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Vladimir Aleksandrovich Kondrat'ev

(Obituary)

On March 11, 2010, Vladimir Aleksandrovich Kondrat'ev, a prominent mathematician, Professor of Moscow M. V. Lomonosov State University, Doctor of Physical and Mathematical Sciences, suddenly passed away at the age of 75.

V. A. Kondrat'ev was born on July 2, 1935, in the city of Samara (Kuybyshev). His father, Aleksandr Sergeyevich Kondrat'ev was a professor of mechanics at the Kuybyshev Industrial Institute, while his mother Evgeniya Vasil'evna was a teacher of mathematics at a secondary school. In 1952 V. A. Kondrat'ev graduated from the school No. 6 of Kuybyshev with Golden Medal and entered the Faculty of Mechanics and Mathematics of Moscow M. V. Lomonosov State University which he graduated in 1957. In 1959, under supervision of S. A. Gal'pern, V. A. Kondrat'ev defended his Candidate of Science Thesis "On Zeros of Solutions of Linear Differential Equations of Order Higher than Two", while in 1965 he defended his

Doctor of Science Thesis "Boundary Value Problems for Elliptic and Parabolic Equations with Singularities at the Boundary". V. A. Kondrat'ev was deeply influenced by I. G. Petrovskiĭ in choosing the area of his scientific interests. Since 1961, V. A. Kondrat'ev had been working at the Chair of Differential Equations of the Faculty of Mechanics and Mathematics of the Moscow State University.

V. A. Kondrat'ev obtained first scientific results in his undergraduate years, and they dealt with investigation of oscillation of solutions of linear ordinary differential equations. He obtained a nonoscillation criterion for second order linear differential equations which easily implied all nonoscillation criteria known by that time. The papers of V. A. Kondrat'ev which laid the basis of his Candidate thesis include elegant proofs of Sturm-type theorems on separation of zeros, as well as oscillation and nonoscillation criteria for solutions of third and fourth order linear differential equations. Later on, he generalized these results for the case of linear differential equations of arbitrary order and obtained a depending on equations' order estimate of number of zeros of a solution as the right end of the interval tends to infinity.

V. A. Kondrat'ev initiated a systematic investigation of elliptic and parabolic problems in domains with nonsmooth boundaries. The first result he obtained in this direction concerned parabolic equations in a noncylindrical domain with characteristic points at the boundary. V. A. Kondrat'ev obtained a solvability criterion for boundary value problems in weighted Sobolev spaces and found the asymptotics of solutions in the vicinity of a characteristic point. A theory of elliptic equations in domains with conic points at the boundary is another important achievement of V. A. Kondrat'ev in this direction. In his papers devoted to this theory a universal method is developed which is applicable to a wide range of equations in domains with isolated singularities at the boundary. These results provided a basis for his doctoral thesis. In a series of papers that have already become classical, V. A. Kondrat'ev introduced and studied the notion of capacity for higher order elliptic equations. His results have served as a starting point for many investigations. Due to those works the notion of capacity was widely applied to Sobolev's imbedding theorems as well as to the theory of higher order elliptic equations - the issues of the unique solvability of the first boundary value problem, smoothness of solutions near the boundary, removable singularities of solutions.

V. A. Kondrat'ev (jointly with O. A. Oleĭnik and I. Kopaček) investigated the regularity of solutions of elliptic equations in the vicinity of a boundary point, and established best values of the Holder exponents for second order elliptic equations.

In sixties, while dealing with asymptotic behavior of solutions of elliptic equations at angular points, V. A. Kondrat'ev decided to use a product of polynomials by logarithms of polynomials for transformation of variables for linearization of a system of ordinary differential equation in the vicinity of a singular point. This approach gave rise to a series of investigations which resulted in elaboration of the theory of finitely smooth equivalence and linearization of systems of ordinary differential equations in vicinity of a non-degenerate singular point.

V.A. Kondrat'ev (jointly with Yu. V. Egorov) obtained fundamental results dedicated to the boundary value problem with oblique derivative for elliptic equations.

V. A. Kondrat'ev (jointly with E. M. Landis) obtained a series of important results for divergent and non-divergent second order elliptic equations with nonsmooth coefficients. In their famous work a theorem on removable character of isolated singularity of solutions was obtained. Besides, the authors found sufficient conditions for each entire nonnegative solution to be trivial. Earlier similar results were known only in the case where the left hand side of the equation is the Laplace operator.

Jointly with L. Veron, V. A. Kondrat'ev obtained results on asymptotic properties of solutions of nonlinear elliptic and parabolic equations in unbounded domains.

V. A. Kondrat'ev investigated the problem on completeness of the system of eigen- and adjoint functions of elliptic operators. He found conditions to be imposed on the principal part of the operator for guaranteeing the completeness of eigen- and adjoint functions of the Dirichlet problem for second order elliptic operator of divergent type in the spaces $\overset{\circ}{W}{}^{1}_{p}$, $p \geq 1$, and weighted Sobolev spaces.

V. A. Kondrat'ev (jointly with Yu. V. Egorov and B. Schultze) established completeness of systems of eigen- and adjoint functions of boundary value problems for $2m$th order elliptic operators in the space $W_2^{2m}(\Omega)$ with Lopatinskiĭ type boundary conditions in a bounded domain whose boundary is everywhere smooth except for neighborhoods of a finite number of points where it is a conic surface.

V. A. Kondrat'ev, jointly with V. G. Maz'ya and M.A. Shubin, extended A.M Molchanov's discrete spectrum criterion to the case of an operator of more general type than the Schrodinger operator.

In the last years of his life, V. A. Kondrat'ev fruitfully worked in the sphere of the theory of nonlinear problems for equations of mathematical physics. He developed (jointly with L. Veron) methods enabling one to obtain asymptotic expansions of solutions of such problems. These methods initiated many scientific investigations both in Russia and abroad.

V. A. Kondrat'ev was actively engaged in the blow-up problem, that is, the problem of absence of nontrivial global solutions of nonlinear equations (jointly with V. A. Galaktionov, Yu. V. Egorov and S. I. Pokhozhaev).

The last remarkable work of V. A. Kondrat'ev "On positive solutions of the heat conduction equation satisfying a nonlinear boundary condition" will appear in the journal "Differentsial'nye Uravneniya", v. 46, 2010.

V. A. Kondrat'ev devoted much attention to the work with his pupils. He created a scientific school on qualitative theory of differential equations. The investigations in the sphere of qualitative theory of ordinary differential equations and partial differential equations whose basis was laid by V. A. Kondrat'ev were continued in the works of his pupils. Among his pupils there are 6 Doctors and 35 Candidates of Science.

The name of Vladimir Aleksandrovich Kondrat'ev will always remain in the history of mathematics, while his memory will live in our hearts.

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EULER CASE FOR A CLASS OF THIRD-ORDER DIFFERENTIAL EQUATION

Abstract. We deal with an Euler-Case for a class of third-order differential equation. A theorem on asymptotic behaviour at the infinity of three linearly independent solutions is proved. This theorem coveres different class of coefficients.

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Key words and phrases. Differential equations, asymptotic form of solutions, Euler case.

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1. Introduction

In this paper we investigate the form of three linearly independent solutions for a class of the third-order differential equation

$$(q(qy')')' - (py')' - ry = 0 \tag{1}$$

as $x \to \infty$, where x is the independent variable and the prime denotes d/dx. The functions q, p and r are defined on the interval $[a, \infty)$, are not necessarily real-valued and continuously differentiable, and all are non-zero everywhere in this interval. In this situation where p is sufficiently small compared to q and r as $x \to \infty$, (1) can be considered as a perturbation of the equation investigated by Eastham. In this paper,we consider the opposite situation where p is large compared to q and r. In this situation, we identify the Euler case:

$$\begin{aligned} \frac{(pr)'}{pr} &\sim const. \times \frac{p}{q^2}, \\ \frac{(pq^{-1})'}{pq^{-1}} &\sim const. \times \frac{p}{q^2} \end{aligned} \tag{2}$$

as $x \to \infty$. The various conditions imposed on the coefficients will be introduced when they are required in the development of the method. Al-Hammadi [1] considers (1) in the case where the solutions all have a similar exponential factor. A third-order equation similar to (1) has been considered previously by Unsworth [11] and Pfeiffer[10]. Eastham [6] considered the Euler case for a fourth-order differential equation and showed that this case represents a border line between situations where all solutions have a certain exponential character as $x \to \infty$ and where only two solutions have this character. The case (2) will appear in the method in Sections 4–6, where we use the recent asymptotic theorem of Eastham [4, Section 2] to obtain the solutions of (1). Two examples are considered in Section 6.

2. The General Method

We write (1) in the standard way [8] as a first order system

$$Y' = AY, \tag{3}$$

where the first component of Y is y and

$$A = \begin{pmatrix} 0 & q^{-1} & 0 \\ 0 & pq^{-2} & q^{-1} \\ r & 0 & 0 \end{pmatrix}. \tag{4}$$

As in [2], we express A in its diagonal form

$$T^{-1}AT = \Lambda \tag{5}$$

and we therefore require the eigenvalues λ_j and eigenvectors ν_j ($1 \le j \le 3$) of A, with the eigenvalues λ_j are chosen as continuously differentiable function.

Writing

$$q^2 = s, \tag{6}$$

we obtain the characteristic equation of A as

$$s\lambda^3 - p\lambda^2 - r = 0. \tag{7}$$

An eigenvector ν_j of A corresponding to λ_j is

$$\nu_j = \left(1, s^{\frac{1}{2}}\lambda_j, r\lambda_j^{-1}\right)^t, \tag{8}$$

where the superscript denotes the transpose. We assume at this stage that the λ_j are distinct, and we define the matrix T in (5) by

$$T = \begin{pmatrix} m_1^{-1}v_1 & m_2^{-1}v_2 & m_3^{-1}v_3 \end{pmatrix}, \tag{9}$$

where the m_j $(1 \le j \le 3)$ are scalar factors to be specified according to the following procedure. Now from (4), we note that EA is symmetric, where

$$E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{10}$$

Hence, by [7, Section 2(i)], the v_j have the orthogonality property

$$(Ev_k)^t v_j = 0 \ \ (k \ne j). \tag{11}$$

We then define the scalars

$$m_j = (Ev_j)^t v_j \tag{12}$$

and the row vectors

$$r_j = (Ev_j)^t. \tag{13}$$

Hence by [7, Section 2]

$$T^{-1} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}, \tag{14}$$

$$m_j = 3s\lambda_j^2 - 2p\lambda_j = s\lambda_j^2 + 2r\lambda_j^{-1}. \tag{15}$$

By (5), the transformation

$$Y = TZ \tag{16}$$

takes (3) into

$$Z' = (\Lambda - T^{-1}T')Z, \tag{17}$$

where

$$\Lambda = dg(\lambda_1, \lambda_2, \lambda_3). \tag{18}$$

From (8)–(12), we obtain $T^{-1}T' = (t_{jk})$, where

$$t_{jj} = -\frac{1}{2}\frac{m_j'}{m_j} \tag{19}$$

and, for $j \ne k$,

$$t_{jk} = \frac{1}{2}\frac{m_k'}{m_k} + \frac{\lambda_j - \lambda_k}{m_k}\left(s\lambda_k' + \frac{1}{2}\lambda_k s'\right) - \frac{m_k'}{m_k^2}\left(r\lambda_j^{-1} + s\lambda_j\lambda_k + r\lambda_k^{-1}\right). \tag{20}$$

Now we need to work out (19) and (20) in some detail in terms of s, p and r in order to determine the form of (17).

3. The Matrices Λ and $T^{-1}T'$

In our analysis, we impose a basic condition on the coefficients as follows: (I) p, r and s are all nowhere zero in some interval $[a, \infty)$, and

$$\left(\frac{r}{p}\right)^{\frac{1}{2}} = o\left(\frac{p}{s}\right) \quad (x \to \infty), \tag{21}$$

If we write

$$\delta = \frac{sr^{\frac{1}{2}}}{p^{\frac{3}{2}}}\,, \tag{22}$$

then by (21)

$$\delta = o(1) \quad (x \to \infty). \tag{23}$$

Now as in [1,2], we can solve the characteristic equation (7) asymptotically as $x \to \infty$. Using (21) and (23), we obtain the distinct eigenvalues λ_j as

$$\lambda_1 = i\left(\frac{r}{p}\right)^{\frac{1}{2}}(1 + \delta_1), \tag{24}$$

$$\lambda_2 = -i\left(\frac{r}{p}\right)^{\frac{1}{2}}(1 + \delta_2), \tag{25}$$

$$\lambda_3 = \left(\frac{p}{s}\right)(1 + \delta_3), \tag{26}$$

where

$$\delta_1 = O(\delta), \quad \delta_2 = O(\delta), \quad \delta_3 = O(\delta^2). \tag{27}$$

By(21), the ordering of λ_j is such that

$$\lambda_j = o(\lambda_3) \quad (x \to \infty, \;\; j = 1, 2). \tag{28}$$

Now substituting (24)–(26) into (7) and differentiating, we obtain

$$\lambda_1' = \frac{1}{2}\, i\left(\frac{r}{p}\right)^{\frac{1}{2}}\left\{\frac{r'}{r} - \frac{p'}{p} + O(\varepsilon)\right\}, \tag{29}$$

$$\lambda_2' = -\frac{1}{2}\, i\left(\frac{r}{p}\right)^{\frac{1}{2}}\left\{\frac{r'}{r} - \frac{p'}{p} + O(\varepsilon)\right\}, \tag{30}$$

$$\lambda_3' = \left(\frac{p}{s}\right)\left\{\frac{p'}{p} - \frac{s'}{s} + O(\delta\varepsilon)\right\}. \tag{31}$$

Now we work out m_j $(1 \leq j \leq 3)$ asymptotically as $x \to \infty$; hence by (24)–(27), (15) gives,

$$m_1 = -2i(pr)^{\frac{1}{2}}\left\{1 + O(\delta)\right\}, \tag{32}$$

$$m_2 = 2i(pr)^{\frac{1}{2}}\left\{1 + O(\delta)\right\}, \tag{33}$$

$$m_3 = \left(\frac{p^2}{s}\right)\left\{1 + O(\delta^2)\right\}. \tag{34}$$

Also by substituting λ_j $(j = 1, 2, 3)$ into (15) and using (24), (25) and (26) respectively, and differentiating, we obtain

$$m_1' = -i(rp)^{\frac{1}{2}}\Big\{\frac{r'}{r} + \frac{p'}{p} + O(\varepsilon)\Big\}, \tag{35}$$

$$m_2' = i(rp)^{\frac{1}{2}}\Big\{\frac{r'}{r} + \frac{p'}{p} + O(\varepsilon)\Big\}, \tag{36}$$

$$m_3' = \Big(\frac{p^2}{s}\Big)\Big\{2\frac{p'}{p} - \frac{s'}{s} + O(\delta\varepsilon)\Big\}, \tag{37}$$

where

$$\varepsilon = \Big|\frac{r'}{r}\,\delta\Big| + \Big|\frac{s'}{s}\delta\Big| + \Big|\frac{p'}{p}\delta\Big|. \tag{38}$$

At this stage we also require the following condition:

(II)

$$\delta\,\frac{r'}{r}\,,\ \delta\,\frac{s'}{s}\,,\ \delta\,\frac{p'}{p}\ \text{ are all }\ L(a,\infty). \tag{39}$$

Now by (22)

$$\delta' = O\Big(\frac{r'}{r}\,\delta\Big) + O\Big(\frac{s'}{s}\,\delta\Big) + O\Big(\frac{p'}{p}\,\delta\Big). \tag{40}$$

Also by substituting (24)–(25) into (7) and differentiating, we obtain

$$\delta_j' = O\Big(\frac{r'}{r}\,\delta\Big) + O\Big(\frac{s'}{s}\,\delta\Big) + O\Big(\frac{p'}{p}\delta\Big)\quad (j = 1, 2) \tag{41}$$

and

$$\delta_3' = O\Big(\frac{r'}{r}\,\delta^2\Big) + O\Big(\frac{s'}{s}\,\delta^2\Big) + O\Big(\frac{p'}{p}\delta^2\Big). \tag{42}$$

Hence by (38), (40), (41), (42) and (39)

$$\varepsilon,\ \delta',\ \delta_j' \in L(a,\infty). \tag{43}$$

We can now substitute the estimates (24)–(27), (32)–(37) and (29)–(31) into (19) and (20) as in [1], we obtain the following expressions for t_{jk},

$$\begin{gathered}
t_{11} = -\rho + O(\varepsilon), \quad t_{22} = -\rho + O(\varepsilon),\\
t_{33} = -\eta + O(\delta\varepsilon), \quad t_{12} = \rho + O(\varepsilon),\\
t_{21} = \rho + O(\varepsilon), \quad t_{13} = O(\varepsilon), \quad t_{23} = O(\varepsilon)\\
t_{31} = \frac{1}{2}\,\eta + O(\varepsilon), \quad t_{32} = \frac{1}{2}\,\eta + O(\varepsilon)
\end{gathered} \tag{44}$$

with

$$\rho = \frac{1}{4}\,\frac{(rp)'}{rp}\,, \quad \eta = \frac{(ps^{-1/2})'}{ps^{-1/2}}\,. \tag{45}$$

It follows from (43) the O-terms in (44) are $L(a,\infty)$, and we can therefore write (17)

$$Z' = (\Lambda + R + S)Z, \tag{46}$$

where

$$R = \begin{bmatrix} \rho & -\rho & 0 \\ -\rho & \rho & 0 \\ -\frac{1}{2}\,\eta & -\frac{1}{2}\,\eta & \eta \end{bmatrix} \tag{47}$$

and $S \in L(a,\infty)$ by (43).

4. The Euler Case

Now we deal with (2) more generally. So we write (2) as

$$\frac{(pr)'}{pr} = 4\sigma\,\frac{p}{s}\,(1+\phi), \tag{48}$$

$$\frac{(ps^{-1/2})'}{ps^{-1/2}} = w\,\frac{p}{s}\,(1+\psi), \tag{49}$$

where σ and w are non zero constants, and $\phi(x) \to 0$, $\psi(x) \to 0$ $(x \to \infty)$. At this stage we let

$$\phi',\, \psi' \in L(a,\infty). \tag{50}$$

We note that by (48) and (49), the matrix Λ no longer dominates the matrix R and so Eastham's theorem [4, Section 2] is not satisfied which means that we have to carry out a second diagnolization of the system(46). First we write

$$\Lambda + R = \lambda_3\{S_1 + S_2\} \tag{51}$$

and we need to work out the two matrices $S_1 = \mathit{const.}$ with the matrix $S_2(x) = o(1)$ as $x \to \infty$ using (24), (25), (26) and Euler case (48) and (49). Hence after some calculations, we obtain

$$S_1 = \begin{pmatrix} \sigma & -\sigma & 0 \\ -\sigma & \sigma & 0 \\ -\frac{1}{2}\,\omega & -\frac{1}{2}\,\omega & 1+\omega \end{pmatrix}, \tag{52}$$

$$S_2(x) = \begin{pmatrix} u_1 & u_2 & 0 \\ u_2 & u_3 & 0 \\ u_4 & u_4 & u_5 \end{pmatrix}, \tag{53}$$

where

$$\begin{gathered} u_1 = \lambda_1\lambda_3^{-1} - u_2, \quad u_2 = -\sigma(1+\delta_3)^{-1}(\phi - \delta_3), \\ u_3 = \lambda_2\lambda_3^{-1} - u_2, \quad u_4 = -\frac{1}{2}\,\omega(1+\delta_3)^{-1}(\psi - \delta_3), \quad u_5 = -2u_4. \end{gathered} \tag{54}$$

It is clear that by (28) and (27), $S_2(x) \to 0$ as $x \to \infty$. Hence we diagonalize the constant matrix S_1. Now the eigenvalues $\alpha_j (1 \le j \le 3)$ of the matrix S_1 are given by

$$\alpha_1 = 0, \quad \alpha_2 = 2\sigma, \quad \alpha_3 = 1+\omega. \tag{55}$$

Let

$$\omega \ne -1 \;\text{ and }\; 2\sigma - \omega \ne 1. \tag{56}$$

Hence by (56), the eigenvalues α_j are distinct. Thus we use the transformation

$$Z = T_1 W \tag{57}$$

in (46), where T_1 diagonalizes the constant matrix S_1. Then (46) transforms to

$$W' = (\Lambda_1 + M + T_1^{-1} S T_1) W, \tag{58}$$

where

$$\begin{gathered} \Lambda_1 = \lambda_3 T_1^{-1} S_1 T_1 = dg(v_1, v_2, v_3) = \lambda_3 dg(\alpha_1, \alpha_2, \alpha_3), \\ M = \lambda_3 T_1^{-1} S_2 T_1, \quad T_1^{-1} S T_1 \in L(a, \infty). \end{gathered} \tag{59}$$

Now we can apply the asymptotic theorem of Eastham [4, Section 2] to (58) provided only that Λ_1 and M satisfy the conditions in [4, Section 2]. We first require that the v_j $(1 \le j \le 3)$ are distinct, and this holds because α_j $(1 \le j \le 3)$ are distinct. Second, we need to show that

$$\frac{M}{v_i - v_j} \to 0 \ \ (x \to \infty) \tag{60}$$

for $i \ne j$ and $1 \le i, j \le 3$. Now

$$\frac{M}{v_i - v_j} = (\alpha_i - \alpha_j)^{-1} T_1^{-1} S_2 T_1 = o(1) \ \ (x \to \infty). \tag{61}$$

Thus (60) holds. Third, we need to show that

$$S_2' \in L(a, \infty). \tag{62}$$

Thus it suffices to show that

$$u_i'(x) \in L(a, \infty) \ \ (1 \le i \le 5). \tag{63}$$

Now by (24), (25), (26) and (54)

$$\begin{aligned} u_1' &= O(\delta') + O(\delta_1' \delta) + O(\delta_3') + O(\phi'), \\ u_2' &= O(\delta_3') + O(\phi'), \\ u_3' &= O(\delta') + O(\delta_2' \delta) + O(\delta_3') + O(\phi'), \\ u_4' &= O(\delta_3') + O(\psi'), \\ u_5' &= O(\delta_3') + O(\psi'). \end{aligned} \tag{64}$$

Thus, by (64), (43) and (50), we see that (63) holds and consequently (62) holds. Now we state our main theorem for (1).

5. The Main Result

Theorem 5.1. *Let the coefficients p, r and s are $C^{(2)}[a, \infty)$. Let* (21), (38)*,* (48)*,* (49) *and* (55) *hold. Let*

$$Re\, I(x), \tag{65}$$

$$Re\Big[\lambda_3 + \eta - \frac{1}{2}\,(2\rho + \lambda_1 + \lambda_2 \pm I)\Big] \tag{66}$$

be of one sign in $[a,\infty)$*, where*

$$I(x) = \left[4\rho^2 + (\lambda_1 - \lambda_2)^2\right]^{\frac{1}{2}}. \tag{67}$$

Then (1) *has the solutions*

$$\begin{aligned}
y_1(x) &= o\left\{(r(x)p(x))^{\frac{-1}{4}} \exp\left(\frac{1}{2}\int_a^x \left[\lambda_1(t) + \lambda_2(t) - I(t)\right] dt\right)\right\},\\
y_2(x) &= [-i + o(1)](r(x)p(x))^{\frac{-1}{4}} \times\\
&\qquad \times \exp\left(\frac{1}{2}\int_a^x \left[\lambda_1(t) + \lambda_2(t) + I(t)\right] dt\right),\\
y_3(x) &= o\left\{(r(x)s(x))^{\frac{-1}{2}} p^{1/2}(x) \exp\left(\int_a^x \lambda_3(t)\, dt\right)\right\}.
\end{aligned} \tag{68}$$

Proof. Before applying the theorem in [4, Section 2], we show that the eigenvalues μ_k $(1 \le k \le 3)$ of $\Lambda_1 + M$ satisfy the dichotomy condition [9]. As in [2], the dichotomy condition holds if

$$Re(\nu_j - \nu_k) = f + g \quad (j \ne k,\ \ 1 \le k \le 3), \tag{69}$$

where f has one sign in $[a,\infty)$ and g belongs to $L(a,\infty)$ [4, (1.5)]. Now since the eigenvalues of $\Lambda_1 + M$ are the same as the eigenvalues of $\Lambda + R$, by (18) and (47) we have

$$\begin{aligned}
\mu_k &= \frac{1}{2}\left[2\rho + \lambda_1 + \lambda_2 + (-1)^k I\right] \quad (k = 1, 2),\\
\mu_3 &= \lambda_3 + \eta.
\end{aligned} \tag{70}$$

Thus by (70) and (66), we see that (69) holds. Since (58) satisfies all the conditions for the asymptotic result [4, Section 2], it follows that, as $x \to \infty$, (58) has three linearly independent solutions

$$W_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t)\, dt\right), \tag{71}$$

where μ_k are given by (70) and e_k are the coordinate vectors with kth component unity and other components zero. Now we transform back to Y by means of (16) and (57), where T_1 in (57) is given by

$$T_1 = \begin{pmatrix} 1 & -1 & 0\\ 1 & 1 & 0\\ \dfrac{\omega}{1+\omega} & 0 & 1 \end{pmatrix}. \tag{72}$$

We obtain

$$Y_k(x) = T(x)T_1W_k(x) \quad (1 \le k \le 3). \tag{73}$$

Now using (9), (32), (33), (34), (71), (72) and (45) in (73) and carrying out the integration of $\frac{(ps^{\frac{-1}{2}})'}{ps^{\frac{-1}{2}}}$ and $(\frac{1}{4})\,\frac{(rp)'}{rp}$, for $1 \le k \le 3$, we obtain (68). $\square$

6. Discussion

(1) In a familiar case, the coefficients covered by Theorem 5.1 are

$$s(x) = Ax^{\alpha}, \quad p(x) = Bx^{\beta}, \quad r(x) = Cx^{\gamma}, \tag{74}$$

where α, β, γ, $A(\neq 0)$, $B(\neq 0)$ and $C(\neq 0)$ are real constants. Then the Euler case (48)–(49) is given by

$$\alpha - \beta = 1. \tag{75}$$

The values of σ and ω are given by

$$\sigma = \frac{1}{4}\,\frac{(B+\gamma)A}{B}, \quad \omega = \frac{(\beta - \frac{1}{2}\,\alpha)A}{B}\,. \tag{76}$$

Also in this example $\phi(x) = \psi(x) = 0$ in (48) and (49).

(2) Theorem 5.1 coveres also the following class of coefficients

$$s = Ax^{\alpha}e^{x^b}, \quad p = Bx^{\beta}e^{x^b}, \quad r = Cx^{\gamma}e^{\frac{1}{2}x^b}, \tag{77}$$

where α, β, γ, $A(\neq 0)$, $B(\neq 0)$, $C(\neq 0)$ and $b(> 0)$ are real constants. Then the Euler case (48)–(49) is given by

$$b - 1 = \beta - \alpha. \tag{78}$$

The values of σ and ω are given by

$$\sigma = \frac{3}{8}\,\frac{bA}{B}, \quad \omega = \frac{1}{2}\,\frac{bA}{B}\,. \tag{79}$$

Also

$$\phi(x) = \frac{2}{3}\,b^{-1}(\beta + \gamma)x^{-b}, \tag{80}$$

$$\psi(x) = 2b^{-1}\Big(\beta - \frac{1}{2}\,\alpha\Big)x^{-b}. \tag{81}$$

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LOCAL VARIATION FORMULAS FOR SOLUTION OF DELAY CONTROLLED DIFFERENTIAL EQUATION WITH MIXED INITIAL CONDITION

Abstract. In this work the variation formulas are proved for solution of non-linear controlled differential equation with variable delays and mixed initial condition.

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Introduction

In the present paper the differential equation

$$\dot{x}(t) = f\big(t, y(\tau_1(t)), \dots, y(\tau_s(t)), z(\sigma_1(t)), \dots, z(\sigma_m(t)), u(t)\big) \qquad (1)$$

with the mixed initial condition

$$x(t) = \big(y(t), z(t)\big)^T = \big(\varphi(t), g(t)\big)^T, \;\; t \in [\tau, t_0), \quad x(t_0) = \big(y_0, g(t_0)\big)^T \quad (2)$$

is considered.

The condition (2) is called the mixed initial condition. It consists of two parts: the first one is the discontinuous part, $y(t) = \varphi(t)$, $t \in [\tau, t_0)$, $y(t_0) = y_0$, because in general $\varphi(t_0) \neq y_0$; the second part is the continuous part $z(t) = g(t)$, $t \in [\tau, t_0]$ because, always $z(t_0) = g(t_0)$.

The local formula of variation of solution, that is, a linear representation of variation of the solution of the problem (1)–(2) in a neighborhood of the right end of the main interval with respect to initial data and perturbation of control $u(t)$ is proved by the scheme given in [1].

An analogous formula for the equation

$$\dot{x}(t) = f\Big(t, y(\tau_1(t)), \dots, y(\tau_s(t)), z(\sigma_1(t)), \dots, z(\sigma_m(t))\Big) \qquad (3)$$

with the initial condition (2) when variation of initial data and right-hand side of equation occurs is proved in [1].

It is important to note that the formula of variation which is proved in the present work doesn't follow from the formula proved in [1].

Formulas of variation for differential equations with delays for concrete cases of continuous and discontinuous initial conditions are obtained in [2]–[6].

Formulas of variation for controlled differential equations with delays, with continuous and discontinuous initial conditions are proved in [7], [8].

Formulas of variation of solution play an important role in the proof of necessary conditions of optimality [6], [9]–[12].

1. Formulation of Main Results

Let R_x^n be the n-dimensional vector space of points $x = (x^1, \dots, x^n)^T$, T means transpose; $O_1 \subset R_y^k$, $O_2 \subset R_z^e$, $G \subset R_u^r$ be open sets, $x = (y, z)^T$, $n = k + e$; $\tau_i(t)$, $i = \overline{1, s}$, $\sigma_j(t)$, $j = \overline{1, m}$, $t \in R_t^1$ be absolutely continuous scalar-valued functions and satisfy the following conditions:

$$\tau_i(t) \le t, \;\; \dot{\tau}_i(t) > 0; \quad \sigma_j(t) \le t, \;\; \dot{\sigma}_j(t) > 0.$$

Let $f(t, y_1, \dots, y_s, z_1, \dots, z_m, u)$ be an n-dimensional function satisfying the following conditions: for almost all $t \in I = [a, b]$ the function $f(t, \cdot) : O_1^s \times O_2^m \times G \to R_x^n$ is continuously differentiable; for any

$$(y_1, \dots, y_s, z_1, \dots, z_m, u) \in O_1^s \times O_2^m \times G$$

the functions f, f_{y_i}, $i = \overline{1, s}$, f_{z_j}, $j = \overline{1, m}$, f_u, are measurable on I; for any compacts $K \subset O_1^s \times O_2^m$ and $M \subset G$ there exists a function $m_{K,M}(\cdot) \in$

$L(I,R_+)$, $R_+=[0,\infty)$, such that for any $(y_1,\dots,y_s,z_1,\dots,z_m,u)\in K\times M$ and for almost all $t\in I$ we have

$$\big|f(t,y_1,\dots,y_s,z_1,\dots,z_m,u)\big|+\\ +\sum_{i=1}^{s}\big|f_{y_i}(\,\cdot\,)\big|+\sum_{j=1}^{m}\big|f_{z_j}(\,\cdot\,)\big|+\big|f_u(\,\cdot\,)\big|\le m_{K,M}(t).$$

Let $E_\varphi^{(k)}=E_\varphi(I_1,R_y^k)$ be the space of piecewise continuous functions $\varphi:I_1=[\tau,b]\to R_y^k$ with a finite number of discontinuity points of the first kind, equipped with the norm $\|\varphi\|=\sup\{|\varphi(t)|:\ t\in I_1\}$, $\tau=\min\{\tau_1(a),\dots,\tau_s(a),\sigma_1(a),\dots,\sigma_m(a)\}$.

Next, $\Delta_1=\{\varphi\in E_\varphi^{(k)}:\ \operatorname{cl}\varphi(I_1)\subset O_1\}$, $\Delta_2=\{g\in E_g^{(e)}=E_g^{(e)}(I_1;R_z^e):\operatorname{cl}g(I_1)\subset O_2\}$ are sets of initial functions, where $\varphi(I_1)=\{\varphi(t),\ t\in I_1\}$; let E_u be the space of measurable functions $u:I\to R_u^r$, satisfying the following condition: the set $\operatorname{cl}u(I)$ is compact in R_u^r, $\|u\|=\sup\{|u(t)|:\ t\in I\}$, $\Omega=\{u\in E_u:\ \operatorname{cl}u(I)\subset G\}$ is the set of controls.

To any element $\mu=(t_0,y_0,\varphi,g,u)\in A=I\times O_1\times\Delta_1\times\Delta_2\times\Omega$ we put in correspondence the differential equation

$$\dot x(t)=f\big(t,y(\tau_1(t)),\dots,y(\tau_s(t)),z(\sigma_1(t)),\dots,z(\sigma_m(t)),u(t)\big) \tag{1.1}$$

with the mixed initial condition

$$x(t)=\big(y(t),z(t)\big)^T=\big(\varphi(t),g(t)\big)^T,\ \ t\in[\tau,t_0),\ \ x(t_0)=\big(y_0,g(t_0)\big)^T. \tag{1.2}$$

Definition 1.1. Let $\mu=(t_0,y_0,\varphi,g,u)\in A$, $t_0<b$. A function $x(t;\mu)=\big(y(t;\mu),z(t;\mu)\big)^T$, $t\in[\tau,t_1]$, $t_1\in(t_0,b]$, where $y(t,\mu)\in O_1$, $z(t,\mu)\in O_2$, is called a solution, corresponding to the element μ, and defined on the interval $[\tau,t_1]$, if it satisfies the condition (1.2) on the interval $[\tau,t_0]$, it is absolutely continuous on the interval $[t_0,t_1]$ and almost everywhere on $[t_0,t_1]$ satisfies the equation (1.1).

In the space $E_\mu=R\times R_y^k\times E_\varphi^{(k)}\times E_g^{(e)}\times E_u$ we introduce the set of variations

$$V=\Big\{\delta\mu=(\delta t_0,\delta y_0,\delta\varphi,\delta g,\delta u)\in E_\mu:\ |\delta t_0|\le c,\ |\delta y_0|\le c,\ \|\delta\varphi\|\le c,\\ \delta g=\sum_{i=1}^{l}\lambda_i\delta g_i,\ |\lambda_i|\le c,\ i=\overline{1,l},\ \|\delta u\|\le c\Big\},$$

where $c>0$ is a fixed number and $\delta g_i\in E_g^{(e)}$, $i=\overline{1,l}$ are fixed points.

Lemma 1.1. *Let $x_0(t)$ be the solution corresponding to the element $\mu_0=(t_{00},y_{00},\varphi_0,g_0,u_0)\in A$, and defined on the interval $[\tau,t_{10}]$, t_{00}, $t_{10}\in(a,b)$. There exist numbers $\varepsilon_1>0$ and $\delta_1>0$, such that for any $(\varepsilon,\delta\mu)\in[0,\varepsilon_1]\times V$ we have $\mu_0+\varepsilon\delta\mu\in A$. In addition, to this element corresponds a solution $x(t;\mu_0+\varepsilon\delta\mu)$, defined on the interval $[\tau,t_{10}+\delta_1]\subset I_1$.*

This lemma follows from Theorem 1.3.2 (see [6, p. 17]).

Due to uniqueness, the solution $x(t;\mu_0)$, which is defined on $[\tau, t_{10}+\delta_1]$ is a continuation of the solution $x_0(t)$. Therefore we can assume that the solution $x_0(t)$ is defined on the whole interval $[\tau, t_{10}+\delta_1]$.

Lemma 1.1 allows us to introduce the increment of the solution $x_0(t) = x(t;\mu_0)$:

$$\Delta x(t) = \Delta x(t;\varepsilon\delta\mu) = x(t;\mu_0+\varepsilon\delta\mu) - x_0(t),$$
$$(t,\varepsilon,\delta\mu) \in [\tau, t_{10}+\delta_1]\times[0,\varepsilon_1]\times V.$$

In order to formulate main results, consider the following notation:

$$\omega_{0i}^- = \big(t_{00}, \underbrace{y_{00},\dots,y_{00}}_{i}, \underbrace{\varphi_0(t_{00}-),\dots,\varphi_0(t_{00}-)}_{p-i}, \varphi_0(\tau_{p+1}(t_{00}-)),\dots,$$
$$\varphi_0(\tau_s(t_{00}-)), g_0(\sigma_1(t_{00}-)),\dots,g_0(\sigma_m(t_{00}-))\big),\ \ i=\overline{0,p},$$
$$\omega_{0i}^- = \big(\gamma_i, y_0(\tau_1(\gamma_i)),\dots,y_0(\tau_{i-1}(\gamma_i)), y_{00}, \varphi_0(\tau_{i+1}(\gamma_i-)),\dots,\varphi_0(\tau_s(\gamma_i-)),$$
$$z_0(\sigma_1(\gamma_i-)),\dots,z_0(\sigma_m(\gamma_i-))\big),$$
$$\omega_{1i}^- = \big(\gamma_i, y_0(\tau_1(\gamma_i)),\dots,y_0(\tau_{i-1}(\gamma_i)), \varphi_0(t_{00}-), \varphi_0(\tau_{i+1}(\gamma_i-)),\dots,$$
$$\varphi_0(\tau_s(\gamma_i-)), z_0(\sigma_1(\gamma_i-)),\dots,z_0(\sigma_m(\gamma_i-))\big),\ \ i=\overline{p+1,s},$$
$$\gamma_i(t) = \tau_i^{-1}(t),\quad \gamma_i=\gamma_i(t_{00}),\quad \rho_j(t)=\sigma_j^{-1}(t),\quad \dot\gamma_i^- = \dot\gamma_i(t_{00}-);$$
$$\omega = (t, y_1,\dots,y_s, z_1,\dots,z_m),$$
$$f_0[t] = f\big(t, y_0(\tau_1(t)),\dots,y_0(\tau_s(t)), z_0(\sigma_1(t)),\dots,z_0(\sigma_m(t))u_0(t)\big);$$
$$f_0(\omega) = f\big(\omega, u_0(t)\big).$$

$$\lim_{\omega\to\omega_{0i}^-} f_0(\omega) = f_i^-,\quad \omega\in(t_{00}-\delta, t_{00}]\times O_1^s\times O_2^m,\quad i=\overline{0,p},\ \ \delta>0,$$
$$\lim_{(\omega_1,\omega_2)\to(\omega_{0i}^-,\omega_{1i}^-)} \big[f_0(\omega_1)-f_0(\omega_2)\big] = f_i^-,$$
$$\omega_1,\,\omega_2\in(\gamma_i-\delta,\gamma_i]\times O_1^s\times O_2^m,\ \ i=\overline{p+1,s}.$$

Similarly we can define ω_{0i}^+, ω_{1i}^+, $\dot\gamma_i^+$, f_i^+. In this case we have $t_{00}+$, γ_i+, and the right semi-intervals of points t_{00}, γ_i.

Theorem 1.1. *Let the following conditions hold:*

(1) $\gamma_i = t_{00}$, $i=\overline{1,p}$, $\gamma_{p+1}<\cdots<\gamma_s<t_{10}$;
(2) *there exists a number* $\delta>0$ *such that* $\gamma_1(t)\le\cdots\le\gamma_p(t)$, $t\in(t_{00}-\delta, t_{00}]$;
(3) *the quantities* $\dot\gamma_i^-$, f_i^-,$i=\overline{1,s}$ *are finite;*
(4) *the function* $g_0(t)$ *is absolutely continuous on the interval* $(t_{00}-\delta, t_{00}]$ *and there exists a finite limit* $\dot g_0^-$.

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$, $\delta_2\in(0,\delta_1)$ *such that for any*

$$(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2, t_{10}+\delta_2]\times[0,\varepsilon_2]\times V^-,$$

where $V^- = \{\delta\mu \in V: \ \delta t_0 \le 0\}$, *we have*

$$\Delta x(t) = \varepsilon\delta x(t;\delta\mu) + o(t;\varepsilon\delta\mu), \tag{1.3}$$

where

$$\delta x(t;\delta\mu) = Y(t_{00};t)\big[Y_0\delta y_0 + Y_1\delta g(t_{00}-)\big] + \\ + \Big\{Y(t_{00};t)\Big[Y_1\dot g_0{}^- + \sum_{i=0}^{p}\big(\widehat\gamma\,{}^-_{i+1} - \widehat\gamma\,{}^-_i\big)f_i^-\Big] - \\ - \sum_{i=p+1}^{s} Y(\gamma_i;t)f_i^-\dot\gamma\,{}^-_i\Big\}\delta t_0 + \beta(t;\delta\mu), \tag{1.4}$$

$$\beta(t;\delta\mu) = \sum_{i=p+1}^{s}\int_{\tau_i(t_{00})}^{t_{00}} Y(\gamma_i(\xi);t)f_{0y_i}[\gamma_i(\xi)]\dot\gamma_i(\xi)\delta\varphi(\xi)\,d\xi + \\ + \sum_{j=1}^{m}\int_{\sigma_j(t_{00})}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot\rho_j(\xi)\delta g(\xi)\,d\xi + \\ + \int_{t_{00}}^{t} Y(\xi;t)f_{0u}[\xi]\delta u(\xi)\,d\xi, \tag{1.5}$$

$\widehat\gamma\,{}^-_0 = 1$, $\widehat\gamma\,{}^-_i = \dot\gamma\,{}^-_i$, $i = \overline{1,p}$, $\widehat\gamma\,{}^-_{p+1} = 0$; *next,* $\lim\limits_{\varepsilon\to 0}\frac{o(t;\varepsilon\delta\mu)}{\varepsilon} = 0$ *uniformly with respect to* $(t,\delta\mu) \in [t_{10}-\delta_2, t_{10}+\delta_2]\times V^-$;

$$f_{0y_i}[t] = f_{y_i}\Big(t, y_0(\tau_1(t)),\dots,y_0(\tau_s(t)), z_0(\sigma_1(t)),\dots,z_0(\sigma_m(t)), u_0(t)\Big);$$

$Y(\xi;t)$ *is an* $n\times n$ *matrix-valued function satisfying the equation*

$$Y_\xi(\xi;t) = -\sum_{i=1}^{s} Y(\gamma_i(\xi);t)F_{y_i}[\gamma_i(\xi)]\dot\gamma_i(\xi) - \\ - \sum_{j=1}^{m} Y(\rho_j(\xi);t)F_{z_j}[\rho_j(\xi)]\dot\rho_j(\xi), \quad \xi \in [t_{00},t], \tag{1.6}$$

and the condition

$$Y(\xi,t) = \begin{cases} I_{n\times n}, & \xi = t, \\ \Theta_{n\times n}, & \xi > t, \end{cases} \tag{1.7}$$

where $I_{n\times n}$ *and* $\Theta_{n\times n}$ *are the identity and zero* $n\times n$ *matrices,* $F_{y_i} = (f_{0y_i}, \Theta_{n\times e})$, $F_{z_j} = (\Theta_{n\times k}, f_{0z_j})$, $Y_0 = (I_{k\times k}, \Theta_{e\times k})^T$, $Y_1 = (\Theta_{k\times e}, I_{e\times e})^T$.

The function $\delta x(t;\delta\mu)$ *is called the variation of the solution* $x_0(t)$, $t \in [t_{10}-\delta_2, t_{10}+\delta_2]$ *and the formula* (1.4) *is called the variation formula.*

Theorem 1.2. *Let the condition* (1) *and the following conditions hold:*

(5) *there exists a number* $\delta > 0$ *such that* $\gamma_1(t) \le \dots \le \gamma_p(t)$, $t \in [t_{00}, t_{00}+\delta)$;

(6) *the quantities* $\dot\gamma_i^+$, f_i^+, $i=\overline{1,s}$ *are finite*

(7) *the function* $g_0(t)$ *is absolutely continuous on the interval* $[t_{00},t_{00}+\delta)$ *and there exists a finite limit* $\dot g_0{}^+$.

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ *and* $\delta_2\in(0,\delta_1)$ *such that for any* $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times[0,\varepsilon_2]\times V^+$, *where* $V^+=\{\delta\mu\in V:\delta t_0\geq 0\}$, *the formula* (1.3) *holds, where*

$$\begin{aligned}\delta x(t;\delta\mu)=Y(t_{00};t)\big[Y_0\delta y_0+Y_1\delta g(t_{00}+)\big]+&\\ +\bigg\{Y(t_{00};t)\Big[Y_1\dot g_0{}^+ +\sum_{i=0}^{p}(\widehat\gamma_{i+1}^{\,+}-\widehat\gamma_i^{\,+})f_i^+\Big]-&\\ -\sum_{i=p+1}^{s}Y(\gamma_i;t)f_i^+\dot\gamma_i^+\bigg\}\delta t_0+\beta(t;\delta\mu),&\end{aligned} \tag{1.8}$$

$$\widehat\gamma_0^{\,+}=1,\quad \widehat\gamma_i^{\,+}=\dot\gamma_i^+,\quad i=\overline{1,p},\quad \widehat\gamma_{p+1}^{\,+}=0.$$

Theorems 1.1 and 1.2 immediately imply the following assertion.

Theorem 1.3. *Let the conditions* (1)–(7) *and the following conditions hold:*

(8) $\displaystyle\sum_{i=0}^{p}(\widehat\gamma_{i+1}^{\,-}-\widehat\gamma_i^{\,-})f_i^-+Y_1\dot g_0{}^-=\sum_{i=0}^{p}(\widehat\gamma_{i+1}^{\,+}-\widehat\gamma_i^{\,+})f_i^++Y_1\dot g_0{}^+=:f_0,$

$$f_i^-\dot\gamma_i^-=f_i^+\dot\gamma_i^+=:f_i,\quad i=\overline{p+1,s};$$

(9) *the functions* $\delta g_i(t)$, $i=\overline{1,l}$ *are continuous at the point* t_{00}.

Then there exist numbers $\varepsilon_2>0$, $\delta_2>0$ *such that for any* $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times[0,\varepsilon_2]\times V$ *the formula* (1.3) *holds, where*

$$\begin{aligned}\delta x(t;\delta\mu)=Y(t_{00};t)\big[Y_0\delta y_0+Y_1\delta g(t_{00})\big]+&\\ +\Big\{Y(t_{00};t)f_0-\sum_{i=p+1}^{s}Y(\gamma_i;t)f_i\Big\}\delta t_0+\beta(t;\delta\mu).&\end{aligned}$$

Some comments: Theorems 1.1 and 1.2 correspond to the case where at the point t_{00} right-hand and left-hand variations, respectively, take place. Theorem 1.3 corresponds to the case where at the point t_{00} double-sided variation takes place.

In the formula of variation proved in [1], for the equation (3) instead of the expression

$$\int_{t_{00}}^{t}Y(\xi;t)f_{0u}[\xi]\delta u(\xi)\,d\xi$$

(see (1.5)), we have

$$\int_{t_{00}}^{t}Y(\xi;t)\delta f[\xi]\,d\xi.$$

The formula (1.4) follows from the formula of variation obtained in [1] if the function f additionally satisfies the condition: $f_u(t, y_1, \dots, y_s, z_1, \dots, z_m, u)$ is continuously differentiable with respect to the variables $y_i \in O_1$, $i = \overline{1, s}$ and $z_j \in O_2$, $j = \overline{1, m}$.

In the present work formulas of variation are proved without of these conditions.

2. Auxiliary Lemmas

To any element $\mu = (t_0, y_0, \varphi, g, u) \in A$, let us correspond the functional-differential equation

$$\dot{\omega}(t) = f\Big(t, h(t_0, \varphi, q)(\tau_1(t)), \dots, h(t_0, \varphi, q)(\tau_s(t)), h(t_0, g, v)(\sigma_1(t)), \dots, h_0(t_0, g, v)(\sigma_m(t)), u(t)\Big) \tag{2.1}$$

with the initial condition

$$\omega(t_0) = \big(q(t_0), v(t_0)\big)^T = x_0 = \big(y_0, g(t_0)\big)^T, \tag{2.2}$$

where the operator $h(\cdot)$ is defined by the formula

$$h(t_0, \varphi, q)(t) = \begin{cases} \varphi(t), & t \in [\tau, t_0), \\ q(t), & t \in [t_0, b]. \end{cases} \tag{2.3}$$

Definition 2.1. Let $\mu = (t_0, y_0, \varphi, g, u) \in A$. An absolutely continuous function $\omega(t) = \omega(t; \mu) = (q(t; \mu), v(t; \mu))^T \in (O_1, O_2)^T$, $t \in [r_1, r_2] \subset I$, where $(O_1, O_2)^T = \{x = (y, z)^T \in R_x^n : \ y \in O_1, \ z \in O_2\}$, is called a solution corresponding to the element $\mu \in A$, defined on the interval $[r_1, r_2]$, if $t_0 \in [r_1, r_2]$, the function $\omega(t)$ satisfies the condition (2.2) and the equation (2.1) almost everywhere on $[r_1, r_2]$.

Remark 2.1. Let $\omega(t; \mu)$, $t \in [r_1, r_2]$ be the solution corresponding to the element $\mu \in A$. Then the function

$$x(t; \mu) = \big(y(t; \mu), z(t; \mu)\big)^T = \big(h(t_0, \varphi, q(\cdot\,; \mu))(t), h(t_0, g, v(\cdot\,; \mu))(t)\big)^T, \ \ t \in [\tau, r_2] \tag{2.4}$$

is a solution of the equation (1.1) with the initial condition (1.2) (see (2.3)).

Lemma 2.1. *Let $\omega_0(t)$, $t \in [r_1, r_2] \subset (a, b)$ be the solution corresponding to the element $\mu_0 \in A$; let $K \subset (O_1, O_2)^T$ be a compact set containing some neighborhood of the set $((\varphi_0(I_1) \cup q_0([r_1, r_2])), (g_0(I_1) \cup v_0([r_1, r_2])))^T$ and let $M \subset G$ be a compact set containing some neighborhood of the set $\operatorname{cl} u_0(I)$. Then there exist numbers $\varepsilon_1 > 0$, $\delta_1 > 0$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V$ to the element $\mu_0 + \varepsilon\delta\mu \in A$ there corresponds a solution*

$\omega(t;\mu_0+\varepsilon\delta\mu)$ defined on $[r_1-\delta_1, r_2+\delta_1]\subset I$. Moreover,

$$\begin{gathered}\big(\varphi(t), g(t)\big)=\big(\varphi_0(t)+\varepsilon\delta\varphi(t), g_0(t)+\varepsilon g(t)\big)\in K,\ \ t\in I_1,\\ u(t)=u_0(t)+\varepsilon\delta u(t)\in M,\ \ t\in I,\\ \omega(t;\mu_0+\varepsilon\delta\mu)\in K,\ \ t\in[r_1-\delta_1, r_2+\delta_1],\\ \lim_{\varepsilon\to 0}\omega(t;\mu+\varepsilon\delta\mu)=\omega(t,\mu_0)\\ \text{uniformly for }\ (t,\delta\mu)\in[r_1-\delta_1, r_2+\delta_1]\times V.\end{gathered}\tag{2.5}$$

This lemma follows from Lemma 1.3.2 (see [6, p. 18]).

Due to uniqueness, the solution $\omega(t;\mu_0)$ on the interval $[r_1-\delta_1, r_2+\delta_1]$ is a continuation of the solution $\omega(t;\mu_0)$, therefore the solution $\omega_0(t)$ is assumed to be defined on the whole interval $[r_1-\delta_1, r_2+\delta_1]$.

Let us define the increment of the solution $\omega_0(t)=\omega(t;\mu_0)$,

$$\begin{gathered}\Delta\omega(t)=\big(\Delta q(t),\Delta v(t)\big)^T=\Delta\omega(t;\varepsilon\delta\mu)=\omega(t;\mu_0+\varepsilon\delta\mu)-\omega_0(t),\\ (t,\varepsilon,\delta\mu)\in[r_1-\delta_1, r_2+\delta_1]\times[0,\varepsilon_1]\times V.\end{gathered}\tag{2.6}$$

It is obvious that

$$\lim_{\varepsilon\to 0}\Delta\omega(t;\varepsilon\delta\mu)=0 \tag{2.7}$$
$$\text{uniformly with respect to }\ (t,\delta\mu)\in[r_1-\delta_1, r_2+\delta_1]\times V.$$

Lemma 2.2. *Let $\gamma_i=t_{00}$, $i=\overline{1,p}$, $\gamma_{p+1}<\cdots<\gamma_s\le r_2$ and let the conditions 2)–4) of Theorem 1.1 hold. Then there exist numbers $\varepsilon_2>0$ and $\delta_2>0$ such that for any $(\varepsilon,\delta\mu)\in[0,\varepsilon_2]\times V^-$ we have*

$$\max_{t\in[t_{00}, r_2+\delta_2]}|\Delta\omega(t)|=O(\varepsilon\delta\mu).\tag{2.8}$$

Moreover,

$$\begin{aligned}\Delta\omega(t_{00})=\varepsilon\big[Y_0\delta y_0+Y_1\delta g(t_{00}-)\big]+&\\ +\varepsilon\Big[Y_1\dot g_0^{\,-}+\sum_{i=0}^{p}(\widehat\gamma_{\,i+1}^{\,-}-\widehat\gamma_i)f_i^-\Big]&\delta t_0+o(\varepsilon\delta\mu).\end{aligned}\tag{2.9}$$

Lemma 2.3. *Let $\gamma_i=t_{00}$, $i=\overline{1,p}$; $\gamma_{p+1}<\cdots<\gamma_s\le r_2$, and let*

conditions (5)–(7) of Theorem 1.2 hold. Then there exist numbers $\varepsilon_2>0$ and $\delta_2>0$ such that for any $(\varepsilon,\delta\mu)\in[0,\varepsilon_2]\times V^+$ we have

$$\max_{t\in[t_0, r_2+\delta_2]}|\Delta\omega(t)|=O(\varepsilon\delta\mu).\tag{2.10}$$

In addition,

$$\Delta\omega(t_0)=\varepsilon\big[Y_0\delta y_0+Y_1\delta g(t_{00}+)+(Y_1\dot g_0^{\,+}-f_p^+)\delta t_0\big]+o(\varepsilon\delta\mu).\tag{2.11}$$

Lemmas 2.2 and 2.3 are proved in analogue way as Lemmas 2.2 and 3.1, respectively (see [1]).

3. Proof of Theorem 1.1

Let $r_1 = t_{00}$, $r_2 = t_{10}$. Then for an arbitrary element $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V^-$ the corresponding solution $\omega(t; \mu_0 + \varepsilon\delta\mu)$ is defined on the interval $[t_{00} - \delta_1, t_{10} + \delta_1]$ and the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ is defined on the interval $[\tau, t_{10} + \delta_1]$. Moreover,

$$\omega(t; \mu_0 + \varepsilon\delta\mu) = x(t, \mu_0 + \varepsilon\delta\mu), \quad t \in [t_{00}, t_{10} + \delta_1]$$

(see Lemma 1.1 , 2.1 and Remark 2.1).

Therefore

$$\Delta y(t) = \begin{cases} \varepsilon\delta\varphi(t), & t \in [\tau, t_0), \\ q(t; \mu_0 + \varepsilon\delta\mu) - \varphi_0(t), & t \in [t_0, t_{00}), \\ \Delta q(t), & t \in [t_{00}, t_{00} + \delta_1], \end{cases} \tag{3.1}$$

$$\Delta z(t) = \begin{cases} \varepsilon\delta g(t), & t \in [\tau, t_0), \\ v(t; \mu_0 + \varepsilon\delta\mu) - g_0(t), & t \in [t_0, t_{00}), \\ \Delta v(t), & t \in [t_{00}, t_{00} + \delta_1] \end{cases} \tag{3.2}$$

(see(2.6)).

By Lemma 2.2, there exist numbers

$$\varepsilon_2 \in (0, \varepsilon_1), \quad \delta_2 \in \big(0, \min(\delta_1, t_{10} - \gamma_s)\big) \tag{3.3}$$

such that the following inequalities hold

$$|\Delta y(t)| \le O(\varepsilon\delta\mu), \ \ \forall\, (t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times [0, \varepsilon_2] \times V^-, \tag{3.4}$$

$$|\Delta z(t)| \le O(\varepsilon\delta\mu), \ \ \forall\, (t, \varepsilon, \delta\mu) \in [\tau, t_{10} + \delta_2] \times [0, \varepsilon_2] \times V^- \tag{3.5}$$

(see (2.8), (3.1), (3.2)),

$$\Delta x(t_{00}) = \Delta\omega(t_{00}) = \varepsilon\bigg(Y_0\delta y_0 + Y_1\delta g(t_{00}-) + \\ + \Big[Y_1\, \dot g_{\,0}^{\,-} + \sum_{i=0}^{p} \big(\widehat{\gamma}_{\,i+1}^{\,-} - \widehat{\gamma}_{\,i}^{\,-}\big) f_i^-\Big]\delta t_0\bigg) + o(\varepsilon\delta\mu) \tag{3.6}$$

(see (2.9)).

The function $\Delta x(t)$ on the interval $[t_{00}, t_{10} + \delta_2]$ satisfies the equation

$$\frac{d}{dt}\,\Delta x(t) = \sum_{i=1}^{s} f_{0y_i}[t]\Delta y(\tau_i(t)) + \\ \sum_{j=1}^{m} f_{0z_j}[t]\Delta z(\sigma_j(t)) + \varepsilon f_{0u}[t]\delta u(t) + R(t; \varepsilon\delta\mu), \tag{3.7}$$

where

$$R(t; \varepsilon\delta\mu) = f\Big(t, y_0(\tau_1(t)) + \Delta y(\tau_1(t)), \ldots, y_0(\tau_s(t)) + \Delta y(\tau_s(t)), \\ z_0(\sigma_1(t)) + \Delta z(\sigma_1(t)), \ldots, z_0(\sigma_m(t)) + \Delta z(\sigma_m(t)), u_0(t)\Big) -$$

$$f_0[t] - \sum_{i=1}^{s} f_{0y_i}[t]\Delta y(\tau_i(t)) - \sum_{j=1}^{m} f_{0z_j}[t]\Delta z(\sigma_j(t)) - \varepsilon f_{0u}[t]\delta u(t). \quad (3.8)$$

We can represent the solution of (3.7) by the Cauchy formula in the following form:

$$\Delta x(t) = Y(t_{00};t)\Delta x(t_{00}) + \varepsilon \int_{t_{00}}^{t} Y(\xi;t) f_{0u}[t]\delta u(\xi)\, d\xi + \\ + \sum_{i=0}^{2} h_i(t;t_0,\varepsilon\delta\mu), \quad t \in [t_{00}, t_{10}+\delta_2], \quad (3.9)$$

where

$$\begin{cases} h_0 = \sum_{i=p+1}^{s} \int_{\tau_i(t_{00})}^{t_{00}} Y(\gamma_i(\xi);t) f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\Delta y(\xi)\, d\xi, \\ h_1 = \sum_{j=1}^{m} \int_{\tau_i(t_{00})}^{t_{00}} Y(\rho_j(\xi);t) f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\Delta z(\xi)\, d\xi, \\ h_2 = \int_{t_{00}}^{t} Y(\xi;t)R(\xi;\varepsilon\delta\mu)\, d\xi. \end{cases} \quad (3.10)$$

$Y(\xi,t)$ is a matrix-valued function satisfying (1.6) and the condition (1.7).

The function $Y(\xi,t)$ is continuous on the set $\Pi = \{(\xi,t):\ a \le \xi \le t \le b\}$. Therefore

$$Y(t_{00},t)\Delta x(t_{00}) = \varepsilon Y(t_{00};t)\Big\{ Y_0\delta y_0 + Y_1\delta g(t_{00}-) + \\ + \Big[Y_1\, \dot{g}_0^{\,-} + \sum_{i=0}^{p} \big(\widehat{\gamma}_{i+1}^{\,-} - \widehat{\gamma}_{i}^{\,-}\big) f_i^- \Big]\delta t_0 \Big\} + o(t;\varepsilon\delta\mu) \quad (3.11)$$

(see (3.6)).

For $h_0(t;t_0,\varepsilon\delta\mu)$ we have

$$h_0(t;t_0,\varepsilon\delta\mu) = \sum_{i=p+1}^{s} \Big[\varepsilon \int_{\tau_i(t_{00})}^{t_0} Y(\gamma_i(\xi);t) f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\delta\varphi(\xi)\, d\xi + \\ + \int_{t_0}^{t_{00}} Y(\gamma_i(\xi);t) f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\Delta y(\xi)\, d\xi \Big] = \\ = \varepsilon \sum_{i=p+1}^{s} \int_{\tau_i(t_{00})}^{t_{00}} Y(\gamma_i(\xi);t) f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\delta\varphi(\xi)\, d\xi +$$

$$+\sum_{i=p+1}^{s}\int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)f_{0y_i}[\xi]\Delta y(\tau_i(\xi))\,d\xi+o(t;\varepsilon\delta\mu), \qquad (3.12)$$

where

$$o(t;\varepsilon\delta\mu)=-\varepsilon\sum_{i=p+1}^{s}\int\limits_{t_0}^{t_{00}} Y(\gamma_i(\xi);t)f_{0y_i}[\gamma_i(\xi)]\dot\gamma_i(\xi)\delta\varphi(\xi)\,d\xi.$$

Further, for $h_1(t;t_0,\varepsilon\delta\mu)$ we have

$$h_1(t;t_0,\varepsilon\delta\mu)=\sum_{j\in I_1\cup I_2}\int\limits_{\tau_j(t_{00})}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot\rho_j(\xi)\Delta z(\xi)\,d\xi=$$

$$=\sum_{j\in I_1\cup I_2}\Bigg[\varepsilon\int\limits_{\tau_j(t_{00})}^{t_0} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot\rho_j(\xi)\delta g(\xi)\,d\xi+$$

$$+\int\limits_{t_0}^{t_{00}} Y(\rho_i(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot\rho_j(\xi)\Delta z(\xi)\,d\xi\Bigg]=$$

$$=\sum_{j\in I_1\cup I_2}\big[\varepsilon\alpha_j(t)+\beta_j(t)\big],$$

where

$$\alpha_j(t)=\int\limits_{\sigma_j(t_{00})}^{t_0} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot\rho_j(\xi)\delta g(\xi)\,d\xi,$$

$$\beta_j(t)=\int\limits_{t_0}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot\rho_j(\xi)\Delta z(\xi)\,d\xi.$$

It is easy to see that

$$\alpha_j(t)=\int\limits_{\sigma_j(t_{00})}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot\rho_j(\xi)\delta g(\xi)\,d\xi-$$

$$-\int\limits_{t_0}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot\rho_j(\xi)\delta g(\xi)\,d\xi,$$

$$\beta_j(t)=o(t;\varepsilon\delta\mu)$$

(see (3.5)). Therefore

$$h_1(t;t_0,\varepsilon\delta\mu)=\varepsilon\sum_{i=1}^{m}\int\limits_{\sigma_j(t_{00})}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot\rho_j(\xi)\delta g(\xi)\,d\xi+ \\ +o(t;\varepsilon\delta\mu). \tag{3.13}$$

For $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$ we have

$$h_2(t;t_0,\varepsilon\delta\mu)=\sum_{k=1}^{4}\alpha_k(t;\varepsilon\delta\mu), \tag{3.14}$$

where

$$\alpha_1(t;\varepsilon\delta\mu)=\int\limits_{t_{00}}^{\gamma_{p+1}(t_0)}\overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi,\quad \alpha_2(t;\varepsilon\delta\mu)=\sum_{i=p+1}^{s}\int\limits_{\gamma_i(t_0)}^{\gamma_i}\overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi,$$

$$\alpha_3(t;\varepsilon\delta\mu)=\sum_{i=p+1}^{s-1}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)}\overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi,\quad \alpha_4(t;\varepsilon\delta\mu)=\int\limits_{\gamma_s}^{t}\overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi$$

(see (3.10)),

$$\overline{\omega}(\xi;t,\varepsilon\delta\mu)=Y(\xi;t)R(\xi;\varepsilon\delta\mu).$$

Let us estimate $\alpha_1(t;\varepsilon\delta\mu)$

$$\big|\alpha_1(t;\varepsilon\delta\mu)\big|\le\|Y\|\int\limits_{t_{00}}^{\gamma_{p+1}(t_0)}\Big[\Big|f\Big(t,y_0(\tau_1(t))+\Delta y(\tau_1(t)),\dots, \\ y_0(\tau_p(t))+\Delta y(\tau_p(t)),\varphi(\tau_{p+1}(t)),\dots,\varphi(\tau_s(t)), \\ z_0(\sigma_1(t))+\Delta z(\sigma_1(t)),\dots,z_0(\sigma_m(t))+\Delta z(\sigma_m(t)),u_0(t)+\varepsilon\delta u(t)\Big)- \\ -f\Big(t,y_0(\tau_1(t)),\dots,y_0(\tau_p(t)),\varphi_0(\tau_{p+1}(t)),\dots,\varphi_0(\tau_s(t)), \\ z_0(\sigma_1(t)),\dots,z_0(\sigma_m(t)),u_0(t)\Big)- \\ -\sum_{i=1}^{p}f_{0y_i}[t]\Delta y(\tau_i(t))-\varepsilon\sum_{i=p+1}^{s}f_{0y_i}[t]\delta\varphi(\tau_i(t))- \\ -\sum_{j=1}^{m}f_{0z_j}[t]\Delta z(\sigma_j(t))-\varepsilon f_{0u}[t]\Big|\Big]\,dt\le$$

$$\le\|Y\|\int\limits_{t_{00}}^{t_{10}+\delta_2}\bigg\{\int\limits_0^1\Big|\frac{d}{d\xi}\,f\Big(t,y_0(\tau_1(t))+\xi\Delta y(\tau_1(t)),\dots,y_0(\tau_p(t))+\xi\Delta y(\tau_p(t)), \\ \varphi(\tau_{p+1}(t))+\xi\varepsilon\delta\varphi_0(\tau_{p+1}(t)),\dots,\varphi_0(\tau_s(t))+\xi\varepsilon\delta\varphi(\tau_s(t)),$$

$$z_0(\sigma_1(t))+\xi\Delta z(\sigma_1(t)),\dots,z_0(\sigma_s(t))+\xi\Delta z(\sigma_s(\xi)),u_0(t)+\xi\varepsilon\delta u(t)\Big)\Big|-$$
$$-\sum_{i=1}^{p}f_{0y_i}[t]\Delta y(\tau_i(t))-\varepsilon\sum_{i=p+1}^{s}f_{0y_i}[t]\delta\varphi(\tau_i(t))-$$
$$-\sum_{j=1}^{m}f_{0z_j}[t]\Delta z(\sigma_j(t))-\varepsilon f_{0u}[t]\delta u(t)\Big|\Big]\,d\xi\Big\}\,dt\le$$
$$\le\|Y\|\int\limits_{t_{00}}^{t_{10}+\delta_2}\Big\{\int\limits_0^1\Big[\sum_{i=1}^{p}\Big|f_{y_i}\big(t,y_0(\tau_1(t))+\xi\Delta y(\tau_1(t)),\dots\big)-f_{0y_i}[t]\Big|\,\big|\Delta y(\tau_i(t))\big|+$$
$$+\varepsilon\sum_{i=p+1}^{s}\Big|f_{y_i}\big(t,y_0(\tau_1(t))+\xi\Delta y(\tau_1(t)),\dots\big)-f_{0y_i}[t]\Big|\,\big|\delta\varphi(\tau_i(t))\big|+$$
$$+\sum_{j=1}^{m}\Big|f_{z_j}\big(t,y_0(\tau_1(t))+\xi\Delta y(\tau_1(t)),\dots\big)-f_{0z_j}[t]\Big|\,\big|\delta z(\sigma_j(t))\big|+$$
$$+\varepsilon\Big|f_u\big(t,y_0(\tau_1(t))+\xi\Delta y(\tau_1(t)),\dots\big)-f_{0u}[t]\Big|\,\big|\delta u(t)\big|\Big]\,d\xi\Big\}\,dt\le$$
$$\le\|Y\|\Big[O(\varepsilon\delta\mu)\sum_{i=1}^{p}\vartheta_i(t_{00};\varepsilon\delta\mu)+\varepsilon c\sum_{i=p+1}^{s}\vartheta_i(t_{00};\varepsilon\delta\mu)+$$
$$+O(\varepsilon\delta\mu)\sum_{j=1}^{m}\eta_j(t_{00};\varepsilon\delta\mu)+\varepsilon c\delta(t_{00};\varepsilon\delta\mu)\Big],\qquad(3.15)$$

where

$$\|Y\|=\sup_{(\xi,t)\in\Pi}|Y(\xi,t)|,$$
$$\vartheta_i(t_{00};\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{10}+\delta_2}\Big[\int\limits_0^1\Big|f_{y_i}\big(t,y_0(\tau_1(t))+\xi\Delta y(\tau_1(t)),\dots\big)-f_{0y_i}[t]\Big|\,d\xi\Big]\,dt,\quad i=\overline{1,s},$$
$$\eta_j(t_{00};\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{10}+\delta_2}\Big[\int\limits_0^1\Big|f_{z_j}\big(t,y_0(\tau_1(t))+\xi\delta y(\tau_1(t)),\dots\big)-f_{0z_j}[t]\Big|\,d\xi\Big]\,dt,\quad j=1,\dots,m,$$
$$\delta(t_{00};\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{10}+\delta_2}\Big[\int\limits_0^1\Big|f_u\big(t,y_0(\tau_1(t))+\xi\Delta y(\tau_1(t)),\dots\big)-f_{0u}[t]\Big|\,d\xi\Big]\,dt.$$

We have

$$\varphi(t)=\varphi_0(t)+\varepsilon\delta\varphi(t)\to\varphi_0(t);\quad \Delta y(\tau_i(t))\to 0,\quad i=\overline{1,p},$$
$$\Delta z(\sigma_j(t))\to 0,\quad j=\overline{1,m};$$

$$u_0(t) + \xi\varepsilon\delta u(t) \to u_0(t)$$

as $\varepsilon \to 0$ uniformly with respect to

$$(\xi, t, \delta\mu) \in [0,1] \times [t_{00}, t_{10} + \delta_2] \times V^-.$$

By the Lebesque theorem we obtain that

$$\lim_{\varepsilon\to 0} \vartheta_i(t_{00}; \varepsilon\delta\mu) = 0, \quad i = \overline{1,s}, \quad \lim_{\varepsilon\to 0} \eta_j(t_{00}; \varepsilon\delta\mu) = 0, \quad j = \overline{1,m},$$
$$\lim_{\varepsilon\to 0} \delta(t_{00}; \varepsilon\delta\mu) = 0$$

uniformly with respect to $\delta\mu \in V^-$.

Therefore

$$\alpha_1(t; \varepsilon\delta\mu) = o(t; \varepsilon\delta\mu).$$

Consider $\alpha_2(t; \varepsilon\delta\mu)$. It is easy to see that for $i \in p+1, \dots, s$ and $t \in [\gamma_i(t_0), \gamma_i]$ we have

$$\begin{aligned} \big|\Delta y(\tau_j(t))\big| &\le O(\varepsilon\delta\mu), \quad j = \overline{1, i-1}; \\ \Delta y(\tau_j(t)) &= \varepsilon\delta\varphi(\tau_j(t)), \quad j = \overline{i+1, s} \end{aligned} \tag{3.16}$$

(see (3.1), (3.4)). Therefore

$$\int\limits_{\gamma_i(t_0)}^{\gamma_i} \overline{\omega}(\xi; t, \varepsilon\delta\mu)\, d\xi = \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi; t)\beta_i(\xi)\, d\xi -$$
$$- \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi; t) f_{0y_i}[\xi]\Delta y(\tau_i(\xi))\, d\xi + o(t; \varepsilon\delta\mu),$$

where

$$\begin{aligned} \beta_i(\xi) = f\Big(&\xi, y_0(\tau_1(\xi)) + \Delta y(\tau_1(\xi)), \dots, y_0(\tau_i(\xi)) + \Delta y(\tau_i(\xi)), \\ &\varphi(\tau_{i+1}(\xi)), \dots, \varphi(\tau_s(\xi)), z_0(\sigma_1(\xi)) + \Delta z(\sigma_1(\xi)), \dots, \\ &z_0(\sigma_m(\xi)) + \Delta z(\sigma_m(\xi)), u_0(\xi) + \delta u(\xi)\Big) - f_0[\xi], \end{aligned}$$

$$\begin{aligned} o(t; \varepsilon\delta\mu) = &- \sum_{j=1}^{i-1} \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi; t) f_{0y_j}[\xi]\Delta y(\tau_j(\xi))\, d\xi - \\ &- \varepsilon \sum_{j=i+1}^{s} \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi; t) f_{0y_j}[\xi]\delta\varphi(\tau_j(\xi))\, d\xi - \\ &- \sum_{j=1}^{m} \int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi; t) f_{0z_j}[\xi]\Delta z(\sigma_j(\xi))\, d\xi - \varepsilon \int\limits_{\gamma_i(t_0)}^{\gamma_i} f_{0u}[\xi]\delta u(\xi)\, d\xi \end{aligned}$$

(see (3.5), (3.16)). Clearly,

$$\int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)\beta_i(\xi)\,d\xi = \alpha_5(t;\varepsilon\delta\mu)+\alpha_6(t;\varepsilon\delta\mu),$$

where

$$\alpha_5(t;\varepsilon\delta\mu)=\int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)[\beta_i(\xi)-f_i^-]\,d\xi,\quad \alpha_6(t;\varepsilon\delta\mu)=\int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)f_i^-\,d\xi.$$

Further, if $i\in\{p{+}1,\dots,s\}$ and $\xi\in[\gamma_i(t_0),\gamma_i]$, then $\tau_j(\xi)\ge t_{00}$, $j=\overline{1,i-1}$. Hence

$$\lim_{\varepsilon\to 0}\big(y_0(\tau_j(\xi))+\Delta y(\tau_j(\xi))\big)=\lim_{\xi\in\gamma_i-} y_0(\tau_j(\xi))=y_0(\tau_j(\gamma_i)),\quad j=\overline{1,i-1}.$$

We have $\tau_i(\xi)\in[t_0,t_{00}]$ for $\xi\in[\gamma_i(t_0),\gamma_i]$. Therefore

$$y_0(\tau_i(\xi))+\Delta y(\tau_i(\xi))=y(\tau_i(\xi),\mu_0+\varepsilon\delta\mu)=q_0(\tau_i(\xi))+\Delta q(\tau_i(\xi))$$

(see (2.4), (2.5)).

Therefore, taking into account the continuity of the function $q_0(t)$, $t\in[t_{00}-\delta_2,t_{10}+\delta_2]$, (2.6), and the condition $q_0(t_{00})=y_{00}$, we have

$$\lim_{\varepsilon\to 0}\big(y_0(\tau_i(\xi))+\Delta y(\tau_i(\xi))\big)=\lim_{\xi\in\gamma_i-} q_0(\tau_i(\xi))=y_{00}.$$

Hence, we see that for $\varepsilon\to 0$, $i\in\{p+1,\dots,s\}$ and $\xi\in[\gamma_i(t_0),\gamma_i]$, we have

$$\lim_{\varepsilon\to 0}\Big(\xi,y_0(\tau_1(\xi))+\Delta y(\tau_1(\xi)),\dots,y_0(\tau_i(\xi))+\Delta y(\tau_i(\xi)),\varphi(\tau_{i+1}(\xi)),\dots,$$
$$\varphi(\tau_s(\xi)),z_0(\sigma_1(\xi))+\Delta z(\sigma_1(\xi)),\dots,z_0(\sigma_m(\xi))+\Delta z(\sigma_m(\xi))\Big)=\omega_{0i}^-.$$

On the other hand,

$$\lim_{\varepsilon\to 0}\Big(\xi,y_0(\tau_1(\xi)),\dots,y_0(\tau_{i-1}(\xi)),$$
$$\varphi_0(\tau_i(\xi)),\dots,\varphi_0(\tau_s(\xi)),z_0(\sigma_1(\xi)),\dots,z_0(\sigma_m(\xi))\Big)=\omega_{1i}^-.$$

Therefore,

$$\lim_{\varepsilon\to 0}\sup_{\xi\in[\gamma_i(t_0),\gamma_i]}|\beta_i(\xi)-f_i^-|=0$$

uniformly with respect to $\delta\mu\in V^-$.

The function $Y(\xi;t)$ is continuous on the set

$$[\gamma_i(t_0),\gamma_i]\times[t_{10}-\delta_2,t_{10}+\delta_2]\subset\Pi$$

and, moreover

$$\gamma_i-\gamma_i(t_0)=-\varepsilon\dot\gamma_i^-\delta t_0+o(\varepsilon\delta\mu).$$

Therefore $\alpha_5(t;\varepsilon\delta\mu)=o(t;\delta\mu)$ and

$$\alpha_6(t;\varepsilon\delta\mu)=-\varepsilon\sum_{i=p+1}^{s}Y(\gamma_i;t)f_i^-\,\dot\gamma_i^-\,\delta t_0+o(t;\varepsilon\delta\mu).$$

Finally,

$$\alpha_2(t;\varepsilon\delta\mu) = -\,\varepsilon\sum_{i=p+1}^{s} Y(\gamma_i;t)f_i^- \dot{\gamma}_i^-\,\delta t_0 - \\ -\sum_{i=p+1}^{s}\int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\gamma_i;t)f_{0y_i}[\xi]\Delta y(\tau_i(\xi))\,d\xi + o(t;\varepsilon\delta\mu).$$

Similarly, we can prove the relations

$$\alpha_i(t;\varepsilon\delta\mu) = o(t;\varepsilon\delta\mu), \quad i=3,4$$

(see (3.15)).

For $h_2(t;t_{00},\varepsilon\delta\mu)$ we have the final formula

$$h_2(t;t_{00},\varepsilon\delta\mu) = -\,\varepsilon\sum_{i=p+1}^{s} Y(\gamma_i;t)f_i^- \dot{\gamma}_i^-\,\delta t_0 - \\ -\sum_{i=p+1}^{s}\int\limits_{\gamma_i(t_0)}^{\gamma_i} Y(\xi;t)f_{0y_i}[\xi]\Delta y(\tau_i(\xi))\,d\xi + o(t;\varepsilon\delta\mu) \quad (3.17)$$

(see (3.14)).

Taking into account (3.9)–(3.13) and (3.17), we obtain (1.3), where $\delta x(t;\varepsilon\delta\mu)$ has the form (1.4).

4. Proof of Theorem 1.2

Assume that in Lemma 2.3 $r_1 = t_{00}$ and $r_2 = t_{10}$. Then for any element $(\varepsilon,\delta\mu)\in[0,\varepsilon_1]\times V^+$, the corresponding solution $\omega(t;\mu_0+\varepsilon\delta\mu)$ is defined on $[t_{10}-\delta_1, t_{10}+\delta_1]$. The solution $x(t;\mu_0+\varepsilon\delta\mu)$ is defined on $[\tau, t_{10}+\delta_1]$ and

$$\omega(t;\mu_0+\varepsilon\delta\mu) = x(t;\mu_0+\varepsilon\delta\mu), \;\; t\in[t_0,t_{10}+\delta_1]$$

(see Lemma 1.1 and 2.1). It is easy to see that

$$\Delta y(t) = \begin{cases} \varepsilon\delta\varphi(t), & t\in[\tau,t_{00}],\\ \varphi(t)-y_0(t), & t\in[t_{00},t_0),\\ \Delta q(t), & t\in[t_0,t_{10}+\delta_1], \end{cases} \quad (4.1)$$

$$\Delta z(t) = \begin{cases} \varepsilon\delta g(t), & t\in[\tau,t_{00}],\\ g(t)-v_0(t), & t\in[t_{00},t_0),\\ \Delta v(t), & t\in[t_0,t_{10}+\delta_1]. \end{cases} \quad (4.2)$$

Let numbers $\delta_2\in(0,\delta_1)$ and $\varepsilon_2\in(0,\varepsilon_1)$ be sufficiently small so that for an arbitrary $(\varepsilon,\delta\mu)\in[0,\varepsilon_2]\times V^+$ the inequality $\gamma_s(t_0) < t_{10}-\delta_2$ holds. By Lemma 3.1 we have

$$|\Delta y(t)| \le O(\varepsilon\delta\mu), \;\; \forall\,(t,\varepsilon,\delta\mu)\in[t_0,t_{10}+\delta_1]\times[0,\varepsilon_2]\times V^+, \quad (4.3)$$

$$|\Delta z(t)| \le O(\varepsilon\delta\mu), \;\; \forall\,(t,\varepsilon,\delta\mu)\in[\tau,t_{10}+\delta_1]\times[0,\varepsilon_2]\times V^+ \quad (4.4)$$

(see (4.1), (4.2), (2.10)). Moreover,

$$\Delta x(t_0) = \Delta\omega(t_0) = \varepsilon\big[Y_0\delta y_0 + Y_1\delta g(t_{00}+) + (Y_1\dot{g}_0{}^{+} - f_p^{+})\delta t_0\big] + o(\varepsilon\delta\mu) \quad (4.5)$$

(see (2.11)).

The function $\Delta x(t)$ on the interval $[t_0, t_{10}+\delta_2]$ satisfies (3.7) and hence it can be represented by the Cauchy formula

$$\Delta x(t) = Y(t_{00}, t)\Delta x(t_0) + \varepsilon\int_{t_0}^{t} Y(\xi; t) f_{0u}[t]\delta u(\xi)\, d\xi + \sum_{i=0}^{2} h_i(t; t_0, \varepsilon\delta\mu), \quad (4.6)$$

where

$$h_0(t; t_0, \varepsilon\delta\mu) = \sum_{i=1}^{s}\int_{\tau_i(t_0)}^{t_0} Y(\gamma_i(\xi); t) f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\Delta y(\xi)\, d\xi$$

and the functions $h_i(t; t_0, \varepsilon\delta\mu)$, $i = 1, 2$ are defined by the formulas (3.10).

The function $Y(\xi; t)$ is continuous on the set $[t_{00}, \tau_s(t_{10}-\delta_2)] \times [t_{10} - \delta_2, t_{10}+\delta_2]$. Since $t_0 \in [t_{00}, \tau_s(t_{10}-\delta_2)]$, we have

$$\begin{aligned} Y(t_{00}; t)\Delta x(t_0) = \varepsilon Y(t_{00}; t)\big[Y_0\delta y_0 + Y_1\delta g(t_{00}+) + (Y_1\dot{g}_0^{+} - f_p^{+})\delta t_0\big] + \\ + o(t; \varepsilon\delta\mu). \end{aligned} \quad (4.7)$$

(see(4.5)).

Consider $h_0(t; t_0, \varepsilon\delta\mu)$. We have

$$\begin{aligned} h_0(t; t_0, \varepsilon\delta\mu) &= \sum_{i=1}^{p}\int_{\tau_i(t_0)}^{t_0} Y(\gamma_i(\xi); t) f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\Delta y(\xi)\, d\xi + \\ &+ \sum_{i=p+1}^{s}\Big[\varepsilon\int_{\tau_i(t_0)}^{t_{00}} Y(\gamma_i(\xi); t) f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\delta\varphi(\xi)\, d\xi + \\ &+ \int_{t_{00}}^{t_0} Y(\gamma_i(\xi); t) f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\Delta y(\xi)\, d\xi\Big] = \\ &= \sum_{i=1}^{p}\int_{t_0}^{\gamma_i(t_0)} Y(\xi; t) f_{0y_i}[\xi]\Delta y(\tau_i(\xi))\, d\xi + \\ &+ \varepsilon\sum_{i=p+1}^{s}\int_{\tau_i(t_{00})}^{t_{00}} Y(\gamma_i(\xi); t) f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\delta\varphi(\xi)\, d\xi + \\ &+ \sum_{i=p+1}^{s}\int_{\gamma_i}^{\gamma_i(t_0)} Y(\xi; t) f_{0y_i}[\xi]\Delta y(\tau_i(\xi))\, d\xi + o(t; \varepsilon\delta\mu), \end{aligned} \quad (4.8)$$

where

$$o(t;\varepsilon\delta\mu) = -\varepsilon\sum_{i=1}^{s}\int\limits_{\tau_i(t_{00})}^{\tau_i(t_0)} Y(\gamma_i(\xi);t)f_{0y_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\delta\varphi(\xi)\,d\xi.$$

This implies

$$\sum_{i=1}^{p}\int\limits_{t_0}^{\gamma_i(t_0)} Y(\xi;t)f_{0y_i}[\xi]\Delta y(\tau_i(\xi))\,d\xi =$$

$$= \sum_{i=1}^{p}\sum_{j=0}^{i-1}\int\limits_{\gamma_j(t_0)}^{\gamma_{j+1}(t_0)} Y(\xi;t)f_{0y_i}[\xi]\Delta y(\tau_i(\xi))\,d\xi =$$

$$= \sum_{i=0}^{p-1}\sum_{j=i+1}^{p}\int\limits_{\gamma_i(t_0)}^{\gamma_{i-1}(t_0)} Y(\xi;t)f_{0y_j}[\xi]\Delta y(\tau_j(\xi))\,d\xi,\quad \gamma_0(t_0)=t_0. \tag{4.9}$$

Further,

$$h_1(t;t_0,\varepsilon\delta\mu) = \sum_{j\in I_1\cup I_2}\Bigg[\varepsilon\int\limits_{\sigma_j(t_0)}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\delta g(\xi)\,d\xi +$$

$$+ \int\limits_{t_{00}}^{t_0} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\Delta z(\xi)\,d\xi\Bigg] +$$

$$+ \sum_{j\in I_3}\int\limits_{\sigma_j(t_0)}^{t_0} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\Delta z(\xi)\,d\xi =$$

$$= \sum_{j\in I_1\cup I_2}(\varepsilon\alpha_j(t)+\beta_j(t)) + \sum_{j\in I_3}\eta_j(t),$$

where

$$\alpha_j(t) = \int\limits_{\sigma_j(t_0)}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\delta g(\xi)\,d\xi,$$

$$\beta_j(t) = \int\limits_{t_{00}}^{t_0} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\Delta z(\xi)\,d\xi,$$

$$\eta_j(t) = \int\limits_{\sigma_j(t_0)}^{t_0} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot{\rho}_j(\xi)\Delta z(\xi)\,d\xi.$$

Obviously $\beta_j(t)=o(t;\varepsilon\delta\mu)$, $\eta_j(t)=o(t;\varepsilon\delta\mu)$, so we have

$$\alpha_j(t)=\int\limits_{\sigma_j(t_{00})}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot\rho_j(\xi)\delta g(\xi)\,d\xi-$$
$$-\int\limits_{\sigma_j(t_{00})}^{\sigma_j(t_0)} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot\rho_j(\xi)\delta g(\xi)\,d\xi.$$

Therefore

$$h_1(t;t_0,\varepsilon\delta\mu)=\varepsilon\sum_{j=1}^{s}\int\limits_{\sigma_j(t_{00})}^{t_{00}} Y(\rho_j(\xi);t)f_{0z_j}[\rho_j(\xi)]\dot\rho_j(\xi)\delta g(\xi)\,d\xi+$$
$$+o(t;\varepsilon\delta\mu). \tag{4.10}$$

$h_2(t;t_0,\varepsilon\delta\mu)$ for $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$ can be represented by the form

$$h_2(t;t_0,\varepsilon\delta\mu)=\sum_{i=1}^{5}\beta_i(t;\varepsilon\delta\mu), \tag{4.11}$$

where

$$\beta_1(t;\varepsilon\delta\mu)=\sum_{i=1}^{p-1}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)}\overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi,$$

$$\beta_2(t;\varepsilon\delta\mu)=\int\limits_{\gamma_p(t_0)}^{\gamma_{p+1}}\overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi,$$

$$\beta_3(t;\varepsilon\delta\mu)=\sum_{i=p+1}^{s}\int\limits_{\gamma_i}^{\gamma_i(t_0)}\overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi,$$

$$\beta_4(t;\varepsilon\delta\mu)=\sum_{i=p+1}^{s}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}}\overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi,$$

$$\beta_5(t;\varepsilon\delta\mu)=\int\limits_{\gamma_s(t_0)}^{t}\overline{\omega}(\xi;t,\varepsilon\delta\mu)\,d\xi.$$

For $\beta_1(t;\varepsilon\delta\mu)$ we have

$$\beta_1(t;\varepsilon\delta\mu)=\beta_{11}(t;\varepsilon\delta\mu)-\beta_{12}(t;\varepsilon\delta\mu), \tag{4.12}$$

where

$$\beta_{11}(t;\varepsilon\delta\mu)=\sum_{i=0}^{p-1}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)}Y(\xi;t)\Big[f\Big(\xi,y_0(\tau_1(\xi))+\Delta y(\tau_1(\xi)),\dots,$$
$$y_0(\tau_i(\xi))+\Delta y(\tau_i(\xi)),\varphi(\tau_{i+1}(\xi)),\dots,\varphi(\tau_s(\xi)),$$
$$z_0(\sigma_1(\xi))+\Delta z(\sigma_1(\xi)),\dots,z_0(\sigma_m(\xi))+\Delta z(\sigma_m(\xi)),$$
$$u_0(\xi)+\varepsilon\delta u(\xi)\Big)-$$
$$-f\Big(\xi,y_0(\tau_1(\xi)),\dots,y_0(\tau_p(\xi)),\varphi_0(\tau_{p+1}(\xi)),\dots,\varphi_0(\tau_s(\xi)),$$
$$z_0(\sigma_1(\xi)),\dots,z_0(\sigma_m(\xi)),u_0(\xi)\Big)\Big]\,d\xi,$$

$$\beta_{12}(t;\varepsilon\delta\mu)=\sum_{i=0}^{p-1}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)}Y(\xi;t)\Big[\sum_{j=1}^{s}f_{0y_j}[\xi]\Delta y(\tau_j(\xi))+$$
$$+\sum_{j=1}^{m}f_{0z_j}[\xi]\Delta z(\tau_j(\xi))\Big]\,d\xi.$$

Let $\xi\in[\gamma_i(t_0),\gamma_{i+1}(t_0)]$. Then

$$\tau_j(\xi)\geq t_0,\ \ j=\overline{1,i},\quad \tau_j(\xi)\leq t_0,\ \ j=\overline{i+1,p},\quad \tau_j(\xi)<t_{00},\ \ j=\overline{p+1,s},$$

and hence

$$\begin{aligned}\big|\Delta y(\tau_j(\xi))\big|&\leq O(\varepsilon\delta\mu),\ \ j=\overline{1,i},\\ \Delta y(\tau_j(\xi))&=\varepsilon\delta\varphi(\tau_j(\xi)),\ \ j=\overline{p+1,s},\end{aligned}\tag{4.13}$$

(see (4.1), (4.3)).

For any $i\in\{0,\dots,p-1\}$, the function $\gamma_{i+1}(t_0)-\gamma_i(t_0)$ tends to zero as $\varepsilon\to 0$. Therefore, taking into account (4.13) and (4.4) we have

$$\beta_{12}(t;\varepsilon\delta\mu)=\sum_{i=0}^{p-1}\sum_{j=i+1}^{p}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)}Y(\xi;t)f_{0y_i}[\xi]\Delta y(\tau_j(\xi))\,d\xi+o(t;\varepsilon\delta\mu).\tag{4.14}$$

Further

$$\lim_{\varepsilon\to 0}\sup_{\xi\in[\gamma_i(t_0),\gamma_{i+1}(t_0)]}\Big|f\Big(\xi,y_0(\tau_1(\xi))+\Delta y(\tau_1(\xi)),\dots,y_0(\tau_i(\xi))+\Delta y(\tau_i(\xi)),$$
$$\varphi(\tau_{i+1}(\xi)),\dots,\varphi(\tau_s(\xi)),$$
$$z_0(\sigma_1(\xi))+\Delta z(\sigma_1(\xi)),\dots,z_0(\sigma_m(\xi))+\Delta z(\sigma_m(\xi)),u_0(\xi)+\varepsilon\delta u(\xi)\Big)$$
$$-f_i^{+}+f_p^{+}-f\Big(\xi,y_0(\tau_1(\xi)),\dots,y_0(\tau_p(\xi)),\varphi_0(\tau_{p+1}(\xi)),\dots,\varphi_0(\tau_s(\xi)),$$
$$z_0(\sigma_1(\xi)),\dots,z_0(\sigma_m(\xi)),u_0(\xi)\Big)\Big|=0,\ \ i=\overline{0,p-1},\tag{4.15}$$

uniformly with respect of to $\delta\mu\in V^{+}$.

The properties of the functions $Y(\xi;t)$ and $\gamma_i(t)$, $i=\overline{1,p}$ imply that

$$\lim_{\varepsilon\to 0}\sup_{\xi\in[\gamma_i(t_0),\gamma_{i+1}(t_0)]}\big|Y(\xi;t)-Y(t_{00};t)\big|=0,\quad i=\overline{0,p-1} \tag{4.16}$$

uniformly with respect to $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$ and

$$\gamma_{i+1}(t_0)-\gamma_i(t_0)=\varepsilon\big(\dot\gamma^{\,+}_{i+1}-\dot\gamma^{\,+}_{i}\big)\delta t_0+o(\varepsilon\delta\mu),\ \ i=\overline{0,p-1},\ \ \dot\gamma_0=1. \tag{4.17}$$

From (4.13)–(4.15) we have

$$\beta_{11}(t;\varepsilon\delta\mu)=\varepsilon Y(t_{00},t)\sum_{i=0}^{p}(f_i^+-f_p^+)(\dot\gamma^{\,+}_{i+1}-\dot\gamma^{\,+}_{i})\delta t_0+o(t;\varepsilon\delta\mu). \tag{4.18}$$

From (4.12), (4.14) and (4.18) we have

$$\beta_1(t;\varepsilon\delta\mu)=\varepsilon Y(t_{00},t)\Big[\sum_{i=0}^{p}(\dot\gamma^{\,+}_{i+1}-\dot\gamma^{\,+}_{i})f_i^+ + f_p^+\Big]\delta t_0-$$
$$-\sum_{i=0}^{p-1}\sum_{j=i+1}^{p}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)}Y(\xi;t)f_{0y_j}[\xi]\Delta y(\tau_j(\xi))\,d\xi+o(t;\varepsilon\delta\mu). \tag{4.19}$$

It is easy to see that

$$\beta_2(t;\varepsilon\delta\mu)=\int\limits_{\gamma_p(t_0)}^{\gamma_{p+1}}Y(\xi;t)\Big[f\Big(\xi,y_0(\tau_1(\xi))+\Delta y(\tau_1(\xi)),\dots,y_0(\tau_p(\xi))+\Delta y(\tau_p(\xi)),$$
$$\varphi(\tau_{p+1}(\xi)),\dots,\varphi(\tau_s(\xi)),z_0(\sigma_1(\xi))+\Delta z(\sigma_1(\xi)),\dots,z_0(\sigma_m(\xi))+\Delta z(\sigma_m(\xi)),$$
$$u_0(\xi)+\varepsilon\delta u(\xi)\Big)-$$
$$-f\Big(\xi,y_0(\tau_1(\xi)),\dots,y_0(\tau_p(\xi)),\varphi_0(\tau_{p+1}(\xi)),\dots,\varphi_0(\tau_s(\xi)),$$
$$z_0(\sigma_1(\xi)),\dots,z_0(\sigma_m(\xi)),u_0(\xi)\Big)-$$
$$-\sum_{j=1}^{p}f_{0y_j}[\xi]\Delta y(\tau_j(\xi))-\varepsilon\sum_{j=p+1}^{s}f_{0y_j}[\xi]\delta\varphi(\tau_j(\xi))-$$
$$-\sum_{j=1}^{m}f_{0z_j}[\xi]\Delta z(\sigma_j(\xi))-\varepsilon f_{0u}[\xi]\delta u(\xi)\Big]\,d\xi.$$

It is easy to prove that

$$\beta_2(t;\varepsilon\delta\mu)=o(t;\varepsilon\delta\mu) \tag{4.20}$$

(see (4.3) and (4.4)).

Consider the other terms of (4.11). We have

$$\beta_3(t;\varepsilon\delta\mu)=\sum_{i=p+1}^{s}\int\limits_{\gamma_i}^{\gamma_i(t_0)}Y(\xi;t)\Big[f\Big(\xi,y_0(\tau_1(\xi))+\Delta y(\tau_1(\xi)),\dots,$$

$$y_0(\tau_{i-1}(\xi)) + \Delta y(\tau_{i-1}(\xi)), \varphi(\tau_i(\xi)), \dots, \varphi(\tau_s(\xi)),$$
$$z_0(\sigma_1(\xi)) + \Delta z(\sigma_1(\xi)), \dots, z_0(\sigma_m(\xi)) + \Delta z(\sigma_m(\xi)), u_0(\xi) + \varepsilon\delta u(\xi)\Big) -$$
$$- f\Big(\xi, y_0(\tau_1(\xi)), \dots, y_0(\tau_i(\xi)), \varphi(\tau_{i+1}(\xi)), \dots, \varphi(\tau_s(\xi)),$$
$$z_0(\sigma_1(\xi)), \dots, z_0(\sigma_m(\xi)), u_0(\xi)\Big)\Big]\, d\xi -$$
$$- \sum_{i=p+1}^{s} \Bigg[\sum_{j=1}^{i-1} \int_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t) f_{0y_j}[\xi] \Delta y(\tau_j(\xi))\, d\xi +$$
$$+ \int_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t) f_{0y_i}[\xi] \Delta y(\tau_i(\xi))\, d\xi + \varepsilon \sum_{j=i+1}^{s} \int_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t) f_{0y_j}[\xi] \delta\varphi(\tau_j(\xi))\, d\xi \Bigg] -$$
$$- \sum_{i=p+1}^{s} \int_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t) \sum_{j=1}^{m} f_{0z_j}[\xi] \Delta z(\sigma_j(\xi))\, d\xi.$$

By the condition (6) we have

$$\lim_{\varepsilon\to 0} \sup_{\xi\in[\gamma_i,\gamma_i(t_0)]} \Big| f\Big(\xi, y_0(\tau_1(\xi)) + \Delta y(\tau_1(\xi)), \dots, y_0(\tau_{i-1}(\xi)) + \Delta y(\tau_{i-1}(\xi)),$$
$$\varphi(\tau_i(\xi)), \dots, \varphi(\tau_s(\xi)), z_0(\sigma_1(\xi)) + \Delta z(\sigma_1(\xi)), \dots,$$
$$z_0(\sigma_m(\xi)) + \Delta z(\sigma_m(\xi)), u_0(\xi) + \varepsilon\delta u(\xi)\Big) -$$
$$- f\Big(\xi, y_0(\tau_1(\xi)), \dots, y_0(\tau_i(\xi)), \varphi(\tau_{i+1}(\xi)), \dots, \varphi(\tau_s(\xi)),$$
$$z_0(\sigma_1(\xi)), \dots, z_0(\sigma_m(\xi)), u_0(\xi)\Big) + f_i^+ \Big| = 0, \quad i = \overline{p+1, s}$$

uniformly with respect to $\delta\mu \in V^+$.

Further,

$$\big|\Delta y(\tau_j(\xi))\big| \le O(\varepsilon\delta\mu), \quad j = \overline{1, i-1}, \quad \xi \in [\gamma_i, \gamma_i(t_0)],$$
$$\lim_{\varepsilon\to 0} \sup_{\xi\in[\gamma_i,\gamma_i(t_0)]} \big|Y(\xi;t) - Y(\gamma_i;t)\big| = 0, \quad i = \overline{p+1, s}$$

uniformly with respect to $t \in [t_{10} - \delta_2, t_{10} + \delta_2]$.

Now, we obtain for the function $\beta_3(t;\varepsilon\delta\mu)$ the representation

$$\beta_3(t;\varepsilon\delta\mu) = -\,\varepsilon \sum_{i=p+1}^{s} Y(\gamma_i;t) f_i^+ \delta t_0 -$$
$$- \sum_{i=p+1}^{s} \int_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t) f_{0y_i}[\xi] \Delta y(\tau_j(\xi))\, d\xi + o(t;\varepsilon\delta\mu). \qquad (4.21)$$

Similarly we can prove (see (3.16)) that

$$\beta_i(t;\varepsilon\delta\mu) = o(\varepsilon\delta\mu), \quad i = 4, 5. \tag{4.22}$$

Taking into account (4.19)–(4.22), we obtain

$$\begin{aligned} h_1(t;t_0,\varepsilon\delta\mu) = \varepsilon\Big\{ & Y(t_{00},t)\sum_{i=0}^{p}(\dot\gamma^{\,+}_{i+1}-\dot\gamma^{\,+}_{i}) - \sum_{i=p+1}^{s} Y(\gamma_i;t)f_i^+\Big\}\delta t_0 - \\ & -\sum_{i=0}^{p-1}\sum_{j=i+1}^{p}\int\limits_{\gamma_i(t_0)}^{\gamma_{i+1}(t_0)} Y(\xi;t)f_{0y_i}[\xi]\Delta y(\tau_j(\xi))\,d\xi - \\ & -\sum_{i=p+1}^{s}\int\limits_{\gamma_i}^{\gamma_i(t_0)} Y(\xi;t)f_{0y_i}[\xi]\Delta y(\tau_i(\xi))\,d\xi + o(t;\varepsilon\delta\mu) \end{aligned} \tag{4.23}$$

(see (4.11)).

From (4.6), taking into account (4.7)–(4.10) and (4.23), we obtain (1.3), where $\delta x(t;\delta\mu)$ has the form (1.8).

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TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS OF THE THEORY OF CONSOLIDATION WITH DOUBLE POROSITY

Abstract. The purpose of this paper is to consider two-dimensional version of quasistatic Aifantis' equation of the theory of consolidation with double porosity and to study the uniqueness and existence of solutions of basic boundary value problems (BVPs). The fundamental and some other matrices of singular solutions are constructed in terms of elementary functions for the steady-state quasistatic equations of the theory of consolidation with double porosity. Using the fundamental matrix we construct the simple and double layer potentials and study their properties near the boundary. Using these potentials, for the solution of the first basic BVP we construct Fredholm type integral equation of the second kind and prove the existence theorem of solution for the finite and infinite domains.

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რეზიუმე. ნაშრომის მიზანია განვიხილოთ აიფანტის კვაზისტატიკის განტოლებები ორგვარი ფოროვნობის მქონე ორგანზომილებიანი სხეულებისათვის. დამტკიცებულია ძირითადი სასაზღვრო ამოცანების ამონახსნის ერთადერთობის თეორემები კვაზისტატიკის განტოლებებისათვის ორგვარი ფოროვნების გათვალისწინებით. აგებულია ამონახსნთა ფუნდამენტური და სხვა მატრიცები ელემენტარული ფუნქციების საშუალებით. ამ მატრიცების საშუალებით შედგენილია მარტივი და ორმაგი ფენის პოტენციალები და შესწავლილია მათი თვისებები. ამ პოტენციალების გამოყენებით პირველი სასაზღვრო ამოცანისათვის აგებულია ფრედჰოლმის მეორე გვარის ინტეგრალური განტოლებები და დამტკიცებულია მისი ამონახსნის არსებობის თეორემა როგორც სასრული, ისე უსასრულო არისათვის.

Introduction

A theory of consolidation with double porosity has been proposed by Aifantis. This theory unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt for seepage in undeformable media with two degrees of porosity. In a material with two degrees of porosity, there are two pore systems, the primary and the secondary. For example, in a fissured rock (i.e. a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or the secondary porosity. When fluid flow and deformations processes occur simultaneously, three coupled partial differential equations can be derived [1], [2] to describe the relationships governing pressure in the primary and secondary pores (and therefore the mass exchange between them) and the displacement of the solid. Inertia effects are neglected as they are in Biot's theory.

The physical and mathematical foundations of the theory of double porosity were considered in the papers [1]–[3]. In part I of a series of papers on the subject, R. K. Wilson and E. C. Aifantis [1] gave detailed physical interpretations of the phenomenological coefficients appearing in the double porosity theory. They also solved several representative boundary value problems. In part II of this series, uniqueness and variational principles were established by D. E. Beskos and E. C. Aifantis [2] for the equations of double porosity, while in part III Khaled, Beskos and Aifantis [3] provided a related finite element to consider the numerical solution of Aifantis' equations of double porosoty (see [1]–[3] and the references cited therein). The basic results and the historical information on the theory of porous media were summarized by Boer [4].

The purpose of this paper is to consider a two-dimensional version of quasistatic Aifantis' equation of the theory of consolidation with double porosity and to study the uniqueness and existence of solutions of basic boundary value problems (BVPs). The fundamental and some other matrices of singular solutions are constructed in terms of elementary functions for the steady-state quasistatic equations of the theory of consolidation with double porosity. Using the fundamental matrix, we construct the simple and double layer potentials and study their properties near the boundary. Using these potentials, for solving the first basic BVP we construct a Fredholm type integral equation of the second kind and prove the existence theorem of solution for the finite and infinite domains.

1. Basic Equations, Boundary Value Problems and Uniqueness Theorems

The basic steady-state quasistatic Aifantis' equations of the theory of consolidation with double porosity in the case of plane deformation are given by partial differential equations of the form [1], [2]

$$\begin{aligned} &\mu\Delta u + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u - \mathrm{grad}\,(\beta_1 p_1 + \beta_2 p_2) = 0,\\ &\frac{i\omega\beta_1}{m_1}\,\mathrm{div}\,u + \left(\Delta + \frac{\alpha_3}{m_1}\right)p_1 + \frac{k}{m_1}\,p_2 = 0,\\ &\frac{i\omega\beta_2}{m_2}\,\mathrm{div}\,u + \frac{k}{m_2}\,p_1 + \left(\Delta + \frac{\alpha_4}{m_2}\right)p_2 = 0, \end{aligned} \tag{1.1}$$

where $u=(u_1,u_2)$ is the displacement vector, p_1 is the fluid pressure within the primary pores and p_2 is the fluid pressure within the secondary pores. $\alpha_3 = i\omega\alpha_1 - k$, $\alpha_4 = i\omega\alpha_2 - k$, $m_j = \frac{k_j}{\mu^*}$, $j=1,2$. The constant λ is the Lame modulus, μ is the shear modulus and the constants β_1 and β_2 measure the change of porosities due to an applied volumetric strain. The constants α_1 and α_2 measure the compressibilities of primary and secondary pores filled with pore fluid.The constants k_1 and k_2 are the permeabilities of the primary and secondary systems of pores, the constant μ^* denotes the viscosity of the pore fluid and the constant k measures the transfer of fluid from the secondary pores to the primary pores. The quantities λ, μ, α_j, β_j, k_j $(j=1,2)$ and μ^* are all positive constants. $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the two-dimensional Laplace operator, ω is the oscilation frequency $(\omega > 0)$.

We also rewrite the equation (1.1) in the matrix form

$$B(\partial x)U = 0, \tag{1.2}$$

where

$$\begin{gathered} B(\partial x) = \| \, B_{pq}(\partial x) \, \|_{4x4}, \quad p,q = 1,2,3,4,\\ B_{jj}(\partial x) = \mu\Delta + (\lambda+\mu)\,\frac{\partial^2}{\partial x_j^2}, \quad j=1,2,\\ B_{12}(\partial x) = B_{21}(\partial x) = (\lambda+\mu)\,\frac{\partial^2}{\partial x_1 \partial x_2}\,,\\ B_{j3}(\partial x) = -\beta_1\,\frac{\partial}{\partial x_j}\,, \quad B_{j4}(\partial x) = -\beta_2\,\frac{\partial}{\partial x_j}\,, \quad j=1,2,\\ B_{3j}(\partial x) = \frac{i\omega\beta_1}{m_1}\,\frac{\partial}{\partial x_j}\,, \quad B_{4j}(\partial x) = \frac{i\omega\beta_2}{m_2}\,\frac{\partial}{\partial x_j}\,, \quad j=1,2,\\ B_{33}(\partial x) = \Delta + \frac{\alpha_3}{m_1}\,, \quad B_{34}(\partial x) = \frac{k}{m_1}\,, \quad B_{43}(\partial x) = \frac{k}{m_2}\,,\\ B_{44}(\partial x) = \Delta + \frac{\alpha_4}{m_2}\,, \quad U(u_1,u_2,p_1,p_2). \end{gathered}$$

The conjugate system of the equation (2) is

$$\widetilde{B}(\partial x)U = B^T(-\partial x)U = 0.$$

Throughout this paper "T" denotes transposition.

Now we write the expressions for the components of the stress vector, which acts on elements of the arc with the normal $n=(n_1,n_2)$. Denoting the stress vector by $P(\partial x, n)u$, we have

$$P(\partial x, n)u = T(\partial x, n)u - n(\beta_1 p_1 + \beta_2 p_2), \tag{1.3}$$

where [9]

$$\begin{aligned} &T(\partial x, n) = \| T_{kj}(\partial x, n) \|_{2x2}, \\ &T_{kj}(\partial x, n) = \mu\delta_{kj}\frac{\partial}{\partial n} + \lambda n_k \frac{\partial}{\partial x_j} + \mu n_j \frac{\partial}{\partial x_k}, \quad k, j = 1, 2. \end{aligned} \tag{1.4}$$

Let $D^+(D^-)$ be a finite (an infinite) two-dimensional region bounded by the contour S. Suppose that $S \in C^{1,\beta}$, $0 < \beta \leq 1$, i.e., S is a Lyapunov curve.

Introduce the definition of a regular vector-function.

Definition 1. A vector-function $U(x) = (u_1, u_2, p_1, p_2)$ defined in the domain $D^+(D^-)$ is called regular if it has integrable continuous second derivatives in D^+ (D^-), and U itself and its first order derivatives are continuously extendable at every point of the boundary of $D^+(D^-)$, i.e., $U \in C^2(D^+) \cap C^1(\overline{D^+})$, $(U \in C^2(D^+) \cap C^1(\overline{D^+}))$. Note that for the infinite domain D^- the vector $U(x)$ additionally satisfies the following conditions at infinity:

$$U(x) = O(1), \quad \frac{\partial U_k}{\partial x_j} = O(|x|^{-2}), \quad |x|^2 = x_1^2 + x_2^2, \ \ j = 1, 2, \tag{1.5}$$

where $O(1)$ denotes a bounded function.

For the equation (1.1) we pose the following boundary value problems:

Find a regular vector U satisfying in D^+ (D^-) the equation (1.1), and on the boundary S one of the following conditions:

Problem 1. The displacement vector and the fluid pressures are given in the form

$$u^{\pm}(z) = f(z)^{\pm}, \quad p_1^{\pm}(z) = f_3^{\pm}, \quad p_2^{\pm}(z) = f_4^{\pm}(z), \ \ z \in S;$$

Problem 2. The stress vector and the normal derivatives of the preasure functions $\frac{\partial p_j}{\partial n}$ are given in the form

$$(Pu)^{\pm} = f(z)^{\pm}, \quad \Big(\frac{\partial p_1(z)}{\partial n}\Big)^{\pm} = f_3^{\pm}, \quad \Big(\frac{\partial p_2(z)}{\partial n}\Big)^{\pm} = f_4^{\pm}(z), \ \ z \in S;$$

Problem 3.

$$u^{\pm}(z) = f(z)^{\pm}, \quad \Big(\frac{\partial p_1(z)}{\partial n}\Big)^{\pm} = f_3^{\pm}(z), \quad \Big(\frac{\partial p_2(z)}{\partial n}\Big)^{\pm} = f_4^{\pm}(z), \ \ z \in S;$$

Problem 4.

$$(Pu(z))^{\pm} = f(z)^{\pm}, \quad p_1^{\pm}(z) = f_3^{\pm}(z), \quad p_2^{\pm}(z) = f_4^{\pm}(z), \ \ z \in S,$$

where $(\,\cdot\,)^{\pm}$ denotes the limiting values on S from $D^{\pm}$ and $f = (f_1, f_2)$, f_3, f_4 are given functions.

Generalized Green's Formulas. Let u and $\overline{u}$ be two regular solutions of the equation (1.1) in D^+. Multiply the first equation of (1.1) by $\overline{u}$, the second one by $\overline{p_1}$ and the third one by $\overline{p_2}$, where $\overline{u}$, $\overline{p_1}$ and $\overline{p_2}$ are the complex conjugate functions of u, p_1 and p_2 respectively, integrate over D^+ and sum to obtain

$$\int\limits_{D^+}\bigg[E(u,\bar{u})+\alpha_1|p_1|^2+\alpha_2|p_2|^2+ \\ +\frac{k}{i\omega}\,|p_1-p_2|^2+\frac{m_1}{i\omega}\,|\operatorname{grad}\,p_1|^2+\frac{m_2}{i\omega}\,|\operatorname{grad}\,p_2|^2\bigg]\,dx= \\ =\int\limits_S\bigg[\overline{u}P(\partial x,n)u+\frac{m_1}{i\omega}\,p_1\,\frac{\partial\overline{p_1}}{\partial n}+\frac{m_2}{i\omega}\,p_2\,\frac{\partial\overline{p_2}}{\partial n}\bigg]\,ds, \quad (1.6)$$

where

$$E(u,u)=(\lambda+\mu)(\operatorname{div}u)^2+\mu\Big(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\Big)^2+\mu\Big(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\Big)^2.$$

For positive definiteness of the potential energy the inequalities $\lambda+\mu>0$, $\mu>0$ are necessary and sufficient.

One can generalize the formula (1.6) to the infinite domain D^-, provided the condition

$$\lim_{R\to\infty}\int\limits_{S(0,R)}\bigg[\overline{u}P(\partial x,n)u+\frac{m_1}{i\omega}\,p_1\,\frac{\partial\overline{p_1}}{\partial n}+\frac{m_2}{i\omega}\,p_2\,\frac{\partial\overline{p_2}}{\partial n}\bigg]\,ds=0 \quad (1.7)$$

is fulfilled, where $S(0,R)$ is a circumference of radius R with center at the point O lying inside D^+. The radius R is taken so large that the region D^+ lies entirely inside the circumference $S(0,R)$.

Obviously, the condition (1.7) is fulfilled if the vector u and $\overline{u}$ satisfy the conditions (1.5).

If (1.7) is fulfilled, then Green's formula for the domain D^- takes the form

$$\int\limits_{D^-}\bigg[E(u,u)+\alpha_1|p_1|^2+\alpha_2|p_2|^2+ \\ +\frac{k}{i\omega}\,|p_1-p_2|^2+\frac{m_1}{i\omega}\,|\operatorname{grad}\,p_1|^2+\frac{m_2}{i\omega}\,|\operatorname{grad}\,p_2|^2\bigg]\,dx= \\ =-\int\limits_S\bigg[\overline{u}P(\partial x,n)u+\frac{m_1}{i\omega}\,p_1\,\frac{\partial\overline{p_1}}{\partial n}+\frac{m_2}{i\omega}\,p_2\,\frac{\partial\overline{p_2}}{\partial n}\bigg]\,ds. \quad (1.8)$$

The Uniqueness Theorems. In this subsection we investigate the question of uniqueness of solutions of the above-mentioned problems.

Now let us prove the following theorems.

Theorem 1. *The first boundary value problem has at most one regular solution in the finite domain* D^+.

Proof. Let the first BVP have in the domain D^+ two regular solutions $U^{(1)}$ and $U^{(2)}$. Denote $u = U^{(1)} - U^{(2)}$. Evidently, the vector u satisfies (1.1) and the boundary condition $u^+ = 0$ on S. Note that if u is a regular solution of the equation (1.1), we have Green's formula (1.6). Using (1.6) and taking into account the fact that the potential energy is positive definite, we conclude that $U = C, x \in D^+$, where $C = const$. Since $U^+ = 0$, we have $C = 0$ and $U(x) = 0$, $x \in D^+$. □

Theorem 2. *The first boundary value problem has at most one regular solution in the infinite domain* D^-.

Proof. The vectors $U^{(1)}$ and $U^{(2)}$ in the domain D^- must satisfy the condition (1.5). In this case the formula (1.8) is valid and $U(x) = C$, $x \in D^-$, where C is again a constant vector. But U on the boundary satisfies the condition $U^- = 0$, which implies that $C = 0$ and $U(x) = 0$, $x \in D^-$. □

Theorem 3. *A regular solution of the second boundary value problem is not unique in the domain* D^+. *Two regular solutions may differ by the vector* (u, p_1, p_2), *where* u *is a rigid displacement vector and* $p_j = 0$, $j = 1, 2$.

Proof. Let

$$(P(\partial x, n)u)^+ = 0, \quad \left(\frac{\partial p_1}{\partial n}\right)^+ = 0, \quad \left(\frac{\partial p_2}{\partial n}\right)^+ = 0, \quad x \in S.$$

The positive definiteness of the potential energy implies

$$u_1 = c_1 - \varepsilon x_2, \;\; u_2 = c_2 + \varepsilon x_1, \;\; p_1 = 0, \;\; p_2 = 0, \;\; x \in D^+. \qquad \square$$

Theorem 4. *Two regular solutions of the second boundary value problem in the domain* D^- *may differ by the vector* (u, p_1, p_2), *where* u *is a constant vector and* $p_j = 0$, $j = 1, 2$.

Proof. For the exterior second homogeneous boundary value problem the vector u must satisfy the condition at infinity (1.5). In this case, the formula (1.8) is valid for a regular u. Using this formula, we obtain

$$u_1 = c_1 - \varepsilon x_2, \quad u_2 = c_2 + \varepsilon x_1, \;\; p_1 = 0, \;\; p_2 = 0, \;\; x \in D^-.$$

Bearing in mind (1.5), we have $\varepsilon = 0$ and

$$u_1 = c_1, \;\; u_2 = c_2, \;\; p_1 = 0, \;\; p_2 = 0, \;\; x \in D^-. \qquad \square$$

Analogously, the following theorems are valid:

Theorem 5. *The boundary value problems* $(III)^{\pm}$ *have in the domains* $D^{\pm}$ *at most one regular solution.*

Theorem 6. *Two regular solutions of the boundary value problem $(IV)^+$ may differ by the vector $U(u,p_1,p_2)$, where u is a rigid displacement and $p_j=0$, $j=1,2$. Two regular solutions of the boundary value problem $(IV)^-$ may differ by the vector (u,p_1,p_2), where u is a constant vector and $p_j=0$, $j=1,2$.*

2. Matrix of Fundamental Solutions

Here we construct the matrix of fundamental solutions for the system (1.1).

Let

$$B^*=\frac{1}{a\mu}\begin{pmatrix} B_{11}^*-B_{12}^*\xi_1^2 & -B_{12}^*\xi_1\xi_2 & \mu B_{13}^*\xi_1 & \mu B_{14}^*\xi_1\\ -B_{12}^*\xi_1\xi_2 & B_{11}^*-B_{12}^*\xi_2^2 & \mu B_{13}^*\xi_2 & \mu B_{14}^*\xi_2\\ -\iota\omega\mu B_{31}^*\xi_1 & -\iota\omega\mu B_{31}^*\xi_2 & \mu B_{33}^*\Delta\Delta & -\mu B_{34}^*\Delta\Delta\\ -\iota\omega\mu B_{41}^*\xi_1 & -\iota\omega\mu B_{41}^*\xi_2 & -\mu B_{43}^*\Delta\Delta & \mu B_{44}^*\Delta\Delta \end{pmatrix},$$

where

$$B_{11}^*=a\Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2),$$
$$B_{12}^*=a(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)-\mu\Big[\Delta\Delta+\Big(\frac{\alpha_4}{m_2}+\frac{\alpha_3}{m_1}\Big)\Delta+\frac{\alpha_3\alpha_4-k^2}{m_1m_2}\Big],$$
$$B_{13}^*=\beta_1\Delta\Delta+\Delta\,\frac{\alpha_4\beta_1-k\beta_2}{m_2}\,,\quad B_{14}^*=\beta_2\Delta\Delta+\Delta\,\frac{\alpha_3\beta_2-k\beta_1}{m_1}\,,$$
$$B_{31}^*=\frac{\beta_1}{m_1}\Delta\Delta+\Delta\,\frac{\alpha_4\beta_1-k\beta_2}{m_1m_2}\,,\quad B_{41}^*=\frac{\beta_2}{m_2}\Delta\Delta+\Delta\,\frac{\alpha_3\beta_2-k\beta_1}{m_1m_2}\,,$$
$$B_{33}^*=a\Big(\Delta+\frac{\alpha_4}{m_2}\Big)+\frac{i\omega\beta_2^2}{m_2}\,,\quad B_{34}^*=\frac{ka+i\omega\beta_2\beta_1}{m_1}\,,$$
$$B_{43}^*=\frac{ka+i\omega\beta_2\beta_1}{m_2}\,,\quad B_{44}^*=a\Big(\Delta+\frac{\alpha_3}{m_1}\Big)+\frac{i\omega\beta_1^2}{m_1}\,.$$

Supposing

$$U(x)=B^*(\partial x)\Psi, \tag{2.1}$$

where $\Psi=(\Psi_1,\Psi_2,\Psi_3,\Psi_4)$ is a four-dimensional vector function, we can write the equation (1.1) as

$$\mu a\Delta\Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)\Psi=0; \tag{2.2}$$

here λ_j^2, $j=1,2$ are the roots of the characteristic equation

$$x^2-\Big[\frac{\alpha_4}{m_2}+\frac{\alpha_3}{m_1}+\frac{i\omega}{a}\Big(\frac{\beta_2^2}{m_2}+\frac{\beta_1^2}{m_1}\Big)\Big]x+\frac{\alpha_3\alpha_4-k^2}{m_1m_2}+$$
$$+\frac{i\omega}{am_1m_2}\big(\alpha_4\beta_1^2+\alpha_3\beta_2^2-2k\beta_1\beta_2\big)=0,\quad a=\lambda+2\mu. \tag{2.3}$$

We assume that $\lambda_1^2\neq\lambda_2^2$. Without loss of generality we assume that $Im\lambda_j>0$, $\quad j=1,2$.

From (2.2) it follows that

$$\Psi(x)=-\frac{2i}{\pi}\,\frac{\lambda_1^2+\lambda_2^2}{\lambda_1^4\lambda_2^4}\,\ln r+\frac{2i}{\pi}\,\frac{r^2(\ln r-1)}{4\lambda_1^2\lambda_2^2}-\frac{H_0^{(1)}(\lambda_1 r)}{\lambda_1^4(\lambda_1^2-\lambda_2^2)}+\frac{H_0^{(1)}(\lambda_2 r)}{\lambda_2^4(\lambda_1^2-\lambda_2^2)}\,, \tag{2.4}$$

$H_0^{(1)}(\lambda r)$ is the first kind Hankel function of zero order [5]

$$\begin{aligned}H_0^{(1)}(\lambda r) &= \frac{2i}{\pi}\,\ln r+\frac{2i}{\pi}\,\big[J_0(\lambda r)-1\big]\ln r+\\ &\quad+\frac{2i}{\pi}\,J_0(\lambda r)\Big(\ln\frac{\lambda}{2}+C-\frac{i\pi}{2}\Big)-\\ &\quad-\frac{2i}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}\,\Big(\frac{\lambda r}{2}\Big)^{2k}\Big(\frac{1}{k}+\frac{1}{k-1}+\cdots+1\Big), \qquad (2.5)\\ J_0(\lambda r) &= \sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\,\Big(\frac{\lambda r}{2}\Big)^{2k}.\end{aligned}$$

Substituting $\Psi(x)$ in (2.1), after some calculations we obtain the fundamental matrix of solutions for the equation (1.1) which is denoted by $\Gamma(x-y)$

$$\Gamma(x-y)=\begin{pmatrix}\frac{2i}{\pi\mu}\,\ln r+\frac{\partial^2\Psi_{11}}{\partial x_1^2} & \frac{\partial^2\Psi_{11}}{\partial x_1\partial x_2} & \frac{\partial\Psi_{13}}{\partial x_1} & \frac{\partial\Psi_{14}}{\partial x_1}\\ \frac{\partial^2\Psi_{11}}{\partial x_1\partial x_2} & \frac{2i}{\pi\mu}\,\ln r+\frac{\partial^2\Psi_{11}}{\partial x_2^2} & \frac{\partial\Psi_{13}}{\partial x_2} & \frac{\partial\Psi_{14}}{\partial x_2}\\ -\frac{i\omega}{m_1}\,\frac{\partial\Psi_{13}}{\partial x_1} & -\frac{i\omega}{m_1}\,\frac{\partial\Psi_{13}}{\partial x_2} & \Psi_{33} & \Psi_{34}\\ -\frac{i\omega}{m_2}\,\frac{\partial\Psi_{14}}{\partial x_1} & -\frac{i\omega}{m_2}\,\frac{\partial\Psi_{14}}{\partial x_2} & \frac{m_1}{m_2}\Psi_{34} & \Psi_{44}\end{pmatrix}, \tag{2.6}$$

where

$$\begin{aligned}\Psi_{11} &= \alpha_{11}\ln r+\alpha_{12}\,\frac{r^2(\ln r-1)}{4}+\alpha_{21}H_0^{(1)}(\lambda_1 r)+\alpha_{22}H_0^{(1)}(\lambda_2 r),\\ \Psi_{13} &= \beta_{11}\ln r+\beta_{12}H_0^{(1)}(\lambda_1 r)+\beta_{13}H_0^{(1)}(\lambda_2 r),\\ \Psi_{14} &= \gamma_{11}\ln r+\gamma_{12}H_0^{(1)}(\lambda_1 r)+\gamma_{13}H_0^{(1)}(\lambda_2 r),\\ \Psi_{31} &= \frac{1}{m_1}\,\Psi_{13},\quad \Psi_{33}=\delta_{11}H_0^{(1)}(\lambda_1 r)+\delta_{12}H_0^{(1)}(\lambda_2 r), \qquad (2.7)\\ \Psi_{41} &= \frac{1}{m_2}\,\Psi_{14},\quad \Psi_{34}=\delta_{34}\big[H_0^{(1)}(\lambda_2 r)-H_0^{(1)}(\lambda_1 r)\big],\\ \Psi_{43} &= \frac{m_1}{m_2}\,\Psi_{34},\quad \Psi_{44}=\delta_{41}H_0^{(1)}(\lambda_1 r)+\delta_{42}H_0^{(1)}(\lambda_2 r),\\ \alpha_{11} &= \frac{2i}{\pi a\lambda_1^2\lambda_2^2}\,\Big[\frac{\alpha_3}{m_1}+\frac{\alpha_4}{m_2}-\frac{(\lambda_1^2+\lambda_2^2)(\alpha_3\alpha_4-k^2)}{m_1m_2\lambda_1^2\lambda_2^2}\Big],\end{aligned}$$

$$\alpha_{12}=\frac{2i}{\pi}\left[\frac{\alpha_3\alpha_4-k^2}{am_1m_2\lambda_1^2\lambda_2^2}-\frac{1}{\mu}\right],\quad \delta_{34}=-\frac{ka+i\omega\beta_1\beta_2}{m_1a(\lambda_1^2-\lambda_2^2)}\,,$$

$$\alpha_{2k}=\frac{(-1)^k}{a(\lambda_1^2-\lambda_2^2)}\left[1-\frac{1}{\lambda_k^2}\left(\frac{\alpha_3}{m_1}+\frac{\alpha_4}{m_2}\right)+\frac{\alpha_3\alpha_4-k^2}{m_1m_2\lambda_k^4}\right],\quad k=1,2,$$

$$\beta_{11}=\frac{2i(\alpha_4\beta_1-k\beta_2)}{\pi m_2a\lambda_1^2\lambda_2^2}\,,\quad \gamma_{11}=\frac{2i(\alpha_3\beta_2-k\beta_1)}{\pi m_1a\lambda_1^2\lambda_2^2}\,,$$

$$\beta_{1k}=\frac{(-1)^k}{a(\lambda_1^2-\lambda_2^2)}\left[-\beta_1+\frac{\alpha_4\beta_1-k\beta_2}{m_2\lambda_{k-1}^2}\right],\quad k=2,3,$$

$$\gamma_{1k}=\frac{(-1)^k}{a(\lambda_1^2-\lambda_2^2)}\left[-\beta_2+\frac{\alpha_3\beta_2-k\beta_1}{m_1\lambda_{k-1}^2}\right],\quad k=2,3,$$

$$\delta_{1k}=\frac{1}{\lambda_1^2-\lambda_2^2}\left[-\lambda_k^2+\frac{\alpha_4a+i\omega\beta_2^2}{m_2a}\right],\quad k=1,2,$$

$$\delta_{4k}=\frac{1}{\lambda_1^2-\lambda_2^2}\left[-\lambda_k^2+\frac{\alpha_3a+i\omega\beta_1^2}{m_2a}\right],\quad k=1,2,$$

$$\alpha_{11}+\frac{2i}{\pi}\left[\alpha_{21}+\alpha_{22}\right]=0,\quad \beta_{11}+\frac{2i}{\pi}\left[\beta_{12}+\beta_{13}\right]=0,$$

$$\gamma_{11}+\frac{2i}{\pi}\left[\gamma_{12}+\gamma_{13}\right]=0,$$

$$\delta_{11}+\delta_{33}=1,\quad \delta_{22}+\delta_{44}=1,\quad r^2=(x_1-y_1)^2+(x_2-y_2)^2.$$

Moreover, on the basis of the identity

$$H_0^{(1)}(\lambda r)=\frac{2i}{\pi}\ln r-\frac{2i}{4\pi}r^2\ln r+const+O(r^2)$$

we easily conclude that $\Gamma(x-y)$ has a logarithmic singularity. It can be shown that the columns of the matrix $\Gamma(x-y)$ are solutions to the equation (1.1) with respect to x for any $x\neq y$.

Denote $\widetilde{\Gamma}(x)=\Gamma^T(-x)$. Hence we have proved the following

Theorem. *The matrix* $\Gamma(x)$ *is a solution of the system* (1.1) *and the matrix* $\widetilde{\Gamma}(x)$ *is a solution of the adjoint system* $\widetilde{B}(\partial x)U=0$.

3. Matrix of Singular Solutions

In solving boundary value problems of the theory of consolidation with double porosity by the method of potential theory, the fundamental matrix and some other matrices of singular solutions to the equation (1.1) are of great importance. These matrices will be constructed explicitly in the present section with the help of elementary functions. Using the basic fundamental matrix, we will construct the so-called singular matrices of solutions. For simplicity, we will introduce the special generalized stress vector.

Write now the expressions for the components of the generalized stress vector, which acts on elements of the arc with the normal $n=(n_1,n_2)$.

Denoting the generalized stress vector by $\overset{\kappa}{\mathrm{P}}(\partial x, n)u$, where κ is an arbitrary constant, we have

$$\overset{\kappa}{\mathrm{P}}(\partial x, n)u = \overset{\kappa}{\mathrm{T}}(\partial x, n)u - n(\beta_1 p_1 + \beta_2 p_2), \tag{3.1}$$

where

$$\overset{\kappa}{\mathrm{T}}(\partial x, n)u = \begin{pmatrix} \mu \dfrac{\partial}{\partial n} + (\lambda+\mu)n_1 \dfrac{\partial}{\partial x_1} & (\lambda+\mu)n_1 \dfrac{\partial}{\partial x_2} - \kappa \dfrac{\partial}{\partial s} \\ (\lambda+\mu)n_2 \dfrac{\partial}{\partial x_1} + \kappa \dfrac{\partial}{\partial s} & \mu \dfrac{\partial}{\partial n} + (\lambda+\mu)n_2 \dfrac{\partial}{\partial x_2} \end{pmatrix} u,$$

$$\frac{\partial}{\partial s} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}.$$

If $\kappa = \mu$, then we have the stress vector $P(\partial x, n)u$. The operator which will be obtained from $\overset{\kappa}{\mathrm{P}}(\partial x, n)$ for $\kappa = \kappa_n = \frac{\mu(\lambda+\mu)}{\lambda+3\mu}$ will be called the operator $N(\partial x, n)$, and the vector $N(\partial x, n)u$ will be called the pseudo-stress vector. The pseudo-stress operator succeeded in obtaining the Fredholm integral equation of the second kind for the first boundary value problem.

We introduce the following notation $\overset{\kappa}{\mathrm{R}}(\partial x, n)$, $\widetilde{\mathrm{R}}^{\kappa}(\partial x, n)$

$$\overset{\kappa}{\mathrm{R}}(\partial x, n) = \begin{pmatrix} \overset{\kappa}{\mathrm{T}}(\partial x, n)_{11} & \overset{\kappa}{\mathrm{T}}(\partial x, n)_{12} & -\beta_1 n_1 & -\beta_1 n_1 \\ \overset{\kappa}{\mathrm{T}}(\partial x, n)_{21} & \overset{\kappa}{\mathrm{T}}(\partial x, n)_{22} & -\beta_1 n_2 & -\beta_2 n_2 \\ 0 & 0 & \dfrac{\partial}{\partial n} & 0 \\ 0 & 0 & 0 & \dfrac{\partial}{\partial n} \end{pmatrix},$$

$$\widetilde{\mathrm{R}}^{\kappa}(\partial x, n) = \begin{pmatrix} \overset{\kappa}{\mathrm{T}}(\partial x, n)_{11} & \overset{\kappa}{\mathrm{T}}(\partial x, n)_{12} & -i\omega n_1 \dfrac{\beta_1}{m_1} & -i\omega n_1 \dfrac{\beta_2}{m_2} \\ \overset{\kappa}{\mathrm{T}}(\partial x, n)_{21} & \overset{\kappa}{\mathrm{T}}(\partial x, n)_{22} & -i\omega n_2 \dfrac{\beta_1}{m_1} & -i\omega n_2 \dfrac{\beta_2}{m_2} \\ 0 & 0 & \dfrac{\partial}{\partial n} & 0 \\ 0 & 0 & 0 & \dfrac{\partial}{\partial n} \end{pmatrix}.$$

By Applying the operator $\overset{\kappa}{\mathrm{R}}(\partial x, n)$ to the matrix $\Gamma(x)$, we will construct the so-called singular matrix of solutions. Let us consider the matrix $[\overset{\kappa}{\mathrm{R}}(\partial y, n)\Gamma(y-x)]^*$ which is obtained from $\overset{\kappa}{\mathrm{R}}(\partial x, n)\Gamma(x-y) = (\overset{\kappa}{\mathrm{R}}_{pq})_{4x4}$ by transposition of the columns and rows and the variables x and y. We can easily prove that every column of the matrix $[\overset{\kappa}{\mathrm{R}}(\partial y, n)\Gamma(y-x)]^*$ is a solution of the system $\widetilde{B}(\partial x)U = 0$ with respect to the point x, if $x \neq y$.

The elements $\overset{\kappa}{\mathrm{R}}_{pq}$ are as follows:

$$\overset{\kappa}{\mathrm{R}}_{pp} = \frac{2i}{\pi} \frac{\partial}{\partial n} \ln r + (-1)^p (\kappa + \mu) \frac{\partial}{\partial s} \frac{\partial^2 \Psi_{11}}{\partial x_1 \partial x_2}, \quad p = 1, 2,$$

$$\overset{\kappa}{\mathrm{R}}_{12} = -\frac{\partial}{\partial s}\left[\frac{2i}{\pi}\,\frac{\kappa}{\mu}\,\ln r + (\kappa+\mu)\,\frac{\partial^2\Psi_{11}}{\partial x_2^2}\right],$$

$$\overset{\kappa}{\mathrm{R}}_{21} = \frac{\partial}{\partial s}\left[\frac{2i}{\pi}\,\frac{\kappa}{\mu}\,\ln r + (\kappa+\mu)\,\frac{\partial^2\Psi_{11}}{\partial x_1^2}\right],$$

$$\overset{\kappa}{\mathrm{R}}_{13} = -(\kappa+\mu)\,\frac{\partial}{\partial s}\,\frac{\partial\Psi_{13}}{\partial x_2}, \quad \overset{\kappa}{\mathrm{R}}_{14} = -(\kappa+\mu)\,\frac{\partial}{\partial s}\,\frac{\partial\Psi_{14}}{\partial x_2}\,, \tag{3.2}$$

$$\overset{\kappa}{\mathrm{R}}_{23} = (\kappa+\mu)\,\frac{\partial}{\partial s}\,\frac{\partial\Psi_{13}}{\partial x_1}, \quad \overset{\kappa}{\mathrm{R}}_{24} = (\kappa+\mu)\,\frac{\partial}{\partial s}\,\frac{\partial\Psi_{14}}{\partial x_1}\,,$$

$$\overset{\kappa}{\mathrm{R}}_{3j} = -\frac{i\omega}{m_1}\,\frac{\partial}{\partial n}\,\frac{\partial\Psi_{13}}{\partial x_j}, \quad \overset{\kappa}{\mathrm{R}}_{4j} = -\frac{i\omega}{m_2}\,\frac{\partial}{\partial n}\,\frac{\partial\Psi_{14}}{\partial x_j}\,, \ \ j=1,2,$$

$$\overset{\kappa}{\mathrm{R}}_{33} = \frac{\partial\Psi_{33}}{\partial n}\,, \quad \overset{\kappa}{\mathrm{R}}_{34} = \frac{\partial\Psi_{34}}{\partial n}\,, \quad \overset{\kappa}{\mathrm{R}}_{43} = \frac{m_1}{m_2}\,\frac{\partial\Psi_{34}}{\partial n}\,, \quad \overset{\kappa}{\mathrm{R}}_{44} = \frac{\partial\Psi_{44}}{\partial n}\,,$$

Analogously, we obtain the matrix

$$\widetilde{\mathrm{R}}^{\kappa}(\partial y, n)\widetilde{\Gamma}(y-x) = \big([\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}]_{pq}\big)_{4x4},$$

where

$$\big[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\big]_{pq} = \overset{\kappa}{\mathrm{R}}_{pp}, \ \ p=1,2, \quad \big[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\big]_{12} = \overset{\kappa}{\mathrm{R}}_{12}, \quad \big[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\big]_{21} = \overset{\kappa}{\mathrm{R}}_{21},$$

$$\big[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\big]_{13} = \frac{i\omega}{m_1}\overset{\kappa}{\mathrm{R}}_{13}, \quad \big[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\big]_{14} = \frac{i\omega}{m_2}\overset{\kappa}{\mathrm{R}}_{14}, \quad \big[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\big]_{23} = \frac{i\omega}{m_1}\overset{\kappa}{\mathrm{R}}_{23},$$

$$\big[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\big]_{24} = \frac{i\omega}{m_2}\overset{\kappa}{\mathrm{R}}_{24}, \quad \big[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\big]_{3j} = -\frac{\partial}{\partial n}\,\frac{\partial\Psi_{13}}{\partial x_j},$$

$$\big[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\big]_{4j} = -\frac{\partial}{\partial n}\,\frac{\partial\Psi_{14}}{\partial x_j}, \ \ j=1,2,$$

$$\big[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\big]_{33} = \frac{\partial}{\partial n}\,\Psi_{33}, \quad \big[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\big]_{34} = \frac{m_1}{m_2}\,\frac{\partial}{\partial n}\,\Psi_{34},$$

$$\big[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\big]_{43} = \frac{\partial}{\partial n}\,\Psi_{34}, \quad \big[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\big]_{44} = \frac{\partial}{\partial n}\,\Psi_{44},$$

The matrix $\big[\widetilde{\mathrm{R}}^{\kappa}(\partial y, n)\widetilde{\Gamma}(y-x)\big]^*$ is a solution of the system (1.1). It shows, that the matrices $\big[\widetilde{\mathrm{R}}^{\kappa}(\partial x, n)\widetilde{\Gamma}\big]^*$ and $\big[\overset{\kappa}{\mathrm{R}}(\partial x, n)\Gamma\big]^*$ contain a singular part, which is integrable in the sense of the principal Cauchy value.

4. Potentials and Their Properties

Introduce the following definitions:

Definition 2. The vector-functions defined by the equalities

$$V^{(1)}(x) = \frac{1}{4i}\int\limits_S \Gamma(y-x)h(y)\,dy,$$
$$V^{(2)}(x) = \frac{1}{4i}\int\limits_S \widehat{\Gamma}(x-y)h(y)\,ds, \tag{4.1}$$

where $\Gamma(x,y)$ is the fundamental matrix, $\widetilde{\Gamma}(x)=\Gamma^T(-x)$, h is a continuous (or Holder continuous) vector and S is a closed Lyapunov curve, will be called simple layer potentials.

Definition 3. The vector-function defined by the equalities

$$\begin{aligned} U^{(1)}(x) &= \frac{1}{4i}\int\limits_S [\widetilde{N}(\partial y,n)\widetilde{\Gamma}(y-x)]^* h(y)\,dy,\\ U^{(2)}(x) &= \frac{1}{4i}\int\limits_S [N(\partial y,n)\Gamma(y-x)]^* h(y)\,dy, \end{aligned} \tag{4.2}$$

will be called double layer potentials.

The potentials $V^{(1)}$, $U^{(1)}$ are solutions of the system (1.1) and the potentials $V^{(2)}$, $U^{(2)}$ are solutions of the system $\widetilde{B}(\partial x)U=0$ both in the domains D^+ and D^-. When the point x tends to a point $z\in S$, the potential (4.2) has the discontinuity as the harmonic double layer potential

$$\begin{aligned} U^{(1)\pm} &= \pm h(z)+\frac{1}{4i}\int\limits_S [\widetilde{N}(\partial y,n)\widetilde{\Gamma}(y-z)]^* h(y)\,dy,\\ U^{(2)\pm} &= \pm h(z)+\frac{1}{4i}\int\limits_S [N(\partial y,n)\Gamma(y-z)]^* h(y)\,dy. \end{aligned} \tag{4.3}$$

Now let us investigate properties of the operation $\overset{\kappa}{\mathrm{R}}(\partial x,n)$ acting on a simple layer potential. We obtain

$$\overset{\kappa}{\mathrm{R}}(\partial x,n)V(x)=\frac{1}{4i}\int\limits_S \overset{\kappa}{\mathrm{R}}(\partial x,n)\Gamma(y-x)h(y)\,dy. \tag{4.4}$$

When $\kappa=\kappa_n$ we obtain

$$\begin{aligned} \big[N(\partial y,n)V^{(1)}(z)\big]^{\mp} &= \mp h(z)+\frac{1}{4i}\int\limits_S N(\partial y,n)\Gamma(z-y)h(y)\,dy,\\ \big[\widetilde{N}(\partial y,n)V^{(2)}(z)\big]^{\mp} &= \mp h(z)+\frac{1}{4i}\int\limits_S \widetilde{N}(\partial y,n)\widehat{\Gamma}(z-y)h(y)\,dy. \end{aligned} \tag{4.5}$$

It is well-known ([8]) that in the case of a Lyapunov curve $S\in C^{1,\alpha}$ the function $\frac{\partial \ln r}{\partial n}$ for $x,y\in S$ has a week singularity and $\frac{\partial \ln r}{\partial n}$ is integrable in the sense of the principal Cauchy value. Consequently, $\frac{\partial \ln r}{\partial n}$ is a singular kernel on S.

It is obvious that $\big[\overset{\kappa}{\mathrm{R}}(\partial y,n)\Gamma(y-x)\big]^*$ is a singular kernel (in the sense of Cauchy). Note that if $\kappa=\kappa_n=\frac{\mu(\lambda+\mu)}{\lambda+3\mu}$, then $\big[\overset{\kappa}{\mathrm{R}}(\partial x,n)\Gamma(x-y)\big]^*$ is a weakly singular kernel.

5. Solution of the First Boundary Value Problem

Problem $(I)^+$. Let us first prove the existence of solution of the first boundary value problem in the domain D^+. A solution is sought in the form of the double layer potential

$$U(x) = \frac{1}{4i} \int_S \big[\widetilde{N}(\partial y, n)\widetilde{\Gamma}(y-x)\big]^* h(y)\,dy. \tag{5.1}$$

Then for determining the unknown real vector function h we obtain the following Fredholm integral equation of the second kind

$$-h(z) + \frac{1}{4i} \int_S \big[\widetilde{N}(\partial y, n)\widetilde{\Gamma}(y-z)\big]^* h(y)\,dy = f^+. \tag{5.2}$$

Let us prove that the equation (5.2) is solvable for any continuous right-hand side. Consider the associated to (5.2) homogeneous equation

$$-h(z) + \frac{1}{4i} \int_S N(\partial y, n)\Gamma(y-z)h(y)\,dy = 0 \tag{5.3}$$

and prove that it has only the trivial solution.Assume the contrary and denote by $\varphi(z)$ a nonzero solution of (5.3). Compose the simple layer potential

$$V(x) = \frac{1}{4i} \int_S \Gamma(y-x)\varphi(y)\,dy. \tag{5.4}$$

It is obvious from (5.3), that

$$[N(\partial z, n)V(z)]^- = 0, \quad \int_S \varphi(y)\,ds = 0.$$

Using the formula (1.8) for $\kappa = \kappa_n$ in D^-, we obtain $V(x) = 0$, $x \in D^-$.

Now taking into account the continuity of the simple layer potential and using the uniqueness theorem for the solution of the first boundary value problem, we have $V(x) = 0$, $x \in D^+$.

Note that $[NV]^+ - [NV]^- = 2\varphi(x) = 0$ and hence the equation (5.3) has only the trivial solution. This implies that the associated to (5.3) homogeneous equation also has only the trivial solution, and the equation(5.2) is solvable for any continuous right-hand side (according to the first Fredholm theorem).

For the regularity of the double layer potential in the domain D^+ it is sufficient to assume that $S \in C^{2,\beta}$, $(0 < \beta < 1)$ and $\frac{\partial f}{\partial s}$ is Holder continuous $f \in C^{1,\alpha}(S)$ $(0 < \alpha < \beta)$.

Problem $(I)^-$. Consider now the first boundary value problem in the domain D^-. Its solution is sought in the form

$$U(x) = \frac{1}{4i} \int_S \Big(\big[\widetilde{N}(\partial y, n)\widetilde{\Gamma}(y-x)\big]^* - \big[\widetilde{N}(\partial y, n)\widetilde{\Gamma}(y)\big]^*\Big)\psi(y)\,dy. \tag{5.5}$$

Then for determining the unknown real valued vector function ψ we obtain the following Fredholm integral equation of the second kind

$$\psi(z)+\frac{1}{4i}\int\limits_S\Big(\big[\widetilde{N}(\partial y,n)\widetilde{\Gamma}(y-z)\big]^*-\big[\widetilde{N}(\partial y,n)\widetilde{\Gamma}(y)\big]^*\Big)\psi(y)\,dy=f^-. \quad (5.6)$$

Prove that the equation (5.6) is solvable for any continuous right-hand side. We consider the associated to (5.6) homogeneous equation

$$h(z)+\frac{1}{4i}\int\limits_S\big[N(\partial y,n)\Gamma(z-y)+N(\partial y,n)\Gamma(y)\big]h(y)\,dy=0. \quad (5.7)$$

Let us prove that (5.7) has only the trivial solution. Suppose that it has a nonzero solution $h(z)$. From (5.7) by integration we obtain

$$\int\limits_S h\,ds=0.$$

In this case the equation (5.7) corresponds to the boundary condition $[N(\partial x,n)V]^+=0$, where

$$V(x)=\frac{1}{4i}\int\limits_S\Gamma(y-x)h(y)\,dy. \quad (5.8)$$

We find that $V=C$, $x\in D^+$, where C is a constant vector.

Taking into account the equation $\int\limits_S h\,ds=0$ and the fact that the single layer potential is continuous while passing through the boundary, and using Green's formula for $\kappa=\kappa_n$, we obtain $V=0$, $x\in D^-$. Since $[NV]^+-[NV]^-=2h(x)=0$, and $[NV]^+=0$, $[NV]^-=0$, we get $h(x)=0$.

Thus we conclude that the associated to (5.7) homogeneous equation has only the trivial solution, and the equation (5.6) is solvable for any continuous right-hand side.

To prove the regularity of the potential (5.5) in the domain D^-, it is sufficient to assume that $S\in C^{2,\beta}$ $(0<\beta<1)$ and $f\in C^{1,\alpha}(S)$ $(0<\alpha<\beta)$.

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ON AN INTEGRAL EQUATION WITH MONOTONIC NONLINEARITY

Abstract. We prove the existence of a nonnegative and bounded solution of a type of homogeneous integral equations with monotonic nonlinearity. Under certain assumptions on the kernel, the properties of the obtained solutions are investigated. Some particular examples which arise in applications are demonstrated.

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რეზიუმე. ნაშრომში დამტკიცებულია მონოტონური არაწრფივობის შემცველი ერთგვაროვანი ინტეგრალური განტოლების ერთი ტიპისთვის არაუარყოფითი და შემოსაზღვრული ამონახსნის არსებობა. განტოლების გულზე დადებულ გარკვეულ პირობებში გამოკვლეულია მიღებული ამონახსნების ასიმპტოტიკა. განხილულია რამდენიმე კერძო მაგალითი, რომლებიც გამოყენებაში გვხვდება.

1. Introduction

We consider the following nonlinear integral equation:

$$\varphi^p(x) = \int_0^\infty K(x,t)\varphi(t)\,dt, \quad x > 0, \tag{1}$$

in regard to unknown function $\varphi(x) \geq 0$. Here $p > 1$ is a real number, $0 \leq K(x,t)$ is a measurable function defined on $(0,+\infty)\times(0,+\infty)$ satisfying the condition

$$\sup_{x>0} \int_0^\infty K(x,t)\,dt = 1. \tag{2}$$

We will also consider the general integral equation of Hammerstein type:

$$f(x) = \int_0^\infty K(x,t)Q(f(t))\,dt, \tag{1*}$$

where the function $Q(x)$ is defined on $(-\infty,+\infty)$ and satisfies some additional conditions (see Theorem 6).

The problems (1), (2) and (1*), (2) are of considerable interest not only in mathematics, but also in the theory of nonlocal interactions, string filed theory, cosmology, kinetic theory of gases (see [1]–[6]).

In the present paper, under certain assumptions on the kernel $K(x,t)$ we prove the existence of a nontrivial, nonnegative and bounded solution of nonlinear homogenous equations (1) and (1*). The properties of the obtained solutions are investigated (see Theorems 1–3, 6). We also undertake mathematical investigation of a special case which arises in applications, particularly in the dynamics of P-adic closed string field theory (see Theorems 4–5). Some particular examples of the function $Q(x)$ are listed.

2. Convolution type nonlinear integral equation

2.1. **Symmetric kernel.** First, we consider the equation (1), in particular, the case where

$$K(x,t) = k_0(x-t); \quad 0 \leq k_0 \in L_1(-\infty,+\infty).$$

We have

$$\psi^p(x) = \int_0^\infty k_0(x-t)\psi(t)\,dt, \quad x > 0, \quad p > 1. \tag{3}$$

The condition (2) takes the form of

$$\int_{-\infty}^{+\infty} k_0(x)\,dx = 1. \tag{4}$$

We also assume that

$$k_0(-x) = k_0(x), \quad \forall\, x > 0. \tag{5}$$

Denoting $f(x) = \psi^p(x)$, we have

$$f(x) = \int_0^\infty k_0(x-t)\sqrt[p]{f(t)}\,dt, \quad x > 0, \quad p > 1. \tag{6}$$

We will consider the following iteration process

$$f^{(n+1)}(x) = \int_0^\infty k_0(x-t)\sqrt[p]{f^{(n)}(t)}\,dt, \quad f^{(0)}(x) \equiv 1, \quad n = 0, 1, 2 \dots . \tag{7}$$

The following statements are valid.

Statement 1. *The sequence of functions $\{f^{(n)}(x)\}_0^\infty$ is monotonously decreasing as n increases.*

Proof. Indeed, for $n = 0$ we have

$$f^{(1)}(x) \le \int_{-\infty}^{+\infty} k_0(t)dt = 1 \equiv f^{(0)}(x).$$

Assuming that the analogous inequality holds for n and using the monotonicity of the function $y = \sqrt[p]{x}$ on $(0, +\infty)$, from (7) we obtain

$$f^{(n+1)}(x) \le f^{(n)}(x).$$ □

Statement 2. *The following inequality is valid*

$$f^{(n)}(x) \ge \left(\frac{1}{2}\right)^{\frac{p}{p-1}}, \quad n = 0, 1, 2, \dots . \tag{8}$$

Proof. For $n = 0$ this estimate is obvious. Let $f^{(n)}(x) \ge (\frac{1}{2})^{\frac{p}{p-1}}$ be true. Taking into account (4) and (5), from (7) we get

$$f^{(n+1)}(x) \ge \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \int_{-\infty}^{x} k_0(t)\,dt \ge \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \int_{-\infty}^{0} k(t)\,dt = \left(\frac{1}{2}\right)^{\frac{p}{p-1}}. \tag{9}$$

The statement is proved. □

Statements 1 and 2 imply that almost everywhere the limit of the sequence of functions $\{f^{(n)}(x)\}_0^\infty$ exists:

$$\lim_{n\to\infty} f^{(n)}(x) = f(x). \tag{10}$$

Furthermore,

$$\left(\frac{1}{2}\right)^{\frac{p}{p-1}} \le f(x) \le 1. \tag{11}$$

Using Levi's limit theorems, we conclude that $f(x)$ is a solution of the equation (6).

Statement 3. *The solution $f(x)$ of the equation* (6) *is monotonously increasing as x increases.*

Proof. First, we prove that the sequence of functions $\{f^{(n)}(x)\}_{n=0}^{\infty}$ is increasing in x. Indeed, for $n = 0$ this is obvious. Suppose that $f^{(n-1)}(x) \uparrow$ as x increases. Let $x_1, x_2 \in (0, +\infty)$, $x_1 > x_2$, are two arbitrary numbers. We have

$$f^{(n)}(x_1) - f^{(n)}(x_2) =$$
$$= \int_{-\infty}^{x_1} k_0(t)[\sqrt[p]{f^{(n-1)}(x_1 - t)}\, dt - \int_{-\infty}^{x_2} k_0(t) \sqrt[p]{f^{(n-1)}(x_2 - t)}]\, dt \geq$$
$$\geq \int_{-\infty}^{x_2} k_0(t) \left[\sqrt[p]{f^{(n-1)}(x_1 - t)} - \sqrt[p]{f^{(n-1)}(x_2 - t)}\right] dt \geq 0.$$

Therefore $f^{(n)}(x_1) \geq f^{(n)}(x_2)$, which implies that $f(x_1) \geq f(x_2)$. □

Statement 4. *The limit of the function $f(x)$ exists:*

$$\lim_{x\to+\infty} f(x) = 1. \tag{12}$$

Proof. Denote $\lim\limits_{x\to+\infty} f(x) = \delta$.

It is easy to check that

$$\lim_{x\to+\infty} \sqrt[p]{f(x)} = \lim_{x\to+\infty} \psi(x) = \sqrt[p]{\delta}. \tag{13}$$

We show that

$$\lim_{x\to+\infty} \int_{0}^{\infty} k_0(x - t) \sqrt[p]{f(t)}\, dt = \sqrt[p]{\delta}. \tag{14}$$

Indeed,

$$\left| \int_{0}^{\infty} k_0(x - t) \sqrt[p]{f(t)}\, dt - \sqrt[p]{\delta} \int_{-\infty}^{+\infty} k_0(t)\, dt \right| =$$
$$= \left| \int_{-\infty}^{x} k_0(t) \sqrt[p]{f(x - t)}\, dt - \sqrt[p]{\delta} \int_{-\infty}^{x} k_0(t) dt - \int_{x}^{\infty} \sqrt[p]{\delta} k_0(t) dt \right| \leq$$
$$\leq \int_{-\infty}^{x} k_0(t) \left| \sqrt[p]{f(x - t)} - \sqrt[p]{\delta} \right| dt + \sqrt[p]{\delta} \int_{x}^{\infty} k_0(t)\, dt = J_1 + J_2.$$

It is obvious that $\lim_{x\to+\infty} J_2 = 0$. We have

$$J_1 = \int_{-\infty}^{x} k_0(t) \left| \sqrt[p]{f(x-t)} - \sqrt[p]{\delta} \right| dt \leq$$

$$\leq \int_{-\infty}^{\frac{x}{2}} k_0(t) \left| \sqrt[p]{f(x-t)} - \sqrt[p]{\delta} \right| dt + \int_{\frac{x}{2}}^{x} k_0(t) \left| \sqrt[p]{f(x-t)} - \sqrt[p]{\delta} \right| dt =$$

$$= J_3 + J_4,$$

$$J_3 = \int_{\frac{x}{2}}^{\infty} k_0(x-t) \left| \sqrt[p]{f(t)} - \sqrt[p]{\delta} \right| dt \leq \sup_{t\geq\frac{x}{2}} \left| \sqrt[p]{f(t)} - \sqrt[p]{\delta} \right| dt \int_{-\infty}^{+\infty} k_0(t)\, dt \to 0$$

as $x \to +\infty$.

$$J_4 = (1 + \sqrt[p]{\delta}) \int_{\frac{x}{2}}^{x} k_0(t)\, dt \to 0$$

as x tends to ∞. Thus the formula (13) holds. Passing in (6) to limit, we obtain $\delta = \sqrt[p]{\delta} \Rightarrow \delta = 1$. From (14) it follows that

$$\lim_{x\to+\infty} \psi(x) = 1. \tag{15}$$

The statement is proved. □

Statement 5. *Let $f_1(x)$ and $f_2(x)$ be the constructed solutions of the equation* (6) *for the integers p_1 and p_2, respectively. If $p_1 > p_2$, then $f_1(x) \geq f_2(x)$.*

Proof. We consider the iterations for $p = p_1$ and $p = p_2$ separately.

$$f_i^{(n+1)}(x) = \int_0^{\infty} k_0(x-t) \sqrt[p_i]{f_i^{(n)}(t)}\, dt, \quad f_i^{(0)} \equiv 1, \quad i=1,2, \quad n=0,1,2,\ldots. \tag{16}$$

We will prove that

$$f_1^{(n)}(x) \geq f_2^{(n)}(x). \tag{17}$$

Indeed, for $n = 0$ the inequality (17) is obvious. Assuming that (17) holds for n, we check it for $n+1$. Taking into account the estimates $0 < f^{(n)}(x) \leq 1$, from (16) we get

$$f_1^{(n+1)}(x) \geq \int_0^{\infty} k_0(x-t) \sqrt[p_1]{f_2^{(n)}(t)}\, dt \geq$$

$$\geq \int_0^{\infty} k_0(x-t) \sqrt[p_2]{f_2^{(n)}(t)}\, dt = f_2^{(n+1)}(x), \tag{18}$$

which implies that

$$f_1(x) \geq f_2(x). \tag{19}$$

Thus we have proved the statement. □

Theorem 1. *Under the conditions* (4), (5) *the equation* (3) *has a positive and bounded solution* $\psi(x)$ *which possesses the following properties:*

a) $\psi(x) \uparrow$ *in* x*;*
b) *the estimates* $(\frac{1}{2})^{\frac{1}{p-1}} \leq \psi(x) \leq 1$ *are valid;*
c) *there exists the limit* $\lim\limits_{x\to+\infty} \psi(x) = 1$.

Remark 1. The linear equation (3)–(5) ($p = 1$) represents the well-known homogeneous conservative Wiener–Hopf equation. Many works are devoted to the investigation of the corresponding linear equation (3) (see [7]–[9] and the literature therein). It is known (see [7]) that the corresponding linear equation in the symmetric case $k_0(-x) = k_0(x)$ has a positive solution, possessing the asymptotic $O(x)$ at $x \to +\infty$. Thus we confirm that there is a quantitative difference between solutions of nonlinear ($p > 1$) and linear ($p = 1$) equations.

2.2. **Nonsymmetric kernel.** We will assume that

$$\nu(k_0) = \int\limits_{-\infty}^{+\infty} x k_0(x)\,dx < 0. \tag{20}$$

The convergence of the integral (20) is understood in the Cauchy v.p. sense. Together with the equation (3) we consider the corresponding linear equation

$$S(x) = \int\limits_0^\infty k_0(x-t) S(t)\,dt, \quad x > 0. \tag{21}$$

It is well–known that if the function $k_0(x)$ satisfies the conditions (4), (20), then the equation (21) has a positive monotonously increasing and bounded solution $S(x)$ (see [8,9]). We denote $C = \sup\limits_{x>0} S(x)$. Due to the linearity of (21), the function $S^* = \dfrac{1}{C} S(x)$ will also satisfy the equation (21). Furthermore, $S^*(x) \uparrow 1$ as $x \to +\infty$. We consider the equation (7) with the kernel (4), (20).

Analogously, it is easy to verify that $f^{(n)}(x) \downarrow$ as n increases. We prove $f^{(n)}(x) \geq S^*(x)$. For $n = 0$ this is obvious. Taking into account (21) and $0 < S^*(x) \leq 1$, from (7) we obtain

$$f^{(n+1)}(x) \geq \int\limits_0^\infty k_0(x-t) \sqrt[p]{S^*(t)}dt \geq \int\limits_0^\infty k_0(x-t) S^*(t)\,dt = S^*(x).$$

Thus, there exists $f(x) = \lim\limits_{x\to+\infty} f^{(n)}(x)$. Moreover,

$$S^*(x) \le f(x) \le 1. \tag{22}$$

From Levi's theorem it follows that the limit function $f(x)$ satisfies the equation (3).

Acting analogously as in Theorem 1, we obtain that $f(x) \uparrow$ as x increases. Since $S^*(x) \to 1$ as $x \to +\infty$, it follows from (22) that

$$\lim_{x\to\infty} f(x) = 1.$$

Thus the following theorem holds.

Theorem 2. *Under the conditions* (4), (20) *the equation* (3) *has a positive monotonically increasing and bounded solution* $\psi(x)$. *Moreover,*

$$\lim_{x\to\infty} \psi(x) = 1, \quad S^*(x) \le \psi(x) \le 1.$$

Acting analogously we will be able to prove the following general theorem.

Theorem 3. *Let there exist* $k_0(x)$, $k_0(x) \ge 0$, $\int\limits_{-\infty}^{+\infty} k_0(x)\,dx = 1$, *such that* $K(x,t) \ge k_0(x-t)\ \forall\, x,t \in R^+ \times R^+$.

1) *if* $k_0(-x) = k_0(x)$, *then the equation* (1) *has a positive and bounded solution* $\varphi(x)$:

$$\left(\frac{1}{2}\right)^{\frac{1}{p-1}} \le \psi(x) \le \varphi(x) \le 1; \quad \lim_{x\to+\infty} \varphi(x) = 1;$$

2) *if* $\nu(k_0) < 0$, *then the equation* (1) *has a positive and bounded solution* $\varphi(x)$:

$$S^*(x) \le \psi(x) \le \varphi(x) \le 1; \quad \lim_{x\to+\infty} \varphi(x) = 1.$$

2.3. **Examples.** We bring two particular examples of the equation (1) satisfying the conditions of Theorem 3:

$$\begin{aligned} &1)\quad \varphi^p(x) = \int_0^\infty k_0(x-t)\varphi(t)\,dt + \int_0^\infty k_1(x+t)\varphi(t)\,dt, \quad \text{where} \\ &\quad 0 \le k_1 \in L_1(0,+\infty), \quad \int_x^\infty k_1(t)\,dt \le \int_x^\infty k_0(t)\,dt, \quad \forall\, x > 0; \end{aligned} \tag{23}$$

$$2)\quad \varphi^p(x) = \mu(x) \int_0^\infty k_0(x-t)\varphi(t)\,dt, \tag{24}$$

where $\mu(x)$ is a measurable function on $(0,+\infty)$ satisfying the condition $1 \le \mu(x) \le \frac{1}{\int\limits_{-\infty}^{x} k_0(t)\,dt}$.

3. On a Special Case Arising in Applications

We consider the equation (1) in the case where

$$K(x,t) = k_0(x-t) - k_1(x+t) \geq 0. \tag{25}$$

It should be noted that the condition $K(x,t) \geq k_0(x-t)$ doesn't work for the kernel (25) and it is necessary to develop a new approach for studying the problem of solvability of the equation (1), (25). We should also note that the nonlinear equation (1) with the kernel

$$K(x,t) = \frac{1}{\sqrt{\pi}}\,(e^{-(x-t)^2} - e^{-(x+t)^2}) \tag{26}$$

describes the dynamics (rolling) of $P-$adic closed strings for a scalar tachyon field (see [2], [3]).

First we consider the corresponding linear equation ($p = 1$)

$$\eta(x) = \int\limits_0^\infty k_0(x-t)\eta(t)\,dt - \int\limits_0^\infty k_1(x+t)\eta(t)\,dt, \quad x > 0, \tag{27}$$

where $\eta(x)$ is the unknown function.

We rewrite the equation (27) in the operator form

$$(I - \widehat{K}_0 + \widehat{K}_1)\eta = 0, \tag{28}$$

where I is the unit operator, $\widehat{K}_0$ is a Wiener–Hopf integral operator, and $\widehat{K}_1$ is a Henkel operator. Let E be one of the following Banach spaces: $L_p(0,+\infty)$, $1 \leq p \leq \infty$, $M(0,+\infty)$, $C_u(0,+\infty)$, $C_0(0,+\infty)$, where $C_u(0,+\infty)$ is the space of continuous functions having a finite limit at infinity.

It is known (see [10]) that if $\nu(k_0) \leq 0$ and $m_2(k_1) = \int\limits_0^\infty x^2 k_1(x)\,dx < +\infty$, then the operator $I - \widehat{K}_0 + \widehat{K}_1$ admits the following three factor decomposition

$$I - \widehat{K}_0 + \widehat{K}_1 = (I - \widehat{V_-})(I + \widehat{W})(I - \widehat{V_+}), \tag{29}$$

where $\widehat{V}_\pm$ are Volterra operators:

$$(\widehat{V}_- f)(x) = \int\limits_x^\infty v_-(t-x) f(t)\,dt, \quad f \in E, \tag{30}$$

$$(\widehat{V}_+ f)(x) = \int\limits_0^x v_+(x-t) f(t)\,dt, \quad f \in E, \tag{31}$$

$0 \leq v_\pm \in L_1(0,+\infty)$, $\gamma_\pm = \int\limits_0^\infty v_\pm(x)\,dx \leq 1$, and $\widehat{W}$ is a Henkel type integral operator

$$(\widehat{W} f)(x) = \int\limits_0^\infty W(x+t) f(t) dt, \quad f \in E, \tag{32}$$

$0 \le W \in L_1(0, +\infty)$. It should be noted that (see [8])

i) if $\nu(k_0) < 0$, then $\gamma_- = 1$, $\gamma_+ < 1$;

ii) if $\nu(k_0) = 0$, then $\gamma_\pm = 1$.

At the same time, if the functions k_0 and k_1 are bounded, then $W \in M(0, +\infty)$, $v_\pm \in M(0, +\infty)$.

It is well known that $\widehat{W}$ is a compact operator in the spaces $L_1(0, +\infty)$ and $C_u(0, +\infty)$ (and in other natural functional spaces).

Taking into account the factorization (29), we rewrite the equation (28) in the form

$$(I - \widehat{V_-})(I + \widehat{W})(I - \widehat{V_+})\eta = 0. \tag{33}$$

Solving the equation (33) is equivalent to solving the following three coupled equations

$$(I - \widehat{V_-})\eta_1 = 0, \tag{34}$$

$$(I + \widehat{W})\eta_2 = \eta_1, \tag{35}$$

$$(I - \widehat{V_+})\eta = \eta_2. \tag{36}$$

Statement 6. *Let $\nu(k_0) < 0$. Then the equation* (27) *has a nontrivial solution $\eta(x) \in C_u(0, +\infty)$.*

Proof. Let us consider the following possibilities:

a) $\varepsilon = -1$ is an eigenvalue for the operator $\widehat{W}$;

b) $\varepsilon = -1$ is not an eigenvalue for the operator $\widehat{W}$.

a) We choose the trivial solution of the equation (34). Inserting it in (35), we obtain

$$\eta_2(x) = -\int_0^\infty W(x + t)\eta_2(t)\, dt. \tag{37}$$

Since $\varepsilon = -1$ is an eigenvalue for the operator $\widehat{W}$, the equation (37) has a nontrivial solution $\eta_2 \in C_u(0, +\infty)$. Furthermore, from the estimate

$$|\eta_2(x)| \le \sup_{t>0} |\eta_2(t)| \int_x^\infty W(\tau)\, d\tau$$

it follows that $\eta_2 \in C_0(0, +\infty)$.

Now we consider the equation (36)

$$\eta(x) = \eta_2(x) + \int_0^x v_+(x - t)\eta(t)\, dt. \tag{38}$$

Since $\gamma_+ < 1$, the equation (38) in the space $C_0(0, +\infty)$ has a unique solution (see [9]).

b) It is easy to check that $\eta_1(x) = const \ne 0$ satisfies the equation (34) because $\gamma_- = 1$.

We choose $\eta_1(x) \equiv 1$ as η_1. Substituting it in (35), using the fact that $\varepsilon = -1$ is not an eigenvalue for $\widehat{W}$ and taking into account that $\widehat{W}$ is completely continuous (in $C_u(0,+\infty)$), we conclude that the equation (35) has a bounded solution $\eta_2 \in C_u(0,+\infty)$. Since $\gamma_+ < 1$, the equation (38) has a solution belonging to $C_u(0,+\infty)$. □

Statement 7. *Let $\nu(k_0) = 0$, $k_0 \in L_1(-\infty,+\infty) \cap M(-\infty,+\infty)$, $k_1 \in L_1(0,+\infty) \cap M(0,+\infty)$. If $\varepsilon = -1$ is an eigenvalue for the operator $\widehat{W}$, then the equation* (27) *has a nontrivial bounded solution.*

Proof. First we note that under the above-mentioned conditions and from the results of [9], [10] it follows that $W \in M(0,+\infty) \cap L_1(0,+\infty)$, $v_\pm \in M(0,+\infty) \cap L_1(0,+\infty)$. Choosing the trivial solution of the equation (34) and taking into account that $\varepsilon = -1$ is an eigenvalue for the completely compact operator $\widehat{W}$ (in $L_1(0,+\infty)$), we conclude that the equation (35) in $L_1(0,+\infty)$ has a nontrivial solution. Since $W \in M(0,+\infty) \cap L_1(0,+\infty)$, from the inequality

$$|\eta_2(x)| \le \sup_{x>0} |W(x)| \int_0^\infty |\eta_2(t)|\, dt$$

it follows that $\eta_2 \in M(0,+\infty)$. Thus we have proved that $\eta_2 \in L_1(0,+\infty) \cap M(0,+\infty)$. Now we consider the equation (36) in the conservative case (when $\gamma_+ = 1$). Using the results of the work [11], we conclude that the equation (36) has a bounded solution $\eta(x)$. Below we assume that one of the conditions of Statements 6 or 7 is fulfilled. Denote $C = \sup_{x>0} |\eta(x)|$. Due to the linearity of the equation (27), the function $\widetilde{\eta} = \frac{1}{C}\eta$ will be a nontrivial solution of the equation (27). Furthermore,

$$\sup_{x>0} |\widetilde{\eta}(x)| = 1. \tag{39}$$

Let us consider the following iteration

$$f^{(n+1)}(x) = \int_0^\infty K(x,t) \sqrt[p]{f^{(n)}(t)}\, dt, \quad f^{(n)}(x) \equiv 1, \quad n = 0,1,2,\dots, \tag{40}$$

where $K(x,t)$ is given by the formula (25).

It is easy to check that for arbitrary $n = 0,1,2,\dots$ the inequality

$$f^{(n)}(x) \ge |\widetilde{\eta}(x)| \tag{41}$$

holds. Indeed, for $n = 0$ it is obvious (see (39)). Assuming that the inequality (41) holds for some n, we will prove that it is true for $n+1$. Since $|\widetilde{\eta}(x)| \le 1$, we have

$$f^{(n+1)}(x) \ge \int_0^\infty K(x,t) \sqrt[p]{|\widetilde{\eta}(t)|} dt \ge \int_0^\infty K(x,t) |\widetilde{\eta}(t)| dt \ge |\widetilde{\eta}(x)|.$$

Hence the sequence of functions $\{f^{(n)}(x)\}_0^\infty$ has a limit as $n \to +\infty$,

$$\lim_{n\to\infty} f^{(n)}(x) = f(x). \tag{42}$$

At the same time,

$$|\widetilde{\eta}(x)| \le f(x) \le 1. \tag{43}$$

Using Levi's theorem, we conclude that $f(x)$ is a solution of the equation

$$f(x) = \int_0^\infty K(x,t)\sqrt[p]{f(t)}\,dt. \qquad \square$$

Statement 8. *$f(x) \uparrow$ as x increases.*

Proof. Let x_1, $x_2 \in (0,+\infty)$, $x_1 < x_2$, be arbitrary numbers and consider the following iteration process

$$f^{(n+1)}(x) = \int_{-\infty}^{x} k_0(t)\sqrt[p]{f^{(n)}(x-t)}\,dt - \int_x^\infty k_1(t)\sqrt[p]{f^{(n)}(t-x)}\,dt.$$

We have

$$\begin{aligned}
&f^{(n+1)}(x_1) - f^{(n+1)}(x_2) = \\
&\quad = \int_{-\infty}^{x_1} k_0(t)\sqrt[p]{f^{(n)}(x_1-t)}\,dt - \int_{x_1}^{\infty} k_1(t)\sqrt[p]{f^{(n)}(t-x_1)}\,dt - \\
&\quad - \int_{-\infty}^{x_2} k_0(t)\sqrt[p]{f^{(n)}(x_2-t)}\,dt + \int_{x_2}^{\infty} k_1(t)\sqrt[p]{f^{(n)}(t-x_2)}\,dt \ge \\
&\quad \ge \int_{-\infty}^{x_2} k_0(t)\left[\sqrt[p]{f^{(n)}(x_1-t)} - \sqrt[p]{f^{(n)}(x_2-t)}\right]dt + \\
&\qquad + \int_{x_2}^{\infty} k_1(t)\left[\sqrt[p]{f^{(n)}(t-x_2)} - \sqrt[p]{f^{(n)}(t-x_1)}\right]dt \ge 0.
\end{aligned}$$

Therefore $f(x) \uparrow$ as x increases. From (39) and (43) it follows that $\lim\limits_{x\to\infty} f(x) = 1$. $\square$

Thus the following theorems are valid.

Theorem 4. *Let*

1) $0 \le k_0 \in L_1(-\infty;+\infty)$, $\int\limits_{-\infty}^{+\infty} k_0(t)\,dt = 1$, $K(x,t) = k_0(x-t) - k_1(x+t) \ge 0$, $0 \le k_1 \in L_1(0,+\infty)$, $m_2(k_1) = \int\limits_0^\infty x^2 k_1(x)\,dx < +\infty$;
2) $\nu(k_0) < 0$.

Then the equation (1) *has a nontrivial nonnegative solution* $\varphi(x)$ *and* $\lim\limits_{x\to\infty}\varphi(x)=1$.

Theorem 5. *Let*

1) *the condition* 1) *of Theorem* 4 *be fulfilled;*
2) *if* $\nu(k_0)=0$ *and* $\varepsilon=-1$ *is an eigenvalue for the operator* $\widehat{W}$, *and* $k_0\in M(-\infty,+\infty)\cap L_1(-\infty,+\infty)$, *then the equation* (1) *has a nontrivial, nonnegative solution* $\varphi(x)$ *and* $\lim\limits_{x\to\infty}\varphi(x)=1$.

Remark 2. We note that Theorems 4, 5 are true for the kernels $K(x,t)$ satisfying the condition $K(x,t)\geq k_0(x-t)-k_1(x+t)$.

4. General Equation

We consider the general nonlinear equation (1^*). Acting analogously as in Theorem 1 and leaving out the details, we will formulate the following theorem.

Theorem 6. *Let the following conditions be fulfilled:*

1) *there exists* $k_0(x):\ k_0(-x)=k_0(x),\ \int\limits_{-\infty}^{+\infty}k_0(x)\,dx=1$, *such that*

$$K(x,t)\geq k_0(x-t)\quad\forall\, x,\,t\in R^+\times R^+; \tag{44}$$

2) *there exist* $\eta,\ \zeta,\ \eta>2\zeta$, *such that* $Q(\eta)=\eta,\ Q(\zeta)=2\zeta,\ Q(x)\uparrow$ *on* $[\zeta,\eta],\ Q\in C[\zeta,\eta]$,

where η *is the first positive root of the equation* $Q(x)=x$. (45)

Then the equation (1^*) *has a nonnegative and bounded solution* $f(x)$:

$$\lim_{x\to\infty}f(x)=\eta.$$

Moreover, if $K(x,t)\equiv k_0(x-t)$, *then the solution possesses the following properties:*

i) $\zeta\leq f(x)\leq\eta$;
ii) $f(x)\uparrow$ *as* x *increases.*

Examples. We bring some particular examples of the function $Q(x)$ (see below) which arise in applications:

(1) $Q(x)=x^{\frac{1}{p}},\ x>0,\ \zeta=(\frac{1}{2})^{\frac{p}{p-1}},\ \eta=1$;
(2) $Q(x)=sinx+x+1,\ x>0,\ \zeta\in(0,\frac{3}{4}\pi),\ \eta=\frac{3}{2}\pi$;
(3) $Q(x)=ae^{-(x-a)^2},\ x>0,\ \zeta\in(0,\frac{\eta}{2})$, where η is the first positive root of the equation $ae^{-(x-a)^2}=x$;
(4) $Q(x)=e^{x-1},\ x>0,\ \zeta\in(0,\frac{1}{4}),\ \eta=1$.

Summarizing, let us demonstrate one sample example. So, let $K(x,t)=k_0(x-t)$, $k_0(x)=\frac{1}{2}e^{-|x|}$, $Q(x)=e^{x-1}$, $\eta=1$, ζ be the solution of equation $e^{x-1}=2x$.

From (1*) we obtain

$$f''(x) - f(x) + e^{f(x)-1} = 0. \tag{46}$$

In spite of the fact that it is impossible to solve the obtained nonlinear differential equation analytically, the equation (46) has a positive and bounded solution $f(x) \not\equiv 1$ which has the following properties:

i) $\zeta \le f(x) \le 1$;
ii) $\lim_{x\to\infty} f(x) = 1$;
iii) $f(x) \uparrow$ as x increases.

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ZAREMBA'S BOUNDARY VALUE PROBLEM IN THE SMIRNOV CLASS OF HARMONIC FUNCTIONS IN DOMAINS WITH PIECEWISE-SMOOTH BOUNDARIES

Abstract. Zaremba's problem is studied in weighted Smirnov classes of harmonic functions in domains bounded by arbitrary simple smooth curves as well as in some domains with piecewise-smooth boundaries. The conditions of solvability are obtained and the solutions are written in quadratures.

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Key words and phrases. Harmonic functions of Smirnov type, Zaremba's problem, mixed problem, weighted functions, Poisson integral, singular integral equation in a weight Lebesgue space.

რეზიუმე. ზარემბას სასაზღვრო ამოცანა შესწავლილია ჰარმონიულ ფუნქციათა სმირნოვის წონიან კლასებში გლუვი წირებით შემოსაზღვრულ არეებში და აგრეთვე ზოგიერთ უბან-უბან გლუვ საზღვრიან არეებში. დადგენილია ამოხსნადობის პირობები და აგებულია ამონახსნები კვადრატურებში.

Boundary value problems for harmonic functions of two variables are well-studied under various assumptions regarding the unknown functions and the domains in which they are considered. In particular, problems are studied for harmonic functions of the class $e^p(D)$ being real parts of analytic in a simlpy connected domain D functions of the Smirnov class $E^p(D)$ (for their definition see, e.g., [1, Ch. IX–X], or [2]). In these classes the Dirichlet, Neumann and Riemann–Hilbert problems are investigeted in domains with piecewise-smooth boundaries (see, e.g., [3]–[7]). The boundary value problems are considered in some analogous classes, as well ([7]–[9]).

Of special interest is the investigation of a mixed boundary value problem of Smirnov type, when values of unknown functions are prescribed on a part L_1 of the boundary L of the domain D, while the values of its normal derivative are given on the supplementary part $L_2 = L \setminus L_1$.

S. Zaremba was the first who studied this problem ([10]) and hence in literature it frequently is called Zaremba's problem (see, e.g., [11]).

In [12] we have introduced the weighted Smirnov classes of harmonic functions $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$ and investigated Zaremba's problem in the abovementioned classes when D is a bounded domain with Lyapunov boundary L, and ρ_1 and ρ_2 are power functions. The same problem has been considered in [13] for some domains with piecewise-Lyapunov boundaries. However, we did not succeed in covering the case of domains with smooth boundaries because when reducing, by means of a conformal mapping, the problem to the case of a circle, we obtain a problem in the class $e(L_{1p}(\omega_1), L'_{2q}(\omega_2))$, where ω_1 and ω_2 are not power functions, and hence the emerged Smirnov classes need further investigation.

In the present work we show that the method of investigation of Zaremba's problem suggested by us in [12] and [13] allows us to obtain a picture of solvability of the problem in domains with arbitrary smooth boundaries and also in some domains with piecewise-smooth boundaries. Towards this end, we use properties of the conformal mapping of a unit circle onto the domain with a piecewise-smooth boundary and of its derivative (see, e.g., [14] and [5, Ch. III]). On the basis of these properties we manage to show that the functions of the class $e(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$ for $p > 1$, $q > 1$, are representable by the Poisson integral. We also succeed in extending to the case of the emerged nonpower weights ω_1 and ω_2 some needed for investigation properties of the Smirnov class stated in [12] for power weights. Next, using Stein's interpolation theorem on weight functions ([15]) for singular integrals with Cauchy kernel, we reveal such properties of the functions ω_1 and ω_2 which allow us to solve the characteristic Cauchy singular integral equation in the weighted Lebesgue classes with the weight ω_2, rather important for investigation of Zaremba's problem.

1^0. Definitions, Notation and Some Auxiliary Statements

Let D be a simply connected domain with Jordan smooth oriented boundary L. Let $\mathcal{L}_k = [A_k, B_k]$, $k = \overline{1, n}$, be arcs lying separately on L (the points

$A_1, B_1, A_2, B_2, \dots, A_m B_m$ lie separately on L following each other in the positive direction), and let $[A_k', B_k']$ be the arcs lying on $\mathcal{L}_k$. Assume

$$L_1 = \bigcup_{k=1}^{m} \mathcal{L}_k, \quad \widetilde{L} = \bigcup_{k=1}^{m} \big([A_k, A_k'] \cup [B_k', B_k]\big), \quad L_2 = L \setminus L_1. \tag{1}$$

Let $z = z(w)$ be a conformal mapping of the circle $\cup = \{w : \ |w| < 1\}$ onto the domain D, and $w = w(z)$ be the inverse mapping. Assume

$$\Gamma_1 = w(L_1), \quad \widetilde{\Gamma} = w(\widetilde{L}), \quad \Gamma_2 = w(L_2), \quad \gamma = \{w : \ |w| = 1\}, \tag{2}$$

$$\Theta(E) = \big\{\vartheta : \ 0 \le \vartheta \le 2\pi, \ e^{i\vartheta} \in E, \ E \subset \gamma\big\},$$

$$\Gamma_j(r) = \big\{w : \ w = re^{i\vartheta}, \ \vartheta \in \Theta(\Gamma_j)\big\}, \ \ j = 1, 2, \quad L_j(r) = z(\Gamma_j(r)).$$

Let $C_1, C_2, \dots, C_{2m}$ be the points $A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_m$ taken arbitrarily, and $D_1, D_2, \dots, D_n$ be points on $L \setminus \widetilde{L}$, different from C_k. Note that the points $D_1, D_2, \dots, D_{n_1}$ lie on L_1 and the points $D_{n_1+1}, \dots, D_n$ lie on L_2.

Let p and q be numbers from the interval $(1, \infty)$, and we assume that

$$\rho_1(z) = \prod_{k=1}^{n_1} (z - D_k)^{\alpha_k}, \ \ -\frac{1}{p} < \alpha_k < \frac{1}{p'}\,, \ \ p' = \frac{p}{p-1}\,, \tag{3}$$

$$\rho_2(z) = \prod_{k=1}^{m_1} (z - C_k)^{\nu_k} \prod_{k=m_1+1}^{2m} (z - C_k)^{\lambda_k} \prod_{k=n_1+1}^{n} (z - D_k)^{\beta_k}, \tag{4}$$

$$-\frac{1}{q} < \nu_k \le 0, \ \ 0 \le \lambda_k < \frac{1}{q'}\,, \ \ -\frac{1}{q} < \beta_k < \frac{1}{q'}\,, \ \ q' = \frac{q}{q-1}\,.$$

Definition 1 ([12])**.** Let $r_1(z)$, $r_2(z)$ be analytic functions given in D. We say that the function $u(z)$, $z = x + iy$, harmonic in the domain D, belongs to the class $e(L_{1p}(r_1), L_{2q}'(r_2))$, if

$$\sup_r \Bigg[\int_{L_1(r)} \big|u(z) r_1(z)\big|^p \, |dz| + \int_{L_2(r)} \Big(\Big|\frac{\partial u}{\partial x}\Big|^q + \frac{\partial u}{\partial y}\Big|^q \Big) |r_2(z)|^q \, |dz| \Bigg] < \infty. \tag{5}$$

Assume $e(L_{1p}, L_{2q}'(r_2)) \equiv e(L_{1p}(1), L_{2q}'(r_2))$. If $L = L_1 = \gamma = \gamma_1$, then the class $e(\gamma_{1p}(1))$ coincides with the class of harmonic functions h_p. For $p > 1$, the functions of that class are representable by the Poisson integral (see, e.g., [1, Ch. IX]).

Definition 2. Let E be a finite union of closed intervals lying on the real straight line. By $A(E)$ we denote the set of functions $f(t)$ absolutely continuous on E, that is, the functions f for which for an arbitrary $\varepsilon > 0$ there is a number $\tau > 0$ such that if $\cup(\alpha_k, \beta_k)$ is an arbitrary finite union of nonintersecting intervals from E such that $\sum(\beta_k - \alpha_k) < \delta$, then the inequality $\sum |f(\beta_k) - f(\alpha_k)| < \varepsilon$ is fulfilled.

If $f(z)$ is a function defined on the subset E of the curve L and $z = z(s)$ is the equation of the curve L with respect to the arc coordinate, then we say that $f(z)$ is absolutely continuous on E and write $f \in A(E)$, if the function $f(z(s))$ is absolutely continuous on the set $\{s: \ z(s) \in E\}$.

Statement 1 ([12, Lemma 9]). *If $f(t) \in A(L_2 \cup \widetilde{L})$, then the function $f(z(\tau))$, where $z(\tau)$ is the restriction on γ of the conformal mapping of $\overline{\cup}$ onto $\overline{D}$, belongs to $A(\Gamma_2 \cup \widetilde{\Gamma})$, and vice versa, if $\varphi \in A(\Gamma_2 \cup \widetilde{\Gamma})$, then $\varphi(w(t)) \in A(L_2 \cup \widetilde{L})$.*

Statement 2 ([12, Lemma 8]). *If $U(z) = U(x,y)$ belongs to the class $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$, then the function $u(w) = U(z(w)) = U(x(\xi,\eta), y(\xi,\eta))$ belongs to the class $e\big(\Gamma_{1p}(\rho_1(z(w))\sqrt[p]{z'(w)}), \Gamma'_{2q}(\rho_2(z(w))\sqrt[q]{z'(w)})\big)$.*

Thus by means of substitution $z = z(w)$, where $z = z(w)$ is the conformal mapping of $\cup$ onto D, the function $U(z)$ of the class $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$ transforms into the function $u(w)$ of the class $e(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$, where

$$\omega_1(w) = \rho_1(z(w))\sqrt[p]{z'(w)}, \tag{6}$$

$$\omega_2(w) = \rho_2(z(w))\sqrt[q]{z'(w)}. \tag{7}$$

2^0. Formulation of a Mixed Problem and Its Reduction to a Problem in the Circle

Consider the following mixed problem (Zaremba's problem in Smirnov class of harmonic functions): Find a function $U(z)$ satisfying the conditions

$$\begin{cases} \Delta U = 0, \ \ U \in e(L_{1p}(\rho_1), L'_{2q}(\rho_2)), \ \ p > 1, \ \ q > 1, \\ U^+\big|_{L_1 \setminus \widetilde{L}} = F, \ \ F \in L^p(L_1 \setminus \widetilde{L}, \rho_1), \ \ U^+ \in A(L_2 \cup \widetilde{L}), \\ U^+\big|_{\widetilde{L}} = \Psi, \ \ \Psi' \in L^q(\widetilde{L}, \rho_2), \ \ \Big(\dfrac{\partial U}{\partial n}\Big)^+\Big|_{L_2} = G, \ \ G \in L^q(L_2, \rho_2). \end{cases} \tag{8}$$

Relying on Statements 1 and 2, the following theorem is valid.

Theorem 1. *Let ρ_1, ρ_2, ω_1, ω_2 be the functions given by the equalities* (3)–(4) *and* (6)–(7).

If $U = U(z)$ is a solution of the problem (8) *and*

$$f(\tau) = F(z(\tau)), \quad \psi(\tau) = \Psi(z(\tau)), \quad g(\tau) = G(z(\tau)), \tag{9}$$

then the function $u(w) = U(z(w))$ is a solution of the problem

$$\begin{cases} \Delta u = 0, \ \ u \in e(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2)), \\ u^+\big|_{\Gamma_1 \setminus \widetilde{\Gamma}} = f, \ \ f \in L^p(\Gamma_1 \setminus \widetilde{\Gamma}, \omega_1), \ \ u^+ \in A(\Gamma_2 \cup \widetilde{\Gamma}), \\ u^+\big|_{\widetilde{\Gamma}} = \psi, \ \ \psi' \in L^q(\widetilde{L}, \rho_2), \ \ \Big(\dfrac{\partial u}{\partial u}\Big)^+\Big|_{\Gamma_2} = g, \ \ g \in L^q(\Gamma_2; \omega_2). \end{cases} \tag{10}$$

Conversely, if $u = u(w)$ is a solution of the problem (10), *then $U(z) = u(w(z))$ is a solution of the problem* (8).

3^0. The Weight Properties of the Functions ω_1 and ω_2

By $W^p(\Gamma)$ we denote the set of all functions $r(t)$ given on the set Γ which is a finite union of simple rectifiable curves for which the operator

$$T: f \to Tf, \quad (Tf)(t) = \frac{r(t)}{\pi i} \int\limits_{\Gamma} \frac{1}{r(\zeta)} \, \frac{f(\zeta)}{\zeta - t} \, d\zeta, \quad t \in \Gamma,$$

is bounded in $L^p(\Gamma)$. Assume that $W^p = W^p(\gamma)$. Obviously, if Γ is a finite union of nonintersecting closed arcs on Γ and $r \in W^p$, then the restriction on Γ of the functions r (i.e., $\chi_\Gamma(t) r(t)$) belongs to $W^p(\Gamma)$.

We will need the following results.

Statement 3 (see, e.g., [5, p. 104]). *If $G(t)$ is a continuous on γ function such that $[\operatorname{ind} G]_\gamma = \frac{1}{2\pi} [\arg G(t)]_\gamma = 0$, then the function*

$$r(\tau) = \exp \bigg\{ \frac{1}{2\pi i} \int\limits_{\gamma} \frac{\ln G(\zeta)}{\zeta - \tau} \, d\zeta \bigg\}, \quad \tau \in \gamma, \tag{11}$$

belongs to the set $\bigcap\limits_{\delta > 1} W^\delta$.

Corollary 1. *For any real number a we have*

$$r^a(\tau) \in \bigcap_{\delta > 1} W^\delta. \tag{12}$$

Corollary 2. *If μ is a real continuous on γ function, then*

$$\exp \bigg\{ \frac{1}{2\pi} \int\limits_{\gamma} \frac{\mu(\zeta)}{\zeta - \tau} \, d\zeta \bigg\} = r(\tau) \in \bigcap_{\delta > 1} W^\delta. \tag{13}$$

Statement 4. *If the domain D is bounded by a simple closed smooth curve L and $z(w)$ is a conformal mapping of U onto D, then:*

(a) *$z'(\tau) \in \bigcap\limits_{\delta > 1} W^\delta$, and $[z'(w)]^{\pm 1} \in \bigcap\limits_{\delta > 1} H^\delta$, where H^δ is the class of Hardy;*

(b) *if $c \in \gamma$, then $z(w) - z(c) = (w - c) z_c(w)$, $[z_c(w)]^{\pm 1} \in \bigcap\limits_{\delta > 1} H^\delta$ and*

$$z(\tau) - z(c) = (\tau - c) z_c(\tau), \quad \textit{where} \ \ z_c(\tau) \in \bigcap_{\delta > 1} W^\delta. \tag{14}$$

Statements (a) and (b) are particular cases of theorems stated in [14] (see also [5, Ch. III]). In particular, Statement (a) can be found in [5, Ch. III, Theorem 1.1, Corollary 1], and Statement (b) is also therein, Ch. III, Theorem 3.1. In this connection, as it follows from the proofs, both $z'(\tau)$ and $z_c(\tau)$ are representable by equalities of the type (11) (see, respectively, [5, p. 139, the equality (1.14) and p. 154, the equalities (3.16) and (3.18)]).

By virtue of Corollaries 1 and 2 of Statement 3, for any $a \in \mathbb{R}$ we have

$$[z'(\tau)]^a,\ [z_c(\tau)]^a,\ z_0(\tau) = \prod_{k=1}^{n} [z_{c_k}(\tau)]^a \in \bigcap_{\delta>1} W^\delta, \tag{15}$$
$$c_k \in \gamma,\ \ c_j \neq c_k,\ \ j \neq k.$$

Consequently, we also have

$$[\sqrt[p]{z'(\tau)}]^a,\ [\sqrt[q]{z'(\tau)}]^a \in \bigcap_{\delta>1} W^\delta. \tag{16}$$

Theorem 2. *If the functions ρ_1 and ρ_2 are given by the equalities* (3) *and* (4)*, then the functions ω_1 and ω_2 defined by the equalities* (6) *and* (7) *belong, respectively, to W^p and W^q.*

Proof. We have

$$\rho_1(z(\tau)) = \prod_{k=1}^{n} \big(z(\tau) - z(d_k)\big)^{\alpha_k},$$

where $d_k = w(D_k)$, $-\frac{1}{p} < \alpha_k < \frac{1}{p'}$. From the equalities (14) it follows that

$$\rho_1(z(\tau)) = \prod_{k=1}^{n_1} (\tau - d_k)^{\alpha_k} \prod_{k=1}^{n_1} z_{d_k}(\tau) = r_1(\tau) r_2(\tau).$$

By means of the above assumptions regarding α_k, we can find numbers a, $b \in (0,1)$ such that

$$r_1^{\frac{1}{(1-a)(1-b)}} = \Big[\prod_{k=1}^{n_1} (z - d_k)^{\alpha_k}\Big]^{\frac{1}{(1-a)(1-b)}} \in W^p.$$

Moreover, by virtue of (15) we have $r_2^{\frac{1}{a(1-b)}} \in \bigcap_{\delta>1} W^\delta$.

Here we use the following Stein's theorem ([15]).

Let M be a linear operator acting from one space of measurable functions to the other,

$$1 \le l_1, l_2, s_1, s_2 \le \infty, \quad l^{-1} = (1-a) l_1^{-1} + a l_2^{-1},$$
$$s^{-1} = (1-a) s_1^{-1} + a s_2^{-1},\ \ 0 \le a \le 1,$$
$$\big\|(Mf)k_i\big\|_{s_i} \le C_i \|f u_i\|_{l_i}.$$

Then

$$\|(Mf)k\|_s \le C \|fu\|_l,$$

where

$$k = k_1^{1-a} k_2^a, \quad u = u_1^{1-a} u_2^a, \quad C = C_1^{1-a} C_2^a.$$

Assuming in this theorem

$$k_1(\tau) = u_1(\tau) = \Big[\prod_{k=1}^{n_1} (\tau - d_k)^{\alpha_k}\Big]^{\frac{1}{(1-a)(1-b)}},\ \ k_2(\tau) = u_2(\tau) = \prod_{k=1}^{n_1} [z_{d_k}(\tau)]^{\frac{1}{a(1-b)}},$$
$$s_1 = s_2 = s = p > 1,$$

we find that the function

$$r_1^{1-a} r_2^a = \Big[\prod_{k=1}^{n_1} (\tau - d_k)^{\alpha_k} \prod_{k=1}^{n_1} z_{d_k} \Big]^{\frac{1}{1-b}} \Big(= \rho_1^{\frac{1}{1-b}}(z(\tau)) \Big)$$

belongs to W^p.

Further, taking in the above theorem

$$k_1(\tau) = u_1(\tau) = \rho_1^{\frac{1}{1-b}}(z(\tau)), \quad k_2(\tau) = u_2(\tau) = \big(\sqrt[p]{z'}\big)^{\frac{1}{b}}, \ \ s_1 = s_2 = s = p,$$

we find that

$$\Big(\big[\rho_1(z(\tau))\big]^{\frac{1}{1-b}}\Big)^{1-b} \Big(\big[\sqrt[p]{z'}\big]^{\frac{1}{b}}\Big)^b = \rho_1(z(\tau))\sqrt[p]{z'(\tau)} = \omega_1(\tau) \in W^p.$$

Taking into account that $-\frac{1}{q} < \beta_k < \frac{1}{q'}$, $-\frac{1}{q} < \nu_k \le 0$, $0 \le \lambda_k < \frac{1}{q'}$, we analogously see that $\omega_2(\tau) \in W^q$. □

4^0. One Property of Functions of the Class $e(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$ for $p > 1$, $q > 1$

Theorem 3. *If $u \in e(\Gamma_{1p}, \Gamma'_{2q}(\omega_2))$, where $p > 1$, $q > 1$, then:*

(i) *if $p < q$, then $u \in h_p$;*
(ii) *if $p > q$, then $u \in h_{q_1}$ for any $q_1 \in [0, q]$;*
(iii) *if $p = q$, then $u \in h_{p_1}$ for any $p_1 \in (0, p)$.*

Proof. (i) Let

$$I(r) = \int_0^{2\pi} \big|u(re^{i\vartheta})\big|^p \, d\vartheta.$$

We have

$$\begin{aligned} I(r) &= \int_{\Theta(\Gamma_1)} \big|u(re^{i\vartheta})\big|^p \, d\vartheta + \int_{\Theta(\Gamma_2)} \big|u(re^{i\vartheta})\big|^p \, d\vartheta \le \\ &\le \sup_r \int_{\Theta(\Gamma_1)} \big|u(re^{i\vartheta})\big|^p \, d\vartheta + \int_{\Theta(\Gamma_2)} \bigg| \int_0^r \frac{\partial u}{\partial r} \, dr - u(0) \bigg|^p d\vartheta \le \\ &\le M_1 + 2^p \bigg(\int_{\Theta(\Gamma_2)} \bigg| \int_0^r \frac{\partial u}{\partial r} \, dr \bigg|^p d\vartheta + |u(0)|^p \, 2\pi \bigg) = \\ &= M_2 + 2^p \int_{\Theta(\Gamma_2)} \bigg(\int_0^r \Big| \frac{\partial u}{\partial r} \Big| \, dr \bigg)^p d\vartheta = \\ &= M_2 + 2^p I_1(r). \end{aligned} \tag{17}$$

Since $\left|\frac{\partial u}{\partial r}\right| \le \left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial u}{\partial y}\right|$, we have

$$
\begin{aligned}
I_1(r) &= \int\limits_{\Theta(\Gamma_2)} \Big|\int\limits_0^r \Big|\frac{\partial u}{\partial r}\Big|\,dr\Big|^p d\vartheta \le \int\limits_{\Theta(\Gamma_2)} \Big[\int\limits_0^r \Big|(\Big|\frac{\partial u}{\partial x}\Big| + \Big|\frac{\partial u}{\partial y}\Big|\Big)\,dr\Big]^p d\vartheta = \\
&= \int\limits_{\Theta(\Gamma_2)} \Big[\int\limits_0^r \Big(\Big|\frac{\partial u}{\partial x}\Big| + \Big|\frac{\partial u}{\partial y}\Big|\Big)|\omega_2|\,\frac{1}{|\omega_2|}\,dr\Big]^p d\vartheta \le \\
&\le \int\limits_{\Theta(\Gamma_2)} \Big[\Big(\int\limits_0^r \Big(\Big|\frac{\partial u}{\partial x}\Big| + \Big|\frac{\partial u}{\partial y}\Big|\Big)^q |\omega_2|^q\,dr\Big)^{\frac{p}{q}} \Big(\int\limits_0^r \frac{dr}{|\omega_2|^{q'}}\Big)^{\frac{p}{q'}}\Big] d\vartheta \le \\
&\le (2^q)^{\frac{p}{q}} \int\limits_{\Theta(\Gamma_2)} \Big(\int\limits_0^r \Big(\Big|\frac{\partial u}{\partial x}\Big|^q + \Big|\frac{\partial u}{\partial y}\Big|^q\Big)|\omega_2|^q\,dr\Big)^{\frac{p}{q}} \Big(\int\limits_0^r \frac{dr}{|\omega_2|^{q'}}\Big)^{\frac{p}{q'}} d\vartheta = \\
&= 2^p \int\limits_{\Theta(\Gamma_2)} \Big(\int\limits_0^r \Big(\Big|\frac{\partial u}{\partial x}\Big|^q + \Big|\frac{\partial u}{\partial y}\Big|^q\Big)|\omega_2|^q\,dr\Big)^{\frac{p}{q}} (J(\vartheta))^{\frac{p}{q'}}\,d\vartheta, \qquad (18)
\end{aligned}
$$

where we have put

$$J(\vartheta) = \int\limits_0^r \frac{dr}{|\omega_2(re^{i\vartheta})|^{q'}}\,.$$

Estimate the value $\sup\limits_{\Theta(\Gamma_2)} J(\vartheta)$. We have

$$\frac{1}{|\omega_2(re^{i\vartheta})|^{q'}} \le \frac{M_3}{\prod\limits_{k=m_1+1}^{2m} |re^{i\vartheta} - c_k|^{\lambda_k q'} \prod\limits_{\beta_k>0} |re^{i\vartheta} - d_k|^{\beta_k q'} |z_0(re^{i\vartheta})|^{q'}}$$

(for definition of z_0, see (15)).

Assume that $\alpha = \sup\limits_k(\lambda_k q', \beta_k q')$. By virtue of the inequalities (7), we have $0 \le \alpha < 1$. Since $|re^{i\vartheta} - c_k| \ge 1 - r$, $|re^{i\vartheta} - d_k| \ge 1 - r$, $\Theta(\Gamma_2) = \bigcup_{k=1}^m (\beta_k, \alpha_{k+1})$ with $\alpha_{m+1} = \alpha_1$, and on every interval (β_k, α_{k+1}) there are no more than three points from the set $\cup\{z_k\} = \cup\{c_k\} \cup \cup\{d_k\}$, then dividing the corresponding intervals into three or two parts, we will represent $\Theta(\Gamma_2)$ as the union of no more than $6m$ intervals, and on every interval

$$\frac{1}{|\omega_2(re^{i\vartheta})|^{q'}} \le \frac{M_4}{(1-r)^\alpha |z_0(re^{i\vartheta})|^{q'}}\,, \quad M_4 = \max_{k \ne j} \frac{3}{|z_k - z_j|}\,.$$

Thus

$$\sup_{\Theta(\Gamma_2)} J(\vartheta) \le (6m) M_4 \sup_{\Theta(\Gamma_2)} \int\limits_0^r \frac{dr}{(1-r)^\alpha |z_0(re^{i\vartheta})|^{q'}} =$$

$$= M_5 \sup_{\Theta(\Gamma_2)} \int_0^r \frac{dr}{(1-r)^\alpha |z_0(re^{i\vartheta})|^{q'}}. \tag{19}$$

Applying in the last integral Hölder's inequality with exponent $\frac{1+\alpha}{2\alpha}$, we obtain

$$(J(\vartheta))^{\frac{p}{q'}} \le M_6 \Big(\int_0^r \frac{dr}{(1-r)^{\frac{1+\alpha}{2}}} \Big)^{\frac{p}{q'} \frac{2\alpha}{1+\alpha}} \Big(\int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^{q' \frac{1+\alpha}{1-\alpha}}} \Big)^{\frac{p}{q'} \frac{1-\alpha}{1+\alpha}} \le$$

$$\le M_7 \Big(\int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^{q' \frac{1+\alpha}{1-\alpha}}} \Big)^{\frac{p}{q'} \frac{1-\alpha}{1+\alpha}}. \tag{20}$$

Show that the integral

$$J_1(\vartheta) = \Big(\int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^{q' \frac{1+\alpha}{1-\alpha}}} \Big)^{\frac{p}{q'} \frac{1-\alpha}{1+\alpha}}$$

is a function integrable in any degree on γ and hence on $\Theta(\Gamma_2)$.

Towards this end, we note that if $\frac{p}{q'} \frac{1-\alpha}{1+\alpha} \le 1$, then

$$J_1(\vartheta) \le 1 + \int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^{q' \frac{1+\alpha}{1-\alpha}}}.$$

If, however, $\frac{p}{q'} \frac{1-\alpha}{1+\alpha} > 1$, then using Hölder's inequality with the above exponent, we have

$$J_1(\vartheta) \le \int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^p}.$$

From the above estimates we can see that $J_1(\vartheta) \in \bigcap\limits_{\delta>1} L^\delta([0,2\pi])$ if we prove that for arbitrary $\delta > 1$ the function $\int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^\mu}$ is integrable in the δ-th degree for any μ.

We have

$$\int_0^{2\pi} \Big(\int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^\mu} \Big)^\delta d\vartheta \le \int_0^{2\pi} \int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^{\mu\delta}} \, d\vartheta \le$$

$$\le \int_0^1 \int_0^{2\pi} \frac{d\vartheta}{|z_0(re^{i\vartheta})|^{\mu\delta}} \, dr = M_8 < \infty.$$

This inequality is valid since $\frac{1}{z_0} \in \bigcap\limits_{\delta>1} H^\delta$ (see Statement 4). Thus we have proved that $J(\vartheta) \in \bigcap\limits_{\delta>1} L^\delta[0,2\pi]$.

Applying now to the right-hand side of (18) Hölder's inequality with exponent $\frac{q}{p} > 1$, we obtain

$$I_1(r) \leq 2^p \int\limits_{\Theta(\Gamma_2)} \Big[\int\limits_0^r \Big(\Big|\frac{\partial u}{\partial x}\Big|^q + \Big|\frac{\partial u}{\partial y}\Big|^q \Big) |\omega_2|^q \, dr \Big] \, d\vartheta \Big(\int\limits_{\Theta(\Gamma_2)} |J(\vartheta)|^{\frac{p}{q'} \frac{q}{q-p}} \Big)^{\frac{q-p}{q}}.$$

But $u \in e(\Gamma_{1p}, \Gamma'_{2q}(\omega_2))$, whence it follows that $\sup\limits_r I_1(r) < \infty$, and from (17) we can conclude that $\sup\limits_r I(r) < \infty$ and hence $u \in h_p$.

(ii) It can be easily verified that if $p_1 < p_2$, then $u \in e(\Gamma_{1p_2}, \Gamma'_{2q}(\omega_2)) \subset u \in e(\Gamma_{1p_1}, \Gamma'_{2q}(\omega_2))$. Therefore if $p > q > q_1$ and $u \in e(\Gamma_{1p}, \Gamma'_{2q}(\omega_2))$, then $u \in e(\Gamma_{1q_1}, \Gamma'_{2q}(\omega_2))$ and $u \in h_{q_1}$.

(iii) If $u \in e(\Gamma_{1p}, \Gamma'_{2p}(\omega_2))$, then for any $1 < p_1 < p$, we have $u \in e(\Gamma_{1p_1}, \Gamma'_{2q}(\omega_2))$, and hence $u \in h_{p_1}$. □

Let now $u \in e(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$, $p > 1$, $q > 1$. Since $\frac{1}{\omega_1} \in H^{p'+\varepsilon}$, $\varepsilon > 0$, there exists $\eta > 0$ such that $u \in e(L_{1+\eta}, L'_{2q}(\omega_2))$, and therefore $\sup\limits_r \int\limits_{\Theta(\Gamma_1)} |u(re^{i\vartheta})|^{1+\eta} \, d\vartheta < \infty$. Assuming $1 + \eta < q$, by Theorem 3 we can conclude that $u \in h_{1+\eta}$. As far as the functions of the class $h_{1+\eta}$ are representable by the Poisson integral, we state the following

Theorem 4. *If $u \in e(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$, $p > 1$, $q > 1$, then u is likewise representable by the Poisson integral.*

5^0. Reduction of the Problem (10) to a Singular Integral Equation

If $w(A_k) = a_k$, $w(B_k) = b_k$, $w(A'_k) = a'_k$, $w(B'_k) = b'_k$, we have

$$\Gamma_1 = w(L_1) = \bigcup_{k=1}^m (a_k, b_k), \quad \widetilde{\Gamma} = \bigcup [a_k, a'_k] \cup [b'_k, b_k], \quad \Gamma_2 = \gamma \setminus \Gamma_1.$$

Following the way of investigation of the problem (10) carried out in Section 3^0 of [12], we can state that if u is a solution of the problem (10) and $u^+(e^{i\vartheta})$ is its boundary function, then the function $\frac{\partial u^+}{\partial \vartheta}$ is a solution of the integral equation

$$\frac{1}{2\pi} \int\limits_{\Theta(\Gamma_2)} \frac{\partial u^+}{\partial \vartheta} \operatorname{ctg} \frac{\vartheta - \varphi}{2} \, d\vartheta = \mu(\varphi), \quad e^{i\varphi} \in \Gamma_2, \tag{21}$$

where

$$\mu(\varphi) = -g(\varphi) - \frac{1}{2\pi} \int\limits_{\Theta(\Gamma_1 \setminus \widetilde{\Gamma})} f(\vartheta) \frac{d\vartheta}{\sin^2 \frac{\vartheta - \varphi}{2}} - \frac{1}{2\pi} \int\limits_{\Theta(\Gamma_2)} \psi(\vartheta) \frac{d\vartheta}{\sin^2 \frac{\vartheta - \varphi}{2}} +$$

$$+ \sum_{k=1}^m \Big[\psi(a'_k) \operatorname{ctg} \frac{\alpha'_k - \varphi}{2} - \psi(b'_k) \operatorname{ctg} \frac{\beta'_k - \varphi}{2} \Big] - \widetilde{\psi}(\varphi), \tag{22}$$

$$\widetilde{\psi}(\varphi)=\frac{1}{2\pi}\int\limits_{\gamma}\chi_{\widetilde{\Gamma}}(\vartheta)\,\frac{\partial\psi}{\partial\vartheta}\,\mathrm{ctg}\,\frac{\vartheta-\varphi}{2}\,d\vartheta. \tag{23}$$

Here $\chi_{\widetilde{\Gamma}}$ is the characteristic function of the set $\widetilde{\Gamma}$, we write $f(\vartheta)$, $\psi(\vartheta)$, $g(\varphi)$ instead of $f(e^{i\vartheta})$, $\psi(e^{i\vartheta})$, $g(e^{i\varphi})$ and put $a_k'=e^{i\alpha_k}$, $b_k'=e^{i\beta_k'}$.

Let us show that under the adopted assumptions the functions $\frac{\partial u^+}{\partial\vartheta}$ and μ belong to the class $L^q(\Gamma_2;\omega_2)$.

We start with the function $\frac{\partial u^+}{\partial\vartheta}$. Tracing the proof of Lemma 1 in [12], we establish that the condition $u\in e(\Gamma_{1p}(\omega_1),\Gamma_{2q}'(\omega_2))$ is equivalent to the condition

$$\sup_r\bigg[\int\limits_{\Theta(\Gamma_1)}\big|u(re^{i\vartheta})\omega_1(re^{i\vartheta})\big|^p\,d\vartheta+$$
$$+\int\limits_{\Theta(\Gamma_2)}\bigg|\sqrt{\Big(\frac{\partial u}{\partial x}\,(re^{i\vartheta})\Big)^2+\Big(\frac{\partial u}{\partial y}\,(re^{i\vartheta})\Big)^2}\,\omega_2(r^{i\vartheta})\bigg|^q\,d\vartheta\bigg]<\infty \tag{24}$$

for the functions ω_1, ω_2, as well (and not only for the power functions). It is now not difficult to see that the statement below is valid.

Statement 5. *If $u\in e(\Gamma_{1p}(\omega_1),\Gamma_{2q}'(\omega_2))$, $p>1$, $q>1$, and $u^+\in A(\Gamma_2)$ (in particular, if u is a solution of the problem* (10)*), then $\big(\frac{\partial u}{\partial\vartheta}\big)^+$ and $\frac{\partial u^+}{\partial\vartheta}$ belong to $L^q(\Gamma_2;\omega_2)$.*

The proof of the above statement is analogous to that of Lemma 5 in [12] if in the appropriate place we take advantage of the fact that the condition (5) is equivalent to the condition (24).

For the function μ to belong to $L^q(\Gamma_2;\omega_2)$, as is seen from the equality (22), it suffices to show that $\widetilde{\psi}\in L^q(\Gamma_2;\omega_2)$. This follows from Theorem 2 since $\lambda(\vartheta)=\chi_{\widetilde{\Gamma}}(\vartheta)\,\frac{\partial\psi}{\partial\vartheta}\in L^q(\gamma,\omega_2)$ (because $\frac{\partial\psi}{\partial\vartheta}\in L^q(\Gamma_2;\omega_2)$), while the operator

$$\lambda\to\widetilde{\lambda},\quad \widetilde{\lambda}(\varphi)=\frac{1}{\pi}\int\limits_0^{2\pi}\lambda(\vartheta)\,\mathrm{ctg}\,\frac{\vartheta-\varphi}{2}\,d\vartheta$$

is bounded in $L^q(\gamma,\omega_2)$ if the singular Cauchy operator is bounded in it, and latter is bounded in $L^q(\gamma,\omega_2)$ since $\omega_2\in W^q$ by Theorem 2.

6^0. The Solution of the Equation (21) in the Space $L^q(\Gamma_2;\omega_2)$

Assuming $\tau=e^{i\vartheta}$, $t=e^{i\varphi}$ and taking into account that

$$\frac{d\tau}{\tau-t}=\Big(\frac{1}{2}\,\mathrm{ctg}\,\frac{\vartheta-\varphi}{2}+\frac{i}{2}\Big)\,d\vartheta,$$

the equation (21) can be written in the form

$$\frac{1}{\pi i}\int\limits_{\Gamma_2}\frac{\partial u^+}{\partial\vartheta}\,\frac{d\tau}{\tau-e^{i\varphi}}=i\mu(\varphi)+a,\quad a=\frac{1}{2\pi}\int\limits_{\Gamma_2}\frac{\partial u^+}{\partial\vartheta}\,d\vartheta. \tag{25}$$

Since u^+ is a boundary value of a solution of the problem (10), we see

$$a=\frac{1}{2\pi}\sum_{k=1}^{m}\big(u(a_{k+1})-u(b_k)\big)=\frac{1}{2\pi}\sum_{k=1}^{m}\big[\psi(a_{k+1})-\psi(b_k)\big],\ \ a_{m+1}=a_1.$$

Thus the function $\frac{\partial u^+}{\partial\vartheta}$ is a solution of the singular integral equation

$$\frac{1}{\pi i}\int\limits_{\Gamma_2}\frac{\partial u^+}{\partial\vartheta}\,\frac{d\tau}{\tau-e^{i\varphi}}=i\mu(\varphi)+\frac{1}{2\pi}\sum_{k=1}^{m}\big[\psi(a_{k+1})-\psi(b_k)\big] \tag{26}$$

belonging to $L^q(\Gamma_2;\omega_2)$.

Let Γ be a finite union of arcs $[a_k,b_k]\subset\gamma$, ρ be a weight function from W^q and

$$S_\Gamma:\varphi\to S_\Gamma\varphi,\quad (S_\Gamma\varphi)(t)=\frac{1}{\pi i}\int\limits_{\Gamma}\frac{\varphi(\tau)}{\tau-t}\,d\tau\,,\ \ t\in\Gamma.$$

Let $\lambda\in L^q(\Gamma;\rho)$. Consider the singular integral equation

$$S_\Gamma\varphi=\lambda \tag{27}$$

in the class $L^q(\Gamma;\rho)$.

This equation has been investigated in different classes of functions by many authors. In our formulation, when ρ is a power function of definite type, it is solved in [16] (see also [17, Ch. III, § 7, pp. 103–109]; a history of the question can be found therein). In connection with investigation of Zaremba's problem, in [12] we showed that this result from [17] was generalized to a general case of power weight functions. We will now show that the property of solvability of the equation (27) in the classes $L^q(\Gamma;\rho)$ for power weights preserves for wider classes of weights, as well.

The points $a_1a_2,\dots,a_m;b_1,b_2,\dots,b_m$ taken arbitrarily are denoted below by $c_1,c_2,\dots,c_{2m}$. Let

$$\Pi_1(z)=\sqrt{\prod_{k=1}^{m_1}(z-c_k)},\quad \Pi_2(z)=\sqrt{\prod_{k=m_1+1}^{2m}(z-c_k)},\quad R(z)=\frac{\Pi_1(z)}{\Pi_2(z)}\,,$$

where the branch of the first function is taken arbitrarily and that of the second one is selected in such a way that the function $R(z)$ in the neighborhood of $z=\infty$ expands in the series $z^{m-m_1}+A_1z^{m-m_1-1}+\cdots$.

Assume

$$R(\tau)=\frac{\Pi_1(\tau)}{\Pi_2(\tau)}\,,\ \ \tau\in\Gamma. \tag{28}$$

Let

$$\rho(\tau)=\prod_{k=1}^{m_1}(\tau-c_k)^{\nu_k}\prod_{k=m_1+1}^{2m}(\tau-c_k)^{\lambda_k},\ -\frac{1}{q}<\nu_k\le 0,\ \ 0\le\lambda_k<\frac{1}{q'}\,. \quad (29)$$

Moreover, we assume that $-\frac{1}{q}<\frac{1}{2}+\nu_k<\frac{1}{q'}$, $-\frac{1}{q}<\lambda_k-\frac{1}{2}<\frac{1}{q'}$, i.e.,

$$-\frac{1}{q}<\nu_k<\min\Big(0;\frac{1}{q'}-\frac{1}{2}\Big),\quad \max\Big(0;\frac{1}{2}-\frac{1}{q}\Big)\le\lambda_k<\frac{1}{q'}\,. \quad (30)$$

Finally, let

$$\omega(\tau)=\rho(z(\tau))\rho_0(\tau), \quad (31)$$

where $\rho_0\in\bigcap\limits_{\delta>1}W^\delta$.

Suppose

$$U_\Gamma\varphi=RS_\Gamma\,\frac{1}{R}\,\varphi. \quad (32)$$

If $\varphi(\tau)\equiv q(\tau)$ is an arbitrary polynomial, then $q\in L^p(\Gamma;\Pi_1^{-1}\Pi_2^{p-1})$, and by Lemma 1 of [17, p. 105] we obtain

$$(U_\Gamma S_\Gamma q)(\tau)=q(\tau),\ \text{ when }\ m_1\ge m. \quad (33)$$

However, if $m>m_1$, then

$$(U_\Gamma S_\Gamma q)(\tau)=q(\tau)+R(\tau)Q_{r-1}(\tau), \quad (34)$$

where $Q_{r-1}(\tau)$ is a polynomial of degree not higher than $r-1$, $r=m-m_1-1$.

Since $\omega(\tau)=\rho(z(\tau))\rho_0(\tau)$, according to Theorem 2 $\omega\in W^q$. Moreover, since the conditions (30) are fulfilled, the function $\widetilde{\omega}(\tau)=R(\tau)\omega(\tau)$ belongs to W^q. Since the set of polynomials $\{q_n\}$ is dense in $L^q(\Gamma;\widetilde{\omega})$ for any $\widetilde{\omega}\in W^q$, passing in the equalities (33) and (34) to the limit as $q=q_n\to\varphi\in L^q(\Gamma;\omega)$, we find that

$$(U_\Gamma S_\Gamma\varphi)=\varphi\ \text{ for }\ m\le m_1,\ \text{ and }\ (U_\Gamma S\varphi)=\varphi+RQ_{r-1}\ \text{ for }\ m>m_1. \quad (35)$$

On the basis of the above equalities, just as in [17, pp. 107–108] (see also [12, p. 46]) we prove

Theorem 5. *Let for the weight ρ given by the equality* (29) *the conditions* (30) *be fulfilled, and $\omega(\tau)=\rho(z(\tau))\sqrt[q]{z'(\tau)}$, where $z=z(w)$ is a conformal mapping of the circle $\cup$ onto a simply connected domain bounded by a simple closed smooth curve L, and let Γ be a finite union of arcs from γ. Then the equation*

$$S_\Gamma\varphi=\lambda$$

(i) *is solvable for $m_1\le m$ and all its solutions are given by the equality*

$$\varphi(\tau)=(U_\Gamma\lambda)(\tau)+RQ_{r-1}(\tau), \quad (36)$$

where $Q_{r-1}(\tau)$ is an arbitrary polynomial of order $r-1$, $r=m-m_1$ $(Q_{-1}(\tau)\equiv 0)$.

(ii) *for $m_1 > m$, the equality is solvable if and only if*

$$\int_{\Gamma} \tau^k R(\tau)\lambda(\tau)\,d\tau = 0, \quad k = \overline{0, l-1}, \quad l = m_1 - m, \tag{37}$$

and if these conditions are fulfilled it is uniquely solvable and the solution is given by the equality (36), *where $Q_{r-1}(\tau) \equiv 0$.*

7^0. The Solution of the Problem (8)

Having at hand Theorem 5, we are able to investigate the equation (21): find the conditions of its solvability and write out all solutions. By virtue of the same theorem, solving the equation (26) and hence (21), we can find the function $\frac{\partial u^+}{\partial \vartheta}$ on Γ_2; integrating it, we find $u^+(\tau)$ on Γ_2. There appear arbitrary constants which (or a part of which) are defined by the conditions of absolute continuity of u^+ on $\Gamma_2 \cup \widetilde{\Gamma}$ (see (10)). Having found the values u^+ on Γ_2, we will have u^+ on the entire neighborhood, because it was given on Γ_1 beforehand. By virtue of Theorem 4, all the above-said allows us to find $u(w)$ by using the Poisson formula with density $u^+(e^{i\vartheta})$. Having known $u(w)$, by Theorem 1 we find a solution $U(z) = u(w(z))$, $z \in D$, of the problem (8).

Detailed calculations are analogous to those carried out in [12] (Sections 5^0–7^0). Omitting them, we can formulate the final result.

Theorem 6. *Let:*

(a) *the domain D, the curve L, and its parts L_1, $\widetilde{L}$, L_2 be defined according to Section 1^0 and the equalities* (1), *while the weight functions $\rho_1(z)$, $\rho_2(z)$ be defined by the conditions* (3)–(4);

(b) *$z = z(w)$ be a conformal mapping of the unit circle $\cup$ onto D; $w = w(z)$ be the inverse mapping; the sets Γ_1, $\widetilde{\Gamma}$, Γ_2 be defined by* (2) *and the functions ω_1, ω_2 by the equalities* (6)–(7);

(c) *$a_k = w(A_k) = e^{i\alpha_k}$, $b_k = w(B_k) = e^{i\beta_k}$, $a'_k = w(A'_k) = e^{i\alpha'_k}$, $b'_k = w(B'_k) = e^{i\beta'_k}$, $0 \le m_1 \le 2m$, $c_k = w(C_k)$, $d_k = w(D_k)$;*

(d) *the function $R(\tau)$ be defined by the equality* (28).

If the problem (8) *is considered in the class $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$, $p > 1$, $q > 1$, the functions f, ψ, g are defined by the equalities* (9) *and we assume that for the exponents of the weights the conditions* (30) *are fulfilled, then:*

I. *If $m_1 \le m$, then for the solvability of the problem* (8) *it is necessary and sufficient that the conditions*

$$\int_{\beta_k}^{\alpha_{k+1}} \operatorname{Re}\left[\frac{R(e^{i\alpha})}{\pi i} \int_{\Theta(\Gamma_2)} \frac{i\mu(\tau) + a}{R(\tau)(\tau)(\tau - e^{i\alpha})}\,d\tau\right] d\alpha =$$
$$= \psi(e^{i\alpha_{k+1}}) - \psi(e^{i\beta_k}), \quad k = \overline{1, m}, \tag{38}$$

be fulfilled, where

$$\mu(\tau)=\mu(e^{i\varphi})\equiv\mu(\varphi)=$$

$$=-g(\varphi)+\frac{1}{2\pi}\sum_{k=1}^{m}\Big[\psi(e^{i\alpha_{k+1}})\operatorname{ctg}\frac{\alpha_{k+1}-\varphi}{2}-\psi(e^{i\alpha_k})\operatorname{ctg}\frac{\beta_k-\varphi}{2}\Big]-$$

$$-\frac{1}{2\pi}\int\limits_{\Theta(\Gamma\setminus\widetilde{\Gamma})}f(\vartheta)\,\frac{d\vartheta}{2\sin^2\frac{\vartheta-\varphi}{2}}-\frac{1}{2\pi}\int\limits_{\Theta(\widetilde{\Gamma})}\psi(\vartheta)\,\frac{d\vartheta}{2\sin^2\frac{\vartheta-\varphi}{2}}\,, \tag{39}$$

$$a=\frac{1}{2\pi}\sum_{k=1}^{m}\big[\psi(e^{i\alpha_{k+1}})-\psi(e^{i\beta_k})\big],\ \ \alpha_{m+1}=\alpha_1. \tag{40}$$

II. *If $m_1>m$, then for the solvabilty of the problem* (8) *it is necessary and sufficient that the conditions* (38) *and*

$$\int\limits_{\Gamma_2}\frac{i\mu(\tau)+a}{R(\tau)}\,\tau^k\,d\tau=0,\ \ k=\overline{0,l-1},\ \ l=m_1-m, \tag{41}$$

be fulfilled.

III. *If the above conditions are fulfilled, then a solution of the problem* (8) *is given by the equality*

$$U(z)=u^*(w(z))+u_0(w(z)), \tag{42}$$

where

$$u^*(w)=u(re^{i\vartheta})=\frac{1}{2\pi}\int\limits_{\Theta(\widetilde{\Gamma})}\psi(\vartheta)P(r,\vartheta-\varphi)\,d\vartheta+$$

$$+\frac{1}{2\pi}\int\limits_{\Theta(\Gamma_1\setminus\widetilde{\Gamma})}f(\vartheta)P(r,\vartheta-\varphi)\,d\vartheta+\frac{1}{2\pi}\int\limits_{\Theta(\Gamma_2)}W_{\Gamma_2}(\vartheta)P(r,\vartheta-\varphi)\,d\vartheta \tag{43}$$

in which

$$P(r,x)=\frac{1-r^2}{1+r^2-2r\cos x}\,,$$

$$W_{\Gamma_2}(\vartheta)=\int\limits_{\beta_1}^{\vartheta}\chi_{\Theta(\Gamma_2)}(\alpha)\Big[\operatorname{Re}\,\frac{R(e^{i\alpha})}{\pi i}\int\limits_{\Gamma_2}\frac{i\mu(\tau)+a}{R(\tau)(\tau-e^{i\alpha})}\,d\tau\Big]\,d\alpha+B_k,$$

$\Theta(E)=\{\vartheta:\ e^{i\vartheta}\in E\}$, *and χ_{E} denotes characteristic function of the set E,*

$$B_k=\psi(e^{i\alpha_{k+1}})-\int\limits_{\beta_i}^{\alpha_{k+1}}\chi_{\Theta(\Gamma_2)}(\alpha)\operatorname{Re}\Big[\frac{R(e^{i\alpha})}{\pi i}\int\limits_{\Gamma_2}\frac{i\mu(\tau)+a}{R(\tau)(\tau-e^{i\alpha})}\,d\tau\Big]\,d\alpha, \tag{44}$$

and

$$u_0(re^{i\vartheta}) = \begin{cases} 0, \quad when \;\; m_1 > m, \\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} W^*_{\Gamma_2}(\vartheta) P(r, \vartheta - \varphi)\, d\vartheta, \\ W^*_{\Gamma_2}(\vartheta) = \displaystyle\int_{\beta_1}^{\vartheta} \chi_{\Theta(\Gamma_2)}(\alpha) \operatorname{Re}\big[R(e^{i\alpha}) Q_{r-1}(e^{i\alpha})\big]\, d\alpha + A_k, \\ \qquad\qquad\qquad e^{i\vartheta} \in (b_k, a_{k+1}), \end{cases} \tag{45}$$

$$A_k = -\int_{\beta_k}^{\alpha_{k+1}} \operatorname{Re}\big[R(e^{i\alpha}) Q_{r-1}(e^{i\alpha})\big]\, d\alpha, \tag{46}$$

$Q_{r-1}(\tau) \equiv 0$, *and for* $m_1 < m$,

$$Q_{r-1}(e^{i\vartheta}) = \sum_{j=0}^{r-1} (x_j + i y_j) e^{ij\vartheta}, \tag{47}$$

where the coefficients x_j, y_j, $j = \overline{0, r-1}$, *are defined from the system*

$$\begin{cases} \displaystyle\sum_{j=0}^{r-1} \int_{\beta_k}^{\alpha_{k+1}} \Big[x_j R_1(e^{i\vartheta}) \cos j\vartheta - y_j R_2(e^{i\vartheta}) \sin j\vartheta\Big]\, d\vartheta = 0, \\ \displaystyle\sum_{j=0}^{r-1} \int_{\beta_k}^{\alpha_{k+1}} \Big[x_j R_2(e^{i\vartheta}) \cos j\vartheta + y_j R_1(e^{i\vartheta}) \sin j\vartheta\Big]\, d\vartheta = 0 \end{cases} \tag{48}$$

and we put $R_1(e^{i\vartheta}) = \operatorname{Re} R(e^{i\vartheta})$, $R_2(e^{i\vartheta}) = \operatorname{Im} R(\vartheta)$.

If the rank of the matrix composed by the coefficients of the system (48) *is equal to* ν, *then among the numbers* $x_0, x_1, \dots, x_{r-1}, y_0, y_1, \dots, y_{r-1}$ *there are* $2(m - m_1) - \nu$ *arbitrary constants, and hence the general solution of the problem* (8) *contains* $2(m - m_1) - \nu$ *arbitrary real parameters.*

8^0. On a Mixed Problem in Domains with Piecewise-Smooth Boundaries

In [13], the problem (8) is investigated in domains with piecewise-Lyapunov boundaries. For curves with arbitrary nonzero angles there is Theorem 1 in [13] which shows relations between the values p, q, α_k, β_k, ν_k, λ_k, and if they are fulfilled, the statements of type I–III in Theorem 6 of the present work remain valid. A detailed analysis of cases where the above relations are realized is given. When investigating the problem we have used the results obtained by S. Warschawskiĭ ([18]) on conformal mappings of a circle on a domain with piecewise-Lyapunov boundary.

Consider the case where L is a piecewise-smooth curve. Assume that on L there are angular points $t_1, t_2, \dots, t_s$ with the values $\mu_k\pi$, $0 < \mu_k \le 2$, $k = \overline{1,s}$ of the interior angles at these points. In this case, for conformal mapping we use some results from [14] (see also [5, Ch. III]) according to which

$$z'(w) = \prod_{k=1}^{s} (w-\tau_k)^{\mu_k - 1} z_1(w), \quad \tau_k = w(t_k),$$
$$z(w) = \prod_{k=1}^{s} (w-\tau_k)^{\mu_k} z_2(w),$$

where $[z_1(w)]^{\pm 1}$, $[z_2(w)]^{\pm 1}$ belong to $\bigcap\limits_{\delta>0} H^\delta$, while the functions $z_1(\tau)$, $z_2(\tau)$ have the form (11) and hence belong to $\bigcap\limits_{\delta>1} W^\delta$.

On the basis of the above-said, taking into account the results of Sections 3^0 and 4^0 of the present work and following the reasoning from [13], we can see that the basic result obtained in [13, Theorem 1] for the problem (8), when L is a piecewise-smooth curve, remains valid.

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INVARIANT REGIONS AND THE GLOBAL EXISTENCE FOR REACTION-DIFFUSION SYSTEMS WITH A TRIDIAGONAL MATRIX OF DIFFUSION COEFFICIENTS

Abstract. The aim of this study is to prove the global existence of solutions for reaction-diffusion systems with a tridiagonal matrix of diffusion coefficients and nonhomogeneous boundary conditions. In so doing, we make use of the appropriate techniques which are based on invariant domains and Lyapunov functional methods. The nonlinear reaction term has been supposed to be of polynomial growth. This result is a continuation of that by Kouachi [12].

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Key words and phrases. reaction diffusion systems, invariant domains, Lyapunov functionals, global existence.

რეზიუმე. ნაშრომის მიზანია დამტკიცებულ იქნეს არაერთგვაროვანი სასაზღვრო ამოცანების ამონახსნთა გლობალური არსებობა ისეთი რეაქციულ-დიფუზიური სისტემებისათვის, რომელთა დიფუზიის კოეფიციენტები ქმნიან ტრიდიაგონალურ (იაკობის) მატრიცს. ამისათვის გამოყენებულია შესაბამისი ტექნიკა, რომელიც ეფუძნება ინვარიანტული არეების და ლიაპუნოვის ფუნქციონალის მეთოდებს. არაწრფივი რეაქციის წევრზე დადებულია პოლინომიალური ზრდის პირობა. ნაშრომში მოყვანილი შედეგი წარმოადგენს კუაშის [12] ერთი შედეგის განვრცობას.

1. Introduction

We consider the reaction-diffusion system

$$\frac{\partial u}{\partial t} - a_{11}\Delta u - a_{12}\Delta v - a_{23}\Delta w = f(u,v,w) \ \text{ in } \ \mathbb{R}^+ \times \Omega, \tag{1.1}$$

$$\frac{\partial v}{\partial t} - a_{21}\Delta u - a_{22}\Delta v - a_{23}\Delta w = g(u,v,w) \ \text{ in } \ \mathbb{R}^+ \times \Omega, \tag{1.2}$$

$$\frac{\partial w}{\partial t} - a_{21}\Delta u - a_{32}\Delta v - a_{11}\Delta w = h(u,v,w) \ \text{ in } \ \mathbb{R}^+ \times \Omega, \tag{1.3}$$

with the boundary conditions

$$\begin{aligned} \lambda u + (1-\lambda)\frac{\partial u}{\partial \eta} &= \beta_1, \\ \lambda v + (1-\lambda)\frac{\partial v}{\partial \eta} &= \beta_2 \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \\ \lambda w + (1-\lambda)\frac{\partial w}{\partial \eta} &= \beta_3, \end{aligned} \tag{1.4}$$

and the initial data

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad w(0,x) = w_0(x) \ \text{ in } \ \Omega, \tag{1.5}$$

where:

- (i) $0 < \lambda < 1$ and $\beta_i \in \mathbb{R}$, $i = 1,2,3$, for nonhomogeneous Robin boundary conditions.
- (ii) $\lambda = \beta_i = 0$, $i = 1,2,3$, for homogeneous Neumann boundary conditions.
- (iii) $1 - \lambda = \beta_i = 0$, $i = 1,2,3$, for homogeneous Dirichlet boundary conditions.

Ω is an open bounded domain of the class $\mathbb{C}^1$ in $\mathbb{R}^N$ with boundary $\partial\Omega$, and $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on $\partial\Omega$. The diffusion terms a_{ij} ($i,j = 1,2,3$ and $(i,j) \neq (1,3), (3,1)$) are supposed to be positive constants with $a_{11} = a_{33}$ and $(a_{12}+a_{21})^2 + (a_{23}+a_{32})^2 < 4a_{11}a_{22}$, which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{11} \end{pmatrix}$$

is positive definite. The eigenvalues λ_1, λ_2 and λ_3 ($\lambda_1 < \lambda_2$, $\lambda_3 = a_{11}$) of A are positive. If we put

$$\underline{a} = \min\{a_{11}, a_{22}\} \ \text{ and } \ \overline{a} = \max\{a_{11}, a_{22}\},$$

then the positivity of a_{ij}'s implies that

$$\lambda_1 < \underline{a} \leq \lambda_3 \leq \overline{a} < \lambda_2.$$

The initial data are assumed to be in the domain

$$\Sigma = \begin{cases} \Big\{(u_0,v_0,w_0)\in\mathbb{R}^3 : \mu_2 v_0 \le a_{21}u_0 + a_{23}w_0 \le \mu_1 v_0,\ a_{32}u_0 \le a_{12}w_0\Big\} \\ \quad \text{if } \mu_2\beta_2 \le a_{21}\beta_1 + a_{23}\beta_3 \le \mu_1\beta_2,\ a_{32}\beta_1 \le a_{12}\beta_3, \\ \Big\{(u_0,v_0,w_0)\in\mathbb{R}^3 : \mu_2 v_0 \le a_{21}u_0 + a_{23}w_0 \le \mu_1 v_0,\ a_{12}w_0 \le a_{32}u_0\Big\} \\ \quad \text{if } \mu_2\beta_2 \le a_{21}\beta_1 + a_{23}\beta_3 \le \mu_1\beta_2,\ a_{12}\beta_3 \le a_{32}\beta_1, \\ \left\{\begin{array}{l}(u_0,v_0,w_0)\in\mathbb{R}^3 : \\ \frac{1}{\mu_2}(a_{21}u_0+a_{23}w_0)\le v_0 \le \frac{1}{\mu_1}(a_{21}u_0+a_{23}w_0),\ a_{32}u_0\le a_{12}w_0\end{array}\right\} \\ \quad \text{if } \frac{1}{\mu_2}(a_{21}\beta_1+a_{23}\beta_3)\le\beta_2\le\frac{1}{\mu_1}(a_{21}\beta_1+a_{23}\beta_3),\ a_{32}\beta_1\le a_{12}\beta_3, \\ \left\{\begin{array}{l}(u_0,v_0,w_0)\in\mathbb{R}^3 : \\ \frac{1}{\mu_2}(a_{21}u_0+a_{23}w_0)\le v_0 \le \frac{1}{\mu_1}(a_{21}u_0+a_{23}w_0),\ a_{32}u_0\ge a_{12}w_0\end{array}\right\} \\ \quad \text{if } \frac{1}{\mu_2}(a_{21}\beta_1+a_{23}\beta_3)\le\beta_2\le\frac{1}{\mu_1}(a_{21}\beta_1+a_{23}\beta_3),\ a_{32}\beta_1\ge a_{12}\beta_3, \end{cases}$$

where

$$\mu_1 = \underline{a} - \lambda_1 > 0 > \mu_2 = \underline{a} - \lambda_2.$$

Since we use the same methods to treat all the cases, we will tackle only with the first one. We suppose that the reaction terms f, g and h are continuously differentiable, polynomially bounded on Σ,

$$\Big(f(r_1,r_2,r_3), g(r_1,r_2,r_3), h(r_1,r_2,r_3)\Big)$$

is in Σ for all (r_1,r_2,r_3) in $\partial\Sigma$ (we say that (f,g,h) points into Σ on $\partial\Sigma$), i.e.,

$$a_{21}f(r_1,r_2,r_3) + a_{23}h(r_1,r_2,r_3) \le \mu_1 g(r_1,r_2,r_3) \tag{1.6}$$

for all r_1, r_2 and r_3 such that $\mu_2 r_2 \le a_{21}r_1 + a_{23}r_3 = \mu_1 r_2$ and $a_{32}r_1 \le a_{12}r_3$, and

$$\mu_2 g(r_1,r_2,r_3) \le a_{21}f(r_1,r_2,r_3) + a_{23}h(r_1,r_2,r_3) \tag{1.6a}$$

for all r_1, r_2 and r_3 such that $\mu_2 r_2 = a_{21}r_1 + a_{23}r_3 \le \mu_1 r_2$ and $a_{32}r_1 \le a_{12}r_3$, and

$$a_{32}f(r_1,r_2,r_3) \le a_{12}h(r_1,r_2,r_3) \tag{1.6b}$$

for all r_1, r_2 and r_3 such that $\mu_2 r_2 \le a_{21}r_1 + a_{23}r_3 \le \mu_1 r_2$ and $a_{32}r_1 = a_{12}r_3$, and for positive constants E and D, we have

$$(Ef + Dg + h)(u,v,w) \le C_1(u+v+w+1), \tag{1.7}$$

for all (u,v,w) in Σ, where C_1 is a positive constant.

In the two-component case, where $a_{12} = 0$, Kouachi and Youkana [13] generalized the method of Haraux and Youkana [4] with the reaction terms $f(u,v) = -\lambda F(u,v)$ and $g(u,v) = \mu F(u,v)$ with $F(u,v) \ge 0$, requiring the condition

$$\lim_{s\to+\infty}\left[\frac{\ln(1+F(r,s))}{s}\right] < \alpha^* \ \text{ for any } \ r \ge 0,$$

with

$$\alpha^* = \frac{2a_{11}a_{22}}{n\,(a_{11}-a_{22})^2\,\|u_0\|_\infty}\,\min\left\{\frac{\lambda}{\mu},\frac{a_{11}-a_{22}}{a_{21}}\right\},$$

where the positive diffusion coefficients a_{11}, a_{22} satisfy $a_{11} > a_{22}$, and a_{21}, λ, μ are positive constants. This condition reflects the weak exponential growth of the reaction term F. Kanel and Kirane [6] proved the global existence in the case where $g(u,v) = -f(u,v) = uv^n$ and n is an odd integer, under the embarrassing condition

$$|a_{12} - a_{21}| < C_p,$$

where C_p contains a constant from Solonnikov's estimate [18]. Later they improved their results in [7] to obtain the global existence under the restrictions

H$_1$. $a_{22} < a_{11} + a_{21}$,

H$_2$. $a_{12} < \varepsilon_0 = \dfrac{a_{11}a_{22}\,(a_{11}+a_{21}-a_{22})}{a_{11}a_{22} + a_{21}\,(a_{11}+a_{21}-a_{22})}$ if $a_{11} \le a_{22} < a_{11} + a_{21}$,

H$_3$. $a_{12} < \min\left\{\dfrac{1}{2}\,(a_{11}+a_{21})\,,\varepsilon_0\right\}$ if $a_{22} < a_{11}$,

and

$$|F(v)| \le C_F\left(1+|v|^{1-\varepsilon}\right),\quad vF(v) \ge 0 \quad \text{for all} \quad v \in \mathbb{R},$$

where ε and C_F are positive constants with $\varepsilon < 1$ and

$$g(u,v) = -f(u,v) = uF(v)\,.$$

Kouachi [12] has proved global existence for solutions of two-component reaction-diffusion systems with a general full matrix of diffusion coefficients and nonhomgeneous boundary conditions.

Many chemical and biological operations are described by reaction-diffusion systems with a tridiagonal matrix of diffusion coefficients. The components $u(t,x)$, $v(t,x)$ and $w(t,x)$ can represent either chemical concentrations or biological population densities (see, e.g., Cussler [1] and [2]).

We note that the case of strongly coupled systems which are not triangular in the diffusion part is more difficult. As a consequence of the blow-up of the solutions found in [16], we can indeed prove that there is a blow-up of the solutions in finite time for such nontriangular systems even though the initial data are regular, the solutions are positive and the nonlinear terms are negative, a structure that ensured the global existence in the diagonal case. For this purpose, we construct invariant domains in which we can demonstrate that for any initial data in these domains, the problem (1.1)–(1.5) is equivalent to the problem for which the global existence follows from the usual techniques based on Lyapunov functionals (see Kirane and Kouachi [8], Kouachi and Youkana [13] and Kouachi [12]).

2. Local Existence and Invariant Regions

This section is devoted to proving that if (f, g, h) points into Σ on $\partial\Sigma$, then Σ is an invariant domain for the problem (1.1)–(1.5), i.e., the solution remains in Σ for any initial data in Σ. Once the invariant domains are constructed, both problems of the local and global existence become easier to be established. For the first problem we demonstrate that the system (1.1)–(1.3) with the boundary conditions (1.4) and the initial data in Σ is equivalent to a problem for which the local existence throughout the time interval $[0, T^*[$ can be obtained by the known procedure, and for the second one we need invariant domains as explained in the preceeding section.

The main result of this section is

Proposition 1. *Suppose that* (f, g, h) *points into* Σ *on* $\partial\Sigma$*. Then for any* (u_0, v_0, w_0) *in* Σ *the solution* (u, v, w) *of the problem* (1.1)–(1.5) *remains in* Σ *for all* t*'s in* $[0, T^*[$.

Proof. Let $(x_{i1}, x_{i2}, x_{i3})^t$, $i = 1, 2, 3$, be the eigenvectors of the matrix A^t associated with its eigenvalues λ_i, $i = 1, 2, 3$ $(\lambda_1 < \lambda_3 < \lambda_2)$. Multiplying the equations (1.1), (1.2) and (1.3) of the given reaction-diffusion system by x_{i1}, x_{i2} and x_{i3}, respectively, and summing the resulting equations, we get

$$\frac{\partial}{\partial t} z_1 - \lambda_1 \Delta z_1 = F_1(z_1, z_2, z_3) \quad \text{in} \quad]0, T^*[\times \Omega, \tag{2.1}$$

$$\frac{\partial}{\partial t} z_2 - \lambda_2 \Delta z_2 = F_2(z_1, z_2, z_3) \quad \text{in} \quad]0, T^*[\times \Omega, \tag{2.2}$$

$$\frac{\partial}{\partial t} z_3 - \lambda_3 \Delta z_3 = F_3(z_1, z_2, z_3) \quad \text{in} \quad]0, T^*[\times \Omega, \tag{2.3}$$

with the boundary conditions

$$\lambda z_i + (1 - \lambda)\frac{\partial z_i}{\partial \eta} = \rho_i, \quad i = 1, 2, 3, \quad \text{on} \quad]0, T^*[\times \partial\Omega, \tag{2.4}$$

and the initial data

$$z_i(0, x) = z_i^0(x), \quad i = 1, 2, 3, \quad \text{in} \quad \Omega, \tag{2.5}$$

where

$$z_i = x_{i1}u + x_{i2}v + x_{i3}w, \quad i = 1, 2, 3, \quad \text{in} \quad]0, T^*[\times \Omega, \tag{2.6}$$
$$\rho_i = x_{i1}\beta_1 + x_{i2}\beta_2 + x_{i3}\beta_3, \quad i = 1, 2, 3,$$

and

$$F_i(z_1, z_2, z_3) = x_{i1}f + x_{i2}g + x_{i3}h, \quad i = 1, 2, 3, \tag{2.7}$$

for all (u, v, w) in Σ.

First, as has been mentioned above, note that the condition of the parabolicity of the system (1.1)–(1.3) implies the parabolicity of the system

(2.1)–(2.3) since

$$(a_{12}+a_{21})^2+(a_{23}+a_{32})^2<4a_{11}a_{22}\Longrightarrow$$
$$\Longrightarrow(\det A>0 \text{ and } a_{11}a_{22}-a_{23}a_{32}>0).$$

Since λ_1, λ_2 and λ_3 ($\lambda_1<\lambda_3<\lambda_2$) are the eigenvalues of the matrix A^t, the problem (1.1)–(1.5) is equivalent to the problem (2.1)–(2.5) and to prove that Σ is an invariant domain for the system (1.1)–(1.3) it suffices to prove that the domain

$$\left\{\left(z_1^0,z_2^0,z_3^0\right)\in\mathbb{R}^3:\ z_i^0\geq 0,\ i=1,2,3\right\}=\left(\mathbb{R}^+\right)^3 \tag{2.8}$$

is invariant for the system (2.1)–(2.3) and that

$$\Sigma=\left\{(u_0,v_0,w_0)\in\mathbb{R}^3:\ z_i^0=x_{i1}u_0+x_{i2}v_0+x_{i3}w_0\geq 0,\ i=1,2,3\right\}. \tag{2.9}$$

Since $(x_{i1},x_{i2},x_{i3})^t$ is an eigenvector of the matrix A^t associated to the eigenvalue λ_i, $i=1,2,3$, we have

$$\begin{aligned}(a_{11}-\lambda_i)\,x_{i1}+a_{21}x_{i2}&=0,\\ a_{12}x_{i1}+(a_{22}-\lambda_i)\,x_{i2}+a_{32}x_{i3}&=0\ ,\quad i=1,2,3,\\ a_{23}x_{i2}+(a_{11}-\lambda_i)\,x_{i3}&=0.\end{aligned}$$

If we assume, without loss of generality, that $a_{11}\leq a_{22}$ and choose $x_{12}=\mu_1$, $x_{22}=-\mu_2$ and $x_{33}=a_{12}$, then we have

$$x_{i1}u_0+x_{i2}v_0+x_{i3}w_0\geq 0,\quad i=1,2,3\Longleftrightarrow\begin{cases}-a_{21}u_0+\mu_1v_0-a_{23}w_0\geq 0,\\ a_{21}u_0-\mu_2v_0+a_{23}w_0\geq 0,\\ -a_{32}u_0+a_{12}w_0\geq 0.\end{cases}\Longleftrightarrow$$
$$\Longleftrightarrow\mu_2v_0\leq a_{21}u_0+a_{23}w_0\leq\mu_1v_0,\quad a_{32}u_0\leq a_{12}w_0.$$

Thus (2.9) is proved and (2.6) can be written as

$$\begin{cases}z_1=-a_{21}u+\mu_1v-a_{23}w,\\ z_2=a_{21}u-\mu_2v+a_{23}w,\\ z_3=-a_{32}u+a_{12}w.\end{cases} \tag{2.6a}$$

Now, to prove that the domain $\left(\mathbb{R}^+\right)^3$ is invariant for the system (2.1)–(2.3), it suffices to show that $F_i\left(z_1,z_2,z_3\right)\geq 0$ for all (z_1,z_2,z_3) such that $z_i=0$ and $z_j\geq 0$, $j=1,2,3$ $(j\neq i)$, $i=1,2,3$, thanks to the invariant domain method (see Smoller [17]). Using the expressions (2.7), we get

$$\begin{cases}F_1=-a_{21}f+\mu_1g-a_{23}h,\\ F_2=a_{21}f-\mu_2g+a_{23}h,\\ F_3=-a_{32}f+a_{12}h\end{cases} \tag{2.7a}$$

for all (u,v,w) in Σ. Since from (1.6), (1.6a) and (1.6b) we have $F_i\left(z_1,z_2,z_3\right)\geq 0$ for all (z_1,z_2,z_3) such that $z_i=0$ and $z_j\geq 0$, $j=1,2,3$ $(j\neq i)$, $i=1,2,3$, we obtain $z_i\left(t,x\right)\geq 0$, $i=1,2,3$, for all $(t,x)\in[0,T^*[\times\Omega$. Then Σ is an invariant domain for the system (1.1)–(1.3). □

In addition, the system (1.1)–(1.3) with the boundary conditions (1.4) and initial data in Σ is equivalent to the system (2.1)–(2.3) with the boundary conditions (2.4) and positive initial data (2.5). As has been mentioned at the beginning of this section and since ρ_i, $i = 1, 2, 3$, given by

$$\begin{cases} \rho_1 = -a_{21}\beta_1 + \mu_1\beta_2 - a_{23}\beta_3, \\ \rho_2 = a_{21}\beta_1 - \mu_2\beta_2 + a_{23}\beta_3, \\ \rho_3 = -a_{32}\beta_1 + a_{12}\beta_3, \end{cases}$$

are positive, we have for any initial data in $\mathbb{C}\left(\overline{\Omega}\right)$ or $\mathbb{L}^p(\Omega)$, $p \in \left]1, +\infty\right]$, the local existence and uniqueness of solutions to the initial value problem (2.1)–(2.5) and consequently those of the problem (1.1)–(1.5) follow from the basic existence theory for abstract semilinear differential equations (see Friedman [3], Henry [5] and Pazy [15]). These solutions are classical on $[0, T^*[\times \Omega$, where T^* denotes the eventual blow up time in $\mathbb{L}^\infty(\Omega)$. A local solution is continued globally by a priori estimates.

Once invariant domains are constructed, one can apply the Lyapunov technique and establish the global existence of unique solutions for (1.1)–(1.5).

3. Global Existence

As the determinant of the linear algebraic system (2.6), with respect to the variables u, v and w, is different from zero, to prove the global existence of solutions of the problem (1.1)–(1.5) one needs to prove it for the problem (2.1)–(2.5). To this end, it suffices (see Henry [5]) to derive a uniform estimate of $\|F_i(z_1, z_2, z_3)\|_p$, $i = 1, 2, 3$ on $[0, T]$, $T < T^*$, for some $p > N/2$, where $\|\cdot\|_p$ denotes the usual norms in spaces $\mathbb{L}^p(\Omega)$ defined by

$$\|u\|_p^p = \frac{1}{|\Omega|}\int_\Omega |u(x)|^p\, dx, \quad 1 \le p < \infty, \quad \text{and} \quad \|u\|_\infty = \operatorname*{esssup}_{x\in\Omega} |u(x)|.$$

Let θ and σ be two positive constants such that

$$\theta > A_{12}, \tag{3.1}$$

$$\left(\theta^2 - A_{12}^2\right)\left(\sigma^2 - A_{23}^2\right) > \left(A_{13} - A_{12}A_{23}\right)^2, \tag{3.2}$$

where

$$A_{ij} = \frac{\lambda_i + \lambda_j}{2\sqrt{\lambda_i\lambda_j}}, \quad i, j = 1, 2, 3 \quad (i < j),$$

and let

$$\theta_q = \theta^{(p-q+1)^2} \quad \text{and} \quad \sigma_p = \sigma^{p^2}, \quad \text{for} \quad q = 0, 1, \ldots, p \quad \text{and} \quad p = 0, 1, \ldots, n, \tag{3.3}$$

where n is a positive integer. The main result of this section is

Theorem 1. *Let* $(z_1(t,\cdot), z_2(t,\cdot), z_3(t,\cdot))$ *be any positive solution of* (2.1)–(2.5). *Introduce the functional*

$$t \longmapsto L(t) = \int_\Omega H_n\big(z_1(t,x), z_2(t,x), z_3(t,x)\big)\, dx, \tag{3.4}$$

where

$$H_n(z_1, z_2, z_3) = \sum_{p=0}^{n}\sum_{q=0}^{p} C_n^p C_p^q \theta_q \sigma_p z_1^q z_2^{p-q} z_3^{n-p}, \tag{3.5}$$

with n *being a positive integer and* $C_n^p = \frac{n!}{(n-p)!p!}$.

Then the functional L *is uniformly bounded on the interval* $[0,T]$, $T < T^*$.

For the proof of Theorem 1 we need some preparatory Lemmas.

Lemma 1. *Let* H_n *be the homogeneous polynomial defined by* (3.5). *Then*

$$\frac{\partial H_n}{\partial z_1} = n\sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_{q+1}\sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-1)-p}, \tag{3.6}$$

$$\frac{\partial H_n}{\partial z_2} = n\sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_q\sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-1)-p}, \tag{3.7}$$

$$\frac{\partial H_n}{\partial z_3} = n\sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_q\sigma_p z_1^q z_2^{p-q} z_3^{(n-1)-p}. \tag{3.8}$$

Proof. Differentiating H_n with respect to z_1 and using the fact that

$$qC_p^q = pC_{p-1}^{q-1} \text{ and } pC_n^p = nC_{n-1}^{p-1} \tag{3.9}$$

for $q = 1, 2, \ldots, p$, $p = 1, 2, \ldots, n$, we get

$$\frac{\partial H_n}{\partial z_1} = n\sum_{p=1}^{n}\sum_{q=1}^{p} C_{n-1}^{p-1} C_{p-1}^{q-1} \theta_q\sigma_p z_1^{q-1} z_2^{p-q} z_3^{n-p}.$$

Replacing in the sums the indexes $q-1$ by q and $p-1$ by p, we deduce (3.6). For the formula (3.7), differentiating H_n with respect to z_2, taking into account

$$C_p^q = C_p^{p-q}, \quad q = 0, 1, \ldots, p-1 \text{ and } p = 1, 2, \ldots, n, \tag{3.10}$$

using (3.9) and replacing the index $p-1$ by p, we get (3.7).

Finally, we have

$$\frac{\partial H_n}{\partial z_3} = \sum_{p=0}^{n-1}\sum_{q=0}^{p} (n-p)\, C_n^p C_p^q \theta_q\sigma_p z_1^q z_2^{p-q} z_3^{n-p-1}.$$

Since $(n-p)\, C_n^p = (n-p)\, C_n^{n-p} = nC_{n-1}^{n-p-1} = nC_{n-1}^p$, we get (3.8). □

Lemma 2. *The second partial derivatives of H_n are given by*

$$\frac{\partial^2 H_n}{\partial z_1^2} = n\,(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^{p} C_p^q \theta_{q+2}\sigma_{p+2} z_1^q z_2^{p-q} z_3^{(n-2)-p}, \tag{3.11}$$

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_2} = n\,(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^{p} C_p^q \theta_{q+1}\sigma_{p+2} z_1^q z_2^{p-q} z_3^{(n-2)-p}, \tag{3.12}$$

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_3} = n\,(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^{p} C_p^q \theta_{q+1}\sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-2)-p}, \tag{3.13}$$

$$\frac{\partial^2 H_n}{\partial z_2^2} = n\,(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^{p} C_p^q \theta_{q}\sigma_{p+2} z_1^q z_2^{p-q} z_3^{(n-2)-p}, \tag{3.14}$$

$$\frac{\partial^2 H_n}{\partial z_2 \partial z_3} = n\,(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^{p} C_p^q \theta_{q}\sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-2)-p}, \tag{3.15}$$

$$\frac{\partial^2 H_n}{\partial z_3^2} = n\,(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^{p} C_p^q \theta_{q}\sigma_{p} z_1^q z_2^{p-q} z_3^{(n-2)-p}. \tag{3.16}$$

Proof. Differentiating $\dfrac{\partial H_n}{\partial z_1}$ given by (3.6) with respect to z_1 yields

$$\frac{\partial^2 H_n}{\partial z_1^2} = n\sum_{p=1}^{n-1}\sum_{q=1}^{p} q C_{n-1}^{p} C_p^q \theta_{q+1}\sigma_{q+1} z_1^{q-1} z_2^{p-q} z_3^{(n-1)-p}.$$

Using (3.9), we get (3.11).

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_2} = n\sum_{p=1}^{n-1}\sum_{q=0}^{p-1} (p-q)\, C_{n-1}^{p} C_p^q \theta_{q+1}\sigma_{p+1} z_1^{q} z_2^{p-q-1} z_3^{(n-1)-p}.$$

Applying (3.10) and then (3.9), we get (3.12).

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_3} = n\sum_{p=0}^{n-2}\sum_{q=0}^{p} ((n-1)-p)\, C_{n-1}^{p} C_p^q \theta_{q+1}\sigma_{p+1} z_1^{q} z_2^{p-q} z_3^{(n-2)-p}.$$

Applying successively (3.10), (3.9) and (3.10) for the second time, we deduce (3.13).

$$\frac{\partial^2 H_n}{\partial z_2^2} = n\sum_{p=1}^{n-1}\sum_{q=0}^{p-1} (p-q)\, C_{n-1}^{p} C_p^q \theta_{q}\sigma_{p+1} z_1^{q} z_2^{p-q-1} z_3^{(n-1)-p}.$$

The application of (3.10) and then of (3.9) yields (3.14).

$$\frac{\partial^2 H_n}{\partial z_2 \partial z_3} = n\sum_{p=0}^{n-2}\sum_{q=0}^{p} ((n-1)-p)\, C_{n-1}^{p} C_p^q \theta_{q}\sigma_{p} z_1^{q} z_2^{p-q} z_3^{(n-2)-p}.$$

Applying (3.10) and then (3.9) yields (3.15). Finally we get (3.16) by differentiating $\frac{\partial H_n}{\partial z_3}$ with respect to z_3 and applying successively (3.10), (3.9) and (3.10) for the second time. □

Proof of Theorem 1. Differentiating L with respect to t yields

$$\begin{aligned}L'(t) &= \int_\Omega \left(\frac{\partial H_n}{\partial z_1} \frac{\partial z_1}{\partial t} + \frac{\partial H_n}{\partial z_2} \frac{\partial z_2}{\partial t} + \frac{\partial H_n}{\partial z_3} \frac{\partial z_3}{\partial t} \right) dx = \\ &= \int_\Omega \left(\lambda_1 \frac{\partial H_n}{\partial z_1} \Delta z_1 + \lambda_2 \frac{\partial H_n}{\partial z_2} \Delta z_2 + \lambda_3 \frac{\partial H_n}{\partial z_3} \Delta z_3 \right) dx + \\ &\quad + \int_\Omega \left(\frac{\partial H_n}{\partial z_1} F_1 + \frac{\partial H_n}{\partial z_2} F_2 + \frac{\partial H_n}{\partial z_3} F_3 \right) dx = \\ &=: I + J.\end{aligned}$$

Using Green's formula in I, we get $I = I_1 + I_2$, where

$$I_1 = \int_{\partial\Omega} \left(\lambda_1 \frac{\partial H_n}{\partial z_1} \frac{\partial z_1}{\partial \eta} + \lambda_2 \frac{\partial H_n}{\partial z_2} \frac{\partial z_2}{\partial \eta} + \lambda_3 \frac{\partial H_n}{\partial z_3} \frac{\partial z_3}{\partial \eta} \right) ds,$$

where ds denotes the $(n-1)$-dimensional surface element, and

$$\begin{aligned}I_2 = -\int_\Omega \Big[&\lambda_1 \frac{\partial^2 H_n}{\partial z_1^2} |\nabla z_1|^2 + (\lambda_1 + \lambda_2) \frac{\partial^2 H_n}{\partial z_1 \partial z_2} \nabla z_1 \nabla z_2 \\ &+ (\lambda_1 + \lambda_3) \frac{\partial^2 H_n}{\partial z_1 \partial z_3} \nabla z_1 \nabla z_3 + \lambda_2 \frac{\partial^2 H_n}{\partial z_2^2} |\nabla z_2|^2 \\ &+ (\lambda_2 + \lambda_3) \frac{\partial^2 H_n}{\partial z_2 \partial z_3} \nabla z_2 \nabla z_3 + \lambda_3 \frac{\partial^2 H_n}{\partial z_3^2} |\nabla z_3|^2 \Big] dx.\end{aligned}$$

We prove that there exists a positive constant C_2 independent of $t \in [0, T^*[$ such that

$$I_1 \le C_2 \ \text{ for all } \ t \in [0, T^*[\,, \tag{3.17}$$

and that

$$I_2 \le 0. \tag{3.18}$$

To see this, we follow the same reasoning as in [11].

(*i*) If $0 < \lambda < 1$, using the boundary conditions (2.4) we get

$$I_1 = \int_{\partial\Omega} \left(\lambda_1 \frac{\partial H_n}{\partial z_1} (\gamma_1 - \alpha z_1) + \lambda_2 \frac{\partial H_n}{\partial z_2} (\gamma_2 - \alpha z_2) + \lambda_3 \frac{\partial H_n}{\partial z_3} (\gamma_3 - \alpha z_3) \right) ds,$$

where $\alpha = \frac{\lambda}{1-\lambda}$ and $\gamma_i = \frac{\rho_i}{1-\lambda}$, $i = 1, 2, 3$. Since

$$\begin{aligned}H(z_1, z_2, z_3) &= \lambda_1 \frac{\partial H_n}{\partial z_1} (\gamma_1 - \alpha z_1) + \lambda_2 \frac{\partial H_n}{\partial z_2} (\gamma_2 - \alpha z_2) + \lambda_3 \frac{\partial H_n}{\partial z_3} (\gamma_3 - \alpha z_3) \\ &= P_{n-1}(z_1, z_2, z_3) - Q_n(z_1, z_2, z_3),\end{aligned}$$

where P_{n-1} and Q_n are polynomials with positive coefficients and respective degrees $n-1$ and n, and since the solution is positive, we obtain

$$\limsup_{(|z_1|+|z_2|+|z_3|)\to+\infty} H(z_1,z_2,z_3) = -\infty, \tag{3.19}$$

which proves that H is uniformly bounded on $(\mathbb{R}^+)^3$, and consequently (3.17).

(ii) If $\lambda = 0$, then $I_1 = 0$ on $[0,T^*[$.

(iii) The case of the homogeneous Dirichlet conditions is trivial since the positivity of the solution on $[0,T^*[\times\Omega$ implies $\frac{\partial z_1}{\partial\eta}\le 0$, $\frac{\partial z_2}{\partial\eta}\le 0$ and $\frac{\partial z_3}{\partial\eta}\le 0$ on $[0,T^*[\times\partial\Omega$. Consequently, one again gets (3.17) with $C_2=0$.

Now, we prove (3.18). Applying Lemma 1 and Lemma 2, we get

$$I_2 = -n(n-1)\int_\Omega \sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^p C_p^q \left[(B_{pq}z)\cdot z\right] dx,$$

where

$$B_{pq} = \begin{pmatrix} \lambda_1\theta_{q+2}\sigma_{p+2} & \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \frac{\lambda_1+\lambda_3}{2}\theta_{q+1}\sigma_{p+1} \\ \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \lambda_2\theta_q\sigma_{p+2} & \frac{\lambda_2+\lambda_3}{2}\theta_q\sigma_{p+1} \\ \frac{\lambda_1+\lambda_3}{2}\theta_{q+1}\sigma_{p+1} & \frac{\lambda_2+\lambda_3}{2}\theta_q\sigma_{p+1} & \lambda_3\theta_q\sigma_p \end{pmatrix},$$

for $q=0,1,\ldots,p$, $p=0,1,\ldots,n-2$ and $z=(\nabla z_1,\nabla z_2,\nabla z_3)^t$.

The quadratic forms (with respect to $\nabla z_1,\nabla z_2$ and ∇z_3) associated with the matrices B_{pq}, $q=0,1,\ldots,p$, $p=0,1,\ldots,n-2$, are positive since their main determinants Δ_1, Δ_2 and Δ_3 are positive too, according to the Sylvester criterion. To see this, we have

1. $\Delta_1 = \lambda_1\theta_{q+2}\sigma_{p+2} > 0$ for $q=0,1,\ldots,p$ and $p=0,1,\ldots,n-2$.
2. $$\Delta_2 = \begin{vmatrix} \lambda_1\theta_{q+2}\sigma_{p+2} & \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} \\ \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \lambda_2\theta_q\sigma_{p+2} \end{vmatrix} = \lambda_1\lambda_2\theta_{q+1}^2\sigma_{p+2}^2\left(\theta^2-A_{12}^2\right),$$
   for $q=0,1,\ldots,p$ and $p=0,1,\ldots,n-2$.

   Using (3.1), we get $\Delta_2>0$.
3. $$\Delta_3 = \begin{vmatrix} \lambda_1\theta_{q+2}\sigma_{p+2} & \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \frac{\lambda_1+\lambda_3}{2}\theta_{q+1}\sigma_{p+1} \\ \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \lambda_2\theta_q\sigma_{p+2} & \frac{\lambda_2+\lambda_3}{2}\theta_q\sigma_{p+1} \\ \frac{\lambda_1+\lambda_3}{2}\theta_{q+1}\sigma_{p+1} & \frac{\lambda_2+\lambda_3}{2}\theta_q\sigma_{p+1} & \lambda_3\theta_q\sigma_p \end{vmatrix}$$
   $$=\lambda_1\lambda_2\lambda_3\theta_{q+1}^2\theta_q\sigma_{p+2}\sigma_{p+1}^2\left[(\theta^2-A_{12}^2)(\sigma^2-A_{23}^2)-(A_{13}-A_{12}A_{23})^2\right],$$
   for $q=0,1,\ldots,p$ and $p=0,1,\ldots,n-2$.

Using (3.2), we get $\Delta_3 > 0$. Consequently we have (3.18).

Substitution of the expressions of the partial derivatives given by Lemma 1 in the second integral yields

$$J = \int_\Omega \left[n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_{n-1}^p C_p^q z_1^q z_2^{p-q} z_3^{(n-1)-p} \right] \times$$
$$\times \left(\theta_{q+1}\sigma_{p+1}F_1 + \theta_q\sigma_{p+1}F_2 + \theta_q\sigma_p F_3\right) dx.$$

Using the expressions (2.7a), we get

$$\begin{aligned}
&\theta_{q+1}\sigma_{p+1}F_1 + \theta_q\sigma_{p+1}F_2 + \theta_q\sigma_p F_3 = \\
&= \big(-\theta_{q+1}\sigma_{p+1}a_{21} + a_{21}\theta_q\sigma_{p+1} - a_{32}\theta_q\sigma_p \big) f + \left(\theta_{q+1}\sigma_{p+1}\mu_1 - \mu_2\theta_q\sigma_{p+1}\right) g + \\
&\quad + \big(-\theta_{q+1}\sigma_{p+1}a_{23} + a_{23}\theta_q\sigma_{p+1} + a_{12}\theta_q\sigma_p \big) h = \\
&= \Big(a_{23}(\theta_q\sigma_{p+1} - \theta_{q+1}\sigma_{p+1}) + a_{12}\theta_q\sigma_p \Big) \left(\frac{a_{21}\left(\theta_q\sigma_{p+1} - \theta_{q+1}\sigma_{p+1}\right) - a_{32}\theta_q\sigma_p}{a_{23}(\theta_q\sigma_{p+1} - \theta_{q+1}\sigma_{p+1}) + a_{12}\theta_q\sigma_p} f + \right. \\
&\quad \left. + \frac{\theta_{q+1}\sigma_{p+1}\mu_1 - \mu_2\theta_q\sigma_{p+1}}{a_{23}\left(\theta_q\sigma_{p+1} - \theta_{q+1}\sigma_{p+1}\right) + a_{12}\theta_q\sigma_p} g + h \right) = \\
&\quad = \theta_{q+1}\sigma_p \left(a_{23}\frac{\sigma_{p+1}}{\sigma_p}\left(\frac{\theta_q}{\theta_{q+1}} - 1\right) + a_{12}\frac{\theta_q}{\theta_{q+1}} \right) \times \\
&\times \left(\frac{a_{21}\frac{\sigma_{p+1}}{\sigma_p}\left(\frac{\theta_q}{\theta_{q+1}} - 1\right) - a_{32}\frac{\theta_q}{\theta_{q+1}}}{a_{23}\frac{\sigma_{p+1}}{\sigma_p}\left(\frac{\theta_q}{\theta_{q+1}} - 1\right) + a_{12}\frac{\theta_q}{\theta_{q+1}}} f + \frac{\mu_1\frac{\sigma_{p+1}}{\sigma_p} - \mu_2\frac{\theta_q}{\theta_{q+1}}\frac{\sigma_{p+1}}{\sigma_p}}{a_{23}\frac{\sigma_{p+1}}{\sigma_p}\left(\frac{\theta_q}{\theta_{q+1}} - 1\right) + a_{12}\frac{\theta_q}{\theta_{q+1}}} g + h \right).
\end{aligned}$$

Since $\frac{\theta_q}{\theta_{q+1}}$ and $\frac{\sigma_{p+1}}{\sigma_p}$ are sufficiently large if we choose θ and σ sufficiently large, using the condition (1.7) and the relation (2.6a) successively we get, for an appropriate constant C_3,

$$J \le C_3 \int_\Omega \left[\sum_{p=0}^{n-1} \sum_{q=0}^{p} \left(z_1 + z_2 + z_3 + 1\right) C_{n-1}^p C_p^q z_1^q z_2^{p-q} z_3^{(n-1)-p} \right] dx.$$

To prove that the functional L is uniformly bounded on the interval $[0, T]$, we first write

$$\sum_{p=0}^{n-1} \sum_{q=0}^{p} \left(z_1 + z_2 + z_3 + 1\right) C_{n-1}^p C_p^q z_1^q z_2^{p-q} z_3^{(n-1)-p} =$$
$$= R_n\left(z_1, z_2, z_3\right) + S_{n-1}\left(z_1, z_2, z_3\right),$$

where $R_n\left(z_1, z_2, z_3\right)$ and $S_{n-1}\left(z_1, z_2, z_3\right)$ are two homogeneous polynomials of degrees n and $n-1$, respectively. First, since the polynomials H_n and R_n are of degree n, there exists a positive constant C_4 such that

$$\int_\Omega R_n\left(z_1, z_2, z_3\right) dx \le C_4 \int_\Omega H_n\left(z_1, z_2, z_3\right) dx.$$

Applying Hölder's inequality to the integral $\int_\Omega S_{n-1}(z_1,z_2,z_3)\,dx$, one gets

$$\int\limits_\Omega S_{n-1}(z_1,z_2,z_3)\,dx \le (meas\,\Omega)^{\frac{1}{n}}\left(\int\limits_\Omega (S_{n-1}(z_1,z_2,z_3))^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}}.$$

Since for all $z_1 \ge 0$ and $z_2, z_3 > 0$

$$\frac{(S_{n-1}(z_1,z_2,z_3))^{\frac{n}{n-1}}}{H_n(z_1,z_2,z_3)} = \frac{(S_{n-1}(\xi_1,\xi_2,1))^{\frac{n}{n-1}}}{H_n(\xi_1,\xi_2,1)},$$

where $\xi_1 = \frac{z_1}{z_2}$, $\xi_2 = \frac{z_2}{z_3}$ and

$$\lim_{\substack{\xi_1\to+\infty\\ \xi_2\to+\infty}} \frac{(S_{n-1}(\xi_1,\xi_2,1))^{\frac{n}{n-1}}}{H_n(\xi_1,\xi_2,1)} < +\infty,$$

one asserts that there exists a positive constant C_5 such that

$$\frac{(S_{n-1}(z_1,z_2,z_3))^{\frac{n}{n-1}}}{H_n(z_1,z_2,z_3)} \le C_5 \text{ for all } z_1,z_2,z_3 \ge 0.$$

Hence the functional L satisfies the differential inequality

$$L'(t) \le C_6 L(t) + C_7 L^{\frac{n-1}{n}}(t),$$

which for $Z = L^{\frac{1}{n}}$ can be written as

$$nZ' \le C_6 Z + C_7.$$

A simple integration gives a uniform bound of the functional L on the interval $[0,T]$. This completes the proof of Theorem 1. □

Corollary 1. *Suppose that the functions $f(r_1,r_2,r_3)$, $g(r_1,r_2,r_3)$ and $h(r_1,r_2,r_3)$ are continuously differentiable on Σ, point into Σ on $\partial\Sigma$ and satisfy the condition* (1.7)*. Then all uniformly bounded on Ω solutions of* (1.1)–(1.5) *with the initial data in Σ are in $L^\infty(0,T;L^p(\Omega))$ for all $p \ge 1$.*

Proof. The proof of this Corollary is an immediate consequence of Theorem 1, the trivial inequality

$$\int\limits_\Omega (z_1+z_2+z_3)^p\,dx \le L(t) \quad \text{on} \quad [0,T^*[,$$

and (2.6a). □

Proposition 2. *Under the hypothesis of Corollary* 1*, if $f(r_1,r_2,r_3)$, $g(r_1,r_2,r_3)$ and $h(r_1,r_2,r_3)$ are polynomially bounded, then all uniformly bounded on Ω solutions of* (1.1)–(1.4) *with the initial data in Σ are global in time.*

Proof. As has been mentioned above, it suffices to derive a uniform estimate of $\|F_1(z_1,z_2,z_3)\|_p$, $\|F_2(z_1,z_2,z_3)\|_p$ and $\|F_3(z_1,z_2,z_3)\|_p$ on $[0,T]$, $T<T^*$ for some $p>\frac{N}{2}$. Since the reactions $f(u,v,w)$, $g(u,v,w)$ and $h(u,v,w)$ are polynomially bounded on Σ, by using relations (2.6a) and (2.7a) we get that so are $F_1(z_1,z_2,z_3)$, $F_2(z_1,z_2,z_3)$ and $F_3(z_1,z_2,z_3)$, and the proof becomes an immediate consequence of Corollary 1. □

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POSITIVE PERIODIC SOLUTIONS FOR A NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATION

Abstract. In this paper, sufficient conditions have been obtained for the existence of at least two positive periodic solutions of the Nicholson's Blowflies model

$$x'(t) = -a(t)x(t) + p(t)x^m(t-\tau(t))e^{-\gamma(t)x^n(t-\tau(t))}.$$

The Leggett–Williams multiple fixed point theorem has been used to prove our results.

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$$x'(t) = -a(t)x(t) + p(t)x^m(t-\tau(t))e^{-\gamma(t)x^n(t-\tau(t))}$$

ჰქონდეს სულ მცირე ორი დადებითი პერიოდული ამონახსნი. შედეგების დასამტკიცებლად გამოყენებულია ლექეტ–ვილიამსის მრავალი უძრავი წერტილის თეორემა.

1. Introduction

In this paper, we study the existence of two positive periodic solutions of a nonlinear functional differential equation of the form

$$x'(t) = -a(t)x(t) + p(t)x^m(t-\tau(t))e^{-\gamma(t)x^n(t-\tau(t))}, \tag{1}$$

where a, p, γ and $\tau \in C(R, R^+)$ are T-periodic functions, $m > 1$ and $n > 0$ are reals and T is a positive constant.

If $m = 1$ and $n = 1$, then (1) yields the Nicholson's Blowflies model

$$x'(t) = -a(t)x(t) + p(t)x(t-\tau(t))e^{-\gamma(t)x(t-\tau(t))}. \tag{2}$$

When all the parameters are positive constants, (2) reduces to an original model developed by Gurney et al. [6] to describe the population of Australian sheep-blowfly that agrees well with the experimental data of Nicholson [11]. One may note that the equation explains Nicholson's data of blowfly quite accurately and hence we refer (2) as the Nicholson's Blowflies model.

The variation of the environment plays an important role in many biological and ecological dynamical systems. In particular, the effects of a periodically varying environment are important for evolutionary theories, as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumption of periodicity of parameters of the system (in a way) incorporates the periodicity of the environment (e.g., seasonal effects of weather, food supplies, mating habits, etc.). In fact, it has been suggested by Nicholson [12] that any periodic change of climate tends to improve it's periodicity upon oscillations of internal origin or to cause such oscillations to have a harmonic relation to periodic climate changes. In view of the above fact, it is realistic to assume the periodicity on the parameters or on the coefficient functions of (1) and (2). Thus, the existence of periodic solutions of (1) or (2) are naturally expected.

Many authors have studied the existence of at least one positive periodic solution of (2). For this, one may refer the papers in [5], [7], [16], [23], [24], [27]–[29]. Krasnoselskiĭ fixed point theorem [3] have been used to prove the results. Although the existence of at least one periodic solution of (2) is largely studied in the literature, studies on the existence of at least two periodic solutions of (1) and (2) are relatively scarce.

In this paper, we have made an attempt to study the existence of at least two positive periodic solutions of (1). We have used Leggett–Williams multiple fixed point theorem [10] to prove our theorem. This theorem have been used by the authors in [19]–[22] to study the existence of three periodic solution of the following differential equations:

$$x'(t) = -a(t)x(t) + \lambda f(t, x(h(t))),$$

and

$$x'(t) = a(t)x(t) - \lambda f(t, x(h(t))),$$

where λ is a positive parameter. The results obtained for the above equations were applied to (1) with constant coefficients of the form

$$x'(t) = -ax(t) + px^m(t-\tau)e^{-\gamma x^n(t-\tau)}, \tag{3}$$

We state the results obtained in [20], [21] in the form of theorems.

Theorem 1.1 ([20])**.** *Let $m > 1$ and $2e(\delta-1)\delta^{m-1}\gamma^{\frac{(m-1)}{n}} \le 1$. Then the equation (3) has at least three positive T-periodic solutions for $\frac{1}{2T} < p < \frac{1}{T}$.*

Theorem 1.2 ([21])**.** *Assume that $m > 1$ and that*

$$\int_0^T p(t)\,dt > \delta(\delta-1)\Big(\frac{\gamma\delta^2 e}{m-1}\Big)^{m-1}. \tag{4}$$

Then the equation

$$x'(t) = -a(t)x(t) + p(t)x^m(t-\tau(t))e^{-\gamma x(t-\tau(t))} \tag{5}$$

has at least three nonnegative T-periodic solutions, where $\gamma > 0$ is a constant and $\delta = \exp\big(\int_0^T a(s)\,ds\big)$.

For the last two decades, there has been a rich literature on the use of fixed point theorems on the existence of positive solutions of boundary value problems. The existence of periodic solutions of this type equation is closely related to the existence of solutions of general boundary value problems. The ideas in this paper have come from those for general boundary value problem.

In the next section, we will state the well known Leggett–Williams multiple fixed point theorem [10] and then we will apply the theorem to the model (1). The obtained result improves our previous result.

2. Main Results

From the periodicity of the solution and the assumption that x is known on the nonlinear parts of (1), one can construct a Green's Kernel. In fact, (1) is equivalent to

$$x(t) = \int_t^{t+T} G(t,s)p(s)x^m(s-\tau(s))e^{-\gamma(s)x^n(s-\tau(s))}\,ds,$$

where $G(t,s) = \dfrac{e^{\int_t^s a(\theta)\,d\theta}}{e^{\int_0^T a(\theta)\,d\theta}-1}$ is Green's Kernel, which is bounded by

$$\alpha = \frac{1}{\delta-1} \le G(t,s) \le \frac{\delta}{\delta-1} = \beta, \quad \delta = e^{\int_0^T a(\theta)\,d\theta}.$$

The following concept from the Leggett–Williams multiple fixed point theorem [10] is needed. Let X be a Banach space and K be a cone in X.

For $a > 0$, define $K_a = \{x \in K; \|x\| < a\}$. A mapping ψ is said to be a concave nonnegative continuous functional on K if $\psi : K \to [0, \infty)$ is continuous and

$$\psi(\mu x + (1-\mu)y) \geq \mu\psi(x) + (1-\mu)\psi(y), \quad x, y \in K, \quad \mu \in [0,1].$$

Let $b, c > 0$ be constants with K and X as defined above. Define

$$K(\psi, b, c) = \big\{x \in K; \psi(x) \geq b, \|x\| \leq c\big\}.$$

Theorem 2.1 (Leggett–Williams multiple fixed point theorem [10, Theorem 3.3])**.** *Let $X = (X, \|\cdot\|)$ be a Banach space and $K \subset X$ a cone, and $c_4 > 0$ a constant. Suppose there exists a concave nonnegative continuous functional ψ on K with $\psi(u) \leq \|u\|$ for $u \in \overline{K}_{c_4}$ and let $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$ be a continuous compact map. Assume that there are numbers c_1, c_2 and c_3 with $0 < c_1 < c_2 < c_3 \leq c_4$ such that*

(i) $\big\{u \in K(\psi, c_2, c_3);\ \psi(u) > c_2\big\} \neq \phi$ *and* $\psi(Au) > c_2$ *for all* $u \in K(\psi, c_2, c_3)$*;*
(ii) $\|Au\| < c_1$ *for all* $u \in \overline{K}_{c_1}$*;*
(iii) $\psi(Au) > c_2$ *for all* $u \in K(\psi, c_2, c_4)$ *with* $\|Au\| > c_3$*.*

Then A has at least three fixed points u_1, u_2 and u_3 in $\overline{K}_{c_4}$. Furthermore, we have $u_1 \in \overline{K}_{c_1}$, $u_2 \in \{u \in K(\psi, c_2, c_4); \psi(u) > c_2\}$, $u_3 \in \overline{K}_{c_4} \setminus \{K(\psi, c_2, c_4) \cup \overline{K}_{c_1}\}$.

In this article, X will denote the set of continuous T-periodic functions, which forms a Banach space under the norm $\|x\| = \sup_{0 \leq t \leq T} |x(t)|$. Define an operator A on X by

$$(Ax)(t) = \int_t^{t+T} G(t,s)p(s)x(s-\tau(s))e^{-\gamma(s)x(s-\tau(s))}\, ds$$

and a cone K on X by

$$K = \Big\{x \in X; x(t) \geq \frac{1}{\delta}\,\|x\|\Big\}.$$

It is easy to verify that $A(K) \subset K$ and A is a completely continuous operator on K. Further, the existence of a positive periodic solution of (1) is equivalent to the existence of a fixed point of A in K.

According to the localization of the fixed points in Theorem 2.1, one of them is possibly a zero (namely $u_1 \in \overline{K}_{c_1}$). Thus, the operator A has at least two positive fixed points and a zero fixed point as can be easily observed. Accordingly, (1) has two positive T-periodic solutions and a possible trivial solution (if the conditions of Theorem 1 are satisfied).

On the cone K, we define a nonnegative concave functional ψ as

$$\psi(x) = \inf_{0 \leq t \leq T} x(t)$$

and let

$$\gamma = \max_{0\le t\le T} \gamma(t).$$

Now, we are ready to prove our main results in this paper.

Theorem 2.2. *Let $m > 1$, $a(t) > 0$ and $\gamma(t) > 0$ for $t \in R$, and*

$$\int_0^T p(t)\,dt > e(\delta - 1)\delta^{m-1}\gamma^{\frac{m-1}{n}} \tag{6}$$

hold. Then (1) *has at least two positive T-periodic solutions.*

Proof. From

$$\limsup_{x\to\infty} \max_{0\le t\le T} \frac{p(t)x^{m-1}e^{-\gamma(t)x^n}}{a(t)} = 0$$

it follows that there exist constants $0 < \mu_1 < 1$ and $\eta > 0$ such that

$$\frac{p(t)x^m e^{-\gamma(t)x^n}}{a(t)} < \mu_1 x \ \text{ for } \ 0 \le t \le T, \ \ x \ge \eta.$$

Let

$$\mu_2 = \max_{0\le t\le T, 0\le x\le \eta} \frac{p(t)x^m e^{-\gamma(t)x^n}}{a(t)}.$$

Then

$$\frac{p(t)x^m e^{-\gamma(t)x^n}}{a(t)} < \mu_1 x + \mu_2, \ \text{ for } \ x \ge 0 \ \text{ and } \ 0 \le t \le T.$$

Choose $c_4 > 0$ such that

$$c_4 > \max\left\{\frac{\mu_2}{1-\mu_1}, \frac{1}{\gamma^{\frac{1}{n}}}\right\}.$$

Then for $x \in \overline{K}_{c_4}$, we have

$$\begin{aligned}\|Ax\| &\le \sup_{0\le t\le T} \int_t^{t+T} G(t,s)p(s)x^m(s-\tau(s))e^{-\gamma(s)x^n(s-\tau(s))}\,ds \le \\ &\le \sup_{0\le t\le T} \int_t^{t+T} G(t,s)a(s)(\mu_1 x(s-\tau(s)) + \mu_2)\,ds \le \\ &\le \sup_{0\le t\le T} \int_t^{t+T} G(t,s)a(s)(\mu_1\|x\| + \mu_2)\,ds \le \\ &\le \mu_1 c_4 + \mu_2 \le c_4.\end{aligned}$$

Hence $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$. Set $c_2 = \frac{1}{\delta\gamma^{\frac{1}{n}}}$ and $c_3 = \frac{1}{\gamma^{\frac{1}{n}}}$. Clearly $c_2 < \delta c_2 = c_3 \le c_4$. Setting $\phi_0(t) = \phi_0 = \frac{c_2+c_3}{2}$, we have that $\phi_0 \in \{x; x \in$

$K(\psi, c_2, c_3), \psi(x) > c_2\} \neq \phi$. Now, for $x \in K(\psi, c_2, c_3)$ we obtain

$$\psi(Ax) = \min_{0\le t\le T} \int\limits_t^{t+T} G(t,s)p(s)x^m(s-\tau(s))e^{-\gamma(s)x^n(s-\tau(s))} \ge$$

$$\ge \frac{1}{\delta-1} c_2^m e^{-\gamma\delta^n c_2^n} \int\limits_0^T p(s)\,ds > c_2.$$

Hence the condition (i) of Theorem 2.1 is satisfied. Since $m > 1$, we have that

$$\limsup_{x\to 0} \max_{0\le t\le T} \frac{p(t)x^m e^{-\gamma(t)x^n}}{a(t)x} = 0$$

implies that there exists a constant $c_1 \in (0, c_2)$ small enough such that

$$\frac{p(t)x^m e^{-\gamma(t)x^n}}{a(t)x} < 1 \text{ for } 0 \le x \le c_1.$$

Thus for $x \in \overline{K}_{c_1}$, we have

$$\|Ax\| \le \sup_{0\le t\le T} \int\limits_t^{t+T} G(t,s)p(s)x^m(s-\tau(s))e^{-\gamma(s)x^n(s-\tau(s))}\,ds <$$

$$< \sup_{0\le t\le T} \int\limits_t^{t+T} G(t,s)a(s)\|x\|\,ds \le c_1,$$

that is, $A : \overline{K}_{c_1} \to \overline{K}_{c_1}$. Thus the property (ii) of Theorem 2.1 is satisfied.

Finally, for $x \in K(\psi, c_2, c_4)$ with $\|Ax\| > c_3$,

$$c_3 < \|Ax\| \le \frac{\delta}{\delta-1} \int\limits_0^T p(s)x^m(s-\tau(s))e^{-\gamma(s)x^n(s-\tau(s))}\,ds$$

implies that

$$\psi(Ax) \ge \frac{1}{\delta-1} \int\limits_0^T p(s)x^m(s-\tau(s))e^{-\gamma(s)x^n(s-\tau(s))}\,ds >$$

$$> \frac{1}{\delta} c_3 = c_2.$$

This shows that the condition (iii) of Theorem 2.1 is satisfied. By Theorem 2.1, the equation (1) has at least two positive T-periodic solutions. This completes the proof of the theorem. □

The following corollary can be obtained as an immediate consequence of Theorem 2.2.

Corollary 2.3. *If $m > 1$, $a > 0$, $\gamma > 0$ and*

$$pT > e(\delta - 1)\delta^{m-1}\gamma^{\frac{m-1}{n}} \tag{7}$$

hold, then (3) has at least two positive T-periodic solutions, where $\delta = e^{aT}$.

Remark 2.4. The conditions of Theorem 1.1 imply the conditions of Corollary 2.3. However, Corollary 2.3 gives two positive T-periodic solutions where as Theorem 1.1 yields three positive T-periodic solutions. Although the range on p defined in Theorem 1.1 forces us to assume that $pT < 1$ and $2e(\delta - 1)\delta^{m-1}\gamma^{\frac{m-1}{n}} \leq 1$ must hold. On the other hand, the condition (7) is sufficient in corollary 2.3 for the existence of two positive periodic solutions of (1).

In what follows, we prove another theorem on the existence of two positive periodic solutions of (1).

Theorem 2.5. *Let $m > 1$, $a(t) > 0$ and $\gamma(t) > 0$ for $t \in R$, and*

$$\min_{0 \leq t \leq T} \left\{ \frac{p(t)}{a(t)} \right\} > e\delta^{m-1}\gamma^{\frac{m-1}{n}} \tag{8}$$

hold. Then (1) has at least two positive T-periodic solutions.

Proof. Set $c_2 = \frac{1}{\delta\gamma^{\frac{1}{n}}}$ and $c_3 = \frac{1}{\gamma^{\frac{1}{n}}}$. Choose $c_4 > 0$ as in Theorem 2.2. One may proceed as in Theorem 2.2 to prove that $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$. Clearly, $\phi_0 = \phi_0(t) = \frac{c_2+c_3}{2} \in \{x,\ x \in K(\psi, c_2, c_3), \psi(x) > c_2\} \neq 0$. For $x \in K(\psi, c_2, c_3)$, we have

$$\psi(Ax) > \min_{0 \leq t \leq T} \left\{ \frac{p(t)}{a(t)} \right\} c_2^m e^{-\gamma\delta^n c_2^n} \int_t^{t+T} G(t,s)a(s)\, ds > c_2.$$

Choose $c_1 = \frac{1}{\max\{\frac{p(t)}{a(t)}\}^{\frac{1}{m-1}}}$. Using (8) we have $c_1 < c_2$. Now, for $x \in \overline{K}_{c_1}$ we obtain

$$\|Ax\| < \max_{0 \leq t \leq T} \left\{ \frac{p(t)}{a(t)} \right\} c_1^m = c_1.$$

The third condition of Theorem 2.1 is easy to verify and hence we omit it. The theorem is proved. □

The following corollary follows from Theorem 2.5 as a direct application to equation (3).

Corollary 2.6. *Let $m > 1$, $a > 0$, $\gamma > 0$ and*

$$p > ae^{1+(m-1)aT}\gamma^{\frac{m-1}{n}} \tag{9}$$

hold. Then (3) has at least two positive T-periodic solutions.

Remark 2.7. Since $aT < e^{aT} - 1$, Corollary 2.6 gives a better sufficient condition than the one in Corollary 2.3.

3. Conclusion

In this paper, we have been able to find sufficient conditions for the existence of multiple periodic solutions of (1) when $m > 1$. We have not obtained any result concerning the existence of multiple periodic solutions of (1) when $0 \le m \le 1$. As mentioned earlier, many authors [5], [7], [16], [23], [24], [27]–[29] have used Krasnoselskiĭ and other fixed point theorems for the existence of one periodic solution of (1) when $m = 1$, that is, of equation (2). From the literature, it seems that no results have been obtained regarding the existence of multiple periodic solutions of (1) with $0 \le m \le 1$. Thus, it would be interesting to obtain sufficient conditions for the existence of multiple periodic solutions of (1) when $0 \le m \le 1$. This is left as an open problem.

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ON FUNDAMENTAL SOLUTION OF STEADY STATE OSCILLATION EQUATIONS

Abstract. The system of differential equations of steady state oscillations of anisotropic elasticity are considered. By the generalized Fourier transform technique and with the help of the limiting absorbtion principle, we construct maximally decaying at infinity matrices of fundamental solutions explicitly. Their expressions contain surface integral over a certain semi-sphere and a line integral along the edge boundary of the semi-sphere. We investigate near field and far field properties of the fundamental matrices and show that they satisfy the generalized Sommerfeld–Kupradze type radiation conditions at infinity.

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Key words and phrases: Elliptic systems, fundamental solution, steady state oscillations.

რეზიუმე. განიხილება ანიზოტროპული დრეკადობის თეორიის მდგრადი დრეკადი რხევების განტოლებათა სისტემა. განზოგადებული ფურიეს გარდაქმნისა და ზღვრული ქრობის პრინციპის გამოყენებით ცხადი სახით აგებულია ფუნდამენტურ ამონახსნთა მატრიცა, რომლის ელემენტების წარმოდგენაში შედის ზედაპირული ინტეგრალი ერთეულოვან ნახევარსფეროზე და მრუდწირული ინტეგრალი ამ ნახევარსფეროს საზღვარზე, ხოლო საინტეგრო ფუნქციები ჩაწერილია ცხადი სახით ელემენტარული ფუნქციებით, რომლებიც დაკავშირებულია სიმბოლურ მატრიცასთან.

დადგენილია ფუნდამენტურ ამონახსნთა ასიმპტოტიკა. ნაჩვენებია, რომ ფუნდამენტური ამონახსნები აკმაყოფილებს ზომერფელდ-კუპრაძის განზოგადებული სახის გამოსხივების პირობებს.

1. Introduction

Fundamental solutions play an important role in investigation of boundary value problems for partial differential equations.

For isotropic bodies the matrix of fundamental solutions of steady state oscillation equations satisfying the so-called Sommerfeld–Kupradze radiation conditions at infinity is constructed in [5], where it is written explicitly in terms of standard functions.

In the paper, using the generalized Fourier transform method and the limiting absorbtion principle (see [1]), we represent the fundamental solution of steady state oscillation equations of anisotropic elasticity under the assumption that the characteristic surfaces satisfy some specific restrictions.

The fundamental solution is constructed by means of surface and curvilinear integrals. In the surface integral the integration manifold is a hemisphere, while in the curvilinear integral the integration line is a unit circumference. On the basis of these representations we define the generalized Sommerfeld–Kupradze radiation conditions in anisotropic elasticity. Similar results can be found in the references [2], [3], [6]–[9].

2. Representation of the Fundamental Solution

2.1. **Equations.** The homogeneous system of differential equations of steady state oscillations of anisotropic elasticity reads as follows (see, e.g., [6], [7])

$$\mathbb{C}(\partial,\omega)u := C(\partial)u + \omega^2 u = c_{kjpq}\partial_j\partial_q u_p + \omega^2 u = 0, \tag{2.1}$$

where $u = (u_1, u_2, u_3)^\top$ is the displacement vector (amplitude), $\omega > 0$ is the oscillation (frequency) parameter,

$$\mathbb{C}(\partial,\omega) := C(\partial) + \omega^2 I_3 = \left[c_{kjpq}\partial_j\partial_q + \delta_{kp}\omega^2\right]_{3\times 3},$$
$$C(\partial) = [c_{kjpq}\partial_j\partial_q]_{3\times 3}.$$

Here $\partial_j = \frac{\partial}{\partial x_j}$, I_3 stands for the unit 3×3 matrix, δ_{kp} is the Kroneker delta, the superscript $(\cdot)^\top$ denotes transposition, c_{kjpq} are elastic constants

$$c_{kjpq} = c_{jkpq} = c_{pqkj}, \quad k,j,p,q = 1,2,3.$$

Let $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse generalized Fourier transform in the space of tempered distributions (Schwartz space $S'(\mathbb{R}^3)$) which for regular summable functions f and g read as follows

$$\mathcal{F}_{x\to\xi}[f] = \int\limits_{\mathbb{R}^3} f(x)e^{ix\cdot\xi}dx, \quad \mathcal{F}^{-1}_{\xi\to x}[g] = \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} g(\xi)e^{-ix\cdot\xi}d\xi,$$

where $x = (x_1, x_2, x_3)$, $\xi = (\xi_1, \xi_2, \xi_3)$ and $x\cdot\xi = x_k\xi_k$. Note that for an arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $f \in S'(\mathbb{R}^3)$

$$\mathcal{F}[\partial^\alpha f] = (-i\xi)^\alpha \mathcal{F}[f], \quad \mathcal{F}^{-1}[\xi^\alpha g] = (i\partial)^\alpha \mathcal{F}^{-1}[g].$$

Denote by $\Psi(x.\omega)$ the matrix of fundamental solutions of the operator $\mathbb{C}(\partial,\omega)$

$$\mathbb{C}(\partial,\omega)\Psi(x,\omega)=I_3\delta(x).$$

Here $\delta(\cdot)$ is the Dirac's delta distribution. By standard arguments we can show that

$$\Psi(x,\omega)=\mathcal{F}^{-1}[\mathbb{C}^{-1}(-i\xi,\omega)]=\mathcal{F}^{-1}\left[\frac{\mathbb{C}^*(-i\xi,\omega)}{H(\xi,\omega)}\right]=$$
$$=N(\partial_x,\omega)\mathcal{F}^{-1}\left[\frac{1}{H(\xi,\omega)}\right]=N(\partial_x,\omega)\Gamma(x,\omega), \tag{2.2}$$

where $\mathbb{C}^{-1}(-i\xi,\omega)$ is the inverse to the symbol matrix $\mathbb{C}(-i\xi,\omega)$, $\mathbb{C}^*(-i\xi,\omega)$ is the corresponding matrix of cofactors, $H(\xi,\omega)\ :=\ \det\mathbb{C}(-i\xi,\omega)$, $N(\partial_x,\omega)=[N_{kj}(\partial_x,\omega)]_{3\times3}$ is the formally adjoint matrix to the matrix $\mathbb{C}(\partial,\omega)$, i.e.,

$$N(\partial_x,\omega)\mathbb{C}(\partial,\omega)=\mathbb{C}(\partial,\omega)N(\partial_x,\omega)=H(x,\omega)I_3.$$

It is clear that N_{kj} is a nonhomogeneous differential operator of order 4 containing 0th, 2nd and 4th order differential operators.

Assume that for any $\eta\in\Sigma_1$, where

$$\Sigma_1=\{\eta\in\mathbb{R}^3\mid|\eta|=1\},$$

the equation $H(\xi,\omega)=0$ written in spherical coordinates

$$\xi_1=\rho\cos\varphi\sin\theta,$$
$$\xi_2=\rho\sin\varphi\sin\theta,$$
$$\xi_3=\rho\cos\theta,\quad 0\le\theta\le\pi,\quad 0\le\varphi<2\pi,\quad \rho=\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}=|\xi|,$$

has three different roots t_1, t_2, t_3 with respect to $t=\frac{\rho^2}{\omega^2}$, so

$$H(\xi,\omega)=-a(\eta)\prod_{j=1}^{3}(\rho^2-\omega^2\mu_j^2(\eta)),$$

where $t_j=\mu_j^2(\eta)$, $j=1,2,3$, and

$$a(\eta)=[\mu_1^2(\eta)\mu_2^2(\eta)\mu_3^2(\eta)]^{-1},\quad \eta\in\Sigma_1;\quad \mu_j(-\eta)=\mu_j(\eta),\quad a(-\eta)=a(\eta).$$

It is clear that

$$\mathbb{C}(-i\xi,\omega)=-C(\xi)+I_3\omega^2,$$

where $C(\xi)=[c_{kjpq}\xi_k\xi_j]_{3\times3}$ and $C(\xi)$ is a positive definite matrix, which means that there exists $\delta>0$ such that

$$C(\xi)a\cdot\overline{a}\ge\delta|a|^2|\xi|^2\quad\text{for all }\ a\in\mathbb{C}^3.$$

Note that $a(\eta)=\det C(\eta)\ge\delta_1>0$, $\eta\in\Sigma_1$, and $H(-\xi,\omega)=H(\xi,\omega)$.

Lemma 2.1. *Let $\tau=\omega+i\varepsilon$ with $\varepsilon\neq0$ and $\omega>0$. Then*

$$H(\xi,\tau)=\det(-i\xi,\tau)\neq0\quad\textit{for all }\ \xi\in\mathbb{R}^3\setminus\{0\}.$$

Proof. Assume that $H(\xi,\tau)=0$ for some $\xi\in\mathbb{R}^3\setminus\{0\}$ and a complex τ. There exists $a_0\in\mathbb{C}^3$, $a_0\neq 0$, such that

$$\mathbb{C}(-i\xi,\tau)a_0=-C(\xi)a_0+\tau^2a_0=0.$$

Multiplying the last equation by $\overline{a}_0$ (in scalar sense) we have

$$\tau^2|a_0|^2=C(\xi)a_0\cdot\overline{a}_0,$$

or

$$\tau^2=\frac{1}{|a_0|^2}\,C(\xi)a_0\cdot\overline{a}_0>0$$

due to the positive definiteness of $C(\xi)$. But τ is a complex number. This contradiction completes the proof. □

2.2. **Fundamental solution of pseudooscillation.** First we consider the situation of complex $\tau=\omega+i\varepsilon$, $\varepsilon\neq 0$ instead of $\omega>0$ and construct the fundamental solution of the corresponding system of pseudooscillation.

Theorem 2.2. *The fundamental solution of* (2.1) *for a complex* $\tau=\omega+i\varepsilon$ *have the following form:*

$$\underset{(\varepsilon>0)}{\Psi(x,\tau)}=N(\partial_x,\tau)\left[-\frac{i}{16\pi^2\tau^3}\int\limits_{\Sigma_1}\left\{\sum_{q=1}^{3}\frac{e^{i|(x\cdot\eta)|\tau\mu_q}\mu_q}{\prod\limits_{j=1,\,j\neq q}^{3}(\mu_q^2-\mu_j^2)}\right\}\frac{d\Sigma_1}{a(\eta)}\right],\qquad(2.3)$$

or

$$\underset{(\varepsilon<0)}{\Psi(x,\tau)}=N(\partial_x,\tau)\left[\frac{i}{16\pi^2\tau^3}\int\limits_{\Sigma_1}\left\{\sum_{q=1}^{3}\frac{e^{-i|(x\cdot\eta)|\tau\mu_q}\mu_q}{\prod\limits_{j=1,\,j\neq q}^{3}(\mu_q^2-\mu_j^2)}\right\}\frac{d\Sigma_1}{a(\eta)}\right].\qquad(2.4)$$

Proof. Taking a complex $\tau=\omega+i\varepsilon$, $\varepsilon\neq 0$, we have $H(\xi,\tau)\neq 0$ due to Lemma 2.1 and

$$\Gamma(x,\tau)=\mathcal{F}^{-1}[H^{-1}(\xi,\tau)]=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}\frac{e^{-ix\cdot\xi}}{H(\xi,\tau)}\,d\xi\quad(\text{cf. }(2.2)).$$

It is easy to check that

$$\Gamma(x,\tau)=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}\frac{e^{-ix\cdot\xi}}{H(\xi,\tau)}\,d\xi=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}\frac{e^{ix\cdot\xi}}{H(-\xi,\tau)}\,d\xi=$$
$$=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}\frac{e^{ix\cdot\xi}}{H(\xi,\tau)}\,d\xi.$$

Taking into account that $x\cdot\xi=|x|\cdot\rho\cos\gamma=(x\cdot\eta)\rho$, $\cos\gamma=\left(\frac{x}{|x|}\cdot\eta\right)=\left(\frac{x}{|x|}\cdot\frac{\xi}{|\xi|}\right)$, we have

$$\Gamma(x,\tau)=(2\pi)^{-3}\int\limits_{\Sigma_1}\int\limits_{0}^{\infty}\left\{\frac{e^{-i|x|\rho\cos\gamma}\rho^2d\rho\,d\Sigma_1}{-a(\eta)\prod\limits_{j=1}^{3}[\rho-\tau\mu_j(\eta)][\rho+\tau\mu_j(\eta)]}\right\}=$$

$$= (2\pi)^{-3} \int\limits_{\Sigma_1} \int\limits_0^\infty \left\{ \frac{e^{i|x|\rho\cos\gamma}\rho^2 d\rho\, d\Sigma_1}{-a(\eta) \prod\limits_{j=1}^3 [\rho - \tau\mu_j(\eta)][\rho + \tau\mu_j(\eta)]} \right\} =$$

$$= -(2\pi)^{-3} \int\limits_{\Sigma_1} \frac{d\Sigma_1}{a(\eta)} \left\{ \int\limits_0^\infty \frac{e^{\pm i|x|\rho\cos\gamma}}{\prod\limits_{j=1}^3 [\rho - \tau\mu_j(\eta)][\rho + \tau\mu_j(\eta)]}\, \rho^2\, d\rho \right\}. \quad (2.5)$$

From (2.5) we can write that

$$\Gamma(x,\tau) = -\frac{1}{2(2\pi)^3} \int\limits_{\Sigma_1} \frac{d\Sigma_1}{a(\eta)} \left\{ \int\limits_0^\infty \frac{e^{i(x\cdot\eta)\rho}}{\prod\limits_{j=1}^3 [\rho^2 - \tau^2\mu_j^2(\eta)]}\, \rho^2\, d\rho + \right.$$

$$\left. + \int\limits_0^\infty \frac{e^{-i(x\cdot\eta)\rho}}{\prod\limits_{j=1}^3 [\rho^2 - \tau^2\mu_j^2(\eta)]}\, \rho^2\, d\rho \right\}. \quad (2.6)$$

Taking into account

$$\int\limits_0^\infty \frac{e^{-i(x\cdot\eta)\rho}}{\prod\limits_{j=1}^3 [\rho^2 - \tau^2\mu_j^2(\eta)]}\, \rho^2\, d\rho = \begin{bmatrix} \rho = -r \\ d\rho = -dr \end{bmatrix} =$$

$$= \int\limits_0^{-\infty} \frac{e^{i(x\cdot\eta)r}}{\prod\limits_{j=1}^3 [r^2 - \tau^2\mu_j^2(\eta)]}\, r^2\, (-dr) = \int\limits_{-\infty}^0 \frac{e^{i(x\cdot\eta)r}}{\prod\limits_{j=1}^3 [r^2 - \tau^2\mu_j^2(\eta)]}\, r^2\, dr,$$

(2.6) can be rewritten as

$$\Gamma(x,\tau) = -\frac{1}{2(2\pi)^3} \int\limits_{\Sigma_1} \frac{d\Sigma_1}{a(\eta)} \int\limits_{-\infty}^\infty \frac{e^{i(x\cdot\eta)\rho}\rho^2\, d\rho}{\prod\limits_{j=1}^3 [\rho - \tau\mu_j(\eta)][\rho + \tau\mu_j(\eta)]}\, . \quad (2.7)$$

Assume that ρ is a complex variable $\rho = \rho' + i\rho''$, $\tau\mu_j(\eta) = \omega\mu_j(\eta) + i\varepsilon\mu_j(\eta)$, $\varepsilon \neq 0$, $j = 1, 2, 3$.

In (2.7) the integrand is an analytic function with respect to ρ and (see Fig. 2.1)

$$\int\limits_{-\infty}^\infty \left\{ \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^3 [\rho - \tau\mu_j(\eta)][\rho + \tau\mu_j(\eta)]} \right\} d\rho = \int\limits_{\ell_\varepsilon} \left\{ \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^3 [\rho - \tau\mu_j(\eta)][\rho + \tau\mu_j(\eta)]} \right\} d\rho,$$

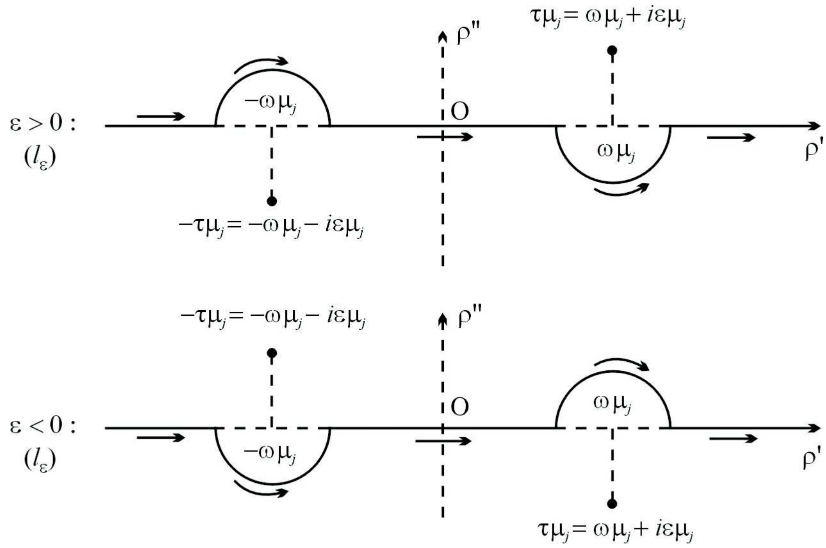


FIGURE 2.1.

or

$$\Gamma(x,\tau) = -\frac{1}{2(2\pi)^3}\int\limits_{\Sigma_1}\frac{d\Sigma_1}{a(\eta)}\int\limits_{\ell_\varepsilon}\frac{e^{i(x\cdot\eta)\rho}\rho^2\,d\rho}{\prod\limits_{j=1}^{3}[\rho-\tau\mu_j(\eta)][\rho+\tau\mu_j(\eta)]}\,. \tag{2.8}$$

Let us denote by C_R^+ and C_R^- the upper and the lower half-part of the circumference with radius $R \gg 1$ on the plane $0\rho'\rho''$. If $(x\cdot\eta)\geq 0$, then $i(x\cdot\eta)\rho = i(x\cdot\eta)\rho' - (x\cdot\eta)\rho''$ and in this case $\mathrm{Re}\{i(x\cdot\eta)\rho\}\leq 0$.

Clearly, for $(x\cdot\eta)\geq 0$

$$\int\limits_{C_R^+}\frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}[\rho^2-\tau^2\mu_j^2(\eta)]}\,d\rho \to 0 \quad \text{as} \quad R\to+\infty,$$

because the integrand is $O(\rho^{-4})$.

Similarly, if $(x\cdot\eta)\leq 0$, then $\mathrm{Re}\,\{i(x\cdot\eta)\rho\}_{\rho\in C_R^-}\leq 0$ and

$$\int\limits_{C_R^-}\frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}[\rho^2-\tau^2\mu_j^2(\eta)]}\,d\rho \to 0 \quad \text{as} \quad R\to+\infty.$$

We have the following situations:

a) $(x \cdot \eta) \geq 0$, $\varepsilon > 0$;

$$\int\limits_{\ell_{3,R}\circlearrowleft} + \int\limits_{C_R^+\circlearrowleft} + \int\limits_{C_{j,\delta}\circlearrowleft} = 0.$$

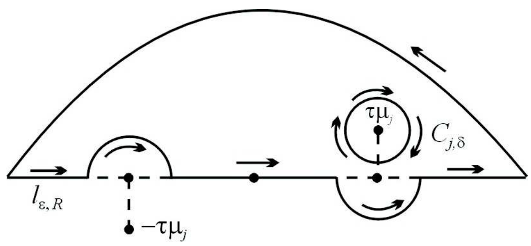


FIGURE 2.2.

Choosing $\delta > 0$ sufficiently small and taking limit as $R \to +\infty$, we get

$$\int\limits_{\ell_\varepsilon} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2 - \tau^2\mu_j^2)}\, d\rho = \sum_{q=1}^{3} \int\limits_{C_{q,\delta}(\tau\mu_q)\circlearrowleft} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2 - \tau^2\mu_j^2)}\, d\rho. \tag{2.9}$$

b) $(x \cdot \eta) \leq 0$, $\varepsilon > 0$;

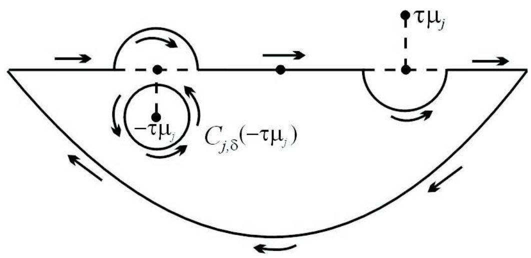


FIGURE 2.3.

$$\int\limits_{\ell_\varepsilon} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2 - \tau^2\mu_j^2)}\, d\rho = -\sum_{q=1}^{3} \int\limits_{C_{q,\delta}(-\tau\mu_q)\circlearrowleft} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2 - \tau^2\mu_j^2)}\, d\rho. \tag{2.10}$$

c) $(x \cdot \eta) \geq 0$, $\varepsilon < 0$;

$$\int\limits_{\ell_\varepsilon} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2 - \tau^2\mu_j^2)}\, d\rho = \sum_{q=1}^{3} \int\limits_{C_{q,\delta}(-\tau\mu_q)\circlearrowleft} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2 - \tau^2\mu_j^2)}\, d\rho. \tag{2.11}$$

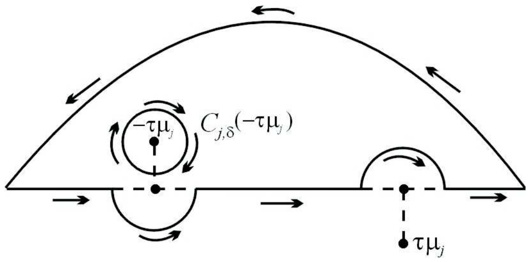


FIGURE 2.4.

d) $(x \cdot \eta) \le 0$, $\varepsilon < 0$;

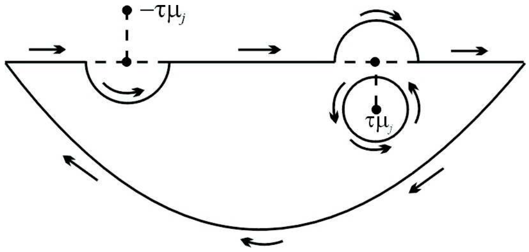


FIGURE 2.5.

$$\int\limits_{\ell_\varepsilon} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2-\tau^2\mu_j^2)}\,d\rho = -\sum_{q=1}^{3}\int\limits_{C_{q,\delta}(\tau\mu_q)} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2-\tau^2\mu_j^2)}\,d\rho. \tag{2.12}$$

In what follows, we use the following notation (see Fig. 2.6)

$$\Sigma_x^+ = \{\eta \in \Sigma_1 \ : \ (x\cdot\eta) \ge 0\},$$
$$\Sigma_x^- = \{\eta \in \Sigma_1 \ : \ (x\cdot\eta) \le 0\}.$$

From the relations (2.9)–(2.12) and (2.8) we conclude that for $\varepsilon > 0$

$$\Gamma(x,\tau) = -\frac{1}{2(2\pi)^3}\Bigg[\int\limits_{\Sigma_x^+}\Bigg\{\sum_{q=1}^{3}\int\limits_{C_{q,\delta}(\tau\mu_q)} \frac{e^{i(x\cdot\eta)\rho}\rho^2\,d\rho}{\prod\limits_{j=1}^{3}(\rho^2-\tau^2\mu_j^2)}\Bigg\}\frac{d\Sigma_1}{a(\eta)} - \int\limits_{\Sigma_x^-}\Bigg\{\sum_{q=1}^{3}\int\limits_{C_{q,\delta}(-\tau\mu_q)} \frac{e^{i(x\cdot\eta)\rho}\rho^2\,d\rho}{\prod\limits_{j=1}^{3}(\rho^2-\tau^2\mu_j^2)}\Bigg\}\frac{d\Sigma_1}{a(\eta)}\Bigg], \tag{2.13}$$

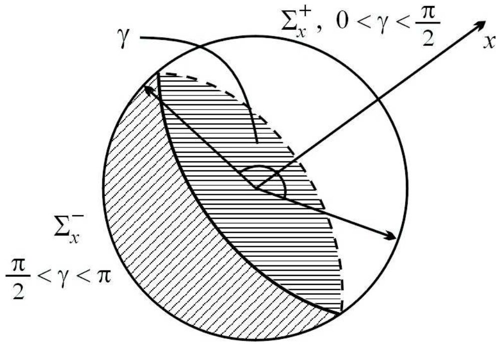


FIGURE 2.6.

and for $\varepsilon < 0$

$$\Gamma(x,\tau) = -\frac{1}{2(2\pi)^3}\left[\int\limits_{\Sigma_x^+}\left\{\sum_{q=1}^{3}\int\limits_{C_{q,\delta}(-\tau\mu_q)\circlearrowleft}\frac{e^{i(x\cdot\eta)\rho}\rho^2\,d\rho}{\prod\limits_{j=1}^{3}(\rho^2-\tau^2\mu_j^2)}\right\}\frac{d\Sigma_1}{a(\eta)}\right]. \tag{2.14}$$

Using the Cauchy integral formula, we can write

$$\int\limits_{C_{q,\delta}(\tau\mu_q)\circlearrowleft}\frac{e^{i(x\cdot\eta)\rho}\rho^2\,d\rho}{\prod\limits_{j=1}^{3}(\rho^2-\tau^2\mu_j^2)} = 2\pi i\frac{e^{i(x\cdot\eta)\tau\mu_q}\tau^2\,\mu_q^2}{\prod\limits_{j=1}^{3}[\tau\mu_q+\tau\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[\tau\mu_q-\tau\mu_j]}\,;$$

$$\int\limits_{C_{q,\delta}(-\tau\mu_q)\circlearrowleft}\frac{e^{i(x\cdot\eta)\rho}\rho^2\,d\rho}{\prod\limits_{j=1}^{3}(\rho^2-\tau^2\mu_j^2)} = 2\pi i\frac{e^{-i(x\cdot\eta)\tau\mu_q}\tau^2\,\mu_q^2}{\prod\limits_{j=1}^{3}[-\tau\mu_q-\tau\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[-\tau\mu_q+\tau\mu_j]}\,.$$

Due to these relations, we can rewrite (2.13) and (2.14) as follows

$$\underset{\varepsilon>0}{\Gamma(x,\tau)} = -\frac{i}{8\pi^2}\left[\int\limits_{\Sigma_x^+}\left\{\sum_{q=1}^{3}\frac{e^{i(x\cdot\eta)\tau\mu_q}(\tau^2\,\mu_q^2)}{\prod\limits_{j=1}^{3}[\mu_q+\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[\mu_q-\mu_j]\tau^5}\right\}\frac{d\Sigma_1}{a(\eta)}-\right.$$
$$\left.-\int\limits_{\Sigma_x^-}\left\{\sum_{q=1}^{3}\frac{e^{-i(x\cdot\eta)\tau\mu_q}(\tau^2\,\mu_q^2)}{\prod\limits_{j=1}^{3}[-\mu_q-\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[-\mu_q+\mu_j]\tau^5}\right\}\frac{d\Sigma_1}{a(\eta)}\right] \tag{2.15}$$

and

$$\underset{\varepsilon<0}{\Gamma(x,\tau)} = -\frac{i}{8\pi^2}\left[\int\limits_{\Sigma_x^+}\left\{\sum_{q=1}^{3}\frac{e^{-i(x\cdot\eta)\tau\mu_q}(\tau^2\,\mu_q^2)}{\prod\limits_{j=1}^{3}[-\mu_q-\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[-\mu_q+\mu_j]\tau^5}\right\}\frac{d\Sigma_1}{a(\eta)}-\right.$$

$$- \int\limits_{\Sigma_x^-} \left\{ \sum_{q=1}^{3} \frac{e^{i(x\cdot\eta)\tau\mu_q}(\tau^2\,\mu_q^2)}{\prod\limits_{j=1}^{3}[\mu_q+\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[\mu_q-\mu_j]\tau^5} \right\} \frac{d\Sigma_1}{a(\eta)} \Bigg]. \tag{2.16}$$

Clearly, (2.15) and (2.16) decay at infinity faster than any negative power of $|x|$.

Taking into account (2.15) and (2.16), we get

$$\underset{\varepsilon>0}{\Gamma(x,\tau)} = -\frac{i}{8\pi^2} \int\limits_{\Sigma_1} \left\{ \sum_{q=1}^{3} \frac{e^{i|(x\cdot\eta)|\tau\mu_q}(\tau^2\,\mu_q^2)}{\prod\limits_{j=1}^{3}[\mu_q+\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[\mu_q-\mu_j]\tau^5} \right\} \frac{d\Sigma_1}{a(\eta)} \tag{2.17}$$

and

$$\underset{\varepsilon<0}{\Gamma(x,\tau)} = \frac{i}{8\pi^2} \int\limits_{\Sigma_1} \left\{ \sum_{q=1}^{3} \frac{e^{-i|(x\cdot\eta)|\tau\mu_q}(\tau^2\,\mu_q^2)}{\prod\limits_{j=1}^{3}[\mu_q+\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[\mu_q-\mu_j]\tau^5} \right\} \frac{d\Sigma_1}{a(\eta)}. \tag{2.18}$$

Finally, from (2.17) and (2.18) we obtain (2.3) and (2.4). □

2.3. **Fundamental solution of steady state oscillation.** Using Theorem 2.2 and limiting procedure, we can prove

Theorem 2.3. *The fundamental solution of* (2.1) *has the following form*

$$\Psi(x,\omega,1) = N(\partial_x,\omega) \int\limits_{\Sigma_x^+} \sum_{q=1}^{3} F_q(\eta) e^{i(x\cdot\eta)\rho_q(\eta)} d\Sigma_1, \tag{2.19}$$

or

$$\Psi(x,\omega,2) = -N(\partial_x,\omega) \int\limits_{\Sigma_x^+} \sum_{q=1}^{3} F_q(\eta) e^{-i(x\cdot\eta)\rho_q(\eta)} d\Sigma_1, \tag{2.20}$$

where

$$F_q(\eta) = -\frac{i}{8\pi^2} \frac{\rho_q(\eta)}{\left\{\prod\limits_{j=1,\,j\neq q}^{3}[\rho_q^2(\eta)-\rho_j^2(\eta)]\right\} a(\eta)}. \tag{2.21}$$

Proof. Taking limit in (2.17) and (2.18) as $|\varepsilon| \to 0$, we get

$$\lim_{\varepsilon\to 0+} \Gamma(x,\tau) = -\frac{i}{16\pi^2\omega^3} \int\limits_{\Sigma_1} \sum_{q=1}^{3} \frac{e^{i|(x\cdot\eta)|\omega\mu_q(\eta)}\mu_q(\eta)}{\prod\limits_{j=1\,j\neq q}^{3}[\mu_q^2(\eta)-\mu_j^2(\eta)]} \frac{d\Sigma_1}{a(\eta)} =$$
$$=: \Gamma(x,\omega,1); \tag{2.22}$$

$$\lim_{\varepsilon\to 0-} \Gamma(x,\tau) = -\frac{i}{16\pi^2\omega^3} \int\limits_{\Sigma_1} \sum_{q=1}^{3} \frac{e^{-i|(x\cdot\eta)|\omega\mu_q(\eta)}\mu_q(\eta)}{\prod\limits_{j=1,\,j\neq q}^{3}[\mu_q^2(\eta)-\mu_j^2(\eta)]} \frac{d\Sigma_1}{a(\eta)} =$$
$$=: \Gamma(x,\omega,2). \tag{2.23}$$

Clearly, $\Gamma(x,\omega,2)=\overline{\Gamma(x,\omega,1)}$.

(2.22) and (2.23) are the formulae similar to those in [4], but they are not identical. Another difference is that (2.22) and (2.23) satisfy the radiation conditions.

We can rewrite (2.22) as

$$\Gamma(x,\omega,1)=-\frac{i}{16\pi^2\omega^3}\Bigg\{\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}\frac{e^{i(x\cdot\eta)\omega\mu_q(\eta)}\mu_q(\eta)}{\prod\limits_{j=1\,j\neq q}^{3}[\mu_q^2(\eta)-\mu_j^2(\eta)]}\,\frac{d\Sigma_1}{a(\eta)}+$$
$$+\int\limits_{\Sigma_x^-}\sum_{q=1}^{3}\frac{e^{-i(x\cdot\eta)\omega\mu_q(\eta)}\mu_q(\eta)}{\prod\limits_{j=1\,j\neq q}^{3}[\mu_q^2(\eta)-\mu_j^2(\eta)]}\,\frac{d\Sigma_1}{a(\eta)}\Bigg\}. \tag{2.24}$$

Using the substitution $\eta=-\widetilde{\eta}$ in the second integral of (2.24), we obtain ($\mu_q(-\eta)=\mu_q(\eta)$, $d\Sigma_{1\eta}=d\Sigma_{1\widetilde{\eta}}$, $\Sigma_x^-\to\Sigma_x^+$, $a(-\eta)=a(\eta)$)

$$\Gamma(x,\omega,1)=-\frac{i}{8\pi^2\omega^3}\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}\frac{e^{i(x\cdot\eta)\omega\mu_q(\eta)}\mu_q(\eta)}{\prod\limits_{j=1\,j\neq q}^{3}[\mu_q^2(\eta)-\mu_j^2(\eta)]}\,\frac{d\Sigma_1}{a(\eta)}\,.$$

$\Gamma(x,\omega,2)$ can be written in a similar form

$$\Gamma(x,\omega,2)=\frac{i}{8\pi^2\omega^3}\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}\frac{e^{-i(x\cdot\eta)\omega\mu_q(\eta)}\mu_q(\eta)}{\prod\limits_{j=1\,j\neq q}^{3}[\mu_q^2(\eta)-\mu_j^2(\eta)]}\,\frac{d\Sigma_1}{a(\eta)}\,.$$

Taking into account the notation (2.21) and the fact that $\rho_\eta(\eta)=\omega\mu_q(\eta)$, $q=1,2,3$, we get

$$\Gamma(x,\omega,1)=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)e^{i(x\cdot\eta)\rho_q}\,d\Sigma_1 \tag{2.25}$$

and

$$\Gamma(x,\omega,2)=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}(-F_q(\eta))e^{-i(x\cdot\eta)\rho_q}\,d\Sigma_1. \tag{2.26}$$

Evidently, (2.25) and (2.26) imply (2.19) and (2.20). □

Denote by S_q the characteristic surface given by the equation $\rho=\rho_q(\eta)$, $\eta\in\Sigma_1$ $(q=1,2,3)$. We assume that S_q is a star-shaped surface with respect to the origin and it is convex; it means that $\xi\cdot\eta(\xi)\geq 0$ for all $\xi\in S_q$, where $n(\xi)$ is the outward unit normal vector at $\xi\in S_q$.

Note that $\eta\rho_q(\eta)=\xi\in S_q$ and

$$\rho_q^2 d\Sigma_1=\left(\frac{\xi}{|\xi|}\cdot n(\xi)\right)dS_q=\frac{1}{\rho_q}\,(\xi\cdot n(\xi))dS_q.$$

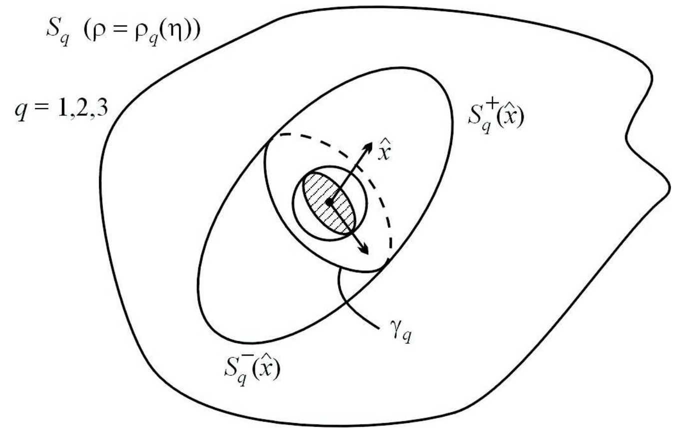


FIGURE 2.7.

Therefore we can rewrite (2.19) and (2.20) in the equivalent form

$$\Psi(x,\omega,1) = N(\partial_x,\omega)\sum_{q=1}^{3}\int\limits_{S_q^+(\hat{x})} \frac{F_q(\eta)e^{i(x\cdot\xi)}(\xi\cdot n(\xi))}{\rho_q^3(\eta)}\,dS_q;$$

$$\Psi(x,\omega,2) = -N(\partial_x,\omega)\sum_{q=1}^{3}\int\limits_{S_q^+(\hat{x})} \frac{F_q(\eta)e^{-i(x\cdot\xi)}(\xi\cdot n(\xi))}{\rho_q^3(\eta)}\,dS_q.$$

3. ASYMPTOTICS

3.1. **Singularity in Vicinity of the Origin.** Let S be a regular surface in $\mathbb{R}^3$. Then

$$\frac{\partial}{\partial S_k(\xi)} = \partial_k(n,\nabla_\xi) = [n\times\nabla_\xi]_k, \quad k=1,2,3,$$

i.e.,

$$\frac{\partial}{\partial S_1(\xi)} = \partial_1(n,\nabla_\xi) = n_2\,\frac{\partial}{\partial\xi_3} - n_3\,\frac{\partial}{\partial\xi_2}\,,$$
$$\frac{\partial}{\partial S_2(\xi)} = \partial_2(n,\nabla_\xi) = n_3\,\frac{\partial}{\partial\xi_1} - n_1\,\frac{\partial}{\partial\xi_3}\,,$$
$$\frac{\partial}{\partial S_3(\xi)} = \partial_3(n,\nabla_\xi) = n_1\,\frac{\partial}{\partial\xi_2} - n_2\,\frac{\partial}{\partial\xi_1}\,,$$

where $\nabla_\xi = \left(\frac{\partial}{\partial\xi_1},\frac{\partial}{\partial\xi_2},\frac{\partial}{\partial\xi_3}\right)$, $n(\xi)$ is the outward unit normal vector at $\xi\in S$ and $\times$ denotes the vector product.

If S is a closed regular surface and f, g are smooth functions, then by the Stokes theorem

$$\int\limits_S [\partial_k(n,\nabla_\xi)f(\xi)]\, f(\xi)\, dS = -\int\limits_S f(\xi)\, [\partial_k(n,\nabla_\xi)g(\xi)]\, dS.$$

Let us consider a special type of the function $\psi_*(\xi)=\psi\left(\frac{\xi}{r}\right)$, where $r=|\xi|$ and $\frac{\xi}{r}=\eta\in\Sigma_1$. We have

$$\begin{aligned}[\nabla_\xi\psi_*(\xi)]_j &= \left[\nabla_\xi\psi\left(\frac{\xi}{r}\right)\right]_j = [\nabla_\xi\psi(\eta)]_j = \frac{\partial}{\partial\xi_j}\,\psi_*(\xi) = \frac{\partial}{\xi_j}\,\psi(\eta) = \\ &= \sum_{p=1}^{3}\frac{\partial\psi(\eta)}{\partial\eta_p}\cdot\frac{\partial\eta_p}{\partial\xi_j} = \sum_{p=1}^{3}\frac{\partial\psi(\eta)}{\partial\eta_p}\,\frac{\partial}{\partial\xi_j}\left[\frac{\xi_p}{r}\right] = \\ &= \sum_{p=1}^{3}\frac{\partial\psi(\eta)}{\partial\eta_p}\left[\frac{\delta_{jp}}{r}-\frac{\xi_p\xi_j}{r^3}\right] = \frac{1}{r}\left[\frac{\partial\psi(\eta)}{\partial\eta_j}-\eta_j(\eta\cdot\nabla_\eta\psi(\eta))\right],\end{aligned}$$

i.e.,

$$\nabla_\xi\psi_*(\xi) = \nabla_\xi\psi(\eta) = \nabla_\xi\psi\left(\frac{\xi}{r}\right) = \frac{1}{r}\left[\nabla_\eta\psi(\eta)-\eta(\eta\cdot\nabla_\eta\psi(\eta))\right]. \qquad (3.1)$$

It follows from (3.1) that for the case of Σ_1 $(\eta=n)$

$$\begin{aligned}\partial_k(n,\nabla_\xi)\psi(\eta) &= \partial_k(n,\nabla_\xi)\psi\left(\frac{\xi}{r}\right) = \partial_k(\eta,\nabla_\xi)\psi(\eta) = \\ &= \frac{1}{r}\,[\eta\times\nabla_\eta\psi(\eta)]_k = \frac{1}{r}\,\partial_k(\eta,\nabla_\eta)\psi(\eta),\end{aligned}$$

or

$$[\eta\times\nabla_\eta\psi(\eta)]_k = \partial_k(\eta,\nabla_\eta)\psi(\eta).$$

Hence

$$\partial_1(\eta,\nabla_\eta)\eta = \begin{bmatrix}0\\ -\eta_3\\ \eta_2\end{bmatrix}, \quad \partial_2(\eta,\nabla_\eta)\eta = \begin{bmatrix}\eta_3\\ 0\\ -\eta_1\end{bmatrix}, \quad \partial_3(\eta,\nabla_\eta)\eta = \begin{bmatrix}-\eta_2\\ \eta_1\\ 0\end{bmatrix}. \qquad (3.2)$$

Let us consider $\psi(\eta)=e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)}$ with $\frac{x}{|x|}=\widehat{x}\in\Sigma_1$, $\eta\in\Sigma_1$, $\lambda=const$ and $\rho(\eta)=\rho_k(\eta)$, $k=1,2,3$. We easily derive

$$\begin{aligned}&\partial_k(\eta,\nabla_\eta)e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)} = \\ &\quad = i\lambda e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)}\left[(\widehat{x}\cdot\partial_k(\eta,\nabla_\eta)\eta)\rho(\eta)+(\widehat{x}\cdot\eta)\partial_k(\eta,\nabla_\eta)\rho(\eta)\right]. \qquad (3.3)\end{aligned}$$

It is evident from (3.2) that

$$\widehat{x}\cdot\partial_1(\eta,\nabla_\eta)\eta = \begin{pmatrix}\widehat{x}_1\\ \widehat{x}_2\\ \widehat{x}_3\end{pmatrix}\cdot\begin{pmatrix}0\\ -\eta_3\\ \eta_2\end{pmatrix} = \widehat{x}_3\eta_2-\widehat{x}_2\eta_3 = [\eta\times\widehat{x}]_1,$$

$$\widehat{x}\cdot\partial_2(\eta,\nabla_\eta)\eta=\begin{pmatrix}\widehat{x}_1\\ \widehat{x}_2\\ \widehat{x}_3\end{pmatrix}\cdot\begin{bmatrix}\eta_3\\ 0\\ -\eta_1\end{bmatrix}=\widehat{x}_1\eta_3-\widehat{x}_3\eta_1=[\eta\times\widehat{x}]_2,$$

$$\widehat{x}\cdot\partial_3(\eta,\nabla_\eta)\eta=\begin{pmatrix}\widehat{x}_1\\ \widehat{x}_2\\ \widehat{x}_3\end{pmatrix}\cdot\begin{bmatrix}-\eta_2\\ \eta_1\\ 0\end{bmatrix}=\widehat{x}_2\eta_1-\widehat{x}_1\eta_2=[\eta\times\widehat{x}]_3,$$

i.e.,

$$(\widehat{x}\cdot\partial_k(\eta,\nabla_\eta)\eta)=[\eta\times\widehat{x}]_k,\quad k=1,2,3. \tag{3.4}$$

Denoting

$$\Phi_k(\widehat{x},\eta)=[\eta\times\widehat{x}]_k\rho(\eta)+(\widehat{x}\cdot\eta)\partial_k(\eta,\nabla_\eta)\rho(\eta)=$$
$$=\eta\times(\widehat{x}\rho(\eta)+(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)),\quad k=1,2,3, \tag{3.5}$$

we can rewrite (3.3) as

$$\partial_k(\eta,\nabla_\eta)e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)}=i\lambda e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)}\Phi_k(\widehat{x},\eta),\quad k=1,2,3. \tag{3.6}$$

Lemma 3.1. *The following conditions are equivalent:*
i) $\Phi(\widehat{x},\eta)=\eta\times[\widehat{x}\rho(\eta)+(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)]\neq 0$*;*
ii) $\widehat{x}\rho(\eta)+(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)\nparallel\eta$*;*
iii) $\eta\times\Phi(\widehat{x},\eta)=-\widehat{x}\rho(\eta)-(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)\neq 0$.

Proof. Since $\sum\limits_{k=1}^{3}\eta_k\partial_k(\eta,\nabla_\eta)\equiv 0$, from (3.5) and (3.6) we obtain

$$\sum_{k=1}^{3}\eta_k\Phi_k(\widehat{x},\eta)\equiv 0,\quad \text{i.e.,}\quad \eta\cdot\Phi(\widehat{x},\eta)\equiv 0,\quad \eta\in\Sigma_1,$$

where $\Phi(\widehat{x},\eta)=(\Phi_1(\widehat{x},\eta),\Phi_2(\widehat{x},\eta),\Phi_1(\widehat{x},\eta))$ and $\eta=(\eta_1,\eta_2,\eta_3)$.

If $\Phi(\widehat{x},\eta)\neq 0$, then this condition is equivalent to $[\eta\times\Phi(\widehat{x},\eta)]\neq 0$.

On the other hand,

$$\Phi(\widehat{x},\eta)\neq 0\Longleftrightarrow\Phi(\widehat{x},\eta)=\eta\times(\widehat{x}\rho(\eta)+(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta))\neq 0,$$

i.e., the vector $\rho(\eta)\widehat{x}+(\widehat{x}\cdot\eta)\nabla_2\rho(\eta)$ is not parallel to η. Thus, i)$\Leftrightarrow$ii).

In the particular case under consideration it is clear that

$$\rho_q(t\eta)=\omega\mu_q(t\eta)=\frac{1}{t}\,\omega\mu_q(\eta)=\frac{1}{t}\,\rho_q(\eta),\quad t>0.$$

The functions $\rho_q(\eta)$, $q=1,2,3$, are homogeneous functions of order (-1) for $\eta\in\Sigma_1$. Ttherefore

$$(\eta\cdot\nabla_\eta\rho(\eta))=-\rho(\eta). \tag{3.7}$$

Taking into account (3.7) and the fact that for arbitrary vectors a,b and c, $a\times[b\times c]=b(a\cdot c)-c(a\cdot b)$, we have

$$\eta\times\Phi(\widehat{x},\eta)=\eta\times\{\eta\times(\widehat{x}\rho(\eta)+(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta))\}=$$
$$=\eta\,\{(\eta\cdot\widehat{x})\rho(\eta)+(\widehat{x}\cdot\eta)(\eta\cdot\nabla_\eta\rho(\eta))\}-(\widehat{x}\rho(\eta)+(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta))=$$
$$=(\eta\cdot\widehat{x})\{\rho(\eta)-\rho(\eta)\}\eta-\{\widehat{x}\rho(\eta)+(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)\}=$$

$$= -\widehat{x}\rho(\eta) - (\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta),$$

hence

$$\eta\Phi(\widehat{x},\eta) = -\widehat{x}\rho(\eta) - (\widehat{x},\eta)\nabla_\eta\rho(\eta).$$

Using (3.4), we conclude that i)⇔iii). □

Note that if $(\widehat{x}\cdot\eta)=0$, then $\widehat{x}\perp\eta$, $|\widehat{x}\times\eta|=1$ and

$$|\Phi(\widehat{x},\eta)| = |\eta\times\widehat{x}|\rho(\eta) = \rho(\eta) > 0,$$

i.e., if $(\widehat{x}\cdot\eta)=0$, then $\Phi(\widehat{x},\eta)\neq 0$.

From Lemma 3.1 we conclude that

$$\Phi(\widehat{x},\eta) = \eta\times[\widehat{x}\rho(\eta) + (\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)] = 0 \Longleftrightarrow \Longleftrightarrow \widehat{x}\rho(\eta) + (\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta) = 0. \quad (3.8)$$

Since

$$\eta\cdot(\widehat{x}\rho(\eta) + (\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)) = (\eta\cdot\widehat{x})\rho(\eta) + (\widehat{x}\cdot\eta)(\eta\cdot\nabla_\eta\rho(\eta)) = = (\eta\cdot\widehat{x})\rho(\eta) - (\widehat{x}\cdot\eta)\rho(\eta) = 0,$$

this means that $\{\widehat{x}\rho(\eta) + (\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)\}\perp\eta$ and

$$|\Phi(\widehat{x},\eta)| = |\widehat{x}\rho(\eta) + (\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)|.$$

The points $\eta\in\Sigma_1$ satisfying the equation (3.8) will be called critical points on Σ_1 corresponding to the direction $\widehat{x}$.

Denote by $\widetilde{S}$ the surface defined by the equation $\rho=\rho(\eta)$. Clearly,

$$\rho(\eta):\Sigma_1\to\widetilde{S}.$$

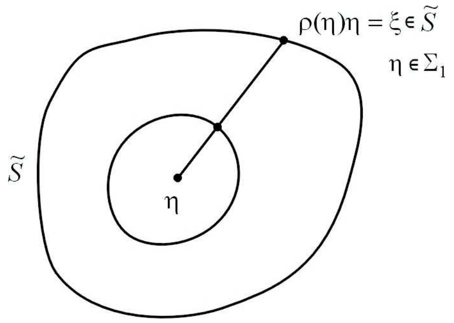


Figure 3.1.

Lemma 3.2. *$\eta_0\in\Sigma_1$ is a critical point corresponding to the direction $\widehat{x}\in\Sigma_1$ if and only if $\eta(\xi_0)=\pm\widehat{x}$, where $\xi_0=\rho(\eta_0)\eta_0\in\widetilde{S}$.*

Proof. Let us consider the function

$$F(\xi) = |\xi| - \rho\left(\frac{\xi}{|\xi|}\right), \quad \xi \in \mathbb{R}^3 \setminus \{0\},$$

where $\rho(\eta)$ is a positive function defined on Σ_1 as a function of η, is differentiable with respect to η and homogeneous of order -1.

It is evident that $F(\xi) = 0$ is an equation for $\widetilde{S}$, i.e., $\widetilde{S}$ is a level surface for the function F. Therefore $\nabla_\xi F(\xi)$ defines the field of outward normal directions on $\widetilde{S}$: $n(\xi) = \left.\dfrac{\nabla_\xi F(\xi)}{|\nabla_\xi F(\xi)|}\right|_{\xi\in\widetilde{S}}$ is the outward unit normal vector to $\widetilde{S}$ at the point $\xi \in \widetilde{S}$.

Elementary calculations show

$$\nabla_\xi F(\xi) = \frac{\xi}{|\xi|} - \nabla_\xi \rho\left(\frac{\xi}{|\xi|}\right) =$$
$$= \frac{\xi}{|\xi|} - \frac{1}{|\xi|}\left[\nabla_\xi \rho(\eta) - \eta(\eta\cdot\nabla_\eta\rho(\eta))\right] = \eta - \frac{1}{|\xi|}\left[\nabla_\eta\rho(\eta) + \eta\rho(\eta)\right].$$

Therefore

$$\nabla_\xi F(\xi)\big|_{\xi\in\widetilde{S}} = \eta - \frac{1}{\rho(\eta)}\left[\nabla_\eta\rho(\eta) + \eta\rho(\eta)\right] = -\frac{1}{\rho(\eta)}\nabla_\eta\rho(\eta).$$

Note that the surface $\widetilde{S} = S_q$, $q = 1,2,3$, are star shape with respect to the origin point 0, i.e., if $n(\xi)$ is the outward unit normal vector to $\widetilde{S}$ at $\xi \in \widetilde{S}$, then $(\eta \cdot n(\xi)) \geq 0$.

Since $(\eta\cdot n(\xi)) = -\dfrac{1}{\nabla_\eta\rho(\eta)}\,(\eta\cdot\nabla_\eta\rho(\eta)) = \dfrac{\rho(\eta)}{|\nabla_\eta\rho(\eta)|} > 0$, we conclude that

$$n(\xi) = -\frac{\nabla_\eta\rho(\eta)}{|\nabla_\eta\rho(\eta)|} \quad \text{for } \xi \in \widetilde{S} \tag{3.9}$$

defines the outward unit normal vector.

If $\eta_0 \in \Sigma_1$ is a critical point corresponding to $\widehat{x} \in \Sigma_1$, then using (3.8) and (3.9) we conclude that $\eta(\xi_0) = \pm\widehat{x}$, where $\xi_0 = \rho(\eta_0)\eta_0 \in \widetilde{S}$.

On the other hand, let $n(\xi_0)\,\|\,\widehat{x}$, i.e., $n(\xi_0) = \pm\widehat{x}$, or due to (3.9) $\widehat{x} = \pm\dfrac{\nabla_\eta\rho(\eta_0)}{|\nabla_\eta\rho(\eta_0)|}$.

Let us write (3.8) for η_0

$$\widehat{x}\rho(\eta_0) + (\widehat{x}\cdot\eta_0)\nabla_\eta\rho(\eta_0) =$$
$$= \pm\frac{1}{|\nabla_\eta\rho(\eta_0)|}\left\{(\nabla_\eta\rho(\eta_0))\rho(\eta_0) + (\nabla_\eta\rho(\eta_0)\cdot\eta_0)\nabla_\eta\rho(\eta_0)\right\} =$$
$$= \pm\frac{1}{|\nabla_\eta\rho(\eta_0)|}\left\{\rho(\eta_0)\nabla_\eta\rho(\eta_0) - \rho(\eta_0)\nabla_\eta\rho(\eta_0)\right\} = 0.$$

Therefore we get that $\Phi(\widehat{x},\eta_0) = 0$, i.e., η_0 is a critical point. □

Remark 3.3. If the surface $\widetilde{S}$ does not contain a plane two-dimensional part (i.e., curvature of the surface $\widetilde{S}$ does not vanish on a subset of $\widetilde{S}$ of positive 2-dimensional measure), then the set of critical points consists of isolated points or lines on $\widetilde{S}$.

Using Lemmas 3.1 and 3.2, one can easily prove the following

Theorem 3.4. i) *If $\eta_0 \in \Sigma_1$ is not a critical point corresponding to the direction $\widehat{x} \in \Sigma_1$, then*

$$\Phi(\widehat{x},\eta) = \eta \times [\rho(\eta)\widehat{x} + (\widehat{x}\cdot\eta)\nabla_\eta \rho(\eta)] \neq 0$$

and

$$|\Phi(\widehat{x},\eta)| = |\rho(\eta)\widehat{x} + (\widehat{x},\eta)\nabla_\eta \rho(\eta)| > 0.$$

ii) *If $(\widehat{x}\cdot\eta) = 0$, then $|\Phi(\widehat{x},\eta)| = \rho(\eta) > 0$.*
iii) *$\Phi(\widehat{x},\eta) = 0$ only at critical points.*

From ii) of Theorem 3.4 it follows

Corollary 3.5. *There exists a neighborhood $U(\delta,\partial\Sigma_x^\pm)$ of the circumference $\partial\Sigma_x^\pm$ with $|\Phi(\widehat{x},\eta)| \geq \delta > 0$ for $\eta \in U(\delta,\partial\Sigma_x^\pm)$.*

Using the Stokes theorem for $f \in C^1(\Sigma_1)$, we can write

$$\int\limits_{\Sigma^*} \partial_k(\eta,\nabla_\eta) f(\eta)\, d\Sigma_1 = \int\limits_{\gamma} f(\eta)\ell_k(\eta)\, d\gamma, \tag{3.10}$$

where $\Sigma^* \subset \Sigma_1$, $\partial\Sigma^* = \gamma$, $n = \eta$ on Σ_1 and $\ell = (\ell_1,\ell_2,\ell_3)$ is the unit tangent vector to γ.

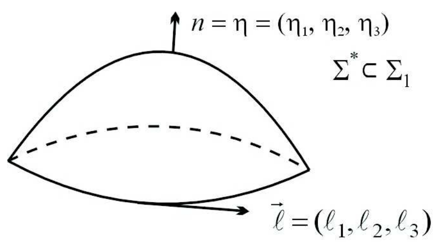


FIGURE 3.2.

As a result, from (3.10) we have

$$\int\limits_{\Sigma^*} [\partial_k(\eta,\nabla_\eta) f(\eta)] g(\eta)\, d\Sigma_1 = $$

$$= -\int\limits_{\Sigma^*} f(\eta)[\partial_k(\eta,\nabla_\eta) g(\eta)]\, d\Sigma_1 + \int\limits_{\gamma} f(\eta) g(\eta) \ell_k(\eta)\, d\gamma. \tag{3.11}$$

If either $f|_\gamma = 0$ or $g|_\gamma = 0$, then

$$\int\limits_{\Sigma^*} [\partial_k(\eta, \nabla_\eta) f(\eta)] g(\eta)\, d\Sigma_1 = -\int\limits_{\Sigma^*} f(\eta)[\partial_k(\eta, \nabla_\eta) g(\eta)]\, d\Sigma_1.$$

Lemma 3.6. *If $\eta \in \Sigma_1$ is not a critical point corresponding to the direction $\widehat{x} \in \Sigma_1$, then*

$$e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)} = \frac{1}{i\lambda} \sum_{k=1}^{3} \frac{\Phi_k(\widehat{x}, \eta)}{|\Phi_k(\widehat{x}, \eta)|} \left[\partial_k(\eta, \nabla_\eta) e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)}\right]. \tag{3.12}$$

Proof. Multiplying both sides of the formula (3.6) by $\Phi_k(\widehat{x}, \eta)$ and summing, we obtain the equation

$$\sum_{k=1}^{3} \Phi_k(\widehat{x}, \eta) \left[\partial_k(\eta, \nabla_\eta) e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)}\right] = i\lambda e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)} |\Phi(\widehat{x}, \eta)|^2 \tag{3.13}$$

(for $\rho(\eta) = \rho_q(\eta)$ we will use the notation $\Phi_k^{(q)}(\widehat{x}, \eta)$ and $\Phi^{(q)}(\widehat{x}, \eta)$).

If η is not a critical point, then $\Phi(\widehat{x}, \eta) \neq 0$, and (3.13) can be rewritten in the form (3.12). □

In what follows, we essentially use the following

Lemma 3.7. *If $\Phi(x) = \int\limits_{\Sigma_x^+} \varphi(x, \eta) d_\eta \Sigma_1$ and $\varphi(\cdot, \eta) \in C^1(\mathbb{R}^3)$, $\Sigma_x^+ = \{\eta \in \Sigma_1 : (x \cdot \eta) \geq 0\}$, then*

$$\frac{\partial \Phi(x)}{\partial x_k} = \int\limits_{\Sigma_x^+} \frac{\partial \varphi(x, \eta)}{\partial x_k} d_\eta \Sigma_1 + \frac{1}{|x|} \int\limits_{\gamma_x} \varphi(x, \eta) \eta_k d_\eta \gamma_x, \tag{3.14}$$

where $\gamma_x = \partial \Sigma_x^+$.

Proof. First let us calculate the derivative of $\Phi(x)$ in the direction $e_0 = (e_{01}, e_{02}, e_{03})$, $|e_0| = 1$,

$$\frac{\partial \Phi(x)}{\partial e_0} = \lim_{t \to 0} \frac{\Phi(x + te_0) - \Phi(x)}{t}.$$

It is clear that

$$\Phi(x + te_0) - \Phi(x) = \int\limits_{\Sigma_{x+te_0}^+} \varphi(x + te_0, \eta) d_\eta \Sigma_1 - \int\limits_{\Sigma_x^+} \varphi(x, \eta) d_\eta \Sigma_1 =$$

$$= \int\limits_{\Sigma_x^+} [\varphi(x + te_0, \eta) - \varphi(x, \eta)] d_\eta \Sigma_1 + \int\limits_{\Sigma_{x+te_0}^+} \varphi(x + te_0, \eta) d_\eta \Sigma_1 -$$

$$- \int\limits_{\Sigma_x^+} \varphi(x + te_0, \eta) d_\eta \Sigma_1 = \int\limits_{\Sigma_x^+} [\varphi(x + te_0, \eta) - \varphi(x, \eta)] d_\eta \Sigma_1 +$$

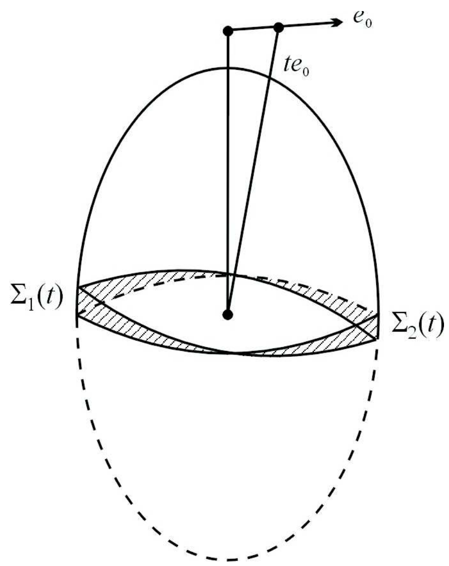


FIGURE 3.3.

$$+ \int\limits_{\Sigma_2(t)} \varphi(x+te_0,\eta)d_\eta\Sigma_1 - \int\limits_{\Sigma_1(t)} \varphi(x+te_0,\eta)d_\eta\Sigma_1. \quad (3.15)$$

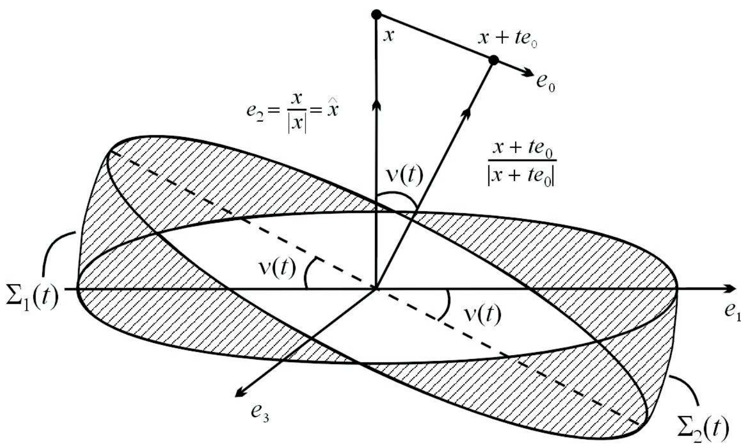


FIGURE 3.4.

$$e_3 = -\frac{\widehat{x}\times e_0}{|\widehat{x}\times e_0|} = \frac{e_0\times\widehat{x}}{|\widehat{x}\times e_0|}; \quad e_2 = \widehat{x}; \quad e_1 = -e_3\times\widehat{x} = e_2\times e_3.$$

From (3.15) we get

$$\frac{1}{t}\left[\Phi(x+te_0)-\Phi(x)\right]=\int\limits_{\Sigma_x^+}\frac{\varphi(x+te_0,\eta)-\varphi(x,\eta)}{t}\,d_\eta\Sigma_1+$$
$$+\frac{1}{t}\int\limits_{\Sigma_2(t)}\varphi(x+te_0,\eta)\,d\Sigma_1-\frac{1}{t}\int\limits_{\Sigma_1(t)}\varphi(x+te_0,\eta)\,d\Sigma_1=$$
$$=\Phi_1(x,t)+\Phi_2(x,t)+\Phi_3(x,t), \quad (3.16)$$

where

$$\Phi_1(x,t)=\int\limits_{\Sigma_x^+}\frac{\varphi(x+te_0,\eta)-\varphi(x,t)}{t}\,d_\eta\Sigma_1,$$
$$\Phi_2(x,t)=\frac{1}{t}\int\limits_{\Sigma_2(t)}\varphi(x+te_0,\eta)\,d\Sigma_1,\quad \Phi_3(x,t)=-\frac{1}{t}\int\limits_{\Sigma_1(t)}\varphi(x+te_0,\eta)\,d\Sigma_1.$$

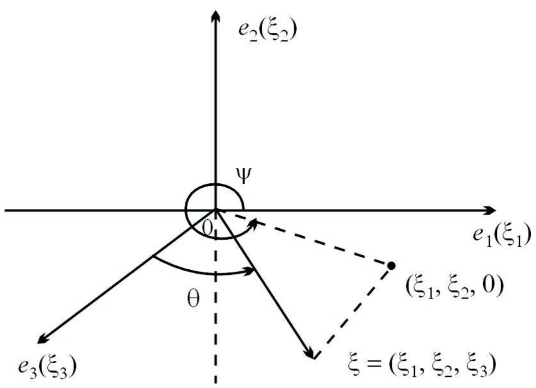
Evidently,

$$\lim_{t\to 0}\Phi_1(x,t)=\int\limits_{\Sigma_x^+}\frac{\partial\varphi(x,\eta)}{\partial e_0(x)}\,d\Sigma_1. \quad (3.17)$$

Let us make an orthogonal transform of the initial system such that $0\xi_1$ coincides with e_1, $0\xi_2$ with e_2 and $0\xi_3$ with e_3 (see Fig. 3.4). Denote by $B := B(x,e)$ the orthogonal matrix of this transform ($B\xi=\eta$)

$$B=\begin{bmatrix} e_{11} & e_{21} & e_{31}\\ e_{12} & e_{22} & e_{32}\\ e_{13} & e_{23} & e_{33}\end{bmatrix},\quad e_k=\begin{bmatrix} e_{k1}\\ e_{k2}\\ e_{k3}\end{bmatrix},\quad k=1,2,3.$$

Using the spherical coordinates, we have



$$\xi_1=|\xi|\cos\psi\sin\theta,$$
$$\xi_2=|\xi|\sin\psi\sin\theta,$$
$$\xi_3=|\xi|\cos\theta,$$

$$0\le\theta\le\pi \quad \text{and}$$
$$0\le\psi<2\pi.$$

Figure 3.5

As it is seen from Fig. 3.4, $2\pi - \nu(t) < \psi < 2\pi$ for $\Sigma_2(t)$ and $\pi - \nu(t) < \psi < \pi$ for $\Sigma_1(t)$, i.e., for both surfaces $0 < \theta < \pi$. Let us estimate the angle $\nu(t)$ ($\nu(t) \geq 0$ is sufficiently small)

$$\cos\nu(t) = \widehat{x}\cdot\frac{x+te_0}{|x+te_0|} = \frac{x\cdot(x+te_0)}{|x|\,|x+te_0|} =$$
$$= \frac{|x|^2 + t(e_0\cdot x)}{|x|\,|x+te_0|} = 1 + \frac{|x|^2 + t(e_0\cdot x) - |x|\,|x+e_0t|}{|x|\,|x+e_0t|} =$$
$$= 1 + \frac{|x|^4 + 2t|x|^2(e_0\cdot x)^2 + t^2(e_0\cdot x)^2 - |x|^2[|x|^2 + 2t(e_0\cdot x) + t^2]}{|x|\,|x+e_0t|[|x|^2 + t(e_0\cdot x) + |x|\,|x+e_0t|]} =$$
$$= 1 - \frac{t^2[|x|^2 - (e_0\cdot x)^2]}{|x|\,|x+e_0t|[|x|^2 + t(e_0\cdot x) + |x|\,|x+e_0t|]}\,,$$

i.e.,

$$2\sin^2\frac{\nu(t)}{2} = t^2\,\frac{|x|^2 - (e_0\cdot x)^2}{|x|\,|x+e_0t|[|x|^2 + t(e_0\cdot x) + |x|\,|x+e_0t|]} =$$
$$= t^2\left\{\frac{|x|^2 - (e_0\cdot x)^2}{2|x|^4}\right\} + O(t^3).$$

Hence

$$\lim_{t\to 0}\frac{\nu(t)}{t} = \sqrt{\frac{|x|^2 - (e_0\cdot x)^2}{|x|^4}}\,. \tag{3.18}$$

If $B\Sigma_2(t) = \widetilde{\Sigma}_2(t)$, then

$$\Phi_2(x,t) = \frac{1}{t}\int\limits_{\Sigma_2(t)}\varphi(x+te_0,\eta)\,d\Sigma_1 = \frac{1}{t}\int\limits_{\widetilde{\Sigma}_2(t)}\varphi(x+te_0,B\xi)\,d\Sigma_1 =$$
$$= \frac{1}{t}\int\limits_{2\pi-\nu(t)}^{2\pi}d\psi\int\limits_0^{\pi}\varphi(x+te_0,B\xi)\sin\theta\,d\theta. \tag{3.19}$$

Using the mean value theorem in (3.19), we obtain

$$\Phi_2(x,t) = \frac{1}{t}\,\nu(t)\int\limits_0^{\pi}\varphi(x+te_0,B\xi')\sin\theta\,d\theta, \tag{3.20}$$

where

$$\begin{aligned}\xi' = (\xi_1',\xi_2',\xi_3')\ :\ &\xi_1' = \cos\psi'\sin\theta,\\ &\xi_2' = \sin\psi'\sin\theta,\\ &\xi_3' = \cos\theta\quad\text{and}\\ &2\pi - \nu(t) \leq \psi' < 2\pi.\end{aligned} \tag{3.21}$$

Similarly, if $B\Sigma_1(t) = \widetilde{\Sigma}_1(t)$, then

$$\Phi_1(x,t) = -\frac{1}{t}\int\limits_{\Sigma_1(t)} \varphi(x+te_0,\eta)\,d\Sigma_1 = -\frac{1}{t}\int\limits_{\widetilde{\Sigma}_1(t)} \varphi(x+te_0,B\xi)\,d\Sigma_1 =$$

$$= -\frac{1}{t}\int\limits_{\pi-\nu(t)}^{\pi} d\psi \int\limits_0^{\pi} \varphi(x+te_0,B\xi)\sin\theta\,d\theta =$$

$$= -\frac{\nu(t)}{t}\int\limits_0^{\pi} \varphi(x+te_0,B\xi'')\sin\theta\,d\theta, \tag{3.22}$$

where

$$\begin{aligned} \xi'' = (\xi_1'',\xi_2'',\xi_3'') \ : \ & \xi_1'' = \cos\psi''\sin\theta, \\ & \xi_2'' = \sin\psi''\sin\theta, \\ & \xi_3'' = \cos\theta, \\ & 2\pi - \nu(t) \le \psi'' < 2\pi. \end{aligned} \tag{3.23}$$

Due to (3.18)–(3.23) we find

$$\lim_{t\to 0}\Phi_2(x,t) = \frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int\limits_0^{\pi}\varphi(x,B\xi_0')\sin\theta\,d\theta, \tag{3.24}$$

where $\xi_0' = \begin{bmatrix}\sin\theta\\ 0\\ \cos\theta\end{bmatrix}$, and

$$\lim_{t\to 0}\Phi_3(x,t) = -\frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int\limits_0^{\pi}\varphi(x,B\xi_0'')\sin\theta\,d\theta, \tag{3.25}$$

where $\xi_0'' = \begin{bmatrix}-\sin\theta\\ 0\\ \cos\theta\end{bmatrix}$.

The substitution $\theta = \pi - \widetilde{\theta}$ in (3.25) leads to

$$\int\limits_0^{\pi}\varphi(x,B\xi_0'')\sin\theta\,d\theta = -\int\limits_{\pi}^{0}\varphi(x,B(-\xi_0'))\sin\theta\,d\theta = \int\limits_0^{\pi}\varphi(x,B\xi_0')\sin\theta\,d\theta.$$

If $\theta = \widetilde{\theta}-\pi$, then $\sin\theta = \sin(\widetilde{\theta}-\pi) = -\sin\widetilde{\theta}$, $\cos\theta = \cos(\widetilde{\theta}-\pi) = -\cos\widetilde{\theta}$, $0 \le \theta \le \pi$, $\pi \le \widetilde{\theta} \le 2\pi$,

$$-B\xi_0' = -B\begin{bmatrix}\sin\theta\\ 0\\ \cos\theta\end{bmatrix} = B\begin{bmatrix}\sin\overline{\theta}\\ 0\\ \cos\overline{\theta}\end{bmatrix} = B\overline{\xi}_0'$$

and

$$\int_0^{\pi} \varphi(x,-B\xi_0')\sin\theta\,d\theta = \int_{\pi}^{2\pi} \varphi(x,-B\overline{\xi}_0')\sin\overline{\theta}\,d\overline{\theta} = \int_{\pi}^{2\pi} \varphi(x,B\xi_0')\sin\theta\,d\theta.$$

Hence from (3.24) and (3.25)

$$\lim_{t\to 0}\Phi_2(x,t) = \frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int_0^{\pi}\varphi\left(x, B\begin{bmatrix}\sin\theta\\0\\\cos\theta\end{bmatrix}\right)\sin\theta\,d\theta, \tag{3.26}$$

$$\lim_{t\to 0}\Phi_3(x,t) = -\frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int_0^{\pi}\varphi\left(x, -B\begin{bmatrix}\sin\theta\\0\\\cos\theta\end{bmatrix}\right)\sin\theta\,d\theta =$$

$$= \frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int_{\pi}^{2\pi}\varphi\left(x, B\begin{bmatrix}\sin\theta\\0\\\cos\theta\end{bmatrix}\right)\sin\theta\,d\theta. \tag{3.27}$$

Note that

$$B\begin{bmatrix}\sin\theta\\0\\\cos\theta\end{bmatrix} = e_1\sin\theta + e_3\cos\theta = \begin{pmatrix}e_{11}\\e_{12}\\e_{13}\end{pmatrix}\sin\theta + \begin{pmatrix}e_{31}\\e_{32}\\e_{33}\end{pmatrix}\cos\theta = \zeta,$$

i.e., $\zeta = e_1\sin\theta + e_3\cos\theta$.

Clearly, $\zeta\in\gamma_x = \partial\Sigma_x^{\pm}$, and when θ varies from 0 to 2π, then ζ moves on γ_x in positive direction. Moreover,

$$\sin\theta = (e_1\cdot\zeta). \tag{3.28}$$

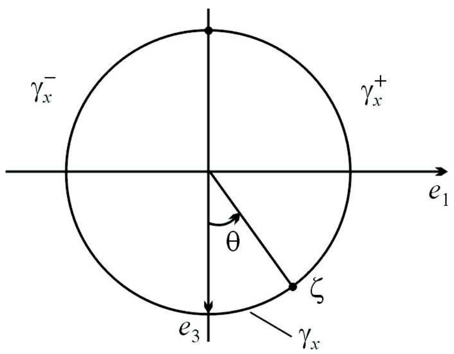


FIGURE 3.6.

Taking into account (3.26)–(3.28), we get

$$\lim_{t\to 0}\Phi_2(x,t) = \frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int_0^{\pi}\varphi(x,\zeta)(e_1\cdot\zeta)\,d\theta, \tag{3.29}$$

$$\lim_{t\to 0}\Phi_3(x,t)=\frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int\limits_{\pi}^{2\pi}\varphi(x,\zeta)(e_1\cdot\zeta)\,d\theta. \tag{3.30}$$

$d\theta=d\gamma_x$ on $\gamma_x=\partial\Sigma_x^{\pm}$ (see Fig. 3.6), and hence

$$\int\limits_0^{\pi}\varphi(x,\zeta)(e_1\cdot\zeta)\,d\theta=\int\limits_{\gamma_x^+}\varphi(x,\zeta)(e_1\cdot\zeta)\,d_\zeta\gamma_x \tag{3.31}$$

$$\int\limits_{\pi}^{2\pi}\varphi(x,\zeta)(e_1\cdot\zeta)\,d\theta=\int\limits_{\gamma_x^-}\varphi(x,\zeta)(e_1\cdot\zeta)\,d_\zeta\gamma_x. \tag{3.32}$$

Using (3.16), (3.17) and (3.29)–(3.32), we get

$$\begin{aligned}\frac{\partial\Phi}{\partial e_0}=\lim_{t\to 0}\frac{1}{t}[\Phi(x+te_0)-\Phi(x)]=\int\limits_{\Sigma_x^+}\frac{\partial\varphi(x,\eta)}{\partial e_0}\,d_\eta\Sigma_1+\\ +\frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int\limits_{\gamma_x^+}\varphi(x,\zeta)(e_1\cdot\zeta)\,d_\zeta\gamma_x.\end{aligned} \tag{3.33}$$

Note that the vector $e_0=(\delta_{1k},\delta_{2k},\delta_{3k})$ corresponds to $\frac{\partial}{\partial x_k}$:

I) If $e_0=(1,0,0)\sim\frac{\partial}{\partial x_1}$, then

$$[e_0\times\widehat{x}]=\begin{vmatrix} i & j & k\\ 1 & 0 & 0\\ \widehat{x}_1 & \widehat{x}_2 & \widehat{x}_3\end{vmatrix}=(0,-\widehat{x}_3,\widehat{x}_2)^\top,\quad r_1=\sqrt{\widehat{x}_2^2+\widehat{x}_3^2},$$

$$e_2^{(1)}=(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3),\quad e_3^{(1)}=\frac{e_0\times\widehat{x}}{|e_0\times\widehat{x}|}=e_3^{(1)}=\frac{1}{r_1}(0,-\widehat{x}_3,\widehat{x}_2),$$

$$e_1^{(1)}=[e_2^{(i)}\times e_3^{(1)}]=\begin{vmatrix} i & j & k\\ \widehat{x}_1 & \widehat{x}_2 & \widehat{x}_3\\ 0 & -\frac{\widehat{x}_3}{r_1} & -\frac{\widehat{x}_2}{r_1}\end{vmatrix}=\left(r_1,-\frac{\widehat{x}_1\widehat{x}_2}{r_1},-\frac{\widehat{x}_1\widehat{x}_3}{r_1}\right)=$$

$$=\frac{1}{r_1}\,(\widehat{x}_2^2+\widehat{x}_3^2+x_1^2-\widehat{x}_1^2;-\widehat{x}_1\widehat{x}_2,-\widehat{x}_1\widehat{x}_3)^\top=\frac{1}{r_1}\{(1,0,0)^\top-\widehat{x}_1\widehat{x}\};$$

II) If $e_0=(0,1,0)\sim\frac{\partial}{\partial x_2}$, then

$$[e_0\times\widehat{x}]=\begin{vmatrix} i & j & k\\ 0 & 1 & 0\\ \widehat{x}_1 & \widehat{x}_2 & \widehat{x}_3\end{vmatrix}=(\widehat{x}_3,0,-\widehat{x}_2)^\top,\quad r_2=\sqrt{\widehat{x}_1^2+\widehat{x}_3^2},$$

$$e_3^{(2)}=\frac{1}{r_2}(\widehat{x}_3,0,-\widehat{x}_2)^\top,\ \ e_2^{(2)}=\widehat{x}\ \text{ and }\ e_1^{(2)}=\frac{1}{r_2}\{(0,1,0)^\top-\widehat{x}_2\widehat{x}\};$$

III) If $e_0=(0,0,1)\sim\frac{\partial}{\partial x_3}$, then

$$[e_0\times\widehat{x}]=\begin{vmatrix} i & j & k\\ 0 & 1 & 1\\ \widehat{x}_1 & \widehat{x}_2 & \widehat{x}_3\end{vmatrix}=(-\widehat{x}_2,\widehat{x}_1,0)^\top,\quad r_3=\sqrt{\widehat{x}_1^2+\widehat{x}_2^2},$$

$$e_3^{(3)}=\frac{1}{r_3}(-\widehat{x}_2,\widehat{x}_1,0)^\top,\ \ e_2^{(3)}=\widehat{x}\ \text{ and }\ e_1^{(3)}=\frac{1}{r_3}\{(0,0,1)^\top-\widehat{x}_3\widehat{x}\}.$$

The parametric equation of $\gamma_x=\partial\Sigma_x^\pm$ is

$$\zeta=e_1^{(k)}\sin\theta+e_3^{(k)}\cos\theta,\quad k=1,2,3.$$

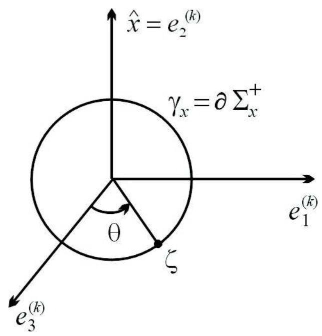


Figure 3.7.

Here the coordinates of $\zeta=(\zeta_1,\zeta_2,\zeta_3)$ correspond to the initial system.

Applying (3.33), we have

$$\frac{\partial\Phi}{\partial x_k}=\int\limits_{\Sigma_x^+}\frac{\partial\varphi(x,\eta)}{\partial x_k}\,d_\eta\Sigma_1+\frac{\sqrt{|x|^2-x_k^2}}{|x|^2}\int\limits_{\gamma_x}\varphi(x,\zeta)(e_1^{(k)}\cdot\zeta)d_\zeta\gamma_x.$$

Clearly,

$$\frac{\sqrt{|x|^2-x_k^2}}{|x|^2}=\frac{\sqrt{1-\widehat{x}^2}}{|x|}=\frac{r_k}{|x|},\ \ \zeta\cdot\widehat{x}=0,\ \text{ and }\ r_k(e_1^{(k)}\cdot\zeta)=\zeta_k,$$

so

$$\frac{\partial\Phi}{\partial x_k}=\int\limits_{\Sigma_x^+}\frac{\partial\varphi(x,\eta)}{\partial x_k}\,d_\eta\Sigma_1+\frac{1}{|x|}\int\limits_{\gamma_x}\varphi(x,\zeta)\zeta_k d_\zeta\gamma_x. \tag{3.34}$$

We can write η and η_k instead of ζ and ζ_k in (3.34) to get (3.14). □

Now we can prove the following

Theorem 3.8. *The fundamental solution* $\Psi(x,\omega,1)$ *of the equation* (2.1) *is represented as*

$$\Psi(x,\omega,1)=\Psi^{(1)}(x)+\Psi^{(0)}(x), \tag{3.35}$$

where

$$\Psi^{(1)}(x) = \int\limits_{\Sigma_x^+} \sum_{q=1}^{3} F_q(\eta) N(i\eta\rho_q, \omega) e^{i(x\cdot\eta)\rho_q} d\Sigma_1, \tag{3.36}$$

$$\Psi^{(0)}(x) = -\frac{1}{8\pi^2 |x|} \int\limits_{\gamma_x} C^{-1}(\eta)\, d\gamma. \tag{3.37}$$

Here $C^{-1}(\eta)$*is the inverse matrix of* $C(\eta)$ *(see* (2.1)*),* $d\gamma = d_\eta \gamma_x$*,* $F_q(\eta)$ *is defined by* (2.21)*.*

Moreover, if $|x| \to 0$*, then*

$$\frac{\partial}{\partial x_k}[\Psi^{(1)}(x)] = O(1); \quad \frac{\partial^2}{\partial x_k \partial x_j}[\Psi^{(1)}] = O\left(\frac{1}{|x|}\right) \tag{3.38}$$

and

$$\lim_{t\to 0} \Psi^{(1)}(x) = \int\limits_{\Sigma_x^+} \sum_{q=1}^{3} F_q(\eta) N(i\eta\rho_q, \omega)\, d\Sigma_1. \tag{3.39}$$

Proof. Note that $a(\eta) > 0$ and $\rho_q(\eta) > 0$, $\eta \in \Sigma_q$, are even functions. Therefore

$$F_q(-\eta) = F_q(\eta).$$

It is easy to check that

$$\sum_{q=1}^{3} F_q(\eta) \rho_q^{\pm 1}(\eta) = 0. \tag{3.40}$$

Due to Lemma 3.7, we have

$$\frac{\partial}{\partial x_k} \Gamma(x, \omega, 1) = \int\limits_{\Sigma_x^+} \sum_{q=1}^{3} F_q(\eta) e^{i(x\cdot\eta)\rho_q(\eta)} i\eta_k \rho_q(\eta)\, d\Sigma_1 + \\ + \frac{1}{|x|} \int\limits_{\gamma_x} \sum_{q=1}^{3} F_q(\eta) e^{i(x\cdot\eta)\rho_q(\eta)} \eta_k\, d\gamma. \tag{3.41}$$

We know that $F_q(\eta)$ is an even function and $(x \cdot \eta) = 0$ on γ_x. Therefore

$$\int\limits_{\gamma_x} \sum_{q=1}^{3} F_q(\eta) e^{i(x\cdot\eta)\rho_q(\eta)} \eta_k\, d\gamma = \int\limits_{\gamma_x} \sum_{q=1}^{3} F_q(\eta) \eta_k\, d\gamma = \\ = \int\limits_{\gamma_x} \sum_{q=1}^{3} F_q(-\eta)(-\eta_k)\, d\gamma = -\int\limits_{\gamma_x} \sum_{q=1}^{3} F_q(\eta) \eta_k\, d\gamma = 0.$$

Now we can rewrite (3.41) as

$$\frac{\partial}{\partial x_k}\,\Gamma(x,\omega,1)=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)i\eta_k\rho_q(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1. \tag{3.42}$$

With the help of (3.14) and (3.40) we have

$$\begin{aligned}\frac{\partial^2}{\partial x_k\partial x_j}\,\Gamma(x,\omega,1)&=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)(i\eta_k)(i\eta_j)\rho_q^2(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1+\\&\quad+\frac{1}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)i\eta_k\rho_q(\eta)\eta_j\,d\gamma=\\&=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)i^2\eta_k\eta_j\rho_q^2(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1.\end{aligned} \tag{3.43}$$

Similarly,

$$\begin{aligned}\frac{\partial^3}{\partial x_k\partial x_j\partial x_m}\,\Gamma(x,\omega,1)&=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)i^3\eta_k\eta_j\eta_m\rho_q^3(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1+\\&\quad+\frac{1}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)i^2\eta_k\eta_j\rho_q^2(\eta)\eta_m\,d\gamma.\end{aligned} \tag{3.44}$$

The curvilinear integral in (3.44) vanishes since the integrand is an odd function, i.e.

$$\frac{\partial^3}{\partial x_k\partial x_j\partial x_m}\,\Gamma(x,\omega,1)=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)i^3\eta_k\eta_j\eta_m\rho_q^3(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1. \tag{3.45}$$

Another use of (3.14) gives

$$\begin{aligned}\frac{\partial^4}{\partial x_k\partial x_j\partial x_m\partial x_p}\,\Gamma(x,\omega,1)&=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)i^4\eta_k\eta_j\eta_m\eta_p\rho_q^4(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1+\\&\quad+\frac{1}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)i^3\eta_k\eta_j\eta_m\rho_q^3(\eta)\eta_p\,d\gamma=\\&=i^4\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)\eta_k\eta_j\eta_m\eta_p e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1+\\&\quad+\frac{i^3}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)\eta_k\eta_j\eta_m\rho_q^3(\eta)\,d\gamma,\end{aligned} \tag{3.46}$$

where the curvilinear integral does not vanish, it is a homogeneous function of order -1. Clearly, the first integral in (3.46) is bounded in a vicinity of the origin.

Using (3.42)–(3.46), we can write

$$\Psi(x,\omega,1) = N(\partial_x,\omega)\int\limits_{\Sigma_x^+}\sum_{q=1}^{3} F_q(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1 = \\ = \int\limits_{\Sigma_x^+}\sum_{q=1}^{3} F_q(\eta)N(i\eta\rho_q,\omega)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1 + \\ + \frac{i^3}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3} F_q(\eta)\rho_q^3(\eta)N^0(\eta)\,d\gamma, \tag{3.47}$$

where $N^0(\eta)$ is the principle part of the matrix $N(\eta,\omega)$.

Let us calculate

$$\sum_{q=1}^{3} F_q(\eta)\rho_q^3(\eta) = -\frac{i}{8\pi^2 a(\eta)}\left\{\frac{\rho_1^4}{(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)} + \right. \\ \left. + \frac{\rho_2^4}{(\rho_2^2-\rho_1^2)(\rho_2^2-\rho_3^2)} + \frac{\rho_3^4}{(\rho_3^2-\rho_1^2)(\rho_3^2-\rho_2^2)}\right\} = \\ = -\frac{i}{8\pi^2 a(\eta)}\left\{\frac{\rho_1^4(\rho_2^2-\rho_3^2)-\rho_2^4(\rho_1^2-\rho_3^2)-\rho_3^4(\rho_1^2-\rho_2^2)}{(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)(\rho_2^2-\rho_3^2)}\right\} = -\frac{i}{8\pi^2 a(\eta)}\,.$$

Now we can rewrite (3.47) as

$$\Psi(x,\omega,1) = \int\limits_{\Sigma_x^+} F_q(\eta)N(i\eta\rho_q,\omega)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1 - \frac{1}{8\pi^2|x|}\int\limits_{\gamma_x} C^{-1}(\eta)\,d\gamma,$$

where $C^{-1}(\eta) = \dfrac{1}{a(\eta)}\,N^0(\eta)$ is the matrix inverse to $C(\eta)$.

Using the notation (3.36) and (3.37), we arrive to (3.35). Note that $\Psi^0(x)$ is the fundamental solution of the static equation ($\omega = 0$)

$$C(\partial)\Psi^{(0)}(x) = \delta(x)I_3.$$

We know that

$$N(\partial_x,\omega) = [N_{kj}(\partial_x,\omega)]_{3\times 3} \quad \text{and} \\ N_{kj}(i\eta\rho_q,\omega) = N_{kj}^0(\eta)i^4\rho_q^4 - N_{kj}^1(\eta)\rho_q^2\omega^2 + \omega^4\delta_{kj},$$

where $N_{kj}^0(\eta)$ is a 4th order polynomial with respect to η, $N_{kl}^1(\eta)$ is a second order polynomial,

$$\Psi^{(1)}(x) = \int\limits_{\Sigma_x^+}\sum_{q=1}^{3} N^0(\eta)F_q(\eta)\rho_q^4(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1 -$$

$$-\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}N^1(\eta)F_q(\eta)\rho_q^2(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+$$

$$+\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}I_3F_q(\eta)\omega^4e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1=$$

$$=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}\frac{\rho_q^5(\eta)}{\prod\limits_{j=1\, j\neq q}^{3}[\rho_q^2(\eta)-\rho_j^2(\eta)]}N^0(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+$$

$$-\int\limits_{\Sigma_x^+}\omega^2\sum_{q=1}^{3}\frac{\rho_q^3(\eta)}{\prod\limits_{j=1\, j\neq q}^{3}[\rho_q^2(\eta)-\rho_j^2(\eta)]}N^1(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+$$

$$+\int\limits_{\Sigma_x^+}\omega^4\sum_{q=1}^{3}\frac{\rho_q(\eta)}{\prod\limits_{j=1\, j\neq q}^{3}[\rho_q^2(\eta)-\rho_j^2(\eta)]}I_3e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1$$

$$=O(\omega)\to 0\quad\text{as}\quad\omega\to 0,$$

i.e., for any $x\in\mathbb{R}^3\setminus\{0\}$

$$\lim_{\omega\to 0}\Psi(x,\omega,1)=\Psi^{(0)}(x)$$

uniformly for all $|x|>\delta>0$.

Clearly, $\Psi^0(x)=O(1)$ as $|x|\to 0$,

Using (3.14) and the fact that $F_q(-\eta)=F_q(\eta)$, we obtain

$$\frac{\partial\Psi^{(1)}(x)}{\partial x_k}=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)[i\eta_k\rho_q(\eta)]e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+$$

$$+\frac{1}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)\eta_k\,d\gamma=$$

$$=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)[i\eta_k\rho_q(\eta)]e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1=O(1)$$

and

$$\frac{\partial^2\Psi^{(1)}(x)}{\partial x_k\partial x_j}=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)[i^2\eta_k\eta_j]\rho_q^2(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+$$

$$+\frac{1}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)i\eta_k\rho_q(\eta)\eta_j\,d\gamma=O\left(\frac{1}{|x|}\right).$$

Taking into account that $F_q(\eta)N(i\eta\rho_q,\omega)$ is an even function, we derive (3.39). □

3.2. **Asymptotics at infinity and the radiation conditions.** Using Lemma 3.7, we can prove

Theorem 3.9. *For* $|x|\to+\infty$

$$\frac{\partial^5}{\partial x_k\partial x_j\partial x_m\partial x_p\partial x_n}\Gamma(x,\omega,1)=$$
$$=i^5\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)\eta_k\eta_j\eta_m\eta_p\eta_n\rho_q^5(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+O(|x|^{-2}),\tag{3.48}$$

$$\frac{\partial^4}{\partial x_k\partial x_j\partial x_m\partial x_p}\Gamma(x,\omega,1)=$$
$$=i^4\int\limits_{\Sigma_x^+}\psi_1(\eta)\sum_{q=1}^{3}F_q(\eta)\eta_k\eta_j\eta_m\eta_p e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+O(|x|^{-2}),\tag{3.49}$$

where $\psi_1\in C^\infty(\Sigma_1)$*,* $\psi_1(\eta)=0$ *for* $\eta\in\gamma_x$*.*

Proof. Due to (3.14) and (3.46), we get

$$\frac{\partial^5}{\partial x_k\partial x_j\partial x_m\partial x_p\partial x_n}\Gamma(x,\omega,1)=$$
$$=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)i^5\eta_k\eta_j\eta_m\eta_p\eta_n\rho_q^5(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+$$
$$+i^4\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)\rho_q^4(\eta)\eta_k\eta_j\eta_m\eta_p\eta_n d\gamma+$$
$$+\frac{\partial}{\partial x_n}\left[\frac{i^3}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)\eta_k\eta_j\eta_m\eta_p\rho_q^3(\eta)d\gamma\right].\tag{3.50}$$

The second integral in(3.50) vanishes since the integrand is an odd function. The third integral in(3.50) is $O(|x|^{-2})$ as $|x|\to\infty$ (or $|x|\to0$), more precisely, it is a homogeneous function of order -2. Hence we can rewrite (3.50) as (3.48).

Let us consider the function

$$\Phi(x)=\int\limits_{\Sigma_x^+}\varphi(\eta)e^{i(x\cdot\eta)\rho(\eta)}d\Sigma_1,\quad\varphi\in C^1(\Sigma_1).\tag{3.51}$$

Due to Theorem 3.4 and Corollary 3.3, there exists $\varepsilon>0$ such that there is no critical point in $\Sigma_x^+(\varepsilon)$ (see Fig. 3.8).

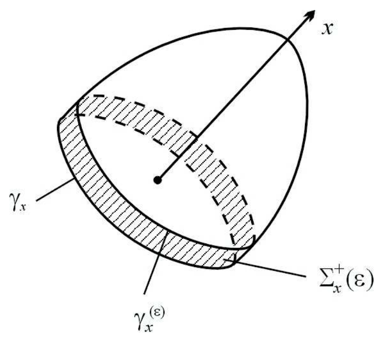


FIGURE 3.8.

Let us rewrite (3.51) as

$$\Phi(x)=\int\limits_{\Sigma_x^+}(\psi_0(\eta)+\psi_1(\eta))\varphi(\eta)e^{i(x\cdot\eta)\rho(\eta)}d\Sigma_1=\Phi^*(x)+\Phi^{**}(x),$$

where

$$\Phi^*(x)=\int\limits_{\Sigma_x^+}\psi_0(\eta)\varphi(\eta)e^{i(x\cdot\eta)\rho(\eta)}d\Sigma_1,$$
$$\Phi^{**}(x)=\int\limits_{\Sigma_x^+}\psi_1(\eta)\varphi(\eta)e^{i(x\cdot\eta)\rho(\eta)}d\Sigma_1;$$

here $\psi_0(\eta)+\psi_1(\eta)=0$, $\eta\in\Sigma_1$, $\psi_0,\psi_1\in C^\infty(\Sigma_1)$, $\psi_0\geq 0$, $\psi_0(\eta)=0$ in vicinity of γ_x, $\operatorname{supp}\psi_0\subset\Sigma_x^+(\varepsilon)$, $\psi_1(\eta)=1-\psi_0(\eta)$ vanishes on γ_x and in $\Sigma_x^+(\varepsilon)$.

Applying (3.11) and (3.12), we have

$$\Phi^*(x)=\int\limits_{\Sigma_x^+}\psi_0(\eta)\varphi(\eta)e^{i|x|(\widehat{x}\cdot\eta)\rho(\eta)}d\Sigma_1=$$
$$=\int\limits_{\Sigma_x^+}\psi_0(\eta)\varphi(\eta)\Big[\frac{1}{i|x|}\sum_{k=1}^3\frac{\Phi_k(\widehat{x},\eta)}{|\Phi(\widehat{x},\eta)|^2}\,\partial_k(\eta,\nabla_\eta)e^{i|x|(\widehat{x}\cdot\eta)\rho(\eta)}\Big]d\Sigma_1=$$
$$=\frac{1}{i|x|}\bigg\{-\int\limits_{\Sigma_x^+}\bigg(\sum_{k=1}^3\partial_k(\eta,\nabla_\eta)\Big[\psi_0(\eta)\varphi(\eta)\,\frac{\Phi_k(\widehat{x},\eta)}{|\Phi(\widehat{x},\eta)|^2}\Big]\bigg)e^{i|x|(\widehat{x}\cdot\eta)\rho(\eta)}d\Sigma_1+$$
$$+\int\limits_{\gamma_x}\psi_0(\eta)\varphi(\eta)\sum_{k=1}^3\frac{\Phi_k(\widehat{x},\eta)}{|\Phi(\widehat{x},\eta)|^2}\,\ell_k(\eta)e^{i|x|(\widehat{x}\cdot\eta)\rho(\eta)}d\gamma\bigg\}. \quad (3.52)$$

Applying the same procedure in the first integral of (3.52), we see that it is $O(|x|^{-2})$.

On the other hand, $(\widehat{x}\cdot\eta)=0$, $\eta\in\gamma_x$,

$$\Phi(\widehat{x},\eta)=[\eta\times\widehat{x}]\rho(\eta)=-\ell(\eta)\rho(\eta),\quad \Phi_k(\widehat{x},\eta)=-\rho(\eta)\ell_k(\eta)$$

(here $\ell(\eta)$ is the tangent vector to γ_x), so

$$\Phi^*(x)=-\frac{1}{i|x|}\int\limits_{\gamma_x}\varphi(\eta)\sum_{k=1}^{3}\frac{\rho(\eta)\ell_k(\eta)}{\rho^2(\eta)}\,\ell_k\,d\gamma+O(|x|^{-2})=$$
$$=-\frac{1}{i|x|}\int\limits_{\gamma_x}\varphi(\eta)\frac{1}{\rho(\eta)}\,d\gamma+O(|x|^{-2}).$$

For $x\gg 1$

$$\Phi(x)=\int\limits_{\Sigma_x^+}\varphi(\eta)e^{i(\widehat{x}\cdot\eta)\rho(\eta)}d\Sigma_1=$$
$$=-\frac{1}{i|x|}\int\limits_{\gamma_x}\varphi(\eta)\frac{1}{\rho(\eta)}\,d\gamma+\Phi^{**}(x)+O(|x|^{-2}). \tag{3.53}$$

Using (3.53) in (3.46), we can write

$$\frac{\partial^4}{\partial x_k\partial x_j\partial x_m\partial x_p}\Gamma(x,\omega,1)=-\frac{i^4}{i|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)\rho_q^4(\eta)\eta_k\eta_j\eta_m\eta_p\,\frac{1}{\rho_q(\eta)}\,d\gamma+$$
$$+O(|x|^{-2})+\frac{i^3}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)\rho_q^3(\eta)\eta_k\eta_j\eta_m\eta_p\,d\gamma+$$
$$+i^4\int\limits_{\Sigma_x^+}\psi_1(\eta)\sum_{q=1}^{3}F_q(\eta)\eta_k\eta_j\eta_m\eta_p\rho_q^4(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1.$$

From this relation we obtain (3.49). □

Theorem 3.10. *For* $|x|\gg 1$

$$\Psi(x,\omega,1)=\sum_{q=1}^{3}\overset{(q)}{\Psi}(x,\omega,1),\quad \overset{(q)}{\Psi}(x,\omega,1)=O(|x|^{-1}), \tag{3.54}$$

$$\frac{\partial\overset{(q)}{\Psi}(x,\omega,1)}{\partial x_k}-i\xi_k^{(q)}\overset{(q)}{\Psi}(x,\omega,1)=O(|x|^{-2}), \tag{3.55}$$

where $\xi_k^{(q)}\in S_q$ *and* $\eta(\xi_k^{(q)})=\frac{x}{|x|}$.

These conditions are called the generalized Sommerfeld–Kupradze type radiation conditions.

Proof. Taking into account the form of N_{kj}, we can write

$$\Psi(x,\omega,1)=\int\limits_{\Sigma_x^+}(\psi_0(\eta)+\psi_1(\eta))\sum_{q=1}^{3}F_q(\eta)[N^0(\eta)i^4\rho_q^4(\eta)-$$

$$-N^1(\eta)\rho_q^2(\eta)\omega^2+\omega^2 I]e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+\frac{i^3}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)\rho_q^3(\eta)N^0\,d\gamma=$$

$$=\int\limits_{\Sigma_x^+}\psi_1(\eta)\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1-$$

$$-\frac{1}{i|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}\Big[F_q(\eta)i^4\rho_q^4(\eta)N^0\,\frac{1}{\rho_1(\eta)}-$$

$$-F_q(\eta)N^1(\eta)\rho_q^2(\eta)\omega^2\,\frac{1}{\rho_q(\eta)}+F_q(\eta)\omega^2\,\frac{1}{\rho_q(\eta)}\Big]d\gamma+$$

$$+\frac{i^3}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)\rho_q^3(\eta)N^0 d\gamma+O(|x|^{-2}). \tag{3.56}$$

Here ψ_0 and ψ_1 are the same as in the previous theorem.

Due to (3.40)

$$\Psi(x,\omega,1)=\int\limits_{\Sigma_x^+}\psi_1(\eta)\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+$$
$$+O(|x|^{-2}),\quad |x|\gg 1. \tag{3.57}$$

Let us calculate $\dfrac{\partial\Psi(x,\omega,1)}{\partial x_k}$ with the help of (3.46) and (3.35)–(3.37)

$$\frac{\partial\Psi(x,\omega,1)}{\partial x_k}=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)[\eta_k\rho_q(\eta)]e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+$$

$$+\frac{1}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)\eta_k\,d\gamma+\frac{\partial}{\partial x_k}\,\Psi^{(0)}(x). \tag{3.58}$$

The last term in (3.58) is $O(|x|^{-2})$. If we apply (3.53) in (3.58), we have

$$\frac{\partial\Psi(x,\omega,1)}{\partial x_k}=-\frac{1}{i|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)[\eta_k\rho_q(\eta)]\,\frac{1}{\rho_q(\eta)}\,d\gamma+$$

$$+\int\limits_{\Sigma_x^+}\psi_1(\eta)\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)[\eta_k\rho_q(\eta)]e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+O(|x|^{-2})=$$

$$= \int\limits_{\Sigma_x^+} \psi_1(\eta) \sum_{q=1}^{3} F_q(\eta) N(i\eta\rho_q, \omega)[\eta_k \rho_q(\eta)] e^{i(x\cdot\eta)\rho_q(\eta)} d\Sigma_1 + O(|x|^{-2}). \quad (3.59)$$

Note that

$$\rho_q^2 d\Sigma_1 = \cos\gamma \, dS_q = (\eta \cdot \eta(\xi))\, dS_q = \left(\eta \cdot \frac{-\nabla_\eta \rho_q(\eta)}{|\nabla_\eta \rho_q(\eta)|}\right) dS_q = \frac{\rho_q(\eta)}{|\nabla_\eta \rho_q(\eta)|}\, dS_q,$$

so we can rewrite (3.57) as follows

$$\Psi(x,\omega,1) = \sum_{q=1}^{3} \int\limits_{S_q^+(x)} \psi_1(\eta) F_q(\eta) N(i\xi,\omega) e^{i(x\cdot\xi)} \frac{1}{\rho_q(\eta)|\nabla_\eta \rho_q(\eta)|}\, dS_q + $$
$$+ O(|x|^{-2}) =$$
$$= \sum_{q=1}^{3} \left\{ -\frac{1}{8\pi^2} \int\limits_{S_q^+(x)} \psi_1(\eta) \frac{N(i\xi,\omega)}{a(\eta) \prod\limits_{j=1,\, j\neq q}^{3} [\rho_q^2(\eta) - \rho_j^2(\eta)]} \cdot \frac{e^{i(x\cdot\xi)}}{|\nabla_\eta \rho_q(\eta)|}\, dS_q \right\} +$$
$$+ O(|x|^{-2});$$

here $\eta = \frac{\xi}{|\xi|} = \frac{\xi}{\rho_q(\eta)}$.

Now we can apply the results obtained in [8] and [9] to get

$$\Psi(x,\omega,1) = \sum_{q=1}^{3} \left\{ -\frac{i}{8\pi^2} \frac{N(i\xi^{(q)},\omega) e^{-i\frac{\pi}{4}} \cdot 2\pi}{a(\eta^{(q)}) \prod\limits_{j=1,\, j\neq q}^{3} [\rho_q^2(\eta^{(q)}) - \rho_j^2(\eta^{(q)})]} \times \right.$$
$$\left. \times \frac{e^{i(x\cdot\xi^{(q)})}}{|\nabla_\eta \rho_q(\eta^{(q)})| \sqrt{K_q(\xi^{(q)})}} \right\} + O(|x|^{-2}),$$

i.e.

$$\Psi(x,\omega,1) = \sum_{q=1}^{3} \left\{ -\frac{1}{4\pi} \frac{N(i\xi^{(q)},\omega)}{a(\eta^{(q)}) \prod\limits_{j=1,\, j\neq q}^{3} [\rho_q^2(\eta^{(q)}) - \rho_j^2(\eta^{(q)})]} \times \right.$$
$$\left. \times \frac{e^{i(x\cdot\xi^{(q)})}}{|\nabla_\eta \rho_q(\eta^{(q)})| \sqrt{k_q(\xi^{(q)})}} \right\} + O(|x|^{-2}), \quad (3.60)$$

where $\eta^{(q)} = \dfrac{\xi^{(q)}}{|\xi^{(q)}|}$, $\eta_k \rho_q(\eta^{(q)}) = \xi_k^{(q)}$ and k_q is the Gaussian curvature of S_q.

With the help of (3.60), (3.56) and (3.59) we obtain the radiation conditions (3.54), (3.55). □

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Short Communications

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ON TWO-POINT BOUNDARY VALUE PROBLEMS FOR TWO-DIMENSIONAL LINEAR DIFFERENTIAL SYSTEMS WITH SINGULAR COEFFICIENTS

Abstract. Two-point boundary value problems for two-dimensional systems of linear differential equations with singular coefficients are considered. The cases are optimally described when the above-mentioned problems have the Fredholm property, and unimprovable in a certain sense conditions are established guaranteeing the unique solvability of those problems.

რეზიუმე. განხილულია ორწერტილოვანი სასაზღვრო ამოცანები წრფივ დიფერენციალურ განტოლებათა ორგანზომილებიანი სისტემებისათვის სინგულარული კოეფიციენტებით. ოპტიმალურადაა აღწერილი შემთხვევები, როცა აღნიშნულ ამოცანებს გააჩნიათ ფრედჰოლმის თვისება, და დადგენილია გარკვეული აზრით არაგაუმჯობესებადი პირობები, რომლებიც უზრუნველყოფენ ამ ამოცანების ცალსახად ამოხსნადობას.

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Boundary value problems for second and higher order linear differential equations, whose coefficients have nonintegrable singularities at the points bearing the boundary data, are investigated in full detail (see, e.g., [1], [2], [5]–[7], [9]–[16] and the references therein).

From the theorems proven by R. P. Agarwal and I. Kiguradze [10] for the second order differential equation

$$u'' = p(t)u + q(t),$$

it follow unimprovable in a certain sense results on the unique solvability of the boundary value problems

$$u(a) = 0, \quad u(b) = 0, \quad \int\limits_a^b u'^2(t)\,dt < +\infty$$

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and

$$u(a) = 0, \quad u'(b) = 0, \quad \int\limits_a^b u'^2(t)\,dt < +\infty.$$

These results cover the cases where the concerned differential equation is strongly singular, more precisely, when the order of singularity of the function $t \to (|p(t)| - p(t))/2$ at the points a and b is equal to 2. In the present paper, the above-mentioned results are generalized for two-dimensional linear differential systems.

By $L_{loc}(]a,b[)$ we denote the space of functions $p :]a,b[\to \mathbb{R}$ Lebesgue integrable in the interval $[a+\varepsilon, b-\varepsilon]$ for arbitrarily small $\varepsilon > 0$. Analogously, by $L_{loc}(]a,b])$ we denote the space of functions $p :]a,b] \to \mathbb{R}$ Lebesgue integrable in the interval $[a+\varepsilon, b]$ for arbitrarily small $\varepsilon > 0$.

It is clear that the functions from the space $L_{loc}(]a,b[)$ may have nonintegrable singularities at the points a and b. As for the functions from the space $L_{loc}(]a,b])$, they may have nonintegrable singularities only at the point a.

For an arbitrary number x we set

$$[x]_- = \frac{|x| - x}{2}\,.$$

We consider the two-dimensional linear differential system

$$u_i' = p_{i1}(t)u_1 + p_{i2}(t)u_2 + p_{i0}(t) \ \ (i = 1,2) \tag{1}$$

with locally integrable coefficients $p_{ik} \in L_{loc}(]a,b[)$ $(i = 1,2;\ k = 0,1,2)$.

We do not exclude from consideration the cases where some (or all) of the coefficients of that system are not integrable on $[a,b]$, having singularities at the points a and b. In that sense the system (1) is singular.

It is naturally admitted the possibility that the functions p_{12} and p_{21} be equal to zero on the sets of positive measure. This is the most interesting case since in that case the system (1) cannot be reduced to a second order linear differential equation.

Denote

$$a_0 = \frac{a+b}{2}, \quad r_i(t) = \exp\Bigg(\int\limits_{a_0}^t p_{ii}(s)\,ds\Bigg) \ \ (i = 1,2), \quad r(t) = \frac{|p_{12}(t)|}{r_1(t)r_2(t)}\,;$$

$$p_1(t) = \frac{p_{12}(t)r_2(t)}{r_1(t)}\,, \quad p_2(t) = \frac{p_{21}(t)r_1(t)}{r_2(t)}\,; \quad q_i(t) = \frac{p_{i0}(t)}{r_i(t)} \ \ (i = 1,2).$$

For the system (1) we consider the boundary value problems

$$\lim_{t\to a} \frac{u_1(t)}{r_1(t)} = 0, \quad \lim_{t\to b} \frac{u_1(t)}{r_1(t)} = 0, \quad \int\limits_a^b r(t)u_2^2(t)\,dt < +\infty \tag{2}$$

and

$$\lim_{t\to a}\frac{u_1(t)}{r_1(t)}=0,\quad \lim_{t\to b}\frac{u_2(t)}{r_2(t)}=0,\quad \int_a^b r(t)u_2^2(t)\,dt<+\infty. \tag{3}$$

Note that if the functions p_{11} and p_{22} are integrable on $[a,b]$, then the conditions (2) and (3), respectively, are equivalent to the conditions

$$u_1(a)=0,\quad u_1(b)=0,\quad \int_a^b |p_{12}(t)|u_2^2(t)\,dt<+\infty$$

and

$$u_1(a)=0,\quad u_2(b)=0,\quad \int_a^b |p_{12}(t)|u_2^2(t)\,dt<+\infty,$$

where by $u_i(a)$ and $u_i(b)$ it is understood, respectively, the right and the left limits of the function u_i at the points a and b.

Both the problems (1), (2) and (1), (3) we investigate in the case where the condition

$$0\le \sigma p_1(t)\le \ell_0 \ \text{ for } \ a<t<b,\quad \int_a^b |p_1(t)|\,dt>0 \tag{4}$$

is satisfied. Here $\sigma\in\{-1,1\}$ and ℓ_0 is a positive number.

Along with (1) we consider the corresponding homogeneous differential system

$$u_i'=p_{i1}(t)u_1+p_{i2}(t)u_2 \ \ (i=1,2), \tag{1_0}$$

and we introduce

Definition 1. We say that the problem (1), (2) has *the Fredholm property* if the unique solvability of the corresponding homogeneous problem (1_0), (2) guarantees the unique solvability of the problem (1), (2) for any $p_{i0}\in L_{loc}(]a,b[)$ $(i=1,2)$ satisfying the conditions

$$q_1\in L([a,b]),\quad \int_a^b (t-a)(b-t)\bigg(p_2(t)\int_a^t |q_1(s)|\,ds\int_t^b |q_1(s)|\,ds\bigg)^2 dt<+\infty; \tag{5}$$

$$\int_a^b |p_1(t)|\bigg|\int_{a_0}^t q_2(s)\,ds\bigg|^2 dt<+\infty. \tag{6}$$

The following theorem is valid.

Theorem 1. *If along with* (4) *the inequalities*

$$\limsup_{t\to a}\left((t-a)\int_t^{a_0}[\sigma p_2(s)]_-\,ds\right)<\frac{1}{4\ell_0}\,,$$
$$\limsup_{t\to b}\left((b-t)\int_{a_0}^{t}[\sigma p_2(s)]_-\,ds\right)<\frac{1}{4\ell_0} \tag{7}$$

are fulfilled, then the problem (1), (2) *has the Fredholm property.*

From this theorem it follows

Corollary 1. *If along with* (4) *the inequalities*

$$\liminf_{t\to a}\left(\sigma(t-a)^2p_2(t)\right)>-\frac{1}{4\ell_0}\,,\quad \liminf_{t\to b}\left(\sigma(b-t)^2p_2(t)\right)>-\frac{1}{4\ell_0} \tag{8}$$

are fulfilled, then the problem (1), (2) *has the Fredholm property.*

On the basis of Theorem 1 the following theorem can be proved.

Theorem 2. *Let along with* (4) *the inequality*

$$\left|\int_{a_0}^{t}[\sigma p_2(s)]_-\,ds\right|\le\frac{\ell(b-a)}{(t-a)(b-t)}\ \ \text{for}\ \ a<t<b$$

be fulfilled, where ℓ *is a non-negative constant such that*

$$\ell<\frac{1}{4\ell_0}\,. \tag{9}$$

If, moreover, the conditions (5) *and* (6) *are satisfied, then the problem* (1), (2) *has a unique solution.*

Theorem 2 yields

Corollary 2. *Let along with* (4) *the inequality*

$$\sigma p_2(t)\ge-\ell\Big(\frac{1}{(t-a)^2}+\frac{1}{(b-t)^2}\Big)\ \ \text{for}\ \ a<t<b$$

be fulfilled, where ℓ *is a non-negative constant, satisfying the inequality* (9)*. If, moreover, the conditions* (5) *and* (6) *are satisfied, then the problem* (1), (2) *has a unique solution.*

Note that the conditions of Theorems 1 and 2 as well as the conditions of Corollary 1 and 2 are unimprovable. More precisely, none of the strict inequalities (7) and (8) can be replaced by the non-strict ones, and the inequality (9) cannot be replaced by the equality

$$\ell=\frac{1}{4\ell_0}\,.$$

As an example, we consider the differential system

$$u_1' = g_1(t)u_2 + (t-a)^\alpha (b-t)^\alpha g_{10}(t),$$
$$u_2' = \Big(\frac{g_2(t)}{(t-a)^\beta (b-t)^\beta} - \frac{\ell}{(t-a)^2} - \frac{\ell}{(b-t)^2}\Big)u_1 + \frac{g_{20}(t)}{(t-a)^\gamma (b-t)^\gamma}\,, \tag{10}$$

where $g_i : [a,b] \to [0,+\infty[$ and $g_{i0} : [a,b] \to \mathbb{R}$ $(i = 1,2)$ are continuous functions, and α, β, γ, and ℓ are positive constants. Moreover, $g_1(t) \not\equiv 0$ and

$$0 \le g_1(t) \le \Big(\frac{t-a}{b-a}\Big)^\lambda \Big(\frac{b-t}{b-a}\Big)^\lambda \quad \text{for} \ \ a < t < b,$$

where $\lambda > 0$.

The system (10), generally speaking, cannot be reduced to a second order linear differential equation since the restrictions, imposed on the functions g_1 and g_2, do not exclude, for example, the cases where

$$g_1(t) = g_2(t) = 0 \quad \text{for} \ \ t \in I =$$
$$= \bigcup_{k=1}^{\infty} \Big[a + \frac{b-a}{4k+1}, a + \frac{b-a}{4k}\Big] \bigcup \Big[b - \frac{b-a}{4k}, b - \frac{b-a}{4k+1}\Big],$$
$$\text{and} \ \ g_1(t) > 0, \ \ g_2(t) > 0 \ \ \text{for} \ \ t \in [a,b] \setminus I.$$

From Corollary 2 it follows

Corollary 3. *If*

$$\ell < \frac{1}{4}\,, \ \ \alpha > 0, \ \ \beta < 2 + \alpha, \ \ \textit{and} \ \ \gamma < \frac{3+\lambda}{2}\,, \tag{11}$$

then the system (10) *has a unique solution satisfying the conditions*

$$u_1(a) = 0, \ \ u_1(b) = 0, \ \ \int\limits_a^b g_1(t)u_2^2(t)\,dt < +\infty.$$

According to Corollary 3, the second equation in the system (10) may have the singularity of an arbitrary order. More precisely, β and γ may be arbitrarily large numbers if α and λ are also large.

Note that Corollary 3 does not follow from the previous well-known results on the unique solvability of two-point boundary value problems for linear differential systems (see [3], [4], [8], [17]).

Now we consider the problem (1), (3). First of all we introduce

Definition 2. We say that the problem (1), (3) has *the Fredholm property* if the unique solvability of the corresponding homogeneous problem (1_0), (3) guarantees the unique solvability of the problem (1), (3) for any

$p_{i0} \in L_{loc}(]a,b[)$ $(i=1,2)$ satisfying the conditions

$$q_1 \in L([a,b]), \quad \int\limits_a^b (t-a)\bigg(p_2(t)\int\limits_a^t |q_1(s)|\,ds\bigg)^2 dt < +\infty, \tag{12}$$

$$q_2 \in L_{loc}(]a,b]), \quad \int\limits_a^b |p_1(t)|\,\bigg|\int\limits_t^b q_2(s)\,ds\bigg|^2 dt < +\infty. \tag{13}$$

The following theorem is valid.

Theorem 3. *Let $p_2 \in L_{loc}(]a,b])$, and let along with (4) the inequality*

$$\limsup_{t\to a}\bigg(\sigma(t-a)\int\limits_t^b [\sigma p_2(s)]_-\,ds\bigg) < \frac{1}{4\ell_0} \tag{14}$$

be fulfilled. Then the problem (1), (3) has the Fredholm property.

Corollary 4. *Let $p_2 \in L_{loc}(]a,b])$, and let along with (4) the inequality*

$$\liminf_{t\to a}\big(\sigma(t-a)^2 p_2(t)\big) > -\frac{1}{4\ell_0} \tag{15}$$

be fulfilled. Then the problem (1), (3) has the Fredholm property.

Theorem 4. *Let $p_2 \in L_{loc}(]a,b])$, and let along with (4) the inequality*

$$\int\limits_t^b [\sigma p_2(s)]_-\,ds \le \frac{\ell}{t-a} \quad \text{for } a<t<b, \quad \text{where } \ell < \frac{1}{4\ell_0}\,, \tag{16}$$

be fulfilled. If, moreover, the conditions (12) and (13) are satisfied, then the problem (1), (3) has a unique solution.

Corollary 5. *Let $p_2 \in L_{loc}(]a,b])$, and let along with (4) the inequality*

$$\sigma p_2(t) \ge -\frac{\ell}{(t-a)^2} \quad \text{for } a<t<b, \quad \text{where } \ell < \frac{1}{4\ell_0}\,, \tag{17}$$

be fulfilled. If, moreover, the conditions (12) and (13) are satisfied, then the problem (1), (3) has a unique solution.

Note that the conditions (14)–(17) in Theorems 3, 4 and Corollaries 4, 5 are unimprovable.

As an example, we consider the differential system

$$\begin{aligned} u_1' &= g_1(t)u_2 + (t-a)^\alpha g_{10}(t),\\ u_2' &= \Big(\frac{g_2(t)}{(t-a)^\beta} - \frac{\ell}{(t-a)^2}\Big)u_1 + \frac{g_{20}(t)}{(t-a)^\gamma}\,, \end{aligned} \tag{18}$$

where $g_i : [a,b] \to [0,+\infty[$ and $g_{i0} : [a,b] \to \mathbb{R}$ $(i=1,2)$ are continuous functions, α, β, γ, and ℓ are positive constants. Moreover, $g_1(t) \not\equiv 0$ and

$$0 \le g_1(t) \le \Big(\frac{t-a}{b-a}\Big)^\lambda \quad \text{for } a<t<b,$$

where $\lambda > 0$.

From Corollary 5 it follows

Corollary 6. *If the condition* (11) *is fulfilled, then the system* (18) *has a unique solution satisfying the conditions*

$$u_1(a) = 0, \quad u_2(b) = 0, \quad \int\limits_a^b g_1(t) u_2^2(t)\, dt < +\infty.$$

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ON PERTURBED MULTI-POINT PROBLEMS FOR NONLINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

Abstract. For nonlinear functional differential systems unimprovable conditions of solvability of perturbed multi-point boundary value problems are established.

რეზიუმე. არაწრფივი ფუნქციონალურ დიფერენციალური სისტემებისათვის დადგენილია შეშფოთებული მრავაწერტილოვანი სასაზღვრო ამოცანების ამოხსნადობის არაგაუმჯობესებადი პირობები.

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Consider the boundary value problem

$$\frac{dx_i(t)}{dt} = f_i(x_1,\dots,x_n)(t) \quad (i=1,\dots,n), \tag{1}$$

$$x_i(t_i) = \varphi_i(x_1,\dots,x_n)(t) \quad (i=1,\dots,n), \tag{2}$$

where $t_1,\dots,t_n$ are points from the segment $I=[a,b]$, while $f_i : C(I;\mathbb{R}^n) \to L(I;\mathbb{R})$ $(i=1,\dots,n)$ and $\varphi_i : C(I;\mathbb{R}^n) \to \mathbb{R}$ $(i=1,\dots,n)$ are, respectively, continuous operators and functionals.

A vector function $(x_i)_{i=1}^n : I \to \mathbb{R}^n$ with absolutely continuous components $x_i : I \to \mathbb{R}$ $(i=1,\dots,n)$ is said to be a solution of the system (1) if it satisfies this system almost everywhere on I.

A solution of the system (1), satisfying the boundary conditions (2), is said to be a solution of the problem (1),(2).

Particular cases of (1) are systems of ordinary differential equations

$$\frac{dx_i(t)}{dt} = f_{0i}\big(t, x_1(t),\dots,x_n(t)\big) \quad (i=1,\dots,n) \tag{3}$$

and systems of differential equations with deviated arguments

$$\frac{dx_i(t)}{dt} = g_i\big(t, x_1(\tau_i(t)),\dots,x_n(\tau_n(t)), x_i(t)\big) \quad (i=1,\dots,n), \tag{4}$$

where $f_{0i} : I\times\mathbb{R}^n \to \mathbb{R}$ and $g_i : I\times\mathbb{R}^{n+1} \to \mathbb{R}$ $(i=1,\dots,n)$ are functions from the Carathéodory class, and $\tau_i : I \to I$ $(i=1,\dots,n)$ are measurable functions.

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Particular cases of (2) are the boundary conditions of periodic type

$$x_i(a) = \alpha_i x_i(b) \ \ (i = 1, \dots, n) \tag{2_1}$$

and the multi-point boundary conditions

$$x_i(t_i) = \sum_{k=1}^{n} \sum_{j=1}^{m} \ell_{ijk} x_k(t_{ijk}) + c_i \ \ (i = 1, \dots, n). \tag{2_2}$$

Boundary value problems for systems of the type (1) have been investigating intensively and are the subject of numerous works (see, e.g., [1]–[5], [12] and the references therein).

In the case where $\varphi_i = c_i = const$ $(i = 1, \dots, n)$, the problem (3), (2), i.e. the system (3) with the boundary conditions

$$x_i(t_i) = c_i \ \ (i = 1, \dots, n)$$

is called the Cauchy–Nicoletti problem. Optimal, in a certain sense, sufficient conditions for the solvability and unique solvability of that problem are contained in the papers [6], [7], [14].

In the paper [8] I. Kiguradze proposed a new method of investigation of boundary value problems of the type (3), (2) which is based on a priori estimates of solutions of systems of one-sided differential inequalities. This method allows us to study from the unified viewpoint a sufficiently large class of perturbed multi-point boundary value problems and the periodic problem (see [8] and [10]).

In our paper, new sufficient conditions for the solvability of boundary value problems of the type (1), (2) are given, which, in contrast to previous results, cover the cases where the system (1) is superlinear or sublinear or some equations of this system are superlinear, while others are sublinear.

Throughout the paper, the use will be made of the following notation:

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$;

$\mathbb{R}^n$ is the n-dimensional real Euclidian space;

$y = (y_i)_{i=1}^n$ and $Y = (y_{ik})_{i,k=1}^n$ are an n-dimensional column vector and an $n \times n$-matrix with elements y_i and $y_{ik} \in \mathbb{R}$ $(i = 1, \dots, n)$;

Y^{-1} is the inverse matrix to Y; $r(Y)$ is the spectral radius of Y;

E is the unit matrix;

$C(I; \mathbb{R}^n)$ is the space of n-dimensional continuous vector functions $x = (x_i)_{i=1}^n : I \to \mathbb{R}^n$ with the norm

$$\|x\|_C = \max\Big\{ \sum_{i=1}^{n} |x_i(t)| : \ t \in I \Big\};$$

$L(I; \mathbb{R})$ is the space of Lebesgue integrable functions $x : I \to \mathbb{R}$ with the norm $\|x\|_L = \int_a^b |x(s)|\, ds$;

$L(I; \mathbb{R}_+)$ is the set all nonnegative functions from $L(I; \mathbb{R})$.

We will say that the operator $p : C(I;\mathbb{R}^n) \to L(I;\mathbb{R})$ belongs to the Carathéodory class if it is continuous and

$$\sup\Big\{|p(x)(\cdot)| : \ x \in C(I;\mathbb{R}^n), \ \|x\|_C \le \rho\Big\} \in L(I;\mathbb{R}_+) \ \text{ for } \ \rho \in \mathbb{R}_+.$$

Everywhere below, when we discuss the boundary value problem (1), (2), it is supposed that the operators $f_i : C(I;\mathbb{R}^n) \to L(I;\mathbb{R})$ $(i = 1,\dots,n)$ belong to the Carathéodory class, and the functionals $\varphi_i : C(I;\mathbb{R}^n) \to \mathbb{R}$ $(i = 1,\dots,n)$ are continuous.

We will consider the case where for arbitrary $(x_i)_{i=1}^n \in C(I;\mathbb{R}^n)$ and for almost all $t \in I$ the inequalities

$$f_i(x_1,\dots,x_n)(t)\,\mathrm{sgn}\,\big((t-t_i)x_i(t)\big) \le$$
$$\le p_i(x_1,\dots,x_n)(t)\Big(-|x_i(t)| + \sum_{k=1}^n h_{ik}\|x_k\|_C + h_i\Big) +$$
$$+\delta_i(x_1,\dots,x_n)\Big(\sum_{k=1}^n q_{ik}(t)\|x_k\|_C + q_i(t)\Big) \ \ (i = 1,\dots,n), \tag{5}$$
$$\big|\varphi_i(x_1,\dots,x_n)\big| \le$$
$$\le \varphi_{0i}(|x_i|) + \delta_i(x_1,\dots,x_n)\Big(\sum_{k=1}^n \ell_{ik}\|x_k\|_C + \ell_i\Big) \ \ (i = 1,\dots,n) \tag{6}$$

are satisfied, where $p_i : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}_+)$ $(i = 1,\dots,n)$, $\delta_i : C(I;\mathbb{R}^n) \to \mathbb{R}_+$ are any nonlinear operators and functionals; $\varphi_{0i} : C(I;\mathbb{R}) \to \mathbb{R}$ $(i = 1,\dots,n)$ are linear non-negative functionals; h_{ik}, h_i, ℓ_{ik} and ℓ_i are non-negative constants;

$$q_{ik} \in L(I;\mathbb{R}_+), \ \ q_i \in L(I;\mathbb{R}_+) \ \ (i,k = 1,\dots,n).$$

Suppose

$$\widetilde{p}_i(x_1,\dots,x_n)(t) = \exp\bigg(-\bigg|\int_{t_i}^t p_i(x_1,\dots,x_n)(s)\,ds\bigg|\bigg) \ \ (i = 1,\dots,n), \tag{7}$$
$$H = \Big(h_{ik} + (1 + \varphi_{0i}(1))\|q_{ik}\|_L + \ell_{ik}\Big)_{i,k=1}^n. \tag{8}$$

Theorem 1. *Let along with* (5) *and* (6) *the conditions*

$$\varphi_{0i}(1) \le 1, \quad 1 - \varphi_{0i}\big(\widetilde{p}_i(x_1,\dots,x_n)\big) \ge \delta_i(x_1,\dots,x_n) \ \ (i = 1,\dots,n), \tag{9}$$
$$r(H) < 1 \tag{10}$$

be fulfilled, where $\widetilde{p}_i$ $(i=1,\dots,n)$ *and* H *are operators and a matrix, given by the equalities* (7) *and* (8)*. Then the problem* (1), (2) *has at least one solution.*

Consider now the boundary value problem of periodic type (1), (2_1), where $\alpha_1,\dots,\alpha_n$ are arbitrary real constants. In particular, if $\alpha_1 = \dots = \alpha_n = 1$, then (1), (2_1) is a periodic problem.

The following theorem is valid.

Theorem 2. *Let there exist operators* $p_i : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}_+)$ $(i = 1,\dots,n)$ *and numbers* $\sigma_i \in \{-1,1\}$, $h_{ik} \geq 0$, $h_i \geq 0$ *such that for any* $(x_i)_{i=1}^n \in C(I;\mathbb{R}^n)$ *and for almost all* $t \in I$ *the inequalities*

$$f_i(x_1,\dots,x_n)\,\mathrm{sgn}(\sigma_i x_i(t)) \leq$$
$$\leq p_i(x_1,\dots,x_n)(t)\Big(-|x_i(t)| + \sum_{k=1}^n h_{ik}\|x_k\|_C + h_i\Big) \ \ (i = 1,\dots,n),$$
$$\int_a^b p_i(x_1,\dots,x_n)(s)\,ds > 0 \ \ (i = 1,\dots,n)$$

hold. If, moreover, the numbers α_i, σ_i *satisfy the inequalities*

$$(1 - |\alpha_i|)\sigma_i \geq 0 \ \ (i = 1,\dots,n),$$

and the matrix $H = (h_{ik})_{i,k=1}^n$ *satisfies the condition* (10)*, then the problem* (1), (2_1) *has at least one solution.*

Note that in both Theorems 1 and 2 the condition (10) is unimprovable in the sense that it cannot be replaced by the non-strict inequality

$$r(H) \leq 1. \tag{10'}$$

Indeed, it is clear that the periodic problem

$$\frac{dx_i(t)}{dt} = -\sigma_i x_i(t) + \|x_i\|_C + 1, \quad x_i(a) = x_i(b) \ \ (i = 1,\dots,n)$$

has no solution, though for this problem all the conditions of Theorem 2 are satisfied except the condition (10), instead of which the inequality (10′) holds, since in that case $H = E$, $r(H) = 1$.

Let us now consider the boundary value problem (4), (2_1).

For this problem from Theorem 2 we get

Corollary 1. *Let on the set* $I \times \mathbb{R}^{n+1}$ *the inequalities*

$$g_i(t, y_1,\dots,y_n,y_{n+1})\,\mathrm{sgn}(\sigma_i y_{n+1}) \leq$$
$$\leq p_i(t, y_1,\dots,y_n,y_{n+1})\Big(-|y_{n+1}| + \sum_{k=1}^n h_{ik}|y_k| + h_i\Big) \ \ (i = 1,\dots,n)$$

hold, where $p_i : I \times \mathbb{R}^{n+1} \to]-\infty,[$ $(i = 1,\dots,n)$ *are functions from the Carathéodory class,* h_{ik} *and* h_i *are non-negative constants, and* $\sigma_i \in \{-1,1\}$*. If, moreover, the inequalities*

$$(1 - |\alpha_i|)\sigma_i \geq 0 \ \ (i = 1,\dots,n), \quad r(H) < 1$$

are satisfied, where $H = (h_{ik})_{i,k=1}^n$*, then the problem* (4), (2_1) *has at least one solution.*

As it is noted above, the theorems proven by us cover the cases where the system (1) is superlinear or sublinear or some of equations of these systems are superlinear, and others are sublinear.

Indeed, suppose the equalities

$$g_i(t, y_1, \dots, y_n, y_{n+1}) =$$

$$= p_i(t) \exp\Big(\beta_i \sum_{k=1}^{n+1} |y_k|\Big)\big(- \sigma_i y_{n+1} + g_{0i}(t, y_1, \dots, y_n, y_{n+1})\big) \ \ (i = 1, \dots, n)$$

hold, where $\beta_i \in \mathbb{R}$, $\sigma_i \in \{-1, 1\}$, $p_i \in L(I; R_+)$ $(i = 1, \dots, n)$, and $g_{0i} : I \times \mathbb{R}^{n+1} \to \mathbb{R}$ $(i = 1, \dots, n)$ are continuous bounded functions. If, moreover, $\sigma_i(1 - |\alpha_i| \geq 0$ $(i = 1, \dots, n)$, then according to Corollary 1 the problem $(4), (2_1)$ has at least one solution. On the other hand, the i-th equation of the system (1) is superlinear if $\beta_i > 0$, and sublinear if $\beta_i < 0$. Note that in these cases the problem $(4), (2_1)$, generally speaking, is a problem at resonance since if $\alpha_i = 1$ for some $i \in \{1, \dots, n\}$, then the linear homogeneous problem $\frac{dx_i(t)}{dt} = 0, \quad x_i(a) = \alpha_i x_i(b) \ \ (i = 1, \dots, n)$ has an infinite set of solutions.

Finally, consider the problem (1),(2_2), where $t_{ijk} \in I$, $\ell_{ij} \in R$, $c_i \in R$. Put

$$\ell_{ik} = \sum_{j=1}^{n} |\ell_{ijk}|.$$

For this problem Theorem 1 takes the form

Theorem 3. *Let there exist operators $p_i : C(I; \mathbb{R}^n) \to L(I; \mathbb{R}_+)$ $(i = 1, \dots, n)$, non-negative numbers h_{ik}, h_i $(i = 1, \dots, n)$, and functions q_{ik} and $q_i \in L(I; \mathbb{R}_+)$ $(i, k = 1, \dots, n)$ such that for any $(x_k)_{k=1}^n \in C(I; \mathbb{R}^n)$ almost everywhere on I, the inequalities*

$$f_i(x_1, \dots, x_n) \operatorname{sgn}(\sigma_i x_i(t)) \leq p_i(x_1, \dots, x_n)(t)\Big(- |x_i(t)| + \\ + \sum_{k=1}^{n} h_{ik} \|x_k\|_C + h_i\Big) + \sum_{k=1}^{n} q_{ik}(t) \|x_k\|_C + q_i(t) \ \ (i = 1, \dots, n)$$

hold. If, moreover, the matrix $H = (h_{ik} + \|q_{ik}\|_L + \ell_{ik})_{i,k=1}^n$ satisfies the condition (10)*, then the problem* $(1), (2_2)$ *has at least one solution.*

For the boundary value problem $(4), (2_2)$ this theorem yields

Corollary 2. *Let on the set $I \times \mathbb{R}^n$ the inequalities*

$$g_i(t, y_1, \dots, y_n, y_{n+1}) \operatorname{sgn}\big((t - t_i) y_{n+1}\big) \leq p_i(t, y_1, \dots, y_n, y_{n+1})\Big(-|y_{n+1}| + \\ + \sum_{k=1}^{n} h_{ik} \|y_k\| + h_i\Big) + \sum_{k=1}^{n} q_{ik}(t) \|y_k\| + q_i(t) \ \ (i = 1, \dots, n)$$

be fulfilled, where $p_i : I \times \mathbb{R}^n \to]-\infty, 0[$ $(i = 1, \dots, n)$ are functions from the Carathéodory class, h_{ik} and h_i are non-negative constants, q_{ik} and $q_i \in L(I; \mathbb{R}_+)$. If, moreover, the matrix $H = (h_{ik} + \|q_{ik}\|_L + \ell_{ik})_{i,k=1}^n$ satisfies the condition (10)*, then the problem* $(1), (2_2)$ *has at least one solution.*

The above-formulated theorems are a generalization of I. Kiguradze's results [10] for the system (1). They are proved using the results of the papers [9], [11], [13].

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