j^{(m)} \;\text{ on }\; \Gamma_C^{(m)},\;\; j = \overline{1,4}, \tag{1.54}$$
$$r_{\Gamma_C^{(m)}} \{ [\, \mathcal{T}(\partial, n)\, U \,]_k \}^+ = \widetilde{Q}_k \;\text{ on }\; \Gamma_C^{(m)},\;\; k = \overline{1,5}, \tag{1.55}$$

where $n=-\nu$ on $\Gamma^{(m)}$,

$$
\begin{gathered}
Q_k\in B^{-1/p}_{p,p}(S_N),\;\; Q^{(m)}_j\in B^{-1/p}_{p,p}(S^{(m)}_N),\;\; f_k\in B^{1/p'}_{p,p}(S_D),\\
f^{(m)}_k\in B^{1/p'}_{p,p}(\Gamma^{(m)}_T),\;\; F^{(m)}_j\in B^{-1/p}_{p,p}(\Gamma^{(m)}_T),\\
\widetilde{Q}^{(m)}_j,\, \widetilde{Q}_k\in B^{-1/p}_{p,p}(\Gamma^{(m)}_C),\;\; \frac{1}{p'}+\frac{1}{p}=1,\;\; k=\overline{1,5},\;\; j=\overline{1,4}.
\end{gathered}
\tag{1.56}
$$

Note that the functions $F^{(m)}_j$, Q_j, $\widetilde{Q}_j$, $\widetilde{Q}^{(m)}_j$ and $Q^{(m)}_j$ have to satisfy some evident compatibility conditions (see Subsection 5.1, inclusion (5.17)). We set

$$
\begin{aligned}
Q&=(Q_1,Q_2,Q_3,Q_4,Q_5)^\top\in\big[\,B^{-\frac{1}{p}}_{p,p}(S_N)\,\big]^5,\\
\widetilde{Q}&=(\widetilde{Q}_1,\widetilde{Q}_2,\widetilde{Q}_3,\widetilde{Q}_4,\widetilde{Q}_5)^\top\in\big[\,B^{-\frac{1}{p}}_{p,p}(\Gamma^{(m)}_C)\,\big]^5,\\
Q^{(m)}&=\big(Q^{(m)}_1,Q^{(m)}_2,Q^{(m)}_3,Q^{(m)}_4\big)^\top\in\big[\,B^{-\frac{1}{p}}_{p,p}(S^{(m)}_N)\,\big]^4,\\
\widetilde{Q}^{(m)}&=\big(\widetilde{Q}^{(m)}_1,\widetilde{Q}^{(m)}_2,\widetilde{Q}^{(m)}_3,\widetilde{Q}^{(m)}_4\big)^\top\in\big[\,B^{-\frac{1}{p}}_{p,p}(\Gamma^{(m)}_C)\,\big]^4,\\
f&=(f_1,f_2,f_3,f_4,f_5)^\top\in\big[\,B^{\frac{1}{p'}}_{p,p}(S_D)\,\big]^5,\\
f^{(m)}&=\big(f^{(m)}_1,f^{(m)}_2,f^{(m)}_3,f^{(m)}_4,f^{(m)}_5\big)^\top\in\big[\,B^{\frac{1}{p'}}_{p,p}(\Gamma^{(m)}_T)\,\big]^5,\\
F^{(m)}&=\big(F^{(m)}_1,F^{(m)}_2,F^{(m)}_3,F^{(m)}_4\big)^\top\in\big[\,B^{-\frac{1}{p}}_{p,p}(\Gamma^{(m)}_T)\,\big]^4.
\end{aligned}
\tag{1.57}
$$

A pair $(U^{(m)},U)\in[W^1_p(\Omega^{(m)})]^4\times[W^1_p(\Omega)]^5$ will be called a solution to the boundary-transmission problem (ICP-A) (1.46)–(1.55).

The differential equations (1.46) and (1.47) are understood in the distributional sense, in general. But note that if $U^{(m)}\in[W^1_p(\Omega^{(m)})]^4$ and $U\in[W^1_p(\Omega)]^5$ solve the homogeneous differential equations, then actually $U^{(m)}\in[C^\infty(\Omega^{(m)})]^4$ and $U\in[C^\infty(\Omega)]^5$ due to the ellipticity of the corresponding differential operators. In fact, $U^{(m)}$ and U are complex valued analytic vectors of spatial real variables (x_1,x_2,x_3) in $\Omega^{(m)}$ and Ω, respectively.

The Dirichlet-type conditions (1.50), (1.51), and (1.52) involving boundary limiting values of the vectors $U^{(m)}$ and U are understood in the usual trace sense, while the Neumann-type conditions (1.48), (1.49), (1.5), (1.54) and (1.55) involving boundary limiting values of the vectors $\mathcal{T}^{(m)}u^{(m)}$ and $\mathcal{T}U$ are understood in the functional sense defined by Green's formulae (1.37) and (1.40)

$$
\begin{aligned}
&\Big\langle \{\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}\}^+,\{V^{(m)}\}^+\Big\rangle_{\partial\Omega^{(m)}}:=\\
&=\int\limits_{\Omega^{(m)}}A^{(m)}(\partial,\tau)U^{(m)}\cdot V^{(m)}\,dx+\int\limits_{\Omega^{(m)}}\Big[E^{(m)}(u^{(m)},\overline{v^{(m)}})+\\
&\quad+\varrho^{(m)}\tau^2\,u^{(m)}\cdot v^{(m)}+\varkappa^{(m)}_{jl}\,\partial_j u^{(m)}_4\,\overline{\partial_l v^{(m)}_4}+
\end{aligned}
$$

$$+\gamma_{jl}^{(m)}\Big(\tau\, T_0^{(m)}\partial_j u_l^{(m)}\,\overline{v_4^{(m)}} - u_4^{(m)}\,\overline{\partial_j v_l^{(m)}}\Big) + \tau\,\alpha^{(m)}\,u_4^{(m)}\,\overline{v_4^{(m)}}\Big]\,dx, \quad (1.58)$$

$$\big\langle \{\mathcal{T}(\partial,n)U\}^+\,,\,\{V\}^+\big\rangle_{\partial\Omega} := \int\limits_{\Omega} A(\partial,\tau)\,U\cdot V\,dx +$$

$$+\int\limits_{\Omega}\Big[\,E(u,\overline{v}) + \varrho\,\tau^2\,u\cdot v + \gamma_{jl}\,\big(\tau\,T_0\,\partial_j u_l\,\overline{v_4} - u_4\,\overline{\partial_j v_l}\big) +$$

$$+\varkappa_{jl}\,\partial_j u_4\,\overline{\partial_l v_4} + \tau\,\alpha\,u_4\,\overline{v_4} + \varepsilon_{jl}\,\partial_j u_5\,\overline{\partial_l v_5} +$$

$$+e_{lij}\,(\partial_l u_5\,\overline{\partial_i v_j} - \partial_i u_j\,\overline{\partial_l v_5}) - g_l\,\big(\tau\,T_0\,\partial_l u_5\,\overline{v_4} + u_4\,\overline{\partial_l v_5}\big)\Big]\,dx, \quad (1.59)$$

where $V^{(m)} = (v^{(m)}, v_4^{(m)})^\top \in [W^1_{p'}(\Omega^{(m)})]^4$ and $V = (v, v_4, v_5)^\top \in [W^1_{p'}(\Omega)]^5$ are arbitrary vector-functions with $v = (v_1, v_2, v_3)^\top$ and $v^{(m)} = (v_1^{(m)}, v_2^{(m)}, v_3^{(m)})^\top$, while

$$E^{(m)}(u^{(m)}, \overline{v^{(m)}}) = c_{ijlk}^{(m)}\,\partial_i u_j^{(m)}\,\overline{\partial_l v_k^{(m)}}, \quad E(u,\overline{v}) = c_{ijlk}\,\partial_i u_j\,\overline{\partial_l v_k}.$$

Here $\langle\cdot\,,\cdot\rangle_{\partial\Omega^{(m)}}$ (respectively $\langle\cdot\,,\cdot\rangle_{\partial\Omega}$) denotes the duality between the function spaces $[B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(m)})]^4$ and $[B_{p',p'}^{\frac{1}{p}}(\partial\Omega^{(m)})]^4$ (respectively $[B_{p,p}^{-\frac{1}{p}}(\partial\Omega)]^5$ and $[B_{p',p'}^{\frac{1}{p}}(\partial\Omega)]^5$) which extends the usual L_2 inner product

$$\langle\, f\,,\,g\,\rangle_{\mathcal{M}} = \int\limits_{\mathcal{M}}\sum_{j=1}^{N} f_j\,\overline{g_j}\,d\mathcal{M} \ \text{ for } \ f,g\in[L_2(\mathcal{M})]^N,$$

where $\mathcal{M}\in\{\partial\Omega^{(m)}, \partial\Omega\}$.

By standard arguments it can easily be shown that the functionals, from now on called "generalized traces", $\{\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}\}^+ \in [B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(m)})]^4$ and $\{\mathcal{T}(\partial,n)U\}^+ \in [B_{p,p}^{-\frac{1}{p}}(\partial\Omega)]^5$, are well defined by the above relations, provided that $A(\partial,\tau)\,U\in\big[\,L_p(\Omega)\,\big]^5$ and $A^{(m)}(\partial,\tau)\,U^{(m)}\in\big[\,L_p(\Omega^{(m)})\,\big]^4$.

Problem (ICP-B) - the crack gap is thermally and electrically conductive: Find vector-functions

$$U^{(m)} = (u_1^{(m)},\ldots,u_4^{(m)})^\top \ :\ \Omega^{(m)}\to\mathbb{C}^4,$$
$$U = (u_1,\ldots,u_5)^\top \ :\ \Omega\to\mathbb{C}^5$$

belonging respectively to the spaces $[W^1_p(\Omega^{(m)})]^4$ and $[W^1_p(\Omega)]^5$ with $1 < p < \infty$ and satisfying

(i) *the systems of partial differential equations*:

$$\big[\,A^{(m)}(\partial_x,\tau)\,U^{(m)}\,\big]_j = 0 \ \text{ in } \ \Omega^{(m)}, \ \ j=\overline{1,4}, \quad (1.60)$$

$$[\,A(\partial_x,\tau)\,U\,]_k = 0 \ \text{ in } \ \Omega, \ \ k=\overline{1,5}, \quad (1.61)$$

(ii) *the boundary conditions*:

$$r_{S_N^{(m)}}\big\{\big[\,\mathcal{T}^{(m)}(\partial,\nu)\,U^{(m)}\,\big]_j\big\}^+ = Q_j^{(m)} \ \text{ on } \ S_N^{(m)}, \ \ j=\overline{1,4}, \tag{1.62}$$

$$r_{S_N}\big\{\big[\,\mathcal{T}(\partial,n)\,U\,\big]_k\big\}^+ = Q_k \ \text{ on } \ S_N, \ \ k=\overline{1,5}, \tag{1.63}$$

$$r_{S_D}\{u_k\}^+ = f_k \ \text{ on } \ S_D, \ \ k=\overline{1,5}, \tag{1.64}$$

$$r_{\Gamma^{(m)}}\{u_5\}^+ = f_5^{(m)} \ \text{ on } \ \Gamma^{(m)}, \tag{1.65}$$

(iii) *the transmission conditions for* $l=\overline{1,3}$:

$$r_{\Gamma_T^{(m)}}\,\{u_l\}^+ - r_{\Gamma_T^{(m)}}\big\{u_l^{(m)}\big\}^+ = f_l^{(m)} \text{ on } \Gamma_T^{(m)}, \tag{1.66}$$

$$r_{\Gamma_T^{(m)}}\big\{[\mathcal{T}(\partial,n)U]_l\big\}^+ + r_{\Gamma_T^{(m)}}\big\{[\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}]_l\big\}^+ = F_l^{(m)} \text{ on } \Gamma_T^{(m)}, \tag{1.67}$$

$$r_{\Gamma^{(m)}}\,\{u_4\}^+ - r_{\Gamma^{(m)}}\big\{u_4^{(m)}\big\}^+ = f_4^{(m)} \text{ on } \Gamma^{(m)}, \tag{1.68}$$

$$r_{\Gamma^{(m)}}\big\{[\mathcal{T}(\partial,n)U]_4\big\}^+ + r_{\Gamma^{(m)}}\big\{[\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}]_4\big\}^+ = F_4^{(m)} \text{ on } \Gamma^{(m)}, \tag{1.69}$$

(iv) *the interface crack conditions* $\Gamma_C^{(m)}$ *for* $l=\overline{1,3}$:

$$r_{\Gamma_C^{(m)}}\big\{\big[\,\mathcal{T}^{(m)}(\partial,\nu)\,U^{(m)}\,\big]_l\big\}^+ = \widetilde{Q}_l^{(m)} \ \text{ on } \ \Gamma_C^{(m)}, \tag{1.70}$$

$$r_{\Gamma_C^{(m)}}\big\{\big[\,\mathcal{T}(\partial,n)\,U\,\big]_l\big\}^+ = \widetilde{Q}_l \ \text{ on } \ \Gamma_C^{(m)}. \tag{1.71}$$

Problem (ICP-C) - the crack gap is thermally insulated and electrically conductive:

Find vector-functions

$$U^{(m)} = (u_1^{(m)},\dots,u_4^{(m)})^\top \ : \ \Omega^{(m)} \to \mathbb{C}^4,$$
$$U = (u_1,\dots,u_5)^\top \ : \ \Omega \to \mathbb{C}^5$$

belonging respectively to the spaces $[W_p^1(\Omega^{(m)})]^4$ and $[W_p^1(\Omega)]^5$ with $1<p<\infty$ and satisfying

(i) *the systems of partial differential equations*:

$$\big[\,A^{(m)}(\partial_x,\tau)\,U^{(m)}\,\big]_j = 0 \ \text{ in } \ \Omega^{(m)}, \ \ j=\overline{1,4}, \tag{1.72}$$

$$\big[\,A(\partial_x,\tau)\,U\,\big]_k = 0 \ \text{ in } \ \Omega, \ \ k=\overline{1,5}, \tag{1.73}$$

(ii) *the boundary conditions*:

$$r_{S_N^{(m)}}\big\{\big[\,\mathcal{T}^{(m)}(\partial,\nu)\,U^{(m)}\,\big]_j\big\}^+ = Q_j^{(m)} \ \text{ on } \ S_N^{(m)}, \ \ j=\overline{1,4}, \tag{1.74}$$

$$r_{S_N}\big\{\big[\,\mathcal{T}(\partial,n)\,U\,\big]_k\big\}^+ = Q_k \ \text{ on } \ S_N, \ \ k=\overline{1,5}, \tag{1.75}$$

$$r_{S_D}\{u_k\}^+ = f_k \ \text{ on } \ S_D, \ \ k=\overline{1,5}, \tag{1.76}$$

$$r_{\Gamma^{(m)}}\{u_5\}^+ = f_5^{(m)} \ \text{ on } \ \Gamma^{(m)}, \tag{1.77}$$

(iii) *the transmission conditions on* $\Gamma_T^{(m)}$ *for* $j=\overline{1,4}$:

$$r_{\Gamma_T^{(m)}}\{u_j\}^+ - r_{\Gamma_T^{(m)}}\{u_j^{(m)}\}^+ = f_j^{(m)} \text{ on } \Gamma_T^{(m)}, \tag{1.78}$$

$$r_{\Gamma_T^{(m)}}\{[\mathcal{T}(\partial,n)U]_j\}^+ + r_{\Gamma_T^{(m)}}\{[\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}]_j\}^+ = F_j^{(m)} \text{ on } \Gamma_T^{(m)}, \tag{1.79}$$

(iv) *the interface crack conditions on* $\Gamma_C^{(m)}$:

$$r_{\Gamma_C^{(m)}}\{[\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}]_j\}^+ = \widetilde{Q}_j^{(m)} \text{ on } \Gamma_C^{(m)}, \ \ j=\overline{1,4}, \tag{1.80}$$

$$r_{\Gamma_C^{(m)}}\{[\mathcal{T}(\partial,n)U]_j\}^+ = \widetilde{Q}_j \text{ on } \Gamma_C^{(m)}, \ \ j=\overline{1,4}. \tag{1.81}$$

Problem (ICP-D) - the crack gap is heat-conducting dielectric:
Find vector-functions

$$U^{(m)} = (u_1^{(m)},\dots,u_4^{(m)})^\top : \Omega^{(m)} \to \mathbb{C}^4,$$
$$U = (u_1,\dots,u_5)^\top : \Omega \to \mathbb{C}^5$$

belonging respectively to the spaces $[W_p^1(\Omega^{(m)})]^4$ and $[W_p^1(\Omega)]^5$ with $1<p<\infty$ and satisfying

(i) *the systems of partial differential equations*:

$$[A^{(m)}(\partial_x,\tau)U^{(m)}]_j = 0 \text{ in } \Omega^{(m)}, \ \ j=\overline{1,4}, \tag{1.82}$$

$$[A(\partial_x,\tau)U]_k = 0 \text{ in } \Omega, \ \ k=\overline{1,5}, \tag{1.83}$$

(ii) *the boundary conditions*:

$$r_{S_N^{(m)}}\{[\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}]_j\}^+ = Q_j^{(m)} \text{ on } S_N^{(m)}, \ \ j=\overline{1,4}, \tag{1.84}$$

$$r_{S_N}\{[\mathcal{T}(\partial,n)U]_k\}^+ = Q_k \text{ on } S_N, \ \ k=\overline{1,5}, \tag{1.85}$$

$$r_{S_D}\{u_k\}^+ = f_k \text{ on } S_D, \ \ k=\overline{1,5}, \tag{1.86}$$

$$r_{\Gamma_T^{(m)}}\{u_5\}^+ = f_5^{(m)} \text{ on } \Gamma_T^{(m)}, \tag{1.87}$$

(iii) *the transmission conditions for* $l=\overline{1,3}$:

$$r_{\Gamma_T^{(m)}}\{u_l\}^+ - r_{\Gamma_T^{(m)}}\{u_l^{(m)}\}^+ = f_l^{(m)} \text{ on } \Gamma_T^{(m)}, \tag{1.88}$$

$$r_{\Gamma_T^{(m)}}\{[\mathcal{T}(\partial,n)U]_l\}^+ + r_{\Gamma_T^{(m)}}\{[\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}]_l\}^+ = F_l^{(m)} \text{ on } \Gamma_T^{(m)}, \tag{1.89}$$

$$r_{\Gamma^{(m)}}\{u_4\}^+ - r_{\Gamma^{(m)}}\{u_4^{(m)}\}^+ = f_4^{(m)} \text{ on } \Gamma^{(m)}, \tag{1.90}$$

$$r_{\Gamma^{(m)}}\{[\mathcal{T}(\partial,n)U]_4\}^+ + r_{\Gamma^{(m)}}\{[\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}]_4\}^+ = F_4^{(m)} \text{ on } \Gamma^{(m)}, \tag{1.91}$$

(iv) *the interface crack conditions on* $\Gamma_C^{(m)}$:

$$r_{\Gamma_C^{(m)}}\{[\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}]_l\}^+ = \widetilde{Q}_l^{(m)} \text{ on } \Gamma_C^{(m)}, \ \ l=1,2,3, \tag{1.92}$$

$$r_{\Gamma_C^{(m)}}\big\{\big[\mathcal{T}(\partial,n)U\big]_l\big\}^+ = \widetilde{Q}_l \ \text{ on } \ \Gamma_C^{(m)}, \ \ l=1,2,3,5. \tag{1.93}$$

The boundary data in all the above formulated problems satisfy the inclusions (1.56).

1.6. **Uniqueness results.** Here we prove the following uniqueness theorem for $p=2$. The similar uniqueness theorem for $p\neq 2$ will be proved later.

Theorem 1.1. *Let $\Omega^{(m)}$ and Ω be Lipschitz and either $\tau=\sigma+i\omega$ with $\sigma>0$ or $\tau=0$. The above formulated interface crack problems (ICP-A)-(ICP-D) have at most one solution in the space $[W_2^1(\Omega^{(m)})]^4\times[W_2^1(\Omega)]^5$, provided* $\operatorname{mes} S_D>0$.

Proof. It suffices to show that the corresponding homogeneous problems have only the trivial solution. Let a pair $(U^{(m)},U)\in[W_2^1(\Omega^{(m)})]^4\times[W_2^1(\Omega)]^5$ be a solution to one of the above formulated homogeneous interface crack problem.

Green's formulae (1.37) and (1.40) with $V^{(m)}=U^{(m)}$, $V=U$ and $T_0=T_0^{(m)}$ along with the homogeneous boundary and transmission conditions then imply (see Subsection 1.4, formulae (1.38) and (1.41))

$$\begin{gathered}\int\limits_{\Omega^{(m)}}\Big[E^{(m)}(u^{(m)},\overline{u^{(m)}})+\varrho^{(m)}\tau^2|u^{(m)}|^2+\frac{\tau}{|\tau|^2T_0}\varkappa_{lj}^{(m)}\,\partial_l u_4^{(m)}\,\overline{\partial_j u_4^{(m)}}+\\ +\frac{\alpha^{(m)}}{T_0}|u_4^{(m)}|^2\Big]\,dx+\int\limits_{\Omega}\Big[E(u,\overline{u})+\varrho\,\tau^2|u|^2+\frac{\alpha}{T_0}|u_4|^2+\varepsilon_{jl}\,\partial_l u_5\,\overline{\partial_j u_5}+\\ +\frac{\tau}{|\tau|^2T_0}\varkappa_{jl}\,\partial_l u_4\,\overline{\partial_j u_4}-2\mathrm{Re}\,\{g_l\,u_4\,\overline{\partial_l u_5}\}\Big]\,dx=0. \end{gathered}\tag{1.94}$$

Note that due to the relations (1.8), (1.29), and (1.30) we have

$$\begin{gathered}E^{(m)}(u^{(m)},\overline{u^{(m)}})\geqslant 0,\quad \varkappa_{lj}^{(m)}\,\partial_l u_4^{(m)}\,\overline{\partial_j u_4^{(m)}}\geqslant 0,\\ \varkappa_{jl}\,E(u,\overline{u})\geqslant 0,\quad \partial_l u_4\,\overline{\partial_j u_4}\geqslant 0,\quad \varepsilon_{jl}\,\partial_l u_5\,\overline{\partial_j u_5}\geqslant 0\end{gathered}\tag{1.95}$$

with the equality only for complex rigid displacement vectors, constant temperature distributions and a constant electric potential field,

$$\begin{gathered}u^{(m)}=a^{(m)}\times x+b^{(m)},\ \ u_4^{(m)}=a_4^{(m)},\\ u=a\times x+b,\ \ u_4=a_4,\ \ u_5=a_5,\end{gathered}\tag{1.96}$$

where $a^{(m)},b^{(m)},a,b\in\mathbb{C}^3$, $a_4^{(m)},a_4,a_5\in\mathbb{C}$, and $\times$ denotes the usual cross product of two vectors.

Take into account the above inequalities and separate the real and imaginary parts of (1.94) to obtain

$$\int\limits_{\Omega^{(m)}}\Big[E^{(m)}(u^{(m)},\overline{u^{(m)}})+\varrho^{(m)}(\sigma^2-\omega^2)|u^{(m)}|^2+$$

$$+\frac{\alpha^{(m)}}{T_0}\,|u_4^{(m)}|^2+\frac{\sigma}{|\tau|^2\,T_0}\,\varkappa_{lj}^{(m)}\,\partial_l u_4^{(m)}\,\overline{\partial_j u_4^{(m)}}\,\Big]\,dx+$$

$$+\int\limits_{\Omega}\Big[\,E(u,\overline{u})+\varrho\,(\sigma^2-\omega^2)\,|u|^2+\frac{\alpha}{T_0}\,|u_4|^2+$$

$$+\frac{\sigma}{|\tau|^2\,T_0}\,\varkappa_{jl}\,\partial_l u_4\,\overline{\partial_j u_4}-2\mathrm{Re}\,\{g_l\,u_4\,\overline{\partial_l u_5}\}+\varepsilon_{jl}\,\partial_l u_5\,\overline{\partial_j u_5}\Big]\,dx=0, \qquad (1.97)$$

$$\int\limits_{\Omega^{(m)}}\Big[\,2\,\varrho^{(m)}\,\sigma\,\omega\,|u^{(m)}|^2+\frac{\omega}{|\tau|^2\,T_0}\,\varkappa_{lj}^{(m)}\partial_l u_4^{(m)}\,\overline{\partial_j u_4^{(m)}}\,\Big]\,dx+$$

$$+\int\limits_{\Omega}\Big[\,2\,\varrho\,\sigma\,\omega\,|u|^2+\frac{\omega}{|\tau|^2\,T_0}\,\varkappa_{jl}\,\partial_l u_4\,\overline{\partial_j u_4}\,\Big]\,dx=0. \qquad (1.98)$$

First, let us assume that $\sigma>0$ and $\omega\neq 0$. With the help of the homogeneous boundary and transmission conditions we easily derive from (1.98) that $u_j^{(m)}=0$ in $\Omega^{(m)}$ and $u_j=0$ in Ω, $j=\overline{1,4}$. From (1.97) we then conclude $u_5=const$ in Ω, whence $u_5=0$ in Ω follows due to the homogeneous boundary condition on S_D.

Thus $U^{(m)}=0$ in $\Omega^{(m)}$ and $U=0$ in Ω.

The proof for the case $\sigma>0$ and $\omega=0$ is quite similar. The only difference is that now, in addition to the above relations, we have to apply the inequality in (1.31) as well.

For $\tau=0$, by adding the relations (1.42) and (1.43) with c/T_0 for c_1 and c, we arrive at the equality

$$\int\limits_{\Omega^{(m)}}\Big[E^{(m)}(u^{(m)},\overline{u^{(m)}})+\frac{c}{T_0}\,\varkappa_{lj}^{(m)}\partial_l u_4^{(m)}\overline{\partial_j u_4^{(m)}}-\gamma_{jl}^{(m)}\,u_4^{(m)}\,\overline{\partial_j u_j^{(m)}}\,\Big]\,dx+$$

$$+\int\limits_{\Omega}\Big[\,E(\,u,\overline{u}\,)+\frac{c}{T_0}\,\varkappa_{jl}\,\partial_l u_4\,\overline{\partial_j u_4}-\gamma_{jl}\,u_4\,\overline{\partial_l u_j}-g_l\,\overline{u_4}\,\partial_l u_5+$$

$$+\,\varepsilon_{jl}\,\partial_l u_5\,\overline{\partial_j u_5}\Big]\,dx=0, \qquad (1.99)$$

where c is an arbitrary constant parameter.

Dividing the equality by c and sending c to infinity we conclude that $u_4^{(m)}=0$ in $\Omega^{(m)}$ and $u_4=0$ in Ω due to the homogeneous boundary and transmission conditions for the temperature distributions. In view of (1.99), this easily yields that $U^{(m)}=0$ in $\Omega^{(m)}$ and $U=0$ in Ω due to the homogeneous boundary conditions on S_D. □

Note that for $\tau=i\,\omega$ (i.e., for $\sigma=0$ and $\omega\neq 0$) the homogeneous problem may possess a nontrivial solution, in general. These values of the frequency parameter ω correspond to resonance regimes and the corresponding exterior steady state oscillation problems need special consideration related to generalized Sommerfeld radiation conditions (cf. [29, 30, 51]).

Below we apply the potential method and the theory of pseudodifferential equations to study the existence of solutions to the pseudo-oscillation problems in different function spaces and to establish their regularity properties.

2. Pseudodifferential Equations and Local Principle

In the present section, for the readers convenience we collect some results from the theory of pseudodifferential equations which we need in the study of the above formulated mixed transmission-boundary vale problems. Note that some results exposed below are known and are dispersed in scientific papers (for details and historical notes see e.g. [12, 20]), but some results, in particular, Theorem 2.31 is new and plays a crucial role in our analysis.

2.1. ΨDOs: definition and basic properties.

Let $\mathbb{S}(\mathbb{R}^n)$ denote the Schwartz space of rapidly decaying smooth functions endowed with the semi-norms

$$\mathbb{S}(\mathbb{R}^n)=\Big\{f\in C^\infty(\mathbb{R}^n):\ p_m(f)<\infty,\ m=0,1,\dots\Big\},$$
$$p_m(f):=\sup_{x\in\mathbb{R}^n}\langle x\rangle^m\sum_{|\alpha|\leqslant m}|\partial^\alpha f(x)|,\ \ \langle x\rangle:=(1+|x|^2)^{1/2}. \tag{2.1}$$

The dual space $\mathbb{S}'(\mathbb{R}^n)$ to $\mathbb{S}(\mathbb{R}^n)$, the Frechet space of functionals over $\mathbb{S}(\mathbb{R}^n)$, is known as the space of tempered distributions.

It is well known that the Frechet spaces $\mathbb{S}(\mathbb{R}^n)$ and $\mathbb{S}'(\mathbb{R}^n)$ are both invariant with respect to the Fourier direct and inverse transforms

$$\mathcal{F}^{\pm1}\ :\ \mathbb{S}(\mathbb{R}^n)\to\mathbb{S}(\mathbb{R}^n)\quad\text{and}\quad\mathcal{F}^{\pm1}\ :\ \mathbb{S}'(\mathbb{R}^n)\to\mathbb{S}'(\mathbb{R}^n), \tag{2.2}$$

which are continuous operators there. For absolutely integrable functions on $\mathbb{R}^n$ they are defined as follows

$$\mathcal{F}_{x\to\xi}[f]=\int\limits_{\mathbb{R}^n}f(x)\,e^{ix\xi}\,dx,\quad \mathcal{F}^{-1}_{\xi\to x}[g]=\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}g(\xi)\,e^{-ix\xi}\,d\xi.$$

A partial differential operator (PDO)

$$\mathbf{P}(x,D)=\sum_{|\alpha|\leqslant m}a_\alpha(x)D^\alpha=\sum_{|\alpha|\leqslant m}a_\alpha(x)(i\partial)^\alpha \tag{2.3}$$

with scalar or matrix coefficients $a_\alpha(x)$, can also be written as follows

$$\mathbf{P}(x,D_x)=\mathcal{F}^{-1}_{\xi\mapsto x}P(x,\xi)\mathcal{F}_{y\mapsto\xi}, \tag{2.4}$$

where $D=i\partial=(i\partial_1,i\partial_2,i\partial_3)$,

$$\mathcal{P}(x,\xi):=\sum_{|\alpha|\leqslant m}a_\alpha(x)\xi^\alpha \tag{2.5}$$

is the characteristic polynomial or the symbol of the operator $\mathbf{P}(x,D)$.

In this section, sometimes we do not distinguish between scalar and vector spaces of functions when it does not lead to misunderstanding and it is clear from the context which space is appropriate for the operators under consideration.

Another class of similar operators are convolutions: With a given tempered distribution $a \in \mathbb{S}'(\mathbb{R}^n)$ we associate the convolution operator

$$\mathbf{a}(D)\varphi = W_a^0\varphi := \mathcal{F}^{-1}a\mathcal{F}\varphi \quad \text{for} \quad \varphi \in \mathbb{S}(\mathbb{R}^n), \tag{2.6}$$

which is a bounded transform

$$W_a^0 \,:\, \mathbb{S}(\mathbb{R}^n) \to \mathbb{S}'(\mathbb{R}^n). \tag{2.7}$$

Indeed, $\mathbf{a}(D) = W_a^0$ represents a composition of three bounded operators (see (2.2)):

$$\begin{aligned} \mathcal{F} \,&:\, \mathbb{S}(\mathbb{R}^n) \to \mathbb{S}(\mathbb{R}^n),\\ aI \,&:\, \mathbb{S}(\mathbb{R}^n) \to \mathbb{S}'(\mathbb{R}^n),\\ \mathcal{F}^{-1} \,&:\, \mathbb{S}'(\mathbb{R}^n) \to \mathbb{S}'(\mathbb{R}^n). \end{aligned}$$

The distribution $a \in \mathbb{S}'(\mathbb{R}^n)$ is referred to as the symbol of W_a^0.
For the composition of convolution operators we have:

$$W_a^0 W_b^0 = W_b^0 W_a^0 = W_{ab}^0 \tag{2.8}$$

whenever $a, b \in \mathbb{S}'(\mathbb{R}^n)$ and the product $ab = ba$ is a well defined distribution $ab \in \mathbb{S}'(\mathbb{R}^n)$.

Indeed, $\mathcal{F}W_b^0\varphi = \mathcal{F}\mathcal{F}^{-1}b\mathcal{F}\varphi = b\mathcal{F}\varphi$ for all $\varphi \in \mathbb{S}(\mathbb{R}^n)$. The product $ab\mathcal{F}\varphi$ is well defined since $ab \in \mathbb{S}'(\mathbb{R}^n)$ and $\mathcal{F}\varphi \in \mathbb{S}(\mathbb{R}^n)$. Moreover, $ab\mathcal{F}\varphi \in \mathbb{S}'(\mathbb{R}^n)$ and the final result follows

$$W_a^0 W_b^0\varphi = \mathcal{F}^{-1}[ab\mathcal{F}\varphi] = W_{ab}^0\varphi \in \mathbb{S}'(\mathbb{R}^n) \;\text{ for }\; \varphi \in \mathbb{S}(\mathbb{R}^n).$$

Equalities (2.4) and (2.6) demonstrate a similarity of PDOs and convolution operators and justifies the following preliminary definition of a pseudodifferential operator

$$\begin{aligned} \mathbf{a}(x, D)u(x) &:= \mathcal{F}^{-1}_{\xi\mapsto x}\big\{a(x,\xi)\mathcal{F}_{y\mapsto\xi}[u(y)]\big\} = \\ &= \int\limits_{\mathbb{R}^n} e^{-ix\xi} a(x,\xi)(\mathcal{F}u)(\xi)\,đ\xi\,, \quad u \in \mathbb{S}(\mathbb{R}^n)\,, \end{aligned} \tag{2.9}$$

where

$$đ\xi := \frac{1}{(2\pi)^n}\, d\xi,$$

To make the definition (2.9) rigorous, we formulate conditions on the symbol a.

Definition 2.1. For $m \in \mathbb{R}$ the notation $\mathbb{S}^m(\mathbb{R}^n \times \mathbb{R}^n) = \mathbb{S}^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ refers to the Hörmander class of functions $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ which admit the following estimate

$$\big|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\big| \leqslant C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}, \tag{2.10}$$

for all $x, \xi \in \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{N}_0^n$ with $\mathbb{N}_0 := \{0, 1, 2, 3, \cdots\}$.

Definition 2.2. Let $a \in \mathbb{S}^m(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$. The operator $\mathbf{a}(x, D)$ in (2.9) is a pseudodifferential operator (abbreviation-ΨDO) of order m and $a(x, \xi)$ is the symbol of $\mathbf{a}(x, D)$.

The notation $\mathbb{OPS}^m$ refers to the set of all ΨDOs with symbols from the class $\mathbb{S}^m(\mathbb{R}^n \times \mathbb{R}^n)$.

In what follows, for the symbol of a pseudodifferential operator $\mathbf{a}(x, D) \in \mathbb{OPS}^m$ we will use also another notation $\mathfrak{S}_{\mathbf{a}} \in \mathbb{S}^m(\mathbb{R}^n \times \mathbb{R}^n)$ and we write

$$\mathbf{a} = \mathcal{F}^{-1}_{\xi\mapsto x}\mathfrak{S}_{\mathbf{a}}(x, \xi)\mathcal{F}_{y\mapsto\xi}.$$

A simplest boundedness result for a ΨDO is the following.

Proposition 2.3. *Let $m \in \mathbb{R}$, $a \in \mathbb{S}^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $\mathbb{S}(\mathbb{R}^n)$ be the Fréchet-Schwartz space of fast decaying test functions. The corresponding ΨDO is a bounded operator in the space of fast decaying Fréchet-Schwartz test functions*

$$\mathbf{a}(x, D) \ : \ \mathbb{S}(\mathbb{R}^n) \to \mathbb{S}(\mathbb{R}^n)$$

and in the dual space of tempered distributions

$$\mathbf{a}(x, D) \ : \ \mathbb{S}'(\mathbb{R}^n) \to \mathbb{S}'(\mathbb{R}^n).$$

For the proof we refer to the monographs on ΨDOs, e.g. to [28, vol. 3, Theorem 18.1.6, Theorem 18.1.7].

For a rigorous definition of a ΨDOs one can apply oscillatory integrals.

Let us consider a special cut off function $\chi(\,\cdot\,,\,\cdot\,) \in C_0^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, $\chi(x, \xi) = 1$ in some neighborhood of the diagonal $x = \xi$. Let $g \in \mathbb{S}^m(\mathbb{R}^n \times \mathbb{R}^n)$. If the limit

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^n_\xi}\int\limits_{\mathbb{R}^n_x}\chi(\varepsilon x, \varepsilon\xi)g(x, \xi)e^{\pm ix\xi}dx\,đ\xi\,,$$

exists, it is called the oscillatory integral and is denoted by $\mathrm{Os}(g(x, \xi)e^{\pm ix\xi})$

Note, that the oscillatory integral for $g \in \mathbb{L}_1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ coincides with the usual one

$$\mathrm{Os}(g(x, \xi)e^{\pm ix\xi}) = \int\limits_{\mathbb{R}^n_\xi}\int\limits_{\mathbb{R}^n_x}g(x, \xi)e^{\pm ix\xi}dx\,đ\xi\,.$$

Proposition 2.4. *For arbitrary $a \in \mathbb{S}^m(\mathbb{R}^n \times \mathbb{R}^n)$ the oscillatory integral $\mathrm{Os}(a(x, \xi)e^{\pm ix\xi})$ exists and is independent of the choice of a cut-off function $\chi(x, \xi)$.*

For the proof we refer, e.g., to the monograph [67, § 1].

Let $a \in \mathbb{S}^m(\mathbb{R}^n \times \mathbb{R}^n)$. The corresponding ΨDO

$$\mathbf{a}(x, D)u(x) := \int\limits_{\mathbb{R}^n}e^{-ix\xi}a(x, \xi)(\mathcal{F}u)(\xi)đ\xi =$$

$$= \int\limits_{\mathbb{R}^n_\xi}\int\limits_{\mathbb{R}^n_y}e^{i\xi(y-x)}a(x, \xi)u(y)đ\xi\,dy =$$

$$= \int\limits_{\mathbb{R}^n_\xi} \int\limits_{\mathbb{R}^n_y} e^{iy\xi} a(x,\xi) u(x+y) đ\xi \, dy \,, \quad u \in \mathbb{S}(\mathbb{R}^n) \,. \tag{2.11}$$

exists as a x-parameter dependent oscillatory integral:

$$\mathbf{a}(x,D)u(x) = \mathrm{Os}(e^{\pm iy\xi}\mathbf{a}(x,\xi)u(x+y)) \,, \quad x \in \mathbb{R}^n \,. \tag{2.12}$$

The oscillatory integral in (2.11) extends to all smooth functions with polynomial growth at infinity

$$u \in \mathbb{C}^\infty_{\mathrm{POL}}(\mathbb{R}^n) := \\ := \Big\{ v \in C^\infty(\mathbb{R}^n) : \ \langle x \rangle^{-N_{v,\alpha}} |\partial^\alpha_x v(x)| \leqslant M_{v,\alpha} < \infty, \ N_{v,\alpha} \in \mathbb{N}_0 \Big\}.$$

Let the dotted Euclidean space $\dot{\mathbb{R}}^n$ denote the one point compactification of $\mathbb{R}^n$ with neighborhoods of infinity, defined as the complementary domains $U^c := \mathbb{R}^n \backslash \overline{U}$ to compact domains $U \subset \mathbb{R}^n$. The Hörmander class of symbols $\mathbb{S}^m(\dot{\mathbb{R}}^n \times \mathbb{R}^n)$ consists of those functions $a(x,\xi)$ from $\mathbb{S}^m(\mathbb{R}^n \times \mathbb{R}^n)$, which have limits $\lim\limits_{|x|\to\infty} a(x,\xi)$ uniformly with respect to $\xi \in \mathbb{R}^n$.

Proposition 2.5 (Calderon–Vaillancourt). *Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $m, s \in \mathbb{R}$ and $a \in \mathbb{S}^m(\dot{\mathbb{R}}^n \times \mathbb{R}^n)$. Then the ΨDOs*

$$\begin{aligned} \mathbf{a}(x,D) \ &: \ H^s_p(\mathbb{R}^n) \to H^{s-m}_p(\mathbb{R}^n), \\ &: \ B^s_{p,q}(\mathbb{R}^n) \to B^{s-m}_{p,q}(\mathbb{R}^n) \end{aligned} \tag{2.13}$$

are bounded.

For the proof we refer to [60] and, for the case $p = 2$, the monographs [28, vol. 3, Theorem 18.1.13] and [67, § 7].

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the Lipschitz boundary $S := \partial\Omega \neq \varnothing$ and r_Ω be the restriction operator to the domain Ω. Let ℓ_Ω be the extension by 0 from Ω to $\mathbb{R}^n$. Then, by definition,

$$\widetilde{H}^s_p(\Omega) := \big\{ \varphi \in H^s_p(\mathbb{R}^n) : \ \mathrm{supp}\, \varphi \subset \overline{\Omega} \big\}. \tag{2.14}$$

$\widetilde{H}^s_p(\Omega)$ is a subspace of $H^s_p(\mathbb{R}^n)$ and inherits the norm from the ambient space.

The space $H^s_p(\Omega)$ represents restrictions of functions from $H^s_p(\mathbb{R}^n)$, i.e., $H^s_p(\Omega) := r_\Omega H^s_p(\mathbb{R}^n)$. The space is endowed with the norm of the factor-space $H^s_p(\mathbb{R}^n)/\widetilde{H}^s_p(\Omega^c)$, where $\Omega^c := \mathbb{R}^n \setminus \overline{\Omega}$ is the complemented domain to Ω. It means that the norm of $\varphi \in H^s_p(\Omega)$ is defined by the equality

$$\|\varphi\|_{H^s_p(\Omega)} \equiv \big\|\varphi | H^s_p(\Omega)\big\| := \inf_{\mathrm{ext}_\Omega \varphi \in H^s_p(\mathbb{R}^n)} \big\| \mathrm{ext}_\Omega\, \varphi | H^s_p(\mathbb{R}^n)\big\|, \tag{2.15}$$

where ext_Ω is an extension operator from Ω to $\mathbb{R}^n$ preserving the space.

The spaces $\widetilde{B}^s_{p,q}(\Omega)$ and $B^s_{p,q}(\Omega)$ are defined Similarly.

Let $\mathbb{S}^m(\dot{\Omega}\times\mathbb{R}^n)$ denote the Hórmander class $\mathbb{S}^m(\Omega\times\mathbb{R}^n)$ if Ω is compact and consist of those functions $a(x,\xi)$ from $\mathbb{S}^m(\Omega\times\mathbb{R}^n)$, which have limits $\lim\limits_{x\in\Omega,\ |x|\to\infty} a(x,\xi)$ uniformly with respect to $\xi\in\mathbb{R}^n$ if Ω is unbounded.

Corollary 2.6. *Let $1<p<\infty$, $1\leqslant q\leqslant\infty$, $m,s\in\mathbb{R}$ and $a\in\mathbb{S}^m(\dot{\Omega}\times\mathbb{R}^n)$. Then the ΨDO*

$$\begin{aligned} r_\Omega\mathbf{a}(x,D) \ &: \ \widetilde{H}^s_p(\Omega)\to H^{s-m}_p(\Omega),\\ &: \ \widetilde{B}^s_{p,q}(\Omega)\to B^{s-m}_{p,q}(\Omega), \end{aligned} \tag{2.16}$$

is bounded. Moreover, if the symbol of the ΨDO $\mathbf{a}(x,D)$ is a rational function in ξ and $s>-1/p$, then the operators

$$\begin{aligned} r_\Omega\mathbf{a}(x,D)\ell_\Omega \ &: \ H^s_p(\Omega)\to H^{s-m}_p(\Omega),\\ &: \ B^s_{p,q}(\Omega)\to B^{s-m}_{p,q}(\Omega), \end{aligned} \tag{2.17}$$

are bounded as well.

The boundedness result (2.16) is a direct consequence of Proposition 1.8 and the above definition of the spaces. The boundedness result (2.17) follows as a particular case of the boundedness result for ΨDOs which possess the transmission property, because the ΨDOs with rational symbols have the transmission property (see [2, 3, 22, 26]).

Corollary 2.7. *If $a\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ has a compact support in the second variable, i.e., $a(x,\xi)=0$ for $|\xi|>M$ for some $M>0$ and all $x\in\mathbb{R}^n$, the ΨDO $\mathbf{a}(x,D)$ is infinitely smoothing and maps the spaces*

$$\begin{aligned} \mathbf{a}(x,D) \ &: \ H^s_p(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n),\\ &: \ B^s_{p,q}(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n) \end{aligned} \tag{2.18}$$

for all $1<p<\infty$, $1\leqslant q\leqslant\infty$ and $s\in\mathbb{R}$.

Definition 2.8. A symbol $a\in\mathbb{S}^m(\mathbb{R}^n\times\mathbb{R}^n)$ will be referred to as a classical of order m and the corresponding ΨDO-a classical ΨDO if there exist homogeneous symbols

$$a_j(x,\lambda\xi)=\lambda^{m-j}a_j(x,\xi) \ \ \forall\,\lambda>0, \ \ j=0,1,\dots,$$

such that for arbitrary non-negative integer $N\in\mathbb{N}_0$, the remainder term

$$a^0_{N+1}(x,\xi):=a(x,\xi)-\sum_{j=0}^{N}\psi(\xi)a_j(x,\xi)$$

satisfies the inclusion $a^0_{N+1}\in\mathbb{S}^{m-N-1}(\mathbb{R}^n\times\mathbb{R}^n)$, where $\psi\in C^\infty(\mathbb{R}^n)$ is a smooth cut off function: $\psi(\xi)=1$ for $|\xi|>1$ and $\psi(\xi)=0$ for $|\xi|<\frac{1}{2}$.

In such a case we write

$$a(x,\xi)\sim\sum_{j=0}^{\infty}a_j(x,\xi) \tag{2.19}$$

and call the leading homogeneous of order m term in the asymptotic expansion $a_{\rm pr}(x,\xi)=a_0(x,\xi)$ the principal symbol of $\mathbf{a}(x,D)$.

Denote the class of above introduced symbols by $\mathbb{CLS}^m(\mathbb{R}^n\times\mathbb{R}^n)$.

An important subclass of ΨDOs is the algebra of all partial differential operators (PDOs) with C^∞–smooth $N\times N$ matrix coefficients. If all derivatives of coefficients of a PDO of order m are uniformly bounded, symbol of such PDO belongs obviously to the class $\mathbb{S}^m(\mathbb{R}^n\times\mathbb{R}^n)$.

An important subclass of ΨDOs are elliptic operators:

Definition 2.9. A ΨDO $\mathbf{a}(x,D)$ with a symbol $a(x,\xi)$ in $\mathbb{S}^m(\mathbb{R}^n\times\mathbb{R}^n)$ is called elliptic if

$$\lim_{R\to\infty}\inf_{|\xi|\geqslant R}\frac{|\det\, a(x,\xi)|}{\langle\xi\rangle^m}\neq 0\ \ \forall\, x\in\mathbb{R}^n. \tag{2.20}$$

The most important property of elliptic operators is the existence of a parametrix.

Definition 2.10. An operator $\mathbf{R}(x,D)$ is called a parametrix for a ΨDO $\mathbf{a}(x,D)$ with a symbol $a(x,\xi)$ in $\mathbb{S}^m(\mathbb{R}^n\times\mathbb{R}^n)$ if

$$\mathbf{R}(x,D)\,\mathbf{a}(x,D)=I+\mathbf{T}_1(x,D),\quad \mathbf{a}(x,D)\,\mathbf{R}(x,D)=I+\mathbf{T}_2(x,D), \tag{2.21}$$

where $\mathbf{T}_1(x,D),\mathbf{T}_2(x,D)$ are infinitely smoothing operators and map spaces $\mathbf{T}_1(x,D),\mathbf{T}_2(x,D)\ :\ H^s_p(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n)$ for arbitrary $1<p<\infty$ and $s\in\mathbb{R}$.

For a compact manifold $\mathcal{M}$ existence of a parametrix implies that $\mathbf{a}(x,D)$ is a Fredholm operator (see Theorem 2.22), while in the case of $\mathbb{R}^n$ it helps, for example, to prove local regularity of a solution.

Proposition 2.11. *Let $m\in\mathbb{R}$ and $a\in\mathbb{S}^m(\dot{\mathbb{R}}^n\times\mathbb{R}^n)$ be elliptic. Then the ΨDO $\mathbf{a}(x,D)$ has a parametrix.*

We drop the proof and refer the reader for details to [28, vol. 3, § 18.1], [67, S 5.4] and [60].

Very important subclass of pseudodifferential operators are differential operators with C^∞-smooth uniformly bounded matrix coefficients

$$\mathbf{A}(x,D):=\sum_{|\alpha|\leqslant m}a_\alpha(x)\partial^\alpha,\ \ a_\alpha\in C^\infty(\mathbb{R}^n). \tag{2.22}$$

Assume that the differential operator $\mathbf{A}(x,D)$ in (2.22) has a fundamental solution $\mathcal{K}_{\mathbf{A}}(x,y)$

$$\mathbf{A}(x,D)\mathcal{K}_{\mathbf{A}}(x,y)=\delta(x-y)\,I,\ \ x,y\in\mathbb{R}^n. \tag{2.23}$$

Then this operator has the inverse, written with the help of the fundamental solution

$$\mathbf{A}^{-1}(x,D)\varphi(x):=\int_{\mathbb{R}^n}\mathcal{K}_{\mathbf{A}}(x,y)\varphi(y)\,dy,\ \ \varphi\in C^\infty_0(\mathbb{R}^n). \tag{2.24}$$

If the operator $\mathbf{A}(x,D)=\mathbf{A}(D)$ has constant coefficients and is not zero identically, the fundamental solution $\mathcal{K}_{\mathbf{A}}(x,y)=\mathcal{K}_{\mathbf{A}}(x-y)$ exists and depends on the difference of variables (see, e.g., [28, §10, Theorem 10.2.1]). Moreover, inverse $\mathcal{A}^{-1}(\xi)$ of the symbol

$$\mathcal{A}(\xi)=\sum_{|\alpha|\leqslant m}a_\alpha(-i)^{|\alpha|}\xi^\alpha,\ \ \xi\in\mathbb{R}^n,$$

of elliptic PDO $\mathbf{A}(D)$ with constant coefficients and the Fourier transform of the fundamental solution are equal $\mathcal{A}^{-1}(\xi)=\mathcal{F}_{x\to\xi}[\mathcal{K}_{\mathbf{A}}(x)]$. Thus, the inverse operator $\mathbf{A}^{-1}(x,D)$ looks like a ΨDO, but does not belong to the class of ΨDOs defined above, since the symbol $\mathcal{A}^{-1}(x,\xi)$ in elliptic case might be unbounded, but is bounded for $|\xi|>M_{\mathbf{A}}$ if $M_{\mathbf{A}}$ is sufficiently large. Let us consider a C^∞-smooth cut off function $\psi(x)=0$ for $|\xi|<M_{\mathbf{A}}$ and $\psi(x)=1$ for $|\xi|>M_{\mathbf{A}}+1$. Then

$$\begin{gathered}\mathcal{A}^{-1}(x,\xi):=\mathcal{A}_0^{-1}(x,\xi)+\mathcal{A}_1^{-1}(x,\xi),\\ \mathcal{A}_0^{-1}(x,\xi):=\psi(\xi)\mathcal{A}^{-1}(x,\xi),\quad \mathcal{A}_1^{-1}(x,\xi)=[1-\psi(\xi)]\mathcal{A}^{-1}(x,\xi),\end{gathered}\tag{2.25}$$

where $\mathcal{A}_0^{-1}\in\mathbb{S}^m(\mathbb{R}^n\times\mathbb{R}^n)$, while $\mathcal{A}_1^{-1}(x,\xi)$ has a compact support in ξ. The corresponding ΨDO $\mathbf{A}_1(x,D)$ is infinitely smoothing, like the operator in (2.18) (cf. Corollary 2.7).

Definition 2.12. For $m\in\mathbb{R}$ by $\widetilde{\mathbb{S}}^m(\mathbb{R}^n\times\mathbb{R}^n)$ denote the extension of Hörmander's class of symbols

$$a(x,\xi)=a_0(x,\xi)+a_1(x,\xi),\ \ a_0\in\mathbb{S}^m(\mathbb{R}^n\times\mathbb{R}^n),\tag{2.26}$$

where the symbol $a_1(x,\xi)$ is such that the corresponding ΨDO $\mathbf{a}_1(x,D)$, defined by equality (2.11), is infinitely smoothing

$$\begin{aligned}\mathbf{a}_1(x,D)\ &:\ H^s_p(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n),\\ &:\ B^s_{p,q}(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n)\end{aligned}\tag{2.27}$$

for all $1<p<\infty$, $1\leqslant q\leqslant\infty$ and $s\in\mathbb{R}$.

The Freshet space $H^s_{p,\mathrm{loc}}(\Omega)$ on a non-compact domain Ω (including the case $\Omega=\mathbb{R}^n$) is defined as the space of functions φ which belong to $H^s_p(\Omega)$ locally: $\chi\varphi\in H^s_p(\Omega)$ for all $\chi\in C^\infty_0(\Omega)$.

The Freshet spaces $H^s_{p,\mathrm{com}}(\Omega)$ on a non-compact domain Ω (including the case $\Omega=\mathbb{R}^n$) is defined as the subspace of $H^s_p(\Omega)$ consisting of all functions with compact supports.

If Ω is compact, then evidently $H^s_{p,\mathrm{com}}(\Omega)=H^s_{p,\mathrm{loc}}(\Omega)=H^s_p(\Omega)$.

The spaces $B^s_{p,q,\mathrm{loc}}(\Omega)$, $B^s_{p,q,\mathrm{com}}(\Omega)$, $\widetilde{H}^s_{p,\mathrm{com}}(\Omega)$, $\widetilde{B}^s_{p,q,\mathrm{com}}(\Omega)$ are defined similarly.

Corollary 2.13. *Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $m, s \in \mathbb{R}$ and $a \in \widetilde{\mathbb{S}}^m(\Omega \times \mathbb{R}^n)$. Then the ΨDOs*

$$\begin{aligned} r_\Omega \mathbf{a}(x,D) \ &: \ \widetilde{H}^s_{p,\mathrm{com}}(\Omega) \to H^{s-m}_{p,\mathrm{loc}}(\Omega), \\ &: \ \widetilde{B}^s_{p,q,\mathrm{com}}(\Omega) \to B^{s-m}_{p,q,\mathrm{loc}}(\Omega), \end{aligned} \tag{2.28}$$

are bounded. Moreover, if the symbol of the ΨDO $\mathbf{a}(x,D)$ is a rational function in ξ and $s > -1/p$, then the operators

$$\begin{aligned} r_\Omega \mathbf{a}(x,D)\ell_\Omega \ &: \ H^s_{p,\mathrm{com}}(\Omega) \to H^{s-m}_{p,\mathrm{loc}}(\Omega), \\ &: \ B^s_{p,q,\mathrm{com}}(\Omega) \to B^{s-m}_{p,q,\mathrm{loc}}(\Omega), \end{aligned} \tag{2.29}$$

are bounded as well.

If $a \in \widetilde{\mathbb{S}}^m(\overset{\bullet}{\Omega} \times \mathbb{R}^n)$, then the ΨDOs

$$\begin{aligned} r_\Omega \mathbf{a}(x,D) \ &: \ \widetilde{H}^s_p(\Omega) \to H^{s-m}_{p,\mathrm{loc}}(\Omega), \\ &: \ \widetilde{B}^s_{p,q}(\Omega) \to B^{s-m}_{p,q,\mathrm{loc}}(\Omega), \end{aligned} \tag{2.30}$$

are bounded. Moreover, if the symbol of the ΨDO $\mathbf{a}(x,D)$ is a rational function in ξ and $s > -1/p$, then the operators

$$\begin{aligned} r_\Omega \mathbf{a}(x,D)\ell_\Omega \ &: \ H^s_p(\Omega) \to H^{s-m}_{p,\mathrm{loc}}(\Omega), \\ &: \ B^s_{p,q}(\Omega) \to B^{s-m}_{p,q,\mathrm{loc}}(\Omega). \end{aligned} \tag{2.31}$$

are bounded as well.

2.2. **ΨDOs on manifolds.** Let us proceed by the definition of a manifold.

Definition 2.14. A topological space $\mathcal{M}$ is called a closed manifold (or a manifold without boundary $\partial\mathcal{M} = \varnothing$) if it is covered by a finite number of coordinate patches $\mathcal{M} = \bigcup_{j=1}^M U_j$ which are homeomorphic to subsets in $\mathbb{R}^m$

$$\varkappa_j \ : \ V_j \to U_j, \quad V_j \subset \mathbb{R}^m, \ \ j = 1, \dots, M. \tag{2.32}$$

Here m is the dimension of $\mathcal{M}$, $\varkappa_j$ are called coordinate homeomorphisms, the pairs $\{U_j, \varkappa_j\}$-the coordinate charts and the collection $\big\{\{U_j, \varkappa_j\}\big\}_{j=1}^M$-the coordinate atlas.

If all domains $V_1, \dots, V_M$ in (2.32) are compact (bounded), then $\mathcal{M}$ is a compact manifold.

If $\mathcal{X} = \varkappa_j(x) \in \mathcal{M}$, the Euclidean coordinates of $x \in \mathbb{R}^n$ are called the Cartesian coordinates of $\mathcal{X} \in \mathcal{M}$.

If $\varkappa_j^{-1} \circ \varkappa_k \in C^\mu(V_j \cap V_k)$ for some $0 < \mu < \infty$ and for all pairs (j,k) whenever $V_j \cap V_k \neq \varnothing$, manifold $\mathcal{M}$ is μ-smooth. C^∞-smooth manifold is called smooth.

Two coordinate atlases $\{U_j, \varkappa_j\}$ and $\{\widetilde{U}_j, \widetilde{\varkappa}_j\}$ on a manifold $\mathcal{M}$ are equivalent if there exists a third atlas $\{\widehat{U}_j, \widehat{\varkappa}_j\}$ on $\mathcal{M}$, which contains both atlases (or, the merged set $\{U_j, \varkappa_j\} \bigcup \{\widetilde{U}_j, \widetilde{\varkappa}_j\}$ is an atlas on $\mathcal{M}$).

A compact manifold without boundary will be referred to as Manifold.

For the definition of a function spaces on $\mathcal{M}$, including the spaces of distributions, and for many other purposes it is convenient to have a partition of unity $\{\psi_j\}_{j=1}^M$ subordinated to a given covering $\{U_j\}_{j=1}^M$:

$$\sum_{j=1}^{M} \psi_j(\mathcal{x}) \equiv 1\,, \quad \text{supp}\ \psi_j \subset U_j\,, \quad j = 1, \dots, M\,. \tag{2.33}$$

By $\mathbb{D}'(\mathcal{M})$ we denote the the space of Schwartz distributions on a smooth manifold $\mathcal{M}$.

Definition 2.15. An operator

$$\mathbf{A}\ :\ C^\infty(\mathcal{M}) \to \mathbb{D}'(\mathcal{M}) \tag{2.34}$$

is called pseudodifferential $\mathbf{A} = \mathbf{a}(\mathcal{x}, D)$ with a symbol $a \in \widetilde{\mathbb{S}}^m(\mathcal{M} \times \mathbb{R}^n)$, if:

i. $\chi_1 \mathbf{A} \chi_2 I\ :\ H^s(\mathcal{M}) \to C^\infty(\mathcal{M})$ are continuous for all $s \in \mathbb{R}$ and all pairs of functions with disjoint supports supp $\chi_1 \cap$ supp $\chi_2 = \varnothing$; in other words, $\chi_1 \mathbf{A} \chi_2 I$ has order $-\infty$;

ii. the pull back operator

$$\varkappa_{j,*} \psi_j \mathbf{A} \psi_j \varkappa_{j,*}^{-1} u = \mathbf{a}^{(j)}(x, D) u, \ \ u \in C_0^\infty(\mathbb{R}^n), \tag{2.35}$$

are ΨDOs for all $j = 1, \dots, M$ with the "pull back" symbols

$$a^{(j)}(\varkappa_j(x), \xi) = \psi_j(\varkappa_j(x)) a\big(x, [\varkappa'_j(x)]^\top \xi\big), \ \ a^{(j)} \in \widetilde{\mathbb{S}}^m(\mathcal{M} \times \mathbb{R}^n),$$
$$\mathcal{x} = \varkappa_j(x) \in V_j \subset \mathcal{M}\,, \ \ x \in U_j \subset \mathbb{R}^n, \ \ \xi \in \mathbb{R}^n$$

and $\varkappa_{j,*}\psi(x) := \psi(\varkappa_j(x))$, $\varkappa_{j,*}^{-1}\varphi(\mathcal{x}) := \varphi(\varkappa_j^{-1}(\mathcal{x}))$, while $\varkappa'_j(x)$ denotes the corresponding Jacobian.

More precisely, the symbol $a(\mathcal{x}, \xi)$ of a ΨDO $\mathbf{A} = \mathbf{a}(\mathcal{x}, D)$ on a manifold $\mathcal{M}$ is defined on the cotangent bundle $\mathcal{T}^*\mathcal{M}$ and is independent of the choice of the coordinate diffeomorphisms and charts. For details of the definition we refer to the monographs [28, vol. 3, Definition 18.1.20] and [67, § 4.3].

Definition 2.16. If $\mathbb{X}(\mathbb{R}^n)$ is a function space on $\mathbb{R}^n$ (e.g., the Bessel potential space $H_p^s(\mathbb{R}^n)$ or the Besov space $B_{p,q}^s(\mathbb{R}^n)$), the corresponding function space $\mathbb{X}(\mathcal{M})$ on a sufficiently smooth manifold $\mathcal{M}$ (e.g. the Bessel potential space $H_p^s(\mathcal{M})$ or the Besov space $B_{p,q}^s(\mathcal{M})$) consists of functions $\varphi \in \mathbb{X}(\mathcal{M})$ for which $\varkappa_{j,*}[\psi\varphi](x) := \psi(\varkappa_j(x))\varphi(\varkappa_j(x)) \in \mathbb{X}(\mathbb{R}^n)$ for all $j = 1, \dots, M$, and is endowed with the norm

$$\|\varphi\|_{\mathbb{X}(\mathcal{M})} \equiv \big\|\varphi|\mathbb{X}(\mathcal{M})\big\| := \sum_{j=1}^{M} \big\|\varkappa_{j,*}[\psi_j\varphi]|\mathbb{X}(\mathbb{R}^n)\big\|. \tag{2.36}$$

As a byproduct of Definition 2.15 and the Calderon–Vaillancourt Proposition 2.5 we have the following.

Theorem 2.17. *Let $1 < p < \infty$ and $s, m \in \mathbb{R}$. Let $a(\mathcal{x}, D)$ be a ΨDO on a manifold $\mathcal{M}$ with a symbol $a \in \widetilde{\mathbb{S}}^m(\mathcal{M}, \mathbb{R}^n)$. Then the operator*

$$\mathbf{a}(\mathcal{x}, D) \;:\; H_p^s(\mathcal{M}) \to H_p^{s-m}(\mathcal{M})$$

is continuous.

The next assertion states a boundedness result for a ΨDO with non-classical symbol. The proof can be found in [65].

Proposition 2.18. *Let $1 < p < \infty$, $r \in \mathbb{R}$, $s > n/2$ and*

$$\sum_{|\alpha| \leqslant [\frac{n}{2}]+1} \sum_{|\beta| \leqslant m} \sup_{\xi \in \mathbb{R}^n} \left\| \langle \xi \rangle^{-r} \xi^\alpha a^{(\alpha)}_{(\beta)}(\,\cdot\,, \xi) \big| H_p^s(\mathbb{R}^n) \right\| < +\infty \quad \text{for } p \neq 2,$$

$$\sum_{|\beta| \leqslant m} \sup_{\xi \in \mathbb{R}^n} \left\| \langle \xi \rangle^{-r} a_{(\beta)}(\,\cdot\,, \xi) \big| H_2^s(\mathbb{R}^n) \right\| < +\infty \quad \text{for } p = 2, \tag{2.37}$$

where

$$a^{(\alpha)}_{(\beta)}(x, \xi) := \partial_\xi^\alpha \partial_x^\beta a(x, \xi) \ \text{ for } \ \alpha, \beta \in \mathbb{N}_0^n\,.$$

Then the operator

$$\mathbf{a}(x, D) \;:\; H_p^{\sigma+r}(\mathbb{R}^n) \to H_p^\sigma(\mathbb{R}^n)$$

is bounded for arbitrary $-m \leq \sigma \leq m$.

Now we formulate the well known Sobolev's compact embedding lemma (see, e.g., [67, § 7.6], [22, Theorem 4.3], for $p = 2$ and [72,73] for $1 < p < \infty$).

Proposition 2.19. *Let $1 < p < \infty$, $s, \sigma \in \mathbb{R}$, $\sigma < s$, and $\mathcal{M}$ be a compact manifold. Then the embedding $H_p^s(\mathcal{M}) \subset H_p^\sigma(\mathcal{M})$ is compact.*

As a byproduct of Theorem 2.17 and Proposition 2.19 we have the following.

Lemma 2.20. *Let $1 < p < \infty$, $s, m \in \mathbb{R}$, $\varepsilon > 0$, and $\mathcal{M}$ be a compact manifold.*

Then the operator $\mathbf{a}(\mathcal{x}, D) \;:\; H_p^s(\mathcal{M}) \to H_p^{s-m}(\mathcal{M})$ with a symbol $a \in \widetilde{\mathbb{S}}^{m-\varepsilon}(\mathcal{M}, \mathbb{R}^n)$ is compact.

Next proposition shows that the set of ΨDOs is an algebra, i.e., a composition of ΨDOs is again a ΨDO.

Proposition 2.21. *Let $1 < p < \infty$ and $s, m_1, m_2, \sigma \in \mathbb{R}$.*

Then the composition $\mathbf{a}(\mathcal{x}, D)\mathbf{b}(\mathcal{x}, D) =: \mathbf{c}(\mathcal{x}, D)$ of ΨDOs with symbols $a \in \widetilde{\mathbb{S}}^{m_1}(\mathcal{M} \times \mathbb{R}^n)$ and $b \in \widetilde{\mathbb{S}}^{m_2}(\mathcal{M} \times \mathbb{R}^n)$ is a ΨDO with the symbol

$$c(\mathcal{x}, \xi) = a(\mathcal{x}, \xi) b(\mathcal{x}, \mathcal{x}) + c_{m_1+m_2-1}(\mathcal{x}, \xi)\,,$$

$$c_{m_1+m_2-1} \in \widetilde{\mathbb{S}}^{m_1+m_2-1}(\mathcal{M} \times \mathbb{R}^n).$$

If, in particular, $\mathcal{M}$ is compact, the operator

$$\mathbf{c}_{m_1+m_2-1}(\mathcal{x}, D) := \mathbf{c}(\mathcal{x}, D) - \mathbf{a}(\mathcal{x}, D)\mathbf{b}(\mathcal{x}, D)$$

is compact between the spaces

$$\mathbf{c}_{m_1+m_2-1}(\mathcal{X}, D) \ : \ H^s_p(\mathcal{M}) \to H^{s-m_1-m_2}_p(\mathcal{M}) \,. \tag{2.38}$$

For the proof we refer to monograph on ΨDOs, e.g. to [67, § 7].

The next Theorem 2.22 is actually a consequence of the foregoing Proposition 2.21, but concerning only the sufficiency of ellipticity of the symbol for the Fredholm property of a ΨDO. Necessity of the condition is proved with the help of a local principle, exposed below in Subsection 2.4. We drop the proof and refer the reader for details to [12, 17, 21, 64].

Theorem 2.22. *Let $1 < p < \infty$, $s, m \in \mathbb{R}$, and $\mathcal{M}$ be a compact manifold. The operator*

$$\mathbf{a}(\mathcal{X}, D) \ : \ [H^s_p(\mathcal{M})]^N \to [H^{s-m}_p(\mathcal{M})]^N \tag{2.39}$$

with a $N \times N$ matrix symbol $a \in \widetilde{\mathbb{S}}^m(\mathcal{M} \times \mathbb{R}^n)$ is Fredholm if and only if the symbol is elliptic

$$\lim_{R\to\infty} \inf_{|\xi|\geqslant R} \frac{|\det a(\mathcal{X}, \xi)|}{\langle\xi\rangle^m} \neq 0 \ \ \forall\, \mathcal{X} \in \mathcal{M} \,. \tag{2.40}$$

If $\mathbf{a}(\mathcal{X}, D)$ *is Fredholm, it has a* regularizer (a parametrix) $\mathbf{P}(\mathcal{X}, D)$*, such that*

$$\begin{aligned} \mathbf{P}(\mathcal{X}, D)\mathbf{a}(\mathcal{X}, D) &= I + \mathbf{T}_1(\mathcal{X}, D) \,, \\ \mathbf{a}(\mathcal{X}, D)\mathbf{P}(\mathcal{X}, D) &= I + \mathbf{T}_2(\mathcal{X}, D) \,, \end{aligned} \tag{2.41}$$

where $\mathbf{T}_1(\mathcal{X}, D)$ and $\mathbf{T}_2(\mathcal{X}, D)$ are infinitely smoothing compact operators

$$\mathbf{T}_1(\mathcal{X}, D), \ \mathbf{T}_2(\mathcal{X}, D) \ : \ [H^s_p(\mathcal{M})]^N \to [C^\infty(\mathcal{M}))]^N \ \ \forall\, s \in \mathbb{R} \,. \tag{2.42}$$

Let $s \in \mathbb{R}$, $m \in \mathbb{N}_0$ and $1 < p < \infty$. By $\mathbb{H}^{s,m}_p(\mathbb{R}^n_+)$ we denote the Banach space of functions (of distributions for $s < 0$) endowed with the norm

$$\big\| u | \mathbb{H}^{s,m}_p(\mathbb{R}^n_+) \big\| := \sum_{k=0}^m \big\| x^k_n u | H^{s+k}_p(\mathbb{R}^n_+) \big\|. \tag{2.43}$$

Obviously, $\mathbb{H}^{s,0}_p(\mathbb{R}^n_+) = H^s_p(\mathbb{R}^n_+)$.

The spaces $\mathbb{H}^{s,m}_p(\Omega)$ are defined similarly, by replacing x^k_n in (2.43) with $\operatorname{dist}(x, \partial\Omega)$.

The spaces

$$\mathbb{H}^{s,\infty}_p(\Omega) := \bigcap_{m\in\mathbb{N}_0} \mathbb{H}^{s,m}_p(\Omega) \,, \tag{2.44}$$

endowed with an appropriate topology, are Freshet spaces.
The Besov weighted spaces $\mathbb{B}^{s,m}_{p,q}$ are defined similarly.

Theorem 2.23. *Let $1 < p < \infty$, $s, r \in \mathbb{R}$, $m \in \mathbb{N}_0$, $a \in \mathbb{S}^r(\overset{\bullet}{\Omega}, \mathbb{R}^n)$. Then the operators*

$$\begin{aligned} r_\Omega \mathbf{a}(x, D) \ &: \ \widetilde{\mathbb{H}}^{s,m}_p(\Omega) \to \mathbb{H}^{s-r,m}_p(\Omega) \,, \\ &: \ \widetilde{\mathbb{B}}^{s,m}_{p,q}(\Omega) \to \mathbb{B}^{s-r,m}_{p,q}(\Omega) \,, \end{aligned} \tag{2.45}$$

are continuous. Moreover, if the symbol of the ΨDO $\mathbf{a}(x,D)$ *is a rational function in* ξ *and* $s > -1/p$*, then the operators*

$$\begin{aligned} r_\Omega \mathbf{a}(x,D)\ell_\Omega \ &: \ \mathbb{H}^{s,m}_p(\Omega) \to \mathbb{H}^{s-r,m}_p(\Omega)\,, \\ &: \ \mathbb{B}^{s,m}_{p,q}(\Omega) \to \mathbb{B}^{s-r,m}_{p,q}(\Omega)\,, \end{aligned} \tag{2.46}$$

are bounded as well.

If $a \in \widetilde{\mathbb{S}}^r(\Omega\times\mathbb{R}^n)$*, then the operators*

$$\begin{aligned} r_\Omega \mathbf{a}(x,D) \ &: \ \widetilde{\mathbb{H}}^{s,m}_{p,\mathrm{com}}(\Omega) \to \mathbb{H}^{s-r,m}_{p,\mathrm{loc}}(\Omega), \\ &: \ \widetilde{\mathbb{B}}^{s,m}_{p,q,\mathrm{com}}(\Omega) \to \mathbb{B}^{s-r,m}_{p,q,\mathrm{loc}}(\Omega), \end{aligned} \tag{2.47}$$

are bounded.

Moreover, if the symbol of the ΨDO $\mathbf{a}(x,D)$ *is a rational function in* ξ *and* $s > -1/p$*, then the operators*

$$\begin{aligned} r_\Omega \mathbf{a}(x,D)\ell_\Omega \ &: \ \mathbb{H}^{s,m}_{p,\mathrm{com}}(\Omega) \to \mathbb{H}^{s-r,m}_{p,\mathrm{loc}}(\Omega), \\ &: \ \mathbb{B}^{s,m}_{p,q,\mathrm{com}}(\Omega) \to \mathbb{B}^{s-r,m}_{p,q,\mathrm{loc}}(\Omega), \end{aligned} \tag{2.48}$$

are bounded as well.

Proof. Let us prove the continuity properties (2.45), (2.46). The continuity properties (2.47), (2.48) are proved similarly.

The continuity (2.45) is a local property and it suffices to prove the theorem for $\Omega = \mathbb{R}^n_+$. To this end, let us apply the equality

$$x_n^k \mathbf{a}(x,D)u(x) = \sum_{l=0}^{k} \frac{i^l k!}{l!(k-l)!} (\partial^l_{\xi_n}\mathbf{a})(x,D)x_n^{k-l}u(x), \quad u \in C_0^\infty(\mathbb{R}^n), \tag{2.49}$$

which is easy to verify directly. Applying (2.49) we proceed as follows

$$\begin{aligned} \big\|\mathbf{a}(x,D)u|\mathbb{H}^{s-r,m}_p(\mathbb{R}^n_+)\big\| &= \sum_{k=0}^{m} \big\|x_n^k\mathbf{a}(x,D)u|H^{s-r+k}_p(\mathbb{R}^n_+)\big\| \leqslant \\ &\leqslant \sum_{k=0}^{m}\sum_{l=0}^{k} \frac{k!}{l!(k-l)!}\Big\|(\partial^l_{\xi_n}\mathbf{a})(x,D)x_n^{k-l}u\big|H^{s-r+k}_p(\mathbb{R}^n)\Big\| \leqslant \\ &\leqslant M'_m \sum_{k=0}^{m}\sum_{l=0}^{k} \big\|x_n^{k-l}u|H^{s+l}_p(\mathbb{R}^n)\big\| \leqslant M_m\big\|u|\mathbb{H}^{s,m}_p(\mathbb{R}^n)\big\| \end{aligned}$$

since $\partial^l_{\xi_n} a \in \widetilde{\mathbb{S}}^{r-l}(\dot{\Omega}\times\mathbb{R}^n)$.

The continuity property (2.46) is a similar consequence of Corollary 2.6. □

Very important role in the operator theory and applications belong to the interpolation of operators.

Definition 2.24. Let Banach spaces $\mathfrak{G} := \{\mathfrak{B}_\alpha\}_{\alpha\in\mathcal{A}}$ be embedded in one Banach space $\mathfrak{B}_\alpha \subset \mathfrak{B}$ for all $\alpha \in \mathcal{A}$.

An Interpolation functor $\mathbb{F}$ of type θ, $0 \leqslant \theta \leqslant 1$, assigns to all pairs $\mathfrak{B}_{\alpha_0}, \mathfrak{B}_{\alpha_1} \in \mathfrak{G}$ a new space $\mathbb{F}(\mathfrak{B}_{\alpha_0}, \mathfrak{B}_{\alpha_1}) = \mathfrak{B}_\alpha$, i.e., $\mathbb{F} \,:\, \mathfrak{G}\times\mathfrak{G} \to \mathfrak{G}$, so that:

i. $\mathfrak{B}_{\alpha_0} \cap \mathfrak{B}_{\alpha_1} \subset \mathfrak{B}_\alpha \subset \mathfrak{B}_{\alpha_0} + \mathfrak{B}_{\alpha_1}$;

ii. an operator, bounded in arbitrary two pairs of spaces

$$A \,:\, \mathfrak{B}_{\alpha_0} \to \mathfrak{B}_{\alpha_1}, \quad \mathfrak{B}_{\alpha_0} \in \mathfrak{G}, \;\; \mathfrak{B}_{\alpha_1} \in \mathfrak{G},$$
$$A \,:\, \mathfrak{B}_{\beta_0} \to \mathfrak{B}_{\beta_1}, \quad \mathfrak{B}_{\beta_0} \in \mathfrak{G}, \;\; \mathfrak{B}_{\beta_1} \in \mathfrak{G},$$

after being restricted to the interpolated space $\mathfrak{B}_\alpha := \mathbb{F}(\{\mathfrak{B}_{\alpha_0}, \mathfrak{B}_{\beta_0}\}$ maps this space to the interpolated space $\mathfrak{B}_\beta := \mathbb{F}(\{\mathfrak{B}_{\alpha_1}, \mathfrak{B}_{\beta_1}\}$ and

$$A \,:\, \mathfrak{B}_\alpha \to \mathfrak{B}_\beta$$

is bounded.

iii. there is a constant $C > 0$ such that the inequality

$$\big\|A|\mathcal{L}(\mathfrak{B}_\alpha, \mathfrak{B}_\beta)\big\| \leqslant \; C\big\|A|\mathcal{L}(\mathfrak{B}_{\alpha_0}, \mathfrak{B}_{\alpha_1})\big\|^{1-\theta} \, \big\|A|\mathcal{L}(\mathfrak{B}_{\beta_0}, \mathfrak{B}_{\beta_1})\big\|^{\theta} \qquad (2.50)$$

holds, where $\mathcal{L}(\mathfrak{B}_0, \mathfrak{B}_1)$ denotes the set of all bounded linear operators from $\mathfrak{B}_0$ into $\mathfrak{B}_1$.

In the next proposition we expose interpolation properties of the spaces defined in the present section. For the proof and further details we refer to [73, § 2.4.2, § 2.4.7].

Proposition 2.25. *Let*

$$s_0, s_1 \in \mathbb{R}, \quad 0 < \theta < 1, \quad 1 \leqslant p_0, p_1, \nu, q_0, q_1, \leqslant \infty,$$
$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad s = (1-\theta)s_0 + \theta s_1\,.$$

For the real $(\cdot,\cdot)_{\theta,p}$, the complex $(\cdot,\cdot)_\theta$ and the modified complex $[\cdot,\cdot]_\theta$ interpolation functors the following holds:

i. $(H^s_{p_0}(\mathbb{R}^n), H^s_{p_1}(\mathbb{R}^n))_{\theta,p} = H^s_p(\mathbb{R}^n)$ *provided* $1 < p_0, p_1 < \infty$*;*

ii. $(H^{s_0}_{p_0}(\mathbb{R}^n), H^{s_1}_{p_1}(\mathbb{R}^n))_\theta = [H^{s_0}_{p_0}(\mathbb{R}^n), H^{s_1}_{p_1}(\mathbb{R}^n)]_\theta = H^s_p(\mathbb{R}^n)$ *provided* $1 < p_0, p_1 < \infty$*;*

iii. $(H^{s_0}_r(\mathbb{R}^n), H^{s_1}_r(\mathbb{R}^n))_{\theta,\nu} = B^s_{r,\nu}(\mathbb{R}^n)$ *provided* $s_0 \neq s_1$, $1 < r < \infty$*;*

iv. $(\mathbb{B}^{s_0}_{p_0,q_0}(\mathbb{R}^n), \mathbb{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n))_\theta = \mathbb{B}^s_{p,q}(\mathbb{R}^n)$.

Remark 2.26. The interpolation between loc-spaces $H^s_{p,\mathrm{loc}}(\mathbb{R}^n)$, $W^s_{p,\mathrm{loc}}(\mathbb{R}^n)$ and $B^s_{p,q,\mathrm{loc}}(\mathbb{R}^n)$ holds as well: it suffices to apply the above interpolations to operators $\chi\mathbf{A}$ with cut-off functions $\chi \in C^\infty_0(\mathbb{R}^n)$.

The interpolation between Bessel potential and Sobolev–Slobodetskii spaces on a domain $H^s_p(\Omega)$, $W^s_p(\Omega)$ and $\widetilde{H}^s_p(\Omega)$, $\widetilde{W}^s_p(\Omega)$ and on a manifold $H^s_p(\mathcal{M})$, $W^s_p(\mathcal{M})$ holds as well (see [73, § 2.4.2, § 2.4.7]).

The interpolation holds also between weighted spaces on the Euclidean half space $\mathbb{H}^{s,m}_p(\mathbb{R}^n_+)$, $\mathbb{W}^{s,m}_p(\mathbb{R}^n_+)$ and on a domain $\mathbb{H}^{s,m}_p(\Omega)$, $\mathbb{W}^{s,m}_p(\Omega)$ and

$\widetilde{\mathbb{H}}_p^{s,m}(\Omega)$, $\widetilde{\mathbb{W}}_p^{s,m}(\Omega)$. To justify such interpolation, just note, that we can interpolate the operator $\rho^k A$, $k = 1, \dots, m$, instead of the operator A, where $\rho = x_n$ for $\mathbb{R}^n_+$ and $\rho(x) = \text{dist}(x, \partial\Omega)$ for a domain Ω.

2.3. **Fredholm properties of ΨDOs on manifolds with boundary.** Let us commence the present subsection with the definition of a manifold with boundary.

Definition 2.27. A topological space $\mathcal{M}$ is called an open manifold with boundary $\partial\mathcal{M}$ if there exist two types of coordinate charts $\{U_j, \varkappa_j\}$:

i. The inner patches U_j, when domains in the Euclidean space $V_j \subset \mathbb{R}^n$ are transformed by $\varkappa_j$ into U_j;

ii. The boundary patches U_j, when domains in the Euclidean half space $V_j \subset \mathbb{R}^n_+$ are transformed by $\varkappa_j$ into U_j.

Let $\mathcal{M}$ be a compact, n-dimensional, smooth, nonselfintersecting manifold with the smooth boundary $\partial\mathcal{M} \neq \varnothing$ and let $\mathbf{A}(x, D)$ be a strongly elliptic $N \times N$ matrix ΨDO of order $\nu \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\mathcal{A}(x, \xi)$ the principal homogeneous symbol matrix of the operator $\mathbf{A}(x, D)$ in some local coordinate system ($x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n \setminus \{0\}$).

Let $\lambda_1(x), \dots, \lambda_N(x)$ be the eigenvalues of the matrix

$$\big[\,\mathcal{A}(x, 0, \dots, 0, +1)\,\big]^{-1}\,\big[\,\mathcal{A}(x, 0, \dots, 0, -1)\,\big], \quad x \in \partial\mathcal{M}, \tag{2.51}$$

and introduce the notation

$$\delta_j(x) = \text{Re}\,\big[\,(2\pi\, i)^{-1} \ln \lambda_j(x)\,\big], \quad j = 1, \dots, N. \tag{2.52}$$

Here $\ln\zeta$ denotes the branch of the logarithmic function analytic in the complex plane cut along $(-\infty, 0]$. Note that the numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system and the strong inequality $-1/2 < \delta_j(x) < 1/2$ holds for all $x \in \overline{\mathcal{M}}$, $j = \overline{1, N}$, due to the strong ellipticity of $\mathcal{A}$. In a particular case, when $\mathcal{A}(x, \xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, we have $\delta_1(x) = \dots = \delta_N(x) = 0$ since the eigenvalues $\lambda_1(x), \dots, \lambda_N(x)$ are positive for all $x \in \overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudo-differential operators on manifolds with boundary are characterized by the following theorem (see [22, 64]).

Theorem 2.28. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, and let $\mathbf{A}(x, D)$ be a ΨDO of order $\nu \in \mathbb{R}$ with the strongly elliptic symbol $\mathcal{A}(x, \xi)$, that is, there is a positive constant c_0 such that*

$$\text{Re}\,\mathcal{A}(x, \xi)\,\eta \cdot \eta \geqslant c_0\,|\eta|^2 \tag{2.53}$$

for $x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\eta \in \mathbb{C}^N$.

Then the operators

$$\begin{aligned} \mathbf{A} \ &: \ \big[\widetilde{H}^s_p(\mathcal{M})\big]^N \to \big[H^{s-\nu}_p(\mathcal{M})\big]^N \\ &: \ \big[\widetilde{B}^s_{p,q}(\mathcal{M})\big]^N \to \big[B^{s-\nu}_{p,q}(\mathcal{M})\big]^N \end{aligned} \tag{2.54}$$

are Fredholm and have the trivial index $\operatorname{Ind}\mathbf{A}=0$ *if*

$$\frac{1}{p}-1+\sup_{x\in\partial\mathcal{M},\,1\leqslant j\leqslant N}\delta_j(x)<s-\frac{\nu}{2}<\frac{1}{p}+\inf_{x\in\partial\mathcal{M},\,1\leqslant j\leqslant N}\delta_j(x). \tag{2.55}$$

Moreover, the null-spaces and indices of the operators (2.54) *coincide for all values of the parameter* $q\in[1,+\infty]$ *provided* p *and* s *satisfy inequality* (2.55).

Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be Banach spaces and $\mathfrak{B}:=\mathfrak{B}_1\times\mathfrak{B}_2$ be their direct product, consisting of pairs $U=(u',u'')^\top\in\mathfrak{B}$, where $u'\in\mathfrak{B}_1$ and $u''\in\mathfrak{B}_2$. Further, let $\mathfrak{B}_j^*$ be the adjoint spaces to $\mathfrak{B}_j$, $j=1,2$, and $\mathfrak{B}^*:=\mathfrak{B}_1^*\times\mathfrak{B}_2^*$. The notation $\langle F,\,u\rangle$ with $F\in\mathfrak{B}_j^*$ and $u\in\mathfrak{B}_j$ (or $F\in\mathfrak{B}^*$ and $u\in\mathfrak{B}$) is used for the duality pairing between the adjoint spaces.

It is obvious that the bounded operator $\mathbf{A}\;:\;\mathfrak{B}\to\mathfrak{B}^*$ has the form

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}, \tag{2.56}$$

where the operators

$$\begin{aligned}\mathbf{A}_{11}\;:\;\mathfrak{B}_1\to\mathfrak{B}_1^*,\quad \mathbf{A}_{12}\;:\;\mathfrak{B}_2\to\mathfrak{B}_1^*,\\ \mathbf{A}_{21}\;:\;\mathfrak{B}_1\to\mathfrak{B}_2^*,\quad \mathbf{A}_{22}\;:\;\mathfrak{B}_2\to\mathfrak{B}_2^*\end{aligned} \tag{2.57}$$

are all bounded.

Lemma 2.29. *Let the operator* $\mathbf{A}$ *in* (2.56) *be strongly coercive, i.e., there is a constant* $C>0$ *such that*

$$\operatorname{Re}\langle\mathbf{A}U,U\rangle\geqslant C\|U\|^2_{\mathfrak{B}}\ \ \forall\,U\in\mathfrak{B}\,. \tag{2.58}$$

Then the operators $\mathbf{A}_{11}$ *and* $\mathbf{A}_{22}$ *are both strongly coercive*

$$\begin{aligned}\operatorname{Re}\langle\mathbf{A}_{11}u,\,u\rangle\geqslant C\|u\|^2_{\mathfrak{B}_1}\ \ \forall\,u\in\mathfrak{B}_1,\\ \operatorname{Re}\langle\mathbf{A}_{22}v,\,v\rangle\geqslant C\|v\|^2_{\mathfrak{B}_2}\ \ \forall\,v\in\mathfrak{B}_2\end{aligned} \tag{2.59}$$

and, thus, invertible. Moreover, the operators

$$\mathbf{B}:=\mathbf{A}_{11}-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\;:\;\mathfrak{B}_1\to\mathfrak{B}_1^*\,, \tag{2.60}$$

$$\mathbf{D}:=\mathbf{A}_{22}-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\;:\;\mathfrak{B}_2\to\mathfrak{B}_2^*\,, \tag{2.61}$$

are strongly coercive

$$\operatorname{Re}\langle\mathbf{B}u,\,u\rangle\geqslant C\|u\|^2_{\mathfrak{B}_1}\qquad\forall\,u\in\mathfrak{B}_1, \tag{2.62}$$

$$\operatorname{Re}\langle\mathbf{D}v,\,v\rangle\geqslant C\|v\|^2_{\mathfrak{B}_2}\qquad\forall\,v\in\mathfrak{B}_2 \tag{2.63}$$

with the same constant $C>0$ *as in* (2.58) *and* (2.59) *and, thus, invertible.*

Proof. The strong coercivity (2.59) follows by taking in (2.58) consecutively $U=(u,0)^\top$ and $U=(0,v)^\top$. The strong coercivity implies the invertibility.

To prove (2.62) we proceed as follows. For $U=(u',u'')^\top$ we have

$$\begin{aligned}&C\|U\|^2_{\mathfrak{B}}\leqslant\operatorname{Re}\langle\mathbf{A}U,U\rangle=\\ &\quad=\operatorname{Re}\Big[\langle\mathbf{A}_{11}u',u'\rangle+\langle\mathbf{A}_{12}u'',u'\rangle+\langle\mathbf{A}_{21}u',u''\rangle+\langle\mathbf{A}_{22}u'',u''\rangle\Big].\end{aligned} \tag{2.64}$$

Since

$$\|u'\|^2_{\mathfrak{B}_1} \leqslant \|u'\|^2_{\mathfrak{B}_1} + \|u''\|^2_{\mathfrak{B}_2} = \|U\|^2_{\mathfrak{B}},$$

by introducing $u'' = -\mathbf{A}_{22}^{-1}\mathbf{A}_{21}u$ and $u' = u$ into the inequality (2.64) we get

$$C\|u\|^2_{\mathfrak{B}_1} \leqslant C\|U\|^2_{\mathfrak{B}} \leqslant \mathrm{Re}\Big[\langle \mathbf{A}_{11}u, u\rangle - \langle \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}u, u\rangle - \\ - \langle \mathbf{A}_{21}u, \mathbf{A}_{22}^{-1}\mathbf{A}_{21}u\rangle + \langle \mathbf{A}_{21}u, \mathbf{A}_{22}^{-1}\mathbf{A}_{21}u\rangle\Big] = \mathrm{Re}\,\langle \mathbf{B}u, u\rangle$$

and (2.62) is proven. Similarly, by introducing $u' = -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}u$ and $u'' = u$ into the inequality (2.64), we get (2.63). □

Remark 2.30. We will actually apply the foregoing Lemma 2.29 not only to Banach spaces, but also to a finite dimensional case when $\mathfrak{B}_1 = \mathbb{C}^n$ and $\mathfrak{B}_2 = \mathbb{C}^m$ are finite dimensional spaces and

$$\mathbf{A}(x,\xi) = \begin{bmatrix} \mathbf{A}_{11}(x,\xi) & \mathbf{A}_{12}(x,\xi) \\ \mathbf{A}_{21}(x,\xi) & \mathbf{A}_{22}(x,\xi) \end{bmatrix},$$

$$\mathbf{A}_{11}(x,\xi) = \big[\,\mathcal{A}_{11}^{jk}(x,\xi)\,\big]_{n\times n}, \quad \mathbf{A}_{12}(x,\xi) = \big[\,\mathcal{A}_{12}^{jk}(x,\xi)\,\big]_{n\times m},$$
$$\mathbf{A}_{21}(x,\xi) = \big[\,\mathcal{A}_{21}^{jk}(x,\xi)\,\big]_{m\times n}, \quad \mathbf{A}_{22}(x,\xi) = \big[\,\mathcal{A}_{22}^{jk}(x,\xi)\,\big]_{m\times m},$$

are the matrix-symbols of ΨDOs (cf. Theorem 2.31 and Theorem 6.3).

In particular, it follows that if the matrix $\mathbf{A}$ is strongly elliptic, then the matrices $\mathbf{A}_{11}$, $\mathbf{A}_{22}$, $\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}$ and $\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$ are strongly elliptic as well.

Further, we treat an important example to demonstrate the local principle and Lemma 2.29 for the investigation of a nonclassical system of ΨDOs on overlapping manifolds which is essentially employed in our analysis in Section 6.

Let $\mathcal{S}$ be a closed smooth manifold of dimension n and $\mathcal{M}$, $\mathcal{C}_0$ be a couple of embedded open submanifolds $\mathcal{C}_0 \subset \mathcal{M} \subset \mathcal{S}$ with the smooth disjoint boundaries $\partial\mathcal{C}_0$ and $\partial\mathcal{M}$, $\partial\mathcal{C}_0 \cap \partial\mathcal{M} = \varnothing$. Then the complemented surface $\mathcal{C} := \mathcal{M}\setminus\overline{\mathcal{C}}_0$ has the boundary $\partial\mathcal{C} = \partial\mathcal{M}\cup\partial\mathcal{C}_0$. Clearly, $\mathcal{M}$ and $\mathcal{C}$ is another couple of embedded open submanifolds $\mathcal{C} \subset \mathcal{M} \subset \mathcal{S}$.

Let us consider a ΨDO

$$\mathbf{N}(x,D) := \begin{bmatrix} r_{\mathcal{C}}[\mathbf{A}_{lk}(x,D)]_{M\times N} \\ r_{\mathcal{M}}[\mathbf{A}_{tk}(x,D)]_{L\times N} \end{bmatrix}_{N\times N} \tag{2.65}$$

with $N = M + L$, $k = \overline{1,N}$, $l = 1,\dots,M$, and $t = M+1,\dots,N$, where

$$\mathbf{A}(x,D) := \big[\,\mathbf{A}_{lk}(x,D)\,\big]_{N\times N} \tag{2.66}$$

is a $N\times N$ matrix pseudodifferential operator of order ν on $\mathcal{S}$ with an elliptic principal homogeneous symbol $\mathcal{A}(x,\xi) := [\mathcal{A}_{lk}(x,\xi)]_{N\times N}$. Let us treat the

ΨDO $\mathbf{N}(x,D)$ in the following settings

$$\begin{aligned}\mathbf{N}(x,D) \ &: \ \widetilde{\mathbb{H}}^s_p \to \mathbb{H}^{s-\nu}_p, \\ &: \ \widetilde{\mathbb{B}}^s_{p,q} \to \mathbb{B}^{s-\nu}_{p,q}, \ \ 1<p,q<\infty, \ \ s,\nu\in\mathbb{R},\end{aligned} \tag{2.67}$$

where

$$\begin{aligned}\widetilde{\mathbb{H}}^s_p &:= \big[\widetilde{H}^s_p(\mathcal{C})\big]^M \times \big[\widetilde{H}^s_p(\mathcal{M})\big]^L, \\ \mathbb{H}^{s-\nu}_p &:= \big[H^{s-\nu}_p(\mathcal{C})\big]^M \times \big[H^{s-\nu}_p(\mathcal{M})\big]^L, \\ \widetilde{\mathbb{B}}^s_{p,q} &:= \big[\widetilde{B}^s_{p,q}(\mathcal{C})\big]^M \times \big[\widetilde{B}^s_{p,q}(\mathcal{M})\big]^L, \\ \mathbb{B}^{s-\nu}_p &:= \big[B^{s-\nu}_{p,q}(\mathcal{C})\big]^M \times \big[B^{s-\nu}_{p,q}(\mathcal{M})\big]^L.\end{aligned} \tag{2.68}$$

Now, let us represent the operator $\mathbf{A}(x,D)$ given by (2.66) and its symbol $\mathcal{A}(x,\xi)$ in the following block wise form

$$\mathbf{A}(x,D) = \begin{bmatrix}\mathbf{A}_{11}(x,D) & \mathbf{A}_{12}(x,D) \\ \mathbf{A}_{21}(x,D) & \mathbf{A}_{22}(x,D)\end{bmatrix}, \tag{2.69}$$

$$\mathbf{A}_{11}(x,D) = \big[\,\mathbf{A}^{jk}_{11}(x,D)\,\big]_{M\times M}, \quad \mathbf{A}_{12}(x,D) = \big[\,\mathbf{A}^{jk}_{12}(x,D)\,\big]_{M\times L},$$
$$\mathbf{A}_{21}(x,D) = \big[\,\mathbf{A}^{jk}_{21}(x,D)\,\big]_{L\times M}, \quad \mathbf{A}_{22}(x,D) = \big[\,\mathbf{A}^{jk}_{22}(x,D)\,\big]_{L\times L},$$

$$\mathcal{A}(x,\xi) = \begin{bmatrix}\mathcal{A}_{11}(x,\xi) & \mathcal{A}_{12}(x,\xi) \\ \mathcal{A}_{21}(x,\xi) & \mathcal{A}_{22}(x,\xi)\end{bmatrix}, \tag{2.70}$$

$$\mathcal{A}_{11}(x,\xi) = \big[\,\mathcal{A}^{jk}_{11}(x,\xi)\,\big]_{M\times M}, \quad \mathcal{A}_{12}(x,\xi) = \big[\,\mathcal{A}^{jk}_{12}(x,\xi)\,\big]_{M\times L},$$
$$\mathcal{A}_{21}(x,\xi) = \big[\,\mathcal{A}^{jk}_{21}(x,\xi)\,\big]_{L\times M}, \quad \mathcal{A}_{22}(x,\xi) = \big[\,\mathcal{A}^{jk}_{22}(x,\xi)\,\big]_{L\times L}.$$

If $\mathbf{A}(x,D)$ is a strongly coercive ΨDO, then the symbol $\mathcal{A}(x,\xi)$ is strongly elliptic (cf. e.g. [21]) which, in view of Remark 2.30 and Lemma 2.29, implies that the symbol

$$\mathcal{D}(x,\xi) := \mathcal{A}_{11}(x,\xi) - \mathcal{A}_{12}(x,\xi)[\mathcal{A}_{22}(x,\xi)]^{-1}\mathcal{A}_{21}(x,\xi) \tag{2.71}$$

as well as the symbols $\mathcal{A}_{11}(x,\xi)$ and $\mathcal{A}_{22}(x,\xi)$ are strongly elliptic.

Denote by $\lambda^{\mathcal{A}}_1(x),\dots,\lambda^{\mathcal{A}}_N(x)$ and $\lambda^{\mathcal{D}}_1(x),\dots,\lambda^{\mathcal{D}}_M(x)$ the eigenvalues of the matrices

$$\big[\,\mathcal{A}(x,0,\dots,0,+1)\,\big]^{-1}\big[\,\mathcal{A}(x,0,\dots,0,-1)\,\big], \ \ x\in\partial\mathcal{M}\,, \tag{2.72a}$$

$$\big[\,\mathcal{D}(x,0,\dots,0,+1)\,\big]^{-1}\big[\,\mathcal{D}(x,0,\dots,0,-1)\,\big], \ \ x\in\partial\mathcal{C}_0\,, \tag{2.72b}$$

respectively, and define

$$\delta^{\mathcal{A}}_j(x) = \mathrm{Re}\,\big[\,(2\pi\, i)^{-1}\ln\lambda^{\mathcal{A}}_j(x)\,\big], \quad x\in\partial\mathcal{M}, \tag{2.72c}$$

$$\delta^{\mathcal{D}}_k(x) = \mathrm{Re}\,\big[\,(2\pi\, i)^{-1}\ln\lambda^{\mathcal{D}}_k(x)\,\big], \quad x\in\partial\mathcal{C}_0, \tag{2.72d}$$

$$j=1,\dots,N, \quad k=1,\dots,M.$$

Theorem 2.31. *Let a nonclassical ΨDO* $\mathbf{N}(x,D)$ *in* (2.65) *be compiled of a classical ΨDO* $\mathbf{A}(x,D)$ *in* (2.66) *with a strongly elliptic symbol* $\mathcal{A}(x,\xi)$.

The operator $\mathbf{N}(x,D)$ *in* (2.67) *is Fredholm and has the trivial index* $\operatorname{Ind}\mathbf{N}(x,D)=0$ *provided the following constraints hold*

$$\frac{1}{p}-1+\frac{\nu}{2}+\gamma''<s<\frac{1}{p}+\frac{\nu}{2}+\gamma', \tag{2.73}$$

with $\gamma''=\max\{\gamma''_{\mathcal{A}},\gamma''_{\mathcal{D}}\}$ *and* $\gamma'=\min\{\gamma'_{\mathcal{A}},\gamma'_{\mathcal{D}}\}$, *where*

$$\gamma'_{\mathcal{A}}:=\inf_{x\in\partial\mathcal{M},\,1\leqslant j\leqslant N}\delta^{\mathcal{A}}_j(x),\quad \gamma''_{\mathcal{A}}:=\sup_{x\in\partial\mathcal{M},\,1\leqslant j\leqslant N}\delta^{\mathcal{A}}_j(x), \tag{2.74a}$$

$$\gamma'_{\mathcal{D}}:=\inf_{x\in\partial\mathcal{C}_0,\,1\leqslant k\leqslant M}\delta^{\mathcal{D}}_k(x),\quad \gamma''_{\mathcal{D}}:=\sup_{x\in\partial\mathcal{C}_0,\,1\leqslant k\leqslant M}\delta^{\mathcal{D}}_k(x). \tag{2.74b}$$

Moreover, the null-spaces and indices of the operators (2.67) *coincide for all values of the parameter* $q\in[1,+\infty]$ *and all* p, s *which satisfy the inequality* (2.73).

In particular, if the operator (*see* (2.67) *and* (2.68))

$$\mathbf{N}(x,D)\,:\,\widetilde{\mathbb{H}}^{\frac{\nu}{2}}_2\to\mathbb{H}^{-\frac{\nu}{2}}_2$$

is strongly coercive, i.e., for all $W=(U,V)^\top\in\widetilde{\mathbb{H}}^{\frac{\nu}{2}}_2$ *there is a constant* $C_0>0$, *such that*

$$\operatorname{Re}\big\langle\mathbf{N}(x,D)W,W\big\rangle\geqslant C_0\|W\|_{\widetilde{\mathbb{H}}^{\frac{\nu}{2}}_2}\,, \tag{2.75}$$

$$\|W\|^2_{\widetilde{\mathbb{H}}^{\frac{\nu}{2}}_2}=\|U\|^2_{[\widetilde{H}^{\frac{\nu}{2}}_2(\mathcal{C})]^M}+\|V\|^2_{[\widetilde{H}^{\frac{\nu}{2}}_2(\mathcal{M})]^L}\,,$$

then it is invertible in the space setting (2.67) *under the constraints* (2.73).

Proof. Since $\mathcal{C}$ is a proper part of $\mathcal{M}$ we can not apply Theorem 2.28 directly to characterize the Fredholm properties of the operator (2.65). It is a proper place to address the local principle for para-algebras. To this end, let either $\mathbb{Z}^s_p:=\mathbb{H}^s_p$ ($\widetilde{\mathbb{Z}}^s_p:=\widetilde{\mathbb{H}}^s_p$) or $\mathbb{Z}^s_p:=\mathbb{B}^s_{p,q}$ ($\widetilde{\mathbb{Z}}^s_p:=\widetilde{\mathbb{B}}^s_{p,q}$). Consider the quotient para-algebra

$$\Psi'(\widetilde{\mathbb{Z}}^s_p,\mathbb{Z}^{s-\nu}_p)=\big[\Psi(\widetilde{\mathbb{Z}}^s_p,\mathbb{Z}^{s-\nu}_p)/\mathfrak{C}(\widetilde{\mathbb{Z}}^s_p,\mathbb{Z}^{s-\nu}_p)\big]_{2\times2}$$

of all bounded ΨDOs $\Psi(\widetilde{\mathbb{Z}}^s_p,\mathbb{Z}^{s-\nu}_p)$ in the indicated space pairs factored by the ideal of all compact operators $\mathfrak{C}(\widetilde{\mathbb{Z}}^s_p,\mathbb{Z}^{s-\nu}_p)$. Further, for arbitrary point $y\in\overline{\mathcal{M}}$ we define the following localizing class

$$\Delta_y:=\Big\{[g_yI_N],\ g_y\in C^\infty(\mathcal{M}),\ \operatorname{supp}\,g_y\subset W_y,\ g_y(x)=1\ \forall\,x\in\widetilde{W}_y\Big\}, \tag{2.76}$$

where $\widetilde{W}_y\subset W_y\subset\overline{\mathcal{M}}$ is arbitrary pair of small embedded neighborhoods of y. The symbol $[\mathbf{A}]$ stands for the quotient class containing the operator $\mathbf{A}$. It is obvious that the system $\{\Delta_y\}_{y\in\overline{\mathcal{M}}}$ is covering and all its elements $[g_yI_N]$ commute with the class $[\mathbf{B}(x,D)]$ for arbitrary ΨDO $\mathbf{B}(x,D)\in\Psi(\widetilde{\mathbb{Z}}^s_p,\mathbb{Z}^{s-\nu}_p)$ (to verify the commutativity recall that a commutant $\mathbf{B}(x,D)gI_N-g\mathbf{B}(x,D)$ is compact for an arbitrary smooth function g).

ΨDOs are operators of local type: if g_1 and g_2 are functions with disjoint supports $\operatorname{supp}\,g_1\cap\operatorname{supp}\,g_2=\varnothing$, then the operator $g_1\mathbf{B}(x,D)g_2I$ is compact in the spaces where $\mathbf{B}(x,D)$ is a bounded ΨDO. Applying the local property of ΨDOs we can check the following local equivalence

$$[\mathbf{N}(x,D)] \overset{\Delta_y}{\sim} [\mathbf{N}_y(x,D)] \ \ \forall\, y\in\overline{\mathcal{M}}, \tag{2.77}$$

where the local representatives $\mathbf{N}_y(x,D)$ in (2.77) are defined as follows:

$$\mathbf{N}_y(x,D) := \mathbf{A}(x,D)\ :\ [H^s_p(\mathcal{S})]^N \to [H^{s-\nu}_p(\mathcal{S})]^N \ \text{ for } \ y\in\mathcal{C}, \tag{2.78a}$$

$$\mathbf{N}_y(x,D) := \mathbf{A}_{22}(x,D)\ :\ [H^s_p(\mathcal{S})]^L \to [H^{s-\nu}_p(\mathcal{S})]^L \quad \text{for } \ y\in\mathcal{C}_0=\mathcal{M}\setminus\overline{\mathcal{C}}, \tag{2.78b}$$

$$\mathbf{N}_y(x,D) := r_{\mathcal{M}}\mathbf{A}(x,D)\ :\ [\widetilde{H}^s_p(\mathcal{M})]^N \to [H^{s-\nu}_p(\mathcal{M})]^N \quad \text{for } \ y\in\partial\mathcal{M}, \tag{2.78c}$$

$$\mathbf{N}_y(x,D) := \begin{bmatrix} [r_{\mathcal{C}_0^c}\mathbf{A}^{jk}_{11}(x,D)]_{M\times M} & [r_{\mathcal{C}_0^c}\mathbf{A}^{jk}_{12}(x,D)]_{M\times L}\\ [\mathbf{A}^{jk}_{21}(x,D)]_{L\times M} & [\mathbf{A}^{jk}_{22}(x,D)]_{L\times L}\end{bmatrix}_{N\times N} = \\ = \begin{bmatrix} r_{\mathcal{C}_0^c}\mathbf{A}_{11}(x,D) & r_{\mathcal{C}_0^c}\mathbf{A}_{12}(x,D)\\ [\mathbf{A}_{21}(x,D) & \mathbf{A}_{22}(x,D)\end{bmatrix}_{N\times N} :\ \widetilde{\mathbb{V}}^s_p \to \mathbb{V}^{s-\nu}_p, \quad \text{for } \ y\in\partial\mathcal{C}_0, \tag{2.78d}$$

where $\mathcal{C}_0^c$ is the complement surface $\mathcal{C}_0^c := \mathcal{S}\setminus\overline{\mathcal{C}_0}$ with the boundary $\partial\mathcal{C}_0^c = \partial\mathcal{C}_0$,

$$\begin{gathered}\widetilde{\mathbb{V}}^s_p := \big[\widetilde{X}^s_p(\mathcal{C}^c_0)\big]^M \times \big[X^s_p(\mathcal{S})\big]^L,\\ \mathbb{V}^{s-\nu}_p := \big[X^{s-\nu}_p(\mathcal{C}^c_0)\big]^M \times \big[X^{s-\nu}_p(\mathcal{S})\big]^L\end{gathered} \tag{2.79}$$

and either $X^s_p = H^s_p$ or $X^s_p = B^s_{p,q}$.

Due to Theorem 2.45, formulated below, the operator $\mathbf{N}(x,D)$ in (2.67) is Fredholm if and only if the operators $\mathbf{N}_y(x,D)$ in (2.78a)–(2.78d) are Fredholm for all $y\in\overline{\mathcal{M}}$.

Since the ΨDO $\mathbf{N}(x,D)$ is strongly elliptic by the assumption, it has strongly elliptic symbol ΨDOs $\mathcal{N}(x,\xi)$ (see e.g., [21]) and the symbols $\mathcal{A}_{11}(x,\xi)$, $\mathcal{A}_{22}(x,\xi)$ in (2.70) are strongly elliptic due to Remark 2.30.

The ΨDOs $\mathbf{N}_y(x,D)$ in (2.78a) for $y\in\mathcal{C}$ and in (2.78b) for $y\in\mathcal{C}_0$ on the closed manifold S have strongly elliptic symbols and are Fredholm for all $y\in\mathcal{C}\cup\mathcal{C}_0$.

The ΨDO $\mathbf{N}_y(x,D)$ in (2.78c) has strongly elliptic symbol as well, but restricted to the surface with the smooth boundary $\mathcal{M}$ needs the following additional constraints to be Fredholm

$$\frac{1}{p}-1+\frac{\nu}{2}+\gamma'' < s < \frac{1}{p}+\frac{\nu}{2}+\gamma', \tag{2.80}$$

with γ' and γ'' defined in Theorem 2.31.

To investigate the elliptic ΨDO $\mathbf{N}_y(x,D)$ in (2.78d), first we remind that, as noted above, the ΨDO

$$\mathbf{A}_{22}(x,D) = \big[\mathbf{A}^{jk}_{22}(x,D)\big]_{L\times L}$$

has strongly elliptic homogeneous principal symbol due to Lemma 2.29 and Remark 2.30. Since $\mathbf{A}_{22}(x,D)$ is defined on the closed manifold S, it is Fredholm with index 0 and there exists a compact operator $\mathbf{T}$ such that $\mathbf{A}_{22}(x,D)+\mathbf{T}$ is invertible. For the quotient classes the equalities $[\mathbf{A}_{22}(x,D)+\mathbf{T}] = [\mathbf{A}_{22}(x,D)]$ and $[\mathbf{A}_{22}(x,D)+\mathbf{T}]^{-1} = [\mathbf{A}_{22}(x,D)]^{-1}$ hold.

Note that the quotient classes

$$[\mathbf{F}_{\pm}(x,D)] := \begin{bmatrix} [I_{M\times M}] & [[0]_{M\times L}] \\ \pm[\mathbf{A}_{22}(x,D)]^{-1}[\mathbf{A}_{21}(x,D)] & [I_{L\times L}] \end{bmatrix}_{N\times N}$$

are invertible

$$[\mathbf{F}_{-}(x,D)]\,[\mathbf{F}_{+}(x,D)] = [\mathbf{F}_{+}(x,D)]\,[\mathbf{F}_{-}(x,D)] = [I_{N\times N}]$$

and composing the quotient class $[\mathbf{N}_y(x,D)]$ with this invertible quotient class we get

$$[\widetilde{\mathbf{N}}_y(x,D)] := [\mathbf{N}_y(x,D)]\,[\mathbf{F}_{-}(x,D)] := \\ := \begin{bmatrix} [\mathbf{D}(x,D)] & \left[r_{\mathcal{C}_0^c}[\mathbf{A}_{12}(x,D)]_{M\times L}\right] \\ [[0]_{L\times M}] & [\mathbf{A}_{22}(x,D)] \end{bmatrix}_{N\times N}, \qquad (2.81)$$

where

$$\mathbf{D}_y(x,D) = r_{\mathcal{C}_0^c}\Big\{\mathbf{A}_{11}(x,D) - \mathbf{A}_{12}(x,D)[\mathbf{A}_{22}(x,D)]^{-1}\mathbf{A}_{21}(x,D)\Big\} \qquad (2.82)$$

is a strongly elliptic ΨDO of order ν due to Lemma 2.29. It is sufficient to prove that the composition $[\widetilde{\mathbf{N}}_y(x,D)]$ is an invertible class.

Note that $[\widetilde{\mathbf{N}}_y(x,D)]$ is upper block-triangular and the diagonal entry $[\mathbf{A}_{22}(x,D)]$ is an invertible class. Moreover, the entries $[\mathbf{D}(x,D)]$ and $[\mathbf{A}_{22}(x,D)]$ on the diagonal, being ΨDOs, commute (actually, these matrix entries might have different dimension $M\times M$ and $L\times L$, but we can extend the entire matrix $[\widetilde{\mathbf{N}}_y(x,D)]$ by identities on the diagonal and zeros on the off-diagonal entries in the corresponding rows and columns, which does not change the invertibility properties of the matrix and which will equate the dimensions of the diagonal entries). Therefore $[\widetilde{\mathbf{N}}_y(x,D)]$ is invertible if and only if the quotient class $[\mathbf{D}_y(D,x)]$ is invertible. This is interpreted as follows: the operator

$$\widetilde{\mathbf{N}}_y(x,D) \;:\; \widetilde{\mathbb{Z}}_p^s \to \mathbb{Z}_p^{s-\nu}$$

is Fredholm if and only if the operator

$$\mathbf{D}_y(D,x) \;:\; [\widetilde{X}_p^s(\mathcal{C}_0^c)]^M \to [X_p^{s-\nu}(\mathcal{C}_0^c)]^M \qquad (2.83)$$

is Fredholm. Since the principal homogeneous symbol of $\mathbf{D}_y(x,D)$ is $\mathcal{D}(x,\xi)$ defined in (2.71), the operators $\mathbf{D}_y(D,x)$ in (2.73) is Fredholm provided the following constraints are fulfilled

$$\frac{1}{p} - 1 + \frac{\nu}{2} + \gamma''_{\mathcal{D}} < s < \frac{1}{p} + \frac{\nu}{2} + \gamma'_{\mathcal{D}}, \qquad (2.84)$$

where $\gamma'_{\mathcal{D}}$ and $\gamma''_{\mathcal{D}}$ are defined in (2.74b) (see Theorem 2.31).

Summarizing the above we conclude that the ΨDO $\mathbf{N}(x,D)$ in (2.67) is Fredholm provided the system of inequalities (2.80), (2.84) hold, which can be rewritten in the form (2.73).

Next we have to prove that the ΨDO $\mathbf{N}(x,D)$ in (2.67) has zero index, $\operatorname{Ind}\mathbf{N}(x,D)=0$. It suffices to prove this for a particular case $s=\nu/2$ and $p=2$ (cf., e.g., [7,19]). To this end, we consider the homotopy

$$\mathbf{N}_\lambda(x,D):=\lambda\,\mathbf{R}(x,D)+(1-\lambda)\,\mathbf{N}(x,D)\ :\ \widetilde{\mathbb{H}}{}_2^{\frac{\nu}{2}}\to\mathbb{H}_2^{-\frac{\nu}{2}},$$
$$\mathbf{R}(x,D):=\begin{bmatrix}\Lambda_{\mathcal{C}}^{(\frac{\nu}{2},\nu)}(x,D)I_M & [0]_{M\times L}\\ [0]_{L\times M} & \Lambda_{\mathcal{M}}^{(\frac{\nu}{2},\nu)}(x,D)I_L\end{bmatrix}]_{N\times N},$$

where $0\leqslant\lambda\leqslant1$,

$$\Lambda_{\mathcal{C}}^{(\frac{\nu}{2},\nu)}(x,D):=\Lambda_{\mathcal{C}}^{\frac{\nu}{2}}(x,D)\widetilde{\Lambda}_{\mathcal{C}}^{\frac{\nu}{2}}(x,D)$$

and

$$\widetilde{\Lambda}_{\mathcal{C}}^{\frac{\nu}{2}}(x,D)\ :\ \widetilde{H}_2^{\frac{\nu}{2}}(\mathcal{C})\to\widetilde{H}_2^{0}(\mathcal{C})=H_2^0(\mathcal{C}),$$
$$\Lambda_{\mathcal{C}}^{\frac{\nu}{2}}(x,D)\ :\ H_2^0(\mathcal{C})\to H_2^{-\frac{\nu}{2}}(\mathcal{C})$$

are the Bessel potential operators, arranging isomorphism of the spaces. The principal homogeneous symbol of the operator $\Lambda_{\mathcal{C}}^{(\frac{\nu}{2},\nu)}(x,D)$ is positive definite and the operator $\Lambda_{\mathcal{C}}^{(\frac{\nu}{2},\nu)}(x,D):\widetilde{H}_2^{\frac{\nu}{2}}(\mathcal{C})\to H_2^{-\frac{\nu}{2}}(\mathcal{C})$ is invertible (cf., e.g., [21], [22, § 4]).

The definition and the properties of the isomorphism $\Lambda_{\mathcal{M}}^{(\frac{\nu}{2},\nu)}(x,D)$ are similar.

Thus, $\mathbf{R}(x,D):\widetilde{\mathbb{H}}{}_2^{\frac{\nu}{2}}\to\mathbb{H}_2^{-\frac{\nu}{2}}$ has a positive definite symbol and is invertible.

The continuous homotopy $\mathbf{N}_\lambda(x,D)$ connects the initial operator $\mathbf{N}_0(x,D)=\mathbf{N}(x,D)$ with the invertible one

$$\mathbf{N}_1(x,D)=\mathbf{R}(x,D)\ :\ \widetilde{\mathbb{H}}{}_2^{\frac{\nu}{2}}\to\mathbb{H}_2^{-\frac{\nu}{2}}.$$

Moreover, the operator $\mathbf{N}_\lambda(x,D)$ is strongly elliptic for all $0\leqslant\lambda\leqslant1$ since it represents the sum of the operators with positive definite and strongly elliptic symbols (see Remark 2.30). Then the operator $\mathbf{N}_\lambda(x,D)$ is Fredholm for all $0\leqslant\lambda\leqslant1$. Therefore,

$$\operatorname{Ind}\mathbf{N}(x,D)=\operatorname{Ind}\mathbf{N}_0(x,D)=\operatorname{Ind}\mathbf{N}_1(x,D)=\operatorname{Ind}\mathbf{R}(x,D)=0.$$

From the results obtained above it follows that the ΨDO $\mathbf{N}(x,D)$ in (2.67) is Fredholm with index zero.

Now, if $\mathbf{N}(x,D)$ is strongly coercive (see (2.75)), it has a trivial kernel in the space $\widetilde{\mathbb{H}}{}^{\frac{\nu}{2}}$ and the operator

$$\mathbf{N}(x,D)\ :\ \widetilde{\mathbb{H}}{}^{\frac{\nu}{2}}\to\mathbb{H}^{-\frac{\nu}{2}}$$

is invertible. Then, the operator $\mathbf{N}(x, D)$ has the trivial null space in the space setting (2.67) and is invertible for all p and s if the conditions (2.73) are fulfilled. The proof is complete. $\square$

Remark 2.32. To achieve the invertibility of the operator $\mathbf{N}(x, D)$ in the space setting (2.67) under the conditions (2.73) we need less than the strong coercivity property (2.75). It suffices to know that the operator is Fredholm and its null space $\operatorname{Ker}\mathbf{N}(x, D)$ is trivial only in one space, say in $\widetilde{\mathbb{H}}_2^{\frac{\nu}{2}} = [\widetilde{H}_2^{\frac{\nu}{2}}(\mathcal{C})]^M \times [\widetilde{H}_2^{\frac{\nu}{2}}(\mathcal{M})]^L$. Due to the concluding part of Theorem 2.28 this implies that $\operatorname{Ker}\mathbf{N}(x, D)$ is trivial and the operator $\mathbf{N}(x, D)$ is invertible in the space settings (2.67) provided the constraints (2.73) hold (see, e.g., Lemma 6.1).

Further, if the operator (see (2.67) and (2.68))

$$\mathbf{N}(x, D) \,:\, \widetilde{\mathbb{H}}_2^{\frac{\nu}{2}} \to \mathbb{H}_2^{-\frac{\nu}{2}}$$

is coercive, i.e., for all $W = (U, V)^\top \in \widetilde{\mathbb{H}}_2^{\frac{\nu}{2}}$ there are constants $C_0 > 0$ and $C_1 > 0$, such that

$$\operatorname{Re}\langle \mathbf{N}(x, D)W, W\rangle \geqslant C_0\, \|W\|^2_{\widetilde{\mathbb{H}}_2^{\frac{\nu}{2}}} - C_1\, \|W\|^2_{\widetilde{\mathbb{H}}_2^{\kappa}} \tag{2.85}$$

for $\kappa < \nu/2$, then $\mathbf{N}(x, D)$ is Fredholm in the space setting (2.67) and has the trivial index $\operatorname{Ind}\mathbf{N}(x, D) = 0$, provided the conditions (2.73) hold.

2.4. **ΨDOs on hypersurfaces in $\mathbb{R}^n$.** We remind that $S \subset \mathbb{R}^n$ is a C^k−smooth, compact hypersurface in $\mathbb{R}^n$ and $\boldsymbol{\nu}(t)$ is the outward unit normal vector field.

Let a surface $S \subset \mathbb{R}^n$, which is a particular case of a manifold (see Definition 2.14) be given by the coordinate diffeomorphisms (cf. (2.14))

$$\varkappa_j \,:\, V_j \to S_j, \qquad V_j \subset \mathbb{R}^{n-1},\;\; S_j \subset S,\;\; j = 1, \dots, N,$$

where $S = \bigcup_{j=1}^N S_j$ is a covering of S. Let $\mathcal{G}_{\varkappa_j}$ be the square root of Gram's determinant

$$\mathcal{G}_{\varkappa_j}(y) := \sqrt{\det\, \big[\langle \partial_\ell \varkappa_j, \partial_m \varkappa_j\rangle\big]_{(n-1)\times(n-1)}}\,, \tag{2.86}$$

which is responsible for the integration on S (see [62, § IV.10.38], [66, § 4.6] and (2.89) below).

Now we prove the following assertion.

Theorem 2.33. *Let $m < -1$ and $a \in \mathbb{CLS}^m(\mathbb{R}^n)$ be a classical $N \times N$ matrix-symbol* (*see Definition* 2.8*; for $m = 1$ cf. Remark* 2.35 *below*),

$$a(x, \xi) = \mathcal{F}_{z\mapsto\xi}[k(x, z)] \sim a_m(x, \xi) + a_{m-1}(x, \xi) + \cdots\,,$$
$$a_{m-k}(x, \lambda\xi) = \lambda^{m-k} a_{m-k}(x, \xi),\;\; \xi \in \mathbb{R}^n,\;\; \lambda > 0.$$

Then the trace on the surface[*]

$$\mathbf{a}_S(\mathcal{X}, D)\varphi(\mathcal{X}) := \gamma_S \mathbf{a}(x, D)(\varphi \otimes \delta_S)(\mathcal{X}) = \\ = \int\limits_S k(\mathcal{X}, \mathcal{X} - \mathcal{Y})\varphi(\mathcal{Y})dS\,, \quad \mathcal{X} \in S, \tag{2.87}$$

is a pseudodifferential operator

$$\mathbf{a}_S(\mathcal{X}, D) \; : \; H^s_p(S) \to H^{s-m-1}_p(S)$$

with the classical symbol:

$$a_S(\mathcal{X}, \xi') \sim \sum_{k=0}^{\infty} a_{S,m+1-k}(\mathcal{X}, \xi'), \; a_{S,m+1-k} \in \mathbb{S}^{m+1-k}(\mathcal{T}^* S), \; \xi' \in \mathbb{R}^{n-1}. \tag{2.88}$$

Proof. Let us check that the operator $\mathbf{a}_S(t, D)$ in (2.87) is pseudodifferential. First note that,

$$\int\limits_S g(\mathcal{Y})\, dS = \sum_{j=1}^{M} \int\limits_{\mathbb{R}^{n-1}} \psi^0_j(y)\mathcal{G}_{\varkappa_j}(y)g(\varkappa_j(y))\, dy, \quad \psi^0_j := \varkappa_{j,*}\psi_j, \tag{2.89}$$

(cf. [62, §IV.10.38], [66, §4.6]). Therefore

$$\mathbf{a}^{(j)}(x, D)\varphi(x) = \varkappa_{j,*}\psi_j \mathbf{a}_S(\mathcal{X}, D)\psi_j \varkappa^{-1}_{j,*}\varphi(x) = \\ = \psi^0_j(x) \int\limits_{\mathbb{R}^{n-1}} \psi_j(y)\mathcal{G}_{\varkappa_j}(y)k(\varkappa_j(x), \varkappa_j(x) - \varkappa_j(y))\varphi(y)\, dy \sim \\ \sim \sum_{m=0}^{\infty} \psi^0_j(x) \int\limits_{\mathbb{R}^{n-1}} \psi^0_j(y)\mathcal{G}_{\varkappa_j}(y)k_{m-l}(\varkappa_j(x), \varkappa_j(x) - \varkappa_j(y))\varphi(y)\, dy\,,$$

where $\psi^0_j(x) := \psi_j(\varkappa_j(x))$ are pull back of cut-off functions and

$$\mathcal{F}_{x\to\xi} k_{m-l}(z, \xi) = a_{m-l}(z, \xi)\,, \quad k_{m-l}(z, \lambda\, x) = \lambda^{n-m+l} k_{m-l}(z, x)$$

for all $\lambda > 0$ and all $x \in \mathbb{R}^{n-1}$. By the Taylor formula at $y \in \mathbb{R}^{n-1}$

$$\varkappa_j(x) - \varkappa_j(y) \; = \varkappa'_j(x)(x - y) + \\ + \sum_{2\leqslant|\delta|\leqslant N} \frac{1}{\delta!}(\partial^\delta \varkappa_j)(x)(x-y)^\delta + \varkappa_{j,K+1}(x, x-y)\,, \tag{2.90}$$

$$\varkappa_{j,K+1}(x, x - y) := \sum_{|\delta|=K+1} \frac{(N + 1)z^\delta}{\delta!} \int\limits_0^1 (1 - t)^K \partial^\delta_y \varkappa_j(x, y + t(x - y))\, dt\,,$$

where $\varkappa'_j(x)$ is the Jacobi matrix of $\varkappa_j(x)$. By inserting (2.90) into the kernels $k_{m-l}(\varkappa_j(x), \varkappa_j(x) - \varkappa_j(y))$ and by applying the Taylor formula, now at $\varkappa'_j(x)(x - y) \in \mathbb{R}^{n-1}$, we get

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[*]For the definition of the surface delta-function see (3.75).

$$k_{m-l}(\varkappa_j(x),\varkappa_j(x)-\varkappa_j(y))=\sum_{|\alpha|=0}^{K-l}\frac{1}{\alpha!}k^{(\alpha)}_{m-l}(\varkappa_j(x),\varkappa'_j(x)(x-y))\times$$

$$\times\Big[\sum_{|\delta|=2}^{K-l}\frac{1}{\delta!}(\partial^\delta\varkappa_j)(x)(x-y)^\delta\Big]^\alpha+k^1_{m-l,K+1}(x,x-y)=$$

$$=\sum_{|\alpha|=0}^{K-l}\sum_{|\beta|=2|\alpha|}b_{\alpha,\beta}(x)k^{(\alpha)}_{m-l}(\varkappa_j(x),\varkappa'_j(x)(x-y))(x-y)^\beta$$

$$+k^2_{m-l,K+1}(x,x-y)\,,\quad k^{(\alpha)}_{m-l}(x,z):=\partial^\alpha_z k_{m-l}(x,z),\quad (2.91)$$

where α, β, and δ are multi-indices, $b_{0,\beta}=1$ and the other coefficients $b_{\alpha,\beta}(x)$, $|\alpha|>0$, are defined from the equality

$$\frac{1}{\alpha!}\Big[\sum_{|\delta|=2}^{K-l}\frac{z^\delta}{\delta!}\partial^\delta\varkappa_j(x)\Big]^\alpha=\sum_{|\beta|=2|\alpha|}^{(K-l)|\alpha|}b_{\alpha,\beta}(x)z^\beta\,. \quad (2.92)$$

Obviously,

$$|\partial^\gamma_x\partial^\mu_z k^q_{m-l,K+1}(x,z)|\leqslant C_N|z|^{K+1-|\mu|}\ \ \forall\,\gamma,\mu\in\mathbb{N}^{n-1}_0\,,\quad q=1,2\,. \quad (2.93)$$

Applying the Taylor formula to the product $\widetilde{\psi}^0_j(y)\mathcal{G}_{\varkappa_j}(y)$ we write

$$\widetilde{\psi}^0_j(y)\mathcal{G}_{\varkappa_j}(y)=\mathcal{G}_{\varkappa_j}(x)+$$
$$+\sum_{|\gamma|=1}^{K}\frac{(-1)^{|\gamma|}}{\gamma!}\partial^\gamma\mathcal{G}_{\varkappa_j}(x)(x-y)^\gamma+\Psi_{\varkappa_j,K+1}(x,x-y) \quad (2.94)$$

with the remainder $\Psi_{\varkappa_j,K+1}(x,x-y)$ which has the form similar to that $\varkappa_{j,K+1}$ in (2.90), with an estimate similar to (2.93). Then the remainder $k^3_{m-l,K+1}(x,z):=\varkappa_{j,K+1}(x,x-y)+\Psi_{\varkappa_j,K+1}(x,z)$ has the estimate (2.93).

Note that the cut off function $\widetilde{\psi}^0_j(y)$ does not appears in the right-hand side of (2.94) since $\widetilde{\psi}^0_j(x)=1$ and all derivatives vanish in a neighborhood of x.

With (2.90)–(2.94) at hand we get the asymptotic decomposition (2.88) with the following entries:

$$a_{S,m+1-k}(\varkappa_j(x),\xi')=\sum_{l=0}^{k}\sum_{\substack{2\alpha\leqslant\beta\\|\beta|+|\gamma|-|\alpha|=k-m}}\frac{b_{\alpha,\beta}(x)(-\partial_x)^\gamma\mathcal{G}_{\varkappa_j}(x)}{\gamma!}\times$$

$$\times\mathcal{F}_{z\mapsto\xi'}\big[z^{\beta+\gamma}k^{(\alpha)}_{m-l}(\varkappa_j(x),\varkappa'_j(x)z)\big]=$$

$$= \sum_{l=0}^{k} \sum_{\substack{2\alpha\leqslant\beta \\ |\beta|+|\gamma|-|\alpha|=k-m}} \frac{(-i)^{|\alpha+\beta+\gamma|} b_{\alpha,\beta}(x)(-\partial_x)^\gamma \mathcal{G}_{\varkappa_j}(x)}{2\pi \det \varkappa_j'(x,0)\gamma!} \times$$
$$\times \int_{-\infty}^{\infty} \widetilde{a}^{(\beta+\gamma)}_{m-l+|\alpha|} \big(\varkappa_j(x), [\varkappa_j'(x,0)]^\top(\xi',\lambda)\big)\, d\lambda \;, \quad (2.95)$$

where

$$\widetilde{a}^{(\delta)}_{m-l+|\alpha|}(x,\xi) := \xi^\alpha \partial^\delta_{\xi'} a_{m-l}(x,\xi)\,, \quad (2.96)$$
$$[\varkappa_j'(x,0)^\top]^{-1}\xi = \big((\partial_1\varkappa_j,\xi),\dots,(\partial_1\varkappa_j,\xi),(\boldsymbol{\nu},\xi)\big)\,, \quad \xi=(\xi',\lambda)\in\mathbb{R}^n.$$

Indeed, we proceed as follows:

$$\mathcal{F}_{z\mapsto\xi'}\big[z^{\beta+\gamma}k^{(\alpha)}_{m-l}\big(\varkappa_j(x),\varkappa_j'(x)z\big)\,\big] =$$
$$= \int_{\mathbb{R}^{n-1}} e^{i\xi' z}\frac{z^{\beta+\gamma}}{(2\pi)^n}\int_{\mathbb{R}^n}(-i\eta)^\alpha e^{-i\varkappa_j'(x)z\eta}a_{m-l}(\varkappa_j(x),\eta)\,d\eta =$$
$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{n-1}} e^{i\xi' z} z^{\beta+\gamma}\int_{\mathbb{R}^n} e^{-i\varkappa_j'(x,0)(z,0)\eta}\widetilde{a}_{m-l+|\alpha|}(\varkappa_j(x),\eta)\,d\eta\,dz =$$
$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{n-1}} e^{i\xi' z} z^{\beta+\gamma}\int_{\mathbb{R}^n} e^{-i(0,z)\varkappa_j'(x,0)^\top\eta}\widetilde{a}_{m-l+|\alpha|}(\varkappa_j(x),\eta)\,d\eta\,dz =$$
$$= \frac{\gamma_1}{(2\pi)^n}\,\partial^{\beta+\gamma}_{\xi'}\int_{\mathbb{R}^{n-1}} e^{i\xi' z}\int_{\mathbb{R}^n} e^{-iz\eta}\widetilde{a}_{m-l+|\alpha|}(\varkappa_j(x),[\varkappa_j'(x,0)^\top]^{-1}\eta)\,d\eta\,dz =$$
$$= \frac{\gamma_1}{2\pi}\,\partial^{\beta+\gamma}_{\xi'}\mathcal{F}_{z\mapsto\xi'}\mathcal{F}^{-1}_{\eta'\mapsto z}\bigg[\int_{-\infty}^{\infty}\widetilde{a}_{m-l+|\alpha|}(\varkappa_j(x),[\varkappa_j'(x,0)^\top]^{-1}(\eta',\lambda))\,d\lambda\bigg] =$$
$$= \frac{\gamma_1}{2\pi}\partial^{\beta+\gamma}_{\xi'}\int_{-\infty}^{\infty}\widetilde{a}_{m-l+|\alpha|}(\varkappa_j(x),[\varkappa_j'(x,0)^\top]^{-1}(\xi',\lambda))d\lambda\,,$$

where

$$\gamma_1 = \frac{(-i)^{|\alpha+\beta+\gamma|}}{\det\varkappa_j'(x,0)}\,.$$

For the remainder term in the asymptotic (2.88) with the kernel admitting the estimate (2.93), we easily derive that it belongs to the class $\mathbb{S}^{m-N}(\mathcal{T}^*S)$. □

Remark 2.34. As a byproduct of the proof we can identify the symbol $a_{S,m+1-k}$ as a homogeneous in the variable ξ function of order $m+1-k$ and

$$a_{S,m+1-k}(\varkappa_j(x),\xi') = \sum_{l=0}^{k} \sum_{\substack{|\beta|+|\gamma|-|\alpha|=k-l \\ 2\alpha\leqslant\beta}} \frac{(-i)^{|\alpha+\beta+\gamma|} b_{\alpha,\beta}(x)\partial_x^\gamma \mathcal{G}_{\varkappa_j}(x)}{2\pi \det \varkappa_j'(x,0)\gamma!} \times$$

$$\times \int_{-\infty}^{\infty} \widetilde{a}^{(\beta+\gamma)}_{m-l+|\alpha|}\big(x,[\varkappa_j'(x,0)^\top]^{-1}(\xi',\lambda)\big)\,d\lambda\,, \quad \xi'\in\mathbb{R}^{n-1}\,, \quad (2.97)$$

where the coefficients $b_{\alpha,\beta}(x)$ for $|\alpha|>0$ are defined in (2.92), the symbol $\widetilde{a}^{(\delta)}_{m-l+|\alpha|}(x,\xi)$ in (2.96) and the Gram determinant $\mathcal{G}_{\varkappa_j}(x)$ in (2.86).

In particular, the homogeneous principal symbol reads

$$a_{S,pr}(\varkappa_j(x),\xi') = \frac{\mathcal{G}_{\varkappa_j}(x)}{2\pi\det\varkappa_j'(x,0)} \int_{-\infty}^{\infty} a_m\big(x,[\varkappa_j'(x,0)^\top]^{-1}(\xi',\lambda)\big)\,d\lambda =:$$

$$=: a_{S,m+1}(\varkappa_j(x),\xi'), \quad x\in Y_j\,, \quad \xi'\in\mathbb{R}^{n-1}\,. \quad (2.98)$$

Remark 2.35. If $m=-1$ we can not write (2.87), although formulae (2.88)–(2.98) hold with some modification. The difference emerges because

$$\mathbf{a}_S(\mathcal{x},D)\varphi(\mathcal{x}) = \gamma_S\mathbf{a}(D)(\varphi\otimes\delta_S)(\mathcal{x}) =$$

$$= c_0(\mathcal{x})\varphi(\mathcal{x}) + \int_S k_0(\mathcal{x},\mathcal{x}-\mathcal{y})\varphi(\mathcal{y})\,dS \quad (2.99)$$

is a pseudodifferential operator of order zero $\mathbf{a}_S(\mathcal{x},D)\ :\ H^s_p(S)\to H^s_p(S)$, i.e., it is a singular integral operator, the integral in (2.87) is understood in the Cauchy principal value sense and

$$c_0(\mathcal{x}) = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n-1}{2}}} \int_{|\mathcal{y}|=1} a_{S,\mathrm{pr}}(\mathcal{x},\mathcal{y})\,dS\,. \quad (2.100)$$

The kernel $k_0(t,\tau)$ satisfies the cancelation condition (cf. [45, Ch. IX, § 1], [22, formula (4.26)]):

$$k_0(\mathcal{x},\mathcal{y}) = \mathcal{F}^{-1}_{\xi\mapsto\mathcal{y}}\big[a_{S,pr}(\mathcal{x},\xi)-c_0(\mathcal{x})\big]\,, \quad \mathcal{x},\mathcal{y}\in S\,. \quad (2.101)$$

The further proof is verbatim to the case $m<-1$.

2.5. **The local principle.** "Freezing coefficients" is a common method of investigation in the theory of integro-differential and pseudodifferential equations. It is convenient to formalize the method as a local principle. There exists many different versions of a local principle (Allan's, Simonenko's, Gohberg–Krupnik's etc.). Most convenient for us is the local principle for Banach para-algebras from [15] which is based on the Gohberg–Krupnik's local principle (also see [17] for an earlier version).

Definition 2.36. Let $\mathfrak{A}$ be a Banach algebra. A set $\Delta\subset\mathfrak{A}$ is called a localizing class if:

(i) $0\notin\Delta$;

(ii) for a pair of elements $a_1, a_2 \in \Delta$ an element $a \in \Delta$ exists such that $a_m a = a a_m = a$, $m = 1, 2$.

It is clear that any subset of $\mathfrak{A}$ containing 0, has the property (ii), but the property (i) in the definition excludes such a trivial localizing class.

Definition 2.37. Let $\Delta \in \mathfrak{A}$ be a localizing class in a Banach algebra $\mathfrak{A}$. Two elements $a, b \in \mathfrak{A}$ are called Δ-equivalent from the left (Δ-equivalent from the right) and we write $a \overset{L-\Delta}{\sim} b$ ($a \overset{R-\Delta}{\sim} b$, respectively), provided

$$\inf_{u\in\Delta} \|(a-b)u\|_{\mathfrak{A}} = 0 \quad \left(\inf_{u\in\Delta} \|u(a-b)\|_{\mathfrak{A}} = 0, \ \text{ respectively}\right).$$

If a and b are both the left and the right equivalent, we say that these elements are Δ-equivalent and write $a \overset{\Delta}{\sim} b$.

Lemma 2.38. *The relations of local equivalences* (*left, right, two-sided*) *are all linear, continuous and multiplicative:*

(i) *Let $\Delta \subset \mathfrak{A}$ be a bounded set, $a_k, b_k \in \mathfrak{A}$, $a_k \overset{R-\Delta}{\sim} b_k$, ($a_k \overset{L-\Delta}{\sim} b_k$), $k = 1, 2$. Then*

$$\alpha_1 a_1 + \alpha_2 a_2 \overset{R-\Delta}{\sim} \alpha_1 b_1 + \alpha_2 b_2 \ \ (\alpha_1 a_1 + \alpha_2 a_2 \overset{L-\Delta}{\sim} \alpha_1 b_1 + \alpha_2 b_2) \ \ \forall\, \alpha_1,\, \alpha_2 \in \mathbb{C};$$

(ii) *let $\Delta \subset \mathfrak{A}$ be a bounded set, $a_m, b_m \in \mathfrak{A}$, $a_m \overset{R-\Delta}{\sim} b_m$, ($a_m \overset{L-\Delta}{\sim} b_m$), $m \in \mathbb{N}$, $\lim\limits_{m\to\infty} a_m = a$ and $\lim\limits_{m\to\infty} b_m = b$. Then $a \overset{R-\Delta}{\sim} b$ ($a \overset{L-\Delta}{\sim} b$);*

(iii) *if $a, b, c \in \mathfrak{A}$ and $a \overset{L-\Delta}{\sim} b$ ($a \overset{R-\Delta}{\sim} b$), then $ca \overset{L-\Delta}{\sim} cb$ ($ac \overset{R-\Delta}{\sim} bc$).*

Definition 2.39. Let Δ be a localizing class in $\mathfrak{A}$. An element $a \in \mathfrak{A}$ is called Δ-invertible from the left (Δ-invertible from the right) if there exist $d \in \mathfrak{A}$ and $u \in \Delta$ such that $dau = u$ ($uad = u$, respectively). If $a \in \mathfrak{A}$ is Δ-invertible from the left and is Δ-invertible from the right, we call it Δ-invertible.

Lemma 2.40. *Let $\Delta \subset \mathfrak{A}$ be a localizing class, $a, b \in \mathfrak{A}$, $a \overset{L-\Delta}{\sim} b$ ($a \overset{R-\Delta}{\sim} b$). If a is Δ-invertible from the left* (*is Δ-invertible from the right*) *then b is Δ-invertible from the left* (*is Δ-invertible from the right*).

Definition 2.41. A system $\{\Delta_y\}_{y\in\Omega}$ of localizing classes in $\mathfrak{A}$ is said to be covering if from arbitrary collection $\{u_y\}_{y\in\Omega}$ of elements $u_y \in \Delta_y$ there can be selected a finite collection $\{u_{y_j}\}_{j=1}^N$ so that the sum $\sum\limits_{j=1}^N u_{y_j}$ is invertible in $\mathfrak{A}$.

Lemma 2.42. *Let $\{\Delta_y\}_{y\in\Omega}$ be a covering system of localizing classes $\Delta_y \subset \mathfrak{A}$, $a \in \mathfrak{A}$ and let $ua = au$ for all $u \in \bigcup\limits_{y\in\Omega} \Delta_y$.*

Then a is invertible from the left (*is invertible from the right*) *if and only if a is Δ_y-invertible from the left* (*is Δ_y-invertible from the right*) *for all $y \in \Omega$.*

The next theorem is an immediate consequence of the two foregoing lemmata.

Theorem 2.43 (Local principle). *Let $\{\Delta_y\}_{y\in\Omega}$ be a covering systems of localizing classes in $\mathfrak{A}$. Let elements $a \in \mathfrak{A}$ and $a_y \in \mathfrak{A}$ be Δ_y-equivalent from the left (be Δ_y-equivalent from the right) for all $y \in \Omega$.*

Let $au = ua$ for all $u \in \Delta_y$, $y \in \Omega$. Then a is invertible from the left (a is invertible from the right) if and only if the element a_y is Δ_y-invertible from the left (is Δ_y-invertible from the right) for all $y \in \Omega$.

The formulated local principle can not be applied to an operator $A \in \mathcal{L}(\mathfrak{B}_1, \mathfrak{B}_2)$ which maps different Banach spaces $\mathfrak{B}_1 \neq \mathfrak{B}_2$. To involve such cases the method needs certain modification. An option is to consider para-algebras.

Definition 2.44. A quadruple $\mathfrak{A} = [\mathfrak{A}_{jk}]_{2\times 2}$ of Banach spaces is called a Banach para-algebra if there exists a binary mapping (a multiplication)

$$\mathfrak{A}_{jk} \times \mathfrak{A}_{kr} \to \mathfrak{A}_{jr}$$

for each choice of $j, k, r = 1, 2$, which is continuous, associative and bilinear.

The definition implies that the spaces $\mathfrak{A}_{11}$ and $\mathfrak{A}_{22}$ from a Banach para-algebra $\mathfrak{A} = [\mathfrak{A}_{jk}]_{2\times 2}$ are Banach algebras.

For a pair of Banach spaces $\mathfrak{B}_1$ and $\mathfrak{B}_2$ the quadruple

$$\mathfrak{A}_0(\mathfrak{B}_1, \mathfrak{B}_2) := [\mathcal{L}(\mathfrak{B}_j, \mathfrak{B}_k)]_{2\times 2}$$

represents a Banach para-algebra. Moreover, the quotient algebras factored by the space of all compact operators $\mathfrak{C}(\mathfrak{B}_j, \mathfrak{B}_k)$,

$$\mathfrak{A}'_0(\mathfrak{B}_1, \mathfrak{B}_2) = [\mathfrak{A}'_{jk}]_{2\times 2} = \big[\mathcal{L}(\mathfrak{B}_j, \mathfrak{B}_k)/\mathfrak{C}(\mathfrak{B}_j, \mathfrak{B}_k)\big]_{2\times 2}$$

represents a Banach para-algebra as well. For simplicity we dwell on these particular para-algebras.

Let $\mathfrak{A} = [\mathfrak{A}_{jk}]_{2\times 2}$ be a Banach para-algebra of operators $\mathfrak{A}_{jk} = \mathcal{L}(\mathfrak{B}_j, \mathfrak{B}_k)$ or quotient algebras $\mathfrak{A}_{jk} = \mathcal{L}(\mathfrak{B}_j, \mathfrak{B}_k)/\mathfrak{C}(\mathfrak{B}_j, \mathfrak{B}_k)$. Let $\{\Delta_y\}_{y\in\Omega}$ be a common covering system of localizing classes in $\mathcal{L}(\mathfrak{B}_1)$ and in $\mathcal{L}(\mathfrak{B}_2)$:

$$\{\Delta_y\}_{y\in\Omega} \subset \mathcal{L}(\mathfrak{B}_1) \cap \mathcal{L}(\mathfrak{B}_2)\,. \tag{2.102}$$

The local equivalence and the local invertibility are defined for para-algebras as in the case $\mathfrak{B}_1 = \mathfrak{B}_2$ (see Definition 2.39 and Definition 2.37). The following theorem is proved by a minor modification of Theorem 2.43.

Theorem 2.45. *Let $\mathfrak{A} = [\mathfrak{A}_{jk}]_{2\times 2}$ be a Banach para-algebra of operators $\mathfrak{A}_{jk} = \mathcal{L}(\mathfrak{B}_j, \mathfrak{B}_k)$ or quotient algebras $\mathfrak{A}_{jk} = \mathcal{L}(\mathfrak{B}_j, \mathfrak{B}_k)/\mathfrak{C}(\mathfrak{B}_j, \mathfrak{B}_k)$. Let $\{\Delta_y\}_{y\in\Omega}$ be a common covering system of localizing classes in $\mathcal{L}(\mathfrak{B}_1)$ and in $\mathcal{L}(\mathfrak{B}_2)$ (see (2.102)).*

Let $A \in \mathfrak{A}_{jk}$ and $A_y \in \mathcal{L}(\mathfrak{B}_{jk})$ be Δ_y-equivalent from the left (be Δ_y-equivalent from the right) for all $y \in \Omega$.

If $AA_y = A_yA$ for all $A_y \in \Delta_y$, $y \in \Omega$, then A is invertible from the left (is invertible from the right) if and only if A_y is Δ_y-invertible from the left (is Δ_y-invertible from the right) for all $y \in \Omega$.

3. Layer Potentials

In the present section we expose some well known results about properties of layer potentials for a second order partial differential operators, enriched with some simplified results from [18] and adapted to the present purposes.

Throughout the section we assume that $\Omega^+ \subset \mathbb{R}^n$ is a bounded domain with the boundary $\partial\Omega^+ = S$, which is C^k-smooth, compact surface in $\mathbb{R}^n$ and $\Omega^- := \mathbb{R}^n \backslash \overline{\Omega^+}$ is the exterior unbounded domain; $\boldsymbol{\nu}(\mathcal{x}) = (\nu_1(\mathcal{x}), \dots, \nu_n(\mathcal{x}))$ is the outward unit normal vector to the surface S at the point $\mathcal{x} \in S$.

By Ω we denote either of the domains Ω^- and Ω^+ in cases if there is no need to distinguish them.

3.1. **Green's formulae for a general second order PDO.** Let $\mathbf{A}(x, D)$ be a second order partial differential operator with smooth $N \times N$ matrix coefficients

$$\mathbf{A}(x, D) := \sum_{|\alpha|\leqslant 2} a_\alpha(x)\partial^\alpha, \quad a_\alpha \in C^\infty(\overset{\bullet}{\Omega}). \tag{3.1}$$

The operator

$$\mathbf{A}^*(x, D) := \sum_{|\alpha|\leqslant 2} (-1)^\alpha \partial^\alpha \big[\,\overline{a_\alpha(x)}\,\big]^\top I_N\,, \tag{3.2}$$

is the formally adjoint to $\mathbf{A}(x, D)$

$$\begin{gathered}(\mathbf{AU}, \mathbf{V})_\Omega = (\mathbf{U}, \mathbf{A}^*\mathbf{V})_\Omega \ \ \forall\, \mathbf{U}, \mathbf{V} \in C^2(\Omega)\,, \\ \mathbf{U} = (U_1, \dots, U_N)^\top, \quad \mathbf{V} = (V_1, \dots, V_N)^\top, \quad \operatorname{supp} \mathbf{V} \subset \Omega,\end{gathered} \tag{3.3}$$

with respect to the sesquilinear form

$$(\mathbf{U}, \mathbf{V})_\Omega := \int\limits_\Omega \langle \mathbf{U}(y), \mathbf{V}(y)\rangle\, dy\,, \tag{3.4}$$

$$\langle \mathbf{U}(x), \mathbf{V}(x)\rangle \equiv \mathbf{U}(x)\cdot\mathbf{V}(x) := \sum_{j=1}^N U_j(x)\overline{V_j(x)}\,.$$

Definition 3.1. The operator $\mathbf{A}(x, D)$ in (3.1) is called normal on S if

$$\inf_{\mathcal{X}\in S} \big|\det \mathcal{A}_{\rm pr}(\mathcal{x}, \boldsymbol{\nu}(\mathcal{x}))\big| \neq 0 \tag{3.5}$$

and is called elliptic on the domain Ω if

$$\inf_{x\in\Omega,\ |\xi|=1} \big|\det \mathcal{A}_{\rm pr}(x, \xi)\big| \neq 0, \tag{3.6}$$

where $\mathcal{A}_{\rm pr}(x,\xi)$ is the principal homogeneous symbol of $\mathbf{A}(x,D)$

$$\mathcal{A}_{\rm pr}(x,\xi) := \sum_{|\alpha|=2} a_\alpha(x)(-i\xi)^\alpha\,, \quad (x,\xi)\in\overline{\Omega}\times\mathbb{R}^n\,. \tag{3.7}$$

Lemma 3.2. *A partial differential operator* $\mathbf{A}(x,D)$ *given by* (3.1) *is a normal operator if and only if the formally adjoint operator* $\mathbf{A}^*(x,D)$ *in* (3.2) *is normal.*

If $\mathbf{A}(x,D)$ *is elliptic on the surface* S *then it is normal.*

For the operator $\mathbf{A}(x,D)$ *with constant coefficients the inverse is valid: if* $\mathbf{A}(x,D)$ *is normal, it is elliptic.*

Proof. The first assertion follows since the homogeneous principal symbol of the formally adjoint operator reads as follows (cf. (3.7)):

$$\mathcal{A}^*_{\rm pr}(x,\xi) = \sum_{|\alpha|=2}\big[\,\overline{a_\alpha(x)}\,\big]^\top\xi^\alpha = \Big[\,\overline{\mathcal{A}_{\rm pr}(x,\xi)}\,\Big]^\top, \quad (x,\xi)\in\overline{\Omega}\times\mathbb{R}^n\,. \tag{3.8}$$

The second assertion is also trivial since the ellipticity condition on S

$$\inf_{x\in S\,,\ |\xi|=1}|\det\mathcal{A}_{\rm pr}(x,\xi)|\neq 0 \tag{3.9}$$

implies the condition (3.5).

To prove the third and the last claim of the lemma note that if t ranges over the smooth surface S without boundary then the corresponding unit normal vector $\boldsymbol{\nu}(t)$ ranges through the entire unit sphere. This, obviously, implies that normal operator with constant coefficients is elliptic. □

Let us consider a boundary value problem

$$\begin{cases}\mathbf{A}(x,D)u(x) = f(x), & x\in\Omega,\\ \gamma_S\mathbf{B}_0u(\varkappa) = G(t), & \varkappa\in S,\end{cases} \tag{3.10}$$

where $\mathbf{A}(x,D)$ is the basic operator written in (3.1),

$$\mathbf{B}_0(x,D) = \sum_{|\alpha|\leqslant 1} b_\alpha(x)\partial^\alpha\,, \quad b_\alpha\in C^\infty(U^\pm_S)$$

is a boundary operator of order 0 or 1 with $N\times N$ matrix coefficients and $U^\pm_S$ stand for one-sided neighbourhoods of S.

Along with (3.10) we consider a BVP for the formally adjoint operator

$$\begin{cases}\mathbf{A}^*(x,D)v(x) = d(x), & x\in\Omega,\\ \gamma_S\mathbf{C}_0v(\varkappa) = H(\varkappa), & \varkappa\in S,\end{cases} \tag{3.11}$$

(see (3.2)), where $\mathbf{C}_0(x,D)$ is a boundary differential operator

$$\mathbf{C}_0(x,D) = \sum_{|\alpha|\leqslant\mu_j} c_\alpha(x)\partial^\alpha\,, \quad c_\alpha\in C^\infty(U^\pm_S),$$

of order 0 or 1 such that $\operatorname{ord}\mathbf{C}_0(x,D)+\operatorname{ord}\mathbf{B}_0(x,D)=1$.

A pair of boundary differential operators $\{\mathbf{B}_0(x,D), \mathbf{B}_1(x,D)\}$ is called a Dirichlet system if both operators are normal and have different orders $\operatorname{ord}\mathbf{B}_0(x,D)=0$, $\operatorname{ord}\mathbf{B}_1(x,D)=1$.

A simplest example of a Dirichlet system of boundary operators is

$$\mathbf{B}_0 = I_N, \quad \mathbf{B}_1 = \partial_{\boldsymbol{\nu}}\, I_N = \sum_{j=1}^{n} \nu_j \partial_j\, I_N. \tag{3.12}$$

Definition 3.3. The BVP (3.11) is called formally adjoint to the BVP (3.10) if there exist operators $\mathbf{B}_1(x,D)$ and $\mathbf{C}_1(x,D)$ such that Green's formula

$$\int\limits_{\Omega^+} \big[\langle \mathbf{A}u, v\rangle - \langle u, \mathbf{A}^* v(y)\rangle\big]\, dy = \sum_{j=0}^{1} \int\limits_{S} \langle \mathbf{B}_j u, \mathbf{C}_j v\rangle\, dS \tag{3.13}$$

holds for all pairs $u, v \in C^2(\overline{\Omega^+})$ of smooth functions with compact supports if the domain is unbounded.

For some classes of operators and under additional constraints on behaviour of functions, Green's formula (3.13) can be written for unbounded domains.

Theorem 3.4. *For a pair of normal boundary operators* $\mathbf{B}_0(x,D)$ *and* $\mathbf{C}_0(x,D)$*, with the property* $\operatorname{ord}\mathbf{C}_0 + \operatorname{ord}\mathbf{B}_0 = 1$ *there exist another pair of operators* $\mathbf{B}_1(x,D)$ *and* $\mathbf{C}_1(x,D)$ *such that Green's formula* (3.13) *holds and*

$$\operatorname{ord}\mathbf{B}_0 \neq \operatorname{ord}\mathbf{B}_1, \quad \operatorname{ord}\mathbf{C}_0 \neq \operatorname{ord}\mathbf{C}_1, \quad \operatorname{ord}\mathbf{C}_1 + \operatorname{ord}\mathbf{B}_1 = 1. \tag{3.14}$$

The boundary operator $\mathbf{C}_1(x,D)$ *is unique if the boundary operator* $\mathbf{B}_1(x,D)$ *is chosen already and both of them are normal operators if the basic operator* $\mathbf{A}(x,D)$ *is normal.*

We will prove Theorem 3.4 later in this section, to expose prior some auxiliary lemmata. Moreover, as a byproduct of the proof we write explicit formulae for $\mathbf{C}_1(x,D)$ whenever $\mathbf{B}_1(x,D)$ is given (see Corollary 3.12).

Most elliptic systems which appear in applications (e.g., in elasticity, thermo-elasticity, electro elasticity, micropolar elasticity, hydrodynamics etc.) have second order and some of them are self-adjoint. Therefore we consider some simplification of Green's formula for self-adjoint systems.

Assume that the operator in (3.1) is represented in the form

$$\mathbf{A}(x,D) = \sum_{|\alpha|,|\beta|\leqslant 1} (-1)^{|\alpha|} \partial^\alpha a_{\alpha,\beta}(x)\partial^\beta, \ \ a_{\alpha,\beta} \in C^\infty(\overline{\Omega^+}), \tag{3.15}$$

and consider the associated sesquilinear form

$$A(u,v) := \sum_{|\alpha|,|\beta|\leqslant 1} \int\limits_{\Omega^+} \big\langle a_{\alpha,\beta}(y)\partial_y^\beta u(y), \partial_y^\alpha v(y)\big\rangle\, dy, \ \ u,v \in C^1(\overline{\Omega^+}). \tag{3.16}$$

Theorem 3.5. *For arbitrary basic differential operator* (3.15) *of order* 2 *with matrix* $N \times N$ *coefficients there exists a boundary operator* $\mathbf{B}(x, D)$ *of order* $\operatorname{ord} \mathbf{B} = 1$ *such that*

$$A(u,v) = \int\limits_{\Omega^+} \langle \mathbf{A}u, v\rangle\, dy + \int\limits_{S} \langle \mathbf{B}u, v\rangle\, dS\,, \quad u, v \in C^2(\overline{\Omega^+})\,. \tag{3.17}$$

If $\mathbf{A}$ *is formally self-adjoint,* $\mathbf{A} = \mathbf{A}^*$, *then Green's second formula* (3.13)*acquires the following form*

$$\int\limits_{\Omega^+} \big[\langle \mathbf{A}u, v\rangle - \langle u, \mathbf{A}v\rangle\big]\, dy = \int\limits_{S} \big[\langle \mathbf{B}u, v\rangle - \langle u, \mathbf{B}v\rangle\big]\, dS, \tag{3.18}$$

i.e., $\mathbf{B_0} = -\mathbf{C_1} = I$ *and* $\mathbf{B_1} = -\mathbf{C_0} = \mathbf{B}$.

The proof will be exposed later in this section.

We remind that $\partial_{\boldsymbol{\nu}}$ is the normal derivative, defined on the boundary surface S of the domain Ω (see (3.12)). We can extend the normal vector filed $\boldsymbol{\nu}(t)$, $t \in S$ in a neighbourhood $U_S \subset \overline{\Omega}$ of the surface S with the same smoothness and denote the extended filed by $\boldsymbol{\nu}(x)$, $x \in U_S$. Then the derivative with respect to the extended field $\partial_{\boldsymbol{\nu}}$ can be applied to functions in the neighborhood U_S.

Definition 3.6. A first order partial differential operator with scalar coefficients on the surface S

$$\mathbf{a}(\mathcal{x}, D)u(\mathcal{x}) := \sum_{j=1}^{n} a_j(\mathcal{x})\partial_j u(\mathcal{x})\,, \quad a_j \in C(S), \quad \mathcal{x} \in S, \tag{3.19}$$

is called tangential if the vector field compiled of the coefficients $a = (a_1, \dots, a_n)$ is orthogonal to the normal vector field on the boundary:

$$\langle a(\mathcal{x}), \boldsymbol{\nu}(\mathcal{x})\rangle = \sum_{j=1}^{n} a_j(\mathcal{x})\nu_j(\mathcal{x}) \equiv 0 \;\text{ on }\; S. \tag{3.20}$$

A tangential differential operator can be applied to a function $\varphi(\mathcal{x})$ which is defined only on the surface S. The simplest definition is to take such derivative in direction of the tangential vector $a(\mathcal{x})$:

$$\partial_a \varphi(\mathcal{x}) = \mathbf{a}(\mathcal{x}, D)\varphi(\mathcal{x}) := \lim_{h\to 0} \frac{\varphi(\mathcal{x} + h\, a(\mathcal{x})) - \varphi(\mathcal{x})}{h}\,. \tag{3.21}$$

More precise definition involves the orbit of the vector field $\boldsymbol{\nu}(\mathcal{x})$ and can be found, for example, in [68].

The simplest tangential derivatives are Gunter's derivatives

$$\mathcal{D}_j := \partial_j - \nu_j \partial_{\boldsymbol{\nu}}, \quad j = 1, \dots, n, \tag{3.22}$$

which can be applied to a function φ defined only on the surface S. Also the differential operator of high order, compiled of Gunter's derivatives

$$\mathbf{A}(\mathcal{x}, \mathcal{D}) := \sum_{|\alpha|\leqslant 2} a_\alpha(\mathcal{x})\mathcal{D}^\alpha, \quad \mathcal{D}^\alpha := \mathcal{D}_1^{\alpha_1} \cdots \mathcal{D}_n^{\alpha_n}, \tag{3.23}$$

can be applied to a function φ defined only on the surface S.

Lemma 3.7. *For a first order differential operator*

$$\mathbf{G}(x,D)u(x) := \sum_{j=1}^{n} g_j(x)\,\partial_j u(x)\,, \quad g_j \in C(\Omega^+), \quad x \in \Omega^+, \tag{3.24}$$

the following integration by parts formulae is valid

$$\int_{\Omega^+} \langle \mathbf{G}(y,D)u(y), v(y)\rangle\, dy = \int_S \langle \mathbf{G}(\mathcal{y},\boldsymbol{\nu}(\mathcal{y}))u(\mathcal{y}), v(\mathcal{x})\rangle\, dS - \int_{\Omega^+} \langle u(y), \mathbf{G}^*(y,D)v(y)\rangle\, dy, \quad u,v \in C^1(\overline{\Omega^+}), \tag{3.25}$$

where $\mathbf{G}^*(y,D)$ *is the formally adjoint operator to* $\mathbf{G}(y,D)$ *and*

$$\mathbf{G}(\mathcal{x},\boldsymbol{\nu}(\mathcal{x})) := \sum_{j=1}^{n} g_j(\mathcal{x})\nu_j(\tau)\,, \quad \mathcal{x} \in S. \tag{3.26}$$

In particular,

$$\int_{\Omega^+} \langle \partial_{\boldsymbol{\nu}} u(y), v(y)\rangle\, dy = \int_S \langle u(\mathcal{y}), v(\mathcal{y})\rangle\, dS - \int_{\Omega^+} \langle u(y), \partial^*_{\boldsymbol{\nu}} v(y)\rangle\, dy, \tag{3.27}$$

where

$$\partial^*_{\boldsymbol{\nu}} u(x) := -\sum_{k=1}^{n} \partial_k[\boldsymbol{\nu}_k(x)u(x)] = -\partial_{\boldsymbol{\nu}} u(x) - \sum_{k=1}^{n} [\partial_k \nu_k(x)]u(x)\,, \tag{3.28}$$

for $u,v \in C^1(\overline{\Omega^+})$ *and if* $\mathbf{G}(x,D)$ *is a scalar tangential on the boundary* S *operator* (*see Definition* 3.6)*, then the integral on the boundary* S *in* (3.25) *disappears and the formula acquires the form*

$$\int_{\Omega^+} \langle \mathbf{G}(y,D)u(y), v(y)\rangle\, dy = -\int_{\Omega^+} \langle u(y), \mathbf{G}^*(y,D)v(y)\rangle\, dy\,. \tag{3.29}$$

Proof. Formula (3.25) is a direct consequence of the celebrated Gauß formula

$$\int_{\Omega^+} [\partial_k u(y)]\,\overline{v(y)}\, dy = \int_S \nu_k(\mathcal{y})u(\mathcal{y})\,\overline{v(\mathcal{y})}\, dS - \int_{\Omega^+} u(y)\,\overline{\partial_k v(y)}\, dy, \quad k=1,\dots,n.$$

Formulae (3.27) follow from (3.25) since in such a case $\mathbf{G}(x,D) = \langle \boldsymbol{\nu}, \mathrm{grad}\rangle$ and, therefore, $\mathbf{G}(\mathcal{x},\boldsymbol{\nu}(\mathcal{x})) = \langle \boldsymbol{\nu}(\mathcal{x}), \boldsymbol{\nu}(\mathcal{x})\rangle \equiv 1$.

Formula (3.29) follows from (3.25) as well: if $\mathbf{G}(x,D) = \langle \mathbf{G}(x), \mathrm{grad}\rangle$ is a tangential operator, $\mathbf{G}(\mathcal{x},\boldsymbol{\nu}(\mathcal{x})) = \langle \mathbf{G}(\mathcal{x}), \boldsymbol{\nu}(\mathcal{x})\rangle \equiv 0\ \forall\, \mathcal{x} \in S$. $\square$

Successive application of integration by parts (3.25) to the basic operator $\mathbf{A}(x,D)$ in (3.1) (not necessarily normal one) provides a Green formula

(3.13) with some boundary operators $\{\mathbf{B}_j\}_{j=0}^1$ and $\{\mathbf{C}_j\}_{j=0}^1$. It is easy to trace down that these operators have proper orders

$$\operatorname{ord}\mathbf{B}_j + \operatorname{ord}\mathbf{C}_j = j\,, \quad j = 0, 1, \tag{3.30}$$

but it is difficult to control their principal symbols, because we need to replace these "random" boundary operators by those prescribed in BVPs (3.10) and (3.11). To this end we should derive the special Green formula in Theorem 3.8.

The operator $\mathbf{A}(x, D)$ in (3.1) can be rewritten in the form

$$\begin{aligned} \mathbf{A}(x, D) &= \mathcal{A}_{\rm pr}(x, \boldsymbol{\nu}(x))\partial_{\boldsymbol{\nu}}^2 I_N + \mathbf{A}_1(x, \mathcal{D})\partial_{\boldsymbol{\nu}} I_N + \mathbf{A}_2(x, \mathcal{D})\,, \\ \mathbf{A}_k(x, \mathcal{D}) &= \sum_{|\alpha|\leqslant k} a_{k,\alpha}^0(x)\mathcal{D}^\alpha\,, \quad x \in \Omega\,, \quad k = 1, 2\,, \end{aligned} \tag{3.31}$$

where $\partial_{\boldsymbol{\nu}}$ is the directional derivative, defined in (3.12), and $\mathbf{A}_1(x, \mathcal{D})$ and $\mathbf{A}_2(x, \mathcal{D})$ are tangential partial differential operators in the plane orthogonal to the vector field $\boldsymbol{\nu}$ of order 1 and 2, respectively (see (3.23)); $\mathcal{A}_{\rm pr}(x, \xi)$ is the principal homogeneous symbol of $\mathbf{A}(x, D)$ (see (3.7)).

The representation (3.31) follows easily if we substitute partial derivatives ∂_j by (see (3.22))

$$\partial_j = \mathcal{D}_j + \boldsymbol{\nu}_j\partial_{\boldsymbol{\nu}}\,, \quad j = 1, \dots, n. \tag{3.32}$$

For arbitrary operator $\mathbf{A}(x, D)$ of order m we arrange a mathematical induction. Substituting each derivative from clusters ∂^α, $|\alpha| = m$, by the sum in (3.32), we certainly deduce the formula (3.31) modulo operators of order $m - 1$, which are written in the form (3.31) by the assumption.

Theorem 3.8. *Let* $\mathbf{A}(x, D)$ *be defined in* (3.1) *and* $\mathbf{A}_0(x, \mathcal{D}) := \mathbf{A}_{\rm pr}(x, \boldsymbol{\nu}(x))$*,* $\mathbf{A}_1(x, \mathcal{D})$*, and* $\mathbf{A}_2(x, \mathcal{D})$ *be tangential operators from the representation* (3.31)*. Then Green's formula* (3.13) *holds with the following boundary operators:*

$$\begin{gathered} \mathbf{B}_0(x, D) := I_N, \quad \mathbf{B}_1(x, D) := \mathcal{A}_{\rm pr}(x, \boldsymbol{\nu}(x))\partial_{\boldsymbol{\nu}} I_N, \quad \mathbf{C}_1(x, D) := I_N, \\ \mathbf{C}_0(x, D) := \partial_{\boldsymbol{\nu}}^*\mathcal{A}_{\rm pr}^*(x, \boldsymbol{\nu}(x)) + \mathbf{A}_1^*(x, \mathcal{D}), \end{gathered} \tag{3.33}$$

where $\mathbf{A}_1^*(x, \mathcal{D})$ *and* $\partial_{\boldsymbol{\nu}}^*$ *are the formally adjoint operators to* $\mathbf{A}_1(x, \mathcal{D})$ *and* $\partial_{\boldsymbol{\nu}}$ *respectively.*

The operators $\mathbf{C}_0(x, D)$ *and* $\mathbf{B}_1(x, D)$ *are normal.*

Proof. By taking $u, v \in C^2(\overline{\Omega^+})$ and applying (3.27) successively 2 times we get the following:

$$\int\limits_{\Omega^+} \langle \mathbf{A}u, v\rangle\, dy = \int\limits_S \langle u^+, [(A_1^* + \partial_{\boldsymbol{\nu}}^*\mathcal{A}_{\rm pr}^*)v]^+\rangle\, dS + \int\limits_S \langle [\partial_{\boldsymbol{\nu}} u]^+, v^+\rangle\, dS + \int\limits_{\Omega^+} \langle u, \mathbf{A}^* v\rangle\, dy.$$

The Green formula (3.13) for BVPs (3.10), (3.11) with operators (3.33) is proved.

Note that for the symbol $\partial^*_{\boldsymbol{\nu}}(\mathcal{x},\xi)$ of the formally adjoint operator to the normal derivative $\partial_{\boldsymbol{\nu}}$ (see (3.28)) and the symbols $\mathcal{A}^*_{\rm pr}(\mathcal{x})$, $\mathcal{A}^*_j(\mathcal{x},\xi)$, $j=1,2$, have the properties:

$$(\partial^*_{\boldsymbol{\nu}})_{\rm pr}(\mathcal{x},\boldsymbol{\nu}(\mathcal{x})):=-\sum_{j=1}\nu_j(\mathcal{x})\nu_j(\mathcal{x})\equiv 1,\ \ \mathcal{A}^*_{\rm pr}(\mathcal{x})=\big[\,\overline{\mathcal{A}_{\rm pr}(\mathcal{x},\boldsymbol{\nu}(\mathcal{x}))}\,\big]^\top,$$
$$(\mathcal{A}^*_j)_{\rm pr}(\mathcal{x},\boldsymbol{\nu}(\mathcal{x}))\equiv 0 \ \text{ for } \ j=1,2.$$

Then

$$(\mathcal{B}_1)_{\rm pr}(\mathcal{x},\boldsymbol{\nu}(\mathcal{x}))=(\mathcal{C}_0)_{\rm pr}(\mathcal{x},\boldsymbol{\nu}(\mathcal{x}))=[\overline{\mathcal{A}_{\rm pr}(\mathcal{x},\boldsymbol{\nu}(\mathcal{x}))}]^\top$$

are normal since

$$\overline{\det\,\mathcal{A}_{\rm pr}(\mathcal{x},\boldsymbol{\nu}(\mathcal{x}))}\neq 0. \qquad \square$$

Let us introduce some shortened notation. Consider boundary operator systems arranged as vectors of length 2:

$$\begin{aligned}\mathbf{B}^{(2)}(x,D)&:=\big\{\mathbf{B}_0(x,D),\mathbf{B}_1(x,D)\big\}^\top,\\ \mathbf{C}^{(2)}(x,D)&:=\big\{\mathbf{C}_1(x,D),\mathbf{C}_0(x,D)\big\}^\top.\end{aligned} \tag{3.34}$$

Note, that in (3.34) we have arranged the vector-operators in ascending orders:

$$\operatorname{ord}\mathbf{B}_0(x,D)=\operatorname{ord}\mathbf{C}_1(x,D)=0,\quad \operatorname{ord}\mathbf{B}_1(x,D)=\operatorname{ord}\mathbf{C}_0(x,D)=1.$$

Applied to a N-vector-function they produce vector-functions of length $2N$:

$$\mathbf{B}^{(2)}(x,D)u:=\big(\mathbf{B}_0(x,D)u,\mathbf{B}_1(x,D)u\big)^\top.$$

Without restriction of generality we suppose that orders are arranged as follows

$$\operatorname{ord}\mathbf{B}_0=\operatorname{ord}\mathbf{C}_1=0,\quad \operatorname{ord}\mathbf{B}_1=\operatorname{ord}\mathbf{C}_0=1. \tag{3.35}$$

Under the notation (3.34) Green's formula (3.13) is written in the form

$$\int\limits_{\Omega^+}\big[\mathbf{A}u\cdot v-u\cdot\mathbf{A}^*v\big]\,dy=\int\limits_S\mathbf{B}^{(2)}u\cdot\mathbf{C}^{(2)}v\,dS\,. \tag{3.36}$$

Moreover, the system of boundary operators $\mathbf{B}^{(2)}(\tau,D)$ is written as follows

$$\mathbf{B}^{(2)}(x,D)=\mathbf{b}^{(2\times 2)}(x,\mathcal{D})\partial^{(2)}_{\boldsymbol{\nu}}(x,D)\,, \tag{3.37}$$

where

$$\partial^{(2)}_{\boldsymbol{\nu}}:=\begin{bmatrix}I_N & 0\\ 0 & \partial_{\boldsymbol{\nu}}I_N\end{bmatrix}, \tag{3.38}$$

$\mathbf{b}^{(2\times 2)}(x,\mathcal{D})$ is a $2N\times 2N$ lower block-triangular matrix-operator

$$\mathbf{b}^{(2\times 2)}(x,\mathcal{D})=\begin{bmatrix}\mathcal{B}_{0,\rm pr}(x,\boldsymbol{\nu}(x)) & 0\\ \mathbf{B}_{1,1}(x,\mathcal{D}) & \mathcal{B}_{1,\rm pr}(x,\boldsymbol{\nu}(x))\end{bmatrix} \tag{3.39}$$

The entry $\mathbf{B}_{1,1}(x,\mathcal{D})$ is a first order tangential differential operator from the representation of the boundary operator

$$\mathbf{B}_1(x,D) = \mathcal{B}_{1,\mathrm{pr}}(x,\boldsymbol{\nu}(x))\partial_{\boldsymbol{\nu}} I_N + \mathbf{B}_{1,1}(x,\mathcal{D}), \tag{3.40}$$
$$\mathbf{B}_{1,1}(x,\mathcal{D}) = \sum_{|\alpha|\leqslant 1} b_\alpha(x)\mathcal{D}^\alpha I_N\,, \quad x\in\Omega$$

and $\mathcal{B}_{1,\mathrm{pr}}(x,\xi)$ stands for the homogeneous principal symbol of $\mathbf{B}_1(x,D)$.

A similar representation is available for the system $\mathbf{C}^{(2)}(\tau,D)$:

$$\mathbf{C}^{(2)}(x,D) = \mathbf{c}^{(2\times 2)}(x,\mathcal{D})\partial^{(2)}_{\boldsymbol{\nu}}(x,D)\,, \tag{3.41}$$

where $\mathbf{c}^{(2\times 2)}(x,\mathcal{D})$ is a $2N\times 2N$ lower block-triangular matrix-operator, similar to $\mathbf{b}^{(2\times 2)}(x,\mathcal{D})$ in (3.39).

Invertible block matrix-operators of type (3.38) will be referred to as admissible operators. The set of admissible matrix-operators is a multiplicative ring: finite compositions and inverses of admissible matrix-operators are admissible again. The listed properties are trivial, except the last one. To check this note that for a Dirichlet system $\mathbf{B}^{(2)}(\tau,D)$ the corresponding operator matrix $\mathbf{b}^{(2\times 2)}(x,\mathcal{D})$ in (3.38) is admissible in a neighborhood of S. Indeed, the boundary operators $\mathbf{B}_0(\tau,D)$, $\mathbf{B}_1(\tau,D)$ are normal and, therefore, the entries of the principal diagonal in (3.38) are non-degenerate in a small neighborhood of the boundary

$$\det\mathcal{B}_{j,\mathrm{pr}}(x,\boldsymbol{\nu}(x)) \neq 0, \;\; x\in U^{\pm}_S, \;\; j=0,1.$$

This allows to invert the matrix and the inverse reads:

$$\left[\mathbf{b}^{(2\times 2)}(x,\mathcal{D})\right]^{-1} = \begin{bmatrix} \mathcal{B}^{-1}_{0,0}(x,\boldsymbol{\nu}) & 0 \\ -\mathcal{B}^{-1}_{1,0}(x,\boldsymbol{\nu})\mathbf{B}_{1,1}(x,\mathcal{D})\mathcal{B}^{-1}_{0,0}(x,\boldsymbol{\nu}) & \mathcal{B}^{-1}_{1,0}(x,\boldsymbol{\nu}) \end{bmatrix}. \tag{3.42}$$

Now we are in a position to prove the following.

Lemma 3.9. *Two Dirichlet systems of partial differential operators* $\mathbf{B}^{(2)}(x,D)$ *and* $\mathbf{C}^{(2)}(x,D)$ (*see* (3.34)) *are related as follows*

$$\mathbf{B}^{(2)}(x,D) = (\mathbf{bc^{-1}})^{(2\times 2)}(x,\mathcal{D})\mathbf{C}^{(2)}(x,D)\,, \tag{3.43}$$

where the matrix

$$(\mathbf{bc^{-1}})^{(2\times 2)}(x,\mathcal{D}) := \mathbf{b}^{(2\times 2)}(x,\mathcal{D})\big[\,\mathbf{c}^{(2\times 2)}(x,\mathcal{D})\,\big]^{-1} \tag{3.44}$$

is admissible lower triangular (*cf.* (3.39)).

Proof. The proof follows immediately from the representations (3.37), (3.41) and the invertibility of the corresponding admissible matrices. □

Corollary 3.10. *If* $\{\mathbf{B}_j(x,D)\}_{j=0}^1$ *and* $\{\mathbf{C}_j(x,D)\}_{j=0}^1$ *are Dirichlet systems of partial differential operators and the traces*

$$(\mathbf{C}_j u)^{\pm}(\mathcal{x}) = G^{\pm}_j(\mathcal{x}), \;\; j=0,1, \;\; \mathcal{x}\in S, \tag{3.45}$$

for some function $u \in H_p^{s+1}(\Omega)$, $s > 1+1/p$, *are known. Then the boundary values* $\{\gamma_S^{\pm}\mathbf{B}_j u(\mathcal{x})\}_{j=0}^{1}$ *are well defined. In particular, the normal derivative* $\partial_{\boldsymbol{\nu}} u(\mathcal{x})$ *is well defined.*

Proof. Indeed, let us apply (3.43) and write

$$\big(\mathbf{B}^{(2)}(\mathcal{x}, D)u\big)^{\pm}(\mathcal{x}) = (\mathbf{bc^{-1}})^{(2\times 2)}(\mathcal{x}, \mathcal{D})\big(\mathbf{C}^{(2)}(\mathcal{x}, D)u\big)^{\pm}(\mathcal{x}). \tag{3.46}$$

We get a well defined operation: the tangential operator $(\mathbf{bc^{-1}})^{(2\times 2)}(\mathcal{x}, \mathcal{D})$ applied to a vector-function $(\mathbf{C}^{(2)}(\mathcal{x}, D)u)^{\pm}(\mathcal{x})$ which is defined on the surface (see (3.21)–(3.23)). □

As a corollary we can derive the order reduction for a boundary operator $\mathbf{B}_j(x, D)$ with high order $\operatorname{ord}\mathbf{B}_j(x, D) \geqslant 2$.

Corollary 3.11. *Let the basic operator* $\mathbf{A}(x, D)$ *of order* 2 *in* (3.1) *be normal. If the order of a boundary operator* $\mathbf{B}_j(x, D)$ *in* (3.10) *is bigger than* 1, $r_j := \operatorname{ord}\mathbf{B}_j(x, D) \geqslant 2 = \operatorname{ord}\mathbf{A}(x, D)$, *than*

$$\mathbf{B}_j(x, D)u(x) = \mathbf{B}_j^0(x, D)u(x) + \mathbf{C}(x, D)\mathbf{A}(x, D)u(x)\,, \quad x \in \Omega\,, \tag{3.47}$$

where $\operatorname{ord}\mathbf{C}(x, D) = r_j - 2$ *and* $r_j^0 = \operatorname{ord}\mathbf{B}_j^0(x, D) \leqslant 1$, *i.e.,*

$$\mathbf{B}_j^0(x, D) = \sum_{j=0}^{1} \mathbf{b}_j(x, \mathcal{D})\,\partial_{\boldsymbol{\nu}}^j I_N \tag{3.48}$$

and $\mathbf{b}_0(x, \mathcal{D})$, $\mathbf{b}_1(x, \mathcal{D})$ *are tangential operators.*

In particular, if u *is a solution of the equation* $\mathbf{A}(x, D)u(x) = f(x)$ *in* Ω *and* $\big\{\mathbf{B}_j^0(\mathcal{x}, D)u\big\}^{\pm}(\mathcal{x}) = G_j^{\pm}(\mathcal{x})$ *are given, from* (3.47) *we get*

$$\big\{\mathbf{B}_j(\mathcal{x}, D)u\big\}^{\pm}(\mathcal{x}) = G_j^{\pm}(\mathcal{x}) + \{\mathbf{C}(\mathcal{x}, D)f\}^{\pm}(\mathcal{x})\,, \quad \mathcal{x} \in S. \tag{3.49}$$

Proof. Since

$$\mathbf{B}_j(x, D) = \sum_{j=0}^{r_j} \mathbf{B}_{jk}(x, \mathcal{D})\partial_{\boldsymbol{\nu}}^k I_N \tag{3.50}$$

it suffices to have the representation (3.47) only for the Dirichlet data $\mathbf{B}_j(x, D) = \partial_{\boldsymbol{\nu}}^j I_N$ for $j = 2, 3, \dots$.

The operator $\mathbf{A}(x, D)$ in (3.1) is normal and due to (3.31) we get the representation for $j = 2$:

$$\partial_{\boldsymbol{\nu}}^2 I_N = [\mathcal{A}_{\rm pr}(x, \boldsymbol{\nu}(x))]^{-1}\Big[\mathbf{A}(x, D) - \sum_{k=0}^{1} \mathbf{A}_{2-j}(x, \mathcal{D})\partial_{\boldsymbol{\nu}}^j I_N\Big]. \tag{3.51}$$

Further we proceed by mathematical induction: if the representation is known for $j = 2 + k$

$$\begin{gathered}\partial_{\boldsymbol{\nu}}^{2+k} I_N = \mathbf{G}_{2+k}(x, D) + \mathbf{C}_{2+k}(x, D)\mathbf{A}(x, D)\,, \quad x \in \Omega\,,\\ \mathbf{G}_{2+k}(x, D) := \sum_{j=0}^{1} \mathbf{G}_{2+k,j}(x, \mathcal{D})\partial_{\boldsymbol{\nu}}^j I_N\,,\end{gathered} \tag{3.52}$$

we write

$$\partial_{\boldsymbol{\nu}}^{2+k+1} I_N = \mathbf{G}_{2+k+1}(x,D) + \mathbf{C}_{2+k+1}(x,D)\mathbf{A}(x,D)\,, \quad x \in \Omega\,,$$
$$\mathbf{C}_{2+k+1}(x,D) = \partial_{\boldsymbol{\nu}} \mathbf{C}_{2+k}(x,D),$$
$$\mathbf{G}_{2+k+1}(x,\mathcal{D}) := \partial_{\boldsymbol{\nu}} \mathbf{G}_{2+k}(x,\mathcal{D}) =$$
$$= \mathbf{G}^0_{2+k1,0}(x,\mathcal{D})\partial^2_{\boldsymbol{\nu}(x)} I_N + \sum_{j=0}^{1} \mathbf{G}^0_{2+k+1,j}(x,\mathcal{D})\partial^k_{\boldsymbol{\nu}} I_N\,.$$

By inserting in the latter $\partial^2_{\boldsymbol{\nu}} I_N$ from (3.51), we get a representation for $\partial_{\boldsymbol{\nu}}^{2+k+1} I_N$ similar to (3.52). □

Proof of Theorem 3.4. Let $\mathbf{B}^{(2)}(x,D) = (\mathbf{B}_0(x,D), \mathbf{B}_1(x,D))$ be a Dirichlet system and $\operatorname{ord}\mathbf{B}_j = j$, $j = 0, 1$ (cf. (3.35)).

Our starting point is the green formula from Theorem 3.8, which we rewrite in the form:

$$\int\limits_{\Omega^+} \big[\langle \mathbf{A}u, v\rangle - \langle u, \mathbf{A}^* v\rangle\big]\, dy = \int\limits_{S} \big\langle (\partial^{(2)}_{\boldsymbol{\nu}} u)^+, (\mathbf{G}^{(2)} v)^+ \big\rangle \, dS \tag{3.53}$$

(cf. (3.36)), where $\partial^{(2)}_{\boldsymbol{\nu}} u = (u, \partial_{\boldsymbol{\nu}} u)^\top$ (see (3.38)) and

$$\mathbf{G}^{(2)}(x,D)v := \big\{\mathbf{G}_0(x,D)v, \mathbf{G}_1(x,D)v\big\}^\top = \\ = (\partial^{(2)}_{\boldsymbol{\nu}})^* \big(\mathbf{A}^{(2\times 2)}(x,\mathcal{D})\big)^* \; (v,v)^\top \tag{3.54}$$

(see (3.33)). The matrix operator

$$\mathbf{A}^{(2\times 2)}(x,\mathcal{D}) = \begin{bmatrix} \mathcal{A}_{\rm pr}(x,\boldsymbol{\nu}(x)) & 0 \\ \mathbf{A}_1(x,\mathcal{D}) & \mathcal{A}_{\rm pr}(x,\boldsymbol{\nu}(x)) \end{bmatrix} \tag{3.55}$$

is composed of tangential differential operators of the representation (3.31).

$\mathbf{A}^{(2\times 2)}(x,\mathcal{D})$ is admissible if and only if $\mathbf{A}(x,D)$ is a normal operator. Due to (3.37),

$$\partial^{(2)}_{\boldsymbol{\nu}} I_N = \big[\mathbf{b}^{(2\times 2)}(\varkappa,\mathcal{D})\big]^{-1} \mathbf{B}^{(2)}(\varkappa,D)\,, \quad \varkappa \in S\,. \tag{3.56}$$

Inserting (3.56) into (3.53), taking into account that the admissible matrix operator $\mathbf{b}^{(2\times 2)}(x,\mathcal{D})$ is tangential and, thus, possesses the surface dual, we get

$$\int\limits_{\Omega^+} \big[\langle \mathbf{A}u, v\rangle - \langle u, \mathbf{A}^* v\big]\, dy = \\ = \int\limits_{S} \big\langle (\mathbf{b}^{(2\times 2)})^{-1}\mathbf{B}^{(2)} u, \mathbf{G}^{(2)} v\big\rangle \, dS = \int\limits_{S} \big\langle \mathbf{B}^{(2)} u, \mathbf{C}^{(2)} v\big\rangle \, dS\,, \tag{3.57}$$

where $\mathbf{C}^{(2)}(x,D)$ is uniquely defined by the relation

$$\mathbf{C}^{(2)}(x,D) = \big[(\mathbf{b}^{(2\times 2)})^*(x,\mathcal{D})\big]^{-1} \big[(\partial^{(2)}_{\boldsymbol{\nu}})^*\big]^\top (\mathbf{A}^{(2\times 2)})^*(x,\mathcal{D})\mathbb{S}_2. \tag{3.58}$$

The matrix operators $(\mathbf{b}^{(2\times 2)})^*(x,\mathcal{D})$ and $(\mathbf{A}^{(2\times 2)})^*(x,\mathcal{D})$ are formally adjoint (see (3.2), (3.3)) to the corresponding admissible matrix operators $\mathbf{b}^{(2\times 2)}(x,\mathcal{D})$ in (3.39) and $\mathbf{A}^{(2\times 2)}(x,\mathcal{D})$ in (3.55).

The claimed relation (3.14) between orders follows from (3.58) and we leave details to the reader. □

Remark 3.12. Let the Dirichlet system $\mathbf{B}^{(2)}(x,D)$ be fixed, the basic operator be normal and the convention (3.35) holds. Then the system $\mathbf{C}^{(2)}(x,D)$ in Green's formula (3.13) (see (3.36)) is found by the formula (3.58).

Proof of Theorem 3.5. If we apply (3.25) to (3.16) we get the formula (3.17) but with systems $\{\widetilde{\mathbf{B}}_j\}_{j=0}^{\ell-1}$ and $\{\widetilde{\mathbf{C}}_j\}_{j=0}^{\ell-1}$, which we can not control. Therefore we commence by the representations

$$\partial_x^\alpha I_N=\boldsymbol{\nu}^\alpha(x)\partial_{\boldsymbol{\nu}}I_N+\mathbf{a}_\alpha(x,\mathcal{D})\,,\quad \alpha\in\mathbb{N}_0^n,\quad |\alpha|\leqslant 1,$$
$$\boldsymbol{\nu}^\alpha(x):=\boldsymbol{\nu}_1^{\alpha_1}(x)\dots\boldsymbol{\nu}_n^{\alpha_n}(x)$$

(cf. (3.31)); by inserting them into (3.16) and applying (3.27) we get

$$\begin{aligned}\mathcal{A}(u,v):=&\sum_{|\alpha|,|\beta|\leqslant 1}\int\limits_{\Omega^+}\Big\langle\partial^\beta u,a_{\alpha,\beta}\big[\boldsymbol{\nu}^\alpha\partial_{\boldsymbol{\nu}}I_N+\mathbf{a}_\alpha(y,\mathcal{D})\big]v\Big\rangle\,dy=\\ &=\sum_{|\alpha|,|\beta|\leqslant 1}\int\limits_{\Omega^+}\Big[\big\langle\boldsymbol{\nu}^\alpha a_{\alpha,\beta}\partial^\beta u,\partial_{\boldsymbol{\nu}}v\big\rangle+\big\langle\mathbf{a}_\alpha^*(y,\mathcal{D})a_{\alpha,\beta}\partial^\beta u,v\big\rangle\Big]\,dy=\\ &=\int\limits_{\Omega^+}\langle\mathbf{A}(y,D)u,v\rangle\,dy+\int\limits_S\big\langle\mathbf{C}(\tau,D)u,v\big\rangle\,dS\,,\end{aligned}\tag{3.59}$$

$$\mathbf{C}(x,D):=\sum_{|\alpha|,|\beta|\leqslant 1}\boldsymbol{\nu}^\alpha(x)a_{\alpha,\beta}(x)\partial^\beta I_N,$$

Thus, we get Green's formula (3.17).

If $\mathbf{A}$ is formally self-adjoint, $\mathbf{A}=\mathbf{A}^*$, then $\mathcal{A}(u,v)=\overline{\mathcal{A}(v,u)}$ and from (3.17) written for pairs u, v and v, u we get the simplified Green formula (3.18). □

3.2. **On traces of functions.** Besides the classical Sobolev $W_p^m(\mathbb{R}_+^n)$, Bessel potential $H_p^s(\mathbb{R}_+^n)$ and Besov $B_{p,q}^s(\mathbb{R}_+^n)$ spaces on the half-space $\mathbb{R}_+^n$ and on a domain Ω with the boundary, we will treat weighted spaces introduced in Subsection 2.1, which are well-adapted to consideration of potential operators.

Let us define the Trace operator:

$$\begin{aligned}&\mathcal{R}_k u=\gamma_S\partial_{\boldsymbol{\nu}}^{(k)}u:=\big\{\gamma_S^0u,\gamma_S^1u,\dots,\gamma_S^ku\big\}^\top,\\ &\gamma_S^0:=\gamma_S\,,\quad\gamma_S^j:=\gamma_S\partial_{\boldsymbol{\nu}}^jI_N,\quad u\in C_0^\infty(\overline{\Omega})\,,\end{aligned}\tag{3.60}$$

Theorem 3.13. *Let* $1 \leqslant p, q \leqslant \infty$, $m, k \in \mathbb{N}_0$, $k < s - \frac{1}{p} \notin \mathbb{N}_0$. *The trace operator*

$$\begin{aligned}\mathcal{R}_k \; : \; \mathbb{H}^{s,m}_{p,loc}(\Omega) \to \bigotimes_{j=0}^{k} B^{s-\frac{1}{p}-j}_{p,p}(S)\,,\\ \mathcal{R}_k \; : \; \mathbb{B}^{s,m}_{p,q,loc}(\Omega) \to \bigotimes_{j=0}^{k} B^{s-\frac{1}{p}-j}_{p,q}(S)\,.\end{aligned} \tag{3.61}$$

is a retraction*, i.e., it is continuous and has a continuous right inverse, called a* coretraction*:*

$$\begin{aligned}(\mathcal{R}_k)^{-1} \; : \; \bigotimes_{j=0}^{k} \mathbb{B}^{s-\frac{1}{p}-j}_{p,p}(S) \to \mathbb{H}^{s,m}_{p,loc}(\Omega)\,,\\ (\mathcal{R}_k)^{-1} \; : \; \bigotimes_{j=0}^{k} \mathbb{B}^{s-\frac{1}{p}-j}_{p,q}(S) \to \mathbb{B}^{s,m}_{p,q,loc}(\Omega)\,,\\ \mathcal{R}_k(\mathcal{R}_k)^{-1}\Phi = \Phi\,, \;\; \forall\, \Phi \in \bigotimes_{j=0}^{k} \mathbb{B}^{s-\frac{1}{p}-j}_{p,q}(S)\,,\end{aligned} \tag{3.62}$$

Proof. The proof represents a slight modification of the proof for the case $m = 0$. We will carry out the proof for the space $\mathbb{H}^{s,m}_{p,loc}(\Omega)$. For the space $\mathbb{B}^{s,m}_{p,q,loc}(\Omega)$ the proof is similar.

Since the assertion has local character, we can dwell on the case of the half-spaces $\Omega = \mathbb{R}^n_{\pm}$ and $k = 0$ (cf. [72, Theorem 2.7.2, Steps 6-7] and [72, Theorem 3.3.3] for details when $k \neq 0$ and Ω is arbitrary).

Let us recall an alternative definition of (equivalent) norms in the spaces $B^s_{p,p}(\mathbb{R}^n)$ and $H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$:

$$\begin{aligned}\big\|\varphi|B^s_{p,p}(\mathbb{R}^n)\big\| = \Big\|\big\{2^{sj}\mathcal{F}^{-1}\chi_j\mathcal{F}\varphi\big\}_{j=0}^{\infty}\big|\ell_p(L_p(\mathbb{R}^n))\Big\|\,,\\ \big\|\varphi|H^s_p(\mathbb{R}^n)\big\| = \Big\|\big\{2^{sj}\mathcal{F}^{-1}\chi_j\mathcal{F}\varphi\big\}_{j=0}^{\infty}\big|L_p(\mathbb{R}^n, \ell_2)\Big\|\end{aligned} \tag{3.63}$$

(see [72, §§ 2.3.1,2.5.6]), where

$$\chi_j \in C^\infty_0(\mathbb{R}^n)\,, \quad \operatorname{supp}\,\chi_0 \subset \big\{x \in \mathbb{R}^n : \; |x| \leqslant 2\big\}\,,$$

$$\operatorname{supp}\chi_j \subset \big\{x \in \mathbb{R}^n : \; 2^{j-1} < |x| < 2^{j+1}\big\}\,, \quad \sum_{j=0}^{\infty}\chi_j(x) \equiv 1\,.$$

In [72, § 2.3.1, Step 5] the coretraction $\mathcal{R}_0^{-1}$ is defined as follows

$$\mathcal{R}_0^{-1}\varphi(x', x_n) = \sum_{j=0}^{\infty} 2^{-j}\mathcal{F}^{-1}_{\lambda_n\to x_n}\psi_j(\lambda_n)\mathcal{F}^{-1}_{\lambda'\to x'}\chi_j(\lambda')\mathcal{F}_{y'\to\lambda'}[\varphi(y')]\,, \tag{3.64}$$

where

$$\psi_j(\lambda_n) = \psi(2^{-j}\lambda_n)\,, \;\; j \in \mathbb{N}\,, \;\; \psi_0, \psi \in C^\infty_0(\mathbb{R})\,,$$

$$\operatorname{supp}\,\psi_0 \in (0,1)\,, \;\; \operatorname{supp}\,\psi \in (1,2)\,, \;\; \mathcal{F}^{-1}\psi_0(0) = \mathcal{F}^{-1}\psi(0) = 1\,.$$

Then $\mathcal{F}^{-1}\psi_j(0) = 2^j$ which yields $(\mathcal{R}_0^{-1}\varphi)(x',0) = \psi(x',0)$. We proceed as in [72, § 2.7.2-(30)]

$$\begin{aligned}\big\|x_n^m\mathcal{R}_0^{-1}\varphi\big|B_{p,p}^{s+m+\frac{1}{p}}\big\| \leqslant \\ \leqslant C_1\Big\|\Big\{2^{(s+m+\frac{1}{p})j}\mathcal{F}^{-1}_{\lambda_n\to x_n}\big[(-i\partial_{\lambda_n})^m\psi_j(\lambda_n)\big]\times \\ \times\mathcal{F}^{-1}_{\lambda'\to x'}\chi_j(x)\mathcal{F}_{y'\to\lambda'}[\varphi(y')]\Big\}_{j=0}^{\infty}\Big|\ell_p(L_p)\Big\| = \\ = C_1\Big\|\Big\{2^{(s+\frac{1}{p})j}\mathcal{F}^{-1}_{\lambda_n\to x_n}\psi_j^{(m)}(\lambda_n)\mathcal{F}^{-1}_{\lambda'\to x'}\chi_j(x)\mathcal{F}_{y'\to\lambda'}[\varphi(y')]\Big\}_{j=0}^{\infty}\Big|\ell_p(L_p)\Big\| \leqslant \\ \leqslant C_2\big\|\big\{\mathcal{F}^{-1}\chi_j\mathcal{F}\varphi\big\}_{j=0}^{\infty}\big|\ell_p(L_p)\| = \big\|\varphi\big|B_{p,p}^s\big\|,\end{aligned}$$

where $\psi^{(m)}(t) := \partial_t^m\psi(t)$. Similarly we find

$$\begin{aligned}\big\|x_n^m\mathcal{R}_0^{-1}\varphi\big|H_p^{s+m+1/p}\big\| \leqslant \\ \leqslant C_3\Big\|\big\{2^{sj}\mathcal{F}^{-1}\chi_j\mathcal{F}\varphi\big\}_{j=0}^{\infty}\big|L_p(\mathbb{R}^n,\ell_2)\Big\| \leqslant C_3\big\|\varphi\big|H_p^s\big\|\end{aligned}$$

and the proof is complete. □

Corollary 3.14. *Let* $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p}+k < s < \frac{1}{p}+k+1$. *Then*

$$\begin{aligned} r_\Omega\widetilde{\mathbb{H}}_p^{s,m}(\Omega) = \big\{u\in\mathbb{H}_p^{s,m}(\Omega):\ \mathcal{R}_k u = 0\big\}, \\ r_\Omega\widetilde{\mathbb{B}}_{p,q}^{s,m}(\Omega) = \big\{u\in\mathbb{B}_{p,q}^{s,m}(\Omega):\ \mathcal{R}_k u = 0\big\}. \end{aligned} \tag{3.65}$$

Lemma 3.15. *Let* $\mathbf{A}(x,D)$ *in* (3.1) *be a normal operator,* $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $s > 1/p$, *and* $m = 0,1,\dots$; *let further* $\mathbf{B}^{(2)}(x,D) := \{\mathbf{B}_0(x,D),\mathbf{B}_1(x,D)\}^\top$ *be a Dirichlet system and*

$$\Phi = (\varphi_0,\varphi_1)^\top \in \bigotimes_{j=0}^{1} B_{p,p}^{s+1-j}(S), \quad \Psi = (\psi_0,\psi_1)^\top \in \bigotimes_{j=0}^{1} B_{p,q}^{s+1-j}(S),$$

be given vector-functions. Then, for arbitrary integer $m\in\mathbb{N}_0$ *there exists a continuous linear operator*

$$\begin{aligned} \mathcal{P}_{\mathbf{A}} \ :\ \bigotimes_{j=0}^{1} B_{p,p}^{s+1-j}(S) \to \mathbb{H}_{p,loc}^{s+1+\frac{1}{p},m}(\Omega) \\ \mathcal{P}_{\mathbf{A}} \ :\ \bigotimes_{j=0}^{1} B_{p,q}^{s+1-j}(S) \to \mathbb{B}_{p,q,loc}^{s+1+\frac{1}{p},m}(\Omega) \end{aligned} \tag{3.66}$$

such that

$$\gamma_S\mathbf{B}_j\mathcal{P}_{\mathbf{A}}\Phi = \varphi_j, \ \ \gamma_S\mathbf{B}_j\mathcal{P}_{\mathbf{A}}\Psi = \psi_j, \ \ j=0,1, \tag{3.67}$$

$$\mathbf{A}\mathcal{P}_{\mathbf{A}}\Phi \in \widetilde{\mathbb{H}}_{p,loc}^{s-1+\frac{1}{p},m}(\Omega), \ \ \mathbf{A}\mathcal{P}_{\mathbf{A}}\Psi \in \widetilde{\mathbb{B}}_{p,q,loc}^{s-1+\frac{1}{p},m}(\Omega). \tag{3.68}$$

Proof. There exists an integer $k\in\mathbb{N}_0$ such that $\frac{1}{p}<s-k\leqslant\frac{1}{p}+1$. Due to (3.65), the condition (3.68) can be reformulated as follows (cf. (3.60)):

$$\mathcal{R}_k\mathbf{A}\mathcal{P}\Phi=\big\{\gamma_S^0\mathbf{A}\mathcal{P}\Phi,\gamma_S^1\mathbf{A}\mathcal{P}\Phi\big\}=0. \tag{3.69}$$

The operators

$$\mathbf{B}_{2+j}(x,D):=\partial_{\boldsymbol{\nu}}^j\mathbf{A}(x,D)\,,\quad \operatorname{ord}\mathbf{B}_{2+j}=2+j\,,\ \ j=0,1,\dots,k$$

are normal

$$\mathcal{B}_{2+j,0}(\varkappa,\boldsymbol{\nu}(\varkappa))=\Big(-i\sum_{s=1}^n\boldsymbol{\nu}_s^2(\varkappa)\Big)^j\mathcal{A}_0(\varkappa,\boldsymbol{\nu}(\varkappa))=(-i)^j\mathcal{A}_0(\varkappa,\boldsymbol{\nu}(\varkappa))\,,$$

$$\det\mathcal{B}_{2+j,0}(\varkappa,\boldsymbol{\nu}(\varkappa))\neq0\,,\ \ \varkappa\in S\,,\ \ j=0,1,\dots,k$$

and adding them to the Dirichlet system $\mathbf{B}^{(m)}(x,D)$ we get the extended Dirichlet system $\mathbf{B}^{(3+k)}(x,D):=\{\mathbf{B}_0(x,D),\dots,\mathbf{B}_{1+k}(x,D)\}^\top$. On defining

$$\Phi_0:=(\varphi_0,\varphi_1,\underbrace{0,\dots,0}_{(k+1)-times})\in\bigotimes_{j=0}^{2+k}B_{p,p}^{s+1-j}(S)\,, \tag{3.70}$$

we can match conditions (3.67) and (3.69) (which replaces (3.68)) and reformulate the problem as follows: let us look for a continuous linear operator

$$\mathcal{P}_{\mathbf{A}}^{(0)}\ :\ \bigotimes_{j=0}^{2+k}B_{p,p}^{s+1-j}(S)\to\mathbb{H}_{p,loc}^{s+3+\frac{k}{p},k}(\overline{\Omega}) \tag{3.71}$$

such that

$$\gamma_S^{\pm}\mathbf{B}^{(3+k)}\mathcal{P}_{\mathbf{A}}^{(0)}\Phi_0=\Phi_0 \tag{3.72}$$

for Φ_0 given in (3.70).

Since $\mathbf{B}^{(3+k)}(x,D)$ is a Dirichlet system, there exists an admissible matrix operator $\mathbf{b}^{((3+k)\times(3+k))}(x,\mathcal{D})$ such that

$$\mathbf{B}^{(3+k)}(x,D)=\mathbf{b}^{((3+k)\times(3+k))}(x,\mathcal{D})\partial_{\boldsymbol{\nu}}^{(3+k)}(x,D) \tag{3.73}$$

(see (3.38), (3.42)). Let us define the coretraction

$$\mathcal{P}_{\mathbf{A}}^{(0)}:=\mathcal{R}_{3+k}^{-1}\big[\mathbf{b}^{((3+k)\times(3+k))}(x,\mathcal{D})\big]^{-1} \tag{3.74}$$

based on the coretraction in (3.62). The inverse $[\mathbf{b}^{((3+k)\times(3+k))}(x,\mathcal{D})]^{-1}$ is an admissible and tangential differential operator, and can be applied to the function Φ_0 defined on the boundary only. Thus, the operator $\mathcal{P}_{\mathbf{A}}^{(0)}$ in (3.74) is well defined and continuous in the setting (3.71).

Applying (3.73) and (3.74) we find that

$$\gamma_S^{\pm}\mathbf{B}^{(3+k)}\mathcal{P}_{\mathbf{A}}^{(0)}\Phi_0=\mathbf{b}^{((3+k)\times(3+k))}(x,\mathcal{D})\gamma_S\partial_{\boldsymbol{\nu}}^{(3+k)}(x,D)\mathcal{R}_{3+k}^{-1}\times$$
$$\times\big[\mathbf{b}^{((3+k)\times(3+k))}(x,\mathcal{D})\big]^{-1}\Phi_0=\Phi_0$$

because

$$\gamma_S\mathbf{b}^{((3+k)\times(3+k))}(x,\mathcal{D})\Psi=\mathbf{b}^{((3+k)\times(3+k))}(x,\mathcal{D})\gamma_S\Psi$$

$$\text{and } \gamma_S \partial_{\boldsymbol{\nu}}^{(3+k)}(x,D)\mathcal{R}_{3+k}^{-1}\Psi = \Psi$$

due to (3.62). Equation (3.72) is thus solved. □

For regular case, the counterpart of Lemma 3.15 is proved in [49, Ch. 2, § 16], where the operator $\mathcal{P}_{\mathbf{A}}$ is efficiently constructed by means of the special potential type operators.

Let us consider the following surface δ-function

$$\langle g\otimes\delta_S, v\rangle_{\mathbb{R}^n} := \int\limits_S g(\tau)\gamma_S^{\pm} v(\tau)dS, \quad g\in C^\infty(S), \quad v\in C_0^\infty(\mathbb{R}^n) \tag{3.75}$$

and its normal derivatives $\delta_S^{(k)} := \partial_{\boldsymbol{\nu}}^k \delta_S$:

$$\langle g\otimes\delta_S^{(k)}, v\rangle_{\mathbb{R}^n} := \int\limits_S g(\tau)\gamma_S^{\pm}((\partial_{\boldsymbol{\nu}}^*)^k v)(\tau)dS\,, \quad k=1,2,\dots \tag{3.76}$$

(see (3.12) for the normal derivatives and (3.28) for the adjoint). Obviously, $\operatorname{supp}(g\otimes\delta_S^{(k)}) = \operatorname{supp} g\subset S$ for arbitrary $k\in\mathbb{N}_0$.

Definitions (3.75)–(3.76) can be extended to Bessel potential and Besov spaces.

Lemma 3.16. *Let $1\leqslant p,q\leqslant\infty$, $s<0$, $g\in B^s_{p,p}(S)$ and $h\in B^s_{p,q}(S)$. Then*

$$\begin{gathered}
g\otimes\delta_S^{(k)} \in H^{s-k-1/p'}_{p,com}(\mathbb{R}^n)\cap B^{s-k-1/p'}_{p,p,com}(\mathbb{R}^n)\,,\\
g\otimes\delta_S^{(k)} \in \widetilde{\mathbb{H}}^{s-k-1/p',m}_{p,com}(\Omega)\cap \widetilde{\mathbb{B}}^{s-k-1/p',m}_{p,p,com}(\Omega)\,,\\
h\otimes\delta_S^{(k)} \in B^{s-k-1/p'}_{p,q,com}(\mathbb{R}^n)\,,\\
h\otimes\delta_S^{(k)} \in \widetilde{\mathbb{B}}^{s-k-1/p',m}_{p,q,com}(\Omega)\,,
\end{gathered}$$

where $\Omega=\Omega^\pm$, $p'=p/(p-1)$ and $k,m\in\mathbb{N}_0$ are arbitrary.

Proof. The distribution $g\otimes\delta_S^{(k)}$ in (3.75) and (3.76) is a properly defined functional on the space $\mathbb{X}_p^{-s+k}(\mathbb{R}^n)$, where, for conciseness, $\mathbb{X}^\mu_p(\mathbb{R}^n)$ denotes either $H^\mu_p(\mathbb{R}^n)$ or $B^\mu_{p,p}(\mathbb{R}^n)$ (see Theorem 3.13). Moreover, due to the same Theorem 3.13 we get the inequalities

$$|\langle g\otimes\delta_S^{(k)}, v\rangle_{\mathbb{R}^n}| \leqslant C_k(g)\|g|B^s_{p,p}(S)\|\|\chi v|\mathbb{X}^{-s+k+1/p'}_{p'}(\mathbb{R}^n)\|\,,$$

recording the continuity property of the corresponding functionals; here $\chi\in C_0^\infty(\mathbb{R}^n)$ is a cut-off function, which equals 1 in a neighborhood of S. Therefore, by duality, $g\otimes\delta_S^{(k)}\in\mathbb{X}^{s-k-\frac{1}{p'}}_{p,com}(\mathbb{R}^n)$.

To prove the result for the weighted spaces let us note that, for arbitrary $m, k \in \mathbb{N}_0$,

$$\begin{aligned} \partial_{\boldsymbol{\nu}}^k \rho^m(x) &= \frac{m!}{k!}\rho^{m-k}(x)\,, \\ (\partial_{\boldsymbol{\nu}}^*)^k \rho^m(x) &= \big(-\operatorname{div}\boldsymbol{\nu}(x) - \partial_{\boldsymbol{\nu}}\big)^k \rho^m(x) = \sum_{j=0}^{k} h_j(x)\partial_{\boldsymbol{\nu}}^j \rho^m(x) = \\ &= \sum_{j=0}^{k} \frac{m!}{j!} h_j(x)\rho^{m-j}(x)\,, \end{aligned} \tag{3.77}$$

where $\rho = \rho(x) := \operatorname{dist}(x, S)$, $x \in \Omega$ and $h_0, \dots, h_m \in C^\infty(\Omega)$. Indeed, if $t_x \in S$ is a point for which the distance $\rho(x) := \operatorname{dist}(x, S) = \operatorname{dist}(x, t_x)$ from $x \in \Omega$ to the boundary S is attained then

$$\partial_{\boldsymbol{\nu}(t_x)}\rho(x) = \lim_{h\to 0} \frac{\rho(x + h\boldsymbol{\nu}(t_x)) - \rho(x)}{h} = 1\,,$$

because $\rho(x + h\boldsymbol{\nu}(t_x)) - \rho(x) = h$. For arbitrary $m, k \in \mathbb{N}_0$ the first formula in (3.77) follows by a standard approach and is used to prove the second one.

Now we apply definition (3.76):

$$\begin{aligned} \big\langle \rho^\ell (g \otimes \delta_S^{(k)}), v\big\rangle_{\mathbb{R}^n} &:= \int_S g(\tau)\gamma_S^{\pm}\big[(\partial_{\boldsymbol{\nu}}^*)^k(\rho^\ell v)(y)\big]\, dS \\ &= \int_S g(y)\gamma_S^{\pm}\big[(-\operatorname{div}\boldsymbol{\nu}(x) - \partial_{\boldsymbol{\nu}})^k(\rho^\ell v)(y)\big]\, dS = \\ &= \sum_{m=0}^{k}\sum_{j=0}^{m} \frac{(-1)^k j!}{m!(k-j)!}\int_S h_m(y)g(y)\gamma_S^{\pm}\big[\partial_{\boldsymbol{\nu}}^j \rho^\ell \partial_{\boldsymbol{\nu}}^{m-j} v\big](y)\, dS = \\ &= \sum_{m=0}^{k}\sum_{j=0}^{\min\{m,\ell\}} \frac{(-1)^k \ell!}{m!(k-j)!}\int_S h_m(y)g(y)\gamma_S^{\pm}\big[\rho^{\ell-j}\partial_{\boldsymbol{\nu}}^{m-j} v\big](y)\, dS = \\ &= \begin{cases} \displaystyle\sum_{m=\ell}^{k} \frac{(-1)^k \ell!}{m!(k-\ell)!}\int_S h_m(y)g(y)\gamma_S^{\pm}\big[\partial_{\boldsymbol{\nu}}^{m-\ell} v\big](y)\, dS & \text{if } \ell \leqslant k\,, \\ 0, & \text{if } \ell > k \end{cases} \\ &= \begin{cases} \displaystyle\sum_{m=\ell}^{k} \frac{(-1)^k \ell!}{m!(k-\ell)!}\langle h_m g \otimes \delta_S^{(m-\ell)}, v\rangle_{\mathbb{R}^n} & \text{if } \ell \leqslant k\,, \\ 0, & \text{if } \ell > k\,. \end{cases} \end{aligned} \tag{3.78}$$

According to the proved part of the lemma from (3.78) we get the inclusion $\rho^\ell (g \otimes \delta_S^{(k)}) \in \widetilde{\mathbb{X}}_{p,com}^{s-k+\ell-1/p',m}(\overline{\Omega})$; this yields the inclusion $g \otimes \delta_S^{(k)} \in$

$\widetilde{\mathbb{X}}^{s-k-1/p',m}_{p,com}(\overline{\Omega})$ for arbitrary $m \in \mathbb{N}$ due to the definition of the weighted space.

For the function $h \otimes \delta_S^{(k)}$ the proof is verbatim. $\square$

3.3. **Integral representation formulae and layer potentials.** Throughout the present section we assume that the differential operator $\mathbf{A}(x,D)$ in (3.1) is extendible onto entire $\mathbb{R}^n$ and the extension has a fundamental solution $\mathcal{K}_{\mathbf{A}}(x,y) \in C^\infty(\mathbb{R}^n, S'(\mathbb{R}^n))$

$$\mathbf{A}(x,D)\mathcal{K}_{\mathbf{A}}(x,y) := \delta(x-y)I_N. \tag{3.79}$$

Moreover, we assume that the formally adjoint operator possesses the fundamental solution $\mathcal{K}_{\mathbf{A}^*}$ and $\mathcal{K}_{\mathbf{A}^*}(x,y) = \overline{\mathcal{K}_{\mathbf{A}}(y,x)^\top}$.

If the operator $\mathbf{A}(x,D) = \mathbf{A}(D)$ has constant coefficients, the fundamental solution $\mathcal{K}_{\mathbf{A}}(x,y) = \mathcal{K}_{\mathbf{A}}(x-y)$ exists, depends on the difference of variables and $\mathcal{K}_{\mathbf{A}^*}(x-y) = \overline{\mathcal{K}_{\mathbf{A}}(y-x)^\top}$ (see, e.g., [28, § 10, Theorem 10.2.1]). Moreover, the fundamental solution is smooth outside the diagonal set: $\mathcal{K}_{\mathbf{A}} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta_{\mathbb{R}^n})$, where $\Delta_{\mathbb{R}^n} := \{(x,x) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ \forall\, x \in \mathbb{R}^n\}$ (see [67, § 2.5, Proposition 2.4] and [28, vol. 3, Theorem 18.1.16]).

If $\mathbf{A}(x,D)$ is elliptic of order 2, $\operatorname{ord}\mathbf{A} = 2$ and has C^∞-smooth uniformly bounded coefficients, then the symbols $\mathcal{A}(x,\xi)$ belongs to the class $\mathbb{S}^2(\mathbb{R}^n \times \mathbb{R}^n)$ and the inverse symbol $\mathcal{A}^{-1}(x,\xi)$ belongs to the class $\widetilde{\mathbb{S}}^{-2}(\dot{\mathbb{R}}^n \times \mathbb{R}^n)$ (cf. Definition 2.12 and (2.25)).

As a first application of Green's formula (3.13) we can derive the representation of a solution to the BVP (3.10). Since the boundary operators $\mathbf{B}_0(x,D)$ and $\mathbf{C}_1(x,D)$ in Green's formula (3.13) have order 0 (i.e., represent multiplications by functions), we will suppose, for simplicity, that

$$\begin{gathered}\mathbf{B}_0(x,D) = \mathbf{C}_1(x,D) = I_N, \quad \mathbf{B}_1(x,D) = \mathbf{B}(x,D),\\ \mathbf{C}_0(x,D) = \mathbf{C}(x,D), \quad \operatorname{ord}\mathbf{B}(x,D) = \operatorname{ord}\mathbf{C}(x,D) = 1.\end{gathered} \tag{3.80}$$

Then the following Green's third integral representation formula is valid

$$\chi_{\Omega^+}(x)u(x) = \mathbf{N}_{\Omega^+}f(x) + (\mathbf{W}\gamma_S^+u)(x) + (\mathbf{V}\gamma_S^+\mathbf{B}u)(x)\,, \tag{3.81}$$

where χ_{Ω^+} is the indicator function of the domain $\Omega^+ \subset \mathbb{R}^n$ and

$$\mathbf{V}\varphi(x) := \int\limits_S \big[\overline{\mathcal{K}_{\mathbf{A}^*}(y,x)}\big]^\top \varphi(y)\,dS = \int\limits_S \mathcal{K}_{\mathbf{A}}(x,y)\varphi(y)\,dS, \tag{3.82}$$

$$\begin{aligned}\mathbf{W}\varphi(x) &:= \int\limits_S \big[\overline{\mathbf{C}(y,D)\mathcal{K}_{\mathbf{A}^*}(y,x)}\big]^\top \varphi(y)\,dS =\\ &= \sum_{|\alpha|\leqslant 1}\int\limits_S \partial_y^\alpha \mathcal{K}_{\mathbf{A}}(x,y)\overline{c_{j\alpha}^\top(y)}\varphi(y)\,dS\end{aligned} \tag{3.83}$$

(cf. (3.11)) are the single and the double layer potentials, respectively. The operator

$$\mathbf{N}_{\Omega^+}\varphi(x) := \int\limits_{\Omega^+} \left[\overline{\mathcal{K}_{\mathbf{A}^*}(y,x)}\right]^\top \varphi(y)\,dy = \int\limits_{\Omega^+} \mathcal{K}_{\mathbf{A}}(x,y)\varphi(y)\,dy \tag{3.84}$$

is called Newton's potential and represents a ΨDO of order -2, restricted to the domain Ω.

The following operators

$$\begin{aligned} \mathbf{N}_{\Omega^+} \ &: \ \widetilde{H}^s_p(\Omega^+) \to H^{s+2}_{p,\mathrm{loc}}(\mathbb{R}^n)\,, \\ &: \ \widetilde{B}^s_{p,q}(\Omega^+) \to B^{s+2}_{p,q,\mathrm{loc}}(\mathbb{R}^n) \end{aligned} \tag{3.85}$$

are bounded for all $s \in \mathbb{R}$, $1 < p < \infty$, and $1 \leqslant q \leqslant \infty$ (see Corollary 2.13).

The following assertions hold true.

Proposition 3.17. *The single and double layer potentials with $\varphi \in H^s_p(S)$, $s \in \mathbb{R}$, $1 < p < \infty$, are solutions to the homogeneous equation*

$$\mathbf{A}(x,D)\,\mathbf{V}\varphi(x) = \mathbf{A}(x,D)\mathbf{W}\varphi(x) = 0, \ \ x \in \Omega^+ \cup \Omega^-\,. \tag{3.86}$$

Proposition 3.18. *Let $m \in \mathbb{N}_0$, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. Then Newton's potential operator has following continuous mapping properties:*

$$\begin{aligned} \mathbf{N}_{\Omega^+} \ &: \ \widetilde{\mathbb{H}}^{s,m}_p(\Omega^+) \to \mathbb{H}^{s+2,m}_{p,\mathrm{loc}}(\mathbb{R}^n)\,, \\ &: \ \widetilde{\mathbb{B}}^{s,m}_{p,q}(\Omega^+) \to \mathbb{B}^{s+2,m}_{p,q,\mathrm{loc}}(\mathbb{R}^n) \ \ \textit{for} \ \ s \in \mathbb{R}, \end{aligned} \tag{3.87}$$

$$\begin{aligned} \mathbf{N}_{\Omega^+} = r_{\Omega^+}\mathbf{N}_{\mathbb{R}^n}\ell_{\Omega^+} \ &: \ \mathbb{H}^{s,m}_p(\Omega^+) \to \mathbb{H}^{s+2,m}_{p,\mathrm{loc}}(\Omega^\pm)\,, \\ &: \ \mathbb{B}^{s,m}_{p,q}(\Omega^+) \to \mathbb{B}^{s+2,m}_{p,q,\mathrm{loc}}(\Omega^\pm) \ \ \textit{for} \ \ s > -\frac{1}{p}\,. \end{aligned} \tag{3.88}$$

Applying the definition (3.75) we can represent the layer potentials (3.82) and (3.83) in the form of volume potentials:

$$\mathbf{V}\varphi(x) = \int\limits_{\mathbb{R}^n} \mathcal{K}_{\mathbf{A}}(x,y)(\varphi\otimes\delta_S)(y)\,dy = \mathbf{N}_{\mathbb{R}^n}(\varphi\otimes\delta_S)(x), \tag{3.89}$$

$$\mathbf{W}\varphi(x) = \int\limits_{\mathbb{R}^n} \left[\overline{\mathbf{C}(y,D)\mathcal{K}_{\mathbf{A}^*}(y,x)}\right]^\top (\varphi\otimes\delta_S)(y)\,dy. \tag{3.90}$$

On the other hand, due to Lemma 3.16,

$$\begin{aligned} \varphi\otimes\delta_S &\in \widetilde{\mathbb{H}}^{s-1+\frac{1}{p},m}_{p,com}(\Omega) \quad \text{for} \quad \varphi \in B^s_{p,p}(S), \\ \varphi\otimes\delta_S &\in \widetilde{\mathbb{B}}^{s-1+\frac{1}{p},m}_{p,q,com}(\Omega) \quad \text{for} \quad \varphi \in B^s_{p,q}(S)\,, \end{aligned} \tag{3.91}$$

for arbitrary $s < 0$. Therefore continuity properties of layer potentials can be derived from the foregoing Proposition 3.18. But this approach has a clear shortcoming: we can not conclude the continuity for $s \geqslant 0$ because $\varphi\otimes\delta_S \notin \mathbb{X}^s_{p,loc}(\overline{\Omega})$ for $s > -1+1/p$ even for $\varphi \in C^\infty(S)$ (i.e., Lemma 3.16 is optimal and can not be improved). Indeed, locally S can be interpreted as $\mathbb{R}^{n-1}$ and Ω as $\mathbb{R}^n_+$. Then $1\otimes\delta_{\mathbb{R}^{n-1}}=\delta(x_n)\notin\mathbb{X}^s_{p,loc}(\overline{\mathbb{R}^n_+})$ if $s>-1+1/p$ (see [22] for $p=2$ and [72, 73] for $1<p<\infty$).

In the next theorem we choose yet another approach to the continuity of generalized layer potentials, which enables to prove the continuity property for positive $s>0$.

Theorem 3.19. *Let $s\in\mathbb{R}$, $1<p<\infty$, $1\leqslant q\leqslant\infty$, and $m\in\mathbb{N}_0$. Let $\mathbf{A}(x,D)$ be an elliptic second order partial differential operator as in* (3.1).

The single and double layer potentials map continuously the following spaces:

$$\begin{aligned}
\mathbf{V}\ :&\ \ W^s_p(S)\to\mathbb{H}^{s+1+\frac1p,m}_{p,loc}(\Omega)\quad provided\quad s>0\\
&\qquad\qquad and\quad s\neq 2,3,\dots,\quad if\quad 1<p<2,\\
:&\ \ H^s_p(S)\to\mathbb{H}^{s+1+\frac1p,m}_{p,loc}(\Omega)\quad provided\quad s>0,\quad p\geqslant 2,\\
:&\ \ B^s_{p,p}(S)\to\mathbb{H}^{s+1+\frac1p,m}_{p,loc}(\Omega),\\
:&\ \ B^s_{p,q}(S)\to\mathbb{B}^{s+1+\frac1p,m}_{p,q,loc}(\Omega),
\end{aligned}\tag{3.92}$$

$$\begin{aligned}
\mathbf{W}\ :&\ \ W^s_p(S)\to\mathbb{H}^{s+\frac1p,m}_{p,loc}(\Omega)\quad provided\quad s>0\\
&\qquad\qquad and\quad s\neq 2,3,\dots,\quad if\quad 1<p<2,\\
:&\ \ H^s_p(S)\to\mathbb{H}^{s+\frac1p,m}_{p,loc}(\Omega)\quad provided\quad s>0,\quad p\geqslant 2,\\
:&\ \ B^s_{p,p}(S)\to\mathbb{H}^{s+\frac1p,m}_{p,loc}(\Omega),\\
:&\ \ B^s_{p,q}(S)\to\mathbb{B}^{s+\frac1p,m}_{p,q,loc}(\Omega).
\end{aligned}\tag{3.93}$$

Proof. Due to Theorem 3.4 we can suppose that Green's formula (3.13) is valid and let

$$\{\mathbf{B_0}(x,D),\mathbf{B_1}(x,D)\}\,,\ \{\mathbf{C_0}(x,D),\mathbf{C_1}(x,D)\}\,,\ \mathbf{B_0}(x,D)=\mathbf{C_1}(x,D)=\mathbf{I}_N,$$

be the Dirichlet systems from formula (3.13), where $\operatorname{ord}\mathbf{C_0}=\operatorname{ord}\mathbf{B_1}=1$ (see (3.14)).

The continuity results (3.92) and (3.93) for $s<0$ in the case of the spaces $B^s_{p,p}$ and $B^s_{p,q}$ follow from Lemma 3.16 and Proposition 3.18 (see representations (3.89) and (3.90)).

Next we take $s>0$, $s\notin\mathbb{N}$ and define the operators

$$\mathcal{P}_j\varphi:=\mathcal{P}_{\mathbf{A}}\Psi_j,\quad \Psi_0:=(\varphi,0),\quad \Psi_1:=(0,\varphi);$$

$\mathcal{P}_{\mathbf{A}}$ is from Lemma 3.15. From the same Lemma 3.15 we derive the following continuity properties

$$\mathcal{P}_j\ :\ B^{s+1-j}_{p,p}(S)\to\mathbb{X}^{s+1+1/p,m}_{p,loc}(\Omega)\,,\tag{3.94}$$

$$\mathbf{A}\mathcal{P}_j\ :\ B^{s+1-j}_{p,p}(S)\to\widetilde{\mathbb{X}}^{s-1+1/p,m}_{p,loc}(\Omega)\tag{3.95}$$

for arbitrary $m = 0, 1, \dots$, where either $\mathbb{X}^{\nu}_{p,loc}(\Omega) = H^{\nu}_{p,loc}(\Omega)$ or $\mathbb{X}^{\nu}_{p,loc}(\Omega) = B^{\nu}_{p,p,loc}(\Omega)$, and similarly for $\widetilde{\mathbb{X}}^{\nu}_{p,loc}(\Omega)$. Moreover,

$$\gamma^{\pm}_{S}\mathbf{B}_k\mathcal{P}_j = \delta_{jk}\mathbf{I}_N \quad \text{for} \quad j,k = 0,1, \tag{3.96}$$

where the signs $\pm$ stand for the traces from $\Omega = \Omega^{\pm}$.

Let us consider $v_{\varepsilon,x}(y) := \chi_{\varepsilon}(x-y)\mathcal{K}_{\mathbf{A}}(y,x)$, where $\mathcal{K}_{\mathbf{A}}(x,y)$ is the fundamental solution of $\mathbf{A}(x,D)$ and $\chi_{\varepsilon} \in C^{\infty}(\mathbb{R}^n)$ is a cut off function: $\chi_{\varepsilon}(x) = 1$ for $|x| > \varepsilon$ and $\chi_{\varepsilon}(x) = 0$ for $|x| < \varepsilon/2$. By inserting

$$v(y) = v_{\varepsilon,x}(y)\,, \quad u = \mathcal{P}_1\varphi\,, \quad \varphi \in B^{s+\mu_j}_{p,p}(S)$$

into Green's formula (3.13) and sending $\varepsilon \to 0$, similarly to (3.81) we find the following

$$\begin{aligned}\pm\mathbf{W}\varphi(x) &= \chi_{\Omega^{\pm}}(x)\mathcal{P}_1\varphi(x) - \mathbf{N}_{\Omega^{\pm}}\mathbf{A}\mathcal{P}_1\varphi(x) + \\ &+ \sum_{\alpha+\beta\leqslant 2}\int\limits_{\Omega^{\pm}} c^1_{\alpha\beta}(y)(\partial^{\beta}_y\mathcal{K}_{\mathbf{A}})(x,y)c^2_{\alpha\beta}(y)\mathcal{P}_1\varphi(y)dy,\end{aligned} \tag{3.97}$$

where $c^1_{\alpha\beta},\ c^2_{\alpha\beta} \in C^{\infty}(\mathbb{R}^n)$.

Applying Lemma 3.15 and Proposition 3.18 from (3.97) we derive the following continuity results:

$$\begin{aligned}\mathbf{W} \ &: \ B^{s-1}_{p,p}(S) \to \mathbb{H}^{s-1+\frac{1}{p},\,m}_{p,p,loc}(\Omega), \\ &: \ B^{s-1}_{p,q}(S) \to \mathbb{B}^{s-1+\frac{1}{p},\,m}_{p,q,loc}(\Omega),\end{aligned} \tag{3.98}$$

provided $s > 0$, $s \neq 1, 2, \dots$, $1 < p < \infty$. The continuity

$$\begin{aligned}\mathbf{V} \ &: \ B^{s}_{p,p}(S) \to \mathbb{H}^{s+1+\frac{1}{p},\,m}_{p,loc}(\Omega), \\ &: \ B^{s}_{p,q}(S) \to \mathbb{B}^{s+1+\frac{1}{p},\,m}_{p,q,loc}(\Omega)\end{aligned} \tag{3.99}$$

for $s > 1$, $s \neq 1, 2, \dots$ and $1 < p < \infty$, is proved similarly.

The gaps $0 \leqslant s \leqslant 1$ and $s = 1, 2, \dots$ for the spaces $B^s_{p,q}$ and $B^s_{p,p}$ are filled by the interpolation: the complex interpolation method gives

$$(B^{s_0}_{p,p}(S), B^{s_1}_{p,q}(S))_{\theta} = B^{s}_{p,q}(S), \quad 1 < p < \infty, \quad 1 \leqslant q \leqslant \infty,$$

$$(\mathbb{B}^{s_0}_{p,q,\mathrm{loc}}(\Omega), \mathbb{B}^{s_1}_{p,q,\mathrm{loc}}(\Omega))_{\theta} = \mathbb{B}^{s}_{p,q,\mathrm{loc}}(\Omega), \quad s_0 \neq s_1, \quad s = (1-\theta)s_0 + \theta s_1$$

for arbitrary $s_0, s_1 \in \mathbb{R}$ (see Proposition 2.25 and Remark 2.26).

The boundedness results for the Bessel spaces H^s_p, $s > 0$ in (3.92) and (3.93) follow trivially, since

$$\|\chi\mathbf{V}\varphi|H^{s+1+\frac{1}{p},m}_{p}(\Omega)\| \leqslant M\|\varphi|B^{s}_{p,p}(S)\| \leqslant M_0\|\varphi|H^{s}_{p}(S)\|$$

for arbitrary smooth cut off function $\chi \in C^{\infty}_0(\overline{\Omega})$ and some constants $M > 0$, $M_0 > 0$, due to the continuous embedding $H^s_p \subset B^s_{p,p}$, which is true for

$p \geqslant 2$. The gap $s = 1, 2, \ldots$ for the spaces H^s_p is filled by the interpolation: the complex interpolation method gives

$$(H^{s_0}_p(S), H^{s_1}_p(S))_\theta = H^s_p(S),$$

$$(\mathbb{H}^{s_0}_{p,\mathrm{loc}}(\Omega), \mathbb{H}^{s_1}_{p,\mathrm{loc}}(\Omega))_\theta = \mathbb{H}^s_{p,\mathrm{loc}}(\Omega), \quad s_0 \neq s_1, \quad s = (1-\theta)s_0 + \theta s_1$$

(see Proposition 2.25 and Remark 2.26).

The boundedness results for the Sobolev-Slobodetskii spaces W^s_p in (3.92) and (3.93) follow trivially, since $B^s_{p,p} = W^s_p$ for all $s > 0$, $s \neq 1, 2, \ldots$ and $H^\ell_p = W^\ell_p$ for all $2 \leqslant p < \infty$, $\ell = 1, 2, \ldots$. □

Remark 3.20. *The continuity properties in the Bessel potential spaces*

$$\begin{aligned} \mathbf{V} \quad &: \quad H^s_p(S) \to \mathbb{H}^{s+1+\frac{1}{p},m}_{p,loc}(\Omega)\,, \\ \mathbf{W} \quad &: \quad H^s_p(S) \to \mathbb{H}^{s+\frac{1}{p},m}_{p,loc}(\Omega) \end{aligned} \tag{3.100}$$

(see (3.92) *and* (3.93)*) for* $1 < p < 2$ *are false.*

Indeed, let us take $s > 0$ *and* $1 < p < 2$. *Then* $B_{p,p}(S)$ *is a proper subset of* $H^s_p(S)$ *and we can choose* $\varphi \in H^s_p(S) \setminus B_{p,p}(S)$. *If* (3.100) *holds,* $\mathbf{W}\varphi \in \mathbb{H}^{s+\frac{1}{p},m}_{p,loc}(\Omega)$. *The trace*

$$\varphi := (\mathbf{W}\varphi)^+ - (\mathbf{W}\varphi)^-$$

(see Plemelji's formulae (3.126) *below) should then belong to* $B_{p,p}(S)$ *(see Theorem 3.13), which is false by the assumption. For the operator* $\mathbf{V}$ *we should face a similar contradiction if the boundary operator* $\mathbf{B_1}(x,D)$ *and formulae* (3.126) *below are applied:*

$$\varphi := (\mathbf{B_1}(x,D)\mathbf{V}\varphi)^+ - (\mathbf{B_1}(x,D)\mathbf{V}\varphi)^-.$$

Moreover, the continuity result for the Sobolev-Slobodetskii spaces W^s_p *in* (3.92) *and* (3.93) *for* $s = 1, 2, \ldots$, $1 < p < 2$, *does not hold, because* $W^s_p = H^s_p$ *for integer* $s = 1, 2, \ldots$ *and, after interpolation, we end up with the false boundedness results* (3.100).

3.4. **Traces of generalized potentials.** Let $\mathbf{A}(x,D)$ in (3.1) be an elliptic differential operator of second order $\operatorname{ord}\mathbf{A}(x,D) = 2$ and $\mathcal{K}_\mathbf{A}(x,y)$ be its fundamental solution. Let us consider a Potential-type operator

$$\mathbf{V}_{\mathbf{B},\mathbf{C}} := \mathbf{B}(x,D)\mathbf{V}\mathbf{C}\,, \quad x \in \Omega\,, \tag{3.101}$$

where $\mathbf{V}$ is the single layer potential (see (3.82)) and

$$\begin{aligned} \mathbf{B}(x,D) &= \sum_{|\alpha| \leqslant k} b_\alpha(x)\partial^\alpha_x, \quad b_\alpha \in C^\infty(\overset{\bullet}{\Omega})\,, \quad x \in \Omega\,, \\ \mathbf{C} = \mathbf{C}(\mathcal{y}, \mathcal{D}) &= \sum_{|\alpha| \leqslant \mu} c_\alpha(\mathcal{y})\mathcal{D}^\alpha, \quad c_\alpha \in C^\infty(S)\,, \quad \mathcal{y} \in S, \end{aligned} \tag{3.102}$$

are some differential operators on the domain Ω and on the boundary surface S, respectively. $\mathbf{C}(\mathcal{y}, \mathcal{D})$ is a tangential differential operator and it can be applied to functions defined on the boundary surface S only (see (3.22) and (3.23)).

Theorem 3.21. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $k, \mu \in \mathbb{N}_0$ and $1 \leqslant q \leqslant \infty$. Then the potential-type operators*

$$\begin{aligned} \mathbf{V}_{\mathbf{B},\mathbf{C}}(x, D) \;&:\; B^s_{p,p}(S) \to H^{s+1-k-\mu+\frac{1}{p},\,m}_{p,loc}(\Omega), \\ &:\; B^s_{p,q}(S) \to B^{s+1-k-\mu+\frac{1}{p},\,m}_{p,q,loc}(\Omega) \end{aligned} \tag{3.103}$$

are bounded for all $m = 0, 1, \dots, \infty$.

Moreover, the traces $\gamma^{\pm}_S \mathbf{V}_{\mathbf{B},\mathbf{C}}(x, D)$ exist and are classical pseudodifferential operators with symbols

$$\begin{aligned} \mathcal{V}_{\mathcal{B},\mathcal{C}}(\tau, \xi) \simeq \sum_{k=0}^{N} \mathcal{V}_{\mathcal{B},\mathcal{C},k}(\tau, \xi) + \widetilde{\mathcal{V}}_{\mathcal{B},\mathcal{C}.N+1}(\tau, \xi)\,, \\ \widetilde{\mathcal{V}}_{\mathcal{B},\mathcal{C},N+1} \in \mathbb{S}^{-2\ell+1+m+\mu-N-1}(S)\,, \end{aligned} \tag{3.104}$$

where $N \in \mathbb{N}_0$ is arbitrary and $\mathcal{V}_{\mathcal{B},\mathcal{C},k}(\tau, \xi)$ are homogeneous functions in ξ of order $-2\ell + 1 + m + \mu - |\beta| - k$ $(\tau \in S,\ \xi \in \mathbb{R}^n \setminus \{0\},\ k = 0, 1, \dots, N)$.

Proof. The continuity properties (3.103) follow from Theorem 3.19 and the boundedness of the differential operators (see the second part of Theorem 2.23)

$$\begin{aligned} \mathbf{C}(\varkappa, \mathcal{D}) \;:\; H^s_p(S) \to H^{s-k}_p(S), \quad \mathbf{B}(x, D) \;:\; \mathbb{H}^{s,m}_{p,loc}(\Omega) \to \mathbb{H}^{s-\mu,m}_{p,loc}(\Omega), \\ \mathbf{C}(\varkappa, \mathcal{D}) \;:\; B^s_{p,p}(S) \to B^{s-k}_{p,p}(S), \quad \mathbf{B}(x, D) \;:\; \mathbb{B}^{s,m}_{p,p,loc}(\Omega) \to \mathbb{B}^{s-\mu,m}_{p,p,loc}(\Omega). \end{aligned}$$

We shall concentrate on the existence of the traces $\gamma^{\pm}_S \mathbf{V}_{\mathbf{B},\mathbf{C}}(x, D)$.

Without loss of generality we can suppose $\mathbf{C}(\varkappa, \mathcal{D}) = I_N$ because a composition of classical ΨDOs is classical. Decomposing the operator

$$\mathbf{B}(x, D) = \sum_{j=0}^{k} \mathbf{B}^{(k-j)}(x, \mathcal{D}) \partial^j_{\boldsymbol{\nu}} I_N, \qquad \mathbf{B}^{(0)}(x, \mathcal{D}) = \mathcal{B}(x, \boldsymbol{\nu}(x))$$

(cf. (3.31)), where $\mathbf{B}^{(r)}(x, \mathcal{D})$ is a tangential differential operator of order r, we find

$$\begin{aligned} \mathbf{V}_{\mathbf{B}} = \sum_{j=0}^{k} \mathbf{B}^{(k-j)}(x, \mathcal{D}) \widetilde{\mathbf{V}}_j, \\ \widetilde{\mathbf{V}}_j := \mathbf{V}_{\partial^j_{\nu}} := \partial^j_{\boldsymbol{\nu}} \mathbf{V}(x, D)\,. \end{aligned} \tag{3.105}$$

The generalized potentials $\widetilde{\mathbf{V}}_0 = \mathbf{V}$ coincide with the single layer potential, while for $k = 1$ they behave like the double layer potential, where the boundary operator is $\mathbf{B}(x, D) = \partial_{\boldsymbol{\nu}} I_N$. These are ΨDOs due to Theorem 2.33, Remark 2.35, and the traces $\gamma^{\pm}_S \widetilde{\mathbf{V}}_k(x, D)$, $k = 0, 1$, are well defined classical ΨDOs on S.

Therefore, in the representation (3.105) both the continuity results (3.103) and the existence of the traces $\gamma_S^{\pm}\mathbf{V}_{\mathbf{B},\mathbf{C}}(x,D)$ are guaranteed.

Now let ord $\mathbf{B}\geqslant 2$ and consider the representation (3.31):

$$\begin{gathered}\mathbf{A}(x,D)=\mathcal{A}_{\rm pr}(x,\boldsymbol{\nu}(x))\partial^2_{\boldsymbol{\nu}}I_N+\mathbf{A}_1(x,\mathcal{D})\partial_{\boldsymbol{\nu}}I_N+\mathbf{A}_2(x,\mathcal{D})\,,\\ \mathbf{A}_k(x,\mathcal{D})=\sum_{|\alpha|\leqslant k}a^0_{k,\alpha}(x)\mathcal{D}^\alpha I_N\,,\quad x\in\Omega,\quad k=1,2\,,\end{gathered}\tag{3.106}$$

where $\mathcal{A}_{\rm pr}(x,\xi)$ is the principal symbol of $\mathbf{A}(x,D)$ (cf. (3.7)) and the operators $\mathbf{A}_1$, $\mathbf{A}_2$, restricted to the surface $\gamma_S\mathbf{A}_k(x,\mathcal{D})$, $x\in S$, $k=1,2$, are tangential differential operators. Since $\mathcal{K}_{\mathbf{A}}(x,y)$ is the fundamental solution, $\mathbf{A}(x,D))\mathcal{K}_{\mathbf{A}}(x,y)=\delta(x-y)I_N$. On the other hand, by invoking (3.106), we find

$$\begin{aligned}\mathbf{A}(x,D)\mathcal{K}_{\mathbf{A}}(x,y)&=\mathcal{A}_{\rm pr}(x,\boldsymbol{\nu}(x))\partial^2_{\boldsymbol{\nu}}\mathcal{K}_{\mathbf{A}}(x,y)+\\ &+\mathbf{A}_1(x,\mathcal{D})\partial_{\boldsymbol{\nu}}\mathcal{K}_{\mathbf{A}}(x,y)+\mathbf{A}_2(x,\mathcal{D})\mathcal{K}_{\mathbf{A}}(x,y)=\delta(x-y)I_N.\end{aligned}\tag{3.107}$$

Now we recall that $\mathbf{A}(x,D)$ is elliptic, which implies $\det\mathcal{A}_{\rm pr}(x,\boldsymbol{\nu}(x))\neq 0$ in a neighborhood of the boundary S (see Lemma 3.2). This ensures solvability of equation (3.107) and we find:

$$\begin{aligned}\partial^2_{\boldsymbol{\nu}}\mathcal{K}_{\mathbf{A}}(x,y)&=\delta(x-y)\mathcal{A}^{-1}_{\rm pr}(x,\boldsymbol{\nu}(x))+\\ &+\sum_{j=1}^{2}\mathcal{A}^{-1}_{\rm pr}(x,\boldsymbol{\nu}(x)\mathbf{A}_j(x,\mathcal{D})\partial^{2-j}_{\boldsymbol{\nu}}\mathcal{K}_{\mathbf{A}}(x,y)\,.\end{aligned}\tag{3.108}$$

Applying the mathematical induction and invoking (3.108) we obtain the representation

$$\begin{aligned}\partial^r_{\boldsymbol{\nu}}\mathcal{K}_{\mathbf{A}}(x,y)&=\delta(x-y)\mathcal{A}^{-1}_{\rm pr}(x,\boldsymbol{\nu}(x))+\\ &+\sum_{j=1}^{r}\mathcal{A}^{-1}_{\rm pr}(x,\boldsymbol{\nu}(x)\mathbf{A}_j(x,\mathcal{D})\partial^{r-j}_{\boldsymbol{\nu}}\mathcal{K}_{\mathbf{A}}(x,y)\\ &=\delta(x-y)\mathcal{B}(x)+\sum_{j=0}^{1}B_k(x,\mathcal{D})\partial^k\mathcal{K}_A(x,y)\end{aligned}\tag{3.109}$$

for arbitrary operator $A(x,D)$ of order $r=2,3,\dots$.

The representation (3.109), inserted into (3.105), diminishes the order of the operator (i.e., the order with respect to the normal derivative) to $k\leqslant 1$. As we have already noted, this guarantees both, the continuity results (3.103) and the existence of traces. □

3.5. **Calderón's projections.** Throughout this section it is assumed that the hypotheses of Theorem 3.4 hold and Green's formula (3.13) is valid also for unbounded domain Ω^-

$$\int\limits_{\Omega^\pm}\big[\langle\mathbf{A}u,v\rangle-\langle u,\mathbf{A}^*v(y)\rangle\big]\,dy=\pm\sum_{j=0}^{1}\int\limits_S\langle\mathbf{B}_ju,\mathbf{C}_jv\rangle\,dS\tag{3.110}$$

under certain constraints on functions $u, v \in C^2(\overline{\Omega^\pm})$ at infinity (e.g, for $u(x), v(x) = \mathcal{O}(\langle x\rangle^{-\gamma})$, $\gamma > n/2$, as $|x| \to \infty$).

Let

$$\begin{aligned} H_p^{s,\pm}(\mathbf{A}, \mathbf{B}_j, S) &:= \Big\{\gamma_S^\pm \mathbf{B}_j\varphi :\ \varphi \in H_p^{s+j+\frac{1}{p}}(\Omega^\pm),\ \mathbf{A}(x,D)\varphi = 0\Big\}, \\ B_{p,q}^{s,\pm}(\mathbf{A}, \mathbf{B}_j, S) &:= \Big\{\gamma_S^\pm \mathbf{B}_j\varphi :\ \varphi \in B_{p,q}^{s+j+\frac{1}{p}}(\Omega^\pm),\ \mathbf{A}(x,D)\varphi = 0\Big\}, \end{aligned} \tag{3.111}$$

for $j = 0, 1$, $s > 0$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, where $\gamma_S^\pm u$ denote traces.

Theorem 3.22. *Let $j = 0, 1$, $s > 0$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. The functional spaces are decomposed in the following direct sums*

$$\begin{gathered} H_p^s(S) = H_p^{s,-}(\mathbf{A}, \mathbf{B_j}, S) \oplus H_p^{s,+}(\mathbf{A}, \mathbf{B_j}, S)\,, \\ B_{p,q}^s(S) = B_{p,q}^{s,-}(\mathbf{A}, \mathbf{B_j}, S) \oplus B_{p,q}^{s,+}(\mathbf{A}, S)\,, \\ H_p^{s,-}(\mathbf{A}, \mathbf{B_j}, S) \cap H_p^{s,+}(\mathbf{A}, \mathbf{B_j}, S) = \{0\}\,, \\ B_{p,q}^{s,-}(\mathbf{A}, \mathbf{B_j}, S) \cap B_{p,q}^{s,+}(\mathbf{A}, \mathbf{B_j}, S) = \{0\} \end{gathered} \tag{3.112}$$

and the corresponding Calderón projections

$$\begin{aligned} \mathbf{P}_{\mathbf{A},\mathbf{j}}^\pm \ &:\ H_p^s(S) \to H_p^{s,\pm}(\mathbf{A}, \mathbf{B_j}, S)\,, \\ &:\ B_{p,q}^s(S) \to B_{p,q}^{s,\pm}(\mathbf{A}, \mathbf{B_j}, S) \end{aligned} \tag{3.113}$$

are defined as follows

$$\mathbf{P}_{\mathbf{A},\mathbf{0}}^\pm = \pm\gamma_S^\pm \mathbf{B_0}\mathbf{W}, \quad \mathbf{P}_{\mathbf{A},\mathbf{1}}^\pm = \pm\gamma_S^\pm \mathbf{B_1}\mathbf{V}. \tag{3.114}$$

Proof. Let $\mathbb{X}_p^s(S)$ stand either for $H_p^s(S)$ or for $B_{p,q}^s(S)$. We will prove (3.112) and (3.113) for the Sobolev–Slobodetskii spaces. For the Bessel potential spaces we have to prove only the continuity property (3.113) while the others (including (3.114)) follow from the embedding $B_{p,q}^s(S) \subset H_r^s(S)$ for $1 < r < p < \infty$, $s \in \mathbb{R}$ (see [72, 73]).

The continuity (3.113) follow from Theorem 3.21 since $\mathbf{P}_{\mathbf{A},\mathbf{j}}^\pm$ are ΨDOs of order 0 (see Theorem 2.33 and Remark 2.35) and

$$\operatorname{ord} \mathbf{P}_{\mathbf{A},\mathbf{0}}^\pm = \operatorname{ord} \mathbf{B_0} + \operatorname{ord} \mathbf{W} = 0, \quad \operatorname{ord} \mathbf{P}_{\mathbf{A},\mathbf{1}}^\pm = \operatorname{ord} \mathbf{B_1} + \operatorname{ord} \mathbf{V} = 1 - 1 = 0.$$

By inserting $u = \mathcal{P}_j\varphi$, $f = \mathbf{A}u = \mathbf{A}\mathcal{P}_j\varphi$ into (3.81), where $\mathbf{B_0} = \mathbf{I}$, $\mathbf{B_1} = \mathbf{B}$ (see (3.80)) and involving (3.96), we get

$$\begin{aligned} \chi_{\Omega^\pm}\mathcal{P}_0\varphi(x) &= \mathbf{N}_{\Omega^\pm}\mathbf{A}\mathcal{P}_0\varphi(x) \pm \mathbf{W}\mathbf{B_0}\mathcal{P}_0\varphi(x) \pm \mathbf{V}\mathbf{B_1}\mathcal{P}_0\varphi(x) + \\ &= \mathbf{N}_{\Omega^\pm}\mathbf{A}\mathcal{P}_0\varphi(x) \pm \mathbf{W}\varphi(x), \\ \chi_{\Omega^\pm}\mathcal{P}_1\varphi(x) &= \mathbf{N}_{\Omega^\pm}\mathbf{A}\mathcal{P}_1\varphi(x) \pm \mathbf{W}\mathbf{B_0}\mathcal{P}_1\varphi(x) \pm \mathbf{V}\mathbf{B_1}\mathcal{P}_1\varphi(x) + \\ &= \mathbf{N}_{\Omega^\pm}\mathbf{A}\mathcal{P}_1\varphi(x) \pm \mathbf{V}\varphi(x), \quad x \in \Omega^\pm. \end{aligned} \tag{3.115}$$

Since the first summand in (3.115) and its derivatives are continuous across the surface S,

$$(\gamma_S^- \partial_x^\alpha \mathbf{N}_{\Omega^-}\mathbf{A}\mathcal{P}_j\varphi)(x) = (\gamma_S^+ \partial_x^\alpha \mathbf{N}_{\Omega^+}\mathbf{A}\mathcal{P}_j\varphi)(x), \ \ x \in S, \tag{3.116}$$

for arbitrary multi-index $\alpha \in \mathbb{N}_0^n$. Taking the sum and invoking (3.96) we obtain the following

$$(\gamma_S^+ \mathbf{B_k} \mathbf{V}_j \varphi)(x) - (\gamma_S^- \mathbf{B_k} \mathbf{V}_j \varphi)(x) = \mathbf{B_k} \mathcal{P}_j \varphi(x) = \delta_{kj} \varphi(x) \,, \qquad (3.117)$$

where $j, k = 0, 1$, $\mathbf{V}_0 = \mathbf{W}$, $\mathbf{V}_1 = \mathbf{V}$ and $\mathbf{B_0} = I$, $\mathbf{B_1} = \mathbf{B}$ are chosen as in (3.80). Formula (3.117) yields

$$\mathbf{P}^+_{\mathbf{A},\mathbf{j}} \varphi + \mathbf{P}^-_{\mathbf{A},\mathbf{j}} \varphi = \gamma_S^+ \mathbf{B}_j \mathbf{V}_j \varphi - \gamma_S^- \mathbf{B}_j \mathbf{V}_j \varphi = \varphi \,, \;\; \varphi \in \mathbb{X}^s_p(S) \qquad (3.118)$$

and with (3.113) they imply (3.112).

To prove (3.112) let us apply formula (3.115), written for the homogeneous equation $\mathbf{A} u = \mathbf{A} \mathcal{P}_j \varphi = 0$:

$$\chi_{\Omega^\pm} \mathcal{P}_0 \varphi(x) = \pm \mathbf{W} \varphi(x) \,, \quad \chi_\Omega \mathcal{P}_1 \varphi(x) = \pm \mathbf{V} \varphi(x) \,, \;\; x \in \Omega^- \cup \Omega^+ . \;\; (3.119)$$

Now assume, that $\varphi = \mathbf{P}^+_{\mathbf{A},\mathbf{j}} \varphi = \mathbf{P}^-_{\mathbf{A},\mathbf{j}} \varphi$, which means that the function φ is in the intersection $\varphi \in \mathbb{X}^{s,-}_p(\mathbf{A}, \mathbf{B_j}, S) \cap \mathbb{X}^{s,+}_p(\mathbf{A}, \mathbf{B_j} S)$. Then from (3.119), by applying the operator $\mathbf{B}_j$ and invoking (3.96), (3.95) we find that

$$\begin{aligned} \varphi^\pm(x) &= \{\mathbf{B}_0 \mathcal{P}_0 \varphi\}^\pm(x) = \pm \{\mathbf{B}_0 \mathbf{W} \varphi\}^\pm(x) = \mathbf{P}^\pm_{\mathbf{A},\mathbf{0}} \varphi(x), \\ \varphi^\pm(x) &= \{\mathbf{B}_1 \mathcal{P}_1 \varphi\}^\pm(x) = \pm \{\mathbf{B}_1 \mathbf{V} \varphi\}^\pm(x) = \mathbf{P}^\pm_{\mathbf{A},\mathbf{1}} \varphi(x), \;\; x \in S, \end{aligned}$$

and since $\varphi = \mathbf{P}^+_{\mathbf{A},\mathbf{j}} \varphi = \mathbf{P}^-_{\mathbf{A},\mathbf{j}} \varphi$, we get $[\varphi](x) := \varphi^+(x) - \varphi^-(x) = 0$. On the other hand, by taking the sum of traces, we derive from (3.119) that

$$\mathcal{P}_0 \varphi(x) = \mathbf{W}[\varphi](x) \equiv 0 \,, \quad \mathcal{P}_1 \varphi(x) = \mathbf{V}[\varphi](x) \equiv 0 \,, \;\; x \in \Omega^- \cup \Omega^+ , \;\; (3.120)$$

which implies $\varphi(x) \equiv 0$. Thus

$$\mathbf{P_{A,j}}^{\pm} \mathbf{P_{A,j}}^{\mp} \psi = 0 \;\; \forall\, \psi \in \mathbb{X}^s_p(S). \qquad (3.121)$$

From (3.118) and (3.121) we get that $\mathbf{P}^\pm_{\mathbf{A},\mathbf{j}}$ are projections:

$$(\mathbf{P}^\pm_{\mathbf{A},\mathbf{j}})^2 = \mathbf{P}^\pm_{\mathbf{A},\mathbf{j}} (\mathbf{P}^\pm_{\mathbf{A},\mathbf{j}} + \mathbf{P}^\mp_{\mathbf{A},\mathbf{j}}) = \mathbf{P}^\pm_{\mathbf{A},\mathbf{j}} \,. \qquad \square$$

3.6. **Plemelji's formulae for layer potentials.** Let

$$\begin{aligned} \mathbf{B}_j(x, D) \mathbf{V}_k \varphi(x) \; &= \int_S \mathbf{B}_j(x, D) \big[\overline{\mathbf{C}_k(y, D)} \mathcal{K}^\top_{\mathbf{A}}(y, x)\big]^\top \varphi(y) \, dS, \;\; x \in \Omega^\pm, \\ \mathbf{V}_{j,k}(x, D) \varphi(x) &:= \frac{1}{2} \big[(\mathbf{B}_j \mathbf{V}_k \varphi(x))^+ + (\mathbf{B}_j \mathbf{V}_k \varphi(x))^-\big], \;\; x \in S. \quad (3.122) \end{aligned}$$

According to Theorems 3.19, 3.21 and 3.13, $\mathbf{V}_{j,k}$ is a pseudodifferential operator, $\operatorname{ord} \mathbf{B}_j(x, D) = j$, $\operatorname{ord} \mathbf{C}_k(x, D) = 1 - k$, and maps the spaces

$$\begin{aligned} \mathbf{V}_{j,k} \;&:\; H^s_p(S) \to H^{s+k-j}_p(S), \\ &:\; B^s_{p,q}(S) \to B^{s+k-j}_{p,q}(S), \;\; j, k = 0, 1, \end{aligned} \qquad (3.123)$$

continuously, provided that $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leqslant p \leqslant \infty$.

We have already explained in Corollary 3.11 in what sense the operator $\mathbf{V}_{j,k}$ should be understood when its order is strictly positive, i.e. $\operatorname{ord} \mathbf{V}_{0,1} = 1$. Since $\operatorname{ord} \mathbf{V}_{j,j} = 0$ (see (3.14)), $\mathbf{V}_{j,j}$ represents a Calderón–Zygmund singular integral operator and the integral in (3.122) is understood in the Cauchy principal value sense:

$$\mathbf{V}_{j,j}(\mathcal{x},D)\varphi(\mathcal{x}) := \\ := \lim_{\varepsilon\to 0}\int\limits_{S\setminus S(\mathcal{X},\varepsilon)} \mathbf{B}_j(\mathcal{x},D)\big[\overline{\mathbf{C}_j(\mathcal{y},D)}\mathcal{K}^\top_{\mathbf{A}}(\mathcal{y},\mathcal{x})\big]^\top\varphi(\mathcal{y})\,dS\,. \tag{3.124}$$

Here $S(\mathcal{x},\varepsilon)$ is the part of the surface S inside the sphere $S^{n-1}(\mathcal{x},\varepsilon)$ with radius ε centered at $\mathcal{x}\in S$. Then $\mathbf{V}_{j,j}$ is continuous in the spaces $H^s_p(S)$ and $B^s_{p,q}(S)$ (see (3.123)).

Theorem 3.23. *Let the BVP* (3.11) *be formally adjoint to* (3.10) *and suppose that Green's formula* (3.110) *holds. Then, for the traces* $\gamma^\pm_S\mathbf{B}_j\mathbf{V}_k$ *the following Plemelji's formulae are valid:*

$$(\gamma^-_S\mathbf{B}_j(\mathcal{x},D)\mathbf{V}_k\varphi)(\mathcal{x}) = (\gamma^+_S\mathbf{B}_j(\mathcal{x},D)\mathbf{V}_k\varphi)(\mathcal{x}) \;\; \textit{for} \;\; k\neq j\,, \tag{3.125}$$

$$(\gamma^\pm_S\mathbf{B}_j(\mathcal{x},D)\mathbf{V}_j\varphi)(\mathcal{x}) = \pm\frac{1}{2}\,\varphi(\mathcal{x}) + \mathbf{V}_{j,j}(\mathcal{x},D)\varphi(\mathcal{x})\,, \tag{3.126}$$

$$\mathcal{x}\in S\,,\quad k,j=0,1\,,\quad \varphi\in H^s_p(S)\,.$$

We remind, that $\mathbf{V}_1=\mathbf{V}$ *is the single layer and* $\mathbf{V}_0=\mathbf{W}$ *is the double layer potential.*

Proof. (3.125) follows directly from (3.117).

Let $\mathcal{x}\in S$ be the projection of $x\in\Omega$, i.e. $x=\mathcal{x}\pm c_x\boldsymbol{\nu}(\mathcal{x})$ (recall that $\boldsymbol{\nu}(\mathcal{x})$ is the unit exterior normal vector to S). The potential-type operator

$$\mathbf{V}_{j,j}\varphi(x) := \int\limits_S \mathcal{K}_{j,\mathbf{A}}(x,\mathcal{y})\varphi(\mathcal{y})dS\,, \tag{3.127}$$

$$\mathcal{K}_{j,\mathbf{A}}(x,\mathcal{y}) := \mathbf{B}_j(\mathcal{y},D)\big[\overline{\mathbf{C}_j(\mathcal{y},D)}\mathcal{K}^\top_{\mathbf{A}}(\mathcal{y},x)\big]^\top,\quad x\in\Omega\,,$$

restricted to S, has order 0 and has the following Calderón–Zygmund kernel

$$\mathcal{K}_{j,\mathbf{A}}\in C^\infty(\mathbb{R}^n\otimes\mathbb{R}^n\setminus\Delta_{\mathbb{R}^n})\,, \tag{3.128}$$

$$\big|\mathcal{K}_{j,\mathbf{A}}(x,\mathcal{y})\big| \leqslant M_0|\mathcal{y}|^{1-n}\,,\quad \mathcal{y}\in\mathbb{R}^n\,,\quad \mathcal{y}\neq 0\,. \tag{3.129}$$

Then the truncated operator

$$\mathbf{V}^0_{j,j,\varepsilon}\varphi(x) := \int\limits_{S\setminus S(\mathcal{X},\varepsilon)} \mathcal{K}_{j,\mathbf{A}}(x,\mathcal{y})\varphi(\mathcal{y})dS\,,\quad \varepsilon>0 \tag{3.130}$$

(see (3.124)) has C^∞-smooth kernel (see (3.128)) and

$$\lim_{\varepsilon\to 0}(\gamma^-_S\mathbf{V}^0_{j,j,\varepsilon}\varphi)(\mathcal{x}) = \lim_{\varepsilon\to 0}(\gamma^+_S\mathbf{V}^0_{j,j,\varepsilon}\varphi)(\mathcal{x})\,. \tag{3.131}$$

Due to the definition (3.124) and the continuity property (3.131),

$$(\gamma_S^{\pm}\mathbf{B}_j(x,D)\mathbf{V}_j\varphi)(\mathcal{X}) = (\mathbf{V}_{j,j}(\mathcal{X},D)\varphi)(\mathcal{X}) + \lim_{\varepsilon\to 0}(\gamma_S^{\pm}\mathbf{V}_{j,j,\varepsilon}\varphi)(\mathcal{X}),$$
$$\mathbf{V}_{j,j,\varepsilon}\varphi(x) = \int\limits_{S(\mathcal{X},\varepsilon)} \mathcal{K}_{j,\mathbf{A}}(x,\mathcal{Y})\varphi(\mathcal{Y})dS\,,\quad x\in\Omega\,,\quad \varphi\in C^\infty(S)\,. \tag{3.132}$$

Since $\varepsilon>0$ is sufficiently small there exists a diffeomorphism

$$\begin{gathered}\varkappa\,:\,S_0(\mathcal{X},\varepsilon)\to S(\mathcal{X},\varepsilon)\,,\quad \varkappa(x')=(x',g(x'))\in S(\mathcal{X},\varepsilon)\subset S\,,\\ x'=(x_1,\dots,x_{n-1})\in S_0(\mathcal{X},\varepsilon)\subset\mathbb{R}^{n-1}_{\mathcal{X}}\,,\\ g(\mathcal{X})=\mathcal{X}\in S\,,\ \ (\partial_k g)(\mathcal{X})=0\,,\ \ k=1,\dots,n-1\end{gathered} \tag{3.133}$$

and $S_0(\mathcal{X},\varepsilon)$ is the projection of the part $S(\mathcal{X},\varepsilon)$ into the tangential plane $\mathbb{R}^{n-1}_x$ to S at $\mathcal{X}\in S$. By the variable transformastion $\mathcal{X}=\varkappa(y')$, $y'\in S_0(\mathcal{X},\varepsilon)$ in the integral (3.6) we get the following

$$\mathbf{V}_{j,j,\varepsilon}\varphi(x) := \int\limits_{\mathbb{R}^{n-1}} \mathcal{K}_{j,\mathbf{A}}(x,x-\varkappa(y'))\mathcal{G}_\varkappa(y')\chi_\varepsilon(y')\varphi(\varkappa(y'))dy'\,,$$
$$|x-y'|<2\varepsilon\,,\quad x\neq y'\,,$$

where χ_ε is the indicator function of the part $S_0(\mathcal{X},\varepsilon)\subset\mathbb{R}^{n-1}$ and

$$\mathcal{G}_\varkappa(y') := \sqrt{|\partial\, g(y')|^2+1} = 1+\mathcal{O}|y'-\mathcal{X}| \tag{3.134}$$

is the Gram determinant (see (2.86) in § 6.3).

Next we note that

$$\mathbf{V}_{j,\varepsilon}\varphi(x) := \int\limits_{\mathbb{R}^{n-1}} \mathcal{K}_{j,\mathbf{A}}(x,y')\chi_\varepsilon(y')\varphi(\varkappa(y'))\,dy' + o(1) \tag{3.135}$$

as $\varepsilon\to 0$ uniformly for $x\in\mathbb{R}^n$.

Indeed, the remainder kernel

$$\mathcal{K}^0_{j,\mathbf{A}}(x,y') := \mathcal{K}_{j,\mathbf{A}}(x,\varkappa(y'))\mathcal{G}_\varkappa(y') - \mathcal{K}_{j,\mathbf{A}}(x,y')$$

is weakly singular

$$\left|\mathcal{K}^0_{j,\mathbf{A}}(x,y')\right| \leqslant M_1|x-y'|^{2-n}\,,\ \ x,y'\in\mathbb{R}^n\,,\ \ x\neq y' \tag{3.136}$$

(cf. (3.128); see (3.131)) and, almost obviously,

$$\lim_{\varepsilon\to 0}\gamma_S^{\pm}\int\limits_{S_0(\mathcal{X},\varepsilon)} \mathcal{K}^0_{j,\mathbf{A}}(x,y')\mathcal{G}_\varkappa(y')\chi_\varepsilon(y')\varphi(\varkappa(y'))\,dS = 0$$

for arbitrary $\varphi\in C^\infty(S)$. By the same reason

$$\mathbf{V}_{j,\varepsilon}\varphi(x) := \varphi(\mathcal{X})\int\limits_{S_0(\mathcal{X},\varepsilon)} \mathcal{K}_{j,\mathbf{A}}(x,x-y')dS + o(1)\ \text{ as }\ \varepsilon\to 0\,, \tag{3.137}$$

because $|\varphi(\varkappa(y'))-\varphi(\mathcal{X})|\leqslant M_2|y'-\mathcal{X}|$.

The difference between the kernel $\mathcal{K}_{j,\mathbf{A}}(x,y')$ in (3.127) defined by the differential operators $\mathbf{B}_j(x,D)$, $\mathbf{C}_j(x,D)$ and $\mathbf{A}(x,D)$ and the kernel

$\mathcal{K}^0_{j,\mathbf{A}}(x,y')$ defined by the principal parts $\mathbf{B}_{j,0}(\mathcal{X},D)$, $\mathbf{C}_{j,0}(\mathcal{X},D)$ is weakly singular and admits an estimate similar to (3.136). Therefore, as in (3.137),

$$\mathbf{V}_{j,\varepsilon}\varphi(x) := \varphi(\mathcal{X}) \int\limits_{S_0(\mathcal{X},\varepsilon)} \mathcal{K}_{j,0,\mathbf{A}}(x,y')dy' + o(1) \ \text{ as } \ \varepsilon \to 0\,. \tag{3.138}$$

We can further simplify the integral in (3.138):

(1) Replace the domain of integration $S_0(\mathcal{X},\varepsilon)$ by the ball

$$\mathcal{B}(\mathcal{X},\varepsilon) := \big\{|y' - \mathcal{X}| \leqslant \varepsilon :\ y' \in \mathbb{R}^{n-1}\big\}.$$

Observe that $\operatorname{mes}\mathcal{B}(\mathcal{X},\varepsilon) - \operatorname{mes} S_0(\mathcal{X},\varepsilon) = \mathcal{O}(\varepsilon)$, while the corresponding integrals differ by $o(1)$ as $\varepsilon \to 0$.

(2) Freeze coefficients at $\mathcal{X}_0 \in S$ as $\varepsilon \to 0$, to consider a pure convolution kernel $\mathcal{K}_{j,\mathcal{X}_0,\mathbf{A}}(x-y')$, which is translation invariant; the remainder has a weak singularity and contributes the summand $o(1)$ in (3.138).

(3) Due to the described simplifications, the domain of integration in (3.138), $|y' - \mathcal{X}| \leqslant \varepsilon$ can be translated (shifted) to the origin and stretched to the unit ball $|y'| \leqslant 1$; the integral is invariant with respect to translations and dilations (stretching).

Finally, taking the traces, we get the following

$$(\gamma_S^{\pm}\mathbf{V}_{j,\varepsilon}\varphi)(\mathcal{X}) := \pm c_0\ \varphi(\mathcal{X}) + o(1) \ \text{ as } \ \varepsilon \to 0\,, \tag{3.139}$$

where $\gamma^{\pm}$ denote the traces on different faces of the surface; the integral

$$c_0 := \int\limits_{|y'|\leqslant 1} \mathcal{K}_{j,x_0,\mathbf{A}}(y')\,dy'$$

is independent of $\varepsilon > 0$ and $\mathcal{X}_0 \in S$. By invoking (3.120) we find $c_0 = 1/2$. Now (3.6) and (3.139) yield (3.126). □

4. Representation Formulae in Thermoelasticity and Piezo-Thermoelasticity

In this section we apply the general results established in the previous two sections to the differential equations of the theory of thermoelasticity and piezo-thermoelasticity. In particular, we derive general representation formulas of solutions to some special BVS which are very important in our analysis. Moreover, we calculate explicitly the principal homogeneous symbol matrices of the boundary integral (pseudodifferential) operators generated by the corresponding single and double layer potentials and study their properties which are essentially applied in the subsequent sections in the qualitative analysis of solutions of the mixed boundary-transmission problems.

4.1. **Fundamental solutions in thermoelasticity and piezo-thermoelasticity.** Recall that $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ stand for the generalized direct and inverse Fourier transforms. Denote by $\Psi^{(m)}(\cdot\,,\tau)=[\,\Psi^{(m)}_{kj}(\cdot\,,\tau)\,]_{4\times4}$ and $\Psi(\cdot\,,\tau)=[\,\Psi_{kj}(\cdot\,,\tau)\,]_{5\times5}$ the fundamental matrices of the operators $A^{(m)}(\partial,\tau)$ and $A(\partial,\tau)$,

$$\begin{aligned}A^{(m)}(\partial_x,\tau)\Psi^{(m)}(x-y,\tau)&=\delta(x-y)\,I_4,\\ A(\partial_x,\tau)\Psi(x-y,\tau)&=\delta(x-y)\,I_5,\end{aligned}$$

where $\delta(\cdot)$ denotes Dirac's delta function. We have then the following representation formulas

$$\begin{aligned}\Psi^{(m)}(x,\tau)&=\mathcal{F}^{-1}_{\xi\to x}\big([A^{(m)}(-i\xi,\tau)]^{-1}\big)=\\ &=\frac{1}{(2\pi)^3}\lim_{R\to\infty}\int\limits_{|\xi|<R}[A^{(m)}(-i\xi,\tau)]^{-1}\,e^{-ix\xi}\,d\xi,\end{aligned}\tag{4.1}$$

$$\begin{aligned}\Psi(x,\tau)&=\mathcal{F}^{-1}_{\xi\to x}\big([A(-i\xi,\tau)]^{-1}\big)=\\ &=\frac{1}{(2\pi)^3}\lim_{R\to\infty}\int\limits_{|\xi|<R}[A(-i\xi,\tau)]^{-1}\,e^{-ix\xi}\,d\xi.\end{aligned}\tag{4.2}$$

Recall that $A^{(m,0)}(\partial)$ and $A^{(0)}(\partial)$ are the principal homogeneous parts of the differential operators $A^{(m)}(\partial,\tau)$ and $A(\partial,\tau)$, respectively (see (1.13) and (1.28)). The principal singular parts of the matrices $\Psi^{(m)}(\,\cdot\,,\tau)$ and $\Psi(\,\cdot\,,\tau)$ can be represented as (see [7])

$$\begin{aligned}\Psi^{(m,0)}(x)&=\mathcal{F}^{-1}_{\xi'\to x'}\bigg(\frac{1}{2\pi}\int\limits_{\ell^{\pm}}\big[A^{(m,0)}(-i\xi)\big]^{-1}e^{-i\,\xi_3\,x_3}\,d\xi_3\bigg)\\ &=-\frac{1}{8\pi^2\,|x|}\int\limits_0^{2\pi}\big[A^{(m,0)}(\Lambda\,\eta)\big]^{-1}\,d\theta,\end{aligned}\tag{4.3}$$

$$\begin{aligned}\Psi^{(0)}(x)&=\mathcal{F}^{-1}_{\xi'\to x'}\bigg(\frac{1}{2\pi}\int\limits_{\ell^{\pm}}\big[A^{(0)}(-i\xi)\big]^{-1}e^{-i\,\xi_3\,x_3}\,d\xi_3\bigg)\\ &=-\frac{1}{8\pi^2\,|x|}\int\limits_0^{2\pi}\big[A^{(0)}(\Lambda\,\eta)\big]^{-1}\,d\theta,\end{aligned}\tag{4.4}$$

where $x=(x_1,x_2,x_3)$, $\xi=(\xi_1,\xi_2,\xi_3)$, $x'=(x_1,x_2)$, $\xi'=(\xi_1,\xi_2)$, the sign "$-$" corresponds to the case $x_3>0$, and the sign "$+$" to the case $x_3<0$; ℓ^+ (respectively ℓ^-) is a closed simple contour in the complex half-plane $\operatorname{Im}\xi_3>0$ (respectively $\operatorname{Im}\xi_3<0$) orientated counterclockwise (respectively clockwise) and enveloping all the roots of the corresponding polynomials $\det A^{(m,0)}(-i\,\xi)$ and $\det A^{(0)}(-i\,\xi)$ with respect to ξ_3 with positive (respectively negative) imaginary parts; here $\Lambda=[\Lambda_{kj}]_{3\times3}$ is an orthogonal matrix associated with x and possessing the property $\Lambda^\top x = (0,0,|x|)^\top$, and $\eta = (\cos\theta, \sin\theta, 0)^\top$.

Note that

$$\Psi^{(m,0)}(x,\tau) = \Psi^{(m,0)}(-x,\tau) = \big[\Psi^{(m,0)}(x,\tau)\big]^\top,$$
$$\Psi^{(0)}(x,\tau) = \Psi^{(0)}(-x,\tau) \neq \big[\Psi^{(0)}(x,\tau)\big]^\top.$$

These matrices have the singularity of type $O(|x|^{-1})$ in a neighbourhood of the origin and at infinity decay as $O(|x|^{-1})$. Moreover, there are positive constants $c_0^{(m)} > 0$ and $c_0 > 0$ (depending on τ and on the material parameters) such that in a neighbourhood of the origin (say $|x| < 1/2$) there hold the estimates

$$\big|\,\Psi_{kj}^{(m)}(x,\tau) - \Psi_{kj}^{(m,0)}(x)\,\big| \leqslant c_0^{(m)}\ \log|x|^{-1},$$
$$\big|\,\partial^\alpha\big[\,\Psi_{kj}^{(m)}(x,\tau) - \Psi_{kj}^{(m,0)}(x)\,\big]\,\big| \leqslant c_0^{(m)}\ |x|^{-|\alpha|}$$
$$\text{for } |\alpha| = 1,2, \text{ and } k,j = \overline{1,4},$$

$$\big|\,\Psi_{pq}(x,\tau) - \Psi_{pq}^{(0)}(x)\,\big| \leqslant c_0\ \log|x|^{-1},$$
$$\big|\,\partial^\alpha\big[\,\Psi_{pq}(x,\tau) - \Psi_{pq}^{(0)}(x)\,\big]\,\big| \leqslant c_0\ |x|^{-|\alpha|}$$
$$\text{for } |\alpha| = 1,2, \text{ and } p,q = \overline{1,5},$$

where $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ is a multi-index and $|\alpha| = \alpha_1+\alpha_2+\alpha_3$. Moreover,

$$\Psi_{k4}^{(m,0)}(x) = \Psi_{4k}^{(m,0)}(x) = 0, \quad \Psi_{j4}^{(0)}(x) = \Psi_{4j}^{(0)}(x) = 0,$$
$$k = 1,2,3,\ j = 1,2,3,5,$$

and the kernels $\varkappa_{jl}^{(m)}\,\nu_j(y)\,\partial_l\Psi_{44}^{(m,0)}(x-y)$ and $\varkappa_{jl}\,n_j(y)\,\partial_l\Psi_{44}^{(0)}(x-y)$, associated with the co-normal derivatives corresponding to the thermoconductive operators $\varkappa_{jl}^{(m)}\,\partial_j\,\partial_l$ and $\varkappa_{jl}\,\partial_j\,\partial_l$, have weak singularity on $\partial\Omega^{(m)}$ and $\partial\Omega$ respectively.

For the Newton type volume potentials

$$N_\tau^{(m)}(\Phi^{(m)})(x) := \int\limits_{\Omega^{(m)}} \Psi^{(m)}(x-y,\tau)\Phi^{(m)}(y)\,dy,$$

$$N_\tau(\Phi)(x) := \int\limits_{\Omega} \Psi(x-y,\tau)\,\Phi(y)\,dy,$$

the following theorem holds.

Theorem 4.1. *Let $\Omega^{(m)}$ and Ω be Lipschitz domains and $0 < \beta' < 1$. Then the operators*

$$N_\tau^{(m)}\ :\ [L_2(\Omega)]^4 \to [W_2^2(\Omega)]^4\,, \tag{4.5}$$
$$:\ [C^{0,\beta'}(\overline{\Omega})]^4 \to [C^{2,\beta'}(\Omega)]^4 \cap [C^{1,\beta'}(\overline{\Omega})]^4\,,$$

$$N_\tau \ : \ [L_2(\Omega)]^5 \to [W_2^2(\Omega)]^5 \,, \qquad (4.6)$$
$$\ : \ [C^{\,0,\beta'}(\overline{\Omega})]^5 \to [C^{\,2,\beta'}(\Omega)]^5 \cap [C^{\,1,\beta'}(\overline{\Omega})]^5$$

are bounded. Moreover,

$$A^{(m)}(\partial,\tau)\, N_\tau^{(m)}(\Phi^{(m)})(x) = \Phi^{(m)}(x), \quad x \in \Omega^{(m)}, \qquad (4.7)$$
$$A(\partial,\tau)\, N_\tau(\Phi)(x) = \Phi(x), \quad x \in \Omega, \qquad (4.8)$$

for almost all $x \in \Omega^{(m)}$ *and for almost all* $x \in \Omega$ *respectively. In addition, if* $\Phi^{(m)} \in [C^{\,0,\beta'}(\overline{\Omega^{(m)}})]^4$ *and* $\Phi \in [C^{\,0,\beta'}(\overline{\Omega})]^5$*, then the relations* (1.44) *and* (1.45) *hold for all* $x \in \Omega^{(m)}$ *and for all* $x \in \Omega$*, respectively.*

4.2. **Layer potentials of thermoelasticity and piezo-thermoelasticity.** Let $\Psi^{(m)}(\cdot\,,\tau) = \big[\,\Psi_{kj}^{(m)}(\cdot\,,\tau)\big]_{4\times 4}$ and $\Psi(\cdot\,,\tau) = \big[\,\Psi_{kj}(\cdot\,,\tau)\big]_{5\times 5}$ be the fundamental matrix-functions of the differential operators $A^{(m)}(\partial_x,\tau)$ and $A(\partial_x,\tau)$ constructed above and introduce the single and double layer potentials:

$$V_\tau^{(m)}(h^{(m)})(x) = \int\limits_{\partial\Omega^{(m)}} \Psi^{(m)}(x-y,\tau)\; h^{(m)}(y)\, d_yS, \qquad (4.9)$$

$$W_\tau^{(m)}(h^{(m)})(x) = \int\limits_{\partial\Omega^{(m)}} \Big[\widetilde{\mathcal{T}}^{(m)}(\partial_y,\nu(y),\overline{\tau})\big[\Psi^{(m)}(x-y,\tau)\big]^\top\Big]^\top h^{(m)}(y)\, d_yS, \qquad (4.10)$$

$$V_\tau(h)(x) = \int\limits_{\partial\Omega} \Psi(x-y,\tau)\; h(y)\, d_yS, \qquad (4.11)$$

$$W_\tau(h)(x) = \int\limits_{\partial\Omega} \Big[\,\widetilde{\mathcal{T}}(\partial_y,n(y),\overline{\tau})\,\big[\Psi(x-y,\tau)\big]^\top\Big]^\top h(y)\, d_yS, \qquad (4.12)$$

where $h^{(m)} = (h_1^{(m)}, h_2^{(m)}, h_3^{(m)}, h_4^{(m)})^\top$ and $h = (h_1,h_2,h_3,h_4,h_5)^\top$ are densities of the potentials.

For the boundary integral (pseudodifferential) operators generated by the layer potentials we will employ the following notation:

$$\mathcal{H}_\tau^{(m)}(h^{(m)})(x) := \int\limits_{\partial\Omega^{(m)}} \Psi^{(m)}(x-y,\tau)\; h^{(m)}(y)\, d_yS,$$

$$\mathcal{K}_\tau^{(m)}(h^{(m)})(x) := \int\limits_{\partial\Omega^{(m)}} \Big[\,\mathcal{T}^{(m)}(\partial_x,\nu(x))\Psi^{(m)}(x-y,\tau)\Big] h^{(m)}(y)\, d_yS,$$

$$\widetilde{\mathcal{K}}_\tau^{(m)\,*}(h^{(m)})(x) := \int\limits_{\partial\Omega^{(m)}} \Big[\,\widetilde{\mathcal{T}}^{(m)}(\partial_y,\nu(y),\overline{\tau})\,\big[\Psi^{(m)}(x-y,\tau)\big]^\top\Big]^\top h^{(m)}(y)\, d_yS,$$

$$\mathcal{H}_\tau(h)(x) := \int\limits_{\partial\Omega} \Psi(x-y,\tau)\; h(y)\, d_yS,$$

$$\mathcal{K}_\tau(h)(x) := \int\limits_{\partial\Omega} \big[\, \mathcal{T}(\partial_x, n(x))\Psi(x-y,\tau) \,\big]\; h(y)\, d_yS,$$

$$\widetilde{\mathcal{K}}^*_\tau(h)(x) := \int\limits_{\partial\Omega} \Big[\, \widetilde{\mathcal{T}}(\partial_y, n(y), \overline{\tau})\big[\Psi(x-y,\tau)\big]^\top \,\Big]^\top h(y)\, d_yS,$$

$$\mathcal{L}^{(m)}_\tau(h^{(m)})(x) := \Big\{\, \mathcal{T}^{(m)}(\partial_x, \nu(x)) W^{(m)}(h^{(m)})(x) \,\Big\}^\pm, \;\; x \in \partial\Omega^{(m)},$$

$$\mathcal{L}_\tau(h)(x) := \big\{\, \mathcal{T}(\partial_x, n(x)) W_\tau(h)(x) \,\big\}^\pm, \;\; x \in \partial\Omega.$$

The boundary operators $\mathcal{H}^{(m)}_\tau$, $\mathcal{H}_\tau$ and $\mathcal{L}^{(m)}_\tau$, $\mathcal{L}_\tau$ are pseudodifferential operators of order -1 and 1, respectively, while the operators $\mathcal{K}^{(m)}_\tau$, $\widetilde{\mathcal{K}}^{(m)\,*}_\tau$, $\mathcal{K}_\tau$ and $\widetilde{\mathcal{K}}^*_\tau$ are singular integral operators (pseudodifferential operators of order 0) (for details see [4–7, 9, 29, 30, 53]).

4.3. **Properties of layer potentials of thermoelasticity and piezo-thermoelasticity.** Recall that n and ν stand for the unite outward normal vectors to $\partial\Omega$ and $\partial\Omega^{(m)}$, respectively, and that $\partial\Omega, \partial\Omega^{(m)} \in C^\infty$. We describe here mapping properties of the layer potentials and the boundary integral operators generated by them which actually have been proved in the previous two sections. However, we note that for the potentials $V^{(m)}_\tau$ and $W^{(m)}_\tau$ with regular densities the proofs can be found in [33], in the isotropic case, and in [29, 30, 54], in the anisotropic case, while for the potentials V_τ and W_τ the proofs can be found in [4, 6, 7, 9]. Note that the main ideas for generalization to the scale of Bessel potential and Besov spaces are based on the duality and interpolation technique and is described in the references [10, 17–19, 27, 63], using the theory of pseudodifferential operators on smooth manifolds without boundary.

For similar properties in the case of general Lipschitz domains see [25, 44, 50].

Theorem 4.2. *Let $1 < p < \infty$, $1 \leqslant t \leqslant \infty$, and $s \in \mathbb{R}$. The operators*

$$\begin{aligned}
V^{(m)}_\tau \;&:\; \big[B^s_{p,p}(\partial\Omega^{(m)})\big]^4 \to \big[H^{s+1+\frac1p}_p(\Omega^{(m)})\big]^4\\
&:\; \big[B^s_{p,t}(\partial\Omega^{(m)})\big]^4 \to \big[B^{s+1+\frac1p}_{p,t}(\Omega^{(m)})\big]^4,\\
W^{(m)}_\tau \;&:\; \big[B^s_{p,p}(\partial\Omega^{(m)})\big]^4 \to \big[H^{s+\frac1p}_p(\Omega^{(m)})\big]^4,\\
&:\; \big[B^s_{p,t}(\partial\Omega^{(m)})\big]^4 \to \big[B^{s+\frac1p}_{p,t}(\Omega^{(m)})\big]^4,\\
V_\tau \;&:\; \big[B^s_{p,p}(\partial\Omega)\big]^5 \to \big[H^{s+1+\frac1p}_p(\Omega)\big]^5\\
&:\; \big[B^s_{p,t}(\partial\Omega)\big]^5 \to \big[B^{s+1+\frac1p}_{p,t}(\Omega)\big]^5,\\
W_\tau \;&:\; \big[B^s_{p,p}(\partial\Omega)\big]^5 \to \big[H^{s+\frac1p}_p(\Omega)\big]^5,\\
&:\; \big[B^s_{p,t}(\partial\Omega)\big]^5 \to \big[B^{s+\frac1p}_{p,t}(\Omega)\big]^5
\end{aligned}$$

are continuous.

Theorem 4.3. *Let $1 < p < \infty$, $1 \leqslant t \leqslant \infty$,*

$$h^{(m)} \in \big[\, B_{p,t}^{-\frac{1}{p}}(\partial\Omega^{(m)})\,\big]^4, \quad g^{(m)} \in \big[\, B_{p,t}^{1-\frac{1}{p}}(\partial\Omega^{(m)})\,\big]^4,$$
$$h \in \big[\, B_{p,t}^{-\frac{1}{p}}(\partial\Omega)\,\big]^5, \quad g \in \big[\, B_{p,t}^{1-\frac{1}{p}}(\partial\Omega)\,\big]^5.$$

Then

$$\begin{aligned}
\big\{V_\tau^{(m)}(h^{(m)})\big\}^+ &= \big\{V_\tau^{(m)}(h^{(m)})\big\}^- = \mathcal{H}_\tau^{(m)}\, h^{(m)} \ \ on \ \ \partial\Omega^{(m)},\\
\big\{\mathcal{T}^{(m)}(\partial,\nu)V_\tau^{(m)}(h^{(m)})\big\}^\pm &= \big[\, \mp 2^{-1} I_4 + \mathcal{K}_\tau^{(m)}\,\big]\, h^{(m)} \ \ on \ \ \partial\Omega^{(m)},\\
\big\{W_\tau^{(m)}(g^{(m)})\big\}^\pm &= \big[\, \pm 2^{-1} I_4 + \widetilde{\mathcal{K}}_\tau^{(m)*}\,\big]\, g^{(m)} \ \ on \ \ \partial\Omega^{(m)},\\
\big\{V_\tau(h)\big\}^+ &= \big\{V_\tau(h)\big\}^- = \mathcal{H}_\tau\, h \ \ on \ \ \partial\Omega,\\
\big\{\mathcal{T}(\partial,n)V_\tau(h)\big\}^\pm &= \big[\, \mp 2^{-1} I_5 + \mathcal{K}_\tau\,\big]\, h \ \ on \ \ \partial\Omega,\\
\big\{W_\tau(g)\big\}^\pm &= \big[\, \pm 2^{-1} I_5 + \widetilde{\mathcal{K}}_\tau^{*}\,\big]\, g \ \ on \ \ \partial\Omega,
\end{aligned}$$

where I_k stands for the $k \times k$ unit matrix. Moreover,

$$\big\{\mathcal{T}^{(m)}(\partial,\nu)W_\tau^{(m)}(g^{(m)})\big\}^+ = \big\{\mathcal{T}^{(m)}(\partial,\nu)W_\tau^{(m)}(g^{(m)})\big\}^- \ \ on \ \ \partial\Omega^{(m)}$$

and

$$\big\{\mathcal{T}(\partial,n)W_\tau(g)\,\big\}^+ = \big\{\mathcal{T}(\partial,n)W_\tau(g)\big\}^- \ \ on \ \ \partial\Omega.$$

Theorem 4.4. *Let $1 < p < \infty$, $1 \leqslant t \leqslant \infty$, $s \in \mathbb{R}$. The operators*

$$\begin{aligned}
\mathcal{H}_\tau^{(m)} \ &: \ \big[H_p^s(\partial\Omega^{(m)})\big]^4 \to \big[H_p^{s+1}(\partial\Omega^{(m)})\big]^4,\\
&: \ \big[B_{p,t}^s(\partial\Omega^{(m)})\big]^4 \to \big[B_{p,t}^{s+1}(\partial\Omega^{(m)})\big]^4,\\
\mathcal{K}_\tau^{(m)},\ \widetilde{\mathcal{K}}_\tau^{(m)*} \ &: \ \big[H_p^s(\partial\Omega^{(m)})\big]^4 \to \big[H_p^{s}(\partial\Omega^{(m)})\big]^4,\\
&: \ \big[B_{p,t}^s(\partial\Omega^{(m)})\big]^4 \to \big[B_{p,t}^{s}(\partial\Omega^{(m)})\big]^4,\\
\mathcal{L}_\tau^{(m)} \ &: \ \big[H_p^{s+1}(\partial\Omega^{(m)})\big]^4 \to \big[H_p^{s}(\partial\Omega^{(m)})\big]^4,\\
&: \ \big[B_{p,t}^{s+1}(\partial\Omega^{(m)})\big]^4 \to \big[B_{p,t}^{s}(\partial\Omega^{(m)})\big]^4,\\
\mathcal{H}_\tau \ &: \ \big[H_p^s(\partial\Omega)\big]^5 \to \big[H_p^{s+1}(\partial\Omega)\big]^5,\\
&: \ \big[B_{p,t}^s(\partial\Omega)\big]^5 \to \big[B_{p,t}^{s+1}(\partial\Omega)\big]^5,\\
\mathcal{K}_\tau,\ \widetilde{\mathcal{K}}_\tau^{*} \ &: \ \big[H_p^s(\partial\Omega)\big]^5 \to \big[H_p^{s}(\partial\Omega)\big]^5,\\
&: \ \big[B_{p,t}^s(\partial\Omega)\big]^5 \to \big[B_{p,t}^{s}(\partial\Omega)\big]^5,\\
\mathcal{L}_\tau \ &: \ \big[H_p^{s+1}(\partial\Omega)\big]^5 \to \big[H_p^{s}(\partial\Omega)\big]^5,\\
&: \ \big[B_{p,t}^{s+1}(\partial\Omega)\big]^5 \to \big[B_{p,t}^{s}(\partial\Omega)\big]^5
\end{aligned}$$

are continuous.

Moreover, the following operator equalities hold in appropriate function spaces:

$$\widetilde{\mathcal{K}}_\tau^{(m)*}\,\mathcal{H}_\tau^{(m)} = \mathcal{H}_\tau^{(m)}\,\mathcal{K}_\tau^{(m)}, \quad \mathcal{L}_\tau^{(m)}\,\widetilde{\mathcal{K}}_\tau^{(m)*} = \mathcal{K}_\tau^{(m)}\,\mathcal{L}_\tau^{(m)},$$

$$\mathcal{L}_\tau^{(m)}\,\mathcal{H}_\tau^{(m)} = -4^{-1}I_4 + \big[\,\mathcal{K}_\tau^{(m)}\,\big]^2, \quad \mathcal{H}_\tau^{(m)}\,\mathcal{L}_\tau^{(m)} = -4^{-1}I_4 + \big[\,\widetilde{\mathcal{K}}_\tau^{(m)*}\,\big]^2,$$

$$\widetilde{\mathcal{K}}_\tau^{*}\,\mathcal{H}_\tau = \mathcal{H}_\tau\,\mathcal{K}_\tau, \quad \mathcal{L}_\tau\,\widetilde{\mathcal{K}}_\tau^{*} = \mathcal{K}_\tau\,\mathcal{L}_\tau,$$

$$\mathcal{L}_\tau\,\mathcal{H}_\tau = -4^{-1}I_5 + \big[\,\mathcal{K}_\tau\,\big]^2, \quad \mathcal{H}_\tau\,\mathcal{L}_\tau = -4^{-1}I_5 + \big[\,\widetilde{\mathcal{K}}_\tau^{*}\,\big]^2.$$

Theorem 4.5. *Let $1 < p < \infty$, $1 \leqslant t \leqslant \infty$, $s \in \mathbb{R}$ and $\tau = \sigma + i\,\omega$. The operators*

$$\begin{aligned}
\mathcal{H}_\tau^{(m)} \ &:\ \big[H_p^s(\partial\Omega^{(m)})\big]^4 \to \big[H_p^{s+1}(\partial\Omega^{(m)})\big]^4,\\
&:\ \big[B_{p,t}^s(\partial\Omega^{(m)})\big]^4 \to \big[B_{p,t}^{s+1}(\partial\Omega^{(m)})\big]^4,\\
\mathcal{H}_\tau \ &:\ \big[H_p^s(\partial\Omega)\big]^5 \to \big[H_p^{s+1}(\partial\Omega)\big]^5,\\
&:\ \big[B_{p,t}^s(\partial\Omega)\big]^5 \to \big[B_{p,t}^{s+1}(\partial\Omega)\big]^5
\end{aligned}$$

are invertible if $\sigma > 0$ or $\tau = 0$.

The operators

$$\begin{aligned}
\pm\frac{1}{2}\,I_4 + \mathcal{K}_\tau^{(m)} \ &:\ \big[H_p^s(\partial\Omega^{(m)})\big]^4 \to \big[H_p^s(\partial\Omega^{(m)})\big]^4,\\
&:\ \big[B_{p,t}^s(\partial\Omega^{(m)})\big]^4 \to \big[B_{p,t}^s(\partial\Omega^{(m)})\big]^4,\\
\pm\frac{1}{2}\,I_4 + \widetilde{\mathcal{K}}_\tau^{(m)*} \ &:\ \big[H_p^s(\partial\Omega^{(m)})\big]^4 \to \big[H_p^s(\partial\Omega^{(m)})\big]^4,\\
&:\ \big[B_{p,t}^s(\partial\Omega^{(m)})\big]^4 \to \big[B_{p,t}^s(\partial\Omega^{(m)})\big]^4
\end{aligned}$$

are invertible if $\sigma > 0$.

The operators

$$\begin{aligned}
\pm\frac{1}{2}\,I_5 + \mathcal{K}_\tau \ &:\ \big[\,H_p^s(\partial\Omega)\,\big]^5 \to \big[\,H_p^s(\partial\Omega)\,\big]^5,\\
&:\ \Big[\big[\,B_{p,t}^s(\partial\Omega)\,\big]^5 \to \big[\,B_{p,t}^s(\partial\Omega)\,\big]^5\Big]
\end{aligned}$$

are Fredholm with the index equal to zero for any $\tau \in \mathbb{C}$.

4.4. **Explicit expressions for symbol matrices.** Here we present the explicit expressions for the principal homogeneous symbol matrices of the pseudodifferential operators introduced in Subsection 4.2, and establish their properties. Recall that the principal homogeneous symbol matrix of the pseudodifferential operator $\mathcal{A}$ on a manifold S is denoted by $\mathfrak{S}_{\mathcal{A}}(x,\xi_1,\xi_2)$, $x \in S$, $(\xi_1,\xi_2) \in \mathbb{R}^2 \setminus \{0\}$. With the help of the relations (1.13), (1.16), (1.28), (1.34), (1.35), (4.3) and (4.4) we can derive the following formulas for the principal homogeneous symbol matrices of the operators

$\mathcal{H}_\tau^{(m)}$, $-2^{-1}I_4+\mathcal{K}_\tau^{(m)}$, $\mathcal{H}$, and $-2^{-1}I_5+\mathcal{K}_\tau$:

$$\widetilde{M}^{(m)}(x,\xi_1,\xi_2):=\mathfrak{S}_{\mathcal{H}_\tau^{(m)}}(x,\xi_1,\xi_2)=\big[\,\widetilde{M}_{kj}^{(m)}(x,\xi_1,\xi_2)\,\big]_{4\times 4}=$$
$$=\begin{bmatrix}[M_{kj}^{(m)}(x,\xi_1,\xi_2)]_{3\times 3} & [\,0\,]_{3\times 1}\\ [\,0\,]_{1\times 3} & M_{44}^{(m)}(x,\xi_1,\xi_2)\end{bmatrix}_{4\times 4}=$$
$$=-\frac{1}{2\pi}\int\limits_{\ell^{\pm}}[\,A^{(m,0)}(\,B_\nu\,\xi\,)\,]^{-1}\,d\xi_3,\quad \xi=(\xi_1,\xi_2,\xi_3), \tag{4.13}$$

$$\widetilde{N}_{\pm}^{(m)}(x,\xi_1,\xi_2):=\mathfrak{S}_{\pm 2^{-1}I_4+\mathcal{K}_\tau^{(m)}}(x,\xi_1,\xi_2)=\big[\,\widetilde{N}_{kj,\pm}^{(m)}(x,\xi_1,\xi_2)\,\big]_{4\times 4}=$$
$$=\begin{bmatrix}[N_{kj,\pm}^{(m)}(x,\xi_1,\xi_2)]_{3\times 3} & [\,0\,]_{3\times 1}\\ [\,0\,]_{1\times 3} & \pm 2^{-1}\end{bmatrix}_{4\times 4}=$$
$$=\frac{i}{2\pi}\int\limits_{\ell^{\mp}}\mathcal{T}^{(m,0)}(\,B_\nu\,\xi,\nu)\big[\,A^{(m,0)}(\,B_\nu\,\xi\,)\,\big]^{-1}\,d\xi_3, \tag{4.14}$$

$$\widetilde{M}(x,\xi_1,\xi_2):=\mathfrak{S}_{\mathcal{H}_\tau}(x,\xi_1,\xi_2)=\big[\,\widetilde{M}_{kj}(x,\xi_1,\xi_2)\,\big]_{5\times 5}=$$
$$=\begin{bmatrix}[M_{kj}(x,\xi_1,\xi_2)]_{3\times 3} & [\,0\,]_{3\times 1} & [M_{k5}(x,\xi_1,\xi_2)]_{3\times 1}\\ [\,0\,]_{1\times 3} & M_{44}(x,\xi_1,\xi_2) & 0\\ [M_{5j}(x,\xi_1,\xi_2)]_{1\times 3} & 0 & M_{55}(x,\xi_1,\xi_2)\end{bmatrix}_{5\times 5}=$$
$$=-\frac{1}{2\pi}\int\limits_{\ell^{\pm}}\big[\,A^{(0)}(\,B_n\,\xi\,)\,\big]^{-1}\,d\xi_3, \tag{4.15}$$

$$\widetilde{N}_{\pm}(x,\xi_1,\xi_2):=\mathfrak{S}_{\pm 2^{-1}I_5+\mathcal{K}_\tau}(x,\xi_1,\xi_2)=\big[\,\widetilde{N}_{kj,\pm}(x,\xi_1,\xi_2)\,\big]_{5\times 5}=$$
$$=\begin{bmatrix}[N_{kj,\pm}(x,\xi_1,\xi_2)]_{3\times 3} & [\,0\,]_{3\times 1} & [N_{k5,\pm}(x,\xi_1,\xi_2)]_{3\times 1}\\ [\,0\,]_{1\times 3} & \pm 2^{-1} & 0\\ [N_{5j,\pm}(x,\xi_1,\xi_2)]_{1\times 3} & 0 & N_{55,\pm}(x,\xi_1,\xi_2)\end{bmatrix}_{5\times 5}=$$
$$=\frac{i}{2\pi}\int\limits_{\ell^{\mp}}\mathcal{T}^{(0)}(\,B_n\,\xi,n)\big[\,A^{(0)}(\,B_n\,\xi\,)\,\big]^{-1}\,d\xi_3, \tag{4.16}$$

$$B_\nu=\begin{bmatrix}l'_1 & l''_1 & \nu_1\\ l'_2 & l''_2 & \nu_2\\ l'_3 & l''_3 & \nu_3\end{bmatrix}\ \text{ for }\ x\in\partial\Omega^{(m)},\quad B_n=\begin{bmatrix}l'_1 & l''_1 & n_1\\ l'_2 & l''_2 & n_2\\ l'_3 & l''_3 & n_3\end{bmatrix}\ \text{ for }\ x\in\partial\Omega,$$

where $B_\nu(x)$ and $B_n(x)$ are orthogonal matrices with $\det B_\nu(x)=1$ and $\det B_n(x)=1$, $\nu(x)$ for $x\in\partial\Omega^{(m)}$ and $n(x)$ for $x\in\partial\Omega$ are the exterior unit normal vectors, respectively, and $l'(x)$ and $l''(x)$ are orthogonal unit vectors in the tangent plane associated with some local chart; ℓ^+ (respectively ℓ^-) is a closed simple contour in the complex half-plane $\operatorname{Im}\xi_3>0$ (respectively $\operatorname{Im}\xi_3<0$) , orientated counterclockwise (clockwise) and circumventing all the roots with positive (respectively negative) imaginary parts of the equations $\det A^{(m,0)}(B_\nu\,\xi) = 0$ and $\det A^{(0)}(B_n\,\xi) = 0$, respectively, with respect to ξ_3, while $(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$ play the role of parameters.

In equations (4.14) and (4.16) we employed that the operators $[\mathcal{K}_\tau^{(m)}]_{44}$ and $[\mathcal{K}_\tau]_{44}$ are weakly singular integral operators due to the remark at the end of Subsection 4.1.

The matrix $-\widetilde{M}^{(m)}(x,\xi_1,\xi_2)$ is positive definite, while $-\widetilde{M}(x,\xi_1,\xi_2)$ is strongly elliptic (for details see [6,29,30]), that is there are positive constants $c^{(m)}$ and c depending on the material parameters such that

$$-\widetilde{M}^{(m)}(x,\xi_1,\xi_2)\eta\cdot\eta \geqslant c^{(m)}\,|\xi|^{-1}\,|\eta|^2 \quad \text{for all } x\in\partial\Omega^{(m)},\ (\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\},\ \eta\in\mathbb{C}^4, \tag{4.17}$$

$$\mathrm{Re}\,\{-\widetilde{M}(x,\xi_1,\xi_2)\eta\cdot\eta\} \geqslant c\,|\xi|^{-1}\,|\eta|^2 \quad \text{for all } x\in\partial\Omega,\ (\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\},\ \eta\in\mathbb{C}^5. \tag{4.18}$$

In particular, $-M_{44}^{(m)}(x,\xi_1,\xi_2)>0$ for $x\in\partial\Omega^{(m)}$, $(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$, and $-M_{44}(x,\xi_1,\xi_2)>0$ for $x\in\partial\Omega$, $(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$.

The entries of the matrices $\widetilde{M}^{(m)}(x,\xi_1,\xi_2)$ and $\widetilde{M}(x,\xi_1,\xi_2)$ are even functions in (ξ_1,ξ_2).

The matrices (4.14) and (4.16) are nondegenerate, that is

$$\det\widetilde{N}^{(m)}(x,\xi_1,\xi_2)\neq 0 \ \text{ for all } \ x\in\partial\Omega^{(m)},\ \ (\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$$

and

$$\det\widetilde{N}_\pm(x,\xi_1,\xi_2)\neq 0 \ \text{ for all } \ x\in\partial\Omega,\ \ (\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}.$$

It is evident that the principal homogeneous symbol matrix of the operator $\mathcal{P}_\tau$, given by (4.41), reads as

$$\mathfrak{S}_{\mathcal{P}_\tau}(x,\xi_1,\xi_2)=\mathfrak{S}_{-2^{-1}I_5+\mathcal{K}_\tau}(x,\xi_1,\xi_2)=\widetilde{N}_-(x,\xi_1,\xi_2)=:\widetilde{N}(x,\xi_1,\xi_2) \tag{4.19}$$

and is nondegenerate.

Further, for the principal homogeneous symbol matrix of the operator $\mathcal{A}_\tau=\mathcal{H}_\tau\,[\mathcal{P}_\tau]^{-1}$ we have

$$\mathfrak{S}_{\mathcal{A}_\tau}(x,\xi_1,\xi_2)=\mathfrak{S}_{\mathcal{H}_\tau}(x,\xi_1,\xi_2)\left[\mathfrak{S}_{\mathcal{P}_\tau}(x,\xi_1,\xi_2)\right]^{-1}= =\widetilde{M}(x,\xi_1,\xi_2)\left[\widetilde{N}(x,\xi_1,\xi_2)\right]^{-1}. \tag{4.20}$$

Clearly, this matrix is nondegenerate as well.

Let us introduce the matrices obtained from (4.15) and (4.19) by deleting the fourth column and fourth row (see (4.16))

$$M(x,\xi_1,\xi_2):=\begin{bmatrix}[M_{kj}(x,\xi_1,\xi_2)]_{3\times3} & [M_{k5}(x,\xi_1,\xi_2)]_{3\times1}\\ [M_{5j}(x,\xi_1,\xi_2)]_{1\times3} & M_{55}(x,\xi_1,\xi_2)\end{bmatrix}_{4\times4},$$

$$N(x,\xi_1,\xi_2):=\begin{bmatrix}[N_{kj}(x,\xi_1,\xi_2)]_{3\times3} & [N_{k5}(x,\xi_1,\xi_2)]_{3\times1}\\ [N_{5j}(x,\xi_1,\xi_2)]_{1\times3} & N_{55}(x,\xi_1,\xi_2)\end{bmatrix}_{4\times4}.$$

Note that these non-degenerate matrices represent the principal homogeneous symbol matrices of the corresponding operators of piezoelastostatics and it is shown in [6] that the symbol

$$D(x,\xi_1,\xi_2):=[D_{kj}(x,\xi_1,\xi_2)]_{4\times 4}=M(x,\xi_1,\xi_2)\,[N(x,\xi_1,\xi_2)]^{-1} \qquad (4.21)$$

is a strongly elliptic matrix, that is

$$\operatorname{Re}\big\{D(x,\xi_1,\xi_2)\eta\cdot\eta\big\}\geqslant c\,|\xi|^{-1}\,|\eta|^2$$
$$\text{for all } \ x\in\partial\Omega,\ \ (\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\},\ \ \eta\in\mathbb{C}^4.$$

As an easy consequence we conclude that the symbol

$$\widetilde{\mathfrak{S}}_1(x,\xi_1,\xi_2):=\mathfrak{S}_{\mathcal{A}_\tau}(x,\xi_1,\xi_2)=\widetilde{M}(x,\xi_1,\xi_2)\,[\widetilde{N}(x,\xi_1,\xi_2)]^{-1}=$$
$$=\begin{bmatrix}[D_{kj}(x,\xi_1,\xi_2)]_{3\times 3} & [0]_{3\times 1} & [D_{k4}(x,\xi_1,\xi_2)]_{3\times 1}\\ [0]_{1\times 3} & -2\,M_{44}(x,\xi_1,\xi_2) & 0\\ [D_{4j}(x,\xi_1,\xi_2)]_{1\times 3} & 0 & D_{44}(x,\xi_1,\xi_2)\end{bmatrix}_{5\times 5} \qquad (4.22)$$

is strongly elliptic, since $-2M_{44}(x,\xi_1,\xi_2)>0$. Moreover, since $M_{44}(x,\xi_1,\xi_2)$ is an even function with respect to (ξ_1,ξ_2) we derive

$$\big[\widetilde{\mathfrak{S}}_1(x,0,+1)\big]^{-1}\,\widetilde{\mathfrak{S}}_1(x,0,-1)=$$
$$=\begin{bmatrix}[\mathcal{D}_{kj}(x)]_{3\times 3} & [0]_{3\times 1} & [\mathcal{D}_{k4}(x)]_{3\times 1}\\ [0]_{1\times 3} & 1 & 0\\ [\mathcal{D}_{4j}(x)]_{1\times 3} & 0 & \mathcal{D}_{44}(x)\end{bmatrix}_{5\times 5}, \qquad (4.23)$$

where

$$\mathcal{D}(x):=[\mathcal{D}_{kj}(x)]_{4\times 4}=[D(x,0,+1)]^{-1}\ D(x,0,-1),\ \ x\in\partial\Omega. \qquad (4.24)$$

Denote by $\lambda_j^{(1)}(x)$, $j=\overline{1,4}$, the eigenvalues of the matrix (4.24), that is the roots of the equation

$$\det\ \big[\mathcal{D}(x)-\lambda\,I_4\big]=0 \qquad (4.25)$$

with respect to λ. Then $\lambda_j^{(1)}(x)$, $j=\overline{1,4}$, and $\lambda_5^{(1)}=1$ are eigenvalues of the matrix (4.23). From the strong ellipticity property of the symbol matrix (4.21) it follows that $\lambda_j^{(1)}(x)$, $j=\overline{1,4}$, are complex numbers, in general, and $-\pi<\arg\lambda_j^{(1)}(x)<\pi$, that is $\lambda_j^{(1)}(x)\notin(-\infty,0]$. Remark, that the numbers $\lambda_j^{(1)}(x)$, $j=\overline{1,4}$, coincide with the eigenvalues corresponding to piezoelastostatics without taking into consideration thermal effects (see [6]).

Quite analogously for the homogeneous principal symbol matrix of the operator $\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)}$ at a point $x\in\Gamma^{(m)}$ we get (see (5.18))

$$\widetilde{\mathfrak{S}}_2(x,\xi_1,\xi_2)=\mathfrak{S}_{\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)}}(x,\xi_1,\xi_2)=\mathfrak{S}_{\mathcal{A}_\tau}(x,\xi_1,\xi_2)+\mathfrak{S}_{\mathcal{B}_\tau^{(m)}}(x,\xi_1,\xi_2)=$$
$$=\widetilde{\mathfrak{S}}_1(x,\xi_1,\xi_2)+\widetilde{\mathfrak{S}}_1^{(m)}(x,\xi_1,\xi_2),\ \ (\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}, \qquad (4.26)$$

where $\widetilde{\mathfrak{S}}_1(x,\xi_1,\xi_2)$ is given by (4.22) and

$$\widetilde{\mathfrak{S}}_1^{(m)}(x,\xi_1,\xi_2) := \mathfrak{S}_{\mathcal{B}_\tau^{(m)}}(x,\xi_1,\xi_2) = \\ = \begin{bmatrix} [D_{kj}^{(m)}(x,\xi_1,\xi_2)]_{3\times3} & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{1\times3} & -2\,M_{44}^{(m)}(x,\xi_1,\xi_2) & 0 \\ [0]_{1\times3} & 0 & 0 \end{bmatrix}_{5\times5}, \quad (4.27)$$

$$D^{(m)}(x,\xi_1,\xi_2) := [D_{kj}^{(m)}(x,\xi_1,\xi_2)]_{3\times3} = \\ = M^{(m)}(x,\xi_1,\xi_2)\,[N^{(m)}(x,\xi_1,\xi_2)]^{-1} \quad (4.28)$$

with

$$\begin{aligned} M^{(m)}(x,\xi_1,\xi_2) &:= \big[M_{kj}^{(m)}(x,\xi_1,\xi_2)\big]_{3\times3}, \\ N^{(m)}(x,\xi_1,\xi_2) &:= \big[(N_-^{(m)})_{kj}(x,\xi_1,\xi_2)\big]_{3\times3}. \end{aligned} \quad (4.29)$$

Here $M_{kj}^{(m)}(x,\xi_1,\xi_2)$ and $(N_-^{(m)})_{kj}(x,\xi_1,\xi_2)$ are the entries of the matrices (4.13) and (4.14). The matrices $M^{(m)}(x,\xi_1,\xi_2)$ and $N^{(m)}(x,\xi_1,\xi_2)$ correspond to the operators of the classical elastostatics, while (4.28) represents the homogeneous symbol matrix of the so called Steklov–Poincaré operator and is positive definite (see [54]). Therefore, it is clear that (4.27) is a nonnegative definite matrix due to the inequality $-2\,M_{44}^{(m)}(x,\xi_1,\xi_2) > 0$ and consequently (4.26) is strongly elliptic symbol matrix due to the strong ellipticity of $\widetilde{\mathfrak{S}}_1(x,\xi_1,\xi_2)$.

Thus we have

$$\widetilde{\mathfrak{S}}_2(x,\xi_1,\xi_2) = \\ = \begin{bmatrix} [D_{kj}(x,\xi_1,\xi_2)]_{3\times3} & [0]_{3\times1} & [D_{k4}(x,\xi_1,\xi_2)]_{3\times1} \\ [0]_{1\times3} & -2\,M_{44}(x,\xi_1,\xi_2) & 0 \\ [D_{4j}(x,\xi_1,\xi_2)]_{1\times3} & 0 & D_{44}(x,\xi_1,\xi_2) \end{bmatrix}_{5\times5} + \\ + \begin{bmatrix} [D_{kj}^{(m)}(x,\xi_1,\xi_2)]_{3\times3} & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{1\times3} & -2\,M_{44}^{(m)}(x,\xi_1,\xi_2) & 0 \\ [0]_{1\times3} & 0 & 0 \end{bmatrix}_{5\times5} = \\ = \begin{bmatrix} [D_{kj}^{*}(x,\xi_1,\xi_2)]_{3\times3} & [0]_{3\times1} & [D_{k4}(x,\xi_1,\xi_2)]_{3\times1} \\ [0]_{1\times3} & -2\,D_{44}^{*}(x,\xi_1,\xi_2) & 0 \\ [D_{4j}(x,\xi_1,\xi_2)]_{1\times3} & 0 & D_{44}(x,\xi_1,\xi_2) \end{bmatrix}_{5\times5}, \quad (4.30)$$

where

$$D_{kj}^{*}(x,\xi_1,\xi_2) = D_{kj}(x,\xi_1,\xi_2) + D_{kj}^{(m)}(x,\xi_1,\xi_2), \quad k,j,=1,2,3,$$
$$D_{44}^{*}(x,\xi_1,\xi_2) = M_{44}(x,\xi_1,\xi_2) + M_{44}^{(m)}(x,\xi_1,\xi_2).$$

Denote

$$T(x,\xi_1,\xi_2) := \begin{bmatrix} [D^*_{kj}(x,\xi_1,\xi_2)]_{3\times 3} & [D_{k4}(x,\xi_1,\xi_2)]_{3\times 1} \\ [D_{4j}(x,\xi_1,\xi_2)]_{1\times 3} & D_{44}(x,\xi_1,\xi_2) \end{bmatrix}_{4\times 4},$$

$$\mathcal{D}^{(m)}(x) := \big[\, \mathcal{D}^{(m)}_{kj}(x) \,\big]_{4\times 4} = [T(x,0,+1)]^{-1}\, T(x,0,-1). \tag{4.31}$$

One can easily check that

$$\big[\widetilde{\mathfrak{S}}^{(m)}_1(x,0,+1)\big]^{-1}\, \widetilde{\mathfrak{S}}^{(m)}_1(x,0,-1) = \\ = \begin{bmatrix} [\mathcal{D}^{(m)}_{kj}(x)]_{3\times 3} & [0]_{3\times 1} & [\mathcal{D}^{(m)}_{k4}(x)]_{3\times 1} \\ [0]_{1\times 3} & 1 & 0 \\ [\mathcal{D}^{(m)}_{4j}(x)]_{1\times 3} & 0 & \mathcal{D}^{(m)}_{44}(x) \end{bmatrix}_{5\times 5}. \tag{4.32}$$

Denote by $\lambda^{(2)}_j(x)$, $j=\overline{1,4}$, the eigenvalues of the matrix (4.31), that is the roots of the equation

$$\det\,\big[\,\mathcal{D}^{(m)}(x) - \lambda\, I_4\,\big] = 0 \tag{4.33}$$

with respect to λ. Then $\lambda^{(2)}_j(x)$, $j=\overline{1,4}$, and $\lambda^{(2)}_5 = 1$ are eigenvalues of the matrix (4.32). From the strong ellipticity of the symbol matrix (4.30) it follows that $\lambda^{(2)}_j(x)$, $j=\overline{1,4}$, are complex numbers, in general, and $-\pi < \arg \lambda^{(2)}_j(x) < \pi$, that is $\lambda^{(2)}_j(x) \notin (-\infty, 0]$. Remark, that again the numbers $\lambda^{(2)}_j(x)$, $j=\overline{1,4}$, coincide with the eigenvalues corresponding to the piezoelastostatics case without taking into consideration thermal effects (see [6]).

4.5. **Auxiliary problems and representation formulas of solutions.** Here we assume that $\operatorname{Re}\tau = \sigma > 0$ and consider two auxiliary boundary value problems needed for our further purposes.

Auxiliary problem I: Find a vector function $U^{(m)} : \Omega^{(m)} \to \mathbb{C}^4$ which belongs to the space $[\,W^1_2(\Omega^{(m)})\,]^4$ and satisfies the following conditions:

$$A^{(m)}(\partial,\tau)\, U^{(m)} = 0 \ \text{ in } \ \Omega^{(m)}, \tag{4.34}$$

$$\big\{\,\mathcal{T}^{(m)}\, U^{(m)}\,\big\}^+ = \chi^{(m)} \ \text{ on } \ \partial\Omega^{(m)}, \tag{4.35}$$

where $\chi^{(m)} = (\,\chi^{(m)}_1, \chi^{(m)}_2, \chi^{(m)}_3, \chi^{(m)}_4)^\top \in [\,H_2^{-\frac{1}{2}}(\partial\Omega^{(m)})\,]^4$. With the help of Green's formula it can easily be shown that the homogeneous version of this auxiliary BVP possesses only the trivial solution. Moreover, we have the following result concerning the representation of solutions of equation (4.34).

Lemma 4.6. *Let* $\operatorname{Re}\tau = \sigma > 0$ *and* $1 < p < \infty$. *An arbitrary solution vector* $U^{(m)} \in [\,W^1_p(\Omega^{(m)})\,]^4$ *to the homogeneous equation* (4.34) *can be* *uniquely represented by the single layer potential*

$$U^{(m)}(x) = V_\tau^{(m)}\big(\big[\mathcal{P}_\tau^{(m)}\big]^{-1}\chi^{(m)}\big)(x), \quad x \in \Omega^{(m)}, \tag{4.36}$$

where

$$\begin{aligned} \mathcal{P}_\tau^{(m)} &:= -2^{-1} I_4 + \mathcal{K}_\tau^{(m)}, \\ \chi^{(m)} &= \big\{ \mathcal{T}^{(m)} U^{(m)} \big\}^+ \in \big[B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(m)}) \big]^4. \end{aligned} \tag{4.37}$$

Proof. Clearly, if

$$\chi^{(m)} = \big(\chi_1^{(m)}, \dots, \chi_4^{(m)}\big)^\top = \big\{ \mathcal{T}^{(m)} U^{(m)} \big\}^+ \in \big[B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(m)}) \big]^4$$

then the vector function (4.36) solves the auxiliary BVP and belongs to the space $[W_p^1(\Omega^{(m)})]^4$ by Theorems 4.2, 4.3 and 4.5. The uniqueness follows from the following general integral representation formula for an arbitrary solution vector $U^{(m)} \in [W_p^1(\Omega^{(m)})]^4$ of the homogeneous equation (4.34)

$$U^{(m)}(x) = W_\tau^{(m)}\big(\{U^{(m)}\}^+\big)(x) - V_\tau^{(m)}\big(\{\mathcal{T}^{(m)}U^{(m)}\}^+\big)(x), \quad x \in \Omega^{(m)},$$

and invertibility of the operator

$$-2^{-1} I_4 + \widetilde{\mathcal{K}}_\tau^{(m)*} : \big[B_{p,p}^s(\partial\Omega^{(m)})\big]^4 \to \big[B_{p,p}^s(\partial\Omega^{(m)})\big]^4$$

(see Theorem 4.5). □

Auxiliary problem II: Find a vector function $U : \Omega \to \mathbb{C}^5$ which belongs to the space $[W_2^1(\Omega)]^5$ and satisfies the following conditions:

$$A(\partial, \tau)U = 0 \ \text{ in } \ \Omega, \tag{4.38}$$

$$\big\{ \mathcal{T}U \big\}^+ + \beta\,\{U\}^+ = \chi \ \text{ on } \ \partial\Omega, \tag{4.39}$$

where $\chi := (\chi_1, \chi_2, \chi_3, \chi_4, \chi_5)^\top \in [H_2^{-\frac{1}{2}}(\partial\Omega)]^5$, β is a smooth real valued scalar function which does not vanish identically and

$$\beta \geqslant 0, \ \ \text{supp}\,\beta \subset S_D. \tag{4.40}$$

By the same arguments as in the proof of Theorem 1.1 we can easily show that the homogeneous version of this boundary value problem possesses only the trivial solution in the space $[W_2^1(\Omega)]^5$.

We look for a solution to the auxiliary BVP (4.38)-(4.39) as a single layer potential, $U(x) = V_\tau(f)(x)$, where $f = (f_1, f_2, f_3, f_4, f_5)^\top \in [H_2^{-\frac{1}{2}}(\partial\Omega)]^5$ is an unknown density. The boundary condition (4.39) leads then to the system of equations:

$$\big(-2^{-1} I_5 + \mathcal{K}_\tau\big) f + \beta\,\mathcal{H}_\tau f = \chi \ \text{ on } \ \partial\Omega.$$

Denote the matrix operator generated by the left hand side expression of this equation by $\mathcal{P}_\tau$ and rewrite the system as

$$\mathcal{P}_\tau f = \chi \ \text{ on } \ \partial\Omega,$$

where

$$\mathcal{P}_\tau := -2^{-1} I_5 + \mathcal{K}_\tau + \beta\,\mathcal{H}_\tau. \tag{4.41}$$

Lemma 4.7. *Let* $\operatorname{Re}\tau=\sigma>0$. *The operators*

$$\mathcal{P}_\tau \ : \ \big[\,H^s_p(\partial\Omega)\,\big]^5 \to \big[\,H^s_p(\partial\Omega)\,\big]^5, \tag{4.42}$$

$$: \ \big[\,B^s_{p,t}(\partial\Omega)\,\big]^5 \to \big[\,B^s_{p,t}(\partial\Omega)\,\big]^5, \tag{4.43}$$

are invertible for all $1<p<\infty$, $1\leqslant t\leqslant\infty$, *and* $s\in\mathbb{R}$.

Proof. From the uniqueness result for the auxiliary BVP (4.38)–(4.39) it follows that the operator (4.42) is injective for $p=2$ and $s=-1/2$. The operator $\mathcal{H}_\tau \ : \ [\,H_2^{-\frac12}(\partial\Omega)\,]^5 \to [\,H_2^{-\frac12}(\partial\Omega)\,]^5$ is compact. By Theorem 4.5 we then conclude that the index of the Fredholm operator (4.42) equals to zero. Since $\mathcal{P}_\tau$ is an injective singular integral operator of normal type with zero index it follows that it is surjective. Thus the operator (4.42) is invertible for $p=2$ and $s=-1/2$.

The invertibility of the operators (4.42) and (4.43) for all $1<p<\infty$, $1\leqslant t\leqslant\infty$, and $s\in\mathbb{R}$ then follows by standard duality and interpolation arguments for the $C^\infty-$regular surface $\partial\Omega$ (see, e.g., [1, 63]). □

As a consequence we have the following representation formula.

Lemma 4.8. *Let* $\operatorname{Re}\tau=\sigma>0$ *and* $1<p<\infty$. *An arbitrary solution* $U\in[\,W^1_p(\Omega)\,]^5$ *to the homogeneous equation* (4.38) *can be uniquely represented by the single layer potential* $U(x)=V_\tau\big(\mathcal{P}_\tau^{-1}\,\chi\big)(x)$, *where* $\chi=\{\mathcal{T}U\}^+ +\beta\,\{U\}^+\in[\,B_{p,p}^{-\frac1p}(\partial\Omega)\,]^5$.

Remark 4.9. By standard arguments it can be shown that Lemmata 4.6, 4.7 and 4.8 with $p=2$ remain true for Lipschitz domains $\Omega^{(m)}$ and Ω (cf. [44]).

5. Existence and Regularity Results for Problem (ICP-A)

5.1. **Reduction to boundary equations.** Let us return to the interface crack problem (1.46)–(1.55) and derive the equivalent boundary integral formulation of this problem. Keeping in mind (1.57), let

$$G:=\begin{cases} Q & \text{on } S_N,\\ \widetilde{Q} & \text{on } \Gamma^{(m)}_C,\end{cases} \quad G^{(m)}:=\begin{cases} Q^{(m)} & \text{on } S^{(m)}_N,\\ \widetilde{Q}^{(m)} & \text{on } \Gamma^{(m)}_C,\end{cases} \tag{5.1}$$

$$G\in\big[\,B^{-1/p}_{p,p}(S_N\cup\Gamma^{(m)}_C)\,\big]^5,\quad G^{(m)}\in\big[\,B^{-1/p}_{p,p}(S^{(m)}_N\cup\Gamma^{(m)}_C)\,\big]^4,$$

and

$$G_0=(G_{01},\dots,G_{05})^\top\in\big[\,B_{p,p}^{-\frac1p}(\partial\Omega)\,\big]^5,$$

$$G^{(m)}_0=(G^{(m)}_{01},\dots,G^{(m)}_{04})^\top\in\big[\,B_{p,p}^{-\frac1p}(\partial\Omega^{(m)})\big]^4$$

be some fixed extensions of the vector-function G and $G^{(m)}$ respectively onto $\partial\Omega$ and $\partial\Omega^{(m)}$ preserving the space. It is evident that arbitrary extensions of the same vector functions can be represented then as

$$G^*=G_0+\psi+h,\quad G^{(m)*}=G^{(m)}_0+h^{(m)}, \tag{5.2}$$

where

$$\begin{aligned} \psi &:= (\psi_1,\dots,\psi_5)^\top \in \big[\, \widetilde{B}^{-\frac{1}{p}}_{p,p}(S_D)\,\big]^5, \\ h &:= (h_1,\dots,h_5)^\top \in \big[\, \widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma^{(m)}_T)\,\big]^5, \\ h^{(m)} &:= (h^{(m)}_1,\dots,h^{(m)}_4)^\top \in \big[\, \widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma^{(m)}_T)\,\big]^4 \end{aligned} \tag{5.3}$$

are arbitrary vector-functions.

We develop here the so-called indirect boundary integral equations method. In accordance with Lemmas 4.6 and 4.8 we look for a solution pair $(U^{(m)}, U)$ of the interface crack problem (1.46)–(1.55) in the form of single layer potentials,

$$\begin{aligned} U^{(m)} &= (u^{(m)}_1,\dots,u^{(m)}_4)^\top = \\ &= V^{(m)}_\tau\big(\,[\,\mathcal{P}^{(m)}_\tau\,]^{-1}\,\big[\,G^{(m)}_0 + h^{(m)}\big]\,\big) \ \text{ in } \ \Omega^{(m)}, \end{aligned} \tag{5.4}$$

$$U = (u_1,\dots,u_5)^\top = V_\tau\big(\,\mathcal{P}^{-1}_\tau\,\big[\,G_0+\psi+h\big]\,\big) \ \text{ in } \ \Omega, \tag{5.5}$$

where $\mathcal{P}^{(m)}_\tau$ and $\mathcal{P}_\tau$ are given by (4.37) and (4.41), and $h^{(m)}$, h and ψ are unknown vector-functions satisfying the inclusions (5.3).

By Lemmas 4.6, 4.8 and the property (4.40), we see that the homogeneous differential equations (1.46)–(1.47), boundary conditions (1.48)–(1.49) and crack conditions (1.54)–(1.55) are satisfied automatically.

The remaining boundary and transmission conditions (1.50)–(1.5) lead to the equations

$$r_{S_D}\,\big[\,\mathcal{H}_\tau\,\mathcal{P}^{-1}_\tau\,\big(\,G_0+\psi+h\big)\,\big]_k = f_k \ \text{ on } \ S_D, \ \ k=\overline{1,5}, \tag{5.6}$$

$$r_{\Gamma^{(m)}_T}\,\big[\,\mathcal{H}_\tau\,\mathcal{P}^{-1}_\tau\,(\,G_0+\psi+h)\,\big]_5 = f^{(m)}_5 \ \text{ on } \ \Gamma^{(m)}_T, \tag{5.7}$$

$$\begin{aligned} r_{\Gamma^{(m)}_T}\big[\mathcal{H}_\tau\,\mathcal{P}^{-1}_\tau\,(G_0+\psi+h)\big]_j - r_{\Gamma^{(m)}_T}\big[\mathcal{H}^{(m)}_\tau\,[\mathcal{P}^{(m)}_\tau]^{-1}(\,G^{(m)}_0+h^{(m)})\big]_j = \\ = f^{(m)}_j \ \text{ on } \ \Gamma^{(m)}_T, \ \ j=\overline{1,4}, \end{aligned} \tag{5.8}$$

$$r_{\Gamma^{(m)}_T}[G_0+\psi+h]_j + r_{\Gamma^{(m)}_T}\big[G^{(m)}_0+h^{(m)}\big]_j = F^{(m)}_j \ \text{ on } \ \Gamma^{(m)}_T, \ \ j=\overline{1,4}. \tag{5.9}$$

After some evident simplification we arrive at the system of pseudodifferential equations with the unknown vector-functions ψ, h and $h^{(m)}$

$$r_{S_D}\,\big[\,\mathcal{H}_\tau\,\mathcal{P}^{-1}_\tau\,(\,\psi+h)\,\big]_k = \widetilde{f}_k \ \text{ on } \ S_D, \ \ k=\overline{1,5}, \tag{5.10}$$

$$r_{\Gamma^{(m)}_T}\,\big[\,\mathcal{H}_\tau\,\mathcal{P}^{-1}_\tau(\psi+h)\,\big]_5 = \widetilde{f}^{(m)}_5 \ \text{ on } \ \Gamma^{(m)}_T, \tag{5.11}$$

$$\begin{aligned} r_{\Gamma^{(m)}_T}\,\big[\,\mathcal{H}_\tau\,\mathcal{P}^{-1}_\tau(\psi+h)\,\big]_j - r_{\Gamma^{(m)}_T}\,\big[\,\mathcal{H}^{(m)}_\tau\,[\mathcal{P}^{(m)}_\tau]^{-1}\,h^{(m)}\,\big]_j = \\ = \widetilde{f}^{(m)}_j \ \text{ on } \ \Gamma^{(m)}_T, \ \ j=\overline{1,4}, \end{aligned} \tag{5.12}$$

$$r_{\Gamma^{(m)}_T}\,h^{(m)}_j + r_{\Gamma^{(m)}_T}\,h_j = \widetilde{F}^{(m)}_j \ \text{ on } \ \Gamma^{(m)}_T, \ \ j=\overline{1,4}, \tag{5.13}$$

where

$$\widetilde{f}_k := f_k - r_{_{S_D}} \big[\mathcal{H}_\tau \, \mathcal{P}_\tau^{-1} \, G_0 \big]_k \in B^{1-\frac{1}{p}}_{p,p}(S_D), \quad k = \overline{1,5}, \tag{5.14}$$

$$\widetilde{f}_5^{(m)} := f_5^{(m)} - r_{_{\Gamma_T^{(m)}}} \big[\mathcal{H}_\tau \, \mathcal{P}_\tau^{-1} \, G_0 \big]_5 \in B^{1-\frac{1}{p}}_{p,p}(\Gamma_T^{(m)}), \tag{5.15}$$

$$\widetilde{f}_j^{(m)} := f_j^{(m)} + r_{_{\Gamma_T^{(m)}}} \big[\mathcal{H}_\tau^{(m)} \, [\mathcal{P}_\tau^{(m)}]^{-1} \, G_0^{(m)} \big]_j - \\ - r_{_{\Gamma_T^{(m)}}} \big[\mathcal{H}_\tau \, \mathcal{P}_\tau^{-1} \, G_0 \big]_j \in B^{1-\frac{1}{p}}_{p,p}(\Gamma_T^{(m)}), \quad j = \overline{1,4}, \tag{5.16}$$

$$\widetilde{F}_j^{(m)} := F_j^{(m)} - r_{_{\Gamma_T^{(m)}}} G_{0j} - r_{_{\Gamma_T^{(m)}}} G_{0j}^{(m)} \in r_{_{\Gamma_T^{(m)}}} \widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma_T^{(m)}), \quad j = \overline{1,4}. \tag{5.17}$$

The last inclusions are the *compatibility conditions* for Problem (ICP-A). Therefore, in what follows we assume that $\widetilde{F}_j^{(m)}$ are extended from $\Gamma_T^{(m)}$ onto the whole of $\partial\Omega^{(m)} \cup \partial\Omega$ by zero, i.e., $\widetilde{F}_j^{(m)} \in \widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma_T^{(m)})$, $j = \overline{1,3}$.

Let us introduce the Steklov–Poincaré type 5×5 matrix pseudodifferential operators

$$\mathcal{A}_\tau := \mathcal{H}_\tau \, \mathcal{P}_\tau^{-1}, \quad \mathcal{B}_\tau^{(m)} := \begin{bmatrix} [\mathcal{H}_\tau^{(m)} \, [\mathcal{P}_\tau^{(m)}]^{-1} \,]_{4\times 4} & [\, 0 \,]_{4\times 1} \\ [\, 0 \,]_{1\times 4} & [\, 0 \,]_{1\times 1} \end{bmatrix}_{5\times 5}, \tag{5.18}$$

and rewrite equations (5.10)–(5.13) as

$$r_{_{S_D}} \mathcal{A}_\tau \, (\psi + h) = \widetilde{f} \ \text{ on } \ S_D, \tag{5.19}$$

$$r_{_{\Gamma_T^{(m)}}} \mathcal{A}_\tau \, (\psi + h) + r_{_{\Gamma_T^{(m)}}} \mathcal{B}_\tau^{(m)} h = \widetilde{g}^{\,(m)} \ \text{ on } \ \Gamma_T^{(m)}, \tag{5.20}$$

$$r_{_{\Gamma_T^{(m)}}} h_j + r_{_{\Gamma_T^{(m)}}} h_j^{(m)} = \widetilde{F}_j^{(m)} \ \text{ on } \ \Gamma_T^{(m)}, \quad j = \overline{1,4}, \tag{5.21}$$

where

$$\widetilde{f} := (\widetilde{f}_1, \dots, \widetilde{f}_5)^\top \in \big[B^{1-\frac{1}{p}}_{p,p}(S_D) \big]^5, \tag{5.22}$$

$$\widetilde{g}^{\,(m)} := (\widetilde{g}_1^{(m)}, \dots, \widetilde{g}_5^{(m)})^\top \in \big[B^{1-\frac{1}{p}}_{p,p}(\Gamma_T^{(m)}) \big]^5, \tag{5.23}$$

$$\widetilde{g}_j^{(m)} := \widetilde{f}_j^{(m)} + r_{_{\Gamma_T^{(m)}}} \big[\mathcal{H}_\tau^{(m)} [\, \mathcal{P}_\tau^{(m)} \,]^{-1} \, \widetilde{F}^{(m)} \big]_j, \quad j = \overline{1,4}, \tag{5.24}$$

$$\widetilde{g}_5^{(m)} \ = \widetilde{f}_5^{(m)}, \ \ \widetilde{F}^{(m)} := (\widetilde{F}_1^{(m)}, \dots, \widetilde{F}_4^{(m)})^\top \in \big[\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma_T^{(m)}) \big]^4. \tag{5.25}$$

We note here that since the unknown vector function h is supported on $\Gamma_T^{(m)}$, the operator $\mathcal{B}_\tau^{(m)} h$ is defined correctly provided h is extended by zero on $S_N^{(m)} \cup \Gamma_C^{(m)}$ (see Figure 1). For this extended vector function we will keep the same notation h. So, actually, in what follows we can assume that h is a vector function defined on $\partial\Omega \cup \partial\Omega^{(m)}$ and is supported on $\Gamma_T^{(m)}$.

It is easy to see that the simultaneous equations (5.6)–(5.9) and (5.19)–(5.21), where the right hand sides are related by the equalities (5.14)–(5.17) and (5.22)–(5.24), are equivalent in the following sense: if the triplet

$(\psi, h, h^{(m)}) \in [\widetilde{B}_{p,p}^{-\frac{1}{p}}(S_D)]^5 \times [\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)})]^5 \times [\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)})]^4$ solves the system (5.19)–(5.21), then the pair $(G_0+\psi+h, G_0^{(m)}+h^{(m)})$ solves the system (5.6)–(5.9), and vice versa.

5.2. **Existence theorems and regularity of solutions.** Here we show that the system of pseudodifferential equations (5.19)–(5.21) is uniquely solvable in appropriate function spaces. To this end, let us put

$$\mathcal{N}_\tau^{(A)} := \begin{bmatrix} r_{S_D}\,\mathcal{A}_\tau & r_{S_D}\,\mathcal{A}_\tau & r_{S_D}\,[0]_{5\times 4} \\ r_{\Gamma_T^{(m)}}\,\mathcal{A}_\tau & r_{\Gamma_T^{(m)}}[\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}] & r_{\Gamma_T^{(m)}}\,[0]_{5\times 4} \\ r_{\Gamma_T^{(m)}}\,[0]_{4\times 5} & r_{\Gamma_T^{(m)}}\,I_{4\times 5} & r_{\Gamma_T^{(m)}}\,I_4 \end{bmatrix}_{14\times 14}, \tag{5.26}$$

$$I_{4\times 5} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{5.27}$$

Further, let

$$\begin{aligned} \Phi &:= (\psi,\ h,\ h^{(m)})^\top, \quad Y := (\widetilde{f},\ \widetilde{g}^{(m)},\ \widetilde{F}^{(m)})^\top, \\ \mathbf{X}_p^s &:= \big[\widetilde{B}_{p,p}^s(S_D)\big]^5 \times \big[\widetilde{B}_{p,p}^s(\Gamma_T^{(m)})\big]^5 \times \big[\widetilde{B}_{p,p}^s(\Gamma_T^{(m)})\big]^4, \\ \mathbf{Y}_p^s &:= \big[B_{p,p}^{s+1}(S_D)\big]^5 \times \big[B_{p,p}^{s+1}(\Gamma_T^{(m)})\big]^5 \times \big[\widetilde{B}_{p,p}^s(\Gamma_T^{(m)})\big]^4, \\ \mathbf{X}_{p,q}^s &:= \big[\widetilde{B}_{p,q}^s(S_D)\big]^5 \times \big[\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})\big]^5 \times \big[\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})\big]^4, \\ \mathbf{Y}_{p,q}^s &:= \big[B_{p,q}^{s+1}(S_D)\big]^5 \times \big[B_{p,q}^{s+1}(\Gamma_T^{(m)})\big]^5 \times \big[\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})\big]^4. \end{aligned}$$

Due to Theorems 4.4 and 4.5, the operator $\mathcal{N}_\tau^{(A)}$ has the following mapping properties

$$\mathcal{N}_\tau^{(A)}\ :\ \mathbf{X}_p^s \to \mathbf{Y}_p^s \quad \big[\mathbf{X}_{p,q}^s \to \mathbf{Y}_{p,q}^s\big], \tag{5.28}$$

for all $s \in \mathbb{R}$, $1 < p < \infty$ and all $1 \leqslant q \leqslant \infty$.

Clearly, we can rewrite the system (5.19)-(5.21) as

$$\mathcal{N}_\tau^{(A)}\,\Phi = Y, \tag{5.29}$$

where $\Phi \in \mathbf{X}_p^s$ is the unknown vector introduced above and $Y \in \mathbf{Y}_p^s$ is a given vector.

As it will become clear later the operator (5.28) is not invertible for all $s \in \mathbb{R}$. The interval $a < s < b$ of invertibility depends on p and on some parameters γ' and γ'' which are determined by the eigenvalues of special matrices constructed by means of the principal homogeneous symbol matrices of the operators $\mathcal{A}_\tau$ and $\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}$ (see (5.18) and (5.39)). Note that the numbers γ' and γ'' define also Hölder smoothness exponents for the solutions to the original interface crack problem in a neighbourhood of the exceptional curves ∂S_D, $\partial\Gamma_C^{(m)}$ and $\partial\Gamma^{(m)}$. We start with the following theorem.

Theorem 5.1. *Let the conditions*

$$1 < p < \infty, \quad 1 \leqslant q \leqslant \infty, \quad \frac{1}{p} - 1 + \gamma'' < s + \frac{1}{2} < \frac{1}{p} + \gamma' \tag{5.30}$$

be satisfied with γ' *and* γ'' *given by* (5.36)*,* (5.37)*, and* (5.39)*. Then the operators in* (5.28) *are invertible.*

Proof. We prove the theorem in several steps. First we show that the operators (5.28) are Fredholm with zero index and afterwards we establish that the corresponding null-spaces are trivial.

Step 1. First of all let us note that the operators

$$\begin{aligned} r_{S_D}\mathcal{A}_\tau \ &: \ \big[\widetilde{B}^s_{p,q}(\Gamma^{(m)}_T)\big]^5 \to \big[B^{s+1}_{p,q}(S_D)\big]^5, \\ r_{\Gamma^{(m)}_T}\mathcal{A}_\tau \ &: \ \big[\widetilde{B}^s_{p,q}(S_D)\big]^5 \to \big[B^{s+1}_{p,q}(\Gamma^{(m)}_T)\big]^5, \end{aligned} \tag{5.31}$$

are compact since S_D and $\Gamma^{(m)}_T$ are disjoint, $\overline{S_D} \cap \overline{\Gamma^{(m)}_T} = \varnothing$. Further, we establish that the operators

$$\begin{aligned} r_{S_D}\mathcal{A}_\tau \ &: \ \big[\widetilde{H}^{-\frac{1}{2}}_2(S_D)\big]^5 \to \big[[H^{\frac{1}{2}}_2(S_D)\big]^5, \\ r_{\Gamma^{(m)}_T}\big[\mathcal{A}_\tau + \mathcal{B}^{(m)}_\tau\big] \ &: \ \big[\widetilde{H}^{-\frac{1}{2}}_2(\Gamma^{(m)}_T)\big]^5 \to \big[H^{\frac{1}{2}}_2(\Gamma^{(m)}_T)\big]^5 \end{aligned} \tag{5.32}$$

are strongly elliptic Fredholm pseudodifferential operators of order -1 with index zero. We note that the principal homogeneous symbol matrices of these operators are strongly elliptic.

Using Green's formula (1.40) and the Korn's inequality (see, e.g., [23]), for an arbitrary solution vector $U \in [H^1_2(\Omega)]^5 \equiv [W^1_2(\Omega)]^5$ to the homogeneous equation $A(\partial, \tau)U = 0$ in Ω by standard arguments we derive

$$\operatorname{Re}\big\langle [U]^+, [\mathcal{T}U]^+\big\rangle_{\partial\Omega} \geqslant c_1 \,\|\, U \,\|^2_{[H^1_2(\Omega)]^5} - c_2 \,\|\, U \,\|^2_{[H^0_2(\Omega)]^5}\,. \tag{5.33}$$

Substitute here $U = V_\tau(\mathcal{P}^{-1}_\tau\zeta)$ with $\zeta \in [H^{-\frac{1}{2}}_2(\partial\Omega)]^5$. Due to the equality $\zeta = \mathcal{P}_\tau\mathcal{H}^{-1}_\tau\{U\}^+$ and boundedness of the operators involved, we have $\|\zeta\|^2_{[H^{-\frac{1}{2}}_2(\partial\Omega)]^5} \leqslant c^*\|\{U\}^+\|^2_{[H^{\frac{1}{2}}_2(\partial\Omega)]^5}$ with some positive constant c^*. Therefore, by the trace theorem from (5.33) we easily obtain

$$\begin{aligned} \operatorname{Re}\langle \mathcal{H}_\tau\mathcal{P}^{-1}_\tau\zeta, \zeta\rangle_{\partial\Omega} \geqslant c'_1 \,\|\, \zeta \,\|^2_{[H^{-\frac{1}{2}}_2(\partial\Omega)]^5} + \\ + \big\|\, \beta\,\mathcal{H}\mathcal{P}^{-1}_\tau\zeta\big\|_{[H^{-\frac{1}{2}}_2(\partial\Omega)]^5} - c'_2 \,\big\|\, V_\tau(\mathcal{P}^{-1}_\tau\zeta)\big\|^2_{[H^0_2(\Omega)]^5}. \end{aligned} \tag{5.34}$$

In particular, in view of Theorem 4.2, for arbitrary $\zeta \in [\widetilde{H}^{-\frac{1}{2}}_2(S_D)]^5$ we have

$$\|\, U \,\|^2_{[H^0_2(\Omega)]^5} \leqslant c^{**} \,\|\, \zeta \,\|^2_{[H^{-\frac{3}{2}}_2(S_D)]^5}\,,$$

and, consequently,

$$\operatorname{Re}\big\langle r_{S_D}\mathcal{H}_\tau\mathcal{P}^{-1}_\tau\zeta, \zeta\big\rangle_{\partial\Omega} \geqslant c''_1 \,\|\, \zeta \,\|^2_{[\widetilde{H}^{-\frac{1}{2}}_2(S_D)]^5} - c''_2 \,\|\, \zeta \,\|^2_{[\widetilde{H}^{-\frac{3}{2}}_2(S_D)]^5}\,. \tag{5.35}$$

From (5.35) it follows that

$$r_{S_D}\,\mathcal{A}_\tau = r_{S_D}\,\mathcal{H}_\tau\mathcal{P}_\tau^{-1}\ :\ \big[\,\widetilde{H}_2^{-\frac{1}{2}}(S_D)\,\big]^5 \to \big[\,H_2^{\frac{1}{2}}(S_D)\,\big]^5$$

is a strongly elliptic pseudodifferential Fredholm operator with index zero.

Then the same is true for the operator (5.32) since the principal homogeneous symbol matrix of the operator $\mathcal{B}_\tau^{(m)}$ is nonnegative (see formula (4.27)).

Therefore, due to the compactness of the operators (5.31), the operator (5.28) is Fredholm with index zero for $s=-1/2$, $p=2$ and $q=2$.

Step 2. With the help of the uniqueness Theorem 1.1 via representation formulas (5.4) and (5.5) with $G_0^{(m)}=0$ and $G_0=0$ we can easily show that the operator (5.28) is injective for $s=-1/2$, $p=2$ and $q=2$. Since its index is zero, we conclude that it is surjective. Thus the operator (5.28) is invertible for $s=-1/2$, $p=2$ and $q=2$.

Step 3. To complete the proof for the general case we proceed as follows. The following lower triangular operator

$$\mathcal{N}_\tau^{(A,0)} := \begin{bmatrix} r_{S_D}\,\mathcal{A}_\tau & r_{S_D}\,[\,0\,]_{5\times5} & r_{S_D}\,[\,0\,]_{5\times4} \\ r_{\Gamma_T^{(m)}}\,[\,0\,]_{5\times5} & r_{\Gamma_T^{(m)}}[\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)}] & r_{\Gamma_T^{(m)}}\,[\,0\,]_{5\times4} \\ r_{\Gamma_T^{(m)}}\,[\,0\,]_{4\times5} & r_{\Gamma_T^{(m)}}\,I_{4\times5} & r_{\Gamma_T^{(m)}}\,I_4 \end{bmatrix}_{14\times14}$$

is a compact perturbation of the operator $\mathcal{N}_\tau^{(A)}$. Therefore it suffices to establish the properties of the diagonal entries

$$r_{S_D}\mathcal{A}_\tau\ :\ [\,\widetilde{B}^s_{p,q}(S_D)\,]^5 \to \big[\,B^{s+1}_{p,q}(S_D)\,\big]^5,$$
$$r_{\Gamma_T^{(m)}}\big[\,\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)}\,\big]\ :\ [\,\widetilde{B}^s_{p,q}(\Gamma_T^{(m)})]^5 \to \big[\,B^{s+1}_{p,q}(\Gamma_T^{(m)})\,\big]^5.$$

To this end, we apply the results presented in Section 3. Let

$$\mathfrak{S}_1(x,\xi_1,\xi_2) := \mathfrak{S}(\mathcal{A}_\tau)(x,\xi_1,\xi_2)$$

be the principal homogeneous symbol matrix of the operator $\mathcal{A}_\tau$ and $\lambda_j^{(1)}(x)$ $(j=\overline{1,5})$ be the eigenvalues of the matrix

$$\mathcal{D}_1(x) := \big[\mathfrak{S}_1(x,0,+1)\big]^{-1}\,\mathfrak{S}_1(x,0,-1)$$

for $x\in\partial S_D$ (see (4.20) and (4.23)).

Similarly, let

$$\mathfrak{S}_2(x,\xi_1,\xi_2) = \mathfrak{S}(\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)})(x,\xi_1,\xi_2)$$

be the principal homogeneous symbol matrix of the operator $\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)}$ and $\lambda_j^{(2)}(x)$ $(j=\overline{1,5})$ be the eigenvalues of the corresponding matrix

$$\mathcal{D}_2(x) := \big[\mathfrak{S}_2(x,0,+1)\big]^{-1}\,\mathfrak{S}_2(x,0,-1)$$

for $x \in \partial\Gamma_T^{(m)}$ (see (4.20) and (4.23)). Note that the curve $\partial\Gamma_T^{(m)}$ is the union of the curves where the interface intersects the exterior boundary, $\partial\Gamma^{(m)}$, and the crack edge, $\partial\Gamma_C^{(m)}$.

Further, we set

$$\begin{aligned} \gamma_1' &:= \inf_{x\in\partial S_D,\,1\leqslant j\leqslant 5} \frac{1}{2\pi} \arg \lambda_j^{(1)}(x), \\ \gamma_1'' &:= \sup_{x\in\partial S_D,\,1\leqslant j\leqslant 5} \frac{1}{2\pi} \arg \lambda_j^{(1)}(x), \end{aligned} \tag{5.36}$$

$$\begin{aligned} \gamma_2' &:= \inf_{x\in\partial\Gamma_T^{(m)},\,1\leqslant j\leqslant 5} \frac{1}{2\pi} \arg \lambda_j^{(2)}(x), \\ \gamma_2'' &:= \sup_{x\in\partial\Gamma_T^{(m)},\,1\leqslant j\leqslant 5} \frac{1}{2\pi} \arg \lambda_j^{(2)}(x). \end{aligned} \tag{5.37}$$

As is shown in Subsection 4.4, one of the eigenvalues equals to one, namely, $\lambda_5^{(1)} = 1$. Therefore

$$\gamma_1' \leqslant 0, \quad \gamma_1'' \geqslant 0. \tag{5.38}$$

Note that γ_j' and γ_j'' $(j = 1, 2)$ depend on the material parameters, in general, and belong to the interval $(-\frac{1}{2}, \frac{1}{2})$. We put

$$\gamma' := \min\{\gamma_1', \gamma_2'\}, \quad \gamma'' := \max\{\gamma_1'', \gamma_2''\}. \tag{5.39}$$

In view of (5.38) we have

$$-\frac{1}{2} < \gamma' \leqslant 0 \leqslant \gamma'' < \frac{1}{2}\,. \tag{5.40}$$

From Theorem 2.28 we conclude that if the parameters $r_1, r_2 \in \mathbb{R}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, satisfy the conditions

$$\frac{1}{p} - 1 + \gamma_1'' < r_1 + \frac{1}{2} < \frac{1}{p} + \gamma_1', \quad \frac{1}{p} - 1 + \gamma_2'' < r_2 + \frac{1}{2} < \frac{1}{p} + \gamma_2',$$

then the operators

$$\begin{aligned} r_{S_D}\mathcal{A}_\tau \ &: \ \big[\widetilde{H}_p^{r_1}(S_D)\big]^5 \to \big[H_p^{r_1+1}(S_D)\big]^5, \\ &: \ \big[\widetilde{B}_{p,q}^{r_1}(S_D)\big]^5 \to \big[[B_{p,q}^{r_1+1}(S_D)\big]^5, \\ r_{\Gamma_T^{(m)}}\big[\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}\big] \ &: \ \big[\widetilde{H}_p^{r_2}(\Gamma_T^{(m)})\big]^5 \to \big[H_p^{r_2+1}(\Gamma_T^{(m)})\big]^5, \\ &: \ \big[\widetilde{B}_{p,q}^{r_2}(\Gamma_T^{(m)})\big]^5 \to \big[B_{p,q}^{r_2+1}(\Gamma_T^{(m)})\big]^5, \end{aligned}$$

are Fredholm operators with index zero.

Therefore, if the conditions (5.30) are satisfied, then the above operators are Fredholm with zero index. Consequently, the operators (5.28) are Fredholm with zero index and are invertible due to the results obtained in Step 2. □

Now we formulate the basic existence and uniqueness results for the interface crack problem under consideration.

Theorem 5.2. *Let the inclusions* (1.56) *and compatibility conditions* (5.17) *hold and let*

$$\frac{4}{3-2\gamma''} < p < \frac{4}{1-2\gamma'} \tag{5.41}$$

with γ' *and* γ'' *defined in* (5.39). *Then the interface crack problem* (1.46)–(1.55) *has a unique solution*

$$(U^{(m)}, U) \in \big[\, W^1_p(\Omega^{(m)})\,\big]^4 \times [\, W^1_p(\Omega)\,]^5,$$

which can be represented by formulas

$$U^{(m)} = V^{(m)}_\tau\big(\,\big[\,\mathcal{P}^{(m)}_\tau\,\big]^{-1}\big[\,G^{(m)}_0 + h^{(m)}\big]\,\big) \ \ in\ \ \Omega^{(m)}, \tag{5.42}$$

$$U = V_\tau\big(\,\mathcal{P}^{-1}_\tau\big[\,G_0 + \psi + h\big]\,\big) \ \ in\ \ \Omega, \tag{5.43}$$

where the densities ψ, h, *and* $h^{(m)}$ *are to be determined from the system* (5.10)-(5.13) (*or from the system* (5.19)–(5.21)).

Moreover, the vector functions $G_0 + \psi + h$ *and* $G^{(m)}_0 + h^{(m)}$ *are defined uniquely by the above systems and are independent of the extension operators.*

Proof. From Theorems 4.2, 4.3 and 5.1 with p satisfying (5.41) and $s = -1/p$ it follows immediately that the pair $(U^{(m)}, U) \in [W^1_p(\Omega^{(m)})]^4 \times [W^1_p(\Omega)]^5$ given by (5.42)–(5.43) represents a solution to the interface crack problem (1.46)–(1.55). Next we show the uniqueness of solutions.

Due to the inequalities (5.40)

$$p = 2 \in \Big(\frac{4}{3-2\gamma''}\,,\ \frac{4}{1-2\gamma'}\Big).$$

Therefore the unique solvability for $p = 2$ is a consequence of Theorem 1.1.

To show the uniqueness result for all other values of p from the interval (5.41) we proceed as follows. Let a pair

$$(U^{(m)}, U) \in \big[\, W^1_p(\Omega^{(m)})\,\big]^4 \times [\, W^1_p(\Omega)\,]^5 \tag{5.44}$$

with p satisfying (5.41) be a solution to the homogeneous boundary-transmission problem. Then, it is evident that

$$\begin{aligned} &\big\{\, U^{(m)}\,\big\}^+ \in \big[\, B^{1-\frac{1}{p}}_{p,p}(\partial\Omega^{(m)})\,\big]^4, \quad \big\{\, U\,\big\}^+ \in \big[\, B^{1-\frac{1}{p}}_{p,p}(\partial\Omega)\,\big]^5, \\ &\big\{\, \mathcal{T}^{(m)}U^{(m)}\,\big\}^+ \in \big[\, B^{-\frac{1}{p}}_{p,p}(\partial\Omega^{(m)})\,\big]^4, \quad \big\{\, \mathcal{T}U\,\big\}^+ \in \big[\, B^{-\frac{1}{p}}_{p,p}(\partial\Omega)\,\big]^5, \end{aligned} \tag{5.45}$$

and the vectors $U^{(m)}$ and U in $\Omega^{(m)}$ and Ω respectively are representable in the form

$$U^{(m)} = V^{(m)}_\tau\big(\,\big[\,\mathcal{P}^{(m)}_\tau\big]^{-1} h^{(m)}\,\big) \ \text{ in }\ \Omega^{(m)}, \quad h^{(m)} = \big\{\mathcal{T}^{(m)}U^{(m)}\,\big\}^+, \tag{5.46}$$

$$U = V_\tau\big(\,\mathcal{P}^{-1}_\tau\,\chi\,\big) \ \text{ in }\ \Omega, \quad \chi = \big\{\mathcal{T}U\big\}^+ + \beta\,\big\{U\big\}^+, \tag{5.47}$$

due to Lemmas 4.6 and 4.8. Moreover, due to the homogeneous boundary and transmission conditions we have

$$\begin{gathered} h^{(m)} \in \big[\,\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma^{(m)}_T)\,\big]^4, \;\; \chi = h + \psi, \\ \psi \in \big[\,\widetilde{B}^{-\frac{1}{p}}_{p,p}(S_D)\,\big]^5, \quad h \in \big[\,\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma^{(m)}_T)\,\big]^5. \end{gathered} \tag{5.48}$$

By the same arguments as above we arrive at the homogeneous system

$$\mathcal{N}^{(A)}_\tau\, \Phi = 0 \;\text{ with }\; \Phi := (\psi,\; h,\; h^{(m)})^\top \in \mathbf{X}_p^{-\frac{1}{p}}\,.$$

Due to Theorem 5.1, $\Phi = 0$ and we conclude that $U^{(m)} = 0$ in $\Omega^{(m)}$ and $U = 0$ in Ω.

The last assertion of the theorem is trivial and is an easy consequence of the fact that if the single layer potentials (5.42) and (5.43) vanish identically in $\Omega^{(m)}$ and Ω, then the corresponding densities vanish as well. □

Remark 5.3. Theorems 5.1 and 5.2 remain valid with $p = 2$ and $s = -1/2$ for Lipschitz domains $\Omega^{(m)}$ and Ω. Indeed, one can easily verify that the arguments, applied in the first two steps of the proof of Theorem 5.1 and in the proof of Theorem 5.2, hold true in the case of Lipschitz domains.

Finally, we can prove the following regularity result for the solution of Problem (ICP).

Theorem 5.4. *Let the inclusions* (1.56) *and compatibility conditions* (5.17) *hold and let* $1 < r < \infty$, $1 \leqslant q \leqslant \infty$,

$$\frac{4}{3 - 2\gamma''} < p < \frac{4}{1 - 2\gamma'}\,, \quad \frac{1}{r} - \frac{1}{2} + \gamma'' < s < \frac{1}{r} + \frac{1}{2} + \gamma', \tag{5.49}$$

with γ' *and* γ'' *defined in* (5.39).

Further, let $U^{(m)} \in [W^1_p(\Omega^{(m)})]^4$ *and* $U \in [W^1_p(\Omega)]^5$ *be a unique solution pair to the interface crack problem* (1.46)–(1.55). *Then the following hold:*

i) *if*

$$\begin{gathered} Q_k \in B^{s-1}_{r,r}(S_N), \quad Q^{(m)}_j \in B^{s-1}_{r,r}(S^{(m)}_N), \quad f_k \in B^{s}_{r,r}(S_D), \\ f^{(m)}_k \in B^{s}_{r,r}(\Gamma^{(m)}_T), \quad F^{(m)}_j \in B^{s-1}_{r,r}(\Gamma^{(m)}_T), \quad \widetilde{Q}^{(m)}_j \in B^{s-1}_{r,r}(\Gamma^{(m)}_C), \\ \widetilde{Q}_k \in B^{s-1}_{r,r}(\Gamma^{(m)}_C), \;\; k = \overline{1,5}, \;\; j = \overline{1,4}, \end{gathered}$$

and the compatibility conditions

$$\widetilde{F}^{(m)}_j := F^{(m)}_j - r_{_{\Gamma^{(m)}_T}} G_{0j} - r_{_{\Gamma^{(m)}_T}} G^{(m)}_{0j} \in r_{_{\Gamma^{(m)}_T}} \widetilde{B}^{s-1}_{r,r}(\Gamma^{(m)}_T), \;\; j = \overline{1,4},$$

are satisfied, then $U^{(m)} \in [\,H^{s+\frac{1}{r}}_r(\Omega^{(m)})\,]^4$ *and* $U \in [\,H^{s+\frac{1}{r}}_r(\Omega)\,]^5$*;*

ii) *if*

$$\begin{gathered} Q_k \in B^{s-1}_{r,q}(S_N), \quad Q^{(m)}_j \in B^{s-1}_{r,q}(S^{(m)}_N), \quad f_k \in B^{s}_{r,q}(S_D), \\ f^{(m)}_k \in B^{s}_{r,q}(\Gamma^{(m)}_T), \quad F^{(m)}_j \in B^{s-1}_{r,q}(\Gamma^{(m)}_T), \quad \widetilde{Q}^{(m)}_j \in B^{s-1}_{r,q}(\Gamma^{(m)}_C), \\ \widetilde{Q}_k \in B^{s-1}_{r,q}(\Gamma^{(m)}_C), \;\; k = \overline{1,5}, \;\; j = \overline{1,4}, \end{gathered}$$

and the compatibility conditions

$$\widetilde{F}_j^{(m)} := F_j^{(m)} - r_{_{\Gamma_T^{(m)}}} G_{0j} - r_{_{\Gamma_T^{(m)}}} G_{0j}^{(m)} \in r_{_{\Gamma_T^{(m)}}} \widetilde{B}^{s-1}_{r,q}(\Gamma_T^{(m)}), \quad j = \overline{1,4},$$

are satisfied, then

$$U^{(m)} \in \big[\, B^{s+\frac{1}{r}}_{r,q}(\Omega^{(m)})\,\big]^4, \quad U \in \big[\, B^{s+\frac{1}{r}}_{r,q}(\Omega)\,\big]^5; \tag{5.50}$$

iii) *if* $\alpha > 0$ *is not integer and*

$$\begin{gathered} Q_k \in B^{\alpha-1}_{\infty,\infty}(S_N), \quad Q_j^{(m)} \in B^{\alpha-1}_{\infty,\infty}(S_N^{(m)}), \quad f_k \in C^{\alpha}(\overline{S_D}), \\ f_k^{(m)} \in C^{\alpha}(\overline{\Gamma_T^{(m)}}), \quad F_j^{(m)} \in B^{\alpha-1}_{\infty,\infty}(\Gamma_T^{(m)}), \quad \widetilde{Q}_j^{(m)} \in B^{\alpha-1}_{\infty,\infty}(\Gamma_C^{(m)}), \\ \widetilde{Q}_k \in B^{\alpha-1}_{\infty,\infty}(\Gamma_C^{(m)}), \quad k = \overline{1,5}, \quad j = \overline{1,4}, \end{gathered} \tag{5.51}$$

and the compatibility conditions

$$\widetilde{F}_j^{(m)} := F_j^{(m)} - r_{_{\Gamma_T^{(m)}}} G_{0j} - r_{_{\Gamma_T^{(m)}}} G_{0j}^{(m)} \in r_{_{\Gamma_T^{(m)}}} \widetilde{B}^{\alpha-1}_{\infty,\infty}(\Gamma_T^{(m)}), \quad j = \overline{1,4},$$

are satisfied, then

$$U^{(m)} \in \bigcap_{\alpha' < \kappa} \big[\, C^{\alpha'}(\overline{\Omega^{(m)}})\,\big]^4, \quad U \in \bigcap_{\alpha' < \kappa} \big[\, C^{\alpha'}(\overline{\Omega})\,\big]^5,$$

where $\kappa = \min\{\alpha,\ \gamma' + \frac{1}{2}\} > 0$.

Proof. The proofs of items i) and ii) follow easily from Theorems 5.1, 5.2, and 4.2.

To prove (iii) we use the following embedding (see, e.g., [72])

$$\begin{aligned} C^{\alpha}(\mathcal{M}) = B^{\alpha}_{\infty,\infty}(\mathcal{M}) \subset B^{\alpha-\varepsilon}_{\infty,1}(\mathcal{M}) \subset \\ \subset B^{\alpha-\varepsilon}_{\infty,q}(\mathcal{M}) \subset B^{\alpha-\varepsilon}_{r,q}(\mathcal{M}) \subset C^{\alpha-\varepsilon-\frac{k}{r}}(\mathcal{M}), \end{aligned} \tag{5.52}$$

where ε is an arbitrary small positive number, $\mathcal{M} \subset \mathbb{R}^3$ is a compact k-dimensional ($k = 2, 3$) smooth manifold with smooth boundary, $1 \leqslant q \leqslant \infty$, $1 < r < \infty$, $\alpha - \varepsilon - k/r > 0$, and α and $\alpha - \varepsilon - k/r$ are not integers.

From (5.51) and the embedding (5.52) the condition (5.50) follows with any $s \leqslant \alpha - \varepsilon$.

Bearing in mind (5.49) and taking r sufficiently large and ε sufficiently small, we can put

$$s = \alpha - \varepsilon \ \text{ if } \ \frac{1}{r} - \frac{1}{2} + \gamma'' < \alpha - \varepsilon < \frac{1}{r} + \frac{1}{2} + \gamma', \tag{5.53}$$

and

$$s \in \Big(\frac{1}{r} - \frac{1}{2} + \gamma'', \frac{1}{r} + \frac{1}{2} + \gamma'\Big) \ \text{ if } \ \frac{1}{r} + \frac{1}{2} + \gamma' < \alpha - \varepsilon. \tag{5.54}$$

By (5.50) for the solution vectors we have $U^{(m)} \in [\, B^{s+\frac{1}{r}}_{r,q}(\Omega^{(m)})\,]^4$ and $U \in [\, B^{s+\frac{1}{r}}_{r,q}(\Omega)\,]^5$ with

$$s + \frac{1}{r} = \alpha - \varepsilon + \frac{1}{r}$$

if (5.53) holds, and with

$$s+\frac{1}{r}\in\Big(\frac{2}{r}-\frac{1}{2}+\gamma'',\frac{2}{r}+\frac{1}{2}+\gamma'\Big)$$

if (5.54) holds. In the last case we can take

$$s+\frac{1}{r}=\frac{2}{r}+\frac{1}{2}+\gamma'-\varepsilon\,.$$

Therefore, we have either

$$U^{(m)}\in\big[\,B_{r,q}^{\alpha-\varepsilon+\frac{1}{r}}(\Omega^{(m)})\,\big]^4,\quad U\in\big[\,B_{r,q}^{\alpha-\varepsilon+\frac{1}{r}}(\Omega)\,\big]^5,$$

or

$$U^{(m)}\in\big[\,B_{r,q}^{\frac{1}{2}+\frac{2}{r}+\gamma'-\varepsilon}(\Omega^{(m)})\,\big]^4,\quad U\in\big[\,B_{r,q}^{\frac{1}{2}+\frac{2}{r}+\gamma'-\varepsilon}(\Omega)\,\big]^5,$$

in accordance with the inequalities (5.53) and (5.54). The last embedding in (5.52) (with $k=3$) yields then that either

$$U^{(m)}\in\big[\,C^{\alpha-\varepsilon-\frac{2}{r}}(\overline{\Omega^{(m)}})\,\big]^4,\quad U\in\big[\,C^{\alpha-\varepsilon-\frac{2}{r}}(\overline{\Omega})\,\big]^5,$$

or

$$U^{(m)}\in\big[\,C^{\frac{1}{2}-\varepsilon+\gamma'-\frac{1}{r}}(\overline{\Omega}_1)\,\big]^4,\quad U\in\big[\,C^{\frac{1}{2}-\varepsilon+\gamma'-\frac{1}{r}}(\overline{\Omega})\,\big]^5.$$

These relations lead to the inclusions

$$U^{(m)}\in\big[\,C^{\kappa-\varepsilon-\frac{2}{r}}(\overline{\Omega^{(m)}})\,\big]^4,\quad U\in\big[\,C^{\kappa-\varepsilon-\frac{2}{r}}(\overline{\Omega})\,\big]^5,\tag{5.55}$$

where $\kappa=\min\{\alpha,\ \gamma'+\frac{1}{2}\}$ and $\kappa>0$ due to the inequality (5.40). Since r is sufficiently large and ε is sufficiently small, the inclusions (5.55) accomplish the proof. □

Regularity results for $u_4=\vartheta$ and $u_4^{(m)}=\vartheta^{(m)}$ can be refined. Namely, we can assert the following

Proposition 5.5. *Let conditions of Theorem* 5.4 *hold. Then,*

$$u_4\in C^{\frac{1}{2}-\varepsilon}(\overline{\Omega}),\quad u_4^{(m)}\in C^{\frac{1}{2}-\varepsilon}(\overline{\Omega^{(m)}}),\tag{5.56}$$

where ε is an arbitrarily small positive number.

Indeed, $u_4=\vartheta$ and $u_4^{(m)}=\vartheta^{(m)}$ solve the following transmission problem

$$\begin{cases}\varkappa_{ij}\partial_i\partial_j u_4=Q^* \ \text{ in } \ \Omega,\\ \varkappa_{ij}^{(m)}\partial_i\partial_j u_4^{(m)}=Q^{(m)*} \ \text{ in } \ \Omega^{(m)},\\ r_{\Gamma_T^{(m)}}\{u_4\}^+-r_{\Gamma_T^{(m)}}\{u_4^{(m)}\}^+=f_4^{(m)} \ \text{ on } \ \Gamma_T^{(m)},\\ r_{\Gamma_T^{(m)}}\big\{[T(\partial,n)U]_4\big\}^+ +\\ \qquad\qquad +r_{\Gamma_T^{(m)}}\big\{[T^{(m)}(\partial,\nu)U^{(m)}]_4\big\}^+=F_4^{(m)} \ \text{ on } \ \Gamma_T^{(m)},\\ r_{S_N\cup\Gamma_C^{(m)}}\big\{[T(\partial,n)U]_4\big\}^+=\widetilde{Q}_4 \ \text{ on } \ S_N\cup\Gamma_C^{(m)},\\ r_{S_N^{(m)}\cup\Gamma_C^{(m)}}\big\{[T^{(m)}(\partial,\nu)U^{(m)}]_4\big\}^+=\widetilde{Q}_4^{(m)} \ \text{ on } \ S_N^{(m)}\cup\Gamma_C^{(m)},\\ r_{S_D}\{u_4\}^+=f_4 \ \text{ on } \ S_D,\end{cases}\tag{5.57}$$

Due to Theorem 5.4.(i) we deduce

$$U \in \big[\, H_r^{s+\frac{1}{r}}(\Omega)\,\big]^5, \quad U^{(m)} \in \big[\, H_r^{s+\frac{1}{r}}(\Omega^{(m)})\,\big]^4,$$

whence

$$\begin{gathered}\big[\, T(\partial, n) U\,\big]_4 = \varkappa_{il}\, n_i\, \partial_l \vartheta, \quad \big[\, T^{(m)}(\partial,\nu) U^{(m)}\,\big]_4 = \varkappa_{il}^{(m)}\, \nu_i\, \partial_l \vartheta^{(m)},\\ Q^* = \tau\, T_0\, \gamma_{ie}\, \partial_l u_i - \tau\, T_0\, g_i \partial_i \varphi + \tau\, \alpha\, \vartheta \in H_r^{s+\frac{1}{r}-1}(\Omega),\\ Q^{(m)*} = \tau\, T_0^{(m)}\, \gamma_{ie}^{(m)}\, \partial_l u_i^{(m)} + \tau\, \alpha^{(m)}\, \vartheta^{(m)} \in H_r^{s+\frac{1}{r}-1}(\Omega^{(m)}),\\ f_4^{(m)} \in B_{r,r}^{s}(\Gamma_T^{(m)}), \quad F_4^{(m)} \in B_{r,r}^{s-1}(\Gamma_T^{(m)}), \quad f_4 \in B_{r,r}^{s}(S_D),\\ Q_4 \in B_{r,r}^{s-1}(S_N \bigcup \Gamma_C^{(m)}), \quad Q_4^{(m)} \in B_{r,r}^{s-1}(S_N^{(m)} \bigcup \Gamma_C^{(m)}),\\ \frac{1}{r} - \frac{1}{2} + \gamma'' < s < \frac{1}{r} + \frac{1}{2} + \gamma', \;\; 1 < r < \infty.\end{gathered}$$

Since the symbols of the differential operators $-\varkappa_{ij}\partial_i\partial_j$ and $-\varkappa_{ij}^{(m)}\partial_i\partial_j$ are positive, the above problem can be reduced to the strongly elliptic system of pseudodifferential equations. Moreover, the corresponding pseudodifferential operator is positive definite. Therefore (see [54])

$$u_4 \in H_r^{s+\frac{1}{r}}(\Omega), \;\; u_4^{(m)} \in H_r^{s+\frac{1}{r}}(\Omega^{(m)}), \;\; \frac{1}{r} - \frac{1}{2} < s < \frac{1}{r} + \frac{1}{2}\,, \;\; 1 < r < \infty.$$

Due to the embedding theorem (see [72]), for sufficiently small $\delta > 0$, sufficiently large r and $s > 1/2 + 1/r - \delta$ we have

$$H_r^{s+\frac{1}{r}}(\Omega) \subset C^{\frac{1}{2}-\frac{1}{r}-\delta}(\overline{\Omega}), \quad H_r^{s+\frac{1}{r}}(\Omega^{(m)}) \subset C^{\frac{1}{2}-\frac{1}{r}-\delta}(\overline{\Omega^{(m)}}).$$

Whence (5.56) follows with $\varepsilon = 1/r + \delta$.

5.3. **Asymptotic formulas for solutions of Problem (ICP-A).** Here we study general asymptotic properties of solutions to the problem (ICP-A) near the exceptional curves. Namely, we investigate in detail the asymptotic expansion of solutions at the interface crack edge $\partial\Gamma_C^{(m)}$ and at the curve $\partial\Gamma^{(m)}$ where the interface intersects the exterior boundary. Note that $\partial\Gamma_C^{(m)} \cup \partial\Gamma^{(m)} = \partial\Gamma_T^{(m)} =: \ell_m$.

For simplicity of description of the method, we assume that the boundary data of the problem are infinitely smooth. Namely,

$$\begin{gathered}Q_k \in C^\infty(\overline{S}_N), \;\; Q_j^{(m)} \in C^\infty(\overline{S_N^{(m)}}), \;\; f_k \in C^\infty(\overline{S}_D),\\ f_k^{(m)} \in C^\infty(\overline{\Gamma_T^{(m)}}), \;\; F_j^{(m)} \in C^\infty(\overline{\Gamma_T^{(m)}}), \;\; \widetilde{Q}_j^{(m)} \in C^\infty(\overline{\Gamma_C^{(m)}}),\\ \widetilde{F}_j^{(m)} := F_j^{(m)} - r_{\Gamma_T^{(m)}} G_{0j} - r_{\Gamma_T^{(m)}} G_{0j}^{(m)} \in C_0^\infty(\overline{\Gamma_T^{(m)}}),\\ Q_j^{(m)} \in C^\infty(\overline{\Gamma_C^{(m)}}), \;\; j = \overline{1,4}, \;\; k = \overline{1,5},\end{gathered} \tag{5.58}$$

where $C_0^\infty(\overline{\Gamma_T^{(m)}})$ denotes a space of infinitely differentiable functions vanishing on $\partial\Gamma_T^{(m)}$ along with all tangential derivatives.

We have already shown that the interface crack problem (ICP-A) is uniquely solvable and the pair of solution vectors $(U^{(m)}, U)$ are represented by (5.42)-(5.43) with the densities defined by the system of pseudodifferential equations (5.10)–(5.13) (or (5.19)–(5.21)).

Let $\Phi := (\psi, h, h^{(m)})^\top \in \mathbf{X}_p^s$ be a solution of the system (5.19)-(5.21) which is written in matrix form as

$$\mathcal{N}_\tau^{(A)}\Phi = Y,$$

(see Subsections 5.1 and 5.2) where

$$Y \in \big[\, C^\infty(\overline{S}_D)\,\big]^5 \times \big[\, C^\infty(\overline{\Gamma_T^{(m)}})\,\big]^5 \times \big[\, C_0^\infty(\overline{\Gamma_T^{(m)}})\,\big]^4.$$

To establish asymptotic properties of the solution vectors $U^{(m)}$ and U near the exceptional curve $\partial\Gamma_T^{(m)}$, we rewrite the representations (5.42)–(5.43) in the form

$$U^{(m)} = V_\tau^{(m)}([\mathcal{P}_\tau^{(m)}]^{-1}h^{(m)}) + R^{(m)} \ \text{ in } \ \Omega^{(m)},$$
$$U = V_\tau(\mathcal{P}_\tau^{-1}\psi) + V_\tau(\mathcal{P}_\tau^{-1}h) + R \ \text{ in } \ \Omega,$$

where $h^{(m)} = -(h_1, \cdots, h_4)^\top + \big(\widetilde{F}_1^{(m)}, \ldots, \widetilde{F}_4^{(m)}\big)^\top$ on $\Gamma_T^{(m)}$,

$$R^{(m)} := V_\tau^{(m)}([\mathcal{P}_\tau^{(m)}]^{-1}G_0^{(m)}) \in \big[\, C^\infty(\overline{\Omega^{(m)}})\,\big]^4,$$
$$R := V_\tau(\mathcal{P}_\tau^{-1}G_0) \in \big[\, C^\infty(\overline{\Omega})\,\big]^5.$$

The vectors $h = (h_1, \ldots, h_5)^\top$ and $\psi = (\psi_1, \ldots, \psi_5)^\top$ solve the following strongly elliptic system of pseudodifferential equations:

$$r_{_{S_D}}\,\mathcal{A}_\tau\psi = \Phi^{(1)} \ \text{ on } \ S_D,$$
$$r_{_{\Gamma_T^{(m)}}}(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})h = \Phi^{(2)} \ \text{ on } \ \Gamma_T^{(m)},$$

where

$$\Phi^{(1)} = (\Phi_1^{(1)}, \ldots, \Phi_5^{(1)})^\top \in \big[\, C^\infty(\overline{S}_D)\big]^5,$$
$$\Phi_k^{(1)} = f_k - r_{_{S_D}}[\mathcal{A}_\tau G_0]_k - r_{_{S_D}}[\mathcal{A}_\tau h]_k, \quad k = \overline{1,5},$$
$$\Phi^{(2)} = (\Phi_1^{(2)}, \cdots, \Phi_5^{(2)})^\top \in \big[\, C^\infty(\overline{\Gamma_T^{(m)}})\,\big]^5,$$
$$\Phi_j^{(2)} = f_j^{(m)} + r_{_{\Gamma_T^{(m)}}}\big[\,\mathcal{H}_\tau^{(m)}(\mathcal{P}_\tau^{(m)})^{-1}G_0^{(m)}\,\big]_j - r_{_{\Gamma_T^{(m)}}}[\mathcal{A}_\tau G_0]_j +$$
$$+\, r_{_{\Gamma_T^{(m)}}}\big[\,\mathcal{H}_\tau^{(m)}(\mathcal{P}_\tau^{(m)})^{-1}\widetilde{F}^{(m)}\,\big]_j - r_{_{\Gamma_T^{(m)}}}[\mathcal{A}_\tau\psi]_j, \quad j = \overline{1,4},$$
$$\Phi_5^{(2)} = f_5^{(m)} - r_{_{\Gamma_T^{(m)}}}[\mathcal{A}_\tau G_0]_5 - r_{\Gamma_T^{(m)}}[\mathcal{A}_\tau\psi]_5\,.$$

Applying a partition of unity, natural local co-ordinate systems and standard rectifying technique based on canonical diffeomorphisms, we can assume that $\partial\Gamma_T^{(m)}$ is rectified. Then we identify a one-sided neighbourhood on $\Gamma_T^{(m)}$ of an arbitrary point $\widetilde{x} \in \partial\Gamma_T^{(m)}$ as a part of the half-plane $x_2 > 0$.

Thus we assume that $(x_1, 0) = \widetilde{x} \in \partial\Gamma_T^{(m)} = \ell_m$ and $(x_1, x_{2,+}) \in \Gamma_T^{(m)}$ for $0 < x_{2,+} < \varepsilon$ with some positive ε.

Consider the 5×5 matrix $\mathcal{D}_2(x_1)$ related to the principal homogeneous symbol $\mathfrak{S}_2(x, \xi)$ of the operator $\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}$ (see Subsection 5.2) for $\widetilde{x} = (x_1, 0) \in \ell_m$

$$\mathcal{D}_2(x_1) := \big[\mathfrak{S}_2(x_1, 0, 0, +1)\big]^{-1}\, \mathfrak{S}_2(x_1, 0, 0, -1).$$

We introduce the short notation $\lambda_j(x_1)$ for the eigenvalues $\lambda_j^{(2)}(x_1)$, $j = \overline{1, 5}$, of the matrix $\mathcal{D}_2(x_1)$ and denote by m_j the algebraic multiplicities of $\lambda_j(x_1)$. Let $\mu_1(x_1), \dots, \mu_l(x_1)$, $1 \leqslant l \leqslant 5$, be the distinct eigenvalues. Evidently, m_j and l depend on x_1, in general, and $m_1 + \cdots + m_l = 5$.

It is well known that the matrix $\mathcal{D}_2(x_1)$ admits the following decomposition (see, e.g., [24])

$$\mathcal{D}_2(x_1) = \mathcal{D}(x_1)\, \mathcal{J}_{\mathcal{D}_2}(x_1)\, \mathcal{D}^{-1}(x_1), \quad (x_1, 0) \in \partial\Gamma_T^{(m)},$$

where $\mathcal{D}$ is 5×5 nondegenerate matrix with infinitely differentiable entries and $\mathcal{J}_{\mathcal{D}_2}$ is block diagonal

$$\mathcal{J}_{\mathcal{D}_2}(x_1) := \operatorname{diag}\Big\{\mu_1(x_1) B^{(m_1)}(1), \dots, \mu_l(x_1) B^{(m_l)}(1)\Big\}.$$

Here $B^{(\nu)}(t)$, $\nu \in \{m_1, \dots, m_l\}$, are upper triangular matrices:

$$B^{(\nu)}(t) = \big\|b_{jk}^{(\nu)}(t)\big\|_{\nu\times\nu}, \quad b_{jk}^{(\nu)}(t) = \begin{cases} \dfrac{t^{k-j}}{(k-j)!}, & j < k, \\ 1, & j = k, \\ 0, & j > k, \end{cases}$$

i.e.,

$$B^{(\nu)}(t) = \begin{pmatrix} 1 & t & \dfrac{t^2}{2!} & \dots & \dfrac{t^{\nu-2}}{(\nu-2)!} & \dfrac{t^{\nu-1}}{(\nu-1)!} \\ 0 & 1 & t & \dots & \dfrac{t^{\nu-3}}{(\nu-3)!} & \dfrac{t^{\nu-2}}{(\nu-2)!} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & t \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}_{\nu\times\nu}.$$

Denote

$$B_0(t) := \operatorname{diag}\big\{B^{(m_1)}(t),\, \dots,\, B^{(m_l)}(t)\big\}.$$

Applying the results from the reference [12], we derive the following asymptotic expansion

$$\begin{aligned} h(x_1, x_{2,+}) &= \mathcal{D}(x_1)\, x_{2,+}^{-\frac{1}{2}+\Delta(x_1)}\, B_0\Big(-\frac{1}{2\pi i}\log x_{2,+}\Big)\, \mathcal{D}^{-1}(x_1)\, b_0(x_1) \\ &\quad + \sum_{k=1}^{M} \mathcal{D}(x_1)\, x_{2,+}^{-\frac{1}{2}+\Delta(x_1)+k}\, B_k\big(x_1, \log x_{2,+}\big) + h_{M+1}(x_1, x_{2,+}), \end{aligned} \tag{5.59}$$

where $b_0 \in \big[C^\infty(\ell_m)\big]^5$, $h_{M+1} \in [\,C^\infty(\ell^+_{m,\varepsilon})\,]^5$, $\ell^+_{m,\varepsilon} = \ell_m \times [0,\varepsilon]$, and

$$B_k(x_1,t) = B_0\Big(-\frac{t}{2\pi i}\Big) \sum_{j=1}^{k(2m_0-1)} t^j\, d_{kj}(x_1).$$

Here $m_0 = \max\{m_1,\dots,m_l\}$, the coefficients $d_{kj} \in [C^\infty(\ell_m)]^5$ and $\Delta := (\Delta_1,\dots,\Delta_5)^\top$,

$$\Delta_j(x_1) = \frac{1}{2\pi i}\log\lambda_j(x_1) = \frac{1}{2\pi}\,\arg\lambda_j(x_1) + \frac{1}{2\pi i}\,\log|\lambda_j(x_1)|,$$
$$-\pi < \arg\lambda_j(x_1) < \pi, \quad (x_1,0)\in\ell_m, \quad j=\overline{1,5}.$$

Furthermore, let

$$x_{2,+}^{-\frac12+\Delta(x_1)} := \operatorname{diag}\Big\{x_{2,+}^{-\frac12+\Delta_1(x_1)},\,\dots,\,x_{2,+}^{-\frac12+\Delta_5(x_1)}\Big\}.$$

Now, having in hand the above asymptotic expansion for the density vector function h, we can apply the results of the reference [13] and write the spatial asymptotic expansions of the solution vectors U and $U^{(m)}$

$$\begin{aligned} U(x) = \sum_{\mu=\pm1}\bigg\{\sum_{s=1}^{l_0}\sum_{j=0}^{n_s-1} x_3^j\Big[d_{sj}(x_1,\mu) z_{s,\mu}^{\frac12+\Delta(x_1)-j}\,B_0(\zeta)\Big]\,c_j(x_1)+\\ +\sum_{k,l=0}^{M+2}\sum_{j+p=0}^{M+2-l} x_2^l\,x_3^j d_{sljp}(x_1,\mu)\,z_{s,\mu}^{\frac12+\Delta(x_1)+p+k}\,B_{skjp}(x_1,\log z_{s,\mu})\bigg\}+\\ +U_{M+1}(x)\,,\quad x_3>0,\quad \zeta := -\frac{1}{2\pi i}\,\log z_{s,\mu}, \end{aligned} \tag{5.60}$$

$$\begin{aligned} U^{(m)}(x) = \\ = \sum_{\mu=\pm1}\bigg\{\sum_{s=1}^{l_0^{(m)}}\sum_{j=0}^{n_s^{(m)}-1} x_3^j\Big[d_{sj}^{(m)}(x_1,\mu)(z_{s,\mu}^{(m)})^{\frac12+\Delta(x_1)-j}\,B_0(\zeta^{(m)})\Big]\,c_j(x_1)+\\ +\sum_{k,l=0}^{M+2}\sum_{j+p=0}^{M+2-l} x_2^l\,x_3^j d_{sljp}^{(m)}(x_1,\mu)\,(z_{s,\mu}^{(m)})^{\frac12+\Delta(x_1)+p+k}\,B_{skjp}^{(m)}(x_1,\log z_{s,\mu}^{(m)})\bigg\}+\\ +U_{M+1}^{(m)}(x)\,,\quad x_3>0,\quad \zeta^{(m)} := -\frac{1}{2\pi i}\,\log z_{s,\mu}^{(m)}\,. \end{aligned} \tag{5.61}$$

The coefficients $d_{sj}(\cdot\,,\mu)$, $d_{sj}^{(m)}(\cdot\,,\mu)$, $d_{sljp}(\cdot\,,\mu)$ and $d_{sljp}^{(m)}(\cdot\,,\mu)$ are matrices with entries from the space $C^\infty(\ell_m)$, $B_{skjp}(x_1,t)$ and $B_{skjp}^{(m)}(x_1,t)$ are polynomials in t with vector coefficients which depend on the variable x_1 and have the order $\nu_{kjp} = k(2m_0-1)+m_0-1+p+j$ with $m_0 = \max\{m_1, \dots, m_l\}$,

$$
\begin{gathered}
c_j \in [C^\infty(\ell_m)]^5, \ \ U_{M+1} \in \left[C^{M+1}(\overline{\Omega}) \right]^5, \ \ U^{(m)}_{M+1} \in \left[C^{M+1}(\overline{\Omega^{(m)}}) \right]^4, \\
z^{\kappa+\Delta(x_1)}_{s,\mu} := \operatorname{diag}\Big\{ z^{\kappa+\Delta_1(x_1)}_{s,\mu}, \, \dots, \, z^{\kappa+\Delta_5(x_1)}_{s,\mu} \Big\}, \\
(z^{(m)}_{s,\mu})^{\kappa+\Delta(x_1)} := \operatorname{diag}\Big\{ (z^{(m)}_{s,\mu})^{\kappa+\Delta_1(x_1)}, \, \dots, \, (z^{(m)}_{s,\mu})^{\kappa+\Delta_5(x_1)} \Big\}, \\
\kappa \in \mathbb{R}, \ \ \mu = \pm 1, \ \ (x_1, \, 0) \in \ell_m, \\
z_{s,+1} = -x_2 - x_3 \tau_{s,+1}, \ \ z_{s,-1} = x_2 - x_3 \tau_{s,-1}, \\
z^{(m)}_{s,+1} = -x_2 - x_3 \tau^{(m)}_{s,+1}, \ \ z^{(m)}_{s,-1} = x_2 - x_3 \tau^{(m)}_{s,-1}, \\
-\pi < \arg z_{s,\pm 1} < \pi, \quad -\pi < \arg z^{(m)}_{s,\pm 1} < \pi, \\
\tau_{s,\pm 1} \in C^\infty(\ell_m), \quad \tau^{(m)}_{s,\pm 1} \in C^\infty(\ell_m).
\end{gathered}
\tag{5.62}
$$

Here $\{\tau_{s,\pm 1}\}_{s=1}^{l_0}$ (respectively $\{\tau^{(m)}_{s,\pm 1}\}_{s=1}^{l_0^{(m)}}$) are the different roots of multiplicity n_s, $s=1, \dots, l_0$, (respectively $n_s^{(m)}$, $s=1, \dots, l_0^{(m)}$) of the polynomial in ζ, $\det A^{(0)}([J^\top_\varkappa(x_1,0,0)]^{-1}\eta_\pm)$ (respectively $\det A^{(m,0)}([J^\top_{\varkappa_m}(x_1,0,0)]^{-1}\eta_\pm)$) with $\eta_\pm = (0, \pm 1, \zeta)^\top$, satisfying the condition $\operatorname{Re} \tau_{s,\pm 1} < 0$ (respectively $\operatorname{Re} \tau^{(m)}_{s,\pm 1} < 0$). The matrix $J_\varkappa$ (respectively $J_{\varkappa_m}$) stands for the Jacobi matrix corresponding to the canonical diffeomorphism $\varkappa$ (respectively $\varkappa_m$) related to the local co-ordinate system. Under this diffeomorphism the curve ℓ_m is locally rectified and we assume that $(x_1, 0, 0) \in \ell_m$, $x_2 = \operatorname{dist}(x^{(m)}_T, \ell_m)$, $x_3 = \operatorname{dist}(x, \Gamma^{(m)}_T)$, where $x^{(m)}_T$ is the projection of the reference point $x \in \Omega$ (respectively $x \in \Omega^{(m)}$) on the plane corresponding to the image of $\Gamma^{(m)}_T$ under the diffeomorphism $\varkappa$ (respectively $\varkappa_m$).

Note that the coefficients $d_{sj}(\,\cdot\,, \mu)$ and $d^{(m)}_{sj}(\,\cdot\,, \mu)$ can be calculated explicitly, whereas the coefficients c_j can be expressed by means of the first coefficient b_0 in the asymptotic expansion of (5.59) (see [13])

$$
\begin{aligned}
d_{sj}(x_1, +1) &= \frac{1}{2\pi} \, G_\varkappa(x_1, 0) \, P^+_{sj}(x_1) \, \mathcal{D}(x_1), \\
d_{sj}(x_1, -1) &= \frac{1}{2\pi} \, G_\varkappa(x_1, 0) \, P^-_{sj}(x_1) \, \mathcal{D}(x_1) \, e^{i\pi(\frac{1}{2} - \Delta(x_1))}, \\
& \qquad\qquad s = \overline{1, l_0}, \ \ j = \overline{0, n_s - 1}, \\
d^{(m)}_{sj}(x_1, \, +1) &= \frac{1}{2\pi} \, G_{\varkappa_m}(x_1, 0) \, P^{+(m)}_{sj}(x_1) \, \widetilde{\mathcal{D}}(x_1), \\
d^{(m)}_{sj}(x_1, \, -1) &= \frac{1}{2\pi} \, G_{\varkappa_m}(x_1, 0) \, P^{-(m)}_{sj}(x_1) \, \widetilde{\mathcal{D}}(x_1) \, e^{i\pi(\frac{1}{2} - \Delta(x_1))}, \\
& \qquad\qquad s = \overline{1, l_0^{(m)}}, \ \ j = \overline{0, n_s^{(m)} - 1},
\end{aligned}
$$

where $\widetilde{\mathcal{D}} = \|\mathcal{D}_{kj}\|_{4\times 5}$,

$$
P^\pm_{sj}(x_1) := V^{(s)}_{-1,\, j}(x_1, 0, 0, \pm 1) \mathfrak{S}^{-1}_{-\frac{1}{2} I + \mathcal{K}}(x_1, 0, 0, \pm 1),
$$

$$P_{sj}^{\pm(m)}(x_1) := V_{-1,\,j}^{(m),s}(x_1,0,0,\pm 1)\mathfrak{S}^{-1}_{-\frac{1}{2}\,I+\mathcal{K}^{(m)}}(x_1,0,0,\pm 1),$$

$$V_{-1,\,j}^{(s)}(x_1,0,0,\pm 1) := -\frac{i^{j+1}}{j!(n_s-1-j)!}\,\frac{d^{n_s-1-j}}{d\tau^{n_s-1-j}}\,(\zeta-\tau_{s,\pm 1})^{n_s}\times$$
$$\times\Big(A^{(0)}\big((J_{\varkappa}^{\top}(x_1,0))^{-1}\big)\cdot(0,\pm 1,\zeta)^{\top}\Big)^{-1}\bigg|_{\zeta=\tau_{s,\pm 1}},$$

$$V_{-1,\,j}^{(m),s}(x_1,0,0,\pm 1) := -\frac{i^{j+1}}{j!(n_s^{(m)}-1-j)!}\,\frac{d^{n_s^{(m)}-1-j}}{d\tau^{n_s^{(m)}-1-j}}\,(\zeta-\tau^{(m)}_{s,\pm 1})^{n_s^{(m)}}\times$$
$$\times\Big(A^{(m,0)}\big((J_{\varkappa_m}^{\top}(x_1,0))^{-1}\big)\cdot(0,\pm 1,\zeta)^{\top}\Big)^{-1}\bigg|_{\zeta=\tau^{(m)}_{s,\pm 1}},$$

$G_{\varkappa}(x_1,0)$ and $G_{\varkappa_m}(x_1,0)$ are the square roots of the Gram's determinant of $\varkappa$ and $\varkappa_m$ respectively, and

$$c_j(x_1)=a_j(x_1)\,B_0^{-}\Big(-\frac{1}{2}+\Delta(x_1)\Big)\,\mathcal{D}^{-1}(x_1)b_0(x_1), \qquad (5.63)$$
$$j=0,\dots,n_s-1, \quad (j=0,\dots,n_s^{(m)}-1),$$

where

$$B_0^{-}\Big(-\frac{1}{2}+\Delta(x_1)\Big)=$$
$$=\operatorname{diag}\Big\{B_-^{m_1}\Big(-\frac{1}{2}+\Delta_1(x_1)\Big),\dots,B_-^{m_l}\Big(-\frac{1}{2}+\Delta_l(x_1)\Big)\Big\},$$
$$B_-^{m_q}(t)=\big\|\widetilde{b}_{kp}^{m_q}(t)\big\|_{m_q\times m_q}, \ \ q=1,\dots,l,$$
$$\widetilde{b}_{kp}^{m_q}(t)=\begin{cases}\Big(\dfrac{1}{2\pi i}\Big)^{p-k}\,\dfrac{(-1)^{p-k}}{(p-k)!}\,\dfrac{d^{p-k}}{dt^{p-k}}\,\Gamma(t+1)\,e^{\frac{i\pi(t+1)}{2}}, & \text{for } \ k\leqslant p,\\ 0, & \text{for } \ k>p,\end{cases}$$

and $\Gamma(t+1)$ is the Euler function,

$$a_j(x_1)=\operatorname{diag}\Big\{a^{m_1}(\alpha_1^{(j)}),\dots,a^{m_l}(\alpha_l^{(j)})\Big\},$$
$$\alpha_q^{(j)}(x_1)=-\frac{3}{2}-\Delta_q(x_1)+j, \ \ q=\overline{1,l}, \ \ j=\overline{0,n_s-1} \ \ (j=\overline{0,n_s^{(m)}-1})$$
$$a^{m_q}(\alpha_q^{(j)})=\big\|a_{kp}^{m_q}(\alpha_q^{(j)})\big\|_{m_q\times m_q},$$
$$a_{kp}^{m_q}(\alpha_q^{(j)})=\begin{cases}-i\displaystyle\sum_{l=k}^{p}\frac{(-1)^{p-k}(2\pi i)^{l-p}\widetilde{b}_{kl}^{m_q}(\mu_q)}{(\alpha_q^{(0)}+1)^{p-l+1}}\,, \ \ j=0, \ \ k\leqslant p,\\ (-1)^{p-k}\widetilde{b}_{kp}^{m_q}(\alpha_q^{(j)}), \ \ j=\overline{1,n_s-1} \ \ (j=\overline{1,n_s^{(m)}-1}), \ \ k\leqslant p,\\ 0, \ \ k>p,\end{cases}$$
$$\mu_q=-\frac{1}{2}-\Delta_q(x_1), \ \ -1<\operatorname{Re}\mu_q<0.$$

Analogous investigation for basic mixed and interior crack problems for homogeneous piezoelectric bodies has been carried out in the reference [8], where the asymptotic properties of solutions have been established near the crack edges and the curves where the different boundary conditions collide. In [8] it is shown that the stress singularity exponents at the interior crack edges are independent of the material parameters and equal to -0.5, while they essentially depend on the material parameters at the curves where different boundary conditions collide.

As it is evident from the above exposed results the stress singularity exponents at the interface crack edges and at the curves where the interface intersects the exterior boundary essentially depend on the material parameters. More precise results for particular cases are presented in the next subsection, where these exponents are calculated explicitly for some particular values of the material parameters.

5.4. **Analysis of singularities of solutions to Problem (ICP-A).** As in the previous subsection, let $\ell_m = \partial\Gamma_T^{(m)}$. For $x' \in \ell_m$ by $\Pi_{x'}^{(m)}$ we denote the plane passing trough the point x' and orthogonal to the curve ℓ_m. We introduce the polar coordinates (r,α), $r \geqslant 0$, $-\pi \leqslant \alpha \leqslant \pi$, in the plane $\Pi_{x'}^{(m)}$ with pole at the point x'. Denote by $\Gamma_T^{(m)\pm}$ the two different faces of the surface $\Gamma_T^{(m)}$. It is evident that $(r,\pm\pi) \in \Gamma_T^{(m)\pm}$.

The intersection of the plane $\Pi_{x'}^{(m)}$ and Ω is identified with the half-plane $r \geqslant 0$ and $-\pi \leqslant \alpha \leqslant 0$, while the intersection of the plane $\Pi_{x'}^{(m)}$ and $\Omega^{(m)}$ is identified with the half-plane $r \geqslant 0$ and $0 \leqslant \alpha \leqslant \pi$.

The roots given by (5.62) are represented as follows

$$z_{s,+1} = -r\big[\cos\alpha + \tau_{s,+1}(x')\sin\alpha\big], \quad z_{s,-1} = r\big[\cos\alpha - \tau_{s,-1}(x')\sin\alpha\big],$$
$$s = 1,\dots,l_0, \ \ x' \in \ell_m,$$
$$z_{s,+1}^{(m)} = -r\big[\cos\alpha + \tau_{s,+1}^{(m)}(x')\sin\alpha\big], \quad z_{s,-1}^{(m)} = r\big[\cos\alpha - \tau_{s,-1}^{(m)}(x')\sin\alpha\big],$$
$$s = 1,\dots,l_0^{(m)}, \ \ x' \in \ell_m.$$

From the asymptotic expansions (5.60) and (5.61) we get

$$U(x) = \sum_{\mu=\pm1}\sum_{s=1}^{l_0}\sum_{j=0}^{n_s-1} c_{sj\mu}(x',\,\alpha)\, r^{\gamma+i\delta} B_0(\zeta)\, \widetilde{c}_{sj\mu}(x',\,\alpha) + \cdots, \tag{5.64}$$

$$U^{(m)}(x) = \sum_{\mu=\pm1}\sum_{s=1}^{l_0^{(m)}}\sum_{j=0}^{n_s^{(m)}-1} c_{sj\mu}^{(m)}(x',\,\alpha)\, r^{\gamma+i\delta} B_0(\zeta)\, \widetilde{c}_{sj\mu}^{(m)}(x',\,\alpha) + \cdots, \tag{5.65}$$

where

$$r^{\gamma+i\delta} = \operatorname{diag}\big\{r^{\gamma_1+i\delta_1},\,\dots,\,r^{\gamma_5+i\delta_5}\big\}, \ \ \zeta = -\frac{1}{2\pi i}\,\log r,$$

$$\gamma_j = \frac{1}{2} + \frac{1}{2\pi}\,\arg\lambda_j(x'), \ \ \delta_j = -\frac{1}{2\pi}\,\log|\lambda_j(x')|, \ \ x' \in \ell_m, \ \ j = \overline{1,5}, \tag{5.66}$$

and λ_j, $j = \overline{1,5}$, are eigenvalues of the matrix

$$\mathcal{D}_2(x') = \big[\mathfrak{S}_2(x',0,+1)\big]^{-1}\,\mathfrak{S}_2(x',0,-1), \ \ x' \in \ell_m. \tag{5.67}$$

Note that the subsequent terms in the expansion (5.64) and (5.65) have higher regularity, i.e., the real parts of the corresponding exponents are greater than γ_j.

The coefficients $c_{sj\mu}$, $\widetilde{c}_{sj\mu}$, $c^{(m)}_{sj\mu}$ and $\widetilde{c}^{(m)}_{sj\mu}$ in asymptotic expansions (5.64) and (5.65) read as

$$c_{sj\mu}(x',\,\alpha)=\sin^j\alpha\;d_{sj}(x',\,\mu)\big[\psi_{s,\mu}(x',\,\alpha)\big]^{\gamma+i\delta-j},$$

$$\widetilde{c}_{sj\mu}(x',\,\alpha)=B_0\Big(-\frac{1}{2\pi i}\log\psi_{s,\mu}(x',\,\alpha)\Big)\,c_j(x'),$$

$$j=\overline{0,n_s-1},\quad \mu=\pm1,\quad s=\overline{1,l_0},$$

$$c^{(m)}_{sj\mu}(x',\,\alpha)=\sin^j\alpha\;d^{(m)}_{sj}(x',\,\mu)\big[\psi^{(m)}_{s,\mu}(x',\,\alpha)\big]^{\gamma+i\delta-j},$$

$$\widetilde{c}^{(m)}_{sj\mu}(x',\,\alpha)=B_0\Big(-\frac{1}{2\pi i}\log\psi^{(m)}_{s,\mu}(x',\,\alpha)\Big)\,c_j(x'),$$

$$j=\overline{0,n_s-1},\quad \mu=\pm1,\quad s=\overline{1,l^{(m)}_0},$$

where

$$\psi_{s,\mu}(x',\,\alpha)=-\mu\cos\alpha-\tau_{s,\mu}(x')\sin\alpha,\quad s=\overline{1,l_0},$$

$$\psi^{(m)}_{s,\mu}(x',\,\alpha)=-\mu\cos\alpha-\tau^{(m)}_{s,\mu}(x')\sin\alpha,\quad s=\overline{1,l^{(m)}_0},$$

$$c_{sj\mu}(x',\,\alpha)=\big\|c^{(kp)}_{j\mu}(x',\,\alpha)\big\|_{5\times5},\qquad c^{(m)}_{sj\mu}(x',\,\alpha)=\big\|c^{((m)kp)}_{sj\mu}(x',\,\alpha)\big\|_{4\times5}.$$

Remark 5.6. If B_0 is the identity matrix, then the coefficients $\widetilde{c}_{sj\mu}$ and $\widetilde{c}^{(m)}_{sj\mu}$ take simpler form

$$\widetilde{c}_{sj\mu}(x',\,\alpha)=c_j(x'),\quad j=\overline{0,n_s-1},$$

$$\widetilde{c}^{(m)}_{sj\mu}(x',\,\alpha)=c_j(x'),\quad j=\overline{0,n^{(m)}_s-1},$$

where

$$c_j(x')=i^{\,j}\,\Gamma(j-\gamma-i\delta)\Gamma(\gamma+i\delta)\mathcal{D}^{-1}(x')b_0(x'). \tag{5.68}$$

In what follows for particular piezoelectric elastic materials we will analyze the exponents $\gamma_j+i\delta_j$, which determine the behaviour of U and $U^{(m)}$ near the line ℓ_m. Non-zero parameters δ_j lead to the so called oscillating singularities for the first order derivatives of U and $U^{(m)}$, in general. In turn, this yields oscillating stress singularities, which sometimes lead to mechanical contradictions, for example, to an overlapping of materials. So, from the practical point of view, it is important to single out classes of solids for which the oscillating effects do not occur.

To this end, let us consider the case when the domain Ω is occupied by a special class of solids belonging to the **422** (Tetragonal) or **622** (Hexagonal) class of crystals for which the corresponding system of differential equations reads as follows (see, e.g., [16])

$$(c_{11}\,\partial_1^2+c_{66}\,\partial_2^2+c_{44}\,\partial_3^2)u_1+(\,c_{12}+c_{66}\,)\,\partial_1\partial_2u_2+(\,c_{13}+c_{44}\,)\,\partial_1\partial_3u_3-$$
$$-\widetilde{\gamma}_1\,\partial_1\vartheta-e_{14}\,\partial_2\partial_3\varphi-\varrho\,\tau^2\,u_1=0,$$

$$(c_{12}+c_{66})\,\partial_2\partial_1 u_1 + (c_{66}\,\partial_1^2 + c_{11}\,\partial_2^2 + c_{44}\,\partial_3^2)\,u_2 + (c_{13}+c_{44})\,\partial_2\partial_3 u_3 - \\ -\widetilde{\gamma}_1\partial_2\vartheta + e_{14}\partial_1\partial_3\varphi - \varrho\,\tau^2\,u_2 = 0,$$

$$(c_{13}+c_{44})\,\partial_3\partial_1 u_1 + (c_{13}+c_{44})\,\partial_3\partial_2 u_2 + (c_{44}\,\partial_1^2 + c_{44}\,\partial_2^2 + c_{33}\,\partial_3^2)\,u_3 - \\ -\widetilde{\gamma}_3\,\partial_3\vartheta - \varrho\,\tau^2\,u_3 = 0,$$

$$-\tau\,T_0\,(\widetilde{\gamma}_1\,\partial_1 u_1 + \widetilde{\gamma}_1\,\partial_2 u_2 + \widetilde{\gamma}_3\,\partial_3 u_3) + (\varkappa_{11}\,\partial_1^2 + \varkappa_{11}\,\partial_2^2 + \varkappa_{33}\,\partial_3^2)\,\vartheta + \\ +\tau\,T_0\,g_3\,\partial_3\varphi - \tau\,\alpha\,\vartheta = 0,$$

$$e_{14}\partial_2\partial_3 u_1 - e_{14}\partial_1\partial_3 u_2 - g_3\,\partial_3\vartheta + (\varepsilon_{11}\,\partial_1^2 + \varepsilon_{11}\,\partial_2^2 + \varepsilon_{33}\,\partial_3^2)\,\varphi = 0,$$

where c_{11}, c_{12}, c_{13}, c_{33}, c_{44}, and $c_{66} = (c_{11} - c_{12})/2$ are the elastic constants, e_{14} is the piezoelectric constant, ε_{11} and ε_{33} are the dielectric constants, $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_3$ are the thermal strain constants, $\varkappa_{11}$ and $\varkappa_{33}$ are the thermal conductivity constants, g_3 is the pyroelectric constant.

It turned out that some important polymers and bio-materials (for example, the *collagen-hydroxyapatite* and TeO_2) are modelled by the above partial differential equations. These materials are widely used in biology and medicine (see [69]). In this model the thermoelectromechanical stress operator is defined as $\mathcal{T}(\partial, n) = \|\mathcal{T}_{jk}(\partial, n)\|_{5\times 5}$ with

$$\mathcal{T}_{11}(\partial,n) = c_{11}\,n_1\,\partial_1 + c_{66}\,n_2\,\partial_2 + c_{44}\,n_3\,\partial_3,$$
$$\mathcal{T}_{12}(\partial,n) = c_{12}\,n_1\,\partial_2 + c_{66}\,n_2\,\partial_1, \quad \mathcal{T}_{13}(\partial,n) = c_{13}\,n_1\,\partial_3 + c_{44}\,n_3\,\partial_1,$$
$$\mathcal{T}_{14}(\partial,n) = -\widetilde{\gamma}_1\,n_1, \quad \mathcal{T}_{15}(\partial,n) = -e_{14}\,n_3\,\partial_2,$$
$$\mathcal{T}_{21}(\partial,n) = c_{66}\,n_1\,\partial_2 + c_{12}\,n_2\,\partial_1,$$
$$\mathcal{T}_{22}(\partial,n) = c_{66}\,n_1\,\partial_1 + c_{11}\,n_2\,\partial_2 + c_{44}\,n_3\,\partial_3,$$
$$\mathcal{T}_{23}(\partial,n) = c_{13}\,n_2\,\partial_3 + c_{44}\,n_3\,\partial_2,$$
$$\mathcal{T}_{24}(\partial,n) = -\widetilde{\gamma}_1\,n_2, \quad \mathcal{T}_{25}(\partial,n) = e_{14}\,n_3\,\partial_1,$$
$$\mathcal{T}_{31}(\partial,n) = c_{44}\,n_1\,\partial_3 + c_{13}\,n_3\,\partial_1, \quad \mathcal{T}_{32}(\partial,n) = c_{44}\,n_2\,\partial_3 + c_{13}\,n_3\,\partial_2,$$
$$\mathcal{T}_{33}(\partial,n) = c_{44}\,n_1\,\partial_1 + c_{44}\,n_2\,\partial_2 + c_{33}\,n_3\,\partial_3,$$
$$\mathcal{T}_{34}(\partial,n) = -\widetilde{\gamma}_3\,n_3, \quad \mathcal{T}_{35}(\partial,n) = 0,$$
$$\mathcal{T}_{4j}(\partial,n) = 0 \ \text{ for } \ j = 1,2,3,5,$$
$$\mathcal{T}_{44}(\partial,n) = \varkappa_{11}\,(n_1\,\partial_1 + n_2\,\partial_2) + \varkappa_{33}\,n_3\,\partial_3,$$
$$\mathcal{T}_{51}(\partial,n) = e_{14}\,n_2\,\partial_3, \quad \mathcal{T}_{52}(\partial,n) = -e_{14}\,n_1\,\partial_3,$$
$$\mathcal{T}_{53}(\partial,n) = e_{14}\,(n_2\,\partial_1 - n_1\,\partial_2), \quad \mathcal{T}_{54}(\partial,n) = -g_3\,n_3,$$
$$\mathcal{T}_{55}(\partial,n) = \varepsilon_{11}\,(n_1\,\partial_1 + n_2\,\partial_2) + \varepsilon_{33}\,n_3\,\partial_3.$$

The material constants satisfy the following inequalities

$$c_{11} > |c_{12}|, \ \ c_{44} > 0, \ \ c_{66} > 0, \ \ c_{33}(c_{11}+c_{12}) > 2c_{13}^2,$$
$$\varepsilon_{11} > 0, \ \ \varepsilon_{33} > 0, \ \ \varkappa_{11} > 0, \ \ \varkappa_{33} > 0, \ \ \frac{\varepsilon_{33}\alpha}{T_0} > g_3^2,$$

which follow from the positive definiteness of the internal energy form (see Subsection 1.3).

Further, we assume that the domain $\Omega^{(m)}$ is occupied by an isotropic material modeled by the Lamé equations

$$\mu\Delta u^{(m)} + (\lambda^{(m)} + \mu^{(m)})\,\mathrm{grad}\,\mathrm{div}\,u^{(m)} - \gamma^{(m)}\,\mathrm{grad}\,\vartheta^{(m)} - \varrho^{(m)}\tau^2 u^{(m)} = 0,$$
$$\Delta\vartheta^{(m)} - \tau\alpha^{(m)}\vartheta^{(m)} - \tau T_0^{(m)} u^{(m)} = 0,$$
$$\mu^{(m)} > 0, \quad 3\lambda^{(m)} + 2\mu^{(m)} > 0, \quad \gamma^{(m)} > 0, \quad \alpha^{(m)} > 0.$$

Furthermore, we assume that the interface crack edge $\partial\Gamma_C^{(m)}$ is parallel to the plane of isotropy (i.e., to the plane $x_3 = 0$). In this case the symbol matrix $\mathfrak{S}_2(x', 0, \pm 1)$ is calculated explicitly and has the form $\mathfrak{S}_2(x', 0, \pm 1) = [\,D_{kj}^{\pm}\,]_{5\times 5}$, where

$$D_{12}^{\pm} = D_{21}^{\pm} = D_{13}^{\pm} = D_{31}^{\pm} = D_{14}^{\pm} = D_{41}^{\pm} = D_{24}^{\pm} = D_{42}^{\pm} =$$
$$= D_{25}^{\pm} = D_{52}^{\pm} = D_{34}^{\pm} = D_{43}^{\pm} = D_{35}^{\pm} = D_{53}^{\pm} = D_{45}^{\pm} = D_{54}^{\pm} = 0,$$
$$D_{11}^{\pm} = \frac{2a_{11}}{b_1^*} + \frac{1}{\mu^{(m)}}, \quad D_{15}^{\pm} = \pm\frac{i4a_{11}A_{15}}{b_1^*}, \quad D_{22}^{\pm} = \frac{2a_{22}}{b_2^*} + a^{(m)},$$
$$D_{23}^{\pm} = \pm\frac{i4a_{22}A_{23}}{b_2^*} \pm i\,b^{(m)}, \quad D_{32}^{\pm} = \pm\frac{i4a_{33}A_{32}}{b_2^*} \mp i\,b^{(m)},$$
$$D_{33}^{\pm} = \frac{2a_{33}}{b_2^*} + a^{(m)}, \quad D_{44}^{\pm} = -2a_{44} + 1,$$
$$D_{51}^{\pm} = \pm\frac{i4a_{55}A_{51}}{b_1^*}, \quad D_{55}^{\pm} = \frac{2a_{55}}{b_1^*},$$

with

$$A_{15} = \frac{e_{14}\,c_{66}\,(b_1 - b_2)}{2\,b_1\,b_2\sqrt{B}}\,, \quad A_{51} = \frac{e_{14}\,\varepsilon_{33}\,(b_1 - b_2)}{2\sqrt{B}}\,,$$
$$b_1 = \sqrt{\frac{A - \sqrt{B}}{2\,c_{44}\,\varepsilon_{33}}}\,, \quad b_2 = \sqrt{\frac{A + \sqrt{B}}{2\,c_{44}\,\varepsilon_{33}}}\,,$$
$$a_{11} = \frac{(b_1 - b_2)(\varepsilon_{11} + \varepsilon_{33}b_1b_2)}{2b_1b_2\sqrt{B}}\,, \quad a_{44} = -\frac{1}{2\sqrt{\varkappa_{11}\varkappa_{33}}}\,,$$
$$a_{55} = \frac{(b_1 - b_2)(c_{66} + c_{44}b_1b_2)}{2b_1b_2\sqrt{B}}\,, \quad d_1 = \sqrt{\frac{C - \sqrt{D}}{2\,c_{44}\,c_{33}}}\,, \quad d_2 = \sqrt{\frac{C + \sqrt{D}}{2\,c_{44}\,c_{33}}}\,,$$

$$a_{22} = \begin{cases} \dfrac{(d_1 - d_2)\,(c_{44} + c_{33}\,d_1 d_2)}{2\,d_1\,d_2\sqrt{D}} & \text{for } D > 0, \\ -\dfrac{a\sqrt{c_{33}}}{\sqrt{-D}\sqrt{c_{11}}}(c_{44} + \sqrt{c_{11}}\sqrt{c_{33}}) & \text{for } D < 0, \end{cases}$$

$$a_{33} = \begin{cases} \dfrac{(d_1 - d_2)\,(c_{11} + c_{44}\,d_1\,d_2)}{2\,d_1\,d_2\sqrt{D}} & \text{for } D > 0, \\ -\dfrac{a}{\sqrt{-D}}\,(c_{44} + \sqrt{c_{11}}\,\sqrt{c_{33}}) & \text{for } D < 0, \end{cases}$$

$$A_{23} = \begin{cases} \dfrac{c_{44}\,(d_2 - d_1)\,(c_{11} - c_{13}\,d_1 d_2)}{2\,d_1\,d_2\sqrt{D}} & \text{for } D > 0, \\ \dfrac{a\,c_{44}(\sqrt{c_{11}\,c_{33}} - c_{13})}{\sqrt{-D}} & \text{for } D < 0, \end{cases}$$

$$A_{32} = \begin{cases} -\dfrac{c_{44}\,(d_2 - d_1)\,(c_{33}\,d_1\,d_2 - c_{13})}{2\,d_1\,d_2\sqrt{D}} & \text{for } D > 0, \\ -\dfrac{a\,c_{44}(\sqrt{c_{11}\,c_{33}} - c_{13})\sqrt{c_{33}}}{\sqrt{-D}\sqrt{c_{11}}} & \text{for } D < 0, \end{cases}$$

$$a^{(m)} = \frac{2(\lambda^{(m)} + 2\mu^{(m)})\mu^{(m)}}{\lambda^{(m)} + \mu^{(m)}}\,, \quad b^{(m)} = \frac{(\mu^{(m)})^2}{\lambda^{(m)} + 3\mu^{(m)}}\,,$$

$$a = \frac{1}{2}\sqrt{\frac{-C + 2c_{44}\sqrt{c_{11}c_{33}}}{c_{44}c_{33}}} > 0\,, \quad A = e_{14}^2 + c_{44}\,\varepsilon_{11} + c_{66}\,\varepsilon_{33} > 0,$$

$$B = A^2 - 4\,c_{44}\,c_{66}\,\varepsilon_{11}\,\varepsilon_{33} > 0, \quad b_1^* = -4A_{15}A_{51} - 1 < 0,$$

$$b_2^* = -4A_{32}A_{23} - 1 < 0, \quad D = C^2 - 4\,c_{44}^2\,c_{33}c_{11},$$

$$C = c_{11}\,c_{33} - c_{13}^2 - 2\,c_{13}\,c_{44}.$$

It is easy to see that

$$a_{33}A_{32} = -a_{22}A_{23}, \ \ a_{11}A_{15} = a_{55}A_{51}, \ \ A > \sqrt{B},$$
$$A_{15}\,A_{51} > 0, \ \ A_{23}\,A_{32} < 0.$$

The characteristic polynomial of the matrix $\mathcal{D}_2(x')$ for $x' \in \partial\Gamma_C^{(m)}$ can be represented in the following form

$$\det \begin{bmatrix} \kappa_1\,d_{11} & 0 & 0 & 0 & \kappa_2\,d_{15} \\ 0 & \kappa_1\,d_{22} & \kappa_2\,d_{23} & 0 & 0 \\ 0 & -\kappa_2\,d_{23} & \kappa_1\,d_{33} & 0 & 0 \\ 0 & 0 & 0 & \kappa_1\,d_{44} & 0 \\ \kappa_2\,d_{15} & 0 & 0 & 0 & \kappa_1\,d_{55} \end{bmatrix}, \tag{5.69}$$

where

$$\kappa_1 = 1 - \lambda, \;\; \kappa_2 = -1 - \lambda,$$
$$d_{11} = \frac{2a_{11}}{b_1^*} + \frac{1}{\mu^{(m)}}\,, \;\; d_{15} = \frac{i\,4a_{11}A_{15}}{b_1^*}\,, \;\; d_{22} = \frac{2a_{22}}{b_2^*} + a^{(m)}\,,$$
$$d_{23} = \frac{i\,4a_{22}A_{23}}{b_2^*} + i\,b^{(m)}\,, \;\; d_{33} = \frac{2a_{33}}{b_2^*} + a^{(m)}\,,$$
$$d_{44} = -2a_{44} + 1, \;\; d_{55} = \frac{2a_{55}}{b_1^*}\,.$$

It can easily be verified that we have the following expressions for the eigenvalues of the matrix $\mathcal{D}_2(x')$ (i.e., the roots of the polynomial (5.69) with respect to λ):

$$\lambda_1 = \frac{1-i\,p}{1+i\,p}\,, \;\; \lambda_2 = \frac{1}{\lambda_1}\,, \;\; \lambda_3 = \frac{1-q}{1+q}\,, \;\; \lambda_4 = \frac{1}{\lambda_3}\,, \;\; \lambda_5 = 1,$$

where

$$p = \frac{|d_{15}|}{\sqrt{d_{11}\,d_{55}}} > 0, \quad q = \frac{|d_{23}|}{\sqrt{d_{22}\,d_{33}}} > 0\,.$$

Note that $|\lambda_1| = |\lambda_2| = 1$, $\lambda_3 > 0$ and $\lambda_4 > 0$.

Applying the above results we can write the exponents of the first dominant terms of the asymptotic expansions of solutions explicitly (see (5.64)–(5.66))

$$\gamma_1 = \frac{1}{2} + \frac{1}{2\pi}\arg\lambda_1 = \frac{1}{2} + \frac{1}{2\pi}\big[\arg(1-i\,p) - \arg(1+i\,p)\big] = \frac{1}{2} - \frac{1}{\pi}\arctan p\,,$$
$$\delta_1 = 0, \quad \gamma_2 = \frac{1}{2} + \frac{1}{\pi}\arctan p\,, \;\; \delta_2 = 0; \quad \gamma_3 = \gamma_4 = \frac{1}{2}\,,$$
$$\delta_3 = -\delta_4 = \widetilde{\delta} = \frac{1}{2\pi}\,\ln\frac{1-q}{1+q}\,, \quad \gamma_5 = \frac{1}{2}\,, \;\; \delta_5 = 0.$$

Clearly, $0 < \gamma_1 < 1/2$ and $1/2 < \gamma_2 < 1$ and we can draw the following conclusions:

(1) In view of Proposition 5.5, solutions of the problems under consideration have the following asymptotic behaviour near the curve $\partial\Gamma_C^{(m)}$

$$(u,\varphi)^\top = c_0\,r^{\gamma_1} + c_1\,r^{\frac{1}{2}+i\,\widetilde{\delta}} + c_2\,r^{\frac{1}{2}-i\,\widetilde{\delta}} + c_3\,r^{\frac{1}{2}} + c_4\,r^{\gamma_2} + \cdots\,,$$
$$\vartheta = \widetilde{b}_0\,r^{\frac{1}{2}} + \widetilde{b}_1\,r^{\gamma_2} + \cdots\,, \tag{5.70}$$
$$u^{(m)} = c_0^{(m)}\,r^{\gamma_1} + c_1^{(m)}\,r^{\frac{1}{2}+i\,\widetilde{\delta}} + c_2^{(m)}\,r^{\frac{1}{2}-i\,\widetilde{\delta}} + {}$$
$$+ c_3^{(m)}\,r^{\frac{1}{2}} + c_4^{(m)}\,r^{\gamma_2} + \cdots\,,$$
$$\vartheta^{(m)} = -b_0^{(m)}\,r^{\frac{1}{2}} + b_1^{(m)}\,r^{\gamma_2} + \cdots\,, \tag{5.71}$$

where γ_1, γ_2 and $\widetilde{\delta}$ are defined above.

(2) As we can see, the exponent γ_1, characterizing the behaviour of U and $U^{(m)}$ near the line $\partial\Gamma_C^{(m)}$, belongs to the interval (0, 1/2). It depends only on the elastic constants c_{44}, c_{66}, $\mu^{(m)}$, dielectric constants ε_{11}, ε_{33}, and piezoelectric constant e_{14}, and does not depend on the thermal constants.

(3) In the above asymptotic expansions, the first five terms of u, φ, and $u^{(m)}$ and the first two terms of ϑ and $\vartheta^{(m)}$ do not contain logarithmic factors due to the equality $B_0(t) = I$.

(4) Since $\gamma_1 < 1/2$, there do not appear oscillating singularities for physical fields in some vicinity of the curve $\partial\Gamma_C^{(m)}$. Recall that in the classical elasticity theory (for both isotropic and anisotropic solids) for interface crack problems the dominant exponents are 1/2 and $1/2 \pm i\,\beta$ with $\beta \neq 0$ and, consequently, the corresponding stress tensor possesses oscillating singularities, in general.

(5) In the considered case,

$$B_0(t) = I, \quad l_0^{(m)} = 1, \quad n_1^{(m)} = 4,$$
$$\tau_{1,\mu}^{(m)} = -i, \quad l_0 = 5, \quad n_s = 1, \quad s = \overline{1,5},$$
$$\tau_{1,\mu} = -ib_1, \quad \tau_{2,\mu} = -ib_2, \quad \tau_{3,\mu} = -id_1,$$
$$\tau_{4,\mu} = -id_2, \quad \tau_{5,\mu} = -i\sqrt{\kappa_{11}/\kappa_{33}}, \quad \mu = \pm 1.$$

Note, that if $D > 0$, then the roots $\tau_{3\mu}$ and $\tau_{4\mu}$ are pure imaginary. For $D < 0$ the roots are complex numbers with opposite real parts and equal imaginary parts (see [8]). Therefore, in view of Remark 5.6 the coefficients of the dominant terms in the asymptotic expansions (5.70) and (5.71) read as

$$c_0 = c_0(x', \alpha) = \sum_{\mu=\pm1} \sum_{s=1}^{5} c_0^{(1)}(x') c_{s0\mu}^{(1)}(x', \alpha), \tag{5.72}$$

$$\widetilde{b}_0 = \widetilde{b}_0(x', \alpha) = \sum_{\mu=\pm1} \sum_{s=1}^{5} c_0^{(5)}(x') c_{s0\mu}^{(45)}(x', \alpha), \tag{5.73}$$

$$c_0^{(m)} = c_0^{(m)}(x', \alpha) = \sum_{\mu=\pm1} \sum_{j=0}^{3} c_j^{(1)}(x') c_{1j\mu}^{((m),1)}(x', \alpha), \tag{5.74}$$

$$b_0^{(m)} = b_0^{(m)}(x', \alpha) = \sum_{\mu=\pm1} \sum_{j=0}^{3} c_j^{(5)}(x') c_{1j\mu}^{((m),45)}(x', \alpha), \tag{5.75}$$

where the vector $c_{s0\mu}^{(1)}$ is composed of the first three and fifth entries of the first column of the matrix $c_{s0\mu}$, while the vector $c_{1j\mu}^{((m),1)}$ is composed of the first three entries of the first column of the matrix $c_{1j\mu}^{(m)}$. The coefficients $c_j^{(1)}$ and $c_j^{(5)}$ are the first and the fifth components, respectively, of the vector c_j, $j = \overline{0,3}$. In our case they read as

$$c_j^{(1)}(x') = i^j\,\Gamma(j-\gamma_1)\,\Gamma(\gamma_1)\,\widetilde{b}_0^{(1)}(x'),$$
$$c_j^{(5)}(x') = \sqrt{\pi}\,i^j\,\Gamma(j-1/2)\,\widetilde{b}_0^{(5)}(x'),$$

where $\widetilde{b}_0^{(1)}$ and $\widetilde{b}_0^{(5)}$ are the first and the fifth components of the vector $\mathcal{D}^{-1}b_0$ respectively.

As we will see below, there exists a class of piezoelectric media for which the dominant stress singularity exponent near the line $\partial\Gamma_C^{(m)}$ does not depend on the material constants and equals to $-1/2$.

Let us consider the class of piezoelectric media with cubic anisotropy. The corresponding system of differential equations are:

$$\begin{gathered}
(c_{11}\,\partial_1^2 + c_{44}\,\partial_2^2 + c_{44}\,\partial_3^2)u_1 + (\,c_{12}+c_{44}\,)\,\partial_1\partial_2u_2 + \\
+(\,c_{12}+c_{44}\,)\,\partial_1\partial_3u_3 - \widetilde{\gamma}_1\,\partial_1\vartheta + 2e_{14}\,\partial_2\partial_3\varphi - \varrho\,\tau^2\,u_1 = F_1, \\
(\,c_{12}+c_{44}\,)\,\partial_2\partial_1u_1 + (\,c_{44}\,\partial_1^2 + c_{11}\,\partial_2^2 + c_{44}\,\partial_3^2\,)\,u_2 + \\
+(\,c_{12}+c_{44}\,)\,\partial_2\partial_3u_3 - \widetilde{\gamma}_1\partial_2\vartheta + 2e_{14}\partial_1\partial_3\varphi - \varrho\,\tau^2\,u_2 = F_2, \\
(\,c_{12}+c_{44}\,)\,\partial_3\partial_1u_1 + (\,c_{12}+c_{44}\,)\,\partial_3\partial_2u_2 + \\
+(\,c_{44}\,\partial_1^2 + c_{44}\,\partial_2^2 + c_{11}\,\partial_3^2\,)\,u_3 - \widetilde{\gamma}_3\,\partial_3\vartheta + 2e_{14}\partial_1\partial_2\varphi - \varrho\,\tau^2\,u_3 = F_3, \\
-\tau\,T_0\,(\,\widetilde{\gamma}_1\,\partial_1u_1 + \widetilde{\gamma}_1\,\partial_2u_2 + \widetilde{\gamma}_3\,\partial_3u_3\,) + \\
+(\,\varkappa_{11}\,\partial_1^2 + \varkappa_{11}\,\partial_2^2 + \varkappa_{33}\,\partial_3^2\,)\,\vartheta - \tau\,\alpha\,\vartheta + \tau\,T_0\,g_3\,\partial_3\varphi = F_4, \\
-2e_{14}\partial_2\partial_3u_1 - 2e_{14}\partial_1\partial_3u_2 - 2e_{14}\partial_1\partial_2u_3 - g_3\,\partial_3\vartheta + \\
+(\,\varepsilon_{11}\,\partial_1^2 + \varepsilon_{11}\,\partial_2^2 + \varepsilon_{11}\,\partial_3^2\,)\,\varphi = F_5,
\end{gathered} \tag{5.76}$$

where the elastic, piezoelectric and thermal constants involved in the governing equations satisfy the conditions:

$$\begin{gathered}
c_{11} > 0, \quad c_{44} > 0, \quad -\frac{1}{2} < \frac{c_{12}}{c_{11}} < 1, \\
\varepsilon_{11} > 0, \quad \frac{\varepsilon_{11}\alpha}{T_0} > g_3^2, \quad \varkappa_{11} > 0, \quad \varkappa_{33} > 0.
\end{gathered} \tag{5.77}$$

In this case, the matrix $\mathfrak{S}_2(x',\,0,\,\pm 1)$ is self-adjoint

$$\mathfrak{S}_2(x',0,\pm 1) =$$
$$= \begin{bmatrix}
-2a_{11} + [\mu^{(m)}]^{-1} & 0 & 0 & 0 & 0 \\
0 & Q_1 & \pm i\,Q_2 & 0 & 0 \\
0 & \mp i\,Q_2 & Q_1 & 0 & 0 \\
0 & 0 & 0 & -2a_{44}+1 & 0 \\
0 & 0 & 0 & 0 & -2a_{55}
\end{bmatrix},$$

where

$$Q_1 = \frac{2a_{22}}{b^*} + a^{(m)}, \quad Q_2 = \frac{4a_{22}A_{23}}{b^*} + b^{(m)}, \quad b^* = 4\,A_{23}^2 - 1,$$

$$a_{22} = \begin{cases} \dfrac{(d_1 - d_2)\,(c_{11} + c_{44})}{2\sqrt{D}} & \text{for } D > 0, \\ -\dfrac{a\,(c_{11} + c_{44})}{\sqrt{-D}} & \text{for } D < 0, \end{cases}$$

$$a_{11} = \frac{(b_1 - b_2)\varepsilon_{11}}{\sqrt{B}}, \quad a_{44} = -\frac{1}{2}\,\frac{\sqrt{\varkappa_{33}}}{\sqrt{\varkappa_{11}}}, \quad a_{55} = \frac{(b_1 - b_2)c_{44}}{\sqrt{B}},$$

$$A_{23} = \begin{cases} \dfrac{c_{44}\,(d_2 - d_1)(c_{11} - c_{12})}{2\sqrt{D}} & \text{for } D > 0, \\ \dfrac{c_{44}\,a\,(c_{11} - c_{12})}{2\sqrt{-D}} & \text{for } D < 0, \end{cases}$$

$$b_1 = \sqrt{\frac{A - \sqrt{B}}{2c_{44}\varepsilon_{11}}}, \quad b_2 = \sqrt{\frac{A + \sqrt{B}}{2c_{44}\varepsilon_{11}}},$$

$$d_1 = \sqrt{\frac{C - \sqrt{D}}{2c_{44}c_{11}}}, \quad d_2 = \sqrt{\frac{C + \sqrt{D}}{2c_{44}c_{11}}},$$

$$D = C^2 - 4\,c_{11}^2\,c_{44}^2, \quad a = \frac{1}{2}\sqrt{\frac{-C + 2c_{44}\sqrt{c_{11}}}{c_{44}c_{11}}} > 0,$$

$$A = 2c_{44}\varepsilon_{11} + 4e_{14}^2, \;\; B = A^2 - 4c_{44}^2\varepsilon_{11}^2, \;\; C = c_{11}^2 - c_{12}^2 - 2\,c_{12}\,c_{44}.$$

The eigenvalues of the matrix

$$\mathcal{D}_2(x') = \big[\mathfrak{S}_2(x', 0, +1)\big]^{-1}\mathfrak{S}_2(x', 0, -1), \;\; x' \in \partial\Gamma_C^{(m)},$$

read as

$$\lambda_j = 1, \;\; j = 1, 2, 5; \;\; \lambda_3 = \frac{1 - q}{1 + q} > 0, \;\; \lambda_4 = \frac{1}{\lambda_3} > 0, \;\; \gamma_j = \frac{1}{2}, \;\; j = \overline{1, 5},$$

$$\delta_j = 0, \;\; j = 1, 2, 5, \;\; \delta_3 = -\delta_4 = \widetilde{\delta} = \frac{1}{2\pi}\,\ln\frac{1 - q}{1 + q},$$

$$q = \left|\frac{2a_{22}\,A_{23} + b^{(m)}b^*}{2a_{22} + a^{(m)}b^*}\right|.$$

The matrix $\mathcal{D}_2$ is self-adjoint and, consequently, is similar to a diagonal matrix, i.e., there is a unitary matrix $\mathcal{D}$ such that $\mathcal{D}\,\mathcal{D}_2\,\mathcal{D}^{-1}$ is diagonal. In turn, this implies that $B_0(t) = I$ and the leading terms of the asymptotic expansion near the curve $\partial\Gamma_C^{(m)}$ do not contain logarithmic factors. As a result we obtain the asymptotic expansion

$$\begin{aligned}(u,\varphi)^\top &= c_0\, r^{\frac12} + c_1\, r^{\frac12+i\,\widetilde{\delta}} + c_2\, r^{\frac12-i\,\widetilde{\delta}} + \cdots\,,\\ \vartheta &= b_0\, r^{\frac12} + \cdots\,,\\ u^{(m)} &= c_0^{(m)}\, r^{\frac12} + c_1^{(m)}\, r^{\frac12+i\,\widetilde{\delta}} + c_2^{(m)}\, r^{\frac12-i\,\widetilde{\delta}} + \cdots\,,\\ \vartheta^{(m)} &= b_0^{(m)}\, r^{\frac12} + \cdots\,,\end{aligned} \tag{5.78}$$

where the first coefficients have the same structure as in (5.72)–(5.75). Consequently, the solution is $C^{1/2}$-smooth in a one sided closed neighbourhood of the curve $\partial\Gamma_C^{(m)}$.

5.5. **Numerical results for stress singularity exponents.** The above analysis based on the asymptotic expansions of solutions (see [12,13]) shows that for sufficiently smooth boundary data (e.g., C^∞-smooth data say) the principal dominant singular terms of the solution vectors $U^{(m)}$ and U near the exceptional curves ∂S_D and $\partial\Gamma_T^{(m)}$ can be represented as a product of a smooth vector-function and a singular factor of the form $[\ln\varrho(x)]^{m_k-1}[\varrho(x)]^{\gamma_k+i\,\delta_k}$. Note that the crack edge $\partial\Gamma_C^{(m)}$ is a proper part of the curve $\partial\Gamma_T^{(m)}$. Here $\varrho(x)$ is the distance from a reference point x to the exceptional curves. Therefore, near these curves the dominant singular terms of the corresponding generalized stress vectors $\mathcal{T}^{(m)}U^{(m)}$ and $\mathcal{T}U$ are represented as a product of a smooth vector-function and the factor $[\ln\varrho(x)]^{m_k-1}[\varrho(x)]^{-1+\gamma_k+i\,\delta_k}$. The numbers δ_k are different from zero, in general, and display the oscillating character of the stress singularities.

The exponents $\gamma_k+i\,\delta_k$ and the corresponding eigenvalues of the matrices (4.23) are related by the equalities

$$\gamma_k = \frac12 + \frac{\arg\lambda_k}{2\pi}\,, \quad \delta_k = -\frac{\ln|\lambda_k|}{2\pi}\,.$$

Here either $\lambda_k \in \{\lambda_j^{(1)}(x)\}_{j=1}^5$ for $x\in\partial S_D$ or $\lambda_k\in\{\lambda_j^{(2)}(x)\}_{j=1}^5$ for $x\in\partial\Gamma_T^{(m)}$. In the above expressions the parameter m_k denotes the algebraic multiplicity of the eigenvalue λ_k. It is evident that at the exceptional curves the components of the generalized stress vector behave like $\mathcal{O}([\ln\varrho(x)]^{m_0-1}[\varrho(x)]^{-\frac12+\gamma'})$, where m_0 denotes the maximal multiplicity of the eigenvalues. This is a global singularity effect for the first order derivatives of the vectors $U^{(m)}$ and U. Note that γ_k, δ_k and γ' depend on the material parameters (see (5.36)-(5.39)). Moreover, γ' is non-positive and $\delta_k\neq 0$, in general. This is related to the fact that the eigenvalues λ_k are complex and $|\lambda_k|\neq 1$, in general.

For numerical calculations, we have considered particular cases when the domain $\Omega^{(m)}$ is occupied by the isotropic metallic material *silver-palladium alloy* whereas the domain Ω is occupied by one of the following piezoelectric materials: $BaTiO_3$ (with the crystal symmetry of the class **4mm**), PZT-4 and PZT-5A (with the crystal symmetry of the class **6mm**). Calculations have shown that the parameters γ'_k and γ''_k depend on the material parameters. In particular, $\gamma'_k = -\gamma''_k$ and we have the following values for them

	$BaTiO_3$	PZT-4	PZT-5A
γ'_1	-0.12	-0.12	-0.13
γ'_2	-0.06	-0.08	-0.09

(5.79)

Therefore, for $\gamma' := \min\{\gamma'_1, \gamma'_2\}$ we have (see (5.36)-(5.39))

	$BaTiO_3$	PZT-4	PZT-5A
γ'	-0.12	-0.12	-0.13

Consequently, if the boundary data of the transmission problem under consideration are sufficiently smooth (e.g., satisfy the conditions of Theorem 5.4. iii with $\alpha > 0.5$), then for the Hölder smoothness exponent κ, involved in Theorem 5.4.iii, we derive

	$BaTiO_3$	PZT-4	PZT-5A
κ	0.38	0.38	0.37

Thus, in the closed domains the solution vectors have $C^{\kappa-\delta}$-smoothness , where $\delta > 0$ is an arbitrarily small number. This shows that the Hölder smoothness exponents depend on the material parameters. Moreover, for these particular cases, from the table (5.79) it follows that $\gamma'_1 < \gamma'_2$, which yields that the stress singularities at the curve ∂S_D are higher than the singularities near the curve $\partial\Gamma_T^{(m)}$.

The graphs presented below show the significant influence of the piezoelectric constants on the stress singularity exponents and on the oscillating stress singularity effects. We have calculated the deviation $\gamma^{(j)}$ of the stress singularity exponents γ'_j from the value $-0,5$ (the value for the materials without piezoelectric properties): $\gamma^{(j)} = |-0,5-\gamma'_j|,\ j = 1,2$, and parameters $\delta^{(1)} = \max\limits_{1\leqslant k\leqslant 5}\ \sup\limits_{x\in\partial S_D} |\delta_k|$ and $\delta^{(2)} = \max\limits_{1\leqslant k\leqslant 5}\ \sup\limits_{x\in\partial\Gamma_T^{(m)}} |\delta_k|$, which determine stress oscillating singularity effects at the exceptional curves ∂S_D and $\partial\Gamma_T^{(m)}$ respectively. We carried out calculations for PZT-4 with the constants te_{kj} instead of e_{kj}, where $1 < t < 2$. The corresponding graphs are presented in Figures 2 and 3.

We see that the stress singularity exponents essentially depend on the piezoelectric constants. In particular, when the piezoelectric constants are sufficiently small, the stress singularity exponents are equal to -0.5, similar to the materials without piezoelectric properties. Starting from some threshold value of t (which is different for ∂S_D and $\partial\Gamma_T^{(m)}$) the stress singularities differ from -0.5 and simultaneously we have no oscillating singularities any

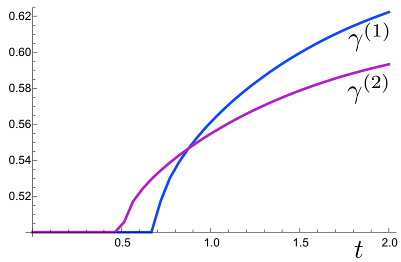

FIGURE 2

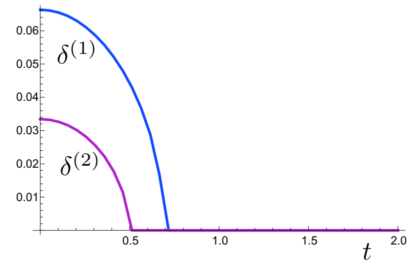

FIGURE 3

more (see Figures 4 and 5). Note that the threshold value of the parameter t corresponding to the curve $\partial\Gamma_T^{(m)}$ is smaller than the threshold value corresponding to the curve ∂S_D. However, when t grows, the stress singularity exponent near the curve ∂S_D increases more rapidly and starting from some value of the parameter t it exceeds the stress singularity exponent corresponding to the curve $\partial\Gamma_T^{(m)}$.

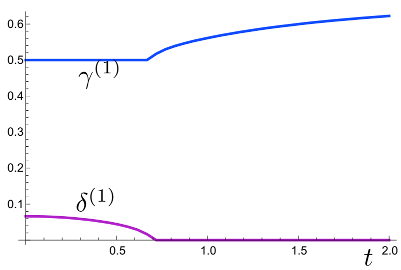

FIGURE 4

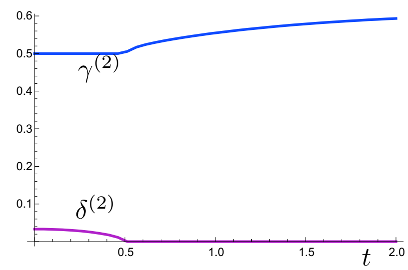

FIGURE 5

The graphs presented below in Figures 6-9 show the dependence of the stress singularity exponents and the oscillation parameters $\delta^{(1)}$ and $\delta^{(2)}$ on the angle $\beta\pi$ between the symmetry axis of the piezoelectric material and the normal of surface at the reference point $x \in \partial S_D \cup \partial\Gamma_T^{(m)}$. As we see, the stress singularity exponents essentially depend on the angle $\beta\pi \in [0, 2\pi]$ as well.

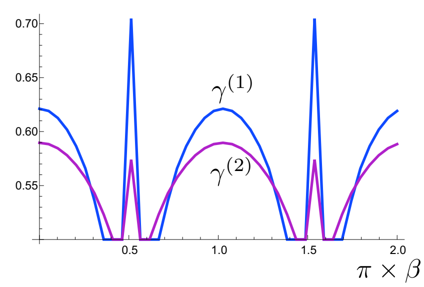

FIGURE 6

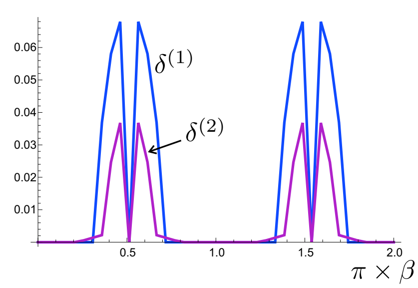

FIGURE 7

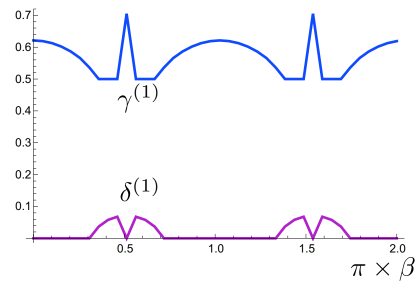

FIGURE 8

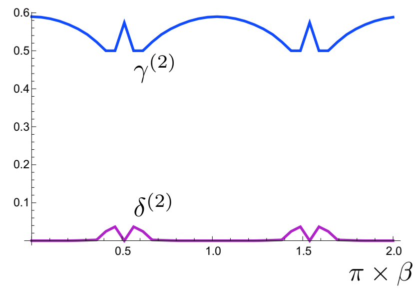

FIGURE 9

6. Existence and Regularity Results for Problem (ICP-B)

Here we will consider the interface crack problem (ICP-B), see (1.60)–(1.71). As we will see this problem is reduced to a more complicated, non-classical system of boundary pseudodifferential equations which needs a special analysis.

6.1. **Reduction to boundary integral equations.** For the data of the problem (ICP-B) we assume that

$$
\begin{aligned}
Q_j^{(m)} &\in B_{p,p}^{-\frac{1}{p}}(S_N^{(m)}), \quad j=\overline{1,4},\\
Q_k &\in B_{p,p}^{-\frac{1}{p}}(S_N), \quad f_k \in B_{p,p}^{\frac{1}{p'}}(S_D), \quad k=\overline{1,5},\\
f_l^{(m)} &\in B_{p,p}^{1/p'}(\Gamma_T^{(m)}), \quad l=1,2,3,\\
f_t^{(m)} &\in B_{p,p}^{\frac{1}{p'}}(\Gamma^{(m)}), \quad t=4,5,\\
F_l^{(m)} &\in B_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)}), \quad l=1,2,3,\\
F_4^{(m)} &\in B_{p,p}^{-\frac{1}{p}}(\Gamma^{(m)}),\\
\widetilde{Q}_l &\in B_{p,p}^{-\frac{1}{p}}(\Gamma_C^{(m)}), \quad l=1,2,3,\\
\widetilde{Q}_l^{(m)} &\in B_{p,p}^{-\frac{1}{p}}(\Gamma_C^{(m)}), \quad l=1,2,3,
\end{aligned} \tag{6.1}
$$

Further, let

$$
\begin{aligned}
G_l &:= \begin{cases} Q_l & \text{on } S_N, \\ \widetilde{Q}_l & \text{on } \Gamma_C^{(m)}, \end{cases} \quad l=1,2,3,\\
G_l^{(m)} &:= \begin{cases} Q_l^{(m)} & \text{on } S_N^{(m)}, \\ \widetilde{Q}_l^{(m)} & \text{on } \Gamma_C^{(m)}, \end{cases} \quad l=1,2,3,\\
G_t &:= Q_t \ \text{ on } S_N, \quad t=4,5,\\
G_4^{(m)} &:= Q_4^{(m)} \text{ on } S_N^{(m)}.
\end{aligned} \tag{6.2}
$$

Denote by

$$G_{0k} \in B_{p,p}^{-\frac{1}{p}}(\partial\Omega), \quad k=\overline{1,5}, \quad \text{and} \quad G_{0j}^{(m)} \in B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(m)}), \quad j=\overline{1,4},$$

some fixed extensions of the functions G_k and $G_j^{(m)}$ respectively onto $\partial\Omega$ and $\partial\Omega^{(m)}$ preserving the space,

$$\begin{aligned} G_0 &:= (G_{01}, \dots, G_{05})^\top \in \big[B_{p,p}^{-\frac{1}{p}}(\partial\Omega) \big]^5, \\ G_0^{(m)} &:= (G_{01}^{(m)}, \dots, G_{04}^{(m)})^\top \in \big[B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(m)}) \big]^4. \end{aligned} \tag{6.3}$$

It is clear that arbitrary extensions G_j^* and $G_j^{(m)*}$ of the same functions can be represented then as

$$G_k^* = G_{0k} + \psi_k + h_k, \quad k=\overline{1,5}, \quad G_j^{(m)*} = G_{0j}^{(m)} + h_j^{(m)}, \quad j=\overline{1,4}, \tag{6.4}$$

where

$$\psi_k \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(S_D), \quad k=\overline{1,5}, \quad h_l \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)}), \quad l=\overline{1,3}, \tag{6.5}$$

$$h_t \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma^{(m)}), \quad t=4,5, \quad h_l^{(m)} \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)}), \quad l=1,2,3, \tag{6.6}$$

$$h_4^{(m)} \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma^{(m)}),$$

are arbitrary functions. We set

$$\begin{aligned} \psi &:= (\psi_1, \dots, \psi_5)^\top \in \big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(S_D) \big]^5, \\ h &:= (h_1, \dots, h_5)^\top \in \big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)}) \big]^3 \times \big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma^{(m)}) \big]^2, \\ h^{(m)} &:= (h_1^{(m)}, \dots, h_4^{(m)})^\top \in \big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)}) \big]^3 \times \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma^{(m)}). \end{aligned} \tag{6.7}$$

As in the previous subsection, we develop here the indirect boundary integral equations method, and in accordance with Lemmata 4.6 and 4.8, we look for a solution pair $(U^{(m)}, U)$ of the interface crack problem (1.60)–(1.71) in the form of single layer potentials,

$$\begin{aligned} U^{(m)} &= (u^{(m)}, \dots, u_4^{(m)})^\top = \\ &= V_\tau^{(m)}\big(\big[\mathcal{P}_\tau^{(m)} \big]^{-1} \big[G_0^{(m)} + h^{(m)} \big] \big) \ \text{ in } \ \Omega^{(m)}, \end{aligned} \tag{6.8}$$

$$U = (u_1, \dots, u_5)^\top = V_\tau\big(\mathcal{P}_\tau^{-1} [G_0 + \psi + h] \big) \ \text{ in } \ \Omega, \tag{6.9}$$

where $\mathcal{P}_\tau^{(m)}$ and $\mathcal{P}_\tau$ are given by (4.37) and (4.41), G_0 and $G_0^{(m)}$ are the above introduced known vector-functions, and $h^{(m)}$, h and ψ are unknown vector-functions satisfying the inclusions (6.7).

By Lemmata 4.6, 4.8 and the property (4.40) we see that the homogeneous differential equations (1.60)-(1.61), the boundary conditions (1.62)–(1.63) and the crack conditions (1.70)–(1.71) are satisfied automatically.

The remaining boundary and transmission conditions (1.64)–(1.69) lead to the equations

$$r_{S_D}\big[\mathcal{H}_\tau\mathcal{P}_\tau^{-1}(G_0+\psi+h)\big]_k=f_k \ \text{ on } \ S_D, \ \ k=\overline{1,5}, \tag{6.10}$$

$$r_{\Gamma^{(m)}}\big[\mathcal{H}_\tau\mathcal{P}_\tau^{-1}(G_0+\psi+h)\big]_5=f_5^{(m)} \ \text{ on } \ \Gamma^{(m)}, \tag{6.11}$$

$$r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau\mathcal{P}_\tau^{-1}(G_0+\psi+)\big]_l-r_{\Gamma_T^{(m)}}\Big[\mathcal{H}_\tau^{(m)}[\mathcal{P}_\tau^{(m)}]^{-1}(G_0^{(m)}+h^{(m)})\Big]_l= \\ =f_l^{(m)} \ \text{ on } \ \Gamma_T^{(m)}, \ \ l=\overline{1,3}, \tag{6.12}$$

$$r_{\Gamma_T^{(m)}}\big[G_0+\psi+h\big]_l+r_{\Gamma_T^{(m)}}\big[G_0^{(m)}+h^{(m)}\big]_l= \\ =F_l^{(m)} \ \text{ on } \ \Gamma_T^{(m)}, \ \ l=\overline{1,3}, \tag{6.13}$$

$$r_{\Gamma^{(m)}}\big[\mathcal{H}_\tau\mathcal{P}_\tau^{-1}(G_0+\psi+h)\big]_4-r_{\Gamma^{(m)}}\Big[\mathcal{H}_\tau^{(m)}[\mathcal{P}_\tau^{(m)}]^{-1}(G_0^{(m)}+h^{(m)})\Big]_4= \\ =f_4^{(m)} \ \text{ on } \ \Gamma^{(m)}, \tag{6.14}$$

$$r_{\Gamma^{(m)}}\big[G_0+\psi+h\big]_4+r_{\Gamma^{(m)}}\big[G_0^{(m)}+h^{(m)}\big]_4=F_4^{(m)} \ \text{ on } \ \Gamma^{(m)}. \tag{6.15}$$

We can rewrite these system as the following simultaneous pseudodifferential equations with respect to the unknown vector-functions ψ, h and $h^{(m)}$:

$$r_{S_D}\big[\mathcal{H}_\tau\mathcal{P}_\tau^{-1}(\psi+h)\big]_k=\widetilde{f}_k \ \text{ on } \ S_D, \ \ k=\overline{1,5}, \tag{6.16}$$

$$r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau\mathcal{P}_\tau^{-1}(\psi+h)\big]_l-r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau^{(m)}[\mathcal{P}_\tau^{(m)}]^{-1}h^{(m)}\big]_l= \\ =\widetilde{f}_l^{(m)} \ \text{ on } \ \Gamma_T^{(m)}, \ \ l=\overline{1,3}, \tag{6.17}$$

$$r_{\Gamma^{(m)}}\big[\mathcal{H}_\tau\mathcal{P}_\tau^{-1}(\psi+h)\big]_4-r_{\Gamma^{(m)}}\Big[\mathcal{H}_\tau^{(m)}[\mathcal{P}_\tau^{(m)}]^{-1}h^{(m)}\Big]_4= \\ =\widetilde{f}_4^{(m)} \ \text{ on } \ \Gamma^{(m)}, \tag{6.18}$$

$$r_{\Gamma^{(m)}}\big[\mathcal{H}_\tau\mathcal{P}_\tau^{-1}(\psi+h)\big]_5=\widetilde{f}_5^{(m)} \ \text{ on } \ \Gamma^{(m)}, \tag{6.19}$$

$$r_{\Gamma_T^{(m)}}h_l^{(m)}+r_{\Gamma_T^{(m)}}h_l=\widetilde{F}_l^{(m)} \ \text{ on } \ \Gamma_T^{(m)}, \ \ l=\overline{1,3}, \tag{6.20}$$

$$r_{\Gamma^{(m)}}h_4^{(m)}+r_{\Gamma^{(m)}}h_4=\widetilde{F}_4^{(m)} \ \text{ on } \ \Gamma^{(m)}, \tag{6.21}$$

where

$$\widetilde{f}_k:=f_k-r_{S_D}\big[\mathcal{H}_\tau\mathcal{P}_\tau^{-1}G_0\big]_k\in B_{p,p}^{1-\frac{1}{p}}(S_D), \ \ k=\overline{1,5}, \tag{6.22}$$

$$\widetilde{f}_l^{(m)}:=f_l^{(m)}+r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau^{(m)}[\mathcal{P}_\tau^{(m)}]^{-1}G_0^{(m)}\big]_l- \\ -r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau\mathcal{P}_\tau^{-1}G_0\big]_l\in B_{p,p}^{1-\frac{1}{p}}(\Gamma_T^{(m)}), \ \ l=\overline{1,3}, \tag{6.23}$$

$$\widetilde{f}_4^{(m)}:=f_4^{(m)}+r_{\Gamma^{(m)}}\big[\mathcal{H}_\tau^{(m)}[\mathcal{P}_\tau^{(m)}]^{-1}G_0^{(m)}\big]_4- \\ -r_{\Gamma^{(m)}}\big[\mathcal{H}_\tau\mathcal{P}_\tau^{-1}G_0\big]_4\in B_{p,p}^{1-\frac{1}{p}}(\Gamma^{(m)}), \tag{6.24}$$

$$\widetilde{f}_5^{(m)}:=f_5^{(m)}-r_{\Gamma^{(m)}}\big[\mathcal{H}_\tau\mathcal{P}_\tau^{-1}G_0\big]_5\in B_{p,p}^{1-\frac{1}{p}}(\Gamma^{(m)}), \tag{6.25}$$

$$\widetilde{F}_l^{(m)} := F_l^{(m)} - r_{\Gamma_T^{(m)}} G_{0l} - r_{\Gamma_T^{(m)}} G_{0l}^{(m)} \in r_{\Gamma_T^{(m)}} \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)}), \quad l = \overline{1,3}, \tag{6.26}$$

$$\widetilde{F}_4^{(m)} := F_4^{(m)} - r_{\Gamma^{(m)}} G_{04} - r_{\Gamma^{(m)}} G_{04}^{(m)} \in r_{\Gamma^{(m)}} \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma^{(m)}). \tag{6.27}$$

The inclusions (6.26) and (6.27) are the *compatibility conditions* for Problem (ICP-B) due to the relations (6.7). Therefore, in what follows we assume that $\widetilde{F}_l^{(m)}$ and $\widetilde{F}_4^{(m)}$ are extended from $\Gamma_T^{(m)}$ and $\Gamma^{(m)}$, respectively, onto $\partial\Omega^{(m)}$ by zero, i.e., $\widetilde{F}_l^{(m)} \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)})$, $l = \overline{1,3}$, and $\widetilde{F}_4^{(m)} \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma^{(m)})$.

We employ the notation (5.18) to rewrite equations (6.16)–(6.21) as (see the remark after formula (5.25))

$$r_{S_D}\big[\mathcal{A}_\tau \psi\big]_k + r_{S_D}\big[\mathcal{A}_\tau h\big]_k = \widetilde{f}_k \ \text{ on } \ S_D, \ \ k = \overline{1,5}, \tag{6.28}$$

$$r_{\Gamma_T^{(m)}}\big[\mathcal{A}_\tau \psi\big]_l + r_{\Gamma_T^{(m)}}\big[(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})h\big]_l = \widetilde{g}_l^{(m)} \ \text{ on } \ \Gamma_T^{(m)}, \ \ l = 1,2,3, \tag{6.29}$$

$$r_{\Gamma^{(m)}}\big[\mathcal{A}_\tau \psi\big]_4 + r_{\Gamma^{(m)}}\big[(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})h\big]_4 = \widetilde{g}_4^{(m)} \ \text{ on } \ \Gamma^{(m)}, \tag{6.30}$$

$$r_{\Gamma^{(m)}}\big[\mathcal{A}_\tau \psi\big]_5 + r_{\Gamma^{(m)}}\big[\mathcal{A}_\tau h\big]_5 = \widetilde{g}_5^{(m)} \ \text{ on } \ \Gamma^{(m)}, \tag{6.31}$$

$$r_{\Gamma_T^{(m)}} h_l^{(m)} + r_{\Gamma_T^{(m)}} h_l = \widetilde{F}_l^{(m)} \ \text{ on } \ \Gamma_T^{(m)}, \ \ l = 1,2,3, \tag{6.32}$$

$$r_{\Gamma^{(m)}} h_4^{(m)} + r_{\Gamma^{(m)}} h_4 = \widetilde{F}_4^{(m)} \ \text{ on } \ \Gamma^{(m)}, \tag{6.33}$$

with

$$\begin{aligned}
&\widetilde{g}_l^{(m)} := \widetilde{f}_l^{(m)} + r_{\Gamma_T^{(m)}}\Big[\mathcal{H}_\tau^{(m)}[\mathcal{P}_\tau^{(m)}]^{-1}\widetilde{F}^{(m)}\Big]_l \in B_{p,p}^{1-\frac{1}{p}}(\Gamma_T^{(m)}), \ l = \overline{1,3},\\
&\widetilde{g}_4^{(m)} := \widetilde{f}_4^{(m)} + r_{\Gamma^{(m)}}\Big[\mathcal{H}_\tau^{(m)}[\mathcal{P}_\tau^{(m)}]^{-1}\widetilde{F}^{(m)}]_4 \in B_{p,p}^{1-\frac{1}{p}}(\Gamma^{(m)}),\\
&\widetilde{g}_5^{(m)} = \widetilde{f}_5^{(m)} \in B_{p,p}^{1-\frac{1}{p}}(\Gamma^{(m)}).
\end{aligned} \tag{6.34}$$

It is easy to see that the simultaneous equations (6.10)–(6.15) and (6.28)–(6.33), where the right hand sides are related by the equalities (6.22)–(6.27) and (6.34), are equivalent in the following sense: if the triplet

$$\begin{aligned}
(\psi, h, h^{(m)}) \in \big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(S_D)\big]^5 \times \big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)})\big]^3 \times\\
\times \big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma^{(m)})\big]^2 \times \big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T^{(m)})\big]^3 \times \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma^{(m)})
\end{aligned}$$

solves the system (6.28)–(6.33), then the pair $(G_0 + \psi + h, G_0^{(m)} + h^{(m)})$ solves the system (6.10)–(6.15), and vice versa.

Note that the above simultaneous equations are not classical systems of pseudodifferential equations since the sub-manifolds $\Gamma_T^{(m)}$ and $\Gamma_C^{(m)}$ are proper parts of $\Gamma^{(m)}$. We will discuss this problem in detail in the next subsection.

6.2. **Existence theorems for problem (ICP-B).** Here we show that the system of pseudodifferential equations (6.28)-(6.33) is uniquely solvable. To this end, let us denote by $\mathcal{N}_\tau^{(B)}$ the operator generated by the left hand side expressions of the equations (6.28)–(6.33),

$$\mathcal{N}_\tau^{(B)} :=$$

$$:= \begin{bmatrix} r_{S_D}\mathcal{A}_\tau & r_{S_D}\mathcal{A}_\tau & r_{S_D}[\,0\,]_{5\times 4} \\ r_{\Gamma_T^{(m)}}[(\mathcal{A}_\tau)_{l,k}]_{3\times 5} & r_{\Gamma_T^{(m)}}[(\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)})_{l,k}]_{3\times 5} & r_{\Gamma_T^{(m)}}[\,0\,]_{3\times 4} \\ r_{\Gamma^{(m)}}[(\mathcal{A}_\tau)_{t,k}]_{2\times 5} & r_{\Gamma^{(m)}}[(\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)})_{t,k}]_{2\times 5} & r_{\Gamma^{(m)}}[\,0\,]_{2\times 4} \\ r_{\Gamma_T^{(m)}}[\,0\,]_{3\times 5} & r_{\Gamma_T^{(m)}} I_{3\times 5} & r_{\Gamma_T^{(m)}} I_{3\times 4} \\ r_{\Gamma^{(m)}}[\,0\,]_{1\times 5} & r_{\Gamma^{(m)}} I_{1\times 5} & r_{\Gamma^{(m)}} I_{1\times 4} \end{bmatrix}_{14\times 14}, \qquad (6.35)$$

where the operators $\mathcal{A}_\tau$ and $\mathcal{B}_\tau^{(m)}$ are defined in (5.18), the subindexes involved in the block matrices take the following values $k=\overline{1,5}$, $l=1,2,3$, and $t=4,5$; the symbol $[\,0\,]_{N\times M}$ stands for the zero matrix of dimension $N\times M$, while

$$I_{3\times 5} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad I_{3\times 4} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$I_{1\times 5} = (0,\,0,\,0,\,1,\,0), \quad I_{1\times 4} = (0,\,0,\,0,\,1).$$

Further, let

$$\begin{aligned}
\mathbb{X}_{p,q}^s &:= \big[\,\widetilde{B}_{p,q}^s(S_D)\,\big]^5 \times \big[\,\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})\,\big]^3 \times \big[\,\widetilde{B}_{p,q}^s(\Gamma^{(m)})\,\big]^2 \times \\
&\quad \times \big[\,\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})\,\big]^3 \times \widetilde{B}_{p,q}^s(\Gamma^{(m)})\,, \\
\mathbb{Y}_{p,q}^s &:= \big[\,B_{p,q}^{s+1}(S_D)\,\big]^5 \times \big[\,B_{p,q}^{s+1}(\Gamma_T^{(m)})\,\big]^3 \times \big[\,B_{p,q}^{s+1}(\Gamma^{(m)})\,\big]^2 \times \\
&\quad \times \big[\,\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})\,\big]^3 \times \widetilde{B}_{p,q}^s(\Gamma^{(m)})\,, \\
\mathbb{X}_p^s &:= \big[\,\widetilde{H}_p^s(S_D)\,\big]^5 \times \big[\,\widetilde{H}_p^s(\Gamma_T^{(m)})\,\big]^3 \times \big[\,\widetilde{H}_p^s(\Gamma^{(m)})\,\big]^2 \times \\
&\quad \times \big[\,\widetilde{H}_p^s(\Gamma_T^{(m)})\,\big]^3 \times \widetilde{H}_p^s(\Gamma^{(m)})\,, \\
\mathbb{Y}_p^s &:= \big[\,H_p^{s+1}(S_D)\,\big]^5 \times \big[\,H_p^{s+1}(\Gamma_T^{(m)})\,\big]^3 \times \big[\,H_p^{s+1}(\Gamma^{(m)})\,\big]^2 \times \\
&\quad \times \big[\,\widetilde{H}_p^s(\Gamma_T^{(m)})\,\big]^3 \times \widetilde{H}_p^s(\Gamma^{(m)})\,.
\end{aligned}$$

Note that $\mathbb{X}_{2,2}^s = \mathbb{X}_2^s$ and $\mathbb{Y}_{2,2}^s = \mathbb{Y}_2^s$.

Employing the notation (6.2), we rewrite then the system (6.28)–(6.33) as follows

$$\mathcal{N}_\tau^{(B)}\,\Phi = Y, \qquad (6.36)$$

where the vector $\Phi := (\psi,\ h,\ h^{(m)})^\top \in \mathbb{X}_{p,p}^{-\frac{1}{p}}$ is unknown, while $Y := (\widetilde{f},\, \widetilde{g}^{(m)},\, \widetilde{F}^{(m)})^\top \in \mathbb{Y}_{p,p}^{-\frac{1}{p}}$ is a given vector with $\widetilde{f} = (\widetilde{f}_1,\, \dots,\, \widetilde{f}_5)^\top$, $\widetilde{g}^{(m)} := (\widetilde{g}_1^{(m)},\, \dots,\, \widetilde{g}_5^{(m)})^\top$, and $\widetilde{F}^{(m)} := (\widetilde{F}_1^{(m)},\, \dots,\, \widetilde{F}_4^{(m)})^\top$.

In accordance with Theorems 4.2, 4.3, and Lemma 4.7 we have the following mapping properties

$$\mathcal{N}_\tau^{(B)} \;:\; \mathbb{X}_p^s \to \mathbb{Y}_p^s \quad \big[\mathbb{X}_{p,q}^s \to \mathbb{Y}_{p,q}^s\big], \quad s\in\mathbb{R}, \;\; 1<p<\infty, \;\; 1\leqslant q\leqslant\infty. \tag{6.37}$$

Our goal is to establish Fredholm properties and invertibility of the operator (6.37). To this end, first we prove the following lemma.

Lemma 6.1. *The operator*

$$\mathcal{N}_\tau^{(B)} \;:\; \mathbb{X}_2^{-\frac{1}{2}} \to \mathbb{Y}_2^{-\frac{1}{2}} \tag{6.38}$$

is invertible.

Proof. We prove the theorem in several steps. First we show that the operator (6.38) is Fredholm with zero index and afterwards we establish that the corresponding null-space is trivial.

Step 1. First of all let us remark that the operators

$$\begin{aligned} r_{S_D}\mathcal{A}_\tau \;&:\; \big[\widetilde{H}_2^{-\frac{1}{2}}(\Gamma^{(m)})\big]^5 \to \big[H_2^{\frac{1}{2}}(S_D)\big]^5, \\ r_{\Gamma^{(m)}}\mathcal{A}_\tau \;&:\; \big[\widetilde{H}_2^{-\frac{1}{2}}(S_D)\big]^5 \to \big[H_2^{\frac{1}{2}}(\Gamma^{(m)})\big]^5, \end{aligned} \tag{6.39}$$

are compact since S_D and $\Gamma^{(m)}$ are disjoint, $\overline{S_D}\cap\overline{\Gamma^{(m)}} = \varnothing$. Therefore the operator

$$\mathcal{N}_\tau^{(B,0)} :=$$

$$:= \begin{bmatrix} r_{S_D}\,\mathcal{A}_\tau & r_{S_D}\,[\,0\,]_{5\times5} & r_{S_D}\,[\,0\,]_{5\times4} \\ r_{\Gamma_T^{(m)}}\,[\,0\,]_{3\times5} & r_{\Gamma_T^{(m)}}[(\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)})_{l,k}]_{3\times5} & r_{\Gamma_T^{(m)}}\,[\,0\,]_{3\times4} \\ r_{\Gamma^{(m)}}\,[\,0\,]_{2\times5} & r_{\Gamma^{(m)}}[(\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)})_{t,k}]_{2\times5} & r_{\Gamma^{(m)}}\,[\,0\,]_{2\times4} \\ r_{\Gamma_T^{(m)}}\,[\,0\,]_{3\times5} & r_{\Gamma_T^{(m)}}\,I_{3\times5} & r_{\Gamma_T^{(m)}}\,I_{3\times4} \\ r_{\Gamma^{(m)}}\,[\,0\,]_{1\times5} & r_{\Gamma^{(m)}}\,I_{1\times5} & r_{\Gamma^{(m)}}\,I_{1\times4} \end{bmatrix}_{14\times14} \tag{6.40}$$

is a compact perturbation of the operator $\mathcal{N}_\tau^{(B)}$. As above, here $k=\overline{1,5}$, $l=1,2,3$, and $t=4,5$. More precisely, the operator

$$\mathcal{N}_\tau^{(B)} - \mathcal{N}_\tau^{(B,0)} \;:\; \mathbb{X}_2^{-\frac{1}{2}} \to \mathbb{Y}_2^{-\frac{1}{2}} \tag{6.41}$$

is compact. Clearly the operator $\mathcal{N}_\tau^{(B,0)}$ has the following mapping property

$$\mathcal{N}_\tau^{(B,0)} \;:\; \mathbb{X}_2^{-\frac{1}{2}} \to \mathbb{Y}_2^{-\frac{1}{2}}. \tag{6.42}$$

Further, as we have shown in the proof of Theorem 5.1, the operator

$$r_{S_D}\mathcal{A}_\tau \;:\; \big[\widetilde{H}_2^{-\frac{1}{2}}(S_D)\big]^5 \to \big[H_2^{\frac{1}{2}}(S_D)\big]^5 \tag{6.43}$$

is invertible. Therefore, in view of (6.40), it remains to investigate the operator

$$\mathcal{N}_\tau^{(2)} \;:\; \widetilde{\mathbb{H}}_2^{-\frac{1}{2}} \to \mathbb{H}_2^{\frac{1}{2}}, \tag{6.44}$$

where

$$\begin{aligned} \widetilde{\mathbb{H}}_2^{-\frac{1}{2}} &:= \big[\, \widetilde{H}_2^{-\frac{1}{2}}(\Gamma_T^{(m)})\,\big]^3 \times \big[\, \widetilde{H}_2^{-\frac{1}{2}}(\Gamma^{(m)})\,\big]^2, \\ \mathbb{H}_2^{\frac{1}{2}} &:= \big[\, H_p^{\frac{1}{2}}(\Gamma_T^{(m)})\,\big]^3 \times \big[\, H_2^{\frac{1}{2}}(\Gamma^{(m)})\,\big]^2, \end{aligned} \tag{6.45}$$

and

$$\mathcal{N}_\tau^{(2)} := \begin{bmatrix} r_{_{\Gamma_T^{(m)}}}[(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})_{l,k}]_{3\times 5} \\ r_{_{\Gamma^{(m)}}}[(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})_{t,k}]_{2\times 5} \end{bmatrix}_{5\times 5} \tag{6.46}$$

with $k = \overline{1,5}$, $l = 1,2,3$, and $t = 4,5$.

In what follows, on the basis of the Lax–Milgram theorem, we show that the operator (6.44) is invertible. This is equivalent to the unique solvability of the simultaneous equations:

$$\mathcal{N}_\tau^{(2)}\, h = f^*, \tag{6.47}$$

or componentwise

$$\begin{aligned} r_{_{\Gamma_T^{(m)}}}\big[\,(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})\, h\,\big]_l &= f_l^* \ \text{ on } \ \Gamma_T^{(m)}, \ \ l = 1,2,3, \\ r_{_{\Gamma^{(m)}}}\big[\,(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})\, h\,\big]_t &= f_t^* \ \text{ on } \ \Gamma^{(m)}, \ \ t = 4,5, \end{aligned} \tag{6.48}$$

where

$$h = (h_1,\dots,h_5) \in \widetilde{\mathbb{H}}_2^{-\frac{1}{2}} = \big[\, \widetilde{H}_2^{-\frac{1}{2}}(\Gamma_T^{(m)})\,\big]^3 \times \big[\, \widetilde{H}_2^{-\frac{1}{2}}(\Gamma^{(m)})\,\big]^2 \tag{6.49}$$

is unknown and

$$f^* = (f_1^*,\cdots,f_5^*) \in \mathbb{H}_2^{\frac{1}{2}} = \big[\, H_2^{\frac{1}{2}}(\Gamma_T^{(m)})\,\big]^3 \times \big[\, H_2^{\frac{1}{2}}(\Gamma^{(m)})\,\big]^2 \tag{6.50}$$

is an arbitrary right hand side.

Step 2. Here first we show that the operator (6.44) is injective. Indeed, let $h \in \widetilde{\mathbb{H}}_2^{-\frac{1}{2}}$ be a solution to the homogeneous equation $\mathcal{N}_\tau^{(2)}\, h = 0$. Construct the vectors

$$U^{(m)} = -V_\tau^{(m)}\big(\big[\,\mathcal{P}_\tau^{(m)}\,\big]^{-1} h^{(m)}\big) \ \text{ in } \ \Omega^{(m)}, \tag{6.51}$$

$$U = V_\tau\big(\mathcal{P}_\tau^{-1}\, h\big) \ \text{ in } \ \Omega, \tag{6.52}$$

where $h^{(m)} = (h_1^{(m)},\dots,h_4^{(m)}) \in [\widetilde{H}_2^{-\frac{1}{2}}(\Gamma_T^{(m)})]^3 \times \widetilde{H}_2^{-\frac{1}{2}}(\Gamma^{(m)})$ and, moreover,

$$r_{_{\Gamma_T^{(m)}}} h_l^{(m)} = r_{_{\Gamma_T^{(m)}}} h_l \ \text{ and } \ r_{_{\Gamma^{(m)}}} h_4^{(m)} = r_{_{\Gamma^{(m)}}} h_4. \tag{6.53}$$

By Lemmas 4.6 and 4.8, then we have

$$\begin{aligned} -h^{(m)} &= \big\{\mathcal{T}^{(m)} U^{(m)}\big\}^+ \ \text{ on } \ \partial\Omega^{(m)}, \\ h &= \{\mathcal{T}U\}^+ + \beta\{U\}^+ \ \text{ on } \ \partial\Omega. \end{aligned} \tag{6.54}$$

It can easily be verified that the pair $(U^{(m)}, U)$ solves the homogeneous interface crack problem with homogeneous boundary, transmission and interface crack conditions just as in Problem (ICP-B), but with the homogeneous Robin type condition

$$r_{_{S_D}}\big[\,\{\mathcal{T}U\}^+ + \beta\{U\}^+\,\big] = 0 \ \text{ on } \ S_D,$$

for the Dirichlet homogeneous condition (1.64) on S_D. Therefore by Green's formulae, as in the proof of Theorem 1.1, we derive that $U^{(m)}=0$ in $\Omega^{(m)}$ and $U=0$ in Ω, which implies that $h=0$, i.e. the null space of the operator (6.44) is trivial.

Step 3. Further we show that the operator (6.44) is Fredholm with zero index. First we derive an auxiliary coercivity inequality. By summing of Green's formulae (1.37) and (1.40) with $V^{(m)}=U^{(m)}$ and $V=U$ we obtain

$$\begin{aligned}
&\int\limits_{\partial\Omega}\{\mathcal{T}\,U\}^+\cdot\{U\}^+\,dS+\int\limits_{\partial\Omega^{(m)}}\{\mathcal{T}^{(m)}\,U^{(m)}\}^+\cdot\{U^{(m)}\}^+\,dS=\\
&\quad=\int\limits_{\Omega}\Big[E(u,\overline{u})+\varrho\,\tau^2\,|u|^2+\gamma_{jl}\,(\tau\,T_0\,\partial_j u_l\,\overline{u_4}-u_4\,\overline{\partial_j u_l}\,)+\\
&\quad+\varkappa_{jl}\,\partial_j u_4\,\overline{\partial_l u_4}+\tau\,\alpha\,|u_4|^2+e_{lij}\,(\partial_l u_5\,\overline{\partial_i u_j}-\partial_i u_j\,\overline{\partial_l u_5})-\\
&\quad-g_l\,(\tau\,T_0\,\partial_l u_5\,\overline{u_4}+u_4\,\overline{\partial_l u_5})+\varepsilon_{jl}\,\partial_j u_5\,\overline{\partial_l u_5}\Big]\,dx+\\
&+\int\limits_{\Omega^{(m)}}\Big[E^{(m)}(u^{(m)},\overline{u^{(m)}})+\varrho^{(m)}\tau^2\,|u^{(m)}|^2+\varkappa_{jl}^{(m)}\partial_j u_4^{(m)}\,\overline{\partial_l u_4^{(m)}}+\\
&+\tau\alpha^{(m)}\,|u_4^{(m)}|^2+\gamma_{jl}^{(m)}\Big(\tau T_0^{(m)}\partial_j u_l^{(m)}\overline{u_4^{(m)}}-u_4^{(m)}\,\overline{\partial_j u_l^{(m)}}\Big)\Big]\,dx. \qquad (6.55)
\end{aligned}$$

With the help of relations (6.51), (6.52), (6.53), (6.54) and (6.49) we can show that the left hand side expression can be rewritten as

$$\begin{aligned}
&\int\limits_{\partial\Omega}\{\mathcal{T}\,U\}^+\cdot\{U\}^+\,dS+\int\limits_{\partial\Omega^{(m)}}\{\mathcal{T}^{(m)}\,U^{(m)}\}^+\cdot\{U^{(m)}\}^+\,dS=\\
&\quad=\big\langle\{\mathcal{T}\,U\}^++\beta\,\{U\}^+,\,\{U\}^+\big\rangle_{\partial\Omega}-\big\langle\beta\,\{U\}^+,\,\{U\}^+\big\rangle_{\partial\Omega}+\\
&\quad+\Big\langle\{\mathcal{T}^{(m)}\,U^{(m)}\}^+,\,\{U^{(m)}\}^+\Big\rangle_{\partial\Omega^{(m)}}=\\
&=\big\langle h,\,\{U\}^+\big\rangle_{\partial\Omega}-\big\langle\beta\,\{U\}^+,\,\{U\}^+\big\rangle_{\partial\Omega}-\big\langle h^{(m)},\,\{U^{(m)}\}^+\big\rangle_{\partial\Omega^{(m)}}=\\
&\quad=\big\langle h,\,\mathcal{H}_\tau[\mathcal{P}_\tau]^{-1}\,h\big\rangle_{\Gamma^{(m)}}-\big\langle\beta\,\{U\}^+,\,\{U\}^+\big\rangle_{S_D}+\\
&\quad+\Big\langle h^{(m)},\,\mathcal{H}_\tau^{(m)}[\mathcal{P}_\tau^{(m)}]^{-1}\,h^{(m)}\}^+\Big\rangle_{\Gamma^{(m)}}=\\
&=\langle h,\,\mathcal{A}_\tau\,h\rangle_{\Gamma^{(m)}}-\big\langle\beta\,\{U\}^+,\,\{U\}^+\big\rangle_{S_D}+\big\langle h,\,\mathcal{B}_\tau^{(m)}\,h\big\rangle_{\Gamma^{(m)}}=\\
&\quad=\big\langle h,\,[\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)}]\,h\big\rangle_{\Gamma^{(m)}}-\big\langle\beta\,\{U\}^+,\,\{U\}^+\big\rangle_{S_D}=\\
&\qquad=\langle h,\,\mathcal{N}_\tau^{(2)}\,h\rangle_{\Gamma^{(m)}}-\int\limits_{S_D}\beta\,|\{U\}^+|^2\,dS.
\end{aligned}$$

On the other hand, with the help of Korn's inequality and evident standard manipulations, the real part of the right hand side in (6.55) can be estimated from below by the expression

$$C_1 \Big\{ \|U\|^2_{[H^1_2(\Omega)]^5} + \|U^{(m)}\|^2_{[H^1_2(\Omega^{(m)})]^4} \Big\} - \\ - C_2 \Big\{ \|U\|^2_{[H^0_2(\Omega)]^5} + \|U^{(m)}\|^2_{[H^0_2(\Omega^{(m)})]^4} \Big\},$$

where C_1 and C_2 are some positive constants depending on the material parameters and the complex parameter τ. Actually, we can choose C_1 independent of τ and $C_2 = \mathcal{O}(|\tau|^2)$ for sufficiently large $|\tau|$. Therefore we finally derive the following inequality

$$\mathrm{Re}\,\langle \mathcal{N}^{(2)}_\tau h,\, h\rangle_{\Gamma^{(m)}} \geqslant C' \|h\|^2_{[H_2^{-\frac{1}{2}}(\partial\Omega)]^5} - C'' \|\Lambda\, h\|^2_{[H_2^{-\frac{1}{2}}(\partial\Omega)]^5}, \tag{6.56}$$

where C' and C'' are some positive constants depending on the material parameters and the complex parameter τ, and $\Lambda \ :\ \widetilde{\mathbb{H}}_2^{-\frac{1}{2}} \to \mathbb{H}_2^{\frac{1}{2}}$ is a compact operator. Note that $\widetilde{\mathbb{H}}_2^{-\frac{1}{2}}$ and $\mathbb{H}_2^{\frac{1}{2}}$ are mutually adjoint spaces.

Now from (6.56) we conclude that the operator (6.44) is Fredholm with zero index (see, e.g, [44, Ch. 2]), and consequently, it is invertible, since its null space is trivial.

Step 4. From the results obtained above it follows that the operator

$$\mathcal{N}^{(B,0)}_\tau \ :\ \mathbb{X}_2^{-\frac{1}{2}} \to \mathbb{Y}_2^{-\frac{1}{2}} \tag{6.57}$$

is invertible. Therefore the operator (6.38) is Fredholm with index zero due to the compactness of the operator (6.41). It remains to prove that $\ker \mathcal{N}^{(B)}_\tau$ is trivial. We proceed as follows. Let a triplet $\Phi = (\psi, h, h^{(m)})$ solve the homogeneous equation $\mathcal{N}^{(B)}_\tau \Phi = 0$ and construct the vectors $U^{(m)}$ and U by formulae

$$U^{(m)} = V^{(m)}_\tau \big(\big[\mathcal{P}^{(m)}_\tau \big]^{-1} h^{(m)} \big) \ \text{ in } \ \Omega^{(m)}, \tag{6.58}$$

$$U = V_\tau \big(\mathcal{P}^{-1}_\tau [\psi + h] \big) \ \text{ in } \ \Omega. \tag{6.59}$$

These vectors solve the homogeneous interface crack problem (ICP-B) and $U^{(m)} = 0$ in $\Omega^{(m)}$ and $U = 0$ in Ω by the uniqueness Theorem 1.1. These equations imply $\Phi = 0$, which shows that $\ker \mathcal{N}^{(B)}_\tau$ is trivial. Consequently, the operator (6.38) is invertible. The proof is complete. □

Remark 6.2. One can easily verify that all arguments applied in the proof of Lemma 6.1 remain valid for Lipschitz domains $\Omega^{(m)}$ and Ω.

Now we prove the following basic theorem (see Theorem 2.31 for a generalized version).

Theorem 6.3. *The operator* $\mathcal{N}^{(B)}_\tau$ *in* (6.2) *and* (6.37) *is invertible provided the following constraints hold*

$$\frac{1}{p} - \frac{3}{2} + \max\{\gamma''_1, \gamma''_3, \gamma''_4\} < r < \frac{1}{p} - \frac{1}{2} + \min\{\gamma'_1, \gamma'_3, \gamma'_4\}, \tag{6.60}$$

where γ_1', γ_1'' *are the same as in Theorem* 5.1 *and are defined in* (5.36),

$$\begin{aligned} \gamma_3' &:= \inf_{x\in\partial\Gamma^{(m)},\,1\leqslant j\leqslant 5} \frac{1}{2\pi}\ \arg\,\lambda_j^{(2)}(x), \\ \gamma_3'' &:= \sup_{x\in\partial\Gamma^{(m)},\,1\leqslant j\leqslant 5} \frac{1}{2\pi}\ \arg\,\lambda_j^{(2)}(x), \end{aligned} \tag{6.61}$$

$$\begin{aligned} \gamma_4' &:= \inf_{x\in\partial\Gamma^{(m)},\,1\leqslant j\leqslant 5} \frac{1}{2\pi}\ \arg\,\mu_j(x), \\ \gamma_4'' &:= \sup_{x\in\partial\Gamma^{(m)},\,1\leqslant j\leqslant 5} \frac{1}{2\pi}\ \arg\,\mu_j(x), \end{aligned} \tag{6.62}$$

$\mu_j(x)$ *are the eigenvalues of the matrix in* (6.79) *and* $\lambda_j^{(2)}(x)$ *are the same as in* (5.37).

Proof. First of all let us remark that the the operator $\mathcal{N}_\tau^{(B,0)}$, defined by (6.40), has the same mapping property as $\mathcal{N}_\tau^{(B)}$

$$\begin{aligned} \mathcal{N}_\tau^{(B,0)}\ &:\ \mathbb{X}_p^s \to \mathbb{Y}_p^s, \\ &:\ \mathbb{X}_{p,q}^s \to \mathbb{Y}_{p,q}^s \end{aligned} \tag{6.63}$$

for all $s\in\mathbb{R}$, $1<p<\infty$, and $1\leqslant q\leqslant\infty$. Moreover, the operators

$$\begin{aligned} r_{S_D}\mathcal{A}_\tau\ &:\ \big[\widetilde{B}_{p,q}^s(\Gamma^{(m)})\big]^5 \to \big[B_{p,q}^{s+1}(S_D)\big]^5, \\ r_{\Gamma^{(m)}}\mathcal{A}_\tau\ &:\ \big[\widetilde{B}_{p,q}^s(S_D)\big]^5 \to \big[B_{p,q}^{s+1}(\Gamma^{(m)})\big]^5 \end{aligned} \tag{6.64}$$

are compact for $1<p<+\infty$, $s\in\mathbb{R}$ and $1\leqslant q\leqslant+\infty$ since the domains are disjoint $\overline{S_D}\cap\overline{\Gamma^{(m)}}=\varnothing$. Therefore $\mathcal{N}_\tau^{(B,0)}$ represents a compact perturbation of the operator $\mathcal{N}_\tau^{(B)}$, i.e., the operators

$$\begin{aligned} \mathcal{N}_\tau^{(B)}-\mathcal{N}_\tau^{(B,0)}\ &:\ \mathbb{X}_p^s \to \mathbb{Y}_p^s, \\ &:\ \mathbb{X}_{p,q}^s \to \mathbb{Y}_{p,q}^s \end{aligned} \tag{6.65}$$

are compact.

The operator $\mathcal{N}_\tau^{(B,0)}$ in (6.40) is of block-lower triangular form

$$\mathcal{N}_\tau^{(B,0)} := \begin{bmatrix} r_{S_D}\mathcal{A}_\tau & 0 & 0 \\ 0 & \mathcal{N}_\tau^{(2)} & 0 \\ 0 & I_{4\times 5} & I_4 \end{bmatrix}_{14\times 14}, \tag{6.66}$$

where $\mathcal{N}_\tau^{(2)}$ is defined in (6.46) and $I_{4\times 5}$ is as in (5.26). Further, as we have shown in the proof of Theorem 5.1, the operators

$$\begin{aligned} r_{S_D}\mathcal{A}_\tau\ &:\ \big[\widetilde{H}_p^r(S_D)\big]^5 \to \big[H_p^{r+1}(S_D)\big]^5, \\ &:\ \big[\widetilde{B}_{p,q}^r(S_D)\big]^5 \to \big[B_{p,q}^{r+1}(S_D)\big]^5 \end{aligned} \tag{6.67}$$

are invertible if

$$\frac{1}{p}-1+\gamma_1''<r+\frac{1}{2}<\frac{1}{p}+\gamma_1', \tag{6.68}$$

where γ_1' and γ_1'' are determined in (5.36).

To prove the invertibility of $\mathcal{N}_\tau^{(B,0)}$ in (6.42) it remains to investigate the operators

$$\begin{aligned} \mathcal{N}_\tau^{(2)} \ &: \ \widetilde{\mathbb{H}}_p^r \to \mathbb{H}_p^{r+1}, \\ &: \ \widetilde{\mathbb{B}}_{p,q}^r \to \mathbb{B}_{p,q}^{r+1} \end{aligned} \tag{6.69}$$

in the following space settings

$$\begin{aligned} \widetilde{\mathbb{H}}_p^r &:= \big[\, \widetilde{H}_p^r(\Gamma_T^{(m)})\,\big]^3 \times \big[\, \widetilde{H}_p^r(\Gamma^{(m)})\,\big]^2, \\ \mathbb{H}_p^{r+1} &:= \big[\, H_p^{r+1}(\Gamma_T^{(m)})\,\big]^3 \times \big[\, H_p^{r+1}(\Gamma^{(m)})\,\big]^2, \\ \widetilde{\mathbb{B}}_{p,q}^r &:= \big[\, \widetilde{B}_{p,q}^r(\Gamma_T^{(m)})\,\big]^3 \times \big[\, \widetilde{B}_{p,q}^r(\Gamma^{(m)})\,\big]^2, \\ \mathbb{B}_{p,q}^{r+1} &:= \big[\, B_{p,q}^{r+1}(\Gamma_T^{(m)})\,\big]^3 \times \big[\, B_{p,q}^{r+1}(\Gamma^{(m)})\big]^2 . \end{aligned} \tag{6.70}$$

Since $\Gamma_T^{(m)}$ is a proper part of $\Gamma^{(m)}$ we can not apply Theorem 2.28 to characterize the Fredholm properties of the operators (6.69). Instead we will apply the local principle for para-algebras, exposed in Section 2. To this end, let either $\mathbb{Z}_p^r := \mathbb{H}_p^r$ ($\widetilde{\mathbb{Z}}_p^r := \widetilde{\mathbb{H}}_p^r$) or $\mathbb{Z}_p^r := \mathbb{B}_{p,q}^r$ ($\widetilde{\mathbb{Z}}_p^r := \widetilde{\mathbb{B}}_{p,q}^r$). Consider the quotient para-algebra

$$\Psi'(\widetilde{\mathbb{Z}}_p^r, \mathbb{Z}_p^{r+1}) = \big[\Psi(\widetilde{\mathbb{Z}}_p^r, \mathbb{Z}_p^{r+1})/\mathfrak{C}(\widetilde{\mathbb{Z}}_p^r, \mathbb{Z}_p^{r+1})\big]_{2\times 2},$$

of all ΨDOs $\Psi(\widetilde{\mathbb{Z}}_p^r, \mathbb{Z}_p^{r+1})$ acting between the indicated spaces factored by the space of all compact operators $\mathfrak{C}(\widetilde{\mathbb{Z}}_p^r, \mathbb{Z}_p^{r+1})$. Further, for arbitrary point $y \in \overline{\Gamma^{(m)}}$ we define the following localizing class

$$\Delta_y := \Big\{[g_y I_5],\ g_y \in \mathbb{C}^\infty(\Gamma^{(m)}),\ \mathrm{supp}\ g_y \subset W_y,\ g_y(x) = 1\ \forall\, x \in \widetilde{W}_y\Big\}, \tag{6.71}$$

where $\widetilde{W}_y \subset W_y \subset \overline{\Gamma^{(m)}}$ are arbitrarily small embedded neighborhoods of y. The symbol $[A]$ stands for the quotient class containing the operator A. It is obvious that the system $\{\Delta_y\}_{y\in\overline{\Gamma^{(m)}}}$ is covering and all its elements $[g_y I_5]$ commute with the class $[A]$ for arbitrary ΨDO $A \in \Psi(\widetilde{\mathbb{Z}}_p^r, \mathbb{Z}_p^{r+1})$ (to justify the commutativity recall that a commutant $AgI - gA$, with the identity operator I, is compact for an arbitrary smooth function g).

The ΨDO $\mathcal{A}_\tau = \mathcal{H}_\tau\, \mathcal{P}_\tau^{-1}$ "lives" on the surface $\partial\Omega$ (see (5.18) and Section 2). Let us consider a similar operator $\mathcal{A}_\tau^{(m)} := \mathcal{H}_\tau^{(m)}\, g(\mathcal{P}_\tau^{(m)})^{-1}$ which "lives" on the surface $\partial\Omega^{(m)}$, where the ΨDOs $\mathcal{H}_\tau^{(m)}$ and $\mathcal{P}_\tau^{(m)}$ are the direct values of potential operators, defined in Section 2. The closed surfaces $\partial\Omega$ and $\partial\Omega^{(m)}$, where the operators $\mathcal{A}_\tau$ and $\mathcal{A}_\tau^{(m)}$ are defined, have in common the open surface $\Gamma^{(m)} = \partial\Omega \cap \partial\Omega^{(m)}$. On the other hand, an arbitrary ΨDO $A(x, D)$ and, in particular the operators $\mathcal{A}_\tau$ and $\mathcal{A}_\tau^{(m)}$, are of local type: if g_1 and g_2 are functions with disjoint supports $\mathrm{supp}\ g_1 \cap \mathrm{supp}\ g_2 = \varnothing$, then the operator $g_1 A(x, D) g_2 I$ is compact in the spaces where $A(x, D)$ is bounded. Applying the mentioned property, it is easy to check the following local equivalences

$$[\mathcal{A}_\tau] \overset{\Delta_y}{\sim} [\mathcal{A}_{\tau,y}]\,, \qquad [\mathcal{A}_\tau^{(m)}] \overset{\Delta_y}{\sim} [\mathcal{A}_{\tau,y}^{(m)}] \qquad \text{for all} \quad y \in \Gamma^{(m)}.$$

Consequently,

$$[\,\mathcal{N}_\tau^{(2)}\,] \overset{\Delta_y}{\sim} [\,\mathcal{N}_{\tau,y}^{(2)}\,], \tag{6.72}$$

where

$$\mathcal{N}_{\tau,y}^{(2)} := \mathcal{A}_\tau + \mathcal{B}_\tau^{(m)} \,:\, \big[\,H_p^r(\partial\Omega^{(m)})\,\big]^5 \to \big[\,H_p^{r+1}(\partial\Omega^{(m)})\,\big]^5 \text{ for } y \in \Gamma_T^{(m)}, \tag{6.73a}$$

$$\mathcal{N}_{\tau,y}^{(2)} := \big[(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})_{t,q}\big]_{2\times 2} : \big[\,H_p^r(\partial\Omega^{(m)})\big]^2 \to \big[H_p^{r+1}(\partial\Omega^{(m)})\,\big]^2 \quad \text{for } \ y \in \Gamma_C^{(m)}, \tag{6.73b}$$

$$\mathcal{N}_{\tau,y}^{(2)} := \big[\,r_{\Gamma^{(m)}}(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})\,\big]_{5\times 5} : \big[\,\widetilde{H}_p^r(\Gamma^{(m)})\,\big]^5 \to \big[\,H_p^{r+1}(\Gamma^{(m)})\,\big]^5 \quad \text{for } \ y \in \partial\Gamma^{(m)}, \tag{6.73c}$$

$$\mathcal{N}_{\tau,y}^{(2)} := \\ := \begin{bmatrix} r_{\Gamma^c_{(m)}}[(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})_{l,k}]_{3\times 3} & r_{\Gamma^c_{(m)}}[(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})_{l,q}]_{3\times 2} \\ [(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})_{t,k}]_{2\times 3} & [(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})_{t,q}]_{2\times 2} \end{bmatrix}_{5\times 5} : \widetilde{\mathbb{V}}_p^r \to \mathbb{V}_p^{r+1} \tag{6.73d}$$

$$\text{for } \ y \in \partial\Gamma_C^{(m)}, \ \ l,k = 1,2,3, \ \ t,q = 4,5.$$

Here $\partial\Omega^{(m)}$ is a closed surface and

$$\begin{aligned} \Gamma^c_{(m)} &:= \partial\Omega^{(m)} \setminus \overline{\Gamma_C^{(m)}} = \Gamma_T^{(m)} \cup \overline{S_N^{(m)}}, \\ \widetilde{\mathbb{V}}_p^r &:= \big[\,\widetilde{X}_p^r(\Gamma^c_{(m)})\,\big]^3 \times \big[\,X_p^r(\partial\Omega^{(m)})\,\big]^2, \\ \mathbb{V}_p^{r+1} &:= \big[\,X_p^{r+1}(\Gamma^c_{(m)})\,\big]^3 \times \big[\,X_p^{r+1}(\partial\Omega^{(m)})\,\big]^2 \end{aligned} \tag{6.74}$$

with either $X_p^r = H_p^r$ or $X_p^r = B_{p,q}^r$.

Due to Theorem 2.45 the operator $\mathcal{N}_\tau^{(2)}$ in (6.69) is Fredholm if and only if the operators $\mathcal{N}_{\tau,y}^{(2)}$ in (6.73a)-(6.73d) are Fredholm for all $y \in \overline{\Gamma^{(m)}}$.

The strongly elliptic ΨDOs $\mathcal{N}_{\tau,y}^{(2)}$ in (6.73a) and in (6.73b) on the closed surface $\partial\Omega^{(m)}$ are Fredholm with index 0 for all $y \in \Gamma_C^{(m)} \cup \Gamma_T^{(m)}$.

The same strongly elliptic ΨDO $\mathcal{N}_{\tau,y}^{(2)}$ in (6.73c) but on the surface $\Gamma^{(m)}$ with the smooth boundary $\partial\Gamma^{(m)} \neq \varnothing$ is Fredholm if the following constraints hold

$$\frac{1}{p} - \frac{3}{2} + \gamma_3'' < r < \frac{1}{p} - \frac{1}{2} + \gamma_3' \tag{6.75}$$

with γ_3' and γ_3'' defined in (6.61) (see Section2, Theorem 2.28).

To investigate the elliptic ΨDO $\mathcal{N}_{\tau,y}^{(2)}$ in (6.73d) for $y \in \partial\Gamma_C^{(m)}$, first note that $\mathcal{G}_y := [(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})_{t,q}]_{2\times 2}$ is defined on the closed surface $\partial\Omega^{(m)}$, has a strongly elliptic symbol due to Remark 2.30 and, therefore, is Fredholm. Then the quotient class $[\mathcal{G}_y]$ is invertible and since $\operatorname{Ind}\mathcal{G}_y = 0$ (see Theorem 2.28), there exists a compact operator T_y such that $\mathcal{G}_y + T_y$ is invertible for all $y \in \partial\Gamma_C^{(m)}$. For the quotient classes the equalities $[\mathcal{G}_y + T_y] = [\mathcal{G}_y]$ and $[\mathcal{G}_y + T_y]^{-1} = [\mathcal{G}_y]^{-1}$ hold.

Note that the quotient classes

$$[\mathcal{F}_\pm] := \begin{bmatrix} [I_{3\times 3}] & [\,[0]_{3\times 2}\,] \\ \pm[\mathcal{G}_y]^{-1}[\,[(\mathcal{N}_{\tau,y}^{(2)})_{t,k}]_{2\times 3}\,] & [I_{2\times 2}] \end{bmatrix}_{5\times 5}$$

are invertible

$$[\mathcal{F}_-]\,[\mathcal{F}_+] = [\mathcal{F}_+]\,[\mathcal{F}_-] = [I_{5\times 5}]$$

and composing the quotient class $\big[\mathcal{N}_{\tau,y}^{(2)}\big]$ with this invertible quotient class we get

$$\big[\widetilde{\mathcal{N}}_{\tau,y}^{(2)}\big] := \big[\mathcal{N}_{\tau,y}^{(2)}\big]\,[\mathcal{F}_-] = \begin{bmatrix} D_{\tau,y} & r_{\Gamma^c_{(m)}}[(\mathcal{N}_{\tau,y}^{(2)})_{l,q}]_{3\times 2} \\ [0]_{2\times 3} & \mathcal{G}_y \end{bmatrix}_{5\times 5}, \tag{6.76}$$

where

$$D_{\tau,y} :=$$

$$:= r_{\Gamma^c_{(m)}}\Big([(\mathcal{N}_{\tau,y}^{(2)})_{l,k}]_{3\times 3} - [(\mathcal{N}_{\tau,y}^{(2)})_{l,k}]_{3\times 2}[\mathcal{G}_y + T_y]^{-1}[(\mathcal{N}_{\tau,y}^{(2)})_{t,k}]_{2\times 3}\Big) \tag{6.77}$$

is the strongly elliptic ΨDO of order -1 due to Lemma 2.29. It is sufficient to prove that the composition $[\widetilde{\mathcal{N}}_{\tau,y}^{(2)}]$ is an invertible class.

$[\widetilde{\mathcal{N}}_{\tau,y}^{(2)}]$ in upper block-triangular and the entry $[\mathcal{G}_y]$ on the diagonal is an invertible class. Moreover, the entries on the diagonal $D_{\tau,y}$ and $\mathcal{G}_y$ are ΨDOs and the corresponding quotient classes commute (actually, these entries are matrices of different dimension 3×3 and 2×2, but we can extend the entire matrix $[\widetilde{\mathcal{N}}_{\tau,y}^{(2)}]$ by identity on the diagonal and by zeros on the off-diagonal entries in the last row and the last column, without change the invertibility properties of the entire matrix and the diagonal entries. Then $[\mathcal{G}_y]$ extends to the matrix of the same dimension 3×3 as $[\mathbf{D}_y(D, x)]$). Therefore $[\widetilde{\mathbf{N}}_y(x, D)]$ is invertible if and only if the quotient class $[\mathbf{D}_y(D, x)]$ is invertible. This is interpreted as follows: the operator

$$\widetilde{\mathcal{N}}_{\tau,y}^{(2)} \;:\; \widetilde{\mathbb{Z}}_p^r \to \mathbb{Z}_p^{r+1}$$

is Fredholm if and only if the operator

$$D_{\tau,y} \;:\; \big[\,\widetilde{X}_p^r(\Gamma^c_{(m)})\,\big]^3 \to \big[\,X_p^{r+1}(\Gamma^c_{(m)})\,\big]^3 \tag{6.78}$$

is Fredholm.

Let $\mathfrak{S}_{D_{\tau,y}}(x, \xi_1, \xi_2)$ be the principal homogeneous symbol matrix of the operator $D_{\tau,y}$ and $\mu_j(x)$ $(j = 1, 2, 3)$ be the eigenvalues of the matrix

$$\mathcal{D}_{\tau,y}(x) := [\mathfrak{S}_{D_{\tau,y}}(x, 0, +1)]^{-1}\,\mathfrak{S}_{D_{\tau,y}}(x, 0, -1) \tag{6.79}$$

for $x \in \partial\Gamma_C^{(m)}$.

The operators $D_{\tau,y}$ in (6.78) and, therefore, the operator $\mathcal{N}_\tau^{(2)}$ in (6.69) are Fredholm if the following constraints are fulfilled

$$\frac{1}{p} - \frac{3}{2} + \gamma_4'' < r < \frac{1}{p} - \frac{1}{2} + \gamma_4', \tag{6.80}$$

where γ_4' and γ_4'' are defined in (6.62) (cf. Section 2, Theorem 2.31).

The system of inequalities (6.68), (6.75) and (6.80) are equivalent to (6.60).

Therefore the operator $\mathcal{N}_\tau^{(B)}$ is Fredholm if the conditions (6.60) hold.

Next we note that the operator $\mathcal{N}_\tau^{(B)}$ in (6.38) is invertible due to Lemma 6.1.

Therefore, the operator $\mathcal{N}_\tau^{(B)}$ is invertible for all p and r if the conditions (6.60) are fulfilled (cf. Theorem 2.31). □

Theorem 6.3 yields the following existence result.

Theorem 6.4. *Let the inclusions* (6.1) *and the compatibility conditions* (6.26) *and* (6.27) *hold and*

$$\frac{4}{3 - 2\gamma''} < p < \frac{4}{1 - 2\gamma'} \tag{6.81}$$

with

$$\gamma' := \min\{\gamma_1', \gamma_2', \gamma_3'\}, \quad \gamma'' := \max\{\gamma_1'', \gamma_2'', \gamma_3''\}. \tag{6.82}$$

Then the interface crack problem (1.60)–(1.71) *has a unique solution*

$$(U^{(m)}, U) \in \left[W_p^1(\Omega^{(m)}) \right]^4 \times \left[W_p^1(\Omega) \right]^5,$$

which can be represented by formulae

$$U^{(m)} = V_\tau^{(m)}\big(\left[\mathcal{P}_\tau^{(m)} \right]^{-1} \left[G_0^{(m)} + h^{(m)} \right] \big) \ \ in \ \ \Omega^{(m)}, \tag{6.83}$$

$$U = V_\tau\big(\mathcal{P}_\tau^{-1} \left[G_0 + \psi + h \right] \big) \ \ in \ \ \Omega, \tag{6.84}$$

where the densities ψ, h, *and* $h^{(m)}$ *are to be determined from the system* (6.16)-(6.21).

Moreover, the vector functions $G_0 + \psi + h$ *and* $G_0^{(m)} + h^{(m)}$ *are defined uniquely by the above systems.*

Proof. It is word for word of the proof of Theorem 5.2. □

Remark 6.5. Theorem 6.4 with $p = 2$ remains valid for Lipschitz domains $\Omega^{(m)}$ and Ω. This immediately follows from Remark 6.2 and Theorem 1.1.

One can easily formulate the regularity results, similar to Theorem 5.4, for solutions of the interface crack problem (ICP-B) (see (1.60)–(1.71)).

6.3. **Asymptotic formulas for solutions of problem (ICP-B).** In this section we study asymptotic properties of solutions to the problem (ICP-B) near the exceptional curve $\partial\Gamma_T^{(m)}$. We assume, that the boundary data of the problem are infinitely smooth, namely,

$$
\begin{gathered}
Q_j^{(m)} \in C^\infty(\overline{S}_N^{(m)}), \ \ j=\overline{1,4}, \quad Q_k \in C^\infty(\overline{S}_N), \ \ k=\overline{1,5}, \\
f_k \in C^\infty(\overline{S}_D), \ \ k=\overline{1,5}, \quad f_5^{(m)} \in C^\infty(\overline{\Gamma^{(m)}}), \\
f_l^{(m)}, F_l^{(m)} \in C^\infty(\overline{\Gamma_T^{(m)}}), \ \ l=1,2,3, \quad f_4^{(m)}, F_4^{(m)} \in C^\infty(\overline{\Gamma^{(m)}}), \\
\widetilde{Q}_l^{(m)}, \widetilde{Q}_l \in C^\infty(\overline{\Gamma_C^{(m)}}), \ \ l=1,2,3, \\
\widetilde{F}_l^{(m)} = F_l^{(m)} - r_{\Gamma_T^{(m)}} G_{0l} - r_{\Gamma_T^{(m)}} G_{0l}^{(m)} \in C_0^\infty(\overline{\Gamma_T^{(m)}}), \ \ l=1,2,3, \\
\widetilde{F}_4^{(m)} = F_4^{(m)} - r_{\Gamma^{(m)}} G_{04} - r_{\Gamma^{(m)}} G_{04}^{(m)} \in C_0^\infty(\overline{\Gamma^{(m)}}).
\end{gathered}
$$

Let $\Phi = (\psi, h, h^{(m)})^T \in \mathbb{X}_p^s$ be a solution of the system (6.28)–(6.33) which can be written in the following form

$$
\mathcal{N}_\tau^{(B)} \Phi = \mathcal{Y},
$$

where

$$
\begin{aligned}
\mathcal{Y} \in \big[\, C^\infty(\overline{S}_D) \big]^5 \times \big[\, C^\infty(\overline{\Gamma_T^{(m)}}) \big]^3 \times \\
\times \big[\, C^\infty(\overline{\Gamma^{(m)}}) \big]^2 \times \big[\, C_0^\infty(\overline{\Gamma_T^{(m)}}) \big]^3 \times C_0^\infty(\overline{\Gamma^{(m)}})
\end{aligned}
$$

To establish asymptotic properties of the solution vectors $U^{(m)}$ and U near the exceptional curve $\partial\Gamma_T^{(m)}$ we rewrite the representations (6.83)–(6.84) in the following form

$$
U^{(m)} = V_\tau^{(m)}\big(\big[\mathcal{P}_\tau^{(m)}\big]^{-1} h^{(m)}\big) + \mathcal{R}^{(m)} \ \text{ in } \ \Omega^{(m)}, \tag{6.85}
$$

$$
U = V_\tau\big(\mathcal{P}_\tau^{-1} h\big) + V_\tau\big(\mathcal{P}_\tau^{-1} \psi\big) + \mathcal{R} \ \text{ in } \ \Omega, \tag{6.86}
$$

where

$$
\begin{aligned}
h^{(m)} &= (h_1^{(m)}, \dots, h_4^{(m)})^\top, \quad h = (h_1, \dots, h_5)^\top, \\
h_l^{(m)} &= -h_l + \widetilde{F}_l^{(m)}, \ \ l=1,2,3, \ \text{ on } \ \Gamma_T^{(m)}, \\
h_4^{(m)} &= -h_4 + \widetilde{F}_4^{(m)} \ \text{ on } \ \Gamma^{(m)},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{R}^{(m)} &:= V_\tau^{(m)}\big(\, \big[\, \mathcal{P}_\tau^{(m)} \big]^{-1} G_0^{(m)} \big) \in \big[\, C^\infty(\overline{\Omega^{(m)}}) \big]^4, \\
\mathcal{R} &:= V_\tau\big(\, \mathcal{P}_\tau^{-1} G_0 \big) \in \big[\, C^\infty(\overline{\Omega}) \big]^5,
\end{aligned}
$$

the vectors $h=(h_1,h_2,h_3,h_4,h_5)^\top$ and $\psi=(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5)^\top$ solve the following strongly elliptic system of pseudodifferential equations

$$\begin{cases} r_{S_D}\mathcal{A}_\tau\psi=\Phi^{(1)} & \text{on } S_D,\\ r_{\Gamma_T^{(m)}}\big[\,(\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)})_{l,k}\,\big]_{3\times5}h=\Phi^{(2)} & \text{on } \Gamma_T^{(m)}, \quad l=1,2,3;\ k=\overline{1,5};\ t=4,5.\\ r_{\Gamma^{(m)}}\big[\,(\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)})_{t,k}\,\big]_{2\times5}h=\Phi^{(3)} & \text{on } \Gamma^{(m)}, \end{cases}$$

where

$$\begin{aligned}
&\Phi^{(1)}=\big(\Phi_1^{(1)},\dots,\Phi_5^{(1)}\big)^\top\in\big[C^\infty(\overline{S_D})\big]^5,\\
&\Phi_k^{(1)}=f_k-r_{S_D}[\mathcal{A}_\tau G_0]_k-r_{S_D}\big[\mathcal{A}_\tau h\big]_k,\ \ k=\overline{1,5},\\
&\Phi^{(2)}=\big(\Phi_1^{(2)},\Phi_2^{(2)},\Phi_3^{(2)}\big)^\top\in\big[\,C^\infty(\overline{\Gamma_T^{(m)}})\,\big]^3,\\
&\Phi_l^{(2)}=f_l^{(m)}+r_{\Gamma_T^{(m)}}\big[\,\mathcal{A}_\tau^{(m)}G_0^{(m)}\,\big]_l-r_{\Gamma_T^{(m)}}\big[\,\mathcal{A}_\tau G_0\,\big]_l+\\
&\qquad\quad+r_{\Gamma_T^{(m)}}\big[\mathcal{A}_\tau^{(m)}\widetilde{F}^{(m)}\big]_l-r_{\Gamma_T^{(m)}}\big[\mathcal{A}_\tau^{(m)}\psi\big]_l,\ \ l=\overline{1,3},\\
&\Phi^{(3)}=\big(\Phi_1^{(3)},\Phi_2^{(3)}\big)^\top\in\big[\,C^\infty(\overline{\Gamma^{(m)}})\,\big]^2,\\
&\Phi_1^{(3)}=f_4^{(m)}+r_{\Gamma^{(m)}}\big[\mathcal{A}_\tau^{(m)}G_0^{(m)}\big]_4-r_{\Gamma^{(m)}}\big[\mathcal{A}_\tau G_0\big]_4+\\
&\qquad\quad+r_{\Gamma^{(m)}}\big[\,\mathcal{A}_\tau^{(m)}\widetilde{F}^{(m)}\,\big]_4-r_{\Gamma^{(m)}}\big[\,\mathcal{A}_\tau\psi\,\big]_4,\\
&\Phi_2^{(3)}=f_5^{(m)}+r_{\Gamma^{(m)}}\big[\,\mathcal{A}_\tau G_0\,\big]_5-r_{\Gamma^{(m)}}\big[\,\mathcal{A}_\tau\psi\,\big]_5.
\end{aligned}$$

If $y\in\partial\Gamma^{(m)}$, then $h=(h_1,\dots,h_5)^\top$ solves the pseudodifferential equation on surface $\Gamma^{(m)}$ with boundary $\partial\Gamma^{(m)}$

$$\mathcal{N}_{\tau,y}^{(2)}h=r_{\Gamma^{(m)}}\big(\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)}\big)h=F\ \text{ on }\ \Gamma^{(m)}.$$

The solution of this equation has the same asymptotics as (5.59) and, consequently, the solution of the problem (ICP-B) in the neighborhood of the boundary of $\partial\Gamma^{(m)}$ has the same asymptotics as the solution of the problem (ICP-A) (see (5.64)–(5.65), (5.70)–(5.71), (5.78)).

Now consider the case, when $y\in\partial\Gamma_C^m$. Then $h=(h',h'')^\top$, where $h'=(h_1,h_2,h_3)^\top$ and $h''=(h_4,h_5)^\top$ satisfy the following system of equations

$$r_{\Gamma_{(m)}^C}\mathcal{N}_{3\times3}h'+r_{\Gamma_{(m)}^C}\mathcal{N}_{3\times2}h''=F_1\ \text{ on }\ \Gamma_{(m)}^C, \tag{6.87}$$

$$\mathcal{N}_{2\times3}h'+\mathcal{N}_{2\times2}h''=F_2\ \text{ on }\ \partial\Omega_m, \tag{6.88}$$

where

$$\begin{gathered}
F_1\in\big[\,C^\infty\big(\overline{\Gamma_{(m)}^C}\big)\,\big]^3,\quad F_2\in\big[\,C^\infty(\partial\Omega_m)\,\big]^2,\\
\mathcal{N}_{3\times3}=\big[\,\big(\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)}\big)_{l,k}\,\big]_{3\times3},\quad \mathcal{N}_{3\times2}=\big[\,\big(\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)}\big)_{l,q}\,\big]_{3\times2},\\
\mathcal{N}_{2\times3}=\big[\,\big(\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)}\big)_{t,k}\,\big]_{2\times3},\quad \mathcal{N}_{2\times2}=\big[\,\big(\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)}\big)_{t,q}\,\big]_{2\times2},\\
l,k=1,2,3,\ \ t,q=4,5.
\end{gathered}$$

The operator $\mathcal{N}_{2\times 2}$ is a Fredholm operator with zero index, therefore it has a regularizer $\mathcal{R}_{2\times 2}$ such, that

$$\mathcal{R}_{2\times 2}\circ\mathcal{N}_{2\times 2}=I_{2\times 2}+\mathcal{T}_{2\times 2}, \tag{6.89}$$

where $\mathcal{T}_{2\times 2}$ is a compact operator of order $-\infty$.

Now, using equality (6.89), the second equation of the system (6.87)–(6.88) can be written as

$$(\mathcal{R}_{2\times 2}\circ\mathcal{N}_{2\times 3})\,h'+(I_{2\times 2}+\mathcal{T}_{2\times 2})\,h''=\mathcal{R}_{2\times 2}F_2,$$

whence we get

$$h''=-\,(\mathcal{R}_{2\times 2}\circ\mathcal{N}_{2\times 3})\,h'-\mathcal{T}_{2\times 2}h''+\mathcal{R}_{2\times 2}F_2.$$

Inserting the obtained expression of h'' into the (6.87) we get

$$r_{\Gamma^C_{(m)}}\widetilde{\mathcal{D}}_{\tau,y}h'=F\quad\text{on}\quad\Gamma^C_{(m)}, \tag{6.90}$$

where

$$\widetilde{\mathcal{D}}_{\tau,y}=\mathcal{N}_{3\times 3}-\mathcal{N}_{3\times 2}\circ\mathcal{R}_{2\times 2}\circ\mathcal{N}_{2\times 3},$$
$$F=F_1+\big(\mathcal{N}_{3\times 2}\circ\mathcal{T}_{2\times 2}\big)h''-\big(\mathcal{N}_{3\times 2}\circ\mathcal{R}_{2\times 2}\big)F_2\in\big[\,C^\infty\big(\overline{\Gamma^C_{(m)}}\big)\,\big]^3.$$

The principal homogeneous symbol of the operator $\widetilde{\mathcal{D}}_{\tau,y}$ reads as

$$\mathfrak{S}\big(\widetilde{\mathcal{D}}_{\tau,y}\big)(\xi')=\mathfrak{S}\big(\mathcal{N}_{3\times 3}\big)(y,\xi')-$$
$$-\mathfrak{S}\big(\mathcal{N}_{3\times 2}\big)(y,\xi')\mathfrak{S}^{-1}\big(\mathcal{N}_{2\times 2}\big)(y,\xi')\mathfrak{S}\big(\mathcal{N}_{2\times 3}\big)(y,\xi'),\ \ \xi'=(\xi_1,\xi_2).$$

Let $\mu_j,\ j=1,2,3$, be the eigenvalues of the matrix

$$\big[\mathfrak{S}_{\widetilde{\mathcal{D}}_{\tau,y}}(0,+1)\big]^{-1}\mathfrak{S}_{\widetilde{\mathcal{D}}_{\tau,y}}(0,-1),\ \ y\in\partial\Gamma^{(m)}_C,$$

and denote by m_j the algebraic multiplicities of $\mu_j,\ m_1+\cdots+m_l=3$; then the asymptotic expansion of solutions of the strongly elliptic equation (6.90) reads as [12]

$$h'=(h_1,h_2,h_3)^\top=D(y)r^{-\frac12+\gamma+i\delta}B_0\Big(-\frac{1}{2\pi i}\log r\Big)D^{-1}(y)b_0(y)+$$
$$+\sum_{k=1}^{M}D(y)r^{-\frac12+\gamma+i\delta+k}B_k(y,\log r)+h'_{M+1}(x), \tag{6.91}$$

where

$$r^{-\frac12+\gamma+i\delta}=\operatorname{diag}\Big\{r^{-\frac12+\gamma_1+i\delta_1},r^{-\frac12+\gamma_2+i\delta_2},r^{-\frac12+\gamma_3+i\delta_3}\Big\},$$
$$\gamma_j=\frac{1}{2\pi}\arg\mu_j(y),\quad \delta_j=-\frac{1}{2\pi}\log|\mu_j(y)|,\ \ j=1,2,3,\ \ y\in\partial\Gamma^{(m)}_C,$$
$$B_k(y,t)=B_0\Big(-\frac{t}{2\pi i}\sum_{j=1}^{k(2m_0-1)}t^jd_{kj}(y)\Big),\ \ m_0=\max\{m_1,\dots,m_l\},$$
$$b_0\in\big[C^\infty\big(\partial\Gamma^{(m)}_C\big)\big]^3,\ \ h'_{M+1}\in\big[C^\infty\big(l^+_{m,\varepsilon}\big)\big]^3,\ \ l^+_{m,\varepsilon}=\partial\Gamma^{(m)}_C\times[0,\,\varepsilon],$$

while the 3×3-dimensional matrices D and B_0 are defined likewise to the respective 5×5-dimensional matrices in Section 5.

Denote by C the matrix

$$C=[C_{lk}]_{5\times 3}=\begin{bmatrix} I_{3\times 3}\\ -\mathcal{R}_{2\times 2}\circ\mathcal{N}_{2\times 3}\end{bmatrix}_{5\times 3}, \quad l=\overline{1,5}, \quad k=\overline{1,3},$$

and by C' the matrix composed from the first four rows of matrix C

$$C'=[C_{t,k}]_{4\times 3}, \quad t=\overline{1,4}, \quad k=\overline{1,3}.$$

Then we can rewrite formulae (6.85) and (6.86) as follows

$$U^{(m)}=V_\tau^{(m)}\big[\big(\mathcal{P}_\tau^{(m)}\big)^{-1}C'\big]_{4\times 3}h'+\mathcal{R}_1^{(m)} \text{ in } \Omega_m,$$
$$U=V_\tau\big[\mathcal{P}_\tau^{-1}C\big]_{5\times 3}h'+\mathcal{R}_1 \text{ in } \Omega,$$

where $\mathcal{R}_1^{(m)}\in[C^\infty(\overline{\Omega_m})]^4$, $\mathcal{R}_1\in[C^\infty(\overline{\Omega})]^5$.

Now we can apply the asymptotic expansion (6.91) and the asymptotic expansion of the potential-type functions (see [13]) and obtain the following asymptotic expansion of solutions of the problem (ICP-B) near the crack edge $\partial\Gamma_C^{(m)}$

$$U^{(m)}(y,\alpha,r)=$$
$$=\sum_{\mu=\pm 1}\sum_{s=1}^{l_0^{(m)}}\sum_{j=0}^{n_s^{(m)}-1}c_{sj\mu}^{(m)}(y,\,\alpha)\,r^{\frac12+\gamma+i\delta}B_0(\zeta)\,\widetilde{c}_{sj\mu}^{(m)}(y,\,\alpha)+\cdots, \tag{6.92}$$
$$U(y,\alpha,r)=$$
$$=\sum_{\mu=\pm 1}\sum_{s=1}^{l_0}\sum_{j=0}^{n_s-1}c_{sj\mu}(y,\,\alpha)\,r^{\frac12+\gamma+i\delta}B_0(\zeta)\,\widetilde{c}_{sj\mu}(y,\,\alpha)+\cdots, \tag{6.93}$$

where $\zeta=-\frac{1}{2\pi i}\log r$, and n_s, $n_s^{(m)}$, l_0 and $l_0^{(m)}$ are defined in the Section 5.

Coefficients $c_{sj\mu}$, $\widetilde{c}_{sj\mu}$, $c_{sj\mu}^{(m)}$ and $\widetilde{c}_{sj\mu}^{(m)}$ in asymptotic expansions (6.92) and (6.93) read as

$$c_{sj\mu}(y,\,\alpha)=\sin^j\alpha\,d_{sj}(y,\mu)\big[\,\psi_{s,\mu}(y,\alpha)\,\big]^{\frac12+\gamma+i\delta-j},$$
$$\widetilde{c}_{sj\mu}(y,\,\alpha)=B_0\Big(-\frac{1}{2\pi i}\,\log\psi_{s,\mu}(y,\alpha)\Big)c_j(y),$$
$$j=\overline{0,n_s-1}, \quad \mu=\pm 1, \quad s=1,\dots,l_0,$$
$$c_{sj\mu}^{(m)}(y,\,\alpha)=\sin^j\alpha\,d_{sj}^{(m)}(y,\mu)\Big[\psi_{s,\mu}^{(m)}(y,\alpha)\Big]^{\frac12+\gamma+i\delta-j},$$
$$\widetilde{c}_{sj\mu}^{(m)}(y,\,\alpha)=B_0\Big(-\frac{1}{2\pi i}\log\psi_{s,\mu}^{(m)}(y,\alpha)\Big)c_j(y)$$
$$j=\overline{0,n_s^{(m)}-1}, \quad \mu=\pm 1, \quad s=1,\dots,l_0^{(m)},$$

with $\psi_{s,\mu}$, $\psi^{(m)}_{s,\mu}$ defined in the Section 5, and

$$\begin{aligned}
c_{sj\mu} &= \big[c^{(l,k)}_{sj\mu}\big]_{5\times 3}, \quad c^{(m)}_{sj\mu} = \big[c^{(m),(l,k)}_{sj\mu}\big]_{4\times 3},\\
\widetilde{c}_{sj\mu} &= \big(c^{(1)}_{sj\mu}, c^{(2)}_{sj\mu}, c^{(3)}_{sj\mu}\big)^\top, \quad \widetilde{c}^{(m)}_{sj\mu} = \big(c^{(m)(1)}_{sj\mu}, c^{(m)(2)}_{sj\mu}, c^{(m)(3)}_{sj\mu}\big)^\top,\\
d_{sj}(y,-1) &= \frac{1}{2\pi}\, G_{\varkappa}(y,0)P^-_{sj}(y)D(y)e^{i\pi(\frac12-\gamma-i\delta)},\\
d_{sj}(y,+1) &= \frac{1}{2\pi}\, G_{\varkappa}(y,0)P^+_{sj}(y)D(y), \quad s=\overline{1,l_0}, \ \ j=\overline{0,n_s-1},\\
d^{(m)}_{sj}(y,-1) &= \frac{1}{2\pi}\, G_{\varkappa_m}(y,0)P^{-(m)}_{sj}(y)D(y)e^{i\pi(\frac12-\gamma-i\delta)},\\
d^{(m)}_{sj}(y,+1) &= \frac{1}{2\pi}\, G_{\varkappa_m}(y,0)P^{+(m)}_{sj}(y)D(y), \ \ s=\overline{1,l^{(m)}_0}, \ \ j=\overline{0,n^{(m)}_s-1},
\end{aligned}$$

where

$$\begin{aligned}
d_{sj} &= [d^{(l,k)}_{sj}]_{5\times 3}, \quad d^{(m)}_{sj} = \big[d^{(m)(l,k)}_{sj}\big]_{4\times 3},\\
P^{\pm}_{sj} &= V^{(s)}_{-1,j}(y,0,\pm1)\mathfrak{S}_{\mathcal{P}^{-1}\cdot C'}(y,0,\pm1),\\
P^{\pm(m)}_{sj} &= V^{(m)(s)}_{-1,j}(y,0,\pm1)\mathfrak{S}_{(\mathcal{P}^{(m)})^{-1}\cdot C'}(y,0,\pm1),
\end{aligned}$$

and $G_{\varkappa}$, $G_{\varkappa_m}$, $V^{(s)}_{-1,j}$, $V^{(m)(s)}_{-1,j}$, are defined in Section 5, whereas the coefficients $c_j = (c^{(1)}_j, c^{(2)}_j, c^{(3)}_j)^\top$ are defined similarly to the comparable coefficients from (5.63).

Consider the media possessed tetragonal or hexagonal symmetry. In this case

$$\mathfrak{S}_{\widetilde{D}_{\tau,y}}(0,\pm1) = \begin{bmatrix} D^{\pm}_{11} - D^{\pm}_{15}(D^{\pm}_{55})^{-1}D^{\pm}_{51} & 0 & 0\\ 0 & D^{\pm}_{22} & D^{\pm}_{23}\\ 0 & D^{\pm}_{32} & D^{\pm}_{33}\end{bmatrix},$$

where $D^{\pm}_{jk}$, $j,k=1,2,3,5$ are defined in Section 5. To find the eigenvalues of the matrix

$$\big[\,\mathfrak{S}_{\widetilde{D}_{\tau,y}}(0,+1)\,\big]^{-1}\mathfrak{S}_{\widetilde{D}_{\tau,y}}(0,-1), \ \ y\in\partial\Gamma^{(m)}_C, \tag{6.94}$$

take into account that $D^+_{kk} = D^-_{kk}$, $k=1,2,3,5$, $D^{\pm}_{15} = D^{\pm}_{51}$ and $D^{\pm}_{23} = -D^{\pm}_{32}$. Then the characteristic equation of the matrix (6.94) reads

$$\det\begin{bmatrix} (1-\mu)D^-_{11} - D^-_{15}\left(D^-_{55}\right)^{-1}D^-_{51} & 0 & 0\\ 0 & (1-\mu)D^-_{22} & (1+\mu)D^-_{23}\\ 0 & (1+\mu)D^-_{32} & (1-\mu)D^-_{33}\end{bmatrix} = 0.$$

From where we get that the eigenvalues of the matrix (6.94) are

$$\mu_1 = 1, \quad \mu_2 = \frac{1-q}{1+q} > 0, \quad \mu_3 = \frac{1}{\mu_2} > 0,$$

where

$$q = \frac{|D_{23}^-|^2}{D_{22}^- D_{33}^-} \,,$$

and

$$\gamma_j = 0, \;\; j = 1,2,3, \quad \delta_1 = 0, \;\; \delta_2 = -\delta_3 = \widetilde{\delta} = \frac{1}{2\pi} \log \frac{1-q}{1+q} \,.$$

The matrix (6.94) is self-adjoint and therefore is similar to a diagonal matrix. Then $B_0(t) = I_{3\times 3}$ and we obtain the asymptotic expansion

$$U = c_0 r^{\frac{1}{2}} + c_1 r^{\frac{1}{2}+i\widetilde{\delta}} + c_2 r^{\frac{1}{2}-i\widetilde{\delta}} + \cdots \,,$$
$$U^{(m)} = c_0^{(m)} r^{\frac{1}{2}} + c_1^{(m)} r^{\frac{1}{2}+i\widetilde{\delta}} + c_2^{(m)} r^{\frac{1}{2}-i\widetilde{\delta}} + \cdots \,.$$

Consequently, the solution is $C^{\frac{1}{2}}$-smooth in a one sided closed neighborhood of the curve $\partial\Gamma_C^{(m)}$. The first coefficients have the same structure as in (5.72)–(5.75), in particular

$$c_0 = c_0(y,\alpha) = \sum_{\mu=\pm 1} \sum_{s=1}^{5} c_0^{(1)}(y) c_{s0\mu}^{(1)}(y,\alpha),$$
$$c_0^{(m)} = c_0^{(m)}(y,\alpha) = \sum_{\mu=\pm 1} \sum_{j=1}^{3} c_j^{(1)}(y) c_{1j\mu}^{((m),1)}(y,\alpha),$$

where the vector $c_{s0\mu}^{(1)}$ is composed of the first column of the matrix $c_{s0\mu} = [c_{s0\mu}^{(l,k)}]_{5\times 3}$, while the vector $c_{1j\mu}^{((m),1)}$ is composed of the first column of the matrix $c_{1j\mu} = [c_{1j\mu}^{(m)(l,k)}]_{4\times 3}$. The coefficients $c_0^{(1)}$ and $c_j^{(1)}$ are the first coefficients of the vector c_0 and c_j, respectively. In our case they read as

$$c_0^{(1)}(y) = -2\pi\, \widetilde{b}_0^{(1)}(y), \quad c_j^{(1)}(y) = \sqrt{\pi}\, i^j\, \Gamma(j-1/2)\, \widetilde{b}_0^{(1)}(y),$$

where $\widetilde{b}_0^{(1)}$ is the first component of the vector $D^{-1} b_0$.

Remark, that we have the same asymptotic expansion of the solution in the case, when the piezoelectric medium belongs to the class of cubic anisotropy.

For numerical calculations consider the same example as in the previous section: the domain $\Omega^{(m)}$ is occupied by the isotropic metallic material *silver-palladium alloy* whereas the domain Ω is occupied by the piezoelectric material possessing the crystal symmetry either of the class **4mm** or the class **6mm** (e.g. $BaTiO_3$, PZT-4 or PZT-5A).

The graphs presented below show that the stress singularity exponents as well as the stress oscillation parameters depend on the piezoelectric constants. We have calculated the deviations $\gamma^{(j)}$ and $\gamma^{(3)}$ of the stress singularity exponents γ' and γ from the value $-0,5$:

$$\gamma^{(j)} = |-0,5 - \gamma'_j|, \;\; j = 1,2, \quad \gamma^{(3)} = |-0,5 - \gamma|,$$

and oscillation parameters

$$\delta^{(1)} = \max_{1\leqslant k\leqslant 5} \sup_{x\in\partial S_D} |\delta_k|,$$

$$\delta^{(2)} = \max_{1\leqslant k\leqslant 5} \sup_{x\in\partial \Gamma_T^{(m)}} |\delta_k|,$$

$$\delta^{(3)} = \delta = \max_{1\leqslant k\leqslant 3} \sup_{x\in\partial \Gamma_C^{(m)}} |\delta_k|,$$

which determine stress oscillating singularity effects at the exceptional curves ∂S_D and $\partial\Gamma_T^{(m)}$ respectively.

We carried out calculations for PZT-4 with the constants te_{kj} instead of e_{kj}, where $1 < t < 2$. The corresponding graphs are presented in Figures 10 and 11.

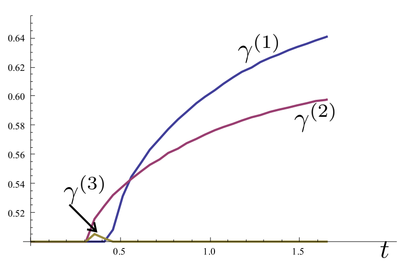

FIGURE 10

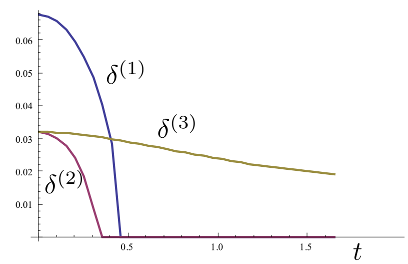

FIGURE 11

We see that the deviation of the stress singularity exponents from -0.5 (the value for the materials without piezoelectric properties) near the crack edge $\partial\Gamma_C$ is significantly less then near ∂S_D or $\partial\Gamma_T^{(m)}$ and differs from zero only in a small range of t whereas the corresponding oscillation parameter $\delta^{(3)}$ is nonzero in full range of t.

The graphs in Figures 12-13 reveal that the stress singularity exponent $\gamma^{(3)}$ and the oscillation parameter $\delta^{(3)}$ depend on the angle β between the symmetry axis of the piezoelectric material and the normal of surface at the reference point $x \in \partial\Gamma_C$ as well.

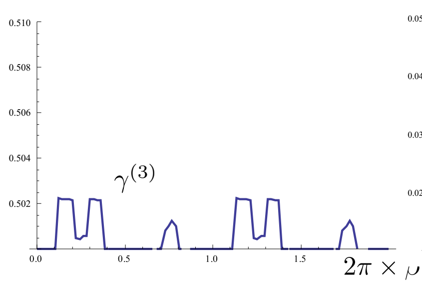

FIGURE 12

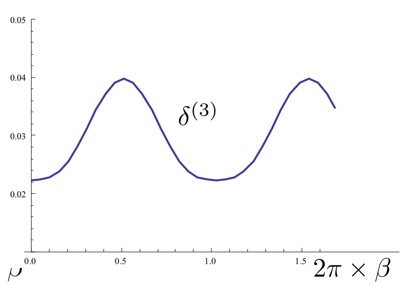

FIGURE 13

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