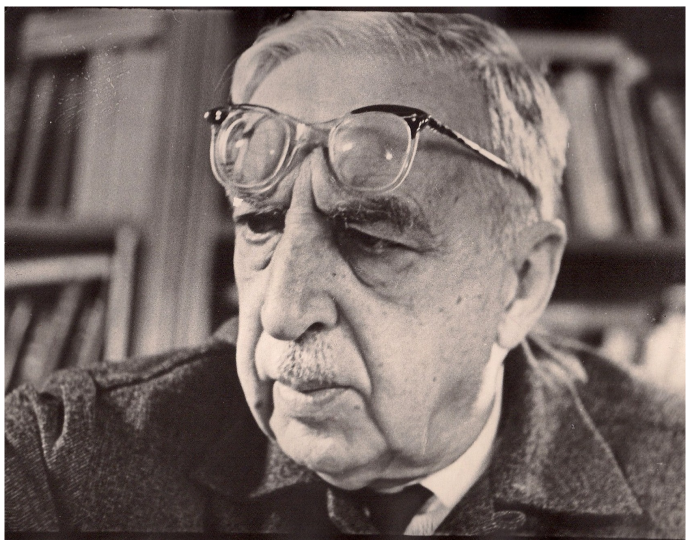
This issue is dedicated
to the memory of Academician Niko Muskhelishvili
on the occasion of his 120th birthday anniversary



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UPPER AND LOWER SOLUTIONS METHOD FOR IMPULSIVE DIFFERENTIAL EQUATIONS INVOLVING THE CAPUTO FRACTIONAL DERIVATIVE

Abstract. For impulsive differential equations involving the Caputo fractional derivative, sufficient conditions for the solvability of initial value problem are established using the lower and upper solutions method and Schauder's fixed point theorem.

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Key words and phrases. Initial value problem, fractional differential equations, impulses, Caputo fractional derivative, fractional integral, lower and upper solution, fixed point.

რეზიუმე. კაპუტოს წილადწარმოებულიანი იმპულსური დიფერენციალური განტოლებებისათვის ქვედა და ზედა ამონახსნების მეთოდის და შაუდერის უძრავი წერტილის პრინციპის გამოყენებით დადგენილია საწყისი ამოცანის ამოხსნადობის საკმარისი პირობები.

1. Introduction

This paper is concerned the existence of solutions for the initial value problems (IVP for short), for impulsive fractional order differential equation

$$
\begin{aligned}
&{}^cD^\alpha y(t) = f(t, y(t)) \\
&\text{for each } \ t \in J = [0, T], \ \ t \neq t_k, \ \ k = 1, \dots, m, \ \ 0 < \alpha \leq 1,
\end{aligned} \tag{1}
$$

$$
\Delta y|_{t=t_k} = I_k(y(t_k^-)), \ \ k = 1, \dots, m, \tag{2}
$$

$$
y(0) = y_0, \tag{3}
$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R}$ is a continuous function, $I_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \dots, m$ and $y_0 \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y\big|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim\limits_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim\limits_{h \to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$, $k = 1, \dots, m$.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [12, 16, 17, 19, 25, 26, 28]). There has been a significant development in fractional differential and partial differential equations in recent years; see the monographs of Kilbas *et al.* [21], Kiryakova [22], Lakshmikantham *et al.* [24], Miller and Ross [27], Samko *et al.* [32] and the papers of Agarwal *et al.* [1, 2], Belarbi *et al.* [5, 6], Benchohra *et al.* [7, 8, 10], Diethelm *et al.* [12, 13, 14], Furati and Tatar [15], Kilbas and Marzan [20], Mainardi [25], Podlubny *et al.* [31], and the references therein.

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $y(0)$, $y'(0)$, etc. the same requirements of boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann–Liouville and Caputo types see [18, 30].

Integer order impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance the monographs by Bainov and Simeonov [4], Benchohra *et al.* [9], Lakshmikantham *et al.* [23], and Samoilenko and Perestyuk [33] and the references therein. In [3, 11] Agarwal *et al.* and Benchohra and Slimani have initiated the study of fractional differential equations with impulses.

By means of the concept of upper and lower solutions combined with Schauder's fixed point theorem, we present an existence result for the problem (1)–(3). This paper initiates the application of the upper and lower solution method to impulsive fractional differential equations at fixed moments of impulse.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $[a,b]$ be a compact interval. $C([a,b],\mathbb{R})$ be the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\left\{|y(t)| : \ a \le t \le b\right\},$$

and we let $L^1([a,b],\mathbb{R})$ the Banach space of functions $y : [a,b] \longrightarrow \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_a^b |y(t)|\, dt.$$

Definition 2.1 ([21, 29])**.** The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.2 ([21, 29])**.** For a function h given on the interval $[a,b]$, the αth Riemann–Liouville fractional-order derivative of h, is defined by

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)\, ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3 ([21])**.** For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of h, is defined by

$$({}^c D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)\, ds,$$

where $n = [\alpha] + 1$.

3. Main Result

Consider the following space

$$PC(J,\mathbb{R}) = \Big\{ y : \ J \to \mathbb{R} : y \in C((t_k, t_{k+1}],\mathbb{R}), \ k = 0,\dots,m+1 \\ \text{and there exist } y(t_k^-) \text{ and } y(t_k^+), \ k = 1,\dots,m \text{ with } y(t_k^-) = y(t_k) \Big\}.$$

$C(J,\mathbb{R})$ is a Banach space with norm

$$\|y\|_{PC} = \sup_{t\in J} |y(t)|.$$

Set $J' := [0,T] \setminus \{t_1,\dots,t_m\}$.

Definition 3.1. A function $y \in PC(J,\mathbb{R}) \cap C^1(J',\mathbb{R})$ is said to be a solution of (1)–(3) if satisfies the differential equation ${}^cD^\alpha y(t) = f(t,y(t))$ on J', and conditions

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m,$$

and

$$y(0) = y_0$$

are satisfied.

Definition 3.2. A function $u \in PC(J,\mathbb{R}) \cap C^1(J',\mathbb{R})$ is said to be a lower solution of (1)–(3) if ${}^cD^\alpha u(t) \le f(t,u(t))$ on J', $\Delta u|_{t=t_k} \le I_k(u(t_k^-))$, $k = 1,\dots,m$, and $u(0) \le y_0$. Similarly, a function $v \in PC(J,\mathbb{R}) \cap C^1(J',\mathbb{R})$ is said to be an upper solution of (1)–(3) if ${}^cD^\alpha v(t) \ge f(t,v(t))$ on J', $\Delta v|_{t=t_k} \ge I_k(v(t_k^-))$, $k = 1,\dots,m$, and $v(0) \ge y_0$.

For the existence of solutions for the problem (1)–(3), we need the following auxiliary lemmas:

Lemma 3.3 ([21])**.** *Let $\alpha > 0$. Then the differential equation*

$${}^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1}t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0,1,2,\dots,n-1$, $n = [\alpha]+1$.

Lemma 3.4 ([21])**.** *Let $\alpha > 0$. Then*

$$I^{\alpha\,c}D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1}t^{n-1} + I^\alpha h(t)$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\dots,n-1$, $n = [\alpha]+1$.

As a consequence of Lemma 3.3 and Lemma 3.4 we have the following result which is useful in what follows. The proof may be found in [11]. For the completeness we present it.

Lemma 3.5. *Let $0 < \alpha \le 1$ and let $\rho \in PC(J,\mathbb{R})$. A function $y \in PC(J,\mathbb{R})$ is a solution of the fractional integral equation*

$$y(t) = \begin{cases} y_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}\rho(s)\,ds \quad \text{if } t \in [0,t_1], \\ y_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\rho(s)\,ds + \dfrac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}\rho(s)\,ds + \\ \qquad + \displaystyle\sum_{i=1}^k I_i(y(t_i^-)), \quad \text{if } t \in (t_k,t_{k+1}],\ k = 1,\dots,m \end{cases} \tag{4}$$

if and only if $y \in PC(J,\mathbb{R}) \cap C^1(J',\mathbb{R})$ *is a solution of the fractional IVP*

$$^cD^\alpha y(t) = \rho(t) \ \ \text{for each} \ \ t \in J', \tag{5}$$

$$\Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{6}$$

$$y(0) = y_0. \tag{7}$$

Proof. Assume that y satisfies (5)–(7). If $t \in [0,t_1]$, then

$$^cD^\alpha y(t) = \rho(t).$$

Lemma 3.4 implies

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\rho(s)\,ds.$$

If $t \in (t_1,t_2]$, then Lemma 3.4 implies

$$y(t) = y(t_1^+) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\rho(s)\,ds =$$

$$= \Delta y\big|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\rho(s)\,ds =$$

$$= I_1(y(t_1^-)) + y_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}\rho(s)\,ds +$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\rho(s)\,ds.$$

If $t \in (t_2,t_3]$, then from Lemma 3.4 we get

$$y(t) = y(t_2^+) + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t (t-s)^{\alpha-1}\rho(s)\,ds =$$

$$= \Delta y\big|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t (t-s)^{\alpha-1}\rho(s)\,ds =$$

$$= I_2(y(t_2^-)) + I_1(y(t_1^-)) + y_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}\rho(s)\,ds +$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\rho(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t (t-s)^{\alpha-1}\rho(s)\,ds.$$

If $t \in (t_k,t_{k+1}]$, then again from Lemma 3.4 we get (4).

Conversely, assume that y satisfies the impulsive fractional integral equation (4). If $t \in [0, t_1]$, then $y(0) = y_0$ and using the fact that ${}^cD^\alpha$ is the left inverse of I^α we get

$$ {}^cD^\alpha y(t) = \rho(t) \quad \text{for each} \quad t \in [0, t_1]. $$

If $t \in [t_k, t_{k+1})$, $k = 1, \ldots, m$ and using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we get

$$ {}^cD^\alpha y(t) = \rho(t) \quad \text{for each} \quad t \in [t_k, t_{k+1}). $$

Also, we can easily show that

$$ \Delta y\Big|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m. \qquad \square $$

For the study of this problem we first list the following hypotheses:

(H1) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is jointly continuous;

(H2) There exist u and $v \in PC \cap C^1(J', \mathbb{R})$, lower and upper solutions for the problem (1)–(3) such that $u \leq v$.

(H3)

$$ u(t_k^+) \leq \min_{y \in [u(t_k^-), v(t_k^-)]} I_k(y) \leq \max_{y \in [u(t_k^-), v(t_k^-)]} I_k(y) \leq v(t_k^+), \quad k = 1, \ldots, m. $$

Theorem 3.6. *Assume that hypotheses* (H1)–(H3) *hold. Then the problem* (1)–(3) *has at least one solution* y *such that*

$$ u(t) \leq y(t) \leq v(t) \quad \textit{for all} \quad t \in J. $$

Proof. Transform the problem (1)–(3) into a fixed point problem. Consider the following modified problem,

$$ {}^cD^\alpha y(t) = f_1(t, y(t)), \quad t \in J, \quad t \neq t_k, \quad k = 1, \ldots, m, \quad 0 < \alpha \leq 1, \tag{8} $$

$$ \Delta y|_{t=t_k} = I_k(\tau(t_k^-, y(t_k^-)), \quad k = 1, \ldots, m, \tag{9} $$

$$ y(0) = y_0, \tag{10} $$

where

$$ f_1(t, y) = f(t, \tau(t, y)), $$
$$ \tau(t, y) = \max\{u(t), \min(y, v(t))\}. $$

A solution for (8)–(10) is a fixed point of the operator $N : PC(J, \mathbb{R}) \longrightarrow PC(J, \mathbb{R})$ defined by

$$ N(y)(t) = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f_1(s, \overline{y}(s))\, ds + $$
$$ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} f_1(s, \overline{y}(s))\, ds + \sum_{0<t_k<t} I_k\big(\tau(t_k^-, y(t_k^-))\big). $$

Note that f_1 is a continuous function and from (H2) there exists $M > 0$ such that

$$|f_1(t,y)| \leq M \text{ for each } t \in J \text{ and } y \in \mathbb{R}. \tag{11}$$

Also, by the definition of τ and from (H3) we have

$$u(t_k^+) \leq I_k(\tau(t_k, y(t_k))) \leq v(t_k^+), \quad k = 1, \dots, m. \tag{12}$$

Set

$$\eta = |y_0| + \frac{M}{\Gamma(\alpha+1)} \sum_{k=1}^{m} (t_k - t_{k-1})^\alpha + + \frac{MT^\alpha}{\Gamma(\alpha+1)} + \sum_{k=1}^{m} \max\left\{|u(t_k^+)|, |v(t_k^+)|\right\}$$

and consider the subset

$$D = \left\{y \in PC(J,\mathbb{R}) : \|y\|_{PC} \leq \eta\right\}.$$

Clearly D is a closed, convex subset of $PC(J,\mathbb{R})$ and N maps D into D. We shall show that N satisfies the assumptions of Schauder's fixed point theorem. The proof will be given in several steps.

Step 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in D. Then for each $t \in J$

$$\begin{aligned}
&\left|N(y_n)(t) - N(y)(t)\right| \leq \\
&\leq \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \left|f_1(s, \overline{y}_n(s)) - f_1(s, \overline{y}(s))\right| ds + \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \left|f_1(s, \overline{y}_n(s)) - f_1(s, \overline{y}(s))\right| ds \\
&+ \sum_{0<t_k<t} \left|I_k\big(\tau\big(t_k^-, y_n(t_k^-)\big)\big) - I_k\big(\tau\big(t_k^-, y(t_k^-)\big)\big)\right|.
\end{aligned}$$

Since f_1, I_k, $k = 1, \dots, m$, and τ are continuous functions, we have

$$\|N(y_n) - N(y)\|_{PC} \to 0 \text{ as } n \to \infty.$$

Step 2: *$N(D)$ is bounded.*

This is clear since $N(D) \subset D$ and D is bounded.

Step 3: *$N(D)$ is equicontinuous.*

Let τ_1, $\tau_2 \in J$, $\tau_1 < \tau_2$, and $y \in D$. Then

$$\left|N(\tau_2) - N(\tau_1)\right| = \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<\tau_2-\tau_1} \int_{t_{k-1}}^{t_k} \left|(t_k - s)^{\alpha-1}\right| \left|f_1(s, \overline{y}(s))\right| ds +$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\big|f_1(s,\overline{y}(s))v|\,ds+$$

$$+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\big|(\tau_2-s)^{\alpha-1}\big|\,\big|f_1(s,\overline{y}(s))\big|\,ds+\sum_{0<t_k<\tau_2-\tau_1}\Big|I_k\big(\tau\big(t_k^-,y(t_k^-)\big)\big)\Big|\leq$$

$$\leq\frac{M}{\Gamma(\alpha+1)}\,(t_k-t_{k-1})^\alpha+\frac{M}{\Gamma(\alpha)}\int_0^{\tau_1}\big|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\big|\,ds+$$

$$+\frac{M}{\Gamma(\alpha+1)}\,(\tau_2-\tau_1)^\alpha+\sum_{0<t_k<\tau_2-\tau_1}\Big|I_k\big(\tau\big(t_k^-,y(t_k^-)\big)\big)\Big|.$$

As $\tau_1 \longrightarrow \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1–3 together with the Arzelá–Ascoli theorem, we can conclude that $N : D \to D$ is continuous and compact. From Schauder's theorem we deduce that N has a fixed point y which is a solution of the problem (8)–(10).

Step 4: *The solution y of* (8)–(10) *satisfies*

$$u(t)\leq y(t)\leq v(t)\ \text{ for all }\ t\in J.$$

Let y be the above solution of (8)–(10). We prove that

$$y(t)\leq v(t)\ \text{ for all }\ t\in J.$$

Assume that $y-v$ attains a positive maximum on $[t_k^+,t_{k+1}^-]$ at $\overline{t}_k\in[t_k^+,t_{k+1}^-]$ for some $k=0,\dots,m$; that is,

$$(y-v)(\overline{t}_k)=\max\big\{y(t)-v(t):\ t\in[t_k^+,t_{k+1}^-]\big\}>0\ \text{ for some }\ k=0,\dots,m.$$

We distinguish the following cases.

Case 1. If $\overline{t}_k\in(t_k^+,t_{k+1}^-)$, there exists $t_k^*\in(t_k^+,t_{k+1}^-)$ such that

$$y(t_k^*)-v(t_k^*)\leq 0, \tag{13}$$

and

$$y(t)-v(t)>0\ \text{ for all }\ t\in(t_k^*,\overline{t}_k]. \tag{14}$$

By the definition of τ one has

$${}^cD^\alpha y(t)=f(t,v(t))\ \text{ for all }\ t\in[t_k^*,\overline{t}_k].$$

An integration on $[t_k^*,t]$ for each $t\in[t_k^*,\overline{t}_k]$ yields

$$y(t)-y(t_k^*)=\frac{1}{\Gamma(\alpha)}\int_{t_k^*}^{t}(t-s)^{\alpha-1}f(s,v(s))\,ds. \tag{15}$$

From (15) and using the fact that v is an upper solution to (1)–(3) we get

$$y(t)-y(t_k^*)\leq v(t)-v(t_k^*). \tag{16}$$

Thus from (13), (14) and (16) we obtain the contradiction

$$0 < y(t) - v(t) \leq y(t_k^*) - v(t_k^*) \leq 0 \text{ for all } t \in [t_k^*, \bar{t}_k].$$

Case 2. If $\bar{t}_k = t_k^+$, $k = 1, \ldots, m$, then

$$v(t_k^+) < I_k(\tau(t_k^-, y(t_k^-)) \leq v(t_k^+),$$

which is a contradiction. Thus

$$y(t) \leq v(t) \text{ for all } t \in [0, T].$$

Analogously, we can prove that

$$y(t) \geq u(t) \text{ for all } t \in [0, T].$$

This shows that the problem (8)–(10) has a solution in the interval $[u, v]$ which is solution of (1)–(3). □

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THREE-DIMENSIONAL BOUNDARY VALUE PROBLEMS OF THE THEORY OF CONSOLIDATION WITH DOUBLE POROSITY

Abstract. The purpose of this paper is to consider three-dimensional version of Aifantis' equations of statics of the theory of consolidation with double porosity and to study the uniqueness and existence of solutions of basic boundary value problems (BVPs). In this work we intend to extend the potential method and the theory of integral equation to BVPs of the theory of consolidation with double porosity. Using these equations, the potential method and generalized Green's formulas, we prove the existence and uniqueness theorems of solutions for the first and second BVPs for bounded and unbounded domains. For Aifantis' equation of statics we construct one particular solution and we reduce the solution of basic BVPs of the theory of consolidation with double porosity to the solution of the basic BVPs for the equation of an isotropic body.

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Key words and phrases. Porous media, double porosity, consolidation, fundamental solution.

რეზიუმე. ნაშრომის მიზანია აიფანტისის ორგვარი ფოროვნობის კონსოლიდაციის თეორიის სტატიკის განტოლებების განხილვა სამი განზომილების შემთხვევაში და ძირითადი სასაზღვრო ამოცანებისათვის ამონახსნების არსებობისა და ერთადერთობის საკითხების შესწავლა. ამ ამოცანების შესასწავლად ნაშრომში გამოყენებულია პოტენციალთა მეთოდი და ინტეგრალური განტოლებები ორგვარი ფოროვნობის კონსოლიდაციის თეორიისათვის. პირველი და მეორე სასაზღვრო ამოცანებისათვის ფრედჰოლმის ინტეგრალური განტოლებებისა და გრინის ფორმულების გამოყენებით დამტკიცებულია მათი ამონახსნების არსებობა და ერთადერთობა, როგორც სასრული, ისე უსასრულო არეებისათვის. აიფანტისის ორგვარი ფოროვნობის კონსოლიდაციის თეორიის სტატიკის განტოლებებისათვის ერთი კერძო ამონახსნის აგებით სტატიკის ამოცანების ამოხსნა მიყვანილია იზოტროპული დრეკადი ტანის სტატიკის ძირითადი ამოცანების ამოხსნაზე.

Introduction

In a material with two degrees of porosity, there are two pore systems, the primary and the secondary. For example, in a fissured rock (i.e., a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or secondary porosity. When fluid flow and deformation processes occur simultaneously, three coupled partial differential equations can be derived [1], [2] to describe the relationships governing pressure in the primary and secondary pores (and therefore the mass exchange between them) and the displacement of the solid.

A theory of consolidation with double porosity has been proposed by Aifantis. The physical and mathematical foundations of the theory of double porosity were considered in the papers [1]–[3], where analytical solutions of the relevant equations are also given. In part I of a series of papers on the subject, R. K. Wilson and E. C. Aifantis [1] gave detailed physical interpretations of the phenomenological coefficients appearing in the double porosity theory. They also solved several representative boundary value problems. In part II of that series, uniqueness and variational principles were established by D. E. Beskos and E. C. Aifantis [2] for the equations of double porosity, while in part III M. Y. Khaled, D. E. Beskos and E. C. Aifantis [3] provided a related finite element to consider the numerical solution of Aifantis' equations of double porosity (see [1]–[3] and the references therein). The basic results and the historical information on the theory of porous media were summarized by R. de. Boer in [4]. The fundamental solution in the theory of consolidation with double porosity is given in [5].

In this work we prove the existence and uniqueness theorems of solutions of basic BVPs of the theory of consolidation with double porosity for bounded and unbounded domains. For the proof of all theorems we used the method given in [6].

1. Formulation of Boundary Value Problems and Uniqueness Theorems

The basic equations of statics of the theory of consolidation with double porosity are given by the partial differential equations in the form ([1], [2])

$$A(\partial x)u = \operatorname{grad}(\beta_1 p_1 + \beta_2 p_2), \tag{1.1}$$

$$(m_1\Delta - k)p_1 + kp_2 = 0, \quad kp_1 + (m_2\Delta - k)p_2 = 0, \tag{1.2}$$

$$A(\partial x)u = \mu\Delta u + (\lambda + \mu)\operatorname{grad}\operatorname{div} u, \tag{1.3}$$

where $u = (u_1, u_2, u_3)$ is the displacement vector, p_1 is the fluid pressure within the primary pores and p_2 is the fluid pressure within the secondary pores, $m_j = \frac{k_j}{\mu^*}$, $j = 1, 2$. The constant λ is the Lamé modulus, μ is the shear modulus and the constants β_1 and β_2 measure the change of porosities due to an applied volumetric strain. The constant μ^* denotes the viscosity of the pore fluid and the constant k measures the transfer of fluid from the secondary pores to the primary pores. All quantities λ, μ, β_j, k $(j=1,2)$ and μ^* are positive constants; $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the Laplace operator.

Let $D^+(D^-)$ be a bounded (an unbounded) three-dimensional domain surrounded by the surface S. $\overline{D}^+ = D^+ \cup S$; $D^- = E_2 \setminus \overline{D}^+$. Suppose that $S \in C^{1,\alpha}$, $0 < \alpha \leq 1$.

First of all, we introduce the definition of a regular vector-function.

Definition 1. A vector-function $U(U_1,U_2,U_3,U_4,U_5) = (u_1,u_2,u_3,p_1,p_2)$ defined in the domain $D^+(D^-)$ is called regular if it has integrable continuous second order derivatives in $D^+(D^-)$, and U and its first order derivatives are continuously extendable at every point of the boundary of $D^+(D^-)$, i.e., $U \in C^2(D^+) \cap C^1(\overline{D^+})$ $(U \in C^2(D^-) \cap C^1(\overline{D^-}))$. Note that for the infinite domain D^- the vector $U(x)$ additionally satisfies the following conditions at infinity:

$$\begin{gathered} U_k(x) = O(|x|^{-1}), \quad \frac{\partial U_k}{\partial x_j} = O(|x|^{-2}), \\ |x|^2 = x_1^2 + x_2^2 + x_3^2, \ \ k = 1,2,\dots,5, \ \ j = 1,2,3. \end{gathered} \tag{1.4}$$

For the equations (1.1)–(1.2) we pose the following boundary value problems:

Find a regular vector U satisfying in $D^+(D^-)$ the equations (1.1)–(1.2), and on the boundary S one of the following conditions is given:

Problem 1. The displacement vector and the fluid pressures are given on S :

$$u^\pm(z) = f(z)^\pm, \quad p_1^\pm(z) = f_4^\pm(z), \quad p_2^\pm(z) = f_5^\pm(z), \ \ z \in S.$$

Problem 2. The stress vector and the normal derivatives of the pressure functions $\frac{\partial p_j}{\partial n}$ are given on S :

$$(Pu(z))^\pm = f(z)^\pm, \quad \Big(\frac{\partial p_1(z)}{\partial n}\Big)^\pm = f_4^\pm(z), \quad \Big(\frac{\partial p_2(z)}{\partial n}\Big)^\pm = f_5^\pm(z), \ \ z \in S.$$

Problem 3.

$$u^\pm(z) = f(z)^\pm, \quad \Big(\frac{\partial p_1(z)}{\partial n}\Big)^\pm = f_4^\pm(z), \quad \Big(\frac{\partial p_2(z)}{\partial n}\Big)^\pm = f_5^\pm(z), \ \ z \in S.$$

Problem 4.

$$(Pu(z))^\pm = f(z)^\pm, \quad p_1^\pm(z) = f_4^\pm(z), \quad p_2^\pm(z) = f_5^\pm(z), \ \ z \in S,$$

where $(\cdot)^\pm$ denote the limiting values on S from $D^\pm$ and $f = (f_1, f_2, f_3)$, f_4, f_5 are given functions. $Pu(x)$ is the stress vector which acts on an element of the surface with the exterior to D^+ unit normal vector $n(x) = (n_1(x), n_2(x), n_3(x))$ at the point $x \in S$,

$$P(\partial x, n)u = T(\partial x, n)u - n(\beta_1 p_1 + \beta_2 p_2), \tag{1.5}$$

where [6]

$$\begin{aligned} &T(\partial x, n) = \| T_{kj}(\partial x, n) \|_{3\times 3}, \\ &T_{kj}(\partial x, n) = \mu\delta_{kj}\,\frac{\partial}{\partial n} + \lambda n_k\,\frac{\partial}{\partial x_j} + \mu n_j\,\frac{\partial}{\partial x_k}\,, \quad k, j, = 1, 2, 3, \\ &\qquad \frac{\partial}{\partial n} = n_1\,\frac{\partial}{\partial x_1} + n_2\,\frac{\partial}{\partial x_2} + n_3\,\frac{\partial}{\partial x_3}\,. \end{aligned} \tag{1.6}$$

Now we introduce the generalized stress vector. Denoting the generalized stress vector by $\overset{\kappa}{\mathrm{P}}(\partial x, n)u$, we have

$$\overset{\kappa}{\mathrm{P}}(\partial x, n)u = \overset{\kappa}{\mathrm{T}}(\partial x, n)u - n(\beta_1 p_1 + \beta_2 p_2),$$

where κ is an arbitrary positive constant and

$$\overset{\kappa}{\mathrm{T}}(\partial x, n)u = (2\mu - \kappa)\,\frac{\partial u}{\partial n} + (\lambda + \kappa)n \operatorname{div} u + (\kappa - \mu)n \times \operatorname{rot} u, \tag{1.7}$$

with $a \times b$ denoting the cross product of two vectors a and b. Further, let us introduce the generalized stress tensor, $\|\sigma_{kj}(\partial x, n)\|_{3\times 3}$: [6]

$$\begin{aligned} &\sigma_{jj} = (\lambda + \kappa) \operatorname{div} u + (2\mu - \kappa)\,\frac{\partial u_j}{\partial x_j}\,, \quad j = 1, 2, 3, \\ &\sigma_{12} = \mu\,\frac{\partial u_2}{\partial x_1} + (\mu - \kappa)\,\frac{\partial u_1}{\partial x_2}, \sigma_{21} = \mu\,\frac{\partial u_1}{\partial x_2} + (\mu - \kappa)\,\frac{\partial u_2}{\partial x_1}\,, \\ &\sigma_{13} = \mu\,\frac{\partial u_3}{\partial x_1} + (\mu - \kappa)\,\frac{\partial u_1}{\partial x_3}, \sigma_{31} = \mu\,\frac{\partial u_1}{\partial x_3} + (\mu - \kappa)\,\frac{\partial u_3}{\partial x_1}\,, \\ &\sigma_{23} = \mu\,\frac{\partial u_3}{\partial x_2} + (\mu - \kappa)\,\frac{\partial u_2}{\partial x_3}, \quad \sigma_{32} = \mu\,\frac{\partial u_2}{\partial x_3} + (\mu - \kappa)\,\frac{\partial u_3}{\partial x_2}\,. \end{aligned} \tag{1.8}$$

If $\kappa = 0$, from (1.7) we have $\overset{\kappa}{\mathrm{T}}(\partial x, n)u = T(\partial x, n)u$. We set $\overset{\kappa}{\mathrm{T}}(\partial x, n)u = N(\partial x, n)u$ for $\kappa = \frac{2\lambda+3\mu}{\lambda+3\mu}$.

Generalized Green's formulas. Let us write the generalized Green's formulas for the domains D^+ and D^-. Let u be a regular solution of the equation (1.1) in D^+. Multiply first equation of (1.1) by u. Integrate the result over D^+ and apply the integration by parts formula to obtain

$$\int\limits_{D^+} \Big[\overset{\kappa}{\mathrm{E}}(u, u) - (\beta_1 p_1 + \beta_2 p_2) \operatorname{div} u\Big]\, dx = \int\limits_{S} u\ \overset{\kappa}{\mathrm{P}}(\partial x, n)u\, dS, \tag{1.9}$$

where

$$\begin{aligned} \overset{\kappa}{\mathrm{E}}(u, u) = \frac{3\lambda + 2\mu - \kappa}{2}\,(\operatorname{div})^2 + \frac{\kappa}{2}\,(\operatorname{rot} u)^2 + \frac{2\mu - \kappa}{4} \sum_{k\neq j} \Big(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}\Big)^2 + \\ + \frac{2\mu - \kappa}{6}\Big[\Big(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\Big)^2 + \Big(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_3}{\partial x_3}\Big)^2 + \Big(\frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3}\Big)^2\Big]. \end{aligned}$$

If the vector u satisfies the conditions (1.4), Green's formula for the domain D^- takes the form

$$\int\limits_{D^-}\Big[\overset{\kappa}{\mathrm{E}}(u,u)-(\beta_1p_1+\beta_2p_2)\operatorname{div}u\Big]\,dx=-\int\limits_S u\,\overset{\kappa}{\mathrm{P}}(\partial x,n)u\,dS. \tag{1.10}$$

For the positive definiteness of the potential energy, the inequalities

$$3\lambda+2\mu-\kappa>0,\quad \kappa>0,\quad \kappa<2\mu$$

are necessary and sufficient.

Analogously we obtain Green's formula for p_j, $j=1,2$,

$$\begin{aligned}&\int\limits_{D^+}\Big[m_1(\operatorname{grad}p_1)^2+m_2(\operatorname{grad}p_2)^2+k(p_1-p_2)^2\Big]\,dx=\\&\qquad=\int\limits_S\Big[m_1p_1\,\frac{\partial p_1}{\partial n}+m_2p_2\,\frac{\partial p_2}{\partial n}\Big]\,dS,\\&\int\limits_{D^-}\Big[m_1(\operatorname{grad}p_1)^2+m_2(\operatorname{grad}p_2)^2+k(p_1-p_2)^2\Big]\,dx=\\&\qquad=-\int\limits_S\Big[m_1p_1\,\frac{\partial p_1}{\partial n}+m_2p_2\,\frac{\partial p_2}{\partial n}\Big]\,dS.\end{aligned} \tag{1.11}$$

Note that if $\beta_1p_1+\beta_2p_2=const$, in view of the equality $\int\limits_{D^+}\operatorname{div}u\,dx=\int\limits_S nu\,dS$ from (1.9) we get

$$\int\limits_{D^+}\overset{\kappa}{\mathrm{E}}(u,u)\,dx=\int\limits_S u\,\overset{\kappa}{\mathrm{T}}(\partial x,n)u\,dS. \tag{1.12}$$

Uniqueness theorems. In this subsection we prove the uniqueness theorems of solutions to the above formulated problems. Let the above formulated problems have two regular solutions $U^{(1)}$ and $U^{(2)}$, where $U^{(k)}=(u_1^{(k)},u_2^{(k)},u_3^{(k)},p_1^{(k)},p_2^{(k)})$, $k=1,2$. We put

$$U=U^{(1)}-U^{(2)}.$$

Evidently, the vector U satisfies the equations (1.1)–(1.2) and the homogeneous boundary conditions

1. $u^\pm(z)=0$, $p_j^\pm(z)=0$, $j=1,2$, $z\in S$,
2. $(P(\partial z,n)u(z))^\pm=0$, $\Big(\dfrac{\partial p_j(z)}{\partial n}\Big)^\pm=0$, $j=3,4$, $z\in S$,
3. $u^\pm(z)=0$, $\Big(\dfrac{\partial p_j(z)}{\partial n}\Big)^\pm=0$, $j=1,2$, $z\in S$,
4. $\big(P(\partial z,n)u\big)^\pm(z)=0$, $p_j^\pm(z)=0$, $j=1,2$, $z\in S$.

Now we prove the following theorems.

Theorem 1. *The first boundary value problem has at most one regular solution in the bounded domain D^+.*

Proof. Evidently, the vector U satisfies (1.1)–(1.2) and the boundary condition $U^+ = 0$ on S. Note that if U is a regular solution of (1.1)–(1.2), we have Green's formulas (1.9), (1.11). Taking into account the fact that the potential energy $\overset{\kappa}{\mathrm{E}}(u,u)$ is positive definite, we conclude that $U = C$, $x \in D^+$, where $C = const$. Since $U^+ = 0$, we have $C = 0$ and $U(x) = 0$, $x \in D^+$. □

Theorem 2. *The first boundary value problem has at most one regular solution in the infinite domain D^-.*

Proof. The vectors $U^{(1)}$ and $U^{(2)}$ in the domain D^- must satisfy the condition (1.4). In this case the formulas (1.11) are valid and $U(x) = C$, $x \in D^-$, where C is again a constant vector. But U on the boundary satisfies the condition $U^- = 0$, which implies that $C = 0$ and $U(x) = 0$, $x \in D^-$. □

Analogously can be proved the following theorems.

Theorem 3. *A regular solution of the second boundary value problem is not unique in the domain D^+. Two regular solutions may differ by a vector (u, p_1, p_2), where $u(x) = a + b \times x + c(\beta_1 + \beta_2)x$, $p_j(x) = c$, $j = 1, 2$, $x \in D^+$, with a and b constant vectors, and c be an arbitrary constant.*

Theorem 4. *Two regular solutions of the boundary value problem $(III)^+$ may differ by the vector (u, p_1, p_2), where $u = 0$ and $p_j = c$, $j = 1, 2$, with c be an arbitrary constant.*

Theorem 5. *Two regular solutions of the boundary value problem $(IV)^+$ may differ by the vector (u, p_1, p_2), where u is a rigid displacement and $p_j = 0$, $j = 1, 2$.*

Theorem 6. *The boundary value problems $(II)^-$, $(III)^-$, $(IV)^-$ have at most one regular solution in the domain D^-.*

Note that from the equation (1.2) one may define the functions $p_j(x)$, $j = 1, 2$. Further we assume that p_j is known, when $x \in D^+$ or $x \in D^-$. Substitute $\beta_1 p_1 + \beta_2 p_2$ in (1.1) and search a particular solution of the following equation

$$\mu \Delta u + (\lambda + \mu) \,\mathrm{grad}\,\mathrm{div}\, u = \mathrm{grad}(\beta_1 p_1 + \beta_2 p_2).$$

We put

$$u_0 = -\frac{1}{4\pi} \int\limits_D \Gamma(x - y) \,\mathrm{grad}(\beta_1 p_1 + \beta_2 p_2)\, dx, \tag{1.13}$$

where

$$\Gamma(x-y) =$$
$$= \frac{1}{4\mu(\lambda+2\mu)} \left\| \frac{(\lambda+3\mu)\delta_{kj}}{r} + \frac{(\lambda+\mu)(x_k-y_k)(x_j-y_j)}{r^3} \right\|_{3\times 3}, \quad r = |x-y|.$$

Substituting the volume potential u_0 into (1.1), we obtain (see [6])

$$\mu\Delta u_0 + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u_0 = \mathrm{grad}(\beta_1 p_1 + \beta_2 p_2). \tag{1.14}$$

Thus we have proved that $u_0(x)$ is a particular solution of the equation (1.1). In (1.13) D denotes either D^+ or D^-, $\mathrm{grad}(\beta_1 p_1+\beta_2 p_2)$ is a continuous vector in D^+ along with its first order derivatives. When $D = D^-$, the vector $\mathrm{grad}(\beta_1 p_1+\beta_2 p_2)$ has to satisfy the following decay condition at infinity

$$\mathrm{grad}(\beta_1 p_1+\beta_2 p_2) = O(|x|^{-2-\alpha}), \quad \alpha > 0.$$

Thus the general solution of the equation (1.1) is representable in the form $u = V + u_0$, where

$$A(\partial x)V = \mu\Delta V + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,V = 0. \tag{1.15}$$

This equation is the equation of an isotropic elastic body. Thus we have reduced the solution of basic BVPs of the theory of consolidation with double porosity to the solution of the basic BVPs for the equation of an isotropic elastic body.

First of all we will construct a fundamental matrix of solutions for the equation (1.2). We look for p_j in the form

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} m_2\Delta - k & -k \\ -k & m_1\Delta - k \end{pmatrix}\psi, \tag{1.16}$$

where the vector $\psi(x)$ is the fundamental solution of the scalar equation

$$\Delta(\Delta-\lambda_0^2)\psi = 0, \quad \lambda_0^2 = \frac{k}{m_1} + \frac{k}{m_2}\,, \quad \psi = \frac{e^{-\lambda_0 r}-1}{\lambda_0^2 r}\,.$$

From (1.16) it follows that the fundamental matrix of solutions of the equation (1.2) is the following matrix

$$M(x-y) =$$
$$= \begin{pmatrix} m_2\dfrac{e^{-\lambda_0 r}}{r} - \dfrac{k}{\lambda_0^2}\,\dfrac{e^{-\lambda_0 r}-1}{r} & -\dfrac{k}{\lambda_0^2}\,\dfrac{e^{-\lambda_0 r}-1}{r} \\ -\dfrac{k}{\lambda_0^2}\,\dfrac{e^{-\lambda_0 r}-1}{r} & m_1\dfrac{e^{-\lambda_0 r}}{r} - \dfrac{k}{\lambda_0^2}\,\dfrac{e^{-\lambda_0 r}-1}{r} \end{pmatrix}. \tag{1.17}$$

The following theorem is valid:

Theorem 7. *Each column of the matrix $M(x-y)$ is a solution to the equation* (1.2) *with respect to x for $x \neq y$.*

2. Integral Equations of BVPs

A solution of the first boundary value problem ($p_1^{\pm} = f_4^{\pm}$, $p_2^{\pm} = f_5^{\pm}$, $V^{\pm} = F^{\pm}$) in the domains $D^{\pm}$ for the systems (1.2), (1.15) will be sought in the form of the double layer potential

$$\begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix} = \frac{1}{2\pi} \int\limits_S \frac{\partial}{\partial n(y)} M(x-y)\varphi(y)\, d_y S, \tag{2.1}$$

$$V(x) = \frac{1}{\pi} \int\limits_S \big[N(\partial y, n)\Gamma(x-y)\big]^T g(y)\, d_y S, \tag{2.2}$$

where $S \in C^{1,\alpha}$, $\varphi \in C^{0,\beta}$, $g \in C^{0,\beta}$, $0 < \beta < \alpha \le 1$, $M(x-y)$ is given by (1.17),

$$\big[N(\partial y, n)\Gamma(x-y)\big]^T_{kj} =$$

$$= \frac{\partial}{\partial n} \frac{\delta_{kj}}{r} + \sum_{k=1}^{3} M_{kj}(\partial y, n)\Big[\frac{(\lambda+\mu)(x_k-y_k)(x_j-y_j)}{(\lambda+3\mu)r^3}\Big],$$

$$M_{kj} = n_j \frac{\partial}{\partial x_k} - n_k \frac{\partial}{\partial x_j}\,.$$

Then for determining the unknown vectors φ and g we obtain the following system of Fredholm integral equations of the second kind on S

$$\pm(\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix} \varphi(z) + \frac{1}{2\pi} \int\limits_S \frac{\partial}{\partial n} M(z-y)\varphi(y)\, d_y S = \begin{pmatrix} f_4^{\pm}(z) \\ f_5^{\pm}(z) \end{pmatrix}, \tag{2.3}$$

$$\mp g(z) + \frac{1}{\pi} \int\limits_S \big[T(\partial y, n)\Gamma(y-z)\big] g(y)\, d_y S = F^{\pm}(z). \tag{2.4}$$

If a solution of the first BVP is sought in the form

$$V(x) = \frac{1}{\pi} \int\limits_S \big[T(\partial y, n)\Gamma(x-y)\big]^T g(y)\, d_y S, \tag{2.5}$$

for determining of the unknown vector g we obtain the following singular integral equation of the second kind

$$\mp g(z) + \frac{1}{\pi} \int\limits_S \big[T(\partial y, n)\Gamma(y-z)\big] g(y)\, d_y S = F^{\pm}(z). \tag{2.6}$$

A solution of the second boundary value problem $((\frac{\partial p_1}{\partial n})^{\pm} = f_4^{\pm}$, $(\frac{\partial p_2}{\partial n})^{\pm} = f_5^{\pm}$, $(T(\partial x, n)V)^{\pm} = \Phi^{\pm})$ in the domains $D^{\pm}$ for the systems (1.2)–(1.15) will be sought in terms of the single layer potential

$$\begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix} = \frac{1}{2\pi} \int\limits_S M(x-y)\varphi(y)\, d_y S, \tag{2.7}$$

$$V(x) = \frac{1}{\pi} \int\limits_S \Gamma(x-y)h(y)\, d_y S, \tag{2.8}$$

Then for determining the unknown vectors φ and g we obtain the following system of Fredholm integral equations of the second kind

$$\mp\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix}\varphi(z)+\frac{1}{2\pi}\int\limits_S \frac{\partial}{\partial n(z)}\,M(z-y)\varphi(y)\,d_yS=\begin{pmatrix} f_4^{\pm}(z) \\ f_5^{\pm}(z)\end{pmatrix}, \tag{2.9}$$

$$\pm h(z)+\frac{1}{\pi}\int\limits_S T(\partial z,n)\Gamma(z-y)h(y)\,d_yS=\Phi^{\pm}(z), \tag{2.10}$$

where [6]

$$T(\partial y,n)\Gamma(x-y)=$$
$$=\left\|\frac{\partial}{\partial n}\,\frac{\delta_{kj}}{r}+\sum_{k=1}^{3}M_{kj}(\partial y,n)\Big[\frac{2\delta_{kj}}{(\lambda+2\mu)r}+\frac{2(\lambda+\mu)(x_k-y_k)(x_j-y_j)}{(\lambda+2\mu)r^3}\Big]\right\|_{3\times 3}.$$

3. Analysis of the Basic BVPs in the Domains D^+ and D^-

Problem $(I)^+$**.** First let us prove the existence of solution of the first boundary value problem in the domain D^+. Consider the equation (2.3)

$$-\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix}\varphi+\frac{1}{2\pi}\int\limits_S \frac{\partial}{\partial n}\,M(z-y)\varphi(y)\,d_yS=\begin{pmatrix} f_4^{+}(z) \\ f_5^{+}(z)\end{pmatrix}, \tag{3.1}$$

Let us prove that the equation (3.1) is solvable for any continuous right-hand side. To this end, consider the associated to (3.1) homogeneous equation

$$-\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix}\psi(z)+\frac{1}{2\pi}\int\limits_S \frac{\partial}{\partial n}\,M(z-y)\psi(y)\,d_yS=0, \tag{3.2}$$

and prove that it has only the trivial solution. Assume the contrary and denote by ψ_0 a nontrivial solution of (3.2). The equation (3.2) corresponds to the boundary conditions

$$\Big(\frac{\partial p_1}{\partial\nu}\Big)^-=0,\quad \Big(\frac{\partial p_2}{\partial\nu}\Big)^-=0,$$

whence we have $\int\limits_S \psi_k\,ds=0$, $k=4,5$.

Now taking into account the continuity of the simple layer potential and using the uniqueness theorem for the solution of the first boundary value problem, we will have $p_k(x)=c$, $x\in D^-$.

Note that

$$\Big(\frac{\partial p_1}{\partial\nu}\Big)^- - \Big(\frac{\partial p_1}{\partial\nu}\Big)^+ = 2m_2\psi_4=0,\quad \Big(\frac{\partial p_2}{\partial\nu}\Big)^- - \Big(\frac{\partial p_2}{\partial\nu}\Big)^+ = 2m_1\psi_5=0,$$

hence the equation (3.2) has only the trivial solution. This implies that the associated to (3.2) homogeneous equation also has only the trivial solution, and the equation (3.1) is solvable for any continuous right-hand side.

For the regularity of the double layer potential in the domain D^+ it is sufficient to assume that $S \in C^{2,\beta}$ $(0 < \beta < 1)$ and $f_k \in C^{1,\alpha}(S)$ $(0 < \alpha < \beta)$, $k = 4, 5$.

Problem $(I)^-$**.** Now consider the first boundary value problem in the domain D^-. Consider the equation (2.3)

$$\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix} \varphi + \frac{1}{2\pi} \int\limits_S \frac{\partial}{\partial n} M(z-y)\varphi(y)\, d_y S = \begin{pmatrix} f_4^-(z) \\ f_5^-(z) \end{pmatrix}. \tag{3.3}$$

Prove that the equation (3.3) is solvable for any continuous right-hand side. We consider the associated to (3.3) homogeneous equation

$$\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix} \varphi + \frac{1}{2\pi} \int\limits_S \frac{\partial}{\partial \nu} M(z-y)\varphi(y)\, d_y S = 0. \tag{3.4}$$

Let us prove that (3.4) has only the trivial solution. Suppose that it has a nonzero solution $\varphi(z)$. From (3.4) by integration we obtain

$$\int\limits_S \varphi\, dS = 0.$$

In this case the equation (3.4) corresponds to the boundary condition

$$\left(\frac{\partial p_k}{\partial \nu}\right)^+ = 0.$$

We find that $p_k = c$, $x \in D^+$, where c is a constant vector.

Taking into account the equation $\int\limits_S \varphi\, ds = 0$ and the fact that the single layer potential is continuous while passing through the boundary, and using Green's formula for $\kappa = \kappa_n$, we obtain $p_k = 0$, $x \in D^-$. Since

$$\left(\frac{\partial p_1}{\partial \nu}\right)^- - \left(\frac{\partial p_1}{\partial \nu}\right)^+ = 2m_2\varphi_4 = 0, \quad \left(\frac{\partial p_2}{\partial \nu}\right)^- - \left(\frac{\partial p_2}{\partial \nu}\right)^+ = 2m_1\varphi_5 = 0,$$

we have $\varphi(x) = 0$.

Thus we conclude that the associated to (3.4) homogeneous equation has only the trivial solution, and the equation (3.3) is solvable for any continuous right-hand side.

To prove the regularity of the potential (2.1) in the domain D^-, it is sufficient to assume that $S \in C^{2,\beta}(0 < \beta < 1)$ and $f_k \in C^{1,\alpha}(S)$ $(0 < \alpha < \beta)$, $k = 4, 5$.

4. Problems $(1)^+$ and $(2)^-$

Consider the equations (2.4), (2.10)

$$-g(z) + \frac{1}{\pi} \int\limits_S \left[T(\partial y, n)\Gamma(y-z)\right]^T g(y)\, d_y S = F^+(z), \tag{4.1}$$

$$-h(z)+\frac{1}{\pi}\int\limits_S T(\partial z,\nu)\Gamma(y-z)h(y)\,d_yS=\Phi^-(z), \tag{4.2}$$

where $F^+\in C^{1,\beta}(S)$, $\Phi^-\in C^{1,\beta}(S)$, $0<\alpha<\beta$ are given vectors on the boundary.

Let us prove that the homogeneous equation corresponding to (4.2) has only the trivial solution. Assume that it has a nontrivial solution denoted by $h_0(z)$. Compose the simple layer potential

$$V(x)=\frac{1}{4\pi}\int\limits_S \Gamma(y-x)h_0(y)\,dS. \tag{4.3}$$

It is obvious, that $[T(\partial z,n)V(z)]^-=0$, $\int\limits_S h_0(y)\,ds=0$. $V\in C^{0,\beta}(D^-)$ and satisfies the conditions (1.4). This implies that $V(z)=0$, $z\in D^-$, whence $V^+=V^-=0$. Now taking into account the continuity of the simple layer potential and using the uniqueness theorem for the solution of the first boundary value problem, we will have $V(x)=0$, $x\in D^+$. Thus $V(x)$ vanishes on the whole space and therefore $h_0(x)=0$. Due to the fredholm theorem we conclude that the nonhomogeneous equation is solvable for an arbitrary Hölder continuous vector Φ^-.

Finally, from the solvability of the equations (4.1) and (4.2) it follows that the solutions of BVPs $(1)^+$ and $(2)^-$ are representable in the form of second kind double and single-layer potentials, respectively. On the basis of the general theory, the following theorems are valid.

Theorem 8. *If $S\in C^{2,\beta}(S)$ and $F^+\in C^{1,\beta}$, then the BVP* $(1)^+$ *has unique solution. Moreover, this solution is given in the form of the double-layer potential* (2.5)*, where g is a solution of the equation* (4.1).

Theorem 9. *If $S\in C^{2,\beta}(S)$ and $\Phi^-\in C^{1,\beta}$, then the BVP* $(2)^-$ *has unique solution satisfying the conditions* (1.4) *in the neighborhood of infinity. Moreover, this solution is given in the form of the single-layer potential* (2.8)*, where h is a solution of the equation* (4.2).

5. Problems $(1)^-$ and $(2)^+$

Consider the first external BVP (when on S it is given $V^-=F^-$). A solution of the equation (1.15) is sought in the form

$$V(x,g)=\frac{1}{2\pi}\int\limits_S \big[N(\partial y,n)\Gamma(z-x)\big]^*g(y)\,d_yS+\frac{1}{2}\,\Gamma(x)\alpha, \tag{5.1}$$

where

$$\alpha=\frac{1}{2\pi}\int\limits_S \big[N(\partial y,n)\Gamma(y)\big]^*g(y)\,d_yS.$$

The origin is assumed to lie in the domain D^+. Taking into account the boundary behavior of the potential $V(x)$ and the boundary condition, to define the unknown vector g from (5.1) we obtain the Fredholm integral equation of the second kind

$$g(z) + \frac{1}{2\pi} \int\limits_S \big[N(\partial y, n)\Gamma(y - z)\big]^* g(y)\, d_y S + \frac{1}{2}\,\Gamma(z)\alpha = F^-(z). \tag{5.2}$$

The conjugate equation is

$$h(z) + \frac{1}{2\pi} \int\limits_S \Big[N(\partial y, n)\Gamma(y - z) + \frac{1}{2}\, N(\partial z, n)\Gamma(z)\alpha\Big] h(y)\, d_y S = \Phi^+(z). \tag{5.3}$$

Let us show that the equation (5.3) is always solvable. For this it is sufficient to show that the homogeneous equation corresponding to (5.3) has only the trivial solution. Denote the homogeneous equation by $(5.3)_0$ and assume that it has a solution h_0 different from zero.

From $(5.3)_0$ we get

$$\frac{1}{2\pi} \int\limits_S \Gamma(y)h(y)\, d_y S = 0. \tag{5.4}$$

and the equation $(5.3)_0$ obtain the form

$$h(z) + \frac{1}{2\pi} \int\limits_S N(\partial y, n)\Gamma(y - z)h(y)\, d_y S = 0. \tag{5.5}$$

Construct now the potential

$$V(x) = \frac{1}{2\pi} \int\limits_S \Gamma(x - y)h_0(y)\, d_y S.$$

Here $N(V)^+ = 0$ and $V(0) = 0$. From this we get $V(x) = 0$, $x \in D^-$. Since $h_0(x) = 0$. Thus our assumption is not valid. The equation (5.2) has a solution for an arbitrary right-hand side. Note that a solution of the equation (5.2) exists if $S \in C^{2,\beta}(S)$, $F^-(z) \in C^{1,\beta}(S)$, $0 < \beta < \alpha \le 1$.

Consider the second BVP. The solution of the equation (1.15) is sought in the form (when on S it is given $(TV)^+ = \Phi^+$)

$$V(x) = \frac{1}{2\pi} \int\limits_S \Gamma(y - z)h(y)\, d_y S - \frac{1}{2}\,\Gamma(x)A - \frac{1}{2}\,\Gamma_0(x)B, \tag{5.6}$$

where A and B are defined as follows:

$$A = \frac{1}{2\pi} \int\limits_S \Gamma(y)h(y)\, d_y S, \quad B = \frac{1}{2\pi} \int\limits_S \Gamma_0(y)h(y)\, d_y S, \tag{5.7}$$

$$\Gamma_0 = \operatorname{rot}_x \Gamma(x - y)_{x=0}.$$

To define $h(z)$, we have the integral equation

$$h(z)+\frac{1}{2\pi}\int\limits_S T(\partial y,n)\Gamma(y-z)h(y)\,d_yS-$$

$$\frac{1}{4}\,T(\partial z,n)\Gamma(z)A-\frac{1}{4}\,T(\partial z,n)\Gamma_0(z)B=\Phi^+(z), \quad (5.8)$$

Let us now show that the integral equation (5.8) is always solvable. Let $h(y)\neq 0$. From (5.8) we have

$$A=\int\limits_S \Phi^+(z)\,dS, \quad (5.9)$$

$$B=\frac{1}{2\pi}\int\limits_S r(y)\times\Phi^+\,dS, \quad (5.10)$$

where $r(y)=(y_1,y_2,y_3)$. If $\Phi^+=0$, then $A=0$, $B=0$, $(Tu)^+=0$, $u=a+[b,r]$. If the principal vector A and the principal moment B are equal to zero, we have $u=0$, $h=0$. Thus (5.8) is solvable for any right-hand side.

Consider the conjugate equation

$$g(z)+\frac{1}{2\pi}\int\limits_S T(\partial y,n)\Gamma(y-z)^*g(y)\,d_yS-\frac{1}{2}\,\Gamma(z)\alpha-\frac{1}{4}\,\Gamma_0(z)\beta=$$

$$=F^-(z), \quad (5.11)$$

where

$$\alpha=\frac{1}{2\pi}\int\limits_S \big[T(\partial y,n)\Gamma(y)\big]^*g(y)\,d_yS,$$

$$\beta=\frac{1}{2\pi}\int\limits_S \big[T(\partial y,n)\Gamma_0(y)\big]^*g(y)\,d_yS.$$

The equation (5.11) is always solvable if $F^-\in C^{1,\alpha}(S)$, $S\in C^{1,\alpha}(S)$, $0<\beta<\alpha\leq 1$.

If the solution of BVP $(1)^-$ is sought in the form

$$V(x)=\frac{1}{2\pi}\int\limits_S \big[T(\partial y,n)\Gamma(y-z)\big]^*g(y)\,dS-\frac{1}{2}\,\Gamma(x)\alpha-\frac{1}{4}\,\Gamma_0(x)\beta, \quad (5.12)$$

then to define the unknown vector g we obtain the integral equation (5.11).

Therefore we formulate the final result.

Theorem 10. *The problem* $(1)^-$ *is solvable for an arbitrary vector* $F^-\in C^{1,\beta}(S)$ *for* $S\in C^{2,\alpha}(S)$, *and the solution is represented by the formula* (5.12).

Theorem 11. *The problem* $(2)^+$ *is solvable for the vector* $\Phi^+ \in C^{0,\beta}(S)$, *only if the principal vector and the principal moment of external stresses are equal to zero,* $A = 0$ *and* $B = 0$. *The solution is represented by the formula* (5.6). *The solution is defined to within rigid displacement.*

The existence theorems for the third and fourth BVPs will be proved analogously.

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ON THE CHOICE OF INITIAL CONDITIONS OF DIFFERENCE SCHEMES FOR PARABOLIC EQUATIONS

Abstract. We consider the first initial-boundary value problem for linear heat conductivity equation with constant coefficient in $\Omega\times(0,T]$, where Ω is a unit square. A high order accuracy ADI two level difference scheme is constructed on a 18-point stencil using Steklov averaging operators. We prove that the finite difference scheme converges in the discrete L_2-norm with the convergence rate $O(h^s+\tau^{s/2})$, when the exact solution belongs to the anisotropic Sobolev space $W_2^{s,s/2}$, $s\in(2,\,4]$.

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Key words and phrases. Heat equation, ADI difference scheme, high order convergence rate.

რეზიუმე. $\Omega\times(0,\,T]$ არეში, სადაც Ω ერთეულოვანი კვადრატია, განხილულია მუდმივკოეფიციენტიანი სითბოგამტარობის წრფივი განტოლებისათვის დასმული პირველი საწყის-სასაზღვრო ამოცანა. სტეკლოვის გასაშუალების ოპერატორების გამოყენებით 18 წერტილიან შაბლონზე აგებულია მაღალი რიგის სიზუსტის ორშრიანი ცვალებადი მიმართულებით არაცხადი სხვაობიანი სქემა. დამტკიცებულია, რომ თუ ზუსტი ამონახსნი მიეკუთვნება სობოლევის ანიზოტროპულ $W_2^{s,s/2}$, $s\in(2,4]$ სივრცეს, მაშინ სასრულ-სხვაობიანი სქემის დისკრეტული L_2-ნორმით კრებადობის სიჩქარეა $O(h^s+\tau^{s/2})$.

1. Introduction

The purpose of this paper is to study the difference schemes approximating the first initial-boundary value problem for linear second order parabolic equations and to obtain some convergence rate estimates.

The finite difference method is a basic tool for the solution of partial differential equations. When studying the convergence of the finite difference schemes, Taylor's expansion was used traditionally. Often, the Bramble-Hilbert lemma [1], [2] takes the role of Taylor's formula for the functions from the Sobolev spaces.

As a model problem, we consider the first initial-boundary value problem for linear second-order parabolic equations with constant coefficients. We suppose that the generalized solution of this problem belongs to the anisotropic Sobolev space $W_2^{s,s/2}(Q)$, $s > 2$.

In the case of difference schemes constructed for the mentioned problem, when obtaining convergence rate estimate compatible with smoothness of the solution, various authors assume that the solution of the problem can be extended to the exterior of the domain of integration, preserving the Sobolev class.

Our investigations have shown that if instead of the exact initial condition its certain approximation is taken, then this restriction can be removed.

A high order alternating direction implicit (ADI) difference scheme is constructed in the paper for which the convergence rate estimate

$$\|y-u\|_{L_2(Q_{h,\tau})} \le c(h^s+\tau^{s/2})\|u\|_{W_2^{s,s/2}(Q)}, \quad s \in (2,\,4],$$

is obtained. Here y is a solution to the difference scheme, $Q_{h,\tau}$ is a mesh in Q, c is a positive constant independent of h, τ and u, and h and τ are space and time steps, respectively.

2. The Problem and Its Approximation

Let $\Omega = \{x=(x_1,x_2):\ 0 < x_\alpha < 1,\ \alpha = 1,2\}$ be the unit square in R^2 with boundary Γ and let T denote a positive real number. In $Q = \Omega \times (0,T]$ we consider the equation of heat conductivity

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - au + f(x,t), \quad a = const \ge 0, \quad (x,t) \in Q_T, \tag{1}$$

under the initial and first kind boundary conditions

$$u(x,0) = u^0(x), \quad x \in \Omega, \quad u(x,t) = 0, \quad x \in \Gamma, \quad t \in [0,T]. \tag{2}$$

We mean that the solution to the problem (1), (2) belongs to the anisotropic Sobolev space $W_2^{s,s/2}(Q)$, $s > 2$.

Throughout the paper $\|\cdot\|_{W_2^{\lambda,\lambda/2}(Q)}$ will denote the norms and $|\cdot|_{W_2^{\lambda,\lambda/2}(Q)}$ the highest semi norms of corresponding Sobolev spaces [6].

We assume that $\bar\omega$ is a uniform mesh in Ω with the step $h = 1/n$. $\omega = \overline{\omega} \cap \Omega$, $\gamma = \overline{\omega} \setminus \omega$. We cover the segment $[0,T]$ with a uniform mesh $\overline{\omega}_\tau$

(with the mesh step $\tau = T/N$). Let $\omega_\tau = \overline{\omega}_\tau \cap (0,T)$, $\omega_\tau^+ = \overline{\omega}_\tau \cap (0,T]$, $\omega_\tau^- = \overline{\omega}_\tau \cap [0,T)$, $Q_{h,\tau} = \omega \times \overline{\omega}_\tau$. We assume that there exist two positive constants $c_1 h^2 \le \tau \le c_2 h^2$. For functions defined on the mesh cylinder $\overline{\omega} \times \overline{\omega}_\tau$ we use the notation:

$$y = y(x,t) = y^j, \quad x \in \overline{\omega}, \quad t = t_j \in \overline{\omega}_\tau,$$
$$\widehat{y}(x,t) = y(x,t+\tau), \quad \check{y}(x,t) = y(x,t-\tau),$$
$$y_t = \frac{\widehat{y} - \check{y}}{\tau}, \quad y_{x_\alpha} = \frac{(I^{(+\alpha)} - I)y}{h}, \quad y_{\overline{x}_\alpha} = \frac{(I - I^{(-\alpha)})y}{h}, \quad \varkappa := \frac{h^2}{12},$$

where $Iy := y$, $I^{\pm\alpha} y := y(x \pm h r_\alpha, t)$ and r_α represents the unit vector of the axis x_α.

We define also the Steklov averaging operators:

$$T_1 u(x,t) = \frac{1}{h^2} \int\limits_{x_1-h}^{x_1+h} \big(h - |x_1 - \xi|\big) u(\xi, x_2, t)\, d\xi,$$
$$\widehat{S} u(x,t) = \frac{1}{\tau} \int\limits_{t}^{t+\tau} u(x,\zeta)\, d\zeta.$$

The operator T_2 is defined similarly. Note that these operators are commutative and

$$T_\alpha \frac{\partial^2 u}{\partial x_\alpha^2} = \Lambda_\alpha u, \quad \widehat{S} \frac{\partial u}{\partial t} = u_t.$$

If we apply the operator $\widehat{S} T_1 T_2$ to the eq. (1), we will get

$$(T_1 T_2 u)_t = \Lambda_1(\widehat{S} T_2 u) + \Lambda_2(\widehat{S} T_1 u) - a \widehat{S} T_1 T_2 u + \widehat{S} T_1 T_2 f. \tag{3}$$

It is easy to check that on the set of sufficiently smooth functions the following operators:

$$T_\alpha \sim I + \varkappa \Lambda_\alpha \ \text{ with errors of order } \ O(h^4),$$
$$\widehat{S} \sim (I + \widehat{I})/2 \ \text{ with errors of order } \ O(\tau^2)$$

are equivalent and, therefore, within the accuracy $O(h^4 + \tau^2)$ we obtain

$$T_1 T_2 \sim (I + \varkappa\Lambda_1)(I + \varkappa\Lambda_2), \tag{4}$$
$$\widehat{S} T_1 T_2 \sim (I + \varkappa\Lambda_1 + \varkappa\Lambda_2)\, \frac{\widehat{I} + I}{2}, \tag{5}$$
$$\widehat{S} T_\alpha \sim (I + \varkappa\Lambda_\alpha)\, \frac{\widehat{I} + I}{2}. \tag{6}$$

Taking into the account the relations (4)–(6), we denote:

$$\eta_0 = T_1T_2u - (I + \varkappa\Lambda_1)(I + \varkappa\Lambda_2)u - (\tau^2/4)\Lambda_1\Lambda_2u, \tag{7}$$

$$\eta_\alpha = \widehat{S}T_{3-\alpha}u - (I + \varkappa\Lambda_{3-\alpha})\,\frac{\widehat{u}+u}{2}, \quad \alpha = 1, 2, \tag{8}$$

$$\eta = \widehat{S}T_1T_2u - (I + \varkappa\Lambda_1 + \varkappa\Lambda_2)\,\frac{\widehat{u}+u}{2} + \\ + \Big(\frac{\tau\varkappa}{4} + \frac{\tau^2}{8}\Big)\Big(\Lambda_1 + \Lambda_2\Big)u_t - \frac{a\tau^2}{16}\,u_t. \tag{9}$$

In the equalities (7), (9) the additional terms are introduced with the aim that the resulting difference scheme operator should be factorizable.

Due to (7)–(9), from (3) we get

$$(I + \varkappa\Lambda_1)(I + \varkappa\Lambda_2)u_t + \frac{\tau^2}{4}\,\Lambda_1\Lambda_2u_t + (\eta_0)_t = \\ = \Lambda_1(I + \varkappa\Lambda_2)\,\frac{\widehat{u}+u}{2} + \Lambda_2(I + \varkappa\Lambda_1)\,\frac{\widehat{u}+u}{2} + \Lambda_1\eta_1 + \Lambda_2\eta_2 - \\ -a\bigg((I + \varkappa\Lambda_1 + \varkappa\Lambda_2)\,\frac{\widehat{u}+u}{2} - \Big(\frac{\tau\varkappa}{4} + \frac{\tau^2}{8}\Big)(\Lambda_1 + \Lambda_2)u_t + \frac{a\tau^2}{16}\,u_t + \eta\bigg) + \\ + \widehat{S}T_1T_2f,$$

that is,

$$\Big(I + \varkappa\Lambda_1 - \frac{\tau}{2}\,\Lambda_1 + \frac{a\tau}{4}\,I\Big)\Big(I + \varkappa\Lambda_2 - \frac{\tau}{2}\,\Lambda_2 + \frac{a\tau}{4}\,I\Big)u_t = \\ = \Big(\Lambda_1(I + \varkappa\Lambda_2) + \Lambda_2(I + \varkappa\Lambda_1) - a(I + \varkappa\Lambda_1 + \varkappa\Lambda_2)\Big)u + \\ + \widehat{S}T_1T_2f + \psi, \tag{10}$$

where

$$\psi = \Lambda_1\eta_1 + \Lambda_2\eta_2 - a\eta - (\eta_0)_t. \tag{11}$$

Finally, if in the equation (10) we reject the remainder term and change u by the mesh function y, we will come to the difference scheme

$$By_t + Ay = \varphi, \quad (x,t) \in \omega \times \omega_\tau^-, \tag{12}$$

where

$$A := A_1(I - \varkappa A_2) + A_2(I - \varkappa A_1) + a(I - \varkappa A_1 - \varkappa A_2), \\ B := \Big(I - \varkappa A_1 + \frac{\tau}{2}\,A_1 + \frac{a\tau}{4}\,I\Big)\Big(I - \varkappa A_2 + \frac{\tau}{2}\,A_2 + \frac{a\tau}{4}\,I\Big).$$

We define the initial and boundary conditions as follows:

$$By^0 = T_1T_2u_0 + \frac{\tau}{2}\,Au_0, \quad x \in \omega, \quad y(x,t) = 0, \quad (x,t) \in \gamma \times \overline{\omega}_\tau. \tag{13}$$

3. An a Priori Estimate of the Solution Error

Let H be the space of mesh functions defined on $\overline{\omega}$ and vanishing on γ, with inner product and norm

$$(y,v)=\sum_{x\in\omega}h^2y(x)v(x),\quad \|y\|=\|y\|_{L_2(\omega)}=(y,y)^{1/2}.$$

Besides, let

$$\|y\|_0=\|y\|_{L_2(Q_{h,\tau})}=\Big(\sum_{t\in\overline{\omega}_\tau}\tau\|y(\cdot,t)\|^2_{L_2(\omega)}\Big)^{1/2}.$$

In the case of self-conjugate positive operators we will use the notation

$$(y,v)_D:=(Dy,v),\ \ \|y\|_D:=\sqrt{(Dy,y)},\ \ D=D^*>0.$$

Let

$$C:=B-\frac{\tau}{2}\,A. \tag{14}$$

It is easy to verify that

$$C=(I-\varkappa A_1)(I-\varkappa A_2)+\Big(\frac{a\tau^2}{8}+\frac{a\tau\varkappa}{4}\Big)(A_1+A_2)+ \\ +\frac{a^2\tau^2}{16}\,I+\frac{\tau^2}{4}\,A_1A_2\geq\frac{4}{9}\,I+\frac{\tau^2}{4}\,A_1A_2>0. \tag{15}$$

The following lemma plays a significant role in getting the needed a priori estimate of the solution of the difference scheme.

Lemma 1. *Let $A=A^*>0$, $B=B^*>0$ be arbitrary independent on t operators and $B>(\tau/2)A$. Then for the solution of the problem*

$$Bv_t+Av=\psi_t,\ \ (x,t)\in\omega\times\omega_\tau^-, \tag{16}$$

$$Bv^0=\psi^0,\ x\in\omega \tag{17}$$

the estimate

$$\|v\|_{L_2(Q_{h,\tau})}\leq\|C^{-1}\psi\|_{L_2(Q_{h,\tau})}$$

is valid with C defined in (14).

Proof. Summing up by $t=0,\tau,\dots,(k-1)\tau$, from (16) we find

$$Bv^k-Bv^0+\sum_{j=0}^{k-1}\tau Av^j=\psi^k-\psi^0,\ \ k=1,2,\dots,$$

that is, taking into account the initial condition (17),

$$Bv^k+\sum_{j=0}^{k-1}\tau Av^j=\psi^k,\ \ k=1,2,\dots\,. \tag{18}$$

Since $C = C^* > 0$, the inverse operator $C^{-1} = (C^{-1})^* > 0$ exists. Multiply (18) scalarly by $C^{-1}v^k$:

$$(Bv^k, C^{-1}v^k) + \Big(\sum_{j=0}^{k-1} \tau A v^j, C^{-1}v^k\Big) = (\psi^k, C^{-1}v^k), \quad k = 1, 2, \dots . \tag{19}$$

Denote

$$\chi^0 = 0, \quad \chi^k = \sum_{j=0}^{k-1} \tau v^j, \quad k = 1, 2, \dots .$$

Then (19) yields

$$(Bv^k, C^{-1}v^k) + \Big(A\chi^k, C^{-1}\frac{\chi^{k+1} - \chi^k}{\tau}\Big) = (\psi^k, C^{-1}v^k),$$

from which, after some transformations, we obtain

$$\tau\bigg(\Big(B - \frac{\tau}{2}A\Big)v^k, C^{-1}v^k\bigg) + \frac{1}{2}\,\|\chi^{k+1}\|^2_{AC^{-1}} - \frac{1}{2}\,\|\chi^k\|^2_{AC^{-1}} = \\ = \tau(\psi^k, C^{-1}v^k)$$

or

$$2\tau\|v^k\|^2 + \|\chi^{k+1}\|^2_{AC^{-1}} - \|\chi^k\|^2_{AC^{-1}} = 2\tau(C^{-1}\psi^k, v^k), \quad k = 1, 2, \dots . \tag{20}$$

Using the Cauchy–Bunyakovski inequality, we estimate the right-hand side of (20)

$$2\tau(C^{-1}\psi^k, v^k) \le \tau\|C^{-1}\psi^k\|^2 + \tau\|v^k\|^2$$

and sum up the obtained result by $k = 1, 2, \dots, N$. We get

$$\sum_{k=1}^{N} \tau\|v^k\|^2 + \|\chi^{N+1}\|^2_{AC^{-1}} - \|\chi^1\|^2_{AC^{-1}} \le \sum_{k=1}^{N} \tau\|C^{-1}\psi^k\|^2. \tag{21}$$

From (14) we have

$$B^2 = C^2 + \tau AC + \frac{\tau^2}{4}\,A^2 > C^2 + \tau AC.$$

Hence

$$\tau AC^{-1} \le B^2C^{-2} - I.$$

Using this inequality and taking into account the relation $\chi^1 = \tau v^0$, we get

$$\|\chi^1\|^2_{AC^{-1}} = (\tau AC^{-1}v^0, \tau v^0) \le \big((B^2C_I^{-2})v^0, \tau v^0\big) = \\ = \tau\|BC^{-1}v^0\|^2 - \tau\|v^0\|^2 = \tau\|C^{-1}\psi^0\|^2 - \tau\|v^0\|^2,$$

which together with (21) proves the lemma. □

Consider the error $z = y - u$. From (10)–(13) we get the following problem for it:

$$Bz_t + Az = A_1\eta_1 + A_2\eta_2 + a\eta + (\eta_0)_t, \quad (x, t) \in \omega \times \omega_\tau^-, \\ Bz^0 = \eta_0^0, \quad x \in \omega, \quad z \in H. \tag{22}$$

We define the functions η_1, η_2 to be zeros on $t = T$ and substitute z in (22) by the following expression

$$z = v + A^{-1}(A_1\eta_1 + A_2\eta_2 + a\eta). \tag{23}$$

Then for v we obtain the problem (16), (17), where

$$\psi = \eta_0 - BA^{-1}(A_1\eta_1 + A_2\eta_2 + a\eta).$$

Using Lemma 1 for v, we get the estimate

$$\sum_{k=0}^{N} \tau \|v^k\|^2 \le \sum_{k=0}^{N} \tau J_k^2, \qquad J_k := \big\|C^{-1}\eta_0^k - C^{-1}BA^{-1}(A_1\eta_1^k + A_2\eta_2^k + a\eta^k)\big\|. \tag{24}$$

Because of (14), (15) we have

$$C^{-1}BA^{-1} = A^{-1} + \frac{\tau}{2}\,C^{-1} \le A^{-1} + \frac{9\tau}{8}\,I, \quad C^{-1} \le (9/4)I.$$

Therefore

$$J_k \le \frac{9}{4}\,\|\eta_0^k\| + \big\|A^{-1}(A_1\eta_1^k + A_2\eta_2^k + a\eta^k)\big\| + \frac{9\tau}{8}\,\big\|A_1\eta_1^k + A_2\eta_2^k + a\eta^k\big\|.$$

Taking into account the operator inequalities

$$A \ge \frac{2}{3}\,(A_1 + A_2), \quad A \ge \frac{32}{3}\,I, \quad A^{-1}A_\alpha \le \frac{3}{2}\,I,$$

we get

$$J_k \le \frac{9}{4}\,\|\eta_0^k\| + \frac{3}{2}\,\Big(\|\eta_1^k\| + \|\eta_2^k\| + \frac{a}{16}\,\|\eta^k\|\Big) + \frac{9\tau}{8}\,\big\|A_1\eta_1^k + A_2\eta_2^k + a\eta^k\big\|.$$

On the basis of this and the following algebraic inequalities

$$\bigg\{\sum_k \Big(\sum_i a_{ik}\Big)^2\bigg\}^{1/2} \le \sum_i \Big(\sum_k a_{ik}^2\Big)^{1/2}, \quad a_{ik} \ge 0,$$

we get from (24)

$$\|v\|_0 \le \frac{9}{4}\,\|\eta_0\|_0 + \frac{3}{2}\Big(\|\eta_1\|_0 + \|\eta_2\|_0 + \frac{a}{16}\,\|\eta\|_0\Big) + \\ + \frac{9\tau}{8}\,\big(\|A_1\eta_1\|_0 + \|A_2\eta_2\|_0 + a\|\eta\|_0\big). \tag{25}$$

(23), (25) enable us to assert the validity of the following

Theorem 1. *For the solution of the difference problem* (22) *the following a priori estimate is true*

$$\|z\|_0 \le \frac{9}{4}\,\|\eta_0\|_0 + 3(\|\eta_1\|_0 + \|\eta_2\|_0) + \frac{9\tau}{8}\,\big(\|A_1\eta_1\|_0 + \|A_2\eta_2\|_0\big). \tag{26}$$

4. Convergence of the Finite-Difference Scheme

Let E denote a bounded open set in R^2 with Lipschitz continuous boundary, and let $G = E \times (0,1)$. We introduce the set of multi-indices

$$\mathcal{B}_k = \Big\{(\alpha_1, \alpha_2, \beta) : \ \alpha_i, \beta = 0, 1, 2, \dots ; \ \alpha_1 + \alpha_2 + 2\beta \le k\Big\}.$$

Further, let $[s]^-$ denote the largest integer less than s. The convergence analysis of our finite difference scheme is based on the following lemma.

Lemma 2. *If φ is a bounded linear functional on $W_2^{s,s/2}(G)$ such that*

$$\varphi(x_1^{\alpha_1} x_2^{\alpha_2} t^\beta) = 0, \ \ \forall\, (\alpha_1, \alpha_2, \beta) \in \mathcal{B}_{[s]^-},$$

then there exists a positive constant $c = c(G, s)$ such that

$$|\varphi(v)| \le c|v|_{W_2^{s,s/2}(G)}, \ \ \forall\, v \in W_2^{s,s/2}(G).$$

Lemma 2 is an easy consequence of the Dupont–Scott approximation theorem [4] (see also [5]).

If we use Lemma 2 and the well-known techniques (see, e.g., [1]–[3], [5]) for estimation of the terms in the right-hand side of the equation (26), we will get convinced in the validity of the following

Theorem 2. *Assume that the solution u to the problem* (1), (2) *belongs to the space $W_2^{s,s/2}(Q_{h,\tau})$, $2 < s \le 4$. Then the rate of convergence of the difference scheme* (12), (13) *in the L_2 grid norm is described by the estimate*

$$\|y - u\|_{L_2(Q_{h,\tau})} \le ch^s \|u\|_{W_2^{s,s/2}(Q)}, \ \ s \in (2, 4],$$

where the constant c does not depend on h and u.

Remark. A more detailed analysis enables us to obtain the estimate

$$\|y - u\|_{L_2(Q_{h,\tau})} \le c(h^s + \tau^{s/2}) \|u\|_{W_2^{s,s/2}(Q)}, \ \ s \in (2,\, 4],$$

as well without restriction $\tau \sim h^2$.

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WIENER–HOPF AND WIENER–HOPF–HANKEL OPERATORS WITH PIECEWISE-ALMOST PERIODIC SYMBOLS ON WEIGHTED LEBESGUE SPACES

Dedicated to the memory of
Professor Nikolai Ivanovich Muskhelisvili
on the occasion of his 120th birthday

Abstract. We consider Wiener–Hopf, Wiener–Hopf plus Hankel, and Wiener–Hopf minus Hankel operators on weighted Lebesgue spaces and having piecewise almost periodic Fourier symbols. The main results concern conditions to ensure the Fredholm property and the lateral invertibility of these operators. In addition, under the Fredholm property, conclusions about the Fredholm index of those operators are also discussed.

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Key words and phrases. Wiener–Hopf operator, Hankel operator, Fredholm operator, index, weighted Lebesgue space, piecewise almost periodic symbols.

რეზიუმე. ჩვენ ვიხილავთ ვინერ-ჰოფის, ვინერ-ჰოფ პლუს ჰენლელისა და ვინერ-ჰოფ მინუს ჰენკელის ოპერატორებს უბან-უბან თითქმის პერიოდული ფურიეს სიმბოლოებით წონიან ლებეგის სივრცეებში. ძირითადი შედეგები ეხება პირობებს, რომლებიც უზრუნველყოფენ ამ ოპერატორების ფრედჰოლმურობასა და შებრუნებადობას. დამატებით, ფრედჰოლმურობასთან ერთად, განხილულია ან ოპერატორების ფრედჰოლმის ინდექსი.

1. Introduction

Wiener–Hopf and Hankel operators are known to be very important objects in the modeling of a great variety of applied problems. In fact, since their first appearance in the first half of the twentieth century, advances on the knowledge of their theory, consequent generalizations and their use have been continuously increasing. This circumstance is not indifferent of the interplay between these operators and singular integral operators – which can be identified in different monographs on the subject (cf., e.g., [3], [10], [17], [18]). Additionally, certain combinations of Wiener–Hopf and Hankel operators have also proved to be quite useful in the applications (and several examples of this can be seen e.g. in some wave diffraction problems when analysed by an operator theory approach [15], [16], [22]). A great part of the study in this kind of operators is concentrated in the description of their Fredholm and invertibility properties. In particular, for several classes of the so-called Fourier symbols of the operators, their Fredholm and invertibility properties are already characterized (see e.g. [1]–[5], [7]–[14], [19]–[21] and the references given there). Despite these advances, for some other classes of Fourier symbols and more general spaces, a complete description of the Fredholm and invertibility properties is still missing.

Within this scope, in the present paper we would like to consider Wiener–Hopf, Wiener–Hopf plus Hankel and Wiener–Hopf minus Hankel operators on weighted Lebesgue spaces and having piecewise-almost periodic Fourier symbols (i.e., a certain combination of piecewise continuous elements with almost periodic elements). The main efforts will be devoted to obtain invertibility and Fredholm descriptions of these operators. In view of stating the formal definitions of the operators under study, we will now introduce some preliminary notation.

Let E be a connected subspace of $\mathbb{R}$. A (Lebesgue) measurable function $w : E \to [0, \infty]$ is called a weight if $w^{-1}(\{0, \infty\})$ has (Lebesgue) measure zero. For $1 < p < \infty$, we denote by $L^p(\mathbb{R}, w)$ the usual Lebesgue space with the norm

$$\|f\|_{p,w} := \left(\int\limits_{\mathbb{R}} |f(x)|^p w(x)^p \, dx \right)^{\frac{1}{p}}.$$

Additionally, $A_p(\mathbb{R})$ will denote the set of all weights w on $\mathbb{R}$ for which the Cauchy singular integral operator $S_{\mathbb{R}}$ given by

$$(S_{\mathbb{R}} f)(x) = \lim_{\epsilon \to 0} \frac{1}{\pi i} \int\limits_{\mathbb{R} \setminus (x-\epsilon, x+\epsilon)} \frac{f(t)}{t-x} \, dt, \quad x \in \mathbb{R},$$

is bounded on the space $L^p(\mathbb{R}, w)$. The weights $w \in A_p(\mathbb{R})$ are called *Muckenhoupt weights.*

Let $\mathcal{F}$ denote the Fourier transformation. A function $\phi \in L^\infty(\mathbb{R})$ is a *Fourier multiplier* on $L^p(\mathbb{R}, w)$ if the map $f \mapsto \mathcal{F}^{-1} \phi \cdot \mathcal{F} f$ maps $L^2(\mathbb{R}) \cap L^p(\mathbb{R}, w)$ into itself and extends to a bounded operator on $L^p(\mathbb{R}, w)$ (notice that $L^2(\mathbb{R}) \cap L^p(\mathbb{R}, w)$ is dense in $L^p(\mathbb{R}, w)$ whenever w belongs to $A_p(\mathbb{R})$). We let $\mathcal{M}_{p,w}$ stand for the set of all Fourier multipliers on $L^p(\mathbb{R}, w)$. We will denote by $A_p^0(\mathbb{R})$ the set of all weights $w \in A_p(\mathbb{R})$ for which the functions $e_\lambda : x \mapsto e^{i\lambda x}$ belong to $\mathcal{M}_{p,w}$ for all $\lambda \in \mathbb{R}$. Let J be the reflection operator given by the rule $J\varphi(x) = \widetilde{\varphi}(x) = \varphi(-x)$, $x \in \mathbb{R}$. We denote by $A_p^e(\mathbb{R})$ the subspace of all weights $w \in A_p(\mathbb{R})$ for which $Jw = w$. Additionally, let $A_p^{e,0}(\mathbb{R}) := A_p^0(\mathbb{R}) \cap A_p^e(\mathbb{R})$.

We shall use $L_+^p(\mathbb{R}, w)$ to denote the subspace of $L^p(\mathbb{R}, w)$ formed by all the functions supported in the closure of $\mathbb{R}_+ = (0, +\infty)$.

In what follows we will consider Wiener–Hopf operators defined by

$$W_\phi = r_+ \mathcal{F}^{-1} \phi \cdot \mathcal{F} : L_+^p(\mathbb{R}, w) \to L^p(\mathbb{R}_+, w),$$

and so-called Wiener–Hopf–Hankel operators [5], [14], [16], [22] (i.e., Wiener–Hopf plus Hankel and Wiener–Hopf minus Hankel operators) of the form

$$W_\phi \pm H_\phi : L_+^p(\mathbb{R}, w) \to L^p(\mathbb{R}_+, w) \tag{1.1}$$

with H_ϕ being the Hankel operator defined by

$$H_\phi = r_+ \mathcal{F}^{-1} \phi \cdot \mathcal{F} J.$$

Here, r_+ represents the operator of restriction from $L^p(\mathbb{R}, w)$ into $L^p(\mathbb{R}_+, w)$, $w \in A_p^e(\mathbb{R})$ and $\phi \in \mathcal{M}_{p,w}$ is the so-called Fourier symbol. For such Fourier symbol and weight, the operators in (1.1) are bounded.

2. Auxiliary Material

2.1. **The algebra of piecewise-almost periodic elements.** In this subsection we will introduce the piecewise almost periodic elements (which will take the role of Fourier symbols of our main operators), and consider already some of their characteristics.

The smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all functions $e_\lambda := e^{i\lambda x}$ $(x \in \mathbb{R})$ is denoted by AP and called the algebra of almost periodic functions.

For $\phi \in AP$, there exists a number

$$M(\phi) := \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} \phi(x)\, dx$$

which is called the (Bohr) mean value of ϕ.

Let $\mathcal{G}B$ denote the group of all invertible elements of a Banach algebra B.

Theorem 2.1 (Bohr)**.** *If $\phi \in \mathcal{G}AP$, then there exists a real number $k(\phi)$ and a function $\psi \in AP$ such that*

$$\phi(x) = e^{ik(\phi)x} e^{\psi(x)} \quad \textit{for all } x \in \mathbb{R}.$$

The number $k(\phi)$ is uniquely determined and it is called the *mean motion* of ϕ. Considering $\phi \in \mathcal{G}AP$, the mean motion of ϕ can be obtained by

$$k(\phi) = \lim_{T\to\infty} \frac{(\arg\phi)(T) - (\arg\phi)(0)}{T}\,, \tag{2.2}$$

where $\arg\phi$ is any continuous argument of ϕ. The *geometric mean value* of the function ϕ is defined by $\mathbf{d}(\phi) = e^{M(\psi)}$.

For $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, we denote by PC or $PC(\dot{\mathbb{R}})$ the algebra of all functions $\varphi \in L^\infty(\mathbb{R})$ for which the one-sided limits $\varphi(x_0 - 0) = \lim_{x\to x_0-0} \varphi(x)$, $\varphi(x_0 + 0) = \lim_{x\to x_0+0} \varphi(x)$ exist for each $x_0 \in \dot{\mathbb{R}}$, and by $C(\dot{\mathbb{R}})$ the set of all (bounded and) continuous functions φ on the real line for which the two limits $\varphi(-\infty) := \lim_{x\to-\infty} \varphi(x)$, $\varphi(+\infty) := \lim_{x\to+\infty} \varphi(x)$ exist and coincide. Let $C(\overline{\mathbb{R}}) := C(\mathbb{R}) \cap PC(\dot{\mathbb{R}})$ and $PC_0 := \{\varphi \in PC : \varphi(\pm\infty) = 0\}$. We denote by $C_{p,w}(\dot{\mathbb{R}})$ ($PC_{p,w}(\dot{\mathbb{R}})$) the closure in $\mathcal{M}_{p,w}$ of the set of all functions $\phi \in C(\dot{\mathbb{R}})$ (resp. $\phi \in PC(\dot{\mathbb{R}})$) with finite total variation.

We define $AP_{p,w}$ as the closure of the set of all almost periodic functions in $\mathcal{M}_{p,w}$. Let $SAP_{p,w}$ denote the smallest closed subalgebra of $\mathcal{M}_{p,w}$ that contains $C_{p,w}(\overline{\mathbb{R}})$ and $AP_{p,w}$, and denote by $PAP_{p,w}$ the smallest closed subalgebra of $\mathcal{M}_{p,w}$ that contains $PC_{p,w}$ and $AP_{p,w}$.

2.2. **Operator relations.** In order to relate operators and to transfer certain operator properties between different operators, we will be also using some known operator relations.

Definition 2.2. Consider two bounded linear operators $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$ acting between Banach spaces. We say that T and S are equivalent, and denote this by $T \sim S$, if there are two boundedly invertible linear operators, $E : Y_2 \to X_2$ and $F : X_1 \to Y_1$, such that

$$T = E\, S\, F. \tag{2.3}$$

If two operators are equivalent, then they belong to the same *invertibility class*. More precisely, one of these operators is invertible, left-invertible, right-invertible or only generalized invertible, if and only if the other operator enjoys the same property.

Definition 2.3 ([6])**.** Let $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$ be bounded linear operators. We say that T is Δ*–related after extension to* S if there is a bounded linear operator acting between Banach spaces $T_\Delta : X_{1\Delta} \to X_{2\Delta}$ and invertible bounded linear operators E and F such that

$$\begin{bmatrix} T & 0 \\ 0 & T_\Delta \end{bmatrix} = E \begin{bmatrix} S & 0 \\ 0 & I_Z \end{bmatrix} F, \tag{2.4}$$

where Z is an additional Banach space and I_Z represents the identity operator in Z. In the particular case when $T_\Delta : X_{1\Delta} \to X_{2\Delta} = X_{1\Delta}$ is the identity operator, we say that the operators T and S are *equivalent after extension* and in such a case we will use the notation $T \overset{*}{\sim} S$.

In the following result, we describe a relation between Wiener–Hopf plus Hankel operators and Wiener–Hopf operators within the present framework. This result is well-known for non-weighted spaces (cf., e.g., [7, Theorem 2.1]) and the corresponding proof in the present case runs in a similar way. Anyway, we choose to present here a complete proof of it for the reader convenience.

Theorem 2.4. *Let $\phi \in \mathcal{GM}_{p,w}$ with $w \in A_p^e(\mathbb{R})$ and $1 < p < \infty$. The Wiener–Hopf plus Hankel operator*

$$W_\phi + H_\phi : L_+^p(\mathbb{R}, w) \to L^p(\mathbb{R}_+, w)$$

is Δ-related after extension to the Wiener–Hopf operator

$$W_{\phi\widetilde{\phi^{-1}}} : L_+^p(\mathbb{R}, w) \to L^p(\mathbb{R}_+, w).$$

Proof. We shall use the characteristic functions $\chi_\pm$ to the positive/negative half-line.

Extending $W_\phi + H_\phi$ on the left by the zero extension operator, $\ell_0 : L^p(\mathbb{R}_+, w) \to L_+^p(\mathbb{R}, w)$, we obtain

$$W_\phi + H_\phi \sim \ell_0(W_\phi + H_\phi) : L_+^p(\mathbb{R}, w) \to L_+^p(\mathbb{R}, w).$$

After this we will extend

$$\ell_0(W_\phi + H_\phi) = \chi_+ \mathcal{F}^{-1}(\phi + \phi J)\mathcal{F}_{|\chi_+ L^p(\mathbb{R},w)}$$

to the full $L^p(\mathbb{R}, w)$ space by using the identity in $L_-^p(\mathbb{R}, w)$. Next, we will extend the obtained operator to $[L^p(\mathbb{R}, w)]^2$ with the help of an auxiliary paired operator:

$$L_\phi = \mathcal{F}^{-1}(\phi - \phi J)\mathcal{F}\chi_+ + \chi_- : L^p(\mathbb{R}, w) \to L^p(\mathbb{R}, w).$$

Altogether, we have

$$\begin{bmatrix} \ell_0(W_\phi + H_\phi) & 0 & 0 \\ 0 & I_{\chi_- L^p(\mathbb{R},w)} & 0 \\ 0 & 0 & L_\phi \end{bmatrix} = E_1 \mathcal{W}_\Phi F_1$$

with

$$E_1 = \frac{1}{2} \begin{bmatrix} I_{L^p(\mathbb{R},w)} & J \\ I_{L^p(\mathbb{R},w)} & -J \end{bmatrix},$$

$$F_1 = \begin{bmatrix} I_{L^p(\mathbb{R},w)} & I_{L^p(\mathbb{R},w)} \\ J & -J \end{bmatrix} \begin{bmatrix} I_{L^p(\mathbb{R},w)} - \chi_- \mathcal{F}^{-1}(\phi - \phi J)\mathcal{F}\chi_+ & 0 \\ 0 & I_{L^p(\mathbb{R},w)} \end{bmatrix},$$

$$\begin{aligned}
\mathcal{W}_\phi &= \begin{bmatrix} \mathcal{F}^{-1}\phi\mathcal{F} & 0 \\ \mathcal{F}^{-1}\widetilde{\phi}\mathcal{F} & 1 \end{bmatrix} \chi_+ + \begin{bmatrix} 1 & \mathcal{F}^{-1}\phi\mathcal{F} \\ 0 & \mathcal{F}^{-1}\widetilde{\phi}\mathcal{F} \end{bmatrix} \chi_- = \\
&= \begin{bmatrix} 1 & \mathcal{F}^{-1}\phi\mathcal{F} \\ 0 & \mathcal{F}^{-1}\widetilde{\phi}\mathcal{F} \end{bmatrix} \left(\mathcal{F}^{-1}\Psi\mathcal{F}\chi_+ + \chi_-\right) = \\
&= \begin{bmatrix} 1 & \mathcal{F}^{-1}\phi\mathcal{F} \\ 0 & \mathcal{F}^{-1}\widetilde{\phi}\mathcal{F} \end{bmatrix} \left(\chi_+\mathcal{F}^{-1}\Psi\mathcal{F}\chi_+ + \chi_-\right)\left(I_{[L^p(\mathbb{R},w)]^2} + \chi_-\mathcal{F}^{-1}\Psi\mathcal{F}\chi_+\right),
\end{aligned}$$

where the operators χ_+ and χ_- are here defined on $[L^p(\mathbb{R}, w)]^2$ and

$$\Psi = \begin{bmatrix} 0 & -\phi\widetilde{\phi^{-1}} \\ 1 & \widetilde{\phi^{-1}} \end{bmatrix}.$$

The paired operator

$$I_{[L^p(\mathbb{R},w)]^2} + \chi_- \mathcal{F}^{-1}\Psi\mathcal{F}\chi_+ : [L^p(\mathbb{R}, w)]^2 \to [L^p(\mathbb{R}, w)]^2$$

is an invertible operator with inverse given by

$$I_{[L^p(\mathbb{R},w)]^2} - \chi_- \mathcal{F}^{-1}\Psi\mathcal{F}\chi_+ : [L^p(\mathbb{R}, w)]^2 \to [L^p(\mathbb{R}, w)]^2.$$

Thus, we have demonstrated that $W_\phi + H_\phi$ is Δ-related after extension with

$$W_\Psi = r_+\mathcal{F}^{-1}\Psi\mathcal{F} : [L^p_+(\mathbb{R}, w)]^2 \to [L^p(\mathbb{R}_+, w)]^2.$$

Furthermore, we have

$$\begin{bmatrix} W_{\phi\widetilde{\phi^{-1}}} & 0 \\ 0 & I_{[L^p(\mathbb{R}_+,w)]} \end{bmatrix} =$$
$$= W_\Psi \ell_0 r_+ \mathcal{F}^{-1} \begin{bmatrix} \widetilde{\phi^{-1}} & 1 \\ -1 & 0 \end{bmatrix} \mathcal{F}\ell_0 : [L^p(\mathbb{R}_+, w)]^2 \to [L^p(\mathbb{R}_+, w)]^2$$

which shows an explicit equivalence after extension relation between $W_{\phi\widetilde{\phi^{-1}}}$ and W_Ψ. This, together with the Δ-relation after extension between $W_\phi + H_\phi$ and W_Ψ, concludes the proof. □

Remark 2.5. From the proof of the last theorem we can also realize the last result as an equivalence after extension between the diagonal matrix operator $\operatorname{diag}[W_\phi + H_\phi, W_\phi - H_\phi]$ and $W_{\phi\widetilde{\phi^{-1}}}$.

3. Wiener–Hopf Operators on Weighted Lebesgue Spaces

3.1. **Fredholm theory for Wiener–Hopf operators with piecewise continuous symbols on weighted Lebesgue spaces.** In the present subsection we will recall a Fredholm characterization of Wiener–Hopf operators with piecewise continuous Fourier symbols on weighted Lebesgue spaces (which we will use later on).

Let $\nu \in (0, 1)$. The set $\{e^{2\pi(x+i\nu)} : x \in \mathbb{R}\}$ is a ray starting at the origin and making the angle $2\pi\nu \in (0, 2\pi)$ with the positive real half-line. For z_1, $z_2 \in \mathbb{C}$, the Möbius transform

$$M_{z_1,z_2}(\zeta) := \frac{z_2\zeta - z_1}{\zeta - 1}$$

maps 0 to z_1 and ∞ to z_2. Thus,

$$\mathcal{A}(z_1, z_2; \nu) := \left\{M_{z_1,z_2}(e^{2\pi(x+i\nu)}) : x \in \mathbb{R}\right\} \cup \{z_1, z_2\}$$

is a circular arc between z_1 and z_2 (which contains its endpoints z_1, z_2). Finally, given $0 < \nu_1 \le \nu_2 < 1$, we put

$$\mathcal{H}(z_1, z_2; \nu_1, \nu_2) := \bigcup_{\nu\in[\nu_1,\nu_2]} \mathcal{A}(z_1, z_2; \nu),$$

and refer to $\mathcal{H}(z_1, z_2; \nu_1, \nu_2)$ as the horn between z_1 and z_2 determined by ν_1 and ν_2.

Let $1 < p < \infty$ and $w \in A_p(\mathbb{R})$. Then, each of the sets

$$\begin{aligned} I_x(p, w) &:= \big\{\lambda \in \mathbb{R} : \ |(\xi - x)/(\xi + i)|^{\lambda} w(\xi) \in A_p(\mathbb{R})\big\}, \quad x \in \mathbb{R}, \\ I_\infty(p, w) &:= \big\{\lambda \in \mathbb{R} : \ |\xi + i|^{-\lambda} w(\xi) \in A_p(\mathbb{R})\big\} \end{aligned} \tag{3.5}$$

is an open interval of length no grater than 1 which contains the origin:

$$I_x(p, w) = (-\nu_x^-(p, w), 1 - \nu_x^+(p, w)), \quad x \in \dot{\mathbb{R}}, \tag{3.6}$$

with $0 < \nu_x^-(p, w) \le \nu_x^+(p, w) < 1$.

Theorem 3.1 ([3, Theorem 17.7])**.** *Let $1 < p < \infty$, $w \in A_p(\mathbb{R})$, and let $\nu_x^\pm(p, w)$ be defined by (3.5)–(3.6). If $\psi \in PC_{p,w}$, then the operator W_ψ is Fredholm on the space $L^p(\mathbb{R}_+, w)$ if and only if*

$$0 \notin \psi^{\#}_{p,w}(\dot{\mathbb{R}}) := \left(\bigcup_{x\in\mathbb{R}} \mathcal{H}\Big(\psi(x-0), \psi(x+0); \nu_\infty^-(p, w), \nu_\infty^+(p, w)\Big)\right) \cup \\ \cup\, \mathcal{H}\Big(\psi(+\infty), \psi(-\infty); \nu_0^-(p, w), \nu_0^+(p, w)\Big).$$

If W_ψ is Fredholm on the space $L^p(\mathbb{R}_+, w)$, then

$$\operatorname{Ind} W_\psi = -\operatorname{wind}_{p,w}\psi, \tag{3.7}$$

where $\operatorname{wind}_{p,w}\psi$ *is the winding number about the origin of the naturally oriented curve*

$$\psi^0_{p,w}(\dot{\mathbb{R}}) := \left(\bigcup_{x\in\mathbb{R}} \mathcal{A}\Big(\psi(x-0), \psi(x+0); \nu^0_\infty(p, w)\Big)\right) \cup \\ \cup\, \mathcal{A}\Big(\psi(+\infty), \psi(-\infty); \nu^0_0(p, w)\Big),$$

with

$$\nu_x^0(p, w) := \frac{\nu_x^-(p, w) + \nu_x^+(p, w)}{2}\,. \tag{3.8}$$

Suppose that $\psi \in PC_{p,w}$ has only finitely many jumps at the points $\Lambda_\psi \subset \mathbb{R}$ and possibly at ∞. If $0 \notin \psi^{\#}_{p,w}(\dot{\mathbb{R}})$, then the *Cauchy index* $\operatorname{ind}_{p,w}\psi$ of ψ with respect to p and w is defined by

$$\operatorname{ind}_{p,w}\psi := \sum_l \operatorname{ind}_l\psi + \sum_{x\in\Lambda_\psi} \left(-\nu_x^0(p, w) + \left\{\nu_x^0(p, w) + \frac{1}{2\pi}\arg\frac{\psi(x+0)}{\psi(x-0)}\right\}\right),$$

where l ranges over the connected components of $\mathbb{R} \setminus \Lambda_\psi$, $\{c\}$ denotes the fractional part of the real number c and $\operatorname{ind}_l\psi$ stands for the increment of

$\frac{1}{2\pi}\arg\psi$ on l, with $\arg\psi$ being any continuous argument of ψ on l. Additionally, we have that

$$\mathrm{wind}_{p,w}\psi = \mathrm{ind}_{p,w}\psi + \left(-\nu^0_\infty(p,w) + \left\{ \nu^0_\infty(p,w) + \frac{1}{2\pi}\arg\frac{\psi(-\infty)}{\psi(+\infty)} \right\} \right).$$

Thus, we can also write (3.7) in the form

$$\begin{aligned} \mathrm{Ind} W_\psi = & -\sum_l \mathrm{ind}_l \psi + \\ & + \sum_{x\in\Lambda_\psi\cup\{\infty\}} \left(\nu^0_x(p,w) - \left\{ \nu^0_x(p,w) + \frac{1}{2\pi}\arg\frac{\psi(x+0)}{\psi(x-0)} \right\} \right), \quad (3.9) \end{aligned}$$

where $\psi(\infty \pm 0) := \psi(\mp\infty)$.

3.2. **Wiener–Hopf operators with semi-almost periodic symbols on weighted Lebesgue spaces.**

3.2.1. *Representation of semi-almost periodic functions.* The following theorem is an analogue of the corresponding classic Sarason's result.

Theorem 3.2 ([13, Theorem 3.1.]). *Let $1 < p < \infty$, $w \in A^0_p(\mathbb{R})$ and let u be a monotonically increasing real-valued function in $C(\overline{\mathbb{R}})$ such that $u(-\infty) = 0$ and $u(+\infty) = 1$. Then, every function $\phi \in SAP_{p,w}$ can be uniquely represented in the form:*

$$\phi = (1-u)\phi_\ell + u\phi_r + \phi_0,$$

where ϕ_ℓ, $\phi_r \in AP_{p,w}$, $\phi_0 \in C_{p,w}(\dot{\mathbb{R}})$ and $\phi_0(\infty) = 0$. The maps $\phi \mapsto \phi_\ell$ and $\phi \mapsto \phi_r$ are (continuous) Banach algebra homomorphisms of $SAP_{p,w}$ onto $AP_{p,w}$ of norm 1, where $\|\phi\|_{p,w} = \|\mathcal{F}^{-1}\phi\cdot\mathcal{F}\|_{\mathcal{L}(L^p(\mathbb{R},w))}$.

3.2.2. *Fredholm theory for Wiener–Hopf operators with semi-almost periodic symbols on weighted Lebesgue spaces.* Let us recall an analogue of *Duduchava–Saginashvili Theorem* for weighted Lebesgue spaces $L^p(\mathbb{R}_+, w)$ with Muckenhoupt weights $w \in \mathcal{A}^0_p(\mathbb{R})$.

Theorem 3.3 ([13, Proposition 4.7]). *Let $\phi \in SAP_{p,w} \setminus \{0\}$, with $1 < p < \infty$ and $w \in A^0_p(\mathbb{R})$.*

(a) *If $\phi \notin \mathcal{G}SAP$, then W_ϕ is not semi-Fredholm on $L^p_+(\mathbb{R}, w)$.*
(b) *If $\phi \in \mathcal{G}SAP$ and $k(\phi_\ell)k(\phi_r) < 0$, then W_ϕ is not semi-Fredholm on $L^p_+(\mathbb{R}, w)$.*
(c) *If $\phi \in \mathcal{G}SAP$, $k(\phi_\ell)k(\phi_r) \geq 0$ and $k(\phi_\ell) + k(\phi_r) > 0$, then W_ϕ is properly n-normal on $L^p_+(\mathbb{R}, w)$ and left-invertible.*
(d) *If $\phi \in \mathcal{G}SAP$, $k(\phi_\ell)k(\phi_r) \geq 0$ and $k(\phi_\ell) + k(\phi_r) < 0$, then W_ϕ is properly d-normal on $L^p_+(\mathbb{R}, w)$ and right-invertible.*
(e) *If $\phi \in \mathcal{G}SAP$, $k(\phi_\ell) = k(\phi_r) = 0$ and*

$$0 \notin \mathcal{H}\big(\mathbf{d}(\phi_r), \mathbf{d}(\phi_\ell); \nu^-_0(p,w), \nu^+_0(p,w)\big),$$

then W_ϕ is Fredholm on $L^p_+(\mathbb{R}, w)$.

(f) *If* $\phi \in \mathcal{G}SAP$, $k(\phi_\ell) = k(\phi_r) = 0$ *and*

$$0 \in \mathcal{H}\big(\mathbf{d}(\phi_r), \mathbf{d}(\phi_\ell); \nu_0^-(p,w), \nu_0^+(p,w)),$$

then W_ϕ *is not semi-Fredholm on* $L^p_+(\mathbb{R}, w)$.

We would like to point out that although in [13, Proposition 4.7] do not appear the above left and right-invertibility conclusions (here added in propositions (c) and (d)), these lateral invertibility properties arise directly from the use of *Coburn-Simonenko Theorem* (since we are considering scalar Wiener–Hopf operators).

Lemma 3.4 ([3, Lemma 3.12])**.** *Let* $A \subset (0, \infty)$ *be an unbounded set and consider* $\{I_\alpha\}_{\alpha\in A} := \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ *to be a family of intervals such that* $x_\alpha \geq 0$ *and* $|I_\alpha| = y_\alpha - x_\alpha \to \infty$ *as* $\alpha \to \infty$. *If* $\phi \in \mathcal{G}SAP$ *is such that* $k(\phi_\ell) = k(\phi_r) = 0$ *and* $\arg\phi$ *is any continuous argument of* ϕ, *then the limit*

$$\frac{1}{2\pi} \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} ((\arg\phi)(x) - (\arg\phi)(-x))\, dx \tag{3.10}$$

exists, is finite and is independent of the particular choices of $\{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ *and* $\arg\phi$.

For $\phi \in \mathcal{G}SAP$ such that $k(\phi_\ell) = k(\phi_r) = 0$, the value (3.10) is denoted by $\operatorname{ind}\phi$ and called the *Cauchy index* of ϕ. Following [19, Section 4.3] we can generalize this notion of Cauchy index for SAP functions with $k(\phi_\ell) + k(\phi_r) = 0$.

The following theorem provides a formula for the Fredholm index of Wiener–Hopf operators with semi-almost periodic symbols on $L^p(\mathbb{R}_+, w)$.

Theorem 3.5 ([13, Theorem 4.8])**.** *If* $\phi \in \mathcal{G}SAP_{p,w}$, $k(\phi_\ell) = k(\phi_r) = 0$ *and*

$$0 \notin \mathcal{H}\big(\mathbf{d}(\phi_r), \mathbf{d}(\phi_\ell); \nu_0^-(p,w), \nu_0^+(p,w)\big),$$

then the operator W_ϕ *is Fredholm and*

$$\operatorname{Ind} W_\phi = -\operatorname{ind}\phi + \nu_0^0(p,w) - \Big\{\nu_0^0(p,w) + \frac{1}{2\pi}\arg\frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)}\Big\}, \tag{3.11}$$

where

$$\nu_0^0(p,w) := \frac{\nu_0^-(p,w) + \nu_0^+(p,w)}{2}.$$

3.3. **Wiener–Hopf operators with piecewise-almost periodic symbols on weighted Lebesgue spaces.** Motivated by the material in the previous subsections, the main purpose of the present subsection will be to establish an analogue invertibility and Fredholm description for Wiener–Hopf operators acting between L^p spaces ($1 < p < \infty$) with Muckenhoupt weights $w \in A_p^0(\mathbb{R})$, and with $PAP_{p,w}$ Fourier symbols.

3.3.1. *Representation of $PAP_{p,w}$ piecewise-almost periodic functions.*

Theorem 3.6. *Let $w \in A_p^0(\mathbb{R})$, and let u be a monotonically increasing real-valued function in $C(\overline{\mathbb{R}})$ such that $u(-\infty) = 0$ and $u(+\infty) = 1$.*

(i) *If $\phi \in PAP_{p,w}$, then there are uniquely determined functions φ_r, $\varphi_\ell \in AP_{p,w}$ and $\phi_0 \in PC^0_{p,w}$ such that*

$$\phi = (1 - u)\varphi_\ell + u\varphi_r + \phi_0. \tag{3.12}$$

(ii) *If $\phi \in \mathcal{G}PAP_{p,w}$, then there exists $\varphi \in \mathcal{G}SAP_{p,w}$ and $\psi \in \mathcal{G}PC_{p,w}$ satisfying $\psi(-\infty) = \psi(+\infty) = 1$, such that $\phi = \varphi\psi$ and*

$$W_\phi = W_\varphi W_\psi + K_1 = W_\psi W_\varphi + K_2, \tag{3.13}$$

with compact operators K_1 and K_2.

(iii) *In addition, the φ_ℓ and φ_r elements used in* (i) *coincide with the local representatives of $\varphi \in \mathcal{G}SAP_{p,w}$ used in* (ii) *and their unique existence is ensured by Theorem* 3.2.

Proof. Part (i) is an immediate consequence of Theorem 3.2.

To prove part (ii), suppose that ϕ is in $\mathcal{G}PAP_{p,w}$ and put $f := (1 - u)\varphi_\ell + u\varphi_r$ where the elements u, φ_ℓ and φ_r have the properties described in (3.12). Then, $\phi = f + \phi_0$ (with $\phi_0 \in PC^0_{p,w}$). From the hypothesis there is a constant $C \in (0, \infty)$ such that $|f(x)|$ is bounded away from zero for $|x| > C$, and therefore, we can find a function $f_0 \in C^0_{p,w}(\dot{\mathbb{R}})$ such that $\varphi := f + f_0 \in \mathcal{G}SAP_{p,w}$. Consequently, we have

$$\phi = \varphi + \phi_0 - f_0 = \varphi\big(1 + \varphi^{-1}(\phi_0 - f_0)\big) =: \varphi\psi,$$

and it is clear that $\psi = \varphi^{-1}\phi \in \mathcal{G}PC_{p,w}$ and $\psi(-\infty) = \psi(+\infty) = 1$. Since ϕ is continuous on $\mathbb{R}$ and ψ is continuous at ∞, we deduce that (3.13) holds with compact operators K_1 and K_2.

The proposition (iii) follows immediately from the construction performed for (ii). □

3.3.2. *Fredholm theory of Wiener–Hopf operators with piecewise-almost periodic functions on weighted Lebesgue spaces.* We are now in condition to derive a Fredholm characterization for Wiener–Hopf operators with $PAP_{p,w}$ Fourier symbols on weighted Lebesgue spaces.

Theorem 3.7. *Consider $w \in A_p^0(\mathbb{R})$ and $\phi \in PAP_{p,w}$ such that ϕ is not identically zero.*

(a) *If $\phi \in \mathcal{G}PAP_{p,w}$, $k(\phi_\ell) = k(\phi_r) = 0$ and*

$$0 \notin \phi^{\#}_{p,w}(\dot{\mathbb{R}}) \cup \mathcal{H}\big(\mathbf{d}(\phi_r), \mathbf{d}(\phi_\ell); \nu_0^-(p,w), \nu_0^+(p,w)\big),$$

then W_ϕ is Fredholm on $L^p(\mathbb{R}_+, w)$.

(b) *If $\phi \in \mathcal{G}PAP_{p,w}$, $k(\phi_\ell)k(\phi_r) \geq 0$, $k(\phi_\ell) + k(\phi_r) > 0$ and $0 \notin \phi^{\#}_{p,w}(\dot{\mathbb{R}})$, then W_ϕ is properly n-normal and left-invertible.*

(c) *If $\phi \in \mathcal{G}PAP_{p,w}$, $k(\phi_\ell)k(\phi_r) \geq 0$, $k(\phi_\ell) + k(\phi_r) < 0$ and $0 \notin \phi^{\#}_{p,w}(\dot{\mathbb{R}})$, then W_ϕ is properly d-normal and right-invertible.*

(d) *In all other cases, W_ϕ is not normally solvable.*

Proof. If $\phi \notin \mathcal{G}PAP_{p,w}$, we see from [3, Corollary 2.8], that W_ϕ is not normally solvable.

So, let us now assume that $\phi \in \mathcal{G}PAP_{p,w}$. Then, we can write $\phi = \varphi\psi$, $\varphi \in \mathcal{G}SAP_{p,w}$ and $\psi \in \mathcal{G}PC_{p,w}$ (satisfying $\psi(-\infty) = \psi(+\infty) = 1$). Taking into account (3.13), we see that W_ϕ is Fredholm if and only if both operators W_φ and W_ψ are Fredholm –which by Theorem 3.1 and Theorem 3.3 happens if conditions stated in part (a) are satisfied.

Having in mind (3.13) and since W_ψ is Fredholm under the conditions of parts (b) and (c) (cf. Theorem 3.1), we deduce that W_ϕ is properly n-normal (resp. properly d-normal) if and only if so is W_φ. Therefore, we obtain part (b) (resp. part (c)) from Theorem 3.3 and *Coburn–Simonenko Theorem*.

To complete the proof, we use the following fact: considering linear and bounded operators A and B acting between Banach spaces (such that AB can be computed), if AB is n-normal (resp. d-normal) then B is n-normal (resp. A is d-normal). This, [3, Theorem 2.2] and (3.13) show that W_ϕ is n-normal (resp. d-normal) if and only if so are both W_φ and W_ψ, and hence we get part (d) for $\phi \in \mathcal{G}PAP$ as a consequence of *Coburn–Simonenko Theorem* and Theorems 3.1 and 3.3. □

Corollary 3.8. *Let $\phi \in \mathcal{G}PAP_{p,w}$. If W_ϕ is a Fredholm operator, then*

$$\begin{aligned}\mathrm{Ind}W_\phi = \mathrm{Ind}W_\varphi + \mathrm{Ind}W_\psi = -\sum_l \mathrm{ind}_l\psi - \mathrm{ind}\varphi + \\ + \sum_{x\in\Lambda_\psi\cup\{\infty\}} \left(\nu_x^0(p,w) - \left\{\nu_x^0(p,w) - \frac{1}{2\pi}\arg\frac{\psi(x-0)}{\psi(x+0)}\right\}\right) + \\ + \nu_0^0(p,w) - \left\{\nu_0^0(p,w) + \frac{1}{2\pi}\arg\frac{\mathbf{d}(\varphi_\ell)}{\mathbf{d}(\varphi_r)}\right\},\end{aligned} \tag{3.14}$$

where $\phi = \psi\varphi$ is a corresponding factorization in the sense of Theorem 3.6 (ii).

Proof. This is obtained by jointing together Theorem 3.6(ii) and formulas (3.9) and (3.11). □

3.3.3. *Example of an invertible Wiener–Hopf operator with a piecewise-almost periodic Fourier symbol on weighted Lebesgue spaces.* Let $p = 2$ and choose the weight function $w(x) = |x|^{\frac{1}{4}}$. We will consider the function ϕ (see Figure 1), given by

$$\phi(x) = (1-u(x))3e^{e^{ix}}g(x) + u(x)e^{e^{-2ix}}g(x) + \frac{g(x)}{x^2+1}\,, \tag{3.15}$$

where u is the real-valued function

$$u(x) = \frac{1}{2} + \frac{1}{\pi}\tanh(x)$$

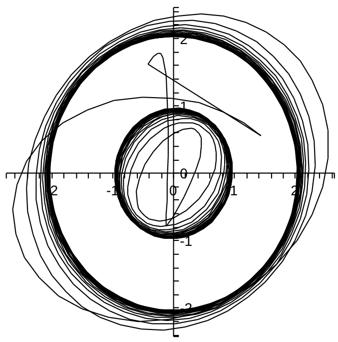

FIGURE 1. The range of $\phi(x)$ for x between -100 and 100.

and $g(x) = \begin{cases} e^x + 1, & \text{if } x < 0 \\ e^{\frac{2i}{x-i}}, & \text{if } x \geq 0 \end{cases}$.

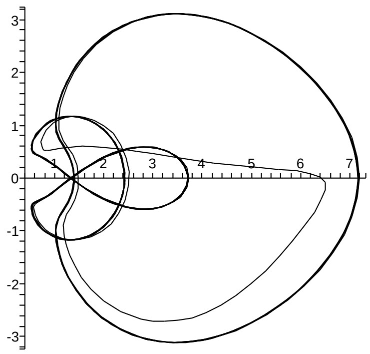

FIGURE 2. The range of $\varphi(x)$ for x between -100 and 100.

It is clear that ϕ admits a factorization $\phi = \varphi\psi$ in the sense of Theorem 3.6 (ii) with

$$\varphi(x) = (1 - u(x))3e^{e^{ix}} + u(x)e^{e^{-2ix}} + \frac{1}{x^2+1}, \tag{3.16}$$

and $\psi(x) = g(x)$.

The function φ (cf. Figure 2) is invertible and we have $\varphi \in \mathcal{G}SAP_{2,w}$.

The element ψ is also an invertible function (see Figure 3). Moreover, $\psi(-\infty) = \psi(+\infty) = 1$. Observing that φ and ψ are invertible, one obtains that ϕ is invertible.

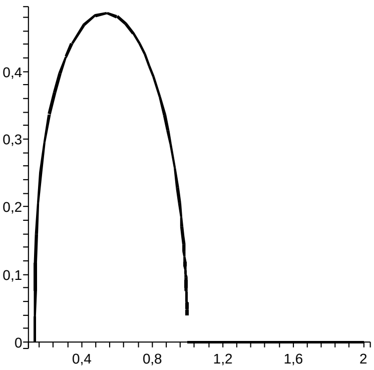

FIGURE 3. The range of $\psi(x)$ for x between -100 and 100.

From Theorem 3.6 (iii), we have that the almost periodic representatives of $\phi \in \mathcal{G}PAP_{2,w}$ coincide with the almost periodic representatives of $\varphi \in \mathcal{G}SAP_{2,w}$. From the definition of $k(\phi)$, it results that $k(\phi_\ell) = k(\phi_r) = 0$.

We have that

$$\begin{aligned} I_0(2, |x|^{\frac{1}{4}}) &= \Big\{\mu \in \mathbb{R} : \Big|\frac{\xi}{\xi + i}\Big|^{\mu} |\xi|^{\frac{1}{4}} \in A_2(\mathbb{R})\Big\} = \\ &= \Big\{\mu \in \mathbb{R} : -\frac{1}{2} < \mu + \frac{1}{4} < \frac{1}{2}\Big\} = \\ &= \Big(-\frac{3}{4}, 1 - \frac{3}{4}\Big), \end{aligned}$$

whence $\nu_0^-(2, |x|^{\frac{1}{4}}) = \nu_0^+(2, |x|^{\frac{1}{4}}) = \frac{3}{4}$. In the same way, we obtain

$$\nu_\infty^-(2, |x|^{\frac{1}{4}}) = \nu_\infty^+(2, |x|^{\frac{1}{4}}) = \frac{1}{4}\,.$$

Consequently, and observing that the only discontinuity point of ϕ is 0, we have

$$\begin{aligned} \phi^{\#}_{p,w}(\dot{\mathbb{R}}) &= \mathcal{H}\Big(\phi(0-0), \phi(0+0); \nu_\infty^-(2, |x|^{\frac{1}{4}}), \nu_\infty^+(2, |x|^{\frac{1}{4}})\Big) \cup \\ &\quad \cup \mathcal{H}\Big(\phi(+\infty), \phi(-\infty); \nu_0^-(2, |x|^{\frac{1}{4}}), \nu_0^+(2, |x|^{\frac{1}{4}})\Big) = \\ &= \mathcal{H}\Big(2e+1, 2e^3+e^2; \frac{1}{4}, \frac{1}{4}\Big) \cup \mathcal{H}\Big(1, 3; \frac{3}{4}, \frac{3}{4}\Big) = \\ &= \mathcal{A}\Big(2e+1, 2e^3+e^2; \frac{1}{4}\Big) \cup \mathcal{A}\Big(1, 3; \frac{3}{4}\Big). \end{aligned}$$

Since $\mathbf{d}(\phi_r) = 1$ and $\mathbf{d}(\phi_\ell) = 3$, it also results that

$$\mathcal{H}\Big(\mathbf{d}(\phi_r), \mathbf{d}(\phi_\ell); \nu_0^-(2, |x|^{\frac{1}{4}}), \nu_0^+(2, |x|^{\frac{1}{4}})\Big) = \mathcal{H}\Big(1, 3; \frac{3}{4}, \frac{3}{4}\Big) = \mathcal{A}\Big(1, 3; \frac{3}{4}\Big).$$

Therefore, we have to consider the arcs $\mathcal{A}(2e{+}1, 2e^3{+}e^2; \frac{1}{4})$ and $\mathcal{A}(1, 3; \frac{3}{4})$ (see Figure 4).

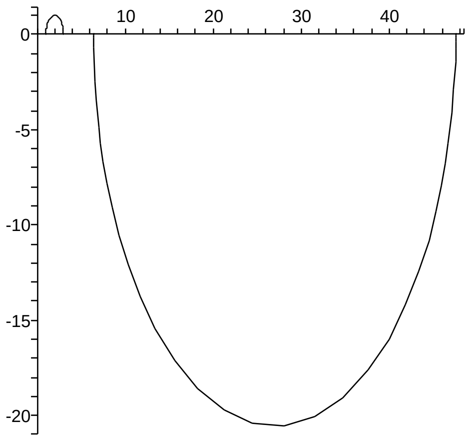

FIGURE 4. The arcs $\mathcal{A}(2e + 1, 2e^3 + e^2; \frac{1}{4})$ and $\mathcal{A}(1, 3; \frac{3}{4})$.

Since these arcs do not contain the origin, the operator

$$W_\phi : L^2_+(\mathbb{R}, |x|^{\frac{1}{4}}) \to L^2(\mathbb{R}_+, |x|^{\frac{1}{4}})$$

is a Fredholm operator (cf. Theorem 3.7 (a)).

Let us now compute the Fredholm index of this operator.

From the definition of $\psi(x)$, we have that if $x < 0$ then $\arg \psi = 0$, and if $x \geq 0$ then $\arg \psi = \frac{2x}{x^2+1}$. Thus, $\mathrm{ind}_l \psi = 0$ and consequently $\sum\limits_{l} \mathrm{ind}_l \psi = 0$.

The only point of discontinuity of ψ is zero and

$$\nu_0^0(2, |x|^{\frac{1}{4}}) = \frac{\nu_0^-(2, |x|^{\frac{1}{4}}) + \nu_0^+(2, |x|^{\frac{1}{4}})}{2} = \frac{3}{4}\,.$$

Additionally, $\arg \frac{\psi(x+0)}{\psi(x-0)} = \arg \frac{e^2}{2} = 0$.

On the other hand, we have that $\mathrm{ind}\varphi = 0$ and

$$\arg \frac{\mathbf{d}(\varphi_\ell)}{\mathbf{d}(\varphi_r)} = \arg(3) = 0.$$

Using these results and substituting on formula (3.14), we obtain that

$$\mathrm{Ind} W_\phi = 0.$$

Consequently, putting together this information with *Coburn–Simonenko Theorem*, we conclude that the Wiener–Hopf operator of this example is invertible.

4. Fredholm and Lateral Invertibility of Wiener–Hopf–Hankel Operators with Piecewise-Almost Periodic Functions on Weighted Lebesgue Spaces

We will turn now to Wiener–Hopf–Hankel operators with piecewise-almost periodic symbols on Lebesgue spaces with Muckenhoupt weights $w \in A_p^{0,e}(\mathbb{R})$. Here, we are also looking for corresponding possible invertibility and Fredholm properties. In fact, we will be able to identify conditions under which the Wiener–Hopf plus/minus Hankel operators are left or right-invertible (and not Fredholm) or have the Fredholm property.

4.1. **Fredholm theory of Wiener–Hopf–Hankel operators with piecewise-almost periodic functions on $L^p(\mathbb{R}_+, w)$.** We will now identify conditions to ensure the Fredholm and lateral invertibility of our Wiener–Hopf plus/minus Hankel operators.

Theorem 4.1. *Let $w \in A_p^{e,0}(\mathbb{R})$ and $\phi \in \mathcal{G}PAP_{p,w}$ $(1 < p < \infty)$.*

(a) *If $k(\phi_\ell) + k(\phi_r) = 0$ and*

$$0 \notin (\phi\widetilde{\phi^{-1}})^{\#}_{p,w}(\dot{\mathbb{R}}) \cup \mathcal{H}\Big(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)}, \frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)}; \nu_0^-(p,w), \nu_0^+(p,w)\Big)$$

then $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are Fredholm operators.

(b) *If $k(\phi_\ell) + k(\phi_r) > 0$ and $0 \notin (\phi\widetilde{\phi^{-1}})^{\#}_{p,w}(\dot{\mathbb{R}})$, then both operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are left-invertible* (*and at least one of the operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ is properly n-normal*).

(c) *If $k(\phi_\ell) + k(\phi_r) < 0$ and $0 \notin (\phi\widetilde{\phi^{-1}})^{\#}_{p,w}(\dot{\mathbb{R}})$, then both operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are right-invertible* (*and at least one of the operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ is properly d-normal*).

(d) *If $k(\phi_\ell) + k(\phi_r) = 0$ and*

$$0 \in (\phi\widetilde{\phi^{-1}})^{\#}_{p,w}(\dot{\mathbb{R}}) \cup \mathcal{H}\Big(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)}, \frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)}; \nu_0^-(p,w), \nu_0^+(p,w)\Big),$$

then at least one of the operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ is not normally solvable on $L^p_+(\mathbb{R}, w)$.

Proof. From the definition of $PAP_{p,w}$, we have the following representation of ϕ:

$$\phi = (1-u)\phi_\ell + u\phi_r + \phi_0,$$

where ϕ_ℓ, $\phi_r \in AP_{p,w}$, $\phi_0 \in PC_{p,w}(\dot{\mathbb{R}})$, $\phi_0(\infty) = 0$ and u is a monotonically increasing real-valued function in $C(\overline{\mathbb{R}})$ satisfying $u(-\infty) = 0$ and $u(+\infty) = 1$.

Taking into consideration Bohr's theorem and the definition of the geometric mean value, it follows that

$$\phi_\ell = e_{k(\phi_\ell)}\mathbf{d}(\phi_\ell)e^{w_\ell},$$
$$\phi_r = e_{k(\phi_r)}\mathbf{d}(\phi_r)e^{w_r},$$

with w_ℓ, $w_r \in AP_{p,w}$, $M(w_\ell) = M(w_r) = 0$ (and $\mathbf{d}(\phi_\ell)\mathbf{d}(\phi_r) \neq 0$). Thus,

$$\phi = (1-u)\mathbf{d}(\phi_\ell)e_{k(\phi_\ell)}e^{w_\ell} + u\mathbf{d}(\phi_r)e_{k(\phi_r)}e^{w_r} + \phi_0. \tag{4.17}$$

Due to the transfer of regularity properties from the Wiener–Hopf operator $W_{\phi\tilde{\phi}^{-1}}$ to the Wiener–Hopf plus and minus Hankel operators $W_\phi \pm H_\phi$, we will study the regularity properties of the Wiener–Hopf operator $W_{\phi\tilde{\phi}^{-1}}$: $L^p_+(\mathbb{R}, w) \to L^p(\mathbb{R}_+, w)$. In view of this, we obtain

$$\phi\widetilde{\phi^{-1}} = \frac{(1-u)\mathbf{d}(\phi_\ell)e_{k(\phi_\ell)}e^{w_\ell} + u\mathbf{d}(\phi_r)e_{k(\phi_r)}e^{w_r} + \phi_0}{(1-\widetilde{u})\mathbf{d}(\phi_\ell)e_{-k(\phi_\ell)}e^{\widetilde{w_\ell}} + \widetilde{u}\mathbf{d}(\phi_r)e_{-k(\phi_r)}e^{\widetilde{w_r}} + \widetilde{\phi_0}} \tag{4.18}$$

being the almost periodic representatives of $\phi\widetilde{\phi^{-1}}$ given by

$$(\phi\widetilde{\phi^{-1}})_\ell = \frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)}\, e_{k(\phi_\ell)+k(\phi_r)}e^{w_\ell - \widetilde{w_r}},$$

$$(\phi\widetilde{\phi^{-1}})_r = \frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)}\, e_{k(\phi_\ell)+k(\phi_r)}e^{w_r - \widetilde{w_\ell}}.$$

From this, taking into account that w_ℓ, $w_r \in AP_{p,w}$ are such that $M(w_\ell) = M(w_r) = 0$ (which additionally implies that $M(\widetilde{w_\ell}) = M(\widetilde{w_r}) = 0$), we have

$$k\big((\phi\widetilde{\phi^{-1}})_\ell\big) = k\big((\phi\widetilde{\phi^{-1}})_r\big) = k(\phi_\ell) + k(\phi_r), \tag{4.19}$$

$$\mathbf{d}\big((\phi\widetilde{\phi^{-1}})_\ell\big) = \frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)}, \quad \mathbf{d}\big((\phi\widetilde{\phi^{-1}})_r\big) = \frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)}. \tag{4.20}$$

Applying now Theorem 3.7 to the Wiener–Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ and having in mind (4.19)–(4.20), it follows that:

(a) If $k(\phi_\ell) + k(\phi_r) < 0$ and $0 \notin (\phi\widetilde{\phi^{-1}})^{\#}_{p,w}(\dot{\mathbb{R}})$, then $W_{\phi\widetilde{\phi^{-1}}}$ is properly d-normal and right-invertible on $L^p_+(\mathbb{R}, w)$;

(b) If $k(\phi_\ell) + k(\phi_r) > 0$ and $0 \notin (\phi\widetilde{\phi^{-1}})^{\#}_{p,w}(\dot{\mathbb{R}})$, then $W_{\phi\widetilde{\phi^{-1}}}$ is properly n-normal and left-invertible on $L^p_+(\mathbb{R}, w)$;

(c) If $k(\phi_\ell) + k(\phi_r) = 0$ and

$$0 \notin (\phi\widetilde{\phi^{-1}})^{\#}_{p,w}(\dot{\mathbb{R}}) \cup \mathcal{H}\Big(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)}, \frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)}; \nu_0^-(p,w), \nu_0^+(p,w)\Big), \tag{4.21}$$

then $W_{\phi\widetilde{\phi^{-1}}}$ is a Fredholm operator on $L^p_+(\mathbb{R}, w)$;

(d) If $k(\phi_\ell) + k(\phi_r) = 0$ and

$$0 \in (\phi\widetilde{\phi^{-1}})^{\#}_{p,w}(\dot{\mathbb{R}}) \cup \mathcal{H}\Big(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)}, \frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)}; \nu_0^-(p,w), \nu_0^+(p,w)\Big),$$

then $W_{\phi\widetilde{\phi^{-1}}}$ is not normally solvable on $L^p_+(\mathbb{R}, w)$.

To arrive at the final assertion, we can interpret the Δ-relation after extension between the Wiener–Hopf plus Hankel operator $W_\phi + H_\phi$ and the Wiener–Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ as an equivalence after extension between $\mathrm{diag}[W_\phi + H_\phi, W_\phi - H_\phi]$ and $W_{\phi\widetilde{\phi^{-1}}}$ (cf. Remark 2.5).

In this way, we get in cases (a) and (b) that $\mathrm{diag}[W_\phi + H_\phi, W_\phi - H_\phi]$ is properly d-normal and right-invertible or properly n-normal and left-invertible, respectively. This implies that – in the case (a) – at least one of the operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ is properly d-normal and both are right-invertible; in the case (b), at least one of the operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ is properly n-normal and both operators are right-invertible.

The case (c) leads to the Fredholm property for both $W_\phi \pm H_\phi$.

In case (d), we have that $\mathrm{diag}[W_\phi + H_\phi, W_\phi - H_\phi]$ is not normally solvable, which implies that at least one of the operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ is not normally solvable. □

4.2. **A formula for the sum of the indices of Fredholm Wiener–Hopf plus and minus Hankel operators.**

Theorem 4.2. *Let $\phi \in \mathcal{G}PAP_{p,w}$. If $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are both Fredholm operators, then*

$$\mathrm{Ind}[W_\phi + H_\phi] + \mathrm{Ind}[W_\phi + H_\phi] = \mathrm{Ind} W_\theta + \mathrm{Ind} W_\zeta = -\sum_l \mathrm{ind}_l \zeta - \mathrm{ind}\theta +$$
$$+ \sum_{x \in \Lambda_\zeta \cup \{\infty\}} \left(\nu_x^0(p,w) - \left\{ \nu_x^0(p,w) - \frac{1}{2\pi} \arg \frac{\zeta(x-0)}{\zeta(x+0)} \right\} \right) +$$
$$+ \nu_0^0(p,w) - \left\{ \nu_0^0(p,w) + \frac{1}{2\pi} \arg \frac{\mathbf{d}(\theta_\ell)}{\mathbf{d}(\theta_r)} \right\}, \qquad (4.22)$$

where $\phi\widetilde{\phi^{-1}} = \zeta\theta$ is a corresponding factorization in the sense of Theorem 3.6 (ii).

Proof. Let $\phi \in \mathcal{G}PAP_{p,w}$ such that $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are both Fredholm.

Recalling that $\mathrm{diag}[W_\phi + H_\phi, W_\phi - H_\phi]$ is equivalent after extension with $W_{\phi\widetilde{\phi}^{-1}}$ (cf. Remark 2.5), it holds that

$$\mathrm{Ind} W_{\phi\widetilde{\phi^{-1}}} = \mathrm{Ind}(W_\phi + H_\phi) + \mathrm{Ind}(W_\phi - H_\phi). \qquad (4.23)$$

From the Fredholm index formula for the Wiener–Hopf operators with $PAP_{p,w}$ Fourier symbols presented in Corollary 3.8, we have

$$\mathrm{Ind} W_{\phi\widetilde{\phi^{-1}}} = \mathrm{Ind} W_\theta + \mathrm{Ind} W_\zeta, \qquad (4.24)$$

where $\phi\widetilde{\phi^{-1}} = \zeta\theta$. Thus, combining (4.23) and (4.24), it follows

$$\mathrm{Ind}[W_\phi + H_\phi] + \mathrm{Ind}[W_\phi + H_\phi] = \mathrm{Ind} W_\theta + \mathrm{Ind} W_\zeta = -\sum_l \mathrm{ind}_l \zeta - \mathrm{ind}\theta +$$
$$+ \sum_{x \in \Lambda_\zeta \cup \{\infty\}} \left(\nu_x^0(p,w) - \left\{ \nu_x^0(p,w) - \frac{1}{2\pi} \arg \frac{\zeta(x-0)}{\zeta(x+0)} \right\} \right) +$$
$$+ \nu_0^0(p,w) - \left\{ \nu_0^0(p,w) + \frac{1}{2\pi} \arg \frac{\mathbf{d}(\theta_\ell)}{\mathbf{d}(\theta_r)} \right\}. \quad \square$$

We would like to remark that due to the method here used we are not able to separate the Fredholm indices of both Wiener–Hopf plus and minus Hankel operators. In view of this, we have the above dependence of both symbols by means of the sum of the corresponding Fredholm indices.

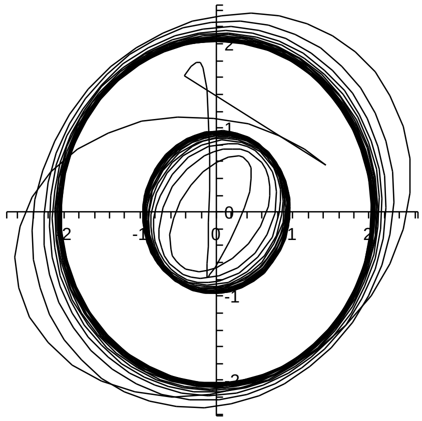

FIGURE 5. The range of $\phi(x)$ defined in (4.25) (for x between -50 and 50).

4.3. **An example within the Wiener–Hopf–Hankel framework.** Let $p = 2$, $w(x) = |x|^{\frac{1}{5}}$ and consider the function ϕ (see Figure 5) given by

$$\phi(x) = (1 - u(x))g(x)e^{-i\pi x} + u(x)\,2i\,g(x)e^{i\pi x}, \qquad (4.25)$$

where

$$u(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x) \text{ and } g(x) = \begin{cases} 1 + \dfrac{1}{x+i}\,, & x \geq 0 \\ 1 + \dfrac{1}{x-i}\,, & x < 0 \end{cases}.$$

It is clear that ϕ admits a factorization $\phi = \varphi\psi$ in sense of Theorem 3.6 (ii), with

$$\varphi(x) = (1 - u(x))e^{-i\pi x} + u(x)\,2i\,e^{i\pi x},$$
$$\psi(x) = g(x).$$

We observe that φ is an invertible function ($\varphi \in \mathcal{G}SAP_{2,w}$), cf. Figure 6, and it is clear that ψ is also an invertible function ($\psi \in \mathcal{G}PC_{2,w}$); see Figure 7. Moreover, $\psi(\pm\infty) = 1$. It therefore follows that ϕ is invertible.

From the definition of mean motion, we have that $k(\phi_\ell) + k(\phi_r) = 0$.

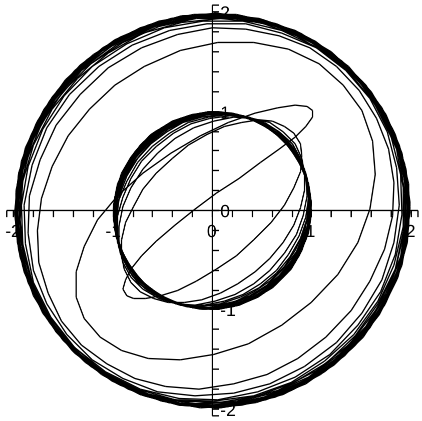

FIGURE 6. The range of $\varphi(x)$ (for x between -50 and 50).

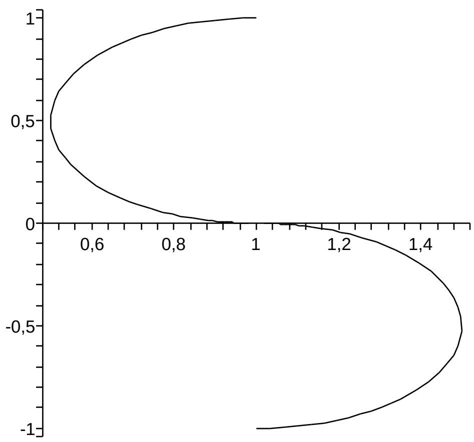

FIGURE 7. The range of $\psi(x)$ (for x between -50 and 50).

Since $\phi = \varphi\psi$, it results that $\phi\widetilde{\phi^{-1}} = \varphi\widetilde{\varphi^{-1}}\psi\widetilde{\psi^{-1}}$, with

$$\psi\widetilde{\psi^{-1}}(x) = \begin{cases} \dfrac{x^2+2i}{x^2-2x+2}, & x<0 \\ 1, & x=0 \\ \dfrac{x^2-2i}{x^2-2x+2}, & x>0 \end{cases}.$$

Recalling that $p=2$ and $w(x)=|x|^{\frac{1}{5}}$, we have

$$\begin{aligned} I_0(x) &= \left\{\mu\in\mathbb{R}:\ \left|\frac{\xi}{\xi+i}\right|^{\mu}|\xi|^{\frac{1}{5}}\in A_2(\mathbb{R})\right\} = \\ &= \left\{\mu\in\mathbb{R}:\ -\frac{1}{2}<\mu+\frac{1}{5}<\frac{1}{2}\right\} = \\ &= \left\{\mu\in\mathbb{R}:\ -\frac{7}{10}<\mu<1-\frac{7}{10}\right\}. \end{aligned}$$

Thus, $\nu_0^-(2,|x|^{\frac{1}{5}})=\nu_0^+(2,|x|^{\frac{1}{5}})=\frac{7}{10}$. In the same way,

$$\nu_\infty^-(2,|x|^{\frac{1}{5}})=\nu_\infty^+(2,|x|^{\frac{1}{5}})=\frac{3}{10}\,.$$

The only discontinuity point of ϕ and $\phi\widetilde{\phi^{-1}}$ is 0. Then, we have

$$\begin{aligned} (\phi\widetilde{\phi^{-1}})^{\#}_{p,w}(\dot{\mathbb{R}}) &:= \mathcal{H}\Big(\phi\widetilde{\phi^{-1}}(0-0),\phi\widetilde{\phi^{-1}}(0+0);\nu_\infty^-(2,|x|^{\frac{1}{5}}),\nu_\infty^+(2,|x|^{\frac{1}{5}})\Big)\cup \\ &\cup\mathcal{H}\Big(\phi\widetilde{\phi^{-1}}(+\infty),\phi\widetilde{\phi^{-1}}(-\infty);\nu_0^-(2,|x|^{\frac{1}{5}}),\nu_0^+(2,|x|^{\frac{1}{5}})\Big) = \\ &= \mathcal{H}\Big(i,-i;\frac{3}{10},\frac{3}{10}\Big)\cup\mathcal{H}\Big(2i,-\frac{1}{2}i;\frac{7}{10},\frac{7}{10}\Big) = \\ &= \mathcal{A}\Big(i,-i;\frac{3}{10}\Big)\cup\mathcal{A}\Big(2i,-\frac{1}{2}i;\frac{7}{10}\Big). \end{aligned}$$

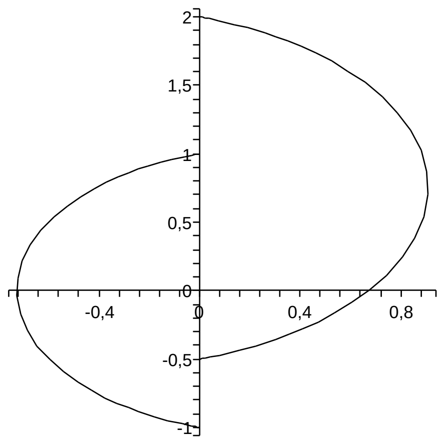

FIGURE 8. The arcs $\mathcal{A}(i,-i;\frac{3}{10})$ and $\mathcal{A}(2i,-\frac{1}{2}i;\frac{7}{10})$.

Additionally, since $\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)} = 2i$ and $\frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)} = -\frac{1}{2}\, i$, we obtain

$$\mathcal{H}\Big(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_\ell)}, \frac{\mathbf{d}(\phi_\ell)}{\mathbf{d}(\phi_r)}; \nu_0^-(2, |x|^{\frac{1}{5}}), \nu_0^+(2, |x|^{\frac{1}{5}})\Big) = \\ = \mathcal{H}\Big(2i, -\frac{1}{2}i; \frac{7}{10}, \frac{7}{10}\Big) = \mathcal{A}\Big(2i, -\frac{1}{2}i; \frac{7}{10}\Big).$$

Figure 8 shows the arcs $\mathcal{A}(i, -i; \frac{7}{10})$ and $\mathcal{A}(2i, -\frac{1}{2}i; \frac{7}{10})$. Since these arcs do not contain the origin, the operators

$$W_\phi \pm H_\phi : L^2_+(\mathbb{R}, |x|^{\frac{1}{5}}) \to L^2(\mathbb{R}_+, |x|^{\frac{1}{5}})$$

have the Fredholm property.

Let us calculate their Fredholm index sum.

If $x < 0$, we have $\arg(\psi\widetilde{\psi^{-1}}) = \arctan(\frac{2}{x^2})$, if $x > 0$, then

$$\arg(\psi\widetilde{\psi^{-1}}) = \arctan\Big(-\frac{2}{x^2}\Big) = -\arctan\Big(\frac{2}{x^2}\Big)$$

and for $x = 0$, $\arg(\psi\widetilde{\psi^{-1}}) = 0$. Therefore,

$$\sum_\ell \mathrm{ind}_\ell \psi\widetilde{\psi^{-1}} = 0.$$

Additionally, $\arg \frac{\psi\widetilde{\psi^{-1}}(0-0)}{\psi\widetilde{\psi^{-1}}(0+0)} = \arg \frac{i}{-i} = 0$. On the other hand, we have $\mathrm{ind}\varphi\widetilde{\varphi^{-1}} = 0$ and

$$\arg \frac{\mathbf{d}((\varphi\widetilde{\varphi^{-1}})_\ell)}{\mathbf{d}((\varphi\widetilde{\varphi^{-1}})_r)} = \arg\Big(\Big(\frac{\mathbf{d}(\varphi_\ell)}{\mathbf{d}(\varphi_r)}\Big)^2\Big) = \arg\Big(\frac{1}{4}\Big) = 0.$$

Finally, using this data in the formula (4.22), we obtain

$$\mathrm{Ind}[W_\phi + H_\phi] + \mathrm{Ind}[W_\phi + H_\phi] = 0.$$

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RECONSTRUCTION OF ELASTIC OBSTACLES FROM THE FAR-FIELD DATA OF SCATTERED ACOUSTIC WAVES

Abstract. We consider the inverse problem for an elastic body emerged in a fluid due to an acoustic wave. The shape of this obstacle is to be reconstructed from the far-field pattern of the scattered wave. For the numerical solution in the two-dimensional case, we compare a simple Newton type iteration method with the Kirsch–Kress algorithm. Our computational tests reveal that the Kirsch–Kress method converges faster for obstacles with very smooth boundaries. The simple Newton method, however, is more stable in the case of not so smooth domains and more robust with respect to measurement errors.

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Key words and phrases. Acoustic and elastic waves, inverse scattering, simple Newton iteration, Kirsch–Kress method.

რეზიუმე. ჩვენ განვიხილავთ სითხეში აკუსტიკური ტალღების მოქმედებით ჩაძირული დრეკადი სხეულისთვის დასმულ შებრუნებულ ამოცანას. ამ დაბრკოლების ფორმა აღდგენილი უნდა იქნას გაბნეული ტალღების შორეული ველის მონაცემებით. ორგანზომილებიან შემთხვევაში მარტივი ნიუტონის ტიპის იტერაციით მიღებულ რიცხვით ამონახსნს ვადარებთ კირშ-კრესის ალგორითმით მიღებულ ამონახსნს. ტესტური გამოთვლები გვიჩვენებენ, რომ მაღალი სიგლუვის მქონე საზღვრიანი დაბრკოლებებისათვის კირშ-კრესის მეთოდი უზრუნველყოფს უფრო სწრაფ კრებადობას. მეორეს მხრივ, მარტივი ნიუტონის ტიპის მეთოდი გაცილებით უფრო სტაბილურია იმ შემთხვევაში, როდესაც დაბრკოლების საზღვარს არ გააჩნია მაღალი სიგლუვე, ის აგრეთვე უფრო მდგრადია გაზომვის ცდომილებების მიმართ.

1. Introduction

If an elastic body is subject to an acoustic wave propagating through the surrounding fluid, then an elastic wave is generated inside the body, and the acoustic wave is perturbed (cf. Figure 1). The wave perturbation is characterized by the asymptotics of the scattered field, namely, the far-field pattern. Suppose the material properties of body and surrounding fluid are known. Then the usual inverse problem of obstacle scattering is to determine the shape of the body from measured far-field data generated by plane waves incident from one or from a finite number of directions. This problem is extremely ill-posed such that regularization techniques are needed for the solution.

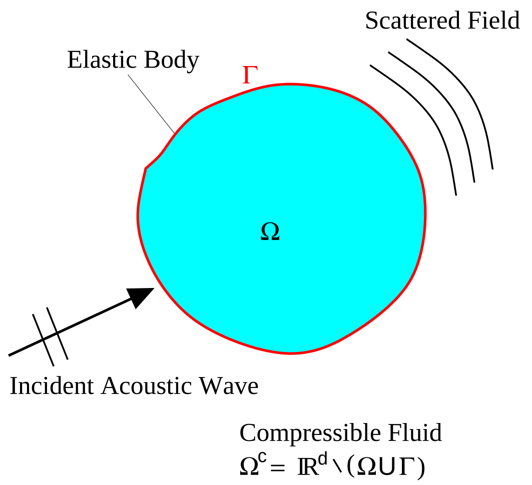

Figure 1. Acoustic wave and obstacle.

Clearly, the same numerical methods used for the inverse problems for obstacles with sound-hard and sound-soft boundaries or for penetrable obstacles can be adapted to the scattering by elastic bodies. Among the available numerical methods, in recent years sampling and factorization methods are very popular (cf. e.g. [12]). Without any a priori information about geometrical details like connectivity components or holes, these methods provide good approximations for the shape of the obstacle. The case of acoustic scattering by elastic bodies in [17] is treated by the linear sampling method. Classical methods such as in [3], [15] (cf. [4] for the case of scattering by elastic obstacles) generally require more information on the geometry of the obstacle. For instance, the boundary of the obstacle is required to be homeomorphic to a circle for 2-D and to a sphere for 3-D problems, respectively. Starting from a reasonable initial guess, the parametrization of the obstacle boundary is approximated in a Newton type iteration. Though the accuracy of the reconstructed solution is always limited by the ill-posedness, we expect the classical Newton approach to be more accurate than the factorization methods. To avoid the solution of direct problems in each step of iteration, besides the boundaries also the wave field can be included into the components of the iterative solutions. For instance, a method proposed by Kirsch and Kress (cf. e.g. [13], [3], [25] and cf. [5] for the case of scattering by elastic obstacles) represents the waves by potentials with generating layer functions defined over artificial curves. Note that, for inverse problems in acoustic scattering by elastic obstacles, difficulties with unpleasant eigensolutions of the direct problem, referred to as Jones modes, can be avoided if the Kirsch–Kress method is applied.

In this paper we consider the two-dimensional case and compare the simple Newton method of [4] with the Kirsch–Kress method of [5] for which we present numerical results for the first time. We implement the same parametrization for the approximate boundary curves iterated by both numerical methods. For a simple egg shaped domain and for a nonconvex domain, we apply the Newton method and the Kirsch–Kress algorithm. The numerical tests show that the Kirsch–Kress method is more accurate due to the better approximation of the fields by potentials in the case of analytic boundaries. Unfortunately, this method is related to an integral equation approach for the direct problem. If the latter integral equation is severely ill-posed, then the Kirsch–Kress algorithm is divergent. Consequently, this method diverges if the curves for the potential representations are too far from the boundary curve of the true obstacle or if the latter curve has large Fourier coefficients. In particular, for the reconstruction of the nonconvex obstacle, the Kirsch–Kress method is divergent. To obtain a convergent version of this method, we use a variant with updated curves for the potential representations during the iteration. For transmission and boundary value problems in acoustic scattering, a comparable update of curves has been proposed in [24] (cf. also the curve updates in [21, Chapter 5] and [15]). Furthermore, our numerical examples reveal that the Kirsch–Kress method is more sensitive with respect to noise in the far-field data, which is also typical for a higher degree of ill-posedness. Finally, we present an example for the reconstruction of an obstacle with Jones modes. Both methods converge for this case.

We start discussing the solution of the direct problem in Section 2. Using the direct solution, we introduce the two numerical schemes for the inverse problem in Section 3. Then we recall the convergence results from [4], [5]. In Section 4 we discuss some details of the implementation. For the least squares problem of the Kirsch–Kress method, we give the formulas for the functional and its gradients in the appendix. Finally, we present the numerical results in Section 5.

2. Direct Problem: Elastic Obstacle in Fluid

Suppose a bounded elastic body is emerged in a homogeneous compressible inviscid fluid. We denote the domain of the body by Ω, its boundary

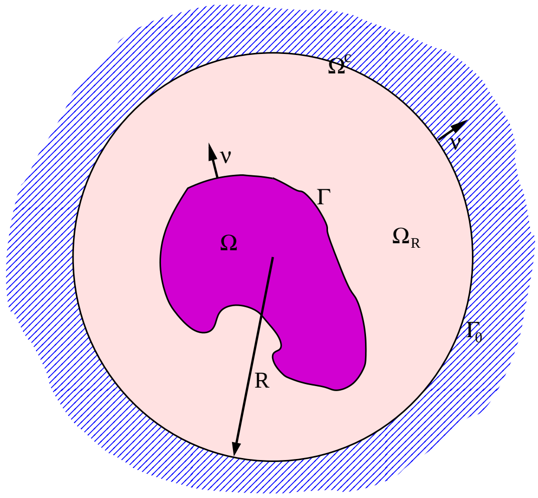

FIGURE 2. Domains.

curve by Γ (cf. Figure 2), and assume that an incoming plane wave is moving in the exterior $\Omega^c := \mathbb{R}^2 \setminus \overline{\Omega}$ toward the body. This wave is scattered by the body and generates an elastic wave inside the body. Mathematically, the acoustic wave is described by the pressure perturbation p over Ω^c and by the displacement function u on Ω. The displacement fulfills the Navier (time-harmonic Lamé) equation

$$\Delta^* u(x) + \varrho\omega^2 u(x) = 0, \quad x \in \Omega, \tag{1}$$
$$\Delta^* u(x) := \mu\Delta u(x) + (\lambda + \mu)\nabla[\nabla \cdot u(x)].$$

Here ω is the frequency, ϱ the density of body, and λ, μ are the Lamé constants. The total pressure p is the sum of the incoming wave p^{inc} and the scattered wave p^s which satisfies the Helmholtz equation and the radiation condition at infinity

$$\Delta p^s(x) + {k_w}^2 p^s(x) = 0, \quad x \in \Omega^c, \tag{2}$$
$$\frac{x}{|x|} \cdot \nabla p^s(x) - \mathbf{i} k_w p^s(x) = o(|x|^{-1/2}), \quad |x| \to \infty, \tag{3}$$

where $k_w^2 = \omega^2/c^2$ is the wave number and c the speed of sound. The pressure and the displacement field are coupled through the transmission conditions

$$u(x) \cdot \nu(x) = \frac{1}{\varrho_f \omega^2} \left\{ \frac{\partial p^s(x)}{\partial \nu} + \frac{\partial p^{inc}(x)}{\partial \nu} \right\}, \quad x \in \Gamma, \tag{4}$$
$$t[u](x) = -\left\{ p^s(x) + p^{inc}(x) \right\} \nu(x), \quad x \in \Gamma, \tag{5}$$
$$t[u](x) := 2\mu \frac{\partial u}{\partial \nu}\Big|_\Gamma + \lambda[\nabla \cdot u]\nu\big|_\Gamma + \mu\nu \times [\nabla \times u]\big|_\Gamma,$$

$$\nu \times [\nabla \times u]\big|_\Gamma := \begin{pmatrix} \nu_2(\partial_{x_1} u_2 - \partial_{x_2} u_1) \\ \nu_1(\partial_{x_2} u_1 - \partial_{x_1} u_2) \end{pmatrix}\Bigg|_\Gamma .$$

Here ϱ_f is the density of the fluid and ν denotes the unit normal at the points of Γ exterior with respect to Ω.

For numerical computations, we truncate the exterior domain Ω^c to the annular domain Ω_R with the outer boundary Γ_0 (cf. Figure 2). The Helmholtz equation (2) is solved over Ω_R, and a non-local boundary condition is imposed on Γ_0 (cf. the boundary integral equation techniques in [11]). Using standard techniques, the boundary value problem can be reformulated in a variational form and solved by the finite element method (cf. [10], [16], [4]). Suppose the boundary Γ of the obstacle is piecewise smooth and choose the auxiliary curve Γ_0 such that the corresponding interior domain has no Dirichlet eigenvalue equal to k_ω^2 for the negative Laplacian. Then existence and uniqueness of the variational solutions as well as the convergence of the finite element method (cf. [4]) can be shown whenever there is no nontrivial solution u_0 of

$$\begin{aligned} \Delta^* u_0(x) + \rho\omega^2 u_0(x) = 0, &\quad x \in \Omega, \\ t[u_0](x) = 0, &\quad x \in \Gamma, \\ u_0(x) \cdot \nu = 0, &\quad x \in \Gamma. \end{aligned} \tag{6}$$

Note that nontrivial solutions of (6) are called Jones modes, and a frequency ω, for which the given domain Ω has a nontrivial solution of (6), is called Jones frequency. It is known that domains with Jones frequencies exist but are exceptional. More precisely, Hargé [8] has shown that the set of domains with Jones frequencies is nowhere dense in a certain metric, and Natroshvili et al. [19] have proved that domains with two non-parallel flat faces have no Jones frequencies. An example of a two-dimensional domain with Jones frequency ω is the disk $\Omega_J := \{x \in \mathbb{R}^2 : \ |x| < r_J\}$ with $r_J = \frac{1}{\omega}\sqrt{\mu/\varrho}\, r_J^0$. Here r_J^0 is any of the positive roots of the equation $rJ_1'(r) = J_1(r)$, and J_1 is the Bessel function of order one. One Jones mode over Ω_J is defined by

$$u_0(x) = J_1\Big(\omega\sqrt{\frac{\varrho}{\mu}}\,|x|\Big)\begin{pmatrix} -x_2/|x| \\ x_1/|x| \end{pmatrix}, \quad x \in \Omega_J. \tag{7}$$

Note that the smallest positive root of $rJ_1'(r) = J_1(r)$ is $r_J^0 = 5.135622\ldots$. Three-dimensional Jones modes are described, e.g., in [19].

Alternatively to the finite element solution, the complete pressure function and the displacement field can be approximated by potentials with sources over auxiliary curves (cf. Figure 3). We introduce the curve Γ_i "close" to Γ, but inside Ω, and the curve Γ_e in Ω_R surrounding Γ. We represent the pressure and the displacement by

$$p^s(x) = \big[V_{\Gamma_i}^{ac}\varphi_i\big](x), \quad x \in \Omega^c, \quad u(x) = \big[V_{\Gamma_e}^{el}\vec{\varphi}_e\big](x), \quad x \in \Omega \tag{8}$$

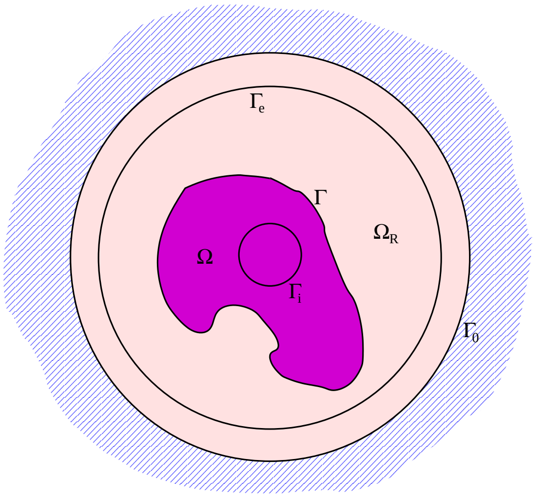

FIGURE 3. Domains and auxiliary curves

with a scalar layer function φ_i and a vector layer $\vec{\varphi}_e$. The potentials are defined by

$$\big[V_\Lambda^{ac} p\big](x) := \int_\Lambda p(y) G(x,y;k_\omega) \mathrm{d}_\Lambda y, \quad x \in \mathbb{R}^2, \tag{9}$$

$$G(x,y;k_w) := \frac{\mathbf{i}}{4} H_0^{(1)}(k_w|x-y|), \tag{10}$$

$$\big[V_\Lambda^{el} u\big](x) := \int_\Lambda G^{el}(y,x) u(y) \mathrm{d}_\Lambda y, \quad x \in \mathbb{R}^2, \tag{11}$$

$$G^{el}(y,x) := \frac{1}{\mu}\left(G(x,y;k_s)\delta_{ij} + \frac{1}{k_s^2} \frac{\partial^2\big(G(x,y;k_s) - G(x,y;k_p)\big)}{\partial x_i \partial x_j} \right)_{i,j=1}^2,$$

where the wave numbers k_p and k_s are defined by $\varrho\omega^2 = (\lambda+2\mu)k_p^2 = \mu k_s^2$ and $H_0^{(1)}$ is the Hankel function of the first kind and of order 0. The layer functions in (8) are chosen such that the corresponding pressure and displacements fields satisfy the transmission conditions (4) and (5). In other words, to get a good approximate solution we have to solve the integral equations

$$t[V_{\Gamma_e}^{el}\vec{\varphi}_e](x) + [V_{\Gamma_i}^{ac}\varphi_i](x)\nu(x) = -p^{inc}\nu(x), \quad x \in \Gamma, \tag{12}$$

$$\varrho_f \omega^2 \nu(x) \cdot [V_{\Gamma_e}^{el}\vec{\varphi}_e](x) - \partial_\nu [V_{\Gamma_i}^{ac}\varphi_i](x) = \partial_\nu p^{inc}(x), \quad x \in \Gamma. \tag{13}$$

Numerical methods based on the discretization of (9), (11), (12) and (13) are well-known to exhibit high rates of convergence (cf. e.g. Sect. 9.8 in [6] and [2], [7], [9]). However, for not so simple geometries, an appropriate choice of Γ_i and Γ_e and an appropriate quadrature of the integrals is not

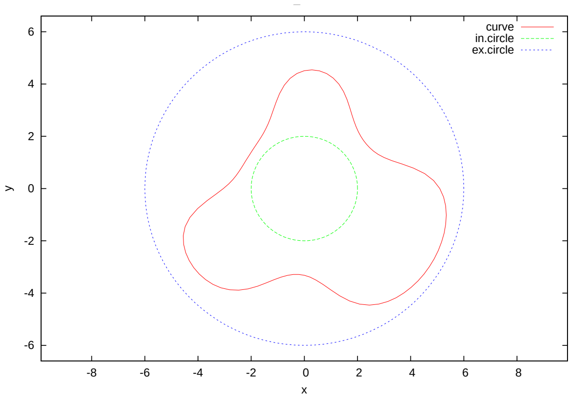

FIGURE 4. Geometry of scatterer.

trivial. A bad choice may lead to extremely ill-posed equations (12), (13) and to false results.

For a point x tending to infinity, the scattered pressure field $p^s(x)$ is known to have the following asymptotics

$$p^s(x) = \frac{e^{\mathbf{i}k_w|x|}}{|x|^{1/2}}\, p^\infty\Big(\frac{x}{|x|}\Big) + \mathcal{O}\Big(\frac{1}{|x|^{3/2}}\Big), \quad |x| \to \infty, \tag{14}$$
$$p^\infty(e^{\mathbf{i}t}) = -\frac{e^{\mathbf{i}\pi/4}}{\sqrt{8\pi k_\omega}} \int_{\Gamma_i} e^{-\mathbf{i}k_w y\cdot e^{\mathbf{i}t}} \varphi_i(y)\, \mathrm{d}_{\Gamma_i} y.$$

The function $\mathcal{F}[p^s](t) := p^\infty(e^{\mathbf{i}t})$ is called the far-field pattern of the scattered field. This is the entity which can be measured.

In order to prepare the numerical results for the inverse problem, we conclude this section by the computation of the corresponding direct problem. If we choose the nonconvex domain with boundary curve Γ according to Figure 4, the constants

$$\begin{aligned} \omega &= \frac{\pi}{2} \text{ kHz}, & \varrho &= 6.75 \cdot 10^{-8} \text{ kg/m}^3, \\ \lambda &= 1.287373095 \text{ Pa}, & \mu &= 0.66315 \text{ Pa}, \\ c &= 1500 \text{ m/s}, & \varrho_f &= 2.5 \cdot 10^{-8} \text{ kg/m}^3, \end{aligned} \tag{15}$$

and the direction of the incoming plane wave equal to $v = (1,0)^\top$, then we get by the finite element method [4] the far-field pattern plotted in Figure 5.

3. Inverse Problem and Iterative Approximation

Now we suppose that the boundary curve Γ of the obstacle is star-shaped and included between the inner curve $\Gamma_i := \{x \in \mathbb{R}^2 : \ |x| = r_i\}$ and the

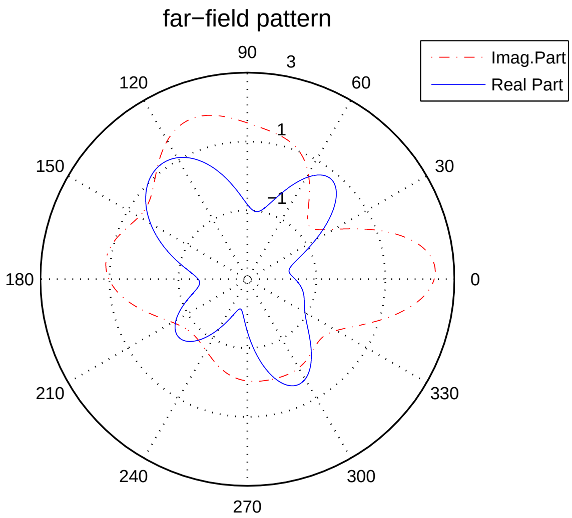

FIGURE 5. Far-field pattern.

outer curve $\Gamma_e := \{x \in \mathbb{R}^2 : \ |x| = r_e\}$, i.e.,

$$\Gamma := \left\{\mathbf{r}(t)e^{\mathbf{i}t} : \ 0 \le t \le 2\pi\right\}, \quad \mathbf{r}(t) = \widehat{a}_0 + \sum_{j=1}^{\infty} \left\{\widehat{a}_j \cos(jt) + \widehat{b}_j \sin(jt)\right\} \quad (16)$$

with the constraint $r_i < \mathbf{r}(t) < r_e$, $0 \le t \le 2\pi$. To avoid this constraint, we can use the parametrization $\Gamma = \Gamma^{\mathbf{r}}$

$$\Gamma^{\mathbf{r}} := \left\{\widetilde{\mathbf{r}}(t)e^{\mathbf{i}t} : \ 0 \le t \le 2\pi\right\}, \quad \widetilde{\mathbf{r}}(t) := \frac{r_e + r_i}{2} + \frac{r_e - r_i}{\pi} \arctan(\mathbf{r}(t)). \quad (17)$$

Having in mind this representation, the star-shaped curve is uniquely determined by the real valued function $\mathbf{r}$ or, equivalently, by the Fourier coefficients $\{\widehat{a}_j, \widehat{b}_j\}$. The direct problem of the previous section defines a continuous mapping (cf. [4])

$$F : H^{1+\varepsilon}_{per}[0, 2\pi] \longrightarrow L^2_{per}[0, 2\pi], \quad \mathbf{r} \mapsto p^\infty,$$

where $p^\infty = \mathcal{F}[p^s]$ is the far-field of the scattered field p^s, and p^s is the pressure part of the solution (p^s, u) to the direct problem (1), (2), (4), (5), and (3) including the interface $\Gamma = \Gamma^{\mathbf{r}}$ and a fixed incoming plane wave p^{inc}. The space $H^{1+\varepsilon}_{per}[0, 2\pi]$ is the periodic Sobolev space of order $1 + \varepsilon > 1$ over the interval $[0, 2\pi]$. $L^2[0, 2\pi]$ is the corresponding Lebesgue space. Now the inverse problem is the following: For a given far-field pattern p^∞, find the shape of the obstacle with boundary $\mathbf{r}_{sol}$ such that the scattered field corresponding to the fixed incoming plane wave p^{inc} has the far-field pattern p^∞, i.e., such that $F(\mathbf{r}_{sol}) = p^\infty$. To our knowledge, results on the uniqueness of the solution $\mathbf{r}_{sol}$ are not known yet. For the case of far-field data given in all directions of incidence, we refer to the theoretical results in [18], [17].

We now define three different optimization problems equivalent to the inverse problem. Numerical algorithms for the inverse problem can be derived simply by applying numerical minimization schemes to the optimization problems. More precisely, the minimization schemes are applied to regularized modifications of the optimization problems.

The first optimization problem is to find a least-squares solution $\mathbf{r}_{min}$, i.e., a minimizer of the following problem

$$\inf_{\mathbf{r}\in H^{1+\varepsilon}_{per}[0,2\pi]} \mathcal{J}_\gamma(\mathbf{r}), \quad \mathcal{J}_\gamma(\mathbf{r}) := \left\| F(\mathbf{r}) - p^\infty \right\|^2_{L^2[0,2\pi]}.$$

Since the inverse problem is ill-posed and since the measured far-field data is given with noise, we replace the last optimization problem by

$$\inf_{\mathbf{r}\in H^{1+\varepsilon}_{per}[0,2\pi]} \mathcal{J}^1_\gamma(\mathbf{r}), \quad \mathcal{J}^1_\gamma(\mathbf{r}) := \left\| F(\mathbf{r}) - p^\infty_{noisy} \right\|^2_{L^2[0,2\pi]} + \gamma \|\mathbf{r}\|^2_{H^{1+\varepsilon}_{per}[0,2\pi]}, \tag{18}$$

where γ is a small positive regularization parameter. As usual this parameter is to be chosen in dependence on the noise level. To guarantee convergence for noise level tending to zero and for $\gamma \to 0$, we suppose

$$\left\| p^\infty - p^\infty_{noisy} \right\|^2_{L^2[0,2\pi]} \leq c\gamma \tag{19}$$

for a constant c independent of γ. The first numerical algorithm (cf. [4]) consists now in discretizing the mapping F by finite elements and applying a Gauss–Newton method to determine a minimizer of (18). This is a modified Newton method for the operator equation $F(\mathbf{r}_{sol}) = p^\infty_{noisy}$ which we shall call the simple Newton iteration.

Theorem 3.1 ([4]). *Suppose Γ_0 is chosen such that the corresponding interior domain has no Dirichlet eigenvalue equal to k^2_ω for the negative Laplacian. Then we have:*

(i) *For any $\gamma > 0$, there is a minimizer $\mathbf{r}^\gamma$ of* (18).

(ii) *Suppose the far-field pattern p^∞ is the exact pattern for a fixed solution $\mathbf{r}^*$ of the inverse problem, i.e., $F(\mathbf{r}^*) = p^\infty$ and $\mathcal{J}^1_0(\mathbf{r}^*) = 0$. Then, for $\varepsilon > 0$ and for any set of minimizers $\mathbf{r}^\gamma$, there exists a subsequence $\mathbf{r}^{\gamma_n}$ converging weakly in $H^{1+\varepsilon}_{per}[0,2\pi]$ and strongly in $H^{1+\varepsilon'}_{per}[0,2\pi]$, $0 < \varepsilon' < \varepsilon$, to a solution $\mathbf{r}^{**}$ of* (18) *with $\gamma = 0$ and, therewith, to a solution of the inverse problem.*

(iii) *If, additionally to the assumptions of* (iii)*, the solution $\mathbf{r}^*$ of the inverse problem is unique, then we even get that $\mathbf{r}^\gamma$ tends to $\mathbf{r}^*$ weakly in $H^{1+\varepsilon}_{per}[0,2\pi]$ and strongly in $H^{1+\varepsilon'}_{per}[0,2\pi]$, $0 < \varepsilon' < \varepsilon$.*

Unfortunately, for the first method the computation of F requires a solution of a direct problem. In particular, if the curve Γ is the boundary of a domain with Jones frequency or close to such a boundary, the direct solution by finite elements is not easy. One way would be to compute with slightly modified frequencies. However, it might be difficult to check whether the curve is "close" to the boundary of a domain with Jones frequency and to choose a modified frequency appropriately.

In order to motivate the second numerical method, the Kirsch–Kress algorithm, which corresponds to a third optimization problem, we introduce a second intermediate optimization method first. The plan is to define a method, where a solution of the direct method is not needed. Therefore, besides the unknown curve Γ the pressure p^s and the displacement field u are included into the set of optimization "parameters". Additionally to the term of the least squares deviation of $\mathcal{F}[p^s]$ from p^∞_{noisy}, new terms are needed which enforce the fulfillment of the equations (1), (2), (4), (5), and (3) at least approximately. Hence, the regularized second optimization problem is to find a minimizer $(\mathbf{r}_{min}, u_{min}, p_{min})$ of

$$\inf_{\mathbf{r}\in H^{2+\varepsilon}_{per}[0,2\pi],\ u\in[H^1(\Omega)]^2,\ p^s\in H^1(\Omega_R)} \mathcal{J}^2_\gamma(\mathbf{r},u,p^s), \tag{20}$$

$$\begin{aligned}
\mathcal{J}^2_\gamma(\mathbf{r},u,p^s) := {} & \left\|\mathcal{F}[p^s]-p^\infty_{noisy}\right\|^2_{L^2[0,2\pi])} + \left\|\Delta^* u+\varrho\omega^2 u\right\|^2_{[H^{-1}(\Omega)]^2} + \\
& + \left\|\Delta p^s + {k_w}^2 p^s\right\|^2_{H^{-1}(\Omega_R)} + \\
& + \left\|t[u]+\{p^s+p^{inc}\}\nu\right\|^2_{[H^{-1/2}(\Gamma)]^2} + \\
& + \left\|u\cdot\nu - \frac{1}{\varrho_f\omega^2}\left\{\frac{\partial p^s}{\partial\nu}+\frac{\partial p^{inc}}{\partial\nu}\right\}\right\|^2_{H^{-1/2}(\Gamma)} + \\
& + \left\|V^{ac}_{\Gamma_0}[\partial_\nu p^s] + \frac{1}{2}\,[p^s] - K^{ac}_{\Gamma_0}[p^s]\right\|^2_{H^{1/2}(\Gamma_0)} + \\
& + c_1\gamma\|\mathbf{r}\|^2_{H^{2+\varepsilon}_{per}[0,2\pi]} + c_2\gamma\|u\|^2_{H^1(\Omega)} + c_3\gamma\|p^s\|^2_{H^1(\Omega_R)},
\end{aligned}$$

$$K^{ac}_{\Gamma_0}[p^s](x) := \int_{\Gamma_0} \frac{\partial G(x,y;k_w)}{\partial\nu(y)}\, p^s(y)\,\mathrm{d}_{\Gamma_0} y$$

where $c_i > 0$, $i = 1,2,3$, are calibration constants and γ is a small positive regularization parameter. Of course, this is a theoretical optimization problem only. For a numerical realization, the operators should be replaced by those of the variational formulation. However, it is clearly seen that the price for avoiding a solution of the direct problem is an increase in the number of the optimization "parameters". The numerical solution of the discretized optimization problem (20) is higher dimensional and might be more involved than that for the case of (18).

The third optimization problem is a modification of (20). The optimization "parameters" u and p^s are replaced by the layer functions φ_i and $\vec{\varphi}_e$ of the potential representations (8). In other words, in the numerical discretization the finite elements over the domains Ω and Ω_R are replaced by lower dimensional boundary elements over the curves Γ_i and Γ_e. Instead of the terms in $\mathcal{J}^2_\gamma$ enforcing the conditions (1), (2), (4), (5), and (3), we only need terms enforcing (12) and (13). Hence, the regularized third optimization problem is to find a minimizer $(\mathbf{r}_{min}, \varphi_{i,min}, \vec{\varphi}_{e,min})$ of

$$\inf_{\mathbf{r}\in H^{2+\varepsilon}_{per}[0,2\pi],\ \varphi_i\in H^{-1}(\Gamma_i),\ \vec{\varphi}_e\in[H^{-1}(\Gamma_e)]^2} \mathcal{J}^3_\gamma(\mathbf{r},\varphi_i,\vec{\varphi}_e), \tag{21}$$

$$\begin{aligned}\mathcal{J}^3_\gamma(\mathbf{r},\varphi_i,\vec{\varphi}_e) := c\Big\|\mathcal{F}\big[V^{ac}_{\Gamma_i}\varphi_i\big] - p^\infty_{noisy}\Big\|^2_{L^2[0,2\pi]} + \\ + \gamma\|\varphi_i\|^2_{H^{-1}(\Gamma_i)} + \gamma\|\vec{\varphi}_e\|^2_{[H^{-1}(\Gamma_e)]^2} + \\ + \Big\|t\big[V^{el}_{\Gamma_e}\vec{\varphi}_e\big] + \big[V^{ac}_{\Gamma_i}\varphi_i\big]\nu + p^{inc}\nu\Big\|^2_{L^2(\Gamma^{\mathbf{r}})} + \\ + \Big\|\varrho_f\omega^2\nu\cdot\big[V^{el}_{\Gamma_i}\vec{\varphi}_e\big] - \partial_\nu\big[V^{ac}_{\Gamma_i}\varphi_i\big] - \partial_\nu p^{inc}\Big\|^2_{L^2(\Gamma^{\mathbf{r}})},\end{aligned} \tag{22}$$

where γ is a small positive regularization parameter and c a positive calibration constant. We choose the layers $\varphi_{i,min}$ and $\vec{\varphi}_{e,min}$ in an unusual Sobolev space of negative order to enable approximations by Dirac-delta functionals, i.e., by the method of fundamental solutions. Though the number of optimization parameters in a discretization of (21) is larger than that in a discretization of (18), the objective functional $\mathcal{J}^3_\gamma$ is simpler than $\mathcal{J}^1_\gamma$. Applying an optimization scheme like the conjugate gradient method or the Levenberg–Marquardt algorithm to (21), we arrive at the Kirsch–Kress method. Note that the accuracy of the solution of this method is limited by the accuracy of solving the integral equations (12) and (13) with a Tikhonov regularization. To improve this, the curves Γ_i and Γ_e can be updated during the iterative solution of the optimization problem (compare the iterative schemes in [21], [24], [15]).

Theorem 3.2 ([5]). *Suppose k^2_ω is not a Dirichlet eigenvalue for the negative Laplacian in the interior of Γ_i and that p^∞ is the exact far-field pattern of a scattered field p^s corresponding to some $\Gamma^{\mathbf{r}^*}$. Then we have:*

(i) *For any $\gamma > 0$, there is a minimizer $(\mathbf{r}^\gamma, \varphi^\gamma_i, \vec{\varphi}^\gamma_e)$ of (21).*

(ii) *For any set of minimizers $(\mathbf{r}^\gamma, \varphi^\gamma_i, \vec{\varphi}^\gamma_e)$, there exists a subsequence $(\mathbf{r}^{\gamma_n}, \varphi^{\gamma_n}_i, \vec{\varphi}^{\gamma_n}_e)$ such that $\mathbf{r}^{\gamma_n}$ converges weakly in $H^{1+\varepsilon}_{per}[0,2\pi]$ and strongly in $H^{1+\varepsilon'}_{per}[0,2\pi]$, $0<\varepsilon'<\varepsilon$, to a solution $\mathbf{r}^{**}$ of the inverse problem.*

(iii) *If, additionally, the solution $\mathbf{r}^*$ of the inverse problem is unique, then we even get that $\mathbf{r}^\gamma$ tends to $\mathbf{r}^*$ weakly in $H^{1+\varepsilon}_{per}[0,2\pi]$ and strongly in $H^{1+\varepsilon'}_{per}[0,2\pi]$, $0<\varepsilon'<\varepsilon$.*

Formulas for the discretization of the optimization problem (21) and for the derivatives of the objective functional are presented in Section 6.

4. Some Details of the Implementation

For the solution of the optimization problems, a lot of **numerical optimization schemes** are available (cf. [20]). Unfortunately, global methods which yield the global minimum are often very slow. We recommend gradient based local optimization schemes. They provide local minimizers, i.e. solutions with minimal value of the objective functional in a neighbourhood of the minimizer. In general, it cannot be guaranteed that the local minimizer is the global minimizer. However, using a good initial guess, the local minimizer will coincide with the global. In particular, we have tested the Gauss–Newton method, the Levenberg–Marquardt algorithm (cf. [14]), and the conjugate gradient method. The last method has been tested for the Kirsch–Kress method to avoid the solution of linear systems in the size of the direct problem.

In order to compute **derivatives** of the objective functionals in case of the simple Newton iteration the calculus of shape derivatives can be applied. The derivatives result from solving the finite element system of the direct problem with new right-hand side vectors. This is fast if the finite element system is solved by an LU factorization for sparse systems (cf. [22], [4]). The derivatives for the Kirsch–Kress method can be obtained by a simple differentiation of the kernel functions in the potential representations. Since the elasticity kernel contains second-order derivatives of the acoustic kernel and since the terms enforcing the transmission conditions contain first-order derivatives of the elastic potential, we need fourth order derivatives of the acoustic kernel. We present the needed formulas in Section 6.

Normally, **quadrature** rules are needed if the layer functions φ_i and $\vec{\varphi}_e$ in the potential representation (8) are approximated by functions of a finite dimensional space. The potential integrals of these functions must be approximated by appropriate quadratures. However, in the case of the Kirsch–Kress method we can approximate the layer functions by linear combinations of Dirac delta functions

$$\varphi_i \sim \varphi_{i,M} := \sum_{\kappa=1}^{M} b_\kappa \delta_{x_{i,\kappa}}, \quad b_\kappa \in \mathbb{C}, \quad x_{i,\kappa} := r_i e^{\mathbf{i} t_\kappa}, \quad t_\kappa := \frac{2\pi\kappa}{M}, \tag{23}$$

$$\vec{\varphi}_e \sim \vec{\varphi}_{e,M} := \sum_{\kappa=1}^{M} c_\kappa \delta_{x_{e,\kappa}}, \quad c_\kappa \in \mathbb{C}^2, \quad x_{e,\kappa} := r_e e^{\mathbf{i} t_\kappa}. \tag{24}$$

This works since the potential operators are smoothing operators from the curve Γ_e, Γ_i to Γ. Only in the case that Γ_e or Γ_i is close to Γ, a trigonometric or spline approximation of φ_i and $\vec{\varphi}_e$ together with an accurate quadrature must be employed.

Another important issue is the **scaling** of the optimization scheme. Indeed, the number of necessary iterations depends on the conditioning of the optimization problem. Using an appropriate scaling, the conditioning can be essentially improved. The first choice is, of course, the natural scaling. The far-field values should be scaled such that the measurement uncertainties of the scaled far-field values coincide, and the parameters should be scaled in accordance with the accuracy requirements. A scaling different from the natural one is chosen not to improve the reconstruction operator, but to speed up the optimization algorithm. This calibration may include different constants in front of the individual terms in the objective functional (cf. the factors c and γ in the definition of $\mathcal{J}^3_\gamma$) and the replacement of the optimization parameters by the products of these parameters with convenient constants. The constants can be chosen, e.g., to minimize the conditioning of the Jacobian of the mapping that maps the parameters to the far-field values. Alternatively, the constants can be chosen by checking typical test examples with known solution. To improve the conditioning of the optimization in the Kirsch–Kress method, we have replaced the "optimization parameters" $\mathbf{r}$, φ_i, and $\vec{\varphi}_e$ by the parameters

$$\mathbf{r}' = \mathbf{r}/c_{\mathbf{r}}, \quad \varphi_i' = \varphi_i/c_i, \quad \vec{\varphi}_e' = \vec{\varphi}_e/c_e. \tag{25}$$

5. Numerical Results

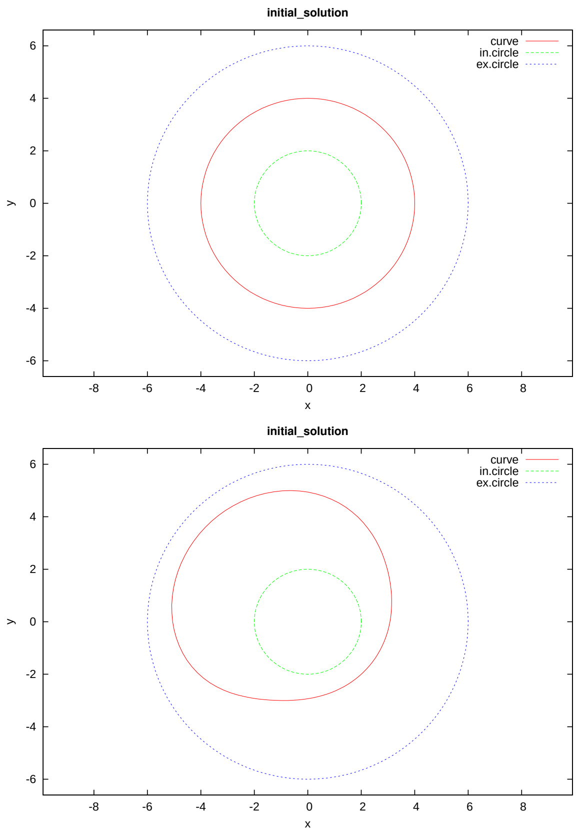

Figure 6. Initial solution and egg shaped domain.

5.1. **The curves for the numerical examples and some technical details.** We have employed (i) the simple Newton iteration and (ii) the

Kirsch–Kress method, both with a circle as initial solution, to reconstruct two different obstacles. The first is an easy egg shaped domain (cf. Figure 6) with a boundary given by (17), by $r_i = 2$, $r_e = 6$, and by the fast decaying Fourier coefficients

$$\begin{array}{lllll} \widehat{a}_0 = 0, & & & & \\ \widehat{a}_1 = -1, & \widehat{a}_2 = 0.1, & \widehat{a}_3 = 0.01, & \widehat{a}_4 = -0.001, & \widehat{a}_5 = 0.0001, \\ \widehat{b}_1 = 1, & \widehat{b}_2 = 0.1, & \widehat{b}_3 = 0.01, & \widehat{b}_4 = 0.001, & \widehat{b}_5 = 0.0001. \end{array} \qquad (26)$$

The second body is the nonconvex obstacle from the end of Section 2 (cf. Figure 7), and its boundary is given by $r_i = 2$, $r_e = 6$ and by the Fourier coefficients

$$\begin{array}{lllll} \widehat{a}_0 = 0, & & & & \\ \widehat{a}_1 = 1, & \widehat{a}_2 = 0.10, & \widehat{a}_3 = 0.040, & \widehat{a}_4 = 0.016, & \widehat{a}_5 = 0.008, \\ \widehat{b}_1 = -1, & \widehat{b}_2 = 0.02, & \widehat{b}_3 = -1.500, & \widehat{b}_4 = -0.010, & \widehat{b}_5 = 0.008. \end{array} \qquad (27)$$

Clearly, both obstacles are defined by a truncated Fourier series and are analytic. However, the egg shaped domain is smoother since the nonzero Fourier coefficients have the strong decay property $|\widehat{a}_j| \le 10^{-j}$ and $|\widehat{b}_j| \le 10^{-j}$. More precisely, the norms

$$\|\mathbf{r}\|_r := \sqrt{|\widehat{a}_0|^2 + \frac{1}{2}\sum_{j=1}^{\infty} r^{-2j}|\widehat{a}_j|^2 + \frac{1}{2}\sum_{j=1}^{\infty} r^{-2j}|\widehat{b}_j|^2}\,, \quad r > 1,$$

of analytic functions are smaller for the egg shaped domain than for the nonconvex example. Note that $\|\mathbf{r}\|_r$ is the norm

$$\sqrt{|\widehat{a}_0|^2 + \sum_{j=1}^{\infty} r^{-2j}\Big|\frac{\widehat{a}_j - \mathbf{i}\widehat{b}_j}{2}\Big|^2 + \sum_{j=1}^{\infty} r^{-2j}\Big|\frac{\widehat{a}_j + \mathbf{i}\widehat{b}_j}{2}\Big|^2}\,,$$

of the analytic extension

$$z = \varrho e^{\mathbf{i}t} \mapsto \widehat{a}_0 + \sum_{j=1}^{\infty}\Big[\frac{\widehat{a}_j - \mathbf{i}\widehat{b}_j}{2}\Big]\varrho^j e^{\mathbf{i}jt} + \sum_{j=1}^{\infty}\Big[\frac{\widehat{a}_j + \mathbf{i}\widehat{b}_j}{2}\Big]\varrho^{-j} e^{-\mathbf{i}jt}$$

of the function $e^{\mathbf{i}t} \mapsto \mathbf{r}(t) = \sum_j \widehat{a}_j \cos(jt) + \sum_j \widehat{b}_j \sin(jt)$ onto the annular domain $\{z \in \mathbb{C} : 1/r < |z| < r\}$.

In all computations, we have chosen the physical constants in accordance with (15). The incoming plane wave has been fixed to $p^{inc}(x) := e^{\mathbf{i}(1,0)^\top \cdot x}$. Moreover, for all initial curves and all iterative solutions, we have fixed the zeroth Fourier coefficient $\widehat{a}_0$ to zero. The "measured" far-field data $\{p^\infty(k/M''), k = 1, \ldots, M''\}$, $M'' = 80$ (cf. Figure 5) has been simulated by the piecewise linear finite element method (FEM) described in Section 2. To avoid what is called an inverse crime, we have chosen the meshsize of the FEM grid (determined by NETGEN [23]) for the far-field computation by a factor of at least 0.25 smaller than that of the FEM grids involved in the

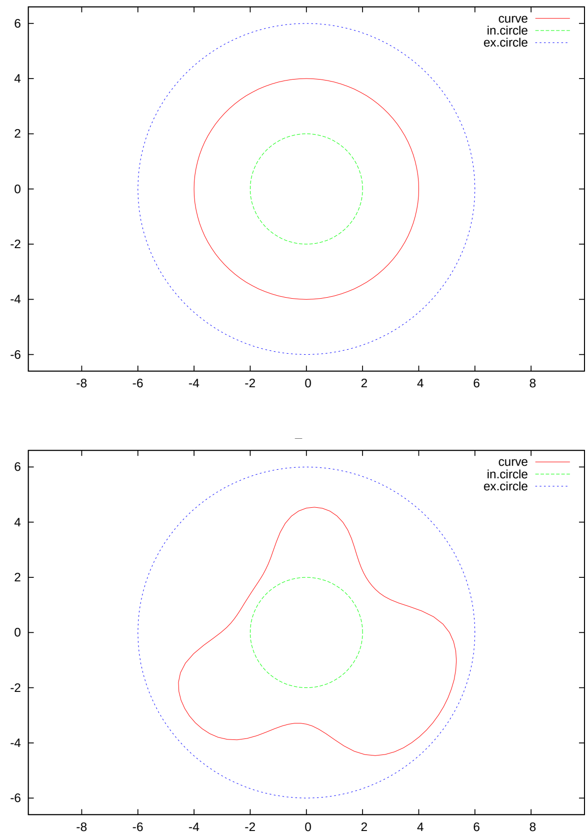

FIGURE 7. Initial solution and nonconvex domain.

inverse algorithms. Our tests have revealed that the far-field of the FEM method is more reliable than that computed by the regularized system (12)–(13). The scaling parameters c, $c_{\mathbf{r}}$, c_i, and c_e for the Kirsch–Kress method (cf. (25) and the definition of $\mathcal{J}^3_\gamma$) have been determined experimentally such that the reconstruction by Gauss–Newton iteration converges with the smallest number of iteration steps. For example, for the egg shaped domain and $M = M' = 44$ points of discretization for the approximate integration over Γ, Γ_i, Γ_e (cf. the discretized objective functional in (A.6)), these values are $c = 4000$, $c_{\mathbf{r}} = 1$, $c_i = 0.1$, and $c_e = 0.005$.

5.2. **Convergence of the simple Newton iteration.** The results for the egg shaped domain and for the simple Newton iteration have been similar to those presented in [4], where the constants where slightly different and the obstacle was similar to our nonconvex body. After a small number (≤ 20) of iterations, the algorithm reconstructs the obstacle. The regularization parameter γ can even be set to zero, which is not surprising since only

10 unknown real parameters are reconstructed from 160 real measurement values. The left Table 1 exhibits the meshsize h, the number of Gauss–Newton iterations it, and the accuracy $err := \|\widetilde{\mathbf{r}} - \widetilde{\mathbf{r}}_{FEM}\|_{L^\infty[0,2\pi]}$ of the reconstruction $\mathbf{r}_{FEM}$. The first row contains the accuracy of the initial guess. For the nonconvex obstacle, the results are similar (cf. right Table 1). Most of the computing time is spent on the evaluation of the objective functional including the solution of a direct problem. Therefore, it is not necessary to replace the expensive Gauss–Newton iteration by a different optimization scheme.

h	err	it
	1.2596	0
0.5	0.0759	6
0.25	0.0247	8
0.125	0.00876	8
0.0625	0.00329	10
0.03125	0.00156	10

h	err	it
	1.5733	0
0.25	1.1435	20
0.125	0.00924	17
0.0625	0.00401	15
0.03125	0.00157	18

Table 1. Reconstruction by simple Newton iteration for egg shaped domain (left) and for nonconvex domain (right).

5.3. **Convergence of the Kirsch–Kress algorithm.** We have started the tests of the Kirsch–Kress method with the nonconvex domain. However, the optimization algorithms did not converge. To fix the problem, we have checked the solution of the corresponding direct problem. We have observed that the far-field of the solution computed by (8), (12), and (13) did not match that of the FEM. Even a Tikhonov regularization in accordance with the last four terms of the functional $\mathcal{J}^3_\gamma$ did not help. Only a regularization with a truncated singular value decomposition and a well-chosen truncation parameter led to the correct far field. In other words, the reason for the divergence of the Kirsch–Kress method is the high degree of ill-posedness of the system (12), (13). On the other hand, if we commit the inverse crime and take the incorrect far-field data computed by solving (12), (13), then the Kirsch–Kress algorithm does converge.

To show the convergence of the Kirsch–Kress method with FEM generated far-field data, we consider the egg shaped domain. This time the solution curve has a higher degree of smoothness, and the direct solution of (12), (13) together with a Tikhonov regularization yields a far-field solution close to that of the FEM. Table 2 shows that the Kirsch–Kress method converges for the egg shaped domain. Indeed, the table shows the regularization parameter γ, the error $\|\widetilde{\mathbf{r}} - \widetilde{\mathbf{r}}_{KK}\|_{L^\infty_{per}[0,2\pi]}$ of the Kirsch–Kress reconstruction $\mathbf{r}_{KK}$, and the number of necessary iteration steps. These depend on the number of discretization points $M = M'$ for the approximate integration over Γ, Γ_i, Γ_e (cf. the discretized objective functional in

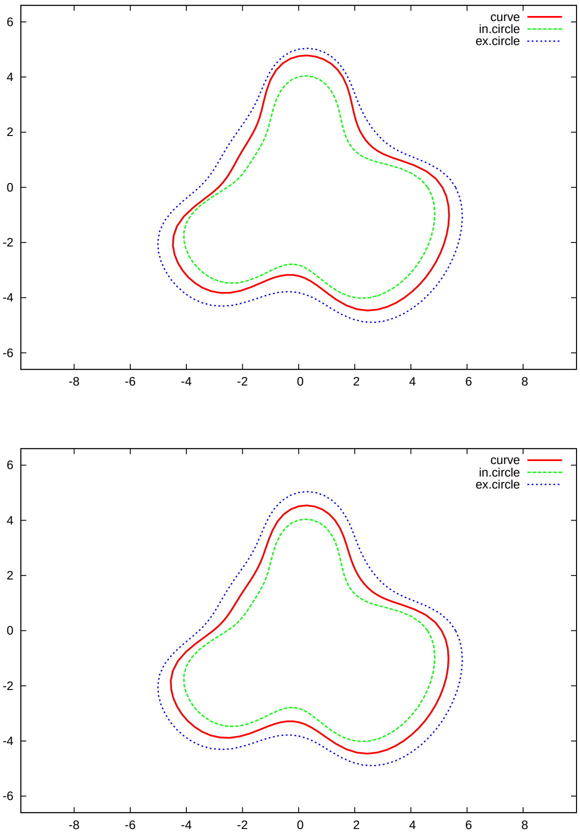

FIGURE 8. Initial solution with Fourier coefficients $\widehat{a}_i^1, \widehat{b}_i^1$ and nonconvex domain with modified curves Γ_i and Γ_e.

(A.6)) and on the choice of the optimization method. In particular, we have checked the Gauss–Newton method with experimentally chosen regularization parameter γ (GNw), the Levenberg–Marquardt method with the same regularization parameter (LMw), and the Levenberg–Marquardt method without regularization (LMo). The results show much better approximations than for the simple Newton iteration. Unfortunately, the conjugate gradient method did not converge.

To get convergence of the Kirsch–Kress method also for the nonconvex domain of Figure 7, we have changed the curves Γ_i and Γ_e (cf. Figure 8). If these are closer to the curve $\Gamma^{\mathbf{r}}$, then the degree of ill-posedness of the operators in (12), (13) is reduced. We have chosen the initial guess of the Fourier coefficients as

$$\begin{array}{lllll} \widehat{a}_0^0 = \ \ 0.0, & & & & \\ \widehat{a}_1^0 = \ \ 1.3, & \widehat{a}_2^0 = -0.10, & \widehat{a}_3^0 = \ \ 0.1, & \widehat{a}_4^0 = -0.05, & \widehat{a}_5^0 = \ \ 0.018, \\ \widehat{b}_1^0 = -0.8, & \widehat{b}_2^0 = \ \ 0.05, & \widehat{b}_3^0 = -1.7, & \widehat{b}_4^0 = \ \ 0.03, & \widehat{b}_5^0 = -0.020. \end{array} \qquad (28)$$

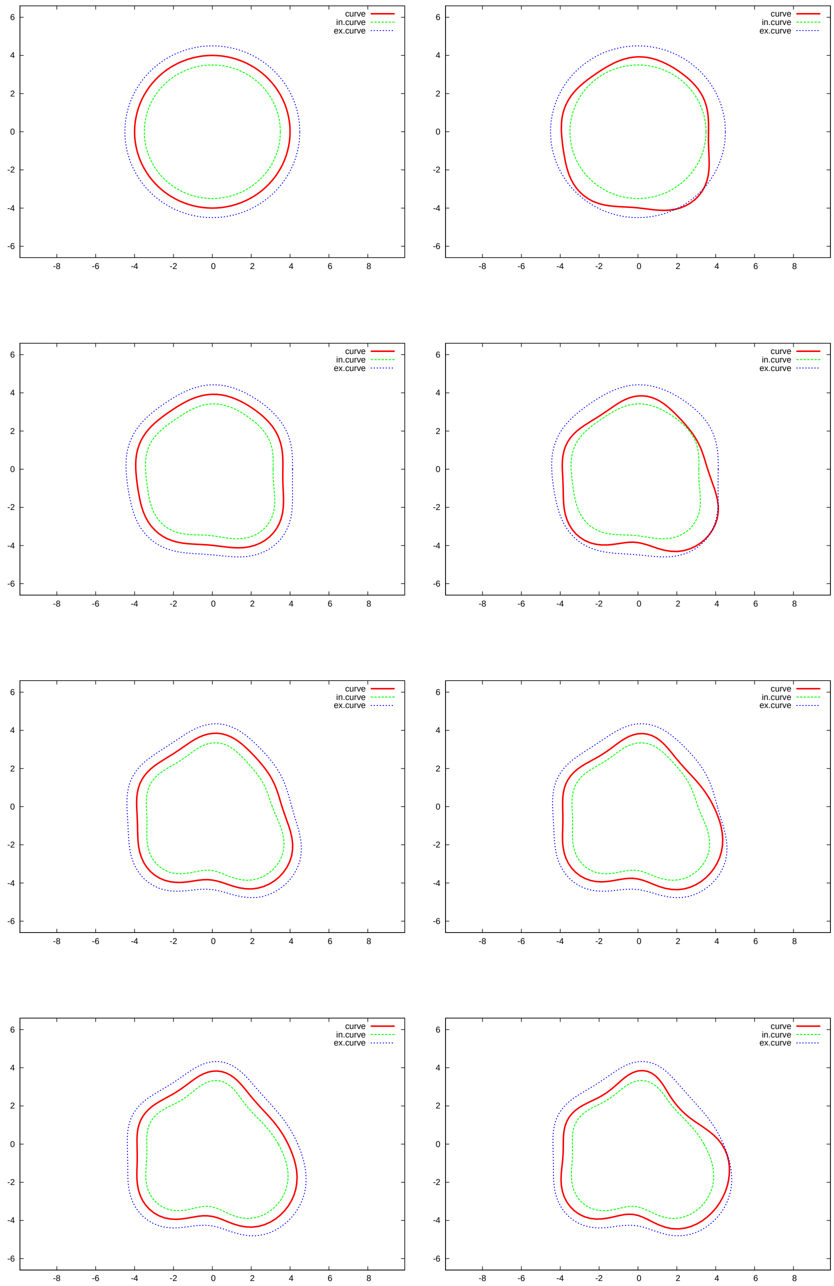

FIGURE 9. Kirsch–Kress steps 1-4 to reconstruct nonconvex domain.

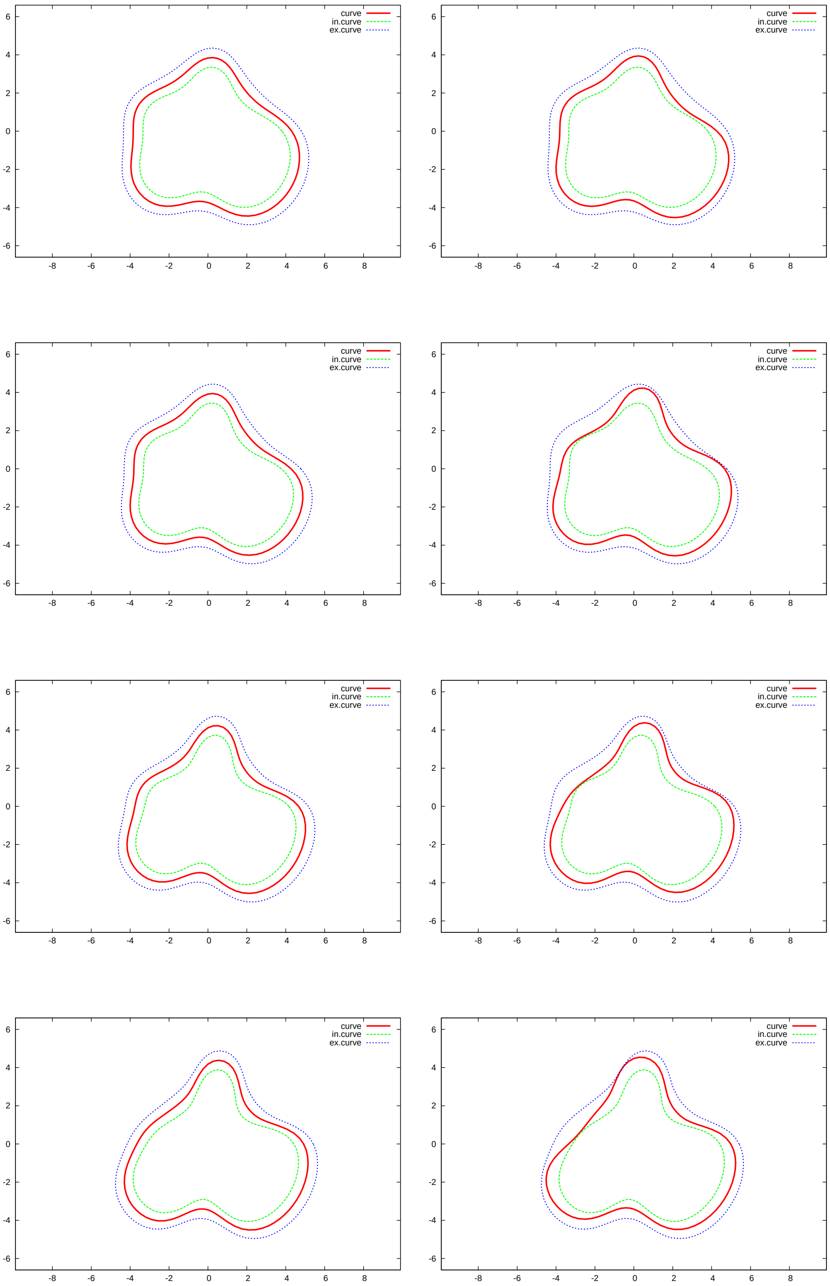

FIGURE 10. Kirsch–Kress steps 5-8 to reconstruct nonconvex domain.

number of pnts. $M=M'$	γ	GNw		LMw		LMo	
		1.2596	(0)	1.2596	(0)	1.2596	(0)
22	$4\cdot 10^{-8}$	0.05427	(13)	0.05461	(30)	0.06793	(30)
44	$0.25\cdot 10^{-12}$	0.002136	(13)	0.002007	(320)	0.002095	(320)
88	$4\cdot 10^{-14}$	0.0002126	(13)	0.0002107	(80)	0.0001997	(160)

Table 2. Reconstruction accuracy (number of iterations) in dependence on the optimization method and on the number of discretization points for the egg shaped domain.

Since the iteration, with this initial solution, converged to a false local minimum, we have introduced an initial solution closer to the true solution in (27). We have checked the initial solution

$$\widehat{a}_i^1 := \frac{1}{2}\big(\widehat{a}_i^0 + \widehat{a}_i\big),\quad i=0,\dots,5,\quad \widehat{b}_i^1 := \frac{1}{2}\big(\widehat{b}_i^0 + \widehat{b}_i\big),\quad i=1,\dots,5$$

and observed convergence. In particular, we had to choose a larger number of discretization points on the curves Γ, Γ_i, Γ_e, namely $M=M'=352$. We have set the regularization parameter $\gamma = 10^{-8}$ and the scaling constants to $c=10000$, $c_{\mathbf{r}}=1$, $c_i=1$, and $c_e=0.2$. For the initial solution $\{\widehat{a}_i^1, \widehat{b}_i^1\}$, we got the reconstructed curve within 11 iterations of the Gauss-Newton method. The error $\|\widetilde{\mathbf{r}} - \widetilde{\mathbf{r}}_{ini}\|_{L^\infty[0,2\pi]} = 0.296$ of the initial parametrization $\mathbf{r}_{ini}$ with Fourier coefficients $\widehat{a}_i^1, \widehat{b}_i^1$ has been reduced to $\|\widetilde{\mathbf{r}} - \widetilde{\mathbf{r}}_{KK}\|_{L^\infty[0,2\pi]} = 0.000279$.

5.4. **Kirsch–Kress algorithm with updated representation curves.** Now we suppose that, for the reconstruction of the nonconvex domain, we have an initial solution like the disk on the left in Figure 7. In order to have the curves Γ_i and Γ_e close to the iterate $\Gamma^{\mathbf{r}_n}$, we have to update Γ_i and Γ_e during the iteration process. More precisely, in each step of the iteration, we proceed as follows:

- We choose $\Gamma_i = \Gamma^{\mathbf{r}_i}$ and $\Gamma_e = \Gamma^{\mathbf{r}_e}$ with $\mathbf{r}_i = \mathbf{r}_{n-1} - 0.5$ and $\mathbf{r}_e = \mathbf{r}_{n-1} + 0.5$ (cf. (17)). Thus Γ_i and Γ_e deviate from the curve $\Gamma^{\mathbf{r}_{n-1}}$ of the previous step by the same amount as the fixed curves Γ_i and Γ_e from the true solution $\Gamma^{\mathbf{r}}$ on the right in Figure 8.
- With these Γ_i and Γ_e we perform a single step of the Gauss-Newton iteration and get the new solution $\Gamma^{\mathbf{r}'_n}$.
- If the resulting $\Gamma^{\mathbf{r}'_n}$ is enclosed between Γ_i and Γ_e, then we choose the new iterate $\mathbf{r}_n = \mathbf{r}'_n$. If not, then we reduce the step of iteration. In other words, we choose $\mathbf{r}_n = \mathbf{r}_{n-1} + 2^{-m}[\mathbf{r}'_n - \mathbf{r}_{n-1}]$ with $m \geq 1$ the smallest integer such that $\Gamma^{\mathbf{r}_n}$ is enclosed between Γ_i and Γ_e.

If the iterative solutions $\Gamma^{\mathbf{r}'_n}$ stay between Γ_i and Γ_e and if the steps of iteration $[\mathbf{r}'_n - \mathbf{r}_{n-1}]$ are small, then we fix the actual Γ_i and Γ_e and perform a larger number of Gauss–Newton steps.Applying this strategy to the

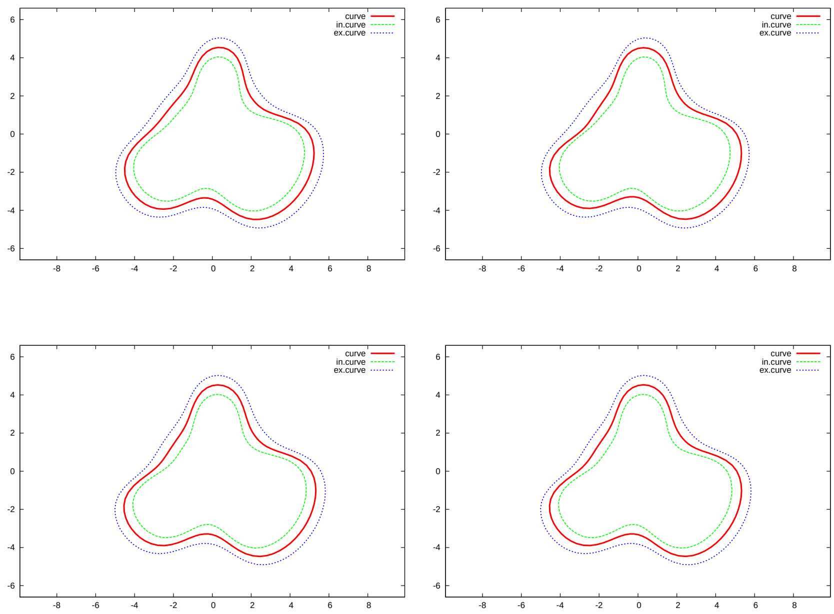

FIGURE 11. Kirsch–Kress steps 9-10 to reconstruct nonconvex domain.

reconstruction of our nonconvex domain, we need 9 Gauss–Newton steps with updated Γ_i and Γ_e and a final Gauss–Newton step (7 iterations) with fixed Γ_i and Γ_e. The Kirsch–Kress method reduces the initial deviation $\|\widetilde{\mathbf{r}} - \widetilde{\mathbf{r}}_{ini}\|_{L^\infty[0,2\pi]} = 1.57$ to a reconstruction accuracy $\|\widetilde{\mathbf{r}} - \widetilde{\mathbf{r}}_{KK}\|_{L^\infty[0,2\pi]} = 0.00032$. The initial solutions and the next iterative solutions of each step are shown in Figures 9–11.

5.5. **Reconstruction of curve with reduced number of nonzero Fourier coefficients.** Surely, one reason for the good reconstruction is that the boundary of the unknown obstacle (cf. (16) and (17)) can be exactly represented by the numerical ansatz for the parametrization including ten nonzero Fourier coefficients (cf. (26) and (27)). In many applications, the boundary of the obstacle can only be approximated by the numerical ansatz. To check our method for such a situation, we have slightly modified the nonconvex curve by adding the small Fourier coefficients

$$\widehat{a}_6 = 0.004, \quad \widehat{a}_7 = 0.001, \quad \widehat{b}_6 = -0.004, \quad \widehat{b}_7 = 0.001$$

to the set of nonzero coefficients in (27). With this boundary curve, we have generated far-field data. For the numerical reconstruction, however, we still use the ten nonzero Fourier coefficients $\widehat{a}_i$, $\widehat{b}_i$, $i = 1, \ldots, 5$. Note that the radial deviation of the unknown curve with fourteen nonzero coefficients from that with the ten is 0.0075. The reconstruction error for the simple Newton iteration is shown in Table 3 and is only slightly larger then that in the right Table 1. Note that the initial solution for the results of Table 3 was chosen as $\widehat{a}_i^0 := 0.75\widehat{a}_i$ and $\widehat{b}_i^0 := 0.75\widehat{b}_i$, $i = 1, \dots, 5$. A reconstruction with a similar accuracy but starting from the initial solution $\widehat{a}_i^0 := 0$ and $\widehat{b}_i^0 := 0$ was possible only over the finest grid with meshsize $h = 0.03125$. If the far-field data for the nonconvex obstacle with the fourteen nonzero Fourier coefficients is used in the Kirsch–Kress method based on ten nonzero Fourier coefficients, then the starting error 0.296 of the initial solution (cf. the left picture in Figure 8) is reduced to 0.00898 after 12 iterations.

h		0.5	0.25	0.125	0.0625	0.03125
err	1.57	0.1147	0.03812	0.01878	0.01688	0.01678
it	0	7	8	7	7	7

TABLE 3. Reconstruction of nonconvex domain by simple Newton iteration, far-field data generated from 14 Fourier coefficients.

5.6. **Reconstruction of obstacle with Jones mode.** Next we check the convergence of our methods in the case of a domain with Jones modes. For $r_J^0 = 5.135622\dots$ as well as ω, μ, and ϱ from (15), we reconstruct the disk $\Omega_J := \{x \in \mathbb{R}^2 : \ |x| < r_J\}$, $r_J = \frac{1}{\omega}\sqrt{\mu/\varrho}\ r_J^0$ (cf. the Jones mode in (7)). We choose the curves $\Gamma_i = \Gamma^{\mathbf{r}_i}$ and $\Gamma_e = \Gamma^{\mathbf{r}_e}$ by $\mathbf{r}_i = r_J - 2$, $\mathbf{r}_e = r_J + 2$, and define the initial solution by (27). The initial and the true solution curves are shown in Figure 12. Applying the Kirsch–Kress algorithm with 176 discretization points per curve, with $\gamma = 4 \cdot 10^{-14}$, and with the scaling constants $c = 200$, $c_{\mathbf{r}} = 200$, $c_i = 5$, $c_e = 0.05$, the true solution is reconstructed after 8 iterations. The starting error 1.26 of the initial solution is reduced to 0.000814. The simple Newton type iteration method should converge only, if the included solver of the direct problem provides a partial solution for domains with Jones modes and an accurate solution for domains close to domains with Jones modes. In particular, an iterative solver might diverge. We have employed the direct solver of [22]. Due to discretization errors, the FEM matrices have small eigenvalues, but are not singular. The stable solver provides good solutions, and the simple Newton type iteration converges even for the reconstruction of the domain Ω_J. Choosing the regularization parameter $\gamma = 0$, we get a reconstruction accuracy of 0.000492 after 13 iterations.

5.7. **Noisy far-field data.** Finally, we have checked perturbed far-field data. For different values of ε, we have added a random number, uniformly distributed in $[-\varepsilon, \varepsilon]$, to the far-field values of the egg shaped domain. Tables 4 show the reconstruction accuracy depending on ε for the simple Newton iteration with FEM stepsize 0.03125 and for the Kirsch–Kress method

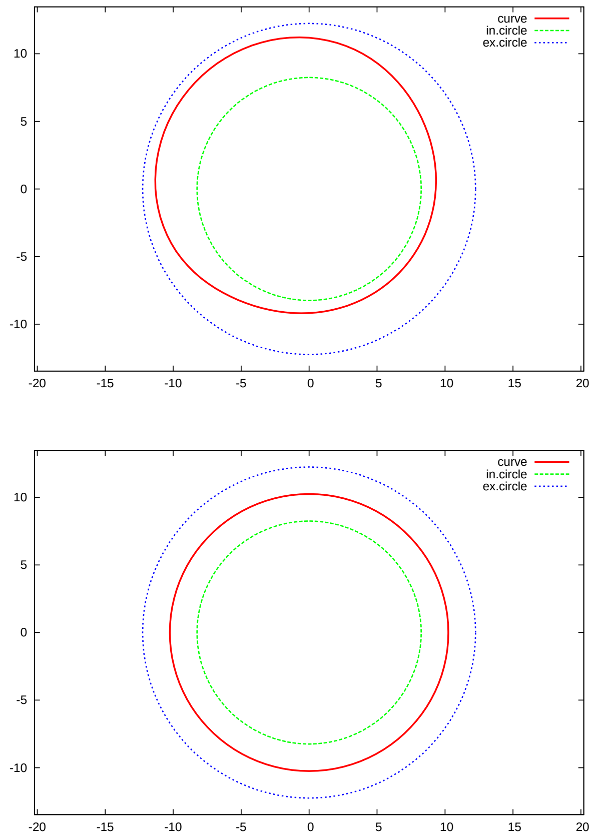

FIGURE 12. Initial solution with Fourier coefficients (27) and disk with Jones frequency.

with a number of discretization points $M = M' = 44$, respectively. Obviously, the simple Newton iteration is much more robust with respect to random perturbations. For the Kirsch–Kress method with $M = M' = 352$ points applied to the nonconvex domain (cf. Figure 8), the test results are shown in Table 5.

5.8. **Conclusions.** Summarizing the results, the advantage of the Kirsch–Kress method is the high accuracy of reconstruction for obstacles with smooth boundaries and, consequently, the fast computation time. Moreover, the method works well even if domains with Jones mode solutions appear. Note that the simple Newton method is based on the solution of the direct problem, which leads to singular or almost singular linear systems if the domain is an obstacle having Jones modes or if it is close to such an obstacle. The solver for this system must return a particular solution. An iterative scheme with preconditioner might diverge.

ε	$\|\widetilde{\mathbf{r}} - \widetilde{\mathbf{r}}_{FEM}\|_{L^\infty}$
0.	0.001568
0.001	0.002637
0.005	0.007156
0.01	0.01368
0.05	0.05433
0.1	0.1087
0.2	0.2339

ε	γ	$\|\widetilde{\mathbf{r}} - \widetilde{\mathbf{r}}_{KK}\|_{L^\infty}$
0.	$0.25 \cdot 10^{-12}$	0.002136
0.0001	$0.25 \cdot 10^{-10}$	0.003640
0.001	$0.25 \cdot 10^{-7}$	0.02041
0.003	$0.25 \cdot 10^{-6}$	0.05686
0.005	$1 \cdot 10^{-6}$	0.09997

Table 4. Reconstruction error of the egg shaped domain depending on the stochastic perturbation of the far-field data for simple Newton iteration and $\gamma = 0$ (left) and for Kirsch–Kress method (right).

ε	γ	$\|\widetilde{\mathbf{r}} - \widetilde{\mathbf{r}}_{KK}\|_{L^\infty}$	it
0.00000	10^{-8}	0.00028	11
0.00010	10^{-8}	0.0141	13
0.00025	10^{-8}	0.0345	11
0.00100	10^{-8}	0.113	8
0.00250	10^{-6}	0.187	9

Table 5. Reconstruction error of the Kirsch–Kress method depending on the stochastic perturbation of the far-field data, nonconvex domain.

Unfortunately, a successful run of the Kirsch–Kress method requires an optimal choice of the scaling constants. Additionally, the curves for the potential representations must be chosen properly, i.e., sufficiently close to the boundary of the iterative solution or to the boundary of the true obstacle. Heuristically, the closeness requirement depends on the degree of smoothness measured by the norms of analyticity of the parametrization functions. Eventually, the curves of the potential representation must be updated during the iteration. However, the closer these curves are the larger is the number of subdivision points and the number of degrees of freedom needed for the numerical discretization. The actual curves for the potential representation, the actual scaling constants, and the actual number of discretization points should be determined beforehand by test computations for known obstacles. A final disadvantage of the Kirsch–Kress method is its higher sensitivity with respect to noisy far-field data.

6. Derivatives of the 2D Discretized Objective Functional

6.1. **Derivatives of the points at the parameterized curve and of the normal vector with respect to the Fourier coefficients.** To define the objective functional of the discretized optimization problem (cf. (A.6)) and to get formulas for its derivatives, we need formulas for the parametrization, the normal, the incoming wave, the Green kernels, and for their derivatives. Since the derivation is straightforward, we only present the results.

Here we start with formulas for the parameterization point $x_{\mathbf{r}}(\zeta) := \widetilde{\mathbf{r}}(\zeta)\exp(\mathbf{i}\zeta)$ (cf. 17) on the approximate interface, for the normal ν at $x_{\mathbf{r},\kappa}$, and for their derivatives with respect to the Fourier coefficients. Clearly, the set of coefficients is to be truncated such that we can compute with a finite set of parameters $\{\widehat{a}_0, \widehat{a}_j, \widehat{b}_j : \; j = 1, 2, \ldots, n\}$. To simplify the formulas, we set $N = 2n+1$ and collect these Fourier coefficients in the set $\{a_\iota : \; \iota \in I_N\}$ and write the parametric representation as

$$x_{\mathbf{r}}(\zeta) := \mathbf{r}(\zeta)e^{\mathbf{i}\zeta}, \quad \mathbf{r}(\zeta) := \frac{r_e + r_i}{2} + \frac{r_e - r_i}{\pi} \arctan\Big(\sum_{\iota \in I_N} a_\iota \psi_\iota(\zeta)\Big). \tag{A.1}$$

Here $\psi_\iota(\zeta) = \cos(j\zeta)$ if $a_\iota = \widehat{a}_j$ and $\psi_\iota(\zeta) = \sin(j\zeta)$ if $a_\iota = \widehat{b}_j$. For the derivatives, we arrive at

$$\mathbf{r}'(\zeta) = \frac{r_e - r_i}{\pi} \frac{\sum_{\iota \in I_N} a_\iota \psi_\iota'(\zeta)}{1 + \big(\sum_{\iota \in I_N} a_\iota \psi_\iota(\zeta)\big)^2},$$

$$\frac{\partial}{\partial a_\iota}\mathbf{r}(\zeta) = \frac{r_e - r_i}{\pi} \frac{\psi_\iota(\zeta)}{1 + \big(\sum_{\iota' \in I_N} a_{\iota'} \psi_{\iota'}(\zeta)\big)^2},$$

$$\frac{\partial}{\partial a_\iota}x_{\mathbf{r}}(\zeta) = \frac{r_e - r_i}{\pi} \frac{\psi_\iota(\zeta)}{1 + \big(\sum_{\iota' \in I_N} a_{\iota'} \psi_{\iota'}(\zeta)\big)^2} e^{i\zeta},$$

$$\frac{\partial}{\partial a_\iota}\mathbf{r}'(\zeta) = \frac{r_e - r_i}{\pi} \frac{\psi_\iota'(\zeta)}{1 + \big(\sum_{\iota' \in I_N} a_{\iota'} \psi_{\iota'}(\zeta)\big)^2} - 2\,\frac{r_e - r_i}{\pi} \frac{\big[\sum_{\iota' \in I_N} a_{\iota'} \psi_{\iota'}'(\zeta)\big]\big[\sum_{\iota' \in I_N} a_{\iota'} \psi_{\iota'}(\zeta)\big]\psi_\iota(\zeta)}{\big[1 + \big(\sum_{\iota' \in I_N} a_{\iota'} \psi_{\iota'}(\zeta)\big)^2\big]^2}.$$

A normal $\widetilde{\nu}$ to the curve at $x_{\mathbf{r}}(\zeta)$ and the unit normal ν are given by

$$\begin{aligned}
\widetilde{\nu}(x_{\mathbf{r}}(\zeta)) &= e^{-\mathbf{i}\pi/2} \frac{\partial}{\partial\zeta}\big[x_{\mathbf{r}}(\zeta)\big] = e^{-\mathbf{i}\pi/2} \frac{\partial}{\partial\zeta}\big[\mathbf{r}(\zeta)e^{\mathbf{i}\zeta}\big] = \\
&= e^{-\mathbf{i}\pi/2}\Big[\mathbf{r}'(\zeta)e^{\mathbf{i}\zeta} + \mathbf{r}(\zeta)e^{\mathbf{i}\pi/2}e^{\mathbf{i}\zeta}\Big] = \\
&= \Big[e^{-\mathbf{i}\pi/2}\mathbf{r}'(\zeta) + \mathbf{r}(\zeta)\Big]e^{\mathbf{i}\zeta},
\end{aligned}$$

$$\nu(x_{\mathbf{r}}(\zeta)) = \frac{\big[e^{-\mathbf{i}\pi/2}\mathbf{r}'(\zeta) + \mathbf{r}(\zeta)\big]e^{\mathbf{i}\zeta}}{\big|e^{-\mathbf{i}\pi/2}\mathbf{r}'(\zeta) + \mathbf{r}(\zeta)\big|} = \frac{\mathbf{r}'(\zeta)e^{\mathbf{i}(\zeta-\pi/2)} + \mathbf{r}(\zeta)e^{\mathbf{i}\zeta}}{\mathbf{s}(\zeta)}, \tag{A.2}$$

$$\mathbf{s}(\zeta) := \sqrt{\mathbf{r}'(\zeta)^2 + \mathbf{r}(\zeta)^2}. \tag{A.3}$$

The derivatives of these entities can be computed by the formulas

$$\frac{\partial}{\partial a_\iota}\,\nu(x_{\mathbf{r}}(\zeta)) = \frac{\partial}{\partial a_\iota}\,\frac{\mathbf{r}'(\zeta)}{\mathbf{s}(\zeta)}\,e^{\mathbf{i}(\zeta-\pi/2)} + \frac{\partial}{\partial a_\iota}\,\frac{\mathbf{r}(\zeta)}{\mathbf{s}(\zeta)}\,e^{\mathbf{i}\zeta}\,,$$

$$\frac{\partial}{\partial a_\iota}\,\frac{\mathbf{r}'(\zeta)}{\mathbf{s}(\zeta)} = \frac{\partial_{a_\iota}\mathbf{r}'(\zeta)}{\mathbf{s}(\zeta)} - \frac{1}{2}\,\frac{\mathbf{r}'(\zeta)\big\{2\mathbf{r}'(\zeta)\partial_{a_\iota}\mathbf{r}'(\zeta) + 2\mathbf{r}(\zeta)\partial_{a_\iota}\mathbf{r}(\zeta)\big\}}{\mathbf{s}(\zeta)^3} =$$

$$= \frac{\partial_{a_\iota}\mathbf{r}'(\zeta)}{\mathbf{s}(\zeta)} - \frac{\mathbf{r}'(\zeta)\big\{\mathbf{r}'(\zeta)\partial_{a_\iota}\mathbf{r}'(\zeta) + \mathbf{r}(\zeta)\partial_{a_\iota}\mathbf{r}(\zeta)\big\}}{\mathbf{s}(\zeta)^3}\,,$$

$$\frac{\partial}{\partial a_\iota}\,\frac{\mathbf{r}(\zeta)}{\mathbf{s}(\zeta)} = \frac{\partial_{a_\iota}\mathbf{r}(\zeta)}{\mathbf{s}(\zeta)} - \frac{\mathbf{r}(\zeta)\big\{\mathbf{r}'(\zeta)\partial_{a_\iota}\mathbf{r}'(\zeta) + \mathbf{r}(\zeta)\partial_{a_\iota}\mathbf{r}(\zeta)\big\}}{\mathbf{s}(\zeta)^3}\,.$$

6.2. **Values and derivatives of incoming wave and kernel functions.** Suppose v^{inc} is the direction of the incoming wave, then

$$p^{inc}(x) = e^{\mathbf{i}k_\omega v^{inc}\cdot x}\,,$$
$$\partial_{x_j}p^{inc}(x) = \mathbf{i}k_\omega e^{\mathbf{i}k_\omega v^{inc}\cdot x}[v^{inc}]_j\,,$$
$$\partial_{x_j}\partial_{x_l}p^{inc}(x) = -k_\omega^2 e^{\mathbf{i}k_\omega v^{inc}\cdot x}[v^{inc}]_j[v^{inc}]_l.$$

For the derivatives of the acoustic Green kernel, we obtain (cf. [1])

$$G(x,y) = \frac{\mathbf{i}}{4}\,H_0^{(1)}(k|x-y|),$$
$$H_0^{(1)}(t) := J_0(t) + \mathbf{i}Y_0(t),$$
$$\partial_{x_j}G(x,y) = \frac{\mathbf{i}k}{4}\,[H_0^{(1)}]'(k|x-y|)\,\frac{(x_j-y_j)}{|x-y|}\,,$$
$$[H_0^{(1)}]'(t) := -\,J_1(t) - \mathbf{i}Y_1(t),$$
$$\partial_{y_j}G(x,y) = \frac{\mathbf{i}k}{4}\,[H_0^{(1)}]'(k|x-y|)\,\frac{(y_j-x_j)}{|x-y|}\,,$$
$$\partial_{x_j}\partial_{x_l}G(x,y) = \frac{\mathbf{i}k}{4}\,[H_0^{(1)}]'(k|x-y|)\,\frac{|x-y|^2\delta_{j,l} - 2(x_j-y_j)(x_l-y_l)}{|x-y|^3}\,,$$
$$-\,\frac{\mathbf{i}k^2}{4}\,H_0^{(1)}(k|x-y|)\,\frac{(x_j-y_j)(x_l-y_l)}{|x-y|^2}\,.$$

For the third order derivatives, we observe

$$\partial_{x_m}\partial_{x_j}\partial_{x_l}G(x,y) =$$
$$= \mathbf{i}k^2 H_0^{(1)}(k|x-y|)\Bigg\{\frac{(x_j-y_j)(x_l-y_l)(x_m-y_m)}{|x-y|^4} -$$
$$-\frac{(x_m-y_m)\delta_{j,l} + (x_l-y_l)\delta_{j,m} + (x_j-y_j)\delta_{l,m}}{4|x-y|^2}\Bigg\} +$$
$$+\frac{\mathbf{i}k}{2}[H_0^{(1)}]'(k|x-y|)\Bigg\{\frac{(x_m-y_m)(x_j-y_j)(x_l-y_l)}{2|x-y|^5}\,\big[8-k^2|x-y|^2\big] -$$

$$-\frac{(x_m-y_m)\delta_{j,l}+(x_l-y_l)\delta_{j,m}+(x_j-y_j)\delta_{l,m}}{|x-y|^3}\Bigg\},$$

$$\partial_{y_m}\partial_{x_j}\partial_{x_l}G(x,y)=$$

$$=\mathbf{i}k^2H_0^{(1)}(k|x-y|)\Bigg\{\frac{(y_j-x_j)(y_l-x_l)(y_m-x_m)}{|x-y|^4}-$$

$$-\frac{(y_m-x_m)\delta_{j,l}+(y_l-x_l)\delta_{j,m}+(y_j-x_j)\delta_{l,m}}{4|x-y|^2}\Bigg\}+$$

$$+\frac{\mathbf{i}k}{2}[H_0^{(1)}]'(k|x-y|)\Bigg\{\frac{(y_m-x_m)(y_j-x_j)(y_l-x_l)}{2|x-y|^5}\left[8-k^2|x-y|^2\right]-$$

$$-\frac{(y_m-x_m)\delta_{j,l}+(y_l-x_l)\delta_{j,m}+(y_j-x_j)\delta_{l,m}}{|x-y|^3}\Bigg\}.$$

The fourth order derivatives take the form

$$\partial_{y_n}\partial_{y_m}\partial_{x_j}\partial_{x_l}G(x,y)=\mathbf{i}k^2H_0^{(1)}(k|x-y|)\times$$

$$\times\Bigg\{\frac{(y_l-x_l)(y_m-x_m)\delta_{n,j}}{|x-y|^4}+\frac{(y_j-x_j)(y_m-x_m)\delta_{n,l}}{|x-y|^4}+$$

$$+\frac{(y_j-x_j)(y_l-x_l)\delta_{n,m}}{|x-y|^4}-\frac{\delta_{j,l}\delta_{n,m}+\delta_{j,m}\delta_{n,l}+\delta_{l,m}\delta_{n,j}}{4|x-y|^2}+$$

$$+(y_n-x_n)\,\frac{(y_m-x_m)\delta_{j,l}+(y_l-x_l)\delta_{j,m}+(y_j-x_j)\delta_{l,m}}{|x-y|^4}-$$

$$-\frac{(y_n-x_n)(y_m-x_m)(y_j-x_j)(y_l-x_l)}{4|x-y|^6}\left[24-k^2|x-y|^2\right]\Bigg\}+$$

$$+\mathbf{i}k[H_0^{(1)}]'(k|x-y|)\times$$

$$\times\Bigg\{\left[2k^2|x-y|^2-12\right]\frac{(y_j-x_j)(y_l-x_l)(y_m-x_m)(y_n-x_n)}{|x-y|^7}+$$

$$+[8-k^2|y-x|^2]\times$$

$$\times(y_n-x_n)\frac{(y_m-x_m)\delta_{j,l}+(y_l-x_l)\delta_{j,m}+(y_j-x_j)\delta_{l,m}}{4|x-y|^5}+$$

$$+\left[8-k^2|x-y|^2\right]\times$$

$$\times\frac{(y_j-x_j)(y_l-x_l)\delta_{n,m}+(y_m-x_m)(y_l-x_l)\delta_{n,j}\delta_{n,l}}{4|x-y|^5}+$$

$$+\left[8-k^2|x-y|^2\right]\frac{(y_m-x_m)(y_j-x_j)\delta_{n,l}}{4|x-y|^5}-$$

$$-\frac{\delta_{j,l}\delta_{n,m}+\delta_{j,m}\delta_{n,l}+\delta_{l,m}\delta_{n,j}}{2|x-y|^3}\Bigg\}.$$

For the derivatives of the elastic Green kernel, we conclude

$$
\begin{aligned}
[G^{el}(x,y)]_{j,l} &= \frac{1}{\mu}\,G(x,y;k_s)\delta_{j,l} + \frac{1}{\mu k_s^2}\,\partial_{x_j}\partial_{x_l}G(x,y;k_s) - \\
&\quad - \frac{1}{\mu k_s^2}\,\partial_{x_j}\partial_{x_l}G(x,y;k_p),\\
\partial_{y_m}[G^{el}(x,y)]_{j,l} &= \frac{1}{\mu}\,\partial_{y_m}G(x,y;k_s)\delta_{j,l} + \frac{1}{\mu k_s^2}\,\partial_{y_m}\partial_{x_j}\partial_{x_l}G(x,y;k_s) - \\
&\quad - \frac{1}{\mu k_s^2}\,\partial_{y_m}\partial_{x_j}\partial_{x_l}G(x,y;k_p),\\
\partial_{y_n}\partial_{y_m}[G^{el}(x,y)]_{j,l} &= \frac{1}{\mu}\,\partial_{y_n}\partial_{y_m}G(x,y;k_s)\delta_{j,l} + \\
&\quad + \frac{1}{\mu k_s^2}\,\partial_{y_n}\partial_{y_m}\partial_{x_j}\partial_{x_l}G(x,y;k_s) - \\
&\quad - \frac{1}{\mu k_s^2}\,\partial_{y_n}\partial_{y_m}\partial_{x_j}\partial_{x_l}G(x,y;k_p),\\
t_y[G^{el}(x,y)]_{\cdot,l} &= 2\mu \begin{pmatrix} \sum_{j=1}^{2} \nu_j \partial_{y_j}[G^{el}(x,y)]_{1,l} \\ \sum_{j=1}^{2} \nu_j \partial_{y_j}[G^{el}(x,y)]_{2,l} \end{pmatrix} + \\
&\quad + \lambda\Big[\partial_{y_1}[G^{el}(x,y)]_{1,l} + \partial_{y_2}[G^{el}(x,y)]_{2,l}\Big]\nu + \\
&\quad + \mu \begin{pmatrix} \nu_2\Big(\partial_{y_1}[G^{el}(x,y)]_{2,l} - \partial_{y_2}[G^{el}(x,y)]_{1,l}\Big) \\ \nu_1\Big(\partial_{y_2}[G^{el}(x,y)]_{1,l} - \partial_{y_1}[G^{el}(x,y)]_{2,l}\Big) \end{pmatrix},\\
\partial_{y_m} t_y[G^{el}(x,y)]_{\cdot,l} &= 2\mu \begin{pmatrix} \sum_{j=1}^{2} \nu_j \partial_{y_m}\partial_{y_j}[G^{el}(x,y)]_{1,l} \\ \sum_{j=1}^{2} \nu_j \partial_{y_m}\partial_{y_j}[G^{el}(x,y)]_{2,l} \end{pmatrix} + \\
&\quad + \lambda\Big[\partial_{y_m}\partial_{y_1}[G^{el}(x,y)]_{1,l} + \partial_{y_m}\partial_{y_2}[G^{el}(x,y)]_{2,l}\Big]\nu + \\
&\quad + \mu \begin{pmatrix} \nu_2\Big(\partial_{y_m}\partial_{y_1}[G^{el}(x,y)]_{2,l} - \partial_{y_m}\partial_{y_2}[G^{el}(x,y)]_{1,l}\Big) \\ \nu_1\Big(\partial_{y_m}\partial_{y_2}[G^{el}(x,y)]_{1,l} - \partial_{y_m}\partial_{y_1}[G^{el}(x,y)]_{2,l}\Big) \end{pmatrix},\\
\partial_{\nu_m} t_y[G^{el}(x,y)]_{\cdot,l} &= 2\mu \begin{pmatrix} \partial_{y_m}[G^{el}(x,y)]_{1,l} \\ \partial_{y_m}[G^{el}(x,y)]_{2,l} \end{pmatrix} + \\
&\quad + \lambda\Big[\partial_{y_1}[G^{el}(x,y)]_{1,l} + \partial_{y_2}[G^{el}(x,y)]_{2,l}\Big] \begin{pmatrix} \delta_{1,m} \\ \delta_{2,m} \end{pmatrix} +
\end{aligned}
$$

$$+\,\mu\begin{pmatrix}\delta_{2,m}\Big(\partial_{y_1}[G^{el}(x,y)]_{2,l}-\partial_{y_2}[G^{el}(x,y)]_{1,l}\Big)\\ \delta_{1,m}\Big(\partial_{y_2}[G^{el}(x,y)]_{1,l}-\partial_{y_1}[G^{el}(x,y)]_{2,l}\Big)\end{pmatrix}.$$

6.3. **Least squares approach for the Gauss-Newton algorithm.** Suppose I_N is the index set of the Fourier coefficients from 6.1 and the layer functions of the Kirsch–Kress method are approximated by (23), (24). Furthermore, suppose the L^2 norms on $\Gamma^{\mathbf{r}}$ and $[0,2\pi]$ in $\mathcal{J}^3_\gamma$ are discretized by

$$\|f\|^2_{L^2(\Gamma^{\mathbf{r}})}\sim\sum_{\kappa=1}^{M'}\big|f(x_{\mathbf{r},\kappa})\big|^2,\;\; x_{\mathbf{r},\kappa}:=\mathbf{r}(\tau_\kappa)e^{\mathbf{i}\tau_\kappa},\;\; \tau_\kappa:=\frac{2\pi\kappa}{M'}\,, \tag{A.4}$$

$$\|g\|^2_{L^2[0,2\pi]}\sim\sum_{\kappa=1}^{M''}\big|g(\sigma_\kappa)\big|^2,\;\; \sigma_\kappa:=\frac{2\pi\kappa}{M''}\,. \tag{A.5}$$

Then the discretized objective functional for (21) is of the form

$$\mathcal{J}_{N,M,M',\gamma}\big(\varphi_{i,M},\vec{\varphi}_{e,M},\mathbf{r}_N\big)=\Big\|\mathcal{M}\big((b_\kappa)_{\kappa=1}^M,(c_\kappa)_{\kappa=1}^M,(a_\iota)_{\iota\in I_N}\big)-\mathcal{R}\Big\|^2_{\ell^2}, \tag{A.6}$$
$$\mathcal{M}:=\Big((\mathcal{M}_{1,\kappa}),(\mathcal{M}_{2,\kappa}),(\mathcal{M}_{3,\kappa,l}),(\mathcal{M}_{4,\kappa,l}),(\mathcal{M}_{5,\kappa})\Big),$$
$$\mathcal{R}:=\Big((\mathcal{R}_{1,\kappa}),(\mathcal{R}_{2,\kappa}),(\mathcal{R}_{3,\kappa,l}),(\mathcal{R}_{4,\kappa,l}),(\mathcal{R}_{5,\kappa})\Big),$$

where

$$\begin{aligned}\mathcal{R}_{1,\kappa}&:=\frac{1}{\sqrt{M''}}\,p^\infty_{noisy}(\sigma_{\kappa'}),\\ \mathcal{R}_{2,\kappa'}&:=0,\\ \mathcal{R}_{3,\kappa',l}&:=0,\\ \mathcal{R}_{4,\kappa',l}&:=0,\\ \mathcal{R}_{5,\kappa'}&:=0,\end{aligned} \tag{A.7}$$

$$\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{1,\kappa'}:=\frac{e^{\mathbf{i}\pi/4}}{\sqrt{8\pi k_\omega}\sqrt{M''}}\sum_{\kappa=1}^M b_\kappa e^{-\mathbf{i}k_\omega\exp(\mathbf{i}\sigma_{\kappa'})\cdot x_{i,\kappa}}, \tag{A.8}$$
$$\kappa'=1,\dots,M'',$$

$$\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{2,\kappa'}:=\frac{\sqrt{\gamma}}{\sqrt{M}}\sum_{\kappa=1}^M\log\sin^2\Big(\frac{\pi[\kappa'-\kappa]}{M}\Big)b_\kappa, \tag{A.9}$$
$$\kappa'=1,\dots,M,$$

$$\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{3,\kappa',l}:=\frac{\sqrt{\gamma}}{\sqrt{M}}\sum_{\kappa=1}^M\log\sin^2\Big(\frac{\pi[\kappa'-\kappa]}{M}\Big)[c_\kappa]_l, \tag{A.10}$$
$$\kappa'=1,\dots,M,\;\; l=1,2,$$

$$\begin{aligned}\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{4,\kappa',l} := \frac{1}{\sqrt{M'}}\sum_{\kappa=1}^{M}\Big[t_{x_{\mathbf{r},\kappa'}}\big[G^{el}(x_{e,\kappa},x_{\mathbf{r},\kappa'})c_\kappa\big]\Big]_l + \\ + \frac{1}{\sqrt{M'}}\sum_{\kappa=1}^{M} b_\kappa G\big(x_{\mathbf{r},\kappa'},x_{i,\kappa}\big)\big[\nu(x_{\mathbf{r},\kappa'})\big]_l + \\ + \frac{p^{inc}(x_{\mathbf{r},\kappa'})[\nu(x_{\mathbf{r},\kappa'})]_l}{\sqrt{M'}}, \\ \kappa'=1,\dots,M', \quad l=1,2,\end{aligned} \tag{A.11}$$

$$\begin{aligned}\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{5,\kappa'} := \frac{1}{\sqrt{M'}}\sum_{\kappa=1}^{M}\nu(x_{\mathbf{r},\kappa'})\cdot G^{el}(x_{e,\kappa},x_{\mathbf{r},\kappa'})c_\kappa - \\ - \frac{1}{\sqrt{M'}\,\varrho_f\omega^2}\sum_{\kappa=1}^{M} b_\kappa \partial_{\nu(x_{\mathbf{r},\kappa'})}G(x_{\mathbf{r},\kappa'},x_{i,\kappa}) - \\ - \frac{\partial_\nu p^{inc}(x_{\mathbf{r},\kappa'})}{\sqrt{M'}\,\varrho_f\omega^2}, \quad \kappa'=1,\dots,M'.\end{aligned} \tag{A.12}$$

Here we define the expression $\{\log\sin^2(\pi 0/M)\}$ as 0. This leads to the following formulas for the derivatives. For the first components, we get

$$\frac{\partial}{\partial[\Re\mathrm{e}\, b_\kappa]}\,\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{1,\kappa'} = \frac{e^{\mathbf{i}\pi/4}}{\sqrt{8\pi k_\omega}\sqrt{M''}}\begin{pmatrix}\cos\big(-k_\omega e^{\mathbf{i}\sigma_{\kappa'}}\cdot x_{i,\kappa}\big)\\ \sin\big(-k_\omega e^{\mathbf{i}\sigma_{\kappa'}}\cdot x_{i,\kappa}\big)\end{pmatrix}^\top,$$

$$\frac{\partial}{\partial[\Im\mathrm{m}\, b_\kappa]}\,\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{1,\kappa'} = \frac{e^{\mathbf{i}\pi/4}}{\sqrt{8\pi k_\omega}\sqrt{M''}}\begin{pmatrix}-\sin\big(-k_\omega e^{\mathbf{i}\sigma_{\kappa'}}\cdot x_{i,\kappa}\big)\\ \cos\big(-k_\omega e^{\mathbf{i}\sigma_{\kappa'}}\cdot x_{i,\kappa}\big)\end{pmatrix}^\top.$$

For the second components, we obtain

$$\frac{\partial}{\partial[\Re\mathrm{e}\, b_\kappa]}\,\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{2,\kappa'} = \frac{\sqrt{\gamma}}{\sqrt{M}}\begin{pmatrix}\log\sin^2\Big(\dfrac{\pi[\kappa'-\kappa]}{M}\Big)\\ 0\end{pmatrix}^\top,$$

$$\frac{\partial}{\partial[\Im\mathrm{m}\, b_\kappa]}\,\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{2,\kappa'} = \frac{\sqrt{\gamma}}{\sqrt{M}}\begin{pmatrix}0\\ \log\sin^2\Big(\dfrac{\pi[\kappa'-\kappa]}{M}\Big)\end{pmatrix}^\top.$$

For the third components, we have

$$\frac{\partial}{\partial[\Re\mathrm{e}\,[c_\kappa]_l]}\,\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{3,\kappa',l'} = \frac{\sqrt{\gamma}}{\sqrt{M}}\begin{pmatrix}\log\sin^2(\dfrac{\pi[\kappa'-\kappa]}{M})\delta_{l,l'}\\ 0\end{pmatrix}^\top,$$

$$\frac{\partial}{\partial[\Im\mathrm{m}\,[c_\kappa]_l]}\,\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{3,\kappa',l'} = \frac{\sqrt{\gamma}}{\sqrt{M}}\begin{pmatrix}0\\ \log\sin^2\Big(\dfrac{\pi[\kappa'-\kappa]}{M}\Big)\delta_{l,l'}\end{pmatrix}^\top.$$

The derivatives of the fourth components take the form

$$\frac{\partial}{\partial[\Re\mathrm{e}\, b_\kappa]}\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{4,\kappa',l} =$$

$$= \frac{1}{\sqrt{M'}}\begin{pmatrix}\Re\mathrm{e}\, G(x_{\mathbf{r},\kappa'},x_{i,\kappa})[\nu(x_{\mathbf{r},\kappa'})]_l\\ \Im\mathrm{m}\, G(x_{\mathbf{r},\kappa'},x_{i,\kappa})[\nu(x_{\mathbf{r},\kappa'})]_l\end{pmatrix}^\top,$$

$$\frac{\partial}{\partial[\Im\mathrm{m}\, b_\kappa]}\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{4,\kappa',l} =$$

$$= \frac{1}{\sqrt{M'}}\begin{pmatrix}-\Im\mathrm{m}\, G(x_{\mathbf{r},\kappa'},x_{i,\kappa})[\nu(x_{\mathbf{r},\kappa'})]_l\\ \Re\mathrm{e}\, G(x_{\mathbf{r},\kappa'},x_{i,\kappa})[\nu(x_{\mathbf{r},\kappa'})]_l\end{pmatrix}^\top,$$

$$\frac{\partial}{\partial[\Re\mathrm{e}\,[c_\kappa]_l]}\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{4,\kappa',l'} =$$

$$= \frac{1}{\sqrt{M'}}\begin{pmatrix}\Big[t_{x_{\mathbf{r},\kappa'}}\Big(\big[\Re\mathrm{e}\, G^{el}(x_{e,\kappa},x_{\mathbf{r},\kappa'})\big]_{m,l}\Big)_m\Big]_{l'}\\ \Big[t_{x_{\mathbf{r},\kappa'}}\Big(\big[\Im\mathrm{m}\, G^{el}(x_{e,\kappa},x_{\mathbf{r},\kappa'})\big]_{m,l}\Big)_m\Big]_{l'}\end{pmatrix}^\top,$$

$$\frac{\partial}{\partial[\Im\mathrm{m}\,[c_\kappa]_l]}\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{4,\kappa',l'} =$$

$$= \frac{1}{\sqrt{M'}}\begin{pmatrix}-\Big[t_{x_{\mathbf{r},\kappa'}}\Big(\big[\Im\mathrm{m}\, G^{el}(x_{e,\kappa},x_{\mathbf{r},\kappa'})\big]_{m,l}\Big)_m\Big]_{l'}\\ \Big[t_{x_{\mathbf{r},\kappa'}}\Big(\big[\Re\mathrm{e}\, G^{el}(x_{e,\kappa},x_{\mathbf{r},\kappa'})\big]_{m,l}\Big)_m\Big]_{l'}\end{pmatrix}^\top,$$

$$\frac{\partial}{\partial a_\iota}\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{4,\kappa',l} =$$

$$= \frac{1}{\sqrt{M'}}\sum_{\kappa=1}^{M}\mathrm{grad}_{\;x_{\mathbf{r},\kappa'}}\Big[t_{x_{\mathbf{r},\kappa'}}\big[G^{el}\big(x_{e,\kappa},x_{\mathbf{r},\kappa'}\big)c_\kappa\big]\Big]_l\frac{\partial}{\partial a_\iota}[x_{\mathbf{r},\kappa'}]+$$

$$+\frac{1}{\sqrt{M'}}\sum_{\kappa=1}^{M}b_\kappa\mathrm{grad}_{\;x_{\mathbf{r},\kappa'}}\big[G(x_{\mathbf{r},\kappa'},x_{i,\kappa})\big]\frac{\partial}{\partial a_\iota}[x_{\mathbf{r},\kappa'}]\big[\nu(x_{\mathbf{r},\kappa'})\big]_l+$$

$$+\frac{1}{\sqrt{M'}}\sum_{\kappa=1}^{M}b_\kappa G(x_{\mathbf{r},\kappa'},x_{i,\kappa})\frac{\partial}{\partial a_\iota}[\nu(x_{\mathbf{r},\kappa'})]_l+$$

$$+\frac{1}{\sqrt{M'}}\mathrm{grad}_{\;x_{\mathbf{r},\kappa'}}\big[p^{inc}(x_{\mathbf{r},\kappa'})\big]\frac{\partial}{\partial a_\iota}[x_{\mathbf{r},\kappa'}]\big[\nu(x_{\mathbf{r},\kappa'})\big]_l+$$

$$+\frac{1}{\sqrt{\#K_{M'}}}p^{inc}(x_{\mathbf{r},\kappa'})\frac{\partial}{\partial a_\iota}\big[\nu(x_{\mathbf{r},\kappa'})\big]_l.$$

Finally, for the fifth components, we obtain

$$\frac{\partial}{\partial[\Re\mathrm{e}\, b_\kappa]}\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{5,\kappa'} =$$

$$= -\frac{1}{\sqrt{M'}\,\varrho_f\omega^2}\begin{pmatrix}\partial_{\nu(x_{\mathbf{r},\kappa'})}\Re\mathrm{e}\,G(x_{\mathbf{r},\kappa'},x_{i,\kappa})\\ \partial_{\nu(x_{\mathbf{r},\kappa'})}\Im\mathrm{m}\,G(x_{\mathbf{r},\kappa'},x_{i,\kappa})\end{pmatrix}^{\top},$$

$$\frac{\partial}{\partial[\Im\mathrm{m}\,b_\kappa]}\,\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{5,\kappa'} =$$

$$= -\frac{1}{\sqrt{M'}\,\varrho_f\omega^2}\begin{pmatrix}-\partial_{\nu(x_{\mathbf{r},\kappa'})}\Im\mathrm{m}\,G\big(x_{\mathbf{r},\kappa'},x_{i,\kappa}\big)\\ \partial_{\nu(x_{\mathbf{r},\kappa'})}\Re\mathrm{e}\,G\big(x_{\mathbf{r},\kappa'},x_{i,\kappa}\big)\end{pmatrix}^{\top},$$

$$\frac{\partial}{\partial[\Re\mathrm{e}\,[c_\kappa]_l]}\,\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{5,\kappa'} =$$

$$= \frac{1}{\sqrt{M'}}\begin{pmatrix}\nu(x_{\mathbf{r},\kappa'})\cdot\Big(\Re\mathrm{e}\,\big[G^{el}(x_{\mathbf{r},\kappa'},x_{i,\kappa})\big]_{m,l}\Big)_m\\ \nu(x_{\mathbf{r},\kappa'})\cdot\Big(\Im\mathrm{m}\,\big[G^{el}(x_{\mathbf{r},\kappa'},x_{i,\kappa})\big]_{m,l}\Big)_m\end{pmatrix}^{\top},$$

$$\frac{\partial}{\partial[\Im\mathrm{m}\,[c_\kappa]_l]}\,\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{5,\kappa'} =$$

$$= \frac{1}{\sqrt{M'}}\begin{pmatrix}-\nu(x_{\mathbf{r},\kappa'})\cdot\Big(\Im\mathrm{m}\,\big[G^{el}(x_{\mathbf{r},\kappa'},x_{i,\kappa})\big]_{m,l}\Big)_m\\ \nu(x_{\mathbf{r},\kappa'})\cdot\Big(\Re\mathrm{e}\,\big[G^{el}(x_{\mathbf{r},\kappa'},x_{i,\kappa})\big]_{m,l}\Big)_m\end{pmatrix}^{\top},$$

$$\frac{\partial}{\partial a_\iota}\,\mathcal{M}\big((b_\kappa),(c_\kappa),(a_\iota)\big)_{5,\kappa'} =$$

$$= \frac{1}{\sqrt{M'}}\sum_{\kappa=1}^{M}{}^{\top}\big[G^{el}(x_{e,\kappa},x_{\mathbf{r},\kappa'})c_\kappa\big]\,\frac{\partial}{\partial a_\iota}\,[\nu(x_{\mathbf{r},\kappa'})]+$$

$$+\frac{1}{\sqrt{M'}}\sum_{\kappa=1}^{M}\nu(x_{\mathbf{r},\kappa'})\cdot\mathrm{grad}\,_{x_{\mathbf{r},\kappa'}}\big[G^{el}(x_{e,\kappa},x_{\mathbf{r},\kappa'})c_\kappa\big]\,\frac{\partial}{\partial a_\iota}\,[x_{\mathbf{r},\kappa'}]-$$

$$-\frac{1}{\sqrt{M'}\,\varrho_f\omega^2}\sum_{\kappa=1}^{M}{}^{\top}\big[b_\kappa\mathrm{grad}\,_{x_{\mathbf{r},\kappa'}}G(x_{\mathbf{r},\kappa'},x_{i,\kappa})\big]\,\frac{\partial}{\partial a_\iota}\,[\nu(x_{\mathbf{r},\kappa'})]-$$

$$-\frac{1}{\sqrt{M'}\,\varrho_f\omega^2}\sum_{\kappa=1}^{M}b_\kappa\nu(x_{\mathbf{r},\kappa'})\cdot\mathrm{grad}\,_{x_{\mathbf{r},\kappa'}}\times$$

$$\times\big[\mathrm{grad}\,_{x_{\mathbf{r},\kappa'}}G(x_{\mathbf{r},\kappa'},x_{i,\kappa})\big]\,\frac{\partial}{\partial a_\iota}[x_{\mathbf{r},\kappa'}]-$$

$$-\frac{1}{\sqrt{M'}\,\varrho_f\omega^2}\,{}^{\top}\big[\mathrm{grad}\,_{x_{\mathbf{r},\kappa'}}p^{inc}(x_{\mathbf{r},\kappa'})\big]\,\frac{\partial}{\partial a_\iota}\,[\nu(x_{\mathbf{r},\kappa'})]-$$

$$-\frac{1}{\sqrt{M'}\,\varrho_f\omega^2}\,\nu(x_{\mathbf{r},\kappa'})\cdot\mathrm{grad}\,_{x_{\mathbf{r},\kappa'}}\times$$

$$\times\big[\mathrm{grad}\,_{x_{\mathbf{r},\kappa'}}p^{inc}(x_{\mathbf{r},\kappa'})\big]\,\frac{\partial}{\partial a_\iota}\,[x_{\mathbf{r},\kappa'}].$$

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INVESTIGATION OF INTERIOR AND EXTERIOR NEUMANN-TYPE STATIC BOUNDARY-VALUE PROBLEMS OF THERMO-ELECTRO-MAGNETO ELASTICITY THEORY

Abstract. We investigate the three-dimensional interior and exterior Neumann-type boundary-value problems of statics of the thermo-electro-magneto-elasticity theory. We construct explicitly the fundamental matrix of the corresponding strongly elliptic non-self-adjoint 6×6 matrix differential operator and study their properties near the origin and at infinity. We apply the potential method and reduce the corresponding boundary-value problems to the equivalent system of boundary integral equations. We have found efficient asymptotic conditions at infinity which ensure the uniqueness of solutions in the space of bounded vector functions. We analyze the solvability of the resulting boundary integral equations in the Hölder and Sobolev-Slobodetski spaces and prove the corresponding existence theorems. The necessary and sufficient conditions of solvability of the interior Neumann-type boundary-value problem are written explicitly.

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Key words and phrases. Thermo-electro-magneto-elasticity, boundary-value problem, potential method, boundary integral equations, uniqueness theorems, existence theorems.

რეზიუმე. სტატიაში გამოკვლეულია ნეიმანის შიგა და გარე სამგანზომილებიანი ამოცანები თერმო-ელექტრო-მაგნეტო დრეკადობის თეორიის სტატიკის განტოლებებისათვის. შესაბამისი ოპერატორისათვის, რომელიც წარმოადგენს 6×6 განზომილების მატრიცულ არათვითშეუღლებულ, ძლიერად ელიფსურ დიფერენციალურ ოპერატორს, ცხადი სახითაა აგებული ფუნდამენტურ ამონახსნთა მატრიცა და დადგენილია მისი ასიმპტოტური თვისებები სათავისა და უსასრულობის მიდამოში. პოტენციალთა მეთოდის გამოყენებით სასაზღვრო ამოცანები დაყვანილია ეკვივალენტურ სასაზღვრო ინტეგრალურ (ფსევდოდიფერენციალურ) განტოლებათა სისტემაზე. გამოკვლეულია ამ ინტეგრალური განტოლების ამოხსნადობის საკითხი და დამტკიცებულია შესაბამისი სასაზღვრო ამოცანების ამონახსნების არსებობის თეორემები ჰელდერისა და სობოლევ–სლობოდეცკის ფუნქციურ სივრცეებში. აღსანიშნავია, რომ ცხადი სახითაა ამოწერილი ნეიმანის შიგა ამოცანის არსებობის აუცილებელი და საკმარისი პირობები.

1. Introduction

Modern industrial and technological processes apply widely, on the one hand, composite materials with complex microstructure and, on the other hand, complex composed structures consisting of materials having essentially different physical properties (for example, piezoelectric, piezomagnetic, hemitropic materials, two- and multi-component mixtures, nanomaterials, bio-materials, and solid structures constructed by composition of these materials, such as, e.g., Smart Materials and other meta-materials). Therefore the investigation and analysis of mathematical models describing the mechanical, thermal, electric, magnetic and other physical properties of such materials have a crucial importance for both fundamental research and practical applications. In particular, the investigation of correctness of corresponding mathematical models (namely, existence, uniqueness, smoothness, asymptotic properties and stability of solutions) and construction of appropriate adequate numerical algorithms have a crucial role for fundamental research.

In the study of active material systems, there is significant interest in the coupling effects between elastic, electric, magnetic and thermal fields. The mathematical model of statics of the thermo-electro-magneto-elasticity theory is described by the non-self-adjoint 6×6 system of second order partial differential equations with appropriate boundary conditions. The problem is to determine three components of the elastic displacement vector, the electric and magnetic scalar potential functions and the temperature distribution. Other field characteristics (e.g., mechanical stresses, electric and magnetic fields, electric displacement vector, magnetic induction vector, and heat flux vector) can be then determined by the gradient and constitutive equations (for details see [2], [3], [4], [5], [6], [16], [21], [24], [27]).

For the equations of dynamics the uniqueness theorems of solutions for some initial-boundary-value problems are well studied. In particular, in the reference [16] the uniqueness theorem is proved without making restrictions on the positive definiteness on the elastic moduli, while the uniqueness theorems for the basic boundary-value problems (BVP) of statics of the thermo-electro-magneto-elasticity theory are proved in [20]. Existence theorems for the Dirichlet-type boundary-value problems are established in [19]. To the best of our knowledge, the existence of solutions to the Neumann-type BVPs of statics are not treated in the scientific literature.

In this paper, with the help of the potential method we reduce the three-dimensional interior and exterior Neumann-type boundary-value problems of the thermo-electro-magneto-elasticity theory to the equivalent 6×6 systems of integral equations and analyze their solvability in the Hölder and Sobolev-Slobodetski spaces and prove the corresponding uniqueness and existence theorems.

Essential difficulties arise in the study of exterior BVPs for unbounded domains. The case is that one has to consider the problem in a class of vector functions which are bounded at infinity. This complicates the proof of uniqueness and existence theorems since Green's formulas do not hold for such vector functions and analysis of null spaces of the corresponding integral operators needs special consideration. We have found efficient and natural asymptotic conditions at infinity which ensure the uniqueness of solutions in the space of bounded vector functions. Moreover, for the interior Neumann-type boundary-value problem, the complete system of linearly independent solutions of the corresponding homogeneous adjoint integral equation is constructed in polynomials and the necessary and sufficient conditions of solvability of the problem are written explicitly.

2. Formulation of Problems

Here we collect the basic field equations of the thermo-electro-magneto-elasticity theory and formulate the interior and exterior Neumann-type boundary-value problems of statics.

2.1. Field equations. Throughout the paper $u=(u_1,u_2,u_3)^\top$ denotes the displacement vector, σ_{ij} is the mechanical stress tensor, $\varepsilon_{kj}=2^{-1}(\partial_k u_j+\partial_j u_k)$ is the strain tensor, the vectors $E=(E_1,E_2,E_3)^\top$ and $H=(H_1,H_2,H_3)^\top$ are electric and magnetic fields respectively, $D=(D_1,D_2,D_3)^\top$ is the electric displacement vector and $B=(B_1,B_2,B_3)^\top$ is the magnetic induction vector, φ and ψ stand for the electric and magnetic potentials and $E=-\operatorname{grad}\varphi$, $H=-\operatorname{grad}\psi$, ϑ is the temperature increment, $q=(q_1,q_2,q_3)^\top$ is the heat flux vector, and $\mathcal{S}$ is the entropy density.

We employ also the notation $\partial=\partial_x=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial/\partial x_j$, $\partial_t=\partial/\partial t$; the superscript $(\cdot)^\top$ denotes transposition operation. In what follows the summation over the repeated indices is meant from 1 to 3, unless stated otherwise.

In this subsection we collect the field equations of the linear theory of thermo-electro-magneto-elasticity for a general anisotropic case and introduce the corresponding matrix partial differential operators. To this end, we recall here the basic relations of the theory:

Constitutive relations:

$$\sigma_{rj}=\sigma_{jr}=c_{rjkl}\varepsilon_{kl}-e_{lrj}E_l-q_{lrj}H_l-\lambda_{rj}\vartheta,\quad r,j=1,2,3, \tag{2.1}$$

$$D_j=e_{jkl}\varepsilon_{kl}+\varkappa_{jl}E_l+a_{jl}H_l+p_j\vartheta,\quad j=1,2,3, \tag{2.2}$$

$$B_j=q_{jkl}\varepsilon_{kl}+a_{jl}E_l+\mu_{jl}H_l+m_j\vartheta,\quad j=1,2,3, \tag{2.3}$$

$$\mathcal{S}=\lambda_{kl}\varepsilon_{kl}+p_kE_k+m_kH_k+\gamma\vartheta. \tag{2.4}$$

Fourier Law:

$$q_j=-\eta_{jl}\partial_l\vartheta,\quad j=1,2,3. \tag{2.5}$$

Equations of motion:

$$\partial_j\sigma_{rj}+X_r=\varrho\partial_t^2u_r,\quad r=1,2,3. \tag{2.6}$$

Quasi-static equations for electro-magnetic fields where the rate of magnetic field is small (*electric field is curl free*) *and there is no electric current* (*magnetic field is curl free*)*:*

$$\partial_j D_j = \varrho_e, \quad \partial_j B_j = 0. \tag{2.7}$$

Linearized equation of the entropy balance:

$$T_0 \partial_t \mathcal{S} - Q = -\partial_j q_j. \tag{2.8}$$

Here ϱ is the mass density, ϱ_e is the electric density, c_{rjkl} are the elastic constants, e_{jkl} are the piezoelectric constants, q_{jkl} are the piezomagnetic constants, $\varkappa_{jk}$ are the dielectric (permittivity) constants, μ_{jk} are the magnetic permeability constants, a_{jk} are the coupling coefficients connecting electric and magnetic fields, p_j and m_j are constants characterizing the relation between thermodynamic processes and electromagnetic effects, λ_{jk} are the thermal strain constants, η_{jk} are the heat conductivity coefficients, $\gamma = \varrho c T_0^{-1}$ is the thermal constant, T_0 is the initial reference temperature, that is the temperature in the natural state in the absence of deformation and electromagnetic fields, c is the specific heat per unit mass, $X = (X_1, X_2, X_3)^\top$ is a mass force density, Q is a heat source intensity.

The constants involved in these equations satisfy the symmetry conditions:

$$\begin{gathered} c_{rjkl} = c_{jrkl} = c_{klrj}, \quad e_{klj} = e_{kjl}, \\ q_{klj} = q_{kjl}, \quad \varkappa_{kj} = \varkappa_{jk}, \quad \lambda_{kj} = \lambda_{jk}, \quad r,j,k,l = 1,2,3. \\ \mu_{kj} = \mu_{jk}, \quad \eta_{kj} = \eta_{jk}, \quad a_{kj} = a_{jk}, \end{gathered} \tag{2.9}$$

From physical considerations it follows that (see, e.g., [16], [27]):

$$\begin{gathered} c_{rjkl}\xi_{rj}\xi_{kl} \geq c_0 \xi_{kl}\xi_{kl}, \quad \varkappa_{kj}\xi_k\xi_j \geq c_1 |\xi|^2, \\ \mu_{kj}\xi_k\xi_j \geq c_2 |\xi|^2, \quad \eta_{kj}\xi_k\xi_j \geq c_3|\xi|^2, \\ \text{for all } \ \xi_{kj} = \xi_{jk} \in \mathbb{R} \ \text{ and for all } \ \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3, \end{gathered} \tag{2.10}$$

where c_0, c_1, c_2, and c_3 are positive constants.

It is easy to see that due to the symmetry conditions (2.9)

$$\begin{gathered} c_{rjkl}\xi_{rj}\overline{\xi_{kl}} \geq c_0 \xi_{kl}\overline{\xi_{kl}}, \quad \varkappa_{kj}\xi_k\overline{\xi_j} \geq c_1 |\xi|^2, \\ \mu_{kj}\xi_k\overline{\xi_j} \geq c_2 |\xi|^2, \quad \eta_{kj}\xi_k\overline{\xi_j} \geq c_3|\xi|^2, \\ \text{for all } \ \xi_{kj} = \xi_{jk} \in \mathbb{C} \ \text{ and for all } \ \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3. \end{gathered}$$

More careful analysis related to the positive definiteness of the potential energy and thermodynamical laws insure that for arbitrary ζ', $\zeta'' \in \mathbb{C}^3$ and $\theta \in \mathbb{C}$ there is a positive constant δ_0 depending on the material constants such that (cf. [27])

$$\begin{aligned} \varkappa_{kj}\zeta'_k\overline{\zeta'_j} + a_{kj}\big(\zeta'_k\overline{\zeta''_j} + \overline{\zeta'_k}\zeta''_j\big) + \mu_{kj}\zeta''_k\overline{\zeta''_j} \pm 2\Re\big[\overline{\theta}(p_j\zeta'_j + m_j\zeta''_j)\big] + \gamma|\theta|^2 \geq \\ \geq \delta_0\big(|\zeta'|^2 + |\zeta''|^2 + |\theta|^2\big). \end{aligned} \tag{2.11}$$

This condition is equivalent to positive definiteness of the matrix

$$\Xi := \begin{bmatrix} [\varkappa_{kj}]_{3\times 3} & [a_{kj}]_{3\times 3} & [p_j]_{3\times 1} \\ [a_{kj}]_{3\times 3} & [\mu_{kj}]_{3\times 3} & [m_j]_{3\times 1} \\ [p_j]_{1\times 3} & [m_j]_{1\times 3} & \gamma \end{bmatrix}_{7\times 7} .$$

In particular, it follows that the matrix

$$\Lambda := \begin{bmatrix} [\varkappa_{kj}]_{3\times 3} & [a_{kj}]_{3\times 3} \\ [a_{kj}]_{3\times 3} & [\mu_{kj}]_{3\times 3} \end{bmatrix}_{6\times 6} \tag{2.12}$$

is positive definite, i.e.,

$$\varkappa_{kj}\zeta'_k\overline{\zeta'_j} + a_{kj}\big(\zeta'_k\overline{\zeta''_j} + \overline{\zeta'_k}\zeta''_j\big) + \mu_{kj}\zeta''_k\overline{\zeta''_j} \geq \kappa(|\zeta'|^2 + |\zeta''|^2)$$

with some positive constant κ depending on the material parameters involved in (2.12). A sufficient condition for the quadratic form in the left hand side of (2.11) to be positive definite then reads as $\nu^2 < \frac{\kappa\gamma}{6}$ with $\nu = \max\big\{|p_1|, |p_2|, |p_3|, |m_1|, |m_2|, |m_3|\big\}$.

With the help of the symmetry conditions (2.9) we can rewrite the constitutive relations (2.1)–(2.4) as follows

$$\begin{aligned} \sigma_{rj} &= c_{rjkl}\partial_l u_k + e_{lrj}\partial_l\varphi + q_{lrj}\partial_l\psi - \lambda_{rj}\vartheta, \quad r,j = 1,2,3, \\ D_j &= e_{jkl}\partial_l u_k - \varkappa_{jl}\partial_l\varphi - a_{jl}\partial_l\psi + p_j\vartheta, \quad j = 1,2,3, \\ B_j &= q_{jkl}\partial_l u_k - a_{jl}\partial_l\varphi - \mu_{jl}\partial_l\psi + m_j\vartheta, \quad j = 1,2,3, \\ \mathcal{S} &= \lambda_{kl}\partial_l u_k - p_l\partial_l\varphi - m_l\partial_l\psi + \gamma\vartheta. \end{aligned}$$

In the theory of thermo-electro-magneto-elasticity the components of the three-dimensional mechanical stress vector acting on a surface element with a unit normal vector $n = (n_1, n_2, n_3)$ have the form

$$\sigma_{rj}n_j = c_{rjkl}n_j\partial_l u_k + e_{lrj}n_j\partial_l\varphi + q_{lrj}n_j\partial_l\psi - \lambda_{rj}n_j\vartheta, \quad r = 1,2,3,$$

while the normal components of the electric displacement vector, magnetic induction vector and heat flux vector read as

$$\begin{aligned} D_j n_j &= e_{jkl}n_j\partial_l u_k - \varkappa_{jl}n_j\partial_l\varphi - a_{jl}n_j\partial_l\psi + p_j n_j\vartheta, \\ B_j n_j &= q_{jkl}n_j\partial_l u_k - a_{jl}n_j\partial_l\varphi - \mu_{jl}n_j\partial_l\psi + m_j n_j\vartheta, \\ q_j n_j &= -\eta_{jl}n_j\partial_l\vartheta. \end{aligned}$$

For convenience we introduce the following matrix differential operator

$$\mathcal{T}(\partial, n) = \big[\mathcal{T}_{pq}(\partial, n)\big]_{6\times 6} :=$$

$$:= \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [e_{lrj}n_j\partial_l]_{3\times 1} & [q_{lrj}n_j\partial_l]_{3\times 1} & [-\lambda_{rj}n_j]_{3\times 1} \\ [-e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & -p_j n_j \\ [-q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & -m_j n_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6\times 6} . \tag{2.13}$$

Evidently, for a six vector $U := (u, \varphi, \psi, \vartheta)^\top$ we have

$$\mathcal{T}(\partial, n)U = \big(\sigma_{1j}n_j, \sigma_{2j}n_j, \sigma_{3j}n_j, -D_jn_j, -B_jn_j, -q_jn_j\big)^\top. \tag{2.14}$$

The components of the vector $\mathcal{T}U$ given by (2.14) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of thermo-electro-magneto-elasticity, the forth, fifth and sixth ones are respectively the normal components of the electric displacement vector, magnetic induction vector and heat flux vector with opposite sign.

As we see, all the thermo-mechanical and electro-magnetic characteristics can be determined by the six functions: the three displacement components u_j, $j = 1, 2, 3$, temperature distribution ϑ, and the electric and magnetic potentials φ and ψ. Therefore, all the above field relations and the corresponding boundary-value problems we reformulate in terms of these six functions.

First of all from the equations (2.1)–(2.8) we derive the basic linear system of dynamics of the theory of thermo-electro-magneto-elasticity:

$$\begin{gathered}
c_{rjkl}\partial_j\partial_l u_k(x,t) + e_{lrj}\partial_j\partial_l\varphi(x,t) + q_{lrj}\partial_j\partial_l\psi(x,t) - \lambda_{rj}\partial_j\vartheta(x,t) - \\
-\varrho\partial_t^2 u_r(x,t) = -X_r(x,t), \quad r = 1,2,3, \\
-e_{jkl}\partial_j\partial_l u_k(x,t) + \varkappa_{jl}\partial_j\partial_l\varphi(x,t) + a_{jl}\partial_j\partial_l\psi(x,t) - p_j\partial_j\vartheta(x,t) = -\varrho_e(x,t), \\
-q_{jkl}\partial_j\partial_l u_k(x,t) + a_{jl}\partial_j\partial_l\varphi(x,t) + \mu_{jl}\partial_j\partial_l\psi(x,t) - m_j\partial_j\vartheta(x,t) = 0, \\
-T_0\lambda_{kl}\partial_t\partial_l u_k(x,t) + T_0 p_l\partial_t\partial_l\varphi(x,t) + T_0 m_l\partial_t\partial_l\psi(x,t) + \eta_{jl}\partial_j\partial_l\vartheta(x,t) - \\
-T_0\gamma\partial_t\vartheta(x,t) = -Q(x,t).
\end{gathered}$$

If all the functions involved in these equations are harmonic time dependent, that is they can be represented as the product of a function of the spatial variables (x_1, x_2, x_3) and the multiplier $\exp\{\tau t\}$, where $\tau = \sigma + i\omega$ is a complex parameter, we have then the *pseudo-oscillation equations* of the theory of thermo-electro-magneto-elasticity. Note that the pseudo-oscillation equations can be obtained from the corresponding dynamical equations by the Laplace transform. If τ is a pure imaginary number, $\tau = i\omega$ with the so called frequency parameter $\omega \in \mathbb{R}$, we obtain the *steady state oscillation equations*. Finally, if $\tau = 0$ we get the *equations of statics*:

$$\begin{gathered}
c_{rjkl}\partial_j\partial_l u_k(x) + e_{lrj}\partial_j\partial_l\varphi(x) + q_{lrj}\partial_j\partial_l\psi(x) - \lambda_{rj}\partial_j\vartheta(x) = \\
= -X_r(x), \quad r = 1,2,3, \\
-e_{jkl}\partial_j\partial_l u_k(x) + \varkappa_{jl}\partial_j\partial_l\varphi(x) + a_{jl}\partial_j\partial_l\psi(x) - p_j\partial_j\vartheta(x) = -\varrho_e(x), \\
-q_{jkl}\partial_j\partial_l u_k(x) + a_{jl}\partial_j\partial_l\varphi(x) + \mu_{jl}\partial_j\partial_l\psi(x) - m_j\partial_j\vartheta(x) = 0, \\
\eta_{jl}\partial_j\partial_l\vartheta(x) = -Q(x).
\end{gathered} \tag{2.15}$$

In matrix form these equations can be written as

$$A(\partial)U(x) = \Phi(x),$$

where

$$U = (u_1, u_2, u_3, u_4, u_5, u_6)^\top := (u, \varphi, \psi, \vartheta)^\top,$$

$$\Phi = (\Phi_1, \dots, \Phi_6)^\top := (-X_1, -X_2, -X_3, -\varrho_e, 0, -Q)^\top,$$

and $A(\partial)$ is the matrix differential operator generated by equations (2.15),

$$A(\partial) = [A_{pq}(\partial)]_{6\times 6} :=$$

$$:= \begin{bmatrix} [c_{rjkl}\partial_j\partial_l]_{3\times 3} & [e_{lrj}\partial_j\partial_l]_{3\times 1} & [q_{lrj}\partial_j\partial_l]_{3\times 1} & [-\lambda_{rj}\partial_j]_{3\times 1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -p_j\partial_j \\ [-q_{jkl}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -m_j\partial_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\partial_j\partial_l \end{bmatrix}_{6\times 6}. \tag{2.16}$$

2.2. **Formulation of the boundary-value problems.** Let Ω^+ be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $S = \partial\Omega^+$, $\overline{\Omega^+} = \Omega^+ \cup S$, and $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. Assume that the domains $\overline{\Omega^\pm}$ are filled by an anisotropic homogeneous material with thermo-electro-magneto-elastic properties.

Throughout the paper $n = (n_1, n_2, n_3)$ stands for the outward unit normal vector with respect to Ω^+ at the point $x \in \partial\Omega^+$.

Neumann-type problems $(N)^\pm$**:** Find a regular solution vector $U = (u,\varphi,\psi,\vartheta)^\top \in [C^1(\overline{\Omega^+})]^6 \cap [C^2(\Omega^+)]^6$ (resp. $U \in [C^1(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6$), to the system of equations

$$A(\partial)U = \Phi \ \text{ in } \ \Omega^\pm,$$

satisfying the Neumann-type boundary conditions

$$\{\mathcal{T}U\}^\pm = f \ \text{ on } \ S,$$

where $A(\partial)$ is a nonselfadjoint strongly elliptic matrix partial differential operator generated by the equations of statics of the theory of thermo-electro-magneto-elasticity defined in (2.16), while $\mathcal{T}(\partial, n)$ is the matrix boundary operator defined in (2.13). The symbols $\{\cdot\}^\pm$ denote the one sided limits (the trace operators) on $\partial\Omega^\pm$ from $\Omega^\pm$.

In our analysis we need special asymptotic conditions at infinity in the case of unbounded domains [20].

Definition 2.1. We say that a continuous vector $U = (u, \varphi, \psi, \vartheta)^\top \equiv (U_1, \cdots, U_6)^\top$ in the domain Ω^- has the property $Z(\Omega^-)$ if the following conditions are satisfied

$$\left.\begin{aligned} \widetilde{U}(x) := (u(x), \varphi(x), \psi(x))^\top = \mathcal{O}(1), \\ U_6(x) = \vartheta(x) = \mathcal{O}(|x|^{-1}), \end{aligned}\right. \quad \text{as } \ |x| \to \infty,$$

$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma_R} U_k(x)\, d\Sigma_R = 0, \quad k = \overline{1,5},$$

where Σ_R is a sphere centered at the origin and radius R.

In what follows we always assume that in the case of exterior boundary-value problem a solution possesses $Z(\Omega^-)$ property.

2.3. **Potentials and their properties.** Denote by $\Gamma(x)=[\Gamma_{kj}(x)]_{6\times 6}$ the matrix of fundamental solutions of the operator $A(\partial)$, $A(\partial)\Gamma(x)=I_6\,\delta(x)$, where $\delta(\cdot)$ is the Dirac's delta distribution and I_6 stands for the unit 6×6 matrix. Applying the generalized Fourier transform technique, the fundamental matrix can be constructed explicitly,

$$\Gamma(x)=\mathcal{F}^{-1}_{\xi\to x}[A^{-1}(-i\,\xi)]\,, \tag{2.17}$$

where $\mathcal{F}^{-1}$ is the generalized inverse Fourier transform and $A^{-1}(-i\,\xi)$ is the matrix inverse to $A(-i\,\xi)$. The properties of the fundamental matrix near the origin and at infinity are established in [23]. The entries of the fundamental matrix $\Gamma(x)$ are homogeneous functions in x and at the origin and at infinity the following asymptotic relations hold

$$\Gamma(x)=\begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{5\times 5} & [\mathcal{O}(1)]_{5\times 1} \\ [0]_{1\times 5} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{6\times 6}.$$

Moreover, the columns of the matrix $\Gamma(x)$ possess the property $Z(\mathbb{R}^3\setminus\{0\})$. With the help of the fundamental matrix we construct the generalized single and double layer potentials, and the Newton-type volume potentials,

$$V(h)(x)=\int\limits_S \Gamma(x-y)\,h(y)\,dS_y,\quad x\in\mathbb{R}^3\setminus S,$$

$$W(h)(x)=\int\limits_S [\mathcal{P}(\partial_y,n(y))\Gamma^\top(x-y)]^\top\,h(y)\,dS_y,\quad x\in\mathbb{R}^3\setminus S,$$

$$N_{\Omega^\pm}(g)(x)=\int\limits_{\Omega^\pm} \Gamma(x-y)\,g(y)\,dy,\quad x\in\mathbb{R}^3,$$

where $S=\partial\Omega^\pm\in C^{m,\kappa}$ with integer $m\geq 1$ and $0<\kappa\leq 1$; $h=(h_1,\dots,h_6)^\top$ and $g=(g_1,\cdots,g_6)^\top$ are density vector-functions defined respectively on S and in $\Omega^\pm$; the so called *generalized stress operator* $\mathcal{P}(\partial,n)$, associated with the adjoint differential operator $A^*(\partial)=A^\top(-\partial)$, reads as

$$\mathcal{P}(\partial,n)=\big[\mathcal{P}_{pq}(\partial,n)\big]_{6\times 6}=$$

$$=\begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [-e_{lrj}n_j\partial_l]_{3\times 1} & [-q_{lrj}n_j\partial_l]_{3\times 1} & [0]_{3\times 1} \\ [e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & 0 \\ [q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & 0 \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}. \tag{2.18}$$

The following properties of layer potentials immediately follow from their definition.

Theorem 2.2. The generalized single and double layer potentials solve the homogeneous differential equation $A(\partial)U=0$ in $\mathbb{R}^3\setminus S$ and possess the property $Z(\Omega^-)$.

In what follows by L_p, W_p^r, H_p^s, and $B_{p,q}^s$ (with $r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) we denote the well-known Lebesgue, Sobolev–Slobodetski, Bessel potential, and Besov function spaces, respectively (see, e.g., [29]). Recall that $H_2^r = W_2^r = B_{2,2}^r$, $H_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$, and $H_p^k = W_p^k$, for any $r \geq 0$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

With the help of Green's formulas, one can derive general integral representations of solutions to the homogeneous equation $A(\partial)U = 0$ in $\Omega^\pm$. In particular, the following theorems hold.

Theorem 2.3. Let $S = \partial\Omega^+ \in C^{1,\kappa}$ with $0 < \kappa \leq 1$ and U be a regular solution to the homogeneous equation $A(\partial)U = 0$ in Ω^+ of the class $[C^1(\overline{\Omega^+})]^6 \cap [C^2(\Omega^+)]^6$. Then there holds the integral representation formula

$$W(\{U\}^+)(x) - V(\{\mathcal{T}U\}^+)(x) = \begin{cases} U(x) & \text{for } x \in \Omega^+, \\ 0 & \text{for } x \in \Omega^-. \end{cases}$$

Theorem 2.4. Let $S = \partial\Omega^-$ be $C^{1,\kappa}$-smooth with $0 < \kappa \leq 1$ and let U be a regular solution to the homogeneous equation $A(\partial)U = 0$ in Ω^- of the class $[C^1(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6$ having the property $Z(\Omega^-)$. Then there holds the integral representation formula

$$-W(\{U\}^-)(x) + V(\{\mathcal{T}U\}^-)(x) = \begin{cases} 0 & \text{for } x \in \Omega^+, \\ U(x) & \text{for } x \in \Omega^-. \end{cases}$$

By standard limiting procedure, these formulas can be extended to Lipschitz domains and to solution vectors from the spaces $[W_p^1(\Omega^+)]^6$ and $[W_{p,loc}^1(\Omega^-)]^6 \cap Z(\Omega^-)$ with $1 < p < \infty$ (cf., [12], [17], [25]).

The qualitative and mapping properties of the layer potentials are described by the following theorems (cf. [7], [9], [15], [17], [23]).

Theorem 2.5. Let $S = \partial\Omega^\pm \in C^{m,\kappa}$ with integers $m \geq 1$ and $k \leq m-1$, and $0 < \kappa' < \kappa \leq 1$. Then the operators

$$V : [C^{k,\kappa'}(S)]^6 \to [C^{k+1,\kappa'}(\overline{\Omega^\pm})]^6, \quad W : [C^{k,\kappa'}(S)]^6 \to [C^{k,\kappa'}(\overline{\Omega^\pm})]^6 \quad (2.19)$$

are continuous.

For any $g \in [C^{0,\kappa'}(S)]^6$, $h \in [C^{1,\kappa'}(S)]^6$, and any $x \in S$ we have the following jump relations:

$$\{V(g)(x)\}^\pm = V(g)(x) = \mathcal{H}g(x), \quad (2.20)$$

$$\big\{\mathcal{T}(\partial_x, n(x))V(g)(x)\big\}^\pm = \big[\mp 2^{-1}I_6 + \mathcal{K}\big]g(x), \quad (2.21)$$

$$\{W(g)(x)\}^\pm = [\pm 2^{-1}I_6 + \mathcal{N}]g(x), \quad (2.22)$$

$$\big\{\mathcal{T}(\partial_x, n(x))W(h)(x)\big\}^+ = \\ = \{\mathcal{T}(\partial_x, n(x))W(h)(x)\}^- = \mathcal{L}h(x), \quad m \geq 2, \quad (2.23)$$

where $\mathcal{H}$ is a weakly singular integral operator, $\mathcal{K}$ and $\mathcal{N}$ are singular integral operators, and $\mathcal{L}$ is a singular integro-differential operator,

$$\begin{aligned}
\mathcal{H}g(x) &:= \int\limits_S \Gamma(x-y)g(y)\,dS_y,\\
\mathcal{K}g(x) &:= \int\limits_S \mathcal{T}(\partial_x, n(x))\Gamma(x-y)\ g(y)\,dS_y,\\
\mathcal{N}g(x) &:= \int\limits_S \big[\mathcal{P}(\partial_y, n(y))\Gamma^\top(x-y)\big]^\top g(y)\,dS_y,\\
\mathcal{L}h(x) &:= \lim_{\Omega^\pm\ni z\to x\in S}\mathcal{T}(\partial_z, n(x))\int\limits_S\big[\mathcal{P}(\partial_y, n(y))\Gamma^\top(z-y)\big]^\top h(y)\,dS_y.
\end{aligned}\tag{2.24}$$

Theorem 2.6. Let S be a Lipschitz surface. The operators V and W can be extended to the continuous mappings

$$V:[H_2^{-\frac12}(S)]^6\to[H_2^1(\Omega^+)]^6,\quad V\ :\ [H_2^{-\frac12}(S)]^6\to[H^1_{2,loc}(\Omega^-)]^6\cap Z(\Omega^-),$$
$$W:[H_2^{\frac12}(S)]^6\to[H_2^1(\Omega^+)]^6,\quad W:[H_2^{\frac12}(S)]^6\to[H^1_{2,loc}(\Omega^-)]^6\cap Z(\Omega^-).$$

The jump relations (2.20)–(2.23) on S remain valid for the extended operators in the corresponding function spaces.

Theorem 2.7. Let S, m, κ, κ' and k be as in Theorem 2.5. Then the operators

$$\mathcal{H}:[C^{k,\kappa'}(S)]^6\to[C^{k+1,\kappa'}(S)]^6,\ \ m\ge1, \tag{2.25}$$
$$:[H_2^{-\frac12}(S)]^6\to[H_2^{\frac12}(S)]^6,\ \ m\ge1, \tag{2.26}$$
$$\mathcal{K}:[C^{k,\kappa'}(S)]^6\to[C^{k,\kappa'}(S)]^6,\ \ m\ge1, \tag{2.27}$$
$$:[H_2^{-\frac12}(S)]^6\to[H_2^{-\frac12}(S)]^6,\ \ m\ge1, \tag{2.28}$$
$$\mathcal{N}:[C^{k,\kappa'}(S)]^6\to[C^{k,\kappa'}(S)]^6,\ \ m\ge1, \tag{2.29}$$
$$:[H_2^{\frac12}(S)]^6\to[H_2^{\frac12}(S)]^6,\ \ m\ge1, \tag{2.30}$$
$$\mathcal{L}:[C^{k,\kappa'}(S)]^6\to[C^{k-1,\kappa'}(S)]^6,\ \ m\ge2,\ \ k\ge1, \tag{2.31}$$
$$:[H_2^{\frac12}(S)]^6\to[H_2^{-\frac12}(S)]^6,\ \ m\ge2, \tag{2.32}$$

are continuous. The operators (2.26), (2.28), (2.30), and (2.32) are bounded if S is a Lipschitz surface.

Proofs of the above formulated theorems are word for word proofs of the similar theorems in [8], [10], [11], [13], [14], [15], [22], [26].

The next assertion is a consequence of the general theory of elliptic pseudodifferential operators on smooth manifolds without boundary (see, e.g., [1], [5], [9], [12], [28], and the references therein).

Theorem 2.8. Let V, W, $\mathcal{H}$, $\mathcal{K}$, $\mathcal{N}$ and $\mathcal{L}$ be as in Theorems 2.5 and let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, $S \in C^\infty$. The layer potential operators (2.19) and the boundary integral (pseudodifferential) operators (2.25)–(2.32) can be extended to the following continuous operators

$$\begin{gathered}
V : [B^s_{p,p}(S)]^6 \to [H_p^{s+1+\frac{1}{p}}(\Omega^+)]^6, \quad W : [B^s_{p,p}(S)]^6 \to [H_p^{s+\frac{1}{p}}(\Omega^+)]^6,\\
V : [B^s_{p,p}(S)]^6 \to [H_{p,loc}^{s+1+\frac{1}{p}}(\Omega^-)]^6, \quad W : [B^s_{p,p}(S)]^6 \to [H_{p,loc}^{s+\frac{1}{p}}(\Omega^-)]^6,\\
\mathcal{H} : [H^s_p(S)]^6 \to [H^{s+1}_p(S)]^6, \quad \mathcal{K} : [H^s_p(S)]^6 \to [H^s_p(S)]^6,\\
\mathcal{N} : [H^s_p(S)]^6 \to [H^s_p(S)]^6, \quad \mathcal{L} : [H^{s+1}_p(S)]^6 \to [H^s_p(S)]^6.
\end{gathered}$$

The jump relations (2.20)–(2.23) remain valid for arbitrary $g \in [B^s_{p,q}(S)]^6$ with $s \in \mathbb{R}$ if the limiting values (traces) on S are understood in the sense described in [28].

Remark 2.9. Let either $\Phi \in [L_p(\Omega^+)]^6$ or $\Phi \in [L_{p,comp}(\Omega^-)]^6$, $p > 1$. Then the Newtonian volume potentials $N_{\Omega^\pm}(\Phi)$ possess the following properties (see, e.g., [18]):

$$\begin{gathered}
N_{\Omega^+}(\Phi) \in [W^2_p(\Omega^+)]^6, \quad N_{\Omega^-}(\Phi) \in [W^2_{p,loc}(\Omega^-)]^6,\\
A(\partial)N_{\Omega^\pm}(\Phi) = \Phi \ \text{ almost everywhere in } \ \Omega^\pm.
\end{gathered}$$

Therefore, without loss of generality, we can assume that in the formulation of the Neumann-type problems the right hand side function in the differential equations vanishes, $\Phi(x) = 0$ in $\Omega^\pm$.

3. Investigation of the Exterior Neumann BVP

Let us consider the exterior Neumann-type BVP for the domain Ω^-:

$$A(\partial)U(x) = 0, \quad x \in \Omega^-, \tag{3.1}$$

$$\big\{\mathcal{T}(\partial, n)U(x)\big\}^- = F(x), \quad x \in S. \tag{3.2}$$

We assume that $S \in C^{1,\kappa}$ and $F \in C^{0,\kappa'}(S)$ with $0 < \kappa' < \kappa \leq 1$. We investigate this problem in the space of regular vector functions $[C^{1,\kappa'}(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6 \cap Z(\Omega^-)$. In [20] it is shown that the homogeneous version of the exterior Neumann-type problem possesses only the trivial solution.

To prove the existence result, we look for a solution of the problem (3.1)–(3.2) as the single layer potential

$$U(x) \equiv V(h)(x) = \int_S \Gamma(x-y)h(y)\,dS_y, \tag{3.3}$$

where Γ is defined by (2.17) and $h = (h_1, \dots, h_6)^\top \in [C^{0,\kappa'}(S)]^6$ is unknown density. By Theorem 2.5 and in view of the boundary condition (3.2), we get the following integral equation for the density vector h

$$[2^{-1}I_6 + \mathcal{K}]h = F \ \text{ on } \ S, \tag{3.4}$$

where $\mathcal{K}$ is a singular integral operator defined by (2.24). Note that the operator $2^{-1}I_6 + \mathcal{K}$ has the following mapping properties

$$2^{-1}I_6 + \mathcal{K} : [C^{0,\kappa'}(S)]^6 \to [C^{0,\kappa'}(S)]^6, \tag{3.5}$$

$$: [L_2(S)]^6 \to [L_2(S)]^6. \tag{3.6}$$

These operators are compact perturbations of their counterpart operators associated with the pseudo-oscillation equations which are studied in [23]. Applying the results obtained in [23] one can show that $2^{-1}I_6 + \mathcal{K}$ is a singular integral operator of normal type (i.e., its principal homogeneous symbol matrix is non-degenerate) and its index equals to zero.

Let us show that the operators (3.5) and (3.6) have trivial null spaces. To this end, it suffices to prove that the corresponding homogeneous integral equation

$$[2^{-1}I_6 + \mathcal{K}]h = 0 \ \text{ on } \ S, \tag{3.7}$$

has only the trivial solution in the appropriate space. Let $h^{(0)} \in [L_2(S)]^6$ be a solution to equation (3.7). By the embedding theorems (see, e.g., [15], Ch.4), we actually have that $h^{(0)} \in [C^{0,\kappa'}(S)]^6$. Now we construct the single layer potential $U_0(x) = V(h^{(0)})(x)$. Evidently, $U_0 \in [C^{1,\kappa'}(\overline{\Omega^\pm})]^6 \cap [C^2(\Omega^\pm)]^6 \cap Z(\Omega^-)$ and the equation $A(\partial)U_0 = 0$ in $\Omega^\pm$ is automatically satisfied. Since $h^{(0)}$ solves equation (3.7), we have $\{\mathcal{T}(\partial,n)U_0\}^- = [2^{-1}I_6 + \mathcal{K}]h^{(0)} = 0$ on S. Therefore U_0 is a solution to the homogeneous exterior Neumann problem satisfying the property $Z(\Omega^-)$. Consequently, due to the uniqueness theorem [20], $U_0 = 0$ in Ω^-. Applying the continuity property of the single layer potential we find: $0 = \{U_0\}^- = \{U_0\}^+$ on S, yielding that the vector $U_0 = V(h^{(0)})$ represents a solution to the homogeneous interior Dirichlet problem. Now by the uniqueness theorem for the Dirichlet problem [20], we deduce that $U_0 = 0$ in Ω^+. Thus $U_0 = 0$ in $\Omega^\pm$. By virtue of the jump formula

$$\big\{\mathcal{T}(\partial,n)U_0\big\}^+ - \big\{\mathcal{T}(\partial,n)U_0\big\}^- = -h^{(0)} = 0 \ \text{ on } \ S,$$

whence it follows that the null space of the operator $2^{-1}I_6 + \mathcal{K}$ is trivial and the operators (3.5) and (3.6) are invertible. As a ready consequence, we finally conclude that the non-homogeneous integral equation (3.4) is solvable for arbitrary right hand side vector $F \in [C^{0,\kappa'}(S)]^6$, which implies the following existence result.

Theorem 3.1. Let $m \geq 0$ be a nonnegative integer and $0 < \kappa' < \kappa \leq 1$. Further, let $S \in C^{m+1,\kappa}$ and $F \in [C^{m,\kappa'}(S)]^6$. Then the exterior Neumann-type BVP (3.1)–(3.2) is uniquely solvable in the space of regular vector functions, $[C^{m+1,\kappa'}(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6 \cap Z(\Omega^-)$, and the solution is representable by the single layer potential $U(x) = V(h)(x)$ with the density $h = (h_1, \dots, h_6)^\top \in [C^{m,\kappa'}(S)]^6$ being a unique solution of the integral equation (3.4).

Remark 3.2. Let S be Lipschitz and $F \in \big[H^{-1/2}(S)\big]^6$. Then by the same approach as in the reference [17], the following propositions can be established:

(i) the integral equation (3.4) is uniquely solvable in the space $[H^{-1/2}(S)]^6$;

(ii) the exterior Neumann-type BVP (3.1)–(3.2) is uniquely solvable in the space $[H^1_{2,loc}(\Omega^-)]^6 \cap Z(\Omega^-)$ and the solution is representable by the single layer potential (3.3), where the density vector $h \in [H^{-1/2}(S)]^6$ solves the integral equation (3.4).

4. Investigation of the Interior Neumann BVP

Before we go over to the interior Neumann problem we prove some preliminary assertions needed in our analysis.

4.1. **Some auxiliary results.** Let us consider the adjoint operator $A^*(\partial)$ to the operator $A(\partial)$

$$A^*(\partial) :=$$
$$:= \begin{bmatrix} [c_{kjrl}\partial_j\partial_l]_{3\times 3} & [-e_{jkl}\partial_j\partial_l]_{3\times 1} & [-q_{jkl}\partial_j\partial_l]_{3\times 1} & [0]_{3\times 1} \\ [e_{lrj}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & 0 \\ [q_{lrj}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & 0 \\ [\lambda_{rj}\partial_j]_{1\times 3} & p_j\partial_j & m_j\partial_j & \eta_{jl}\partial_j\partial_l \end{bmatrix}_{6\times 6}. \quad (4.1)$$

The corresponding matrix of fundamental solutions $\Gamma^*(x-y) = [\Gamma(y-x)]^\top$ has the following property at infinity

$$\Gamma^*(x-y) = \Gamma^\top(y-x) := \begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{5\times 5} & [0]_{5\times 1} \\ [\mathcal{O}(1)]_{1\times 5} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{6\times 6}$$

as $|x| \to \infty$. With the help of the fundamental matrix $\Gamma^*(x-y)$ we construct the single and double layer potentials, and the Newtonian volume potentials

$$V^*(h^*)(x) \equiv \int\limits_S \Gamma^*(x-y)h^*(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \quad (4.2)$$

$$W^*(h^*)(x) \equiv \int\limits_S \big[\mathcal{T}(\partial_y, n(y))[\Gamma^*(x-y)]^\top\big]^\top h^*(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \quad (4.3)$$

$$N^*_{\Omega^\pm}(g^*)(x) \equiv \int\limits_{\Omega^\pm} \Gamma^*(x-y)g^*(y)\,dy, \quad x \in \mathbb{R}^3,$$

where the density vector $h^* = (h^*_1, \dots, h^*_6)^\top$ is defined on S, while $g^* = (g^*_1, ..., g^*_6)^\top$ is defined in $\Omega^\pm$. We assume that in the case of the domain Ω^- the vector g^* has a compact support.

It can be shown that the layer potentials V^* and W^* possess exactly the same mapping properties and jump relations as the potentials V and W (see Theorems 2.5–2.8). In particular,

$$\{V^*(h^*)\}^+ = \{V^*(h^*)\}^- = \mathcal{H}^* h^*,$$
$$\{W^*(h^*)\}^\pm = \pm\, 2^{-1}\, h^* + \mathcal{K}^* h^*, \tag{4.4}$$
$$\big\{\mathcal{P}V^*(h^*)\big\}^\pm = \mp\, 2^{-1}\, h^* + \mathcal{N}^* h^*, \tag{4.5}$$

where $\mathcal{H}^*$ is a weakly singular integral operator, while $\mathcal{K}^*$ and $\mathcal{N}^*$ are singular integral operators,

$$\begin{aligned} \mathcal{H}^* h^*(x) &:= \int\limits_S \Gamma^*(x-y) h^*(y)\, dS_y, \\ \mathcal{K}^* h^*(x) &:= \int\limits_S \big[\mathcal{T}(\partial_y, n(y))[\Gamma^*(x-y)]^\top\big]^\top h^*(y)\, dS_y, \\ \mathcal{N}^* h^*(x) &:= \int\limits_S [\mathcal{P}(\partial_x, n(x))\Gamma^*(x-y)] h^*(y)\, dS_y. \end{aligned} \tag{4.6}$$

Now we introduce a special class of vector functions which is a counterpart of the class $Z(\Omega^-)$.

Definition 4.1. We say that a continuous vector function $U^* = (u^*, \varphi^*, \psi^*, \vartheta^*)^\top$ has the property $Z^*(\Omega^-)$ in the domain Ω^-, if the following conditions are satisfied

$$\widetilde{U}^*(x) = \big(u^*(x), \varphi^*(x), \psi^*(x)\big)^\top = \mathcal{O}(|x|^{-1}) \ \text{ as } \ |x| \to \infty,$$
$$\vartheta^*(x) = \mathcal{O}(1) \text{ as } |x| \to \infty,$$
$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma_R} \vartheta^*(x)\, d\Sigma_R = 0,$$

where Σ_R is a sphere centered at the origin and radius R.

As in the case of usual layer potentials here we have the following

Theorem 4.2. The generalized single and double layer potentials, defined by (4.2) and (4.3), solve the homogeneous differential equation $A^*(\partial)U^* = 0$ in $\mathbb{R}^3 \setminus S$ and possess the property $Z^*(\Omega^-)$.

For an arbitrary regular solution to the equation $A^*(\partial)U^*(x) = 0$ in Ω^+ one can derive the following integral representation formula

$$W^*(\{U^*\}^+)(x) - V^*\big(\{\mathcal{P}U^*\}^+\big)(x) = \begin{cases} U^*(x) & \text{for } \ x \in \Omega^+, \\ 0 & \text{for } \ x \in \Omega^-. \end{cases} \tag{4.7}$$

Similar representation formula holds also for an arbitrary regular solution to the equation $A^*(\partial)U^*(x)=0$ in Ω^- which possesses the property $Z^*(\Omega^-)$:

$$-W^*\big(\{U^*\}_S^-\big)(x)+V^*\big(\{\mathcal{P}U^*\}_S^-\big)(x)=\begin{cases}U^*(x), & x\in\Omega^-,\\ 0, & x\in\Omega^+.\end{cases} \tag{4.8}$$

To derive this representation we denote $\Omega_R^- := B(0,R)\backslash\overline{\Omega^+}$, where $B(0,R)$ is a ball centered at the origin and radius R. Then in view of (4.7) we have

$$U^*(x)=-W_S^*\big(\{U^*\}_S^-\big)(x)+V_S^*\big(\{\mathcal{P}U^*\}_S^-\big)(x)+\Phi_R^*(x),\ \ x\in\Omega_R^-, \tag{4.9}$$

$$0=-W_S^*\big(\{U^*\}_S^-\big)(x)+V_S^*\big(\{\mathcal{P}U^*\}_S^-\big)(x)+\Phi_R^*(x),\ \ x\in\Omega^+, \tag{4.10}$$

where

$$\Phi_R^*(x):=W_{\Sigma_R}^*\big(U^*\big)(x)-V_{\Sigma_R}^*\big(\mathcal{P}U^*\big)(x). \tag{4.11}$$

Here $V_{\mathcal{M}}^*$ and $W_{\mathcal{M}}^*$ denote the single and double layer potential operators with integration surface $\mathcal{M}$. Evidently

$$A^*(\partial)\Phi_R^*(x)=0,\ \ |x|<R. \tag{4.12}$$

In turn, from (4.9) and (4.10) we get

$$\begin{aligned}&\Phi_R^*(x)=U^*(x)+W_S^*\big(\{U^*\}_S^-\big)(x)-V_S^*\big(\{\mathcal{P}U^*\}_S^-\big)(x),\ \ x\in\Omega_R^-,\\ &\Phi_R^*(x)=W_S^*\big(\{U^*\}_S^-\big)(x)-V_S^*\big(\{\mathcal{P}U^*\}_S^-\big)(x),\ \ x\in\Omega^+,\end{aligned} \tag{4.13}$$

whence the equality $\Phi_{R_1}^*(x)=\Phi_{R_2}^*(x)$ follows for $|x|<R_1<R_2$. We assume that R_1 and R_2 are sufficiently large numbers. Therefore, for an arbitrary fixed point $x\in\mathbb{R}^3$ the following limit exists

$$\Phi^*(x):=\lim_{R\to\infty}\Phi_R^*(x)=$$

$$=\begin{cases}U^*(x)+W_S^*\big(\{U^*\}_S^-\big)(x)-V_S^*\big(\{\mathcal{P}U^*\}_S^-\big)(x), & x\in\Omega^-,\\ W_S^*\big(\{U^*\}_S^-\big)(x)-V_S^*\big(\{\mathcal{P}U^*\}_S^-\big)(x), & x\in\Omega^+,\end{cases} \tag{4.14}$$

and $A^*(\partial)\Phi^*(x)=0$ for all $x\in\Omega^+\cup\Omega^-$. On the other hand, for arbitrary fixed point $x\in\mathbb{R}^3$ and a number R_1, such that $|x|<R_1$ and $\overline{\Omega^+}\subset B(0,R_1)$, from (4.13) we have

$$\Phi^*(x)=\lim_{R\to\infty}\Phi_R^*(x)=\Phi_{R_1}^*(x).$$

Now from (4.11)–(4.12) we deduce

$$A^*(\partial)\Phi^*(x)=0\ \ \forall\, x\in\mathbb{R}^3. \tag{4.15}$$

Since U^*, W^*, $V^*\in Z^*(\Omega^-)$ we conclude from (4.14) that $\Phi^*(x)\in Z^*(\mathbb{R}^3)$. In particular, we have

$$\lim_{R\to\infty}\frac{1}{4\pi R^2}\int\limits_{\Sigma_R}\Phi^*(x)\,d\Sigma_R=0. \tag{4.16}$$

Our goal is to show that

$$\Phi^*(x)=0\ \ \forall\, x\in\mathbb{R}^3.$$

Applying the generalized Fourier transform to equation (4.15) we get

$$A^*(-i\xi)\widehat{\Phi}^*(\xi) = 0, \ \ \xi \in \mathbb{R}^3,$$

where $\widehat{\Phi}^*(\xi)$ is the Fourier transform of Φ^*. Taking into account that $\det A^*(-i\xi) \neq 0$ for all $\xi \in \mathbb{R}^3 \setminus \{0\}$, we conclude that the support of the generalized functional $\widehat{\Phi}^*(\xi)$ is the origin and consequently

$$\widehat{\Phi}^*(\xi) = \sum_{|\alpha|\leq M} c_\alpha \delta^{(\alpha)}(\xi),$$

where α is a multi-index, c_α are arbitrary constant vectors and M is some nonnegative integer. Then it follows that $\Phi^*(x)$ is polynomial in x and due to the inclusion $\Phi^* \in Z^*(\Omega^-)$, $\Phi^*(x)$ is bounded at infinity, i.e., $\Phi^*(x) = const$ in $\mathbb{R}^3$. Therefore (4.16) implies that $\Phi^*(x)$ vanishes identically in $\mathbb{R}^3$. This proves that the formula (4.8) holds.

Theorem 4.3. Let $S \in C^{2,\kappa}$ and $h \in \big[C^{1,\kappa'}(S)\big]^6$ with $0 < \kappa' < \kappa \leq 1$. Then for the double layer potential W^* defined by (4.3) there holds the following formula (generalized Lyapunov–Tauber relation)

$$\big\{\mathcal{P}W^*(h)\big\}^+ = \big\{\mathcal{P}W^*(h)\big\}^- \ \text{ on } \ S, \tag{4.17}$$

where the operator $\mathcal{P}$ is given by (2.18).

For $h \in [H_2^{\frac{1}{2}}(S)]^6$ the relation (4.17) also holds in the space $[H_2^{-\frac{1}{2}}(S)]^6$.

Proof. Since $h \in \big[C^{1,\kappa'}(S)\big]^6$, evidently $U^* := W^*(h) \in [C^{1,\kappa'}(\overline{\Omega^\pm})]^6$. It is clear that the vector U^* is a solution of the homogeneous equation $A^*(\partial)U^*(x) = 0$ in $\Omega^+ \cup \Omega^-$, where the operator $A^*(\partial)$ is defined by (4.1). With the help of (4.7) and (4.8), for the vector function U^* we derive the following representation formula

$$U^*(x) = W^*([U^*]_S)(x) - V^*\big([\mathcal{P}U^*]_S\big)(x), \ \ x \in \Omega^+ \cup \Omega^-, \tag{4.18}$$

where

$$[U^*]_S \equiv \{U^*\}^+ - \{U^*\}^- \ \text{ and } \ [\mathcal{P}U^*]_S \equiv \{\mathcal{P}U^*\}^+ - \{\mathcal{P}U^*\}^- \ \text{ on } \ S.$$

In view of the equality $U^* = W^*(h)$, from (4.18) we get

$$W^*(h)(x) = W^*([W^*(h)]_S)(x) - V^*([\mathcal{P}W^*(h)]_S)(x), \ \ x \in \Omega^+ \cup \Omega^-.$$

Using the jump relation (4.4), we find

$$[U^*]_S = [W^*(h)]_S = \{W^*(h)\}^+ - \{W^*(h)\}^- = h.$$

Therefore

$$W^*(h)(x) = W^*(h)(x) - V^*([\mathcal{P}W^*(h)]_S)(x), \ \ x \in \Omega^+ \cup \Omega^-,$$

i.e., $V^*(\Phi^*)(x) = 0$ in $\Omega^+ \cup \Omega^-$, where $\Phi^* := [\mathcal{P}W^*(h)]_S$. With the help of the jump relation (4.5) finally we arrive at the equation

$$0 = \{\mathcal{P}V^*(\Phi^*)\}^- - \{\mathcal{P}V^*(\Phi^*)\}^+ =$$
$$= \Phi^* = [\mathcal{P}W^*(h)]_S = \{\mathcal{P}W^*(h)\}^+ - \{\mathcal{P}W^*(h)\}^-$$

on S, which completes the proof for the regular case.

The second part of the theorem can be proved by standard limiting procedure. □

Let us consider the interior and exterior homogeneous Dirichlet BVPs for the adjoint operator $A^*(\partial)$

$$A^*(\partial)U^* = 0 \ \text{ in } \ \Omega^{\pm}, \tag{4.19}$$

$$\{U^*\}^{\pm} = 0 \ \text{ on } \ S. \tag{4.20}$$

In the case of the interior problem, we assume that either U^* is a regular vector of the class $[C^{1,\kappa'}(\overline{\Omega^+})]^6$ or $U^* \in [W_2^1(\Omega^+)]^6$, while in the case of the exterior problem, we assume that either $U^* \in [C^{1,\kappa'}(\overline{\Omega^-})]^6 \cap Z^*(\Omega^-)$ or $U^* \in [W^1_{2,loc}(\Omega^-)]^6 \cap Z^*(\Omega^-)$.

Theorem 4.4. The interior and exterior homogeneous Dirichlet type BVPs (4.19)–(4.20) have only the trivial solution in the appropriate spaces.

Proof. First we treat the exterior Dirichlet problem. In view of the structure of the operator $A^*(\partial)$, it is easy to see that we can consider separately the BVP for the vector function $\widetilde{U}^* = (u^*, \varphi^*, \psi^*)^\top$, constructed by the first five components of the solution vector U^*,

$$\widetilde{A}^*(\partial)\widetilde{U}^*(x) = 0, \ \ x \in \Omega^-, \tag{4.21}$$

$$\{\widetilde{U}^*(x)\}^- = 0, \ \ x \in S, \tag{4.22}$$

where $\widetilde{A}^*(\partial)$ is the 5×5 matrix differential operator, obtained from $A^*(\partial)$ by deleting the sixth column and the sixth row,

$$\widetilde{A}^*(\partial) := \begin{bmatrix} [c_{kjrl}\partial_j\partial_l]_{3\times 3} & [-e_{jkl}\partial_j\partial_l]_{3\times 1} & [-q_{jkl}\partial_j\partial_l]_{3\times 1} \\ [e_{lrj}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l \\ [q_{lrj}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l \end{bmatrix}_{5\times 5}. \tag{4.23}$$

With the help of Green's identity in $\Omega_R^- = B(0,R)\backslash\overline{\Omega^+}$, we have

$$\int\limits_{\Omega_R^-} \Big[\widetilde{U}^* \cdot \widetilde{A}^*(\partial)\widetilde{U}^* + \widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*)\Big]\, dx =$$

$$= -\int\limits_S \{\widetilde{U}^*\}^- \cdot \{\widetilde{P}(\partial,n)\widetilde{U}^*\}^- \, dS + \int\limits_{\Sigma_R} \widetilde{U}^* \cdot \widetilde{P}(\partial,n)\widetilde{U}^* \, d\Sigma_R, \tag{4.24}$$

where

$$\widetilde{\mathcal{P}}(\partial,n) := \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [-e_{lrj}n_j\partial_l]_{3\times 1} & [-q_{lrj}n_j\partial_l]_{3\times 1} \\ [e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l \\ [q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l \end{bmatrix}_{5\times 5}, \tag{4.25}$$

and

$$\widetilde{\mathcal{E}}(\widetilde{U}^*,\widetilde{U}^*) = c_{rjkl}\partial_l u_k^* \partial_j u_r^* + \varkappa_{jl}\partial_l\varphi^*\partial_j\varphi^* + \\ + a_{jl}(\partial_l\varphi^*\partial_j\psi^* + \partial_j\psi^*\partial_l\varphi^*) + \mu_{jl}\partial_l\psi^*\partial_j\psi^*. \quad (4.26)$$

Due to the fact that U^* has the property $Z^*(\Omega^-)$, it follows that $\widetilde{U}^* = \mathcal{O}(|x|^{-1})$ and $\partial_j\widetilde{U}^* = \mathcal{O}(|x|^{-2})$ as $|x|\to\infty$, $j=1,2,3$. Therefore,

$$\Big|\int\limits_{\Sigma_R} \widetilde{U}^*\cdot\widetilde{P}(\partial,n)\widetilde{U}^*\,d\Sigma_R\Big| \leq \\ \leq \int\limits_{\Sigma_R}\frac{C}{R^3}\,d\Sigma_R = \frac{C}{R^3}\,4\pi R^2 = \frac{4\pi C}{R} \to 0 \ \text{ as } \ R\to\infty. \quad (4.27)$$

Taking into account that $\widetilde{\mathcal{E}}(\widetilde{U}^*,\widetilde{U}^*) \geq 0$, applying the relations (4.21), (4.22), and (4.27), from (4.24) we conclude that $\widetilde{\mathcal{E}}(\widetilde{U}^*,\widetilde{U}^*) = 0$. Hence in view of (2.10)-(2.11) it follows that $\widetilde{U}^* = (a\times x + b, b_4, b_5)$, where a and b are arbitrary constant vectors, and b_4 and b_5 are arbitrary scalar constants. Here the symbol $\times$ denotes the cross product operation. Due to the boundary condition (4.22) we get then $a=b=0$ and $b_4=b_5=0$, from which we derive that $\widetilde{U}^*=0$. Since $\widetilde{U}^*$ vanishes in Ω^-, from (4.19)–(4.20) we arrive at the following boundary-value problem for ϑ^*,

$$\begin{aligned} \eta_{kj}\partial_k\partial_j\vartheta^* &= 0 \ \text{ in } \ \Omega^-, \\ \{\vartheta^*\}^- &= 0 \ \text{ on } \ S. \end{aligned} \quad (4.28)$$

From boundedness of ϑ^* at infinity and from (4.28) one can derive that $\vartheta^*(x) = C + \mathcal{O}(|x|^{-1})$, where C is an arbitrary constant. In view of $U^*\in Z^*(\Omega^-)$ we have $C=0$ and $\vartheta^*(x) = \mathcal{O}(|x|^{-1})$, $\partial_j\vartheta^*(x) = \mathcal{O}(|x|^{-2})$, $j=1,2,3$. Therefore we can apply Green's formula

$$\int\limits_{\Omega_R^-}\Big[\vartheta^*\,\eta_{kj}\partial_k\partial_j\vartheta^* + \eta_{kj}\partial_k\vartheta^*\,\partial_j\vartheta^*\Big]\,dx = \\ = -\int\limits_S \{\vartheta^*\}^-\big\{\eta_{kj}n_k\partial_j\vartheta^*\big\}^-\,dS + \int\limits_{\Sigma_R}\vartheta^*\,\eta_{kj}n_k\partial_j\vartheta^*\,d\Sigma_R.$$

Passing to the limit as $R\to\infty$, we get

$$\int\limits_{\Omega^-}\eta_{kj}\partial_k\vartheta^*\partial_j\vartheta^*\,dx = 0.$$

Using the fact that the matrix $[\eta_{kj}]_{3\times 3}$ is positive definite, we conclude that $\vartheta^* = C_1 = const$ and since $\vartheta^*(x) = \mathcal{O}(|x|^{-1})$ as $|x|\to\infty$, finally we get that $\vartheta^*=0$ in Ω^-. Thus $U^*=0$ in Ω^- which completes the proof for the exterior problem.

The interior problem can be treated quite similarly. □

4.2. **Investigation of the interior Neumann BVP.** First let us treat the uniqueness question. To this end we consider the homogeneous interior Neumann-type BVP

$$A(\partial)U(x)=0, \quad x\in\Omega^+, \tag{4.29}$$

$$\big\{\mathcal{T}(\partial,n)U(x)\big\}^+=0, \quad x\in S=\partial\Omega^+. \tag{4.30}$$

It can be shown that a general solution to the problem (4.29)-(4.30) can be represented in the form (for details see [20])

$$U=\sum_{k=1}^{9} C_k U^{(k)} \ \text{ in } \ \Omega^+, \tag{4.31}$$

where C_k are arbitrary scalar constants and $\{U^{(k)}\}_{k=1}^{9}$ is the basis in the space of solution vectors of the homogeneous problem (4.29)–(4.30). They can be constructed explicitly and read as

$$U^{(k)}=\big(\widetilde{V}^{(k)},0\big)^\top, \ \ k=\overline{1,8}, \quad U^{(9)}=\big(\widetilde{V}^{(9)},1\big)^\top, \tag{4.32}$$

where $U^{(k)}=(u^{(k)},\varphi^{(k)},\psi^{(k)},\vartheta^{(k)})^\top$, $\widetilde{V}^{(k)}=(u^{(k)},\varphi^{(k)},\psi^{(k)})^\top$,

$$\widetilde{V}^{(1)}=(0,-x_3,x_2,0,0)^\top, \quad \widetilde{V}^{(2)}=(x_3,0,-x_1,0,0)^\top,$$
$$\widetilde{V}^{(3)}=(-x_2,x_1,0,0,0)^\top, \quad \widetilde{V}^{(4)}=(1,0,0,0,0)^\top,$$
$$\widetilde{V}^{(5)}=(0,1,0,0,0)^\top, \quad \widetilde{V}^{(6)}=(0,0,1,0,0)^\top,$$
$$\widetilde{V}^{(7)}=(0,0,0,1,0)^\top, \quad \widetilde{V}^{(8)}=(0,0,0,0,1)^\top,$$

and $\widetilde{V}^{(9)}$ is defined as

$$\widetilde{V}^{(9)}=(u^{(9)},\varphi^{(9)},\psi^{(9)})^\top, \quad u^{(9)}_k=b_{kq}x_q, \ \ k=1,2,3,$$
$$\varphi^{(9)}=c_qx_q, \quad \psi^{(9)}=d_qx_q,$$

with the twelve coefficients $b_{kq}=b_{qk}$, c_q and d_q, $k,q=1,2,3$, defined by the uniquely solvable linear algebraic system of equations

$$c_{rjkl}b_{kl}+e_{lrj}c_l+q_{lrj}d_l=\lambda_{rj}, \ \ r,j=1,2,3,$$
$$-e_{jkl}b_{kl}+\varkappa_{jl}c_l+a_{jl}d_l=p_j, \ \ j=1,2,3,$$
$$-q_{jkl}b_{kl}+a_{jl}c_l+\mu_{jl}d_l=m_j, \ \ j=1,2,3.$$

From (4.31) it follows that U can be alternatively written as

$$U=(\widetilde{V},0)^\top+b_6(\widetilde{V}^{(9)},1)^\top$$

with $\widetilde{V}=(a\times x+b,b_4,b_5)^\top$, where $a=(a_1,a_2,a_3)^\top$ and $b=(b_1,b_2,b_3)^\top$ are arbitrary constant vectors and b_4, b_5, b_6 are arbitrary scalar constants.

Now, we start the investigation of the non-homogeneous interior Neumann-type BVP

$$A(\partial)U(x)=0, \quad x\in\Omega^+, \tag{4.33}$$

$$\big\{\mathcal{T}(\partial,n)U(x)\big\}^+=F(x), \quad x\in S, \tag{4.34}$$

where $U \in [C^{1,\kappa'}(\overline{\Omega^+})]^6 \cap [C^2(\Omega^+)]^6$ is a sought for vector and $F \in [C^{0,\kappa'}(S)]^6$ is a given vector-function. It is clear that if the problem (4.33)–(4.34) is solvable, then a solution is defined within a summand vector of type (4.31).

We look for a solution to the problem (4.33)–(4.34) in the form of the single layer potential,

$$U(x) = V(h)(x), \quad x \in \Omega^+, \tag{4.35}$$

where $h = (h_1, \dots, h_6)^\top \in [C^{0,\kappa'}(S)]^6$ is an unknown density. From the boundary condition (4.34) and by virtue of the jump relation (2.21) (see Theorem 2.5) we get the following integral equation for the density vector h

$$[-2^{-1} I_6 + \mathcal{K}]h = F \ \text{ on } \ S, \tag{4.36}$$

where $\mathcal{K}$ is a singular integral operator defined by (2.24). Note that $-2^{-1}I_6 + \mathcal{K}$ is a singular integral operator of normal type with index zero (cf. [23]). Now we investigate the null space $\operatorname{Ker}(-2^{-1}I_6 + \mathcal{K})$. To this end, we consider the homogeneous equation

$$[-2^{-1} I_6 + \mathcal{K}]h = 0 \ \text{ on } \ S \tag{4.37}$$

and assume that a vector $h^{(0)}$ is a solution to (4.37), i.e., $h^{(0)} \in \operatorname{Ker}(-2^{-1}I_6 + \mathcal{K})$. Since $h^{(0)} \in [C^{0,\kappa'}(S)]^6$, it is evident that the corresponding single layer potential $U_0(x) = V(h^{(0)})(x)$ belongs to the space of regular vector functions and solves the homogeneous equation $A(\partial)U_0(x) = 0$ in Ω^+. Moreover, $\{\mathcal{T}(\partial, n)U_0(x)\}^+ = -2^{-1}h^{(0)} + \mathcal{K}h^{(0)} = 0$ on S due to (4.37), i.e., $U_0(x)$ solves the homogeneous interior Neumann problem. Therefore, in accordance to the above results, we can write $U_0(x) = \sum\limits_{k=1}^{9} C_k U^{(k)}(x)$ in Ω^+, where C_k, $k = \overline{1,9}$, are some constants, and the vectors $U^{(k)}(x)$ are defined by (4.32). Hence we have

$$V(h^{(0)})(x) = \sum_{k=1}^{9} C_k U^{(k)}(x), \quad x \in \Omega^+.$$

If we take into account the jump relation (2.20), we derive that

$$\big\{V(h^{(0)})(x)\big\}^+ \equiv \mathcal{H}(h^{(0)})(x) = \sum_{k=1}^{9} C_k U^{(k)}(x), \quad x \in S. \tag{4.38}$$

The operators

$$\begin{aligned} \mathcal{H} : [H^{-\frac{1}{2}}(S)]^6 &\to [H^{\frac{1}{2}}(S)]^6, \\ : [C^{0,\kappa'}(S)]^6 &\to [C^{1,\kappa'}(S)]^6 \end{aligned}$$

are invertible ([19], [23]). Therefore from (4.38) we obtain

$$h^{(0)} = \sum_{k=1}^{9} C_k h^{(k)}(x), \quad x \in S,$$

with

$$h^{(k)} := \mathcal{H}^{-1}(U^{(k)}), \quad k = \overline{1,9}. \tag{4.39}$$

Further we show that the system of vectors $\{h^{(k)}\}_{k=1}^{9}$ is linearly independent. Let us assume the opposite. Then there exist constants c_k, $k = \overline{1,9}$, such that $\sum\limits_{k=1}^{9} |c_k| \neq 0$ and the following equation

$$\sum_{k=1}^{9} c_k h^{(k)} = 0 \ \text{ on } \ S$$

holds, i.e., $\sum\limits_{k=1}^{9} c_k \mathcal{H}^{-1}(U^{(k)}) = 0$ on S. Hence we get

$$\mathcal{H}^{-1}\Big(\sum_{k=1}^{9} c_k U^{(k)}\Big) = 0 \ \text{ on } \ S,$$

and, consequently,

$$\sum_{k=1}^{9} c_k U^{(k)}(x) = 0, \quad x \in S. \tag{4.40}$$

Now consider the vector

$$U^*(x) \equiv \sum_{k=1}^{9} c_k U^{(k)}(x), \quad x \in \Omega^+.$$

Since the vectors $U^{(k)}$ are solutions of the homogeneous equation (4.33), in view of (4.40) we have

$$A(\partial)U^*(x) = 0, \quad x \in \Omega^+,$$

$$\{U^*(x)\}^+ = \bigg\{\sum_{k=1}^{9} c_k U^{(k)}(x)\bigg\}^+ = 0, \quad x \in S.$$

That is, U^* is a solution of the homogeneous interior Dirichlet problem and in accordance with the uniqueness theorem for the interior Dirichlet BVP we conclude $U^*(x) = 0$ in Ω^+, i.e.,

$$\sum_{k=1}^{9} c_k U^{(k)}(x) = 0, \quad x \in \Omega^+.$$

This contradicts to linear independence of the system $\{U^{(k)}\}_{k=1}^{9}$. Thus, the system of the vectors $\{h^{(k)}\}_{k=1}^{9}$ is linearly independent which implies that

$$\dim \operatorname{Ker}(-2^{-1}I_6 + \mathcal{K}) \geq 9.$$

Next we show that

$$\dim \operatorname{Ker}(-2^{-1}I_6 + \mathcal{K}) \leq 9.$$

Let the equation $(-2^{-1}I_6 + \mathcal{K})h = 0$ have a solution $h^{(10)}$ which is not representable in the form of a linear combination of the system $\{h^{(k)}\}_{k=1}^{9}$. Then the system $\{h^{(k)}\}_{k=1}^{10}$ is linearly independent. It is easy to show that the system of the corresponding single layer potentials $V^{(k)}(x) := V(h^{(k)})(x)$, $k = \overline{1,10}$, $x \in \Omega^+$, is linearly independent as well. Indeed, let us assume the opposite. Then there are constants a_k, such that

$$U(x) := \sum_{k=1}^{10} a_k V^{(k)}(x) = 0, \quad x \in \Omega^+, \tag{4.41}$$

with $\sum\limits_{k=1}^{10} |a_k| \neq 0$. From (4.41) we then derive that $\{U(x)\}^+ = 0$, $x \in S$. Therefore,

$$\{U\}^+ = \sum_{k=1}^{10} a_k \{V^{(k)}\}^+ = \sum_{k=1}^{10} a_k \mathcal{H}(h^{(k)}) = \mathcal{H}\Big(\sum_{k=1}^{10} a_k h^{(k)}\Big) = 0 \text{ on } S.$$

Whence, due to the invertibility of the operator $\mathcal{H}$, we get

$$\sum_{k=1}^{10} a_k h^{(k)} = 0 \text{ on } S.$$

which contradicts to the linear independence of the system $\{h^{(k)}\}_{k=1}^{10}$.

Thus the system $\{V(h^{(k)})(x)\}_{k=1}^{10}$ is linearly independent.

On the other hand, we have

$$A(\partial)V^{(k)}(x) = 0, \quad x \in \Omega^+,$$
$$\big\{\mathcal{T}V^{(k)}\big\}^+ = (-2^{-1}I_6 + \mathcal{K})h^{(k)} = 0, \quad x \in S,$$

since $h^{(k)}$, $k = \overline{1,10}$, are solutions to the homogeneous equation (4.37). Therefore, the vectors $V^{(k)}$, $k = \overline{1,10}$, are solutions to the homogeneous interior Neumann-type BVP and they can be expressed by linear combinations of the vectors $U^{(j)}$, $j = \overline{1,9}$, defined in (4.32). Whence it follows that the system $\{V^{(k)}\}_{k=1}^{10}$ is linearly dependent and so is the system $\{h^{(k)}\}_{k=1}^{10}$ for an arbitrary solution $h^{(10)}$ of the equation (4.37). Consequently, $\dim \operatorname{Ker}(-2^{-1}I_6 + \mathcal{K}) \leq 9$ implying that $\dim \operatorname{Ker}(-2^{-1}I_6 + \mathcal{K}) = 9$. We can consider the system $h^{(1)}, \dots, h^{(9)}$ defined in (4.39) as basis vectors of the null space of the operator $-2^{-1}I_6 + \mathcal{K}$. If h_0 is a particular solution to the nonhomogeneous integral equation (4.36), then a general solution of the same equation is represented as

$$h = h_0 + \sum_{k=1}^{9} c_k h^{(k)},$$

where c_k are arbitrary constants.

For our further analysis we need also to study the homogeneous interior Neumann-type BVP for the adjoint operator $A^*(\partial)$, which reads as follows

$$A^*(\partial)U^* = 0 \text{ in } \Omega^+, \tag{4.42}$$
$$\{\mathcal{P}U^*\}^+ = 0 \text{ on } S = \partial\Omega^+; \tag{4.43}$$

here the adjoint operator $A^*(\partial)$ and the boundary operator $\mathcal{P}$ are defined by (4.1) and (2.18) respectively.

Note that in the case of the problem (4.42)–(4.43) we get also two separated problems:

a) For the vector function $\widetilde{U}^* \equiv (u^*, \varphi^*, \psi^*)^\top$,

$$\widetilde{A}^*(\partial)\widetilde{U}^* = 0 \ \text{ in } \ \Omega^+, \tag{4.44}$$

$$\big\{\widetilde{\mathcal{P}}\widetilde{U}^*\big\}^+ = 0 \ \text{ on } \ S, \tag{4.45}$$

where $\widetilde{A}^*$ and $\widetilde{\mathcal{P}}$ are defined by (4.23) and (4.25) respectively, and

b) For the function $U_6^* \equiv \vartheta^*$

$$\lambda_{rj}\partial_j u_r^* + p_j\partial_j\varphi^* + m_j\partial_j\psi^* + \eta_{jl}\partial_j\partial_l\vartheta^* = 0 \ \text{ in } \ \Omega^+, \tag{4.46}$$

$$\eta_{jl}n_j\partial_l\vartheta^* = 0 \ \text{ on } \ S. \tag{4.47}$$

For a regular solution vector $\widetilde{U}^*$ of the problem (4.44)–(4.45) we can write the following Green's identity

$$\int\limits_{\Omega^+} \Big[\widetilde{U}^* \cdot \widetilde{A}^*(\partial)\widetilde{U}^* + \widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*)\Big]\,dx = \int\limits_{\partial\Omega^+} \{\widetilde{U}^*\}^+ \cdot \big\{\widetilde{\mathcal{P}}(\partial, n)\widetilde{U}^*\big\}^+\,dS, \tag{4.48}$$

where $\widetilde{\mathcal{E}}$ is given by (4.26). If we take into account the conditions (4.44)–(4.45), from (4.48) we get

$$\int\limits_{\Omega^+} \widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*)\,dx = 0.$$

Hence we have that $\partial_j\varphi^* = 0$, $\partial_j\psi^* = 0$, $j = 1, 2, 3$, and $\partial_l u_k^* + \partial_j u_r^* = 0$ in Ω^+. Therefore, $u^*(x) = a \times x + b$ is a rigid displacement vector, $\varphi^* = b_4$ and $\psi^* = b_5$ are arbitrary constants in Ω^+. It is evident that

$$\lambda_{rj}\partial_j u_r^* = \frac{1}{2}\,\lambda_{rj}(\partial_j u_r^* + \partial_r u_j^*) = 0$$

and $p_j\partial_j\varphi^* = m_j\partial_j\psi^* = 0$. Then from (4.46)–(4.47) we get the following BVP for the scalar function ϑ^*,

$$\eta_{jl}\partial_j\partial_l\vartheta^* = 0 \ \text{ in } \ \Omega^+,$$
$$\eta_{jl}n_j\partial_l\vartheta^* = 0 \ \text{ on } \ S.$$

Using the following Green's identity

$$\int\limits_{\Omega^+}\eta_{jl}\partial_j\partial_l\vartheta^*\,\vartheta^*\,dx = -\int\limits_{\Omega^+}\eta_{jl}\partial_l\vartheta^*\,\partial_j\vartheta^*\,dx + \int\limits_{\partial\Omega^+}\{\eta_{jl}n_j\partial_l\vartheta^*\}^+\{\partial_j\vartheta^*\}^+\,dS,$$

we find

$$\int\limits_{\Omega^+} \eta_{jl}\partial_l\vartheta^*\,\partial_j\vartheta^*\,dx = 0,$$

and by the positive definiteness of the matrix $[\eta_{jl}]_{3\times3}$ we get $\partial_j\vartheta^* = 0$, $j = \overline{1,3}$, in Ω^+, i.e., $\vartheta^* = b_6 = const$ in Ω^+. Consequently, a general solution $U^* = (u^*, \varphi^*, \psi^*, \vartheta^*)^\top$ of the adjoint homogeneous BVP (4.42)–(4.43) can be represented as

$$U^*(x) = \sum_{k=1}^{9} C_k U^{*(k)}(x), \quad x \in \Omega^+,$$

where C_k are arbitrary scalar constants, while

$$\begin{gathered} U^{*(1)} = (0, -x_3, x_2, 0, 0, 0)^\top, \quad U^{*(2)} = (x_3, 0, -x_1, 0, 0, 0)^\top, \\ U^{*(3)} = (-x_2, x_1, 0, 0, 0, 0)^\top, \quad U^{*(4)} = (1, 0, 0, 0, 0, 0)^\top, \\ U^{*(5)} = (0, 1, 0, 0, 0, 0)^\top, \quad U^{*(6)} = (0, 0, 1, 0, 0, 0)^\top, \\ U^{*(7)} = (0, 0, 0, 1, 0, 0)^\top, \quad U^{*(8)} = (0, 0, 0, 0, 1, 0)^\top, \\ U^{*(9)} = (0, 0, 0, 0, 0, 1)^\top. \end{gathered} \tag{4.49}$$

As we see, $U^{*(k)} = U^{(k)}$, $k = \overline{1, 8}$, where $U^{(k)}$, $k = \overline{1, 8}$, is given in (4.32). One can easily check that the system $\{U^{*(k)}\}_{k=1}^9$ is linearly independent. As a result we get the following

Proposition 4.5. The space of solutions of the adjoint homogeneous BVP (4.42)–(4.43) is nine dimensional and an arbitrary solution can be represented as a linear combination of the vectors $\{U^{*(k)}\}_{k=1}^9$, i.e., the system $\{U^{*(k)}\}_{k=1}^9$ is a basis in the space of solutions to the homogeneous BVP (4.42)–(4.43).

Now, we return to equation (4.36) and consider the corresponding homogeneous adjoint equation

$$(-2^{-1} I_6 + \mathcal{K}^*) h^* = 0 \ \text{ on } \ S,$$

where $\mathcal{K}^*$ is the adjoint operator to $\mathcal{K}$ defined by the duality relation,

$$(\mathcal{K}h, h^*)_{L_2(S)} = (h, \mathcal{K}^* h^*)_{L_2(S)}, \quad \forall h, h^* \in [L_2(S)]^6.$$

It is easy to show that the operator $\mathcal{K}^*$ is the same as the operator given by (4.6). In what follows we prove that $\dim \operatorname{Ker} \big(-\frac{1}{2} I_6 + \mathcal{K}^* \big) = 9$.

Indeed, in accordance with Proposition 4.5 we have that $A^*(\partial) U^{*(k)} = 0$ in Ω^+ and $\{\mathcal{P} U^{*(k)}\}^+ = 0$ on S. Therefore from (4.7) we have

$$U^{*(k)}(x) = W^*\big(\{U^{*(k)}\}^+\big)(x), \ \ x \in \Omega^+. \tag{4.50}$$

By the jump relations (4.4) we get

$$h^{*(k)} = 2^{-1} h^{*(k)} + \mathcal{K}^* h^{*(k)} \ \text{ on } \ S,$$

where

$$h^{*(k)} := \{U^{*(k)}\}^+, \ \ k = \overline{1, 9}. \tag{4.51}$$

Whence it follows that

$$\big(-2^{-1} I_6 + \mathcal{K}^* \big) h^{*(k)} = 0, \ \ k = \overline{1, 9}.$$

By Theorem 4.4 and the relations (4.50) and (4.51) we conclude that the system $\big\{h^{*(k)}\big\}_{k=1}^{9}$ is linearly independent, and therefore

$$\dim\operatorname{Ker}\big(-2^{-1}\,I_6+\mathcal{K}^*\big)\geq 9.$$

Now, let $h^{*(0)}\in\operatorname{Ker}\big(-\,2^{-1}\,I_6+\mathcal{K}^*\big)$, i.e., $\big(-\,2^{-1}\,I_6+\mathcal{K}^*\big)h^{*(0)}=0$. The corresponding double layer potential $U_0^*(x):=W^*(h^{*(0)})(x)$ is a solution to the homogeneous equation $A^*(\partial)U_0^*=0$ in Ω^+. Moreover, $\{W^*(h^{*(0)})\}^-=-2^{-1}\,h^{*(0)}+\mathcal{K}^*h^{*(0)}=0$ on S. Consequently, U_0^* is a solution of the homogeneous exterior Dirichlet BVP possessing the property $Z^*(\Omega^-)$. With the help of the uniqueness Theorem 4.4 we conclude that $W^*(h^{*(0)})=0$ in Ω^-. Further, $\big\{\mathcal{P}W^*(h^{*(0)})\big\}^+=\{\mathcal{P}W^*(h^{*(0)})\}^-=0$ due to Theorem 4.3, and for the vector function U_0^* we arrive at the following BVP,

$$A^*(\partial)U_0^*=0\ \text{ in }\ \Omega^+,$$
$$\big\{\mathcal{P}U_0^*\big\}^+=0\ \text{ on }\ S.$$

Using Proposition 4.5 we can write

$$U_0^*(x)=W^*(h^{*(0)})(x)=\sum_{k=1}^{9}c_kU^{*(k)}(x),\ \ x\in\Omega^+,$$

where c_k are some constants. The jump relation for the double layer potential then gives

$$\big\{W^*(h^{*(0)})(x)\big\}^+-\big\{W^*(h^{*(0)})(x)\big\}^-$$
$$=h^{*(0)}(x)=\sum_{k=1}^{9}c_k\big\{U^{*(k)}(x)\big\}^+=\sum_{k=1}^{9}c_kh^{*(k)}(x),\ \ x\in S,$$

which implies that the system $\big\{h^{*(k)}\big\}_{k=1}^{9}$ represents a basis of the null space $\operatorname{Ker}\big(-2^{-1}\,I_6+\mathcal{K}^*\big)$. Whence it follows that $\dim\operatorname{Ker}\big(-2^{-1}\,I_6+\mathcal{K}^*\big)=9$.

Now we can formulate the following basic existence theorem for the integral equation (4.36) and the interior Neumann-type BVP.

Theorem 4.6. Let $m\geq 0$ be a nonnegative integer and $0<\kappa'<\kappa\leq 1$. Further, let $S\in C^{m+1,\kappa}$ and $F\in[C^{m,\kappa'}(S)]^6$. The necessary and sufficient conditions for the integral equation (4.36) and the interior Neumann-type BVP (4.33)–(4.34) to be solvable read as

$$\int\limits_S F(x)\cdot h^{*(k)}(x)\,dS=0,\ \ k=\overline{1,9},\tag{4.52}$$

where the system $\{h^{*(k)}\}_{k=1}^{9}$ is defined explicitly by (4.51) and (4.49).

If these conditions are satisfied, then a solution vector to the interior Neumann-type BVP is representable by the single layer potential (4.35), where the density vector $h\in[C^{m,\kappa'}(S)]^6$ is defined by the integral equation (4.36).

A solution vector function $U \in [C^{m+1,\kappa'}(\overline{\Omega^+})]^6$ is defined modulo a linear combination of the vector functions $\{U^{(k)}\}_{k=1}^9$ given by (4.32).

Remark 4.7. Similar to the exterior problem, if S is a Lipschitz surface, $F \in \left[H^{-1/2}(S)\right]^6$, and the conditions (4.52) is fulfilled, then

(i) the integral equation (4.36) is solvable in the space $\left[H^{-1/2}(S)\right]^6$;

(ii) the interior Neumann-type BVP (4.33)-(4.34) is solvable in the space $\left[H_2^1(\Omega^+)\right]^6$ and solutions are representable by the single layer potential (4.35), where the density vector $h \in \left[H^{-1/2}(S)\right]^6$ solves the integral equation (4.36);

(iii) A solution $U \in \left[H_2^1(\Omega^+)\right]^6$ to the interior Neumann-type BVP (4.33)-(4.34) is defined modulo a linear combination of the vector functions $\left\{U^{(k)}\right\}_{k=1}^9$ given by (4.32).

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PSEUDODIFFERENTIAL OPERATORS WITH OPERATOR VALUED SYMBOLS. FREDHOLM THEORY AND EXPONENTIAL ESTIMATES OF SOLUTIONS

Dedicated to 120 Birthday of Academician,
Professor N. Muskhelishvili

Abstract. We consider a class of pseudodifferential operators with operator-valued symbols $a = a(x,\xi)$ having power growth with respect to the variables x and ξ. Moreover we consider the symbols analytically extended with respect to ξ onto a tube domain in $\mathbb{C}^n$ with a base being a ball in $\mathbb{R}^n$ with a radius depending on the variable x.

The main results of the paper are the Fredholm theory of pseudodifferential operators with operator valued symbols and exponential estimates at infinity of solutions of pseudodifferential equations $Op(a)u = f$.

We apply these results to Schrödinger operators with operator-valued potentials and to the spectral properties of Schrödinger operators in quantum waveguides.

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Key words and phrases. pseudodifferential operators with operator-valued symbols, Fredholmness, exponential estimates of solutions, quantum waveguides.

რეზიუმე. ჩვენ განვიხილავთ ფსევდოდიფერენციალურ ოპერატორებს ოპერატორულ მნიშვნელობებიანი სიმბოლოებით $a = a(x,\xi)$, რომელთაც აქვთ ხარისხოვანი ზრდა x და ξ ცვლადების მიმართ. უფრო მეტიც, ჩვენ განვიხილავთ სიმბოლოებს, რომლებიც უშვებენ ანალიზურ გაგრძელებას ξ ცვლადის მიმართ მილისებრ არეზე $\mathbb{C}^n$-ში, რომლის ფუძე წარმოადგენს ბირთვს $\mathbb{R}^n$-ში და ამ ბირთვის რადიუსი დამოკიდებულია x ცვლადზე.

ნაშრომის ძირითადი შედეგია ოპერატორულ მნიშვნელობებიანი სიმბოლოების მქონე ფსევდოდიფერენციალური ოპერატორებისთვის ფრედჰოლმის თეორია და $Op(a)u = f$ ფსევდოდიფერენციალური განტოლებების ამონახსნების ექსპონენციალური შეფასებები უსასრულობაში.

მიღებულ შედეგებს ვიყენებთ ოპერატორულ მნიშვნელობებიაბი პოტენციალების მქონე შროდინგერის ოპერატორებისათვის და კვანტური ტალღების გამტარებში შროდინგერის ოპერატორების სპექტრალური თვისებებისათვის.

1. Introduction

We consider the class of pseudodifferential operators

$$(Op(a)u)(x) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} d\xi \int\limits_{\mathbb{R}^n} a(x,\xi)u(y)e^{i(x-y)\cdot\xi}\, dy, \quad u \in S(\mathbb{R}^n, \mathcal{H}_1), \quad (1)$$

with symbols a with values in the space of bounded linear operators acting from a Hilbert space $\mathcal{H}_1$ into a Hilbert space $\mathcal{H}_2$. In (1), $S(\mathbb{R}^n, \mathcal{H}_1)$ is the space of $\mathcal{H}_1$-valued infinitely differentiable functions rapidly decreasing with all their derivatives. We consider the symbols which can have a power growth at infinity with respect to the variables x and ξ. Moreover, we suppose that the symbol a can be analytically extended with respect to ξ onto a tube domain $\mathbb{R}^n + i\{\eta \in \mathbb{R}^n : \ |\eta| < b(x)\}$, where b is a continuous positive function.

The main results of the paper are the Fredholm theory of pseudodifferential operators and exponential estimates at infinity of solutions of pseudodifferential equations $Op(a)u = f$. We apply these results to the Schrödinger operators with operator-valued potentials and discuss applications to quantum waveguides.

Our approach is based on the construction of the local inverse operator at infinity and on estimates of commutators of pseudodifferential operators with exponential weights (First the idea of this approach for scalar pseudodifferential operators with bounded symbols appeared in the paper [20], and later also for scalar pseudodifferential operators with symbols admitting a power, exponential and super-exponential grows and local discontinuities in [31], [32], [34]. [35].)

Estimates of exponential decay are intensively studied in the literature. We would like to emphasize Agmon's monograph [1] where the exponential estimates of the behavior of solutions of second order elliptic operators have been obtained in terms of a special metric (now called the Agmon metric). See also [4], [18], [19], [16], [20], [24], [25], [28], [31], [32], [5], [6], [35]. In [36], [37] the authors established the relation between the essential spectrum of pseudodifferential operators and exponential decay of their solutions at infinity. The recent paper [33] is devoted to local exponential estimates of solutions of finite-dimensional h-pseudodifferential operators with applications to the tunnel effect for Schrödinger, Dirac and Klein–Gordon operators.

It turns out that many problems in mathematical physics are reduced to the study of associated pseudodifferential operators with operator-valued symbols. In particular, this happens for problems of wave propagation in acoustic, electromagnetic and quantum waveguides (see for instance [3] and references cited there).

This paper is organized as follows. In Section 2 we present some auxiliary facts on operator-valued pseudodifferential operators. Some standard references for the theory of pseudodifferential operators are [17], [39], [40], whereas operator-valued pseudodifferential operators have been studied in [21], [22]. The approach in the latter books follows ideas by Hörmander and employs a special partition of unity connected with a metric defining the class of pseudodifferential operators. We will follow here the approach of [30], which based on the notion of a formal symbol. A main point is the representation of the symbol of a product of pseudodifferential operators and of a double pseudodifferential operators in form of an operator-valued double oscillatory integral. This approach allows us to extend the theory of scalar pseudodifferential operators to pseudodifferential operators with operator-valued symbols, and it provides us with an pseudodifferential operator calculus which is convenient for applications.

In Section 3 we examine the local invertibility at infinity of operator-valued pseudodifferential operators in suitable spaces and discuss their Fredholm property. Section 4 is devoted to the exponential estimates at infinity of solutions of operator-valued pseudodifferential operators. In the concluding Section 5 we study the Fredholm property of Schrödinger operators and derive exponential estimates at infinity of solutions of Schrödinger equations with operator-valued increasing potentials. These general results are then applied to the Fredholm property of Schrödinger operators with increasing potentials for quantum waveguides, for which we obtain exponential estimates of eigenfunctions. Note that spectral problems for quantum waveguides have attracted many attention in the last time. See, for instance, [3], [10], [13], [9].

2. Pseudodifferential Operators with Operator Valued Symbols and its Fredholm Properties

2.1. Notations.

- Given Banach spaces X, Y, we denote the Banach space of all bounded linear operators acting from X in Y by $\mathcal{L}(X,Y)$. In case $X = Y$, we simply write $\mathcal{L}(X)$.
- Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then we denote by $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ the points of the dual space with respect to the scalar product $\langle x, \xi \rangle = x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$.
- For $j = 1, \dots, n$, let $\partial_{x_j} := \frac{\partial}{\partial x_j}$ and $D_{x_j} := -i\,\frac{\partial}{\partial x_j}$. More generally, given a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, set $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and

$$\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \quad \text{and} \quad D_x^\alpha := D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}.$$

- Let $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ for $\xi \in \mathbb{R}^n$.
- Let X be a Banach space. We denote by
  - (i) $C^\infty(\mathbb{R}^n, X)$ the set of all infinitely differentiable functions on $\mathbb{R}^n$ with values in X;
  - (ii) $C_0^\infty(\mathbb{R}^n, X)$ the set of all functions in $C^\infty(\mathbb{R}^n, X)$ with compact supports;

(iii) $C^\infty_{b,N}(\mathbb{R}^n, X)$ the set of all functions $a \in C^\infty(\mathbb{R}^n, X)$ such that for some $N \geq 0$

$$\sup_{x\in\Omega} \sum_{|\alpha|\leq k} \langle x\rangle^{-N} \left\|(\partial_x^\alpha a)(x)\right\|_X < \infty$$

for every $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We will write $C^\infty_b(\mathbb{R}^n, X)$ if $N = 0$.

(iv) $S(\mathbb{R}^n, X)$ the set of all functions $a \in C^\infty(\mathbb{R}^n, X)$ such that

$$\sup_{x\in\mathbb{R}^n} \langle x\rangle^k \sum_{|\alpha|\leq k} \left\|(\partial_x^\alpha a)(x)\right\|_X < \infty$$

for every $k \in \mathbb{N}_0$.

In each case, we omit X whenever $X = \mathbb{C}$.

- Let $\mathcal{H}$ be a Hilbert space and $u \in S(\mathbb{R}^n, \mathcal{H})$. Then we denote by

$$\widehat{u}(\xi) = (Fu)(\xi) := \int_{\mathbb{R}^n} u(x)e^{-ix\cdot\xi}\, dx$$

the Fourier transform of u. Note that $F : S(\mathbb{R}^n, \mathcal{H}) \to S(\mathbb{R}^n, \mathcal{H})$ is an isomorphism with inverse

$$(F^{-1}\widehat{u})(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{u}(\xi)e^{ix\cdot\xi}\, d\xi.$$

We write $S'(\mathbb{R}^n, \mathcal{H})$ for the space of distributions over $S(\mathbb{R}^n, \mathcal{H})$ and define the Fourier transform of distributions in $S'(\mathbb{R}^n, \mathcal{H})$ via duality. Note that $F : S'(\mathbb{R}^n, \mathcal{H}) \to S'(\mathbb{R}^n, \mathcal{H})$ is an isomorphism.

- In what follows we consider separable Hilbert spaces $\mathcal{H}$ only.

2.2. **Oscillatory vector-valued integrals.** 1^0. Let B be a Banach space, and let a be a function in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n, B)$ for which there exist m_1, $m_2 \in \mathbb{R}$ such that

$$|a|_{r,t} := \sum_{|\alpha|\leq r,\, |\beta|\leq t} \sup_{\mathbb{R}^n\times\mathbb{R}^n} \left\|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right\|_B \langle x\rangle^{-m_1} \langle \xi\rangle^{-m_2} < \infty \tag{2}$$

for all r, $t \in \mathbb{N}_0$. Further let $\chi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be such that $\chi(x,\xi) = 1$ for all points (x,ξ) in a neighborhood of the origin. Let $R > 0$. In what follows we call $\chi_R(x,\xi) := \chi(x/R, \xi/R)$ a cut-off function.

Proposition 1. Let $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n, B)$ satisfy the estimates (2). Then the limit

$$\mathcal{I}(a) := \lim_{R\to\infty} (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} \chi_R(x,\xi) a(x,\xi) e^{-ix\cdot\xi}\, dx\, d\xi$$

exists in the norm topology of B and

$$\mathcal{I}(a) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} \langle\xi\rangle^{-2k_2} \langle D_x\rangle^{2k_2} \big\{\langle x\rangle^{-2k_1} \langle D_\xi\rangle^{2k_1} a(x,\xi)\big\} e^{-ix\cdot\xi}\, dx\, d\xi$$

for all

$$2k_1 > n + m_1, \quad 2k_2 > n + m_2. \tag{3}$$

This limit is independent on k_1, k_2 satisfying (3) and the choice of χ. Moreover,

$$\begin{aligned}\|\mathcal{I}(a)\|_B \leq C \sum_{|\alpha|\leq 2k_1,\, |\beta|\leq 2k_2} \sup_{\mathbb{R}^n\times\mathbb{R}^n} \left\|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right\|_B \langle x\rangle^{-m_1}\langle \xi\rangle^{-m_2} = \\ = C|a|_{2k_1,2k_2}.\end{aligned} \tag{4}$$

The element $\mathcal{I}(a) \in B$ is called the *oscillatory integral.*

In what follows the double integral

$$\iint\limits_{\mathbb{R}^{2n}} a(x,\xi)e^{-ix\cdot\xi}\, dx\, d\xi$$

is understood as oscillatory.

Proposition 2. Let $a \in C^\infty(\mathbb{R}^n, B)$ and for all β

$$\|\partial_x^\beta a(x)\|_B \leq C_\beta \langle x\rangle^N, \quad N > 0.$$

Then, for each $x \in \mathbb{R}^n$,

$$(2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a(x+y)e^{-iy\cdot\xi}\, dy\, d\xi = a(x). \tag{5}$$

Propositions 1 and 2 are proved as in the scalar case by integrating by parts (see for instance [30]).

2.3. **Pseudodifferential operators.** Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces. A function $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}(\mathcal{H}', \mathcal{H}))$ is said to be a *weight function* in the class $O(\mathcal{H}, \mathcal{H}')$ if the operator $p(x,\eta)$ is invertible for each $(x,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$ and for all α, β there are constants $C_{\alpha\beta} > 0$ such that

$$\begin{aligned}\left\|p(y,\eta)^{-1}\partial_x^\beta\partial_\xi^\alpha p(x+y,\xi+\eta)\right\|_{\mathcal{L}(\mathcal{H}')} \leq C_{\alpha\beta}\big(1+|y|+|\eta|\big)^N, \\ \left\|\big(\partial_x^\beta\partial_\xi^\alpha p(x+y,\xi+\eta)\big)p^{-1}(y,\eta)\right\|_{\mathcal{L}(\mathcal{H})} \leq C_{\alpha\beta}\big(1+|y|+|\eta|\big)^N\end{aligned} \tag{6}$$

for some $N > 0$ and arbitrary pairs (x,ξ), $(y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$.

Example 3. We give an important example of a weight function. Let L be an unbounded self-adjoint positive operator in a Hilbert space $\mathcal{H}$ with a dense in $\mathcal{H}$ domain D_L and $L \geq \delta I$, E_μ, $\mu \in [\delta, \infty)$ be the family of the spectral projectors of the self-adjoint operator L. Then the operator $L^m, m \geq 0$ is defined by means of the spectral decomposition as

$$L^m u = \int\limits_\delta^{+\infty} \mu^m\, dE_\mu u$$

with domain

$$D_{L^m} = \left\{ u \in \mathcal{H} : \int\limits_{\delta}^{+\infty} \mu^{2m} \, \|dE_\mu u\|^2_{\mathcal{H}} < \infty \right\}.$$

One can introduce in D_{L^m} the structure of the Hilbert space $\mathcal{H}_{L^m}$ by the scalar product

$$\langle u, v\rangle_{\mathcal{H}_{L^m}} = \int\limits_{\delta}^{+\infty} \mu^{2m} \, \langle dE_\mu u, v\rangle_{\mathcal{H}}.$$

We denote by $\mathcal{H}_{L^{-m}}$ the dual space to $\mathcal{H}_{L^m}, m > 0$ with respect to the scalar product $\langle u, v\rangle_{\mathcal{H}}$. Note that the operator $L^m : \mathcal{H}_{L^m} \to \mathcal{H}$ is an isomorphism of the Hilbert spaces with inverse $L^{-m} : \mathcal{H} \to \mathcal{H}_{L^m}$. Let

$$p^m(x,\xi) = \Big(\big(\langle \xi \rangle + q(x) \big) I + L \Big)^m, m \in \mathbb{R},$$

where $q(x) \geq 1$ for all $x \in \mathbb{R}^n$ and

$$\left| \partial_x^\beta q(x+y) q^{-1}(x) \right| \leq C_\beta q(x) \langle y \rangle^r, \;\; r > 0, \;\; C > 0. \tag{7}$$

Inequality (7) implies that for every $\mu \geq 0$

$$\mu + q(x+y) \leq \mu + Cq(x)\langle y\rangle^r \leq C\langle y\rangle^r (\mu + q(x)), \tag{8}$$

and

$$\langle \xi + \eta\rangle + \mu \leq \sqrt{2}\,\langle\xi\rangle\langle\eta\rangle + \mu \leq \sqrt{2}\,\langle\eta\rangle\big(\langle\xi\rangle + \mu\big). \tag{9}$$

Applying (8) and (9) we obtain that for every $\mu \geq 0$

$$\langle \xi + \eta\rangle + q(x+y) + \mu \leq C\langle\eta\rangle\langle y\rangle^r\big(\langle\xi\rangle + q(x) + \mu\big). \tag{10}$$

It follows from (10) that for every $m \in \mathbb{R}$

$$\big(\langle \xi + \eta\rangle + q(x+y) + \mu\big)^m \leq C\langle\eta\rangle^{|m|}\langle y\rangle^{|m|r}\big(\langle\xi\rangle + q(x) + \mu\big)^m. \tag{11}$$

The spectral representation for $p^m(x,\xi)$, $m \in \mathbb{R}$

$$p^m(x,\xi) = \int\limits_{\mathbb{R}_+} \big(\langle\xi\rangle + q(x) + \mu\big)^m \, dE_\mu$$

yields the estimates

$$\begin{aligned} &\left\| p^m(x,\xi)^{-1} p^m(x+y,\xi+\eta) \right\|^2_{\mathcal{L}(\mathcal{H}_m)} = \\ = &\left\| L^m p^m(x,\xi) p^{-m}(x+y,\xi+\eta) L^{-m} \right\|^2_{\mathcal{L}(\mathcal{H})} \leq \\ &\leq \sup_{\mu\in[\delta,\infty)} \left| \frac{\big(\langle\xi+\eta\rangle + q(x+y) + \mu\big)^m}{\big(\langle\xi\rangle + q(x) + \mu\big)^m} \right| \leq C\langle\eta\rangle^{|m|}\langle y\rangle^{|m|r}. \end{aligned} \tag{12}$$

In the same way we obtain that

$$\left\| p^{-m}(x,\xi) p^m(x+y,\xi+\eta) \right\|^2_{\mathcal{L}(\mathcal{H})} \leq C\langle\eta\rangle^{|m|}\langle y\rangle^{|m|r}$$

and corresponding estimates (6) for derivatives. Hence $p^m \in O(\mathcal{H}_{L^m}, \mathcal{H})$ for every $m \in \mathbb{R}$.

Let now $\mathcal{H}_1$, $\mathcal{H}_1'$, $\mathcal{H}_2$ and $\mathcal{H}_2'$ be Hilbert spaces and $p_1 \in O(\mathcal{H}_1, \mathcal{H}_1')$ and $p_2 \in O(\mathcal{H}_2, \mathcal{H}_2')$. We say that a function $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ belongs to $S(p_1, p_2)$ if

$$|a|_{l_1,l_2} :=$$
$$:= \sum_{|\alpha|\leq l_1,\, |\beta|\leq l_2} \sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n} \left\| p_2^{-1}(x,\xi)\partial_x^\beta \partial_\xi^\alpha a(x,\xi) p_1(x,\xi)\right\|_{\mathcal{L}(\mathcal{H}_1',\mathcal{H}_2')} < \infty \quad (13)$$

for every l_1, $l_2 \in \mathbb{N}_0$. The semi-norms $|a|_{l_1,l_2}$ define a Frechet topology on $S(p_1, p_2)$. The (operator-valued) functions in $S(p_1, p_2)$ are called symbols.

With each symbol $a \in S(p_1, p_2)$, we associate the pseudodifferential operator $Op(a)$ which acts at $u \in S(\mathbb{R}^n, \mathcal{H}_1)$ by

$$Op(a)u(x) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} a(x,\xi)\widehat{u}(\xi) e^{ix\cdot\xi}\, d\xi =$$
$$= (2\pi)^{-n} \int\limits_{\mathbb{R}^n} d\xi \int\limits_{\mathbb{R}^n} a(x,\xi) u(y) e^{i(x-y)\cdot\xi}\, dy. \quad (14)$$

We denote the set of all pseudodifferential operators with symbols in $S(p_1, p_2)$ by $OPS(p_1, p_2)$.

We will also need double symbols and their associated double pseudodifferential operators. Let again $p_1 \in O(\mathcal{H}_1, \mathcal{H}_1')$ and $p_2 \in O(\mathcal{H}_2, \mathcal{H}_2')$. A function $a : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is said to belong to the class $S_d(p_1, p_2)$ of *double symbols* if there exist $N > 0$ such that

$$|a|_{l_1,l_2,l_3} = \sum_{|\alpha|\leq l_1,\, |\beta|\leq l_2,\, |\gamma|\leq l_3} \sup_{(x,y,\xi)\in\mathbb{R}^{3n}} \langle y\rangle^{-N} \times$$
$$\times \left\| p_2(x,\xi)^{-1} \partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha a(x, x+y, \xi) p_1(x,\xi)\right\|_{\mathcal{L}(\mathcal{H}_1',\mathcal{H}_2')} < \infty \quad (15)$$

for each l_1, l_2, $l_3 \in \mathbb{N}_0$. We correspond to each double symbol $a \in S_d(p_1, p_2)$ the *double pseudodifferential operator*

$$Op_d(a)u(x) := (2\pi)^{-n} \int\limits_{\mathbb{R}^n} d\xi \int\limits_{\mathbb{R}^n} a(x,y,\xi) u(y) e^{i(x-y)\cdot\xi}\, dy, \quad (16)$$

$u \in S(\mathbb{R}^n, \mathcal{H}_1)$ and denote the class of all double pseudodifferential operators by $OPS_d(p_1, p_2)$. Note that the estimates (6) and (13) imply that if $a \in S(p_1, p_2)$ or $S_d(p_1, p_2)$ there exist $M > 0$, $N > 0$ and constants $C_{\alpha\beta}$ and $C_{\alpha\beta\gamma}$ such that

$$\left\|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \leq C_{\alpha\beta}\big(1+|x|+|\xi|\big)^N \quad (17)$$

and

$$\left\|\partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha a(x,y,\xi)\right\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \leq C_{\alpha\beta\gamma}\big(1+|x|+|\xi|\big)^N \langle y\rangle^M \quad (18)$$

for all multiindeces α, β, γ.

Integrating by parts one can prove as in the scalar case that the pseudodifferential operators (14) and (16) can be written of the form of double oscillatory integrals depending on the parameter $x \in \mathbb{R}^n$,

$$(Op(a)u)(x) = (2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a(x,\xi)u(x+y)e^{-iy\cdot\xi}\, d\xi\, dy, \tag{19}$$

$$(Op_d(a)u)(x) = (2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a(x,x+y,\xi)u(x+y)e^{-iy\cdot\xi}\, d\xi\, dy, \tag{20}$$

and that the operators $Op(a)$ and $Op_d(a)$ in (19) and (20) are defined on $C_b^\infty(\mathbb{R}^n, \mathcal{H}_1)$.

For $\xi \in \mathbb{R}^n$, define $e_\xi : \mathbb{R}^n \to \mathbb{C}$ by $e_\xi(x) := e^{ix\cdot\xi}$. Let now A be a continuous linear operator from $C_b^\infty(\mathbb{R}^n, \mathcal{H}_1)$ to $C_{b,N}^\infty(\mathbb{R}^n, \mathcal{H}_2)$, $N \geq 0$, and let $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Then there is a bounded linear operator $\sigma_A(x,\xi) : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$e_{-\xi}(x)\big[A(e_\xi \otimes \varphi)\big](x) = \sigma_A(x,\xi)\varphi \tag{21}$$

for every $\varphi \in \mathcal{H}_1$. The function $\sigma_A : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is then called the *formal symbol* of A.

We will suppose that there exists $N \geq 0, C > 0$ such that

$$\|\sigma_A(x,\xi)\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \leq C\big(1+|x|+|\xi|\big)^N. \tag{22}$$

Proposition 4. Let $A : C_b^\infty(\mathbb{R}^n, \mathcal{H}_1) \to S'(\mathbb{R}^n, \mathcal{H}_2)$ be a continuous linear operator with a formal symbol σ_A. Then A acts at functions $u \in S(\mathbb{R}^n, \mathcal{H}_1)$ via

$$(Au)(x) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma_A(x,\xi)\widehat{u}(\xi)\, d\xi. \tag{23}$$

Proof. Let $u \in S(\mathbb{R}^n, \mathcal{H}_1)$. Then

$$u(x) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \widehat{u}(\xi)e_\xi(x)\, d\xi.$$

Let $\{\phi_j\}$ be an orthonormal basis of $\mathcal{H}_1$ and write $\widehat{u}(\xi) = \sum\limits_{j=1}^{\infty} \widehat{u}_j(\xi)\phi_j$ with Fourier coefficients $\widehat{u}_j(\xi) = \langle \widehat{u}(\xi), \phi_j\rangle_{\mathcal{H}_1}$. Hence,

$$\begin{aligned}(Au)(x) &= (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \sum_{j=1}^{\infty} \widehat{u}_j(\xi)(A(e_\xi \otimes \phi_j))(x)\, d\xi = \\ &= (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \sum_{j=1}^{\infty} \widehat{u}_j(\xi)e^{ix\cdot\xi}\sigma_A(x,\xi)\phi_j\, d\xi = \\ &= (2\pi)^{-n} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma_A(x,\xi)\widehat{u}(\xi)\, d\xi. \end{aligned} \tag{24}$$

The last integral exists according to estimate (22). □

Proposition 5. Let $A = Op(a) \in OPS(p_1, p_2)$. Then A has a formal symbol σ_A which coincides with a.

Proof. Let $\xi \in \mathbb{R}^n$ and $\varphi \in \mathcal{H}_1$. Then, by (19),

$$\big(A(e_\xi \otimes \varphi)\big)(x) = (2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a(x,\eta)\varphi e^{i(x+y)\cdot\xi} e^{-iy\cdot\eta}\, d\eta\, dy =$$

$$= e^{ix\cdot\xi}(2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a(x,\xi+\eta)\varphi e^{-iy\cdot\eta}\, d\eta\, dy. \qquad (25)$$

Using equality (5) we obtain from (25)

$$\sigma_A(x,\xi)\varphi = e^{-ix\cdot\xi} A(e_\xi \otimes \varphi)(x) = a(x,\xi)\varphi$$

which gives the assertion. □

The next propositions describe the main properties of pseudodifferential operators with operator-valued symbols.

Proposition 6. Every operator in $OPS(p_1, p_2)$ is bounded from $S(\mathbb{R}^n, \mathcal{H}_1)$ to $S(\mathbb{R}^n, \mathcal{H}_2)$.

The proof makes use of estimates (17) and runs completely similar to the proof for scalar pseudodifferential operators (see, for instance, [30]).

Hence the composition of pseudodifferential operators is well defined. But below we will prove that the product of pseudodifferential operators is a pseudodifferential operator again.

Proposition 7.

(i) Let $A^1 = Op(a_1) \in OPS(p_1, p_2)$ and $A^2 = Op(a_2) \in OPS(p_2, p_3)$. Then $A^2A^1 \in OPS(p_1, p_3)$, and the symbol of A^2A^1 is given by

$$\sigma_{A^2A^1}(x,\xi) = (2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a_2(x,\xi+\eta)a_1(x+y,\xi)e^{-iy\cdot\eta}\, dy\, d\eta. \qquad (26)$$

(ii) Let $A = Op_d(a) \in OPS_d(p_1, p_2)$. Then $A \in OPS(p_1, p_2)$, and the symbol of A is given by

$$\sigma_A(x,\xi) = (2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a(x, x+y, \xi+\eta)e^{-iy\cdot\eta}\, dy\, d\eta. \qquad (27)$$

The double integrals in (26), (27) are understood as oscillatory integrals.

Proof. The proof mimics the proof for the scalar case (see [30]).

(i) Let $\varphi \in \mathbb{H}_1$. Then, applying formula (5) we obtain

$$\begin{aligned}\sigma_{A^2A^1}(x,\xi)\phi &= e^{-ix\cdot\xi}A_2\big[A_1(e_\xi\phi)\big](x) = \\ &= e^{-ix\cdot\xi}A_2(a_1(\cdot,\xi)e_\xi\phi)(x) = \\ &= (2\pi)^{-n}\iint\limits_{\mathbb{R}^{2n}} a_2(x,\eta)a_1(y,\xi)e^{-i(x-y)\cdot(\xi-\eta)}\phi\,dy\,d\eta = \\ &= (2\pi)^{-n}\iint\limits_{\mathbb{R}^{2n}} a_2(x,\xi+\eta)a_1(x+y,\xi)e^{-iy\cdot\eta}\phi\,dy\,d\eta.\end{aligned}$$

Hence, formula (26) holds. Further we have to show that

$$\sigma_{A^2A^1}(x,\xi) = (2\pi)^{-n}\times$$
$$\times\iint\limits_{\mathbb{R}^{2n}}\langle y\rangle^{-2k_1}\langle D_\eta\rangle^{2k_1}\Big\{\langle\eta\rangle^{-2k_2}\langle D_y\rangle^{2k_2}a_2(x,\xi+\eta)a_1(x+y,\xi)\Big\}e^{-iy\cdot\eta}dyd\eta. \quad (28)$$

Application of the Leibnitz formula leads to the estimates

$$p_3^{-1}(x,\xi)\mathcal{I}_{\gamma,\delta}(x,\xi)p_1(x,\xi) = (2\pi)^{-n}\iint\limits_{\mathbb{R}^{2n}}\langle y\rangle^{-2k_1}\langle\eta\rangle^{-2k_2}p_3^{-1}(x,\xi)\times$$
$$\times\,\partial_\eta^\gamma a_2(x,\xi+\eta)\partial_y^\delta a_1(x+y,\xi)p_1(x,\xi)e^{-iy\cdot\eta}\,dy\,d\eta. \quad (29)$$

Applying the next estimates following from (6)

$$\begin{aligned}\big\|p_3^{-1}(x,\xi)p_3(x,\xi+\eta)\big\| &\le C\langle\eta\rangle^{M_3},\\ \big\|p_2^{-1}(x,\xi+\eta)p_2(x,\xi)\big\| &\le C\langle\eta\rangle^{M_2},\\ \big\|p_2^{-1}(x+y,\xi)p_2(x,\xi)\big\| &\le C\langle y\rangle^{M_2},\\ \big\|p_1^{-1}(x+y,\xi)p_1(x,\xi)\big\| &\le C\langle y\rangle^{M_1},\end{aligned} \quad (30)$$

and choosing $2k_1 > n+M_1+M_2, 2k_2 > n+M_2+M_3$, we obtain the estimate

$$\big\|p_3^{-1}(x,\xi)\mathcal{I}_{\gamma,\delta}(x,\xi)p_1(x,\xi)\big\|_{\mathcal{B}(\mathcal{H}_1',\mathcal{H}_3')} \le C|a_2|_{l_1,l_2}|a_1|_{l_1,l_2},$$

for some $l_1, l_2 \in \mathbb{N}$. In the same way one can show that

$$\big\|p_3^{-1}(x,\xi)\partial_x^\beta\partial_\xi^\alpha\sigma_{A^2A^1}(x,\xi)p_1(x,\xi)\big\|_{\mathcal{B}(\mathcal{H}_1',\mathcal{H}_3')} \le C|a_2|_{l_1,l_2}|a_1|_{l_1,l_2},$$

for some l_1, $l_2 \in \mathbb{N}$.

(ii) Following the proof of (i) we have to estimate the integrals

$$p_2^{-1}(x,\xi)\mathcal{I}_{\gamma,\delta}(x,\xi)p_1(x,\xi) = (2\pi)^{-n}\times$$
$$\times\iint\limits_{\mathbb{R}^{2n}}\langle y\rangle^{-2k_1}\langle\eta\rangle^{-2k_2}p_2^{-1}(x,\xi)\partial_\eta^\gamma\partial_y^\delta a(x,x+y,\xi+\eta)p_1(x,\xi)e^{-iy\cdot\eta}\,dy\,d\eta. \quad (31)$$

Applying (30) and the estimate

$$\big\|p_2^{-1}(x,\xi+\eta)\partial_\eta^\gamma\partial_x^\delta a(x,x+y,\xi+\eta)p_1(x,\xi+\eta)\big\|_{\mathcal{L}(\mathcal{H}_1',\mathcal{H}_2')} \le C|a|_{l_1,0,l_3}\langle y\rangle^N,$$

and choosing $2k_1 > n+N$, $2k_2 > n+M_1+M_2$ we obtain

$$\left\|p_2^{-1}(x,\xi)\mathcal{I}_{\gamma,\delta}(x,\xi)p_1(x,\xi)\right\|_{\mathcal{L}(\mathcal{H}_1',\mathcal{H}_2')} \leq C|a|_{l_1,0,l_3}.$$

In the same way we obtain the estimate

$$\left\|p_2^{-1}(x,\xi)\partial_x^\beta\partial_\xi^\alpha\sigma_A(x,\xi)p_1(x,\xi)\right\|_{\mathcal{L}(\mathcal{H}_1',\mathcal{H}_3')} \leq C|a|_{l_1,l_2,l_3}. \qquad \square$$

An operator A^* is called the *formal adjoint* to the operator $A \in OPS(p_1,p_2)$ if, for arbitrary functions $u \in S(\mathbb{R}^n,\mathcal{H}_1)$ and $v \in S(\mathbb{R}^n,\mathcal{H}_2)$,

$$\langle Au,v\rangle_{L^2(\mathbb{R}^n,\mathcal{H}_2)} = \langle u,A^*v\rangle_{L^2(\mathbb{R}^n,\mathcal{H}_1)}. \tag{32}$$

Proposition 8. Let $A = Op(a)\in OPS(p_1,p_2)$. Then $A^* \in OPS(p_2^*,p_1^*)$, and the symbol of A^* is given by

$$\sigma_{A^*}(x,\xi) = (2\pi)^{-n}\iint\limits_{\mathbb{R}^{2n}} a^*(x+y,\xi+\eta)e^{i(x-y)\cdot\xi}\,dy\,d\xi, \tag{33}$$

where

$$\langle a(x,\xi)u,v\rangle_{\mathcal{H}_2} = \langle u,a^*(x,\xi)v\rangle_{\mathcal{H}_1}$$

for all $u \in \mathcal{H}_1$ and $v \in \mathcal{H}_2$. The double integrals in (33) are understood as oscillatory integrals.

The assertion of Proposition 8 follows from Proposition 7 (ii).

By Proposition 8 and formula (32), one can think of operators in $OPS(p_1,p_2)$ as acting from $S'(\mathbb{R}^n,\mathcal{H}_1)$ to $S'(\mathbb{R}^n,\mathcal{H}_2)$.

Theorem 9 (Calderon–Vaillancourt)**.** *If* $A= Op(a) \in OPS(I_{\mathcal{H}_1},I_{\mathcal{H}_2}) := OPS(\mathcal{H}_1,\mathcal{H}_2)$, *then* A *is bounded as operator from* $L^2(\mathbb{R}^n,\mathcal{H}_1)$ *to* $L^2(\mathbb{R}^n,\mathcal{H}_2)$, *and there exists constants* $C>0$ *and* $2k_1,2k_2>n$ *such that*

$$\|A\|_{\mathcal{L}(L^2(\mathbb{R}^n,\mathcal{H}_1),L^2(\mathbb{R}^n,\mathcal{H}_2))} \leq C\sum_{|\alpha|\leq 2k_1,|\beta|\leq 2k_2}\sup_{(x,\xi)\in\mathbb{R}^{2n}}\left\|a_{(\alpha)}^{(\beta)}(x,\xi)\right\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}.$$

Proposition 10 (Beals)**.** Let $A = Op(a) \in OPS(\mathcal{H}_1,\mathcal{H}_2)$ be invertible as operator from $L^2(\mathbb{R}^n,\mathcal{H}_1)$ to $L^2(\mathbb{R}^n,\mathcal{H}_2)$. Then $A^{-1} \in OPS(\mathcal{H}_2,\mathcal{H}_1)$.

2.4. **Sobolev spaces** $H(\mathbb{R}^n,p_h)$**.** Let $p \in O(\mathcal{H}',\mathcal{H})$. We denote by $p_h, h > 0$ the symbol $p_h(x,\xi) = p(x,h\xi)$.

Proposition 11. . Let $p \in O(\mathcal{H}_1,\mathcal{H}_2)$. Then for every $h>0$

$$\begin{aligned} Op(p_h)Op(p_h^{-1}) &= I_{\mathcal{H}_2} + hOp(r_h^2),\\ Op(p_h^{-1})Op(p_h) &= I_{\mathcal{H}_1} + hOp(r_h^1), \end{aligned} \tag{34}$$

where $Op(r_h^j) \in OPS(\mathcal{H}_j,\mathcal{H}_j)$, $j=1,2$, and

$$\sup_{h>0}\|Op(r_h^j)\|_{\mathcal{L}(\mathcal{H}_j)} < \infty, \quad j=1,2.$$

For the proof see [33], Proposition 7 and Corollary 14.

Corollary 12. *For $h > 0$ small enough*

$$Op(p_h)Op(p_h)^{-1} = I_{\mathcal{H}_2}, Op(p_h)^{-1}Op(p_h) = I_{\mathcal{H}_1}, \tag{35}$$

where

$$Op(p_h)^{-1} = Op(p_h^{-1})(I_{\mathcal{H}_2} + hOp(r_h^2))^{-1} = (I_{\mathcal{H}_2} + hOp(r_h^1))^{-1}Op(p_h^{-1}).$$

In what follows for $p \in O(\mathcal{H}_1, \mathcal{H}_2)$ we fix $h > 0$, such that there exists $Op(p_h)^{-1}$.

We denote by $H(\mathbb{R}^n, p_h)$ the Banach space which is the closure of $S(\mathbb{R}^n, \mathcal{H})$ with respect to the norm

$$\|u\|_{H(\mathbb{R}^n,p_h)} := \|Op(p_h)u\|_{L^2(\mathbb{R}^n,\mathcal{H}')}.$$

It turns out that then $Op(p_h) : H(\mathbb{R}^n, p_h) \to L^2(\mathbb{R}^n, \mathcal{H}_1)$ is an isomorphism. Using these facts one easily gets the following versions of Proposition 9 and 10, respectively.

Proposition 13. Let $Op(a) \in OPS(p_1, p_2)$. Then $Op(a)$ is bounded as operator from $H(\mathbb{R}^n, p_{1,h})$ to $H(\mathbb{R}^n, p_{2,h})$, and

$$\|A\|_{\mathcal{L}(H(\mathbb{R}^n,p_{1,h}),H(\mathbb{R}^n,p_{2,h}))} \le C|a|_{l_1,l_2},$$

where $C > 0$ and $l_1, l_2 \in \mathbb{N}$ are independent of A.

Proposition 14. Let $A = Op(a) \in OPS(p_1, p_2)$ be invertible as operator from $H(\mathbb{R}^n, p_{1,h})$ to $H(\mathbb{R}^n, p_{2,h})$. Then $A^{-1} \in OPS(p_2, p_1)$.

Let $a \in C_b^\infty(\mathbb{R}^n)$ and $\mathcal{H}$ be a Hilbert space. In what follows we write $aI_{\mathcal{H}}$ for the operator of multiplication by a acting on $S'(\mathbb{R}^N, \mathcal{H})$. Note that this operator is bounded on $H(\mathbb{R}^n, p_h)$ for every weight function $p \in O(\mathcal{H}, \mathcal{H}')$.

We note one more import property of operators in $OPS(p_1, p_2)$ which follows easily from Propositions 7 (i) and 13.

Proposition 15. Let $A = Op(a) \in OPS(p_1, p_2)$. Further let $\varphi \in C_b^\infty(\mathbb{R}^n)$ and set $\varphi_R(x) := \varphi(x/R)$. Then, with $[A, \varphi_R] := A\varphi_R I_{\mathcal{H}_1} - \varphi_R I_{\mathcal{H}_2} A$

$$\lim_{R\to\infty} \big\|[A, \varphi_R]\big\|_{\mathcal{L}(H(\mathbb{R}^n,p_{1,h}),H(\mathbb{R}^n,p_{2,h}))} = 0. \tag{36}$$

2.5. **Pseudodifferential operators with slowly oscillating symbols.** We say that a symbol $a \in S(p_1, p_2)$ is *slowly oscillating at infinity* if, for all multi-indices α, β,

$$\big\|p_2^{-1}(x, \xi)\partial_x^\beta \partial_\xi^\alpha a(x, \xi)p_1(x, \xi)\big\|_{\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')} \le C_{\alpha\beta}^a(x), \tag{37}$$

where

$$\lim_{x\to\infty} C_{\alpha\beta}^a(x) = 0 \tag{38}$$

for all multi-indices α, β with $\beta \neq 0$. We denote this class of symbols by $S_{sl}(p_1, p_2)$ and write $OPS_{sl}(p_1, p_2)$ for the corresponding class of pseudodifferential operators. Furthermore, let $S^0(p_1, p_2)$ refer to the subset of $S_{sl}(p_1, p_2)$ of all symbols such that (38) holds for all multi-indices α, β.

Similarly, a double symbol $a \in S_d(p_1, p_2)$ is called *slowly oscillating* at infinity if, for all mutli-indices α, β and some $N > 0$

$$\left\| p_2^{-1}(x,\xi)\partial_x^\beta \partial_\xi^\alpha a(x, x+y, \xi) p_1(x,\xi) \right\|_{\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')} \leq C_{\alpha\beta}^a(x) \langle y \rangle^N ,$$

where

$$\lim_{x\to\infty} C_{\alpha\beta}^a(x) = 0$$

for all multi-indices α, β with $\beta \neq 0$. We denote the set of all slowly oscillating double symbols by $S_{d,sl}(p_1, p_2)$ and write $OPS_{d,sl}(p_1, p_2)$ for the corresponding class of double pseudodifferential operators.

The next proposition describes some properties of pseudodifferential operators with operator-valued slowly oscillating at infinity symbols which will be needed in what follows.

Proposition 16.

(i) Let $A^1 = Op(a_1) \in OPS_{sl}(p_1, p_2)$ and $A^2 = Op(a_2) \in OPS_{sl}(p_2, p_3)$. Then $A^2 A^1 \in OPS_{sl}(p_1, p_3)$, and

$$\sigma_{A^2A^1}(x,\xi) = a_2(x,\xi) a_1(x,\xi) + r(x,\xi),$$

where $r \in S^0(p_1, p_3)$.

(ii) Let $A = Op_d(a) \in OPS_{d,sl}(p_1, p_2)$. Then $A \in OPS_{sl}(p_1, p_2)$, and

$$\sigma_A(x,\xi) = a(x,x,\xi) + r(x,\xi),$$

where $r \in S^0(p_1, p_2)$.

(iii) Let $A = Op(a) \in OPS(p_1, p_2)$. Then $A^* \in OPS(p_2^*, p_1^*)$, and

$$\sigma_{A^*}(x,\xi) = a^*(x,x,\xi) + r(x,\xi),$$

where $r \in S^0(p_2^*, p_1^*)$.

Proof. We prove (i). Statements (ii), (iii) are proved in the similar way. We use the representation (26) for $\sigma_{A^2A^1}$

$$\sigma_{A^2A^1}(x,\xi) = (2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} a_2(x, \xi+\eta) a_1(x+y, \xi) e^{-iy\cdot\eta} \, dy \, d\eta. \tag{39}$$

For obtain estimate (37) for $\sigma_{A^2A^1}$ we have to estimate the integrals

$$\mathcal{I}_{\alpha,\beta,\gamma,\delta}(x,\xi) =$$

$$= (2\pi)^{-n} \iint\limits_{\mathbb{R}^{2n}} \langle y \rangle^{-2k_1} \langle \eta \rangle^{-2k_2} \partial_x^\beta \partial_\xi^\alpha a_2(x, \xi+\eta) \partial_x^\gamma \partial_\xi^\delta a_1(x+y, \xi) e^{-iy\cdot\eta} \, dy \, d\eta,$$

for $|\beta| \geq 1$ or $|\gamma| \geq 1$. Let $2k_1 > n+1+M_1+M_2$, $2k_2 > n+1+M_2+M_3$, Then similar to the proof of Proposition 7 we obtain

$$\begin{aligned} &\left\| p_3^{-1}(x,\xi)\mathcal{I}_{\alpha,\beta,\gamma,\delta}(x,\xi)p_1(x,\xi)\right\|_{\mathcal{L}(\mathcal{H}_1',\mathcal{H}_2')} \leq \\ &\leq C \iint\limits_{\mathbb{R}^{2n}} \langle y\rangle^{-2k_1-M_1-M_2} \langle\eta\rangle^{-2k_2-M_3-M_2} \times \\ &\times \left\| p_3^{-1}(x,\xi+\eta)\partial_x^\beta\partial_\xi^\alpha a_2(x,\xi+\eta)p_2(x,\xi+\eta)\right\| \times \\ &\times \left\| p_2^{-1}(x+y,\xi)\partial_x^\gamma\partial_\xi^\delta a_2(x+y,\xi)p_1(x+y,\xi)\right\| \, dy\, d\eta \leq \\ &\leq C C_{\alpha\beta}^{a_2}(x) \sup_{y\in\mathbb{R}^n} \frac{C_{\gamma\delta}^{a_1}(x+y)}{\langle y\rangle} . \end{aligned} \tag{40}$$

Estimate (40) shows that

$$\lim_{x\to\infty} \sup_{\xi\in\mathbb{R}^n} \left\| p_3^{-1}(x,\xi)\mathcal{I}_{\alpha,\beta,\gamma,\delta}(x,\xi)p_1(x,\xi)\right\|_{\mathcal{L}(\mathcal{H}_1',\mathcal{H}_2')} = 0.$$

Hence $\sigma_{A^2A^1} \in OPS_{sl}(p_1,p_3)$. Further, by the Lagrange formula

$$a_2(x,\xi+\eta) = a_2(x,\xi) + \sum_{j=1}^n \eta_j \int_0^1 \partial_{\xi_j} a_2(x,\xi+\theta\eta)\, d\theta. \tag{41}$$

Substituting (41) in (39) and applying formula (5) we obtain

$$\sigma_{A^2A^1}(x,\xi) = a_2(x,\xi)a_1(x,\xi) + r(x,\xi),$$

where

$$\begin{aligned} r(x,\xi) &= (2\pi)^{-n} \times \\ \times \sum_{j=1}^n \int_0^1 d\theta &\iint\limits_{\mathbb{R}^{2n}} \partial_{\xi_j} a_2(x,\xi+\theta\eta) D_{x_j} a_1(x+y,\xi) e^{-iy\cdot\eta}\, dy\, d\eta. \end{aligned} \tag{42}$$

Because the integral (42) contains the derivative of $a_1(\in S_{sl}(p_1,p_2))$ with respect to x one can prove that $r \in S^0(p_1,p_3)$ following to the proof that $\sigma_{A^2A^1} \in OPS_{sl}(p_1,p_3)$. □

3. Invertibility at Infinity and Fredholm Property of Pseudodifferential Operators

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\chi(x) = 1$ if $|x| \leq 1$ and $\chi(x) = 0$ if $|x| \geq 2$. Set $\phi := 1 - \chi$ and, for $R > 0$, $\chi_R(x) := \chi(x/R)$ and $\phi_R(x) := \phi(x/R)$. Further let

$$B_R := \{x \in \mathbb{R}^n : |x| < R\} \text{ and } B_R' := \{x \in \mathbb{R}^n : |x| > R\}.$$

We say that an operator $A : H(\mathbb{R}^n, p_1) \to H(\mathbb{R}^n, p_2)$ is *locally invertible at infinity* if there is an $R_0 > 0$ such that, for every $R > R_0$, there are operators $\mathcal{L}_R$ and $\mathcal{R}_R$ such that

$$\mathcal{L}_R A\phi_R I_{\mathcal{H}_1} = \phi_R I_{\mathcal{H}_1} \text{ and } \phi_R A\mathcal{R}_R = \phi_R I_{\mathcal{H}_2}. \tag{43}$$

Operators $\mathcal{L}_R$ and R_R with these properties are called *locally left and right inverses of* A, respectively.

Theorem 17. *Let* $A = Op(a) \in OPS_{sl}(p_1, p_2)$. *Assume there is a constant* $R_0 > 0$ *such that the operator* $a(x,\xi) : \mathcal{H}_1 \to \mathcal{H}_2$ *is invertible for every* $(x,\xi) \in B'_{R_0} \times \mathbb{R}^n$ *and that*

$$\sup_{(x,\xi)\in B'_{R_0}\times\mathbb{R}^n} \left\| p_1^{-1}(x,\xi) a(x,\xi)^{-1} p_2(x,\xi) \right\|_{\mathcal{L}(\mathcal{H}'_2,\mathcal{H}'_1)} < \infty.$$

Then the operator $A : H(\mathbb{R}^n, p_{1,h}) \to H(\mathbb{R}^n, p_{2,h})$ *is locally invertible at infinity.*

Proof. Given ϕ as above, choose $\varphi \in C_b^\infty(\mathbb{R}^n)$ such that $\varphi\phi = \phi$, and set $\varphi_R(x) := \varphi(x/R)$ for $R > R_0$. Condition (43) implies that the function $b_R(x,\xi) := \varphi_R(x) a(x,\xi)^{-1}$ belongs to $S(p_2, p_1)$. Hence, and by Proposition 16 (i),

$$Op(b_R) Op(a) \phi_R I_{\mathcal{H}_1} = (I_{\mathcal{H}_1} + Op(q_R)\psi_R I_{\mathcal{H}_1})\phi_R I_{\mathcal{H}_1},$$

where $q_R \in S^0(p_1, p_2)$. Moreover, one can prove that for all multi-indices α, β,

$$\lim_{x\to\infty} \sup_{\xi\in\mathbb{R}^n} \left\| p_1^{-1}(x,\xi) \partial_x^\beta \partial_\xi^\alpha q_R(x,\xi) p_1(x,\xi) \right\|_{\mathcal{L}(\mathcal{H}'_1)} = 0$$

uniformly with respect to $R > R_0$. It follows from Proposition 13 that there exists an $R' > R_0$ such that

$$\|Op(q_R)\psi_R I_{\mathcal{H}_1}\|_{\mathcal{L}(H(\mathbb{R}^n, p_1))} < 1$$

for every $R > R'$. Hence,

$$\left(I_{\mathcal{H}_1} + Op(q_R)\psi_R I_{\mathcal{H}_1}\right)^{-1} Op(b_R) Op(a) \phi_R I_{\mathcal{H}_1} = \phi_R I_{\mathcal{H}_1}, \tag{44}$$

and $Op(a)$ is locally invertible from the left at infinity, with a local left inverse operator given by

$$\mathcal{L}_R := \left(I_{\mathcal{H}_1} + Op(q_R)\psi_R I_{\mathcal{H}_1}\right)^{-1} Op(b_R) \in OPS(p_2, p_1).$$

In the same way, a local right inverse operator $\mathcal{R}_R \in OPS(p_2, p_1)$ can be constructed. It follows from the definition of the operators $\mathcal{L}_R$ and $\mathcal{R}_R$ that

$$\begin{aligned} &\sup_{R>R_0} \|\mathcal{L}_R\|_{\mathcal{L}(H(\mathbb{R}^n, p_{2,h}), H(\mathbb{R}^n, p_{1,h}))} < \infty, \\ &\sup_{R>R_0} \|\mathcal{R}_R\|_{\mathcal{L}(H(\mathbb{R}^n, p_{2,h}), H(\mathbb{R}^n, p_{1,}))} < \infty \end{aligned} \tag{45}$$

which finishes the proof. □

We say that a linear operator $A : H(\mathbb{R}^n, p_{1,h}) \to H(\mathbb{R}^n, p_{2,h})$ is *locally Fredholm* if, for every $R > 0$, there exist bounded linear operators $\mathcal{L}_R, \mathcal{D}_R : H(\mathbb{R}^n, p_{2,h}) \to H(\mathbb{R}^n, p_{1,h})$ and compact operators $T'_R : H(\mathbb{R}^n, p_{1,h}) \to H(\mathbb{R}^n, p_{1,h})$ and $T''_R : H(\mathbb{R}^n, p_{2,h}) \to H(\mathbb{R}^n, p_{2,h})$ such that

$$\mathcal{L}_R A \phi_R I_{\mathcal{H}_1} = \phi_R I_{\mathcal{H}_1} + T'_R \quad \text{and} \quad \phi_R A \mathcal{D}_R = \phi_R I_{\mathcal{H}_2} + T''_R. \tag{46}$$

Theorem 18. *Let $A = Op(a) \in OPS_{sl}(p_1, p_2)$ an operator which satisfies the conditions of Theorem 17. If A is a locally Fredholm operator, then A has the Fredholm property as operator from $H(\mathbb{R}^n, p_{1,h})$ to $H(\mathbb{R}^n, p_{2,h})$.*

Proof. Let R_0 be such that for every $R > R_0$ there exist local inverse operators $\mathcal{L}_R, \mathcal{R}_R \in OPS(p_2, p_1)$ of A. Set $\Lambda_R := \mathcal{B}_R \phi_R I_{\mathcal{H}_2} + \mathcal{L}_R \chi_R I_{\mathcal{H}_2}$. Then $\Lambda_R A = I_{\mathcal{H}_1} + T'_R + Q_R$ where $Q_R := \mathcal{B}_R[\phi_R, A] + \mathcal{B}_R[\chi_R, A]$ and where $T'_R : H(\mathbb{R}^n, p_{1,h}) \to H(\mathbb{R}^n, p_{1,h})$ is compact. Proposition 7 implies that

$$\lim_{R\to 0} \left\|[\phi_R, A]\right\|_{\mathcal{L}(H(\mathbb{R}^n, p_{1,h}), H(\mathbb{R}^n, p_{2,h}))} = \\ = \lim_{R\to 0} \left\|[\chi_R, A]\right\|_{\mathcal{L}(H(\mathbb{R}^n, p_{1,h}), H(\mathbb{R}^n, p_{2,h}))} = 0. \quad (47)$$

From (47) and (45) we conclude that $\|Q_R\|_{\mathcal{L}(H(\mathbb{R}^n, p_1))} < 1$ for large enough $R > 0$. Hence, $\Lambda'_R := (I_{\mathcal{H}_1} + Q_R)^{-1} \Lambda_R$ is a left regularizator of A whenever R_0 is large enough. In the same way, a regularizator from the right-hand side can be found. □

4. Pseudodifferential Operators with Analytical Symbols and Exponential Estimates

4.1. **Operators and weight spaces.**

In this section we consider the weight functions of the form

$$p_T(x, \xi) = \big(\langle \xi \rangle + q(x)\big) I + T, \quad (48)$$

where T is a self-adjoint operator in a Hilbert space $\mathcal{H}$ with a dense domain D_T. We suppose that T is positively defined. Let $\mathcal{H}_{T^m}, m \in \mathbb{R}$ be the Hilbert spaces introduced in Example 7, $q(x) \geq 1$ for every $x \in \mathbb{R}^n$. Moreover, $q \in C^\infty(\mathbb{R}^n)$ and

$$\left|\partial_x^\alpha q(x + y) q^{-1}(x)\right| \leq C_\alpha \langle y \rangle^r, r \geq 0. \quad (49)$$

The estimate (49) implies the estimate

$$\left|\partial_x^\alpha q(x)\right| \leq C_\alpha q(x). \quad (50)$$

In what follows we consider the weight functions of the form $p(x, \xi) = p_T^m(x, \xi)$. We say that the such weight function $p \in O(T^m, q)$.

Let $a \in S(p_1, p_2)$ where $p_j \in O(T_j^{m_j}, q)$, $j = 1, 2$. We denote by $S(p_1, p_2, B_{dq(x)})$ the class of symbols such that:

(1) for every $x \in \mathbb{R}^n$ the operator-valued function $\xi \mapsto a(x, \xi)$ can be extended analytically with respect to ξ into the tube domain $\mathbb{R}^n + iB_{dq(x)}$, where $B_{dq(x)} = \{\eta \in \mathbb{R}^n : |\eta| < dq(x)\}$, $d > 0$.

(2) for arbitrary multi-indices α, β there exists a constant $C_{\alpha\beta}$ such that

$$\left\|p_2^{-1}(x, \xi + i\eta) \partial_x^\beta \partial_\xi^\alpha a(x, \xi + i\eta) p_1(x, \xi + i\eta)\right\|_{\mathcal{L}(\mathcal{H}'_1, \mathcal{H}'_2)} \leq \\ \leq C_{\alpha\beta} \langle \xi + i\eta \rangle^{-|\alpha|} \quad (51)$$

for all $(x, \xi + i\eta) \in \mathbb{R}^n \times (\mathbb{R}^n + iB_{dq(x)})$, where

$$p_j(x, \xi + i\eta) = \Big(\big(1 + |\xi|^2 + |\eta|^2\big)^{1/2} + q_j(x) + T_j\Big)^m.$$

We denote by $OPS(p_1, p_2, B_{dq(x)})$ the class of pseudodifferential operators with symbols in $S(p_1, p_2, B_{dq(x)})$.

(3) If in estimates (51) $C_{\alpha\beta} = C_{\alpha\beta}(x)$ and $\lim\limits_{x\to\infty} C_{\alpha\beta}(x) = 0$ for $\beta \neq 0$ then we denote the corresponding classes of symbols and operators by $S_{sl}(p_1, p_2, B_{dq(x)})$.

(4) We say that a positive C^∞-function $w(x) = e^{v(x)}$ is *a weight in the class* $\mathcal{R}(dq)$ if $v \in C^\infty(\mathbb{R}^n)$ and

$$\big|\partial_x^\alpha(\nabla v(x))\big| < C_\alpha dq(x), \;\; C_0 = 1 \tag{52}$$

for every α and every point $x \in \mathbb{R}^n$. We say that a weight w is slowly oscillating if there exists $\delta \in (0, 1]$ such that

$$\big|\partial_x^\alpha(\nabla v(x))\big| \le C_\alpha dq^{1-\delta|\alpha|}(x). \tag{53}$$

We denote by $\mathcal{R}_{sl}(dq)$ the class of slowly oscillating weights.

Theorem 19.

(i) *Let* $a \in S(p_1, p_2, B_{dq(x)})$ *where* $p_j \in O(T_j^{m_j}, q), j = 1, 2$ *and* $w = \exp v \in \mathcal{R}(dq)$. *Then* $w^{-1}Op(a)wI = Op_d(a_w) \in OPS_d(p_1, p_2)$, *where*

$$a_w(x, y, \xi) = a(x, \xi + i\theta_w(x, y)),$$

and

$$\theta_w(x, y) = \int_0^1 (\nabla v)((1 - t)x + ty)\, dt.$$

(ii) *Let* $a \in S_{sl}(p_1, p_2, B_{dq(x)})$ *where* $p_j \in O^{m_j}(T_j, q_j), j = 1, 2$ *and* $w = \exp v \in \mathcal{R}_{sl}(dq)$. *Then* $w^{-1}Op(a)wI = Op(\tilde{a}_w) \in OPS_{sl}(p_1, p_2)$ *where*

$$\widetilde{a}_w(x, \xi) = a\big(x, \xi + i\nabla v(x)\big) + r(x, \xi), \tag{54}$$

and $r \in S^0(p_1, p_2)$.

Proof. (i) Let $w = \exp v \in \mathcal{R}(\mu)$. By the theorem of the mean value there exists $t_0 \in [0, 1]$ such that

$$\theta_w(x, y) = (\nabla v)((1 - t_0)x + t_0 y).$$

Hence $\theta_w(x, y) \in B_{\mu(x)}$ for every pair (x, y). As in the scalar case (see [32]) we prove that

$$(w^{-1}Op(a)w)\varphi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a\big(x, \xi + i\theta_w(x, y)\big)u(y)e^{i(x-y)\cdot\xi}\, dy$$

for $u \in S(\mathbb{R}^n, \mathcal{H}_1)$. The next step is to prove that the function $(x, y, \xi) \to a_w(x, y, \xi)$ satisfies estimates (15). Applying formulas

$$\partial_{x_k}\big(a_w(x, \xi + i\nabla v(x + t_0 y))\big) = \partial_{x_k} a_w(x, \xi + i\theta_w(x, y)) + \\ + i\sum_{k=1}^{n} \partial_{\xi_k} a_w\big(x, \xi + i\nabla v(x + t_0 y)\big)\, \frac{\partial \nabla v(x + t_0 y)}{\partial x_k}\,, \tag{55}$$

$$\partial_{y_k}\big(a_w(x, \xi + i\nabla v(x + t_0 y))\big) = \\ = i\sum_{k=1}^{n} \partial_{\xi_k} a_w\big(x, \xi + i\nabla v(x + t_0 y)\big)\, \frac{\partial \nabla v(x + t_0 y)}{\partial y_k}\,. \tag{56}$$

Taking into account that $\theta_w(x, x + y) = \nabla v(x + t_0 y)$, estimates (51), and the Leibnitz formula we obtain

$$\Big\| p_2^{-1}\big(x, \xi + i\nabla v(x + t_0 y)\big)\partial_x^\beta \partial_\xi^\alpha a\big(x, \xi + i\nabla v(x + t_0 y)\big)\times \\ \times p_1\big(x, \xi + i\nabla v(x + t_0 y)\big)\Big\|_{\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')} \le \\ \le C'_{\alpha\beta}\big\langle \xi + i\nabla v(x + t_0 y)\big\rangle^{-|\beta|} |\nabla v(x + t_0 y)|^\beta \le C'_{\alpha\beta} \tag{57}$$

for all α, β with some constants $C'_{\alpha\beta}$. Estimate (49) and spectral decomposition for the operator T yield the estimate

$$\big\| p\big(x, \xi + i\nabla v(x + t_0 y)\big) p^{-1}(x, \xi)\big\|_{\mathcal{L}(\mathcal{H})} \le C\langle y\rangle^N, \tag{58}$$

for some $C > 0$ and $N > 0$. Then estimates (57), (58) imply that

$$\big\| p_2^{-1}(x, \xi)\partial_x^\beta \partial_\xi^\alpha a_w(x, x + y, \xi) p_1(x, \xi)\big\|_{\mathcal{L}(\mathcal{H})} \le C_{\alpha\beta}\langle y\rangle^M$$

for some $C_{\alpha\beta} > 0$ and $M > 0$. Hence $a_w \in S_d(p_1, p_2)$.

(ii) Let now $a \in S_{sl}(p_1, p_2, \mu)$ and $w \in \mathcal{R}_{sl}(dq)$. Again applying the definition of $S_{sl}(p_1, p_2, \mu)$ and estimate (53) we obtain as in (57)

$$\Big\| p_2^{-1}\big(x, \xi + i\nabla v(x + t_0 y)\big)\partial_x^\beta \partial_\xi^\alpha a\big(x, \xi + i\nabla v(x + t_0 y)\big)\times \\ \times p_1\big(x, \xi + i\nabla v(x + t_0 y)\big)\Big\|_{\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')} \le \\ \le C'_{\alpha\beta}(x)\big\langle \xi + i\nabla v(x + t_0 y)\big\rangle^{-|\beta|} |\nabla v(x + t_0 y)|^\beta \le C'_{\alpha\beta}(x), \tag{59}$$

where

$$\lim_{x\to\infty} C'_{\alpha\beta}(x) = 0,$$

if $\beta \ne 0$. Estimates (58), (59) imply that

$$\big\| p_2^{-1}(x, \xi)\partial_x^\beta \partial_\xi^\alpha a_w(x, x + y, \xi) p_1(x, \xi)\big\|_{\mathcal{L}(\mathcal{H})} \le C_{\alpha\beta}(x)\langle y\rangle^M,$$

where $\lim\limits_{x\to\infty} C_{\alpha\beta}(x) = 0$ if $\beta \ne 0$. Formula (54) now follows from Proposition 16 (ii). □

4.2. **Exponential estimates.** For a C^∞-weight w, let $H(\mathbb{R}^n, p_h, w)$ denote the space of distributions with norm

$$\|u\|_{H(\mathbb{R}^n,p_h,w)} := \|wu\|_{H(\mathbb{R}^n,p_h)} < \infty. \tag{60}$$

Theorem 20. *Let $a \in S(p_{1,h}, p_{2,h}, B_{dq(x)})$ where $p_j \in O(T_j^{m_j}, q)$, $j = 1, 2$ and $w = \exp v \in \mathcal{R}(dq)$. Then the operator $Op(a) : H(\mathbb{R}^n, p_{1,h}, w) \to H(\mathbb{R}^n, p_{2,h}, w)$ is bounded.*

Theorem 21. *Let $a \in S_{sl}(p_1, p_2, B_{dq(x)})$ where $p_j \in O(T_j^{m_j}, q)$, $j = 1, 2$ and $w = \exp v \in \mathcal{R}_{sl}(\mu)$ be a weight with $\lim_{x\to\infty} v(x) = +\infty$. Assume that the operators $a(x, x, \xi + it\nabla v(x))$ are invertible for all enough large x, all $\xi \in \mathbb{R}^n$, $t \in [-1, 1]$, and*

$$\lim_{x\to\infty} \sup_{(\xi,t)\in\times\mathbb{R}^n\times[-1,1]} \left\|p_1^{-1}(x,\xi)a^{-1}(x, \xi + it\nabla v(x))p_2(x, \xi)\right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{61}$$

Finally, let $A = Op(a)$ be locally Fredholm as operator from $H(\mathbb{R}^n, p_{1,h})$ to $H(\mathbb{R}^n, p_{2,h})$.

If $f \in H(\mathbb{R}^n, p_{2,h}, w)$ then every solution of the equation $Au = f$, which a priori belongs to $H(\mathbb{R}^n, p_{1,h}, w^{-1})$, a posteriori belongs to $H(\mathbb{R}^n, p_{1,h}, w)$.

Proof. Condition (61) implies that the operators A_{w^t} are locally invertible at infinity, and the local Fredholm property of A moreover implies that these operators are locally Fredholm for each $t \in [-1, 1]$. Hence, by Theorem 18, each operator $A_{w^t} : H(\mathbb{R}^n, p_{1,h}) \to L^2(\mathbb{R}^n, p_{2,h})$ has the Fredholm property. Note that the symbol of A_{w^t} is given by

$$\sigma_{A_{w^t}}(x, \xi) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} a(x, y, \xi + it\theta_w(x, y))e^{-iy\cdot\xi}\, dy\, d\xi. \tag{62}$$

This formula shows that the mapping $[-1, 1] \to S(p_1, p_2)$, $t \mapsto \sigma_{A_{w^t}}$ is continuous. Thus, and by Proposition 13, the mapping

$$[-1, 1] \to \mathcal{L}(H(\mathbb{R}^n, p_{1,h}), H(\mathbb{R}^n, p_{2,h})), \quad t \mapsto A_{w^t}$$

is continuous. This shows that the Fredholm index of the operator $A_{w^t} : H(\mathbb{R}^n, p_1) \to H(\mathbb{R}^n, p_2)$ does not depend on $t \in [-1, 1]$. Hence, the operator A, considered as operator from $H(\mathbb{R}^n, p_{1,h}, w)$ to $H(\mathbb{R}^n, p_{2,h}, w)$, and the same operator A, but now considered as operator from $H(\mathbb{R}^n, p_{1,h}, w^{-1})$ to $H(\mathbb{R}^n, p_{2,h}, w^{-1})$, are Fredholm with the same Fredholm indices. Further, since $H(\mathbb{R}^n, p_h, w)$ is a dense subset of $H(\mathbb{R}^n, p_h, w^{-1})$ for $j = 1, 2$, we conclude that the kernel of A, considered as operator from $H(\mathbb{R}^n, p_{1,h}, w)$ to $H(\mathbb{R}^n, p_{2,h}, w)$, coincides with the kernel of A, now considered as operator from $H(\mathbb{R}^n, p_{1,h}, w^{-1})$ to $H(\mathbb{R}^n, p_{2,h}, w^{-1})$. Finally, if $u \in H(\mathbb{R}^n, p_{1,h}, w^{-1})$ is a solution of the equation $Au = f$ with $f \in H(\mathbb{R}^n, p_{2,h}, w)$, then $u \in H(\mathbb{R}^n, p_{1,h}, w^{-1})$ (see, for instance, [23, p. 308]). □

5. Schrödinger Operators with Operator-Valued Potentials

5.1. **Fredholm property.** Let T be a positive self-adjoint operator on a separable Hilbert space $\mathcal{H}$ with a dense domain D_T. Suppose that, for each $x \in \mathbb{R}^n$, we are given a bounded linear operator $L(x) : D_{T^{1/2}} \to D_{T^{-1/2}}$ which is symmetric on $D_{T^{1/2}}$, i.e.,

$$\big\langle L(x)\varphi, \psi\big\rangle_{\mathcal{H}} = \big\langle \varphi, L(x)\psi\big\rangle_{\mathcal{H}} \ \text{ for all } \ \varphi,\, \psi \in D_{T^{1/2}}.$$

We assume that the function $x \mapsto L(x)$ is strongly differentiable and that

$$\sup_{x\in\mathbb{R}^n} \left\| \big(T + \langle x\rangle^m I\big)^{-1/2} \partial_x^\beta L(x) \big(T + \langle x\rangle^m I\big)^{-1/2} \right\|_{\mathcal{L}(\mathcal{H})} < \infty, \ \ m \geq 0 \quad (63)$$

for every multiindex β. Moreover, we suppose that

$$\lim_{x\to\infty} \left\| \big(T + \langle x\rangle^m I\big)^{-1/2} \partial_x^\beta L(x) \big(T + \langle x\rangle^m I\big)^{-1/2} \right\|_{\mathcal{L}(\mathcal{H})} = 0 \qquad (64)$$

if $\beta \neq 0$.

We consider the Schrödinger operator

$$(\mathbf{H}u)(x) := -\partial_{x_j} \rho^{jk}(x) \partial_{x_k} u(x) + L(x)u(x), \ \ x \in \mathbb{R}^n, \qquad (65)$$

on the Hilbert space $L^2(\mathbb{R}^n, \mathcal{H})$ of vector-functions with values in $\mathcal{H}$. In (65) and in what follows, we make use of the Einstein summation convention. We will assume that $\rho^{jk} \in C_b^\infty(\mathbb{R}^n, \mathcal{L}(\mathcal{H}))$ and

$$\lim_{x\to\infty} \partial_{x_l} \rho^{jk}(x) = 0 \ \text{ for } \ l = 1, \dots, n; \qquad (66)$$

$\rho^{kj}(x) = (\rho^{jk}(x))^*$, and there is a $C > 0$ such that, for every $\varphi \in \mathcal{H}$,

$$\langle \rho^{jk}(x)\xi_j\xi_k\varphi, \varphi\rangle_{\mathcal{H}} \geq C|\xi|^2 \|\varphi\|_{\mathcal{H}}^2. \qquad (67)$$

Let

$$p(x,\xi) := \Big(\big(|\xi|^2 + \langle x\rangle^m\big)I + T\Big)^{1/2},$$

and write $H(\mathbb{R}^n, p)$ for the Hilbert space with norm

$$\|u\|_{H(\mathbb{R}^n, p_h)} := \|Op(p_h)u\|_{L^2(\mathbb{R}^n,\mathcal{H})},$$

for fixed $h > 0$ enough small. The estimates (63), (64) and (66) imply that $\mathbf{H}$ is a pseudodifferential operator in the class $OPS_{sl}(p^{-1}, p)$ with symbol

$$\sigma_{\mathbb{H}}(x,\xi) = \rho^{jk}(x)\xi_j\xi_k + i\frac{\partial \rho^{jk}(x)}{\partial x_j}\xi_k + L(x).$$

The following theorem states conditions of the Fredholmness of the operator $\mathbf{H} : H(\mathbb{R}^n, p_h) \to H(\mathbb{R}^n, p_h^{-1})$.

Theorem 22. *Let conditions* (63)–(67) *hold, and assume there are constants* $R > 0$ *and* $C > 0$ *such that*

$$\Re\langle L(x)\varphi, \varphi\rangle_{\mathcal{H}} \geq \gamma\big\langle \big(T + \langle x\rangle^m I\big)\varphi, \varphi\big\rangle_{\mathcal{H}}, \ \ \gamma > 0 \qquad (68)$$

for every $x \in B_R'$ *and every vector* $\varphi \in D_{T^{1/2}}$. *If the operator* $\mathbf{H} : H(\mathbb{R}^n, p_h) \to H(\mathbb{R}^n, p_h^{-1})$ *is locally Fredholm, then it is already a Fredholm operator.*

Proof. Conditions (67) and (68) imply that there exist $C > 0$ and $R > 0$ such that, for every $x \in B'_R$ and every $\varphi \in D_{T^{1/2}}$,

$$\Re\langle\sigma_{\mathbf{H}}(x,\xi)\varphi,\varphi\rangle_{\mathcal{H}} \geq C\Big\langle\big((|\xi|^2 + \langle x\rangle^m)I + T\big)\varphi,\varphi\Big\rangle_{\mathcal{H}}. \tag{69}$$

It follows from estimate (69) that, for every $x \in B'_R$ and every $\psi \in \mathcal{H}$,

$$\Re\Big\langle\big((|\xi|^2 + \langle x\rangle^m)I + T\big)^{-1/2}\sigma_{\mathbf{H}}(x,\xi)\big((|\xi|^2 + \langle x\rangle^m)I + T\big)^{-1/2}\psi,\psi\Big\rangle_{\mathcal{H}} \geq \\ \geq C\|\psi\|^2_{\mathcal{H}}. \tag{70}$$

This estimate yields that the operator

$$\Big((|\xi|^2 + \langle x\rangle^m I) + T\Big)^{-1/2}\sigma_{\mathbf{H}}(x,\xi)\Big((|\xi|^2 + \langle x\rangle^m I)I + T\Big)^{-1/2}$$

is invertible on $\mathcal{H}$ for every $x \in B'_R$ and every $\xi \in \mathbb{R}^n$ and that

$$\sup_{(x,\xi)\in B'_R\times\mathbb{R}^n}\Big\|\big((|\xi|^2+\langle x\rangle^m)I+T\big)^{1/2}\sigma^{-1}_{\mathbf{H}}(x,\xi)\big((|\xi|^2+\langle x\rangle^m)I+T\big)^{1/2}\Big\|_{\mathcal{L}(\mathcal{H})} < \\ < C^{-1}. \tag{71}$$

Hence, the conditions of Theorem 18 are satisfied, and $\mathbf{H}$ has the Fredholm property as operator from $H(\mathbb{R}^n, p_h)$ to $H(\mathbb{R}^n, p_h^{-1})$. □

5.2. Exponential estimates.

Theorem 23. *Let*

$$\mathbf{H}u(x) = -\Delta u(x) + L(x)u(x) = f(x), \tag{72}$$

be the Schrödinger equation with potential $x \to L(x)$ satisfies conditions (63), (64) *and* (68). *Let $w(x) = \exp d\langle x\rangle^{\frac{m+2}{2}}$ be the weight, where*

$$d = \frac{\sqrt{\gamma}}{\frac{m}{2}+1} - \varepsilon, \ \ \varepsilon > 0$$

and $f \in H(\mathbb{R}^n, p_h, w)$. Then every solution of the equation (72) a priory in the space $H(\mathbb{R}^n, p_h, w^{-1})$ a posteriori belongs to the space $H(\mathbb{R}^n, p_h, w)$.

Proof. We have

$$\Re\langle\sigma_{\mathbb{H}}(x,\xi+it\nabla v(x))\varphi,\varphi\rangle \geq \Big\langle\Big(|\xi|^2 - t^2d^2\Big(\frac{m}{2}\Big)^2\langle x\rangle^m\Big)I + L(x)\varphi,\varphi\Big\rangle \geq \\ \geq \Big\langle\Big(|\xi|^2 + \Big(\gamma - t^2d^2\Big(\frac{m}{2}\Big)^2\Big)\langle x\rangle^m\Big)I + T\varphi,\varphi\Big\rangle \geq \\ \geq C\Big\langle\Big(|\xi|^2 + \langle x\rangle^m\Big)I + T\varphi,\varphi\Big\rangle, \tag{73}$$

for some $C > 0$ and for every $\varphi \in D_{T^{1/2}}$. As in the proof of Theorem 22, we conclude from (73) that

$$\sup_{(x,\xi,t)\in B'_R\times\mathbb{R}^n\times[-1,1]}\Big\|\big((|\xi|^2 + \langle x\rangle^m)I + T\big)^{1/2}\sigma^{-1}_{\mathbb{H}}\big(x,\xi + it\nabla v(x)\big)\times \\ \times\big((|\xi|^2 + \langle x\rangle^m)I + T\big)^{1/2}\Big\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Thus, all conditions of Theorem 21 are satisfied. □

5.3. **Quantum waveguides.** Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^m_y$ with a sufficiently regular boundary, and let Φ be a real valued function in the space $C^\infty(\Pi)$, where $\Pi = \mathbb{R}^n \times \mathcal{D}$. We suppose that for all β, γ there exist $C_{\beta\gamma} > 0$ such that

$$\left|\partial_x^\beta \partial_y^\gamma \Phi(x, y)\right| \leq C_{\beta\gamma} \langle x\rangle^{m-\delta|\beta|}, \quad \delta \in (0, 1]. \tag{74}$$

We consider the spectral problem for the Schrödinger equation in the quantum waveguide, i.e. the problem

$$\big((\mathbf{H} - \lambda I)u\big)(x, y) = \big(- \Delta_x - \Delta_y + \Phi(x, y) - \lambda\big)u(x, y) = 0, \tag{75}$$
$$(x, y) \in \mathbb{R}^n \times \mathcal{D} =: \Pi, \quad u\big|_{\partial\mathcal{D}} = 0, \quad k \in \mathbb{N}.$$

This problem describes the bound states of a quantum system with the electric potential Φ on the configuration space Π. We suppose that

$$\liminf_{x\to\infty} \inf_{y\in\mathcal{D}} \Phi(x, y)\langle x\rangle^{-m} \geq \gamma > 0. \tag{76}$$

The operator $\mathbf{H} - \lambda I$ can be realized as a pseudodifferential operator with operator-valued symbol $\sigma_{\mathbf{H}-\lambda I}(x, \xi) = |\xi|^2 I + L_\lambda(x)$, where

$$\big(L_\lambda(x)\varphi\big)(y) = \big(- \Delta_y + \tilde{\Phi}(x) - \lambda I\big)\varphi(y) \;\text{ for }\; y \in \mathcal{D}, \;\; \varphi\big|_{\partial\mathcal{D}} = 0$$

is the operator of the Dirichlet problem in $\mathcal{D}$ depending on the parameter $x \in \mathbb{R}^n$, where $(\tilde{\Phi}(x)\varphi)(x) := \Phi(x, y)\varphi(y)$ for $y \in \mathcal{D}$.

Let T be the operator of the Dirichlet problem for the Laplacian $-\Delta_y$ in the domain $\mathcal{D}$, considered as an unbounded operator on $\mathcal{H} = L^2(\mathcal{D})$ with domain $\mathring{H}^2(\mathcal{D}) = \big\{\varphi \in H^2(\mathcal{D}) : \; \varphi|_{\partial\mathcal{D}} = 0\big\}$ where $H^2(\mathcal{D})$ is the standard Sobolev space on $\mathcal{D}$. It is well-known that T is a positive define operator.

We set $p(x, \xi) = ((\xi^2 + \langle x\rangle^m)I + T)^{1/2}$. Then

$$\left\|p^{-1}(x, \xi)\partial_x^\beta \partial_\xi^\alpha \sigma_{\mathbf{H}-\lambda I}(x, \xi)p^{-1}(x, \xi)\right\|_{\mathcal{L}(L^2(\mathcal{D}))} \leq C_{\alpha\beta}$$

for all α, β. Hence $\sigma_{\mathbf{H}-\lambda I} \in S(p^{-1}, p)$. Moreover one can prove that condition (76) provides that $\sigma_{\mathbf{H}-\lambda I} \in S_{sl}(p^{-1}, p)$.

Let $H_h(\mathbb{R}, p)$ is the set of the distributions $u \in S'(\mathbb{R}^n, \mathcal{H})$ such that

$$\|u\|_{H_h(\mathbb{R}^n,p)} := \left\|(-h^2\Delta_x + \langle x\rangle^m + T)^{1/2}u\right\|_{L^2(\mathbb{R}^n,\mathcal{H})} < \infty,$$

where $h > 0$ is small enough such that $Op(h^2|\xi|^2 + \langle x\rangle^m + T)^{1/2}$ is invertible operator. One can prove that the $H_h(\mathbb{R}^n, p)$ within equivalent norms coincides with the closure of $C_0^\infty(\Pi)$ in the norm

$$\|u\|_{H(\mathbb{R}^n,p)} = \Big(\|u\|^2_{\mathring{H}^1(\Pi)} + \|\langle x\rangle^m u\|_{L^2(\Pi)}\Big)^{1/2}.$$

Consider now the problem of Fredholmness of the operator

$$\mathbf{H} - \lambda I : H_h(\mathbb{R}^n, p) \to H_h(\mathbb{R}^n, p^{-1}).$$

Theorem 24. *The operator* $\mathbf{H} - \lambda I : H_h(\mathbb{R}^n, p) \to H_h(\mathbb{R}^n, p^{-1})$ *is a Fredholm operator for every* $\lambda \in \mathbb{C}$.

Proof. It follows from standard local elliptic estimates for the Dirichlet problem in bounded domains that the operator $\mathbf{H} - \lambda I : H(\mathbb{R}^n, p_h) \to H(\mathbb{R}^n, p_h^{-1})$ is locally Fredholm. Conditions (76) implies condition (68) of Theorem 22. Hence $\mathbf{H} - \lambda I$ is locally invertible at infinity for every $\lambda \in \mathbb{C}$. It implies by Theorem 18 that $\mathbf{H} - \lambda I : H(\mathbb{R}^n, p_h) \to H(\mathbb{R}^n, p_h^{-1})$ is the Fredhom operator for every $\lambda \in \mathbb{C}$. □

Note that the operator $\mathbf{H}$ can be considered as an unbounded closed operator in $L^2(\Pi)$ with the domain $H(\mathbb{R}^n, p_h)$. Theorem 24 has the following corollary.

Corollary 25. *The operator* $\mathbf{H}$ *as unbounded has a discrete spectrum.*

Proof. Let $\lambda < \mu = \inf_\Pi \Phi(x, y)$. Then $\mathbf{H} - \lambda I : H(\mathbb{R}^n, p_h) \to H(\mathbb{R}^n, p_h^{-1})$ is inverible. Hence by the Theorem on the Analytic Fredholmness $\mathbf{H} - \lambda I : H(\mathbb{R}^n, p_h) \to H(\mathbb{R}^n, p_h^{-1})$ is invertible for all $\lambda \in \mathbb{R}$ except of a discrete set Λ of points λ for which $\ker(\mathbf{H} - \lambda I)$ has a finite dimension. Taking into account that the spectrum of $\mathbf{H}$ as unbounded operator coincides with the spectrum of $\mathbf{H}$ as a bounded operator acting from $H(\mathbb{R}^n, p_h)$ in $H(\mathbb{R}^n, p_h^{-1})$, and that $\mathbf{H} - \lambda I$ is a Fredholm operator as unbounded if and only if $\mathbf{H} - \lambda I : H(\mathbb{R}^n, p_h) \to H(\mathbb{R}^n, p_h^{-1})$ is a Fredholm operator we obtain the assertion of the corollary. □

Theorem 23 implies the exponential estimates of eigenfunctions of $\mathbf{H}$.

Theorem 26. *Every eigenfunction* u_λ *of the operator* $\mathbf{H}$ *belongs to* $H(\mathbb{R}^n, p_h, w)$, *where* $w(x) = \exp d\langle x\rangle^{\frac{m+2}{2}}$ *with*

$$d = \frac{\sqrt{\gamma}}{\frac{m}{2} + 1} - \varepsilon, \quad \varepsilon > 0.$$

In particular

$$\int_\Pi \left|u_\lambda(x, y)\right|^2 e^{2d\langle x\rangle^{\frac{m+2}{2}}}\, dx\, dy < \infty.$$

Example 27. Let the potential Φ be of the form

$$\Phi(x, y) = \Psi(x, y) + |x|^2,$$

where $\Psi \in C_b^\infty(\Pi)$. Hence (75) is a spectral problem for a perturbed Harmonic oscillator in the wavegide Π. In this case $p(x, \xi) = (1 + |\xi|^2 + |x|^2 + T)^{1/2}$. The unbounded operator $\mathbf{H}$ with domain $H(\mathbb{R}^n, p_h)$ has a discrete spectrum and the eigenfunctions u_λ satisfies the estimates

$$\int_\Pi |u_\lambda(x, y)|^2 e^{(1-\varepsilon)|x|^2}\, dx\, dy < \infty.$$

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Short Communication

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THE PRINCIPLE OF A PRIORI BOUNDEDNESS FOR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF NONLINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. A general theorem (principle of a priori boundedness) on solvability of the boundary value problem

$$dx(t) = dA(t) \cdot f(t, x(t)), \quad h(x) = 0$$

is established, where $A : [a, b] \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a nondecreasing matrix-function, $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector-function belonging to the Carathéodory class corresponding to the matrix-function A, and $h : \mathrm{BV}_s([a, b], \mathbb{R}^n) \to \mathbb{R}^n$ is a continuous operator.

რეზიუმე. მოყვანილია ზოგადი თეორემა (აპრიორული შემოსაზღვრულობის პრინციპი)

$$dx(t) = dA(t) \cdot f(t, x(t)), \quad h(x) = 0$$

სასაზღვრო ამოცანის ამოხსნადობის შესახებ, სადაც $A : [a, b] \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ არაკლებადი მატრიცული ფუნქციაა, $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ არის A მატრიცის შესაბამისი კარათეოდორის კლასის ფუნქცია, ხოლო $h : \mathrm{BV}_s([a, b], \mathbb{R}^n) \to \mathbb{R}^n$ კი უწყვეტი ოპერატორია.

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Let n be a natural number, $[a, b]$ be a closed interval of real axis, $A = (a_{ik})_{i,k=1}^{n} : [a, b] \to \mathbb{R}^{n\times m}$ be a nondecreasing matrix-function, f be a vector-function belonging to the Carathéodory class corresponding to the matrix-function A, and let $h : \mathrm{BV}_s([a, b], \mathbb{R}^n) \to \mathbb{R}^n$ be a continuous operator satisfying the condition

$$\sup\Big\{\|h(x)\| : \; x \in \mathrm{BV}_s([a, b], \mathbb{R}^n), \; \|x\|_s \le \rho\Big\} < +\infty$$

for every $\rho \in \,]0, +\infty[\,$.

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Consider the nonlinear system of generalized ordinary differential equations

$$dx(t) = dA(t) \cdot f(t, x(t)) \tag{1}$$

with the boundary condition

$$h(x) = 0. \tag{2}$$

The theorem on the existence of a solution of the problem (1), (2) which will be given below and be called the principle of a priori boundedness, generalizes the well-known Conti–Opial type theorems (see [8], [16]) and supplements earlier known criteria for the solvability of nonlinear boundary value problems for systems of generalized ordinary differential equations ([1], [2], [5], [6], [16]).

Analogous and related questions are investigated in [9]–[14] for the boundary value problems for the nonlinear systems of ordinary differential and functional differential equations. In the paper we use the methods of investigation given in [10] and [11].

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see, i.e., [1]–[7], [15], [17] and the references therein).

Throughout the paper the following notation and definitions will be used.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$, $[a, b]$, $]a, b[$, $[a, b[$ and $]a, b]$ $(a, b \in R)$ are, respectively, a closed, an open and semi-open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|;$$

$$\mathbb{R}_+^{n\times m} = \Big\{(x_{ij})_{i,j=1}^{n,m} : \ x_{ij} \geq 0 \ (i = 1, \dots, n; \ j = 1, \dots, m)\Big\}.$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = R_+^{n\times 1}$.

$\overset{b}{\underset{a}{\vee}}(X)$ is the total variation of the matrix-function $X : [a, b] \to R^{n\times m}$, i.e., the sum of total variations of the latter's components x_{ij} $(i = 1, \dots, n;\ j = 1, \dots, m)$; $V(X)(t) = (V(x_{ij})(t))_{i,j=1}^{n,m}$, where $V(x_{ij})(a) = 0$, $V(x_{ij})(t) = \overset{t}{\underset{a}{\vee}}(x_{ij})$ for $a < t \leq b$;

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$ at the point t (we will assume $X(t) = X(a)$ for $t \leq a$ and $X(t) = X(b)$ for $t \geq b$, if necessary);

$$d_1X(t) = X(t) - X(t-), \quad d_2X(t) = X(t+) - X(t);$$

$$\|X\|_s = \sup\big\{\|X(t)\| : \ t \in [a, b]\big\},$$

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all matrix-functions of bounded total variations $X:[a,b]\to\mathbb{R}^{n\times m}$ (i.e., such that $\overset{b}{\underset{a}{\vee}}(X)<+\infty$);

$\mathrm{BV}_s([a,b],\mathbb{R}^n)$ is the normed space $(\mathrm{BV}([a,b],\mathbb{R}^n),\|\cdot\|_s)$;

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If $I\subset\mathbb{R}$ is an interval, then $C(I,\mathbb{R}^{n\times m})$ is the set of all continuous matrix-functions $X:I\to\mathbb{R}^{n\times m}$.

$s_j:\mathrm{BV}([a,b],\mathbb{R})\to\mathrm{BV}([a,b],\mathbb{R})$ $(j=0,1,2)$ are the operators defined, respectively, by

$$s_1(x)(a)=s_2(x)(a)=0,$$

$$s_1(x)(t)=\sum_{a<\tau\le t}d_1x(\tau)\ \text{ and }\ s_2(x)(t)=\sum_{a\le\tau<t}d_2x(\tau)\ \text{ for }\ a<t\le b,$$

and

$$s_0(x)(t)=x(t)-s_1(x)(t)-s_2(x)(t)\ \text{ for }\ t\in[a,b].$$

If $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, $x:[a,b]\to\mathbb{R}$ and $a\le s<t\le b$, then

$$\int\limits_s^t x(\tau)\,dg(\tau)=$$

$$=\int\limits_{]s,t[}x(\tau)\,ds_0(g)(\tau)+\sum_{s<\tau\le t}x(\tau)d_1g(\tau)+\sum_{s\le\tau<t}x(\tau)d_2g(\tau),$$

where $\int\limits_{]s,t[}x(\tau)\,ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu(s_0(g))$ corresponding to the function $s_0(g)$; if $a=b$, then we assume $\int\limits_a^b x(t)\,dg(t)=0$;

$L([a,b],R;g)$ is the space of all functions $x:[a,b]\to\mathbb{R}$ measurable and integrable with respect to the measures $\mu(g)$ with the norm

$$\|x\|_{L,g}=\int\limits_a^b|x(t)|\,dg(t).$$

If $G=(g_{ik})_{i,k=1}^{l,n}:[a,b]\to\mathbb{R}^{l\times n}$ is a nondecreasing matrix-function and $D\subset R^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X=(x_{kj})_{k,j=1}^{n,m}:[a,b]\to D$ such that $x_{kj}\in L([a,b],\mathbb{R};g_{ik})$ $(i=1,\dots,l;\ k=1,\dots,n;\ j=1,\dots,m)$;

$$\int\limits_s^t dG(\tau)\cdot X(\tau)=\bigg(\sum_{k=1}^n\int\limits_s^t x_{kj}(\tau)dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}\ \text{ for }\ a\le s\le t\le b,$$

$$S_j(G)(t)\equiv\big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n}\ \ (j=0,1,2).$$

If $D_1 \subset R^n$, $D_2 \subset \mathbb{R}^{n\times m}$ and $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to \mathbb{R}^{l\times n}$, then $K([a,b]\times D_1, D_2; G)$ is the Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that for each $i \in \{1,\dots,l\}$, $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$:

(i) the function $f_{kj}(\cdot\,,x) : [a,b] \to \mathbb{R}$ is $\mu(g_{ik})$ measurable for every $x \in D_1$;

(ii) the function $f_{kj}(t,\cdot) : D_1 \to \mathbb{R}$ is continuous, $\mu(g_{ik})$ almost for every $t \in [a,b]$, and

$$\sup\big\{|f_{kj}(.,x)| : \ x \in D_0\big\} \in L([a,b],\mathbb{R}; g_{ik})$$

for every compact $D_0 \subset D_1$.

If $G(t) \equiv \mathrm{diag}(t,\dots,t)$, then we omit G in the notation containing G.

The inequalities between the vectors and between the matrices are understood componentwise.

A vector-function $x \in \mathrm{BV}([a,b],\mathbb{R}^n)$ is said to be a solution of the system (1) if

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot f(\tau,x(\tau)) \ \text{ for } \ a \le s \le t \le b.$$

Under the solution of the problem (1), (2) we mean solutions of the system (1) satisfying (2).

We assume that $g(t) \equiv \|A(t)\|$.

Definition 1. The pair (P,l) of a matrix-function $P \in K([a,b]\times\mathbb{R}^n, \mathbb{R}^{n\times n}; A)$ and a continuous operator $l : \mathrm{BV}_s([a,b],\mathbb{R}^n)\times\mathrm{BV}_s([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ is said to be consistent if:

(i) for any fixed $x \in \mathrm{BV}_s([a,b],\mathbb{R}^n)$ the operator $l(x,.) : \mathrm{BV}_s([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ is linear;

(ii) for any $z \in \mathbb{R}^n$, x and $y \in \mathrm{BV}_s([a,b],\mathbb{R}^n)$ and for $\mu(g)$ almost all $t \in [a,b]$ the inequalities

$$\|P(t,z)\| \le \alpha(t,\|z\|), \ \ \|l(x,y)\| \le \alpha_0(\|x\|_s)\cdot\|y\|_s$$

are fulfilled, where $\alpha_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function, and $\alpha : [a,b]\times\mathbb{R}_+ \to \mathbb{R}_+$ is a function, measurable and integrable with respect to the measure $\mu(g)$ in the first argument and nondecreasing in the second one;

(iii) there exists a positive number β such that for any $y \in \mathrm{BV}_s([a,b],\mathbb{R}^n)$, $q \in L([a,b],\mathbb{R}^n; A)$ and $c_0 \in \mathbb{R}^n$, for which the condition

$$\det\big(I_n + (-1)^j d_jA(t)\cdot P(t,y(t))\big) \ne 0 \ \text{ for } \ t \in [a,b] \ \ (j=1,2)$$

holds, an arbitrary solution y of the boundary value problem

$$dx(t) = dA(t)\cdot\big(P(t,y(t))\cdot x(t) + q(t)\big), \ \ l(x,y) = c_0$$

admits the estimate

$$\|y\|_s \le \beta(\|c_o\| + \|q\|_{L,g}).$$

Theorem 1. *Let there exist a positive number ρ and a consistent pair (P, l) of a matrix-function $P \in K([a,b] \times \mathbb{R}^n, \mathbb{R}^{n\times n}; A)$ and a continuous operator $l : \mathrm{BV}_s([a,b],\mathbb{R}^n) \times \mathrm{BV}_s([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ such that for any $\lambda \in]0,1[$ an arbitrary solution of the problem*

$$dx(t) = dA(t) \cdot \Big(P(t,x(t)) \cdot x(t) + \lambda\big[f(t,x(t)) - P(t,x(t)) \cdot x(t)\big]\Big), \quad (3)$$

$$l(x,y) = \lambda[l(x,x) - h(x)] \quad (4)$$

admits the estimate

$$\|x\|_s \le \rho. \quad (5)$$

Then the problem (1), (2) *is solvable.*

Definition 2. Let $P \in K([a,b] \times \mathbb{R}^n, \mathbb{R}^{n\times n}; A)$. We say that a matrix-function $B_0 \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ belongs to the set $\mathcal{E}_{A,P}$ if the condition

$$\det\big(I_n + (-1)^j d_j B_0(t)\big) \ne 0 \ \text{ for } \ t \in [a,b] \ \ (j=1,2) \quad (6)$$

holds and there exists a sequence $x_k \in \mathrm{BV}([a,b],\mathbb{R}^n)$ $(k = 1,2,\dots)$ such that

$$\lim_{k\to+\infty} \int\limits_a^t dA(\tau) \cdot P(\tau, x_k(\tau)) = B_0(t) \ \text{ uniformly on } \ [a,b]. \quad (7)$$

Definition 3. We say that the pair (P, l) of the matrix-function $P \in K([a,b] \times \mathbb{R}^n, \mathbb{R}^{n\times n}; A)$ and the continuous operator $l : \mathrm{BV}_s([a,b],\mathbb{R}^n) \times \mathrm{BV}_s([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ belongs to the Opial class $\mathcal{O}_0^A$ with respect to the matrix-function A if:

(i) for any fixed $x \in \mathrm{BV}_s([a,b],\mathbb{R}^n)$ the operator $l(x,.) : \mathrm{BV}_s([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ is linear;

(ii) for any $z \in \mathbb{R}^n$, x and $y \in \mathrm{BV}_s([a,b],\mathbb{R}^n)$ and for $\mu(g)$ almost all $t \in [a,b]$ the inequalities

$$\|P(t,z)\| \le \alpha(t), \quad (8)$$
$$\|l(x,y)\| \le \alpha_0 \|y\|_s$$

are fulfilled, where $\alpha_0 \in \mathbb{R}_+$, and $\alpha : I \to \mathbb{R}_+$ is a function measurable and integrable with respect to the measure $\mu(g)$;

(iii) for every matrix-function $B_0 \in \mathcal{E}_{A,P}$ the following condition holds: if y is a solution of the system

$$dy(t) = dB_0(t) \cdot y(t),$$

and, in addition,

$$\lim_{k\to+\infty} l(x_k, y) = 0$$

for some sequence $x_k \in \mathrm{BV}_s([a,b],\mathbb{R}^n)$ $(k=1,2,\dots)$, then $y(t) \equiv 0$.

Remark 1. By equalities (7) and (8) the condition

$$\|d_j A(t)\| \cdot \alpha(t) < 1 \ \text{ for } \ t \in [a,b] \ \ (j=1,2)$$

guarantees the condition (6).

Corollary 1. *Let there exist a positive number ρ and a pair $(P,l)\in\mathcal{O}_0^A$ such that for any $\lambda\in\,]0,1[$ an arbitrary solution of the problem* (3), (4) *admits the estimate* (5). *Then the problem* (1), (2) *is solvable.*

The following result belongs to Z. Opial (see, [9], [16]).

Corollary 2. *Let the pair $(P,l)\in\mathcal{O}_0^A$ be such that*

$$\big|f(t,x)-P(t,x)x\big|\le\alpha(t,\|x\|)\ \ for\ \ t\in[a,b],\ \ x\in\mathbb{R}^n, \tag{9}$$

$$|h(x)-l(x)|\le l_0(|x|)+l_1(\|x\|_s)\ \ for\ \ x\in\mathrm{BV}_s([a,b],\mathbb{R}^n), \tag{10}$$

where $\alpha\in K([a,b]\times\mathbb{R}_+,\mathbb{R}_+^n;A)$ is a nondecreasing in second variable vector-function, $l_0:\mathrm{BV}_s([a,b],\mathbb{R}_+^n)\to\mathbb{R}_+^n$ is a positive homogeneous continuous operator, $l_1\in C(\mathbb{R}_+,\mathbb{R}_+^n)$. Let, moreover,

$$\lim_{k\to+\infty}\frac{1}{\rho}\int\limits_a^b dV(A)(\tau)\cdot\alpha(\tau,\rho)=\lim_{\rho\to+\infty}\frac{\|l_1(\rho)\|}{\rho}=0.$$

Then the problem (1), (2) *is solvable.*

By $Y_P(x)$ we denote the fundamental matrix of the system

$$dy(t)=dA(t)\cdot P(t,x(t))y(t))$$

for every $x\in\mathrm{BV}_s([a,b],\mathbb{R}^n)$, satisfying the condition $Y_P(x)(a)=I_n$.

Corollary 3. *Let conditions* (9) *and* (10) *hold, where P and l are, respectively, matrix-function and operator, satisfying the conditions* (i) *and* (ii) *of Definition* 3; *$l_0:\mathrm{BV}_s([a,b],\mathbb{R}_+^n)\to\mathbb{R}_+^n$ is a positive homogeneous continuous operator, and a nondecreasing in second variable vector-function $\alpha\in K([a,b]\times\mathbb{R}_+,\mathbb{R}_+^n;A)$ and a vector-function $l_1\in C(\mathbb{R}_+,\mathbb{R}_+^n)$ are such that the condition*

$$\inf\Big\{\big|\det\big(l(x,Y_P(x))\big)\big|:\ x\in\mathrm{BV}_s([a,b],\mathbb{R}^n)\Big\}>0$$

holds. Then the problem (1), (2) *is solvable.*

Corollary 4. *Let $P(t,x)\equiv P_0(t)$ and $l(x,y)\equiv l_0(y)$, where $P_0\in L([a,b],\mathbb{R}^{n\times n};A)$, and $l_0:\mathrm{BV}_s([a,b],R^n)\to\mathbb{R}^n$ is a bounded linear operator such that*

$$\det\big(I_n+(-1)^j d_jA(t)\cdot P_0(t)\big)\ne0\ \ for\ \ t\in[a,b]\ \ (j=1,2)$$

and the problem

$$dy(t)=dA(t)\cdot P_0(t)y(t),\ \ l_0(y)=0$$

has only the trivial solution. Let, moreover, there exist a positive number ρ such that for every $\lambda\in\,]0,1[$ an arbitrary solution of the problem

$$dx(t)=dA(t)\cdot\big(P_0(t)\cdot x(t)+\lambda[f(t,x(t))-P_0(t)\cdot x(t)]\big),$$
$$l_0(x)=\lambda[l_0(x)-h(x)]$$

admits the estimate (5). *Then the problem* (1), (2) *is solvable.*

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CORRECTIONS

to the paper [M. A. Grekov and N. F. Morozov, Solution of the Kirsch Problem in View of Surface Stresses. *Mem. Differential Equations Math. Phys.* **52** (2011), 123–129]

The corresponding formulae from the published paper should be replaced by the following ones:

$$\sigma_{rr} + i\sigma_{r\theta} = \sigma^s_{\theta\theta} - i\,\frac{\partial\sigma^s_{\theta\theta}}{\partial\theta} \equiv t^s, \quad r = 1, \tag{3}$$

$$I^{\pm}(\zeta) = \pm\frac{\tau(\zeta)}{2} \pm \frac{\zeta\tau'(\zeta)}{2} + \frac{1}{2\pi i}\int\limits_{|\eta|=1} \frac{\tau(\eta) + \eta\tau'(\eta)}{\eta - \zeta}\,\eta, \tag{20}$$

$$\begin{aligned}&[2r - M(\varkappa - 1)]\tau(\zeta) - M(\varkappa + 1)\times\\ &\quad\times\left[\frac{1}{2\pi i}\int\limits_{|\eta|=1} \frac{\tau(\eta) + \eta\tau'(\eta)}{\eta - \zeta}\,d\eta - \frac{1}{2\pi i}\int\limits_{|\eta|=1} \frac{\overline{\tau(\eta)} + \overline{\eta\tau'(\eta)}}{\bar\eta - \bar\zeta}\,d\bar\eta\right] =\\ &\qquad\qquad = \frac{Mr(\varkappa+1)}{2}\,\sigma\big(1 - \zeta^2 - \zeta^{-2}\big).\end{aligned} \tag{21}$$

$$\begin{aligned}&[2r - M(\varkappa - 1)]\tau(\zeta) - M(\varkappa + 1)\times\\ &\quad\times\left[\frac{1}{2\pi i}\int\limits_{|\eta|=1} \frac{\tau(\eta) + \eta\tau'(\eta)}{\eta - \zeta}\,d\eta - \frac{\zeta}{2\pi i}\int\limits_{|\eta|=1} \frac{\eta^{-1}\tau(\eta) - \tau'(\eta)}{\eta - \zeta}\,d\eta\right] =\\ &\qquad\qquad = \frac{Mr(\varkappa+1)}{2}\,\sigma\big(1 - \zeta^2 - \zeta^{-2}\big).\end{aligned} \tag{22}$$

$$d_0 = \frac{Mr(1+\varkappa)}{4(r+M)}\sigma, \quad d_2 = \overline{d_{-2}} = -\frac{Mr(1+\varkappa)}{2[2r + M(3+\varkappa)]}\,\sigma, \quad d_k = 0, \quad k \neq 0, -2, 2. \tag{24}$$

$$\sigma_{\theta\theta} = -\frac{d_0}{r} - \frac{6d_2}{r}\,\sigma\cos 2\theta + (1 - 2\cos 2\theta)\sigma. \tag{26}$$

$$\sigma_{\theta\theta}\big|_{\theta=\pi/2} = 3\sigma - \frac{M(1+\varkappa)[14r + M(15+\varkappa)]}{4(r+M)[2r + M(3+\varkappa)]}\,\sigma. \tag{27}$$

From (27) it follows that in case $M > 0$ for $r/M \sim 1$ or $r/M < 1$, where r is the radius of a hole, the surface stresses $\sigma^s_{\theta\theta}$ reduce concentration of the hoop stresses $\sigma_{\theta\theta}$. For a big value of the ratio r/M, this effect disappears.

The item [6] in the References of the corrected paper reads as follows:

6. R. V. Goldstein, V. A. Gorodtsov, and K. B. Ustinov, Effect of residual surface stress and surface elasticity on deformation of nanometer spherical inclusions in an elastic matrix. (Russian) *Phys. Mesomechanics.* **13** (2010), No. 5, 127–138.

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International Conference

“CONTINUUM MECHANICS AND RELATED PROBLEMS OF ANALYSIS”

September 9-14, 2011, Tbilisi, Georgia

The conference was dedicated to 70 year of the Georgian National Academy of Sciences and 120-th birthday of its first president academician Nikoloz (Niko) Muskhelishvili (16.02.1891 – 16.07.1976).

The conference was organized by:

- Ministry of Education and Science of Georgia,
- The Georgian National Academy of Sciences,
- I. Javakhishvili Tbilisi State University,
  Andrea Razmadze Mathematical Institute,
  I. Vekua Institute of Applied Mathematics,
- Georgian Technical University,
  N. Muskhelishvili Institute of Computational Mathematics,
- Georgian Mathematical Union.

Academician N. Muskhelishvili (http:/rmi.ge/person/muskhel/), the most famous Georgian mathematician and mechanist, was the author of outstanding scientific works in the fields of singular integral equations and theory of elasticity, the first and lifetime Director of Andrea Razmadze Mathematical Institute (1933–1976), and the first and almost lifetime (1941–1971) President of the Georgian Academy of Sciences (at present Georgian National Academy of sciences). His contribution to mathematics and mechanics is widely acknowledged and respected by scientists from all over the world.

The conference covered the following topics: Mechanics of Continua; Singular Integral and Pseudodifferential Equations; Differential Equations and Applications; Real, Complex and Stochastic Analysis; Mathematical Physics; Numerical Analysis and Mathematical Modeling.

In the conference participated up to 150 scientists from 27 countries, among them 80 from abroad and about 70 from Georgia.

The participants delivered about 130 30 minute reports on Sections, 4 plenary and 20 semi-plenary 1 hour talks. The plenary talks were delivered by: Gamkrelidze, Revaz (Russia), Mang, Herbert (Austria), Toland, John (UK), Shatashvili, Samson (Ireland/France). The semi-plenary talks were delivered by: Bancuri, Revaz & Shavlakadze Nugzar (Georgia), Bojarsky, Bogdan (Poland), Chobanjan, Sergey (Georgia), Epremidze Lasha (Georgia), Elishakoff, Isaac (USA), Giorgadze, Grigor & Khimshiashvili, George (Georgia), Hsiao, George (USA), Jaiani, George (Georgia), Kinzler, Reinhold (Germany), Kokilashvili, Vakhtang & Paatashvili, Vakhtang (Georgia), Lanza de Cristoforis, Massimo (Italy), Meskhi, Alexander (Georgia), Meunargia Tengiz (Georgia), Persson, Lars-Erik (Sweden), Podio-Guidugli, Paolo (Italy), Speck, Frank (Portugal), Shargorodsky, Eugene (UK), Spitkovsky, Ilya (USA), Stephan, Ernst (Germany), Vasilevski, Nikolay (Mexico)

More detailed information about the conference, posters, program, abstract, the list of plenary speakers and participants are available on the WEBs:

`http://www.rmi.ge/muskhelishvili120/` and/or
`http://www.science.org.ge/muskhelishvili120/`

Prof. Roland Duduchava
Chairman of the Organizing Committee,
President of the Georgian Mathematical Union

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შ ი ნ ა ა რ ს ი

მოკლე წერილი