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CONTENTS

Dedicated to Professor Archil Kharadze,
prominent mathematician, devoted teacher and distinguished personality,
on the occasion of his 125 birthday anniversary

(21.04.1895 – 17.12.1976)

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PROFESSOR ARCHIL KHARADZE

NIKOLOZ VAKHANIA

Professor Archil Kharadze - prominent mathematician, devoted teacher and distinguished personality - was born in April 21, 1895, in a village of Western Georgia. He attended his elementary school in the same rural area, and then in 1912 successfully finished, being awarded the Silver medal, his middle school education at a gymnasium (grammar school) in Tbilisi. His father Kirile Kharadze who had a strong appreciation towards education but did not have enough funds, still managed to send his gifted son to Moscow for higher education, and in the same year of 1912 Archil Kharadze became a student of Department of Physics and Mathematics of the famous MGU, Moscow State University. That period, like probably all other periods for Moscow State University, was excellent for the history of this university. Mathematics courses have been conducted by famous mathematicians including D. Egorov, B. Mlodzeevski, L. Larkin, N. Luzin. Very soon young A. Kharadze became one of the advanced students at the department. Being in his third year, in 1915, he made his first research and was marked by the university with official Certificate of Approval for it. He has finished university education by the end of 1916, and was officially graduated, after passing State examinations, in March of 1917 with first grade diploma and an official offer (by recommendation of Prof. D. Egorov) to remain at the university for preparing to professorship. However, because of the financial shortage, this offer was not realized, and young mathematician A. Kharadze returned to Tbilisi by fall of the same year 1917. After a few months, in May of 1918, at the age of 23, he started to work at Tbilisi University which was officially inaugurated just a few months before he became a university lecturer. In 1930 he was appointed a university professor and the head of Chair of mathematical analysis. In 1975, after several persistent applications made by him to the rector of the university, he left the position of the head of chair.

During the long-lasting pedagogical and research career, Prof. A. Kharadze had a considerable influence on Georgian mathematicians and mathematics due to his excellent many-sided research in mathematics, his devotion to teaching mathematics, his exemplary personal properties and the general attitude in diverse problems usually arising in the social life of any community of people.

Starting to teach at the university, A. Kharadze immediately faced two major problems: lack (or full absence) of mathematical terminology in the Georgian language and full absence of Georgian text-books in basic mathematical disciplines. Both of these problems caused serious difficulties in teaching, and required the urgent care. The full responsibility for this direction, as well as for many others which usually appear in any pioneering undertaking, fell naturally on the “magnificent four” of the Georgian mathematicians of the first generation Georgi Nikoladze (born in 1888, graduated from Technological Institute of St. Petersburg in 1913), Andria Razmadze (1889, Moscow State University, 1910), Nikoloz Muskhelishvili (1891, St. Petersburg University, 1915) and the youngest of them Archil Kharadze (1895, Moscow State University, 1917). They, these four, made the principal contribution to the establishment of Georgian mathematical terminology significantly improving the possibility to teach and write mathematics in Georgian. Of course, this job could not be, and by no means was, a single work done by one attempt. It was a constant care of the just mentioned founders of this initiative as well as of their followers in the next generations. And I want to mention in this respect the name of Prof. G. Chogoshvili. The work on terminology is, of course, closely connected with the writing of textbooks. One of the first Georgian mathematical textbooks was the manual “A theory of determinants” by A. Kharadze, first issued in 1920. In subsequent years two more editions of this book and also several editions of the two large textbooks in mathematical analysis and foundations of higher mathematics for non-mathematical specialities have been published by him and also in cooperation with Prof. A. Rukhadze. Later, in forties of the last century, Prof. A. Kharadze invited professors

V. Chelidze, B. Khvedelidze and I. Kartsivadze to work together with him on the project of writing a fundamental course of mathematical analysis for mathematical specialities. This work lasted several years and was completed successfully in 1950. Since than this capital book in two volumes had four editions, and still remains a good source for mathematics students to go deep in the subject.

Now I want to sketch out Prof. A. Kharadze's mathematical inheritance. I want to give an idea about scope of his mathematical interest. I cannot speak about technical details, and will just try to show some areas of his research and give some of his results only for those particular cases which allow simple formulations.

It is my very pleasant obligation to say here that during the preparation of the mathematical part of this communication I was essentially using the very interesting small book written by Prof. I. Kartsivadze and Prof. B. Khvedelidze, and published by Tbilisi State University in 1985 on the occasion of Prof. A. Kharadze's 90 anniversary.

1. We start with a qualitative approach to explicit solutions of algebraic equations of third and fourth order based on what is known as the notion of circulant determinant. A circular determinant is the determinant of a circular matrix which means a matrix any of whose row, starting from the second is a circular permutation of the first one. It was noticed by Prof. A. Kharadze that any algebraic equation of order $k = 3$ and $k = 4$ can be written as a circulant determinant $\Delta_k(x)$, and moreover, the determinant $\Delta_k(x)$ can be expressed as the product of linear forms. These two arguments for the case $k = 4$ give

$$\Delta_4(x) = \begin{vmatrix} x, & a, & b, & c \\ c, & x, & a, & b \\ b, & c, & x, & a \\ a, & b, & c, & x \end{vmatrix}$$

and $\Delta_4(x) = (x + a + b + c)(x + ia - b - ic)(x - a + b - c)(x - ia - b + ic)$ with a, b, c depending on the coefficients of the equation. Therefore, the roots of the equation can be expressed in an explicit elementary form if this is the case for the dependence of a, b, c on the coefficients which happens for some classes of the equations. The analysis of reasons why this approach did not work for $k > 4$ was given as well.

2. We continue with the introduction of special numerical sequences and closely related with them polynomials which can be regarded as a generalization of some classical objects. These polynomials have an independent interest and, besides, they are used by A. Kharadze in other areas of his research, and we will be talking about that a bit later.

3. Among geometrical investigations of Prof. A. Kharadze we note a contribution to the theory of generalized evolutes and their applications. He gave an extension of the notion of pedal of plane curve to the case of two-dimensional surfaces in three dimensional space. These ideas and results he then successfully used to establish the from of the general solution of some partial differential equations. A particular case of such equations for three independent variables is the following one

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^3} + \frac{\partial^3 u}{\partial z^3} - 3\frac{\partial^3 u}{\partial x \partial y \partial z} = 0.$$

4. Many interesting and valuable results were obtained by Prof. A. Kharadze in the area of classical mathematical analysis. We mention a few of them starting with a simple elegant result concerning the generalization of the well-known Leibnitz's criterion for alternating signs series.

Let θ be a primitive root of the equation $x^k = 1$, and $(a_n)_{n \geq 1}$ be a decreasing sequence of positive numbers tending to zero. Then the series

$$\sum_n a_n \theta^n$$

is converging and the following inequality for its remainder is valid

$$|r_n| \leq \frac{a_n}{\sin \frac{\pi}{k}} \quad \text{if} \quad k \quad \text{is even}$$

and

$$|r_n| \leq \frac{a_n}{2 \sin \frac{\pi}{2k}} \quad \text{if} \quad k \quad \text{is odd.}$$

5. Prof. A. Kharadze gave a condition for convergence of continuous functions which reminds a criterion for normal systems of holomorphic function. This could be considered as a result of his significant interest to the Montel's theory of normal families of functions.

6. It is particularly noteworthy a wide circle of papers by Prof. A. Kharadze on the localization of an intermediate point ξ in the intermediate value theorems (of Lagrange, Cauchy, Taylor, Rolley). On this subject the monograph entitled as "On application of intermediate value theorems for polynomials" was issued by him. This area is closely related with the theory of orthogonal polynomials, and Prof. A. Kharadze successfully used some generalizations of the sequences of classical orthogonal polynomials for finding the precise subintervals of localization of intermediate points ξ for diverse families of polynomials. In this directions some deep generalizations of Chakalov's and Favard's results were obtained. A very special case of an excellent theorem of Prof. A. Kharadze is Chakalov's theorem which can be formulated as follows: for the class of polynomials of degree $2n$ or $2n-1$ $(n=2,3,\dots)$ defined on the interval $(-1,1)$, intermediate points for Lagrange theorem belong to the inner interval made up by the endroots of the Legendre polynomial of order n and this is the smallest interval with this property. For example, for any polynomial of the third order $(n=2)$ these bounds are $-\frac{1}{\sqrt{3}} \le \xi \le \frac{1}{\sqrt{3}}$ (note that $\frac{1}{\sqrt{3}}$ and $-\frac{1}{\sqrt{3}}$ are the only two zeros and therefore are the endroots of the Legendre polynomial $P_2(x)=\frac{1}{2}(3x^2-1)$, $-1\le x\le 1$).

7. Another circle of papers by Prof. A. Kharadze deals with algebraic and analytical theory of polynomials in one and several variables. He studied new problems in the theory of classical orthogonal polynomials and also found some notable applications of results obtained by him in this area to diverse problems of mathematical analysis. We will mention here only two of them.

(i) It is known that every sequence of orthogonal polynomials is a Hamel basis in the linear space of polynomials. Choosing for Hamel basis generalized orthogonal polynomials introduced by Prof. A. Kharadze himself, he found areas for all zeros of polynomials in the complex plane. For example, if (H_n) denotes the sequence of generalized Hermitian polynomials, and polynomial of degree is represented as

$$\varphi(z)=\sum_{n=0}^{m} a_{nk}H_{nk}(z), \quad a_{mk}\neq 0,$$

then all zeros of the function φ are situated in the area of the complex plane described by the following inequality

$$r^k|\sin k\alpha| \le b_k\Big(1+\frac{M}{|a_{mk}|}\Big),$$

where $z=re^{ia}$, $M=\max|a_{nk}|$ $(n=0,1,\dots,m-1)$ and b_k are real (positive) numbers depending explicitly on the coefficients of the expansion of φ .

One of the consequences of the general theorem of Kharadze in this direction is the following result first proved by P. Turan: if is an even polynomial on the complex plane, and its representation by Hermitian polynomials is

$$f(z)=\sum_{k=0}^{m} c_{2k}H_{2k}(z), \quad c_{2m}\neq 0,$$

then all roots of f are situated in the area described by the inequality

$$|xy|\le \frac{5}{4}\Big(1+\frac{M}{|c_{2m}|}\Big), \quad M=\max|c_{2k}|, \quad k=0,1,\dots,m-1, \quad x+iy=z.$$

(ii) In the analytic theory of polynomials it is known Grace phenomenon meaning the following: any linear relation between the coefficients of a polynomial characterizes in a certain sense the area of the complex plane where all zeros of the derivative of the polynomial are situated. Several refinements of this classical result belong to Prof. A. Kharadze. One of them is the following: if a polynomial of degree n satisfies the condition

$$f(i)-f(0)=i[f(-i)-f(0)],$$

then the derivative of f has at least one root in the circle with radius $\operatorname{ctg}\frac{\pi}{4n}$ and center at zero.

Now let me finish my very schematic account on the mathematical inheritance of Prof. A. Kharadze, and go back to his personality. I could speak much about Archil Kharadze, his very non-trivial personality, his moral self-restriction (Solzhenitsin's expression). Yes, I could speak on this matter as much as my English would let me go on. But now I will be concise and try to express my attitude in short.

In some occasional cases professional communities have their spiritual leaders. Spiritual leaders usually do not have any official positions on top levels of the administrative staircase, neither are officially elected for the leadership. Unlike the case of official elections, when we sometimes make mistakes, in choosing the spiritual leaders mistakes are very rare, if at all. Spiritual leaders gain only new heavy duties and no benefits. And still, it is the highest moral position in the society. There is no sufficient condition for getting it but there are many necessary conditions that can be groupped around professional level, devotion and ability to serve public interests, spotless honesty. No meetings, no negotiations, no debates are needed to decide the choice. Guration date exists since no inauguration exists at all for such cases. The decision matures gradually, bit by bit to arrive to the full consensus. People believe that spiritual leaders are the conscience of their communities. Professor Archil Kharadze undoubtedly was the conscience and the honour of the Georgian mathematical community for many years. I am fully aware that no one from our mathematical community, in past years or afterwards, would question this statement.

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ON AN APPLICATION OF POWER INCREASING SEQUENCES

HÜSEYİN BOR

Abstract. In this paper we prove a general new summability factor theorem for infinite series involving quasi-power increasing sequences. Some new results are also deduced.

1. Introduction

A positive sequence (X_n) is said to be a quasi-σ-power increasing sequence if there exists a constant $K = K(\sigma, X) \geq 1$ such that $Kn^\sigma X_n \geq m^\sigma X_m$ for all $n \geq m \geq 1$ (see [18]). For any sequence (λ_n) we write that $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α and t_n^α the nth Cesàro means of order α, $\alpha > -1$, of the sequences (s_n) and (na_n), respectively, that is (see [13]),

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \quad \text{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \quad ({t_n}^1 = t_n),$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0.$$

A series $\sum a_n$ is said to be summable $|C, \alpha; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [15])

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} |u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} n^{\delta k-1} |t_n^\alpha|^k < \infty.$$

If we set δ=0, then we get the $|C, \alpha|_k$ summability (see [14]). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0,\ i \geq 1).$$

The sequence-to-sequence transformation

$$v_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v$$

defines the sequence (v_n) of the Riesz mean, or simply, the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [16]). The series $\sum a_n$ is said to be the $|\bar{N}, p_n; \delta|_k$ summable, $k \geq 1$ and $\delta \geq 0$, if (see [7])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k+k-1} |v_n - v_{n-1}|^k < \infty.$$

If we set $\delta = 0$, then we obtain the $|\bar{N}, p_n|_k$ summability (see [1]). If we take $p_n = 1$ for all n, then we get the $|C, 1; \delta|_k$ summability. Finally, if we set $\delta = 0$ and $k = 1$, then we get the $|\bar{N}, p_n|$ summability (see [20]).

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Key words and phrases. Absolute summability; Summability factors; Riesz mean; Power increasing sequences; Hölder's inequality; Minkowski's inequality.

2. Known Result

Several theorems have been proved dealing with the $|\bar{N}, p_n; \delta|_k$ summability factors of infinite series (see [3, 5–12, 17]). Among them, in [10], the following theorem has been proved.

Theorem A. *Let (X_n) be a quasi-σ-power increasing sequence for some σ $(0 < \sigma < 1)$. Suppose that there exist the sequences (β_n) and (λ_n) such that*

$$|\Delta\lambda_n| \le \beta_n, \tag{1}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{2}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n| X_n < \infty, \tag{3}$$

$$|\lambda_n| X_n = O(1). \tag{4}$$

If

$$\sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{|s_v|^k}{v X_v^{k-1}} = O(X_n) \quad as \quad n \to \infty \tag{5}$$

and (p_n) is a sequence such that

$$P_n = O(np_n), \tag{6}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{7}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\Big(\Big(\frac{P_v}{p_v}\Big)^{\delta k} \frac{1}{P_v}\Big) \quad as \quad m \to \infty, \tag{8}$$

hold, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is the $|\bar{N}, p_n; \delta|_k$ summable, $k \ge 1$ and $0 \le \delta < 1/k$.

3. The Main Result

The aim of this paper is to prove Theorem A under weaker conditions. Now, we prove the following

Theorem. *Let (X_n) be a quasi-σ-power increasing sequence for some σ $(0 < \sigma < 1)$. If the conditions (1), (2), (3), (4), (6), (7), (8) and*

$$\sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{|t_v|^k}{v X_v^{k-1}} = O(X_n) \quad as \quad n \to \infty \tag{9}$$

hold, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is the $|\bar{N}, p_n; \delta|_k$ summable, $k \ge 1$ and $0 \le \delta < 1/k$.

Remark. It should be noted that the condition (5) implies the condition (9) but the converse is need not be true (see [4, 19]).

To prove our theorem, we need the following lemmas.

Lemma 1 ([18])**.** *Under the conditions on (X_n), (β_n) and (λ_n) as as expressed in the statement of the theorem, we have the following:*

$$n X_n \beta_n = O(1),$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{10}$$

Lemma 2 ([2])**.** *If the conditions (6) and (7) are satisfied, then*

$$\Delta\left(\frac{P_n}{n^2 p_n}\right) = O\left(\frac{1}{n^2}\right)$$

4. Proof of the theorem

Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{n p_n}$. Then, by the definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{r p_r} = \frac{1}{P_n}\sum_{v=1}^{n}(P_n - P_{v-1})\frac{a_v P_v \lambda_v}{v p_v},$$

and hence

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}P_v a_v \lambda_v}{v p_v}, \quad n \geq 1, \quad (P_{-1} = 0).$$

Using Abel's transformation, we get

$$\begin{aligned}
T_n - T_{n-1} =& \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1}\Delta\left(\frac{P_{v-1}P_v\lambda_v}{v^2 p_v}\right)\sum_{r=1}^{v} r a_r + \frac{\lambda_n}{n^2}\sum_{v=1}^{n} v a_v \\
=& \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{p_v}(v+1)t_v p_v \frac{\lambda_v}{v^2} \\
+& \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v (v+1)\frac{t_v}{v^2 p_v} - \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} P_v \lambda_{v+1}(v+1)t_v \Delta(P_v/v^2 p_v) \\
+& \lambda_n t_n (n+1)/n^2 = T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |T_{n,r}|^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4.$$

Now, applying Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k + k - 1}|T_{n,1}|^k =& O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k - 1}\frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}p_v|t_v||\lambda_v|\frac{1}{v}\right\}^k \\
=& O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k - 1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k p_v|t_v|^k|\lambda_v|^k\frac{1}{v^k} \\
&\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1} \\
=& O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k p_v|t_v|^k|\lambda_v|^k\frac{1}{v^k}\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k - 1}\frac{1}{P_{n-1}} \\
=& O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k|\lambda_v|^{k-1}|\lambda_v|p_v|t_v|^k\frac{1}{v^k}\frac{1}{P_v}\left(\frac{P_v}{p_v}\right)^{\delta k} \\
=& O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}|\lambda_v|\left(\frac{1}{X_v}\right)^{k-1}|t_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^{\delta k} \\
=& O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k} v^{k-1}\frac{1}{v^k}\left(\frac{1}{X_v}\right)^{k-1}|\lambda_v||t_v|^k \\
=& O(1)\sum_{v=1}^{m}|\lambda_v|\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|t_v|^k}{v {X_v}^{k-1}} \\
=& O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k}\frac{|t_r|^k}{r {X_r}^{k-1}}
\end{aligned}$$

$$
\begin{aligned}
&+O(1)|\lambda_m|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|t_v|^k}{v{X_v}^{k-1}}\\
&=O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m\\
&=O(1)\sum_{v=1}^{m-1}\beta_v X_v+O(1)|\lambda_m|X_m=O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}
$$

by the hypotheses of the theorem and Lemma 1. Now, using (6), we obtain

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,2}|^k =&O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}p_v|\Delta\lambda_v||t_v|\right\}^k\\
=&O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k p_v|\Delta\lambda_v|^k|t_v|^k\\
&\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}\\
=&O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k p_v(\beta_v)^k|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\\
=&O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\left(\frac{P_v}{p_v}\right)^{k-1}(\beta_v)^k|t_v|^k\\
=&O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}v^{k-1}(\beta_v)^{k-1}(\beta_v)|t_v|^k\\
=&O(1)\sum_{v=1}^{m}v\beta_v\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|t_v|^k}{v{X_v}^{k-1}}\\
=&O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k}\frac{|t_r|^k}{r{X_r}^{k-1}}+O(1)m\beta_m\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|t_v|^k}{v{X_v}^{k-1}}\\
=&O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v+O(1)\sum_{v=1}^{m-1}\beta_v X_v+O(1)m\beta_m X_m\\
=&O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}
$$

by the hypotheses of the theorem and Lemma 1. Again, using Lemma 1 and Lemma 2, as in $T_{n,1}$, we have

$$
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,3}|^k=O(1)\quad\text{as}\quad m\to\infty.
$$

Finally, as in $T_{n,1}$, we have

$$
\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,4}|^k =&O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\frac{n+1}{n}\right)^k\frac{1}{n^k}|\lambda_n|^k|t_n|^k\\
=&O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}n^{k-1}\frac{1}{n^k}|\lambda_n|^{k-1}|\lambda_n||t_n|^k\\
=&O(1)\sum_{n=1}^{m}|\lambda_n|\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{|t_n|^k}{n{X_n}^{k-1}}=O(1)\quad\text{as}\quad m\to\infty.
\end{aligned}
$$

This completes the proof of the theorem. If we set δ=0, then we have a result dealing with $|\bar{N},p_n|_k$ summability factors of infinite series. Also, if we take $p_n=1$ for all n, then we obtain a new result concerning the $|C,1;\delta|_k$ summability factors of infinite series. Finally, if we set $\delta=0$ and $k=1$, then we get a result related to the $|\bar{N},p_n|$ summability factors of infinite series.

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A SYMMETRIZATION IN π-REGULAR RINGS

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Abstract. We introduce and study the so-called (m, n)-*regularly nil clean rings* by showing that these rings are, in fact, a non-trivial generalization of the classical π-regular rings. Our results somewhat supply a recent publication of the author in Turk. J. Math. (2019) and some recent assertions from an own draft (2020).

1. Introduction and Background

Throughout this paper, all rings are assumed to be associative and unital. Our standard terminology and notations are in the most part in agreement with those in [8, 9]. Specifically, we let $U(R)$ denote the set of all units in R, $Id(R)$ the set of all idempotents in R, $Nil(R)$ the set of all nilpotents in R, $J(R)$ the Jacobson radical of R, and $C(R)$ the center of R. About some of the more specific notions, we shall state them in detail below.

Let us recollect that a ring R is called *von Neumann regular* or just *regular* for short if, for every $a \in R$, there is $b \in R$ such that $a = aba$. If, in addition, $b \in U(R)$, the ring R is said to be *unit-regular*. If, however, $b \in Id(R)$, we surprisingly arrive at the so-called *Boolean* rings in which every element is an idempotent. Indeed, it is pretty easy to see that $U(R) = \{1\}$ whence R is reduced (that is, it does not possess any non-trivial nilpotent) and thus abelian (that is, each its idempotent is central). Therefore, as both $b, ab \in Id(R)$, it must be $a = a^2b = ab$, and hence $a = a.a = a^2$, as required.

In that direction, we recall also that a ring R is called π-*regular* if, for each $a \in R$, there are $n \in \mathbb{N}$ and $b \in R$, both depending on a, such that $a^n = a^nba^n$; if $b \in U(R)$, we say that R is *unit* π-*regular*. In case $b = d^n$ for some $d \in R$ and, possibly, $n \geq 2$, R is then called *perfectly regular*, as well as if $d \in U(R)$, we call R *perfectly unit-regular*. In the same vein, we recall that a ring R is said to be *strongly* π-*regular* if, for each $a \in R$, there are $n \in \mathbb{N}$ and $c \in R$, both depending on a, with the property that $a^n = a^{n+1}c = ca^{n+1}$ or, equivalently, $a^n = a^{2n}c^n$. This leads to the fact that any strongly π-regular ring is perfectly unit-regular and the latter one is obviously unit π-regular. It was established in [1] that strongly π-regular rings are always π-regular, whereas the converse is not generally true; however, it holds for abelian rings and for rings with a bounded index of nilpotence. Another interesting class of rings is the class of the so-termed π-*boolean* rings that are rings R for which, for every $a \in R$, there is $i \in \mathbb{N}$ with $a^i = a^{i+1}$. These are, certainly, strongly π-regular by taking $c = 1$. Likewise, it is a principal fact that strongly π-regular rings are unit-regular, provided they are regular.

It is worthwhile noticing that π-regularity was successfully generalized to some non-elementary ways in [6], [7] and [3], [4], respectively. In this connection, as a non-trivial extension of the aforementioned π-regular rings, it was recently defined in [4] the class of the so-called *regularly nil clean rings* as those rings R having the property that for any $r \in R$, there exists $e \in Rr \cap Id(R)$ with $r(1-e) \in Nil(R)$ (or, equivalently, $(1-e)r \in Nil(R)$. In [4, Proposition 1.3] is given the following left-right symmetric property, namely that there exists $f \in rR$ with the property $r(1-f) \in Nil(R)$ (or, in an equivalent form, $(1-f)r \in Nil(R)$. Likewise, it is proved in [4, Proposition 2.1] that π-regular rings are by themselves regularly nil clean.

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On the other hand, referring to [5], a ring R is said to be *double regularly nil clean* or just *D-regularly nil clean* for short if, for each $a \in R$, there exists $e \in (aRa) \cap Id(R)$ such that $a(1-e) \in Nil(R)$ (and hence, $(1-e)a \in Nil(R)$).

Certainly, the requirement $e \in (aRa) \cap Id(R)$ is obviously equivalent to the relation $e \in (aR) \cap (Ra) \cap Id(R)$ as the idempotent $e \in aR \cap Ra$ makes sense that $e = e.e \in aRa$.

Apparently, D-regularly nil clean rings are always regularly nil clean. Reformulating [4, Problem 3.1], an intriguing question is of whether or not the properties of being regularly nil clean and D-regularly nil clean are independent of each other, i.e., does there exist a regularly nil clean ring that is *not* D-regularly nil clean?

By what we have discussed so far, our further work is mainly motivated by the following new and more general concept:

Definition 1.1. A ring R is called *(m,n)-regularly nil clean* if, for any $a \in R$, there exist two nonnegative integers m,n and an idempotent $e \in a^m R a^n$ such that $a^m(1-e)a^n \in Nil(R)$.

It is clear that the condition $a^m(1-e)a^n \in Nil(R)$ is equivalent to $a^{m+n}(1-e) \in Nil(R)$, that is, $(1-e)a^{m+n} \in Nil(R)$ as for any two elements x, y of a ring R it follows that $xy \in Nil(R) \iff yx \in Nil(R)$.

Moreover, if $m = 0$ and $n \geq 1$, we just obtain the R-version (marked for right), while if $m \geq 1$ and $n = 0$, we obtain the L-version (marked for left), which both versions were discussed above.

2. Preliminary and Main Results

We begin here with the following technicality, which could be useful for further applications.

Lemma 2.1. *Suppose that R is a ring and m, n are nonnegative integers. Then R is (m,n)-regularly nil clean, if and only if $R/J(R)$ is (m,n)-regularly nil clean and $J(R)$ is nil.*

Proof. Before we proceed to proving this assertion, we need the following folklore fact:

If P is a ring with a nil-ideal I and if $d \in P$ with $d + I \in Id(P/I)$, then $d + I = e + I$ for some $e \in Id(P) \cap dPd$ such that $de = ed$.

The left-to-right implication, being valid in the same manner as in [4, Theorem 2.9], we will deal with the right-to-left one. So, given an arbitrary element a of R, there exists $b+J(R) \in Id(R/J(R)) \cap (a+J(R))(R/J(R))(a+J(R))$ with $(a+J(R))(1+J(R)-(b+J(R)) \in Nil(R/J(R))$. Consequently, bearing in mind the above folklore fact, there is $r \in R$ such that $b+J(R) = (a+J(R))(r+J(R))(a+J(R)) = ara + J(R) = e + J(R)$ for some $e \in Id(R) \cap (ara)R(ara) \subseteq Id(R) \cap aRa$. Furthermore,

$$\begin{aligned}(a + J(R))(1 + J(R) - (e + J(R))) &= (a + J(R))(1 - e + J(R)) \\ &= a(1-e) + J(R) \in Nil(R/J(R))\end{aligned}$$

and, therefore, there exists $m \in \mathbb{N}$ having the property that $[a(1-e)]^m \in J(R) \subseteq Nil(R)$. This means that $a(1-e) \in Nil(R)$, as required. □

Although it has been long ago known that the center of an exchange ring need not to be again exchange, the following statement is somewhat curious even in the light of [4, Proposition 2.7].

Proposition 2.2. *The center of an (m,n)-regularly nil clean ring is again an (m,n)-regularly nil clean ring.*

Proof. Letting R be such a ring and given $c \in C(R)$, we can write that $(c(1-e))^m = c^m(1-e) = 0$ for some $e \in Id(R) \cap cRc = c^2R$. What suffices to prove is that $e \in C(R)$. To do that, for all $r \in R$, it must be that $er(1-e) \in c^m R(1-e) = Rc^m(1-e) = 0$ as $e \in c^2R$ implies at once that $e = e^m \in c^m R$. Thus $er = ere$ and, by a reason of similarity, we also have $re = ere$. Hence, it now immediately follows that $er = re$, proving the claim about the centrality of e.

What remains to be shown is just that $e \in cC(R)c = c^2C(R)$. Indeed, write $e = c^2b$ for some $b \in R$. This forces that $e = c^2be = c^2y$, where $y = be = eb$ as e is central. We claim that $y \in C(R)$, as needed. In fact, for any $z \in R$, one derives that $yz(1-e) = (1-e)yz = (1-e)ebz = 0$ and that $(1-e)zy = zy(1-e) = zbe(1-e) = 0$, because $1-e \in C(R)$, which tells us that $yz = yze$ and $zy = ezy$. Further, $yz = yzc^2y = c^2yzy$ and $zy = c^2yzy$ and, finally, $yz = zy$, as claimed. □

It was proved in [4, Proposition 2.6] that the corner subring of any regularly nil clean ring is again regularly nil clean as well as in [5] the same claim was proved for D-regularly nil clean ring. The next assertion parallels to these two statements.

Proposition 2.3. *Given two integers $m, n \geq 0$, if R is an (m,n)-regularly nil clean ring, then so is the corner ring eRe for any $e \in Id(R)$. In particular, if $\mathbb{M}_n(R)$ is (m,n)-regularly nil clean, then so does R.*

Proof. Choose an arbitrary element $ere \in eRe$ for some $r \in R$. Since $ere \in R$, it follows that there is an idempotent f in R with $f \in (ere)R(ere)$ such that $(1-f)ere \in Nil(R)$. But this could be written as $ere - fere = (e-f)ere = (e-fe)ere = q \in Nil(R)$. Thus $(e-efe)ere = eq = eqe \in Nil(eRe)$ with $efe \in Id(eRe) \cap (ere)(eRe)(ere) = Id(eRe) \cap (ere)R(ere)$, because $efe = f$ and $qe = q$ so that $eq \in Nil(R)$, as expected.

The second part-half appears to be a direct consequence of the first part-half as R is always isomorphic to a corner subring of $\mathbb{M}_n(R)$. $\square$

An important, but seemingly rather difficult problem is the reciprocal implication of the last assertion, namely, if both eRe and $(1-e)R(1-e)$ are (m,n)-regularly nil clean rings, does the same hold for R, too?

Let us now denote by $\mathbb{T}_n(R)$ the upper triangular matrix ring over a ring R, where n runs over $\mathbb{N}$. The next result sheds some more light on the structure of this ring.

Proposition 2.4. *The ring $\mathbb{T}_n(R)$ is (m,n)-regularly nil clean, if and only if the ring R is (m,n)-regularly nil clean.*

Proof. It is well known that

$$\mathbb{T}_n(R)/I \cong \underbrace{R \times \cdots \times R}_{n-times}$$

for a proper nil-ideal I of $\mathbb{T}_n(R)$. So, the claim follows at once by using the standard arguments, leaving the check to the interested readers. $\square$

The next two tricky technicalities are pivotal.

Lemma 2.5. *If R is a ring and $x, y \in R$ with $x = xyx$, then for the element $y' := yxy$ the following two relations*

$(*)$ $x = xy'x$;

$(**)$ $y' = y'xy'$

are fulfilled.

Proof. About the first relationship, $xy'x = x(yxy)x = (xyx)yx = xyx$. As for the second one, $y'xy' = (yxy)x(yxy) = y(xyx)yxy = y(xyx)y = yxy = y'$, as promised. $\square$

It is worthwhile noticing that in [4] it was showed that if a is a π-regular element, that is, a^n is regular for some $n \in \mathbb{N}$, then a is regularly nil clean, too. Nevertheless, this pleasant implication perhaps cannot be happen in the situation of D-regular nil cleanness. Specifically, the following critical assertion is valid:

Proposition 2.6. *If R is a ring having an element a such that a^n is regular for some $n \geq 2$, then a is $(1,1)$-regularly nil clean of index not greater than n.*

Proof. Writing $a^n = a^n b a^n$ for some existing $b \in R$, then with Lemma 2.5 at hand, we can also write that $b = ba^n b$. Indeed, setting $b' = ba^n b$, by consulting with the cited lemma we will have $a^n = a^n b' a^n$ and $b' = b'a^n b'$, so that without loss of generality, we could replace b' via b. Furthermore, letting $e := aba^{n-1}$, we easily check that $e \in Id(R) \cap (aRa)$ – by a way of similarity we may also consider the idempotent $f = a^{n-1}ba$. By a direct inspection, one verifies that $[a(1-e)a]^n = 0$. In fact, first of all, one finds that $a(1-e)a = a^2 - a^2ba^n$ and that $[a(1-e)a]^2 = a^4 - a^4ba^n$. So, by induction, $[a(1-e)a]^n = a^{2n} - a^{2n}ba^n = 0$, as expected. $\square$

We are now ready to proceed by proving with the following

Theorem 2.7. *All π-regular rings are $(1,1)$-regularly nil clean.*

Proof. For such a ring R, letting $r \in R$, if r^2 is a regular element, we are set applying Proposition 2.6. However, if not, since $r^2 \in R$, there is an integer $k > 1$ such that $(r^2)^k = r^{2k}$ is a regular element. As $2k > 2$, again Proposition 2.6 is applicable to conclude the claim. $\square$

Now, the same idea as that in [5] can be adopted to get the following statement which could be of independent interest, as well.

Lemma 2.8. *Let V be a vector space over an arbitrary field K, let $R = End_K(V)$ whose elements are being written to the left of elements of V, and let $a \in R$. Then there exists an idempotent $e \in aRa$ such that $(a(1-e)a)^2 = 0$.*

We are now in a position to show the existence of a concrete construction of an $(1,1)$-regularly nil clean ring that is surely not π-regular.

Example 2.9. For any pair of natural numbers (m,n), there exists an (m,n)-regularly nil clean ring which is *not* π-regular.

Proof. We will restrict our attention to the pair $(1,1)$ as the argumentation follows by analogy with the situation in [4] bearing in mind Lemma 2.8. The general construction can be exhibited by induction, but it is a rather technical matter and so we leave all details to the interested reader. $\square$

In regard to our considerations alluded to above, we state the following

Problem 2.10. What is the relationship between D-regularly nil clean rings and (m,n)-regularly nil clean rings? Are they independent to each other or not?

In the case where the elements a and e from Definition 1.1 commute, these two notions are deducible one of other, i.e., they coincide.

Another question of interest could be the following: It is long ago known that a ring R is *exchange* if, for any $r \in R$, there exists $e \in Id(R) \cap rR$ such that $1-e \in (1-r)R$. This, curiously, is equivalent to the existence of an integer $k > 1$ with the properties that $e \in r^k R r^k$ and $1-e \in (1-r)^k R (1-r)^k$. This can be directly proved observing that both elements r^k and $(1-r)^k$ are commuting and co-maximal.

In that aspect, we define a ring R to be *strongly π-exchange* if, for every $x \in R$, there exists an idempotent $e \in x^n R^n$ for some natural $n > 1$ such that $1 - e \in (1-x)^n R^n$, and we define R to be *π-exchange* if, for every $x \in R$, there exists an idempotent $e \in ex^n R^n$ for some natural $n > 1$ such that $1-e \in (1-e)(1-x)^n R^n$, where $R^n = \{r^n \mid r \in R\}$ is a subset of R consisting of all n-th powers of elements from R.

So, we come to our final problem.

Problem 2.11. Determine the structure of strongly π-exchange and π-exchange rings.

In view of [2], there are rather unexpected and non-trivial examples of strongly π-exchange rings, so that the posed question seems to be hard. Indeed, in virtue of our discussion in the introductional section, are perfectly regular rings always π-exchange?

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BOUNDARY VALUE PROBLEMS OF THERMOELASTIC DIFFUSION THEORY WITH MICROTEMPERATURES AND MICROCONCENTRATIONS

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Abstract. The paper deals with the linear theory of thermoelastic diffusion for elastic isotropic and homogeneous materials with microtempeatures and microconcetrations. For the system of the corresponding differential equations of pseudo-oscillations the fundamental matrix is constructed explicitly in terms of elementary functions. With the help of Green's identities the general integral representation formula of solutions is derived by means of generalized layer and Newtonian potentials. The basic Dirichlet and Neumann type boundary value problems are formulated in appropriate function spaces and the uniqueness theorems are proved. The existence theorems for classical solutions are established by using the potential method.

1. Introduction

Construction of a refined mathematical model of continuum mechanics with regard for different physical fields and their investigation is a very important problem from the theoretical and practical points of view, due to the rapidly increasing use of composite materials in modern technological processes, as well as in geology, biology, medicine, etc.

One such refined model, a thermoelastic diffusion theory with microtemperatures and microconcentrations, is proposed by M. Aouadi, M. Ciarletta, and V, Tibullo [1]. In this paper, the dynamical problems for a thermoelastic material with diffusion, whose microelements are assumed to possess microtemperatures and microconcentrations, are considered. The constitutive and field equations of the thermodynamic for the homogeneous and isotropic bodies are derived. Using the semigroup theory for linear operators, they show that a wide class of mixed problems with appropriate initial and boundary conditions are well posed, and the asymptotic behavior of solutions is established for a sufficiently large time parameter.

Recently, in [2], a linear dynamical problem involving a thermoelastic material with diffusion, whose microelements are assumed to possess microtemperatures and microconcentrations, has been analyzed. The problem is studied from the numerical point of view, introducing a fully discrete approximation by using the finite element method and the implicit Euler scheme. A discrete stability property is established and some a priori error estimates are obtained.

The system of differential equations of thermodynamic diffusion linear theory for isotropic homogeneous elastic materials with microtemperatures and microconcentrations with respect to the displacement vector, microconcentration vector, microtemperature vector, chemical potential function and temperature function, represents a fully coupled complex system of second order partial differential equations (see [1]).

If the physical characteristics involved in the dynamical system of differential equations are time harmonic dependent (i.e., they are represented as the product of the time dependent exponential function $exp(-i\sigma t)$ with a complex parameter $\sigma = \sigma_1 + i\sigma_2$, $\sigma_1 \in \mathbb{R}$, $\sigma_2 > 0$, and a function of the spatial variable $x \in \mathbb{R}^3$), then we have the so- called *system of pseudo-oscillation equations.* The corresponding matrix differential operator is strongly elliptic, formally non-self-adjoint operator with constant coefficients.

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The present paper is devoted to the investigation of the basic boundary value problems for the system of pseudo-oscillation equations for homogeneous isotropic materials by using the potential method.

To this end, we construct the matrix of fundamental solutions explicitly in terms of elementary functions for the pseudo-oscillation equations and investigate mapping properties of the corresponding volume and layer potential operators.

Using the approaches developed in [5,6,8,12,15], with the help of the potential method we reduce the Dirichlet and Neumann type boundary value problems to the corresponding system of singular integral equations and prove the existence theorems in the space of regular vector functions.

2. Constitutive Relations and Basic Differential Equations

Denote by $\mathbb{R}^3$ the three-dimensional Euclidean space and let $\Omega^+ \subset \mathbb{R}^3$ be a bounded domain with boundary $S := \partial\Omega^+$, $\overline{\Omega^+} = \Omega^+ \cup S$. Further, let $\Omega^- = \mathbb{R}^3 \backslash \overline{\Omega^+}$. We assume that $\overline{\Omega} \in \{\overline{\Omega^+}, \overline{\Omega^-}\}$ is filled with a thermoelastic diffusion isotropic and homogeneous material with microtemperatures and microconcentration. Denote by $u = (u_1, u_2, u_3)^\top$, $C = (C_1, C_2, C_3)^\top$, and $T = (T_1, T_2, T_3)^\top$ the displacement vector, the microconcentration vector and the microtemperatures vector, respectively. By P we denote the chemical potential of material and by ϑ the temperature, measured from fixed absolute temperature T_0. We assume that T_0 is a given positive constant. The symbol $(\cdot)^\top$ denotes transposition.

Denote by t_{ij}, η_{ij}, q_{ij}, η_j, and q_j the stress tensor, the first mass diffusion flux moment tensor, the first heat flux moment tensor, the flux vector of mass diffusion, and the heat flux vector, respectively. By C^*, S^*, σ_i^*, ζ_i^*, Ω_i^*, and ϵ_i^* we denote the concentration of the diffusive material, the microentropy, the micromass, the microheat flux average, the first moment of mass diffusion, and the first moment of energy vector, respectively.

In the case of an isotropic and homogeneous thermoelastic diffusion material, with microtemperatures and microconcentration, the constitutive equations read as follows [1]

$$t_{ij} = t_{ij}(U) := \mu\,(\partial_j\, u_i + \partial_i\, u_j) + \delta_{ij}\,(\lambda_0 \,\operatorname{div} u - \gamma_2\, P - \gamma_1\, \vartheta), \tag{2.1}$$

$$\eta_{ij} = \eta_{ij}(U) := -h_4\, \delta_{ij}\, \operatorname{div} C - h_5\, \partial_j\, C_i - h_6\, \partial_i\, C_j, \tag{2.2}$$

$$q_{ij} = q_{ij}(U) := -k_4\, \delta_{ij}\, \operatorname{div} T - k_5\, \partial_j\, T_i - k_6\, \partial_i\, T_j, \tag{2.3}$$

$$\eta_i = \eta_i(U) := h_1\, C_i + h\, \partial_i\, P, \tag{2.4}$$

$$q_i = q_i(U) := k_1\, T_i + k\, \partial_i\, \vartheta, \tag{2.5}$$

$$\begin{aligned}
&\rho\, S^*(U) := \gamma_1\, \operatorname{div} u + c\vartheta + \varkappa\, P,\\
&C^*(U) := \gamma_2\, \operatorname{div} u + \varkappa\, \vartheta + m\, P,\\
&\sigma_i^*(U) := (h - h_3)\, \partial_i\, P + (h_1 - h_2)\, C_i\,,\\
&\zeta_i^*(U) := (k - k_3)\, \partial_i\, \vartheta + (k_1 - k_2)\, T_i\,,\\
&\rho\Omega_i^*(U) := -m_1\, C_i - \varkappa_1\, T_i\,,\\
&\rho\, \epsilon_i^*(U) := -\varkappa_1\, C_i - c_1\, T_i\,,
\end{aligned}$$

where $U = (u, C, T, P, \vartheta)^\top$, δ_{ij} is the Kronecker delta, $\partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, $j = 1, 2, 3$;

$$\lambda_0 = \lambda - \frac{\beta_2^2}{\varrho},\ \ \gamma_1 = \beta_1 + \frac{\overline{\omega}\beta_2}{\varrho},\ \ \gamma_2 = \frac{\beta_2}{\varrho},$$

λ and μ are Lame's constants, $\beta_1 = (3\lambda + 2\mu)\alpha_t$, $\beta_2 = (3\lambda + 2\mu)\alpha_c$, where α_t is the coefficient of linear thermal expansion and α_c is the coefficient of linear diffusion expansion; $\overline{\omega}$ and ϱ are the measures of thermodiffusion and diffusive effects, respectively; ρ is the mass density and h, k, h_j, k_j, $j = 1, 2, \ldots, 6$, are the thermoelastic material constants;

$$c = \frac{\rho c_E}{T_0} + \frac{\overline{\omega}^2}{\varrho},\quad \varkappa = \frac{\overline{\omega}}{\varrho},\quad m = \frac{1}{\varrho},$$

where c_E is the specific heat at constant strain; c_1 and m_1 are the constants of microthermal and microdiffusion conductivity, respectively; $\varkappa_1$ is measure of microthermodiffusion.

In the sequel, we assume that the above constitutive coefficients satisfy the following assumptions [1]

$$\begin{aligned}
&\rho>0,\ \ \mu>0,\ \ 3\lambda_0+2\mu>0,\ \ c>0,\ \ c_1>0,\ \ cm-\varkappa^2>0,\ \ c_1m_1-\varkappa_1^2>0,\\
&h>0,\ \ 3h_4+h_5+h_6\geq 0,\ \ h_6\pm h_5\geq 0,\ \ 4h\,h_2-(h_1+h_3)^2\geq 0,\\
&k>0,\ \ 3k_4+k_5+k_6\geq 0,\ \ k_6\pm k_5\geq 0,\ \ 4\,T_0\,k\,k_2-(\,k_1+T_0\,k_3)^2\geq 0.
\end{aligned}\tag{2.6}$$

The linear field equations of dynamics of the thermoelasticity diffusion theory with microtemperatures and microconcentrations of homogeneous and isotropic bodies have the form [1]

$$\begin{aligned}
&\mu\Delta u(x,t)+(\lambda_0+\mu)\,\mathrm{grad}\,\mathrm{div}\,u(x,t)-\gamma_2\ \mathrm{grad}\,P(x,t)-\gamma_1\ \mathrm{grad}\,\vartheta(x,t)+\rho\,F(x,t)=\rho\frac{\partial^2u(x,t)}{\partial t^2},\\
&h_6\Delta C(x,t)+(h_4+h_5)\,\mathrm{grad}\,\mathrm{div}\,C(x,t)-h_2C(x,t)-h_3\,\mathrm{grad}\,P(x,t)=m_1\frac{\partial C(x,t)}{\partial t}+\varkappa_1\frac{\partial T(x,t)}{\partial t},\\
&k_6\Delta T(x,t)+(k_4+k_5)\,\mathrm{grad}\,\mathrm{div}\,T(x,t)-k_2T(x,t)-k_3\,\mathrm{grad}\ \vartheta(x,t)-\rho\,G(x,t)\\
&\qquad\qquad\qquad=\varkappa_1\frac{\partial C(x,t)}{\partial t}+c_1\frac{\partial T(x,t)}{\partial t},\\
&-\gamma_2\,\frac{\partial}{\partial t}\ \mathrm{div}\,u(x,t)+h_1\ \mathrm{div}\,C(x,t)+h\,\Delta P(x,t)=m\frac{\partial P(x,t)}{\partial t}+\varkappa\frac{\partial\vartheta(x,t)}{\partial t},\\
&-\gamma_1\,\frac{\partial}{\partial t}\ \mathrm{div}\,u(x,t)+\frac{k_1}{T_0}\ \mathrm{div}\,T(x,t)+\frac{k}{T_0}\,\Delta\,\vartheta(x,t)+\frac{\rho}{T_0}\,s(x,t)=\varkappa\frac{\partial P(x,t)}{\partial t}+c\frac{\partial\vartheta(x,t)}{\partial t},
\end{aligned}\tag{2.7}$$

where Δ is the Laplace operator, t is the time variable, $F=(F_1,F_2,F_3)^\top$ is the body force vector per unit mass, $G=(G_1,G_2,G_3)^\top$ is the first moment of the heat source vector, s is the heat source per unit mass.

If all the vector and scalar functions in (2.7) are harmonic time dependent, i.e.,

$$\begin{aligned}
u(x,t)=&u(x)\ \exp\{-i\,t\sigma\},\ \ C(x,t)=C(x)\ \exp\{-i\,t\sigma\},\ \ T(x,t)=T(x)\ \exp\{-i\,t\sigma\},\\
P(x,t)=&P(x)\ \exp\{-i\,t\sigma\},\ \ \vartheta(x,t)=\vartheta(x)\ \exp\{-i\,t\sigma\},\\
F(x,t)=&F(x)\ \exp\{-i\,t\sigma\},\ \ G(x,t)=G(x)\ \exp\{-i\,t\sigma\},\ \ s(x,t)=s(x)\ \exp\{-i\,t\sigma\},
\end{aligned}$$

with $\sigma\in\mathbb{R}$ and $i=\sqrt{-1}$, we obtain the system of steady state oscillation equations of the thermoelastic diffusion linear theory with microtemperatures and microconcentrations:

$$\mu\Delta u(x)+(\lambda_0+\mu)\,\mathrm{grad}\,\mathrm{div}\,u(x)+\rho\sigma^2u(x)-\gamma_2\ \mathrm{grad}\,P(x)-\gamma_1\ \mathrm{grad}\,\vartheta(x)=-\rho\,F(x),\tag{2.8}$$

$$h_6\Delta C(x)+(h_4+h_5)\,\mathrm{grad}\,\mathrm{div}\,C(x)+\delta\,C(x)+i\sigma\varkappa_1T(x)-h_3\ \mathrm{grad}\,P(x)=0,\tag{2.9}$$

$$k_6\Delta T(x)+(k_4+k_5)\,\mathrm{grad}\,\mathrm{div}\,T(x)+\varkappa_0\,T(x)+i\sigma\varkappa_1C(x)-k_3\ \mathrm{grad}\,\vartheta(x)=\rho G(x),\tag{2.10}$$

$$i\sigma\gamma_2\,\mathrm{div}\,u(x)+h_1\ \mathrm{div}\,C(x)+h\,\Delta\,P(x)+i\sigma m\,P(x)+i\sigma\varkappa\,\vartheta(x)=0,\tag{2.11}$$

$$i\sigma\gamma_1\,T_0\ \mathrm{div}\,u(x)+k_1\ \mathrm{div}\,T(x)+i\sigma\varkappa\,T_0\,P(x)+k\,\Delta\,\vartheta(x)+i\sigma c\,T_0\,\vartheta(x)=-\rho\,s(x),\tag{2.12}$$

where

$$\delta=i\sigma m_1-h_2,\quad \varkappa_0=i\sigma c_1-k_2;$$

u, C, T, F, and G are complex-valued vector functions, while P, ϑ, and s are complex-valued scalar functions, and σ is a frequency parameter. If $\sigma=\sigma_1+i\,\sigma_2$ is a complex parameter with $\sigma_2\neq 0$, then the above equations are called the *pseudo–oscillation equations*, while for $\sigma=0$, they represent the *equilibrium equations of statics*. Note that the pseudo–oscillation equations are obtained from the equations of dynamical system (2.7) by the Laplace transform with the complex parameter σ.

Throughout the paper, we assume that σ is a complex parameter,

$$\sigma=\sigma_1+i\,\sigma_2,\quad \sigma_1\in\mathbb{R},\quad \sigma_2>0.\tag{2.13}$$

Let us introduce the matrix differential operator

$$L(\partial,\sigma) := \begin{bmatrix} L^{(1)}(\partial,\sigma) & L^{(6)}(\partial,\sigma) & L^{(11)}(\partial,\sigma) & L^{(16)}(\partial,\sigma) & L^{(21)}(\partial,\sigma) \\ L^{(2)}(\partial,\sigma) & L^{(7)}(\partial,\sigma) & L^{(12)}(\partial,\sigma) & L^{(17)}(\partial,\sigma) & L^{(22)}(\partial,\sigma) \\ L^{(3)}(\partial,\sigma) & L^{(8)}(\partial,\sigma) & L^{(13)}(\partial,\sigma) & L^{(18)}(\partial,\sigma) & L^{(23)}(\partial,\sigma) \\ L^{(4)}(\partial,\sigma) & L^{(9)}(\partial,\sigma) & L^{(14)}(\partial,\sigma) & L^{(19)}(\partial,\sigma) & L^{(24)}(\partial,\sigma) \\ L^{(5)}(\partial,\sigma) & L^{(10)}(\partial,\sigma) & L^{(15)}(\partial,\sigma) & L^{(20)}(\partial,\sigma) & L^{(25)}(\partial,\sigma) \end{bmatrix}_{11\times 11}, \tag{2.14}$$

where

$$\begin{aligned} L^{(1)}(\partial,\sigma) :=&(\mu\,\Delta+\rho\sigma^2)I_3+(\lambda_0+\mu)Q(\partial), \;\; L^{(2)}(\partial,\sigma) := [0]_{3\times 3}, \\ L^{(3)}(\partial,\sigma) :=&[0]_{3\times 3}, \quad L^{(4)}(\partial,\sigma) := i\sigma\gamma_2\,\nabla, \quad L^{(5)}(\partial,\sigma) := i\sigma\gamma_1\,T_0\,\nabla, \\ L^{(6)}(\partial,\sigma) :=&[0]_{3\times 3}, \quad L^{(7)}(\partial,\sigma) := (h_6\,\Delta+\delta)I_3+(h_4+h_5)Q(\partial), \\ L^{(8)}(\partial,\sigma) :=&i\sigma\varkappa_1\,I_3, \quad L^{(9)}(\partial,\sigma) := h_1\nabla, \quad L^{(10)}(\partial,\sigma) := [0]_{1\times 3}, \\ L^{(11)}(\partial,\sigma) :=&[0]_{3\times 3}, \quad L^{(12)}(\partial,\sigma) := i\sigma\varkappa_1\,I_3, \\ L^{(13)}(\partial,\sigma) :=&(k_6\Delta+\varkappa_0)I_3+(k_4+k_5)Q(\partial), \quad L^{(14)}(\partial,\sigma) := [0]_{1\times 3}, \\ L^{(15)}(\partial,\sigma) :=&k_1\,\nabla, \quad L^{(16)}(\partial,\sigma) := -\gamma_2\,\nabla^\top, \quad L^{(17)}(\partial,\sigma) := -h_3\,\nabla^\top, \\ L^{(18)}(\partial,\sigma) :=&[0]_{3\times 1}, \quad L^{(19)}(\partial,\sigma) := h\Delta+i\sigma m, \quad L^{(20)}(\partial,\sigma) := i\sigma\varkappa\,T_0, \\ L^{(21)}(\partial,\sigma) :=&-\gamma_1\nabla^\top, \quad L^{(22)}(\partial,\sigma) := [0]_{3\times 1}, \quad L^{(23)}(\partial,\sigma) := -k_3\nabla^\top, \\ L^{(24)}(\partial,\sigma) :=&i\sigma\varkappa, \quad L^{(25)}(\partial,\sigma) := k\,\Delta+i\sigma c\,T_0. \end{aligned} \tag{2.15}$$

Here and in the sequel, I_k stands for the $k\times k$ unit matrix and

$$Q(\partial) := [\partial_k\partial_j]_{3\times 3}, \; \nabla := [\partial_1,\partial_2,\partial_3], \; \partial_k = \partial/\partial_{x_k}.$$

It is easy to show that for $V = (V_1,V_2,V_3)^\top$,

$$Q(\partial)V = \operatorname{grad}\operatorname{div} V, \;\; Q(\partial) = [Q(\partial)]^\top, \;\; [Q(\partial)]^2 = \Delta Q(\partial). \tag{2.16}$$

Due to the above notation, system (2.8)–(2.12) can be rewritten in a matrix form as

$$L(\partial,\sigma)U(x) = \Phi(x),$$

where $U = (u,C,T,P,\vartheta)^\top$, $\Phi(x) = \big(-\rho\,F(x),\ 0,\ \rho\,G(x),\ 0,\ -\rho\,s(x)\big)^\top$. The operator $L(\partial,\sigma)$ is not formally self-adjoint differential operator.

Here, the central dot denotes the real scalar product $a\cdot b = \sum_{k=1}^N a_k b_k$ for $a,b\in\mathbb{C}^N$, and $[c\times d]$ denotes the cross product of two vectors $c,d\in\mathbb{C}^3$.

In view of the constitutive equations (2.1)–(2.3), the components of the stress vector $t^{(n)}(U)$, the first mass diffusion flux moment vector $\eta^{(n)}(U)$, and the first heat flux moment vector $q^{(n)}(U)$, acting on a surface element with a unit outward normal vector $n = (n_1,n_2,n_3)^\top$, read as

$$t_j^{(n)}(U) = \sum_{p=1}^3 t_{pj}(U)\,n_p, \quad \eta_j^{(n)}(U) = \sum_{p=1}^3 \eta_{pj}(U)\,n_p, \quad q_j^{(n)}(U) = \sum_{p=1}^3 q_{pj}(U)\,n_p, \quad j=1,2,3. \tag{2.17}$$

It is easy to see that (2.17) can be rewritten as

$$\begin{aligned} t^{(n)}(U) &= 2\mu\,\partial_n\,u+\lambda_0\,n\,\operatorname{div} u+\mu[n\times\operatorname{curl}\,u]-\gamma_2\,n\,P-\gamma_1\,n\,\vartheta, \\ \eta^{(n)}(U) &= -(h_5+h_6)\,\partial_n\,C-h_4\,n\,\operatorname{div} C-h_5[n\times\operatorname{curl}\,C], \\ q^{(n)}(U) &= -(k_5+k_6)\,\partial_n\,T-k_4\,n\,\operatorname{div} T-k_5[n\times\operatorname{curl}\,T], \end{aligned}$$

where $\partial_n = \partial/\partial n$ stands for the normal derivative.

Due to the constitutive equation (2.4) and (2.5), the normal components of the flux vector of mass diffusion and the heat flux vector across a surface element with a unit outward normal vector $n=(n_1,n_2,n_3)^\top$, are expressed as follows:

$$\eta_n(U)=\sum_{j=1}^{3}\eta_j(U)\,n_j=h_1\,n\cdot C+h\;\partial_n P,\qquad q_n(U)=\sum_{j=1}^{3}q_j(U)\,n_j=k_1\,n\cdot T+k\;\partial_n\vartheta.$$

Throughout the paper, we will refer the eleventh vector $(t^{(n)},\,\eta^{(n)},\,q^{(n)},\,\eta_n,\,q_n)^\top$ as the *generalized stress vector*. Further, let us introduce the generalized stress operator

$$\mathcal{P}(\partial,n):=\begin{bmatrix}\mathcal{P}^{(1)}(\partial,n) & [0]_{3\times3} & [0]_{3\times3} & -\gamma_2 n & -\gamma_1 n\\ [0]_{3\times3} & \mathcal{P}^{(2)}(\partial,n) & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{3\times3} & [0]_{3\times3} & \mathcal{P}^{(3)}(\partial,n) & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{1\times3} & h_1\,n^\top & [0]_{1\times3} & h\,\partial_n & 0\\ [0]_{1\times3} & [0]_{1\times3} & k_1\,n^\top & 0 & k\,\partial_n\end{bmatrix}_{11\times11},\tag{2.18}$$

where

$$\begin{aligned}&\mathcal{P}^{(l)}(\partial,n)=\Big[\mathcal{P}^{(l)}_{kj}(\partial,n)\Big]_{3\times3},\ l=1,2,3,\\ &\mathcal{P}^{(1)}_{kj}(\partial,n)=\mu\,\delta_{kj}\,\partial_n+\lambda_0\,n_k\,\partial_j+\mu\,n_j\,\partial_k,\\ &\mathcal{P}^{(2)}_{kj}(\partial,n)=h_6\,\delta_{kj}\,\partial_n+h_4\,n_k\,\partial_j+h_5\,n_j\,\partial_k,\\ &\mathcal{P}^{(3)}_{kj}(\partial,n)=k_6\,\delta_{kj}\,\partial_n+k_4\,n_k\,\partial_j+k_5\,n_j\,\partial_k.\end{aligned}\tag{2.19}$$

Note that for an arbitrary vector $U=(u,C,T,P,\vartheta)^\top$, the eleventh vector $\mathcal{P}(\partial,n)\,U$ is related to the components of the generalized stress vector as follows:

$$\mathcal{P}(\partial,n)\,U=(t^{(n)},\,-\eta^{(n)},\,-q^{(n)},\,\eta_n,\,q_n)^\top.$$

Let us introduce the associated boundary operator which is related to the adjoint differential operator $L^*(\partial,\sigma):=L^\top(-\partial,\sigma)$,

$$\mathcal{P}^*(\partial,n):=\begin{bmatrix}\mathcal{P}^{(1)}(\partial,n) & [0]_{3\times3} & [0]_{3\times3} & -i\sigma\gamma_2 n & -i\sigma\gamma_1 T_0 n\\ [0]_{3\times3} & \mathcal{P}^{(2)}(\partial,n) & [0]_{3\times1} & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{3\times3} & [0]_{3\times1} & \mathcal{P}^{(3)}(\partial,n) & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{1\times3} & h_3\,n^\top & [0]_{1\times3} & h\,\partial_n & [0]_{1\times3}\\ [0]_{1\times3} & [0]_{1\times3} & k_3\,n^\top & [0]_{1\times3} & k\,\partial_n\end{bmatrix}_{11\times11},\tag{2.20}$$

where $\mathcal{P}^{(j)}(\partial,n)$, $j=1,2,3$, are given by (2.19).

3. Green's Formulas

Here we assume that the boundary $\partial\Omega^+$ of Ω^+ is a Lyapunov surface and n stands for the outward unit normal vector to $\partial\Omega^+$.

Definition 3.1. A vector function $U=(u,C,T,P,\vartheta)^\top$ is said to be regular in the domain Ω^+ if $U\in C^2(\Omega^+)\cap C^1(\overline{\Omega^+})$.

For regular vector functions $U=(u,C,T,P,\vartheta)^\top$ and $U'=(u',C',T',P',\vartheta')^\top$ in the domain Ω^+, we have the following Green's formulas:

$$\int\limits_{\Omega^+}U'\cdot L(\partial,\sigma)U\,dx=\int\limits_{\partial\Omega^+}\{U'\}^+\cdot\{\mathcal{P}(\partial,n)U\}^+dS-\int\limits_{\Omega^+}E(U',U)dx,\tag{3.1}$$

$$\int\limits_{\Omega^+}U\cdot L^*(\partial,\sigma)U'\,dx=\int\limits_{\partial\Omega^+}\{U\}^+\cdot\{\mathcal{P}^*(\partial,n)U'\}^+dS-\int\limits_{\Omega^+}E(U',U)dx,\tag{3.2}$$

where the differential operator $L(\partial,\sigma)$ is given by (2.14), $L^*(\partial,\sigma)=L^\top(-\partial,\sigma)$ is the formally adjoint operator to $L(\partial,\sigma)$, the boundary operators $\mathcal{P}(\partial,n)$ and $\mathcal{P}^*(\partial,n)$ are defined by (2.18) and (2.20), respectively; the symbols $\{\cdot\}^\pm$ denote one-sided limiting values on $\partial\Omega^+$ from $\Omega^\pm$, respectively; $E(\cdot,\cdot)$ is the so-called energy bilinear form

$$\begin{aligned}E(U',U)=\ & E^{(1)}(u',u)+E^{(2)}(C',C)+E^{(3)}(T',T)-\rho\sigma^2\,u'\cdot u-(\gamma_2\,P+\gamma_1\,\vartheta)\operatorname{div}u'-\delta\,C'\cdot C\\ & -i\sigma\,\varkappa_1\,C'\cdot T+h_3\,C'\cdot\operatorname{grad}P-\varkappa_0\,T'\cdot T-i\sigma\,\varkappa_1\,T'\cdot C+k_3\,T'\cdot\operatorname{grad}\vartheta\\ & -i\sigma\,m\,P'\,P-i\sigma\,\gamma_2\,P'\operatorname{div}u-i\sigma\,\varkappa\,P'\,\vartheta+h_1\,C\cdot\operatorname{grad}P'+h\operatorname{grad}P'\cdot\operatorname{grad}P\\ & +k\operatorname{grad}\vartheta'\cdot\operatorname{grad}\vartheta-i\sigma\,c\,T_0\,\vartheta'\,\vartheta-i\sigma\,\gamma_1\,T_0\,\vartheta'\operatorname{div}u+k_1\,T\cdot\operatorname{grad}\vartheta'-i\sigma\,\varkappa\,T_0\,P\,\vartheta',\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}E^{(1)}(u',u)=&\frac{3\lambda_0+2\mu}{3}\operatorname{div}u'\operatorname{div}u\\ &+\frac{\mu}{3}\sum_{k,j=1}^{3}\Big(\frac{\partial u_k'}{\partial x_k}-\frac{\partial u_j'}{\partial x_j}\Big)\Big(\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}\Big)\\ &+\frac{\mu}{2}\sum_{k,j=1,\,k\neq j}^{3}\Big(\frac{\partial u_k'}{\partial x_j}+\frac{\partial u_j'}{\partial x_k}\Big)\Big(\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}\Big),\end{aligned}\tag{3.4}$$

$$\begin{aligned}E^{(2)}(C',C)=&\frac{3h_4+h_5+h_6}{3}\operatorname{div}C'\operatorname{div}C+\frac{h_6-h_5}{2}\operatorname{curl}C'\cdot\operatorname{curl}C\\ &+\frac{h_5+h_6}{4}\sum_{k,j=1,\,k\neq j}^{3}\Big(\frac{\partial C_k'}{\partial x_j}+\frac{\partial C_j'}{\partial x_k}\Big)\Big(\frac{\partial C_k}{\partial x_j}+\frac{\partial C_j}{\partial x_k}\Big)\\ &+\frac{h_5+h_6}{6}\sum_{k,j=1}^{3}\Big(\frac{\partial C_k'}{\partial x_k}-\frac{\partial C_j'}{\partial x_j}\Big)\Big(\frac{\partial C_k}{\partial x_k}-\frac{\partial C_j}{\partial x_j}\Big),\end{aligned}\tag{3.5}$$

$$\begin{aligned}E^{(3)}(T',T)=&\frac{3k_4+k_5+k_6}{3}\operatorname{div}T'\operatorname{div}T+\frac{k_6-k_5}{2}\operatorname{curl}T'\cdot\operatorname{curl}T\\ &+\frac{k_5+k_6}{4}\sum_{k,j=1,\,k\neq j}^{3}\Big(\frac{\partial T_k'}{\partial x_j}+\frac{\partial T_j'}{\partial x_k}\Big)\Big(\frac{\partial T_k}{\partial x_j}+\frac{\partial T_j}{\partial x_k}\Big)\\ &+\frac{k_5+k_6}{6}\sum_{k,j=1}^{3}\Big(\frac{\partial T_k'}{\partial x_k}-\frac{\partial T_j'}{\partial x_j}\Big)\Big(\frac{\partial T_k}{\partial x_k}-\frac{\partial T_j}{\partial x_j}\Big).\end{aligned}\tag{3.6}$$

With the help of relations (3.1) and (3.2) we can show that the following second Green's identity

$$\begin{aligned}&\int\limits_{\Omega^+}[U'\cdot L(\partial,\sigma)U-U\cdot L^*(\partial,\sigma)U']\,dx\\ &=\int\limits_{\partial\Omega^+}\Big[\{U'\}^+\cdot\{\mathcal{P}(\partial,n)U\}^+-\{U\}^+\cdot\{\mathcal{P}^*(\partial,n)U'\}^+\Big]\,dS\end{aligned}\tag{3.7}$$

holds.

Let us note that the differential operator

$$L(\partial):=L(\partial,0)\tag{3.8}$$

corresponds to the static equilibrium case, while the formally self-adjoint differential operator

$$L_0(\partial):=\begin{bmatrix}L_0^{(1)}(\partial) & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{3\times3} & L_0^{(7)}(\partial) & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{3\times3} & [0]_{3\times3} & L_0^{(13)}(\partial) & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & h\,\Delta & 0\\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & 0 & k\Delta\end{bmatrix}_{11\times11}\tag{3.9}$$

with

$$\begin{aligned} L_0^{(1)}(\partial) &:= \mu\Delta I_3 + (\lambda_0+\mu)Q(\partial),\\ L_0^{(7)}(\partial) &:= h_6\Delta I_3 + (h_4+h_5)Q(\partial),\\ L_0^{(13)}(\partial) &:= k_6\Delta I_3 + (k_4+k_5)Q(\partial), \end{aligned} \tag{3.10}$$

represents the principal homogeneous part of operators (2.14) and (3.8). With the help of inequalities (2.6), one can show that the differential operators $L_0(\partial)$ and $L(\partial,\sigma)$ are strongly elliptic and the following inequality

$$D_2|\xi|^2|\zeta|^2 \geq L_0(\xi)\zeta\cdot\zeta = \sum_{k,j=1}^{11} L_0(\xi)_{kj}\zeta_j\overline{\zeta_k} \geq D_1|\xi|^2|\zeta|^2$$

holds with some constants $D_k>0$ $(k=1,2)$ for an arbitrary $\xi\in\mathbb{R}^3$ and arbitrary complex vector $\zeta\in\mathbb{C}^{11}$.

4. The Matrix of Fundamental Solutions

Note that the construction of the fundamental matrix is carried out by the same method as indicated in [6,7,14]. Let $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse distributional Fourier transform in the space of tempered distributions (Schwartz space $\mathcal{S}'(\mathbb{R}^3)$), which for regular summable functions f and $\widehat{f}$ reads as follows:

$$\mathcal{F}_{x\to\xi}[f] = \int\limits_{\mathbb{R}^3} f(x)\,e^{i\,x\cdot\xi}dx = \widehat{f}(\xi), \quad \mathcal{F}^{-1}_{\xi\to x}[\widehat{f}\,] = \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}\widehat{f}(\xi)\,e^{-i\,x\cdot\xi}d\xi = f(x),$$

where $x=(x_1,x_2,x_3)$ and $\xi=(\xi_1,\xi_2,\xi_3)$. Note that for an arbitrary multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $f\in\mathcal{S}'(\mathbb{R}^3)$

$$\mathcal{F}[\partial^\alpha f] = (-i\,\xi)^\alpha\mathcal{F}[f], \quad \mathcal{F}^{-1}[\xi^\alpha\widehat{f}\,] = (i\,\partial)^\alpha\mathcal{F}^{-1}[\widehat{f}\,], \tag{4.1}$$

where $|\alpha|=\alpha_1+\alpha_2+\alpha_3$ and $\xi^\alpha=\xi_1^{\alpha_1}\,\xi_2^{\alpha_2}\,\xi_3^{\alpha_3}$. Denote by $\Gamma(x,\sigma)=[\Gamma_{kj}(x,\sigma)]_{11\times 11}$ the matrix of fundamental solutions of the operator $L(\partial,\sigma)$ (see (2.14), (2.15))

$$L(\partial,\sigma)\,\Gamma(x,\sigma) = \delta(x)\,I_{11}, \tag{4.2}$$

where $\delta(\,\cdot\,)$ is Dirac's distribution.

We represent the matrix $\Gamma(x,\sigma)$ in the blockwise form

$$\Gamma(x,\sigma) = \begin{bmatrix} \Gamma^{(1)}(x,\sigma) & \Gamma^{(2)}(x,\sigma) & \Gamma^{(3)}(x,\sigma) & \Gamma^{(4)}(x,\sigma) & \Gamma^{(5)}(x,\sigma)\\ \Gamma^{(6)}(x,\sigma) & \Gamma^{(7)}(x,\sigma) & \Gamma^{(8)}(x,\sigma) & \Gamma^{(9)}(x,\sigma) & \Gamma^{(10)}(x,\sigma)\\ \Gamma^{(11)}(x,\sigma) & \Gamma^{(12)}(x,\sigma) & \Gamma^{(13)}(x,\sigma) & \Gamma^{(14)}(x,\sigma) & \Gamma^{(15)}(x,\sigma)\\ \Gamma^{(16)}(x,\sigma) & \Gamma^{(17)}(x,\sigma) & \Gamma^{(18)}(x,\sigma) & \Gamma^{(19)}(x,\sigma) & \Gamma^{(20)}(x,\sigma)\\ \Gamma^{(21)}(x,\sigma) & \Gamma^{(22)}(x,\sigma) & \Gamma^{(23)}(x,\sigma) & \Gamma^{(24)}(x,\sigma) & \Gamma^{(25)}(x,\sigma) \end{bmatrix}_{11\times 11},$$

where

$$\begin{aligned} \Gamma^{(j)}(x,\sigma) &= \Big[\Gamma^{(j)}_{pq}(x,\sigma)\Big]_{3\times 3}, & j&=1,2,3,6,7,8,11,12,13,\\ \Gamma^{(j)}(x,\sigma) &= \Big[\Gamma^{(j)}_{pq}(x,\sigma)\Big]_{3\times 1}, & j&=4,5,9,10,14,15,\\ \Gamma^{(j)}(x,\sigma) &= \Big[\Gamma^{(j)}_{pq}(x,\sigma)\Big]_{1\times 3}, & j&=16,17,18,21,22,23, \end{aligned}$$

and $\Gamma^{(19)}(x,\sigma)$, $\Gamma^{(20)}(x,\sigma)$, $\Gamma^{(24)}(x,\sigma)$, and $\Gamma^{(25)}(x,\sigma)$ are scalar functions. By $\widehat{\Gamma}(\xi,\sigma)$ and $\widehat{\Gamma}^{(k)}(\xi,\sigma)$ we denote the Fourier transforms of the matrices $\Gamma(x,\sigma)$ and $\Gamma^{(k)}(x,\sigma)$, $k=1,2,\dots,25$. Applying the Fourier transform to equation (4.2) and taking into consideration (4.1) and the equality $\mathcal{F}[\delta(\cdot)]=1$, we get

$$L(-i\,\xi,\sigma)\,\widehat{\Gamma}(\xi,\sigma) = I_{11}. \tag{4.3}$$

We have to find $\widehat{\Gamma}(\xi,\sigma)$ from (4.3) and afterwards with the help of the inverse Fourier transform construct the fundamental matrix $\Gamma(x,\sigma)$ explicitly in terms of the standard elementary functions.

First of all, we have to represent the matrix $\widehat{\Gamma}(\xi,\sigma)=[L(-i\,\xi,\sigma)]^{-1}$ in such a form which is convenient for calculation of the inverse Fourier transform. To this end, we proceed as follows. We set $r:=|\xi|=\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}$ and introduce the notation

$$\begin{aligned}
A(\xi) :=&L^{(1)}(-i\,\xi,\sigma)=(\rho\,\sigma^2-\mu\,r^2)\,I_3-(\lambda_0+\mu)\,Q(\xi),\\
B(\xi) :=&L^{(7)}(-i\,\xi,\sigma)=(\delta-h_6\,r^2)\,I_3\,-(h_4+h_5)Q(\xi),\\
D(\xi) :=&L^{(13)}(-i\,\xi,\sigma)=(\varkappa_0-k_6r^2)\,I_3-(k_4+k_5)\,Q(\xi),
\end{aligned}\tag{4.4}$$

where $Q(\cdot)$ is defined by (2.16). Applying the relations (2.16) and (4.4) we can easily show that

$$A(\xi)=A(-\xi)=A^\top(\xi),\quad B(\xi)=B(-\xi)=B^\top(\xi),$$
$$D(\xi)=D(-\xi)=D^\top(\xi),\quad Q(\xi)=Q^\top(\xi),\quad [Q(\xi)]^2=r^2Q(\xi),$$

and the matrices A,B, and D commute to each other.

In view of (2.14)–(2.16) from (4.3) we derive

$$\begin{aligned}
&A(\xi)\,\widehat{\Gamma}^{(j)}(\xi,\sigma)+i\gamma_2\xi^\top\,\widehat{\Gamma}^{(j+15)}(\xi,\sigma)+i\gamma_1\xi^\top\,\widehat{\Gamma}^{(j+20)}(\xi,\sigma)=\delta_{1j}I_3\,,\\
&B(\xi)\,\widehat{\Gamma}^{(j+5)}(\xi,\sigma)+i\sigma\,\varkappa_1\,\widehat{\Gamma}^{(j+10)}(\xi,\sigma)+i\,h_3\,\xi^\top\,\widehat{\Gamma}^{(j+15)}(\xi,\sigma)=\delta_{2j}I_3\,,\\
&i\sigma\,\varkappa_1\,\widehat{\Gamma}^{(j+5)}(\xi,\sigma)+D(\xi)\,\widehat{\Gamma}^{(j+10)}(\xi,\sigma)+i\,k_3\,\xi^\top\,\widehat{\Gamma}^{(j+20)}(\xi,\sigma)=\delta_{3j}I_3\,,\\
&\sigma\,\gamma_2\xi\,\widehat{\Gamma}^{(j)}(\xi,\sigma)-i\,h_1\xi\,\widehat{\Gamma}^{(j+5)}(\xi,\sigma)+(i\sigma\,m-h\,r^2)\,\widehat{\Gamma}^{(j+15)}(\xi,\sigma)+i\sigma\,\varkappa\,\widehat{\Gamma}^{(j+20)}(\xi,\sigma)=\delta_{4j},\\
&\sigma\,\gamma_1\,T_0\,\xi\,\widehat{\Gamma}^{(j)}(\xi,\sigma)-i\,k_1\,\xi\,\widehat{\Gamma}^{(j+10)}(\xi,\sigma)+i\sigma\,\varkappa\,T_0\,\widehat{\Gamma}^{(j+15)}(\xi,\sigma)+(i\sigma\,c\,T_0\,-k\,r^2)\,\widehat{\Gamma}^{(j+20)}(\xi,\sigma)=\delta_{5j}\,,\\
&\qquad j=1,2,\dots,5.
\end{aligned}\tag{4.5}$$

From the system (4.5), by direct calculations, we can show that the elements of the matrix $\widehat{\Gamma}(\xi,\sigma)$ have the form

$$\begin{aligned}
\widehat{\Gamma}^{(j)}(\xi,\sigma)&=\frac{1}{\Lambda(\xi)}[a_j(\xi)I_3+b_j(\xi)Q(\xi)],\quad j=1,7,8,12,13,\\
\widehat{\Gamma}^{(j)}(\xi,\sigma)&=\frac{1}{\Lambda(\xi)}b_j(\xi)Q(\xi),\quad j=2,3,6,11,\\
\widehat{\Gamma}^{(j)}(\xi,\sigma)&=\frac{1}{\Lambda(\xi)}c_j(\xi)\xi^\top,\quad j=4,5,9,10,14,15,\\
\widehat{\Gamma}^{(j)}(\xi,\sigma)&=\frac{1}{\Lambda(\xi)}c_j(\xi)\xi,\quad j=16,17,18,21,22,23,\\
\widehat{\Gamma}^{(j)}(\xi,\sigma)&=\frac{1}{\Lambda(\xi)}a_j(\xi),\quad j=19,20,24,25.
\end{aligned}$$

Here,

$$\begin{aligned}
&\Lambda(\xi)=\det L(-i\xi,\sigma)=a'(\xi)\left(a'(\xi)+b'(\xi)r^2\right)a(\xi)\left(a(\xi)+b(\xi)r^2\right)\Lambda_0(\xi)=d_1\prod_{j=1}^{11}(r^2-\lambda_j^2),\\
&\quad a'(\xi)=\rho\,\sigma^2-\mu\,r^2=-\mu(r^2-\lambda_1^2),\quad \lambda_1^2=\rho\,\sigma^2\,\mu^{-1},\quad b'(\xi)=-(\lambda_0+\mu),\\
&\quad a'(\xi)+b'(\xi)\,r^2=\rho\,\sigma^2-(\lambda_0+2\mu)\,r^2=-(\lambda_0+2\mu)(r^2-\lambda_2^2),\quad \lambda_2^2=\rho\,\sigma^2(\lambda_0+2\mu)^{-1},\\
&\quad d_1=-\mu\,(\lambda_0+2\mu)h_0\,k_0\,h_6\,k_6\,d,\ \ d=(\lambda_0+2\mu)h_0\,k_0\,h\,k,\ \ h_0=h_4+h_5+h_6,\quad k_0=k_4+k_5+k_6;
\end{aligned}\tag{4.6}$$

$\pm\lambda_3$, $\pm\lambda_4$ and $\pm\lambda_5$, $\pm\lambda_6$ are the roots, with respect to $r=|\xi|$, of the equations $a(\xi)=0$ and $a(\xi)+b(\xi)\,r^2=0$, respectively;

$$\begin{aligned}
a(\xi)&=(h_6\,r^2-\delta)(k_6\,r^2-\varkappa_0)+\sigma^2\varkappa_1^2=h_6\,k_6\,(r^2-\lambda_3^2)(r^2-\lambda_4^2),\\
b(\xi)&=(h_4+h_5)(k_4+k_5)r^2+(k_4+k_5)(h_6\,r^2-\delta)+(h_4+h_5)(k_6\,r^2-\varkappa_0),\\
a(\xi)+b(\xi)\,r^2&=(h_0\,r^2-\delta)(k_0\,r^2-\varkappa_0)+\sigma^2\,\varkappa_1^2=\,h_0\,k_0\,(r^2-\lambda_5^2)(r^2-\lambda_6),
\end{aligned}\tag{4.7}$$

$\pm\lambda_j,\ j=7,\,8,\,\dots,11$, are the roots of the equation $\Lambda_0(\xi)=0$ with respect to $r=|\xi|$, where

$$\begin{aligned}\Lambda_0(\xi) = {}&\big[h_1\,h_3(\varkappa_0 - k_0\,r^2)\,r^2 - (i\sigma\,m - h\,r^2)(a(\xi)+b(\xi)\,r^2)\big]\,\big[i\,\sigma\,T_0\,\gamma_1^2\,r^2 - (i\,\sigma\,c\,T_0 - k\,r^2)(\rho\,\sigma^2 -\\ &- (\lambda_0+2\mu)\,r^2)\big] + \big[\varkappa\,T_0\,(a(\xi)+b(\xi)\,r^2) + \varkappa_1\,(h_1\,k_3\,T_0 + k_1\,h_3)r^2\big]\big[\sigma^2\,\varkappa\,(\,\rho\,\sigma^2 - (\lambda_0+2\mu)\,r^2) -\\ &- \sigma^2\,\gamma_1\,\gamma_2\,r^2\big] + k_1\,k_3(\,\rho\,\sigma^2 - (\lambda_0+2\mu)\,r^2)\big[h_1\,h_3\,r^2 - (i\sigma\,m - h\,r^2)(\delta - h_0\,r^2)\big]r^2 -\\ &- \sigma^2\,\gamma_1\,\gamma_2\,\varkappa\,T_0\,(a(\xi)+b(\xi)\,r^2) + i\sigma\,\gamma_2^2\,r^2\big[k_1\,k_3\,(\delta - h_0\,r^2)r^2 -\\ &- (i\,\sigma\,c\,T_0 - k\,r^2)\,(a(\xi)+b(\xi)\,r^2)\big] = -d\,\prod_{j=7}^{11}(r^2-\lambda_j^2);\end{aligned} \tag{4.8}$$

$$\begin{aligned}a_1(\xi) &= (a'(\xi)+b'(\xi)r^2)\,a(\xi)\,(a(\xi)+b(\xi)r^2)\Lambda_0(\xi),\\ a_7(\xi) &= a'(\xi)\,(a'(\xi)+b'(\xi)r^2)\,(a(\xi)+b(\xi)r^2)(\varkappa_0 - k_6\,r^2)\Lambda_0(\xi),\\ a_8(\xi) &= -i\sigma\,\varkappa_1\,a'(\xi)\,(a'(\xi)+b'(\xi)r^2)\,(a(\xi)+b(\xi)r^2)\Lambda_0(\xi),\\ a_{12}(\xi) &= -i\sigma\,\varkappa_1\,a'(\xi)\,(a'(\xi)+b'(\xi)r^2)\,(a(\xi)+b(\xi)r^2)\Lambda_0(\xi),\\ a_{13}(\xi) &= a'(\xi)\,(a'(\xi)+b'(\xi)r^2)\,(a(\xi)+b(\xi)r^2)(\delta - h_6\,r^2)\Lambda_0(\xi),\end{aligned} \tag{4.9}$$

$$\begin{aligned}a_{19}(\xi) &= a'(\xi)\,(a'(\xi)+b'(\xi)r^2)\,a(\xi)\,(a(\xi)+b(\xi)r^2)\,\gamma_{44}(\xi),\\ a_{20}(\xi) &= a'(\xi)\,(a'(\xi)+b'(\xi)r^2)\,a(\xi)\,(a(\xi)+b(\xi)r^2)\,\gamma_{45}(\xi),\\ a_{24}(\xi) &= a'(\xi)\,(a'(\xi)+b'(\xi)r^2)\,a(\xi)\,(a(\xi)+b(\xi)r^2)\,\gamma_{54}(\xi),\\ a_{25}(\xi) &= a'(\xi)\,(a'(\xi)+b'(\xi)r^2)\,a(\xi)\,(a(\xi)+b(\xi)r^2)\,\gamma_{55}(\xi),\end{aligned} \tag{4.10}$$

$$\begin{aligned}\gamma_{44}(\xi) = {}&(a(\xi)+b(\xi)r^2)\big[(a'(\xi)+b'(\xi)r^2)(i\sigma\,c\,T_0 - k\,r^2) - i\sigma\,\gamma_1^2\,T_0\,r^2\big]\\ &- k_1\,k_3(a'(\xi)+b'(\xi)r^2)(\delta - h_0\,r^2)\,r^2,\\ \gamma_{45}(\xi) = {}&i\sigma\,(a(\xi)+b(\xi)r^2)\big[\gamma_1\,\gamma_2\,r^2 - \varkappa\,(a'(\xi)+b'(\xi)r^2)\big] - i\sigma\,\varkappa_1\,h_1\,k_3\,(a'(\xi)+b'(\xi)r^2)\,r^2,\\ \gamma_{54}(\xi) = {}&i\sigma\,T_0(a(\xi)+b(\xi)r^2)\big[\gamma_1\,\gamma_2\,r^2 - \varkappa\,(a'(\xi)+b'(\xi)r^2)\big] - i\sigma\,\varkappa_1\,k_1\,h_3\,(a'(\xi)+b'(\xi)r^2)\,r^2,\\ \gamma_{55}(\xi) = {}&(a(\xi)+b(\xi)r^2)\big[(a'(\xi)+b'(\xi)r^2)(i\sigma\,m - h\,r^2) - i\sigma\,\gamma_2^2\,r^2\big]\\ &- h_1\,h_3\,(a'(\xi)+b'(\xi)r^2)(\varkappa_0 - k_0\,r^2)\,r^2.\end{aligned} \tag{4.11}$$

$$\begin{aligned}b_1(\xi) &= -a(\xi)(a(\xi)+b(\xi)r^2)\,\big\{b'(\xi)\,\Lambda_0(\xi) + i\,a'(\xi)\big[\gamma_2\,\gamma_{41}(\xi) + \gamma_1\,\gamma_{51}(\xi)\big]\big\},\\ b_2(\xi) &= -ia'(\xi)a(\xi)(a(\xi)+b(\xi)r^2)\,\big[\gamma_2\,\gamma_{42}(\xi) + \gamma_1\,\gamma_{52}(\xi)\big],\\ b_3(\xi) &= -ia'(\xi)a(\xi)(a(\xi)+b(\xi)r^2)\,\big[\gamma_2\,\gamma_{43}(\xi) + \gamma_1\,\gamma_{53}(\xi)\big],\\ b_6(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)\,\big[i\,h_3\,(k_0\,r^2 - \varkappa_0)\,\gamma_{41}(\xi) - \sigma\,\varkappa_1\,k_3\,\gamma_{51}(\xi)\big],\\ b_7(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)\,\big\{\Lambda_0(\xi)\,\big[(k_6\,r^2 - \varkappa_0)\,b(\xi) - (k_4+k_5)\,a(\xi)\big]\\ &\qquad + a(\xi)\big[ih_3\,(k_0\,r^2 - \varkappa_0)\gamma_{42}(\xi) - \sigma\,\varkappa_1\,k_3\,\gamma_{52}(\xi)\big]\big\},\\ b_8(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)\,\big\{i\sigma\,\varkappa_1\,b(\xi)\Lambda_0(\xi)\\ &\qquad + a(\xi)\big[ih_3\,(k_0\,r^2 - \varkappa_0)\gamma_{43}(\xi) - \sigma\,\varkappa_1\,k_3\,\gamma_{53}(\xi)\big]\big\},\end{aligned} \tag{4.12}$$

$$\begin{aligned}b_{11}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)\,a(\xi)\big[i\ k_3\,(h_0\,r^2 - \delta)\gamma_{51}(\xi) - \sigma\,\varkappa_1\,h_3\,\gamma_{41}(\xi)\big],\\ b_{12}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)\big\{a(\xi)\big[ik_3\,(h_0\,r^2 - \delta)\gamma_{52}(\xi)\\ &\qquad - \sigma\,\varkappa_1\,h_3\,\gamma_{42}(\xi)\big] - i\sigma\,\varkappa_1\,b(\xi)\,\Lambda_0(\xi)\big\},\\ b_{13}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)\big\{a(\xi)\big[ik_3\,(h_0\,r^2 - \delta)\gamma_{53}(\xi) - \sigma\,\varkappa_1\,h_3\,\gamma_{43}(\xi)\big]\\ &\qquad - \big[(h_4+h_5)\,a(\xi) + (\delta - h_6\,r^2)\,b(\xi)\big]\,\Lambda_0(\xi)\big\},\end{aligned} \tag{4.13}$$

$$
\begin{aligned}
c_4(\xi) &= -ia'(\xi)a(\xi)(a(\xi)+b(\xi)r^2)\left[\gamma_2\,\gamma_{44}(\xi)+\gamma_1\,\gamma_{54}(\xi)\right],\\
c_5(\xi) &= -ia'(\xi)a(\xi)(a(\xi)+b(\xi)r^2)\left[\gamma_2\,\gamma_{45}(\xi)+\gamma_1\,\gamma_{55}(\xi)\right],\\
c_9(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)\left[i\,h_3\,(k_0\,r^2-\varkappa_0)\,\gamma_{44}(\xi)-\sigma\,\varkappa_1\,k_3\,\gamma_{54}(\xi)\right],\\
c_{10}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)\left[i\,h_3\,(k_0\,r^2-\varkappa_0)\,\gamma_{45}(\xi)-\sigma\,\varkappa_1\,k_3\,\gamma_{55}(\xi)\right],\\
c_{14}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)\left[i\,k_3\,(h_0\,r^2-\delta)\,\gamma_{54}(\xi)-\sigma\,\varkappa_1\,h_3\,\gamma_{44}(\xi)\right],\\
c_{15}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)\left[i\,k_3\,(h_0\,r^2-\delta)\,\gamma_{55}(\xi)-\sigma\,\varkappa_1\,h_3\,\gamma_{45}(\xi)\right],\\
c_{16}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)(a(\xi)+b(\xi)r^2)\gamma_{41}(\xi),\\
c_{17}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)(a(\xi)+b(\xi)r^2)\gamma_{42}(\xi),\\
c_{18}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)(a(\xi)+b(\xi)r^2)\gamma_{43}(\xi),\\
c_{21}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)(a(\xi)+b(\xi)r^2)\gamma_{51}(\xi),\\
c_{22}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)(a(\xi)+b(\xi)r^2)\gamma_{52}(\xi),\\
c_{23}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)(a(\xi)+b(\xi)r^2)\gamma_{53}(\xi),
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
\gamma_{41}(\xi) =&\big(a(\xi)+b(\xi)r^2\big)\big[i\sigma^2\,\varkappa\,\gamma_1\,T_0-\sigma\,\gamma_2(i\sigma\,c\,T_0-k\,r^2)\big]\\
&+\big[i\sigma^2\,\varkappa_1\,\gamma_1\,h_1\,k_3\,T_0+\sigma k_1\,k_3\,\gamma_2(\delta-h_0\,r^2)\big]r^2,\\
\gamma_{42}(\xi) =&\sigma\,\gamma_1^2\,h_1\,T_0\,r^2(\varkappa_0-k_0\,r^2)+i\sigma^2\,\varkappa_1\,k_1\,\gamma_1\,\gamma_2\,r^2\\
&-(a'(\xi)+b'(\xi)r^2)\left[i\sigma^2\,\varkappa\,\varkappa_1\,k_1+i\,h_1\,k_1\,k_3\,r^2-i\,h_1\,(\varkappa_0-k_0\,r^2)(i\sigma\,c\,T_0-k\,r^2)\right],\\
\gamma_{43}(\xi) =&\sigma\,k_1\,\gamma_1\,\gamma_2\,(h_0\,r^2-\delta)r^2-i\sigma^2\,\varkappa_1\,h_1\,\gamma_1^2\,T_0\,r^2\\
&+(a'(\xi)+b'(\xi)r^2)\left[\sigma\,\varkappa\,k_1\,(\delta-h_0\,r^2)+\sigma\,\varkappa_1\,h_1\,\big(i\sigma\,c\,T_0-k\,r^2\big)\right],\\
\gamma_{51}(\xi) =&T_0\,(a(\xi)+b(\xi)r^2)\big[i\sigma^2\,\varkappa\,\gamma_2-\sigma\,\gamma_1(i\sigma\,m-h\,r^2)\big]\\
&+\sigma\,\gamma_1\,h_1\,h_3\,T_0(\varkappa_0-k_0\,r^2)r^2+i\sigma^2\,\varkappa_1\,k_1\,h_3\,\gamma_2\,r^2,\\
\gamma_{52}(\xi) =&\sigma\,h_1\,\gamma_1\,\gamma_2\,T_0\,(k_0\,r^2-\varkappa_0)\,r^2-i\,\sigma^2\,\varkappa_1\,k_1\,\gamma_2^2\,r^2\\
&-(a'(\xi)+b'(\xi)r^2)\left[\sigma\,\varkappa_1\,k_1\,(h\,r^2-i\sigma\,m)-\sigma\,\varkappa\,h_1\,T_0(k_0\,r^2-\varkappa_0)\right],\\
\gamma_{53}(\xi) =&\sigma\,k_1\,\gamma_2^2\,(\delta-h_0\,r^2)\,r^2+i\sigma^2\,\varkappa_1\,h_1\,\gamma_1\,\gamma_2\,T_0\,r^2+\\
&+(a'(\xi)+b'(\xi)r^2)\left[i\,k_1(\delta-h_0\,r^2)(i\sigma\,m-h\,r^2)-i\,k_1\,h_1\,h_3\,r^2-i\sigma^2\,\varkappa\,\varkappa_1\,h_1\,T_0\right],
\end{aligned}
\tag{4.15}
$$

Now, we can represent the matrix $\widehat{\Gamma}(\xi,\sigma)$ in the form

$$
\widehat{\Gamma}(\xi,\sigma)=[L(-i\,\xi,\sigma)]^{-1}=\frac{1}{\Lambda(\xi)}\,\mathcal{M}(\xi,\sigma),
\tag{4.16}
$$

where

$$
\begin{aligned}
\mathcal{M}(\xi,\sigma) :=& \begin{bmatrix}
a_1(\xi)\,I_3 & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1}\\
[0]_{3\times3} & a_7(\xi)\,I_3 & a_8(\xi)\,I_3 & [0]_{3\times1} & [0]_{3\times1}\\
[0]_{3\times3} & a_{12}(\xi)\,I_3 & a_{13}(\xi)\,I_3 & [0]_{3\times1} & [0]_{3\times1}\\
[0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & a_{19}(\xi) & a_{20}(\xi)\\
[0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & a_{24}(\xi) & a_{25}(\xi)
\end{bmatrix}\\
&+\begin{bmatrix}
b_1(\xi)\,Q(\xi) & b_2(\xi)\,Q(\xi) & b_3(\xi)\,Q(\xi) & [0]_{3\times1} & [0]_{3\times1}\\
b_6(\xi)\,Q(\xi) & b_7(\xi)\,Q(\xi) & b_8(\xi)\,Q(\xi) & [0]_{3\times1} & [0]_{3\times1}\\
b_{11}(\xi)\,Q(\xi) & b_{12}(\xi)\,Q(\xi) & b_{13}(\xi)\,Q(\xi) & [0]_{3\times1} & [0]_{3\times1}\\
[0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & 0 & 0\\
[0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & 0 & 0
\end{bmatrix}
\end{aligned}
\tag{4.17}
$$

$$+\begin{bmatrix} [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & c_4(\xi)\,\xi^\top & c_5(\xi)\,\xi^\top \\ [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & c_9(\xi)\,\xi^\top & c_{10}(\xi)\,\xi^\top \\ [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & c_{14}(\xi)\,\xi^\top & c_{15}(\xi)\,\xi^\top \\ c_{16}(\xi)\,\xi & c_{17}(\xi)\,\xi & c_{18}(\xi)\,\xi & 0 & 0 \\ c_{21}(\xi)\,\xi & c_{22}(\xi)\,\xi & c_{23}(\xi)\,\xi & 0 & 0 \end{bmatrix}.$$

Note that the entries of the matrix $\mathcal{M}(\xi,\sigma)$ are polynomials in ξ. Therefore, to invert the Fourier transform and find an explicit form for the fundamental matrix $\Gamma(x,\sigma)$ we need the roots with respect to $r=|\xi|$ of the equation

$$\Lambda(\xi) = \det L(-i\,\xi,\sigma) = 0. \tag{4.18}$$

Due to the evenness of the function $\Lambda(\xi)$ with respect to $r=|\xi|$, it is clear that if $r=r_0$ is a root of the equation $\Lambda(\xi)=0$, then so is $r=-r_0$. In view of (4.6) the roots of the equation $\Lambda(\xi)=0$ are $\pm\lambda_j$, $j=1,2,\dots,11$. For the sake of simplicity, we assume that $\lambda_j\neq\lambda_k$, for $j\neq k$, $\operatorname{Im}\lambda_j>0$, and if $\operatorname{Im}\lambda_j=0$, then $\lambda_j>0$, (see Appendix A). Therefore, in view of (4.16) we can represent the fundamental solution as

$$\Gamma(x,\sigma) = \mathcal{F}^{-1}_{\xi\to x}\Big[\widehat{\Gamma}(\xi,\sigma)\Big] = \frac{1}{d_1}\,\mathcal{F}^{-1}_{\xi\to x}\Big[\mathcal{M}(\xi,\sigma)\,\frac{1}{\Phi(r)}\Big] = \frac{1}{d_1}\,\mathcal{M}(i\,\partial,\sigma)\,\mathcal{F}^{-1}_{\xi\to x}\Big[\,\frac{1}{\Phi(r)}\,\Big], \tag{4.19}$$

where

$$\Phi(r) = \prod_{j=1}^{11}(r^2-\lambda_j^2), \qquad d_1 = -\mu\,(\lambda_0+2\mu)\,h_0\,k_0\,h_6\,k_6\,d.$$

Note that

$$\frac{1}{\Phi(r)} = \sum_{j=1}^{11}\frac{p_j}{r^2-\lambda_j^2},$$

where the parameters $p_1,p_2,\dots,p_{11}$ solve the system of linear algebraic equations

$$\lambda_1^{2m}\,p_1+\lambda_2^{2m}\,p_2+\cdots+\lambda_{11}^{2m}\,p_{11}=0, \quad m=0,1,\dots,9,$$

$$\lambda_1^{20}\,p_1+\lambda_2^{20}\,p_2+\cdots+\lambda_{11}^{20}\,p_{11}=1.$$

They can be represented as follows:

$$p_j = \left[\prod_{l=1,\,l\neq j}^{11}(\lambda_l^2-\lambda_j^2)\right]^{-1}.$$

Note that if $\operatorname{Im}\lambda_j\geq 0$, then

$$\mathcal{F}^{-1}_{\xi\to x}\left[\frac{1}{r^2-\lambda_j^2}\right] = \frac{e^{i\,\lambda_j|x|}}{4\pi|x|}.$$

Therefore,

$$\mathcal{F}^{-1}_{\xi\to x}\Big[\,\frac{1}{\Phi(r)}\,\Big] = \frac{1}{4\pi}\sum_{j=1}^{11}p_j\,\frac{e^{i\,\lambda_j|x|}}{|x|}.$$

Now, from (4.19), we deduce

$$\Gamma(x,\sigma) = \frac{1}{4\,\pi\,d_1}\,\mathcal{M}(i\,\partial,\sigma)\,\sum_{j=1}^{11}p_j\,\frac{e^{i\,\lambda_j|x|}}{|x|}, \tag{4.20}$$

or

$$\Gamma(x,\sigma) = \frac{1}{4\,\pi\,d_1}\mathcal{M}(i\,\partial,\sigma)\,\Psi(x,\sigma)\,,$$

where the differential operator $\mathcal{M}(i\,\partial,\sigma)$ is given by (4.17) with $i\,\partial$ for ξ and

$$\Psi(x,\sigma)=\sum_{j=1}^{11} p_j\,\frac{e^{i\,\lambda_j|x|}}{|x|}\,.$$

We can simplify $\mathcal{M}(i\,\partial,\sigma)\Psi(x,\sigma)$ and rewrite the fundamental solution in a more explicit form. To this end, let us note that

$$\Delta\,\frac{e^{i\,\lambda_j|x|}}{|x|}=-\lambda_j^2\,\frac{e^{i\,\lambda_j|x|}}{|x|},\quad |x|\neq 0,$$

and apply formulas (4.7)–(4.15) to obtain

$$a(i\,\partial)\Psi(x,\sigma)=\sum_{j=1}^{11} p_j\,a^{(j)}\,\frac{e^{i\,\lambda_j|x|}}{|x|},\qquad b(i\,\partial)\Psi(x,\sigma)=\sum_{j=1}^{11} p_j\,b^{(j)}\,\frac{e^{i\,\lambda_j|x|}}{|x|},$$

$$a_l(i\,\partial)\Psi(x,\sigma)=\sum_{j=1}^{11} p_j a_l^{(j)}\,\frac{e^{i\,\lambda_j|x|}}{|x|}\,,\quad l=1,7,8,12,13,19,20,24,25,$$

$$b_l(i\,\partial)\Psi(x,\sigma)=\sum_{j=1}^{11} p_j b_l^{(j)}\,\frac{e^{i\,\lambda_j|x|}}{|x|}\,,\quad l=1,2,3,6,7,8,11,12,13,$$

$$c_l(i\,\partial)\Psi(x,\sigma)=\sum_{j=1}^{11} p_j c_l^{(j)}\,\frac{e^{i\,\lambda_j|x|}}{|x|}\,,\quad l=4,5,9,10,14,15,16,17,18,21,22,23,$$

where

$$a^{(j)}=h_6\,k_6(\lambda_j^2-\lambda_3^2)(\lambda_j^2-\lambda_4^2),$$

$$b^{(j)}=(h_4+h_5)(k_4+k_5)\lambda_j^2+(k_4+k_5)(h_6\,\lambda_j^2-\delta)+(h_4+h_5)(k_6\,\lambda_j^2-\varkappa_0),$$

$$a_1^{(j)}=(\lambda_0+2\mu)h_0\,k_0\,h_6\,k_6\,d\,\prod_{l=2}^{11}(\lambda_j^2-\lambda_l^2),$$

$$a_7^{(j)}=-\mu\,(\lambda_0+2\mu)h_0\,k_0\,d\,(\varkappa_0-k_6\,\lambda_j^2)\,\prod_{l=1}^{2}(\lambda_j^2-\lambda_l^2)\,\prod_{l=5}^{11}(\lambda_j^2-\lambda_l^2),$$

$$a_8^{(j)}=i\sigma\,\varkappa_1\,\mu\,(\lambda_0+2\mu)h_0\,k_0\,d\,\prod_{l=1}^{2}(\lambda_j^2-\lambda_l^2)\,\prod_{l=5}^{11}(\lambda_j^2-\lambda_l^2),$$

$$a_{12}^{(j)}=a_8^{(j)},$$

$$a_{13}^{(j)}=-\mu\,(\lambda_0+2\mu)h_0\,k_0\,d\,(\delta-h_6\,\lambda_j^2)\prod_{l=1}^{2}(\lambda_j^2-\lambda_l^2)\,\prod_{l=5}^{11}(\lambda_j^2-\lambda_l^2),$$

$$a_{19}^{(j)}=\mu\,(\lambda_0+2\mu)h_0\,k_0\,h_6\,k_6\,\prod_{l=1}^{6}(\lambda_j^2-\lambda_l^2)\,\gamma_{44}^{(j)},$$

$$a_{20}^{(j)}=\mu\,(\lambda_0+2\mu)h_0\,k_0\,h_6\,k_6\,\prod_{l=1}^{6}(\lambda_j^2-\lambda_l^2)\,\gamma_{45}^{(j)},$$

$$a_{24}^{(j)}=\mu\,(\lambda_0+2\mu)h_0\,k_0\,h_6\,k_6\,\prod_{l=1}^{6}(\lambda_j^2-\lambda_l^2)\,\gamma_{54}^{(j)},$$

$$a_{25}^{(j)}=\mu\,(\lambda_0+2\mu)h_0\,k_0\,h_6\,k_6\,\prod_{l=1}^{6}(\lambda_j^2-\lambda_l^2)\,\gamma_{55}^{(j)},$$

$$\gamma_{44}^{(j)}=h_0\,k_0\,(\lambda_j^2-\lambda_5^2)(\lambda_j^2-\lambda_6^2)\big[(\lambda_0+2\mu)\,(\lambda_j^2-\lambda_2^2)\,(k\,\lambda_j^2-i\sigma\,c\,T_0)-i\sigma\,\gamma_1^2\lambda_j^2\big]$$

$$
\begin{aligned}
&\quad + k_1\, k_3\, (\lambda_0+2\mu)\, (\lambda_j^2-\lambda_2^2)(\delta-h_0\,\lambda_j^2)\lambda_j^2 \\
\gamma_{45}^{(j)} &= i\sigma\, h_0\, k_0\, (\lambda_j^2-\lambda_5^2)(\lambda_j^2-\lambda_6^2)\big[\gamma_1\,\gamma_2\lambda_j^2+\varkappa\,(\lambda_0+2\mu)\,(\lambda_j^2-\lambda_2^2)\big] \\
&\quad + i\sigma\,\varkappa_1\, h_1\, k_3(\lambda_0+2\mu)\,(\lambda_j^2-\lambda_2^2)\lambda_j^2, \\
\gamma_{54}^{(j)} &= i\sigma\, h_0\, k_0\, T_0\, (\lambda_j^2-\lambda_5^2)(\lambda_j^2-\lambda_6^2)\big[\gamma_1\,\gamma_2\lambda_j^2+\varkappa\,(\lambda_0+2\mu)\,(\lambda_j^2-\lambda_2^2)\big] \\
&\quad + i\,\sigma\,\varkappa_1\, k_1\, h_3(\lambda_0+2\mu)\,(\lambda_j^2-\lambda_2^2)\lambda_j^2, \\
\gamma_{55}^{(j)} &= h_0\, k_0\, (\lambda_j^2-\lambda_5^2)(\lambda_j^2-\lambda_6^2)\big[(\lambda_0+2\mu)\,(\lambda_j^2-\lambda_2^2)(h\,\lambda_j^2-i\,\sigma\, m)-i\sigma\,\gamma_2^2\,\lambda_j^2\big] \\
&\quad + h_1\, h_3(\lambda_0+2\mu)\,(\lambda_j^2-\lambda_2^2)(\varkappa_0-k_0\,\lambda_j^2)\lambda_j^2, \\
b_1^{(j)} &= -h_0\, k_0\, h_6\, k_6\,\Big\{(\lambda_0+\mu)\, d\prod_{l=3}^{11}(\lambda_j^2-\lambda_l^2)-i\mu(\lambda_j^2-\lambda_1^2)\prod_{l=3}^{6}(\lambda_j^2-\lambda_l^2)\big[\gamma_2\,\gamma_{41}^{(j)}+\gamma_1\,\gamma_{51}^{(j)}\big]\Big\}, \\
b_2^{(j)} &= i\,\mu\, h_0\, k_0\, h_6\, k_6\,(\lambda_j^2-\lambda_1^2)\prod_{l=3}^{6}(\lambda_j^2-\lambda_l^2)\big[\gamma_2\,\gamma_{42}^{(j)}+\gamma_1\,\gamma_{52}^{(j)}\big], \\
b_3^{(j)} &= i\,\mu\, h_0\, k_0\, h_6\, k_6\,(\lambda_j^2-\lambda_1^2)\prod_{l=3}^{6}(\lambda_j^2-\lambda_l^2)\big[\gamma_2\,\gamma_{43}^{(j)}+\gamma_1\,\gamma_{53}^{(j)}\big], \\
b_6^{(j)} &= \mu\,(\lambda_0+2\mu)\, h_6\, k_6\,\prod_{l=1}^{4}(\lambda_j^2-\lambda_l^2)\big[ih_3(k_0\lambda_j^2-\varkappa_0)\gamma_{41}^{(j)}-\sigma\,\varkappa_1\, k_3\,\gamma_{51}^{(j)}\big], \\
b_7^{(j)} &= \mu\,(\lambda_0+2\mu)(\lambda_j^2-\lambda_1^2)(\lambda_j^2-\lambda_2^2)\Big\{-d\prod_{l=7}^{11}(\lambda_j^2-\lambda_l^2)\big[(k_6\lambda_j^2-\varkappa_0)\, b^{(j)}-(k_4+k_5)a^{(j)}\big] \\
&\quad + a^{(j)}\big[ih_3(k_0\lambda_j^2-\varkappa_0)\gamma_{42}^{(j)}-\sigma\,\varkappa_1\, k_3\,\gamma_{52}^{(j)}\big]\Big\}, \\
b_8^{(j)} &= \mu\,(\lambda_0+2\mu)(\lambda_j^2-\lambda_1^2)(\lambda_j^2-\lambda_2^2)\Big\{-i\sigma\,\varkappa_1\, d\, b^{(j)}\prod_{l=7}^{11}(\lambda_j^2-\lambda_l^2) \\
&\quad + a^{(j)}\big[ih_3(k_0\lambda_j^2-\varkappa_0)\gamma_{43}^{(j)}-\sigma\,\varkappa_1\, k_3\,\gamma_{53}^{(j)}\big]\Big\}, \\
b_{11}^{(j)} &= \mu\,(\lambda_0+2\mu)h_6\, k_6\,\prod_{l=1}^{4}(\lambda_j^2-\lambda_l^2)\big[ik_3(h_0\lambda_j^2-\delta)\gamma_{51}^{(j)}-\sigma\,\varkappa_1\, h_3\,\gamma_{41}^{(j)}\big], \\
b_{12}^{(j)} &= \mu\,(\lambda_0+2\mu)(\lambda_j^2-\lambda_1^2)(\lambda_j^2-\lambda_2^2)\Big\{a^{(j)}\big[ik_3(h_0\lambda_j^2-\delta)\gamma_{52}^{(j)}-\sigma\,\varkappa_1\, h_3\,\gamma_{42}^{(j)}\big]+i\sigma\varkappa_1 d\, b^{(j)}\prod_{l=7}^{11}(\lambda_j^2-\lambda_l^2)\Big\}, \\
b_{13}^{(j)} &= \mu\,(\lambda_0+2\mu)(\lambda_j^2-\lambda_1^2)(\lambda_j^2-\lambda_2^2)\Big\{a^{(j)}\big[ik_3(h_0\lambda_j^2-\delta)\gamma_{53}^{(j)}-\sigma\,\varkappa_1\, h_3\,\gamma_{43}^{(j)}\big] \\
&\quad + d\prod_{l=7}^{11}(\lambda_j^2-\lambda_l^2)\big[(h_4+h_5)a^{(j)}+(\delta-h_6\lambda_j^2)b^{(j)}\big]\Big\}, \\
c_4^{(j)} &= i\,\mu\, h_0\, k_0\, h_6\, k_6\,(\lambda_j^2-\lambda_1^2)\prod_{l=3}^{6}(\lambda_j^2-\lambda_l^2)\big[\gamma_2\,\gamma_{44}^{(j)}+\gamma_1\,\gamma_{54}^{(j)}\big], \\
c_5^{(j)} &= i\,\mu\, h_0\, k_0\, h_6\, k_6\,(\lambda_j^2-\lambda_1^2)\prod_{l=3}^{6}(\lambda_j^2-\lambda_l^2)\big[\gamma_2\,\gamma_{45}^{(j)}+\gamma_1\,\gamma_{55}^{(j)}\big], \\
c_9^{(j)} &= \mu\,(\lambda_0+2\mu)\, h_6\, k_6\,\prod_{l=1}^{4}(\lambda_j^2-\lambda_l^2)\big[ih_3\,(k_0\lambda_j^2-\varkappa_0)\,\gamma_{44}^{(j)}-\sigma\,\varkappa_1\, k_3\,\gamma_{54}^{(j)}\big], \\
c_{10}^{(j)} &= \mu\,(\lambda_0+2\mu)\, h_6\, k_6\,\prod_{l=1}^{4}(\lambda_j^2-\lambda_l^2)\big[ih_3\,(k_0\lambda_j^2-\varkappa_0)\,\gamma_{45}^{(j)}-\sigma\,\varkappa_1\, k_3\,\gamma_{55}^{(j)}\big],
\end{aligned}
$$

$$c_{14}^{(j)} = \mu\,(\lambda_0 + 2\mu)\,h_6\,k_6\,\prod_{l=1}^{4}(\lambda_j^2 - \lambda_l^2)\big[ik_3\,(h_0\lambda_j^2 - \delta)\,\gamma_{54}^{(j)} - \sigma\,\varkappa_1\,h_3\,\gamma_{44}^{(j)}\big],$$

$$c_{15}^{(j)} = \mu\,(\lambda_0 + 2\mu)\,h_6\,k_6\,\prod_{l=1}^{4}(\lambda_j^2 - \lambda_l^2)\big[ik_3\,(h_0\lambda_j^2 - \delta)\,\gamma_{55}^{(j)} - \sigma\,\varkappa_1\,h_3\,\gamma_{45}^{(j)}\big],$$

$$c_{16}^{(j)} = \mu\,(\lambda_0 + 2\mu)\,h_6\,k_6\,h_0\,k_0\,\prod_{l=1}^{6}(\lambda_j^2 - \lambda_l^2)\,\gamma_{41}^{(j)},$$

$$c_{17}^{(j)} = \mu\,(\lambda_0 + 2\mu)\,h_6\,k_6\,h_0\,k_0\prod_{l=1}^{6}(\lambda_j^2 - \lambda_l^2)\,\gamma_{42}^{(j)},$$

$$c_{18}^{(j)} = \mu\,(\lambda_0 + 2\mu)\,h_6\,k_6\;h_0\,k_0\prod_{l=1}^{6}(\lambda_j^2 - \lambda_l^2)\,\gamma_{43}^{(j)},$$

$$c_{21}^{(j)} = \mu\,(\lambda_0 + 2\mu)\,h_6\,k_6\;h_0\,k_0\prod_{l=1}^{6}(\lambda_j^2 - \lambda_l^2)\,\gamma_{51}^{(j)},$$

$$c_{22}^{(j)} = \mu\,(\lambda_0 + 2\mu)\,h_6\,k_6\;h_0\,k_0\prod_{l=1}^{6}(\lambda_j^2 - \lambda_l^2)\,\gamma_{52}^{(j)},$$

$$c_{23}^{(j)} = \mu\,(\lambda_0 + 2\mu)\,h_6\,k_6\;h_0\,k_0\prod_{l=1}^{6}(\lambda_j^2 - \lambda_l^2)\,\gamma_{53}^{(j)},$$

$$\begin{aligned}
\gamma_{41}^{(j)} &= h_0\,k_0\,(\lambda_j^2 - \lambda_5^2)(\lambda_j^2 - \lambda_6^2)\big[i\sigma^2\,\varkappa\,\gamma_1\,T_0 - \sigma\gamma_2(i\sigma\,c\,T_0 - k\lambda_j^2)\big]\\
&\quad + \big[i\sigma^2\,\varkappa_1\,\gamma_1\,h_1\,k_3T_0 + \sigma\,k_1\,k_3\,\gamma_2(\delta - h_0\,\lambda_j^2)\big]\,\lambda_j^2,\\
\gamma_{42}^{(j)} &= \sigma\,h_1\,\gamma_1^2\,T_0\,\lambda_j^2(\varkappa_0 - k_0\,\lambda_j^2) + i\sigma^2\,\varkappa_1\,k_1\,\gamma_1\gamma_2\,\lambda_j^2\\
&\quad + (\lambda_0 + 2\mu)(\lambda_j^2 - \lambda_2^2)\big[i\sigma^2\,\varkappa\,\varkappa_1\,k_1 + i\,h_1\,k_1\,k_3\lambda_j^2 - ih_1(\varkappa_0 - k_0\lambda_j^2)(i\sigma\,c\,T_0 - k\,\lambda_j^2)\big],\\
\gamma_{43}^{(j)} &= \sigma\,k_1\,\gamma_1\,\gamma_2\,(h_0\,\lambda_j^2 - \delta)\lambda_j^2 - i\sigma^2\,\varkappa_1\,h_1\,\gamma_1^2\,T_0\,\lambda_j^2\\
&\quad - (\lambda_0 + 2\mu)(\lambda_j^2 - \lambda_2^2)\,\big[\sigma\,\varkappa\,k_1\,(\delta - h_0\,\lambda_j^2) + \sigma\,\varkappa_1\,h_1\,\big(i\sigma\,c\,T_0 - k\,\lambda_j^2\big)\big],\\
\gamma_{51}^{(j)} &= h_0\,k_0\,T_0\,(\lambda_j^2 - \lambda_5^2)(\lambda_j^2 - \lambda_6^2)\big[i\sigma^2\,\varkappa\,\gamma_2 - \sigma\,\gamma_1(i\sigma\,m - h\,\lambda_j^2)\big]\\
&\quad + \sigma\,\gamma_1\,h_1\,h_3\,T_0(\varkappa_0 - k_0\,\lambda_j^2)\lambda_j^2 + i\sigma^2\,\varkappa_1\,k_1\,h_3\,\gamma_2\,\lambda_j^2,\\
\gamma_{52}^{(j)} &= \sigma\,h_1\,\gamma_1\,\gamma_2\,T_0\,(k_0\,\lambda_j^2 - \varkappa_0)\,\lambda_j^2 - i\,\sigma^2\,\varkappa_1\,k_1\,\gamma_2^2\,\lambda_j^2\\
&\quad + (\lambda_0 + 2\mu)(\lambda_j^2 - \lambda_2^2)\,\big[\sigma\,\varkappa_1\,k_1\,(h\,\lambda_j^2 - i\sigma\,m) - \sigma\,\varkappa\,h_1\,T_0(k_0\,\lambda_j^2 - \varkappa_0)\big],\\
\gamma_{53}^{(j)} &= \sigma\,k_1\,\gamma_2^2\,(\delta - h_0\,\lambda_j^2)\,\lambda_j^2 + i\sigma^2\,\varkappa_1\,h_1\,\gamma_1\,\gamma_2\,T_0\,\lambda_j^2\\
&\quad - (\lambda_0 + 2\mu)(\lambda_j^2 - \lambda_2^2)\,\big[i\,k_1(\delta - h_0\,\lambda_j^2)(i\sigma\,m - h\,\lambda_j^2) - i\,k_1\,h_1\,h_3\,\lambda_j^2 - i\sigma^2\,\varkappa\,\varkappa_1\,h_1\,T_0\big],
\end{aligned}$$

From (4.17) and (4.19), for the fundamental matrix, we get the following representation:

$$\Gamma(x,\sigma) = \frac{1}{4\,\pi\,d_1}\left\{\begin{bmatrix} \Psi_1(x,\sigma)\,I_3 & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{3\times3} & \Psi_7(x,\sigma)\,I_3 & \Psi_8(x,\sigma)\,I_3 & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{3\times3} & \Psi_{12}(x,\sigma)\,I_3 & \Psi_{13}(x,\sigma)I_3 & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & \Psi_{19}(x,\sigma) & \Psi_{20}(x,\sigma)\\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & \Psi_{24}(x,\sigma) & \Psi_{25}(x,\sigma) \end{bmatrix}\right.$$

$$
+\begin{bmatrix} Q(\partial)\widetilde{\Psi}_1(x,\sigma) & Q(\partial)\widetilde{\Psi}_2(x,\sigma) & Q(\partial)\widetilde{\Psi}_3(x,\sigma) & [0]_{3\times 1} & [0]_{3\times 1} \\ Q(\partial)\widetilde{\Psi}_6(x,\sigma) & Q(\partial)\widetilde{\Psi}_7(x,\sigma) & Q(\partial)\widetilde{\Psi}_8(x,\sigma) & [0]_{3\times 1} & [0]_{3\times 1} \\ Q(\partial)\widetilde{\Psi}_{11}(x,\sigma) & Q(\partial)\widetilde{\Psi}_{12}(x,\sigma) & Q(\partial)\widetilde{\Psi}_{13}(x,\sigma) & [0]_{3\times 1} & [0]_{3\times 1} \\ [0]_{1\times 3} & [0]_{1\times 3} & [0]_{1\times 3} & 0 & 0 \\ [0]_{1\times 3} & [0]_{1\times 3} & [0]_{1\times 3} & 0 & 0 \end{bmatrix} \tag{4.21}
$$

$$
\left.+\begin{bmatrix} [0]_{3\times 3} & [0]_{3\times 3} & [0]_{3\times 3} & \nabla^\top \Psi'_4(x,\sigma) & \nabla^\top \Psi'_5(x,\sigma) \\ [0]_{3\times 3} & [0]_{3\times 3} & [0]_{3\times 3} & \nabla^\top \Psi'_9(x,\sigma) & \nabla^\top \Psi'_{10}(x,\sigma) \\ [0]_{3\times 3} & [0]_{3\times 3} & [0]_{3\times 3} & \nabla^\top \Psi'_{14}(x,\sigma) & \nabla^\top \Psi'_{15}(x,\sigma) \\ \nabla\Psi'_{16}(x,\sigma) & \nabla\Psi'_{17}(x,\sigma) & \nabla\Psi'_{18}(x,\sigma) & 0 & 0 \\ \nabla\Psi'_{21}(x,\sigma) & \nabla\Psi'_{22}(x,\sigma) & \nabla\Psi'_{23}(x,\sigma) & 0 & 0 \end{bmatrix}\right\},
$$

where

$$
\begin{aligned}
\Psi_l(x,\sigma) &= \sum_{j=1}^{11} p_j\, a_l^{(j)} \frac{e^{i\,\lambda_j |x|}}{|x|}, \qquad l = 1,7,8,12,13,19,20,24,25, \\
\widetilde{\Psi}_l(x,\sigma) &= -\sum_{j=1}^{11} p_j\, b_l^{(j)} \frac{e^{i\,\lambda_j |x|}}{|x|}, \qquad l = 1,2,3,6,7,8,11,12,13, \\
\Psi'_l(x,\sigma) &= i\sum_{j=1}^{11} p_j\, c_l^{(j)} \frac{e^{i\,\lambda_j |x|}}{|x|}, \quad l = 4,5,9,10,14,15,16,17,18,21,22,23.
\end{aligned}
$$

Remark 4.1. Note that (4.20) can be rewritten in the form

$$
\Gamma(x,\sigma) = \sum_{j=1}^{11} \Phi^{(j)}(x,\sigma), \tag{4.22}
$$

where

$$
\Phi^{(j)}(x,\sigma) = \frac{p_j}{4\pi\, d_1}\, \mathcal{M}(i\,\partial,\sigma)\, \frac{e^{i\lambda_j |x|}}{|x|}, \tag{4.23}
$$

and $\mathcal{M}(i\,\partial,\sigma)$ is defined by (4.17). Since $\mathcal{M}(i\,\partial,\sigma)$ is a matrix differential operator with constant coefficients, from the representation (4.23) it follows that the entries of the matrix $\Phi^{(j)}(x,\sigma) = \Big[\Phi^{(j)}_{pq}(x,\sigma)\Big]_{11\times 11}$ are metaharmonic functions corresponding to the wave number λ_j, i.e., they are solutions of the Helmholtz equation

$$
(\Delta + \lambda_j^2)\, \Phi^{(j)}_{pq}(x,\sigma) = 0, \quad |x| \neq 0,
$$

and decay exponentially at infinity:

$$
\frac{\partial}{\partial |x|}\, \Phi^{(j)}_{pq}(x,\sigma) - i\,\lambda_j\, \Phi^{(j)}_{pq}(x,\sigma) = \exp\{-\mathrm{Im}\,\lambda_j\, |x|\}\, O(|x|^{-2}), \quad p,q = \overline{1,11},
$$

as $|x| \to +\infty$. The entries of the matrix $\Phi^{(j)}(x,\sigma)$ and its derivatives likewise satisfy at infinity the following decay conditions [16]:

$$
\begin{aligned}
&\Phi^{(j)}_{pq}(x,\sigma) = \exp\{-\mathrm{Im}\,\lambda_j\, |x|\}\, O(|x|^{-1}), \\
&\frac{\partial}{\partial x_l}\, \Phi^{(j)}_{pq}(x,\sigma) - i\,\lambda_j\, \frac{x_l}{|x|}\, \Phi^{(j)}_{pq}(x,\sigma) = \exp\{-\mathrm{Im}\,\lambda_j\, |x|\}\, O(|x|^{-2}), \quad l = 1,2,3.
\end{aligned}
$$

These asymptotic relations can be differentiated any times with respect to the variable x.

In accordance with formulas (4.22), (4.23) and Corollary A.2 (see Appendix A) we see that for $\operatorname{Im}\sigma = \sigma_2 > 0$ the entries of the matrix $\Gamma(x,\sigma)$ decay exponentially as $|x| \to \infty$, since $\operatorname{Im}\lambda_j > 0$, $j = \overline{1,11}$.

Remark 4.2. Note that the matrix $\Gamma^*(x,\sigma) := [\Gamma(-x,\sigma)]^\top$ represents a fundamental solution to the formally adjoint differential operator $L^*(\partial,\sigma) \equiv \big[L(-\partial,\sigma)\big]^\top$,

$$L^*(\partial,\sigma)\big[\Gamma(-x,\sigma)\big]^\top = I_{11}\,\delta(x).$$

In the case of repeated roots the fundamental solution can be obtained from (4.20) by the standard limiting procedure.

5. Principal Singular Part of the Fundamental Matrix

The principal singular part $\Gamma_0(x)$ of the fundamental matrix (4.21) represents an 11×11 fundamental matrix of the operator $L_0(\partial)$ defined by (3.9), (3.10) and solves the equation

$$L_0(\partial)\Gamma_0(x) = \delta(x)I_{11}.$$

It is easy to show that

$$\Gamma_0(x) = \begin{bmatrix} \Gamma_0^{(1)}(x) & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{3\times3} & \Gamma_0^{(7)}(x) & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{3\times3} & [0]_{3\times3} & \Gamma_0^{(13)}(x) & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & \Gamma_0^{(19)}(x) & 0 \\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & 0 & \Gamma_0^{(25)}(x) \end{bmatrix}_{11\times11},$$

where

$$\Gamma_0^{(1)}(x) = -\frac{1}{8\pi\,\mu}\Big\{\frac{2}{|x|}\,I_3 - \frac{\lambda_0+\mu}{\lambda_0+2\mu}Q(\partial)|x|\Big\},$$

$$\Gamma_0^{(7)}(x) = -\frac{1}{8\pi\,h_6}\Big\{\frac{2}{|x|}\,I_3 - \frac{h_4+h_5}{h_0}Q(\partial)|x|\Big\},$$

$$\Gamma_0^{(13)}(x) = -\frac{1}{8\pi\,k_6}\Big\{\frac{2}{|x|}\,I_3 - \frac{k_4+k_5}{k_0}Q(\partial)|x|\Big\},$$

$$\Gamma_0^{(19)}(x) = -\frac{1}{4\pi\,h\,|x|},$$

$$\Gamma_0^{(25)}(x) = -\frac{1}{4\pi\,k\,|x|}.$$

Note that $\Gamma_0(x) = \Gamma_0^\top(x) = \Gamma_0(-x)$ and the entries of the matrix $\Gamma_0(x)$ are homogeneous functions of order -1. For an arbitrary multi-index $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ and an arbitrary complex number σ it can easily be shown that in a neighbourhood of the origin (i.e., for small $|x|$)

$$\partial^\alpha[\Gamma(x,\sigma) - \Gamma_0(x)] = \mathcal{O}(|x|^{-\alpha}), \quad |\alpha| = \alpha_1+\alpha_2+\alpha_3,$$

which shows that $\Gamma_0(x)$ is a principal singular part of the matrix $\Gamma(x,\sigma)$.

6. Potentials and Their Properties

Let us introduce the generalized single and double-layer potentials, and the Newton type volume potential,

$$V(\varphi)(x) = \int\limits_S \Gamma(x-y,\sigma)\,\varphi(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{6.1}$$

$$W(\varphi)(x) = \int\limits_S \left[\mathcal{P}^*(\partial_y, n(y))\Gamma^\top(x-y,\sigma)\right]^\top \varphi(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{6.2}$$

$$N_{\Omega^\pm}(\psi)(x) = \int\limits_{\Omega^\pm} \Gamma(x-y,\sigma)\, \psi(y)\, dy, \quad x \in \mathbb{R}^3, \tag{6.3}$$

where $\Gamma(\cdot;\sigma)$ is the fundamental matrix given by (4.20) or (4.21), $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_{11})^\top$ is a density vector-function defined on S, while a density vector-function $\psi = (\psi_1, \dots, \psi_{11})^\top$ is defined on $\Omega^\pm$ and we assume that in the case of Ω^- the support of the density vector-function ψ of the Newtonian potential (6.3) is a compact set, $\mathcal{P}^*(\partial_y, n(y))$ is the boundary differential operator defined by (2.20). It can be checked that the potentials defined by (6.1) and (6.2) are C^∞-smooth in $\mathbb{R}^3 \setminus S$ and solve the homogeneous equation $L(\partial,\sigma)U = 0$ in $\mathbb{R}^3 \setminus S$ for an arbitrary continuous vector function φ. The volume potential solves the nonhomogeneous equation

$$L(\partial,\sigma) N_{\Omega^\pm}(\psi) = \psi \ \text{ in } \ \Omega^\pm \ \text{ for } \ \psi \in C^{0,\alpha}(\overline{\Omega^\pm}). \tag{6.4}$$

Theorem 6.1. *Let $S = \partial\Omega^+$ be $C^{1,\gamma'}$smooth with $0 < \gamma' \le 1$, $\sigma = \sigma_1 + i\,\sigma_2$ with $\sigma_2 > 0$, and let U be a regular vector function of the class $C^2(\overline{\Omega^+})$. Then the integral representation formula*

$$W(\{U\}^+)(x) - V(\{\mathcal{P}U\}^+)(x) + N_{\Omega^+}(L(\partial,\sigma)U)(x) = \begin{cases} U(x) & \text{for } x \in \Omega^+, \\ 0 & \text{for } x \in \Omega^-. \end{cases}$$

holds.

This follows from Green's formula (3.7) (see [4, Appendix D]).

Similar representation formula holds in the exterior domain Ω^- if the vector U and its derivatives possess some asymptotic properties at infinity. In particular, the following assertion holds.

Theorem 6.2. *Let $S = \partial\Omega^-$ be $C^{1,\gamma'}$smooth with $0 < \gamma' \le 1$ and let U be a regular vector of the class $C^2(\overline{\Omega^-})$ such that for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $0 \le |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \le 2$, the function $\partial^\alpha U_j$ is polynomially bounded at infinity, i.e., for sufficiently large $|x|$,*

$$|\partial^\alpha U_j(x)| \le C_0\, |x|^m, \ \ j = 1, 2, \dots, 11,$$

with some constants m and $C_0 > 0$. Then the integral representation formula

$$-W(\{U\}^-)(x) + V(\{\mathcal{P}U\}^-)(x) + N_{\Omega^-}(L(\partial,\sigma)U)(x) = \begin{cases} 0 & \text{for } x \in \Omega^+, \\ U(x) & \text{for } x \in \Omega^-, \end{cases}$$

where $\sigma = \sigma_1 + i\,\sigma_2$ with $\sigma_2 > 0$, holds.

The proof immediately follows from Theorem 6.1 and Remark 4.1.

From Theorem 6.2, it follows immediately that if $U \in C^2(\overline{\Omega^-})$ grows at infinity polynomially, and $L(\partial,\sigma)U$ possesses a compact support, then actually U and its all partial derivatives decay exponentially at infinity and the following Green's formula

$$\int\limits_{\Omega^-} U' \cdot L(\partial,\sigma) U dx = -\int\limits_{\partial\Omega^-} \{U'\}^- \cdot \{\mathcal{P}(\partial,n)U\}^- dS - \int\limits_{\Omega^-} E(U',U) dx \tag{6.5}$$

holds for all polynomially bounded vector functions $U' \in C^1(\overline{\Omega^-})$.

Now let us consider the mapping and regularity properties of the single and double-layer potentials and the boundary pseudodifferential operators generated by them in the Hölder $C^{m,\gamma'}$ spaces. They can be established by standard methods. We remark only that the layer potentials corresponding to the fundamental matrices with different values of the parameter σ have the same smoothness properties and possess the same jump relations. Therefore, using the word for word arguments given in [3,4,8,9,11–14], we can prove the following theorems concerning the above-introduced layer potentials. Unless otherwise stated, for simplicity, we assume that

$$\begin{aligned} &S = \partial\Omega^\pm \in C^{m,\gamma'} \text{ with integer } m \ge 2 \ \text{ and } \ 0 < \gamma' \le 1; \\ &\sigma = \sigma_1 + i\,\sigma_2, \ \ \sigma_1 \in \mathbb{R}, \quad \operatorname{Im}\sigma = \sigma_2 > 0. \end{aligned} \tag{6.6}$$

Theorem 6.3. *Let S, m, and $\gamma\,'$ be as in* (6.6), $0<\delta'<\gamma\,'$, *and let $k\le m-1$ be an integer. Then the operators*

$$V\;:\;C^{k,\,\delta'}(S)\to C^{k+1,\,\delta'}(\overline{\Omega^{\pm}}),\quad W\;:\;C^{k,\,\delta'}(S)\to C^{k,\,\delta'}(\overline{\Omega^{\pm}}) \tag{6.7}$$

are continuous. For any $g\in C^{\,0,\,\delta'}(S)$, $h\in C^{\,1,\,\delta'}(S)$, and any $x\in S$,

$$[\,V(g)(x)\,]^{\pm}=V(g)(x)=\mathcal{H}\,g(x), \tag{6.8}$$

$$\{\,\mathcal{P}(\partial_x,n(x))\,V(g)(x)\,\}^{\pm}=\big[\,\mp 2^{-1}I_{11}+\mathcal{K}\,\big]\,g(x), \tag{6.9}$$

$$\{\,W(g)(x)\,\}^{\pm}=\big[\,\pm 2^{-1}I_{11}+\mathcal{N}\,\big]\,g(x), \tag{6.10}$$

$$\{\mathcal{P}(\partial_x,n(x))\,W(h)(x)\,\}^{+}=\{\,\mathcal{P}(\partial_x,n(x))\,W(h)(x)\,\}^{-}=\mathcal{L}\,h(x), \tag{6.11}$$

where

$$\mathcal{H}\,g(x):=\int\limits_S\Gamma(x-y,\sigma)\,g(y)\,dS_y\,, \tag{6.12}$$

$$\mathcal{K}\,g(x):=\int\limits_S\big[\,\mathcal{P}(\partial_x,n(x))\,\Gamma(x-y,\sigma)\,\big]\;g(y)\,dS_y\,, \tag{6.13}$$

$$\mathcal{N}\,g(x):=\int\limits_S\big[\,\mathcal{P}^{*}(\partial_y,n(y))\,\Gamma^{\top}(x-y,\sigma)\,\big]^{\top}\;g(y)\,dS_y\,, \tag{6.14}$$

$$\mathcal{L}\,h(x):=\lim_{\Omega^{\pm}\ni z\to x\in S}\mathcal{P}(\partial_z,n(x))\int\limits_S\big[\,\mathcal{P}^{*}(\partial_y,n(y))\,\Gamma^{\top}(z-y,\sigma)\,\big]^{\top}\,h(y)\,dS_y\,. \tag{6.15}$$

The proof of the relations (6.7)–(6.11) can be performed by standard arguments (see, e.g., [6,8,12]). The relation (6.11) is called the *Liapunov-Tauber type theorem.*

With the help of the explicit form of the fundamental matrix $\Gamma(x-y,\sigma)$ it can be shown that the operators $\mathcal{K}$ and $\mathcal{N}$ are singular integral operators, $\mathcal{H}$ is a smoothing (weakly singular) integral operator, while $\mathcal{L}$ is a singular integro-differential operator.

Theorem 6.4. *Let S, m, γ', δ' and k be as in Theorem* 6.3. *Then the operators*

$$\mathcal{H}:C^{k,\,\delta'}(S)\to C^{k+1,\,\delta'}(S)\,, \tag{6.16}$$

$$\mathcal{K}:C^{k,\,\delta'}(S)\to C^{k,\,\delta'}(S)\,, \tag{6.17}$$

$$\mathcal{N}:C^{k,\,\delta'}(S)\to C^{k,\,\delta'}(S)\,, \tag{6.18}$$

$$\mathcal{L}:C^{k,\,\delta'}(S)\to C^{k-1,\,\delta'}(S)\,, \tag{6.19}$$

are continuous. Moreover, 1) *the principal homogeneous symbol matrices of the operators $\pm2^{-1}I_{11}+\mathcal{K}$ and $\pm2^{-1}I_{11}+\mathcal{N}$ are non-degenerate, while the principal homogeneous symbol matrices of the operators $-\mathcal{H}$ and $\mathcal{L}$ are positive definite;* 2) *the operators $\mathcal{H}$, $\pm2^{-1}I_{11}+\mathcal{K}$, $\pm2^{-1}I_{11}+\mathcal{N}$, and $\mathcal{L}$ are elliptic pseudodifferential operators* (*of order* -1, 0, 0, *and* 1, *respectively*) *with zero index; in appropriate function spaces, the following equalities* 3)

$$\begin{aligned}&\mathcal{N}\,\mathcal{H}=\mathcal{H}\,\mathcal{K}, &&\mathcal{L}\,\mathcal{N}=\mathcal{K}\,\mathcal{L},\\ &\mathcal{H}\,\mathcal{L}=-4^{-1}\,I_{11}+\mathcal{N}^{\,2}, &&\mathcal{L}\,\mathcal{H}=-4^{-1}\,I_{11}+\mathcal{K}^{2}.\end{aligned} \tag{6.20}$$

hold.

The mapping properties (6.16)–(6.19) are standard and can be proved as their counterparts in [8,11,13,14]. Items 1) and 2) are based on the positive definiteness of the potential energy functional and positive definiteness of the symbol matrix $L_0(\xi)$ for $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3\setminus\{0\}$ (see (3.9), (3.10)), (cf. [3,4,10,11,14] and [6]). Item 3) follows from the jump relations for the layer potentials and the general integral representation formulas of solutions to the homogeneous equation $L(\partial,\sigma)U=0$.

7. Formulation of Boundary Value Problems and Uniqueness Theorems

Let us formulate the basic interior and exterior boundary value problems for the domains Ω^+ and Ω^-. We assume that $S = \partial\Omega^+ \in C^{1,\gamma\,'}$, $0 < \gamma\,' \leq 1$.

Problem $(I^{(\sigma)})^{\pm}$ (The Dirichlet problem). Find a regular solution vector function $U = (u, C, T, P, \vartheta)^\top$ to the system of differential equations

$$L(\partial, \sigma)\, U(x) = \Phi^{\pm}(x), \quad x \in \Omega^{\pm}, \tag{7.1}$$

satisfying the boundary condition

$$\{U(z)\}^{\pm} = f(z), \quad z \in S. \tag{7.2}$$

Problem $(II^{(\sigma)})^{\pm}$ (The Neumann problem). Find a regular solution vector function $U = (u, C, T, P, \vartheta)^\top$ to system (7.1), satisfying the boundary condition

$$\{\mathcal{P}(\partial, n)\, U(z)\}^{\pm} = F(z), \quad z \in S. \tag{7.3}$$

We assume that the data of the boundary value problems belong to the appropriate classes,

$$\Phi^{\pm} \in C^{0,\alpha'}(\overline{\Omega})^{\pm}, \quad f \in C^{1,\alpha'}(S), \quad F \in C^{0,\alpha'}(S), \quad 0 < \alpha' < \gamma\,' \leq 1.$$

In addition, in the case of exterior problems we assume that the vector function Φ^- is compactly supported in Ω^-. Now we prove the following uniqueness theorem.

Theorem 7.1. *Let $\sigma = \sigma_1 + i\sigma_2$, with $\sigma_1 \in \mathbb{R}$ and $\sigma_2 > 0$. Then the homogeneous boundary value problems $(I^{(\sigma)})^{\pm}$ and $(II^{(\sigma)})^{\pm}$ have only the trivial solution in the class of regular vector functions.*

Proof. Let $U = (u, C, T, P, \vartheta)^\top$ be a regular solution of the homogeneous boundary value problem $(I^{(\sigma)})^{\pm}$ or $(II^{(\sigma)})^{\pm}$. Apply Green's formula (3.1) or (6.5) for the vector functions U and U', where

$$U' = \Big(i\,\overline{\sigma}\,\overline{u},\, \overline{C},\, \overline{T},\, \overline{P},\, \frac{1}{T_0}\,\overline{\vartheta}\Big)^\top.$$

Keeping in mind (3.3)–(3.6), we get the following relation:

$$\pm \int\limits_{\partial\Omega^{\pm}} \{U'\}^{\pm} \cdot \{\mathcal{P}(\partial, n) U\}^{\pm} dS - \int\limits_{\Omega^{\pm}} E(U', U) dx = 0, \tag{7.4}$$

where

$$\begin{aligned} E(U', U) =& i\,\overline{\sigma} E^{(1)}(\overline{u}, u) + E^{(2)}(\overline{C}, C) + E^{(3)}(\overline{T}, T) - i\,\overline{\sigma}\,(\gamma_2\, P + \gamma_1 \vartheta)\operatorname{div}\overline{u} - i\,\rho\,\sigma\,|\sigma|^2\,|u|^2 \\ & - \delta\,|C|^2 - i\,\sigma\,\varkappa_1 \overline{C}\cdot T + h_3 \overline{C}\cdot \operatorname{grad} P - \varkappa_0\,|T|^2 - i\,\sigma\,\varkappa_1 \overline{T}\cdot C + k_3 \overline{T}\cdot \operatorname{grad}\vartheta \\ & - i\,\sigma\, m |P|^2 - i\,\sigma\,\gamma_2\,\overline{P}\operatorname{div} u - i\,\sigma\,\varkappa\,\overline{P}\,\vartheta + h_1\, C\cdot \operatorname{grad}\overline{P} + h\,|\operatorname{grad} P|^2 \\ & + \frac{k}{T_0}|\operatorname{grad}\vartheta|^2 - i\,\sigma\, c\,|\vartheta|^2 - i\,\sigma\,\gamma_1\,\overline{\vartheta}\,\operatorname{div} u - i\,\sigma\,\varkappa\, P\,\overline{\vartheta} + \frac{k_1}{T_0} T\cdot \operatorname{grad}\overline{\vartheta}; \end{aligned}$$

$$E^{(1)}(\overline{u}, u) = \frac{3\lambda_0 + 2\mu}{3}|\operatorname{div} u|^2 + \frac{\mu}{3}\sum_{k,j=1}^{3}\left|\frac{\partial u_k}{\partial x_k} - \frac{\partial u_j}{\partial x_j}\right|^2 + \frac{\mu}{2}\sum_{k,j=1,\ k\neq j}^{3}\left|\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}\right|^2, \tag{7.5}$$

$$\begin{aligned} E^{(2)}(\overline{C}, C) =& \frac{3h_4 + h_5 + h_6}{3}|\operatorname{div} C|^2 + \frac{h_6 - h_5}{2}\,|\operatorname{curl} C|^2 \\ & + \frac{h_5 + h_6}{4}\sum_{k,j=1,\ k\neq j}^{3}\left|\frac{\partial C_k}{\partial x_j} + \frac{\partial C_j}{\partial x_k}\right|^2 + \frac{h_5 + h_6}{6}\sum_{k,j=1}^{3}\left|\frac{\partial C_k}{\partial x_k} - \frac{\partial C_j}{\partial x_j}\right|^2, \end{aligned} \tag{7.6}$$

$$\begin{aligned} E^{(3)}(\overline{T}, T) =& \frac{3k_4 + k_5 + k_6}{3}|\operatorname{div} T|^2 + \frac{k_6 - k_5}{2}\,|\operatorname{curl} T|^2 \\ & + \frac{k_5 + k_6}{4}\sum_{k,j=1,\ k\neq j}^{3}\left|\frac{\partial T_k}{\partial x_j} + \frac{\partial T_j}{\partial x_k}\right|^2 + \frac{k_5 + k_6}{6}\sum_{k,j=1}^{3}\left|\frac{\partial T_k}{\partial x_k} - \frac{\partial T_j}{\partial x_j}\right|^2. \end{aligned} \tag{7.7}$$

Since $U=(u,C,T,P,\vartheta)^\top$ solves the homogeneous boundary value problem $(I^{(\sigma)})^\pm$, or $(II^{(\sigma)})^\pm$, the surface integral in (7.4) vanishes and we arrive at the equation

$$\int\limits_{\Omega^\pm} E(U',U)dx=0.$$

The real part of this equation reads as

$$\begin{gathered}\int\limits_{\Omega^\pm}\big\{\sigma_2\,E^{(1)}(\overline{u},u)+E^{(2)}(\overline{C},C)+E^{(3)}(\overline{T},T)+\rho\,\sigma_2\,|\sigma|^2|u|^2\\
+\sigma_2\left[m\,|P|^2+\varkappa\,(P\overline{\vartheta}+\overline{P}\vartheta)+c|\vartheta|^2\right]+\sigma_2\left[m_1|C|^2+\varkappa_1\,(C\cdot\overline{T}+\overline{C}\cdot T)+c_1\,|T|^2\right]\\
+\frac{1}{2}\left[2\,h_2|C|^2+(h_1+h_3)\,(\overline{C}\cdot\operatorname{grad}P+C\cdot\operatorname{grad}\overline{P})+2h\,|\operatorname{grad}P|^2\,\right]\\
+\frac{1}{2T_0}\left[2\,k_2\,T_0\,|T|^2+(k_1+T_0\,k_3)\,(\overline{T}\cdot\operatorname{grad}\vartheta+T\cdot\operatorname{grad}\overline{\vartheta})+2\,k\,|\operatorname{grad}\vartheta|^2\right]\big\}dx=0,\end{gathered}\tag{7.8}$$

By means of relations (7.5)–(7.7) we see that $E^{(1)}(\overline{u},u)\geq 0$, $E^{(2)}(\overline{C},C)\geq 0$, and $E^{(3)}(\overline{T},T)\geq 0$. Transforming the integrand and taking into account conditions (2.6), we establish

$$m|P|^2+\varkappa\,(P\overline{\vartheta}+\overline{P}\vartheta)+c|\vartheta|^2=\frac{1}{c}\left[(m\,c-\varkappa^2)\,|P|^2+|\varkappa\,P+c\vartheta|^2\right]\geq 0,$$

$$m_1|C|^2+\varkappa_1\,(C\cdot\overline{T}+\overline{C}\cdot T)+c_1\,|T|^2=\frac{1}{c_1}\left[(m_1\,c_1-\varkappa_1^2)|C|^2+|\varkappa_1\,C+c_1\,T|^2\right]\geq 0,$$

$$\begin{aligned}&\frac{1}{2}\left[2\,h_2|C|^2+(h_1+h_3)(\overline{C}\cdot\operatorname{grad}P+C\cdot\operatorname{grad}\overline{P})+2h\,|\operatorname{grad}P|^2\,\right]\\
&\qquad=\frac{1}{4h}\left\{\left[4h\,h_2-(h_1+h_3)^2\right]|C|^2+|(h_1+h_3)\,C+2h\ \operatorname{grad}P|^2\right\}\geq 0,\\
&\frac{1}{2T_0}\left[2\,k_2\,T_0\,|T|^2+(k_1+T_0\,k_3)(\overline{T}\cdot\operatorname{grad}\vartheta+T\cdot\operatorname{grad}\overline{\vartheta})+2\,k\,|\operatorname{grad}\vartheta|^2\right]\\
&\qquad=\frac{1}{4k\,T_0}\left\{\left[4T_0\,k\,k_2-(k_1+T_0\,k_3)^2\right]|T|^2+|(k_1+T_0\,k_3)\,T+2k\ \operatorname{grad}\vartheta|^2\right\}\geq 0.\end{aligned}$$

Consequently, from (7.8) we derive $\operatorname{Re}E(U',U)\geq 0$ in $\Omega^\pm$, implying $U=0$ for $x\in\Omega^\pm$.

□

8. Existence Results

Now, we apply the potential method and prove the existence theorems for the above formulated Dirichlet and Neumann type boundary value problems. We reduce these problems to the equivalent integral equations on the boundary of the elastic body under consideration and investigate their Fredholm properties. We show that the corresponding integral operators are invertible. Without loss of generality, we consider the boundary value problems for the homogeneous differential equation $L(\partial,\sigma)\,U=0$, since a particular solution to the nonhomogeneous equation (7.1) can be written explicitly in the form of the volume potential $N_{\Omega^\pm}(\Phi^\pm)$ (see (6.4)). Moreover, throughout this section we assume that the conditions (6.6) are fulfilled, unless otherwise stated.

8.1. **Investigation of the interior and exterior Dirichlet problems.** We assume that $\Phi^{(\pm)}=0$ and look for solutions in $\Omega^\pm$ in the form of the double-layer potential $U=W(h)$ (see (6.2)). Applying the jump relations for the double-layer potential (see Theorem 6.3) and taking into accoun the boundary conditions (7.2), for the unknown density vector function $h=(h_1,h_2,\dots,h_{11})^\top$ we get the following boundary integral equations:

$$\left[\,2^{-1}I_{11}+\mathcal{N}\,\right]h=f\ \text{ on }\ S,\tag{8.1}$$

in the case of Problem $(I^{(\sigma)})^+$, and

$$\left[\,-2^{-1}I_{11}+\mathcal{N}\,\right]h=f\ \text{ on }\ S,\tag{8.2}$$

in the case of Problem $(I^{(\sigma)})^-$.
Here, the operator $\mathcal{N}$ is given by (6.14). Due to Theorem 6.4, the operators $\pm 2^{-1}I_{11}+\mathcal{N}$ are singular integral operators of normal type with index zero. This leads to the following existence theorems.

Theorem 8.1. *Let $S \in C^{2,\nu}$ and $f \in C^{1,\tau}(S)$ with $0<\tau<\nu\le 1$. Then the boundary value problem $(I^{(\sigma)})^+$ is uniquely solvable in the space $C^{1,\tau}(\overline{\Omega^+})$ and the solution can be represented by the double-layer potential $W(h)$ defined by* (6.2)*, where the density $h \in C^{1,\tau}(S)$ is uniquely defined from the integral equation* (8.1)*.*

Proof. The uniqueness follows from Theorem 7.1. Now, let us show that the singular integral operator

$$2^{-1}I_{11}+\mathcal{N}\ :\ C^{1,\tau}(S)\to C^{1,\tau}(S) \tag{8.3}$$

is invertible. Due to Theorem 6.4, we conclude that (8.3) is a Fredholm operator with zero index. Further, we show that ker $\big[\,2^{-1}I_{11}+\mathcal{N}\,\big]$ is trivial. Indeed, let $h_0 \in C^{1,\tau}(S)$ be a solution of the homogeneous equation

$$\big[\,2^{-1}I_{11}+\mathcal{N}\,\big]\,h_0=0 \ \text{ on } \ S. \tag{8.4}$$

We construct the double-layer potential $W(h_0)$. Evidently, $W(h_0)\in C^{1,\tau}(\overline{\Omega^{\pm}})$ by Theorem 6.3. In view of equation (8.4), we have $\{W(h_0)(x)\}^+=0$ for $x\in S$ and by the uniqueness Theorem 7.1, we get $W(h_0)(x)=0$ for $x\in\Omega^+$. Consequently, $\{\,\mathcal{P}(\partial,n)\,W(h_0)(x)\,\}^+=0$ for $x\in S$. By the Liapunov-Tauber theorem (see Theorem 6.3)

$$\{\,\mathcal{P}(\partial,n)\,W(h_0)(x)\}^+=\{\mathcal{P}(\partial,n)\,W(h_0)(x)\,\}^-=0,\ \ x\in S,$$

i.e., $W(h_0)$ solves the homogeneous exterior Neumann type boundary value problem $(II^{(\sigma)})^-$ and decays at infinity exponentially. Therefore, $W(h_0)(x)=0$ in Ω^- by Theorem 7.1. Since

$$\{W(h_0)(x)\}^+-\{W(h_0)(x)\}^-=2\,h_0(x),\quad x\in S,$$

we conclude that $h_0=0$ on S, which shows that the null space of the operator $2^{-1}I_{11}+\mathcal{N}$ is trivial. Therefore, (8.3) is invertible. □

Quite similarly, with the help of Theorem 7.1, we can show that the operator

$$-2^{-1}I_{11}+\mathcal{N}\ :\ C^{1,\tau}(S)\to C^{1,\tau}(S) \tag{8.5}$$

is invertible, which leads to the existence theorem for the Dirichlet type exterior boundary value problem.

Theorem 8.2. *Let $S\in C^{2,\nu}$ and $f\in C^{1,\nu}(S)$ with $0<\tau<\nu\le 1$. Then the boundary value problem $(I^{(\sigma)})^-$ is uniquely solvable in the class of vector functions belonging to the space $C^{1,\tau}(\overline{\Omega^-})$ and decaying at infinity, and the solution is represented by the double-layer potential $W(h)$ defined by* (6.2)*, where $h\in C^{1,\tau}(S)$ is defined by the integral equation* (8.2)*.*

8.2. **Investigation of the interior and exterior Neumann problems.** These problems are formulated in Section 7 as problems $(II^{(\sigma)})^+$ and $(II^{(\sigma)})^-$. As above, we assume that $\Phi^{(\pm)}=0$ and look for solutions in $\Omega^{\pm}$ in the form of the single-layer potential $U=V(g)$ (see (6.1)). Taking into consideration the boundary conditions (7.3), for the unknown density vector function $g=(g_1,g_2,\dots,g_{11})^\top$ we get the following boundary integral equations:

$$\big[-\,2^{-1}I_{11}+\mathcal{K}\,\big]\,g=F \ \text{ on } \ S, \tag{8.6}$$

in the case of Problem $(II^{(\sigma)})^+$, and

$$\big[\,2^{-1}I_{11}+\mathcal{K}\,\big]\,g=F \ \text{ on } \ S, \tag{8.7}$$

in the case of Problem $(II^{(\sigma)})^-$. Here, the operator $\mathcal{K}$ is given by (6.13). Due to Theorem 6.4, the operators $\pm 2^{-1}I_{11}+\mathcal{K}$ are singular integral operators of normal type with index zero. This leads to the following existence theorems.

Theorem 8.3. *Let $S\in C^{1,\nu}$ and $F\in C^{0,\tau}(S)]$ with $0<\tau<\nu\le 1$. Then the boundary value problem $(II^{(\sigma)})^+$ is uniquely solvable in the space $C^{1,\tau}(\overline{\Omega^+})$ and the solution is represented by the single-layer potential $V(g)$ defined by* (6.1)*, where $g\in C^{0,\tau}(S)$ is uniquely defined by the integral equation* (8.6)*.*

Proof. The uniqueness is a consequence of the uniqueness Theorem 7.1. Now, we show that the operator

$$-\,2^{-1}I_{11}+\mathcal{K}\ :\ C^{0,\tau}(S)\to C^{0,\tau}(S) \tag{8.8}$$

is invertible. Due to Theorem 6.4, the operator (8.8) is a Fredholm operator with zero index. Therefore, it remains to show that the null space of the operator $-2^{-1}I_{11}+\mathcal{K}$ is trivial. Let $g_0\in C^{0,\tau}(S)$ solve the homogeneous equation

$$[-\,2^{-1}I_{11}+\mathcal{K}\,]\,g_0=0\ \text{ on }\ S.$$

Construct the single-layer potential $V(g_0)$. Evidently, $V(g_0)\in C^{1,\tau}(\overline{\Omega^+})$ due to Theorem 6.3. Moreover, $V(g_0)$ solves the homogeneous Problem $(II^{(\sigma)})^+$ and therefore it vanishes identically in Ω^+, due to Theorem 7.1. Further, by Theorem 6.3, we have $\{\,V(g_0)(x)\,\}^+=\{\,V(g_0)(x)\,\}^-=0$ for $x\in S$, and since it exponentially decays at infinity, by the uniqueness theorem for the Dirichlet exterior boundary value problem, we conclude $V(g_0)(x)\,=\,0$ for $x\in\Omega^-$. Finally, with the help of the jump relation

$$\{\,\mathcal{P}(\partial,n)\,V(g_0)(x)\,\}^- - \{\,\mathcal{P}(\partial,n)\,V(g_0)(x)\,\}^+ = 2\,g_0\,(x),\ \ x\in S,$$

we derive $g_0=0$ on S. Thus, the operator (8.8) is invertible. □

By the word for word arguments we can prove that the operator

$$2^{-1}I_{11}+\mathcal{K}\ :\ C^{0,\tau}(S)\to C^{0,\tau}(S) \tag{8.9}$$

is invertible, which leads to the existence theorem for the Neumann type exterior boundary value problem.

Theorem 8.4. *Let $S\in C^{1,\nu}$ and $F\in C^{0,\tau}(S)$ with $0<\tau<\nu\le 1$. Then the boundary value problem $(II^{(\sigma)})^-$ is uniquely solvable in the class of vector functions belonging to the space $C^{1,\tau}(\overline{\Omega^-})$ and decaying at infinity, and the solution is represented by the single-layer potential $V(g)$ defined by* (6.1)*, where $g\in C^{0,\tau}(S)$ is a unique solution of the integral equation* (8.7)*.*

8.3. **Investigation of the basic boundary value problems by the first kind integral equations.** Here we apply an alternative approach and reduce the basic interior and exterior boundary value problems, considered in the previous subsections, to the first kind integral equations (cf. [14]). These results play a crucial role in the study of mixed boundary value problems.

9.3.1. Investigation of the Dirichlet problem with the help of the first kind integral equations. We look for a solution to the problems $(I^{(\sigma)})^+$ and $(I^{(\sigma)})^-$ (see (7.1)–(7.2) with $\Phi^{(\pm)}=0$) in the form of the single-layer potential $U=V(g)$ (see (6.1)). In both cases, for the interior and exterior boundary value problems, we arrive at the equation

$$\mathcal{H}\,g=f\ \text{ on }\ S, \tag{8.10}$$

where $\mathcal{H}$ is defined by (6.12). We have the following existence theorem.

Theorem 8.5. *Let $S\in C^{2,\nu}$ and $f\in C^{1,\tau}(S)$ with $0<\tau<\nu\le 1$. Then the boundary value problems $(I^{(\sigma)})^\pm$ are uniquely solvable in the class of vector functions belonging to the space $C^{1,\tau}(\overline{\Omega^\pm})$ and decaying at infinity, and the solution is represented by the single-layer potential $V(g)$ defined by* (6.1)*, where $g\in C^{0,\tau}(S)$ is a unique solution of the integral equation* (8.10)*.*

Proof. The uniqueness follows from Theorem 7.1. Evidently, it remains to show the invertibility of the operator

$$\mathcal{H}\ :\ C^{0,\tau}(S)\to C^{1,\tau}(S). \tag{8.11}$$

To this end, we apply the operator $\mathcal{L}$ (see (6.15)) to both sides of equation (8.10) and take into consideration the operator equalities (6.20),

$$\mathcal{L}\mathcal{H}g\equiv\big[-\,4^{-1}I_1+\mathcal{K}^2\,\big]\,g=\mathcal{L}f\ \text{ on }\ S. \tag{8.12}$$

Clearly, $\mathcal{L}f\ \in C^{0,\tau}(S)$ due to Theorem 6.4. Since the operators (8.8) and (8.9) are invertible, we conclude that the singular integral operator

$$\mathcal{L}\mathcal{H}\ =\ \big[-\,2^{-1}I_{11}+\mathcal{K}\,\big]\,\big[\,2^{-1}I_{11}+\mathcal{K}\,\big]\ :\ C^{0,\tau}(S)\to C^{0,\tau}(S)$$

is invertible, as well. Therefore, from (8.12), we get the following representation of a solution of equation (8.10),

$$g = \big[-4^{-1} + \mathcal{K}^2 \big]^{-1} \mathcal{L} f \in C^{0,\tau}(S).$$

With the help of the uniqueness Theorem 7.1, one can easily show that the operators

$$\mathcal{H} \ : \ C^{0,\tau}(S) \to C^{1,\tau}(S), \quad \mathcal{L} \ : \ C^{1,\tau}(S) \to C^{0,\tau}(S) \tag{8.13}$$

are injective. Therefore, equations (8.10) and (8.12) are equivalent and the operator (8.11) is invertible, which completes the proof. □

Corollary 8.6. *A solution* $U \in C^{1,\tau}(\overline{\Omega^{\pm}})$ *of the boundary value problems* $(I^{(\sigma)})^{\pm}$ *with* $\Phi^{(\pm)} = 0$ *is uniquely representable in the form*

$$U(x) = V(\mathcal{H}^{-1} f)(x), \quad x \in \Omega^{\pm},$$

where $f = \{U\}^{\pm}$ *on* S *and*

$$\mathcal{H}^{-1} \ : \ C^{1,\tau}(S) \to C^{0,\tau}(S)$$

is the inverse to the operator (8.11).

This representation plays a crucial role in the investigation of mixed boundary value problems (cf. [14]).

9.3.2. Investigation of the Neumann problem with the help of the first kind integral equations. We look for a solution to the problems $(II^{(\sigma)})^{+}$ and $(II^{(\sigma)})^{-}$ (see (7.1), (7.3) with $\Phi^{\pm} = 0$) in the form of the double-layer potential $U = W(h)$ (see (6.2)). In both cases, for the interior and exterior boundary value problems, we arrive at the equation

$$\mathcal{L}\, h = F \ \text{ on } \ S, \tag{8.14}$$

where $\mathcal{L}$ is defined by (6.15). We have the following existence theorem.

Theorem 8.7. *Let* $S \in C^{2,\nu}$ *and* $F \in C^{0,\tau}(S)$ *with* $0 < \tau < \nu \le 1$. *Then the boundary value problems* $(II^{(\sigma)})^{\pm}$ *are uniquely solvable in the class of vector functions belonging to the space* $C^{1,\tau}(\overline{\Omega^{\pm}})$ *and decaying at infinity, and the solution is represented by the double-layer potential* $W(h)$ *defined by* (6.2), *where* $h \in C^{1,\tau}(S)$ *is a unique solution of the integral equation* (8.14).

Proof. The uniqueness follows from Theorem 7.1. Evidently, it remains to show the invertibility of the operator

$$\mathcal{L} \ : \ C^{1,\tau}(S) \to C^{0,\tau}(S). \tag{8.15}$$

To this end, we apply the operator $\mathcal{H}$ (see (6.12)) to both sides of equation (8.14) and take into consideration the operator equalities (6.20),

$$\mathcal{H}\mathcal{L}h \equiv \big[-4^{-1} I_{11} + \mathcal{N}^2 \big]\, h = \mathcal{H}F \ \text{ on } \ S. \tag{8.16}$$

Clearly, $\mathcal{H}F \in C^{1,\tau}(S)$ due to Theorem 6.4. Since the operators (8.3) and (8.5) are invertible, we conclude that the singular integral operator

$$\mathcal{H}\mathcal{L} = \big[-2^{-1} I_{11} + \mathcal{N} \big] \big[2^{-1} I_{11} + \mathcal{N} \big] \ : \ C^{1,\tau}(S) \to C^{1,\tau}(S)$$

is invertible, as well. Therefore, from (8.16), for a solution of equation (8.14), we get the following representation formula:

$$h = \big[-4^{-1} I_{11} + \mathcal{N}^2 \big]^{-1} \mathcal{H}F \in C^{1,\tau}(S).$$

Since the operators (8.13) are injective, we conclude that equations (8.14) and (8.16) are equivalent and the operator (8.15) is invertible, which completes the proof. □

Corollary 8.8. *A solution* $U \in C^{1,\tau}(\overline{\Omega^{\pm}})$ *of the boundary value problems* $(II^{(\sigma)})^{\pm}$ *with* $\Phi^{\pm} = 0$ *is uniquely representable in the form*

$$U(x) = W(\mathcal{L}^{-1} F)(x), \quad x \in \Omega^{\pm},$$

where $F = \{\mathcal{P}(\partial, n)U\}^{\pm}$ *on* S *and*

$$\mathcal{L}^{-1} \ : \ C^{0,\tau}(S) \to C^{1,\tau}(S)$$

is the inverse to the operator (8.15).

9. Appendix A: Properties of the Characteristic Roots

Here, we investigate the properties of roots of equation (4.6) with respect to r. In particular, we prove the following assertion.

Lemma A.1. *Let us assume that $\sigma=\sigma_1+i\,\sigma_2$ is a complex parameter, where $\sigma_1\in\mathbb{R}$ and $\sigma_2>0$. Then*

$$\det L(-i\,\xi,\sigma)\neq 0$$

for arbitrary $\xi\in\mathbb{R}^3$.

Proof. We prove the lemma by contradiction. Let $\det L(-i\,\xi,\sigma)=0$, $\xi\in\mathbb{R}^3$. Then the system of linear equations $L(-i\,\xi,\sigma)\,X=0$ has a nontrivial solution $X\in\mathbb{C}^{11}\setminus\{0\}$ which can be written as $X=(X^{(1)},X^{(2)},X^{(3)},X^{(4)},X^{(5)})^\top$, where $X^{(j)}=(X_1^{(j)},X_2^{(j)},X_3^{(j)})^\top\in\mathbb{C}^3$, $j=1,2,3$ and $X^{(j)}\in\mathbb{C}$, $j=4,5$, are scalars. Taking into consideration (2.14), the system $L(-i\,\xi,\sigma)\,X=0$ can be rewritten as follows:

$$\begin{gathered}L^{(j)}(-i\,\xi,\sigma)\,X^{(1)}+L^{(j+5)}(-i\,\xi,\sigma)\,X^{(2)}+L^{(j+10)}(-i\,\xi,\sigma)\,X^{(3)}\\ +L^{(j+15)}(-i\,\xi,\sigma)\,X^{(4)}+L^{(j+20)}(-i\,\xi,\sigma)\,X^{(5)}=0,\\ j=1,2,3,4,5,\end{gathered}$$

implying

$$\big[(-\mu\,|\xi|^2+\rho\,\sigma^2)\,I_3-(\lambda_0+\mu)\,Q(\xi)\big]\,X^{(1)}+i\gamma_2\xi^\top X^{(4)}+i\gamma_1\,\xi^\top X^{(5)}=0, \tag{A.1}$$

$$\big[(\delta-h_6\,|\xi|^2)\,I_3-(h_4+h_5)\,Q(\xi)\big]X^{(2)}+i\,\sigma\,\varkappa_1\,X^{(3)}+i\,h_3\,\xi^\top X^{(4)}=0, \tag{A.2}$$

$$i\,\sigma\,\varkappa_1\,X^{(2)}+\big[(\varkappa_0-k_6|\xi|^2)\,I_3-(k_4+k_5)\,Q(\xi)\big]X^{(3)}+i\,k_3\,\xi^\top X^{(5)}=0, \tag{A.3}$$

$$\sigma\,\gamma_2\,\xi\cdot X^{(1)}-i\,h_1\,\xi\cdot X^{(2)}+(i\,\sigma\,m-h\,|\xi|^2)\,X^{(4)}+i\,\sigma\,\varkappa\,X^{(5)}=0, \tag{A.4}$$

$$\sigma\,\gamma_1\,T_0\,\xi\cdot X^{(1)}-ik_1\,\xi\cdot X^{(3)}+i\,\sigma\,T_0\,\varkappa X^{(4)}+\big(i\,\sigma\,c\,T_0-k\,|\xi|^2\big)\,X^{(5)}=0. \tag{A.5}$$

Let us take the dot products of equations (A.1), and (A.2) by the vectors $-i\,\overline{\sigma}\,\overline{X^{(1)}}$ and $-\overline{X^{(2)}}$ respectively, multiply equality (A.3) by the vector $-\overline{X^{(3)}}$, then multiply complex conjugates of equations (A.4) and (A.5) by the functions $-X^{(4)}$ and $-\frac{1}{T_0}\,X^{(5)}$ respectively and sum up the results to obtain

$$\begin{aligned}&i\,\overline{\sigma}\big[\mu\,|\xi|^2-\rho\,\sigma^2\big]\,\big|X^{(1)}\big|^2+i\,\overline{\sigma}\,(\lambda_0+\mu)\,\big|\xi\cdot X^{(1)}\big|^2+\big[h_6\,|\xi|^2-\delta\big]\,\big|X^{(2)}\big|^2+(h_4+h_5)\,\big|\xi\cdot X^{(2)}\big|^2\\ &-i\,\sigma\,\varkappa_1\big[X^{(2)}\cdot\overline{X^{(3)}}+\overline{X^{(2)}}\cdot X^{(3)}\big]-i\,h_3\,\big(\xi\cdot\overline{X^{(2)}}\big)\,X^{(4)}+\big[k_6\,|\xi|^2-\varkappa_0\big]\,\big|X^{(3)}\big|^2\\ &+(k_4+k_5)\,\big|\xi\cdot X^{(3)}\big|^2-i\,k_3\,\big(\xi\cdot\overline{X^{(3)}}\big)X^{(5)}-i\,h_1\,\big(\xi\cdot\overline{X^{(2)}}\big)X^{(4)}+\big[i\,\overline{\sigma}\,m+h\,|\xi|^2\big]\,\big|X^{(4)}\big|^2\\ &+i\,\overline{\sigma}\,\varkappa\,\big[X^{(4)}\,\overline{X^{(5)}}+\overline{X^{(4)}}\,X^{(5)}\big]-\frac{i\,k_1}{T_0}\,\big(\xi\cdot\overline{X^{(3)}}\big)\,X^{(5)}+\big[i\,\overline{\sigma}\,c+\frac{k}{T_0}\,|\xi|^2\big]\,\big|X^{(5)}\big|^2=0.\end{aligned}$$

By separating the real part from this equation, we deduce

$$\begin{aligned}&\sigma_2\big[\mu\,|\xi|^2+\rho\,|\sigma|^2\big]\big|X^{(1)}\big|^2+\sigma_2(\lambda_0+\mu)\,\big|\xi\cdot X^{(1)}\big|^2+h_6\,|\xi|^2\,\big|X^{(2)}\big|^2+(h_4+h_5)\,\big|\xi\cdot X^{(2)}\big|^2\\ &+k_6\,|\xi|^2\,\big|X^{(3)}\big|^2+(k_4+k_5)\,\big|\xi\cdot X^{(3)}\big|^2+\sigma_2\big[m_1\,\big|X^{(2)}\big|^2+\varkappa_1\,\big(X^{(2)}\cdot\overline{X^{(3)}}+\overline{X^{(2)}}\cdot X^{(3)}\big)+c_1\,\big|X^{(3)}\big|^2\big]\\ &+\frac{1}{2}\big\{2h_2\,\big|X^{(2)}\big|^2-i\,(h_1+h_3)\,\big[\big(\xi\cdot\overline{X^{(2)}}\big)X^{(4)}-\big(\xi\cdot X^{(2)}\big)\,\overline{X^{(4)}}\big]+2h\,|\xi|^2\,\big|X^{(4)}\big|^2\big\}\\ &+\frac{1}{2T_0}\big\{2T_0\,k_2\,\big|X^{(3)}\big|^2-i\,(k_1+T_0\,k_3)\,\big[\big(\xi\cdot\overline{X^{(3)}}\big)X^{(5)}-\big(\xi\cdot X^{(3)}\big)\,\overline{X^{(5)}}\,\big]+2k\,|\xi|^2\,\big|X^{(5)}\big|^2\big\}\\ &+\sigma_2\big[m\,\big|X^{(4)}\big|^2+\varkappa\,\big(X^{(4)}\cdot\overline{X^{(5)}}+\overline{X^{(4)}}\cdot X^{(5)}\big)+c\,\big|X^{(5)}\big|^2\,\big]=0.\end{aligned} \tag{A.6}$$

With the help of the following relations and inequalities (2.6),

$$\begin{aligned}
&\big|\xi\big|^2\big|X^{(j)}\big|^2-\big|\xi\cdot X^{(j)}\big|^2=\big|\,\xi\times X^{(j)}\,\big|^2,\ \ j=1,2,3,\\
&h_6\big|\xi\big|^2\big|X^{(2)}\big|^2+(h_4+h_5)\big|\xi\cdot X^{(2)}\big|^2=h_0\big|\xi\cdot X^{(2)}\big|^2+h_6\big|\,\big[\xi\times X^{(2)}\big]\,\big|^2\geq 0,\\
&k_6\big|\xi\big|^2\big|X^{(3)}\big|^2+(k_4+k_5)\big|\xi\cdot X^{(3)}\big|^2=k_0\,\big|\xi\cdot X^{(3)}\big|^2+k_6\big|\,\big[\xi\times X^{(3)}\big]\,\big|^2\geq 0,\\
&m_1\,\big|X^{(2)}\big|^2+\varkappa_1\big(X^{(2)}\cdot\overline{X^{(3)}}+\overline{X^{(2)}}\cdot X^{(3)}\big)+c_1\,\big|X^{(3)}\big|^2\\
&\qquad=\frac{1}{c_1}\big\{\big[m_1\,c_1-\varkappa_1^2\big]\big|X^{(2)}\big|^2+\big|\varkappa_1\,X^{(2)}+c_1\,X^{(3)}\big|^2\big\}\geq 0,\\
&m\,\big|X^{(4)}\big|^2+\varkappa\big(X^{(4)}\cdot\overline{X^{(5)}}+\overline{X^{(4)}}\cdot X^{(5)}\big)+c\,\big|X^{(5)}\big|^2\\
&\qquad=\frac{1}{c}\big\{\big[m\,c-\varkappa^2\big]\big|X^{(4)}\big|^2+\big|\varkappa\,X^{(4)}+c\,X^{(5)}\big|^2\big\}\geq 0,\\
&2h_2\,\big|X^{(2)}\big|^2-i\,(h_1+h_3)\,\big[\big(\xi\cdot\overline{X^{(2)}}\big)X^{(4)}-\big(\xi\cdot X^{(2)}\big)\,\overline{X^{(4)}}\big]+2h\,|\xi|^2\,\big|X^{(4)}\big|^2\\
&\qquad=\frac{1}{2h}\,\big\{\big[4h\,h_2-(h_1+h_3)^2\big]\big|X^{(2)}\big|^2+\big|(h_1+h_3)\,X^{(2)}-2i\,h\,\xi^\top X^{(4)}\big|^2\big\}\geq 0,\\
&2T_0\,k_2\,\big|X^{(3)}\big|^2-i\,(k_1+T_0\,k_3)\,\big[\big(\xi\cdot\overline{X^{(3)}}\big)X^{(5)}-\big(\xi\cdot X^{(3)}\big)\,\overline{X^{(5)}}\big]+2k\,|\xi|^2\,\big|X^{(5)}\big|^2\\
&\qquad=\frac{1}{2k}\,\big\{\big[4T_0\,k\,k_2-(k_1+T_0\,k_3)^2\big]\big|X^{(3)}\big|^2+\big|(k_1+T_0\,k_3)\,X^{(3)}-2i\,k\,\xi^\top X^{(5)}\big|^2\big\}\geq 0,
\end{aligned}$$

from (A.6), we conclude that

$$X^{(j)}=0,\qquad j=1,2,3,4,5.$$

Thus, the system $L(-i\,\xi,\sigma)\,X=0$ possesses only the trivial solution for arbitrary $\xi\in\mathbb{R}^3$. This contradiction proves the lemma. □

Corollary A.2. *Let $\sigma=\sigma_1+i\,\sigma_2$ be a complex parameter with $\sigma_1\in\mathbb{R}$ and $\sigma_2>0$. Then the equation*

$$\Lambda(\xi)=\det L(-i\,\xi,\sigma)=0$$

with respect to $|\xi|$ possesses only complex roots $\pm\lambda_j$, $j=\overline{1,11}$ with $\operatorname{Im}\lambda_j>0$, $j=\overline{1,11}$.

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BOUNDARY-TRANSMISSION PROBLEMS OF THE THEORY OF ACOUSTIC WAVES FOR PIECEWISE INHOMOGENEOUS ANISOTROPIC MULTI-COMPONENT LIPSCHITZ DOMAINS

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Abstract. We consider the time-harmonic acoustic wave scattering by a bounded anisotropic inhomogeneous obstacle embedded in an unbounded anisotropic homogeneous medium assuming that the boundary of the obstacle and the interface are Lipschitz surfaces. We assume that the obstacle contains a cavity and the material parameters may have discontinuities across the interface between the inhomogeneous interior and homogeneous exterior regions. The corresponding mathematical model is formulated as a boundary-transmission problem for a second order elliptic partial differential equation of Helmholtz type with piecewise Lipschitz-continuous variable coefficients. The problem is studied by the so-called nonlocal approach which reduces the problem to a variational-functional equation containing sesquilinear forms over a bounded region occupied by the inhomogeneous obstacle and over the interfacial surface. This is done with the help of the theory of layer potentials on Lipschitz surfaces. The coercivity properties of the corresponding sesquilinear forms are analyzed and the unique solvability of the boundary transmission acoustic problem in appropriate Sobolev-Slobodetskii and Bessel potential spaces is established.

1. Introduction

The paper deals with the time-harmonic acoustic wave scattering by a bounded *anisotropic inhomogeneous obstacle* embedded in an unbounded *anisotropic* homogeneous medium. We assume that the bounded obstacle contains an interior cavity. The boundary of the cavity will be referred to as *interior boundary of the obstacle*. We require that the interior boundary of the obstacle and the interface between the inhomogeneous interior and homogeneous exterior regions are the Lipschitz surfaces. The physical wave scattering problem with a frequency parameter $\omega \in \mathbb{R}$ is formulated mathematically as a boundary-transmission problem for a second order elliptic partial differential equation with variable Lipschitz-continuous coefficients, $A_2(x,\partial_x,\omega)\,u(x) \equiv \partial_{x_k}\big(a^{(2)}_{kj}(x)\,\partial_{x_j}u(x)\big) + \omega^2\,\kappa_2(x)\,u(x) = f_2(x)$, in the bounded region $\Omega_2 \subset \mathbb{R}^3$ occupied by an inhomogeneous anisotropic obstacle and for a Helmholtz type equation with constant coefficients, $A_1(\partial_x,\omega)u(x) \equiv a^{(1)}_{kj}\,\partial_{x_k}\partial_{x_j}u(x) + \omega^2\kappa_1\,u(x) = f_1(x)$, in the unbounded region Ω_1 occupied by the homogeneous anisotropic medium. The material parameters $a^{(q)}_{kj}$ and κ_q, $q = 1,2$, are not assumed to be continuous across the interface. Note that in the case of *isotropic medium* occupying the domain Ω_q, we have only one material coefficient $a^{(q)}$, i.e., the corresponding material parameters satisfy the relations $a^{(q)}_{kj} = a^{(q)}\delta_{kj}$, where δ_{kj} is the Kronecker symbol.

We analyse the case when the transmission conditions relating the interior and exterior traces of the wave amplitude u and its conormal derivatives are prescribed on the interface surface, while on the interior boundary of the inhomogeneous obstacle there are given the Dirichlet or Neumann or mixed Dirichlet-Neumann boundary conditions.

The transmission problems for the Helmholtz equation in the case of *the whole piecewise homogenous isotropic space* $\mathbb{R}^3 = \overline{\Omega}_2 \cup \overline{\Omega}_1$ with *a smooth interface surface* $S = \partial\Omega_1 = \partial\Omega_2$, when $A_2(\partial) = \Delta + \kappa_2\omega^2$ and $A_1(\partial) = \Delta + \kappa_1\omega^2$, $\kappa_q = \text{const}$, $q = 1,2$, are well studied in [14, 24–26] (see also references therein). In these papers, using the method of standard direct and indirect boundary integral equations method the transmission problem is reduced to a uniquely solvable coupled pair of boundary integral

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equations for a pair of unknowns. Moreover, in [25], by coupling the direct and indirect approaches, the transmission problem is reduced to a uniquely solvable single integral equation for a single unknown.

Using the harmonic analysis technique and the approach employed in the reference [26], the same transmission problem for the whole piecewise homogenous isotropic space $\mathbb{R}^3 = \overline{\Omega}_2 \cup \overline{\Omega}_1$ with a *Lipschitz interface* is considered in [40] using the potential method. Note that the harmonic analysis approach gives the optimal L_2 results, establishes the nontangential almost everywhere convergence of the solution to the boundary values, guarantees the boundedness of the corresponding nontangential maximal function, which in turn give better regularity results (see, e.g., [22]).

Similar acoustic scattering problems for the whole *isotropic composed space* $\mathbb{R}^3 = \overline{\Omega}_2 \cup \overline{\Omega}_1$ with smooth interface and with a variable continuous refractive index $\kappa(x)$, when $\kappa(x) = 1$ in the exterior domain Ω_1, are also well presented in the literature. In this case, $A_2(x, \partial_x, \omega) = \Delta + \omega^2\kappa(x)$ in the *isotropic inhomogeneous obstacle* region and $A_1(\partial_x, \omega) = \Delta + \omega^2$ in the unbounded homogeneous isotropic region. The problem is reduced to the Lippmann-Schwinger equation which is unconditionally solvable Fredholm type integral equation on the bounded obstacle region Ω_1 (see [12,35] and references therein).

Analogous acoustic transmission problem in the whole composed space $\mathbb{R}^3 = \overline{\Omega}_2 \cup \overline{\Omega}_1$ with a smooth interface, corresponding to a more general isotropic case, when $A_2(x, \partial_x, \omega) = \partial_{x_k}\big(a(x)\,\partial_{x_j}\big) + \omega^2$ with a sufficiently smooth function $a(x)$ and $A_1(\partial_x, \omega) = \Delta + \omega^2$, was analysed by the indirect boundary-domain integral equation method in the references [28, 44, 45].

The transmission problem for the whole composed anisotropic space $\mathbb{R}^3 = \overline{\Omega}_2 \cup \overline{\Omega}_1$ in the case of a smooth interface and sufficiently smooth in Ω_2 material coefficients $a^{(2)}_{kj}$ and κ_2 is studied in [10] by a special direct method based on the application of localized harmonic parametrix. This approach reduces the transmission problem to the uniquely solvable system of localized boundary-domain integral equations.

In this paper, we investigate more general anisotropic boundary-transmission problems using the so-called nonlocal approach when the interior boundary of the obstacle and the interface surface are Lipschitz manifolds, and the coefficients $a^{(2)}_{kj}$ and κ_2 are Lipschitz-continuous. Moreover, we consider in detail the case when the mixed Dirichlet-Neumann conditions are prescribed on the interior boundary.

We apply the theory of layer potentials on Lipschitz surfaces and reduce equivalently the boundary-transmission problem to the variational-functional equation containing sesquilinear forms over the interfacial surface and over a bounded domain occupied by the inhomogeneous obstacle. To substantiate our approach, we use essentially the results of [13, 21, 22], and the so-called *combined field integral equations* approach described in [6, 8, 27, 36] (see also [7]).

The paper is organized as follows. In Section 2, we introduce the generalized radiation conditions for anisotropic media, formulate the acoustic transmission problems for multi-component piecewise anisotropic structures with Lipschitz-continuous boundaries and interfaces, and prove the uniqueness theorems in appropriate function spaces. In Section 3, we construct the generalized Steklov–Poincaré type integral operator in the case of Lipschitz surfaces and derive the corresponding Dirichlet-to-Neumann relations for the acoustic equation in an unbounded anisotropic region. In Section 4, the transmission problems are equivalently reformulated as variational-functional equations containing sesquilinear forms which live on a bounded domain occupied by the obstacle and the interface surface. The boundedness and coercivity properties for the sesquilinear forms are proved in the appropriately chosen function spaces which eventually lead to the unique solvability of the original acoustic transmission problems. Finally, for the readers convenience, in Appendix we collect some auxiliary material related to anisotropic radiating layer potentials over Lipschitz surfaces.

2. Formulation of the Problems and Uniqueness Theorems

2.1. **Some auxiliary definitions and relations.** Let $\Omega_1 := \Omega^-$ be an unbounded domain in $\mathbb{R}^3$ with a simply connected compact boundary $\partial\Omega_1 = S_1$ and $\Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega}_1$. Further, let $\Omega_2 := \Omega^+ \setminus \overline{\Omega}_3$, where Ω_3 is a subdomain of Ω^+ such that $\overline{\Omega}_3 \subset \Omega^+$. Put $S_2 = \partial\Omega_3$. Evidently, $\partial\Omega_2 = S_1 \cup S_2$. Throughout the paper, $n = (n_1, n_2, n_3)$ denotes the outward unit normal vector to S_q, $q = 1, 2$.

In what follows, we assume that the interface S_1 and the interior boundary S_2 are arbitrary Lipschitz surfaces, unless otherwise stated, and the following condition holds:

$$\text{the interface } S_1 \text{ contains a } C^2\text{-smooth open submanifold } S_1^*. \tag{2.1}$$

By $H^s(\Omega) = H_2^s(\Omega)$, $H^s_{\mathrm{loc}}(\Omega) = H^s_{2,\,\mathrm{loc}}(\Omega)$, $H^s_{\mathrm{comp}}(\Omega) = H^s_{2,\,\mathrm{comp}}(\Omega)$ and $H^s(S) = H_2^s(S)$, $s \in \mathbb{R}$, we denote the L_2-based Bessel potential spaces of complex-valued functions on an open domain $\Omega \subset \mathbb{R}^3$ and on a closed manifold S without boundary, while $\mathcal{D}(\Omega)$ stands for the space of infinitely differentiable test functions with support in Ω. Recall that $H^0(\Omega) = L_2(\Omega)$ is a space of square integrable functions on Ω.

Further, let us define the following classes of functions:

$$\begin{aligned}
&H^{1,\,0}(\Omega_2; A_2) := \{\, v \in H^1(\Omega_2) \,:\, A_2 v \in H^0(\Omega_2)\,\}\,,\\
&H^{1,\,0}_{\mathrm{loc}}(\Omega_1; A_1) := \{\, v \in H^1_{\mathrm{loc}}(\Omega_1) \,:\, A_1 v \in H^0_{\mathrm{loc}}(\Omega_1)\,\}\,,\\
&\widetilde{H}^s(\Omega_2) := \{v :\ v \in H^s(\mathbb{R}^3),\ \operatorname{supp} v \subset \overline{\Omega_2}\}\,,\\
&\widetilde{H}^s(\mathcal{M}) := \{g :\ g \in H^s(S_2),\ \operatorname{supp} g \subset \overline{\mathcal{M}}\}\,,\\
&H^s(\mathcal{M}) := \{r_{\mathcal{M}} g :\ g \in H^s(S_2)\}\,,
\end{aligned}$$

where $\mathcal{M} \subset S_2$ is an open submanifold of the Lipschitz surface S_2 with a Lipschitz boundary curve $\partial\mathcal{M}$ and $r_{\mathcal{M}}$ stands for the restriction operator onto $\mathcal{M}$.

We assume that the propagation region of a time harmonic acoustic wave u^{tot} is the domain $\mathbb{R}^3 \setminus \Omega_3 = \overline{\Omega}_1 \cup \overline{\Omega}_2$, which consists of the homogeneous part Ω_1 and the inhomogeneous part Ω_2.

Acoustic wave propagation is governed by a uniformly elliptic second order scalar partial differential equation

$$A(x, \partial_x, \omega)\, u^{\mathrm{tot}}(x) \equiv \partial_k \big(a_{kj}(x)\, \partial_j u^{\mathrm{tot}}(x)\big) + \omega^2\, \kappa(x)\, u^{\mathrm{tot}}(x) = f(x), \quad x \in \Omega_1 \cup \Omega_2,$$

where $\partial_x \equiv \partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial_{x_j} = \partial/\partial x_j$, $a_{kj}(x) = a_{jk}(x)$ and $\kappa(x)$ are real-valued functions, $\omega \in \mathbb{R}$ is a frequency parameter, while f is a square integrable function in $\mathbb{R}^3$ with a compact support, $f \in L_{2,\mathrm{comp}}(\mathbb{R}^3)$. Here and in what follows, the Einstein summation by repeated indices from 1 to 3 is assumed.

Note that in the mathematical model of an inhomogeneous absorbing medium the function κ is complex-valued, with nonzero real and imaginary parts, in general (see, e.g., [12, Ch. 8]). Here we treat only the case when the refractive index κ is a real-valued function, but it should be mentioned that the complex-valued case can also be considered by the approach developed in the present paper.

In our further analysis, it is assumed that the real-valued variable coefficients a_{kj} and κ are the constants in the homogeneous unbounded region Ω_1,

$$a_{kj}(x) = a_{jk}(x) = \begin{cases} a^{(1)}_{kj} & \text{for} \quad x \in \Omega_1, \\ a^{(2)}_{kj}(x) & \text{for} \quad x \in \Omega_2, \end{cases} \qquad \kappa(x) = \begin{cases} \kappa_1 > 0 & \text{for} \quad x \in \Omega_1, \\ \kappa_2(x) > 0 & \text{for} \quad x \in \Omega_2, \end{cases} \tag{2.2}$$

where $a^{(1)}_{kj}$ and κ_1 are the constants, while $a^{(2)}_{kj}$ and κ_2 are the Lipschitz-continuous functions in $\overline{\Omega}_2$,

$$a^{(2)}_{kj},\ \kappa_2 \in C^{0,\,1}(\overline{\Omega}_2), \quad j, k = 1, 2, 3. \tag{2.3}$$

Moreover, the matrices $\mathbf{a}_q = \big[\, a^{(q)}_{kj} \,\big]_{k,j=1}^{3}$ are uniformly positive definite, i.e., there are positive constants c_1 and c_2 such that

$$c_1\, |\xi|^2 \le a^{(q)}_{kj}(x)\, \xi_k\, \xi_j \le c_2\, |\xi|^2 \qquad \forall\, x \in \overline{\Omega}_q,\ \ \forall\, \xi \in \mathbb{R}^3,\ \ q = 1, 2. \tag{2.4}$$

We do not assume that the coefficients a_{kj} and κ are continuous across the interface S_1, in general, i.e., the case $a^{(2)}_{kj}(x) \ne a^{(1)}_{kj}$ and $\kappa_2(x) \ne \kappa_1$ for $x \in S_1$ is covered by our analysis.

Further, we denote

$$\begin{aligned}
&r_{\Omega_1} A(x, \partial_x, \omega) u(x) \equiv A_1(\partial_x, \omega) u(x) := a^{(1)}_{kj}\, \partial_k \partial_j u(x) + \omega^2 \kappa_1\, u(x) \ \text{ for } \ x \in \Omega_1, \qquad (2.5)\\
&r_{\Omega_2} A(x, \partial_x, \omega) u(x) \equiv A_2(x, \partial_x, \omega) u(x) := \partial_{x_k} \big(a^{(2)}_{kj}(x)\, \partial_j u(x)\big) + \omega^2\, \kappa_2(x)\, u(x) \ \text{ for } \ x \in \Omega_2.
\end{aligned}$$

We will often drop the arguments and write A_1 and A_2 instead of $A_1(\partial_x,\omega)$ and $A_2(x,\partial_x,\omega)$, respectively, when this does not lead to misunderstanding.

For a function u_q, sufficiently smooth in Ω_q (say, $u_1 \in H^2_{\rm loc}(\Omega_1)$ or $u_2 \in H^2(\Omega_2)$), the *classical conormal derivative operators* $T^{\pm}_q$ are well defined as

$$T^{\pm}_q u_q(x) := a^{(q)}_{kj}\, n_k(x)\, \gamma^{\pm}_{S_m}(\partial_j u_q(x)), \quad x \in S_m, \quad q,m=1,2, \tag{2.6}$$

where the symbols $\gamma^{+}_{S_m}$ and $\gamma^{-}_{S_m}$ denote one-sided boundary trace operators on S_m from the interior and exterior domains, respectively.

Motivated by the first Green identity, the classical conormal derivative operators (2.6) can be extended by continuity to the functions $u_1 \in H^{1,0}_{\rm loc}(\Omega_1;A_1)$ and $u_2 \in H^{1,0}(\Omega_2;A_2)$ giving well defined *canonical conormal derivatives* $T^-_1 u_1 \in H^{-\frac12}(S_1)$, $T^+_2 u_2 \in H^{-\frac12}(S_1)$, and $T^-_2 u_2 \in H^{-\frac12}(S_2)$, defined for arbitrary $g_1 \in H^{\frac12}(S_1)$ and $g_2 \in H^{\frac12}(S_2)$ by the following relations:

$$\big\langle T^-_1 u_1\,,\,\overline{g_1}\big\rangle_{S_1} := -\int\limits_{\Omega_1} A_1 u_1(x)\,\overline{w_1(x)}\,dx - \int\limits_{\Omega_1}\big[E_1(u_1,\overline{w_1}) - \omega^2\kappa_1 u_1(x)\,\overline{w_1(x)}\big]\,dx, \tag{2.7}$$

$$\begin{aligned}\big\langle T^+_2 u_2\,,\,\overline{g_1}\big\rangle_{S_1} - \big\langle T^-_2 u_2\,,\,\overline{g_2}\big\rangle_{S_2} &:= \int\limits_{\Omega_2} A_2 u_2(x)\;\overline{w_2(x)}\,dx\\ &\quad + \int\limits_{\Omega_2}\big[E_2(u_2,\overline{w_2}) - \omega^2\kappa_2(x)u_2(x)\,\overline{w_2(x)}\,\big]\,dx,\end{aligned} \tag{2.8}$$

where the angular brackets $\langle\cdot\,,\,\cdot\rangle_{S_m}$ are understood as duality pairing of $H^{-\frac12}(S_m)$ with $H^{\frac12}(S_m)$ which extends the usual bilinear $L_2(S_m)$ inner product, $w_1 \in H^1_{\rm comp}(\Omega_1)$ with $\gamma^-_{S_1} w_1 = g_1$, $w_2 \in H^1(\Omega_2)$ with $\gamma^+_{S_1} w_2 = g_1$ and $\gamma^-_{S_2} w_2 = g_2$, and

$$E_1(u_1,\overline{w_1}) := a^{(1)}_{kj}\,\partial_j u_1(x)\,\overline{\partial_k w_1(x)}, \qquad E_2(u_2,\overline{w_2}) := a^{(2)}_{kj}(x)\,\partial_j u_2(x)\,\overline{\partial_k w_2(x)}. \tag{2.9}$$

Evidently, there is a constant $C>0$ such that

$$\begin{aligned}\|T^-_1 u_1\|_{H^{-\frac12}(S_1)} &\leqslant C\big(\|A_1 u_1\|_{H^0(\Omega_1^*)} + \|u_1\|_{H^1(\Omega_1^*)}\big),\\ \|T^+_2 u_2\|_{H^{-\frac12}(S_1)} &\leqslant C\big(\|A_2 u_2\|_{H^0(\Omega_2)} + \|u_2\|_{H^1(\Omega_2)}\big),\\ \|T^-_2 u_2\|_{H^{-\frac12}(S_2)} &\leqslant C\big(\|A_2 u_2\|_{H^0(\Omega_2)} + \|u_2\|_{H^1(\Omega_2)}\big),\end{aligned} \tag{2.10}$$

where Ω_1^* is an arbitrary one-sided exterior neighbourhood of the surface $S_1 = \partial\Omega_1$ located in Ω_1. For the properties of the trace operator in the case of Lipschitz domains and for the corresponding conormal derivatives see [13, 15], [29, Ch. 4], [30].

Recall that for arbitrary functions $u_1 \in H^{1,0}_{\rm loc}(\Omega_1;A_1)$ and $u_2 \in H^{1,0}(\Omega_2;A_2)$, the Green first identities associated with the operators A_1 and A_2 (see, e.g., [13, Section 3], [29, Ch. 4], [30, Theorem 3.9])

$$\begin{aligned}&\int\limits_{\Omega_1(R)} A_1 u_1(x)\,\overline{v_1(x)}\,dx + \int\limits_{\Omega_1(R)}\big[E_1(u_1,\overline{v_1}) - \omega^2\kappa_1 u_1(x)\,\overline{v_1(x)}\big]\,dx\\ &\qquad = \big\langle T^+_1 u_1\,,\,\overline{\gamma^+_{\Sigma(R)} v_1}\big\rangle_{\Sigma(R)} - \big\langle T^-_1 u_1\,,\,\overline{\gamma^-_{S_1} v_1}\big\rangle_{S_1} \qquad \forall\, v_1 \in H^1_{\rm loc}(\Omega_1),\end{aligned} \tag{2.11}$$

$$\begin{aligned}&\int\limits_{\Omega_2} A_2 u_2(x)\;\overline{v_2(x)}\,dx + \int\limits_{\Omega_2}\big[E_2(u_2,\overline{v_2}) - \omega^2\kappa_2(x)u_2(x)\,\overline{v_2(x)}\,\big]\,dx\\ &\qquad = \big\langle T^+_2 u_2\,,\,\overline{\gamma^+_{S_1} v_2}\big\rangle_{S_1} - \big\langle T^-_2 u_2\,,\,\overline{\gamma^-_{S_2} v_2}\big\rangle_{S_2} \qquad \forall\, v_2 \in H^1(\Omega_2)\end{aligned} \tag{2.12}$$

hold, where $\Omega_1(R) := \Omega_1 \cap B(R)$ with $B(R)$ being a ball centered at the origin and radius R such that $\overline{\Omega}_2 \subset B(R)$, $\Sigma(R) := \partial B(R)$, $E_q(u_q,\overline{v_q})$, $q=1,2$, are defined in (2.9).

By $Z(\Omega_1)$ we denote the sub-class of complex-valued functions from $H^1_{\rm loc}(\Omega_1)$ satisfying the Sommerfeld radiation conditions at infinity (see [12,37,42] for the Helmholtz operator and [19,20,34,41] for the "anisotropic" operator A_1 defined by (2.5)). Denote by S_ω the characteristic ellipsoid associated with the operator $A_1(\partial_x,\omega)$,

$$a^{(1)}_{kj}\xi_k\,\xi_j-\omega^2\kappa_1=0,\quad \xi\in\mathbb{R}^3,\quad \omega\neq 0.$$

For an arbitrary vector $\eta\in\mathbb{R}^3$ with $|\eta|=1$ there exists only one point $\xi(\eta)\in S_\omega$ such that the outward unit normal vector $n(\xi(\eta))$ to S_ω at the point $\xi(\eta)$ has the same direction as η, i.e., $n(\xi(\eta))=\eta$. Note that $\xi(-\eta)=-\xi(\eta)\in S_\omega$ and $n(-\xi(\eta))=-\eta$.

It can easily be verified that

$$\xi(\eta)=\omega\,\sqrt{\kappa_1}\,(\mathbf{a}_1^{-1}\eta\cdot\eta)^{-\frac{1}{2}}\ \mathbf{a}_1^{-1}\eta, \tag{2.13}$$

where $\mathbf{a}_1^{-1}$ is the matrix, inverse to $\mathbf{a}_1:=\big[\,a^{(1)}_{kj}\,\big]_{k,j=1}^3$, and the central dot denotes the scalar product in $\mathbb{R}^3$.

Definition 2.1. A complex-valued function v belongs to the class $Z(\Omega_1)$ if there exists a ball $B(R)$ of radius R centered at the origin such that $v\in C^1(\Omega_1\setminus B(R))$, and v satisfies the Sommerfeld radiation conditions associated with the operator $A_1(\partial_x,\omega)$ for sufficiently large $|x|$,

$$v(x)=\mathcal{O}(|x|^{-1}),\quad \partial_k v(x)-i\xi_k(\eta)v(x)=\mathcal{O}(|x|^{-2}),\quad k=1,2,3, \tag{2.14}$$

where $\xi(\eta)\in S_\omega$ corresponds to the vector $\eta=x/|x|$ (i.e., $\xi(\eta)$ is given by (2.13) with $\eta=x/|x|$).

Notice that due to the ellipticity of the operator $A_1(\partial_x,\omega)$, any solution to the constant coefficient homogeneous equation $A_1(\partial_x,\omega)v(x)=0$ in an open region $\Omega\subset\mathbb{R}^3$ is a real analytic function of x in Ω.

Conditions (2.14) are equivalent to the classical Sommerfeld radiation conditions for the Helmholtz equation if $A_1(\partial_x,\omega)=\Delta(\partial)+\omega^2$, i.e., if $\kappa_1=1$ and $a^{(1)}_{kj}=\delta_{kj}$, where δ_{kj} is the Kronecker delta. The following analogue of the classical Rellich-Vekua lemma holds (for details see [19, 34]).

Lemma 2.2. *Let $v\in Z(\Omega_1)$ be a solution of the equation $A_1(\partial_x,\omega)v=0$ in Ω_1 and let*

$$\lim_{R\to+\infty}\operatorname{Im}\Bigg\{\int\limits_{\Sigma(R)}\overline{v(x)}\,T_1(x,\partial_x)v(x)\,d\Sigma(R)\Bigg\}=0, \tag{2.15}$$

where $\Sigma(R)$ is the sphere of radius R centered at the origin. Then $v=0$ in Ω_1.

Remark 2.3. For $x\in\Sigma(R)$ and $\eta=x/|x|$, we have $n(x)=\eta$ and, in view of (2.6) and (2.14), for a function $v\in Z(\Omega_1)$, we get

$$T_1(x,\partial_x)v(x)=a^{(1)}_{kj}n_k(x)\,[\,i\,\xi_j(\eta)\,v(x)]+\mathcal{O}(|x|^{-2})=i\,a^{(1)}_{kj}\eta_k\,\xi_j(\eta)\,v(x)+\mathcal{O}(|x|^{-2})\,.$$

Therefore, by (2.13) and the symmetry condition $a^{(1)}_{kj}=a^{(1)}_{jk}$, we arrive at the relation

$$\begin{aligned}\overline{v(x)}\,T_1(x,\partial_x)v(x)&=i\,\omega\,\sqrt{\kappa_1}\,|v(x)|^2\,(\mathbf{a}_1^{-1}\eta\cdot\eta)^{-\frac{1}{2}}\,\mathbf{a}_1\eta\cdot\mathbf{a}^{-1}\eta+\mathcal{O}(|x|^{-3})\\&=i\,\omega\,\sqrt{\kappa_1}\,(\mathbf{a}_1^{-1}\eta\cdot\eta)^{-\frac{1}{2}}\,|v(x)|^2+\mathcal{O}(|x|^{-3}).\end{aligned}$$

On the other hand, the matrix $\mathbf{a}_1$ is positive definite (cf. (2.4)), which implies positive definiteness of the inverse matrix $\mathbf{a}_1^{-1}$. Hence there are positive constants δ_0 and δ_1 such that for all $\eta\in\Sigma(1)$,

$$0<\delta_0\leqslant(\mathbf{a}_1^{-1}\eta\cdot\eta)^{-\frac{1}{2}}\leqslant\delta_1<\infty\,.$$

Consequently, for $\omega\neq 0$, condition (2.15) is equivalent to the following relation:

$$\lim_{R\to+\infty}\int\limits_{\Sigma(R)}|v(x)|^2\,d\Sigma(R)=0,$$

which is the well known Rellich-Vekua condition in the theory of Helmholtz equation (for details see [12, 37, 42]).

2.2. **Formulation of the transmission problems.** In the unbounded region Ω_1, we have a total wave field $u^{\rm tot} = u^{\rm inc} + u^{\rm sc}$, where $u^{\rm inc}$ is a wave motion initiating the known incident field and u^{sc} is a radiating unknown scattered field. It is often assumed that the incident field is defined in the whole of $\mathbb{R}^3$, being, for example, a corresponding plane wave which solves the homogeneous equation $A_1 u^{\rm inc} = 0$ in $\mathbb{R}^3$ but does not satisfy the Sommerfeld radiation conditions at infinity. Motivated by relations (2.2), we set $u_1(x) := u^{sc}(x)$ for $x \in \Omega_1$ and $u_2(x) := u^{\rm tot}(x)$ for $x \in \Omega_2$.

Now we formulate the transmission problem associated with the time-harmonic acoustic wave scattering by a bounded anisotropic inhomogeneity embedded in an unbounded anisotropic homogeneous medium:

Find complex-valued functions $u_1 \in H^{1,0}_{\rm loc}(\Omega_1; A_1) \cap Z(\Omega_1)$ *and* $u_2 \in H^{1,0}(\Omega_2; A_2)$ *satisfying the differential equations*

$$A_1(\partial_x, \omega) u_1(x) = f_1(x) \ \ \text{for} \ \ x \in \Omega_1, \tag{2.16}$$

$$A_2(x, \partial_x, \omega) u_2(x) = f_2(x) \ \ \text{for} \ \ x \in \Omega_2, \tag{2.17}$$

the transmission conditions on the interface S_1,

$$\gamma^+_{S_1} u_2 - \gamma^-_{S_1} u_1 = \varphi_1 \ \ \text{on} \ \ S_1, \tag{2.18}$$

$$T^+_2 u_2 - T^-_1 u_1 = \psi_1 \ \ \text{on} \ \ S_1, \tag{2.19}$$

and one of the following boundary conditions on S_2*:*
The Dirichlet condition

$$\gamma^-_{S_2} u_2 = 0 \ \ \text{on} \ \ S_2, \tag{2.20}$$

The Neumann condition

$$T^-_2 u_2 = \psi_2 \ \ \text{on} \ \ S_2, \tag{2.21}$$

The mixed type conditions

$$\gamma^-_{S_{2D}} u_2 = 0 \ \ \text{on} \ \ S_{2D}, \qquad T^-_2 u_2 = \psi_{2N} \ \ \text{on} \ \ S_{2N}, \tag{2.22}$$

where $S_{2D} \cap S_{2N} = \varnothing$, $\overline{S}_{2D} \cup \overline{S}_{2N} = S_2$, *and*

$$\begin{gathered} f_2 := r_{\Omega_2} f \in H^0(\Omega_2), \quad f_1 := r_{\Omega_1} f \in H^0_{\rm comp}(\Omega_1), \quad f \in H^0_{\rm comp}(\mathbb{R}^3), \\ \varphi_1 \in H^{\frac12}(S_1), \quad \psi_1 \in H^{-\frac12}(S_1), \quad \psi_2 \in H^{-\frac12}(S_2), \quad \psi_{2N} \in H^{-\frac12}(S_{2N}). \end{gathered} \tag{2.23}$$

In the above setting, equations (2.16) and (2.17) are understood in the distributional sense, the Dirichlet type conditions in (2.18), (2.20) and (2.22) are understood in the usual trace sense, while the Neumann type conditions in (2.19), (2.21) and (2.22) are understood in the canonical conormal derivative sense defined by relations (2.7)–(2.8).

If the total field $u^{\rm tot}$ and its conormal derivative are continuous across the interface, then $\varphi_1 = \gamma^-_{S_1} u^{\rm inc}$ and $\psi_1 = T^-_1 u^{\rm inc}$.

The above-formulated boundary-transmission problems with the Dirichlet, Neumann, and mixed type conditions will be referred to as Problem (TD), (TN) and (TM), respectively.

2.3. **Uniqueness theorems.** Here we prove the uniqueness theorem.

Theorem 2.4. *The boundary-transmission problems (TD), (TN) and (TM) possess at most one solution.*

Proof. Due to the linearity of the problems, we have to show that the corresponding homogeneous problems possess only the trivial solution.

Let a pair (u_2, u_1) with $u_2 \in H^{1,0}(\Omega_2; A_2)$ and $u_1 \in H^{1,0}_{\rm loc}(\Omega_1; A_1) \cap Z(\Omega_1)$ be a solution to the homogeneous boundary-transmission problem (TD) or (TN) or (TM). Note that $u_1 \in C^\infty(\Omega_1)$ due to ellipticity of the constant coefficient operator A_1.

Let R be an arbitrary positive number such that $\overline{\Omega}_2 \subset B(R)$. We can write Green's first identities (2.11) and (2.12) for the functions u_1 and u_2 in the domains $\Omega_1(R) := \Omega_1 \cap B(R)$ and Ω_2. In view of the homogeneity of the boundary conditions on S_2, we arrive at the relations

$$\int\limits_{\Omega_1(R)} [a_{kj}^{(1)}\, \partial_j u_1(x)\, \overline{\partial_k u_1(x)} - \omega^2 \kappa_1 |u_1(x)|^2]\, dx = -\langle T_1^- u_1\, ,\, \overline{\gamma_{S_1}^- u_1} \rangle_{S_1} + \langle T_1^+ u_1\, ,\, \overline{\gamma_{\Sigma(R)}^+ u_1} \rangle_{\Sigma(R)}, \tag{2.24}$$

$$\int\limits_{\Omega_2} [a_{kj}^{(2)}(x)\, \partial_j u_2(x)\, \overline{\partial_k u_2(x)} - \omega^2 \kappa_2(x) |u_2(x)|^2]\, dx = \langle T_2^+ u_2\, ,\, \overline{\gamma_{S_1}^+ u_2} \rangle_{S_1}. \tag{2.25}$$

Due to the homogeneous transmission conditions and since the matrices $\mathbf{a}_q = [a_{kj}^{(q)}]_{k,j=1}^3$ are symmetric and positive definite, after adding (2.24) and (2.25) and separating the imaginary part, we get

$$\operatorname{Im} \Big\{ \int\limits_{\Sigma(R)} \overline{u_1(x)}\, T_1(x, \partial_x) u_1(x)\, d\Sigma(R) \Big\} = 0,$$

whence by Lemma 2.2, we deduce that $u_1 = 0$ in Ω_1.

Therefore, in view of (2.16)–(2.22), the function $u_2 \in H^{1,\,0}(\Omega_2; A_2)$ satisfies the homogeneous differential equation

$$A_2(x, \partial_x, \omega) u_2(x) = 0 \ \text{ in } \ \Omega_2,$$

the homogeneous Cauchy type conditions

$$\gamma_{S_1}^+ u_2 = 0 \quad \text{and} \quad T_2^+ u_2 = 0 \quad \text{on} \quad S_1,$$

and one of the homogeneous boundary conditions (2.20)–(2.22) on S_2.

Keeping in mind the relations (2.1) and (2.3), by the interior and boundary regularity properties of solutions to a strongly elliptic partial differential equation, we deduce $u_2 \in C^2(\Omega_2 \cup S_1^*)$ (see, e.g., [18, Lemmas 6.16, 6.18], [29, Theorem 4.18]). Thus, the Cauchy data of the function u_2 vanish continuously on $S_1^* \subset S_1$ and due to [39, Theorem 2.9], we conclude that $u_2 = 0$ in Ω_2, which completes the proof. □

3. Integral Relations for Radiating Function in the Domain Ω_1

For any radiating solution $u_1 \in H^{1,\,0}_{\mathrm{loc}}(\Omega_1; A_1) \cap Z(\Omega_1)$ with $A_1 u_1 \in H^0_{\mathrm{comp}}(\Omega_1)$ the Green third identity (for details see [13, 19, 29, 34])

$$u_1 + V(T_1^- u_1) - W(\gamma_{S_1}^- u_1) = \mathcal{P}(A_1 u_1) \quad \text{in} \quad \Omega_1 \tag{3.1}$$

holds, where V, W, and $\mathcal{P}$ denote, respectively, the single layer potential, double layer potential and volume potential associated with the operator $A_1(\partial_x, \omega)$,

$$V g(y) := -\int\limits_{S_1} \Gamma(x - y, \omega)\, g(x)\, dS_x, \quad y \in \mathbb{R}^3 \setminus S_1, \tag{3.2}$$

$$W g(y) := -\int\limits_{S_1} [T_1(x, \partial_x) \Gamma(x - y, \omega)]\, g(x)\, dS_x, \quad y \in \mathbb{R}^3 \setminus S_1, \tag{3.3}$$

$$\mathcal{P} h(y) := \int\limits_{\Omega_1} \Gamma(x - y, \omega)\, h(x)\, dx, \quad y \in \mathbb{R}^3. \tag{3.4}$$

Here g and h are densities of the potentials, $T_1(x, \partial_x) = a_{kj}^{(1)} n_k(x) \partial_{x_j}$, $n(x)$ is the outward unit normal vector to S_1 at the point $x \in S_1$, and

$$\Gamma(x, \omega) = -\frac{\exp\{ i\omega \sqrt{\kappa_1}\, (\mathbf{a}_1^{-1} x \cdot x)^{1/2} \}}{4\pi (\det \mathbf{a}_1)^{1/2}\, (\mathbf{a}_1^{-1} x \cdot x)^{1/2}} \tag{3.5}$$

is a radiating fundamental solution of the operator $A_1(\partial_x, \omega)$ (see, e.g., Lemma 1.1 in [20]).

Remark 3.1. In a neighbourhood of the origin, e.g., for $|x| < 1$, we have the decomposition

$$\Gamma(x,\omega) = -\frac{1}{4\pi\,(\det \mathbf{a}_1)^{1/2}}\Big\{\frac{1}{(\mathbf{a}_1^{-1}x\cdot x)^{1/2}} + i\omega\sqrt{\kappa_1} - \frac{1}{2}\,\omega^2\kappa_1\,(\mathbf{a}_1^{-1}x\cdot x)^{1/2} + \cdots\Big\}, \tag{3.6}$$

while for sufficiently large $|y|$, we have the following asymptotic formula:

$$\Gamma(y-x,\omega) = c(\xi)\frac{\exp\{i\xi\cdot(y-x)\}}{|y|} + O(|y|^{-2}), \quad c(\xi) = -\frac{|\mathbf{a}_1\,\xi|}{4\pi\omega\sqrt{\kappa_1}\,(\det\mathbf{a}_1)^{1/2}}, \tag{3.7}$$

where x varies in a bounded subset of $\mathbb{R}^3$, $\xi = \xi(\eta) \in S_\omega$ corresponds to the direction $\eta = y/|y|$ and is given by (2.13). The asymptotic formula (3.7) can be differentiated arbitrarily many times with respect to x and y. Both formulas, (3.5) and (3.6), hold true for an arbitrary complex parameter $\omega = \omega_1 + i\omega_2$ with $\omega_j \in \mathbb{R}$, $j = 1,2$. Evidently, the function $\Gamma(x) := \Gamma(x,0)$ is a fundamental solution of the operator $A_1(\partial_x) := A_1(\partial_x, 0)$, while $\Gamma(x,i)$ is the exponentially decaying real-valued fundamental solution of the operator $A_1(\partial_x, i)$. In view of (3.6), we have

$$\Gamma(x,\omega) - \Gamma(x,i) = -\frac{1}{4\pi\,(\det\mathbf{a}_1)^{1/2}}\Big\{(1+i\omega)\sqrt{\kappa_1} - \frac{1}{2}(\omega^2+1)\,\kappa_1\,(\mathbf{a}_1^{-1}x\cdot x)^{1/2} + \cdots\Big\}, \tag{3.8}$$

implying for $|x| < 1$ the following relations:

$$\frac{\partial}{\partial x_k}\big[\Gamma(x,\omega) - \Gamma(x,i)\big] = \mathcal{O}(1), \quad \frac{\partial^2}{\partial x_j\partial x_k}\big[\Gamma(x,\omega) - \Gamma(x,i)\big] = \mathcal{O}(|x|^{-1}), \;\; k,j = 1,2,3. \tag{3.9}$$

The mapping properties of these potentials and the boundary operators generated by them in the case of Lipschitz surface S_1 are collected in Appendix A. Note that the mapping properties of layer potentials associated with Lipschitz and smooth surfaces are essentially different (cf., e.g., [3–5, 13, 29, 43] and references cited therein).

Evidently, the layer potentials Vg and Wg solve the homogeneous differential equation (2.16), i.e.,

$$A_1Vg = A_1Wg = 0 \;\text{ in }\; \mathbb{R}^3\setminus S_1, \tag{3.10}$$

while for $f_1 \in H^0_{comp}(\Omega_1)$, the volume potential $\mathcal{P}f_1 \in H^2_{\rm loc}(\mathbb{R}^3)$ solves the following nonhomogeneous equation (see Lemma A.1)

$$A_1\mathcal{P}f_1 = \begin{cases} f_1 & \text{in} \quad \Omega_1, \\ 0 & \text{in} \quad \mathbb{R}^3\setminus\overline{\Omega}_1. \end{cases} \tag{3.11}$$

Using the properties of layer and volume potentials (see Lemma A.1(iii)), for the exterior traces of Green's third identity (3.1) and its conormal derivative on S_1, we get

$$\mathcal{V}(T_1^- u_1) + (2^{-1}I - \mathcal{W})(\gamma^-_{S_1}u_1) = \gamma^-_{S_1}\mathcal{P}(A_1u_1) \quad \text{on} \quad S_1, \tag{3.12}$$

$$(2^{-1}I + \mathcal{W}')(T_1^- u_1) - \mathcal{L}(\gamma^-_{S_1}u_1) = T_1^-\mathcal{P}(A_1u_1) \quad \text{on} \quad S_1, \tag{3.13}$$

where the integral operators $\mathcal{V}$, $\mathcal{W}$, $\mathcal{W}'$ and $\mathcal{L}$ are defined in Appendix A by formulas (A.2)–(A.5). Recall that the operators $\mathcal{V}$, $2^{-1}I - \mathcal{W}$, $2^{-1}I + \mathcal{W}'$ and $\mathcal{L}$ involved in (3.12)–(3.13) are not invertible for resonant values of the frequency parameter ω. The set of these resonant values is countable and consists of eigenfrequencies of the interior Dirichlet and Neumann boundary value problems for the operator A_1 in the bounded domain surrounded by the surface S_1 (see [42, Section 4], [11, Ch. 3], [9, Section 7.7], [7]). Therefore, to obtain Dirichlet-to-Neumann or Neumann-to-Dirichlet mappings for arbitrary values of the frequency parameter ω, we apply *the combined-field integral equations* approach and proceed as follows. Multiply equation (3.12) by $i\,\alpha$ with some fixed positive α and add to equation (3.13) to obtain (cf., [6, 8, 27, 36])

$$\mathcal{K}(T_1^- u_1) - \mathcal{M}(\gamma^-_{S_1}u_1) = \Phi(A_1u_1) \quad \text{on} \quad S_1, \tag{3.14}$$

where

$$\mathcal{K}g := \Big(\frac{1}{2}\,I + \mathcal{W}' + i\,\alpha\,\mathcal{V}\Big)g = \big(T_1^+ + i\,\alpha\,\gamma^+_{S_1}\big)V\,g \quad \text{on} \quad S_1, \tag{3.15}$$

$$\mathcal{M}h := \Big[\mathcal{L} + i\,\alpha\,\Big(-\frac{1}{2}\,I + \mathcal{W}\Big)\Big]h = \big(T_1^+ + i\,\alpha\,\gamma^+_{S_1}\big)W\,h \quad \text{on} \quad S_1, \tag{3.16}$$

$$\Phi\, f_1 := \big(T_1^- + i\,\alpha\,\gamma^-_{S_1}\big)\mathcal{P}\, f_1 = \big(T_1^+ + i\,\alpha\,\gamma^+_{S_1}\big)\mathcal{P}\, f_1 \quad \text{on} \quad S_1, \tag{3.17}$$

for $f_1 \in H^0_{\text{comp}}(\Omega_1)$, $g \in H^{-\frac{1}{2}}(S_1)$ and $h \in H^{\frac{1}{2}}(S_1)$. The relation (3.17) follows from the imbedding $\mathcal{P}\, f_1 \in H^2_{\text{loc}}(\mathbb{R}^3)$ for $f_1 \in H^0_{\text{comp}}(\mathbb{R}^3)$.

In view of Lemma A.2, from (3.14), for arbitrary $u_1 \in H^{1,0}_{\text{loc}}(\Omega_1; A_1) \cap Z(\Omega_1)$, we derive the following analogue of the Steklov-Poincaré type relation:

$$T_1^- u_1 = \mathcal{K}^{-1}\big[\mathcal{M}\,(\gamma^-_{S_1} u_1) + \Phi(A_1 u_1)\big] \quad \text{on} \quad S_1, \tag{3.18}$$

where $\mathcal{K}^{-1} : H^{-\frac{1}{2}}(S_1) \to H^{-\frac{1}{2}}(S_1)$ is the inverse to the operator $\mathcal{K} : H^{-\frac{1}{2}}(S_1) \to H^{-\frac{1}{2}}(S_1)$.

4. Weak Formulation of the Mixed Boundary-transmission Problems and the Existence Results

Here we apply the so-called non-local approach to obtain the variational-functional formulation of the transmission problem under consideration. To this end, let us assume that a pair $(u_2, u_1) \in H^{1,0}(\Omega_2; A_2) \times \big(H^{1,0}_{\text{loc}}(\Omega_1; A_1) \cap Z(\Omega_1)\big)$ solves the mixed transmission problem (TM) (see (2.16)–(2.19)) and (2.22). Applying relation (3.18), transmission conditions (2.18)–(2.19) and mixed boundary conditions (2.22) in the Green first identity (2.12), for the domain Ω_2, we arrive at the equation

$$\begin{gathered} \mathfrak{B}(u_2, v) = \mathfrak{F}(v) \\ \forall\; v \in H^1(\Omega_2; S_{2D}) := \{w \in H^1(\Omega_2) \,:\, r_{S_{2D}}\gamma^-_{S_{2D}} w = 0\}, \end{gathered} \tag{4.1}$$

where $\mathfrak{B}$ is a sesquilinear form and $\mathfrak{F}$ is an antilinear functional defined, respectively, as

$$\mathfrak{B}(u_2, v) := \mathfrak{B}^{(1)}(u_2, v) + \mathfrak{B}^{(2)}(u_2, v), \tag{4.2}$$

$$\mathfrak{B}^{(1)}(u_2, v) := \int\limits_{\Omega_2} \big[a^{(2)}_{kj}(x)\,\partial_j u_2(x)\,\overline{\partial_k v(x)} - \omega^2 \kappa_2(x) u_2(x)\,\overline{v(x)}\,\big]\,dx, \tag{4.3}$$

$$\mathfrak{B}^{(2)}(u_2, v) := -\big\langle \mathcal{K}^{-1}\,\mathcal{M}(\gamma^+_{S_1} u_2)\,,\, \overline{\gamma^+_{S_1} v}\big\rangle_{S_1}, \tag{4.4}$$

$$\begin{aligned} \mathfrak{F}(v) := -\int\limits_{\Omega_2} f_2(x)\,\overline{v(x)}\,dx + \langle \mathcal{K}^{-1}\Phi f_1\,,\, \overline{\gamma^+_{S_1} v}\rangle_{S_1} + \big\langle \psi_1, \overline{\gamma^+_{S_1} v}\big\rangle_{S_1} - \big\langle \mathcal{K}^{-1}\,\mathcal{M}\varphi_1\,,\, \overline{\gamma^+_{S_1} v}\big\rangle_{S_1} \\ - \big\langle \psi_{2N}, \overline{\gamma^-_{S_{2N}} v}\big\rangle_{S_{2N}}, \end{aligned} \tag{4.5}$$

with the operators $\mathcal{K}$, $\mathcal{M}$, and Φ defined by relations (3.15)–(3.17). We associate with equation (4.1) the following variational-functional problem (in a wider space).

Problem (VMT1)**.** Find a function $u_2 \in H^1(\Omega_2; S_{2D})$ satisfying variational-functional equation (4.1) for all $v \in H^1(\Omega_2; S_{2D})$.

Now, let us first prove the following equivalence

Theorem 4.1. *Let conditions* (2.23) *be fulfilled.*

(i) *If a pair* $(u_2, u_1) \in H^{1,0}(\Omega_2; A_2) \times \big(H^{1,0}_{\text{loc}}(\Omega_1; A_1) \cap Z(\Omega_1)\big)$ *solves the mixed transmission problem (TM), then the function* u_2 *solves variational-functional equation* (4.1) *and the following relation holds:*

$$u_1(y) = \mathcal{P} f_1(y) - V(T_2^+ u_2 - \psi_1)(y) + W(\gamma^+_{S_1} u_2 - \varphi_1)(y), \quad y \in \Omega_1. \tag{4.6}$$

(ii) *Vice versa, if a function* $u_2 \in H^1(\Omega_2; S_{2D})$ *solves variational-functional equation* (4.1)*, then the pair* (u_2, u_1) *with* u_1 *defined by* (4.6) *belongs to the class* $H^{1,0}(\Omega_2; A_2) \times \big(H^{1,0}_{\text{loc}}(\Omega_1; A_1) \cap Z(\Omega_1)\big)$ *and solves the mixed transmission problem (TM).*

Proof. (i) The first part of the theorem follows from the derivation of variational-functional equation (4.1).

(ii) To prove the second part, we proceed as follows. If u_2 solves variational-functional equation (4.1), then for $v \in \mathcal{D}(\Omega_2)$ the equation

$$\int\limits_{\Omega_2} \big[a_{kj}^{(2)}(x)\,\partial_j u_2(x)\,\overline{\partial_k v(x)} - \omega^2 \kappa_2(x) u_2(x)\,\overline{v(x)}\,\big]\,dx = -\int\limits_{\Omega_2} f_2(x)\,\overline{v(x)}\,dx,$$

particularly holds and implies that u_2 is a solution of equation (2.17), $A_2(x, \partial_x, \omega)u_2 = f_2$ in Ω_2 in the sense of distributions and, evidently, $u_2 \in H^{1,\,0}(\Omega_2; A_2)$ in view of (2.23). Therefore the canonical conormal derivatives $T_2^+ u_2 \in H^{-\frac{1}{2}}(S_1)$ and $T_2^- u_2 \in H^{-\frac{1}{2}}(S_2)$ are well-defined in the sense of (2.8).

Further, it is easy to see that function (4.6) is well-defined, solves the differential equation (2.16) in Ω_1 due to (3.10)–(3.11), and belongs to the space $H^{1,\,0}_{\rm loc}(\Omega_1; A_1) \cap Z(\Omega_1)$ in view of (2.23) and properties of the volume and layer potentials. Therefore, the canonical conormal derivative $T_1^- u_1 \in H^{-\frac{1}{2}}(S_1)$ is also well-defined in the sense of (2.7).

Now we show that mixed boundary conditions (2.22) on S_2 and transmission conditions (2.18)–(2.19) on S_1 are satisfied. To this end, we write Green's identity (2.12) for u_2 and arbitrary $v \in H^1(\Omega_2; S_{2D})$,

$$\begin{aligned}\int\limits_{\Omega_2} \big[E_2(u_2, \overline{v}) - \omega^2 \kappa_2(x) u_2(x)\,\overline{v(x)}\,\big]\,dx = -\int\limits_{\Omega_2} f_2(x)\,\overline{v(x)}\,dx \\ + \big\langle T_2^+ u_2\,,\,\overline{\gamma_{S_1}^+ v}\big\rangle_{S_1} - \big\langle T_2^- u_2\,,\,\overline{\gamma_{S_2}^- v}\big\rangle_{S_{2N}}.\end{aligned} \tag{4.7}$$

Comparing (4.7) and (4.1) leads to the relation

$$\begin{aligned}\big\langle \mathcal{K}^{-1}\,\mathcal{M}(\gamma_{S_1}^+ u_2)\,,\,\overline{\gamma_{S_1}^+ v}\big\rangle_{S_1} + \langle \mathcal{K}^{-1}\Phi f_1\,,\,\overline{\gamma_{S_1}^+ v}\rangle_{S_1} + \big\langle \psi_1, \overline{\gamma_{S_1}^+ v}\big\rangle_{S_1} - \big\langle \mathcal{K}^{-1}\,\mathcal{M}\varphi_1\,,\,\overline{\gamma_{S_1}^+ v}\big\rangle_{S_1} \\ -\big\langle \psi_{2N}, \overline{\gamma_{S_{2N}}^- v}\big\rangle_{S_{2N}} = \langle T_2^+ u_2\,,\,\overline{\gamma_{S_1}^+ v}\rangle_{S_1} - \langle T_2^- u_2\,,\,\overline{\gamma_{S_2}^- v}\rangle_{S_{2N}}\end{aligned} \tag{4.8}$$

for all $v \in H^1(\Omega_2; S_{2D})$.

If we take an arbitrary function $v \in H^1(\Omega_2; S_{2D})$ such that $\gamma_{S_1}^+ v = 0$, from (4.8), we get

$$\big\langle \psi_{2N}, \overline{\gamma_{S_{2N}}^- v}\big\rangle_{S_{2N}} = \langle T_2^- u_2\,,\,\overline{\gamma_{S_2}^- v}\rangle_{S_{2N}} \tag{4.9}$$

implying the boundary condition $T_2^- u_2 = \psi_{2N}$ on S_{2N}. Consequently, due to the inclusion $u_2 \in H^1(\Omega_2; S_{2D})$, it is evident that the mixed boundary conditions (2.22) on S_2 are satisfied.

In view of (4.9), from (4.8), we deduce

$$\mathcal{K}^{-1}\,\mathcal{M}(\gamma_{S_1}^+ u_2) + \mathcal{K}^{-1}\Phi f_1 + \psi_1 - \mathcal{K}^{-1}\,\mathcal{M}\varphi_1 = T_2^+ u_2 \ \text{ on } \ S_1.$$

Applying the operator $\mathcal{K}$ to this equation and taking into account (3.17), we arrive at the relation

$$\mathcal{M}(\gamma_{S_1}^+ u_2 - \varphi_1) - \mathcal{K}(T_2^+ u_2 - \psi_1) = -\big(T_1^+ + i\,\alpha\,\gamma_{S_1}^+\big)\mathcal{P}\,f_1 \ \text{ on } \ S_1.$$

By (3.15), (3.16) and (3.17), the later equation can be rewritten as

$$\big(T_1^+ + i\,\alpha\,\gamma_{S_1}^+\big)\big[W(\gamma_{S_1}^+ u_2 - \varphi_1) - V(T_2^+ u_2 - \psi_1) + \mathcal{P}\,f_1\big] = 0 \ \text{ on } \ S_1. \tag{4.10}$$

Let us introduce the function

$$w := W(\gamma_{S_1}^+ u_2 - \varphi_1) - V(T_2^+ u_2 - \psi_1) + \mathcal{P}\,f_1 \ \text{ in } \ \mathbb{R}^3 \setminus S_1.$$

Evidently, in view of the mapping properties of the layer and volume potentials (see Lemma A.1), on the one hand, $r_{\Omega_1} w = u_1 \in H^{1,0}_{\rm loc}(\Omega_1; A_1) \cap Z(\Omega_1)$ due to (4.6), and on the other hand, $r_{\Omega^+} w \in H^{1,\,0}(\Omega^+; A_1)$, where $\Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega}_1$.

Further, by (3.10), (3.11) and (4.10), we deduce that w solves the homogeneous interior Robin's problem

$$A_1(\partial,\omega)w = 0 \ \text{ in } \ \Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega}_1,$$
$$\big(T_1^+ + i\,\alpha\,\gamma_{S_1}^+\big)w = 0 \ \text{ on } \ S_1 = \partial\Omega^+,$$

where α is a positive number. Therefore, by the uniqueness theorem, for the interior Robin's problem we infer $w = 0$ in Ω^+. Thus,

$$w = W(\gamma_{S_1}^+ u_2 - \varphi_1) - V(T_2^+ u_2 - \psi_1) + \mathcal{P}\, f_1 = \begin{cases} u_1 & \text{in } \ \Omega_1, \\ 0 & \text{in } \ \Omega^+. \end{cases} \tag{4.11}$$

Using the inclusion $\mathcal{P}\, f_1 \in H^2_{\rm loc}(\mathbb{R}^3)$, relation (3.17) and the jump relations, for the layer potentials across the surface S_1 (see Lemma A.1), we find from (4.11) that

$$\gamma_{S_1}^- w - \gamma_{S_1}^+ w = \gamma_{S_1}^+ u_2 - \varphi_1 = \gamma_{S_1}^- u_1 \ \text{ on } \ S_1,$$
$$T_1^- w - T_1^+ w = T_2^+ u_2 - \psi_1 = T_1^- u_1 \ \text{ on } \ S_1,$$

which show that the transmission conditions (2.18)–(2.19) hold. This completes the proof. □

Theorem 4.2. *The homogeneous variational-functional Problem (VMT1) possesses only the trivial solution in the space $H^1(\Omega_2; S_{2D})$.*

Proof. Let $u_2 \in H^1(\Omega_2; S_{2D})$ be a solution of the homogeneous variational-functional Problem (VMT1),

$$\begin{aligned}\mathfrak{B}(u_2, v) \equiv \int\limits_{\Omega_2} \big[a_{kj}^{(2)}(x)\,\partial_j u_2(x)\,\overline{\partial_k v(x)} - \omega^2 \kappa_2(x) u_2(x)\,\overline{v(x)}\,\big]\,dx \\ -\big\langle \mathcal{K}^{-1}\,\mathcal{M}(\gamma_{S_1}^+ u_2)\,,\,\overline{\gamma_{S_1}^+ v}\big\rangle_{S_1} = 0 \qquad \forall\, v \in H^1(\Omega_2; S_{2D}).\end{aligned} \tag{4.12}$$

By the word for word arguments applied in the proof of Theorem 4.1, we can show that u_2 is a solution of the homogeneous equation $A_2(x,\partial_x,\omega)u_2 = 0$ in Ω_2 in the distributional sense and, evidently, $u_2 \in H^{1,\,0}(\Omega_2; A_2)$. Therefore the canonical conormal derivatives $T_2^+ u_2 \in H^{-\frac{1}{2}}(S_1)$ and $T_2^- u_2 \in H^{-\frac{1}{2}}(S_2)$ are well-defined in the sense of (2.8) and for u_2 and arbitrary $v \in H^1(\Omega_2; S_{2D})$, Green's identity

$$\int\limits_{\Omega_2} \big[E_2(u_2,\overline{v}) - \omega^2\kappa_2(x)u_2(x)\,\overline{v(x)}\,\big]\,dx = \big\langle T_2^+ u_2\,,\,\overline{\gamma_{S_1}^+ v}\big\rangle_{S_1} - \big\langle T_2^- u_2\,,\,\overline{\gamma_{S_2}^- v}\big\rangle_{S_{2N}} \tag{4.13}$$

holds. Comparing (4.12) and (4.13) leads to the relation

$$\big\langle \mathcal{K}^{-1}\,\mathcal{M}(\gamma_{S_1}^+ u_2)\,,\,\overline{\gamma_{S_1}^+ v}\big\rangle_{S_1} = \langle T_2^+ u_2\,,\,\overline{\gamma_{S_1}^+ v}\rangle_{S_1} - \langle T_2^- u_2\,,\,\overline{\gamma_{S_2}^- v}\rangle_{S_{2N}} \tag{4.14}$$

for all $v \in H^1(\Omega_2; S_{2D})$. If we take an arbitrary function $v \in H^1(\Omega_2; S_{2D})$ such that $\gamma_{S_1}^+ v = 0$, from (4.14), we find

$$T_2^- u_2 = 0 \ \text{ on } \ S_{2N}.$$

Therefore from (4.14), we deduce

$$\mathcal{M}(\gamma_{S_1}^+ u_2) - \mathcal{K}(T_2^+ u_2) = 0 \ \text{ on } \ S_1,$$

which can be rewritten as

$$\big(T_1^+ + i\,\alpha\,\gamma_{S_1}^+\big)\big[W(\gamma_{S_1}^+ u_2) - V(T_2^+ u_2)\big] = 0 \ \text{ on } \ S_1. \tag{4.15}$$

Let

$$u_1 := W(\gamma_{S_1}^+ u_2) - V(T_2^+ u_2) \ \text{ in } \ \mathbb{R}^3 \setminus S_1. \tag{4.16}$$

Note that in view of Lemma A.1, $r_{\Omega_1} u_1 \in H^{1,0}_{\mathrm{loc}}(\Omega_1; A_1) \cap Z(\Omega_1)$ and $r_{\Omega^+} u_1 \in H^{1,0}(\Omega^+; A_1)$ with $\Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega}_1$. Moreover, by (3.10), (4.15) and (4.16), we see that u_1 solves the homogeneous interior Robin's problem

$$A_1(\partial, \omega) u_1 = 0 \ \text{ in } \ \Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega}_1,$$
$$\big(T_1^+ + i\,\alpha\,\gamma_{S_1}^+\big) u_1 = 0 \ \text{ on } \ S_1 = \partial\Omega^+,$$

where α is a positive number. Consequently, $u_1 = 0$ in Ω^+ and due to the jump relations, for the layer potentials, from (4.16), we deduce

$$\gamma_{S_1}^- u_1 = \gamma_{S_1}^- u_1 - \gamma_{S_1}^+ u_1 = \gamma_{S_1}^+ u_2 \ \text{ on } \ S_1,$$
$$T_1^- u_1 = T_1^- u_1 - T_1^+ u_1 = T_2^+ u_2 \ \text{ on } \ S_1.$$

Combining the above obtained results, we finally see that the pair $(u_2, u_1) \in H^{1,0}(\Omega_2; A_2) \times \big(H^{1,0}_{\mathrm{loc}}(\Omega_1; A_1) \cap Z(\Omega_1)\big)$ solves the mixed homogeneous transmission problem and by the uniqueness Theorem 2.4, we have $u_2 = 0$ in Ω^+, which completes the proof. □

Now let us consider the following variational-functional problem.

Problem (VMT2)**.** Find a pair $(u_2, u_1) \in H^1(\Omega_2; S_{2D}) \times \big(H^1_{\mathrm{loc}}(\Omega_1) \cap Z(\Omega_1)\big)$ satisfying the system of equations

$$\mathfrak{B}(u_2, v) = \mathfrak{F}(v) \quad \text{for all} \quad v \in H^1(\Omega_2; S_{2D}), \tag{4.17}$$

$$u_1(y) + V(T_2^+ u_2)(y) - W(\gamma_{S_1}^+ u_2)(y) = \mathcal{P} f_1(y) + V\psi_1(y) - W\varphi_1(y), \quad y \in \Omega_1, \tag{4.18}$$

where $\mathfrak{B}$ and $\mathfrak{F}$ are defined in (4.2)–(4.5) and conditions (2.23) are satisfied.

Corollary 4.3. *System* (4.17)–(4.18) *is equivalent to the mixed transmission problem (TM) in the following sense: if a pair* $(u_2, u_1) \in H^1(\Omega_2; S_{2D}) \times \big(H^1_{\mathrm{loc}}(\Omega_1) \cap Z(\Omega_1)\big)$ *solves system* (4.17)–(4.18), *then it is unique and solves the mixed transmission problem (TM), and vice versa.*

Proof. In view of Theorems 2.4, 4.1 and 4.2, it suffices to show that the right-hand sides of system (4.17)–(4.18) vanish if and only if

$$f_1 = 0, \quad f_2 = 0, \quad \varphi_{_1} = 0, \quad \psi_{_1} = 0, \quad \psi_{_{2N}} = 0. \tag{4.19}$$

Let

$$\mathfrak{F}(v) = 0 \quad \forall\, v \in H^1(\Omega_2; S_{2D}), \tag{4.20}$$

$$\mathcal{P} f_1 + V\psi_1 - W\varphi_1 = 0 \ \text{ in } \ \Omega_1. \tag{4.21}$$

By the same arguments as in the proof of Theorem 4.1 (see the derivation of formula (4.11)), from (4.20), we obtain

$$\mathcal{P}\, f_1 + V\psi_1 - W\varphi_1 = 0 \ \text{ in } \ \Omega_2. \tag{4.22}$$

From relations (4.21) and (4.22) the equalities $f_1 = 0$, $\varphi_{_1} = 0$, and $\psi_{_1} = 0$ follow immediately in view of Lemma A.1. In accordance with (4.5), then (4.20) takes the form

$$-\int\limits_{\Omega_2} f_2(x)\,\overline{v(x)}\,dx - \big\langle \psi_{_{2N}}, \overline{\gamma_{S_{2N}}^- v} \big\rangle_{S_{2N}} = 0, \quad \forall\, v \in H^1(\Omega_2; S_{2D}),$$

implying $f_2 = 0$ and $\psi_{_{2N}} = 0$, which completes the proof. □

Remark 4.4. Note that only equality (4.20) separately leads to (4.22) and does not imply relations (4.19).

Now we prove the following boundedness and coercivity theorem.

Theorem 4.5. *For the sesquilinear form* $\mathfrak{B}$ *defined by* (4.2)–(4.4) *and the antilinear functional* $\mathfrak{F}$ *defined in* (4.5) *under conditions* (2.23), *there are real constants* $C_j^* > 0$, $j = 1, 2, 3, 4$ *such that*

$$|\mathfrak{B}(u_2, v)| \le C_1^* \|u_2\|_{H^1(\Omega_2)} \|v\|_{H^1(\Omega_2)} \qquad \forall\, u_2,\, v \in H^1(\Omega_2; S_{2D}), \tag{4.23}$$

$$|\mathfrak{F}(v)| \le C_2^* \|v\|_{H^1(\Omega_2)} \qquad \forall\, v \in H^1(\Omega_2; S_{2D}),$$

$$\operatorname{Re} \mathfrak{B}(u_2, u_2) \ge C_3^* \|u_2\|^2_{H^1(\Omega_2)} - C_4^* \|u_2\|^2_{H^0(\Omega_2)} \qquad \forall\, u_2 \in H^1(\Omega_2; S_{2D}). \tag{4.24}$$

Proof. The boundedness of the sesquilinear form $\mathfrak{B}^{(1)}(u_2, v)$ follows directly from the Cauchy-Schwartz inequality, $\big|\mathfrak{B}^{(1)}(u_2, v)\big| \leqslant C_5 \|u_2\|_{H^1(\Omega_2)} \|v\|_{H^1(\Omega_2)}$, while the boundedness of the sesquilinear form $\mathfrak{B}^{(2)}(u_2, v)$ can be shown by the duality inequality, Lemma A.2, and trace theorem,

$$\begin{aligned}
\big|\mathfrak{B}^{(2)}(u_2, v)\big| &= \big|\big\langle \mathcal{K}^{-1}\, \mathcal{M}(\gamma^+_{S_1} u_2)\,,\, \overline{\gamma^+_{S_1} v}\big\rangle_{S_1}\big| \\
&\leqslant C_1 \|\mathcal{K}^{-1}\, \mathcal{M}(\gamma^+_{S_1} u_2)\|_{H^{-\frac12}(S_1)} \|\gamma^+_{S_1} v\|_{H^{\frac12}(S_1)} \\
&\leqslant C_2 \|\mathcal{M}(\gamma^+_{S_1} u_2)\|_{H^{-\frac12}(S_1)} \|v\|_{H^1(\Omega_2)} \leqslant C_3 \|\gamma^+_{S_1} u_2\|_{H^{\frac12}(S_1)} \|v\|_{H^1(\Omega_2)} \\
&\leqslant C_4 \|u_2\|_{H^1(\Omega_2)} \|v\|_{H^1(\Omega_2)},
\end{aligned}$$

where C_j, $j = 1, \dots, 4$, are some positive constants. Consequently, (4.23) holds.

Keeping in mind conditions (2.23), relations (2.10), (3.11), (3.17), (4.5) and the estimate

$$\begin{aligned}
\big|\langle \mathcal{K}^{-1} \Phi f_1\,,\, \overline{\gamma^+_{S_1} v}\rangle_{S_1}\big| &\leqslant C_5 \|\big(T_1^+ + i\,\alpha\, \gamma^+_{S_1}\big)\mathcal{P} f_1\|_{H^{-\frac12}(S_1)} \|\gamma^+_{S_1} v\|_{H^{\frac12}(S_1)} \\
&\leqslant C_6\, \big(\|A_1 \mathcal{P} f_1\|_{H^0(\Omega_2)} + \|\mathcal{P} f_1\|_{H^1(\Omega_2)}\big)\, \|v\|_{H^1(\Omega_2)} \\
&\leqslant C_7 \|f_1\|_{H^0(\Omega_1)} \|v\|_{H^1(\Omega_2)},
\end{aligned}$$

the boundedness of the functional $\mathfrak{F}$ can be proved by the arguments similar to the above ones,

$$\begin{aligned}
|\mathfrak{F}(v)| \le\, & C_8 \Big(\|f_1\|_{L_2(\Omega_1)} + \|f_2\|_{L_2(\Omega_2)} + \|\varphi_1\|_{H^{\frac12}(S_1)} + \|\psi_1\|_{H^{-\frac12}(S_1)} \\
& + \|\psi_{S_{2N}}\|_{H^{-\frac12}(S_{2N})}\Big) \|v\|_{H^1(\Omega_2)} \quad \text{for all} \quad v \in H^1(\Omega_2; S_{2D}).
\end{aligned}$$

Now we prove inequality (4.24). In view of the positive definiteness of the matrix $\mathbf{a}_2 = \big[\, a^{(2)}_{kj} \,\big]^3_{k,j=1}$, we have

$$\operatorname{Re} \mathfrak{B}^{(1)}(u_2, u_2) \geqslant C_9 \|u_2\|^2_{H^1(\Omega_2)} - C_{10} \|u_2\|^2_{H^0(\Omega_2)},$$

where $C_9 > 0$ and $C_{10} = \omega^2 \max\limits_{\overline{\Omega}_2} \kappa_2(x)$.

Further, by Lemma A.6, we deduce

$$\begin{aligned}
\operatorname{Re} \mathfrak{B}^{(2)}(u_2, u_2) &= -\operatorname{Re} \big\langle \mathcal{K}^{-1}\, \mathcal{M}(\gamma^+_{S_1} u_2)\,,\, \overline{\gamma^+_{S_1} u_2}\big\rangle_{S_1} \\
&\geqslant C_1' \|\gamma^+_{S_1} u_2\|^2_{H^{\frac12}(S_1)} - C_2' \|\gamma^+_{S_1} u_2\|^2_{H^0(S_1)} \\
&\geqslant C_1' \|\gamma^+_{S_1} u_2\|^2_{H^{\frac12}(S_1)} - C_2' \|\gamma^+_{S_1} u_2\|^2_{H^\delta(S_1)} \\
&\geqslant C_1' \|\gamma^+_{S_1} u_2\|^2_{H^{\frac12}(S_1)} - C_2' \|u_2\|^2_{H^{\frac12+\delta}(\Omega_2)} \\
&\geqslant -C_2' \|u_2\|^2_{H^{\frac12+\delta}(\Omega_2)},
\end{aligned} \tag{4.25}$$

where $C_1' > 0$, $C_2' > 0$, and δ is an arbitrarily small positive number. By Ehrling's lemma (see, e.g., [38, Theorem 7.30]), for an arbitrarily small positive number ε there is a positive constant $C(\varepsilon)$

such that

$$\|w\|^2_{H^{\frac{1}{2}+\delta}(\Omega_2)} \leqslant \varepsilon \|w\|^2_{H^1(\Omega_2)} + C(\varepsilon)\|w\|^2_{H^0(\Omega_2)} \quad \text{for all} \quad w \in H^1(\Omega_2), \quad 0 < \delta < \frac{1}{2}.$$

Therefore from (4.25), we have

$$\operatorname{Re} \mathfrak{B}^{(2)}(u_2, u_2) \geqslant -C_2' \left(\varepsilon \|u_2\|^2_{H^1(\Omega_2)} + C(\varepsilon)\|u_2\|^2_{H^0(\Omega_2)} \right),$$

with ε such that $C_9 - \varepsilon C_2' > 0$, which completes the proof. □

Now we can prove the following existence results.

Theorem 4.6. *Let conditions* (2.23) *be fulfilled.*

(i) *Variational-functional equation* (4.1) *is uniquely solvable in the space* $H^1(\Omega_2; S_{2D})$ *for arbitrary antilinear bounded functional* $\mathfrak{F}$ *defined on* $H^1(\Omega_2; S_{2D})$.

(ii) *System* (4.17)–(4.18) *with* $\mathfrak{F}$ *defined in* (4.5)*, is uniquely solvable with respect to the unknown pair* $(u_2, u_1) \in H^1(\Omega_2; S_{2D}) \times \big(H^1_{\mathrm{loc}}(\Omega_1) \cap Z(\Omega_1)\big)$.

(iii) *The mixed transmission problem (TM) is uniquely solvable in the space* $H^1(\Omega_2; S_{2D}) \times \big(H^1_{\mathrm{loc}}(\Omega_1) \cap Z(\Omega_1)\big)$.

Proof. Item (i) follows directly from Theorem 4.2, Theorem 4.5 and the Lax–Milgram lemma (see, e.g., [29, Theorem 2.33]).

Further, as we have already shown, equation (4.17) with $\mathfrak{F}$ given by (4.5) uniquely defines the sought function u_2 and, consequently, equation (4.18) defines explicitly and uniquely the sought function u_1 in Ω_1 which proves Item (ii).

Item (iii) follows from the uniqueness Theorem 2.4, Corollary 4.3 and Item (ii). □

Remark 4.7. Investigation of the transmission problems with Dirichlet or Neumann boundary conditions on the interior surface S_2 can be carried out quite similarly by using the above-employed arguments. Under conditions (2.23), they are uniquely solvable in the spaces $H^1(\Omega_2; S_2) \times \big(H^1_{\mathrm{loc}}(\Omega_1) \cap Z(\Omega_1)\big)$ and $H^1(\Omega_2) \times \big(H^1_{\mathrm{loc}}(\Omega_1) \cap Z(\Omega_1)\big)$ respectively.

5. Appendix A: Properties of Radiating Potentials

Here, we present some results concerning the properties of the layer potentials defined by (3.2), (3.3), and the volume potential (cf. (3.4))

$$\mathbf{P} f(y) := \int\limits_{\mathbb{R}^3} \Gamma(x - y, \omega)\, f(x)\, dx, \quad y \in \mathbb{R}^3,$$

in the case of Lipschitz domains which are employed in the main text of the paper. Evidently, $\mathcal{P} f_1(y) = \mathbf{P} \widetilde{f}_1(y)$, where $\widetilde{f}_1$ is the extension by zero of the function f_1 form Ω_1 onto its complement $\mathbb{R}^3 \setminus \overline{\Omega}_1$.

We start with the following well known results (for more specific properties see [2–5, 13, 16, 17, 29, 32, 43] and references cited therein).

Lemma A.1. (i) **[13, Theorem 1(i),(ii)]** *For all* $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, *the following layer potential operators*

$$V \,:\, H^{-\frac{1}{2}+\sigma}(S_1) \to H^{1+\sigma}(\mathbb{R}^3 \setminus \overline{\Omega}_1),$$

$$V \,:\, H^{-\frac{1}{2}+\sigma}(S_1) \to H^{1+\sigma}_{\mathrm{loc}}(\Omega_1) \cap Z(\Omega_1),$$

$$W \,:\, H^{\frac{1}{2}+\sigma}(S_1) \to H^{1+\sigma}(\mathbb{R}^3 \setminus \overline{\Omega}_1),$$

$$W \,:\, H^{\frac{1}{2}+\sigma}(S_1) \to H^{1+\sigma}_{\mathrm{loc}}(\Omega_1) \cap Z(\Omega_1)$$

are continuous.

(ii) **[31, Ch.XI, Theorem 11.2]; [16, Proposition 2.1]** *If $f \in H^0_{\rm comp}(\mathbb{R}^3)$, then $\mathbf{P}f \in H^2_{\rm loc}(\mathbb{R}^3)\cap Z(\mathbb{R}^3)$ and*

$$A_1\mathbf{P}f = f \ \text{ in } \mathbb{R}^3, \qquad \|\mathbf{P}f\|_{H^2(\Omega^*)} \leqslant C^*\,\|f\|_{H^0(\Omega_f)},$$

where Ω^* is an arbitrary bounded domain in $\mathbb{R}^3$, $\Omega_f := \operatorname{supp} f$, and $C^* > 0$ is a constant which depends on the diameter of the domain Ω^*.

(iii) **[13, Lemma 4.1]; [17, Theorem 1.1]** *For $h \in H^{-\frac{1}{2}}(S_1)$ and $g \in H^{\frac{1}{2}}(S_1)$, the following jump relations*

$$\gamma^+_{S_1} V\,h = \gamma^-_{S_1} V(h) = \mathcal{V}(h), \quad T_1^{\pm} V\,h = \Big(\pm\,\frac{1}{2}I + \mathcal{W}'\Big)h \quad on \quad S_1, \tag{A.1}$$

$$\gamma^{\pm}_{S_1} W\,g = \Big(\mp\,\frac{1}{2}I + \mathcal{W}\Big)g, \quad T_1^{+} W\,g = T_1^{-} W\,g =: \mathcal{L}\,g \quad on \quad S_1, \tag{A.2}$$

hold true, where I stands for the identity operator, and

$$\mathcal{V}\,h(y) := -\int\limits_{S_1} \Gamma(x-y,\omega)\,h(x)\,dS_x, \quad y \in S_1, \tag{A.3}$$

$$\mathcal{W}\,g(y) := -\int\limits_{S_1} [T_1(x,\partial_x)\Gamma(x-y,\omega))]\,g(x)\,dS_x, \quad y \in S_1, \tag{A.4}$$

$$\mathcal{W}'\,h(y) := -\int\limits_{S_1} [T_1(y,\partial_y)\Gamma(x-y,\omega))]\,h(x)\,dS_x, \quad y \in S_1, \tag{A.5}$$

$\Gamma(x,\omega)$ is the radiating fundamental solution defined by (3.5). *The operators* (A.4) *and* (A.5) *are to be understood in the Cauchy principal value sense, while* (A.3) *is a weakly singular integral operator.*

(iv) **[13, Theorem 1(iii)–(vi)]; [32, Theorems 7.1, 7.2]; [16, Theorems 3.1 & 4.1]; [5, Corollary 3.6, Theoem 3.10]** *For all $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, the operators*

$$\mathcal{V} : H^{-\frac{1}{2}+\sigma}(S_1) \to H^{\frac{1}{2}+\sigma}(S_1), \qquad \pm\tfrac{1}{2}\,I + \mathcal{W}' : H^{-\frac{1}{2}+\sigma}(S_1) \to H^{-\frac{1}{2}+\sigma}(S_1),$$

$$\pm\tfrac{1}{2}\,I + \mathcal{W} : H^{\frac{1}{2}+\sigma}(S_1) \to H^{\frac{1}{2}+\sigma}(S_1), \qquad \mathcal{L} : H^{\frac{1}{2}+\sigma}(S_1) \to H^{-\frac{1}{2}+\sigma}(S_1),$$

are continuous Fredholm operators with zero index.

Lemma A.2. *Let $\mathcal{K}$ and $\mathcal{M}$ be defined by* (3.15) *and* (3.16) *with $\alpha > 0$. For all $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$ the following operators*

$$\mathcal{K} \equiv \frac{1}{2}\,I + \mathcal{W}' + i\,\alpha\,\mathcal{V} : H^{-\frac{1}{2}+\sigma}(S_1) \to H^{-\frac{1}{2}+\sigma}(S_1), \tag{A.6}$$

$$\mathcal{M} \equiv \mathcal{L} + i\,\alpha\Big(-\frac{1}{2}\,I + \mathcal{W}\Big) : H^{\frac{1}{2}+\sigma}(S_1) \to H^{-\frac{1}{2}+\sigma}(S_1), \tag{A.7}$$

are invertible.

Proof. Due to Lemma A.1(iv), we need only to prove that the operators (A.6) and (A.7) have the trivial null-spaces. First, we consider the case $\sigma = 0$ and let $g \in H^{-\frac{1}{2}}(S_1)$ be a solution of the homogeneous equation

$$\mathcal{K}\,g = 0 \quad \text{on} \ \ S_1, \tag{A.8}$$

and construct the function $v := Vg$ in $\mathbb{R}^3$, where Vg is the single layer potential defined by (3.2). Evidently, $v \in H^{1,0}(\mathbb{R}^3 \setminus \overline{\Omega}_1; A_1)$, $v \in H^{1,0}_{\rm loc}(\Omega_1; A_1) \cap Z(\Omega_1)$, $A_1(\partial_x,\omega)v = A_1(\partial_x,\omega)V(g) = 0$ in $\mathbb{R}^3 \setminus S_1$, and $T_1^{\pm}v = T_1^{\pm}Vg \in H^{-\frac{1}{2}}(S_1)$ is well defined. In accordance with relation (3.15), equation (A.8) is equivalent to the condition

$$\big(T_1^{+} + i\,\alpha\,\gamma^{+}_{S_1}\big)v = 0 \quad \text{on} \quad S_1, \quad \alpha > 0. \tag{A.9}$$

Therefore v solves the homogeneous interior Robin problem in $\mathbb{R}^3\setminus\overline{\Omega}_1$. Boundary condition (A.9) and Green's formula

$$\int\limits_{\mathbb{R}^3\setminus\overline{\Omega}_1} A_1 v\,\overline{v}\,dx + \int\limits_{\mathbb{R}^3\setminus\overline{\Omega}_1}\big[E_1(v,\overline{v}) - \omega^2\kappa_1|v|^2\,\big]\,dx = \big\langle T_1^+ v\,,\,\overline{\gamma^+_{S_1} v}\big\rangle_{S_1}$$

lead to the equality

$$\int\limits_{\mathbb{R}^3\setminus\overline{\Omega}_1}\big[E_1(v,\overline{v}) - \omega^2\kappa_1|v|^2\,\big]\,dx + i\,\alpha\int\limits_{S_1}|\gamma^+_{S_1} v|^2\,dS = 0.$$

By separating imaginary part, we deduce $\gamma^+_{S_1} v = 0$ on S_1, implying $T_1^+ v = 0$ on S_1. Therefore, with the help of the general integral representation formula of solutions of the homogeneous differential equation $A_1 v = 0$, $v = V(T_1^+ v) - W(\gamma^+_{S_1} v)$, we finally deduce $v = V(g) = 0$ in $\mathbb{R}^3\setminus\overline{\Omega}_1$. By the continuity property of the single layer potential across the surface S_1 (see the first equation in (A.1)), we have $\gamma^+_{S_1} v = \gamma^-_{S_1} v = 0$ on S_1. Consequently, the radiating function $v = Vg$ solves the homogeneous exterior Dirichlet problem for the operator $A_1(\partial_x,\omega)$ and therefore vanishes identically in Ω_1. Consequently, by the jump relations (A.1) for the conormal derivative of the single layer potential, we find that $g = 0$ on S_1 implying that the null-space of the operator (A.6) is trivial.

Now let $\sigma\in[-\frac{1}{2},\,\frac{1}{2}]$. Recall that for $-\frac{1}{2}\leqslant\sigma_1<\sigma_2\leqslant\frac{1}{2}$, the inclusion $H^{-\frac{1}{2}+\sigma_2}(S_1)\subset H^{-\frac{1}{2}+\sigma_1}(S_1)$ is continuous and dense. Therefore the null-space of the Fredholm operator (A.6) is the same for all $\sigma\in[-\frac{1}{2},\,\frac{1}{2}]$ (see, e.g., [33, Lemma 11.40], [1, Proposition 10.6]). This completes the proof for operator (A.6).

The proof for operator (A.7) is quite similar. $\square$

Introduce the boundary operators $\widetilde{\mathcal{V}}$, $\widetilde{\mathcal{W}}$, $\widetilde{\mathcal{W}}'$, $\widetilde{\mathcal{L}}$, $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{M}}$ generated by the single and double layer potentials $\widetilde{V}$ and $\widetilde{W}$ constructed by the exponentially decaying real-valued fundamental solution $\Gamma(x-y,i)$ (see (3.5)). Evidently, they are defined by the same formulas as their counterpart operators $\mathcal{V}$, $\mathcal{W}$, $\mathcal{W}'$, $\mathcal{L}$, $\mathcal{K}$, $\mathcal{M}$, V and W with $\Gamma(x-y,i)$ for $\Gamma(x-y,\omega)$ and have all the mapping and jump properties described in Lemmas A.1 and A.2. In addition, for these "tilde" operators we have the following assertion.

Lemma A.3. *For all* $\sigma\in[-\frac{1}{2},\,\frac{1}{2}]$ *and* $\alpha>0$, *the following operators*

$$\widetilde{\mathcal{V}} : H^{-\frac{1}{2}+\sigma}(S_1)\to H^{\frac{1}{2}+\sigma}(S_1),$$

$$\pm\frac{1}{2}\,I+\widetilde{\mathcal{W}}' : H^{-\frac{1}{2}+\sigma}(S_1)\to H^{-\frac{1}{2}+\sigma}(S_1),$$

$$\pm\frac{1}{2}\,I+\widetilde{\mathcal{W}} : H^{\frac{1}{2}+\sigma}(S_1)\to H^{\frac{1}{2}+\sigma}(S_1),$$

$$\widetilde{\mathcal{L}} : H^{\frac{1}{2}+\sigma}(S_1)\to H^{-\frac{1}{2}+\sigma}(S_1),$$

$$\widetilde{\mathcal{K}}\equiv\frac{1}{2}\,I+\widetilde{\mathcal{W}}'+i\,\alpha\,\widetilde{\mathcal{V}} : H^{-\frac{1}{2}+\sigma}(S_1)\to H^{-\frac{1}{2}+\sigma}(S_1),$$

$$\widetilde{\mathcal{M}}\equiv\widetilde{\mathcal{L}}+i\,\alpha\Big(-\frac{1}{2}\,I+\widetilde{\mathcal{W}}\Big) : H^{\frac{1}{2}+\sigma}(S_1)\to H^{-\frac{1}{2}+\sigma}(S_1)$$

are invertible.

Proof. All the operators stated in the lemma are Fredholm ones with zero index and their null-spaces are the same for all $\sigma\in[-\frac{1}{2},\,\frac{1}{2}]$ (see, e.g., [33, Lemma 11.40], [1, Proposition 10.6]). Therefore it suffices to show that the null-spaces of the operators are trivial for $\sigma=0$.

Recall that $\Omega_1 := \Omega^-$ and $\Omega^+ = \mathbb{R}^3\setminus\overline{\Omega}_1$.

First, let us prove that the null-space of the operator $\widetilde{\mathcal{V}}$ is trivial. Let $g\in H^{-\frac{1}{2}}(S_1)$ be a solution to the homogeneous equation $\widetilde{\mathcal{V}}g = 0$ on S_1. Then the single layer potential $u=\widetilde{V}(g)$ belongs to

$H^1(\Omega^\pm, \widetilde{A}_1)$ with $\widetilde{A}_1 := A_1(\partial_x, i)$, exponentially decays at infinity, and solves the homogeneous interior and exterior Dirichlet problems

$$A_1(\partial_x, i)u = a^{(1)}_{kj}\, \partial_k\partial_j u - \kappa_1\, u = 0 \text{ in } \Omega^\pm, \qquad \gamma^\pm_{S_1} u = 0 \text{ on } S_1 = \partial\Omega^\pm.$$

Consequently, with the help of Green's formulas (cf. (2.11))

$$\int\limits_{\Omega^\pm} A_1(\partial_x, i)u(x)\,\overline{u(x)}\,dx + \int\limits_{\Omega^\pm} [\, E_1(u, \overline{u}) + \kappa_1 |u(x)|^2\,]\,dx = \pm\big\langle T_1^\pm u\,,\, \overline{\gamma^\pm_{S_1} u}\big\rangle_{S_1}, \tag{A.10}$$

we deduce $u = 0$ in $\Omega^\pm$, whence $g = 0$ on S_1 follows due to the jump relations for the conormal derivative of the single layer potential (see Lemma A.1(iii)) which completes the proof for the operator $\widetilde{\mathcal{V}}$.

Now, let us consider the operator $\widetilde{\mathcal{M}}$ and let $h \in H^{\frac12}(S_1)$ be a solution to the homogeneous equation $\widetilde{\mathcal{M}}h = 0$ on S_1. Then the double layer potential $w = \widetilde{W}(h)$ belongs to $H^1(\Omega^\pm; \widetilde{A}_1)$, exponentially decays at infinity and solves the homogeneous interior Robin's problem

$$A_1(\partial_x, i)w = a^{(1)}_{kj}\, \partial_k\partial_j w - \kappa_1\, w = 0 \text{ in } \Omega^+, \qquad T_1^+ w + i\,\alpha\,\gamma^+_{S_1} w = 0 \text{ on } S_1.$$

Therefore by Green's formula (A.10), we deduce $w = 0$ in Ω^+ and by Lemma A.1(iii), we have $T_1^+ w = T_1^- w = 0$. Thus, w solves the homogeneous exterior Neumann problem and, consequently, $w = 0$ in Ω^- in view of (A.10). The jump properties of the double layer potential complete the proof for the operator $\widetilde{\mathcal{M}}$. $\square$

For the other operators stated in the lemma the proofs are word for word.

Lemma A.4. *For $\sigma \in [-\frac12, \frac12]$, the operators*

$$\mathcal{V} - \widetilde{\mathcal{V}} : H^{-\frac12+\sigma}(S_1) \to H^{\frac12+\sigma}(S_1), \tag{A.11}$$

$$\mathcal{W}' - \widetilde{\mathcal{W}}' : H^{-\frac12+\sigma}(S_1) \to H^{-\frac12+\sigma}(S_1), \tag{A.12}$$

$$\mathcal{W} - \widetilde{\mathcal{W}} : H^{\frac12+\sigma}(S_1) \to H^{\frac12+\sigma}(S_1), \tag{A.13}$$

$$\mathcal{L} - \widetilde{\mathcal{L}} : H^{\frac12+\sigma}(S_1) \to H^{-\frac12+\sigma}(S_1), \tag{A.14}$$

$$\mathcal{K} - \widetilde{\mathcal{K}} : H^{-\frac12+\sigma}(S_1) \to H^{-\frac12+\sigma}(S_1), \tag{A.15}$$

$$\mathcal{M} - \widetilde{\mathcal{M}} : H^{\frac12+\sigma}(S_1) \to H^{-\frac12+\sigma}(S_1). \tag{A.16}$$

are compact.

Proof. In view of Remark 3.1 and relations (3.8) and (3.9), the potential type operators $V - \widetilde{V}$ and $W - \widetilde{W}$ for $\sigma \in [-\frac12, \frac12]$ have the following mapping properties:

$$V - \widetilde{V} : H^{-\frac12+\sigma}(S_1) \to H^{3+\sigma}(\Omega_2),$$

$$W - \widetilde{W} : H^{\frac12+\sigma}(S_1) \to H^{3+\sigma}(\Omega_2).$$

Therefore the traces on S_1 of the functions $V(h) - \widetilde{V}(h)$ and $W(g) - \widetilde{W}(g)$ with $h \in H^{-\frac12+\sigma}(S_1)$ and $g \in H^{\frac12+\sigma}(S_1)$ belong to $H^1(S_1)$ in view of the Lipschitz character of the surface S_1. Recall that in the case of Lipschitz surfaces, the space $H^s(S_1)$ is well-defined only for $-1 \leqslant s \leqslant 1$. Moreover, in general, the trace of a function from the space $H^s(\Omega^\pm)$ belongs either to the space $H^{s-\frac12}(\partial\Omega^\pm)$ if $\frac12 < s < \frac32$, or to the space $H^1(\partial\Omega^\pm)$ if $s > \frac32$ (see, e.g., [13,15], [23, Section 3]).

Consequently, for $\sigma \in (-\frac12, \frac12)$, the operators (A.11) and (A.13) are smoothing operators with the range in $H^1(S_1)$ which is compactly imbedded in $H^{\frac12+\sigma}(S_1)$, while operators (A.12), (A.14), (A.15) and (A.16) are smoothing operators with the range in $H^0(S_1)$ which is compactly imbedded in $H^{-\frac12+\sigma}(S_1)$ for arbitrary $\sigma \in (-\frac12, \frac12)$.

For $\sigma = \pm\frac{1}{2}$, the claim can be proved again using relations (3.8) and (3.9). For illustration, we consider operator (A.11) for $\sigma = \frac{1}{2}$, i.e., we show the compactness of the operator

$$\mathcal{V} - \widetilde{\mathcal{V}} \,:\, H^0(S_1) \to H^1(S_1).$$

Let M_0 be a bounded subset in $H^0(S_1)$, i.e. $\|g\|_{H^0(S_1)} \leqslant C_0$ for all $g \in M_0$. Let $\{g_n\}_{n=1}^{\infty} \in M_0$ be an arbitrary sequence and $Q(y,x) := \Gamma(y-x,\omega) - \Gamma(y-x,i)$ be defined by (3.8). Then the sequence

$$v_n(y) = \mathcal{V}(g_n)(y) - \widetilde{\mathcal{V}}(g_n)(y) \equiv \mathcal{Q}\, g_n(y) := \int\limits_{S_1} Q(y,x)\, g_n(x)\, dS_x, \quad y \in S_1,$$

contains a fundamental subsequence in the norm of the space $H^0(S_1)$ since the Hilbert-Schmidt integral operator $\mathcal{Q} : H^0(S_1) \to H^0(S_1)$ is compact. We denote the fundamental subsequence by $v_n^{(1)} = \mathcal{Q}\, g_n^{(1)}$. It is evident that the same arguments can be applied to the sequence

$$D_{y_j} v_n^{(1)}(y) = D_{y_j} \mathcal{Q}\, g_n^{(1)}(y) = \int\limits_{S_1} D_{y_j} Q(y,x)\, g_n(x)\, dS_x, \quad y \in S_1,$$

where D_{y_j} denotes a tangential differentiation. We again conclude that this sequence contains a fundamental subsequence in the norm of the space $H^0(S_1)$. Denote this subsequence by $v_n^{(2)} = \mathcal{Q}\, g_n^{(2)}$. Thus we have shown that the sequence $v_n = \mathcal{Q}\, g_n$ contains a fundamental subsequence $v_n^{(2)}$ in the norm of the space $H^{(1)}(S_1)$ which implies that the operator $\mathcal{Q} : H^0(S_1) \to H^1(S_1)$, i.e., operator (A.11) for $\sigma = \frac{1}{2}$ is compact. For $\sigma = -\frac{1}{2}$, the claim follows from the duality arguments.

Now let us consider operator (A.13) for $\sigma = \frac{1}{2}$,

$$\mathcal{R} := \mathcal{W} - \widetilde{\mathcal{W}} : H^1(S_1) \to H^1(S_1). \tag{A.17}$$

Further, let $M_1 \subset H^1(S_1)$ be a bounded set and $\{g_n\}_{n=1}^{\infty} \in M_1$ be an arbitrary sequence. It is evident that the kernel function $T_1(x,\partial_x)Q(y,x)$ of the weakly singular integral operator

$$\mathcal{R}\, g_n(y) := \int\limits_{S_1} T_1(x,\partial_x)Q(y,x)\, g_n(x)\, dS_x, \quad y \in S_1, \tag{A.18}$$

is bounded on $S_1 \times S_1$ in view of (3.8)–(3.9). Moreover, the kernel function $D_{y_j} T_1(x,\partial_x)Q(y,x)$ of the operator $D_{y_j} \mathcal{R}\, g_n(y)$ has a weak singularity of type $\mathcal{O}(|x-y|^{-1})$. Therefore, by the same arguments as above, we again can show that the sequence $\{\mathcal{R}\, g_n\}_{n=1}^{\infty}$ contains a fundamental subsequence in the norm of the space $H^1(S_1)$ which completes the proof for operator (A.17), i.e., for operator (A.13) for $\sigma = \frac{1}{2}$.

The duality arguments imply the compactness of operator (A.12) for $\sigma = -\frac{1}{2}$.

The compactness of operator (A.12) for $\sigma = \frac{1}{2}$ and operator (A.13) for $\sigma = -\frac{1}{2}$ is trivial.

Next, we consider operator (A.14) for $\sigma = \frac{1}{2}$,

$$\mathcal{N} := \mathcal{L} - \widetilde{\mathcal{L}} : H^1(S_1) \to H^0(S_1).$$

We have

$$\mathcal{N}\, g(y) := \int\limits_{S_1} T_1(y,\partial_y) T_1(x,\partial_x) Q(y,x)\, g(x)\, dS_x, \quad y \in S_1.$$

It is evident that the kernel function $T_1(y,\partial_y)T_1(x,\partial_x)Q(y,x)$ is symmetric and possesses a weak singularity of type $\mathcal{O}(|x-y|^{-1})$ due to (3.8)–(3.9). Therefore the Hilbert-Schmidt operator $\mathcal{N} : H^0(S_1) \to H^0(S_1)$ is compact, implying the compactness of operator (A.14). By the duality arguments, we conclude the compactness of operator (A.14) for $\sigma = -\frac{1}{2}$.

The above results along with relations (A.6)–(A.7) and their counterparts for the "tilde" operators imply directly the compactness of operators (A.15) and (A.16) for $\sigma = \pm\frac{1}{2}$, which completes the proof. $\square$

Remark A.5. Actually, in the proof of Lemma A.4 we have shown the following mapping properties (cf. [5]):

$$\mathcal{V}-\widetilde{\mathcal{V}}: H^{-\frac{1}{2}}(S_1) \to H^1(S_1),$$

$$\mathcal{W}'-\widetilde{\mathcal{W}}': H^{-\frac{1}{2}}(S_1) \to H^0(S_1),$$

$$\mathcal{W}-\widetilde{\mathcal{W}}: H^{\frac{1}{2}}(S_1) \to H^1(S_1),$$

$$\mathcal{L}-\widetilde{\mathcal{L}}: H^{\frac{1}{2}}(S_1) \to H^0(S_1),$$

$$\mathcal{K}-\widetilde{\mathcal{K}}: H^{-\frac{1}{2}}(S_1) \to H^0(S_1),$$

$$\mathcal{M}-\widetilde{\mathcal{M}}: H^{\frac{1}{2}}(S_1) \to H^0(S_1).$$

For the operator $\mathcal{K}$ defined by (A.6), we have the following representation $\mathcal{K}=\widetilde{\mathcal{T}}+\mathcal{C}$ with $\widetilde{\mathcal{T}}=\frac{1}{2}I+\widetilde{\mathcal{W}}'$ and $\mathcal{C}=\mathcal{W}'-\widetilde{\mathcal{W}}'+i\alpha\mathcal{V}$ and by Lemmas A.2 and A.3, we deduce

$$\begin{aligned}\mathcal{K}^{-1}&=\widetilde{\mathcal{T}}^{-1}-\mathcal{K}^{-1}\mathcal{C}\widetilde{\mathcal{T}}^{-1},\\ \mathcal{K}^{-1}\mathcal{M}&=\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}+\mathcal{G},\end{aligned}\tag{A.19}$$

where

$$\mathcal{G}:=-\mathcal{K}^{-1}\mathcal{C}\widetilde{\mathcal{T}}^{-1}\mathcal{M}+\widetilde{\mathcal{T}}^{-1}\Big[\mathcal{L}-\widetilde{\mathcal{L}}+i\alpha\Big(-\frac{1}{2}I+\mathcal{W}\Big)\Big].$$

By Lemmas A.2, A.3 and A.4, the following operators

$$\begin{aligned}\mathcal{K}^{-1}\mathcal{C}\widetilde{\mathcal{T}}^{-1}&: H^{-\frac{1}{2}}(S_1) \to H^0(S_1),\\ \mathcal{G}&: H^{\frac{1}{2}}(S_1) \to H^0(S_1)\end{aligned}\tag{A.20}$$

are continuous.

Lemma A.6. *There are positive constants $C_1'>0$ and $C_2'>0$ such that*

$$\mathrm{Re}\,\langle -\mathcal{K}^{-1}\mathcal{M}\psi\,,\,\overline{\psi}\rangle_{S_1} \geq C_1'\,\|\psi\|^2_{H^{\frac{1}{2}}(S_1)}-C_2'\,\|\psi\|^2_{H^0(S_1)}\ \ \textit{for all}\ \ \psi\in H^{\frac{1}{2}}(S_1).$$

Proof. In view of (A.19), (A.20) and the Schwartz inequality for all $\psi\in H^{\frac{1}{2}}(S_1)$, we have

$$\begin{aligned}\mathrm{Re}\,\big\langle -\mathcal{K}^{-1}\mathcal{M}\psi\,,\,\overline{\psi}\big\rangle_{S_1}&=\mathrm{Re}\,\big\langle -\big[\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}+\mathcal{G}\big]\psi\,,\,\overline{\psi}\big\rangle_{S_1}\\ &\geqslant \mathrm{Re}\,\big\langle -\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}\psi\,,\,\overline{\psi}\big\rangle_{S_1}-|\big\langle\mathcal{G}\psi\,,\,\overline{\psi}\big\rangle_{S_1}|\\ &=\mathrm{Re}\,\big\langle -\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}\psi\,,\,\overline{\psi}\big\rangle_{S_1}-\Big|\int\limits_{S_1}\overline{\psi}\;\mathcal{G}\psi\,dS\Big|\\ &\geqslant \mathrm{Re}\,\big\langle -\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}\psi\,,\,\overline{\psi}\big\rangle_{S_1}-\|\mathcal{G}\psi\|_{H^0(S_1)}\;\|\overline{\psi}\|_{H^0(S_1)}\\ &\geqslant \mathrm{Re}\,\big\langle -\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}\psi\,,\,\overline{\psi}\big\rangle_{S_1}-c_1\,\|\psi\|_{H^{\frac{1}{2}}(S_1)}\;\|\psi\|_{H^0(S_1)}\end{aligned}\tag{A.21}$$

with some positive constant c_1.

To estimate the first summand from below, we proceed as follows. The general integral representation formula for an exponentially decaying solution to the homogeneous equation $A_1(\partial_x,i)w=0$ in Ω_1 reads as

$$w+\widetilde{V}(T_1^-w)-\widetilde{W}(\gamma^-_{S_1}w)=0 \quad \text{in} \quad \Omega_1.$$

Substituting here $w=\widetilde{W}\varphi$ with arbitrary $\varphi\in H^{\frac{1}{2}}(S_1)$ and taking the generalized trace of the conormal derivative on S_1, we obtain

$$\widetilde{\mathcal{L}}\Big(\frac{1}{2}I+\widetilde{\mathcal{W}}\Big)\varphi=\Big(\frac{1}{2}I+\widetilde{\mathcal{W}}'\Big)\,\widetilde{\mathcal{L}}\varphi \ \text{ on }\ S_1.$$

This implies the following operator relation with the domain of definition $H^{\frac{1}{2}}(S_1)$ and the range $H^{-\frac{1}{2}}(S_1)$,

$$\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}} = \Big(\frac{1}{2}I + \widetilde{\mathcal{W}}'\Big)^{-1}\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}\Big(\frac{1}{2}I + \widetilde{\mathcal{W}}\Big)^{-1}. \tag{A.22}$$

Further, substituting $u = \widetilde{W}g$ with $g = \big(\frac{1}{2}I + \widetilde{\mathcal{W}}\big)^{-1}\varphi$ and $\varphi \in H^{\frac{1}{2}}(S_1)$ in (A.10) for $\Omega^- = \Omega_1$ and taking into consideration the equalities $T_1^- u = \widetilde{\mathcal{L}}\Big(\frac{1}{2}I + \widetilde{\mathcal{W}}\Big)^{-1}\varphi$ and (A.22), we get

$$-\big\langle T_1^- u\,,\,\overline{\gamma_{S_1}^- u}\big\rangle_{S_1} = \big\langle -\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}\varphi\,,\,\overline{\varphi}\big\rangle_{S_1} = \int\limits_{\Omega_1}\big[\,E_1(u,\overline{u}) + \kappa_1|u(x)|^2\,\big]\,dx, \tag{A.23}$$

where E_1 is defined in (2.9). Since the matrix $\mathbf{a}_1 = \big[\,a_{kj}^{(1)}\,\big]_{k,j=1}^3$ is positive definite, $\kappa_1 > 0$ and $\gamma_{S_1}^- u = \varphi$, with the help of the trace theorem, from (A.23), we deduce

$$\big\langle -\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}\varphi\,,\,\overline{\varphi}\big\rangle_{S_1} \geqslant c_2\,\|u\|^2_{H^1(\Omega_1)} \geqslant c_3\,\|\gamma_{S_1}^- u\|^2_{H^{\frac{1}{2}}(S_1)} = c_3\,\|\varphi\|^2_{H^{\frac{1}{2}}(S_1)}, \tag{A.24}$$

where c_2 and c_3 are some positive constants.

Now, using the inequalities (A.24) and

$$\|\psi\|_{H^{\frac{1}{2}}(S_1)}\,\|\psi\|_{H^0(S_1)} \leqslant \varepsilon\,\|\psi\|^2_{H^{\frac{1}{2}}(S_1)} + \frac{1}{4\varepsilon}\,\|\psi\|^2_{H^0(S_1)},$$

from (A.21), we finally obtain

$$\begin{aligned}\mathrm{Re}\big\langle -\mathcal{K}^{-1}\mathcal{M}\psi\,,\,\overline{\psi}\big\rangle_{S_1} &\geqslant \mathrm{Re}\big\langle -\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}\psi\,,\,\overline{\psi}\big\rangle_{S_1} - c_1\,\|\psi\|_{H^{\frac{1}{2}}(S_1)}\,\|\psi\|_{H^0(S_1)}\\ &\geqslant c_3\,\|\psi\|^2_{H^{\frac{1}{2}}(S_1)} - c_1\,\|\psi\|_{H^{\frac{1}{2}}(S_1)}\,\|\psi\|_{H^0(S_1)}\\ &\geqslant (c_3 - \varepsilon c_1)\,\|\psi\|^2_{H^{\frac{1}{2}}(S_1)} - (4\varepsilon)^{-1}c_1\,\|\psi\|^2_{H^0(S_1)},\end{aligned}$$

where ε is an arbitrarily small positive number. This completes the proof. □

Remark A.7. In many papers, the one-sided boundary traces of layer potentials and their conormal derivaives are understood in the nontangential limit sense (for details see, e.g., [17, 32, 43]). Note that in the case of a bounded Lipschitz domain Ω, a single layer potential $V(h)$ with a density $h \in H^{-\frac{1}{2}}(\partial\Omega)$, as well as a double layer potential $W(g)$ with a density $g \in H^{\frac{1}{2}}(\partial\Omega)$, belong to the space $H^1(\Omega)$ and possess the Sobolev boundary traces belonging to the space $H^{\frac{1}{2}}(\partial\Omega)$ (see Lemma A.1). Therefore, for these potentials the nontangential boundary values exist almost everywhere on $\partial\Omega$ and the corresponding nontangential maximal functions are square integrable (see [32, 43]). Consequently, for these potentials the Sobolev traces and the nontangential traces on $\partial\Omega$ coincide (see, e.g., [2, Remark 6.7]).

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BOUNDEDNESS OF HIGHER ORDER COMMUTATORS OF G-FRACTIONAL INTEGRAL AND G-FRACTIONAL MAXIMAL OPERATORS WITH $G-BMO$ FUNCTIONS

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Abstract. In this paper we introduce the Gegenbauer BMO $(G-BMO)$ space and study its basic properties, analogous to the classical case. The John-Nirenberg type theorem is proved for $f \in BMO_G(\mathbb{R}_+)$. Moreover, the notions of a higher order commutator of Gegenbauer fractional (G-fractional) integral $J_G^{b,\alpha,k}$ and Gegenbauer fractional (G-fractional) maximal operator $M_G^{b,\alpha,k}$ with $G-BMO$ function are studied. When commutator b is a $(G-BMO)$ function, the necessary and sufficient conditions for $(L_p; L_q)$ boundedness of commutators $J_G^{b,\alpha,k}$ and $M_G^{b,\alpha,k}$ are obtained.

Introduction

The boundedness of the fractional maximal operator, fractional integral and its commutators plays an important role in harmonic analysis and their applications. In recent decades, many authors have proved the boundedness of the commutators with BMO functions of fractional maximal operator and fractional integral operator on some function spaces (see, e.g., [1–4, 6–8, 13, 19]).

The fractional integral operator I_α and fractional maximal operator M_α are defined as follows:

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}, \quad n \geq 1, \quad 0 < \alpha < n,$$

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int\limits_{|x-y|\leq r} |f(y)| dy.$$

Let $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$, then the commutator is generated by the function $b(x)$ and I_α is defined as the form

$$[b, I_\alpha] f(x) = b(x) I_\alpha(x) - I_\alpha(bf)(x) = \int\limits_{\mathbb{R}^n} \frac{[b(x)-b(y)]}{|x-y|^{n-\alpha}} f(y) dy.$$

In [2] and [19], the following theorem is proved by a somewhat different method.

Theorem A. *Let $0 < \alpha < n$, $1 < p < \dfrac{n}{\alpha}$ and $\dfrac{1}{p} - \dfrac{1}{q} = \dfrac{\alpha}{n}$. Then $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.*

Define the commutator $[b, M_\alpha]$ of the fractional maximal operator M_α as

$$[b, M_\alpha](f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int\limits_{|x-y|\leq r} |b(x)-b(y)||f(y)| dy.$$

In [19], it is proved that under the conditions of Theorem A $[b, M_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.

In the present paper, we prove theorems on the boundedness of commutators both of the G-fractional integral and of the G-fractional maximal operator on $G-BMO$ space. The results obtained here are analogous to the corresponding theorem obtained for the $[b, I_\alpha]$ and $[b, M_\alpha]$ in [2] and [19].

The paper is organized as follows.

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In Section 1, we present some definitions, notations and auxiliary results. In Section 2, the $G{-}BMO$ space is introduced and its properties are proved. In Sections 3 and 4 we prove the $(L_{p,\lambda};\, L_{q,\lambda})$ boundedness of the commutator of G-fractional integrals and the $(L_{p,\lambda};\, L_{q,\lambda})$ boundedness of the commutator of G-fractional maximal operator on $G-BMO$ space, respectively.

1. Definitions, Notations and Auxiliary Results

Our investigation is based on the Gegenbauer differential operator G_λ (see [5])

$$G_\lambda \equiv G = (x^2-1)^{\frac{1}{2}-\lambda}\frac{d}{dx}(x^2-1)^{\lambda+\frac{1}{2}}\frac{d}{dx}, x\in(1,\infty), \lambda\in(0,\frac{1}{2}).$$

The shift operator $A^\lambda_{\mathrm{ch}y}$ generated by G_λ is given in the form (see [10,11])

$$A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})}\int\limits_0 f(\mathrm{ch}x\mathrm{ch}y - \mathrm{sh}x\mathrm{sh}y\cos\varphi)(\sin\varphi)^{2\lambda-1}d\varphi$$

and it possesses all properties of the generalized shift operator given in the monograph due to B.M.Levitan [16,17].

Let $H = H(0,r) = (0,r)$. For any measurable set E, $\mu E = |E|_\lambda = \int\limits_E \mathrm{sh}^{2\lambda}ydy$. For $1\le p<\infty$, let $L_p(\mathbb{R}_+, G) = L_{p,\lambda}(\mathbb{R}_+)$, $\mathbb{R}_+ = (0,\infty)$ be the space of measurable functions on $\mathbb{R}_+$ with the finite norm

$$\|f\|_{L_{p,\lambda}} = \Bigg(\int\limits_{\mathbb{R}_+} |f(\mathrm{ch}y)|^p \mathrm{sh}^{2\lambda}ydy\Bigg)^{\frac{1}{p}}, \quad 1\le p<\infty,$$

$$\|f\|_{\infty,\lambda} \equiv \|f\|_\infty = ess\sup_{x\in\mathbb{R}_+}|f(\mathrm{ch}x)|, \quad p=\infty.$$

For $f\in L^{\mathrm{loc}}_{1,\lambda}(\mathbb{R}_+)$, the G-fractional maximal operator M^α_G and the G-fractional integral J^α_G are defined in [14] as follows:

$$M^\alpha_G f(\mathrm{ch}x) = \sup_{r>0}\frac{1}{|H|_\lambda^{1-\frac{\alpha}{2\lambda+1}}}\int\limits_H A^\lambda_{\mathrm{ch}y}|f(\mathrm{ch}x)|\mathrm{sh}^{2\lambda}ydy.$$

Here $|H(0,r)|_\lambda = \int\limits_0^r \mathrm{sh}^{2\lambda}ydy$ is the measure that is absolutely continuous with respect to the Lebesgue measure of the interval H

$$J^\alpha_G f(\mathrm{ch}x) = \int\limits_0^\infty \frac{A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)}{(\mathrm{sh}y)^{2\lambda+1-\alpha}}\mathrm{sh}^{2\lambda}ydy.$$

The next result has been obtained in [14] and gives us the $(L_{p,\lambda}, L_{q,\lambda})$ boundedness of M^α_G and J^α_G (see also [13,15]).

Theorem B. *Suppose that* $0<\lambda<\dfrac{1}{2}, 0<\alpha<2\lambda+1,$ *and* $1\le p<\dfrac{2\lambda+1}{\alpha}$.

(a) *If* $1<p<\frac{2\lambda+1}{\alpha}$, *then the condition* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\lambda+1}$ *is necessary and sufficient for the boundedness of* M^α_G *and* J^α_G *from* $L_{p,\lambda}(\mathbb{R}_+)$ *to* $L_{q,\lambda}(\mathbb{R}_+)$.

(b) *If* $p=1$, *then the condition* $1-\frac{1}{q}=\frac{\alpha}{2\lambda+1}$ *is necessary and sufficient for the boundedness of* M^α_G *and* J^α_G *from* $L_{1,\lambda}(\mathbb{R}_+)$ *to* $WL_{q,\lambda}(\mathbb{R}_+)$.

We denote by $WL_{q,\lambda}(\mathbb{R}_+)$ the spaces of all locally integrable functions $f(\mathrm{ch}x)$, $x\in\mathbb{R}_+$, with the finite norm

$$\|f\|_{WL_{q,\lambda}(\mathbb{R}_+)} = \sup_{r>0} r|\{x\in\mathbb{R}_+ : |f(\mathrm{ch}x)|>r\}|_\lambda^{\frac{1}{p}},\ 1\le p<q.$$

Throughout the paper $A\lesssim B$ mean that $A\le CB$ with some positive constant C, which may depend on some parameters. If $A\lesssim B$ and $B\lesssim A$, we write $A\approx B$ and say that A and B are equivalent.

Let $H(x,r) = (x-r, x+r) \cap [0,\infty)$, $r \in (0,\infty)$, $x \in [0,\infty)$. Thus,

$$H(x,r) = \begin{cases} (0, x+r), & 0 \le x < r, \\ (x-r, x+r), & x \ge r. \end{cases}$$

We will need the following lemmas.

Lemma 1.1 ([14]). *For any $\mu > 0$, the following relation is true:*

$$|H(x,r)|_{\frac{\mu}{2}} \approx \begin{cases} \left(\mathrm{sh}\dfrac{x+r}{2}\right)^{\mu+1}, & 0 < x+r < 2, \\ \left(\mathrm{sh}\dfrac{x+r}{2}\right)^{2\mu}, & 2 \le x+r < \infty. \end{cases}$$

For $x = 0$ and $\mu = 2\lambda$, we have

$$|H(0,r)|_\lambda \approx \left(\mathrm{sh}\frac{r}{2}\right)^\gamma,$$

where $\gamma = \gamma_\lambda(r) = \begin{cases} 2\lambda+1, & \text{if} \quad 0 < r < 2, \\ 4\lambda, & \text{if} \quad 2 \le r < \infty. \end{cases}$

Lemma 1.2 ([11]). *If $f \in L_{p,\lambda}(\mathbb{R}_+)$, then for any $y \in [0,\infty)$, the inequality*

$$\|A_{\mathrm{ch}y} f\|_{L_{p,\lambda}} \le \|f\|_{L_{p,\lambda}}, 1 \le p \le \infty \tag{1.1}$$

holds.

2. The Gegenbauer BMO-Space

The space of functions of bounded mean oscillation, or BMO_G, naturally arises as the class of functions whose deviation from their means over intervals is bounded. The L_∞ functions have this property, but there exist unbounded functions with a bounded mean oscillation. Such functions are slowly growing, and they typically have at most logarithmic blow up. The space BMO_G shares similar properties with the space L_∞ and often serves as its substitute. What exactly is a bounded mean oscillation and what kind of functions have this property?

The mean of a locally integrable function over a set is another word for its average over that set. The oscillation of a function over a set is the absolute value of the difference of the function from its mean over this set. The mean oscillation is therefore the average of this oscillation over a set. A function is said to be of bounded mean oscillation if its mean oscillation over all intervals is bounded. Precisely, given a locally integrable function f on $\mathbb{R}_+ = (0,\infty)$, denote by

$$f_H(\mathrm{ch}x) = \frac{1}{|H|_\lambda} \int\limits_H A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) \mathrm{sh}^{2\lambda} y dy,$$

where $H = H(0,r)$, the mean (or average) of f over H. Then the oscillation of f over H are the functions $|A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)|$, and the mean oscillation of f over H is

$$\frac{1}{|H|_\lambda} \int\limits_H |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy.$$

2.1. Definition and some properties of the $G - BMO$ space.

Definition 2.1. We denote by $BMO_G(\mathbb{R}_+)$ the Gegenbauer-BMO space ($G - BMO$ space) as the set of locally integrable functions on $\mathbb{R}_+ = (0,\infty)$ such that

$$\|f\|_{BMO_G(\mathbb{R}_+)} = \sup_{x,r \in \mathbb{R}_+} \frac{1}{|H|_\lambda} \int\limits_H |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy < \infty.$$

We set

$$BMO_G(\mathbb{R}_+) = \left\{ f \in L^{\mathrm{loc}}_{1,\lambda}(\mathbb{R}_+) : \|f\|_{BMO_G(\mathbb{R}_+)} < \infty \right\}.$$

Several remarks are in order. First, it is a simple fact that $BMO_G(\mathbb{R}_+)$ is a linear space, that is, if $f, g \in BMO_G(\mathbb{R}_+)$ and $\mu \in R$, then $f+g$ and μf in $BMO_G(\mathbb{R}_+)$, and

$$\|f+g\|_{BMO_G} \le \|f\|_{BMO_G} + \|g\|_{BMO_G}, \ \|\mu f\|_{BMO_G} = |\mu| \|f\|_{BMO_G}.$$

But $\|\cdot\|_{BMO_G}$ is not a norm. The problem is that if $\|\cdot\|_{BMO_G} = 0$, this does not imply that $f = 0$, but that f is a constant. From Proposition 2.2, every constant function C satisfies $\|C\|_{BMO_G} = 0$, then the functions f and $f + c$ have the same BMO_G norms. In the sequel, we keep in mind that elements of BMO_G whose difference is a constant are identified. Although $\|\cdot\|_{BMO_G}$ is only a seminorm, we occasionally refer to it as a norm when there is no possibility of confusion.

We begin with the basic properties of BMO_G.

Proposition 2.2. *The following properties of the $BMO_G(\mathbb{R}_+)$ space are valid:*

1) *If $\|f\|_{BMO_G} = 0$, then f is a.e. equal to a constant.*

2) *$L_\infty(\mathbb{R}_+)$ is contained in $BMO_G(\mathbb{R}_+)$ and $\|f\|_{BMO_G} \le 2\|f\|_{L_\infty}$.*

3) *Suppose that there exist a constant $A > 0$ and for all intervals H in $\mathbb{R}_+$ a constant C_H such that*

$$\sup_{x,r\in\mathbb{R}_+} \frac{1}{|H|_\lambda} \int\limits_H |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - C_H| \mathrm{sh}^{2\lambda} y dy \le A, \tag{2.1}$$

then $f \in BMO_G(\mathbb{R}_+)$ and $\|f\|_{BMO_G} \le 2A$.

4) *If $f \in BMO_G(\mathbb{R}_+)$, $y \in \mathbb{R}_+$, then $A^\lambda_{\mathrm{ch}y} f$ is also in $BMO_G(\mathbb{R}_+)$ and*

$$\|A^\lambda_{\mathrm{ch}y} f\|_{BMO_G} \le \|f\|_{BMO_G}.$$

5) *Let f be in $BMO_G(\mathbb{R}_+)$. Given an interval H and a positive integer m, we have*

$$|b_H(\mathrm{ch}x) - b_{2^m H}(\mathrm{ch}x)| \le 2m\|b\|_{BMO_G}.$$

Proof. To prove 1), we note that f is a.e. equal to its average C_N over every segment $[0, N]$. Since $[0, N] \subset [0, N + 1]$, it follows that $C_N = C_{N+1}$ for all N. This implies the required conclusion.

To prove 2), we using (1.1). Then

$$\begin{aligned} &A^\lambda_{\mathrm{ch}y} |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)| \le A^\lambda_{\mathrm{ch}y} \left(|A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x)| + |f_H(\mathrm{ch}x)|\right) \\ \le& 2A^\lambda_{\mathrm{ch}y} |f(\mathrm{ch}x)| \le 2\|f\|_{L_\infty}. \end{aligned}$$

For item 3), we get

$$\begin{aligned} &|A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)| \le |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - C_H| + |f_H(\mathrm{ch}x) - C_H| \\ \le& |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - C_H| + \frac{1}{|H|_\lambda} \int\limits_H |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - C_H| \mathrm{sh}^{2\lambda} y dy. \end{aligned}$$

Averaging over H and using (2.1), one has

$$\|f\|_{BMO_G} \le 2A.$$

Let us prove property 4). Applying Lemma 1.2, we have

$$\begin{aligned} \|A^\lambda_{\mathrm{ch}y} f\|_{BMO_G} &\le \sup_{x,r\in\mathbb{R}_+} \frac{1}{|H|_\lambda} \int\limits_H |A^\lambda_{\mathrm{ch}y} A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - A^\lambda_{\mathrm{ch}y} f_H(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy \\ \le& \sup_{x,r\in\mathbb{R}_+} \frac{1}{|H|_\lambda} \int\limits_H A^\lambda_{\mathrm{ch}y} |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy \\ \le& \sup_{x,r\in\mathbb{R}_+} \frac{1}{|H|_\lambda} \int\limits_H |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy = \|f\|_{BMO_G}. \end{aligned}$$

Finally, we prove 5). In fact,

$$|b_H(\mathrm{ch}x) - b_{2H}(\mathrm{ch}x)| \le \frac{1}{|H|_\lambda}\left|\int\limits_H \left(A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_{2H}(\mathrm{ch}x)\right)\mathrm{sh}^{2\lambda} y dy\right|$$

$$\le \frac{2}{|2H|_\lambda}\int\limits_H \left|A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - b_{2H}(\mathrm{ch}x)\right|\mathrm{sh}^{2\lambda} y dy \le 2\|f\|_{BMO_G}.$$

Then A_n iteration yields

$$|b_H - b_{2H} + b_{2H} - b_{2^2H} + \cdots + b_{2^{m-1}H} - b_{2^m H}| \le 2m\|f\|_{BMO_G}. \qquad \square$$

Example. We show that $L_\infty(\mathbb{R}_+)$ is a proper subspace of $BMO_G(\mathbb{R}_+)$. We claim that the function $\log(\mathrm{sh}x)$ is in $BMO_G(\mathbb{R}_+)$, but not in $L_\infty(\mathbb{R}_+)$. To prove that it is in $BMO_G(\mathbb{R}_+)$, for every $x_0 \in \mathbb{R}_+$ and $r > 0$, we choose a constant $C_{x_0,r}$ such that the average $|A^\lambda_{\mathrm{ch}y}\log(\mathrm{sh}x) - C_{x_0,r}|$ for all $y \in [0, x_0+r]$ is uniformly bounded.

Consider the integral

$$\frac{1}{|H(0,x_0+r)|_\lambda}\int\limits_0^{x_0+r}\left|A^\lambda_{\mathrm{ch}y}\log(\mathrm{sh}x) - C_{x_0,r}\right|\mathrm{sh}^{2\lambda} y dy,$$

where $C_{x_0,r} = (\log r)(\log x_0)$, $0 \le x_0 \le 2$ and $0 \le x_0 \le \mathrm{arcsh}1$. We may take $r = 1$, then

$$\frac{1}{|H(0,x_0+1)|_\lambda}\int\limits_0^{x_0+1}\left|A^\lambda_{\mathrm{ch}y}\log(\mathrm{sh}x)\right|\mathrm{sh}^{2\lambda} y dy$$

$$= \frac{1}{|H(0,x_0+1)|_\lambda}\int\limits_0^{x_0+1}\left|A^\lambda_{\mathrm{ch}y}\log(\mathrm{ch}^2x - 1)^{\frac{1}{2}}\right|\mathrm{sh}^{2\lambda} y dy$$

$$= \frac{1}{|H(0,x_0+1)|_\lambda}\int\limits_0^{x_0+1}\left|\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})}\int\limits_0^\pi \log[(\mathrm{ch}x\mathrm{ch}y - \mathrm{sh}x\mathrm{sh}y\cos\varphi)^2 - 1]^{\frac{1}{2}}\right|\mathrm{sh}^{2\lambda} y dy$$

$$\le \frac{1}{|H(0,x_0+1)|_\lambda}\int\limits_0^{x_0+1}|\log\mathrm{sh}(x+y)|\,\mathrm{sh}^{2\lambda} y dy \le \log\mathrm{sh}(x+x_0+1)$$

$$\le \log\mathrm{sh}(x_0+1+\mathrm{arcsh}1) \le \log\mathrm{sh}(x_0+2) \le \log\mathrm{sh}4.$$

Now, let $C_{x_0,1} = \log(2x_0)$, $\mathrm{arcsh}1 \le x \le x_0$, $x_0 > 2$. In this case, we have

$$\frac{1}{|H(0,x_0+1)|_\lambda}\int\limits_0^{x_0+1}\left|A^\lambda_{\mathrm{ch}y}\log(\mathrm{sh}x) - \log(2x_0)\right|\mathrm{sh}^{2\lambda} y dy$$

$$\le \log\frac{\mathrm{sh}(x+x_0+1)}{\mathrm{sh}(2x_0)} < \log\frac{\mathrm{sh}(2x_0+2)}{\mathrm{sh}2x_0} = \log\frac{(\mathrm{sh}2x_0)\mathrm{ch}2 + (\mathrm{ch}2x_0)\mathrm{sh}2}{\mathrm{sh}2x_0}$$

$$= \log\left(\mathrm{ch}2 + \frac{\mathrm{ch}2x_0}{\mathrm{sh}(2x_0)}\mathrm{sh}2\right) \le \log(\mathrm{ch}2 + 2\mathrm{sh}2) \le \log(3\mathrm{ch}2),$$

since $\mathrm{ch}x \le 2\mathrm{sh}x$ if $x \le 1$.

Thus, according to property 3), $\log(\mathrm{sh}x)$ is in $BMO_G(\mathbb{R}_+)$. It is obvious that $\log(\mathrm{sh}x)$ is not in $L_\infty(\mathbb{R}_+)$.

Below, we will need some property of $BMO_G(\mathbb{R}_+)$ functions. Observe that if an interval H_1 is contained in the interval H_2, then

$$|f_{H_1} - f_{H_2}| \leq \frac{1}{|H_1|_\lambda} \int\limits_{H_1} |A^\lambda_{\text{ch}y} f(\text{ch}x) - f_{H_2}(\text{ch}x)| \text{sh}^{2\lambda} y dy$$

$$\leq \frac{1}{|H_1|_\lambda} \int\limits_{H_2} |A^\lambda_{\text{ch}y} f(\text{ch}x) - f_{H_2}(\text{ch}x)| \text{sh}^{2\lambda} y dy$$

$$\leq \frac{|H_2|_\lambda}{|H_1|_\lambda} \|f\|_{BMO_G}.$$

Theorem 2.3. *$BMO_G(\mathbb{R}_+)$ is a complete space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $BMO_G(\mathbb{R}_+)$. Thus $\|f_n - f_m\|_{BMO_G} \to 0$, for $n, m \to \infty$. We choose a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\|f_{n_{k+1}} - f_{n_k}\|_{BMO_G} < \frac{1}{2^k}$ for all $k \geq 1$. From this it follows that

$$\sum_{k=1}^\infty \|f_{n_{k+1}} - f_{n_k}\|_{BMO_G} < \sum_{k=1}^\infty \frac{1}{2^k} = 1.$$

Then for a.e. $x \in \mathbb{R}_+$,

$$\sum_{k=1}^\infty |f_{n_{k+1}} - f_{n_k}| < \infty,$$

and, consequently, the series

$$f_{n_1}(\text{ch}x) + \sum_{k=1}^\infty \{f_{n_{k+1}}(\text{ch}x) - f_{n_k}(\text{ch}x)\}$$

converges, this is equivalent to the existence of

$$\lim_{k\to\infty} f_{n_k}(\text{ch}x), \text{for a.e. } x \in \mathbb{R}_+.$$

We define the function f as follows:

$$f(\text{ch}x) = \begin{cases} \lim\limits_{k\to\infty} f_{n_k}(\text{ch}x), & \text{for a.e.} \quad x \in \mathbb{R}_+, \\ 0, & \text{otherwise.} \end{cases}$$

Thus we prove that

$$\lim_{k\to\infty} f_{n_k}(\text{ch}x) = f(\text{ch}x), \text{a.e. } x \in \mathbb{R}_+.$$

By the triangle inequality,

$$\|f_{n_k}\|_{BMO_G} = \left\| f_{n_1} + \sum_{\nu=1}^{k-1} \left(f_{n_{k+1}} - f_{n_k}\right) \right\|_{BMO_G}$$

$$\leq \left\| |f_{n_1}| + \sum_{\nu=1}^{k-1} \left|f_{n_{k+1}} - f_{n_k}\right| \right\|_{BMO_G}$$

$$\leq \|f_{n_1}\|_{BMO_G} + \left\| \sum_{\nu=1}^{k-1} \left|f_{n_{k+1}} - f_{n_k}\right| \right\|_{BMO_G} \leq \|f_{n_1}\|_{BMO_G} + 1.$$

From this it follows that

$$\|f_{n_k}\|_{BMO_G} \leq \text{const}, \quad \text{at} \quad k \to \infty,$$

i.e., $f \in BMO_G(\mathbb{R}_+)$.

Now, we show that

$$\|f - f_{n_k}\|_{BMO_G} \to 0, \quad \text{at} \quad k \to \infty.$$

In fact,

$$\|f - f_{n_k}\|_{BMO_G} = \Big\| \sum_{\nu=k}^{\infty} (f_{n_{\nu+1}} - f_{n_\nu}) \Big\|_{BMO_G}$$
$$\leq \Big\| \sum_{\nu=k}^{\infty} |f_{n_{\nu+1}} - f_{n_\nu}| \Big\|_{BMO_G} \leq \sum_{\nu=1}^{\infty} \|f_{n_{\nu+1}} - f_{n_\nu}\|_{BMO_G} < 1.$$

By the dominated convergence theorem,

$$\|f - f_{n_k}\|_{BMO_G} \to 0, \quad \text{at} \quad k \to \infty.$$

Finally, we have to show that $\{f_n\}$ is the Cauchy. Given $\varepsilon > 0$, there exists N_ε so, for all $n, m > N_\varepsilon$, we have

$$\|f_n - f_m\|_{BMO_G} < \frac{\varepsilon}{2}.$$

We choose a number $n_k > N_\varepsilon$ such that

$$\|f - f_{n_k}\|_{BMO_G} < \frac{\varepsilon}{2}.$$

Then we have

$$\|f - f_n\|_{BMO_G} \leq \|f - f_{n_k}\|_{BMO_G} + \|f_n - f_{n_k}\|_{BMO_G} < \varepsilon.$$

This completes the proof. □

The next section needs the following statement.

Theorem 2.4 (Calderon–Zygmund decomposition of $\mathbb{R}_+$)**.** *Suppose that f is a non-negative integrable function on $\mathbb{R}_+$. Then for any fixed number $\beta > 0$, there exists a sequence $\{(j-1)r, jr\} = \{H_j\}$ of disjoint intervals such that*

(1) $f(\mathrm{ch}x) \leq \beta$, $x \notin \bigcup\limits_j H_j$;

(2) $\big|\bigcup\limits_j H_j\big|_\lambda \leq \frac{1}{\beta}\|f\|_{L_{1,\lambda}}$;

(3) $\beta < \dfrac{1}{|H_j|_\lambda} \int\limits_{H_j} A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x)\mathrm{sh}^{2\lambda} y dy \leq 2^{(2\lambda+1)n}\beta$, $n = 1, 2, \ldots$.

Proof. Since $f \in L_{1,\lambda}(\mathbb{R}_+)$, by Lemma 1.2, $A^\lambda_{\mathrm{ch}y} f \in L_{1,\lambda}(\mathbb{R}_+)$ and by the integral continuity, we can decompose $\mathbb{R}_+$ into a net of equal intervals (by the Lindelöf covering theorem (see [18]), this is possible)) such that for every H from the net

$$\frac{1}{|H|_\lambda} \int\limits_H A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x)\mathrm{sh}^{2\lambda} y dy \leq \beta. \tag{2.2}$$

In fact, for any $\beta > 0$, there exists $\delta = \delta(\beta) > 0$ such that for every H_j with measure $|H_j|_\lambda = |H|_\lambda < \delta$,

$$\int\limits_{H_j} A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x)\mathrm{sh}^{2\lambda} y dy < \beta, \quad (j = 1, 2, \ldots),$$

where

$$|H_j|_\lambda = \int\limits_{H_j} \mathrm{sh}^{2\lambda} y dy, \quad (j = 1, 2, \ldots).$$

First, we prove (3). Let $H_1 = (0, r)$ be a fixed interval in the net. Then by (2.2), we can write

$$\frac{1}{H_1} \int\limits_{H_1} A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x)\mathrm{sh}^{2\lambda} y dy \leq \beta. \tag{2.3}$$

We divide the interval H_1 into 2^n equal intervals and let $H_1' = \left(0, \frac{r}{2^n}\right)$ be one from this intervals. By Lemma 1.1 (then $\mu = 2\lambda$), one has

$$|H_1'|_\lambda = \int_0^{\frac{r}{2^n}} \mathrm{sh}^{2\lambda} y dy \approx \left(\mathrm{sh}\frac{r}{2^{n+1}}\right)^{2\lambda+1}, \quad 0 < \frac{r}{2^n} < 2.$$

Since for $0 < t < 1$, $\mathrm{sh}t \approx t$, we have

$$|H_1'|_\lambda \approx \left(\mathrm{sh}\frac{r}{2^{n+1}}\right)^{2\lambda+1} \approx \left(\frac{r}{2^{n+1}}\right)^{2\lambda+1} \approx \left(\frac{1}{2^n}\mathrm{sh}\frac{r}{2}\right)^{2\lambda+1} \approx 2^{-(2\lambda+1)n}|H_1'|_\lambda. \tag{2.4}$$

Concerning H_1', there may possibly be two cases:

(A) $\frac{1}{|H_1'|_\lambda}\int_{H_1'} A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x)\mathrm{sh}^{2\lambda} y dy > \beta$.

(B) $\frac{1}{|H_1'|_\lambda}\int_{H_1'} A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x)\mathrm{sh}^{2\lambda} y dy \le \beta$.

For case (A), from (2.4) and (2.3), we have

$$\begin{aligned}\beta &< \frac{1}{|H_1'|_\lambda}\int_{H_1'} A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x)\mathrm{sh}^{2\lambda} y dy \\ &\approx \frac{2^{(2\lambda+1)n}}{|H_1'|_\lambda}\int_{H_1'} A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x)\mathrm{sh}^{2\lambda} y dy \\ &\lesssim \frac{2^{(2\lambda+1)n}}{|H_1|_\lambda}\int_{H_1} A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x)\mathrm{sh}^{2\lambda} y dy \lesssim 2^{(2\lambda+1)n}\beta.\end{aligned}$$

Here H_1' we choose as one of the sequences $\{H_j\}$.

We consider case (B). Suppose that $H_1' = H_2(r, 2r)$. Dividing the interval into 2^n equal partials and reasoning however, we obtain

$$\begin{aligned}\beta &< \frac{1}{|H_2'|_\lambda}\int_{H_2'} A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x)\mathrm{sh}^{2\lambda} y dy \\ &\lesssim \frac{2^{(2\lambda+1)n}}{|H_1|_\lambda}\int_{H_1} A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x)\mathrm{sh}^{2\lambda} y dy \lesssim 2^{(2\lambda+1)n}\beta,\end{aligned}$$

where H_2' we choose as one of the sequences $\{H_j\}$. Further reasoning of the process, we obtain a sequence of disjoint $\{H_j\}$ such that

$$\beta < \frac{1}{|H_j|_\lambda}\int_{H_j} A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x)\mathrm{sh}^{2\lambda} y dy \lesssim 2^{(2\lambda+1)n}\beta, \quad (n = 1, 2, \ldots).$$

Proof of (1). Taking into account (2.4), from the Lebesgue differentiation theorem (see [12, Corollary 2.1]), we have

$$f(\mathrm{ch}x) = \lim_{r\to 0}\frac{1}{|H(0,r)|_\lambda}\int_{H(0,r)} A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x)\mathrm{sh}^{2\lambda} y dy \le \beta$$

for a.e. $x \notin \bigcup_j H_j$. It remains to prove (2). Passing to the limit by $n \to \infty$ in the inequality

$$\left|\bigcup_{j=1,2,\ldots,n} H_j\right|_\lambda \le \sum_{j=1}^n |H_j|_\lambda \le \frac{1}{\beta}\sum_{j=1}^n \int_{H_j} f(\mathrm{ch}x)\mathrm{sh}^{2\lambda} x dx,$$

which is contained in the proof of Theorem 2.4 in [12], we obtain the assertion (2). □

Remark 2.5. The Calderon–Zygmund decomposition stay valid if we replace $\mathbb{R}_+$ by a fixed interval H_0 for $f \in L_{p,\lambda}(H_0)$.

2.2. **The John-Nirenberg type theorem.** Having stated some basic facts about BMO_G, we now turn to a deeper property of BMO_G functions, that is, their exponential integrability. As we saw in Example 2.5, the function $f(\mathrm{ch}x) = \log(\mathrm{sh}x)$ is in BMO_G.

This function is exponentially integrable over any segment $[a, b]$ of $\mathbb{R}_+$ in the sense that

$$\int_a^b e^{|f(\mathrm{ch}x)|}\mathrm{sh}^{2\lambda}x dx < \infty.$$

It turns out that this is a general property of BMO_G functions, and this is the content of the next theorem.

Theorem 2.6. *For all $f \in BMO_G(\mathbb{R}_+)$, for all interval $H = H(0, r)$ and $\alpha > 0$, we have*

$$\begin{aligned} &|\{x \in H : |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)| > \alpha\}|_\lambda \\ &\le e|H|e^{-\frac{A\alpha}{\|f\|_{BMO_G}}} \text{ with } A = \left(2^{(2\lambda+1)n}e\right)^{-1}. \end{aligned}$$

The proof of this theorem is based on the Calderon–Zygmund decomposition and is the same as that of Theorem 7.1.6 in [9].

Corollary 2.7. *For all $0 < p < \infty$ and $H = H(0, r)$, one has*

$$\sup_{r>0}\left(\frac{1}{|H|_\lambda}\int_H |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)|^p \mathrm{sh}^{2\lambda}y dy\right)^{\frac{1}{p}} \lesssim \|f\|_{BMO_G}. \tag{2.5}$$

Proof. In fact

$$\begin{aligned} &\frac{1}{|H|_\lambda}\int_H |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)|^p \mathrm{sh}^{2\lambda}y dy \\ =&\frac{p}{|H|_\lambda}\int_0^\infty \Bigg(\int_0^{|A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)|} \alpha^{p-1}dx\Bigg)\mathrm{sh}^{2\lambda}y dy \\ =&\frac{p}{|H|_\lambda}\int_0^\infty \alpha^{p-1}\Bigg(\int_{\{x\in H:|A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)|>\alpha\}} \mathrm{sh}^{2\lambda}y dy\Bigg)dx \\ =&\frac{p}{|H|_\lambda}\int_0^\infty \alpha^{p-1}\left|\{x \in H : |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)| > \alpha\}\right|_\lambda dx \\ \le&\frac{p}{|H|_\lambda}e|H|_\lambda\int_0^\infty \alpha^{p-1}e^{-\frac{A\alpha}{\|f\|_{BMO_G}}}dx \\ =&pe\frac{\Gamma(p)}{A^p}\|f\|_{BMO_G} = \frac{e}{A^p}\Gamma(p+1)\|f\|_{BMO_G}, \end{aligned}$$

where $A = \left(e^{(2\lambda+1)n}e\right)^{-1}$. □

Since inequality (2.7) can be reversed for $p > 1$ via Hölder's inequality (see [14, Theorem 3.3]), we obtain the following important $L_{p,\lambda}$ characterization of BMO_G norms.

Corollary 2.8. *For all $1 < p < \infty$, we have*

$$\sup_{x,r\in\mathbb{R}_+}\left(\frac{1}{|H|_\lambda}\int_H |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)|^p \mathrm{sh}^{2\lambda}y dy\right)^{\frac{1}{p}} \approx \|f\|_{BMO_G}.$$

3. Commutators of Gegenbauer Fractional Integrals

In this section we study the $(L_{p,\lambda}, L_{q,\lambda})$ boundedness of commutators of the Gegenbauer fractional integrals J_G^α, where

$$J_G^\alpha f(\mathrm{ch}x) = \int_0^\infty \frac{A_{\mathrm{ch}y}^\lambda f(\mathrm{ch}x)}{(\mathrm{sh}y)^{\gamma-\alpha}} \mathrm{sh}^{2\lambda} y dy, \ \alpha < \gamma \le 2\lambda + 1.$$

We will also illustrate that the boundedness of commutators of J_G^α may characterize the $BMO_G(\mathbb{R}_+)$ spaces. First, we will give some related results. Suppose that $b \in L_{1,\lambda}^{\mathrm{loc}}(\mathbb{R}_+)$, then the commutator generated by the function b and the J_G^α is defined as follows:

$$\begin{aligned} J_G^{b,\alpha} f(\mathrm{ch}x) &= b(\mathrm{ch}x) J_G^\alpha f(\mathrm{ch}x) - J_G^\alpha (bf)(\mathrm{ch}x) \\ &= \int_0^\infty \frac{[A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b(\mathrm{ch}x)]}{(\mathrm{sh}y)^{\gamma-\alpha}} A_{\mathrm{ch}y}^\lambda f(\mathrm{ch}x) \mathrm{sh}^{2\lambda} y dy. \end{aligned}$$

This implies that

$$\begin{aligned} J_G^{b,\alpha} f(\mathrm{ch}x) = \lim_{r \to 0} \Bigg\{ [b_H(\mathrm{ch}x) - b(\mathrm{ch}x)] \int_r^\infty \frac{A_{\mathrm{ch}y}^\lambda f(\mathrm{ch}x)}{(\mathrm{sh}y)^{\gamma-\alpha}} \mathrm{sh}^{2\lambda} y dy \\ + \int_r^\infty \frac{A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)}{(\mathrm{sh}y)^{\gamma-\alpha}} A_{\mathrm{ch}y}^\lambda f(\mathrm{ch}x) \mathrm{sh}^{2\lambda} y dy \Bigg\}, \end{aligned}$$

where $H = H(0, r)$.

Since $b \in BMO_G(\mathbb{R}_+)$, by Theorem 4.1 and Corollary 2.1 in [12], the first term tends to zero a.e. and

$$J_G^{b,\alpha} f(\mathrm{ch}x) = \int_0^\infty \frac{[A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)]}{(\mathrm{sh}y)^{\gamma-\alpha}} A_{\mathrm{ch}y}^\lambda f(\mathrm{ch}x) \mathrm{sh}^{2\lambda} y dy.$$

The $k-th$ order commutator of the J_G^α we define as follows:

$$J_G^{b,\alpha,k} f(\mathrm{ch}x) = \int_0^\infty \frac{[A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)]^k}{(\mathrm{sh}y)^{\gamma-\alpha}} A_{\mathrm{ch}y}^\lambda f(\mathrm{ch}x) \mathrm{sh}^{2\lambda} y dy.$$

Theorem 3.1. *Suppose that $0 < \alpha < \gamma \le 2\lambda + 1$, $1 < p < \frac{\gamma}{\alpha}$ and let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$. Then $J_G^{b,\alpha,k}$ is bounded from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$, if and only if $b \in BMO_G(\mathbb{R}_+)$.*

Proof. Sufficiency. Let $0 < \alpha < \gamma \le 2\lambda + 1$, $1 < p < \frac{\gamma}{\alpha}$ and $b \in BMO_G(\mathbb{R}_+)$, we get

$$\begin{aligned} J_G^{b,\alpha,k} f(\mathrm{ch}x) &= \left(\int_0^r + \int_r^\infty \right) \frac{[A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)]^k}{(\mathrm{sh}y)^{\gamma-\alpha}} A_{\mathrm{ch}y}^\lambda f(\mathrm{ch}x) \mathrm{sh}^{2\lambda} y dy \\ &= J_1(r) + J_2(r). \end{aligned} \tag{3.1}$$

Consider $J_1(r)$. By Hölder's inequality, we have

$$\begin{aligned} |J_1(r)| &\le \left(\int_0^r \frac{|A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)|^{kq}}{(\mathrm{sh}y)^{\gamma-\alpha}} \mathrm{sh}^{2\lambda} y dy \right)^{\frac{1}{q}} \left(\int_0^r \frac{A_{\mathrm{ch}y}^\lambda |f(\mathrm{ch}x)|^p}{(\mathrm{sh}y)^{\gamma-\alpha}} \mathrm{sh}^{2\lambda} y dy \right)^{\frac{1}{p}} \\ &= J_{1.1}(r) \cdot J_{1.2}(r). \end{aligned} \tag{3.2}$$

We estimate $J_{1.1}(r)$. One has

$$J_{1.1}(r) \leq \Bigg(\sum_{k=0}^{\infty} \int\limits_{2^{-(k+1)}r}^{2^{-k}r} \frac{|A^{\lambda}_{\text{ch}y} b(\text{ch}x) - b_H(\text{ch}x)|^{kq}}{(\text{sh}y)^{\gamma-\alpha}} \text{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{q}}$$

$$\leq \Bigg(\sum_{k=0}^{\infty} \frac{(\text{sh}\frac{r}{2^{k+1}})^{\alpha}}{(\text{sh}\frac{r}{2^{k+1}})^{\gamma}} \int\limits_{0}^{2^{-k}r} |A^{\lambda}_{\text{ch}y} b(\text{ch}x) - b_H(\text{ch}x)|^{kq} \text{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{q}}$$

$$\leq (\text{sh}r)^{\frac{\alpha}{q}} \|b\|^k_{BMO_G} \Bigg(\sum_{k=0}^{\infty} 2^{-(k+1)\alpha} \Bigg) \lesssim (\text{sh}r)^{\frac{\alpha}{q}} \|b\|_{BMO_G}. \tag{3.3}$$

Now we estimate $J_{1.2}(r)$. One has

$$J_{1.2}(r) \leq \Bigg(\sum_{k=0}^{\infty} \frac{1}{(\text{sh}\frac{r}{2^{k+1}})^{\gamma-\alpha}} \int\limits_{2^{-(k+1)}r}^{2^{-k}r} A^{\lambda}_{\text{ch}y} |f(\text{ch}x)|^p \text{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{p}}$$

$$\leq (\text{sh}r)^{\frac{\alpha}{p}} \left(M_G |f(\text{ch}x)|^p \right)^{\frac{1}{p}}. \tag{3.4}$$

Taking into account (3.3) and (3.4) in (3.2), we get

$$|J_1(r)| \lesssim (\text{sh}r)^{\alpha} \|b\|^k_{BMO_G} \left(M_G |f(\text{ch}x)|^p \right)^{\frac{1}{p}}.$$

Consider $J_2(r)$. By Hölder's inequality, we have

$$|J_2(r)| \leq \int\limits_{r}^{\infty} |A^{\lambda}_{\text{ch}y} b(\text{ch}x) - b_H(\text{ch}x)|^k \frac{A^{\lambda}_{\text{ch}y} |f(\text{ch}x)|}{(\text{sh}y)^{\gamma-\alpha}} \text{sh}^{2\lambda} y dy$$

$$\leq \Bigg(\int\limits_{r}^{\infty} \frac{|A^{\lambda}_{\text{ch}y} b(\text{ch}x) - b_H(\text{ch}x)|^{kq}}{(\text{sh}y)^{(\gamma-\alpha)q}} \text{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{q}}$$

$$\times \Bigg(\int\limits_{r}^{\infty} A^{\lambda}_{\text{ch}y} |f(\text{ch}x)|^p \text{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{p}} \leq J_2'(r) \|f\|_{L_{p,\lambda}}. \tag{3.5}$$

For $J_2'(r)$, we have

$$J_2'(r) \leq \Bigg(\sum_{k=0}^{\infty} \int\limits_{2^k r}^{2^{k+1}r} \frac{|A^{\lambda}_{\text{ch}y} b(\text{ch}x) - b_H(\text{ch}x)|^{kq}}{(\text{sh}y)^{(\gamma-\alpha)q}} \text{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{q}}$$

$$\leq \Bigg(\sum_{k=0}^{\infty} \frac{(\text{sh}2^k)^{\gamma-(\gamma-\alpha)q}}{(\text{sh}2^k)^{\gamma}} \int\limits_{0}^{2^{k+1}r} |A^{\lambda}_{\text{ch}y} b(\text{ch}x) - b_H(\text{ch}x)|^{kq} \text{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{q}}.$$

By property (5), we have $|b_H(\text{ch}x) - b_{2^k H}(\text{ch}x)| \leq 2k\|b\|_{BMO_G}$. Then

$$J_2'(r) \lesssim (\text{sh}r)^{\frac{\gamma}{q}+\alpha-\gamma} \Bigg(\sum_{k=0}^{\infty} \frac{(2^k)^{\gamma-(\gamma-\alpha)q}}{(\text{sh}2^k)^{\gamma}} \int\limits_{0}^{2^{k+1}r} |A^{\lambda}_{\text{ch}y} b(\text{ch}x) - b_{2^k H}(\text{ch}x)|^{kq} \text{sh}^{2\lambda} y dy$$

$$+ \sum_{k=0}^{\infty} \frac{(2^k)^{\gamma-(\gamma-\alpha)q}}{(\text{sh}2^k)^{\gamma}} \int\limits_{0}^{2^{k+1}r} |b_H(\text{ch}x) - b_{2^k H}(\text{ch}x)|^{kq} \text{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{q}}$$

$$\lesssim (\text{sh}r)^{\alpha-\frac{\gamma}{p}} \|b\|_{BMO_G} \Bigg(\sum_{k=0}^{\infty} \frac{k}{(2^k)^{(\gamma-\alpha)q-\gamma}} \Bigg)^{\frac{1}{q}} \lesssim (\text{sh}r)^{\alpha-\frac{\gamma}{q}} \|b\|_{BMO_G}, \tag{3.6}$$

since

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma} \Leftrightarrow \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma} \Leftrightarrow \frac{1}{q} = \frac{\gamma - \alpha p}{\gamma p}$$
$$\Leftrightarrow q = \frac{\gamma p}{\gamma - \alpha p} \Leftrightarrow (\gamma - \alpha)q - \gamma = \frac{(\gamma - \alpha)\gamma p}{\gamma - \alpha p} - \gamma > 0 \Leftrightarrow \frac{(\gamma - \alpha)\gamma p}{\gamma - \alpha p} > \gamma$$
$$\Leftrightarrow (\gamma - \alpha)p > \gamma - \gamma p \Leftrightarrow \gamma p > \gamma \Leftrightarrow p > 1.$$

From (3.6) and (3.5), we have

$$|J_2(r)| \lesssim (\mathrm{sh} r)^{\alpha - \frac{\gamma}{q}} \|f\|_{L_{p,\lambda}} \|b\|_{BMO_G}. \tag{3.7}$$

Taking into account (3.5) and (3.7) in (3.1), we obtain

$$|J_G^{b,\alpha,k} f(\mathrm{ch} x)| \lesssim \left[(\mathrm{sh} r)^\alpha \left(M_G |f(\mathrm{ch} x)|^p\right)^{\frac{1}{p}} + (\mathrm{sh} r)^{\alpha - \frac{\gamma}{p}} \|f\|_{L_{p,\lambda}}\right] \|b\|^k_{BMO_G}.$$

The right-hand side attains its minimum for

$$\mathrm{sh} r = \left(\frac{\gamma - \alpha p}{\alpha} \frac{\|f\|_{L_{p,\lambda}}}{(M_G |f|^p(\mathrm{ch} x))^{\frac{1}{p}}}\right)^{\frac{p}{\gamma}},$$

and we have

$$\begin{aligned} |J_G^{b,\alpha,k} f(\mathrm{ch} x)| &\lesssim \left\{ \left[\frac{\|f\|_{L_{p,\lambda}}}{(M_G |f|^p(\mathrm{ch} x))^{\frac{1}{p}}}\right]^{\frac{\alpha p}{\gamma}} (M_G |f|^p(\mathrm{ch} x))^{\frac{1}{p}} \right. \\ &\quad \left. + \left[\frac{\|f\|_{L_{p,\lambda}}}{(M_G |f|^p(\mathrm{ch} x))^{\frac{1}{p}}}\right]^{-\frac{p}{q}} \|f\|_{L_{p,\lambda}} \right\} \|b\|_{BMO_G} \\ &= (M_G |f|^p(\mathrm{ch} x))^{\frac{1}{p}} \|f\|^{1 - \frac{p}{q}}_{L_{p,\lambda}} \|b\|^k_{BMO_G}, \end{aligned}$$

since

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma} \Leftrightarrow 1 - \frac{p}{q} = \frac{\alpha p}{\gamma}.$$

From this and Theorem 2.2 in [12], we have

$$\begin{aligned} \int_0^\infty |J_G^{b,\alpha,k} f(\mathrm{ch} x)|^q \mathrm{sh}^{2\lambda} x dx &\lesssim \|M_G |f|^p\|_{L_{p,\lambda}} \|f\|^{q-p}_{L_{p,\lambda}} \|b\|^{kq}_{BMO_G} \\ &\lesssim \|f\|^q_{L_{p,\lambda}} \|b\|^{kq}_{BMO_G}. \end{aligned}$$

Thus, we obtain

$$\|J_G^{b,\alpha,k} f(\mathrm{ch} x)\|_{L_{q,\lambda}} \lesssim \|f\|_{L_{p,\lambda}} \|b\|^k_{BMO_G}.$$

Necessity. Let $1 < p < \frac{\gamma}{\alpha}$, $f \in L_{p,\lambda}(\mathbb{R}_+)$, and let $J_G^{b,\alpha,k}$ act boundedly from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$, i.e.,

$$\|J_G^{b,\alpha,k} f(\mathrm{ch} x)\|_{L_{q,\lambda}} \lesssim \|f\|_{L_{p,\lambda}}. \tag{3.8}$$

In what follows, the function f will be assumed positive and monotonically increasing. The dilation function $f_t(\mathrm{ch} x)$ will be defined as follows:

$$\begin{aligned} f(\mathrm{ch}(\mathrm{th} t)x) \le f_t(\mathrm{ch} x) \le f(\mathrm{ch}(\mathrm{cth} t)x), &\quad 0 < t < 1, \\ f(\mathrm{ch}(\mathrm{th} t)x) \le f_t(\mathrm{ch} x) \le f(\mathrm{ch}(\mathrm{sh} t)x), &\quad 1 \le t < \infty. \end{aligned} \tag{3.9}$$

Using (3.9) for $0 < t < 1$, we obtain

$$\|f_t\|_{L_{p,\lambda}} = \left(\int_0^\infty |f_t(\mathrm{ch} x)|^p \mathrm{sh}^{2\lambda} x dx\right)^{\frac{1}{p}} \le \left(\int_0^\infty |f(\mathrm{ch}(\mathrm{cth} t)x)|^p \mathrm{sh}^{2\lambda} x dx\right)^{\frac{1}{p}}$$

$[(\mathrm{cth} t)x = u, x = (\mathrm{th} t)u]$

$$
\begin{aligned}
&=(\mathrm{th}t)^{\frac{1}{p}}\left(\int\limits_0^\infty |f(\mathrm{ch}u)|^p\mathrm{sh}^{2\lambda}(\mathrm{th}t)udu\right)^{\frac{1}{p}}\\
&\leq(\mathrm{th}t)^{\frac{2\lambda+1}{p}}\left(\int\limits_0^\infty |f(\mathrm{ch}u)|^p\mathrm{sh}^{2\lambda}udu\right)^{\frac{1}{p}}\\
&=(\mathrm{th}t)^{\frac{2\lambda+1}{p}}\|f\|_{L_{p,\lambda}}=\left(\frac{\mathrm{sh}t}{\mathrm{ch}t}\right)^{\frac{2\lambda+1}{p}}\|f\|_{L_{p,\lambda}}\\
&\lesssim\frac{1}{(\mathrm{ch}t)^{\frac{2\lambda+1}{p}-(\alpha+\frac{2\lambda+1-\gamma}{p})}}\|f\|_{L_{p,\lambda}}\lesssim(\mathrm{sh}t)^{\alpha-\frac{\gamma}{q}}\|f\|_{L_{p,\lambda}}.
\end{aligned}\tag{3.10}
$$

On the other hand,

$$
\|f_t\|_{L_{p,\lambda}}=\left(\int\limits_0^\infty |f_t(\mathrm{ch}x)|^p\mathrm{sh}^{2\lambda}xdx\right)^{\frac{1}{p}}\geq\left(\int\limits_0^\infty |f(\mathrm{ch}(\mathrm{th}t)x)|^p\mathrm{sh}^{2\lambda}xdx\right)^{\frac{1}{p}}
$$

$[(\mathrm{th}t)x=u, x=(\mathrm{cth}t)u]$

$$
\begin{aligned}
&=(\mathrm{cth}t)^{\frac{1}{p}}\left(\int\limits_0^\infty |f(\mathrm{ch}u)|^p\mathrm{sh}^{2\lambda}(\mathrm{cth}t)udu\right)^{\frac{1}{p}}\\
&\leq(\mathrm{cth}t)^{\frac{2\lambda+1}{p}}\left(\int\limits_0^\infty |f(\mathrm{ch}u)|^p\mathrm{sh}^{2\lambda}udu\right)^{\frac{1}{p}}\\
&=(\mathrm{cth}t)^{\frac{2\lambda+1}{p}}\|f\|_{L_{p,\lambda}}=\left(\frac{\mathrm{ch}t}{\mathrm{sh}t}\right)^{\frac{2\lambda+1}{p}}\|f\|_{L_{p,\lambda}}\\
&\lesssim\frac{1}{(\mathrm{sh}t)^{\frac{2\lambda+1}{p}-(\alpha+\frac{2\lambda+1-\gamma}{p})}}\|f\|_{L_{p,\lambda}}\lesssim(\mathrm{sh}t)^{\alpha-\frac{\gamma}{q}}\|f\|_{L_{p,\lambda}}.
\end{aligned}\tag{3.11}
$$

From (3.10) and (3.11), we have

$$
\|f_t\|_{L_{p,\lambda}}\approx(\mathrm{sh}t)^{\alpha-\frac{\gamma}{q}}\|f\|_{L_{p,\lambda}},\quad 0<t<1.\tag{3.12}
$$

Now let $1\leq t<\infty$. Then from (3.9), we have

$$
\|f_t\|_{L_{p,\lambda}}=\left(\int\limits_0^\infty |f_t(\mathrm{ch}x)|^p\mathrm{sh}^{2\lambda}xdx\right)^{\frac{1}{p}}\geq\left(\int\limits_0^\infty |f(\mathrm{ch}(\mathrm{th}t)x)|^p\mathrm{sh}^{2\lambda}xdx\right)^{\frac{1}{p}}
$$

$[(\mathrm{th}t)x=u, x=(\mathrm{cth}t)u]$

$$
\begin{aligned}
&=(\mathrm{cth}t)^{\frac{1}{p}}\left(\int\limits_0^\infty |f(\mathrm{ch}u)|^p\mathrm{sh}^{2\lambda}(\mathrm{cth}t)udu\right)^{\frac{1}{p}}\\
&\leq(\mathrm{cth}t)^{\frac{2\lambda+1}{p}}\left(\int\limits_0^\infty |f(\mathrm{ch}u)|^p\mathrm{sh}^{2\lambda}udu\right)^{\frac{1}{p}}\\
&\lesssim(\mathrm{sh}t)^{\alpha-\frac{\gamma}{q}}\|f\|_{L_{p,\lambda}}.
\end{aligned}\tag{3.13}
$$

On the other hand,

$$
\|f_t\|_{L_{p,\lambda}}\leq\left(\int\limits_0^\infty |f_t(\mathrm{ch}(\mathrm{sh}t)x)|^p\mathrm{sh}^{2\lambda}xdx\right)^{\frac{1}{p}}
$$

$[(\mathrm{sh}t)x=u, x=\frac{u}{\mathrm{sh}t}]$

$$=(\mathrm{sh}t)^{-\frac{1}{p}}\left(\int\limits_0^\infty |f(\mathrm{ch}u)|^p \mathrm{sh}^{2\lambda}\frac{u}{\mathrm{sh}t}du\right)^{\frac{1}{p}}$$

$$\leq(\mathrm{sh}t)^{-\frac{2\lambda+1}{p}}\left(\int\limits_0^\infty |f(\mathrm{ch}u)|^p \mathrm{sh}^{2\lambda}udu\right)^{\frac{1}{p}}$$

$$\leq(\mathrm{sh}t)^{\alpha+\frac{2\lambda+1-\gamma}{p}-\frac{2\lambda+1}{p}}\|f\|_{L_{p,\lambda}}=(\mathrm{sh}t)^{\alpha-\frac{\gamma}{q}}\|f\|_{L_{p,\lambda}}. \tag{3.14}$$

From (3.13) and (3.14), we have

$$\|f_t\|_{L_{p,\lambda}}\approx(\mathrm{sh}t)^{\alpha-\frac{\gamma}{q}}\|f\|_{L_{p,\lambda}},\ 1\leq t<\infty. \tag{3.15}$$

From (3.12) and (3.15),

$$\|f_t\|_{L_{p,\lambda}}\approx(\mathrm{sh}t)^{\alpha-\frac{\gamma}{q}},\ 0<t<\infty. \tag{3.16}$$

Further, from (3.9) for $0<t<1$, we have

$$\|J_G^{b,\alpha,k}f_t\|_{L_{q,\lambda}}=\left(\int\limits_0^\infty |J_G^{b,\alpha,k}f_t(\mathrm{ch}x)|^q\mathrm{sh}^{2\lambda}xdx\right)^{\frac{1}{q}}$$

$$\leq\left(\int\limits_0^\infty |J_G^{b,\alpha,k}f(\mathrm{ch}(\mathrm{cth}t)x)|^q\mathrm{sh}^{2\lambda}xdx\right)^{\frac{1}{q}}$$

$$[(\mathrm{cth}t)x=u, x=(\mathrm{th}t)u]$$

$$=(\mathrm{th}t)^{\frac{1}{q}}\left(\int\limits_0^\infty |J_G^{b,\alpha,k}f(\mathrm{ch}u)|^q\mathrm{sh}^{2\lambda}(\mathrm{th}t)udu\right)^{\frac{1}{q}}$$

$$\leq(\mathrm{th}t)^{\frac{2\lambda+1}{q}}\left(\int\limits_0^\infty |J_G^{b,\alpha,k}f(\mathrm{ch}u)|^q\mathrm{sh}^{2\lambda}udu\right)^{\frac{1}{q}}$$

$$=(\mathrm{th}t)^{\frac{2\lambda+1}{q}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}=\left(\frac{\mathrm{sh}t}{\mathrm{ch}t}\right)^{\frac{2\lambda+1}{q}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}$$

$$\lesssim\frac{1}{(\mathrm{ch}t)^{\frac{2\lambda+1}{q}}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}\lesssim\frac{1}{(\mathrm{ch}t)^{\frac{\gamma}{q}}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}$$

$$\lesssim(\mathrm{sh}t)^{-\frac{\gamma}{q}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}. \tag{3.17}$$

On the other hand,

$$\|J_G^{b,\alpha,k}f_t\|_{L_{q,\lambda}}\geq\left(\int\limits_0^\infty |J_G^{b,\alpha,k}f(\mathrm{ch}(\mathrm{th}t)x)|^q\mathrm{sh}^{2\lambda}xdx\right)^{\frac{1}{q}}$$

$$[(\mathrm{th}t)x=u, x=(\mathrm{cth}t)u]$$

$$=(\mathrm{cth}t)^{\frac{1}{q}}\left(\int\limits_0^\infty |J_G^{b,\alpha,k}f(\mathrm{ch}u)|^q\mathrm{sh}^{2\lambda}(\mathrm{cth}t)udu\right)^{\frac{1}{q}}$$

$$\geq(\mathrm{cth}t)^{\frac{2\lambda+1}{q}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}=\left(\frac{\mathrm{ch}t}{\mathrm{sh}t}\right)^{\frac{2\lambda+1}{q}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}$$

$$\geq\frac{1}{(\mathrm{sh}t)^{\frac{2\lambda+1}{q}}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}\lesssim(\mathrm{sh}t)^{-\frac{\gamma}{q}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}. \tag{3.18}$$

From (3.17) and (3.18), we have

$$\|J_G^{b,\alpha,k}f_t\|_{L_{q,\lambda}}\approx(\mathrm{sh}t)^{-\frac{\gamma}{q}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}},\quad 0<t<1. \tag{3.19}$$

Now let $1 \leq t < \infty$. Then from (3.9), we get

$$\|J_G^{b,\alpha,k} f_t\|_{L_{q,\lambda}} \geq \left(\int_0^\infty |J_G^{b,\alpha,k} f(\mathrm{ch}(\mathrm{sh}t)x)|^q \mathrm{sh}^{2\lambda} x dx \right)^{\frac{1}{q}}$$

$$[(\mathrm{sh}t)x = u, x = \frac{u}{\mathrm{sh}t}]$$

$$=(\mathrm{sh}t)^{-\frac{1}{q}} \left(\int_0^\infty |J_G^{b,\alpha,k} f(\mathrm{ch}u)|^q \mathrm{sh}^{2\lambda} \frac{u}{\mathrm{sh}t} du \right)^{\frac{1}{q}}$$

$$\leq (\mathrm{sh}t)^{-\frac{2\lambda+1}{q}} \|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}} \leq (\mathrm{sh}t)^{-\frac{\gamma}{q}} \|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}}. \tag{3.20}$$

On the other hand,

$$\|J_G^{b,\alpha,k} f_t\|_{L_{q,\lambda}} \geq \left(\int_0^\infty |J_G^{b,\alpha,k} f(\mathrm{ch}(\mathrm{th}t)x)|^q \mathrm{sh}^{2\lambda} x dx \right)^{\frac{1}{q}}$$

$$[(\mathrm{th}t)x = u, x = (\mathrm{cth}t)u]$$

$$=(\mathrm{cth}t)^{\frac{1}{q}} \left(\int_0^\infty |J_G^{b,\alpha,k} f(\mathrm{ch}u)|^q \mathrm{sh}^{2\lambda} (\mathrm{cth}t) u du \right)^{\frac{1}{q}}$$

$$\geq (\mathrm{cth}t)^{\frac{2\lambda+1}{q}} \|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}} \geq (\mathrm{sh}t)^{-\frac{\gamma}{q}} \|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}}. \tag{3.21}$$

From (3.20) and (3.21), we have

$$\|J_G^{b,\alpha,k} f_t\|_{L_{q,\lambda}} \approx (\mathrm{sh}t)^{-\frac{\gamma}{q}} \|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}},\ 1 \leq t < \infty. \tag{3.22}$$

Combining (3.19) and (3.22), we obtain

$$\|J_G^{b,\alpha,k} f_t\|_{L_{q,\lambda}} \approx (\mathrm{sh}t)^{-\frac{\gamma}{q}} \|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}},\ 0 < t < \infty. \tag{3.23}$$

Taking into account inequality (3.8), as well as (3.23) and (3.16), we obtain

$$\|J_G^{b,\alpha,k} f_t\|_{L_{q,\lambda}} \approx (\mathrm{sh}t)^{\frac{\gamma}{q}} \|J_G^{b,\alpha,k} f_t\|_{L_{q,\lambda}}$$

$$\lesssim (\mathrm{sh}t)^{\frac{\gamma}{q}} \|f_t\|_{L_{q,\lambda}} \lesssim (\mathrm{sh}t)^{\alpha - \frac{\gamma}{q} + \frac{\gamma}{q}} \|f\|_{L_{q,\lambda}} = (\mathrm{sh}t)^{\alpha - \gamma(\frac{1}{q} - \frac{1}{q})} \|f\|_{L_{q,\lambda}}.$$

If $\frac{1}{q} - \frac{1}{q} < \frac{\alpha}{\gamma}$, then, as $t \to 0$, we have

$$\|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}} = 0 \text{ for all } f \in L_{q,\lambda}(\mathbb{R}_+).$$

If $\frac{1}{q} - \frac{1}{q} > \frac{\alpha}{\gamma}$ then, as $t \to \infty$,

$$\|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}} = 0 \text{ for all } f \in L_{q,\lambda}(\mathbb{R}_+),$$

which cannot be true.

Therefore,

$$\frac{1}{q} - \frac{1}{q} = \frac{\alpha}{\gamma}.$$

□

4. Commutators of the Gegenbauer Fractional Maximal Operator

Let $b \in L_{1,\lambda}^{\mathrm{loc}}(\mathbb{R}_+)$, then the $k-th$ order commutator $M_G^{b,\alpha,k}$ generated by the function b and M_G^α is defined as follows:

$$M_G^{b,\alpha,k} f(\mathrm{ch}x) =$$

$$= \sup_{r \in \mathbb{R}_+} \frac{1}{|H|_\lambda^{1-\frac{\alpha}{\gamma}}} \int_H |A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)|^k A_{\mathrm{ch}y}^\lambda |f(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy, k = 1, 2, \ldots,$$

where $H = H(0, r)$.

Theorem 4.1. *Suppose that $0<\alpha<\gamma\leq 2\lambda+1$, $1<p<\frac{\gamma}{\alpha}$ and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{\gamma}$. Then the commutator $M_G^{b,\alpha,k}$ is bounded from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$, if and only if $b\in BMO_G(\mathbb{R}_+)$.*

Proof. Let $b\in BMO_G(\mathbb{R}_+)$. For the fixed $x\in\mathbb{R}_+$ and $r>0$, we have

$$J_G^{b,\alpha,k}|f(\mathrm{ch}x)|=\int\limits_{\mathbb{R}_+}\frac{|A^\lambda_{\mathrm{ch}y}b(\mathrm{ch}x)-b_H(\mathrm{ch}x)|^k}{(\mathrm{sh}y)^{\gamma-\alpha}}A^\lambda_{\mathrm{ch}y}|f(\mathrm{ch}x)|\mathrm{sh}^{2\lambda}ydy$$

$$\geq\int\limits_0^r\frac{|A^\lambda_{\mathrm{ch}y}b(\mathrm{ch}x)-b_H(\mathrm{ch}x)|^k}{(\mathrm{sh}y)^{\gamma-\alpha}}A^\lambda_{\mathrm{ch}y}|f(\mathrm{ch}x)|\mathrm{sh}^{2\lambda}ydy$$

$$\geq\frac{1}{(\mathrm{sh}y)^{\gamma-\alpha}}\int\limits_0^r|A^\lambda_{\mathrm{ch}y}b(\mathrm{ch}x)-b_H(\mathrm{ch}x)|^kA^\lambda_{\mathrm{ch}y}|f(\mathrm{ch}x)|\mathrm{sh}^{2\lambda}ydy$$

$$\approx\frac{1}{|H|_\lambda^{1-\frac{\alpha}{\gamma}}}\int\limits_H|A^\lambda_{\mathrm{ch}y}b(\mathrm{ch}x)-b_H(\mathrm{ch}x)|^kA^\lambda_{\mathrm{ch}y}|f(\mathrm{ch}x)|\mathrm{sh}^{2\lambda}ydy. \tag{4.1}$$

Taking supremum for $r>0$ on both sides of (4.1), we obtain

$$M_G^{b,\alpha,k}f(\mathrm{ch}x)\lesssim J_G^{b,\alpha,k}(|f|)(\mathrm{ch}x),\quad\forall\in\mathbb{R}_+.$$

Thus, when $b\in BMO_G(\mathbb{R}_+)$, from this and Theorem 3.1, we have

$$\|M_G^{b,\alpha,k}f(\mathrm{ch}x)\|_{L_{q,\lambda}(\mathbb{R}_+)}\lesssim\|f\|_{L_{p,\lambda}(\mathbb{R}_+)}.$$

On the other hand, suppose that $M_G^{b,\alpha,k}$ is bounded from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$. Choose any interval H in $\mathbb{R}_+$,

$$\frac{1}{|H|_\lambda}\int\limits_H|A^\lambda_{\mathrm{ch}y}b(\mathrm{ch}x)-b_H(\mathrm{ch}x)|\mathrm{sh}^{2\lambda}ydy$$

$$\approx\frac{1}{|H|^2_\lambda}\int\limits_H|A^\lambda_{\mathrm{ch}y}b(\mathrm{ch}x)-b_H(\mathrm{ch}x)|\mathrm{sh}^{2\lambda}ydy\cdot\int\limits_HA^\lambda_{\mathrm{ch}y}\chi_H(\mathrm{ch}x)\mathrm{sh}^{2\lambda}xdx$$

$$\approx\frac{1}{|H|_\lambda^{1+\frac{\alpha}{\gamma}}}\int\limits_H\left(\frac{1}{|H|_\lambda^{1-\frac{\alpha}{\gamma}}}\int\limits_H|A^\lambda_{\mathrm{ch}y}b(\mathrm{ch}x)-b_H(\mathrm{ch}x)|\cdot A^\lambda_{\mathrm{ch}y}\chi_H(\mathrm{ch}x)\mathrm{sh}^{2\lambda}xdx\right)\mathrm{sh}^{2\lambda}ydy$$

$$\approx\frac{1}{|H|_\lambda^{1+\frac{\alpha}{\gamma}}}\int\limits_HM_G^{b,\alpha}(\chi_H(\mathrm{ch}x))\mathrm{sh}^{2\lambda}ydy$$

$$\lesssim\frac{1}{|H|_\lambda^{1+\frac{\alpha}{\gamma}}}\left(\int\limits_H\mathrm{sh}^{2\lambda}ydy\right)^{\frac{1}{q'}}\left(\int\limits_HM_G^{b,\alpha}(\chi_H(\mathrm{ch}x))\mathrm{sh}^{2\lambda}ydy\right)^{\frac{1}{q}}$$

$$\lesssim\frac{1}{|H|_\lambda^{1+\frac{\alpha}{\gamma}}}|H_\lambda|^{\frac{1}{q'}}\|M_G^{b,\alpha}\chi_H\|_{L_{q,\lambda}(H)}\lesssim\frac{1}{|H|_\lambda^{1+\frac{\alpha}{\gamma}}}|H_\lambda|^{\frac{1}{q'}}\|\chi_H\|_{L_{q,\lambda}(H)}$$

$$\lesssim\frac{1}{|H|_\lambda^{1+\frac{\alpha}{\gamma}}}|H_\lambda|^{\frac{1}{q'}}|H_\lambda|^{\frac{1}{p}}\lesssim 1.$$

Thus $b\in BMO_G(\mathbb{R}_+)$. □

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SOME MODULAR INEQUALITIES IN LEBESGUE SPACES WITH A VARIABLE EXPONENT

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Abstract. Our aim is to study the modular inequalities for some operators, for example, the Bergman projection in Lebesgue spaces with a variable exponent. Under proper assumptions on the variable exponent, we prove that the modular inequalities hold, if and only if the exponent almost everywhere is equal to a constant. In order to get the main results, we establish a lower pointwise bound for these operators of a characteristic function.

1. Introduction

The study on variable exponent analysis has been rapidly developed after the work [18] where Kováčik and Rákosník have established fundamental properties of variable Lebesgue spaces (see also [4, 14, 21]). In particular, the theory of variable function spaces in connection with the boundedness of the Hardy–Littlewood maximal operator M has been deeply studied. Cruz-Uribe, Fiorenza and Neugebauer [6, 7] and Diening [9] have independently obtained the log-Hölder continuous conditions that guarantee the boundedness of M on variable Lebesgue spaces. We also note that the recent development of variable exponent analysis has the extrapolation theorem from weighted inequalities to norm inequalities on variable Lebesgue spaces [5, 8].

In general, the boundedness of M on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ describes that the norm inequality

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\,\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{1.1}$$

holds for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$, where C is a positive constant independent of f. Lerner [19] has pointed out the crucial difference between the norm inequality (1.1) and the following modular inequality

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx \le C \int_{\mathbb{R}^n} |f(x)|^{p(x)}dx. \tag{1.2}$$

More precisely, Lerner has proved that $p(\cdot)$ must be a constant function whenever $1 < \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) \le \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty$ and the modular inequality (1.2) holds. Izuki [11] has considered the difference for some operators arising from the wavelet theory. Izuki, Nakai and Sawano [13, 14] have given an alternative proof of Lerner's result. They have also studied the problem in the weighted case [15].

Recently, Izuki, Koyama, Noi and Sawano [12] have considered some modular inequalities for some operators. In this paper, we focus on three operators below. First, we investigate the Bergman projection operator on the unit disc $\mathbb{D}$ in the complex plane. The generalization of holomorphic function spaces in terms of variable exponent and the boundedness of Bergman projection operators on variable exponent spaces have been studied [1–3, 16, 17]. Among them we focus on the work [1] due to Chacón and Rafeiro. They defined Bergman spaces $A^{p(\cdot)}(\mathbb{D})$ with variable exponent $p(\cdot)$ on the open unit disk $\mathbb{D}$. Applying the local log-Hölder continuous condition and the extrapolation theorem, they proved the density of the set of polynomials in $A^{p(\cdot)}(\mathbb{D})$ and the boundedness of the Bergman projection $P : L^{p(\cdot)}(\mathbb{D}) \to A^{p(\cdot)}(\mathbb{D})$. In particular, Chacón and Rafeiro [1] have obtained the norm inequality

$$\|Pf\|_{L^{p(\cdot)}(\mathbb{D})} \le C\,\|f\|_{L^{p(\cdot)}(\mathbb{D})} \tag{1.3}$$

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for all $f \in L^{p(\cdot)}(\mathbb{D})$.

Our second target operator is

$$B_{\mathbb{R}^2_+} f(z) = \frac{-1}{\pi} \int\limits_{\mathbb{R}^2_+} \frac{f(w)}{(z-\overline{w})^2} dA(w), \quad z = x + iy \in \mathbb{R}^2_+,$$

where $dA(w)$ denotes the Lebesgue measure and $\mathbb{R}^2_+$ is the upper half-space over $\mathbb{R}^2 \simeq \mathbb{C}$. Via this identification of $\mathbb{R}^2$ and $\mathbb{C}$, the space $A^{p(\cdot)}(\mathbb{R}^2_+)$ is defined to be the set of all holomorphic functions which belong to $L^{p(\cdot)}(\mathbb{R}^2_+)$. Karapetyants and Samko [17] proved that $B_{\mathbb{R}^2_+}$ is a projection from $L^{p(\cdot)}(\mathbb{R}^2_+)$ onto $A^{p(\cdot)}(\mathbb{R}^2_+)$ if $p(\cdot) \in \mathcal{P}(\mathbb{R}^2_+)$, the set of all measurable functions $p(\cdot) : \mathbb{R}^2_+ \to (0, \infty)$ such that $\log \log p(\cdot) \in L^\infty(\mathbb{R}^2_+)$, satisfies the log-Hölder condition and the log-decay condition [17, Theorem 3.1 (1)]. So, they have obtained the norm inequality

$$\|B_{\mathbb{R}^2_+} f\|_{L^{p(\cdot)}(\mathbb{R}^2_+)} \le C \|f\|_{L^{p(\cdot)}(\mathbb{R}^2_+)} \tag{1.4}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^2_+)$.

Finally, we consider $b_{\mathbb{R}^n_+}$, the harmonic projection in $\mathbb{R}^n_+$. Let $\mathbb{R}^n_+$ stand for the upper half-space over $\mathbb{R}^n$ with $n \ge 2$. For $x = (x_1, x_2, \ldots, x_n)$, we write $x' = (x_1, x_2, \ldots, x_{n-1})$ and $\bar{x} = (x', -x_n)$. As usual, $h^p(\mathbb{R}^n_+)$ stands for the harmonic Bergman space of harmonic functions that belong to $L^p(\mathbb{R}^n_+)$. Once again $dA(x)$ denotes the Lebesgue measure. The corresponding Bergman projection $b_{\mathbb{R}^n_+}$ defined by

$$\begin{aligned} b_{\mathbb{R}^n_+} f(x) &= \int\limits_{\mathbb{R}^n_+} R(x, y) f(y) dA(y) \\ &= \frac{2}{\pi^{\frac{n}{2}}} \Gamma\left(\frac{n}{2}\right) \int\limits_{\mathbb{R}^n_+} \frac{n(x_n + y_n) - |x - \bar{y}|^2}{|x - \bar{y}|^{n+2}} f(y) dA(y), \end{aligned}$$

is bounded from $L^p(\mathbb{R}^n_+)$ onto $h^p(\mathbb{R}^n_+)$ [22]. Namely, $b_{\mathbb{R}^n_+} f \in h^p(\mathbb{R}^n_+)$ and the norm inequality

$$\|b_{\mathbb{R}^n_+} f\|_{L^p(\mathbb{R}^n_+)} \le C \|f\|_{L^p(\mathbb{R}^n)} \tag{1.5}$$

hold for all $f \in L^p(\mathbb{R}^n_+)$. Karapetyants and Samko have extended (1.5) to the variable exponent settings [17, Theorem 5.1].

In the present paper, we consider the modular inequalities corresponding to the norm inequalities (1.3), (1.4) and (1.5). More precisely, for example, if $p(\cdot)$ satisfies

$$1 < \operatorname*{ess\,sup}_{z \in \mathbb{D}} p(z) \le \operatorname*{ess\,sup}_{z \in \mathbb{D}} p(z) < \infty$$

and the modular inequality

$$\int\limits_{\mathbb{D}} |Pf(z)|^{p(z)} dA(z) \le C \int\limits_{\mathbb{D}} |f(z)|^{p(z)} dA(z)$$

holds for all $f \in L^{p(\cdot)}(\mathbb{D})$, then the variable exponent $p(\cdot)$ must be a constant function. We can prove similar results for $B_{\mathbb{R}^n_+}$ and $b_{\mathbb{R}^n_+}$. In order to prove them, we need a lower bound for the image of the characteristic function of a certain set. We will show a key lemma for the lower bound before the statement of the main results.

In the present paper we will use the following notation.

1. Given a measurable set E, we denote the Lebesgue measure of E by $|E|$. We define the characteristic function of E by χ_E.

2. A symbol C always stands for a positive constant, independent of the main parameters.

2. Function Spaces with Variable Exponent

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, that is,

$$\mathbb{D} := \{z \in \mathbb{C} \,:\, |z| < 1\}.$$

Let also $\mathbb{R}^n_+$ be the upper half plane, that is,

$$\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n \,:\, x' \in \mathbb{R}^{n-1}, x_n > 0\}.$$

In the present paper we concentrate on the theory on function spaces defined on $\mathbb{D}$ or $\mathbb{R}^n_+$ with $n \geq 2$. We first define some fundamental notation on variable exponents. Let X denote either $\mathbb{D}$ or $\mathbb{R}^n_+$.

Definition 2.1.
1. Given a measurable function $p(\cdot) : X \to [1, \infty)$, we define

$$p_+ := \operatorname*{ess\,sup}_{z \in X} p(z), \quad p_- := \operatorname*{ess\,inf}_{z \in X} p(z).$$

2. The set $\mathcal{P}(X)$ consists of all measurable functions $p(\cdot) : X \to [1, \infty)$ satisfying $1 < p_-$ and $p_+ < \infty$.

Chacón and Rafeiro [1] defined generalized Lebesgue spaces and Bergman spaces on $\mathbb{D}$ with a variable exponent.

Definition 2.2. Let $dA(z)$ be the normalized Lebesgue measure on X and $p(\cdot) \in \mathcal{P}(X)$. The Lebesgue space $L^{p(\cdot)}(X)$ consists of all measurable functions f on X satisfying that the modular

$$\rho_p(f) := \int_X |f(z)|^{p(z)} \, dA(z)$$

is finite. The Bergman space $A^{p(\cdot)}(\mathbb{D})$ is the set of all holomorphic functions f on $\mathbb{D}$ such that $f \in L^{p(\cdot)}(\mathbb{D})$.

We note that $L^{p(\cdot)}(X)$ is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(X)} := \inf \left\{\lambda > 0 \,:\, \rho_p(f/\lambda) \leq 1\right\}.$$

The projection $P : L^2(\mathbb{D}) \to A^2(\mathbb{D})$ is called the Bergman projection and given by

$$Pf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \bar{w}z)^2} dA(w).$$

It is known that $P : L^p(\mathbb{D}) \to A^p(\mathbb{D})$ is bounded in the case where $p(\cdot) = p \in (0, \infty)$ is a constant exponent [10, 22]. See also [20] for the case of $p = 2$.

Chacón and Rafeiro [1, Theorem 4.4] proved the following boundedness

Theorem 2.3. Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{D})$ satisfies the local log-Hölder continuous condition

$$|p(z_1) - p(z_2)| \leq \frac{C}{\log(\mathrm{e} + 1/|z_1 - z_2|)} \quad (z_1,\, z_2 \in \mathbb{D}).$$

Then the Bergman projection P is bounded from $L^{p(\cdot)}(\mathbb{D})$ to $A^{p(\cdot)}(\mathbb{D})$, in particular, the norm inequality

$$\|Pf\|_{L^{p(\cdot)}(\mathbb{D})} \leq C \, \|f\|_{L^{p(\cdot)}(\mathbb{D})}$$

holds for all $f \in L^{p(\cdot)}(\mathbb{D})$.

In the following sections, we consider the modular inequalities corresponding to the norm inequalities (1.3), (1.4) and (1.5).

3. Bergman Projection on $\mathbb{D}$

Theorem 3.1. Let $p(\cdot) \in \mathcal{P}(\mathbb{D})$. If the modular inequality

$$\int_{\mathbb{D}} |Pf(z)|^{p(z)} dA(z) \le C \int_{\mathbb{D}} |f(z)|^{p(z)} dA(z) \tag{3.1}$$

holds for all $f \in L^{p(\cdot)}(\mathbb{D})$, then $p(z)$ equals to a constant for almost every $z \in \mathbb{D}$.

In order to prove this theorem, we apply the following lower pointwise estimate for the Bergman projection.

Lemma 3.2. Let $\tau \in \mathbb{D}$. Then there exists a compact neighborhood K_τ of τ such that

$$\mathrm{Re}(P\chi_E(z)) \ge c_\tau |E|$$

for all measurable sets $E \subset K_\tau$, where c_τ is a positive constant depending only on τ.

Proof. Note that there exists a compact neighborhood K_τ of τ such that

$$c_\tau := \inf_{z,w \in K_\tau} \mathrm{Re}\left(\frac{1}{(1-\bar{w}z)^2}\right) > 0.$$

Thus,

$$\mathrm{Re}(P\chi_E(z)) = \int_E \mathrm{Re}\left(\frac{1}{(1-\bar{w}z)^2}\right) dA(w) \ge c_\tau \int_E dA(w) = c_\tau |E|,$$

as required. □

Now we prove Theorem 3.1.

Proof of Theorem 3.1. Let $\tau \in \mathbb{D}$ and K_τ be the compact neighborhood appearing in Lemma 3.2. Assume that $p(z)$ does not equal to any constant for almost every $z \in K_\tau$. Then we can find subsets $K_\tau^\pm$ of K_τ such that

$$\sup_{z \in K_\tau^-} p(z) < \inf_{z \in K_\tau^+} p(z). \tag{3.2}$$

Using Lemma 3.2 and modular inequality (3.1), we have

$$\int_{K_\tau^+} (kc_\tau |K_\tau^-|)^{p(z)}\, dA(z) \le \int_{K_\tau^+} |kP\chi_{K_\tau^-}(z)|^{p(z)}\, dA(z) \le C \int_{\mathbb{D}} (k\chi_{K_\tau^-})^{p(z)}\, dA(z)$$

for all $k > 0$. Consequently, if $kc_\tau|K_\tau^-| > 1$ and $k > 1$, then we obtain

$$|K_\tau^+|(kc_\tau|K_\tau^-|)^{\operatorname{ess\,inf}_{z \in K_\tau^+} p(z)} \le C|K_\tau^-| k^{\operatorname{ess\,sup}_{z \in K_\tau^-} p(z)}.$$

This contradicts (3.2). Consequently, it follows that for all $\tau \in \mathbb{D}$ there exists a compact neighborhood K_τ such that $p(z)$ is equal to a constant for almost every $z \in K_\tau$. Since $\mathbb{D}$ is connected, it follows that $p(z)$ is equal to a constant for almost every $z \in \mathbb{D}$. □

4. Bergman Projection onto $\mathbb{R}^2_+$

As the following lemma shows, $B_{\mathbb{R}^2_+}$ is not degenerate.

Lemma 4.1. Let $\tau \in \mathbb{R}^2_+$. Then there exists a compact neighborhood K_τ of τ such that

$$\mathrm{Re}\left(B_{\mathbb{R}^2_+}(\chi_E)(z)\right) \ge C_\tau |E|$$

for all measurable sets $E \subset K_\tau$.

Proof. Let $\tau = \alpha + \beta i \in \mathbb{C} \simeq \mathbb{R}^2_+$. Firstly, we prove that there exist C_τ and a compact neighborhood K_τ of τ such that

$$\mathrm{Re}\left(\frac{1}{(z-\overline{w})^2}\right) \le -C_\tau < 0$$

holds for any $z, w \in K_\tau$. To do this, we consider the real part of $(\overline{z}-w)^2$ keeping in mind that

$$\mathrm{Re}\left(\frac{1}{(z-\overline{w})^2}\right) = \mathrm{Re}\left(\frac{(\overline{z}-w)^2}{|z-\overline{w}|^4}\right).$$

We can take $\gamma > 0$ so that $\beta - \gamma > 0$ because $\beta > 0$. We learn that

$$K_\tau = \{x + yi \,:\, \alpha - (\beta-\gamma)/2 \le x \le \alpha + (\beta-\gamma)/2,\ \beta-\gamma \le y \le \beta+\gamma\} (\subset \mathbb{R}^2_+)$$

makes the job. In fact, let $z = a + bi$, $w = c + di \in K_\tau$. It is easy to see that $\mathrm{Re}(\overline{z}-w)^2 < 0$, since

$$(\overline{z}-w)^2 = (a-c)^2 - (b+d)^2 - 2(a-c)(b+d)i$$

and $|a-c| \le \beta-\gamma < 2(\beta-\gamma) \le |b+d|$.

Consequently, from the property of K_τ, we have

$$\mathrm{Re}(B_{\mathbb{R}^2_+}(\chi_E(z))) = \frac{-1}{\pi}\int_E \mathrm{Re}\left(\frac{1}{(z-\overline{w})^2}\right) dA(w) \ge C_\gamma \int_E dA(w) = c_\gamma |E|$$

for any $E \subset K_\tau$. $\square$

Using Lemma 4.1 and an argument similar to the proof of Theorem 3.1, we obtain the following theorem. So we omit the proof.

Theorem 4.2. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^2_+)$. If the modular inequality

$$\int_{\mathbb{R}^2_+} \left|B_{\mathbb{R}^2_+} f(z)\right|^{p(z)} dA(z) \le C \int_{\mathbb{R}^2_+} |f(z)|^{p(z)}\, dA(z)$$

holds for all $f \in L^{p(\cdot)}(\mathbb{R}^2_+)$, then $p(z)$ is equal to a constant for almost every $z \in \mathbb{R}^2_+$.

5. Harmonic Projection in $\mathbb{R}^n_+$

The same technique can be applied to the harmonic projection over $\mathbb{R}^n_+$.

Theorem 5.1. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_+)$. If the modular inequality

$$\int_{\mathbb{R}^n_+} \left|b_{\mathbb{R}^n_+} f(z)\right|^{p(z)} dA(z) \le C \int_{\mathbb{R}^n_+} |f(z)|^{p(z)}\, dA(z)$$

holds for all $f \in L^{p(\cdot)}(\mathbb{R}^n_+)$, then $p(z)$ is equal to a constant for almost every $z \in \mathbb{R}^n_+$.

Proof. Let $x = (x', x_n) \in \mathbb{R}^n_+$ be fixed. Then we have

$$\frac{n(x_n+z_n) - |x-\bar{z}|^2}{|x-\bar{z}|^{n+2}} = \frac{n-2x_n}{2^{n+1}} {x_n}^{-n-1}$$

for $z = (z', z_n) = x = (x', x_n)$. Based on this equality, we will prove that $p(z)$ is equal to a constant for almost every $z \in \mathbb{R}^n_+$ via three steps.

1. If $x_n < \frac{n}{2}$, then we obtain

$$\frac{n(x_n+y_n) - |x-\bar{y}|^2}{|x-\bar{y}|^{n+2}} > \frac{n-2x_n}{2^{n+3}} {x_n}^{-n-1} > 0$$

as long as $y = (y', y_n)$ belongs to an open neighborhood U of x. Thus, if we go through the same argument as before, we see that $p(z)$ is equal to a constant p_1 for almost every $z \in \mathbb{R}^n_+$ with $z_n > \frac{n}{2}$.

2. If $x_n > \frac{n}{2}$ instead, then we obtain

$$\frac{n(x_n+y_n) - |x-\bar{y}|^2}{|x-\bar{y}|^{n+2}} < \frac{n-2x_n}{2^{n+3}} {x_n}^{-n-1} < 0$$

as long as $y = (y', y_n)$ belongs to an open neighborhood U of x. Thus, if we go through the same argument as before, we see that $p(z)$ equals to a constant p_2 for almost every $z \in \mathbb{R}^n_+$ with $z_n < \frac{n}{2}$.

3. Finally, we prove that $p_1 = p_2$. To this end, we consider a small neighborhood U at $(0, \frac{n}{4})$ and a small neighborhood V at $(0, 3n)$. Since

$$\frac{n(x_n + z_n) - |x - \bar{z}|^2}{|x - \bar{z}|^{n+2}} < 0$$

if $x = (0, \frac{n}{4})$ and $z = (0, 3n)$,

$$\frac{n(x_n + z_n) - |x - \bar{z}|^2}{|x - \bar{z}|^{n+2}} \leq -c_n$$

for any $x \in U$ and $z \in V$ for some $c_n > 0$. Thus, we can through the same argument as before, to conclude that $p_1 = p_2$. □

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COMPLEX REPRESENTATION IN THE PLANE THEORY OF VISCOELASTICITY AND ITS APPLICATIONS

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Abstract. The complex representation in the plane theory of viscoelasticity and Kolosov–Muskhelishvili's type formulas in the conditions of plane deformation and in the plane stressed state are obtained. Investigation of various possible forms of viscoelastic correlations can be found in [1–4, 6–9, 11]. Certain contact problems of viscoelastic bodies and the corresponding integro-differential equations are studied in [5, 12, 13]. The present paper considers the problem of a rigid punch on the boundary of a half-plane in the presence of fraction.

1. Introduction

Basic equations of the creep theory expressing the connection between stresses and deformations of hereditary aging media under small deformations have the form [1, 4, 11]

$$\begin{gathered} 2e_{ij}(t,r) = \frac{s_{ij}(t,r)}{G(t)} - \int\limits_{t_0}^{t} s_{ij}(\tau,r)\,K_1(t,\tau)\,d\tau \quad ((i,j)=1,2,3), \\ \varepsilon(t,r) = \frac{\sigma(t,r)}{E^*(t)} - \int\limits_{t_0}^{t} \sigma(\tau,r)\,K_2(t,\tau)\,d\tau, \end{gathered} \tag{1.1}$$

where t is time, r is the radius-vector of the point, t_0 is the age of the material element at the moment of loading, $s_{ij}(t,r)$ and $e_{ij}(t,r)$ are, respectively, the tensor deviator components of stress and deformation, $G(t)$ is the instantaneous shear modulus, $E^*(t)$ is the instantaneous volumetric deformation, $\varepsilon(t,r)$ is the mean deformaton, $\sigma(t,r)$ is the mean stress, $K_1(t,\tau)$ and $K_2(t,\tau)$ are the kernels of shearing and volumetric creep deformation, respectively, which can be represented in the form

$$K_1(t,\tau) = \frac{\partial}{\partial\tau}\left[\frac{1}{G(\tau)} + \omega(t,\tau)\right], \qquad K_2(t,\tau) = \frac{\partial}{\partial\tau}\left[\frac{1}{E^*(\tau)} + C^*(t,\tau)\right],$$

where $\omega(t,\tau)$ and $C^*(t,\tau)$, are the creep measures of shearing and volumetric deformation. As is known, the components of stress and deformation tensors σ_{ij} and ε_{ij} are connected with the components of the corresponding deviator as follows:

$$s_{ij} = \sigma_{ij} - \sigma\delta_{ij}, \quad \sigma = \frac{1}{3}\quad \sigma_{ii}, \quad e_{ij} = \varepsilon_{ij} - \varepsilon\delta_{ij}, \quad \varepsilon = \frac{1}{3}\varepsilon_{ii};$$

here, δ_{ij} is the Kronecker symbol.

For one-dimensional stressed state of tension-compression we have

$$\varepsilon_{ii}(t,r) = \frac{\sigma_{ii}(t,r)}{E(t)} - \int\limits_{t_0}^{t} \sigma_{ii}(\tau,r)\,K(t,\tau)\,d\tau, \tag{1.2}$$

$K(t,\tau) = \frac{\partial}{\partial\tau}\left[\frac{1}{E(\tau)} + C(t,\tau)\right]$ is the creep kernel of tension-compression deformation, $E(t)$ is the instantaneous Young modulus, $C(t,\tau)$ is the creep measure of tension-compression deformation. The

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following correlations are known:

$$G(t)=\frac{E(t)}{2(1+\nu_1(t))},\quad E^*(t)=\frac{E(t)}{1-2\nu_1(t)},$$
$$\omega(t,\tau)=2\left[1+\nu_2(t,\tau)\right]C(t,\tau),\quad C^*(t,\tau)=\left[1+2\nu_2(t,\tau)\right]C(t,\tau),$$

where $\nu_1(t)$ is the Poisson coefficient of elasto-instantaneous deformation, $\nu_2(t,\tau)$ is the Poisson coefficient of creep deformation.

Elasto-instantaneous modules are the positive, continuous, bounded and monotonically increasing functions on every $t_0\le\tau<\infty$, therefore they may satisfy the following conditions:

$$\frac{dE(\tau)}{d\tau}>0\quad(\tau<\infty),\quad E(\tau)\sim E_0<\infty\quad(\tau\to\infty),\quad E(t_0)>0,$$

where E_0 is an elastic modulus of the material, rather large in age. Creep measures are the nonnegative, continuous functions of two variables with the following properties: $t_0\le\tau\le t\le\infty$.

$$C(t,t)=0,\quad C(t,\tau)\sim\varphi(\tau)\quad(t\to\infty),$$
$$C(t,\tau)\sim\psi(t-\tau)\quad(\tau\to\infty,\quad\tau\le t),\quad\frac{\partial C(t,\tau)}{\partial t}>0,\quad\frac{\partial C(t,\tau)}{\partial\tau}<0\quad(\tau\le t<\infty).$$

$\varphi(\tau)$ defines the aging process of the material, and the function $\psi(y)$ characterizes hereditary properties of the material, moreover,

$$\frac{d\varphi(\tau)}{d\tau}<0\quad(\tau<\infty),\quad\varphi(\tau)\sim C_0>0\quad(\tau\to\infty),\quad\varphi(t_0)<\infty,$$
$$\frac{d\psi(y)}{dy}>0\quad(y<\infty),\quad\psi(y)\sim C_0\quad(y\to\infty),\quad\psi(0)=0,$$

where C_0 is the limiting creeping measure for the material, highly large in age.

In view of the above-mentioned properties, the creeping measure $C(t,\tau)$ is usually representable in the form [4]:

$$C(t,\tau)=\varphi(\tau)\left(1-e^{-\gamma(t-\tau)}\right),\qquad\gamma=\text{const}. \tag{1.3}$$

The correlations expressing stress components through deformation components are obtained from (1.1) and (1.2) by solving the Volterra integral equations. From (1.2) we get [11]:

$$\frac{\sigma_{ii}(t,r)}{E(t)}=\varepsilon_{ii}(t,r)+\int_{t_0}^{t}\varepsilon_{ii}(\tau,r)R(t,\tau)d\tau.$$

Here, $R(t,\tau)$ is called a kernel of relaxation, or in other words, the resolvent of creeping kernel $K(t,\tau)$.

2. Complex Representations in the Plane Theory of Viscoelasticity

(a) For a plane stressed state $\sigma_{13}=\sigma_{23}=\sigma_{33}=0$, all the rest components of stresses together with the components of deformation are the functions of variables (t,x,y), therefore correlations (1.1) take the form

$$\varepsilon_{ij}(t,x,y)=\frac{1+\nu_1(t)}{E(t)}\sigma_{ij}-\int_{t_0}^{t}\sigma_{ij}\frac{\partial}{\partial\tau}\left[\frac{1+\nu_1(\tau)}{E(\tau)}+(1+\nu_2(t,\tau))C(t,\tau)\right]d\tau$$
$$-\delta_{ij}\frac{\nu_1(t)}{E(t)}(\sigma_{11}+\sigma_{22})+\delta_{ij}\int_{t_0}^{t}\frac{\partial}{\partial\tau}\left[\frac{\nu_1(\tau)}{E(\tau)}+\nu_2(t,\tau)C(t,\tau)\right](\sigma_{11}+\sigma_{22})d\tau,\quad i,j=1,2, \tag{2.1}$$
$$\varepsilon_{33}(t,x,y)=-\frac{\nu_1(t)}{E(t)}(\sigma_{11}+\sigma_{22})+\int_{t_0}^{t}\frac{\partial}{\partial\tau}\left[\frac{\nu_1(\tau)}{E(\tau)}+\nu_2(t,\tau)C(t,\tau)\right](\sigma_{11}+\sigma_{22})d\tau.$$

(b) For a plane deformation, ε_{11} and ε_{22} are independent of z, $\varepsilon_{33}(t,x,y)=0$. Assuming $E(t)=E=\text{const}$, $\nu_1(t)=\nu_2(t,\tau)=\nu=\text{const}$, we obtain $\sigma_{33}=\nu(\sigma_{11}+\sigma_{22})$, and equalities (1.1) take the form

$$\varepsilon_{ij}(t,x,y)=\frac{1+\nu}{E}\sigma_{ij}-(1+\nu)\int_{t_0}^{t}\sigma_{ij}\frac{\partial}{\partial\tau}C(t,\tau)\,d\tau-\delta_{ij}\frac{\nu(1+\nu)}{E}(\sigma_{11}+\sigma_{22})$$
$$+\delta_{ij}\nu(1+\nu)\int_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)(\sigma_{11}+\sigma_{22})\,d\tau,\quad i,j=1,2. \tag{2.2}$$

Expressions (2.1) and (2.2) are the analogues of Hook's law in the theory of viscoelasticity, i.e., they establish a connection between the components of deformation and stress tensors in the conditions of plane deformation and plane stressed state, respectively.

In the absence of body forces, the equilibrium equations take the form

$$\frac{\partial\sigma_{11}(t,x,y)}{\partial x}+\frac{\partial\sigma_{12}(t,x,y)}{\partial y}=0,\qquad \frac{\partial\sigma_{21}(t,x,y)}{\partial x}+\frac{\partial\sigma_{22}(t,x,y)}{\partial y}=0.$$

As is known [10], these equalities result in

$$\sigma_{11}(t,x,y)=\frac{\partial^2U(t,x,y)}{\partial y^2},\quad \sigma_{22}(t,x,y)=\frac{\partial^2U(t,x,y)}{\partial x^2},\quad \sigma_{12}(t,x,y)=-\frac{\partial^2U(t,x,y)}{\partial x\partial y},$$

where $U(t,x,y)$ is the stress function or the Airy function. $\Delta\Delta U=0$, $\Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

Equalities (2.2) yield

$$\begin{aligned}\varepsilon_{11}(t,x,y)&=\frac{(1+\nu)\left(\Delta U-\frac{\partial^2U}{\partial x^2}\right)}{E}-(1+\nu)\int_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\left(\Delta U-\frac{\partial^2U}{\partial x^2}\right)d\tau-\frac{\nu(1+\nu)}{E}\Delta U\\&\quad+\nu(1+\nu)\int_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\,\Delta U\,d\tau,\\ \varepsilon_{22}(t,x,y)&=\frac{(1+\nu)\left(\Delta U-\frac{\partial^2U}{\partial y^2}\right)}{E}-(1+\nu)\int_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\left(\Delta U-\frac{\partial^2U}{\partial y^2}\right)d\tau-\frac{\nu(1+\nu)}{E}\Delta U\\&\quad+\nu(1+\nu)\int_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\,\Delta U\,d\tau.\end{aligned} \tag{2.3}$$

Introducing the notation $\Delta U\equiv P$ and considering holomorphic functions $F(z,t)=P+iQ$ ($\Delta P=0$, $\Delta Q=0$) and $\varphi(z,t)=p+iq=\frac{1}{4}\int F(z,t)\,dz$, we have $P=4\frac{\partial p}{\partial x}=4\frac{\partial q}{\partial y}$, whence (2.3) takes the form

$$\varepsilon_{11}(t,x,y)=\frac{\partial u_1}{\partial x}=\frac{(1+\nu)\left(4\frac{\partial p}{\partial x}-\frac{\partial^2 U}{\partial x^2}\right)}{E}-(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\left(4\frac{\partial p}{\partial x}-\frac{\partial^2 U}{\partial x^2}\right)d\tau$$
$$-\frac{4\nu(1+\nu)}{E}\frac{\partial p}{\partial x}+4\nu(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\frac{\partial p}{\partial x}d\tau,$$
$$\varepsilon_{22}(t,x,y)=\frac{\partial u_2}{\partial y}=\frac{(1+\nu)\left(4\frac{\partial q}{\partial y}-\frac{\partial^2 U}{\partial y^2}\right)}{E}-(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\left(4\frac{\partial q}{\partial y}-\frac{\partial^2 U}{\partial y^2}\right)d\tau$$
$$-\frac{4\nu(1+\nu)}{E}\frac{\partial q}{\partial y}+4\nu(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\frac{\partial q}{\partial y}d\tau, \tag{2.4}$$

where u_1, u_2 are displacement components.

As a result of integration of each of the correlations (2.4), we get

$$u_1=\frac{(1+\nu)\left(4p-\frac{\partial U}{\partial x}\right)}{E}-(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\left(4p-\frac{\partial U}{\partial x}\right)d\tau$$
$$-\frac{4\nu(1+\nu)}{E}p+4\nu(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\,p\,d\tau+f_1(y,t),$$
$$u_2=\frac{(1+\nu)\left(4q-\frac{\partial U}{\partial y}\right)}{E}-(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\left(4q-\frac{\partial U}{\partial y}\right)d\tau$$
$$-\frac{4\nu(1+\nu)}{E}q+4\nu(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\,q\,d\tau+f_2(x,t). \tag{2.5}$$

Taking into account the third equality of (2.2), it follows that

$$f'_{1y}(y,t)+f'_{2x}(x,t)=0,$$

from which $f_1(y,t)=\varepsilon yt+\alpha$, $f_2(x,t)=-\varepsilon xt+\beta$, i.e., $f_1(y,t)$ and $f_2(x,t)$ provide a rigid displacement of the body which can be neglected. From equality (2.5) we have

$$u_1+iu_2=\frac{(1+\nu)}{E}\left(4\varphi(z,t)-\left(\frac{\partial U}{\partial x}+i\frac{\partial U}{\partial y}\right)\right)$$
$$-(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\left[4\varphi(z,\tau)-\left(\frac{\partial U}{\partial x}+i\frac{\partial U}{\partial y}\right)\right]d\tau$$
$$-\frac{4\nu(1+\nu)}{E}\varphi(z,t)+4\nu(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\varphi(z,\tau)d\tau. \tag{2.6}$$

As is known [10], from a general solution of biharmonic equation, the Goursat formula $U=\mathrm{Re}\,[\overline{z}\varphi(z,t)+\chi(z,t)]$, we find that $\frac{\partial U}{\partial x}+i\frac{\partial U}{\partial y}=\varphi(z,t)+z\overline{\varphi'(z,t)}+\overline{\psi(z,t)}$, where $\frac{\partial\chi(z,t)}{\partial z}=\psi(z,t)$, $\varphi(z,t)$ and $\psi(z,t)$ are holomorphic functions of the variable $z=x+iy$.

If we introduce the notation

$$(I-L)\,g\,(t)=\frac{g\,(t)}{E}-\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C\,(t,\tau)\,g\,(\tau)\,d\tau,$$

then expression (2.6) can be written as

$$u_1+iu_2=(1+\nu)\,(I-L)\left((3-4\nu)\varphi\,(z,t)-(z\overline{\varphi'\,(z,t)}+\overline{\psi\,(z,t)})\right), \tag{2.7}$$

and for a plane stressed state, an analogous reasoning results in

$$u_1+iu_2=(I-L)\left((3-\nu)\varphi\,(z,t)-(1+\nu)\,(z\overline{\varphi'\,(z,t)}+\overline{\psi\,(z,t)})\right). \tag{2.8}$$

Correlations (2.7) and (2.8) together with the relations

$$\sigma_{11}+\sigma_{22}=4\left[\Phi\,(z,t)+\overline{\Phi\,(z,t)}\right],\qquad \sigma_{22}-\sigma_{11}+2i\sigma_{12}=2\left[\overline{z}\Phi'\,(z,t)+\Psi\,(z,t)\right],$$

where $\Phi\,(z,t)=\varphi'\,(z,t)$, $\Psi\,(z,t)=\psi'\,(z,t)$, are the analogues of the well-known Kolosov–Muskhelishvili's formulas in the theory of viscoelasticity.

3. Solution of the Punch Problem for a Half-plane

Let in the conditions of plane deformation a viscoelastic body occupy a half-plane $y<0$ which we denote by S^-, so the body S^- leaves on the right when moving along the ox-axis in a positive direction. We denote the upper half-plane by S^+ and the ox-axis by L.

Assume also that the principal vector (X,Y) of outer forces applied to the boundary is finite, stresses and rotations vanish at infinity. Thus, for large $|z|$, we have

$$\Phi\,(z,t)=\frac{X+iY}{2\pi z}+o\left(\frac{1}{z}\right),\quad \Phi'\,(z,t)=-\frac{X+iY}{2\pi z^2}+o\left(\frac{1}{z^2}\right),\quad \Psi\,(z,t)=\frac{X-iY}{2\pi z}+o\left(\frac{1}{z}\right).$$

For a half-plane, Kolosov–Muskhelishvili's formulas take the form [10]:

$$\sigma_{12}-\sigma_{11}+2i\sigma_{12}=2\left[\Phi'\,(z,t)\,(\overline{z}-z)-\overline{\Phi}\,(z,t)-\Phi\,(z,t)\right],$$

$$\sigma_{22}-i\sigma_{12}=\Phi\,(z,t)-\Phi\,(\overline{z},t)+(z-\overline{z})\,\overline{\Phi'\,(z,t)}, \tag{3.1}$$

$$u'_1+iu'_2=(1+\nu)\,(I-L)\left[(3-4\nu)\,\Phi\,(z,t)+\Phi\,(\overline{z},t)+(\overline{z}-z)\,\overline{\Phi'\,(z,t)}\right]. \tag{3.2}$$

Equality (3.2) is written for the case of plane deformation. The prime denotes the derivative with respect to the variable z, and in the sequel, the dot will denote the derivative with respect to the variable t.

A punch with a base of given shape, with a force directed vertically downwards, acts along the segment $L'=[a;b]$ of the boundary. Let the punch displacement along the boundary normal be translational (vertically downwards) in the conditions of friction. The boundary conditions have the form

$$T\,(x,t)=kP\,(x,t)\,,\qquad x\in L', \tag{3.3}$$

$$v^-\,(x,t)=f\,(x,t)+\text{const},\qquad x\in L', \tag{3.4}$$

$$T\,(x,t)=P\,(x,t)=0,\qquad x\in L-L', \tag{3.5}$$

where $f\,(x,t)$ is the given function defining the punch profile at the moment $t=t_0$, i.e., $y=f\,(x,t_0)$ is the punch profile equation.

Let $f'\,(x,t)$ satisfy the Hölder (H) condition with respect to the variable x, and $P_0\,(t)=\int\limits_a^b P\,(x,t)\,dx$, $T_0\,(t)=kP_0\,(t)$. From (3.1) and (3.2), passing to the boundary values as $y\to 0-$, we obtain

$$Y_y-iX_y=\Phi^-\,(x,t)-\Phi^+\,(x,t)\,,$$

$$u'+iv'=(1+\nu)\,(I-L)\left[(3-4\nu)\,\Phi^-\,(x,t)+\Phi^+\,(x,t)\right],$$

whence, in view of the boundary conditions (3.3)–(3.5), we have

$$(1-ik)\,\Phi^{+}(x,t)+(1+ik)\,\overline{\Phi}^{+}(x,t)=(1-ik)\,\Phi^{-}(x,t)+(1+ik)\,\overline{\Phi}^{-}(x,t)\,, \tag{3.6}$$

$$(1+\nu)\,(I-L)\Big[(3-4\nu)\,\Phi^{-}(x,t)+\Phi^{+}(x,t)-(3-4\nu)\,\overline{\Phi}^{+}(x,t)-\overline{\Phi}^{-}(x,t)\Big]=2if'(x,t)\,. \tag{3.7}$$

From (3.6), according to the Liouville theorem, $(1-ik)\,\Phi(z,t)+(1+ik)\,\overline{\Phi}(z,t)=0$. Taking into account the last correlation in (3.7), we obtain

$$(I-L)\left[\Phi^{+}(x,t)-g\Phi^{-}(x,t)\right]=f_0(x,t)\,, \tag{3.8}$$

where

$$g=-\frac{(3-4\nu)\,(1+ik)+1-ik}{1+ik+(3-4\nu)\,(1-ik)},\quad f_0(x,t)=\frac{2i\,(1+ik)}{(1+\nu)\,(1+ik+(3-4\nu)\,(1-ik))}\,f'(x,t)\,.$$

Introducing the notation

$$\Gamma(x,t)=\Phi^{+}(x,t)-g\Phi^{-}(x,t)\,, \tag{3.9}$$

the Volterra integral equation (3.8) takes the form

$$(I-L)\,\Gamma(x,t)=f_0(x,t)\,. \tag{3.10}$$

Based on (1.3), the integral equation (3.10) reduces to the ordinary differential equation of second order

$$\ddot{\Gamma}(x,t)+\gamma\alpha(t)\,\dot{\Gamma}(x,t)=A(x,t) \tag{3.11}$$

with the following initial conditions

$$\begin{cases}\Gamma(x,t_0)=Ef_0(x,t_0)\,,\\ \dot{\Gamma}(x,t_0)=E\dot{f}_0(x,t_0)-\gamma E^2\varphi(t_0)\,f_0(x,t_0)\,,\end{cases} \tag{3.12}$$

where $\alpha(t)\equiv 1+E\varphi(t)$, $A(x,t)\equiv E\left[\ddot{f}_0(x,t)+\gamma\dot{f}_0(x,t)\right]$.

A solution of equations (3.11) and (3.12) is represented in the form

$$\Gamma(x,t)=C(x)\int_{t_0}^{t}\delta(\tau)\,d\tau+\int_{t_0}^{t}\delta(\tau)\left(\int_{t_0}^{\tau}\frac{A(x,s)\,ds}{\delta(s)}\right)d\tau+C_1(x)\,, \tag{3.13}$$

where

$$C(x)=E\dot{f}_0(x,t_0)-\gamma E^2\varphi(t_0)\,f_0(x,t_0)\,,\quad C_1(x)=Ef_0(x,t_0)\,,$$

$$\delta(t)=\exp\left\{-\gamma\int_{t_0}^{t}\alpha(\tau)\,d\tau\right\}.$$

Respectively, from (3.9) we obtain the following problem of linear conjugation:

$$\Phi^{+}(x,t)=g\Phi^{-}(x,t)+\Gamma(x,t)\,, \tag{3.14}$$

where $\Gamma(x,t)$ is defined by equality (3.13).

Introducing the constant α defined by the equality

$$tg\pi\alpha=k\frac{1-2\nu}{2\,(1-\nu)}\quad 0<\alpha<\frac{1}{2},\quad \text{we get}\quad g=-e^{2\pi i\alpha}.$$

Any solution of the homogeneous problem will be [10]

$$\chi_0(z)=(z-a)^{-\frac{1}{2}-\alpha}(b-z)^{-\frac{1}{2}+\alpha}.$$

Finally, a general solution of problem (3.14) takes the form

$$\Phi(z,t)=\frac{\chi_0(z)}{2\pi i}\int_{a}^{b}\frac{\Gamma(x,t)\,dx}{\chi_0^{+}(x)\,(x-z)}+\chi_0(z)\,\widetilde{C}(t), \tag{3.15}$$

where the function $\widetilde{C}(t)$ to be determined.

Under the expression $(z-a)^{-\frac{1}{2}-\alpha}(b-z)^{-\frac{1}{2}+\alpha}$ we mean a branch which is holomorphic on the segment $[a,b]$ and takes a real positive value $(x-a)^{\frac{1}{2}+\alpha}(b-x)^{\frac{1}{2}-\alpha}$ on the upper boundary of that segment. This branch is characterized by the fact that

$$\lim_{z\to\infty}\frac{(z-a)^{-\frac{1}{2}-\alpha}(b-z)^{-\frac{1}{2}+\alpha}}{z}=-ie^{\pi i\alpha}.$$

$\widetilde{C}(t)$ can be defined from the following formula:

$$\lim_{z\to\infty} z\Phi\left(z,t\right)=\frac{-T_0\left(t\right)+iP_0\left(t\right)}{2\pi}=\frac{iP_0\left(t\right)\left(1+ik\right)}{2\pi},$$

whence by virtue of (3.15), we get $\widetilde{C}\left(t\right)=\dfrac{P_0\left(t\right)\left(1+ik\right)e^{\pi i\alpha}}{2\pi}$.

Finally,

$$\Phi\left(z,t\right)=\frac{\chi_0\left(z\right)}{2\pi i}\int\limits_a^b\frac{\Gamma\left(x,t\right)dx}{\chi_0^+\left(x\right)\left(x-z\right)}+\chi_0\left(z\right)\frac{P_0\left(t\right)\left(1+ik\right)e^{\pi i\alpha}}{2\pi}.$$

It can be easily verified that all the conditions of the problem will be satisfied if $\Gamma\left(x,t\right)$ satisfies Hölder's condition condition (H) with respect to the variable x on the segment $[a,b]$.

Since

$$P\left(x,t\right)+iT\left(x,t\right)=P\left(x,t\right)\left(1+ik\right)=\Phi^+\left(x,t\right)-\Phi^-\left(x,t\right),$$

therefore the pressure under the punch is calculated by the formula

$$P\left(x,t\right)=\frac{\chi_0\left(x\right)}{\pi i}\int\limits_a^b\frac{\Gamma\left(y,t\right)dy}{\chi_0^+\left(y\right)\left(y-x\right)}+\chi_0\left(x\right)\frac{2P_0\left(t\right)\left(1+ik\right)e^{\pi i\alpha}}{2\pi}\,.$$

For $k=0$ $(\alpha=0)$, we obtain a solution corresponding to the case without friction.

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A MEASURE ZERO SET IN THE PLANE WITH ABSOLUTELY NONMEASURABLE LINEAR SECTIONS

ALEXANDER KHARAZISHVILI

Abstract. It is proved that there exists a translation invariant extension μ of the two-dimensional Lebesgue measure λ_2 on the plane $\mathbf{R}^2$ such that μ is metrically isomorphic to λ_2 and all linear sections of some μ-measure zero set are absolutely nonmeasurable.

Throughout this paper, we use the following fairly standard notation.
$X \triangle Y$ is the symmetric difference of two sets X and Y;
$\mathrm{dom}(f)$ is the domain of a function f;
$\mathrm{card}(X)$ is the cardinality of a set X;
ω is the least infinite ordinal (cardinal) number;
$\mathbf{R}$ is the real line equipped with the group of all its translations;
$\mathbf{c}$ is the cardinality of the continuum, i.e., $\mathbf{c}$ is $\mathrm{card}(\mathbf{R})$;
λ is the standard one-dimensional Lebesgue measure on $\mathbf{R}$;
$\mathbf{R}^n$ is the Euclidean n-dimensional space equipped with the group of all its translations;
λ_n is the standard n-dimensional Lebesgue measure on $\mathbf{R}^n$ (in particular, $\lambda_1 = \lambda$).

As is widely known, if Z is a λ_2-measure zero subset of the Euclidean plane $\mathbf{R}^2$, then almost all (with respect to λ) linear sections of Z, parallel to the coordinate axes, i.e., λ-almost all sets of the form

$$\{y : (x, y) \in Z\} \qquad (x \in \mathbf{R}),$$
$$\{x : (x, y) \in Z\} \qquad (y \in \mathbf{R}),$$

are of λ-measure zero. This fact is a direct consequence of Fubini's classical theorem. More generally, it follows from the same theorem that if l is any straight line in $\mathbf{R}^2$, then λ-almost all linear sections of Z, parallel to l, are of λ-measure zero.

The main goal of the present paper is to show that for a certain translation invariant extension μ of λ_2, which is metrically isomorphic to λ_2, the above-mentioned fact fails to be true in a very strong sense.

For our further purposes, we need some auxiliary notions from the general theory of invariant (quasi-invariant) measures (see, e.g., [1, 6, 11]).

Let E be an infinite ground set and let G be a group of transformations of E.

A nonzero complete σ-finite measure θ on E is called quasi-invariant with respect to G (in short, G-quasi-invariant) if the domain of θ is a G-invariant σ-algebra of subsets of E and the family of all θ-measure zero sets is a G-invariant σ-ideal of subsets of E.

A set $X \subset E$ is called almost G-invariant in E if for every transformation $g \in G$ one has

$$\mathrm{card}(g(X) \triangle X) < \mathrm{card}(E).$$

Almost G-invariant subsets of E play an important role in many topics of general topology and of the theory of invariant (quasi-invariant) measures (see, e.g., [1–4, 6, 10, 11]).

A set $Y \subset E$ is called G-absolutely nonmeasurable if for every nonzero σ-finite G-quasi-invariant measure μ on E one has $Y \notin \mathrm{dom}(\mu)$.

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In other words, $Y \subset E$ is G-absolutely nonmeasurable if Y is absolutely nonmeasurable with respect to the class of all nonzero σ-finite G-quasi-invariant measures on E.

In particular, if E is a group, then one can take as G the group of all left translations of E. In such a case, identifying E and G, one can speak of E-absolutely nonmeasurable subsets of E.

Lemma 1. *Let $(G, +)$ be an uncountable commutative group identified with the group of all its translations, and let Y be a subset of G.*

The following two assertions are equivalent:

(1) *there exists a countable family $\{g_j : j \in J\}$ of elements of G such that*

$$\cup\{g_j + Y : j \in J\} = G;$$

(2) *there exists a G-absolutely nonmeasurable set entirely contained in Y.*

For a detailed proof of Lemma 1, see [7].

We shall use this lemma in the special case where G is a group, isomorphic to the additive group of $\mathbf{R}$.

More precisely, let l be any straight line in the plane $\mathbf{R}^2$. For l, we may consider the family G_l of all those translations g of $\mathbf{R}^2$ which satisfy $g(l) = l$. In other words, G_l is the stabilizer of l in the group of all translations of $\mathbf{R}^2$. Also, l is equipped with the isomorphic image μ_l of λ and μ_l is invariant with respect to G_l. But there are many other measures on l which are invariant (or, more generally, quasi-invariant) under G_l. Let us denote by $\mathcal{M}_l$ the class of all nonzero σ-finite G_l-quasi-invariant measures on l (notice that the domains of such measures are various G_l-invariant σ-algebras of subsets of l).

According to the general definition presented above, we say that a set $Y \subset l$ is G_l-absolutely nonmeasurable in l if Y is nonmeasurable with respect to each measure from the class $\mathcal{M}_l$.

Using Lemma 1, it is not hard to show the validity of the next auxiliary statement.

Lemma 2. *Let l be a straight line in the plane $\mathbf{R}^2$ and let X be a set in l such that* $\mathrm{card}(l \setminus X) < \mathbf{c}$.

Then X contains a G_l-absolutely nonmeasurable subset of l.

Proof. Since $\mathrm{card}(l \setminus X) < \mathbf{c}$, there is an element $g \in G_l$ such that

$$(g + (l \setminus X)) \cap (l \setminus X) = \emptyset$$

or, equivalently,

$$(g + X) \cup X = l.$$

Now, taking into account Lemma 1, we conclude that X contains some G_l-absolutely nonmeasurable set. □

Lemma 3. *There exists a set $Z \subset \mathbf{R}^2$ which satisfies the following three conditions:*

(1) *Z is almost $\mathbf{R}^2$-invariant, i.e.,* $\mathrm{card}((h + Z) \triangle Z) < \mathbf{c}$ *for every $h \in \mathbf{R}^2$;*

(2) *the inner λ_2-measure of the set Z is equal to zero;*

(3) *for any straight line l in $\mathbf{R}^2$, the set $l \setminus Z$ has cardinality strictly less than* $\mathbf{c}$.

Proof. We follow the argument used in [5].

Let α be the least ordinal number of cardinality $\mathbf{c}$. We introduce the following notation.

$\{l_\xi : \xi < \alpha\}$ is the injective family of all straight lines in $\mathbf{R}^2$.

$\{F_\xi : \xi < \alpha\}$ is the family of all closed subsets of $\mathbf{R}^2$ having strictly positive λ_2-measure.

$\{G_\xi : \xi < \alpha\}$ is a family of groups of translations of $\mathbf{R}^2$ such that:

(a) $\{G_\xi : \xi < \alpha\}$ is increasing by the standard inclusion relation;

(b) $\mathrm{card}(G_\xi) \leq \mathrm{card}(\xi) + \omega$ for each ordinal $\xi < \alpha$;

(c) $\cup\{G_\xi : \xi < \alpha\}$ is the group of all translations of $\mathbf{R}^2$.

Further, we construct by transfinite recursion a family $\{z'_\xi : \xi < \alpha\}$ of points of $\mathbf{R}^2$.

Suppose that for an ordinal $\xi < \alpha$, the partial family $\{z'_\zeta : \zeta < \xi\}$ has already been defined. Let us put

$$L_\xi = G_\xi(\cup\{l_\zeta : \zeta < \xi\}) \cup G_\xi(\{z'_\zeta : \zeta < \xi\}).$$

Keeping in mind the fact that $\lambda_2(F_\xi) > 0$, it is not hard to show that there exists a point $z' \in F_\xi \setminus L_\xi$. Then we define $z'_\xi = z'$.

Proceeding in this manner, we obtain the required α-sequence $\{z'_\xi : \xi < \alpha\}$ of points of $\mathbf{R}^2$. It follows from the above construction that the set

$$Z' = \cup\{G_\xi(z'_\xi) : \xi < \alpha\}$$

is almost $\mathbf{R}^2$-invariant and λ_2-thick in $\mathbf{R}^2$. Moreover, it is not difficult to check that

$$\mathrm{card}(Z' \cap l) < \mathbf{c}$$

for every straight line l in $\mathbf{R}^2$. These properties of Z' imply that the set

$$Z = \mathbf{R}^2 \setminus Z'$$

satisfies all conditions (1), (2) and (3) of Lemma 3, so is as required. □

Lemma 4. *Let Z be a subset of $\mathbf{R}^2$ as in Lemma* 3.

There exists a complete translation invariant measure μ on $\mathbf{R}^2$ such that:

(1) *μ is an extension of λ_2;*

(2) *$Z \in \mathrm{dom}(\mu)$ and $\mu(Z) = 0$;*

(3) *every μ-measurable set $X \subset \mathbf{R}^2$ admits a representation in the form*

$$X = (X_0 \cup A) \setminus B,$$

where $X_0 \in \mathrm{dom}(\lambda_2)$ and $\mu(A) = \mu(B) = 0$ (in particular, the measures μ and λ_2 are metrically isomorphic).

Proof. Since Z satisfies conditions (1), (2) and (3) of Lemma 3, the required measure μ is obtained in the standard manner, by applying Marczewski's method of extending measures (see, e.g., [8, 9, 11]). Moreover, slightly modifying the transfinite construction of Z, it can be established that μ is a measure invariant under the group of all isometric transformations of $\mathbf{R}^2$. □

Using the above lemmas, we can prove the following statement.

Theorem 1. *For the measure μ indicated in Lemma* 4*, there exists a set $W \subset \mathbf{R}^2$ such that:*

(1) *$W \subset Z$ and, consequently, $\mu(W) = 0$;*

(2) *for any straight line l in $\mathbf{R}^2$, the set $l \cap W$ is G_l-absolutely nonmeasurable.*

Let α be the least ordinal number of cardinality $\mathbf{c}$. We again denote by $\{l_\xi : \xi < \alpha\}$ the injective family of all straight lines in $\mathbf{R}^2$.

Using the method of transfinite recursion, we construct a disjoint family $\{W_\xi : \xi < \alpha\}$ of sets which fulfil the following two conditions:

(a) $W_\xi \subset l_\xi \cap Z$ for each ordinal $\xi < \alpha$;

(b) W_ξ is G_{l_ξ}-absolutely nonmeasurable for each ordinal $\xi < \alpha$.

Assume that, for an ordinal $\xi < \alpha$, the partial disjoint family $\{W_\zeta : \zeta < \xi\}$ of sets has already been constructed so that

$$W_\zeta \subset l_\zeta \ (\zeta < \xi).$$

Take the straight line l_ξ and consider the set

$$P_\xi = (Z \cap l_\xi) \setminus \cup\{l_\zeta : \zeta < \xi\}.$$

Since $\mathrm{card}(l_\xi \setminus Z) < \mathbf{c}$, it is not difficult to verify that

$$\mathrm{card}(l_\xi \setminus P_\xi) < \mathbf{c}.$$

According to Lemma 2, there exists a set $T \subset P_\xi$ which is G_{l_ξ}-absolutely nonmeasurable. We then define $W_\xi = T$.

Proceeding in this manner, we get the disjoint family of sets $\{W_\xi : \xi < \alpha\}$. Finally, putting

$$W = \cup\{W_\xi : \xi < \alpha\},$$

we obtain the set W satisfying conditions (1) and (2) of Theorem 1.

The next auxiliary statement generalizes Lemma 2 to the case of $\mathbf{R}^n$.

Lemma 5. *Let $n \geq 1$ be a natural number and let $\{\Gamma_j : j \in J\}$ be a family of affine hyperplanes in the Euclidean space $\mathbf{R}^n$ such that* $\mathrm{card}(J) < \mathbf{c}$.

Then the set $\mathbf{R}^n \setminus \cup\{\Gamma_j : j \in J\}$ contains an $\mathbf{R}^n$-absolutely nonmeasurable subset.

This lemma can be deduced from the general Lemma 1.

Using Lemma 5, we obtain an analog of Theorem 1 for the space $\mathbf{R}^n$ and for the Lebesgue measure λ_n, where $n \geq 3$.

Theorem 2. *For any natural number $n \geq 3$, there exist a complete measure ν on $\mathbf{R}^n$ and a set $V \subset \mathbf{R}^n$ such that:*

(1) *ν extends λ_n and is invariant under the group of all isometric transformations of $\mathbf{R}^n$;*

(2) *ν is metrically isomorphic to λ_n;*

(3) $\nu(V) = 0$*;*

(4) *for every affine hyperplane Γ in $\mathbf{R}^n$, the set $V \cap \Gamma$ is absolutely nonmeasurable with respect to the class of all nonzero σ-finite translation quasi-invariant measures on Γ.*

A set $U \subset \mathbf{R}^n$ is called $\mathbf{R}^n$-negligible in $\mathbf{R}^n$ if U satisfies the following two relations:

(i) there exists at least one nonzero σ-finite $\mathbf{R}^n$-quasi-invariant measure θ such that $U \in \mathrm{dom}(\theta)$ (equivalently, U is not $\mathbf{R}^n$-absolutely nonmeasurable);

(ii) for every σ-finite $\mathbf{R}^n$-quasi-invariant measure θ' such that $U \in \mathrm{dom}(\theta')$, the equality $\theta'(U) = 0$ holds true.

Some structural properties of $\mathbf{R}^n$-negligible sets are considered in [4] and [6].

It would be interesting to study the question of whether there exists an $\mathbf{R}^n$-negligible set $U \subset \mathbf{R}^n$ such that, for any affine hyperplane Γ in $\mathbf{R}^n$, the set $U \cap \Gamma$ is absolutely nonmeasurable with respect to the class of all nonzero σ-finite translation quasi-invariant measures on Γ.

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DYNAMICS OF $2D$ SOLITONS IN MEDIA WITH VARIABLE DISPERSION: SIMULATION AND APPLICATIONS

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Abstract. Dynamics of multidimensional solitons in media with variable dispersion is studied numerically. The application of the obtained results to the dynamics of FMS waves in a magnetized plasma, and the 2-dimensional surface waves on shallow water are discussed.

In this paper we consider the problem of dynamics the multidimensional solitons which are described by the Kadomtsev–Petviashvili (KP) equation

$$\partial_t u + \alpha u \partial_x u + \beta \partial_x^3 u = \varkappa \int\limits_{-\infty}^{x} \Delta_\perp u dx, \tag{1}$$

in complex media with the varying in time and/or space dispersive parameter $\beta = \beta(t, \mathbf{r})$. This problem is mainly interesting from the point of view of its evident applications in physics of real complex media with the dispersion. For example, such situation can have place in the problems of the propagation of the 2-dimensional (2D) gravity and gravity-capillary waves on the surface of "shallow" water [5, 7] when β is defined respectively as

$$\beta = c_0 H^2/6$$

and

$$\beta = (c_0/6)[H^2 - 3\sigma/\rho g]$$

where H is the depth, ρ is the density, and σ is the coefficient of surface tension of fluid. If $H = H(t, x, y)$, β also becomes the function of the coordinates and time. Similar situation may have place on studying of the evolution of the 3D fast magnetosonic (FMS) waves in magnetized plasma [1, 6] in case of the inhomogeneous and/or non-stationary plasma and magnetic field when β is a function of the Alfv'en velocity $v_A = f[B(t, \mathbf{r}), n(t, \mathbf{r})]$ and angle $\theta = (\mathbf{k}\hat{\ }\mathbf{B})$:

$$\beta = v_A(c^2/2\omega_{0i}^2)(\cot^2\theta - m/M)$$

where m and M are the masses of electron and ion, respectively. It is well known [8] that the 1D solutions of the Korteweg-de Vries (KdV) equation (equation (1) with $\varkappa = 0$) with $\beta = \text{const}$ in dependence on value of β are divided into two classes: at $|\beta| < u_0(0, x)l/12$ (l is the characteristic wave length) they have soliton character, in an opposite case they are the wave packets with asymptotes being proportional to the derivative of the Airy function [7,8]. In these cases, the KdV equation can be integrated by the inverse scattering transform (IST) method [5, 7]. But, if $\beta = \beta(x, t)$ it is impossible principally, and it is necessary to resort to a numerical simulation. Similar situation has place for the multidimensional KP equation: in case $\beta = \beta(t, \mathbf{r})$ the dispersive term becomes quasi-linear and the model being not exactly integrable [7].

Here, the problem of study of structure and evolution of the nonlinear waves described by the KP equation with $\beta = \beta(t, \mathbf{r})$ is considered distracting from a specific type of the propagation medium. The numerical experiments were conducted for several model types of function β when at $t < t_{cr}$

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$\beta = \beta_0 = \text{const}$, and at $t \geq t_{cr}$

$$1)\ \beta(x) = \begin{cases} \beta, & x \leq a; \\ \beta_0 + c, & x > a; \end{cases} \tag{2}$$

$$2)\ \beta(x,t) = \begin{cases} \beta_0, & x \leq a; \\ \beta_0 + nc, n = (t - t_{cr})/\tau = 1, 2 \ldots; & x > a \end{cases} \tag{3}$$

$$3)\ \beta(t) = \beta_0(1 + k_0\overline{\beta}\sin\omega t), \quad \overline{\beta} = (\beta_{\max} - \beta_{\min})/2, \tag{4}$$
$$0 < k_0 < l, \quad \pi/2\tau < \omega < 2\pi/\tau,$$

a and c are constants. In terms of the propagation of the waves on shallow water that means respectively, that after reaching of time t_{cr}: 1) sharp "break of bottom"; 2) gradual "change of a height" of a segment of bottom; and 3) the "oscillations of bottom" with time take place.

In the first series of numerical experiments we investigated the evolution of initial pulse in case when at t_{cr} the spasmodic change of $\beta = \beta(t, x, y)$ has a place behind soliton ["negative" step when $c < 0$ in (2), (3)]. At this, the dependence of spatial structure of solution on parameter a was studied. The obtained results (see Figure 1) showed that in all cases the evolution leads to the formation of waving tail which is not connected with soliton going away and caused only by local influence of sudden change of the "relief" $\beta(t, x, y)$. Consequently, the formation of oscillatory structure is connected not so much with decreasing of a role of the dispersion effects behind soliton as with the spasmodic changing of β in space.

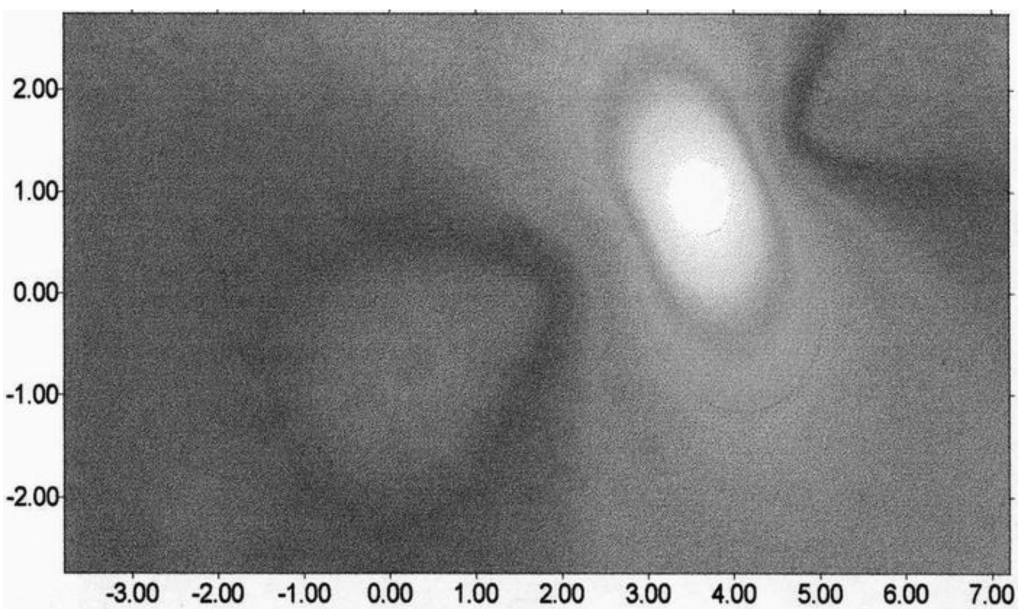

FIGURE 1. Evolution of a 2D soliton of equation (1) for the dispersion change law (3) at $a = 5.0$, $c = -0.0038$ for $t = 0.6$.

In the next series of simulations we considered a case when the sudden change of β takes place directly under or in front of an initial pulse ("negative" step). An example of the results is shown in Figure 2. One can see that for such character of the "relief" the disturbance caused by sudden change of β has also local character, i.e. it doesn't propagate together with the going away soliton.

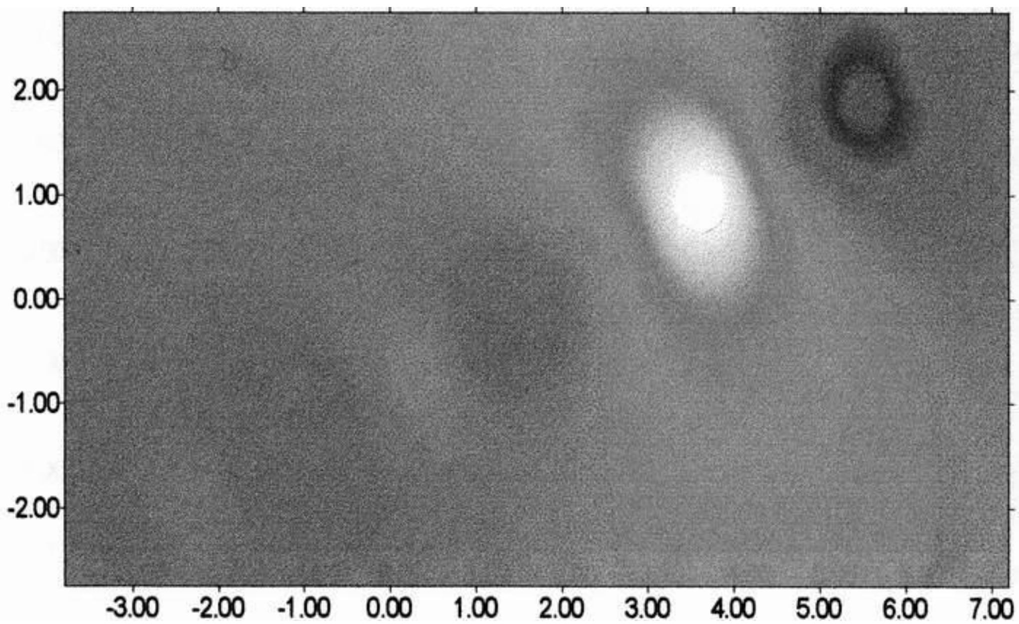

FIGURE 2. Evolution of a 2D soliton of equation 1 for the dispersion change law (2) at $a = 4.0$, $c = -0.0038$ for $t = 0.6$.

But, unlike the cases of the first series, the asymptotes of leaving soliton become oscillating, besides, against a background of the long-wave oscillations of the waving tail we can also see the appearance of the wave fluctuations. The effects noted can be interpreted as a result of those that for the areas of the wave surface with different values of local wave number k_x the value of the dispersive effects is different. As a result, the dispersive confusion of the Fourier-harmonics phases takes place in the (x, y)-region not equally intensity everywhere and, consequently, it counteracts with different extent of activity to the generation due to nonlinearity of the harmonics with big k_x.

In the next series of simulations with β changing with the laws (2) and (3) we considered the cases of "positive" step ($c > 0$) being both in front of and behind of initial pulse for the wide range of values of a. The examples of the most interesting results are shown in Figure 3.

One can see that when "positive" step is far in front of maximum of function $u(0, x, y)$ the soliton evolution on the initial stage does not differ qualitatively from that for $\beta = \text{const}$ (Figure 3a), but in the future their character is defined by presence of the step, namely the processes, caused by the same causes which have been noted for the results of the second series, begin to be developed (Figure 3b).

As we can see, the appreciable change of the soliton structure which can lead to wave falling is observed owing to intensive generation of the harmonics with big k_x in the soliton front region, even for rather small height of the step. Thus, the disturbance of the propagating 2D soliton has also local character.

As to equation (4), the simulation for different $k_0 = \text{const}$ and variable frequency ω showed that for some values of ω the stationary (locally) standing waves can be formed, in another cases the formation of the stationary periodical wave structures is possible, and in the intermediate cases a chaotic regime is usually realized.

In conclusion, we studied propagation of 2D solitons in complex media with variable dispersion, considering as a concrete example evolution of 2D solitary waves on shallow water. Let us note that such approach can be useful and effective in the problems of nonlinear dynamics of the FMS waves and wave beams in a magnetized plasma [1, 5–7], and also in problems of investigation of evolution and transformation of the internal gravity waves (IGW) and travelling ionospheric disturbances at

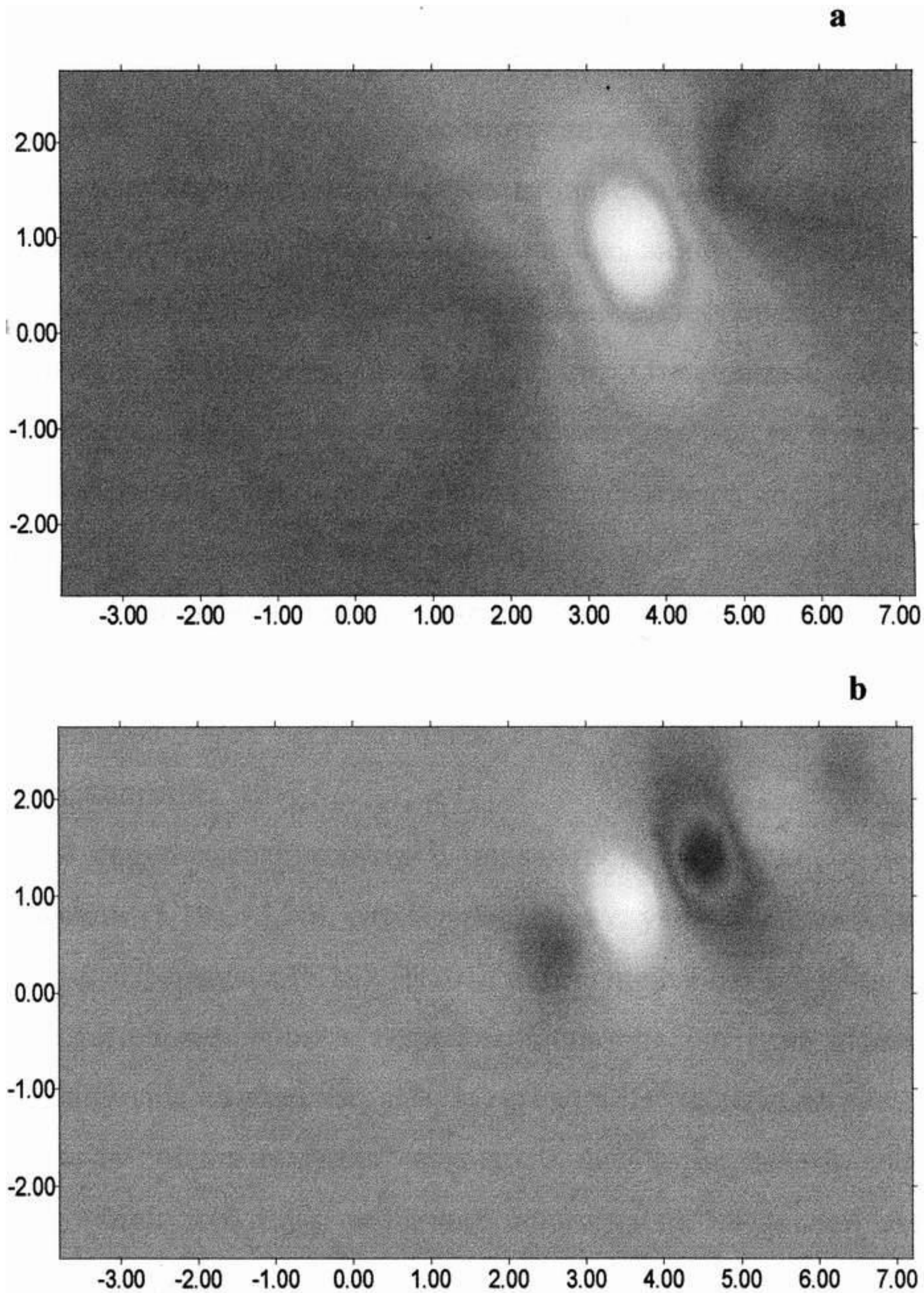

FIGURE 3. Evolution of a 2D soliton of equation (1) for the dispersion change law (3) at $a = 5.0$, $c = 0.0038$: (a) $t = 0.6$, (b) $t = 0.8$.

heights of the ionosphere F-region on fronts of the solar terminator and the solar eclipse spot [2, 4] and in regions where basic ionosphere characteristics are changed in time and space [3].

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ON THE INFLUENCE OF BOUNDARY CONDITIONS OF RIGID FIXING ON EIGEN-OSCILLATIONS AND THERMOSTABILITY OF SHELLS OF REVOLUTION, CLOSE BY THEIR FORM TO CYLINDRICAL ONES, WITH AN ELASTIC FILLER, UNDER THE ACTION OF PRESSURE AND TEMPERATURE

SERGO KUKUDZHANOV

Abstract. The influence of boundary conditions of rigid fixing on eigen-oscillations and thermostability of shells of revolution which by their form are close to cylindrical ones, with an elastic filler, under the action of external pressure and temperature, is investigated. We consider closed shells of middle length whose form of midsurface generatrix is defined by a parabolic function. The shells of positive and negative Gaussian curvature are studied. Formulas and graphs of dependence of the least frequency and form of wave formation on the type of boundary conditions, external pressure, temperature, rigidity of an elastic filler, as well as on the amplitude of shell deviation from the cylinder, are presented. Comparison of the given parameters with the situation when the shell ends are freely supported, is carried out. The question of thermostability is considered and the formula for finding critical pressure is given.

In the present paper we investigate the influence of boundary conditions of rigid fixing, temperature, external pressure and rigidity of an elastic filler on eigen-oscillations and stability of closed shells of revolution, close by their form to cylindrical ones. We consider a light filler for which the influence of tangential stresses on the contact surface and inertia forces may be neglected. The shell is assumed to be thin and elastic. Temperature is uniformly distributed in the shell body. An elastic filler is modelled by Winkler's base; its extension upon heating comes out of account. We investigate the shells of middle length whose form of the midsurface generatrix is defined by a parabolic function. We consider the shells of positive, as well as of negative Gaussian curvature. Formulas and universal curves of dependence of the least frequency and critical load on the Gaussian curvature, type of boundary conditions, temperature, rigidity of an elasic filler, as well as on the amplitude of shell deviation from the cylinder, are obtained. The question of thermostability is also considered and the formula for determination of critical pressure is given.

1. We consider the shell whose middle surface is formed by the rotation of square parabola around the z-axis of the rectangular system of coordinates x, y, z with the origin at the bisecting point of a segment of the axis of revolution. It is assumed that the radius R of the midsurface cross-section is defined by the equality $R = r + \delta_0[1 - \xi^2(r/\ell)^2]$, where r is the end-wall cross-section, δ_0 is the maximal deviation from the cylindrical form (for $\delta_0 > 0$, the shell is convex and for $\delta_0 < 0$, it is concave), $L = 2\ell$ is the shell length, $\xi = z/r$. We consider the shells of middle length [6] and it is assumed that

$$\big(\delta_0/r\big)^2,\ \big(\delta_0/\ell\big)^2 \ll 1. \tag{1}$$

For the shells of middle length, the forms of oscillation corresponding to the lower frequencies are accompanied by a weakly-marked wave formation in longitudinal direction as compared with the circumferential one, therefore the relation

$$\partial^2 f/\partial\xi^2 \ll \partial^2 f/\partial\varphi^2 \quad (f = u, v, w), \tag{2}$$

is valid, where u, v, w are, respectively, meridional, circumferential and radial displacement components characterizing oscillation form. Hence, according to Novozhilov's statement [3], as the basic

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equations of oscillations we can take those corresponding to Vlasov's semimomentless theory [5]. As a result of simplification, the system of equations takes the form (due to the adopted assumption, temperature terms are equal to zero [4])

$$\begin{gathered}\frac{\partial^2 u}{\partial \varphi^2} = -\big[1 + 2(1+\nu)\delta\big]\frac{\partial w}{\partial \xi}, \quad \frac{\partial^2 v}{\partial \varphi^2} = (1+2\nu\delta)\frac{\partial w}{\partial \varphi}\,, \\ \varepsilon\,\frac{\partial^8 w}{\partial \varphi^8} + \frac{\partial^4 w}{\partial \xi^4} + 4\delta\frac{\partial^4 w}{\partial \xi^2\,\partial \varphi^2} + 4\delta^2\frac{\partial^4 w}{\partial \varphi^4} - t_1^0\frac{\partial^6 w}{\partial \xi^2\,\partial \varphi^4} - t_2^0\frac{\partial^6 w}{\partial \varphi^6} \\ - 2s^0\frac{\partial^6 w}{\partial \xi\,\partial \varphi^5} + \gamma\frac{\partial^4 w}{\partial \varphi^4} + \frac{\rho r^2}{E}\frac{\partial^2}{\partial t^2}\Big(\frac{\partial^4 w}{\partial \varphi^4}\Big) = 0, \\ \varepsilon = h^2/12r^2(1-\nu^2), \quad \delta = \delta_0 r/\ell^2, \\ t_i = T_i^0/Eh \quad (i=1,2), \quad s^0 = S^0/Eh, \quad \gamma = \beta r^2/Eh,\end{gathered} \tag{3}$$

where E, ν is an elastic module and the Poison coefficient; T_1^0 and T_2^0 are, respectively, meridional and circumferential stresses of the initial state, S^0 is a shearing stress of the initial state; ρ is density of the shell material; β is the "bed" coefficient of the elastic filler (characterizing elastic rigidity of the filler); φ is angular coordinate, t is time.

The initial state is assumed to be momentless. With rigid fixing of the shell ends there are no meridional displacements at the ends. On the basis of a corresponding solution, taking into account the filler reaction, temperature and also equations (1), we obtain the following approximate expressions

$$\begin{aligned} T_1^0 &= -qr\left\{\nu + \frac{\delta_0}{r}\left[\frac{1+\nu}{3} + 2(1-2\nu^2)\big(r/\ell\big)^2 - (1-\nu^2)\xi^2\big(r/\ell\big)^2\right]\right\} - \frac{\alpha T E h}{1-\nu}, \\ T_2^0 &= -qr\left[1 - 2\nu\frac{\delta_0}{r}\Big(\frac{r}{\ell}\Big)^2\right] + w_0\beta_0 r, \quad S^0 = 0, \end{aligned} \tag{4}$$

where w_0 and β_0 are, respectively, deflection and "bed" coefficient of the filler in the initial state; α is the coefficient of linear extension; T is temperature; q is external pressure ($q > 0$).

Taking into account relations (1) and (2), we find that

$$\frac{\delta_0}{r}\left[\frac{1+\nu}{3} + 2(1-2\nu^2)\big(r/\ell\big)^2 - (1-\nu^2)\xi^2\big(r/\ell\big)^2\right]\frac{\partial^2 w}{\partial \xi^2} \ll \frac{\partial^2 w}{\partial \varphi^2}\,, \quad \nu\frac{\partial^2 w}{\partial \xi^2} \ll \frac{\partial^2 w}{\partial \varphi^2}\,.$$

Therefore expressions (4) after substitution into (3) can be simplified and they take the form

$$T_1^0 = -\frac{\alpha T E h}{1-\nu}\,, \quad T_2^0 = -qr\left[1 - 2\nu\frac{\delta_0}{r}\Big(\frac{r}{\ell}\Big)^2\right] + w_0\beta_0 r, \quad T_i^0 = \sigma_i^0 h \quad (i=1,2). \tag{4$'$}$$

Bering in mind that in the initial state the shell deformation in a circumferential direction is defined by the equalities

$$\varepsilon_\varphi^0 = \frac{\sigma_2^0 - \nu\sigma_1^0}{E} + \alpha T, \quad \varepsilon_\varphi^0 = -\frac{w_0}{r}$$

we get

$$w_0 = \big(-\sigma_2^0 + \nu\sigma_1^0\big)\frac{r}{E} - \alpha T r. \tag{5}$$

Substituting (5) into (4′), we obtain

$$\begin{aligned} \frac{T_1^0}{Eh} &= \frac{\sigma_1^0}{E} = -\frac{\alpha T}{1-\nu}\,, \\ \frac{T_2^0}{Eh} &= \frac{\sigma_2^0}{E} = -\frac{qr}{Eh}\left[1 - 2\nu\frac{\delta_0}{r}\Big(\frac{r}{\ell}\Big)^2\right] + \frac{\beta_0 r}{Eh}\Big[\big(-\sigma_2^0 + \nu\sigma_1^0\big)\frac{r}{E} - \alpha T r\Big]. \end{aligned} \tag{6}$$

Introduce the notation

$$\overline{q} = \frac{qr}{Eh}\,, \quad \delta = \frac{\delta_0}{r}\Big(\frac{r}{\ell}\Big)^2, \quad \gamma_0 = \frac{\beta_0 r^2}{Eh}\,, \quad g = 1 + \gamma_0.$$

Then (6) takes the form

$$\frac{\sigma_1^0}{E} = -\frac{\alpha T}{1-\nu}, \quad \frac{\sigma_2^0}{E} = -\overline{q}(1-2\nu\delta) + \left[\left(-\frac{\sigma_2^0}{E} + \nu\,\frac{\sigma_1^0}{E}\right)\gamma_0 - \alpha T\gamma_0\right], \tag{7}$$

whence we arrive at

$$\frac{\sigma_2^0}{E}(1+\gamma_0) = -\overline{q}(1-2\nu\delta) + \nu\gamma_0\,\frac{\sigma_1^0}{E} - \alpha T\gamma_0. \tag{8}$$

Substituting into (8) the first expression of (7), we obtain

$$\frac{\sigma_2^0}{E} = -\left[\overline{q}(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\right]g^{-1}.$$

Consequently,

$$-\frac{\sigma_1^0}{E} = \frac{\alpha T}{1-\nu}, \quad -\frac{\sigma_2^0}{E} = \left[\overline{q}(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\right]g^{-1}. \tag{9}$$

In view of the fact that R is close to r, in the expressions for stresses (9) we adopted $R \approx r$.

As a result, the third equation of system (3) takes the form

$$\begin{aligned} &\varepsilon\,\frac{\partial^8 w}{\partial\varphi^8} + \frac{\partial^4 w}{\partial\xi^4} + 4\delta\,\frac{\partial^4 w}{\partial\xi^2\,\partial\varphi^2} + 4\delta^2\,\frac{\partial^4 w}{\partial\varphi^4} + \left[\overline{q}(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\right]g^{-1}\frac{\partial^6 w}{\partial\varphi^6} \\ &\quad + \frac{\alpha T}{1-\nu}\,\frac{\partial^6 w}{\partial\xi^2\,\partial\varphi^4} + \gamma\,\frac{\partial^4 w}{\partial\varphi^4} + \frac{\rho r^4}{E}\,\frac{\partial^2}{\partial\varphi^2}\left(\frac{\partial^4 w}{\partial\varphi^4}\right) = 0. \end{aligned} \tag{10}$$

A solution of system (3) for harmonic oscillations of closed shells will be sought in the form

$$\begin{aligned} u &= U(\xi)\sin n\varphi\,\cos\omega t, \\ v &= V(\xi)\cos n\varphi\,\cos\omega t, \\ w &= W(\xi)\sin n\varphi\,\cos\omega t. \end{aligned}$$

From the first two equations of system (3) we obtain

$$n^2 U = \bigl[1 - 2(1-\nu)\delta\bigr]W', \tag{11}$$

$$nV = \bigl(1 + 2\nu\delta\bigr)W. \tag{12}$$

Note certain simplifications of boundary conditions of rigid fixing for the shells (for $\xi = \text{const}$) having the form

$$u = v = w = w'_\xi = 0. \tag{13}$$

On the basis of equality (12) we find that the fulfilment of the condition $w = 0$ leads to that of the condition $v = 0$, while in view of (11), the fulfilment of the condition $w'_\xi = 0$ leads to that of the condition $u = 0$.

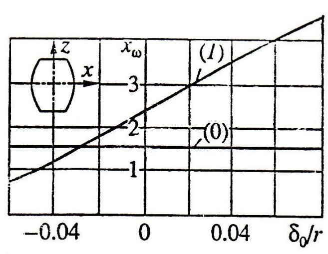

Figure 1

Thus if conditions $w = w'_\xi = 0$ ($\xi = \text{const}$) are fulfilled, then all conditions (13) are likewise fulfilled.

Let the shell edges be rigidly fixed. In addition, the solution should satisfy the condition of periodicity with respect to φ and also the following boundary conditions with respect to the coordinate ξ,

$$w = 0 \quad (\xi = \pm\,\ell/r), \quad w'_\xi = 0 \quad (\xi = \pm\,\ell/r). \tag{14}$$

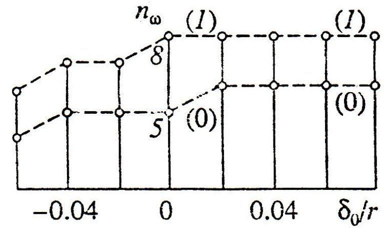

FIGURE 2

Solution w of equation (10), as is mentioned above, for harmonic oscillations is sought in the form

$$w = W \sin n\varphi \cos \omega t. \tag{15}$$

From (10) and (15) follows

$$W^{(4)} - \Big(4\delta n^2 - \frac{\alpha T}{1-\nu}\, n^4\Big) W^{(2)} - n^4 \Big\{ \frac{\rho r^2}{E}\, \omega^2 + \Big[\overline{q}(1-2\nu\delta) + \frac{\alpha T \gamma_0}{1-\nu}\Big] g^{-1} n^2 - \varepsilon n^4 - 4\overline{\delta}^{\,2} \Big\} W = 0, \quad \overline{\delta}^{\,2} = \delta^2 + \gamma/4. \tag{16}$$

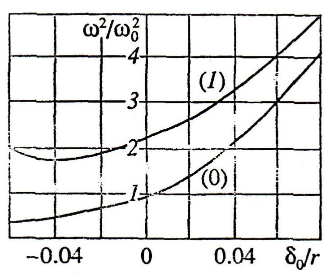

FIGURE 3

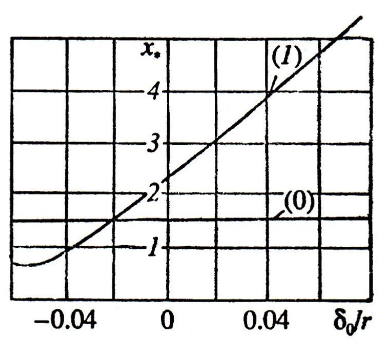

FIGURE 4

Assuming $W = Ce^{\alpha\xi}$, we obtain the following characteristic equation

$$\alpha^4 - \Big(4\delta n^2 - \frac{\alpha T}{1-\nu}\, n^4\Big)\alpha^2 - n^4 \Big\{ \frac{\rho r^2}{E}\, \omega^2 + \Big[\overline{q}(1-2\nu\delta) + \frac{\alpha T \gamma_0}{1-\nu}\Big] g^{-1} n^2 - \varepsilon n^4 - 4\overline{\delta}^{\,2} \Big\} = 0$$

which can be written as

$$p^2 - ap - b = 0, \quad \Omega = \rho r^2/E, \tag{17}$$

$$\begin{aligned} &p = \alpha^2, \quad a = 4\delta n^2 + \frac{\alpha T}{1-\nu}\, n^4, \\ &b = n^4\left\{\Omega\omega^2 + \left[\overline{q}(1-2\delta\nu) + \frac{\alpha T\gamma_2}{1-\nu}\right]g^{-1}n^2 - \varepsilon n^4 - 4\overline{\delta}^{\,2}\right\}. \end{aligned} \tag{18}$$

Proceeding from the condition $b > 0$, from (17) and (18) we have

$$\begin{gathered} \alpha_{1,2} = \pm\sqrt{p_1}, \quad \alpha_{3,4} = \pm\, i\sqrt{-p_2}, \\ p_1 = \frac{a}{2} + \sqrt{\frac{a^2}{4} + b} > 0, \quad p_2 = \frac{a}{2} - \sqrt{\frac{a^2}{4} + b} < 0. \end{gathered} \tag{19}$$

General solution of equation (16) takes the form

$$\begin{gathered} W = A\,\mathrm{ch}\,k_1\xi + B\,\mathrm{sh}\,k_1\xi + C\cos k_2\xi + D\sin k_2\xi, \\ k_1 = \sqrt{p_1}, \quad k_2 = \sqrt{-p_2}. \end{gathered}$$

Satisfying boundary conditions (14), we obtain the system of four homogeneous equations.
Since the determinant of that system is equal to zero, we get

$$\mathrm{th}\,k_1\overline{\ell} = \frac{k_1}{k_2}\,\mathrm{tg}\,k_2\overline{\ell} = -\frac{k_2}{k_1}\,\mathrm{tg}\,k_2\overline{\ell}, \quad \overline{\ell} = \ell/r. \tag{20}$$

Consequently, this system falls into two independent systems and hence a solution falls into odd and even functions. To the even function there correspond symmetric with respect to ξ forms of oscillations, while to the odd function there correspond skew-symmetric ones. Thus we obtain

$$\begin{aligned} W &= D\left(\sin k_2\xi - \frac{\sin k_2\overline{\ell}}{\mathrm{sh}\,k_1\overline{\ell}}\,\mathrm{sh}\,k_1\xi\right), \\ W &= C\left(\cos k_2\xi - \frac{\cos k_2\overline{\ell}}{\mathrm{ch}\,k_1\overline{\ell}}\,\mathrm{ch}\,k_1\xi\right). \end{aligned}$$

First, let us consider the case $\delta = 0$, $\overline{q} = \gamma = T = 0$ where $p_1 = -p_2 = \sqrt{b}$, $k_1 = k_2 = \sqrt[4]{b} = k$. Equation (20) corresponding to skew-symmetric forms of oscillations takes the form

$$\mathrm{th}\,k\overline{\ell} = \mathrm{tg}\,k\overline{\ell}.$$

To the lower root of that equation there corresponds the value

$$k = 3,927\,r/\ell,$$

where as equation (20) corresponding to the symmetric forms of oscillation take for $\delta = 0$, $q = \gamma = T = 0$ the form

$$\mathrm{th}\,k\overline{\ell} = -\,\mathrm{tg}\,k\overline{\ell}.$$

To the lower root of that equation there corresponds the value

$$k = 2,365\,r/\ell = 0,75\pi\,r/\ell, \tag{21}$$

i.e., the lower value k corresponds to the symmetric form of oscillation. Therefore in the sequel we will consider oscillations with symmetric form of deflection with respect to ξ. Taking into account that

$$-p_1p_2 = b, \quad b = n^4(\Omega\omega^2 - \varepsilon n^4),$$

for $\delta = 0$, $q = \gamma = T = 0$, we get

$$k^4 = n^4(\Omega\omega^2 - \varepsilon n^4).$$

This implies that to the lower root (21) for fixed n there corresponds the least value of eigen-frequency defined by the expression

$$\Omega\omega^2 = \varepsilon n^4 + (d_1\lambda_1)^4 n^{-4}, \quad d_1 = 1,55, \quad \lambda_1 = \pi r/2\ell.$$

The least frequency value depending on n is realized for

$$n_0^2 = d_1\lambda_1\varepsilon^{-1/4}. \tag{22}$$

For $n = n_0$, from (22), for the least frequency of cylindrical shell of middle length with rigidly fixing ends we obtain the known formula [1]

$$\Omega\omega_{01}^2 = 2\,d_1^2\lambda_1^2\varepsilon^{1/2}.$$

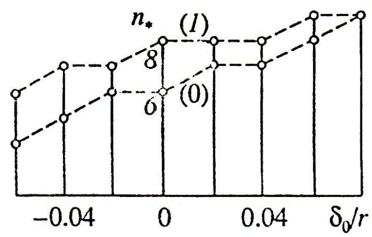

FIGURE 5

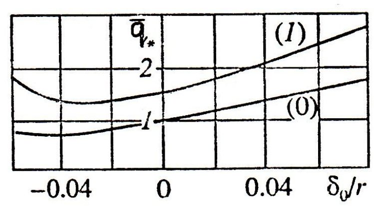

FIGURE 6

For freely supported ends, the least frequency of cylindrical shell is, as is known, defined by the formula

$$\Omega\omega_0^2 = 2\lambda_1^2\varepsilon^{1/2}. \tag{23}$$

Let us turn now to the general case and investigate axially symmetric forms of oscillations corresponding to lower frequencies. Relying on (19), we have

$$-p_2 = p_1 - a, \quad a = \Big(4\delta - \frac{\alpha T}{1-\nu}\,n^2\Big)n^2.$$

from which, putting $x = \bar{\ell}\sqrt{p_1}$, we obtain

$$-p_2\bar{\ell}^{\,2} = x^2 - \beta, \quad \beta = 4\,n^2\,\frac{\delta_0}{r} - \frac{\alpha T}{1-\nu}\Big(\frac{\ell}{r}\Big)^2 n^4. \tag{24}$$

Then equation (20) corresponding to symmetric forms of oscillations can be represented as

$$x\,\mathrm{th}\,x = -\sqrt{x^2-\beta}\;\mathrm{tg}\,\sqrt{x^2-\beta}. \tag{25}$$

On the basis of the first equality of (19), we have $p_1(p_1 - a) = b$ from which we find that

$$\Omega\omega^2 = \varepsilon n^4 + x^2(x^2-\beta)\Big(\frac{r}{\ell}\Big)^4 n^{-4} + 4\bar{\delta}^{\,2} - \Big[\bar{q}(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\Big]g^{-1}n^2. \tag{26}$$

Consequently, in a general case, the eigen-frequencies ω for the shells under consideration are defined by formula (26), where x is any root of equation (25). The least frequency ω is obtained by minimizing the right-hand side of (26) with respect to n, when as x we take the least root of equation (25) which we denote by x_ω. On the basis of (24) and (25), it is not difficult to see that x_ω depends both on δ_0/r, T and on n. Such a minimization is realized by sorting out natural values n in the neighbourhood n_0 defined by equality (22). Below we present results of our calculations for the shells with geometric dimensions $\ell = r$, $h/r = 10^{-2}$, $\nu = 0,3$ for different values δ_0/r (for $\overline{q} = \gamma = T = 0$). In Figure 1 we can see dependence of x_ω on δ_0/r (curve (1) corresponds to the rigid shell ends fixing; straight line (0) corresponds to the freely supported ends). Figure 2 presents dependence of n_ω on δ_0/r ((1) corresponds to rigidly fixing ends and (0) to freely supported ends). In Figure 3 we can see the curves of dependence of the least frequencies ω^2/ω_0^2 on δ_0/r ((1) is the case of rigidly fixing ends and (0) for fxreely supported ends [2]), ω_0^2 is defined by expression (23).

For $\omega = 0$, from (26) we obtain

$$\overline{q}(1-2v\delta) = \left[\varepsilon n^2 + x^2(x^2-\beta)n^{-6}\left(\frac{r}{\ell}\right)^4 + 4\overline{\delta}^{\,2} n^{-2}\right] g - \frac{\alpha T\gamma_0}{1-v}\,. \tag{27}$$

The least value $\overline{q}$ is obtained after minimization of the right-hand side of equality (28) depending on n, when as x we take the least positive root of equation (26) which is denoted by x_*. It is not difficult to see that on the basis of the value x_* depends on n_*. Corresponding values x_*, n_*, $\overline{q}_*/\overline{q}_{0*}$ are critical and presented depending on δ_0/r by the curves (1) in Figures 4, 5, 6 for $\gamma_0 = T = 0$. In Figure 6, over the Oy-axis is drawn the dimensionless critical pressure $\overline{q}_*/\overline{q}_{0*}$ ($\overline{q}_{0*}$ characterizes critical pressure for freely supported cylindrical shell and is defined by the equality $\overline{q}_{0*} = 0,855(1-\nu^2)^{-3/4}(h/r)^{3/2} r/L$ [6]). Comparing curves 1 in Figures 3, 6, it is not difficult to notice that their behaviour is qualitativly close: if for $\delta_0 > 0$ the values of the least frequency and of critical pressure increase, then for $\delta_0 < 0$ they first decrease up to $\delta_0/r \approx -(0,03 \div 0,04)$ and then increase. According to (27), the formula for finding critical pressure $\overline{q}_*$ has the form

$$\overline{q}_* = \frac{1+\gamma_0}{1-2\nu\delta}\left[\varepsilon n_*^2 + x_*^2(x_*^2-\beta)n^{-6}\big(r/\ell\big)^4 + 4\overline{\delta}^{\,2} n_*^{-2}\right] - \frac{\alpha T\gamma_0}{(1-\nu)(1-2\nu\delta)}\,.$$

Thus we have obtained formulas for determination of lower frequencies for the shells of revolution which by their form are close to cylindrical ones, depending on the boundary conditions of rigid fixing, amplitude of cylinder deviation, rigidity of an elastic filler, external pressure and temperature. The formula for determination of critical pressure depending on the above-mentioned factors, is also given.

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ON THE VECTOR FIBER SURFACE OF THE SPACE $Lm(Vn)$ OF TRIPLET CONNECTEDNESS

GOCHA TODUA

Abstract. In this paper, we consider the theory of the surface of metric vector fibers for the space $Lm(Vn)$ with triplet connectedness. It is proved that in metric vector fibers there always exists an internal triplet connectedness. Analogues of Gauss–Weingarten derivation formulas and also analogues of generalized Gauss, Peterson–Codazzi–Mainardi equations are found.

Let us consider the vector fiber space $Lm(Vn)$, where the local coordinates of a point transform by the law [2]

$$\overline{x^i} = \overline{x^l}(x^k); \qquad \overline{y^\alpha} = A^\alpha_\beta(x) y^\beta;$$
$$\det \left\| \frac{\partial \overline{x^i}}{\partial \overline{x^k}} \right\| \neq 0; \quad \det \|A^\alpha_\beta\| \neq 0; \quad i,j,k = 1,\dots,n; \quad \alpha,\beta,\gamma = 1,\dots,m. \tag{1}$$

Assume that the tensor field $G_{AB}(A,B,C = 1,2,\dots,n+m)$ is given on the space $Lm(Vn)$, i.e.

$$\overline{G_{AB}} = \overset{*}{\mathfrak{X}}{}^C_A \overset{*}{\mathfrak{X}}{}^D_B G_{CD},$$

where

$$\mathfrak{X}^A_B = \left\| \frac{\partial \overline{\mathfrak{X}}^A}{\partial \overline{\mathfrak{X}}^B} \right\| = \left\| \begin{matrix} \dfrac{\partial \overline{x}^i}{\partial x^j} & \dfrac{\partial \overline{x}^i}{\partial x^\alpha} \\[2ex] \dfrac{\partial \overline{x}^\alpha}{\partial x^i} & \dfrac{\partial \overline{x}^\alpha}{\partial x^\beta} \end{matrix} \right\| = \left\| \begin{matrix} x^i_j & 0 \\ A^\alpha_{\beta k} y^\beta & A^\alpha_\beta \end{matrix} \right\| .$$

An inverse matrix of the matrix has the form

$$\overset{*}{\mathfrak{X}}{}^A_B = \left\| \frac{\partial \overline{\mathfrak{X}}^A}{\partial \overline{\mathfrak{X}}^B} \right\| = \left\| \begin{matrix} \dfrac{\partial x^i}{\partial \overline{x}^k} & \dfrac{\partial x^i}{\partial \overline{x}^\alpha} \\[2ex] \dfrac{\partial x^\alpha}{\partial \overline{x}^i} & \dfrac{\partial x^\alpha}{\partial \overline{x}^\beta} \end{matrix} \right\| = \left\| \begin{matrix} \overset{*}{x}{}^i_j & 0 \\ \overset{*}{A}{}^\alpha_{\beta k} A^\beta_\gamma y^\gamma & \overset{*}{A}{}^\alpha_\beta \end{matrix} \right\| .$$

Since $G^{\alpha\beta} = A^\alpha_\gamma A^\beta_\delta G^{\gamma\delta}$, where $G_{\beta\gamma} G^{\alpha\beta} = \delta^\alpha_\gamma$ and $G_{\beta i} = \overset{*}{A}{}^\gamma_\beta \overset{*}{x}{}^k_i G_{\gamma k} + \overset{*}{A}{}^\gamma_\beta \overset{*}{A}{}^\alpha_{\varepsilon i} A^\varepsilon_\rho y^\rho G_{\gamma\alpha}$, we can use them to construct the values Γ^α_i as follows: $\Gamma^\alpha_i = G^{\alpha\beta} G_{\beta i}$.

Furthermore,

$$G^{\alpha\beta} G_{\beta i} = A^\alpha_\gamma A^\beta_\delta G^{\gamma\delta} \overset{*}{A}{}^\rho_\beta \overset{*}{x}{}^k_i G_{\rho k} + A^\beta_\gamma G^{\gamma\delta} \overset{*}{A}{}^\omega_\beta \overset{*}{A}{}^\sigma_{\varepsilon i} A^\varepsilon_\rho y^\rho G_{\omega\sigma}.$$

Since $A^\alpha_\gamma \overset{*}{A}{}^\alpha_\beta = \delta^\alpha_\beta$, $A_\gamma k^\alpha \overset{*}{A}{}^\gamma_\varepsilon + A^\alpha_\gamma \overset{*}{A}{}^\gamma_{\varepsilon i} x^i_k = 0$, $-\overset{*}{x}{}^k_i A^k_{\gamma k} A^\alpha_{\gamma k} \overset{*}{A}{}^\gamma_\varepsilon = A^\alpha_\gamma \overset{*}{A}{}^\gamma_{\varepsilon i}$, we observe that the values Γ^α_i form an object of linear connectedness with the following transformation law

$$\overset{*}{\Gamma}{}^\alpha_i = A^\alpha_\gamma \overset{*}{x}{}^k_i \Gamma^\gamma_k - \overset{*}{x}{}^k_i A^\alpha_{\gamma k} y^\gamma,$$

and the values

$$g_{ij} = G_{ij} - \Gamma^\alpha_i G_{\alpha j} - \Gamma^\beta_j G_{\beta i} + \Gamma^\alpha_i \Gamma^\beta_j G_{\alpha\beta}$$

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are a double covariant symmetric tensor so that we can construct an object of affine connectedness Γ^i_{jk} in the following manner

$$\Gamma^i_{jk} - \frac{1}{2} g^{ip}(\nabla_k g_{pj} + \nabla_j g_{kp} - \nabla_p g_{jk}).$$

Note that the linear connectedness Γ^α_i induces the vertical affine connectedness defined by the object $\nabla_\beta \Gamma^\alpha_i \equiv \Gamma^\alpha_{\beta i}$ with the following transformation law

$$\overline{\Gamma^\alpha_{\beta i}} = \overset{*}{x}{}^k_i \overset{*}{A}{}^\gamma_\beta A^\alpha_\delta \Gamma^\delta_{\gamma k} - A^\alpha_{\beta i}.$$

Structural equations of the space $Lm(Vn)$ with triplet connectedness have the form [3,4]:

$$\begin{cases} D\omega^i = \omega^k \wedge \widetilde{\omega}^i_k, \\ D\widetilde{\theta}^\alpha = \widetilde{\theta}^\beta \wedge \widetilde{\omega}^\alpha_\beta + R^\alpha_{ik}\omega^i \wedge \omega^k, \\ D\widetilde{\omega}^\alpha_\beta = \widetilde{\omega}^\gamma_\beta \wedge \widetilde{\omega}^\alpha_\gamma + R^\alpha_{\beta ik}\omega^i \wedge \omega^k + R^\alpha_{\beta i\gamma}\omega^i \wedge \widetilde{\theta}^\gamma, \\ D\widetilde{\omega}^i_j = \widetilde{\omega}^k_j \wedge \widetilde{\omega}^i_k + R^i_{jpq}\omega^p \wedge \omega^q + R^i_{jp\gamma}\omega^p \wedge \widetilde{\theta}^\gamma. \end{cases} \tag{2}$$

Assume that a hypersurface $\mathfrak{N}$ is given on the space $Lm(Vn)$

$$\omega^i = M^i_a \psi^a \tag{3}$$

and the 1-forms ψ^a re such that

$$\begin{cases} D\psi^a = \psi^b \wedge \psi^a_b, \\ D\psi^a_b = \psi^c_b \wedge \psi^a_c + \psi^c \wedge \psi^a_{bc}. \end{cases}$$

Note that

$$D\widetilde{\theta}^\alpha = \widetilde{\theta}^\beta \wedge \widetilde{\omega}^\alpha_\beta + R^\alpha_{ik} \wedge \omega^k = \widetilde{\theta}^\beta \wedge \widetilde{\omega}^\alpha_\beta + R^\alpha_{ik} M^i_a \psi^a \wedge M^k_a \psi^a = R^\alpha_{ab}\psi^a \wedge \psi^b,$$

where $R^\alpha_{ab} = R^\alpha_{ik} M^i_a M^k_b$.

The extension of system (3) is given by

$$\begin{cases} \nabla M^i_a = M^i_{ab}\psi^b, \quad \nabla M^i_{ab} + M^i_c \psi^c_{ab} = M^i_{abc}\psi^c, \\ \nabla M^i_{abc} + 2M^i_{(a|d|}\psi^d_{b)c} - M^i_d \psi^d_{bc} = M^i_{abd}\psi^d, \end{cases}$$

where

$$\begin{cases} M^i_{[ab]} = 0, \quad M^i_{a[bc]} = -R^i_{jpq} M^p_a M^q_b M^j_c, \\ M^i_{ab[cd]} = -R^i_{pqj} M^p_{ab} M^q_c M^j_d. \end{cases}$$

The values M^i_a, M^i_{ab} and M^i_{abc} form a fundamental third-order difference-geometric object of the surface $\mathfrak{N}$.

The normal vector of the hypersurface $\mathfrak{N}$ at the point T satisfies the equations

$$g_{ij} n^i M^i_a = 0, \qquad g_{ij} n^i n^j = 1.$$

A metric tensor of the hypersurface $\mathfrak{N}$ is written in the form

$$g_{ab} = g_{ij} M^i_a M^j_b$$

and $\nabla g_{ab} = g_{abc}\psi^c$, where $g_{abc} = g_{ij} M^i_a M^j_{bc} + g_{ij} M^i_{ac} M^j_b$.

The vectors $M^i_{ab} e_i$ and $n^i_a e_i$ admit representations in the form of a linear combination of vectors of the reference point $\{T, M_a, n\}$:

$$M^i_{ab} e_i = \mathbb{Q}_{ab}{}^c M_c + \mathcal{L}_{ab} n, \tag{4}$$

$$n^i_a e_i = \mathcal{L}^b_a M_b + n_a n, \tag{5}$$

where

$$\mathbb{Q}^c_{ab} = g^{cd} g_{ik} M^i_{ab} M^k_d, \quad \mathcal{L}_{ab} = g_{ki} n^k M^i_{ab}, \quad \mathcal{L}^b_a = -g^{cb}\mathcal{L}_{ca}, \quad n_a = g_{ki} n^k n^i_a. \tag{6}$$

We call equations (4) and (5) the Gauss-Weingarten formulas of the hypersurface $\mathfrak{N}$. From (6) we obtain

$$\nabla \mathbb{Q}^c_{ab} + M^c_{ab} = \mathbb{Q}^c_{abd}\psi^d, \tag{7}$$

$$\nabla \mathcal{L}_{ab} = \mathcal{L}_{abc}\psi^c, \tag{8}$$

where

$$\begin{cases} \mathbb{Q}^c_{abd} = g^{ec}_{\dots,d} g_{ik} M^i_{ab} M^k_e + g^{ei} g_{ik} M^i_{abd} M^k_e + g^{ec} g_{ik} M^i_{ab} M^k_{ed}, \\ \mathcal{L}_{abc} = g_{ki} n^k_c M^i_{ab} + g_{ki} n^k M^i_{abc}, \\ g^{ec}_{\dots,d} = -g^{ea} g^{bc} g_{abd}. \end{cases}$$

From (7) and (8) it follows that $\mathbb{Q}^c_{ab}$ is the object of affine connectedness and $\mathcal{L}_{ab}$ is the tensor.

We call the object $\mathbb{Q}^c_{ab}$ the object of induced affine connectedness of the hypersurface $\mathfrak{N}$. It is easy to prove that the induced affine connectedness and the internal affine connectedness coincide. The 1-forms of this connectedness have the form

$$\widetilde{\psi}^a_b = \psi^a_b + \mathbb{Q}^a_{bc}\psi^c.$$

It is obvious that

$$D\psi^a = \psi^b \wedge \widetilde{\psi}^a_b, \quad D\widetilde{\psi}^a_b = \widetilde{\psi}^c_b \wedge \widetilde{\psi}^a_c + M^a_{bcd}\psi^c \wedge \psi^d,$$

where

$$M^a_{bcd} = \mathbb{Q}^a_{b[cd]} - \mathbb{Q}^a_{e[c}\mathbb{Q}^e_{|b|d]}.$$

The values M^a_{bcd} form the tensor which we call the curvature tensor of the hypersurface $\mathfrak{N}$. By extending equation (4) we obtain

$$R^i_{jpq} M^k_a M^p_b M^q_c = \big(M^d_{abc} - \mathcal{L}_{a[b}\mathcal{L}^d_{c]}\big) M^i_d - \big(\overset{k}{\nabla}_{[c}\mathcal{L}_{|a|b]} - M^d_{bc}\mathcal{L}_{ad} + \mathcal{L}_{a[b} n_{c]}\big) n^i,$$

where $\overset{k}{\nabla}_c$ is the symbol of nonholomorphic covariant differentiation.

From the above equalities we obtain the generalized Gauss equations

$$R_{ipq} M^j_a M^p_b M^q_c M^i_e = M^i_{abce} + \mathcal{L}_{a[b}\mathcal{L}_{c]d} \tag{9}$$

and the generalized Peterson–Codazzi–Mainardi equations

$$R_{kpqr} M^k_a M^q_b M^r_c n^p = M^d_{bc}\mathcal{L}_{ad} - \overset{k}{\nabla}_{[c}\mathcal{L}_{|a|b]} - \mathcal{L}_{a[b} n_{c]}, \tag{10}$$

where

$$R_{ipqr} = q_{ij} R^j_{pqr}, \quad M_{abce} = g_{de} M^d_{abc}.$$

Equations (9) and (10) establish the connection between the curvature tensor of the space $Lm(Vn)$ and the curvature tensor of the hypersurface $\mathfrak{N}$ in $Lm(Vn)$ [1,5].

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EXTENSION OPERATORS ON SOBOLEV SPACES WITH DECREASING INTEGRABILITY

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Abstract. We study extension operators on Sobolev spaces with decreasing integrability on the base of set functions associated with the operator norms. Sharp necessary conditions are given in terms of the generalized measure density condition and in terms of a weak integral equivalence of the Euclidean metric and an intrinsic metric.

1. Introduction

Let Ω be a domain in the Euclidean space $\mathbb{R}^n$, $n \geq 2$. Recall that the operator

$$E : W^1_p(\Omega) \to W^1_q(\mathbb{R}^n), \quad 1 \leq q \leq p \leq \infty,$$

is called an extension operator on Sobolev spaces (with decreasing integrability in the case $q < p$), if $E(f)\big|_{\Omega} = f$ for any function $f \in W^1_p(\Omega)$ and

$$\|E\| = \sup_{f \in W^1_p(\Omega) \setminus \{0\}} \frac{\|E(f) \mid W^1_q(\mathbb{R}^n)\|}{\|f \mid W^1_p(\Omega)\|} < \infty.$$

Sobolev extension operators arise in the analysis of PDE (see, for example, [15, 21]) and play an important role in the Sobolev spaces theory. In the present article we prove the sharp Ahlfors type necessary generalized (p, q)-measure density condition for extension operators of seminormed Sobolev spaces: *Let there exist a continuous linear extension operator $E : L^1_p(\Omega) \to L^1_q(\mathbb{R}^n)$, $n < q \leq p < \infty$, then*

$$\Phi(B(x, r))^{p-q} |B(x, r) \cap \Omega|^q \geq c_0 |B(x, r)|^p, \ 0 < r < 1, \tag{1.1}$$

where Φ is an additive set function associated with the extension operator and a constant $c_0 = c_0(p, q, n)$ depends on p, q and n only. In the case $p = q$ the measure density condition was introduced in [9] (see, also [24]) and the study of the case $q < p$ requires the use of set functions associated with the extension operators [23, 28].

It is well known [3, 21] that if $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, then there exists the bounded extension operator $E : W^1_p(\Omega) \to W^1_p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. In [11], the notion of (ε, δ)-domains was introduced and it was proved that in every (ε, δ)-domain there exists the bounded extension operator $E : W^k_p(\Omega) \to W^k_p(\mathbb{R}^n)$, for all $k \geq 1$ and $p \geq 1$.

The complete description of extension operators of the homogeneous Sobolev space $L^1_2(\Omega)$, $\Omega \subset \mathbb{R}^2$, was obtained in [26] in terms of the quasi-hyperbolic (quasiconformal) geometry of domains. Namely, it was proved that a simply connected domain $\Omega \subset \mathbb{R}^2$ is the L^1_2-extension domain iff Ω is an Ahlfors domain (quasi-disc). In the case of spaces $L^k_p(\Omega)$, $2 < p < \infty$, defined in domains $\Omega \subset \mathbb{R}^2$, the necessary and sufficient conditions were obtained in [20] and formulated in terms of sub-hyperbolic metrics. Note that extension operators on Sobolev spaces $W^k_p(\Omega)$ were intensively studied in the last decade (see, for example, [4, 9, 12, 13, 19]), but the problem of the complete characterization of Sobolev extension domains in the general case remains still open.

In the case $p > n$, the necessary conditions on W^1_p-extension domains written in terms of an intrinsic metric and a measure density were obtained in [24]. Extension operators of Sobolev spaces defined in domains of Carnot groups $E : W^1_p(\Omega) \to W^1_p(\mathbb{G})$ were considered in [8] and extensions of Sobolev spaces on metric measure spaces can be found in [10].

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The results of [9, 24] state that the extension operator

$$E : W^1_p(\Omega) \to W^1_p(\mathbb{R}^n), \quad 1 \leq p < \infty,$$

does not exist in Hölder cusp domains $\Omega \subset \mathbb{R}^n$. In [7], the extension operators with decreasing integrability from domains with the Hölder cusps were constructed by using the method of reflections. Later, the more general theory of composition operators on Sobolev spaces with decreasing integrability was founded in [22, 27]. Using another technique, the extension operators in such type of domains were considered in [16, 17]. An extension operator from Hölder singular domains with decreasing smoothness was studied in [2]. The detailed study of extension operators on Sobolev spaces defined in non-Lipschitz domains is given in [18].

Extension operators with decreasing integrability are considered in [23], where for the first time was introduced a set function (measure) associated with extension operators and were obtained the necessary conditions in integral terms. In the present article, we give a sharp necessary condition of the existence of extension operators on Sobolev spaces with decreasing integrability in capacitary terms and prove the generalized (p, q)-measure density condition that refines the results of [23] and generalized [9] in the case $n < q < p < \infty$.

The necessary conditions written in terms of intrinsic metrics are considered also. On this base, the lower estimates of norms of extension operators are obtained. The norm estimates of extension operators have applications in the spectral theory of non-linear elliptic operators and give estimates of Neumann eigenvalues in terms of operator's norms [5].

2. Set Functions Associated with the Extension Operator

Let Ω be a domain in the Euclidean space $\mathbb{R}^n$, $n \geq 2$, then the Sobolev space $W^1_p(\Omega)$, $1 \leq p \leq \infty$, is defined as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W^1_p(\Omega)\| = \|f \mid L_p(\Omega)\| + \|\nabla f \mid L_p(\Omega)\|,$$

where ∇f is the weak gradient of the function f and $\|\nabla f \mid L_p(\Omega)\|$ its norm in the Lebesgue space $L_p(\Omega)$. The homogeneous seminormed Sobolev space $L^1_p(\Omega)$, $1 \leq p \leq \infty$, is considered with the seminorm

$$\|f \mid L^1_p(\Omega)\| = \|\nabla f \mid L_p(\Omega)\|.$$

We consider the Sobolev spaces as Banach spaces of equivalence classes of functions up to a set of p-capacity zero [15].

2.1. **Set functions and capacity.**

Let $A \subset \mathbb{R}^n$ be an open bounded set such that $A \cap \Omega \neq \emptyset$. Denote by $W_0(A; \Omega)$ the class of continuous functions $f \in L^1_p(\Omega)$ such that $f\eta$ belongs to $L^1_p(A \cap \Omega) \cap C_0(A \cap \Omega)$ for all smooth functions $\eta \in C^\infty_0(\Omega)$. We define the set function

$$\Phi(A) = \sup_{f \in W_0(A;\Omega)} \left(\frac{\|E(f) \mid L^1_q(A)\|}{\|f \mid L^1_p(A \cap \Omega)\|} \right)^\kappa, \quad \frac{1}{\kappa} = \frac{1}{q} - \frac{1}{p}.$$

This set function was introduced in [23] in connection with the lower estimates of norms of extension operators on Sobolev spaces. For readers convenience we give the detailed proof of the following theorem announced in [23]:

Theorem 2.1. *Let there exist a continuous linear extension operator*

$$E : L^1_p(\Omega) \to L^1_q(\mathbb{R}^n), \quad 1 \leq q < p < \infty.$$

Then the function $\Phi(A)$ is a bounded monotone countably additive set function defined on open bounded subsets $A \subset \mathbb{R}^n$.

Proof. Let $A_1 \subset A_2$ be open subsets of $\mathbb{R}^n$. Then extending functions of $C_0(A_1)$ by zero, we have $C_0(A_1) \subset C_0(A_2)$ and obtain

$$\Phi(A_1) = \sup_{f\in W_0(A_1;\Omega)} \left(\frac{\|E(f) \mid L^1_q(A_1)\|}{\|f \mid L^1_p(A_1\cap\Omega)\|}\right)^{\kappa} \leq \sup_{f\in W_0(A_1;\Omega)} \left(\frac{\|E(f) \mid L^1_q(A_2)\|}{\|f \mid L^1_p(A_2\cap\Omega)\|}\right)^{\kappa}$$
$$\leq \sup_{f\in W_0(A_2;\Omega)} \left(\frac{\|E(f) \mid L^1_q(A_2)\|}{\|f \mid L^1_p(A_2\cap\Omega)\|}\right)^{\kappa} = \Phi(A_2).$$

Hence Φ is the monotone set function.

Consider open bounded disjoint sets A_k, $k = 1, 2, \dots$ such that $A_0 = \bigcup_{k=1}^{\infty} A_k$. We choose arbitrary functions $f_k \in W_0(A_k;\Omega)$ such that

$$\|E(f_k) \mid L^1_q(A_k)\| \geq \left(\Phi(A_k)\left(1-\frac{\varepsilon}{2^k}\right)\right)^{\frac{1}{\kappa}} \|f_k \mid L^1_p(A_k\cap\Omega)\|,$$
$$\|f_k \mid L^1_p(A_k\cap\Omega)\|^p = \Phi(A_k)\left(1-\frac{\varepsilon}{2^k}\right),$$

where $k = 1, 2, \dots$ and $\varepsilon \in (0,1)$ is a fixed number. Setting $g_N = \sum\limits_{k=1}^{N} f_k$, we find that

$$\left\|E(g_N)\middle| L^1_q\left(\bigcup_{k=1}^{N} A_k\right)\right\| \geq \left(\sum_{k=1}^{N}\left(\Phi(A_k)\left(1-\frac{\varepsilon}{2^k}\right)\right)^{\frac{q}{\kappa}} \|g_N \mid L^1_p(A_k\cap\Omega)\|^q\right)^{\frac{1}{q}}$$
$$= \left(\sum_{k=1}^{N}\Phi(A_k)\left(1-\frac{\varepsilon}{2^k}\right)\right)^{\frac{1}{\kappa}} \left\|g_N\middle| L^1_p\left(\left(\bigcup_{k=1}^{N} A_k\right)\cap\Omega\right)\right\|$$
$$\geq \left(\sum_{k=1}^{N}\Phi(A_k) - \varepsilon\Phi(A_0)\right)^{\frac{1}{\kappa}} \left\|g_N\middle| L^1_p\left(\left(\bigcup_{k=1}^{N} A_k\right)\cap\Omega\right)\right\|,$$

since the sets, where $\nabla E(f_k)$ do not vanish, are disjoint. By the last inequality, we have

$$\Phi(A_0)^{\frac{1}{\kappa}} \geq \sup \frac{\left\|E(g_N)\middle| L^1_q\left(\bigcup_{k=1}^{N} A_k\right)\right\|}{\left\|g_N\middle| L^1_p\left(\left(\bigcup_{k=1}^{N} A_k\right)\cap\Omega\right)\right\|} \geq \left(\sum_{k=1}^{N}\Phi(A_k) - \varepsilon\Phi(A_0)\right)^{\frac{1}{\kappa}},$$

where the upper bound is taken over all the above functions $g_N \in W_0\left(\left(\bigcup_{k=1}^{N} A_k\right);\Omega\right)$. Since both N and ε are arbitrary, we have

$$\sum_{k=1}^{\infty}\Phi(A_k) \leq \Phi\left(\bigcup_{k=1}^{\infty} A_k\right).$$

The inverse inequality can be proved directly. □

Corollary 2.2. *Let there exist a continuous linear extension operator*

$$E : L^1_p(\Omega) \to L^1_q(\mathbb{R}^n), \quad 1 \leq q < p < \infty.$$

Then

$$\|E(f) \mid L^1_q(A)\| \leq \Phi(A)^{\frac{1}{\kappa}} \|f \mid L^1_p(A\cap\Omega)\|, \quad \frac{1}{\kappa} = \frac{1}{q} - \frac{1}{p}, \tag{2.1}$$

for any function $f \in W^1_\infty(A) \cap C_0(A)$.

Recall the notion of a variational p-capacity [6]. The condenser in the domain $\Omega \subset \mathbb{R}^n$ is the pair (F_0, F_1) of connected, closed relatively to Ω, sets $F_0, F_1 \subset \Omega$. A continuous function $f \in L^1_p(\Omega)$ is called an admissible function for the condenser (F_0, F_1), if the set $F_i \cap \Omega$ is contained in some connected component of the set $\mathrm{Int}\{x \in \Omega : f(x) = i\}$, $i = 0, 1$. We call as the p-capacity of the condenser (F_0, F_1) relatively to the domain Ω the following quantity:

$$\mathrm{cap}_p(F_0, F_1; \Omega) = \inf \|f | L^1_p(\Omega)\|^p. \tag{2.2}$$

Here the greatest lower bond is taken over all functions, admissible for the condenser $(F_0, F_1) \subset \Omega$. If the condenser has no admissible functions, we put the capacity equal to infinity.

Let $F_1 = E$ be a subset of open set $U \subset \Omega$ and $F_0 = \Omega \setminus U$, then the condenser $R = (E, U) = (\Omega \setminus U, E)$ is called a ring condenser, or a ring. Note that the infimum in (2.2) can be taken on over functions $f \in C_0^\infty(\Omega)$ such that $f = 1$ on E and $f = 0$ on $\Omega \setminus U$.

Theorem 2.3. *Let there exist a continuous linear extension operator*

$$E : L^1_p(\Omega) \to L^1_q(\mathbb{R}^n), \quad 1 \leq q < p < \infty.$$

Then for any compact set $E \subset (U \cap \Omega)$ the inequality

$$\mathrm{cap}_q^{\frac{1}{q}}(E, U) \leq \Phi(U)^{\frac{1}{\kappa}} \mathrm{cap}_p^{\frac{1}{p}}(E, (U \cap \Omega)), \quad \frac{1}{\kappa} = \frac{1}{q} - \frac{1}{p}, \tag{2.3}$$

holds for any open set $U \subset \mathbb{R}^n$.

Proof. Let a smooth function $u \in L^1_p(\Omega)$ be an admissible function for the condenser $(E, (U \cap \Omega)) \subset \Omega$. Then, extending u by zero on the set $U \setminus \Omega$ we obtain the function $E(u) \in L^1_q(\mathbb{R}^n)$ which is an admissible function for the condenser $(E, U) \subset \mathbb{R}^n$. Hence, by inequality (2.1), we have

$$\mathrm{cap}_q^{\frac{1}{q}}(E, U) \leq \Phi(U)^{\frac{1}{\kappa}} \|u \mid L^1_p(\Omega)\|.$$

Since u is an arbitrary admissible function for the condenser $(E, (U \cap \Omega)) \subset \Omega$, therefore

$$\mathrm{cap}_q^{\frac{1}{q}}(E, U) \leq \Phi(U)^{\frac{1}{\kappa}} \mathrm{cap}_p^{\frac{1}{p}}(E, (U \cap \Omega)). \qquad \square$$

2.2. **Generalized (p, q)-measure density conditions.** Consider measure density conditions in domains allowing the extension of operators with decreasing integrability.

Theorem 2.4. *Let there exist a continuous linear extension operator*

$$E : L^1_p(\Omega) \to L^1_q(\mathbb{R}^n), \quad n < q < p < \infty.$$

Then the domain Ω satisfies the generalized (p, q)-measure density condition

$$\Phi(B(x, r))^{p-q} |B(x, r) \cap \Omega|^q \geq c_0 |B(x, r)|^p, \quad 0 < r < 1,$$

where $x \in \overline{\Omega}$ and a constant $c_0 = c_0(p, q, n)$ depends on p, q and n only.

Proof. Fix a smooth test function $\eta : \mathbb{R}^n \to \mathbb{R}$ with $\mathrm{supp}(\eta) \subset B(0, 1)$ such that η is equal to 1 in the neighborhood of $0 \in \mathbb{R}^n$ and $0 \leq \eta(x) \leq 1$ for all $x \in \mathbb{R}^n$. Consider the points $x \in \overline{\Omega}$, $y \in \Omega$, and denote by $r := |x - y|$. Then the function

$$f(z) = \eta\left(\frac{x - z}{r}\right)$$

is a smooth function such that $f = 1$ in the neighborhood of $x \in \overline{\Omega}$, $f(y) = 0$ and

$$|\nabla f(z)| \leq \frac{\widetilde{C}}{r} \quad \text{for all} \quad z \in \mathbb{R}^n.$$

Substituting this test function f into inequality (2.1), we obtain

$$\begin{aligned} \|f \mid L^1_q(B(x, r)\| &\leq \Phi(B(x, r))^{\frac{p-q}{pq}} \|f \mid L^1_p(B(x, r) \cap \Omega)\| \\ &\leq \Phi(B(x, r))^{\frac{p-q}{pq}} \frac{\widetilde{C}}{r} |B(x, r) \cap \Omega|^{\frac{1}{p}}. \end{aligned}$$

Because $q > n$, applying the embedding theorem of the space of compact supported Sobolev functions to the space of Hölder continuous functions (see, for e.g., [15])

$$L^1_q(B(x, r)) \hookrightarrow H^\gamma(B(x, r)), \quad \gamma = 1 - n/q,$$

we have

$$\frac{1}{|x-y|^{1-\frac{n}{q}}} = \frac{|f(x)-f(y)|}{|x-y|^{1-\frac{n}{q}}} \leq \|f \mid H^{\gamma}B(x,r)\| \leq C\|f \mid L^1_q(B(x,r)\|.$$

So, using these inequalities, we obtain

$$\frac{(r^n)^{\frac{1}{q}}}{r} = \frac{1}{|x-y|^{1-\frac{n}{q}}} \leq \Phi(B(x,r))^{\frac{p-q}{pq}} C\frac{\widetilde{C}}{r}|B(x,r)\cap\Omega|^{\frac{1}{p}}.$$

Hence

$$(r^n)^{\frac{1}{q}} \leq \Phi(B(x,r))^{\frac{p-q}{pq}} C\widetilde{C}|B(x,r)\cap\Omega|^{\frac{1}{p}},$$

and the required inequality is proved. □

To prove the sharpness of condition (1.1), we consider as an example the Hölder singular domain Ω_α, $\alpha > 1$, [7, 14, 18]:

$$\Omega_\alpha = \left\{(x_1,x_2)\in\mathbb{R}^2 : 0 < x_1 \leq 1, |x_2| < x_1^\alpha\right\} \cup B((2,0),\sqrt{2}).$$

Then $|B(0,r)\cap\Omega_\alpha| = cr^{\alpha+1}$ and, substituting it into inequality (1.1), we obtain

$$\Phi(B(0,r))^{p-q}r^{(\alpha+1)q} \geq Cr^{2p}, \quad 0 < r < 1.$$

Hence $1 \leq q < 2p/(\alpha+1)$ that coincide with the sufficient condition of the existence of (p,q)-extension operators [7, 14, 18]. So, the necessary condition of Theorem 2.4 is sharp.

Recall that a bounded domain $\Omega\subset\mathbb{R}^n$ is called α-integral regular domain [23] if the function

$$K(x) = \limsup_{r\to 0}\frac{|B(x,r)|}{|B(x,r)\cap\Omega|}$$

belongs to the Lebesgue spaces $L_\alpha(\overline{\Omega})$. From Theorem 2.4 follows the assertion which was originally formulated in [23]:

Theorem 2.5. *Let there exist a continuous linear extension operator*

$$E : L^1_p(\Omega)\to L^1_q(\mathbb{R}^n), \quad n < q < p < \infty.$$

Then the domain Ω is α-integral regular for $\alpha = q/(p-q)$ and

$$\|E\| \geq c_1\|K \mid L_\alpha(\overline{\Omega})\|^{\frac{1}{p}},$$

where a constant $c_1 = c_1(p,q,n)$ depends on p, q and n only.

Proof. Rewrite inequality (1.1) in the form

$$\left(\frac{|B(x,r)|}{|B(x,r)\cap\Omega|}\right)^{\frac{q}{p-q}} \leq \frac{1}{c_1^\kappa}\frac{\Phi(B(x,r))}{|B(x,r)|}.$$

Putting $r\to 0$ and using the Lebesgue type differentiability theorem, we have

$$K(x)^\alpha \leq \frac{1}{c_1^\kappa}\Phi'(x), \text{ for almost all } x\in\overline{\Omega}.$$

Integrating the last inequality on the closed domain $\overline{\Omega}$, we find that for any bounded open set $\overline{\Omega}\subset U\subset\mathbb{R}^n$,

$$\int\limits_{\overline{\Omega}} K(x)^\alpha\, dx \leq \frac{1}{c_1^\kappa}\int\limits_{\overline{\Omega}}\Phi'(x)\, dx \leq \frac{1}{c_1^\kappa}\int\limits_U \Phi'(x)\, dx = \frac{1}{c_1^\kappa}\Phi(U) \leq \frac{1}{c_1^\kappa}\|E\|^\kappa.$$

□

2.3. **Intrinsic metrics in extension domains.** Let $\gamma : [a,b] \to \Omega$ be a rectifiable curve, then the length $l(\gamma)$ can be calculated by the formula

$$l(\gamma) = \int\limits_a^b \langle \dot\gamma(t), \dot\gamma(t)\rangle^{\frac{1}{2}}\ dt.$$

In the domain $\Omega \subset \mathbb{R}^n$, we define an intrinsic metric in Alexandrov's sense [1]:

$$d_\Omega(x,y) = \inf l(\gamma(x,y)), \quad x,y \in \Omega,$$

where infimum is taken over all rectifiable curves $\gamma \subset \Omega$, joint points $x, y \in \Omega$.

We use the following lemma [8,25].

Lemma 2.6. *For any points $x, y \in \Omega$ there exists a function $f \in W^1_\infty(\Omega)$ such that:*
(1) $0 \leq f(t) \leq 1$ *for any* $t \in \Omega$, $f(x) = 1$ *and* $f(y) = 0$,
(2) $|f(t) - f(s)| \leq d_\Omega(t,s)/d_\Omega(x,y)$,
(3) $\operatorname{supp}(f) \subset B(x, d_\Omega(x,y)) := B(x,R)$,
(4) $|\nabla f| \leq 1/d_\Omega(x,y)$ *a.e. in* Ω.

Note that the proof of this lemma is based on the test function

$$f(t) = \frac{d_\Omega(t,\Omega_x)}{d_\Omega(x,y)}, \quad t \in \Omega,$$

for fixed $x,y \in \Omega$, which was introduced in [25] (see, also [8]). The sets Ω_x and $d_\Omega(t,\Omega_x)$ are defined for the fixed points $x,y \in \Omega$ by the formulas

$$\Omega_x = \{s \in \Omega : d_\Omega(x,s) \geq d_\Omega(x,y)\}$$

and

$$d_\Omega(t,\Omega_x) = \inf\{d_\Omega(t,s) : s \in \Omega_x\}.$$

In the following theorem we give the relation between the intrinsic metric and the Euclidean metric in (p,q)-extension domains that can be considered as a generalized Ahlfors type metric condition.

Theorem 2.7. *Let there exist a continuous linear extension operator*

$$E : L^1_p(\Omega) \to L^1_q(\mathbb{R}^n), \quad n < q \leq p < \infty.$$

Then in the domain Ω, the intrinsic metric is (p,q)-equivalent to the Euclidean metric

$$d_\Omega(x,y)^{1-\frac{n}{p}} \leq C_0 \Phi(B(x,R))^{\frac{1}{\kappa}} |x-y|^{1-\frac{n}{q}}, \quad R = d_\Omega(x,y), \tag{2.4}$$

for all $|x-y| < 1$, where a constant $C_0 = C_0(p,q,n)$ depends on p, q and n only.

Proof. Substituting the test function f from Lemma 2.6 into inequality (2.1), we obtain

$$\|E(f) \mid L^1_q(B(x,R)\| \leq \Phi(B(x,R))^{\frac{1}{\kappa}} \cdot \frac{1}{d_\Omega(x,y)^{1-\frac{n}{p}}}, \tag{2.5}$$

because of

$$\|f \mid L^1_p(\Omega)\| \leq \left(\int\limits_{B(x,R)} |\nabla f(z)|^p\ dz\right)^{\frac{1}{p}}$$
$$\leq \left(\int\limits_{B(x,R)} \left(\frac{1}{R}\right)^p\ dz\right)^{\frac{1}{p}} = \frac{1}{R^{1-\frac{n}{p}}} = \frac{1}{d_\Omega(x,y)^{1-\frac{n}{p}}}.$$

In the left-hand side of inequality (2.5) we apply the embedding theorem of the space of compact supported Sobolev functions to the space of Hölder continuous functions $L^1_q(B) \hookrightarrow H^\gamma(B)$, $\gamma = 1 - n/q$. So, we obtain

$$\frac{1}{|x-y|^{1-\frac{n}{q}}} = \frac{|f(x)-f(y)|}{|x-y|^{1-\frac{n}{q}}} \leq \|f \mid H^\gamma(B(x,R))\| \leq C_0 \|f \mid L^1_q(B(x,R)\|.$$

Hence

$$\frac{1}{|x-y|^{1-\frac{n}{q}}} \leq C_0 \Phi(B(x,R))^{\frac{1}{\kappa}} \cdot \frac{1}{d_\Omega(x,y)^{1-\frac{n}{p}}}, \ |x-y|<1.$$

The theorem is proved. □

Let $x \in \Omega$, we define the value [23]

$$M(x) = \limsup_{r\to 0} M(x,r) := \limsup_{r \to 0} \left\{ \inf_{|x-y|\leq r} \left\{ m : d_{\Omega(x,y)} \leq m|x-y| \right\} \right\}.$$

Inequality (2.4) leads to the following lower estimate of the extension operator formulated in [23]:

Theorem 2.8. *Let there exist a continuous linear extension operator*

$$E : L^1_p(\Omega) \to L^1_q(\mathbb{R}^n), \quad n < q \leq p < \infty.$$

Then

$$\|E\| \geq C_0 \|M \mid L_\alpha(\Omega)\|^{1-\frac{n}{q}}, \tag{2.6}$$

where $\alpha = (pq - pn)/(p-q)$ and a constant $C_0 = C_0(p,q,n)$ depends on p, q and n only.

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THE UNIFORM SUBSETS OF THE EUCLIDEAN PLANE

MARIAM BERIASHVILI

Abstract. We consider some measurability properties of the uniform subsets of the Euclidean plane $\mathbf{R^2}$. Furthermore, it is shown that there exists an uniform subset of the plane which is simultaneously a Hamel basis of the plane.

Many years ago, Luzin posed the so-called graph problem, in particular, he asked whether there exists a function

$$\phi : \mathbf{R} \to \mathbf{R}$$

such that the whole plane $\mathbf{R}^2$ may be covered by countable isometric copies of the graph of ϕ.

Let us define the standard terminology which was introduced by Luzin (see, e.g., [7,9]).

Let e be an arbitrary nonzero vector in the Euclidean plane and ω be the first infinite cardinal number (i.e., $\omega = card(N)$).

- A set $A \subset \mathbf{R^2}$ is called uniform in direction e if $card(l \cap A) \leq 1$ for any straight line $l \subset \mathbf{R^2}$, parallel to e.
- A set $B \subset \mathbf{R^2}$ is called finite in direction e if $card(l \cap B) < \omega$ for any straight line $l \subset \mathbf{R^2}$, parallel to e.
- A set $C \subset \mathbf{R^2}$ is called countable in direction e if $card(l \cap C) \leq \omega$ for any straight line $l \subset \mathbf{R^2}$, parallel to e.

After this definition we can reduce the equivalent formulation of Luzin's problem:

There exists a countable family of uniform sets, whose union is identical to $\mathbf{R^2}$.

The Luzin's problem has found interesting applications for the mathematicians, in particular, this topic has a close connection with Sierpinski,s partition of the plane $\mathbf{R^2}$. Furthermore, under the assumption of the Continuum Hypothesis (**CH**) Sierpinski has solved positively the question (see, e.g., [8,9]).

Sierpinski's Theorem. *Assuming Continuum Hypothesis in* $\mathbf{R^2}$, *there exist two subsets* A *and* B *such that*

- *The set* A *is uniform with respect to the axis* $\mathbf{R} \times 0$*;*
- *The set* B *is uniform with respect to the axis* $0 \times \mathbf{R}$*;*
- *There exists a countable family* $\{h_n : n > \omega\}$ *of translations of* $\mathbf{R}^2$, *for which we have*

$$\cup\{h_n(A \cup B) : n < \omega\} = \mathbf{R}^2.$$

Note that the converse assertion holds true. In particular, the existence of the sets A and B satisfying the above-mentioned properties, implies the validity of **CH**.

The theorem of Sierpinski and the problem of Luzin have found interesting connections with the measure extension problem. The study of the measurability properties of uniform sets is an essential topic of our research. In the measure theory, the standard concept of measurability of sets and functions with respect to a fixed measure μ on a base (ground) set E is well known. We now introduce the concept of measurability of sets and functions not with respect to a fixed measure , but with respect to certain classes of measures, which are defined on different σ-algebras of subsets of the base space E (see [4,5]).

Let E be a set and let M be a class of measures on E (in general, we do not assume that measures belonging to M are defined on one and the same σ-algebra of subset of E).

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- We say that a function $f : E \to \mathbf{R}$ is absolutely (or universally) measurable with respect to M if f is measurable with respect to all measures from M.
- We say that a function $f : E \to \mathbf{R}$ is relatively measurable with respect to M if there exists at least one measure μ from M such that f is μ-measurable.
- We say that a function $f : E \to \mathbf{R}$ is absolutely nonmeasurable with respect to M if there exists no measure μ from M such that f is μ-measurable.

Accordingly, we say that a set $X \subset E$ is relatively measurable (absolutely measurable, absolutely nonmeasurable) with respect to M if its characteristic function χ_X is relatively measurable (absolutely measurable, absolutely nonmeasurable) with respect to M.

Example 1. There exists μ Π_2-invariant extension of the Lebesgue measure λ_2 such that all uniform sets in direction of the Oy-axis are measurable with respect to μ.

Example 2. There exist the A uniform set in direction of the Oy-axis and the B uniform set in direction of the Ox-axis such that $A \cup B$ is absolutely nonmeasurable with respect to the class of all Π_2-invariant extensions of the two-dimensional Lebesgue measure.

Let $M(\mathbf{R}^2)$ be a class of all nonzero σ-finite translation invariant measures on $\mathbf{R}^2$. A set $X \subset \mathbf{R}^2$ is called *negligible* with respect to $M(\mathbf{R}^2)$ if these two conditions are satisfied for X:

- there exists a measure $\nu \in M(\mathbf{R}^2)$ such that $X \in dom(\nu)$;
- for any measure $\mu \in M(\mathbf{R}^2)$, the relation $X \in dom(\mu)$ implies the equality $\mu(X) = 0$.

A set $X \subset \mathbf{R}^2$ is called *absolutely negligible* with respect to $M(\mathbf{R}^2)$ if for every measure $\mu \in M(\mathbf{R}^2)$, there exists a measure $\mu' \in M(\mathbf{R}^2)$ such that the relations

$$\mu' \ extends\ \mu, Y \in dom(\mu'), \mu'(Y) = 0$$

hold true.

Let us notice that any $\mathbf{R}^2$-absolutely negligible set is also $\mathbf{R}^2$-negligible, but the converse assertion fails to be valid.

Example 3. In 1914, Mazurkiewicz presented a transfinite construction to show that there exists a point subset M of the Euclidean plane $\mathbf{R^2}$ such that every straight line in the plane meets M at exactly two points. After his result it is natural to say that a set $Z \subset \mathbf{R^2}$ is a Mazurkiewicz subset of $\mathbf{R^2}$ if $card(Z \cap l) = 2$ for every straight line l lying in $\mathbf{R^2}$. The above definition immediately implies that for any nonzero vector $e \in \mathbf{R^2}$, the Mazurkiewicz set Z is finite in direction e. If $M \subset \mathbf{R}^2$ is finite in some direction l, then M is negligible with respect to the class $M(\mathbf{R}^2)$. In particular, every Mazurkiewicz set is negligible with respect to the same class of measures.

Example 4. By the definition, a Hamel basis for $\mathbf{R}$ is any of its bases construed as a vector space over $\mathbf{Q}$. It is a well-known fact that in the theory $\mathbf{ZF}+\mathbf{DC}$, where $\mathbf{DC}$ denotes the so-called Axiom of Dependent Choice, the existence of a Hamel basis implies the existence of a subset of $\mathbf{R}$, nonmeasurable in the Lebesgue sense. Moreover, every Hamel basis of the space $\mathbf{R}^n$ is an absolutely negligible subset of $\mathbf{R}^n$.

We recall that a subset X of $\mathbf{R^n}$ is λ_n-*thick* (or λ_n massive) in $\mathbf{R^n}$ if for each λ_n-measurable set $Z \subseteq \mathbf{R^n}$ with $\lambda_n(Z) > 0$, we have

$$X \cap Z \neq \emptyset.$$

In other words, X is λ_n-*thick* in $\mathbf{R^n}$ if and only if the equality

$$(\lambda_n)_*(\mathbf{R^n} \setminus \mathbf{X}) = \mathbf{0}$$

is satisfied.

Example 5. In the $\mathbf{R^n}$ Euclidian space, there exists the set Y such that: (i) Y is finite in direction of any $e \in \mathbf{R^n}$ vector. (ii)There exists a countable family $\{h_n : n > \omega\}$ of translations of $\mathbf{R}^n$ for which the intersection of sets $(h_k(Y))_k \in N$ is λ_n-thick (massive) set in $\mathbf{R}^n$.

As is mentioned above, under **CH**, the Sierpinski theorem yields a positive solution to the Luzin problem, but in the frame of **ZFC**, the final result was obtained by Davis (see, e.g., [2]).

Davis Theorem. *There exist a function*

$$\phi : \mathbf{R} \to \mathbf{R}$$

and a countable family $(g_n)_{n<\omega}$ *of motions of the Euclidean plane* $\mathbf{R^2}$ *such that*

$$\cup\{g_n(\Gamma_\phi) : n < \omega\} = \mathbf{R^2},$$

where Γ_ϕ *denotes the graph of* ϕ.

Example 6. The graph of a function $\phi : \mathbf{R} \to \mathbf{R}$, which yields a positive solution of Luzins problem, is an absolutely nonmeasurable subset of $E = \mathbf{R^2}$ with respect to the class of all nonzero σ-finite measures on $\mathbf{R^2}$ that are invariant under the group of all isometries of $\mathbf{R^2}$.

It is shown in the above-presented example that there is a finite set which is simultaneously a Hamel basis. The proof of this statement can be find in [6]. This fact motivated us to prove the following theorem.

Theorem. *There exists an uniform subset of* $\mathbf{R^2}$ *which is a Hamel basis of* $\mathbf{R^2}$.

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WEIGHTED MULTILINEAR HARDY AND RELLICH INEQUALITIES

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Abstract. Multilinear variants of weighted Rellich inequalities are derived on the real line. Weighted estimates for multilinear Hardy operators are also discussed.

1. Introduction and Preliminaries

A considerable effort has been made in recent years to establish the (weighted) boundedness of integral operators in Lebesgue spaces. Such problems have been studied extensively in Harmonic Analysis, especially in the last two decades (see, e.g., the monograph [9] and references therein). Our aim is to establish an $m-$ linear weighted Rellich inequality

$$\Big\| \prod_{j=1}^{m} u_j \Big\|_{L^p_{w(\delta(\cdot))}(I)} \le C \prod_{j=1}^{m} \Big\| u_j'' \Big\|_{L^{p_j}(I)}, \quad I := (a,b), \quad -\infty \le a < b \le +\infty, \tag{1}$$

with a certain positive constant c, independent of $u_k \in C_0^\infty(I)$, $k = 1, \dots, m$, where $\delta(x)$ is the distance function on I given by the formula

$$\delta(x) = \min\{x-a, b-x\}, \tag{2}$$

and p is defined as follows:

$$\frac{1}{p} := \sum_{k=1}^{m} \frac{1}{p_k}, \quad 1 < p_k < \infty, \quad k = 1, \dots, m. \tag{3}$$

Throughout the paper, we assume that m is a positive integer, and p is determined by (3). Note that in this case $0 < p < \infty$.

Let v be an a.e. positive function (i.e., a weight) on the interval $I := (a,b)$, $-\infty \le a < b \le \infty$. We denote by $L^r_v(I)$ (or by $L^r_v(a,b)$), $0 < r < \infty$, the Lebesgue space defined by the norm for $r \ge 1$ (quasi-norm if $0 < r < 1$):

$$\|g\|_{L^r_v(I)} = \Big(\int_a^b |g(x)|^r v(x) dx \Big)^{1/r}.$$

If $v \equiv$ *const*, then $L^r_v(I)$ will be denoted by $L^r(I)$ (or by $L^r(a,b)$).

We establish (1) by deriving appropriate multilinear weighted Hardy inequalities

$$\Big\| \prod_{j=1}^{m} \int_a^x f_j(t)dt \Big\|_{L^p_v(a,b)} \le c \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(a,b)}, \tag{4}$$

$$\Big\| \prod_{j=1}^{m} \int_x^b f_j(t)dt \Big\|_{L^p_v(a,b)} \le c \prod_{j=1}^{m} \|f_j\|_{L^{p}(a,b)}. \tag{5}$$

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It should be noted that the necessary and sufficient conditions governing the two-weight bilinear Hardy inequality

$$\left(\int_a^b \left(\int_a^b f\right)^q \left(\int_a^x g\right)^q w(x)dx\right)^{1/q} \le C\left(\int_a^b f^{p_1} w_1\right)^{1/p_1} \left(\int_a^b g^{p_2} w_1\right)^{1/p_2},$$

for non-negative f and g were found in [10] under different conditions on weights for various ranges of p_1, p_2 and q, with $q > 1$.

The Rellich inequality in the linear setting has first appeared in [14]. The papers [2, 4–8] (see also the monograph [1]) were devoted to this problem, generally speaking, in a higher-dimensional setting.

Here we formulate the following statements which are inherited from [5].

Theorem A (the case $n = 1$)**.** *Suppose that $-\infty < a < b \le \infty$ and let $r \in (1, \infty)$; put $\delta(t) = \min\{t - a, b - t\}$. Then for all $u \in C_0^2(a, b)$,*

$$\int_a^b \frac{|u(t)|^r}{\delta(t)^{2r}} dt \le \left(\frac{r}{2r-1}\right)^r \left(\frac{r}{r-1}\right)^r \int_a^b |u''(t)|^r \, dt.$$

Theorem B (the higher-dimensional case)**.** *Let Ω be a non-empty, proper open subset of $\mathbb{R}^n$ and let $r \in (1, \infty)$; suppose that $u \in C_0^2(\Omega)$. If $r = 2$, then*

$$\int_\Omega \frac{|u(x)|^2}{\delta_{M,4}(x)^4} dx \le \frac{16}{9} \int_\Omega |\Delta u(x)|^2 \, dx,$$

while if $r \in (1, \infty)\backslash\{2\}$, then for some explicit constant $K(r, n)$,

$$\int_\Omega \frac{|u(x)|^r}{\delta_{M,2r}(x)^{2r}} dx \le K(r, n) \int_\Omega |\Delta u(x)|^r \, dx.$$

Here, $\delta_{M,4}$ and $\delta_{M,2r}$ are the mean distance functions obtained by averaging, in a certain sense, the distance to the boundary of Ω in all possible directions.

2. Results

We have proved the following statements.

Theorem 2.1. *Let $-\infty < a < b < \infty$, $I := (a, b)$, and let w be a weight function on the interval $\big(0, (b - a)\big)$. If*

$$\widetilde{D}_{a,b} := \sup_{0<\tau<b-a} \left(\int_\tau^b w(x) x^{mp} dx\right)^{1/p} \tau^{m-1/p} < \infty,$$

then for all $u_j \in C_0^2(I)$, $j = 1, \dots, m$, inequality (1) holds with the constant C given by the formula

$$C = 4^{1/p} \widetilde{D}_{a,b} \left[1 + 2^{mp-2} \prod_{i=1}^m (p_i')^p\right]^{1/p}. \tag{6}$$

The next statement deals with the cases $b = \infty$ and $a = -\infty$, respectively.

Theorem 2.2. *Let $-\infty < a < \infty$. Suppose that $I := (a, \infty)$. Let w be a weight function on $(0, \infty)$. If*

$$\widetilde{D} := \sup_{t>0} \left(\int_t^\infty w(x) x^{mp} dx\right)^{1/p} t^{m-1/p} < \infty, \tag{7}$$

then for all $u_j \in C_0^2(I)$, $j = 1, \dots, m$, inequality (1) holds, where

$$C = 2^{m-1/p} \widetilde{D} \prod_{i=1}^m p_i'. \tag{8}$$

Theorem 2.3. *Let $-\infty < b < \infty$ and $I := (-\infty, b)$. Suppose that w is a positive function on $(0, \infty)$. If condition (7) is satisfied, then for all $u_j \in C_0^2(I)$, $j = 1, \ldots, m$, inequality (1) holds, where*

$$C = 2^{m-1/p} \widetilde{D} \prod_{i=1}^{m} p_i', \tag{9}$$

where $\widetilde{D}$ is defined by (7).

By applying Theorems 2.1, 2.2 and 2.3, we can easily deduce the following statements.

Corollary 2.4. *Let $-\infty < a < b < \infty$ and $I := (a, b)$. Then for all $u_j \in C_0^2(I)$, $j = 1, \ldots, m$, the inequalty*

$$\left(\int_I \Big| \prod_{j=1}^{m} u_j(x) \Big|^p \delta(x)^{-2mp} dx \right)^{1/p} \leq C \prod_{j=1}^{m} \|u_j''\|_{L^{p_j}(I)}, \tag{10}$$

holds, where

$$C = (2mp-1)^{-1/p} \left[1 + 2^{mp-2} \prod_{i=1}^{m} (p_i')^p \right]^{1/p}.$$

Corollary 2.5. *Let $-\infty < a < \infty$ and let $I := (a, \infty)$. Then inequality (10) holds for all $u_j \in C_0^2(I)$, $j = 1, \ldots, m$, where*

$$C = 2^{m-1/p} (2mp-1)^{-1/p} \prod_{j=1}^{m} p_j'.$$

Corollary 2.6. *Let $-\infty < b < \infty$. Suppose that $I := (-\infty, b)$. Then inequality (10) holds for all $u_j \in C_0^2(I)$, $j = 1, \ldots, m$, where C is defined by*

$$C = 2^{m-1/p} (2mp-1)^{-1/p} \prod_{j=1}^{m} p_j'.$$

To get the main results of this paper, we obtain the following statements about the weighted multilinear Hardy inequalities in which by H_a and H_b' are denoted the Hardy-type operators of the following form:

$$H_a f(x) = \frac{1}{x-a} \int_a^x f(t)dt, \quad x \in (a, b), \quad -\infty < a < b \leq \infty;$$

$$H_b' f(x) = \frac{1}{b-x} \int_x^b f(t)dt; \quad x \in (a, b), \quad -\infty \leq a < b < \infty.$$

Theorem 2.7. *Let $-\infty < a < b \leq \infty$, v be a weight function on (a, b). Then inequality (4) with a positive constant c, independent of f_j, $f_j \in L^{p_j}(a, b)$, $j = 1, \ldots, m$, holds if and only if*

$$A_{a,b} := \sup_{a<t<b} \left(\int_t^b v(x)dx \right)^{1/p} \Big(t-a \Big)^{m-1/p} < \infty.$$

Moreover, if c is the best possible constant in (4), then

$$A_{a,b} \leq c \leq C A_{a,b},$$

where

$$C = \begin{cases} \left(2 + 2^{mp-1} \prod_{i=1}^{m} \|H_a\|_{L^{p_i}(a,b)}^p \right)^{1/p}, & \textit{if} \quad b < \infty, \\ 2^{m-1/p} \prod_{j=1}^{m} \|H_a\|_{L^{p_i}(a,\infty)}, & \textit{if} \quad b = \infty. \end{cases}$$

Theorem 2.8. *Let $-\infty \le a < b < \infty$, v be a weight function on (a,b). Then inequality* (5) *with a positive constant c, independent of f_j, $f_j \in L^{p_j}(a,b)$, $j=1,\dots,m$, holds if and only if*

$$B_{a,b} := \sup_{a<t<b} \left(\int_a^t v(x)dx \right)^{1/p} \Big(b-t\Big)^{m-1/p} < \infty.$$

Moreover, if c is the best possible constant in (5), *then*

$$B_{a,b} \le c \le C B_{a,b},$$

where

$$C = \begin{cases} \Big(2 + 2^{mp-1} \prod_{i=1}^m \|H_b'\|^p_{L^{p_i}(a,b)}\Big)^{1/p}, & \text{if} \quad a > -\infty, \\ 2^{m-1/p} \prod_{j=1}^m \|H_b'\|_{L^{p_i}(-\infty,b)}, & \text{if} \quad a = -\infty. \end{cases}$$

Historically, in the linear case the two-weight problem for the Hardy operator was solved by B. Muckenhoupt [13] for the diagonal case, and by J. Bradly [3] and V. Kokilashvili [11] for the off-diagonal case (see also the monograph [12], Ch.1 and references therein).

Finally, we mention that the results of this note with proofs will appear separately.

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WEIGHTED NORM ESTIMATES FOR ONE-SIDED MULTILINEAR INTEGRAL OPERATORS

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Abstract. In this note one-sided and two-weight inequalities for one-sided multilinear fractional integrals are derived. One-weight estimates are based on Welland's type pointwise estimates which are also presented. Integral operators studied in this note involve one-sided multi(sub)linear fractional maximal operators, multilinear Riemann-Liouville and Weyl integral transforms.

In this note one– and two-weight norm inequalities for one-sided multilinear fractional integrals are presented. One-weight estimates are based on Welland's type pointwise inequalities which are also derived. Integral operators involve one-sided multisublinear fractional maximal operators, multilinear Riemann–Liouville and Weyl integral transforms.

Let $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, m$, be measurable functions and let

$$\overrightarrow{f} := (f_1, \dots, f_m).$$

Throughout the note, it will be assumed that p is a constant satisfying the condition

$$\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}, \tag{1}$$

where $1 < p_i < \infty$, $i = 1, \dots, m$.

Multilinear fractional integrals were introduced and studied in the papers by L. Grafakos [4], C. Kenig and E. Stein [7], L. Grafakos and N. Kalton [5]. In particular, these works deal with the operator

$$B_\gamma(f,g)(x) = \int\limits_{\mathbb{R}^n} \frac{f(x+t)g(x-t)}{|t|^{n-\gamma}}\,dt, \quad x \in \mathbb{R}^n,$$

where γ is a constant parameter satisfying the condition $0 < \gamma < n$.

In the above-mentioned papers it was proved that if $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, then B_γ is bounded from $L^{p_1} \times L^{p_2}$ to L^q.

As a tool to understand B_γ, the operator

$$\mathcal{I}_\gamma(\overrightarrow{f})(x) = \int\limits_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1| + \cdots + |x-y_m|)^{mn-\gamma}}\,d\overrightarrow{y},$$

where $x \in \mathbb{R}^n$, γ is a constant satisfying the condition $0 < \gamma < nm$, $\overrightarrow{f} := (f_1, \dots, f_m)$, $\overrightarrow{y} := (y_1, \dots, y_m)$, was studied, as well. The corresponding multisublinear maximal operator is given by (see [11]) the formula

$$\mathcal{M}_\gamma(\overrightarrow{f})(x) = \sup_{Q \ni x} \prod_{i=1}^{m} \frac{1}{|Q|^{1-\frac{\gamma}{mn}}} \int\limits_Q |f_i(y_i)|dy_i,$$

where the supremum is taken over all cubes Q containing x. It can be immediately checked that

$$\mathcal{I}_\gamma(\overrightarrow{f})(x) \geq c_{n,\gamma} \mathcal{M}_\gamma(\overrightarrow{f})(x),$$

where $f_i \geq 0$, $i = 1, \dots, m$ and c is a positive constant, depends only on n and γ. If $m = 1$, then $\mathcal{I}_\gamma$ will be denoted by I_γ.

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Let $0 < r < \infty$ and let w be a weight function (i.e., w be an a.e. positive function) on $\mathbb{R}^n$. We denote by $L^r_w(\mathbb{R}^n)$ the class of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^r_w(\mathbb{R}^n)} := \Bigg(\int\limits_{\mathbb{R}^n} |f(x)|^r w(x) dx \Bigg)^{1/r} < \infty.$$

In 1974, Muckenhoupt and Wheeden [12] showed that the weighted Sobolev-type inequality

$$||I_\gamma(f)||_{L^s_{w^s}(\mathbb{R}^n)} \le C||f||_{L^r_{w^r}(\mathbb{R}^n)},$$

where $1 < r < \infty$, $0 < \gamma < 1/r$, $1/s = 1/r - \gamma/n$, holds if and only if $w \in A_{r,s}$. A locally integrable non-negative function (weight) w on $\mathbb{R}^n$ is said to belong to $A_{r,s}$ $(1 < r, s < \infty)$ if and only if

$$\sup_Q \Bigg(\frac{1}{|Q|} \int\limits_Q w^s(x) dx \Bigg)^{1/s} \Bigg(\frac{1}{|Q|} \int\limits_Q w^{-r'}(x) dx \Bigg)^{1/r'} < \infty, \quad \frac{1}{r} + \frac{1}{r'} = 1,$$

where the supremum is taken over all n-dimensional cubes Q with sides, parallel to the coordinate axes.

We say that a vector of weights $\overrightarrow{w} = (w_1, \dots, w_m)$ satisfies the $A_{\overrightarrow{p},q}$ condition ($\overrightarrow{p} = (p_1, \dots, p_m)$) if

$$\sup_Q \Bigg(\frac{1}{|Q|} \int\limits_Q \Bigg(\prod_{i=1}^m w_i(x) \Bigg)^q dx \Bigg)^{1/q} \prod_{i=1}^m \Bigg(\frac{1}{|Q|} \int\limits_Q {w_i}^{-p_i'}(x) dx \Bigg)^{1/p_i'} < \infty.$$

Theorem A ([11]). *Suppose that $0 < \gamma < nm$ and $1 < p_1, \dots, p_m < \infty$ are exponents with $1/m < p < n/\gamma$ and q is the exponent defined by $1/q = 1/p - \gamma/n$. Then the inequality*

$$\Bigg(\int\limits_{\mathbb{R}^n} \Bigg(\big|\mathcal{I}_\gamma(\overrightarrow{f})(x)\big| \Bigg(\prod_{i=1}^m w_i(x) \Bigg) \Bigg)^q dx \Bigg)^{1/q} \le C \prod_{i=1}^m \Bigg(\int\limits_{\mathbb{R}^n} \big(|f_i(x)| w_i(x)\big)^{p_i} dx \Bigg)^{1/p_i}$$

holds for every $\overrightarrow{f} \in L^{p_1}(w_1^{p_1}) \times \dots \times L^{p_m}(w_m^{p_m})$ if and only if $\overrightarrow{w}$ satisfies the $A_{\overrightarrow{p},q}$ condition.

In [14], the authors derived the following different type one-weighted result.

Theorem B. *Let $0 < \gamma < nm$, suppose that $f_i \in L^{p_i}_{w^{p_i}}(\mathbb{R}^n)$ with $1 < p_i < mn/\gamma$ $(i = 1, \dots, m)$ and $w \in \bigcap\limits_{i=1}^m A_{p_i,q_i}$ i.e.,*

$$\prod_{i=1}^m \sup_Q \Bigg(\frac{1}{|Q|} \int\limits_Q w^{q_i}(x) dx \Bigg)^{1/q_i} \Bigg(\frac{1}{|Q|} \int\limits_Q w^{-p_i'}(x) dx \Bigg)^{1/p_i'} < \infty,$$

where $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\gamma}{mn}$. We set $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i}$. Then there is a constant $C > 0$, independent of f_i such that

$$||\mathcal{I}_\gamma(\overrightarrow{f})||_{L^q_{w^q}(\mathbb{R}^n)} \le C \prod_{i=1}^m ||f_i||_{L^{p_i}_{w^{p_i}}(\mathbb{R}^n)}.$$

The one-weight problem for multisublinear maximal functions and multilinear singular integrals was studied in [8] under the $A_{\overrightarrow{p}}$ condition. Various types of Fefferman–Stein multisublinear inequalities for fractional maximal functions were established in [13] and [6].

We introduce the following one-sided multisublinear fractional maximal functions:

$$\mathcal{M}^-_\alpha(\overrightarrow{f})(x) = \sup_{h>0} \prod_{i=1}^m \frac{1}{h^{1-\alpha/m}} \int\limits_{x-h}^{x} |f_i(y_i)| dy_i, \quad 0 < \alpha < m,$$

$$\mathcal{M}^+_\alpha(\overrightarrow{f})(x) = \sup_{h>0} \prod_{i=1}^m \frac{1}{h^{1-\alpha/m}} \int\limits_{x}^{x+h} |f_i(y_i)| dy_i, \quad 0 < \alpha < m,$$

which play an important role in the study of multilinear variants of the Riemann-Liouville and Weyl integral transforms

$$\mathcal{R}_\alpha(\overrightarrow{f})(x) = \int_{-\infty}^{x} \cdots \int_{-\infty}^{x} \frac{f_1(y_1)\cdots f_m(y_m)}{((x-y_1)+\cdots+(x-y_m))^{m-\alpha}} d\overrightarrow{y}, \quad 0<\alpha<m, \quad x\in\mathbb{R},$$

$$\mathcal{W}_\alpha(\overrightarrow{f})(x) = \int_{x}^{\infty} \cdots \int_{x}^{\infty} \frac{f_1(y_1)\cdots f_m(y_m)}{((y_1-x)+\cdots+(y_m-x))^{m-\alpha}} d\overrightarrow{y}, \quad 0<\alpha<m, \quad x\in\mathbb{R},$$

respectively.

If $m=1$, then the operators $\mathcal{R}_\alpha$, $\mathcal{W}_\alpha$, $\mathcal{M}_\alpha^-$ and $\mathcal{M}_\alpha^+$ will be denoted by R_α, W_α, M_α^- and M_α^+, respectively.

For the linear one-sided fractional integral operators the one-weight problem was solved in [1] (see also [2] Ch. 2 for related topics). In particular, the following statement holds.

Theorem C. *If $0\le\alpha<1$, $1<p<1/\alpha$ ($1/\alpha=\infty$, if $\alpha=0$), $1/q=1/p-\alpha$, $1/p+1/p'=1$. Then*

$$\Big[\int_{-\infty}^{\infty} |T(f)(x)u(x)|^q dx\Big]^{1/q} \le C\Big[\int_{0}^{\infty} |f(x)u(x)|^p dx\Big]^{1/p}$$

holds.

(a) *for $T=M_\alpha^-$ or $T=R_\alpha$ ($\alpha>0$) if and only if $u\in A_{p,q}^-$ i.e.,*

$$\Big[\frac{1}{h}\int_{a}^{a+h} u^q(x)dx\Big]^{1/q}\Big[\frac{1}{h}\int_{a-h}^{a} u^{-p'}(x)dx\Big]^{1/p'} \le C$$

for some constant C and all a, h with $a\in\mathbb{R}$, $h>0$;

(b) *for $T=M_\alpha^+$ or $T=W_\alpha$ ($\alpha>0$) if and only if $u\in A_{p,q}^+$ i.e.*

$$\Big[\frac{1}{h}\int_{a-h}^{a} u^q(x)dx\Big]^{1/q}\Big[\frac{1}{h}\int_{a}^{a+h} u^{-p'}(x)dx\Big]^{1/p'} \le C$$

for some constant C and all a, h with $a\in\mathbb{R}$, $h>0$.

For the two-weight theory for linear one-sided fractional integral operators under different types of conditions on weights we refer to the papers [3,9,10] (see also the monograph [2, ch. 2] and references cited therein).

Now we formulate the main statements of this note.

Welland-type Inequalities

Theorem 1. *Let $0<\alpha<m$ and $0<\epsilon<min\{\alpha,m-\alpha\}$. Then there exists a positive constant C depending only on m, α and ϵ such that the following pointwise inequality*

$$\Big|\mathcal{R}_\alpha(\overrightarrow{f})(x)\Big| \le C\Big[\Big(\mathcal{M}_{\alpha-\epsilon}^-(\overrightarrow{f})(x)\Big)\Big(\mathcal{M}_{\alpha+\epsilon}^-(\overrightarrow{f})(x)\Big)\Big]^{\frac{1}{2}}$$

holds for all $\overrightarrow{f}:=(f_1,\dots,f_m)$, where f_i, $i=1,\dots,m$, are bounded functions with a compact support.

The similar theorem can be written for the Weyl integral transform.

Theorem 2. *Let $0<\alpha<m$ and $0<\epsilon<\min\{\alpha,m-\alpha\}$. Then if $\overrightarrow{f}:=(f_1,\dots,f_m)$,*

$$\Big|\mathcal{W}_\alpha(\overrightarrow{f})(x)\Big| \le C\Big[\Big(\mathcal{M}_{\alpha-\epsilon}^+(\overrightarrow{f})(x)\Big)\Big(\mathcal{M}_{\alpha+\epsilon}^+(\overrightarrow{f})(x)\Big)\Big]^{\frac{1}{2}},$$

where f_i, $i=1,\dots,m$, are bounded functions with compact support and C depends only on m, α and ϵ.

One-weighted Inequalities

Theorem 3. *Let* $0 < \alpha < m$, *suppose that* $f_i \in L^{p_i}_{w^{p_i}}(\mathbb{R})$ *with* $1 < p_i < m/\alpha$ $(i = 1, \dots, m)$ *and* $w \in \bigcap_{i=1}^{m} A^{-}_{p_i,q_i}$ *i.e.,*

$$\prod_{i=1}^{m} \sup_{\substack{h>0\\ x\in\mathbb{R}}} \left(\frac{1}{h}\int_{x}^{x+h} w^{q_i}(t)dt\right)^{1/q_i} \left(\frac{1}{h}\int_{x-h}^{x} w^{-p_i'}(t)dt\right)^{1/p_i'} < \infty,$$

where $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{m}$. *We set* $\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i}$. *Then there is a constant* $C > 0$, *independent of* f_i *such that*

$$\|\mathcal{R}_\alpha(\overrightarrow{f})\|_{L^q_{w^q}(\mathbb{R})} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}_{w^{p_i}}(\mathbb{R})}.$$

Similar theorem for the Weyl integral transform holds.

Theorem 4. *Let* $0 < \alpha < m$, *suppose that* $f_i \in L^{p_i}_{w^{p_i}}(\mathbb{R})$ *with* $1 < p_i < m/\alpha$ $(i = 1, \dots, m)$ *and* $w \in \bigcap_{i=1}^{m} A^{+}_{p_i,q_i}$ *i.e.,*

$$\prod_{i=1}^{m} \sup_{\substack{h>0\\ x\in\mathbb{R}}} \left(\frac{1}{h}\int_{x-h}^{x} w^{q_i}(t)dt\right)^{1/q_i} \left(\frac{1}{h}\int_{x}^{x+h} w^{-p_i'}(t)dt\right)^{1/p_i'} < \infty,$$

where $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{m}$. *We set* $\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i}$. *Then there is a constant* $C > 0$, *independent of* f_i *such that*

$$\|\mathcal{W}_\alpha(\overrightarrow{f})\|_{L^q_{w^q}(\mathbb{R})} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}_{w^{p_i}}(\mathbb{R})}.$$

Fefferman–Stein Two-weighted Inequalities

In the two-weighted setting, we proved the following Fefferman-Stein type inequalities:

Theorem 5. *Let* $0 < \alpha < m$ *and let* $1 < \min\{p_1, \dots, p_m\} \le \max\{p_1, \dots, p_m\} < \min\{q, m/\alpha\}$. *Suppose that* p *is defined by* (1). *Let* v_i *be weights on* $\mathbb{R}$, $i = 1, \dots, m$. *We set* $v(x) = \prod_{i=1}^{m} v_i^{p/p_i}(x)$. *Then the inequalities*

$$\left\|\Big(\mathcal{M}^{-}_{\alpha}(\overrightarrow{f})\Big)v^{1/q}\right\|_{L^q(\mathbb{R})} \le C \prod_{i=1}^{m} \left\|f_i\big(\mathcal{M}^{+}_{\alpha,p_i,q} v_i\big)^{1/q}\right\|_{L^{p_i}(\mathbb{R})},$$

$$\left\|\Big(\mathcal{M}^{+}_{\alpha}(\overrightarrow{f})\Big)v^{1/q}\right\|_{L^q(\mathbb{R})} \le C \prod_{i=1}^{m} \left\|f_i\big(\mathcal{M}^{-}_{\alpha,p_i,q} v_i\big)^{1/q}\right\|_{L^{p_i}(\mathbb{R})}$$

hold, where C *is a constant, independent of* f_i, $i = 1, \dots, m$, *and*

$$\mathcal{M}^{+}_{\alpha,p_i,q} v_i(x) = \sup_{h>0} \left(\frac{1}{h^{(1-\alpha p_i/m)q/p}} \int_{x}^{x+h} v_i(y)dy\right)^{p/p_i},$$

$$\mathcal{M}^{-}_{\alpha,p_i,q} v_i(x) = \sup_{h>0} \left(\frac{1}{h^{(1-\alpha p_i/m)q/p}} \int_{x-h}^{x} v_i(y)dy\right)^{p/p_i}.$$

Corollary 1. *Let* α, p_i, q *and* m *satisfy the conditions of Theorem* 5.

If

$$\prod_{i=1}^{m} \sup_{I} \left(\frac{1}{|I|^{(1-\alpha p_i/m)q/p}} \int_{I} v_i(y)dy\right)^{p/p_i} < \infty,$$

then the following trace-type inequalities hold:

(i)

$$\left\|\mathcal{M}_\alpha^{-}(\overrightarrow{f})\right\|_{L_v^q(\mathbb{R})} \le C\prod_{i=1}^{m}\left\|f_i\right\|_{L^{p_i}(\mathbb{R})},$$

$$\left\|\mathcal{M}_\alpha^{+}(\overrightarrow{f})\right\|_{L_v^q(\mathbb{R})} \le C\prod_{i=1}^{m}\left\|f_i\right\|_{L^{p_i}(\mathbb{R})}.$$

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THE PUNCH PROBLEM OF THE PLANE THEORY OF VISCOELASTICITY WITH A FRICTION

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Abstract. The paper considers the problem of pressure of a rigid punch onto a viscoelastic half-plane in the presence of friction. The problems of the linear theory of viscoelasticity attracted the attention of many scientists first of all due to the fact that building and composite materials (concrete, plastic polymers, wood, human fabric, etc.) exhibit significant viscoelastic properties and, thus, calculations of constructions for strength, with regard for the viscoelastic properties, are now becoming increasingly important. Thanks to this fact, various methods of calculating the above-mentioned problems were proposed, one of which is the Kelvin–Voigt differential model on which the present paper is based.

Using the methods of a complex analysis elaborated in the plane theory of elasticity by N. I. Muskhelishvili and his followers, the unknown complex potentials, characterizing viscoelastic equilibrium of a half-plane, are constructed effectively and the tangential and normal stresses under the punch are defined.

Introduction

The theory of viscoelasticity originated in the works by Boltzmann [3] and developed in his works by Volterra [10] finds applications not only in mechanics of deformable solid bodies, but also in other branches of mathematical physics. Viscoelasticity combines the properties of materials to be viscous or elastic during deformation. In addition, elastic bodies and viscous liquids, as is known, differ significantly in their properties under the deformation; the former after removal of applied loads return to their undeformed state and the latter (for example, incompressible liquids) are deprived of this property. Moreover, stresses in an elastic body are connected directly with strains, but in viscous liquids (with some exception) they are connected with deformation velocities (for details, see [2,4,5,8,9]. For viscoelastic materials, the ordinary equilibrium equations, the boundary conditions and compatibility equations written in terms of stresses remain valid for purely elastic bodies under the condition that the constants E and σ obtained in the equations are replaced by the functions $E(t)$ and $\sigma(t)$. Moreover, unlike purely elastic materials (steel, aluminium, quartz) whose behavior does not deviate much from the linear elasticity, such materials as synthetic polymers, wood, metals, human fabric, etc., exhibit under high temperatures significant viscoelastic properties.

Of great importance in the development of the theory of viscoelasticity are synthetic materials worked out at the end of the twentieth century and also their widespread applications in various fields.

Subsequently, various models of material properties evaluation for viscoelasticity have been elaborated (see [1]).

In the theory of linear viscoelasicity, Hook's law can be represented either by the Volterra equation (integral model), or by the dependence where there occur both the deformations and their derivatives in time (differential model).

In the present work the use is made of the Kelvin–Voigt differential model in which Hook's law is of the form [8]

$$\begin{aligned} X_x &= \lambda\theta + 2\mu e_{xx} + \lambda^*\dot{\theta} + 2\mu^*\dot{e}_{xx}, \\ Y_y &= \lambda\theta + 2\mu e_{yy} + \lambda^*\dot{\theta} + 2\mu^*\dot{e}_{yy}, \end{aligned} \tag{1}$$

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$$X_y=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\mu^*\left(\frac{\partial \dot v}{\partial x}+\frac{\partial \dot u}{\partial y}\right),$$

where $\vartheta=e_{xx}+e_{yy}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$, X_x, Y_y, X_y, u, v, e_{xx},e_{yy},e_{xy} are the functions of the variables x, y, t. Under t we will always mean the time parameter and the dots in the expressions $\dot\theta,\dots,\dot u$ (unlike dashes) will denote time derivatives t; λ, μ are elastic and λ^*, μ^* are viscoelasticity constants.

We cite here the certain well-known Kolosov–Muskhelishvili's formulas which can, as is known, be attributed to any solid bodies (see [6])

$$\begin{gathered}X_x+Y_y=4\mathrm{Re}[\Phi(z,t)]=4\mathrm{Re}[\varphi'(z,t)],\\ Y_y-X_x+2iX_y=2[\overline{z}\varphi''(z,t)+\psi'(z,t)]=2[\overline{z}\Phi'(z,t)+\Psi(z,t)].\end{gathered}\tag{2}$$

In the sequel, we will also use the formula following from formulas (2),

$$Y_y-iX_y=\Phi(z,t)+\overline{\Phi(z,t)}+z\overline{\Phi'(z,t)}+\overline{\Psi(z,t)}.\tag{3}$$

We assume that the resultant vector (X,Y) of outer forces applied to the punch is finite, and stresses and rotation vanish at infinity, thus for large $|z|$, we have

$$\Phi(z,t)=-\frac{X+iY}{2\pi z}+o\left(\frac{1}{z}\right);\quad \Psi(z,t)=\frac{X-iY}{2\pi z}+o\left(\frac{1}{z}\right).\tag{4}$$

It can be easily seen that from the correlations (1) and (2), for the function $\vartheta(z,t)=e_{xx}+e_{yy}$ we obtain the following differential equation

$$\dot\vartheta(z,t)+k\vartheta(z,t)=\frac{2}{\lambda^*+\mu^*}\mathrm{Re}[\varphi'(z,t)],\quad \Big(k=\frac{\lambda+\mu}{\lambda^*+\mu^*}\Big),$$

whose solution is of the form (assuming $\vartheta(z;0)=0$)

$$\vartheta(z,t)=\frac{2}{\lambda^*+\mu^*}\int\limits_0^t\mathrm{Re}[\varphi'(z,\tau)]e^{k(\tau-t)}d\tau.\tag{5}$$

Similarly, from the same correlations (1) and (2), for the function $\gamma(z,t)=e_{xx}-e_{yy}$ we have

$$\dot\gamma(z,t)+m\gamma(z,t)=-\frac{1}{\mu^*}\mathrm{Re}[\overline{z}\varphi''(z,t)+\psi'(z,t)],\quad \Big(m=\frac{\mu}{\mu^*}\Big),$$

whose solution under zero initial conditions has the form

$$\gamma(z,t)=-\frac{1}{\mu^*}\int\limits_0^t\mathrm{Re}\,[\overline{z}\varphi''(z,\tau)+\psi'(z,\tau)]e^{m(\tau-t)}d\tau.\tag{6}$$

From (5) and (6), we get

$$\begin{aligned}2\mu^*e_{xx}&=\int\limits_0^t\mathrm{Re}\left[\varkappa^*\varphi'(z,\tau)e^{k(\tau-t)}-\left(\overline{z}\varphi''(z,\tau)+\psi'(z,\tau)\right)e^{m(\tau-t)}\right]d\tau,\\ 2\mu^*e_{yy}&=\int\limits_0^t\mathrm{Re}\left[\varkappa^*\varphi'(z,\tau)e^{k(\tau-t)}+\left(\overline{z}\varphi''(z,\tau)+\psi'(z,\tau)\right)e^{m(\tau-t)}\right]d\tau,\end{aligned}\tag{7}$$

where

$$\varkappa^*=\frac{2\mu^*}{\lambda^*+\mu^*}.$$

Taking into account equalities $dx=dz$, $dx=d\overline{z}$, $dy=-idz$, $dy=id\overline{z}$,, from (7), by integration with respect to x and y, respectively, we obtain the formula

$$2\mu^*(u+iv)=\int\limits_0^t\left[\varkappa^*\varphi(z,\tau)e^{k(\tau-t)}+\Big(\varphi(z,\tau)-z\overline{\varphi'(z,\tau)}-\overline{\psi(z,\tau)}\Big)e^{m(\tau-t)}\right]d\tau+2\mu^*(u_0+iv_0),\tag{8}$$

where $u_0=u(z,0)$, $v_0=v(z,0)$.

Formula (8) is an analogue of Kolosov–Muskhelishvili's formula for the second basic problem of the plane theory of elasticity ([6]) for viscoelastic isotropic body.

From formula (8), by differentiation with respect to x, we obtain

$$2\mu^* v'(x,y,t) = \operatorname{Im}\Big[\int\limits_0^t \varkappa^* e^{k(\tau-t)}\Phi(z,\tau)d\tau\Big]$$

$$+\operatorname{Im}\Big[\int\limits_0^t e^{m(\tau-t)}\Big(\Phi(z,\tau)-\overline{\Phi(z,\tau)}-z\overline{\Phi'(z,\tau)}-\overline{\Psi(z,\tau)}\Big)d\tau\Big]+2\mu^* {v'}_0(x,y,0). \tag{9}$$

Statement of the Problem. Let a viscoelastic body occupy a lower half-plane S^-. By L we denote the boundary of that domain (i.e., the Ox-axis) and assume that a portion $L'=[-1;1]$ comes in contact with the punch of prescribed base shape and the punch goes into the half-plane by a given force acting onto the punch and directed vertically downwards. We will also assume that the displacement of the punch is translatory in the direction, normal to the boundary, in the presence of friction. In this case, the boundary conditions can be written in the form

$$\begin{aligned} X_y^-(x,t) =&\alpha p(x,t), \qquad \alpha=\text{const}>0, \quad x\in L';\\ X_y^-(x,t) =&Y_y^-(x,t)=0, \quad x\in L''=L-L';\\ v^-(x,t) =&f(x,t)+c, \qquad x\in L', \quad (c=\text{const}), \end{aligned} \tag{10}$$

where $f(x,0)=f(x)$ is the given function defining the base shape of the punch before pressing into the half-plane. In (10), by $X_y^-(x,t),\dots,v^-(x,t)$ we have denoted the expressions $X_y^-(x,0,t),\dots,$ $v^-(x,0,t)$, and the same writing will be retained in the sequel.

The total tangential stress in the case under consideration has the form $T_0=\alpha N_0$, where $N_0=\int\limits_{-1}^{1}N(x,t)dx$ $N(x,t)$ is a normal stress at the point $x\in L'$, and hence, the resultant vector of outer forces acting onto the punch (which are assumed to be prescribed) is of the kind $(X;Y)=(\alpha N_0;-N_0)$.

Relying on (3), formula (9) is written as follows:

$$\operatorname{Im}\Big[\varkappa^*\int\limits_0^t e^{k(\tau-t)}\Phi(z,\tau)d\tau+2\int\limits_0^t e^{m(\tau-t)}\Phi(z,\tau)d\tau\Big]$$

$$+\int\limits_0^t e^{m(\tau-t)}X_y(z,\tau)d\tau =2\mu^*[v'(x,y,t)-v'(x,y,0)]. \tag{11}$$

Passing in (11) to the limit as $z\to x$ $(z\in S^-)$, we obtain

$$\operatorname{Im}\Big[\varkappa^*\int\limits_0^t e^{k(\tau-t)}\Phi^-(x,\tau)d\tau+2\int\limits_0^t e^{m(\tau-t)}\Phi^-(x,\tau)d\tau\Big]$$

$$+\int\limits_0^t e^{m(\tau-t)}X_y^-(x,\tau)d\tau =f_1(x,t), \tag{12}$$

where

$$f_1(x,t)=2\mu^*[f'(x,t)-f'(x)].$$

Differentiating (12) with respect to t and adding the obtained equality with (12), multiplied by m, we have

$$\operatorname{Im}\Big[(\text{m-k})\varkappa^*\int\limits_0^t e^{k\tau}\Phi^-(x,\tau)d\tau+(\varkappa^*+2)e^{kt}\Phi^-(x,t)\Big]+e^{kt}X_y^-(x,t)=f_2(x,t), \tag{13}$$

where

$$f_2(x,t) = e^{kt}\left[\dot{f}_1(x,t) + m f_1(x,t)\right].$$

After the differentiation with respect to t, it follows from (13) that

$$\operatorname{Im}\left[(\varkappa^* m + 2k)\Phi^-(x,t) + (\varkappa^* + 2)\dot{\Phi}^-(x,t)\right] + \dot{X}_y^-(x,t) + kX_y^-(x,t) = \dot{f}_2(x,t)e^{-kt}. \tag{14}$$

Following N. I. Muskhelishvili (see [6]), we extend the function $\Phi(z,t)$ to the upper half-plane (i.e., S^+) so as to continue analytically the values of $\Phi(z,t)$ into the lower half-plane through the unloaded sections (i.e., to L'')).

In our case, on the basis of the boundary conditions (10) and formula (3), we define $\Phi(z,t)$ in S^+ as follows:

$$\Phi(z,t) = -\Phi_*(z,t) - z{\Phi'}_*(z,t) - \Psi_*(z,t), \quad z \in S^+, \tag{15}$$

where $\Phi_*(z,t) = \overline{\Phi(\overline{z},t)}$; $\Psi_*(z,t) = \overline{\Psi(\overline{z},t)}$.

Taking into account that $[\Phi_*(z,t)]_* = \Phi(z,t)$, $[\Psi_*(z,t)]_* = \Psi(z,t)$, from (15) we have

$$\Phi_*(z,t) = -\Phi(z,t) - z\Phi'(z,t) - \Psi(z,t). \tag{16}$$

The obtained in such a way piecewise holomorphic function we denote again by $\Phi(z,t)$, and then to find the function $\Psi(z,t)$ by $\Phi(z,t)$, from (16) we get

$$\Psi(z,t) = -\Phi(z,t) - \Phi_*(z,t) - z\Phi'(z,t) \tag{17}$$

Thus, the stress and displacement components are expressed in terms of one piecewise holomorphic function $\Phi(z,t)$.

Introducing the value (17) into (3), we have

$$Y_y - iX_y = \Phi(z,t) - \Phi(\overline{z},t) + (z-\overline{z})\overline{\Phi'(z,t)},$$

whence

$$Y_y^-(x,t) - i\,X_y^-(x,t) = \Phi^-(x,t) - \Phi^+(x,t), \quad x \in L'. \tag{18}$$

Owing to the fact that $X_y^- = -\alpha Y_y^-(x,t)$, from (18) we get

$$X_y^-(x,t) = \frac{\alpha}{1+i\alpha}\left[\Phi^+(x,t) - \Phi^-(x,t)\right] \tag{19}$$

Taking into account equalities $\overline{\Phi^-(x,t)} = \Phi_*^+(x,t)$ and $\overline{\Phi^+(x,t)} = \Phi_*^-(x,t)$ and bearing in mind that $X_y^-(x,t) = \overline{X_y^-(x,t)}$, from (19) we obtain

$$(1-i\alpha)\Phi^-(x,t) + (1+i\alpha)\Phi_*^-(x,t) = (1-i\alpha)\Phi^+(x,t) + (1+i\alpha)\Phi_*^+(x,t),$$

and thus we conclude that the vanishing at infinity function

$$(1-i\alpha)\Phi(z,t) + (1+i\alpha)\Phi_*(z,t)$$

is holomorphic on the whole plane and, consequently,

$$\Phi(z,t) = -\frac{1+i\alpha}{1-i\alpha}\Phi_*(z,t)$$

whence we obtain

$$\Phi^-(x,t) = -\frac{1+i\alpha}{1-i\alpha}\overline{\Phi^+(x,t)}; \quad \Phi^+(x,t) = -\frac{1+i\alpha}{1-i\alpha}\overline{\Phi^-(x,t)}. \tag{20}$$

On the basis of (20) and (19), we get

$$\begin{aligned} X_y^-(x,t) &= -\left[\frac{\alpha}{1+i\alpha}\Phi^-(x,t) + \frac{\alpha}{1-i\alpha}\overline{\Phi^-(x,t)}\right] \\ &= -\operatorname{Re}\left[\frac{2\alpha}{1+i\alpha}\Phi^-(x,t)\right] = -\operatorname{Im}\left[\frac{2i\alpha}{1+i\alpha}\Phi^-(x,t)\right]. \end{aligned} \tag{21}$$

From (2) and (13), with regard for the equality $X_y^- = -\alpha Y_y^-$, we have

$$\operatorname{Re}\left[\Phi^-(x,0)\right] = -\frac{X_y^-(x,0)}{4\alpha},$$
$$\operatorname{Im}\left[\Phi^-(x,0)\right] = -\frac{1}{\varkappa^*+2}\left[X_y^-(x,0) - f_2(x,0)\right]. \tag{22}$$

Thus, for $\Phi^-(x,0)$ we obtain the formula

$$\Phi^-(x,0) = -\frac{X_y^-(x,0)}{4\alpha} - \frac{i}{\varkappa^*+2}\left[X_y^-(x,0) - f_2(x,0)\right]. \tag{23}$$

Taking into account the fact that

$$\frac{2i\alpha}{1+i\alpha}\Phi^-(x,0) = \frac{1}{1+\alpha^2}\left[2\alpha^2 \operatorname{Re}\Phi^-(x,0) - 2\alpha \operatorname{Im}\Phi^-(x,0)\right]$$
$$+\frac{i}{1+\alpha^2}\left[2\alpha \operatorname{Re}\Phi^-(x,0) + 2\alpha^2 \operatorname{Im}\Phi^-(x,0)\right],$$

from (21) follows

$$X_y^-(x,0) = -\frac{2\alpha}{1+\alpha^2}\left[\operatorname{Re}\Phi^-(x,0) + \alpha \operatorname{Im}\Phi^-(x,0)\right]. \tag{24}$$

Substituting into (24) the values $\operatorname{Re}\Phi^-(x,0)$ and $\operatorname{Im}\Phi^-(x,0)$ from (22), after not complicated calculations, we obtain

$$X_y^-(x,0) = -\frac{4\alpha^2 f_2(x,0)}{\varkappa^*(1+2\alpha^2)+2}. \tag{25}$$

After the appropriate calculations, it follows from (25) and (23) that

$$\Phi^-(x,0) = \frac{f_2(x,0)}{\varkappa^*(1+2\alpha^2)+2}[\alpha + i(1+2\alpha^2)]. \tag{26}$$

For the tangential and normal stresses under the punch we have

$$T(x,t) = X_y^-(x,t) = -2\alpha \operatorname{Im}\Big[\frac{i}{1+i\alpha}\Phi^-(x,t)\Big],$$
$$P(x,t) = Y_y^-(x,t) = -\frac{1}{\alpha}T(x,t) = 2\operatorname{Im}\Big[\frac{i}{1+i\alpha}\Phi^-(x,t)\Big], \tag{27}$$

respectively.

Thus the problem reduces to finding of the function $\Phi^-(x,t)$. Relying on (21), from (14) we get

$$\operatorname{Im}\Big\{\Big[(\varkappa^*+2) - \frac{2i\alpha}{1+i\alpha}\Big]\dot{\Phi}^-(x,t) + \Big[\varkappa^* m + 2k - \frac{2ik\alpha}{1+i\alpha}\Big]\Phi^-(x,t)\Big\} = e^{-kt}\dot{f}_2(x,t).$$

We write the obtained equation in the form

$$\operatorname{Im}\left[(a+ib)\Phi^-(x,t) + (c+id)\Phi^-(x,t)\right] = f_3(x,t), \tag{28}$$

where

$$a = (\varkappa^*+2)(1+\alpha^2) - 2\alpha^2; \quad c = (\varkappa^* m + 2k)(1+\alpha^2);$$
$$b = -2\alpha; \qquad d = -2\alpha k; \tag{29}$$
$$f_3(x,t) = (1+\alpha^2)e^{-kt}\dot{f}_2(x,t)$$

In view of (20), from (28), after simple transformations, we obtain

$$\Big[\dot{\Phi}^+(x,t) + \frac{c-id}{a-ib}\Phi^+(x,t)\Big] = -\frac{(1+i\alpha)(a+ib)}{(1-i\alpha)(a-ib)}\Big[\dot{\Phi}^-(x,t) + \frac{c+id}{a+ib}\Phi^-(x,t)\Big]$$
$$-\frac{2i(1+i\alpha)^2}{(a-ib)(1+\alpha^2)}f_3(x,t) \tag{30}$$

Considering the piecewise holomorphic function $\Omega(z,t)$ defined by the formula

$$\Omega(z,t)=\begin{cases}\dot{\Phi}(z,t)+\frac{c-id}{a-ib}\Phi(z,t), & z\in S^+,\\ \dot{\Phi}(z,t)+\frac{c+id}{a+ib}\Phi(x,t), & z\in S^-,\end{cases}$$

from (30) we obtain the following boundary value problem of linear conjugation:

$$\Omega^+(x,t)=g\,\Omega^-(x,t)+F(x,t), \tag{31}$$

where

$$g=-\frac{(1+i\alpha)(a+ib)}{(1-i\alpha)(a-ib)};\qquad F(x,t)=-\frac{2i(1+i\alpha)^2}{(a-ib)(1+\alpha^2)}f_3(x,t).$$

Taking into account that on the basis of (29),

$$(1+i\alpha)(a+ib)=(\varkappa^*+2+i\alpha\varkappa^*)(1+\alpha^2),$$

we can write the constant g in the form

$$g=-\frac{1+i\beta_0}{1-i\beta_0}, \tag{32}$$

where

$$\beta_0=\frac{\alpha\varkappa^*}{\varkappa^*+2}.$$

Bearing in mind that $\alpha>0$, $\varkappa^*>0$ and introducing the constant δ defined by the conditions

$$\operatorname{tg}\,\pi\delta=\beta_0,\qquad 0\le\delta<\frac{1}{2}, \tag{33}$$

due to (32), the coefficient of problem (31) is written in the form

$$g=e^{2\pi i\gamma}, \tag{34}$$

where $\gamma=\frac{1}{2}+\delta$.

As a canonical function of problem (31) we can take the function

$$\chi(z)=(1+z)^{\frac{1}{2}+\delta}(1-z)^{\frac{1}{2}-\delta},$$

where under the right-hand side is meant the certain branch which is holomorphic outside of L', adopts on the upper side of the segment the positive values and takes at infinity the form

$$\chi(z)=(1+z)^{\frac{1}{2}+\delta}(1-z)^{\frac{1}{2}-\delta}=-ize^{\pi i\delta}+O(1). \tag{35}$$

Relying on the above reasoning, we obtain factorization of the coefficient of problem (31) in the form

$$g=\frac{\chi^-(x)}{\chi^+(x)},\quad x\in L'. \tag{36}$$

Further, the vanishing at infinity solution of problem (31) of the class h_0 (for that class, see [7]) is of the form

$$\Omega(z,t)=\frac{1}{2\pi i\chi(z)}\int_{-1}^{1}\frac{\chi^+(\sigma)F(\sigma,t)}{\sigma-z}d\sigma+\frac{D_0}{\chi(z)}, \tag{37}$$

where D_0 is the constant defined from the conditions (4) and (35), having the form

$$D_0=\frac{(1+i\alpha)N_0}{2\pi}\,e^{\pi i\delta}.$$

Owing to (33), (34), (36) and (37), we have

$$\Omega^-(x,t)=\frac{e^{-2\pi i\delta}}{2}\left[F(x,t)-\frac{1}{\pi i\chi^+(x)}\int_{-1}^{1}\frac{\chi^+(\sigma)F(\sigma,t)}{\sigma-x}d\sigma\right]-\frac{D_0e^{-2\pi i\delta}}{\chi^+(x)}.$$

Having defined $\Omega^-(x,t)$, to find $\Phi^-(x,t)$, we obtain the following differential equation

$$\dot{\Phi}^-(x,t)+\lambda\Phi^-(x,t)=\Omega^-(x,t), \tag{38}$$

where

$$\lambda = \frac{c - id}{a - ib} = \lambda_1 + i\lambda_2; \quad \lambda_1 = \frac{[\varkappa^* m(1+\alpha^2) + 2k][\varkappa^*(1+\alpha^2)+2] + 4k\alpha^2}{[\varkappa^*(1+\alpha^2)+2]^2 + 4\alpha^2},$$
$$\lambda_2 = \frac{2\alpha\varkappa^*\,(m-k)\,(1+\alpha^2)}{[\varkappa^*(1+\alpha^2)+2]^2 + 4\alpha^2}. \tag{39}$$

The solution of equation (38) is represented by the formula

$$\Phi^-(x,t) = e^{-(\lambda_1+i\lambda_2)t}\left[\Phi^-(x,0) + \int\limits_0^t e^{(\lambda_1+i\lambda_2)\tau}\Omega^-(x,\tau)d\tau\right], \tag{40}$$

where $\Phi^-(x,0)$ is of the form (26).

On the basis of the above-obtained results, we can conclude that in our case (i.e., in the case of pressure of a rigid punch with friction) the tangential and normal stresses defined by formula (27) have, as is seen from (40), the character of damping oscillations with respect to time t. Also, taking into account (39), we can conclude that oscillations are absent in the following cases:

(1) for $\alpha = 0$ (i.e., without friction);

(2) for $m = k$ (i.e., the constants $\lambda, \dots, \mu^*$ are connected by the relation $\frac{\lambda}{\lambda^*} = \frac{\mu}{\mu^*}$).

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ON ONE NEUMANN TYPE PROBLEM FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

IVAN KIGURADZE

Dedicated to the Blessed Memory of Professor A. Kharadze

Abstract. Optimal in a certain sense conditions guaranteeing the existence of a unique solution of the differential equation

$$u'' = p(t)u + q(t),$$

satisfying the Neumann type boundary conditions

$$u'(a) = \ell_1 u(a) + c_1, \quad u'(b) = \ell_2 u(b) + c_2,$$

are established.

On a finite interval $[a, b]$, we consider the differential equation

$$u'' = p(t)u + q(t) \tag{1}$$

with the boundary conditions

$$u'(a) = \ell_1 u(a) + c_1, \quad u'(b) = \ell_2 u(b) + c_2, \tag{2}$$

where $p, q \in [a, b] \to \mathbb{R}$ are Lebesgue integrable functions, $c_i \in \mathbb{R}$ $(i = 1, 2)$,

$$\ell_1 \geq 0, \quad \ell_2 \leq 0.$$

For $\ell_i = 0$ $(i = 1, 2)$, the boundary conditions (2) are the Neumann ones. In this case, problem $(1), (2)$ is studied in detail (see, e.g., [1, 3–5] and the references therein). However, this problem in a general case remains still insufficiently studied. The present paper is devoted to fill up this gap.

Assume

$$p_+(t) \equiv \big(|p(t)| + p(t)\big)/2, \quad p_-(t) \equiv \big(|p(t)| - p(t)\big)/2,$$

$$\mathcal{P}_+ = \int_a^b p_+(t)\, dt, \quad \mathcal{P}_- = \int_a^b p_-(t)\, dt.$$

Theorem 1. *Let $\ell_1 \geq 0$, $\ell_2 \leq 0$,*

$$\ell_1 - \ell_2 + \operatorname{mes}\big\{t \in [a, b] :\ p(t) \neq 0\big\} > 0, \tag{3}$$

$$\ell_1 - \ell_2 + \mathcal{P}_+ \leq \mathcal{P}_-, \tag{4}$$

and there exist a number $\lambda \geq 1$ such that

$$\int_a^b [p(t)]_-^\lambda\, dt \leq \frac{4}{b - a} \Big(\frac{\pi}{b - a}\Big)^{2\lambda - 2}. \tag{5}$$

Then problem $(1), (2)$ has one and only one solution.

To prove this theorem, we need the following

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Lemma 1. *Let $\ell_1 \geq 0$, $\ell_2 \leq 0$, conditions* (3), (4) *be fulfilled and the homogeneous problem*

$$u'' = p(t)u, \tag{1_0}$$

$$u'(a) = \ell_1 u(a), \quad u'(b) = \ell_2 u(b) \tag{2_0}$$

have a nontrivial solution u. Then there exist points $t_0 \in \,]a,b[\,$, $t_1 \in [a,t_0[\,$, and $t_2 \in \,]t_0,b]$ such that

$$u(t_0) = 0, \quad u'(t_1) = 0, \quad u'(t_2) = 0. \tag{6}$$

Proof. In view of (2_0), it is obvious that

$$u(a) \neq 0, \quad u(b) \neq 0. \tag{7}$$

First, let us show that the solution u in the interval $]a,b[$ has at least one zero. Assume the contrary, i.e., $u(t) \neq 0$ for $a < t < b$. Then by virtue of inequality (7), we have

$$u(t) \neq 0 \ \text{ for } \ a \leq t \leq b.$$

Consequently,

$$\frac{u''(t)}{u(t)} + p(t) = 0 \ \text{ for almost all } \ t \in [a,b]. \tag{8}$$

Integrating this identity from a to b, and taking into account equality (2_0), we find

$$\int_a^b \frac{u'^2(t)}{u^2(t)}\,dt = \ell_1 - \ell_2 + \mathcal{P}_+ - \mathcal{P}_-,$$

whence, owing to conditions (2)–(4) and (8), it follows that $u'(t) \equiv 0$,

$$p(t) = 0 \ \text{ for almost all } \ t \in \,]a,b], \ \ \ell_1 - \ell_2 > 0,$$

and either $u(a) = 0$, or $u(b) = 0$. But this contradicts condition (7). The obtained contradiction proves that for some $t_0 \in \,]a,b[$ the equality

$$u(t_0) = 0 \tag{9}$$

is fulfilled.

Without loss of generality, we can assume that

$$u'(t_0) > 0. \tag{10}$$

If $\ell_1 = 0$, then

$$u'(t_1) = 0, \tag{11}$$

where $t_1 = a$. Let us show that if $\ell_1 > 0$, then this equality is fulfilled for some $t_1 \in \,]a,t_0[\,$. Assume the contrary, i.e.,

$$u'(t) > 0 \ \text{ for } \ a < t \leq t_0.$$

Then, in view of (9), we have

$$u(t) < 0 \ \text{ for } \ a \leq t < t_0.$$

But this is impossible since

$$u(a) = u'(a)/\ell_1 \geq 0.$$

Thus we have proved that for some $t_1 \in [a,t_0[$ equality (11) is fulfilled.

Analogously we can show that for some $t_2 \in \,]t_0,b]$, the equality

$$u'(t_2) = 0$$

is fulfilled. □

Lemma 2 (T. Kiguradze [2])**.** *Let for some $a_0 \in [a, b[$ and $b_0 \in]a_0, b]$ the differential equation* (1_0) *have a nontrivial solution u satisfying either the boundary conditions*

$$u'(a_0) = 0, \quad u(b_0) = 0,$$

or the boundary conditions

$$u(a_0) = 0, \quad u'(b_0) = 0.$$

Then

$$(b_0 - a_0)^{2\lambda-1} \int\limits_{a_0}^{b_0} [p(t)]_-^\lambda \, dt > \Big(\frac{\pi}{2}\Big)^{2\lambda-2} \quad \textit{for} \;\; \lambda \geq 1.$$

This lemma is a corollary of Theorem 1.3 from [2].

Proof of Theorem 1. Assume that the theorem is not true. Then, owing to the Fredholmicity of problem $(1), (2)$, the homogeneous problem $(1_0), (2_0)$ has a nontrivial solution u.

According to Lemma 1, there exist points $t_0 \in]a, b[$, $t_1 \in [a, t_0[$ and $t_2 \in]t_0, b]$ such that the solution u satisfies equalities (6). Thus by Lemma 2, we have the inequalities

$$(t_0 - t_1)^{2\lambda-1} \int\limits_{t_1}^{t_0} [p(t)]_-^\lambda \, dt > \Big(\frac{\pi}{2}\Big)^{2\lambda-2}, \quad (t_2 - t_0)^{2\lambda-1} \int\limits_{t_0}^{t_2} [p(t)]_-^\lambda \, dt > \Big(\frac{\pi}{2}\Big)^{2\lambda-2}.$$

Consequently,

$$\big[(t_0 - t_1)(t_2 - t_0)\big]^{2\lambda-1} \bigg(\int\limits_{t_1}^{t_0} [p(t)]_-^\lambda \, dt\bigg)\bigg(\int\limits_{t_0}^{t_2} [p(t)]_-^\lambda \, dt\bigg) > \Big(\frac{\pi}{2}\Big)^{4\lambda-4}.$$

On the other hand,

$$(t_0 - t_1)(t_2 - t_0) \leq \frac{(t_2 - t_1)^2}{4} \leq \frac{(b - a)^2}{4},$$

$$\bigg(\int\limits_{t_1}^{t_0} [p(t)]_-^\lambda \, dt\bigg)\bigg(\int\limits_{t_0}^{t_2} [p(t)]_-^\lambda \, dt\bigg) \leq \bigg(\int\limits_{t_1}^{t_2} [p(t)]_-^\lambda \, dt\bigg)^2 \Big/ 4 \leq \bigg(\int\limits_{a}^{b} [p(t)]_-^\lambda \, dt\bigg)^2 \Big/ 4.$$

Therefore,

$$\frac{1}{4}\Big(\frac{b-a}{2}\Big)^{4\lambda-2} \bigg(\int\limits_{a}^{b} [p(s)]_-^\lambda \, ds\bigg)^2 > \Big(\frac{\pi}{2}\Big)^{4\lambda-4}.$$

However, this inequality contradicts inequality (5). The obtained contrdiction proves the theorem. □

Remark 1. If $\ell_1 \geq 0$ and $\ell_2 \leq 0$, then for problem $(1), (2)$ to be uniquely solvable, it is necessary that inequality (3) is fulfilled. Indeed, if the above-mentioned inequality is violated, then $p(t) = 0$ for almost all $t \in]a, b[$, $\ell_1 = \ell_2 = 0$ and, consequently, the homogeneous problem $(1_0), (2_0)$ has an infinite set of solutions.

Remark 2. Examples 1 and 2 below show that conditions (4) and (5) in Theorem 1 are unimprovable and they cannot be replaced by the conditions

$$\ell_1 - \ell_2 + \mathcal{P}_+ < \mathcal{P}_- + \varepsilon, \tag{12}$$

$$\int\limits_a^b [p(t)]_-^\lambda \, dt < \frac{4}{b-a}\Big(\frac{\pi + \varepsilon}{b - a}\Big)^{2\lambda-2} \tag{13}$$

no matter how small $\varepsilon > 0$ is.

Example 1. Let ε be an arbitrary positive constant,

$$p(t) \equiv -\Big(\frac{2x}{b-a}\Big)^2, \quad \ell_1 = \frac{2x}{b-a}\ \mathrm{tg}(x), \quad \ell_2 = -\frac{2x}{b-a}\ \mathrm{tg}(x), \tag{14}$$

and $x \in]0,1[$ be so small that

$$\frac{4x}{b-a}\ \mathrm{tg}(x) < \frac{4x^2}{b-a} + \varepsilon.$$

Then $\ell_1 > 0$, $\ell_2 < 0$,

$$\ell_1 - \ell_2 + \mathcal{P}_+ = \frac{4x}{b-a}\ \mathrm{tg}(x) < \mathcal{P}_- + \varepsilon,$$

$$\int_a^b [p(t)]_-^\lambda\, dt = (b-a)\Big(\frac{2x}{b-a}\Big)^{2\lambda} < (b-a)\Big(\frac{2}{b-a}\Big)^{2\lambda} < \frac{4}{b-a}\,\Big(\frac{\pi}{b-a}\Big)^{2\lambda-2}.$$

Consequently, all the conditions of Theorem 1 are fulfilled, except inequality (4), instead of which inequality (12) holds. Nevertheless, in this case the homogeneous problem $(1_0), (2_0)$ has the nontrivial solution

$$u(t) \equiv \cos\Big(\frac{2x(t-a)}{b-a} - x\Big).$$

Example 2. Let $\varepsilon \in]0, \frac{\pi}{4}[$, $x \in]0, \varepsilon[$,

$$p(t) \equiv -\Big(\frac{\pi+x}{b-a}\Big)^2, \quad \ell_1 = \frac{\pi+x}{b-a}\ \mathrm{tg}(x), \quad \ell_2 = 0,$$

and the number $\lambda \in [1, +\infty[$ be such that

$$\frac{\pi+x}{b-a} < \Big(\frac{2}{b-a}\Big)^{\frac{1}{\lambda}}\Big(\frac{\pi+\varepsilon}{b-a}\Big)^{1-\frac{1}{\lambda}}.$$

Then

$$\ell_1 - \ell_2 + \mathcal{P}_+ = \frac{\pi+x}{b-a}\ \mathrm{tg}(x) < \frac{\pi+x}{b-a} < \frac{(\pi+x)^2}{b-a} = \mathcal{P}_-,$$

$$\int_a^b [p(t)]_-^\lambda\, dt = (b-a)\Big(\frac{\pi+x}{b-a}\Big)^{2\lambda} < \frac{4}{b-a}\,\Big(\frac{\pi+\varepsilon}{b-a}\Big)^{2\lambda-2}.$$

Consequently, all the conditions of Theorem 1 are fulfilled, except inequality (5), instead of which inequality (13) holds. On the other hand, the homogeneous problem $(1_0), (2_0)$ has the nontrivial solution

$$u(t) \equiv \cos\Big(\frac{(\pi+x)(t-a)}{b-a} - x\Big).$$

Theorem 2. *If $\ell_1 \geq 0$, $\ell_2 \leq 0$, and*

$$\mathcal{P}_- < \frac{\ell_1 - \ell_2 + \mathcal{P}_+}{1 + (b-a)(\ell_1 - \ell_2 + \mathcal{P}_+)}\,, \tag{15}$$

then problem $(1), (2)$ *has one and only one solution.*

Proof. First note that inequality (15) yields the following inequalities

$$\delta = \ell_1 - \ell_2 + \mathcal{P}_+ - \mathcal{P}_- > 0, \tag{16}$$

$$r = 1 - (b-a)(\mathcal{P}_- + \delta^{-1}\mathcal{P}_-^2) > 0. \tag{17}$$

Assume that the theorem is not true. Then the homogeneous problem $(1_0), (2_0)$ has a nontrivial solution u. Put

$$x = \min\big\{|u(t)| :\ a \leq t \leq b\big\}, \quad y = \Big(\int_a^b u'^2(t)\, dt\Big)^{\frac{1}{2}}.$$

Then

$$x^2 \le u^2(t) \le x^2 + 2(b-a)^{\frac{1}{2}} y + (b-a)y^2 \ \text{ for } \ a \le t \le b. \tag{18}$$

On the other hand, in view of inequality (16), it is obvious that

$$y > 0. \tag{19}$$

Integrating both sides of the identity

$$u''(t)u(t) = p(t)u^2(t) \ \text{ for almost all } \ t \in [a,b]$$

from a to b and taking into account the boundary conditions (2_0), we obtain

$$\ell_1 u^2(a) - \ell_2 u^2(b) + \int\limits_a^b [p(t)]_+ u^2(t)\,dt + \int\limits_a^b u'^2(t)\,dt = \int\limits_a^b [p(t)]_- u^2(t)\,dt.$$

Thus, by inequality (18), it follows that

$$(\ell_1 - \ell_2 + \mathcal{P}_+)x^2 + y^2 \le \mathcal{P}_-\big(x^2 + 2(b-a)^{\frac{1}{2}} xy + (b-a)y^2\big),$$

that is,

$$\big(\delta^{\frac{1}{2}} x - (b-a)^{\frac{1}{2}} \delta^{-\frac{1}{2}} \mathcal{P}_- y\big)^2 + ry^2 \le 0.$$

However, this inequality contradicts inequalities (17) and (19). The obtained contradiction proves the theorem. □

Remark 3. Condition (15) is unimprovable and it cannot be replaced by the condition

$$\mathcal{P}_- < \frac{\ell_1 - \ell_2 + \mathcal{P}_+}{1 + (b-a)(\ell_1 - \ell_2 + \mathcal{P}_+)} + \varepsilon \tag{20}$$

no matter how small is $\varepsilon > 0$.

Indeed, if

$$0 < \varepsilon < \frac{4}{b-a}\,, \ \ 0 < x < \varepsilon^{\frac{1}{2}} (b-a)^{\frac{1}{2}}/2,$$

and the function p and numbers ℓ_i $(i = 1,2)$ are defined by equalities (14), then instead of (15) inequality (20) holds, but nevertheless, the homogeneous problem $(1_0), (2_0)$ has the nontrivial solution

$$u(t) \equiv \cos\Big(\frac{2x(t-a)}{b-a} - x\Big).$$

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ON THE BOUNDEDNESS IN GENERALIZED WEIGHTED GRAND LEBESGUE SPACES OF SOME INTEGRAL OPERATORS ASSOCIATED TO THE SCHRÖDINGER OPERATOR

VAKHTANG KOKILASHVILI

Abstract. In the present note, in generalized weighted grand Lebesgue spaces on R^d, $d \geq 3$, we consider the boundedness of diffusion semi-group maximal functions, Riesz transforms and their adjoints, as well as the Littlewood-Paley quadratic functions related to the Schrödinger differential operator $-\Delta + V$, where the potential V satisfies a reverse Hölder inequality with an exponent, greather than $d/2$. The class of weights, more general than that of Muchenhoupt' one, is used.

1. Introduction

Our note deals with the mapping properties of certain integral operators associated with the Schrödinger differential operator

$$\mathcal{L} = -\Delta + V(x), \quad x \in R^d, \quad d \geq 3,$$

where Δ is the Laplasian and $V(x)$ is non-negative, non-identicaly zero and for some $q > \frac{d}{2}$ satisfies the reverse Hölder inequality

$$\left(\frac{1}{|B|} \int\limits_B v^q(y)dy\right)^{1/q} \leq \frac{c}{|B|} \int\limits_B v(y)dy,$$

for every ball $B \subset R^d$.

We consider the boundedness problems relating to the Schrödinger integral operators in some nonstandard Banach function space.

The mapping properties in L^p of several types Schrödinger–Riesz transforms have been studied in a pioneer work by Z. W. Shen [9], in which he introduced the following critical radius function associated with the potential V:

$$\rho(x) = \sup\left\{r > 0 : \ r^{\frac{1}{d-q}} \int\limits_{B(x,r)} \leq 1\right\}, \quad x \in R^d.$$

This notion has played an essential role in the extensive study of the boundedness of Schrödinger integral operators in weighted L^p spaces with weights, larger, in general, than Muchenhoupt's ones.

Here, we present the definition of generalized weighted grand Lebesgue spaces $L_v^{p),\phi}(R^n, w)$.

Let $1 < p < \infty$, ϕ be a positive non-decreasing function on $(0, p-1)$ satisfying $\phi(0+) = 0$. The generalized weighted grand Lebesgue space $L_v^{p),\phi}(R^n, w)$ is defined as the set of all everywhere finite measurable functions for which

$$\|f\|_{L_v^{p),\phi}(R^n,w)} = \sup_{0<\varepsilon<p-1} \left(\phi(\varepsilon) \int\limits_{R^n} |f(x)|^{p-\varepsilon} w(x) v^{\varepsilon}(x) dx\right)^{\frac{1}{p-\varepsilon}} < \infty,$$

where $wv^{\varepsilon} \in L^1_{\text{loc}}(R^n)$ for all ε, $0 < \varepsilon < p-1$.

Further, we follow the definitions given in [3].

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Key words and phrases. Grand Lebesgue spaces; Weights; Critical radius; Schrödinger's integral operators; Reverse Hölder inequality.

Definition 1. Let $1 < p < \infty$. A weight function $w \in A_p^{\text{loc}}$ if there exists a constant $c > 0$ such that

$$\left(\int_B w(y)dy\right)^{1/p}\left(\int_B w^{-\frac{1}{p-1}}(y)dy\right) \le c|B|$$

for every ball $B = B(x,r)$, where $0 \le r \le \rho(x)$, $x \in R^d$.

Definition 2. Given $p > 1$, the class

$$A_p^\rho := \bigcup_{\theta \ge 0} A_p^{\rho,\theta},$$

where $A^{\rho,\theta}$ is defined as the weights w such that

$$\left(\int_B w(y)dy\right)^{1/p}\left(\int_B w^{-\frac{1}{p-1}}(y)dy\right)^{1/p'} \le c|B|\left(1+\frac{r}{\rho(x)}\right)^\theta$$

for all balls $B(x,r)$.

The following proper inclusions

$$A_p \subset A_p^\rho \subset A_p^{\text{loc}}$$

are valid.

In the case for $\rho \equiv 1$, the function $w(x) = 1 + |x|^\gamma$, $\gamma > d(p-1)$ belongs to A_p^ρ, but it is not in A_p.

Here, we establish the weighted inequalities in $L_v^{p),\phi}(R^n, w)$ for the following Schrödinger operators:

i) Maximal operator of the diffusion semi-group

$$\mathcal{M}^* f(x) = \sup_{t>0} e^{-t\mathcal{L}} f(x);$$

ii) $\mathcal{L}$- Riesz transform

$$R = \nabla \mathcal{L}^{-\frac{1}{2}},$$

and its adjoint

$$R^* = \mathcal{L}^{-\frac{1}{2}}\nabla;$$

iii) $\mathcal{L}$-Littlewood-Paley function

$$g(f)(x) = \left(\int_0^\infty \left|\frac{d}{dt} e^{-t\mathcal{L}}(f)(x)\right|^2 t dt\right)^{\frac{1}{2}}.$$

Let T stand for any of the above operators.

Now we present one of the main results of our note.

Theorem 1. *Let $1 < p < \infty$, $w \in A_p^\rho$ and let $v \in L^p(R^n, w)$, $v^\gamma \in A_p^\rho$ for some $\gamma > 0$. Then the operator T is bounded in $L_v^{p),\phi}(R^n, w)$.*

By T_{loc} we denote the ρ-localization of T:

$$T_\rho f(x) = T\left(f\chi_{B(x,\rho(x)}(x)\right).$$

Theorem 2. *Let $1 < p < \infty$, $w \in A_p^{\rho,\text{loc}}$ and let $v \in L^p(R^n, w)$, $v^\gamma \subset A_p^{\rho,\text{loc}}$ for some $\gamma > 0$. Then the operator T_{loc} is bounded in $L_v^{p),\phi}(R^n, w)$.*

The boundedness problems for the classical versions of the above-mentioned integral operators when $\mathcal{L} = -\Delta$ in weighted grand Lebesgue spaces in the framework of Muckenhoupt's A_p classes were studied in [4–7] (see also the monograph [8, Chapter 7], and references therein).

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TWO-POINT BOUNDARY VALUE PROBLEMS FOR SINGULAR TWO-DIMENSIONAL LINEAR DIFFERENTIAL SYSTEMS

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Abstract. For two-dimensional systems of ordinary linear differential equations with singular coefficients, unimprovable in a certain sense conditions are established guaranteeing, respectively, the Fredholmicity and unique solvability of the Dirichlet and the Nicoletti problems.

On an open interval $]a,b[$, we consider the two-dimensional linear differential system

$$u_i' = p_i(t)u_{3-i} + q_i(t) \quad (i = 1,2) \tag{1}$$

with the Dirichlet boundary conditions

$$u_1(a+) = 0, \quad u_1(b-) = 0, \tag{2_1}$$

and the Nicoletti boundary conditions

$$u_1(a+) = 0, \quad u_2(b-) = 0, \tag{2_2}$$

where p_1 and $q_1 :]a,b[\to \mathbb{R}$ are Lebesgue integrable functions, while the functions p_2 and $q_2 :]a,b[\to \mathbb{R}$ are Lebesgue integrable on every closed interval contained in $]a,b[$.

We are mainly interested in the case where the functions p_2 and q_2 have nonintegrable singularities at the points a and b, i.e. the case, where

$$\int_a^b \big(|p_2(t)| + |q_2(t)|\big)dt = +\infty.$$

System (1) is singular in that sense.

In the case, where $p_1(t) \equiv 1$ and $q_1(t) \equiv 0$, i.e., when system (1) is equivalent to a second order linear differential equation, the singular problems $(1),(2_1)$ and $(1),(2_2)$ are investigated in sufficient detail (see, [1–6] and the references therein). In the general case the above mentioned problems are still not well studied. The present paper is devoted exactly to this case.

Theorems 1_1 and 1_2 below contain conditions guaranteeing, respectively, the Fredholmicity of the singular problems $(1),(2_1)$ and $(1),(2_2)$. Based on these theorems we have established unimprovable in a certain sense conditions for the unique solvability of these problems (see, Theorems 2_1 and 2_2, and their corollaries). They are generalizations of some results by T. Kiguradze [4] concerning the unique solvability of the Dirichlet and the Nicoletti problems for singular second order linear differential equations.

We use the following notation.

$$[x]_+ = \frac{|x|+x}{2}, \quad [x]_- = \frac{|x|-x}{2};$$

$u(t_0+)$ and $u(t_0-)$ are the right and the left limits, respectively, of the function u at the point t_0;

$L([a,b])$ is the space of Lebesgue integrable on $[a,b]$ real functions;

$L_{loc}(]a,b[)$ and $L_{loc}(]a,b])$ are the spaces of real functions which are Lebesgue integrable on every closed interval contained in $]a,b[$ and $]a,b]$, respectively;

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If $p \in L([a,b])$, then

$$I_a(p)(t) = \int_a^t p(s)ds, \quad I_{a,b}(p)(t) = \int_a^t p(s)ds \int_t^b p(s)ds \quad \text{for } a \le t \le b.$$

A vector-function $(u_1, u_2) :]a,b[\to \mathbb{R}^2$ is said to be **a solution of system** (1) if its components are absolutely continuous on every closed interval contained in $]a,b[$ and satisfy system (1) almost everywhere on $]a,b[$.

A solution of system (1) satisfying the boundary conditions (2_1) (the boundary conditions (2_2)) is said to be **a solution of problem** $(1), (2_1)$ **(of problem** $(1), (2_2)$**)**.

We investigate problem $(1), (2_1)$ in the case where the functions p_i and q_i $(i = 1, 2)$ satisfy the conditions

$$p_1 \in L([a,b]), \quad q_1 \in L([a,b]), \quad p_2 \in L_{loc}(]a,b[), \quad q_2 \in L_{loc}(]a,b[), \tag{3}$$

$$p_1(t) \ge 0 \ \text{ for } \ a < t < b, \quad \delta = \int_a^b p_1(t)dt > 0. \tag{4}$$

Along with system (1) we consider the corresponding homogeneous system

$$u_i' = p_i(t)u_{3-i} \ \ (i = 1, 2). \tag{1_0}$$

The following theorem is valid.

Theorem 1_1. *Let along with* (3) *and* (4) *the conditions*

$$\int_a^b I_{a,b}(p_1)(t)[p_2(t)]_- dt < +\infty$$

and

$$\int_a^b I_{a,b}(p_1)(t)\Big(I_{a,b}(|q_1|)(t)[p_2(t)]_+ + |q_2(t)|\Big)dt < +\infty \tag{5}$$

be satisfied. Then for the unique solvability of problem $(1), (2_1)$ *it is necessary and sufficient that the corresponding homogeneous problem* $(1_0), (2_1)$ *to have only the trivial solution.*

Theorem 2_1. *Let there exist a constant* $\lambda \ge 1$ *and a measurable function* $p :]a,b[\to [0, +\infty[$ *such that along with* (3)–(5) *the conditions*

$$[p_2(t)]_- = p(t)p_1^{1-\frac{1}{\lambda}}(t) \quad \textit{for } \ a < t < b,$$

and

$$\int_a^b I_{a,b}(p_1)(t)p^\lambda(t)dt \le \Big(\frac{\pi}{\delta}\Big)^{2\lambda-2}\delta \tag{6}$$

are satisfied. Then problem $(1), (2_1)$ *has one and only one solution.*

Corollary 1_1. *If along with* (3)–(5) *the condition*

$$\int_a^b I_{a,b}(p_1)(t)[p_2(t)]_- dt \le \delta \tag{7}$$

holds, then problem $(1), (2_1)$ *has one and only one solution.*

Corollary 2_1. *If along with* (3)–(5) *the conditions*

$$p_2(t) \geq -\Big(\frac{\pi}{\delta}\Big)^2 p_1(t) \quad \textit{for } a < t < b, \tag{8}$$

$$\operatorname{mes}\Big\{t \in]a,b[\,:\, p_2(t) > -\Big(\frac{\pi}{\delta}\Big)^2 p_1(t)\Big\} > 0 \tag{9}$$

hold, then problem $(1),(2_1)$ *has one and only one solution.*

Example 1_1. If

$$0 \leq p_1(t) \leq \exp\left(-\frac{b-a}{(t-a)(b-t)}\right) \text{ for } a < t < b, \quad \delta = \int_a^b p_1(t)dt > 0,$$

$$-\frac{\delta}{(b-a)(t-a)^2(b-t)^2}\exp\left(\frac{b-a}{(t-a)(b-t)}\right) \leq p_2(t) \leq 0 \text{ for } a < t < b,$$

$$|q_2(t)| \leq \frac{\ell}{(t-a)^\mu(b-t)^\mu}\exp\left(\frac{b-a}{(t-a)(b-t)}\right) \text{ for } a < t < b,$$

where $\ell > 0$, $\mu < 3$, then all the conditions of Corollary 1_1 are fulfilled, and therefore problem $(1),(2_1)$ has a unique solution.

The above example shows that the functions p_2 and q_2 in the conditions of Theorems 1_1 and 2_1 may have singularities of arbitrary order at the points a and b.

Remark 1_1. Inequalities (6) and (7) in Theorem 2_1 and Corollary 1_1 are unimprovable and they cannot be replaced, respectively, by the conditions

$$\int_a^b I_{a,b}(p_1)(t)p^\lambda(t)dt \leq \Big(\frac{\pi}{\delta}\Big)^{2\lambda-2}\delta + \varepsilon$$

and

$$\int_a^b I_{a,b}(p_1)(t)[p_2(t)]_- \, dt \leq \delta + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

Remark 2_1. Inequalities (8) and (9) in Corollary 2_1 are unimprovable as well since if along with (4) the conditions

$$p_2(t) \equiv -\Big(\frac{\pi}{\delta}\Big)^2 p_1(t), \quad q_i(t) \equiv 0 \ \ (i = 1, 2)$$

hold, then problem $(1),(2_1)$ has an infinite set of solutions.

In contrast to problem $(1),(2_1)$, we investigate problem $(1),(2_2)$ in the case where instead of (3) the conditions

$$p_1 \in L([a,b]), \quad q_1 \in L([a,b]), \quad p_2 \in L_{loc}(]a,b]), \quad q_2 \in L_{loc}(]a,b]) \tag{10}$$

are satisfied.

Theorem 1_2. *Let along with* (4) *and* (10) *the conditions*

$$\int_a^b I_a(p_1)(t)[p_2(t)]_- \, dt < +\infty$$

and

$$\int_a^b I_a(p_1)(t)\left(I_a(|q_1|)(t)[p_2(t)]_+ + |q_2(t)|\right)dt < +\infty \tag{11}$$

be satisfied. Then for the unique solvability of problem $(1),(2_2)$ *it is necessary and sufficient that the corresponding homogeneous problem* $(1_0),(2_2)$ *to have only the trivial solution.*

Theorem 2_2. *Let there exist a constant* $\lambda \geq 1$ *and a measurable function* $p :]a,b[\to [0,+\infty[$ *such that along with* (4), (10), *and* (11) *the conditions*

$$[p_2(t)]_- = p(t)p_1^{1-\frac{1}{\lambda}}(t) \quad \text{for } a < t < b,$$

and

$$\int_a^b I_a(p_1)(t)p^\lambda(t)dt \leq \Big(\frac{\pi}{2\delta}\Big)^{2\lambda-2} \tag{12}$$

are satisfied. Then problem $(1),(2_2)$ *has one and only one solution.*

Corollary 1_2. *If along with* (4), (10), *and* (11) *the condition*

$$\int_a^b I_a(p_1)(t)[p_2(t)]_- dt \leq 1 \tag{13}$$

holds, then problem $(1),(2_2)$ *has one and only one solution.*

Corollary 2_2. *If along with* (4), (10), *and* (11) *the conditions*

$$p_2(t) \geq -\Big(\frac{\pi}{2\delta}\Big)^2 p_1(t) \quad \text{for } a < t < b, \tag{14}$$

$$\operatorname{mes}\Big\{t \in]a,b[\, : p_2(t) > -\Big(\frac{\pi}{2\delta}\Big)^2 p_1(t)\Big\} > 0 \tag{15}$$

hold, then problem $(1),(2_2)$ *has one and only one solution.*

Example 1_2. If

$$0 \leq p_1(t) \leq (t-a)^{-2}\exp\left(-\frac{1}{t-a}\right) \quad \text{for } a < t < b, \quad \delta = \int_a^b p_1(t)dt > 0,$$

$$-\frac{1}{b-a}\exp\left(\frac{1}{(t-a)(b-t)}\right) \leq p_2(t) \leq 0, \quad |q_2(t)| \leq \frac{\ell}{(t-a)^\mu}\exp\left(\frac{1}{t-a}\right) \quad \text{for } a < t < b,$$

where $\ell > 0$, $\mu < 1$, then all the conditions of Corollary 1_2 are fulfilled, and therefore problem $(1),(2_2)$ has a unique solution.

Remark 1_2. Inequalities (12) and (13) in Theorem 1_2 and Corollary 1_2 are unimprovable and they cannot be replaced, respectively, by the conditions

$$\int_a^b I_a(p_1)(t)p^\lambda(t)dt \leq \Big(\frac{\pi}{2\delta}\Big)^{2\lambda-2} + \varepsilon$$

and

$$\int_a^b I_a(p_1)(t)[p_2(t)]_- dt \leq 1 + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

Remark 2_2. Inequalities (14) and (15) in Corollary 2_2 are unimprovable as well since if along with (4) the conditions

$$p_2(t) \equiv -\Big(\frac{\pi}{2\delta}\Big)^2 p_1(t), \quad q_i(t) \equiv 0 \ \ (i = 1,2)$$

hold, then problem $(1),(2_2)$ has an infinite set of solutions.

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ON THE SECONDARY COHOMOLOGY OPERATIONS

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Abstract. The new secondary cohomology operations are constructed. These operations together with the Adams operations are intended to calculate the mod p cohomology algebra of loop spaces. In particular, the kernel of the loop suspension map is explicitly described.

1. Introduction

Let X be a topological space and $H^*(X;\mathbb{Z}_p)$ be the cohomology algebra in the coefficients $\mathbb{Z}_p=\mathbb{Z}/p\mathbb{Z}$ where $\mathbb{Z}$ is the integers and p is a prime. Given $n\geq 1$, let $P^*_n(X)\subset H^*(X;\mathbb{Z}_p)$ be the subset of elements of finite height

$$P^*_n(X)=\{x\in H^*(X;\mathbb{Z}_p)\mid x^{n+1}=0,\, n\geq 1\}.$$

Let $\mathcal{P}_1: H^m(X;\mathbb{Z}_p)\to H^{pm-p+1}(X;\mathbb{Z}_p)$ denote the Steenrod cohomology operation. Given $n,r\geq 1$, we construct the maps

$$\psi_{r,1}: H^{2m+1}(X;\mathbb{Z}_p)\to H^{2mp^{r+1}+1}(X;\mathbb{Z}_p)/\operatorname{Im}\mathcal{P}_1,\quad p>2, \tag{1.1}$$

and

$$\psi_{r,n}: P^m_n(X)\to H^{(m(n+1)-2)p^r+1}(X;\mathbb{Z}_p)/\operatorname{Im}\mathcal{P}_1 \quad (m \text{ is even when } p>2) \tag{1.2}$$

in which $\psi_{1,p^k-1}=\psi_k$ is the Adams secondary cohomology operation for p odd or $p=2$ and $k>1$ (cf. [1–3]). Note that when $n>1$, these maps are linear for $n+1=p^k$, $k\geq 1$ (e.g., $H^*(X;\mathbb{Z}_p)$ is a Hopf algebra). Let ΩX be the (based) loop space on X. Let $\sigma: H^*(X;\mathbb{Z}_p)\to H^{*-1}(\Omega X;\mathbb{Z}_p)$ be the loop suspension map. Theorem 2 (cf. [3]) explicitly describes $\operatorname{Ker}\sigma$ in terms of the operations $\mathcal{P}_1$ and $\psi_{1,n}$ and higher order Bockstein homomorphisms β_k associated with the short exact sequence

$$0\to\mathbb{Z}_p\to\mathbb{Z}_{p^{k+1}}\to\mathbb{Z}_{p^k}\to 0.$$

The calculation of the loop space cohomology algebra $H^*(\Omega X;\mathbb{Z}_p)$ in terms of generators and relations will appear elsewhere.

2. The Secondary Cohomology Operations $\psi_{r,n}$

The secondary cohomology operations are constructed by using the integral filtered model of a space X considered in [4].

2.1. **The Hirsch filtered models of a space.** Given a commutative graded algebra (cga) H, there are two kinds of Hirsch resolutions

$$\rho_a:(R_aH,d)\to H \quad\text{and}\quad \rho:(RH,d)\to H,$$

the absolute Hirsch resolution R_aH and the minimal Hirsch resolution RH, respectively. The first R_aH is endowed, besides the Steenrod cochain operation $E_{1,1}=\smile_1$, the cup-one product, with the higher order operations $E_{p,q}$, $p,q\geq 1$, as they usually exist in the cochain complex $C^*(X;\mathbb{Z})$; the second RH is, in fact, endowed only with the cup-one measuring the non-commutativity of the cup product $\cdot:=\smile$. In general, the operations $E_{p,q}$ appear to measure the deviations of the cup-one product from being the left and right derivations with respect to the cup product. But in RH the freeness of the multiplicative structure enables us to fix the relationship between the cup and cup-one

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products by explicit formulas, while the relation between RH and the cochain complex $C^*(X;\mathbb{Z})$ is fixed via zig-zag Hirsch maps

$$(RH,d_h)\xleftarrow{g}(R_aH,d_h)\xrightarrow{f}C^*(X;\mathbb{Z}). \tag{2.1}$$

In fact, $RH=R_aH/J$ for a certain Hirsch ideal $J\subset R_aH$. Thus, the Hirsch algebra (RH,d_h), being generated only by the $\smile_1$-product, becomes an efficient tool for calculating of the loop cohomology algebra.

Denote $H^*=H^*(X;\mathbb{Z})$. Given a prime p, let $t_{\mathbb{Z}_p}:RH\to RH\otimes\mathbb{Z}_p$ be the standard map. For $z=[c]\in H^*(X;\mathbb{Z}_p)$ with $c\in RH\otimes\mathbb{Z}_p$, let $x_0:=t^{-1}_{\mathbb{Z}_p}(c)$. If $c\in P^*_n(X)$, then in RH there is the equality $dx_1=\lambda x_0^{n+1}=0 \mod p$, some $x_1\in RH$, and p does not divide λ. Note that the essential idea can be seen for $n=1$ (the case $n>1$ is somewhat technically difficult only). Each $z\in P^*_n$ produces an infinite sequence of elements $(x_m)_{m\geq 0}$ in RH given by the following formulas:

$$dx_{2k+1}=\sum_{i_1+\cdots+i_{n+1}=k}(-1)^{|z|}\,\lambda\,x_{2i_1}\dots x_{2i_{n+1}}+\sum_{i+j=2k-1}x_{2i+1}x_{2j+1}+p\tilde{x}_{2k+1},$$

$$dx_{2k}=\sum_{i+j=2k-1}(-1)^{|x_i|+1}x_ix_j+p\tilde{x}_{2k},\qquad i_m,i,j,k\geq 0.$$

(The signs are fixed for $|z|$ and $n+1$ to be not simultaneously odd above.) In particular, when z is odd dimensional and $n,\lambda=1$, one gets for $k,i,j\geq 0$:

$$dx_k=\sum_{i+j=k-1}x_ix_j+p\tilde{x}_k.$$

In turn, the sequence $(x_m)_{m\geq 0}$ by means of the $\smile_1$- product induces four kinds of infinite sequences $b^{i_1,i_2}_{k,\ell}\in\left\{b^{1,n}_{k,\ell},\,b^{n,1}_{k,\ell},\,b^{n,n}_{k,\ell},\,b^{1,1}_{k,\ell}\right\}$ in RH for $n\geq 1$ (more precisely, one sequence $(b_{k,\ell})_{k,\ell\geq 1}$ when $n=1$) with $b_{k,\ell}:=b^{1,1}_{k,\ell}=b^{1,1}_{\ell,k}$ ($k,l\geq 2$ when $n>1$, while $k,l\geq 1$ when $n=1$), $b^{1,n}_{2i,2j}=b^{n,1}_{2i,2j},\ i,j\geq 1$, defined by the recursive formulas: $b^{1,n}_{1,1}=-(-1)^{|z|}b^{n,1}_{1,1}$ for $(k,\ell)=(1,1)$, and

$$db^{1,n}_{1,1}=\begin{cases}2x_1+\lambda\,x_0\smile_1 x_0^n, & |z| \text{ is odd},\\ x_0\smile_1 x_0^n, & |z| \text{ is even},\end{cases}$$

(in the latter case, we, in fact, have $b^{1,n}_{1,1}=\sum\limits_{i+j=n-1}x_0^i(x_0\cup_2 x_0)x_0^j$),

$b^{n,n}_{1,1}=\sum\limits_{i+j=n-1}x_0^i\,b^{1,n}_{1,1}\,x_0^j$, and for $k,\ell\geq 1$:

$$\begin{aligned}db^{*,*}_{k,\ell}&=-(-1)^{|z|}\,\alpha^{*,*}_{k,\ell}\,x^{*,*}_{k+\ell-1}+x^{(*)}_{k-1}\smile_1 x^{(*)}_{\ell-1}\\&+\sum_{\substack{0\leq r<k\\0\leq m<\ell}}\Big((-1)^{\epsilon_1+|z|}\,\alpha^{*,*}_{r,m}\,b^{*,*}_{k-r,\ell-m}\,x^{*,*}_{r+m-1}-(-1)^{\epsilon_2}\big(x^{(*)}_{r-1}\smile_1 x^{(*)}_{m-1}\big)b^{*,*}_{k-r,\ell-m}\Big)+p\,\tilde{b}^{*,*}_{k,\ell}\end{aligned}\tag{2.2}$$

with the convention $x_{-1}\smile_1 x_m=x_m\smile_1 x_{-1}=-x_m$, and $\alpha_{s,t}:=\alpha^{1,1}_{s,t}=\alpha^{n,n}_{s,t}$, $\alpha^{1,n}_{s,t}=\alpha^{n,1}_{s,t}$; in particular, for $|x|$ odd:

$$\alpha_{s,t}=\begin{cases}\binom{s+t}{s}, & n=1,\\ \binom{(s+t)/2}{s/2}, & n>1 \text{ and } s,\ell \text{ are even}, \qquad \mod p,\\ \binom{(s+t-1)/2}{s/2}, & n>1 \text{ and } s \text{ is even and } t \text{ is odd}, \qquad \mod p,\\ 0, & n>1 \text{ and } s,t \text{ are odd}, \qquad \mod p,\end{cases}$$

and for $|x|$ even:

$$\alpha_{s,t} = \begin{cases} \binom{(s+t)/2}{s/2}, & n \geq 1 \text{ and } s, \ell \text{ are even}, \quad \bmod p, \\ \binom{(s+t-1)/2}{s/2}, & n \geq 1 \text{ and } s \text{ is even and } t \text{ is odd}, \quad \bmod p, \\ 0, & n \geq 1 \text{ and } s, t \text{ are odd}, \quad \bmod p. \end{cases}$$

Therefore, when $|z|$ is odd and $n, \lambda = 1$, formula (2.2) takes the form

$$db_{k,\ell} = \binom{k+\ell}{k} x_{k+\ell-1} + x_{k-1} \smile_1 x_{\ell-1}$$

$$- \sum_{\substack{1 \leq r < k \\ 1 \leq m < \ell}} \left(\binom{r+m}{r} b_{k-r,\ell-m}\, x_{r+m-1} + (x_{r-1} \smile_1 x_{m-1})\, b_{k-r,\ell-m} \right)$$

$$- \sum_{1 \leq r < k,\ell} ((b_{k-r,\ell} + b_{k,\ell-r}) x_{r-1} - x_{r-1}(b_{k-r,\ell} + b_{k,\ell-r})) + p\, \tilde{b}_{k,\ell}.$$

The values of the perturbation h on x_q and $b^{*,*}_{k,\ell}$ are, in fact, purely determined by the *transgressive* terms $y_{q+1} := hx_q|_{R^0H \oplus R^{-1}H}$ and $c^{*,*}_{k,\ell} := h(b^{*,*}_{k,\ell})|_{R^0H\oplus R^{-1}H}$, respectively. Namely,

$$hx_q = \sum_{\substack{ir_i = q-m,\, r_i \geq 1 \\ jr_j = q+1, \quad r_j \geq 1 \\ 0 \leq m < q}} - x_m \smile_1 y_i^{\cup_2 r_i} + y_j^{\cup_2 r_j} + p\, h\tilde{x}_q$$

and denoting $\gamma_{k,\ell} = \alpha^{*,*}_{\alpha_0,\ell_0} \dots \alpha^{*,*}_{k_s,\ell_s}$ and $m_{[s]} = m_1 + \cdots + m_s$,

$$h\Big(b^{*,*}_{k,\ell}\Big) = \sum_{\substack{1 \leq k_i < k_{i+1} \\ 1 \leq \ell_i < \ell_{i+1}}} - \gamma_{k,\ell}\, x^{*,*}_{k_0+\ell_0-1} \smile_1 c^{*,*}_{k_1-k_0,\ell_1-\ell_0} \smile_1 \cdots \smile_1 c^{*,*}_{k-k_s,\ell-\ell_s}$$

$$- \sum_{k=k_{[t]}\,;\, \ell=\ell_{[t]}} c^{*,*}_{k_1,\ell_1} \smile_1 \cdots \smile_1 c^{*,*}_{k_t,\ell_t} + \sum_{\substack{1 \leq r < k \\ 1 \leq m < \ell}} b^{*,*}_{r,m}\, h\Big(b^{*,*}_{k-r,\ell-m}\Big) + c^{*,*}_{k,\ell} + p\, h\Big(\tilde{b}^{*,*}_{k,\ell}\Big). \tag{2.3}$$

Furthermore, by means of $b_{k,\ell}$, we define the elements $\mathfrak{b}_{k,\ell} \in RH$ as follows. Fix the integer $k \geq 1$. Denote $\mathfrak{b}_{k,k} = b_{k,k}$ and $\varrho_{k,k} = 1$. If $\mathfrak{b}_{k,mk}$ has already been constructed for $1 \leq m < q$ and $\varrho_{k,qk} := \alpha_{k,(q-1)k} \dots \alpha_{k,2k}\, \alpha_{k,k}$, let

$$\mathfrak{b}_{k,qk} = \varrho_{k,qk} b_{k,qk} - x_{k-1} \smile_1 \mathfrak{b}_{k,(q-1)k} = \varrho_{k,qk}\, b_{k,qk}$$

$$- \varrho_{k,(q-1)k}\, x_{k-1} \smile_1 b_{k,(q-1)k} - \cdots - \varrho_{k,2k}\, x_{k-1}^{\smile_1 q} \smile_1 b_{k,2k} - x_{k-1}^{\smile_1 (q+1)} \smile_1 b_{k,k}.$$

Then

$$\begin{aligned} d_h \mathfrak{b}_{k,qk} &= \varrho_{k,qk}\, x_{k+qk-1} + x_{k-1}^{\smile_1 (q+1)} + u_{k,qk} + p\, \tilde{\mathfrak{b}}_{k,qk} + h\mathfrak{b}_{k,qk} \\ &= \varrho_{k,qk}\, x_{k+qk-1} + x_{k-1}^{\smile_1 (q+1)} + w_{k,qk} + \varrho_{k,qk} c_{k,qk}, \end{aligned} \tag{2.4}$$

where $w_{k,qk} := u_{k,qk} + p\, \tilde{\mathfrak{b}}_{k,qk} + (h\mathfrak{b}_{k,qk} - \varrho_{k,qk}\, c_{k,qk})$ and $u_{k,qk}$ is expressed by x_i and $b_{s,t}$ with $(s,t) \leq (k, qk)$.

a) Let p be odd. Set $k = p^r$ and $q = p-1$ in (2.4), and define (1.1) for $z \in H^{2m+1}(X; \mathbb{Z}_p)$ and $r \geq 1$ by

$$\psi_{r,1}(z) = \Big[t_{Z_p}\big(x_{p^r-1}^{\smile_1 p} + w_{p^r,(p-1)p^r}\big) \Big];$$

b) Let p and m be not odd simultaneously. Set $k = 2p^{r-1}$ and $q = p-1$ in (2.4), and define (1.2) for $z \in P_n^m(X)$ and $r, n \geq 1$ by

$$\psi_{r,n}(z) = \Big[t_{Z_p}\big(x_{2p^{r-1}-1}^{\smile_1 p} + w_{2p^{r-1},2(p-1)p^{r-1}}\big) \Big].$$

Theorem 1. *For any map $f : X \to Y$, the following diagrams*

$$\begin{array}{ccc} H^{2m+1}(X;\mathbb{Z}_p) & \xrightarrow{\psi_{r,1}} & H^{2mp^{r+1}+1}(X;\mathbb{Z}_p)/\operatorname{Im}\mathcal{P}_1 \\ f^*\uparrow & & f^*\uparrow \\ H^{2m+1}(Y;\mathbb{Z}_p) & \xrightarrow{\psi_{r,1}} & H^{2mp^{r+1}+1}(Y;\mathbb{Z}_p)/\operatorname{Im}\mathcal{P}_1 \end{array}$$

and

$$\begin{array}{ccc} P^m_n(X) & \xrightarrow{\psi_{r,n}} & H^{(m(n+1)-2)p^r+1}(X;\mathbb{Z}_p)/\operatorname{Im}\mathcal{P}_1 \\ f^*\uparrow & & f^*\uparrow \\ P^m_n(Y) & \xrightarrow{\psi_{r,n}} & H^{(m(n+1)-2)p^r+1}(Y;\mathbb{Z}_p)/\operatorname{Im}\mathcal{P}_1 \end{array}$$

commute.

Sketch of the proof. Define the cohomology operations on $H^*(C^*(X;\mathbb{Z}_p))$ by means of the canonical operations $\{E_{p,q}\}_{p,q\geq 1}$ on the cochain complex $C^*(X;\mathbb{Z}_p)$ ([4]) that agree with $\psi_{r,n}$ on $H^*(RH,d_h)$ via zig-zag maps (2.1). $\square$

Let $\mathcal{D}^* := H^+(X;\mathbb{Z}_p)\cdot H^+(X;\mathbb{Z}_p) \subset H^*(X;\mathbb{Z}_p)$ be the decomposables and $\mathcal{P}_1^{(m)}$ denote m-fold composition $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_1$.

Theorem 2. *Let $H^*(X;\mathbb{Z}_p)$ be a Hopf algebra. Given $r \geq 1$, let $p(r)$ denote the largest integer such that $p^{p(r)}$ divides the factorial $p^r!$. Let $\mathcal{I}^* \subset H^*(X;\mathbb{Z}_p)$ be the subset of indecomposables defined for $a \in \mathcal{I}^*$, $z \in H^*(X;\mathbb{Z}_p)$ and the integer $\kappa_z \geq 1$ such that $\beta_{p(t)}\,\mathcal{P}_1^{(t)}(z) = \beta_{p(t)}\,\mathcal{P}_1^{(t-1)}\psi_{1,n}(z) = 0$* mod $\mathcal{D}^*$ *for $t < \kappa_z$ and*

a) *For $p > 2$:*

$$a = \begin{cases} \beta_{p(\kappa_z)}\,\mathcal{P}_1^{(\kappa_z)}(z), & n = 1 \text{ and } z \text{ is odd dimensional}, \\ \beta_{p(\kappa_z)}\,\mathcal{P}_1^{(\kappa_z-1)}\psi_{1,n}(z), & n > 1 \text{ and } z \text{ is even dimensional}; \end{cases}$$

b) *For $p = 2$:*

$$a = \beta_{2(\kappa_z)}\,Sq_1^{(\kappa_z-1)}\psi_{1,n}(z), \qquad n \geq 1.$$

Then $\operatorname{Ker}\sigma = \mathcal{I}^* \cup \mathcal{D}^*$.

Proof. The map $\tau : RH\otimes\mathbb{Z}_p \to \bar{V}\otimes\mathbb{Z}_p$, $a\otimes 1 \to \overline{a|_V}\otimes 1$ realizes the loop suspension map σ as (cf. [4])

$$\sigma : H^m(X;\mathbb{Z}_p) \approx H^m(RH\otimes\mathbb{Z}_p, d_h) \xrightarrow{\tau^*} H^{m-1}(\bar{V}\otimes\mathbb{Z}_p, \bar{d}_h) \approx H^{m-1}(\Omega X;\mathbb{Z}_p).$$

The inclusion $\mathcal{D}^* \subset \operatorname{Ker}\sigma$ immediately follows from the above definition of σ. Let $a \in \operatorname{Ker}\sigma$ be indecomposable. Then for $y \in RH$ with $[t_{\mathbb{Z}_p}(y)] = a$, there is the sequence $(x_m)_{m\geq 0}$ in RH and $r \geq 1$ such that

$$d_h(x_{m-1}) = y + u_{m-1} \qquad \text{mod } p,$$

$$d_h(x_i) = u_i \qquad \text{mod } p, \quad u_i \in \mathcal{D}^*, \quad i < m \quad \text{for}$$

$$m = \begin{cases} p^r, & p \quad \text{and} \quad |x_0| \quad \text{are odd}, \\ 2p^{r-1}, & \text{otherwise}. \end{cases}$$

Let $z = \frac{p^{p(r)}}{p^r!}[t_{\mathbb{Z}_p}(x_0)]$. Denote $\kappa_z := r$. Then taking into account (2.3) and the coefficients $\varrho_{k,qk}$ of x_{k+qk-1} in (2.4) for $q = p-1$ and $k = p^t$ and $k = 2p^{t-1}$, $1 \leq t \leq \kappa_z$, we establish the equalities of Items a) – b) as desired. Hence, $a \subset \mathcal{I}^*$. The implication $\mathcal{I}^* \cup \mathcal{D}^* \subset \operatorname{Ker}\sigma$ is obvious. $\square$

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ON THE EXISTENCE OF UNIVERSAL SERIES WITH SPECIAL PROPERTIES

SHAKRO TETUNASHVILI

Abstract. An arbitrary system of Lebesgue measurable and almost everywhere finite functions $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ such that there exists a universal series with respect to Φ is considered. A theorem asserting that for any sequence of real numbers $(c_n)_{n=1}^{\infty}$ there exist two universal series with respect to Φ such that every c_n is a product of two corresponding coefficients of these two universal series is formulated.

Let $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ be an arbitrary system of Lebesgue measurable and almost everywhere finite functions defined on $[a, b]$.

Definition 1. A series

$$\sum_{n=1}^{\infty} \alpha_n \varphi_n(x) \tag{1}$$

is called a universal series with respect to Φ in the sense of subsequences of partial sums of this series, if for any Lebesgue measurable function $f(x)$ defined on $[a, b]$ and finite or infinite at any point of $[a, b]$ there exists a strictly increasing sequence of natural numbers $(m_k)_{k=1}^{\infty} \uparrow \infty$ such that the equality:

$$\lim_{k\to\infty} \sum_{n=1}^{m_k} \alpha_n \varphi_n(x) = f(x)$$

holds for almost all $x \in [a, b]$.

In what follows, for the sake of brevity, a universal series (1) with respect to Φ in the sense of subsequences of partial sums of (1) is called a universal series with respect to Φ and measurable is applied instead of Lebesgue measurable.

D. E. Menshoff was the first who established the existence of universal trigonometric series and proved that any trigonometric series is a sum of two universal trigonometric series (see [3]). Namely, he proved that for any sequence of real numbers $(c_n)_{n=1}^{\infty}$ there exist two universal trigonometric series with coefficients $\left(\alpha_n^{(1)}\right)_{n=1}^{\infty}$ and $\left(\alpha_n^{(2)}\right)_{n=1}^{\infty}$, respectively, such that for every natural number $n \geq 1$ the following equality

$$c_n = \alpha_n^{(1)} + \alpha_n^{(2)}$$

holds.

A. A. Talaljan proved (see [2, Theorem 9.2.11]) that for any complete and orthonormal system Φ there exists a universal series (1) with respect to Φ such that $\lim_{n\to\infty} \alpha_n = 0$.

Various aspects of the theory of universal series are presented in the article of K. G. Grosse-Erdman [1].

In [4], the above-mentioned result of Menshoff on trigonometric series is generalized for the series with respect to any system Φ of measurable and almost everywhere finite functions such that there exists a universal series with respect to Φ, in particular, for the series with respect to any complete and orthonormal system Φ (see [4, Theorem 1 and Theorem 2]).

The above-indicated results of [3] and [4] hold true not only for the sums of corresponding coefficients of the above-mentioned two universal series, but also for the products of corresponding coefficients of certain two universal series. Namely, the following theorem holds.

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Key words and phrases. System of measurable functions; Universal series; Trigonometric series.

Theorem 1. *Let* $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ *be an arbitrary system of measurable and almost everywhere finite functions defined on* $[a, b]$ *and* $(c_n)_{n=1}^{\infty}$ *be any sequence of real numbers, then a necessary and sufficient condition for the validity of the equality*

$$c_n = \alpha_n^{(1)} \cdot \alpha_n^{(2)}$$

for every natural number $n \geq 1$*, where* $\sum\limits_{n=1}^{\infty} \alpha_n^{(1)} \varphi_n(x)$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n^{(2)} \varphi_n(x)$ *are certain universal series with respect to* Φ*, is the existence of a universal series with respect to* Φ*.*

A consequence of Theorem 1 and of the above-mentioned theorem of A. A. Talaljan is the following

Theorem 2. *Let* $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ *be any complete and orthonormal system of functions defined on* $[a, b]$*, then for any sequence of real numbers* $(c_n)_{n=1}^{\infty}$*, there exist two universal series* $\sum\limits_{n=1}^{\infty} \alpha_n^{(1)} \varphi_n(x)$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n^{(2)} \varphi_n(x)$ *with respect to* Φ *such that the equality*

$$c_n = \alpha_n^{(1)} \cdot \alpha_n^{(2)}$$

holds for every natural number $n \geq 1$*.*

Proposition 1. *For any system* Φ *of measurable and almost everywhere finite functions defined on* $[a, b]$ *such that there exists a universal series with respect to* Φ*, in particular, for any complete and orthonormal system* Φ*, there exist two series*

$$\sum_{n=1}^{\infty} \alpha_n \varphi_n(x) \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{1}{\alpha_n} \varphi_n(x)$$

with respect to Φ *such that every one of them is a universal series with respect to* Φ*.*

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ON SETS OF UNIQUENESS OF SOME FUNCTION SERIES

SHAKRO TETUNASHVILI[1,2] AND TENGIZ TETUNASHVILI[2,3]

Abstract. Uniqueness theorems for function series with respect to systems of finite functions, Lebesgue measurable and finite functions, and some orthonormal systems of functions are formulated.

1. Notation and Definitions

Let $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ be a system of finite functions defined on $[0,1]$, $a = (a_1, a_2, \ldots, a_n, \ldots)$ be a sequence of real numbers, and $\theta = (0,0,0,\ldots)$ be a constant sequence of zeros. $a \neq \theta$ means that there exists a natural number $n_0 \geq 1$ such that $a_{n_0} \neq 0$.

Let S be the set of all sequences of real numbers, $S = \{a: \ a = (a_1, a_2, \ldots, a_n, \ldots)\}$. Let S_0 be the set $S \backslash \{\theta\}$, i.e., $S_0 = \{a: \ (a \in S) \ \& \ (a \neq \theta)\}$.

Consider a series with respect to Φ:

$$\sum_{n=1}^{\infty} a_n \varphi_n(x). \tag{1}$$

For every fixed $x \in [0,1]$ let

$$A(x) = \left\{a: \ (a \in S_0) \ \& \left(\sum_{n=1}^{\infty} a_n \varphi_n(x) \neq 0\right)\right\}$$

and for every fixed $a \in S_0$ let

$$E(a) = \left\{x: \ (x \in [0,1]) \ \& \left(\sum_{n=1}^{\infty} a_n \varphi_n(x) \neq 0\right)\right\}.$$

Definition 1. A set $H \subset [0,1]$ is called a U-set if the convergence of a series $\sum\limits_{n=1}^{\infty} a_n \varphi_n(x)$ to zero for every $x \in [0,1] \backslash H$ implies that $a_n = 0$ for every natural number $n \geq 1$.

2. A Uniqueness Theorem for Series with Respect to Systems of Finite Functions

Let $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ be a system of finite functions defined on $[0,1]$, then the following assertions hold true:

Theorem 1. *A set $H \subset [0,1]$ is a U-set if and only if*

$$\bigcup_{x \in [0,1] \backslash H} A(x) = S_0.$$

Proposition 1. *A set $H \subset [0,1]$ is a U-set if and only if*

$$E(a) \bigcap ([0,1] \backslash H) \neq \emptyset \qquad \textit{for any} \qquad a \in S_0.$$

Proposition 2. *If the empty set is a U-set, then a nonempty set $H \subset [0,1]$ is a U-set if and only if*

$$\bigcup_{x \in H} A(x) \subset \bigcup_{x \in [0,1] \backslash H} A(x).$$

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Key words and phrases. Set of uniqueness; Function series, Null-series.

3. A Uniqueness Theorem for Series with Respect to Systems of Lebesgue Measurable and Finite Functions

Let $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ be a system of Lebesgue measurable and finite functions defined on $[0,1]$.

In what follows, μ_* and μ^* stand for Lebesgue inner and outer linear measures of a set, respectively, and measurable is applied instead of Lebesgue measurable for the sake of brevity.

Definition 2. A series $\sum\limits_{n=1}^{\infty} a_n\varphi_n(x)$ is called a null-series with respect to Φ if $\sum\limits_{n=1}^{\infty} a_n\varphi_n(x) = 0$ for almost all $x \in [0,1]$ and there exists a natural number $n_0 \geq 1$ such that $a_{n_0} \neq 0$.

Definition 3. An orthonormal system of functions $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ defined on $[0,1]$ is called a strictly convergence system if $\sum\limits_{n=1}^{\infty} a_n^2 < \infty$ implies that a series $\sum\limits_{n=1}^{\infty} a_n\varphi_n(x)$ converges almost everywhere on $[0,1]$ and $\sum\limits_{n=1}^{\infty} a_n^2 = \infty$ implies that a series $\sum\limits_{n=1}^{\infty} a_n\varphi_n(x)$ diverges on a subset of $[0,1]$ of positive Lebesgue measure.

It is well known that if Φ is a strictly convergence system, then there is no null-series with respect to Φ.

Note that examples of strictly convergence systems defined on $[0,1]$, are lacunar trigonometric systems (see [3, Ch. 5, §6]), Rademacher's system (see [1, Ch. 4, §5]), Kashin's complete and orthonormal system [2].

The following assertions hold true.

Theorem 2. *If there is no null-series with respect to the system $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$, then any set $H \subset [0,1]$ such that $\mu_* H = 0$ is a U-set.*

Note that if a set $H \subset [0,1]$ is such that $\mu_* H = 0$ and $\mu^* H = 1$, then $\mu_*\left([0,1]\backslash H\right) = 0$ and therefore, according to Theorem 2, we have

Corollary 1. *If there is no null-series with respect to the system $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$, and a set $H \subset [0,1]$ is such that $\mu_* H = 0$ and $\mu^* H = 1$, then both H and $[0,1]\backslash H$ are U-sets.*

Corollary 1 implies:

Corollary 2. *If $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ is a strictly convergence system and a set $H \subset [0,1]$ is such that $\mu_* H = 0$ and $\mu^* H = 1$, then both H and $[0,1]\backslash H$ are U-sets.*

Remark. It can be proved that after appropriate modifications of the notation and definitions presented in Section 1, the assertions formulated in Section 2 remain true for multiple function series, too.

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