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Original article

On the Opial type criterion for the well-posedness of the Cauchy problem for linear systems of ordinary differential equations

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Abstract

There are obtained necessary and sufficient conditions for the well-posedness of the Cauchy problem for the systems of linear ordinary differential equations, analogous to the sufficient condition by Z. Opial for the problem one. Moreover, there are given the efficient sufficient conditions for the problem one.

Keywords: Linear systems of ordinary differential equations; The Cauchy problem; Well-posedness; The Opial type condition; Necessary and sufficient conditions; Efficient sufficient conditions

1. Statement of the problem and basic notation

Let $P_0 \in L_{loc}(I, \mathbb{R}^{n\times n})$, $q_0 \in L_{loc}(I, \mathbb{R}^n)$ and $t_0 \in I$, where I is an arbitrary interval from $\mathbb{R}$ non-degenerated in the point. Let x_0 be a unique solution of the Cauchy problem

$$\frac{dx}{dt} = \mathcal{P}_0(t)\,x + q_0(t), \tag{1.1}$$

$$x(t_0) = c_0, \tag{1.2}$$

where $c_0 \in \mathbb{R}^n$ is a constant vector.

Consider sequences of matrix- and vector-functions $P_k \in L_{loc}(I, \mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$ and $q_k \in L_{loc}(I, \mathbb{R}^n)$ $(k = 1, 2, \dots)$, respectively; sequence of points t_k $(k = 1, 2, \dots)$ and sequence of constant vectors $c_k \in \mathbb{R}^n$ $(k = 1, 2, \dots)$.

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In [1–8] (see, also the references therein), the sufficient conditions are given such that a sequence of unique solutions x_k $(k = 1, 2, \dots)$ of the Cauchy problems

$$\frac{dx}{dt} = \mathcal{P}_k(t)\,x + q_k(t), \tag{1.1$_k$}$$

$$x(t_k) = c_k \tag{1.2$_k$}$$

$(k = 1, 2, \dots)$ satisfy the condition

$$\lim_{k\to+\infty} x_k(t) = x_0(t) \quad \text{uniformly on } I. \tag{1.3}$$

In the present paper necessary and sufficient conditions are established for the sequence of the Cauchy problems (1.1_k), (1.2_k) $(k = 1, 2, \dots)$ to have the above-mentioned property. The obtained criterion are based on the concept by Z. Opial, concerning to the sufficient condition considered in [8], and it differs from analogous one given in [1].

The Opial type sufficient conditions are investigated in [5] for the well-posedness problem of the Cauchy problem for linear functional-differential equations.

In the paper the use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$; $[a, b]$ and $]a, b[(a, b \in \mathbb{R})$ are, respectively, closed and open intervals.

I is an arbitrary, non-degenerated in the point, finite or infinite interval from $\mathbb{R}$.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$O_{n\times m}$ is the zero $n \times m$-matrix.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; o_n is the zero n-vector.

$\mathbb{R}^{n\times n}$ is the space of all real quadratic $n \times n$-matrices $X = (x_{ij})_{i,j=1}^n$;

I_n is the identity $n \times n$-matrix; $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_n$; δ_{ij} is the Kronecker symbol, i.e. $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j\,(i, j = 1, \dots)$;

If $X \in \mathbb{R}^{n\times n}$, then X^{-1} and $\det(X)$ are, respectively, the matrix inverse to X and the determinant of X; $\operatorname{diag}X = \operatorname{diag}(x_{11}, \dots, x_{nn})$ is the diagonal matrix corresponding to X.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

We say that the matrix-function $X \in L_{loc}(I, \mathbb{R}^{n\times n})$ satisfies the Lappo-Danilevskiĭ condition if for every $\tau \in I$ the following condition holds

$$X(t) \int_\tau^t X(\tau)\,d\tau = \int_\tau^t X(\tau) d\tau \cdot X(t) \quad \text{for a. a. } t \in I.$$

$\overset{b}{\underset{a}{\mathrm{V}}}(X)$ is the sum total variation of the components $x_{ij}(i = 1, \dots, n;\ j = 1, \dots, m)$ of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$; $\overset{a}{\underset{b}{\mathrm{V}}}(X) = -\overset{b}{\underset{a}{\mathrm{V}}}(X)$;

$\underset{I}{\mathrm{V}}(X) = \lim_{a\to\alpha+,b\to\beta-} \overset{b}{\underset{a}{\mathrm{V}}}(X)$, where $\alpha = \inf I$ and $\beta = \sup I$.

$C(I; \mathbb{R}^{m\times n})$ is a space of continuous and bounded matrix-functions $X : I \to \mathbb{R}^{m\times n}$ with the norm

$$\|X\|_c = \sup\{\|X(t)\| : t \in I\};$$

$C(I; D)$, where $D \subset \mathbb{R}^{m\times n}$, is the set of continuous and bounded matrix-functions $X : I \to D$;

$C_{loc}(I; D)$ is the set of continuous matrix-functions $X : I \to D$;

$\widetilde{C}(I; D)$ is the set of absolutely continuous matrix-functions $X : I \to D$;

$\widetilde{C}_{loc}(I; D)$ is the set of matrix-functions $X : I \to D$ which are absolutely continuous on the every closed interval $[a, b]$ from I.

$L(I; D)$, where $D \subset \mathbb{R}^{m\times n}$, is the set of matrix-functions $X : I \to D$ whose components are Lebesgue-integrable;

$L_{loc}(I; D)$ is the set of matrix-functions $X : I \to D$ whose components are Lebesgue-integrable on the every closed interval $[a, b]$ from I.

We introduce the operators. If $G \in L(I; \mathbb{R}^{l\times n})$, $X \in L(I; \mathbb{R}^{n\times m})$, $Y \in L(I; \mathbb{R}^{n\times n})$, and $H \in \widetilde{C}(I; \mathbb{R}^{n\times n})$ is nonsingular, then

$$\mathcal{B}_c(G, X)(t) = \int_\alpha^t G(\tau)\, X(\tau) d\tau \quad \text{for } t \in I,$$

$$\mathcal{I}_c(H, Y)(t) = \int_\alpha^t (H'(\tau) + H(\tau)\, Y(\tau))\, H^{-1}(\tau) d\tau \quad \text{for } t \in I.$$

The vector-function $x : I \to \mathbb{R}^n$ is said to be a solution of the system (1.1) if it belongs to $\widetilde{C}_{loc}(I; \mathbb{R}^n)$ and satisfies the equality $x'(t) = \mathcal{P}_0(t)x(t) + q_0(t)$ at almost all $t \in I$.

Under a solution of the Cauchy problem (1.1), (1.2) we understand a solution of system (1.1) satisfying condition (1.2).

We will assume that $P_k = (p_{kil})_{i,l=1}^n$ and $q_k = (q_{kl})_{l=1}^n (k = 0, 1, \ldots)$.

Along with systems (1.1) and (1.1_k) we consider the corresponding homogeneous systems

$$\frac{dx}{dt} = P_0(t)\, x \tag{1.1_0}$$

and

$$\frac{dx}{dt} = P_k(t)\, x \tag{1.1_{k0}}$$

$(k = 1, 2, \ldots)$.

2. Formulation of the main results

Definition 2.1. We say that the sequence $(P_k, q_k; t_k)$ $(k = 1, 2, \ldots)$ belongs to the set $\mathcal{S}(P_0, q_0; t_0)$ if for every $c_0 \in \mathbb{R}^n$ and a sequence $c_k \in \mathbb{R}^n$ $(k = 1, 2, \ldots)$ satisfying the condition

$$\lim_{k\to+\infty} c_k = c_0, \tag{2.1}$$

condition (1.3) holds, where x_k is the unique solution of problem (1.1_k), (1.2_k) for every natural k.

Theorem 2.1. *Let $P_0 \in L(I, \mathbb{R}^{n\times n})$, $q_0 \in L(I, \mathbb{R}^n)$ and $t_k \in I (k = 0, 1, \ldots)$ be such that*

$$\lim_{k\to+\infty} t_k = t_0. \tag{2.2}$$

Then

$$((P_k, q_k; t_k))_{k=1}^{+\infty} \in \mathcal{S}(P_0, q_0; t_0) \tag{2.3}$$

if and only if there exists a sequence of matrix-functions $H_k \in \widetilde{C}(I; \mathbb{R}^{n\times n})$ $(k = 0, 1, \ldots)$ such that

$$\inf\Big\{|\det(H_0(t))| : t \in I\Big\} > 0, \tag{2.4}$$

and the conditions

$$\lim_{k\to+\infty} H_k(t) = H_0(t), \tag{2.5}$$

$$\lim_{k\to+\infty} \left\{ \left\| \mathcal{I}_c(H_k, P_k)(\tau)\Big|_{t_k}^t - \mathcal{I}_c(H_0, P_0)(\tau)\Big|_{t_0}^t \right\| \times \left(1 + \left|\bigvee_{t_k}^t(\mathcal{I}_c(H_k, P_k))\right|\right) \right\} = 0 \tag{2.6}$$

and

$$\lim_{k\to+\infty} \left\{ \left\| \mathcal{B}_c(H_k, q_k)(\tau)\Big|_{t_k}^t - \mathcal{B}_c(H_0, q_0)(\tau)\Big|_{t_0}^t \right\| \times \left(1 + \left|\bigvee_{t_k}^t(\mathcal{I}_c(H_k, P_k))\right|\right) \right\} = 0 \tag{2.7}$$

hold uniformly on I.

Theorem 2.2. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$, $c_k \in \mathbb{R}^n$ and $t_k \in I$ $(k = 0, 1, \ldots)$ be such that conditions* (2.1) *and* (2.2) *hold, and the conditions*

$$\lim_{k\to+\infty}\left\{\left\|\int_{t_k}^{t} P_k(\tau)d\tau - \int_{t_0}^{t} P_0(\tau)d\tau\right\|\left(1+\left|\int_{t_k}^{t}\|P_k(\tau)\|d\tau\right|\right)\right\} = 0 \tag{2.8}$$

and

$$\lim_{k\to+\infty}\left\{\left\|\int_{t_k}^{t} q_k(\tau)d\tau - \int_{t_0}^{t} q_0(\tau)d\tau\right\|\left(1+\left|\int_{t_k}^{t}\|P_k(\tau)\|d\tau\right|\right)\right\} = 0 \tag{2.9}$$

are fulfilled uniformly on I. Then condition (1.3) *holds.*

Theorem 2.3. *Let x_0^* be a unique solution of the Cauchy problem*

$$\frac{dx}{dt} = \mathcal{P}_0^*(t)\,x + q_0^*(t), \tag{2.10}$$

$$x(t_0) = c_0^*, \tag{2.11}$$

where $P_0^* \in L(I, \mathbb{R}^{n\times n})$, $q_0^* \in L(I, \mathbb{R}^n)$, $c_0^* \in \mathbb{R}^n$, $t_0 \in I$. Let, moreover, $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$, $c_k \in \mathbb{R}^n$ and $t_k \in I\,(k = 1, 2, \ldots)$ be such that conditions (2.2),

$$\inf\{|\det(H_k(t))| : t \in I_{t_k}\} > 0 \quad \textit{for every sufficiently large } k, \tag{2.12}$$

and

$$\lim_{k\to+\infty} c_k^* = c_0^* \tag{2.13}$$

hold, and conditions (2.6) *and*

$$\lim_{k\to+\infty}\left\{\left\|\int_{t_k}^{t} q_k^*(\tau)d\tau - \int_{t_0}^{t} q_0^*(\tau)d\tau\right\|\left(1+\left|\mathop{\mathrm{V}}_{t_k}^{t}(\mathcal{I}_c(H_k, P_k))\right|\right)\right\} = 0 \tag{2.14}$$

are fulfilled uniformly on I, where $H_k \in \widetilde{C}(I; \mathbb{R}^{n\times n})$, $h_k \in \widetilde{C}(I; \mathbb{R}^n)$ $(k = 1, 2, \ldots)$,

$$q_k^*(t) = H_k(t)\,q_k(t) + h_k'(t) - (H_k'(t) + H_k(t)\,P_k(t))\,H_k^{-1}(t)\,h_k(t) \quad \textit{for } t \in I \ (k = 1, 2, \ldots)$$

and

$$c_k^* = H_k(t_k)\,c_k + h_k(t_k) \quad (k = 1, 2, \ldots).$$

Then

$$\lim_{k\to+\infty}(H_k(t)\,x_k(t) + h_k(t)) = x_0^*(t) \quad \textit{uniformly on } I. \tag{2.15}$$

Remark 2.1. In Theorem 2.3, the vector function $x_k^*(t) = H_k(t)\,x_k(t) + h_k(t)$ is a solution of problem

$$\frac{dx}{dt} = \mathcal{P}_k^*(t)\,x + q_k^*(t), \tag{2.10$_k$}$$

$$x(t_k) = c_k^* \tag{2.11$_k$}$$

for every natural k.

Corollary 2.1. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$, $c_k \in \mathbb{R}^n$ and $t_k \in I\,(k = 0, 1, \ldots)$ be such that conditions* (2.2), (2.4) *and*

$$\lim_{k\to+\infty}(c_k - \varphi_k(t_k)) = c_0 \tag{2.16}$$

hold, and conditions (2.5), (2.6) *and*

$$\lim_{k\to+\infty}\left\{\left\|\int_{t_k}^{t} H_k(\tau)\big(q_k(\tau)-\varphi_k'(\tau)+P_k(\tau)\,\varphi_k(\tau)\big)d\tau-\int_{t_0}^{t} H_0(\tau)\,q_0(\tau)d\tau\right\|\right.$$
$$\left.\times\left(1+\left|\bigvee_{t_k}^{t}(\mathcal{I}_c(H_k,P_k))\right|\right)\right\}=0$$

are fulfilled uniformly on I, *where* $H_k\in\widetilde{C}(I;\mathbb{R}^{n\times n})$ *and* $\varphi_k\in\widetilde{C}(I;\mathbb{R}^n)$ $(k=0,1,\dots)$. *Then*

$$\lim_{k\to+\infty}(x_k(t)-\varphi_k(t))=x_0(t)\quad\textit{uniformly on } I. \tag{2.17}$$

Below, we give some sufficient conditions guaranteeing inclusion (2.3). To this connection we give a theorem different from Theorem 2.1 concerning the necessary and sufficient condition for inclusion (2.3), as well, and corresponding propositions.

Theorem 2.1′. *Let* $P_0\in L(I,\mathbb{R}^{n\times n})$, $q_0\in L(I,\mathbb{R}^n)$, $t_0\in I$, *and* $t_k\in I$ $(k=1,2,\dots)$ *be such that condition* (2.2) *hold. Then inclusion* (2.3) *holds if and only if there exists a sequence of matrix-functions* $H_k\in\widetilde{C}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ *such that conditions* (2.4) *and*

$$\lim_{k\to+\infty}\sup\int_I\|H_k'(\tau)+H_k(\tau)\,P_k(\tau)\|d\tau<+\infty \tag{2.18}$$

hold, and conditions (2.5),

$$\lim_{k\to+\infty}\int_{t_k}^{t} H_k(\tau)\,P_k(\tau)d\tau=\int_{t_0}^{t} H_0(\tau)\,P_0(\tau)d\tau \tag{2.19}$$

and

$$\lim_{k\to+\infty}\int_{t_k}^{t} H_k(\tau)\,q_k(\tau)d\tau=\int_{t_0}^{t} H_0(\tau)\,q_0(\tau)d\tau \tag{2.20}$$

are fulfilled uniformly on I.

Remark 2.2. Due to (2.4), (2.5), there exists a positive number r such that

$$\sup\left\{\left|\bigvee_{t_k}^{t}(\mathcal{I}_c(H_k,P_k))\right|:t\in I\right\}\le r\int_I\|H_k'(\tau)+H_k(\tau)\,P_k(\tau)\|d\tau\quad(k=0,1,\dots).$$

In addition, in view of Lemma 3.2 (see below), by conditions (2.18) and (2.19) we get

$$\lim_{k\to+\infty}(\mathcal{I}_c(H_k,P_k)(t)-\mathcal{I}_c(H_k,P_k)(t_k))=\mathcal{I}_c(H_0,P_0)(t)-\mathcal{I}_c(H_0,P_0)(t_0)$$

uniformly on I. Therefore, thanks to this, (2.18) and (2.20), conditions (2.6) and (2.7) are fulfilled uniformly on I

Theorem 2.2′. *Let* $P_k\in L(I,\mathbb{R}^{n\times n})$, $q_k\in L(I,\mathbb{R}^n)$, $c_k\in\mathbb{R}^n$ *and* $t_k\in I$ $(k=0,1,\dots)$ *be such that conditions* (2.1), (2.2) *and*

$$\lim_{k\to+\infty}\sup\int_I\|P_k(\tau)\|d\tau<+\infty \tag{2.21}$$

hold, and the conditions

$$\lim_{k\to+\infty}\int_{t_k}^{t} P_k(\tau)d\tau=\int_{t_0}^{t} P_0(\tau)d\tau \tag{2.22}$$

and

$$\lim_{k\to+\infty}\int_{t_k}^{t} q_k(\tau)d\tau=\int_{t_0}^{t} q_0(\tau)d\tau \tag{2.23}$$

are fulfilled uniformly on I. *Then condition* (1.3) *holds.*

Theorem 2.3′. *Let x_0^* be a unique solution of the Cauchy problem* (2.10), (2.11), *where $P_0^* \in L(I, \mathbb{R}^{n\times n})$, $q_0^* \in L(I, \mathbb{R}^n)$, $c_0^* \in \mathbb{R}^n$, $t_0 \in I$. Let, moreover, $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$, $c_k \in \mathbb{R}^n$ and $t_k \in I$ $(k = 1, 2, \dots)$ be such that conditions* (2.2), (2.12), (2.18) *and*

$$\lim_{k\to+\infty} (H_k(t_k)\, c_k + h_k(t_k)) = c_0^* \tag{2.24}$$

hold, and the conditions

$$\lim_{k\to+\infty} (\mathcal{I}_c(H_k, P_k)(t) - \mathcal{I}_c(H_k, P_k)(t_k)) = \mathcal{I}_c(H_0, P_0^*)(t) - \mathcal{I}_c(H_0, P_0^*)(t_0), \tag{2.25}$$

and

$$\lim_{k\to+\infty} \int_{t_k}^{t} q_k^*(\tau)d\tau = \int_{t_0}^{t} q_0^*(\tau)d\tau \tag{2.26}$$

are fulfilled uniformly on I, where $H_k \in \widetilde{C}(I; \mathbb{R}^{n\times n})$, $h_k \in \widetilde{C}(I; \mathbb{R}^n)(k = 1, 2, \dots)$, and the vector-functions $q_k^*(k = 1, 2, \dots)$ are defined as in Theorem 2.3. *Then condition* (1.3) *holds.*

Corollary 2.1′. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$, $c_k \in \mathbb{R}^n$ and $t_k \in I$ $(k = 0, 1, \dots)$ be such that conditions* (2.2), (2.4), (2.16) *and* (2.18) *hold, and conditions* (2.5), (2.19) *and*

$$\lim_{k\to+\infty} \int_{t_k}^{t} H_k(\tau) \big(q_k(\tau) - \varphi_k'(\tau) + P_k(\tau)\, \varphi_k(\tau)\big)d\tau = \int_{t_0}^{t} H_0(\tau)\, q_0(\tau)d\tau$$

are fulfilled uniformly on I, where $H_k \in \widetilde{C}(I; \mathbb{R}^{n\times n})$ and $\varphi_k \in \widetilde{C}(I; \mathbb{R}^n)(k = 0, 1, \dots)$. Then condition (2.17) *holds.*

Corollary 2.2. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$ and $t_k \in I(k = 0, 1, \dots)$ be such that conditions* (2.2), (2.4) *and* (2.18) *hold, and conditions* (2.5), (2.22), (2.23),

$$\lim_{k\to+\infty} \int_{t_k}^{t} H_k'(\tau) \left(\int_{t_k}^{\tau} P_k(s)ds\right) d\tau = \int_{t_0}^{t} P^*(\tau)d\tau \tag{2.27}$$

and

$$\lim_{k\to+\infty} \int_{t_k}^{t} H_k'(\tau) \left(\int_{t_k}^{\tau} q_k(s)ds\right) d\tau = \int_{t_0}^{t} q^*(\tau)d\tau \tag{2.28}$$

are fulfilled uniformly on I, where $H_0(t) = I_n$, $H_k \in \widetilde{C}(I; \mathbb{R}^{n\times n})(k = 1, 2, \dots)$, $P^* \in L(I, \mathbb{R}^{n\times n})$, $q^* \in L(I, \mathbb{R}^n)$. Then

$$((P_k, q_k; t_k))_{k=1}^{+\infty} \in \mathcal{S}(P_0 - P^*, q_0 - q^*; t_0).$$

Corollary 2.3. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$ and $t_k \in I(k = 0, 1, \dots)$ be such that condition* (2.2) *holds and let there exist a natural number m and matrix-functions $P_{0l} \in L(I; \mathbb{R}^{n\times n})(l = 1, \dots, m-1)$ such that*

$$\limsup_{k\to+\infty} \int_{I} \|H_{k\,m-1}'(t) + H_{k\,m-1}(t)\, P_k(t)\|dt < +\infty, \tag{2.29}$$

and the conditions

$$\lim_{k\to+\infty} H_{k\,m-1}(t) = I_n, \tag{2.30}$$

$$\lim_{k\to+\infty} \int_{t_k}^{t} H_{k\,m-1}(\tau)\, P_k(\tau)d\tau = \int_{t_0}^{t} P_0(\tau)d\tau, \tag{2.31}$$

$$\lim_{k\to+\infty} \int_{t_k}^{t} H_{k\,m-1}(\tau)\, q_k(\tau)d\tau = \int_{t_0}^{t} q_0(\tau)d\tau \tag{2.32}$$

hold uniformly on I*, where*

$$H_{k0}(t)=I_n,\qquad H_{k\,j+1}(t)=\left(I_n-\int_{t_k}^{t}(P_{k\,j+1}(\tau)-P_{0l}(\tau))d\tau\right)H_{kj}(t),$$
$$P_{kj+1}(t)=H'_{kj}(t)+H_{kj}(t)\,P_k(t),\qquad q_{kj+1}(t)=H_{kj}(t)\,q_k(t)$$
$$\text{for } t\in I\ (j=0,\dots,m-1;\ k=0,1,\dots).$$

Then inclusion (2.3) *holds.*

If $m=1$, then Corollary 2.3 coincides to Theorem 2.2′.
If $m=2$, then Corollary 2.3 has the following form.

Corollary 2.3′. *Let* $P_k\in L(I,\mathbb{R}^{n\times n})$*,* $q_k\in L(I,\mathbb{R}^n)$*,* $c_k\in\mathbb{R}^n$ *and* $t_k\in I$ $(k=0,1,\dots)$ *be such that condition* (2.2) *holds and let there exist a matrix-function* $P_{01}\in L(I;\mathbb{R}^{n\times n})$ *such that*

$$\lim_{k\to+\infty}\sup\int_I\left\|P_{01}(t)-\int_{t_k}^{t}(P_k(\tau)-P_{01}(\tau))d\tau\cdot P_k(t)\right\|dt<+\infty,$$

and the conditions

$$\lim_{k\to+\infty}\int_{t_k}^{t}P_k(\tau)d\tau=\int_{t_0}^{t}P_{01}(\tau)d\tau,$$
$$\lim_{k\to+\infty}\int_{t_k}^{t}\left((P_k(\tau)-P_{01}(\tau))\int_{t_k}^{\tau}P_k(s)ds\right)d\tau=\int_{t_0}^{t}(P_0(\tau)-P_{01}(\tau))d\tau$$

and

$$\lim_{k\to+\infty}\left\{\int_{t_k}^{t}q_k(\tau)d\tau+\int_{t_k}^{t}\left((P_k(\tau)-P_{01}(\tau))\int_{t_k}^{\tau}q_k(s)ds\right)d\tau\right\}=\int_{t_0}^{t}q_0(\tau)d\tau$$

are fulfilled uniformly on I*. Then inclusion* (2.3) *holds.*

Corollary 2.4. *Let* $P_0\in L(I,\mathbb{R}^{n\times n})$*,* $q_0\in L(I,\mathbb{R}^n)$*,* $t_0\in I$*, and* $t_k\in I(k=1,2,\dots)$ *be such that condition* (2.2) *holds. Then inclusion* (2.3) *holds if and only if there exists a sequence of matrix-functions* $Q_k\in L(I;\mathbb{R}^{n\times n})(k=0,1,\dots)$ *such that the condition*

$$\lim_{k\to+\infty}\sup\int_I\|P_k(\tau)-Q_k(\tau)\|d\tau<+\infty \tag{2.33}$$

holds, and the conditions

$$\lim_{k\to+\infty}Z_k^{-1}(t)=Z_0^{-1}(t), \tag{2.34}$$

$$\lim_{k\to+\infty}\int_{t_k}^{t}Z_k^{-1}(\tau)\,P_k(\tau)d\tau=\int_{t_0}^{t}Z_0^{-1}(\tau)\,P_0(\tau)d\tau \tag{2.35}$$

and

$$\lim_{k\to+\infty}\int_{t_k}^{t}Z_k^{-1}(\tau)\,q_k(\tau)d\tau=\int_{t_0}^{t}Z_0^{-1}(\tau)\,q_0(\tau)d\tau \tag{2.36}$$

are fulfilled uniformly on I*, where* $Z_k(Z_k(t_k)=I_n)$ *is a fundamental matrices of the homogeneous problems*

$$\frac{dx}{dt}=Q_k(t)x \tag{2.37}$$

for every $k\in\{0,1,\dots\}$.

Corollary 2.5. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$ and $t_k \in I (k = 0, 1, \ldots)$ be such that condition* (2.2) *holds and let there exist a sequence of matrix-functions $Q_k \in L(I; \mathbb{R}^{n\times n})$ $(k = 0, 1, \ldots)$, satisfying the Lappo-Danilevskiĭ condition, such that condition* (2.33) *holds, and the conditions*

$$\lim_{k\to+\infty} \int_{t_k}^{t} Q_k(\tau)d\tau = \int_{t_0}^{t} Q_0(\tau)d\tau,$$

$$\lim_{k\to+\infty} \int_{t_k}^{t} \exp\left(-\int_{t_k}^{\tau} Q_k(s)ds\right) P_k(\tau)d\tau = \int_{t_0}^{t} \exp\left(-\int_{t_0}^{\tau} Q_0(s)ds\right) P_0(\tau)d\tau \tag{2.38}$$

and

$$\lim_{k\to+\infty} \int_{t_k}^{t} \exp\left(-\int_{t_k}^{\tau} Q_k(s)ds\right) q_k(\tau)d\tau = \int_{t_0}^{t} \exp\left(-\int_{t_0}^{\tau} Q_0(s)ds\right) q_0(\tau)d\tau \tag{2.39}$$

are fulfilled uniformly on I. Then inclusion (2.3) *holds.*

Corollary 2.6. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$ and $t_k \in I (k = 0, 1, \ldots)$ be such that condition* (2.2) *holds, the matrix functions $P_k (k = 0, 1, \ldots)$ satisfy the Lappo-Danilevskiĭ condition, and the conditions*

$$\lim_{k\to+\infty} \int_{t_k}^{t} P_k(\tau)d\tau = \int_{t_0}^{t} P_0(\tau)d\tau, \tag{2.40}$$

and

$$\lim_{k\to+\infty} \int_{t_k}^{t} \exp\left(-\int_{t_k}^{\tau} P_k(s)ds\right) q_k(\tau)d\tau = \int_{t_0}^{t} \exp\left(-\int_{t_0}^{\tau} P_0(s)ds\right) q_0(\tau)d\tau \tag{2.41}$$

are fulfilled uniformly on I. Then inclusion (2.3) *holds.*

Corollary 2.7. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$ and $t_k \in I (k = 0, 1, \ldots)$ be such that conditions* (2.2) *and*

$$\lim_{k\to+\infty} \sup \sum_{i,l=1;\, i\neq l}^{n} \int_I \|p_{kil}(\tau)\|d\tau < +\infty$$

hold, and the conditions

$$\lim_{k\to+\infty} \int_{t_k}^{t} p_{kii}(\tau)d\tau = \int_{t_0}^{t} p_{0ii}(\tau)d\tau \quad (i = 1, \ldots, n)$$

$$\lim_{k\to+\infty} \int_{t_k}^{t} z_{kii}^{-1}(\tau)p_{kil}(\tau)d\tau = \int_{t_0}^{t} z_{0ii}^{-1}(\tau)p_{0il}(\tau)d\tau \quad (i \neq l;\ i, l = 1, \ldots, n)$$

and

$$\lim_{k\to+\infty} \int_{t_k}^{t} z_{kii}^{-1}(\tau)q_{ki}(\tau)d\tau = \int_{t_0}^{t} z_{0ii}^{-1}(\tau)q_{0i}(\tau)d\tau \quad (i = 1, \ldots, n)$$

are fulfilled uniformly on I, where

$$z_{kii}(t) = \exp\left(\int_{t_k}^{t} p_{kii}(s)ds\right) \quad \textit{for } t \in I\ (i = 1, \ldots, n;\ k = 1, 2, \ldots).$$

Then inclusion (2.3) *holds.*

Remark 2.3. In Theorems 2.1′–2.3′ and Corollaries 2.1′, 2.2–2.7, we can assume $H_0(t) = I_n$, without loss of generality. It is evident that

$$\mathcal{I}_c(H_0, Y)(t) - \mathcal{I}_c(H_0, Y)(s) = \int_s^t Y(\tau)d\tau \quad \text{for } Y \in L(I; \mathbb{R}^{n\times n}) \text{ and } s, t \in I,$$

in this case.

Remark 2.4. In Theorem 2.2′, condition (2.21) is essential and it cannot be removed. In connection with this we give the example from [4].

Example 2.1. Let $I = [0, 2\pi]$, $n = 1$, $c_k = c_0 = 0$, $P_0(t) = q_0(t) = 0$, $P_k(t) = k\cos^2 k^2 t$, $q_k(t) = -k\sin k^2 t$, $t_0 = t_k = 0 (k = 1, 2, \ldots)$. Then

$$x_0(t) \equiv 0, \qquad x_k(t) \equiv -k\int_0^t \exp\Big(\frac{\sin k^2 t}{k} - \frac{\sin k^2\tau}{k}\Big)\sin k^2\tau d\tau \quad (k = 1, 2, \ldots)$$

and

$$\lim_{k\to+\infty} x_k(t) = x_0(t) + \frac{t}{2} \quad \text{uniformly on } [0, 2\pi].$$

It is evident that, in the case, all conditions of Theorem 2.2′ are valid except of (2.21). On the other hand, the case coordinates to Corollary 2.2 because its conditions hold and the function $x_0^*(t) = t/2$ is a solution of problem (2.10), (2.11), where $P_0^*(t) = 0$, $q_0^*(t) = t/2$, and

$$H_k(t) = \exp\Big(-\frac{\sin k^2 t}{k}\Big) \quad (k = 1, 2, \ldots).$$

Example 2.2. Let $I = [0, 2\pi]$, $n = 2$, $t_0 = t_k = 0$ $(k = 1, 2, \ldots)$,

$$c_0 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad c_k = \begin{pmatrix}1\\1/k\end{pmatrix} \quad (k = 1, 2, \ldots);$$

$$P_0(t) = \begin{pmatrix}0 & 0\\-1/2 & 0\end{pmatrix}, \qquad P_k(t) = \begin{pmatrix}k\cos k^2 t & 0\\-k\sin k^2 t & 0\end{pmatrix} \quad (k = 1, 2, \ldots);$$

$$q_0(t) = q_k(t) = \begin{pmatrix}0\\0\end{pmatrix} \quad (k = 1, 2, \ldots).$$

Then

$$x_0(t) \equiv \begin{pmatrix}1\\-t/2\end{pmatrix}, \qquad x_k(t) \equiv \begin{pmatrix}x_{1k}(t)\\x_{2k}(t)\end{pmatrix} \quad (k = 1, 2, \ldots),$$

where

$$x_{1k}(t) \equiv \exp\Big(\frac{\sin k^2 t}{k}\Big), \qquad x_{2k}(t) \equiv \frac{1}{k} - k\int_0^t \exp\Big(\frac{\sin k^2\tau}{k}\Big)\sin k^2\tau d\tau \quad (k = 1, 2, \ldots).$$

It is not difficult to verify that condition (1.3) is fulfilled uniformly on I. Note that, in the case, condition (2.21) is not hold. But, all conditions of Theorem 2.1′ hold if we assume $H_k(t) = Y_k(t) (k = 0, 1, \ldots)$ therein, where Y_0 and $Y_k (k = 1, 2, \ldots)$, $Y_0(0) = Y_k(0) = I_2$, are is the fundamental matrix of the systems (1.1_0) and (1.1_{k0}) $(k = 1, 2, \ldots)$, respectively.

Remark 2.5. As compared with Theorem 2.1′ and Theorem 2.2′, it is not assumed, in Theorem 2.1′, that the equalities (2.22) and (2.23) hold uniformly on I. Below we will give an example of a sequence of initial value problems for which inclusion (2.3) holds but condition (2.22) is not fulfilled uniformly on I.

Example 2.3. Let $I = [0, \pi]$, $n = 2$, $t_0 = t_k = 0$ $(k = 1, 2, \ldots)$,

$$c_0 = c_k = \begin{pmatrix}0\\0\end{pmatrix} \quad (k = 1, 2, \ldots);$$

$$P_0(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_k(t) = \begin{pmatrix} 0 & p_{k1}(t) \\ 0 & p_{k2}(t) \end{pmatrix} \quad (k = 1, 2, \dots);$$

$$q_0(t) = q_k(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (k = 1, 2, \dots);$$

$$p_{k1}(t) = \begin{cases} (\sqrt{k} + \sqrt[4]{k}) \sin kt & \text{for } t \in I_k, \\ \sqrt{k} \sin kt & \text{for } t \in [0, 2\pi] \setminus I_k \ (k = 1, 2, \dots); \end{cases}$$

$$p_{k2}(t) = \begin{cases} -\alpha'_k(t)\,(1 - \alpha_k(t))^{-1} & \text{for } t \in I_k, \\ 0 & \text{for } t \in [0, 2\pi] \setminus I_k \ (k = 1, 2, \dots); \end{cases}$$

$$\beta_k(t) = \int_0^t (1 - \alpha_k(\tau))\, p_{k1}(\tau)\, d\tau \quad (k = 1, 2, \dots);$$

$$\alpha_k(t) = \begin{cases} 4\pi^{-1}(\sqrt[4]{k} + 1)^{-1} \sin kt & \text{for } t \in I_k, \\ 0 & \text{for } t \in [0, 2\pi] \setminus I_k \ (k = 1, 2, \dots); \end{cases}$$

where

$$I_k = \bigcup_{m=0}^{k-1}]2mk^{-1}\pi, (2m+1)k^{-1}\pi[\quad (k = 1, 2, \dots).$$

Let, moreover, Y_0 and $Y_k (k = 1, 2, \dots)$, $Y_0(0) = Y_k(0) = I_2$, be the fundamental matrix of the systems (1.1_0) and (1.1_{k0}) $(k = 1, 2, \dots)$, respectively. It can easily be shown that

$$Y_0(t) \equiv I_2, \qquad Y_k(t) \equiv \begin{pmatrix} 1 & \beta_k(t) \\ 0 & 1 - \alpha_k(t) \end{pmatrix} \quad (k = 1, 2, \dots)$$

and

$$\lim_{k \to +\infty} Y_k(t) = Y_0(t) \quad \text{uniformly on } [0, 2\pi],$$

since

$$\lim_{k \to +\infty} \|\alpha_k\|_c = \lim_{k \to +\infty} \|\beta_k\|_c = 0.$$

Note that

$$\lim_{k \to +\infty} \int_0^{2\pi} p_{k1}(t)\, dt = 2 \lim_{k \to +\infty} \sqrt[4]{k} = +\infty$$

and

$$\lim_{k \to +\infty} \sup \int_0^{2\pi} |p_{k2}(t)|\, dt = +\infty.$$

Therefore, condition (2.22) is not fulfilled uniformly on I.

On the other hand, if we assume that $H_0(t) = I_n$ and $H_k(t) = Y_k^{-1}(t) (k = 1, 2, \dots)$, then all conditions of Theorem 2.1′ hold.

3. Auxiliary propositions

We will use the following simple lemma.

Lemma 3.1. *Let $h \in \widetilde{C}_{loc}(I; \mathbb{R}^n)$, and $H \in \widetilde{C}_{loc}(I; \mathbb{R}^{n \times n})$ be a nonsingular matrix-function. Then the mapping*

$$x \to y = H\,x + h$$

establishes a one-to-one corresponding between the solution between the solutions x and y of systems

$$\frac{dx}{dt} = \mathcal{P}(t)\,x + q(t)$$

and

$$\frac{dy}{dt} = \mathcal{P}_*(t)\, y + q_*(t)$$

respectively, where the matrix- and vector-functions P_* and q_* are defined, respectively, by

$$P_*(t) \equiv (H'(t) + H(t)P(t))\, H^{-1}(t), \qquad q_*(t) = H(t)\, q(t) + h'(t) - P^*(t)\, h(t).$$

Lemma 3.2. *Let $\alpha_k, \beta_k \in L(I; \mathbb{R})$ $(k = 0, 1, \dots)$ be such that*

$$\lim_{k\to+\infty} \|\beta_k - \beta_0\|_s = 0, \qquad \lim_{k\to+\infty} \sup \int_I |\alpha_k(t)| dt < +\infty,$$

and the condition

$$\lim_{k\to+\infty} \int_a^t \alpha_k(\tau) d\tau = \int_a^t \alpha_0(\tau) d\tau$$

hold uniformly on I, where $a \in I$ is some fixed point. Then

$$\lim_{k\to+\infty} \int_a^t \beta_k(\tau)\alpha_k(\tau) d\tau = \int_a^t \beta_0(\tau)\alpha_0(\tau) d\tau$$

uniformly on I, as well.

The proof of the lemma one can find in [3,6].

4. Proof of the main results

Proof of Theorem 2.2. Let $z_k(t) = x_k(t) - x_0(t)$ for $t \in I (k = 1, 2, \dots\}$.

It is not difficult to check that

$$z_k(t) = z_k(t_k) + \int_{t_k}^t P_0(s)\, z_k(s) ds + \int_{t_k}^t \bar{P}_k(s)\, x_k(s) ds + \int_{t_k}^t \bar{q}_k(s) ds \quad \text{for } t \in I \ (k = 1, 2, \dots),$$

where

$$\bar{P}_k(t) = P_k(t) - P_0(t), \qquad \bar{q}_k(t) = q_k(t) - q_0(t) \quad (k = 1, 2, \dots).$$

Using the integration-by-parts formula we conclude

$$\begin{aligned}\int_{t_k}^t \bar{P}_k(s)\, x_k(s) ds &= \int_{t_k}^t \bar{P}_k(s) ds \cdot x_k(t) - \int_{t_k}^t \left(\int_{t_k}^s \bar{P}_k(\tau) d\tau\right) x_k'(s) ds \\ &= \int_{t_k}^t \bar{P}_k(s) ds \cdot x_k(t) - \int_{t_k}^t \left(\int_{t_k}^s \bar{P}_k(\tau) d\tau\right) (P_k(s)\, x_k(s) + q_k(s)) ds \quad \text{for } t \in I \ (k = 1, 2, \dots).\end{aligned}$$

Therefore,

$$z_k(t) = z_k(t_k) + \mathcal{J}_k(t) + \mathcal{Q}_k(t) + \int_{t_k}^t P_0(s)\, z_k(s) ds \quad \text{for } t \in I \ (k = 1, 2, \dots) \tag{4.1}$$

where

$$\mathcal{J}_k(t) = \int_{t_k}^t \bar{P}_k(s) ds \cdot x_k(t) - \int_{t_k}^t \left(\int_{t_k}^s \bar{P}_k(\tau) d\tau\right) P_k(s)\, x_k(s) ds \quad (k = 1, 2, \dots),$$

and

$$\mathcal{Q}_k(t) = \int_\tau^t \bar{q}_k(s) ds - \int_{t_k}^t \left(\int_{t_k}^s \bar{P}_k(\tau) d\tau\right) q_k(s) ds \quad (k = 1, 2, \dots).$$

Due to (4.1) we get

$$\|z_k(t)\| \le \|z_k(t_k)\| + \|\mathcal{J}_k(t)\| + \|\mathcal{Q}_k(t)\| + \left| \int_{t_k}^t \|P_0(s)\|\, \|z_k(s)\| ds \right| \quad \text{for } t \in I \ (k = 1, 2, \dots). \tag{4.2}$$

Let

$$\alpha_k = \sup_{t \in I} \left\| \int_{t_k}^{t} \bar{P}_k(s)ds \right\|, \qquad \beta_k = \sup_{t \in I} \left\| \int_{t_k}^{t} \bar{q}_k(s)ds \right\|$$

and

$$\gamma_k = \sup_{t \in I} \left| \int_{t_k}^{t} \|P_k(s)\|ds \right| \quad (k = 1, 2, \ldots).$$

Then by (2.8) and (2.9) we have

$$\lim_{k \to +\infty} \alpha_k(1 + \gamma_k) = \lim_{k \to +\infty} \beta_k(1 + \gamma_k) = 0. \tag{4.3}$$

It is evident that

$$\|\mathcal{J}_k(t)\| \le \varepsilon_k \|x_k\|_c \quad \text{for } t \in I \ (k = 1, 2, \ldots) \tag{4.4}$$

where $\varepsilon_k = \alpha_k(1 + \gamma_k)(k = 1, 2, \ldots)$.

Further, we have

$$\left\| \int_{t_k}^{t} \left(\int_{t_k}^{s} \bar{P}_k(\tau)d\tau \right) q_0(s)ds \right\| \le r_0\alpha_k \quad \text{for } t \in I \ (k = 1, 2, \ldots)$$

and, in addition, using the integration-by-parts formulae we get

$$\left\| \int_{t_k}^{t} \left(\int_{t_k}^{s} \bar{P}_k(\tau)d\tau \right) \bar{q}_k(s)ds \right\| \le \alpha_k\beta_k + \beta_k(\gamma_k + r_1) \quad \text{for } t \in I \ (k = 1, 2, \ldots),$$

where

$$r_0 = \int_I \|q_0(t)\|dt, \qquad r_1 = \int_I \|P_0(t)\|dt.$$

Due to the last two estimates, thanks to the inequalities

$$\begin{aligned} \left\| \int_{t_k}^{t} \left(\int_{t_k}^{s} \bar{P}_k(\tau)d\tau \right) q_k(s)ds \right\| &\le \left\| \int_{t_k}^{t} \left(\int_{t_k}^{s} \bar{P}_k(\tau)d\tau \right) \bar{q}_k(s)ds \right\| \\ &\quad + \left\| \int_{t_k}^{t} \left(\int_{t_k}^{s} \bar{P}_k(\tau)d\tau \right) q_0(s)ds \right\| \quad \text{for } t \in I \ (k = 1, 2, \ldots), \end{aligned}$$

we conclude

$$\|\mathcal{Q}_k(t)\| \le \delta_k \quad \text{for } t \in I \ (k = 1, 2, \ldots), \tag{4.5}$$

where $\delta_k = \alpha_k(\beta_k + r_0) + \beta_k(\gamma_k + r_1)$.

From (4.2), by (4.4) and (4.5) we find

$$\|z_k(t)\| \le \|z_k(t_k)\| + \varepsilon_k\|x_k\|_c + \delta_k + \left| \int_{t_k}^{t} \|P_0(s)\| \, \|z_k(s)\|ds \right| \quad \text{for } t \in I \ (k = 1, 2, \ldots).$$

Hence, according to the Gronwall inequality (see [4])

$$\|z_k\|_c \le \left(\|z_k(t_k)\| + \varepsilon_k\|x_k\|_c + \delta_k\right) \exp(r_1) \quad (k = 1, 2, \ldots). \tag{4.6}$$

In virtue of (4.3) we have

$$\lim_{k \to +\infty} \varepsilon_k = 0. \tag{4.7}$$

Therefore, there exists a natural k_0 such that

$$\varepsilon_k < \frac{1}{2}\exp(-r_1) \quad \text{for } k > k_0.$$

From this and (4.6) it follows

$$\|x_k\|_c \le \|x_0\|_c + \|z_k\|_c \le \|x_0\|_c + \big(\|z_k(t_k)\| + \varepsilon_k \|x_k\|_c + \delta_k\big) \exp(r_1) \quad (k > k_1).$$

So, the sequence $\|x_k\|_c (k = 1, 2, \dots)$ is bounded. In addition, in view of conditions (2.8) and (2.9) we have

$$\lim_{k\to+\infty} \delta_k = 0, \tag{4.8}$$

and using (2.1) we conclude

$$\lim_{k\to+\infty} z_k(t_k) = \lim_{k\to+\infty} (x_k(t_k) - x_0(t_k)) = \lim_{k\to+\infty} c_k - x_0(t_0) = 0.$$

Therefore, by this, (4.7) and (4.8), it follows from (4.6)

$$\lim_{k\to+\infty} \|z_k\|_c = 0,$$

since the sequence $\|x_k\|_c (k = 1, 2, \dots)$ is bounded. □

Proof of Theorem 2.3. According to Theorem 2.2 the mapping $x \to H_k x + h_k$ establishes a one-to-one corresponding between the solution x_k of problem (1.1_k), (1.2_k) and the solution x_k^* of the Cauchy problem (2.10_k), (2.11_k) and, in addition, $x_k^*(t) \equiv H_k(t)\, x_k(t) + h_k(t)$ for every natural k.

Conditions (2.12)–(2.14) guarantee the fulfillment of the conditions of Theorem 2.2 for the Cauchy problem (2.10), (2.11) and sequence of the Cauchy problems (2.10_k), (2.11_k) $(k = 1, 2, \dots)$. Therefore, according to Theorem 2.2

$$\lim_{k\to+\infty} x_k^*(t) = x_0^*(t) \quad \text{uniformly on } I.$$

So, condition (2.15) holds. □

Proof of Corollary 2.1. Verifying the conditions of Theorem 2.3. From (2.4) and (2.5) it follows that condition (2.12) holds, and the condition

$$\lim_{k\to+\infty} H_k^{-1}(t) = H_0^{-1}(t) \quad \text{uniformly on } I. \tag{4.9}$$

Put

$$h_k(t) = -H_k(t)\, \varphi_k(t) \quad \text{for } t \in I \ (k = 1, 2, \dots).$$

Due to (2.2) and (2.5) we get

$$\lim_{k\to+\infty} H_k(t_k) = H_0(t_0).$$

By this and (2.16) condition (2.13) is fulfilled for $c_0^* = H_0(t_0)\, c_0$.

Let $q_k^* (k = 1, 2, \dots)$ are the vector-functions given in Theorem 2.3. It is not difficult to verify that

$$q_k^*(t) \equiv q_k(t) - \varphi_k'(t) + P_k(t)\, \varphi_k(t) \quad (k = 1, 2, \dots)$$

in the case. Further, by (2.6) and (2.1) condition (2.14) holds uniformly on I for the functions q_k^* $(k = 1, 2, \dots)$ given above, $q_0^*(t) = H_0(t)\, q_0(t)$ and $c_k^* = H_k(t_k)\, (c_k - \varphi_k(t)) (k = 1, 2, \dots)$. In view of Lemma 3.1, the vector-function $x_0^*(t) = H_0(t)\, x_0(t)$ is the unique solution of problem (2.10), (2.11). By Theorem 2.3 we have

$$\lim_{k\to+\infty} (H_k(t)\, x_k(t) - H_k(t)\, \varphi_k(t)) = x_0^*(t) \quad \text{uniformly on } I.$$

Therefore, by (2.5) and (4.9), condition (2.17) holds. □

Proof of Theorem 2.1. Sufficiency follows from Corollary 2.1 if we assume $\varphi_k(t) = o_n (k = 1, 2, \dots)$ therein.

Let us show necessity. Let $c_k \in \mathbb{R}^n (k = 0, 1, \dots)$ be an arbitrary sequence of constant vectors satisfying (2.1) and let $e_j = (\delta_{ij})_{i=1}^n \delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \ne j (i, j = 1, \dots, n)$.

Let x_k be a unique solution of problem (1.1_k), (1.2_k) for every natural k.

For any $k \in \{0, 1, \dots\}$ and $j \in \{1, \dots, n\}$ let us denote

$$y_{kj}(t) = x_k(t) - x_{kj}(t),$$

where x_{kj} is a unique solution of the system (1.1_k) under the Cauchy condition

$$x(t_k) = c_k - e_j.$$

Moreover, let $Y_k(t)$ be matrix-function whose columns are $y_{k1}(t), \dots, y_{kn}(t)$.

It can be easily shown that Y_0 and $Y_k (k = 1, 2, \dots)$ satisfy, respectively, of homogeneous systems (1.1_0) and (1.1_{k0}) $(k = 1, 2, \dots)$ and

$$y_{kj}(t_k) = e_j \quad (k = 0, 1, \dots) \tag{4.10}$$

for every $j \in \{1, \dots, n\}$. If for some natural k and $\alpha_j \in \mathbb{R} (j = 1, \dots, n)$

$$\sum_{j=1}^{n} \alpha_j y_{kj}(t) \equiv o_n,$$

then using (4.10) we have

$$\sum_{j=1}^{n} \alpha_j e_j = o_n$$

and, therefore,

$$\alpha_1 = \cdots = \alpha_n = 0,$$

i.e., Y_0 and $Y_k (k = 1, 2, \dots)$ are the fundamental matrices, respectively, of homogeneous systems (1.1_0) and (1.1_{k0}) $(k = 1, 2, \dots)$.

Thanks to Corollary 2.1 we have

$$\lim_{k \to +\infty} Y_k(t) = Y_0(t) \quad \text{uniformly on } I$$

and, consequently,

$$\lim_{k \to +\infty} Y_k^{-1}(t) = Y_0^{-1}(t) \quad \text{uniformly on } I, \tag{4.11}$$

as well.

We may assume without loss of generality that

$$Y_k(t_k) = I_n \quad (k = 0, 1, \dots).$$

We put

$$H_k(t) = Y_k^{-1}(t) \quad \text{for } t \in I \ (k = 0, 1, \dots)$$

and verify conditions (2.4)–(2.7) of the theorem.

Condition (2.4) is evident, and condition (2.5) coincides to (4.11).

Using the equality

$$(Y_k^{-1}(t))' = -Y_k^{-1}(t)\, P_k(t) \quad \text{for } t \in I \ (k = 0, 1, \dots), \tag{4.12}$$

we show

$$\mathcal{I}_c(H_k, A_k)(t) - \mathcal{I}_c(H_k, A_k)(t_k) = \int_{t_k}^{t} \Big((Y_k^{-1}(t))' + Y_k^{-1}(t)\, P_k(t) \Big) d\tau = O_{n\times n} \quad \text{for } t \in I \ (k = 0, 1, \dots).$$

Thus condition (2.6) is evident.

On the other hand, using integration-by-parts formulae we find

$$\mathcal{B}_c(H_k, q_k)(t) - \mathcal{B}_c(H_k, q_k)(t_k) = \int_{t_k}^{t} Y_k^{-1}(\tau) q_k(\tau) d\tau = \int_{t_k}^{t} Y_k^{-1}(\tau) \left(x_k'(\tau) - P_k(\tau)\, x_k(\tau) \right) d\tau$$
$$= Y_k^{-1}(t)\, x_k(t) - Y_k^{-1}(t_k)\, x_k(t_k) = Y_k^{-1}(t)\, x_k(t) - c_k \quad \text{for } t \in I \ (k = 0, 1, \dots).$$

Hence,

$$\int_{t_k}^{t} Y_k^{-1}(\tau)\, q_k(\tau) d\tau - \int_{t_0}^{t} Y_k^{-1}(\tau)\, q_0(\tau) d\tau = (Y_k^{-1}(t)\, x_k(t) - Y_0^{-1}(t)\, x_0(t))$$
$$- (c_k - c_0) \quad \text{for } t \in I \ (k = 1, 2, \ldots). \tag{4.13}$$

By this, (2.1), (4.11) and (4.13), if we take account that due to necessity of theorem condition (1.3) holds uniformly on I, we conclude that condition (2.7) holds uniformly on I, as well. □

Proof of Theorem 2.2′. It is evident that doe to conditions (2.21), (2.22) and (2.23) conditions (2.8) and (2.9) are valid. So, the theorem follows from Theorem 2.2. □

Proof of Theorem 2.3′. In the case, condition (2.24) is equivalent to condition (2.13). Moreover, due to conditions (2.18), (2.25) and (2.26) conditions (2.6) and (2.14) are fulfilled uniformly on I. So, the theorem follows from Theorem 2.3. □

Proof of Corollary 2.1′. From (2.4) and (2.5) it follows that conditions (2.12) and (4.9) are valid. By (4.9) there exists a positive number is r such that

$$\|H_k^{-1}(t)\| \le r \quad \text{for } t \in I \ (k = 0, 1, \ldots).$$

Therefore, due to Remark 2.2 and (2.18) we get

$$\sup\left\{\left|\bigvee_{t_k}^{t}(\mathcal{I}_c(H_k, P_k))\right| : t \in I\right\} \le r r_0 < +\infty \quad (k = 0, 1, \ldots),$$

where r_0 is the right hand of inequality (2.18). So, thanks to this, the uniform fulfillment on I of conditions (2.19) and (2.20), guarantees, respectively, the same property for conditions (2.6) and (2.7). Hence, the corollary follows from Corollary 2.1. □

Proof of Theorem 2.1′. Sufficiency follows from Corollary 2.1′ if we assume $\varphi_k(t) = o_n$ $(k = 1, 2, \ldots)$ therein. The proof of the necessity is the same as in the proof of Theorem 2.1. We only note that by condition (2.5) and equality (4.12) condition (2.18) is valid, and condition (2.19) is fulfilled uniformly on I. Moreover, according to Remark 2.2, it is evident that the sufficiency immediately follows from Theorem 2.1. □

Proof of Corollary 2.2. In virtue of the integration-by-parts formula, conditions (2.5), (2.22), (2.23), (2.27) and (2.28) yield that the conditions

$$\lim_{k\to+\infty} \int_{t_k}^{t} H_k(\tau)\, P_k(\tau) d\tau = \int_{t_0}^{t} (P_0(\tau) - P^*(\tau)) d\tau$$

and

$$\lim_{k\to+\infty} \int_{t_k}^{t} H_k(\tau)\, q_k(\tau) d\tau = \int_{t_0}^{t} (q_0(\tau) - q^*(\tau)) d\tau$$

are fulfilled uniformly on I. Corollary 2.2 follows from Theorem 2.1′. □

Proof of Corollary 2.3. Let

$$C_{kl}(t) = I_n - \int_{t_k}^{t} (P_{kl}(\tau) - P_{0l}(\tau)) d\tau \quad (l = 1, \ldots, m;\ k = 1, 2, \ldots).$$

Thanks to (2.30), without loss of generality we can assume that the matrix-functions H_{kl} and $C_{kl}(l = 1, \ldots, m)$ are nonsingular for every natural k.

Based on the definitions of the operators $\mathcal{B}_c$ ad $\mathcal{I}_c$, it is not difficult to verify the equality

$$\mathcal{B}_c(C_{kj}, H_{kj-1} P_k)(\tau)\Big|_{t_k}^{t} \equiv \mathcal{B}_c(H_{kj}, P_k)(\tau)\Big|_{t_k}^{t},$$
$$\mathcal{B}_c(C_{kj}, H_{kj-1} f_k)(\tau)\Big|_{t_k}^{t} \equiv \mathcal{B}_c(H_{kj}, f_k)(\tau)\Big|_{t_k}^{t}$$

and

$$\mathcal{I}_c(C_{kj}, (H'_{kj-1} + H_{kj-1}P_k)H^{-1}_{kj-1})(\tau)\Big|_{t_k}^{t} \equiv \mathcal{I}_c(H_{kj}, P_k)(\tau)\Big|_{t_k}^{t} \quad (j = 1, \dots, m;\ k = 1, 2, \dots).$$

In addition, by conditions (2.29)–(2.32) conditions (2.4) and (2.18) hold, and conditions (2.5) and (2.19) and (2.20) are fulfilled uniformly on I, where $H_0(t) = I_n$ and $H_k(t) = H_{km-1}(t)(k = 1, 2, \dots)$. So, the corollary follows from Theorem 2.1′. □

Proof of Corollary 2.4. Let us show the sufficiency. Let $H_k(t) = Z_k^{-1}(t)(k = 0, 1, \dots)$ in Theorem 2.1′. Thanks to (2.34), there exists a positive number r such that

$$\|Z_k^{-1}(t)\| \le r \quad \text{for } t \in I \ (k = 0, 1, \dots).$$

Using this estimate and the equality

$$(Z_k^{-1}(t))' = -Z_k^{-1}(t)\, Q_k(t) \quad \text{for } t \in I \ (k = 0, 1, \dots),$$

by the integration-by-parts formulae we have

$$\left\| Z_k^{-1}(t) - Z_k^{-1}(s) + \int_s^t Z_k^{-1}(\tau)P_k(\tau)d\tau \right\| = \left\| \int_s^t Z_k^{-1}(\tau)(P_k(\tau) - Q_k(\tau))d\tau \right\|$$
$$\le r \int_s^t \|P_k(\tau) - Q_k(\tau)\| d\tau \quad \text{for } s < t \ (k = 0, 1, \dots).$$

Therefore,

$$\int_I \|H_k'(\tau) + H_k(\tau)P_k(\tau)\| d\tau \le r \int_I \|P_k(\tau) - Q_k(\tau)\| d\tau \quad (k = 0, 1, \dots)$$

and due to (2.33) estimate (2.18) holds. Moreover, conditions (2.19) and (2.20) coincide to conditions (2.35) and (2.36), respectively. So, the sufficiently follows from Theorem 2.1′.

Let us show the necessity. Let $Q_k(t) = P_k(t)(k = 0, 1, \dots)$. Then $Z_k(t) \equiv Y_k(t)(k = 0, 1, \dots)$, where Y_0 and Y_k $(k = 1, 2, \dots)$ are fundamental matrices, respectively, of the homogeneous systems (1.1_0) and (1.1_{k0}). Analogously, as in the proof of Theorem 2.1, conditions (2.34) and equality (4.13) are valid. In addition, condition (2.35) coincides to condition (2.19), and condition (2.36) follows from equality (4.13). □

Proof of Corollary 2.5. The corollary immediately follows from Corollary 2.4 if we note the fundamental matrix of $Z_k(t)(Z_k(t_k) = I_n)$ of system (2.37), in the case, has the form

$$Z_k(t) \equiv \exp\left(\int_{t_k}^t Q_k(\tau)d\tau\right) \quad (k = 0, 1, \dots). \quad \square$$

Proof of Corollary 2.6. The corollary follows from Corollary 2.5 if we assume that therein $Q_k(t) = P_k(t)$ $(k = 0, 1, \dots)$ and, in addition, we note that condition (2.38) is equivalent to condition (2.40), and condition (2.39) coincides to (2.41). □

Proof of Corollary 2.7. The corollary follows from Corollary 2.4 if we assume therein that $Q_k(t) = \mathrm{diag}(P_k(t))(k = 0, 1, \dots)$. □

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Original article

The Hardy–Littlewood–Sobolev theorem for Riesz potential generated by Gegenbauer operator

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Abstract

In this paper we introduced and studied the maximal function (G-maximal function) and the Riesz potential (G-Riesz potential) generated by Gegenbauer differential operator

$$G_\lambda = \left(x^2-1\right)^{\frac{1}{2}-\lambda} \frac{d}{dx} \left(x^2-1\right)^{\lambda+\frac{1}{2}} \frac{d}{dx}.$$

The $L_{p,\lambda}$ boundedness of the G-maximal operator is obtained. Hardy–Littlewood–Sobolev theorem of G-Riesz potential on $L_{p,\lambda}$ spaces is established.

Keywords: G-Riesz potential; G-maximal function; G-BMO space

0. Introduction

The Hardy–Littlewood maximal function is an important tool of harmonic analysis. It was first introduced by Hardy and Littlewood in 1930 (see [1]) for 2π-periodical functions, and later it was extended to the Euclidean spaces, some weighted measure spaces (see [2–4]), symmetric spaces (see [5,6]), various Lie groups [7], for the Jacobi-type hypergroups [8,9], for Chebli–Trimeche hypergroups [10], for the one-dimensional Bessel–Kingman hypergroups [11–13], for the n-dimensional Bessel–Kingman hypergroups ($n \geq 1$) [14–18], and for Laguerre hypergroup [19–22]. The structure of the paper is as follows. In Section 1 we present some definitions, notation and auxiliary results. In Section 2 the $L_{p,\lambda}$ boundedness of the G-maximal operator is proved. In Section 3 we introduce definition of G-Riesz potential. In Section 4 it is proved for the Sobolev type theorem.

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1. Definitions, notation and auxiliary results

Let $H(x,r) = (x-r, x+r) \cap [0,\infty)$, $r \in (0,\infty)$, $x \in [0,\infty)$. For all measurable sets $E \subset [0,\infty)$, $\mu E \equiv |E|_\lambda = \int_E sh^{2\lambda}\ tdt$. For $1 \le p \le \infty$ let $L_p([0,\infty), G) \equiv L_{p,\lambda}[0,\infty)$ be the space of functions measurable on $[0,\infty)$ with the finite norm

$$\|f\|_{L_{p,\lambda}} = \left(\int_0^\infty |f(ch\ t)|^p sh^{2\lambda}\ tdt\right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$
$$\|f\|_{\infty,\lambda} = \operatorname*{ess\,sup}_{t\in[0,\infty)} |f(ch\ t)|, \quad p = \infty.$$

Analogy by [9] we define Gegenbauer maximal functions as follows:

$$M_G f\ (ch\ x) = \sup_{r>0} \frac{1}{|H(0,r)|_\lambda} \int_0^r A^\lambda_{ch\ t} |f\ (ch\ x)|\, d\mu_\lambda(t),$$
$$M_\mu f\ (ch\ x) = \sup_{r>0} \frac{1}{|H\ (x,r)|_\lambda} \int_{H(x,r)} |f(ch\ t)|\, d\mu_\lambda(t),\ d\mu_\lambda(t) = sh^{2\lambda}\ tdt,$$
$$|H(0,r)|_\lambda = \int_0^r sh^{2\lambda}\ tdt, \quad |H(x,r)|_\lambda = \int_{H(x,r)} sh^{2\lambda}\ tdt, \quad 0 < \lambda < \frac{1}{2},$$

where

$$H(x,r) = \begin{cases} (0,\ x+r), & x < r, \\ (x-r, x+r), & x > r. \end{cases}$$

Here (see [22])

$$A^\lambda_{ch\ t} f(ch\ x) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^\pi f(ch\ xch\ t - sh\ xsh\ t\cos\ \varphi)(\sin\ \varphi)^{2\lambda-1} d\varphi$$

denote the generalized shift operator, associated with the Gegenbauer differential operator

$$G = \left(x^2-1\right)^{1/2-\lambda} \frac{d}{dx} \left(x^2-1\right)^{\lambda+1/2} \frac{d}{dx}, \quad x \in (1,\ \infty).$$

Further we will need some auxiliary assertions.

Lemma 1.1. *For $0 < \lambda < 1/2$ the following correlations are true:*

$$|H(0,r)|_\lambda \sim \begin{cases} \left(sh\ \frac{r}{2}\right)^{2\lambda+1}, & 0 < r < 2, \\ \left(ch\ \frac{r}{2}\right)^{4\lambda}, & 2 \le r < \infty, \end{cases}$$

where c denotes a positive constant.

Here $f \sim g$ denotes that $c_{1,\lambda} g \le f \le c_{2,\lambda} g$ for some positive constants $c_{1,\lambda}$ and $c_{2,\lambda}$ depending on λ.

Proof. Let first $0 < r < 2$, then

$$\begin{aligned} |H(0,r)|_\lambda &= \int_0^r sh^{2\lambda}\ tdt = \int_0^r (sh\ t)^{2\lambda-1} d(ch\ t) = \int_0^r (ch^2\ t-1)^{\lambda-\frac{1}{2}} d(ch\ t) \\ &= \int_1^{ch\ r} (t-1)^{\lambda-\frac{1}{2}} (t+1)^{\lambda-\frac{1}{2}} dt \ge (ch\ r+1)^{\lambda-\frac{1}{2}} \int_1^{ch\ r} (t-1)^{\lambda-\frac{1}{2}} dt \\ &\ge (ch1+1)^{\lambda-\frac{1}{2}} \frac{(t-1)^{\lambda+\frac{1}{2}}}{\lambda+\frac{1}{2}} \Big|_1^{ch\ r} = \frac{2(ch\ r-1)^{\lambda+\frac{1}{2}}}{(2\lambda+1)(1+ch1)^{\frac{1}{2}-\lambda}} \\ &= \frac{2^{\lambda+\frac{3}{2}}}{(2\lambda+1)(1+ch1)^{\frac{1}{2}-\lambda}} \left(sh\ \frac{r}{2}\right)^{2\lambda+1}. \end{aligned} \tag{1.1}$$

On the other hand,

$$|H(0,r)|_\lambda = \int_0^r sh^{2\lambda}\ t dt = \int_1^{ch\ r} (t-1)^{\lambda-\frac{1}{2}}(t+1)^{\lambda-\frac{1}{2}} dt \le 2^{\lambda-\frac{1}{2}} \int_1^{ch\ r} (t-1)^{\lambda-\frac{1}{2}} dt$$
$$= \frac{2^{\lambda+\frac{1}{2}}}{2\lambda+1}(t-1)^{\lambda+\frac{1}{2}}\Big|_1^{ch\ r} = \frac{2^{\lambda+\frac{1}{2}}}{2\lambda+1}(ch\ r-1)^{\lambda+\frac{1}{2}} = \frac{2^{2\lambda+1}}{2\lambda+1}\left(sh\ \frac{r}{2}\right)^{2\lambda+1}.$$

Let now $2 \le r < \infty$. Then

$$|H(0,r)|_\lambda = \int_0^r sh^{2\lambda}\ t dt = \int_0^r (sh\ t)^{2\lambda-1} d(ch\ t) = \int_0^r (ch^2\ t-1)^{\lambda-\frac{1}{2}} d(ch\ t)$$
$$= \int_1^{ch\ r} \frac{(t-1)^{\lambda-\frac{1}{2}}}{(t+1)^{\frac{1}{2}-\lambda}} dt \ge (ch\ r+1)^{\lambda-\frac{1}{2}} \int_1^{ch\ r} (t-1)^{\lambda-\frac{1}{2}} dt$$
$$= (ch\ r+1)^{\lambda-\frac{1}{2}} \frac{(t-1)^{\lambda+\frac{1}{2}}}{\lambda+\frac{1}{2}}\Bigg|_1^{ch\ r} = \frac{2}{2\lambda+1}\frac{(ch\ r-1)^{\lambda+\frac{1}{2}}}{(ch\ r+1)^{\frac{1}{2}-\lambda}}$$
$$= \frac{2}{2\lambda+1}\frac{\left(2sh^2\ \frac{r}{2}\right)^{\lambda+\frac{1}{2}}}{\left(2ch^2\ \frac{r}{2}\right)^{\frac{1}{2}-\lambda}} = \frac{2^{2\lambda+1}}{(2\lambda+1)2^{2\lambda+1}}\frac{\left(2sh\ \frac{r}{2}\right)^{2\lambda+1}}{\left(ch\ \frac{r}{2}\right)^{1-2\lambda}}$$
$$\ge \frac{2^{2\lambda+1}}{(2\lambda+1)2^{2\lambda+1}}\left(ch\ \frac{r}{2}\right)^{4\lambda} \Leftrightarrow \left(2sh\ \frac{r}{2}\right)^{2\lambda+1}$$
$$\ge \left(ch\ \frac{r}{2}\right)^{2\lambda+1} \Leftrightarrow 2sh\ \frac{r}{2} \ge ch\ \frac{r}{2} \Leftrightarrow 2\frac{e^{r/2}-e^{-r/2}}{2}$$
$$\ge \frac{e^{r/2}+e^{-r/2}}{2} \Leftrightarrow 2(e^r-1) \ge e^r+1 \Leftrightarrow 2e^r \ge 3,$$

that takes place for $r \ge 2$.

Thus,

$$|H(0,r)|_\lambda \ge \frac{2^{2\lambda+1}}{(2\lambda+1)2^{2\lambda+1}}\left(ch\ \frac{r}{2}\right)^{4\lambda}.$$

Let us obtain an upper bound for $|H(0,r)|_\lambda$.

$$|H(0,r)|_\lambda = \int_0^r sh^{2\lambda}\ t dt = \int_0^r \left(2sh\ \frac{t}{2} ch\ \frac{t}{2}\right)^{2\lambda} dt$$
$$= 2^{2\lambda+1}\int_0^r \left(sh\ \frac{t}{2}\right)^{2\lambda}\left(ch\ \frac{t}{2}\right)^{2\lambda-1} d\left(sh\ \frac{t}{2}\right) \le 2^{2\lambda+1}\int_0^r \left(sh\ \frac{t}{2}\right)^{4\lambda-1} d\left(sh\ \frac{t}{2}\right)$$
$$= \frac{2^{2\lambda+1}}{4\lambda}\left(sh\ \frac{t}{2}\right)^{4\lambda}\Bigg|_0^r = \frac{4^\lambda}{2\lambda}\left(sh\ \frac{r}{2}\right)^{4\lambda} \le \frac{4^\lambda}{2\lambda}\left(ch\ \frac{r}{2}\right)^{4\lambda}. \tag{1.2}$$

Combining (1.1)–(1.2), we obtain assertion of Lemma 1.1. □

Lemma 1.2. *Let $0 < \lambda < 1/2$ and $x \in [0,\infty)$, $r \in (0,\infty)$. Then the following estimates are reasonable for $0 < r < 2$*

$$|H(x,r)|_\lambda \le c_\lambda \begin{cases} \left(sh\ \frac{r}{2}\right)^{2\lambda+1}, & 0 \le x < r, \\ sh\ \frac{r}{2} ch^{2\lambda}\ x, & r \le x < \infty. \end{cases} \tag{a}$$

For $2 \le r < \infty$.

$$|H(x,r)|_\lambda \le c_\lambda \begin{cases} ch^{2\lambda}\ r, & 0 \le x < r, \\ ch^{2\lambda}\ x ch^{2\lambda}\ r, & r \le x < \infty. \end{cases} \tag{b}$$

Here and further $c_\lambda, c_{\alpha,\lambda}, c_{\alpha,\lambda,p}$ will denote some constants, depending only on subscribed indexes and generally speaking different in different formulas.

Proof. First we consider the case $0 < r < 1$ and $x \in [0, \infty)$.

Let $0 \le t \le 2$, Then we have

$$t \le sh\ t \le e \cdot t. \tag{1.3}$$

We prove left-hand part of this estimate. We consider the function $f(t) = sh\ t - t$. Since, $f'(t) = ch\ t - 1 \ge 0$, then $f(t)$ increases on $[0, \infty)$, and that takes the smallest value for $t = 0$, $f(0) = 0$, consequently $f(t) \ge 0$ is equivalent to $sh\ t \ge t$.

We prove right-hand part of estimate (1.3).

$$\frac{e^t - e^{-t}}{2} \le e \cdot t \quad \Leftrightarrow \quad e^{2t} - 1 \le 2 \cdot e^{1+t} \cdot t \quad \Leftrightarrow \quad e^{2t} \le 2 \cdot e^{1+t} \cdot t + 1.$$

We consider the function $f(t) = 2 \cdot e^{1+t} \cdot t + 1 - e^{2t}$.

$$f'(t) = 2 \cdot e^{1+t} + 2 \cdot e^{1+t} \cdot t - 2e^{2t} = 2e^t \left(e + t \cdot e - e^t\right) \ge e(t+1) - e^t \ge 0 \quad \text{as, } t \le 2.$$

Thus, the estimate (1.3) is proved.

Hence it follows that for $0 \le x < r < 2$

$$|H(x,r)|_\lambda = \int_0^{x+r} sh^{2\lambda}\ t dt \le e^{2\lambda} \int_0^{2r} t^{2\lambda} dt = \frac{(2e)^{2\lambda}}{2\lambda+1} \cdot r^{2\lambda+1} \le c_\lambda \left(sh\ \frac{r}{2}\right)^{2\lambda+1}. \tag{1.4}$$

For $r \le x < 2$

$$|H(x,r)|_\lambda = \int_{x-r}^{x+r} sh^{2\lambda}\ t dt \le e^{2\lambda} \int_{x-r}^{x+r} t^{2\lambda} dt \le 2e^{2\lambda} \cdot r \cdot (x+r)^{2\lambda} \le 2e^{2\lambda} \cdot r \cdot (2x)^{2\lambda} \le c_\lambda\, sh\ \frac{r}{2} ch^{2\lambda}\ x.$$

Let now $0 < r < 2 \le x < \infty$, then we have

$$\begin{aligned} |H(x,r)|_\lambda &= \int_{x-r}^{x+r} sh^{2\lambda}\ t dt \le 2r \cdot sh^{2\lambda}(x+r) = 2r(sh\ xch\ r + ch\ xsh\ r)^{2\lambda} \\ &\le 2r(sh\ xch\ 1 + ch\ xsh\ 1)^{2\lambda} \le 2r(2ch\ xch\ 1)^{2\lambda} \le c_\lambda\, sh\ \frac{r}{2} ch^{2\lambda}\ x. \end{aligned} \tag{1.5}$$

Now we consider the case, $2 \le r < \infty$, $x \in [0, \infty)$.

Let $0 \le x < 2 \le r$. As in the proof of the estimate (1.2), we obtain

$$\begin{aligned} |H(x,r)|_\lambda &= \int_0^{x+r} sh^{2\lambda}\ t dt = \frac{4^\lambda}{2\lambda} sh^{4\lambda} \frac{t}{2} \Big|_0^{x+r} = \frac{4^\lambda}{2\lambda} sh^{4\lambda}\ \frac{x+r}{2} \\ &= \frac{4^\lambda}{2\lambda} \left(sh\ \frac{x}{2} ch\ \frac{r}{2} + ch \frac{x}{2} sh\ \frac{r}{2}\right)^{4\lambda} \le c_\lambda \left(sh\ \frac{1}{2}\, ch\ \frac{r}{2} + ch\ \frac{1}{2}\, sh\ \frac{r}{2}\right)^{4\lambda} \le c_\lambda ch^{4\lambda}\ \frac{r}{2}. \end{aligned} \tag{1.6}$$

Let now $2 \le r \le x < \infty$, then

$$\begin{aligned} |H(x,r)|_\lambda &= \int_{x-r}^{x+r} sh^{2\lambda}\ t dt \le \frac{4^\lambda}{2\lambda}\ sh^{4\lambda}\ \frac{t}{2}\Big|_{x-r}^{x+r} = \frac{4^\lambda}{2\lambda} \left(sh^{4\lambda}\ \frac{x+r}{2} - sh^{4\lambda}\ \frac{x-r}{2}\right) \\ &\le \frac{4^\lambda}{2\lambda} sh^{4\lambda}\ \frac{x+r}{2} \le c_\lambda ch^{4\lambda}\ \frac{x}{2} ch^{4\lambda}\ \frac{r}{2} \le c_\lambda ch^{2\lambda}\ xch^{2\lambda}\ r. \end{aligned} \tag{1.7}$$

From (1.6) and (1.7) it follows that at $2 \le r < \infty$ and $0 \le x < \infty$

$$|H(x,r)|_\lambda \le c_\lambda ch^{2\lambda}\ r, \quad 0 \le x < r, \tag{1.8}$$

$$|H(x,r)|_\lambda \le ch^{2\lambda}\ xch^{2\lambda}\ r, \quad r \le x < \infty. \tag{1.9}$$

Assertion of Lemma 1.2 follows from (1.4)–(1.5), (1.8) and (1.9). □

2. $L_{p,\lambda}$-boundedness of the G-maximal operator

Theorem 2.1. *For $0 \le x < \infty$ and $0 < r < \infty$ the following inequality is valid*

$$M_G f(ch\ x) \le c_\lambda M_\mu f(ch\ x),$$

where c_λ is a positive constant.

Proof. Consider the integral

$$\begin{aligned} I(x,r) &= \int_0^r A^\lambda_{ch\ t} |f(ch\ x)| sh^{2\lambda}\ tdt \\ &= \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^r \left\{ \int_0^\pi |f(ch\ x \cdot ch\ t - sh\ x \cdot sh\ t \cos\ \varphi)| (\sin\ \varphi)^{2\lambda-1} d\varphi \right\} sh^{2\lambda}\ tdt. \end{aligned}$$

Making the substitution
$z = ch\ x \cdot ch\ t - sh\ x \cdot sh\ t \cos\ \varphi$, we get that

$$\cos\ \varphi = \frac{ch\ x \cdot ch\ t - z}{sh\ x \cdot sh\ t}, \quad \varphi =$$
$$mboxarccos \frac{ch\ x \cdot ch\ t - z}{sh\ x \cdot sh\ t},$$
$$\begin{aligned} d\varphi &= \frac{dz}{\sqrt{1-(\frac{ch\ x \cdot ch\ t - z}{sh\ x \cdot sh\ t})^2} sh\ x \cdot sh\ t} \\ &= (sh^2\ x \cdot sh^2\ t - ch^2\ x \cdot ch^2\ t + 2 \cdot z \cdot ch\ x \cdot ch\ t - z^2)^{-\frac{1}{2}} dz. \end{aligned}$$

Since,

$$\begin{aligned} sh^2\ x \cdot sh^2\ t - ch^2\ x \cdot ch^2\ t &= (ch^2\ x - 1) sh^2\ t - ch^2\ x \cdot ch^2\ t = ch^2\ x \cdot sh^2\ t - sh^2\ t - ch^2\ x \cdot ch^2\ t \\ &= -sh^2\ t + ch^2\ x (sh^2\ t - ch^2\ t) = -sh^2\ t - ch^2\ x, \end{aligned}$$

that

$$d\varphi = (2z \cdot ch\ x \cdot ch\ t - sh^2\ t - ch^2\ x - z^2)^{-\frac{1}{2}} dz$$

and

$$(\sin\ \varphi)^{2\lambda-1} = (2z \cdot ch\ x \cdot ch\ t - sh^2\ t - ch^2\ x - z^2)^{\lambda-\frac{1}{2}} (sh\ x \cdot sh\ t)^{1-2\lambda}.$$

Then $I(x,r)$ makes a list of form

$$\begin{aligned} I(x,r) &= \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \\ &\times \int_0^r \left\{ \int_{ch\ (x-t)}^{ch\ (x+t)} |f(z)| (2z \cdot ch\ x \cdot ch\ t - sh^2\ t - ch^2\ x - z^2)^{\lambda-1} (sh\ x)^{1-2\lambda} dz \right\} sh\ tdt. \end{aligned} \tag{2.1}$$

Transform expansion

$$\begin{aligned} &2z \cdot ch\ x \cdot ch\ t - sh^2\ t - ch^2\ x - z^2 \\ &= 2z \cdot ch\ x \cdot ch\ t - sh^2\ t (ch^2\ x - sh^2\ x) - ch^2\ x - z^2 \\ &= 2z \cdot ch\ x \cdot ch\ t - sh^2\ t \cdot ch^2\ t + sh^2\ t \cdot sh^2\ x - ch^2\ x - z^2 \\ &= 2z \cdot ch\ x \cdot ch\ t + sh^2\ t \cdot sh^2\ x - (ch^2\ t - 1) ch^2\ x - ch^2\ x - z^2 \\ &= 2z \cdot ch\ x \cdot ch\ t + sh^2\ x \cdot (ch^2\ t - 1) - ch^2\ t \cdot ch^2\ x - z^2 (ch^2\ x - sh^2\ x) \\ &= 2z \cdot ch\ x \cdot ch\ t + sh^2\ x \cdot ch^2\ t - sh^2\ x - ch^2\ t \cdot ch^2\ x - z^2 \cdot ch^2\ x - z^2 \cdot sh^2\ x \end{aligned}$$

$$= 2z \cdot ch\ x \cdot ch\ t - sh^2\ x - ch^2\ t - z^2 ch^2\ x - z^2 sh^2\ x = (z^2-1) sh^2\ x - (ch\ t - z\cdot ch\ x)^2$$

$$= \left(z^2-1\right) sh^2\ x \left[1 - \left(\frac{ch\ t - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}\right)^2\right]. \tag{2.2}$$

Taking into account (2.1) and (2.2) we get

$$I(x,r) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^r \left\{ \int_{ch\ (x-t)}^{ch\ (x+t)} |f(z)|\, (z^2-1)^{\lambda-1} \left[1 - \left(\frac{ch\ t - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}\right)^2\right]^{\lambda-1} dz \right\} \frac{sh\ t}{sh\ x} dt. \tag{2.3}$$

Note that

$$\frac{sh\ t}{sh\ x} = (z^2-1)^{\frac{1}{2}} \frac{\partial}{\partial t}\left(\frac{ch\ t - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}\right),$$

rewrite (2.3) of form

$$I(x,r) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^r \left\{ \int_{ch\ (x-t)}^{ch\ (x+t)} |f(z)|\, (z^2-1)^{\lambda-\frac{1}{2}} \right.$$
$$\left. \times \left[1 - \left(\frac{ch\ t - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}\right)^2\right]^{\lambda-1} \frac{\partial}{\partial t}\left(\frac{ch\ t - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}\right) \right\} dz dt. \tag{2.4}$$

Since $ch\ (x-t) \le z \le ch\ (x+t)$, then we obtain

$$\begin{cases} ch\ (x-r) \le z \le ch\ x \\ x - \mathrm{arc}ch z \le t \le r \end{cases} \quad \text{and} \quad \begin{cases} ch\ x \le z \le ch\ (x+r) \\ \mathrm{arc}ch z - x \le t \le r. \end{cases}$$

That is why changing the order of integration in (2.4), we get

$$I(x,r) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})} \left(\int_{ch\ (x-r)}^{ch\ x} dz \int_{x-\mathrm{arc}ch z}^{r} dt + \int_{ch\ x}^{ch\ (x+r)} dz \int_{\mathrm{arc}ch z - x}^{r} dt \right). \tag{2.5}$$

Consider the integral

$$A(x,z,r) \equiv A(x,r) = \int_{x-\mathrm{arc}ch z}^{r} \left[1 - \left(\frac{ch\ t - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}\right)^2\right]^{\lambda-1} \frac{\partial}{\partial t}\left(\frac{ch\ t - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}\right) dt.$$

Putting $u = \frac{ch\ t - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}$, we get

$$A(x,z,r) \equiv A(x,r) = \int_{-1}^{\frac{ch\ r - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}} (1-u^2)^{\lambda-1} du. \tag{2.6}$$

On the even power of $ch\ t$

$$B(x,r) = \int_{\mathrm{arc}ch z - x}^{r} \left[1 - \left(\frac{ch\ t - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}\right)^2\right]^{\lambda-1} \frac{\partial}{\partial t}\left(\frac{ch\ t - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}\right) dt$$
$$= \int_{-1}^{\frac{ch\ r - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}} (1-u^2)^{\lambda-1} du. \tag{2.7}$$

Taking into account (2.6) and (2.7) in (2.5), we have

$$I(x,r) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_{ch\ (x-r)}^{ch\ (x+r)} |f(z)|\, (z^2-1)^{\lambda-\frac{1}{2}} \int_{-1}^{\frac{ch\ r - z\cdot ch\ x}{\sqrt{z^2-1}\cdot sh\ x}} (1-u^2)^{\lambda-1} du dz. \tag{2.8}$$

Since $ch\ (x-r) \le z \le ch\ (x+r)$, then

$$\frac{ch\ r - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x} \ge \frac{ch\ r - ch\ x \cdot ch\ (x+r)}{sh\ x \cdot sh(x+r)} = \frac{2ch\ r - 2ch\ x \cdot ch\ (x+r)}{2sh\ x \cdot sh(x+r)}$$

$$= \frac{2ch\ r - ch(2x+r) - ch\ r}{ch(2x+r) - ch\ r} = \frac{ch\ r - ch(2x+r)}{ch(2x+r) - ch\ r} = -1. \tag{2.9}$$

On the other hand for $ch\ (x-r) \le z \le ch\ (x+r)$,

$$\frac{ch\ r - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x} \le \frac{ch\ r - ch\ x \cdot ch\ (x-r)}{sh\ x\, |sh(x-r)|} = \frac{2ch\ r - 2ch\ x \cdot ch\ (x-r)}{2sh\ x \cdot sh(r-x)}$$

$$= \frac{2ch\ r - ch(2x-r) - ch\ r}{ch\ r - ch(2x-r)} = \frac{ch\ r - ch(2x-r)}{ch\ r - ch(2x-r)} = 1. \tag{2.10}$$

From (2.9) and (2.10) it follows that for $ch\ (x-r) \le z \le ch\ (x+r)$, and $0 < x < r < 2$

$$-1 \le \frac{ch\ r - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x} \le 1. \tag{2.11}$$

From (2.11) it follows that for $0 < x < r < 2$

$$A(x,r) = \int_{-1}^{\frac{ch\ r - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}} (1-u^2)^{\lambda-1} du \le \int_{-1}^{1} (1-u^2)^{\lambda-1} du = \frac{\Gamma(\frac{1}{2})\Gamma(\lambda)}{\Gamma(\lambda+\frac{1}{2})}. \tag{2.12}$$

But taking into account (2.12) and (2.8), we obtain that for $0 < x < r < 2$

$$I(x,r) \le \int_{ch\ (x-r)}^{ch\ (x+r)} |f(z)|\, (z^2-1)^{\lambda-\frac{1}{2}} dz = \int_{x-r}^{x+r} |f(ch\ t)|\, sh^{2\lambda}\ t dt. \tag{2.13}$$

Now let $2 \le r \le x < \infty$ and $ch\ (x-r) \le z \le ch\ (x+r)$.
Then we have

$$\frac{ch\ r - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x} \le \frac{ch\ x - z \cdot ch\ x}{\sqrt{z^2-1} sh\ x} = \frac{(1-z) ch\ x}{\sqrt{z^2-1} sh\ x} = -\frac{\sqrt{z-1} ch\ x}{\sqrt{z+1} sh\ x} \le 0.$$

From (2.9) it follows that

$$\max(1-u)^{\lambda-1} \le \max_{-1 \le u \le 0} (1-u)^{\lambda-1} = \max(2^{\lambda-1}, 1) = 1,$$

$$-1 \le u \le \frac{ch\ r - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}.$$

Taking into account this circumstance, for the integral $A(x,r)$ we obtain of (2.6)

$$A(x,r) = \int_{-1}^{\frac{ch\ r - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}} \left(1-u^2\right)^{\lambda-1} du$$

$$\le \int_{-1}^{\frac{ch\ r - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}} (1+u)^{\lambda-1} du = \frac{1}{\lambda} (1+u)^{\lambda} \Big|_{-1}^{\frac{ch\ r - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}} = \frac{1}{\lambda} \left(1 + \frac{ch\ r - z \cdot ch\ x}{\sqrt{z^2-1} \cdot sh\ x}\right)^{\lambda}$$

$$= \frac{1}{\lambda} \left(1 - \frac{z \cdot ch\ x - ch\ r}{\sqrt{z^2-1} \cdot sh\ x}\right)^{\lambda} \le \frac{1}{\lambda} \left[1 - \left(\frac{z \cdot ch\ x - ch\ r}{\sqrt{z^2-1} \cdot sh\ x}\right)^2\right]^{\lambda}. \tag{2.14}$$

We find extremum of the function

$$f(z) = 1 - \left(\frac{z \cdot ch\ x - ch\ r}{\sqrt{z^2-1} \cdot sh\ x}\right)^2.$$

$$f'(z) = -2\left(\frac{z \cdot ch\ x - ch\ r}{\sqrt{z^2-1} \cdot sh\ x}\right) \times \frac{(z^2-1)sh\ x \cdot ch\ x - z^2 sh\ x \cdot ch\ x + z \cdot ch\ r \cdot sh\ x}{(z^2-1)^{\frac{3}{2}} sh^2\ x}$$
$$= -2\left(\frac{z \cdot ch\ x - ch\ r}{\sqrt{z^2-1} \cdot sh\ x}\right) \frac{z \cdot ch\ r \cdot sh\ x - ch\ x \cdot sh\ x}{(z^2-1)^{\frac{3}{2}} sh^2\ x} = \frac{2(z \cdot ch\ x - ch\ r)(ch\ x - z \cdot ch\ r)}{(z^2-1)^2 sh^2\ x}.$$

Since $ch\ (x-r) \le z \le ch\ (x+r)$, then the function $f(z)$ for $z = ch\ x/ch\ r$ has a maximum

$$f_{\max}\left(\frac{ch\ x}{ch\ r}\right) = 1 - \left(\frac{ch^2\ x - ch^2\ r}{\sqrt{ch^2\ x - ch^2\ r} \cdot sh\ x}\right)^2 = 1 - \frac{ch^2\ x - ch^2\ r}{sh^2\ x} = \frac{ch^2\ r - 1}{sh^2\ x} = \left(\frac{sh\ r}{sh\ x}\right)^2.$$

From (2.14) we have

$$A(x,r) \le \frac{1}{\lambda}\left(\frac{sh\ r}{sh\ x}\right)^{2\lambda}. \tag{2.15}$$

According to definition of maximal function we have

$$M_G f(chx) \le M_{G,1} f(chx) + M_{G,2} f(chx),$$

where

$$M_{G,1} f(chx) = \sup_{0<r<2} \frac{1}{|H(0,r)|_\lambda} \int_0^r |f(cht)| d\mu_\lambda(t),$$
$$M_{G,2} f(chx) = \sup_{2\le r<\infty} \frac{1}{|H(0,r)|_\lambda} \int_0^r |f(cht)| d\mu_\lambda(t).$$

Let $0 < r < 2$, then taking into account Lemmas 1.1 and 1.2 (a), for (2.13) with $0 \le x < r < 2$ we get

$$\begin{aligned} M_{G,1} f(ch\ x) &= \sup_{0<r<2} \frac{1}{|H(0,r)|_\lambda} \int_0^r A^\lambda_{ch\ t} |f(ch\ x)|\, d\mu_\lambda(t) \\ &= \sup_{0<r<2} \frac{|H(x,r)|_\lambda}{|H(0,r)|_\lambda} \cdot \frac{1}{|H(x,r)|_\lambda} \int_{|x-r|}^{x+r} |f(ch\ t)|\, sh^{2\lambda}\ tdt \\ &\le c_\lambda \sup_{0<r<2} \frac{1}{|H(x,r)|_\lambda} \int_{H(x,r)} |f(ch\ t)|\, d\mu(t) = c_\lambda M_\mu f(ch\ x). \end{aligned} \tag{2.16}$$

For $r < 2 \le x < \infty$ from Lemmas 1.1, 1.2(a), (2.15) and (2.8) we obtain

$$\begin{aligned} M_{G,1} f(ch\ x) &\le \sup_{0<r<2} \frac{A(x,r)\, |H(x,r)|_\lambda}{|H(0,r)|_\lambda\, |H(x,r)|_\lambda} \int_{x-r}^{x+r} |f(ch\ t)|\, sh^{2\lambda}\ tdt \\ &\le c_\lambda \sup_{0<r<2} \frac{sh\ \frac{r}{2} ch^{2\lambda}\ x \cdot sh^{2\lambda}\ r}{|H(x,r)|_\lambda \left(sh\ \frac{r}{2}\right)^{2\lambda+1} sh^{2\lambda}\ x} \int_{x-r}^{x+r} |f(ch\ t)|\, d\mu(t) \\ &\le c_\lambda \left(\frac{ch\ x}{sh\ x}\right)^{2\lambda} \sup_{0<r<2} ch^{2\lambda}\ \frac{r}{2} \cdot \frac{1}{|H(x,r)|_\lambda} \int_{x-r}^{x+r} |f(ch\ t)|\, d\mu(t) \\ &\le c_\lambda \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^{2\lambda} ch^{2\lambda}\ \frac{1}{2} \sup_{0<r<2} \frac{1}{|H(x,r)|_\lambda} \int_{H(x,r)} |f(ch\ t)|\, d\mu(t) \\ &\le c_\lambda \cdot 4^\lambda \cdot e \cdot M_\mu f(ch\ x), \end{aligned} \tag{2.17}$$

as $\frac{e^{2x}+1}{e^{2x}-1} \le 2 \Leftrightarrow e^{2x} + 1 \le 2e^{2x} - 2 \Leftrightarrow e^{2x} \ge 3$ at $x \ge 1$.

From (2.16) and (2.17) it follows that

$$M_{G,1} f(ch\ x) \le c_\lambda M_\mu f(ch\ x), 0 < r < 2, \quad 0 \le x < \infty. \tag{2.18}$$

Now we consider the case $2 \le r < \infty$.

Point that for $ch\ (x-r) \le z \le ch\ (x+r)$ and $x > r$ the function $f(z) = \frac{ch\ r - zch\ x}{\sqrt{z^2-1}sh\ x}$ has maximum equal $-\frac{\sqrt{ch^2\ x - ch^2\ r}}{sh\ x}$.

In fact

$$
\begin{aligned}
f'(z) &= -\frac{\sqrt{z^2-1}sh\ xch\ x + \frac{z}{\sqrt{z^2-1}} sh\ x(ch\ r - zch\ x)}{(z^2-1)sh^2\ x} \\
&= -\frac{(z^2-1)sh\ xch\ x + zsh\ xch\ r - z^2 sh\ xch\ x}{(z^2-1)^{\frac{3}{2}} sh^2\ x} = \frac{ch\ x - zch\ r}{(z^2-1)^{\frac{3}{2}} sh\ x} = 0 \quad \Leftrightarrow \quad z = \frac{ch\ x}{ch\ r}.
\end{aligned}
$$

In this point the function $f(z)$ has a maximum:

$$
\begin{aligned}
f_{\max}(z) &= f\left(\frac{ch\ x}{ch\ r}\right) = \frac{ch^2\ r - ch^2\ x}{\sqrt{ch^2\ x - ch^2\ r} \cdot sh\ x} = -\frac{\sqrt{ch^2\ x - ch^2\ r}}{sh\ x} \\
&= -\frac{ch\ x}{sh\ x}\sqrt{1-\left(\frac{ch\ r}{ch\ x}\right)^2} \sim -\frac{sh\ x}{ch\ x}, \qquad (2.19)
\end{aligned}
$$

as

$$
\lim_{x\to\infty} \frac{sh\ x}{ch\ x} = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1.
$$

From (2.15) and (2.19) we obtain

$$
\begin{aligned}
A(x,r) &\le \int_{-1}^{\frac{ch\ r - zch\ x}{\sqrt{z^2-1}\ sh\ x}} (1+u)^{\lambda-1} du \le \int_{-1}^{-\frac{\sqrt{ch^2\ x - ch^2 z}}{sh\ x}} (1+u)^{\lambda-1} du \\
&\sim \int_{-1}^{-\frac{sh\ x}{ch\ x}} (1+u)^{\lambda-1} du = \frac{1}{\lambda}\left(1 - \frac{sh\ x}{ch\ x}\right)^{\lambda} \le \frac{1}{\lambda}\left(1 - \frac{sh^2\ x}{ch^2\ x}\right)^{\lambda} = \frac{1}{\lambda}\left(ch\ x\right)^{-2\lambda},\ x \to \infty. \qquad (2.20)
\end{aligned}
$$

Now, taking into account Lemmas 1.1 and 1.2(b), also inequalities (2.12) and (2.20), for $2 \le r < \infty$ we get

$$
A(x,\ r)\frac{|H(x,r)|_\lambda}{|H(0,r)|_\lambda} \le c_\lambda \begin{cases} \dfrac{ch^{2\lambda}\ r}{ch^{4\lambda}\ \frac{r}{2}}, \\ \dfrac{ch^{2\lambda}\ xch^{2\lambda}\ r}{ch^{2\lambda}\ xch^{4\lambda}\ \frac{r}{2}} \end{cases} \le c_\lambda. \qquad (2.21)
$$

Applying (2.21) we easily obtain

$$
\begin{aligned}
M_{G,2} f(ch\ x) &= \sup_{r\ge 2} \frac{1}{|H(0,r)|_\lambda} \int_0^r A^\lambda_{ch\ t}\, |f(ch\ x)|\, d\mu_\lambda(t) \\
&= \sup_{r\ge 2} \frac{|H(x,r)|_\lambda}{|H(0,r)|_\lambda} \cdot \frac{A(x,\ r)}{|H(x,r)|_\lambda} \int_{|x-r|}^{x+r} |f(ch\ t)|\, sh^{2\lambda}\ tdt \\
&\le c_\lambda \frac{1}{|H(x,r)|_\lambda} \int_{H(x,r)} |f(ch\ t)|\, d\mu_\lambda(t) = c_\lambda M_\mu f(ch\ x). \qquad (2.22)
\end{aligned}
$$

Combining (2.18) and (2.22), we get

$$
M_G f(ch\ x) \le c_\lambda M_\mu f(ch\ x).
$$

Thus Theorem 2.1 is proved. □

Further we need the following lemma, which is a version of Vitali's covering lemma.

Lemma *([23], Sawano)***.** Suppose we have a family of n intervals $\{H(x_j,\ r_j)\}_{j\in\{1,\dots,n\}}$. Then we can take a subfamily $\{H(x_j,\ r_j)\}_{j\in A}$ such that

(1) $\{H(x_j, r_j)\}_{j\in A}$ is disjoint.

(2) $\bigcup_{j\in\{1,\dots,n\}} H(x_j, r_j) \subset \bigcup_{j\in A} H(x_j, 3r_j)$,

where $A = \{j_1, \dots j_p\}$ and $j_1, \dots j_p \in \{1, \dots n\}$.

The following theorem is valid.

Theorem 2.2. (a) *If $f \in L_{1,\lambda}[0,\infty)$, then for all $\alpha > 0$*

$$|\{x : M_G f(ch\ x) > \alpha\}|_\lambda \le \frac{c_\lambda}{\alpha}\int_0^\infty |f(ch\ t)|\, sh^{2\lambda}\ t dt = \frac{c_\lambda}{\alpha}\|f\|_{L_{1,\lambda}[0,\infty)}$$

holds, where $c_\lambda > 0$ depends only on λ.

(b) *If $f \in L_{p,\lambda}[0,\infty)$, $1 < p \le \infty$, then $M_G f(ch\ x) \in L_{p,\lambda}[0,\infty)$ and $\|M_G f\|_{L_{p,\lambda}[0,\infty)} \le c_{p,\lambda}\|f\|_{L_{p,\lambda}[0,\infty)}$.*

Corollary 2.1. *If $f \in L_{p,\lambda}[0,\infty)$, $1 \le p \le \infty$, then*

$$\lim_{r\to 0}\frac{1}{|H(0,r)|_\lambda}\int_{H(0,r)} A^\lambda_{ch\ t} f(ch\ x) sh^{2\lambda}\ t dt = f(ch\ x)$$

for a. e. $x \in [0,\infty)$.

Proof of Theorem 2.2. We define $E_\alpha = \{t : M_\mu f(ch\, t) > \alpha\}$. We introduce the function $h(\alpha)$ which is equal to the measure of the set E_α, i.e.

$$h(\alpha) = |E_\alpha|_\lambda = \left|\{t : M_\mu f(ch\, t) > \alpha\}\right|_\lambda.$$

By the definition of function $M_\mu f$ it follows that, for all $x_j \in E_\alpha$ there exists an interval $H(x_j, r_j) \subset E_\alpha$ with centered x_j such that

$$\begin{aligned}\int_{2H(x_j,r_j)} |f(ch\, t)|\, sh^{2\lambda}\, t dt &\ge \int_{\{t\in H(x_j,r_j):|f(ch\, t)|>\alpha\}} |f(ch\, t)|\, sh^{2\lambda} t dt\\ &> \alpha\int_{\{t\in H(x_j,r_j):|f(ch\, t)|>\alpha\}} sh^{2\lambda} t dt \ge \alpha\left|H(x_j, r_j)\right|_\lambda.\end{aligned} \tag{2.23}$$

Further, since

$$H(x, r) = \begin{cases}(0,\ x+r) & \text{if } x < r\\ (x-r,\ x+r) & \text{if } x > r,\end{cases}$$

then for $x_j < 3r_j$ we have

$$\left|H(x_j, 3r_j)\right|_\lambda = \int_0^{x_j+3r_j} sh^{2\lambda}\, t dt > \int_0^{x_j+r_j} sh^{2\lambda}\, t dt = \left|H(x_j, r_j)\right|_\lambda. \tag{2.24}$$

Let $x_j > 3r_j$, then

$$\left|H(x_j, 3r_j)\right|_\lambda = \int_{x_j-3r_j}^{x_j+3r_j} sh^{2\lambda}\, t dt > \int_{x_j-r_j}^{x_j+r_j} sh^{2\lambda}\, t dt. \tag{2.25}$$

From (2.24) and (2.25) we find that for all $r_j > 0$, $j \in \{1,\dots,n\}$

$$\left|H(x_j, 3r_j)\right|_\lambda \ge \left|H(x_j, r_j)\right|_\lambda. \tag{2.26}$$

By the previous lemma which was proved by Sawano, there exists a set $A \subset \{1, \dots n\}$ such that $\bigcup_{j=1,\dots n} H(x_j, r_j) \subset \bigcup_{l\in A} H(x_l, 3r_l)$ and the intervals $H(x_l, 3r_l)$ are disjoint, moreover by (2.26)

$$\left|\bigcup_{j=1,\dots n} H(x_j, r_j)\right|_\lambda \le \sum_{j=1}^{n} \left|H(x_j, r_j)\right|_\lambda \le \sum_{l\in A} |H(x_l, 3r_l)|_\lambda .$$

From this and (2.23) we have

$$\sum_{j=1}^{n} \left|H(x_j, r_j)\right|_\lambda \le \frac{1}{\alpha}\sum_{l\in A}\int_{H(x_l,3r_l)} |f(ch\, t)|\, sh^{2\lambda}\, t dt. \tag{2.27}$$

We show that $E_\alpha = \{t \in [0, \infty) : M_\mu f\,(ch\, t) > \alpha\}$ is an open set. For this we need double-sided estimates for the $|H\,(x,\ r)\,|_\lambda$.

At first we consider case $0 < x < r$. Then $H\,(x,\ r) = (0,\ x + r)$.

Let $0 < x + r < 2$, then we have

$$\begin{aligned}
|H\,(x,\ r)\,|_\lambda &= \int_0^{x+r} sh^{2\lambda}\, t dt = 2^{2\lambda+1}\int_0^{x+r} \frac{\left(sh\,\frac{t}{2}\right)^{2\lambda} d\left(sh\,\frac{t}{2}\right)}{\left(ch\,\frac{t}{2}\right)^{1-2\lambda}}\\
&\ge \frac{2^{2\lambda+1}}{\left(ch\,\frac{x+r}{2}\right)^{1-2\lambda}}\int_0^{x+r}\left(sh\,\frac{t}{2}\right)^{2\lambda} d\left(sh\,\frac{t}{2}\right) = \frac{2^{2\lambda-1}}{2\lambda+1}\frac{\left(sh\,\frac{x+r}{2}\right)^{2\lambda+1}}{\left(ch\,\frac{x+r}{2}\right)^{1-2\lambda}}\\
&\ge \frac{2^{2\lambda+1}}{(2\lambda+1)\, ch\, 1}\left(sh\,\frac{x+r}{2}\right)^{2\lambda+1} \ge \frac{1}{(2\lambda+1)\, ch\, 1}\,(x+r)^{2\lambda+1}.
\end{aligned} \tag{2.28}$$

On the other hand, since $ch\,\frac{t}{2} \ge 1$ for $t \ge 0$, then

$$|H\,(x,\ r)\,|_\lambda \le 2^{2\lambda+1}\int_0^{x+r}\left(sh\,\frac{t}{2}\right)^{2\lambda} d\left(sh\,\frac{t}{2}\right) = \frac{2^{2\lambda+1}}{2\lambda+1}\left(sh\,\frac{x+r}{2}\right)^{2\lambda+1} \le \frac{e^{2\lambda+1}}{2\lambda+1}\,(x+r)^{2\lambda+1}. \tag{2.29}$$

At the end we use the inequality (1.3).

Now let $2 \le x + r < \infty$. Then

$$\begin{aligned}
|H\,(x,\ r)\,|_\lambda &= \int_0^{x+r} sh^{\,2\lambda} t dt \ge \int_{\frac{x+r}{2}}^{x+r} \frac{sh^{2\lambda}\, t d\,(sh\, t)}{ch\, t} \ge \frac{1}{2}\int_{\frac{x+r}{2}}^{x+r} (sh\, t)^{2\lambda-1}\, d\,(sh\, t)\\
&= \frac{1}{4\lambda}\left(sh^{2\lambda}\,(x+r) - sh^{2\lambda}\,\frac{x+r}{2}\right) \ge \frac{1}{4\lambda}\left(sh^{2\lambda}\,(x+r) - \frac{1}{4^\lambda} sh^{2\lambda}\,(x+r)\right)\\
&= \frac{1}{4\lambda}\left(1 - \frac{1}{4\lambda}\right) sh^{2\lambda}\,(x+r) = \frac{4^\lambda - 1}{4\lambda\cdot 4^\lambda}\left(2sh\,\frac{x+r}{2} ch\,\frac{x+r}{2}\right)^{2\lambda}\\
&\ge \frac{4^\lambda-1}{4\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda}.
\end{aligned} \tag{2.30}$$

On the other hand

$$\begin{aligned}
|H\,(x,\ r)\,|_\lambda &= \int_0^{x+r} sh^{\,2\lambda} t dt = 2^{2\lambda+1}\int_0^{x+r} \frac{\left(sh\,\frac{t}{2}\right)^{2\lambda} d\left(sh\,\frac{t}{2}\right)}{\left(ch\,\frac{t}{2}\right)^{1-2\lambda}}\\
&\le 2^{2\lambda+1}\int_0^{x+r}\left(sh\,\frac{t}{2}\right)^{4\lambda-1} d\left(sh\,\frac{t}{2}\right) = \frac{2^{2\lambda+1}}{\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda}.
\end{aligned} \tag{2.31}$$

Combining (2.28)–(2.31) we obtain for $0 < x < r$ and $0 < x + r < 2$

$$\frac{1}{(2\lambda+1)\, ch\, 1}\,(x+r)^{2\lambda+1} \le |H\,(x,\ r)\,|_\lambda \le \frac{e^{2\lambda+1}}{2\lambda+1}\,(x+r)^{2\lambda+1}, \tag{2.32}$$

and for $0 < x < r$ and $2 \le x + r < \infty$

$$\frac{4^{\lambda}-1}{4\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda} \le |H(x,\ r)|_{\lambda} \le \frac{2^{2\lambda-1}}{\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda}. \tag{2.33}$$

Now we consider the case $r \le x < \infty$.

Then $H(x,\ r) = (x-r,\ x+r)$.

Let $0 < x + r < 2$. Then we have

$$|H(x,\ r)|_{\lambda} = \int_{x-r}^{x+r} sh^{2\lambda}\,tdt \ge \int_{\frac{x+r}{2}}^{x+r} sh^{2\lambda}\,tdt \ge \frac{x+r}{2}\left(sh\,\frac{x+r}{2}\right)^{2\lambda}$$
$$\ge \frac{1}{2^{2\lambda+1}}(x+r)^{2\lambda+1}. \tag{2.34}$$

At the end we use the inequality (1.3).

On the other hand according to (2.29) we have

$$|H(x,\ r)|_{\lambda} = \int_{x-r}^{x+r} sh^{2\lambda}\,tdt \le \int_{0}^{x+r} sh^{2\lambda}\,tdt \le \frac{1}{4(2\lambda+1)}(x+r)^{2\lambda+1}.$$

It remains to consider the case $2 \le x + r < \infty$.

For inequality (2.30) we have

$$|H(x,\ r)|_{\lambda} = \int_{x-r}^{x+r} sh^{2\lambda}\,tdt \ge \int_{\frac{x+r}{2}}^{x+r} sh^{2\lambda}\,tdt \ge \frac{4^{\lambda}-1}{4\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda}.$$

On the other hand

$$|H(x,\ r)|_{\lambda} = \int_{x-r}^{x+r} sh^{2\lambda}\,tdt = 2^{2\lambda+1}\int_{x-r}^{x+2}\frac{\left(sh\,\frac{t}{2}\right)^{2\lambda} d\left(sh\,\frac{t}{2}\right)}{\left(ch\,\frac{t}{2}\right)^{1-2\lambda}}$$
$$\le 2^{2\lambda+1}\int_{x-r}^{x+r}\left(sh\,\frac{t}{2}\right)^{4\lambda-1}\left(sh\,\frac{t}{2}\right) = \frac{2^{2\lambda-1}}{\lambda}\left(\left(sh\,\frac{x+r}{2}\right)^{4\lambda} - \left(sh\,\frac{x-r}{r}\right)^{4\lambda}\right)$$
$$\le \frac{2^{2\lambda-1}}{\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda}. \tag{2.35}$$

Combining (2.32)–(2.35) we obtain for $r \le x < \infty$ and $0 < x + r < 2$

$$\frac{1}{(2\lambda+1)ch1}(x+r)^{2\lambda+1} \le |H(x,\ r)|_{\lambda} \le \frac{e^{2\lambda+1}}{2\lambda+1}(x+r)^{2\lambda+1}, \tag{2.36}$$

and for $2 \le x + r < \infty$

$$\frac{4^{\lambda}-1}{4\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda} \le |H(x,\ r)|_{\lambda} \le \frac{2^{2\lambda-1}}{\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda}. \tag{2.37}$$

Now from (2.32) and (2.36) for $0 < x + r < 2$ we have

$$\frac{1}{(2\lambda+1)\,ch\,1}(x+r)^{2\lambda+1} \le |H(x,\ r)|_{\lambda} \le \frac{e^{2\lambda+1}}{2\lambda+1}(x+r)^{2\lambda+1}. \tag{2.38}$$

But from (2.33) and (2.35) for $2 \le x + r < \infty$

$$\frac{4^{\lambda}-1}{4\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda} \le |H(x,\ r)|_{\lambda} \le \frac{2^{2\lambda-1}}{\lambda}\left(sh\,\frac{x+r}{2}\right)^{4\lambda}. \tag{2.39}$$

Now we will prove that the set E_α is open. By the definition of the maximal operator there exists $r > 0$ such that for some $u > \alpha$

$$\int_{H(x,r)} |f(ch\,t)|sh^{2\lambda}\,tdt = u\,|H(x,r)|_\lambda.$$

We consider the case $0 < x + r < 2$. There exists $\delta_1 > 0$ such that

$$\frac{u}{\alpha} > e^{2\lambda+1}\,(ch\,1)\left(\frac{r+\delta_1}{r}\right)^{2\lambda+1}. \tag{2.40}$$

Let $|x-y| < \delta_1$, then $H(x,r) \subset H(y, r+\delta_1)$. If $z \in H(x,r)$, then $|z-y| \le |z-x| + |x-y| \le r+\delta_1$, from this it follows that $z \in H(y, r+\delta_1)$.

Then

$$\int_{H(y,r+\delta_1)} |f(ch\,t)|sh^{2\lambda}\,tdt \ge \int_{H(y,r)} |f(ch\,t)|sh^{2\lambda}\,tdt = u|H(y,r)|_\lambda. \tag{2.41}$$

Now by (2.36), we have

$$\begin{aligned}|H(y,r+\delta_1)|_\lambda &\le \frac{e^{2\lambda+1}}{2\lambda+1}(y+r+\delta_1)^{2\lambda+1}\\ &\le \frac{e^{2\lambda+1}}{2\lambda+1}(y+r)^{2\lambda+1}\left(\frac{r+\delta_1}{r}\right)^{2\lambda+1} \le e^{2\lambda+1}\,(ch\,1)\left(\frac{r+\delta_1}{r}\right)^{2\lambda+1}|H(y,r)|_\lambda.\end{aligned}$$

From this it follows that

$$|H(y,r)|_\lambda \ge \left(\frac{r+\delta_1}{r}\right)^{-(2\lambda+1)}\left(e^{2\lambda+1}ch\,1\right)^{-1}|H(y,r+\delta_1)|_\lambda. \tag{2.42}$$

From (2.41) and (2.42) it follows that

$$\frac{1}{|H(y,r+\delta_1)|_\lambda}\int_{H(y,r+\delta_1)}\left|f\left(e^{2\lambda+1}ch\,t\right)\right|sh^{2\lambda}\,tdt \ge (ch\,1)^{-1}\left(\frac{r+\delta_1}{r}\right)^{-(2\lambda+1)}u > \alpha,$$

if

$$\frac{u}{\alpha} > e^{2\lambda+1}\,(ch\,1)\left(\frac{r+\delta_1}{r}\right)^{2\lambda+1}.$$

In the case $0 < x + r < 2$ we obtain that $\exists\delta_1 > 0$ by condition (2.40) such that for $\forall t \in H(y,\delta_1)$ the inequality $M_\mu f(ch\,t) > \alpha$ holds, from this it follows that $H(y,\delta_1) \subset E_\alpha$, that is the set E_α is open.

It remains consider the case $2 \le x + r < \infty$. There exists $\delta_2 > 0$ such that

$$\frac{u}{\alpha} \ge \frac{2^{2\lambda+1}\cdot 3^{4\lambda}}{4^\lambda-1}\left(ch\,(r+1)\frac{r+\delta_2}{r}\right)^{4\lambda} > \frac{2^{2\lambda+1}\cdot 3^{4\lambda}}{4^\lambda-1}\left(\frac{r+\delta_2}{r}\right)^{4\lambda}. \tag{2.43}$$

From (2.39) we have

$$\begin{aligned}|H(y,r+\delta_2)|_\lambda &\le \frac{2^{2\lambda-1}}{\lambda}\left(sh\,\frac{y+r+\delta_2}{2}\right)^{4\lambda} \le \frac{2^{2\lambda-1}}{\lambda}\left[sh\left(\frac{y+r}{2}+\frac{r+\delta_2}{2}\right)\right]^{4\lambda}\\ &\le \frac{2^{2\lambda-1}}{\lambda}\left[sh\left(\frac{y+r}{2}+(r+1)\frac{r+\delta_2}{2}\right)\right]^{4\lambda}\\ &= \frac{2^{2\lambda-1}}{\lambda}\left(sh\,\frac{y+r}{2}ch\,(r+1)\frac{r+\delta_2}{2}+ch\,\frac{y+r}{2}sh\,(r+1)\frac{r+\delta_2}{2}\right)^{4\lambda}\\ &\le \frac{3^{4\lambda}\cdot 2^{2\lambda-1}}{\lambda}\left(sh\,\frac{y+r}{2}\right)^{4\lambda}\left(ch\,(r+1)\frac{r+\delta_2}{2}\right)^{4\lambda}.\end{aligned} \tag{2.44}$$

At the end we use the inequality $ch\, t \leq 2sh\, t$ at $t \geq 1$. From (2.44) and (2.39) we have

$$|H\,(y,\ r+\delta_2)\,|_\lambda \leq \frac{2^{2\lambda+1}\cdot 3^{4\lambda}}{4^\lambda - 1}\left(ch\ (r+1)\,\frac{r+\delta_2}{2}\right)^{4\lambda}|H\,(y,\ r)\,|_\lambda.$$

From this it follows that

$$|H\,(y,\ r)\,|_\lambda \geq \frac{4^\lambda - 1}{2^{2\lambda+1}\cdot 3^{4\lambda}}\left(ch\ (r+1)\,\frac{r+\delta_2}{2}\right)^{-4\lambda}|H\,(y,\ r+\delta_2)\,|_\lambda.$$

Now from (2.41) we have

$$\frac{1}{|H\,(y,\ r+\delta_2)\,|_\lambda}\int_{H(y,\,r+\delta)}|f\,(ch\,t)|sh^{2\lambda}\,tdt \geq \frac{4^\lambda - 1}{2^{2\lambda+1}\cdot 3^{4\lambda}}\left(ch\ (r+1)\,\frac{r+\delta_2}{r}\right)^{-4\lambda}u > \alpha,$$

if (2.43) is true.

In the case $2 \leq x + r < \infty$ we prove that $\exists \delta_2 > 0$ by condition (2.43) such that for $\forall t \in H\,(y,\ \delta_2)$ the inequality $M_\mu f\,(ch\,t) > \alpha$ holds, from this it follows that $H\,(y,\ \delta_2) \subset E_\alpha$, that is the set E_α is open.

As above it follows that $\exists \delta > 0$, where $\delta = \min\{\delta_1,\ \delta_2\}$ such that for $\forall t \in H\,(y,\ \delta)$ the inequality $M_\mu f\,(ch\,t) > \alpha$ holds, from this it follows that $H\,(y,\ \delta) \subset E_\alpha$, that is the set E_α is open.

Since $[0,\ \infty)$ is separable, so with the help of the Lindelöf (see [24]) covering theorem $E_\alpha \subset \bigcup_{j\in N} H(x_j, r_j)$. Then, letting n tends to infinity in (2.27), we obtain

$$|E_\alpha|_\lambda \leq \frac{1}{\alpha}\int_0^\infty |f(ch\,t)|\,sh^{2\lambda}\,tdt,$$

and this is the assertion (a) of theorem.

Further, since by Theorem 2.1 $M_G f(ch\,x) \leq c_\lambda M_\mu f(ch\,x)$, then

$$F_\alpha = \{x : M_G f(ch\,x) > \alpha\} \subset E_\alpha = \left\{x : M_\mu f(ch\,x) > \frac{\alpha}{c_\lambda}\right\};$$

consequently

$$|F_\alpha|_\lambda \leq \left|\left\{x : M_\mu f(ch\,x) > \frac{\alpha}{c_\lambda}\right\}\right|_\lambda \leq \frac{c_\lambda}{\alpha}\|f\|_{p,\,\lambda},$$

and this is the assertion (a) of theorem.

We will prove the approval (b). Suppose

$$f_1(ch\,x) = \begin{cases} f(ch\,x), & \text{if } |f(ch\,x)| \geq \dfrac{\alpha}{2}, \\ 0, & \text{if } 0 \leq |f(ch\,x)| < \dfrac{\alpha}{2}. \end{cases} \tag{2.45}$$

Then we have

$$|f(ch\,x)| \leq |f_1(ch\,x)| + \frac{\alpha}{2} \quad \text{and} \quad M_\mu f(ch\,x) \leq M_\mu f_1(ch\,x) + \frac{\alpha}{2},$$

so $\{x : M_\mu f(ch\,x) > \alpha\} \subset \{x : M_\mu f_1(ch\,x) > \frac{\alpha}{2}\}$ and at last

$$|E_\alpha|_\lambda = \left|\{x : M_\mu f(ch\,x) > \alpha\}\right|_\lambda \leq \frac{2}{\alpha}\|f_1\|_{1,\,\lambda},$$

from here and (2.45) it follows that

$$|E_\alpha|_\lambda = \left|\{x : M_\mu f(ch\,x) > \alpha\}\right|_\lambda \leq \frac{2}{\alpha}\int_{\{x:|f(ch\,x)|>\frac{\alpha}{2}\}}|f(ch\,x)|\,sh^{2\lambda}\,xdx. \tag{2.46}$$

Suppose that the function $f\,(ch\,x)$ is defined on $[0,\ \infty)$. We consider for each $\alpha > 0$ the set E_α such that $|f| > \alpha$; $E_\alpha = \{x : |f(ch\,x)| > \alpha\}$.

Let $h(\alpha)$ be the measure of the set E_α, i.e.

$$h(\alpha) = |E_\alpha|_\lambda = |\{x : |f\,(ch\,x)| > \alpha\}|_\lambda\,.$$

The function $h(\alpha)$ is called the distribution of the function $|f\,(ch\,x)|$. Every quantity, depended only on ëxtent" f, can be expressed over of distribution of function $h(\alpha)$ (see [25], p. 15). For example if $f \in L_{p,\lambda}$, then by Fubini's theorem we obtain

$$\begin{aligned}\int_0^\infty |f\,(ch\,t)|^p\, sh^{2\lambda}\, t dt &= p\int_0^\infty \left(\int_0^{|f(ch\,t)|} \alpha^{p-1} d\alpha\right) sh^{2\lambda}\, t dt \\ &= p\int_0^\infty \alpha^{p-1}\left(\int_{\{t\in[0,\,\infty):|f(ch\,t)|>\alpha\}} sh^{2\lambda}\, t dt\right) \\ &= p\int_0^\infty \alpha^{p-1}|\{t\in[0,\ \infty) : |f(ch\,t)|>\alpha\}|_\lambda d\alpha = p\int_0^\infty \alpha^{p-1} h(\alpha) d\alpha. \qquad (2.47)\end{aligned}$$

Now, if $M_\mu f \in L_{1,\lambda}$, then by (2.46) and (2.47) applying Fubini's theorem, we will have

$$\begin{aligned}\|M_\mu f\|_{p,\lambda}^p &= p\int_0^\infty \alpha^{p-1}\Big|\{x : M_\mu f(ch\,x)>\alpha\}\Big|_\lambda d\alpha \\ &\le p\int_0^\infty \alpha^{p-1}\left(\frac{2}{\alpha}\int_{\{x:|f(ch\,x)|>\frac{\alpha}{2}\}} |f(ch\,x)|\, sh^{2\lambda}\, x dx\right) d\alpha \\ &= 2p\int_0^\infty |f(ch\,x)|\left(\int_0^{2|f(ch\,x)|} \alpha^{p-2} d\alpha\right) sh^{2\lambda}\, x dx \\ &= \frac{2p}{p-1}\int_0^\infty |f(ch\,x)|\left(\alpha^{p-1}\Big|_0^{2|f(ch\,x)|}\right) sh^{2\lambda}\, x dx \\ &= \frac{p\cdot 2^p}{p-1}\int_0^\infty |f(ch\,x)|^p\, sh^{2\lambda}\, x dx = c_p\|f\|_{p,\lambda}^p,\end{aligned}$$

from this it follows that

$$\|M_\mu f\|_{p,\lambda} \le c_p\|f\|_{p,\lambda}, \quad 1<p<\infty. \qquad (2.48)$$

The assertion (b) follows from Theorem 2.1 and inequality (2.48):

$$\|M_G f\|_{p,\lambda} \le c_\lambda\|M_\mu f\|_{p,\lambda} \le c_{p,\lambda}\|f\|_{p,\lambda}.$$

In the case $p=\infty$ last inequality is obtained evidently.

Thus Theorem 2.2 is proved. □

Proof of Corollary 2.1. At first let us show that for any function $f \in L_{p,\lambda}[0,\infty)$, $1 \le p \le \infty$,, representation $ch\ t \mapsto A^\lambda_{ch\ t} f$ from $\mathbb{R}$ into $L_{p,\lambda}$ continuous, that is

$$\|A^\lambda_{ch\ t} f - f\|_{L_{p,\lambda}} \to 0 \quad \text{at } t\to 0. \qquad (2.49)$$

Let $f(x)$ be a continuous function defined for $[a,b]\subset[0,\infty)$. Consider the function

$$y(t,x,\varphi) = ch\ tch\ x - sh\ tsh\ x\cos\ \varphi.$$

Hence we have

$$\begin{aligned}|y(t,x,\varphi)-y(0,x,\varphi)| &= |ch\ tch\ x - sh\ tsh\ x\cos\ \varphi - ch\ x| \\ &= |(ch\ t-1)ch\ x - sh\ tsh\ x\cos\ \varphi - ch\ x| \le 2sh^2\ \frac{t}{2}ch\ x + 2sh\ \frac{t}{2}ch\ \frac{t}{2}sh\ x \\ &\le 2sh\ \frac{t}{2}\left(sh\ \frac{t}{2}ch\ x + ch\ \frac{t}{2}sh\ x\right) \\ &= 2sh\ \frac{t}{2}sh\left(\frac{t}{2}+x\right) \le 2sh\ \frac{t}{2}sh\left(\frac{t}{2}+b\right) \to 0 \quad t\to 0. \qquad (2.50)\end{aligned}$$

On the strength of uniform continuity of the function $f(x)$ on segment $[a, b]$ for any $\varepsilon > 0$ one may choose the number $\delta > 0$, such that

$$|f[y(t,x,\varphi)] - f[y(0,x,\varphi)]| < \varepsilon, \text{ if } |y(t,x,\varphi) - y(0,x,\varphi)| < \delta, \text{ (that follows from (2.50)).}$$

Then we have

$$\left|A^{\lambda}_{ch\ t} f(ch\ x) - f(ch\ x)\right| \leq \frac{\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(\lambda)} \int_0^{\pi} |f[y(t,x,\varphi)] - f[y(0,x,\varphi)]| (\sin\ \varphi)^{2\lambda-1} d\varphi < \varepsilon.$$

It follows, that

$$\|A^{\lambda}_{ch\ t} f - f\|_{\infty,\lambda} = \sup_{x\in[a,b]} |A^{\lambda}_{ch\ t} f(ch\ x) - f(ch\ x)| < \varepsilon.$$

And for $1 \leq p < \infty$

$$\|A^{\lambda}_{ch\ t} f - f\|_{L_{p,\lambda}[a,b]} = \left(\int_a^b |A^{\lambda}_{ch\ t} f(ch\ x) - f(ch\ x)|^p sh^{2\lambda} x dx\right)^{\frac{1}{p}} < \varepsilon \left(\int_a^b sh^{2\lambda}\ x dx\right)^{\frac{1}{p}} < c_{p,\lambda}\varepsilon.$$

Thus for any continuous function defined on the segment $[a, b] \subset [0, \infty)$ and for any number $\varepsilon > 0$ the following inequality is valid:

$$\|A^{\lambda}_{ch\ t} f - f\|_{L_{p,\lambda}[a,b]} < \varepsilon \quad 1 \leq p \leq \infty. \tag{2.51}$$

It is known the set of all continuous functions with compact support in $[0, \infty)$ is dense in $L_{p,\lambda}[0, \infty)$ (see [26], Theorem 4.2). Therefore for any number $\varepsilon > 0$ there exists a continuous function with compact support in $[0, \infty)$, such that

$$\|f - f_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)} < \varepsilon. \tag{2.52}$$

We denote $g_{\varepsilon} = f - f_{\varepsilon}$. Then $g_{\varepsilon} \in L_{p,\lambda}[0, \infty)$ and

$$\|g_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)} < \varepsilon. \tag{2.53}$$

Thus, if $f \in L_{p,\lambda}[0, \infty)$, then for any number $\varepsilon > 0$ there exists a continuous function f_{ε} with the compact support and function $g_{\varepsilon} \in L_{p,\lambda}[0, \infty)$ with condition $\|g_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)} < \varepsilon$, such that $f = f_{\varepsilon} + g_{\varepsilon}$.

Hence we have $A^{\lambda}_{ch\ t} f(ch\ x) = A^{\lambda}_{ch\ t} f_{\varepsilon}(ch\ x) + A^{\lambda}_{ch\ t} g_{\varepsilon}(ch\ x) - f(ch\ x) + f_{\varepsilon}(ch\ x) - f_{\varepsilon}(ch\ x)$, from which it follows that

$$A^{\lambda}_{ch\ t} f - f\|_{L_{p,\lambda}[0,\infty)} \leq \|A^{\lambda}_{ch\ t} f_{\varepsilon} - f_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)} + \|f - f_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)} + \|A^{\lambda}_{ch\ t} g_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)}.$$

Now, taking into account that (see [22], Lemma 2)

$$\|A^{\lambda}_{ch\ t} g_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)} \leq \|g_{\varepsilon}\|_{L_{p,\lambda}[0,\infty)}, \quad t \in [0, \infty), 1 \leq p \leq \infty$$

and also the inequalities (2.51)–(2.53), we get

$$\|A^{\lambda}_{ch\ t} f - f\|_{L_{p,\lambda}[0,\infty)} \leq 3\varepsilon,$$

from which (2.49) follows.

By the locality of the problem, one can account that $f \in L_{1,\lambda}[0, \infty)$. In general case one can multiply f by characteristic function of interval $H(0, r) = [0, r)$ and obtain required convergence almost everywhere interior to this interval and by tending r to infinity one could obtain it on the whole interval $[0, \infty)$.

Suppose for any $r > 0$ and for any $x \in [0, \infty)$

$$f_r(ch\ x) = \frac{1}{|H(0,r)|_{\lambda}} \int_{H(0,r)} A^{\lambda}_{ch\ t} f(ch\ x) sh^{2\lambda}\ tdt.$$

Let $r_0 > 0$, $H = H(0, r_0)$. According to the generalized Minkowski generalized inequality and discount (2.49), we obtain

$$\begin{aligned} f_r - f\|_{L_{1,\lambda}(H)} &= \Big|\frac{1}{|H(0,r)|_\lambda}\int_{H(0,r)} \big(A^\lambda_{ch\ t} f(ch\ x) - f(ch\ x)\big) sh^{2\lambda}\ tdt\Big|_{L_{1,\lambda}(H)} \\ &\le \frac{1}{|H(0,r)|_\lambda}\int_{H(0,r)} \|A^\lambda_{ch\ t} f - f\|_{L_{1,\lambda}(H)} sh^{2\lambda}\ tdt \\ &\le \sup_{|t|\le r_0} \|A^\lambda_{ch\ t} f - f\|_{L_{1,\lambda}(H)} \to 0, \quad \text{at } r_o \to +0. \end{aligned}$$

It means that there is a sequence r_k such that $r_k \to +0$, $(k \to \infty)$ and

$$\lim_{k\to\infty} f_{r_k}(ch\ x) = f(ch\ x)$$

almost everywhere at $x \in [0, \infty)$.

Now, let us prove that $\lim_{r\to+0} f_r(ch\ x)$ exists almost everywhere. For this purpose for any $x \in [0, \infty)$ we consider

$$\Omega_f(ch\ x) = \Big|\overline{\lim_{r\to+0}} f_r(ch\ x) - \underline{\lim_{r\to+0}} f_r(ch\ x)\Big|$$

the oscillation of f_r at the point x as $r \to +0$.

If g is a continuous function with compact support on $[0, \infty)$, then g_r is convergent to g and consequently $\Omega_g \equiv 0$.

Further, if $g \in L_{1,\lambda}[0, \infty)$, then according to the statement of Theorem 2.2 we get

$$|\{x \in [0,\infty) : M_G g(ch\ x) > \varepsilon\}|_\lambda \le \frac{c}{\varepsilon}\|g\|_{L_{1,\lambda}[0,\infty)}, \quad g \in L_{1,\lambda}[0,\infty).$$

On the other hand it is obvious that $\Omega g(ch\ x) \le 2M_G g(ch\ x)$. Thus

$$|\{x \in [0,\infty) : \Omega_g(ch\ x) > \varepsilon\}|_\lambda \le \frac{2c}{\varepsilon}\|g\|_{L_{1,\lambda}[0,\infty)},\ g \in L_{1,\lambda}[0,\infty).$$

By the same way as it was proved above, any function $f \in L_{p,\lambda}[0, \infty)$ can be written in form $f = h + g$, where h is continuous function and has a compact support on $[0, \infty)$, and $g \in L_{p,\lambda}[0, \infty)$, moreover $\|g\|_{L_{p,\lambda}[0,\infty)} < \varepsilon$, for any $\varepsilon > 0$. But $\Omega \le \Omega_h + \Omega_g$ $\Omega_h \equiv 0$, however is continuous by h. Therefore it follows that

$$|\{x \in [0,\infty) : \Omega_g(ch\ x) > \varepsilon\}|_\lambda \le \frac{c}{\varepsilon}\|g\|_{L_{1,\lambda}[0,\infty)}.$$

Taking in inequality $\|g\|_{L_{1,\lambda}[0,\infty)} < \varepsilon$ the number ε arbitrary small, we get $\Omega f = 0$ almost everywhere on $[0, \infty)$. Consequently, $\lim_{r\to0} f_r(ch\ x)$ exists almost everywhere on $[0, \infty)$, which was required to prove. □

Remark 2.1. Theorem 2.2 was proved earlier by W.C. Connett and A.L. Schwartz [8] for the Jacobi-type hypergroups.

Remark 2.2. If $f \in L_{1,\lambda}[0, \infty)$, then (see [27], Theorem 2.1)

$$\lim_{r\to0} \frac{1}{\big(sh\ \frac{r}{2}\big)^{2\lambda+1}} \int_0^r \big|A^\lambda_{ch\ t} f(ch\ x) - f(ch\ x)\big| sh^{2\lambda}\ tdt = 0,$$

almost everywhere for $x \in [0, \infty)$.

This implies that for any $\varepsilon > 0$ one can find $\delta > 0$, such that for all $r < \delta$ the following inequality is just:

$$\frac{1}{\big(sh\ \frac{r}{2}\big)^{2\lambda+1}} \int_0^r \big|A^\lambda_{ch\ t} f(ch\ x) - f(ch\ x)\big| sh^{2\lambda}\ tdt < \varepsilon.$$

Then from Lemma 1.1, we obtain

$$\begin{aligned} &\Big|\frac{1}{|H(0,r)|_\lambda}\int_{H(0,r)} \big[A^\lambda_{ch\ t} f(ch\ x) - f(ch\ x)\big] sh^{2\lambda}\ tdt\Big| \\ &\quad\le \frac{1}{\big(sh\ \frac{r}{2}\big)^{2\lambda+1}} \int_0^r \big|A^\lambda_{ch\ t} f(ch\ x) - f(ch\ x)\big| sh^{2\lambda}\ tdt < \varepsilon, \end{aligned}$$

for all $r < \delta$, which means that Corollary 2.1 is valid under assumption $f \in L_{1,\lambda}[0, \infty)$.

3. *G*-Riesz potential

In this section the concept of Riesz–Gegenbauer potential associated with the Gegenbauer differential operator G is introduced and its integral representation is found. For the functions $f, g \in L_{1,\lambda}[1,\infty)$ in [9], the Gegenbauer transformation is defined as follows:

$$F_P : f(t) \mapsto \hat{f}_P(\gamma) = \int_1^\infty f(t) P_\gamma^\lambda(t) \left(t^2-1\right)^{\lambda-\frac{1}{2}} dt, \tag{3.1}$$

$$F_Q : f(t) \mapsto \hat{f}_Q(\gamma) = \int_1^\infty f(t) Q_\gamma^\lambda(t) \left(t^2-1\right)^{\lambda-\frac{1}{2}} dt,$$

where the functions $P_\gamma^\lambda(x)$ and $Q_\gamma^\lambda(x)$ are eigenfunctions of operator G.

The inverse of the Gegenbauer transformations is defined by the formulas

$$F_P^{-1} : \hat{f}_P(\alpha) \mapsto f(x) = c_\lambda^* \int_1^\infty \hat{f}_P(\gamma) Q_\gamma^\lambda(x) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma, \tag{3.2}$$

$$F_Q^{-1} : \hat{f}_Q(\alpha) \mapsto f(x) = c_\lambda^* \int_1^\infty \hat{f}_Q(\gamma) P_\gamma^\lambda(x) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma, \tag{3.3}$$

where $c_\lambda^* = \frac{2^{\frac{3}{2}-\lambda}\sqrt{\pi}\,\Gamma(\lambda+1)\Gamma\left(\frac{1}{2}-\gamma\right)\Gamma\left(\frac{3+2\lambda}{4}\right)\left(\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\frac{5-2\lambda}{4}\right)\cos\pi\lambda\right)^{-1}}{{}_2F_1\left(1,\frac{1}{2}-\lambda;\frac{5-2\lambda}{4};\frac{1}{2}\right)-{}_2F_1\left(1,\frac{1}{2}-\lambda;\frac{5-2\lambda}{4};\frac{1-2\lambda}{2}\right)}$, and ${}_2F_1(\alpha;\ \beta;\ \gamma;\ x)$ is Gauss function.

For $f \in D(\mathbb{R}_+)$ the transformations (3.1)–(3.3) are defined, where $D(\mathbb{R}_+)$ is the set of infinitely differentiable even functions on $\mathbb{R}_+ = [0,\ \infty)$ with compact supports.

Preliminary we prove the following lemma.

Lemma 3.1. *Let $f, g \in L_{1,\lambda}[1,\infty) \cap L_{2,\lambda}[1,\infty)$. Then the following equality is true:*

$$\int_1^\infty f(x) A_t^\lambda g(x) \left(x^2-1\right)^{\lambda-\frac{1}{2}} dx = c_\lambda^* \int_1^\infty \hat{f}_P(\gamma) \left(\widehat{A_t^\lambda g}\right)_P(\gamma) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma. \tag{3.4}$$

Proof. From (3.4) we have

$$\int_1^\infty f(x) A_t^\lambda g(x) \left(x^2-1\right)^{\lambda-\frac{1}{2}} dx$$
$$= c_\lambda^* \int_1^\infty A_t^\lambda g(x) \left(x^2-1\right)^{\lambda-\frac{1}{2}} dx \int_1^\infty \hat{f}_P(\gamma) Q_\gamma^\lambda(x) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma. \tag{3.5}$$

Since (see the proof of Lemma 8 in [22])

$$\int_1^\infty \hat{f}_P(\gamma) Q_\gamma^\lambda(x) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma \lesssim \|f\|_{L_{2,\lambda}},$$

then taking into account the inequality (see [22], Lemma 1.2)

$$\left\|A_t^\lambda g\right\|_{L_{1,\lambda}} \le \|g\|_{L_{1,\lambda}},$$

we obtain

$$\left|\int_1^\infty A_t^\lambda g(x) \left(x^2-1\right)^{\lambda-\frac{1}{2}} dx \int_1^\infty \hat{f}_P(\gamma) Q_\gamma^\lambda(x) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma\right|$$
$$\lesssim \|f\|_{L_{2,\lambda}} \int_1^\infty \left|A_t^\lambda g(x)\right| \left(x^2-1\right)^{\lambda-\frac{1}{2}} dx = \|f\|_{L_{2,\lambda}} \left\|A_t^\lambda g\right\|_{L_{1,\lambda}} \le \|f\|_{L_{2,\lambda}} \|g\|_{L_{1,\lambda}}.$$

By the Fubini theorem we have

$$c_\lambda^* \int_1^\infty A_t^\lambda g(x) \left(x^2-1\right)^{\lambda-\frac{1}{2}} dx \int_1^\infty \hat{f}_P(\gamma) Q_\gamma^\lambda(x) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma$$

$$= c_\lambda^* \int_1^\infty A_t^\lambda g(x)\, Q_\gamma^\lambda(x) \left(x^2-1\right)^{\lambda-\frac{1}{2}} dx \int_1^\infty \hat{f}_P(\gamma) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma$$

$$= c_\lambda^* \int_1^\infty \left(\widehat{A_t^\lambda g}\right)_Q(\gamma)\, \hat{f}_P(\gamma) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma. \tag{3.6}$$

Taking into account (3.5) in (3.6), we obtain (3.4).

Thus Lemma 3.1 is proved. □

Definition 3.1. For $0 < \alpha < 2\lambda + 1$ Riesz–Gegenbauer potential (G-Riesz potential) $I_G^\alpha f(ch\, x)$ is defined by the equality

$$I_G^\alpha f(ch\ x) = G_\lambda^{-\frac{\alpha}{2}} f(ch\ x). \tag{3.7}$$

Such (see [28], p. 1933)

$$G_\lambda P_\gamma^\lambda(ch\ x) = \gamma(\gamma+2\lambda) P_\gamma^\lambda(ch\ x),$$

then taking into account selfadjoint of operator G (see [21], Lemma 4), we obtain for (3.5)

$$\begin{aligned}\left(\widehat{G_\lambda f}\right)_P(\gamma) &= \int_1^\infty P_\gamma^\lambda(ch\ x)\, G_\lambda f(ch\ x)\, sh^{2\lambda}\ x dx\\ &= \int_0^\infty f(ch\ x)\left(G_\lambda P_\gamma^\lambda(ch\ x)\right) sh^{2\lambda}\ x dx\\ &= \gamma(\gamma+2\lambda)\int_\lambda^\infty f(ch\ x)\, P_\gamma^\lambda(ch\ x)\, sh^{2\lambda} x dx = \gamma(\gamma+2\lambda)\, \hat{f}_P(\gamma).\end{aligned}$$

Obviously, by induction we have

$$\left(\widehat{G_\lambda^k f}\right)_P(\gamma) = (\gamma(\gamma+2\lambda))^k\, \hat{f}_p(\lambda), \quad k = 1, 2, \ldots.$$

This formula is naturally spread for the fractional indexes in the following form:

$$\left(\widehat{G_\lambda^{-\frac{\alpha}{2}} f}\right)_P(\gamma) := (\gamma(\gamma+2\lambda))^{-\frac{\alpha}{2}}\, \hat{f}_P(\lambda). \tag{3.8}$$

But then for (3.7) and (3.8) we have

$$\left(\widehat{I_G^\alpha f}\right)_p(\gamma) = (\gamma(\gamma+2\lambda))^{-\frac{\alpha}{2}}\, \hat{f}_p(\lambda). \tag{3.9}$$

Lemma 3.2. *Let $h_r(ch\ x)$ be the kernel associated with G_λ and $0 < \alpha < 2\lambda + 1$. Then*

$$I_G^\alpha f(ch\ t) = \frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(ch\ x)\, dr\right) A_{ch\ t}^\lambda f(ch\ x)\, sh^{2\lambda} x dx. \tag{3.10}$$

Proof. Let

$$\left(\hat{h}_r\right)_Q(\gamma) = e^{-\gamma(\gamma+2\lambda)r},$$

then from (3.3) it follows, that

$$h_r(ch\ x) = \int_1^\infty e^{-\gamma(\gamma+2\lambda)r} P_\gamma^\lambda(ch\ x) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma.$$

By Lemma 3.1

$$\int_0^\infty h_r(ch\ x)\, A_{ch\ x}^\lambda f(ch\ x)\, sh^{2\lambda}\ x dx = c_\lambda^* \int_1^\infty e^{-\gamma(\gamma+2\lambda)r} \left(\widehat{A_{ch\ t}^\lambda f}\right)_P(\gamma) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma.$$

Thus we have

$$\int_0^\infty \int_0^\infty r^{\frac{\alpha}{2}-1} h_r\,(ch\ x)\, A^\lambda_{ch\ t} f\,(ch\ x)\, sh^{2\lambda}\ xdxdr$$

$$= c^*_\lambda \int_1^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} e^{-\gamma(\gamma+2\lambda)r} dr\right) \left(\widehat{A^\lambda_{ch\ t} f}\right)_P(\gamma) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma$$

$$\left[\gamma\,(\gamma+2\lambda)\, r = t,\, dr = \frac{dt}{\gamma\,(\gamma+2\lambda)}\right]$$

$$= c^*_\lambda \int_1^\infty \left(\int_0^\infty e^{-t} t^{\frac{\alpha}{2}-1} dt\right) (\gamma\,(\gamma+2\lambda))^{-\frac{\alpha}{2}} \left(\widehat{A^\lambda_{ch\ t} f}\right)_P(\gamma) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma$$

$$= c^*_\lambda \Gamma\left(\frac{\alpha}{2}\right) \int_1^\infty (\gamma\,(\gamma+2\lambda))^{-\frac{\alpha}{2}} \left(\widehat{A^\lambda_{ch\ t} f}\right)_P(\gamma) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma.$$

Taking into account that (see [22], Lemma 1.2)

$$\left(\widehat{A^\lambda_{ch\ t} f}\right)_P(\gamma) = \hat{f}_P\,(\gamma)\, Q^\lambda_\gamma\,(ch\, t)$$

for (3.9) and (3.2) we obtain

$$\int_0^\infty \int_0^\infty r^{\frac{\alpha}{2}-1} h_r\,(ch\ x)\, A^\lambda_{ch\ t} f\,(ch\ x)\, sh^{2\lambda}\ xdxdr$$

$$= c^*_\lambda \Gamma\left(\frac{\alpha}{2}\right) \int_1^\infty (\gamma\,(\gamma+2\lambda))^{-\frac{\alpha}{2}}\ \hat{f}_P\,(\gamma)\, Q^\lambda_\gamma\,(ch\ t) \left(\gamma^2-1\right)^{\lambda-\frac{\alpha}{2}} d\gamma$$

$$= \Gamma\left(\frac{\alpha}{2}\right) \int_0^\infty \left(\widehat{I^\alpha_G f}\right)_P(\gamma)\, Q^\lambda_\gamma\,(ch\ t) \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\lambda = \Gamma\left(\frac{\alpha}{2}\right) I^\alpha_G f\,(ch\ t)\,,$$

from this and for (3.2) it follows, that

$$I^\alpha_G f\,(ch\ t) = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r\,(ch\ x)\, dr\right) A^\lambda_{ch\ t} f\,(ch\ x)\, sh^{2\lambda}\ xdx.$$

Thus Lemma 3.2 is proved. □

Corollary 3.1. *The following equality is true*

$$\left|I^\alpha_G f\,(ch\ t)\right| \lesssim \int_0^\infty \left|A^\lambda_{ch\ t} f\,(ch\ x)\right| (sh\ x)^{\alpha-2\lambda-1}\, sh^{2\lambda}\ xdx. \tag{3.11}$$

In fact from formula (see [28], p. 1933)

$$P^\lambda_\gamma\,(ch\ x) = \frac{\Gamma\,(\gamma+2\lambda)\cos\pi\lambda}{\Gamma\,(\gamma)\,\Gamma\,(\gamma+\lambda+1)} (2ch\ x)^{-\gamma-2\lambda} \times_2 F_1\left(\frac{\gamma}{2}+\lambda,\, \frac{\gamma}{2}+\lambda+\frac{1}{2};\, \gamma+\lambda+1;\, \frac{1}{ch^2\ x}\right)$$

we have

$$\left|P^\alpha_\gamma\,(ch\ x)\right| \lesssim (ch\ x)^{-\gamma-2\lambda}.$$

The function of Gauss ${}_2F_1\,(\alpha, \beta; \gamma; x)$ is convergent by appointed importance of parameters on the interval $[0, \infty)$ (see [29], p. 1054).

Taking into account the last inequality, we estimate from above $h_r\,(ch\ x)$

$$|h_r\,(ch\ x)| \lesssim \int_1^\infty e^{-\gamma(\gamma+2\lambda)r}\,(ch\ x)^{-\gamma-2\lambda} \left(\gamma^2-1\right)^{\lambda-\frac{1}{2}} d\gamma$$

$$\lesssim \int_0^\infty e^{-(\gamma+1)(\gamma+1+2\lambda)r} \gamma^{\lambda-\frac{1}{2}} (ch\ x)^{-\gamma-2\lambda-1} d\gamma$$

$$\lesssim e^{-r}\left(ch\ x\right)^{-2\lambda-1}\int_0^\infty \gamma^{\lambda-\frac{1}{2}}\left(ch\ x\right)^{-\gamma}d\gamma\left[\frac{1}{ch\ x}\le\frac{e}{e^{x+1}}\right]$$
$$\lesssim e^{-r}\left(ch\ x\right)^{-2\lambda-1}\int_0^\infty e^{-(x+1)\gamma}\gamma^{\lambda-\frac{1}{2}}d\gamma\left[(x+1)\gamma=u\right]$$
$$\lesssim e^{-r}\left(ch\ x\right)^{-2\lambda-1}\int_0^\infty e^{-u}u^{\lambda-\frac{1}{2}}du=\Gamma\left(\lambda+\frac{1}{2}\right)e^{-r}\left(ch\ x\right)^{-2\lambda-1}.$$

Hence we have

$$\int_0^\infty r^{\frac{\alpha}{2}-1}h_r\left(ch\ x\right)dr\lesssim\left(ch\ x\right)^{-2\lambda-1}\int_0^\infty r^{\frac{\alpha}{2}-1}e^{-r}dr$$
$$=\Gamma\left(\frac{\alpha}{2}\right)\left(ch\ x\right)^{-2\lambda-1}\le\Gamma\left(\frac{\alpha}{2}\right)\left(ch\ x\right)^{\alpha-2\lambda-1}\le\Gamma\left(\frac{\alpha}{2}\right)\left(sh\ x\right)^{\alpha-2\lambda-1}.$$

Taking into account this inequality on (3.10), we obtain our approval.

4. The Hardy–Littlewood–Sobolev theorem for G-Riesz potential

We consider the G-fractional integral

$$\Im_G^\alpha f(ch\ x)=\int_0^\infty A_{ch\ t}^\lambda(sh\ x)^{\alpha-2\lambda-1}f(ch\ t)sh^{2\lambda}\ tdt,\quad 0<\alpha<2\lambda+1.$$

We denote by $WL_{p,\lambda}[0,\ \infty)$ the weak $L_{p,\lambda}$ space of measurable functions f for which

$$\|f\|_{WL_{p,\lambda}[0,\ \infty)}=\sup_{t>0}t\ \left|\{x\in[0,\ \infty):|f\ (ch\,x)|>t\}\right|^{\frac{1}{p}}.$$

The next examples show that for $p\ge\frac{2\lambda+1}{\alpha}$ the integral $\Im_G^\alpha$ does not exist for $f\in L_{p,\lambda}[0,\ \infty)$.

Example 1. Let $x\in[0,\ \infty)$, $0<\alpha<2\lambda+1$, $f\ (x)=\frac{1}{sh^\alpha\ xln^2(sh\,x)}\chi_{\left(0,\ \frac{1}{2}\right)}(x)$. For $p=\frac{2\lambda+1}{\alpha}$ $f\in L_{p,\lambda}[0,\ \infty]$ and $\Im_G^\alpha f\ (x)=+\infty$.

In fact

$$\|f\|_{L_{p,\lambda}}=\int_0^{\frac{1}{2}}\frac{sh^{2\lambda}\ xdx}{(sh\,x)^{\alpha p}\left(ln^2sh\,x\right)^p}\le\int_0^{\frac{1}{2}}\frac{ch\,xdx}{sh\,x\left(ln^2sh\,x\right)^p}$$
$$=-\int_0^{\frac{1}{2}}(-lnsh\,x)^{-2p}\,d\,(-lnsh\,x)=-\frac{1}{1-2p}\left(-lnsh\,x\right)^{1-2p}\Big|_0^{\frac{1}{2}}$$
$$=\frac{1}{2p-1}\left|lnsh\,\frac{1}{2}\right|^{1-2p}.$$

On the other hand

$$\Im_G^\alpha f\ (x)=\int_0^\infty(sh\,t)^{\alpha-2\lambda-1}A_{ch\,t}f\ (x)\,sh^{2\lambda}\ tdt$$
$$=\int_0^\infty A_{ch\,t}\,(sh\,x)^{\alpha-2\lambda-1}\ f\ (t)\,sh^{2\lambda}\ tdt$$
$$=\int_0^\infty\frac{A_{ch\,t}\,(sh\,x)^{\alpha-2\lambda-1}\,sh^{2\lambda}\ tdt}{sh^\alpha\ tln^2\,(sh\,t)}.$$

Since

$$A_{ch\,t}\,(sh\,x)^{\alpha-2\lambda-1} = A_{ch\,t}\left(ch^2\,x-1\right)^{\frac{\alpha-2\lambda-1}{2}}$$
$$= \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\,\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi}\left((ch\,xch\,t - sh\,xsh\,t\cos\varphi)^2-1\right)^{\frac{\alpha-2\lambda-1}{2}}(\sin\varphi)^{2\lambda-1}\,d\varphi$$

(as $ch\;(x-t)\le ch\,xch\,t - sh\,xsh\,t\cos\varphi \le ch\;(x+t)$)

$$\ge \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\,\Gamma\frac{1}{2}}\int_0^{\pi}\left(ch^2\,(x+t)-1\right)^{\frac{\alpha-2\lambda-1}{2}}(\sin\varphi)^{2\lambda-1}\,d\varphi \ge (sh\;(x+t))^{\alpha-2\lambda-1}$$
$$= (sh\,xch\,t + ch\,xsh\,t)^{\alpha-2\lambda-1} \ge (2ch\,xch\,t)^{\alpha-2\lambda-1},$$

then for any fixed $x\in[0,\infty)$

$$\Im_G^{\alpha} f\,(x) \ge (2ch\,x)^{\alpha-2\lambda-1}\int_0^{\frac{1}{2}}\frac{sh^{2\lambda}\,tdt}{(ch\,t)^{2\lambda+1-\alpha}\,sh^{\alpha}\,tln^2\,(sh\,t)}$$
$$\ge \frac{4^{\alpha-2\lambda-1}}{\left(ch\,\frac{1}{2}\right)^{2\lambda+1}}(ch\,x)^{\alpha-2\lambda-1}\int_0^{\frac{1}{2}}\frac{ch^{\alpha}\,tsh^{2\lambda}\,tdt}{sh^{\alpha}\,tln^2\,(sh\,t)}$$
$$\ge c_{\alpha,\lambda}\,(ch\,x)^{\alpha-2\lambda-1}\int_0^{\frac{1}{2}}\frac{sh^{2\lambda}\,tdt}{ln^2\,(sh\,t)}$$
$$\ge c_{\alpha,\,\lambda}\,(ch\,x)^{\alpha-2\lambda-1}\int_0^{\frac{1}{2}}\frac{sh^{2\lambda}\,tdt}{sh^2\,t} \ge c_{\alpha,\,\lambda}\,(ch\,x)^{\alpha-2\lambda-1}\int_0^{\frac{1}{2}} t^{2\lambda-2}dt = +\infty.$$

At the end we use the inequality (1.3).

Example 2. Let $x\in[0,\,\infty)$, $0<\alpha<2\lambda+1$, $f\,(x) = \frac{1}{sh^{\alpha}\,x}\chi_{(2,\,\infty)}\,(x)$. For $p>\frac{2\lambda+1}{\alpha}$ $f\in L_{p,\,\lambda}\,[0,\,\infty)$ and $\Im_G^{\alpha} f\,(x) = +\infty$.

In fact

$$\|f\|_{L_{p,\,\lambda}} = \int_2^{\infty}\frac{sh^{2\lambda}\,xdx}{(sh\,x)^{\alpha p}} \le \int_2^{\infty}\frac{ch\,xsh^{2\lambda}\,xdx}{(sh\,x)^{\alpha p}}$$
$$= \int_2^{\infty}\frac{sh^{2\lambda}\,xd\,(sh\,x)}{(sh\,x)^{\alpha p}} = \left.\frac{(sh\,x)^{2\lambda+1-\alpha p}}{2\lambda+1-\alpha p}\right|_2^{\infty} = \frac{(sh\,2)^{2\lambda+1-\alpha p}}{\alpha p-2\lambda-1}.$$

On other hand for any fixed $x\in[0,\,\infty)$

$$\Im_G^{\alpha} f\,(x) \ge (2ch\,x)^{\alpha-2\lambda-1}\int_2^{\infty}\frac{sh^{2\lambda}\,tdt}{(ch\,t)^{2\lambda+1-\alpha}\,sh^{\alpha}\,t}$$

(as, $\frac{1}{2}ch\,t < sh\,t < ch\,t,\;\; t\ge 2$)

$$\ge 2^{2\alpha-4\lambda-2}\,(ch\,x)^{\alpha-2\lambda-1}\int_2^{\infty}\frac{dt}{sh\,t} \ge 4^{\alpha-2\lambda-2}\,(ch\,x)^{\alpha-2\lambda-1}\int_2^{\infty}\frac{dt}{ch\,t}$$
$$= 2^{2\alpha-4\lambda-2}\,(ch\,x)^{\alpha-2\lambda-1}\int_2^{\infty}\frac{e^t\,dt}{e^{2t}+1} = 2^{2\alpha-4\lambda-1}\,(ch\,x)^{\alpha-2\lambda-1}\left.\mathrm{arct}ge^t\right|_2^{\infty} = +\infty.$$

For the G-Riesz potential the following Hardy–Littlewood–Sobolev theorem is valid.

Theorem 4.1. *Let* $1-2\lambda<\alpha<1+2\lambda$, $1\le p<\frac{2\lambda+1}{\alpha}$ *and* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\lambda+1}$.

(a) *If* $f\in L_{p,\lambda}[0,\infty)$, *then the integral* $\Im_G^\alpha f$ *is convergent absolutely for almost every* $x\in[0,\infty)$.

(b) *If* $1<p<\frac{2\lambda+1}{\alpha}$, $f\in L_{p,\lambda}[0,\infty)$, *then* $\Im_G^\alpha f\in L_{q,\lambda}[0,\infty)$ *and*

$$\|I_G^\alpha f\|_{L_{q,\lambda}[0,\infty)}\lesssim\left\|\Im_G^\alpha f\right\|_{L_{q,\lambda}[0,\infty)}\lesssim\|f\|_{L_{p,\lambda}[0,\infty)},$$

(c) *If* $f\in L_{1,\lambda}[0,\infty)$, $\frac{1}{q}=1-\frac{\alpha}{2\lambda+1}$, *then*

$$\|I_G^\alpha f\|_{WL_{q,\lambda}[0,\infty)}\lesssim\|\Im_G^\alpha f\|_{WL_{q,\lambda}[0,\infty)}\lesssim\|f\|_{L_{1,\lambda}}[0,\infty).\tag{4.1}$$

Proof. Let $f\in L_{p,\lambda}[0,\infty)$, $1\le p<\frac{2\lambda+1}{\alpha}$, $f_1(ch\ x)=f(ch\ x)\chi_{(0,1)}(x)$,

$$f_2(ch\ x)=f(ch\ x)-f_1(ch\ x),\qquad \chi_{(0,1)}(x)=\begin{cases}1, & x\in(0,1),\\ 0, & x\in(1,\infty).\end{cases}$$

Then

$$\Im_G^\alpha f(ch\ x)=\Im_G^\alpha f_1(ch\ x)+\Im_G^\alpha f_2(ch\ x)=\Im_1(ch\ x)+\Im_2(ch\ x).$$

We estimate above the $\Im_1(ch\ x)$.

$$\begin{aligned}|\Im_1(ch\ x)|&\le\int_0^1(sh\ x)^{\alpha-2\lambda-1}A_{ch\ t}^\lambda|f(ch\ x)|\,sh^{2\lambda}\ tdt\\&=\int_0^\infty(sh\ x)^{\alpha-2\lambda-1}\chi_{(0,1)}(t)A_{ch\ t}^\lambda|f(ch\ x)|\,sh^{2\lambda}\ tdt.\end{aligned}$$

By Young inequality (see [22], Lemma 4) we have

$$\|\Im_1(ch)(\cdot)\|_{L_{p,\lambda}[0,\infty)}\le\|f(ch)(\cdot)\|_{L_{p,\lambda}[0,\infty)}\cdot\left\||\cdot|^{\alpha-2\lambda-1}\chi_{(0,1)}\right\|_{L_{1,\lambda}[0,\infty)}.\tag{4.2}$$

Here

$$\begin{aligned}\left\||\cdot|^{\alpha-2\lambda-1}\chi_{(0,1)}\right\|_{L_{1,\lambda}}&=\int_0^1(sh\ t)^{\alpha-2\lambda-1}sh^{2\lambda}\ tdt\\&\le\int_0^1(sh\ t)^{\alpha-1}ch\ tdt=\int_0^1(sh\ t)^{\alpha-1}d(sh\ t)=\frac{1}{\alpha}sh^\alpha 1.\end{aligned}$$

From (4.1) and (4.2) it follows, that $\Im_1(ch\ x)$ for almost every $x\in[0,\infty)$ is convergent absolutely.

By using the Hölder inequality

$$\begin{aligned}|\Im_2(ch\ x)|&\le\int_1^\infty(sh\ t)^{\alpha-2\lambda-1}A_{ch\ t}^\lambda|f(ch\ x)|\,sh^{2\lambda}\ tdt\\&\le\left\|A_{ch\ t}^\lambda f\right\|_{L_{p,\lambda}}\cdot\left(\int_1^\infty(sh\ t)^{(\alpha-2\lambda-1)q}\,sh^{2\lambda}\ tdt\right)^{\frac{1}{q}}\\&\le\|f\|_{L_{p,\lambda}}\left(\int_1^\infty(sh\ t)^{(\alpha-2\lambda-1)q+2\lambda}\,ch\ tdt\right)^{\frac{1}{q}}\\&=\|f\|_{L_{p,\lambda}}\left(\int_1^\infty(sh\ t)^{(\alpha-2\lambda-1)q+2\lambda}\,d(sh\ t)\right)^{\frac{1}{q}}\\&=\left(\frac{(sh1)^{(\alpha-2\lambda-1)q+2\lambda+1}}{(2\lambda+1-\alpha)\,q-2\lambda-1}\right)^{\frac{1}{q}}\cdot\|f\|_{L_{p,\lambda}}=c_{\alpha,\lambda,p}\,\|f\|_{L_{p,\lambda}},\end{aligned}$$

from this it follows the absolutely convergence of $\Im_2(ch\ x)$ for almost every $x\in[0,\infty)$.

Thus, for all $f\in L_{p,\lambda}[0,\infty)$, $1\le p<\frac{2\lambda+1}{\alpha}$, G-Riesz potential $\Im_G^\alpha f(ch\ x)$ is convergent absolutely for almost every $x\in[0,\infty)$.

(b) We have

$$\Im_G^\alpha f(ch\ x) = \left(\int_0^r + \int_r^\infty\right) A_{ch\ t}^\lambda f(ch\ x)(sh\ t)^{\alpha-2\lambda-1} sh^{2\lambda}\ tdt = A_1(x,r) + A_2(x,r). \tag{4.3}$$

We consider $A_1(x,r)$. Let $0 < r < 2$.

$$\begin{aligned}
|A_1(x,r)| &\le \int_0^r \left|A_{ch\ t}^\lambda f(ch\ x)\right| (sh\ t)^{2\lambda}(sh\ t)^{\alpha-2\lambda-1}dt \\
&\le \sum_{k=0}^\infty \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A_{ch\ t}^\lambda |f(ch\ x)|\, sh^{2\lambda}\ tdt}{(sh\ t)^{2\lambda+1-\alpha}} \\
&\le \sum_{k=0}^\infty \left(sh\frac{r}{2^{k+1}}\right)^\alpha \left(sh\frac{r}{2^{k+1}}\right)^{-2\lambda-1} \int_0^{\frac{r}{2^k}} A_{ch\ t}^\lambda |f(ch\ x)|\, sh^{2\lambda}\ tdt \\
&\lesssim M_{G,1} f\ (ch\, x) \sum_{k=1}^\infty \left(\frac{1}{2^k} sh\,\frac{r}{2}\right)^\alpha \lesssim \left(sh\,\frac{r}{2}\right)^\alpha M_{G,1} f\ (ch\, x) \sum_{k=1}^\infty \frac{1}{2^{k\alpha}} \\
&\lesssim \left(sh\ \frac{r}{2}\right)^\alpha M_{G,1} f(ch\ x),
\end{aligned} \tag{4.4}$$

as $sh\,\frac{t}{a} \le \frac{1}{a} sh\, t$ for $a \ge 1$.

We consider $A_2(x,r)$. By Hölder inequality

$$\begin{aligned}
|A_2(x,r)| &\le \left(\int_r^\infty \left|A_{ch\ t}^\lambda f(ch\ x)\right|^p sh^{2\lambda}\ tdt\right)^{\frac{1}{p}} \left(\int_r^\infty (sh\ t)^{(\alpha-2\lambda-1)q} sh^{2\lambda}\ tdt\right)^{\frac{1}{q}} \\
&\le \left\|A_{ch\ t}^\lambda f\right\|_{L_{p,\lambda}} \left(\int_{r/2}^\infty (sh\ t)^{(\alpha-2\lambda-1)q+2\lambda} ch\ tdt\right)^{\frac{1}{q}} \\
&\le \|f\|_{L_{p,\lambda}} \left(\frac{\left(sh\ \frac{r}{2}\right)^{(\alpha-2\lambda-1)q+2\lambda+1}}{(2\lambda+1-\alpha)\,q - 2\lambda - 1}\right)^{\frac{1}{q}} < \|f\|_{L_{p,\lambda}} \left(sh\ \frac{r}{2}\right)^{\alpha-2\lambda-1+\frac{2\lambda+1}{q}} \\
&= \|f\|_{L_{p,\lambda}} \left(sh\ \frac{r}{2}\right)^{\alpha-2\lambda-1+(2\lambda+1)\left(\frac{1}{p}-\frac{\alpha}{2\lambda+1}\right)} \\
&= \|f\|_{L_{p,\lambda}} \left(sh\ \frac{r}{2}\right)^{(2\lambda+1)\left(\frac{1}{p}-1\right)} = \|f\|_{L_{p,\lambda}} \left(sh\ \frac{r}{2}\right)^{-\frac{2\lambda+1}{q}}.
\end{aligned} \tag{4.5}$$

Taking into account (4.4) and (4.5) in (4.3), we obtain

$$\left|\Im_G^\alpha f(ch\ x)\right| \le \left(\left(sh\ \frac{r}{2}\right)^\alpha M_{G,1} f(ch\ x) + \left(sh\ \frac{r}{2}\right)^{-\frac{2\lambda+1}{q}} \|f\|_{L_{p,\lambda}}\right). \tag{4.6}$$

Minimum of the right-hand side of the inequality (4.6) reaches to

$$sh\ \frac{r}{2} = \left(\frac{2\lambda+1}{\alpha q} \cdot \frac{\|f\|_{L_{p,\lambda}}}{M_{G,1} f(ch\ x)}\right)^{\frac{p}{2\lambda+1}}.$$

Then from (4.6) we have

$$\begin{aligned}
\left|\Im_G^\alpha f(ch\ x)\right| &\le \left\{\left(\frac{\|f\|_{L_{p,\lambda}}}{M_{G,1} f(ch\ x)}\right)^{\frac{\alpha p}{2\lambda+1}} M_{G,1} f(ch\ x) + \left(\frac{\|f\|_{L_{p,\lambda}}}{M_{G,1} f(ch\ x)}\right)^{-\frac{p}{q}} \|f\|_{L_{p,\lambda}}\right\} \\
&= \left(M_{G,1} f(ch\ x)\right)^{\frac{p}{q}} \|f\|_{L_{p,\lambda}}^{1-\frac{p}{q}},
\end{aligned}$$

for the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1} \Rightarrow 1 - \frac{p}{q} = \frac{\alpha p}{2\lambda+1}$.

From this we have

$$\int_0^\infty \left|\Im_G^\alpha f(ch\ t)\right|^q sh^{2\lambda}\ tdt \le \left\|M_{G,1} f(ch(\cdot))\right\|_{L_{p,\lambda}}^p \cdot \|f\|_{L_{p,\lambda}}^{q-p} \le \|f\|_{L_{p,\lambda}}^{q-p} \cdot \|f\|_{L_{p,\lambda}}^p = \|f\|_{L_{p,\lambda}}^q,$$

from this it follows that for $0 < r < 2$

$$\|I_G^\alpha\|_{L_{q,\lambda}} \lesssim \left\|\Im_G^\alpha f\right\|_{L_{q,\lambda}} \lesssim \|f\|_{L_{p,\lambda}}. \tag{4.7}$$

Now let $2 \le r < \infty$. Then from (4.3) and by Lemma 1.1 we have

$$\begin{aligned}
|A_1(x, r)| &\le \int_0^r \frac{A_{ch\,t}^\lambda |f(ch\,x)|sh^{2\lambda}\,tdt}{(sh\,t)^{2\lambda+1-\alpha}} = \sum_{k=0}^\infty \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A_{ch\,t}^\lambda\,|f(ch\,t)|\,sh^{2\lambda}\,tdt}{(sh\,t)^{2\lambda+1-\alpha}} \\
&\le \sum_{k=0}^\infty \left(sh\,\frac{r}{2^{k+1}}\right)^{\alpha+2\lambda-1} \left(sh\,\frac{r}{2^{k+1}}\right)^{-4\lambda} \int_0^{\frac{r}{2^k}} A_{ch\,t}^\lambda |f(ch\,x)|sh^{2\lambda}\,tdt \\
&\lesssim \sum_{k=0}^\infty \left(sh\,\frac{r}{2^{k+1}}\right)^{\alpha+2\lambda-1} \left(ch\,\frac{r}{2^{k+1}}\right)^{-4\lambda} \int_0^{\frac{r}{2^k}} A_{ch\,t}^\lambda\,|f(ch\,x)|\,sh^{2\lambda}\,tdt \\
&\lesssim M_{G,2} f(ch\,x) \left(sh\,\frac{r}{2}\right)^{\alpha+2\lambda-1} \sum_{k=1}^\infty \frac{1}{2^{(\alpha+2\lambda-1)k}} \lesssim \left(sh\,\frac{r}{2}\right)^{\alpha+2\lambda-1} M_{G,2} f(ch\,x).
\end{aligned} \tag{4.8}$$

Taking into account Hölder inequality from (4.3) we obtain

$$\begin{aligned}
|A_1(x, r)| &\le \|f\|_{L_{p,\lambda}} \left(\int_r^\infty (sh\,t)^{(\alpha-2\lambda-1)q}\,sh^{2\lambda}\,tdt\right)^{\frac{1}{q}} \\
&\lesssim \|f\|_{L_{p,\lambda}} \left(\int_r^\infty \frac{\left(sh\,\frac{t}{2}\right)^{(\alpha-2\lambda-1)q}}{\left(ch\,\frac{t}{2}\right)^{(2\lambda+1-\alpha)q}} \frac{sh^{2\lambda}\,\frac{t}{2} d\left(sh\,\frac{t}{2}\right)}{\left(ch\,\frac{t}{2}\right)^{1-2\lambda}}\right)^{\frac{1}{q}} \\
&\lesssim \|f\|_{L_{p,\lambda}} \left(\int_r^\infty \left(sh\,\frac{t}{2}\right)^{(\alpha-2\lambda-1)q} \left(sh\,\frac{t}{2}\right)^{4\lambda-1} d\left(sh\,\frac{t}{2}\right)\right)^{\frac{1}{r}} \\
&\lesssim \|f\|_{L_{p,\lambda}} \left(sh\,\frac{r}{2}\right)^{\alpha-2\lambda-1+\frac{4\lambda}{q}}.
\end{aligned} \tag{4.9}$$

Now from (4.3), (4.8) and (4.9) we have

$$|\Im_G^\alpha f(ch\,x)| \lesssim \left(\left(sh\,\frac{r}{2}\right)^{\alpha+2\lambda-1} M_{G,2} f(ch\,x) + \left(sh\,\frac{r}{2}\right)^{\alpha-2\lambda-1+\frac{4\lambda}{q}} \|f\|_{L_{p,\lambda}}\right). \tag{4.10}$$

Minimum of the right-hand side of the inequality (4.10) reaches to

$$sh\,\frac{r}{2} = \left(\frac{(2\lambda+1-\alpha)\,q - 4\lambda}{(\alpha+2\lambda-1)\,q} \frac{\|f\|_{L_{p,\lambda}}}{M_{G,2} f(ch\,x)}\right)^{\frac{p}{4\lambda}}.$$

Then from (4.10) we have

$$\begin{aligned}
|\Im_G^\alpha f(ch\,x)| &\lesssim \left\{\left(\frac{\|f\|_{L_{p,\lambda}}}{M_{G,2} f(ch\,x)}\right)^{\frac{(\alpha+2\lambda-1)p}{4\lambda}} M_{G,2} f(ch\,x) + \left(\frac{\|f\|_{L_{p,\lambda}}}{M_{G,2} f(ch\,x)}\right)^{\frac{(\alpha+2\lambda-1)p-4\lambda}{4\lambda}} \|f\|_{L_{p,\lambda}}\right\} \\
&= \left(M_{G,2} f(ch\,x)\right)^{\frac{(1+2\lambda-\alpha)p+4\lambda}{4\lambda}} \|f\|_{L_{p,\lambda}}^{\frac{(\alpha+2\lambda-1)p}{4\lambda}}.
\end{aligned}$$

From this we obtain

$$\int_0^\infty |\Im_G^\alpha f(ch\,x)|^q sh^{2\lambda}\,xdx \le \|M_G f\|^{\frac{(1-2\lambda-\alpha)q}{4\lambda}+q} \|f\|_{L_{p,\lambda}}^{\frac{(\alpha+2\lambda-1)p}{4\lambda}} \lesssim \|f\|_{L_{p,\lambda}}^{q+\frac{(1-2\lambda-\alpha)q}{4\lambda}} \|f\|_{L_{p,\lambda}}^{\frac{(\alpha+2\lambda-1)q}{4\lambda}} = \|f\|_{L_{p,\lambda}}^q.$$

From this it follows that for $2 \le r < \infty$

$$\|I_G^{\alpha} f\|_{L_{q,\lambda}} \lesssim \|\mathfrak{F}_G^{\alpha} f\|_{L_{q,\lambda}} \lesssim \|f\|_{L_{p,\lambda}}.$$

Combining last inequality and (4.7) we obtain the approval (b).

c Let $f \in L_{1,\lambda}[0,\infty)$ and $0 < r < 2$. By (4.3) we get

$$\left|\left\{x : |\Im_G^{\alpha} f(ch\ x)| > 2\beta\right\}\right|_{\lambda} \le |\{x : |A_1(x,r)| > \beta\}|_{\lambda} + |\{x : |A_2(x,r)| > \beta\}|_{\lambda}.$$

From inequality (4.4) and Theorem 2.2 we have

$$\begin{aligned} \beta\,|\{x \in [0,\infty) : |A_1(x,r)| > \beta\}|_{\lambda} &= \beta \int_{\{x\in[0,\infty):|A_1(x,r)|>\beta\}} sh^{2\lambda}\ x dx \\ &\le \beta \int_{\{x\in[0,\infty):c_{\alpha,\lambda}\left(sh^{\alpha}\ \frac{r}{2}\right)M_G f(ch\ x)>\beta\}} sh^{2\lambda}\ x dx \\ &= \beta \left|\left\{x \in [0,\infty) : M_G f(ch\ x) > \frac{\beta}{sh^{\alpha}\ \frac{r}{2}}\right\}\right|_{\lambda} \\ &\le \beta \cdot \frac{c_{\lambda}}{\beta} sh^{\alpha}\ \frac{r}{2} \int_0^{\infty} |f(ch\ x)|\, sh^{2\lambda}\ x dx = c_{\lambda} sh^{\alpha}\ \frac{r}{2}\, \|f\|_{L_{1,\lambda}}, \end{aligned}$$

and also

$$\begin{aligned} |A_2(x,r)| &\le \int_r^{\infty} \left|A_{ch\ t}^{\lambda} f(ch\ x)\right| (sh\ t)^{\alpha-2\lambda-1} sh^{2\lambda}\ t dt \\ &\le \int_r^{\infty} \frac{\left|A_{ch\ t}^{\lambda} f(ch\ x)\right| sh^{2\lambda}\ t dt}{(sh\ t)^{2\lambda+1-\alpha}} \le \int_r^{\infty} \frac{\left|A_{ch\ t}^{\lambda} f(ch\ x)\right| sh^{2\lambda}\ t dt}{\left(sh\ \frac{t}{2}\right)^{2\lambda+1-\alpha}} \\ &\le \left(sh\ \frac{r}{2}\right)^{\alpha-2\lambda-1} \int_r^{\infty} \left|A_{ch\ t}^{\lambda} f(ch\ x)\right| sh^{2\lambda}\ t dt \le \left(sh\ \frac{r}{2}\right)^{-\frac{2\lambda+1}{q}} \|f\|_{L_{1,\lambda}}. \end{aligned}$$

Suppose $\left(sh\ \frac{r}{2}\right)^{-\frac{2\lambda+1}{q}} \|f\|_{L_{1,\lambda}} = \beta$, we obtain $|A_2(x,r)| \le \beta$ and consequently $|\{x \in [0,\infty) : |A_2(x,r)| > \beta\}|_{\lambda} = 0$.

Finally we get

$$\begin{aligned} &\left|\left\{x \in [0,\infty) : \left|\Im_G^{\alpha} f(ch\ x)\right| > \beta\right\}\right|_{\lambda} \le c_{\alpha,\lambda} \cdot \frac{1}{\beta} sh^{\alpha}\ \frac{r}{2}\, \|f\|_{L_{1,\lambda}} \\ &\le c_{\lambda} \left(sh\ \frac{r}{2}\right)^{\alpha+\frac{2\lambda+1}{q}} = c_{\lambda} \left(sh\ \frac{r}{2}\right)^{2\lambda+1} = c_{\lambda} \left(\frac{1}{\beta} \|f\|_{L_{1,\lambda}}\right)^{q}. \end{aligned}$$

From this and (3.11) for $0 < r < 2$ it follows (4.1). Now we consider the case $2 \le r < \infty$. From the inequality (4.8) and Theorem 2.2 we have

$$\begin{aligned} |\{x \in [0,\ \infty) : |A_1\,(x,\ r) > \beta|\}|_{\lambda} &= \beta \int_{\{x\in[0,\ \infty):A_1(x,r)>\beta\}} sh^{2\lambda}\, x dx \\ &\le \beta \int_{\{x\in[0,\ \infty):\left(sh\,\frac{r}{2}\right)^{\alpha+2\lambda-1} M_G f(ch\, x)>\beta\}} \\ &= \beta \left|\left\{x \in [0,\ \infty) : M_G f\,(ch\, x) > \frac{\beta}{\left(sh\,\frac{r}{2}\right)^{\alpha+2\lambda-1}}\right\}\right|_{\lambda} \\ &\le c_{\lambda} \cdot \beta \cdot \frac{1}{\beta} \left(sh\,\frac{r}{2}\right)^{\alpha+2\lambda-1} \int_0^{\infty} |f\,(ch\, x)\,| sh^{2\lambda}\, x dx \\ &= c_{\lambda} \left(sh\,\frac{r}{2}\right)^{\alpha+2\lambda-1} \|f\|_{L_{1,\lambda}}. \end{aligned}$$

On the other hand

$$|A_2(x, r)| \le \int_r^\infty \frac{|A_{ch\,t} f(ch\,x)\, sh^{2\lambda}\, tdt|}{(sh\,t)^{2\lambda+1-\alpha}} \le \int_r^\infty \frac{A_{ch\,t}|f(ch\,x)|sh^{2\lambda}\, tdt}{(sh\,t)^{2\lambda+1-\alpha}}$$
$$\le \int_r^\infty \frac{A_{ch\,t}|f(ch\,x)|sh^{2\lambda}\, tdt}{\left(sh\,\frac{t}{2}\right)^{\frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha q}}} \le \left(sh\,\frac{r}{2}\right)^{-\frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha q}} \|f\|_{L_{1,\lambda}},$$

since

$$2\lambda+1-\alpha > \frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha q} = \frac{(\alpha+2\lambda-1)(2\lambda+1)(2\lambda+1-\alpha)}{\alpha(2\lambda+1)}$$
$$= \frac{(\alpha+2\lambda-1)(2\lambda+1-\alpha)}{\alpha} \Leftrightarrow \alpha > \alpha+2\lambda-1 \Leftrightarrow \lambda < \frac{1}{2}.$$

Putting $\left(sh\,\frac{r}{2}\right)^{\frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha q}} \|f\|_{L_{1,\lambda}} = \beta$, we obtain $|A_2(x, r)| \le \beta$ and consequently

$$|\{x \in [0, \infty) : |A_2(x, r)| > \beta\}|_\lambda = 0.$$

Now by Theorem 2.2 we have

$$|\{x \in [0, \infty) : |\Im_G^\alpha f(ch\,x)| > 2\beta\}|_\lambda \le c_\lambda \frac{1}{\beta}\left(sh\,\frac{r}{2}\right)^{\alpha+2\lambda-1} \|f\|_{L_{1,\lambda}}$$
$$\le c_\lambda \left(sh\,\frac{r}{2}\right)^{\alpha+2\lambda-1+\frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha q}} \|f\|_{L_{1,\lambda}}$$
$$= c_\lambda \left(sh\,\frac{r}{2}\right)^{\frac{(\alpha+2\lambda-1)(2\lambda+1)}{\alpha}} \|f\|_{L_{1,\lambda}} \le c_\lambda \left(\frac{1}{\beta}\|f\|_{L_{1,\lambda}}\right)^q.$$

From this and (3.11) for $2 \le r < \infty$ it follows (4.10).
Thus, $f \mapsto \Im_G^\alpha f$ is weak type $(1, q)$ and Theorem 4.1 is proved. □

Definition 4.1 (*[14]*)**.** We denote by $BMO_G[0, \infty)$ the BMO-Gegenbauer space (G-BMO space) as the set of functions locally integrable on $[0, \infty)$, with finite norm

$$\|f\|_{BMO_G} = \sup_{x,\, r\in(0,\infty)} \frac{1}{|H(0, r)|_\lambda} \int_{H(0,r)} \left|A_{ch\,t}^\lambda f(ch\,x) - f_{H(0,r)}(ch\,x)\right| sh^{2\lambda}\, tdt,$$

where

$$f_{H(0,r)}(ch\,x) = \frac{1}{|H(0, r)|_\lambda} \int_{H(0,r)} A_{ch\,t}^\lambda\, |f(ch\,x)|\, sh^{2\lambda}\, tdt.$$

As an analogue of [14] we introduce modified fractional Riesz–Gegenbauer integral (G-Riesz integral) by

$$\widetilde{\Im}_G^\alpha f(ch\,x) = \int_0^\infty (A_{ch\,t}(sh\,x)^{\alpha-2\lambda-1} - (sh\,t)^{\alpha-2\lambda-1}\chi_{(\frac{1}{4},\infty)}(ch\,t)) f(ch\,t) sh^{2\lambda}\, tdt$$

and modified Riesz–Gegenbauer potential (G-Riesz potential) by

$$\widetilde{I}_G^\alpha f(ch\,x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(ch\,t)\, dr\right)$$
$$\times (A_{ch\,t}(sh\,x)^{\alpha-2\lambda-1} - (sh\,t)^{\alpha-2\lambda-1}\chi_{(\frac{1}{4},\infty)}(ch\,t)) f(ch\,t) sh^{2\lambda}\, tdt,$$

where $\chi_{\left(\frac{1}{4},\infty\right)}(ch\,t)$ is a characteristic of function of the interval $\left(\frac{1}{4}, \infty\right)$. Also in the proof of inequality (3.11) we obtain that

$$|\widetilde{I}_G^\alpha f(ch\,x)| \lesssim \int_0^\infty \left|A_{ch\,t}(sh\,x)^{\alpha-2\lambda-1} - (sh\,t)^{\alpha-2\lambda-1}\chi_{\left(\frac{1}{4},\infty\right)}(ch\,t)\right| |f(ch\,t)| sh^{2\lambda}\, tdt,$$

from this we have

$$\|\widetilde{I}_G^{\alpha} f\|_{BMO_G} \lesssim \|\widetilde{\Im}_G^{\alpha} f\|_{BMO_G}. \tag{4.11}$$

The following theorem is valid.

Theorem 4.2. *Let* $1-2\lambda < \alpha < 2\lambda+1$, $p\alpha = 2\lambda+1$ *and* $f \in L_{p,\lambda}[0,\infty)$.
Then $\widetilde{\Im}_G^{\alpha} f \in BMO_G[0,\infty)$ *and the inequality*

$$\left\|\widetilde{\Im}_G^{\alpha} f\right\|_{BMO_G} \lesssim \|f\|_{L_{p,\lambda}}$$

is valid.

Proof. Suppose that

$$f_1(ch\ x) = f(ch\ x)\,\chi_{(0,r/4)}(ch\ x), \qquad f_2(ch\ x) = f(ch\ x) - f_1(ch\ x),$$

where $\chi_{(0,r/4)}(ch\ x)$ is the characteristic function of the interval $\left(0,\ \frac{r}{4}\right)$, that is,

$$\chi_{(0,r/4)}(ch\ x) = \begin{cases} 1, & 0 < x < \dfrac{r}{4} \\ 0, & x > \dfrac{r}{4}. \end{cases}$$

Then

$$\widetilde{\Im}_G^{\alpha} f(ch\ x) = \widetilde{\Im}_G^{\alpha} f_1(ch\ x) + \widetilde{\Im}_G^{\alpha} f_2(ch\ x) = F_1(ch\ x) + F_2(ch\ x),$$

where

$$F_1(ch\ x) = \int_0^{r/4} \left(A_{ch\ t}^{\lambda}(sh\ x)^{\alpha-2\lambda-1} - (sh\ t)^{\alpha-2\lambda-1}\chi_{\left(\frac{1}{4},\infty\right)}(ch\ t)\right) f(ch\ t)\, sh^{2\lambda}\ t dt$$

and

$$F_2(ch\ x) = \int_{r/4}^{\infty} \left(A_{ch\ t}^{\lambda}(sh\ x)^{\alpha-2\lambda-1} - (sh\ t)^{\alpha-2\lambda-1}\chi_{\left(\frac{1}{4},\infty\right)}(ch\ t)\right) f(ch\ t)\, sh^{2\lambda}\ t dt.$$

Since the function $f_1(ch\ x)$ has compact support, then the number

$$a_1 = -\int_{(0,r/4)/\left(0,\min\left\{\frac{1}{4},\frac{r}{4}\right\}\right)} (sh\ t)^{\alpha-2\lambda-1} f(ch\ t)\, sh^{2\lambda}\ t dt$$

is finite. We can write

$$\begin{aligned} F_1(ch\ x) - a_1 &= \int_0^{r/4} A_{ch\ t}^{\lambda}(sh\ x)^{\alpha-2\lambda-1} f(ch\ x) sh^{2\lambda}\ t dt \\ &\quad - \int_{(0,r/4)/\left(0,\min\left\{\frac{1}{4},\frac{r}{4}\right\}\right)} (sh\ t)^{\alpha-2\lambda-1} f(ch\ t)\, sh^{2\lambda}\ t dt \\ &\quad + \int_{(0,r/4)\left(0,\min\left\{\frac{1}{4},\frac{r}{4}\right\}\right)} (sh\ t)^{\alpha-2\lambda-1} f(ch\ t)\, sh^{2\lambda}\ t dt \\ &= \int_0^{r/4} A_{ch\ t}^{\lambda}(sh\ x)^{\alpha-2\lambda-1} f(ch\ t)\, sh^{2\lambda}\ t dt \\ &= \int_0^{\infty} A_{ch\ t}^{\lambda}(sh\ x)^{\alpha-2\lambda-1} f_1(ch\ t)\, sh^{2\lambda}\ t dt. \end{aligned} \tag{4.12}$$

Consider the integral

$$\begin{aligned} A_{ch\ t}^{\lambda} f_1(ch\ x) = c_\lambda \int_0^{\pi} & f(ch\ xch\ t - sh\ xsh\ t\cos\ \varphi) \\ & \times \chi_{(0,r/4)}(ch\ xch\ t - sh\ xsh\ t\cos\ \varphi)(\sin\ \varphi)^{2\lambda-1} d\varphi. \end{aligned}$$

So far as, $ch\ (x-t) \le ch\ xch\ t - sh\ xsh\ t\cos\ \varphi \le ch\ (x+t)$, then for $|x-t| > \frac{r}{4}$

$$\chi_{(0,r/4)}\left(ch\ xch\ t - sh\ xsh\ t\cos\ \varphi\right) = 0,$$

and then

$$\begin{aligned} A^{\lambda}_{ch\ t} f_1\left(ch\ x\right) &= c_{\lambda} \int_{\{\varphi\in[0,\pi],|x-t|\le r/4\}} f\left(ch\ xch\ t - sh\ xsh\ t\cos\ \varphi\right)\left(\sin\ \varphi\right)^{2\lambda-1} d\varphi \\ &= A^{\lambda}_{ch\ t} f\left(ch\ x\right). \end{aligned}$$

Then for (4.12) we have

$$\left|F_1\left(ch\ x\right) - a_1\right| \le \int_{\{t\in[0,\infty);|x-t|\le r/4\}} \left(sh\ t\right)^{\alpha-2\lambda-1} A^{\lambda}_{ch\ t}\left|f\left(ch\ x\right)\right| sh^{2\lambda}\ tdt. \tag{4.13}$$

We consider the estimation (4.13).

Let $\left(x-\frac{r}{4}, x+\frac{r}{4}\right)\cap[0,\infty) = \left(0, x+\frac{r}{4}\right)$, then $0 \le x \le r/4$ and we have from (4.13) and (4.14)

$$\begin{aligned} \left|F_1\left(ch\ x\right) - a_1\right| &\le \int_0^{x+r/4} \left(sh\ t\right)^{\alpha-2\lambda-1} A^{\lambda}_{ch\ t}\left|f\left(ch\ x\right)\right| sh^{2\lambda}\ tdt \\ &\le \int_0^{r} \left(sh\ t\right)^{\alpha-2\lambda-1} A^{\lambda}_{ch\ t}\left|f\left(ch\ x\right)\right| sh^{2\lambda}\ tdt \le \left(sh\ \frac{r}{2}\right)^{\alpha} M_{G,1} f\left(ch\ x\right). \end{aligned} \tag{4.14}$$

Let now $\left(x-\frac{r}{4}, x+\frac{r}{4}\right)\cap[0,\infty) = \left(x-\frac{r}{4}, x+\frac{r}{4}\right)$, then $x > \frac{r}{4}$. Consider the case $\frac{r}{4} \le x \le \frac{3r}{4}$. Then

$$\begin{aligned} \left|F_1\left(ch\ x\right) - a_1\right| &\le \int_{x-r/4}^{x+r/4} \left(sh\ t\right)^{\alpha-2\lambda-1} A^{\lambda}_{ch\ t}\left|f\left(ch\ x\right)\right| sh^{2\lambda}\ tdt \\ &\le \int_0^{r} \left(sh\ t\right)^{\alpha-2\lambda-1} A^{\lambda}_{ch\ t}\left|f\left(ch\ x\right)\right| sh^{2\lambda}\ tdt \le \left(sh\ \frac{r}{2}\right)^{\alpha} M_{G,1} f\left(ch\ x\right). \end{aligned} \tag{4.15}$$

Finally, let $\frac{3r}{4} \le x < \infty$, then by Hölder inequality, we have

$$\begin{aligned} \left|F_1\left(ch\ x\right) - a_1\right| &= \int_{x-\frac{r}{4}}^{x+\frac{r}{4}} \left(sh\ t\right)^{\alpha-2\lambda-1} A^{\lambda}_{ch\ t}\left|f\left(ch\ x\right)\right| sh^{2\lambda}\ tdt \\ &\le \left\|A^{\lambda}_{ch\ t} f\right\|_{L_{p,\lambda}} \left(\int_{x-\frac{r}{4}}^{x+\frac{r}{4}} \left(sh\ t\right)^{(\alpha-2\lambda-1)q} sh^{2\lambda}\ tdt\right)^{\frac{1}{q}} \\ &\le \|f\|_{L_{p,\lambda}} \left(sh\left(x-\frac{r}{4}\right)\right)^{\alpha-2\lambda-1} \left(\int_{x-\frac{r}{4}}^{x+\frac{r}{4}} \left(2sh\ \frac{t}{2}ch\ \frac{t}{2}\right)^{2\lambda} dt\right)^{\frac{1}{q}} \\ &\le c_{\lambda,p} \|f\|_{L_{p,\lambda}} \left(sh\left(x-\frac{r}{4}\right)\right)^{\alpha-2\lambda-1} \left(\int_{x-\frac{r}{4}}^{x+\frac{r}{4}} \left(sh\ \frac{t}{2}\right)^{2\lambda} \left(ch\ \frac{t}{2}\right)^{2\lambda-1} d\left(sh\ \frac{t}{2}\right)\right)^{\frac{1}{q}} \\ &\le c_{\lambda,p} \|f\|_{L_{p,\lambda}} \left(sh\left(x-\frac{r}{4}\right)\right)^{\alpha-2\lambda-1} \left(\int_{x-\frac{r}{4}}^{x+\frac{r}{4}} sh^{2\lambda}\ \frac{t}{2} d\left(sh\ \frac{t}{2}\right)\right)^{\frac{1}{q}} \\ &\le c_{\lambda,p} \|f\|_{L_{p,\lambda}} \left(sh\left(x-\frac{r}{4}\right)\right)^{\alpha-2\lambda-1} \left(sh\left(\frac{x}{2}+\frac{r}{8}\right)\right)^{\frac{2\lambda+1}{q}} \\ &\le c_{\lambda,p} \|f\|_{L_{p,\lambda}} \left(sh\left(\frac{x}{2}+\frac{r}{8}\right)\right)^{\alpha-2\lambda-1+(2\lambda+1)\left(1-\frac{\alpha}{2\lambda+1}\right)} = c_{\lambda,p} \|f\|_{L_{p,\lambda}}. \end{aligned} \tag{4.16}$$

Combining (4.14)–(4.16), we obtain that for $0 < r < 2$

$$\left|F_1\left(ch\ x\right) - a_1\right| \le c_{\alpha,\lambda} \left(sh\ \frac{r}{2}\right)^{\alpha} M_{G,1} f\left(ch\ x\right) + c_{\lambda,p} \|f\|_{L_{p,\lambda}}.$$

From this it follows that

$$\begin{aligned}
&\sup_{0<r<2} \frac{1}{|H(0,r)|_\lambda} \int_0^r \left| A_{ch\, t}^\lambda (F_1(ch\ x) - a_1) \right| sh^{2\lambda}\ tdt \\
&\le \sup_{0<r<2} \frac{1}{|H(0,r)|_\lambda} \int_0^r A_{ch\, t}^\lambda |F_1(ch\ x) - a_1| sh^{2\lambda}\ tdt \\
&\le \sup_{0<r<2} \left(sh\ \frac{r}{2} \right)^{\alpha-2\lambda-1} \int_0^r A_{ch\, t}^\lambda \left| M_{G,1} f(ch\ x) \right| sh^{2\lambda}\ tdt + c_{\lambda,p} \frac{\|f\|_{L_{p,\lambda}}}{|H(0,r)|_\lambda} \int_0^r sh^{2\lambda}\ tdt \\
&\le \sup_{0<r<2} \left(sh\ \frac{r}{2} \right)^{\alpha-2\lambda-1} \left(\int_0^r sh^{2\lambda}\ tdt \right)^{\frac{1}{q}} \left\| A_{ch\, t}^\lambda \left| M_{G,1} f(\cdot) \right| \right\|_{L_{p,\lambda}} + c_{\lambda,p} \|f\|_{L_{p,\lambda}} \\
&\le \sup_{0<r<2} \left(sh\ \frac{r}{2} \right)^{\alpha-2\lambda-1} \left(sh\ \frac{r}{2} \right)^{\frac{2\lambda+1}{q}} \left\| M_{G,1} f \right\|_{L_{p,\lambda}} + c_{\lambda,p} \|f\|_{L_{p,\lambda}} \lesssim \|f\|_{L_{p,\lambda}}. \qquad (4.17)
\end{aligned}$$

We consider the case $2 \le r < \infty$. Let $0 \le x \le \frac{3r}{4}$.
Then

$$\begin{aligned}
|F_1(ch\, x) - a_1| &\le \int_0^r (sh\, t)^{\alpha-2\lambda-1} A_{ch\, t} |f(ch\, x)| sh^{2\lambda}\, tdt \le \sum_{k=0}^\infty \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A_{ch\, t} |f(ch\, x)\, sh^{2\lambda} tdt|}{(sh\, t)^{2\lambda+1-\alpha}} \\
&\le \sum_{k=0}^\infty \left(sh\, \frac{r}{2^{k+1}} \right)^{\alpha+2\lambda-1} \left(sh\, \frac{r}{2^{k+1}} \right)^{-4\lambda} \int_0^{\frac{r}{2^k}} A_{ch\, t} |f(ch\, x)| sh^{2\lambda} tdt \\
&\lesssim \left(sh\, \frac{r}{2} \right)^{\alpha+2\lambda-1} M_{G,2} f(ch\, x). \qquad (4.18)
\end{aligned}$$

Using (4.18), and by Lemma 1.1 we obtain

$$\begin{aligned}
&\sup_{r\ge 2} \frac{1}{|H(0,\ r)|_\lambda} \int_0^r |A_{ch\, t} f(ch\, x)| sh^{2\lambda}\, tdt \\
&\le \|M_{G,2} f\|_{L_{p,\lambda}} \sup_{r\ge 2} \left(sh\, \frac{r}{2} \right)^{\alpha-2\lambda-1} \left(\int_0^r sh^{2\lambda}\, tdt \right)^{\frac{1}{q}} + \|f\|_{L_{p,\lambda}} \\
&\lesssim \|f\|_{L_{p,\lambda}} \sup_{r\ge 2} \left(sh\, \frac{r}{2} \right)^{\alpha-2\lambda-1} \left(ch\, \frac{r}{2} \right)^{\frac{4\lambda}{q}} + \|f\|_{L_{p,\lambda}} \lesssim \|f\|_{L_{p,\lambda}} \le \|f\|_{L_{p,\lambda}} \\
&\lesssim \|f\|_{L_{p,\lambda}} \sup_{r\ge 2} \left(ch\, \frac{r}{2} \right)^{\alpha-2\lambda-1+\frac{4\lambda}{q}} \|f\|_{L_{p,\lambda}} \lesssim \|f\|_{L_{p,\lambda}}, \qquad (4.19)
\end{aligned}$$

since $\alpha - 2\lambda - 1 + \frac{4\lambda}{q} = \alpha - 2\lambda - 1 + 4\lambda \left(1 - \frac{1}{2\lambda+1}\right) = \alpha + 2\lambda - 1 - \frac{4\lambda\alpha}{2\lambda+1} = \frac{\alpha-2\lambda}{1+2\lambda}\alpha + 2\lambda - 1 < 0 \Leftrightarrow \frac{1-2\lambda}{1+2\lambda}\alpha < 1 - 2\lambda \Leftrightarrow 0 < \alpha < 2\lambda + 1$.

Combining (4.17) and (4.19) we obtain that

$$\sup_{r>0} \frac{1}{|H(0,\ r)|_\lambda} \int_0^r |A_{ch\, t}(F_1(ch\, x) - a_1)| sh^{2\lambda}\, tdt \lesssim \|f\|_{L_{p,\lambda}}. \qquad (4.20)$$

Suppose $a_2 = \int_{\left(0,\max\left\{\frac{1}{4},\frac{r}{4}\right\}\right)/\left(0,\frac{r}{4}\right)} (sh\ t)^{\alpha-2\lambda-1} f(ch\ t)\, sh^{2\lambda}\ tdt$.

We estimate above the difference

$$\begin{aligned}
|F_2(ch\ x) - a_2| = &\left| \int_{\frac{r}{4}}^\infty \left(A_{ch\ t}^\lambda (sh\ x)^{\alpha-2\lambda-1} - (sh\ t)^{\alpha-2\lambda-1} \chi_{\left(\frac{1}{4},1\right)}(ch\ t) \right) f(ch\ t)\, sh^{2\lambda}\ tdt \right. \\
&- \left. \int_{(0,\max\left\{\frac{1}{4},\frac{r}{4}\right\})/\left(0,\frac{r}{4}\right)} (sh\ t)^{\alpha-2\lambda-1} f(ch\ t)\, sh^{2\lambda}\ td\ t \right|
\end{aligned}$$

$$= \left| \int_{\frac{r}{4}}^{\infty} \left(A^{\lambda}_{ch\ t} (sh\ x)^{\alpha-2\lambda-1} - (sh\ t)^{\alpha-2\lambda-1} \right) f\ (ch\ t)\, sh^{2\lambda}\ tdt \right|$$

$$\leq \int_{\frac{r}{4}}^{\infty} |f\ (ch\ t)|\, B\ (x,t)\, sh^{2\lambda}\ tdt = \tau\ (x,r)\,. \tag{4.21}$$

We consider expansion

$$B\ (x,t) = \left| A^{\lambda}_{ch\ t} (sh\ x)^{\alpha-2\lambda-1} - (sh\ t)^{\alpha-2\lambda-1} \right|$$

$$= c_{\lambda} \left| \int_0^{\pi} \left((ch\ xch\ t - sh\ xsh\ t \cos\ \varphi)^2 - 1 \right)^{\frac{\alpha-2\lambda-1}{2}} - ((sh\ t)^{\alpha-2\lambda-1})\, (\sin\ \varphi)^{2\lambda-1}\, d\varphi \right|$$

$$\leq c_{\lambda} \int_0^{\pi} \left| (\max\,(sh\,(x+t)\,, |sh\,(x-t)|))^{\alpha-2\lambda-1} - (sh\ t)^{\alpha-2\lambda-1} \right| (\sin\ \varphi)^{2\lambda-1}\, d\varphi.$$

We estimate above the expression $B(x,t)$. It is easy to notice that

$$B(x,t) \lesssim \left| \max\,(\{sh\ (x+t), |sh\ (x-t)|\})^{\alpha-2\lambda-1} - (sh\ t)^{\alpha-2\lambda-1} \right| \equiv V(x,t).$$

I. If $0 < t < x - t < \infty$, then $0 < t < \frac{x}{2} < x + t$.
From this it follows, that

$$(sh\ t)^{\alpha-2\lambda-1} > (sh\,(x+t))^{\alpha-2\lambda-1}\,. \tag{4.22}$$

II. If $0 < x - t < t < \infty$, then $\frac{x}{2} < t < x < x + t$, and in this case the inequality (4.22) is just.
III. If $0 < t - x < \infty$, then $x < t < x + t < \infty$.
Again the inequality (4.22) takes place.
IV. If $0 < x + t < \infty$, since $t < x + t$, then (4.22) is valid.
Combining all these cases, we obtain that

$$V\ (x,t) = (sh\ t)^{\alpha-2\lambda-1} - (sh\,(x+t))^{\alpha-2\lambda-1}\,.$$

Applying the Lagrange formula to segment $[t, x+t]$, we obtain

$$V\ (x,t) \equiv V_{\xi}\ (x,t) = \frac{(2\lambda+1-\alpha)\, xch\ \xi}{(sh\ \xi)^{2\lambda+2-\alpha}}, \quad t < \xi < t + x.$$

From this we have

$$\nu_{\xi}\ (x,t) \lesssim x\ (sh\ t)^{\alpha-2\lambda-2} \quad \xi < 1, \tag{4.23}$$
$$\nu_{\xi}\ (x,t) \lesssim x\ (sh\ t)^{\alpha-2\lambda-1} \quad \xi \geq 1,$$

At first we consider the case $\xi < 1$.
Applying the Hölder inequality and also (4.22) and (4.23), from (4.21) for $x \leq r$ we obtain

$$\tau\ (x,r) = \int_{r/4}^{\infty} |f\ (ch\ t)|\, B\ (x,t)\, sh^{2\lambda}\ tdt \lesssim \|f\|_{L_{p,\lambda}}\, x \left(\int_{r/4}^{\infty} \frac{sh^{2\lambda}\ tdt}{(sh\ t)^{(2\lambda+2-\alpha)q}} \right)^{\frac{1}{q}}$$

$$\lesssim \|f\|_{L_{p,\lambda}}\, r \left(\int_{r/4}^{\infty} (sh\ t)^{(\alpha-2\lambda-2)q+2\lambda}\, dsh\ t \right)^{\frac{1}{q}} = \|f\|_{L_{p,\lambda}} \frac{r}{sh\ \frac{r}{4}} \lesssim \|f\|_{L_{p,\lambda}}\,. \tag{4.24}$$

By the hypothesis of theorem $\alpha - 2\lambda - 2 + (2\lambda+1)\,/q = \alpha - 2\lambda - 2 + (2\lambda+1)\left(1 - \frac{\alpha}{2\lambda+1}\right) = \alpha - 2\lambda - 2 + 2\lambda + 1 - \alpha = -1$.
Now we consider the case $\xi \geq 1$.
Let $0 < x \leq 8$. Taking into account (4.22) and acting as above for $x \leq r$ we obtain

$$\tau\ (x,r) = \int_{r/4}^{\infty} |f\ (ch\ t)|\, B\ (x,t)\, sh^{2\lambda}\ tdt \lesssim \|f\|_{L_{p,\lambda}}\, x \left(\int_{r/4}^{\infty} \frac{sh^{2\lambda}\ tdt}{(sh\ t)^{(2\lambda+1-\alpha)q}} \right)^{\frac{1}{q}}$$

$$\lesssim \|f\|_{L_{p,\lambda}} x \left(\int_{x/4}^{\infty} \frac{\left(sh^{2\lambda}\ \frac{t}{2}\right)\left(ch\ \frac{t}{2}\right)^{2\lambda-1} d(sh\ \frac{t}{2})}{\left(2sh\ \frac{t}{2} ch\ \frac{t}{2}\right)^{(2\lambda+1-\alpha)q}} \right)^{\frac{1}{q}} \lesssim \|f\|_{L_{p,\lambda}} x \left(\int_{x/4}^{\infty} \frac{\left(sh\ \frac{t}{2}\right)^{4\lambda-1} d(sh\ \frac{t}{2})}{\left(sh\ \frac{t}{2}\right)^{(2\lambda+1-\alpha)q}} \right)^{\frac{1}{q}}$$

$$= \|f\| x \left(\int_{x/4}^{\infty} \left(sh\ \frac{t}{2}\right)^{(\alpha-2\lambda-1)q+4\lambda-1} d\left(sh\ \frac{t}{2}\right) \right)^{\frac{1}{q}}$$

$$\lesssim \|f\|_{L_{p,\lambda}} x \left(sh\ \frac{x}{8}\right)^{\alpha-2\lambda-1+4\lambda/q} = \|f\|_{L_{p,\lambda}} x \left(sh\ \frac{x}{8}\right)^{\alpha-2\lambda-1+4\lambda\left(1-\frac{\alpha}{2\lambda+1}\right)}$$

$$= \|f\|_{L_{p,\lambda}} x \left(sh\ \frac{t}{8}\right)^{\frac{1-2\lambda}{1+2\lambda}\alpha+2\lambda-1} \lesssim \|f\|_{L_{p,\lambda}} \left(sh\ \frac{x}{8}\right)^{\frac{1-2\lambda}{1+2\lambda}\alpha+2\lambda} \lesssim \|f\|_{L_{p,\lambda}}. \tag{4.25}$$

Let now $8 < x < \infty$. Then as above, we obtain

$$\tau(x,r) \lesssim \|f\|_{L_{p,\lambda}} x \left(\int_{x/4}^{\infty} \frac{sh^{2\lambda}\ t dt}{(sh\ t)^{(2\lambda+1-\alpha)q}} \right)^{\frac{1}{q}} \lesssim \|f\|_{L_{p,\lambda}} x \left(sh\ \frac{x}{8}\right)^{\frac{1-2\lambda}{1+2\lambda}\alpha+2\lambda-1}$$

$$= \|f\|_{L_{p,\lambda}} \frac{x}{\left(sh\ \frac{x}{8}\right)^{1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha}} = \|f\|_{L_{p,\lambda}} \frac{x}{\left(2sh\ \frac{x}{16} ch\ \frac{x}{16}\right)^{1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha}}$$

$$= \|f\|_{L_{p,\lambda}} \frac{x}{\left(2sh\ \frac{x}{2^n}\right)^{2\left(1-2\lambda-\frac{1-2\lambda}{1+2\lambda}x\right)}} \lesssim \cdots \lesssim \|f\|_{L_{p,\lambda}} \frac{x}{\left(2^n sh\ \frac{x}{2^{n+3}}\right)^{2^n\left(1-2\lambda-\frac{1-2\lambda}{1+2\lambda}x\right)}}$$

$$\lesssim \|f\|_{L_{p,\lambda}} \frac{x}{\left(\frac{x}{8}\right)^{2^n\left(1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha\right)}} \lesssim \|f\|_{L_{p,\lambda}} \frac{8}{\left(\frac{x}{8}\right)^{2^n\left(1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha\right)-1}} \lesssim \|f\|_{L_{p,\lambda}}. \tag{4.26}$$

For the sufficiently large $n = n_0$

$$2^{n_0}\left(1-2\lambda-\frac{1-2\lambda}{1+2\lambda}\alpha\right)-1 \geq 0 \Leftrightarrow \frac{1-2\lambda}{1+2\lambda}\alpha \leq 1-2\lambda-\frac{1}{2^{n_0}}$$
$$\frac{1-2\lambda}{1+2\lambda}\alpha < 1-2\lambda \Leftrightarrow \alpha < 2\lambda+1.$$

Combining the estimates (4.24)–(4.26), on (4.21) for $0 < x \leq r$ we obtain

$$|F_2(ch\ x) - a_2| \lesssim \|f\|_{L_{p,\lambda}}.$$

Hence we have

$$\left|A_{ch\ t}^{\lambda} F_2(ch\ x) - a_2\right| \leq A_{ch\ t}^{\lambda} |F_2(ch\ x) - a_2| \lesssim \|f\|_{L_{p,\lambda}}. \tag{4.27}$$

From (4.27) it follows, that

$$\sup_{r>0} \frac{1}{|H(0,r)|_{\lambda}} \int_0^r \left|A_{ch\ t}^{\lambda} F_2(ch\ x - a_2)\right| sh^{2\lambda}\ tdt$$
$$\leq \sup_{r>0} \frac{1}{|H(0,r)|_{\lambda}} \int_0^r A_{ch\ t}^{\lambda} |F_2(ch\ x) - a_2| sh^{2\lambda}\ tdt$$
$$\lesssim \|f\|_{L_{p,\lambda}} \sup_{r>0} \frac{1}{|H(0,r)|_{\lambda}} \int_0^r sh^{2\lambda} tdt \lesssim \|f\|_{L_{p,\lambda}}. \tag{4.28}$$

Denote $a_f = a_1 + a_2 = \int_{\left(0,\max\left\{\frac{1}{4},\frac{r}{4}\right\}\right)} (sh\ t)^{\alpha-2\lambda-1} f(ch\ t) sh^{2\lambda}\ tdt.$

Finally from (4.20) and (4.28) we obtain

$$\begin{aligned}
&\sup_{r>0}\frac{1}{|H(0,r)|_\lambda}\int_0^r\left|A^\lambda_{ch\,t}\tilde{\Im}^\alpha_G f(ch\ x)-a_f\right|sh^{2\lambda}\ tdt\\
&\quad=\sup_{r>0}\frac{1}{|H(0,r)|_\lambda}\int_0^r\left|A^\lambda_{ch\,t}F_1(ch\ x)-a_1+A^\lambda_{ch\,t}F_2(ch\ x)-a_2\right|sh^{2\lambda}\ tdt\\
&\quad\le\sup_{r>0}\frac{1}{|H(0,r)|_\lambda}\int_0^r\left|A^\lambda_{ch\,t}F_1(ch\ x)-a_1\right|sh^{2\lambda}\ tdt\\
&\qquad+\sup_{r>0}\frac{1}{|H(0,r)|_\lambda}\int_0^r\left|A^\lambda_{ch\,t}F_2(ch\ x)-a_2\right|sh^{2\lambda}\ tdt\lesssim\|f\|_{L_{p,\lambda}},
\end{aligned}$$

from this it follows that

$$\left\|\tilde{\Im}^\alpha_G f\right\|_{BMO_G[0,\infty)}\le 2\sup_{x,r>0}\frac{1}{|H(0,r)|_\lambda}\int_0^r\left|A^\lambda_{ch\,t}\tilde{\Im}^\alpha_G f(ch\ x)-a_f\right|sh^{2\lambda}\ tdt\lesssim\|f\|_{L_{p,\lambda}}.$$

Theorem 4.2 is proved. □

Corollary 4.1. *Let $\alpha p=2\lambda+1$, $1-2\lambda<\alpha<2\lambda+1$, $f\in L_{p,\lambda}[0,\infty)$. If the integral $\Im^\alpha_G f$ is absolutely convergent, then $\Im^\alpha_G f\in BMO_G[0,\infty)$ and the inequality*

$$\left\|\Im^\alpha_G f\right\|_{BMO_G[0,\infty)}\lesssim\|f\|_{L_{p,\lambda}[0,\infty)}$$

is valid.

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Original article

On the cardinal number of the family of all invariant extensions of a nonzero σ-finite invariant measure

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Abstract

It is shown that, for any nonzero σ-finite translation invariant (translation quasi-invariant) measure μ on the real line $\mathbf{R}$, the cardinality of the family of all translation invariant (translation quasi-invariant) measures on $\mathbf{R}$ extending μ is greater than or equal to 2^{ω_1}, where ω_1 denotes the first uncountable cardinal number. Some related results are also considered.

Keywords: Invariant measure; Quasi-invariant measure; Metrical transitivity; Nonseparable extension of measure; Ulam's transfinite matrix

Let E be a base (ground) set and let G be a group of transformations of E. The pair (E, G) is usually called a space equipped with a transformation group.

A measure μ defined on some G-invariant σ-algebra of subsets of E is called quasi-invariant with respect to G (briefly, G-quasi-invariant) if, for any μ-measurable set X and for any transformation g from G, the relation

$$\mu(X) = 0 \quad \Leftrightarrow \quad \mu(g(X)) = 0$$

holds true. Moreover, if the equality $\mu(g(X)) = \mu(X)$ is valid for any μ-measurable X and for any g from G, then μ is called an invariant measure with respect to G (briefly, G-invariant measure).

According to these definitions, the triplet of the form (E, G, μ) determines the structure of an invariant (quasi-invariant) measure on E.

Suppose that μ is a nonzero σ-finite G-invariant (G-quasi-invariant) measure on E. It is known that if a group G is uncountable and acts freely in E, then there always exist subsets of E nonmeasurable with respect to μ (see [1]; cf. also [2]). So the domain of μ differs from the family of all subsets of E, i.e., $\mathrm{dom}(\mu) \neq \mathcal{P}(E)$. In this connection, the natural question arises whether there exists a G-invariant (G-quasi-invariant) measure μ' on E strongly extending μ. This question was studied for various types of spaces (E, G, μ). Undoubtedly, the most interesting case for classical Real Analysis is when E coincides with the n-dimensional Euclidean space $\mathbf{R}^n$, a group G is a subgroup of the group of all isometric transformations of $\mathbf{R}^n$, and μ is a G-invariant extension of the standard n-dimensional Lebesgue measure λ_n on $\mathbf{R}^n$ (see, for instance, [3–8]).

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Another important case is when $E = \Gamma$, where Γ is an uncountable σ-compact locally compact topological group, Γ coincides with the group of all left (right) translations of Γ, and μ is a G-invariant extension of the left (right) Haar measure on Γ (cf. [9,5,10,11]).

A more general form of the above question is as follows. For a given space (E, G, μ), denote by $\mathcal{M}_G(\mu)$ the family of all measures on E extending μ and invariant (quasi-invariant) with respect to G. It is natural to try to evaluate the cardinality of $\mathcal{M}_G(\mu)$ in terms of card(E) and card(G). In the present paper, we will be dealing with this problem for the case when E coincides with the real line $\mathbf{R}$ and G is the group of all translations of $\mathbf{R}$. Notice that the method applied in our further considerations is primarily taken from [6].

Below, we will use the following standard notation:

$X \triangle Y =$ the symmetric difference of two sets X and Y;
$\omega =$ the least infinite cardinal (ordinal) number;
$\omega_1 =$ the least uncountable cardinal (ordinal) number;
$\mathbf{c} =$ the cardinality of the continuum.

Let μ be a measure defined on some σ-algebra of subsets of E (here μ is not assumed to be invariant or quasi-invariant under a nontrivial group of transformations of E). The Hilbert space of all square μ-integrable real-valued functions on E is usually denoted by the symbol $L_2(\mu)$. If $L_2(\mu)$ is a separable Hilbert space, then μ is called a separable measure. Otherwise, μ is called a nonseparable measure.

Treating the real line $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$ of all rational numbers and keeping in mind the existence of a Hamel basis in $\mathbf{R}$, it is not difficult to show that the additive group $(\mathbf{R}, +)$ admits a representation in the form

$$\mathbf{R} = G + H \quad (G \cap H = \{0\}),$$

where G and H are some two subgroups of $(\mathbf{R}, +)$ and

$$\mathrm{card}(G) = \omega_1, \quad \mathrm{card}(H) \leq \mathbf{c}.$$

We denote by $\mathcal{I}$ the σ-ideal generated by all those subsets X of $\mathbf{R}$ which are representable in the form $X = Y + H$, where $Y \subset G$ and card$(Y) \leq \omega$.

It can readily be seen that $\mathcal{I}$ is a translation invariant σ-ideal of sets in $\mathbf{R}$.

We begin with the following auxiliary statement.

Lemma 1. *There exists a partition $\{X_\xi : \xi < \omega_1\}$ of $\mathbf{R}$ satisfying these two relations:*

(1) *for any ordinal $\xi < \omega_1$, the set X_ξ belongs to the σ-ideal $\mathcal{I}$;*
(2) *for each subset Ξ of ω_1 and for any $r \in \mathbf{R}$, the relation*

$$(\cup\{X_\xi : \xi \in \Xi\}) \triangle (r + \cup\{X_\xi : \xi \in \Xi\}) \in \mathcal{I}$$

holds true, i.e., the set $\cup\{X_\xi : \xi \in \Xi\}$ is $\mathcal{I}$-almost translation invariant in $\mathbf{R}$.

The proof of this lemma is given in [6].

By combining Lemma 1 with the well-known $(\omega \times \omega_1)$-matrix of Ulam (see, e.g., [12]), the next auxiliary statement can be deduced.

Lemma 2. *Let $\{X_\xi : \xi < \omega_1\}$ be a partition of $\mathbf{R}$ described in Lemma 1 and let μ be a nonzero σ-finite translation invariant (translation quasi-invariant) measure on $\mathbf{R}$.*

There exists a disjoint family $\{\Xi_j : j \in J\}$ of subsets of ω_1 such that:

(1) card$(J) = \omega_1$*;*
(2) *for each index $j \in J$, the set $Z_j = \cup\{X_\xi : \xi \in \Xi_j\}$ is nonmeasurable with respect to μ (where $\{X_\xi : \xi < \omega_1\}$ is a partition of $\mathbf{R}$ described in Lemma 1);*
(3) $\mu_*(\cup\{Z_j : j \in J\}) = 0$ *(where the symbol μ_* denotes the inner measure associated with μ).*

Notice that the proof of Lemma 2 is similar to the argument presented in [6] (cf. also [7]).

Lemma 3. *Let μ be a σ-finite translation invariant (translation quasi-invariant) measure on $\mathbf{R}$. There exists a measure μ' on $\mathbf{R}$ such that:*

(1) *μ' is translation invariant (translation quasi-invariant);*
(2) *μ' extends μ;*
(3) $\mathcal{I} \subset \mathrm{dom}(\mu')$.

Proof. If X is any set belonging to $\mathcal{I}$, then the equality $\mu_*(X) = 0$ is satisfied, because in $\mathbf{R}$ there are uncountably many pairwise disjoint translates of X. So we may apply Marczewski's standard method to μ and $\mathcal{I}$ for extending μ. Namely, introduce the σ-algebra $\mathcal{S}'$ of all those subsets Z of $\mathbf{R}$ which admit a representation

$$Z = (Y \cup X') \setminus X'' \quad (Y \in \mathrm{dom}(\mu), X' \in \mathcal{I}, X'' \in \mathcal{I})$$

and define on $\mathcal{S}'$ the functional μ' by the formula

$$\mu'(Z) = \mu(Y) \quad (Z \in \mathcal{S}').$$

It is not hard to verify that the definition of μ' is correct (i.e., the value $\mu'(Z)$ does not depend on a representation of Z in the above-mentioned form), and μ' satisfies the relations. (1), (2), and (3) of Lemma 3. $\square$

The preceding lemmas enable us to establish the following statement.

Theorem 1. *Let μ be a nonzero σ-finite translation invariant (translation quasi-invariant) measure on $\mathbf{R}$. Then the inequality $\mathrm{card}(\mathcal{M}_{\mathbf{R}}(\mu)) \geq 2^{\omega_1}$ holds true. In particular, there are measures on $\mathbf{R}$ strictly extending μ and invariant (quasi-invariant) under the group of all translations of $\mathbf{R}$.*

Proof. Taking into account Lemma 3, we may assume without loss of generality that the measure μ is complete and $\mathcal{I} \subset \mathrm{dom}(\mu)$.

Let $\{Z_j : j \in J\}$ be the disjoint family of subsets of $\mathbf{R}$ described in Lemma 2. This family has the following properties:

(a) $\mathrm{card}(J) = \omega_1$ and the sets Z_j $(j \in J)$ are pairwise disjoint;
(b) every set Z_j $(j \in J)$ is nonmeasurable with respect to μ;
(c) for each set $J_0 \subset J$ and for every $r \in \mathbf{R}$, the equality

$$\mu((\cup\{Z_j : j \in J_0\}) \triangle (r + \cup\{Z_j : j \in J_0\})) = 0$$

is valid;
(d) $\mu_*(\cup\{Z_j : j \in J\}) = 0$.

Further, take a subset J_1 of J and associate to this J_1 the set

$$Z(J_1) = \cup\{Z_j : j \in J_1\}.$$

By virtue of (c) and (d), we get the relations:
(e) for every $r \in \mathbf{R}$, the set $Z(J_1)$ is μ-almost translation invariant, i.e.,

$$\mu(Z(J_1) \triangle (r + Z(J_1))) = 0;$$

(f) $\mu_*(Z(J_1)) = 0$.

Consequently, applying Marczewski's method of extending invariant and quasi-invariant measures (cf. the proof of Lemma 3), we obtain the measure μ_{J_1} on $\mathbf{R}$ which extends μ, is invariant (quasi-invariant) under the group of all translations of $\mathbf{R}$, and satisfies the equality $\mu_{J_1}(Z(J_1)) = 0$.

Now, let us establish that if J_1 and J_2 are any two distinct subsets of J, then the associated measures μ_{J_1} and μ_{J_2} differ from each other. Indeed, if $J_1 \neq J_2$, then either $J_1 \setminus J_2 \neq \emptyset$ or $J_2 \setminus J_1 \neq \emptyset$. We may suppose that $J_1 \setminus J_2 \neq \emptyset$, so there is an index $j \in J_1 \setminus J_2$. According to the definition of μ_{J_1}, the set Z_j turns out to be of μ_{J_1}-measure zero. On the other hand, the same set Z_j cannot be of μ_{J_2}-measure zero. To see this circumstance, suppose to the contrary that $\mu_{J_2}(Z_j) = 0$. Then, keeping in mind the construction of μ_{J_2}, we must have

$$Z_j = (T \cup T') \setminus T'',$$

where

$$\mu(T) = 0, \qquad T' \subset Z(J_2), \qquad T'' \subset Z(J_2).$$

However, it can easily be verified that the above relations imply the inclusion $Z_j \subset T$ and the equality $\mu(Z_j) = 0$. In particular, we obtain that Z_j is a μ-measurable set, which contradicts (b).

Thus, we have an injective mapping from the power set $\mathcal{P}(\omega_1)$ into the family of all those measures on $\mathbf{R}$ which extend μ and are translation invariant (translation quasi-invariant). The existence of such a mapping trivially yields the desired inequality $\mathrm{card}(\mathcal{M}_{\mathbf{R}}(\mu)) \geq 2^{\omega_1}$, and the proof of Theorem 1 is finished. □

Remark 1. Consider the n-dimensional Euclidean space $\mathbf{R}^n$, where $n \geq 1$. Since there exists an isomorphism between the additive groups $(\mathbf{R}, +)$ and $(\mathbf{R}^n, +)$, the direct analogue of Theorem 1 is valid for the space $\mathbf{R}^n$ (and, more generally, for any uncountable vector space over the field $\mathbf{Q}$ of all rational numbers).

Remark 2. As an immediate consequence of Theorem 1, we get the relation

$$\mathrm{card}(\mathcal{M}_{\mathbf{R}}(\mu)) \geq 2^{\omega_1} \geq 2^{\omega} = \mathbf{c}.$$

This relation is a statement of **ZFC** set theory. Assuming the Continuum Hypothesis (**CH**), we directly come to the much stronger inequality

$$\mathrm{card}(\mathcal{M}_{\mathbf{R}}(\mu)) \geq 2^{\mathbf{c}}.$$

We do not know whether the latter inequality can be proved within the framework of **ZFC** theory.

Let the symbol λ $(= \lambda_1)$ denote the standard Lebesgue measure on the real line $\mathbf{R}$. Kakutani and Oxtoby demonstrated in 1950 that there exist nonseparable measures on $\mathbf{R}$ belonging to the class $\mathcal{M}_{\mathbf{R}}(\lambda)$ (see [13]). Obviously, all those measures are strict extensions of λ. A radically different approach to the problem of the existence of nonseparable measures belonging to $\mathcal{M}_{\mathbf{R}}(\lambda)$ was given in the work by Kodaira and Kakutani (see again [13]).

The method of Kakutani and Oxtoby allows one to conclude that there exist at least $2^{2^{\mathbf{c}}}$ nonseparable measures on $\mathbf{R}$, all of which extend λ and are translation invariant. Thus, for the concrete measure λ on $\mathbf{R}$, the inequality of Theorem 1 can be essentially strengthened and, in fact, we have the following equality:

$$\mathrm{card}(\mathcal{M}_{\mathbf{R}}(\lambda)) = 2^{2^{\mathbf{c}}}.$$

In this context, the natural question arises whether the analogous equality

$$\mathrm{card}(\mathcal{M}_{\mathbf{R}}(\mu)) = 2^{2^{\mathbf{c}}}$$

is valid for any nonzero σ-finite translation invariant (translation quasi-invariant) measure μ on $\mathbf{R}$. We do not know the answer to this question. Nevertheless, assuming the Continuum Hypothesis (**CH**), for a sufficiently wide class of measures μ on $\mathbf{R}$ it can be proved that the last equality holds true, too.

Let (E, G, μ) be a space equipped with a σ-finite G-invariant (G-quasi-invariant) measure μ. Recall that μ is metrically transitive (or ergodic) if, for any μ-measurable set X with $\mu(X) > 0$, there exists a countable family $\{g_k : k < \omega\}$ of transformations from G such that

$$\mu(E \setminus \cup\{g_k(X) : k < \omega\}) = 0.$$

It is well known that metrically transitive (ergodic) measures play an important role in many topics of mathematical analysis and probability theory.

Lemma 4. *Let (E, G) be a space equipped with a transformation group satisfying these two conditions:*

(1) $\mathrm{card}(E) = \omega_1$*;*
(2) *the group G acts freely and transitively in E.*

If μ is a nonzero σ-finite ergodic G-invariant (G-quasi-invariant) measure on E, then there exists a partition $\{X_\xi : \xi < \omega_1\}$ of E such that:

(i) *every set X_ξ $(\xi < \omega_1)$ is μ-thick in E, i.e., $\mu_*(E \setminus X_\xi) = 0$;*
(ii) *for any set $\Xi \subset \omega_1$ and for each transformation $g \in G$, the inequality*

$$\mathrm{card}((\cup\{X_\xi : \xi \in \Xi\}) \triangle (g(\cup\{X_\xi : \xi \in \Xi\}))) \leq \omega$$

is valid.

The proof of Lemma 4 is presented in [10].

Starting with the previous lemma and applying some modified version of the method of Kakutani and Oxtoby, we get the following statement.

Theorem 2. *Assume* **CH** *and let* (E, G) *be a space equipped with a transformation group, satisfying the conditions* (1) *and* (2) *of Lemma* 4.

Then, for every nonzero σ-finite ergodic G-invariant (G-quasi-invariant) measure μ on E, the class $\mathcal{M}_G(\mu)$ contains at least $2^{2^{\mathbf{c}}}$ nonseparable measures.

As has already been mentioned, the proof of Theorem 2 is based on Lemma 4 and on the argument of Kakutani and Oxtoby [13] (cf. also [10]).

Remark 3. Marczewski's method of extending σ-finite invariant (quasi-invariant) measures does not substantially change the structure of an initial measure. On the other hand, the method of Kakutani and Oxtoby allows one to obtain nonseparable translation invariant extensions of λ on $\mathbf{R}$, starting with the separable measure λ (however, those extensions are not ergodic). Further modifications of this method were applied to the Haar measure on an uncountable σ-compact locally compact Polish topological group (see, for instance, [9]). Notice that various properties of invariant and quasi-invariant measures given on algebraic-topological structures are thoroughly discussed in [14].

Theorem 3. *Assume* **CH** *and let* (E, G) *be again a space equipped with a transformation group, satisfying the conditions* (1) *and* (2) *of Lemma* 4.

Then, for every nonzero σ-finite ergodic G-invariant (G-quasi-invariant) measure μ on E, the class $\mathcal{M}_G(\mu)$ contains $2^{2^{\mathbf{c}}}$ ergodic measures.

The proof of Theorem 3 follows the method presented in [7] for a concrete space (E, G, μ). Namely, in [7] the role of (E, G, μ) is played by the triplet $(\mathbf{R}^n, D_n, \lambda_n)$, where $n \geq 1$ and D_n denotes the group of all isometric transformations of $\mathbf{R}^n$. Under **CH**, the argument given in [7] for $(\mathbf{R}^n, D_n, \lambda_n)$ works also for a space (E, G, μ) of Theorem 3.

Remark 4. Both Theorems 2 and 3 show that, supposing **CH**, the cardinality of the class $\mathcal{M}_G(\mu)$ is equal to the cardinality of the class of all measures on E (where a space (E, G) satisfies (1) and (2) of Lemma 4 and μ is a nonzero σ-finite ergodic G-invariant or G-quasi-invariant measure on E).

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Original article

On small sets from the measure-theoretical point of view

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Abstract

For nonzero invariant (quasi-invariant) σ-finite measures on an uncountable group $(G, \cdot)$, the behaviour of small sets with respect to the group operation in G is studied.

Keywords: Invariant measure; Measure zero set; Nonmeasurable set; Absolutely negligible set

Let G be an arbitrary group and μ be a nonzero σ-finite G-invariant (more generally, G-quasi-invariant) measure defined on some σ-algebra of subsets of G. We recall that the symbol $I(\mu)$ denotes the σ-ideal of subsets of G, generated by the family of all μ-measure zero sets. Members of $I(\mu)$ are usually called negligible sets with respect to the given measure μ. Quite often, they are also called small sets with respect to μ.

Let us introduce one important notion concerning the general theory of small (negligible) sets.

Let G be an arbitrary group and let $Y \subset G$. We say that Y is G-absolutely negligible in G if, for any σ-finite G-invariant (G-quasi-invariant) measure μ on G, there exists a G-quasi-invariant measure $\hat{\mu}$ on G extending μ and such that $\hat{\mu}(Y) = 0$.

For more detailed information about the above-mentioned notion see [1–5].

Notice that it is natural to introduce the notion of a small set not only with respect to a given invariant (quasi-invariant) measure but also with respect to a given class of invariant (quasi-invariant) measures (see, for example, [1,5,6]).

The following statement gives a purely geometrical characterization of absolutely negligible sets and plays an essential role the process of studying various properties of these sets.

Lemma 1. *Let $(G, \cdot)$ be an arbitrary uncountable group and let Y be a subset of G. Then the following two relations are equivalent:*

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(1) *Y is a G-absolutely negligible set in G;*
(2) *for each countable family $\{g_i : i \in I\}$ of elements from G, there exists a countable family $\{h_j : j \in J\}$ of elements from G, such that*

$$\cap_{j \in J}(h_j \cdot (\cup_{i \in I}(g_i \cdot Y))) = \emptyset.$$

For the proof of this lemma, see e.g. [1] or [4].

By applying a Hamel basis of the real line **R**, W. Sierpinski has established the following statement.

Proposition. *Let λ be the standard Lebesgue measure on* **R**. *Then there exist two sets $X \subset \mathbf{R}$ and $Y \subset \mathbf{R}$ satisfying the relations*

$$X \in I(\lambda),\ Y \in I(\lambda), \quad X + Y \notin dom(\lambda).$$

For more details, see [6]. Some generalization of this result for uncountable vector spaces over the field **Q** of all rational numbers and for quasi-invariant extensions of measures on such spaces can be found in [7]. Similar properties of algebraic sums of subsets of the real line **R** are also discussed in [4,8].

It is reasonable to ask whether similar statements hold in more general situations when no topology is considered on given group. Namely, it is natural to pose the following question:

Let $(G, \cdot)$ be an uncountable group equipped with a nonzero σ-finite G-invariant (G-quasi-invariant) measure μ. Do there exist two sets $X \in I(\mu)$ and $Y \in I(\mu)$ such that $X \cdot Y$ does not belong to $dom(\mu)$.

For an arbitrary uncountable commutative group $(G, +)$ and for a nonzero σ-finite complete G-invariant (G-quasi-invariant) measure μ we have a direct analogue of the second part of above-mentioned proposition by Sierpinski. In particular, the following statement is valid.

Theorem 1. *Let $(G, +)$ be an uncountable commutative group and let μ be a nonzero σ-finite G-invariant measure on G. There exists a G-invariant complete measure $\hat{\mu}$ on G extending μ and such that, for some two sets $X \in I(\hat{\mu})$ and $Y \in I(\hat{\mu})$, the relation*

$$X + Y \notin dom(\hat{\mu})$$

is satisfied.

The proof of Theorem 1 can be found, for instance in [3].

It seems to be interesting to generalize the above result (i.e. Theorem 1) to a wider class of uncountable groups $(G, \cdot)$ (not necessarily commutative). From this point of view the following statement can be formulated.

Theorem 2. *Let $(G, \cdot)$ be an uncountable solvable group and let μ be a nonzero σ-finite G-invariant measure on G. There exists a G-invariant complete measure $\hat{\mu}$ on G extending μ and such that, for some two sets $X \in I(\hat{\mu})$ and $Y \in I(\hat{\mu})$, the relation*

$$X + Y \notin dom(\hat{\mu})$$

is satisfied.

In connection with Theorem 2, let us remark that its proof is obtained by using the method of surjective homomorphisms (see [4,5] for a detailed description this method).

Finally notice that an analogous problem for arbitrary noncommutative groups is still open.

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Original article

The Riemann–Hilbert problem in the class of Cauchy type integrals with densities of grand Lebesgue spaces

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Abstract

The present paper deals with a solution of the Riemann–Hilbert problem in the class of Cauchy type integrals with densities of certain new nonstandard Banach function spaces. The solvability conditions are explored and the solutions (if any) are constructed explicitly.

Keywords: Grand Lebesgue spaces; Riemann–Hilbert problem; Cauchy type integrals; Weights

1. Introduction

The grand Lebesgue spaces were introduced by T. Iwaniec and C. Sbordone in [1], where they studied the integrability problem of the Jacobian under minimal hypotheses. Later on, the more general Lebesgue grand spaces $L^{p),\theta}(1 < p < \infty, \theta > 0)$ appeared in the paper of L. Greco, T. Iwaniec and S. Sbordone [2] in which they studied the existence and uniqueness of solutions to the inhomogeneous n-harmonic equation div $A(x, \nabla u) = \mu$. The necessity to investigate these spaces has emerged from their rather essential role in various fields, in particular, in nonlinear partial differential equations. It turns our that the spaces $L^{p),\theta}$ are intended to establish the existence and uniqueness, as well as the regularity for various PDEs.

The boundedness in weighted grand Lebesgue spaces of fundamental integral operators in linear and nonlinear harmonic analysis is established in [3–6] (see also [7, Ch. 14] and [8, Ch. 2]).

It should be emphasized that the first author has established the necessary and sufficient conditions for the curve and the weight simultaneously ensuring the boundedness of the operator generated by the Cauchy singular integral

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defined on the rectifiable curve. The Dirichlet and Riemann boundary value problems in the framework of grand Lebesgue spaces are solved in [9] (see also [8, Ch. 4]).

In the present work, we present the solution of the Riemann–Hilbert problem

$$\operatorname{Re}[\lambda(t)\phi^{+}(t)] = b(t) \tag{1}$$

in the class $K^{p),\theta}(D)$, i.e., a set of the Cauchy type integrals

$$\phi(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{\varphi(t)}{t-z}dt, \quad z \in D,$$

where D is a simply-connected bounded domain with the boundary Γ and $\varphi \in L^{p),\theta}(\Gamma)$, $(1 < p < \infty, \theta > 0)$.

The definition of the grand Lebesgue spaces and the conditions for the boundary Γ and for the functions $\lambda(t)$ and $b(t)$ are given in the next section.

2. Preliminaries

Let Γ be a simple rectifiable curve. Suppose that ω is a weight function prescribed on Γ. The weighted grand Lebesgue space $L^{p),\theta}_{\omega}(\Gamma)$ $(1 < p < \infty, \theta > 0)$ is defined by the norm

$$\|f\|_{L^{p),\theta}_{\omega}(\Gamma)} = \sup_{0<\varepsilon<p-1}\left(\varepsilon^{\theta}\int_{\Gamma}|f(t)|^{p-\varepsilon}\omega(t)|dt|\right)^{\frac{1}{p-\varepsilon}}.$$

$L^{p),\theta}_{\omega}(\Gamma)$ is a Banach function space.

Let now D be a simply-connected bounded domain with the boundary Γ and let $z = z(w)$ be conformal mapping of a circle $U = \{w : |w| < 1\}$ onto D. By $w = w(z)$ we denote its inverse mapping. Assume $\gamma = \{\tau : |\tau| = 1\}$.

Here we introduce certain classes of analytic functions.

For $1 < p < \infty, \theta > 0$ we put:

$$K^{p),\theta}(D) = \left\{\phi : \phi(z) = (K_{\Gamma}\varphi)(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{\varphi(t)dt}{t-z}, \quad \varphi \in L^{p),\theta}(\Gamma), \quad z \in D\right\},$$

$$\widetilde{K}^{p),\theta}(\mathbb{C}\setminus\Gamma) = \left\{\phi : \phi(z) = (K_{\Gamma}\varphi)(z) + Q(z), \quad \varphi \in L^{p),\theta}(\Gamma), \quad z \in \mathbb{C}\setminus\Gamma \quad Q \text{ is a polynomial}\right\},$$

$$K^{p),\theta}_{\omega}(U) = \left\{F : F(w) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\tau)}{\tau - w}d\tau, \quad f \in L^{p),\theta}_{\omega}(\gamma), \quad w \in U\right\}$$

and

$$\widetilde{K}^{p),\theta}_{\omega}(\mathbb{C}\setminus\gamma) = \left\{F : F(w) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\tau)d\tau}{\tau - w} + q(w), \quad f \in L^{p),\theta}_{\omega}(\gamma), \quad w \in \mathbb{C}\setminus\gamma, \quad q \text{ is a polynomial of } w\right\}.$$

First of all, we adduce our assumptions for the curve Γ.

In what follows, it will be assumed that

$$1/z'(w) \in \bigcup_{\delta>0} H^{\delta}(U), \quad \text{and} \quad z'(\tau) \in A_p(\gamma)$$

where H^{δ} denotes a class of analytic Hardy class functions and $A_p(\gamma)$ is a class of weighted Muckenhoupt functions, i.e., a set of weight functions ω defined on γ for which

$$\sup\left(\frac{1}{|l|}\int_{l}\omega(\tau)|d\tau|\right)\left(\frac{1}{|l|}\int_{l}\omega^{1-p'}(\tau)|d\tau|\right)^{p-1} < +\infty,$$

where the least upper bound is taken over all arcs l of the unit circumference γ.

As for the coefficients and the right-hand side of (1), it is required that: $\lambda(t) \in C(\Gamma)$, $\lambda(t) \neq 0$ the real function $b \in L^{p),\theta}$, $a(t) = \overline{\lambda}(t)/\lambda(t)$, and the index $\varkappa = \operatorname{ind}_{\Gamma} a(t) = \frac{1}{2\pi}[\arg a(t)]_{\Gamma}$.

3. The statement of the problem and its reduction to the case of a unit disk

Let all the assumptions formulated in the previous section for the curve Γ, coefficients $\lambda(t)$ and the right-hand side $b(t)$ be fulfilled.

We have to define the function $\phi(z) \in K^{p),\theta}(D)$ for which for almost all $t \in \Gamma$ the equality (1) is fulfilled, where ϕ^+ are angular boundary values of the function $\phi(z)$, when z tends along a nontangential path to t.

Find now the function

$$F(w) = \phi(z(w)),$$

where ϕ is a solution of problem (1), and $z = z(w)$ is conformal mapping of the unit circle onto D. Then $F(w)$ will, obviously, be a solution of the problem

$$\mathrm{Re}[\lambda(z(\tau))F^+(\tau)] = b(z(\tau)), \quad \tau \in \gamma. \tag{2}$$

The following theorem is valid.

Theorem 1. *If $\phi \in K^{p),\theta}(D)$ is a solution of problem* (1)*, then F will be a solution of problem* (2) *of the class $K^{p),\theta}_{|z'|}(U)$.*

Conversely, if $F \in K^{p),\theta}_{|z'|}(U)$ is a solution of problem (2)*, then the function $\phi(z) = F(w(z))$ is a solution of problem* (1) *of the class $K^{p),\theta}(D)$.*

4. Reduction of problem (2) to the Riemann problem with an additional requirement on the solution

Following N. Muskhelishvili's method ([10, Ch. 2]), we put

$$\Omega(w) = \begin{cases} F(w), & |w| < 1 \\ \overline{F}\left(\dfrac{1}{\overline{w}}\right), & |w| > 1. \end{cases} \tag{3}$$

The function Ω belongs to the class $\widetilde{K}^{p),\theta}(\mathbb{C} \setminus \gamma)$ and is a solution of the Riemann problem

$$\Omega^+(\tau) = a(\tau)\Omega^-(\tau) + \widetilde{b}(\tau), \tag{4}$$

where $a(\tau) = \overline{\lambda}(z(\tau)) \setminus \lambda(z(\tau))$, $\widetilde{b}(\tau) = 2b(z(\tau)) \setminus \lambda(z(\tau))$.

Owing to the specific construction of the function Ω, it is necessary to require that

$$\Omega(w) = \Omega_*(w), \quad |w| \neq 1 \tag{5}$$

where $\Omega_*(w) = \overline{\Omega}\left(\frac{1}{\overline{w}}\right)$, $|w| \neq 1$.

Theorem 2. *If $F(w) \in K^{p),\theta}(U)$ is a solution of problem* (2)*, then the function $\Omega(w)$ prescribed by the equality* (3) *belongs to the class $\widetilde{K}^{p),\theta}_{|z'|}(\mathbb{C} \setminus \gamma)$ and satisfies the conditions* (4) *and* (5)*.*

Conversely, if $\Omega(w)$ satisfies the conditions (4) *and* (5)*, then the function $\phi(z) = \Omega(w(z))$ is a solution of problem* (1) *of the class $K^{p),\theta}(D)$.*

5. Solution of the Riemann–Hilbert problem

Having solved problems (4), (5), we state that the following basic result is valid.

Theorem 3. *Let all the assumptions formulated in Section* 2 *be fulfilled, then:*

(i) *if $\varkappa \geq 0$, problem* (1) *is solvable, and its general solution is given by the equality*

$$\phi(z) = \Omega(w(z)) = X(w(z))\left[\frac{1}{2\pi i}\int_\gamma \frac{b(z(\tau))}{X^+(z(\tau))\lambda(z(\tau))}\frac{d\tau}{\tau - w(z)} + \overline{\frac{1}{2\pi i}\int_\gamma \frac{b(z(\tau))}{X^+(z(\tau))\lambda(z(\tau))}\frac{d\tau}{\tau - \frac{1}{w(z)}}}\right] + X(w(z))Q_\varkappa(w(z)),$$

where

$$X(w) = \begin{cases} X_0(w), & |w| < 1 \\ (w - w_0)^{-\varkappa} X_0(w), & |w| > 1, \end{cases}$$

$$X_0(w) = \exp\left(\frac{1}{2\pi i}\int_\gamma \frac{\ln a(\tau)(\tau - w_0)^{-\varkappa}}{\tau - w} d\tau\right), \quad w_0 \in U$$

and $Q_\varkappa(x)$ *is an arbitrary polynomial of the type* $Q_\varkappa(w) = \sum_{k=0}^{\varkappa} a_k w^k$ *whose coefficients satisfy the conditions*

$$\overline{a}_k = a_{\varkappa - k}, \quad k = 0, 1, \dots, \varkappa;$$

(ii) *if* $\varkappa < 0$, *then for problem* (1) *to be solvable, it is necessary and sufficient that the conditions*

$$\int_\Gamma \frac{b(t)}{X^+(w(t))\lambda(t)} [w(t)]^k w'(t) dt = 0, \quad k = 0, 1, \dots, |\varkappa| - 1$$

be fulfilled, and if these conditions are fulfilled, then the solution is unique and

$$\phi(z) = X(w(z))\left[\frac{1}{2\pi i}\int_\gamma \frac{b(z(\tau))}{X^+(z(\tau))\lambda(z(\tau))}\frac{d\tau}{\tau - w(z)} + \overline{\frac{1}{2\pi i}\int_\gamma \frac{b(z(\tau))}{X^+(z(\tau))\lambda(z(\tau))}\frac{d\tau}{\tau - \frac{1}{\overline{w(z)}}}}\right].$$

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Original article

Generalized singular integral on Carleson curves in weighted grand Lebesgue spaces

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Abstract

This paper studies the mapping properties of the integral operator generated by that singular integral which arises in the theory of I. Vekua generalized analytic functions. Boundedness problems are explored in weighted grand Lebesgue spaces.

Keywords: Vekua's generalized singular integral; Generalized analytic functions; Carleson curves; Weighted grand Lebesgue spaces

The theory of generalized analytic functions was developed by L. Bers and I. Vekua. We refer to their books [1–3]. Generalized analytic functions of the class $U_{r,2}(A, B; E)$, $r > 2$, in the sense of I. Vekua, are regular solutions of the equation

$$\partial_{\overline{z}}\Phi(z) + A(z)\Phi(z) + B(z)\overline{\Phi}(z) = 0, \tag{1}$$

where $\partial_{\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, $A(z), B(z) \in L^{r,2}(E)$, $r > 2$. Here E denotes the plane. The set of functions f defined on E is called the class $L^{r,2}(E)$ if

$$f(z) \in L^r(U), \quad f_0(z) = z^2 f\left(\frac{1}{z}\right) \in L^r(U), \quad U = \{z : |z| < 1\}.$$

Let Γ be a simple, rectifiable curve of the complex plane.

Let $f \in L^1(\Gamma)$. It is known [3,4] that the integral

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \Omega_1(z,\tau) f(\tau)\,d\tau - \Omega_2(z,\tau)\overline{f}(\tau)\,d\overline{\tau}, \tag{2}$$

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where Ω_1 and Ω_2 are the so-called basic normalized kernels of the class $U_{r,2}(A, B; E)$, is a regular solution of (1) (see [2,3] for details). The integral (2) is called the generalized Cauchy type integral. The corresponding generalized singular integral is introduced as

$$\widetilde{S}_\Gamma f(t) = \frac{1}{2\pi i} \int_\Gamma \Omega_1(t,\tau) f(\tau)\, d\tau - \Omega_2(t,\tau) \overline{f}(\tau)\, d\overline{\tau}. \tag{3}$$

The kernels Ω_1 and Ω_2 have the following structures

$$\Omega_1(z,\tau) = \frac{1}{t-z} + \frac{m_1(z,t)}{|t-z|^\alpha} \text{ and } \Omega_2(z,\tau) = \frac{m_2(z,t)}{|t-z|^\alpha}, \tag{4}$$

where the functions $m_1(z,t)$ and $m_2(z,t)$ are continuous and bounded.

In [4] for the case of L^p the following statement was proved.

Proposition A. *Let Γ be a Carleson curve. The operator $\widetilde{S}_\Gamma$ is bounded in $L^p(\Gamma)$ if*

$$p > \frac{r}{r-2}. \tag{5}$$

In [5] it was established more general result for variable exponent Lebesgue space $L^{p(t)}$ which even in the case of constant p is stronger than the existing result of Proposition A because it was admitted the whole range $1 < p < \infty$ avoiding restriction (5). In the same paper [5] it was also proved the boundedness of the operator $\widetilde{S}_\Gamma$ in weighted variable exponent Lebesgue spaces with a certain class of weights including power type weights.

This paper deals with the boundedness of $\widetilde{S}_\Gamma$ in weighted grand Lebesgue spaces. All possible cases of weighted grand Lebesgue spaces are discussed, namely the case when in the definition of the norm a weight generates absolutely continuous measure and the other case, when a weight plays a role of multiplier. It is known (see, e.g., [6]) that these two spaces are not reducible to each other unlike the classical weighted Lebesgue spaces.

Let us define the aforementioned two weighted grand Lebesgue spaces.

Let $1 < p < \infty, \theta > 0$. Let w be a weight function defined on the rectifiable curve Γ of the complex plane, i.e. w be a.e. positive and integrable function on Γ. We define the weighted grand Lebesgue space $L_w^{p),\theta}(\Gamma)$ as follows:

$$L_w^{p),\theta}(\Gamma) = \left\{ f : \|f\|_{L_w^{p),\theta}(\Gamma)} < \infty \right\},$$

where

$$\|f\|_{L_w^{p),\theta}(\Gamma)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^\theta \int_\Gamma |f(t)|^{p-\varepsilon} w(t)\, |dt| \right)^{\frac{1}{p-\varepsilon}}.$$

Now, we define another type weighted grand Lebesgue space $\mathcal{L}_w^{p),\theta}(\Gamma)$ by the norm

$$\|f\|_{\mathcal{L}_w^{p),\theta}(\Gamma)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^\theta \int_\Gamma |f(t) w(t)|^{p-\varepsilon}\, |dt| \right)^{\frac{1}{p-\varepsilon}}.$$

Both these spaces are non-reflexive, non-separable Banach spaces. It should be noted that in unweighted case when $\theta = 1$ the spaces $L^{p)}$ were introduced by T. Iwaniec and C. Sbordone [7]. More general space $L^{p),\theta}$, $\theta > 0$, was introduced by L. Greco, T. Iwaniec and C. Sbordone [8].

Let Γ be a simple finite rectifiable curve $\Gamma = \{t \in \mathbb{C} : \ t = t(s),\ 0 \le s \le t\}$ with arc-length measure $\nu(t) = s$. In the sequel we use the notation

$$D(t,r) = \Gamma \cap B(t,r), \quad B(t,r) = \left\{ \tau \in \mathbb{C} : \ |\tau - t| < r \right\}, \quad t \in \Gamma, \quad r > 0.$$

Γ is called Carleson if

$$\nu D(t,r) \le c_0 r$$

with $c_0 > 0$ not depending on t and r.

We need also the definition of the Muckenhoupt $\mathcal{A}_p(\Gamma)$ class of weights suited to the curves.
By the definition $w \in \mathcal{A}_p(\Gamma)$ if

$$\sup_{\substack{r>0\\ t\in\Gamma}}\left(\frac{1}{r}\int_{D(t,r)} w(t)\,d|t|\right)\left(\frac{1}{r}\int_{D(t,r)} (w(t))^{1-p'}\,d|t|\right)^{p-1} < \infty.$$

The main results of this paper read as follows:

Theorem 1. *Let $1 < p < \infty$, $\theta > 0$. Assume that $w \in \mathcal{A}_p(\Gamma)$. Then the operator $\widetilde{S}_\Gamma$ is bounded in $L_w^{p),\theta}(\Gamma)$.*

Theorem 2. *Let $1 < p < \infty$, $\theta > 0$. If $w^p \in \mathcal{A}_p(\Gamma)$, then the operator $\widetilde{S}_\Gamma$ is bounded in $\mathcal{L}_w^{p),\theta}(\Gamma)$.*

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Original article

Almost linear functional differential equations with Properties **A** and **B**

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Abstract

For the general functional differential equation

$$u^{(n)}(t) + F(u)(t) = 0,$$

where $F : \mathbf{C}(R_+; R) \to L_{\text{loc}}(R_+; R)$ is a continuous operator, the sufficient conditions in order to have Property **A** (Property **B**) are established. As a particular case, we consider the ordinary differential equation with a deviating argument

$$u^{(n)}(t) + p(t)\big|u(\sigma(t))\big|^{\mu(t)} \operatorname{sign} u(\sigma(t)) = 0, \tag{0.1}$$

where $p \in L_{\text{loc}}(R_+; R)$, $\sigma \in C(R_+; R_+)$, $\mu \in C(R_+; (0, +\infty))$ and $\lim_{t\to+\infty} \sigma(t) = +\infty$. Eq. (0.1) is called almost linear if $\lim_{t\to+\infty} \mu(t) = 1$. For Eq. (0.1), the sufficient conditions are obtained for the solutions to be oscillatory. These criteria cover the well-known results for the linear differential equation ($\mu(t) \equiv 1$).

Keywords: Property **A**; Property **B**; Oscillation

1. Introduction

We study oscillatory properties of solutions of a functional differential equation

$$u^{(n)}(t) + F(u)(t) = 0, \tag{1.1}$$

where $F : \mathbf{C}(R_+; R) \to L_{\text{loc}}(R_+; R)$ is a continuous mapping. Let $\tau \in C(R_+; R_+)$, $\lim_{t\to+\infty} \tau(t) = +\infty$. We denote by $V(\tau)$ the set of continuous mappings F satisfying the following condition: $F(x)(t) = F(y)(t)$ holds for any $t \in R_+$ and $x, y \in C(R_+; R)$ provided $x(s) = y(s)$ for $s \geq \tau(t)$. For any $t_0 \in R_+$, we denote by $H_{t_0,\tau}$ the set

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of all functions $u \in C(R_+; R)$ satisfying $u(t) \neq 0$ for $t \geq t_*$, where $t_* = \min\{t_0, \tau_*(t_0)\}$, $\tau_*(t) = \inf\{\tau(s) : s \geq t\}$. Throughout the work, whenever the notations $V(\tau)$ and $H_{t_0,\tau}$ occur, it will be understood, unless otherwise specified, that the function τ satisfies the conditions stated above.

It will always be assumed that either the condition

$$F(u)(t)\,u(t) \geq 0 \quad \text{for } t \geq t_0,\ u \in H_{t_0,\tau}, \tag{1.2}$$

or the condition

$$F(u)(t)\,u(t) \leq 0 \quad \text{for } t \geq t_0,\ u \in H_{t_0,\tau} \tag{1.3}$$

is fulfilled.

A function $u : [t_0, +\infty) \to R$ is said to be a proper solution of Eq. (1.1), if it is locally absolutely continuous along with its derivatives up to the order $n-1$ inclusive, $\sup\{|u(s)| : s \in [t, +\infty)\} > 0$ for $t \geq t_0$ and there exists a function $\overline{u} \in C(R_+; R)$ such that $\overline{u}(t) \equiv u(t)$ on $[t_0, +\infty)$ and the equality

$$\overline{u}^{(n)}(t) + F(\overline{u})(t) = 0$$

holds for $t \in [t_0, +\infty)$. A proper solution $u : [t_0, +\infty) \to R$ of Eq. (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise, the solution u is said to be nonoscillatory.

Definition 1.1. We say that Eq. (1.1) has Property **A** if any of its proper solutions is oscillatory, when n is even, and is either oscillatory or satisfies

$$\left|u^{(i)}(t)\right| \downarrow 0 \quad \text{as } t \uparrow +\infty\ (i = 0, \ldots, n-1), \tag{1.4}$$

when n is odd.

Definition 1.2. We say that Eq. (1.1) has Property **B** if any of its proper solutions is either oscillatory or satisfies either (1.4) or

$$\left|u^{(i)}(t)\right| \uparrow +\infty \quad \text{as } t \uparrow +\infty\ (i = 0, \ldots, n-1), \tag{1.5}$$

when n is even, and is either oscillatory or satisfies (1.5), when n is odd.

A. Kneser was the first who showed that the condition

$$\liminf_{t \to +\infty} t^{n/2} p(t) > 0$$

is sufficient for the equation

$$u^{(n)}(t) + p(t)\,u(t) = 0 \tag{1.6}$$

to have Property **A** [1]. This result was essentially generalized by Kondratev [2]. His method was based on a comparison theorem which enables one to obtain optimal results for establishing oscillatory properties of solutions of Eq. (1.6). The following comparison theorem was proved. If the inequality

$$p(t) \geq q(t) \geq 0 \quad \text{for } t \geq 0$$

holds and the equation

$$u^{(n)}(t) + q(t)\,u(t) = 0 \tag{1.7}$$

has Property **A**, then Eq. (1.6) has also Property **A**.

T. Chanturia [3] showed that in the case of Property **B**, if

$$p(t) \leq q(t) \leq 0 \quad \text{for } t \in R_+$$

and Eq. (1.7) has Property **B**, then Eq. (1.6) has also Property **B**.

This comparison theorem implies that if $p(t) \geq 0$ ($p(t) \leq 0$) and

$$\liminf_{t \to +\infty} t^n \big|p(t)\big| > M_n \quad (M_n^*), \tag{1.8}$$

then Eq. (1.6) has Property **A** (**B**), where

$$\begin{aligned} &M_n = \max\big\{-\lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda \in [0, n-1]\big\} \\ &\Big(M_n^* = \max\big\{\lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda \in [0, n-1]\big\}\Big). \end{aligned} \tag{1.9}$$

It should be noted that inequality (1.8) cannot be replaced by a constrict one.

The sufficient integral conditions for Eq. (1.6) to have Property **A** or Property **B** were given in [4–6]. Later, T. Chanturia [7] proved the integral comparison theorems which are integral generalizations of the above-mentioned comparison theorems. Using these theorems, he succeeded in improving condition (1.8). Namely, he showed that if $p(t) \geq 0$ ($p(t) \leq 0$) and the inequality

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-2} p(s) ds > M_n \qquad \left(\liminf_{t \to +\infty} t^2 \int_t^{+\infty} s^{n-3} \big|p(s)\big| ds > \frac{M_n^*}{2}\right)$$

is fulfilled, then Eq. (1.6) has Property **A** (**B**), where M_n (M_n^*) is given by (1.9). R. Koplatadze [8,9] proved two types of integral comparison theorems for differential equations with deviating arguments. The theorem of the first type enables one not only to generalize the above-mentioned results for equations with deviated arguments, but also to improve Chanturia's result concerning Property **B** even in the case of Eq. (1.6).

The ordinary differential equation with deviating argument

$$u^{(n)}(t) + p(t)\big|u(\sigma(t))\big|^{\mu(t)} \operatorname{sign} u(\sigma(t)) = 0 \tag{1.10}$$

is a particular case of Eq. (1.1), where $p \in L_{\mathrm{loc}}(R_+; R)$, $\sigma \in C(R_+; R)$, $\mu \in C(R_+; (0, +\infty))$ and $\lim_{t\to+\infty} \sigma(t) = +\infty$. In case $\lim_{t\to+\infty} \mu(t) = 1$, we call the differential equation (1.10) almost linear, while if $\liminf_{t\to+\infty} \mu(t) \neq 1$, or $\limsup_{t\to+\infty} \mu(t) \neq 1$, then we call Eq. (1.10) the essentially nonlinear generalized Emden–Fowler type differential equation.

In the present paper we study both cases of Properties **A** and **B**, when the operator F has an almost linear minorant. It turned out that even in the case of Property **A** (Property **B**) there arises the possibility to improve the results obtained in [10–13]. The method used in this paper enables one to get such statements for a quite general equation which, when specialized to the well studied equations, provide us with qualitatively new results.

2. Some auxiliary lemmas

In the sequel, $\widetilde{C}_{\mathrm{loc}}^{n-1}([t_0, +\infty))$ will denote the set of all functions $u : [t_0, +\infty) \to R$, absolutely continuous on any finite subinterval of $[t_0, +\infty)$, along with their derivatives of order up to and including $n-1$.

Lemma 2.1. *Let $u \in \widetilde{C}_{\mathrm{loc}}^{n-1}([t_0, +\infty))$, $u(t) > 0$, $u^{(n)}(t) \leq 0$ ($u^{(n)}(t) \geq 0$) for $t \geq t_0$ and $u^{(n)} \not\equiv 0$ in any neighborhood of $+\infty$. Then there exist $t_1 \geq t_0$ and $\ell \in \{0, \dots, n\}$ such that $\ell + n$ odd ($\ell + n$ even) and*

$$\begin{aligned} u^{(i)}(t) > 0 \quad &\text{for } t \geq t_1 \ (i = 0, \dots, \ell - 1), \\ (-1)^{\ell+i} u^{(i)}(t) \geq 0 \quad &\text{for } t \geq t_1 \ (i = \ell, \dots, n). \end{aligned} \tag{2.1$_\ell$}$$

Note. In (2.1$_\ell$), if $\ell = 0$, there takes place the second inequality, but if $\ell = n$, there takes place the first one.

Lemma 2.2. *Let $t_0 \in (0, +\infty)$, $u \in \widetilde{C}_{\mathrm{loc}}([t_0, +\infty))$, $u^{(n)}(t) \leq 0$, ($u^{(n)}(t) \geq 0$) and (2.1$_\ell$) be fulfilled for some $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd ($\ell + n$ even). Then*

$$\int_{t_0}^{+\infty} t^{n-\ell-1} \big|u^{(n)}(t)\big| dt < +\infty, \tag{2.2}$$

$$\int_{t_0}^{+\infty} s^{-2} \int_{t_0}^{s} \xi^{n-\ell} \big|u^{(n)}(\xi)\big| d\xi\, ds < +\infty, \tag{2.3}$$

$$\lim_{t\to+\infty} \frac{1}{t} \int_{t_0}^{t} s^{n-\ell} \big|u^{(n)}(s)\big| ds = 0, \tag{2.4}$$

$$u^{(i)}(t) \ge u^{(i)}(t_0) + \frac{1}{(\ell-i-1)!\,(n-\ell-1)!} \int_{t_0}^{t} (t-s)^{\ell-i-1} \int_{s}^{+\infty} (\xi-s)^{n-\ell-1} \big|u^{(n)}(\xi)\big| d\xi\, ds$$
$$\text{for } t \ge t_0,\ (i = 0,\dots,\ell-1). \tag{2.5$_i$}$$

If, moreover,

$$\int_{t_0}^{+\infty} t^{n-\ell} \big|u^{(n)}(t)\big| dt = +\infty, \tag{2.6}$$

then there exists $t_1 \ge t_0$ *such that*

$$u(t) \ge \frac{t^{\ell-1}}{\ell!}\, u^{(\ell-1)}(t) \quad \text{for } t \ge t_1, \tag{2.7}$$

$$\lim_{t\to+\infty} \big(u^{(\ell-1)}(t) - t\,u^{(\ell)}(t)\big) = +\infty, \tag{2.8}$$

$$\frac{u^{(i)}(t)}{t^{\ell-i}} \downarrow, \qquad \frac{u^{(i)}(t)}{t^{\ell-i-1}} \uparrow \quad (i = 0,\dots,\ell-1) \text{ as } t \uparrow +\infty. \tag{2.9$_i$}$$

Proof. For the proof of validity of conditions (2.2), (2.5_i), (2.7), (2.8), (2.9_i) see [10]. Let $\varepsilon > 0$. According to (2.2), choose $t_* > t_0$ such that

$$\int_{t_*}^{+\infty} s^{n-\ell-1} \big|u^{(n)}(s)\big| ds < \varepsilon.$$

Therefore

$$\limsup_{t\to+\infty} \frac{1}{t} \int_{t_*}^{t} s^{n-\ell} \big|u^{(n)}(s)\big| ds = \limsup_{t\to+\infty} \frac{1}{t} \int_{t_0}^{t} s^{n-\ell} \big|u^{(n)}(s)\big| ds \le \int_{t_*}^{+\infty} s^{n-\ell-1} \big|u^{(n)}(s)\big| ds < \varepsilon.$$

Taking into account that ε is arbitrary, from the latter inequality we deduce the validity of condition (2.4). If we take into account (2.1_ℓ) and (2.6), then from the equality

$$\sum_{j=\ell-1}^{n-1} \frac{(-1)^j\, t^{j-\ell+1} u^{(j)}(t)}{(j-\ell+1)!} = \sum_{j=\ell-1}^{n-1} \frac{(-1)^j\, t_0^{j-\ell+1} u^{(j)}(t_0)}{(j-\ell+1)!} + \frac{(-1)^{n-1}}{(n-\ell)!} \int_{t_0}^{t} s^{n-\ell} u^{(n)}(s) ds$$

we can see that there exists $t_2 \ge t_1$ such that

$$\frac{1}{(n-\ell)!} \int_{t_2}^{t} s^{n-\ell} \big|u^{(n)}(s)\big| ds \le u^{(\ell-1)}(t) - t\,u^{(\ell)}(t) \quad \text{for } t \ge t_2.$$

According to ($2.9_{\ell-1}$), we have

$$\frac{1}{(n-\ell)!} \int_{t_2}^{+\infty} s^{-2} \int_{t_2}^{s} \xi^{n-\ell} \big|u^{(n)}(\xi)\big| d\xi\, ds \le -\int_{t_2}^{+\infty} \left(\frac{u^{(\ell-1)}(s)}{s}\right)' ds < +\infty.$$

Hence (2.3) is fulfilled. □

Lemma 2.3 *([11]). Let* $n \ge 2$, $\ell \in \{1,\dots,n-1\}$, $u_0 \in \widetilde{C}_{\text{loc}}^{n-1}([t_0,+\infty))$. *Then*

$$u_i^{(\ell)}(t) = (-1)^i\, t^i\, u_0^{\ell+i}(t) \quad (i = 1,\dots,n-\ell), \tag{2.10}$$

where

$$u_i(t) = (\ell+i-1)u_{i-1}(t) - t\,u_{i-1}'(t) \quad (i = 1,\dots,n-\ell). \tag{2.11}$$

Lemma 2.4. *Let $t_0 \in (0,+\infty)$, $u \in \widetilde{C}_{\rm loc}([t_0,+\infty))$, $u^{(n)}(t) \le 0$ ($u^{(n)}(t) \ge 0$) and for any $\ell \in \{1,\dots,n-1\}$, where $\ell+n$ odd ($\ell+n$ even), conditions (2.1_ℓ) and (2.6) be fulfilled. Then there exists $t_1 > t_0$ such that*

$$u(t) \ge \frac{t^\ell}{(\ell-1)!\,(n-\ell-1)!}\int_t^{+\infty} s^{-n}(s-t)^{n-\ell-1}\int_{t_1}^s (s-\xi)^{\ell-1}\xi^{n-\ell}\big|u^{(n)}(\xi)\big|d\xi\,ds \quad \text{for } t \ge t_1. \tag{2.12}$$

Proof. By (2.1_ℓ) and (2.5_0), we have

$$u(t) \ge \frac{1}{(\ell-1)!\,(n-\ell-1)!}\int_{t_0}^t (t-s)^{\ell-1}\int_s^{+\infty}(\xi-s)^{n-\ell-1}\big|u^{(n)}(\xi)\big|d\xi\,ds \quad \text{for } t \ge t_0. \tag{2.13}$$

Denote

$$u_0(t) = \frac{t^\ell}{(\ell-1)!\,(n-\ell-1)!}\int_t^{+\infty} s^{-n}(s-t)^{n-\ell-1}\int_{t_0}^s (s-\xi)^{\ell-1}\xi^{n-\ell}\big|u^{(n)}(\xi)\big|d\xi\,ds. \tag{2.14}$$

According to (2.3), it is obvious that the integral on the right-hand side of (2.14) exists and equalities (2.11) are fulfilled, where

$$u_i(t) = \frac{t^{\ell+i}}{(\ell-1)!\,(n-\ell-i-1)!}\int_t^{+\infty} s^{-n}(s-t)^{n-\ell-i-1}\int_{t_0}^s (s-\xi)^{\ell-1}\xi^{n-\ell}\big|u^{(n)}(\xi)\big|d\xi\,ds \tag{2.15}$$

$$(i=1,\dots,n-\ell-1),$$

$$u_{n-\ell}(t) = \frac{1}{(\ell-1)!}\int_{t_0}^t (t-s)^{\ell-1}s^{n-\ell}\big|u^{(n)}(s)\big|ds. \tag{2.16}$$

Therefore, by virtue of Lemma 2.3, we have

$$(-1)^{n+\ell}u_0^{(n)}(t) = \big|u^{(n)}(t)\big| \quad \text{for } t \ge t_0, \tag{2.17}$$

where $n+\ell$ odd ($n+\ell$ even) if $u^{(n)}(t) \le 0$ ($u^{(n)}(t) \ge 0$). Therefore, according to (2.17)

$$u_0^{(n)}(t) \le 0 \quad \text{if } \ell+n \text{ odd} \tag{2.18}$$

and

$$u_0^{(n)}(t) \ge 0 \quad \text{if } \ell+n \text{ even}. \tag{2.19}$$

By (2.14) and (2.18) ((2.14) and (2.19)), there exists $\ell' \in \{0,\dots,n-1\}$ such that $\ell'+n$ odd ($\ell'+n$ even) and the conditions

$$u_0^{(i)}(t) > 0 \quad (i=0,\dots,\ell'-1), \qquad (-1)^{i+\ell'}u_0^{(i)}(t) > 0 \quad (i=\ell',\dots,n-1) \text{ for } t \ge t_0 \tag{$2.20_{\ell'}$}$$

are fulfilled.

Let us now show that there exists $t_1 > t_0$ such that the inequality

$$t^{\ell-1} \le u_0(t) \le t^\ell \quad \text{for } t \ge t_1 \tag{2.21}$$

is fulfilled. According to (2.3) and (2.14), the validity of the right inequality in (2.21) is obvious. By virtue of (2.6), there exists $t_1 > t_0$ such that

$$\int_{t_0}^{t_1} s^{n-\ell}\big|u^{(n)}(s)\big|ds = c > (\ell-1)!\,(n-\ell-1)!\,2^{n-1}.$$

Therefore from (2.14), we obtain

$$u_0(t) \ge \frac{t^\ell}{(\ell-1)!\,(n-\ell-1)!}\int_{2t}^{+\infty}(s-t)^{n-\ell-1}s^{-n-1+\ell}\int_{t_0}^{t_1}\Big(1-\frac{\xi}{s}\Big)^{\ell-1}\xi^{n-\ell}\big|u^{(n)}(\xi)\big|d\xi\,ds$$

$$\ge \frac{t^\ell}{2^{\ell-1}(\ell-1)!\,(n-\ell-1)!}\int_{t_0}^{t_1}\xi^{n-\ell}\big|u^{(n)}(\xi)\big|d\xi\int_{2t}^{+\infty}(s-t)^{n-\ell-1}s^{-n-1+\ell}ds$$

$$\geq \frac{c\,t^{\ell}}{2^{\ell-1}(\ell-1)!\,(n-\ell-1)!}\int_{2t}^{+\infty}\Big(1-\frac{t}{s}\Big)^{n-\ell-1}s^{-2}ds$$
$$\geq \frac{c\,t^{\ell}}{(\ell-1)!\,(n-\ell-1)!\,2^{n-2}}\int_{2t}^{+\infty}s^{-2}ds \geq t^{\ell-1}\quad \text{for } t\geq t_1.$$

The latter inequality implies the existence of t_1 such that inequality (2.21) is fulfilled. Therefore, according to (2.1_{ℓ}) and ($2.20_{\ell'}$), since $\ell+\ell'$ even, it follows that $\ell=\ell'$.

Now show that

$$\lim_{t\to+\infty} t^i u_0^{(\ell+i)}(t)=0 \quad (i=0,\dots,n-\ell-1). \tag{2.22}$$

Indeed, using (2.15), we have

$$u_i^{(\ell)}(t)=\frac{1}{(\ell-1)!\,(n-\ell-i-1)!}\sum_{j=0}^{\ell}c_{\ell}^{j}(t^{\ell+i})^{(\ell-j)}$$
$$\times\left(\int_t^{+\infty}s^{-n}(s-t)^{n-\ell-i-1}\int_{t_0}^{s}(s-t)^{\ell-1}\xi^{n-\ell}\big|u^{(n)}(\xi)\big|d\xi\,ds\right)^{(j)}. \tag{2.23}$$

Let $j\leq n-\ell-i-1$. Then according to (2.3), we obtain

$$\rho_{ji}(t)=(t^{\ell+i})^{(\ell-j)}\left|\left(\int_t^{+\infty}s^{-n}(s-t)^{n-\ell-i-1}\int_{t_0}^{s}(s-\xi)^{\ell-1}\xi^{n-\ell}\big|u^{(n)}(\xi)\big|d\xi\,ds\right)^{(j)}\right|$$
$$\leq(\ell+i)!\,(n-\ell-i-1)!\,t^{i+j}\int_t^{+\infty}s^{-n}(s-t)^{n-\ell-i-j-1}\int_{t_0}^{s}(s-\xi)^{\ell-1}\xi^{n-\ell}\big|u^{(n)}(\xi)\big|d\xi\,ds$$
$$\leq(\ell+i)!\,(n-\ell-i-1)!\,t^{i+j}\int_t^{+\infty}s^{-i-j-2}\int_{t_0}^{s}\xi^{n-\ell}\big|u^{(n)}(\xi)\big|d\xi\,ds$$
$$\leq(\ell+i)!\,(n-\ell-i-1)!\int_t^{+\infty}s^{-2}\int_{t_0}^{s}\xi^{n-\ell}\big|u^{(n)}(\xi)\big|d\xi\,ds\to 0$$
$$\text{for } t\to+\infty\ (j=0,\dots,n-\ell-i-1). \tag{2.24}$$

Analogously, we can prove that

$$\rho_{ji}(t)\to 0\quad \text{for } t\to+\infty\ (j=n-\ell-i,\dots,\ell). \tag{2.25}$$

According to (2.23), (2.24), (2.10) and (2.25), it is obvious that conditions (2.22) are fulfilled. Therefore, by (2.16) we have

$$u_0^{(\ell)}(t)=\frac{1}{(n-\ell-1)!}\int_t^{+\infty}(s-t)^{n-\ell-1}\big|u^{(n)}(s)\big|ds\quad \text{for } t\geq t_0.$$

From the above equality, we obtain

$$u_0(t)=\sum_{i=0}^{\ell-1}\frac{(t-t_0)^i}{i!}u_0^{(i)}(t_0)$$
$$+\frac{1}{(\ell-1)!\,(n-\ell-1)!}\int_{t_0}^{t}(t-s)^{\ell-1}\int_s^{+\infty}(\xi-s)^{n-\ell-1}\big|u^{(n)}(\xi)\big|d\xi\,ds\quad \text{for } t\geq t_0.$$

Hence, if we take into account (2.13) and (2.14), we obtain

$$u(t)\geq\frac{t^{\ell}}{(\ell-1)!\,(n-\ell-1)!}\int_t^{+\infty}s^{-n}(s-t)^{n-\ell-1}\int_{t_0}^{s}(s-\xi)^{\ell-1}\xi^{n-\ell}\big|u^{(n)}(\xi)\big|d\xi\,ds-ct^{\ell-1}$$
$$\text{for } t\geq t_0, \tag{2.26}$$

where $c=\sum_{i=0}^{\ell-1}\frac{u_0^{(i)}(t_0)}{\ell!}$.

By virtue of (2.6), we choose $t_2 > t_0$ such that

$$\int_{t_0}^{t_2} s^{n-\ell}\big|u^{(n)}(s)\big|ds > 2^{n-1}(\ell-1)!\,(n-\ell-1)!\,c.$$

Hence from (2.26) follows

$$\begin{aligned} u(t) &\geq \frac{t^\ell}{(\ell-1)!\,(n-\ell-1)!}\int_t^{+\infty} s^{-n}(s-t)^{n-\ell-1}\bigg(\int_{t_0}^{t_2}(s-\xi)^{\ell-1}\xi^{n-\ell}\big|u^{(n)}(\xi)\big|d\xi \\ &\quad + \int_{t_2}^{s}(s-\xi)^{\ell-1}\xi^{n-2}\big|u^{(n)}(\xi)\big|d\xi\bigg)ds - ct^{\ell-1} \\ &\geq \frac{t^\ell}{(\ell-1)!\,(n-\ell-1)!}\int_t^{+\infty} s^{-n}(s-t)^{n-\ell-1}\int_{t_2}^{s}(s-\xi)^{\ell-1}\xi^{n-\ell}\big|u^{(n)}(\xi)\big|d\xi\, ds \\ &\quad + \frac{t^\ell}{2^{n-2}(\ell-1)!\,(n-\ell-1)!}\int_{t_0}^{t_2}\xi^{n-\ell}\big|u^{(n)}(\xi)\big|d\xi\cdot\int_{2t}^{+\infty}s^{-n}ds - ct^{\ell-1} \\ &> \frac{t^\ell}{(\ell-1)!\,(n-\ell-1)!}\int_t^{+\infty} s^{-n}(s-t)^{n-\ell-1}\int_{t_1}^{s}(s-\xi)^{\ell-1}\xi^{n-\ell}\big|u^{(n)}(\xi)\big|d\xi\, ds \quad \text{for } t \geq 2t_2, \end{aligned}$$

which proves the validity of inequality (2.12) with $t_1 = 2t_2$. □

Lemma 2.5 *([11]). Let $t_0 \in R_+$, $\varphi; \psi \in C([t_0, +\infty), (0, +\infty))$,*

$$\liminf_{t\to+\infty}\varphi(t) = 0, \qquad \psi(t) \uparrow +\infty \quad \textit{for } t \uparrow +\infty, \qquad \lim_{t\to+\infty}\psi(t)\,\widetilde{\varphi}(t) = +\infty,$$

where $\widetilde{\varphi}(t) = \min\{\varphi(s) : s \in [t_0, t]\}$. Then there exists a sequence of points $\{t_k\}$ such that $t_k \uparrow +\infty$, as $k \uparrow +\infty$,

$$\widetilde{\varphi}(t_k)\,\psi(t_k) \leq \widetilde{\varphi}(s)\,\psi(s) \quad \textit{for } s \geq t_k,\ \widetilde{\varphi}(t_k) = \varphi(t_k)\ (k = 1, 2, \ldots).$$

Remark 2.1. Lemma 2.5 concerns some properties of nonmonotone positive functions. A lemma of different type likewise concerning some properties of nonmonotone functions can be found in [10, Lemma 7.7]. Everywhere below we assume that the inequality

$$\big|F(u)(t)\big| \geq \sum_{i=1}^{m}\int_{\tau_i(t)}^{\sigma_i(t)}\big|u(s)\big|^{\mu_i(s)}d_s r_i(s,t) \quad \text{for } t \geq t_0,\ u \in H_{t_0,\tau}, \tag{2.27}$$

holds, where

$$\begin{aligned} &\mu_i \in C(R_+; (0, +\infty)), \quad \tau_i; \sigma_i \in C(R_+; R_+),\ \tau_i(t) \leq \sigma_i(t) \text{ for } t \in R_+, \\ &\lim_{t\to+\infty}\tau_i(t) = +\infty \quad (i = 1, \ldots, m), \end{aligned} \tag{2.28}$$

$r_i : R_+ \times R_+ \to R_+$ are measurable in t and nondecreasing in s functions $(i = 1, \ldots, m)$.

3. Necessary conditions of the existence of monotone solutions

In Section 3, the necessary conditions will be established for the existence of a solution of type (2.1_ℓ). Mainly we apply the method used in [11] which still made it possible to improve some results in the case of Properties **A** and **B**.

Throughout the work, the use will be made of the following notation:

Let $t_0 \in R_+$, $\ell \in \{1, \ldots, n-1\}$. By U_{ℓ,t_0} we denote the set of proper solutions of Eq. (1.1) satisfying condition (2.1_ℓ);

$$\Lambda_{\ell,u} = \Big\{\lambda|\lambda \in [\ell-1, \ell],\ \liminf_{t\to+\infty}\frac{u(t)}{t^\lambda} = 0\Big\}, \quad u \in U_{\ell,t_0}; \tag{3.1}$$

$$\sigma_*(t) = \max\Big\{\max\big(s, \sigma_1(s), \ldots, \sigma_m(s)\big) : 0 \leq s \leq t\Big\}; \tag{3.2}$$

$$h_{1\varepsilon}(\lambda)=\begin{cases}0 & \text{for } \lambda=\ell,\\ \varepsilon & \text{for } \lambda\in[\ell-1,\ell),\end{cases}\qquad h_{2\varepsilon}(\lambda)=\begin{cases}0 & \text{for } \lambda=\ell-1,\\ \varepsilon & \text{for } \lambda\in(\ell-1,\ell],\end{cases}\qquad h_{\varepsilon\lambda}=h_{1\varepsilon}(\lambda)+h_{2\varepsilon}(\lambda). \tag{3.3}$$

Remark 3.1. We usually do not distinguish between the notations for a constant and the function which is identically equal to that constant.

Remark 3.2. In the definition of the set $\Lambda_{\ell,u}$ we assume that if there is no $\lambda\in[\ell-1,\ell]$ such that $\liminf_{t\to+\infty} t^{-\lambda} u(t)=0$, then $\Lambda_{\ell,u}=\varnothing$.

Theorem 3.1. *Let $F\in V(\tau)$, the conditions* (1.2) *((1.3)),* (2.27), (2.28) *be fulfilled, $\ell\in\{1,\dots,n-1\}$, where $\ell+n$ is odd and ($\ell+n$ is even),*

$$\int_0^{+\infty} t^{n-\ell}\sum_{i=1}^m\int_{\tau_i(t)}^{\sigma_i(t)} s^{(\ell-1)\mu_i(s)}d_sr_i(s,t)dt=+\infty, \tag{3.4$_\ell$}$$

$$\int_0^{+\infty} t^{n-\ell-1}\sum_{i=1}^m\int_{\tau_i(t)}^{\sigma_i(t)} s^{\ell\,\mu_i(s)}d_sr_i(s,t)dt=+\infty, \tag{3.5$_\ell$}$$

and

$$\liminf_{t\to+\infty}\mu_i(t)>0\quad (i=1,\dots,m). \tag{3.6}$$

Moreover, let $U_{\ell,t_0}\neq\varnothing$ for some $t_0\in R_+$. Then there exists $\lambda_0\in[\ell-1,\ell]$ such that

$$\limsup_{\varepsilon\to0+}\Big(\liminf_{t\to+\infty} g_\ell(t,\lambda_0,\varepsilon)\Big)\le(\ell-1)!\,(n-\ell-1)!, \tag{3.7$_\ell$}$$

where

$$\begin{aligned} g_\ell(t,\lambda,\varepsilon)&=t^{\ell-\lambda+h_{2\varepsilon}(\lambda)}\int_t^{+\infty}s^{-n}(s-t)^{n-\ell-1}(\sigma_*(s))^{-h_\varepsilon(\lambda)}\int_0^s(s-\xi)^{\ell-1}\xi^{n-\ell}\\ &\quad\times\sum_{i=1}^m\int_{\tau_i(\xi)}^{\sigma_i(\xi)}\xi_1^{\mu_*(\xi_1,\lambda,\varepsilon)}d_{\xi_1}r_i(\xi_1,\xi)d\xi\,ds, \end{aligned} \tag{3.8}$$

$$\mu_*(t,\lambda,\varepsilon)=(\lambda+h_{1\varepsilon}(\lambda))\mu(t)+(\lambda-h_{2\varepsilon}(\lambda))(\mu_i(t)-\mu(t)), \tag{3.9}$$

$$\mu(t)=\min\{1,\mu_1(t),\dots,\mu_m(t)\}, \tag{3.10}$$

$h_{1\varepsilon}$, $h_{2\varepsilon}$,, h_ε are given by (3.3).

Proof. Let $t_0\in R_+$ and $U_{\ell,t_0}\neq\varnothing$. By the definition of the set U_{ℓ,t_0}, Eq. (1.1) has a proper solution $u\in U_{\ell,t_0}$ satisfying condition (2.1$_\ell$). According to (1.2) ((1.3)), (2.1$_\ell$), (2.27) and (3.4$_\ell$), it is clear that condition (2.6) holds. Thus by Lemma 2.4, (2.27) and (3.4$_\ell$), there exists $t_*>t_0$ such that

$$\begin{aligned} u(t)&\ge\frac{t^\ell}{(\ell-1)!\,(n-\ell-1)!}\int_t^{+\infty}s^{-n}(s-t)^{n-\ell-1}\int_{t_*}^s(s-\xi)^{\ell-1}\xi^{n-\ell}\\ &\quad\times\sum_{i=1}^m\int_{\tau_i(t)}^{\sigma_i(t)}\big|u(\xi_1)\big|^{\mu_i(\xi_1)}d_{\xi_1}r_i(\xi_1,\xi)d\xi\,ds\quad\text{for } t\ge t_* \end{aligned} \tag{3.11}$$

and

$$\lim_{t\to+\infty}\frac{u(t)}{t^{\ell-1}}=+\infty,\qquad \lim_{t\to+\infty}\frac{u(t)}{t^\ell}=0. \tag{3.12}$$

In view (3.1) and (3.12), we have $\ell\in\Lambda_{\ell,u}$, $\ell-1\notin\Lambda_{\ell,u}$ and $\lambda_0\in[\ell-1,\ell]$, where $\lambda_0=\inf\Lambda_{\ell,u}$. Therefore, $\Lambda_{\ell,u}\subset[\ell-1,\ell]$ and for sufficiently small $\varepsilon>0$, by (3.3), we have

$$\liminf_{t\to+\infty}\frac{u(t)}{t^{\lambda_0+h_{1\varepsilon}(\lambda_0)}}=0\quad\text{and}\quad\lim_{t\to+\infty}\frac{u(t)}{t^{\lambda_0-h_{2\varepsilon}(\lambda_0)}}=+\infty. \tag{3.13}$$

Denote

$$\varphi(t) = \left(\frac{u(t)}{t^{\lambda_0 + h_{1\varepsilon}(\lambda_0)}} \right)^{\mu(t)} \tag{3.14}$$

and

$$\widetilde{\varphi}(t) = \min\{\varphi(s) : t_* \le s \le t\}, \tag{3.15}$$

where μ is defined by (3.10).

By the first condition of (3.13) and (3.6), it is obvious that

$$\lim_{t\to+\infty} \widetilde{\varphi}(t) = 0. \tag{3.16}$$

Let us show that

$$\lim_{t\to+\infty} t^{h_\varepsilon(\lambda_0)}\, \widetilde{\varphi}(t) = +\infty, \tag{3.17}$$

where h_ε is defined by (3.3). Indeed, for any $t > t_*$, by (3.14)–(3.16), there exists $s_t \in [t_*, t]$ such that $s_t \to +\infty$ for $t \to +\infty$ and

$$\widetilde{\varphi}(t) = \left(\frac{u(s_t)}{s_t^{\lambda_0 + h_{1\varepsilon}(\lambda_0)}} \right)^{\mu(s_t)}.$$

From this equality, since $\widetilde{\varphi}(t) \downarrow 0$ for $t \uparrow +\infty$, by (3.3) and the second condition of (3.13), we have

$$t^{h_\varepsilon(\lambda_0)}\widetilde{\varphi}(t) \ge s_t^{h_\varepsilon(\lambda_0)} \frac{u(s_t)}{s_t^{\lambda_0 + h_{1\varepsilon}(\lambda_0)}} = \frac{u(s_t)}{s_t^{\lambda_0 - h_{2\varepsilon}(\lambda_0)}} \to +\infty \quad \text{for } t \to +\infty,$$

that is, (3.17) holds. Using (3.2) and (3.15), from (3.11), we have

$$\begin{aligned}
u(\sigma_*(t)) &\ge \frac{\sigma_*^\ell(t)}{(\ell-1)!\,(n-\ell-1)!} \int_{\sigma_*(t)}^{+\infty} s^{-n}\big(s-\sigma_*(t)\big)^{n-\ell-1} \int_{t_*}^{s} (s-\xi)^{\ell-1}\xi^{n-\ell} \\
&\quad \times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \big|u(\xi_1)\big|^{\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1,\xi)\, d\xi\, ds \\
&\ge \frac{\sigma_*^\ell(t)}{(\ell-1)!\,(n-\ell-1)!} \int_{\sigma_*(t)}^{+\infty} s^{-n}\big(s-\sigma_*(t)\big)^{n-\ell-1} \int_{t_*}^{s} (s-\xi)^{\ell-1}\xi^{n-\ell} \\
&\quad \times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \left(\frac{u(\xi)}{\xi_1^{\lambda_0 - h_{2\varepsilon}(\lambda_0)}} \right)^{\mu(\xi_1)} \xi_1^{(\lambda_0 - h_{2\varepsilon}(\lambda_0))\mu_i(\xi)} d_{\xi_1} r_i(\xi_1,\xi)\, d\xi\, ds \\
&= \frac{\sigma_*^\ell(t)}{(\ell-1)!\,(n-\ell-1)!} \int_{\sigma_*(t)}^{+\infty} s^{-n}\big(s-\sigma_*(t)\big)^{n-\ell-1} \int_{t_*}^{s} (s-\xi)^{\ell-1}\xi^{n-\ell} \\
&\quad \times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \left(\frac{u(\xi_1)}{\xi_1^{\lambda_0 + h_{1\varepsilon}(\lambda_0)}} \right)^{\mu(\xi_1)} \xi_1^{(\lambda_0 + h_{1\varepsilon}(\lambda_0))\mu(\xi_1)} \\
&\quad \times \xi_1^{(\lambda_0 - h_{2\varepsilon}(\lambda_0))(\mu_i(\xi_1) - \mu(\xi_1))} d_{\xi_1} r_i(\xi_1,\xi)\, d\xi\, ds \\
&\ge \frac{\sigma_*^\ell(t)}{(\ell-1)!\,(n-\ell-1)!} \int_{\sigma_*(t)}^{+\infty} s^{-n}\big(s-\sigma_*(t)\big)^{n-\ell-1} \widetilde{\varphi}(\sigma_*(s)) \int_{t_*}^{s} (s-\xi)^{\ell-1}\xi^{n-\ell} \\
&\quad \times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{\mu_*(\xi_1,\lambda_0,\varepsilon)} d_{\xi_1} r_i(\xi_1,\xi)\, d\xi\, ds,
\end{aligned} \tag{3.18}$$

where σ_*, μ_* and μ are defined by (3.2), (3.9) and (3.10), the functions φ and $\widetilde{\varphi}$ are defined by (3.14) and (3.15). In view of (3.6), (3.12) and the second condition of (3.13), it is obvious that the functions

$$\varphi(t) = \left(\frac{u(t)}{t^{\lambda_0 - h_{1\varepsilon}(\lambda_0)}} \right)^{\mu(t)} \quad \text{and} \quad \psi(t) = t^{h_\varepsilon(\lambda_0)}$$

satisfy the conditions of Lemma 2.5. Therefore there exists the sequence $\{t_k\}_{k=1}^{+\infty}$ such that $t_k \uparrow +\infty$ for $k \uparrow +\infty$ and

$$\widetilde{\varphi}(\sigma_*(s))\,(\sigma_*(s))^{h_\varepsilon(\lambda_0)} \geq \widetilde{\varphi}\big(\sigma_*(t_k)\,(\sigma_*(t_k))\big)^{h_\varepsilon(\lambda_0)} \quad \text{for } s \geq t_k, \tag{3.19}$$

$$\widetilde{\varphi}(\sigma_*(t_k)) = \varphi(\sigma_*(t_k)) = \left(\frac{u(\sigma_*(t_k))}{\sigma_*^{\lambda_0+h_{1\varepsilon}(\lambda_0)}(t_k)}\right)^{\mu(\sigma_*(t_k))} \quad (k = 1, 2, \ldots). \tag{3.20}$$

According to (3.19) and (3.20), from (3.18), we get

$$u(\sigma_*(t_k)) \geq \frac{\sigma_*^{\ell}(t_k)(\sigma_*(t_k))^{h_\varepsilon(\lambda_0)} \cdot \widetilde{\varphi}(\sigma_*(t_k))}{(\ell-1)!\,(n-\ell-1)!} \int_{\sigma_*(t_k)}^{+\infty} s^{-n}\big(s-\sigma_*(t_k)\big)^{n-\ell-1}(\sigma_*(s))^{-h_\varepsilon(\lambda_0)}$$
$$\times \int_{t_*}^{s} (s-\xi)^{\ell-1}\xi^{n-\ell} \sum_{i=1}^{m} \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{\mu_*(\xi_1,\lambda_0,\varepsilon)} d_{\xi_1} r_i(\xi_1,\xi)d\xi\, ds \quad (k = 1, 2, \ldots),$$

where σ_*, μ_* and μ are given by (3.2), (3.9) and (3.10).

From this latter inequality, we have

$$\left(\frac{u(\sigma_*(t_k))}{\sigma_*^{\lambda_0+h_{1\varepsilon}(\lambda_0)}(t_k)}\right)^{1-\mu(\sigma_*(t_k))} \geq \frac{(\sigma_*(t_k))^{\ell+h_\varepsilon(\lambda_0)-\lambda_0-h_{1\varepsilon}(\lambda_0)}}{(\ell-1)!\,(n-\ell-1)!} \int_{\sigma_*(t_k)}^{+\infty} s^{-n}\big(s-\sigma_*(t_k)\big)^{n-\ell-1}(\sigma_*(s))^{-h_\varepsilon(\lambda_0)}$$
$$\times \int_{t_*}^{s} (s-\xi)^{\ell-1}\xi^{n-\ell} \sum_{i=1}^{m} \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{\mu_*(\xi_1,\lambda_0,\varepsilon)} d_{\xi_1} r_i(\xi_1,\xi)d\xi\, ds. \tag{3.21}$$

Since $\frac{u(\sigma_*(t_k))}{\sigma_*^{\lambda_0+h_{1\varepsilon}(\lambda_0)}(t_k)} \to 0$ for $k \to +\infty$ and $\mu(\sigma_*(t_k)) \leq 1$, for sufficiently large k, we have

$$\left(\frac{u(\sigma_*(t_k))}{\sigma_*^{\lambda_0+h_{1\varepsilon}(\lambda_0)}(t_k)}\right)^{1-\mu(\sigma_*(t_k))} \leq 1.$$

Therefore, from (3.21), we get

$$\limsup_{t\to+\infty}\big(\sigma_*(t_k)\big)^{\ell-\lambda_0+h_{1\varepsilon}(\lambda_0)} \int_{\sigma_*(t_k)}^{+\infty} s^{-n}\big(s-\sigma_*(t_k)\big)^{n-\ell-1}(\sigma_*(s))^{-h_\varepsilon(\lambda_0)} \int_{t_*}^{s} (s-\xi)^{\ell-1}\xi^{n-\ell}$$
$$\times \sum_{i=1}^{m} \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{\mu_*(\xi_1,\lambda_0,\varepsilon)} d_{\xi_1} r_i(\xi_1,\xi)d\xi\, ds \leq (\ell-1)!\,(n-\ell-1)!$$

Thus

$$\liminf_{t\to+\infty} t^{\ell-\lambda_0+h_{2\varepsilon}(\lambda_0)} \int_{t}^{+\infty} s^{-n}(s-t)^{n-\ell-1}(\sigma_*(s))^{-h_\varepsilon(\lambda_0)} \int_{t_*}^{s} (s-\xi)^{\ell-1}\xi^{n-\ell}$$
$$\times \sum_{i=1}^{m} \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{\mu_*(\xi_1,\lambda_0,\varepsilon)} d_{\xi_1} r_i(\xi_1,\xi)d\xi\, ds \leq (\ell-1)!\,(n-\ell-1)!.$$

From the latter inequality, it is clear that

$$\liminf_{t\to+\infty} t^{\ell-\lambda_0+h_{2\varepsilon}(\lambda_0)} \int_{t}^{+\infty} s^{-n}(s-t)^{(n-\ell-1)}(\sigma_*(s))^{-h_\varepsilon(\lambda_0)} \int_{0}^{s} (s-\xi)^{\ell-1}\xi^{n-\ell}$$
$$\times \sum_{i=1}^{m} \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{\mu_*(\xi_1,\lambda_0,\varepsilon)} d_{\xi_1} r_i(\xi_1,\xi)d\xi\, ds \leq (\ell-1)!\,(n-\ell-1)!.$$

Taking the upper limit, as $\varepsilon \to 0+$, we obtain (3.7_ℓ). $\square$

Corollary 3.1. *Let the condition of Theorem* 3.1 *be fulfilled and*

$$\limsup_{t\to+\infty} \frac{\sigma_i(t)}{t} < +\infty \quad (i = 1, \ldots, m). \tag{3.22}$$

Moreover, for some $t_0 \in R_+$, $U_{\ell,t_0} \neq \varnothing$, *then there exists* $\lambda_0 \in [\ell-1, \ell]$ *such that*

$$\limsup_{\varepsilon\to 0+}\Big(\liminf_{t\to+\infty} g_{\ell 1}(t,\lambda_0,\varepsilon)\Big) \le (\ell-1)!\,(n-\ell-1)!, \tag{3.23$_\ell$}$$

where

$$g_{\ell 1}(t,\lambda,\varepsilon) = t^{\ell-\lambda_0+h_{2\varepsilon}(\lambda)} \int_t^{+\infty} s^{-n-h_\varepsilon(\lambda)}(s-t)^{h-\ell-1} \int_{t_0}^{s} (s-\xi)^{\ell-1}\xi^{n-\ell} \times \sum_{i=1}^{m} \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{\mu_*(\xi_1,\lambda,\varepsilon)} d_{\xi_1} r_i(\xi_1,\xi)d\xi\, ds, \tag{3.24}$$

μ_* *is given by* (3.8)–(3.10).

Proof. In view of Theorem 3.1, to prove Corollary 3.1, it suffices to show that inequality (3.7$_\ell$) implies (3.23$_\ell$), where the functions g_ℓ and $g_{\ell 1}$ are given by (3.8) and (3.24), respectively. Indeed, according to (3.22), there exists $c > 0$ such that $\sigma_*(t) \le ct$ for $t \ge t_1 > 0$ and

$$g_{\ell 1}(t,\lambda_0,\varepsilon) \le c^{h_\varepsilon(\lambda_0)} g_\ell(t,\lambda_0,\varepsilon) \quad \text{for } t \ge t_1.$$

Since $\lim_{\varepsilon\to 0+} h_\varepsilon(\lambda_0) = 0$, we get (3.23$_\ell$). $\square$

4. Sufficient conditions of nonexistence of monotone solutions

Theorem 4.1. *Let* $F \in V(\tau)$, *conditions* (1.2) *((1.3)),* (2.27), (2.28) *and* (3.4$_\ell$), (3.5$_\ell$), (3.6) *be fulfilled,* $\ell \in \{1,\dots,n-1\}$ *with* $\ell+n$ *odd* ($\ell+n$ *even) for any* $\lambda \in [\ell-1,\ell]$,

$$\limsup_{\varepsilon\to 0+}\Big(\liminf_{t\to+\infty} g_\ell(t,\lambda,\varepsilon)\Big) > (\ell-1)!\,(n-\ell-1)!. \tag{4.1$_\ell$}$$

Then for any $t_0 \in R_+$, $U_{\ell,t_0} = \varnothing$, *where the function* g_ℓ *is defined by* (3.8)–(3.10).

Proof. Assume the contrary. Let there exist $t_0 \in R_+$ and $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd, if (1.2) holds ($\ell+n$ even if (1.3) holds), such that $U_{\ell,t_0} \neq \varnothing$ (for the definition of the set U_{ℓ,t_0} see Section 3). Thus Eq. (1.1) has a proper solution $u : [t_0,+\infty) \to (0,+\infty)$ satisfying condition (2.1$_\ell$). Since the conditions of Theorem 3.1 are fulfilled, there exists $\lambda_0 \in [\ell-1,\ell]$ such that condition (3.7$_\ell$) holds, which contradicts condition (4.1$_\ell$). The obtained contradiction proves the validity of the theorem. $\square$

Using Corollary 3.1, we can analogously prove

Corollary 4.1. *Let* $F \in V(\tau)$, *conditions* (1.2) *((1.3)),* (2.27), (2.28), (3.22) *and* (3.4$_\ell$), (3.5$_\ell$), (3.6) *be fulfilled,* $\ell \in \{1,\dots,n-1\}$ *with* $\ell+n$ *odd* ($\ell+n$ *even) and for any* $\lambda \in [\ell-1,\ell]$,

$$\limsup_{\varepsilon\to 0+}\Big(\liminf_{t\to+\infty} g_{\ell 1}(t,\lambda,\varepsilon)\Big) > (\ell-1)!\,(n-\ell-1)!. \tag{4.2$_\ell$}$$

Then for any $t_0 \in R_+$, $U_{\ell,t_0} = \varnothing$, *where* $g_{\ell 1}$ *is defined by* (3.24).

Corollary 4.2. *Let* $F \in V(\tau)$, *conditions* (1.2) *((1.3)),* (2.27), (2.28), (3.22) *and* (3.4$_\ell$), (3.5$_\ell$), (3.6) *be fulfilled,* $\ell \in \{1,\dots,n-1\}$ *with* $\ell+n$ *odd* ($\ell+n$ *even), and for any* $\lambda \in [\ell-1,\ell]$, *there exist* $\delta > 1$ *such that*

$$\liminf_{\varepsilon\to 0+}\left(\liminf_{t\to+\infty} t^{\ell-\lambda-1-h_{1\varepsilon}(\lambda)} \int_0^t s^{n-\ell} \sum_{i=1}^{m} \int_{\tau_i(s)}^{\sigma_i(s)} \xi^{\mu_*(\xi,\lambda,\varepsilon)} d_\xi\, r_i(\xi,s)d\xi\, ds\right) > \delta \prod_{i=0;\,i\neq\ell-1}^{n-1} |\lambda - i|. \tag{4.3$_\ell$}$$

Then for any $t_0 \in R_+$, $U_{\ell,t_0} = \varnothing$, *where* μ_* *is defined by* (3.9) *and* (3.10).

Proof. In view of Corollary 4.1, to prove Corollary 4.2, it suffices to show that inequality (4.3_ℓ) implies (4.2_ℓ). By (4.3_ℓ), there exist $\varepsilon_0 > 0$ and $\delta_1 \in (1, \delta]$, $t_1 > t_0$ such that

$$t^{\ell-1-\lambda-h_{1\varepsilon}(\lambda)} \int_0^t s^{n-\ell} \sum_{i=1}^m \int_{\tau_i(s)}^{\sigma_i(s)} \xi^{\mu_*(\xi,\lambda,\varepsilon)} d_\xi\, r_i(\xi, s) d\xi\, ds$$
$$> \delta_1 \prod_{i=0;i\neq\ell-1}^{n-1} |\lambda - i| \quad \text{for } t \geq t_1, \quad 0 < \varepsilon \leq \varepsilon_0. \tag{4.4}$$

Consider the case $\ell = 1$. According to (3.24) and (4.4), we get

$$g_{\ell 1}(t, \lambda, \varepsilon) \geq \delta_1 \prod_{i=1}^{n-1} |\lambda - i| t^{1-\lambda+h_{2\varepsilon}(\lambda)} \int_t^{+\infty} s^{-n-h_\varepsilon(\lambda)+\lambda+h_{1\varepsilon}(\lambda)} (s-t)^{n-2} ds$$
$$= \delta_1 \prod_{i=1}^{n-1} |\lambda - i| \frac{(n-2)!}{\prod_{i=1}^{n-1} |\lambda - i - h_{2\varepsilon}(\lambda)|}.$$

Since $\delta_1 > 1$ and $h_{2\varepsilon}(\lambda) \to 0$ for $\varepsilon \to 0+$, from the last inequality we get (4.2_ℓ), which in the case $\ell = 1$ proves the validity of our corollary. Assume now that $\ell \in \{2, \ldots, n-1\}$. According to (3.24), we have

$$g_{\ell 1}(t, \lambda, \varepsilon) = t^{\ell-\lambda-h_{2\varepsilon}(\lambda)} \int_t^{+\infty} s^{-n-h_\varepsilon(\lambda)} (s-t)^{n-\ell-1} \int_0^s (s-\xi)^{\ell-1} d \int_0^\xi \xi_1^{n-\ell}$$
$$\times \sum_{i=1}^m \int_{\tau_i(\xi_1)}^{\sigma_i(\xi_1)} \xi_2^{\mu_*(\xi_2,\lambda,\varepsilon)} d_{\xi_2} r_i(\xi_2, \xi_1) d_{\xi_1} ds$$
$$= (\ell-1) t^{\ell-\lambda-h_{2\varepsilon}(\lambda)} \int_t^{+\infty} s^{-n-h_\varepsilon(\lambda)} (s-t)^{n-\ell-1} \int_0^s (s-\xi)^{\ell-2} \int_0^\xi \xi_1^{n-\ell}$$
$$\times \sum_{i=1}^m \int_{\tau_i(\xi_1)}^{\sigma_i(\xi_1)} \xi_2^{\mu_*(\xi_2,\lambda,\varepsilon)} d_{\xi_2} r_i(\xi_2, \xi_1) d_{\xi_1} ds,$$

where μ_* is given by (3.9) and (3.10). By (4.4), this last inequality yields

$$g_{\ell 1}(t, \lambda, \varepsilon) \geq \delta_1 (\ell-1) \prod_{i=0;i\neq\ell-1}^{n-1} |\lambda - i| t^{\ell-\lambda-h_{1\varepsilon}(\lambda)} t^{\ell-\lambda-h_{2\varepsilon}(\lambda)} \int_t^{+\infty} s^{-n-h_\varepsilon(\lambda)} (s-t)^{n-\ell-1}$$
$$\times \int_0^s (s-\xi)^{\ell-2} \xi^{1+\lambda+h_{1\varepsilon}(\lambda)-\ell} d\xi\, ds \quad \text{for } t \geq t_1. \tag{4.5}$$

On the other hand,

$$\int_0^s (s-\xi)^{\ell-2} \xi^{1+\lambda+h_{1\varepsilon}(\lambda)-\ell} d\xi = \frac{(\ell-2)!\, s^{\lambda+h_{1\varepsilon}(\lambda)}}{\prod_{i=1}^{\ell-2} |\lambda - i - h_{1\varepsilon}(\lambda)|}.$$

Therefore, from (4.5), we have

$$g_{\ell 1}(t, \lambda, \varepsilon) \geq \frac{\delta_1(\ell-1) \prod_{i=0;i\neq\ell-1}^{n-1} |\lambda - i|}{\prod_{i=0}^{n-2} |\lambda - h_{1\varepsilon} - i|} t^{\lambda-\ell+h_{2\varepsilon}(\lambda)} \int_t^{+\infty} s^{-n-h_{2\varepsilon}(\lambda)+\lambda} (s-t)^{n-\ell-1} ds$$
$$= \frac{\delta_1 (\ell-1)!\, (n-\ell-1)!}{\prod_{i=1}^{\ell-2} |\lambda - h_{1\varepsilon} - i| \prod_{i=\ell}^{n-1} |\lambda - i - h_{2\varepsilon}(\lambda)|} \prod_{i=0;i\neq\ell-1}^{n-1} |\lambda - i|.$$

Since $\delta_1 > 1$ and $h_{j\varepsilon}(\lambda) \to 0$ for $\varepsilon \to 0+$ $(j = 1, 2)$, the last inequality results in (4.2_ℓ). □

Corollary 4.3. *Let $F \in V(\tau)$, conditions* (1.2) ((1.3)), (2.27), (2.28) *and* (3.4_ℓ), (3.5_ℓ), (3.6) *be fulfilled, $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd ($\ell+n$ even), and for any $\lambda \in [\ell-1,\ell]$ there exist $\delta>1$ such that*

$$\liminf_{\varepsilon\to 0+}\left(\liminf_{t\to+\infty}\frac{1}{t}\int_0^t s^{n-\lambda-h_{1\varepsilon}(\lambda)}\sum_{i=1}^m\int_{\tau_i(s)}^{\sigma_i(s)}\xi^{\mu_*(\xi,\lambda,\varepsilon)}d_\xi\, r_i(\xi,s)d\xi\, ds\right) > \delta\prod_{i=0}^{n-1}|\lambda-i|. \tag{4.6_ℓ}$$

Then for any $t_0\in R_+$, $U_{\ell,t_0}=\varnothing$, where μ_* is given by (3.9) *and* (3.10).

Proof. To prove the corollary, it suffices to show that condition (4.6_ℓ) implies the validity of (4.3_ℓ). By (4.6_ℓ), there exist $t_1>t_0$, $\delta_1\in(1,\delta]$ and $\varepsilon_0>0$ such that

$$\frac{1}{t}\int_0^t s^{n-\lambda-h_{1\varepsilon}(\lambda)}\sum_{i=1}^m\int_{\tau_i(s)}^{\sigma_i(s)}\xi^{\mu_*(\xi,\lambda,\varepsilon)}d_\xi\, r_i(\xi,s)ds > \delta_1\prod_{i=0}^{n-1}|\lambda-i| \quad \text{for } t\ge t_1,\ 0<\varepsilon\le\varepsilon_0. \tag{4.7}$$

Thus

$$\begin{aligned}
&t^{\ell-1-\lambda-h_{1\varepsilon}(\lambda)}\int_0^t s^{n-\ell}\sum_{i=1}^m\int_{\tau_i(s)}^{\sigma_i(s)}\xi^{\mu_*(\xi,\lambda,\varepsilon)}d_\xi\, r_i(\xi,s)ds\\
&\quad= t^{\ell-1-\lambda-h_{1\varepsilon}(\lambda)}\int_0^t s^{\lambda-\ell+h_{1\varepsilon}(\lambda)}d\int_0^s \xi^{n-\lambda-h_{1\varepsilon}(\lambda)}\sum_{i=1}^m\int_{\tau_i(\xi)}^{\sigma_i(\xi)}\xi_1^{\mu_*(\xi_1,\lambda,\varepsilon)}d_{\xi_1} r_i(\xi_1,\xi)d\xi\\
&\quad= t^{-1}\int_0^t s^{n-\lambda-h_{1\varepsilon}(\lambda)}\sum_{i=1}^m\int_{\tau_i(s)}^{\sigma_i(s)}\xi^{\mu_*(\xi_1,\lambda,\varepsilon)}d_{\xi_1} r_i(\xi_1,\xi)ds+(\ell-\lambda-h_{1\varepsilon}(\lambda))t^{\ell-1-\lambda-h_{1\varepsilon}(\lambda)}\\
&\quad\times\int_0^t s^{\lambda-\ell-1+h_{1\varepsilon}(\lambda)}\int_0^s \xi^{n-\lambda-h_{1\varepsilon}(\lambda)}\sum_{i=1}^m\int_{\tau_i(s)}^{\sigma_i(s)}\xi_1^{\mu_*(\xi_1,\lambda,\varepsilon)}d_{\xi_1} r_i(\xi_1,\xi)d\xi\, ds,
\end{aligned}$$

where μ_* is defined by (3.9) and (3.10). According to (4.7), from the last equality we get

$$\begin{aligned}
&t^{\ell-1-\lambda-h_{1\varepsilon}(\lambda)}\int_0^t s^{n-\ell}\sum_{i=1}^m\int_{\tau_i(s)}^{\sigma_i(s)}\xi^{\mu_*(\xi,\lambda,\varepsilon)}\, d_\xi\, r_i(\xi,s)ds > \delta_1\prod_{i=0}^{n-1}|\lambda-i|\big(1+(\ell-\lambda-h_{1\varepsilon}(\lambda)\big)\\
&\quad\times t^{\ell-1-\lambda+h_{1\varepsilon}(\lambda)}\int_0^t s^{\lambda-\ell+h_{1\varepsilon}(\lambda)}ds\Big) = \delta_1\prod_{i=0}^{n-1}|\lambda-i|\left(1+\frac{\ell-\lambda-h_{1\varepsilon}(\lambda)}{\lambda+1-\ell+h_{1\varepsilon}(\lambda)}\right).
\end{aligned}$$

Therefore

$$\liminf_{t\to+\infty} t^{\ell-1-\lambda-h_{1\varepsilon}(\lambda)}\int_0^t s^{n-\ell}\sum_{i=1}^m\int_{\tau_i(s)}^{\sigma_i(s)}\xi^{\mu_*(\xi,\lambda,\varepsilon)}\, d_\xi\, r_i(\xi,s)ds > \frac{\delta_1}{\lambda+1-\ell+h_{1\varepsilon}(\lambda)}\prod_{i=0}^{n-1}|\lambda-i|.$$

Hence condition (4.3_ℓ) holds. This proves our corollary. □

Remark 4.1. It is obvious that if the conditions of Theorem 4.1 and Corollaries 4.1–4.3 are fulfilled, then the differential inequality

$$u^{(n)}(t)\ \operatorname{sign} u(t)+\sum_{i=1}^m\int_{\tau_i(t)}^{\sigma_i(t)}|u(s)|^{\mu_i(s)}d_s\, r_i(s,t)\le 0,$$

$$\left(u^{(n)}(t)\ \operatorname{sign} u(t)-\sum_{i=1}^m\int_{\tau_i(t)}^{\sigma_i(t)}|u(s)|^{\mu_i(s)}d_s\, r_i(s,t)\ge 0\right)$$

has no solution of type (2.1_ℓ), where $\ell\in\{1,\dots,n-1\}$ with $\ell+n$ odd ($\ell+n$ even).

5. Functional differential equations with Property A

Relying on the results obtained in Sections 4–6, we establish sufficient conditions for Eq. (1.1) to have Property **A** (Property **B**).

Theorem 5.1. *Let $F \in V(\tau)$, conditions* (1.2), (2.27), (2.28), (3.6) *and*

$$\limsup_{t\to+\infty} \mu_i(t) < +\infty \quad (i = 1, \dots, m) \tag{5.1}$$

for odd n be fulfilled, and for any $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd and for any $\lambda \in [\ell - 1, \ell]$ conditions (3.4_ℓ), (3.5_ℓ), (3.6) *and* (4.1_ℓ) *hold. If, moreover,* (3.5_0) *holds when n is odd, then Eq.* (1.1) *has Property* **A**.

Proof. Let Eq. (1.1) have a proper nonoscillatory solution $u : [t_0, +\infty) \to (0, +\infty)$ (the case $u(t) < 0$ is similar). By (1.1), (1.2) and Lemma 2.1, there exist $\ell \in \{0, \dots, n-1\}$ such that $\ell + n$ is odd and condition (2.1_ℓ) holds. Since for any $\ell \in \{1, \dots, n-1\}$ with $\ell+n$ is odd condition (4.1_ℓ) holds, according to Theorem 4.1, we have $\ell \notin \{1, \dots, n-1\}$.

Therefore n is odd and $\ell = 0$. We claim that (1.4) holds. If this is not the case, then there exist $c > 0$ and $t_1 > t_0$ such that according to (5.1), $(u(t))^{\mu_i(t)} \ge c$ for $t \ge t_1$ $(i = 1, \dots, m)$. Then according to (2.1_0) and (2.26), we have

$$\sum_{i=1}^{n-1}(n-i-1)\left|u^{(i)}(t_1)\right| \ge c\int_{t_1}^{t} s^{n-1}\sum_{i=1}^{m}\big(r_i(\sigma_i(s)), (s) - r_i(\tau_i(s), s)\big)ds \quad \text{for } t \ge t_1.$$

The latter inequality contradicts condition (3.5_0). This proves that Eq. (1.1) has Property **A**. □

Using Corollary 4.1, we can prove analogously Corollary 5.1.

Corollary 5.1. *Let $F \in V(\tau)$, conditions* (1.2), (2.27), (3.6), (3.22) *and* (5.1) *be fulfilled and for any $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd and $\lambda \in [\ell - 1, \ell]$ conditions* (3.4_ℓ), (3.5_ℓ), (3.6) *and* (3.5_0) *hold when n is odd. Then for Eq.* (1.1) *to have Property* **A**, *it is sufficient that one of the following three conditions* (1) (4.2_ℓ); (2) (4.3_ℓ); (3) (4.6_ℓ) *hold.*

Corollary 5.2. *Suppose $F \in V(\tau)$,* (1.2) *holds and for any $t_0 \in R_+$,*

$$|F(u)(t)| \ge \sum_{i=1}^{m} p_i(t)\int_{\alpha_i t}^{\beta_i t} s^{\gamma_i}|u(s)|^{1+\frac{d_i}{\ln s}}ds \quad for \;\; t \ge t_0.\ u \in H_{t_0,\tau}, \tag{5.2}$$

where $0 < \alpha_i < \beta_i$, $p_i \in L_{\rm loc}(R_+; R_+)$, $\gamma_i \in (-1, +\infty)$, $d_i \in R$ $(i = 1, \dots, m)$. Let, moreover, for any $\ell \in \{1, \dots, n-1\}$ and $\lambda \in [\ell - 1, \ell]$ with $\ell + n$ odd and for some $\delta > 1$

$$\liminf_{\varepsilon\to 0+}\bigg(\liminf_{t\to+\infty} t^{\ell-1-\lambda-h_{1\varepsilon}(\lambda)}\int_0^t s^{n-\ell+1+\lambda+h_{1\varepsilon}(\lambda)}\sum_{i=1}^{m}\frac{e^{\lambda d_i}(\beta_i^{1+\gamma_i+\lambda+h_{1\varepsilon}(\lambda)} - \alpha_i^{1+\gamma_i+\lambda+h_{1\varepsilon}(\lambda)})s^{\gamma_i}p_i(s)}{1+\gamma_i+\lambda+h_{1\varepsilon}(\lambda)}\bigg)ds$$
$$\ge \delta\prod_{i=0;i\ne\ell-1}^{n-1}|\lambda - i|. \tag{5.3_ℓ}$$

Then Eq. (1.1) *has Property* **A**, *where $h_{1\varepsilon}$ is defined by the first condition of* (3.3).

Proof. By Corollary 4.2, according to (5.2), (5.3_ℓ), we can easily show that all conditions of Corollary (5.1) are fulfilled, where $\tau_i(t) = \alpha_i t$, $\sigma_i(t) = \beta_i t$, $r_i(s, t) = \frac{p_i(t)s^{1+\gamma_i}}{1+\gamma_i}$, $\mu_i(t) = 1 + \frac{d_i}{\ln t}$ $(i = 1, \dots, m)$, which proves the validity of the corollary. □

Corollary 5.3. *Suppose $F \in V(\tau)$,* (1.2), (5.2) *are fulfilled and for any $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd and for some $\delta > 1$,*

$$\liminf_{\varepsilon\to 0+}\bigg(\liminf_{t\to+\infty}\frac{1}{t}\int_0^t s^{n+1}\sum_{i=1}^{m}\frac{e^{\lambda d_i}(\beta_i^{1+\gamma_i+\lambda+h_{1\varepsilon}(\lambda)} - \alpha_i^{1+\gamma_i+\lambda+h_{1\varepsilon}(\lambda)})s^{\gamma_i}p_i(s)}{1+\gamma_i+\lambda+h_{1\varepsilon}(\lambda)}\bigg)ds \ge \delta\prod_{i=0}^{n-1}|\lambda - i|. \tag{5.4_ℓ}$$

Then Eq. (1.1) *has Property* **A**, *where $h_{1\varepsilon}$ is defined by the first condition of* (3.3).

Proof. By Corollary 4.3, according to (1.2), (5.2) and (5.4_ℓ), all conditions of Corollary 5.2 are fulfilled, which proves the validity of our corollary, $\tau_i(t) = \alpha_i t$, $\sigma_i(t) = \beta_i(t)$, $r_i(s, t) = \frac{p_i(t)s^{1+\gamma_i}}{1+\gamma_i}$. □

Using the arithmetic mean–geometric mean inequality, from Corollary 5.3, we get

Corollary 5.4. *Let $F \in V(\tau)$, (1.2), (5.2) be fulfilled and*

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t s^{n+1+\frac{\gamma}{m}}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$

$$> \frac{1}{m}\max\left\{\frac{-\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\lambda d}\big(\prod\limits_{i=1}^m(1+\gamma_i+\lambda)\big)^{\frac{1}{m}}}{\big(\prod\limits_{i=1}^m(\beta_i^{1+\gamma_i+\lambda}-\alpha_i^{1+\gamma_i+\lambda})\big)^{\frac{1}{m}}} : \lambda\in[0,n-1]\right\}. \tag{5.5$_\ell$}$$

Then Eq. (1.1) *has Property* **A**, *where*

$$d=\sum_{i=1}^m d_i,\qquad \gamma=\sum_{i=1}^m\gamma_i.$$

Corollary 5.5. *Suppose $0<\alpha_i<\beta_i$, $c_i\in(0,+\infty)$, $\gamma_i\in(-1,+\infty)$. Then for the equation*

$$u^{(n)}(t)+\sum_{i=1}^m\frac{c_i}{t^{1+n+\gamma_i}}\int_{\alpha_i t}^{\beta_i t}s^{\gamma_i}|u(s)|^{1+\frac{d_i}{\ln s}}\operatorname{sign}u(s)ds=0,\quad t\ge a \tag{5.6}$$

to have Property **A**, *it is necessary and sufficient that for any $\ell\in\{1,\dots,n-1\}$ with $\ell+n$ odd,*

$$\max\left\{-\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\sum_{i=1}^m c_i e^{\lambda d_i}\frac{\beta_i^{1+\gamma_i+\lambda}-\alpha_i^{1+\gamma_i+\lambda}}{1+\lambda+\gamma_i}\Big)^{-1} : \lambda\in[\ell-1,\ell]\right\}<1. \tag{5.7}$$

Proof. According to (5.6) and (5.7), the sufficiency follows from Corollary 5.5. Show the necessity. Let (5.7) be violated. Then there exists $\lambda_0\in[\ell-1,\ell]$, where $\ell\in\{1,\dots,n-1\}$, $\ell+n$ is odd such that

$$-\lambda(\lambda-1)\cdots(\lambda-n+1)=\sum_{i=1}^m c_i e^{\lambda_0 d_i}\frac{\beta_i^{1+\gamma_i+\lambda}-\alpha_i^{1+\gamma_i+\lambda}}{1+\lambda+\gamma_i}.$$

If we take into account that $\ell+n$ is odd, it is obvious from the latter equality that $u(t)=t^{\lambda_0}$ is a solution of type (2.1_ℓ) of Eq. (5.6), which proves the necessity. □

6. The functional differential equation with Property B

Theorem 6.1. *Let $F\in V(\tau)$, the conditions* (1.3), (2.27), (2.28), (3.6), (5.1) *be fulfilled and for any $\ell\in\{1,\dots,n-1\}$ with $\ell+n$ even and $\lambda\in[\ell-1,\ell]$ the conditions* (3.4_ℓ), (3.5_ℓ), (3.6) *and* (4.1_ℓ) *hold. If, moreover,* (3.5_0) *hold when n is even, then equation* (1.1) *has Property* **B**, *where g_ℓ is defined by* (3.8).

Proof. Suppose Eq. (1.1) has a proper nonoscillatory solution $u:[t_0,+\infty)\to(0,+\infty)$. By (1.1), (1.3) and Lemma 2.1, there exists $\ell\in\{0,\dots,n\}$ such that $\ell+n$ is even and condition (2.1_ℓ) holds. Since for any $\ell\in\{1,\dots,n-2\}$ with $\ell+n$ even condition (4.1_ℓ) holds, according to Theorem 4.1, we have $\ell\notin\{1,\dots,n-2\}$. Since $\ell+n$ is even, either $\ell=n$ or n is even, and $\ell=0$. In the latter case, as is shown in the proof of Theorem 5.1, using (3.5_0) and (5.1), we can easily show that (1.4) holds. On the other hand, if $\ell=n$, then by (2.1_n), there exist $c>0$ and $t_1>t_0$ such that $u(t)\ge ct^{n-1}$ for $t\ge t_1$. Therefore, by (2.1_n), (2.27), (3.4_n) and (5.1), Eq. (1.1) yields

$$u^{(n-1)}(t)\ge u^{(n-1)}(t_*)+c_0\int_{t_*}^t\sum_{i=1}^m\int_{\tau_i(\xi)}^{\sigma_i(\xi)}s^{(n-1-\mu_i(s))}d_s r_i(s_1,s)d\xi\to+\infty\quad\text{as } t\to+\infty,$$

where $c_0>0$ and $t_*>t_1$ is a sufficiently large number. Thus if n is even and $\ell=0$, then there takes place condition (1.4), but if $\ell=n$, then there takes place condition (1.5). This means that Eq. (1.1) has Property **B** and the theorem is complete. □

Using Corollaries 4.1–4.3, similarly to Corollaries 5.1–5.5, one can prove Corollaries 6.1–6.3.

Corollary 6.1. *Let $F \in V(\tau)$, conditions* (1.3), (2.27), (3.6), (3.22), (5.1) *be fulfilled and for any $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ even and $\lambda \in [\ell-1,\ell]$ conditions* (3.4_ℓ), (3.5_ℓ), (3.6) *and* (3.5_0) *hold when n is even for some $\delta > 1$. Then for Eq.* (1.1) *to have Property* **B***, it is sufficient that one of the following three conditions* (1) (4.2_ℓ)*;* (2) (4.3_ℓ)*;* (3) (4.6_ℓ) *hold.*

Corollary 6.2. *Suppose $F \in V(\tau)$, conditions* (1.3), (5.2) *hold and for any $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ even and $\lambda \in [\ell-1,\ell]$ conditions* (5.3_ℓ) *or* (5.4_ℓ) *are fulfilled. Then Eq.* (1.1) *has Property* **B***, where $h_{1\varepsilon}$ is defined by the first condition of* (3.3).

Corollary 6.3. *Let $F \in V(\tau)$, conditions* (1.3), (5.2) *be fulfilled and*

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t s^{1+n+\frac{\gamma}{m}}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\left\{\frac{\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\lambda d}\big(\prod_{i=1}^m(1+\gamma_i+\lambda)\big)^{\frac{1}{m}}}{\big(\prod_{i=1}^m(\beta_i^{1+\gamma_i+\lambda}-\alpha_i^{1+\gamma_i+\lambda})\big)^{\frac{1}{m}}} : \lambda \in [0,n-1]\right\}.$$

Then Eq. (1.1) *has Property* **B**, *where $d=\sum_{i=1}^m d_i$, $\gamma=\sum_{i=1}^m \gamma_i$.*

Corollary 6.4. *Suppose $0<\alpha_i<\beta_i$, $c_i \in (-\infty,0)$, $\gamma_i \in (-1,+\infty)$. Then for Eq.* (5.6) *to have Property* **B***, it is necessary and sufficient that for any $\ell \in \{1,\dots,n-2\}$ with $\ell+n$ even*

$$\max\left\{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\sum_{i=1}^m |c_i|e^{\lambda d_i}\frac{(\beta_i^{1+\gamma_i+\lambda}-\alpha_i^{1+\gamma_i+\lambda})}{1+\lambda+\gamma_i}\Big)^{-1} : \lambda \in [\ell-1,\ell]\right\} < 1.$$

7. The differential equation with a deviating argument with Property A

Throughout this section, it is assumed that instead of (2.27) the inequality

$$\big|F(u)(t)\big| \ge \sum_{i=1}^m p_i(t)\big|u(\delta_i(t))\big|^{\mu_i(\delta_i(t))} \quad \text{for } t \ge t_0,\ u \in H_{t_0,\tau} \tag{7.1}$$

holds with $t_0 \in R_+$ sufficiently large. Here we assume that

$$\delta_i \in C(R_+;R_+), \qquad \lim_{t\to+\infty}\delta_i(t)=+\infty, \tag{7.2}$$

$$p_i \in L_{\mathrm{loc}}(R_+;R_+), \qquad \mu_i \in C(R_+;(0,+\infty)), \qquad \liminf_{t\to+\infty}\mu_i(t)>0 \quad (i=1,\dots,m). \tag{7.3}$$

Theorem 7.1. *Let $F \in V(\tau)$, conditions* (1.2), (5.1), (7.1)–(7.3) *be fulfilled and for any $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd and $\lambda \in [\ell-1,\ell]$*

$$\int_0^{+\infty} t^{n-\ell}\sum_{i=1}^m p_i(t)(\delta_i(t))^{(\ell-1)\mu_i(\delta_i(t))}dt=+\infty, \tag{7.4_ℓ}$$

$$\int_0^{+\infty} t^{n-\ell-1}\sum_{i=1}^m p_i(t)(\delta_i(t))^{\ell\mu_i(\delta_i(t))}dt=+\infty \tag{7.5_ℓ}$$

and

$$\limsup_{\varepsilon\to0+}\Big(\liminf_{t\to+\infty} g_\ell^\delta(t,\lambda,\varepsilon)\Big) > (\ell-1)!\,(n-\ell-1)!. \tag{7.6_ℓ}$$

Then Eq. (1.1) *has Property* **A**, *where*

$$g_\ell^\delta(t,\lambda,\varepsilon) = t^{\ell-\lambda+h_{2\varepsilon}(\lambda)} \int_t^{+\infty} s^{-n}(s-t)^{n-\ell-1}(\delta_*(s))^{-h_\varepsilon(\lambda)} \int_0^s (s-\xi)^{\ell-1}\xi^{n-\ell} \times \sum_{i=1}^m p_i(\xi)(\delta_i(\xi))^{\mu_*(\delta(\xi),\lambda,\varepsilon)} d\xi\, ds, \tag{7.7$_\ell$}$$

μ_* *is given by* (3.9) *and* (3.10) *and*

$$\delta_*(t) = \max\big\{\max\{s,\delta_1(s),\ldots,\delta_m(s)\} : 0 \le s \le t\big\}. \tag{7.8}$$

Proof. In view of (7.1), inequality (2.27) clearly holds with

$$\tau_i(t) = \delta_i(t) - 1, \qquad \sigma_i(t) = \delta_i(t), \qquad r_i(s,t) = p_i(t)\, e(s-\delta_i(t)), \tag{7.9}$$

where

$$e(t) = \begin{cases} 0 & \text{for } t \in (-\infty,0), \\ 1 & \text{for } t \in [0,+\infty). \end{cases} \tag{7.10}$$

Therefore, taking into account (1.2) and (7.1)–(7.3), (7.4$_\ell$)–(7.7$_\ell$), (7.8)–(7.10), we ascertain that the conditions of Theorem 5.1 are fulfilled, which proves the validity of the theorem. □

Using Corollaries 5.1–5.3, we can analogously prove Corollaries 7.1–7.4 and 7.6.

Corollary 7.1. *Let* $F \in V(\tau)$, *conditions* (1.2), (7.1)–(7.3) *be fulfilled,*

$$\limsup_{t\to+\infty} \frac{\delta_i(t)}{t} < \infty \quad (i = 1,\ldots,m) \tag{7.11}$$

and for any $\ell \in \{1,\ldots,n-1\}$ *with* $\ell+n$ *odd and* $\lambda \in [\ell-1,\ell]$ *conditions* (7.4$_\ell$), (7.5$_\ell$) *and*

$$\limsup_{\varepsilon\to0+}\Big(\liminf_{t\to\infty} g_{\ell 1}^\delta(t,\varepsilon,\lambda)\Big) > (\ell-1)!\,(n-\ell-1)! \tag{7.12$_\ell$}$$

hold, where

$$g_{\ell 1}^\delta(t,\varepsilon,\lambda) = t^{\ell-\lambda+h_{2\varepsilon}(\lambda)} \int_t^{+\infty} s^{-n-h_\varepsilon(\lambda)}(s-t)^{n-\ell-1} \int_{t_*}^s (s-\xi)^{\ell-1}\xi^{n-\ell} \sum_{i=1}^m \big(\delta_i(\xi)\big)^{\mu_*(\delta_i(\xi),\lambda)} d\xi\, ds, \tag{7.13}$$

μ_* *is defined by* (3.9) *and* (3.10).

Corollary 7.2. *Let* $F \in V(\tau)$, *conditions* (1.2), (5.1), (3.6), (7.1)–(7.3), (7.4$_\ell$), (7.5$_\ell$) *and* (7.11) *be fulfilled and for any* $\ell \in \{1,\ldots,n-1\}$ *with* $\ell+n$ *odd and* $\lambda \in [\ell-1,\ell]$ *there exist* $\gamma > 1$ *such that*

$$\liminf_{\varepsilon\to0+}\Big(\liminf_{t\to\infty} t^{\ell-\lambda-1-h_{1\varepsilon}(\lambda)} \int_0^t s^{n-\ell} \sum_{i=1}^m p_i(s)(\delta_i(s))^{\mu_*(s,\lambda,\varepsilon)} ds\Big) > \gamma \prod_{i=0;\, i\ne\ell-1}^{n-1} |\lambda - i|. \tag{7.14$_\ell$}$$

Then Eq. (1.1) *has Property* **A**, *where* μ_* *is given by* (3.9) *and* (3.10).

Corollary 7.3. *Let* $F \in V(\tau)$, *conditions* (1.2), (7.1)–(7.3), (7.4$_\ell$), (7.5$_\ell$) *and* (7.11), *be fulfilled and for any* $\ell \in \{1,\ldots,n-1\}$ *with* $\ell+n$ *odd and* $\lambda \in [\ell-1,\ell]$ *there exist* $\gamma > 1$ *such that*

$$\liminf_{\varepsilon\to0+}\Big(\liminf_{t\to\infty} \frac{1}{t} \int_0^t s^{n-\lambda-h_{1\varepsilon}(\lambda)} \sum_{i=1}^m s^{n-\ell} \sum_{i=1}^m p_i(s)(\delta_i(s))^{\mu_*(s,\lambda,\varepsilon)} ds\Big) > \gamma \prod_{i=0}^{n-1} |\lambda - i|. \tag{7.15$_\ell$}$$

Then Eq. (1.1) *has Property* **A**, *where* μ_* *is given by* (3.9) *and* (3.10).

Corollary 7.4. *Let $F \in V(\tau)$, condition* (1.2) *holds and*

$$\big|F(u)(t)\big| \geq \sum_{i=1}^{m} p_i(t)\big|u(\alpha_i t)\big|^{1+\frac{di}{\ln \alpha_i t}} \quad \text{for } t \geq t_0,\ u \in H_{t_0,\tau}, \tag{7.16}$$

where

$$p_i \in L_{\text{loc}}(R_+;R_+), \qquad \alpha_i \in (0,+\infty), \qquad d_i \in \mathbb{R} \quad (i=1,\ldots,m). \tag{7.17}$$

Moreover, for any $\ell \in \{1,\ldots,n-1\}$ with $\ell+n$ odd and $\lambda \in [\ell-1,\ell]$ there exist $\gamma > 1$ such that

$$\liminf_{\varepsilon\to 0}\bigg(\liminf_{t\to\infty}\frac{1}{t}\int_0^t s^n \sum_{i=1}^{m} p_i(s)\alpha_i^{((\lambda))}e^{\lambda d_i}\,ds\bigg) > \gamma\prod_{i=0}^{n-1}|\lambda - i|. \tag{7.18}$$

Then Eq. (1.1) *has Property* **A**.

Proof. By (7.16)–(7.18), it is obvious that the conditions of Corollary 7.3 hold, which proves the corollary. □

Using arithmetic mean–geometric mean inequality, from Corollary 7.4, we get

Corollary 7.5. *Let $F \in V(\tau)$, conditions* (1.2), (7.16), (7.17) *be fulfilled and*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t s^n\Big(\prod_{i=0}^{m} p_i(s)\Big)^{\frac{1}{m}}ds > \frac{1}{m}\max\Big\{-\Big(\prod_{i=0}^{m}\alpha_i e^{\alpha_i}\Big)^{-\frac{\lambda}{m}}\lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda\in[0,n-1]\Big\}.$$

Then Eq. (1.1) *has Property* **A**.

Corollary 7.6. *Let $c_i, \alpha_i \in (0,+\infty)$, $d_i \in R$ $(i=1,\ldots,m)$. Then the equation*

$$u^{(n)}(t) + \sum_{i=1}^{m}\frac{c_i}{t^n}\,\big|u(\alpha_i t)\big|^{1+\frac{d_i}{\ln \alpha_i t}}\operatorname{sign} u(\alpha_i(t)) = 0 \tag{7.19}$$

has Property **A** *if and only if*

$$\max\Big\{-\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\sum_{i=1}^{m}c_i\alpha_i^{\lambda}e^{\lambda d_i}\Big)^{-1}: \lambda\in[0,n-1]\Big\} < 1.$$

8. Differential equations with a deviating argument with Property B

Theorem 8.1. *Let $F \in V(\tau)$, conditions* (1.3), (7.1)–(7.3) *be fulfilled and for any $\ell \in \{1,\ldots,n\}$ with $\ell+n$ even* (7.4_ℓ), (5.5_ℓ), (7.6_ℓ) *hold. Moreover, if* (7.5_0) *holds when n even, then Eq.* (1.1) *has Property* **B**.

Proof. In view of (7.1), inequality (2.27) holds with the functions μ_i, τ_i, σ_i and r_i $(i=1,\ldots,m)$ defined by (7.9) and (7.10). Therefore, it is obvious that the conditions of Theorem 6.1 are fulfilled, which proves the validity of the theorem. □

Taking into account Corollaries 6.1–6.4, we can quite similarly prove Corollaries 8.1–8.3.

Corollary 8.1. *Let $F \in V(\tau)$, conditions* (1.3), (3.6), (5.1), (7.1)–(7.3) *and* (7.11) *be fulfilled and for any $\ell \in \{1,\ldots,n-2\}$ with $\ell+n$ even* (7.4_ℓ), (7.5_ℓ) *and* (7.4_1) *hold when n is even. Then for Eq.* (1.1) *to have Property* **B**, *it is sufficient that one of the following three conditions* (1) (7.12_ℓ); (2) (7.14_ℓ); (3) (7.15_ℓ) *hold.*

Corollary 8.2. *Let $F \in V(\tau)$, conditions* (1.3), (7.16) *and* (7.17) *be fulfilled and*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t s^n\Big(\prod_{i=0}^{m} p_i(s)\Big)^{\frac{1}{m}}ds > \frac{1}{m}\max\Big\{-\Big(\prod_{i=1}^{m}\alpha_i e^{\lambda d_i}\Big)^{-\frac{\lambda}{m}}\lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda\in[0,n-1]\Big\}.$$

Then Eq. (1.1) *has Property* **B**.

Corollary 8.3. *Let $c_i \in (-\infty, 0)$, $\alpha_i \in (0, +\infty)$, $d_i \in R$ $(i = 1, \dots, m)$. Then Eq.* (7.19) *has Property* **B** *if and only if*

$$\max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\sum_{i=1}^{m}|c_i|\alpha_i^{\lambda}e^{\lambda d_i}\Big)^{-1} : \lambda \in [0, n-1]\Big\} < 1.$$

9. Some auxiliary lemmas for the volterra type differential inequalities

Consider the following differential inequalities:

$$u^{(n)}(t)\operatorname{sign}u(t) + \sum_{i=1}^{m}\int_{\tau_i(t)}^{\sigma_i(t)}|u(s)|^{\mu_i(s)}d_s\,r_i(s,t) \le 0 \quad \text{for } t \ge t_0, \tag{9.1}$$

and

$$u^{(n)}(t)\operatorname{sign}u(t) - \sum_{i=1}^{m}\int_{\tau_i(t)}^{\sigma_i(t)}|u(s)|^{\mu_i(s)}d_s\,r_i(s,t) \ge 0 \quad \text{for } t \ge t_0, \tag{9.2}$$

where $t_0 \in R_+$, the functions r_i, τ_i, σ_i and $\mu_i (i = 1, \dots, m)$ satisfy condition (2.28). Furthermore, everywhere below in this section, we assume that one of the following conditions

$$\sigma_i(t) \le t, \qquad \mu_i(t) \le 1 \quad \text{for } t \ge t_0\ (i = 1, \dots, m), \tag{9.3}$$

or

$$\tau_i(t) \ge t, \qquad \mu_i(t) \ge 1 \quad \text{for } t \ge t_0\ (i = 1, \dots, m) \tag{9.4}$$

is fulfilled.

Lemma 9.1 (*[12]*)**.** *Let condition* (9.3) *be fulfilled. Then for the differential inequality* (9.1) *to have Property* **A** *it is necessary and sufficient that it has no solution of type* (2.1_{n-1}).

Lemma 9.2 (*[12]*)**.** *Let conditions* (9.4) *be fulfilled and*

$$\int_0^{+\infty} t^{n-1}\sum_{i=1}^{m}\Big(r_i\big(\sigma_i(t),t\big) - r_i\big(\tau_i(t),t\big)\Big)dt = +\infty, \tag{9.5}$$

when n is odd. Then for the differential inequality (9.1) *to have Property* **A***, it is necessary and sufficient that it has no solution of type* (2.1_1) *when n is even and of type* (2.1_2) *and* (2.1_{n-1}) *when n is odd.*

Lemma 9.3 (*[13]*)**.** *Let condition* (9.3) *be fulfilled and*

$$\int_0^{+\infty}\sum_{i=1}^{m}\int_{\tau_i(t)}^{\sigma_i(t)} s^{(n-1)\mu_i(s)}d_s r_i(s,t)dt = +\infty. \tag{9.6}$$

Then for differential inequality (9.2) *to have Property* **B***, it is necessary and sufficient that it has no solution satisfying* (2.1_{n-2}) *when n is even and of type* (2.1_1) *and* (2.1_{n-2}) *when n is odd.*

Lemma 9.4 (*[13]*)**.** *Let conditions* (9.4) *and* (9.5) *be fulfilled. Then for differential inequality* (9.2) *to have Property* **B***, it is necessary and sufficient that it has no solution satisfying* (2.1_2) *when n is even and satisfying* (2.1_1) *when n is odd.*

10. Functional differential equations with a volterra type minorant having Property A

Theorem 10.1. *Let $F \in V(\tau)$, conditions* (1.2), (2.27), (2.28), (3.6), (9.3) *and* (9.6) *be fulfilled and* (5.1) *holds when n is odd. Then condition* (4.1_{-1}) *is sufficient for Eq.* (1.1) *to have Property* **A**.

Proof. First of all, we note that (9.3) and (9.6) imply the validity of the conditions

$$\int_0^{+\infty} t^{n-k-1} \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} s^{k\mu_i(s)} d_s\, r_i(s,t)dt = +\infty \quad (k = 0, \dots, n-1). \tag{10.1$_k$}$$

Suppose now that Eq. (1.1) does not have Property **A**. Then by Lemma 2.1, Eq. (1.1) has a nonoscillatory solution $u : [t_0, +\infty) \to R_+$ satisfying condition (2.1$_\ell$), where $\ell \in \{o, \dots, n-1\}$ with $\ell + n$ odd. If n is odd and $\ell = 0$, then according to (9.7$_0$) and (5.1), condition (1.4) is fulfilled. Consequently, since Eq. (1.1) does not have Property **A**, we have $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd. According to (1.2) and (2.27), for sufficiently large t, u is a proper solution of type (2.1$_\ell$) of the differential inequality (9.1). By Lemma 9.1, inequality (9.1) is a solution of type (2.1$_{n-1}$). On the other hand, since the conditions of Theorem 4.1 with $\ell = n-1$ are fulfilled, according to Remark 4.1, inequality (9.1) has no solution of type (2.1$_{n-1}$). The obtained contradiction proves the validity of the theorem. □

Theorem 10.2. *Let $F \in V(\tau)$, conditions* (1.2), (2.27), (2.28), (3.6), (9.3) *and* (9.6) *be fulfilled and* (5.1) *holds when n is odd. Then for Eq.* (1.1) *to have Property* **A**, *it is sufficient that one of the following three conditions* (1) (3.22) *and* (4.2$_{n-1}$)*;* (2) (3.22) *and* (4.3$_{n-1}$)*;* (3) (3.22) *and* (4.5$_{n-1}$) *holds.*

Proof. The proof is analogous to that of Theorem 10.1 with the use of Corollaries 4.1–4.3. □

Corollary 10.1. *Let $F \in V(\tau)$, conditions* (1.2), (5.2) *be fulfilled, where $0 < \alpha_i < \beta_i \le 1$, $d_i \in (-\infty, 0]$ and $\gamma_i \in (-1, +\infty)$ $(i = 1, \dots, m)$. Then the condition*

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t s^{1+n+\frac{\gamma}{m}} \Big(\prod_{i=0}^m p_i(s)\Big)^{\frac{1}{m}} ds$$
$$> \frac{1}{m} \max \left\{ \frac{-\lambda(\lambda-1)\cdots(\lambda-n+1)\big(\prod_{i=0}^m (1+\gamma_i(\lambda))\big)^{\frac{1}{m}} \cdot e^{-\lambda d}}{\big(\prod_{i=0}^m (\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda})\big)^{\frac{1}{m}}} : \lambda \in [n-2, n-1] \right\}$$

is sufficient for Eq. (1.1) *to have Property* **A**, *where $d = \sum_{i=1}^m d_i$, $\gamma = \sum_{i=1}^m \gamma_i$.*

Corollary 10.2. *Let $0 < \alpha_i < \beta_i \le 1$, $c_i \in (0, +\infty)$, $d_i \in (-\infty, 0]$, $\gamma_i \in (-1, +\infty]$ $(i = 1, \dots, m)$. Then the condition*

$$\max\left\{ -\lambda(\lambda-1)\cdots(\lambda-n+1)\left(c_i e^{\lambda d_i} \frac{\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda}}{1+\lambda+\gamma_i}\right)^{-1} : \lambda \in [n-2, n-1] \right\} < 1$$

is necessary and sufficient for Eq. (5.6) *to have Property* **A**.

If we take into account Remark 4.1 and Lemma 9.1, then the validity of Corollaries 10.1 and 10.2 follows from Corollaries 6.3 and 6.4.

Theorem 10.3. *Let $F \in V(\tau)$, conditions* (1.2), (2.27), (2.28), (3.6), (9.4) *and* (9.7$_0$) *be fulfilled and* (5.1) *holds when n is odd. Then for Eq.* (1.1) *to have Property* **A**, *it is sufficient that* (4.1$_1$) *holds when n is even and conditions* (4.1$_2$) *and* (4.1$_{n-1}$) *hold when n is odd.*

Proof. The proof of Theorem 10.3 is analogous to that of Theorem 10.1, and Lemma 9.1 is used instead of Lemma 9.2. □

Analogously, we can prove

Theorem 10.4. *Let $F \in V(\tau)$, conditions* (1.2), (2.27), (2.28), (3.6), (9.4) *and* (9.7$_0$) *be fulfilled and* (5.1) *holds when n is odd. Then for Eq.* (1.1) *to have Property* **A**, *it is sufficient that for even n (for odd n) the conditions* (1) (3.22) *and* (4.2$_1$) *(*(3.22), (4.2$_2$) *and* (4.2$_{n-1}$)*);* (2) (3.22) *and* (4.3$_2$) *(*(3.22), (4.3$_2$) *and* (4.3$_{n-1}$)*);* (3) (3.22) *and* (4.5$_2$) *(*(3.22), (4.5$_2$) *and* (4.5$_{n-1}$)*) holds.*

Corollary 10.3. *Let $F \in V(\tau)$, conditions* (1.2), (5.2) *be fulfilled, where $1 \le \alpha_i < \beta_1$, $d_i \in [0, +\infty)$, $\gamma_i \in (-1, +\infty)$ $(i = 1, \dots, m)$. Then for Eq.* (1.1) *to have Property* **A**, *it is sufficient that the condition*

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t s^{1+n+\frac{\gamma}{m}} \Big(\prod_{i=0}^m p_i(s)\Big)^{\frac{1}{m}} ds > \frac{1}{m} \max \left\{ \frac{-\lambda(\lambda-1)\cdots(\lambda-n+1)\big(\prod_{i=1}^m (1+\gamma_i+\lambda)\big)^{\frac{1}{m}} \cdot e^{-\lambda d}}{\big(\prod_{i=1}^m (\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda})\big)^{\frac{1}{m}}} : \lambda \in [0,1] \right\}$$

holds when n is even and the condition

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t s^{1+n+\frac{\gamma}{m}} \Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}} ds > \frac{1}{m} \max \left\{ \frac{-\lambda(\lambda-1)\cdots(\lambda-n+1)\big(\prod_{i=1}^m (1+\gamma_i(\lambda))\big)^{\frac{1}{m}} \cdot e^{-\lambda d}}{\big(\prod_{i=1}^m (\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda})\big)^{\frac{1}{m}}} : \lambda \in [1,2] \cup [n-2, n-1] \right\}$$

holds when n is odd; here, $d = \sum_{i=1}^m d_i$, $\gamma = \sum_{i=1}^m \gamma_i$.

Corollary 10.4. *Let $1 \le \alpha_i < \beta_i$, $c_i \in [0, +\infty)$, $d_i \in [0, +\infty)$, $\gamma_i \in (-1, +\infty)$ $(i = 1, \dots, m)$. Then for Eq.* (5.6) *to have Property* **A**, *it is necessary and sufficient that*

$$\max\left\{-\lambda(\lambda-1)\cdots(\lambda-n+1)\left(\sum_{i=1}^m c_i e^{\lambda d_i} \frac{\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda}}{1+\lambda+\gamma_i}\right)^{-1} : \lambda \in [0,1]\right\} < 1$$

when n is even and

$$\max\left\{-\lambda(\lambda-1)\cdots(\lambda-n+1)\left(\sum_{i=1}^m c_i e^{\lambda d_i} \frac{\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda}}{1+\lambda+\gamma_i}\right)^{-1} : \lambda \in [1,2] \cup [n-2, n-1]\right\} < 1$$

when n is odd.

If we take into account Remark 4.1 and Lemma 9.2, the validity of Corollaries 10.3 and 10.4 follows from Corollaries 5.4 and 5.5.

11. Functional differential equations with a volterra type minorant having Property B

Theorem 11.1. *Let $F \in V(\tau)$, conditions* (1.3) *and* (2.27), (2.28) *and* (3.6), (9.3) *and* (9.6) *be fulfilled and* (5.1) *holds when n is even. Then for Eq.* (1.1) *to have Property* **B**, *it is sufficient that the condition* (4.1_{n-2}) *holds when n is even and conditions* (4.1_1) *and* (4.1_{n-2}) *hold when n is odd.*

Proof. (9.3) and (9.7_{n-1}) imply the validity of (10.1_k) for any $k \in \{0, \dots, n-1\}$. Suppose now that Eq. (1.1) does not have Property **B**. Then by Lemma 2.1, Eq. (1.1) has a nonoscillatory solution $u : [t_0, +\infty) \to R$ satisfying condition (2.1_ℓ), where $\ell \in \{0, \dots, n\}$ and $\ell + n$ is even. If n is even and $\ell = 0$, then according to (10.1_0), condition (1.4) holds. If $\ell = n$, then according to (9.6), it can be shown that (1.5) holds. Consequently, since Eq. (1.1) does not have Property **B**, we have $\ell \in \{1, \dots, n-2\}$ and $\ell + n$ is even. By (1.3) and (2.27), u is a proper solution of type (2.1_ℓ) of the differential inequality (9.2). By Lemma 9.3, inequality (9.2) has a solution of type (2.1_{n-2}), ((2.1_1) or (2.1_{n-1})) when n is even (n is odd). On the other hand, since the conditions of Theorem 4.1 with $\ell = n-2$ ($\ell = 1$ and $\ell = n-2$) when n is even (n is odd) are fulfilled, according to Remark 4.1, inequality (9.2) for even n (for odd n) has no solution of type (2.1_{n-2}) ((2.1_1) or (2.1_{n-2})). The obtained contradiction proves the validity of the theorem. □

Theorem 11.2. *Let $F \in V(\tau)$, conditions* (1.3), (2.27), (2.28), (3.6), (9.3) *and* (9.6) *be fulfilled and* (5.1) *hold when n is even. Then for Eq.* (1.1) *to have Property* **B**, *it is sufficient that for even n (for odd n) one of the following three conditions* (1) (3.22) *and* (4.2_{n-2}) *((3.22),* (4.2_1) *and* (4.2_{n-2})*);* (2) (3.22) *and* (4.3_{n-2}) *((3.22),* (4.3_1) *and* (4.3_{n-2})*);* (3) (3.22) *and* (4.5_{n-2}) *((3.22),* (4.5_1) *and* (4.5_{n-2})*) hold.*

Proof. The proof is analogous to that of Theorem 11.1 with the use of Corollaries 4.1–4.3. □

Corollary 11.1. *Let $F \in V(\tau)$, conditions* (1.3), (5.2) *be fulfilled, where $0 < \alpha_i < \beta_i \le 1$, $d_i \in [0, +\infty)$, $\gamma_i \in (-1, +\infty)$ $(i = 1, \dots, m)$. Then for Eq.* (1.1) *to have Property* **B**, *it is sufficient that the condition*

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t s^{1+n+\frac{\gamma}{m}} \Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}} ds$$
$$> \frac{1}{m} \max\left\{ \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)\big(\prod_{i=1}^m (1+\gamma_i+\lambda)\big)^{\frac{1}{m}} \cdot e^{-\lambda d}}{\big(\prod_{i=1}^m (\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda})\big)^{\frac{1}{m}}} : \lambda \in [n-3, n-2] \right\}$$

holds when n is even and the condition

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t s^{1+n+\frac{\gamma}{m}} \Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}} ds$$
$$> \frac{1}{m} \max\left\{ \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)\big(\prod_{i=1}^m (1+\gamma_i+\lambda)\big)^{\frac{1}{m}} \cdot e^{-\lambda d}}{\big(\prod_{i=1}^m (\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda})\big)^{\frac{1}{m}}} : \lambda \in [0,1] \cup [n-3, n-2] \right\}$$

holds when n is odd, where $d = \sum_{i=1}^m d_i$, $\gamma = \sum_{i=1}^m \gamma_i$.

Corollary 11.2. *Let $0 < \alpha_i < \beta_i \le 1$, $c_i \in [-\infty, 0)$, $d_i \in [-\infty, 0)$ and $\gamma_i \in (-1, +\infty)$ $(i = 1, \dots, m)$. Then for Eq.* (5.6) *to have Property* **B**, *it is necessary and sufficient that*

$$\max\left\{ \lambda(\lambda-1)\cdots(\lambda-n+1)\left(\sum_{i=1}^m |c_i| e^{\lambda d_i} \frac{\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda}}{1+\lambda+\gamma_i}\right)^{-1} : \lambda \in [n-3, n-2] \right\} < 1,$$

when n is even and

$$\max\left\{ -\lambda(\lambda-1)\cdots(\lambda-n+1)\left(\sum_{i=1}^m |c_i| e^{\lambda d_i} \frac{\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda}}{1+\lambda+\gamma_i}\right)^{-1} : \lambda \in [0,1] \cup [n-3, n-2] \right\} < 1,$$

when n is odd.

If we take into account Remark 4.1 and Lemma 9.3, the validity of Corollaries 11.3 and 11.2 follows from Corollaries 6.3 and 6.4.

Theorem 11.3. *Let $F \in V(\tau)$, conditions* (1.3), (2.27), (2.28), (3.6), (3.22), (9.4) *and* (9.5) *be fulfilled and* (5.1) *hold when n is even. Then for Eq.* (1.1) *to have Property* **B**, *it is sufficient that condition* (4.1_2) *holds when n is even and condition* (4.1_1) *holds when n is odd.*

Proof. The proof is analogous to that of Theorem 11.1 and Lemma 9.3 is used instead of Lemma 9.4. □

Theorem 11.4. *Let $F \in V(\tau)$, conditions* (1.3), (2.27), (2.28), (3.6), (9.4) *and* (9.7_0) *be fulfilled and* (5.1) *hold when n is even. Then for Eq.* (1.1) *to have Property* **B**, *it is sufficient that for even n (for odd n) one of the following three conditions* (1) (3.22) *and* (4.2_2) *((3.22) and* (4.2_1)*);* (2) (3.22) *and* (4.3_2) *((3.22) and* (4.3_1)*);* (3) (3.22) *and* (4.5_2) *((3.22) and* (4.5_1)*) hold.*

Corollary 11.3. *Let $F \in V(\tau)$, conditions* (1.3), (5.2) *be fulfilled, where* $1 \le \alpha_i < \beta_i$, $d_i \in [0, +\infty)$, $\gamma_i \in R$ $\gamma_i \in (-1, +\infty)$ $(i = 1, \dots, m)$. *Then for Eq.* (1.1) *to have Property* **B**, *it is sufficient that*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t s^{1+n+\frac{\gamma}{m}}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}} ds > \frac{1}{m}\max\left\{\frac{\lambda(\lambda-1)\cdots(\lambda-n+1)\big(\prod_{i=1}^m (1+\gamma_i+\lambda)\big)^{\frac{1}{m}}\cdot e^{-\lambda d}}{\big(\prod_{i=1}^m (\beta_i^{1+\gamma_i+\lambda}-\alpha_i^{1+\gamma_i+\lambda})\big)^{\frac{1}{m}}} : \lambda \in [1,2]\right\}$$

holds when n is even and

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t s^{1+n+\frac{\gamma}{m}}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}} ds > \frac{1}{m}\max\left\{\frac{\lambda(\lambda-1)\cdots(\lambda-n+1)\big(\prod_{i=1}^m (1+\gamma_i+\lambda)\big)^{\frac{1}{m}}\cdot e^{-\lambda d}}{\big(\prod_{i=1}^m (\beta_i^{1+\gamma_i+\lambda}-\alpha_i^{1+\gamma_i+\lambda})\big)^{\frac{1}{m}}} : \lambda \in [0,1]\right\}$$

when n is odd, where $d = \sum_{i=1}^m d_i$, $\gamma = \sum_{i=1}^m \gamma_i$.

Corollary 11.4. *Let* $1 \le \alpha_i < \beta_i \le 1$, $c_i \in [-\infty, 0)$, $d_i \in [0, +\infty)$ *and* $\gamma_i \in (-1, +\infty)$ $(i = 1, \dots, m)$. *Then for Eq.* (5.6) *to have Property* **B**, *it is necessary and sufficient that*

$$\max\left\{\lambda(\lambda-1)\cdots(\lambda-n+1)\left(\sum_{i=1}^m |c_i| e^{\lambda d_i}\frac{\beta_i^{1+\gamma_i+\lambda}-\alpha_i^{1+\gamma_i+\lambda}}{1+\lambda+\gamma_i}\right)^{-1} : \lambda \in [1,2]\right\} < 1$$

when n is even and

$$\max\left\{-\lambda(\lambda-1)\cdots(\lambda-n+1)\left(\sum_{i=1}^m |c_i| e^{\lambda d_i}\frac{\beta_i^{1+\gamma_i+\lambda}-\alpha_i^{1+\gamma_i+\lambda}}{1+\lambda+\gamma_i}\right)^{-1} : \lambda \in [0,1]\right\} < 1$$

when n is odd.

If we take into account Remark 4.1 and Lemma 9.4, the validity of Corollaries 11.3 and 11.4 follows from Corollaries 6.3 and 6.4.

12. Functional differential equations with a delay argument type minorant

Theorem 12.1. *Let $F \in V(\tau)$, conditions* (1.2), (7.1), (7.2), (7.3), (7.4_{n-1}) *and*

$$\delta_i(t) \le t, \quad \mu_i(t) \le 1 \quad \text{for } t \ge t_0 \quad (i = 1, \dots, m) \tag{12.1}$$

be fulfilled and (5.1) *hold when n is odd. If, moreover, for any $\lambda \in [n-2, n-1]$, condition* (7.6_{n-1}) *is fulfilled, then Eq.* (1.1) *has Property* **A**, *where ρ^{δ}_{n-1} is defined by* (7.7_{n-1}).

Proof. By (7.1), it is obvious that inequality (2.27) holds, where the functions τ_i, σ_i and r_i $(i = 1, \dots, m)$ are defined by (7.9) and (7.10). Therefore, according to (1.2), (7.1), (7.2), (7.3), (7.4_{n-1}), (5.1), (12.1) and (7.6_{n-1}), every conditions of Theorem 10.1 are fulfilled, which proves the validity of the theorem. □

Taking into account Theorem 10.2, the next theorem can be proved similarly.

Theorem 12.2. *Let $F \in V(\tau)$, conditions* (1.2), (7.1), (7.2), (7.3), (7.4_{n-1}) *and* (12.1) *be fulfilled and* (5.1) *hold when n odd. Then for Eq.* (1.1) *to have Property* **B**, *it is sufficient that one of the following three conditions* (1) (7.13_1); (2) (7.15_1); (3) (7.16_1) *holds.*

Corollary 12.1. *Let $F \in V(\tau)$, conditions* (1.2), (7.10), (7.17) *hold and* $0 < \alpha_i \leq 1$, $d_i \in (-\infty, 0]$ $(i = 1, \dots, m)$ *be fulfilled and*

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t s^n \Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}} ds > \frac{1}{m} \max\Big\{-\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m \alpha_i\, e^{d_i}\Big)^{\frac{-\lambda}{m}} : \lambda \in [n-2, n-1]\Big\}.$$

Then Eq. (1.1) *has Property* **A**.

Corollary 12.2. *Let* $0 < \alpha_i \leq 1$, $c_i \in (0, +\infty)$, $d_i \in (-\infty, 0]$ *and* $\gamma_i \in (-1, +\infty)$ $(i = 1, \dots, m)$. *Then for Eq.* (7.19) *to have Property* **A***, it is necessary and sufficient that*

$$\max\Big\{-\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\sum_{i=1}^m c_i \alpha_i^\lambda\, e^{\lambda d_i}\Big)^{-1} \lambda \in [n-2, n-1]\Big\} < 1.$$

If we take into account Remark 4.1 and Lemma 9.1, the validity of Corollaries 12.1 and 12.2 follows from Corollaries 5.4 and 5.5.

Theorem 12.3. *Let $F \in V(\tau)$, conditions* (1.3), (7.1), (7.2), (7.3), (7.4_{n-1}) *and* (12.1) *be fulfilled and* (5.1) *holds when n is even. If, moreover, when n is even* (*n is odd*)*, for any* $\lambda \in [n-2, n-1]$, *condition* (7.6_{n-2}) (*for any* $\lambda \in [0, 1]$, *condition* (7.6_1) *and for any* $\lambda \in [n-3, n-2]$, *condition* (7.6_{n-2})) *hold, then Eq.* (1.1) *has Property* **B**.

Proof. By (7.1), inequality (2.27) holds, where the functions τ_i, σ_i and r_i $(i = 1, \dots, m)$ are defined by (7.10), (7.11). Therefore, according to (1.3), (7.1), (7.2), (7.3), (7.4_{n-1}), (12.1) and (7.6_{n-2}) ((7.6_1) and (7.6_{n-2})), for even n (for odd n), all conditions of Theorem 11.1 are fulfilled, which proves the validity of the theorem. □

Taking into account Theorem 11.2, the next theorem can be proved similarly.

Theorem 12.4. *Let $F \in V(\tau)$, conditions* (1.3), (7.1), (7.2), (7.3), (7.4_{n-1}) *and* (12.1) *be fulfilled and* (5.1) *hold when n is even. Then for Eq.* (1.1) *to have Property* **B***, it is sufficient that for even n (for odd n) one of the following three conditions* (1) (7.13_{n-2}) ((7.13_1) *and* (7.13_{n-2})); (2) (7.15_{n-2}) ((7.15_1) *and* (7.15_{n-2})); (3) (7.16_{n-2}) ((7.16_1) *and* (7.16_{n-2})) *hold.*

Corollary 12.3. *Let $F \in V(\tau)$, conditions* (1.3), (7.16), (7.17) *hold and* $0 < \alpha_i \leq 1$, $d_i \in (-\infty, 0]$ $(i = 1, \dots, m)$ *be fulfilled and*

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t s^n \Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}} ds > \frac{1}{m} \max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m \alpha_i e^{d_i}\Big)^{-\frac{\lambda}{m}} : \lambda \in [n-3, n-2]\Big\},$$

for even n and

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t s^n \Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}} ds$$
$$> \frac{1}{m} \max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m \alpha_i e^{d_i}\Big)^{-\frac{1}{m}} : \lambda \in [0, 1] \cup [n-3, n-2]\Big\}$$

for odd n. Then Eq. (1.1) *has Property* **B**.

Corollary 12.4. *Let* $0 < \alpha_i \leq 1$, $c_i \in [-\infty, 0)$, $d_i \in [-\infty, 0]$ $(i = 1, \dots, m)$. *Then Eq.* (7.19) *has Property* **B** *if and only if*

$$\max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\sum_{i=1}^m |c_i| \alpha_i^\lambda e^{\lambda d_i}\Big)^{-1} : \lambda \in [n-3, n-2]\Big\} < 1,$$

when n is even and

$$\max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\sum_{i=1}^m |c_i| \alpha_i^\lambda e^{\lambda d_i}\Big)^{-1} : \lambda \in [0, 1] \cup [n-3, n-2]\Big\} < 1,$$

when n is odd.

If we take into account Remark 4.1 and Lemma 9.3, the validity of Corollaries 13.3 and 12.4 follows from Corollaries 8.2 and 8.3.

13. Functional differential equations with an advanced argument type minorant

Theorem 13.1. *Let $F \in V(\tau)$, conditions* (1.2), (7.1), (7.2), (7.3), (7.4$_1$) *and*

$$\delta_i(t) \ge t, \quad \mu_i(t) \ge 1 \quad \text{for } t \ge t_0 \quad (i = 1, \dots, m) \tag{13.1}$$

be fulfilled and for even n (5.1) *hold. Then for Eq.* (1.1) *to have Property* **A**, *it is sufficient that for even n (for odd n) for any $\lambda \in [0, 1]$ (for any $\lambda \in [1, 2]$ and $\lambda \in [n-2, n-12]$), conditions* (7.6$_1$) *(conditions* (7.6$_2$) *and* (7.6$_{n-1}$)*) hold.*

Proof. By (7.1), it is obvious that inequality (2.26) holds, where

$$\tau_i(t) = \delta_i(t), \quad \text{and} \quad \sigma_i(t) = \delta_i(t) + 1, \quad r_i(s, t) = p_i(t)\, e(s - \delta_i(t)) \quad (i = 1, \dots, m) \tag{13.2}$$

where the function e is defined by (7.11). Therefore, according to (1.2), (7.1), (7.2), (7.3), (7.4$_1$), (13.1), (13.2) and (7.6$_1$) ((7.6$_2$) and (7.6$_{n-1}$)), every conditions of Theorem 10.3 are fulfilled, which proves the validity of the theorem. $\square$

Taking into account Theorem 10.4, the next theorem can be proved similarly.

Theorem 13.2. *Let $F \in V(\tau)$, conditions* (1.2), (7.1), (7.2), (7.3), (7.4$_1$) *and* (13.1) *be fulfilled and* (5.1) *hold when n is odd. Then for Eq.* (1.1) *to have Property* **A**, *it is sufficient that for even n (for odd n), one of the following three conditions* (1) (3.22) *and* (7.13$_1$) *((3.22), (7.13$_2$) and (7.13$_{n-1}$));* (2) (3.22) *and* (7.15$_1$) *((3.22), (7.15$_2$) and (7.15$_{n-1}$));* (3) (3.22) *and* (7.16$_1$) *((3.22), (7.16$_2$) and (7.16$_{n-1}$)) hold.*

Corollary 13.1. *Let $F \in V(\tau)$, conditions* (1.2), (7.17), (7.18) *and $\alpha_i \ge 1$, $d_i \in [0, +\infty)$ $(i = 1, \dots, m)$ be fulfilled and*

$$\liminf_{t\to\infty} \frac{1}{t} \int_1^t s^n \Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}} ds > \frac{1}{m} \max\bigg\{ -\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m \alpha_i e^{\lambda d_i}\Big)^{-\frac{\lambda}{m}} : \lambda \in [0, 1] \bigg\},$$

for even n and

$$\liminf_{t\to\infty} \frac{1}{t} \int_1^t s^n \Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}} ds$$
$$> \frac{1}{m} \max\bigg\{ -\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m \alpha_i e^{\lambda d_i}\Big)^{-\frac{\lambda}{m}} : \lambda \in [1, 2] \cup [n-2, n-1] \bigg\},$$

for odd n. Then Eq. (1.1) *has Property* **A**.

Corollary 13.2. *Let $\alpha_i \ge 1$, $c_i \in [0, +\infty)$, $d_i \in [0, +\infty)$ $(i = 1, \dots, m)$. Then Eq.* (7.19) *has Property* **A** *if and only if*

$$\max\bigg\{ -\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\sum_{i=1}^m c_i \alpha_i^{\lambda} e^{\lambda d_i}\Big)^{-1} : \lambda \in [0, 1] \bigg\} < 1,$$

when n is even and

$$\max\bigg\{ -\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\sum_{i=1}^m c_i \alpha_i^{\lambda} e^{\lambda d_i}\Big)^{-1} : \lambda \in [1, 2] \cup [n-2, n-1] \bigg\} < 1,$$

when n is odd.

Theorem 13.3. *Let $F \in V(\tau)$, conditions* (1.3), (7.1), (7.2), (7.3), (7.4$_1$) *and* (13.1) *be fulfilled and* (5.1) *hold when n is even. Then for Eq.* (1.1) *to have Property* **B**, *it is sufficient that the condition* (7.6$_2$) *holds for any $\lambda \in [1, 2]$, when n is even, and the condition* (7.6$_1$) *holds for any $\lambda \in [0, 1]$, when n is odd.*

Proof. According to (7.1), it is obvious that inequality (2.26) holds, where τ_i, σ_i and r_i $(i = 1, \dots, m)$ are defined by (13.2). Therefore, according to (1.3), (13.1), (7.1), (7.2), (7.3), (7.4_1), (7.6_2) and (7.6_1), every conditions of Theorem 10.4 are fulfilled, which proves the validity of the theorem. □

According to Theorem 11.4, the next theorem can be proved similarly.

Theorem 13.4. *Let $F \in V(\tau)$, conditions* (1.3), (7.1), (7.2), (7.3), (7.4_1) *and* (13.1) *be fulfilled and* (5.1) *hold when n is even. Then for Eq.* (1.1) *to have Property* **B**, *it is sufficient that for even n (for odd n) and for any $\lambda \in [1, 2]$ (for any $\lambda \in [0, 1]$), one of the following three conditions* (1) (3.21) *and* (7.13_2) *((3.21) and (7.13_1));* (2) (3.21) *and* (7.15_2) *((3.21) and (7.15_1));* (3) (3.21) *and* (7.16_2) *((3.21) and (7.16_1)) hold.*

Corollary 13.3. *Let $F \in V(\tau)$, conditions* (1.3), (7.17), (7.18) *hold and $\alpha_i \ge 1$, $d_i \in [0, +\infty)$ $(i = 1, \dots, m)$ be fulfilled and*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t s^n \Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}} ds > \frac{1}{m}\max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m \alpha_i e^{\lambda d_i}\Big)^{-\frac{\lambda}{m}} : \lambda \in [1, 2]\Big\},$$

when n is even and

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t s^n \Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}} ds > \frac{1}{m}\max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m \alpha_i e^{\lambda d_i}\Big)^{-\frac{\lambda}{m}} : \lambda \in [0, 1]\Big\},$$

when n is odd. Then Eq. (1.1) *has Property* **B**.

Corollary 13.4. *Let $\alpha_i \ge 1$, $c_i \in [-\infty, 0)$, $d_i \in [0, +\infty)$ $(i = 1, \dots, m)$. Then Eq.* (7.19) *has Property **B** if and only if*

$$\max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\sum_{i=1}^m c_i\alpha_i^{\lambda} e^{\lambda d_i}\Big)^{-1} : \lambda \in [1, 2]\Big\} < 1,$$

when n is even and

$$\max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\sum_{i=1}^m c_i\alpha_i^{\lambda} e^{\lambda d_i}\Big)^{-1} : \lambda \in [0, 1]\Big\} < 1,$$

when n is odd.

14. Ordinary differential equations with Property A (Property B)

Here we give the sufficient conditions for the equation

$$u^{(n)}(t) + p(t)\,\big|u(t)\big|^{1+\frac{d}{\ln t}} \operatorname{sign} u(t) = 0, \quad t \ge a > 1 \tag{14.1}$$

to have Property **A** (Property **B**), where $p \in L_{\mathrm{loc}}(R_+; R)$ and $d \in R$.

The results of this section are the consequences of the previous ones, but we present them because in this case the conditions have a quite simple form.

Theorem 14.1. *Let $p \in L_{\mathrm{loc}}(R_+; R_+)$ and*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t s^n p(s)ds > \max\Big\{-\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\lambda d} : \lambda \in [0, n-1]\Big\}.$$

Then Eq. (14.1) *has Property* **A**.

Theorem 14.2. *Let $p \in L_{\mathrm{loc}}(R_+; R_+)$, $d \in (-\infty, 0]$ and*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t s^n p(s)ds > \max\Big\{-\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\lambda d} : \lambda \in [n-2, n-1]\Big\}.$$

Then Eq. (14.1) *has Property* **A**.

Remark 14.1. For $d = 0$, the above theorem results in Koplatadze's theorem [8] which is an integral generalization of Kondratev's result [2].

Theorem 14.3. *Let* $p \in L_{\mathrm{loc}}(R_+; R_+)$, $d \in [0, +\infty)$ *and*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t s^n p(s)ds > \max\Big\{-\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\lambda d} : \lambda \in [0,1]\Big\},$$

when n is even and

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t s^n p(s)ds > \max\Big\{-\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\lambda d} : \lambda \in [1,2]\cup[n-2,n-1]\Big\},$$

when n is odd. Then Eq. (14.1) *has Property* **A**.

Theorem 14.4. *Let* $p \in L_{\mathrm{loc}}(R_+; R_-)$ *and*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t s^n |p(s)|ds > \max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\lambda d} : \lambda \in [0,n-1]\Big\}.$$

Then Eq. (14.1) *has Property* **B**.

Remark 14.2. For $d = 0$, the above theorem results in Koplatadze's theorem [9].

Theorem 14.5. *Let* $p \in L_{\mathrm{loc}}(R_+; R_-)$, $d \in (-\infty, 0]$ *and*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t s^n |p(s)|ds > \max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\lambda d} : \lambda \in [n-3,n-2]\Big\},$$

when n is even and

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t s^n |p(s)|ds > \max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\lambda d} : \lambda \in [0,1]\cup[n-3,n-2]\Big\},$$

when n is odd. Then Eq. (14.1) *has Property* **B**.

Theorem 14.6. *Let* $p \in L_{\mathrm{loc}}(R_+; R_-)$, $d \in (0, +\infty)$ *and*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t s^n |p(s)|ds > \max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\lambda d} : \lambda \in [1,2]\Big\},$$

when n is even and

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t s^n |p(s)|ds > \max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\lambda d} : \lambda \in [0,1]\Big\},$$

when n is odd. Then Eq. (14.1) *has Property* **B**.

Theorem 14.7. *Let* $c \in (0, +\infty)$, ($c \in (-\infty, 0)$), $d \in R$. *Then for the equation*

$$u^{(n)}(t) + \frac{c}{t^n}\left|u(t)\right|^{1+\frac{d}{\ln t}} \operatorname{sign} u(t) = 0, \quad t \ge a > 1$$

to have Property **A** (**B**), *it is necessary and sufficient that*

$$c > \max\Big\{-\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\lambda d} : \lambda \in [0,n-1]\Big\}$$

$$\left(|c| > \max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\lambda d} : \lambda \in [0,n-1]\Big\}\right).$$

Remark 14.3. To show the difference and similarity between linear and almost linear differential equations we will consider a simple example. Consider the equation

$$u^{(n)}(t)+\frac{M_n}{t^n}\,u(t)=0 \quad \left(u^{(n)}(t)-\frac{M_n^*}{t^n}\,u(t)=0\right) \quad t\geq a>1, \tag{14.2}$$

where

$$M_m=\max\Big\{-\lambda(\lambda-1)\cdots(\lambda-n+1):\lambda\in[0,n-1]\Big\},$$
$$M_m^*=\max\Big\{\lambda(\lambda-1)\cdots(\lambda-n+1):\lambda\in[0,n-1]\Big\}.$$

It is obvious that Eq. (14.2) does not have Property **A** (Property **B**), but for any $d>0$, the equation

$$u^{(n)}(t)+\frac{M_n}{t^n}\,\big|u(t)\big|^{1+\frac{d}{\ln t}}\operatorname{sign}u(t)=0 \quad \left(u^{(n)}(t)-\frac{M_n^*}{t^n}\,\big|u(t)\big|^{1+\frac{d}{\ln t}}\operatorname{sign}u(t)=0\right) \quad t\geq a>1$$

has Property **A** (Property **B**).

On the other hand, for any $d>0$, there exists $\varepsilon=\varepsilon(d)>0$ such that the equation

$$u^{(n)}(t)+\frac{M_n+\varepsilon}{t^n}\,u(t)=0 \quad \left(u^{(n)}(t)-\frac{M_n^*+\varepsilon}{t^n}\,u(t)=0\right) \quad t\geq a>1$$

has Property **A** (Property **B**) and the equation

$$u^{(n)}(t)+\frac{M_n}{t^n}\,\big|u(t)\big|^{1-\frac{d}{\ln t}}\operatorname{sign}u(t)=0 \quad \left(u^{(n)}(t)-\frac{M_n^*}{t^n}\,\big|u(t)\big|^{1-\frac{d}{\ln t}}\operatorname{sign}u(t)=0\right) \quad t\geq a>1$$

does not have Property **A** (Property **B**).

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Original article

Thermal diffusion and diffusion thermo effects on unsteady MHD fluid flow past a moving vertical plate embedded in porous medium in the presence of Hall current and rotating system

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Abstract

In this research paper, numerical study of unsteady magnetohydrodynamic natural convective heat and mass transfer of a viscous, rotating fluid, electrically conducting and incompressible fluid flow past an impulsively moving vertical plate embedded in porous medium in the presence of ramped temperature, thermal radiation, hall current, thermal diffusion and diffusion thermo is investigated. The fundamental governing dimensionless coupled boundary layer partial differential equations are solved by an efficient Element Free Galerkin Method (EFGM). Computations were performed for a wide range of some important governing flow parameters viz., Hall current, rotation, thermal diffusion (Soret) and diffusion thermo (Dufour). The effects of these flow parameters on primary and secondary velocity, temperature and concentration fields for externally heating and cooling of the plate are shown graphically. Finally, the effects of these flow parameters on the rate of heat, mass transfer and shear stress coefficients at the wall are prepared through tabular forms for heating and cooling of the plate. Also, these are all discussed for ramped temperature and isothermal plates. We have shown that some results are in good agreement with earlier reported studies.

Keywords: Heat transfer; MHD; Hall current; Rotation; Element Free Galerkin Method

1. Introduction

The Hall effect is the making of a voltage difference across an electrical conductor, transverse to an electric current in the conductor and an electromagnetic field is perpendicular to the current. It is found by Edwin Hall [1]. The problems on magnetohydrodynamics viscous fluids with hall current has importance in engineering applications

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Nomenclature

List of Variables:

B_0	Uniform applied magnetic field (T)
x', y', z'	Co-ordinate system (m)
x, y, z	Dimensionless coordinates (m)
u'	Fluid velocity along the x'-axis (m s^{-1})
w'	Fluid velocity along the z'-axis (m s^{-1})
u	Non-dimensional fluid velocity along the x'-axis (m)
w	Non-dimensional fluid velocity along the z'-axis (m)
t_0	Characteristic time (s)
Nu	Nusselt number or rate of heat transfer coefficient
Sh	Sherwood number or rate of mass transfer coefficient
c_p	Specific heat at constant pressure (J kg^{-1}K)
Gr	Grashof number for heat transfer
Gm	Grashof number for mass transfer
$\bar{g}$	Acceleration due to gravity, 9.81 (m/s^2)
g	Acceleration due to gravity in magnitude (m/s^2)
K_1	Permeability parameter (K d^{-2})
k_T	Thermal diffusion ratio
T_m	Mean fluid temperature (K)
C_s	Concentration susceptibility (m mol^{-1})
k^*	Mean absorption coefficient (m^{-1})
$\bar{B}$	Magnetic induction vector
M^2	Magnetic parameter
Pr	Prandtl number
p	Fluid pressure (N m^{-2})
q_r	Radiative flux (kg/s^3)
m	Hall current parameter
N	Radiation parameter
Sr	Soret number
C'	Species concentration (kg m^{-3})
C'_∞	Species concentration of the fluid far away from the plate (kg m^{-3})
C'_w	Species concentration at the plate (kg m^{-3})
D_m	Molecular mass diffusivity (m^2 s^{-1})
D_T	Molecular diffusivity (m^2 s^{-1})
E	Electric field (S m^{-1})
Dr	Dufour number
Sc	Schmidt Number
T'_w	Temperature at the plate (K)
T'_∞	Temperature of the fluid far away from the plate (K)
t'	Time (s)
T'	Fluid temperature (K)
U_0	Plate velocity (m s^{-1})
T	Non-dimensional temperature (K)
C	Non-dimensional species concentration (kg m^{-3})

Greek symbols:

ρ	Fluid density (kg m^{-3})

κ	Thermal conductivity (W m^{-1} K^{-1})
σ	Electrical conductivity (S m^{-1})
ν	Kinematic viscosity (m^2 s^{-1})
β'	Coefficient of volume expansion for heat transfer (K^{-1})
Ω	Rotation parameter (degrees)
Ω'	Uniform angular velocity (degrees)
β^*	Coefficient of volume expansion for mass transfer (m^3 kg^{-1})
τ_x	Skin-friction in x'-direction (Pa)
τ_z	Skin-friction in z'-direction (Pa)
σ^*	Stefan–Boltzmann constant (W m^{-2} K^{-4})

Superscript

$/$	Dimensionless properties

Subscripts

w	Wall conditions
∞	Free stream conditions
p	Plate

such as MHD generators and MHD accelerators, laboratory plasmas, the rotating flow of fluids in the presence of magnetic field occurring in geophysical and cosmical fluid dynamics, the solar physics involved in the sunspot development, solar cycle and structure of rotating magnetic stars. The effect of Hall current with rotating system on MHD convection flows have been carried out by many researchers due to application of such studies as in the problems of MHD generators and Hall accelerators. Ajay Kumar Singh et al. [2], Mbeledogu and Ogulu [3], Abuga et al. [4], Jain and Singh [5] have studied rotation/Hall effects on various problems. Ahmed and Dutta [6] discussed transient mass transfer flow past an impulsively started infinite vertical plate in ramped plate velocity and ramped temperature. Seth et al. [7] studied the effects of hall current and rotation on natural convection radiative heat and mass transfer MHD flow past a moving vertical plate for ramped and isothermal plate only in case of externally cooling of the plate by Laplace transform technique with the absence of thermal diffusion and diffusion thermo. Chamkha et al. [8] investigated the influence of hall current on unsteady MHD free convective heat and mass transfer on a vertical porous plate with thermal radiation and chemical reaction. Sivaiah and Srinivasa Raju [9] studied the effects of Hall current and Heat source on MHD heat and mass transfer free convective flow in the presence of viscous dissipation by applying finite element technique. Siva Reddy and Srinivasa Raju [10] studied the effect of viscous dissipation on transient free convection flow past an infinite vertical plate through porous medium in the presence of magnetic field using finite element technique. Anand Rao et al. [11] demonstrated transient flow past an impulsively started infinite flat porous plate in a rotating fluid in the presence of magnetic field with Hall current using finite element technique. Anand Rao et al. [12] investigated the combined effects of heat and mass transfer on unsteady MHD flow past a vertical oscillatory plate suction velocity using finite element method. The combined effects of heat and mass transfer on unsteady MHD natural convective flow past an infinite vertical plate enclosed by porous medium in presence of thermal radiation and Hall Current was investigated by Ramana Murthy et al. [13]. Jithender Reddy et al. [14]. Anand Rao [15] and Srinivasa Raju et al. [16] studied MHD free convection fluid flow problems with various physical conditions using Finite Element Technique. Sheikholeslami et al. [17] investigated the effect of space dependent magnetic field on free convection Fe_3O_4–water nanofluid through control volume based finite element technique. Sheikholeslami et al. [18] employed control volume-based finite element technique to simulate Fe_3O_4–water nanofluid mixed convection heat transfer in a lid-driven semi annulus in the presence of a non-uniform magnetic field. Rashidi et al. [19] investigated the numerical study of magnetic field impact on mixed convection heat transfer of nanofluid in a channel with sinusoidal walls. Rashidi et al. [20] studied the combined heat and mass transfer of magnetohydrodynamic (MHD) convective and slip flow due to a rotating disk with influence of viscous dissipation and Ohmic heating by using the combination of the DTM and the Padé approximants.

The heat and mass transfer simultaneously affect each other and these will cause the cross-diffusion effect. The heat transfer caused by concentration (mass) gradient is called the diffusion-thermo (Dufour effect). On the other hand mass transfer caused by the temperature gradient is called thermal-diffusion (Soret) effect. Alam and Rahman et al. [21] investigated the Dufour (thermal-diffusion) and Soret (diffusion-thermo) effects on mixed convection flow past a vertical porous flat plate with the presence of variable suction. El-Arabawy et al. [22] investigated the Soret and Dufour effect on heat and mass transfer by natural convection from vertical surface embedded in a fluid-saturated porous media considered with variable surface temperature and constant concentration. Kafoussias et al. [23] studied thermal-diffusion and diffusion-thermo effects on mixed natural-forced convective and mass transfer boundary layer flow with the temperature dependent viscosity. Nabil et al. [24] studied thermal diffusion and diffusion thermo effects on the viscous fluid flow with heat and mass transfer through porous medium on a shrinking sheet. Srinivas et al. [25] found thermal diffusion and diffusion thermo effects on MHD viscous fluid flow between expanding rotating porous disks with viscous dissipation. Srinivasacharya et al. [26–28], Ram Reddy et al. [29] and Jithender Reddy et al. [30] studied Soret and Dufour effects on MHD free convection problems with varied physical parameters. Ahmed et al. [31] studied the effect of Soret (thermal diffusion) on unsteady free convective flow of an electrically conducting fluid over an infinite vertical oscillating plate embedded in a porous medium in the presence of a uniform transverse magnetic field. Srinivasa Raju [32] studied the combined effects of thermal-diffusion and diffusion-thermo on unsteady free convection fluid flow past an infinite vertical porous plate in the presence of magnetic field and chemical reaction using finite element technique. Srinivasa Raju et al. [33] found the numerical results for the effects of thermal radiation and heat source on an unsteady free convective flow past an infinite vertical plate with transverse magnetic field in the presence of thermal-diffusion and diffusion-thermo. Srinivasa Raju et al. [34] studied application of finite element method to unsteady MHD free convection flow past a vertically inclined porous plate including thermal diffusion and diffusion thermo effects. The influence of viscous dissipation on free convective flow past a semi-infinite vertical plate in the presence of Soret and Magnetic field was studied by Siva Reddy Sheri et al. [35]. Abdelraheem et al. [36] studied double-diffusive free convective flow over a vertical stretching surface embedded in a porous medium in the presence of a homogeneous first-order chemical reaction, radiation and Soret and Dufour effects. A numerical model was developed by Ahmed and Sibanda [37] for the effects of variable viscosity, and Soret and Dufour numbers on MHD mixed convective flow, heat and mass transfer from an exponentially stretching vertical surface embedded in a porous medium.

In this paper, we studied the hall current and rotation effects on MHD free convection flow past a moving vertical plate with the presence of thermal diffusion and diffusion thermo for isothermal and ramped temperature in both cases externally heating and cooling of the plate. The governing partial differential equations are solved by Element Free Galerkin Method and shown the present results are in good agreement with the results of Seth et al. [7].

2. Mathematical modeling

Consider an unsteady MHD natural convection flow with heat and mass transfer of an optically thick radiating, incompressible and electrically conducting viscous fluid past an infinite vertical plate is embedded in a uniform porous medium with a rotating system taking Hall current into account. Consider x'-axis is along the plate in upward direction and y'-axis is normal to plane of the plate in the fluid. A uniform transverse magnetic field B_0 is applied in a direction which is parallel to y'-axis. The fluid and plate rotate with uniform angular velocity Ω' about the y'-axis. Initially i.e. at time $t' \leq 0$, both the fluid and plate are in rest and these are maintained at a uniform temperature T'_∞. Also species concentration is at the surface of the plate as well as at every point within the fluid and it is maintained at uniform concentration C'_∞. At time $t' > 0$, plate starts moving in x'-direction with uniform velocity U_0 in its own plane. The temperature of the plate is raised or lowered to $T'_\infty + (T'_w - T'_\infty)t'/t_0$ when $0 < t' \leq t_0$, and it is maintained at uniform temperature T'_w when $t' > t_0$.

Also, at time $t' > 0$, species concentration is at the surface of the plate, it is raised to uniform species concentration C'_w and it is maintained thereafter. Geometry of the problem is shown in Fig. 1. Since plate is an infinite extent in x' and z' directions and it is electrically non-conducting, all physical quantities except pressure depends on y' and t' only. Also, no applied or polarized voltages are assumed to exist, so that the effect of polarization of fluid is negligible. The induced magnetic field generated by fluid motion is negligible in comparison to the applied one. This assumption is justified because magnetic Reynolds number is very small for liquid metals and partially ionized fluids which are commonly used in industrial applications (Cramer and Pai [38]). Keeping in view of these assumptions and under the

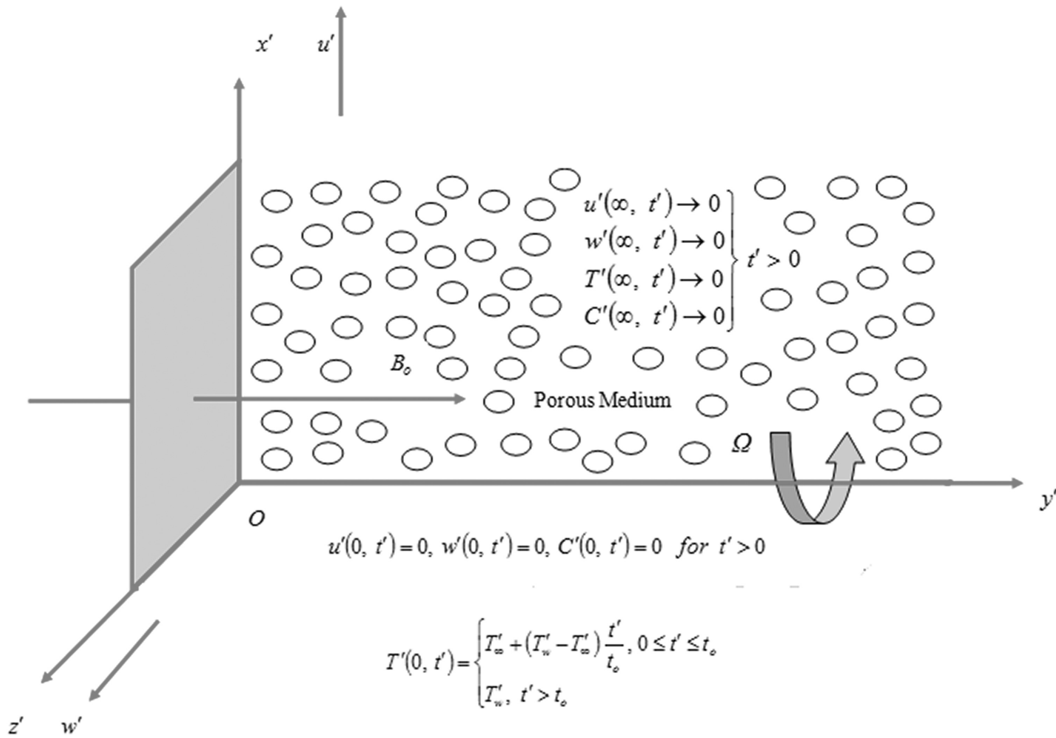

Fig. 1. Geometry of the problem.

Boussinesq's approximation, the governing equations are given by (Seth et al. [7])

$$\frac{\partial u'}{\partial t'} + 2\Omega' w' = \nu \frac{\partial^2 u'}{\partial y'^2} - \frac{\sigma B_o^2}{\rho\left(1+m^2\right)}\left(u' + m w'\right) - \frac{\nu u'}{K_1} + g\beta'\left(T' - T'_\infty\right) + g\beta^*\left(C' - C'_\infty\right) \tag{1}$$

$$\frac{\partial w'}{\partial t'} - 2\Omega' u' = \nu \frac{\partial^2 w'}{\partial y'^2} - \frac{\sigma B_o^2}{\rho\left(1+m^2\right)}\left(m u' - w'\right) - \frac{\nu w'}{K_1} \tag{2}$$

$$\frac{\partial T'}{\partial t'} = \frac{\kappa}{\rho c_p}\frac{\partial^2 T'}{\partial y'^2} - \frac{1}{\rho c_p}\frac{\partial q_r}{\partial y'} + \frac{D_m k_T}{c_s c_p}\frac{\partial^2 C'}{\partial y'^2} \tag{3}$$

$$\frac{\partial C'}{\partial t'} = D\frac{\partial^2 C'}{\partial y'^2} + \frac{D_m k_T}{T_m}\frac{\partial^2 T'}{\partial y'^2}. \tag{4}$$

The boundary conditions for the primary and secondary velocity, temperature and concentration fields are (Seth et al. [7])

$$\forall t' \le 0 : u' = w' = 0,\ T' = T'_\infty,\ C' = C'_\infty \text{ for } y' \ge 0 \tag{5}$$

$$\forall t' > 0 : u' = U_0,\ w' = 0,\ C' = C'_w \text{ at } y' = 0 \tag{6}$$

$$T' = T'_\infty + (T'_w - T'_\infty)t'/t_0 \text{ at } y' = 0 \text{ for } 0 < t' \le t_0 \tag{7}$$

$$\forall t' > t_0 : T' = T'_w \text{ at } y' = 0 \tag{8}$$

$$\forall t' > 0 : u' = 0, w' = 0,\ T' \to T'_\infty,\ C' \to C'_\infty \text{ at } y' \to \infty. \tag{9}$$

The radiative heat flux term by using the Rosseland approximation (Sparrow and Cess [39]) is given by

$$q'_r = -\frac{4\sigma^*}{3k^*}\left(\frac{\partial T'^4}{\partial y'}\right)_{y=0}. \tag{10}$$

It should be noted that by using the Rosseland approximation, present analysis is limited to optically thick fluids. If temperature differences within the flow are sufficiently very small then Eq. (10) can be linearized by expanding T' into the Taylor series about T'_∞ which after neglecting higher order terms take the form

$$T'^4 \cong 4T'^3_\infty - 3T'^4_\infty. \tag{11}$$

Substituting Eqs. (10) and (11), into Eq. (3), we obtain

$$\frac{\partial T'}{\partial t'} = \frac{\kappa}{\rho c_p}\frac{\partial^2 T'}{\partial y'^2} + \frac{1}{\rho c_p}\frac{16\sigma^* T'^3_\infty}{3k^*}\frac{\partial^2 T'}{\partial y'^2} + \frac{D_m k_T}{c_s c_p}\frac{\partial^2 C'}{\partial y'^2}. \tag{12}$$

Introducing the following non-dimensional quantities into the Eqs. (1), (2), (4), (12) and (5)–(9)

$$u = \frac{u'}{U_0},\ w = \frac{w'}{U_0},\ y = \frac{y'U_0}{\nu},\ t = \frac{t'U_0^2}{\nu},\ T = \frac{T' - T'_\infty}{T'_w - T'_\infty},\ C = \frac{C' - C'_\infty}{C'_w - C'_\infty},$$
$$M^2 = \frac{\sigma B_0^2 \nu}{\rho U_0^2},\ \Omega = \frac{\nu \Omega'}{U_0^2},\ K_1 = \frac{K'_1 U_0^2}{\nu^2},\ Gr = \frac{g\beta\nu(T'_w - T'_\infty)}{U_0^3},$$
$$Gm = g\beta^*\nu\frac{C'_w - C'_\infty}{U_0^3},\ Pr = \frac{\nu\rho c_p}{\kappa},\ N = \frac{16\sigma^* T'^3_\infty}{3\kappa k^*},\ S_c = \frac{\nu}{D},$$
$$Sr = \frac{D_m k_T (T'_w - T'_\infty)}{\nu T_m (C'_w - C'_\infty)},\ Dr = \frac{D_m k_T (C'_w - C'_\infty)}{\nu c_s c_p (T'_w - T'_\infty)}$$

then the resultant non-dimensional equations are

$$\frac{\partial u}{\partial t} + 2\Omega w = \frac{\partial^2 u}{\partial y^2} - M^*(u + mw) - \frac{u}{K_1} + GrT + GmC \tag{13}$$

$$\frac{\partial w}{\partial t} - 2\Omega u = \frac{\partial^2 w}{\partial y^2} + M^*(mu - w) - \frac{w}{K_1} \tag{14}$$

$$\frac{\partial T}{\partial t} = R\frac{\partial^2 T}{\partial y^2} + Dr\frac{\partial^2 C}{\partial y^2} \tag{15}$$

$$\frac{\partial C}{\partial t} = \frac{1}{Sc}\frac{\partial^2 C}{\partial y^2} + Sr\frac{\partial^2 T}{\partial y^2} \tag{16}$$

where $M^* = \frac{M^2}{1+m^2}$, $R = \frac{1+N}{Pr}$.

The non-dimensional initial and boundary conditions are

$$\forall t \le 0 : u = w = 0,\ T = 0, C = 0 \text{ for } y \ge 0 \tag{17}$$
$$\forall t > 0 : u = 1,\ w = 0,\ C = 1 \text{ at } y = 0 \tag{18}$$
$$\forall 0 < t \le 1 : T = t \text{ at } y = 0 \tag{19}$$
$$\forall t > 1 : T = 1 \text{ at } y = 0 \tag{20}$$
$$\forall t > 0 :\ u \to 0,\ w \to 0,\ T \to 0,\ C \to 0 \text{ at } y \to \infty. \tag{21}$$

3. Numerical solution by Element Free Galerkin Method (EFGM)

Element Free Galerkin Method (EFGM) is one of the computational method developed by Belytschko et al. [40]. This method is applicable to arbitrary shapes, and it requires only nodal data which is applied to elasticity and heat conduction problems. This method shares essential characteristics with many other numerical methods such as Kernal particle method (Liu et al. [41]), Finite point method (Onate et al. [42]) and Hp-clouds (Duarte and Oden et al. [43]). Previously, the review of these numerical methods was reported by Belytschko et al. [44]. Recently, several authors applied this EFGM in their research problems. In spite of that, Rajesh Sharma and Bhargava [45] found the numerical solutions of unsteady MHD convection heat and mass transfer past a semi-infinite vertical porous moving plate using EFGM. Ryszard [46] applied an EFGM to water wave propagation problems. Very recently, Singh and Bhargava [47] studied the characteristics of heat transfer flow of a phase transition in melting problem using FEM and EFGM, and the results are shown closer to each other. Rajesh Sharma [48] found the numerical simulation of MHD Hiemenz flow of a micropolar fluid on non linear stretching sheet embedded in porous Medium using EFGM. Srinivasa Raju et al. [49] found the numerical and analytical solutions of unsteady MHD free convection on exponential accelerated vertical plate with heat absorption using Element Free Galerkin Method and Laplace Transform Technique respectively. Also they have shown the numerical solutions by FEM are in good agreement with the analytical solutions by LTT.

3.1. Review of Element Free Galerkin Method

The Element Free Galerkin Method (EFGM) requires moving least square (MLS) interpolation functions to approximate an unknown function, which is made up of three components: a weight function associated with each node, a basis function and a set of coefficients that depends on position. The weight function is non-zero over a small neighborhood at a particular node, called support of the node. Using MLS approximation, the unknown velocity component u is approximated over the domain $[0, \infty]$ as

$$u\left(x\right) \cong u^{h}\left(x\right) = \sum_{j=1}^{m} p_{j}\left(x\right) a_{j}\left(x\right) = p^{T}\left(x\right)\, a\left(x\right) \tag{22}$$

where m is the number of terms in the basis, $p_j\left(x\right)$ the monomial basis function, $a_j\left(x\right)$ the non-constant coefficients and $p^T\left(x\right) = [1x]$. The coefficients $a_j\left(x\right)$ are determined by minimizing the functional $J\left(x\right)$ given by

$$J\left(x\right) = \sum_{i=1}^{m} w\left(x - x_i\right) \left\{ \sum_{j=1}^{m} p_j\left(x_i\right) a_j\left(x\right) - u_i \right\}^2 \tag{23}$$

where $w\left(x - x_i\right)$ is a weight function which is non-zero over a small domain, called domain of influence, n is the number of nodes in the domain of influence. The minimization of $J\left(x\right)$ w.r.t $a\left(x\right)$ leads to the following set of equation

$$a\left(x\right) = C^{-1}\left(x\right) D\left(x\right) U^T \tag{24}$$

where C and D are given as

$$C = \sum_{i=1}^{n} w\left(x - x_i\right) p\left(x_i\right) p^T\left(x_i\right) \tag{25}$$

$$D\left(x\right) = \left[w\left(x - x_1\right) p\left(x_1\right), w\left(x - x_2\right) p\left(x_2\right), w\left(x - x_3\right) p\left(x_3\right), \ldots, w\left(x - x_n\right) p\left(x_n\right)\right] \tag{26}$$

$$U^T = \left[U_1, U_2, U_3, \ldots, U_n\right]. \tag{27}$$

Substituting Eq. (24) in Eq. (22), the MLS approximants are obtained as

$$u\left(x\right) \cong u^{h}\left(x\right) = \sum_{i=1}^{n} \Phi_i\left(x\right) u_i = \Phi\left(x\right) u. \tag{28}$$

Similarly $\theta\left(x\right), \phi\left(x\right)$ can be approximated by

$$\theta\left(x\right) \cong \theta^{h}\left(x\right) = \sum_{i=1}^{n} \Phi_i\left(x\right) \theta_i = \Phi\left(x\right) \theta \tag{29}$$

$$\phi\left(x\right) \cong \phi^{h}\left(x\right) = \sum_{i=1}^{n} \Phi_i\left(x\right) \phi_i = \Phi\left(x\right) \phi \tag{30}$$

where the shape function $\Phi_i(x)$ is defined by

$$\Phi_i(x) = \sum_{j=1}^{n} p_j(x)\left(C^{-1}(x)\,D(x)\right)_{ji} = p^T C^{-1} D_i. \tag{31}$$

3.2. Choice of weight function

The weight function is non-zero over a small neighborhood of x_i, called the domain of the influence of node i. The choice of weight function $w(x - x_i)$ affects the resulting approximation in EFGM and other mesh less methods. Singh et al. [50] studied these weight functions and found that cubicspline weight function gives more accurate results as compared to others. Therefore, in the present work, a cubicspline weight function (Singh et al. [50]) has been used.

3.3. Cubic spline weight function

$$w(r - r_i) = w(r) = \begin{Bmatrix} \frac{2}{3} - 4r^2 + 4r^3 & \text{for } r \le \frac{1}{2} \\ \frac{4}{3} - 4r + 4r^2 - \frac{4}{3}r^3 & \text{for } \frac{1}{2} \le r \le 1 \\ 0 & \text{for } r > 1 \end{Bmatrix} \tag{32}$$

where $r_i = \frac{\|x - x_i\|}{d_{ml}}$, d_{ml} are the size of domain of influence which are calculated as $d_{ml} = d_{\max} C_i$, where $d_{\max}$ is a scaling parameter, and C_i is the distance to the nearest neighbors. The size of the domain of influence (d_{ml}) at particular node i is only controlled by scaling parameter ($d_{\max}$) since the distance between nearest neighbors for an evaluation point remains unchanged for a given nodal data distribution. The minimum value of $d_{\max}$ should be greater than 1 so that $n > m$, and the maximum value of $d_{\max}$ should be such that it preserves the local character of MLS approximation. It has been shown in Singh [51] that $1 < d_{\max} < 1.5$ is the optimum range of scaling parameter for heat transfer problem. Therefore $d_{\max}$ has been fixed as 1.01.

The weighted integral forms of Eqs. (13)–(16) can be written as

$$\int_0^{y_{\max}} w_1\left[\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial u}{\partial t}\right) - Nu - M^* m w - 2\Omega w + GrT + GmC\right]dy = 0 \tag{33}$$

$$\int_0^{y_{\max}} w_2\left[\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial w}{\partial t}\right) - Nw + \left(M^*\right)(m)(u) + 2\Omega u\right]dy = 0 \tag{34}$$

$$\int_0^{y_{\max}} w_3\left[R\frac{\partial^2 T}{\partial y^2} - \left(\frac{\partial T}{\partial t}\right) + (Dr)\left(\frac{\partial^2 C}{\partial y^2}\right)\right]dy = 0 \tag{35}$$

$$\int_0^{y_{\max}} w_4\left[\left(\frac{1}{Sc}\right)\frac{\partial^2 C}{\partial y^2} - \left(\frac{\partial C}{\partial t}\right) + (Sr)\left(\frac{\partial^2 T}{\partial y^2}\right)\right]dy = 0 \tag{36}$$

where $N = M^* + \frac{1}{K_1}$ and w_1, w_2, w_3, w_4 are arbitrary test functions and may be viewed as the variations in u, w, T and C, respectively. After reducing the order of integration and non-linearity, the following system of equations are obtained:

$$\begin{aligned}\int_0^{y_{\max}} &\left[\left(\frac{\partial w_1}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right) + (w_1)\left(\frac{\partial u}{\partial t}\right) + N(w_1)\,u + M^* m(w_1)\,w + 2\Omega(w_1)\,w + (Gr)(w_1)\,T\right. \\ &\left. - (Gm)(w_1)\,C\right]dy - \left[(w_1)\left(\frac{\partial u}{\partial y}\right)\right]_0^{y_{\max}} = 0\end{aligned} \tag{37}$$

$$\begin{aligned}\int_0^{y_{\max}} &\left[\left(\frac{\partial w_2}{\partial y}\right)\left(\frac{\partial w}{\partial y}\right) + (w_2)\left(\frac{\partial w}{\partial t}\right) + N(w_2)\,w - M^* m(w_2)\,u - 2\Omega(w_2)\,w\right]dy \\ &- \left[(w_2)\left(\frac{\partial w}{\partial y}\right)\right]_0^{y_{\max}} = 0\end{aligned} \tag{38}$$

$$\int_0^{y_{\max}}\left[R\left(\frac{\partial w_3}{\partial y}\right)\left(\frac{\partial T}{\partial y}\right)+(w_3)\left(\frac{\partial T}{\partial t}\right)+(Dr)(w_3)\left(\frac{\partial w_3}{\partial y}\right)\left(\frac{\partial C}{\partial y}\right)\right]dy$$
$$-\left[R(w_3)\left(\frac{\partial T}{\partial y}\right)+(Dr)(w_3)\left(\frac{\partial C}{\partial y}\right)\right]_0^{y_{\max}}=0 \tag{39}$$
$$\int_0^{y_{\max}}\left(\frac{1}{Sc}\right)\left[\left(\frac{\partial w_4}{\partial y}\right)\left(\frac{\partial C}{\partial y}\right)+(w_4)\left(\frac{\partial C}{\partial t}\right)+(Sr)(w_4)\left(\frac{\partial w_4}{\partial y}\right)\left(\frac{\partial T}{\partial y}\right)\right]dy$$
$$-\left[(w_4)\left(\frac{1}{SC}\right)\left(\frac{\partial C}{\partial y}\right)+(Sr)(w_4)\left(\frac{\partial T}{\partial y}\right)\right]_0^{y_{\max}}=0. \tag{40}$$

Using the essential boundary conditions on w_1, w_2, w_3, w_4 as homogeneous, Eqs. (37)–(40) become

$$\int_0^{y_{\max}}\left[\left(\frac{\partial w_1}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)+(w_1)\left(\frac{\partial u}{\partial t}\right)+N\Big(w_1\Big)u+M^*m(w_1)w\right.$$
$$\left.+2\Omega(w_1)w-(Gr)(w_1)T-(Gm)(w_1)C\right]dy=0 \tag{41}$$
$$\int_0^{y_{\max}}\left[\left(\frac{\partial w_2}{\partial y}\right)\left(\frac{\partial w}{\partial y}\right)+(w_2)\left(\frac{\partial w}{\partial t}\right)+N(w_2)w-M^*m(w_2)u-2\Omega(w_2)w\right]dy=0 \tag{42}$$
$$\int_0^{y_{\max}}\left[R\left(\frac{\partial w_3}{\partial y}\right)\left(\frac{\partial T}{\partial y}\right)+(w_3)\left(\frac{\partial T}{\partial t}\right)+(Dr)(w_3)\left(\frac{\partial w_3}{\partial y}\right)\left(\frac{\partial C}{\partial y}\right)\right]dy=0 \tag{43}$$
$$\int_0^{y_{\max}}\left(\frac{1}{Sc}\right)\left[\left(\frac{\partial w_4}{\partial y}\right)\left(\frac{\partial C}{\partial y}\right)+(w_4)\left(\frac{\partial C}{\partial t}\right)+(Sr)(w_4)\left(\frac{\partial w_4}{\partial y}\right)\left(\frac{\partial T}{\partial y}\right)\right]dy=0. \tag{44}$$

3.4. Essential boundary conditions

Due to lack of Kronecker delta property in EFGM, the shape function Φ_i possesses some difficulty in the imposition of essential boundary conditions. To remove this problem, different numerical techniques have been proposed to enforce the essential boundary condition in EFGM such as Lagrange multiplier technique, modified variational principle approach and penalty approach. The penalty method Zhu and Atluri [52] is applied which is discussed as follows:

Penalty Method (PM):

$$\int_0^{y_{\max}}\left[\left(\frac{\partial w_1}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)+(w_1)\left(\frac{\partial u}{\partial t}\right)+N(w_1)u+M^*m(w_1)w+2\Omega(w_1)w-(Gr)(w_1)T\right.$$
$$\left.-\ (Gm)(w_1)C\right]dy-\alpha(w_1)(u-u_o)|_{y=0}-\alpha(w_1)(u-u_\infty)|_{y\to\infty}=0 \tag{45}$$
$$\int_0^{y_{\max}}\left[\left(\frac{\partial w_2}{\partial y}\right)\left(\frac{\partial w}{\partial y}\right)+(w_2)\left(\frac{\partial w}{\partial t}\right)+N(w_2)w-M^*m(w_2)u-2\Omega(w_2)u\right]dy$$
$$-\alpha(w_2)(w-w_o)|_{y=0}-\alpha(w_2)(w-w_\infty)|_{y\to\infty}=0 \tag{46}$$
$$\int_0^{y_{\max}}\left[R\left(\frac{\partial w_3}{\partial y}\right)\left(\frac{\partial T}{\partial y}\right)+(w_3)\left(\frac{\partial T}{\partial t}\right)+(Dr)(w_3)\left(\frac{\partial w_3}{\partial y}\right)\left(\frac{\partial C}{\partial y}\right)\right]dy$$
$$-\alpha(w_3)(T-T_o)|_{y=0}-\alpha(w_3)(T-T_\infty)|_{y\to\infty}$$
$$-\alpha(Dr)(w_3)(C-C_o)|_{y=0}-\alpha(Dr)(w_3)(C-C_\infty)|_{y\to\infty}=0 \tag{47}$$

$$\int_0^{y_{\max}}\left(\frac{1}{SC}\right)\left[\left(\frac{\partial w_4}{\partial y}\right)\left(\frac{\partial C}{\partial y}\right)+(w_4)\left(\frac{\partial C}{\partial t}\right)+(Sr)(w_4)\left(\frac{\partial w_4}{\partial y}\right)\left(\frac{\partial T}{\partial y}\right)\right]dy$$
$$-\alpha\frac{\alpha}{Sc}(w_4)(C-C_o)\Big|_{y=0}-\alpha\frac{\alpha}{Sc}(w_4)(C-C_\infty)\Big|_{y\to\infty}$$
$$-\alpha(Sr)(w_4)(T-T_o)|_{y=0}-\alpha(Sr)(w_4)(T-T_\infty)|_{y\to\infty}=0 \tag{48}$$

where $\left.\begin{matrix} u_o=1,\ w_o=0, T_o=t \text{ at } 0<t\le 1,\\ T_o=1 \text{ at } t>1\\ C_o=1,\ u_\infty=1,\ w_\infty=0,\ T_\infty=0,\ C_\infty=0\end{matrix}\right\}$ and $w_1=w_2=w_3=w_4=\Phi_i\ (i=1,2,\ldots,n)$.

Thus, Eqs. (45)–(48) can be written as:

$$[K]\{\bar h\}+[\bar M]\{\dot{\bar h}\}=\{F\} \tag{49}$$

where $[K]=\begin{bmatrix}K_{11}&K_{12}&K_{13}&K_{14}\\K_{21}&K_{22}&K_{23}&K_{24}\\K_{31}&K_{32}&K_{33}&K_{34}\\K_{41}&K_{42}&K_{43}&K_{44}\end{bmatrix}$, $[\bar M]=\begin{bmatrix}Mooo\\oMoo\\ooMo\\oooM\end{bmatrix}$, $\{\bar h\}=\begin{bmatrix}\{u\}\\\{w\}\\\{T\}\\\{C\}\end{bmatrix}$, $\{\dot{\bar h}\}=\begin{bmatrix}\{\dot u\}\\\{\dot w\}\\\{\dot T\}\\\{\dot C\}\end{bmatrix}$, $\{F\}=\begin{bmatrix}\{F_1\}\\\{F_2\}\\\{F_3\}\\\{F_4\}\end{bmatrix}$,

$$(K_{11})_{ij}=\int_0^{y_{\max}}\left[\left(\Phi_i^{T'}\right)\left(\Phi_j'\right)\right]dy+N\int_0^{y_{\max}}\left[\left(\Phi_i^{T}\right)\left(\Phi_j\right)\right]dy$$
$$-\left[\alpha\left(\Phi_i^T\right)(\Phi_j)\right]_{y=0}-\left[\alpha\left(\Phi_i^T\right)(\Phi_j)\right]_{y\to\infty},$$
$$(K_{12})_{ij}=\left(M^*m+2\Omega\right)\int_0^{y_{\max}}\left[\left(\Phi_i^T\right)(\Phi_j)\right]dy,\ (K_{13})_{ij}=-(Gr)\int_0^{y_{\max}}\left(\Phi_i^T\right)(\Phi_j)\,dy,$$
$$(K_{14})_{ij}=-(Gm)\int_0^{y_{\max}}\left(\Phi_i^T\right)(\Phi_j)\,dy,\ (M)_{ij}=\int_0^{y_{\max}}\left(\Phi_i^T\right)(\Phi_j)\,dy,\ \forall i=j,(M)_{ij}=0,\forall i\ne j$$
$$(K_{21})_{ij}=\left(M^*m+2\Omega\right)\int_0^{y_{\max}}\left[\left(\Phi_i^T\right)(\Phi_j)\right]dy,$$
$$(K_{22})_{ij}=\int_0^{y_{\max}}\left[\left(\Phi_i^{T'}\right)\left(\Phi_j'\right)\right]dy+N\int_0^{y_{\max}}\left[\left(\Phi_i^{T}\right)(\Phi_j)\right]dy-\left[\alpha\left(\Phi_i^T\right)(\Phi_j)\right]_{y=0}$$
$$-\left[\alpha\left(\Phi_i^T\right)(\phi_j)\right]_{y\to\infty},$$
$$(K_{23})_{ij}=(K_{24})_{ij},\ (K_{31})_{ij}=0=(K_{32})_{ij},$$
$$(K_{33})_{ij}=-R\int_0^{y_{\max}}\left[\left(\Phi_i^{T'}\right)\left(\Phi_j'\right)\right]dy+\left[R\alpha\left(\Phi_i^T\right)(\Phi_j)\right]_{y=0}-\left[R\alpha\left(\Phi_i^T\right)(\Phi_j)\right]_{y\to\infty},$$
$$(K_{34})_{ij}=(Dr)\int_0^{y_{\max}}\left[\left(\Phi_i^{T'}\right)\left(\Phi_j'\right)\right]dy-\left[\alpha(Dr)\left(\Phi_i^T\right)(\Phi_j)\right]_{y=0}-\left[\alpha(Dr)\left(\Phi_i^T\right)(\Phi_j)\right]_{y\to\infty},$$
$$(K_{41})_{ij}=0=(K_{42})_{ij},$$
$$(K_{43})_{ij}=(Sr)\int_0^{y_{\max}}\left[\left(\Phi_i^{T'}\right)\left(\Phi_j'\right)\right]dy-\left[\alpha(Sr)\left(\Phi_i^T\right)(\Phi_j)\right]_{y=0}-\left[\alpha(Sr)\left(\Phi_i^T\right)(\Phi_j)\right]_{y\to\infty},$$
$$(K_{44})_{ij}=\frac{1}{Sc}\int_0^{y_{\max}}\left[\left(\Phi_i^{T'}\right)\left(\Phi_j'\right)\right]dy-\left[\frac{\alpha}{Sc}\left(\Phi_i^T\right)(\Phi_j)\right]_{y=0}-\left[\frac{\alpha}{Sc}\left(\Phi_i^T\right)(\Phi_j)\right]_{y\to\infty},$$
$$(F_1)_i=u_o\alpha\Phi_j'+u_\infty\alpha\Phi_j',\ (F_2)_i=w_o\alpha\Phi_j'+w_\infty\alpha\Phi_j',$$
$$(F_3)_i=T_o\alpha\Phi_j'+T_\infty\alpha\Phi_j' \text{ at } 0<t\le 1,\ (F_3)_i=T_o\alpha\Phi_j'+T_\infty\alpha\Phi_j' \text{ at } t>1,$$
$$(F_4)_i=C_o\alpha\Phi_j'+C_\infty\alpha\Phi_j',$$

Using unconditionally stable Crank–Nicholson scheme (Smith [53]), Eq. (49) at $(s+1)$th level can be written as

$$\left[\hat K\right]_{s+1}\{\overline h\}_{s+1}=\left[\hat K\right]_s\{\overline h\}_s+\left\{\hat F\right\}_{s,s+1} \tag{50}$$

Table 1
The numerical values of u, w, T and C for variation of mesh sizes.

	Mesh size $h = 0.1$				Mesh size $h = 0.2$				Mesh size $h = 0.3$			
	u	w	T	C	u	w	T	C	u	w	T	C
	1.000000	0.000000	0.500000	1.000000	1.000000	0.000000	0.500000	1.000000	1.000000	0.000000	0.500000	1.000000
	0.205745	0.364936	0.329594	0.472351	0.205628	0.364842	0.329485	0.472215	0.205631	0.364853	0.329494	0.472224
	0.081102	0.102188	0.184112	0.158479	0.081852	0.102752	0.184256	0.158542	0.081851	0.102765	0.184264	0.158551
	0.026275	0.024006	0.084624	0.037798	0.026138	0.024158	0.084745	0.037813	0.026145	0.024161	0.084759	0.037824
Time	0.007119	0.004851	0.032037	0.006504	0.007015	0.004784	0.032183	0.006519	0.007019	0.004794	0.032194	0.006521
$t = 0.1$	0.001639	0.000846	0.010157	0.000824	0.001631	0.000850	0.010161	0.000891	0.001642	0.000859	0.010174	0.000912
	0.000324	0.000128	0.002747	7.86E−05	0.000324	0.000126	0.002747	7.86E−05	0.000324	0.000125	0.002758	7.86E−05
	5.54E−05	1.69E−05	0.000644	5.76E−06	5.54E−05	1.69E−05	0.000644	5.76E−06	5.54E−05	1.69E−05	0.000654	5.76E−06
	8.29E−06	1.97E−06	0.000132	3.3E−07	8.29E−06	1.97E−06	0.000132	3.3E−07	8.29E−06	1.97E−06	0.000132	3.3E−07
	1.09E−06	2E−07	2.38E−05	2E−08	1.09E−06	2E−07	2.38E−05	2E−08	1.09E−06	2E−07	2.38E−05	2E−08

where

$$\left[\hat{K}\right]_{s+1} = \left[M\right] + \frac{\Delta t\,[K]_{s+1}}{2},\ \left[\hat{K}\right]_{s} = \left[M\right] - \frac{\Delta t\,[K]_{s}}{2} \text{ and } \left[\hat{F}\right]_{s,s+1} = \frac{\Delta t}{2}\left(\{F\}_{s+1} + \{F\}_s\right). \tag{51}$$

For computational purposes, the coordinate y is varied from 0 to $y_{\max} = 10$, where $y_{\max}$ represents infinity i.e. external to the momentum, energy and concentration boundary layers. The whole domain is divided into 101 nodes. One point Gauss quadrature formula has been used to calculate the integral values. As the systems of equations are non-linear, an iterative scheme is employed to solve the matrix system. This system is linearized by incorporating known function u, which is solved using Gauss elimination method maintaining an accuracy of 0.0000005. The code of the algorithm has been executed in MATLAB running on a PC. Excellent convergence was achieved for all the results.

3.5. Study of grid independence

In general, to study the grid independency/dependency, the mesh size should be varied in order to check the solution at different mesh (grid) sizes and get a range at which there is no variation in the solution. We have shown the numerical values of Primary velocity (u), Secondary velocity (w), temperature (T) and concentration (C) for different values of mesh (grid) size at time $t = 1.0$ in Table 1. From this table, we observed that there is no variation in the values of Primary velocity (u), Secondary velocity (w), temperature (T) and concentration (C) for different values of mesh (grid) size at time $t = 0.1$. Hence, we conclude that the results are independent of mesh (grid) size.

4. Skin-friction, rate of heat and mass transfer coefficients

The skin-friction due to primary velocity at the wall along x'-axis in dimensionless form is given by $\tau_x = \left[\frac{\partial u}{\partial y}\right]_{y=0}$.

The skin-friction due to secondary velocity at the wall along z'-axis in dimensionless form is given by $\tau_z = \left[\frac{\partial u}{\partial z}\right]_{z=0}$.

Rate of heat transfer (Nusselt number) due to temperature profiles in dimensionless form is given by $Nu = -\left[\frac{\partial T}{\partial y}\right]_{y=0}$.

And rate of mass transfer (Sherwood number) due to concentration profiles in dimensionless form is given by $Sh = -\left[\frac{\partial C}{\partial y}\right]_{y=0}$.

5. Code validation

5.1. Comparison with analytical solutions

Comparison of τ_x and τ_z with Seth et al. [7] is shown by the superscript star in Table 2. Ω is replaced by K^2 in Seth et al. [7] with varied values of Hall current and rotation parameters and in the absence of Soret and Dufour number. The present results are in good agreement with the results of Seth et al. [7] for both ramped temperature and isothermal

Table 2
The skin-friction due to primary and secondary velocity with the effect of Hall current and rotation in case of cooling of the plate.

m	Ω	τ_x (Present results)		τ_x^* (Seth et al. [7])		τ_z (Present results)		τ_z^* (Seth et al. [7])	
		Ramped	Isothermal	Ramped	Isothermal	Ramped	Isothermal	Ramped	Isothermal
0.5	5	2.87095	2.29039	2.87124	2.29044	2.35013	2.60663	2.35018	2.60816
1	5	2.49089	1.92395	2.49195	1.92406	2.85873	3.19315	2.85995	3.19048
1.5	5	2.13521	1.56012	2.13659	1.56425	3.40649	3.40037	3.06904	3.45031
0.5	3	2.61221	1.97075	2.61086	1.97001	1.86393	2.09214	1.86439	2.09575
0.5	7	3.14353	2.61096	3.14162	2.61276	2.77062	3.05629	2.77199	3.04199

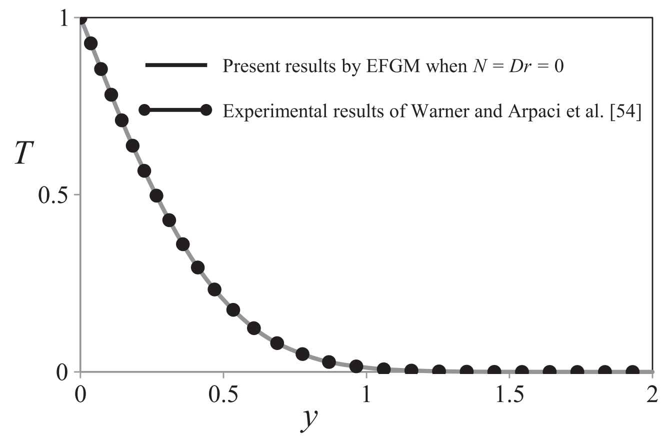

Fig. 2. Comparison of present results with existed experimental results of temperature distribution with an influence of $Pr = 0.71$.

plate in case of externally cooling of the plate. Although it is possible to obtain the exact solution using the Laplace Transform Technique (LTT), it seems to be a laborious process. The present method, EFGM is more economical and flexible in the computational point of view. Therefore, this method is superior than the LTT and other appropriate methods.

5.2. Comparison with experimental results

An experimental investigation of turbulent and laminar natural convection in air on a vertical plate is described by Warner and Arpaci [54]. But this study did not explore the experimental results with the presence of Hall current, rotation, radiation, thermal diffusion and diffusion thermo parameters. We compared the present results with the results of Warner and Arpaci [54] in the absence of these parameters. Fig. 2 shows the comparison of present results with existing experimental results of temperature distribution with an influence of $Pr = 0.71$ (Air). It is evident that the present numerical solutions are in good agreement with experimental results of Warner and Arpaci [54] in the absence of radiation and diffusion thermo parameters. These types of models are useful for validation purpose in view of lab experimental results.

6. Results and discussions

The effects of hall current and rotation on an unsteady radiative MHD free convective heat and mass transfer of an optically thick radiating, incompressible, electrically conducting and viscous fluid past an impulsively moving vertical porous plate with ramped temperature and isothermal were studied taking into account the thermal diffusion and diffusion thermo, and solved by Element Free Galerkin Method. Computations are performed for a wide range of some important governing flow physical parameters *viz*., Hall current (m), Rotation (Ω), Soret (Sr) and Dufour (Dr).

The effects of these flow physical parameters on the primary and secondary velocity, temperature and concentration fields for ramped temperature and isothermal plates in case of both externally cooling ($Gr > 0$) and heating ($Gr < 0$) of the plate are illustrated graphically. We have shown some results are in good agreement with the results of Seth

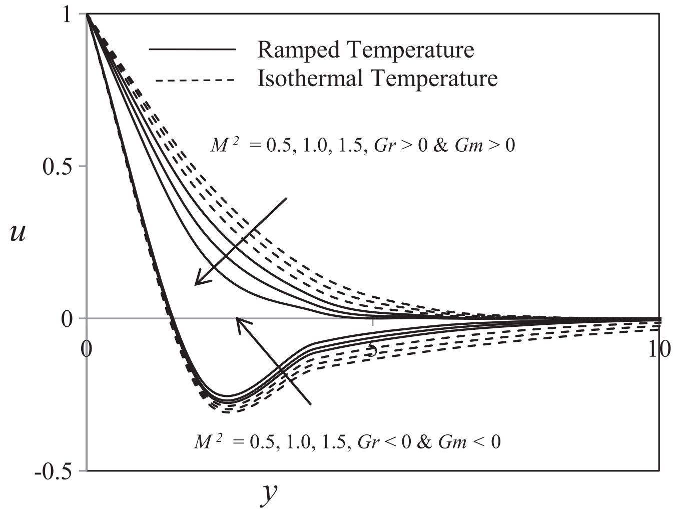

Fig. 3. Magnetic field effect M^2 on primary velocity profiles.

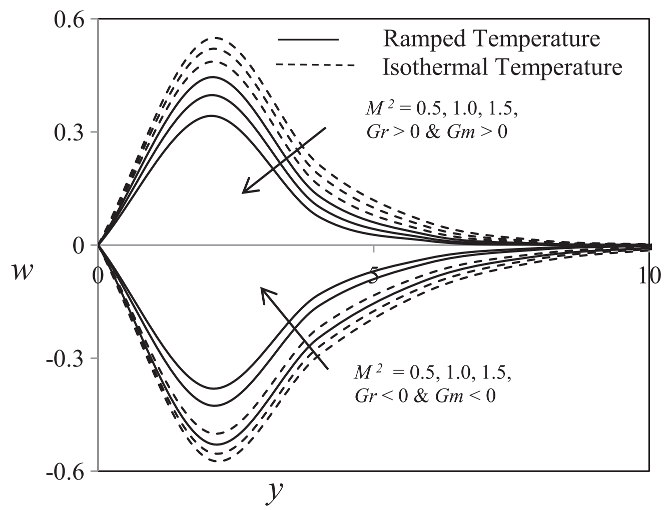

Fig. 4. Magnetic field effect M^2 on secondary velocity profiles.

et al. [7]. Some physical parameters are fixed at real constants with $Gr = 6$, $Gm = 5$, $Gr = -6$, $Gm = -5$, $M^2 = 0.5$, $m = 0.5$, $\Omega = 5$, $N = 5$, $Pr = 0.71$, $Dr = 1$, $Sc = 0.6$ and $Sr = 1$, unless specifically indicated on the appropriate graphs and tables. Figs. 3–12 display the effects of material parameters such as m, $\boldsymbol{\Omega}$, Sr and Dr on the primary and secondary velocity field for both externally cooling ($Gr > 0$) and heating ($Gr < 0$) of the plate. Figs. 13–15 display the effects of material parameters such as N, Pr, and Dr on the temperature profiles and Figs. 16 and 17 display the effects of material parameters such as Sr and Sc on the concentration field.

6.1. Primary and secondary velocity profiles

Figs. 3 and 4 show the effect of Magnetic parameter on the primary and secondary velocity for both ramped temperature and isothermal plate. The primary and secondary velocity decreases with the increase in the magnetic parameter in entire positive quadrant for both ramped and isothermal temperature with externally cooling of the plate while the effect is opposite in case of externally heating of the plate. Velocity profile decreases with increasing Magnetic parameter due to the fact that applied transverse magnetic field produces a drag in the form of Lorentz force thereby decreasing the magnitude of velocity. Figs. 5 and 6 show the effect of Hall current on the primary and secondary velocity for both ramped temperature and isothermal plate. The primary velocity and secondary velocity

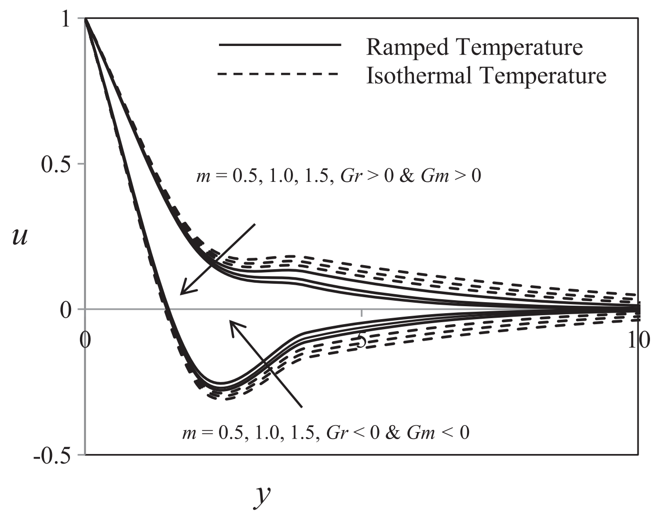

Fig. 5. Hall effect m on primary velocity profiles.

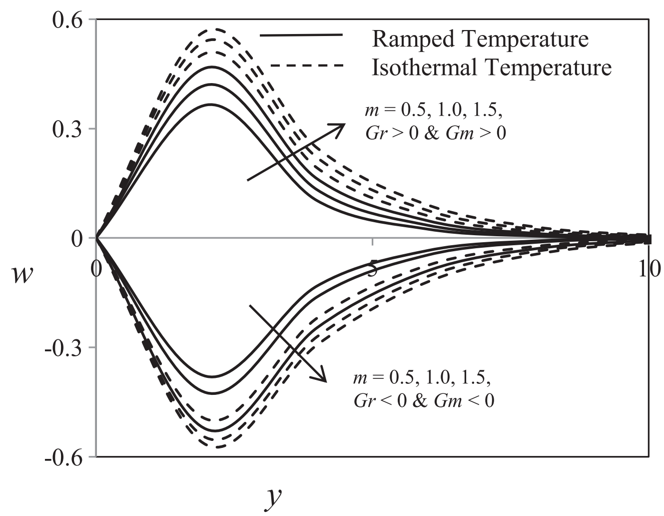

Fig. 6. Hall effect m on secondary velocity profiles.

increases in the entire region with an increase of hall current for ramped temperature and isothermal plates in case of cooling of the plate and the opposite effect in case of externally heating of the plate.

Figs. 7 and 8 show the effect of rotation on the primary and secondary velocity profiles for both ramped temperature and isothermal plate. The primary velocity exponentially decreases in the entire region where as the secondary velocity increases near to the plate and decreases away from the plate with increase of rotational parameter for ramped temperature and isothermal plate in case of cooling of the plate, the opposite effect in case of heating of the plate. Figs. 9 and 10 show the effect of Soret number on primary velocity and secondary velocity for both ramped temperature and isothermal plate. The primary and secondary velocity distribution exponentially increases in the entire region as an increase of Soret number for both ramped temperature and isothermal plate in case of cooling of the plate, and the opposite effect in case of heating of the plate. Figs. 11 and 12 show the effect of Dufour number on primary and secondary velocity distribution for both ramped temperature and isothermal plate. The primary and secondary velocity of the fluid exponentially increases in the entire region with an increase of Dufour number for both ramped temperature and isothermal plate in case of cooling of the plate, and the opposite effect in case of externally heating of the plate.

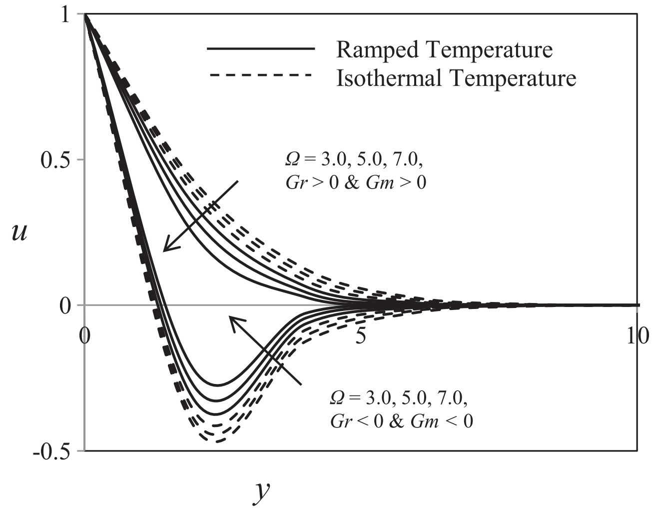

Fig. 7. Rotation effect Ω on primary velocity profiles.

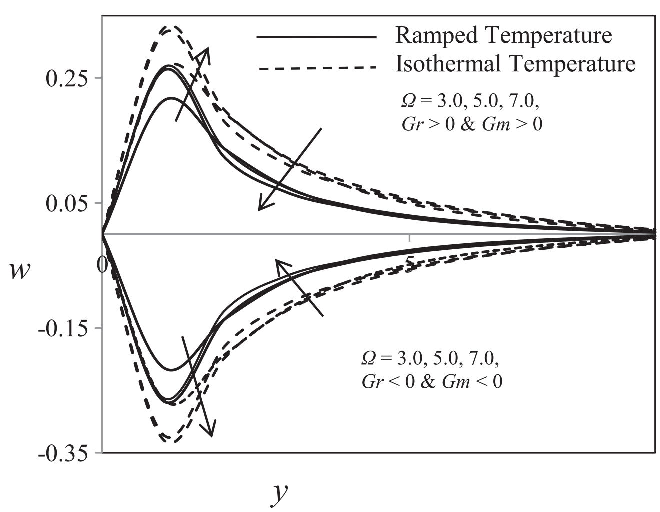

Fig. 8. Rotation effect Ω on secondary velocity profiles.

Tables 3–5 show variation of skin friction coefficient with the various values of Hall current, rotation parameter, Dufour and Soret number. The local skin friction coefficient due to primary velocity decreases with the increase in Hall current, Dufour and Soret number and decreases with increasing of rotation parameter. The local skin friction coefficient due to secondary velocity decreases with increasing of Dufour and Soret number while decreases with increasing of Hall current and rotation parameter for both ramped temperature and isothermal plate in case of cooling of the plate and the opposite effect in case of heating of the plate, it is observed from Tables 3–5.

6.2. Temperature profiles

Figs. 13(a), 13(b) show the effect of thermal radiation on temperature profiles with the absence and presence of Dufour number. The temperature increases in the entire boundary region with an increase of thermal radiation in the absence and presence of Dufour number for ramped temperature and isothermal plate. Figs. 14(a), Fig. 14(b) show the effect of Prandtl number on temperature profiles with the absence and presence of Dufour number. The temperature decreases in the entire boundary region with an increase of Prandtl number with the absence and presence of Dufour number for ramped temperature and isothermal plate. Fig. 15 shows the effect of Dufour number on temperature profiles, the temperature increases in the entire boundary region with an increase of Dufour number for both ramped

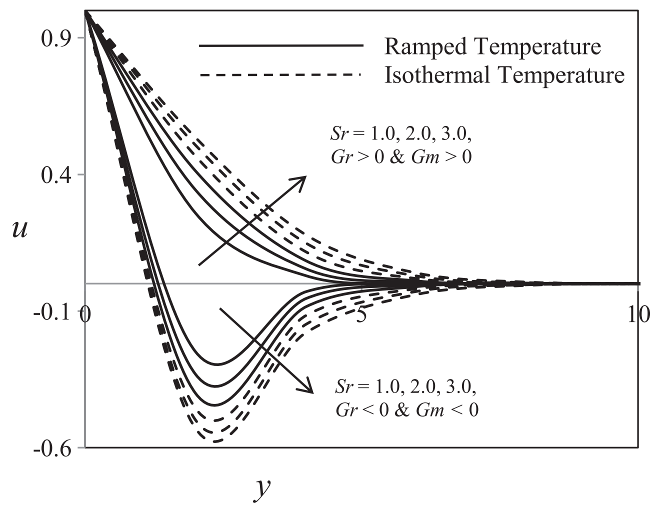

Fig. 9. Soret effect Sr on primary velocity profiles.

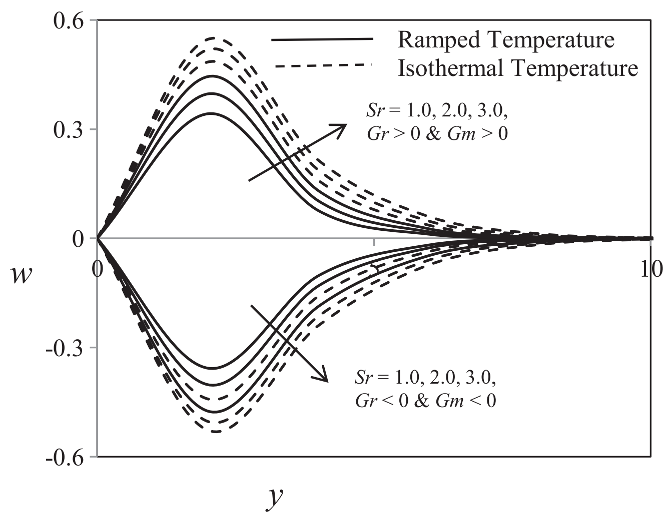

Fig. 10. Soret effect Sr on secondary velocity profiles.

Table 3
The skin-friction due to primary and secondary velocity with the effect of Dufour and Soret number in case of cooling and heating of the plate.

Dr	Sr	τ_x (cooling)		τ_z (cooling)		τ_x (heating)		τ_z (heating)	
		Ramped	Isothermal	Ramped	Isothermal	Ramped	Isothermal	Ramped	Isothermal
1	0	2.91033	2.35922	2.35311	2.63045	5.87581	6.42691	0.84429	0.56695
2	0	2.88051	2.32939	2.38692	2.66426	5.90564	6.45675	0.81048	0.53313
3	0	2.85066	2.29955	2.42074	2.69807	5.93547	6.48658	0.77667	0.49932
0	1	2.91676	2.36565	2.33259	2.62323	5.85768	6.42005	0.86481	0.57417
0	2	2.90506	2.34225	2.34589	2.64983	5.86938	6.44389	0.85151	0.54758
0	3	2.36565	2.31885	2.35918	2.67642	5.88109	6.46733	0.83822	0.52098

temperature and isothermal plate. Table 4 shows the variation of Nusselt number. The rate of heat transfer decreases with increase of thermal radiation and Dufour number and the opposite effect is observed for an increase of Prandtl number.

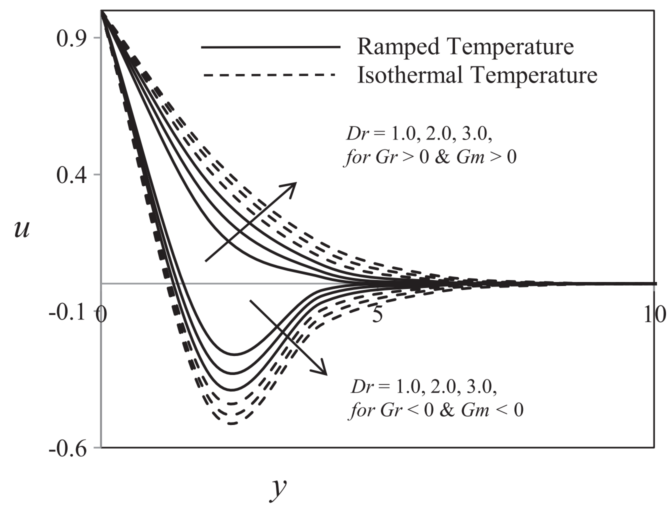

Fig. 11. Dufour effect Dr on primary velocity profiles.

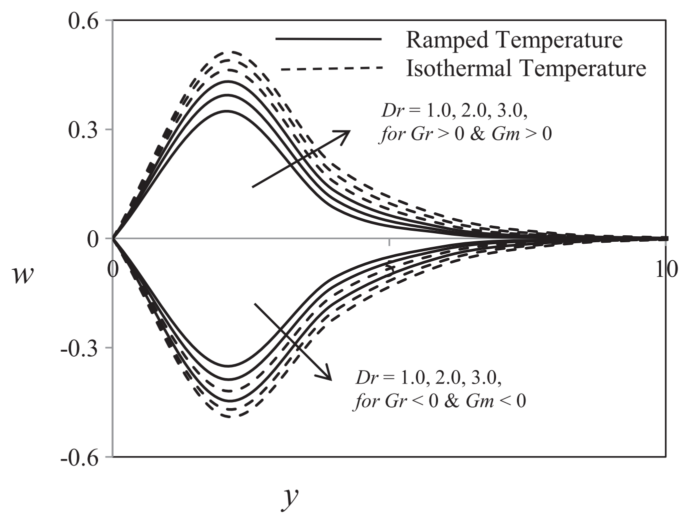

Fig. 12. Dufour effect Dr on secondary velocity profiles.

Table 4
The skin friction due to primary and secondary velocity with the effect of hall current and rotation in case of heating of the plate.

m	Ω	$-\tau_x$ (heating)		τ_z (heating)	
		Ramped	Isothermal	Ramped	Isothermal
0.5	5	−.5.8459	−6.39708	−0.8781	−0.60077
1	5	−5.6173	−6.17274	−0.1574	−0.80637
1.5	5	−5.3605	−5.93414	−1.2705	−0.87509
0.5	3	−5.7324	−6.33896	−1.6526	−0.41375
0.5	7	−6.0063	−6.50614	−1.0776	−0.77461

6.3. Concentration profiles

Fig. 16 shows the effect of Soret number on concentration field, the concentration profile increases in the entire region with an increase of Soret number. Figs. 17(a), 17(b) show the effect of Schmidt number on the concentration

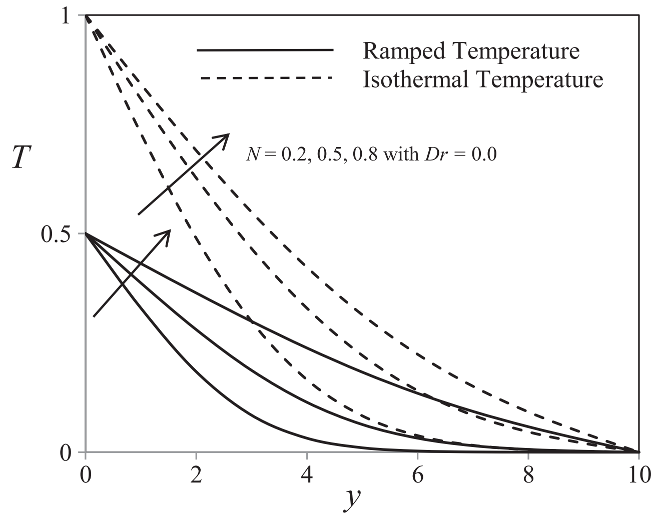

Fig. 13(a). Radiation effect N on temperature profiles with absence of Dufour.

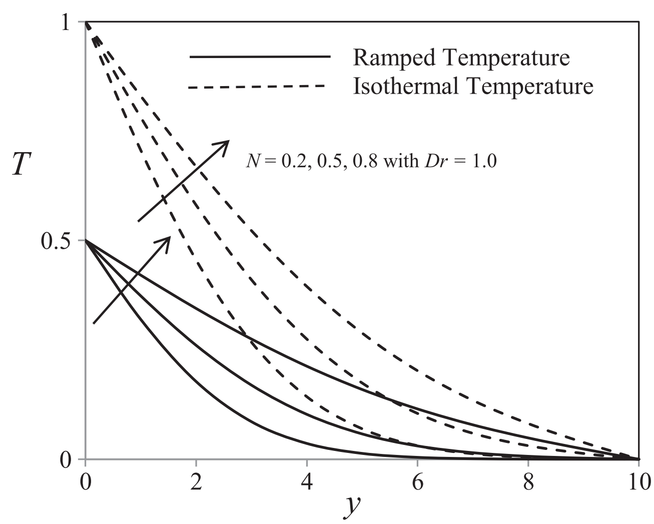

Fig. 13(b). Radiation effect N on temperature profiles with presence of Dufour.

Table 5
Rate of heat transfer near to the plate with the effect of radiation parameter, Prandtl number and Dufour number.

N	Pr	Dr	Nu	
			Ramped	Isothermal
0.2	0.71	0	0.17778	0.35556
0.5	0.71	0	0.16048	0.32096
0.2	0.71	1	0.83946	0.66168
0.5	0.71	1	0.67066	0.51018
5.0	0.71	0	0.12510	0.25021
5	7	0	0.24875	0.49752
5	0.71	1	0.04488	0.16999
5	7	1	0.61451	0.36577
5	0.71	2	0.19577	0.07068
5	0.71	3	0.35621	0.23110

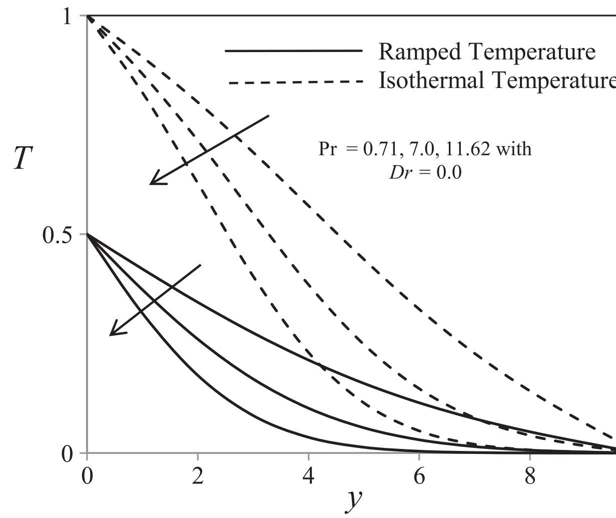

Fig. 14(a). Prandtl number Pr on temperature profiles with absence of Dufour.

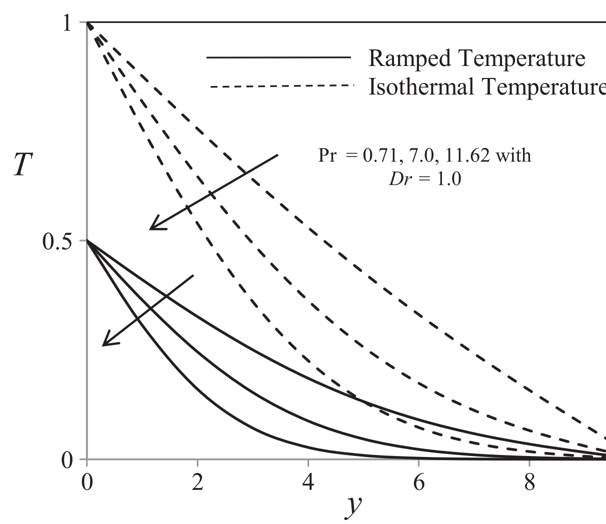

Fig. 14(b). Prandtl number Pr on temperature profiles with presence of Dufour.

profiles with the absence and presence of Soret number. The concentration linearly decreases for small values of Schmidt number and exponentially decreases for large values of Schmidt number in the entire boundary region in the absence of Soret number whereas exponentially decreases with increase of Schmidt number in the presence of Soret number. Table 6 shows that the rate of mass transfer increases with increasing of Schmidt number with the absence and presence of Soret number while decreases with an increase of Soret number.

7. Conclusions

The following conclusions are drawn from the above study, for both ramped temperature and isothermal plate.

1. The primary velocity increases with increasing of Sr and Dr, while decreases with increasing of M^2, m and Ω in case of cooling of the plate and opposite effects in case of heating of the plate.
2. The secondary velocity increases as increasing of m, Sr and Dr while decreases with increasing of M^2 and Ω in case of cooling of the plate and opposite effects in case of heating of the plate.
3. The temperature increases with increasing of N and the opposite effect for Pr while temperature increases with increasing of Dr.

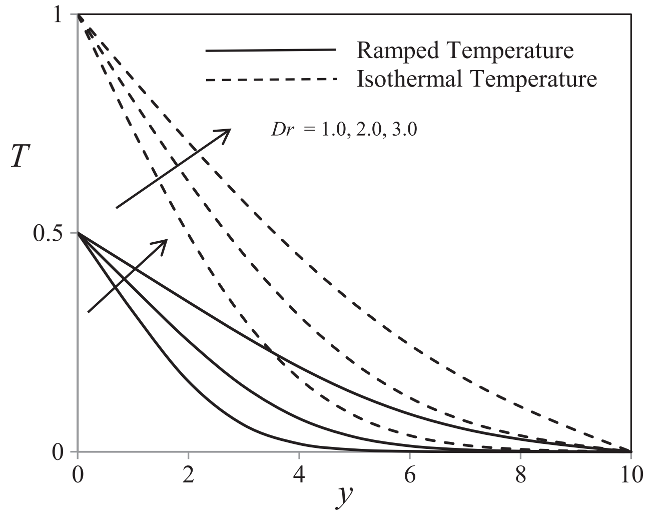

Fig. 15. Dufour effect Dr on temperature profiles.

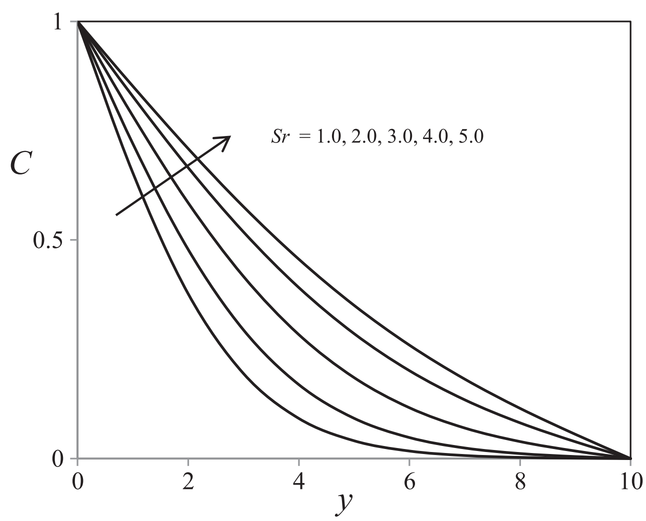

Fig. 16. Soret effect Sr on concentration profiles.

Table 6
Rate of mass transfer near to the plate with the effect of Schmidt number and Soret number.

Sc	Sr	Sh
0.22	0	0.49284
0.60	0	0.79023
0.22	1	0.25744
0.60	1	0.35794
0.22	2	0.20701
0.22	3	0.13156

4. Concentration profile decreases with increasing of Sc while increases with increasing of Sr.
5. Primary skin-friction coefficient increases with an increase of Sr and Dr while decreases with an increase of m and Ω in case of cooling of the plate and opposite effect for heating of the plate.

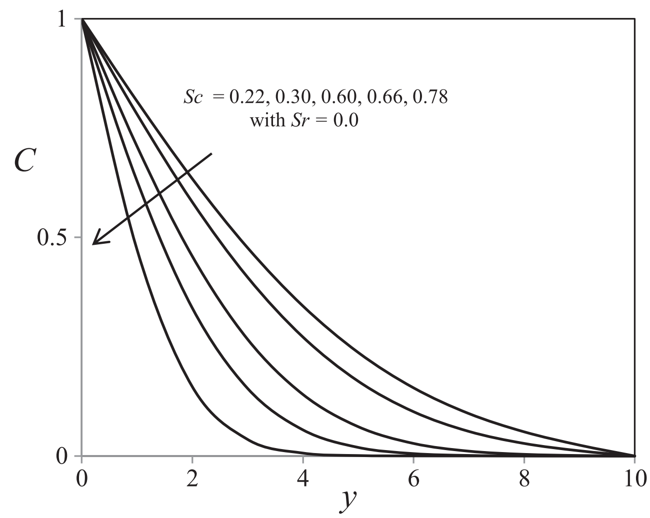

Fig. 17(a). Schmidt number *Sc* on concentration profiles with absence of Soret.

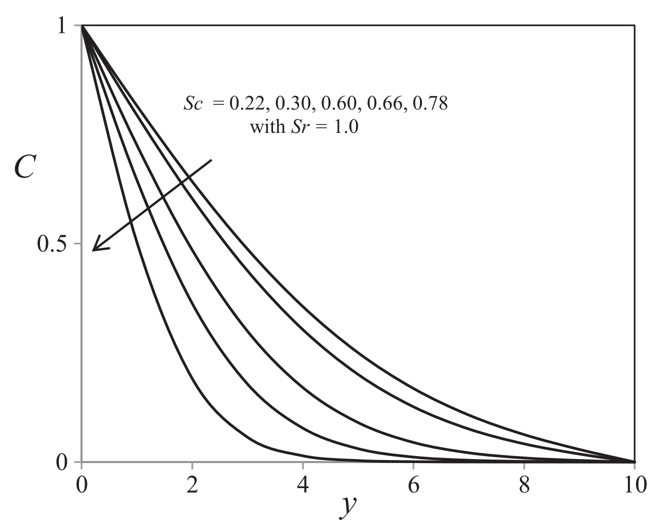

Fig. 17(b). Schmidt number *Sc* on concentration profiles with presence of Soret.

6. Secondary skin-friction coefficient increases with the increase of m, Ω, Dr and Sr in case of cooling of the plate and opposite effect in case of heating of the plate.
7. Heat transfer coefficient increases with increasing of Pr while decreases with increasing of N and Dr.
8. Mass transfer coefficient increases with increasing of Sc while decreases with increasing of Sr.
9. In case of cooling of the plate, the results of primary and secondary velocities and its skin-frictions are in good agreement with the results of Seth et al. [7] with the absence of Soret and Dufour.

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Original article

On the approximate solution of a Kirchhoff type static beam equation

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Abstract

The paper deals with a boundary value problem for the nonlinear integro-differential equation $u'''' - m\left(\int_0^l u'^2\,dx\right)u'' = f(x,u)$, $m(z) \geq \alpha > 0$, $0 \leq z < \infty$, modelling the static state of the Kirchhoff beam. The problem is reduced to an integral equation which is solved using the Picard iteration method. The convergence of the iteration process is established and the error estimate is obtained.

Keywords: Kirchhoff type equation; Green's function; Picard iteration method; Error estimate

1. Statement of the problem

Let us consider the nonlinear beam equation

$$u''''(x) - m\left(\int_0^l u'^2(x)\,dx\right)u''(x) = f(x,u), \quad 0 < x < l, \tag{1}$$

with the conditions

$$u(0) = u(l) = 0, \qquad u''(0) = u''(l) = 0. \tag{2}$$

Here $u = u(x)$ is the displacement function of length l of the beam subjected to the action of a force given by the function $f(x,u)$, the function $m(z)$,

$$m(z) \geq \alpha > 0, \quad 0 \leq z < \infty, \tag{3}$$

describes the type of a relation between stress and strain. Namely, if the function $m(z)$ is linear, this means that this relation is consistent with Hooke's linear law, while otherwise we deal with material nonlinearities.

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Eq. (1) is the stationary problem associated with the equation

$$u_{tt} + u_{xxxx} - m\left(\int_0^l u_x^2\, dx\right)u_{xx} = f(x,t,u),$$
$$m(z) \geq \text{const} > 0,$$

which for the case where $m(z) = m_0 + m_1 z$, $m_0, m_1 > 0$, and $f(x,t,u) = 0$, was proposed by Woinowsky-Krieger [1] as a model of deflection of an extensible dynamic beam with hinged ends. The nonlinear term $\int_0^l u_x^2\, dx$ was for the first time used by Kirchhoff [2] who generalized D'Alembert's classical linear model. Therefore (1) is frequently called a Kirchhoff type equation for a static beam.

The topic of solvability of equations of type (1) is studied in [3–6], while the problem of construction of numerical algorithms and estimation of their accuracy is investigated in [7,4,8–10].

In the present paper, in order to obtain an approximate solution of the problem (1), (2), an approach is used, which differs from those applied in the above-mentioned references. It consists in reducing the problem (1), (2) by means of Green's function to a nonlinear integral equation, to solve for which we use the iteration method. The condition for the convergence of the method is established and its accuracy is estimated.

The Green's function method with a further iteration procedure has been applied by us previously also to a nonlinear problem for the axially symmetric Timoshenko plate [11].

2. Assumptions

Let us assume that besides (3) the function $m(z)$ also satisfies the condition

$$\left|m(z_2) - m(z_1)\right| \leq l_1|z_2 - z_1|, \quad 0 \leq z_1, z_2 < \infty,\ l_1 = \text{const} > 0. \tag{4}$$

As to the function $f(x,u)$, we assume that $f(x,u) \in L_2((0,l);\mathbb{R})$ and, additionally, that the inequalities

$$\begin{aligned} &|f(x,u)| \leq \sigma_1 + \sigma_2|u|, \qquad |f(x,u_2) - f(x,u_1)| \leq l_2|u_2 - u_1|,\\ &0 < x < l, \quad u, u_1, u_2 \in \mathbb{R},\\ &\sigma_i = \text{const}, \quad i = 1,2,\ \sigma_1 > 0,\ \sigma_2 \geq 0,\ l_2 = \text{const} > 0, \end{aligned} \tag{5}$$

are fulfilled.

We impose one more restriction on the beam length l and the parameters α and σ_2 from the conditions (3) and (5) in the form

$$\omega = \alpha + \frac{2}{l^2} - \sigma_2\frac{l^2}{2} > 0. \tag{6}$$

Let us assume that there exists a solution of the problem (1), (2) and $u(x) \in W_0^{2,2}(0,l)$ [12].

3. The method

Applying the Green's function of the problem $u''''(x) - au''(x) = f(x), 0 < x < l, u(0) = u(l) = 0, u''(0) = u''(l) = 0, a = \text{const} > 0$, and performing some transformations, from the problem (1), (2) we come to the nonlinear integral equation

$$u(x) = \int_0^l G(x,\xi) f(\xi, u(\xi))\, d\xi + \frac{1}{\tau}\varphi(x), \tag{7}$$

where

$$G(x,\xi) = \frac{1}{\tau\sqrt{\tau}\,\sinh\left(\sqrt{\tau}\,l\right)} \begin{cases} \sinh\left(\sqrt{\tau}\,(x-l)\right)\sinh\left(\sqrt{\tau}\,\xi\right), & 0 < \xi \leq x < l,\\ \sinh\left(\sqrt{\tau}\,(\xi-l)\right)\sinh\left(\sqrt{\tau}\,x\right), & 0 < x \leq \xi < l, \end{cases}$$

$$\tau = m\left(\int_0^l u'^2(x)\, dx\right),$$

$$\varphi(x)=\frac{1}{l}\left((l-x)\int_0^x \xi f(\xi,u(\xi))\,d\xi+x\int_x^l (l-\xi)f(\xi,u(\xi))\,d\xi\right).$$

Eq. (7) is solved by the method of the Picard iterations. After choosing a function $u_0(x)$, $0\le x\le l$, which together with its second derivative vanishes for $x=0$ and $x=l$, we find subsequent approximations by the formula

$$u_{k+1}(x)=\int_0^l G_k(x,\xi)f(\xi,u_k(\xi))\,d\xi+\frac{1}{\tau_k}\varphi_k(x),\quad 0<x<l,\ k=0,1,\ldots,\tag{8}$$

where

$$G_k(x,\xi)=\frac{1}{\tau_k\sqrt{\tau_k}\,\sinh\left(\sqrt{\tau_k}\,l\right)}\times\begin{cases}\sinh\left(\sqrt{\tau_k}\,(x-l)\right)\sinh\left(\sqrt{\tau_k}\,\xi\right), & 0<\xi\le x<l,\\ \sinh\left(\sqrt{\tau_k}\,(\xi-l)\right)\sinh\left(\sqrt{\tau_k}\,x\right), & 0<x\le\xi<l,\end{cases}$$

$$\tau_k=m\left(\int_0^l u_k'^2(x)\,dx\right),$$

$$\varphi_k(x)=\frac{1}{l}\left((l-x)\int_0^x \xi f(\xi,u_k(\xi))d\xi+x\int_x^l (l-\xi)f(\xi,u_k(\xi))d\xi\right),\quad k=0,1,\ldots,$$

and $u_k(x)$ is the kth approximation of the solution of Eq. (7).

4. The system for the method error

Our aim is to estimate the error of the method, by which we understand the difference between the approximate and the exact solution

$$\delta u_k(x)=u_k(x)-u(x),\quad k=0,1,\ldots.\tag{9}$$

For this, it is advisable to use not formula (8), but the system of equalities

$$u_{k+1}''''(x)-m\left(\int_0^l u_k'^2(x)\,dx\right)u_{k+1}''(x)=f(x,u_k(x)),\quad 0<x<l,\tag{10}$$

$$u_k(0)=u_k(l)=0,\qquad u_k''(0)=u_k''(l)=0,\quad k=0,1,\ldots,\tag{11}$$

which follows from formula (8).

If we subtract the respective equalities in (1) and (2) from (10) and (11), then we get

$$\begin{aligned}\delta u_k''''(x)&-\frac{1}{2}\left(\left(m\left(\int_0^l u_{k-1}'^2(x)\,dx\right)+m\left(\int_0^l u'^2(x)\,dx\right)\right)\delta u_k''(x)\right.\\&\left.+\left(m\left(\int_0^l u_{k-1}'^2(x)\,dx\right)-m\left(\int_0^l u'^2(x)\,dx\right)\right)\left(u_k''(x)+u''(x)\right)\right)\\&=f(x,u_{k-1}(x))-f(x,u(x)),\end{aligned}\tag{12}$$

$$\delta u_k(0)=\delta u_k(l)=0,\qquad \delta u_k''(0)=\delta u_k''(l)=0,\quad k=1,2,\ldots.\tag{13}$$

The system (12) and conditions (13) are the starting point of the estimation of the method error. But preliminarily, we have to derive several a priori estimates. Let us denote the norms in $W_0^{2,2}(0,l)$ as

$$\|u(x)\|_p=\left(\int_0^l\left(\frac{d^pu}{dx^p}(x)\right)^2dx\right)^{\frac{1}{2}},\quad p=0,1,2,\ \|u(x)\|=\|u(x)\|_0.$$

The symbol $(\cdot,\cdot)$ is understood as a scalar product in $L_2(0,l)$.

5. Auxiliary inequalities

Let us derive some estimates.

Lemma 1. *The inequalities*

$$\frac{\sqrt{2}}{l}\|u(x)\| \le \|u(x)\|_1 \le \frac{l}{\sqrt{2}}\|u(x)\|_2 \tag{14}$$

are valid for $u(x) \in W_0^{2,2}(0,l)$.

Proof. Using the equality $u(x) = \int_0^x u'(\xi)\, d\xi$ we obtain

$$|u(x)| \le \left(\int_0^x d\xi\right)^{\frac{1}{2}} \left(\int_0^x u'^2(\xi)\, d\xi\right)^{\frac{1}{2}} \le x^{\frac{1}{2}}\|u(x)\|_1,$$

which implies the left inequality of (14). Applying the latter inequality and taking into account that

$$\|u(x)\|_1^2 = u(x)u'(x)\big|_0^l - \big(u(x), u''(x)\big) = -\big(u(x), u''(x)\big) \le \|u(x)\|\,\|u(x)\|_2,$$

we complete the proof. □

Lemma 2. *The inequality*

$$\|f(x,u(x))\| < \sigma_1 l^{\frac{1}{2}} + \sigma_2 \frac{l}{\sqrt{2}}\|u(x)\|_1 \tag{15}$$

is fulfilled for $u(x) \in W_0^{2,2}(0,l)$.

Proof. By (5) we write

$$\|f(x,u(x))\| \le \sigma_1 \left(\int_0^l dx\right)^{\frac{1}{2}} + \sigma_2\|u(x)\|.$$

Recall also (14). The result is (15). □

Lemma 3. *For the solution of the problem* (1), (2) *we have the inequality*

$$\|u(x)\|_1 \le c_1, \tag{16}$$

where

$$c_1 = \frac{1}{\omega}\sigma_1 l \left(\frac{l}{2}\right)^{\frac{1}{2}}. \tag{17}$$

Proof. We multiply Eq. (1) by $u(x)$ and integrate the resulting equality with respect to x from 0 to l. Using (2), we get

$$\|u(x)\|_2^2 + m(\|u(x)\|_1^2)\|u(x)\|_1^2 = (f(x,u(x)), u(x)).$$

By (14) and (3) we obtain

$$\left(\alpha + \frac{2}{l^2}\right)\|u(x)\|_1^2 \le \frac{l}{\sqrt{2}}\|f(x,u(x))\|\,\|u(x)\|_1.$$

Therefore by (15)

$$\left(\alpha + \frac{2}{l^2} - \frac{1}{2}\sigma_2 l^2\right)\|u(x)\|_1 \le \sigma_1 l \left(\frac{l}{2}\right)^{\frac{1}{2}}.$$

From this relation and (6) follows (16). □

Lemma 4. *Approximations of the iteration method* (8) *satisfy the inequality*

$$\|u_k(x)\|_1 \le c_2, \quad k = 1, 2, \ldots, \tag{18}$$

where

$$c_2 = \begin{cases} c_1, \quad \text{if } \sigma_2 = 0, \\ c_1 + a^{-1} \max(0, \|u_0(x)\|_1 - c_1), \quad a = 1 + \dfrac{2\omega}{\sigma_2 l^2}, \ \text{if } \sigma_2 \neq 0. \end{cases} \tag{19}$$

Proof. Replace k by the index $k-1$ in Eq. (10), multiply the resulting relation by $u_k(x)$ and integrate over x from 0 to l. Taking (11) into account, we get

$$\|u_k(x)\|_2^2 + m\left(\|u_{k-1}(x)\|_1^2\right)\|u_k(x)\|_1^2 = \left(f(x, u_{k-1}(x)), u_k(x)\right), \quad k = 1, 2, \ldots.$$

Applying (3) and (14), we have

$$\left(\alpha + \frac{2}{l^2}\right)\|u_k(x)\|_1^2 \leq \frac{l}{\sqrt{2}} \|f(x, u_{k-1}(x))\| \, \|u_k(x)\|_1,$$

which implies

$$\left(\alpha + \frac{2}{l^2}\right)\|u_k(x)\|_1 \leq \frac{l}{\sqrt{2}} \|f(x, u_{k-1}(x))\|.$$

Hence, using (15), we conclude that

$$\|u_k(x)\|_1 \leq \frac{1}{\alpha + \frac{2}{l^2}} l \left(\frac{l}{2}\right)^{\frac{l}{2}} \left(\sigma_1 + \sigma_2 \left(\frac{l}{2}\right)^{\frac{l}{2}} \|u_{k-1}(x)\|_1\right).$$

Therefore by (17), (6) and (19) we get (18) for the case $\sigma_2 = 0$. In the event $\sigma_2 \neq 0$, again using (17), (6) and (19) we obtain the inequality

$$\|u_k(x)\|_1 \leq c_1(1 - a^{-k}) + a^{-k}\|u_0(x)\|_1 = c_1 + a^{-k}(\|u_0(x)\|_1 - c_1),$$

which implies (18). □

6. Convergence of the method

Multiplying (12) by $\delta u_k(x)$, integrating the resulting equality with respect to x from 0 to l and using (13), we come to the relation

$$\begin{aligned}
&\|\delta u_k(x)\|_2^2 + \frac{1}{2}\left(\left(m\left(\|u_{k-1}(x)\|_1^2\right) + m\left(\|u(x)\|_1^2\right)\right)\|\delta u_k(x)\|_1^2\right. \\
&\qquad \left. + \left(m\left(\|u_{k-1}(x)\|_1^2\right) - m\left(\|u(x)\|_1^2\right)\right)\left(u_k'(x) + u'(x), \delta u_k'(x)\right)\right) \\
&\quad = \left(f(x, u_{k-1}(x)) - f(x, u(x)), \delta u_k(x)\right).
\end{aligned}$$

Using (3)–(5) and (14) we obtain

$$\begin{aligned}
&\|\delta u_k(x)\|_2^2 + \alpha\|\delta u_k(x)\|_1^2 \\
&\quad \leq \frac{1}{2} l_1 \prod_{p=0}^{1} \left|\left(u_{k-p}'(x) + u'(x), \delta u_{k-p}'(x)\right)\right| + l_2\|\delta u_{k-1}(x)\| \, \|\delta u_k(x)\| \\
&\quad \leq \frac{1}{2} l_1 \prod_{p=0}^{1} \left(\|u_{k-p}(x)\|_1 + \|u(x)\|_1\right)\|\delta u_{k-p}(x)\|_1 + \frac{1}{2} l_2 l^2 \prod_{p=0}^{1} \|\delta u_{k-p}(x)\|_1.
\end{aligned}$$

By (16) and (18) we get

$$\|\delta u_k(x)\|_1 \le \frac{1}{2}\left(\alpha+\frac{2}{l^2}\right)^{-1}\left(l_1\prod_{p=0}^{1}\left(\|u_{k-p}(x)\|_1+\|u(x)\|_1\right)+l_2l^2\right)\|\delta u_{k-1}(x)\|_1 \le q\,\|\delta u_{k-1}(x)\|_1,$$

where

$$q=\frac{1}{2}\left(\alpha+\frac{2}{l^2}\right)^{-1}\left(l_1(c_1+c_2)^2+l_2l^2\right).$$

Taking (9), (14), (17) and (19) into consideration, we come to the following result.

Theorem. *Let the assumptions* (3)–(6) *and besides*

$$\|u_0(x)\|_1 \le \frac{1}{\omega}\sigma_1 l\left(\frac{l}{2}\right)^{\frac{1}{2}},$$

$$q=\frac{l^2}{\alpha+\frac{2}{l^2}}\left(ll_1\left(\frac{\sigma_1}{\omega}\right)^2+\frac{l_2}{2}\right)<1$$

be fulfilled.

Then the approximations of the iteration method (8) *converge to the exact solution of the problem* (1), (2) *and for the error the following estimate*

$$\|u_k(x)-u(x)\|_p \le \left(\frac{l}{\sqrt{2}}\right)^{1-p} q^k\|u_0(x)-u(x)\|_1, \quad k=1,2,\ldots,\ p=0,1,$$

is true.

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Original article

Lifts. On tensor structures

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Abstract

The lifts for vector and tensor fields are constructed, some internal tensor structures are considered, the Nijenhuis tensor is found, integrability of these structures is studied, the F-structures are also considered.

Keywords: Vector fibration; Linear connectedness; Lifts; Groups; Tensor structures; Nijenhuis tensor

Let us consider a vector fiber space Lm(Vn) whose local coordinates of the point are transformed by the law [1]

$$\overline{x^i} = \overline{x^i}(x^k); \qquad \overline{y^\alpha} = A^\alpha_\beta(x) y^\beta;$$
$$\det \left\| \frac{\partial x^i}{\partial x^k} \right\| \neq 0; \qquad \det \left\| A^\alpha_\beta \right\| \neq 0; \quad i, j, k = 1, \dots, n; \ \alpha, \beta, \gamma = 1, \dots, m. \tag{1}$$

1. Lifts of the vector and tensor fields

Since the local coordinates (x^i, y^α) of a point of vector fibration Lm(Vn) are transformed by formulas (1), we obtain that the first differential group $\mathrm{Lm}(n, m, R)$ of the vector fibration Lm(Vn) is defined by the matrices of the type

$$\mathcal{L}^A_B = \left\| \frac{\partial \overline{\mathcal{L}}^A}{\partial \overline{\mathcal{L}}^B} \right\| = \left\| \begin{matrix} \dfrac{\partial \overline{x}^i}{\partial x^j} & \dfrac{\partial \overline{x}^i}{\partial x^\alpha} \\ \dfrac{\partial \overline{x}^\alpha}{\partial x^i} & \dfrac{\partial \overline{x}^\alpha}{\partial x^\beta} \end{matrix} \right\| = \left\| \begin{matrix} x^i_j & 0 \\ A^\alpha_{\beta k} y^\beta & A^\alpha_\beta \end{matrix} \right\|.$$

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The matrix, inverse to the above one, is of the form

$$\overset{*}{\mathcal{L}} = \left\| \frac{\partial \mathcal{L}^A}{\partial \overline{\mathcal{L}}^B} \right\| = \left\| \begin{matrix} \dfrac{\partial x^i}{\partial \overline{x}^k} & \dfrac{\partial x^i}{\partial \overline{x}^\alpha} \\ \dfrac{\partial x^\alpha}{\partial \overline{x}^i} & \dfrac{\partial x^\alpha}{\partial \overline{x}^\beta} \end{matrix} \right\| = \left\| \begin{matrix} \overset{*}{x}{}^i_j & 0 \\ \overset{*}{A}{}^\alpha_{\beta k} A^\beta_\gamma y^\gamma & \overset{*}{A}{}^\alpha_\beta \end{matrix} \right\|.$$

Obviously, the first differential group $\mathrm{GL}(n, m, R)$ has always two subgroups $\mathrm{GL}(n, R)$ and $\mathrm{GL}(m, R)$. This implies that with the vector fiber space Lm(Vn) are always connected to five sets of fields of differential geometric objects:

(1) A set of differential geometric objects $\mathfrak{J}$(Vn) (with respect to the group $\mathrm{GL}(n, R)$), whose components are the functions of the point of the base Vn.

(2) A set of differential geometric objects $\mathfrak{J}$(Lm(Vn)) (with respect to the group $\mathrm{GL}(n, R)$), whose components are the functions of the point of vector fibration $\mathrm{Lm}(Vn)$.

(3) A set of differential geometric objects $\overset{*}{\mathfrak{J}}$(Vn) (with respect to the group $\mathrm{GL}(m, R)$), whose components are the functions of the point of the base Vn.

(4) A set of differential geometric objects $\overset{*}{\mathfrak{J}}$(Lm(Vn)) (with respect to the group $\mathrm{GL}(m, R)$), whose components are the functions of the point of vector fibration Lm(Vn).

(5) A set of differential geometric objects $\check{\mathfrak{J}}$ Lm(Vn) (with respect to the group $\mathrm{GL}(n, m, R)$), whose components are the functions of the point of vector fibration Lm(Vn). In the capacity of subsets, these sets have differential geometric objects of the first order. Moreover, among the above-mentioned differential geometric objects we distinguish five graded algebras

$$\mathfrak{J}(\mathrm{Vn}) = \sum_{p,q=0}^{\infty} \mathfrak{J}^p_q(\mathrm{Vn}), \qquad \mathfrak{J}(\mathrm{Lm(Vn)}) = \sum_{p,q=0}^{\infty} \mathfrak{J}^p_q(\mathrm{Lm(Vn)}), \qquad \overset{*}{\mathfrak{J}}(\mathrm{Vn}) = \sum_{p,q=0}^{\infty} \overset{*}{\mathfrak{J}}{}^p_q(\mathrm{Vn}),$$

$$\overset{*}{\mathfrak{J}}(\mathrm{Lm(Vn)}) = \sum_{p,q=0}^{\infty} \overset{*}{\mathfrak{J}}{}^p_q(\mathrm{Lm(Vn)}), \qquad \check{\mathfrak{J}}(\mathrm{Lm(Vn)}) = \sum_{p,q=0}^{\infty} \check{\mathfrak{J}}{}^p_q(\mathrm{Lm(Vn)}).$$

Note that for the tangent vector fibrations, certain graded algebras coincide.

If $T \in \mathfrak{J}^1_1(\mathrm{Lm(Vn)})$, then the law of transformation of that tensor components has the form

$$\mathcal{L}^D_B \, \overline{T^A_D} = \mathcal{L}^A_C \, T^C_B, \quad A, B, C = 1, 2, \dots, n+m. \tag{2}$$

To the tensor T there corresponds the matrix

$$\| T^A_B \| = \left\| \begin{matrix} T^i_j & T^i_\alpha \\ T^\alpha_i & T^\alpha_\beta \end{matrix} \right\|.$$

Obviously, by virtue of (1), we can write formulas (2) as follows:

$$x^p_j \, \overline{T^i_p} = x^i_p \, T^p_j - A^\alpha_{\beta j} \, y^\beta \, \overline{T^i_\alpha}, \qquad A^\beta_\alpha \, \overline{T^i_\beta} = x^i_p \, T^p_\alpha,$$
$$A^\gamma_\beta \, \overline{T^\alpha_\gamma} = A^\alpha_{\gamma p} \, y^\gamma \, T^p_\beta + A^\alpha_\gamma \, T^\gamma_\beta, \qquad x^k_i \, \overline{T^\alpha_k} + A^\beta_{\gamma i} \, y^\gamma \, \overline{T^\alpha_\beta} = A^\alpha_{\gamma k} \, y^\gamma \, T^k_i + A^\alpha_\beta \, T^\beta_i.$$

Thus we can see that the tensor T^A_B has a number of subobjects among which there are the tensor T^i_α (as an element of the algebra $\mathfrak{J}^1_1(\mathrm{Lm}(Vn))$) and the linear homogeneous subobjects (T^i_j, T^i_α), $(T^i_\alpha, T^\alpha_\beta)$. A set of all tensors $T \in \mathfrak{J}^1_1(\mathrm{Lm(Vn)})$, for which the values T^i_α are equal to zero, form a new subalgebra of the algebra $\mathfrak{J}^1_1(\mathrm{Lm(Vn)})$ which we call a triangular subalgebra and denote it by $\overset{*}{\mathfrak{J}}{}^1_1(\mathrm{Lm(Vn)})$. Then to the tensors $T \in \overset{*}{\mathfrak{J}}{}^1_1(\mathrm{Lm(Vn)})$ there correspond the matrices of the form

$$\| T^A_B \| = \left\| \begin{matrix} T^i_j & 0 \\ T^\alpha_i & T^\alpha_\beta \end{matrix} \right\|,$$

and the values T^i_j and T^α_β form tensors.

Analogously, if $T \in \mathfrak{J}_2^1(\mathrm{Lm(Vn)})$, we have

$$\overline{T^A_{BC}} = \overset{*}{\mathcal{L}}{}^D_B \, \overset{*}{\mathcal{L}}{}^E_C \, \mathcal{L}^A_P \, T^P_{DE}.$$

Hence, owing to (1), we obtain

$$T^i_{jk} = \overset{*}{x}{}^q_j \, \overset{*}{x}{}^p_k \, x^i_l \, T^l_{qp} + x^q_j \, x^i_p \, \overset{*}{A}{}^\alpha_{\gamma k} \, A^\gamma_\delta \, y^\delta \, T^p_{q\alpha} + \overset{*}{x}{}^p_k \, x^i_q \, \overset{*}{A}{}^\alpha_{\gamma j} \, A^\gamma_\delta \, y^\delta \, T^q_{\alpha p} + x^i_p \, \overset{*}{A}{}^\alpha_{\gamma j} \, A^\gamma_\delta \, y^\delta \, \overset{*}{A}{}^\beta_{\sigma k} \, A^\sigma_\varepsilon \, y^\varepsilon \, T^p_{\alpha\beta},$$

$$T^i_{j\alpha} = \overset{*}{x}{}^k_j \, \overset{*}{A}{}^\beta_\alpha \, x^i_p \, T^p_{k\beta} + x^i_p \, \overset{*}{A}{}^\gamma_\alpha \, \overset{*}{A}{}^\beta_{\gamma j} \, A^\sigma_\delta \, y^\delta \, T^p_{\beta\sigma}, \qquad T^i_{\alpha\beta} = \overset{*}{A}{}^\gamma_\alpha \, \overset{*}{A}{}^\delta_\beta \, x^i_k \, T^k_{\gamma\delta},$$

$$T^i_{\alpha j} = \overset{*}{A}{}^\beta_\alpha \, \overset{*}{x}{}^p_j \, x^i_k \, T^k_{\beta p} + \overset{*}{A}{}^\beta_\alpha \, x^i_k \, \overset{*}{A}{}^\gamma_{\delta j} \, A^\delta_\sigma \, y^\sigma \, T^k_{\beta\gamma}, \qquad T^\alpha_{\beta\gamma} = \overset{*}{A}{}^\delta_\beta \, \overset{*}{A}{}^\sigma_\gamma \, A^\alpha_\varepsilon \, T^\varepsilon_{\delta\sigma} + \overset{*}{A}{}^\delta_\beta \, \overset{*}{A}{}^\sigma_\gamma \, A^\alpha_{\varepsilon i} \, y^\varepsilon \, T^i_{\delta\sigma},$$

$$T^\alpha_{\beta i} = \overset{*}{A}{}^\gamma_\beta \, \overset{*}{x}{}^p_i \, A^\alpha_{\sigma k} \, y^\sigma \, T^k_{\gamma p} + \overset{*}{A}{}^\gamma_\beta \, \overset{*}{x}{}^p_i \, A^\alpha_\varepsilon \, T^\varepsilon_{\gamma p} + \overset{*}{A}{}^\gamma_\beta \, \overset{*}{A}{}^\delta_{pi} \, y^\sigma \, A^p_\sigma \, A^\alpha_\varepsilon \, T^\varepsilon_{\gamma\delta} + \overset{*}{A}{}^\gamma_\beta \, \overset{*}{A}{}^\delta_{\sigma i} \, A^\sigma_p \, y^p \, A^\alpha_{\varepsilon k} \, y^\varepsilon \, T^k_{\gamma\delta},$$

$$T^\alpha_{i\beta} = \overset{*}{x}{}^k_i \, \overset{*}{A}{}^\gamma_\beta \, A^\alpha_{\sigma p} \, y^\sigma \, T^p_{k\gamma} + \overset{*}{x}{}^k_i \, \overset{*}{A}{}^\gamma_\beta \, A^\alpha_\delta \, T^\delta_{k\gamma} + \overset{*}{A}{}^\gamma_\beta \, \overset{*}{A}{}^\delta_{\sigma i} \, A^\sigma_\varepsilon \, y^\varepsilon \, A^\alpha_{pp} \, T^p_{\delta\gamma} + \overset{*}{A}{}^\gamma_\beta \, A^\alpha_\sigma \, \overset{*}{A}{}^\delta_{\varepsilon i} \, A^\varepsilon_p \, y^p \, T^\sigma_{\delta\gamma},$$

$$\begin{aligned} T^\alpha_{ij} = {} & \overset{*}{x}{}^k_i \, \overset{*}{x}{}^p_j \, A^\alpha_{\beta q} \, y^\beta \, T^q_{kp} + \overset{*}{x}{}^k_i \, \overset{*}{x}{}^p_j \, A^\alpha_\beta \, T^\beta_{kp} + \overset{*}{x}{}^k_i \, \overset{*}{A}{}^\beta_{\gamma j} \, A^\gamma_\delta \, y^\delta \, A^\alpha_{\varepsilon p} \, y^\varepsilon \, T^p_{k\beta} + \overset{*}{x}{}^k_i \, A^\alpha_\gamma \, \overset{*}{A}{}^\beta_{\delta j} \, A^\delta_\sigma \, y^\sigma \, T^\gamma_{k\beta} \\ & + \overset{*}{x}{}^p_j \, \overset{*}{A}{}^\beta_{\gamma i} \, A^\gamma_\delta \, y^\delta \, A^\alpha_{\varepsilon q} \, y^\varepsilon \, T^q_{\beta p} + \overset{*}{x}{}^p_j \, A^\alpha_\gamma \, \overset{*}{A}{}^\beta_{\delta i} \, A^\delta_\sigma \, y^\sigma \, T^\gamma_{\beta p} + \overset{*}{A}{}^\beta_{\delta i} \, A^\delta_\sigma \, y^\sigma \, \overset{*}{A}{}^\gamma_{\varepsilon j} \, A^\varepsilon_p \, y^p \, A^\alpha_{\omega k} \, y^\omega \, T^k_{\beta\gamma} \\ & + A^\alpha_\delta \, \overset{*}{A}{}^\beta_{pi} \, A^p_\sigma \, y^\sigma \, \overset{*}{A}{}^\gamma_{\varepsilon j} \, A^\varepsilon_\omega \, y^\omega \, T^\delta_{\beta\gamma}. \end{aligned}$$

It is not difficult to see that the tensor T^P_{DE} has a number of linear and homogeneous subobjects among which there are the tensor $T^i_{\alpha\beta}$ and the following linear homogeneous subobjects

$$(T^i_{j\alpha}, T^i_{\alpha\beta}), (T^i_{\alpha j}, T^i_{\alpha\beta}), (T^i_{\alpha\beta}, T^\alpha_{\beta\gamma}), (T^i_{jk}, T^i_{j\alpha}, T^i_{\alpha j}, T^i_{\alpha\beta}), (T^i_{\alpha k}, T^i_{\alpha\beta}, T^\alpha_{\beta i}, T^\alpha_{\beta\gamma}), (T^i_{k\alpha}, T^i_{\alpha\beta}, T^\alpha_{k\beta}, T^\alpha_{\beta\gamma}).$$

If components of the tensor $T^i_{\alpha\beta}$ are equal to zero, then the values, respectively, $T^i_{j\alpha}, T^i_{\alpha j}, T^\alpha_{\beta\gamma}$ form tensors (as elements of the algebra $\mathfrak{J}_2^1(\mathrm{Lm(Vn)})$).

If the $\{e_i, e_\alpha\}$-frame of the tangent space T_{n+m} at the point $z = (x, y) \in \mathrm{Lm(Vn)}$, then the vectors $E_i = e_i - \Gamma^\alpha_i e_\alpha$ determine an invariant equipment of the tangent space. The first differential group $\mathrm{GL}(n, m, R)$ has always two differential subgroups $\mathrm{GL}(n, R)$ and $\mathrm{GL}(m, R)$. This implies that on the fiber space Lm(Vn) there exist tensor algebras with respect to the tensor products of the groups $\mathrm{GL}(n, R)$, $\mathrm{GL}(m, R)$, $\mathrm{GL}(n, m, R)$. If on Lm(Vn) is assigned an object of linear connectedness, then every $\mathrm{GL}(n, m, R)$-vector field is uniquely expanded into two vector fields with respect to the groups $\mathrm{GL}(n, R)$ and $\mathrm{GL}(m, R)$. Obviously, to an arbitrary field defined on the base Vn of the space Lm(Vn) there always corresponds the $\mathrm{GL}(n, m, R)$-vector field defined on the whole fiber space. Analogous correspondence takes also place between tensors of another valencies.

Let ξ^A be the $\mathrm{GL}(n, m, R)$-vector field defined on Lm(Vn), i.e.,

$$d\xi^A + \xi^B \, \omega^A_B = \xi^A_k \, \omega^k + \xi^A_\alpha \, \widetilde{\theta}^\alpha, \quad A, B, C = 1, 2, \ldots, n + m.$$

Then

$$\xi = \xi^A \, e_A = \xi^i \, e_i + \xi^\alpha \, e_\alpha = \xi^i \, E_i + (\xi^\alpha + \Gamma^\alpha_k \, \xi^k) e_\alpha.$$

Definition. The vector field ξ^i is called a horizontal projection of the $\mathrm{GL}(n, m, R)$-vector field, and the vector field $\xi^\alpha + \Gamma^\alpha_k \, \xi^k$ is called a vertical projection of the same $\mathrm{GL}(n, m, R)$-vector field.

Upon expansion of the vector field $\mathrm{GL}(n, m, R)$, we have $\xi = \xi_1 \oplus \xi_2$. These vectors in the $\{E_i, e_\alpha\}$-frame have the following coordinates:

$$\xi_1 = \xi^i_1 \, E_i, \qquad \xi_2 = \xi^\alpha_2 \, e_\alpha,$$

that is,

$$\xi_1^i = \xi^i, \qquad \xi_1^\alpha = 0, \qquad \xi_2^i = 0, \qquad \xi_2^\alpha = \xi^\alpha + \Gamma_k^\alpha \xi^k,$$

and in the $\{e_i, e_\alpha\}$-frame, the coordinates

$$\xi_1^i = \xi^i, \qquad \xi_1^\alpha = -\Gamma_k^\alpha \xi^k, \qquad \xi_2^i = 0, \qquad \xi_2^\alpha = \xi^\alpha + \Gamma_k^\alpha \xi^k,$$

that is,

$$\xi_1 = \xi_1^i e_i - \xi_1^k \Gamma_k^\alpha e_\alpha, \qquad \xi_2 = \xi_2^\alpha e_\alpha.$$

If in the base of the space Lm(Vn) the GL(n, R)-vector field η^i is defined, then to that field there always uniquely corresponds the GL(n, m, R)-vector field defined on Lm(Vn). Such a correspondence is assigned as follows:

$$\xi^i = \eta^i, \qquad \xi^\alpha = -\eta^k \Gamma_k^\alpha. \tag{3}$$

Definition. The vector field ξ^A defined by equalities (3) is called a Γ-lift of the vector field η^i.

If T_B^A is the GL$(n, m, R) \times$ GL(n, m, R)-tensor field, then

$$T(\xi) = T_B^A \xi^B e_A \tag{4}$$

is an element of the space T_{n+m}. Since

$$T(\xi) = T_1(\xi) \oplus T_2(\xi),$$

therefore

$$T(\xi) = T_1(\xi_1) \oplus T_1(\xi_2) \oplus T_2(\xi_1) \oplus T_2(\xi_2). \tag{5}$$

Let

$$T_1(\xi_1) = a_j^i \xi_1^j E_i, \qquad T_1(\xi_2) = b_\beta^i E_i, \qquad T_2(\xi_1) = c_j^\alpha \xi_1^j e_\alpha, \qquad T_2(\xi_2) = d_\beta^\alpha \xi_2^\beta e_\alpha,$$

or

$$T_1(\xi_1) = a_j^i \xi^j e_i - a_j^i \xi^j \Gamma_i^\alpha e_\alpha, \qquad T_1(\xi_2) = b_\beta^i(\xi^\beta + \xi^k \Gamma_k^\beta) e_i - b_\beta^i(\xi^\beta + \xi^k \Gamma_k^\beta)\Gamma_i^\alpha e_\alpha,$$
$$T_2(\xi_1) = c_j^\alpha \xi^j e_\alpha, \qquad T_2(\xi_2) = d_\beta^\alpha(\xi^\beta + \xi^k \Gamma_k^\beta) e_\alpha.$$

Since these expansions take place for any vector field ξ^A, therefore, owing to equalities (4) and (5), we find that

$$T_j^i = a_j^i + b_\alpha^i \Gamma_j^\alpha, \qquad T_\alpha^i = b_\alpha^i, \qquad T_\beta^\alpha = d_\beta^\alpha - b_\beta^i \Gamma_i^\alpha, \qquad T_j^\alpha = c_j^\alpha - a_j^i \Gamma_i^\alpha - b_\beta^i \Gamma_j^\beta \Gamma_i^\alpha + d_\beta^\alpha \Gamma_j^\beta. \tag{6}$$

Obviously, the values a_j^i, b_α^i, c_j^α, d_β^α appearing in the above formulas, are the tensors. Thus, formulas (6) can be interpreted as a fully definite correspondence definable by the object of linear connectedness Γ_i^α.

Definition. The GL(n, m, R)-tensor field T_B^A defined by equality (6) is called a Γ-lift of an ordered quadruple of GL(n, R), GL(m, R), GL(n, m, R)-tensor fields a_j^i, b_α^i, c_j^α, d_β^α, defined on Lm(Vn).

2. Internal tensor structures

Definition. The space Lm(Vn) on which is defined the tensor field T_B^A satisfying the conditions

$$T_C^A T_B^C = \lambda \delta_B^A, \tag{7}$$

will be called a fiber space with a tensor structure.

If $\lambda = 0$, then the tensor structure will be called an almost dual structure, if $\lambda = -1$, it will be called an almost complex structure, and if $\lambda = 1$, it will be an almost binary structure [2].

We will focus our attention not on arbitrary tensor structures, but only on those which are generated by an object of linear connectedness and by certain vector fields. Such tensor structures will be called in the sequel internal tensor structures.

Differential equations of the tensor T^A_B have the form

$$\nabla T^A_B = \nabla_C T^A_B \omega^C \equiv \nabla_i T^A_B \omega^i + \nabla_\alpha T^A_B \widetilde{\theta^\alpha}. \tag{8}$$

Let

$$a^i_j = a\,\delta^i_j, \qquad b^i_\alpha = b\,\xi^i\,\eta_\alpha, \qquad c^\alpha_j = c\,\xi^\alpha\,\eta_j, \qquad d^\alpha_\beta = d\,\delta^\alpha_\beta,$$

where a, b, c, d are arbitrary scalars, η_α, η_j are the GL(m, R), GL(n, R)-covector fields, and ξ^i, ξ^α are the GL(n, R), GL(m, R)-vector fields. It is assumed that

$$\xi^i\,\eta_i = 1, \qquad \xi^\alpha\,\eta_\alpha = 1.$$

The lift of that quadruple of tensor fields has the form

$$\begin{aligned} &T^i_j = a\,\delta^i_j + b\,\xi^i\,\eta_\alpha\,\Gamma^\alpha_j, \quad T^i_\alpha = b\,\xi^i\,\eta_\alpha\,, \quad T^\alpha_\beta = d\,\delta^\alpha_\beta - b\,\xi^i\,\eta_\beta\,\Gamma^\alpha_i\,,\\ &T^\alpha_j = c\,\xi^\alpha\,\eta_j - a\,\Gamma^\alpha_j - b\,\xi^k\,\eta_\beta\,\Gamma^\beta_j\,\Gamma^\alpha_k + d\,\Gamma^\alpha_j\,. \end{aligned} \tag{9}$$

Written explicitly, the system of square equations (7) has the form

$$T^i_k\,T^k_j + T^i_\alpha\,T^\alpha_j = \lambda\,\delta^i_j, \qquad T^i_k\,T^k_\alpha + T^i_\beta\,T^\beta_\alpha = 0, \qquad T^\alpha_k\,T^k_j + T^\alpha_\beta\,T^\beta_j = 0, \qquad T^\alpha_k\,T^k_\beta + T^\alpha_\gamma\,T^\gamma_\beta = \lambda\,\delta^\alpha_\beta$$

whence it follows by virtue of (9) that the values are connected by the following relations:

$$\begin{aligned} &(a^2-\lambda)\delta^i_j + bc\,\xi^i\eta_j + b(a+d)\xi^i\,\eta_\gamma\,\Gamma^\gamma_j = 0, \qquad b(a+d)\xi^i\,\eta_\alpha = 0,\\ &(d^2-\lambda)\delta^\alpha_\beta + bc\,\xi^\alpha\,\eta_\beta - b(a+d)\xi^i\,\eta_\beta\Gamma^\alpha_i = 0,\\ &(d^2-a^2)\Gamma^\alpha_j + c(a+d)\xi^\alpha\,\eta_j - b(a+d)\xi^i\,\eta_\beta\,\Gamma^\alpha_i\,\Gamma^\beta_j + bc\,\xi^\alpha\,\eta_\gamma\,\Gamma^\gamma_j - bc\,\xi^k\,\eta_j\,\Gamma^\alpha_k = 0, \end{aligned}$$

that is,

$$\begin{aligned} &a^2 + bc - \lambda = 0, \qquad d^2 + bc - \lambda = 0, \qquad b(a+d) - 0,\\ &(d^2-a^2)\Gamma^\alpha_j + c(a+d)\xi^\alpha\,\eta_j + bc\,\xi^\alpha\,\eta_\gamma\,\Gamma^\gamma_j - bc\,\xi^k\,\eta_j\,\Gamma^\alpha_k = 0. \end{aligned}$$

We will seek only for those solutions which depend on a maximal number of parameters. If $b = 0$, $d + a = 0$, we obtain

$$d = -a, \qquad a^2 = \lambda.$$

In the other case, if $b \neq 0$, we obtain

$$d + a = 0, \qquad c\left(\xi^\alpha\,\eta_\gamma\,\Gamma^\gamma_j - \xi^k\,\eta_j\,\Gamma^\alpha_k\right) = 0.$$

This implies that $d = -a$, $c = 0$, $a^2 = \lambda$.

We have proved that there exist two two-parametric families of internal tensor structures (a, b, c are arbitrary parameters):

$$\left\| \begin{matrix} a\,\delta^i_j & 0\\ c\,\xi^\alpha\,\eta_j - 2a\,\Gamma^\alpha_j & -a\,\delta^\alpha_\beta \end{matrix} \right\|, \tag{10}$$

$$\left\| \begin{matrix} a\,\delta^i_j + b\,\xi^i\,\eta_\alpha\,\Gamma^\alpha_j & b\,\xi^i\,\eta_\alpha\\ -2a\,\Gamma^\alpha_j - b\,\xi^k\,\eta_\beta\,\Gamma^\beta_j\,\Gamma^\alpha_k & -a\,\delta^\alpha_\beta - b\,\xi^i\,\eta_\beta\,\Gamma^\alpha_i \end{matrix} \right\|. \tag{11}$$

It should be, however, noted that each of families consists of different, in the main, tensor structures, since the first family of tensor structures consists of elements of the triangular algebra $\check{\mathfrak{J}}\,$Lm(Vn), and the second one consists of elements of the algebra $\mathfrak{J}$(Lm(Vn)).

Thus, we have proved the following theorems.

Theorem 1. *If in the base* Vn *of space* Lm(Vn) *with linear connectedness* $\Gamma_i^\alpha(x, y)$ *there are the* GL(m, R)*-vector field* $\xi^\alpha(x)$ *and the* GL(n, R)*-covector field* $\eta_i(x)$*, then the tangent bundle of space* Lm(Vn) *has two one-parametric bundles of tensor structures which have no almost complex structures, but have only dual tensor structures and also structures of almost product.*

Theorem 2. *If in the space* Lm(Vn) *with linear connectedness* $\Gamma_i^\alpha(x, y)$ *there are the* GL(m, R)*-vector field* $\xi^\alpha(x, y)$ *and the* GL(n, R)*-covector field* $\eta_i(x, y)$*, then the tangent bundle of space* Lm(Vn) *has two one-parametric bundles of tensor structures which have no almost complex structures, but have only dual tensor structures and also structures of almost product.*

Theorem 3. *If in the base* Vn *of space* Lm(Vn) *with linear connectedness* $\Gamma_i^\alpha(x, y)$ *there are the* GL(n, R)*-vector field* $\xi^i(x)$ *and the* GL(m, R)*-covector field* $\eta_\alpha(x)$*, then the tangent bundle of space* Lm(Vn) *has two one-parametric bundles of tensor structures which have no almost complex structures, but have only dual tensor structures and also structures of almost product.*

Theorem 4. *If in the space* Lm(Vn) *with linear connectedness* $\Gamma_i^\alpha(x, y)$ *there are the* GL(n, R)*-vector field* $\xi^i(x, y)$ *and the* GL(m, R)*-covector field* $\eta_\alpha(x, y)$*, then the tangent bundle of space* Lm(Vn) *has two one-parametric bundles of tensor structures which have no almost complex structures, but have only dual tensor structures and also structures of almost product.*

3. The Nijenhuis tensor

If the tensor field T_B^A is defined by equations $\nabla T_B^A = \nabla_C\, T_B^A\, \omega^C$, then continuing these equations, we obtain

$$\nabla\left(\nabla_C\, T_B^A\right) - T_D^A\, \omega_{CB}^D + T_B^D\, \omega_{DC}^A = \nabla_D\, \nabla_C\, T_B^A\, \omega^D, \tag{12}$$

where

$$\nabla_{[D}\, \nabla_{C]}\, T_B^A = 0.$$

Rolling up Eq. (12) with T_E^C, we get

$$\nabla\left(T_E^C\, \nabla_C\, T_B^A\right) - T_E^C\, T_D^A\, \omega_{CB}^D + T_E^C\, T_B^D\, \omega_{DC}^A = 0.$$

This implies that

$$\nabla\left(T_B^C\, \nabla_C\, T_E^A\right) - T_D^A\, T_B^C\, \omega_{CE}^D + T_E^D\, T_B^C\, \omega_{DC}^A = 0,$$

and composing the difference, we obtain

$$\nabla\left(T_E^C\, \nabla_C\, T_B^A - T_B^C\, \nabla_C\, T_E^A\right) - T_E^C\, T_D^A\, \omega_{CB}^D + T_D^A\, T_B^C\, \omega_{CE}^D = 0.$$

Rolling up Eq. (12) with T_A^E, we find (after the change of indices) that

$$\nabla\left(T_C^A\, \nabla_E\, T_B^C\right) - T_D^C\, T_C^A\, \omega_{EB}^D + T_B^D\, T_C^A\, \omega_{DE}^C = 0,$$
$$\nabla\left(T_C^A\, \nabla_B\, T_E^C\right) - T_D^C\, T_C^A\, \omega_{EB}^D + T_E^D\, T_C^A\, \omega_{DB}^C = 0.$$

The last two equalities yield

$$\nabla\left(T_E^C\, \nabla_C\, T_B^A - T_B^C\, \nabla_C\, T_E^A - T_C^A\, \nabla_E\, T_B^C + T_C^A\, \nabla_B\, T_E^C\right) = 0,$$

and hence we obtain that the values

$$N_{EB}^A = T_E^C\, \nabla_C\, T_B^A - T_B^C\, \nabla_C\, T_E^A - T_C^A\, \nabla_E\, T_B^C + T_C^A\, \nabla_B\, T_E^C$$

form the tensor and we call it the Nijenhuis tensor.

Let us consider the Nijenhuis tensor N^A_{EB} of the internal tensor structure defined by formula (10).

Since the first part of Eq. (8) has the form $\nabla_i\, T^A_B\, \omega^i + \nabla_\alpha\, T^A_B\, \widetilde{\theta^\alpha}$, therefore, according to formula (11) and equalities

$$T^i_j = a\,\delta^i_j, \qquad T^i_\alpha = 0, \qquad T^\alpha_j = c\,\xi^\alpha\,\eta_j - 2a\,\Gamma^\alpha_j\,, \qquad T^\alpha_\beta = -a\,\delta^\alpha_\beta\,,$$
$$\nabla_k\, T^i_j = 0, \qquad \nabla_\alpha\, T^i_j = 0, \qquad \nabla_k\, T^i_\alpha = 0, \qquad \nabla_\beta\, T^i_\alpha = 0, \qquad \nabla_k\, T^\alpha_\beta = 0, \qquad \nabla_\gamma\, T^\alpha_\beta = 0,$$
$$\nabla_k\, T^\alpha_k = c\,\nabla_k(\xi^\alpha\,\eta_i) - 2a\,\nabla_k\,\Gamma^\alpha_i, \qquad \nabla_\beta\, T^\alpha_i = c\,\nabla_\beta(\xi^\alpha\,\eta_i) - 2a\,\nabla_\beta\,\Gamma^\alpha_i\,,$$

we obtain

$$N^i_{jk} = N^i_{j\alpha} = N^i_{\alpha j} = N^i_{\alpha\beta} = N^\alpha_{\beta i} = N^\alpha_{i\beta} = N^\alpha_{\beta\gamma} = 0,$$
$$N^\alpha_{ij} = 4a^2\,R^\alpha_{ij} + 2ac\big(D_j(\xi^\alpha\,\eta_i) - D_i(\xi^\alpha\,\eta_j)\big) + c^2\,\xi^\gamma\,\eta_j\,\nabla_\gamma(\xi^\alpha\,\eta_i) - c^2\,\xi^\gamma\,\eta_i\,\nabla_\gamma(\xi^\alpha\,\eta),$$

where R^α_{ij} is the curvature tensor of connectedness Γ^α_i, and D_j is a nonholonomic covariant product of the first kind [3].

Thus we have the following theorems.

Theorem 5. *If on the base* Vn *of space* Lm(Vn) *there are dual tensor structures and structures of almost product defined by formula* (10)*, then these structures are integrable, if and only if linear connectedness Γ^α_i is plane $R^\alpha_{ij} = 0$, and the vector field ξ^α and covector field η_i are co-constant (the covariant derivative of the first kind is equal to zero).*

Theorem 6. *If on the space* Lm(Vn) *there are dual tensor structures and structures of almost product defined by formula* (10)*, then these structures are integrable, if and only if linear connectedness Γ^α_i is plane $R^\alpha_{ij} = 0$, and the vector field ξ^α and covector field η_i are co-constant (the covariant derivative of the first kind is equal to zero).*

Consider the Nijenhuis tensor N^A_{EB} of the internal tensor structure defined by formula (11). In this case we obtain

$$\begin{aligned}
N^i_{jk} &= 4ab\,\xi^i\,\eta_\alpha\,R^\alpha_{jk} + b^2\,\xi^p\,\eta_\beta\,\Gamma^\beta_j\,\Gamma^\gamma_k\,D_p(\xi^i\,\eta_\gamma) - b^2\,\xi^p\,\eta_\beta\,\Gamma^\gamma_j\,\Gamma^\beta_k\,D_p(\xi^i\,\eta_\gamma)\\
&\quad + b^2\,\xi^i\,\eta_\alpha\,\xi^p\,\eta_\beta\,\Gamma^\beta_j\,R^\alpha_{pk} - b^2\,\xi^i\,\eta_\beta\,\eta_\alpha\,\xi^p\,\Gamma^\beta_k\,R^\alpha_{pj}\,,\\
N^i_{j\alpha} &= b^2\,\xi^p\,\eta_\alpha\,\xi^i\,\eta_\gamma\,R^\gamma_{pj} + b^2\,\xi^p\,\eta_\beta\,\Gamma^\beta_j\,D_p(\xi^i\,\eta_\alpha) - b^2\,\xi^p\,\eta_\alpha\,\Gamma^\beta_j\,D_p(\xi^i\,\eta_\beta),\\
N^i_{\alpha\beta} &= b^2\,\xi^p\,\eta_\alpha\,D_p(\xi^i\,\eta_\beta) - b^2\,\xi^p\,\eta_\beta\,D_p(\xi^i\,\eta_\alpha),\\
N^\alpha_{\beta\gamma} &= b^2\,\xi^p\,\eta_\gamma\,\Gamma^\alpha_i\,D_p(\xi^i\,\eta_\beta) - b^2\,\xi^p\,\eta_\beta\,\Gamma^\alpha_i\,D_p(\xi^i\,\eta_\gamma) + 2ab\,\Gamma^\alpha_p\nabla_\beta(\xi^p\,\eta_\gamma) - 2ab\,\Gamma^\alpha_p\nabla_\gamma(\xi^p\,\eta_\beta),\\
N^\alpha_{\beta i} &= 4ab\,\xi^p\eta_\beta\,R^\alpha_{pi} + b^2\,\xi^k\,\eta_\gamma\,\xi^p\,\eta_\beta\,\Gamma^\beta_j\,R^\gamma_{pi} + b^2\,\xi^p\,\eta_\gamma\,\Gamma^\gamma_i\,\Gamma^\alpha_k\,D_p(\xi^k\,\eta_\beta) - b^2\,\xi^p\,\eta_\beta\,\Gamma^\gamma_i\,\Gamma^\alpha_k\,D_p(\xi^k\,\eta_\gamma)\\
&\quad + 2ab\,\Gamma^\alpha_p\Gamma^\delta_i\,\nabla_\beta(\xi^p\,\eta_\delta) - 2ab\,\Gamma^\alpha_p\,\Gamma^\delta_i\,\nabla_\delta(\xi^p\,\eta_\beta),\\
N^\alpha_{ij} &= 4a^2\,R^\alpha_{ij} - 2ab\,\Gamma^\gamma_i\,\Gamma^\alpha_p\,R^p_{i\gamma} + 2ab\,\Gamma^\gamma_j\,\Gamma^\alpha_p\,R^p_{i\gamma} + b^2\,\xi^k\,\eta_\gamma\,\xi^p\,\eta_\delta\big(\Gamma^\gamma_i\,\Gamma^\delta_j - \Gamma^\gamma_j\,\Gamma^\delta_i\big)R^\alpha_{pk}\\
&\quad + b^2\,\xi^p\,\eta_\gamma\big(\Gamma^\gamma_i\,\Gamma^\delta_j - \Gamma^\gamma_j\,\Gamma^\delta_i\big)D_p(\xi^k\,\eta_\delta) + b^2\,\xi^k\,\eta_\gamma\,\xi^p\,\eta_\delta\,\Gamma^\alpha_k\big(\Gamma^\gamma_i\,R^\delta_{pj} - \Gamma^\gamma_j\,R^\delta_{pi}\big).
\end{aligned}$$

This results in the following theorems.

Theorem 7. *If in the base* Vn *of space* Lm(Vn) *there are dual tensor structures and structures of almost product defined by formula* (11)*, then these structures are integrable, if and only if linear connectedness Γ^α_i is plane $R^\alpha_{ij} = 0$, and the vector field ξ^i and covector field η_α are co-constant (the covariant derivative of the first kind is equal to zero).*

Theorem 8. *If on the space* Lm(Vn) *there are dual tensor structures and structures of almost product defined by formula* (11)*, then these structures are integrable, if and only if linear connectedness Γ^α_i is plane $R^\alpha_{ij} = 0$, and the vector field ξ^i and covector field η_α are co-constant (the covariant derivative of the first kind is equal to zero).*

Theorem 9. *If in the base* Vn *of space* Lm(Vn) *there are dual tensor structures and structures of almost product defined by formula* (11)*, then these structures are not fully integrable, but the subobjects $\{N^i_{\alpha\beta}\}$, $\{N^i_{\alpha\beta}, N^\alpha_{\beta\gamma}\}$ of the Nijenhuis tensor vanish, when the vector field $\xi^i(x)$ and covector field $\eta_\alpha(x)$ are co-constant (the covariant derivative of the first kind is equal to zero).*

Theorem 10. *If in the space* Lm(Vn) *there are dual tensor structures and structures of almost product defined by formula* (11)*, then these structures are not fully integrable, but the subobject* $\{N^i_{\alpha\beta}\}$ *of the Nijenhuis tensor vanishes, when the vector field* $\xi^i(x, y)$ *and covector field* $\eta_\alpha(x, y)$ *are co-constant (the covariant derivative of the first kind is equal to zero).*

Theorem 11. *If on the space* Lm(Vn) *there are dual tensor structures and structures of almost product defined by formula* (11)*, then these structures are not fully integrable, but the subobjects* $\{N^i_{\alpha\beta}\}$, $\{N^i_{j\alpha}, N^i_{\alpha\beta}\}$, $\{N^i_{jk}, N^i_{j\alpha}, N^i_{\alpha\beta}\}$, $\{N^i_{\alpha\beta}, N^\alpha_{ij}, N^i_{jk}, N^i_{j\alpha}, N^\alpha_{\beta\gamma}, N^\alpha_{\beta i}\}$ *of the Nijenhuis tensor vanish, when linear connectedness* Γ^α_i *is plane* $R^\alpha_{ij} = 0$*, and the vector field* $\xi^i(x, y)$ *and covector field* $\eta_\alpha(x, y)$ *are co-constant (the covariant derivative of the firs kind is equal to zero).*

4. *F*-structures

The tensor structure T^A_B is called F-structure if

$$T^A_B\, T^B_C\, T^C_D + \lambda\, T^A_D = 0 \quad (\lambda = \pm 1).$$

Written explicitly, this system has the form

$$\begin{aligned}
T^i_k\, T^k_p\, T^p_j + T^i_\gamma\, T^\gamma_k\, T^k_j + T^i_k\, T^k_\gamma\, T^\gamma_j + T^i_\gamma\, T^\gamma_\beta\, T^\beta_j + \lambda\, T^i_j &= 0,\\
T^\alpha_k\, T^k_p\, T^p_j + T^\alpha_k\, T^k_\gamma\, T^\gamma_j + T^\alpha_\gamma\, T^\gamma_k\, T^k_j + T^\alpha_\gamma\, T^\gamma_\beta\, T^\beta_j + \lambda\, T^\alpha_j &= 0,\\
T^i_k\, T^k_p\, T^p_\alpha + T^i_k\, T^k_\gamma\, T^\gamma_\alpha + T^i_\gamma\, T^\gamma_p\, T^p_\alpha + T^i_\gamma\, T^\gamma_\beta\, T^\beta_\alpha + \lambda\, T^i_\alpha &= 0,\\
T^\alpha_k\, T^k_p\, T^p_\beta + T^\alpha_k\, T^k_\gamma\, T^\gamma_\beta + T^\alpha_\gamma\, T^\gamma_p\, T^p_\beta + T^\alpha_\gamma\, T^\gamma_\delta\, T^\delta_\beta + \lambda\, T^\alpha_\beta &= 0.
\end{aligned}$$

From the above system and from equality (9) follows

$$\begin{aligned}
&b(a^2 + ad + bc + d^2 + \lambda)\xi^i\, \eta_\beta = 0,\\
&(a^3 + \lambda\, a)\delta^i_j + (a^2 b + b^2 c + bd^2 + abd + \eta b)\xi^i\, \eta_\alpha\, \Gamma^\alpha_j + (2abc + bcd)\xi^i\, \eta_j = 0,\\
&(d^3 + \lambda\, d)\delta^\alpha_\beta - (a^2 b + b^2 c + bd^2 + abd + \eta b)\xi^i\, \eta_\beta\, \Gamma^\alpha_i + (abc + 2bcd)\xi^\alpha\, \eta_\beta = 0,\\
&(d - a)(a^2 + ad + d^2 + \lambda)\Gamma^\alpha_j - bc(d + 2a)\xi^i\, \eta_j\, \Gamma^\alpha_i - b(a^2 + ad + bc + d^2 + \lambda)\xi^i\, \eta_\gamma\, \Gamma^\alpha_i\, \Gamma^\alpha_j\\
&\quad + (a^2 c + bc^2 + cd^2 + acd + \lambda c)\xi^\alpha\, \eta_j = 0,
\end{aligned}$$

or

$$\begin{aligned}
&b(a^2 + ad + bc + d^2 + \lambda) = 0, \qquad a^3 + 2abc + bcd + \lambda a = 0, \qquad d^3 + 2bcd + abc + \lambda d = 0,\\
&(d - a)(a^2 + ad + d^2 + \lambda)\Gamma^\alpha_j + c(a^2 + ad + bc + d^2 + \lambda)\xi^\alpha \eta_j\\
&\quad + bc(2d + a)\xi^\alpha \eta_\gamma \Gamma^\gamma_j - bc(d + 2a)\xi^i \eta_j \Gamma^\alpha_i = 0.
\end{aligned}$$

The second and third equalities result in

$$(d - a)(a^2 + ad + d^2 + bc + \lambda) = 0.$$

If $b = 0$, $d = a$, $a^2 + ad + d^2 + bc + \lambda \neq 0$, we obtain $c = 0$ and $a^2 + \lambda = 0$. Analogous results are obtained in the other cases, i.e., the real F-structures exist, if and only if $\lambda = -1$.

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Original article

The nonstationary flow of a conducting fluid in a plane pipe in the presence of a transverse magnetic field

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Abstract

We consider the nonstationary flow of an incompressible viscous conducting fluid in the plane pipe of infinite length in the presence of a transverse magnetic field. Using the Laplace transformation we obtain the expressions for the fluid flow velocity and the electric and magnetic field intensities when the conductivity values of the fluid and pipe walls are arbitrary. Solutions are expressed in terms of complex integrals which are calculated for the particular case of ideally conducting walls.

Keywords: Nonstationary flow; Viscous conducting fluid; Plane pipe; Damped oscillation

1. Introduction

In recent years, nonstationary flows of a conducting incompressible fluid have been considered in a number of works. A class of exact solutions of magnetohydrodynamic equations for laminar flows has been considered in the papers [1–3]. The theoretical statement of nonstationary problems and their solvability were investigated by Ladyzhenskaya and Solonnikov in [4]. In the papers [5–7], an exact solution was obtained for a nonstationary flow of a fluid which is produced by the ideally conducting parallel walls in the presence of a transverse magnetic field. The impulsive motion and oscillations of the plate in a conducting fluid in the presence of a magnetic field are studied in the works [8–12].

2. Main part

In the present paper, an exact solution is obtained for the particular case of a nonstationary flow of a conducting incompressible viscous fluid between the conducting parallel walls of infinite length. An analogous problem was the subject of Regirer's paper, but the induced fields outside the fluid are ignored there.

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Let a stationary fluid, whose conductivity is σ, viscosity coefficient is η, density is ρ and magnetic permeability is μ, occupy an infinitely long plane pipe with parallel walls, the distance between which is $2L$. The pipe walls are assumed to be infinitely thick and characterized by the conductivity σ^*, magnetic permeability μ and dielectric permeability ε. There exists a transverse magnetic field $B_0 = \mu H_0$.

At the initial moment of time $t = 0$, the constant pressure changes suddenly along the pipe and, as a result, the fluid begin to move. The origin of the Cartesian system (right) is chosen between the pipe walls, the x-axis coincides with the fluid motion direction, while the y-axis is directed normally to the walls, i.e. in parallel to the direction of the magnetic field.

To prevent the appearance of electric bulk charges, it is assumed that the conducting walls are grounded at $z \to \pm\infty$.

Let the values L, U_0, $\frac{L}{U_0}$, ρU_0^2, H_0, $\mu H_0 U_0$, $\frac{H_0}{L}$ (U_0 is some typical velocity) denote respectively the radius of the vector $\vec{r}$, fluid velocity $\vec{V}$, time t, pressure ρ, magnetic field intensity $\vec{H}$, electric field intensity $\vec{E}$ and current density $\vec{j}$.

Then the equations of the problem will be written in the non-dimensional form [13–15] as follows: in the domain adjacent to the fluid

$$\operatorname{rot} \vec{H} = \vec{j} \tag{1}$$

$$\operatorname{div} \vec{H} = 0 \tag{2}$$

$$\operatorname{rot} \vec{E} = -\frac{\partial \vec{H}}{\partial t}, \tag{3}$$

$$\operatorname{div} \vec{E} = 0 \tag{4}$$

$$\vec{j} = R_m(\vec{E} + \vec{V} \times \vec{H}) \tag{5}$$

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V}\,\nabla)\vec{V} = -\nabla\rho + S(\operatorname{rot} \vec{H} \times \vec{H}) + \frac{1}{R}\Delta\vec{V} \tag{6}$$

$$\operatorname{div} \vec{V} = 0, \tag{7}$$

in the domain near the pipe walls

$$\operatorname{rot} \vec{H}^* = \vec{j}^* + \beta^2 \frac{\partial \vec{E}^*}{\partial t}, \tag{8}$$

$$\operatorname{div} \vec{H}^* = 0, \tag{9}$$

$$\operatorname{rot} \vec{E}^* = -\frac{\partial \vec{H}^*}{\partial t}, \tag{10}$$

$$\operatorname{div} \vec{E}^* = 0, \tag{11}$$

$$\vec{j}^* = R_m^* \vec{E}^*, \tag{12}$$

where

$$\left.\begin{array}{lll} S = \dfrac{B_0^2}{\mu\rho U_0^2} = \dfrac{M^2}{R R_m}, & M^2 = \dfrac{B_0^2 L^2 \sigma}{\eta}, & \beta^2 = \varepsilon\mu U_0^2, \\ R = \dfrac{U_0 L \rho}{\mu}, & R_m = \sigma\mu U_0 L, & R_m^* = \sigma^*\mu U_0 L, \end{array}\right\}$$

are non-dimensional parameters.

As it is usually done in magnetohydrodynamics, we neglect the displacement current in the fluid.

In the considered problem

$$\vec{V} = \vec{V}[U(y,t), 0, 0], \qquad \vec{H} = \vec{H}[H_x(y,t), 1, 0], \qquad \vec{E} = \vec{E}[0, 0, E_x(y,t)],$$

$$\vec{j} = \vec{j}[0, 0, j_x(y,t)], \qquad p = p(x, y, t),$$

and

$$\frac{\partial p}{\partial x} = \text{const} = -P.$$

Hence Eqs. (1)–(7) reduce to a system

$$\frac{\partial H_x}{\partial t} = \frac{\partial U}{\partial y} + \frac{1}{R_m}\frac{\partial^2 H_x}{\partial y^2},$$

$$\frac{\partial U}{\partial t} = P + S\frac{\partial H_x}{\partial y} + \frac{1}{R}\frac{\partial^2 U}{\partial y^2},$$

$$\frac{\partial p}{\partial y} = -SH_x\frac{\partial H_x}{\partial y}, \tag{13}$$

$$j_z = -\frac{\partial H_x}{\partial y} = R_m(E_z + U), \tag{14}$$

$$\frac{\partial E_z}{\partial y} = -\frac{\partial H_x}{\partial t},$$

while Eqs. (8)–(12) take the form

$$-\frac{\partial H_x^*}{\partial y} = R_m^* E_z^* + \beta^2\frac{\partial E_z^*}{\partial t},$$

$$\frac{\partial E_z^*}{\partial y} = -\frac{\partial H_x^*}{\partial t}.$$

By virtue of the above assumptions, for $t = 0$ the initial conditions are written as

$$\left.\begin{aligned} U = H_x = E_z = j_z = 0, \qquad p = -Px + p_0,\\ H_x^* = E_z^* = j_z^* = 0.\end{aligned}\right\} \tag{15}$$

The boundary conditions imply that the fluid velocity on the pipe walls is zero, while the intensities of the magnetic and electric fields are continuous, i.e. for $t > 0$,

$$y = +1, \qquad U = 0, \qquad H_x = H_x^*, \qquad E_z = E_z^*, \tag{16}$$

and

$$y = -1, \qquad U = 0, \qquad H_x = H_x^*, \qquad E_z = E_z^*. \tag{17}$$

The fields in the upper wall domain remain constant for $y \to +\infty$, and in the lower wall domain they remain constant for $y \to -\infty$.

If we know the intensity of the induced magnetic field $H_x(y,t)$, then the pressure $p(x, y, t)$ is defined from Eq. (13), and the current density $j_x(y,t)$ from the relation (14). Therefore in the sequel we will limit our consideration to finding the unknowns U, H_x and E_z.

Applying the Laplace transformation [16] for the zero initial conditions (15), for the images of the sought functions $\tilde{U}$, $\tilde{H}_x$ and $\tilde{E}_z$ we obtain the equations

$$p\tilde{H}_x = \frac{d\tilde{U}}{dy} + \frac{1}{R_m}\cdot\frac{d^2\tilde{H}_x}{dy^2}, \tag{18}$$

$$p\tilde{U} = S\frac{d\tilde{H}_x}{dy} + \frac{1}{R}\frac{d^2\tilde{U}}{dy^2} + \frac{P}{p}, \tag{19}$$

$$-\frac{d\tilde{H}_x}{dy} = R_m(\tilde{E}_z + \tilde{U}), \tag{20}$$

$$\frac{d\tilde{E}_z}{dy} = -p\tilde{H}_x, \tag{21}$$

where p is the transformation parameter.

From the relations (18) and (19) we find

$$\frac{d^4\tilde{U}}{dy^4} - [M^2 + (R + R_m)p]\frac{d^2\tilde{U}}{dy^2} + RR_m p^2\tilde{U} = RR_m P, \tag{22}$$

$$\frac{d^4\tilde{H}_x}{dy^4} - [M^2 + (R + R_m)p]\frac{d^2\tilde{H}_x}{dy^2} + RR_m p^2\tilde{H}_x = 0. \tag{23}$$

Solutions of Eqs. (22) and (23) have the form

$$\tilde{U} = C_1 chmy + C_2 shmy + C_3 chmy + C_4 shny + \frac{P}{p^2}, \tag{24}$$

$$\tilde{H}_x = C_5 chmy + C_6 shmy + C_7 chny + C_8 shny, \tag{25}$$

where m and n are the roots of the characteristic equations

$$\lambda^4 - [M^2 + (R + R_m)p]\lambda^2 + RR_m p^2 = 0,$$
$$m = \frac{1}{2}\Big[\sqrt{M^2 + \big(\sqrt{R} + \sqrt{R_m}\big)^2 p} + \sqrt{M^2 + \big(\sqrt{R} - \sqrt{R_m}\big)^2 p}\Big],$$
$$n = \frac{1}{2}\Big[\sqrt{M^2 + \big(\sqrt{R} + \sqrt{R_m}\big)^2 p} - \sqrt{M^2 + \big(\sqrt{R} - \sqrt{R_m}\big)^2 p}\Big].$$

Now from (21) we find the electric field intensity

$$\tilde{E}_x = -p\Big[C_5\frac{shmy}{m} + C_6\frac{chmy}{m} + C_7\frac{shny}{n} + C_8\frac{chny}{n}\Big] - \frac{P}{p^2}. \tag{26}$$

The integration constant is defined by Ohm's law (20).

Substituting the solutions (24)–(26) in the initial equations (18)–(20) and assuming that they are identically satisfied, we obtain two systems of equations that connect the integration constants

$$\left.\begin{aligned} C_1\Big(\frac{m^2}{R} - p\Big) + C_6 mS = 0,\\ C_2\Big(\frac{m^2}{R} - p\Big) + C_5 mS = 0,\\ C_3\Big(\frac{n^2}{R} - p\Big) + C_8 nS = 0,\\ C_4\Big(\frac{n^2}{R} - p\Big) + C_7 nS = 0. \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} C_6\Big(\frac{m^2}{R} - p\Big) + C_1 m = 0,\\ C_5\Big(\frac{m^2}{R} - p\Big) + C_2 m = 0,\\ C_8\Big(\frac{n^2}{R} - p\Big) + C_3 n = 0,\\ C_7\Big(\frac{n^2}{R} - p\Big) + C_4 n = 0. \end{aligned}\right\} \tag{27}$$

From the structure of the differential equations (18)–(19) we see that only one of the obtained algebraic systems is independent.

For the final definition of the values $\tilde{U}$, $\tilde{H}_x$ and $\tilde{E}_z$ in the fluid domain it is necessary to define the images of the electric and magnetic field intensities in the pipe walls.

For the upper wall domain the transformed equations have the form

$$-\frac{d\tilde{H}_x^*}{dy} = (R_m^* + \beta^2 p)\tilde{E}_z^*, \tag{28}$$

$$\frac{d\tilde{E}_z^*}{dy} = -p\overset{x}{H}_x. \tag{29}$$

From the system (28)–(29) we obtain the differential equations

$$\frac{d^2\tilde{H}_x^*}{dy^2} = (R_m^* + \beta^2 p)p\tilde{H}_x^*, \tag{30}$$

$$\frac{d^2\tilde{E}_z^*}{dy^2} = (R_m^* + \beta^2 p)p\tilde{E}_z^*. \tag{31}$$

Solutions of Eqs. (30) and (31), which satisfy the initial relations (28) and (29), boundary conditions (16) and remain bounded for $y \to +\infty$, have the form

$$\tilde{H}_x^* = \tilde{H}_x^*\big|_{y=1} e^{\sqrt{p}\sqrt{R_m^*+\beta^2 p}(1-y)}$$

$$\tilde{E}_z^* = \tilde{E}_z^*\big|_{y=1} e^{\sqrt{p}\sqrt{R_m^*+\beta^2 p}(1-y)},$$

where the values of the functions on the wall y for $y = +1$ are related by

$$\tilde{E}_z^*\big|_{y=1} = -\frac{\sqrt{p}}{\sqrt{R_m^* + \beta^2 p}}\,\tilde{H}_x^*\big|_{y=1}.$$

For the lower wall boundary we analogously find

$$\tilde{H}_x^* = \tilde{H}_x^*\big|_{y=-1} e^{\sqrt{p}\sqrt{R_m^*+\beta^2 p}(1+y)}$$

$$\tilde{E}_z^* = \tilde{E}_z^*\big|_{y=-1} e^{\sqrt{p}\sqrt{R_m^*+\beta^2 p}(1+y)}$$

and also

$$\tilde{E}_z^*\big|_{y=-1} = -\frac{\sqrt{p}}{\sqrt{R_m^* + \beta^2 p}}\,\tilde{H}_x^*\big|_{y=-1}. \tag{32}$$

Using now the boundary conditions (16) and (17), the first equation of the system (27) and the relations (31) and (32), we define the integration constants in the solutions (24)–(26) and finally obtain the expressions of the images of $\tilde{U}$, $\tilde{H}_x$ and $\tilde{E}_z$ for the fluid domain

$$\tilde{U} = \frac{Pu(p,y)}{pD(p)}, \qquad \tilde{H}_x = \frac{Ph(p,y)}{pD(p)}, \qquad \tilde{E}_z = -\frac{Pg(p,y)}{p, D(p)}$$

where

$$u(p,y) = \frac{1}{p}\{D(p) + (Schn - F(p))chmy - (Schm - G(p))chny\},$$

$$h(p,y) = \frac{1}{p}\left\{(Schn - F(p))\left(p - \frac{m^2}{R}\right)\frac{shmy}{mS} - (Schm - G(p))\left(p - \frac{n^2}{R}\right)\frac{shny}{nS}\right\},$$

$$g(p,y) = \frac{1}{p}\left\{D(p) + p\left[(Schn - F(p))\left(p - \frac{m^2}{R}\right)\frac{chmy}{m^2 S} - (Schm - G(p))\left(p - \frac{n^2}{R}\right)\frac{chny}{n^2 S}\right]\right\},$$

$$D(p) = F(p)chm - G(p)chn$$

$$F(p) = \left(p - \frac{n^2}{R}\right)\left(\frac{\sqrt{p}shn}{\sqrt{R_m^* + \beta^2 p}} + p\frac{chn}{n}\right)\frac{1}{n},$$

$$G(p) = \left(p - \frac{m^2}{R}\right)\left(\frac{\sqrt{p}shm}{\sqrt{R_m^* + \beta^2 p}} + p\frac{chm}{m}\right)\frac{1}{m}.$$

The fluid velocity and the induced magnetic and electric field intensities are found by means of the Riemann–Mellin formula

$$
\begin{aligned}
U &= \frac{P}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{u(p,y)e^{pt}}{pD(p)}\,dp,\\
H_x &= \frac{P}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{h(p,y)e^{pt}}{pD(p)}\,dp,\\
E_z &= -\frac{P}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{g(p,y)e^{pt}}{pD(p)}\,dp.
\end{aligned}
\tag{33}
$$

In the particular case of the ideally conducting walls ($\sigma^* = \infty$, $R_m^* = \infty$), we manage to calculate the integrals (33) in the general form by using the residue theorem. Indeed, in that case

$$
\begin{aligned}
F(p) &= \left(p-\frac{n^2}{R}\right)\frac{Pchn}{n^2}, \qquad G(p) = \left(p-\frac{m^2}{R}\right)\frac{Pchm}{m^2},\\
D(p) &= \frac{1}{2RR_m}(m+n)(m-n)[ch(m+n)+ch(m-n)]\\
&= \frac{1}{2RR_m}\sqrt{M^2+(\sqrt{R}+\sqrt{R_m})^2p}\sqrt{M^2+(\sqrt{R}-\sqrt{R_m})^2p}\\
&\quad\times\sqrt{M^2+(\sqrt{R}+\sqrt{R_m})^2p}+ch\sqrt{M^2+(\sqrt{R}-\sqrt{R_m})^2p}
\end{aligned}
$$

i.e. the meromorphic functions $\tilde{U}$, $\tilde{H}_x$ and $\tilde{E}_z$ have simple poles at the points

$$
\begin{aligned}
&p = 0,\\
&p_k' = \frac{(2k+1)^2\pi^2}{2RR_m}\left[-(R+R_m)+\sqrt{(R-R_m)^2-\frac{4RR_mM^2}{(2K+1)^2}\pi^2}\right],\\
&p_k'' = \frac{(2k+1)^2\pi^2}{2RR_m}\left[-(R+R_m)-\sqrt{(R-R_m)^2-\frac{4RR_mM^2}{(2K+1)^2}\pi^2}\right],
\end{aligned}
$$

while the points $p = -\frac{M^2}{(\sqrt{R}+\sqrt{R_m})^2}$ and $p = -\frac{M^2}{(\sqrt{R}-\sqrt{R_m})^2}$ are the removable singular points.

Then, by the residue theorem

$$
U = U_{CT} + P\sum_{k=0}^{\infty}\left[\frac{u(p_k,y)e^{p_k't}}{p_k'(\frac{dD}{dp})_{p_k'}}+\frac{u(p_k,y)e^{p_k't}}{p_k''(\frac{dD}{dp})_{p_k''}}\right],
\tag{34}
$$

$$
H_x = H_{x\,CT} + P\sum_{k=0}^{\infty}\left[\frac{h(p_k,y)e^{p_k't}}{p_k'(\frac{dD}{dp})_{p_k'}}+\frac{u(p_k,y)e^{p_k't}}{p_k''(\frac{dD}{dp})_{p_k''}}\right],
\tag{35}
$$

$$
E_z = -P\sum_{k=0}^{\infty}\left[\frac{g(p_k,y)e^{p_k't}}{p_k'(\frac{dD}{dp})_{p_k'}}+\frac{g(p_k,y)e^{p_k't}}{p_k''(\frac{dD}{dp})_{p_k''}}\right],
\tag{36}
$$

where U_{CT} and $H_{x\,CT}$ corresponding to the stationary regimes and calculated as residues for $p = 0$ are defined by the formulas

$$
U_{CT} = \frac{P(chM - chMy)}{SR_mchM}, \qquad H_{x\,CT} = \frac{P(shM - yMchM)}{SMchM}.
\tag{37}
$$

To conclude, it is of interest to note that for the definite ratios of the parameters, the first nonstationary members in the solutions (34)–(36) are damping oscillations, while in ordinary hydrodynamics the transient regime is always of purely aperiodic character.

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Original article

Method of corrections by higher order differences for Poisson equation with nonlocal boundary conditions

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Abstract

We consider the Bitsadze–Samarskii type nonlocal boundary value problem for Poisson equation in a unit square, which is solved by a difference scheme of second-order accuracy. Using this approximate solution, we correct the right-hand side of the difference scheme. It is shown that the solution of the corrected scheme converges at the rate $O(|h|^s)$ in the discrete L_2-norm provided that the solution of the original problem belongs to the Sobolev space with exponent $s \in [2, 4]$.

Keywords: Nonlocal BVP; Difference scheme; Method of corrections; Improvement of accuracy; Compatible estimates of convergence rate

1. Introduction

Finite difference method is a significant tool in the numerical solution of problems posed for differential equations. In order to minimize the amount of calculations it is desirable for the difference scheme to be sufficiently good on coarse meshes, i.e. to have high order accuracy. In the present work, for improving the accuracy of the approximate solution, we study two-stage finite difference method. We consider Bitsadze–Samarskii type nonlocal boundary value problem for Poisson's equation.

At the first stage we solve the difference scheme $\Delta_h \tilde{U} = \varphi$, which has the second order of approximation. Using the solution $\tilde{U}$ the right-hand side of the difference scheme is corrected, $\Delta_h U = \varphi + R\tilde{U}$, and solved again on the same mesh.

This approach for some boundary value problems posed for Poisson and Laplace equations has been studied in Volkov's papers (see, e.g. [1–3]), where the input data were chosen so as to ensure that the exact solution belongs to the Hölder class $C_{6,\lambda}(\bar{\Omega})$.

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For establishing the convergence we use the methodology of obtaining the compatible estimates of convergence rate of difference schemes. This methodology develops from the works of Samarskii, Lazarov and Makarov (see, e.g., [4–6]), and later in the works of other authors (see, e.g., [7,8]). For the elliptic problems such estimates have the form

$$\|U-u\|_{W_2^k(\omega)} \le c|h|^{s-k}\|u\|_{W_2^s(\Omega)}, \quad s>k\ge 0,$$

where u is the solution of original problem, U is the approximate solution, k and s are integer and real numbers, respectively, $W_2^k(\omega)$ and $W_2^s(\Omega)$ are the Sobolev norms on the set of functions with discrete and continuous arguments. Here and below c denotes a positive generic constant, independent of h and u.

It is proved that the solution U of the corrected difference scheme converges at rate $O(h^s)$ in the discrete L_2-norm, when the exact solution belongs to the Sobolev space W_2^s, $s\in[2,4]$.

The generalization of the Bitsadze–Samarskii problem [9] was investigated by many authors (see, e.g., [10–13]).

In [11] for a Poisson equation it is considered a difference scheme, which converges by the rate $O(h^2)$ in the discrete W_2^2-norm to the exact solution from the class $C^4(\bar\Omega)$.

In [13] difference scheme is considered for a second order elliptic equation with variable coefficients and the compatible estimate of convergence rate in discrete W_2^1-norm is obtained.

Results, analogous to those given in the present work, are obtained in [14] for the Dirichlet problem posed for an elliptic equation, and also in [15] for the mixed problem with third kind conditions.

One of the methods for obtaining compact high order approximations is the Mehrstellen method ("Mehrstellenverfahren"), defined by Collatz (see [16]). Instead of approximating only the left hand side of the differential equation, he proposes to take several points of the right hand side as well. In the case of two-dimensional problem, the differential operator is approximated on a 9-point stencil with the fourth order accuracy.

The advantage of the Mehrstellen schemes over ordinary (second order) accuracy schemes on a coarse grid is obvious.

The advantage of our method is:

(a) It needs to approximate the differential operator on minimally acceptable stencil (5-point stencil for a two-dimensional problem). Therefore, the condition number of this operator is better as compared with the Mehrstellen schemes, which is notable on a fine grid.

(b) It is a two-stage method, nevertheless it requires matrix inversion only once (on the second stage we change only the right-hand side of the equation, while the operator is kept unchanged).

(c) The method of correction is handy even in the case when construction of high precision schemes is impossible.

2. Statement of the problem and some auxiliary estimate

As usual, by symbol $W_2^s(\Omega)$, $s\ge 0$ we denote the Sobolev space. For integer s the norm in $W_2^s(\Omega)$ is given by formula

$$\|u\|^2_{W_2^s(\Omega)} = \sum_{j=0}^{s}|u|^2_{W_2^j(\Omega)}, \qquad |u|^2_{W_2^j(\Omega)} = \sum_{|\nu|=j}\|D^\nu u\|^2_{L_2(\Omega)},$$

where $D^\nu = \partial^{|\nu|}/\left(\partial x_1^{\nu_1}\partial x_2^{\nu_2}\right)$, $\nu=(\nu_1,\nu_2)$ is multi-index with non-negative integer components, $|\nu|=\nu_1+\nu_2$.

If $s=\bar s+\varepsilon$, where $\bar s$ is an integer part of s and $0<\varepsilon<1$, then

$$\|u\|^2_{W_2^s(\Omega)} = \|u\|^2_{W_2^{\bar s}(\Omega)} + |u|^2_{W_2^s(\Omega)},$$

where

$$|u|_{W_2^s(\Omega)} = \sum_{|\nu|=\bar s}\int_\Omega\int_\Omega \frac{|D^\nu u(x)-D^\nu u(y)|^2}{|x-y|^{2+2\varepsilon}}\,dx\,dy.$$

Particularly, for $s=0$ we have $W_2^0=L_2$.

Let $\bar{\Omega} = \{(x_1, x_2) : \ 0 \le x_\alpha \le 1, \ \alpha = 1, 2\}$ be a unit square with a boundary Γ; $\Gamma_0 = \Gamma \setminus \{(1, x_2) : 0 < x_2 < 1\}$; ξ_k be fixed points from interval $(0; 1)$, $0 < \xi_1 < \xi_2 < \cdots \xi_m < 1$. Denote $\xi_0 = 0$, $\xi_{m+1} = 1$.

Consider the problem

$$\Delta u = f(x), \quad x \in \Omega, \qquad u\big|_{\Gamma_0} = 0, \qquad u(1, x_2) = \sum_{k=1}^{m} \alpha_k u(\xi_k, x_2), \quad 0 < x_2 < 1 \tag{1}$$

where the coefficients α_k are real numbers satisfying conditions

$$\varkappa := \sum_{k=1}^{m} |\alpha_k| \sqrt{\xi_k} < 1.$$

It was shown in [12] that, for $f(x) \in L_2(\Omega, \rho)$, there exists a unique strong solution of problem (1) in the weighted Sobolev space $W_2^2(\Omega, \rho)$. Throughout the following, we assume that the function $f(x)$ provides the unique solvability of problem (1) in the $W_2^s(\Omega)$, $2 \le s \le 4$.

Consider the following grid domains in $\bar{\Omega}$:

$$\bar{\omega}_k = \{x_k = i_k h : \ i_k = 0, 1, \ldots, n, \ h = 1/n\}, \qquad \omega_k = \bar{\omega}_k \cap (0, 1),$$
$$\omega_k^+ = \bar{\omega}_\alpha \cap (0, 1], \quad k = 1, 2, \qquad \omega = \omega_1 \times \omega_2, \qquad \bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2, \qquad \gamma_0 = \Gamma_0 \cap \bar{\omega}.$$

We assume that the points ξ_k coincide with grid nodes

$$\xi_k = n_k h, \quad k = 1, 2, \ldots, m,$$

where n_k are nonnegative integers $0 < n_1 < n_2 < \cdots < n_m < n$. We suppose also that

$$h/2 \le 1 - \xi_m - \nu, \quad \nu = \text{const} > 0.$$

For grid functions we define difference quotients in x_k directions as follows

$$V_{x_k} = \big(V^{(+1_k)} - V\big)/h, \qquad V_{\bar{x}_k} = \big(V - V^{(-1_k)}\big)/h$$

where

$$V = V(x), \quad V^{(\pm 1_1)} = V(x_1 \pm h, x_2), \quad V^{(\pm 1_2)} = V(x_1, x_2 \pm h).$$

For functions, defined on Ω, we need the following averaging operators:

$$T_1 u(x) := \frac{1}{h^2} \int_{x_1 - h_1}^{x_1 + h_1} (h_1 - |x_1 - t_1|) u(t_1, x_2)\, dt_1.$$

Analogously is defined operator T_2. Note that these operators commute and

$$T_k \frac{\partial^2 u}{\partial x_k^2} = u_{\bar{x}_k x_k}, \quad k = 1, 2.$$

Define the following weight functions

$$r(x_1) = 1 - x_1, \qquad \rho(x_1) = 1 - x_1 - \sum_{k=1}^{m} \varkappa \sigma_k \chi(\xi_k - x_1),$$

where

$$\sigma_k = \frac{|\alpha_k|}{\sqrt{\xi_k}}, \qquad \chi(t) = \begin{cases} t, & \text{if } t \ge 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Let

$$\bar{r} = \big(r + r^{(-1_1)}\big)/2, \qquad \overline{\rho} = \big(\rho + \rho^{(-1_1)}\big)/2.$$

Notice that the following inequality

$$(1-\varkappa^2)r(x_1) \le \rho(x_1) \le r(x_1) \tag{2}$$

holds.

Indeed, the right-hand side inequality is obvious. The left-hand side inequality can be verified as follows:

$$\rho(x_1) = 1 - x_1 - \varkappa \sum_{k=j+1}^{m} \sigma_k(\xi_k - x_1) \ge \left(1 - \varkappa \sum_{k=j+1}^{m} \sigma_k \xi_k\right)(1-x_1)$$
$$\ge (1-\varkappa^2)(1-x_1), \quad x_1 \in (\xi_j, \xi_{j+1}).$$

Remark. Introduction of auxiliary (equivalent to r) weight function ρ gives possibility to state the positive definiteness of the difference scheme operator.

Let $H = H(\omega)$ be the set of grid functions defined on ω with the inner product and norm

$$(U, V)_r = \sum_{x\in\omega} h^2 r(x_1) U(x) V(x), \qquad \|V\|_r = \|V\|_{L_2(\omega,r)} = (V, V)_r^{1/2}.$$

Moreover, let

$$(U, V) = \sum_{x\in\omega} h^2 U(x) V(x), \qquad \|V\| = (V, V)^{1/2}.$$

Inner product and norm, involving ρ in index will make similar to the expression with index r sense.

Denote by $\mathring{H} = \mathring{H}(\bar\omega)$ the set of grid functions $V(x)$, given on $\bar\omega$ and satisfying conditions

$$V(x) = 0, \quad x \in \gamma_0, \qquad V(1, x_2) = \sum_{k=1}^{m} \alpha_k V(\xi_k, x_2), \quad x_2 \in \omega_2. \tag{3}$$

Lemma 1. *For each function, defined on mesh $\bar\omega$, which equals zero on $x_1 = 0$ and satisfies the nonlocal condition from* (3)*, the following inequalities*

$$-\sum_{\omega_1} h\rho Y_{\bar{x}_1 x_1} Y \ge \sum_{\omega_1^+} h\bar\rho Y_{\bar{x}_1}^2, \tag{4}$$

$$\sum_{\omega_1} h r Y^2 \le 4 \sum_{\omega_1^+} h\bar{r}(Y_{\bar{x}_1})^2 \tag{5}$$

hold.

Proof. After simple computations, we obtain

$$-\sum_{\omega_1} h\rho Y_{\bar{x}_1 x_1} Y = \sum_{\omega_1^+} h\bar\rho Y_{\bar{x}_1}^2 - \frac{1}{2} Y^2(1, x_2) - \frac{1}{2} \sum_{\omega_1} h Y^2 \rho_{\bar{x}_1 x_1}.$$

Taking into account

$$\sum_{\omega_1} h Y^2 \rho_{\bar{x}_1 x_1} = -\sum_{\omega_1} h Y^2 \sum_{k=1}^{m} \varkappa \sigma_k \frac{1}{h} \delta(x_1, \xi_k) = -\sum_{k=1}^{m} Y^2(\xi_k, x_2)\sigma_k \varkappa,$$

where $\delta(\cdot,\cdot)$ is the Kronecker delta, and

$$Y^2(1, x_2) \le \left(\sum_{k=1}^{m} \sqrt[4]{\alpha_k^2 \xi_k} \sqrt[4]{\alpha_k^2/\xi_k}\, |Y(\xi_k, x_2)|\right)^2 \le \varkappa \sum_{k=1}^{m} \frac{|\alpha_k|}{\sqrt{\xi_k}} Y^2(\xi_k, x_2), \tag{6}$$

we obtain (4).

One can show that

$$\sum_{\omega_1^+} h\bar{r}^2 (Y^2)_{\bar{x}_1} = \sum_{\omega_1} hrY^2 + \frac{h^2}{8} Y^2(1, x_2). \tag{7}$$

On the other hand,

$$\begin{aligned}
\sum_{\omega_1^+} h\bar{r}^2 (Y^2)_{\bar{x}_1} &= \sum_{\omega_1^+} h\bar{r}^2 Y_{\bar{x}_1}(Y + Y^{(-1_1)}) \\
&\le \Big(\sum_{\omega_1^+} h\bar{r}(Y_{\bar{x}_1})^2\Big)^{1/2} \Big(\sum_{\omega_1^+} h\bar{r}\big(Y + Y^{(-1_1)}\big)^2\Big)^{1/2} \\
&\le \frac{\varepsilon}{2} \sum_{\omega_1^+} h\bar{r}(Y_{\bar{x}_1})^2 + \frac{1}{2\varepsilon} \sum_{\omega_1^+} h\bar{r}(Y + Y^{(-1_1)})^2.
\end{aligned}$$

Whence, choosing $\varepsilon = 4$ we obtain

$$\begin{aligned}
\sum_{\omega_1^+} h\bar{r}^2 (Y^2)_{\bar{x}_1} &\le 2 \sum_{\omega_1^+} h\bar{r}(Y_{\bar{x}_1})^2 + \frac{1}{8} \sum_{\omega_1^+} h\bar{r}(Y + Y^{(-1_1)})^2 \\
&= 2 \sum_{\omega_1^+} h\bar{r}(Y_{\bar{x}_1})^2 + \frac{h^2}{8} Y^2(1, x_2) + \frac{1}{2} \sum_{\omega_1} hrY^2. \qquad (8)
\end{aligned}$$

(7), (8) prove the inequality (5). Lemma 1 is proved. $\square$

3. Difference scheme, correction procedure, and main result

At the *first stage*, we approximate problem (1) by the difference scheme

$$\tilde{U}_{\bar{x}_1 x_1} + \tilde{U}_{\bar{x}_2 x_2} = \varphi(x), \quad x \in \omega, \ \tilde{U} \in \overset{\circ}{H}, \tag{9}$$

where $\varphi = T_1 T_2 f$ is the average of function f.

Define the operators

$$A := A_1 + A_2, \qquad A_k Y := -\overset{\circ}{Y}_{\bar{x}_1 x_1}, \quad k = 1, 2, \ x \in \omega,$$

where

$$Y \in H, \quad \overset{\circ}{Y} \in \overset{\circ}{H} \quad \text{and} \quad Y(x) = \overset{\circ}{Y}(x) \quad \text{for } x \in \omega.$$

The difference scheme (9) can be rewritten in the form of operator equation

$$-A\tilde{U} = \varphi(x), \quad x \in \omega, \ \tilde{U} \in H. \tag{10}$$

Operator A maps H onto H. Indeed, it suffices to show that operator A_1 on near-boundary point $(1 - h, x_2)$ has the form

$$\begin{aligned}
A_1 Y(1 - h, x_2) &= -\overset{\circ}{Y}_{\bar{x}_1 x_1}(1 - h, x_2) \\
&= -\big(\overset{\circ}{Y}(1, x_2) - 2\overset{\circ}{Y}(1 - h, x_2) + \overset{\circ}{Y}(1 - 2h, x_2)\big)/h^2 \\
&= -\Big(\sum_{k=1}^{m} \alpha_k Y(\xi_k, x_2) - 2Y(1 - h, x_2) + Y(1 - 2h, x_2)\Big)/h^2.
\end{aligned}$$

According to the estimates (2), (4) and (5) we obtain the inequality

$$(A_1 Y, Y)_\rho \ge c\|Y\|_\rho^2, \quad Y \in H.$$

In addition, it is well known that A_2 is a self-adjoint and positive definite operator, $A_2 = A_2^*$, $(A_2Y, Y)_\rho \geq c\|Y\|_\rho^2$. Therefore, the operator A is positive definite on the space H,

$$(AY, Y)_\rho \geq \|Y\|_\rho^2,$$

and hence the scheme (10) (i.e. (9)) is uniquely solvable.

At the *second stage*, we use the earlier-found solution of the difference scheme (10), define the correction term

$$\mathcal{R}\tilde{U} := \frac{h^2}{6}\tilde{U}_{\bar{x}_1x_1\bar{x}_2x_2}$$

and solve the difference scheme

$$-AU = \varphi - \mathcal{R}\tilde{U}, \quad x \in \omega, \ U \in H \tag{11}$$

on the same grid.

The following assertion is the main result of the present paper.

Theorem 1. *Let the solution of problem* (1) *belong to the space* $W_2^s(\Omega)$, $s \geq 2$. *Then the convergence rate of the corrected difference scheme* (11) *in the discrete* L_2*-norm is defined by the estimate*

$$\|U - u\|_{L_2(\omega,r)} \leq ch^s\|u\|_{W_2^s(\Omega)}, \quad 2 \leq s \leq 4.$$

4. A priori error estimates. Proof of Theorem 1

Let

$$\zeta_{3-k} = T_k u - u, \qquad \eta_{3-k} = T_k u - u - \frac{h^2}{12}u_{\bar{x}_kx_k}, \quad k = 1, 2.$$

By $\tilde{Z} = \tilde{U} - u$ and $Z = U - u$ we denote the errors in the solution of the schemes (10) and (11) respectively. First, notice that these functions represent solutions of the following problems:

$$-A\tilde{Z} = (\zeta_1)_{\bar{x}_1x_1} + (\zeta_2)_{\bar{x}_2x_2}, \quad x \in \omega, \ \tilde{Z} \in H \tag{12}$$

and

$$-AZ = (\eta_1)_{\bar{x}_1x_1} + (\eta_2)_{\bar{x}_2x_2} - (h^2/6)\tilde{Z}_{\bar{x}_1x_1\bar{x}_2x_2}, \quad x \in \omega, \ Z \in H. \tag{13}$$

Indeed, we have

$$-AZ = -AU + Au = \varphi - \mathcal{R}\tilde{U} + Au = -\mathcal{R}\tilde{Z} + T_1T_2f - \mathcal{R}u + Au,$$

whence using the relation

$$T_1T_2\Delta u = (T_2u)_{\bar{x}_1x_1} + (T_1u)_{\bar{x}_2x_2}$$

and the expressions for the operators Au and $\mathcal{R}u$, we obtain (13). Eq. (12) is obtained analogously.

Lemma 2. *For the solutions of problems* (12), (13) *there hold the following a priori estimates*

$$\|\tilde{Z}_{\bar{x}_1x_1}\|_\rho \leq c\big(\|(\zeta_1)_{\bar{x}_1x_1}\| + \|(\zeta_2)_{\bar{x}_2x_2}\|\big), \tag{14}$$

$$\|Z\|_\rho \leq c\big(\|\eta_1\| + \|\eta_2\| + h^2\|\tilde{Z}_{\bar{x}_1x_1}\|_\rho\big). \tag{15}$$

Proof. From (12) it follows

$$\big(\tilde{Z}_{\bar{x}_1x_1}, \tilde{Z}_{\bar{x}_1x_1}\big)_\rho + \big(\tilde{Z}_{\bar{x}_2x_2}, \tilde{Z}_{\bar{x}_1x_1}\big)_\rho = -\big((\zeta_1)_{\bar{x}_1x_1} + (\zeta_2)_{\bar{x}_2x_2}, \tilde{Z}_{\bar{x}_1x_1}\big)_\rho. \tag{16}$$

Summing up by parts, we get

$$\begin{aligned}\left(\tilde Z_{\bar x_2 x_2}, \tilde Z_{\bar x_1 x_1}\right)_\rho &= \sum_{\omega^+} h^2 \bar\rho (\tilde Z_{\bar x_1 \bar x_2})^2 - \sum_{\omega_2^+} \frac{h}{2} (\tilde Z_{\bar x_2}(1,x_2))^2 - \frac{1}{2}\sum_{\omega_1\times\omega_2^+} h^2 \rho_{\bar x_1 x_1} (\tilde Z_{\bar x_2})^2 \\ &= \sum_{\omega^+} h^2 \bar\rho (\tilde Z_{\bar x_1 \bar x_2})^2 - \sum_{\omega_2^+} \frac{h}{2}\left[(\tilde Z_{\bar x_2}(1,x_2))^2 - \sum_{k=1}^{m} \frac{\varkappa|\alpha_k|}{\sqrt{\xi_k}} (\tilde Z_{\bar x_2}(\xi_k,x_2))^2 \right].\end{aligned}$$

Using analogous to the estimate (6), written for $\tilde Z_{\bar x_2}$, we obtain

$$\left(\tilde Z_{\bar x_2 x_2}, \tilde Z_{\bar x_1 x_1}\right)_\rho \ge \sum_{\omega^+} h^2 \bar\rho (\tilde Z_{\bar x_1 \bar x_2})^2 \ge 0.$$

Therefore, from (16) we obtain the validity of (14).

Now, represent the solution of the problem (13) in the form of sum

$$Z = Z^{(1)} + Z^{(2)},$$

where $Z^{(k)}$, $k = 1, 2$, are the solutions of the following problems

$$-AZ^{(1)} = (\eta_1)_{\bar x_1 x_1}, \quad x \in \omega,\ Z^{(1)} \in H, \tag{17}$$

$$-AZ^{(2)} = (\eta_2)_{\bar x_2 x_2} - \frac{h^2}{6} \tilde Z_{\bar x_1 x_1 \bar x_2 x_2}, \quad x \in \omega,\ Z^{(2)} \in H. \tag{18}$$

From (17) we have

$$Z^{(1)} + A_1^{-1} A_2 Z^{(1)} = -\eta_1,$$
$$\|Z^{(1)}\|_\rho^2 + \left(A_1^{-1} A_2 Z^{(1)}, Z^{(1)}\right)_\rho = -\left(\eta_1, Z^{(1)}\right)_\rho.$$

The operator A_2 is self-adjoint and positive definite, therefore, there exists quadratic root $A_2^{1/2}$, which is self-adjoint and commutable with A_1^{-1}. Thus

$$\left(A_1^{-1} A_2 Z^{(1)}, Z^{(1)}\right)_\rho = \left(A_1^{-1}(A_2^{1/2} Z^{(1)}), (A_2^{1/2} Z^{(1)})\right)_\rho \ge 0$$

and

$$\|Z^{(1)}\|_\rho \le \|\eta_1\|. \tag{19}$$

From (18) it follows

$$A_2^{-1} A_1 Z^{(2)} + Z^{(2)} = -\eta_2 + (h^2/6)\tilde Z_{\bar x_1 x_1},$$

and since

$$\left(A_2^{-1} A_1 Z^{(2)}, Z^{(2)}\right)_\rho = \left(A_1(A_2^{-1/2} Z^{(2)}), (A_2^{-1/2} Z^{(2)})\right)_\rho \ge 0,$$

we obtain

$$\|Z^{(2)}\|_\rho \le \|\eta_2\| + (h^2/6)\|\tilde Z_{\bar x_1 x_1}\|_\rho. \tag{20}$$

(19) and (20) prove (15). □

To determine the rate of convergence of the two-stage finite difference method with the help of Lemma 2, it is sufficient to estimate the terms on the right-hand sides of (18), (19). For that purpose we use the following lemma.

Lemma 3. *Assume that the linear functional $l(u)$ is bounded in $W_2^s(E)$, where $s = \bar s + \varepsilon$, $\bar s$ is an integer, $0 < \varepsilon \le 1$, and $l(P) = 0$ for every polynomial P of degree $\le \bar s$ in two variables. Then, there exists a constant c, independent of u, such that $|l(u)| \le c|u|_{W_2^s(E)}$.*

Table 1
Experimental order of convergence in $L_2(\omega, r)$-norm.

h	$\|\tilde{U}_h - u\|_r$	$\|U_h - u\|_r$	$Ord(\tilde{U})$	$Ord(U)$
1/8	2.53376e−03	3.39828e−05		
			1.9949	3.9896
1/16	6.35699e−04	2.13925e−06		
			1.9987	3.9974
1/32	1.59065e−04	1.33943e−07		
			1.9997	3.9994
1/64	3.977507e−05	8.37520e−09		
			1.9999	3.9998
1/128	9.94431e−06	5.23507e−10		

This lemma is a particular case of the Dupont–Scott approximation theorem [17] and represents a generalization of the Bramble–Hilbert lemma [18].

Quantities $(\zeta_k)_{\bar{x}_k x_k}$, as a linear functionals with respect to u, vanish on the third order polynomials and are bounded in $W_2^s(\Omega)$, $s \geq 2$. Using the well known methodology (see, e.g., [6, Ch. 4, §1]), based on Lemma 3, for them we obtain the estimates

$$\|(\zeta_k)_{\bar{x}_k x_k}\| \leq ch^{s-2}\|u\|_{W_2^s(\Omega)}, \quad k = 1, 2, \tag{21}$$

$$\|\eta_k\| \leq ch^s\|u\|_{W_2^s(\Omega)}, \quad k = 1, 2. \tag{22}$$

Due to Lemma 2

$$\|Z\|_\rho \leq c\left(\|\eta_1\| + \|\eta_2\| + h^2\|(\zeta_1)_{\bar{x}_1 x_1}\| + h^2\|(\zeta_2)_{\bar{x}_2 x_2}\|\right),$$

which together with the estimates (21), (22) accomplishes the proof of Theorem 1.

5. Numerical experiments

Now, we present some numerical results to demonstrate the convergence order of the proposed method. The experimental order of convergence in the discrete $L_2(\omega, r)$ and $L_2(\omega)$ norms is computed by formulas

$$Ord(Y) = \log_2 \frac{\|Y_h - u\|_r}{\|Y_{h/2} - u\|_r}, \qquad Ord(Y) = \log_2 \frac{\|Y_h - u\|}{\|Y_{h/2} - u\|},$$

where u is the exact solution of original problem, while Y_h denotes the solution of the difference scheme on the grid with step h.

Below, in the examples the symbols $\tilde{U}$, U denote solutions of the difference schemes (10), (11), respectively.

The results of calculations are given by Tables 1, 2.

Consider the following problem

$$\Delta u = f, \quad x \in (0, 1)^2, \ u\big|_{\Gamma_0} = 0, \qquad u(1, x_2) = u(0.5, x_2), \quad 0 < x_2 < 1, \tag{23}$$

where

$$f(x) = -\frac{13\pi^2}{9}\sin\Big(\frac{2\pi x_1}{3}\Big)\sin(\pi x_2).$$

The exact solution $u(x) = \sin(\frac{2\pi x_1}{3})\sin(\pi x_2)$ of the problem (23) belongs to the space W_2^4, therefore, theoretical convergence rate of the difference scheme equals 4.

The right-hand side of the scheme is calculated by the formula

$$\varphi(x) = T_1 T_2 f = -\frac{13\pi^2}{9}\lambda_1^2\lambda_2^2\sin\Big(\frac{2\pi ih}{3}\Big)\sin(\pi jh),$$

$$\lambda_1 = \frac{3}{\pi h}\sin\Big(\frac{\pi h}{3}\Big), \quad \lambda_2 = \frac{2}{\pi h}\sin\Big(\frac{\pi h}{2}\Big).$$

Table 2
Experimental order of convergence in $L_2(\omega)$-norm.

h	$\|\tilde{U}_h - u\|$	$\|U_h - u\|$	$Ord(\tilde{U})$	$Ord(U)$
1/8	4.15297$e-$03	5.56995$e-$05		
			1.9654	3.9601
1/16	1.06347$e-$03	3.57879$e-$06		
			1.9844	3.9831
1/32	2.68761$e-$04	2.26315$e-$07		
			1.9926	3.9923
1/64	6.75360$e-$05	1.42207$e-$08		
			1.9964	3.9963
1/128	1.69262$e-$06	8.91061$e-$10		

6. Conclusion

For solution of the Bitsadze–Samarskii type nonlocal problem posed in unit square for Poisson equation it is used a finite-difference scheme. Using the solution, obtained by the method with second order accuracy, we correct the right-hand side of the scheme and solve it again on the same grid. It is proved that if the solution of original problem belongs to the Sobolev space with fractional exponent $s \in [2; 4]$, then the corrected scheme converges with the rate $O(|h|^s)$. The theoretical results are supported by numerical experiments. The obtained results can be extended to the nonlocal problem posed for general elliptic equations, and also to three-dimensional case.

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