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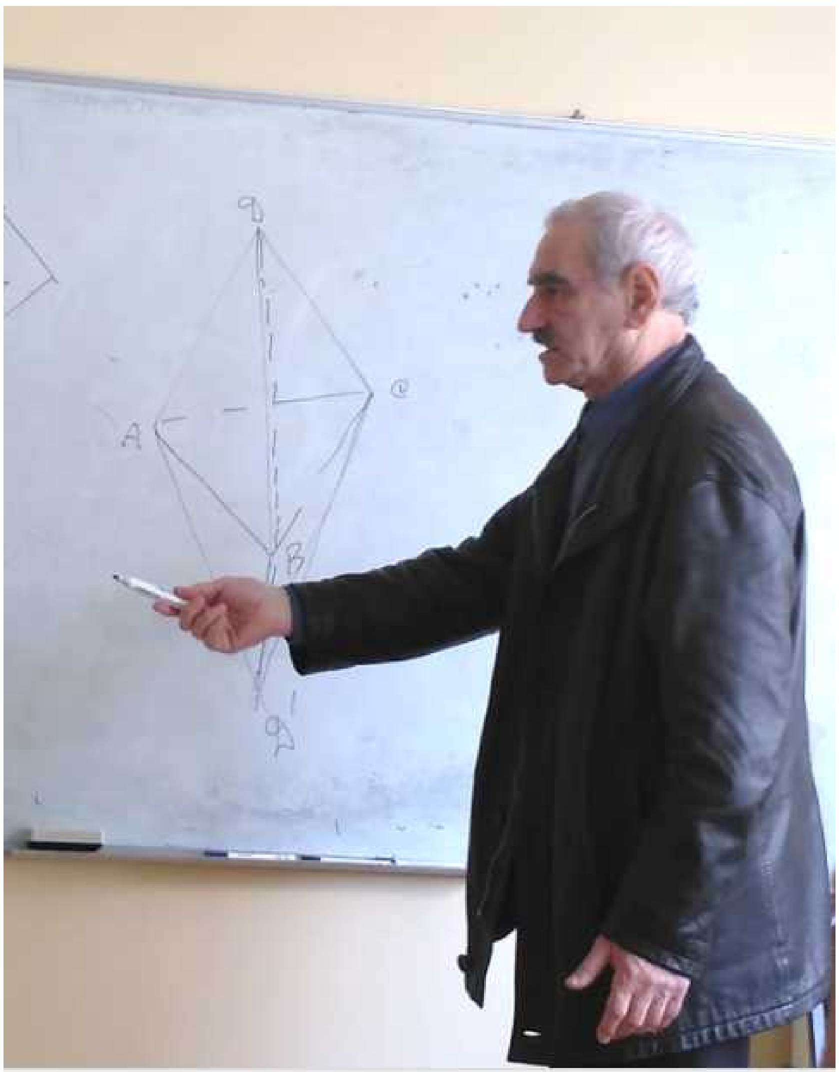
on the occasion of his 70th birthday



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A. KHARAZISHVILI'S SOME RESULTS OF ON THE STRUCTURE OF PATHOLOGICAL FUNCTIONS

ALEKS KIRTADZE[1,2] AND GOGI PANTSULAIA

Dedicated to Professor Alexander Kharazishvili on the occasion of his 70th birthday

Abstract. A brief survey of A. Kharazishvili's some works devoted to the real-valued functions with strange, pathological and paradoxical structural properties is presented. The presentation is primarily focused on the absolutely nonmeasurable functions, Sierpiński–Zygmund functions, sup-measurable and weakly sup-measurable functions of two real variables, and nonmeasurable functions of two real variables for which there exist both iterated integrals.

Professor Alexander B. Kharazishvili's scientific interests cover different fields of mathematics, primarily, the real analysis, measure theory, point set theory, and the geometry of Euclidean spaces. His works in mathematical analysis deal with the study of properties of various pathological (or paradoxical) real-valued functions of a real variable. In this direction, he has published several monographs in the worldwide known International Publishing Houses (see [33], [39], [43], [44]). In particular, his monograph [44] issued in 2000, 2006, 2017 is entirely devoted to those functions of real analysis that have strange structural properties from the viewpoints of continuity, monotonicity, differentiability and integrability. Although such functions have a very bad descriptive structure, quite often they turn out to be helpful for solving delicate questions and problems of mathematical analysis (see, e.g., [12], [15], [44]).

The present article may be considered as a short survey of certain Kharazishvili's results involving the above-mentioned topics.

1. It is well known that in the real analysis and, especially, in the differentiation theory, an important role is played by the notion of a differentiation system consisting of certain types of Lebesgue measurable sets. According to the classical Lebesgue theorem, if a system $\mathcal{L}$ of bounded Lebesgue measurable sets in the n-dimensional Euclidean space $\mathbf{R}^n$ is regular, then it is a differentiation system for the standard Lebesgue measure λ_n on the same space $\mathbf{R}^n$. The latter sentence means that for all absolutely continuous real-valued set functions with respect to λ_n, it becomes possible to reconstruct the Radon–Nikodym derivatives of such functions by using the standard differentiation process with respect to $\mathcal{L}$ (see, e.g., [10, 11], [46], [52]). However, this fundamental and useful result does not hold longer for some measures on $\mathbf{R}^n$ which properly extends λ_n. Indeed, for a certain extension μ of λ_n, Kharazishvili constructed a regular system $\mathcal{S}$ of bounded μ-measurable sets such that $\mathcal{S}$ is not a differentiation system for μ, i.e., there is a real-valued set function, absolutely continuous with respect to μ, for which the Radon-Nikodym derivative cannot be obtained by using the ordinary differentiation process with respect to $\mathcal{S}$. This result and some related ones were published in [24–26]).

2. In [26], [28], [29], [30] and [44], Kharazishvili considered logical aspects of the concept of generalized limits on the real line and also the concepts of generalized derivatives and generalized integrals. He indicated close connections of these concepts with the $\mathbf{ZF}+\mathbf{DC}$ set theory (where $\mathbf{DC}$ stands for the axiom of dependent choices) and established that:

(a) it is consistent with the $\mathbf{ZF}+\mathbf{DC}$ theory that these concepts cover only the first category subspaces of appropriate spaces;

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(b) it is consistent with the $\mathbf{ZF}+\mathbf{DC}$ theory that these concepts are always extendable to a wider subspaces.

The main technical tool for establishing the above results is a clever application of the Kuratowski–Ulam theorem to generalized limits, generalized derivatives and generalized integrals. It was also shown in the same works that the Banach–Steinhaus theorem (or, in another terminology, the principle of condensation of singularities) may be deduced from some special version of the Kuratowski–Ulam theorem. In this context, it should be mentioned that the most interesting situations occur when the generalized limits, derivatives or integrals are described by projective subsets of the appropriate Polish topological vector spaces (cf. [44, Chapter 22]). Kharazishvili's results on this topic were used and cited in [7], [16], [22], [48].

3. A function $f(x,y)$ of two real variables x and y is called sup-measurable if for every Lebesgue measurable function $\phi(x)$ of one real variable, the superposition $f(x,\phi(x))$ is also Lebesgue measurable. Using Luzin's classical C-property, it is not hard to show that in the above definition it suffices to require the Lebesgue measurability of $f(x,\phi(x))$ for only all continuous functions $\phi(x)$. Also, in the literature, there are some analogous versions of the sup-measurability of functions of two variables (cf. [4,5], [14], [21], [27], [44]); one of such versions is formulated in terms of functions having the Baire property.

Motivated by the theory of first-order ordinary differential equations, Kharazishvili introduced the notion of a weakly sup-measurable function of two variables. The definition of weakly sup-measurable functions $f(x,y)$ differs slightly from the definition of sup-measurable functions: it is required that the Lebesgue measurability of superpositions $f(x,\phi(x))$ should be valid for all those continuous functions $\phi(x)$, which are differentiable almost everywhere with respect to the Lebesgue measure $\lambda=\lambda_1$ on $\mathbf{R}$. Assuming some additional set-theoretic hypotheses, e.g., the Continuum Hypothesis (**CH**) or Martins Axiom (**MA**), and starting with the delicate properties of Jarniks continuous nowhere approximately differentiable function [23], Kharazishvili proved that there exist weakly sup-measurable functions, which are not sup-measurable. This result was published in his paper [31]. Also, it was shown in the same paper that there exists a first order ordinary differential equation

$$y'=f(x,y) \qquad ((x,y)\in\mathbf{R}^2),$$

whose right-hand side $f(x,y)$ is a weakly sup-measurable non-Lebesgue measurable function of two real variables and, for any initial condition $(x_0,y_0)\in\mathbf{R}^2$, this equation has a unique solution in the class of all locally absolutely continuous functions on $\mathbf{R}$. It should be especially emphasized that the above-mentioned result is a theorem of the **ZFC** set theory, i.e., it does not appeal to additional set-theoretical assumptions. In other words, there is a first-order ordinary differential equation $y'=f(x,y)$ in which the right-hand side $f(x,y)$ is very bad from the measurability viewpoint but, nevertheless, $f(x,y)$ turns out to be weakly sup-measurable and the corresponding Cauchy problem has a unique solution for any initial condition $(x_0,y_0)\in\mathbf{R}^2$.

In connection with the above results, there was formulated in [27] the question whether is it consistent with the **ZFC** theory that any sup-measurable function of two real variables is Lebesgue measurable. Roslanowski and Shelah constructed a model of **ZFC** in which the answer to this question is positive (see their joint article [52]).

In general, the topic connected with the sup-measurable and weakly sup-measurable functions turned out to be of interest for specialists in the real analysis. For this context, we refer the reader especially to the very recent paper: L. Bernal–Gonzalez, G. A. Munoz–Fernandez, D. L. Rodriguez–Vidanes, J. B. Seoane–Sepulveda, Algebraic genericity within the class of sup-measurable functions, Journal of Mathematical Analysis and Applications, v. 483, 2020.

Further, Kharazishvili investigated certain profound properties of a general superposition operator, he studied particularly generalized step functions with strange descriptive properties from the viewpoint of superposition operators (see [34], [36], [44]). The results obtained by Kharazishvili in this direction were cited in [2], [3], [4], [5], [8], [14], [52].

According to one old result of Sierpiński [55], there exists a real-valued Lebesgue measurable function g on $\mathbf{R}$ such that no Borel function on $\mathbf{R}$ majorizes g.

In [43], one can find another radically different proof of this statement and its further generalization. In order to formulate the generalized result, let us recall two notions.

A function from $\mathbf{R}$ into $\mathbf{R}$ is called a step-function if its range is at most countable.

A function from $\mathbf{R}$ into $\mathbf{R}$ is called universally measurable if it is measurable with respect to the completion of any σ-finite Borel measure on $\mathbf{R}$.

Kharazishvili has proved in [43] that due to Martin's Axiom, there exists a universally measurable step-function $h : \mathbf{R} \to \mathbf{R}$ such that there is no Borel function ϕ above g and, simultaneously, there is no Borel function ψ below h.

Clearly, this statement is a strengthened form of Sierpiński's above-mentioned result. It should be noticed that for obtaining a stronger version of Sierpiński's result in terms of universally measurable functions, the usage of additional set-theoretical assumptions becomes necessary.

4. A series of scientific publications of Kharazishvili is devoted to the concept of absolute non-measurability of real-valued functions. In particular, this concept was introduced and thoroughly examined in his works [35], [39], [43], [44]. It makes sense to give a precise definition of this important concept.

Let E be an uncountable base set and $\mathcal{M}$ be a class of measures on E (in general, the measures from $\mathcal{M}$ are defined on different σ-algebras of subsets of E, but the case is not excluded when all members of $\mathcal{M}$ have the same domain).

A function $f : E \to \mathbf{R}$ is called absolutely nonmeasurable with respect to $\mathcal{M}$ if f turns out to be nonmeasurable with respect to every measure from $\mathcal{M}$.

The symbol $\mathcal{M}(E)$ denotes the class of all those measures μ on E which are nonzero, σ-finite and diffused (i.e., $\mu(\{x\}) = 0$ for each element $x \in E$).

Of course, the most interesting case from the viewpoint of real analysis is when $E = \mathbf{R}$. To illustrate the absolute nonmeasurability of functions, it seems reasonable to give a few examples about this concept.

Example 1. Let $\mathcal{M}$ be the class of all translation invariant measures on $\mathbf{R}$ which extend the Lebesgue measure λ and let V be a Vitali set in $\mathbf{R}$ (in other words, V is a selector of the quotient set $\mathbf{R}/\mathbf{Q}$, where $\mathbf{Q}$ denotes the field of all rational numbers). Let f be the characteristic function of V. It is well known that f is absolutely nonmeasurable with respect to $\mathcal{M}$.

Example 2. Let $\mathcal{M}$ be the class of the completions of all nonzero σ-finite diffused Borel measures on $\mathbf{R}$ and let B be a Bernstein set in $\mathbf{R}$ (by the definition, B and $\mathbf{R} \setminus B$ contain no nonempty perfect sets). Let f denote the characteristic function of B. Then f turns out to be absolutely nonmeasurable with respect to $\mathcal{M}$.

Example 3. According to Martin's Axiom, there exists an additive function $f : \mathbf{R} \to \mathbf{R}$ which is absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$. Actually, it was proved by Kharazishvili that the existence of such f follows from the existence of a generalized Luzin's set which is simultaneously a vector space over $\mathbf{Q}$ (see [35], [39], [43], [44]).

Kharazishvili obtained a characterization of absolutely nonmeasurable functions with respect to the class $\mathcal{M}(E)$ in terms of universal measure zero spaces.

Recall that a topological space T is a universal measure zero space (or an absolute null space) if there exists no nonzero σ-finite diffused Borel measure on T.

It turns out that for a function $f : E \to \mathbf{R}$, the following assertions are equivalent:

(1) f is absolutely nonmeasurable with respect to $\mathcal{M}(E)$;

(2) for each point $r \in \mathbf{R}$, the set $f^{-1}(r)$ is at most countable and the range of f is a universal measure zero subspace of $\mathbf{R}$.

The proof of this equivalence can be found in [45]. It follows from the above characterization that the existence of absolutely nonmeasurable functions with respect to the class $\mathcal{M}(\mathbf{R})$ cannot be established within the **ZFC** set theory.

The equivalence between the assertions (1) and (2) has been applied many times by its author in the process of his studies of different types of pathological real-valued functions.

For instance, owing to the Continuum Hypothesis (**CH**), Kharazishvili has proved that there exists a large group of additive absolutely nonmeasurable functions acting from $\mathbf{R}$ into $\mathbf{R}$. More precisely, he established that assuming **CH**, there is a group $G \subset \mathbf{R}^{\mathbf{R}}$ such that:

(a) $\mathrm{card}(G) > \mathbf{c}$, where $\mathbf{c}$ denotes the cardinality of the continuum;

(b) all functions from G are additive;

(c) all functions from $G \setminus \{0\}$ are absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$.

This result obtained by Kharazishvili can be found in [42]. By Martin's Axiom, he also proved that:

(d) every function from $\mathbf{R}$ into $\mathbf{R}$ is representable as a sum of two injective functions which are absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$;

(e) every additive function from $\mathbf{R}$ into $\mathbf{R}$ is representable as a sum of two injective additive functions which are absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$.

Furthermore, the concept of absolutely nonmeasurable functions was a starting point for obtaining a solution of one problem posed by Pelc and Prikry [49]. The method of Kharazishvili used by him for obtaining statement (e) is insomuch efficient that leads to a positive solution of the above-mentioned problem (see also his paper [32] considering some related questions). Recently, Zakrzewski [60] has introduced and studied the analogue of an absolute nonmeasurability in terms of the Baire property.

5. A cycle of Kharazishvili's publications deal with the Sierpiński–Zygmund functions. As was shown by Blumberg [9], for any function $f : \mathbf{R} \to \mathbf{R}$, there exists a dense subset X of $\mathbf{R}$ such that the restriction $f|X$ is continuous. In particular, the set X is countably infinite. On the other hand, Sierpiński and Zygmund have established in their celebrated paper [58] that there exists a function

$$f_{SZ} : \mathbf{R} \to \mathbf{R}$$

such that the restriction of f_{SZ} to any set $Y \subset \mathbf{R}$ of cardinality continuum is not continuous on Y. Consequently, if one assumes **CH**, then the restriction of f_{SZ} to any uncountable subset Y of $\mathbf{R}$ is not continuous on Y. So, under **CH**, the Sierpiński–Zygmund functions may be treated as totally discontinuous (i.e., discontinuous on all uncountable subsets of $\mathbf{R}$). On the other hand, as demonstrated by Shelah [54], there are models of the **ZFC** theory in which every function from $\mathbf{R}$ into $\mathbf{R}$ has a continuous restriction to some subset of $\mathbf{R}$ of the second category (which trivially is uncountable). A similar result was obtained in [52] for subsets of $\mathbf{R}$ having strictly positive outer Lebesgue measure. This circumstance shows that the existence of totally discontinuous functions from $\mathbf{R}$ into $\mathbf{R}$ cannot be established within the **ZFC** theory.

There are many works devoted to various extraordinary properties of Sierpiński–Zygmund functions (see, e. g., [6], [20], [47], [51]). An extensive survey of such a function is given in [13] with more or less complete list of references.

Naturally, Kharazishvili studied interrelations between Sierpiński–Zygmund type functions and absolutely nonmeasurable functions with respect to certain classes of measures on $\mathbf{R}$. It is known that every Sierpiński–Zygmund function is nonmeasurable with respect to the completion of any nonzero σ-finite diffused Borel measure on $\mathbf{R}$, i.e., every Sierpiński–Zygmund function is absolutely nonmeasurable with respect to the class of such measures and, consequently, every Sierpi'nski–Zygmund function is nonmeasurable in the Lebesgue sense. On the other hand, it was proved by Kharazishvili that there exists a translation invariant extension μ of the Lebesgue measure on $\mathbf{R}$ such that any Sierpiński–Zygmund function becomes measurable with respect to μ. In this connection, see his paper [38]. At the same time, he was able to establish that there exists an additive Sierpiński–Zygmund function, absolutely nonmeasurable with respect to the class of all nonzero σ-finite translation invariant measures on $\mathbf{R}$ (see [37]).

It is not difficult to show that according to **CH**, every Sierpiński–Zygmund function is totally non-monotone, i.e., the restriction of such a function to any uncountable subset of $\mathbf{R}$ is not monotone. There arises the natural question whether any totally non-monotone function is a Sierpiński–Zygmund function. It turns out that the answer is no in certain models of the **ZFC** set theory. Namely, assuming the same **CH** and using some properties of the so-called Luzin's sets on $\mathbf{R}$ with the existence of continuous nowhere differentiable functions, it was established by Kharazishvili that there exists a totally non-monotone function from $\mathbf{R}$ into itself, which is not a Sierpiński–Zygmund function. This principal result shows a substantial difference between the Sierpiński–Zygmund functions and totally non-monotone functions. The result was first presented by the author at the Section of Functional Analysis of Ukrainian Mathematical Congress, (Kyiv, August 27-29, 2009). The title of his report was: "On continuous totally non-monotone functions". Later on, the same result in a more detailed form has been published in [41]. Also, Kharazishvili considered a stronger version of Sierpiński–Zygmund functions. A function $f : \mathbf{R} \to \mathbf{R}$ is called a Sierpiński–Zygmund function in the strong sense if for any set $X \subset \mathbf{R}$ with cardinality continuum, the restriction $f|X$ is not a Borel function on X. Kharazishvili proved that owing to **CH**, there exists a Sierpiński–Zygmund function which is not a Sierpiński–Zygmund function in the strong sense. When constructing such a function, he essentially used the result of Adian and Novikov [1] on the existence of real-valued semicontinuous functions on $\mathbf{R}$ which are not countably continuous. In addition, he observed that supposing Martin's Axiom, it can also be proved that the class of Sierpiński–Zygmund functions differs from the class of Sierpiński–Zygmund functions in the strong sense.

Finally, it should be especially mentioned that Kharazishvili considered a natural class of topologies on the real line $\mathbf{R}$ (more generally, on a set E of cardinality continuum) and proved the existence of a common Sierpiński–Zygmund function for this class (in this context, see, e.g., [39]).

6. If $f : [0,1]^2 \to \mathbf{R}$ is a function of two real variables, then even in the case of very bad descriptive properties of f it may happen that there exist two iterated integrals

$$\int_0^1 \Big(\int_0^1 f(x,y)d \Big) dy, \qquad \int_0^1 \Big(\int_0^1 f(x,y)dy \Big) dx$$

in the Riemann or in the Lebesgue sense. In particular, according to one old result of Sierpiński [57], there exists an injective function $\phi : [0,1] \to [0,1]$ such that the graph of ϕ is λ-massive in the unit square $[0,1]^2$, i.e., this graph meets every λ_2-measurable subset of $[0,1]^2$ with strictly positive λ_2-measure. Obviously, denoting by g the characteristic function of the graph of ϕ, one easily obtains the equalities

$$\int_0^1 \Big(\int_0^1 g(x,y)dx \Big) dy = \int_0^1 \Big(\int_0^1 g(x,y)dy \Big) dx = 0,$$

although g is not a λ_2-measurable function.

If the starting function f is integrable on $[0,1]^2$ in the Lebesgue sense, then no problems occur, because both iterated integrals for f do exist and they are equal to the two-dimensional Lebesgue integral of f.

If a Lebesgue measurable function $f : [0,1]^2 \to \mathbf{R}$ is not assumed to be Lebesgue integrable, then all of the following possibilities are realizable:

(a) none of the iterated integrals for f exists;
(b) one and only one of the iterated integrals does exist;
(c) both iterated integrals exist, but differ from each other;
(d) both iterated integrals exist and are equal to each other.

Moreover, in connection with case (d), G. Fichtenholz [17] constructed an example of a Lebesgue measurable non-integrable function $h : [0,1]^2 \to \mathbf{R}$ such that the equality

$$\int_a^b \Big(\int_c^d h(x,y)dy \Big) dx = \int_c^d \Big(\int_a^b h(x,y)dx \Big) dy$$

holds true for all rectangles $[a,b] \times [c,d] \subset [0,1]^2$.

In this context, it should be remarked that if $f : [0,1]^2 \to \mathbf{R}$ is a bounded function (not necessarily Lebesgue measurable) and both its iterated integrals exist in the Riemann sense, then they are equal to each other (see, e.g., [18]). This statement is compatible with Sierpiński's result [57].

For iterated integrals in the Lebesgue sense the situation is radically different. Sierpiński's famous theorem [56] states that by the Continuum Hypothesis there is a subset S of $[0,1]^2$ such that each set of the form

$$S \cap (\{x\} \times [0,1]) \qquad (x \in [0,1])$$

is at most countable, and each set of the form

$$S \cap ([0,1] \times \{y\}) \qquad (y \in [0,1])$$

is co-countable in $[0,1] \times \{y\}$. Denoting by f the characteristic function of S, it can easily be seen that both iterated integrals for f do exist, but differ from each other (one of them equals 0, while the other equals 1). In fact, Sierpiński established that the existence of S is equivalent to the Continuum Hypothesis. Later on, it was proved by Friedman [19] that an additional set-theoretical assumption is necessary for having iterated integrals with different values. A detailed survey of Sierpiński's theorem [56] and of its numerous consequences is presented in [59].

In the extensive work of Pkhakadze [50], the equality of the iterated integrals was thoroughly investigated from the viewpoint of structural properties of functions $f : [0,1]^2 \to \mathbf{R}$.

Let $\mathcal{F}$ denote the family of all those functions $f : [0,1]^2 \to \mathbf{R}$ for which both iterated integrals do exist and are equal to each other. It is not hard to check that $\mathcal{F}$ is a vector space over $\mathbf{R}$. Moreover, if one has a pointwise convergent sequence $\{f_n : n \in \mathbf{N}\}$ of functions from $\mathcal{F}$ and $|f_n| \leq \phi$ for some fixed nonnegative Lebesgue integrable function $\phi : [0,1]^2 \to \mathbf{R}$ and for all $n \in \mathbf{N}$, then the limit function

$$f = \lim_{n \to \infty} f_n$$

also belongs to $\mathcal{F}$. Pkhakadze has proved that if a set $Z \subset [0,1]^2$ is such that all its vertical and horizontal sections are closed, then the iterated integrals for the characteristic function of Z exist and are equal to each other, i.e., this function belongs to $\mathcal{F}$. This statement is again in coherence with Sierpiński's result [57]. In addition, assuming **CH**, Pkhakadze gave an example of a set $P \subset [0,1]^2$ whose all vertical sections are closed, all horizontal sections are of type F_σ and for which the iterated integrals do exist, but differ from each other. Consequently, the characteristic function of P does not belong to $\mathcal{F}$.

In [40], it was demonstrated that:

(i) by **CH**, there exists a non-negative bounded function $h \in \mathcal{F}$ such that the function h^2 does not belong to $\mathcal{F}$ (in other words, the vector space $\mathcal{F}$ is not an algebra);

(ii) by **CH**, there are two bounded functions $h_1 \in \mathcal{F}$ and $h_2 \in \mathcal{F}$ such that the function $\sup(h_1, h_2)$ does not belong to $\mathcal{F}$ (in other words, the vector space $\mathcal{F}$ is not a lattice).

In fact, Kharazishvili proved in [40] that according to **CH**, there are two sets $A \subset [0,1]^2$ and $B \subset [0,1]^2$ such that both characteristic functions of A and B belong to $\mathcal{F}$, but the characteristic function of $A \cup B$ does not belong to $\mathcal{F}$.

Remark. The authors began to write the present article ten years ago, intending to dedicate it to professor Alexander Kharazishvili on the occasion of his 60th birthday. For some reasons, the process of preparing the article was not finished in due time. Here we present a modified and expanded version of our previous unpublished survey of those results of A. Kharazishvili that are concerned with the structure of various types of pathological real-valued functions.

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ON SINGULAR EXTENSIONS OF CONTINUOUS FUNCTIONALS FROM $C([0,1])$ TO THE VARIABLE LEBESGUE SPACES

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Abstract. Valadier and Hensgen proved independently that the restriction of the functional $\phi(x) = \int_0^1 x(t)dt$, $x \in L^\infty([0,1])$ on the space of continuous functions $C([0,1])$ admits a singular extension back to the whole space $L^\infty([0,1])$. Some general results in this direction for the Banach lattices were obtained by Abramovich and Wickstead. In the present note we investigate analogous problem for the variable exponent Lebesgue spaces, namely, we prove that if the space of continuous functions $C([0,1])$ is a closed subspace in $L^{p(\cdot)}([0,1])$, then every bounded linear functional on $C([0,1])$ is the restriction of a singular linear functional on $L^{p(\cdot)}([0,1])$.

1. Introduction

In the theory of duality of function spaces, the investigation of the space of all singular linear functionals is of importance. It is well known that the topological dual of the Banach function space X can be represented in the form $X^* = X_n^* \oplus X_s^*$, where X_n^* is the order continuous dual of X and X_s^* (the disjoint complement of X_n^* in X^*) is the space of all singular linear functionals on X.

The term "singular functional" is overused in the literature. According to [2], the space of singular functionals X_s^*, is defined, as we have mentioned above, as the band $(X_n^*)^d$, complementary to the band of order continuous functionals. Often the space of singular functionals for Banach function space X is defined as an annihilator

$$(X_a)^\perp = \{x^* \in X^*;\ x^*(x) = 0,\ \text{ for all } x \in X_a\},$$

where X_a is the space of order continuous elements in X. Although, the differences between these definitions are small and they often define the same objects (see, for instance, [1]), for variable exponent Lebesgue spaces the above-mentioned definitions are equivalent. (Note that for $L^\infty([0,1])$, we have $(L^\infty([0,1]))_a = \{0\}$).

Let $L^\infty([0,1])$ and $C([0,1])$ denote, respectively, the Banach space of essentially bounded real-valued functions on the $[0,1]$ with the Lebesgue measure and its subspace of continuous functions. Valadier [12] and Hensgen [8] proved independently that the restriction of the functional

$$\phi(x) = \int\limits_0^1 x(t)dt, \quad x \in L^\infty[0,1]$$

to a fairly large subspace $C([0,1])$ of $L^\infty([0,1])$ admits a singular ("bad") extension back to the $L^\infty([0,1])$. Abramovich and Wickstead showed that every bounded linear functional on $C([0,1])$ is the restriction of a singular functional on $L^\infty([0,1])$. They also generalized this result to the Banach lattice setting ([1], see Theorem 1 and remarks thereafter). Let a finitely additive measure ν represent $f \in (L^\infty([0,1])^*$ and $\widehat{\nu}$ be the Borel measure representing f restricted on $C([0,1])$. Many properties of $\widehat{\nu}$ in terms ν were recently investigated by Toland [11] and Wrobel [13].

Edmunds, Gogatishvili and Kopaliani [6] showed that there is a variable exponent space $L^{p(\cdot)}([0,1])$ with $1 < p(t) < \infty$ a.e., which has in common with $L^\infty([0,1])$ the property that the space $C([0,1])$ is a closed linear subspace in it. Moreover, Kolmogorov's and Marcinkiewicz examples of functions with a.e. divergent Fourier series belong to $L^{p'(\cdot)}([0,1])$, where $p'(\cdot)$ is a function, conjugate to $p(\cdot)$.

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In [10], there is the necessary and sufficient condition on the decreasing rearrangement p^* of the exponent $p(\cdot)$, for the existence of equimeasurable exponent function of $p(\cdot)$ whose corresponding variable Lebesgue space has the property that the space of continuous functions is closed in it. Indeed, let for the functions $p(\cdot) : [0,1] \to [1,\infty)$ we have

$$\limsup_{t\to 0_+} \frac{p^*(t)}{\ln(e/t)} > 0,$$

then there exists equimeasurable with $p(\cdot)$ exponent function $\widetilde{p}(\cdot)$ such that the space $C([0,1])$ is a closed subspace in $L^{\widetilde{p}(\cdot)}([0,1])$.

Let the space $C([0,1])$ be a closed subspace of $L^{p(\cdot)}([0,1])$. In this case, it is interesting to investigate the validity of the analogous Abramovich and Wickstead's theorem mentioned above. We have got the answer to this question. We prove the following

Theorem 1.1. *Let the space $C([0,1])$ be a closed subspace of $L^{p(\cdot)}([0,1])$. Then every bounded linear functional on $C([0,1])$ is the restriction of a linear singular functional on $L^{p(\cdot)}([0,1])$.*

2. Some Properties of Singular Functionals in the Variable Lebesgue Spaces

Let $p(\cdot) : [0,1] \to [1,\infty)$ be a measurable function. Define the modular

$$\rho_{p(\cdot)}(x) = \int\limits_{[0,1]} |x(t)|^{p(t)} dt.$$

Given a measurable function x, we say that $x \in L^{p(\cdot)}([0,1])$ if there exists $\lambda > 0$ such that $\rho_{p(\cdot)}(x/\lambda) < \infty$. This set becomes a Banach function space when equipped with the Luxemburg norm

$$\|x\|_{p(\cdot)} = \inf\{\lambda > 0;\ \rho_{p(\cdot)}(x/\lambda) \le 1\}.$$

The variable Lebesgue spaces were first introduced by Orlicz. They have been widely studied for the past thirty years, both for their interest as function spaces and for their applications to PDEs and the calculus of variation (see [4], [5]).

Define the dual exponent $p'(\cdot)$ pointwise by $1/p(t) + 1/p'(t) = 1,\ t \in [0,1]$.

In the case $p_+ < \infty$, where $p_+ = \operatorname{ess\,sup}_{t\in[0,1]} p(t)$, the dual space of $L^{p(\cdot)}([0,1])$ can be completely characterized, it is isomorphic to $L^{p'(\cdot)}([0,1])$. The problem characterizing the dual of $L^{p(\cdot)}([0,1])$ when $p_+ = \infty$ was considered in [3]. The authors in this case give a decomposition of $(L^{p(\cdot)}([0,1]))^*$ as a direct sum of $L^{p'(\cdot)}([0,1])$ and the dual of a quotient space (we refer to the germ space and denote it by $L^{p(\cdot)}_{\mathrm{germ}}$). Note that the main aspect of this subject was made in a more general setting for Musielak-Orlicz spaces by Hudzik and Zbaszyniak (see [9]). First, we present some basic facts from mentioned paper for a variable Lebesgue setting. We will always assume without loss of generality that $1 < p(t) < \infty$ a.e. (we are interested in characterizing the spaces $L^{p(\cdot)}([0,1])$ close to $L^\infty([0,1])$).

We define the closed subspace $E^{p(\cdot)}([0,1])$ of $L^{p(\cdot)}([0,1])$ by

$$E^{p(\cdot)}([0,1]) = \{x :\ \rho_{p(\cdot)}(\lambda x) < \infty \ \text{ for any } \lambda > 0\}.$$

It is easy to see that $E^{p(\cdot)}([0,1])$ is the subspace of order continuous elements in $L^{p(\cdot)}([0,1])$, i.e., $x \in L^{p(\cdot)}([0,1])$ belongs to $E^{p(\cdot)}([0,1])$ if and only if for any sequence x_n of measurable functions on $[0,1]$ such that $|x_n(t)| \le |x(t)|$ for all $n \in \mathbb{N}$ and $|x_n| \to 0$ a.e. on $[0,1]$ there holds $\|x_n\|_{p(\cdot)} \to 0$. (For the definition of order continuous elements in Banach lattices, see [2]). Note that if $p_+ < \infty$, the spaces $L^{p(\cdot}([0,1])$ and $E^{p(\cdot}([0,1])$ coincide with each other (see [4], [5]).

Let $p_+ = \infty$. Define the sets $\Omega_n = \{t \in [0,1] :\ p(t) \le n\},\ n \in \mathbb{N}$. We will always assume without loss of generality that $|\Omega_n| > 0$ for $n \in \mathbb{N},\ n \ge 2$ and $|\Omega_1| = 0$. For $x \in L^{p(\cdot)}([0,1])$, define the functions $x^{(n)} \in E^{p(\cdot)}([0,1]),\ n \in \mathbb{N}$ as $x^{(n)} = x\chi_{\Omega_n}$ (χ_{Ω_n} denotes the characteristic function of the set Ω_n).

For any $x \in L^{p(\cdot)}([0,1])$, define

$$d(x) = \inf\{\|x-y\|_{p(\cdot)} :\ y \in E^{p(\cdot)}\},$$
$$\theta(x) = \inf\{\lambda > 0;\ \rho_{p(\cdot)}(\lambda x) < +\infty\}.$$

For any $x^* \in (L^{p(\cdot)}([0,1]))^*$, we define the norm in a dual space

$$\|x^*\| = \sup\{x^*(x) : \ \|x\|_{p(\cdot)} \le 1\}.$$

The dual space of $L^{p(\cdot)}([0,1])$ is represented in the following way (see [9]):

$$(L^{p(\cdot)}([0,1]))^* = L^{p'(\cdot)}([0,1]) \oplus (L^{p(\cdot)}([0,1]))^*_s,$$

i.e., every $x^* \in (L^{p(\cdot)}([0,1]))^*$ is uniquely represented in the form $x^* = \xi_v + \varphi$, where ξ_v is the regular functional defined by a function $v \in L^{p'(\cdot)}([0,1])$ by the formula

$$\xi_v(x) = \int\limits_{[0,1]} v(t)x(t)dt, \ \ x \in L^{p(\cdot)}([0,1]), \tag{2.1}$$

and φ is a singular functional, i.e., $\varphi(x) = 0$ for any $x \in E^{p(\cdot)}([0,1])$ (for $p_+ < \infty$, we have $L^{p(\cdot)}[0,1])^*_s = \{0\}$).

Proposition 2.1 ([9], Lemma 1.2)**.** *For any $x \in L^{p(\cdot)}([0,1])$, the equalities*

$$\lim_{n\to\infty} \|x - x_n\|_{p(\cdot)} = \theta(x) = d(x)$$

hold.

Proposition 2.2 ([9], Lemma 1.3)**.** *For any singular functional φ, the equalities*

$$\|\varphi\| = \sup\{\varphi(x) : \ \rho_{p(\cdot)}(x) < \infty\} = \sup_{x\in L^{p(\cdot)}\setminus E^{p(\cdot)}} \varphi(x)/\theta(x)$$

hold.

Proposition 2.3 ([9], Lemma 1.4)**.** *For any functional $x^* = \xi_v + \varphi \in (L^{p(\cdot)}([0,1]))^*$, where ξ_v is defined by* (2.1) *and φ is a singular functional, the equality*

$$\|x^*\| = \|v\|_{p'(\cdot)} + \|\varphi\|$$

holds.

3. Proof of Theorem 1.1

Let the space $C([0,1])$ be a closed subspace in $L^{p(\cdot)}$. Then there exists a positive constant $c > 0$ such that

$$c \le \|\chi_{(a,b)}\|_{p(\cdot)} \ \text{ whenever } \ 0 \le a < b \le 1, \tag{3.1}$$

(see [6]). It is obvious that for some constant $C > 0$,

$$\|\chi_{(a,b)}\|_{p(\cdot)} \le C \ \text{ whenever } \ 0 \le a < b \le 1. \tag{3.2}$$

From (3.1) and (3.2), we can deduce that for some constants c_1, $c_2 > 0$ and for any $x \in C([0,1])$,

$$c_1\|x\|_C \le \|x\|_{p(\cdot)} \le c_2\|x\|_C \tag{3.3}$$

(for more details see [6]).

Denote $X = C([0,1])$ and $Y = E^{p(\cdot)}([0,1])$ (X is the Banach space with both norms $\|\cdot\|_C$ and $\|\cdot\|_{p(\cdot)}$). We have $X \cap Y = \{0\}$ (by (3.1)). Consider the Cartesian product $X \times Y$, equipped with coordinate-wise vector space operations. For this vector space we have the Banach norm

$$\|(u,v)\|_\infty = \max\{\|u\|_{p(\cdot)}, \|v\|_{p(\cdot)}\}.$$

Denote the $X \times Y$ vector space equipped with the norm $\|(\cdot,\cdot)\|_\infty$ as $(X\times Y)_\infty$. Obviously, $(X\times Y)_\infty$ is the Banach space. Our main goal is to prove that the mapping $(X\times Y)_\infty \to X+Y \subset L^{p(\cdot)}([0,1]) :$ $(u,v) \to u+v$ is a (topological) isomorphism. In this case, the vector space $X+Y$ with the norm $\|\cdot\|_{p(\cdot)}$ is the topological direct sum of the Banach spaces X and Y, and it is written as $X+Y = X\oplus Y$. From this fact we find that the vector space $X+Y = C([0,1]) \oplus E^{p(\cdot)}([0,1])$ with the norm $\|\cdot\|_{p(\cdot)}$ is the Banach subspace of $L^{p(\cdot)}([0,1])$, and we have

$$\|x+y\|_{p(\cdot)} \approx \max\{\|x\|_{p(\cdot)}, \|y\|_{p(\cdot)}\}, \ x \in X, \, y \in Y. \tag{3.4}$$

It is obvious that the vector space $X+Y$ with the norm $\|\cdot\|_{p(\cdot)}$ is the topological direct sum of Banach spaces X and Y if the linear projection $P : X+Y \to X$ defined by $P(x+y) = x$ is continuous when $x \in X$ and $y \in Y$. Note that this condition is equivalent to the following: there exists a positive real number δ such that $\|x-y\|_{p(\cdot)} \geq \delta$ whenever $x \in X, y \in Y$ and $\|x\|_{p(\cdot)} = 1$.

Let $x \in X$ and $\|x\|_{p(\cdot)} = 1$. Take $t_0 \in [0,1]$ such that

$$|x(t_0)| = \max_{t\in[0,1]} |x(t)| = \|x\|_C.$$

By (3.3), we have

$$1/c_2 \leq |x(t_0)| \leq 1/c_1. \tag{3.5}$$

We will prove that

$$d(x) = \inf_{y\in Y} \|x-y\|_{p(\cdot)} \geq \delta > 0$$

for some constant δ, independent of x.

By Proposition 2.1, we have

$$d(x) = \lim_{n\to\infty} \|x - x^{(n)}\|_{p(\cdot)}, \tag{3.6}$$

where $x^{(n)} = \chi_{\Omega_n}$, $\Omega_n = \{t : p(t) \leq n\}$.

Denote $O_n = (t_0 - \varepsilon_n, t_0 + \varepsilon_n)$, where the numbers $\varepsilon_n > 0$ will be chosen later.

We have

$$\begin{aligned} \|x - x^{(n)}\|_{p(\cdot)} &= \|x - x^{(n)}\chi_{[0,1]\setminus O_n} - x^{(n)}\chi_{[0,1]\cap O_n}\|_{p(\cdot)} \\ &\geq |\|x - x^{(n)}\chi_{[0,1]\setminus O_n}\|_{p(\cdot)} - \|x^{(n)}\chi_{[0,1]\cap O_n}\|_{p(\cdot)}|. \end{aligned} \tag{3.7}$$

Since for fixed n on the set Ω_n we have $p(t) \leq n$, we may take O_n such that $\|x^{(n)}\chi_{[0,1]\cap O_n}\|_{p(\cdot)}$ is arbitrarily small. Using (3.1) and (3.5), we can choose O_n such small that

$$\|x\chi_{O_n}\|_{p(\cdot)} \geq \frac{1}{2}|x(t_0)|\|\chi_{O_n}\|_{p(\cdot)} \geq \frac{c}{2c_2}. \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$\|x - x^{(n)}\|_{p(\cdot)} \geq \frac{c}{2c_2}$$

and, consequently, by (3.6), we have $d(x) \geq \delta = \frac{c}{2c_2}$.

Let x^* be any continuous linear functional from X^*. It is obvious that x^* is a continuous linear functional on the space X with the norm $\|\cdot\|_{p(\cdot)}$ (by (3.3)). Since the space $X \oplus Y$ is a Banach space with the norm $\|\cdot\|_{p(\cdot)}$, the trivial extension (i.e., $x^*(x) = 0$, for $x \in Y$) of x^* is also a continuous linear functional on $X \oplus Y$ (see (3.4)). For the functional obtained in this way (defined on $X \oplus Y$), there exists a continuous linear extension (non unique) on the whole space $L^{p(\cdot)}([0,1])$. It is obvious that the obtained functional is singular (it is identically 0 on $E^{p(\cdot)}([0,1])$) on $L^{p(\cdot)}([0,1])$. $\square$

Remark 1. A closed subspace Y of the Banach space X is M-ideal in X if $Y^\perp$ is the range of the bounded projection $P : X^* \to X^*$ satisfying

$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\| \quad \text{for all } x^* \in X^*.$$

For more details of the general M-ideal theory and their applications, we refer to [7]. If the subspace Y is M-ideal in X, then Y is proximinal in X (see [7, p. 57, Proposition 1.1]), that is, for any $x \in X$ there exists $y \in Y$ such that

$$d(x) = \inf_{z\in Y} \|x - z\| = \|x - y\|.$$

From Proposition 2.3 we find that the space $E^{p(\cdot)}([0,1])$ is M-ideal in $L^{p(\cdot)}([0,1])$ and, consequently, $E^{p(\cdot)}([0,1])$ is proximinal in $L^{p(\cdot)}([0,1])$; that is, for any $x \in L^{p(\cdot)}([0,1])$, there exists $y \in E^{p(\cdot)}([0,1])$ such that $d(x) = \|x - y\|_{p(\cdot)}$.

Remark 2. Let $C([0,1])$ be a closed subspace in $L^{p(\cdot)}([0,1])$. Denote $I = L^\infty([0,1]) \cap E^{p(\cdot)}([0,1])$. Note that if $x_n \in I, n \in \mathbb{N}$ and $\lim_{n\to\infty} \|x_n - x\|_\infty = 0$, then $x \in E^{p(\cdot)}([0,1])$. It is easy to show that I is an order ideal, which means that it is a closed subspace of $L^\infty([0,1])$ with the ideal property.

Note that there exists $\delta > 0$ such that for $x \in C([0,1])$, $\|x\|_C = 1$ and $y \in I$, we have $\|x-y\|_\infty \geq \delta$. The last inequality can be proved analogously as it has been done in the proof of Theorem 1.1. (It suffices to use the inequality $\|x-y\|_\infty \geq \|x-y\|_{p(\cdot)}$ and the fact that $\|x\|_{p(\cdot)} \approx 1$). Consequently, the vector space $C([0,1]) + I$ is the topological direct sum of Banach spaces $C([0,1])$ and I in $L^\infty([0,1])$.

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EXTENDING THE APPLICABILITY OF AN ULM-NEWTON-LIKE METHOD UNDER GENERALIZED CONDITIONS IN A BANACH SPACE

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Abstract. The aim of this paper is to extend the applicability of an Ulm-Newton-like method for approximating a solution of a nonlinear equation in a Banach space setting. The sufficient local convergence conditions are weaker than those in the earlier works leading to a larger radius of convergence and more precise error estimations on the distances involved. Numerical examples are also provided.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x_* of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet–differentiable operator defined on a convex subset Ω of a Banach space $\mathcal{B}_1$ with values in a Banach space $\mathcal{B}_2$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by the difference or differential equations, and their solutions represent usually the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = R(x)$, for some suitable operator R, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations may be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations) and real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative, that is, when starting from one or several initial approximations, a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

Moser in [13] proposed the following Ulm's-like method for generating a sequence $\{x_n\}$ approximating x_*:

$$x_{n+1} = x_n - B_n F(x_n), \qquad B_{n+1} = 2B_n - B_n F'(x_n) B_n. \tag{1.2}$$

Method (1.2) is useful when the derivative $F'(x_n)$ is not continuously invertible (as in the case of small divisors [1–8, 10, 11, 13–15]). Moser studied the semi-local convergence of method (1.2) and showed that the order of convergence is $1 + \sqrt{2}$ if $F'(x_*) \in L(\mathcal{B}_2, \mathcal{B}_1)$. However, the order of convergence is faster than the Secant method (i.e., $\frac{1+\sqrt{5}}{2}$). The quadratic convergence can be obtained if one uses Ulm's method [14, 15] defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} x_{n+1} &= x_n - B_n F(x_n), \\ B_{n+1} &= 2B_n - B_n F'(x_{n+1}) B_n. \end{aligned} \tag{1.3}$$

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The semi-local convergence of method (1.3) has also been studied in [1–9]. As far as we know, the local convergence analysis of methods (1.2) and (1.3) has not been given. In the present paper, we study the local convergence of Ulm's-like method defined for each $n = 0, 1, 2, 3, \dots$ by

$$x_{n+1} = x_n - B_n F(x_n),\ B_{n+1} = 2B_n - B_n A_{n+1} B_n, \tag{1.4}$$

where A_n is an approximation of $F'(x_n)$. Notice that method (1.4) is inverse free, the computation of $F'(x_n)$ is not required and the method produces successive approximations $\{B_n\} \approx F'(x_*)^{-1}$.

In Section 2, we present the local convergence analysis of method (1.4) and in Section 3, we present the numerical examples.

2. Local Convergence Analysis

The local convergence analysis of method (1.4) is given in this section. Denote by $U(v, \xi)$ and $\overline{U}(x, \xi)$ the open and closed balls in $\mathcal{B}_1$, respectively, with center $v \in \mathcal{B}_1$ and of radius $\xi > 0$.

Let $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ and $w : [0, +\infty) \longrightarrow [0, +\infty)$ be continuous and nondecreasing functions satisfying $w_0(0) = w(0) = 0$. Let also $q \in [0, 1)$ be a parameter. Define functions φ and ψ on the interval $[0, +\infty)$ by

$$\varphi(t) = \left[q\left(\int_0^1 w(\theta t) d\theta + 1 \right) + w_0(t) \right] t$$

and

$$\psi(t) = \varphi(t) - 1.$$

We have that $\psi(0) = -1$ and for sufficiently large $t_0 \geq t$, $\psi(t_0) > 0$. By the intermediate value theorem equation $\psi(t) = 0$ has solutions in the interval $(0, t_0)$. Denote by ρ the smallest such a solution. Then for each $t \in [0, \rho)$, we have

$$0 \leq \psi(t) < 1. \tag{2.1}$$

We need to show an auxiliary perturbation result for method (1.4).

Lemma 2.1. *Let $F : \Omega \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchet-differentiable operator. Suppose that there exist $x_* \in \Omega$, $\{M_n\} \in L(\mathcal{B}_2, \mathcal{B}_1)$, $\{q_n\}$, $q \in \mathbb{R}_0^+$, continuous and nondecreasing functions $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ and $w : [0, +\infty) \longrightarrow [0, +\infty)$ such that for each $x \in \Omega$, $n = 0, 1, 2, \dots$ and $\theta \in [0, 1]$*

$$F(x_*) = 0,\ F'(x_*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1),$$

$$\|F'(x_*)^{-1}(F'(x_* + \theta(x - x_*)) - F'(x_*))\| \leq w(\theta\|x - x_*\|), \tag{2.2}$$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq w_0(\theta\|x - x_*\|), \tag{2.3}$$

$$\|F'(x_*)^{-1}(A_n - F'(x_n))\| \leq q_n \|F'(x_*)^{-1} F(x_n)\| \tag{2.4}$$

$$\text{for each}\, x, x_n \in \Omega_0 := \Omega \cap B(x_*, \rho),$$

$$\sup_{n \geq 0} q_n \leq q, \tag{2.5}$$

$$x_n \in B(x_*, r_0)$$

and

$$B(x_*, r_0) \subseteq \Omega,$$

where

$$r_0 \in (0, \rho). \tag{2.6}$$

Then the following items hold

$$\|F'(x_*)^{-1}F(x_n)\| \leq \left(\int_0^1 w(\theta\|x_n - x_*\|)d\theta + 1\right)\|x_n - x_*\|, \tag{2.7}$$

$$\|F'(x_*)^{-1}[A_n - F'(x_n)]\| \leq q\left(\int_0^1 w(\theta\|x_n - x_*\|)d\theta + 1\right)\|x_n - x_*\|, \tag{2.8}$$

$$A_n^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1) \tag{2.9}$$

and

$$\|A_n^{-1}F'(x_*)\| \leq \frac{1}{1 - \varphi(\|x_n - x_*\|)} \mathit{hold}. \tag{2.10}$$

Proof. We shall show first that estimation (2.8) holds. Using (2.1), we have the identity

$$\begin{aligned} F(x_n) =& F(x_n) - F(x_*) = F(x_n) - F(x_*) - F'(x_*)(x_n - x_*) + F'(x_*)(x_n - x_*) \\ =& \int_0^1 [F'(x_* + \theta(x_n - x_*)) - F'(x_*)](x_n - x_*)d\theta. \end{aligned} \tag{2.11}$$

Then by (2.3) and (2.11), we have

$$\begin{aligned} \|F'(x_*)^{-1}F(x_n)\| \leq & \int_0^1 \|F'(x_*)^{-1}[F'(x_* + \theta(x_n - x_*)) - F'(x_*)]\|d\theta\|x_n - x_*\| \\ & + \|x_n - x_*\| \\ \leq & \left(\int_0^1 w(\theta\|x_n - x_*\|)d\theta + 1\right)\|x_n - x_*\|, \end{aligned}$$

which shows estimation (2.7). Moreover, by (2.4), (2.5) and (2.7), we obtain

$$\|F'(x_*)^{-1}[A_n - F'(x_n)]\| \leq q_n\|F'(x_*)^{-1}F(x_n)\| \leq q\left(\int_0^1 w(\theta\|x_n - x_*\|)d\theta + 1\right)\|x_n - x_*\|,$$

which shows estimation (2.8). Furthermore, using (2.2), (2.3), (2.7), (2.8) and the definition of r_0, we get

$$\begin{aligned} \|F'(x_*)^{-1}[A_n - F'(x_*)]\| \leq & \|F'(x_*)^{-1}[A_n - F'(x_n)]\| \\ & + \|F'(x_*)^{-1}[F'(x_n) - F'(x_*)]\| \\ \leq & \varphi(\|x_n - x_*\|) \\ \leq & \varphi(r_0) < 1. \end{aligned} \tag{2.12}$$

It follows from (2.12) and the Banach lemma on invertible operators [1, 5, 6, 11] that (2.9) and (2.10) hold. □

Remark 2.2. In earlier studies the Lipschitz condition [1–15]

$$\|F'(x_*)^{-1}[F'(x) - F'(y)]\| \leq w_1(\|x - y\|) \text{ for each } x, y, \in \Omega \tag{2.13}$$

is used which is stronger than our conditions (2.2) and (2.3). Notice also that since $\Omega_0 \subseteq \Omega$,

$$w(t) \leq w_1(t) \tag{2.14}$$

and

$$w_0(t) \leq w_1(t), \tag{2.15}$$

where the function w_1 is the same as the function w, but defined on Ω instead of Ω_0. The ratio $\frac{w_0}{w_1}$ may be arbitrarily large [1, 5, 6]. Moreover, if (2.13) is used instead of (2.2) and (2.3) in the proof of

Lemma 2.1, then the conclusions hold provided that r_0 is replaced by r_1 which is the smallest positive solution of the equation

$$\psi_1(t) = 0, \tag{2.16}$$

where $\psi_1(t) = \varphi_1(t) - 1$ and $\varphi_1(t) = [q(\int_0^1 w_1(\theta t)d\theta + 1) + w_1(t)]t$. It follows from (2.7), (2.14), (2.15), (2.16) that

$$r_1 \leq r_0. \tag{2.17}$$

Furthermore, the strict inequality holds in (2.17), if (2.14) or (2.15) hold as strict inequalities. Finally, estimations (2.8) and (2.9) are tighter than the corresponding ones (using (2.13)) given by

$$\|F'(x_*)^{-1}[A_n - F'(x_n)]\| \leq q\Big(\int_0^1 w_1(\theta\|x_n - x_*\|)d\theta + 1\Big)\|x_n - x_*\|.$$

Let λ be a parameter satisfying $\lambda \in [0, 1)$. Let also $w_2 : [0, \rho) \longrightarrow [0, +\infty)$ be a continuous and nondecreasing function. Moreover, define the functions $\alpha : [0, \rho) \longrightarrow [0, +\infty), \beta : [0, \rho) \longrightarrow [0, +\infty), f : [0, \rho) \longrightarrow [0, +\infty)$ and $g : \beta : [0, \rho) \longrightarrow [0, +\infty)$, by $\alpha(t) = \frac{1}{1-\varphi(t)}$, $\beta(t) = 2q(1 + \int_0^1 w(\theta t)d\theta)t + 2w_0(t)$, $f(t) = \alpha(t)\beta(t) - \lambda$ $g(t) = \lambda^2 + (1 + \lambda^2)\alpha(t)\int_0^1 w_2((1-\theta)t)d\theta - 1$, sequences $\alpha_k, \beta_k, \gamma_k$ by $\alpha_k = \frac{1}{1-\varphi(\|x_k - x_*\|)}$, $\beta_k := q(1 + \int_0^1 w(\theta\|x_{k+1} - x_*\|)d\theta)\|x_{k+1} - x_*\| + q(1 + \int_0^1 w(\theta\|x_k - x_*\|)d\theta)\|x_k - x_*\| + w_0(\|x_{k+1} - x_*\|) + w_0(\|x_k - x_*\|)$, $d_0 = \gamma_0$, $\gamma_k = \|I - B_kA_k\|^2 + 2\|I - B_kA_k\|\|A_{k+1} - A_k\| + \|B_k\|^2\|A_{k+1} - A_k\|^2$, parameters α, β by $\alpha = \alpha(r_0)$, $\beta = \beta(r_0)$ and quadratic equation $(1 + \alpha\beta)t^2 + 2\alpha\beta(1 + \alpha\beta)t + (\alpha\beta)^2 - \lambda^2 = 0$. Then we have $f(0) = -\lambda < 0$ and $f(t) \longrightarrow +\infty$ as $t \longrightarrow \rho^-$. Denote by ρ_0 the smallest solution of equation $f(t) = 0$ in $(0, \rho)$. Then we find that for each $t \in (0, \rho_0)$,

$$0 < \alpha(t)\beta(t) < \lambda.$$

In view of the above inequality, the preceding quadratic equation has both a unique positive solution denoted by ρ_+ and a negative solution. Define parameter γ by

$$0 \leq \gamma < \gamma_0 = \min\{\rho_+, \rho_0, r_0\}. \tag{2.18}$$

Then we have

$$(1 + \alpha\beta)\gamma^2 + 2\alpha\beta(1 + \alpha\beta)\gamma + (\alpha\beta)^2 < \lambda^2.$$

Notice that we also have $\alpha_k \leq \alpha$ and $\beta_k \leq \beta$.

Next, we present the local convergence of method (1.4).

Theorem 2.3. *Under the hypotheses of Lemma* 2.1 *and with r_0 given in* (2.6) *for $\lambda \in [0, 1)$, we further suppose that there exists the function $w_2 : [0, r_0) \longrightarrow [0, +\infty)$, continuous and nondecreasing such that for each $x \in B(x_*, r_0)$ $\theta \in [0, 1]$ and*

$$\|A_n^{-1}\| \leq \frac{1}{1 - \varphi_1(\|x_n - x_*\|)} < \varphi_1(r_1)$$

we have

$$\|F'(x_*)^{-1}[F'(x_* + \theta(x - x_*)) - F'(x)]\| \leq w_2((1-\theta)\|x - x_*\|) \tag{2.19}$$

$$\text{for each } x \in \Omega_0 = \Omega \cap B(x_*, r_0),$$

$$\|I - B_0A_0\| \leq d_0 < \lambda^2 \tag{2.20}$$

and

$$B(x_*, \gamma) \subseteq \Omega,$$

where γ is given in (2.18)*. Then the sequence $\{x_n\}$ generated by method* (1.4) *for $x_0 \in B(x_*, \gamma) - \{x_*\}$ is well-defined, remains in $B(x_*, \gamma)$ and converges to x_*.*

Proof. By hypothesis (2.20), we have $\|I - B_0A_0\| \leq \gamma_0 < \lambda^2$, so

$$\|I - B_kA_k\| \leq \gamma_k < \lambda^2 \tag{2.21}$$

is true for $k = 0$. Suppose that (2.21) is true for all integers smaller or equal to k. Using Lemma 2.1, we have the estimations

$$\begin{aligned}\|B_k\| =&\|B_k A_k A_k^{-1}\| \leq \|B_k A_k\|\|A_k^{-1}\|\\ \leq&(1+\|I-B_kA_k\|)\|A_k^{-1}\|\\ \leq&(1+\gamma_k)\frac{1}{1-\varphi(\|x_k-x_*\|)} \leq (1+\gamma_k)\alpha_k.\end{aligned}$$

In view of method (1.4) for $n = k$, we can write in turn that

$$\begin{aligned}x_{k+1}-x_* =&x_k-x_*-B_k(F(x_k)-F(x_*))\\ =&[I-B_kF'(x_k)](x_k-x_*)\\ &+\int_0^1 B_k(F'(x_k)-F'(x_*+\theta(x_k-x_*))(x_k-x_*)d\theta.\end{aligned}\tag{2.22}$$

Using (2.22), we get

$$\|x_{k+1}-x_*\| \leq \|I-B_kF'(x_k)\|\|x_k-x_*\| + \frac{L_2}{2}\|B_k\|\|x_k-x_*\|,$$

since $\|x_k - x_*\| \leq \rho$ and $\|x_* + \theta(x_k - x_*) - x_*\| \leq \theta\|x_k - x_*\| \leq \rho$. We By Lemma 2.1 and the induction hypotheses we also have

$$\begin{aligned}&\|F'(x_*)^{-1}(A_{k+1})-A_k)\|\\ \leq&\|F'(x_*)^{-1}(A_{k+1}-F'(x_{k+1}))\|\\ &+\|F'(x_*)^{-1}(F'(x_{k+1})-F'(x_k))\|+\|F'(x_*)^{-1}(A_k-F'(x_*))\|\\ \leq&\|F'(x_*)^{-1}(A_{k+1}-F'(x_{k+1}))\|+\|F'(x_*)^{-1}(A_k-F'(x_k))\|\\ &+\|F'(x_*)^{-1}(F'(x_{k+1})-F'(x_*))\|+\|F'(x_*)^{-1}(F'(x_k)-F'(x_*))\|\\ \leq&\|F'(x_*)^{-1}(A_{k+1}-F'(x_{k+1}))\|+\|F'(x_*)^{-1}(A_k-F'(x_k))\|\\ &+\|x_{k+1}-x_*\|+\|x_k-x_*\|\\ \leq&q\left(1+\int_0^1 w(\theta\|x_{k+1}-x_*\|)d\theta\right)\|x_{k+1}-x_*\|\\ &+q\left(1+\int_0^1 w(\theta\|x_k-x_*\|)d\theta\right)\|x_k-x_*\|\\ &+w_0(\|x_{k+1}-x_*\|)+w_0(\|x_k-x_*\|)\\ \leq&\beta_k\leq\beta.\end{aligned}$$

By the definition of method (1.4), we have the estimations

$$I - B_{k+1}A_{k+1} = I - (2B_k - B_kA_{k+1}B_k)A_{k+1} = (1 - B_kA_{k+1})^2.\tag{2.23}$$

Then by (2.23) and (2.22) for $n = k$, we get

$$\begin{aligned}\|I-B_{k+1}A_{k+1}\| \leq&(\|I-B_kA_k\|+\|B_k\|\|A_{k+1}-A_k\|)^2\\ \leq&\|I-B_kA_k\|^2+2\|I-B_kA_k\|\|B_k\|\|A_{k+1}-A_k\|\\ &+\|B_k\|^2\|A_{k+1}-A_k\|^2\\ \leq&\gamma_k^2+2\gamma_k(1+\gamma_k)\|A_k^{-1}\|\|A_{k+1}-A_k\|\\ &+(1+\gamma_k)^2\|A_k^{-1}\|^2\|A_{k+1}-A_k\|^2\\ \leq&\gamma_k^2+2\gamma_k(1+\gamma_k)\alpha\beta+(1+\gamma_k)^2\alpha^2\beta^2\\ =&(1+\alpha\beta)^2\gamma_k^2+2\alpha\beta(1+\alpha\beta)\gamma_k+\alpha_k^2\beta_k^2\end{aligned}$$

$$\leq (1+\alpha\beta)^2\gamma^2 + 2\alpha\beta(1+\alpha\beta)\gamma + \alpha^2\beta^2 < \lambda^2,$$

which shows (2.21) for $n = k+1$. Then, using the induction hypotheses, (2.19) and the definition of γ,

$$\begin{aligned}\|x_{k+1} - x_*\| \leq & (\lambda^2 + (1+\lambda^2)\alpha\|x_k - x_*\|) \\ & \times \int_0^1 w_2((1-\theta)\|x_k - x_*\| d\theta \|x_k - x_*\| \\ < & g(\gamma)\|x_k - x_*\| \leq g(\rho_+)\|x_k - x_*\| \leq c\|x_k - x_*\|,\end{aligned}$$

where $c = g(\gamma) \in [0,1)$, so $\lim\limits_{k\longrightarrow\infty} x_k = x_*$ and $x_{k+1} \in B(x_*, \rho)$. □

Remark 2.4.

(a) As is noted in Remark 2.2, conditions (2.3) and (2.4) can be replaced by (2.19).

$$\|F'(x_*)^{-1}[F'(x_* + \theta(x - x_*)) - F'(x)]\| \leq w_3((1-\theta)\|x - x_*\|) \tag{2.24}$$

for each $x \in \Omega$ and $\theta \in [0,1]$, where the function w_3 is the same as w_1.

We have that $w_1(t) \leq w_3(t)$. Then in view of Remark 2.2 and (2.19), the radii of convergence as well as the error bounds are improved under the new approach, since old approaches use only (2.24) with the exception of our approach in [2, 4].

(b) The results obtained here can be used for operators F satisfying autonomous differential equations [1, 5, 6, 11] of the form

$$F'(x) = P(F(x)),$$

where $P : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous operator. Then, since $F'(x_*) = P(F(x_*)) = P(0)$, we can apply the results without actually knowing x_*. For example, let $F(x) = e^x - 1$. Then we can choose $P(x) = x+1$.

(c) The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method (GCR) for combined Newton/finite projection methods, and in connection with the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [1, 5, 6].

(d) Let L_0, L, L_1, L_2, L_3 be positive constants. Researchers choose $w_0(t) = L_0 t$, $w(t) = Lt$, $w_1(t) = L_1 t$, $w_2(t) = L_2 t$ and $w_3(t) = L_3 t$. Moreover, if we choose $\Omega_0 = \Omega$ and $L = L_1$, then our results reduce to the ones where the second order of convergence was shown with the Lipschitz conditions given in non-affine invariant form. In Example 3.1, we show that the radii are extended and the upper bounds on $\|x_n - x_*\|$ are tighter if we use w_0, w, w_2 instead of w_0 and w we have used in [4], or only w_3 as used in [2, 7–15].

3. Numerical Examples

Example 3.1. *Let $X = \mathbb{R}^3$, $D = \bar{U}(0,1)$, $x^* = (0,0,0)^T$. Define the function F on D for $w = (x,y,z)^T$ by*

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then the Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the Lipschitz conditions, we get $w_0(t) = L_0 t$, $w(t) = Lt$, $w_1(t) = L_1 t$, $w_2(t) = L_2 t$ and $w_3(t) = L_3 t$, where $L_0 = L = e-1$, $L_1 = L_3 = e$ and $L_2 = e^{\frac{1}{L_0}}$. Moreover, choose $A_n = \frac{1}{2}F'(x_n)$ to obtain $q_n = q = \frac{1}{2}$. The parameters are

$$\rho = 0.5758, r_1 = 0.4739, \bar{\rho} = 0.5499, \bar{r}_1 = 0.4739,$$

where the bar answers corresponding to the case where only w_3 is used in the derivation of the radii.

Example 3.2. *Let $X = Y = \mathbb{R}^{m-1}$ for a natural integer $n \geq 2$. X and Y are equipped with the max-norm $\mathbf{x} = \max_{1\leq i\leq n-1} x_i$. The corresponding matrix norm is*

$$A = \max_{1\leq i\leq m-1} \sum_{j=1}^{j=m-1} |a_{ij}|$$

for $A = (a_{ij})_{1\leq i,j\leq m-1}$. On the interval $[0,1]$, we consider the following two point boundary value problem

$$\begin{cases} v'' + v^2 = 0 \\ v(0) = v(1) = 0 \end{cases} \tag{3.1}$$

[6,8,9,11]. *To discretize the above equation, we divide the interval $[0,1]$ into m equal parts with length of each part: $h = 1/m$ and coordinate of each point: $x_i = i\,h$ with $i = 0,1,2,\ldots,m$. A second-order finite difference discretization of equation (3.1) results in the following set of nonlinear equations*

$$F(\mathbf{v}) := \begin{cases} v_{i-1} + h^2\, v_i^2 - 2v_i + v_{i+1} = 0 \\ \text{for } \; i = 1,2,\ldots,(m-1) \;\; \text{and from (3.1)} \;\; v_0 = v_m = 0, \end{cases}$$

where $\mathbf{v} = [v_1, v_2, \ldots, v_{(m-1)}]^T$. For the above system-of-nonlinear-equations, we provide the Fréchet derivative

$$F'(\mathbf{v}) = \begin{bmatrix} \frac{2v_1}{m^2} - 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \frac{2v_2}{m^2} - 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \frac{2v_3}{m^2} - 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \frac{2v_{(m-1)}}{m^2} - 2 \end{bmatrix}.$$

We see that for $A_n = \frac{9}{10}F'(x_n)$, $w_0(t) = L_0 t$, $w(t) = Lt$, $w_1(t) = L_1 t$, $w_2(t) = L_2 t$, $w_3(t) = L_3 t$, where $L_0 = L = L_1 = L_2 = 3$, $L_3 = 4$, $q = \frac{1}{10}$ and $\|F'(x_*)^{-1}\| = \frac{1}{2}$. The parameters are

$$\rho = 0.5478, r_1 = 0.5478, \bar{\rho} = 0.4762, \bar{r}_1 = 0.4762,$$

where the bar answers corresponding to the case in which only w_3 is used in the derivation of the radii.

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ON SOME PROPERTIES OF PRIMITIVE POLYHEDRONS

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Abstract. It is shown that in the three-dimensional Euclidean space a convex pentagonal prism is not a primitive polyhedron. Some properties of primitive polyhedrons are investigated and the associated dual graphs are considered.

Decomposing a geometric object into simpler parts is one of the most fundamental topics in geometry (especially in combinatorial, discrete and computational geometry).

In the Euclidean plane $\mathbf{R}^2$ one can consider triangulations of a simple polygon, without adding new vertices, and it is well known that every simple polygon can be triangulated in such a manner (see especially [2]). It is also known that any simple n-gon can be decomposed into $n-2$ triangles by using exactly $n-3$ its interior diagonals. In fact, the natural numbers $n-3$ and $n-2$ turn out to be invariants for triangulations of a simple n-gon without adding new vertices. Also, for any natural number $n \geq 3$, there exists a simple n-gon in $\mathbf{R}^2$ which admits only one triangulation without adding new vertices.

For the Euclidean space $\mathbf{R}^m$ whose dimension m is strictly greater than 2, the situation is radically different. Recall that in [14] one can find an example of a simple three-dimensional polyhedron P in $\mathbf{R}^3$ such that the number of all vertices of P is equal to 6 and P does not admit a triangulation without adding new vertices. At the same time, P admits triangulations via adding the necessary number of new vertices.

It should be mentioned that if Q is a convex polyhedron in the space $\mathbf{R}^3$, with a given number n of its vertices, then, in general, there are no invariants similar to $n-2$ and $n-3$ as in the case of the Euclidean plane $\mathbf{R}^2$. Indeed, it may happen that there are two triangulations of Q, without adding new vertices, such that the total number of tetrahedra in the first triangulation differs from the total number of tetrahedra in the second triangulation. Thus, one may conclude that in the Euclidean space $\mathbf{R}^m$, where $m > 2$, any convex polyhedron Q admits a triangulation without adding new vertices, but the total number of simplexes of the triangulation is not uniquely determined by Q. So, one can only speak of certain lower and upper estimates for this number, e.g., in terms of $v(Q)$, where $v(Q)$ denotes the total number of vertices of Q.

Many works and monographs were devoted to those questions and topics which are connected (more or less) with triangulations and decompositions of simple and convex polyhedrons in the Euclidean space (see, e.g., [1–5], [7–9], [10, 12, 13]).

In this article we would like to consider a certain class of convex polyhedrons (primarily, in the space $\mathbf{R}^3$), which will be called primitive polyhedrons (cf. [8]).

Throughout the article, we use the following standard notation:

$\mathbf{N}$ is the set of all natural numbers;

$\mathbf{R}$ is the set of all real numbers;

$\mathbf{R}^m$ is the m-dimensional Euclidean space, where $m \geq 1$.

For our further purpose, we shall need some notions and lemmas.

If P is an m-dimensional convex polyhedron in the Euclidean space $\mathbf{R}^m$, then $s(P)$ denotes the smallest number of m-dimensional simplexes into which this P can be decomposed.

Let Q be a convex m-dimensional polyhedron in the Euclidean space $\mathbf{R}^m$.

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A convex m-dimensional polyhedron $Q^{'}$ is called a primitive extension of Q if there exist an $(m-1)$-dimensional face D of Q and an m-dimensional simplex T in $\mathbf{R}^m$ such that D is also a face of T and the following two conditions are fulfilled:

$(*)$ $T \cap Q = D$.

$(**)$ The set of all vertices of $Q^{'}$ coincides with the union of the sets of all vertices of Q and T.

In particular, $Q^{'}$ can be obtained by adding to Q some m-dimensional simplex T, whose base is one of the facets of Q (and no vertex of Q is lost after adding T to Q).

In our further considerations we shall say that the above-mentioned simplex T is extreme for the polyhedron Q'. In the case $m = 2$, the term "ear" is commonly used for such T in Q'.

Let now $\{Q_1, Q_2, \dots, Q_k\}$ be a finite sequence of convex m-dimensional polyhedrons in the space $\mathbf{R}^m$.

We shall say that this sequence is primitive if Q_1 is an m-dimensional simplex and, for each natural index $i \in [1, k-1]$, the polyhedron Q_{i+1} is a primitive extension of Q_i.

A convex m-dimensional polyhedron $Q \subset \mathbf{R}^m$ is called primitive if $Q = Q_k$ for some primitive sequence $\{Q_1, Q_2, \dots, Q_k\}$ of convex m-dimensional polyhedrons in $\mathbf{R}^m$.

Some nontrivial properties of primitive polyhedrons, connected with their decompositions into simplexes, are discussed in [8]. In particular, it is shown in Chapter 6 of [8] that for $m > 3$ no m-dimensional cube (parallelepiped) is a primitive polyhedron. On the other hand, it is easy to show that if $m \leq 3$, then all m-dimensional parallelepipeds are primitive polyhedrons.

We also need several simple notions from the theory of finite graphs (see, for example, the well-known monographs [6] or [11]).

Let P be a simple (in particular, convex) m-dimensional polyhedron in the Euclidean space $\mathbf{R}^m$ and let $\{T_i : 1 \leq i \leq n\}$ be a triangulation of P into m-dimensional simplexes.

The dual graph $\Gamma = (V, E)$ of this triangulation is defined as follows. The set V of vertices of Γ is obtained by choosing in every simplex T_i some interior point of T_i (more concretely, one may choose the barycenter t_i of T_i). Two verices t_i and t_j from V are connected by an edge from E if and only if the simplexes T_i and T_j are neighbors, i.e., if and only if there exists a common $(m-1)$-dimensional face of T_i and T_j.

Example 1. Let P be a simple n-gon in the plane $\mathbf{R}^2$ and let $\{T_i : 1 \leq i \leq n-2\}$ be any triangulation of P into triangles, without adding new vertices. It is not hard to see that the dual graph of this triangulation is a tree and every vertex of the dual graph is incident to at most three edges. Conversely, if one has a tree, all vertices of which are incident to at most three edges, then there exist a convex polygon in $\mathbf{R}^2$ and its triangulation, without adding new vertices, such that the dual graph of the triangulation is isomorphic to the given tree. The analogous statement fails to be valid for all convex polyhedrons in the space $\mathbf{R}^3$, but in some another form holds true for primitive polyhedrons (see Theorem 1 below).

Lemma 1. *For any convex three-dimensional polyhedron P in the space $\mathbf{R}^3$ with the number of vertices $v(P)$, the inequality*

$$v(P) - 3 \leq s(P)$$

holds true.

Sketch of the proof. Suppose to the contrary that the above-mentioned inequality fails to be satisfied for some convex three-dimensional polyhedra $P \subset \mathbf{R}^3$. Obviously, in such a case we may choose a convex three-dimensional polyhedron P for which

$$s(P) + 3 < v(P)$$

and the value $s(P)$ is minimal. Consider a dissection

$$\{T_i : 1 \leq i \leq s(P)\}$$

of P into $s(P)$ many tetrahedra.

Only the following two cases are possible.

Case 1. For some natural index $j \in \{1, 2, \dots, s(P)\}$, the tetrahedron T_j has three facets each of which lies in the corresponding facet of P.

Case 2. Every tetrahedron from the family $\{T_i : 1 \leq i \leq s(P)\}$ has at most two facets lying in the corresponding facets of P.

A combinatorial argument based on the classical Euler formula for convex three-dimensional polyhedrons, in both these cases leads to a contradiction with the minimality of $s(P)$ (for more details, see Chapter 7 in [8]). The obtained contradiction shows that neither Case 1 nor Case 2 is possible, so the inequality

$$v(P) - 3 \leq s(P)$$

must be valid for all three-dimensional convex polyhedrons $P \subset \mathbf{R}^3$.

Lemma 2. *Let P be a three-dimensional convex polyhedron in the space $\mathbf{R}^3$ and let $v = v(P)$ denote the number of all vertices of P.*

Then the polyhedron P is primitive if and only if the equality $s(P) = v - 3$ holds true.

The proof of Lemma 2 can be found in [8].

Lemma 3. *Let P be a three-dimensional simple polyhedron in the space $\mathbf{R}^3$ and let $\{T_i : 1 \leq i \leq n\}$ be a triangulation of P into tetrahedra, without adding new vertices.*

Then the dual graph of this triangulation is connected (but, in general, it is not a tree).

We omit an easy proof of this lemma.

Lemma 4. *Let P be a convex m-dimensional primitive polyhedron in the space $\mathbf{R}^m$.*

Then there exists a triangulation of P, without adding new vertices, such that the corresponding dual graph is a tree.

Proof. We use the method of induction according to the complexity of geometric structure of P or, equivalently, we use induction on the total number $v(P)$ of vertices of P.

If P is an m-dimensional simplex, it is clear that its dual graph is a singleton, hence is a trivial tree with only one vertex and without edges.

Suppose now that P is a convex m-dimensional primitive polyhedron in $\mathbf{R}^m$ with $v(P) \geq m + 2$. From the definition of m-dimensional convex primitive polyhedrons it follows that P has at least one extreme simplex. So, we can pick an extreme simplex T of P and consider the reduced polyhedron P which is obtained from P by removing this simplex T. Obviously, we have the inequality $v(P') < v(P)$. Applying the inductive assumption to P', we can construct one of the triangulations of P', without adding new vertices, such that its dual graph is a tree Γ. It is not hard to see, keeping in mind the fact that T has a common facet with P', this tree can be expanded to a tree which will be the dual graph of the initial polyhedron PS. Indeed, it suffices to add to Γ one additional vertex and one additional edge corresponding to the extreme simplex T and incident to this vertex. $\square$

Example 2. In the space $\mathbf{R}^3$, consider an arbitrary trigonal bi-pyramid P which has 5 vertices, i.e., $v(P) = 5$, and which is a primitive polyhedron. It is easy to see that there are two types of the dual graphs that are associated with two triangulations of P, without adding new vertices:

(a) one edge, when P is decomposed into two tetrahedra;

(b) 3-cycle, when P is decomposed into three tetrahedra.

This simple example shows that even for a primitive polyhedron Q in the space $\mathbf{R}^3$, a triangulation of Q, without adding new vertices, should be carefully chosen if one wants to obtain a tree as the dual graph of the triangulation.

The following theorem is valid.

Theorem 1. *Let $\Gamma = (V, E)$ be a tree such that the degrees of all vertices of this tree are less than or equal to $m + 1$.*

Then there exist both a convex m-dimensional primitive polyhedron P in the Euclidean space $\mathbf{R}^m$ and a triangulation of P without adding new vertices such that the dual graph of the triangulation is isomorphic to Γ.

Proof. We use the method of induction on $\mathrm{card}(V)$.

If $\mathrm{card}(V) = 1$, then Γ is trivially isomorphic to the dual graph of an m-dimensional simplex, so there is nothing to prove.

Suppose now that $\mathrm{card}(V) \geq 2$ and that the assertion of this theorem has already been established for all those trees whose cardinalities are strictly less than $\mathrm{card}(V)$. As is well known from the graph theory, there exists at least one vertex $v \in V$ incident to exactly one edge e from E. In our argument below, this vertex will be called a leaf of the tree. So, we can pick a leaf v in (V, E).

Consider the reduced graph $\Gamma' = (V', E')$, where

$$V' = V \setminus \{v\}, \qquad E' = E \setminus \{e\}.$$

Obviously, the graph Γ' is also a tree. Applying the inductive assumption to (V', E'), we can find an m-dimensional convex polyhedron P and its triangulation $\{T_i : 1 \leq i \leq n\}$, without adding new vertices, such that the dual graph of $\{T'_i : 1 \leq i \leq n\}$ is isomorphic to Γ. Moreover, we may assume that all facets of P' are the $(m-1)$-dimensional simplexes. Now, it is easy to see how one can construct a primitive extension P of P' and some triangulation

$$\{T'_i : 1 \leq i \leq n\} \cup \{T\}$$

of P such that the dual graph of this extended triangulation would be isomorphic to Γ. Only one delicate moment should be emphasized here: in order to guarantee the convexity of the required polyhedron P, the new vertex of P, being one of the vertices of T, can be taken in the vicinity of the barycenter of a certain facet of P'. □

Lemma 5. *Let P be a convex polygon with $v = v(P)$ vertices in the Euclidean plane $\mathbf{R}^2$ and let P be decomposed into some finitely many triangles $\{T_i : 1 \leq i \leq n\}$, i.e.,*

$$P = \bigcup \{T_i : 1 \leq i \leq n\}$$

and these triangles pairwise have no common interior points.

Then the inequality $n \geq v - 2$ holds true.

Proof. Consider the sum of all interior angles of the triangles $\{T_i : 1 \leq i \leq n\}$. Clearly, it is equal to $\pi \cdot n$. As is well known, the sum of all interior angles of P is equal to $\pi \cdot (v - 2)$. So, we can write

$$\pi \cdot (v - 2) \leq \pi \cdot n,$$

whence it follows that $v - 2 \leq n$. □

Example 3. For any convex polygon $P \subset \mathbf{R}^2$ denote again by $s(P)$ the minimal cardinality of a dissection of P into triangles. Then we have the equality

$$s(P) = v - 2,$$

where $v = v(P)$ denotes again the number of all vertices of P. This fact is an immediate consequence of Lemma 5. It should be remarked that the analogous statement fails to be true for simple polygons in the plane. For instance, there is a simple polygon in $\mathbf{R}^2$ with 6 vertices which can be decomposed into two triangles.

Lemma 6. *Let P be an arbitrary convex pentagonal prism in the space $\mathbf{R}^3$ and let T be a tetrahedron lying in P.*

Then the inequality $\lambda(T) < \frac{1}{3}\lambda(P)$ holds true, where $\lambda(P)$ denotes the volume of P and $\lambda(T)$ denotes the volume of T.

It is easy to see that any convex quadrilateral prism in the space $\mathbf{R}^3$ is a primitive polyhedron. Moreover, any convex polyhedron in $\mathbf{R}^3$, combinatorially isomorphic to such a prism, is primitive. On the other hand, the next statement is valid.

Theorem 2. *If P is a convex pentagonal prism in the space $\mathbf{R}^3$, then P is not a primitive polyhedron.*

Proof. We use the method of volumes presented and developed in [8]. Suppose to the contrary that the given convex pentagonal prism P is a primitive polyhedron.

By virtue of Lemma 2, we must have the equality

$$s(P) = v(P) - 3,$$

where $v(P)$ denotes the number of vertices of P and $s(P)$ denotes the smallest number of tetrahedra into which the pentagonal prism P can be decomposed. In our case,

$$v(P) = 10, \; s(P) = 10 - 3 = 7.$$

Let us consider the two bases of the prism. By using Lemma 5, we can deduce that each of these bases needs at least 3 tetrahedra of a decomposition. Since no two facets of a tetrahedron are parallel to each other, any tetrahedron corresponding to one base differs from any tetrahedron corresponding to the other base. Consequently, every decomposition of P into tetrahedra contains at least 6 tetrahedra which correspond to the bases of P. The total volume of those tetrahedra does not exceed $(2/3)\lambda(P)$. Now, consider the tetrahedron distinct from all the above-mentioned 6 tetrahedra. Its volume, according to Lemma 6, is strictly less than $(1/3)\lambda(P)$. This circumstance implies that the total volume of seven tetrahedra is strictly less than $\lambda(P)$. Therefore, no seven tetrahedra can constitute a decomposition (dissection) of P. □

Let G be some subgroup of D_m, where D_m denotes the group of all isometric transformations of the space $\mathbf{R}^m$, and let X and Y be two polyhedrons of $\mathbf{R}^m$.

We recall (see, e.g., [1]) that these two polyhedrons are (finitely) G-equidecomposable if there exist two finite disjoint families

$$\{X_k : k \in K\}, \; \{Y_k : k \in K\}$$

of polyhedrons in $\mathbf{R}^m$ such that:

(1) $X = \cup\{X_k : k \in K\}$ and $Y = \cup\{Y_k : k \in K\}$;

(2) for each index $k \in K$, the polyhedron X_k is G-congruent to the polyhedron Y_k;

(3) the polyhedrons X_k (respectively, Y_k) have no pairwise common interior points.

Obviously, if polyhedrons X and Y in $\mathbf{R}^m$ are G-congruent, then they are also (finitely) G-equidecomposable, but the converse assertion is not true, in general.

The finite G-equidecomposability is an equivalence relation in the class of all polyhedrons in the space $\mathbf{R}^m$ (see [1]).

If $G = D_m$, then the G-equicomposability of two polyhedrons X and Y is called simply the equidecomposability of X and Y.

Theorem 3. *Let P and Q be two convex 3-dimensional polyhedrons in $\mathbf{R}^3$ with equal volumes. Then the following six cases can be realized (separately):*

(a) both P and Q are primitive polyhedrons and they are equidecomposable;

(b) both P and Q are primitive polyhedrons and they are not equidecomposable;

(c) P is a primitive polyhedron, Q is not a primitive polyhedron and they are equidecomposable;

(d) P is a primitive polyhedron, Q is not a primitive polyhedron and they are not equidecomposable;

(e) both P and Q are not primitive polyhedrons and they are equidecomposable;

(f) both P and Q are not primitive polyhedrons and they are not equidecomposable.

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A NEW FACTOR THEOREM FOR GENERALIZED ABSOLUTE CESÀRO SUMMABILITY METHODS

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Abstract. In [6], we have proved a main theorem dealing with $\varphi - \mid C, \alpha, \mid_k$ summability factors of infinite series. In this paper, we will generalize this result for the $\varphi - \mid C, \alpha, \beta \mid_k$ summability method. Also, some new and known results are obtained.

1. Introduction

Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α, β), with $\alpha + \beta > -1$, of the sequence (na_n), that is (see [7]),

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1}$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1 \quad \text{and} \quad A_{-n}^{\alpha+\beta} = 0 \quad \text{for} \quad n > 0.$$

Let $(\omega_n^{\alpha,\beta})$ be a sequence defined by (see [3])

$$\omega_n^{\alpha,\beta} = \begin{cases} \left| t_n^{\alpha,\beta} \right|, & \alpha = 1, \beta > -1, \\ \max\limits_{1 \le v \le n} \left| t_v^{\alpha,\beta} \right|, & 0 < \alpha < 1, \beta > -1. \end{cases} \tag{2}$$

Let (φ_n) be a sequence of complex numbers. The series $\sum a_n$ is said to be summable $\varphi - \mid C, \alpha, \beta \mid_k$, $k \ge 1$, if (see [4])

$$\sum_{n=1}^{\infty} n^{-k} \mid \varphi_n t_n^{\alpha,\beta} \mid^k < \infty.$$

In the special case for $\varphi_n = n^{1-\frac{1}{k}}$, the $\varphi - \mid C, \alpha, \beta \mid_k$ summability is the same as $\mid C, \alpha, \beta \mid_k$ summability (see [8]). Also, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then $\varphi - \mid C, \alpha, \beta \mid_k$ summability reduces to $\mid C, \alpha, \beta; \delta \mid_k$ summability (see [5]). If we take $\beta = 0$, then we have $\varphi - \mid C, \alpha \mid_k$ summability (see [1]). If we take $\varphi_n = n^{1-\frac{1}{k}}$ and $\beta = 0$, then we get $\mid C, \alpha \mid_k$ summability (see [9]). Finally, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$ and $\beta = 0$, then we obtain $\mid C, \alpha; \delta \mid_k$ summability (see [10]).

2. The Known Results

The following theorems dealing with the $\varphi - \mid C, \alpha \mid_k$ summability factors of infinite series are known.

Theorem A ([2]). *Let $0 < \alpha \le 1$. Let (X_n) be a positive non-decreasing sequence and let there exist the sequences (β_n) and (λ_n) such that*

$$\mid \Delta\lambda_n \mid \le \beta_n \tag{3}$$

$$\beta_n \to 0 \quad as \quad n \to \infty \tag{4}$$

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$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty \tag{5}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{6}$$

If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} |\varphi_n|^k)$ is non-increasing and if the sequence (ω_n^α) defined by (see [12])

$$\omega_n^\alpha = \begin{cases} |t_n^\alpha| & (\alpha = 1) \\ \max_{1\leq v \leq n} |t_v^\alpha| & (0 < \alpha < 1) \end{cases} \tag{7}$$

satisfies the condition

$$\sum_{n=1}^{m} \frac{(\mid \varphi_n \mid w_n^\alpha)^k}{n^k} = O(X_m) \quad \text{as} \quad m \to \infty,$$

then the series $\sum a_n \lambda_n$ is summable $\varphi - \mid C, \alpha \mid_k$, $k \geq 1$ and $(\alpha + \epsilon) > 1$.

Theorem B ([6])**.** *Let $0 < \alpha \leq 1$. Let (X_n) be a positive non-decreasing sequence and the sequences (β_n) and (λ_n) such that conditions (3), (4), (5), (6) of Theorem A are satisfied. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} |\varphi_n|^k)$ is non-increasing and if the sequence (ω_n^α) defined by (7) satisfies the condition*

$$\sum_{n=1}^{m} \frac{(\mid \varphi_n \mid w_n^\alpha)^k}{n^k {X_n}^{k-1}} = O(X_m) \quad as \quad m \to \infty,$$

then the series $\sum a_n \lambda_n$ is summable $\varphi - \mid C, \alpha \mid_k$, $k \geq 1$ and $(1 + \alpha k + \epsilon - k) > 1$.

3. The Main Result

The aim of this paper is to generalize Theorem B for $\varphi - \mid C, \alpha, \beta \mid_k$ summability method. Now we shall prove the following theorem.

Theorem. *Let $0 < \alpha \leq 1$. Let (X_n) be a positive non-decreasing sequence and the sequences (β_n) and (λ_n) such that conditions (3), (4), (5), (6) of Theorem A are satisfied. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} |\varphi_n|^k)$ is non-increasing and if the sequence $(\omega_n^{\alpha,\beta})$ defined by (2) satisfies the condition*

$$\sum_{n=1}^{m} \frac{(\mid \varphi_n \mid w_n^{\alpha,\beta})^k}{n^k {X_n}^{k-1}} = O(X_m) \quad as \quad m \to \infty,$$

then the series $\sum a_n \lambda_n$ is summable $\varphi - \mid C, \alpha, \beta \mid_k$, $k \geq 1$ and $(1 + (\alpha + \beta)k + \epsilon - k) > 1$.

We need the following lemmas for the proof of our theorem.

Lemma 1 ([3])**.** *If $0 < \alpha \leq 1$, $\beta > -1$, and $1 \leq v \leq n$, then*

$$\left| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^\beta a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^\beta a_p \right|.$$

Lemma 2 ([11])**.** *Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of Theorem A, the conditions*

$$n\beta_n X_n = O(1) \quad as \quad n \to \infty \tag{8}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty.$$

hold, when (5) is satisfied.

4. Proof of the Theorem

Let $(T_n^{\alpha,\beta})$ be the nth (C,α,β) mean of the sequence $(na_n\lambda_n)$.
Then, by (1), we have

$$T_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\lambda_v.$$

Applying first Abel's transformation and then using Lemma 1, we have

$$\begin{aligned}
T_n^{\alpha,\beta}=&\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p+\frac{\lambda_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v,\\
\mid T_n^{\alpha,\beta}\mid\leq&\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\mid\Delta\lambda_v\mid\left|\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p\right|+\frac{\mid\lambda_n\mid}{A_n^{\alpha+\beta}}\left|\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\right|\\
\leq&\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{(\alpha+\beta)}\omega_v^{\alpha,\beta}\mid\Delta\lambda_v\mid+\mid\lambda_n\mid\omega_n^{\alpha,\beta}=T_{n,1}^{\alpha,\beta}+T_{n,2}^{\alpha,\beta}.
\end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it suffices to show that

$$\sum_{n=1}^{\infty}n^{-k}\mid\varphi_nT_{n,r}^{\alpha,\beta}\mid^k<\infty,\quad\text{for}\quad r=1,2.$$

For $k>1$, applying first Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, and then using (8), we obtain

$$\begin{aligned}
\sum_{n=2}^{m+1}n^{-k}\mid\varphi_nT_{n,1}^{\alpha,\beta}\mid^k\leq&\sum_{n=2}^{m+1}n^{-k}(A_n^{\alpha+\beta})^{-k}|\varphi_n|^k\left\{\sum_{v=1}^{n-1}A_v^{\alpha+\beta}\omega_v^{\alpha,\beta}\beta_v\right\}^{k-1}\\
\leq&\sum_{n=2}^{m+1}\frac{1}{n}(A_n^{\alpha+\beta})^{-k}|\varphi_n|^k\sum_{v=1}^{n-1}(A_v^{\alpha+\beta})^k(\omega_v^{\alpha,\beta})^k\beta_v^k\times\left\{\frac{1}{n}\sum_{v=1}^{n-1}1\right\}^{k-1}\\
=&O(1)\sum_{n=2}^{m+1}\frac{|\varphi_n|^k}{n^{1+(\alpha+\beta)k}}\sum_{v=1}^{n-1}v^{(\alpha+\beta)k}(\omega_v^{\alpha,\beta})^k\beta_v^k\\
=&O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}(\omega_v^{\alpha,\beta})^k\beta_v\beta_v^{k-1}\sum_{n=v+1}^{m+1}\frac{n^{\epsilon-k}|\varphi_n|^k}{n^{1+(\alpha+\beta)k+\epsilon-k}}\\
=&O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}(\omega_v^{\alpha,\beta})^k\beta_v\frac{v^{\epsilon-k}|\varphi_v|^k}{v^{k-1}X_v^{k-1}}\int_v^{\infty}\frac{dx}{x^{1+(\alpha+\beta)k+\epsilon-k}}\\
=&O(1)\sum_{v=1}^{m}v\beta_v\frac{(\omega_v^{\alpha,\beta}|\varphi_v|)^k}{v^kX_v^{k-1}}\\
=&O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\frac{(\omega_r^{\alpha,\beta}|\varphi_r|)^k}{r^kX_r^{k-1}}+O(1)m\beta_m\sum_{v=1}^{m}\frac{(\omega_v^{\alpha,\beta}|\varphi_v|)^k}{v^kX_v^{k-1}}\\
=&O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v+O(1)m\beta_mX_m\\
=&O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v+O(1)\sum_{v=1}^{m-1}\beta_vX_v+O(1)m\beta_mX_m\\
=&O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}$$

by the hypotheses of the theorem and Lemma 2. Again, using (6), we have

$$\begin{aligned}\sum_{n=1}^{m} n^{-k} \mid \varphi_n T_{n,2}^{\alpha,\beta} \mid^k &= \sum_{n=1}^{m} n^{-k} |\varphi_n|^k |\lambda_n| |\lambda_n|^{k-1} (\omega_n^{\alpha,\beta})^k = O(1) \sum_{n=1}^{m} |\lambda_n| \frac{(\mid \varphi_n \mid w_n^{\alpha,\beta})^k}{n^k {X_n}^{k-1}} \\ &= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} \frac{(\mid \varphi_v \mid w_v^{\alpha,\beta})^k}{v^k {X_v}^{k-1}} + O(1) |\lambda_m| \sum_{n=1}^{m} \frac{(\mid \varphi_n \mid w_n^{\alpha,\beta})^k}{n^k {X_n}^{k-1}} \\ &= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n + O(1) |\lambda_m| X_m \\ &= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) |\lambda_m| X_m = O(1) \quad \text{as} \quad m \to \infty,\end{aligned}$$

by the hypotheses of the theorem and Lemma 2. This completes the proof of the theorem.

5. Conclusion

If we take $\epsilon = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$, then we obtain a new result concerning the $\mid C, \alpha, \beta \mid_k$ summability factors of infinite series. If we take $\epsilon = 1$, $\beta = 0$ and $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then we have a new result dealing with the $\mid C, \alpha; \delta \mid_k$ summability factors of infinite series. Also, if we take $\beta = 0$, then we obtain Theorem B.

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EXISTENCE RESULTS FOR A CLASS OF NONLINEAR DEGENERATE (p,q)-BIHARMONIC OPERATORS IN WEIGHTED SOBOLEV SPACES

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Abstract. In this paper we are interested in the existence of solutions for Navier problem associated with the degenerate nonlinear elliptic equation

$$\Delta\big[\omega(x)\,|\Delta u|^{p-2}\Delta u+v(x)|\Delta u|^{q-2}\Delta u\big]-\sum_{j=1}^{n}D_j\left[\omega(x)\mathcal{A}_j(x,u,\nabla u)\right]$$
$$=f_0(x)-\sum_{j=1}^{n}D_jf_j(x),\quad\text{in}\quad\Omega$$

in the setting of the weighted Sobolev spaces.

1. Introduction

In this paper we prove the existence of (weak) solutions in the weighted Sobolev space $X = W^{2,p}(\Omega,\omega)\cap W_0^{1,p}(\Omega,\omega)$ (see Definitions 2.3 and 2.4) for the Navier problem

$$(P)\begin{cases} Lu(x)=f_0(x)-\displaystyle\sum_{j=1}^{n}D_jf_j(x), & \text{in}\quad\Omega,\\ u(x)=\Delta u(x)=0, & \text{on}\quad\partial\Omega,\end{cases}$$

where L is the partial differential operator

$$Lu(x)=\Delta\big[\omega(x)\,|\Delta u|^{p-2}\Delta u+v(x)\,|\Delta u|^{q-2}\Delta u\big]-\sum_{j=1}^{n}D_j\big[\omega(x)\mathcal{A}_j(x,u(x),\nabla u(x))\big]$$

where $D_j=\partial/\partial x_j$, Ω is a bounded open set in $\mathbb{R}^n$, ω and v are two weight functions, Δ is the usual Laplacian operator, $2\le q<p<\infty$ and the functions $\mathcal{A}_j:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ $(j=1,\dots,n)$ satisfying the following conditions:

(H1) $x\mapsto\mathcal{A}_j(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$, $(\eta,\xi)\mapsto\mathcal{A}_j(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$;

(H2) there exists a constant $\theta_1>0$ such that $[\mathcal{A}(x,\eta,\xi)-\mathcal{A}(x,\eta',\xi')].\ (\xi-\xi')\ge\theta_1\,|\xi-\xi'|^p$, whenever $\xi,\xi'\in\mathbb{R}^n$, $\xi\neq\xi'$, where $\mathcal{A}(x,\eta,\xi)=(\mathcal{A}_1(x,\eta,\xi),\dots,\mathcal{A}_n(x,\eta,\xi))$ (where the dot denotes here the Euclidean scalar product in $\mathbb{R}^n$);

(H3) $\mathcal{A}(x,\eta,\xi).\xi\ge\lambda_1|\xi|^p$, where λ_1 is a positive constant;

(H4) $|\mathcal{A}(x,\eta,\xi)|\le K_1(x)+h_1(x)|\eta|^{p/p'}+h_2(x)|\xi|^{p/p'}$, where K_1,h_1 and h_2 are positive functions with h_1 and $h_2\in L^\infty(\Omega)$, and $K_1\in L^{p'}(\Omega,\omega)$ (with $1/p+1/p'=1$).

Let Ω be an open set in $\mathbb{R}^n$. By the symbol $\mathcal{W}(\Omega)$ we denote the set of all measurable a.e. in Ω positive and finite functions $\omega=\omega(x)$, $x\in\Omega$. Elements of $\mathcal{W}(\Omega)$ will be called *weight functions*. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will be denoted by μ_ω. Thus, $\mu_\omega(E)=\displaystyle\int_E\omega(x)\,dx$ for measurable sets $E\subset\mathbb{R}^n$.

In general, the Sobolev spaces $\mathrm{W}^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. In the particular case for $p=q=2$ and $\omega=v\equiv1$, we have

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the equation

$$\Delta^2 u - \sum_{j=1}^{n} D_j \mathcal{A}_j(x, u, \nabla u) = f,$$

where $\Delta^2 u$ is the biharmonic operator. If $p = q$, $\omega = v \equiv 1$ and $\mathcal{A}(x, \eta, \xi) = |\xi|^{p-2}\,\xi$, we have the equation

$$\Delta(|\Delta|^{p-2}\,\Delta u) - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f.$$

Biharmonic equations appear in the study of mathematical model in several real-life processes as, among others, radar imaging (see [1]) or incompressible flows (see [17]).

For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [4], [5], [6], [3] and [9]). In various applications, we can meet the boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that there appear some degeneration or singularity appears. There are several very concrete problems from practice which lead to such differential equations, e.g., from glaceology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction and reaction-diffusion problems (see some examples of applications of degenerate elliptic equations in [2] and [8]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [18]). These classes have found many useful applications in harmonic analysis (see [20]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [15]). There are, in fact, many interesting examples of weights (see [14] for p-admissible weights).

In the non-degenerate case (i.e., with $\omega(x) \equiv 1$), for all $f \in L^p(\Omega)$, the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [13]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

is uniquely solvable in $W_0^{1,p}(\Omega)$ (see [7]), where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. In the degenerate case, the weighted p-Biharmonic operator has been studied by many authors (see [19] and the references therein), and the degenerated p-Laplacian was studied in [9].

The following theorem will be proved in Section 3.

Theorem 1.1. *Let $2 \le q < p < \infty$ and assume $(H1) - (H4)$. If*

(H5) *$\omega \in A_p$, $v \in \mathcal{W}(\Omega)$ and $\dfrac{v}{\omega} \in L^r(\Omega, \omega)$, where $r = p/(p-q)$;*

(H6) *$f_j/\omega \in L^{p'}(\Omega, \omega)$ $(j = 0, 1, \ldots, n)$.*

Then the problem (P) has a unique solution $u \in X = W^{2,p}(\Omega, \omega) \cap W_0^{1,p}(\Omega, \omega)$. Moreover, we have

$$\|u\|_X \le \frac{1}{\gamma^{p'/p}} \left(C_\Omega \|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^{n} \|f_j/\omega\|_{L^{p'}(\Omega,\omega)} \right)^{p'/p},$$

where $\gamma = \min\{\lambda_1, 1\}$ and C_Ω is the constant in Theorem 2.2.

2. Definitions and Basic Results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|} \int_B \omega \, dx \right) \left(\frac{1}{|B|} \int_B \omega^{1/(1-p)} \, dx \right)^{p-1} \le C,$$

for all balls $B\subset\mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1<q\leq p$, then $A_q\subset A_p$ (see [12], [14] or [20] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that

$$\mu(B(x;2r))\leq C\,\mu(B(x;r)),$$

for every ball $B=B(x;r)\subset\mathbb{R}^n$, where $\mu(B)=\int_B\omega(x)\,dx$. If $\omega\in A_p$, then μ is doubling (see [14], Corollary 15.7).

As an example of A_p-weight, the function $\omega(x)=|x|^\alpha$, $x\in\mathbb{R}^n$, is in A_p if and only if $-n<\alpha<n(p-1)$ (see [20], Corollary 4.4, Chapter IX, Corollary 4.4).

If $\omega\in A_p$, then

$$\left(\frac{|E|}{|B|}\right)^p\leq C\frac{\mu(E)}{\mu(B)},$$

whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (for *a strong doubling property* see 15.5 in [14]). Therefore, if $\mu(E)=0$, then $|E|=0$. The measure μ and the Lebesgue measure $|\cdot|$ are mutually absolutely continuous, i.e., they have the same zero sets ($\mu(E)=0$ if and only if $|E|=0$); so, there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..

Definition 2.1. Let ω be a weight, and let $\Omega\subset\mathbb{R}^n$ be open. For $0<p<\infty$ we define $L^p(\Omega,\omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)}=\left(\int_\Omega|f|^p\omega\,dx\right)^{1/p}<\infty.$$

If $\omega\in A_p$, $1<p<\infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega)\subset L^1_{\text{loc}}(\Omega)$ for every open set Ω (see [21, Remark 1.2.4]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.2. Let $\Omega\subset\mathbb{R}^n$ be a bounded open set, $1<p<\infty$, k be a nonnegative integer and $\omega\in A_p$. We shall denote by $W^{k,p}(\Omega,\omega)$ the weighted Sobolev spaces, the set of all functions $u\in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u\in L^p(\Omega,\omega)$, $1\leq|\alpha|\leq k$. The norm in the space $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)}=\left(\int_\Omega|u|^p\,\omega\,dx+\sum_{1\leq|\alpha|\leq k}\int_\Omega|D^\alpha u|^p\,\omega\,dx\right)^{1/p}. \tag{2.1}$$

If $\omega\in A_p$, then $W^{1,p}(\Omega,\omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (2.1) (see [21, Theorem 2.1.4]). The spaces $W^{1,p}(\Omega,\omega)$ are Banach spaces.

The space $W^{1,p}_0(\Omega,\omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the norm (2.1). Equipped with this norm, $W^{1,p}_0(\Omega,\omega)$ is a reflexive Banach space (see [16] for more information about the spaces $W^{1,p}(\Omega,\omega)$). The dual of the space $W^{1,p}_0(\Omega,\omega)$ is the space

$$[W^{1,p}_0(\Omega,\omega)]^*=\{T=f_0-\text{div}(F),\ F=(f_1,\ldots,f_n):\ \frac{f_j}{\omega}\in L^{p'}(\Omega,\omega),\ j=0,1,\ldots,n\}.$$

It is evident that a weight function ω which satisfies $0<c_1\leq\omega(x)\leq c_2$ for $x\in\Omega$ (where c_1 and c_2 are constants), gives nothing new (the space $\mathrm{W}^{1,p}_0(\Omega,\omega)$ is then identical with the classical Sobolev space $\mathrm{W}^{1,p}_0(\Omega)$). Consequently, we shall be interested above all in such weight functions ω which either vanish somewhere in $\bar{\Omega}$, or increase at infinity (or both).

In this paper we use the following results.

Theorem 2.1. *Let $\omega\in A_p$, $1<p<\infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m\to u$ in $L^p(\Omega,\omega)$ then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi\in L^p(\Omega,\omega)$ such that*

(i) *$u_{m_k}(x)\to u(x)$, $m_k\to\infty$ a.e. on Ω;*

(ii) *$|u_{m_k}(x)|\leq\Phi(x)$ a.e. on Ω.*

Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [11]. □

Theorem 2.2 (The weighted Sobolev inequality). *Let Ω be an open bounded set in $\mathbb{R}^n$ and $\omega \in A_p$ $(1 < p < \infty)$. There exist the constants C_Ω and δ positive such that for all $u \in W_0^{1,p}(\Omega,\omega)$ and all k satisfying $1 \le k \le n/(n-1)+\delta$,*

$$\|u\|_{L^{kp}(\Omega,\omega)} \le C_\Omega \| \, |\nabla u| \, \|_{L^p(\Omega,\omega)}. \tag{2.2}$$

Proof. It suffices to prove the inequality for the functions $u \in C_0^\infty(\Omega)$ (see [10, Theorem 1.3]). To extend the estimates (2.2) to arbitrary $u \in W_0^{1,p}(\Omega,\omega)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega,\omega)$. Applying the estimates (2.2) to differences $u_{m_1} - u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^{kp}(\Omega,\omega)$. Consequently, the limit function u will lie in the desired spaces and satisfy (2.2). □

Lemma 2.3. *Let $1 < p < \infty$.*

(a) *There exists a constant $\alpha_p > 0$ such that*

$$\Big| \, |x|^{p-2}x - |y|^{p-2}y \Big| \le \alpha_p \, |x-y| (|x|+|y|)^{p-2},$$

for all $x, y \in \mathbb{R}^n$.

(b) *There exist two positive constants β_p, γ_p such that for every $x, y \in \mathbb{R}^n$,*

$$\beta_p \, (|x|+|y|)^{p-2} |x-y|^2 \le (|x|^{p-2}x - |y|^{p-2}y).(x-y) \le \gamma_p \, (|x|+|y|)^{p-2} |x-y|^2.$$

Proof. See [7], Proposition 17.2 and Proposition 17.3. □

Remark 2.4. If $2 \le q < p < \infty$ and $\dfrac{v}{\omega} \in L^r(\Omega,\omega)$ (where $r = p/(p-q)$), then there exists a constant $C_{p,q} = \|v/\omega\|_{L^r(\Omega,\omega)}^{1/q}$ such that

$$\|u\|_{L^q(\Omega,v)} \le C_{p,q} \|u\|_{L^p(\Omega,\omega)}.$$

In fact, by Hölder's inequality $(1/r + q/p = (p-q)/p + q/p = 1)$,

$$\begin{aligned} \|u\|_{L^q(\Omega,v)}^q &= \int_\Omega |u|^q \, v \, dx = \int_\Omega |u|^q \frac{v}{\omega} \, \omega \, dx \\ &\le \left(\int_\Omega |u|^p \omega \, dx \right)^{q/p} \left(\int_\Omega (v/\omega)^r \, \omega \, dx \right)^{1/r} \\ &= \|u\|_{L^p(\Omega,\omega)}^q \|v/\omega\|_{L^r(\Omega,\omega)}. \end{aligned}$$

Hence, $\|u\|_{L^q(\Omega,v)} \le C_{p,q} \|u\|_{L^p(\Omega,\omega)}$, with $C_{p,q} = \|v/\omega\|_{L^r(\Omega,\omega)}^{1/q}$.

Definition 2.3. We denote by $X = W^{2,p}(\Omega,\omega) \cap W_0^{1,p}(\Omega,\omega)$ with the norm

$$\|u\|_X = \left(\int_\Omega |\nabla u|^p \, \omega \, dx + \int_\Omega |\Delta u|^p \, \omega \, dx \right)^{1/p}.$$

Definition 2.4. We say that an element $u \in X = W^{2,p}(\Omega,\omega) \cap W_0^{1,p}(\Omega,,\omega)$ is a (weak) solution of problem (P) if

$$\begin{aligned} &\int_\Omega |\Delta u|^{p-2} \, \Delta u \, \Delta\varphi \, \omega \, dx + \int_\Omega |\Delta u|^{q-2} \, \Delta u \, \Delta\varphi \, v \, dx + \sum_{j=1}^n \int_\Omega \mathcal{A}_j(x,u,\nabla u) \, D_j\varphi \, \omega \, dx \\ &= \int_\Omega f_0 \, \varphi \, dx + \sum_{j=1}^n \int_\Omega f_j \, D_j\varphi \, dx, \end{aligned}$$

for all $\varphi \in X$.

3. Proof of Theorem 1.1

The basic idea is to reduce problem (P) to an operator equation $Au = T$ and apply the theorem below.

Theorem 3.1. *Let $A : X \to X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Then the following assertions hold:*
(a) *for each $T \in X^*$, the equation $Au = T$ has a solution $u \in X$;*
(b) *if the operator A is strictly monotone, then the equation $Au = T$ is uniquely solvable in X.*

Proof. See Theorem 26. A in [23]. □

To prove Theorem 1.1, we define $B, B_1, B_2, B_3 : X \times X \to \mathbb{R}$ and $T : X \to \mathbb{R}$ by

$$B(u,\varphi) = B_1(u,\varphi) + B_2(u,\varphi) + B_3(u,\varphi),$$

$$B_1(u,\varphi) = \sum_{j=1}^{n} \int_\Omega \mathcal{A}_j(x,u,\nabla u) D_j\varphi\,\omega\,dx = \int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi\,\omega\,dx$$

$$B_2(u,\varphi) = \int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx$$

$$B_3(u,\varphi) = \int_\Omega |\Delta u|^{q-2}\Delta u\,\Delta\varphi\,v\,dx$$

$$T(\varphi) = \int_\Omega f_0\,\varphi\,dx + \sum_{j=1}^{n}\int_\Omega f_j\,D_j\varphi\,dx.$$

Then $u \in X$ is a (weak) solution to problem (P) if

$$B(u,\varphi) = B_1(u,\varphi) + B_2(u,\varphi) + B_3(u,\varphi) = T(\varphi),$$

for all $\varphi \in X$.

Step 1. For $j = 1, \dots, n$, we define the operator $F_j : X \to L^{p'}(\Omega,\omega)$ as

$$(F_j u)(x) = \mathcal{A}_j(x, u(x), \nabla u(x)).$$

We now show that the operator F_j is bounded and continuous.

(i) Using (H4), we obtain

$$\begin{aligned}
\|F_j u\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega |F_j u(x)|^{p'}\omega\,dx \\
&= \int_\Omega |\mathcal{A}_j(x,u,\nabla u)|^{p'}\omega\,dx \\
&\le \int_\Omega \left(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\right)^{p'}\omega\,dx \\
&\le C_p \int_\Omega \left[(K_1^{p'} + h_1^{p'}|u|^p + h_2^{p'}|\nabla u|^p)\omega\right]dx \\
&= C_p\left[\int_\Omega K_1^{p'}\,\omega\,dx + \int_\Omega h_1^{p'}|u|^p\,\omega\,dx + \int_\Omega h_2^{p'}|\nabla u|^p\omega\,dx\right],
\end{aligned} \tag{3.1}$$

where the constant C_p depends only on p.

We have, by Theorem 2.2 (with $k = 1$),

$$\int_\Omega h_1^{p'}|u|^p\,\omega\,dx \le \|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega |u|^p\,\omega\,dx$$

$$\begin{aligned}&\le C_\Omega^p\, \|h_1\|_{L^\infty(\Omega)}^{p'} \int_\Omega |\nabla u|^p\,\omega\,dx\\ &\le C_\Omega^p\, \|h_1\|_{L^\infty(\Omega)}^{p'}\, \|u\|_X^p,\end{aligned}$$

and $\int_\Omega h_2^{p'}|\nabla u|^p\omega\,dx \le \|h_2\|_{L^\infty(\Omega)}^{p'}\int_\Omega |\nabla u|^p\,\omega\,dx \le \|h_2\|_{L^\infty(\Omega)}^{p'}\|u\|_X^p$. Therefore, in (3.1) we obtain

$$\|F_j u\|_{L^{p'}(\Omega,\omega)} \le C_p^{1/p'}\Big(\|K\|_{L^{p'}(\Omega,\omega)} + (C_\Omega^{p/p'}\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\,\|u\|_X^{p/p'}\Big).$$

(ii) Let $u_m \to u$ in X as $m\to\infty$. We need to show that $F_j u_m \to F_j u$ in $L^{p'}(\Omega,\omega)$. We will apply the Lebesgue Dominated Convergence Theorem. If $u_m\to u$ in X, then $|\nabla u_m|\to|\nabla u|$ in $L^p(\Omega,\omega)$. Using Theorem 2.1, there exist a subsequence $\{u_{m_k}\}$ and a function Φ_1 such that

$$D_j u_{m_k}(x)\to D_j u(x), \text{ a.e. in } \Omega,$$
$$|\nabla u_{m_k}(x)|\le\Phi_1(x), \text{ a.e. in } \Omega.$$

By Theorem 2.2, we obtain

$$\|u_{m_k}\|_{L^p(\Omega,\omega)} \le C_\Omega\, \|\,|\nabla u_{m_k}|\,\|_{L^p(\Omega,\omega)} \le C_\Omega\,\|\Phi_1\|_{L^p(\Omega,\omega)}.$$

Next, applying (H4), we obtain

$$\begin{aligned}\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega,\omega)}^{p'} &= \int_\Omega |F_j u_{m_k}(x) - F_j u(x)|^{p'}\omega\,dx\\
&= \int_\Omega |\mathcal{A}_j(x,u_{m_k},\nabla u_{m_k}) - \mathcal{A}_j(x,u,\nabla u)|^{p'}\,\omega\,dx\\
&\le C_p \int_\Omega \Big(|\mathcal{A}_j(x,u_{m_k},\nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x,u,\nabla u)|^{p'}\Big)\omega\,dx\\
&\le C_p\Bigg[\int_\Omega \Big(K_1 + h_1|u_{m_k}|^{p/p'} + h_2|\nabla u_{m_k}|^{p/p'}\Big)^{p'}\,\omega\,dx\\
&+ \int_\Omega \Big(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\Big)^{p'}\,\omega\,dx\Bigg]\\
&\le C_p\Bigg[\int_\Omega K_1^{p'}\,\omega\,dx + \|h_1\|_{L^\infty(\Omega)}^{p'}\int_\Omega |u_{m_k}|^p\,\omega\,dx + \|h_2\|_{L^\infty(\Omega)}^{p'}\int_\Omega |\nabla u_{m_k}|^p\,\omega\,dx\\
&+ \int_\Omega K_1^{p'}\,\omega\,dx + \|h_1\|_{L^\infty(\Omega)}^{p'}\int_\Omega |u|^p\,\omega\,dx + \|h_2\|_{L^\infty(\Omega)}^{p'}\int_\Omega |\nabla u|^p\,\omega\,dx\Bigg]\\
&\le 2\,C_p\Bigg[\int_\Omega K_1^{p'}\,\omega\,dx + C_\Omega^p\|h_1\|_{L^\infty(\Omega)}^{p'}\int_\Omega \Phi_1^p\,\omega\,dx + \|h_2\|_{L^\infty(\Omega)}^{p'}\int_\Omega \Phi_1^p\,\omega\,dx\Bigg]\\
&= 2\,C_p\Big[\|K_1\|_{L^{p'}(\Omega,\omega)}^{p'} + \big(C_\Omega^p\|h_1\|_{L^\infty(\Omega)}^{p'} + \|h_2\|_{L^\infty(\Omega)}^{p'}\big)\|\Phi_1\|_{L^p(\Omega,\omega)}^p\Big].\end{aligned}$$

By condition (H1), we have

$$F_j u_{m_k}(x) = \mathcal{A}_j(x,u_{m_k}(x),\nabla u_{m_k}(x))\to\mathcal{A}_j(x,u(x),\nabla u(x)) = F_j u(x),$$

as $m_k\to+\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain

$$\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega,\omega)}\to 0,$$

that is,

$$F_j u_{m_k}\to F_j u \quad\text{in}\quad L^{p'}(\Omega,\omega).$$

We conclude from the Convergence Principle in Banach spaces (see [22, Proposition 10.13]) that

$$F_j u_m \to F_j u \quad \text{in} \quad L^{p'}(\Omega, \omega). \tag{3.2}$$

Step 2. We define the operator $G_1 : X \to L^{p'}(\Omega, \omega)$ by

$$(G_1 u)(x) = |\Delta u(x)|^{p-2}\, \Delta u(x).$$

This operator is continuous and bounded. In fact,
(i) we have

$$\begin{aligned}
\|G_1 u\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega \big|\, |\Delta u|^{p-2}\, \Delta u\big|^{p'}\, \omega\, dx \\
&= \int_\Omega |\Delta u|^{(p-1)\,p'}\, \omega\, dx \\
&= \int_\Omega |\Delta u|^{p}\, \omega\, dx \\
&\le \|u\|^p_X.
\end{aligned}$$

Hence, $\|G_1 u\|_{L^{p'}(\Omega,\omega)} \le \|u\|^{p-1}_X$.

(ii) If $u_m \to u$ in X, then $\Delta u_m \to \Delta u$ in $L^p(\Omega, \omega)$. By Theorem 2.1, there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_2 \in L^p(\Omega, \omega)$ such that

$$\Delta u_{m_k}(x) \to \Delta u(x), \ \text{ a.e. in } \Omega, \tag{3.3}$$

$$|\Delta u_{m_k}(x)| \le \Phi_2(x), \ \text{ a.e. in } \Omega. \tag{3.4}$$

Hence, using Lemma 2.3 (a), $\theta = \dfrac{p}{p'} = p-1$ and $\theta' = \dfrac{(p-2)}{(p-1)}$, we obtain (since $2 \le q < p < \infty$),

$$\begin{aligned}
&\|G_1 u_{m_k} - G_1 u\|^{p'}_{L^{p'}(\Omega,\omega)} = \int_\Omega |\, G_1 u_{m_k} - G_1 u|^{p'}\, \omega\, dx \\
&= \int_\Omega \Big|\, |\Delta u_{m_k}|^{p-2}\, \Delta u_{m_k} - |\, \Delta u|^{p-2}\, \Delta u\Big|^{p'}\, \omega\, dx \\
&\le \int_\Omega \Big[\alpha_p\, |\Delta u_{m_k} - \Delta u|\, (\, |\Delta u_{m_k}| + |\Delta u|)^{(p-2)}\Big]^{p'}\, \omega\, dx \\
&\le \alpha_p^{p'} \int_\Omega |\, \Delta u_{m_k} - \Delta u|^{p'}\, (2\, \Phi_2)^{(p-2)\,p'}\, \omega\, dx \\
&\le 2^{(p-2)p'} \alpha_p^{p'} \left(\int_\Omega |\Delta u_{m_k} - \Delta u|^{p'\theta} \omega\, dx\right)^{1/\theta} \left(\int_\Omega \Phi_2^{(p-2)p'\theta'}\, \omega\, dx\right)^{1/\theta'} \\
&\le \alpha_p^{p'}\, 2^{(p-2)p'} \left(\int_\Omega |\Delta u_{m_k} - \Delta u|^{p}\, \omega\, dx\right)^{p'/p} \left(\int_\Omega \Phi_2^{p}\, \omega\, dx\right)^{(p-2)/(p-1)} \\
&\le \alpha_p^{p'}\, 2^{(p-2)\,p'}\, \|u_{m_k} - u\|^{p'}_X\, \|\Phi_2\|^{(p-2)p'}_{L^p(\Omega,\omega)},
\end{aligned}$$

since $(p-2)p'\theta' = (p-2)\dfrac{p}{(p-1)}\dfrac{(p-1)}{(p-2)} = p$ if $p \ne 2$. Then

$$\|G_1 u_{m_k} - G_1 u\|_{L^{p'}(\Omega,\omega)} \le 2^{(p-2)} \alpha_p\, \|\Phi_2\|^{p-2}_{L^p(\Omega,\omega)} \|u_{m_k} - u\|_X.$$

Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain (as $m_k \to \infty$)

$$\|G_1 u_{m_k} - G_1 u\|_{L^{p'}(\Omega,\omega)} \to 0,$$

that is, $G_1 u_{m_k} \to G_1 u$ in $L^{p'}(\Omega,\omega)$. By the Convergence Principle in Banach spaces, we have

$$G_1 u_m \to G_1 u \text{ in } L^{p'}(\Omega,\omega). \tag{3.5}$$

Step 3. We define the operator $G_2 : X \to L^s(\Omega,\omega)$, where $s = p/(q-1)$, by

$$(G_2 u)(x) = |\Delta u(x)|^{q-2}\Delta u(x).$$

We also have that the operator G_2 is continuous and bounded. In fact,
(i) we have

$$\begin{aligned}
\|G_2 u\|^s_{L^s(\Omega,\omega)} &= \int_\Omega \big||\Delta u|^{q-2}\Delta u\big|^s \,\omega\,dx \\
&= \int_\Omega |\Delta u|^{(q-1)\,s}\,\omega\,dx \\
&= \int_\Omega |\Delta u|^p\,\omega\,dx \\
&\leq \|u\|^p_X,
\end{aligned}$$

and $\|G_2 u\|_{L^s(\Omega,\omega)} \leq \|u\|^{q-1}_X$.

(ii) If $u_m \to u$ in X, then $\Delta u_m \to \Delta u$ in $L^p(\Omega,\omega)$. If $2 < q < p < \infty$, by (3.3), (3.4) and Lemma 2.3(a), we have

$$\begin{aligned}
\|G_2 u_{m_k} - G_2 u\|^s_{L^s(\Omega,\omega)} &= \int_\Omega \Big||\Delta u_{m_k}|^{q-2}\,\Delta u_{m_k} - |\Delta u|^{q-2}\Delta u\Big|^s \omega\,dx \\
&\leq \int_\Omega \Big[\alpha_q |\Delta u_{m_k} - \Delta u| \Big(|\Delta u_{m_k}| + |\Delta u|\Big)^{q-2}\Big]^s \omega\,dx \\
&= \alpha_q^s \int_\Omega |\Delta u_{m_k} - \Delta u|^s \Big(|\Delta u_{m_k}| + |\Delta u|\Big)^{(q-2)s} \omega\,dx \\
&\leq 2^{(q-2)s}\alpha_q^s \int_\Omega |\Delta u_{m_k} - \Delta u|^s\, \Phi_2^{(q-2)s}\,\omega\,dx.
\end{aligned} \tag{3.6}$$

For $\delta = q-1$ and $\delta' = (q-1)/(q-2)$, in (3.6) we have

$$\begin{aligned}
&\|G_2 u_{m_k} - G_2 u\|^s_{L^s(\Omega,\omega)} \\
&\leq 2^{(q-2)s}\alpha_q^s \int_\Omega |\Delta u_{m_k} - \Delta u|^s\,\Phi_2^{(q-2)s}\,\omega\,dx \\
&\leq 2^{(q-2)s}\alpha_q^s \bigg(\int_\Omega |\Delta u_{m_k} - \Delta u|^{s\,\delta}\omega\,dx\bigg)^{1/\delta} \bigg(\int_\Omega \Phi_2^{(q-2)s\,\delta'}\,\omega\,dx\bigg)^{1/\delta'} \\
&= 2^{(q-2)s}\alpha_q^s \bigg(\int_\Omega |\Delta u_{m_k} - \Delta u|^p\,\omega\,dx\bigg)^{1/(q-1)} \bigg(\int_\Omega \Phi_2^p\,\omega\,dx\bigg)^{1/\delta'} \\
&\leq 2^{(q-2)s}\,\alpha_q^s\,\|u_{m_k} - u\|_X^{p/(q-1)}\,\|\Phi_2\|^{p/\delta'}_{L^p(\Omega,\omega)}.
\end{aligned}$$

Hence, $\|G_2 u_{m_k} - G_2 u\|_{L^s(\Omega,\omega} \leq 2^{(q-2)}\,\alpha_q\,\|u_{m_k} - u\|_X\,\|\Phi_2\|^{(q-2)}_{L^p(\Omega,\omega)}$.

In the case $2 = q < p < \infty$, we have $(G_2 u)(x) = \Delta u(x)$ and $s = p$. Hence,

$$\begin{aligned}
&\|G_2 u\|_{L^p(\Omega,\omega)} \leq \|u\|_X, \\
&\|G_2 u_{m_k} - G_2 u\|_{L^p(\Omega,\omega)} \leq \|u_{m_k} - u\|_X.
\end{aligned}$$

Therefore, for $2 \le q < p < \infty$, by the Lebesgue Dominated Convergence Theorem, we obtain (as $m_k \to \infty$)

$$\|G_2 u_{m_k} - G_2 u\|_{L^s(\Omega,\omega)} \to 0,$$

that is, $G_2 u_{m_k} \to G_2 u$ in $L^s(\Omega,\omega)$. By the Convergence Principle in Banach spaces, we have

$$G_2 u_m \to G_2 u \quad \text{in} \quad L^s(\Omega,\omega). \tag{3.7}$$

Step 4. Since $\dfrac{f_j}{\omega} \in L^{p'}(\Omega,\omega)$ $(j = 0,1,\dots,n)$, therefore $T \in [W_0^{1,p}(\Omega,\omega)]^* \subset X^*$. Moreover, by Theorem 2.2, we have

$$\begin{aligned}
|T(\varphi)| &\le \int_\Omega |f_0||\varphi|\,dx + \sum_{j=1}^n \int_\Omega |f_j||D_j\varphi|\,dx \\
&= \int_\Omega \frac{|f_0|}{\omega}|\varphi|\omega\,dx + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\omega}|D_j\varphi|\,\omega\,dx \\
&\le \|f_0/\omega\|_{L^{p'}(\Omega,\omega)}\|\varphi\|_{L^p(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\|D_j\varphi\|_{L^p(\Omega,\omega)} \\
&\le C_\Omega\,\|f_0/\omega\|_{L^{p'}(\Omega,\omega)}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} + \left(\sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\right)\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} \\
&\le \left(C_\Omega\,\|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\right)\|\varphi\|_X.
\end{aligned}$$

Moreover, we also have

$$\begin{aligned}
|B(u,\varphi)| &\le |B_1(u,\varphi)| + |B_2(u,\varphi)| + |B_3(u,\varphi)| \\
&\le \sum_{j=1}^n \int_\Omega |\mathcal{A}_j(x,u,\nabla u)||D_j\varphi|\,\omega\,dx + \int_\Omega |\,\Delta u|^{p-2}\,|\,\Delta u|\,|\,\Delta\varphi|\,\omega\,dx \\
&\quad + \int_\Omega |\Delta u|^{q-2}|\Delta u||\Delta\varphi|\,v\,dx.
\end{aligned} \tag{3.8}$$

In (3.8), by (H4), we have

$$\begin{aligned}
&\int_\Omega |\mathcal{A}(x,u,\nabla u)|\,|\nabla\varphi|\,\omega\,dx \le \int_\Omega \left(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\right)|\nabla\varphi|\,\omega\,dx \\
&\le \|K_1\|_{L^{p'}(\Omega,\omega)}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} + \|h_1\|_{L^\infty(\Omega)}\|u\|_{L^p(\Omega,\omega)}^{p/p'}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} \\
&+\ \|h_2\|_{L^\infty(\Omega)}\|\,|\nabla u|\,\|_{L^p(\Omega,\omega)}^{p/p'}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega)} \\
&\le \left(\|K_1\|_{L^{p'}(\Omega,\omega)} + (C_\Omega^{p/p'}\,\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|_X^{p/p'}\right)\|\varphi\|_X,
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega |\,\Delta u|^{p-2}\,|\,\Delta u|\,|\,\Delta\,\varphi|\,\omega\,dx &= \int_\Omega |\,\Delta u|^{p-1}\,|\,\Delta\,\varphi|\,\omega\,dx \\
&\le \left(\int_\Omega |\,\Delta u|^p\,\omega\,dx\right)^{1/p'}\left(\int_\Omega |\,\Delta\varphi|^p\,\omega\,dx\right)^{1/p} \\
&\le \|u\|_X^{p/p'}\,\|\varphi\|_X,
\end{aligned}$$

and since $s = p/(q-1)$, $r = p/(p-q)$ and $\dfrac{1}{s} + \dfrac{1}{r} + \dfrac{1}{p} = 1$, by the generalized Hölder inequality, we obtain

$$\begin{aligned}
&\int_\Omega |\Delta u|^{q-2}\,|\Delta u|\,|\Delta\varphi|\,v\,dx = \int_\Omega |\Delta u|^{q-1}\,|\Delta\varphi|\,\frac{v}{\omega}\,\omega\,dx\\
&\leq \bigg(\int_\Omega |\Delta u|^{(q-1)s}\,\omega\,dx\bigg)^{1/s}\bigg(\int_\Omega |\Delta\varphi|^p\,\omega\,dx\bigg)^{1/p}\bigg(\int_\Omega \big(\frac{v}{\omega}\big)^r\,\omega\,dx\bigg)^{1/r}\\
&= \bigg(\int_\Omega |\Delta u|^p\,\omega\,dx\bigg)^{(q-1)/p}\bigg(\int_\Omega |\Delta\varphi|^p\,\omega\,dx\bigg)^{1/p}\bigg(\int_\Omega \big(\frac{v}{\omega}\big)^r\,\omega\,dx\bigg)^{1/r}\\
&\leq \|u\|_X^{(q-1)}\,\|\varphi\|_X\,\|v/\omega\|_{L^r(\Omega,\omega)}.
\end{aligned}$$

Hence, in (3.8), for all $u, \varphi \in X$, we obtain

$$\begin{aligned}
&|B(u,\varphi)|\\
&\leq \Big[\|K_1\|_{L^{p'}(\Omega,\omega)} + C_\Omega^{p/p'}\,\|h_1\|_{L^\infty(\Omega)}\|u\|_X^{p/p'} + \|h_2\|_{L^\infty(\Omega,\omega)}\|u\|_X^{p/p'} + \|u\|_X^{p/p'}\\
&+ \|v/\omega\|_{L^r(\Omega,\omega)}\,\|u\|_X^{q-1}\Big]\|\varphi\|_X.
\end{aligned}$$

Since $B(u,.)$ is linear, for each $u \in X$, there exists a linear and continuous functional on X denoted by Au such that $\langle Au, \varphi\rangle = B(u,\varphi)$, for all $u,\ \varphi \in X$ (here $\langle f, x\rangle$ denotes the value of the linear functional f at the point x). Moreover,

$$\begin{aligned}
\|Au\|_* \leq{}& \|K_1\|_{L^{p'}(\Omega,\omega)} + C_\Omega^{p/p'}\,\|h_1\|_{L^\infty(\Omega)}\|u\|_X^{p/p'} + \|h_2\|_{L^\infty(\Omega,\omega)}\|u\|_X^{p/p'} + \|u\|_X^{p/p'}\\
&+ \|v/\omega\|_{L^r(\Omega,\omega)}\|u\|_X^{q-1},
\end{aligned}$$

where $\|Au\|_* = \sup\{|\langle Au,\varphi\rangle| = |B(u,\varphi)| : \varphi \in X, \|\varphi\|_X = 1\}$ is the norm of the operator Au.

Hence, we obtain the operator

$$\begin{aligned}
A :{}& X \to X^*\\
& u \mapsto Au.
\end{aligned}$$

Consequently, problem (P) is equivalent to the operator equation

$$Au = T,\ u \in X.$$

Step 5. Using condition (H2) and Lemma 2.3 (b), we have

$$\begin{aligned}
&\langle Au_1 - Au_2, u_1 - u_2\rangle = B(u_1, u_1 - u_2) - B(u_2, u_1 - u_2)\\
&= \int_\Omega \mathcal{A}(x, u_1, \nabla u_1)\cdot\nabla(u_1 - u_2)\,\omega\,dx + \int_\Omega |\,\Delta u_1|^{p-2}\,\Delta u_1\,\Delta(u_1 - u_2)\,\omega\,dx\\
&+ \int_\Omega |\,\Delta u_1|^{q-2}\,\Delta u_1\,\Delta(u_1 - u_2)\,v\,dx\\
&- \int_\Omega \mathcal{A}(x, u_2, \nabla u_2)\cdot\nabla(u_1 - u_2)\,\omega\,dx - \int_\Omega |\,\Delta u_2|^{p-2}\,\Delta u_2\,\Delta(u_1 - u_2)\,\omega\,dx\\
&- \int_\Omega |\,\Delta u_2|^{q-2}\,\Delta u_2\,\Delta(u_1 - u_2)\,v\,dx\\
&= \int_\Omega \Big(\mathcal{A}(x, u_1, \nabla u_1) - \mathcal{A}(x, u_2, \nabla u_2)\Big)\cdot\nabla(u_1 - u_2)\,\omega\,dx\\
&+ \int_\Omega (|\,\Delta u_1|^{p-2}\,\Delta u_1 - |\,\Delta u_2|^{p-2}\,\Delta u_2)\,\Delta(u_1 - u_2)\,\omega\,dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega (|\,\Delta u_1|^{q-2}\,\Delta u_1 - |\,\Delta u_2|^{q-2}\,\Delta u_2)\,\Delta(u_1-u_2)\,v\,dx\\
\geq&\theta_1\int_\Omega |\nabla(u_1-u_2)|^p\,\omega\,dx + \beta_p\int_\Omega (|\,\Delta u_1| + |\,\Delta u_2|)^{p-2}\,|\Delta u_1-\Delta u_2|^2\,\omega\,dx\\
&+\beta_q\int_\Omega (|\,\Delta u_1| + |\,\Delta u_2|)^{q-2}\,|\Delta u_1-\Delta u_2|^2\,v\,dx\\
\geq&\theta_1\int_\Omega |\nabla(u_1-u_2)|^p\,\omega\,dx + \beta_p\int_\Omega (|\,\Delta u_1 - \Delta u_2|)^{p-2}\,|\Delta u_1-\Delta u_2|^2\,\omega\,dx\\
&+\beta_q\int_\Omega (|\,\Delta u_1 - \Delta u_2|)^{q-2}\,|\Delta u_1-\Delta u_2|^2\,v\,dx\\
=&\theta_1\int_\Omega |\nabla(u_1-u_2)|^p\,\omega\,dx + \beta_p\int_\Omega |\Delta u_1-\Delta u_2|^p\,\omega\,dx\\
&+\beta_q\int_\Omega |\Delta u_1-\Delta u_2|^q\,v\,dx\\
\geq&\theta_1\int_\Omega |\nabla(u_1-u_2)|^p\,\omega\,dx + \beta_p\int_\Omega |\Delta u_1-\Delta u_2|^p\,\omega\,dx\\
\geq&\theta\,\|u_1-u_2\|_X^p,
\end{aligned}
$$

where $\theta = \min\{\theta_1, \beta_p\}$. Therefore, the operator A is strongly monotone, and this implies that A is strictly monotone. Moreover, from (H3), we obtain

$$
\begin{aligned}
&\langle Au, u\rangle = B(u,u) = B_1(u,u) + B_2(u,u) + B_3(u,u)\\
&= \int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla u\,\omega\,dx + \int_\Omega |\,\Delta u|^{p-2}\,\Delta u\,\Delta u\,\omega\,dx \ + \int_\Omega |\,\Delta u|^{q-2}\,\Delta u\,\Delta u\,v\,dx\\
&\geq \int_\Omega \lambda_1|\nabla u|^p\,\omega\,dx + \int_\Omega |\,\Delta u|^p\,\omega\,dx + \int_\Omega |\,\Delta u|^q\,v\,dx\\
&\geq \int_\Omega \lambda_1|\nabla u|^p\,\omega\,dx + \int_\Omega |\,\Delta u|^p\,\omega\,dx\\
&\geq\gamma\,\|u\|_X^p,
\end{aligned}
$$

where $\gamma = \min\{\lambda_1, 1\}$. Hence, since $2 \leq q < p < \infty$, we have

$$
\frac{\langle Au, u\rangle}{\|u\|_X} \to +\infty, \text{ as } \|u\|_X \to +\infty,
$$

that is, A is coercive.

Step 6. We need to show that the operator A is continuous.

Let $u_m \to u$ in X as $m\to\infty$. We have

$$
\begin{aligned}
|B_1(u_m,\varphi) - B_1(u,\varphi)| &\leq \sum_{j=1}^n \int_\Omega |\mathcal{A}_j(x,u_m,\nabla u_m) - \mathcal{A}_j(x,u,\nabla u)||D_j\varphi|\,\omega\,dx\\
&= \sum_{j=1}^n \int_\Omega |F_j u_m - F_j u||D_j\varphi|\,\omega\,dx\\
&\leq \sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega)}\|D_j\varphi\|_{L^p(\Omega,\omega)}
\end{aligned}
$$

$$\leq \bigg(\sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega)} \bigg) \|\varphi\|_X,$$

and

$$\begin{aligned}
&|B_2(u_m,\varphi) - B_2(u,\varphi)| \\
&= \bigg| \int_\Omega |\,\Delta u_m|^{p-2} \Delta u_m \,\Delta\varphi\,\omega\,dx - \int_\Omega |\,\Delta u|^{p-2} \Delta u\,\Delta\varphi\,\omega\,dx \bigg| \\
&\leq \int_\Omega \bigg| |\,\Delta u_m|^{p-2}\,\Delta u_m - |\,\Delta u|^{p-2} \Delta u \bigg|\, |\,\Delta\varphi|\,\omega\,dx \\
&= \int_\Omega |G_1 u_m - G_1 u|\,|\Delta\varphi|\,\omega\,dx \\
&\leq \|G_1 u_m - G_1 u\|_{L^{p'}(\Omega,\omega)}\,\|\varphi\|_X,
\end{aligned}$$

and since $\dfrac{1}{s} + \dfrac{1}{r} + \dfrac{1}{p} = 1$ (remember that $s = p/(q-1)$ (see Step 3) and $r = p/(p-q)$, by (H5)),

$$\begin{aligned}
&|B_3(u_m,\varphi) - B_3(u,\varphi)| \\
&= \bigg| \int_\Omega |\Delta u_m|^{q-2} \Delta u_m \,\Delta\varphi\, v\,dx - \int_\Omega |\Delta u|^{q-2} \Delta u\,\Delta\varphi\, v\,dx \bigg| \\
&\leq \int_\Omega \bigg| |\Delta u_m|^{q-2}\,\Delta u_m - |\Delta u|^{q-2} \Delta u \bigg|\, |\Delta\varphi|\, v\,dx \\
&= \int_\Omega |G_2 u_m - G_2 u|\,|\Delta\varphi|\,\frac{v}{\omega}\,\omega\,dx \\
&\leq \bigg(\int_\Omega |G_2 u_m - G_2 u|^s\,\omega\,dx \bigg)^{1/s} \bigg(\int_\Omega |\Delta\varphi|^p\,\omega\,dx \bigg)^{1/p} \bigg(\int_\Omega \Big(\frac{v}{\omega}\Big)^r\,\omega\,dx \bigg)^{1/r} \\
&\leq \|G_2 u_m - G_2 u\|_{L^s(\Omega,\omega)}\,\|\varphi\|_X\,\|v/\omega\|_{L^r(\Omega,\omega)},
\end{aligned}$$

for all $\varphi \in X$. Hence,

$$\begin{aligned}
&|B(u_m,\varphi) - B(u,\varphi)| \\
&\leq |B_1(u_m,\varphi) - B_1(u,\varphi)| + |B_2(u_m,\varphi) - B_2(u,\varphi)| + |B_3(u_m,\varphi) - B_3(u,\varphi)| \\
&\leq \bigg[\sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega)} + \|G_1 u_m - G_1 u\|_{L^{p'}(\Omega,\omega)} \\
&\quad + \|G_2 u_m - G_2 u\|_{L^s(\Omega,\omega)} \|v/\omega\|_{L^r(\Omega,\omega)} \bigg] \|\varphi\|_X.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\|Au_m - Au\|_* \leq{}& \sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega)} + \|G_1 u_m - G_1 u\|_{L^{p'}(\Omega,\omega)} \\
&+ \|G_2 u_m - G_2 u\|_{L^s(\Omega,\omega)} \|v/\omega\|_{L^r(\Omega,\omega)}.
\end{aligned}$$

Therefore, using (3.2), (3.5) and (3.7), we have $\|Au_m - Au\|_* \to 0$ as $m \to +\infty$, that is, A is continuous and this implies that A is hemicontinuous.

Therefore, by Theorem 3.1, the operator equation $Au = T$ has a unique solution $u \in X$ and it is the unique solution for problem (P).

Step 7. In particular, by setting $\varphi = u$ in Definition 2.4, we have

$$B(u,u) = B_1(u,u) + B_2(u,u) + B_3(u,u) = T(u). \tag{3.9}$$

Hence, using (H3) and $\gamma = \min\{\lambda_1, 1\}$, we obtain

$$\begin{aligned}
&B_1(u,u) + B_2(u,u) + B_3(u,u)\\
&= \int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla u\,\omega\,dx + \int_\Omega |\,\Delta u|^{p-2}\,\Delta u\,\Delta u\,\omega\,dx + \int_\Omega |\,\Delta u|^{q-2}\,\Delta u\,\Delta u\,v\,dx\\
&\geq \int_\Omega \lambda_1\,|\nabla u|^p + \int_\Omega |\,\Delta u|^p\,\omega\,dx + \int_\Omega |\,\Delta u|^q\,v\,dx\\
&\geq \lambda_1 \int_\Omega |\nabla u|^p + \int_\Omega |\,\Delta u|^p\,\omega\,dx\\
&\geq \gamma\|u\|_X^p
\end{aligned}$$

and

$$\begin{aligned}
T(u) &= \int_\Omega f_0\,u\,dx + \sum_{j=1}^n \int_\Omega f_j\,D_j u\,dx\\
&\leq \|f_0/\omega\|_{L^{p'}(\Omega,\omega)}\|u\|_{L^p(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega)}\|D_j u\|_{L^p(\Omega,\omega)}\\
&\leq C_\Omega\,\|f_0/\omega\|_{L^{p'}(\Omega,\omega)}\|\,|\nabla u|\,\|_{L^p(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega)}\|\,|\nabla u|\,\|_{L^p(\Omega,\omega)}\\
&\leq \left(C_\Omega\,\|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega)}\right)\|u\|_X.
\end{aligned}$$

Therefore, in (3.9),

$$\gamma\,\|u\|_X^p \leq \left(C_\Omega\,\|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\right)\|u\|_X,$$

and we obtain

$$\|u\|_X \leq \frac{1}{\gamma^{p'/p}}\left(C_\Omega\,\|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\right)^{p'/p}.$$

Example. Let $\Omega = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 < 1\}$, the weight functions $\omega(x,y) = (x^2+y^2)^{-1/2}$ and $v(x,y) = (x^2+y^2)^{-1/3}$ ($\omega\in A_4$, $v\in A_3$, $p=4$ and $q=3$), and the function

$$\begin{aligned}
&\mathcal{A} : \Omega\times\times\mathbb{R}^2 \to \mathbb{R}^2\\
&\mathcal{A}((x,y),\eta,\xi) = h_2(x,y)\,|\xi|\,\xi,
\end{aligned}$$

where $h(x,y) = 2\,\mathrm{e}^{(x^2+y^2)}$. Let us consider the partial differential operator

$$Lu(x,y) = \Delta\big[\omega(x,y)\,|\Delta u|^2\,\Delta u + v(x,y)\,|\Delta u|\Delta u\big] - \mathrm{div}\,((x^2+y^2)^{-1/2}\,\mathcal{A}((x,y),u,\nabla u)).$$

Therefore, by Theorem 1.1, the problem

$$(P)\begin{cases} Lu(x) = \dfrac{\cos(xy)}{(x^2+y^2)} - \dfrac{\partial}{\partial x}\left(\dfrac{\sin(xy)}{(x^2+y^2)}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\sin(xy)}{(x^2+y^2)}\right), & \text{in } \Omega\\ u(x) = 0, \quad \text{on } \partial\Omega \end{cases}$$

has a unique solution $u\in X = W^{2,4}(\Omega,\omega)\cap W_0^{1,4}(\Omega,\omega)$.

Corollary 3.2. *Under the assumptions of Theorem* 1.1 *with* $2 \le q < p < \infty$, *if* $u_1, u_2 \in X$ *are solutions of*

$$(P_1)\begin{cases} Lu_1 = f_0 - \sum_{j=1}^{n} D_j f_j & \text{in } \Omega, \\ u_1(x) = \Delta u_1(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$(P_2)\begin{cases} Lu_2 = g_0 - \sum_{j=1}^{n} D_j g_j & \text{in } \Omega, \\ u_2(x) = \Delta u_2(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

then

$$\|u_1 - u_2\|_X \le \frac{1}{\alpha^{1/(p-1)}} \left(C_\Omega \left\| \frac{f_0 - g_0}{\omega} \right\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^{n} \left\| \frac{f_j - g_j}{\omega} \right\|_{L^{p'}(\Omega,\omega)} \right)^{1/(p-1)},$$

where α *is a positive constant and* C_Ω *is the constant in Theorem* 2.2.

Proof. If u_1 and u_2 are the solutions of $(P1)$ and $(P2)$, then for all $\varphi \in X$, we have

$$\begin{aligned} &\int_\Omega |\Delta u_1|^{p-2} \Delta u_1\, \Delta\varphi\, \omega\, dx + \int_\Omega |\Delta u_1|^{q-2} \Delta u_1\, \Delta\varphi\, v\, dx + \int_\Omega \mathcal{A}(x, u_1, \nabla u_1) \cdot \nabla\varphi\, \omega\, dx \\ &- \left(\int_\Omega |\Delta u_2|^{p-2} \Delta u_2\, \Delta\varphi\, \omega\, dx + \int_\Omega |\Delta u_2|^{q-2} \Delta u_2\, \Delta\varphi\, v\, dx + \int_\Omega \mathcal{A}(x, u_2, \nabla u_2) \cdot \nabla\varphi\, \omega\, dx \right) \\ &= \int_\Omega (f_0 - g_0)\, \varphi\, dx + \sum_{j=1}^{n} \int_\Omega (f_j - g_j) D_j \varphi\, dx. \end{aligned} \tag{3.10}$$

In particular, for $\varphi = u_1 - u_2$, we obtain in (3.10):

(i) By Lemma 2.3 (b) and since $2 \le q < p < \infty$, there exist two positive constants β_p and β_q such that

$$\begin{aligned} &\int_\Omega \left(|\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2 \right) \Delta(u_1 - u_2)\, \omega\, dx \\ &\ge \beta_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2| \Big)^{p-2} |\Delta u_1 - \Delta u_2|^2\, \omega\, dx \\ &\ge \beta_p \int_\Omega |\Delta u_1 - \Delta u_2|^{p-2} |\Delta u_1 - \Delta u_2|^2\, \omega\, dx = \beta_p \int_\Omega |\Delta(u_1 - u_2)|^p\, \omega\, dx, \end{aligned}$$

and, analogously,

$$\int_\Omega \left(|\Delta u_1|^{q-2} \Delta u_1 - |\Delta u_2|^{q-2} \Delta u_2 \right) \Delta(u_1 - u_2)\, v\, dx \ge \beta_q \int_\Omega |\Delta(u_1 - u_2)|^q\, v\, dx \ge 0.$$

(ii) By condition (H2)

$$\int_\Omega \Big(\mathcal{A}(x, u_1, \nabla u_1) - \mathcal{A}(x, u_2, \nabla u_2) \Big).\nabla(u_1 - u_2)\, \omega\, dx \ge \theta_1 \int_\Omega |\nabla u_1 - \nabla u_2|^p\, \omega\, dx.$$

(iii) By condition (H6) and Theorem 2.2, we also have

$$\begin{aligned} &\left| \int_\Omega (f_0 - g_0)\, (u_1 - u_2)\, dx + \sum_{j=1}^{n} \int_\Omega (f_j - g_j) D_j (u_1 - u_2)\, dx \right| \\ &\le C_\Omega \left\| \frac{f_0 - g_0}{\omega} \right\|_{L^{p'}(\Omega,\omega)} \|\nabla(u_1 - u_2)\|_{L^p(\Omega,\omega)} + \left(\sum_{j=1}^{n} \left\| \frac{f_j - g_j}{\omega} \right\|_{L^{p'}(\Omega,\omega)} \right) \|\nabla(u_1 - u_2)\|_{L^p(\Omega,\omega)} \end{aligned}$$

$$\le\left(C_\Omega\left\|\frac{f_0-g_0}{\omega}\right\|_{L^{p'}(\Omega,\omega)}+\sum_{j=1}^{n}\left\|\frac{f_j-g_j}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right)\|u_1-u_2\|_X.$$

Hence, with $\alpha=\min\{\beta_p,\theta_1\}$, we obtain

$$\begin{aligned}\alpha\,\|u_1-u_2\|_X^p&\le\beta_p\int_\Omega|\Delta(u_1-u_2)|^p\,\omega\,dx+\theta_1\int_\Omega|\nabla u_1-\nabla u_2)|^p\,\omega\,dx\\&\le\left(C_\Omega\left\|\frac{f_0-g_0}{\omega}\right\|_{L^{p'}(\Omega,\omega)}+\sum_{j=1}^{n}\left\|\frac{f_j-g_j}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right)\|u_1-u_2\|_X.\end{aligned}$$

Therefore, since $2\le q<p<\infty$,

$$\|u_1-u_2\|_X\le\frac{1}{\alpha^{1/(p-1)}}\left(C_\Omega\left\|\frac{f_0-g_0}{\omega}\right\|_{L^{p'}(\Omega,\omega)}+\sum_{j=1}^{n}\left\|\frac{f_j-g_j}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right)^{1/(p-1)}.\qquad\square$$

Corollary 3.3. *Assume $2\le q<p<\infty$. Let the assumptions of Theorem 1.1 be fulfilled, and let $\{f_{0m}\}$ and $\{f_{jm}\}$ $(j=1,\dots,n)$ be sequences of functions satisfying $\frac{f_{0m}}{\omega}\to\frac{f_0}{\omega}$ in $L^{p'}(\Omega,\omega)$ and $\frac{f_{jm}}{\omega}\to\frac{f_j}{\omega}$ in $L^{p'}(\Omega,\omega)$ as $m\to\infty$. If $u_m\in X$ is a solution of the problem*

$$(P_m)\begin{cases}Lu_m(x)=f_{0m}(x)-\sum_{j=1}^{n}D_jf_{jm} & \text{in }\ \Omega,\\ u_m(x)=\Delta u_m(x)=0 & \text{on }\ \partial\Omega,\end{cases}$$

then $u_m\to u$ in X and u is a solution of problem (P).

Proof. By Corollary 3.2, we have

$$\|u_m-u_k\|_X\le\frac{1}{\alpha^{1/(p-1)}}\left(C_\Omega\left\|\frac{f_{0m}-f_{0k}}{\omega}\right\|_{L^{p'}(\Omega,\omega)}+\sum_{j=1}^{n}\left\|\frac{f_{jm}-f_{jk}}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right)^{1/(p-1)}.$$

Therefore $\{u_m\}$ is a Cauchy sequence in X. Hence, there is $u\in X$ such that $u_m\to u$ in X. We find that u is a solution of problem (P). In fact, since u_m is a solution of (P_m), for all $\varphi\in X$ we have

$$\begin{aligned}&\int_\Omega|\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\,dx+\int_\Omega|\Delta u|^{q-2}\Delta u\,\Delta\varphi\,v\,dx+\int_\Omega\mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi\,\omega\,dx\\&=\int_\Omega\Big(|\Delta u|^{p-2}\Delta u-|\Delta u_m|^{p-2}\Delta u_m\Big)\Delta\varphi\,\omega\,dx+\int_\Omega\Big(|\Delta u|^{q-2}\Delta u-|\Delta u_m|^{q-2}\Delta u_m\Big)\Delta\varphi\,v\,dx\\&+\int_\Omega\Big(\mathcal{A}(x,u,\nabla u)-\mathcal{A}(x,u_m,\nabla u_m)\Big)\cdot\nabla\varphi\,\omega\,dx\\&+\int_\Omega|\Delta u_m|^{p-2}\Delta u_m\,\Delta\varphi\,\omega\,dx+\int_\Omega|\Delta u_m|^{q-2}\Delta u_m\,\Delta\varphi\,v\,dx+\int_\Omega\mathcal{A}(x,u_m,\nabla u_m)\cdot\nabla\varphi\,\omega\,dx\\&=I_1+I_2+I_3+\int_\Omega f_{0m}\varphi\,dx+\sum_{j=1}^{n}\int_\Omega f_{jm}D_j\varphi\,dx\\&=I_1+I_2+I_3+\int_\Omega f_0\,\varphi\,dx+\sum_{j=1}^{n}\int_\Omega f_jD_j\varphi\,dx\\&+\int_\Omega(f_{0m}-f_0)\,\varphi\,dx+\sum_{j=1}^{n}\int_\Omega(f_{jm}-f_j)D_j\varphi\,dx,\end{aligned}\tag{3.11}$$

where

$$I_1 = \int\limits_\Omega \Big(|\Delta u|^{p-2}\Delta u - |\Delta u_m|^{p-2}\Delta u_m\Big)\,\Delta\varphi\,\omega\,dx,$$

$$I_2 = \int\limits_\Omega \Big(|\Delta u|^{q-2}\Delta u - |\Delta u_m|^{q-2}\Delta u_m\Big)\Delta\varphi\,v\,dx,$$

$$I_3 = \int\limits_\Omega \Big(\mathcal{A}(x,u,\nabla u) - \mathcal{A}(x,u_m,\nabla u_m)\Big)\cdot\nabla\varphi\,\omega\,dx.$$

We find that:
(1) by Lemma 2.3(a), there exists $\alpha_p > 0$ such that

$$\begin{aligned}|I_1| &\leq \int\limits_\Omega \big||\Delta u|^{p-2}\Delta u - |\Delta u_m|^{p-2}\Delta u_m\big|\,|\Delta\varphi|\,\omega\,dx\\ &\leq \alpha_p\int\limits_\Omega |\Delta u - \Delta u_m|\,(|\Delta u| + |\Delta u_m|)^{p-2}\,|\Delta\varphi|\,\omega\,dx.\end{aligned}$$

Let $\delta = p/(p-2)$. Since $\dfrac{1}{p} + \dfrac{1}{p} + \dfrac{1}{\delta} = 1$, by the Generalized Hölder inequality we obtain

$$\begin{aligned}|I_1| &\leq \alpha_p\left(\int\limits_\Omega |\Delta u - \Delta u_m|^p\,\omega\,dx\right)^{1/p}\left(\int\limits_\Omega |\Delta\varphi|^p\,\omega\,dx\right)^{1/p}\left(\int\limits_\Omega (|\Delta u| + |\Delta u_m|)^{(p-2)\delta}\,\omega\,dx\right)^{1/\delta}\\ &\leq \alpha_p\|u-u_m\|_X\,\|\varphi\|_X\||\Delta u| + |\Delta u_m|\|^{(p-2)}_{L^p(\Omega,\omega)}.\end{aligned}$$

Now, since $u_m \to u$ in X, there exists a constant $M > 0$ such that $\|u_m\|_X \leq M$. Hence,

$$\||\Delta u| + |\Delta u_m|\|_{L^p(\Omega,\omega)} \leq \|u\|_X + \|u_m\|_X \leq 2M. \tag{3.12}$$

Therefore,

$$\begin{aligned}|I_1| &\leq \alpha_p\,(2M)^{p-2}\,\|u-u_m\|_X\,\|\varphi\|_X\\ &= C_1\,\|u-u_m\|_X\,\|\varphi\|_X,\end{aligned}$$

where $C_1 = \alpha_p(2M)^{p-2}$.
(2) By Lemma 2.3 (a) there exists a positive constant α_q such that

$$\begin{aligned}|I_2| &\leq \int\limits_\Omega \big||\Delta u|^{q-2}\Delta u - |\Delta u_m|^{q-2}\Delta u_m\big|\,|\Delta\varphi|\,v\,dx\\ &\leq \alpha_q\int\limits_\Omega |\Delta u - \Delta u_m|\,(|\Delta u| + |\Delta u_m|)^{q-2}\,|\Delta\varphi|\,v\,dx.\end{aligned}$$

Let $\varepsilon = q/(q-2)$ (if $2 < q < p < \infty$). Since $\dfrac{1}{q} + \dfrac{1}{q} + \dfrac{1}{\varepsilon} = 1$, by the Generalized Hölder inequality, we obtain

$$\begin{aligned}|I_2| &\leq \alpha_q\left(\int\limits_\Omega |\Delta u - \Delta u_m|^q\,v\,dx\right)^{1/q}\left(\int\limits_\Omega |\Delta\varphi|^q\,v\,dx\right)^{1/q}\left(\int\limits_\Omega (|\Delta u| + |\Delta u_m|)^{(q-2)\varepsilon}\,v\,dx\right)^{1/\varepsilon}\\ &= \alpha_q\,\|\Delta u - \Delta u_m\|_{L^q(\Omega,v)}\,\|\Delta\varphi\|_{L^q(\Omega,v)}\||\Delta u| + |\Delta u_m|\|^{q-2}_{L^q(\Omega,v)}.\end{aligned}$$

Now, by Remark 2.4 and (3.12), we have

$$\begin{aligned}|I_2| &\leq \alpha_q\,C_{p,q}\|\Delta u - \Delta u_m\|_{L^p(\Omega,\omega)}\,C_{p,q}\|\Delta\varphi\|_{L^p(\Omega,\omega)}C_{p,q}^{q-2}\||\Delta u| + |\Delta u_m|\|^{q-2}_{L^p(\Omega,\omega)}\\ &\leq \alpha_q\,C_{p,q}^q\|u-u_m\|_X\|\varphi\|_X\,(2M)^{q-2}\\ &= C_2\,\|u-u_m\|_X\,\|\varphi\|_X,\end{aligned}$$

where $C_2 = (2M)^{q-2}\alpha_q\, C_{p,q}^q$.

In case $q=2$, we have $I_2 = \int_\Omega (\,\Delta u - \Delta u_m)\,\Delta\varphi\, v\, dx$, and by Remark 2.4, we obtain

$$\begin{aligned}|I_2| \leq& \|\Delta u - \Delta u_m\|_{L^2(\Omega,v)}\|\Delta\varphi\|_{L^2(\Omega,v)}\\ \leq& C_{p,2}^2\|\Delta u - \Delta u_m\|_{L^p(\Omega,\omega)}\|\Delta\varphi\|_{L^p(\Omega,\omega)}\\ \leq& C_{p,2}^2\|u-u_m\|_X\|\varphi\|_X.\end{aligned}$$

By Step 1 (Theorem 1.1), we also have

$$\begin{aligned}|I_3| \leq& \sum_{j=1}^n\int_\Omega |\mathcal{A}_j(x,u,\nabla u) - \mathcal{A}_j(x,u_m,\nabla u_m)|\,|D_j\varphi|\,\omega\, dx\\ =& \sum_{j=1}^n\int_\Omega |F_j(u) - F_j(u_m)||D_j\varphi|\,\omega\, dx\\ \leq& \left(\sum_{j=1}^n\|F_j(u)-F_j(u_m)\|_{L^{p'}(\Omega,\omega)}\right)\|\nabla\varphi\|_{L^p(\Omega,\omega)}\\ \leq& \left(\sum_{j=1}^n\|F_j(u)-F_j(u_m)\|_{L^{p'}(\Omega,\omega)}\right)\|\varphi\|_X.\end{aligned}$$

Therefore, we have $I_1, I_2, I_3, \to 0$ as $m\to\infty$.

(3) We also have

$$\left|\int_\Omega (f_{0m} - f_0)\,\varphi\, dx + \sum_{j=1}^n\int_\Omega (f_{jm}-f_j)D_j\varphi\, dx\right|$$
$$\left(C_\Omega\left\|\frac{f_{0m}-f_0}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n\left\|\frac{f_{jm}-f_j}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right)\|\varphi\|_X \to 0,$$

as $m\to\infty$.

Therefore, in (3.11), as $m\to\infty$, we obtain

$$\begin{aligned}&\int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega\, dx + \int_\Omega |\Delta u|^{q-2}\Delta u\,\Delta\varphi\, v\, dx\\ &+ \int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi\,\omega\, dx\\ &= \int_\Omega f_0\,\varphi\, dx + \sum_{j=1}^n\int_\Omega f_j j D_j\varphi\, dx,\end{aligned}$$

i.e., u is a solution of problem (P). □

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NORM CONTINUITY AND COMPACTNESS PROPERTIES FOR SOME PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. In this work, we study the norm continuity and compactness properties to the solution operator for some partial functional integrodifferential equations. The results are established by using the resolvent operator theory suggested by Grimmer in [11].

1. Introduction

The purpose of this paper is to establish some properties of a solution operator for the following partial functional integrodifferential equations with a finite delay

$$\begin{cases} u'(t) = Au(t) + \int\limits_0^t B(t-s)\, u(s)ds + L(u_t) \quad \text{for} \quad t \geq 0, \\ u_0 = \varphi \in C = C\big([-r,0];X\big), \end{cases} \tag{1.1}$$

where $A : D(A) \to X$ is a closed linear operator on a Banach space X, for $t \geq 0$, $B(t)$ is a closed time-independent linear operator on X with domain $D(B) \supset D(A)$, L is a linear bounded operator from $C\big([-r,0];X\big)$ to X. $C([-r,0];X)$ is the Banach space of all continuous functions from $[-r,0]$ to X endowed with the uniform norm topology. For $u \in C\big([-r,+\infty),X\big)$ and for every $t \geq 0$, u_t denotes the history function of C defined by

$$u_t(\theta) = u(t+\theta) \quad \text{for} \quad \theta \in [-r,0].$$

The theory of partial functional integrodifferential equations has been emerging as an important area of investigation in recent years. Many physical and biological models are represented by this class of equations. As a model, one may take the equation arising in the study of heat conductivity in materials with memory [14],

$$\begin{cases} \dfrac{\partial}{\partial t}\, w(t,\xi) = \dfrac{\partial^2}{\partial \xi^2}\, w(t,\xi) + \int\limits_0^t h(t-s)\, \dfrac{\partial^2}{\partial \xi^2}\, w(s,\xi)ds \\ \qquad\qquad + \int\limits_{-r}^0 F\big(w(t+\theta,\xi)\big)d\theta \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \xi \in [0,\pi], \\ w(t,0) = w(t,\pi) = 0 \qquad\qquad \text{for} \quad t \geq 0, \\ w(\theta,\xi) = w_0(\theta,\xi) \qquad\qquad \text{for} \quad \theta \in [-r,0] \quad \text{and} \quad \xi \in [0,\pi], \end{cases} \tag{1.2}$$

where r is a positive number, F, h are two continuous functions and w_0 is a given initial function. Other models arising in viscoelasticity and reaction diffusion problems are given in [4,5,12].

In [15], the authors considered equation (1.1) for $B = 0$. They established some results concerning the existence and stability, and the solutions are studied as a semigroup operator on $C([-r,0];X)$. Due to the importance of this semigroup operator, able to give some information on the stability and growth rate of solutions, many authors studied its properties. The works of Hale [13] for ordinary linear functional differential equations, Webb [16] for ordinary nonlinear functional differential equations, Wu

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[17] and Adimy et al. [1] for partial functional differential equations are worth mentioning. Recently, in [8], the authors established many results on the existence of solutions for equation (1.1). The solutions are studied via the variation of constant formula by using resolvent operators. Similarly, many works have been established in this direction; we refer to [9, 10]. However, the properties of the solution operator for equation (1.1) is an untreated topic and this is the main motivation of the present paper.

In this paper we use the theory of resolvent operators as developed by Grimmer [11] to define the solution operator $(V(t))_{t\geq 0}$ on $C([-r,0];X)$ which solves equation (1.1) in a mild sense (see Section 3). We then show the norm continuity and compactness properties of the solution operator. Our approach and results generalize some results for differential equations ($B = 0$). See, for example, [13, 15, 17].

2. Resolvent operators

Throughout this work, we make the following assumptions:

(**H1**) A is a closed densely defined linear operator in a Banach space $\big(X, |\cdot|\big)$. Since A is closed, $D(A)$ equipped with the graph norm $\|x\| := |Ax| + |x|$ is a Banach space which is denoted by $\big(Y, \|\cdot\|\big)$.

(**H2**) $(B(t))_{t\geq 0}$ is a family of linear operators on X such that $B(t)$ is continuous from Y into X for almost all $t \geq 0$. Moreover, there is a locally integrable function $b : \mathbb{R}^+ \to \mathbb{R}^+$ so that $B(t)y$ is measurable and $\big|B(t)y\big| \leq b(t)\|y\|$ for all $y \in Y$ and $t \geq 0$.

(**H3**) For any $y \in Y$, the map $t \to B(t)y$ belongs to $W^{1,1}_{loc}(\mathbb{R}^+, X)$ and

$$\left|\frac{d}{dt} B(t)y\right| \leq b(t)\|y\| \quad \text{for} \quad y \in Y \quad \text{and a.e.} \quad t \in \mathbb{R}^+.$$

(**H4**) L is a linear bounded operator from $C\big([-r,0];X\big)$ to X.

Now, we consider the following integrodifferential equation

$$\begin{cases} y'(t) = Ay(t) + \displaystyle\int_0^t B(t-s)\,y(s)\,ds \quad \text{for} \quad t \geq 0 \\ y(0) = y_0 \in X. \end{cases} \tag{2.3}$$

Definition 2.1 ([11]). A resolvent operator for equation (2.3) is a bounded linear operator valued function $R(t) \in \mathcal{L}(X)$ for $t \geq 0$ having the following properties:

(a) $R(0) = I$ and $|R(t)| \leq Me^{\beta t}$ for some constants M and β.

(b) For each $x \in X$, $R(t)x$ is strongly continuous for $t \geq 0$.

(c) $R(t) \in \mathcal{L}(Y)$ for $t \geq 0$. For $x \in Y$, $R(\cdot)x \in C^1(\mathbb{R}^+;X) \cap C(\mathbb{R}^+;Y)$ and

$$\begin{aligned} R'(t)x &= A\,R(t)\,x + \int_0^t B(t-s)\,R(s)\,x\,ds \\ &= R(t)\,A\,x + \int_0^t R(t-s)\,B(s)\,x\,ds \quad \text{for} \quad t \geq 0. \end{aligned}$$

For the properties of resolvent operators, we refer the reader to the papers [3, 11]. The following theorem gives an existence result of the resolvent operator for equation (2.3).

Theorem 2.2 ([6]). *Assume that* (**H1**)–(**H3**) *hold. Then equation* (2.3) *admits a resolvent operator if and only if A generates a C_0-semigroup.*

From now, we suppose that

(**H5**) A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on the Banach space X.

Remark 2.3. It is worth noting that assumption (**H5**) ensures the existence of a resolvent operator for equation (2.3). This is a direct consequence of Theorem 2.2.

Lemma 2.4 ([6]). *Assume that* (**H1**)–(**H3**) *and* (**H5**) *hold. Then for all* $a > 0$ *there exists a constant* $H = H(a)$ *such that*

$$\left|R(s+h) - R(h)\,R(s)\right| \le Hh \quad \textit{for} \quad 0 < h \le s \le a.$$

Theorem 2.5 ([6]). *Assume that* (**H1**)–(**H3**) *and* (**H5**) *hold. Let* $T(t)$ *be compact for* $t > 0$. *Then the corresponding resolvent operator* $R(t)$ *of equation* (2.3) *is also compact for* $t > 0$.

The following theorem provides the necessary and sufficient conditions for the resolvent operator to be continuous in the uniform operator topology.

Theorem 2.6 ([7]). *Assume that* (**H1**)–(**H3**) *and* (**H5**) *are satisfied. Then* $T(t)$ *is norm continuous* (*or continuous in the uniform operator topology*) *for* $t > 0$ *if and only if the corresponding resolvent operator* $R(t)$ *of equation* (2.3) *is norm continuous for* $t > 0$.

3. Main Results

We state some relevant definitions and results taken from [8] for the case where L is autonomous.

Definition 3.1 ([8]). A continuous function $u : [-r, +\infty) \to X$ is said to be a mild solution of equation (1.1) if $u_0 = \varphi$ and

$$u(t) = R(t)\varphi(0) + \int_0^t R(t-s)L(u_s)ds \quad \text{for} \quad t \ge 0.$$

Theorem 3.2 ([8]). *Assume that* (**H1**)–(**H5**) *hold. Then for each* $\varphi \in C$, *equation* (1.1) *has a mild solution* $u(\varphi)(\cdot)$ *on* $[-r, +\infty)$ *which is given by*

$$u(\varphi)(t) = \begin{cases} u(\varphi)(t) = R(t)\,\varphi(0) + \displaystyle\int_0^t R(t-s)\,L\big(u_s(\varphi)\big)ds & \textit{for } t \ge 0, \\ u_0(\varphi)(t) = \varphi(t) & \textit{for } t \in [-r, 0]. \end{cases} \tag{3.4}$$

For each $t \ge 0$ define the solution operator $V(t) : C \to C$ by

$$V(t)\varphi = u_t(\varphi).$$

Proposition 3.3. *The family* $\big(V(t)\big)_{t\ge 0}$ *satisfies the translation property*

$$\big(V(t)\varphi\big)(\theta) = \begin{cases} \big(V(t+\theta)\varphi\big)(0) & \textit{for} \quad t+\theta \ge 0, \\ \varphi(t+\theta) & \textit{for} \quad t+\theta \le 0, \end{cases}$$

for $t \ge 0$, $\theta \in [-r, 0]$ *and* $\varphi \in C$.

Proof. For $t \ge 0$ and $\theta \in [-r, 0]$, it follows from (3.4) that

$$u_t(\varphi)(\theta) = \begin{cases} u(\varphi)(t+\theta) = R(t+\theta)\,\varphi(0) + \displaystyle\int_0^{t+\theta} R(t+\theta-s)\,L\big(u_s(\varphi)\big)ds \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad t+\theta \ge 0, \\ u_0(\varphi)(t+\theta) = \varphi(t+\theta) \qquad\qquad\qquad\quad \text{for} \quad t+\theta \le 0. \end{cases}$$

Hence, for $\varphi \in C$, we have

$$\big(V(t)\varphi\big)(\theta) = \big(u_t(\varphi)\big)(\theta) = \begin{cases} \big(V(t+\theta)\varphi\big)(0) & \text{for} \quad t+\theta \ge 0, \\ \varphi(t+\theta) & \text{for} \quad t+\theta \le 0. \end{cases}$$

The proof of the above Proposition is completed. □

Let $B = \big\{\varphi \in C : |\varphi| \le 1\big\}$. Take $N \ge 0$ such that $\big|L(V(s)\varphi)\big| \le N$ for all $s \ge 0$ and $\varphi \in B$.

3.1. **Norm continuity of** $\big(V(t)\big)_{t\geq 0}$**.** To establish the norm continuity of the solution operator, we need the following

Lemma 3.4. *The map*

$$\begin{aligned}&\mathbb{R}^+ \times C \to C\\ &(t,\varphi) \to V(t)\,\varphi \quad \textit{is locally bounded with respect to } t \textit{ and } \varphi.\end{aligned}$$

Proof. Let $0 \leq t \leq a$ and $\varphi \in B$. Then

$$\big|V(t)\,\varphi\big| = \sup_{-r\leq\theta\leq 0}\big|(V(t)\varphi)(\theta)\big|.$$

For $t+\theta \leq 0$, we have

$$\big|V(t)\varphi\big| = \sup_{-r\leq\theta\leq -t}\big|\varphi(t+\theta)\big| \leq \sup_{-r\leq\theta\leq 0}\big|\varphi(\theta)\big| \leq |\varphi|.$$

This implies that

$$\sup_{0\leq t\leq a,\,|\varphi|\leq 1}\big|V(t)\varphi\big| \leq 1.$$

For $t+\theta \geq 0$, we have

$$\begin{aligned}\big|(V(t)\varphi)(\theta)\big| &\leq \big|R(t+\theta)\,\varphi(0)\big| + \Big|\int_0^{t+\theta} R(t+\theta-s)\,L\big(V(s)\varphi\big)ds\Big|\\ &\leq M_a|\varphi| + M_a N\int_0^t \big|V(s)\varphi\big|ds,\end{aligned}$$

where $M_a = \sup\limits_{0\leq s\leq a}|R(s)|$. Thus

$$\big|V(t)\varphi\big| \leq M_a|\varphi| + M_a N\int_0^t \big|V(s)\varphi\big|ds.$$

By Gronwall's Lemma, we deduce that

$$\big|V(t)\varphi\big| \leq M_a e^{M_a N}|\varphi|.$$

Consequently,

$$\sup_{0\leq t\leq a,\,|\varphi|\leq 1}\big|V(t)\varphi\big| \leq M_a e^{M_a N},$$

and the proof of the lemma is completed. □

Theorem 3.5. *Assume that* (**H1**)–(**H5**) are satisfied. If $t \to T(t)$ is norm continuous for $t > 0$. Then the solution operator $t \to V(t)$ is norm continuous on $t > r$.

Proof. Let $t > r$ and $\theta \in [-r, 0]$. Then

$$\big|V(t+h) - V(t)\big| = \sup_{|\varphi|\leq 1}\big|V(t+h)\varphi - V(t)\varphi\big|.$$

For $h < 0$ to be sufficiently small, we have

$$\begin{aligned}\big|\big(V(t+h)\varphi\big)(\theta) - \big(V(t)\varphi\big)(\theta)\big| =& \big|R(t+h+\theta)\varphi(0) - R(t+\theta)\varphi(0)\big|\\ \leq& \sup_{t-r\leq s\leq t}\big|R(s+h) - R(s)\big|\,\big|\varphi(0)\big|.\end{aligned}$$

Let us now fix t, h such that $0 < a < t - r < t + h < b$, then

$$\begin{aligned}\sup_{t-r\le s\le t}\big|R(s+h)-R(s)\big|\,\big|\varphi(0)\big| &\le \sup_{a\le s\le b}\big|R(s+h)-R(s)\big|\,\big|\varphi(0)\big|\\ &\le \sup_{|\varphi|\le 1}\sup_{a\le s\le b}\big|R(s+h)-R(s)\big|\,\big|\varphi(0)\big|\\ &\le \sup_{a\le s\le b}\big|R(s+h)-R(s)\big|.\end{aligned}$$

Theorem 2.6 implies that

$$\big|V(t+h)\varphi - V(t)\varphi\big|$$

tends to 0 as $h \to 0$ uniformly in $\varphi \in B$. Let $h > 0$ be such that $t + h - r > 0$. Then

$$\begin{aligned}\big(V(t+h)\varphi\big)(\theta)-\big(V(t)\varphi\big)(\theta) &= \int_0^{t+\theta}\big(R(t+\theta+h-s)-R(t+\theta-s)\big)L\big(V(s)\varphi\big)ds\\ &\quad+\int_{t+\theta}^{t+\theta+h}R(t+\theta+h-s)\,L\big(V(s)\varphi\big)ds.\end{aligned}$$

By virtue of Lemma 3.4, there exists $\tilde{C}$ such that

$$\begin{aligned}&\Big|\int_0^{t+\theta}R(t+\theta+h-s)-R(t+\theta-s)L\big(V(s)\varphi\big)ds\Big|\\ &\qquad\le\int_0^{t+\theta}\big|R(t+\theta+h-s)-R(t+\theta-s)\big|\,\tilde{C}\,ds.\end{aligned}$$

Thus, there exists $\theta_0 \in [-r, 0]$ such that

$$\begin{aligned}&\sup_{-r\le\theta\le 0}\int_0^{t+\theta}\big|R(t+\theta+h-s)-R(t+\theta-s)\big|ds\\ &\quad=\int_0^{t+\theta_0}\big|R(t+\theta_0+h-s)-R(t+\theta_0-s)\big|ds,\end{aligned}$$

which implies that

$$\lim_{h\to 0}\Big|\int_0^{t+\theta}R(t+\theta+h-s)-R(t+\theta-s)\,L\big(V(s)\varphi\big)ds\Big| = 0.$$

On the other hand, using Definition 2.1 and Lemma 3.4, we deduce that there exists $\delta(h)$ such that

$$\Big|\int_{t+\theta}^{t+\theta+h}R(t+\theta+h-s)L(V(s)\varphi)ds\Big| \le MN\delta(h).$$

This implies that

$$\lim_{h\to 0}\Big|\int_{t+\theta}^{t+\theta+h}R(t+\theta+h-s)\,L(V(s)\varphi)ds\Big| = 0.$$

Thus,

$$\lim_{h\to 0}\big|V(t+h)-V(t)\big| = 0.$$

Hence the map $t \to V(t)$ is norm continuous for $t > r$. $\square$

3.2. **Compactness of the solution operator.** To study the compactness of the solution operator, we introduce the Kuratowski measure of noncompactness $\alpha(\cdot)$ defined on each bounded subset B of the Banach space X by

$$\alpha(B) = \inf\big\{d > 0;\ \ B \text{ can be covered by a finite number of sets of diameter } \ < d\big\}.$$

Some basic properties of $\alpha(\cdot)$ are given in the following.

Lemma 3.6 ([2]). *Let X be a Banach space and B, $C \subseteq X$ be bounded. Then*

(**1**) $\alpha(B) = 0$ *if and only if B is relatively compact;*

(**2**) $\alpha(B) = \alpha(\overline{B}) = \alpha(\overline{co}B)$, *where $\overline{co}B$ is the closed convex hull of B;*

(**3**) $\alpha(B) \le \alpha(C)$, *when $B \subseteq C$;*

(**4**) $\alpha(B + C) \le \alpha(B) + \alpha(C)$;

(**5**) $\alpha(B \cup C) \le \max\{\alpha(B), \alpha(C)\}$;

(**6**) $\alpha(B(0, r)) \le 2r$, *where $B(0, r) = \{x \in X : |x| \le r\}$.*

We need to add the following assumption:

(**H6**) the C_0-semigroup $T(t)$ is compact for $t > 0$.

Theorem 3.7. *Assume that* (**H1**)–(**H6**) are satisfied. Then the solution operator $V(t)$ is compact for $t > r$.

Proof. By the Ascoli-Arzela theorem we prove that $\{V(t)\varphi : \varphi \in B\}$ is relatively compact for each $r < t$. The proof is divided into two steps.

Step 1. We show that $\{(V(t)\varphi)(\theta) : \varphi \in B\}$ is relatively compact in X for every $\theta \in [-r, 0]$. Let $\theta \in [-r, 0]$. Then

$$\big(V(t)\varphi\big)(\theta) = R(t+\theta)\varphi(0) + \int_0^{t+\theta} R(t+\theta-s)L\big(V(s)\varphi\big)ds.$$

Since $t + \theta > 0$, by (**H5**) together with Theorem 2.5, we infer that $R(t+\theta)$ is compact. Thus Lemma 3.6 gives

$$\alpha\Big(\big\{R(t+\theta)\varphi(0) : \varphi \in B\big\}\Big) = 0. \tag{3.5}$$

Now we prove that $\left\{\int_0^{t+\theta} R(t+\theta-s)L\big(V(s)\varphi\big)ds : \varphi \in B\right\}$ is relatively compact in X. Let $0 < \varepsilon < t+\theta$. Then

$$\begin{aligned}\int_0^{t+\theta} R(t+\theta-s)L\big(V(s)\varphi\big)ds &= \int_0^{t+\theta-\varepsilon} R(t+\theta-s)L\big(V(s)\varphi\big)ds \\ &\quad + \int_{t+\theta-\varepsilon}^{t+\theta} R(t+\theta-s)L\big(V(s)\varphi\big)ds \\ &= \int_0^{t\theta-\varepsilon} \big[R(t+\theta-s)-R(\varepsilon)R(t+\theta-s-\varepsilon)\big]L(V(s)\varphi)ds \\ &\quad + R(\varepsilon)\int_0^{t+\theta-\varepsilon} R(t+\theta-s-\varepsilon)L\big(V(s)\varphi\big)ds\end{aligned}$$

$$+\int_{t+\theta-\varepsilon}^{t+\theta} R(t+\theta-s)L\big(V(s)\varphi\big)ds.$$

By Lemma 2.4, we obtain

$$\begin{aligned}&\Big|\int_0^{t+\theta-\varepsilon}\big[R(t+\theta-s)-R(\varepsilon)R(t+\theta-s-\varepsilon)\big]L\big(V(s)\varphi\big)ds\Big|\\&\le\int_0^{t+\theta-\varepsilon}\big|R(t+\theta-s)-R(\varepsilon)R(t+\theta-s-\varepsilon)\big|\big|L(V(s)\varphi)\big|ds\\&\le\varepsilon H\int_0^{t+\theta-\varepsilon}\big|L(V(s)\varphi)\big|ds\le\varepsilon(t-\varepsilon)HN.\end{aligned}$$

Let $t\le b$. Then Lemma 3.6 gives

$$\begin{aligned}&\alpha\bigg(\Big\{\int_0^{t+\theta-\varepsilon}\big[R(t+\theta-s)-R(\varepsilon)R(t+\theta-s-\varepsilon)\big]L\big(V(s)\varphi\big)ds:\varphi\in B\Big\}\bigg)\\&\le 2\varepsilon(b-\varepsilon)HN.\end{aligned}\tag{3.6}$$

Moreover, since $R(\varepsilon)$ is compact, we find that

$$\Big\{R(\varepsilon)\int_0^{t+\theta-\varepsilon}R(t+\theta-s-\varepsilon)L\big(V(s)\varphi\big)ds:\varphi\in B\Big\}$$

is relatively compact in X and, consequently

$$\alpha\bigg(\Big\{R(\varepsilon)\int_0^{t+\theta-\varepsilon}R(t+\theta-s-\varepsilon)L\big(V(s)\varphi\big)ds:\varphi\in B\Big\}\bigg)=0.\tag{3.7}$$

Note that

$$\Big|\int_{t+\theta-\varepsilon}^{t+\theta}R(t+\theta-s)L\big(V(s)\varphi\big)ds\Big|\le MN\delta(\varepsilon).$$

Therefore,

$$\alpha\bigg(\Big\{\int_{t+\theta-\varepsilon}^{t+\theta}R(t+\theta-s)L\big(V(s)\varphi\big)ds:\varphi\in B\Big\}\bigg)\le 2MN\delta(\varepsilon).\tag{3.8}$$

Combining (3.5)–(3.8) and using Lemma 3.6, we obtain

$$\alpha\Big(\big\{(V(t)\varphi)(\theta):\varphi\in B\big\}\Big)\le 2\varepsilon(b-\varepsilon)HN+2MN\delta(\varepsilon).$$

Letting $\varepsilon\to 0$, we deduce that

$$\alpha\Big(\big\{(V(t)\varphi)(\theta):\varphi\in B\big\}\Big)=0.$$

Consequently, $\big\{(V(t)\varphi)(\theta):\varphi\in B\big\}$ is relatively compact in X for all $\theta\in[-r,0]$.

Step 2. We show that $\{V(t)\varphi:\varphi\in B\}$ is equicontinuous on $[-r,0]$. To see this, let $-r\le\theta_1<\theta_2\le 0$. Then

$$\Big|\big(V(t)\varphi\big)(\theta_2)-\big(V(t)\varphi\big)(\theta_1)\Big|\le\Big|\big(R(t+\theta_2)-R(t+\theta_1)\big)\varphi(0)\Big|$$

$$
\begin{aligned}
&+ \int\limits_{t+\theta_1}^{t+\theta_2} \Big| R(t+\theta_2-s) L\big(V(s)\varphi\big) \Big| ds \\
&+ \int\limits_{0}^{t+\theta_1} \Big| \big(R(t+\theta_2-s) - R(t+\theta_1-s)\big) L\big(V(s)\varphi\big) \Big| ds \\
&\le \Big| R(t+\theta_2) - R(t+\theta_1) \Big| \Big| \varphi(0) \Big| + MN\delta(\theta_2-\theta_1) \\
&+ N \int\limits_{0}^{t+\theta_1} \Big| R(t+\theta_2-s) - R(t+\theta_1-s) \Big| ds.
\end{aligned}
$$

Since

$$\big|R(t+\theta_2-s) - R(t+\theta_1-s)\big| \to 0 \quad \text{as} \quad \theta_2 \to \theta_1 \quad \text{for almost all} \quad s \neq t+\theta_1$$

and

$$\big|R(t+\theta_2-s) - R(t+\theta_1-s)\big| \le M\big(e^{\beta(t+\theta_2-s)} + e^{\beta(t+\theta_1-s)}\big) \in L^1\big([0,t+\theta_1]\big),$$

the Lebesgue Dominated Convergence theorem ensures that

$$\int\limits_{0}^{t+\theta_1} \big|R(t+\theta_2-s) - R(t+\theta_1-s)\big| ds \to 0 \quad \text{as} \quad \theta_2 \to \theta_1.$$

Using Theorem 2.6, we obtain

$$\big|(V(t)\varphi)(\theta_2) - (V(t)\varphi)(\theta_1)\big| \to 0 \quad \text{as} \quad \theta_2 \to \theta_1,$$

uniformly in $\varphi \in B$. This implies that $\{V(t)\varphi : \varphi \in B\}$ is equicontinuous. Hence, $\{V(t)\varphi : \varphi \in B\}$ is relatively compact by the Ascoli Arzela theorem and so, $V(t)$ is compact for $t > r$. □

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THE DIRECTED GRAPHS OF SOME FUNCTIONS

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Dedicated to Professor Alexander Kharazishvili on the occasion of his 70th birthday

Abstract. A description of the digraphs associated with Hamel's coordinate functions and with some elementary functions is given. Some cardinal invariants of the corresponding mono-unary algebras are found. It is also proved that the digraph of the function tan is an universal graph for the class of digraphs of functions of a certain type.

1. Introduction

Given a function $f:\mathbb{R}\to\mathbb{R}$, it can be studied from different points of view including analysis, geometry, algebra, graph theory, combinatorics, etc. Many well-known functions, for example, functions constructed by Hamel [2] and having huge values in Linear algebra, Geometry, Functional analysis and Measure theory (see [1], [3], [5], [6], [9–11], [13]), or elementary functions are not sufficiently studied, especially, in algebra and graph theory. Every function f naturally generates the mono-unary algebra and the corresponding functional digraph [12]. In this article, we consider the mono-unary algebras and the corresponding digraphs for coordinate functions of Hamel's basis and for basic elementary functions. A description of the connected components of the digraphs of the above functions is given; the cardinalities of automorphism groups of such digraphs and the cardinality of the set of all mono-unary algebras, isomorphic to a given one, are established also. It is proved that for every coordinate function of Hamel's basis $f:\mathbb{R}\to\mathbb{R}$ there exists an effectively constructed (without the axiom of choice) simple function $g:\mathbb{R}\to\mathbb{R}$ such that the algebras $(\mathbb{R},f)$ and $(\mathbb{R},f)$ are isomorphic. It is also proved that for any basic elementary functions, except for a constant, there are 2^c many functions from $\mathbb{R}$ to $\mathbb{R}$ such that the mono-unary algebras (graphs) generated by any of them are isomorphic to the mono-unary algebras (graphs) generated by the function f. Obviously, most of these functions (in view of cardinality) are discontinuous.

2. Preliminary

We will use the standard algebraic, set-theoretic and graph theoretic notations. A partial mono-unary algebra is a pair (A,f), where A is a non-empty set and f is a map $f:B\to A$ for some subset $B\subset A$. If $B=A$, then the pair (A,f) is called a mono-unary algebra. For each partial mono-unary algebra, the corresponding digraph G_f is determined as follows:

$$G_f=(A,\{(x,f(x)):x\in \mathrm{Dom}(f)\}).$$

If (A,f) is a partial mono-unary algebra, we define a relation E on A in the following way: xEy, if and only if for some natural numbers n and m the equality $f^n(x)=f^m(y)$ holds, where

$$f^0(x)=x, f^{n+1}(x)=f(f^n(x)),\quad \text{for}\quad n\in\omega.$$

Then E is an equivalence relation on A, and we call E-equivalence classes of algebra A with the induced operation, connected components of the partial algebra (A,f). If $\{A_i:i\in I\}$ is the family of all E-equivalence classes, then we have $A=\bigcup\limits_{i\in I}A_i$ and the family $\{(A_i,f_i):i\in I\}$, where $f_i=f_{|A_i}$ is called the injective family of all connected components of partial mono-unary algebra (A,f). The

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corresponding digraphs of the subalgebras $\{A_i, f_i\} : i \in I$ are connected components of the digraph G_f.

The ordinal ω, is the set of all naturals, i.e., of all nonnegative integers, at the same time, ω denotes the cardinality of the set of natural numbers. The cardinality of the continuum is denoted by $\mathbf{c}$. The symbols: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ denote the sets of integers, of rational numbers and the set of reals, respectively. $|X|$ denotes the cardinality of a set X.

(a) If $n \geq 1$ is the smallest integer such that $f^n(x) = x$ for some element x, the following set $\{(x, f(x)), (f(x), f^2(x)), \ldots, (f^{n-1}(x), x)\}$ is called an n-cycle.

(b) If an injective family $\{a_n : n \in \omega\}$ of elements of A such that $f(a_{n+1}) = a_n (f(a_n) = a_{n+1}$ respectively), $n \in \omega$, then we say that the algebra (A, f) contains an ω-chain (ω^*-chain, respectively).

(c) If there is an injective family $\{a_q : q \in \mathbb{Z}\}$ of elements of A such that $f(a_q) = a_{q+1}, q \in \mathbb{Z}$, then we say that the algebra (A, f) contains an $\omega^* + \omega$ -chain.

(d) Every graph, which is considered in this article is a digraph of a partial function, respectively, a root tree is called a tree, in which each vertex is oriented in the direction to the root.

(e) The points, whose in-degree and out-degree is 0, are called isolated points.

3. The Digraphs of Hamel Coordinate Functions

Definition 3.1. Let b be an element of some Hamel basis of the vector space $\mathbb{R}(\mathbb{Q})$. The coordinate function f of Hamel basis is defined as follows: for any $x \in \mathbb{R}$, $f(x)$ is the b-th coordinate of the vector x [1]; The set of all coordinate functions of some Hamel basis of the vector spaces $\mathbb{R}(\mathbb{Q})$ is denoted by H.

The following lemma is trivial to prove.

Lemma 3.2. *If $f \in H$, then the following hold:*

(a) $f(0) = 0$,

(b) *for each $r \in \mathbb{Q}$ the set $f^{-1}(\{r\})$ has the cardinality* $\mathbf{c}$,

(c) *for each $x \in \mathbb{R} \backslash \mathbb{Q}$ we have $f^{-1}(\{x\}) = \varnothing$,*

(d) $|H| = 2^c$.

Definition 3.3.

3.1.1. A root tree of cardinality $\mathbf{c}$ whose root is an incident to any vertex, except the root, is called a tree of type H_0.

3.1.2. If T is a tree of type H_0 and r_0 is its root, then the digraph $T \cup (r_0, r_0)$ is called a graph of type H_1 (see Figure 1).

3.1.3. Let $(T_i)_{i \in \omega}$ be a family of disjoint digraphs of type H_0 and r_i be a root of the tree T_i for each $i \in \omega$, then the following graph $(\cup T_i) \cup \{(r_i, r_0) : i \in \omega\}$ is called a graph of type H_2 (see Figure 2).

3.1.4. Let $(T_k)_{k \in \mathbb{Z}}$ be a family of disjoint graphs of type H_0 and r_k be a root of the tree T_k for each $k \in \mathbb{Z}$, then the following digraph $(\cup_{k \in \mathbb{Z}} T_k) \cup \{(r_k, r_{k+1}) : k \in \mathbb{Z}\}$ is called a graph of type H_3 (see Figure 3).

Remark 3.4. It is easy to verify that the graphs of the same types are pairwise isomorphic, and those of different types are pairwise non-isomorphic.

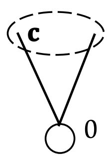

Figure 1

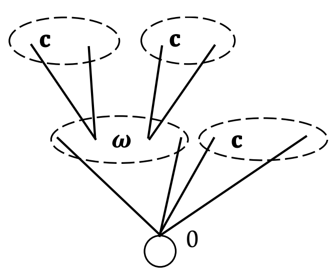

Figure 2

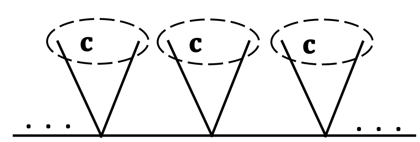

Figure 3

Theorem 3.5. *If $f \in H$, then the following hold:*

(a) *if $f(1) = 0$, then G_f is the graph of type H_2;*

(b) *if $f(1) = 1$, then the graph G_f consists of infinitely countably many connected components of type H_1;*

(c) *if $f(1) \in \mathbb{Q}\backslash\{0; 1\}$, then the graph G_f consists of countably infinitely many connected components, among them, only one component is of type H_1, all other components are of types H_3.*

Proof. (a) It is clear that $f(1) = 0$ iff for any $r \in \mathbb{Q}$ we have $f(r) = 0$. Therefore, from Lemma 3.2 it follows that if $f(1) = 0$, then the digraph G_f of the function f will be of type H_2.

(b) If $f(1) = 1$, then we have $f(r) = r$, $r \in \mathbb{Q}$. Let $\{r_i : i \in \mathbf{Z}\}$ be an injective family of all rational numbers. It is obvious that for each $i \in \mathbb{Z}$ the cardinality of the set $f^{-1}(r_i)$ is $\mathbf{c}$ and the digraph $(f^{-1}(r_i), \{(x, r_i) : x \in f^{-1}(r_i)\})$ is a connected component of type H_1 of the digraph G_f.

(c) Let $f(1) = r \in \mathbb{Q}\backslash\{0; 1\}$. If $r_0 \in \mathbb{Q}\backslash\{0\}$, then for each $k \in \mathbb{Z}$ we have $f(r^k \cdot r_0) = r^{k+1} \cdot r_0$. Therefore $\{(r^k \cdot r_0, r^{k+1} \cdot r_0) : k \in \mathbb{Z}\}$ is an $\omega^* + \omega$ chain in the digraph G_f. If $r_1 \neq r_2$, then $f(r_1) \neq f(r_2)$. Thus, for every nonzero rational number q, the set $f^{-1}(\{q\})$ contains a unique rational number, and the graph G_f contains a unique loop $(0, 0)$. Consequently, each nonzero rational number forms a component of type H_3. It is easy to prove that there are infinitely many different components of type H_3. In addition to the connected components of type H_3, the graph G_f will contain a single-connected component of type H_1 corresponding to the number 0. □

Corollary 3.6. *The maximal family of pairwise non-isomorphic mono-unary algebras $(\mathbb{R}, f)$, where f is an element of H, consists of 3 elements.*

Corollary 3.7. *For $f \in H$, the automorphisms group of mono-unary algebra $(\mathbb{R}, f)$ has the cardinality $2^{\mathbf{c}}$.*

Theorem 3.8. *For each $f \in H$, there exists an effectively (without the axiom of choice) constructed function $g : \mathbb{R} \to \mathbb{R}$ such that the digraphs G_g and G_f are isomorphic.*

Proof. (a) If $f(1) = 0$, consider the function defined as follows:

$$g_a(x) = \begin{cases} 0, & \text{if} \quad x \in \mathbb{Z} \quad \text{or} \quad -1 < x < 1, \\ n, & \text{if} \quad n < x < n+1, \quad n \in \omega\backslash\{0\}, \\ -n, & \text{if} \quad -n-1 < x < -n, \quad n \in \omega\backslash\{0\}. \end{cases}$$

It is easy to prove that in this case the digraphs G_f and G_{g_a} are isomorphic.

(b) If $f(1) = 1$, consider the function defined as follows:

$$g_b(x) = \begin{cases} n, & \text{if} \quad n \leq x < n+1, \quad n \in \omega\backslash\{0\}, \\ 0, & \text{if} \quad -1 < x < 1 \\ -n, & \text{if} \quad -n-1 < x \leq -n, \quad n \in \omega\backslash\{0\}. \end{cases}$$

It is easy to prove that in this case the digraphs G_f and G_{g_b} are isomorphic.

(c) Let $f(1) = r$, for some $r \in Q\backslash\{0; 1\}$.

Then for any prime number p, define the function $g_p(x)$ by the following equality:

$$g_p(x) = \begin{cases} p^{k+1}, & \text{if} \quad x \in (p^k - 1; p^k], \quad \text{for} \quad k \in \omega\backslash\{0\} \\ -p^k, & \text{if} \quad x \in [-p^{k+1}; -p^{k+1} + 1), \quad \text{for} \quad k \in \omega\backslash\{0\}). \end{cases}$$

Evidently, for any prime number p, the digraph G_{g_p} is a digraph of type H_3.

If we now combine the functions g_p for all prime numbers and consider a function g_c defined as follows:

$$g_c(x) = \begin{cases} g_p(x), & \text{if} \quad x \in \mathrm{Dom}(g_p), \quad \text{for some prime number } p, \\ 0, & \text{for all other values of } x \in \mathbb{R}, \end{cases}$$

then it is easy to verify that in this case the digraphs G_f and G_{g_c} are isomorphic. □

Remark 3.9. Despite the fact that the functions g_a, g_b and g_c are constructed effectively, the proof of the existence of the corresponding isomorphisms requires a countable form of the Axiom of Choice.

4. The Digraph of the Algebra $(\mathbb{R}, \sin)$

Definition 4.1.

4.1.1. A countably infinite root tree in which each vertex, except the root, is connected to the root, is called a tree of type S_0 (Figure 4).

4.1.2. If $\mathbf{T}$ is a tree of type S_0 and r_0 is its root, then the digraph $\mathbf{T} \cup \{(r_0, r_0)\}$ is called a digraph of type S_1 (Figure 5).

4.1.3. Let $(\mathbf{T}_i)_{i\in\omega}$ be a family of disjoint digraphs of types S_0 and r_i be a root of the tree $\mathbf{T}_i$, for each $i \in \omega$, the following digraph $\cup\, \{\mathbf{T}_i : i \in \omega\} \cup \{(r_i, r_{i+1}) : i \in \omega\}$ is called a graph of type S_2 (Figure 6).

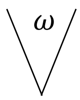

Figure 4

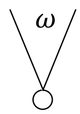

Figure 5

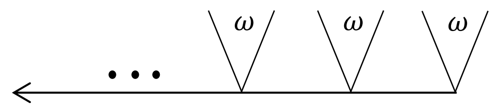

Figure 6

Theorem 4.2. *The set of all connected components of mono-unary algebra* $(\mathbb{R}, \sin)$ *has the cardinality of the continuum, among them one component is a graph of type* S_1 *and all the other components are the graphs of type* S_2.

Proof. It is obvious that the in-degree of each vertex of the graph $G_{\sin}$ is 0 or ω. It is also obvious that the component containing the number 0, is a component of type S_1.

If $x \in \mathbb{R}\backslash\{\pi k : k \in \mathbb{Z}\}$, then it is clear that

$$|x| > \big|\sin|x|\big| > \big|\sin\big|\sin|x|\big|\big| > \cdots$$

holds. Therefore, there aren't any cycles in the component, which doesn't contain 0. It is clear that such components contain an ω^*-chain. From the trivial inequality

$$|x - y| > |\sin x - \sin y|, \quad (x \neq y)$$

it follows that the sequence $x, \arcsin x, \arcsin^2 x, \dots$ is always finite. Therefore, there aren't any ω-chains in the digraph $G_{\sin}$.

Obviously, each connected component of the digraph $G_{\sin}$ contains a countably infinite set of vertices, hence the set of all connected components of $G_{\sin}$ has the cardinality of the continuum. □

5. The digraph of the algebra $(\mathbb{R}, \cos)$

Definition 5.1.

5.1.1. If $\mathbf{T}_0$ and $\mathbf{T}_1$ are two disjoint trees of types S_0, and r_0 and r_1 are their roots, then the digraph $\mathbf{T}_0 \cup \mathbf{T}_1 \cup \{(r_0, r_1)\}$ is a tree of type C_0 (Figure 7).

5.1.2. If $\mathbf{T}$ is a tree of type C_0 and r_0 is its root, then $\mathbf{T} \cup \{(r_0, r_0)\}$ is a digraph of type C_1 (Figure 8).

5.1.3. Let $(\mathbf{T}_i)_{i\in\omega}$ be a family of disjoint root trees of type C_0 and r_i be a root of the tree $\mathbf{T}_i$, $i \in \omega$, the following digraph $\cup\, \{\mathbf{T}_i : i \in \omega\} \cup \{(r_i, r_{i+1}) : i \in \omega\}$ is called a graph of type C_2 (Figure 9).

5.1.4. Let $\mathbf{T}$ be a tree of type S_0 whose root is r, $(\mathbf{T}_i)_{i\in\omega}$ be a family of disjoint root trees of type C_0 and let r_i be the root of a tree $\mathbf{T}_i$, $i \in \omega$. Then the graph $\mathbf{T} \cup \mathbf{T}_i \cup \{(r_i, r_{i+1}) : i \in \omega\} \cup \{(r, r_0)\}$ is called the graph of type C_3 (Figure 10).

Theorem 5.2. *The set of all connected components of mono-unary algebra* $(\mathbb{R}, \cos)$ *has the cardinality of the continuum, among them there is one component of type* C_1*, one component of type* C_2 *and all others are of type* C_3.

Proof. Let d be a fixed point of the function cos. It is obvious that the in-degree of each vertex of the digraph of cos is 0 or ω. It is also obvious that the component containing the fixed point d, is a component of type C_1, in this component only two vertices d and $-d$ will have in-degrees

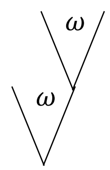

FIGURE 7

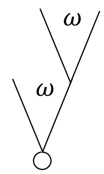

FIGURE 8

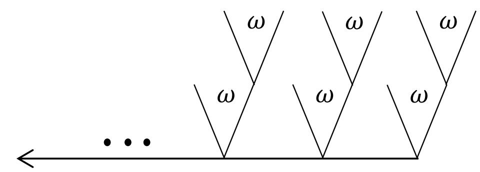

FIGURE 9

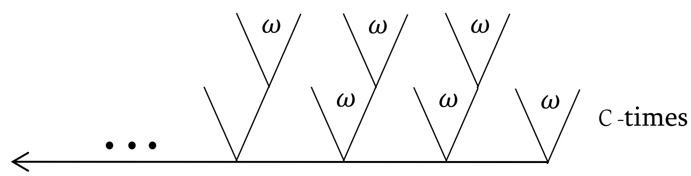

FIGURE 10

equal to ω. From the fact that the function cos has only one fixed point, it follows that the graph has only one component, which has a loop. For any $x \in \mathbb{R}\backslash\{d\}$, the injective sequence of iterations $x, \cos(x), \cos(\cos(x)), \ldots$ converges to d, so the digraph $G_{\cos}$ does not contain cycles, except for a single loop (d, d), and contains an ω^* chain. It follows from the parity of function cos that the other connected components are of type C_2 or of type C_3, clearly that type C_2 will have a single component containing the number 0. Evidently, that the set of all such components has the cardinality $\mathbf{c}$. $\square$

Remark 5.3. It should be remarked that the digraphs $G_{\sin}$ and $G_{\cos}$ are not isomorphic, they don't even have components, isomorphic to each other.

6. The Digraph of the Partial Mono-unary Algebra $(\mathbb{R}, \tan)$

Definition 6.1.

6.1.1. Let's define the component of type Tan_0 as a root tree, whose in-degree of every vertex is countably infinite (Figure 11).

6.1.2. Let $(T_i)_i \in \omega$ be a family of disjoint components of type Tan_0 and r_i be the root of the tree T_i for each $i \in \omega\backslash\{0\}$. For every $n \in \omega\backslash\{0\}$, the following digraph

$$\Big(\bigcup_{i=1}^{n} T_i\Big) \cup \Big(\bigcup_{i=1}^{n-1} \{(r_i, r_{i+1})\}\Big) \cup \{(r_n, r_1)\}$$

is called a component of type Tan_n (Figure 12) and $(\cup_{i\in\omega} T_i) \cup \{(r_i, r_{i+1}) : i \in \omega\}$ is called a component of type Tan_∞ (Figure 13).

Theorem 6.2. *The set of connected components of partial mono-unary algebra* $(\mathbb{R}, \tan)$ *consists of countably infinitely many components of type* Tan_n*, for each* $n \in \omega$*, and continuum-many components of type* Tan_∞*.*

Proof. It is obvious that:

- The in-degree of each vertex of the graph of the function tan is ω.
- For each $r_k = \pi/2 + \pi k$, $k \in \mathbb{Z}$, there is a component T_k in the graph of the partial mono-unary algebra $(\mathbb{R}, \tan)$, which is a component of type Tan_0 and whose root is r_k. There aren't any other components of type Tan_0.
- Due to the reason that for each natural n, the equality-

$$\tan^n(x) = x$$

has countably infinitely many solutions, therefore there are countably infinitely many components of type Tan_n, for each natural n, in the graph of partial mono-unary algebra $(\mathbb{R}, \tan)$.

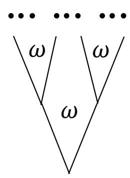

Figure 11

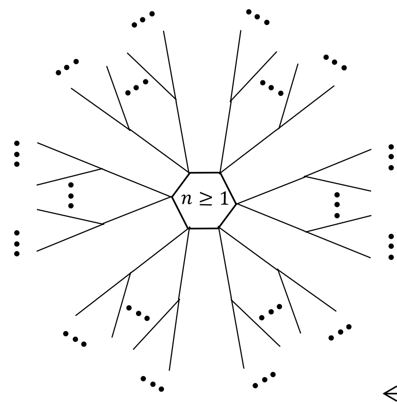

Figure 12

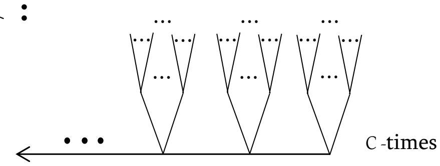

Figure 13

It's obvious that the union of sets of vertices of all components of type Tan_n, $n \in \omega$, is countably infinite. We can say that each of the remaining points is in the set of domains of the function tan as in the set of ranges of the function tan, the cardinality of the set of all such points is c. In addition, we know that the cardinality of the set of vertexes of each component of the graph of $(\mathbb{R}, \tan)$ is countably infinite. Therefore, the components which are not of type Tan_n, $n \in \omega$, are the components of type Tan_∞ and the cardinality of the set of all such components is $\mathbf{c}$. □

Remark 6.3. In the same way we can find out that $(\mathbb{R}, \mathrm{cotan})$ has the same structure of the digraph as $(\mathbb{R}, \tan)$. Therefore, we have $(\mathbb{R}, \tan) \cong (\mathbb{R}, \mathrm{cotan})$.

7. The Digraphs of Some Basic Elementary Functions

It's easy to show, what kinds of graphs have the following functions: see Figure 14

$$F_0 = \{\mathbf{arcsin}, \mathbf{arccos}, \mathbf{arctan}, \mathbf{arccotan}, \boldsymbol{a^x}, \mathbf{log}_{\boldsymbol{a}}\, \boldsymbol{x}, \boldsymbol{x^n}, \boldsymbol{x^{\frac{1}{n}}}\ (\boldsymbol{n} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots)\}$$

8. Universality of the Digraph of the Function tan

Theorem 8.1. *For each mono-unary algebra* $(\mathbb{R}, f)$, $f \in F$, *there is a monomorphism from* $(\mathbb{R}, f)$ *into the* $(\mathbb{R}, \tan)$.

Proof. The corresponding monomorphisms can be easily constructed. We construct a monomorphism of the algebra $(\mathbb{R}, \cos)$ into the partial algebra $(\mathbb{R}, \tan)$. The remaining monomorphisms are constructed more simply.

First, we build a monomorphism from the component C_1^0 of type C_1 with root d, which is the fixed point of cos, into any component T_1 of type $\mathrm{Tan}_{\mathbf{1}}$, whose root is r.

Define the sets: $A_1' = \{x : \cos(x) = d\}$; $B_1' = \{x : \tan(x) = r\}$.

Let f_1' be a bijection between these two sets.

Now we define the following sets: $A_1'' = \{x : \cos(x) = -d\}$ and $B_1'' = \{x : \tan(x) = f_1'(-d)\}$. Let f_1'' be a bijection between these two sets.

So, we can define monomorphism f' from the component of type C_1^0, into the component of type T_1 as follows:

$$f'(x) = \begin{cases} r, & \text{if} \quad x = d; \\ f_1'(x), & \text{if} \quad x \in A_1'; \\ f_1''(x), & \text{if} \quad x \in A_1''. \end{cases}$$

Second, we build a monomorphism from the component C_2^0 of type $\mathbf{C}_2$ into any component T_∞ of type Tan_∞.

Let b be a point from the component T_∞.

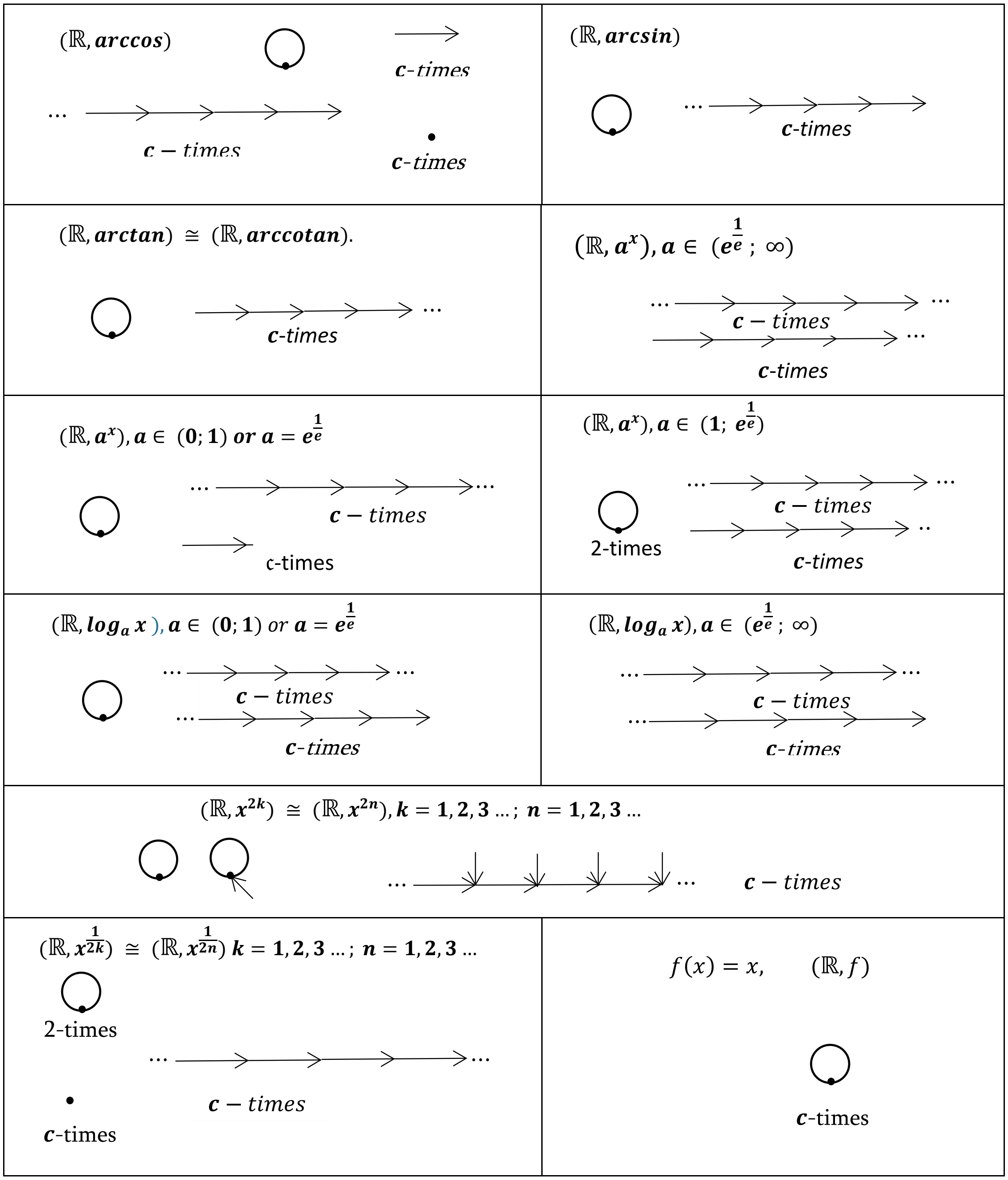

Figure 14

Define the sets

$$A_i = \{x : \cos(x) = \cos^{i+1}(\pi/2) \& x \neq \cos^i(\pi/2)\},\ i \in \omega,$$
$$A_i' = \{x : \cos(x) = -\cos^{i+1}(\pi/2)\},\ i \in \omega \backslash \{0\},$$
$$B_i = \{x : \tan(x) = \tan^{i+1}(b) \& x \neq \tan^i(b)\},\ i \in \omega \backslash \{0\}.$$

For each $i \in \omega\backslash\{0\}$, let b_i be a point for which:

$$\tan(b_i) = \tan^{i+2}(b) \& b_i \neq \tan^{i+1}(b) \quad \text{and let} \quad B'_i = \{x : \tan(x) = b_i\},\ i \in \omega\backslash\{0\}.$$

Let us denote a bijection between the sets A_i and B_i by f_i and the bijection between the sets A'_i and B'_i by f'_i. Let's define a monomorphism from the component C_2^0 into $\mathbf{T}_\infty$ as follows:

$$f_0(x) = \begin{cases} \tan^i(b), & \text{if} \quad x = cos^i(\pi/2)\ i \in \omega; \\ f_i(x), & \text{if} \quad x \in A_i\ i \in \omega; \\ f'_i(x), & \text{if} \quad x \in A'_i\ i \in \omega\backslash\{0\}. \end{cases}$$

Since $\mathbf{C}_2$ can be presented by the union of component of type $\mathbf{C}_3$ and component of type $\mathbf{S}_0$, there is a monomorphism from the component of type $\mathbf{C}_3$ into the component of type Tan_∞.

Therefore, there is a monomorphism from mono-unary algebra $(\mathbb{R}, \cos)$ into the $(\mathbb{R}, \tan)$. For the other mono-unary algebras $(\mathbb{R}, f)$, $f \in F$, the proof is similar. $\square$

Remark 8.2. Since the cardinality of the set of all connected components of $(\mathbb{R}, \cos)$ mono-unary algebra is $\mathbf{c}$, we have used the continuum form of the axiom of choice.

9. Some Cardinal Invariants

Definition 9.1.

9.1.1. Let (E, R) be a relational structure. By $\sigma(E, R)$ we denote the cardinality of the set of all relational structures (E, A) isomorphic to (E, R);

9.1.2. For a partial mono-unary algebra $(\mathbb{R}, f)$, let $\sigma(f)$ denote the cardinality of the set of all partial algebras $(\mathbb{R}, g)$, isomorphic to the $(\mathbb{R}, f)$.

9.1.3. Denote

$$F = \{\sin, \cos, \tan, \mathrm{cotan}, \arcsin, \arccos, \arctan, \mathrm{arccot}, \frac{1}{x}, a^x, \log_a x, x^n, x^{1/n}, \ \ (n = 2, 3, \dots)\}.$$

Finding the cardinal invariants $\sigma(f)$ and $|\mathrm{Aut}(\mathbb{R}, f)|$ for any function f, are special cases of Ulam's product-isomorphism problems (see [14]). In the general case, the problem of finding the cardinal number $\sigma(E, R)$ depends on GCH (see [4]).

Lemma 9.2 ([7]). *Let (E, R) be an infinite relational structure, $|E| = \varepsilon, \Delta_E$ be a diagonal of E^2 and let R be the functional relation with respect to the first or second coordinate. Then:*

1) *if $|\Delta_E \cap R| = \varepsilon \& |\Delta_E \setminus R| = \delta$, then $\sigma(E, R) = \varepsilon^\delta$;*
2) *if $(\exists l)$ $(l \subset E^2 \& (l = \{x\} \times E \vee l = E \times \{x\}) \& |l \cap R| = \varepsilon \& |l \backslash R| = \delta)$, then $\sigma(E, R) = \varepsilon^{\delta+1}$;*
3) *if $\delta = \max\{|pr_1 R|, |pr_2 R|\} < \varepsilon$, then $\sigma(E, R) = \varepsilon^\delta$;*
4) *in all the remaining cases $\sigma(E, R) = 2^\varepsilon$.*

Theorem 9.3. *If $f \in F$, then*

$$\sigma(f) = |\mathrm{Aut}(\mathbb{R}, f)| = 2^c$$

holds.

Proof. If $f \in F$, then:

(i) f has at most countably many fixed points;

(ii) for any l, where $l = \{x\} \times \mathbb{R}$ or $l = \mathbb{R} \times \{x\}$, the function f has at most countable set of intersections with l;

(iii) the cardinalities of the sets $\mathrm{Dom}(f)$ and $\mathrm{Ran}(f)$ are equal to $\mathbf{c}$.

So, from Lemma 9.2 it follows that $\boldsymbol{\sigma}(f) = 2^c$ holds.

If $f \in F$, then the digraph G_f has continuum many pairwise isomorphic components, therefore $|\mathrm{Aut}(\mathbb{R}, f)| = 2^c$. $\square$

Corollary 9.4. *If $f \in F$, then there are 2^c-many discontinuous functions that have isomorphic digraph with the digraph of f.*

Remark 9.5. For the values of cardinalities of the automorphism groups of mono-unary algebras in the general case, see [8].

Remark 9.6. If $f : A \to A$ is a bijection, then (A, f) and (A, f^{-1}) mono-unary algebras are isomorphic and the cardinality of the set of all isomorphisms between the algebras (A, f) and (A, f^{-1}) is $|\mathrm{Aut}(A, f)|$.

Proof. It is not difficult to produce isomorphism between those two mono-unary algebras by building an isomorphism between the digraphs of those two mono-unary algebras, because the components of digraphs of bijections can be only n-cycle for some an $n \in \omega\backslash\{0\}$ or an $\omega^* + \omega$ chain.

If $f : A \to A$ is bijection, h is any automorphism of algebra (A, f) and φ is any isomorphism between (A, f) and (A, f^{-1}) mono-unary algebras, then $\varphi \circ h$ will also be an isomorphism between (A, f) and (A, f^{-1}) mono-unary algebras. □

10. Open Problem

For which function $f \in F$ there exists a non-measurable function $g : \mathbb{R} \to \mathbb{R}$, whose digraph is isomorphic to the digraph of f?

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ASYMPTOTIC ANALYSIS OF AN OVER-REFLECTION EQUATION IN MAGNETIZED PLASMA

GRIGOL GOGOBERIDZE

Abstract. The equation describing the over-reflection of the slow magneto-sonic waves in plasma with background uniform shear flow is derived and analyzed in detail both analytically and numerically. Using the methods of asymptotic analysis, analytical expressions for reflection and transmission coefficients of the waves are obtained for relatively small shear rates.

1. Introduction

The aim of the present paper is to study mathematical aspects of the over-reflection phenomenon [5] in magnetized plasmas which is thought to be one of the possible sources of large scale perturbations in astrophysical and laboratory plasmas [3, 5]. Towards this end, we choose the simplest system, the two-dimensional magneto-hydrodynamical shear flow with domination of plasma energy (the so-called high beta-plasma). This will allow us to study the over-reflection phenomenon in a pure form; in more general cases, the phenomenon is accompanied by a mutual transformation of different wave modes, as well.

Consider the two-dimensional compressible unbounded shear flow along the x-axis with the constant shear parameter, i.e., with the velocity vector $\mathbf{U}_0(Ay, 0)$. We assume that the density ρ_0 and the pressure P_0 are uniform, and the magnetic field $\mathbf{B}_0$ is directed along the streamlines. Assuming $\rho_1,\ u_x,\ u_y,\ b_x, b_y$ are the perturbations of density, velocity and magnetic field, respectively, the linearized equations governing the evolution of the spatial Fourier harmonics of dimensionless perturbations in the uniform shear flow are [3]

$$\dot{d} = v_x + K(T)v_y, \tag{1}$$

$$\dot{v}_x = -Sv_y - \beta d, \tag{2}$$

$$\dot{v}_y = -\beta K(T)d + [1 + K(T)]\, b, \tag{3}$$

$$\dot{b} = -v_y, \tag{4}$$

where $S = A/V_A k_x$ is the dimensionless shear rate, k_x and k_y are parallel and perpendicular wave numbers, respectively, V_A is the Alfvén velocity, $T = V_A k_x t$ is the dimensionless time, $\beta = c_s^2/V_A^2$ is beta-plasma, c_s is the sound speed, $K(T) = k_y/k_x - ST$ is the dimensionless perpendicular wave number, and $d(\mathbf{k}) = i\rho_1(\mathbf{k})/\rho_0,\ b(\mathbf{k}) = ib_y(\mathbf{k})/B_0,\ \mathbf{v}(\mathbf{k}) = \mathbf{u}(\mathbf{k})/V_A$ are dimensionless perturbations of the density, perpendicular component of the magnetic field and the velocity, respectively. In the above equations, the over-dot denotes a derivative with respect to the dimensionless time T.

Equations (1)–(4), along with the over-reflection phenomenon, describe various dynamical effects of the linear perturbations, such as coupling and mutual transformation of different plasma waves [3]. To derive the equations that describe the over-reflection in a pure form, one has to consider evolution of low frequency perturbations in the weakly compressible medium that corresponds to the dynamics of the so-called slow magneto-sonic waves in low beta-plasmas (plasma beta $\beta \gg 1$). Physically this means that in this case we may neglect compressibility of the low frequency waves. From the mathematical point of view, in the case under consideration, equations (3) and (4) decouple and describe the evolution of low frequency perturbations in the shear flow. Then, introducing the new

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variable $\Psi = b[1 + K(T)^2]^{1/2}$, these equations can readily be reduced to the following second-order ordinary differential equation:

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}T^2} + \left[1 - \frac{S}{\left(1 + K(T)^2\right)^2}\right]\Psi = 0. \tag{5}$$

In the next section, we present detailed analysis of this equation. We show that in the vicinity of critical points (where the expression in the squire brackets is undefined or becomes zero) it can be reduced to the specific type of the Bessel equation, thus allowing us to derive analytical expressions for the reflection and transmission coefficients of the wave perturbations.

2. Asymptotic Analysis and Numerical Study

First of all, let us note that in the limit $|K(T)| \gg S$, the Liouville-Green asymptotical solution [2, 6](known also from physical literature [4] as the Wentzel-Kramers-Brillouin or quasi-classical approximation) is applicable, and we have

$$\Psi_\pm = \frac{C_\pm}{\sqrt{\omega(T)}} e^{\pm i \int \omega(T)\mathrm{d}T}, \tag{6}$$

where

$$\omega(T) = \sqrt{1 - \frac{S}{\left[1 + K(T)^2\right]^2}},$$

and $C_\pm$ are some constants determined by the initial conditions.

It is well known [4] that $\Psi_\pm$ correspond to the waves, propagating along and backward with respect to the x-axis, respectively.

If the dimensionless shear rate is high enough, then for the time period, when $|K(T)| \sim S$, the asymptotic solutions (6) are not valid, i.e., physically speaking, evolution of the waves is not adiabatic. Suppose that at the initial moment of time $T = 0$ we have $K(0) \gg S$. When T increases, $K(T)$ decreases and passes through the interval of non-adiabatic evolution, where the condition $|K(T)| \gg S$ is not valid. From the mathematical point of view, this means that the asymptotic amplitudes $C_\pm$ in this interval do not remain constant. On the other hand, when T tends to infinity, the condition $|K(T)| \gg S$ becomes valid again and equation (5) has asymptotic solutions (6) with different amplitudes $C_\pm(\infty)$. Assuming that initially $C_+(0) = 1$ and $C_-(0) = 0$, the reflection (R) and transmission (G) coefficients of the wave can be defined in the usual manner [3–5],

$$R = \left|\frac{C_-(\infty)}{C_+(0)}\right|^2 \text{ and } G = \left|\frac{C_-(\infty)}{C_+(0)}\right|^2. \tag{7}$$

The reflection and transmission coefficients are not independent and the conservation of the wave action implies [4] $1 + R = G$.

First, let us consider the limit $S \ll 1$. In this case, the exact asymptotic solution of equation (5) can be derived. Introducing new variable $\tau = K(T)$, equation (5) can be rewritten as

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\tau^2} + \left[\frac{1}{S^2} - \frac{1}{\left(1 + \tau^2\right)^2}\right]\Psi = 0. \tag{8}$$

From the mathematical point of view, the non-adiabatic evolution (i.e., failure of the solutions (6)) is related to the critical (singular and turning) points of equation (8) [2, 6]. In the case of equation (8), there exist two second-order regular singular points $\tau_{12} = i$ and their complex conjugate, and also four turning points $\tau_{3-6} = \pm i(1 + S)^{1/2}$. In the above-considered limit $S \ll 1$, the turning points tend to coincide with the regular singular points.

Assume that initially there exists only a wave with a positive phase velocity, i.e., $C_-(0) = 0$. To derive the reflection coefficient, one has to consider equation (8) in the complex τ-plane [6] along the

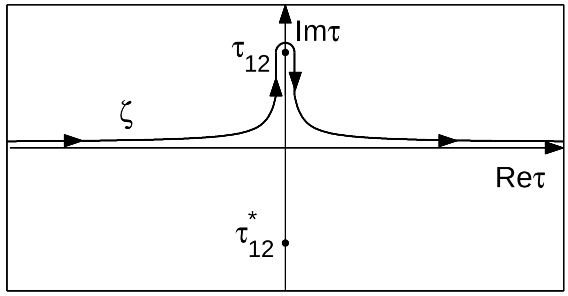

FIGURE 1. The path ζ of integration in the complex τ plane around critical point τ_{12}.

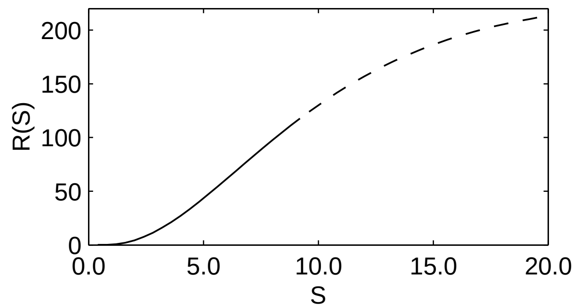

FIGURE 2. The reflection coefficient R as a function of the normalized shear parameter S.

path ζ presented in Figure 1. The evolution is adiabatic everywhere except a small vicinity of the singular point τ_{12}, where equation (8) reduces to the following equation:

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\tau^2} + \left[\frac{1}{S^2} + \frac{1}{(\tau - i)^2}\right]\Psi = 0.$$

This equation represents specific case of the Bessel equation and, as is known [1], its solution can be expressed in terms of the zeroth order Hankel functions

$$\Psi_{1,2}(\tau) = \frac{1}{(\tau - i)^{1/2}} H_{1,2}^{(0)}\left(\frac{\tau - i}{S}\right).$$

Comparing asymptotic expansions of the Hankel functions [1] with equations (6), it can be readily seen that far away from the singular point, for $\tau = 0$, $\Psi_{1,2}$ coincide with $\Psi_{\pm}$. Then, the well-known analytical continuation formulas for the Hankel functions [1], together with the definition of the reflection coefficient (7), give

$$R = e^{-4/S}.$$

This equation represents the exact asymptotic solution for the reflection coefficient in the limit $S \to 0$. As it can be seen from this expression, the reflection coefficient in the limit under consideration is exponentially small with respect to the parameter $1/S$, in accordance with the solutions of similar equations in quantum mechanics [4].

The influence of the velocity shear becomes much more significant if the dimensionless shear parameter S is of the order of unity or higher. The asymptotic mathematical method of the phase integrals [2] is not applicable in this case and, hence, no analytical expression for the reflection coefficient can be obtained and the problem can be solved only numerically.

The dependence of the reflection coefficient R on the normalized shear parameter S obtained by numerical solution of equation (8) is presented in Figure 2. The initial conditions are chosen as the Liouville-Green asymptotical solutions (6). According to the numerical study, the amplitude of the reflected wave exceeds that of the incident wave (i.e., $R > 1$) if $S > 1.4$. This condition indicates that the phenomenon of the over-reflection can take place in plasma with high beta-parameter ($V_A \to 0$) even for small values of the shear parameter A. The amplification of the energy density of perturbations due to the over-reflection is always finite, but it may become arbitrarily large under due increase of the shear parameter.

3. Conclusion

In the presented paper, we have studied the phenomenon of over-reflection in the two-dimensional magneto-hydrodynamical shear flow in high beta-plasma. The equation describing the over-reflection of the slow magneto-sonic waves has been derived and analyzed both analytically and numerically. Using methods of asymptotic analysis analytical expressions for reflection coefficient of the waves are derived for small dimensionless shear parameter. It was shown that in high beta plasmas the over-reflection can take place even for relatively small shear rates.

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S-$\mathcal{I}$-CONVERGENCE OF SEQUENCES

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Abstract. In this article, we use the notions of a semi-open set and topological ideal, in order to define and study a new variant of the classical concept of convergence of sequences in topological spaces, namely, the S-$\mathcal{I}$-convergence. Some basic properties of S-$\mathcal{I}$-convergent sequences and their preservation under certain types of functions are investigated. Also, we study the notions related to compactness and cluster points by using semi-open sets and ideals. Finally, we explore the $\mathcal{I}$-convergence of sequences in the cartesian product space.

1. Introduction and Preliminaries

The ideal theory on a set was established in 1933 by Kuratowski [10]. This theory has recently been used in order to generalize several concepts of Mathematical Analysis and General Topology (see, e.g., see [4], [7], [8], [14]). In particular, in 2000, Kostyrko et al. [9] used ideals on the set $\mathbb{N}$ of the positive integer numbers to introduce the notion of $\mathcal{I}$-convergence on metric spaces, as a generalization of statistical convergence. In 2005, Lahiri and Das [11] extended the notion of $\mathcal{I}$-convergence to the context of topological spaces and established some basic properties. On the other hand, in l963, Levine [12] introduced the notion of semi-open set in topological spaces, which plays an important role in recently researches in General Topology. In this article, we use the notion of a semi-open set, in order to define and study a variant of the classical convergence in topological spaces, namely, the S-$\mathcal{I}$-convergence. Specifically, we investigate some basic properties of S-$\mathcal{I}$-convergent sequences and their preservation under certain types of functions. Also, we study the notions related to compactness and cluster points by using semi-open sets and ideals. In the final part of the work, we explore the $\mathcal{I}$-convergence of sequences in the product space.

Now we will give some definitions and results that will be useful to understand content.

Definition 1.1. Let X be a nonempty set, a family of sets $\mathcal{I} \subset 2^X$ is called an *ideal* [10] on X if the following properties are satisfied:

(1) $\emptyset \in \mathcal{I}$,
(2) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(3) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

An ideal $\mathcal{I}$ on X is called *nontrivial* if $\mathcal{I} \neq \{\emptyset\}$ and $X \notin \mathcal{I}$. A nontrivial ideal $\mathcal{I}$ on X is called *admissible* if $\mathcal{I} \supset \{\{x\} : x \in X\}$. Some examples of admissible ideals can be found in [9].

Throughout this work, (X, τ) stands for a topological space (written frequently as X) and $\mathcal{I}$ is a nontrivial ideal on $\mathbb{N}$, the set of all positive integer numbers.

Definition 1.2. A sequence $\{x_n\}$ in X is called *$\mathcal{I}$-convergent* [11] to a point x_0, if for every nonempty open set U containing x_0, $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$.

Definition 1.3. A subset A of X is said to be *semi-open* [12], if there exists an open set U such that $U \subset A \subset Cl(U)$.

The collection of all semi-open sets of X is denoted by $SO(X)$. The complement of a semi-open set is called a *semi-closed* set. The *semi-closure* of a subset A of X, denoted by $sCl(A)$, is defined as

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the intersection of all semi-closed sets containing A [1]. Obviously, a point $x \in sCl(A)$ if and only if for every semi-open set U containing x, $U \cap A \neq \emptyset$.

In the following definition, we present some well-known in the literature types of functions in the literature, where X and Y are topological spaces.

Definition 1.4. A function $f : X \to Y$ is said to be:
(1) *semi-continuous* [12] if $f^{-1}(A) \in SO(X)$ for each open set A in Y;
(2)*irresolute* [2] if $f^{-1}(A) \in SO(X)$ for each $A \in SO(Y)$.

Theorem 1.5 ([12, Theorem 12])**.** *A function $f : X \to Y$ is semi-continuous if and only if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists $U \in SO(X)$ such that $x \in U$ and $f(U) \subset V$.*

Theorem 1.6. *A function $f : X \to Y$ is irresolute if and only if for each $x \in X$ and each $V \in SO(Y)$ containing $f(x)$, there exists $U \in SO(X)$ such that $x \in U$ and $f(U) \subset V$.*

Definition 1.7. A topological space X is said to be *semi-Hausdorff* [13], if for each pair x, y of distinct points of X, there exist disjoint semi-open sets containing x and y, respectively.

Definition 1.8. Let X be a topological space and A be a subset of X. A point $x \in X$ is said to be a *semi-limit point* [3] of A if for every semi-open set U containing x, $A \cap (U - \{x\}) \neq \emptyset$.

Definition 1.9. A topological space X is said to be:
(1) *semi-compact* [5] if every cover of X by semi-open sets has a finite subcover;
(2) *semi-Lindelöf* [6] if every cover of X by semi-open sets has a countable subcover.

2. The S-$\mathcal{I}$-convergence and its Basic Properties

In this section, we introduce the concept of an S-$\mathcal{I}$-convergent sequence to a point of a topological space and study its relevant properties.

Definition 2.1. A sequence $\{x_n\}$ in X is said to be *S-$\mathcal{I}$-convergent* to a point $x_0 \in X$ if for every nonempty semi-open set U containing x_0, $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$. In this case, x_0 is called the S-$\mathcal{I}$-limit of $\{x_n\}$ and is denoted by S-$\mathcal{I}$-$\lim x_n = x_0$.

Lemma 2.2. *The S-$\mathcal{I}$-convergence implies $\mathcal{I}$-convergence for any nontrivial ideal $\mathcal{I}$ on $\mathbb{N}$.*

Proof. The proof is immediate from the fact that any open set is semi-open and the definition of S-$\mathcal{I}$-convergence. □

The following example shows that the converse of Lemma 2.2 is not necessarily true.

Example 2.3. Let $\mathbb{R}$ be the set of real numbers with the usual topology, $\mathcal{I}$ be an admissible ideal and the sequence $\{x_n\}$ be defined as $x_n = a^n$, where $0 < a < 1$. Observe that the sequence $x_n = a^n$ is $\mathcal{I}$-convergent to 0, since for any open set W containing 0, the set $\{n \in \mathbb{N} : x_n \notin W\}$ is finite. Now consider the semi-open set $U = (-1, 0]$. It is easy to see that the set $\{n \in \mathbb{N} : x_n \notin U\}$ is equal to the set of natural numbers and then the sequence $x_n = a^n$ is not S-$\mathcal{I}$-convergent to 0.

Remark 2.4. If $\mathcal{I}$ is an admissible ideal, then an ordinary convergence implies $\mathcal{I}$-convergence and, in addition, if $\mathcal{I}$ does not contain any infinite set, both concepts coincide (see [11]).

An immediate consequence of Remark 2.4 is the following result.

Proposition 2.5. *If $\mathcal{I}$ is an admissible ideal not containing any infinite set, then S-$\mathcal{I}$-convergence implies convergence.*

The following example shows that the converse of Proposition 2.5 is not necessarily true.

Example 2.6. Let $\mathbb{R}$ be the set of real numbers with the usual topology and the sequence $\{x_n\}$ be defined as $x_n = \frac{1}{n}$. Observe that $\{x_n\}$ converges to 0. Consider the semi-open set $U = (-1, 0]$ and note that $0 \in U$, but $\{n \in \mathbb{N} : x_n \notin U\} = \mathbb{N}$. Therefore $\{n \in \mathbb{N} : x_n \notin U\} \notin \mathcal{I}$ (for any nontrivial ideal $\mathcal{I}$) and so $\{x_n\}$ is not S-$\mathcal{I}$-convergent to 0.

Proposition 2.7. *Let X be a discrete topological space and $\mathcal{I}$ be an admissible ideal, then convergence implies the S-$\mathcal{I}$-convergence.*

Proof. The proof follows from the fact that in the discrete topology the collections of open sets and semi-open sets are the same. □

Example 2.8. Consider $X = \mathbb{R}$ with the usual topology and $\{x_n\}$ the sequence in X defined as $x_n = (-1)^n$. It is clear that $\{x_n\}$ do not converge to any point of X. Now, let $M = \{2j-1 : j \in \mathbb{N}\}$ and take $\mathcal{I} = 2^M$. Then $\mathcal{I}$ is a nontrivial ideal on $\mathbb{N}$, and $\{x_n\}$ is S-$\mathcal{I}$-convergent (also $\mathcal{I}$-convergent) to -1.

Theorem 2.9. *Let X be a semi-Hausdorff space. If $\{x_n\}$ is a S-$\mathcal{I}$-convergent sequence in X, then the point of S-$\mathcal{I}$-convergence is unique.*

Proof. Consider $\{x_n\}$, a sequence that is S-$\mathcal{I}$-convergent in a semi-Hausdorff space X. Suppose that the sequence $\{x_n\}$ has two distinct points of S-$\mathcal{I}$-convergence, say x_0 and y_0. Since X is a semi-Hausdorff space, there exist $U, V \in SO(X)$ such that $x_0 \in U$, $y_0 \in V$ and $U \cap V = \emptyset$. On the other hand, by the definition of the S-$\mathcal{I}$-convergence, we have $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ and $\{n \in \mathbb{N} : x_n \notin V\} \in \mathcal{I}$, which implies that

$$\{n \in \mathbb{N} : x_n \in (U \cap V)^c\} = \{n \in \mathbb{N} : x_n \in U^c\} \cup \{n \in \mathbb{N} : x_n \in V^c\} \in \mathcal{I}.$$

As $\mathcal{I}$ is a nontrivial ideal, then $\{n \in \mathbb{N} : x_n \in (U \cap V)^c\} \neq \mathbb{N}$ and hence there exists $n_0 \in \mathbb{N}$ such that $n_0 \notin \{n \in \mathbb{N} : x_n \in (U \cap V)^c\}$, and so $x_{n_0} \in (U \cap V)$, which is a contradiction. This shows that the point of S-$\mathcal{I}$-convergence is unique. □

Corollary 2.10. *Let X be a Hausdorff space. If $\{x_n\}$ is a S-$\mathcal{I}$-convergent sequence in X, then the point of S-$\mathcal{I}$-convergence is unique.*

Theorem 2.11. *If $\mathcal{I}$ is an admissible ideal and if there exists a sequence $\{x_n\}$ of distinct elements in a subset A of X which is S-$\mathcal{I}$-convergent to $x_0 \in X$, then x_0 is a semi-limit point of A.*

Proof. Let U be any semi-open subset of X containing the point x_0. Since $\{x_n\}$ is S-$\mathcal{I}$-convergent to x_0, therefore $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ and so $\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$, otherwise it would be $\{n \in \mathbb{N} : x_n \notin U\} \cup \{n \in \mathbb{N} : x_n \in U\} = \mathbb{N} \in \mathcal{I}$, which contradicts that $\mathcal{I}$ is nontrivial. As $\mathcal{I}$ is an admissible ideal, it follows that $\{n \in \mathbb{N} : x_n \in U\}$ is an infinite set, otherwise,

$$\{n \in \mathbb{N} : x_n \in U\} = \bigcup_{x_n \in U} \{n\} \in \mathcal{I},$$

which is $\{n \in \mathbb{N} : x_n \in U\}$ would be a finite union of unitary sets, which is a contradiction because $\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$. Choose $n_0 \in \{n \in \mathbb{N} : x_n \in U\}$ such that $x_{n_0} \neq x_0$, then $x_{n_0} \in A \cap (U - \{x_0\})$ and so, $A \cap (U - \{x_0\}) \neq \emptyset$. This shows that for any semi-open set U containing the point x_0, we have $A \cap (U - \{x_0\}) \neq \emptyset$. □

Corollary 2.12. *If $\mathcal{I}$ is an admissible ideal and if there exists a sequence $\{x_n\}$ of distinct elements in a subset $A \subset X$ which is S-$\mathcal{I}$-convergent to $x_0 \in X$, then $x_0 \in sCl(A)$.*

Corollary 2.13. *If $\mathcal{I}$ is an admissible ideal and if there exists a sequence $\{x_n\}$ of distinct elements in a subset $A \subset X$ which is S-$\mathcal{I}$-convergent to $x_0 \in X$, then x_0 is a limit point of A.*

Definition 2.14. Let X be a topological space and $\{x_n\}$ be a sequence in X. We say that a point $x \in X$ is a *semi-cluster point* of the sequence $\{x_n\}$ if for every semi-open set U containing x, there exist infinitely many natural numbers n such that $x_n \in U$.

Theorem 2.15. *If $\mathcal{I}$ is an admissible ideal and $\{x_n\}$ is a sequence having a S-$\mathcal{I}$-convergent subsequence, then $\{x_n\}$ has a semi-cluster point.*

Proof. By the hypothesis, $\{x_n\}$ has a subsequence $\left\{x_{k(n)}\right\}$ which is S-$\mathcal{I}$-convergent, say to x_0. We will show that x_0 is a semi-cluster point of $\{x_n\}$. Let U be any semi-open set containing x_0, then $\left\{n \in \mathbb{N} : x_{k(n)} \notin U\right\} \in \mathcal{I}$, and since $\mathcal{I}$ is an admissible ideal, we have $\left\{n \in \mathbb{N} : x_{k(n)} \in U\right\}$ is an infinite set. Thus, U has infinite terms of the subsequence $\left\{x_{k(n)}\right\}$ and hence, of the sequence $\{x_n\}$. This shows that x_0 is a semi-cluster point of $\{x_n\}$. □

Theorem 2.16. *If $B \subset X$ is a semi-closed set, then for any sequence in B which is S-$\mathcal{I}$-convergent to x_0, we have $x_0 \in B$.*

Proof. Suppose that $B \subset X$ is a semi-closed set and $\{x_n\}$ is any sequence in B which is S-$\mathcal{I}$-convergent to the point x_0, but $x_0 \notin B$. Since B is semi-closed, we have $sCl(B) = B$ and thus, $x_0 \notin sCl(B)$. Then there exists a semi-open set U containing x_0 such that $B \cap U = \emptyset$. By the hypothesis, we have $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ and $\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$, which imply that $\{n \in \mathbb{N} : x_n \in U\} \neq \emptyset$. Thus, there exists $n_0 \in \{n \in \mathbb{N} : x_n \in U\}$, which is $x_{n_0} \in U$. Since $\{x_n\}$ is a sequence in B, hence $x_{n_0} \in B$, as well. Therefore, $x_{n_0} \in B \cap U$ and so $B \cap U \neq \emptyset$, which is a contradiction. □

Corollary 2.17. *If $B \subset X$ is a closed set, then for any sequence in B which is S-$\mathcal{I}$-convergent to x_0, we have $x_0 \in B$.*

Theorem 2.18. *Let $f : X \to Y$ be a semi-continuous function. If $\{x_n\}$ is a sequence in X which is S-$\mathcal{I}$-convergent to $x_0 \in X$, then $\{f(x_n)\}$ is an $\mathcal{I}$-convergent sequence to $f(x_0)$.*

Proof. Assume that $\{x_n\}$ is a sequence in X which is S-$\mathcal{I}$-convergent to $x_0 \in X$ and let V be an open set in Y containing the point $f(x_0)$. By Theorem 1.5, there exists $U \in SO(X)$ containing x_0 such that $f(U) \subset V$. We claim that $\{n \in \mathbb{N} : f(x_n) \notin V\} \subset \{n \in \mathbb{N} : x_n \notin U\}$. In effect, if $n_0 \in \{n \in \mathbb{N} : f(x_n) \notin V\}$, then $f(x_{n_0}) \notin V$ and so $f(x_{n_0}) \notin f(U)$, it follows that $x_{n_0} \notin U$ and hence $n_0 \in \{n \in \mathbb{N} : x_n \notin U\}$. Since $\{x_n\}$ is S-$\mathcal{I}$-convergent to x_0, we have $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ and, consequently, $\{n \in \mathbb{N} : f(x_n) \notin V\} \in \mathcal{I}$. This shows that $\{f(x_n)\}$ is $\mathcal{I}$-convergent to $f(x_0)$. □

It is clear that the condition that $f : X \to Y$ is semi-continuous does not guarantee that if $\{x_n\}$ is an S-$\mathcal{I}$-convergent sequence in X, then $\{f(x_n)\}$ is an S-$\mathcal{I}$-convergent sequence in Y. In the following theorem, we show that the S-$\mathcal{I}$-convergence is preserved by irresolute functions.

Theorem 2.19. *Let $f : X \to Y$ be an irresolute function. If $\{x_n\}$ is a sequence in X which is S-$\mathcal{I}$-convergent to $x_0 \in X$, then $\{f(x_n)\}$ is an S-$\mathcal{I}$-convergent sequence to $f(x_0)$.*

Proof. The proof is similar to that of Theorem 2.18. Just the use is made of the characterization of an irresolute function given in Theorem 1.6. □

Example 2.20. Let $\mathcal{I}$ be the collection of all finite subsets of $\mathbb{N}$, $X = \mathbb{R}$ with the usual topology, $Y = \{0, 1\}$ with the Sierpinski topology, $f : X \to Y$ the function defined by $f(x) = 0$ and $\{x_n\}$ the sequence in X defined as $x_n = (-1)^n$. Note that f is a semi-continuous (resp. irresolute) function such that $\{f(x_n)\}$ is $\mathcal{I}$-convergent (resp. S-$\mathcal{I}$-convergent) to $0 \in Y$, but $\{x_n\}$ do not S-$\mathcal{I}$-converge to any point of X.

3. Compactness and S-$\mathcal{I}$-convergence

Proposition 3.1. *Let X be a topological space and $\mathcal{I}$ be an admissible ideal that does not contain infinite sets. If any sequence $\{x_n\}$ in X has a subsequence which is S-$\mathcal{I}$-convergent, then (X, τ) is a sequentially compact space.*

Proof. This is an immediate consequence of Proposition 2.5. □

Proposition 3.2. *Let X be a topological space and $\mathcal{I}$ be an admissible ideal. If for any infinite subset A of X, there exists a sequence $\{x_n\}$ of distinct elements in A, which is S-$\mathcal{I}$-convergent in X, then (X, τ) is a limit point compact space.*

Proof. This is an immediate consequence of Corollary 2.10. □

Recall that a point p of a topological space X is said to be an *ω-accumulation point* of $A \subset X$ if for every open set U containing p, $U \cap A$ is an infinite set. On the other hand, a point $p \in X$ is said to be an *$\mathcal{I}$-cluster* point [11] of a sequence $\{x_n\}$ in X if for every open set U containing p, $\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$. In the following two definitions we introduce some modifications of these concepts using semi-open sets.

Definition 3.3. Let X be a topological space and $\{x_n\}$ be a sequence in X. A point $p \in X$ is called a *S-$\mathcal{I}$-cluster* point of $\{x_n\}$ if for any semi-open set U containing p, $\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$.

Definition 3.4. Let X be a topological space and $A \subset X$. We say that $p \in X$ is a *semi-ω-accumulation* point of A if for every semi-open set U containing p, $U \cap A$ is an infinite set.

Theorem 3.5. *Let X be a topological space and $\mathcal{I}$ be an admissible ideal. If every sequence $\{x_n\}$ in X has an S-$\mathcal{I}$-cluster point, then every infinite subset of X has a semi-ω-accumulation point. The converse is true if $\mathcal{I}$ does not contain infinite sets.*

Proof. Suppose that every sequence in X has an S-$\mathcal{I}$-cluster point and let A be an infinite subset of X, then there exists a sequence $\{x_n\}$ of distinct points in A. Let p be an S-$\mathcal{I}$-cluster point of $\{x_n\}$ and U be any semi-open set U containing p, then $\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$. Using the fact that $\mathcal{I}$ is an admissible ideal, it follows that $\{n \in \mathbb{N} : x_n \in U\}$ is an infinite subset; as a consequence, U contains infinitely many points of $\{x_n\}$ and hence of A, that is, $U \cap A$ is an infinite set. This shows that p is a semi-ω-accumulation point of A.

Conversely, suppose that every infinite subset of X has a semi-ω-accumulation point. Let $\{x_n\}$ be any sequence in X and let A be the range of $\{x_n\}$. If A is infinite, then by the hypothesis, A has a point of semi-ω-accumulation, say p. Let U be any semi-open set U containing p, then $U \cap A$ is an infinite set, and it follows that U has infinitely many points of A and hence, of the sequence $\{x_n\}$, which implies that $\{n \in \mathbb{N} : x_n \in U\}$ is an infinite set. Since $\mathcal{I}$ is an admissible ideal that does not contain infinite sets, we conclude that $\{n \in \mathbb{N} : x_n \in U\} \notin \mathcal{I}$, that is, p is an S-$\mathcal{I}$-cluster point of $\{x_n\}$. On the other hand, if A is finite, then there exists a point $p \in X$ such that $x_n = p$ for infinitely many subindexes n. Therefore, for every semi-open set U containing p, the set $\{n \in \mathbb{N} : x_n \in U\}$ is infinite and so, $\{n \in \mathbb{N} : x_n \in A\} \notin \mathcal{I}$, which implies that p is an S-$\mathcal{I}$-cluster point of $\{x_n\}$. □

Corollary 3.6. *Let X be a topological space and $\mathcal{I}$ be an admissible ideal. If every sequence $\{x_n\}$ has an S-$\mathcal{I}$-cluster point, then every infinite subset of X has an ω-accumulation point.*

Theorem 3.7. *Let X be a topological space and $\mathcal{I}$ be an admissible ideal. If X is a semi-Lindelöff space such that every sequence in X has an S-$\mathcal{I}$-cluster point, then X is a semi-compact space.*

Proof. Suppose that X is a semi-Lindelöff space such that every sequence in X has an S-$\mathcal{I}$-cluster point and let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be a semi-open cover of X. Since X is a semi-Lindelöff space, $\mathcal{U}$ contains a countable subcover, say $\mathcal{U}' = \{U_1, U_2, \ldots, U_m, \ldots\}$. Proceeding by induction, let $A_1 = U_1$ and for each $m > 1$, let A_m be the first member of the sequence of U's which is not covered by $U_1 \cup U_2 \cup \cdots \cup U_{m-1}$. We claim that in the above selection, there exists m_0 such that for all $m > m_0$ it is impossible to continue with the algorithm. In effect, if in the above selection it is possible to do this for all $n > 1$, we choose a point $a_n \in A_n$ for all $n \in \mathbb{N}$ such that $a_n \notin A_k$ for $k < n$. Now, consider the sequence $\{a_m\}$ and let p be an S-$\mathcal{I}$-cluster point of $\{a_n\}$. Then $p \in A_j$ for some j. By the definition of an S-$\mathcal{I}$-cluster point and the admissibility of the ideal $\mathcal{I}$, we have $\{n \in \mathbb{N} : a_n \in A_j\} \notin \mathcal{I}$ and $\{n \in \mathbb{N} : a_n \in A_j\}$ must be an infinite set of $\mathbb{N}$. Thus, there exists $r > j$ such that $r \in \{n \in \mathbb{N} : a_n \in A_j\}$; that is, there exists some $r > j$ such that $a_r \in A_j$, which is a contradiction. As a consequence, there exists m_0 such that for all $m > m_0$ it is impossible to continue the algorithm and, therefore, $\{A_1, A_2, \ldots, A_{m_0}\}$ is a finite subcover of X. □

Corollary 3.8. *Let X be a topological space and $\mathcal{I}$ be an admissible ideal. If X is a semi-Lindelöff space such that every sequence in X has an S-$\mathcal{I}$-cluster point, then X is a compact space.*

4. The $\mathcal{I}$-convergence in the Product Space

Theorem 4.1. *Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be an indexed family of topological spaces, $\prod_{\alpha \in \Delta} X_\alpha$ be the product space and $\{x_\alpha(n)\}$ be a sequence in X_α for all $\alpha \in \Delta$. Then $\{x_\alpha(n)\}$ is $\mathcal{I}$-convergent to p_α for all $\alpha \in \Delta$ if and only if $\{(x_\alpha(n))_{\alpha \in \Delta}\}$ is $\mathcal{I}$-convergent to $(p_\alpha)_{\alpha \in \Delta}$.*

Proof. Let A be an open set in $\prod_{\lambda \in \Lambda} X_\alpha$ containing the point $(p_\alpha)_{\alpha \in \Delta}$, then there exists a basic open set $B = \prod_{\alpha \in \Delta} B_\alpha$ such that $(p_\alpha)_{\alpha \in \Delta} \in B \subset A$. It follows that $p_\alpha \in B_\alpha$ for all $\alpha \in \Delta$. Since $\prod_{\alpha \in \Delta} B_\alpha$ is a basic open set in the product space $\prod_{\alpha \in \Delta} X_\alpha$, it follows that $B_\alpha = X_\alpha$, except for a finite number of indexes, say $\alpha_1, \dots, \alpha_k$. Thus, $p_{\alpha_i} \in B_{\alpha_i}$ for $i \in \{1, \dots, k\}$ and $p_\alpha \in X_\alpha$ for $\alpha \neq \alpha_1, \dots, \alpha_k$. Since $\{x_\alpha(n)\}$ is $\mathcal{I}$-convergent to p_α for all $\alpha \in \Delta$, therefore $\{n \in \mathbb{N} : x_\alpha(n) \notin B_\alpha\} \in \mathcal{I}$ for all $\alpha \in \Delta$ and hence

$$\bigcup_{\alpha \in \Delta} \{n \in \mathbb{N} : x_\alpha(n) \notin B_\alpha\} = \bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\} \in \mathcal{I}.$$

We claim that $\{n \in \mathbb{N} : (x_\alpha(n))_{\alpha \in \Delta} \notin B\} \subset \bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\}$.
In effect, let $n_0 \in \{n \in \mathbb{N} : (x_\alpha(n))_{\alpha \in \Delta} \notin B\}$, then we have $(x_\alpha(n_0))_{\alpha \in \Delta} \notin B = \prod_{\alpha \in \Delta} B_\alpha$, which implies that there exists $\alpha_0 \in \Delta$ such that $x_{\alpha_0}(n_0) \notin B_{\alpha_0}$ and since $B_\alpha = X_\alpha$ for $\alpha \neq \alpha_1, \dots, \alpha_k$, necessarily $\alpha_0 \in \{\alpha_1, \dots, \alpha_k\}$, otherwise there would be a contradiction; now, as $x_{\alpha_0}(n_0) \notin B_{\alpha_0}$, we have

$$n_0 \in \{n \in \mathbb{N} : x_{\alpha_0}(n) \notin B_{\alpha_0}\} \subset \bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\}.$$

Therefore, $\{n \in \mathbb{N} : (x_\alpha(n))_{\alpha \in \Delta} \notin B\} \subset \bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\}$. On the other hand, the fact that $B = \prod_{\alpha \in \Delta} B_\alpha \subset A$ implies that

$$\begin{aligned} \{n \in \mathbb{N} : (x_\alpha(n))_{\alpha \in \Delta} \notin A\} &\subset \{n \in \mathbb{N} : (x_\alpha(n))_{\alpha \in \Delta} \notin B\} \\ &\subset \bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\}. \end{aligned}$$

Since $\bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\} \in \mathcal{I}$, it follows that

$$\{n \in \mathbb{N} : (x_\alpha(n))_{\alpha \in \Delta} \notin A\} \in \mathcal{I},$$

which shows that $\{(x_\alpha(n))_{\alpha \in \Delta}\}$ is $\mathcal{I}$-convergent to $(p_\alpha)_{\alpha \in \Delta}$.

Conversely, let β be an arbitrary element of Δ and consider the set $\{n \in \mathbb{N} : x_\beta(n) \notin B_\beta\}$, where B_β is an arbitrary open set of X_β containing the point $p_\beta \in X_\beta$. Now, let $B = \prod_{\alpha \in \Delta} B_\alpha$ a basic open set in the product space $\prod_{\alpha \in \Delta} X_\alpha$ containing the point $(p_\alpha)_{\alpha \in \Delta}$ such that $\pi_\beta\big(\prod_{\alpha \in \Delta} B_\alpha\big) = B_\beta$. By the hypothesis, the set $\{n \in \mathbb{N} : (x_\alpha(n))_{\alpha \in \Delta} \notin B\} \in \mathcal{I}$. On the other hand, since $B = \prod_{\alpha \in \Delta} B_\alpha$ is a basic open set, therefore $B_\alpha = X_\alpha$ except for a finite number of indexes, say $\alpha_1, \dots, \alpha_k$. Suppose that $\beta = \alpha_j$ for some $1 \leq j \leq k$ (if $\beta \neq \alpha_j$ for all $1 \leq j \leq k$, the result is trivial). We claim that

$$\bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\} \subset \{n \in \mathbb{N} : (x_\alpha(n))_{\alpha \in \Delta} \notin B\}.$$

In effect, let $n_0 \in \bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\}$, then there exists $\alpha_0 \in \Delta$ such that $n_0 \in \{n \in \mathbb{N} : x_{\alpha_0}(n) \notin B_{\alpha_0}\}$, which implies that $x_{\alpha_0}(n_0) \notin B_{\alpha_0}$ and so, $(x_\alpha(n_0))_{\alpha \in \Delta} \notin B = \prod_{\alpha \in \Delta} B_\alpha$, hence $n_0 \in \{n \in \mathbb{N} : (x_\alpha(n))_{\alpha \in \Delta} \notin B\}$. As $\{n \in \mathbb{N} : (x_\alpha(n))_{\alpha \in \Delta} \notin B\} \in \mathcal{I}$, then $\bigcup_{i=1}^{k} \{n \in \mathbb{N} : x_{\alpha_i}(n) \notin B_{\alpha_i}\} \in \mathcal{I}$ and, consequently, $\{n \in \mathbb{N} : x_\beta(n) \notin A_\beta\} \in \mathcal{I}$. This shows that $\{x_\beta(n)\}$ is $\mathcal{I}$-convergent to p_β, and since $\beta \in \Delta$ is arbitrary, the proof is complete. $\square$

Corollary 4.2. *Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be an indexed family of topological spaces, $\prod_{\alpha\in\Delta} X_\alpha$ be the product space and $\{x_\alpha(n)\}$ be a sequence in X_α for all $\alpha \in \Delta$. If $\{x_\alpha(n)\}$ is S-$\mathcal{I}$-convergent to p_α for all $\alpha \in \Delta$, then $\{(x_\alpha(n))_{\alpha\in\Delta}\}$ is $\mathcal{I}$-convergent to $(p_\alpha)_{\alpha\in\Delta}$.*

Corollary 4.3. *Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be an indexed family of topological spaces, $\prod_{\alpha\in\Delta} X_\alpha$ be the product space and $\{x_\alpha(n)\}$ be a sequence in X_α for all $\alpha \in \Delta$. If $\{(x_\alpha(n))_{\alpha\in\Delta}\}$ is S-$\mathcal{I}$-convergent to $(p_\alpha)_{\alpha\in\Delta}$, then $\{x_\alpha(n)\}$ is S-$\mathcal{I}$-convergent to p_α for all $\alpha \in \Delta$.*

Recall that if $\{I_\alpha\}_{\alpha\in\Delta}$ is a chain of ideals on X, then $\bigcup_{\alpha\in\Delta} I_\alpha$ is an ideal on X [15]. Next, we give two immediate consequences related to a chain of ideals on $\mathbb{N}$.

Corollary 4.4. *Let $\{\mathcal{I}_\alpha\}_{\alpha\in\Delta}$ be a chain of nontrivial ideals on $\mathbb{N}$, $\prod_{\alpha\in\Delta} X_\alpha$ be the product space of a family of topological spaces $\{(X_\alpha, \tau_\alpha) :_{\alpha\in\Delta}\}$ and $\{x_\alpha(n)\}$ be a sequence in X_α for all $\alpha \in \Delta$. If $\{x_\alpha(n)\}$ is $\mathcal{I}_\alpha$-convergent to p_α for all $\alpha \in \Delta$, then $\{(x_\alpha(n))_{\alpha\in\Delta}\}$ is $\mathcal{I}$-convergent to $(p_\alpha)_{\alpha\in\Delta}$, where $\mathcal{I} = \bigcup_{\alpha\in\Delta} \mathcal{I}_\alpha$.*

Corollary 4.5. *Let $\{\mathcal{I}_\alpha\}_{\alpha\in\Delta}$ be a chain of nontrivial ideals on $\mathbb{N}$, $\prod_{\alpha\in\Delta} X_\alpha$ be the product space of a family of topological spaces $\{(X_\alpha, \tau_\alpha) :_{\alpha\in\Delta}\}$ and $\{x_\alpha(n)\}$ be a sequence in X_α for all $\alpha \in \Delta$. If $\{(x_\alpha(n))_{\alpha\in\Delta}\}$ is $\mathcal{I}_\alpha$-convergent to $(p_\alpha)_{\alpha\in\Delta}$, then $\{x_\alpha(n)\}$ is $\mathcal{I}$-convergent to p_α for all $\alpha \in \Delta$, where $\mathcal{I} = \bigcup_{\alpha\in\Delta} \mathcal{I}_\alpha$.*

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MULTILINEAR FEFFERMAN-STEIN INEQUALITY AND ITS GENERALIZATIONS

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Abstract. The Fefferman-Stein type inequalities are established for multilinear fractional maximal operators with a variable parameter defined with respect to the basis $\mathcal{B}$ on $\mathbb{R}^n$ which may be both either $\mathcal{Q}$ or $\mathcal{R}$, where $\mathcal{Q}$ (resp., $\mathcal{R}$) consists of all cubes (resp., of n-dimensional intervals) with sides parallel to the coordinate axes. Some related two-weight boundedness problems are also investigated.

1. Introduction

Let $\mathcal{B}$ in $\mathbb{R}^n$ be a basis which may be both either $\mathcal{Q}$ or $\mathcal{R}$, where $\mathcal{Q}$ (resp., $\mathcal{R}$) is a basis consisting of all cubes (resp., of $n-$dimensional intervals) with sides parallel to the coordinate axes. Further, let

$$\overrightarrow{f} := (f_1, \dots, f_m), \quad \overrightarrow{p} := (p_1, \dots, p_m), \quad \overrightarrow{w} = (w_1, \dots, w_m),$$

where p_i are the constants ($0 < p_i < \infty$) and w_i are a.e. positive functions defined on the Euclidean space. It will also be assumed that

$$\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}. \tag{1}$$

For a given function $\alpha(\cdot)$ on $\mathbb{R}^n$, let

$$\alpha_- := \inf \alpha(\cdot), \quad \alpha_+ := \sup \alpha(\cdot).$$

In this paper we establish the following inequalities: $1 < p_i, q < \infty$, $i = 1, \dots, m$, and $1 < p < q < \infty$, where p is defined by (1). Then

(i)

$$\left\| \Big(\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot)} \overrightarrow{f} \Big) v \right\|_{L^q} \le C \prod_{i=1}^{m} \left\| f_i \big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot), p_i, q} v_i \big)^{1/q} \right\|_{L^{p_i}}, \tag{2}$$

where $v(x) = \prod_{i=1}^m v_i^{p/p_i}(x)$, $\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot)}$ is a strong fractional maximal operator defined with respect to the basis $\mathcal{B}$ given by the formula

$$\mathcal{M}^{(\mathcal{B})}_{\alpha(x)}(\overrightarrow{f})(x) = \sup_{B \ni x, B \in \mathcal{B}} \prod_{i=1}^{m} \frac{1}{|B|^{1-\alpha(x)/(nm)}} \int_B |f_i(y_i)| dy_i, \quad 0 < \alpha_- \le \alpha_+ < mn, \tag{3}$$

and $\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot), p_i, q}$, $i = 1, \dots, m$, are the appropriate fractional maximal operators (see the definition in Theorem 2.1).

(ii)

$$\left\| \mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot),\mu} \overrightarrow{f} \right\|_{L^q_\mu} \le C \prod_{i=1}^{m} \left\| f_i \big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot), p, q, \mu} d\mu \big)^{1/(mq)} \right\|_{L^{p_i}_\mu}, \; 0 < \alpha_- \le \alpha_+ < mn,$$

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with $d\mu(x) = w(x)dx$, where w is a weight function satisfying the doubling condition, the maximal function $\mathcal{M}^{(\mathcal{B})}_{\alpha(x),\mu}$ is defined by

$$\mathcal{M}^{(\mathcal{B})}_{\alpha(x),\mu}(\overrightarrow{f})(x) = \sup_{B\ni x,\ B\in\mathcal{B}} \prod_{i=1}^{m} \frac{|B|^{\alpha(x)/(nm)}}{\mu(B)} \int_B |f_i(y_i)|d\mu,\ 0<\alpha_-\le\alpha_+<mn,\ 1<p<q<\infty,$$

and $\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q,\mu}$ is appropriate fractional maximal operator (see the definition in Theorem 2.2).

We claim that the these results are new even for the linear case ($m = 1$).

For two-weight inequalities and for strong fractional maximal operators with variable parameters we refer to the monograph [19], Chapter 6.

Recall that inequality (2) was derived in [14] for $v_1 = \cdots = v_m = v$ and $\alpha(\cdot) =$ const.

Operator (3) for $\alpha(x) \equiv 0$ and $\mathcal{B} = \mathcal{R}$ was introduced in [10]. In this case we have multi(sub)linear strong maximal operator denoted by $\mathcal{M}^{(S)}$ and defined with respect to rectangles in $\mathbb{R}^k$ with sides parallel to the coordinate axes. In that paper the authors studied one– and two–weight problems for $\mathcal{M}^{(S)}$. In particular, they proved that the one-weight boundedness $\mathcal{M}^{(S)} : L^{p_1}_{w_1} \times \cdots \times L^{p_m}_{w_m} \mapsto L^p_{\nu_{\overrightarrow{w}}}$, $\nu_{\overrightarrow{w}} = \prod_{j=1}^m w_j^{p/p_j}$, holds if and only if $\overrightarrow{w}$ weight satisfies the strong $A_{\overrightarrow{p}}$ condition

$$\sup_{R\in\mathcal{R}} \left(\frac{1}{|R|}\int_R \nu_{\overrightarrow{w}}(x)dx\right)^{1/p} \prod_{i=1}^{m} \left(\frac{1}{|R|}\int_R w_i^{1-p_i'}(x)dx\right)^{1/p_i'} < \infty.$$

Historically, multilinear fractional integrals were introduced in their papers by L. Grafakos [8], C. Kenig and E. Stein [11], L. Grafakos and N. Kalton [9]. In particular, these works deal with the operator

$$B_\gamma(f,g)(x) = \int_{\mathbb{R}^n} \frac{f(x+t)g(x-t)}{|t|^{n-\gamma}}dt,$$

where γ is a constant parameter satisfying the condition $0 < \gamma < n$.

In the above-mentioned papers it was proved that if $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, then B_γ is bounded from $L^{p_1} \times L^{p_2}$ to L^q.

As a tool to understand B_γ, the operator

$$\mathcal{I}_\gamma(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\gamma}}d\overrightarrow{y},$$

where $x \in \mathbb{R}^n$, γ is constant satisfying the condition $0 < \gamma < nm$, $\overrightarrow{f} := (f_1,\dots,f_m)$, $\overrightarrow{y} := (y_1,\dots,y_m)$, was studied as well. The corresponding maximal operator is given by (see [22]) the formula

$$\mathcal{M}_\gamma(\overrightarrow{f})(x) = \sup_{Q\ni x} \prod_{i=1}^{m} \frac{1}{|Q|^{1-\frac{\gamma}{mn}}} \int_Q |f_i(y_i)|dy_i,$$

and the supremum is taken over all cubes Q containing x.

For a variable parameter $\alpha(\cdot)$, let

$$\mathcal{I}_{\alpha(\cdot)}(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\dots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha(x)}}d\overrightarrow{y},$$

$$\mathcal{M}_{\alpha(\cdot)}(\overrightarrow{f})(x) = \sup_{Q\ni x} \prod_{i=1}^{m} \frac{1}{|Q|^{1-\frac{\alpha(x)}{mn}}} \int_Q |f_i(y_i)|dy_i,$$

where $0 < \alpha_- \le \alpha_+ < nm$. The operator $\mathcal{M}_{\alpha(\cdot)}$ for $\alpha \equiv 0$ was introduced and studied in [21].

It can be immediately checked that

$$\mathcal{I}_{\alpha(x)}(\overrightarrow{f})(x) \ge c_{n,\alpha(\cdot)}\mathcal{M}_{\alpha(x)}(\overrightarrow{f})(x), \quad f_i \ge 0, \quad i = 1,\dots,m.$$

Throughout the paper, we use the notation $\mathcal{Q}$ to denote the family of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

Let $0 < r < \infty$ and let μ be a σ- finite measure on $\mathbb{R}^n$. We denote by $L^r_\mu(\mathbb{R}^n)$ the class of all $\mu-$ measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^r_\mu(\mathbb{R}^n)} := \Bigg(\int\limits_{\mathbb{R}^n} |f(x)|^r d\mu(x)\Bigg)^{1/r} < \infty.$$

If $d\mu(x) = w(x)dx$ with a weight function w, then we also use the symbol $L^r_w(\mathbb{R}^n)$ for $L^r_\mu(\mathbb{R}^n)$.

Definition 1.1 (Vector Muckenhoupt condition, [21])**.** Let $1 \le p_i < \infty$ for $i = 1, \dots, m$. Let w_i be weights on $\mathbb{R}^n$, $i = 1, \dots, m$. We say that $\overrightarrow{w} \in A_{\overrightarrow{p}}(\mathbb{R}^n)$ (or simply $\overrightarrow{w} \in A_{\overrightarrow{p}}$) if

$$\sup_{Q \in \mathcal{Q}} \Bigg(\frac{1}{|Q|}\int\limits_Q \prod_{i=1}^m w_i^{p/p_i}(y)dy\Bigg)^{1/p} \prod_{i=1}^m \Bigg(\frac{1}{|Q|}\int\limits_Q w_i^{1-p'_i}(y)dy\Bigg)^{1/p'_i} < \infty.$$

Remark 1.1. In the linear case ($m = 1$) the class $A_{\overrightarrow{p}}$ coincides with the well-known Muckenhoupt class A_p.

Definition 1.2 (Vector Muckenhoupt–Wheeden condition, [22])**.** Let $1 \le p_i < \infty$ for $i = 1, \dots, m$. Suppose that $p < q < \infty$. We say that $\overrightarrow{w} = (w_1, \dots, w_m)$ satisfies $A_{\overrightarrow{p},q}(\mathbb{R}^n)$ condition ($\overrightarrow{w} \in A_{\overrightarrow{p},q}$) if

$$\sup_{Q \in \mathcal{Q}} \Bigg(\frac{1}{|Q|}\int\limits_Q \prod_{i=1}^m w_i^{q}(y)dy\Bigg)^{1/q} \prod_{i=1}^m \Bigg(\frac{1}{|Q|}\int\limits_Q w_i^{-p'_i}(y)dy\Bigg)^{1/p'_i} < \infty.$$

Theorem A ([21])**.** *Let $1 < p_i < \infty$, $i = 1, \dots, m$. Suppose that w_i are weights on $\mathbb{R}^n$. Then the operator M_0 is bounded from $L^{p_1}_{w_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}_{w_m}(\mathbb{R}^n)$ to $L^p_{\prod_{i=1}^m w_i^{p/p_i}}(\mathbb{R}^n)$ if and only if $\overrightarrow{w} \in A_{\overrightarrow{p}}(\mathbb{R}^n)$.*

Theorem B ([22])**.** *Let $1 < p_1, \dots, p_m < \infty$, $0 < \gamma < mn$, $\frac{1}{m} < p < \frac{n}{\gamma}$. Assume that q is an exponent satisfying the condition $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$. Suppose that w_i are a.e. positive functions on $\mathbb{R}^n$ such that $w_i^{p_i}$ are weights. Then the inequality*

$$\Bigg(\int\limits_{\mathbb{R}^n} \Big(\big|N_\gamma(\overrightarrow{f})(x)\big| \prod_{i=1}^m w_i(x)\Big)^q dx\Bigg)^{1/q} \le C \prod_{i=1}^m \Bigg(\int\limits_{\mathbb{R}^n} \big(|f_i(y)|w_i(x)\big)^{p_i} dx\Bigg)^{1/p_i},$$

holds, where N_γ is either I_γ or M_γ, if and only if $\overrightarrow{w} \in A_{\overrightarrow{p},q}(\mathbb{R}^n)$.

Remark 1.2. The two-weight problem for linear fractional integral operators has been already solved. We mention the papers due to E. Sawyer [26] for the conditions involving the operator itself, due to M. Gabidzashvili and V. Kokilashvili [6] (see also [13]) and R. L. Wheeden [32] for integral type conditions.

Finally, we mention that the weighted inequalities for multilinear fractional integrals were also studied in [25], [4], [14], [15]. The study of the boundedness of multi(sub)linear fractional strong maximal operators was initiated in [10] and continued in [15], [2], [3], etc.

1.1. **Preliminaries.** By the symbol $\mathcal{DQ}(\mathbb{R}^n)$ (or shortly, $\mathcal{DQ}$) is denoted a countable collection of dyadic cubes that enjoy the following properties:

(i) $Q \in \mathcal{DQ} \Rightarrow l(Q) = 2^k$ for some $k \in \mathbb{Z}$;

(ii) $Q, P \in \mathcal{DQ} \Rightarrow Q \cap P \in \{\emptyset, P, Q\}$;

(iii) for each $k \in \mathbb{Z}$ the set $\mathcal{DQ}_k = \{Q \in \mathcal{DQ} : l(Q) = 2^k\}$ forms a partition of $\mathbb{R}^n$.

Definition 1.3. We say that a weight function ρ satisfies the dyadic reverse doubling condition with respect to the cubes ($\rho \in \mathcal{RDQ}^{(d)}(\mathbb{R})$) if there exists a constant $d > 1$ such that

$$d\rho(Q') \le \rho(Q),$$

for all $Q', Q \in \mathcal{DQ}$, where Q' is a child interval of Q, i.e., $Q' \subset Q$ and $|Q| = 2^n|Q'|$.

We shall also need the following Carleson–Hörmander type embedding theorem regarding the dyadic intervals.

Theorem C (see, e.g., [29], [31])**.** *Let $1 < r < q < \infty$ and let ρ be a weight function on $\mathbb{R}^n$ such that $\rho^{1-r'}$ satisfies the dyadic reverse doubling condition. Then the Carleson–Hörmander type inequality*

$$\sum_{Q\in\mathcal{DQ}}\bigg(\int\limits_Q \rho^{1-r'}(x)dx\bigg)^{-q/r'}\bigg(\int\limits_Q f(x)dx\bigg)^q \le c\bigg(\int\limits_{\mathbb{R}^n} f^r(x)\rho(x)dx\bigg)^{q/r}$$

holds for all non-negative $f\in L^r_\rho(\mathbb{R}^n)$.

We denote by $\mathcal{DR}$ the family of all dyadic rectangles in $\mathbb{R}^n$ given by the formula

$$\mathcal{DR} := \{2^{-k}(m+[0,1)) : k, m \in \mathbb{Z}\}^n.$$

Definition 1.4. We say that a weight function ρ satisfies the dyadic reverse doubling condition with respect to the rectangles ($\rho \in \mathcal{RDR}^{(d)}(\mathbb{R}^n)$) if there exists a constant $d > 1$ such that

$$d\rho(R') \le \rho(R),$$

for all $R', R \in \mathcal{DR}$, where $R' \subset R$ and $|R| = 2|R'|$.

We denote by $\mathcal{DB}(\mathbb{R}^n)$ (or simply, $\mathcal{DB}$) the dyadic grid which is $\mathcal{DQ}$ for $\mathcal{B} = \mathcal{Q}$ and $\mathcal{DR}$ for $\mathcal{B} = \mathcal{R}$.

In the sequel, under the symbol $\mathcal{DRB}^{(d)}(\mathcal{R}^n)$ (or simply, $\mathcal{DRB}^{(d)}$) we mean the class of weights satisfying the dyadic reverse doubling condition in the sense of cubes if $\mathcal{B} = \mathcal{Q}$, and in the sense of rectangles if $\mathcal{B} = \mathcal{R}$. Further, for $B \in \mathcal{B}$ and $c > 0$ we denote by cB the set in $\mathbb{R}^n$ with the same center but with c times the side-length of B. We say that a measure μ defined on $\mathbb{R}^n$ satisfies the doubling condition with respect to $\mathcal{Q}$ ($\mu \in \mathcal{DCQ}$) if there is a positive constant b_μ such that for all $B \in \mathcal{Q}$ the inequality

$$\mu(2B) \le b_\mu \mu(B) \tag{4}$$

holds; further, we say that μ satisfies the doubling condition with respect to $\mathcal{R}$ ($\mu \in \mathcal{DCR}$) if (4) holds for all $B \in \mathcal{R}$. We write $\mu \in \mathcal{DCB}$ if $\mu \in \mathcal{DCQ}$ for a basis $\mathcal{Q}$, and $\mu \in \mathcal{DCR}$ for the basis $\mathcal{R}$.

Definition 1.5. We say that a measure μ satisfies the reverse doubling condition with respect to $\mathcal{R}$ ($\mu \in \mathcal{RDR}$) if there is a constant $\beta > 1$ such that $\beta\mu(R') \le \mu(R)$ for any $R, R' \in \mathcal{R}$, where R' is the two-equal division of R. Further, μ satisfies the reverse doubling condition with respect to $\mathcal{Q}$ ($\mu \in \mathcal{RDQ}$) if there is a constant $\beta > 1$ such that $\beta\mu(Q') \le \mu(Q)$ for any $Q, Q' \in \mathcal{Q}$, where R' is the 2^n-equal division of Q. We say that $\mu \in \mathcal{RDB}$ if $\mu \in \mathcal{RDR}$ for $\mathcal{B} = \mathcal{DR}$, and $\mu \in \mathcal{RDQ}$ for $\mathcal{B} = \mathcal{DQ}$.

The following fact was noticed in [28]:

Remark 1.3 ([28])**.** The condition $\mu \in \mathcal{DCB}$ is equivalent to the condition $\mu \in \mathcal{RDB}$.

Proposition 1.1 ([2])**.** *Let $1 < r < q < \infty$ and let ρ be a weight function on $\mathbb{R}^n$ such that $\rho^{1-r'} \in \mathcal{RDR}(\mathbb{R}^n)$. Then there is a positive constant C such that the inequality*

$$\sum_{R\in\mathcal{DR}}\bigg(\int\limits_R \rho^{1-r'}(x)dx\bigg)^{-q/r'}\bigg(\int\limits_R f(x)dx\bigg)^q \le C\bigg(\int\limits_{\mathbb{R}^n} f^r(x)\rho(x)dx\bigg)^{q/r}$$

holds for all non-negative $f\in L^r_\rho(\mathbb{R}^n)$.

Proposition 1.1 for the weight $\rho^{(n)}$ having the form $\rho^{(n)}(x_1,\dots,\rho_n) = \rho_1(x_1)\times\cdots\times\rho_n(x_n)$ can also be derived by a simple proof based on the mathematical induction. Indeed, the statement is true owing to Theorem C for $n = 1$. Suppose that it is true for $n-1$–dimensional dyadic rectangles and a weight of the form $\rho^{(n-1)}(x_1,\dots,\rho_{n-1}) = \rho_1(x_1)\times\cdots\times\rho_n(x_{n-1})$. We set $R := I_1\times\cdots\times I_n$, $R_{n-1} := I_1\times\cdots\times I_{n-1}$.

We have

$$\sum_{\rho^{(n)}(R)\in\mathcal{DR}(\mathbb{R}^n)} |R|^{\frac{q}{r'}}\Bigg(\int_R f(x_1,\dots,x_n)\prod_{i=1}^n \rho_i(x_i)dx_1\dots dx_n\Bigg)^q$$

$$\le \sum_{I_n\in\mathcal{DR}(\mathbb{R})} \rho_n(I_n)^{\frac{q}{r'}} \sum_{R_{n-1}\in\mathcal{DR}(\mathbb{R}^{n-1})}\Bigg(\prod_{i=1}^{n-1}\rho_i(x_i)\Bigg)^{-\frac{q}{r'}}$$

$$\times\Bigg(\int_{R_{n-1}}\Bigg(\int_{I_n} f(x_1,\dots,x_n)\rho(x_n)dx_n\Bigg)dx_1\dots dx_{n-1}\rho_1(x_1)\times\dots\times\rho_n(x_{n-1})\Bigg)^q$$

$$\le \sum_{I_n\in\mathcal{DR}(\mathbb{R}^{n-1})} |I_n|^{-\frac{q}{r'}}\Bigg(\int_{R_{n-1}}\Bigg(\int_{I_n} f(x_1,\dots,x_{n-1})\rho_{x_n}dx_1\Bigg)^q$$

$$\times\rho_1(x_1)\dots\rho_n(x_{n-1})dx_1\dots dx_{n-1}\Bigg)^p dx_n\Bigg)^{q/r}$$

$$\le \sum_{I_n\in\mathcal{DR}(\mathbb{R}^{n-1})} |I_n|^{-\frac{q}{p'}}\Bigg(\int_{I_n}\Bigg(f^r(x_1,\dots,x_{n-1})\rho_1(x_1)\dots\rho_n(x_n)dx_1\dots dx_{n-1}\Bigg)^{\frac{1}{r}}\rho_n(x_n)dx_n\Bigg)^q$$

$$\le C\Bigg(\int_{\mathbb{R}^n} f^r(x_1,\dots x_n)\rho(x_1,\dots,x_n)dx_1\dots dx_n\Bigg)^{\frac{q}{r}}.$$

2. Main Results

Now we formulate our main results.

Theorem 2.1. *Let $1<p_i<\infty$, $i=1,\dots,m$. Suppose that $p<q<\infty$ and $0<\alpha_-\le\alpha_+<mn$. Let v_i's be weights on $\mathbb{R}^n$, $i=1,\cdots,m$. We set $v(x)=\prod_{i=1}^m v_i^{p/p_i}(x)$. Then the inequality*

$$\|\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot)}(\overrightarrow{f})\|_{L^q_v(\mathbb{R}^n)}\le C\prod_{i=1}^m \|f_i\big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p_i,q}v_i\big)^{1/q}\|_{L^{p_i}(\mathbb{R}^n)}$$

holds, where

$$\widetilde{M}^{(\mathcal{B})}_{\alpha(x),p_i,q}v_i(x)=\sup_{B\ni x,B\in\mathcal{B}}\prod_{i=1}^n\Bigg(\frac{1}{|B|^{q/p}}\int_B |B|^{\frac{\alpha(y)q}{n}}v_i(y)dy\Bigg)^{p/p_i},\quad i=1,\dots,m.$$

The next two corollaries were proved in [15] for $\alpha(\cdot)\equiv\alpha=\text{const}$.

Corollary 2.1. *Let $1<p_i<\infty$, $i=1,\dots,m$. Suppose that $p<q<\infty$ and $0<\alpha_-\le\alpha_+<mn$. Let v be a weight on $\mathbb{R}^n$. Then the following inequality*

$$\|\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot)}(\overrightarrow{f})\|_{L^q_v(\mathbb{R}^n)}\le C\prod_{i=1}^m \|f_i\big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q}v\big)^{1/q}\|_{L^{p_i}(\mathbb{R}^n)},$$

holds, where

$$\widetilde{M}^{(\mathcal{B})}_{\alpha(x),p,q}v(x)=\sup_{B\ni x,B\in\mathcal{B}}\frac{1}{|B|^{q/p}}\int_B |B|^{\frac{\alpha(y)q}{n}}v(y)dy.$$

Corollary 2.2. *Let the conditions of Corollary 2.1 be satisfied. Then the inequality*

$$\|\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot)}(\overrightarrow{f})\|_{L^q_v(\mathbb{R}^n)}\le C\prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}\tag{5}$$

holds if and only if

$$\sup_{B\in\mathcal{B}}\frac{1}{|B|^{q/p}}\int_B |B|^{\frac{\alpha(y)q}{n}}v(y)dy<\infty.$$

Theorem 2.2. *Let $1 < p_i < \infty$, $i = 1, \dots, m$. Suppose that $p < q < \infty$ and $0 < \alpha_- \le \alpha_+ < mn$. Suppose that a measure μ is doubling, $d\mu(x) = v(x)dx$, where v is a weight on $\mathbb{R}^n$. Then the inequality*

$$\|\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot),\mu}(\overrightarrow{f})\|_{L^q_\mu(\mathbb{R}^n)} \le C \prod_{i=1}^{m} \Big\| f_i \Big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q,\mu}(d\mu)\Big)^{1/(mq)} \Big\|_{L^{p_i}_\mu(\mathbb{R}^n)}$$

holds, where

$$\widetilde{M}^{(\mathcal{B})}_{\alpha(x),p,q,\mu}(d\mu)(x) = \sup_{B \ni x, B \in \mathcal{B}} \frac{1}{\mu(B)^{q/p}} \int_B |B|^{\frac{\alpha(y)q}{n}} d\mu(y). \tag{6}$$

Corollary 2.3. *Let the conditions of Theorem 2.2 hold. Then the inequality*

$$\|\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot),\mu}(\overrightarrow{f})\|_{L^q_\mu(\mathbb{R}^n)} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}_\mu(\mathbb{R}^n)}$$

holds if and only if

$$\sup_{B \in \mathcal{B}} \frac{1}{\mu(B)^{q/p}} \int_B |B|^{\frac{\alpha(y)q}{n}} d\mu(y) < \infty.$$

Let us introduce the following strong fractional maximal operator defined with respect to a measure μ:

$$\mathcal{N}^{(\mathcal{B})}_{\alpha,\mu}(\overrightarrow{f})(x) = \sup_{B \ni x, B \in \mathcal{B}} \prod_{i=1}^{m} \frac{1}{\mu(B)^{1-\alpha/m}} \int_B |f_i(y)| d\mu(y),$$

where α is a constant such that $0 < \alpha < nm$.

We have also proved the following statement.

Theorem 2.3. *Let μ be an infinite measure on $\mathbb{R}^n$ without atoms such that $\mu \in \mathcal{DCB}$, $1 < p_i < \infty$, $i = 1, \dots, m$. Let α be a constant such that $0 < \alpha < n/p$. Then the inequality*

$$\|\mathcal{N}^{(\mathcal{B})}_{\alpha,\mu}(\overrightarrow{f})\|_{L^q_\mu(\mathbb{R}^n)} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}_\mu(\mathbb{R}^n)}$$

holds if and only if $q = \frac{np}{n-\alpha p}$.

It should be mentiond that the necessary and sufficient condition governing the boundedness of the multilinear fractional integral operator

$$T_{\gamma,\mu}\vec{f}(x) = \int_{X^m} \frac{f_1(y_1) \dots f(y_m)}{\big(d(x,y_1) + \dots + d(x,y_m)\big)^{m-\gamma}} d\mu(\vec{y}), \quad d\mu(\vec{y}) := d\mu(y_1) \dots d\mu(y_m)$$

defined with respect to a measure μ on a σ-algebra of Borel sets of quasi-metric space (X, d, μ) from the product $L^{p_1}(X,\mu) \times \dots \times L^{p_m}(X,\mu)$ to $L^q(X,\mu)$ has been established recently in [16].

3. Proofs of the Main Results

In this section we give the proofs of the main results of this paper.

First of all, we will need the following statement.

Lemma 3.1 ([20])**.** *There exist 2^n shifted dyadic grids*

$$\mathcal{D}^\beta := \{2^{-k}([0,1)^n + m + (-1)^k \beta) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}, \quad \beta \in \{0, 1/3\}^n,$$

such that for any given cube Q there are a β and a $Q_\beta \in \mathcal{D}^\beta$ with $Q \subset Q_\beta$ and $l(Q_\beta) \le 6l(Q)$.

As a consequence of this lemma, one has the following pointwise estimate

$$\mathcal{M}_{\alpha(\cdot)}(\overrightarrow{f})(x) \le C \sum_{\beta \in \{0,1/3\}^n} \mathcal{M}^{(d),\mathcal{D}^\beta}_{\alpha(\cdot)}(\overrightarrow{f})(x), \tag{7}$$

where $\mathcal{M}^{(d),\mathcal{D}^\beta}_{\alpha(\cdot)}$ is the dyadic multi(sub)linear fractional maximal operator corresponding to the dyadic grid $\mathcal{D}^\beta$ defined by

$$(\mathcal{M}^{(d),\mathcal{D}^\beta}_{\alpha(\cdot)}\overrightarrow{f})(x)=\sup_{\mathcal{D}^\beta\ni Q, Q\ni x}\prod_{i=1}^m\frac{1}{|Q|^{1-\alpha(\cdot)/(nm)}}\int\limits_Q|f_i(y_i)|dy_i,\quad 0<\alpha_-\le\alpha_+<mn,$$

and the constant C depending only on n, m and α.

Remark 3.1. It can be checked that estimates similar to (7) are also true for the operators $\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot)}$, $\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot),\mu}$ and $\mathcal{N}^{(\mathcal{B})}_{\alpha(\cdot),\mu}$ provided that $\mu\in\mathcal{DCB}$.

Proof of Theorem 2.1. First we show that the two-weight inequality

$$\|\mathcal{M}^{(d),\mathcal{B}}_{\alpha(\cdot)}(\overrightarrow{f})\|_{L^q_v(\mathbb{R}^n)}\le C\prod_{i=1}^m\|f_i\big(\widetilde{M}^{(d),(\mathcal{B})}_{\alpha(\cdot),p_i,q}v_i\big)^{1/q}\|_{L^{p_i}(\mathbb{R}^n)}$$

holds, where $\mathcal{M}^{(d),(\mathcal{B})}_{\alpha(\cdot)}$ is an appropriate to $\mathcal{M}^{\mathcal{B}}_{\alpha(\cdot)}$ dyadic maximal operator and

$$\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p_i,q}v_i(x)=\sup_{B\ni x, B\in(\mathcal{B})}\left(\frac{1}{|B|^{q/p}}\int\limits_B|B|^{\frac{\alpha(y)q}{n}}v_i(y)dy\right)^{p/p_i},\quad i=1,\dots,m.$$

For every $x\in\mathbb{R}^n$, let us take $B_x\in\mathcal{DB}$ such that $B_x\ni x$ and

$$\big(\mathcal{M}^{(d),(\mathcal{B})}_{\alpha(\cdot)}\overrightarrow{f}\big)(x)\le\frac{2}{|B_x|^{m-\alpha(x)/n}}\prod_{i=1}^m\int\limits_{B_x}|f_i(y_i)|dy_i. \tag{8}$$

Without loss of generality, we can assume, for example, that f_i, $i=1,\dots,m$ are non–negative, bounded and have compact supports.

Let us introduce a set

$$F_B=\{x\in\mathbb{R}^n: x\in B \text{ and } (8)\ \ holds\ \ for\ \ B\}.$$

It is obvious that $F_B\subset B$ and $\mathbb{R}^n=\cup_{B\in\mathcal{DB}}F_B$.

Now, applying Hölder's inequality, we have

$$\begin{aligned}
I&=\int\limits_{\mathbb{R}^n}\big(\mathcal{M}^{(d),(\mathcal{B})}_{\alpha(x)}\overrightarrow{f})(x)\big)^q\bigg(\prod_{i=1}^m v_i^{p/p_i}(x)\bigg)dx\le\sum_{B\in\mathcal{DB}}\int\limits_{F_B}\big(\mathcal{M}^{(d),(\mathcal{B})}_{\alpha(x)}\overrightarrow{f}(x)\big)^q\bigg(\prod_{i=1}^m v_i^{p/p_i}(x)\bigg)dx\\
&\le 2^q\sum_{B\in\mathcal{DB}}|B|^{-mq}\bigg(\int\limits_B|B|^{\alpha(x)q/n}\bigg(\prod_{i=1}^m v_i^{p/p_i}(x)\bigg)dx\bigg)\bigg(\prod_{i=1}^m\int\limits_B f_i(y_i)dy_i\bigg)^q\\
&\le 2^q\sum_{B\in\mathcal{DB}}|B|^{-mq}\prod_{i=1}^m\bigg(\int\limits_B|B|^{\alpha(x)q/n}v_i(x)dx\bigg)^{\frac{p}{p_i}}\bigg(\prod_{i=1}^m\int\limits_B f_i(y_i)dy_i\bigg)^q\\
&\le 2^q\sum_{B\in\mathcal{DB}}\prod_{i=1}^m|B|^{-q/p_i'}\bigg(\frac{1}{|B|^{q/p}}\int\limits_B|B|^{\alpha(x)q/n}v_i(x)dx\bigg)^{\frac{p}{p_i}}\bigg(\int\limits_B f_i(y_i)dy_i\bigg)^q\\
&\le 2^q\sum_{B\in\mathcal{DB}}\prod_{i=1}^m|B|^{-q/p_i'}\bigg(\int\limits_B f_i(y_i)\Big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p_i,q}v_i(y_i)\Big)^{1/q}dy_i\bigg)^q.
\end{aligned}$$

Further, by using Hölder's inequality in the form

$$\sum_k a_k^{(1)}\times\cdots\times a_k^{(m)}\le\prod_{j=1}^m\bigg(\sum_k(a_k^{(j)})^{p_j/p}\bigg)^{p/p_j}$$

for positive sequences $\{a_k^{(j)}\}$, $j=1,\dots,m$, we have

$$I \le 2^q \Bigg[\sum_{B\in\mathcal{DB}} |B|^{-(qp_1)/(pp_1')} \Bigg(\int\limits_B f_1(y_1)(\widetilde{M}^{(\mathcal{B})}_{\alpha(y_1),p_1,q} v_1(x))^{1/q} dy_1 \Bigg)^{qp_1/p} \Bigg]^{p/p_1}$$

$$\times \cdots \times \Bigg[\sum_{B\in\mathcal{DB}} |B|^{-(qp_m)/(pp_m')} \Bigg(\int\limits_B f_m(y_m)(\widetilde{M}^{(\mathcal{B})}_{\alpha(y_m),p_m,q} v_m(x))^{1/q} dy_m \Bigg)^{qp_m/p} \Bigg]^{p/p_m}.$$

Finally, Theorem C (for $\mathcal{B}=\mathcal{Q}$) and Proposition 1.1 (for $\mathcal{B}=\mathcal{R}$) for the exponents $(p_i, qp_i/p)$, $i=1,\dots,m$, and weight $\rho\equiv 1$, yield that

$$I \le c \prod_{i=1}^m \| f_i \big(\widetilde{M}^{(\mathcal{B})}_{\alpha(x),p_i,q} v_1(x)\big)^{1/q} \|^q_{L^{p_i}(\mathbb{R}^n)}.$$

At last, taking into account Remark 3.1, we can pass from $\mathcal{M}^{(d),\mathcal{B}}_{\alpha(x)}$ to $\mathcal{M}^{(\mathcal{B})}_{\alpha(x)}$. $\square$

Proof of Corollary 2.2. The proof of the *sufficiency* is a direct consequence of Theorem 2.1. For the *necessity* we take test functions: $f_j=\chi_B$, with $B\in\mathcal{B}$. By applying inequality (5) for these functions, we get the desired condition. $\square$

Proof of Theorem 2.2. Following the proof of Theorem 2.1 we get the inequality

$$\Big\| \mathcal{M}^{(d),(\mathcal{B})}_{\alpha(\cdot),\mu}(\overrightarrow{f}) \Big\|_{L^q_\mu(\mathbb{R}^n)} \le C \prod_{i=1}^m \Big\| f_i \Big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q,\mu}(d\mu) \Big)^{1/(qm)} \Big\|_{L^{p_i}(\mathbb{R}^n)}$$

where $\mathcal{M}^{(d),(\mathcal{B})}_{\alpha(\cdot),\mu}$ is the dyadic analogue of $\mathcal{M}^{(\mathcal{B})}_{\alpha(\cdot),\mu}$ and $\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q,\mu}(d\mu)(x)$ is defined by (6).

Indeed, observe that

$$I = \int\limits_{\mathbb{R}^n} \big(\mathcal{M}^{(d),(\mathcal{B})}_{\alpha(x),\mu} \overrightarrow{f})(x) \big)^q d\mu(x) \le \sum_{B\in(\mathcal{DB})} \int\limits_{F_B} \big(\mathcal{M}^{(d),(\mathcal{B})}_{\alpha(x)} \overrightarrow{f}(x) \big)^q d\mu(x)$$

$$\le 2^q \sum_{B\in\mathcal{DB}} \Big(\int\limits_B |B|^{(\alpha(x)q)/n} d\mu(x) \Big) \prod_{j=1}^m \Big(\frac{1}{\mu(B)} \int\limits_B f_j(y_j) d\mu(y_j) \Big)^q$$

$$= 2^q \sum_{B\in\mathcal{DB}} \prod_{j=1}^m \mu(B)^{-q/p_j'} \Bigg(\int\limits_B f_j(y_j) \Big(\mu(B)^{-q/p} \int\limits_B |B|^{\frac{\alpha(x)q}{n}} d\mu(x) \Big)^{1/(mq)} d\mu(y_j) \Bigg)^q$$

$$\le C \Bigg[\sum_{B\in\mathcal{DB}} \mu(B)^{-qp_1/(pp_1')} \Bigg[\int\limits_B f_1(y_1) \Big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q,\mu}(d\mu)(y_1) \Big)^{1/(mq)} d\mu(y_1) \Bigg]^{qp_1/p} \Bigg]^{p/p_1}$$

$$\times \cdots \times \Bigg[\sum_{B\in\mathcal{DB}} \mu(B)^{-qp_m/(pp_m')} \Bigg[\int\limits_B f_j(y_m) \Big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q,\mu}(d\mu)(y_m) \Big)^{1/(mq)} d\mu(y_m) \Bigg]^{qp_m/p} \Bigg]^{p/p_m}.$$

Now, applying Theorem C (for $\mathcal{B}=\mathcal{Q}$) and Proposition 1.1 (for $\mathcal{B}=\mathcal{R}$) for the weight $\rho\equiv v$ and exponents $(p_i, qp_i/p)$, $i=1,\dots,m$, we can conclude that

$$I \le C \prod_{j=1}^m \Big\| f_j \Big(\widetilde{M}^{(\mathcal{B})}_{\alpha(\cdot),p,q,\mu}(d\mu) \Big)^{1/(mq)} \Big\|^q_{L^{p_j}_\mu(\mathbb{R}^n)}. \qquad \square$$

Proof of Theorem 2.3. The sufficiency follows in the same manner as in the previous theorems by considering dyadic version of the operator $\mathcal{N}^{(\mathcal{B})}_{\alpha,\mu}$ and Remark 3.1; that is why we are focused on the necessity. Let $f_i(x)=\chi_B(x)$. Then the following inequality

$$\|\mathcal{N}^{(\mathcal{B})}_{\alpha,\mu} \overrightarrow{f}\|_{L^q_\mu(\mathbb{R}^n)} \ge \mu(B)^{1/q+\alpha/n}$$

holds.

On the other hand, we notice that

$$\prod_{i=1}^{m} \|f_i\|_{L^{p_i}_{\mu}(\mathbb{R}^n)} = \mu(B)^{1/p},$$

therefore,

$$\mu(B)^{1/q+\alpha/n-1/p} \le C.$$

Since $\mu(\mathbb{R}^n) = \infty$ and μ is a measure without atoms, we conclude that

$$q = \frac{pn}{n - \alpha p}. \qquad \square$$

Remark 3.2. Thus from the above proof we can conclude that in the necessity part of Theorem 2.3 no doubling condition is needed.

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GENERALIZED SCHWARTZ TYPE SPACES AND LCT BASED PSEUDO DIFFERENTIAL OPERATOR

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Abstract. In connection with the LCT, in this paper, we define the Schwartz type spaces $\mathcal{S}_{\Delta,\alpha,A}$, $\mathcal{S}^{\Delta,\beta,B}$, $\mathcal{S}^{\Delta,\beta,B}_{\Delta,\alpha,A}$ and study the mapping properties of LCT between these spaces. Moreover, we define a generalized Δ-pseudo differential operator and investigate its mapping properties in the framework of the above Schwartz type spaces.

1. Introduction

The Fourier transform

$$\hat{f}(\xi) := \mathcal{F}[f;\xi] = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx$$

and the related convolution

$$(f * g)(\xi) = \int_{\mathbb{R}} f(\xi - x) g(x)\, dx$$

have become an essential tool for solving many practical problems over the last few decades. Because of their usefulness, these notions have been generalized and extended by several people to give rise more general transforms and convolutions such as fractional Fourier transform [8], [12], [25], [33]. One such generalizaion is the so-called linear canonical transform (LCT) introduced in 1971 [26] which is connected with the 2×2 matrix M given by

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \text{with } ad - bc = 1.$$

The LCT is defined by

$$\mathcal{L}_M[f;\xi] = \int_{\mathbb{R}} f(x) K_M(x,\xi)\, dx,$$

where the kernel K_M is defined by

$$K_M(x,\xi) = \begin{cases} \frac{1}{\sqrt{2\pi b i}} \exp\left[\frac{i}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\xi + \frac{d}{b}\xi^2\right)\right], & \text{if } b \neq 0 \\ \frac{1}{\sqrt{a}} e^{i\left(\frac{c}{2a}\right)\xi^2} \delta\left(x - \frac{\xi}{a}\right), & \text{if } b = 0. \end{cases}$$

The convolution related to LCT is given by

$$(f \star_M g)(x) = \int_{\mathbb{R}} f(\xi) g(x-\xi) \exp\left[i\frac{a}{b}\xi(x-\xi)\right] d\xi$$

and the inverse LCT is defined by

$$\mathcal{L}_{M^{-1}}[f;x] = \int_{\mathbb{R}} f(\xi) K_{M^{-1}}(\xi, x)\, d\xi,$$

where M^{-1} is the inverse of the matrix M.

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At present, "Fourier Analysis" is usually termed as "Time Frequency Analysis". In this context, the Fourier transform rotates the signals from the time axis to the frequency axis by 90 degrees. It has been observed that certain optical systems rotated the signals by an arbitrary angle which requires the notion of fractional Fourier transforms, i.e., a one-parameter family of transforms. The linear canonical transforms (LCT) form a class of three-parameter family of transforms involving many known transforms. For notational convenience, if we write the matrix M as $(a, b; c, d)$, then the matrices $(0, 1; -1, 0)$ and $(\cos\alpha, \sin\alpha; -\sin\alpha, \cos\alpha)$ correspond, respectively, to the Fourier and fractional Fourier transforms. More special matrices lead to some other known integral transforms, e.g., Fresnel transform, chirp functions etc. Various applications of LCT have been realized in the field of electromagnetic, acoustic and other wave propagation problems. As mentioned in [10], LCT is known under other terminology as well, such as a quadratic phase integral [2], generalized Huygens integral [28], generalized Fresnel transform [9], [13], etc.

Recently, in [23], the authors have studied certain mappings properties of LCT and the associated pseudo-differential operators in a variant of Schwartz space denoted by $\mathcal{S}_M \equiv \mathcal{S}_M(\mathbb{R})$.

In this paper, we first introduce further variants of the space $\mathcal{S}_M$, denoted by $\mathcal{S}_{\Delta,\alpha,A}$, $\mathcal{S}^{\Delta,\beta,B}$ and $\mathcal{S}^{\Delta,\beta,B}_{\Delta,\alpha,A}$, where Δ is a differential operator defined and studied in Section 2, and α, β, A and B are certain constants. These spaces extend the spaces $\mathcal{S}_\alpha$, $\mathcal{S}^\beta$ and $\mathcal{S}^\beta_\alpha$ (see [5]). We study the mapping properties of LCT in the spaces $\mathcal{S}_{\Delta,\alpha,A}$, $\mathcal{S}^{\Delta,\beta,B}$ and $\mathcal{S}^{\Delta,\beta,B}_{\Delta,\alpha,A}$. This is done in Section 3. Finally, in Section 4, we define a generalized Δ-pseudo differential operator and study its mapping properties in the framework of the spaces $\mathcal{S}_{\Delta,\alpha,A}$, $\mathcal{S}^{\Delta,\beta,B}$ and $\mathcal{S}^{\Delta,\beta,B}_{\Delta,\alpha,A}$.

2. LCT Based Convolution and Differential Operators

We begin this section with mentioning that a Young type inequality can be proved for the convolution $\star_M$, and this can be done on lines, similar to those obvious modifications performed in [23]. We only state the result.

Theorem 2.1. *Let $1 \le p < \infty$, $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$. Then $(f \star_M g) \in L^p(\mathbb{R})$ with*

$$\|f \star_M g\|_{L^p(\mathbb{R})} \le \|f\|_{L^1(\mathbb{R})}\|g\|_{L^p(\mathbb{R})}.$$

Next, we prove the following

Theorem 2.2. *Let f be continuous and g be continuous with a compact support. Then $f \star_M g$ is continuous.*

Proof. Let $h \in \mathbb{R}$. Then

$$\begin{aligned}
&|(f \star_M g)(x+h) - (f \star_M g)(x)| \\
&\qquad = \Big| \int_{\mathbb{R}} f(y) g(x+h-y) \exp[i(a/b)y(x+h-y)]\, dy \\
&\qquad\qquad - \int_{\mathbb{R}} f(y) g(x-y) \exp[i(a/b)y(x-y)]\, dy \Big| \\
&\qquad = \Big| \int_{\mathbb{R}} f(y)(g(x+h-y)\exp[i(a/b)yh] - g(x-y)) \exp[i(a/b)y(x-y)]\, dy \Big| \\
&\qquad \le \int_{\mathbb{R}} \Big| f(y)\Big(g(x+h-y)\exp[i(a/b)yh] - g(x-y)\Big)\Big|\, dy \\
&\qquad = \int_{\mathbb{R}} \Big| f(y)\Big(g(x+h-y)\exp[i(a/b)yh] - g(x-y)\exp[i(a/b)yh] \\
&\qquad\qquad + g(x-y)\exp[i(a/b)yh] - g(x-y)\Big)\Big|\, dy
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}} |f(y)||g(x+h-y)-g(x-y)|\,dy \\
&\qquad + \int_{\mathbb{R}} |f(y)||g(x-y)||\exp[i(a/b)yh]-1|\,dy \\
&=: I_1 + I_2.
\end{aligned}
$$

Let $K := \operatorname{supp}(g)$ be compact. Then for any fixed x,

$$
x - K = \{x - y : y \in K\}
$$

is compact and therefore, f is uniformly continuous on $x-K$. Thus, for each $\varepsilon > 0$, there exists $\eta > 0$ such that if $|h| < \eta$, then $I_1 \to 0$ as $h \to 0$. Further, on $x-K$, f, g are bounded, therefore

$$
I_2 \leq \int_{\mathbb{R}} |f(y)||g(x-y)|2|\sin(a/2b)yh|\,dy
$$

which tends to 0 as $h \to 0$. Hence $|(f \star_M g)(x+h) - (f \star_M g)(x)| \to 0$ as $h \to 0$ and the assertion follows. □

A stronger version of Theorem 2.2 is the following

Theorem 2.3. *If $f \in C^\infty(\mathbb{R})$ and g is continuous with a compact support, then $f \star_M g$ is C^∞.*

Proof. We have

$$
\begin{aligned}
&\frac{1}{h}[(f \star_M g)(x+h) - (f \star_M g)(x)] \\
&\quad = \frac{1}{h}\int_{\mathbb{R}} g(y)\Big(f(x+h-y)\exp[i(a/b)yh] - f(x-y)\Big)\exp[i(a/b)y(x-y)]\,dy \\
&\quad = \frac{1}{h}\int_{\mathbb{R}} g(y)\Big(f(x+h-y)\exp[i(a/b)yh] - \exp[i(a/b)yh]f(x-y) \\
&\qquad\quad + \exp[i(a/b)yh]f(x-y)) - f(x-y)\Big)\exp[i(a/b)y(x-y)]\,dy \\
&\quad = \frac{1}{h}\int_{\mathbb{R}} g(y)\Big(f(x+h-y) - f(x-y)\Big)\exp[i(a/b)y(x+h-y)]\,dy \\
&\qquad\quad + \frac{1}{h}\int_{\mathbb{R}} g(y)(\exp[i(a/b)yh]-1)f(x-y)\exp[i(a/b)y(x-y)]\,dy. \\
&\quad \to (Df \star_M g)(x) + (f \star_M (ia/b)(\cdot)g)(x)
\end{aligned}
$$

as $h \to 0$. Therefore, it follows that $f \star_M g$ is differentiable if f is differentiable. It can be proved by induction that

$$
D_x^n(f \star_M g)(x) = \sum_{r=0}^{n} A_{n,r}(D^{n-r}f \star_M (ia/b(\cdot))^r g)(x),
$$

where $A_{n,r}$ are appropriate constants. Hence, $f \star_M g \in C^\infty$. □

Remark 2.4. Since $f \star_M g$ is commutative, therefore, if $g \in C^\infty$ and f is continuous with a compact support, then

$$
D_x^n(f \star_M g)(x) = \sum_{r=0}^{n} A_{n,r}((ia/b(\cdot))^{n-r}f \star_M D^r g)(x)
$$

and, consequently, $f \star_M g \in C^\infty$.

Denote $D_x := \frac{d}{dx}$. Let us define the following generalized differential operators based on the LCT:

$$\Delta_{x,a} = D_x - i\frac{a}{b}x$$
$$\Delta^*_{x,a} = -\left(D_x + i\frac{a}{b}x\right).$$

Remark 2.5. The following can be observed immediately:

(i) $\triangle_{x,a}K_M(x,\xi) = \left(\frac{-i\xi}{b}\right)K_M(x,\xi)$.
(ii) $\triangle_{\xi,d}K_M(x,\xi) = \left(\frac{-ix}{b}\right)K_M(x,\xi)$.
(iii) $\triangle^*_{x,a}K_{M^{-1}}(\xi,x) = \left(\frac{-i\xi}{b}\right)K_{M^{-1}}(\xi,x)$.
(iv) $\triangle^*_{\xi,d}K_{M^{-1}}(\xi,x) = \left(\frac{-ix}{b}\right)K_{M^{-1}}(\xi,x)$.

Let us recall that the Schwartz space $\mathcal{S}(\mathbb{R})$ consists of all functions $\phi \in C^\infty$ such that

$$\sup_{x\in\mathbb{R}}|x^k\phi^{(q)}(x)| \le m_{kq}, \quad k,q = 0,1,2,\dots .$$

Some of the properties of the operator $\triangle_{x,a}$ are given below which can be proved in a way, similar to [23].

Proposition 2.6.

(i) *For $\phi \in \mathcal{S}(\mathbb{R})$, the follwoing*

$$(\triangle_{\xi,d})^n\,\mathcal{L}_M[\phi;\xi] = \mathcal{L}_M\left[\left(\frac{-ix}{b}\right)^n\phi;\xi\right] \textit{holds.}$$

(ii) *For $\phi,\psi \in \mathcal{S}(\mathbb{R})$, the following Leibnitz type rule*

$$\triangle_{x,a}(\phi(x)\psi(x)) = \sum_{r=0}^{n}A_{n,r}D^r_x\phi(x)\cdot\triangle^{n-r}_{x,a}\psi(x)\textit{holds.}$$

Remark 2.7. The results similar to those of Proposition 2.6 can also be proved for $\triangle^*_{x,a}, \triangle_{\xi,d}$ and $\triangle^*_{\xi,d}$.

3. Schwartz Type Spaces Based on LCT

The space S_Δ was defined in [17] (see also [23]) as the space of all $\phi \in C^\infty$ for which

$$\sup_{t\in\mathbb{R}}\left|x^k\triangle^q_{x,a}\phi(x)\right| < \infty, \qquad k,q \in \mathbb{N}_0 \equiv \mathbb{N}\cup\{0\}.$$

When $\triangle_{x,a}$ is the differential operator $\frac{d}{dx}$, the space S_Δ coincides with the standard Schwartz space $\mathcal{S}$. Let us note from the construction of the Schwartz space $\mathcal{S} := \mathcal{S}$ that the sequence m_{kq} depends on both k and q. The Gelfand and Shilov type spaces are the variants of the space $\mathcal{S}$, in which the sequence m_{kq} depends only on k, or only on q, or on both. Such spaces are denoted, respectively, by S_α, S^β and S^β_α. These spaces have further been generalized to give rise to the spaces $S_{\alpha,A}$, $S^{\beta,B}$ and $S^{\beta,B}_{\alpha,A}$. For a systematic study and related results about these spaces, one may refer to [5].

Below, we define and study further generalizations of the spaces $S_{\alpha,A}$, $S^{\beta,B}$ and $S^{\beta,B}_{\alpha,A}$ in which the derivative $\dfrac{d}{dx}$ is replaced by more general operators Δ and Δ^*.

In the literature (see, e.g., [5]), various spaces of type $\mathcal{S}$ such as $\mathcal{S}_\alpha, \mathcal{S}^\beta, \mathcal{S}^\beta_\alpha$ have been defined and studied. In this section, we define and study similar variants of the space $\mathcal{S}_\Delta$.

Definition 3.1. Let $\delta > 0$. We define the space $S_{\Delta,\alpha,A}$ that consists of all $\phi \in C^\infty$ such that

$$|x^k\triangle^q_{x,a}\phi(x)| \le C_{q,\delta}(A+\delta)^k k^{k\alpha},$$

where $k,q \in \mathbb{N}_0$ and $C_{q,\delta}$ depends on ϕ.

Definition 3.2. Let $\rho > 0$. We define the space $\mathcal{S}^{\Delta,\beta,B}$ that consists of all $\phi \in C^\infty$ such that

$$|x^k \triangle_{x,a}^q \phi(x)| \leq C_{k,\rho}(B+\rho)^q q^{q\beta},$$

where $k, q \in \mathbb{N}_0$ and $C_{k,\rho}$ depends upon ϕ.

Definition 3.3. Let $\delta, \rho > 0$. We define the space $\mathcal{S}_{\Delta,\alpha,A}^{\Delta,\beta,B}$ that consists of all $\phi \in C^\infty$ such that

$$|x^k \triangle_{x,a}^q \phi(x)| \leq C_k (A+\delta)^k (B+\rho)^q k^{k,\alpha} q^{q\beta},$$

where $k, q \in \mathbb{N}_0$ and C_k depends on ϕ.

Remark 3.4. We also define the spaces $\mathcal{S}_{\Delta^*,\alpha,A}$, $\mathcal{S}^{\Delta^*,\beta,B}$ and $\mathcal{S}_{\Delta^*,\alpha,A}^{\Delta^*,\beta,B}$, where Δ in Definitions 3.1, 3.2 and 3.3, is replaced by Δ^*.

Theorem 3.5. *Let $\phi \in \mathcal{S}_{\Delta^*,\alpha,A}$. Then $\mathcal{L}_M[\phi;\cdot] \in \mathcal{S}^{\Delta,\alpha,B}$.*

Proof. We have

$$\begin{aligned}
\xi^k \triangle_{\xi,d}^q \mathcal{L}_M[\phi;\xi] &= \xi^k \triangle_{\xi,d}^q \int_{\mathbb{R}} K_M(x,\xi)\phi(x)\,dx \\
&= \xi^k \int_{\mathbb{R}} \triangle_{\xi,d}^q K_M(x,\xi)\phi(x)\,dx \\
&= \xi^k \int_{\mathbb{R}} \left(\frac{-ix}{b}\right)^q K_M(x,\xi)\phi(x)\,dx \\
&= \left(\frac{-i}{b}\right)^{q-k} \int_{\mathbb{R}} \left(\frac{-i\xi}{b}\right)^k K_M(x,\xi)x^q\phi(x)\,dx \\
&= \left(\frac{-i}{b}\right)^{q-k} \int_{\mathbb{R}} (\triangle_{x,a})^k K_M(x,\xi)x^q\phi(x)\,dx \\
&= \left(\frac{-i}{b}\right)^{q-k} \int_{\mathbb{R}} K_M(x,\xi)(\triangle_{x,a}^*)^k (x^q\phi(x))\,dx \\
&= \left(\frac{-i}{b}\right)^{q-k} \int_{\mathbb{R}} K_M(x,\xi)\left(\sum_{r=0}^k A_{k,r} D_x^r x^q (\triangle_{x,a}^*)^{(k-r)}\phi(x)\right)dx \\
&= \left(\frac{-i}{b}\right)^{q-k} \left(\sum_{r=0}^k A_{k,r} \int_{\mathbb{R}} K_M(x,\xi) D_x^r x^q (\triangle_{x,a}^*)^{(k-r)}\phi(x)\,dx\right)
\end{aligned}$$

so that

$$\begin{aligned}
&|\xi^k \triangle_{\xi,d}^q \mathcal{L}_M[\phi;\xi]| \\
&= \left|\left(\frac{-i}{b}\right)^{q-k} \left(\sum_{r=0}^k A_{k,r} \int_{\mathbb{R}} K_M(x,\xi) \frac{q!}{(q-r)!}\psi(x)^{q-r} (\triangle_{x,a}^*)^{(k-r)}\phi(x)\,dx\right)\right|,
\end{aligned}$$

where

$$\psi(x) = \begin{cases} x & \text{if } q-r \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Denote $|A_k| = \sup\limits_r |A_{k,r}|$. Then

$$|\xi^k \triangle_{\xi,d}^q \mathcal{L}_M[\phi;\xi]|$$

$$
\begin{aligned}
&\leq \left(\frac{1}{|b|}\right)^{q-k} \left(\sum_{r=0}^{k} |A_{k,r}| \int_{\mathbb{R}} |K_M(x,\xi)| \frac{q!}{(q-r)!} |\psi(x)|^{q-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)|\, dx\right) \\
&\leq \left(\frac{1}{|b|}\right)^{q-k} |A_k| \left(\sum_{r=0}^{k} \int_{\mathbb{R}} |K_M(x,\xi)| \frac{q!}{(q-r)!} |\psi(x)|^{q-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)|\, dx\right) \\
&= \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! \left(\sum_{r=0}^{k} \int_{\mathbb{R}} |K_M(x,\xi)| \frac{q!}{k!(q-r)!} |\psi(x)|^{q-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)|\, dx\right) \\
&\leq \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! \left(\sum_{r=0}^{k} \int_{\mathbb{R}} |K_M(x,\xi)| \frac{q!}{r!(q-r)!} |\psi(x)|^{q-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)|\, dx\right) \\
&= \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \int_{\mathbb{R}} |\psi(x)|^{q-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)|\, dx\right) \\
&= \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \int_{\mathbb{R}} (1+|x|^2) |\psi(x)|^{q-r}\right. \\
&\qquad \left. \times (\triangle_{x,a}^{*})^{(k-r)} \phi(x)| \frac{dx}{(1+|x|^2)}\right) \\
&= \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \left[\int_{\mathbb{R}} |\psi(x)|^{q-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)| \frac{dx}{(1+|x|^2)}\right.\right. \\
&\qquad \left.\left. + \int_{\mathbb{R}} |\psi(x)|^{q+2-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)| \frac{dx}{(1+|x|^2)}\right]\right) \\
&\leq \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \int_{\mathbb{R}} 2 |\psi(x)|^{q+2-r} |(\triangle_{x,a}^{*})^{(k-r)} \phi(x)| \frac{dx}{(1+|x|^2)}\right) \\
&\leq 2 \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} C_{k-r,\delta} (A+\delta)^{q+2-r} (q+2-r)^{(q+2-r)\alpha}\right. \\
&\qquad \left. \times \int_{\mathbb{R}} \frac{dx}{(1+|x|^2)}\right) \\
&\leq 2 \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \int_{\mathbb{R}} C_{k-r,\delta} (A+\delta)^{q+2-r}\right. \\
&\qquad \left. \times (q+2)^{(q+2)\alpha} \frac{dx}{(1+|x|^2)}\right) \\
&\leq 2 \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} C_{k,\delta} (A+\delta)^{q+2-r}\right. \\
&\qquad \left. \times (q+2)^{(q+2)\alpha} \int_{\mathbb{R}} \frac{dx}{(1+|x|^2)}\right) \\
&\leq 2 \left(\frac{1}{|b|}\right)^{q-k} |A_k| k! |K_M(x,\xi)| C_{k,\delta} \left(\sum_{r=0}^{q+2} \frac{q!}{r!(q-r)!} (A+\delta)^{q+2-r}\right. \\
&\qquad \left. \times (q+2)^{(q+2)\alpha} \int_{\mathbb{R}} \frac{dx}{(1+|x|^2)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\left(\frac{1}{|b|}\right)^{q-k}|A_k|k!|K_M(x,\xi)|C_{k,\delta}(1+A+\delta)^{q+2}(q+2)^{(q+2)\alpha}\int\limits_{\mathbb{R}}\frac{dx}{(1+|x|^2)}\\
&= 2\pi\left(\frac{1}{|b|}\right)^{-k-2}|A_k|k!|K_M(x,\xi)|C_{k,\delta}\left(\frac{1+A}{|b|}+\delta/|b|\right)^{q+2}(q+2)^{(q+2)\alpha}\\
&= 2\pi|b|^{k+2}|A_k|k!|K_M(x,\xi)|C_{k,\delta}\left(\frac{1+A}{|b|}+\delta/|b|\right)^{q+2}(q+2)^{(q+2)\alpha}\\
&= D_{k,\delta}\,(B+\rho)^{q+2}\,(q+2)^{(q+2)\alpha}\\
&= D_{k,\rho}\,(B+\rho)^{q+2}\,(q+2)^{(q+2)\alpha}\\
&= E_{k,\rho}\,(B+\rho)^{q}\,q^{q\alpha}. \qquad (3.1)
\end{aligned}
$$

□

Theorem 3.6. *Let $\phi \in \mathcal{S}^{\Delta^*,\beta,B}$. Then $\mathcal{L}_M[\phi;\cdot] \in \mathcal{S}_{\Delta,\beta,A}$.*

Proof. Let $\phi \in \mathcal{S}_{\Delta^*,\beta,A}$ and $\rho > 0$ be arbitrary. Using (3.1), we get

$$
\begin{aligned}
&|\xi^k \triangle^q_{\xi,d}\mathcal{L}_M[\phi;\xi]|\\
&\quad\le \left(\frac{1}{|b|}\right)^{q-k}|A_k|\Bigg(\sum_{r=0}^{k}\int\limits_{\mathbb{R}}|K_M(x,\xi)|\frac{q!}{(q-r)!}|\psi(x)|^{q-r}|(\triangle^*_{x,a})^{(k-r)}\phi(x)|\,dx\Bigg)\\
&\quad= \left(\frac{1}{|b|}\right)^{q-k}|A_k|\Bigg(\sum_{r=0}^{k}\int\limits_{\mathbb{R}}|K_M(x,\xi)|\frac{q!}{(q-r)!}|\psi(x)|^{q-r}|(\triangle^*_{x,a})^{(k-r)}\phi(x)|\,dx\Bigg)\\
&\quad\le \left(\frac{1}{|b|}\right)^{q-k}|A_k||K_M(x,\xi)|q!\Bigg(\sum_{r=0}^{q}\frac{q!}{r!(q-r)!}\int\limits_{\mathbb{R}}|\psi(x)|^{q-r}|(\triangle^*_{x,a})^{(k-r)}\phi(x)|\,dx\Bigg)\\
&\quad\le \left(\frac{1}{|b|}\right)^{q-k}|A_k||K_M(x,\xi)|q!\Bigg(\sum_{r=0}^{q}\frac{q!}{r!(q-r)!}\int\limits_{\mathbb{R}}(1+|x|^2)|\psi(x)|^{q-r}\\
&\qquad\quad\times|(\triangle^*_{x,a})^{(k-r)}\phi(x)|\,\frac{dx}{(1+|x|^2)}\Bigg)\\
&\quad= \left(\frac{1}{|b|}\right)^{q-k}|A_k||K_M(x,\xi)|q!\Bigg(\sum_{r=0}^{q}\frac{q!}{r!(q-r)!}\Bigg[\int\limits_{\mathbb{R}}|\psi(x)|^{q-r}|(\triangle^*_{x,a})^{(k-r)}\phi(x)|\\
&\qquad\quad\times\frac{dx}{(1+|x|^2)}+\int\limits_{\mathbb{R}}|\psi(x)|^{q+2-r}|(\triangle^*_{x,a})^{(k-r)}\phi(x)|\,\frac{dx}{(1+|x|^2)}\Bigg]\Bigg)\\
&\quad\le \left(\frac{1}{|b|}\right)^{q-k}|A_k||K_M(x,\xi)|q!\Bigg(\sum_{r=0}^{k}\frac{q!}{r!(q-r)!}\int\limits_{\mathbb{R}}2|\psi(x)|^{q+2-r}\\
&\qquad\quad\times|(\triangle^*_{x,a})^{(k-r)}\phi(x)|\,\frac{dx}{(1+|x|^2)}\Bigg)\\
&\quad= 2\left(\frac{1}{|b|}\right)^{q-k}|A_k||K_M(x,\xi)|q!\Bigg(\sum_{r=0}^{q}\frac{q!}{(q-r)!}C_{q+2-r,\rho}(B+\rho)^{k-r}\\
&\qquad\quad\times(k-r)^{(k-r)\alpha}\int\limits_{\mathbb{R}}\frac{dx}{(1+|x|^2)}\Bigg)\\
&\quad\le 2\left(\frac{1}{|b|}\right)^{q-k}|A_k||K_M(x,\xi)|q!\Bigg(\sum_{r=0}^{q}\frac{q!}{r!(q-r)!}\int\limits_{\mathbb{R}}C_{q+2-r,\rho}(B+\rho)^{k}
\end{aligned}
$$

$$
\begin{aligned}
&\times k^{k\beta}\,\frac{dx}{(1+|x|^2)}\Bigg)\\
&\leq 2\left(\frac{1}{|b|}\right)^{q-k}|A_k||K_M(x,\xi)|q!\Bigg(\sum_{r=0}^{q}\frac{q!}{r!(q-r)!}C_{q,\rho}(B+\rho)^k\\
&\qquad \times k^{k\beta}\int_{\mathbb{R}}\frac{dx}{(1+|x|^2)}\Bigg)\\
&\leq 2\left(\frac{1}{|b|}\right)^{q-k}|A_k||K_M(x,\xi)|q!C_{q,\rho}\Bigg(2^q(B+\rho)^k k^{k\beta}\int_{\mathbb{R}}\frac{dx}{(1+|x|^2)}\Bigg)\\
&= 2\pi\left(\frac{1}{|b|}\right)^{q-k}|A_k||K_M(x,\xi)|q!2^qC_{q,\rho}(B+\rho)^k k^{k\beta}\\
&= 2\pi\left(\frac{1}{|b|}\right)^{q}|A_k||K_M(x,\xi)|q!2^qC_{q,\rho}(|b||A_k|^{1/k}(B+\rho))^k k^{k\beta}\\
&= D_{q,\rho}(A+\delta)^k k^{k\beta}.
\end{aligned}
$$

□

Similarly, the following can be proved. We skip the proof for conciseness.

Theorem 3.7. *Let* $\phi\in\mathcal{S}^{\Delta^*,\beta,B}_{\Delta^*,\alpha,A}$. *Then* $\mathcal{L}_M[\phi;\cdot]\in\mathcal{S}^{\Delta,\alpha,A'}_{\Delta,\beta,B'}$.

4. LCT Based Pseudo Differential Operator

Consider the linear differential operator given by

$$P(x,\triangle^*_{x,a})=\sum_{r=0}^{m}a_r(x)(\triangle^*_{x,a})^r,$$

where $a_r(x)$ are the functions on $\mathbb{R}$. We also consider the polynomial given by

$$P_m(x,\xi)=\sum_{r=0}^{m}a_r(x)\left(\frac{-i\xi}{b}\right)^r$$

Let $\phi\in\mathcal{S}$. Then we have

$$
\begin{aligned}
(P(x,\triangle^*_{x,a})\phi)(x) &= \sum_{r=0}^{m}a_r(x)(\triangle^*_{x,a})^r\phi(x)\\
&= \sum_{r=0}^{m}a_r(x)\mathcal{L}_{M^{-1}}\mathcal{L}_M[(\triangle^*_{x,a})^r\phi(x);\xi]\\
&= \sum_{r=0}^{m}a_r(x)\mathcal{L}_{M^{-1}}\left[\left(\frac{-i\xi}{b}\right)^r\mathcal{L}_M[\phi;\xi]\right]\\
&= \mathcal{L}_{M^{-1}}\left[\left(\sum_{r=0}^{m}a_r(x)\left(\frac{-i\xi}{b}\right)^r\right)\mathcal{L}_M[\phi;\xi]\right]\\
&= \mathcal{L}_{M^{-1}}\left[P(x,\xi)\mathcal{L}_M\phi(\xi)\right]\\
&= \int_{\mathbb{R}}K_{M^{-1}}(\xi,x)P_m(x,\xi)\mathcal{L}_M[\phi;\xi]\,d\xi.
\end{aligned}
$$

We replace $P_m(x,\xi)$, the polynomial in $,\xi$ by a more general symbol $a(x,\xi)$, which need not to be a polynomial. This motivates the need to define a more general pseudo-differential operator which will be defined below. First, let us recall the following

Definition 4.1 ([31])**.** Let $m \in \mathbb{R}$. We define S^m to be the set of all functions $\sigma(x,\xi) \in C^\infty(\mathbb{R}\times\mathbb{R})$ such that for any two $k, q \in \mathbb{N}_0$, there is a positive constant c depending on k and q (which, without loss of generality, can be taken > 1) such that

$$\left|(D_x^k D_\xi^q)\sigma(x,\xi)\right| \;\leq\; c^{k+q}(1+|\xi|)^{m-q}. \tag{4.1}$$

It is customary to call the function $\sigma \in S^m$ a symbol. Now, we define the following

Definition 4.2. Let σ be a symbol. Define the $\Delta-$pseudo-differential operator $T_{\sigma,M}$ associated with σ by

$$(T_{\sigma,M}\phi)(x) = \int_{\mathbb{R}} K_{M^{-1}}(\xi,x)\sigma(x,\xi)\mathcal{L}_M(\phi)(\xi)\,d\xi, \quad \phi \in \mathcal{S}.$$

Remark 4.3. The mapping properties of pseudo-differential operators between Schwartz spaces are well-known in the literature (see, e.g., [31]). Recently, in [19], pseudo-differential operators have been studied in the framework of a fractional Fourier transform and in [23] they have been studied in the spaces $\mathcal{S}_\Delta$ and in the corresponding space of tempered distribution $\mathcal{S}'_\Delta$. Below, we prove the mapping properties of the operator $T_{\sigma,M}$ between the generalized Gelfand-Shilov type spaces defined in Section 3.

Theorem 4.4. *The $\Delta-$pseudo-differential operator $T_{\sigma,M}$ maps $\mathcal{S}_{\Delta^*,\alpha,A}$ into $\mathcal{S}_{\Delta^*,1+\alpha,A'}$ for some $A' > 0$ depending on A.*

Proof. Let $\phi \in \mathcal{S}_{\Delta^*,\alpha,A}$. Then for each $\delta > 0$,

$$|(x^k(\triangle^*_{x,a})^q\phi)| \leq C_{q,\delta}(A+\delta)^k k^{k\alpha}, \qquad k,q \in \mathbb{N}_0.$$

Now,

$$\begin{aligned}
&|x^k(\triangle^*_{x,a})^q(T_{\sigma,M}\phi)(x)| \\
&\quad= \left|x^k(\triangle^*_{x,a})^q \int_{\mathbb{R}} K_{M^{-1}}(\xi,x)\sigma(x,\xi)\mathcal{L}_M[\phi](\xi)\,d\xi\right| \\
&\quad= \left|x^k \int_{\mathbb{R}} (\triangle^*_{x,a})^q[K_{M^{-1}}(\xi,x)\sigma(x,\xi)]\mathcal{L}_M[\phi](\xi)\,d\xi\right| \\
&\quad= \left|x^k \int_{\mathbb{R}} \left(\sum_{r=0}^{q} A_{q,r}(\triangle^*_{x,a})^r K_{M^{-1}}(\xi,x)D_x^{q-r}\sigma(x,u)\right)\mathcal{L}_M[\phi](\xi)\,d\xi\right| \\
&\quad= \left|\int_{\mathbb{R}} \left(\sum_{r=0}^{q} A_{q,r}x^k\left(\frac{-i\xi}{b}\right)^r K_{M^{-1}}(\xi,x)D_x^{q-r}\sigma(x,\xi)\right)\mathcal{L}_M[\phi](\xi)\,d\xi\right| \\
&\quad= \left|\left(\frac{-i}{b}\right)^{-k}\int_{\mathbb{R}} \left(\sum_{r=0}^{q} A_{q,r}\left(\frac{-i\xi}{b}\right)^r (\triangle^*_{\xi,d})^k K_{M^{-1}}(\xi,x)D_x^{q-r}\sigma(x,\xi)\right)\mathcal{L}_M[\phi](\xi)\,d\xi\right| \\
&\quad= \left|\left(\frac{-i}{b}\right)^{-k}\sum_{r=0}^{q} A_{q,r}\left(\frac{-i}{b}\right)^r\int_{\mathbb{R}} (\triangle^*_{\xi,d})^k K_{M^{-1}}(\xi,x)D_x^{q-r}\sigma(x,\xi)\xi^r\mathcal{L}_M[\phi](\xi)\,d\xi\right| \\
&\quad= \left|\sum_{r=0}^{q} A_{q,r}\left(\frac{-i}{b}\right)^{-k+r}\int_{\mathbb{R}} K_{M^{-1}}(\xi,x)(\triangle_{\xi,d})^k[D_x^{q-r}\sigma(x,\xi)\xi^r\mathcal{L}_M[\phi](\xi)]\,d\xi\right| \\
&\quad= \left|\sum_{r=0}^{q} A_{q,r}\left(\frac{-i}{b}\right)^{-k+r}\int_{\mathbb{R}} K_{M^{-1}}(\xi,x)\sum_{j=0}^{k} B_{k,j}D_\xi^j D_x^{q-r}\sigma(x,\xi)(\triangle_{\xi,d})^{(k-j)}[\xi^r\mathcal{L}_M[\phi](\xi)]\,d\xi\right| \\
&\quad= \left|\sum_{r=0}^{q} A_{q,r}\left(\frac{-i}{b}\right)^{-k+r}\int_{\mathbb{R}} K_{M^{-1}}(\xi,x)\sum_{j=0}^{k} B_{k,j}D_\xi^j D_x^{q-r}\sigma(x,\xi)\right.
\end{aligned}$$

$$\times \sum_{i=0}^{(k-j)} C_{k-j,i} D_\xi^i \xi^r (\triangle_{\xi,d}^*)^{(k-j)-i} \mathcal{L}_M[\phi](\xi)]\, d\xi \Bigg|$$

$$\leq \sum_{r=0}^{q} |A_{q,r}||b|^{k-r} \int_{\mathbb{R}} |K_{M^{-1}}(x,\xi)| \sum_{j=0}^{k} |B_{k,j}||D_\xi^j D_x^{q-r}\sigma(x,\xi)|$$

$$\times \sum_{i=0}^{(k-j)} |C_{k-j,i}||D_\xi^i \xi^r||(\triangle_{\xi,d})^{(k-j)-i} \mathcal{L}_M[\phi](\xi)|\, d\xi. \tag{4.2}$$

Writing

$$|A_q| = \sup_r |A_{q,r}|, \quad |B_k| = \sup_j |B_{k,j}|, \quad |C_k| = \sup_{i,j} |C_{k-j,i}|, \tag{4.3}$$

the last estimate by using (4.1) gives

$$|x^k (\triangle_{x,a}^*)^q (T_{\sigma,M}\phi)(x)|$$

$$\leq |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} |c^{(q-r)+j}|(1+|\xi|)^{m-j}$$

$$\times \sum_{i=0}^{(k-j)} |D_\xi^i \xi^r||(\triangle_{\xi,d})^{(k-j)-i} \mathcal{L}_M[\phi](\xi)|\, d\xi$$

$$\leq |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} |c^{(q-r)+j}|(1+|\xi|)^{m-j}$$

$$\times \sum_{i=0}^{(k-j)} \frac{r!}{(r-i)!} |\xi^{r-i}||(\triangle_{\xi,d})^{(k-j)-i} \mathcal{L}_M[\phi](\xi)|\, d\xi$$

$$\leq |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} c^{q+k}(1+|\xi|)^{m-j}$$

$$\times \sum_{i=0}^{(k-j)} \frac{r!}{(r-i)!} C_{r-i,\delta}(A+\delta)^{k-j-i}(k-j-i)^{(k-j-i)\alpha}\, d\xi$$

$$\leq |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} c^{q+k}(1+|\xi|)^{m-j}$$

$$\times (k-j)! \sum_{i=0}^{(k-j)} \frac{r!}{i!(r-i)!} C_{r-i,\delta}(A+\delta)^k (k)^{k\alpha}\, d\xi$$

$$\leq C_{q,\delta}|A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} c^{q+k}(1+|\xi|)^{m-j}(k-j)! 2^r\, d\xi$$

$$\times (A+\delta)^k (k)^{k\alpha}$$

$$\leq C_{q,\delta}|A_q||B_k||C_k||b|^k \sum_{r=0}^{q} |b|^{-r} k! 2^q c^{q+k} \sum_{j=0}^{k} \int_{\mathbb{R}} (1+|\xi|)^{m-j}\, d\xi$$

$$\times (A+\delta)^k (k)^{k\alpha}$$

$$\leq 2^q c^q C_{q,\delta}|A_q| \left(\sum_{r=0}^{q} |b|^{-r} \right) |B_k||C_k||b|^k k! c^k I_k (A+\delta)^k (k)^{k\alpha},$$

where

$$I_k = \sum_{j=0}^{k} \int_{\mathbb{R}} (1 + |\xi|)^{m-j} \, d\xi$$

in which the integral converges by choosing $m - k + 1 < 0$. Thus we have

$$\begin{aligned}
&|x^k (\triangle^*_{x,a})^q (T_{\sigma,M}\phi)(x)| \\
&\quad \leq 2^q c^q C_{q,\delta} |A_q| \left(\sum_{r=0}^{q} |b|^{-r} \right) |B_k||C_k||b|^k k^k C^k I_k (A + \delta)^k (k)^{k\alpha} \\
&\quad = E_{q,\delta} (A' + \delta')^k k^{(k+1)\alpha} \\
&\quad = E_{q,\delta'} (A' + \delta')^k k^{k(1+\alpha)},
\end{aligned}$$

where

$$E_{q,\delta} = 2^q c^q C_{q,\delta} |A_q| \left(\sum_{r=0}^{q} |b|^{-r} \right), \; A' = (|B_k||C_k||b|^k c^k I_k)^{1/k} A.$$

□

Theorem 4.5. *The* $\Delta-$*pseudo-differential operator* $T_{\sigma,M}$ *maps* $\mathcal{S}^{\Delta^*,\beta,B}$ *into* $\mathcal{S}^{\Delta^*,1+\beta,B'}$ *for some* $B' > 0$ *depending on* B.

Proof. Let $\phi \in \mathcal{S}_{\Delta^*,\alpha,A}$. Then for each $\beta > 0$,

$$|(x^k (\triangle^*_{x,a})^q \phi)| \leq C_{k,\beta} (B + \beta)^q q^{q\beta}, \quad k, q \in \mathbb{N}_0.$$

Now, using (4.1) and (4.2), we have

$$\begin{aligned}
&|x^k (\triangle^*_{x,a})^q (T_{\sigma,M}\phi)(x)| \\
&\quad \leq \sum_{r=0}^{q} |A_{q,r}||b|^{k-r} \int_{\mathbb{R}} |K_{M^{-1}}(x,\xi)| \sum_{j=0}^{k} |B_{k,j}||D^j_\xi D^{q-r}_x \sigma(x,\xi)| \\
&\qquad \times \sum_{i=0}^{(k-j)} |C_{k-j,i}||D^i_\xi \xi^r||(\triangle_{\xi,d})^{(k-j)-i} \mathcal{L}_M[\phi](\xi)| \, d\xi \\
&\quad \leq |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} |c^{(q-r)+j}|(1 + |\xi|)^{m-j} \\
&\qquad \times \sum_{i=0}^{(k-j)} |D^i_\xi \xi^r||(\triangle_{\xi,d})^{(k-j)-i} \mathcal{L}_M[\phi](\xi)| \, d\xi \\
&\quad \leq |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} |c^{(q-r)+j}|(1 + |\xi|)^{m-j} \\
&\qquad \times \sum_{i=0}^{(k-j)} \frac{r!}{(r-i)!} |\xi^{r-i}||(\triangle_{\xi,d})^{(k-j)-i} \mathcal{L}_M[\phi](\xi)| \, d\xi \\
&\quad \leq |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} c^{q+k} (1 + |\xi|)^{m-j} \\
&\qquad \times \sum_{i=0}^{(k-j)} \frac{r!}{(r-i)!} C_{k-j-i,\beta} (B + \beta)^{r-i} (r-i)^{(r-i)\beta} \, d\xi \\
&\quad \leq |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} c^{q+k} (1 + |\xi|)^{m-j}
\end{aligned}$$

$$
\begin{aligned}
&\times C_{k,\beta} \sum_{i=0}^{(k-j)} \frac{r!}{(r-i)!}(B+\beta)^{r-i}(r-i)^{(r-i)\beta}\, d\xi \\
&\leq |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \int_{\mathbb{R}} \sum_{j=0}^{k} c^{q+k}(1+|\xi|)^{m-j} \\
&\quad \times C_{k,\beta}\, r! \sum_{i=0}^{r} \frac{r!}{i!(r-i)!}(B+\beta)^{r-i} q^{q\beta}\, d\xi \\
&\leq |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} \sum_{j=0}^{k} c^{q+k} \int_{\mathbb{R}} (1+|\xi|)^{m-j} \\
&\quad \times C_{k,\beta}\, r! \sum_{i=0}^{r} \frac{r!}{i!(r-i)!}(B+\beta)^{r-i} q^{q\beta}\, d\xi \\
&= |A_q||B_k||C_k| \sum_{r=0}^{q} |b|^{k-r} c^{q+k} \sum_{j=0}^{k} \int_{\mathbb{R}} (1+|\xi|)^{m-j} \\
&\quad \times C_{k,\beta}\, r!(1+B+\beta)^r q^{q\beta}\, d\xi \\
&\leq |A_q||B_k||C_{k,\beta}||C_k| \sum_{r=0}^{q} |b|^{k-r} c^{q+k} I_k q!(1+B+\beta)^q q^{q\beta}\, d\xi,
\end{aligned}
$$

where

$$
I_k = \sum_{j=0}^{k} \int_{\mathbb{R}} (1+|\xi|)^{m-j}\, d\xi
$$

in which the integral converges by choosing $m-k+1<0$. Thus we have

$$
\begin{aligned}
&|x^k(\triangle^*_{x,a})^q(T_{\sigma,M}\phi)(x)| \\
&\quad \leq |b|^k C^k I_k |B_k||C_{k,\beta}||C_k||A_q| \left(\sum_{r=0}^{q} |b|^{-r} \right) c^q q^q (1+B+\beta)^q q^{q\beta} \\
&\quad = E_{k,\beta}(B'+\beta')^q q^{q(1+\beta)} \\
&\quad = E_{k,\beta'}(B'+\beta')^q q^{q(1+\beta)},
\end{aligned}
$$

where

$$
E_{k,\beta} = |b|^k C^k I_k |B_k||C_{k,\beta}||C_k|,\ B' = \left(|A_q| \left(\sum_{r=0}^{q} |b|^{-r} \right) c^q \right)^{1/q} A
$$

and

$$
\beta' = \left(|A_q| \left(\sum_{r=0}^{q} |b|^{-r} \right) c^q \right) \beta.
$$

□

In a similar way we can prove the following

Theorem 4.6. *The* $\Delta-$*pseudo-differential operator* $T_{\sigma,M}$ *maps* $\mathcal{S}^{\Delta^*,\beta,B}_{\Delta^*,\alpha,A}$ *into* $\mathcal{S}^{\Delta^*,1+\beta,B'}_{\Delta^*,1+\alpha,A'}$.

5. Concluding Remark

Harmonic oscillators occupy an important place in several science and engineering fields. Many mechanical systems such as vibrating string with small amplitude about an equilibrium point can be modeled as a simple harmonic oscillator. In [18], the author points out that one of the most important features of an harmonic oscillator is its energy eigenstates which can be described in terms of their coordinate space wave functions $\psi_n(x)$ as

$$\frac{d^2\psi_n}{dX^2} + \frac{2M}{h^2}\left(E_n - \frac{M\Omega^2 X^2}{2}\right)\psi_n = 0, \tag{5.1}$$

where X is the spatial coordinate, E_n is the energy of the nth stationary state of the oscillator, M is the mass and Ω is the frequency. The author in [18] solved equation (4.1) by using a simple harmonic transformation, although many classical approaches already exist (see, e.g., [4], [27]).

Similarly, in [11], the authors describe the relationship of fractional Fourier transform and certain variants of equation (4.1). Since the LCT is more general than the fractional Fourier transform, it is of interest to work out the relationship between LCT and some further generalized equations as dealt with in [18] or [11].

In the recent paper [3], the authors have defined and studied a transform more general than the LCT, the so-called Special Affine Fourier Transform (SAFT). It will be of interest if the results of the present paper are extended in the framework of SAFT.

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A NOTE ON THE MAXIMAL OPERATORS OF THE NÖRLUND LOGARITMIC MEANS OF VILENKIN-FOURIER SERIES

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Abstract. The main aim of this paper is to investigate the (H_p, L_p)- type inequalities for the maximal operators of Nörlund logarithmic means for $0 < p < 1$.

1. Introduction

It is well-known that (see e.g., [1], [8] and [16]) Vilenkin systems do not form bases in the Lebesgue space $L_1(G_m)$. Moreover, there exists a function in the Hardy space H_1 such that the partial sums of f are not bounded in L_1-norm.

In [19] (see also [21]), it was proved that the following is true:

Theorem T1. Let $0 < p < 1$. Then the maximal operator

$$\widetilde{S}_p^* f := \sup_{n\in\mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}}$$

is bounded from the Hardy space $H_p(G_m)$ to the space $L_p(G_m)$. Here, S_n denotes the n-th partial sum with respect to the Vilenkin system. Moreover, it was proved that the rate of the factor $(n+1)^{1/p-1}$ is in a sense sharp.

In the case $p = 1$, it was proved that the maximal operator $\widetilde{S}^*$ defined by

$$\widetilde{S}^* := \sup_{n\in\mathbb{N}} \frac{|S_n|}{\log(n+1)}$$

is bounded from the Hardy space $H_1(G_m)$ to the space $L_1(G_m)$. Moreover, the rate of the factor $\log(n+1)$ is in a sense sharp. Similar problems for the Nörlund logarithmic means in the case, where $p = 1$, was considered in [15].

Móricz and Siddiqi [9] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of $L_p(G_m)$ functions in L_p-norm. Fridli, Manchanda and Siddiqi [5] improved and extended the results of Móricz and Siddiqi [9] to the Martingale Hardy spaces. However, the case for $\{q_k = 1/k : k \in \mathbb{N}_+\}$ was excluded, since the methods are not applicable to the Nörlund logarithmic means. In [6], Gt and Goginava proved some convergence and divergence properties of Walsh-Fourier series of the Nörlund logarithmic means of functions in the Lebesgue space $L_1(G_m)$. In particular, they proved that there exists a function in the space $L_1(G_m)$ such that

$$\sup_{n\in\mathbb{N}} \|L_n f\|_1 = \infty.$$

In [2] (see also [15, 17]), it was proved that there exists a martingale $f \in H_p(G_m)$, $\quad (0 < p < 1)$ such that

$$\sup_{n\in\mathbb{N}} \|L_n f\|_p = \infty.$$

Analogous problems for the Nörlund means with respect to Walsh, Kaczmarz and unbounded Vilenkin systems were considered in Blahota, and Tephnadze, [3,4], Goginava and Nagy [7], Nagy and Tephnadze [10–12], Persson, Tephnadze and Wall [13,14], Tephnadze [18,20,21], Tutberidze [22].

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In this paper, we discuss the boundedness of the weighted maximal operators from the Hardy space $H_p(G_m)$ to the Lebesgue space $L_p(G_m)$ for $0<p<1$.

2. Definitions and Notation

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N}:=\mathbb{N}_+\cup\{0\}$.

Let $m:=(m_0,m_1,\dots)$ denote a sequence of the positive integers, not less than 2.

Denote by

$$Z_{m_k}:=\{0,1,\dots,m_k-1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j}.

The direct product μ of the measures

$$\mu_k(\{j\}):=1/m_k \qquad (j\in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m)=1$.

If $\sup\limits_{n\in\mathbb{N}} m_n<\infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded one. **In this paper we discuss the bounded Vilenkin groups only.**

The elements of G_m are represented by the sequences

$$x:=(x_0,x_1,\dots,x_j,\dots) \qquad (x_k\in Z_{m_k}).$$

It is easy to give a base for the neighborhood of G_m,

$$I_0(x):=G_m,$$

$$I_n(x):=\{y\in G_m \mid y_0=x_0,\dots,y_{n-1}=x_{n-1}\}\ (x\in G_m,\ n\in\mathbb{N})$$

Denote $I_n:=I_n(0)$, for $n\in\mathbb{N}$ and $\overline{I_n}:=G_m\backslash I_n$.

If we define the so-called generalized number system based on m in the following way :

$$M_0:=1, \qquad M_{k+1}:=m_kM_k \qquad (k\in\mathbb{N})$$

then every $n\in\mathbb{N}$ can be uniquely expressed as $n=\sum\limits_{k=0}^{\infty}n_jM_j$, where $n_j\in Z_{m_j}$ $(j\in\mathbb{N})$ and only a finite number of n_j's differs from zero. Let $|n|:=\max\{j\in\mathbb{N};\quad n_j\neq 0\}$.

The norm (or quasi-norm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p^p:=\int\limits_{G_m}|f|^p\,d\mu \qquad (0<p<\infty).$$

The space $weak-L_p(G_m)$ consists of all measurable functions f for which

$$\|f\|_{weak-L_p(G_m)}^p:=\sup_{\lambda>0}\lambda^p\mu(x:|f(x)|>\lambda)<+\infty.$$

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. First we define the complex-valued function $r_k(x):G_m\to C$, the generalized Rademacher functions as

$$r_k(x):=\exp(2\pi i x_k/m_k) \qquad \left(i^2=-1,\quad x\in G_m,\quad k\in\mathbb{N}\right).$$

Now, define the Vilenkin system $\psi:=(\psi_n:n\in\mathbb{N})$ on G_m as:

$$\psi_n:=\prod_{k=0}^{\infty}r_k^{n_k}, \qquad (n\in\mathbb{N}).$$

Specifically, we call this system the Walsh-Paley one if m=2.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ [1,23].

Now we introduce analogues of the usual definitions in the Fourier analysis.

If $f\in L_1(G_m)$, we can establish the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system ψ in the usual manner:

$$\widehat{f}(k):=\int\limits_{G_m} f\overline{\psi}_k d\mu, \qquad (k\in\mathbb{N}),$$

$$S_nf:=\sum_{k=0}^{n-1}\widehat{f}(k)\psi_k, \qquad (n\in\mathbb{N}_+, \quad S_0f:=0),$$

$$D_n:=\sum_{k=0}^{n-1}\psi_k, \qquad (n\in\mathbb{N}_+).$$

Recall that (for details see e.g. [1])

$$D_{M_n}(x)=\begin{cases} M_n & x\in I_n \\ 0 & x\notin I_n. \end{cases} \tag{1}$$

The σ-algebra generated by the intervals $\{I_n(x): x\in G_m\}$ will be denoted by F_n $(n\in\mathbb{N})$. Denote by $f=(f_n: n\in\mathbb{N})$ a martingale with respect to F_n $(n\in\mathbb{N})$ (for details see e.g. [24,25]). The maximal function of a martingale f is defined by

$$f^*=\sup_{n\in\mathbb{N}}|f_n|.$$

In the case, where $f\in L_1$, the maximal function is also given by

$$f^*(x)=\sup_{n\in\mathbb{N}}\frac{1}{|I_n(x)|}\left|\int\limits_{I_n(x)} f(u)\mu(u)\right|.$$

For $0<p<\infty$, the Hardy martingale spaces $H_p(G_m)$ consist of all martingales for which

$$\|f\|_{H_p}:=\|f^*\|_p<\infty.$$

If $f\in L_1$, then it is easy to show that the sequence $(S_{M_n}f: n\in\mathbb{N})$ is a martingale. If $f=(f_n: n\in\mathbb{N})$ is a martingale, then the Vilenkin-Fourier coefficients should be defined in a slightly different manner:

$$\widehat{f}(i):=\lim_{k\to\infty}\int\limits_{G_m} f_k\overline{\psi}_i d\mu.$$

The Vilenkin-Fourier coefficients of $f\in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}f: n\in\mathbb{N})$ obtained from f.

Let $\{q_k: k>0\}$ be a sequence of non-negative numbers. The n-th Nörlund means for the Fourier series of f is defined by

$$\frac{1}{Q_n}\sum_{k=1}^{n}q_{n-k}S_kf, \quad \text{where} \quad Q_n:=\sum_{k=1}^{n}q_k.$$

If $q_k=1/k$, then we get the Nörlund logarithmic means

$$L_nf:=\frac{1}{l_n}\sum_{k=0}^{n-1}\frac{S_kf}{n-k}, \quad \text{where} \quad l_n=\sum_{k=0}^{n-1}\frac{1}{n-k}=\sum_{j=1}^{n}\frac{1}{j}.$$

A bounded measurable function a is p-atom, if there exists a dyadic interval I such that

$$\int\limits_I a d\mu=0, \quad \|a\|_\infty\le\mu(I)^{-1/p}, \quad \operatorname{supp}(a)\subset I.$$

3. Formulation of Main Results

Theorem 1. a) *Let* $0<p<1$. *Then the maximal operator*

$$\tilde{L}{}_p^* f := \sup_{n\in\mathbb{N}} \frac{|L_n f|}{(n+1)^{1/p-1}}$$

is bounded from the Hardy space $H_p(G_m)$ *to the space* $L_p(G_m)$.

b) *Let* $0<p<1$ *and* $\varphi:\mathbb{N}_+\to[1,\infty)$ *be a non-decreasing function satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{n^{1/p-1}}{\log n\varphi(n)} = +\infty.$$

Then there exists a martingale $f\in H_p(G_m)$ *such that the maximal operator*

$$\sup_{n\in\mathbb{N}} \frac{|L_n f|}{\varphi(n+1)}$$

is not bounded from the Hardy space $H_p(G_m)$ *to the space* $L_p(G_m)$.

4. Proof of the Theorem

Proof. Since

$$\frac{|L_n f|}{(n+1)^{1/p-1}} \le \frac{1}{(n+1)^{1/p-1}} \sup_{1\le k\le n} |S_k f| \le \sup_{1\le k\le n} \frac{|S_k f|}{(k+1)^{1/p-1}} \le \sup_{n\in\mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}},$$

if we use Theorem T1, we obtain

$$\sup_{n\in\mathbb{N}} \frac{|L_n f|}{(n+1)^{1/p-1}} \le \sup_{n\in\mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}}$$

and

$$\left\|\sup_{n\in\mathbb{N}} \frac{|L_n f|}{(n+1)^{1/p-1}}\right\|_p \le \left\|\sup_{n\in\mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}}\right\|_p \le c_p \|f\|_{H_p}.$$

Now, prove part b) of the Theorem. Let

$$f_{n_k} = D_{M_{2n_k+1}} - D_{M_{2n_k}}.$$

It is evident that

$$\widehat{f}_{n_k}(i) = \begin{cases} 1, & \text{if} \quad i = M_{2n_k},\dots,M_{2n_k+1}-1, \\ 0, & \text{otherwise.} \end{cases}$$

Then we can write that

$$S_i f_{n_k} = \begin{cases} D_i - D_{M_{2n_k}}, & \text{if} \quad i = M_{2n_k}+1,\dots,M_{2n_k+1}-1, \\ f_{n_k}, & \text{if} \quad i \ge M_{2n_k+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

From (1), we get

$$\begin{aligned} \|f_{n_k}\|_{H_p} &= \left\|\sup_{n\in\mathbb{N}} S_{M_n} f_{n_k}\right\|_p = \left\|D_{M_{2n_k+1}} - D_{M_{2n_k}}\right\|_p \\ &\le \left\|D_{M_{2n_k+1}}\right\|_p + \left\|D_{M_{2n_k}}\right\|_p \le c M_{2n_k}^{1-1/p} < c < \infty. \end{aligned} \tag{3}$$

Let $0<p<1$ and $\{\lambda_k : k\in\mathbb{N}_+\}$ be an increasing sequence of the positive integers such that

$$\lim_{k\to\infty} \frac{\lambda_k^{1/p-1}}{\varphi(\lambda_k)} = \infty.$$

Let $\{n_k : k \in \mathbb{N}_+\} \subset \{\lambda_k : k \in \mathbb{N}_+\}$ such that

$$\lim_{k\to\infty} \frac{\left(M_{2n_k}+2\right)^{1/p-1}}{\log\left(M_{2n_k}+2\right)\varphi\left(M_{2n_k+2}\right)} \geq c \lim_{k\to\infty} \frac{\lambda_k^{1/p-1}}{\varphi\left(\lambda_k\right)} = \infty.$$

According to (2), we can conclude that

$$\left|\frac{L_{M_{2n_k}+2} f_{n_k}}{\varphi\left(M_{2n_k+2}\right)}\right| = \frac{\left|D_{M_{2n_k}+1} - D_{M_{2n_k}}\right|}{l_{M_{2n_k}+1}\varphi\left(M_{2n_k+1}\right)}$$

$$= \frac{\left|\psi_{M_{2n_k}}\right|}{l_{M_{2n_k}+2}\varphi\left(M_{2n_k+1}\right)} = \frac{1}{l_{M_{2n_k}+1}\varphi\left(M_{2n_k+2}\right)}.$$

Hence,

$$\mu\left\{x \in G_m : \left|L_{M_{2n_k}+2} f_{n_k}\right| \geq \frac{1}{l_{M_{2n_k}+2}\varphi\left(M_{2n_k+2}\right)}\right\} = \mu\left(G_m\right) = 1. \tag{4}$$

By combining (3) and (4), we get

$$\frac{\frac{1}{l_{M_{2n_k}+2}\varphi\left(M_{2n_k+2}\right)}\left(\mu\left\{x \in G_m : \quad \left|L_{M_{2n_k}+2} f_{n_k}\right| \geq \frac{1}{l_{M_{2n_k}+2}\varphi\left(M_{2n_k+2}\right)}\right\}\right)^{1/p}}{\left\|f_{n_k}\right\|_p}$$

$$\geq \frac{M_{2n_k}^{1/p-1}}{l_{M_{2n_k}+2}\varphi\left(M_{2n_k+2}\right)} \geq \frac{c\left(M_{2n_k}+2\right)^{1/p-1}}{\log\left(M_{2n_k}+2\right)\varphi\left(M_{2n_k+2}\right)} \to \infty, \quad \text{as} \quad k \to \infty. \qquad \square$$

Open Problem. For any $0 < p < 1$, let us find a non-decreasing function $\Theta : \mathbb{N}_+ \to [1,\infty)$ such that the following maximal operator

$$\widetilde{L}_p^* f := \sup_{n\in\mathbb{N}} \frac{|L_n f|}{\Theta\left(n+1\right)}$$

is bounded from the Hardy space $H_p\left(G_m\right)$ to the Lebesgue space $L_p\left(G_m\right)$ and the rate of $\Theta : \mathbb{N}_+ \to [1,\infty)$ is sharp, that is, for any non-decreasing function $\varphi : \mathbb{N}_+ \to [1,\infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\Theta\left(n\right)}{\varphi\left(n\right)} = +\infty,$$

there exists a martingale $f \in H_p\left(G_m\right)$ such that the maximal operator

$$\sup_{n\in\mathbb{N}} \frac{|L_n f|}{\varphi\left(n+1\right)}$$

is not bounded from the Hardy space $H_p\left(G_m\right)$ to the space $L_p\left(G_m\right)$.

Remark 1. *According to Theorem* 1*, we can conclude that there exist absolute constants* C_1 *and* C_2 *such that*

$$\frac{C_1 n^{1/p-1}}{\log(n+1)} \leq \Theta\left(n\right) \leq C_2 n^{1/p-1}.$$

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SHORT COMMUNICATIONS

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ON THE DOUBLE FOURIER–WALSH–PALEY SERIES OF CONTINUOUS FUNCTIONS

ROSTOM GETSADZE

Abstract. The following theorem is proved: There exists a continuous function F on $[0,1]^2$ such that the modulus of continuity

$$\omega(F;\delta) = O\left(\frac{1}{\sqrt{\log_2 \frac{1}{\delta}}}\right), \quad \delta \to 0+,$$

and the double Fourier-Walsh-Paley series of F diverges on a set of positive measure by rectangles.

1. Introduction

Kolmogorov [6] proved that there exists a function $f \in L([0,2\pi])$ with the trigonometric Fourier series that diverges almost everywhere. Later, he constructed a function with everywhere divergent trigonometric Fourier series [7].

Stein [8] established that there exists a function $f \in L([0,1])$ with the Fourier–Walsh–Paley series that diverges almost everywhere.

Carleson [3] proved that if $f \in L^2([0,2\pi])$, then its trigonometric Fourier series converges almost everywhere. Billard [2] showed that if $g \in L^2([0,1])$, then its Fourier–Walsh–Paley series converges almost everywhere.

Fefferman [4] proved that in the contrast to the Carleson's theorem, there exists a continuous function of two variables on $[0,2\pi]^2$ with the double trigonometric Fourier series that diverges almost everywhere by rectangles.

In [5], we have shown that there exists a continuous function on $[0,1]^2$ the with double Fourier–Walsh–Paley series that diverges almost everywhere by rectangles.

The system of Rademacher functions $\{r_n(x)\}_{n=0}^{\infty}$ on $[0,1)$ is defined as follows. Set

$$r_0(x) = \begin{cases} 1 & \text{for } 0 \le x < \frac{1}{2}, \\ -1 & \text{for } \frac{1}{2} \le x < 1. \end{cases}$$

We extend the function $r_0(x)$ on $(-\infty,\infty)$ with period 1. For $n \ge 1$, set

$$r_n(x) = r_0(2^n x).$$

The Walsh-Paley system of functions is defined as follows. Set $W_0(x) = 1$, for all $x \in [0,1)$. For $n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_q}$, $(n \ge 1,\ m_1 > m_2 \cdots > m_q \ge 0,)$ set

$$W_n(x) = r_{m_1}(x) r_{m_2}(x) \ldots r_{m_q}(x) \quad x \in [0,1).$$

Let $f \in L([0,1])^2$ and let

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j}(f) W_i(x) W_j(y)$$

be the Fourier series of f with respect to the double Walsh–Paley system $\{W_i(x)W_j(y)\}_{i,j=0}^{\infty}$ on $[0,1)^2$.

The moduls of continuity $\omega(F;\delta)$ of a continuous function F on $[0,1]^2$ is defined by

$$\omega(F;\delta) = \sup_{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \le \delta} \{|f(x_1,y_1) - f(x_2,y_2)|,\ (x_1,y_1),\ (x_2,y_2) \in [0,1]^2\}.$$

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Key words and phrases. Double Walsh-Paley; Fourier series; Divergence; Modulus of continuity.

We have established that we are able to achieve a certain smoothness in the divergence of the double Fourier–Walsh–Paley series of continuous functions. More precisely the following statement is true.

Theorem 1. *There exists a continuous function F on $[0,1]^2$ such that*

$$\omega\,(F;\delta) = O\left(\frac{1}{\sqrt{\log_2 \frac{1}{\delta}}}\right), \quad \delta \to 0+,$$

and the double Fourier–Walsh–Paley series of F diverges on a set of positive measure by rectangles.

Note that for the double trigonometric system, Bakhbokh and Nikishin [1] established stronger result where instead of $O\left(\frac{1}{\sqrt{\log_2 \frac{1}{\delta}}}\right)$, $\delta \to 0+$, one has $O\left(\frac{1}{\log_2 \frac{1}{\delta}}\right)$, $\delta \to 0+$. We note also that our method of proof in [5] allowed to achieve only a smoothness of order $O\left(\frac{1}{\log_2 \log_2 \frac{1}{\delta}}\right)$, $\delta \to 0+$.

It is important to remark also that in the case of the trigonometric system the n-th kernel can be written as follows

$$\sin nx \cot \frac{t}{2} + \cos nx,$$

that is, as a sum of two terms, each of which is a product of a function in the trigonometric system ($\sin nx$ and $\cos nx$) multiplied by a function that does not depend on n ($\cot \frac{t}{2}$ and 1). This is not the case for the Walsh-Paley system and this fact complicates the proofs of the divergence results for this system.

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THE STRONG UNIQUENESS PROPERTY OF INVARIANT MEASURES IN INFINITE DIMENSIONAL TOPOLOGICAL VECTOR SPACES

MARIKA KHACHIDZE[1] AND ALEKS KIRTADZE[1,2]

Abstract. It is shown that there exists a normalized σ-finite invariant Borel measure in the topological vector space $\mathbf{R}^\omega$, having the strong uniqueness property on $\mathbf{R}^\omega$.

It is well known that the uniqueness property of invariant measures plays an important role in various questions of modern analysis, probability theory, and geometry. The main purpose of the present paper is to consider the uniqueness property of invariant measures from the general point of view and to investigate some interrelations between this and other properties of invariant measures.

Throughout this article, we use the following standard notation:

$\mathbf{N}$ is the set of all natural numbers;

$\mathbf{R}$ is the set of all real numbers;

ω is the first infinite cardinal number (i.e., $\omega = \mathrm{card}(\mathbf{N})$;

$\mathbf{c}$ is the cardinality of the continuum (i.e., $\mathbf{c} = 2^\omega$);

$\mathrm{dom}(\mu)$ is the domain of a given measure μ;

$\mathbf{R}^\omega$ is the space of all real-valued sequences;

$\mathbf{B}(\mathbf{R}^\omega)$ is the Borel σ-algebra on $\mathbf{R}^\omega$.

Let E be a nonempty set, G be a group of transformations of E, and let M be a class of σ-finite G-invariant measures on E (note here that the domains of measures from M may differ from each other).

We say that a set X has the uniqueness property with respect to M if for every two measures $\mu_1 \in M$ and $\mu_2 \in M$ such that $X \in \mathrm{dom}(\mu_1)$ and $X \in \mathrm{dom}(\mu_2)$ we have the following equality:

$$\mu_1(X) = \mu_2(X).$$

We present several simple examples illustrating the above notions.

Example 1. Let M be a class of nonzero σ-finite G-measures in $\mathbf{R}^\mathbf{n}$. Then the unit coordinate cube in $\mathbf{R}^\mathbf{n}$ has the uniqueness property with respect to M.

Example 2. If $X \subset \mathbf{R}^\mathbf{n}$ is an absolutely negligible subset in the class of all G-measures, then X has the uniqueness property with respect to the same class of measures (see [5], [8]).

Example 3. Every subset $X \subset \mathbf{R}^\mathbf{n}$, measurable with respect to the classical Jordan measure, has the uniqueness property in the class of π_n-volumes, where π_n denotes the group of all translations in $\mathbf{R}^\mathbf{n}$.

Let again E be a nonempty set, G be a group of transformations of E, and M be a class of σ-finite G-invariant measures on E.

A measure $\mu \in M$ possesses the strong uniqueness property with respect to M if $\mathrm{dom}(\mu)$ contains only those elements that have the uniqueness property with respect to the same class of measures.

In other words, a measure $\mu \in M$ possesses the strong uniqueness property with respect to M if for every $X \in \mathrm{dom}(\mu)$ and for every two measures $\mu_1 \in M$ and $\mu_2 \in M$ such that $X \in \mathrm{dom}(\mu_1)$ and $X \in \mathrm{dom}(\mu_2)$ we have the following equality:

$$\mu_1(X) = \mu_2(X).$$

2010 *Mathematics Subject Classification.* 28A05, 28D05.

Key words and phrases. Invariant measure; The strong uniqueness property.

For the above-mentioned definitions, see [5], [8].

From the above-mentioned definitions it follows that if a measure μ has the strong uniqueness property with respect to M, then μ has the uniqueness property with respect to the same class of measures, too.

Example 4. From the definition of G-volumes it follows that each volume from the class of all G-volumes on $\mathbf{R^n}$ has the strong uniqueness property.

It is known that in the infinite-dimensional vector spaces there are no analogues of the classical Lebesgue measure. In other words, the above-mentioned spaces do not admit nontrivial σ-finite translation-invariant Borel measure. In this context it should be noted that A. Kharazishvili has constructed a normalized σ-finite metrically transitive Borel measure χ in $\mathbf{R}^\omega$ which is invariant with respect to the dense everywhere vector subspace G, where

$$G = \{x : x \in \mathbf{R}^\omega, \operatorname{card}\{i : i \in \mathbf{N} : x_i \neq 0\} < \omega\}.$$

For a detailed information of measure χ, see [6] or [7].

Let s_0 be the central symmetry of $\mathbf{R}^\omega$ with respect to the origin and let S_ω be the group generated by s_0 and G. It is clear that each element of S_ω can be represented in the form $s_0 \circ g$ or $g \circ s_0$.

Using the method of [6], we will construct a σ-finite Borel measure on $\mathbf{R}^\omega$ which is invariant with respect to the group S_ω.

In particular, we put

$$A_n = \mathbf{R}_1 \times \mathbf{R}_2 \times \cdots \times \mathbf{R}_n \times \Big(\prod_{i>n} \triangle_i\Big),$$

where $n \in \mathbf{N}$ and

$$(\forall i)(i \in \mathbf{N} \Rightarrow \mathbf{R}_i = \mathbf{R} \wedge \triangle_i = [-1, 1]).$$

For an arbitrary natural number $i \in \mathbf{N}$, consider the Lebesgue measure μ_i defined on the space $\mathbf{R}_i$ and satisfying the condition $\mu_i(\triangle_i) = 1$. Let us denote by λ_i the Lebesgue measure defined on the $\triangle_i$ such that $\lambda_i(\triangle_i) = 1$.

For an arbitrary $n \in \mathbf{N}$, let us denote by χ_n the measure defined by

$$\chi_n = \Big(\prod_{1 \le i \le n} \mu_i\Big) \times \Big(\prod_{i>n} \lambda_i\Big),$$

and by $\overline{\chi_n}$ the Borel measure in the space $\mathbf{R}^\omega$ defined by

$$\overline{\chi_n} = \chi_n(X \cap A_n),\ X \in \mathbf{B}(\mathbf{R}^\omega).$$

Lemma 1. *For an arbitrary Borel set $X \in \mathbf{B}(\mathbf{R}^\omega)$ there exists a limit*

$$\chi(X) = \lim_{n\to\infty} \overline{\chi_n}(X).$$

Moreover, the functional χ is a nonzero σ-finite measure on the Borel σ-algebra $\mathbf{B}(\mathbf{R}^\omega)$ which is invariant with respect to the group S_ω.

Let χ_1 denote the completion of measure χ. In other words, χ_1 is the complete S_ω-measure in $\mathbf{R}^\omega$ and has the uniqueness property with respect to the class of all σ-finite S_ω-measures. The following statement is valid.

Theorem 1. *There exists a partition $\{A, B\}$ of $\mathbf{R}^\omega$ satisfying the following three conditions:*

(1) $(\forall F)$ $(F \subset \mathbf{R}^\omega$, *$F$ is a closed subset,* $\chi_1(F) > 0 \Rightarrow \operatorname{card}(A \cap F) = \operatorname{card}(B \cap F) = \mathbf{c})$*;*

(2) $(\forall g)$ $(g \in G \Rightarrow \operatorname{card}(A \triangle g(A)) < \mathbf{c}, \operatorname{card}(B \triangle g(B)) < \mathbf{c})$*;*

(3) $(\forall h)$ $(h \in s_0 \Rightarrow h(B) = A \cup \{0\}$*, where $\{0\}$ is the neutral element of the additive group $\mathbf{R}^\omega$).*

Analogous partitions can be found in [2], [5], [8].

By using these two sets, the measure χ_1 can be extended to a measure which fails to have a strong uniqueness property. For this purpose, it suffices to define a σ-algebra S generated by the union

$$\{A, B\} \cup \operatorname{dom}(\chi_1) \cup \mathbf{F}(\mathbf{R}^\omega),$$

where

$$\mathbf{F}(\mathbf{R}^\omega) = \{Z : Z \subset \mathbf{R}^\omega, \mathrm{card}(Z) < \mathbf{c}\}.$$

It is clear that any element from S can be represented in the form

$$((A \cap X) \cup (B \cap Y) \cup X_1) \setminus X_2,$$

where $X \in \mathrm{dom}(\chi_1)$, $Y \in \mathrm{dom}(\chi_1)$, $X_1 \in \mathbf{F}(\mathbf{R}^\omega)$ and $X_2 \in \mathbf{F}(\mathbf{R}^\omega)$.

Define on the S the functional μ by the formula

$$\mu(((A \cap X) \cup (B \cap Y) \cup X_1) \setminus X_2) = \frac{1}{2}(\chi_1(X) + \chi_1(Y)).$$

This definition of μ is correct and μ is a measure extending χ_1.

The following Theorem is valid.

Theorem 2. *In the space $\mathbf{R}^\omega$, there exists a σ-finite S_ω-measure μ having the strong uniqueness property.*

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ON SOME NEW DECOMPOSITION THEOREMS IN MULTIFUNCTIONAL HERZ ANALYTIC FUNCTION SPACES IN BOUNDED PSEUDOCONVEX DOMAINS

ROMI F. SHAMOYAN

Abstract. Under certain integral condition we present new sharp decomposition theorems for multifunctional Herz spaces in the unit ball and pseudoconvex domains expanding the known results from the unit ball. This expands completely the well-known atomic decomposition theorem for one functional Bergman space in the unit ball.

Introduction and Main Results

The problem we consider is well-known for functional spaces in $\mathbb{R}^n$ (the problem of equivalent norms) (see, e.g., [9]). Let $X, (X_j)$ be a function space in a fixed product domain and (or) in $\mathbb{C}^n$ (normed or quasinormed); our aim is to find an equivalent expression for $\|f_1 \dots f_m\|_X$; $m \in \mathbb{N}$. (Note that they are closely connected with spaces on the product domains, since often if $f(z_1 \dots z_m) = \prod\limits_{j}^{m} f_j(z_j)$, then $\|f\|_X = \prod\limits_{j=1}^{m} \|f_j\|_{X_j}$). The obtained results also, as we'll see below, extend some well-known assertions to the atomic decomposition of A^p_α type spaces.

To study such a group of functions it is natural, for example, to ask about the structure of each $\{f_j\}_{j=1}^m$ of this group.

This can be done, for example, if we turn to the following question of finding conditions on $\{f_1, \dots, f_m\}$ such that $\|f_1, \dots, f_m\|_X \asymp \prod\limits_{i=1}^{m} \|f_j\|_{X_j}$ decomposition is valid. In this case we find that if for some positive constant c, $\prod\limits_{i=1}^{m} \|f_j\|_{X_j} \le c\|f_1 \dots f_m\|_X$, then each $f_j, f_j \in X_j$; $j = 1, \dots, m$, where X_j is a new normed (or quasinormed) function space in the $\mathbb{D}$ domain and, hence, we can now easily get properties of $\{f_j\}$ based on the facts of already known one functional function space theory. (For example, to use the known theorems for each $f_j \in X_j$, $j = 1, \dots, m$ on atomic decompositions). This idea has been applied to Bergman spaces in the unit ball and then to bounded pseudoconvex domains with a smooth boundary (see the recent paper [5]. In this paper, we extend these results in various directions by using modification of the known proof.

We denote as usual by $A^p_\alpha(B)$ the Bergman space in the unit ball B (see [5, 8]), where $0 < p < \infty$, $\alpha > -1$.

We showed in [5] that $\|f_1 \dots f_m\|_{A^p_\tau} \asymp \prod\limits_{i=1}^{m} \|f_j\|_{A^p_{\alpha_j}}$ is valid under certain integral (A) condition (see below) if $p \le 1$ and if $\tau = \tau(p, \alpha_1, \dots, \alpha_m, m)$.

From our discussion above we can see that of interest is to show that

$$\prod_{i=1}^{m} \|f_j\|_{A^p_{\alpha_j}(B)} \le c_1 \|f_1 \dots f_m\|_{A^p_\tau(B)},$$

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since the reverse follows directly from the uniform estimate (see [8, 10])

$$|f(z)|(1-|z|)^{\dfrac{\alpha_j+n+1}{p}} \leq c\|f\|_{A^p_{\alpha_j}};\ \ 0 \leq p < \infty;\ \ \alpha_j > -1;\ \ j = 1, \dots, m$$

and ordinary induction. This leads easily to the fact that τ can be calculated as

$$\tau = (n+1)(m-1) + \left(\sum_{u=1}^{m} \alpha_j\right);\ \ \alpha_j > -1;\ \ 0 \leq p < \infty;$$

It should be noted that similar very simple proof based only on various known uniform estimates can be used in all our proofs below for analogous inequalities in various spaces. So, we are, mainly, concentrated on the reverse estimates (see [8] for various uniform estimates).

Note also that this argument likewise allows get easily even more general version with $|f_1|^{p_1} \dots |f_m|^{p_m}$ instead of $|f_1|^p \dots |f_m|^p$ (that has been discussed above for $0 < p_j < \infty$, $j = 1, \dots, m$).

We denote by dV (or $d\delta$) Lebesgue measure on the unit ball B and by C, C_α, C_1, C_2 various positive constants below. By $H(B)$ we denote the space of all analytic functions on B and by $D(a_k, r)$ or $B(z, r)$ Bergman's ball in $\mathbb{B}$ (see [8, 10]).

Assume that

$$f_1(w_1)\dots f_m(w_m) = c\alpha \int\limits_B \frac{(f_1(z)\dots f_m(z))(1-|z|)^\alpha dV(z)}{\prod\limits_{j=1}^{m}(1-<z,w_j>)^{\frac{n+1+\alpha}{m}}} \tag{A}$$

$$\alpha > \alpha_0,\ \ w_j \in B,\ \ j = 1, \dots, m,$$

where α_0 is large enough.

We define some direct extensions of classical Bergman A^p_α function spaces in the unit ball in Herz spaces.

We fix an r-lattice $\{a_k\}$ in the ball (see [10]) till the end of the paper.

Let

$$K^{p,q}_\alpha = \left\{f \in H(B) : \int\limits_B \left(\int\limits_{B(z,r)} |f(\tilde z)|^p(1-|\tilde z|)^\alpha dV(\tilde z)\right)^{\frac{q}{p}} dV(z) < \infty\right\};$$

$$M^{p,q}_\alpha = \left\{f \in H(B) : \sum_{k\geq 0}\left(\int\limits_{D(a_k,r)} |f(w)|^p(1-|w|)^\alpha dV(w)\right)^{\frac{q}{p}} < \infty\right\};\quad 0 < p, q < \infty, \alpha > -1;$$

(Note $M^{p,p}_\alpha = A^p_\alpha$, $0 < p < \infty$, $\alpha > -1$)

$$K^{p,\infty}_\alpha = \left\{f \in H(B) : \int\limits_B \left(\sup_{z\in B(w,r)}\right)|f(z)|^p(1-|z|)^\alpha dV(w) < \infty\right\};$$

$$M^{p,\infty}_\alpha = \left\{f \in H(B) : \sum_{k\geq 0}\left(\sup_{z\in D(a_k,r)}\right)|f(z)|^p(1-|z|)^\alpha < \infty\right\}.$$

$$0 < p, q < \infty, \quad \alpha \geq 0;$$

These are Banach space for $\min(p, q) \geq 1$ and complete metric spaces for other values.

Theorem 1. *Let X be one of these spaces and $0 < q \leq p \leq 1$; (or $0 < p \leq 1$; $q = \infty$). Then for $f_1, \dots, f_m \in H(B)$; $\alpha_j > -1$, (or $\alpha_j \geq 0$); $j = 1, \dots, m$, we have $\|f_1 \dots f_m\|_{X^{p,q}_\tau} \asymp \prod\limits_{i=1}^{m} \|f_i\|_{X^{p,q}_\alpha}$, if for some $\beta; \beta > \beta_0$ and some $\tau, \tau > -1$ (or $\tau \geq 0$),*

$$f_1(w_1)\dots f_m(w_m) = \int\limits_B \frac{(f_1(z)\dots f_m(z)) \times (1-|z|)^\beta}{\prod\limits_{i=1}^{m}(<1-z,w_j>)^{\frac{\beta+n+1}{m}}} dV(z);\ w_j \in B,\ j = 1, \dots, m, \tag{S}$$

where $\tau = \tau(p, q, n, m, \alpha_1, \dots, \alpha_m)$.

Our theorem extends the known result to the atomic decomposition of Bergman multifunctional space A^p_α (see [5]). For $p = q$, in the unit disk, ball we have $M^{p,p}_\alpha = A^p_\alpha$, $K^{p,p}_\alpha = A^p_\beta$, $0 < p < \infty$ for some $\beta = \beta(p,q)$ (see [2]). If, in addition, $m = 1$, then the integral condition(s) vanishes and we can apply now the atomic decomposition theorem for A^p_α class in the ball, disk (see [10]).

Remark 1. For each space, τ is a special number which can be fixed.

Remark 2. For the mixed norm of $A^{p,q}_\alpha, F^{p,q}_\alpha$ spaces we have found the very similar almost sharp results:

$$A^{p,q}_\alpha = \Big\{ f \in H(B) : \int_0^1 \Big(\int_S |f(z)|^p d\sigma(\xi) \Big)^{\frac{q}{p}} (1-|z|)^\alpha d|z| < \infty \};$$
$$0 < p, \ \ q < \infty, \ \ \alpha > -1,$$

where $S = |z| = 1, and d\sigma$ is a Lebesgue measure on S, and

$$F^{p,q}_\alpha = \Big\{ f \in H(B) : \int_S \Big(\int_0^1 |f(z)|^p (1-|z|)^\alpha d|z| \Big)^{\frac{q}{p}} d\sigma(\xi) < \infty \Big\};$$
$$F^{p,\infty}_\beta = \Big\{ f \in H(B) : \int_S \Big(\sup_{0<r<1} \Big) |f(r\xi)|^p (1-r)^\beta d\sigma(\xi) < \infty \Big\};$$
$$A^{p,\infty}_\alpha = \Big\{ f \in H(B) : \int_0^1 (M_\infty(f,r)^p)\,(1-r)^\alpha dr < \infty \Big\};$$
$$0 < q < \infty, \ \ 0 < p < \infty, \ \ \alpha > -1, \ \ \beta \geq 0.$$

Note now if each (f_i) from one functional (X_i) space can be decomposed into atoms and then, since $\|f_1, \dots, f_m\|_X \asymp \prod_{i=1}^m \|f_i\|_{X_i}$, we can also decompose each (f_i) also as soon as $\|f_1 \dots f_m\|_X < \infty, m > 1$ because integral condition we posed is valid for spaces with infinite or finite indices.

Now we turn to the case of more general spaces on the bounded pseudoconvex domains with a smooth boundary on Ω, using Kobayashi balls $B(z,r)$.

First, we define the spaces and then formulate our theorems.

For the basic definitions ofthe function theory in Ω, we refer to [1], [5], [7], [3].

Let, further, (for some of these spaces see, for example, [3])

$$(A^{p,q}_\alpha)(\Omega) = \Big\{ f \in H(\Omega) : \int_0^\rho \Big(\int_{\partial D_r} |f(\omega)|^p d\sigma(\omega) \Big)^{\frac{q}{p}} \times (r^\alpha) dr < \infty \Big\};$$
$$\alpha > -1; \ \ 0 < p, \ \ q \leq \infty.$$

We refer to [5] for the ∂D_r domain and $d\sigma$ is a Lebesgue measure on ∂D_r, where $H(\Omega)$ is a space of all analytic functions on Ω, $\delta(w) = dist(w, \partial\Omega)$ (for these $A^{p,q}_\alpha$ spaces our result is almost sharp).

We fix an r-lattice in pseudoconvex domains (see [5]).

Let also

$$(M^{p,q}_\alpha)(\Omega) = \Big\{ f \in H(\Omega) : \int_\Omega \Big(\int_{B(z,r)} |f(\omega)|^p (\delta^\alpha(\omega)) dV(\omega) \Big)^{\frac{q}{p}} dV(z) < \infty \Big\};$$
$$\alpha > -1; \ \ 0 < p, \ \ q < \infty;$$
$$(K^{p,q}_\alpha)(\Omega) = \Big\{ f \in H(\Omega) : \sum_{k \geq 0} \Big(\int_{D(a_k,r)} |f(\omega)|^p \times (\delta^\alpha(\omega)) dV(\omega) \Big) < \infty \Big\};$$
$$\alpha > -1; \ \ 0 < p, \ \ q < \infty;$$

$$(K_\alpha^{p,\infty})(\Omega) = \left\{ f \in H(\Omega) : \int_\Omega \left(\sup_{z \in B(w,r)} |f(z)|^p \right) (\delta(z))^\alpha \, dV(\omega) < \infty \right\};$$
$$0 < p \le \infty;$$

$M^{p,\infty}$ can be defined similarly as in the ball.

For these general spaces and domains we, however, impose one additional condition on the weighted Bergman Kernel $K(z,\omega)$ in Ω domain to get a new sharp result.

Theorem 2 (for pseudoconvex domains). *Let $f_i \in H(\Omega)$, $i = 1, \dots, m$.*

Let X be one of $K_\alpha^{p,q}$, or $M_\alpha^{p,q}$ type spaces defined above. Then for some τ, we have

$$\|f_1, \dots, f_m\|_{X_\tau^{p,q}} \asymp \prod_{i=1}^m \|f_i\|_{X_{\alpha_i}^{p,q}},$$

for $0 < q \le p \le 1$ or $p \le 1, q = \infty, \alpha_i > -1$ or $\alpha_i \ge 0$ for $i = 1, \dots, m$, if

$$(f_1(z_1), \dots, f_m(z_m)) = \mathbb{C} \int_\Omega (f_1(\omega), \dots, f_m(\omega)) \left(\prod_{j=1}^m K_{\frac{\tau+n+1}{m}}(z_j, \omega) \right) \delta^\tau(\omega) dV(\omega)$$
$$\tau > \tau_0, \;\; z_i \in \Omega, \;\; i = 1, \dots, m,$$

under one additional condition on Bergman Kernel of t type $K_t(z,w)$

$$\int_{B(\tilde{z},r)} |K_t(z,\omega)|^p \, \delta^{\tilde{\alpha}}(z) dv(z) \le \mathbb{C} \left| K_{tp+\tilde{\alpha}+n+1}(w, \tilde{z}) \right|, \tilde{z}, w \in \Omega$$

for every Kobayashi ball $B(\tilde{z}, r), \tilde{z} \in B, \tilde{\alpha} > -1$, $t > 0$, $r > 0$ (with modification for $p = \infty$).

Theorems 1 and 2 are, probably, valid for $p, q \ge 1$, and we will turn to this problem in our other paper.

Remark 1′. Similar results with very similar proofs are valid for analytic spaces on tubular domains over symmetric cones. Such type spaces in unbounded domains have been studied recently by many authors. (see, for example, [6–8] and various references therein).

Proofs are essentially the same and we will present them in the other separate paper devoted, mainly, to the spaces in such a type of general unbounded domains in $\mathbb{C}^n$.

For example, for $(A_\tau^{p,q})$ spaces in a tubular domain T_Ω, $\|f_1 \dots f_m\|_{A_S^p(T_\Omega)} \asymp \prod_{i=1}^m \|f_i\|_{A_{\tau_i}^p(T_\Omega)}$ is valid for $1 < p < \infty$; $\tau_j > -1$; $S = S(\tau_1, \dots, \tau_m, p, q, m)$; if

$$f_1(w_1) \dots f_m(w_m) = \int_{T_\Omega} \frac{(f_1(z) \dots f_m(z) \triangle^\tau (I_m(z)))}{\prod_{i=1}^m \triangle^{\frac{\tau + \frac{2n}{r}}{m}} \left(\frac{z - w_i}{i}\right)} dV(z)$$

for $w_j \in T_\Omega, \tau > \tau_0, \tau_0$ is large enough, $j = 1, \dots, m$, where $\triangle^\tau$ is a determinant function of T_Ω (see [6], [8]), dv is a Lebesgue measure on T_Ω.

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