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Original article

On sets of singular rotations for translation invariant bases

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Abstract

The following problem is studied: For a summable function f, what kind may be a set of all rotations γ for which $\int f$ is not differentiable with respect to the γ-rotation of the given basis B? In particular, for translation invariant bases on the plane, the topological structure of possible sets of singular rotations is found.

Keywords: Differentiation basis; Translation invariant basis; Rotation; Integral

1. Definitions and notation

A mapping B defined on $\mathbb{R}^n$ is said to be a *differentiation basis* if for every $x \in \mathbb{R}^n$, $B(x)$ is a family of bounded measurable sets with positive measure and containing x, such that there exists a sequence $R_k \in B(x) (k \in \mathbb{N})$ with $\lim_{k\to\infty} \operatorname{diam} R_k = 0$.

For $f \in L(\mathbb{R}^n)$, the numbers

$$\overline{D}_B\left(\int f, x\right) = \overline{\lim_{\substack{R\in B(x)\\ \operatorname{diam} R\to 0}}} \frac{1}{|R|}\int_R f \quad \text{and} \quad \underline{D}_B\left(\int f, x\right) = \underline{\lim_{\substack{R\in B(x)\\ \operatorname{diam} R\to 0}}} \frac{1}{|R|}\int_R f$$

are called *the upper and the lower derivatives,* respectively, *of the integral of* f *at a point* x. If the upper and the lower derivative coincide, then their combined value is called the *derivative of* $\int f$ *at the point* x and we denote it by $D_B(\int f, x)$. We say that the *basis* B *differentiates* $\int f$ (or $\int f$ is differentiable with respect to B) if $\overline{D}_B(\int f, x) = \underline{D}_B(\int f, x) = f(x)$ for almost all $x \in \mathbb{R}^n$. If this is true for each f in the class of functions X, we say that B differentiates X.

Denote by $\mathbf{I} = \mathbf{I}(\mathbb{R}^n)$ the basis of intervals, i.e., the basis for which $\mathbf{I}(x)$ $(x \in \mathbb{R}^n)$ consists of all open n-dimensional intervals containing x. Note that the differentiation with respect to $\mathbf{I}$ is called *strong differentiation*.

For the basis B, by F_B we denote the class of all functions $f \in L(\mathbb{R}^n)$ whose integrals are differentiable with respect to B.

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The basis B is called *translation invariant* (briefly, TI-basis) if $B(x) = \{x + R : R \in B(0)\}$ for every $x \in \mathbb{R}^n$.
Denote by Γ_n the family of all rotations in the space $\mathbb{R}^n$.
Let B be the basis in $\mathbb{R}^n$ and $\gamma \in \Gamma_n$. The *γ-rotated basis B* is defined as follows:

$$B(\gamma)(x) = \{x + \gamma(R - x) : R \in B(x)\} \quad (x \in \mathbb{R}^n).$$

The set of two-dimensional rotations Γ_2 can be identified with the circumference $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, if to the rotation γ we put into correspondence the complex number $z(\gamma)$ from $\mathbb{T}$, the argument of which is equal to the value of the angle by which the rotation about the origin takes place in the positive direction under the action of γ.

The distance $d(\gamma, \sigma)$ between the points $\gamma, \sigma \in \Gamma_2$ is assumed to be equal to the length of the smallest arch of the circumference $\mathbb{T}$ connecting the points $z(\gamma)$ and $z(\sigma)$.

Let B and H be bases in $\mathbb{R}^n$ and $E \subset \Gamma_n$. Let us call E a *$W_{B,H}$-set* (*$W^+_{B,H}$-set*), if there exists a function $f \in L(\mathbb{R}^n)$ ($f \in L(\mathbb{R}^n)$, $f \geq 0$) such that: (1) $f \notin F_{B(\gamma)}$ for every $\gamma \in E$ and (2) $f \in F_{H(\gamma)}$ for every $\gamma \notin E$.

Let B and H be bases in $\mathbb{R}^n$ and $E \subset \Gamma_n$. Let us call E an *$R_{B,H}$-set* (*$R^+_{B,H}$-set*), if there exists a function $f \in L(\mathbb{R}^n)$ ($f \in L(\mathbb{R}^n)$, $f \geq 0$) such that: (1) $\overline{D}_{B(\gamma)}\left(\int f, x\right) = \infty$ almost everywhere for every $\gamma \in E$ and (2) $f \in F_{H(\gamma)}$ for every $\gamma \notin E$.

When $B = H$, we will use the terms W_B (W^+_B, R_B, R^+_B)-set.

Remark 1. It is clear that:

(1) each $W^+_{B,H}$ ($R^+_{B,H}$)-set is $W_{B,H}$ ($R_{B,H}$)-set;

(2) if $B \subset H$, then each W_B (W^+_B, R_B, R^+_B)-set is $W_{B,H}$ ($W^+_{B,H}$, $R_{B,H}$, $R^+_{B,H}$)-set.

The definitions of $R_{\mathbf{I}(\mathbb{R}^2)}$, $R^+_{\mathbf{I}(\mathbb{R}^2)}$ and $W_{\mathbf{I}(\mathbb{R}^2)}$-sets were introduced in [1,2] and [3], respectively.

2. Results

Singularities of an integral of a fixed summable function with respect to the collection of rotated bases $B(\gamma)$ were studied by various authors (see [1–9]). In particular, in [1] and [3], one can find the proof of the following results dealing with the topological structure of $R_{\mathbf{I}(\mathbb{R}^2)}$-sets and $W_{\mathbf{I}(\mathbb{R}^2)}$-sets, respectively.

Theorem A. *Each $R_{\mathbf{I}(\mathbb{R}^2)}$-set has G_δ type.*

Theorem B. *Each $W_{\mathbf{I}(\mathbb{R}^2)}$-set has $G_{\delta\sigma}$ type.*

The following generalizations of Theorems A and B are true.

Theorem 1. *For an arbitrary translation invariant basis B in $\mathbb{R}^2$, each W_B-set has $G_{\delta\sigma}$ type.*

Theorem 2. *For an arbitrary translation invariant basis B in $\mathbb{R}^2$, each R_B-set has G_δ type.*

Theorems 1 and 2 were announced in [10].
We will also prove the following result.

Theorem 3. *For arbitrary bases B and H in $\mathbb{R}^2$ not more than a countable union of $R_{B,H}$-sets ($R^+_{B,H}$-sets) is $W_{B,H}$-set ($W^+_{B,H}$-set).*

Proof of Theorem 1. Let $f \in L(\mathbb{R}^2)$. We have to prove that the set

$$W_B(f) = \{\gamma \in \Gamma_2 : f \notin F_{B(\gamma)}\}$$

is of $G_{\delta\sigma}$ type.
Without loss of generality, let us assume that f is finite everywhere and supp $f \subset (0, 1)^n$.

For the basis H, $x \in \mathbb{R}^2$ and $r > 0$ set

$$l_H(f)(x) = \varlimsup_{\substack{R \in H(x) \\ \operatorname{diam} R \to 0}} \left| \frac{1}{|R|} \int_R f - f(x) \right|,$$

$$l_H^r(f)(x) = \sup_{\substack{R \in H(x) \\ \operatorname{diam} R < r}} \left| \frac{1}{|R|} \int_R f - f(x) \right|.$$

For the numbers $\varepsilon > 0$, $\alpha \in (0, 1]$, $r > 0$ and $\beta \in (0, 1)$ we denote

$$W_B(f, \varepsilon, \alpha) = \{\gamma \in \Gamma_2 : |\{l_{B(\gamma)}(f) \ge \varepsilon\}| \ge \alpha\},$$
$$W_B^r(f, \varepsilon, \beta) = \{\gamma \in \Gamma_2 : |\{l_{B(\gamma)}^r(f) > \varepsilon\}| > \beta\}.$$

First, let us prove that $W_B^r(f, \varepsilon, \beta)$ is an open set for any $r > 0$, $\varepsilon > 0$ and $\beta \in (0, 1)$. Suppose $\gamma \in W_B^r(f, \varepsilon, \beta)$, i.e.,

$$|\{l_{B(\gamma)}^r(f) > \varepsilon\}| > \beta.$$

If $x \in \{l_{B(\gamma)}^r(f) > \varepsilon\}$, then there is $R_x \in B(\gamma)(x)$ with $\operatorname{diam} R_x < r$ such that

$$\left| \frac{1}{|R_x|} \int_{R_x} f - f(x) \right| > \varepsilon.$$

Taking into account absolute continuity of the Lebesgue integral, it is easy to check that performing small enough rotation of R_x around the point x, one derives the set R'_x for which

$$\left| \frac{1}{|R'_x|} \int_{R'_x} f - f(x) \right| > \varepsilon.$$

Therefore, for every $x \in \{l_{B(\gamma)}^r(f) > \varepsilon\}$, we can find $k_x \in \mathbb{N}$ such that

$$l_{B(\gamma')}^r(f)(x) > \varepsilon \quad \text{if } \operatorname{dist}(\gamma', \gamma) < 1/k_x.$$

For every $m \in \mathbb{N}$ by A_m we denote the set of all points from $\{l_{B(\gamma)}^r(f) > \varepsilon\}$ for which $k_x = m$. Obviously,

$$A_1 \subset A_2 \subset \cdots \quad \text{and} \quad \bigcup_{m \in \mathbb{N}} A_m = \{l_{B(\gamma)}^r(f) > \varepsilon\}.$$

Now, using the property of continuity of outer measure from below, we can find $m \in \mathbb{N}$ for which $|A_m|_* > \beta$. The last conclusion implies that

$$|\{l_{B(\gamma')}^r(f) > \varepsilon\}| > \beta \quad \text{if } \operatorname{dist}(\gamma', \gamma) < 1/m.$$

Consequently, $W_B^r(f, \varepsilon, \beta)$ is an open set.

Let us now prove that $W_B(f, \varepsilon, \alpha)$ is of G_δ type for any $\varepsilon > 0$ and $\delta \in (0, 1]$. Let us consider strictly increasing sequences of positive numbers (ε_k) and (α_k) such that $\varepsilon_k \to \varepsilon$ and $\alpha_k \to \alpha$. Taking into account openness of sets $W_B^r(f, \varepsilon, \beta)$, it is easy to see that for every $\gamma \in W_B(f, \varepsilon, \alpha)$ and $k \in \mathbb{N}$ there is a neighbourhood $V_{\gamma,k}$ of γ such that

$$|\{l_{B(\gamma')}^{1/k}(f) > \varepsilon_k\}| > \alpha_k \quad \text{if } \gamma' \in V_{\gamma,k}.$$

Denote

$$G_k = \bigcup_{\gamma \in W_B(f, \varepsilon, \alpha)} V_{\gamma,k} \quad (k \in \mathbb{N}).$$

Since $W_B(f, \varepsilon, \alpha) \subset G_k$ $(k \in \mathbb{N})$, we have $W_B(f, \varepsilon, \alpha) \subset \bigcap_{k \in \mathbb{N}} G_k$. On the other hand, if $\gamma \in \bigcap_{k \in \mathbb{N}} G_k$, then

$$|\{l_{B(\gamma)}(f) \ge \varepsilon\}| = \left| \bigcap_{k \in \mathbb{N}} \{l_{B(\gamma)}^{1/k}(f) > \varepsilon_k\} \right| \ge \lim_{k \to \infty} \alpha_k = \alpha.$$

Consequently, $\gamma \in W_B(f, \varepsilon, \alpha)$. Thus $W_B(f, \varepsilon, \alpha) \supset \bigcap_{k \in \mathbb{N}} G_k$. So we have proved that $W_B(f, \varepsilon, \alpha) = \bigcap_{k \in \mathbb{N}} G_k$, wherefrom follows the needed conclusion.

It is easy to check that

$$W_B(f) = \bigcup_{k\in\mathbb{N}} W_B(f, 1/k, 1/k),$$

wherefrom we conclude that $W_B(f)$ is of $G_{\delta\sigma}$ type. $\square$

Proof of Theorem 2. Let $f \in L(\mathbb{R}^2)$ and supp $f \subset (0,1)^2$. Let us prove that the set

$$R_B(f) = \left\{\gamma \in \Gamma_2 : \overline{D}_{B(\gamma)}\left(\int f, x\right) = \infty \text{ a.e. on } (0,1)^2\right\}$$

is of G_δ type. It is easy to check that this assertion implies the validity of the theorem.

For the basis H, $x \in \mathbb{R}^2$ and $r > 0$ set

$$N_H^r(f)(x) = \sup_{\substack{R\in H(x)\\ \operatorname{diam} R<r}} \frac{1}{|R|}\int_R f.$$

For the numbers $\varepsilon > 0$, $r > 0$ and $\beta \in (0,1)$, we denote

$$R_B^r(f,\varepsilon,\beta) = \{\gamma \in \Gamma_2 : |\{N_{B(\gamma)}^r(f) > \varepsilon\}| > \beta\}.$$

First, let us prove that $R_B^r(f,\varepsilon,\beta)$ is an open set for any $r > 0$, $\varepsilon > 0$ and $\beta \in (0,1)$. Suppose $\gamma \in R_B^r(f,\varepsilon,\beta)$, i.e.,

$$|\{N_{B(\gamma)}^r(f) > \varepsilon\}| > \beta.$$

If $x \in \{N_{B(\gamma)}^r(f) > \varepsilon\}$, then there is $R_x \in B(\gamma)(x)$ with diam $R_x < r$ such that

$$\frac{1}{|R_x|}\int_{R_x} f > \varepsilon.$$

Taking into account absolute continuity of the Lebesgue integral, it is easy to check that performing small enough rotation of R_x around the point x one derives the set R'_x for which

$$\frac{1}{|R'_x|}\int_{R'_x} f > \varepsilon.$$

Therefore, for every $x \in \{N_{B(\gamma)}^r(f) > \varepsilon\}$, we can find $k_x \in \mathbb{N}$ such that

$$N_{B(\gamma')}^r(f)(x) > \varepsilon \quad \text{if } \operatorname{dist}(\gamma',\gamma) < 1/k_x.$$

For every $m \in \mathbb{N}$, by A_m we denote the set of all points from $\{N_{B(\gamma)}^r(f) > \varepsilon\}$ for which $k_x = m$. Obviously,

$$A_1 \subset A_2 \subset \cdots \quad \text{and} \quad \bigcup_{m\in\mathbb{N}} A_m = \{N_{B(\gamma)}^r(f) > \varepsilon\}.$$

Using now the property of continuity of outer measure from below, we can find $m \in \mathbb{N}$ for which $|A_m|_* > \beta$. The last conclusion implies that

$$|\{N_{B(\gamma')}^r(f) > \varepsilon\}| > \beta \quad \text{if } \operatorname{dist}(\gamma',\gamma) < 1/m.$$

Consequently, $R_B^r(f,\varepsilon,\beta)$ is an open set.

Next, let us prove that $R_B(f)$ is of G_δ type. Taking into account openness of sets $R_B^r(f,\varepsilon,\beta)$, it is easy to see that for every $\gamma \in R_B(f)$ and $k \in \mathbb{N}$ there is a neighbourhood $V_{\gamma,k}$ of γ such that

$$|\{N_{B(\gamma')}^{1/k}(f) > k\}| > 1 - 1/k \quad \text{if } \gamma' \in V_{\gamma,k}.$$

Denote

$$G_k = \bigcup_{\gamma\in R_B(f)} V_{\gamma,k} \quad (k \in \mathbb{N}).$$

Since $R_B(f) \subset G_k$ $(k \in \mathbb{N})$, we have $R_B(f) \subset \bigcap_{k\in\mathbb{N}} G_k$. On the other hand, if $\gamma \in \bigcap_{k\in\mathbb{N}} G_k$, then

$$\left|\left\{\overline{D}_{B(\gamma)}\left(\int f, x\right) = \infty \text{ a.e. on } (0,1)^2\right\}\right| = \left|\bigcap_{k\in\mathbb{N}}\{N^{1/k}_{B(\gamma)}(f) > k\}\right| \geq \lim_{k\to\infty}(1-1/k) = 1.$$

Consequently, $\gamma \in R_B(f)$. Thus, $R_B(f) \supset \bigcap_{k\in\mathbb{N}} G_k$. So we have proved that $R_B(f) = \bigcap_{k\in\mathbb{N}} G_k$ from which we obtain the needed conclusion. $\square$

Proof of Theorem 3. Let $N \subset \mathbb{N}$ be a not more than a countable non-empty set and for each $k \in N$, E_k be an $R_{B,H}$-set ($R^+_{B,H}$-set). For every $k \in N$ let us consider a summable function f_k with two properties from the definition of $R_{B,H}$-set ($R^+_{B,H}$-set): (1) $\overline{D}_{B(\gamma)}\left(\int f_k, x\right) = \infty$ almost everywhere for every $\gamma \in E_k$ and (2) $f_k \in F_{H(\gamma)}$ for every $\gamma \notin E_k$. Let us consider also an arbitrary family of pairwise disjoint open squares Q_k $(k \in N)$.

Denote

$$g_k = \frac{f_k \chi_{Q_k}}{2^k \|f_k\|_L} \quad (k \in N),$$
$$f = \sum_{k\in N} g_k.$$

Then we have

$$\|f\|_L = \sum_{k\in N}\|g_k\|_L \leq \sum_{k\in N}\frac{1}{2^k} < \infty.$$

Consequently, f is a summable function.

Using the disjointness of squares Q_k, we find that for every $\gamma \in \Gamma_2$, $k \in N$ and $x \in Q_k$,

$$\overline{D}_{B(\gamma)}\left(\int f, x\right) = \overline{D}_{B(\gamma)}\left(\int g_k, x\right).$$

Therefore, for every $k \in N$ and $\gamma \in E_k$,

$$\overline{D}_{B(\gamma)}\left(\int f, x\right) = \infty \quad \text{for a.e. } x \in Q_k.$$

Thus,

$$f \notin F_{B(\gamma)} \quad \text{for every } \gamma \in \bigcup_{k\in N} E_k. \tag{1}$$

We take now an arbitrary $\gamma \notin \bigcup_{k\in N} E_k$. Then $g_k \in F_{H(\gamma)}$ for every $k \in N$. Consequently, using the disjointness of squares Q_k, we find that for every $k \in N$,

$$D_{H(\gamma)}\left(\int f, x\right) = D_{H(\gamma)}\left(\int g_k, x\right) = g_k(x) = f(x)$$

for a.e. $x \in Q_k$. Thus,

$$D_{H(\gamma)}\left(\int f, x\right) = f(x) \quad \text{for a.e. } x \in \bigcup_{k\in N} Q_k.$$

Now taking into account that $f(x) = 0$, for every $x \notin \bigcup_{k\in N} Q_k$, we write

$$D_{H(\gamma)}\left(\int f, x\right) = f(x) \quad \text{for a.e. } x \in \mathbb{R}^2. \tag{2}$$

(1) and (2) imply that $\bigcup_{k\in N} E_k$ is the $W_{B,H}$-set ($W^+_{B,H}$-set). $\square$

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Original article

On the well-posedness of the Cauchy problem for differential equations with distributed prehistory considering delay function perturbations

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Abstract

Theorems on the continuous dependence of the solution on perturbations of the initial data and the right-hand side of equation are proved. Under initial data we understand the collection of initial moment, of delay function and initial function. Perturbations of the right-hand side of equation are small in the integral sense.

Keywords: Well-posedness of the Cauchy problem; Differential equation with distributed delay

1. Formulation of main results

Let $I = [a, b]$ be a finite interval and $\mathbb{R}^n$ be the n-dimensional vector space of points $x = (x^1, \dots, x^n)^T$, where T denotes transposition. Suppose that $O \subset \mathbb{R}^n$ is an open set, and E_f is the set of functions $f : I \times O^2 \to \mathbb{R}^n$ satisfying the following conditions: for each fixed $(x_1, x_2) \in O^2$ the function $f(\cdot, x_1, x_2) : I \to \mathbb{R}^n$ is measurable; for each $f \in E_f$ and compact set $K \subset O$, there exist functions $m_{f,K}(t), L_{f,K}(t) \in L(I, \mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$, such that for almost all $t \in I$

$$|f(t, x_1, x_2)| \le m_{f,K}(t) \quad \forall (x_1, x_2) \in K^2,$$

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \le L_{f,K}(t) \sum_{i=1}^{2} |x_i - y_i|$$

$$\forall (x_1, x_2) \in K^2 \text{ and } \forall (y_1, y_2) \in K^2.$$

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We introduce a topology in E_f using the following base of neighborhoods of the origin

$$\Big\{V_{K,\delta} : K \subset O \ \text{ is a compact set and } \ \delta > 0 \ \text{ is an arbitrary number}\Big\},$$

where

$$V_{K,\delta} = \Big\{\delta f \in E_f : H(\delta f; K) \le \delta\Big\}$$

$$H(\delta f; K) = \sup\Big\{\Big|\int_{t'}^{t''} \delta f(t, x_1, x_2)dt\Big| : t', t'' \in I, x_i \in K, i = 1, 2\Big\}. \tag{1.1}$$

Let D be the set of continuous differentiable scalar functions (delay functions) $\tau(t)$, $t \in [a, \infty)$, satisfying the conditions:

$$\tau(t) < t,\ \dot\tau(t) > 0, \quad \inf\{\tau(a) : \tau \in D\} := \hat\tau > -\infty.$$

Let $C(I_1)$ be the space of continuous functions $\varphi(t) \in \mathbb{R}^n$, $t \in I_1 = [\hat\tau, b]$ equipped with the norm $\|\varphi\|_{I_1} = \sup\{|\varphi(t)| : t \in I_1\}$. By $\Phi = \{\varphi \in C(I_1) : \varphi(t) \in O, t \in I_1\}$ we denote the set of initial functions.

To each element $\mu = (t_0, \tau, \varphi, f) \in A = [a, b) \times D \times \Phi \times E_f$ we assign the differential equation with distributed prehistory on the interval $[\tau(t), t]$

$$\dot x(t) = \int_{\tau(t)}^{t} f(t, x(t), x(s))ds \tag{1.2}$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [\hat\tau, t_0]. \tag{1.3}$$

Definition 1.1. Let $\mu = (t_0, \tau, \varphi, f) \in A$. A function $x(t) = x(t; \mu) \in O$, $t \in [\hat\tau, t_1]$, $t_1 \in (t_0, b]$, is called a solution of Eq. (1.2) with the initial condition (1.3) or a solution corresponding to the element μ and defined on the interval $[\hat\tau, t_1]$, if it satisfies the condition (1.3), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies Eq. (1.2) almost everywhere on $[t_0, t_1]$.

To formulate the main results, we introduce the following sets:

$$W(K; \alpha) = \left\{\delta f \in E_f : \exists\, m_{\delta f,K}(t), L_{\delta f,K}(t) \in L(I, R_+), \int_I [m_{\delta f,K}(t) + L_{\delta f,K}(t)]dt \le \alpha\right\},$$

where $K \subset O$ is a compact set and $\alpha > 0$ is a fixed number depending on δf;

$$B(t_{00}; \delta) = \{t_0 \in I : \ |t_0 - t_{00}| < \delta\}, \qquad V(\tau_0; \delta) = \{\tau \in D : \ \|\tau - \tau_0\|_I < \delta\},$$
$$V_1(\varphi_0; \delta) = \{\varphi \in \Phi : \ \|\varphi - \varphi_0\|_{I_1} < \delta\},$$

where $t_{00} \in [a, b)$ is a fixed point, $\tau_0 \in D$ and $\varphi_0 \in \Phi$ are fixed functions, $\delta > 0$ is a fixed number.

Theorem 1.1. *Let $x_0(t)$ be the solution corresponding to $\mu_0 = (t_{00}, \tau_0, \varphi_0, f_0) \in A$ and defined on $[\hat\tau, t_{10}]$, $t_{10} < b$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following conditions hold:*

1.1. *there exist numbers $\delta_i > 0$, $i = 0, 1$, such that to each element*

$$\mu = (t_0, \tau, \varphi, f_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha) = B(t_{00}; \delta_0) \times V(\tau_0; \delta_0)$$
$$\times V_1(\varphi_0; \delta_0) \times \Big[f_0 + (W(K_1; \alpha) \cap V_{K_1,\delta_0})\Big]$$

corresponds solution $x(t; \mu)$ defined on the interval $[\hat\tau, t_{10} + \delta_1] \subset I_1$ and satisfying the condition $x(t; \mu) \in K_1$;

1.2. *for an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$:*

$$|x(t; \mu) - x(t; \mu_0)| \le \varepsilon \quad \forall t \in [\theta, t_{10} + \delta_1], \theta = \max\{t_0, t_{00}\};$$

1.3. *for an arbitrary $\varepsilon > 0$, there exists a number $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $\mu \in V(\mu_0; K_1, \delta_3, \alpha)$:*

$$\int_{\hat{\tau}}^{t_{10}+\delta_1} |x(t;\mu) - x(t;\mu_0)|dt \le \varepsilon.$$

Obviously, the solution $x(t;\mu_0)$ is the continuation of the solution $x_0(t)$.

In the space $E_\mu = \mathbb{R} \times D \times C(I_1) \times E_f$, we introduce the set of variations

$$\Im = \Big\{\delta\mu = (\delta t_0, \delta\tau, \delta\varphi, \delta f) \in E_\mu - \mu_0 : \ |\delta t_0| \le \beta, \|\delta\tau\|_I \le \beta,$$
$$\|\delta\varphi\|_{I_1} \le \beta, \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, |\lambda_i| \le \beta, i = \overline{1,k}\Big\},$$

where $\beta > 0$ is a fixed number and $\delta f_i \in E_f - f_0$, $i = \overline{1,k}$ are fixed functions.

Theorem 1.2. *Let $x_0(t)$ be the solution corresponding to $\mu_0 = (t_{00}, \tau_0, \varphi_0, f_0) \in A$ and defined on $[\hat{\tau}, t_{10}]$, $t_{i0} \in (a,b)$, $i = 0, 1$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set K_0. Then the following conditions hold:*

1.4. *there exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for an arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im$, we have $\mu_0 + \varepsilon\delta\mu \in A$ and the solution $x(t;\mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to this element. Moreover, $x(t;\mu_0 + \varepsilon\delta\mu) \in K_1$;*

1.5. *the following relations fulfilled:*

$$\lim_{\varepsilon\to 0}\Big[\sup\Big\{|x(t;\mu_0 + \varepsilon\delta\mu) - x(t;\mu_0)| : t \in [\theta, t_{10} + \delta_1]\Big\}\Big] = 0,$$
$$\lim_{\varepsilon\to 0}\int_{\hat{\tau}}^{t_{10}+\delta_1} |x(t;\mu_0 + \varepsilon\delta\mu) - x(t;\mu_0)|dt = 0$$

uniformly in $\delta\mu \in \Im$, where $\theta = \max\{t_{00}, t_{00} + \varepsilon\delta t_0\}$.

Theorem 1.2 is a simple corollary of Theorem 1.1.

Let $U_0 \subset \mathbb{R}^r$ be an open set and Ω be the set of measurable functions $u(t) \in U_0$, $t \in I$ satisfying the conditions: $clu(I)$ is a compact set in $\mathbb{R}^r$ and $clu(I) \subset U_0$.

To each element $\varrho = (t_0, \tau, \varphi, u) \in A_1 = [a, b) \times D \times \Phi \times \Omega$ we assign the controlled differential equation with distributed prehistory

$$\dot{x}(t) = \int_{\tau(t)}^{t} g(t, x(t), x(s), u(t))ds \tag{1.4}$$

with the initial condition (1.3). Here the function $g(t, x_1, x_2, u)$ is defined on $I \times O^2 \times U_0$ and satisfies the following conditions: for each fixed $(x_1, x_2, u) \in O^2 \times U_0$ the function $g(\cdot, x_1, x_2, u) : I \to \mathbb{R}^n$ is measurable; for each compact sets $K \subset O$ and $U \subset U_0$ there exist functions $m_{K,U}(t), L_{K,U}(t) \in L(I, R_+)$ such that for almost all $t \in I$

$$|g(t, x_1, x_2, u)| \le m_{K,U}(t) \quad \forall (x_1, x_2, u) \in K^2 \times U,$$
$$|g(t, x_1, x_2, u_1) - g(t, y_1, y_2, u_2)| \le L_{K,U}(t)\Big[\sum_{i=1}^{2}|x_i - y_i| + |u_1 - u_2|\Big]$$
$$\forall (x_1, x_2) \in K^2, \ \forall (y_1, y_2) \in K^2 \text{ and } (u_1, u_2) \in U^2.$$

Definition 1.2. Let $\varrho = (t_0, \tau, \varphi, u) \in A_1$. A function $x(t) = x(t;\varrho) \in O$, $t \in [\hat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is called a solution of Eq. (1.4) with the initial condition (1.3) or a solution corresponding to the element ϱ and defined on the interval $[\hat{\tau}, t_1]$, if it satisfies condition (1.3), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies Eq. (1.4) almost everywhere on $[t_0, t_1]$.

Theorem 1.3. *Let $x_0(t)$ be the solution corresponding to $\varrho_0 = (t_{00}, \tau_0, \varphi_0, u_0) \in A_1$ and defined on $[\hat{\tau}, t_{10}]$, $t_{10} < b$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following conditions hold:*

1.6. *there exist numbers $\delta_i > 0$, $i = 0, 1$ such that to each element*

$$\varrho = (t_0, \tau, \varphi, u) \in \hat{V}(\varrho_0; \delta_0) = B(t_{00}; \delta_0) \times V(\tau_0; \delta_0) \times V_1(\varphi_0; \delta_0) \times V_2(u_0; \delta_0)$$

corresponds solution $x(t; \varrho)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ and satisfying the condition $x(t; \varrho) \in K_1$, here $V_2(u_0; \delta_0) = \{u \in \Omega : \|u - u_0\|_I < \delta_0\}$;

1.7. *for an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $\varrho \in \hat{V}(\varrho_0; \delta_2)$:*

$$|x(t; \rho) - x(t; \rho_0)| \le \varepsilon \quad \forall t \in [\theta, t_{10} + \delta_1], \theta = \max\{t_0, t_{00}\};$$

1.8. *for an arbitrary $\varepsilon > 0$, there exists a number $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ such that the following inequality fulfilled for any $\varrho \in \hat{V}(\varrho_0; \delta_3)$:*

$$\int_{\hat{\tau}}^{t_{10}+\delta_1} |x(t; \varrho) - x(t; \varrho_0)| dt \le \varepsilon.$$

Some comments. In Theorem 1.1 perturbations of the right-hand side of Eq. (1.2) are small in the integral sense (see (1.1)). Theorems 1.1–1.3 and their like theorems play an important role in the theory of optimal control, in proving variation formulas of solution, in the sensitivity analysis of equations [1–7]. Theorem analogous to Theorem 1.1 without perturbations of constant delay are proved in [8]. Theorems on the continuous dependence of the solution for various classes of ordinary and functional differential equations for the case in which the perturbation of the right-hand side is small in the integral sense are given in [1,5,9–13,7,14].

2. Proof of Theorem 1.1

On the continuous dependence of solution for a class of functional differential equations. To each element $\mu \in A$ we assign the functional differential equation

$$\dot{y}(t) = \int_{\tau(t)}^{t} f(t, y(t), h(t_0, \varphi, y)(s)) ds \tag{2.1}$$

with the initial condition

$$y(t_0) = \varphi(t_0), \tag{2.2}$$

where $h : I \times \Phi \times C(I) \to C(I_1)$ is the operator given by the formula

$$h(t_0, \varphi, y)(t) = \begin{cases} \varphi(t) & \text{for } t \in [\hat{\tau}, t_0), \\ y(t) & \text{for } t \in [t_0, b]. \end{cases}$$

Definition 2.1. An absolutely continuous function $y(t) = y(t; \mu) \in O, t \in [r_1, r_2] \subset I$, is called a solution of Eq. (2.1) with the initial condition (2.2) or the solution corresponding to the element $\mu \in A$ and defined on $[r_1, r_2]$, if $t_0 \in [r_1, r_2]$, $y(t_0) = \varphi(t_0)$ and satisfies Eq. (2.1) almost everywhere on the interval $[r_1, r_2]$.

Remark 2.1. Let $y(t; \mu), t \in [r_1, r_2], \mu \in A$ be the solution of Eq. (2.1) with the initial condition (2.2). Then, as is easily seen, the function

$$x(t; \mu) = h(t_0, \varphi, y(\cdot; \mu))(t), \quad t \in [\hat{\tau}, r_2]$$

is the solution of Eq. (1.2) with the initial condition (1.3).

Theorem 2.1. *Let $y_0(t)$ be a solution corresponding to $\mu_0 \in A$ defined on $[r_1, r_2] \subset (a, b)$ and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \varphi_0(I_1) \cup y_0([r_1, r_2])$. Then the following conditions hold:*

2.1. *there exist numbers $\delta_i > 0$, $i = 0, 1$ such that a solution $y(t; \mu)$ defined on $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ corresponds to each element*

$$\mu = (t_0, \tau, \varphi, f_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha).$$

Moreover,

$$\varphi(t) \in K_1, \quad t \in I_1; \qquad y(t; \mu) \in K_1, \quad t \in [r_1 - \delta_1, r_2 + \delta_1],$$

for arbitrary $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$;

2.2. *for an arbitrary $\varepsilon > 0$ there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$*

$$|y(t; \mu) - y(t; \mu_0)| \le \varepsilon, \quad \forall t \in [r_1 - \delta_1, r_2 + \delta_1]. \tag{2.3}$$

Proof. Let $\varepsilon_0 > 0$ be so small that a closed ε_0-neighborhood of the set K_0:

$$K(\varepsilon_0) = \{x \in \mathbb{R}^n : \exists \hat{x} \in K_0 \; |x - \hat{x}| \le \varepsilon_0\}$$

lies in $int K_1$. There exist a compact set $Q : K_0^2(\varepsilon_0) \subset Q \subset K_1^2$ and a continuously differentiable function $\chi : \mathbb{R}^{2n} \to [0, 1]$ such that,

$$\chi(x_1, x_2) = \begin{cases} 1 & \text{for } (x_1, x_2) \in Q, \\ 0 & \text{for } (x_1, x_2) \notin K_1^2 \end{cases} \tag{2.4}$$

(see Assertion 3.2 in [1, p. 41]).

To each element $\mu \in A$, we assign the functional differential equation

$$\dot{z}(t) = \int_{\tau(t)}^{t} g(t, z(t), h(t_0, \varphi, z)(s)) ds \tag{2.5}$$

with the initial condition

$$z(t_0) = \varphi(t_0), \tag{2.6}$$

where $g = \chi f$.

The function $g(t, x_1, x_2)$ satisfies the conditions:

$$|g(t, x_1, x_2)| \le m_{f,K_1}(t), \quad \forall x_i \in \mathbb{R}^n, \; i = 1, 2, \tag{2.7}$$

for $\forall x_i', x_i'' \in \mathbb{R}^n$, $i = 1, 2$ and for almost all $t \in I$

$$|g(t, x_1', x_2') - g(t, x_1'', x_2'')| \le L_f(t) \sum_{i=1}^{2} |x_i' - x_i''|, \tag{2.8}$$

where

$$L_f(t) = L_{f,K_1}(t) + \alpha_1 m_{f,K_1}(t), \qquad \alpha_1 = \sup\left\{ \sum_{i=1}^{2} |\chi_{x_i}(x_1, x_2)| : x_i \in \mathbb{R}^n, \quad i = 1, 2 \right\} \tag{2.9}$$

(see [8]).

It is clear that if $f = f_0 + \delta f$ then

$$L_{f,K_1}(t) = L_{f_0,K_1}(t) + L_{\delta f,K_1}(t), \qquad m_{f,K_1}(t) = m_{f_0,K_1}(t) + m_{\delta f,K_1}(t). \tag{2.10}$$

The solution of Eq. (2.5) with the initial condition (2.6) depends on the parameter

$$\mu = (t_0, \tau, \varphi, f_0 + \delta f) \in A_0 = [a, b) \times D \times \Phi \times \Big(f_0 + W(K_1; \alpha)\Big) \subset E_\mu.$$

The topology in A_0 is induced from the vector space E_μ.

On the complete metric space $C(I)$ with the distance $d(y_1, y_2) = \|y_1 - y_2\|_I$ we introduce a family

$$F(\cdot;\mu) : C(I) \to C(I) \tag{2.11}$$

of mapping depending on the parameter $\mu \in A_0$ by the formula

$$\zeta(t) = \zeta(t; z, \mu) = \varphi(t_0) + \int_{t_0}^{t}\left[\int_{\tau(\xi)}^{\xi} g(\xi, z(\xi), h(t_0, \varphi, z)(s))ds\right]d\xi,$$

where $g = \chi(f_0 + \delta f)$.

Clearly, every fixed point $z(t;\mu)$, $t \in I$, of mapping (2.11) is a solution of Eq. (2.5) with the initial condition (2.6).

Define the kth iteration $F^k(z;\mu)$ by

$$\zeta_k(t) = \zeta_k(t; z, \mu) = \varphi(t_0) + \int_{t_0}^{t}\left[\int_{\tau(\xi)}^{\xi} g(\xi, \zeta_{k-1}(\xi), h(t_0, \varphi, \zeta_{k-1})(s))ds\right]d\xi,$$
$$k = 1, 2, \ldots,\quad \zeta_0(t) = z(t).$$

Now let us prove that for a sufficiently large k, the family of mappings $F^k(z;\mu)$ is uniformly contractive. For this purpose, we estimate the difference

$$\begin{aligned}|\zeta_k'(t) - \zeta_k''(t)| &= |\zeta_k(t; z', \mu) - \zeta_k(t; z'', \mu)| \le \int_a^t\left[\int_{\tau(\xi)}^{\xi} |g(\xi, \zeta_{k-1}'(\xi), h(t_0, \varphi, \zeta_{k-1}')(s))\right.\\ &\quad \left. - g(\xi, \zeta_{k-1}''(\xi), h(t_0, \varphi, \zeta_{k-1}'')(s))|ds\right]d\xi \le \int_a^t\left[\int_{\tau(\xi)}^{\xi} L_f(\xi)\Big(|\zeta_{k-1}'(\xi) - \zeta_{k-1}''(\xi)|\right.\\ &\quad \left. + |h(t_0, \varphi, \zeta_{k-1}')(s) - h(t_0, \varphi, \zeta_{k-1}'')(s)|\Big)ds\right]d\xi, \quad i = 1, 2, \ldots\end{aligned} \tag{2.12}$$

(see (2.8)), where a function $L_f(\xi)$ has the form (2.9) i.e.

$$L_f(\xi) = L_{f_0+\delta f, K_1}(\xi) + \alpha_1 m_{f_0+\delta f, K_1}(\xi) = L_{f_0, K_1}(\xi) + L_{\delta f, K_1}(\xi) + \alpha_1[m_{f_0, K_1}(\xi) + m_{\delta f, K_1}(\xi)] \tag{2.13}$$

(see (2.10)).

Here, it is assumed that $\zeta_0' = z'(t)$ and $\zeta_0'' = z''(t)$.

It follows from the definition of the operator $h(\cdot)$ that

$$h(t_0, \varphi, \zeta_{k-1}')(s) - h(t_0, \varphi, \zeta_{k-1}'')(s) = h(t_0, 0, \zeta_{k-1}' - \zeta_{k-1}'')(s).$$

Using the last equality from relation (2.12) it follows that

$$\begin{aligned}|\zeta_k'(t) - \zeta_k''(t)| &\le 2\int_a^t L_f(\xi)(\xi - \tau(\xi)) \max_{\theta\in[a,\xi]} |\zeta_{k-1}'(\theta) - \zeta_{k-1}''(\theta)|d\xi\\ &\le 2(b - \tau(a))\int_a^t L_f(\xi) \max_{\theta\in[a,\xi]} |\zeta_{k-1}'(\theta) - \zeta_{k-1}''(\theta)|d\xi.\end{aligned}$$

Furthermore,

$$\max_{\theta\in[a,\xi]} |\zeta_{k-1}'(\theta) - \zeta_{k-1}''(\theta)| \le 2(b - \tau(a))\int_a^{\xi} L_f(\xi_1) \max_{\theta\in[a,\xi_1]} |\zeta_{k-2}'(\theta) - \zeta_{k-2}''(\theta)|d\xi_1.$$

Therefore

$$|\zeta_k'(t) - \zeta_k''(t)| \le [2(b - \tau(a))]^2\int_a^t L_f(\xi_1)d\xi_1 \int_a^{\xi_1} L_f(\xi_2) \max_{\theta\in[a,\xi_2]} |\zeta_{k-2}'(\theta) - \zeta_{k-2}''(\theta)|d\xi_2.$$

By continuing this procedure, we obtain

$$|\zeta_k'(t) - \zeta_k''(t)| \le [2(b - \tau(a))]^k \alpha_k(t)\|z' - z''\|,$$

where

$$\alpha_k(t) = \int_a^t L_f(\xi_1)d\xi_1 \int_a^{\xi_1} L_f(\xi_2)d\xi_2 \ldots \int_a^{\xi_{k-1}} L_f(\xi_k)d\xi_k.$$

By induction, one can readily show that

$$\alpha_k(t) = \frac{1}{k!}\left(\int_a^t L_f(\xi)d\xi\right)^k.$$

Thus,

$$d(F^k(z';\mu), F^k(z'';\mu)) = \|\zeta_k' - \zeta_k''\|_I \le \frac{[2(b-\tau(a))]^k}{k!}\left(\int_a^b L_f(\xi)d\xi\right)^k \|z'-z''\|_I$$
$$= \hat{\alpha}_k\|z'-z''\|_I.$$

Let us prove the existence of a number $\alpha_2 > 0$ such that

$$\int_I L_f(t)dt \le \alpha_2, \quad \forall f \in f_0 + W(K_1;\alpha).$$

Indeed, by (2.13) we have

$$\int_I L_f(t)dt = \int_I (L_{f,K_1}(t) + \alpha_1 m_{f,K_1}(t))dt = \int_I [L_{f_0,K_1}(t) + L_{\delta f,K_1}(t) + \alpha_1(m_{f_0,K_1}(t)$$
$$+ m_{\delta f,K_1}(t))]dt \le \alpha(\alpha_1+1) + \int_I [L_{f_0,K_1}(t) + \alpha_1 m_{f_0,K_1}(t)]dt = \alpha_2.$$

Taking into account this estimate, we obtain $\hat{\alpha}_k \le [2(b-\tau(a))\alpha_2]^k/k!$. Consequently, there exists a positive integer k_1 such that $\hat{\alpha}_{k_1} < 1$. Therefore, the k_1st iteration of the family (2.11) is contracting. By the fixed point theorem for contraction mappings (see [1, p. 61], [15, p. 110]), the mapping (2.11) has a unique fixed point for each μ. Hence it follows that Eq. (2.5) with the initial condition (2.6) has a unique solution $z(t;\mu)$, $t \in I$.

Let us prove that the mapping $F^k(z(\cdot;\mu_0);\cdot) : A_0 \to C(I)$ is continuous at the point $\mu = \mu_0$ for an arbitrary $k = 1, 2, \ldots$. For this purpose, it suffices to show that if a sequence $\mu_i = (t_{0i}, \tau_i, \varphi_i, f_i) \in A_0$, $i = 1, 2, \ldots$, where $f_i = f_0 + \delta f_i$, converges to $\mu_0 = (t_{00}, \tau_0, \varphi_0, f_0)$, i.e. if

$$\lim_{i\to\infty}\Big(|t_{0i} - t_{00}| + \|\tau_i - \tau_0\|_I + \|\varphi_i - \varphi_0\|_{I_1} + H(\delta f_i; K_1)\Big) = 0$$

then

$$\lim_{i\to\infty} F^k(z(\cdot;\mu_0);\mu_i) = F^k(z(\cdot;\mu_0);\mu_0) = z(\cdot;\mu_0). \tag{2.14}$$

We prove relation (2.14) by induction. Let $k = 1$, then we have

$$|\zeta_1^i(t) - z_0(t)| \le |\varphi_i(t_{0i}) - \varphi_0(t_{00})| + \left|\int_{t_{0i}}^t \left[\int_{\tau_i(\xi)}^{\xi} g_i(\xi, z_0(\xi), h(t_{0i},\varphi_i,z_0)(s))ds\right]d\xi\right.$$
$$\left. - \int_{t_{00}}^t \left[\int_{\tau_0(\xi)}^{\xi} g_0(\xi, z_0(\xi), h(t_{00},\varphi_0,z_0)(s))ds\right]d\xi\right| \le \alpha_1^i + \alpha_2^i(t), \tag{2.15}$$

where

$$\zeta_1^i(t) = \zeta_1(t; z_0, \mu_i), \qquad z_0(t) = z(t;\mu_0), \qquad g_i = \chi f_i = g_0 + \delta g_i, \quad g_0 = \chi f_0,\ \delta g_i = \chi\delta f_i;$$
$$\alpha_1^i = |\varphi_i(t_{0i}) - \varphi_0(t_{00})| + \left|\int_{t_{0i}}^t \left[\int_{\tau_i(\xi)}^{\tau_0(\xi)} |g_i(\xi, z_0(\xi), h(t_{0i},\varphi_i,z_0)(s))|ds\right]d\xi\right|$$
$$+ \left|\int_{t_{00}}^{t_{0i}} \left[\int_{\tau_0(\xi)}^{\xi} |g_0(\xi, z_0(\xi), h(t_{00},\varphi_0,z_0)(s))|ds\right]d\xi\right|,$$

$$\alpha_2^i(t) = \left| \int_{t_{0i}}^{t} \left[\int_{\tau_0(\xi)}^{\xi} |g_i(\xi, z_0(\xi), h(t_{0i}, \varphi_i, z_0)(s)) - g_0(\xi, z_0(\xi), h(t_{00}, \varphi_0, z_0)(s))| ds \right] d\xi \right|.$$

According to (2.7) and (2.9) we have

$$\alpha_1^i \le |\varphi_i(t_{0i}) - \varphi_0(t_{00})| + \left| \int_{t_{0i}}^{t} \left[(\tau_0(\xi) - \tau_i(\xi)) m_{f_i, K_1}(\xi) \right] d\xi \right|$$
$$+ \left| \int_{t_{00}}^{t_{0i}} \left[(\xi - \tau_0(\xi)) m_{f_0, K_1}(\xi) \right] d\xi \right| \le |\varphi_i(t_{0i}) - \varphi_0(t_{00})|$$
$$+ \|\tau_0 - \tau_i\|_I \left[\alpha + \int_I m_{f_0, K_1}(t) dt \right] + (b - \tau_0(a)) \left| \int_{t_{0i}}^{t_{00}} m_{f_0, K_1}(t) dt \right|,$$

therefore,

$$\lim_{i \to \infty} \alpha_1^i = 0. \tag{2.16}$$

After elementary transformation we obtain

$$\alpha_2^i(t) \le \left| \int_{t_{0i}}^{t} \left[\int_{\tau_0(\xi)}^{\xi} |g_0(\xi, z_0(\xi), h(t_{0i}, \varphi_i, z_0)(s)) - g_0(\xi, z_0(\xi), h(t_{00}, \varphi_0, z_0)(s))| ds \right] d\xi \right|$$
$$+ \left| \int_{t_{0i}}^{t} \left[\int_{\tau_0(\xi)}^{\xi} |\delta g_i(\xi, z_0(\xi), h(t_{0i}, \varphi_i, z_0)(s)) - \delta g_i(\xi, z_0(\xi), h(t_{00}, \varphi_0, z_0)(s))| ds \right] d\xi \right|$$
$$+ \left| \int_{t_{0i}}^{t} \left[\int_{\tau_0(\xi)}^{\xi} \delta g_i(\xi, z_0(\xi), h(t_{00}, \varphi_0, z_0)(s)) ds \right] d\xi \right| \le \alpha_3^i + \alpha_4^i + \alpha_5^i(t),$$

where

$$\alpha_3^i = \int_I L_{f_0}(\xi) \left[\int_{\tau_0(\xi)}^{\xi} |h(t_{0i}, \varphi_i, z_0)(s) - h(t_{00}, \varphi_0, z_0)(s)| ds \right] d\xi,$$
$$\alpha_4^i = \int_I L_{\delta f_i}(\xi) \left[\int_{\tau_0(\xi)}^{\xi} |h(t_{0i}, \varphi_i, z_0)(s) - h(t_{00}, \varphi_0, z_0)(s)| ds \right] d\xi,$$
$$\alpha_5^i(t) = \max_{t', t' \in I} \left| \int_{t'}^{t'} \left[\int_{\tau_0(\xi)}^{\xi} \delta g_i(\xi, z_0(\xi), h(t_{00}, \varphi_0, z_0)(s)) ds \right] d\xi \right|.$$

Introduce notation

$$s_{1i} = \min(t_{0i}, t_{00}), \qquad s_{2i} = \max(t_{0i}, t_{00}).$$

It is easy to see that

$$\lim_{i \to \infty} (s_{2i} - s_{1i}) = 0.$$

Now we estimate α_3^i and α_4^i. We have

$$\alpha_3^i \le \beta_i \int_I L_{f_0}(t) dt,$$

where

$$\beta_i = \|\varphi_i - \varphi_0\|_{I_1} (b - \tau(a)) + \int_{s_{1i}}^{s_{2i}} |\varphi_i(s) - z_0(s)| ds;$$
$$\alpha_4^i \le \beta_i \int_I L_{\delta f}(t) dt \le \alpha(1 + \alpha_1) \beta_i.$$

It is clear that $\beta_i \to 0$.

Thus,

$$\lim_{i\to\infty} \alpha_3^i = \lim_{i\to\infty} \alpha_4^i = 0. \tag{2.17}$$

Obviously,

$$H(\delta g_i; K_1) = H(\chi \delta f_i; K_1) \le H(\delta f_i; K_1)$$

(see (2.4)). Since $H(\delta f_i; K_1) \to 0$ as $i \to \infty$, we have

$$\lim_{i\to\infty} H(\delta g_i; K_1) = 0.$$

This allows us to use Lemma 2 given in [8], which in turn, implies

$$\lim_{i\to\infty} \alpha_5^i(t) = 0 \tag{2.18}$$

uniformly in $t \in I$.

Conditions (2.17) and (2.18) yield

$$\lim_{i\to\infty} \alpha_2^i(t) = 0 \tag{2.19}$$

uniformly in $t \in I$.

Taking into account (2.16) and (2.19) we get

$$\|\zeta_1^i - z_0\|_I = 0$$

(see (2.15)).

Relation (2.14) is proved for $k = 1$.

Let (2.14) hold for a certain $k > 1$; we will prove it for $k + 1$. Elementary transformations yield:

$$|\zeta_{k+1}^i(t) - z_0(t)| \le |\varphi_i(t_{0i}) - \varphi_0(t_{00})| + \left| \int_{t_{0i}}^t \left[\int_{\tau_i(\xi)}^{\xi} g_i(\xi, \zeta_k^i(\xi), h(t_{0i}, \varphi_i, \zeta_k^i)(s)) ds \right] d\xi \right.$$
$$\left. - \int_{t_{00}}^t \left[\int_{\tau_0(\xi)}^{\xi} g_0(\xi, z_0(\xi), h(t_{00}, \varphi_0, z_0)(s)) ds \right] d\xi \right| \le \alpha_1^i + \alpha_2^i(t) + \alpha_{3k}^i(t),$$

where

$$\alpha_{3k}^i(t) = \left| \int_{t_{0i}}^t \left[\int_{\tau_0(\xi)}^{\xi} |g_i(\xi, \zeta_k^i(\xi), h(t_{0i}, \varphi_i, \zeta_k^i)(s)) - g_i(\xi, z_0(\xi), h(t_{0i}, \varphi_i, z_0)(s))| ds \right] d\xi \right|.$$

The quantities α_1^i and $\alpha_2^i(t)$ have been estimated in the preceding, and it remains to estimate α_{3k}^i. We have

$$\alpha_{3k}^i \le \left| \int_{t_{0i}}^t L_{f_i}(\xi) \left[\int_{\tau_0(\xi)}^{\xi} \Big(|\zeta_k^i(\xi) - z_0(\xi)| + |h(t_{0i}, 0, \zeta_k^i - z_0)(s)| \Big) ds \right] d\xi \right.$$
$$\le \|\zeta_k^i - z_0\|_I (1 + b - \tau_0(a)) \alpha_2.$$

Since

$$\lim_{i\to\infty} \|\zeta_k^i - z_0\|_I = 0$$

it follows that

$$\lim_{i\to\infty} \alpha_{4k}^i = 0. \tag{2.20}$$

According to (2.16), (2.19) and (2.20), we have

$$\lim_{i\to\infty} \|\zeta_{k+1}^i - z_0\|_I = 0.$$

Relation (2.14) is proved for every $k = 1, 2, \ldots$.

Let a number $\delta_1 > 0$ be so small that $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ and $|z(t; \mu_0) - z(r_1; \mu_0)| \le \varepsilon_0/2$ for $t \in [r_1 - \delta_1, r_1]$ and $|z(t; \mu_0) - z(r_2; \mu_0)| \le \varepsilon_0/2$ for $t \in [r_2, r_2 + \delta_1]$.

We can conclude from the uniqueness of the solution $z(t; \mu_0)$ that $z(t; \mu_0) = y_0(t)$ for $t \in [r_1, r_2]$. Taking into account the above inequalities, we have

$$(z(t; \mu_0), h(t_{00}, \varphi_0, z(\cdot; \mu_0))(s)) \in K^2(\varepsilon_0/2) \subset Q, \quad t \in [r_1 - \delta_1, r_2 + \delta_1],\ s \in [\tau_0(t), t].$$

Hence

$$\chi(z(t; \mu_0), h(t_{00}, \varphi_0, z(\cdot; \mu_0))(s)) = 1, \quad t \in [r_1 - \delta_1, r_2 + \delta_1],\ s \in [\tau_0(t), t]$$

and the function $z(t; \mu_0)$ satisfies the equation

$$\dot{y}(t) = \int_{\tau_0(t)}^{t} f_0(t, y(t),\ h(t_0, \varphi, y)(s))ds, \quad t \in [r_1 - \delta_1, r_2 + \delta_1]$$

and the initial condition

$$y(t_{00}) = \varphi_0(t_{00}).$$

Therefore,

$$y(t; \mu_0) = z(t; \mu_0), \quad t \in [r_1 - \delta_1, r_2 + \delta_1].$$

According to the fixed point theorem, for $\varepsilon_0/2$ there exists a number $\delta_0 \in (0, \varepsilon_0)$ such that a solution $z(t; \mu)$ satisfying the condition

$$|z(t; \mu) - z(t; \mu_0)| \le \varepsilon_0/2, \quad t \in I,$$

corresponds to each element $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$. Therefore, for $t \in [r_1 - \delta_1, r_2 + \delta_1]$

$$z(t; \mu) \in K(\varepsilon_0) \forall \mu \in V(\mu_0; K_1, \delta_0, \alpha).$$

Taking into account that $\varphi(t) \in K(\varepsilon_0)$, we see that for $t \in [r_1 - \delta_1, r_2 + \delta_1]$ and $s \in [\tau(t), t]$ this implies

$$\chi(z(t; \mu), h(t_0, \varphi, z(\cdot; \mu))(s)) = 1 \quad \forall \mu \in V(\mu_0; K_1, \delta_0, \alpha).$$

Hence the function $z(t; \mu)$ satisfies Eq. (2.1) and condition (2.2), i.e.

$$y(t; \mu) = z(t; \mu) \in K_1, \quad t \in [r_1 - \delta_1, r_2 + \delta_1],\ \mu \in V(\mu_0; K_1, \delta_0, \alpha). \tag{2.21}$$

The first part of Theorem 2.1 is proved. By the fixed point theorem, for an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that for each $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$,

$$|z(t; \mu) - z(t; \mu_0)| \le \varepsilon, \quad t \in I.$$

Whence using (2.21), we obtain (2.3). □

Proof of Theorem 1.1. In Theorem 2.1, let $r_1 = t_{00}$ and $r_2 = t_{10}$. Obviously, the solution $x_0(t)$ satisfies the following equation on the interval $[t_{00}, t_{10}]$:

$$\dot{y}(t) = \int_{\tau_0(t)}^{t} f_0(t, y(t), h(t_{00}, \varphi, y)(s))ds.$$

Therefore, in Theorem 2.1, as the solution $y_0(t)$ we can take the function $x_0(t)$, $t \in [t_{00}, t_{10}]$.

By Theorem 2.1, there exist numbers $\delta_i > 0$, $i = 0, 1$, and for an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0]$ such that the solution $y(t; \mu)$, $t \in [t_{00} - \delta_1, t_{10} + \delta_1]$, corresponds to each $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$. Moreover, the following conditions hold:

$$\begin{cases} \varphi(t) \in K_1, & t \in I_1;\ y(t; \mu) \in K_1, \\ |y(t : \mu) - y(t; \mu_0)| \le \varepsilon, & t \in [t_{00} - \delta_1, \quad t_{10} + \delta_1], \\ \mu \in V(\mu_0; K_1, \delta_2, \alpha). \end{cases} \tag{2.22}$$

For an arbitrary $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$, the function

$$x(t; \mu) = \begin{cases} \varphi(t), & t \in [\hat{\tau}, t_0), \\ y(t; \mu), & t \in [t_0, t_1 + \delta_1] \end{cases}$$

is the solution corresponding to μ. Moreover, if $t \in [\theta, t_{10} + \delta_1]$, then $x(t; \mu_0) = y(t; \mu_0)$ and $x(t; \mu) = y(t; \mu)$. Taking into account (2.22), we see that this implies 1.1 and 1.2. It is easy to note that for an arbitrary $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$, we have

$$\begin{aligned} \int_{\hat{\tau}}^{t_{10}+\delta_1} |x(t; \mu) - x(t; \mu_0)| dt &= \int_{\hat{\tau}}^{\theta_0} |\varphi(t) - \varphi_0(t)| dt + \int_{\theta_0}^{\theta} |x(t; \mu) - x(t; \mu_0)| dt \\ &+ \int_{\theta}^{t_{10}+\delta_1} |x(t; \mu) - x(t; \mu_0)| dt \le \|\varphi - \varphi_0\|_{I_1} (b - \hat{\tau}) + M |t_0 - t_{00}| \\ &+ \max_{t \in [\theta, t_{10}+\delta_1]} |x(t; \mu) - x(t; \mu_0)| (b - \hat{\tau}), \end{aligned}$$

where $\theta_0 = \min\{t_0, t_{00}\}$, $M = \sup\{|x' - x''| : x', x'' \in K_1\}$.

By 1.1 and 1.2, this inequality implies 1.3. □

3. Proof of Theorem 1.3

To each element $\varrho \in A_1$ we will set in correspondence the functional differential equation

$$\dot{y}(t) = \int_{\tau(t)}^{t} g(t, y(t), h(t_0, \varphi, y))(s), u(t) ds, \tag{3.1}$$

with the initial condition (2.2).

Theorem 3.1. *Let $y_0(t)$ be a solution corresponding to $\varrho_0 = (t_{00}, \tau_0, \varphi_0, u_0) \in A_1$ defined on $[r_1, r_2] \subset (a, b)$ and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \varphi_0(I_1) \cup y_0([r_1, r_2])$. Then the following conditions hold:*

3.1. *there exist numbers $\delta_i > 0$, $i = 0, 1$ such that to each element*

$$\varrho = (t_0, \tau, \varphi, u) \in \hat{V}(\varrho_0; \delta_0) = B(t_{00}; \delta_0) \times V(\tau_0; \delta_0) \times V_1(\varphi_0; \delta_0) \times V_2(u_0; \delta_0)$$

corresponds solution $y(t; \varrho)$ defined on the interval $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ and satisfying the condition $y(t; \varrho) \in K_1$;

3.2. *for an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $\varrho \in \hat{V}(\varrho_0; \delta_2)$:*

$$|y(t; \rho) - y(t; \rho_0)| \le \varepsilon \quad \forall t \in [r_1 - \delta_1, r_2 + \delta_1].$$

Proof. Rewrite Eq. (3.1) in the form

$$\dot{y}(t) = \int_{\tau(t)}^{t} [g_0(t, y(t), h(t_0, \varphi, y)(s)) + \delta g_u(t, y(t), h(t_0, \varphi, y)(s))] ds,$$

where

$$g_0(t, x_1, x_2) = g(t, x_1, x_2, u_0(t)) \in E_f,$$
$$\delta g_u(t, x_1, x_2) = g(t, x_1, x_2, u(t)) - g_0(t, x_1, x_2) \in E_f.$$

Let $\hat{\delta}_0 > 0$ be a number so small that $V_2(u_0; \hat{\delta}_0) \subset \Omega$. There exists a compact set $M \subset U_0$ such that any function from the neighborhood $V_2(u_0; \hat{\delta}_0)$ assumes its values in M.

Let $K \subset O$ be a compact set. There exists a function $L_K(t) \in L(I, \mathbb{R}_+)$ such that for almost all $t \in I$, the following inequality holds:

$$|g(t, x_1', x_2'', u') - g(t, x_1', x_2'', u'')| \le L_K(t) \left[\sum_{i=1}^{2} |x_i' - x_i''| + |u' - u''| \right]$$
$$\forall x_i', \ x_i'' \in K, \ i = 1, 2, \ u', u'' \in M.$$

Hence

$$|\delta g_u(t,x_1,x_2)| \le L_K(t)|u(t)-u_0(t)| \le \hat{\delta}_0 L_K(t) \quad \forall x_i \in K,\, i=1,2,\ \forall u \in V_2(u_0;\hat{\delta}_0),$$
$$|\delta g_u(t,x_1',x_2') - \delta g_u(t,x_1'',x_2'')| \le 2L_K(t)\sum_{i=1}^{2}|x_i'-x_i''|, \quad \forall x_i', x_i'' \in K,\ i=1,2.$$

It is easy to see that the following inclusions hold for $\delta \in (0,\hat{\delta}_0]$:

$$\{\delta g_u(t,x_1,x_2) : u \in V_2(u_0;\delta)\} \subset W(K;\alpha),$$
$$\{\delta g_u(t,x_1,x_2) : u \in V_2(u_0;\delta)\} \subset V_{K,\hat{\delta}_1},$$

where

$$\alpha = (2+\hat{\delta}_0)\int_I L_K(t)dt, \qquad \hat{\delta}_1 = \delta\int_I L_K(t)dt.$$

Now we can use Theorem 2.1, which, is turn, proves Theorem 3.1. □

Proof of Theorem 1.3. In Theorem 3.1, let $r_1 = t_{00}$ and $r_2 = t_{10}$. Obviously, the solution $x_0(t)$ satisfies the following equation on the interval $[t_{00}, t_{10}]$:

$$\dot{y}(t) = \int_{\tau_0(t)}^{t} g(t, y(t), h(t_0,\varphi_0,y)(s), u_0(t))ds.$$

Therefore, in Theorem 3.1, as the solution $y_0(t)$, we can take the function $x_0(t)$, $t \in [t_{00}, t_{10}]$. After that, the proof of the theorem completely coincides with that of Theorem 1.1; for this purpose, it suffices to replace the element μ by the element ϱ and the set $V(\mu_0; K_1, \delta_0, \alpha)$ by the set $\hat{V}(\varrho_0;\delta_0)$ everywhere. □

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Original article

On the essential unboundedness in measure of sequences of superlinear operators in classes $L\phi(L)$

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Abstract

We establish a general theorem for a wide class of sequences of superlinear operators $\{T_n : L^1(I^2) \to L^0(I^2), n = 1, 2, \ldots\}$ about existence of a function g from a certain class $L\phi(L)$ such that the sequence of functions $\{T_n(g), n = 1, 2, \ldots\}$ is essentially unbounded in measure on I^2. This theorem implies several results about divergence of sequences of classical operators.

Keywords: Essential divergence in measure; Orthogonal Fourier series; Lebesgue functions; Superlinear operators

1. Introduction

We start with the following definitions.

Let μ_N, $N = 1, 2, \ldots$, denote Lebesgue measure in the Euclidean space R^N and I denote the interval [0, 1]. For a number $h \in (0, 1)$, by I_h we will denote the interval $[0, 1 - h]$.

If F is a Lebesgue measurable set in R^N, with $0 < \mu_N F < \infty$, then let $L^0(F)$ denote the set of all Lebesgue measurable functions on F that are a.e. finite on F.

A set Q of Lebesgue measurable functions on F is called bounded in measure on F if for any $\varepsilon > 0$ there is a constant $R > 0$ such that $\mu_2\{(x, y) \in F : |f(x, y)| \geq R\} \leq \varepsilon$ for any function $f \in Q$.

A sequence $\{f_n(x, y) : n = 1, 2, \ldots\}$ of Lebesgue measurable functions on F is called bounded in measure on F if the set Q consisting of the members of the sequence $\{f_n(x, y) : n = 1, 2, \ldots\}$ is bounded in measure.

A sequence $\{g_n(x, y) : n = 1, 2, \ldots\}$ of Lebesgue measurable functions on F is called essentially unbounded in measure on F if for any Lebesgue measurable set $E \subset F$, $\mu_2 E > 0$, the sequence $\{g_n(x, y) : n = 1, 2, \ldots\}$ is not bounded in measure on E.

An operator $T : L^1(I^2) \to L^0(I^2)$ is called superlinear [1, p. 131] if for any $f_0 \in L^1(I^2)$ there is a linear operator $G_{f_0} : L^1(I^2) \to L^0(I^2)$ such that

$$G_{f_0}(f_0)(x, y) = T(f_0)(x, y) \tag{1}$$

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and

$$|G_{f_0}(f)(x,y)| \le |T(f)(x,y)| \quad \text{for any } f \in L^1(I^2) \tag{2}$$

and for almost all points (x, y) in I^2.

We mention the following properties of superlinear operators [1, p. 131]: for all functions $f, g \in L^1(I^2)$ and any real number k we have

$$|T(f+g)(x,y)| \le |T(f)(x,y)| + |T(g)(x,y)|,$$
$$|T(kf)(x,y)| = |k||T(f)(x,y)|,$$
$$|T(f+g)(x,y)| \ge |T(f)(x,y)| - |T(g)(x,y)|$$

for almost all points $(x, y) \in I^2$.

A superlinear operator $T : L^1(I^2) \to L^0(I^2)$ is said to be bounded in measure on I^2 if the set

$$Q := \{T(f) : \|f\|_{L^1} \le 1\}$$

is bounded in measure on I^2.

For each pair of numbers $(\theta, \eta) \in I_h^2$ and a number $h \in (0, 1)$ introduce the function of two variables (x, y) defined on I^2 by

$$\delta_{\theta,\eta,h}(x,y) := \begin{cases} h^{-2}, & \text{if } (x,y) \in [\theta, \theta+h] \times [\eta, \eta+h]; \\ 0, & \text{otherwise on } I^2. \end{cases} \tag{3}$$

The kernel of a superlinear operator $T : L^1(I^2) \to L^0(I^2)$ is the function

$$K(x,y,\theta,\eta) := \lim_{h\to 0} T(\delta_{\theta,\eta,h}(.,.))(x,y), \quad (x,y,\theta,\eta) \in I^4,$$

provided the limit exists for a.e. $(x, y, \theta, \eta) \in I^4$.

The system of Rademacher functions $\{r_n(x)\}_{n=0}^{\infty}$ on $[0, 1)$ is defined as follows

$$r_0(x) := \begin{cases} 1, & \text{if } 0 \le x < \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} \le x < 1 \end{cases} \tag{4}$$

and let $r_0(x)$ be continued to $(-\infty, \infty)$ with period 1. For $n \ge 1$ define

$$r_n(x) := r_0(2^n x). \tag{5}$$

Definition 1 *([2])*. Let (X, Σ, ν) be σ-finite measure space, $E \in \Sigma$ and $\nu(E) > 0$. Let also a sequence of measurable real-valued functions $\{f_n(x)\}_{n=1}^{\infty}$ be defined and a.e. finite on E. Then we say that the sequence $\{f_n(x)\}_{n=1}^{\infty}$ is essentially divergent in measure on E if for every $E_1 \subset E$ with $E_1 \in \Sigma$ and $\nu(E_1) > 0$, the sequence is divergent in measure (that is, does not converge in measure to an a.e. finite and measurable function) on E_1.

Definition 2 *([2])*. Let $\{\varphi_n(x)\}_{n=1}^{\infty}$ be a complete orthonormal system on $I := [0, 1]$ such that $\varphi_1(x) = 1$ on I; each function $\varphi_n(x)$ is a bounded function on I; there exists an integer $N > 1$ such that for every positive integer n there exists a number $k(n)$ such that $\varphi_n(Nx) = \varphi_{k(n)}(x)$ and for any $1 \le n_1 < n_2$ we have $k(n_1) < k(n_2)$. Then we say that the system $\{\varphi_n(x)\}_{n=1}^{\infty}$ is a system of type T.

Note that the trigonometric system (contracted on I) is a system of type T (with arbitrary integer $N \ge 2$). The Walsh system in Paley's numeration also is a system of type T with $N = 2^l$ where l is an arbitrary positive integer.

A.N. Kolmogorov [3, p. 267] proved that all trigonometric Fourier series converge in measure on $[0, 2\pi]$. S.V. Konyagin [4] and the author of this paper [5] constructed a double trigonometric Fourier series that diverges in measure by squares on $[0, 2\pi]^2$.

Later we proved [6] the following.

Theorem 1 (*R. Getsadze*)**.** *Let $\{\varphi_k(x)\}_{k=1}^{\infty}$ be an arbitrary uniformly bounded orthonormal system (ONS) on I. Then there exists an integrable function on I^2 whose Fourier series with respect to the product system $\{\varphi_k(x)\varphi_l(y)\}_{k,l=1}^{\infty}$ diverges in measure by squares on I^2.*

The following theorem was proved in [2] (see p. 27).

Theorem 2 (*M.I. Dyachenko; K.S. Kazaryan, P. Sifuéntes*)**.** *Let $\{\varphi_m(x)\}_{m=1}^{\infty}$ be a uniformly bounded ONS on I that is a system of type* T. *Suppose that there exists a function $g_0 \in L(I^2)$ such that the Fourier series of g_0 with respect to the product system $\{\varphi_m(x)\varphi_n(y)\}_{m,n=1}^{\infty}$ unboundedly diverges in measure by squares on I^2. Then there exists a function $f_0 \in L(I^2)$ such that the Fourier series of f_0 with respect to the product system $\{\varphi_m(x)\varphi_n(y)\}_{m,n=1}^{\infty}$ essentially diverges in measure by squares on I^2.*

From the last two theorems it follows that there exists a function $g \in L([0, 2\pi]^2)$ such that it is double trigonometric Fourier series essentially diverges in measure by squares on $[0, 2\pi]^2$. The theorems imply also similar results for the double Walsh–Paley system on I^2.

On the other hand, using uniform weak $(1, 1)$ type property for the sequence of partial sums of one-dimensional Fourier trigonometric series and using standard method of iteration we can conclude that the double Fourier trigonometric series of functions from the class $L\ln^+ L([0, 2\pi]^2)$ converge in measure by rectangles. Similar statement is valid for the double Walsh–Paley system. It is natural to ask the following question: what is the "exact statement" in the sense of $L\phi(L)$ classes of essential divergence in these cases. In this paper we will prove the following general theorem that will imply the answers on this question.

Theorem 3. *Let $\phi(u)$ be a nonnegative, continuous and nondecreasing function on $[0, \infty)$ such that $u\phi(u)$ is a convex function on $[0, \infty)$.*

Let $\{T_n : L^1(I^2) \to L^0(I^2),\ n = 1, 2, \ldots\}$ be a sequence of superlinear operators that are bounded in measure on I^2. Suppose also that for every $f \in L^2(I^2)$ the sequence $\{T_n(f)(x, y),\ n = 1, 2, \ldots\}$ is bounded in measure on I^2 and let $K_n(x, y, \theta, \eta)$ be the kernel for T_n, that satisfies the following condition

$$\|K_n(x, y, \theta, \eta)\|_\infty < \infty, \tag{6}$$

where the norm is taken with respect to four variables and may depend on n.

Suppose that for any Lebesgue measurable set E, $E \subset I^2$, $\mu_2 E > 0$ and for each integer $n > n_0(E)$ there exist: positive numbers h_n, $\xi_n(E)$, $\epsilon_n(E)$ and a Lebesgue measurable set E_n, $E_n \subset E$, $\mu_2 E_n \geq \gamma_1 \mu_2 E$ such that: For each set $F \subset E_n$, whose Lebesgue measure $\mu_2 F \geq \frac{\gamma_1}{6}\mu_2 E$, there exists a positive number $\lambda_n(F)$ with the property

$$\mu_4\{(x, y, \theta, \eta) \in F \times I^2 : |K_n(x, y, \theta, \eta)| \geq \lambda_n(F)\} \geq \frac{\xi_n(E)}{\lambda_n(F)} > 0, \tag{7}$$

$$\lim_{n\to\infty} \xi_n(E) = \infty \quad \text{and} \quad \lim_{n\to\infty} \epsilon_n(E) = 0, \tag{8}$$

$$\phi(h_n^{-2}) \leq \epsilon_n(E)\xi_n(E), \tag{9}$$

$$\mu_4\{(x, y, \theta, \eta) \in E \times I_{t_n}^2 : |T_n(\delta_{\theta,\eta,h_n})(x, y) - K_n(x, y, \theta, \eta)| > 1\} \leq \frac{\xi_n(E)}{20\|K_n(x, y, \theta, \eta)\|_\infty} \tag{10}$$

and

$$h_n \leq t_n,$$

where

$$t_n := \frac{\xi_n(E)}{50\|K_n(x, y, \theta, \eta)\|_\infty} \tag{11}$$

and γ_1 is a positive constant, independent of n, the set E and (x, y).

Then there exists a function $g \in L^1(I^2)$ *such that*

$$\int_{I^2} |g(x, y)|\phi(|g(x, y)|)dxdy < \infty,$$

and the sequence of functions $\{T_n(g)(x, y), n = 1, 2, \ldots\}$, *is essentially unbounded in measure on* I^2.

Using Theorem 3 we can prove the following two theorems.

Theorem 4. *Let* $\phi(u)$ *be a nonnegative, continuous and nondecreasing function on* $[0, \infty)$ *such that* $u\phi(u)$ *is a convex function on* $[0, \infty)$ *and*

$$\phi(u) = o(\ln u) \quad (u \to \infty). \tag{12}$$

Then there exists a function $g_1 \in L^1(I^2)$ *such that*

$$\int_{I^2} |g_1(x, y)|\phi(|g_1(x, y)|)dxdy < \infty,$$

and the sequence of the square partial sums of the double Fourier–Walsh–Paley series of g_1 *is essentially unbounded in measure on* I^2.

Theorem 5. *Let* $\phi(u)$ *be a nonnegative, continuous and nondecreasing function on* $[0, \infty)$ *such that* $u\phi(u)$ *is a convex function on* $[0, \infty)$ *and*

$$\phi(u) = o(\ln u) \quad (u \to \infty).$$

Then there exists a function $g_2 \in L^1(I^2)$ *such that*

$$\int_{I^2} |g_2(x, y)|\phi(|g_2(x, y)|)dxdy < \infty,$$

and the sequence of the square partial sums of the double trigonometric Fourier series of g_2 *is essentially unbounded in measure on* I^2.

We will give in this paper the proofs of Theorems 3 and 4. The proof of Theorem 5 is similar to the proof of Theorem 4.

Unboundedness in measure of sequences of superlinear operators was studied earlier by us in [7].

2. Proof of Theorem 3

Set

$$\psi(u) := u\phi(u).$$

It is clear that (see (7))

$$\sup_{F:\, F \subset E_n, \mu_2 F \geq \frac{\gamma_1}{6}\mu_2 E} \lambda_n(F) \leq \|K_n(x, y, \theta, \eta)\|_\infty. \tag{13}$$

We will prove the following.

Lemma 1. *Under the conditions of Theorem* 3, *for an arbitrary Lebesgue measurable set* $E \subset I^2$, $\mu_2 E > 0$ *and a number* $n > n_0(E)$ *there exists a function* $\Psi_n(x, y) = \Psi_n(x, y; E) \in L^\infty(I^2)$, $\|\Psi_n\|_{L^1(I^2)} \leq 1$, *such that*

$$\int_{I^2} \psi(\Psi_n(x, y))dxdy \leq \epsilon_n(E)\xi_n(E) \tag{14}$$

and

$$\mu_2\left\{(x, y) \in E :\ |T_n(\Psi_n)(x, y)| \geq \frac{9\xi_n(E)}{20}\right\} \geq \frac{\gamma_1}{36}\mu_2 E, \tag{15}$$

where γ_1 *is a positive constant given in Theorem* 3 *and is independent of* n, *the set* E *and* (x, y).

Introduce the sets

$$P_n(F) := \{(x, y, \theta, \eta) \in F \times I^2 : \ |K_n(x, y, \theta, \eta)| \geq \lambda_n(F)\} \tag{16}$$

and

$$Q_n := \{(x, y, \theta, \eta) \in E \times I_{t_n}^2 : \ |T_n(\delta_{\theta,\eta,h_n})(x, y) - K_n(x, y, \theta, \eta)| > 1\}. \tag{17}$$

We shall show that for each $n > r_0(E)$ ($r_0(E)$ is a positive constant that may depend on E) there exist a positive integer $p(n)$ and the following finite sequences: a sequence of disjoint measurable sets $\{B_i^{(n)}\}_{i=1}^{p(n)}$, $B_i^{(n)} \subset E_n$, $i = 1, 2, \ldots, p(n)$; a sequence of positive numbers $\{\lambda_i^{(n)}\}_{i=1}^{p(n)}$ and a sequence of pairs of numbers $\{(\theta_i^{(n)}, \eta_i^{(n)})\}_{i=1}^{p(n)}$, $(\theta_i^{(n)}, \eta_i^{(n)}) \in I_{t_n}^2$, $i = 1, 2, \ldots, p(n)$, such that

$$|T_n(\delta_{\theta_i^{(n)},\eta_i^{(n)},h_n})(x, y) - K_n(x, y, \theta_i^{(n)}, \eta_i^{(n)})| \leq 1 \quad (x, y) \in B_i^{(n)} \ i = 1, 2, \ldots, p(n), \tag{18}$$

$$\mu_2\{\cup_{i=1}^{p(n)} B_i^{(n)}\} \geq \frac{\gamma_1}{6}\mu_2 E > 0, \tag{19}$$

$$\mu_2\{B_i^{(n)}\} \geq \frac{\xi_n(E)}{\lambda_i^{(n)}} \quad \text{for all } i = 1, 2, \ldots, p(n), \tag{20}$$

$$|K_n(x, y, \theta_i^{(n)}, \eta_i^{(n)})| \geq \frac{9}{10}\lambda_i^{(n)} \quad \text{for all } (x, y) \in B_i^{(n)}, \ i = 1, 2, \ldots, p(n). \tag{21}$$

Indeed, introduce the set

$$A_1^{(n)} := P_n(E_n) \cap ((E \times I_{t_n}^2) \setminus Q_n). \tag{22}$$

It is clear that (see (16), (17))

$$\begin{aligned} P_n(E_n) &= P_n(E_n) \bigcap \ (E \times I^2) \\ &= P_n(E_n) \bigcap \ \left\{[(E \times I^2) \setminus Q_n] \bigcup Q_n\right\} \\ &= \left\{P_n(E_n) \bigcap \ [(E \times I^2) \setminus Q_n]\right\} \bigcup \ \left\{P_n(E_n) \bigcap Q_n\right\}. \end{aligned}$$

Now it follows that (see (16), (17), (7), (10))

$$\begin{aligned} \frac{\xi_n(E)}{\lambda_n(E_n)} &\leq \mu_4\{P_n(E_n)\} = \mu_4\left\{P_n(E_n) \bigcap \ [(E \times I^2) \setminus Q_n]\right\} + \mu_4\left\{P_n(E_n) \bigcap Q_n\right\} \\ &\leq \mu_4\left\{P_n(E_n) \bigcap \ [(E \times I^2) \setminus Q_n]\right\} + \frac{\xi_n(E)}{20\|K_n(x, y, \theta, \eta)\|_\infty} \end{aligned}$$

and, consequently (see (13)),

$$\mu_4\left\{P_n(E_n) \bigcap \ [(E \times I^2) \setminus Q_n]\right\} \geq \frac{\xi_n}{\lambda_n(E_n)} - \frac{\xi_n}{20\lambda_n(E_n)}. \tag{23}$$

We note that

$$P_n(E_n) \bigcap \ [(E \times I^2) \setminus Q_n] \subset \left\{P_n(E_n) \bigcap \ [(E \times I_{t_n}^2) \setminus Q_n]\right\} \bigcup \left\{E \times (I^2 \setminus I_{t_n}^2)\right\},$$

that implies the following inequalities (see (11), (22), (23))

$$\begin{aligned} \frac{\xi_n}{\lambda_n(E_n)} - \frac{\xi_n}{20\lambda_n(E_n)} &\leq \mu_4\left\{P_n(E_n) \bigcap \ [(E \times I^2) \setminus Q_n]\right\} \\ &\leq \mu_4\left\{P_n(E_n) \bigcap \ [(E \times I_{t_n}^2) \setminus Q_n]\right\} + \mu_4\left\{E \times (I^2 \setminus I_{t_n}^2)\right\} \\ &\leq \mu_4\{A_1^{(n)}\} + 2t_n \leq \mu_4\{A_1^{(n)}\} + \frac{2\xi_n}{50\lambda_n(E_n)}. \end{aligned}$$

Now it is obvious that (see (22))

$$\mu_4\{A_1^{(n)}\} = \int_{E_n}\int_{I_{t_n}^2} \chi_{A_1^{(n)}}(x, y, \theta, \eta)dxdyd\theta d\eta \geq \frac{9\xi_n}{10\lambda_n(E_n)},$$

where $\chi_{A_1^{(n)}}(x, y, \theta, \eta)$ is the characteristic function of the set $A_1^{(n)}$.

Using Fubini's theorem we conclude that there exists a pair of numbers

$$(\theta_1^{(n)}, \eta_1^{(n)}) \in I_{t_n}^2$$

such that

$$\mu_2\{(x, y) \in E_n : (x, y, \theta_1^{(n)}, \eta_1^{(n)}) \in A_1^{(n)}\} \geq \frac{9\xi_n}{10\lambda_n(E_n)}.$$

Now we let

$$B_1^{(n)} := \{(x, y) \in E_n : (x, y, \theta_1^{(n)}, \eta_1^{(n)}) \in A_1^{(n)}\}$$

and

$$\lambda_1^{(n)} := \frac{10\lambda_n(E_n)}{9}.$$

From (16) to (22) we see that the first step in the construction is completed.

We now assume that the pth step of the construction is complete. If it happens that

$$\mu_2\{\cup_{i=1}^p B_i^{(n)}\} \geq \frac{\gamma_1}{6}\mu_2(E) > 0,$$

then the construction is complete.

Suppose, in the contrary, that

$$\mu_2\{\cup_{i=1}^p B_i^{(n)}\} < \frac{\gamma_1}{6}\mu_2(E).$$

Then we let

$$F := E_n \setminus \cup_{i=1}^p B_i^{(n)}.$$

According to one of the conditions of Theorem 3 we have $\mu_2 E_n \geq \gamma_1\mu_2(E)$. Consequently, we have

$$\mu_2\{E_n \setminus \cup_{i=1}^p B_i^{(n)}\} \geq \frac{\gamma_1}{6}\mu_2(E) > 0.$$

Introduce the set

$$A_{p+1}^{(n)} := P_n(E_n \setminus \cup_{i=1}^p B_i^{(n)}) \cap ((E \times I_{t_n}^2) \setminus Q_n).$$

It is clear that (see (16), (13), (17), (11))

$$\int_{E_n\setminus\cup_{i=1}^p B_i^{(n)}}\int_{I_{t_n}^2} \chi_{A_{p+1}^{(n)}}(x, y, \theta, \eta)dxdyd\theta d\eta \geq \frac{9\xi_n}{10\lambda_n(E_n \setminus \cup_{i=1}^p B_i^{(n)})}$$

where $\chi_{A_{p+1}^{(n)}}(x, y, \theta, \eta)$ is the characteristic function of the set $A_{p+1}^{(n)}$.

Using Fubini's theorem we conclude that there exists a pair of numbers $(\theta_{p+1}^{(n)}, \eta_{p+1}^{(n)}) \in I_{t_n}^2$ such that

$$\mu_2\{(x, y) \in E_n \setminus \cup_{i=1}^p B_i^{(n)} : (x, y, \theta_{p+1}^{(n)}, \eta_{p+1}^{(n)}) \in A_{p+1}^{(n)}\} \geq \frac{9\xi_n}{10\lambda_n(E_n \setminus \cup_{i=1}^p B_i^{(n)})}. \tag{24}$$

Now we let

$$B_{p+1}^{(n)} := \{(x, y) \in E_n \setminus \cup_{i=1}^p B_i^{(n)} : (x, y, \theta_{p+1}^{(n)}, \eta_{p+1}^{(n)}) \in A_{p+1}^{(n)}\} \tag{25}$$

and

$$\lambda_{p+1}^{(n)} := \frac{10\lambda_n(E_n \setminus \cup_{i=1}^{p} B_i^{(n)})}{9}. \tag{26}$$

From (24), (25), (26), (16), (17), (18), (21), (20) we see that the $p+1$-st step in the construction is complete.

It follows now from the construction (see (13), (20)) that after the pth step we have

$$\mu_2(\cup_{i=1}^{p} B_i^{(n)}) = \sum_{i=1}^{p} \mu_2 B_i^{(n)} \geq \frac{9p\xi_n}{10\|K_n(x,y,\theta,\eta)\|_\infty}$$

and, consequently, this inequality cannot hold for sufficiently large numbers p. We can conclude now that the construction terminates at some finite step $p(n)$.

Now we introduce the functions defined on I^2 by $(i = 1, 2, \ldots, p(n))$ (see (3))

$$f_i^{(n)}(x,y) := \delta_{\theta_i^{(n)},\eta_i^{(n)},h_n}(x,y) = \begin{cases} h_n^{-2}, & \text{if } (x,y) \in [\theta_i^{(n)}, \theta_i^{(n)} + h_n] \times [\eta_i^{(n)}, \eta_i^{(n)} + h_n]; \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

Introduce the functions

$$\Phi_n^{(t)}(x,y) = \sum_{i=1}^{p(n)} \frac{\xi_n(E)}{\lambda_i^{(n)}} f_i^{(n)}(x,y) r_i(t) \quad \text{where } (x,y,t) \in I^2 \times [0,1),\ n > r_0(E), \tag{28}$$

where $\{r_i(t),\ i = 1, 2, \ldots\}$ is the Rademacher system.

Consider the set

$$H_n = \bigcup_{i=1}^{p(n)} B_i^{(n)}. \tag{29}$$

Let (x,y) be any point from H_n. Then for some positive integer $i_0 = i_0(x,y)$, $1 \leq i_0 \leq p(n)$, we have (see (18), (7), (8), (29))

$$|T_n(f_{i_0}^{(n)})(x,y)| \geq \frac{9}{10}\lambda_{i_0}^{(n)} - 1 \geq \frac{9}{20}\lambda_{i_0}^{(n)}, \quad (n > n_0(E)). \tag{30}$$

Clearly (see (28), (27), (20)), $\Phi_n^{(t)}(x,y) \in L^1(I^2)$ for each fixed $t \in [0,1)$. According to the definition of superlinear operators (see (1), (2)) there exists a linear operator $G_{i_0}^{(n)} : L^1(I^2) \to L^0(I^2)$ such that

$$G_{i_0}^{(n)}(f_{i_0}^{(n)})(x,y) = T_n(f_{i_0}^{(n)})(x,y)$$

and

$$|G_{i_0}^{(n)}(f)(x,y)| \leq |T_n(f)(x,y)| \quad \text{for any } f \in L^1(I^2)$$

and a.e.

Further it follows from (28) that for any $t \in [0,1)$

$$\begin{aligned} |T_n(\Phi_n^{(t)})(x,y)| &\geq |G_{i_0}^{(n)}(\Phi_n^{(t)})(x,y)| \\ &= \left| r_{i_0}(t)\frac{\xi_n(E)}{\lambda_{i_0}^{(n)}} T_n(f_{i_0}^{(n)})(x,y) + \sum_{i \neq i_0} r_i(t) \frac{\xi_n(E)}{\lambda_i^{(n)}} G_{i_0}^{(n)}(f_i^{(n)})(x,y) \right|. \end{aligned} \tag{31}$$

The following easily verifiable fact is well known (see for example [8, p. 10]): Let $\sum_{i=1}^{m} a_i r_i(t)$ be an arbitrary polynomial with real coefficients in the Rademacher system and i_0 a fixed integer, $1 \leq i_0 \leq m$. Then

$$\mu_1 \left\{ t \in [0,1) : a_{i_0} r_{i_0}(t) \sum_{i \neq i_0} a_i r_i(t) \geq 0 \right\} \geq \frac{1}{2}.$$

Introduce the set

$$Q := \left\{ (x, y, t) \in H_n \times [0, 1) : \ |T_n(\Phi_n^{(t)})(x, y)| \geq \frac{9}{20}\xi_n(E) \right\}. \tag{32}$$

According to (31) and (30) we conclude that for all $(x, y) \in H_n$ we have the inequality

$$\int_0^1 \chi_Q(x, y, t)dt \geq \frac{1}{2},$$

where $\chi_Q(x, y, t)$ is the characteristic function of Q. Therefore (cf. (19), (29))

$$\int_{H_n} \int_0^1 \chi_Q(x, y, t)dxdydt \geq \frac{\gamma_1}{12}\mu_2(E).$$

Consequently, there exists a number $t_0 \in [0, 1)$ such that

$$\int_{H_n} \chi_Q(x, y, t_0)dxdy \geq \frac{\gamma_1}{12}\mu_2(E).$$

From (32) now we see that

$$\mu_2 \left\{ (x, y) \in H_n : \ |T_n(\Phi_n^{(t_0)})(x, y)| \geq \frac{9}{20}\xi_n(E) \right\} \geq \frac{\gamma_1}{12}\mu_2(E). \tag{33}$$

We observe that (see (28), (27) and (20))

$$\int_0^1 \int_0^1 |\Phi_n^{(t_0)}(x, y)|dxdy \leq \sum_{i=1}^{p(n)} \frac{\xi_n(E)}{\lambda_i^{(n)}} \leq \sum_{i=1}^{p(n)} \mu_2\{B_i^{(n)}\} \leq 1.$$

Set (see (28))

$$\Psi_n(x, y) := \Phi_n^{(t_0)}(x, y), \quad (x, y) \in I^2. \tag{34}$$

From (34) and (33) we conclude that for any positive integer $n > n_0(E)$

$$\mu_2 \left\{ (x, y) \in E : \ |T_n(\Psi_n)(x, y)| \geq \frac{9}{20}\xi_n(E) \right\} \geq \frac{\gamma_1}{12}\mu_2(E). \tag{35}$$

Set

$$\psi(u) := u\phi(u), \quad u \in [0, \infty). \tag{36}$$

Taking account of an assumption on ψ we see that if a number $C \in [0, 1]$ then for any $x > 0$

$$\frac{\psi(Cx)}{Cx} \leq \frac{\psi(x)}{x}$$

and, consequently

$$\psi(Cx) \leq C\psi(x), \quad x > 0.$$

Introduce the sequence of numbers $q_i, \ i = 1, 2, \dots, p(n)$, and a number Q_n defined by

$$q_i := \frac{\xi_n(E)}{\lambda_i^{(n)}}, \quad i = 1, 2, \dots, p(n),$$

$$Q_n := \sum_{l=1}^{p(n)} \frac{\xi_n(E)}{\lambda_l^{(n)}}.$$

It is clear that (see (20)) $0 < Q_n \le 1$. The function ψ is a convex function. From (28), (34) we have that for all $(x, y) \in I^2$

$$\psi(|\Psi_n(x, y)|) \le \psi\left(\sum_{i=1}^{p(n)} q_i |f_i^{(n)}(x, y)|\right) \le Q_n \psi\left(\frac{\sum_{i=1}^{p(n)} q_i (|f_i^{(n)}(x, y)|)}{Q_n}\right) \le Q_n \frac{\sum_{i=1}^{p(n)} q_i \psi(|f_i^{(n)}(x, y)|)}{Q_n}.$$

Now it is obvious that (see (9), (36), (27)) for any $n > n_0(E)$

$$\int_0^1 \int_0^1 \psi(|\Psi_n(x, y)|)dxdy \le \xi_n(E)\epsilon_n(E).$$

The proof of Lemma 1 (see (34), (35), (14), (27)) is completed.

Introduce a sequence $\{a_n(E)\}_{n=1}^{\infty}$ defined by

$$a_n(E) := \min\left(\frac{1}{\sqrt{\epsilon_n(E)}}, \sqrt{\xi_n(E)}\right), \quad n = 1, 2, \ldots. \tag{37}$$

It is clear that (see (8))

$$\lim_{n\to\infty} \frac{a_n(E)}{\xi_n(E)} = 0 \tag{38}$$

and

$$\lim_{n\to\infty} a_n(E) = \infty. \tag{39}$$

By a dyadic interval in I we shall mean an interval of the form

$$\Delta_n^{(k)} := [k2^{-n}, (k+1)2^{-n}), \quad (k = 0, 1, \ldots, 2^n - 1,\ n = 0, 1, 2, \ldots). \tag{40}$$

Let $n,\ i$ and $j,\ 0 \le i, j \le 2^n - 1$, be nonnegative integers. Set

$$\Delta_n^{(i,j)} := \Delta_n^{(i)} \times \Delta_n^{(j)}. \tag{41}$$

Let S_n denote a finite one-dimensional sequence of all intervals $\Delta_k^{(i,j)}$ where $i, j = 0, 1, 2, \ldots, 2^k - 1, k = 0, 1, 2, \ldots, n$. According to the following scheme

$$S_0, S_1, S_2, \ldots, S_n, \ldots,$$

we obtain a sequence of sets

$$E_1, E_2, \ldots, E_k, \ldots, \tag{42}$$

that has the following properties:

(i) For each positive integer k there exists a triple of non negative integers (n, i, j) where $0 \le i, j \le 2^n - 1$, such that

$$E_k = \Delta_n^{(i,j)}$$

and

(ii) for each triple of non negative integers (n, i, j), where $i, j = 0, 1, 2, \ldots, 2^n - 1$, there exists an increasing sequence of positive integers $\{r_p = r_p(n, i, j)\}_{p=1}^{\infty}$ such that

$$E_{r_p} = \Delta_n^{(i,j)} \tag{43}$$

for every $p = 1, 2, \ldots$.

By induction we will define: an increasing sequence of positive integers $\{l_j\}_{1=1}^{\infty}$, a sequence of positive integers $\{R_j\}_{j=1}^{\infty}$ and a sequence of positive integers $\{\delta_j\}_{j=2}^{\infty}$. Let (see (38)) $l_1 > n_0(E_1)$ be an integer, such that

$$\frac{a_{l_1}(E_1)}{\xi_{l_1}(E_1)} \leq \frac{1}{2} \quad \text{and} \quad \epsilon_{l_1}(E_1) \leq \frac{1}{4}. \tag{44}$$

Now let the numbers $l_1, l_2, \dots, l_k$, $R_2, R_3, \dots, R_k$ and $\delta_2, \delta_3, \dots, \delta_k$ be already defined.

According to one of the conditions of Theorem 3 the superlinear operator T_{l_k} is bounded in measure, that is the set

$$Q := \{T_{l_k}(f) : \|f\|_1 \leq 1\}$$

is bounded in measure. According to the definition this means that for each $\epsilon > 0$ there is a constant $R = R(\epsilon, k)$ such that

$$\mu_2\{(x, y) \in I^2 : |T_{l_k}(f)(x, y)| \geq R\} \leq \epsilon$$

for any function f such that

$$\|f\|_1 \leq 1.$$

If now h is any nonzero function in $L^1(I^2)$ then the function $\frac{h}{\|h\|_1}$ has norm 1 and, consequently

$$\mu_2\{(x, y) \in I^2 : |T_{l_k}(h)(x, y)| \geq R\|h\|_1\} \leq \epsilon$$

for any nonzero function $h \in L^1(I^2)$. We let (see ()) $\epsilon = \frac{\gamma_1}{108}\mu_2 E_k$. Then for some positive constant R_{k+1} independent of h we will have

$$\mu_2\{(x, y) \in I^2 : |T_{l_k}(h)(x, y)| \geq R_{k+1}\|h\|_1\} \leq \frac{\gamma_1}{108}\mu_2 E_k \tag{45}$$

for any nonzero function $h \in L^1(I^2)$.

Introduce the function (see (38), (42))

$$\alpha_k(x, y) = \sum_{j=1}^{k} \frac{a_{l_j}(E_j)}{\xi_{l_j}(E_j)} \Psi_{l_j}(x, y; E_j), \quad (x, y) \in I^2, \tag{46}$$

where $\Psi_{l_j}(x, y; E_j)$ is the function in Lemma 1 corresponding to the integer $n = l_j$ and the set $E = E_j$.

It is clear that $\alpha_k(x, y) \in L^\infty(I^2)$ and, consequently, the sequence of functions $\{T_n(\alpha_k)(x, y),\ n = 1, 2, \dots\}$ is bounded in measure on I^2 according to one of the conditions of Theorem 3. Now it is clear that we can find a positive number δ_{k+1} such that for all $n = 1, 2, \dots$ we have

$$\mu_2\{(x, y) \in I^2 : |T_n(\alpha_k)(x, y)| \geq \delta_{k+1}\} \leq \frac{\gamma_1}{108}\mu_2 E_{k+1}. \tag{47}$$

Now we define the number l_{k+1} such that the following inequalities are satisfied (see (38), (39), (8))

$$l_{k+1} > l_k,$$

$$\frac{a_{l_{k+1}}(E_{k+1})}{\xi_{l_{k+1}}(E_{k+1})} \leq \frac{1}{2}\,\frac{a_{l_k}(E_k)}{\xi_{l_k}(E_k)}, \tag{48}$$

$$\epsilon_{n_{k+1}} \leq \frac{1}{4}\,\epsilon_{n_k}, \tag{49}$$

$$\frac{3a_{l_{k+1}}(E_{k+1})}{20} \geq \max(k + 1, \delta_{k+1}), \tag{50}$$

$$2R_{k+1}\frac{a_{l_{k+1}}(E_{k+1})}{\xi_{l_{k+1}}(E_{k+1})} \leq \frac{3}{20}a_{l_k}(E_k). \tag{51}$$

The sequences $\{l_k\}_{k=1}^{\infty}$, $\{R_k\}_{k=1}^{\infty}$ and $\{\delta_k\}_{k=2}^{\infty}$ are now constructed.

Set

$$g(x, y) := \sum_{j=1}^{\infty} \frac{a_{l_j}(E_j)}{\xi_{l_j}(E_j)} \Psi_{l_j}(x, y; E_j), \quad (x, y) \in I^2, \tag{52}$$

$$\beta_k(x, y) := \sum_{j=k+1}^{\infty} \frac{a_{l_j}(E_j)}{\xi_{l_j}(E_j)} \Psi_{l_j}(x, y; E_j), \quad (x, y) \in I^2. \tag{53}$$

It is clear that (see (48))

$$\int_{I^2} |g(x, y)| dx dy \leq \sum_{i=1}^{\infty} \frac{a_{l_j}(E_J)}{\xi_{l_j}(E_j)} \leq 1$$

and for all $k = 1, 2, \ldots$

$$\int_{I^2} |\beta_k(x, y)| dx dy \leq \sum_{i=k+1}^{\infty} \frac{a_{l_j}(E_j)}{\xi_{l_j}(E_j)} \leq 2 \frac{a_{l_{k+1}}(E_{k+1})}{\xi_{l_{k+1}}(E_{k+1})}. \tag{54}$$

Introduce the sequence of numbers b_i, $i = 1, 2, \ldots, k$, and a number P_k defined by

$$b_j := \frac{a_{l_j}(E_j)}{\xi_{l_j}(E_j)}, \quad j = 1, 2, \ldots, k, \tag{55}$$

$$P_k := \sum_{i=1}^{k} b_j. \tag{56}$$

It is now clear that (see (46), (55), (56), (44), (48))

$$\begin{aligned}
\psi(|\alpha_k(x, y)|) &\leq \psi\left(\sum_{j=1}^{k} b_j |\Psi_{l_j}(x, y; E_j)|\right) \leq P_k \psi\left(\frac{\sum_{j=1}^{k} b_j |\Psi_{l_j}(x, y; E_j)|}{P_k}\right) \\
&\leq P_k \frac{\sum_{j=1}^{k} b_j \psi(|\Psi_{l_j}(x, y; E_j)|)}{P_k} = \sum_{j=1}^{k} b_j \psi(|\Psi_{l_j}(x, y; E_j)|), \quad (x, y) \in I^2.
\end{aligned}$$

It is obvious that the sequence of functions $\{\psi(\sum_{j=1}^{k} b_j |\Psi_{l_j}(x, y; E_j)|)\ k = 1, 2, \ldots\}$ is increasing and we have for all $k = 1, 2, \ldots$ that (see (14), (49), (55), (37))

$$\int_{I^2} \psi\left(\sum_{j=1}^{k} b_j |\Psi_{l_j}(x, y; E_j)|\right) dx dy \leq \sum_{j=1}^{k} \xi_{l_j}(E_j) \epsilon_{l_j}(E_j) b_i \leq \sum_{j=1}^{k} \sqrt{\epsilon_{l_j}(E_j)} \leq 1.$$

It follows now that the limit of the sequence

$$\left\{\psi\left(\sum_{j=1}^{k} b_j |\Psi_{l_j}(x, y; E_j)|\right),\ k = 1, 2, \ldots\right\},$$

is integrable on I^2 and this limit is an upper bound of the sequence

$$\{\psi(|\alpha_k(x, y)|),\ k = 1, 2, \ldots\}.$$

Consequently, the limit of the latter, that is the function $\psi(|g(x, y)|)$, is also integrable on I^2.

Now let $E_0 \subset I^2$ be an arbitrary Lebesgue measurable set, $\mu_2 E_0 > 0$. It is clear that there exist a triple of non negative integers (n_0, i_0, j_0), where $0 \leq i_0, j_0, \leq 2^{n_0} - 1$, and an increasing sequence of positive integers $\{k_q\}_{q=1}^{\infty}$

such that (see (40), (41), (43))

$$\mu_2\{E_0 \cap \Delta_{n_0}^{(i_0,j_0)}\} \geq \left(1 - \frac{\gamma_1}{216}\right) \mu_2 \Delta_{n_0}^{(i_0,j_0)} \tag{57}$$

and

$$E_{k_q} = \Delta_{n_0}^{(i_0,j_0)}$$

for all $q = 1, 2, \ldots$.

From (52), (53) and (46) we have for all $q = 2, 3, \ldots$

$$g(x, y) = \alpha_{k_q-1}(x, y) + \frac{a_{l_{k_q}}(E_{k_q})}{\xi_{l_{k_q}}(E_{k_q})} \Psi_{l_{k_q}}(x, y; E_{k_q}) + \beta_{k_q}(x, y), \quad (x, y) \in I^2,$$

It is obvious that (see (15)) for all $q = 1, 2, 3, \ldots$ we have

$$\begin{aligned} \frac{\gamma_1}{36}\mu_2 E_{k_q} &\leq \mu_2\left\{(x, y) \in E_{k_q} : \left|T_{l_{k_q}}\left(\frac{a_{l_{k_q}}(E_{k_q})}{\xi_{l_{k_q}}(E_{k_q})} \Psi_{l_{k_q}}(.,.; E_{k_q})\right)(x, y)\right| \geq \frac{9}{20} a_{l_{k_q}}(E_{k_q})\right\} \\ &\leq \mu_2\{(x, y) \in E_{k_q} : |T_{l_{k_q}}(\alpha_{k_q-1})(x, y)| \geq \frac{3}{20} a_{l_{k_q}}(E_{k_q})\} + \mu_2\{(x, y) \in E_{k_q} : |T_{l_{k_q}}(\beta_{k_q})(x, y)| \\ &\geq \frac{3}{20} a_{l_{k_q}}(E_{k_q})\} + \mu_2\{(x, y) \in E_{k_q} : |T_{l_{k_q}}(g)(x, y)| \geq \frac{3}{20} a_{l_{k_q}}(E_{k_q})\}. \end{aligned} \tag{58}$$

According to (47), (50) we have for any $q = 1, 2, \ldots$

$$\mu_2\left\{(x, y) \in E_{k_q} : |T_{l_{k_q}}(\alpha_{k_q-1})(x, y)| \geq \frac{3}{20} a_{l_{k_q}}(E_{k_q})\right\} \leq \frac{\gamma_1}{108}\mu_2 E_{k_q}. \tag{59}$$

Using (45), (51), (54) we come to the conclusion that

$$\mu_2\left\{(x, y) \in E_{k_q} : |T_{l_{k_q}}(\beta_{k_q})(x, y)| \geq \frac{3}{20} a_{l_{k_q}}(E_{k_q})\right\} \leq \frac{\gamma_1}{108}\mu_2 E_{k_q}. \tag{60}$$

Taking account of (58)–(60) we obtain for all $q = 1, 2, \ldots$

$$\mu_2\left\{(x, y) \in \Delta_{n_0}^{(i_0,j_0)} : |T_{l_{k_q}}(g)(x, y)| \geq \frac{3}{20}(k_q)\right\} \geq \frac{\gamma_1}{108}\mu_2 \Delta_{n_0}^{(i_0,j_0)}.$$

According to (57) now it is easy to see that for all $q = 1, 2, \ldots$

$$\mu_2\left\{(x, y) \in E_0 \cap \Delta_{n_0}^{(i_0,j_0)} : |T_{l_{k_q}}(g)(x, y)| \geq \frac{3}{20}(k_q)\right\} \geq \frac{\gamma_1}{216}\mu 2 \Delta_{n_0}^{(i_0,j_0)}.$$

Consequently, the sequence of functions $\{T_n(g),\ n = 1, 2, \ldots\}$ is not bounded in measure on E_0.

The proof of Theorem 3 is completed.

3. Proof of Theorem 4

The Walsh–Paley system $\{w_n(x),\ n = 0, 1, 2, \ldots\}$ is defined on $[0, 1)$ in the following way (see, for example [9, p. 1]). Given a non-negative integer n it is possible to write n uniquely as

$$n = \sum_{i=0}^{\infty} \alpha_i 2^i, \tag{61}$$

where $\alpha_i = 0$ or $\alpha_i = 1$. Then (see (4), (5))

$$w_n(x) := \Pi_{i=0}^{\infty} r_i^{\alpha_i}(x). \tag{62}$$

We define Dirichlet kernels of the Walsh–Paley system by $D_0(x) := 0$ and

$$D_m(x) := \sum_{l=0}^{m-1} w_l(x), \quad x \in [0, 1), \ m = 1, 2, \dots. \tag{63}$$

Let $S_{m,m}(f; x, y)$ denote the mth square partial sum of the Fourier series of $f \in L^1([0, 1)^2)$ with respect to the double Walsh–Paley system ($m = 1, 2, \dots$):

$$S_{m,m}(f; x, y) := \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \int_0^1 \int_0^1 f(s, t) w_i(s) w_j(t) ds dt w_i(x) w_j(y), \quad (x, y) \in [0, 1)^2. \tag{64}$$

Set for $n = 1, 2, \dots$

$$m_n := 2^{2n} + 2^{2n-2} + \cdots + 2^2 + 2^0. \tag{65}$$

It is known that (see, for example, [10]) if $2 \le p \le 2n - 2$ then

$$|D_{m_n}(x)| \ge 2^{p-2} \quad \text{for all } x \in (2^{-p}, 2^{1-p}).$$

Introduce a set

$$B_n := \bigcup_{p=2}^{2n-2} (2^{-p}, 2^{1-p}) \times (2^{p-2n}, 2^{p-2n+1}).$$

It is clear that

$$|D_{m_n}(\theta) D_{m_n}(\eta)| \ge 2^{2n-4} \quad \text{for all } (\theta, \eta) \in B_n \tag{66}$$

and

$$\mu_2 B_n = \frac{2n - 3}{2^{2n}} \tag{67}$$

for all $n = 2, 3, \dots$.

Let $(x, y) \in I^2$. Consider the set (for the definition and properties of the operation $\dot{+}$ see [9, pp. 10–13])

$$B_n \dot{+} (x, y) := \{(\theta, \eta) \in [0, 1)^2 : \ (\theta, \eta) = (\theta_1 \dot{+} x, \eta_1 \dot{+} y), \ (\theta_1, \eta_1) \in B_n\}. \tag{68}$$

It is clear that if $(\theta, \eta) \in B_n \dot{+} (x, y)$ then there exists a point $(\theta_1, \eta_1) \in B_n$ such that $(\theta, \eta) = (\theta_1 \dot{+} x, \eta_1 \dot{+} y)$ and, consequently, (see (66)) we obtain

$$|D_{m_n}(\theta \dot{+} x) D_{m_n}(\eta \dot{+} y)| = |D_{m_n}(\theta_1) D_{m_n}(\eta_1)| \ge 2^{2n-4} \quad \text{for a.e. } (\theta, \eta) \in B_n \dot{+} (x, y). \tag{69}$$

It is clear that (see (67))

$$\mu_2(B_n \dot{+} (x, y)) = \frac{2n - 3}{2^{2n}}. \tag{70}$$

Let E be an arbitrary Lebesgue measurable set, $E \subset I^2$, $\mu_2 E > 0$.

We will use Theorem 3 to prove Theorem 4.

We set in Theorem 3

$$E_n := E \quad \text{for all } n = 1, 2, \dots,$$

$$\gamma_1 := 1, \tag{71}$$

$$h_n := 2^{-9n} \quad \text{for all } n = 1, 2, \dots, \tag{72}$$

and for each $F \subset E_n$ we set

$$\lambda_n(F) := 2^{2n-4}. \tag{73}$$

It is clear that (see (12)) for each $n = 1, 2, \ldots$ there exists a positive number ϵ'_n such that (see (72))

$$\phi(h_n^{-2}) \le \epsilon'_n n$$

and

$$\lim_{n\to\infty} \epsilon'_n = 0. \tag{74}$$

Now we set in Theorem 3 for each $n = 2, 3, \ldots$

$$\xi_n(E) := \frac{2n-3}{6}\mu_2 E, \tag{75}$$

and

$$\epsilon_n(E) := \frac{n\epsilon'_n}{\xi_n(E)}. \tag{76}$$

It is clear that (see (74)–(76))

$$\lim_{n\to\infty} \epsilon_n(E) = 0 \tag{77}$$

and

$$\lim_{n\to\infty} \xi_n = \infty.$$

We set in Theorem 3 for $n = 1, 2, \ldots$ (see (64))

$$T_n(f)(x, y) := S_{m_n,m_n}(f; x, y) \quad f \in L^1(I^2),\ (x, y) \in I^2 \tag{78}$$

that is clearly superlinear and bounded in measure (see (1), (2)). It is easy to see that the kernel of T_n is (see (61)–(63))

$$K_n(x, y, \theta, \eta) = D_{m_n}(\theta\dot{+}x)D_{m_n}(\eta\dot{+}y) \quad (x, y) \in I^2. \tag{79}$$

It is obvious that (see (64), (78)) for each $f \in L^2(I^2)$ the sequence of functions $\{T_n(f)(x, y),\ n = 1, 2, \ldots\}$ is bounded in measure on I^2.

Now let $F \subset E$ be such that (see (71)) $\mu_2 F \ge \frac{1}{6}\mu_2 E$. Introduce the set

$$\Omega_n^{(1)} = \{(x, y, \theta, \eta) \in F \times I^2 :\ |D_{m_n}(\theta\dot{+}x)D_{m_n}(\eta\dot{+}y)| \ge 2^{2n-4}\} \tag{80}$$

where $n = 1, 2, \ldots$.

It is easy to see from (69), (70) and (80) that for a.e. $(x, y) \in F$

$$\int_{I^2} \chi_{\Omega_n^{(1)}}(x, y, \theta, \eta)d\theta d\eta \ge \frac{2n-3}{2^{2n}}$$

and, consequently, (see (73), (75))

$$\mu_4\Omega_n^{(1)} = \int_F\int_{I^2} \chi_{\Omega_n^{(1)}}(x, y, \theta, \eta)dxdyd\theta d\eta \ge \frac{1}{6}\cdot\frac{2n-3}{2^{2n}}\mu_2 E = \frac{\xi_n(E)}{\lambda_n(F)}. \tag{81}$$

Introduce the set

$$G_n = \bigcup_{i=1}^{2^{2n+1}}\bigcup_{j=1}^{2^{2n+1}}\Big[\frac{i-1}{2^{2n+1}}, \frac{i}{2^{2n+1}} - \frac{1}{2^{8n}}\Big) \times \Big[\frac{j-1}{2^{2n+1}}, \frac{j}{2^{2n+1}} - \frac{1}{2^{8n}}\Big).$$

It is clear that

$$\mu_2 G_n \ge 1 - \frac{2}{2^{4n}}.$$

Now let $(x, y, \theta, \eta) \in I^2 \times G_n$. Then (see (4), (5), (61), (62), (64)) for all $n > 16$ we have

$$S_{m_n,m_n}(\delta_{\theta,\eta,h_n}; x, y) = D_{m_n}(\theta\dot{+}x)D_{m_n}(\eta\dot{+}y).$$

We introduce the set $n = 1, 2, \dots$

$$\Theta_n^{(1)} := \{(x, y, \theta, \eta) \in I^4 : \ |S_{m_n, m_n}(\delta_{\theta,\eta,h_n}; x, y) - D_{m_n}(\theta \dot{+} x) D_{m_n}(\eta \dot{+} y)| > 1\}.$$

It is obvious that $\Theta_n^{(1)} \subset I^2 \times (I^2 \setminus G_n)$ and, consequently,

$$\mu_4 \Theta_n^{(1)} \leq \frac{2}{2^{4n}}.$$

We note also that (see (79), (61)–(63), (65))

$$\|K_n(x, y, \theta, \eta)\|_\infty \leq m_n^2 \leq 2^{4n+2}.$$

Taking account of (77)–(81), (6)–(11)) we can conclude that according to Theorem 3 we have completed the proof of Theorem 4.

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Original article

Hardy type operators on grand Lebesgue spaces for non-increasing functions

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Abstract

We characterize the weights w for which the operator $T_\psi f(x) = \int_0^x \psi(x, y) f(y)\, dy$ is bounded between weighted grand Lebesgue spaces $L_w^{p)}$ for non-increasing functions. The conjugate of T_ψ, for a special ψ, given by $S_\phi^* f(x) := \int_x^\infty f(y) \frac{\phi(y)}{\Phi(y)}\, dy$ is considered. An extrapolation type result giving $L^{p)}$-boundedness of S_ϕ^* for non-increasing functions has been proved. Also its L^p-boundedness has been characterized. Finally, a variant of S_ϕ^* has been considered and discussed.

Keywords: Non-increasing functions; $B_{\phi,\,p}$ class of weights; $B^*_{\phi,\,p}$ class of weights; grand Lebesgue space

1. Introduction

By a weight function or simply a weight, we mean a function which is measurable, positive and finite almost everywhere on the underlying domain. Let $I_b = (0, b)$, $0 < b \leq \infty$ and w be a weight. We denote by $L_w^p(I_b)$, $0 < p < \infty$, the space of all measurable functions f on I_b for which

$$\|f\|_{L_w^p(I_b)} := \left(\int_0^b |f(x)|^p w(x)\, dx\right)^{1/p} < \infty.$$

When $b = \infty$, we shall write L_w^p instead of $L_w^p(I_\infty)$.

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A weight w is said to belong to B_p^b-class if the inequality

$$\int_r^b \left(\frac{r}{t}\right)^p w(t)\, dt \leq c \int_0^r w(t)\, dt$$

holds for all $0 < r < b$. For $b = \infty$, we shall write B_p instead of B_p^∞. It was proved by Arino and Muckenhoupt [1] that a B_p weight characterizes the inequality

$$\|Af\|_{L_w^p} \leq c\|f\|_{L_w^p}, \quad p \geq 1$$

for all non-negative non-increasing functions f (now onwards written $f \downarrow$), where A is the averaging operator

$$Af(x) := \frac{1}{x} \int_0^x f(y)\, dy.$$

Also, they proved (see also [2–4]) a very important property of B_p-class of weights: if $w \in B_p$, $0 < p < \infty$, then there exists $\varepsilon > 0$ such that $w \in B_{p-\varepsilon}$. Also, Carro and Lorente [5] proved an extrapolation result involving B_p weights that deals with more general inequalities.

Carro and Soria [6] considered a more general operator given by

$$S_\phi f(x) := \frac{1}{\Phi(x)} \int_0^x f(y)\phi(y)\, dy, \quad \Phi(x) = \int_0^x \phi(u) du \tag{1.1}$$

and proved the following:

Theorem A. *Let $p > 1$ and ϕ be non-negative, locally integrable and $\downarrow$. Then the inequality*

$$\|S_\phi f\|_{L_w^p} \leq c\|f\|_{L_w^p}$$

holds for all non-negative functions $f \downarrow$ if and only if

$$\int_r^\infty \left(\frac{\Phi(r)}{\Phi(x)}\right)^p w(x)\, dx \leq c \int_0^r w(x)\, dx, \quad r > 0.$$

Lai [7] has considered even more general operator

$$T_\psi f(x) := \int_0^x \psi(x, y) f(y)\, dy,$$

ψ being a function from $\mathbb{R}^+ \times \mathbb{R}^+$ to $\mathbb{R}^+$ and obtained its L^p-boundedness for $f \downarrow$ as follows:

Theorem B. *Let $p \geq 1$. The inequality*

$$\|T_\psi f\|_{L_w^p} \leq c_1\|f\|_{L_w^p}$$

holds for all non-negative functions $f \downarrow$ if and only if

$$\int_0^r \Psi(x, x)^p w(x)\, dx + \int_r^\infty \Psi(x, r)^p w(x)\, dx \leq c_2 \int_0^r w(x)\, dx, \quad r > 0$$

where $\Psi(x, r) = \int_0^r \psi(x, y)\, dy$ satisfies the following:

P1 *$\Psi(x, r) \leq \alpha\, \Psi(x, t)\, \Psi(t, r)$ for some $\alpha > 0$ and all $0 < r \leq t \leq x$;*

P2 *$f \downarrow \Rightarrow T_\psi f \downarrow$.*

The first aim of this paper is to characterize the boundedness of T_ψ in weighted grand Lebesgue spaces for $f \downarrow$ defined as follows:

Let $I := I_1 = (0, 1)$, $1 < p < \infty$ and w be a locally integrable weight function. The weighted grand Lebesgue space $L_w^{p)}(I)$ consists of all measurable functions f for which

$$\|f\|_{L_w^{p)}(I)} := \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^1 |f(x)|^{p-\varepsilon} w(x)\, dx\right)^{1/p-\varepsilon} < \infty.$$

These spaces without weights were introduced by Iwaniec and Sbordone [8] and with weights by Fiorenza, Gupta and Jain [9]. Note that the space $L_w^{p)}(I)$ is not rearrangement invariant except for the trivial case when w is a constant weight. About the Lebesgue spaces, the implications $f \in L_w^p \iff fw^{1/p} \in L^p$ hold. However, the same is not true for grand Lebesgue spaces (see [9]). These facts make the study of weighted grand Lebesgue spaces important. In [9], the authors studied the boundedness of the maximal operator between $L_w^{p)}(I)$-spaces. Later, their technique was used by several authors to study various operators, e.g., one may refer to [10–16].

Lai [7] also studied the adjoint of the operator T_ψ given by

$$T_\psi^* f(x) := \int_x^\infty \psi(x, y) f(y)\, dy$$

and obtained its L^p-boundedness as follows:

Theorem C. *Let $p \geq 1$. The inequality*

$$\|T_\psi^* f\|_{L_w^p} \leq c \|f\|_{L_w^p}$$

holds for all non-negative functions $f \downarrow$ if and only if

$$\int_0^r \Psi^*(x, r)^p w(x)\, dx \leq c \int_0^r w(x)\, dx, \quad r > 0$$

where $\Psi^*(x, r) = \int_x^r \psi(x, y)\, dy + 1$ and satisfies the following:

P1* $\Psi^*(x, y) \leq \alpha \Psi^*(x, t) \Psi^*(t, y)$ *for some $\alpha > 0$ and all $x \leq t \leq y$;*
P2* $f \downarrow \Rightarrow T_\psi^* f \downarrow$.

Of particular interest is the case when $\psi(x, y) = \frac{\phi(y)}{\Phi(y)} \chi_{[x,\infty)}(y)$. In this case, the operator T_ψ^* becomes

$$S_\phi^* f(x) := \int_x^\infty f(y) \frac{\phi(y)}{\Phi(y)}\, dy$$

and its L^p-boundedness for $f \downarrow$ has been obtained by Carro and Soria [6]. We prove in this paper that for $1 < p < \infty$, the L^p-boundedness of S_ϕ^* for $f \downarrow$ is independent of p. Neugebauer [4] proved similar independence for the operator $A^* f(x) := \int_x^\infty \frac{f(y)}{y}\, dy$. Note that A^* is a special case of S_ϕ^* and consequently of T_ψ^*. We then prove the boundedness of S_ϕ^* for $f \downarrow$ between grand Lebesgue spaces.

Another special case of T_ψ^* that we deal with is when $\psi(x, y) = \frac{\phi(y)}{\Phi(y)} \chi_{[x,\infty)}(y)$. In this case, the corresponding operator becomes

$$\tilde{S}_\phi f(x) := \frac{1}{\Phi(x)} \int_x^\infty f(y) \phi(y)\, dy.$$

Although the L^p-boundedness of $\tilde{S}_\phi$ for $f \downarrow$ can obviously be written by Theorem C, but the proof of Theorem C does not provide a precise estimate of the constant which is a key point for studying this boundedness in the framework of grand Lebesgue spaces. Therefore, we provide an alternate proof for the L^p-boundedness of $\tilde{S}_\phi$ for $f \downarrow$ and then study the corresponding boundedness between $L_w^{p)}$-spaces.

All the functions considered in this paper are non-negative and measurable. In order to consider the case of finite intervals as well, all the functions will be defined on $(0, b)$ or $(0, b) \times (0, b)$, $0 < b \leq \infty$ as the case may be. Consequently, the integrals $\int_x^\infty$ mentioned in various operators will be changed to $\int_x^b$ but if there is no ambiguity, we shall still denote the corresponding operators by T_ψ^*, S_ϕ^*, etc.

2. Operator T_ψ on grand Lebesgue spaces

In this section, our aim is to characterize the boundedness of the operator T_ψ between weighted grand Lebesgue spaces $L_w^{p)}$ for non-increasing functions.

For $0 < p < \infty$, we denote by $B^b_{\psi,p}$, the class of weights w for which the inequality

$$\int_r^b \Psi(x,r)^p w(x)\,dx \le C_1 \int_0^r w(x)\,dx, \quad 0 < r < b,$$

holds for some constant $C_1 > 0$.

Remark 2.1. As remarked by Lai [7], for $0 < p \le q < \infty$, the inclusion $B^b_{\psi,p} \subseteq B^b_{\psi,q}$ holds if the following condition is satisfied:

P3 $\Psi(x,x) \le D, \quad x \in (0,b)$

where D is a constant which, without loss of generality, can be taken ≥ 1.

Define

$$H^b_{\psi,p} := \left\{ w : \int_0^r \Psi(x,x)^p w(x)\,dx + \int_r^b \Psi(x,r)^p w(x)\,dx \le c \int_0^r w(x)\,dx,\ 0 < r < b \right\},$$

and

$$\|w\|_{H^b_{\psi,p}} := \inf\left\{ c > 0 : w \in H^b_{\psi,p} \right\}.$$

Remark 2.2. If $0 < p \le q < \infty$ and P3 holds, then $H^b_{\psi,p} \subseteq H^b_{\psi,q}$. Moreover,

$$\|w\|_{H^b_{\psi,q}} \le D^q + D^{q-p}\|w\|_{H^b_{\psi,p}}. \tag{2.1}$$

Lemma 2.3. *If* $0 < p < \infty$ *and* P3 *holds, then*

$$w \in H^b_{\psi,p} \quad \text{with } c = \|w\|_{H^b_{\psi,p}} \text{ if and only if } w \in B^b_{\psi,p}.$$

In view of the above consideration, Theorem B can be restated as

Theorem 2.4. *Let* $1 \le p < \infty$ *and* P1–P3 *hold. Then the inequality*

$$\int_0^b \left(T_\psi f\right)^p (x) w(x)\,dx \le C_2 \int_0^b f^p(x) w(x)\,dx$$

holds for all $f \downarrow$ *if and only if* $w \in B^b_{\psi,p}$.

Remark 2.5. In the above theorem, the constants C_2 and C_1 involved in the inequality and the condition, respectively, are same for the necessary part. But for the sufficiency part, we get $C_2 = (C_1 + D^p)^p \alpha^{p(p-1)}$.

Remark 2.6. In view of Lemma 2.3, the constant C_1 of the condition in Theorem 2.4 can be replaced by $\|w\|_{H^b_{\psi,p}}$.

We shall be using a result from [7] in the following modified form:

Theorem D. *If* $1 \le p < \infty$ *and* P1–P3 *hold, then for* $w \in B^b_{\psi,p}$ *there exists* $\sigma > 0$ *such that* $w \in B^b_{\psi,p-\sigma}$.

Now we give our main result of this section.

Theorem 2.7. *Let* $1 < p < \infty$, *and* P1–P3 *hold. Then the inequality*

$$\|T_\psi f\|_{L^{p)}_w(I)} \le C\, \|f\|_{L^{p)}_w(I)} \tag{2.2}$$

holds for all $f \downarrow$ *if and only if* $w \in B^1_{\psi,p}$.

Proof. For the necessity, taking $f = \chi_{(0,r]}$ for $0 < r < 1$, the R.H.S. of the inequality (2.2) becomes

$$\|f\|_{L_w^{p)}(I)} = \left(\varepsilon_r \int_0^r w(x)\,dx\right)^{1/(p-\varepsilon_r)}$$

for some $\varepsilon_r,\ 0 < \varepsilon_r < p-1$, while its L.H.S. gives

$$\begin{aligned}\|T_\psi f\|_{L_w^{p)}(I)} &= \sup_{0<\varepsilon<p-1} \varepsilon^{1/(p-\varepsilon)} \left(\int_0^r w(x)\left(\int_0^x \psi(x,y)f(y)\,dy\right)^{p-\varepsilon} dx\right.\\ &\quad \left. + \int_r^1 w(x)\left(\int_0^x \psi(x,y)f(y)\,dy\right)^{p-\varepsilon} dx\right)^{1/(p-\varepsilon)}\\ &\ge \varepsilon_r^{1/(p-\varepsilon_r)}\left(\int_0^r \Psi(x,x)^{p-\varepsilon_r} w(x)\,dx + \int_r^1 \Psi(x,r)^{p-\varepsilon_r} w(x)\,dx\right)^{1/(p-\varepsilon_r)}.\end{aligned}$$

The above estimates lead to $w \in B^1_{\psi,p-\varepsilon_r}$ and the necessity follows in view of Remark 2.1. Conversely, let $w \in B^1_{\psi,p}$. Then, by Theorem D, $w \in B^1_{\psi,p-\sigma}$ for some $\sigma > 0$. We assume that $\sigma < p-1$ for otherwise the sufficiency follows easily. Remark 2.1 gives that $w \in B^1_{\psi,p-\varepsilon}$ for all $0 < \varepsilon \le \sigma$. For $\sigma < \varepsilon < p-1$, by using Hölder's inequality with the conjugate exponents $\frac{p-\sigma}{p-\varepsilon}$ and $\frac{p-\sigma}{\varepsilon-\sigma}$, we get

$$\begin{aligned}\|T_\psi f\|_{L_w^{p-\varepsilon}(I)} &\le \|T_\psi f\|_{L_w^{p-\sigma}(I)} \left(\int_0^1 w(x)dx\right)^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}}\\ &\le \|T_\psi f\|_{L_w^{p-\sigma}(I)}\,\beta(p,\sigma) \qquad (2.3)\end{aligned}$$

since $0 < \frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)} < \frac{p-1-\sigma}{p-\sigma}$, where $\beta(p,\sigma) := \left(\int_0^1 w(x)dx\, + 1\right)^{\frac{p-1-\sigma}{p-\sigma}}$. Now by using (2.3), (2.1), Remarks 2.5 and 2.6, we obtain

$$\begin{aligned}\|T_\psi f\|_{L_w^{p)}(I)} &\le \max\left\{\sup_{0<\varepsilon\le\sigma}\left(\varepsilon^{1/(p-\varepsilon)}\|T_\psi f\|_{L_w^{p-\varepsilon}(I)}\right),\ \sup_{\sigma<\varepsilon<p-1} \varepsilon^{1/(p-\varepsilon)}\|T_\psi f\|_{L_w^{p-\sigma}(I)}\beta(p,\sigma)\right\}\\ &\le \max\left\{1,\ \sigma^{-\frac{1}{p-\sigma}}\beta(p,\sigma)\sup_{\sigma<\varepsilon<p-1}\varepsilon^{1/(p-\varepsilon)}\right\}\sup_{0<\varepsilon\le\sigma}\left(\varepsilon^{1/(p-\varepsilon)}\|T_\psi f\|_{L_w^{p-\varepsilon}(I)}\right)\\ &\le \max\left\{1,\ p\sigma^{-\frac{1}{p-\sigma}}\beta(p,\sigma)\right\}\sup_{0<\varepsilon\le\sigma}\left(\|w\|_{H^1_{\psi,p-\varepsilon}} + D^{p-\varepsilon}\right)^{p-\varepsilon}\alpha^{(p-\varepsilon)(p-\varepsilon-1)}\varepsilon^{1/(p-\varepsilon)}\|f\|_{L_w^{p-\varepsilon}(I)}\\ &\le \max\left\{1,\ p\sigma^{-\frac{1}{p-\sigma}}\beta(p,\sigma)\right\}\left(D^{p-\sigma} + D^\sigma\|w\|_{H^1_{\psi,p-\sigma}} + D^p\right)^p\\ &\quad \times\ (\alpha+1)^{p(p-1)}\sup_{0<\varepsilon\le\sigma}\left(\varepsilon^{1/(p-\varepsilon)}\|f\|_{L_w^{p-\varepsilon}(I)}\right)\\ &\le C(p,\sigma,D,\alpha)\,\|f\|_{L_w^{p)}(I)},\end{aligned}$$

where

$$C(p,\sigma,D,\alpha) = \max\left\{1,\ p\sigma^{-\frac{1}{p-\sigma}}\beta(p,\sigma)\right\}\left(D^{p-\sigma} + D^\sigma\|w\|_{H^1_{\psi,p-\sigma}} + D^p\right)^p(\alpha+1)^{p(p-1)},$$

and the result follows. □

Corollary 2.8. *Let $1 < p < \infty$ and ϕ be non-negative locally integrable and ↓. Then the inequality*

$$\|S_\phi f\|_{L_w^{p)}(I)} \le c\,\|f\|_{L_w^{p)}(I)}$$

holds for all $f \downarrow$ *if and only if*

$$\int_r^1 \left(\frac{\Phi(r)}{\Phi(x)}\right)^p w(x)\,dx \le c \int_0^r w(x)\,dx, \quad 0 < r < 1,$$

S_ϕ *being the operator given in* (1.1).

Proof. This is immediate by taking $\psi(x, y) = \frac{\phi(y)}{\Phi(x)} \chi_{(0,x]}(y)$ in Theorem 2.7. $\square$

Corollary 2.9. *Let* $1 < q < \infty$ *and consider the operator*

$$A_q f(x) := \frac{1}{x^{1/q}} \int_0^x \frac{f(t)}{t^{1/q'}}\,dt.$$

For $1 < p < \infty$*, the inequality*

$$\|A_q f\|_{L_w^{p)}(I)} \le c\,\|f\|_{L_w^{p)}(I)}$$

holds for all $f \downarrow$ *if and only if*

$$\int_r^1 \left(\frac{r}{x}\right)^{p/q} w(x)\,dx \le c \int_0^r w(x)\,dx, \quad 0 < r < 1. \tag{2.4}$$

Proof. This can be obtained easily by taking $\phi(t) = \frac{1}{qt^{1/q'}}$ in Corollary 2.8. $\square$

Note that Corollary 2.9 extends a result of Meskhi ([14], Theorem 3.1).

Remark 2.10. In view of Theorem A and Corollary 2.8, we note that L^p-boundedness of S_ϕ is equivalent to its $L^{p)}$-boundedness. The same is true for the operator A_q since L^p-boundedness of A_q is also characterized by (2.4), (see [4]).

3. Conjugate Hardy averaging operator

In this section, we shall deal with the operator

$$S_\phi^* f(x) := \int_x^b f(y)\frac{\phi(y)}{\Phi(y)}\,dy, \qquad \Phi(x) = \int_0^x \phi(u)du, \quad 0 < b \le \infty.$$

Let $1 < p < \infty$. We say that $w \in B_{\phi,p}^*(I_b)$ if for all $0 < r \le b$, the inequality

$$\int_0^r \left(\log\frac{\Phi(r)}{\Phi(x)}\right)^p w(x)\,dx \le c \int_0^r w(x)\,dx$$

holds for some constant $c > 0$. For $b = 1$ we shall write $B_{\phi,p}^*(I)$ instead of $B_{\phi,p}^*(I_1)$ and $B_{\phi,p}^* := B_{\phi,p}^*(I_\infty)$ if $b = \infty$.

We prove the following:

Theorem 3.1. *For* ϕ *non-negative, locally integrable and* $\downarrow$*, the following statements are equivalent.*

(i) *The inequality*

$$\|S_\phi^* f\|_{L_w^1(I_b)} \le c\|f\|_{L_w^1(I_b)} \tag{3.1}$$

holds for all $f \downarrow$.

(ii) *The inequality*

$$\|S_\phi^* f\|_{L_w^p(I_b)} \le c\|f\|_{L_w^p(I_b)} \tag{3.2}$$

holds for all $p \in (1, \infty)$ *and for all* $f \downarrow$.

(iii) *For a given* $p \in (1, \infty)$, *the inequality* (3.2) *holds for all* $f \downarrow$.
(iv) $w \in B^*_{\phi,1}(I_b)$.

Proof. (i) $\Rightarrow$ (ii). Define

$$G(x) = p\left(\int_x^b f(u)\frac{\phi(u)}{\Phi(u)}\,du\right)^{p-1} f(x).$$

Then G is $\downarrow$ and

$$\int_x^b G(t)\frac{\phi(t)}{\Phi(t)}\,dt = \left(\int_x^b f(u)\frac{\phi(u)}{\Phi(u)}\,du\right)^p. \tag{3.3}$$

Using (3.1), (3.3) and Hölder's inequality, we get

$$\begin{aligned}\|S^*_\phi f\|^p_{L^p_w(I_b)} &= \int_0^b (S^*_\phi G)(x)\,w(x)\,dx\\ &\le c\int_0^b G(x)\,w(x)\,dx\\ &\le pc\left(\int_0^b f^p(x)\,w(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^b\left(\int_x^b f(u)\frac{\phi(u)}{\Phi(u)}\,du\right)^p w(x)\,dx\right)^{\frac{1}{p'}}\\ &= pc\,\|f\|_{L^p_w(I_b)}\,\|S^*_\phi f\|^{p/p'}_{L^p_w(I_b)},\end{aligned}$$

i.e.,

$$\|S^*_\phi f\|_{L^p_w(I_b)} \le C\|f\|_{L^p_w(I_b)}$$

with $C = pc$.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (iv). For $0 < r \le b$, take $f = \chi_{[0,r)}$. Then for this choice of f, we have

$$\|S^*_\phi f\|^p_{L^p_w(I_b)} = \int_0^r\left(\log\frac{\Phi(r)}{\Phi(x)}\right)^p w(x)\,dx$$

and

$$\|f\|^p_{L^p_w(I_b)} = \int_0^r w(x)\,dx$$

using which and applying Hölder's inequality, (3.2) gives

$$\int_0^r\left(\log\frac{\Phi(r)}{\Phi(x)}\right)w(x)\,dx \le \left(\int_0^r\left(\log\frac{\Phi(r)}{\Phi(x)}\right)^p w(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^r w(x)\,dx\right)^{\frac{1}{p'}} \le c\int_0^r w(x)\,dx$$

i.e., $w \in B^*_{\phi,1}(I_b)$.

(iv) $\Rightarrow$ (i). Let $w \in B^*_{\phi,1}(I_b)$. Choose $r = \psi(y)$ for some function $\psi\downarrow$ such that $\psi(0) = b$, $\psi(b) = 0$ and integrate from 0 to b so that we obtain

$$\int_0^b\left(\int_0^{\psi(y)}\left(\log\frac{\Phi(\psi(y))}{\Phi(x)}\right)w(x)\,dx\right)dy \le c\int_0^b\left(\int_0^{\psi(y)} w(x)\,dx\right)dy.$$

Interchanging the orders of integration on both the sides of the above inequality, we get

$$\int_0^b w(x)\left(\int_0^{\psi^{-1}(x)}\log\left(\frac{\Phi(\psi(y))}{\Phi(x)}\right)dy\right)dx \le c\int_0^b \psi^{-1}(x)w(x)\,dx. \tag{3.4}$$

By making variable substitution $t = \log\left(\frac{\Phi(\psi(y))}{\Phi(x)}\right)$ and writing $\gamma = \log\left(\frac{\Phi(b)}{\Phi(x)}\right) > 0$, we have

$$\begin{aligned}\int_0^{\psi^{-1}(x)} \log\left(\frac{\Phi(\psi(y))}{\Phi(x)}\right) dy &= \int_0^{\gamma} \psi^{-1}\left(\phi^{-1}(e^t\ \Phi(x))\right) dt \\ &= \int_x^b \psi^{-1}(u)\frac{\phi(u)}{\Phi(u)}\, du,\end{aligned}$$

using which in (3.4) and taking $\psi^{-1}(u) = f(u)$, the assertion follows. □

We immediately have the following result which has been proved by Carro and Soria [6]:

Corollary 3.2. *Let* $1 < p < \infty$ *and* w *be a weight function. The inequality*

$$\|S^*_\phi f\|_{L^p_w} \leq C\|f\|_{L^p_w}$$

holds for all $f \downarrow$ *if and only if* $w \in B^*_{\phi,p}$.

Proof. The necessity follows by taking $f = \chi_{[0,r)}$ for some $0 < r < \infty$. For sufficiency, if $w \in B^*_{\phi,p}$, then by an application of Hölder's inequality, we get that $w \in B^*_{\phi,1}$. The assertion now follows by Theorem 3.1. □

Remark 3.3. Theorem 3.1 gives an extrapolation effect in the sense that if the inequality (3.2) holds for some $p > 1$, then it holds for all $p > 1$. This kind of result, for a special case, has been proved by Neugebauer [4] which we derive below as a corollary to Theorem 3.1:

Corollary 3.4. *Let* $1 < p < \infty$ *and* w *be a weight function. Define* $A^* f(x) := \int_x^\infty \frac{f(t)}{t} dt$. *Then the inequality*

$$\|A^* f\|_{L^p_w} \leq c\|f\|_{L^p_w}$$

holds for all $f \downarrow$ *if and only if* $w \in B^*_{1,1}$.

Proof. For $\phi \equiv 1$, $S^*_\phi = A^*$ and the proof follows. □

Now, we prove the $L^{p)}$-boundedness of S^*_ϕ.

Theorem 3.5. *Let* $1 < p < \infty$ *and* w *be a weight function. Then the inequality*

$$\|S^*_\phi f\|_{L^{p)}_w(I)} \leq c\|f\|_{L^{p)}_w(I)} \tag{3.5}$$

holds for all $f \downarrow$ *if and only if* $w \in B^*_{\phi,1}(I)$.

Proof. First assume that (3.5) holds. Taking $f = \chi_{[0,r)}$ for $0 < r \leq 1$, the R.H.S. of (3.5) gives

$$\|f\|_{L^{p)}_w(I)} = \sup_{0<\varepsilon<p-1}\left(\varepsilon\int_0^r w(x)\,dx\right)^{1/(p-\varepsilon)} = \left(\varepsilon_r\int_0^r w(x)\,dx\right)^{1/(p-\varepsilon_r)}$$

for some $0 < \varepsilon_r < p - 1$, while its L.H.S. gives

$$\begin{aligned}\|S^*_\phi f\|_{L^{p)}_w(I)} &= \sup_{0<\varepsilon<p-1}\left(\varepsilon\int_0^r\left(\int_x^r\frac{\phi(t)}{\Phi(t)}dt\right)^{p-\varepsilon} w(x)\,dx\right)^{1/(p-\varepsilon)} \\ &= \sup_{0<\varepsilon<p-1}\left(\varepsilon\int_0^r\left(\log\frac{\Phi(r)}{\Phi(x)}\right)^{p-\varepsilon} w(x)\,dx\right)^{1/(p-\varepsilon)} \\ &\geq \left(\varepsilon_r\int_0^r\left(\log\frac{\Phi(r)}{\Phi(x)}\right)^{p-\varepsilon_r} w(x)\,dx\right)^{1/(p-\varepsilon_r)}.\end{aligned}$$

Consequently, the inequality (3.5) using Hölder's inequality yields

$$\int_0^r \left(\log \frac{\Phi(r)}{\Phi(x)}\right) w(x)\,dx \le \left(\int_0^r \left(\log \frac{\Phi(r)}{\Phi(x)}\right)^{p-\varepsilon_r} w(x)\,dx\right)^{1/(p-\varepsilon_r)} \times \left(\int_0^r w(x)\,dx\right)^{1-\frac{1}{p-\varepsilon_r}} < C\int_0^r w(x)\,dx,$$

where $C = \max\{c, c^{1/p}\}$, c being the constant in (3.5). Thus $w \in B^*_{\phi,1}(I)$.

Conversely, let $w \in B^*_{\phi,1}(I)$. Then, in view of Theorem 3.1, we find that for all $\varepsilon > 0$ such that $p - \varepsilon > 1$, the inequality

$$\|S^*_\phi f\|_{L^{p-\varepsilon}_w(I)} \le c(p-\varepsilon)\|f\|_{L^{p-\varepsilon}_w(I)}$$

holds, i.e., the inequality

$$\sup_{0<\varepsilon<p-1}\left(\varepsilon^{\frac{1}{p-\varepsilon}}\|S^*_\phi f\|_{L^{p-\varepsilon}_w(I)}\right) \le c\sup_{0<\varepsilon<p-1}\left((p-\varepsilon)\varepsilon^{\frac{1}{p-\varepsilon}}\|f\|_{L^{p-\varepsilon}_w(I)}\right) \le cp\sup_{0<\varepsilon<p-1}\left(\varepsilon^{\frac{1}{p-\varepsilon}}\|f\|_{L^{p-\varepsilon}_w(I)}\right)$$

holds. Hence the inequality (3.5) holds with the constant cp and the proof is complete. □

Corollary 3.6. *Let $1 < p < \infty$ and w be a weight function. The inequality*

$$\|A^* f\|_{L^{p)}_w(I)} \le c\|f\|_{L^{p)}_w(I)}$$

holds for all $f \downarrow$ if and only if $w \in B^*_{1,1}(I)$.

Remark 3.7. In view of Theorem 3.5, it comes out that if the inequality (3.5) holds for some $p \in (1,\infty)$, then it holds for all $q \in (1,\infty)$.

4. Conjugate type operator

This section deals with a variant of the operator S^*_ϕ defined by

$$\tilde{S_\phi} f(x) := \frac{1}{\Phi(x)}\int_x^b f(t)\phi(t)\,dt, \quad \Phi(x) = \int_0^x \phi(u)du,\ 0 < b \le \infty.$$

For $0 < p < \infty$, we denote by $\tilde{B}^b_{\phi,p}$, the class of all weights w for which the inequality

$$\int_0^r \left(\frac{\Phi(r)}{\Phi(x)}\right)^p w(x)\,dx \le c\int_0^r w(x)\,dx, \quad 0 < r \le b \tag{4.1}$$

holds for some constant $c > 0$.

Remark 4.1. Observe that the class $\tilde{B}^b_{\phi,p}$ is monotonic in the index p so that if $w \in \tilde{B}^b_{\phi,p}$, then $w \in \tilde{B}^b_{\phi,q}$ for all $q \le p$.

Define

$$\|w\|_{\tilde{B}^b_{\phi,p}} := \inf\left\{c > 0 : \int_0^r w(x)\,dx + \int_0^r \left(\frac{\Phi(r)}{\Phi(x)}\right)^p w(x)\,dx \le c\int_0^r w(x)\,dx, 0 < r \le b\right\}.$$

Observe that

(i) $\|w\|_{\tilde{B}^b_{\phi,p}} > 1$, and
(ii) $\|w\|_{\tilde{B}^b_{\phi,p}} \le \|w\|_{\tilde{B}^b_{\phi,q}}$ if $p \le q$.

We immediately have the following:

Lemma 4.2. *$w \in \tilde{B}^b_{\phi,p}$ if and only if the inequality*

$$\int_0^r w(x)\,dx + \int_0^r \left(\frac{\Phi(r)}{\Phi(x)}\right)^p w(x)\,dx \le A \int_0^r w(x)\,dx$$

holds for all $0 < r \le b$ with $A = \|w\|_{\tilde{B}^b_{\phi,p}}$.

We prove the following:

Theorem 4.3. *Let $1 \le p < \infty$. Then the inequality*

$$\|\tilde{S}_\phi f\|_{L^p_w(I_b)} \le c\|f\|_{L^p_w(I_b)}, \tag{4.2}$$

holds for all $f \downarrow$ if and only if $w \in \tilde{B}^b_{\phi,p}$, where $c = \|w\|_{\tilde{B}^b_{\phi,p}}$.

Proof. We prove the theorem for $1 < p < \infty$. The proof for the case $p = 1$ is similar. Assume first that (4.2) holds. For $0 < r \le b$, take $f = \chi_{[0,\,r)}$. Then using the inequality $a^p + b^p \ge 2^{1-p}(a+b)^p$, we get that $w \in \tilde{B}^b_{\phi,p}$ with constant $\frac{c^p+1}{2^{1-p}}$.

Conversely, assume that $w \in \tilde{B}^b_{\phi,p}$. Denote by $\lambda_f(y) := |\{x : |f(x)| > y\}|$, the distribution function of f with respect to the Lebesgue measure. Then following ([6], Corollary 2.2), we find that

$$\begin{aligned}\left(\int_x^b f(t)\phi(t)\,dt\right)^p &= p\int_x^b \left(\int_t^b f(s)\phi(s)ds\right)^{p-1} f(t)\phi(t)\,dt\\ &= p\int_0^b g(t)\chi_{[x,b)}(t)\phi(t)(\Phi(t))^{p-1}\,dt\\ &= p\int_0^b \left(\int_0^{\lambda_g(y)} \chi_{[x,b)}(t)\phi(t)(\Phi(t))^{p-1}\,dt\right)dy,\end{aligned}$$

where $g(t) = \left(\frac{1}{\Phi(t)}\int_t^b f(s)\phi(s)ds\right)^{p-1} f(t)$, which is a non-increasing function. By applying Fubini's Theorem, we get

$$\begin{aligned}\|\tilde{S}_\phi f\|^p_{L^p_w(I_b)} &= p\int_0^b \frac{w(x)}{(\Phi(x))^p}\left(\int_0^b\left(\int_0^{\lambda_g(y)} \chi_{[x,b)}(t)\phi(t)(\Phi(t))^{p-1}\,dt\right)dy\right)dx\\ &= p\int_0^b\int_0^b \frac{w(x)}{(\Phi(x))^p}\left(\int_0^{\lambda_g(y)} \chi_{[x,b)}(t)\phi(t)(\Phi(t))^{p-1}\,dt\right)dxdy\\ &= p\int_0^b\int_0^{\lambda_g(y)} \frac{w(x)}{(\Phi(x))^p}\left(\int_x^{\lambda_g(y)} \phi(t)(\Phi(t))^{p-1}\,dt\right)dxdy\\ &= \int_0^b\int_0^{\lambda_g(y)}\left[\left(\frac{\Phi(\lambda_g(y))}{\Phi(x)}\right)^p - 1\right]w(x)\,dxdy.\end{aligned}$$

Since $w \in \tilde{B}^b_{\phi,p}$, taking $r = \lambda_g(y)$ in (4.1) and using Lemma 4.2, we get

$$\|\tilde{S}_\phi f\|^p_{L^p_w(I_b)} \le (\|w\|_{\tilde{B}^b_{\phi,p}} - 1)\int_0^b dy\int_0^{\lambda_g(y)} w(x)\,dx.$$

Consequently, using ([6], Corollary 2.2) and Hölder's inequality, we have

$$\begin{aligned}\|\tilde{S}_\phi f\|^p_{L^p_w(I_b)} &\le \|w\|_{\tilde{B}^b_{\phi,p}} \int_0^b g(x)w(x)\,dx \\ &= \|w\|_{\tilde{B}^b_{\phi,p}} \int_0^b \left(\frac{1}{\Phi(x)}\int_x^b f(s)\phi(s)ds\right)^{p-1} f(x)w(x)\,dx \\ &\le \|w\|_{\tilde{B}^b_{\phi,p}} \|\tilde{S}_\phi f\|^{p-1}_{L^p_w(I_b)} \|f\|_{L^p_w(I_b)}\end{aligned}$$

and the assertion follows. □

Now, we prove the following for the boundedness of $\tilde{S}_\phi$ on $L^{p)}_w(I)$-spaces.

Theorem 4.4. *Let* $1 < p < \infty$ *and* $w \in \tilde{B}^1_{\phi,p}$. *Then the inequality*

$$\|\tilde{S}_\phi f\|_{L^{p)}_w(I)} \le c\|f\|_{L^{p)}_w(I)} \tag{4.3}$$

holds for all $f\downarrow$. *Conversely, if* (4.3) *holds for all functions* $f\downarrow$, *then* $w \in \tilde{B}^1_{\phi,p-\sigma}(I)$ *for some* $\sigma \in (0, p-1)$.

Proof. Assume first that $w \in \tilde{B}^1_{\phi,p}$. Then in view of Remark 4.1, $w \in \tilde{B}^1_{\phi,p-\varepsilon}$ for all $0 < \varepsilon < p-1$. Now, by Theorem 4.3, we get

$$\begin{aligned}\|\tilde{S}_\phi f\|_{L^{p)}_w(I)} &= \sup_{0<\varepsilon<p-1} \varepsilon^{1/(p-\varepsilon)} \|\tilde{S}_\phi f\|_{L^{p-\varepsilon}_w(I)} \\ &\le \sup_{0<\varepsilon<p-1} \varepsilon^{1/(p-\varepsilon)} \|w\|_{\tilde{B}^1_{\phi,p-\varepsilon}} \|f\|_{L^{p-\varepsilon}_w(I)} \\ &\le c\|f\|_{L^{p)}_w(I)}\end{aligned}$$

with $c = \|w\|_{\tilde{B}^1_{\phi,p}}$ and the sufficiency follows.

Conversely, suppose that the inequality (4.3) holds. Take $f = \chi_{[0,\,r)}$ for some $0 < r \le 1$. Then R.H.S. of (4.3) becomes

$$\begin{aligned}\|f\|_{L^{p)}_w(I)} &= \sup_{0<\varepsilon<p-1} \varepsilon^{1/(p-\varepsilon)} \left(\int_0^r w(x)\,dx\right)^{1/(p-\varepsilon)} \\ &= \left(\varepsilon_r \int_0^r w(x)\,dx\right)^{1/(p-\varepsilon_r)}\end{aligned} \tag{4.4}$$

for some ε_r, $0 < \varepsilon_r < p-1$ and the L.H.S. gives

$$\begin{aligned}\|\tilde{S}_\phi f\|_{L^{p)}_w(I)} &= \sup_{0<\varepsilon<p-1} \varepsilon^{1/(p-\varepsilon)} \left(\int_0^r \left(\frac{\Phi(r)}{\Phi(x)} - 1\right)^{p-\varepsilon} w(x)\,dx\right)^{1/(p-\varepsilon)} \\ &\ge \sup_{0<\varepsilon<p-1} \varepsilon^{1/(p-\varepsilon)} \left(2^{1-(p-\varepsilon)} \int_0^r \left(\frac{\Phi(r)}{\Phi(x)}\right)^{p-\varepsilon} w(x)\,dx - \int_0^r w(x)\,dx\right)^{1/(p-\varepsilon)} \\ &\ge \varepsilon_r^{1/(p-\varepsilon_r)} \left(2^{1-(p-\varepsilon_r)} \int_0^r \left(\frac{\Phi(r)}{\Phi(x)}\right)^{p-\varepsilon_r} w(x)\,dx - \int_0^r w(x)\,dx\right)^{1/(p-\varepsilon_r)}.\end{aligned} \tag{4.5}$$

Thus, from (4.3)–(4.5), we get

$$\int_0^r \left(\frac{\Phi(r)}{\Phi(x)}\right)^{p-\varepsilon_r} w(x)\,dx \le \frac{c^{p-\varepsilon_r}+1}{2^{1-(p-\varepsilon_r)}} \int_0^r w(x)\,dx \le C\int_0^r w(x)\,dx, \quad 0 < r \le 1$$

where $C = \left[(c+1)^p + 1\right]2^p$. Therefore, $w \in \tilde{B}^1_{\phi,p-\varepsilon_r}$ which means that $w \in \tilde{B}^1_{\phi,p-\sigma}$ for some $0 < \sigma < p-1$. □

Remark 4.5. The condition $w \in \tilde{B}^b_{\phi,p-\sigma}$ for some $0 < \sigma < p-1$ is not sufficient for (4.3) to hold. To see this, take $\phi \equiv 1$ on $I = (0,1)$ so that the operator $\tilde{S}_\phi$ becomes $\tilde{S}_1 f(x) = \frac{1}{x}\int_x^1 f(t)\,dt$. Take $w(x) = x^\alpha$ for $0 < \ p - \sigma - 1 < \alpha < p-1$. It can be seen that $w \in \tilde{B}^1_{\phi,p-\sigma}$. However, we claim that (4.3) is not satisfied for all $f \downarrow$. Take $f(x) = \chi_{(0,s]}$ for some $0 < s < 1$. Then

$$\|\tilde{S}_1 f\|_{L^{p)}_w(I)} \geq \left(\varepsilon \int_0^s (s-x)^{p-\varepsilon}\, x^{\alpha-p+\varepsilon}\, dx\right)^{\frac{1}{p-\varepsilon}}. \tag{4.6}$$

It is easy to check that $\int_0^s (s-x)^{p-\varepsilon}\, x^{\alpha-p+\varepsilon}\, dx$ converges if and only if $\alpha > p-\varepsilon$ for all $0 < \varepsilon < p-1$, i.e., $\alpha \geq p$. But since $\alpha < p$, the integral $\int_0^s (s-x)^{p-\varepsilon}\, x^{\alpha-p+\varepsilon}\, dx$ diverges for some $\varepsilon \in (0, p-1)$. Consequently, (4.6) gives that $\|\tilde{S}_1 f\|_{L^{p)}_w(I)} = \infty$. On the other hand

$$\|f\|_{L^{p)}_w(I)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^s x^\alpha\, dx\right)^{\frac{1}{p-\varepsilon}} < p,$$

which is finite. Hence our claim is proved.

Remark 4.6. If we assume that $w \in \tilde{B}^b_{\phi,\, p-\sigma}$ for all $\sigma \in (0, p-1)$, then $w \in \tilde{B}^b_{\phi,\, p}$.

5. Concluding remark and result

In this section, we consider functions f which need not be non-increasing. Consider the Hardy operator $Hf(x) := \int_0^x f(y)\,dy$. It can be worked out that H is not bounded between (non-weighted) $L^{p)}$-spaces. However, its adjoint $H^* f(x) := \int_x^1 f(t)\,dt$ is so which we prove below:

Theorem 5.1. *For $1 < p < \infty$, the inequality*

$$\|H^* f\|_{L^{p)}(I)} \leq p\|f\|_{L^{p)}(I)}$$

holds for all $f \geq 0$.

Proof. Take $q = \sigma = p - \varepsilon$. Since $\sigma > q - 1 > 0$ for all $0 < \varepsilon < p-1$, by an application of conjugate Hardy inequality (see, e.g., [17]), we have

$$\begin{aligned}
\int_0^1 (H^* f(x))^{p-\varepsilon}\, dx &= \int_0^1 (H^* f(x))^q x^{\sigma-q}\, dx \\
&\leq \left(\frac{q}{\sigma-q+1}\right)^q \int_0^1 f^q(x) x^\sigma\, dx \\
&\leq \left(\frac{q}{\sigma-q+1}\right)^q \int_0^1 f^q(x)\, dx \\
&= (p-\varepsilon)^{p-\varepsilon} \int_0^1 f^{p-\varepsilon}(x)\, dx
\end{aligned}$$

or

$$\left(\varepsilon \int_0^1 (H^* f(x))^{p-\varepsilon}\, dx\right)^{1/p-\varepsilon} \leq (p-\varepsilon) \left(\varepsilon \int_0^1 f^{p-\varepsilon}(x)\, dx\right)^{1/p-\varepsilon}.$$

Now taking supremum on both the sides of the above inequality over all $\varepsilon \in (0, p-1)$, the result follows. $\square$

Remark 5.2. Consider the Hardy averaging operator

$$S_1 f(x) = \frac{1}{x} \int_0^x f(t)dt, \quad x \in (0,1).$$

It was proved in ([9], Theorem 2.1) that S_1 is bounded between (non-weighted) $L^{p)}$-spaces, where the functions in $L^{p)}$ need not necessarily be non-increasing. Regarding the adjoint of S_1, the two variants have been considered in this paper, namely, S_1^* (precise conjugate of S_1) and $\tilde{S}_1$ (conjugate type of S_1). It can be worked out, by taking $f \equiv 1$ that both S_1^* and $\tilde{S}_1$ are not bounded between (non-weighted) $L^{p)}$-spaces. It is of interest to obtain the weights which characterize the boundedness of S_1^* as well as $\tilde{S}_1$ between $L_w^{p)}$-spaces for general non-negative functions.

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Original article

Long-time behavior of solution and semi-discrete scheme for one nonlinear parabolic integro-differential equation

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Abstract

Long-time behavior of solution and semi-discrete scheme for one nonlinear parabolic integro-differential equation are studied. Initial–boundary value problem with mixed boundary conditions are considered. Attention is paid to the investigation of more wide cases of nonlinearity than already were studied. Considered model is based on Maxwell's system describing the process of the penetration of a magnetic field into a substance.

Keywords: Nonlinear parabolic integro-differential equation; Initial–boundary value problem; Mixed boundary conditions; Long-time behavior; Semi-discrete scheme

1. Introduction

Integro-differential equations of parabolic type arise in the study of various problems in physics, chemistry, technology, economics, etc. (see, for example, [1–3] and references therein). One such model is obtained by mathematical modeling of processes of electromagnetic field penetration in the substance. In the quasi-stationary approximation the corresponding system of Maxwell's equations has the form [4]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}(\nu_m \operatorname{rot} H), \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} = \nu_m \left(\operatorname{rot} H\right)^2, \tag{1.2}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, ν_m characterizes the electro-conductivity of the substance. Eq. (1.1) describes the process of diffusion of the magnetic field and Eq. (1.2)—change of the temperature at the expense of Joule's heating. If ν_m depends on temperature θ, i.e., $\nu_m = \nu_m(\theta)$, then the system (1.1),

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(1.2) can be rewritten in the following form [5]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\left[a\left(\int_0^t \left|\operatorname{rot} H\right|^2 d\tau\right)\operatorname{rot} H\right], \tag{1.3}$$

where function $a = a(S)$ is defined for $S \in [0, \infty)$.

Note that integro-differential parabolic models of (1.3) type are complex and still yields to the investigation only for special cases (see, for example, [6–9,5,10–16] and references therein).

Study of the models of type (1.3) has begun in the work [5]. In particular, for the case $a(S) = 1 + S$ the theorems of existence of solution of the first boundary value problem for scalar and one-dimensional space case and uniqueness for more general cases are proved in this work. One-dimensional scalar variant for the case $a(S) = (1 + S)^p$, $0 < p \le 1$ is studied in [7]. Investigations for multi-dimensional space cases at first are carried out in the work [8]. Multidimensional space cases are also discussed in the following works [11,14].

Asymptotic behavior as $t \to \infty$ of solutions of initial–boundary value problems for (1.3) type models are studied in the work [9,11–13] and in a number of other works as well. In these works main attention is paid to one-dimensional analogs.

Interest to above-mentioned integro-differential models is more and more arising and initial–boundary value problems with different kinds of boundary and initial conditions are considered. Particular attention should be paid to construction of numerical solutions and to their importance for integro-differential models. Finite element analogs and Galerkin method algorithm as well as settling of semi-discrete and finite difference schemes for (1.3) type one-dimensional integro-differential models are studied in [10,13,17,16] and in the other works as well.

Our aim is to study long-time behavior of solution and semi-discrete scheme for numerical solution of initial–boundary value problem with mixed boundary condition for the one-dimensional (1.3) equation. Attention is paid to the investigation of more wide cases of nonlinearity than already were studied.

This article is organized as follows. In Section 2 the formulation of the problem and asymptotic behavior of solution is studied. Main attention is paid to construction and investigation of semi-discrete scheme in Section 3. We conclude the paper with some discussion of future research in this area in Section 4.

2. Long-time behavior of solution

If the magnetic field has the form $H = (0, 0, U)$, $U = U(x, t)$, then from (1.3) we obtain the following nonlinear integro-differential equation

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right], \tag{2.1}$$

where

$$S(x, t) = \int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau. \tag{2.2}$$

In the domain $(0, 1) \times (0, \infty)$ let us consider the following initial–boundary value problem for (2.1), (2.2):

$$U(0, t) = \left.\frac{\partial U(x, t)}{\partial x}\right|_{x=1} = 0,$$
$$U(x, 0) = U_0(x), \tag{2.3}$$

where U_0 is a given function.

The study of long-time behavior of solution of the problem (2.1)–(2.3) is actual.

The following statement shows the exponential stabilization of the solution of problem (2.1)–(2.3) in the norm of the Sobolev space $H^1(0, 1)$.

Theorem 1. *If $a(S) = (1 + S)^p$, $0 < p \le 1$ and $U_0 \in H^2(0, 1)$, $U_0(0) = \left.\frac{\partial U(x,t)}{\partial x}\right|_{x=1} = 0$, then for the solution of problem (2.1)–(2.3) the following estimate holds as $t \to \infty$*

$$\left\|\frac{\partial U}{\partial x}\right\| + \left\|\frac{\partial U}{\partial t}\right\| \le C \exp\left(-\frac{t}{2}\right).$$

Theorem 1 can be proven using analogical method as in [9].

Now let us prove the main result of this section that gives exponential stabilization of solution in the norm of the space $C^1(0, 1)$.

Theorem 2. *$a(S) = (1 + S)^p$, $0 < p \leq 1$ and $U_0 \in H^3(0, 1)$, $U_0(0) = \frac{dU_0(x)}{dx}\Big|_{x=1} = 0$, then for the solution of problem (2.1)–(2.3) the following estimates hold as $t \to \infty$:*

$$\left|\frac{\partial U(x,t)}{\partial x}\right| \leq C \exp\left(-\frac{t}{2}\right), \qquad \left|\frac{\partial U(x,t)}{\partial t}\right| \leq C \exp\left(-\frac{t}{2}\right),$$

uniformly in x on $[0, 1]$.

Proof. From (2.2) it follows that

$$\frac{\partial S}{\partial t} = \left(\frac{\partial U}{\partial x}\right)^2, \quad S(x, 0) = 0. \tag{2.4}$$

Let us multiply (2.4) by $(1 + S)^{2p}$

$$\frac{1}{1+2p}\frac{\partial(1+S)^{1+2p}}{\partial t} = \left(\frac{\partial U}{\partial x}\right)^2 (1+S)^{2p}.$$

Note that Eq. (2.1) can be rewritten as

$$\frac{\partial U}{\partial t} = \frac{\partial \sigma}{\partial x},$$

where

$$\sigma = (1+S)^p \frac{\partial U}{\partial x}. \tag{2.5}$$

We have

$$\frac{1}{1+2p}\frac{\partial(1+S)^{1+2p}}{\partial t} = \sigma^2, \tag{2.6}$$

$$\sigma^2(x,t) = \int_0^1 \sigma^2(y,t)dy + 2\int_0^1\int_y^x \sigma(\xi,t)\frac{\partial U(\xi,t)}{\partial t}d\xi dy. \tag{2.7}$$

Introducing the following notation

$$\varphi(t) = 1 + \int_0^t\int_0^1 \sigma^2 dx d\tau, \tag{2.8}$$

from Theorem 1 and relations (2.6)–(2.8) we get

$$\begin{aligned}\frac{1}{1+2p}(1+S)^{1+2p} &= \int_0^t \sigma^2 d\tau + 2\int_0^t\int_0^1\int_y^x \sigma(\xi,\tau)\frac{\partial U(\xi,\tau)}{\partial \tau}d\xi\, dy\, d\tau + \frac{1}{1+2p}\\ &\leq 2\int_0^t\int_0^1 \sigma^2(y,\tau)dyd\tau + \int_0^t\int_0^1\left(\frac{\partial U(x,\tau)}{\partial \tau}\right)^2 dxd\tau + \frac{1}{1+2p}\\ &\leq 2\int_0^t\int_0^1 \sigma^2(y,\tau)dyd\tau + C_1\int_0^t \exp(-\tau)d\tau + \frac{1}{1+2p} \leq C_2\varphi(t),\end{aligned}$$

i.e.,

$$1 + S(x,t) \leq C\varphi^{\frac{1}{1+2p}}(t). \tag{2.9}$$

Analogously,

$$\frac{1}{1+2p}(1+S)^{1+2p} = \int_0^t \int_0^1 \sigma^2(y,\tau)dyd\tau + 2\int_0^t \int_0^1 \int_y^x \sigma(\xi,\tau)\frac{\partial U(\xi,\tau)}{\partial \tau}d\xi dyd\tau + \frac{1}{1+2p}$$
$$\geq \frac{1}{2}\int_0^t \int_0^1 \sigma^2(y,\tau)dyd\tau - C_2 = \frac{1}{2}\varphi(t) - C_3. \tag{2.10}$$

We have

$$C_3\,(1+S)^{1+2p} \geq C_3. \tag{2.11}$$

From (2.10) and (2.11) we get

$$\left(\frac{1}{1+2p} + C_3\right)(1+S)^{1+2p} \geq \frac{1}{2}\varphi(t),$$

or

$$1 + S(x,t) \geq c\varphi^{\frac{1}{1+2p}}(t). \tag{2.12}$$

Finally, from (2.9) and (2.12) it follows the following estimate

$$c\varphi^{\frac{1}{1+2p}}(t) \leq 1 + S(x,t) \leq C\varphi^{\frac{1}{1+2p}}(t). \tag{2.13}$$

Taking into account relations (2.8), (2.13) and Theorem 1 we have

$$\frac{d\varphi(t)}{dt} = \int_0^1 (1+S)^{2p}\left(\frac{\partial U}{\partial x}\right)^2 dx \leq C\varphi^{\frac{2p}{1+2p}}(t)\exp(-t),$$

or

$$\frac{d}{dt}\left(\varphi^{\frac{1}{1+2p}}(t)\right) \leq C\exp(-t).$$

After integrating from 0 to t, keeping in mind definition (2.8), we get

$$1 \leq \varphi(t) \leq C.$$

From this, using the estimate (2.13), we receive

$$1 \leq 1 + S(x,t) \leq C. \tag{2.14}$$

Using estimation (2.14), the equality (2.5) and Theorem 1 from (2.7) we obtain

$$\sigma^2(x,t) \leq 2\int_0^1 (1+S)^{2p}\left(\frac{\partial U}{\partial x}\right)^2 dx + \int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx \leq C\exp(-t),$$

or

$$|\sigma(x,t)| \leq C\exp\left(-\frac{t}{2}\right).$$

This estimate, taking into account (2.14) and relation $\sigma = (1+S)^p\frac{\partial U}{\partial x}$ gives

$$\left|\frac{\partial U(x,t)}{\partial x}\right| \leq C\exp\left(-\frac{t}{2}\right). \tag{2.15}$$

Thus, the first part of Theorem 2 has been proven.
Now let us prove the second part of Theorem 2. By differentiating (2.1),

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial}{\partial x}\left[\frac{\partial(1+S)^p}{\partial t}\frac{\partial U}{\partial x} + (1+S)^p\frac{\partial^2 U}{\partial t\,\partial x}\right] = 0, \tag{2.16}$$

and multiplying (2.16) scalarly by $\partial U/\partial t$, and using the formula of integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx+\int_0^1(1+S)^p\left(\frac{\partial^2 U}{\partial t\,\partial x}\right)^2 dx+p\int_0^1(1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^3\frac{\partial^2 U}{\partial t\,\partial x}dx=0. \tag{2.17}$$

Using the Cauchy–Schwarz inequality in (2.17) leads to the relation

$$\frac{d}{dt}\int_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx+\int_0^1(1+S)^p\left(\frac{\partial^2 U}{\partial t\partial x}\right)^2 dx\le p^2\int_0^1(1+S)^{p-2}\left(\frac{\partial U}{\partial x}\right)^6 dx. \tag{2.18}$$

Let us take the inner product of (2.18) by $\exp(2t)$ and integrate the resulting equation over the interval $(0,t)$. By taking into account estimations (2.14), (2.15) and Theorem 1 and by performing simple manipulations, we obtain the inequalities:

$$\begin{aligned}&\int_0^t\exp(2\tau)\frac{d}{d\tau}\int_0^1\left(\frac{\partial U}{\partial \tau}\right)^2 dx\,d\tau+\int_0^t\exp(2\tau)\int_0^1(1+S)^p\left(\frac{\partial^2 U}{\partial x\partial \tau}\right)^2 dx\,d\tau\\&\quad\le p^2\int_0^t\exp(2\tau)\int_0^1(1+S)^{p-2}\left(\frac{\partial U}{\partial x}\right)^6 dx\,d\tau,\\&\int_0^t\exp(2\tau)\int_0^1\left(\frac{\partial^2 U}{\partial x\,\partial \tau}\right)^2 dx\,d\tau\le-\exp(2t)\int_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx+\left.\int_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx\right|_{t=0}\\&\qquad+2\int_0^t\exp(2\tau)\int_0^1\left(\frac{\partial U}{\partial \tau}\right)^2 dx\,d\tau+C\int_0^t\exp(-\tau)d\tau,\end{aligned}$$

or

$$\int_0^t\exp(2\tau)\int_0^1\left(\frac{\partial^2 U}{\partial x\partial \tau}\right)^2 dx\,d\tau\le C\exp(t). \tag{2.19}$$

By taking the inner product of (2.16) by $\exp(2t)\partial^2U/\partial t^2$ and by taking into account (2.15), the a priori estimates (2.14) and (2.19), we obtain the relations:

$$\begin{aligned}&\int_0^t\exp(2\tau)\int_0^1\left(\frac{\partial^2 U}{\partial \tau^2}\right)^2 dx\,d\tau+\frac{1}{2}\int_0^t\int_0^1\exp(2\tau)(1+S)^p\frac{\partial}{\partial\tau}\left(\frac{\partial^2 U}{\partial \tau\,\partial x}\right)^2 dx\,d\tau\\&\quad+p\int_0^t\int_0^1\exp(2\tau)(1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^3\frac{\partial}{\partial\tau}\left(\frac{\partial^2 U}{\partial \tau\,\partial x}\right)dx\,d\tau=0,\\&\frac{\exp(2t)}{2}\int_0^1\left(\frac{\partial^2 U}{\partial t\,\partial x}\right)^2 dx\le\left.\frac{1}{2}\int_0^1\left(\frac{\partial^2 U}{\partial t\,\partial x}\right)^2 dx\right|_{t=0}\\&\quad+\int_0^t\int_0^1\exp(2\tau)(1+S)^p\left(\frac{\partial^2 U}{\partial \tau\,\partial x}\right)^2 dx\,d\tau+\frac{p}{2}\int_0^t\int_0^1\exp(2\tau)(1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^2\left(\frac{\partial^2 U}{\partial \tau\,\partial x}\right)^2 dx\,d\tau\\&\quad-p\exp(2t)\int_0^1(1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^3\frac{\partial^2 U}{\partial t\,\partial x}dx+\left.p\int_0^1\left(\frac{\partial U}{\partial x}\right)^3\frac{\partial^2 U}{\partial t\,\partial x}dx\right|_{t=0}\\&\quad+2p\int_0^t\int_0^1\exp(2\tau)(1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^3\frac{\partial^2 U}{\partial \tau\,\partial x}dx\,d\tau\\&\quad+p(p-1)\int_0^t\int_0^1\exp(2\tau)(1+S)^{p-2}\left(\frac{\partial U}{\partial x}\right)^5\frac{\partial^2 U}{\partial \tau\,\partial x}dx\,d\tau\\&\quad+3p\int_0^t\int_0^1\exp(2\tau)(1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^2\left(\frac{\partial^2 U}{\partial \tau\,\partial x}\right)^2 dx\,d\tau\\&\quad\le C_1+C_2\exp(t)+C_3\int_0^t\exp(2\tau)\exp(-\tau)\int_0^1\left(\frac{\partial^2 U}{\partial \tau\,\partial x}\right)^2 dx\,d\tau\end{aligned}$$

$$+\frac{\exp(2t)}{4}\int_0^1\left(\frac{\partial^2 U}{\partial t\,\partial x}\right)^2 dx + C_4\exp(-t) + C_5\int_0^t \exp(2\tau)\int_0^1\left(\frac{\partial^2 U}{\partial \tau\,\partial x}\right)^2 dx\,d\tau$$
$$+C_5\int_0^t \exp(-\tau)d\tau + C_6\int_0^t \exp(2\tau)\int_0^1\left(\frac{\partial^2 U}{\partial \tau\partial x}\right)^2 dxd\tau + C_6\int_0^t \exp(-3\tau)d\tau$$
$$+C_7\int_0^t \exp(2\tau)\exp(-\tau)\int_0^1\left(\frac{\partial^2 U}{\partial \tau\,\partial x}\right)^2 dx\,d\tau,$$

or

$$\int_0^1\left(\frac{\partial^2 U}{\partial t\,\partial x}\right)^2 dx \le C\exp(-t). \tag{2.20}$$

The a priori estimate (2.20), together with the relation

$$\frac{\partial U(x,t)}{\partial t} = \int_0^1 \frac{\partial U(y,t)}{\partial t}dy + \int_0^1\int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi\partial t}d\xi\,dy$$

and Theorem 1, implies that

$$\left|\frac{\partial U(x,t)}{\partial t}\right| \le C\exp\left(-\frac{t}{2}\right).$$

Therefore, Theorem 2 has been completely proven. □

Using the a priori estimates of this article, the compactness method, a modified version of the Galerkin method [2,18] the existence and uniqueness of the solution can be proven.

Let us note that same results as in Theorems 1 and 2 are true for problem with first type homogeneous conditions on whole boundary (see, for example, [11] and references therein).

3. Semi-discrete scheme

Let us again consider in $(0, 1) \times (0, T)$ problem (2.1)–(2.3) written in the following form:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[\left(1+\int_0^t\left(\frac{\partial U}{\partial x}\right)^2 d\tau\right)^p \frac{\partial U}{\partial x}\right], \tag{3.1}$$

$$U(0,t) = \left.\frac{\partial U(x,t)}{\partial x}\right|_{x=1} = 0,$$
$$U(x,0) = U_0(x), \tag{3.2}$$

where $0 < p \le 1$, T is positive number and U_0 is a given function.

On $[0, 1]$ let us introduce a net with mesh points denoted by $x_i = ih$, $i = 0, 1, \dots, M$, with $h = 1/M$. The boundaries are specified by $i = 0$ and $i = M$. In this section the semi-discrete approximation at (x_i, t) is designed by $u_i = u_i(t)$. The exact solution to the problem at (x_i, t) is denoted by $U_i = U_i(t)$. At points $i = 1, 2, \dots, M-1$, the integro-differential equation will be replaced by approximation of the space derivatives by forward and backward differences. We will use the following known notations [19]:

$$u_{x,i}(t) = \frac{u_{i+1}(t)-u_i(t)}{h}, \qquad u_{\bar{x},i}(t) = \frac{u_i(t)-u_{i-1}(t)}{h}.$$

Let us correspond to problem (3.1)–(3.2) the following semi-discrete scheme:

$$\frac{du_i}{dt} = \left\{\left(1+\int_0^t (u_{\bar{x},i})^2 d\tau\right)^p u_{\bar{x},i}\right\}_x, \quad i = 1, 2, \dots, M-1, \tag{3.3}$$

$$u_0(t) = u_{\bar{x},M}(t) = 0, \tag{3.4}$$

$$u_i(0) = U_{0,i}, \quad i = 0, 1, \dots, M. \tag{3.5}$$

So, we obtained Cauchy problem (3.3)–(3.5) for nonlinear system of ordinary integro-differential equations.

Introduce usual inner products and norms [19]:

$$(u, v) = h \sum_{i=1}^{M-1} u_i v_i, \qquad (u, v] = h \sum_{i=1}^{M} u_i v_i,$$
$$\|u\| = (u, u)^{1/2}, \qquad \|u]| = (u, u]^{1/2}.$$

Multiplying Eqs. (3.3) scalarly by $u(t) = (u_1(t), u_2(t), \ldots, u_{M-1}(t))$, after simple transformations we get

$$\frac{d}{dt}\|u(t)\|^2 + h \sum_{i=1}^{M} \left(1 + \int_0^t \left(u_{\bar{x},i}\right)^2 d\tau\right)^p \left(u_{\bar{x},i}\right)^2 = 0.$$

From this we obtain the inequality

$$\|u(t)\|^2 + \int_0^t \|u_{\bar{x}}\|^2 d\tau \le C, \tag{3.6}$$

where, here and below in this section, C denotes a positive constant which does not depend on h.

The a priori estimate (3.6) guarantees the global solvability of the problem (3.1)–(3.2).

The principal aim of the present section is the proof of the following statement.

Theorem 3. *If problem* (3.1)–(3.2) *has a sufficiently smooth solution $U = U(x, t)$, then for $0 < p \le 1$ the solution $u = u(t) = (u_1(t), u_2(t), \ldots, u_{M-1}(t))$ of problem* (3.1)–(3.2) *tends to $U = U(t) = (U_1(t), U_2(t), \ldots, U_{M-1}(t))$ as $h \to 0$ and the following estimate is true*

$$\left\|u(t) - U(t)\right\| \le Ch. \tag{3.7}$$

Proof. For $U = U(x, t)$ we have:

$$\frac{dU_i}{dt} - \left\{\left(1 + \int_0^t (U_{\bar{x},i})^2 d\tau\right)^p U_{\bar{x},i}\right\}_x = \psi_i(t), \quad i = 1, 2, \ldots, M-1, \tag{3.8}$$

$$U_0(t) = U_{\bar{x},M}(t) = 0, \tag{3.9}$$

$$U_i(0) = U_{0,i}, \quad i = 0, 1, \ldots, M, \tag{3.10}$$

where

$$\psi_i(t) = O(h).$$

Let $z_i(t) = u_i(t) - U_i(t)$. From (3.1)–(3.2) and (3.8)–(3.10) we have:

$$\frac{dz_i}{dt} - \left\{\left(1 + \int_0^t (u_{\bar{x},i})^2 d\tau\right)^p u_{\bar{x},i} - \left(1 + \int_0^t (U_{\bar{x},i})^2 d\tau\right)^p U_{\bar{x},i}\right\}_x = -\psi_i(t), \tag{3.11}$$

$$z_0(t) = z_{\bar{x},M}(t) = 0, \quad z_i(0) = 0.$$

Multiplying Eq. (3.11) scalarly by $z(t) = (z_1(t), z_2(t), \ldots, z_{M-1}(t))$, using the discrete analog of the formula of integration by parts we get

$$\frac{1}{2}\frac{d}{dt}\|z\|^2 + \sum_{i=1}^{M}\left\{\left(1 + \int_0^t (u_{\bar{x},i})^2 d\tau\right)^p u_{\bar{x},i} - \left(1 + \int_0^t (U_{\bar{x},i})^2 d\tau\right)^p U_{\bar{x},i}\right\} z_{\bar{x},i} h = -h \sum_{i=1}^{M-1} \psi_i z_i. \tag{3.12}$$

Note that,

$$\left\{\left(1 + \int_0^t (u_{\bar{x},i})^2 d\tau\right)^p u_{\bar{x},i} - \left(1 + \int_0^t (U_{\bar{x},i})^2 d\tau\right)^p U_{\bar{x},i}\right\}(u_{\bar{x},i} - U_{\bar{x},i})$$
$$= p \int_0^1 \left(1 + \int_0^t \left[U_{\bar{x},i} + \xi(u_{\bar{x},i} - U_{\bar{x},i})\right]^2 d\tau\right)^{p-1} \frac{d}{dt}\left(\int_0^t \left[U_{\bar{x},i} + \xi(u_{\bar{x},i} - U_{\bar{x},i})\right](u_{\bar{x},i} - U_{\bar{x},i}) d\tau\right)^2 d\xi$$
$$+ \int_0^1 \left(1 + \int_0^t \left[U_{\bar{x},i} + \xi(u_{\bar{x},i} - U_{\bar{x},i})\right]^2 d\tau\right)^p d\xi \left(u_{\bar{x},i} - U_{\bar{x},i}\right)^2.$$

After substituting this equality in (3.12), integrating received equality on $(0, t)$ and using formula of integration by parts we get

$$\|z\|^2 + 2h\sum_{i=1}^{M}\int_0^t\int_0^1\left(1+\int_0^{t'}\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right]^2 d\tau'\right)^p \left(u_{\bar{x},i}-U_{\bar{x},i}\right)^2 d\xi\, d\tau$$
$$+ ph\sum_{i=1}^{M}\int_0^1\left(1+\int_0^{t}\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right]^2 d\tau\right)^{p-1}\left(\int_0^t\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right](u_{\bar{x},i}-U_{\bar{x},i})d\tau\right)^2 d\xi$$
$$- p(p-1)h\sum_{i=1}^{M}\int_0^1\int_0^t\left(1+\int_0^{t'}\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right]^2 d\tau'\right)^{p-2}\left[U_{\bar{x},i}+\xi(u_{\bar{x},i}-U_{\bar{x},i})\right]^2$$
$$\times\left(\int_0^{t'}\left[U_{\bar{x},i}+xi(u_{\bar{x},i}-U_{\bar{x},i})\right](u_{\bar{x},i}-U_{\bar{x},i})d\tau'\right)^2 d\xi\, d\tau = -2h\sum_{i=1}^{M-1}\psi_i z_i.$$

Taking into account relation $0 < p \le 1$ we have from last equality

$$\|z(t)\|^2 \le \int_0^t \|z(\tau)\|^2 d\tau + \int_0^t \|\psi_i\|^2 d\tau. \tag{3.13}$$

From (3.13) we get (3.7) and Theorem 3 thus is proved. □

4. Conclusions

Nonlinear integro-differential equation associated with the penetration of a magnetic field in a substance is considered. Long-time behavior of solution of initial–boundary value problem with mixed boundary conditions is studied. The semi-discrete scheme is investigated for that model as well. One must note that convergence of the fully discrete scheme for $p = 1$ can be also proven [10]. It is important to construct and investigate fully discrete finite difference schemes and finite element analogs studied in this note type models for more general type nonlinearities and for multi-dimensional cases as well.

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Original article

On the theorem of F. Riesz in variable Lebesgue space

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Abstract

The aim of present paper is to introduce variable exponent bounded Riesz $p(\cdot)$-variation and describe variable exponent Sobolev space $W^{1,p(\cdot)}$.

Keywords: Variable Lebesgue space; Riesz p-variation; Sobolev $W^{1,p(\cdot)}$ space

1. Introduction

Let $f \in L(a, b)$ and for function F we have following representation

$$F(x) = \int_a^x f(t)dt + F(a). \tag{1.1}$$

According to Lebesgue statement: F can be represented as (1.1) if and only if F is absolutely continuous function (see [1]).

F. Riesz [2] proved that function F can be represented as (1.1) where $f \in L^p$, $(1 < p < +\infty)$ if and only if for every partition of the interval (a, b) the sums

$$\sum_k \frac{|F(x_k) - F(x_{k-1})|^p}{|x_k - x_{k-1}|^{p-1}}$$

are uniformly bounded (such F functions are called functions of bounded Riesz p-variation). Besides this

$$\sup_{\Pi} \sum_k \frac{|F(x_k) - F(x_{k-1})|^p}{|x_k - x_{k-1}|^{p-1}} = \int_a^b |f(x)|^p dx,$$

where Π is the set of all finite partitions of (a, b).

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It must be mentioned that in this statement interval may be infinite.

Z. Cybertowicz and W. Matuszewska generalized the Riesz's above result for the functions from Orlicz space (see [3]). Let φ be a convex function which satisfies following conditions

$$\varphi(t)/t \to 0 \quad \text{as } t \to 0+ \text{ and } \varphi(t)/t \to +\infty \text{ as } t \to +\infty,$$

then function F can be represented as (1.1) where $f \in L^{\varphi}$ if and only if the sums

$$\sum_k \varphi\left(\frac{|F(x_k) - F(x_{k-1})|}{|x_k - x_{k-1}|}\right)(x_k - x_{k-1})$$

are uniformly bounded for every partition of the interval (a, b). Besides this

$$\sup_{\Pi} \sum_k \varphi\left(\frac{|F(x_k) - F(x_{k-1})|}{|x_k - x_{k-1}|}\right)(x_k - x_{k-1}) = \int_a^b \varphi(|f(x)|)dx.$$

Let $(\varphi(n))$ and $(p(n))$ be some positive sequences, such that $p(n) \uparrow +\infty$, $n \to +\infty$ and $\varphi(n) \geq 1$, $p(n) > 1$ for any natural n.

On the finite interval (a, b) we consider the class of measurable functions f for which

$$A := \sup_n \frac{1}{\varphi(n)} \left(\int_a^b |f(x)|^{p(n)} dx\right)^{1/p(n)} < +\infty. \tag{1.2}$$

In [4] we considered an analogue of the Riesz statement for the above defined class of functions.

Namely we prove following:

(1) Let $\liminf_{n \to +\infty} \varphi(n) < +\infty$. Function f is essentially bounded if and only if function f satisfies condition (1.2).
(2) Let $\liminf_{n \to +\infty} \varphi(n) = +\infty$. For any sequence $(p(n))$ there exists a function f which is not essentially bounded but satisfies condition (1.2).
(3) Function F can be represented as (1.1) where f satisfies (1.2) if and only if

$$B := \sup_n \frac{1}{\varphi(n)} \left(\sup_{\Pi} \sum_k \frac{|F(x_k) - F(x_{k-1})|^{p(n)}}{|x_k - x_{k-1}|^{p(n)-1}}\right)^{1/p(n)} < +\infty.$$

Besides of this $A = B$. The analogue of Riesz's statement is true for functions of many variables (see [4]).

In the present paper we get the necessary and sufficient condition for that the function F can be represented as (1.1) where f belongs to the variable exponent Lebesgue space.

Throughout the whole paper, we use C as an absolute positive constant, which may have different values in different occurrences.

2. Variable exponent Lebesgue spaces

Given a measurable function $p : [0, 1] \to [1, +\infty)$. $L^{p(\cdot)}[0, 1]$ denotes the set of all measurable functions f on $[0, 1]$ such that for some $\lambda > 0$

$$\int_{[0,1]} (|f(x)|/\lambda)^{p(x)}\, dx < +\infty.$$

This set becomes a Banach function space with the Luxemburg's norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_{[0,1]} (|f(x)|/\lambda)^{p(x)}\, dx \leq 1\right\}.$$

The corresponding Sobolev space $W^{k,p(\cdot)}$ is defined to be the subset of functions f in $L^{p(\cdot)}[0, 1]$ such that its derivative of the order $k - 1$ is absolutely continuous and the function f and its derivatives up to order k have a finite $L^{p(\cdot)}$ norm, for given exponent.

Note that by definition Sobolev space $W^{1,1}$ coincides with class of absolute continuous functions. Also $W^{1,\infty}$ coincides with class of Lipschitz continuous functions (f is called Lipschitz continuous if there exists a real constant $C \geq 0$ such that, for all x and y in $[0, 1]$ $|f(x) - f(y)| \leq C|x - y|$).

The variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and the corresponding variable exponent Sobolev spaces $W^{k,p(\cdot)}$ are of interest for their applications to the problems in fluid dynamics, partial differential equations with non-standard growth conditions, calculus of variations, image processing, etc. (see [5]).

From Riesz's statement follows that for constant exponent $p(1 < p < \infty)$ Sobolev space $W^{1,p}$ is fully described by the Riesz p-variation, namely class of functions bounded Riesz p-variation coincides with $W^{1,p}$.

The aim of our paper is to introduce variable exponent bounded Riesz $p(\cdot)$-variation and describe Sobolev class of functions $W^{1,p(\cdot)}$.

For the given $p(\cdot)$, the conjugate exponent $p'(\cdot)$ is defined pointwise $p'(x) = p(x)/(p(x) - 1)$, $x \in [0, 1]$. Given a set $Q \subset [0, 1]$ we define some standard notations:

$$p_-(Q) := \operatorname*{essinf}_{x\in Q} p(x), \qquad p_+(Q) := \operatorname*{esssup}_{x\in Q} p(x), \qquad p_- := p_-([0, 1]), \qquad p_+ := p_+([0, 1]).$$

Recall that the Hardy–Littlewood maximal operator is defined for any $f \in L^1[0, 1]$ by

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(t)|dt,$$

where the supremum is taken over all $Q \subset [0, 1]$ intervals containing point x (assume that sets like $[0, a)$ and $(a, 1]$ are also intervals) and $|Q|$ denotes the Lebesgue measure of Q.

Denote by $\mathcal{B}$ the class of all measurable exponents $p(\cdot)$, $1 < p_- \leq p_+ < \infty$ for which the Hardy–Littlewood maximal operator is bounded on the space $L^{p(\cdot)}[0, 1]$. Different aspects concerning this class can be found in monographs [6] and [5].

Definition 2.1. Let $\mathcal{Q} \in \Pi$. We define averaging operator with respect to $\mathcal{Q}$ by

$$T_{\mathcal{Q}} f(x) = \sum_{Q\in\mathcal{Q}} |f|_Q \chi_Q(x),$$

where $f_Q = \frac{1}{|Q|} \int_Q f$.

In [7], L. Diening showed that $p \in \mathcal{B}$ if and only if there exists $C > 0$ such that for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\sup_{\Pi} \|T_{\mathcal{Q}} f\|_{p(\cdot)} \leq C \cdot \|f\|_{p(\cdot)}.$$

T. Kopaliani [8] gives following characterization of exponents in $\mathcal{B}$.

Theorem 2.1. *$p(\cdot) \in \mathcal{B}$ if and only if*

$$\|\chi_Q\|_{p(\cdot)} \asymp |Q|^{\frac{1}{|Q|}\int_Q \frac{1}{p(x)}dx} \quad \textit{and} \quad \|\chi_Q\|_{p'(\cdot)} \asymp |Q|^{\frac{1}{|Q|}\int_Q \frac{1}{p'(x)}dx} \tag{2.1}$$

uniformly for all intervals $Q \subset [0, 1]$.

Throughout the paper under the relationship $A \asymp B$ we mean that there exist absolute constants $C_1 > 0$ and $C_2 > 0$ such that $C_1 \cdot A \leq B \leq C_2 \cdot A$.

For interval Q we define $\bar{p}(Q)$ and $\bar{p}'(Q)$ by

$$\bar{p}(Q) := \left(\frac{1}{|Q|}\int_Q \frac{1}{p(x)}dx\right)^{-1}, \qquad \bar{p}'(Q) := \left(\frac{1}{|Q|}\int_Q \frac{1}{p'(x)}dx\right)^{-1}.$$

Let us define discrete variable Lebesgue space $l^{p,\mathcal{Q}}$

$$l^{p,\mathcal{Q}} := \left\{ \{x_Q\}_{Q\in\mathcal{Q}} : \sum_{Q\in\mathcal{Q}} |x_Q|^{\bar{p}(Q)} < +\infty \right\},$$

which is equipped with the Luxemburg's norm

$$\|x\|_{l^{p,\mathcal{Q}}} = \inf\left\{\lambda > 0 : \sum_{Q\in\mathcal{Q}} |x_Q/\lambda|^{\bar{p}(Q)} \le 1\right\}.$$

As usual $\{e_Q\}_{Q\in\mathcal{Q}}$ is the canonical basis of $l^{p,\mathcal{Q}}$ (e_Q has entry 1 at the index Q and 0 otherwise).

For discrete $l^{p,\mathcal{Q}}$ space its associate space of $l^{p,\mathcal{Q}}$ is defined by

$$(l^{p,\mathcal{Q}})' = \left\{\{y_Q\}_{Q\in\mathcal{Q}} : \sum_{Q\in\mathcal{Q}} |x_Q \cdot y_Q| < +\infty, \text{for all } x \in l^{p,\mathcal{Q}}\right\}$$

and endowed with associate norm

$$\|y\|_{(l^{p,\mathcal{Q}})'} = \sup\left\{\sum_{Q\in\mathcal{Q}} |x_Q \cdot y_Q| : \|x\|_{l^{p,\mathcal{Q}}} \le 1\right\}.$$

Note that in this case the associate space of $l^{p,\mathcal{Q}}$ is equal to $l^{p',\mathcal{Q}}$ and norms $\|\cdot\|_{(l^{p,\mathcal{Q}})'}$ and $\|\cdot\|_{l^{p',\mathcal{Q}}}$ are equivalent. Also we have Hölder's inequality

$$\sum_{Q\in\mathcal{Q}} |x_Q \cdot y_Q| \le C\|x\|_{l^{p,\mathcal{Q}}} \cdot \|y\|_{l^{p',\mathcal{Q}}}, \quad x \in l^{p,\mathcal{Q}}, y \in l^{p',\mathcal{Q}}. \tag{2.2}$$

Let us now provide the auxiliary result (see [7, Theorem 4.2], [9, Lemma 2.3]) which we will use in the proof of our result.

Lemma 2.1. *Let $p \in \mathcal{B}$. Then*

$$\left\|\sum_{Q\in\mathcal{Q}} x_Q\chi_Q\right\|_{p(\cdot)} \asymp \left\|\sum_{Q\in\mathcal{Q}} x_Q\|\chi_Q\|_{p(\cdot)}e_Q\right\|_{l^{p,\mathcal{Q}}},$$

uniformly for all $\mathcal{Q} \in \Pi$ and all sequences $\{x_Q\}_{Q\in\mathcal{Q}}$, $x_Q \in \mathbb{R}$.

3. Main result

Let $\mathcal{Q} = \{Q_i\}$ be a finite partition of [0,1]. Let Π denote the set of all possible finite partitions of [0,1].

Let us define the class of functions of bounded Riesz $p(\cdot)$-variation.

Definition 3.1. We say that F is function of bounded Riesz $p(\cdot)$-variation if

$$D(F) := \sup_{\Pi}\sum_i \frac{|F(x_i) - F(x_{i-1})|^{\bar{p}_{Q_i}}}{(x_i - x_{i-1})^{\bar{p}_{Q_i}-1}} < +\infty.$$

It is clear that if exponent is constant $p(x) = p$ for all $x \in [0, 1]$, then the class of functions of bounded Riesz $p(\cdot)$-variation coincides with the class of functions of bounded Riesz p-variation.

The following theorem gives the characterization of Sobolev $W^{1,p(\cdot)}$ space by the bounded Riesz $p(\cdot)$-variation.

Theorem 3.1. *Let $1 < p_- \le p_+ < +\infty$.*

(i) *If $D(F) < +\infty$, then F can be represented as* (1.1) *where $f \in L^{p(\cdot)}[0, 1]$;*
(ii) *If $p(\cdot) \in \mathcal{B}$ and F is represented as* (1.1) *where $f \in L^{p(\cdot)}[0, 1]$, then $D(F) < +\infty$.*

Proof. (i) Let $D(F) < +\infty$ and $\mathcal{Q}$ is some finite family of disjoint intervals $Q_i = (x_i; y_i)$ from (0; 1). Using (2.2) we obtain

$$\sum_i |F(y_i) - F(x_i)| = \sum_i \frac{|F(y_i) - F(x_i)|}{|Q_i|^{(\bar{p}_{Q_i}-1)/\bar{p}_{Q_i}}} |Q_i|^{(\bar{p}_{Q_i}-1)/\bar{p}_{Q_i}}$$
$$\leq C \cdot \left\| \left\{ \frac{|F(y_i) - F(x_i)|}{|Q_i|^{(\bar{p}_{Q_i}-1)/\bar{p}_{Q_i}}} \right\}_{Q_i \in \mathcal{Q}} \right\|_{l^{p,\mathcal{Q}}} \cdot \left\| \left\{ |Q_i|^{1/\bar{p}'_{Q_i}} \right\}_{Q_i \in \mathcal{Q}} \right\|_{l^{p',\mathcal{Q}}}.$$

Without loss of generality suppose that $D(F) \geq 1$. By definition of $D(F)$ we have

$$1 \geq \sum_i \left(\frac{|F(y_i) - F(x_i)|}{|Q_i|^{(\bar{p}_{Q_i}-1)/\bar{p}_{Q_i}}} \cdot \frac{1}{(D(F))^{1/\bar{p}_{Q_i}}} \right)^{\bar{p}_{Q_i}}$$
$$\geq \sum_i \left(\frac{|F(y_i) - F(x_i)|}{|Q_i|^{(\bar{p}_{Q_i}-1)/\bar{p}_{Q_i}}} \cdot \frac{1}{(D(F))^{1/p_-}} \right)^{\bar{p}_{Q_i}},$$

consequently by the definition of the norm in the space $l^{p,\mathcal{Q}}$ we get

$$\left\| \left\{ \frac{|F(y_i) - F(x_i)|}{|Q_i|^{(\bar{p}_{Q_i}-1)/\bar{p}_{Q_i}}} \right\}_{Q_i \in \mathcal{Q}} \right\|_{l^{p,\mathcal{Q}}} \leq (D(F))^{1/p_-}.$$

Analogously it is clear that

$$\left\| \left\{ |Q_i|^{1/\bar{p}'_{Q_i}} \right\}_{Q_i \in \mathcal{Q}} \right\|_{l^{p',\mathcal{Q}}} \leq \left(\sum_i |Q_i| \right)^{1/p'_+}.$$

Finally by the last two inequalities we obtain

$$\sum_i |F(y_i) - F(x_i)| \leq C \cdot (D(F))^{1/p_-} \cdot \left(\sum_i |Q_i| \right)^{1/p'_+}.$$

From this inequality we conclude that F is an absolutely continuous function.

Let us for each natural n divide (0, 1) into 2^n equal intervals $Q_i = (x_{i-1}; x_i)$, ($i \in \{1, \ldots, 2^n\}$) and construct step function f_n

$$f_n(t) = \sum_{i=1}^{2^n} \left| \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} \right|^{\bar{p}_{Q_i}} \chi_{Q_i}(t).$$

Since $p(\cdot)$ is a measurable function and $1 < p_- \leq p_+ < +\infty$, it is integrable in Lebesgue sense, therefore by Lebesgue differentiation theorem we have

$$\lim_{\substack{n\to+\infty \\ t\in Q_i}} \bar{p}_{Q_i} = \lim_{\substack{n\to+\infty \\ t\in Q_i}} \left(\frac{1}{|Q_i|} \int_{Q_i} \frac{dx}{p(x)} \right)^{-1} = p(t), \tag{3.1}$$

for almost all $t \in [0, 1]$. By (3.1) and the fact that F is absolutely continuous function we get that $f_n(t) \to |F'(t)|^{p(t)}$, $n \to +\infty$ for almost all $t \in [0, 1]$. Therefore by Fatou's lemma we obtain

$$\int_0^1 |F'(t)|^{p(t)} dt = \int_0^1 \lim_{n\to+\infty} f_n(t) dt = \int_0^1 \liminf_{n\to+\infty} f_n(t) dt$$
$$\leq \liminf_{n\to+\infty} \int_0^1 f_n(t) dt = \liminf_{n\to+\infty} \sum_{i=1}^{2^n} \int_{Q_i} f_n(t) dt$$
$$= \liminf_{n\to+\infty} \sum_{i=1}^{2^n} \int_{Q_i} \left| \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} \right|^{\bar{p}_{Q_i}} \chi_{Q_i}(t) dt$$

$$= \liminf_{n\to+\infty} \sum_{i=1}^{2^n} \frac{|F(x_i) - F(x_{i-1})|^{\bar{p}_{Q_i}}}{(x_i - x_{i-1})^{\bar{p}_{Q_i}-1}}$$

$$\leq \sup_{\Pi} \sum_{i=1}^{2^n} \frac{|F(x_i) - F(x_{i-1})|^{\bar{p}_{Q_i}}}{(x_i - x_{i-1})^{\bar{p}_{Q_i}-1}} = D(F) < +\infty.$$

This implies that $F' \in L^{p(\cdot)}[0, 1]$.

(ii) Let $p \in \mathcal{B}$, $f \in L^{p(\cdot)}[0, 1]$ and $F(x) = \int_0^x f(t)dt + F(0)$ then

$$\left|\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}\right| = \frac{1}{x_i - x_{i-1}} \left|\int_{x_{i-1}}^{x_i} f(t)dt\right| = |f_{Q_i}| \leq |f|_{Q_i}. \tag{3.2}$$

By (2.1) we have $\|\chi_{Q_i}\|_{p(\cdot)} \asymp |Q_i|^{1/\bar{p}_{Q_i}}$. By (3.2) and the fact that $\bar{p}_{Q_i}$ numbers are uniformly bounded we get

$$\sum_i \frac{|F(x_i) - F(x_{i-1})|^{\bar{p}_{Q_i}}}{(x_i - x_{i-1})^{\bar{p}_{Q_i}-1}} = \sum_i \left(\frac{|F(x_i) - F(x_{i-1})|}{x_i - x_{i-1}}\right)^{\bar{p}_{Q_i}} (x_i - x_{i-1})$$

$$\leq \sum_i \left(|f|_{Q_i}\right)^{\bar{p}_{Q_i}} |Q_i| = \sum_i \left(|f|_{Q_i} |Q_i|^{1/\bar{p}(Q_i)}\right)^{\bar{p}_{Q_i}} \leq C \sum_i \left(|f|_{Q_i} \|\chi_{Q_i}\|_{p(\cdot)}\right)^{\bar{p}_{Q_i}}. \tag{3.3}$$

Since boundedness of the Hardy–Littlewood maximal operator implies the boundedness of the averaging operator $T_{\mathcal{Q}}$ in $L^{p(\cdot)}$ then by Lemma 2.1 we get

$$\left\|\sum_{Q\in\mathcal{Q}} |f|_Q \|\chi_Q\|_{p(\cdot)} e_Q\right\|_{l^{p},\mathcal{Q}} \asymp \left\|\sum |f|_Q \chi_Q\right\|_{p(\cdot)} = \|T_{\mathcal{Q}} f\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}, \tag{3.4}$$

uniformly for all $\mathcal{Q} \in \Pi$. By (3.3) and (3.4) we get that $D(F) < +\infty$.

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Original article

On one problem of the plane theory of elasticity for a circular domain with a rectangular hole

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Abstract

The paper considers a plane problem of elasticity for a circle with a rectangular hole. To find a solution, the use is made both of the method of conformal mappings and of boundary value problems of analytic functions. In particular, relying on the well-known Kolosov–Muskhelishvili's formulas, the problem formulated with respect to unknown complex potentials is reduced to the two Riemann–Hilbert problems for a circular ring, and the solutions of the latter problems allow us to construct potentials effectively (analytically). The estimates of the obtained results in the neighborhood of angular points are given. Analogous results (as a particular case) are obtained for a circular domain with a rectilinear cut.

Keywords: Conformal mapping; Kolosov–Muskhelishvili's formulas; The Riemann–Hilbert problem for a circular ring

Statement of the Problem. Let S be a doubly-connected domain occupied by a plate on the plane $z = x + iy$ of a complex variable, bounded by circumference $L_0 = \{|z| = R_0\}$ and rectangle $B_1B_2B_3B_4$ whose sides are parallel to the coordinate axes. By L_1 we denote the boundary of the rectangle (that is, $L_1 = \cup_{k=1}^{4} L_k^{(1)}$, $L_k^{(1)} = B_kB_{k+1}$, $k = \overline{1,4}$, $B_5 = B_1$) and assume that the sides B_1B_2 and B_3B_4 (parallel to the ox-axis) are under the action of constant, normal compressive forces with the given principal vector P (or normal displacements $v_n(t) = v_n^{(k)} = \text{const}$, $t \in L_k^{(1)}$, $k = \overline{1,4}$ are given), and the rest of the boundary $L = L_0 \cup L_1$ is free from the external forces.

Note that certain simplifications in the statement of the problem concerning the cut forms and external forces are insignificant and motivated only to make the problem more clear, namely, to find elastic equilibrium of the plate for a finite doubly-connected domain.

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Analogous problems of the plane theory of elasticity and plate bending for finite doubly-connected domains bounded by polygons have been considered in [1,2].

Solution of the Problem. As is known, the more effective ways of solving the boundary value problems of the plane theory of elasticity by the methods of complex analysis are based on the construction of a conformally mapping function of the given domain onto canonical domains (circle, circular ring). Therefore the above-mentioned methods are little-suited for the effective solution of problems in multi-connected domains. Nevertheless, for some practically important classes of multi-connected domains one manages to construct effectively (analytically) the conformally mapping function of that domain onto a circular ring. These classes involve doubly-connected domains bounded by polygons and their modifications (polygonal domain with a circular hole, or a circle with a polygonal hole). Moreover, the Kolosov–Muskhelishvili's methods in the above-mentioned case allow one to decompose these problems (with respect to complex potentials $\varphi(z)$ and $\psi(z)$) into two Riemann–Hilbert problems for a circular ring, and by solving the latter problems to construct unknown potential in analytical form.

Here we present some results (see [3]) dealing with conformal mapping of a doubly-connected domain, bounded by a polygon, onto a circular ring.

(1) The Dirichlet Problem for a Circular Ring. Let $\mathcal{D}(1 < |z| < R)$ be a circular ring bounded by circumferences $\ell_0(|z| = R)$ and $\ell_1(|z| = 1)$. We consider the problem: find a holomorphic in the ring $\mathcal{D}$ function $\varphi_*(z) = u + iv$ under the boundary condition

$$\mathrm{Re}[\varphi_*(t)] = f_j(t), \quad t \in \ell_j,\ j = 0, 1. \tag{1}$$

The necessary and sufficient condition for solvability of problem (1) is of the form

$$\int_0^{2\pi} f_0(t)d\vartheta = \int_0^{2\pi} f_1(t)d\vartheta \tag{2}$$

and a solution itself is given by the formula

$$\varphi_*(z) = \frac{1}{\pi i}\sum_{j=-\infty}^{\infty}\left[\int_{\ell_0}\frac{f_0(t)}{t - R^{2j}z}dt + \int_{\ell_1}\frac{f_1(t)}{t - R^{2j}z}dt\right] + ik_1,$$

where k_1 is an arbitrary real constant. Integration on ℓ_0 and ℓ_1 taken in the positive direction leaves the domain D at the left.

(2) Conformal Mapping of a Doubly-Connected Domain, Bounded by Polygons, onto a Circular Ring. Let S^0 be the doubly-connected domain on the plane z of a complex variable, bounded by convex polygons (A) and (B). Assume that (A) is an outer and (B) is an interior boundary of the domain S^0; by A_k $(k = 1, \ldots, n)$ and $B_m (m = 1, \ldots, p)$ we denote the vertices (and their affixes) and by $L_0^{(k)}$ and $L_1^{(k)}$ the sides of polygons (A) and (B). By $\pi\alpha_k^0$ and $\pi\beta_m^0$ we denote the sizes of inner angles S^0 at the vertices A_k and B_m, and the angles lying between the ox-axis and exterior normals to the contours L_0 $\left(L_0 = \cup_{k=1}^{n} L_k^{(0)}\right)$ and L_1 $\left(L_1 = \cup_{m=1}^{p} L_m^{(1)}\right)$ we denote by $\alpha(t)$ and $\beta(t)$; the positive direction on $L = L_0 \cup L_1$ is taken that which leaves the domain S^0 at the left.

Consider the problem: find the type of the function $z = \omega_0(\zeta)$ conformally mapping the circular ring $D(1 < |\zeta| < R)$ onto the domain S_0.

From the equalities

$$t - A_k = i|t - A_k|e^{i\alpha_k(t)}, \quad t \in L_0^{(k)}; \qquad t - B_m = i|t - B_m|e^{i\beta_m(t)},\ t \in L_1^{(m)},$$

we get

$$\begin{aligned} &\mathrm{Re}\big[t \cdot e^{-i\alpha(t)}\big] = \mathrm{Re}\big[A(t) \cdot e^{-i\alpha(t)}\big], \quad t \in L_0; \\ &\mathrm{Re}\big[t \cdot e^{-i\beta(t)}\big] = \mathrm{Re}\big[B(t) \cdot e^{-i\beta(t)}\big], \quad t \in L_1, \end{aligned} \tag{3}$$

where $A(t)$, $B(t)$, $\alpha(t)$ and $\beta(t)$ are the piecewise constant functions;

$$\begin{aligned} &A(t) = A_k; \qquad \alpha(t) = \alpha_k(t), \quad t \in L_0^{(k)}; \\ &B(t) = B_m, \qquad \beta(t) = \beta_m(t), \quad t \in L_1^{(m)}. \end{aligned}$$

From conditions (4) regarding the function $\omega_0(\zeta)$ (after differentiation with respect to the abscissa s), we obtain for the circular ring $\mathcal{D}$ the following Riemann–Hilbert boundary value problem (see [3]):

$$\begin{aligned}
&\mathrm{Re}\big[i\sigma\cdot e^{-i\alpha_0(\sigma)}\omega_0'(\sigma)\big]=0,\quad \sigma\in\ell_0\ (|\zeta|=R);\\
&\mathrm{Re}\big[i\sigma\cdot e^{-i\beta_0(\sigma)}\omega_0'(\sigma)\big]=0,\quad \sigma\in\ell_1\ (|\zeta|=1);\\
&\alpha_0(\sigma)=\alpha[\omega_0(\sigma)];\qquad \beta_0(\sigma)=\beta[\omega_0(\sigma)].
\end{aligned}\tag{4}$$

The boundary value problem (5) with respect to the function $\ln\omega_0'(\zeta)$ is reduced in its turn to the Dirichlet problem (1) whose condition of solvability (2) in the class $h(b_1,\dots,b_p)$ (for this class, see [4], §82) has the form

$$\prod_{k=1}^{n}\Big(\frac{a_k}{R}\Big)^{\alpha_k^0-1}\prod_{m=1}^{p}(b_m)^{\beta_m^0-1}=1,$$

(a_k and b_k are the preimages of the points A_k and B_m), and the solution itself of the given class is given by the formula

$$\omega_0'(\zeta)=K_*^0\prod_{j=-\infty}^{\infty}G(R^{2j}\zeta)\,g(R^{2j}\zeta)R^{2\delta_j},$$

where

$$G(\zeta)=\prod_{k=1}^{n}(\zeta-a_k)^{\alpha_k^0-1};\qquad g(\zeta)=\prod_{m=1}^{p}(\zeta-b_m)^{\beta_m^0-1};\qquad \delta_j=\begin{cases}0, & j\ge 0,\\ 1, & j\le -1;\end{cases}$$

K_*^0 is an arbitrary real constant.

Using the above results for the domain S^0 under the condition that (A) is the right n-angle, and (B) is the given rectangle, and considering the domain S as a limiting case S^0 for $n\to\infty$ (in this case $\alpha_0(\sigma)\to\gamma(\sigma)$, where $\gamma(\sigma)=\arg\sigma$, $\sigma\in\ell_0$, $\alpha_k^0\to 1(k=1,2,\dots,n)$), we find that the derivative of the conformally mapping function $z=\omega(\zeta)$ of the circular ring D $(1<|\zeta|<R)$ onto the domain S is a solution of the Riemann–Hilbert problem

$$\mathrm{Re}\big[i\,\omega'(\sigma)\big]=0,\quad \sigma\in\ell_0;\qquad \mathrm{Re}\big[i\,\sigma\,e^{-i\beta_0(\sigma)}\,\omega'(\sigma)\big]=0,\quad \sigma\in\ell_1,\tag{5}$$

and under the condition

$$\prod_{m=1}^{4}(b_m)^{1/2}=1$$

it has the form

$$\omega'(\zeta)=K^0\prod_{m=1}^{4}\Big(1-\frac{b_m}{\zeta}\Big)^{\frac12}\prod_{j=1}^{\infty}\Big(1-\frac{\zeta}{R^{2j}b_m}\Big)^{\frac12}\Big(1-\frac{b_m}{R^{2j}\zeta}\Big)^{\frac12}.\tag{6}$$

(K^0 is an arbitrary real constant).

We get back now to the problem under consideration. On the basis of the well-known Kolosov–Muskhelishvili's formulas (see [5], §41) for finding complex potentials $\varphi(z)$ and $\psi(z)$ in this case we have the boundary conditions

$$\begin{aligned}
&\mathrm{Re}\big[\varphi(t)+t\,\overline{\varphi'(t)}+\overline{\psi(t)}\big]=\mathcal{D}_1,\quad t\in L_0,\\
&\mathrm{Re}\big[\varkappa\varphi(t)-t\,\overline{\varphi'(t)}-\overline{\psi(t)}\big]=0,\quad t\in L_0,\\
&\mathrm{Re}\big[e^{-i\beta(t)}\big(\varphi(t)+t\,\overline{\varphi'(t)}+\overline{\psi(t)}\big)\big]\\
&\quad=\mathrm{Re}\bigg[e^{-i\beta(t)}\bigg(i\int_0^s(N(t_0)+iT(t_0))e^{i\beta(t_0)}ds_0+c_1+ic_2\bigg)\bigg],\quad t\in L_1,\\
&\mathrm{Re}\big[e^{-i\beta(t)}(\varkappa\varphi(t))-t\,\overline{\varphi'(t)}-\overline{\psi(t)}\big]=2\mu v_n(t),\quad t\in L_1,
\end{aligned}$$

where $\varkappa=\frac{\lambda+3\mu}{\lambda+\mu}$ is the Muskhelishvili's constant, λ and μ are the Lame constants, $N(t)$ and $T(t)$ are normal and tangential stresses, respectively.

The obtained conditions in their turn reduce to the two problems

$$\mathrm{Re}\big[\varphi(t)\big] = F_0(t), \quad t \in L_0; \qquad \mathrm{Re}\big[e^{-i\beta(t)}\varphi(t)\big] = F_1(t), \quad t \in L_1; \tag{7}$$

$$\begin{aligned} &\mathrm{Re}\big[\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)}\big] = \Gamma_0(t), \quad t \in L_0; \\ &\mathrm{Re}\big[e^{-i\beta(t)}\big(\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)}\big)\big] = \Gamma_1(t), \quad t \in L_1, \end{aligned} \tag{8}$$

where

$$F_0(t) = (\varkappa+1)^{-1}D_1, \quad t \in L_0; \qquad F_1(t) = (\varkappa+1)^{-1}\big[c_2 + 2\mu v_n(t)\big], \quad t \in L_1^{(1)};$$

$$F_1(t) = (\varkappa+1)^{-1}(P + c_1), \quad t \in L_2^{(1)};$$

$$F_1(t) = -(\varkappa+1)^{-1}\big[c_2 + 2\mu v_n(t)\big], \quad t \in L_3^{(1)};$$

$$F_1(t) = (\varkappa+1)^{-1}c_1, \quad t \in L_4^{(1)}; \qquad \Gamma_0(t) = D_1, \quad t \in L_0; \qquad \Gamma_1(t) = c_2, \quad t \in L_1^{(1)};$$

$$\Gamma_1(t) = P + c_1, \quad t \in L_2^{(1)}; \qquad \Gamma_1(t) = -c_2, \quad t \in L_3^{(1)}; \qquad \Gamma_1(t) = -c_1, \quad t \in L_4^{(1)}.$$

(D_1, c_1, c_2 are arbitrary real constants, one of which, for example D_1, may be assumed to be zero).

After the domain S is mapped onto the circular ring $D(1 < |\zeta| < R)$, with respect to the function $\varphi_0(\zeta) = \varphi[\omega(\zeta)]$, from (9) we obtain the Riemann–Hilbert boundary value problem for the circular ring

$$\mathrm{Re}\big[\varphi_0(\sigma)\big] = F_{00}(\sigma), \quad \sigma \in \ell_0; \qquad \mathrm{Re}\big[e^{-i\beta_0(\sigma)}\varphi_0(\sigma)\big] = F_{10}(\sigma), \quad \sigma \in \ell_1, \tag{9}$$

where

$$F_{00}(\sigma) = F_0[\omega(\sigma)], \quad \sigma \in \ell_0; \qquad F_{10}(\sigma) = F_1[\omega(\sigma)], \quad \sigma \in \ell_1; \qquad \beta_0(\sigma) = \beta[\omega(\sigma)].$$

Let us consider both the homogeneous problem corresponding to problem (9),

$$\mathrm{Re}\big[\varphi_0(\sigma)\big] = 0, \quad \sigma \in \ell_0; \qquad \mathrm{Re}\big[e^{-i\beta_0(\sigma)}\varphi_0(\sigma)\big] = 0, \quad \sigma \in \ell_1,$$

and the auxiliary problem

$$\mathrm{Re}\big[i\sigma\,\chi_0(\sigma)\big] = 0, \quad \sigma \in \ell_0; \qquad \mathrm{Re}\big[i\sigma\,e^{-i\beta_0(\sigma)}\chi_0(\sigma)\big] = 0, \quad \sigma \in \ell_1. \tag{10}$$

We will seek for solutions of that problem of the class $h(b_1, \dots, b_4)$. The index of problem (10) of that class equals -2.

Taking now into account that the function

$$T(\zeta) = \Big(1 - \frac{\zeta}{R}\Big)^{-2}\prod_{j=1}^{\infty}\Big(1 - \frac{\zeta}{R\cdot R^{2j}}\Big)^{-2}\Big(1 - \frac{R}{R^{2j}\zeta}\Big)^{-2}$$

satisfies the conditions

$$\frac{T(\sigma)}{\overline{T(\sigma)}} = \frac{\overline{\sigma}}{\sigma}, \quad \sigma \in \ell_0; \qquad \frac{T(\sigma)}{\overline{T(\sigma)}} = 1, \quad \sigma \in \ell_1,$$

and writing conditions (8) in an expanded form

$$\omega'(\sigma) - \overline{\omega'(\sigma)} = 0, \quad \sigma \in \ell_0; \qquad \omega'(\sigma) - \frac{\overline{\sigma}}{\sigma}\,e^{2i\beta_0(\sigma)}\overline{\omega'(\sigma)} = 0, \quad \sigma \in \ell_1,$$

we conclude that the function $\chi_0(\zeta)$ can be represented by the formula

$$\chi_0(\zeta) = K_1 T(\zeta)\,\omega'(\zeta)$$

(K_1 is an arbitrary real constant).

Thus with respect to the function $\Theta(\zeta) = K_1\zeta T(\zeta)\,\omega'(\zeta)$, we obtain the equalities

$$\frac{\Theta(\sigma)}{\overline{\Theta(\sigma)}} = 1, \quad \sigma \in \ell_0; \qquad \frac{\Theta(\sigma)}{\overline{\Theta(\sigma)}} = e^{2i\beta_0(\sigma)}, \quad \sigma \in \ell_1$$

and, hence, boundary conditions (9) can be rewritten in the form

$$\begin{aligned} &\mathrm{Re}\left[\frac{\varphi_0(\sigma)}{\sigma\, T(\sigma)\, \omega'(\sigma)}\right] = \frac{F_{00}(\sigma)}{\sigma\, T(\sigma)\, \omega'(\sigma)}, \quad \sigma \in \ell_0; \\ &\mathrm{Re}\left[\frac{\varphi_0(\sigma)}{\sigma\, T(\sigma)\, \omega'(\sigma)}\right] = \frac{F_{10}(\sigma)\, e^{i\beta_0(\sigma)}}{\sigma\, T(\sigma)\, \omega'(\sigma)}, \quad \sigma \in \ell_1. \end{aligned} \tag{11}$$

The condition of solvability of problem (11) has the form (see item 1):

$$\int_{\ell_0} \frac{F_{00}(\sigma)}{\sigma\, T(\sigma)\, \omega'(\sigma)}\, \frac{d\sigma}{\sigma} = \int_{\ell_1} \frac{F_{10}(\sigma)\, e^{i\beta_0(\sigma)}}{\sigma\, T(\sigma)\, \omega'(\sigma)}\, \frac{d\sigma}{\sigma} \tag{12}$$

and the solution itself is given by the formula

$$\varphi_0(\zeta) = \zeta\, T(\zeta)\, \omega'(\zeta)\, M(\zeta),$$

where

$$M(\zeta) = \frac{1}{\pi i} \sum_{j=-\infty}^{\infty} \left[\int_{\ell_0} \frac{F_{00}(\sigma) d\sigma}{(\sigma - R^{2j}\zeta)\sigma\, T(\sigma)\, \omega'(\sigma)} + \int_{\ell_1} \frac{F_{10}(\sigma)\, e^{i\beta_0(\sigma)} d\sigma}{(\sigma - R^{2j}\zeta)\sigma\, T(\sigma)\, \omega'(\sigma)}\right] + E_0 + i E_1, \tag{13}$$

$$E_0 = -\frac{1}{2\pi i} \int_{\ell_0} \frac{F_{00}(\sigma)}{\sigma\, T(\sigma)\, \omega'(\sigma)}\, \frac{s\sigma}{\sigma};$$

E_1 is an arbitrary real constant.

Taking into account that the function $T(\zeta)$ at the point $\zeta = R$ has the pole of the second order, we conclude that for the function $\varphi_0(\zeta)$ to be continuously extendable in the domain $D \cup \ell$, it is necessary and sufficient that the conditions

$$M(R) = 0; \qquad M'(R) = 0 \tag{14}$$

be fulfilled.

Denoting $M_0(\zeta) = T(\zeta)\, M(\zeta)$ and taking into account that $\varphi(z) = \varphi[\omega(\zeta)] = \varphi_0(\zeta)$, and hence $\varphi'(z) = \varphi_0'(\zeta)[\omega'(\zeta)]^{-1}$, we obtain

$$\varphi'(z) = \frac{\varphi_0'(\zeta)}{\omega'(\zeta)} = M_0(\zeta) + \zeta \frac{\omega''(\zeta)}{\omega'(\zeta)}\, M_0(\zeta) + \zeta\, M_0'(\zeta). \tag{15}$$

On the basis of the results given in [4] (§26) concerning the behavior of the Cauchy type integral in the vicinity of points of density discontinuity, we can conclude that in the vicinity of the point $b_k (k = 1, \dots, 4)$ the function $M_0(\zeta)$ can be represented in the form

$$M_0(\zeta) = \frac{k}{(\zeta - b_k)^{1/2}} + \Omega_k^0(\zeta), \quad k = 1, \dots, 4,$$

where k is the definite constant and $\Omega_k^0(\zeta)$ in the vicinity of the point b_k admits the estimate

$$\left|\Omega_0^{(k)}(\zeta)\right| < \frac{C}{|(\zeta - b_k)^{\alpha_0}|}, \quad C = \text{const},\ 0 < \alpha_0 < 1/2.$$

As is known (see [6], §37), for the conformally mapping function $\omega(\zeta)$ in the vicinity of angular points the estimates

$$\begin{aligned} &\omega(\zeta) = B_k + (\zeta - b_k)^{\beta_k^0}\, \Omega_k(\zeta), \\ &\zeta\, \frac{\omega''(\zeta)}{\omega'(\zeta)} = \frac{b_k(\beta_k^0 - 1)}{\zeta - b_k} + \Omega_k^*(\zeta), \quad k = 1, \dots, 4, \end{aligned}$$

hold, where $\Omega_k(b_k) \neq 0$, $\Omega_k^*(\zeta)$ is the right part of the Laurent decomposition of the function $\zeta\, \frac{\omega''(\zeta)}{\omega'(\zeta)}$.

Taking the above-said into account, from (15) we obtain the estimate

$$\varphi'(z) = \frac{K_0}{(\zeta - b_k)^{1/2}} + Q_0^k(\zeta), \quad k = 1, \ldots, 4, \ K_0 = \frac{1}{2} K,$$

and hence, in the vicinity of the point B (B is one of the points $B_k (k = 1, \ldots, 4)$), we have the estimates

$$\left|\varphi'(z)\right| < M_1 \left|z - B\right|^{-1/3}; \qquad \left|\varphi''(z)\right| < M_2 \left|z - B\right|^{-4/3}; \quad M_1, M_2 = \text{const}. \tag{16}$$

After the function $\varphi(z)$ is defined, the finding of the function $\psi(z)$ by virtue of (8) reduces to the problem, analogous to problem (7),

$$\operatorname{Re}\big[R(t)\big] = N_0(t), \quad t \in L_0; \qquad \operatorname{Re}\big[e^{i\beta(t)} R(t)\big] = N_1(t), \quad t \in L_1, \tag{17}$$

where

$$R(z) = \psi(z) + P(z)\, \varphi'(z);$$
$$N_0(t) = \Gamma_0(t) - \operatorname{Re}\big[(\overline{\varphi(t)}) + (\bar{t} - P(t))\varphi'(t)\big], \quad t \in L_0;$$
$$N_1(t) = \Gamma_1(t) - \operatorname{Re}\big[e^{i\beta(t)}(\overline{\varphi(t)}) + (\bar{t} - P(t))\varphi'(t)\big], \quad t \in L_1,$$

$P(z)$ is the interpolated polynomial satisfying the condition $P(B_k) = \overline{B_k} (k = 1, \ldots, 4)$ and having the form

$$P(z) = \frac{(z - B_2) \cdots (z - B_4)}{(B_1 - B_2) \cdots (B_1 - B_4)} \cdot \overline{B_1} + \cdots + \frac{(z - B_1) \cdots (z - B_3)}{(B_4 - B_1) \cdots (B_4 - B_3)} \cdot \overline{B_4}.$$

Insertion of the polynomial $P(z)$ into consideration ensures the boundedness of the right-hand side in the boundary condition (17), and thus, a solution of that problem can be constructed analogously to the previous one (see problem (7)), namely, after the domain D is conformally mapped on S, the factorization of problem (14) is written as follows:

$$1 = \frac{\overline{\sigma}\, \overline{\omega'(\sigma)}\, \overline{T(\sigma)}}{\sigma\, \omega'(\sigma)\, T(\sigma)}, \quad \sigma \in \ell_0; \qquad e^{-2i\beta_0(\sigma)} = \frac{\overline{\sigma}\, \overline{\omega'(\sigma)}\, \overline{T(\sigma)}}{\sigma\, \omega'(\sigma)\, T(\sigma)}, \quad \sigma \in \ell_1,$$

and the condition of solvability will have the form

$$\int_{\ell_0} N_0(t)\, \omega'(t)\, T(t) dt = \int_{\ell_1} N_1(t)\, e^{-i\beta(t)} \omega'(t)\, T(t) dt. \tag{18}$$

If this condition is fulfilled, the solution of problem (17) is represented by the formula

$$R_0(\zeta) = \big[\zeta\, T(\zeta)\, \omega'(\zeta)\big]^{-1} M_1(\zeta),$$

where $R_0(\zeta) = R[\omega(\zeta)]$, and the function $M_1(\zeta)$ having the form, analogous to the function $M(\zeta)$ (see formula (13)) and involving one arbitrary real constant E_2, can be easily written out.

The condition of continuous extendability of the function $R_0(\zeta)$ in the domain $D \cup \ell$ has the form

$$M_1(b_k) = 0, \quad k = 1, \ldots, 4, \tag{19}$$

and thus we have eight conditions (conditions (12), (14), (18) and (19)) with respect to eight real constants: C_1, C_2, E_1, E_2, $v_n^{(k)}(t) (k = 1, \ldots, 4)$. These conditions are, in fact, a system of linear algebraic equations with real coefficients of the form

$$\sum_{k=1}^{8} a_{ik}\, d_k = \ell_i \quad (i = 1, \ldots, 8),$$

where $a_{ik} (i, k = 1, \ldots, 8)$ are the known real constants, independent of the external forces P, and ℓ_i are the constants vanishing for $P = 0$. d_k are the above-mentioned unknown real constants $(c_1, c_2, \ldots)$.

Assuming that the determinant of the system equals zero, we find that the homogeneous system

$$\sum_{k=1}^{8} a_{ik}\, d_k = 0, \quad i = 1, \ldots, 8,$$

has nontrivial solutions which as a consequence implies that the problem under consideration has a solution (different from a rigid displacement), representable by complex potentials $\varphi_1(z)$ and $\psi_1(z)$ for which the equality (see [4], §113)

$$\begin{aligned}&\operatorname{Im}\int_L \left[\overline{\varphi_1(t)} + \bar{t}\,\varphi_1'(t) + \psi_1(t)\right] d\left[\varkappa\varphi_1(t) - t\,\overline{\varphi_1'(t)} - \overline{\psi_1(t)}\right]\\ &\quad = 4\iint_S \left\{2(\varkappa-1)\operatorname{Re}\left[\varphi_1'(z)\right]^2 + \left|\bar{z}\,\varphi_1''(z) + \psi_1'(z)\right|^2\right\} dx\,dy,\end{aligned}$$

is valid. In our case this equality can be rewritten in the form

$$\begin{aligned}&\int_L \left\{\operatorname{Re}\, i\, e^{-i\nu(t)}\left[\varphi_1(t) + t\,\overline{\varphi_1'(t)} + \overline{\psi_1(t)}\right] d\operatorname{Re}\, e^{i\nu(t)}\left[\varkappa\varphi_1(t) - t\,\overline{\varphi_1'(t)} - \overline{\psi_1(t)}\right]\right\}\\ &\quad = 4\iint_S \left\{2(\varkappa-1)\left[\operatorname{Re}\varphi_1'(t)\right]^2 + \left|\bar{z}\,\varphi_1''(z) + \psi_1'(z)\right|^2\right\} dx\,dy. \qquad (20)\end{aligned}$$

When writing this equality, we have taken into account continuous extensions of the expression $\varkappa\varphi_1(z) - z\,\overline{\varphi_1'(z)} - \overline{\psi_1(z)}$ in the domain $S \cup L$ and those of $\varphi_1'(z)$ and $\bar{z}\,\varphi_1''(z) + \psi_1'(z)$ up to the boundary L, except possibly the points $B_k (k = 1, \ldots, 4)$.

Taking into account the boundary conditions $(*)$, from (20) we have

$$\varphi_1'(z) = i H_0, \qquad \varphi_1(z) = i H_0 z + H_1; \qquad \psi_1(z) = H_2,$$

where H_0 is the real and H_1 and H_2 are the complex constants, and thus we have rigid displacement of the body as a whole which contradicts our assumption, and consequently, the system determinant is different from zero and the above-posed problem is uniquely solvable.

Remark. The obtained results can be extended to the case of a circular domain with a rectilinear cut (the cut can be considered as a limiting case of a rectangle under contraction of segments B_2B_3 and B_4B_1 to a point). In this case the conformally mapping function has the form

$$\omega'(\zeta) = K^{00}\prod_{k=1}^{2}\left(1 - \frac{b_k}{\zeta}\right)\prod_{j=1}^{\infty}\prod_{k=1}^{2}\left(1 - \frac{\zeta}{R^{2j}b_k}\right)\left(1 - \frac{b_k}{R^{2j}\zeta}\right),$$

and the estimate (23) can be written as follows:

$$|\varphi'(z)| < M_1|z - B_k|^{-1/2}; \qquad |\varphi''(z)| < M_2|z - B_k|^{-3/2}; \quad k = 1, 2.$$

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Original article

On negligible and absolutely nonmeasurable subsets of uncountable solvable groups

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Abstract

It is proved that every uncountable solvable group contains two negligible sets whose union is an absolutely nonmeasurable subset of the same group.

Keywords: Solvable group; Invariant measure; Quasi-invariant measure; Negligible set; Absolutely nonmeasurable set

In this paper we will be dealing with measures invariant (or, more generally, quasi-invariant) under various transformation groups. We will be interested in the behavior of certain sets with respect to such measures. The notation and terminology used in the paper is primarily taken from [1] and [2]. All basic facts of modern measure theory can be found in [3]. An extensive survey devoted to measures given on different algebraic-topological structures is presented in [4].

Let E be a base (ground) set and let G be some group of transformations of E. In this case, the pair (E, G) is usually called a space equipped with a transformation group.

We shall say that a set $X \subset E$ is G-negligible (in E) if the following two conditions are fulfilled for X:

(a) there exists at least one nonzero σ-finite G-invariant (G-quasi-invariant) measure μ on E such that $X \in \text{dom}(\mu)$;

(b) for every σ-finite G-invariant (G-quasi-invariant) measure ν on E such that $X \in \text{dom}(\nu)$, the equality $\nu(X) = 0$ holds true.

We shall say that a set $Y \subset E$ is G-absolutely nonmeasurable (in E) if, for any nonzero σ-finite G-quasi-invariant measure θ on E, we have $X \notin \text{dom}(\theta)$.

If $(G, \cdot)$ is a group, then we may consider G as a ground set E and take the group of all left translations of G as a group of transformations of E. Obviously, identifying G with the group of all left translations of G, we may speak of left G-invariant (left G-quasi-invariant) measures on E $(=G)$ and, respectively, we may consider G-negligible and G-absolutely nonmeasurable subsets of G.

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Example 1. If $(G, \cdot)$ is an arbitrary uncountable solvable group, then there exists a G-absolutely nonmeasurable subset of G (in this connection, see e.g. [2] and references therein). At the same time, it is still unknown whether there exists a Γ-absolutely nonmeasurable set in any uncountable group $(\Gamma, \cdot)$.

The main goal of this paper is to show (for a certain class of spaces (E, G)) that there exist two G-negligible sets in E, the union of which turns out to be G-absolutely nonmeasurable in E. In particular, if E itself is an uncountable solvable group and G coincides with the group of all left translations of E, then the above-mentioned fact is valid for (E, G). Clearly, this yields some generalization of the statement formulated in Example 1.

It should be noticed that basic technical tools which lead us to the required result are motivated by the method of surjective homomorphisms (cf. [1,2,5]).

For our further purposes, we need several auxiliary propositions. The first of them is essentially contained in [2].

As usual, the symbol $\omega(=\omega_0)$ denotes the least infinite cardinal (ordinal) number and ω_1 denotes the least uncountable cardinal (ordinal) number.

Lemma 1. *Let a space (E, G) satisfy the following two relations:*

(1) $\mathrm{card}(E) = \omega_1$ *and the group G acts freely and transitively in E;*

(2) *there are two subgroups G_0 and G_1 of G such that*

$$\mathrm{card}(G_0) = \omega, \qquad \mathrm{card}(G_1) = \omega_1, \qquad G_0 \cap G_1 = \{\mathrm{Id}_E\},$$

where Id_E is the identity transformation of E.

Then there exist two G-negligible subsets T_1 and T_2 of E such that the set $T_1 \cup T_2$ is G-absolutely nonmeasurable in E.

Proof. We would like to recall one construction of a G-absolutely nonmeasurable subset of E (see [2], Chapter 11, Lemma 3). First, let us observe that relation (1) directly implies the equality

$$\mathrm{card}(G) = \omega_1.$$

So we may take an ω_1-sequence $\{\Gamma_\xi : \xi < \omega_1\}$ of subgroups of G, such that:

(a) $\Gamma_0 = G_0$;

(b) for all ordinals $\xi < \omega_1$, we have $\mathrm{card}(\Gamma_\xi) = \omega$;

(c) for each ordinal $\xi < \omega_1$, the set $\cup\{\Gamma_\zeta : \zeta < \xi\}$ is a proper subset of Γ_ξ (in particular, this ω_1-sequence of subgroups of G is strictly increasing by inclusion);

(d) $\cup\{\Gamma_\xi : \xi < \omega_1\} = G$.

Further, fix a point $y \in E$ and, for any ordinal number $\xi < \omega_1$, put

$$Y_\xi = \Gamma_\xi(y) \setminus \cup\{\Gamma_\zeta(y) : \zeta < \xi\}.$$

A straightforward verification shows that the family of sets $\{Y_\xi : \xi < \omega_1\}$ forms a partition of E and each Y_ξ is a Γ'_ξ-invariant subset of E, where the group Γ'_ξ is defined by the formula

$$\Gamma'_\xi = \cup\{\Gamma_\zeta : \zeta < \xi\}.$$

According to relation (c), the group Γ'_ξ is a proper subgroup of Γ_ξ. Also, by virtue of the free action of G in E, it is not hard to see that

$$\mathrm{card}(Y_\xi) = \omega \quad (\xi < \omega_1).$$

Now, for each ordinal number $\xi < \omega_1$, introduce the group

$$G_{1,\xi} = G_1 \cap \Gamma'_\xi.$$

Obviously, the ω_1-sequence $\{G_{1,\xi} : \xi < \omega_1\}$ of groups is increasing by inclusion and

$$\cup\{G_{1,\xi} : \xi < \omega_1\} = G_1.$$

Fix for a while an ordinal $\xi < \omega_1$ and consider the two partitions of Y_ξ into orbits associated with the groups G_0 and $G_{1,\xi}$, respectively. Taking into account the free action of G in E and the relation

$$G_0 \cap G_1 = \{\mathrm{Id}_E\},$$

we infer that the above-mentioned two partitions of Y_ξ are mutually transversal; in other words, any equivalence class of the first partition has at most one common point with any equivalence class of the second partition. Starting with this fact, we define by recursion an ω-sequence

$$\{x_{\xi,0}, x_{\xi,1}, \dots, x_{\xi,k}, \dots\}$$

of points from Y_ξ, such that:

(i) $G_0(\{x_{\xi,k} : k < \omega\}) = Y_\xi$;

(ii) for any two distinct natural numbers k and m, the point $x_{\xi,k}$ does not belong to the orbit $G_{1,\xi}(x_{\xi,m})$.

Indeed, let $\{Z_{\xi,k} : k < \omega\}$ denote an injective family of all those G_0-orbits which are contained in Y_ξ. Suppose that, for a natural number k, the elements

$$x_{\xi,0} \in Z_{\xi,0}, x_{\xi,1} \in Z_{\xi,1}, \dots, x_{\xi,k-1} \in Z_{\xi,k-1}$$

have already been defined and that they lie in pairwise distinct $G_{1,\xi}$-orbits. Consider the set

$$P_k = G_{1,\xi}(x_{\xi,0}) \cup G_{1,\xi}(x_{\xi,1}) \cup \dots \cup G_{1,\xi}(x_{\xi,k-1}).$$

Clearly, we have

$$\mathrm{card}(P_k \cap Z_{\xi,k}) \le k, \qquad \mathrm{card}(Z_{\xi,k}) = \omega.$$

Consequently, there exists an element $x \in Z_{\xi,k} \setminus P_k$. So we can put $x_{\xi,k} = x$.

Therefore, for each ordinal $\xi < \omega_1$, we get the corresponding ω-sequence $\{x_{\xi,k} : k < \omega\}$ of points from Y_ξ, fulfilling conditions (i) and (ii).

Now, we define $X = \{x_{\xi,k} : \xi < \omega_1,\ k < \omega\}$ and verify that the set X is G-absolutely nonmeasurable in E.

Indeed, on the one hand, we may write

$$G_0(X) = \cup\{G_0(\{x_{\xi,k} : k < \omega\}) : \xi < \omega_1\} = \cup\{Y_\xi : \xi < \omega_1\} = E$$

and the above relation implies that if X is measurable with respect to some nonzero σ-finite G-quasi-invariant measure μ on E, then necessarily $\mu(X) > 0$.

On the other hand, let us take an arbitrary element $g \in G_1 \setminus \{\mathrm{Id}_E\}$. Then there exists an ordinal $\xi_0 < \omega_1$ for which $g \in G_{1,\xi_0}$. Further, for any $\xi < \omega_1$, let us denote

$$X_\xi = \{x_{\xi,k} : k < \omega\}.$$

Evidently, we have

$$(\forall \xi < \omega_1)(\mathrm{card}(X_\xi) = \omega).$$

Also, the equality

$$X = \cup\{X_\xi : \xi < \omega_1\}$$

implies the inclusion

$$g(X) \cap X \subset \cup\{g(X_\zeta) \cap X_\eta : \zeta < \omega_1,\ \eta < \omega_1\}.$$

If $\zeta < \omega_1$ and $\eta < \omega_1$ satisfy the relations $\xi_0 < \zeta$ and $\xi_0 < \eta$, then

$$g(X_\zeta) \cap X_\eta = \emptyset.$$

In addition to this, if $\zeta < \xi_0$ and $\eta > \xi_0$, or, respectively, $\zeta > \xi_0$ and $\eta < \xi_0$, then

$$g(X_\zeta) \cap X_\eta = g(X_\zeta \cap g^{-1}(X_\eta)) \subset g(Y_\zeta \cap Y_\eta) = \emptyset,$$

or, respectively,

$$g(X_\zeta) \cap X_\eta \subset Y_\zeta \cap Y_\eta = \emptyset.$$

We thus get the inclusion

$$g(X) \cap X \subset (\cup\{g(X_\zeta) : \zeta \le \xi_0\}) \cup (\cup\{X_\eta : \eta \le \xi_0\})$$

and, therefore,

$$\mathrm{card}(g(X) \cap X) \leq \omega.$$

Finally, suppose that g and h are any two distinct elements of G_1. Then

$$h^{-1} \circ g \neq \mathrm{Id}_E, \qquad h^{-1} \circ g \in G_1,$$

and, according to the fact established above, we may write

$$\mathrm{card}((h^{-1} \circ g)(X) \cap X) \leq \omega$$

which implies at once that

$$\mathrm{card}(g(X) \cap h(X)) \leq \omega.$$

The last inequality shows that if the set X is measurable with respect to some σ-finite G-quasi-invariant measure μ on E, then $\mu(X) = 0$. So we must have simultaneously $\mu(X) > 0$ and $\mu(X) = 0$. Obviously, this yields a contradiction and hence X is a G-absolutely nonmeasurable subset of E.

Now, let us return to the partition $\{Y_\xi : \xi < \omega_1\}$ of our ground set E and introduce the following two sets:

$$T_1 = \cup\{X \cap Y_\xi : \xi < \omega_1,\ \xi \text{ is an odd ordinal number}\},$$

$$T_2 = \cup\{X \cap Y_\xi : \xi < \omega_1,\ \xi \text{ is an even ordinal number}\}.$$

Clearly, $X = T_1 \cup T_2$ and $T_1 \cap T_2 = \emptyset$. Further, if $\{g_i : i \in I\}$ is an arbitrary countable family of elements of G, then

$$E \setminus \cup\{g_i(T_1) : i \in I\} \neq \emptyset, \qquad E \setminus \cup\{g_i(T_2) : i \in I\} \neq \emptyset.$$

It is not hard to infer from this property of the sets T_1 and T_2 that there exist two probability G-invariant measures μ_1 and μ_2 on E such that

$$T_1 \in \mathrm{dom}(\mu_1), \qquad T_2 \in \mathrm{dom}(\mu_2), \qquad \mu_1(T_1) = \mu_2(T_2) = 0.$$

Finally, keeping in mind the relations $T_1 \subset X$ and $T_2 \subset X$, we conclude that both T_1 and T_2 are G-negligible sets in E. Lemma 1 has thus been proved. □

Lemma 2. *Let $(G, \cdot)$ and $(H, \cdot)$ be two groups and let*

$$\phi : (G, \cdot) \to (H, \cdot)$$

be a surjective homomorphism. The following assertions are valid for any two sets $X \subset H$ and $Y \subset H$:

(1) *if X is an H-negligible subset of H, then $\phi^{-1}(X)$ is a G-negligible subset of G;*

(2) *if Y is an H-absolutely nonmeasurable subset of H, then $\phi^{-1}(Y)$ is a G-absolutely nonmeasurable subset of G.*

The proof of Lemma 2 is not difficult (see, e.g., [1] or [2]).

The next two propositions are purely algebraic and can be deduced from well-known theorems of the general theory of commutative groups (cf. [6,7]).

Lemma 3. *If $(H, +)$ is an uncountable commutative group, then there exist two subgroups H_0 and H_1 of $(H, +)$ such that:*

(1) $\mathrm{card}(H_0) = \omega$ *and* $\mathrm{card}(H_1) = \omega_1$*;*

(2) $H_0 \cap H_1 = \{0\}$, *where 0 stands for the neutral element of H.*

Lemma 4. *If $(G, +)$ is an uncountable commutative group, then there exists a surjective homomorphism*

$$\phi : (G, +) \to (H, +),$$

where $(H, +)$ is some commutative group of cardinality ω_1.

Lemma 5. *Let $(G, \cdot)$ be a group and let H be a normal subgroup of G such that* $\mathrm{card}(G/H) \leq \omega$. *The following two assertions are valid:*

(1) *if a set X is H-absolutely nonmeasurable in H, then X is also G-absolutely nonmeasurable in G;*

(2) *if a set Y is H-negligible in H, then Y is also G-negligible in G.*

The proof of Lemma 5 readily follows from the definitions of negligible and absolutely nonmeasurable sets.

Theorem 1. *If $(G, +)$ is an uncountable commutative group, then there exist two G-negligible subsets Y_1 and Y_2 in G such that their union $Y_1 \cup Y_2$ is G-absolutely nonmeasurable in G.*

Proof. According to Lemma 4, there exists a surjective homomorphism

$$\phi : (G, +) \to (H, +)$$

for some commutative group $(H, +)$ of cardinality ω_1. Applying Lemmas 1 and 3 to $(H, +)$, we obtain two H-negligible subsets X_1 and X_2 of H such that the set $X_1 \cup X_2$ is H-absolutely nonmeasurable in H. Let us denote

$$Y_1 = \phi^{-1}(X_1), \qquad Y_2 = \phi^{-1}(X_2).$$

By virtue of Lemma 2, both sets Y_1 and Y_2 are G-negligible in G. Also, in view of the same lemma, the set

$$Y_1 \cup Y_2 = \phi^{-1}(X_1) \cup \phi^{-1}(X_2) = \phi^{-1}(X_1 \cup X_2)$$

turns out to be G-absolutely nonmeasurable in G. This finishes the proof of Theorem 1. □

Theorem 2. *If $(G, \cdot)$ is an uncountable solvable group, then there exist two G-negligible sets Y_1 and Y_2 in G such that the set $Y_1 \cup Y_2$ is G-absolutely nonmeasurable in G.*

Proof. Since $(G, \cdot)$ is solvable, there exists a finite sequence

$$\{e\} = G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_{n-1} \subset G_n = G$$

of subgroups of G satisfying these two relations:

(i) for each natural index $k \in [1, n]$, the group G_{k-1} is normal in the group G_k;

(ii) for each natural index $k \in [1, n]$, the quotient group G_k/G_{k-1} is commutative.

To demonstrate the validity of our assertion, we argue by induction on n.

If $n = 1$, then the uncountable group $G = G_n$ is commutative, and we may apply Theorem 1 to this G.

Suppose now that the assertion holds true for a natural number $n - 1 \geq 1$ and let us establish its validity for n.

For this purpose, consider the commutative quotient group $H = G_n/G_{n-1}$, where, as above, $G_n = G$. Here only two cases are possible.

(a) the group $H = G_n/G_{n-1}$ is uncountable.

In this case, we take the canonical surjective homomorphism

$$\phi : (G_n, \cdot) \to (H, +).$$

By virtue of Theorem 1, there are two H-negligible subsets X_1 and X_2 in H such that their union $X_1 \cup X_2$ is H-absolutely nonmeasurable in H. We put

$$Y_1 = \phi^{-1}(X_1), \qquad Y_2 = \phi^{-1}(X_2).$$

Then, keeping in mind Lemma 2, we see that both sets Y_1 and Y_2 are G-negligible in G, and we also deduce that the set

$$Y_1 \cup Y_2 = \phi^{-1}(X_1) \cup \phi^{-1}(X_2) = \phi^{-1}(X_1 \cup X_2)$$

turns out to be G-absolutely nonmeasurable in G.

(b) the group $H = G_n/G_{n-1}$ is countable.

In this case, in view of the uncountability of $G_n = G$, the group G_{n-1} is necessarily uncountable, and we can apply the inductive assumption to this G_{n-1}. So there are two G_{n-1}-negligible subsets Y_1 and Y_2 of G_{n-1} such that the set $Y_1 \cup Y_2$ is G_{n-1}-absolutely nonmeasurable in G_{n-1}. Lemma 5 now yields that, simultaneously, Y_1 and Y_2 are G-negligible subsets of G and their union $Y_1 \cup Y_2$ is a G-absolutely nonmeasurable set in G. Theorem 2 has thus been proved. □

Example 2. Let $(G, \cdot)$ be an arbitrary uncountable solvable group. It directly follows from Theorem 2 that there are two G-negligible sets Y_1 and Y_2 in G possessing the following property: for any nonzero σ-finite left G-quasi-invariant measure μ on G, at least one of the sets Y_1 and Y_2 is nonmeasurable with respect to μ.

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Original article

Sharp weighted bounds for multiple integral operators

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Abstract

Sharp weighted bounds for strong maximal functions, multiple potentials and singular integrals are derived in terms of Muckenhoupt type characteristics of weights.

Keywords: Strong maximal functions; Integral operators with product kernels; Boundedness; One-weight inequality; Sharp bound

1. Introduction

In this paper, we establish sharp weighted bounds for strong maximal functions and multiple integral operators. Our derived results involve, in particular, Buckley-type estimates for strong Hardy–Littlewood and fractional maximal functions, potentials and singular integrals with product kernels, and their one-sided analogs.

One of the main problems in Harmonic Analysis is to characterize a weight w for which a given integral operator is bounded in L^p_w (one-weight inequality). An important class of such weights is the well-known A_p class. It is known that A_p condition is necessary and sufficient for the boundedness of Hardy–Littlewood and singular integral operators (see, e.g., [1–3]); however, the sharp dependence of the corresponding L^p_w norms in terms of A_p characteristic of w is known only for some operators. The interest in the sharp weighted norm, for example, for singular integral operators is motivated by applications in partial differential equations (see e.g., [4–7]).

Strong maximal operator different from the usual one is defined with respect to parallelepipeds with sides parallel to the co-ordinate axes; the operators with product kernels, such as multiple singular and potential operators have singularities not only at a single point but on the hyperplanes. That is why to study mapping properties for such

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operators became more complicated; however from the one weight viewpoint it is possible to get one-weight boundedness results as well as sharp weighted bounds by deducing the problem to the single variable result and using repeatedly the latter one uniformly with respect to other variables. In this direction Proposition 2.1 is one of the keys to get the main results. One of the important aspects of this paper is that this point enables us to get sharp one-weight results for a quite large class of multiple operators including one-sided cases.

Let X and Y be two Banach spaces. Given a bounded operator $T : X \to Y$, we denote the operator norm by $\|T\|_{X\to Y}$ which is defined in the standard way i.e. $\|T\|_{X\to Y} := \sup_{\|f\|_X\le 1} \|Tf\|_Y$. If $X = Y$ we use the symbol $\|T\|_X$.

An almost everywhere positive locally integrable function (i.e. weight) w defined on $\mathbb{R}^n$ is said to satisfy $A_p(\mathbb{R}^n)$ condition ($w \in A_p(\mathbb{R}^n)$) for $1 < p < \infty$ if

$$\|w\|_{A_p(\mathbb{R}^n)} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{p-1} < \infty,$$

where $p' = \frac{p}{p-1}$ and supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the co-ordinate axes. We call $\|w\|_{A_p(\mathbb{R}^n)}$ the A_p characteristic of w.

In 1972 B. Muckenhoupt [3] showed that if $w \in A_p(\mathbb{R}^n)$, where $1 < p < \infty$, then the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy$$

is bounded in $L^p_w(\mathbb{R}^n)$.

S. Buckley [8] investigated the sharp A_p bound for the operator M and established the inequality

$$\|M\|_{L^p_w(\mathbb{R}^n)} \le C\|w\|^{\frac{1}{p-1}}_{A_p(\mathbb{R}^n)}, \quad 1 < p < \infty. \tag{1.1}$$

Moreover, he showed that the exponent $\frac{1}{p-1}$ is best possible in the sense that we cannot replace $\|w\|^{\frac{1}{p-1}}_{A_p}$ by $\psi(\|w\|_{A_p})$ for any positive non-decreasing function ψ growing slowly than $x^{\frac{1}{p-1}}$. From here it follows that for any $\lambda > 0$,

$$\sup_{w\in A_p} \frac{\|M\|_{L^p_w}}{\|w\|^{\frac{1}{p-1}-\lambda}_{A_p}} = \infty.$$

To explain better the point of sharp estimates for multiple operators, let us discuss, for example, the strong Hardy–Littlewood maximal operator $M^{(s)}$ defined on $\mathbb{R}^2$. Denote by $A^{(s)}_p(\mathbb{R}^2)$ the Muckenhoupt class taken with respect to the rectangles with sides parallel to the co-ordinate axes (see Section 2 for the definitions). Let $\|w\|_{A^{(s)}_p(\mathbb{R}^2)}$ be $A^{(s)}_p$ characteristic of w. There arises a natural question regarding the sharp bound in the inequality

$$\|M^{(s)}\|_{L^p_w(\mathbb{R}^2)} \le c\|w\|^{\beta}_{A^{(s)}_p(\mathbb{R}^2)}. \tag{1.2}$$

We show that the following estimate is sharp

$$\|M^{(s)}\|_{L^p_w(\mathbb{R}^2)} \le c\Big(\|w\|_{A_p(x_1)}\|w\|_{A_p(x_2)}\Big)^{1/(p-1)}, \tag{1.3}$$

where $\|w\|_{A_p(x_i)}$ is the characteristic of the weight w defined with respect to the ith variable uniformly to another one $i = 1, 2$ (see e.g., [9–11], Ch. IV for the one-weight theory for multiple integral operators). Inequality (1.3) together with the Lebesgue differentiation theorem implies that (1.2) holds for $\beta = \frac{2}{p-1}$; however, unfortunately we do not know whether it is or not sharp.

Under the symbol $A \approx B$ we mean that there are positive constants c_1 and c_2 (depending on appropriate parameters) such that $c_1 A \le B \le c_2 A$; $A \ll B$ means that there is a positive constant c such that $A \le cB$.

Finally we mention that constants (often different constants in one and the same lines of inequalities) will be denoted by c or C. The symbol p' stands for the conjugate number of p: $p' = p/(p-1)$, where $1 < p < \infty$.

2. Strong maximal and multiple integral operators

Let w be a weight function on a domain $\Omega \subseteq \mathbb{R}^n$. We denote by $L^p_w(\Omega)$, $1 < p < \infty$, the set of all measurable functions $f : \Omega \to \mathbb{R}$ for which the norm

$$\|f\|_{L^p_w(\Omega)} = \left(\int_\Omega |f(x)|^p w(x)dx\right)^{\frac{1}{p}}$$

is finite. If $w \equiv$ const, then we denote $L^p_w(\Omega) = L^p(\Omega)$.

In this section, we give sharp weighted bounds for strong maximal and multiple integral operators. Given an operator $T_\mathbb{R}$ acting on function in $\mathbb{R}$, by T^k, $k = 1 \cdots n$, we denote the operators defined on class of functions acting on $\mathbb{R}^n$ by letting $T_\mathbb{R}$ acting on the kth variable and keeping rest of $n-1$ variable fixed. Formally, for every $x \in \mathbb{R}^n$,

$$(T^k f)(x) = T_\mathbb{R}(f(x_1, x_2, \dots, x_{k-1}, \cdot, x_k, \dots, x_n))(x_k). \tag{2.1}$$

Remark 2.1. It can be easily verified (see [11], pg. 450–451) that if $T_\mathbb{R}$ is bounded, then T^k is also bounded and further

$$\|T^k f\|_{L^p(\mathbb{R}^n)} \le c\|T_\mathbb{R}\|\|f\|_{L^p(\mathbb{R}^n)},$$

holds.

Definition 2.1. A weight function w satisfies $A^{(s)}_p(\mathbb{R}^n)$ condition ($w \in A^{(s)}_p(\mathbb{R}^n)$), $1 < p < \infty$, if

$$\|w\|_{A^{(s)}_p(\mathbb{R}^n)} := \sup_P \left(\frac{1}{|P|}\int_P w(x)dx\right)\left(\frac{1}{|P|}\int_P w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty,$$

where the supremum is taken over all parallelepipeds P in $\mathbb{R}^n$ with sides parallel to the co-ordinate axes.

Definition 2.2. Let $1 < p < \infty$. A weight function $w = w(x_1, \dots, x_n)$ defined on $\mathbb{R}^n$ is said to satisfy A_p condition in x_i uniformly with respect to other variables ($w \in A_p(x_i)$) if

$$\begin{aligned}\|w\|_{A_p(x_i)} := &\operatorname*{ess\,sup}_{(x_1,\dots,x_{i-1},x_{i+1}\cdots,x_n)\in\mathbb{R}^{n-1}} \sup_I \left(\frac{1}{|I|}\int_I w(x_1, \dots, x_n)dx_i\right)\\ &\times \left(\frac{1}{|I|}\int_I w^{1-p'}(x_1, \dots, x_n)dx_i\right)^{p-1} < \infty,\end{aligned}$$

where by I we denote a bounded interval in $\mathbb{R}$.

Remark 2.2. $w(x_1, \dots, x_n) \in A^{(s)}_p(\mathbb{R}^n) \Leftrightarrow w \in \bigcap_{i=1}^n A_p(x_i)$ (see e.g., pp. 453–454 of [11,10]).

Proposition 2.1. *Let T^k be the operators given by the formula* (2.1) *and let T be an operator defined for functions on $\mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$,*

$$(Tf)(x) \le (T^1 \circ \cdots \circ T^n)(f)(x)$$

and

$$\|T^k\|_{L^p_w(\mathbb{R})} \le c\|w\|^{\gamma(p)}_{A_p(x_k)} \quad k = 1, \dots, n, \tag{2.2}$$

holds, where $\gamma(p)$ is a constant depending only on p. Then the following estimate

$$\|T\|_{L^p_w(\mathbb{R}^n)} \le c(\|w\|_{A_p(x_1)} \cdots \|w\|_{A_p(x_n)})^{\gamma(p)}$$

holds.

Proof. For simplicity we give proof for $n = 2$ the proof general case is the same. Suppose that $f \geq 0$. Using (2.1) two times and Fubini's theorem we have,

$$\begin{aligned}
\|Tf\|^p_{L^p_w(\mathbb{R}^2)} &= \iint_{\mathbb{R}^2} (Tf(x_1,x_2))^p w(x_1,x_2)dx_1dx_2 \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (T^1(T^2 f(\cdot,x_2)))(x_1)^p w(x_1,x_2)dx_1 \right) dx_2 \\
&\leq c\|w\|^{p\gamma(p)}_{A_p(x_1)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (T^2 f(x_1,x_2))^p w(x_1,x_2)dx_1 \right) dx_2 \\
&= c\|w\|^{p\gamma(p)}_{A_p(x_1)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (T^2 f(x_1,x_2))^p w(x_1,x_2)dx_2 \right) dx_1 \\
&= c(\|w\|_{A_p(x_1)}\|w\|_{A_p(x_2)})^{p\gamma(p)} \|f\|^p_{L^p_w(\mathbb{R}^2)}. \quad \square
\end{aligned}$$

2.1. *Strong Hardy–Littlewood maximal functions and multiple singular integrals*

The following theorem is due to S. Buckley [8].

Theorem A. *If $w \in A_p(\mathbb{R}^n)$, then $\|Mf\|_{L^p_w(\mathbb{R}^n)} \leq c_{n,p}\|w\|^{1/(p-1)}_{A_p(\mathbb{R}^n)}\|f\|_{L^p_w(\mathbb{R}^n)}$. The exponent $1/(p-1)$ is best possible.*

Let f be a locally integrable function on $\mathbb{R}^n$. Then we define strong Hardy–Littlewood maximal operator as

$$\big(M^{(s)}f\big)(x) = \sup_{P \ni x} \frac{1}{|P|} \int_P |f(y)|dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all parallelepipeds $P \ni x$ in $\mathbb{R}^n$ with sides parallel to the co-ordinate axes.

Theorem 2.3. *Let $1 < p < \infty$ and w be a weight function on $\mathbb{R}^n$ such that $w \in A^{(s)}_p(\mathbb{R}^n)$. Then there exists a constant c depending only on n and p such that the following inequality*

$$\|M^{(s)}f\|_{L^p_w(\mathbb{R}^n)} \leq c\left(\prod_{i=1}^n \|w\|_{A_p(x_i)} \right)^{1/(p-1)} \|f\|_{L^p_w(\mathbb{R}^n)} \tag{2.3}$$

holds, for all $f \in L^p_w(\mathbb{R}^n)$. Further, the exponent $1/(p-1)$ in estimate (2.3) *is sharp.*

Proof. For every $x \in \mathbb{R}^n$ we can estimate $M^{(s)}$ as follows

$$\big(M^{(s)}f\big)(x) \leq \big(M^1 \circ M^2 \circ \cdots \circ M^n\big)f(x),$$

where

$$\begin{aligned}
\big(M^k f\big)(x_1,\ldots,x_n) &= M(f(x_1,x_2,\ldots,x_{k-1},\cdot,x_k,\ldots,x_n))(x_k) \\
&= \sup_{I_k \ni x_k} \frac{1}{|I_k|} \int_{I_k} |f(x_1,\ldots,x_{k-1},t,x_{k+1},\ldots,x_n)|dt.
\end{aligned}$$

Now by Theorem A and Proposition 2.1 (for $\gamma(p) = \frac{1}{p-1}$) we find that

$$\|M^{(s)}f\|_{L^p_w(\mathbb{R}^n)} \leq c\left(\prod_{i=1}^n \|w\|_{A_p(x_i)} \right)^{1/(p-1)} \|f\|_{L^p_w(\mathbb{R}^n)}.$$

For sharpness we consider the case for $n = 2$. Observe that when w is of product type, i.e. $w(x_1,x_2) = w_1(x_1)w_2(x_2)$, then

$$\|w\|_{A_p(x_1)} = \|w_1\|_{A_p(\mathbb{R})}, \qquad \|w\|_{A_p(x_2)} = \|w_2\|_{A_p(\mathbb{R})}. \tag{2.4}$$

Let us take $0 < \epsilon < 1$. Suppose that $w(x_1, x_2) = |x_1|^{(1-\epsilon)(p-1)}|x_2|^{(1-\epsilon)(p-1)}$. Then it is easy to check that

$$(\|w\|_{A_{p(x_1)}}\|w\|_{A_{p(x_2)}})^{1/(p-1)} \approx \frac{1}{\epsilon^2}.$$

Observe also that for

$$f(x_1, x_2) = {x_1}^{\epsilon-1}\chi_{(0,1)}(x_1){x_2}^{\epsilon-1}\chi_{(0,1)}(x_2),$$

we have $\|f\|^p_{L^p_w} \approx \frac{1}{\epsilon^2}$. Now let $0 < x_1, x_2 < 1$. Then we find that the following estimate

$$\big(M^{(s)}f\big)(x_1, x_2) \geq \frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2} f(t,\tau)dtd\tau = \frac{1}{\epsilon^2}f(x,y)$$

holds. Finally

$$\|M^{(s)}f\|_{L^p_w} \geq \frac{1}{\epsilon^2}\|f\|_{L^p_w}.$$

Thus we have the sharpness in (2.3). □

Now we present the sharp weighted estimates for multiple singular integrals. S. Buckley, in his celebrated paper [8] showed that for $1 < p < \infty$, convolution Calderón–Zygmund singular operator satisfies

$$\|T\|_{L^p_w(\mathbb{R}^n)} \leq c\|w\|^{\frac{p}{p-1}}_{A_p(\mathbb{R}^n)}$$

and the best possible exponent is at least $\max\{1, \frac{1}{p-1}\}$. S. Petermichl [6,7] proved that the estimate

$$\|S\|_{L^p_w(\mathbb{R}^n)} \leq c\|w\|^{\max\left\{1,\frac{1}{p-1}\right\}}_{A_p(\mathbb{R}^n)}$$

is sharp, where S is either the Hilbert transform or one of the Riesz transforms in $\mathbb{R}^n$

$$R_jf(x) = c_n p\cdot v\cdot \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}}f(y)dy.$$

S. Petermichl obtained the results for $p = 2$. The general case $p \neq 2$ then follows by the sharp version of the Rubio de Francia extrapolation theorem given by O. Dragičević, L. Grafakos, C. Pereyra and S. Petermichl [12] (see also, T. Hytönen [13] regarding the A_2 conjecture for Calderón–Zygmund operators which, in fact, implies appropriate estimate for all exponents $1 < p < \infty$ by applying a sharp version of the Rubio de Francia's extrapolation theorem).

Let us denote by $\mathcal{H}^{(n)}$ the Hilbert transform with product kernels (or n-dimensional Hilbert transform) defined by

$$\big(\mathcal{H}^{(n)}f\big)(x) = \lim_{\substack{\epsilon_1\to 0\\ \dots\\ \epsilon_n\to 0}} \int_{|x_1-t_1|>\epsilon_1}\cdots\int_{|x_n-t_n|>\epsilon_n} \frac{f(t_1,\dots,t_n)}{(x_1-t_1)\cdots(x_n-t_n)}dt_1\cdots dt_n.$$

We denote $\mathcal{H}^{(1)} =: \mathcal{H}$. Notice that for each $x \in \mathbb{R}^n$, we can write

$$\big(\mathcal{H}^{(n)}f\big)(x) = \big(\mathcal{H}^1\circ\cdots\circ\mathcal{H}^n\big)f(x) \tag{2.5}$$

where,

$$\begin{aligned}\big(\mathcal{H}^kf\big)(x) &= \mathcal{H}(f(x_1, x_2, \dots, x_{k-1}, \cdot, x_k, \dots, x_n))(x_k)\\ &= \lim_{\epsilon_k\to 0}\int_{|x_k-y_k|>\varepsilon_k} \frac{f(x_1,\dots,y_k,\dots,x_n)}{x_k-y_k}dy_k.\end{aligned}$$

The following theorem is due to S. Petermichl [6].

Theorem B. *Let $1 < p < \infty$. Then there exists a positive constant c depending only on p such that for all weights $w \in A_p(\mathbb{R})$ we have*

$$\|\mathcal{H} f\|_{L^p_w(\mathbb{R})} \le c\|w\|^{\beta}_{A_p(\mathbb{R})}\|f\|_{L^p_w(\mathbb{R})}, \quad f \in L^p_w(\mathbb{R}), \tag{2.6}$$

where $\beta = \max\{1, p'/p\}$. Moreover, the exponent β in this estimate is sharp.

Theorem 2.4. *Let $1 < p < \infty$ and w be a weight function on $\mathbb{R}^n$ such that $w \in A_p^{(s)}(\mathbb{R}^n)$. Then there exists a constant c depending only on n and p such that the following inequality*

$$\|\mathcal{H}^{(n)} f\|_{L^p_w(\mathbb{R}^n)} \le c(\|w\|_{A_p(x_1)} \cdots \|w\|_{A_p(x_n)})^{\max\{1,p'/p\}}\|f\|_{L^p_w(\mathbb{R}^n)} \tag{2.7}$$

holds for all $f \in L^p_w$. Further the exponent $\max\{1, p'/p\}$ *in estimate* (2.7) *is sharp.*

Proof. Using representation (2.5), Proposition 2.1 and Theorem B, we have that

$$\|\mathcal{H}^{(n)} f\|_{L^p_w(\mathbb{R}^n)} \ll \left(\|w\|_{A_p(x_1)} \cdots \|w\|_{A_p(x_n)}\right)^{\max\{1,p'/p\}}\|f\|_{L^p_w(\mathbb{R}^n)}.$$

Let $n = 2$. For sharpness we observe that when w is of product type i.e. $w(x_1, x_2) = w_1(x_1)w_2(x_2)$, then inequality (2.4) holds. Let us first derive sharpness for $p = 2$. Let us take $0 < \epsilon < 1$ and let $w(x_1, x_2) = w_1(x_1)w_2(x_2)$, where $w_1(x_1) = |x_1|^{1-\epsilon}$ and $w_2(x_2) = |x_2|^{1-\epsilon}$. Then it is easy to check that (2.4) holds. Observe also that for

$$f(x_1, x_2) = x_1{}^{\epsilon-1}\chi_{(0,1)}(x_1)x_2{}^{\epsilon-1}\chi_{(0,1)}(x_2), \tag{2.8}$$

$\|f\|^2_{L^2_w} \approx \frac{1}{\epsilon}$. Now let $0 < x_1, x_2 < 1$. Then we find that

$$\|\mathcal{H}^{(2)} f\|_{L^2_w(\mathbb{R}^2)} \ge 4\epsilon^{-3}.$$

Letting $\epsilon \to 0$ we have sharpness in (2.7) for $p = 2$ i.e., the estimate

$$\|\mathcal{H}^{(2)}\|_{L^2_w(\mathbb{R}^2)} \ll \|w\|_{A_2(x_1)}\|w\|_{A_2(x_2)}$$

is sharp.

Let $1 < p < 2$. Suppose that $0 < \epsilon < 1$ and that $w(x_1, x_2) = |x_1|^{(1-\epsilon)(p-1)}|x_2|^{(1-\epsilon)(p-1)}$. Then it is easy to check that

$$(\|w\|_{A_{p(x_1)}}\|w\|_{A_{p(x_2)}})^{1/(p-1)} \approx \frac{1}{\epsilon^2}.$$

Observe also that for the function defined by (2.8) the relation $\|f\|_{L^p_w} \approx (\frac{1}{\epsilon^2})^{\frac{1}{p}}$ holds. Now let $0 < x_1, x_2 < 1$. Then we find that following estimates

$$\|\mathcal{H}^{(2)} f\|_{L^p_w(\mathbb{R}^2)} \ge \frac{1}{\epsilon^2}\|f\|_{L^p_w(\mathbb{R}^2)} \approx \left(\|w\|_{A_p(x_1)}\|w\|_{A_p(x_2)}\right)^{p'/p}\|f\|_{L^p_w(\mathbb{R}^2)}$$

are fulfilled. Thus we have sharpness in (2.7) for $1 < p < 2$. Using the fact that n-dimensional Hilbert transform is essentially self-adjoint and applying duality argument together with the obvious equality

$$\|u^{1-p'}\|_{A_{p'}} = \|u\|^{1/(p-1)}_{A_p}, \quad u \in A_p,$$

we have sharpness for $p > 2$. This completes the proof. □

Let $x = (x^{(1)}, \dots, x^{(n)}) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n}$, where $d_1, d_2, \dots, d_n \in \mathbb{N}$. Suppose that $x^{(k)}_{j_k}$ are components of $x^{(k)}$, $k = 1, \dots, n$, $1 \le j_k \le d_k$. Then we define n-fold Riesz transform

$$\left(R^{(n)}_{(j_1,\dots,j_n)} f\right)(x) = p.v.\int_{\mathbb{R}^{d_1}} \cdots \int_{\mathbb{R}^{d_n}} \prod_{k=1}^{n} \frac{(x^{(k)}_{j_k} - y^{(k)}_{j_k})}{|x^{(k)} - y^{(k)}|^{d_k+1}} f(y^{(1)}, \dots, y^{(n)})dy^{(1)} \cdots dy^{(n)},$$

where $1 \le j_k \le d_k$, $k = 1, \dots, n$. It can be noticed that

$$\left(R^{(n)}_{(j_1,\dots,j_n)} f\right)(x) = \left(R^1_{j_1} \circ \cdots \circ R^n_{j_n} f\right)(x)$$

where

$$\big(R^k_{(j_1,\dots,j_n)}f\big)(x) = p.v.\int_{\mathbb{R}^{d_k}} \frac{x^{(k)}_{j_k} - y^{(k)}_{j_k}}{|x^{(k)} - y^{(k)}|^{d_k+1}} f(x^{(1)},\dots,x^{(k-1)},y^{(k)},x^{(k+1)}\cdots,x^{(n)})dy^{(k)}.$$

Theorem 2.5. *Let $1 < p < \infty$ and w be a weight function on $\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_n}$ satisfy the condition $w \in A^{(s)}_p(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_n})$. Then there exists a constant c independent of $f \in L^p_w(\mathbb{R}^d)$ and w such that the following inequality*

$$\|R^{(n)}_{(j_1,\dots,j_n)}f\|_{L^p_w(\mathbb{R}^d)} \le c(\|w\|_{A_p(x^{(1)})}\cdots\|w\|_{A_p(x^{(n)})})^{\max\{1,p'/p\}}\|f\|_{L^p_w(\mathbb{R}^d)} \tag{2.9}$$

holds for all $1 \le j_k \le d_k$, $k = 1\cdots n$, where $d = d_1 + \cdots, +d_n$. Further, the exponent $\max\{1, p'/p\}$ *in estimate* (2.9) *is sharp.*

Proof of this statement is similar to that of the previous one; we need to apply Proposition 2.1 and the results of [7]. □

Example 2.1. Let $-1 < \gamma < p-1$. It is known that $w(x) = |x|^\gamma$ belongs to $A^{(s)}_p(\mathbb{R}^n)$. Let $\overline{w}(t) = |t|^\gamma$, $t \in \mathbb{R}$. We set $b_\gamma := \max\{2^{\frac{\gamma}{2}}, 1\}$ and $d_\gamma := \max\{2^{\frac{-\gamma}{2}-p+1}, 1\}$.

(i) It follows from Theorem 2.3 that

$$\|M^{(s)}\|_{L^p_w(\mathbb{R}^n)} \le C^n C_\gamma^{\frac{n}{p-1}}\left(1 + \|\overline{w}\|_{A_p(\mathbb{R})}\right)^{\frac{n}{p-1}} \tag{2.10}$$

where C is the constant from the Buckley's estimate (see (1.1)) and

$$C_\gamma = \begin{cases} b_\gamma, & 0 \le \gamma < p-1, \\ d_\gamma, & -1 < \gamma < 0. \end{cases} \tag{2.11}$$

It is known (see [14] pp 287–289) that C in (2.10) can be taken as $C = 3^{p+p'}2^{p'-p}p'24^{\frac{2}{p}}p^{\frac{1}{p-1}}$.

(ii) It follows from Theorem 2.4 that

$$\|\mathcal{H}^{(n)}\|_{L^p_w(\mathbb{R}^n)} \le c^n C_\gamma^{n\max\{1,p'/p\}}\left(1 + \|\overline{w}\|_{A_p(\mathbb{R})}\right)^{n\max\left\{1,\frac{p'}{p}\right\}}$$

holds, where c is the constant from (2.6) and C_γ is defined in (2.11).

Following [14], pp 285–286, it can be verified that

$$\|\overline{w}\|_{A_p(\mathbb{R})} \le \max\left\{2^{|\gamma|}, \frac{4^p}{(\gamma+1)(\gamma(1-p')+1)^{p-1}}\right\}.$$

We can get also another type of estimate of the norms in L^p_w. By using the same arguments as in [14], pp 285–286, we find that

$$\|w\|_{A_p(x_i)} \le \begin{cases} \Gamma_\gamma, & 0 \le \gamma < p-1, \\ G_\gamma, & -1 < \gamma < 0; \end{cases} \qquad i = 1, 2,$$

where

$$\Gamma_\gamma = \max\left\{((4/3)^2+1)^{\gamma/2}(2/3)^\gamma, b_\gamma 4^p\left((\gamma+1)^{-1}(\gamma(1-p')+1)^{1-p}+1\right)\right\},$$

$$G_\gamma = \max\left\{((4/3)^2+1)^{-\gamma/2}(2/3)^\gamma, d_0 d_\gamma 4^p\left((\gamma+1)^{-1}(\gamma(1-p')+1)^{1-p}+1\right)\right\}.$$

Consequently, using directly Theorems 2.3 and 2.4 we have the following estimate

$$\|M^{(s)}\|_{L^p_w(\mathbb{R}^n)} \le C^n \begin{cases} \Gamma_\gamma^{n/(p-1)}, & 0 \le \gamma < p-1, \\ G_\gamma^{n/(p-1)}, & -1 < \gamma < 0; \end{cases}$$

$$\|\mathcal{H}^{(n)}\|_{L^p_w(\mathbb{R}^n)} \le c^n \begin{cases} \Gamma_\gamma^{n\max\{1,p'/p\}}, & 0 \le \gamma < p-1, \\ G_\gamma^{n\max\{1,p'/p\}}, & -1 < \gamma < 0; \end{cases}$$

where C and c are constants in (1.1) and (2.6) respectively.

2.2. Strong fractional maximal functions and Riesz potentials with product kernels

In this subsection, we state and prove sharp weighted norm estimates for strong fractional maximal and Riesz potential with product kernels. To get the main results we use the ideas of the previous subsection.

In 1974 B. Muckenhoupt and R. Wheeden [15] found necessary and sufficient condition for the one-weight inequality; namely, they proved that the Riesz potential I_α (resp the fractional maximal operator M_α) is bounded from $L^p_{w^p}(\mathbb{R}^n)$ to $L^q_{w^q}(\mathbb{R}^n)$, where $1 < p < \infty$, $0 < \alpha < n/p$, $q = \frac{np}{n-\alpha p}$ if and only if w satisfies the so called $A_{p,q}(\mathbb{R}^n)$ condition (see the definition below). Moreover, from their result it follows that there is a positive constant c depending only on p and α such that

$$\|T_\alpha\|_{L^p_{w^p}(\mathbb{R}^n)\to L^q_{w^q}(\mathbb{R}^n)} \le c\|w\|^\beta_{A_{p,q}(\mathbb{R}^n)}, \tag{2.12}$$

for some positive exponent β, where T_α is I_α (resp. M_α), and $\|w\|_{A_{p,q}(\mathbb{R}^n)}$ is the $A_{p,q}$ characteristic of w:

$$\|w\|_{A_{p,q}(\mathbb{R}^n)} := \sup_Q \left(\frac{1}{|Q|}\int_Q w^q(x)dx\right)\left(\frac{1}{|Q|}\int_Q w^{-p'}(x)dx\right)^{q/p'}.$$

In their paper M. Lacey, K. Moen, C. Perez and R. Torres [16] proved that the best possible value of β in (2.12) is $(1-\alpha/n)\max\{1, p'/q\}$ for I_α (resp. $p'/q(1-\alpha/n)$) for M_α (see also [17] for this and other sharp results).

Definition 2.3. A weight function w satisfies $A^{(s)}_{p,q}$ condition ($w \in A^{(s)}_{p,q}$), $1 < p < \infty$ if

$$\|w\|_{A^{(s)}_{p,q}} := \sup_{P \ni x}\left(\frac{1}{|P|}\int_P w^q(x)dx\right)^{1/q}\left(\frac{1}{|P|}\int_P w(x)^{-p'}dx\right)^{1/p'} < \infty,$$

where the supremum is taken over all parallelepipeds P in $\mathbb{R}^n$ with sides parallel to the co-ordinate axes.

Definition 2.4. Let $1 < p \le q < \infty$. A weight function $w = w(x_1, \dots, x_n)$ defined on $\mathbb{R}^n$ is said to satisfy $A_{p,q}$ condition in x_i uniformly with respect to other variables ($w \in A_{p,q}(x_i)$) if

$$\begin{aligned}\|w\|_{A_{p,q}(x_i)} := \operatorname*{ess\,sup}_{(x_1,\dots,x_{i-1},x_{i+1}\cdots,x_n)\in\mathbb{R}^{n-1}} & \sup_I\left(\frac{1}{|I|}\int_I w^q(x_1,\dots,x_n)dx_i\right)^{1/q} \\ & \times\left(\frac{1}{|I|}\int_I w^{-p'}(x_1,\dots,x_n)dx_i\right)^{1/p'} < \infty,\end{aligned}$$

where I is a bounded interval.

Remark 2.6. Like $A^{(s)}_p(\mathbb{R}^n)$ weights for given $w(x_1, \dots, x_n) \in A^{(s)}_{p,q} \Leftrightarrow w \in \bigcap_{i=1}^n A_{p,q}(x_i)$.

Proposition 2.2. *Let $1 < p \le q < \infty$. Suppose that operators T^k are defined by the formula* (2.1) *and that T is an operator defined for functions on $\mathbb{R}^n$. Suppose that weight w belongs to the class $A^{(s)}_{p,q}$. Let*

$$\|T^k\|_{L^p_{w^p}(\mathbb{R})\to L^q_{w^q}(\mathbb{R})} \le c\|w\|^{\gamma(p,q)}_{A_{p,q}(x_k)} \quad k = 1, \dots, n, \tag{2.13}$$

hold, where $\gamma(p,q)$ is a constant depending only on p and q. Then

$$\|T\|_{L^p_{w^p}(\mathbb{R}^n)\to L^q_{w^q}(\mathbb{R}^n)} \le c(\|w\|_{A_p(x_1)}\cdots\|w\|_{A_p(x_n)})^{\gamma(p,q)}.$$

Proof is similar to that of Proposition 2.1; therefore it is omitted.

The following theorem is from [16].

Theorem C. *Suppose that $0<\alpha<n$, $1<p<n/\alpha$ and q is defined by the relationship $1/q=1/p-\alpha/n$. If $w\in A_{p,q}(\mathbb{R}^n)$, then*

$$\|wM_\alpha f\|_{L^q(\mathbb{R}^n)} \le c\|w\|_{A_{p,q}(\mathbb{R}^n)}^{\frac{p'}{q}(1-\alpha/n)}\|wf\|_{L^p(\mathbb{R}^n)}.$$

Furthermore, the exponent $\frac{p'}{q}(1-\alpha/n)$ is sharp.

Let f be a locally integrable function and let $0<\alpha<1$. The strong fractional maximal operator is defined by

$$\big(M_\alpha^{(s)}f\big)(x)=\sup_{P\ni x}\frac{1}{|P|^{1-\alpha}}\int_P|f(y)|dy,$$

where the supremum is taken over all parallelepipeds P in $\mathbb{R}^n$ with sides parallel to the co-ordinate axes. It is easy to see that

$$\big(M_\alpha^{(s)}f\big)(x)\le\big(M_\alpha^1\circ M_\alpha^2\circ\cdots\circ M_\alpha^n f\big)(x), \tag{2.14}$$

where

$$\begin{aligned}\big(M_\alpha^k f\big)(x_1,\dots,x_n)&=M_\alpha(f(x_1,x_2,\dots,x_{k-1},\cdot,x_k,\dots,x_n))(x_k)\\&=\sup_{I_k\ni x_k}\frac{1}{|I_k|^{1-\alpha}}\int_{I_k}|f(x_1,\dots,x_{k-1},t,x_{k+1},\dots,x_n)|dt,\end{aligned}$$

where I_k are intervals in $\mathbb{R}$ such that $P=I_1\times\cdots\times I_k$.

Theorem 2.7. *Let $0<\alpha<1$, $1<p<\frac{1}{\alpha}$, $q=\frac{p}{1-\alpha p}$ and w be a weight function on $\mathbb{R}^n$ such that $w\in A^{(s)}_{p,q}(\mathbb{R}^n)$. Then there exists a constant c depending only on n, p and α such that the following inequality*

$$\|wM_\alpha^{(s)}f\|_{L^q(\mathbb{R}^n)}\le c\left(\prod_{i=1}^n\|w\|_{A_{p,q}(x_i)}\right)^{\frac{p'}{q}(1-\alpha)}\|wf\|_{L^p(\mathbb{R}^n)} \tag{2.15}$$

holds, for all $f\in L^p_{w^p}(\mathbb{R}^n)$. Further, the exponent $\frac{p'}{q}(1-\alpha)$ in estimate (2.15) *is sharp.*

Proof. Using estimate (2.14), Theorem C and Proposition 2.2 we get easily (2.15). The main “difficulty” here is to derive sharpness. Let, for simplicity, $n=2$. Let us take $0<\epsilon<1$. Suppose that w is of product type $w(x_1,x_2)=w_1(x_1)w_2(x_2)$, where $w_1(x_1)=|x_1|^{(1-\epsilon)/p'}$ and $w_2(x_2)=|x_2|^{(1-\epsilon)/p'}$. Then it is easy to see that

$$\|w\|_{A_{p,q}(x_1)}=\|w_1\|_{A_{1+q/p'}(\mathbb{R})}\approx\epsilon^{-q/p'};\qquad \|w\|_{A_{p,q}(x_2)}=\|w_2\|_{A_{1+q/p'}(\mathbb{R})}\approx\epsilon^{-q/p'}.$$

Further, if

$$f(t_1,t_2)=|t_1|^{\epsilon-1}\chi_{(0,1)}(t_1)|t_2|^{\epsilon-1}\chi_{(0,1)}(t_2),$$

then $\|wf\|_{L^p(\mathbb{R}^2)}\approx\frac{1}{\epsilon^{2/p}}$. Let $0<x_1,x_2<1$. Then we find that

$$M_\alpha^{(s)}f(x_1,x_2)\ge\frac{1}{|x_1|^{1-\alpha}|x_2|^{1-\alpha}}\int_0^{x_1}\int_0^{x_2}f(t_1,t_2)dt_1dt_2\approx c\frac{|x_1|^{\epsilon-1+\alpha}|x_2|^{\epsilon-1+\alpha}}{\epsilon^2}.$$

Finally we conclude that,

$$\|wM_\alpha^{(s)}f\|_{L^q(\mathbb{R}^2)}\ge\epsilon^{-2-2/q}. \tag{2.16}$$

Thus letting $\epsilon\to0$ we have sharpness. □

Let $0 < \alpha < 1$. We define Riesz potential with product kernels on $\mathbb{R}^n$ as follows:

$$(I_\alpha^{(n)} f)(x) = \int_{\mathbb{R}^n} \frac{f(t_1, \dots, t_n)}{\prod_{i=1}^{n} |x_i - t_i|^{1-\alpha}} dt_1 \cdots dt_n, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

When $n = 1$ we use the symbol I_α for $I_\alpha^{(1)}$. The following theorem is from [16].

Theorem D. *Let $0 < \alpha < n$, $1 < p < n/\alpha$. We put $q = \frac{np}{n-\alpha p}$. Suppose that $w \in A_{p,q}(\mathbb{R}^n)$. Then*

$$\|w I_\alpha f\|_{L^q(\mathbb{R}^n)} \le c \|w\|_{A_{p,q}(\mathbb{R}^n)}^{(1-\alpha/n)\max\{1, p'/q\}} \|wf\|_{L^p(\mathbb{R}^n)}.$$

Further, the exponent $(1 - \alpha/n)\max\{1, p'/q\}$ is sharp.

Our result regarding $I_\alpha^{(n)}$ reads as follows:

Theorem 2.8. *Let $0 < \alpha < 1$, $1 < p < 1/\alpha$. We put $q = \frac{p}{1-\alpha p}$. Let w be a weight function on $\mathbb{R}^n$ such that $w \in A_{p,q}^{(s)}(\mathbb{R}^n)$. Then there exists a constant c depending only on n, p and α such that the following inequality*

$$\|w I_\alpha^{(n)} f\|_{L^q(\mathbb{R}^n)} \le c \left(\prod_{i=1}^{n} \|w\|_{A_{p,q}(x_i)} \right)^{\max\{1, \frac{p'}{q}\}(1-\alpha)} \|wf\|_{L^p(\mathbb{R}^n)} \tag{2.17}$$

holds for all $f \in L_{w^p}^p(\mathbb{R}^n)$. Further, the exponent $\max\{1, \frac{p'}{q}\}(1-\alpha)$ in estimate (2.17) is sharp.

Proof of this statement follows using the same arguments as in the proof of Theorem 2.7 together with Theorem D.

3. One-sided operators

In 1986 E. Sawyer proved the following inequality for the right maximal operator M^+:

$$\|M^+ f\|_{L_w^p(\mathbb{R})} \le C_p \|w\|_{A_p^+(\mathbb{R})}^{\beta} \|f\|_{L_w^p(\mathbb{R})}, \quad f \in L_w^p(\mathbb{R}), \tag{3.1}$$

with some positive exponent β, where $\|w\|_{A_p^+(\mathbb{R})}$ is A_p^+ characteristic of a weight w defined by

$$\|w\|_{A_p^+(\mathbb{R})} := \sup_{x\in\mathbb{R}, h>0} \left(\frac{1}{h} \int_{x-h}^{x} w(t)dt \right) \left(\frac{1}{h} \int_{x}^{x+h} w^{1-p'}(t)dt \right)^{p-1},$$

and

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(t)|dt.$$

The authors of this work in [18] showed that the best possible exponent in (3.1) is $\beta = \frac{1}{p-1}$.

In their celebrated work [19], K. Andersen and E. Sawyer completely characterized the one-weight boundedness for one-sided fractional operators. In particular, they proved that if $1 < p < \infty$, $0 < \alpha < 1/p$, $q = \frac{p}{1-\alpha p}$, then

$$\|w \mathcal{N}_\alpha^+ f\|_{L^q(\mathbb{R})} \le C_{p,\alpha} \|w\|_{A_{p,q}^+(\mathbb{R})}^{\beta} \|wf\|_{L^p(\mathbb{R})}, \quad f \in L_{w^p}^p(\mathbb{R}), \tag{3.2}$$

for some positive β, where $\mathcal{N}_\alpha^+$ is either the Weyl transform $\mathcal{W}_\alpha$ or the right fractional maximal operator M_α^+ defined by:

$$M_\alpha^+ f(x) = \sup_{h>0} \frac{1}{h^\alpha} \int_x^{x+h} |f(t)|dt \qquad \mathcal{W}_\alpha(f)(x) = \int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}} dt \quad 0 < \alpha < 1,$$

and $\|w\|_{A^+_{p,q}(\mathbb{R}^n)}$ is the right $A^+_{p,q}$ characteristic of a weight w given by

$$\|w\|_{A^+_{p,q}(\mathbb{R})} := \sup_{\substack{x\in\mathbb{R}\\h>0}}\left(\frac{1}{h}\int_{x-h}^{x} w^q(t)dt\right)\left(\frac{1}{h}\int_{x}^{x+h} w^{-p'}(t)dt\right)^{q/p'}.$$

In [18] the authors proved that the best possible exponent β in (3.2) is $\frac{p'}{q}(1-\alpha)$ for M^+_α, and is $(1-\alpha)\max\{1, \frac{p'}{q}\}$ for $\mathcal{W}_\alpha$.

Now we list these and related results from [18].

Theorem 3.1. *Let* $1 < p < \infty$. *Then*

(i)

$$\|M^+\|_{L^p_w(\mathbb{R})} \le c\|w\|^{\frac{1}{p-1}}_{A^+_p(\mathbb{R})}$$

holds and the exponent $\frac{1}{p-1}$ *is best possible, where* $A^+_p(\mathbb{R})$.

(ii)

$$\|M^-\|_{L^p_w(\mathbb{R})} \le c\|w\|^{\frac{1}{p-1}}_{A^-_p(\mathbb{R})}$$

holds and the exponent $\frac{1}{p-1}$ *is best possible, where* $A^-_p(\mathbb{R})$ *is the left Muckenhoupt characteristic of weight:*

$$\|w\|_{A^-_p(\mathbb{R})} := \sup_{\substack{x\in\mathbb{R}\\h>0}}\left(\frac{1}{h}\int_{x}^{x+h} w(t)dt\right)\left(\frac{1}{h}\int_{x+h}^{x} w^{1-p'}(t)dt\right)^{p-1}.$$

Theorem 3.2. *Suppose that* $0 < \alpha < 1$, $1 < p < 1/\alpha$ *and that* q *is such that* $1/p - 1/q - \alpha = 0$. *Then*

(i) *there exists a positive constant* c *depending only on* p *and* α *such that*

$$\|M^+_\alpha\|_{L^p_{w^p}(\mathbb{R})\to L^q_{w^q}(\mathbb{R})} \le c\|w\|^{\frac{p'}{q}(1-\alpha)}_{A^+_{p,q}(\mathbb{R})}. \tag{3.3}$$

Moreover, the exponent $\frac{p'}{q}(1-\alpha)$ *is best possible.*

(ii) *there exists a positive constant* c *depending only on* p *and* α *such that*

$$\|M^-_\alpha\|_{L^p_{w^p}(\mathbb{R})\to L^q_{w^q}(\mathbb{R})} \le c\|w\|^{\frac{p'}{q}(1-\alpha)}_{A^-_{p,q}(\mathbb{R})}, \tag{3.4}$$

where

$$\|w\|_{A^-_{p,q}(\mathbb{R})} := \sup_{\substack{x\in\mathbb{R}\\h>0}}\left(\frac{1}{h}\int_{x}^{x+h} w^q(t)dt\right)\left(\frac{1}{h}\int_{x+h}^{x} w^{-p'}(t)dt\right)^{q/p'}.$$

Moreover, the exponent $\frac{p'}{q}(1-\alpha)$ *is best possible,*

Theorem 3.3. *Let* $0 < \alpha < 1$, $1 < p < 1/\alpha$ *and let* q *satisfy* $q = \frac{p}{1-\alpha p}$. *Then*

(a) *there is a positive constant* c *depending only on* p *and* α *such that*

$$\|\mathcal{R}_\alpha\|_{L^p_{w^p}(\mathbb{R})\to L^q_{w^q}(\mathbb{R})} \le c\|w\|^{(1-\alpha)\max\{1,p'/q\}}_{A^-_{p,q}(\mathbb{R})}. \tag{3.5}$$

Furthermore, this estimate is sharp;

(b) *there is a positive constant* c *depending only on* p *and* α *such that*

$$\|\mathcal{W}_\alpha\|_{L^p_{w^p}(\mathbb{R})\to L^q_{w^q}(\mathbb{R})} \le c\|w\|^{(1-\alpha)\max\{1,p'/q\}}_{A^+_{p,q}(\mathbb{R})}. \tag{3.6}$$

Moreover, this estimate is sharp.

One of our aims is to apply known results to give sharp estimates for multiple operators.

3.1. Strong one-sided maximal operators

Let f be locally integrable function on $\mathbb{R}^n$. We define one-sided strong fractional maximal operators as

$$M^{+(s)} f(x_1, \ldots, x_n) = \sup_{h_1,\ldots,h_n>0} \frac{1}{\prod_{i=1}^{n} h_i} \int_{x_1}^{x_1+h_1} \cdots \int_{x_n}^{x_n+h_n} |f(y_1, \ldots, y_n)| dy_1 \cdots dy_n, \tag{3.7}$$

$$M_\alpha^{-(s)} f(x_1, \ldots, x_n) = \sup_{h_1,\ldots,h_n>0} \frac{1}{\prod_{i=1}^{n} h_i} \int_{x_1-h_1}^{x_1} \cdots \int_{x_n-h_n}^{x_n} |f(y_1, \ldots, y_n)| dy_1 \cdots dy_n. \tag{3.8}$$

Let $1 < p < \infty$. We say that a weight function w belongs to the class $A_p^{-(s)}(\mathbb{R}^n)$ if

$$\|w\|_{A_p^{-(s)}(\mathbb{R}^n)} := \sup_{\substack{h_1,\ldots,h_n>0 \\ x_1,\ldots,x_n \in \mathbb{R}}} \left(\frac{1}{h_1 \cdots h_n} \int_{x_1}^{x_1+h_1} \cdots \int_{x_n}^{x_n+h_n} w(t_1, \ldots, t_n) dt_1 \cdots dt_n \right)$$
$$\times \left(\frac{1}{h_1 \cdots h_n} \int_{x_1-h_1}^{x_1} \cdots \int_{x_n-h_n}^{x_n} w^{1-p'}(t_1, \ldots, t_n) dt_1 \cdots dt_n \right)^{p-1} < \infty;$$

further, $w \in A_p^{-(s)}(\mathbb{R}^n)$ if

$$\|w\|_{A_p^{+(s)}(\mathbb{R}^n)} := \sup_{\substack{h_1,\ldots,h_n>0 \\ x_1,\ldots,x_n \in \mathbb{R}}} \left(\frac{1}{h_1 \cdots h_n} \int_{x_1-h_1}^{x_1} \cdots \int_{x_n-h_n}^{x_n} w(t_1, \ldots, t_n) dt_1 \cdots dt_n \right)$$
$$\times \left(\frac{1}{h_1 \cdots h_n} \int_{x_1}^{x_1+h_1} \cdots \int_{x_n}^{x_n+h_n} w^{1-p'}(t_1, \ldots, t_n) dt_1 \cdots dt_n \right)^{p-1} < \infty.$$

Definition 3.1. Let $1 < p < \infty$. A weight function $w = w(x_1, \ldots, x_n)$ defined on $\mathbb{R}^n$ is said to satisfy A_p^- condition in x_i uniformly with respect to other variables ($w \in A_p^-(x_i)$) if

$$\|w\|_{A_p^-(x_i)} := \operatorname*{ess\,sup}_{(x_1,\ldots,x_{i-1},x_{i+1}\cdots,x_n)\in\mathbb{R}^{n-1}} \sup_{h_i>0} \left(\frac{1}{h_i} \int_{x_i}^{x_i+h_i} w(x_1, \ldots, x_{i-1}, t, x_{i-1}, \ldots, x_n) dt \right)$$
$$\times \left(\frac{1}{h_i} \int_{x_i-h_i}^{x_i} w(x_1, \ldots, x_{i-1}, t, x_{i-1}, \ldots, x_n)^{-1/(p-1)} dt \right)^{p-1} < \infty.$$

Further, $w \in A_p^+(x_i)$ if

$$\|w\|_{A_p^+(x_i)} := \operatorname*{ess\,sup}_{(x_1,\ldots,x_{i-1},x_{i+1}\cdots,x_n)\in\mathbb{R}^{n-1}} \sup_{h_i>0} \left(\frac{1}{h_i} \int_{x_i-h_i}^{x_i} w(x_1, \ldots, x_{i-1}, t, x_{i-1}, \ldots, x_n) dt \right)$$
$$\times \left(\frac{1}{h_i} \int_{x_i}^{x_i+h_i} w(x_1, \ldots, x_{i-1}, t, x_{i-1}, \ldots, x_n)^{-1/(p-1)} dt \right)^{p-1} < \infty.$$

Remark 3.4. It is known that (see [20], Ch. 5) that $w(x_1, \ldots, x_n) \in A_p^{\pm(s)}(\mathbb{R}^n) \Leftrightarrow w \in \bigcap_{i=1}^n A_p^\pm(x_i)$.

Theorem 3.5. *Let* $1 < p < \infty$.

(i) *Suppose that a weight function w on* $\mathbb{R}^n$ *belongs to the class* $A_p^{+(s)}(\mathbb{R}^n)$. *Then there exists a constant c depending only on n and p such that the following inequality*

$$\|M^{+(s)} f\|_{L_w^p(\mathbb{R}^n)} \leq c \left(\prod_{i=1}^{n} \|w\|_{A_p^+(x_i)} \right)^{1/(p-1)} \|f\|_{L_w^p(\mathbb{R}^n)} \tag{3.9}$$

holds for all $f \in L_w^p(\mathbb{R}^n)$. *Further, the exponent* $1/(p-1)$ *in estimate* (3.9) *is sharp.*

(ii) *Let* $w \in A_p^{-(s)}(\mathbb{R}^n)$. *Then there exists a constant c depending only on n and p such that the following inequality*

$$\|M^{-(s)} f\|_{L^p_w(\mathbb{R}^n)} \le c\left(\prod_{i=1}^n \|w\|_{A_p^-(x_i)}\right)^{1/(p-1)} \|f\|_{L^p_w(\mathbb{R}^n)} \tag{3.10}$$

holds for all $f \in L^p_w(\mathbb{R}^n)$. *Further, the exponent* $1/(p-1)$ *in estimate* (3.10) *is sharp.*

Proof. We show (i). The proof of (ii) is similar. Since the proof of inequality (3.9) follows in the same way as in the case of $M^{(s)}$ (see Theorem 2.3), we show only sharpness. Let $n = 2$. We take $0 < \epsilon < 1$. Let $w(x_1, x_2) = |1 - x_1|^{(1-\epsilon)(p-1)}|1 - x_2|^{(1-\epsilon)(p-1)}$. Then it is easy to check that

$$(\|w\|_{A_p^+(x_1)}\|w\|_{A_p^+(x_2)})^{1/(p-1)} \approx \frac{1}{\epsilon^2}.$$

Observe also that for

$$f(x_1, x_2) = (1 - x_1)^{\epsilon(p-1)-1}\chi_{(0,1)}(x_1)(1 - x_2)^{\epsilon(p-1)-1}\chi_{(0,1)}(x_2),$$

we have $\|f\|_{L^p_w} \approx \frac{1}{\epsilon^2}$. Now let $0 < x_1, x_2 < 1$. Then

$$M^{+(s)} f(x_1, x_2) \ge \frac{1}{(1 - x_1)(1 - x_2)} \int_{x_1}^1 \int_{x_2}^1 f(t, \tau)dtd\tau = c\frac{1}{\epsilon^2} f(x_1, x_2).$$

Finally

$$\|M^{+(s)} f\|_{L^p_w(\mathbb{R}^2)} \ge c\frac{1}{\epsilon^2}\|f\|_{L^p_w}.$$

Thus we have the sharpness in (3.9). □

3.2. One-sided multiple fractional integrals

Now we discuss sharp bounds for one-sided strong maximal potential operators with product kernels.

Let f be a locally integrable function on $\mathbb{R}^n$ and let $0 < \alpha < 1$. We define one-sided strong fractional maximal operators as

$$M_\alpha^{+(s)} f(x_1, \dots, x_n) = \sup_{h_1,\dots,h_n>0} \frac{1}{\prod_{i=1}^n h_i^{1-\alpha}} \int_{x_1}^{x_1+h_1} \cdots \int_{x_n}^{x_n+h_n} |f(y_1, \dots, y_n)|dy_1 \cdots dy_n, \tag{3.11}$$

$$M_\alpha^{-(s)} f(x_1, \dots, x_n) = \sup_{h_1,\dots,h_n>0} \frac{1}{\prod_{i=1}^n h_i^{1-\alpha}} \int_{x_1-h_1}^{x_1} \cdots \int_{x_n-h_n}^{x_n} |f(y_1, \dots, y_n)|dy_1 \cdots dy_n. \tag{3.12}$$

Let $1 < p \le q < \infty$. We say that a weight function w belongs to the class $A_{p,q}^{-(s)}(\mathbb{R}^n)$ if

$$\|w\|_{A_{p,q}^{-(s)}(\mathbb{R}^n)} := \sup_{\substack{h_1,\dots,h_n>0\\ x_1,\dots,x_n\in\mathbb{R}}} \left(\frac{1}{h_1 \cdots h_n} \int_{x_1}^{x_1+h_1} \cdots \int_{x_n}^{x_n+h_n} w^q(t_1, \dots, t_n)dt_1 \cdots dt_n\right)^{1/q}$$
$$\times \left(\frac{1}{h_1 \cdots h_n} \int_{x_1-h_1}^{x_1} \cdots \int_{x_n-h_n}^{x_n} w^{-p'}(t_1, \dots, t_n)dt_1 \cdots dt_n\right)^{1/p'} < \infty;$$

further, $w \in A_{p,q}^{+(s)}(\mathbb{R}^n)$ if

$$\|w\|_{A_{p,q}^{+(s)}(\mathbb{R}^n)} := \sup_{\substack{h_1,\dots,h_n>0\\ x_1,\dots,x_n\in\mathbb{R}}} \left(\frac{1}{h_1 \cdots h_n} \int_{x_1-h_1}^{x_1} \cdots \int_{x_n-h_n}^{x_n} w^q(t_1, \dots, t_n)dt_1 \cdots dt_n\right)^{1/q}$$
$$\times \left(\frac{1}{h_1 \cdots h_n} \int_{x_1}^{x_1+h_1} \cdots \int_{x_n}^{x_n+h_n} w^{-p'}(t_1, \dots, t_n)dt_1 \cdots dt_n\right)^{1/p'} < \infty.$$

Definition 3.2. Let $1 < p \le q < \infty$. A weight function $w = w(x_1, \ldots, x_n)$ defined on $\mathbb{R}^n$ is said to satisfy $A^-_{p,q}$ condition in x_i uniformly with respect to other variables ($w \in A^+_{p,q}(x_i)$) if

$$\|w\|_{A^+_{p,q}(x_i)} := \operatorname*{ess\,sup}_{(x_1,\ldots,x_{i-1},x_{i+1}\ldots,x_n)\in\mathbb{R}^{n-1}} \sup_{h_i>0}\left(\frac{1}{h_i}\int_{x_i}^{x_i+h_i} w^q(x_1,\ldots,x_{i-1},t,x_{i+1}\ldots,x_n)dt\right)^{1/q}$$
$$\times\left(\frac{1}{h_i}\int_{x_i-h_i}^{x_i} w^{-p'}(x_1,\ldots,x_{i-1},t,x_{i+1}\ldots,x_n)dt\right)^{1/p'} < \infty,$$

further, $w \in A^-_{p,q}(x_i)$ if

$$\|w\|_{A^-_{p,q}(x_i)} \equiv \sup_{\substack{(x_1,\ldots,x_{i-1},\\ x_{i+1}\cdots,x_n)\in\mathbb{R}^{n-1}}} \sup_{h_i>0}\left(\frac{1}{h_i}\int_{x_i-h_i}^{x_i} w^q(x_1,\ldots,x_{i-1},t,x_{i+1}\ldots,x_n)dt\right)^{1/q}$$
$$\times\left(\frac{1}{h_i}\int_{x_i}^{x_i+h_i} w^{-p'}(x_1,\ldots,x_{i-1},t,x_{i+1}\ldots,x_n)dt\right)^{1/p'} < \infty.$$

Remark 3.6. It is easy to check that $w(x_1,\ldots,x_n) \in A^{\pm(s)}_{p,q}(\mathbb{R}^n) \Leftrightarrow w \in \bigcap_{i=1}^n A^{\pm}_{p,q}(x_i)$.

Theorem 3.7. *Let $0 < \alpha < 1$, $1 < p < 1/\alpha$. We put $q = \frac{p}{1-\alpha p}$. Suppose that w is a weight function defined on $\mathbb{R}^n$ such that $w \in A^{+(s)}_{p,q}(\mathbb{R}^n)$. Then there exists a constant c depending only on n, p and α such that the following inequality*

$$\|wM^{+(s)}_\alpha f\|_{L^q(\mathbb{R}^n)} \le c\big(\|w\|_{A^+_{p,q}(x_1)}\cdots\|w\|_{A^+_{p,q}(x_n)}\big)^{\frac{p'}{q}(1-\alpha)}\|wf\|_{L^p(\mathbb{R}^n)} \tag{3.13}$$

holds for all $f \in L^p_{w^p}(\mathbb{R}^n)$. Further, the exponent $\frac{p'}{q}(1-\alpha)$ in estimate (3.13) *is sharp.*

Proof. Estimate (3.13) follows in the same way as in the previous cases. For sharpness we take $n = 2$ and $w(x_1,x_2) = w_1(x_1)w_2(x_2)$, where $w_1(x_1) = |1-x_1|^{(1-\epsilon)p'}$; $w_2(x_2) = |1-x_2|^{(1-\epsilon)p'}$, $0 < \epsilon < 1$. Then

$$\|w\|_{A^+_{p,q}(x_1)}\|w\|_{A^+_{p,q}(x_2)} = \|w_1\|_{A^+_{p,q}(\mathbb{R})}\|w\|_{A^+_{p,q}(\mathbb{R})} = \|w_1^q\|_{A^+_{1+q/p'}(\mathbb{R})}\|w_2\|_{A^+_{1+q/p'}(\mathbb{R})}$$
$$\approx \varepsilon^{2q/p'}.$$

If

$$f(t_1,t_2) = (1-t_1)^{\epsilon-1}\chi_{(0,1)}(t_1)(1-t_2)^{\epsilon-1}\chi_{(0,1)}(t_2),$$

then $\|wf\|_{L^p(\mathbb{R})^2} \approx \frac{1}{\epsilon^{2/p}}$. Now let $0 < x < 1$. Then we find that the following estimate

$$M^{+(s)}_\alpha f(x_1,x_2) \ge \frac{1}{|1-x_1|^{1-\alpha}|1-x_2|^{1-\alpha}}\int_{x_1}^1\int_{x_2}^1 f(t_1,t_2)dt_1dt_2$$
$$\approx \frac{|1-x_1|^{\epsilon-1+\alpha}|1-x_2|^{\epsilon-1+\alpha}}{\epsilon^2}$$

holds. Finally

$$\|wM^{+(s)}_\alpha f\|_{L^q(\mathbb{R}^2)} \ge \epsilon^{-2-2/q}. \tag{3.14}$$

Thus, letting $\epsilon \to 0$ we are done. □

The next statement can be proved analogously. Details are omitted.

Theorem 3.8. *Let α, p and q satisfy the condition of Theorem* 3.7. *Let w be a weight function on $\mathbb{R}^n$ such that $w \in A^{-(s)}_{p,q}(\mathbb{R}^n)$. Then there exists a constant c depending only on n, p and α such that the following* *inequality*

$$\|wM_\alpha^{-(s)} f\|_{L^q(\mathbb{R}^n)} \le c(\|w\|_{A^-_{p,q}(x_1)} \cdots \|w\|_{A^+_{p,q}(x_n)})^{\frac{p'}{q}(1-\alpha)} \|wf\|_{L^p(\mathbb{R}^n)} \tag{3.15}$$

holds for all $f \in L^p_{w^p}(\mathbb{R}^n)$. *Further, the exponent* $\frac{p'}{q}(1-\alpha)$ *in estimate* (3.15) *is sharp.*

Let f be a measurable function on $\mathbb{R}^n$ and let $0 < \alpha < 1$. We define one-sided potentials $\mathcal{R}^{(n)}_\alpha$ and $\mathcal{W}^{(n)}_\alpha$ with product kernels

$$\mathcal{R}^{(n)}_\alpha f(x_1, \dots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \frac{f(t_1, \dots, t_n)}{(x_1 - t_1)^{1-\alpha} \cdots (x_n - t_n)^{1-\alpha}} dt_1 \cdots dt_n,$$
$$\mathcal{W}^{(n)}_\alpha f(x_1, \dots, x_n) = \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \frac{f(t_1, \dots, t_n)}{(t_1 - x_1)^{1-\alpha} \cdots (t_n - x_n)^{1-\alpha}} dt_1 \cdots dt_n,$$

where $x_i \in \mathbb{R}$, $i = 1, \dots, n$.

Finally we formulate the "sharp result" for one-sided potentials with product kernels. We do not repeat the arguments using above, and therefore omit the proof of the next statement.

Theorem 3.9. *Let* α, p *and* q *satisfy the conditions of Theorem* 3.7. *Suppose that* w *be a weight function on* $\mathbb{R}^n$ *such that* $w \in A^{-(s)}_{p,q}(\mathbb{R}^n)$. *Then*

(i) *there exists a constant* c *depending only on* n, p *and* α *such that the following inequality*

$$\|w\mathcal{R}^{(n)}_\alpha f\|_{L^q(\mathbb{R}^n)} \le c\left(\prod_{i=1}^n \|w\|_{A^-_{p,q}(x_i)}\right)^{\max\{1, \frac{p'}{q}\}(1-\alpha)} \|wf\|_{L^p(\mathbb{R}^n)} \tag{3.16}$$

holds for all $f \in L^p_{w^p}(\mathbb{R}^n)$. *Further, the exponent* $\max\{1, \frac{p'}{q}\}(1-\alpha)$ *in estimate* (3.16) *is sharp.*

(ii) *There is a constant* c *depending only on* n, p *and* α *such that*

$$\|w\mathcal{W}^{(n)}_\alpha f\|_{L^q(\mathbb{R}^n)} \le c\left(\prod_{i=1}^n \|w\|_{A^+_{p,q}(x_i)}\right)^{\max\{1, \frac{p'}{q}\}(1-\alpha)} \|wf\|_{L^p(\mathbb{R}^n)} \tag{3.17}$$

for all $f \in L^p_{w^p}(\mathbb{R}^n)$. *Further, the exponent* $\max\{1, \frac{p'}{q}\}(1-\alpha)$ *in estimate* (3.17) *is sharp.*

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Original article

Common fixed point results for weakly compatible mappings under contractive conditions of integral type in complex valued metric spaces

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Abstract

In this manuscript, using $(E.A)$ property and (CLR) property common fixed point results for weakly compatible mappings, satisfying integral type contractive condition in complex valued metric spaces are investigated.

Keywords: Complex valued metric spaces; Common fixed points; Weakly compatible mappings; Property $(E.A)$; (CLR) property

1. Introduction

Banach contraction principle [1] is the most powerful result in the field of metric fixed point theory. This principle provides distinctive solution to various mathematical models such as Integral equations, Differential equations and Functional equations. Banach's contraction principle has been extended and generalized for different kinds of contractions in various metric spaces. A significant generalization of Banach principle [1] is the Branciari [2] fixed point theorem for integral type inequality. Afterward, several researchers [3–8] further generalized the result of Branciari in metric spaces.

Azam et al. [9] introduced the notion of complex valued metric space and proved common fixed point theorems for two self-mappings satisfying a rational type inequality. Bhatt et al. [10] initiated the concept of weakly compatible maps to study common fixed point theorem for weakly compatible maps in complex valued metric spaces. Verma and Pathak [11] introduced the notion of property $(E.A)$ and (CLR) property and established common fixed point theorems using these properties in complex valued metric space. Manro et al. [12] generalized the theorem of Branciari [2] for two self-maps under contractive condition of integral type satisfying $(E.A)$ and (CLR) properties in the setting of complex valued metric spaces.

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The aim of this paper is to prove common fixed point theorems for integral type contractive condition using property $(E.A)$ and (CLR) property in complex valued metric spaces.

2. Preliminaries

Definition 2.1 *([9])*. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2 \Leftrightarrow \mathrm{Re}(z_1) \leq \mathrm{Re}(z_2), \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$.

Consequently, one can say that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(1) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2), \mathrm{Im}(z_1) < \mathrm{Im}(z_2)$;
(2) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2), \mathrm{Im}(z_1) = \mathrm{Im}(z_2)$;
(3) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2), \mathrm{Im}(z_1) < \mathrm{Im}(z_2)$;
(4) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2), \mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (1)–(3) is satisfied and we will write $z_1 \prec z_2$ if only (3) is satisfied.

Note that one can easily verifies that

- $a, b \in R$ and $a \leq b \Rightarrow az \precsim bz$ for all $z \in \mathbb{C}$;
- $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$;
- $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Definition 2.2 *([9])*. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following axioms:

(1) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(3) $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X and the pair (X, d) is called complex valued metric space.

Example 2.1 *([13])*. Let $X = \mathbb{C}$ and $d : X \times X \to \mathbb{C}$ be the mapping defined by

$$d(x, y) = e^{im}|x - y|,$$

where $x, y \in X$ and $0 \leq m \leq \frac{\pi}{2}$. Then (X, d) is a complex valued metric space.

Definition 2.3 *([9])*. Let $\{x_n\}$ be a sequence in complex valued metric (X, d) and $x \in X$. Then x is called the limit of $\{x_n\}$ if for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in N$ such that $d(x_n, x) \prec c$ for all $n > n_0$ and we write $\lim_{n\to\infty} x_n = x$.

Lemma 2.1 *([9])*. *Any sequence $\{x_n\}$ in complex valued metric space (X, d) converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Definition 2.4 *([14])*. Let K and L be self maps of a non empty set X. Then

(i) $x \in X$ is said to be fixed point of L if $Lx = x$.
(ii) $x \in X$ is said to be a coincidence point of K and L if $Kx = Lx$.
(iii) $x \in X$ is said to be a common fixed point of K and L if $Kx = Lx = x$.

Definition 2.5 *([10])*. Let X be a complex valued metric space. Then the self-mappings $K, L : X \to X$ are weakly compatible if there exist a point $x \in X$ such that $KLx = LKx$ whenever $Kx = Lx$.

Definition 2.6 *([11])*. Two self-maps K and L on a complex valued metric space X satisfy property $(E.A)$, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Lx_n = x \quad \text{for some } x \in X.$$

Definition 2.7 *([11])*. Two self-maps K and L on a complex valued metric space X satisfy the common limit in the range of L property, denoted by (CLR_K) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Lx_n = Kx \quad \text{for some } x \in X.$$

Lemma 2.2 *([15]). If $\{a_n\}$ is a sequence in $[0,\infty)$, then $\lim_{n\to\infty}\int_0^{a_n}\phi(s)\,ds = 0$ if and only if $a_n \to 0$, as $n\to\infty$.*

3. Main results

From [2], let $\Phi = \{\phi : \phi : [0,\infty[\to [0,\infty[$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,\infty[$, non-negative, non-decreasing and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon}\phi(t)\,dt > 0\}$.

Now, for any $z_1, z_2 \in \mathbb{C}_+$, define

$$[z_1, z_2] = \{r(s) \in \mathbb{C} : r(s) = z_1 + s(z_2 - z_1) \text{ for some } s \in [0,1]\}. \tag{1}$$

$$(z_1, z_2] = \{r(s) \in \mathbb{C} : r(s) = z_1 + s(z_2 - z_1) \text{ for some } s \in (0,1]\}. \tag{2}$$

A set $P = \{z_1 = w_0, w_1, w_2, \ldots, w_n = z_2\}$ is a partition of $[z_1, z_2]$ if and only if the sets $\{[w_{i-1}, w_i)\}_{i=1}^n$ are pairwise disjoint and their union along with z_2 is $[z_1, z_2]$.

Let $\zeta : [z_1, z_2] \to \mathbb{C}$ be defined as:

$$\zeta(x, y) = (\phi_1(x), \phi_2(y)),$$

where $(x, y) \in [z_1, z_2]$ and $\phi_1, \phi_2 \in \Phi$. Now, for a given partition $\hat{P}$ of $[z_1, z_2]$, we define the lower summation as:

$$S_L\left(\zeta, \hat{P}\right) = \sum_{n=0}^{n-1} (\phi_1(x_i), \phi_2(y_i))\,|(x_{i+1}, y_{i+1}) - (x_i, y_i)|.$$

Similarly the upper summation as:

$$S_U\left(\zeta, \hat{P}\right) = \sum_{n=0}^{n-1} \zeta(\phi_1(x_{i+1}), \phi_2(y_i))\,|(x_{i+1}, y_{i+1}) - (x_i, y_i)|.$$

Then the integral $\int_{z_1}^{z_2}\zeta\,d_C$ if exists is defined as:

$$\begin{aligned}\int_{z_1}^{z_2}\zeta\,d_C &= \lim_{n\to\infty}\sum_{n=0}^{n-1} (\phi_1(x_i), \phi_2(y_i))\,|(x_{i+1}, y_{i+1}) - (x_i, y_i)| \\ &= \lim_{n\to\infty}\sum_{n=0}^{n-1} \zeta(\phi_1(x_{i+1}), \phi_2(y_i))\,|(x_{i+1}, y_{i+1}) - (x_i, y_i)|.\end{aligned}$$

For any $\zeta := (\phi_1, \phi_2) : [(a, b), (c, d)] \to \mathbb{C}$, define

$$\int_{z_1=(a,b)}^{z_2=(c,d)}\zeta\,d_C = \left(\int_{C_1}\phi_1(s)\,|z_2 - z_1|\,ds, \int_{C_2}\phi_2(s)\,|z_2 - z_1|\,ds\right).$$

Using (1), we have

$$\int_{z_1=(a,b)}^{z_2=(c,d)}\zeta\,d_C = \left(\int_{C_1}\phi_1(s)\left|\acute{r}(s)\right|ds, \int_{C_2}\phi_2(s)\left|\acute{r}(s)\right|ds\right).$$

Particularly for any $\zeta := (\phi_1, \phi_2) : [(0, 0), (a, b)] \to$, we have

$$\int_{z_1=(0,0)}^{z_2=(a,b)}\zeta\,d_C = \left(\int_0^a\phi_1(s)\left|\acute{r}(s)\right|ds, \int_0^b\phi_2(s)\left|\acute{r}(s)\right|ds\right).$$

We denote the set of all complex integrable functions $\zeta : [z_1, z_2] \to \mathbb{C}$ by $\mathcal{L}^1([z_1, z_2], \mathbb{C})$.

Lemma 3.1. *Let $\zeta \in \mathcal{L}^1([z_1, z_2], \mathbb{C})$ and $\{z_n\}$ be a sequence in $\mathbb{C}_+$, then $\lim_{n\to\infty}\int_0^{z_n} \zeta(s)\,ds = (0,0)$ if and only if $z_n \to (0,0)$, as $n \to \infty$.*

Proof. From (1), $r(s) = (0,0) + s(z_n - (0,0)) \Rightarrow \acute{r}(s) = z_n$. Then

$$\begin{aligned}
\lim_{n\to\infty}\int_0^{z_n} \zeta(s)\,ds = 0 &\Leftrightarrow \lim_{n\to\infty}\left(\int_0^{a_n} \phi_1(s)\,|z_n|ds, \int_0^{b_n} \phi_2(s)\,|z_n|ds\right) = (0,0)\\
\lim_{n\to\infty}\int_0^{z_n} \zeta(s)\,ds = 0 &\Leftrightarrow \lim_{n\to\infty}\int_0^{a_n} \phi_1(s)\,ds = 0 \quad \text{and} \quad \lim_{n\to\infty}\int_0^{b_n} \phi_2(s)\,ds = 0\\
&\Leftrightarrow a_n \to 0 \quad \text{and} \quad b_n \to 0, \quad \text{as } n \to \infty, \text{by Lemma 2.2.}\\
&\Leftrightarrow a_n \to 0 \quad \text{and} \quad b_n \to 0, \quad \text{as } n \to \infty.\\
&\Leftrightarrow z_n \to (0,0), \quad \text{as } n \to \infty. \qquad \square
\end{aligned}$$

Definition 3.1. A complex valued function $\varphi : \mathbb{R}^n \to \mathbb{C}$ is measurable if both its real and imaginary parts are measurable.

Now we extend our ideas to Complex valued measurable functions. Let $E \subset \mathbb{R}^n$ be a measurable set. Suppose $f : E \to \mathbb{C}$. Split f into its real and imaginary parts so that $f = \mathrm{Re}(f) + i\,\mathrm{Im}(f)$. Then we define the lebesgue integral of f to be

$$\int_E f = \int_E \mathrm{Re}(f) + i\int_E \mathrm{Im}(f) = \left(\int_E \mathrm{Re}(f), \int_E \mathrm{Im}(f)\right),$$

provided that $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are Lebesgue integrables. Denote the set of all such complex valued lebesgue integrable functions by $\mathcal{L}^1(E, \mathbb{C})$.

We define $\Phi^* = \{\varphi : \mathbb{R}^n \to \mathbb{C}$ as a complex valued Lebesgue-integrable mapping (i.e., $\varphi \in \mathcal{L}^1(E, \mathbb{C})$), which is summable and non-vanishing on each measurable subset of $\mathbb{R}^n$, such that for each $\varepsilon \succ 0$, $\int_0^{\varepsilon} \varphi(t)\,dt \succ 0\}$.

The following Remark and Lemma are the direct consequences of the above whole discussion.

Remark 3.1. Let $\varphi \in \Phi^*$, such that $\mathrm{Re}(\varphi), \mathrm{Im}(\varphi) \in \Phi$ and $\{z_n\}$ is a sequence in $\mathbb{C}_+$ converges to z, then $\lim_{n\to\infty}\int_0^{z_n} \varphi(s)\,ds = \int_0^{z} \varphi(s)\,ds$.

Lemma 3.2. *Let $\varphi \in \Phi^*$, such that $\mathrm{Re}(\varphi), \mathrm{Im}(\varphi) \in \Phi$ and $\{z_n\}$ is a sequence in $\mathbb{C}_+$, then $\lim_{n\to\infty}\int_0^{z_n} \varphi(s)\,ds = 0$ if and only if $z_n \to (0,0)$, as $n \to \infty$.*

Now, we present our main results.

Theorem 3.1. *Let (X, d) be a complex valued metric space and $K, L : X \to X$ be self-mappings satisfying the following conditions:*

I. *the pair (K, L) satisfies property $(E.A)$ such that $K(X) \subseteq L(X)$ and $L(X)$ is a closed subspace of X;*

II. $\forall\ x, y \in X$

$$\begin{aligned}
\int_0^{d(Kx,Ly)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kx,Lx)} \varphi(t)dt + \lambda_2 \int_0^{d(Ky,Ly)} \varphi(t)dt\\
+ \lambda_3 \int_0^{d(Ky,Lx)} \varphi(t)dt + \lambda_4 \int_0^{d(Kx,Ly)} \varphi(t)dt
\end{aligned}$$

where $\lambda_i \in [0,1)$ for $i = 1,2,3,4$ with $\sum_{i=1}^4 \lambda_i < 1$ and $\varphi \in \Phi^*$.

If the pair (K, L) is weakly compatible, then the mappings K and L have a unique common fixed point in X.

Proof. Assume that the pair (K, L) satisfies $(E.A.)$ property, so there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Lx_n = z \quad \text{for some } z \in X. \tag{3}$$

Since $L(X)$ is a closed subspace of X, then there exists $u \in X$ such that $Lu = z$. Thus from (3), we get

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Lx_n = z = Lu. \tag{4}$$

We show that $Ku = Lu$, for this putting $x = u$ and $y = x_n$ in condition (II) of Theorem 3.1, we have

$$\int_0^{d(Ku,Lx_n)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Ku,Lu)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx_n,Lx_n)} \varphi(t)dt + \lambda_3 \int_0^{d(Kx_n,Lu)} \varphi(t)dt + \lambda_4 \int_0^{d(Ku,Lx_n)} \varphi(t)dt.$$

Taking limit as $n \to \infty$ and using (4), we get

$$\int_0^{d(Ku,Lu)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Ku,Lu)} \varphi(t)dt + \lambda_4 \int_0^{d(Ku,Lu)} \varphi(t)dt,$$

$$(1-\lambda_1-\lambda_4)\int_0^{d(Ku,Lu)} \varphi(t)dt \precsim 0 \Rightarrow \left|(1-\lambda_1-\lambda_4)\int_0^{d(Ku,Lu)} \varphi(t)dt\right| \le 0.$$

But $1-\lambda_1-\lambda_4 > 0$, so that

$$\left|\int_0^{d(Ku,Lu)} \varphi(t)dt\right| \le 0, \quad \text{which implies that} \left|\int_0^{d(Ku,Lu)} \varphi(t)dt\right| = 0, \quad \text{thus } Ku = Lu.$$

Hence from Eq. (4), we get

$$Ku = Lu = z. \tag{5}$$

That is z is the common coincident point of K and L.

Next, we show that z is the common fixed point of K and L. For this, since the pair (K, L) is weakly compatible, therefore

$$Ku = Lu \Rightarrow LKu = KLu \Rightarrow Lz = Kz. \tag{6}$$

Now, we have to show that $Kz = z$. For this, using condition (II) of Theorem 3.1 with $x = z$ and $y = u$, we have

$$\int_0^{d(Kz,z)} \varphi(t)dt = \int_0^{d(Kz,Lu)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kz,Lz)} \varphi(t)dt + \lambda_2 \int_0^{d(Ku,Lu)} \varphi(t)dt + \lambda_3 \int_0^{d(Ku,Lz)} \varphi(t)dt + \lambda_4 \int_0^{d(Kz,Lu)} \varphi(t)dt.$$

Using (5) and (6), we get

$$\int_0^{d(Kz,z)} \varphi(t)dt \precsim \lambda_3 \int_0^{d(z,Kz)} \varphi(t)dt + \lambda_4 \int_0^{d(Kz,z)} \varphi(t)dt.$$

$$(1-\lambda_3-\lambda_4)\int_0^{d(Kz,z)} \varphi(t)dt \precsim 0. \quad \text{But } 1-\lambda_3-\lambda_4 > 0, \text{ so that } \int_0^{d(Kz,z)} \varphi(t)dt \precsim 0,$$

which shows that $\int_0^{d(Kz,z)} \varphi(t)dt = 0$, thus $Kz = z$. Hence from (6), we can get

$Kz = Lz = z.$

That is z is a common fixed point of K and L.

Finally, to check the uniqueness, let z^* be another fixed point of K and L, i.e. $Kz^* = Lz^* = z^*$. Then using condition (II) with $x = z$ and $y = z^*$, we have

$$\int_0^{d(z,z^*)} \varphi(t)dt = \int_0^{d(Kz,Lz^*)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kz,Lz)} \varphi(t)dt + \lambda_2 \int_0^{d(Kz^*,Lz^*)} \varphi(t)dt + \lambda_3 \int_0^{d(Kz^*,Lz)} \varphi(t)dt + \lambda_4 \int_0^{d(Kz,Lz^*)} \varphi(t)dt \precsim (\lambda_3+\lambda_4)\int_0^{d(z,z^*)} \varphi(t)dt.$$

Thus

$$(1-\lambda_3-\lambda_4)\int_0^{d(z,z^*)}\varphi(t)dt \precsim 0, \quad \text{But } 1-\lambda_3-\lambda_4>1, \text{ so that } \int_0^{d(z,z^*)}\varphi(t)dt \precsim 0,$$

thus $z=z^*$. Hence z is a unique common fixed point of K and L. □

Theorem 3.2. *Let (X,d) be a complex valued metric space and $K,L,M,N : X \to X$ be self-mappings satisfying the following conditions:*

I. *one of the pairs (K,N) and (L,M) satisfies $(E.A.)$ property such that $K(X)\subseteq M(X)$ and $L(X)\subseteq N(X)$;*
II. $\forall\, x,y\in X$

$$\int_0^{d(Kx,Ly)}\varphi(t)dt \precsim \lambda_1\int_0^{d(Kx,Nx)}\varphi(t)dt+\lambda_2\int_0^{d(Kx,My)}\varphi(t)dt$$
$$+\lambda_3\int_0^{d(Ly,Nx)}\varphi(t)dt+\lambda_4\int_0^{d(Ly,My)}\varphi(t)dt+\lambda_5\int_0^{d(My,Nx)}\varphi(t)dt,$$

where $\lambda_i\in[0,1)$ for $i=1,2,\ldots,5$ with $\sum_{i=1}^5\lambda_i<1$ and $\varphi\in\Phi^*$.

If one of $M(X)$ and $N(X)$ is a closed subspace of X and the pairs (K,N), (L,M) are weakly compatible, then the mappings K,L,M and N have a unique common fixed point in X.

Proof. Assume that the pair (K,N) satisfies $(E.A)$ property, so there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty}Kx_n=\lim_{n\to\infty}Nx_n=z \quad \text{for some } z\in X. \tag{7}$$

Since $K(X)\subseteq M(X)$, so there exists $\{y_n\}$ in X such that $Kx_n=My_n$ and thus from (7), we get

$$\lim_{n\to\infty}Kx_n=\lim_{n\to\infty}Nx_n=\lim_{n\to\infty}My_n=z. \tag{8}$$

We show that $\lim_{n\to\infty}Ly_n=z$. If $\lim_{n\to\infty}Ly_n=w\neq z$, then putting $x=x_n$ and $y=y_n$ in condition (II) of Theorem 3.2, we have

$$\int_0^{d(Kx_n,Ly_n)}\varphi(t)dt \precsim \lambda_1\int_0^{d(Kx_n,Nx_n)}\varphi(t)dt+\lambda_2\int_0^{d(Kx_n,My_n)}\varphi(t)dt$$
$$+\lambda_3\int_0^{d(Ly_n,Nx_n)}\varphi(t)dt+\lambda_4\int_0^{d(Ly_n,My_n)}\varphi(t)dt+\lambda_5\int_0^{d(My_n,Nx_n)}\varphi(t)dt.$$

Taking limit as $n\to\infty$ and using (8), we get

$$\int_0^{d(z,w)}\varphi(t)dt \precsim \lambda_3\int_0^{d(w,z)}\varphi(t)dt+\lambda_4\int_0^{d(w,z)}\varphi(t)dt=(\lambda_3+\lambda_4)\int_0^{d(w,z)}\varphi(t)dt.$$

That is

$$\left|(1-\lambda_3-\lambda_4)\int_0^{d(w,z)}\varphi(t)dt\right|\le 0,$$

But $1-\lambda_3-\lambda_4>0$, so that $\left|\int_0^{d(w,z)}\varphi(t)dt\right|\le 0,$ which is possible if $z=w$ and

hence $\lim_{n\to\infty}Ly_n=z$. Therefore from (8), we get

$$\lim_{n\to\infty}Kx_n=\lim_{n\to\infty}Nx_n=\lim_{n\to\infty}Ly_n=\lim_{n\to\infty}My_n=z. \tag{9}$$

Further, since $M(X)$ is a closed subspace of X, so there exists $u\in X$ such that $Mu=z$ and hence from (9), we get

$$\lim_{n\to\infty}Kx_n=\lim_{n\to\infty}Nx_n=\lim_{n\to\infty}Ly_n=\lim_{n\to\infty}My_n=z=Mu. \tag{10}$$

Now, we assert that $Lu = Mu$, for this putting $x = x_n$ and $y = u$ in condition (II) of Theorem 3.2, we have

$$\int_0^{d(Kx_n,Lu)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kx_n,Nx_n)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx_n,Mu)} \varphi(t)dt + \lambda_3 \int_0^{d(Lu,Nx_n)} \varphi(t)dt + \lambda_4 \int_0^{d(Lu,Mu)} \varphi(t)dt + \lambda_5 \int_0^{d(Mu,Nx_n)} \varphi(t)dt.$$

Taking limit as $n \to \infty$ and using (10), we get

$$\int_0^{d(z,Lu)} \varphi(t)dt \precsim \lambda_3 \int_0^{d(Lu,z)} \varphi(t)dt + \lambda_4 \int_0^{d(Lu,z)} \varphi(t)dt = (\lambda_3 + \lambda_4) \int_0^{d(Lu,z)} \varphi(t)dt,$$

$$(1 - \lambda_3 - \lambda_4) \int_0^{d(Lu,z)} \varphi(t)dt \precsim 0 \Rightarrow \int_0^{d(Lu,z)} \varphi(t)dt = 0, \quad \text{as } 1 - \lambda_3 - \lambda_4 > 0$$

thus $Lu = z$ and hence (10) becomes

$$Lu = Mu = z. \tag{11}$$

But $L(X) \subseteq N(X)$, so there exists $v \in X$ such that $Lu = Nv$ and hence (11) becomes,

$$Lu = Mu = Nv = z. \tag{12}$$

Also, we assert that $Kv = Nv$, for this setting $x = v$ and $y = u$ in condition (II) of Theorem 3.2, we have

$$\int_0^{d(Kv,Lu)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kv,Nv)} \varphi(t)dt + \lambda_2 \int_0^{d(Kv,Mu)} \varphi(t)dt + \lambda_3 \int_0^{d(Lu,Nv)} \varphi(t)dt + \lambda_4 \int_0^{d(Lu,Mu)} \varphi(t)dt + \lambda_5 \int_0^{d(Mu,Nv)} \varphi(t)dt.$$

Using Eq. (12), we get

$$\int_0^{d(Kv,z)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kv,z)} \varphi(t)dt + \lambda_2 \int_0^{d(Kv,z)} \varphi(t)dt,$$

$$\left| (1 - \lambda_1 - \lambda_2) \int_0^{d(Kv,z)} \varphi(t)dt \right| \le 0.$$

But $1 - \lambda_1 - \lambda_2 > 0$, so that $\left| \int_0^{d(Kv,z)} \varphi(t)dt \right| \le 0$, thus $Kv = z$, that is $Kv = Nv$

and hence from (12), one can write

$$Kv = Lu = Mu = Nv = z, \tag{13}$$

showing that is z is the common coincident point of the pairs (L, M), (K, N).

Next, we have to show that z is the common fixed point of K, L, M and N. For this, using the weak compatibility of the pairs (K, N), (L, M) and Eq. (13), we have

$$Kv = Nv \Rightarrow NKv = KNv \Rightarrow Kz = Nz. \tag{14}$$

$$Lu = Mu \Rightarrow MLu = LMu \Rightarrow Lz = Mz. \tag{15}$$

Now, we claim that $Kz = z$. To support our claim, setting $x = z$ and $y = u$ in condition (II) of Theorem 3.2, we have

$$\int_0^{d(Kz,Lu)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kz,Nz)} \varphi(t)dt + \lambda_2 \int_0^{d(Kz,Mu)} \varphi(t)dt + \lambda_3 \int_0^{d(Lu,Nz)} \varphi(t)dt + \lambda_4 \int_0^{d(Lu,Mu)} \varphi(t)dt + \lambda_5 \int_0^{d(Mu,Nz)} \varphi(t)dt.$$

Using Eqs. (13) and (14), we get

$$\int_0^{d(Kz,z)} \varphi(t)dt \precsim \lambda_2 \int_0^{d(Kz,z)} \varphi(t)dt + \lambda_3 \int_0^{d(z,Kz)} \varphi(t)dt + \lambda_5 \int_0^{d(z,Kz)} \varphi(t)dt.$$

$\left|(1-\lambda_2-\lambda_3-\lambda_5)\int_0^{d(z,Kz)} \varphi(t)dt\right| \leq 0.$ But $1-\lambda_2-\lambda_3-\lambda_5 > 0$, so that

$\left|\int_0^{d(z,Kz)} \varphi(t)dt\right| \leq 0,$ thus $Kz = z$ and hence from Eq. (14), we get

$$Kz = Nz = z. \tag{16}$$

Similarly, setting $x = v$ and $y = z$ in condition (II) of Theorem 3.2 and using Eqs. (13), (15), we get

$$Lz = Mz = z. \tag{17}$$

From (16) and (17), we can write

$$Kz = Lz = Mz = Nz = z.$$

That is z is a common fixed point of K, L, M and N in X.

Uniqueness, let z^* be another fixed point of K, L, M and N, i.e. $Kz^* = Lz^* = Mz^* = Nz^* = z^*$. Then using condition (II) of Theorem 3.2 with $x = z$ and $y = z^*$, we have

$$\begin{aligned}
\int_0^{d(z,z^*)} \varphi(t)dt &= \int_0^{d(Kz,Lz^*)} \varphi(t)dt \\
&\precsim \lambda_1 \int_0^{d(Kz,Nz)} \varphi(t)dt + \lambda_2 \int_0^{d(Kz,Mz^*)} \varphi(t)dt + \lambda_3 \int_0^{d(Lz^*,Nz)} \varphi(t)dt \\
&\quad + \lambda_4 \int_0^{d(Lz^*,Mz^*)} \varphi(t)dt + \lambda_5 \int_0^{d(Mz^*,Nz)} \varphi(t)dt \\
&\precsim \lambda_2 \int_0^{d(z,z^*)} \varphi(t)dt + \lambda_3 \int_0^{d(z^*,z)} \varphi(t)dt + \lambda_5 \int_0^{d(z^*,z)} \varphi(t)dt,
\end{aligned}$$

thus

$$\left|(1-\lambda_2-\lambda_3-\lambda_5)\int_0^{d(z,z^*)} \varphi(t)dt\right| \leq 0 \Rightarrow \left|\int_0^{d(z,z^*)} \varphi(t)dt\right| \leq 0,$$

which is a contradiction, unless $z = z^*$. Hence z is a unique common fixed point of K, L, M and N in X. $\square$

From Theorem 3.2, we can derive the following corollaries:

Corollary 3.1. *Let (X, d) be a complex valued metric space and $L, M, N : X \to X$ be self-mappings satisfying the following conditions:*

I. *one of the pairs (L, N) and (L, M) satisfies $(E.A.)$ property such that $L(X) \subseteq M(X)$ and $L(X) \subseteq N(X)$;*

II. $\forall\, x, y \in X$

$$\begin{aligned}
\int_0^{d(Lx,Ly)} \varphi(t)dt &\precsim \lambda_1 \int_0^{d(Lx,Nx)} \varphi(t)dt + \lambda_2 \int_0^{d(Lx,My)} \varphi(t)dt \\
&\quad + \lambda_3 \int_0^{d(Ly,Nx)} \varphi(t)dt + \lambda_4 \int_0^{d(Ly,My)} \varphi(t)dt + \lambda_5 \int_0^{d(My,Nx)} \varphi(t)dt
\end{aligned}$$

where $\lambda_i \in [0, 1)$ for $i = 1, 2, \ldots, 5$ with $\sum_{i=1}^{5} \lambda_i < 1$ and $\varphi \in \Phi^*$.

If one of $M(X)$ and $N(X)$ is a closed subspace of X and the pairs (L, N), (L, M) are weakly compatible, then the mappings L, M and N have a unique common fixed point in X.

Corollary 3.2. *Let (X, d) be a complex valued metric space and $L, M : X \to X$ be self-mappings satisfying the following conditions:*

I. *the pair (L, M) satisfies property $(E.A)$ such that $L(X) \subseteq M(X)$;*
II. $\forall\ x, y \in X$

$$\int_0^{d(Lx,Ly)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Lx,Mx)} \varphi(t)dt + \lambda_2 \int_0^{d(Lx,My)} \varphi(t)dt + \lambda_3 \int_0^{d(Ly,Mx)} \varphi(t)dt + \lambda_4 \int_0^{d(Ly,My)} \varphi(t)dt + \lambda_5 \int_0^{d(My,Mx)} \varphi(t)dt$$

where $\lambda_i \in [0, 1)$ for $i = 1, 2, \ldots, 5$ with $\sum_{i=1}^{5} \lambda_i < 1$ and $\varphi \in \Phi^*$.

If $M(X)$ is a closed subspace of X and the pair (L, M) is weakly compatible, then the mappings L and M have a unique common fixed point in X.

To illustrate Theorem 3.2, we construct the following example.

Example 3.1. Let $X = (1, 3] \cup [4, 6]$ be a metric space with metric $d : X \times X \to \mathbb{C}$ defined by $d(x, y) = e^{im}|x - y|$, where $x, y \in X$ and $0 \le m \le \frac{\pi}{6}$. Define self-maps K, L, M and N on X by:

$$Kx = \begin{cases} \frac{3}{2} & \text{if } x \in \left(1, \frac{3}{2}\right] \cup [4, 6] \\ 1.1 & \text{if } x \in \left(\frac{3}{2}, 3\right] \end{cases}; \qquad Lx = \begin{cases} \frac{3}{2} & \text{if } x \in \left(1, \frac{3}{2}\right] \cup [4, 6] \\ 2 & \text{if } x \in \left(\frac{3}{2}, 3\right] \end{cases};$$

$$Mx = \begin{cases} 3 - x & \text{if } x \in \left(1, \frac{3}{2}\right) \\ \frac{3}{2} & \text{if } x = \frac{3}{2} \\ 4 & \text{if } x \in \left(\frac{3}{2}, 3\right] \\ 1.1 & \text{if } x \in [4, 6] \end{cases} \quad \text{and} \quad Nx = \begin{cases} 3 & \text{if } x \in \left(1, \frac{3}{2}\right) \\ \frac{3}{2} & \text{if } x = \frac{3}{2} \\ 2x & \text{if } x \in \left(\frac{3}{2}, 3\right] \\ 2 & \text{if } x \in [4, 6]. \end{cases}$$

Also define $\varphi : \mathbb{R}^2 \to \mathbb{C}$ by $\varphi(t) = 3z^2$, where $t = (a, b)$ and $z = a + ib$. Then

$$K(X) = \left\{1.1, \frac{3}{2}\right\}, \qquad L(X) = \left\{\frac{3}{2}, 2\right\}, \qquad M(X) = \left[\frac{3}{2}, 2\right] \cup \{1.1, 4\},$$
$$N(X) = [3, 6] \cup \left\{2, \frac{3}{2}\right\}.$$

First we check condition (I) of Theorem 3.2 for this let $\{x_n\} = \{\frac{9n-2}{6n}\}_{n \ge 1}$ be a sequence in X. Then

$$\lim_{n\to\infty} Lx_n = \lim_{n\to\infty} L\left(\frac{9n-2}{6n}\right) = \lim_{n\to\infty} \frac{3}{2} = \frac{3}{2}$$

and

$$\lim_{n\to\infty} Mx_n = \lim_{n\to\infty} M\left(\frac{9n-2}{6n}\right) = \lim_{n\to\infty} \left(3 - \frac{9n-2}{6n}\right) = \lim_{n\to\infty} \left(\frac{9n+2}{6n}\right) = \frac{3}{2},$$

that is there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Lx_n = \lim_{n\to\infty} Mx_n = \frac{3}{2} \in X$.

Hence (L, M) satisfies $(E.A)$ property.

Next, to check condition (II) of Theorem 3.2, we discuss the following cases:
Case 1. Let $x, y \in (1, \frac{3}{2})$, then $Kx = Ky = \frac{3}{2}$, $My = 3 - x$ and $Nx = 3$. Then

$$\int_0^{d(Kx,Ly)} \varphi(t)dt = 0 \prec \lambda_1 \int_0^{d(Kx,Nx)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx,My)} \varphi(t)dt + \lambda_3 \int_0^{d(Ly,Nx)} \varphi(t)dt + \lambda_4 \int_0^{d(Ly,My)} \varphi(t)dt + \lambda_5 \int_0^{d(My,Nx)} \varphi(t)dt.$$

Case 2. Let $x = y = \frac{3}{2}$, then $Kx = Ly = My = Nx = \frac{3}{2}$ and

$$\int_0^{d(Kx,Ly)} \varphi(t)dt = 0 = \lambda_1 \int_0^{d(Kx,Nx)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx,My)} \varphi(t)dt + \lambda_3 \int_0^{d(Ly,Nx)} \varphi(t)dt + \lambda_4 \int_0^{d(Ly,My)} \varphi(t)dt + \lambda_5 \int_0^{d(My,Nx)} \varphi(t)dt.$$

Case 3. Let $x, y \in (\frac{3}{2}, 3]$, then $Kx = 1.1$, $Ly = 2$, $My = 4$ and $Nx = 2x$.
Now

$$\int_0^{d(Kx,Ly)} \varphi(t)dt = \int_0^{e^{im}} 3z^2 dt = z^3 \Big|_0^{e^{im}} = e^{3im}.$$

Also,
for $\lambda_1 = \frac{1}{9}, \lambda_2 = \frac{1}{13}, \lambda_3 = \frac{1}{5}, \lambda_4 = \frac{1}{11}, \lambda_5 = \frac{1}{2}$, with $\sum_{i=1}^{5} \lambda_i = 0.9789432789 < 1$, we have

$$\begin{aligned} &\lambda_1 \int_0^{d(Kx,Nx)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx,My)} \varphi(t)dt + \lambda_3 \int_0^{d(Ly,Nx)} \varphi(t)dt \\ &\quad + \lambda_4 \int_0^{d(Ly,My)} \varphi(t)dt + \lambda_5 \int_0^{d(My,Nx)} \varphi(t)dt \\ &\quad \succ \frac{1}{9}(6.859e^{3im}) + \frac{1}{13}(24.389e^{3im}) + \frac{1}{5}e^{3im} + \frac{1}{11}8e^{3im} + \frac{1}{2}e^{3im} \\ &\quad \succ 4e^{3im}. \end{aligned}$$

Thus

$$\begin{aligned} \int_0^{d(Kx,Ly)} \varphi(t)dt &= e^{3im} \prec 4e^{3im} \\ &\prec \lambda_1 \int_0^{d(Kx,Nx)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx,My)} \varphi(t)dt + \lambda_3 \int_0^{d(Ly,Nx)} \varphi(t)dt \\ &\quad + \lambda_4 \int_0^{d(Ly,My)} \varphi(t)dt + \lambda_5 \int_0^{d(My,Nx)} \varphi(t)dt. \end{aligned}$$

Case 4. Let $x, y \in [4, 6]$, then $Kx = Ly = \frac{3}{2}$, $Mx = 1.1$ and $Ny = 2$. Thus

$$\int_0^{d(Kx,Ly)} \varphi(t)dt = 0 \prec \lambda_1 \int_0^{d(Kx,Nx)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx,My)} \varphi(t)dt + \lambda_3 \int_0^{d(Ly,Nx)} \varphi(t)dt + \lambda_4 \int_0^{d(Ly,My)} \varphi(t)dt + \lambda_5 \int_0^{d(My,Nx)} \varphi(t)dt.$$

Therefore, in view of the above four cases, the integral contractive condition (II) of Theorem 3.2 is satisfied.

Also $K(X) \subseteq M(X)$ and $L(X) \subseteq N(X)$ such that $M(X)$ is a closed subspace of X and the pairs (K, M), (L, N) are weakly compatible. Hence from Theorem 3.2 we can say that, $\frac{3}{2}$ is a unique common fixed point of K, L, M and N.

Theorem 3.3. *Let (X, d) be a complex valued metric space and $K, L : X \to X$ be self-mappings satisfying the following conditions:*

I. *the pair (K, L) satisfies (CLR_L) property;*
II. $\forall\, x, y \in X$

$$\int_0^{d(Kx,Ly)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kx,Lx)} \varphi(t)dt + \lambda_2 \int_0^{d(Ky,Ly)} \varphi(t)dt + \lambda_3 \int_0^{d(Ky,Lx)} \varphi(t)dt + \lambda_4 \int_0^{d(Kx,Ly)} \varphi(t)dt,$$

where $\lambda_i \in [0, 1)$ for $i = 1, 2, 3, 4$ with $\sum_{i=1}^{4} \lambda_i < 1$ and $\varphi \in \Phi^*$.

If the pair (K, L) is weakly compatible, then the mappings K and L have a unique common fixed point in X.

Proof. Assume that the pair (K, L) satisfies (CLR_L) property, so there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Lx_n = Lx \quad \text{for some } x \in X. \tag{18}$$

We show that $Kx = Lx$. For this putting $y = x_n$ in condition (II) of Theorem 3.3, we have

$$\int_0^{d(Kx,Lx_n)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kx,Lx)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx_n,Lx_n)} \varphi(t)dt + \lambda_3 \int_0^{d(Kx_n,Lx)} \varphi(t)dt + \lambda_4 \int_0^{d(Kx,Lx_n)} \varphi(t)dt.$$

Taking limit as $n \to \infty$ and using (18), we can write

$$\int_0^{d(Kx,Lx)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kx,Lx)} \varphi(t)dt + \lambda_4 \int_0^{d(Kx,Lx)} \varphi(t)dt,$$

$$(1 - \lambda_1 - \lambda_4) \int_0^{d(Kx,Lx)} \varphi(t)dt \precsim 0 \;\Rightarrow\; \left| (1 - \lambda_1 - \lambda_4) \int_0^{d(Kx,Lx)} \varphi(t)dt \right| \le 0.$$

But $1 - \lambda_1 - \lambda_4 > 0$, so that

$$\left| \int_0^{d(Kx,Lx)} \varphi(t)dt \right| \le 0, \quad \text{which is possible if } \left| \int_0^{d(Kx,Lx)} \varphi(t)dt \right| = 0. \text{ Thus } Kx = Lx.$$

Now, let $Kx = Lx = z$. Then since the pair (K, L) is weakly compatible, so that

$$Kx = Lx \;\Rightarrow\; LKx = KLx \;\Rightarrow\; Lz = Kz. \tag{19}$$

Next, show that z is a common fixed point of K and L. For this using condition (II) of Theorem 3.3 with $x = z$ and $y = x$, we get

$$\begin{aligned}\int_0^{d(Kz,z)} \varphi(t)dt = \int_0^{d(Kz,Lx)} \varphi(t)dt &\precsim \lambda_1 \int_0^{d(Kz,Lz)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx,Lx)} \varphi(t)dt \\ &\quad + \lambda_3 \int_0^{d(Kx,Lz)} \varphi(t)dt + \lambda_4 \int_0^{d(Kz,Lx)} \varphi(t)dt \\ &\precsim (\lambda_3 + \lambda_4) \int_0^{d(Kz,z)} \varphi(t)dt.\end{aligned}$$

But $0 < \lambda_3 + \lambda_4 < 1$. Therefore $Kz = z$.

Hence from Eq. (19), we get

$$Kz = Lz = z. \tag{20}$$

Thus z is a common fixed point of K and L.

Finally, to check the uniqueness, proceeding the same steps as in Theorem 3.1, we conclude that z is a unique common fixed point of K and L. $\square$

Our next results are proved with the help of CLR Property.

Theorem 3.4. *Let (X, d) be a complex valued metric space and $K, L, N, M : X \to X$ be self-mappings satisfying the following conditions:*

I. *either the pair (K, N) satisfies (CLR_K) property or the pair (L, M) satisfies (CLR_L) property such that $K(X) \subseteq M(X)$ and $L(X) \subseteq N(X)$;*

II. $\forall\, x, y \in X$

$$\int_0^{d(Kx,Ly)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kx,Nx)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx,My)} \varphi(t)dt + \lambda_3 \int_0^{d(Ly,Nx)} \varphi(t)dt + \lambda_4 \int_0^{d(Ly,My)} \varphi(t)dt + \lambda_5 \int_0^{d(My,Nx)} \varphi(t)dt$$

where $\lambda_i \in [0, 1)$ for $i = 1, 2, \ldots, 5$ with $\sum_{i=1}^{5} \lambda_i < 1$ and $\varphi \in \Phi^*$.

If the pairs (K, N) and (L, M) are weakly compatible, then the mappings K, L, M and N have a unique common fixed point in X.

Proof. Assume that the pair (K, L) satisfies (CLR_L) property, so there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Lx_n = Lx \quad \text{for some } x \in X. \tag{21}$$

We show that $Kx = Lx$. For this putting $y = x_n$ in condition (II) of Theorem 3.3, we have

$$\int_0^{d(Kx,Lx_n)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kx,Lx)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx_n,Lx_n)} \varphi(t)dt + \lambda_3 \int_0^{d(Kx_n,Lx)} \varphi(t)dt + \lambda_4 \int_0^{d(Kx,Lx_n)} \varphi(t)dt.$$

Taking limit as $n \to \infty$ and using (21), we can write

$$\int_0^{d(Kx,Lx)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kx,Lx)} \varphi(t)dt + \lambda_4 \int_0^{d(Kx,Lx)} \varphi(t)dt,$$

$$(1 - \lambda_1 - \lambda_4) \int_0^{d(Kx,Lx)} \varphi(t)dt \precsim 0 \;\Rightarrow\; \left|(1 - \lambda_1 - \lambda_4) \int_0^{d(Kx,Lx)} \varphi(t)dt\right| \le 0.$$

But $1 - \lambda_1 - \lambda_4 > 0$, so that

$$\left|\int_0^{d(Kx,Lx)} \varphi(t)dt\right| \le 0, \quad \text{which is possible if } \left|\int_0^{d(Kx,Lx)} \varphi(t)dt\right| = 0. \text{ Thus } Kx = Lx.$$

Now, let $Kx = Lx = z$. Then since the pair (K, L) is weakly compatible, so that

$$Kx = Lx \;\Rightarrow\; LKx = KLx \;\Rightarrow\; Lz = Kz. \tag{22}$$

Next, show that z is common fixed point of K and L. For this using condition (II) of Theorem 3.3 with $x = z$ and $y = x$, we get

$$\begin{aligned}\int_0^{d(Kz,z)} \varphi(t)dt = \int_0^{d(Kz,Lx)} \varphi(t)dt &\precsim \lambda_1 \int_0^{d(Kz,Lz)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx,Lx)} \varphi(t)dt \\ &\quad + \lambda_3 \int_0^{d(Kx,Lz)} \varphi(t)dt + \lambda_4 \int_0^{d(Kz,Lx)} \varphi(t)dt \\ &\precsim (\lambda_3 + \lambda_4) \int_0^{d(Kz,z)} \varphi(t)dt.\end{aligned}$$

But $0 < \lambda_3 + \lambda_4 < 1$. Therefore $Kz = z$.

Hence from Eq. (22), we get

$$Kz = Lz = z. \tag{23}$$

Thus z is a common fixed point of K and L.

Finally, to check the uniqueness, proceeding the same steps as in Theorem 3.1, we conclude that z is a unique common fixed point of K and L. □

Our next results are proved with the help of CLR Property.

Theorem 3.5. *Let (X, d) be a complex valued metric space and $K, L, N, M : X \to X$ be self-mappings satisfying the following conditions:*

I. *either the pair (K, N) satisfies (CLR_K) property or the pair (L, M) satisfies (CLR_L) property such that $K(X) \subseteq M(X)$ and $L(X) \subseteq N(X)$;*

II. $\forall\, x, y \in X$

$$\begin{aligned}\int_0^{d(Kx,Ly)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kx,Nx)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx,My)} \varphi(t)dt \\ + \lambda_3 \int_0^{d(Ly,Nx)} \varphi(t)dt + \lambda_4 \int_0^{d(Ly,My)} \varphi(t)dt + \lambda_5 \int_0^{d(My,Nx)} \varphi(t)dt\end{aligned}$$

where $\lambda_i \in [0, 1)$ for $i = 1, 2, \ldots, 5$ with $\sum_{i=1}^{5} \lambda_i < 1$ and $\varphi \in \Phi^*$.

If the pairs (K, N) and (L, M) are weakly compatible, then the mappings K, L, M and N have a unique common fixed point in X.

Proof. Assume that the pair (K, N) satisfies (CLR_K) property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Nx_n = Kx \quad \text{for some } x \in X. \tag{24}$$

Since $K(X) \subseteq M(X)$, so there exists $u \in X$ such that $Kx = Mu$.

We show that $Lu = Mu$, for this put $x = x_n$ and $y = u$ in condition (II) of Theorem 3.5, we have

$$\begin{aligned}\int_0^{d(Kx_n,Lu)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kx_n,Nx_n)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx_n,Mu)} \varphi(t)dt \\ + \lambda_3 \int_0^{d(Lu,Nx_n)} \varphi(t)dt + \lambda_4 \int_0^{d(Lu,Mu)} \varphi(t)dt + \lambda_5 \int_0^{d(Mu,Nx_n)} \varphi(t)dt.\end{aligned}$$

Taking limit as $n \to \infty$ and using (24), we get

$$\begin{aligned}\int_0^{d(Kx,Lu)} \varphi(t)dt &\precsim \lambda_1 \int_0^{d(Kx,Kx)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx,Kx)} \varphi(t)dt + \lambda_3 \int_0^{d(Lu,Kx)} \varphi(t)dt \\ &\quad + \lambda_4 \int_0^{d(Lu,Kx)} \varphi(t)dt + \lambda_5 \int_0^{d(Kx,Kx)} \varphi(t)dt \\ &\precsim (\lambda_3 + \lambda_4) \int_0^{d(Lu,Kx)} \varphi(t)dt,\end{aligned}$$

$$(1 - \lambda_3 - \lambda_4) \int_0^{d(Kx,Lu)} \varphi(t)dt \precsim 0 \Rightarrow \left|(1 - \lambda_3 - \lambda_4) \int_0^{d(Kx,Lu)} \varphi(t)dt\right| \leq 0.$$

But $1 - \lambda_3 - \lambda_4 > 0$, so that $\left|\int_0^{d(Kx,Lu)} \varphi(t)dt\right| = 0$, thus $Lu = Kx$ and hence

$$Lu = Nu = Kx. \tag{25}$$

Further, since $L(X) \subseteq N(X)$, so there exists $v \in X$ such that $Lu = Nv$. Thus (25) becomes

$$Lu = Mu = Nv = Kx. \tag{26}$$

Now, we show that $Kv = Nv$, for this setting $x = v$ and $y = u$ in condition (II) of Theorem 3.5, we have

$$\int_0^{d(Kv,Lu)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kv,Nv)} \varphi(t)dt + \lambda_2 \int_0^{d(Kv,Mu)} \varphi(t)dt + \lambda_3 \int_0^{d(Lu,Nv)} \varphi(t)dt + \lambda_4 \int_0^{d(Lu,Mu)} \varphi(t)dt + \lambda_5 \int_0^{d(Mu,Nv)} \varphi(t)dt.$$

Using Eq. (26), we get

$$\int_0^{d(Kv,Kx)} \varphi(t)dt \precsim \lambda_1 \int_0^{d(Kv,Kx)} \varphi(t)dt + \lambda_2 \int_0^{d(Kv,Kx)} \varphi(t)dt = (\lambda_1+\lambda_2) \int_0^{d(Kx,Kv)} \varphi(t)dt.$$

As $\lambda_1 + \lambda_2 > 0$, so that $Kv = Kx$ implies that $Kv = Mv = Kx$. Therefore from Eq. (26), we get

$$Kv = Lu = Mu = Nv = Kx = z \quad (say). \tag{27}$$

Hence z is a common coincident point of K, L, M and N in X.

Finally, proceeding in the same lines as in Theorem 3.2, we conclude that z is a unique common point of K, L, M and N in X. □

Similar to the arguments in Theorem 3.2 one can easily derive corollaries from Theorem 3.5.

To illustrate Theorem 3.5, we construct the following example.

Example 3.2. Let $X = (1, 5)$ be a metric space with metric $d : X \times X \to \mathbb{C}$ defined by $d(x, y) = e^{im}|x - y|$, where $x, y \in X$ and $0 \le m \le \frac{\pi}{4}$. Define self-maps K, L, M and N on X by:

$$Kx = \begin{cases} 3 & \text{if } x \in (1,3] \\ 1.5 & \text{if } x \in (3,5) \end{cases}; \qquad Lx = \begin{cases} 3 & \text{if } x \in (1,3] \\ 2 & \text{if } x \in (3,5) \end{cases};$$

$$Mx = \begin{cases} \dfrac{x}{3}+1 & \text{if } x \in (1,3) \\ 3 & \text{if } x = 3 \\ 5 & \text{if } x \in (3,5) \end{cases} \quad \text{and} \quad Nx = \begin{cases} x & \text{if } x \in (1,3] \\ 4 & \text{if } x \in (3,5) \end{cases}.$$

Also define $\varphi : \mathbb{R}^2 \to \mathbb{C}$ by $\varphi(t) = 2z$, where $t = (a, b)$ and $z = a + ib$. Then

$$K(X) = \{1.5, 3\}, \qquad L(X) = \{2, 3\}, \qquad M(X) = \left(\frac{4}{3}, 2\right) \cup \{3, 5\}, \qquad N(X) = (1, 3] \cup \{4\}.$$

Firstly, we verify condition (I) of Theorem 3.5. For this, let $\{x_n\} = \{3 - \frac{1}{n^2+1}\}_{n\ge1}$ be a sequence in X. Then

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} K\left(3 - \frac{1}{n^2+1}\right) = \lim_{n\to\infty} 3 = 3$$

and

$$\lim_{n\to\infty} Nx_n = \lim_{n\to\infty} N\left(3 - \frac{1}{n^2+1}\right) = \lim_{n\to\infty}\left(3 - \frac{1}{n^2+1}\right) = 3,$$

that is $\exists$ a sequence $\{x_n\}$ in $X \ni \lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Nx_n = 3 = Kx \quad \forall\, x \in (1, 3]$.

That is $\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Nx_n = 3 = Kx$ for some $x \in X$.

Hence (K, N) satisfies (CLR_K) property.

With a view to check condition (II) of Theorem 3.5, we distinguish the following three cases:

Case 1. let $x, y \in (1, 3)$, then $Kx = Ky = 3$, $Mx = \frac{x}{3} + 1$ and $Ny = y$.

Now, for all $\lambda_i \in [0,1)$; $i = 1, 2, \ldots, 5$ with $\sum_{i=1}^{5} \lambda_i < 1$, one can get

$$\int_0^{d(Kx,Ly)} \varphi(t)dt = 0 \precsim \lambda_1 \int_0^{d(Kx,Nx)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx,My)} \varphi(t)dt + \lambda_3 \int_0^{d(Ly,Nx)} \varphi(t)dt + \lambda_4 \int_0^{d(Ly,My)} \varphi(t)dt + \lambda_5 \int_0^{d(My,Nx)} \varphi(t)dt.$$

Case 2. Let $x = y = 3$, then $Kx = Ly = Mx = Ny = 3$ and for all $\lambda_i \in [0,1)$; $i = 1, 2, \ldots, 5$ with $\sum_{i=1}^{5} \lambda_i < 1$, one can get

$$\int_0^{d(Kx,Ly)} \varphi(t)dt = 0 = \lambda_1 \int_0^{d(Kx,Nx)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx,My)} \varphi(t)dt + \lambda_3 \int_0^{d(Ly,Nx)} \varphi(t)dt + \lambda_4 \int_0^{d(Ly,My)} \varphi(t)dt + \lambda_5 \int_0^{d(My,Nx)} \varphi(t)dt.$$

Case 3. Let $x, y \in (3,5)$, then $Kx = 1.5$, $Ly = 2$, $Mx = 5$ and $Ny = 4$.

$$\int_0^{d(Kx,Ly)} \varphi(t)dt = \int_0^{0.5e^{im}} 3z^2 dt = z^2 \Big|_0^{0.5e^{im}} = 0.25e^{2im}.$$

Also, for $\lambda_1 = \frac{1}{13}, \lambda_2 = \frac{1}{17}, \lambda_3 = \frac{1}{11}, \lambda_4 = \frac{1}{15}, \lambda_5 = \frac{1}{3}$, with $\sum_{i=1}^{5} \lambda_i = \frac{7617}{12155} < 1$, we have

$$\begin{aligned}
&\lambda_1 \int_0^{d(Kx,Nx)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx,My)} \varphi(t)dt + \lambda_3 \int_0^{d(Ly,Nx)} \varphi(t)dt \\
&\quad + \lambda_4 \int_0^{d(Ly,My)} \varphi(t)dt + \lambda_5 \int_0^{d(My,Nx)} \varphi(t)dt \\
&= \frac{1}{13}(6.25e^{2im}) + \frac{1}{17}(12.25e^{2im}) + \frac{1}{11}4e^{2im} + \frac{1}{15}9e^{2im} + \frac{1}{3}e^{2im} \\
&\succ 2.4e^{2im}.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_0^{d(Kx,Ly)} \varphi(t)dt &= 0.25e^{2im} \prec 2.4e^{2im} \\
&\prec \lambda_1 \int_0^{d(Kx,Nx)} \varphi(t)dt + \lambda_2 \int_0^{d(Kx,My)} \varphi(t)dt + \lambda_3 \int_0^{d(Ly,Nx)} \varphi(t)dt \\
&\quad + \lambda_4 \int_0^{d(Ly,My)} \varphi(t)dt + \lambda_5 \int_0^{d(My,Nx)} \varphi(t)dt.
\end{aligned}$$

Therefore, in view of foregoing three cases, the integral contractive condition (II) of Theorem 3.5 is satisfied.

Also $K(X) \subseteq M(X)$ and $L(X) \subseteq N(X)$ and the pairs (K, M) and (L, N) are weakly compatible. Thus all the conditions of Theorem 3.5 are satisfied and 3 is a unique common fixed point of K, L, M and N.

Remark 3.2. Theorems 3.2 and 3.5 are still valid, if we put $\lambda_1 = \lambda_4 = \lambda_5 = 0$.

Remark 3.3. If we put $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ in Corollary 3.2, we get Theorem 3.3 of [12].

Remark 3.4. If we put $K = L$, $N = M$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ in Theorem 3.5, we get Theorem 3.4 of [12].

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Original article

The boundary contact problem of electroelasticity and related integral differential equations

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Abstract

The problem of finding mechanical and electrical fields in a homogeneous piezoelectric plate supported by a thin wedged-shaped cover plate is considered. Using the methods of the theory of analytic functions, the problem is reduced to a system of singular integro-differential equations. With the help of integral transformation, the exact solution of the posed by us problem is obtained.

Keywords: Singular integral differential equation; Integral transformation; Piezoelectric medium; Riemann's problem

Earlier we have considered the problems of finding mechanical and electrical fields in a piezoelectric medium (a crystal of hexagonal system 6 mm, polarized ceramics), weakened by tunnel rectilinear [1–3] and curvilinear cuts [4,5]. Various boundary value problems of destruction mechanics for a piezoelectric medium have been considered in the monograph [6].

In the present paper we consider a homogeneous unbounded piezoelectric plate (a crystal of hexagonal system 6 mm) in the conditions of plane deformation. The plate is supported by a finite inclusion which is under the action of tangential strain of intensity $\tau_0(x)$. As for the inclusion having the form of a weakly-curved cover plate of small thickness, we assume that it is rigidly linked with a plate, stretches or shrinks like a rod, lying in a uniaxial stressed state. We adopt the condition of compatibility of horizontal deformations in the inclusion and in elastic homogeneous solid plate loaded with tangential stresses.

The problem is to find jumps of tangential $\tau(x)$ and normal $p(x)$ contact stresses along the contact line and to establish their behavior in the neighborhood of cover plate ends. The problem is formulated as follows: Let an elastic body S occupy the plane of a complex variable $z = x + iy$ which along the segment $\ell = (0, 1)$ contains an elastic inclusion with the modulus of elasticity $E_0(x)$, of thickness $h_0(x)$ and with the Poisson coefficient ν_0. External force $\tau_0(x)$ is the function, integrable on the segment $\overline{\ell} = [0, 1]$.

The boundary values of functions on the upper and lower ends of the inclusion will be denoted by the indices "+" and "−", respectively.

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On the segment ℓ, we have the following conditions:

$$\begin{aligned}
&\frac{du_0(x)}{dx} = \frac{1}{E(x)} \int_0^x \big[\tau(t) - \tau_0(t)\big]dt, \quad x \in \ell, \\
&\frac{dv_0(x)}{dx} = 0, \qquad E(x) = \frac{E_0(x)\, h_0(x)}{1 - \nu_0^2},
\end{aligned} \tag{1}$$

where $u_0(x)$ and $v_0(x)$ are horizontal and vertical displacements of the inclusion points, and the condition of equilibrium of the inclusion is of the form

$$\int_0^1 \big(\tau(t) - \tau_0(t)\big)dt = 0.$$

As is known, in the conditions of plane deformation of the piezoelectric medium in the plane xoy, the system of resolving equations with respect to the stress function φ_1 and electric field potential φ_2 has the form [6]:

$$\ell_{11}\, \varphi_1 + \ell_{12}\, \varphi_2 = 0, \qquad \ell_{21}\, \varphi_1 + \ell_{22}\, \varphi_2 = 0, \tag{2}$$

where

$$\begin{aligned}
&\ell_{11} = a_{10}\, \partial_1^4 + a_{12}\, \partial_1^2\, \partial_2^2 + a_{14}\, \partial_2^4, \quad \partial_1 = \frac{\partial}{\partial x},\ \partial_2 = \frac{\partial}{\partial y}, \\
&\ell_{12} = \ell_{21} = a_{21}\, \partial_1^2\, \partial_2 + a_{23}\, \partial_2^3, \qquad \ell_{22} = a_{20}\, \partial_1^2 + a_{22}\, \partial_2^2, \\
&a_{10} = s_{33} - s_{13}^2\, s_{11}^{-1}, \qquad a_{12} = s_{44} + 2s_{13}(1 - s_{12}\, s_{11}^{-1}), \\
&a_{14} = s_{11} - s_{12}^2\, s_{11}^{-1}, \qquad a_{21} = s_{13}\, d_{13}\, s_{11}^{-1} - d_{33} + d_{15}, \\
&a_{23} = d_{13}(s_{12}\, s_{11}^{-1} - 1), \qquad a_{20} = \varepsilon_{11}, \qquad a_{22} = \varepsilon_{33} - d_{13}^2\, s_{11}^{-1},
\end{aligned}$$

$s_{nk}, d_{nk}, \varepsilon_{nk}$ are, respectively, elastic pliability, piezoelectric modules and dielectric constants appearing in the equations of medium state.

A general solution of system (2) has the form

$$\begin{aligned}
&\varphi_1 = 2\,\mathrm{Re} \sum_{k=1}^{3} \gamma_k \int \Phi_k(z_k)dz_k, \qquad \varphi_2 = -2\,\mathrm{Re} \sum_{k=1}^{3} \lambda_k\, \Phi_k(z_k), \\
&z_k = x + \mu_k\, y, \qquad \mu_{3+k} = \overline{\mu}_k, \qquad \gamma_k = a_{20} + a_{22}\, \mu_k^2, \qquad \lambda_k = a_{21}\, \mu_k + a_{23}\, \mu_k^3,
\end{aligned} \tag{3}$$

μ_k $(k = 1, 2, 3)$ are the roots of the corresponding characteristic equation ($\mathrm{Im}\, \mu_k \neq 0$).

$$c_0\, \mu^6 + c_1\, \mu^4 + c_2\, \mu^2 + c_3 = 0,$$

where

$$\begin{aligned}
&c_0 = a_{14}\, a_{22} - a_{23}^2, \qquad c_1 = a_{12}\, a_{22} + a_{14}\, a_{20} - 2a_{21}\, a_{23}, \\
&c_2 = a_{10}\, a_{22} + a_{12}\, a_{20} - a_{21}, \qquad c_3 = a_{10}\, a_{20}.
\end{aligned}$$

Using equations of the state and formulas (3), we find expressions for stresses, displacements, electric field strength and electric induction components in the medium

$$\begin{aligned}
&\sigma_x = 2\,\mathrm{Re} \sum_{k=1}^{3} \gamma_k\, \mu_k^2\, \Phi_k'(z_k), \qquad \sigma_y = 2\,\mathrm{Re} \sum_{k=1}^{3} \gamma_k\, \Phi_k'(z_k), \\
&\sigma_{xy} = -2\,\mathrm{Re} \sum_{k=1}^{3} \gamma_k\, \mu_k\, \Phi_k'(z_k), \qquad u = 2\,\mathrm{Re} \sum_{k=1}^{3} p_k\, \Phi_k(z_k), \\
&v = 2\,\mathrm{Re} \sum_{k=1}^{3} q_k\, \Phi_k(z_k), \qquad E_x = 2\,\mathrm{Re} \sum_{k=1}^{3} \lambda_k\, \Phi_k'(z_k),
\end{aligned}$$

$$E_y = 2\,\mathrm{Re}\sum_{k=1}^{3} \lambda_k\, \mu_k\, \Phi_k'(z_k), \qquad D_x = 2\,\mathrm{Re}\sum_{k=1}^{3} r_k\, \mu_k\, \Phi_k'(z_k),$$
$$D_y = -\,\mathrm{Re}\sum_{k=1}^{3} r_k\, \Phi_k'(z_k),$$
$$p_k = a_{14}\,\gamma_k\,\mu_k^2 + 1/2(a_{12} - s_{44})\gamma_k - a_{23}\,\lambda_k\,\mu_k,$$
$$q_k = 1/2(a_{12} - s_{44})\gamma_k\,\mu_k + a_{10}\,\gamma_k\,\mu_k^{-1} - (a_{21} - d_{15})\lambda_k,$$
$$r_k = a_{20}\,\lambda_k\,\mu_k^{-1} - d_{15}\,\gamma_k.$$

The boundary values at the edges of the inclusion are of the form

$$\sigma_y^+ - \sigma_y^- = p(x), \qquad \sigma_{xy}^+ - \sigma_{xy}^- = \tau(x),$$
$$\left(\frac{\partial u}{\partial x}\right)^+ - \left(\frac{\partial u}{\partial x}\right)^- = 0, \qquad \left(\frac{\partial v}{\partial x}\right)^+ - \left(\frac{\partial v}{\partial x}\right)^- = 0,$$
$$E_x^+(x) = E_x^-(x), \qquad D_y^+(x) = D_y^-(x), \quad 0 < x < 1.$$

Introducing the notations $H_k(x) = [\Phi_k'(x)]^+ - [\Phi_k'(x)]^-$ $(k = 1, 2, 3)$, we can express them through the newly introduced analytic functions

$$\mathrm{Re}\sum_{k=1}^{3} r_k\, H_k(x) = 0, \qquad \mathrm{Re}\sum_{k=1}^{3} \gamma_k\, H_k(x) = \frac{p(x)}{2}, \qquad \mathrm{Re}\sum_{k=1}^{3} p_k\, H_k(x) = 0, \tag{4a}$$

$$\mathrm{Re}\sum_{k=1}^{3} \gamma_k\,\mu_k\, H_k(x) = -\frac{\tau(x)}{2}, \qquad \mathrm{Re}\sum_{k=1}^{3} \lambda_k\, H_k(x) = 0, \qquad \mathrm{Re}\sum_{k=1}^{3} q_k\, H_k(x) = 0. \tag{4b}$$

Assume $\mu_k = i\beta_k$ $(i = \sqrt{-1},\ \beta_k > 0)$, then the first from the last systems (4a) with the real coefficients can be rewritten as

$$\sum_{k=1}^{3} r_k\,\mathrm{Re}\,H_k(x) = 0, \qquad \sum_{k=1}^{3} \gamma_k\,\mathrm{Re}\,H_k(x) = \frac{p(x)}{2}, \qquad \sum_{k=1}^{3} p_k\,\mathrm{Re}\,H_k(x) = 0. \tag{5}$$

If the corresponding determinant

$$\Delta = \begin{vmatrix} r_1 & r_2 & r_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ p_1 & p_2 & p_3 \end{vmatrix} \neq 0,$$

then a solution of system (5) has the form

$$\mathrm{Re}\,H_k(x) = \frac{\widetilde{\Delta}_k}{2\Delta}\,p(x), \quad k = 1, 2, 3,\ \widetilde{\Delta}_k = A_{2k},$$

A_{2k} are the corresponding cofactors.

Then the second system (4b) with the real coefficients takes the form

$$\sum_{k=1}^{3} i\,\gamma_k\,\mu_k\,\mathrm{Im}\,H_k(x) = -\frac{\tau(x)}{2},$$
$$\sum_{k=1}^{3} i\,\lambda_k\,\mathrm{Im}\,H_k(x) = 0,$$
$$\sum_{k=1}^{3} i\,q_k\,\mathrm{Im}\,H_k(x) = 0.$$

Having solved this system, we arrive at the problem of linear conjugation

$$\left[\Phi_k'(x)\right]^+ - \left[\Phi_k'(x)\right]^- = \frac{\widetilde{\Delta}_k}{2\Delta}\, p(x) - \frac{\Delta_k}{2\Delta_0}\, i\,\tau(x), \quad k = 1, 2, 3, \tag{6}$$

where

$$\Delta_0 = \begin{vmatrix} i\,\gamma_1\,\mu_1 & i\,\gamma_2\,\mu_2 & i\,\gamma_3\,\mu_3 \\ i\,\lambda_1 & i\,\lambda_2 & i\,\lambda_3 \\ i\,q_1 & i\,q_2 & i\,q_3 \end{vmatrix} \neq 0, \qquad \Delta_k \equiv B_{1k},$$

B_{1k} are the corresponding cofactors.

In view of the fact that for $x > 1$, $\tau(x) = 0$, $p(x) = 0$, a general solution of problem (6) is represented in the form

$$\Phi_k'(z_k) = \frac{\widetilde{\Delta}_k}{4\pi\, i\,\Delta} \int_0^1 \frac{p(t)dt}{t - z_k} - \frac{\Delta_k}{4\pi\,\Delta_0} \int_0^1 \frac{\tau(t)dt}{t - z_k}, \quad z_k \in S.$$

From the condition of equality of deformations of elastic planes on the segment occupied by the inclusion and those of inclusion sides described by Eqs. (1), we obtain the following system of integral differential equations:

$$\begin{aligned} &\omega_1 \int_0^1 \frac{\varphi'(t)dt}{t - x} + \omega_2 \int_0^1 \frac{p(t)dt}{t - x} = \frac{\pi\,\varphi(x)}{2E(x)} + \frac{\pi}{2E(x)}\, f_0(x), \\ &\omega_3 \int_0^1 \frac{\varphi'(t)dt}{t - x} + \omega_4 \int_0^1 \frac{p(t)dt}{t - x} = 0, \quad 0 < x < 1, \end{aligned} \tag{7}$$

where

$$\varphi(x) = \int_0^x \tau(t)dt, \qquad f_0(x) = -\int_0^x \tau_0(t)dt, \qquad \varphi(1) = \int_0^1 \tau_0(t)dt \equiv T_0,$$

$$\omega_1 = \sum_{k=1}^3 \frac{p_k\,\Delta_k}{\Delta_0}, \qquad \omega_2 = \sum_{k=1}^3 \frac{p_k\,\widetilde{\Delta}_k}{\Delta}, \qquad \omega_3 = \sum_{k=1}^3 \frac{q_k\,\Delta_k}{\Delta_0}, \qquad \omega_4 = \sum_{k=1}^3 \frac{q_k\,\widetilde{\Delta}_k}{\Delta}.$$

We seek for a solution $\varphi(x)$ of system (7) in the class of functions H satisfying Hölder's condition on the segment [0, 1], while the derivative of the function $\varphi(x)$ and the function $p(x)$ may belong to the class H^* [7].

The system of integral equations (7) reduces to the integral differential equation with respect to the function $\varphi(t)$. If the inclusion rigidity varies according to the linear low, i.e., $E(x) = h\,x$, $x \in (0, 1)$, then performing substitution $t = e^{\zeta}$, $x = e^{\xi}$ and applying the generalized Fourier transformation [8–10], we arrive at the Riemann's problem

$$\Psi^+(s) = G(s)\,\Phi^-(s) + g(s), \quad -\infty < s < \infty, \tag{8}$$

where

$$G(s) = s\,\mathrm{cth}\,\pi s - \omega,$$

$$\Phi^-(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \varphi(e^{\zeta})\, e^{is\zeta} d\zeta, \qquad g(s) = i\,T_0(\mathrm{cth}\,\pi s)_- + g_1(s),$$

$$\Psi^+(s) = -\frac{1}{\pi} \int_{-\infty}^0 \psi(e^{\zeta})\, e^{\zeta(1+is)} d\zeta, \qquad g_1(s) = \omega \int_{-\infty}^0 f_0(e^{\xi})\, e^{\xi(1+is)} d\xi,$$

$$\psi(\xi) = \begin{cases} 0, & \xi < 0, \\ -\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \frac{\varphi'(e^{\zeta})d\zeta}{1 - e^{-(\xi-\zeta)}}, & \xi > 0, \end{cases} \qquad \omega = \frac{\omega_4}{2h(\omega_1\,\omega_4 - \omega_2\,\omega_3)}.$$

The function $g_1(s) = O(s^{-1})$, as $s \to -\infty$.

The functions $\Psi^+(s)$ and $\Phi^-(s)$ represent, by virtue of their definition, the limiting values of functions, holomorphic in the upper and lower half-planes, respectively.

The lower index "-" denotes that for $s = 0$, the generalized function should be understood in the known sense [10].

The function $G(s)$ has at infinity the first order discontinuity, since

$$\widetilde{G}(\infty) = -\widetilde{G}(-\infty) = 1, \qquad \widetilde{G}(s) = s \operatorname{cth} \pi s.$$

Then condition (8) can be represented in the form

$$\frac{\Psi^+(s)}{\sqrt{s+i}} = \frac{G(s)}{\sqrt{1+s^2}}\, \Phi^-(s)\sqrt{s-i} + \frac{g(s)}{\sqrt{s+i}}. \tag{9}$$

Under $\sqrt{z+i}$ and $\sqrt{z-i}$ we mean the branches, analytical, respectively, in the planes with cuts along the rays drawn from the points $z=-i$ and $z=i$ to x which get, respectively, positive and negative values on the upper side of the cut. The function $\sqrt{1+z^2}$ for such a choice of branches is analytic in the strip $-1 < \operatorname{Im} z < 1$ and takes positive value on the real axis.

Thus, the above-posed problem can be formulated as follows: Find both the function $\Psi^+(z)$, holomorphic in the half-plane $\operatorname{Im} z > 0$ and vanishing at infinity, and the function $\Phi^-(z)$, holomorphic in the half-plane $\operatorname{Im} z < 1$, except the points which are the roots of the function $G(z)$, vanishing at infinity and satisfying the condition (9).

Introducing the notation $G_0(s) = G(s)(1+s^2)^{1/2}$, we can show that $\operatorname{Re} G_0(t) > 0$ and $G_0(\infty) = G_0(-\infty) = 1$, therefore $\operatorname{Ind} G_0(t) = 0$.

A solution of problem (9) has the form

$$\begin{aligned}
\Phi^-(z) &= \frac{\widetilde{X}(z)}{\sqrt{z-i}}, \quad \operatorname{Im} z \le 0; \qquad \Psi^+(z) = \widetilde{X}(z)\sqrt{z+i}, \quad \operatorname{Im} z > 0,\\
\Phi^-(z) &= \big(\Psi^+(z) - g(z)\big)G^{-1}(z), \quad 0 < \operatorname{Im} z < 1,
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{X}(z) &= X(z)\left\{-\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g(t)dt}{X^+(t)\sqrt{t+i}\,(t-z)}\right\},\\
X(z) &= \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln G_0(t)dt}{t-z}\right\}.
\end{aligned}$$

We can show that $\Phi^-(s+i\,0) = \Phi^-(s-i\,0)$ and, hence, the function $\Phi^-(z)$ is holomorphic in the half-plane $\operatorname{Im} z < 1$, except the points which are zeros of the function $G(z)$ in the strip $0 < \operatorname{Im} z < 1$.

The boundary value of the function $K(z) = T_0 - i\,z\,\Phi^-(z)$ is the Fourier transform of the function $\varphi'(e^{\xi})$. We have

$$\begin{aligned}
K(z) &= T_0 + \frac{z\,X(z)}{\pi\sqrt{z-i}}\int_{-\infty}^{\infty} \frac{g(t)dt}{X^+(t)\sqrt{t+i}\,(t-z)}\\
&= T_0 - \frac{i\,T_0\,z\,X(z)}{\pi\sqrt{z-i}}\int_{-\infty}^{0} \frac{\operatorname{cth}\pi t\,dt}{X^+(t)\sqrt{t+i}\,(t-z)}\\
&\quad - \frac{X(z)}{2\pi\sqrt{z-i}}\int_{-\infty}^{\infty} \frac{g_1(t)dt}{X^+(t)\sqrt{t+i}} + \frac{X(z)}{2\pi\sqrt{z-i}}\int_{-\infty}^{\infty} \frac{t\,g_1(t)dt}{X^+(t)\sqrt{t+i}\,(t-z)}\\
&= T_0 + \frac{C}{\sqrt{z-i}} + \frac{C(X(z)-X(\infty))}{\sqrt{z-i}} + \frac{X(z)}{2\pi\sqrt{z-i}}\int_{-\infty}^{\infty} \frac{t\,g_1(t)dt}{X^+(t)\sqrt{t+i}\,(t-z)}\\
&\quad - \frac{i\,T_0\,z\,X(z)}{\pi\sqrt{z-i}}\int_{-\infty}^{0} \frac{\operatorname{cth}\pi t\,dt}{X^+(t)\sqrt{t+i}\,(t-z)},
\end{aligned}$$

where

$$C = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g_1(t)dt}{X^+(t)\sqrt{t+i}}.$$

Let us study the behavior of the function

$$K_0(z) = \frac{i\,T_0\,z\,X(z)}{\pi\sqrt{z-i}} \int_{-\infty}^{0} \frac{\operatorname{cth}\pi t\,dt}{X^+(t)\sqrt{t+i}\,(t-z)}$$

at infinity.

The change of variables $z = -\frac{1}{\xi}$ and $t = -\frac{1}{t_0}$ provides us with

$$K_0^*(\xi) = \frac{T_0\,X^*(\xi)\sqrt{\xi}}{\pi\,i\,\sqrt{1+i\,\xi}} \int_0^{\infty} \frac{\operatorname{cth}\frac{\pi}{t_0}\,dt_0}{X^{+*}(t_0)\sqrt{1-i\,t_0}\,\sqrt{t_0}\,(t_0-\xi)},$$

where

$$K_0^*(\xi) = K_0\Big(-\frac{1}{\xi}\Big), \qquad X^*(\xi) = X\Big(-\frac{1}{\xi}\Big).$$

Applying N. Muskhelishvili's formulas [7] in the neighborhood of the point $\xi = 0$, we obtain

$$K_0^*(\xi) = T_0 + o(1).$$

Respectively, the function $K(z)$ vanishes at infinity and its boundary value $K^-(s) = \frac{C}{\sqrt{s-i}} + \widetilde{K}_0^-(s)$ is the Fourier transformation of the function $\varphi'(e^{\xi})$. In addition, $\widetilde{K}_0^-(s)$ is the Fourier transformation of the function which is continuous on the half-axis $x \le 0$, except may be the point $s = 0$ at which it may have logarithmic singularity. The inverse Fourier transformation results in the expression for the unknown function

$$\tau(x) = \varphi'(x) = \frac{1}{\sqrt{2\pi}\,x} \int_{-\infty}^{\infty} K^-(t)\,e^{-it\ln x}\,dt. \tag{10}$$

Its behavior in the neighborhood of the point $x = 1$ has the form

$$\varphi'(x) = O\big(1/\sqrt{1-x}\big), \quad x \to 1-. \tag{11}$$

Let us study now the behavior of the function $\tau(x)$ in the neighborhood of the point $z = 0$. The poles of the function $K^-(z)$ in the domain $D_0 = \{z : 0 < \operatorname{Im} z < 1\}$ may be zeros of the function $G(z)$.

Suppose that $z_0 = x_0 + i\,\tau_0$ is a simple zero of the function $G(z)$ with the smallest imaginary part in the domain D_0. Then, applying to the function $e^{-i\xi z}K^-(z)$ the Cauchy residue theorem for the rectangle $D(N)$ ($z_0 \in D(N)$) with the boundary $L(N)$ consisting of the segments $[-N, N]$, $[N + i\,0, N + i\,\beta_0]$, $[N + i\,\beta_0, -N + i\,\beta_0]$, $[-N + i\,\beta_0, -N + i\,0]$ ($\tau_0 < \beta_0 < \tau_0^1$), $(G_0(x_1 + i\tau_0') = 0)$, we obtain

$$\int_{L(N)} K^-(t)\,e^{-it\xi}dt = \int_{-N}^{N} K^-(t)\,e^{-it\xi}dt - e^{\beta_0\xi}\int_{-N}^{N} K^-(t+i\,\beta_0)\,e^{-it\xi}dt + \rho(N,\xi) = K_1\,e^{\xi(\tau_0 - i x_0)},$$

where $\rho(N, \xi) \to 0$, as $N \to \infty$. Passing to the limit and getting back to the above variables, we obtain

$$\tau(x) = \varphi'(x) = O(x^{\tau_0 - 1 - i x_0}), \quad x \to 0+. \tag{12}$$

Thus the following theorem is proved.

Theorem. *The solution of the integral differential equation with respect to the function $\varphi(x)$ obtained from system* (7) *is representable explicitly by formula* (10) *and satisfies the estimates* (11)–(12).

It can be shown that if the function $G(z)$ has no simple zeros in the domain D_0, and if $z = i$ is a simple pole of the function $\Phi^-(z)$, then the contact stresses are bounded in the neighborhood of zero, but if $z = i$ is the second order pole of the function $\Phi^-(z)$, then the stress will have logarithmic singularity in the neighborhood of zero.

Remark. The normal contact stress (the function $p(x) \in H^*$) can be defined from the second equation of system (7).

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Original article

Filtered Hirsch algebras

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Abstract

Motivated by the cohomology theory of loop spaces, we consider a special class of higher order homotopy commutative differential graded algebras and construct the filtered Hirsch model for such an algebra A. When $x \in H(A)$ with $\mathbb{Z}$ coefficients and $x^2 = 0$, the symmetric Massey products $\langle x\rangle^n$ with $n \geq 3$ have a finite order (whenever defined). However, if $\Bbbk$ is a field of characteristic zero, $\langle x\rangle^n$ is defined and vanishes in $H(A \otimes \Bbbk)$ for all n. If p is an odd prime, the Kraines formula $\langle x\rangle^p = -\beta\mathcal{P}_1(x)$ lifts to $H^*(A \otimes \mathbb{Z}_p)$. Applications of the existence of polynomial generators in the loop homology and the Hochschild cohomology with a G-algebra structure are given.

Keywords: Hirsch algebra; Filtered model; Multiplicative resolution; Symmetric Massey product; Steenrod operation; Hochschild cohomology

1. Introduction

In this paper we investigate a special class of homotopy commutative algebras called *Hirsch algebras* [20]. When the structural operations of a Hirsch algebra A agree component-wise with those of a homotopy G-algebra (HGA), the pre-Jacobi axiom can fail [7,8,19,37] and the induced product on the bar construction BA is not necessarily associative. Indeed, the theory of loop space cohomology suggests that it is impossible in general, to construct a small model for $H^*(\Omega X)$ in the category of HGAs. The investigation here applies a perturbation theory that extends the well-developed perturbation theories for differential graded modules and differential graded algebras (dgas) [3,9,13,11,27,28].

One difficulty encountered when constructing a theory of homological algebra for Hirsch algebras is that the Steenrod cochain product $a \smile_1 b$ fails to be a cocycle even for cocycles a and b. Consequently $a \smile_1 b$ does not necessarily lift to cohomology. We control such difficulties by introducing the notion of a *filtered* Hirsch algebra, which can be thought of as a specialization of a distinguished resolution in the sense of [10] (see also [14]). On the other hand, the filtered Hirsch model $(RH, d+h)$ of a Hirsch algebra A is itself a Hirsch algebra whose structural operations $E_{p,q} : RH^{\otimes p} \otimes RH^{\otimes q} \longrightarrow RH$ are completely determined by the commutative graded algebra (cga) structure of $H = H(A, d_A)$; furthermore, the perturbation $h : RH \to RH$ of the resolution differential d is

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determined by the Hirsch algebra structure on A (Theorem 1). Thus by ignoring the operations $E_{p,q}$ we obtain a multiplicative resolution $(RH, d) \to (H, 0)$ of the cga H thought of as a non-commutative version of its Tate–Jozefiak resolution [35,16] and the filtered model of the dga A is the perturbation $(RH, d+h) \to (A, d_A)$ in [27] (such a filtered model in the category of cdgas over a field of characteristic zero was constructed by Halperin and Stasheff in [11]).

A Hirsch resolution always admits a binary operation $\cup_2$, which can be viewed as *divided* Steenrod $\smile_2$-operation. This leads to the notion of a *quasi-homotopy commutative* Hirsch algebra (QHHA) introduced here. We note that in general, the construction of a Hirsch map $(RH, d+h) \to A$ compatible with a QHHA structure on A is obstructed by the non-free action of Sq_1 on its cohomology $H(A)$.

Every cdga H can be thought of as a trivial Hirsch algebra in which the operations $E_{p,q} \equiv 0$ for all $p, q \geq 1$. However, we exhibit an example of a cohomology algebra $H = H(A)$ with a non-trivial Hirsch algebra structure determined by Sq_1.

For a Hirsch algebra A over the integers, we establish some formulas relating the structural operations $E_{p,q}$ with syzygies in (RH, d) that arise from a single element $x \in H(A)$ with $x^2 = 0$. Whereas the n-fold symmetric Massey product $\langle x\rangle^n$ with $n \geq 3$ is defined in $H(A)$ [23,22], our formulas imply that $\langle x\rangle^n$ has finite order. Note that when A is an algebra over a field $\Bbbk$ of characteristic zero, $\langle x\rangle^n$ is defined and vanishes for all $n \geq 3$ (Theorem 2). As a consequence we have (compare [4]):

Theorem A. *Let X be a simply connected space, let $\Bbbk$ be a field of characteristic zero and let $\sigma_* : H_*(\Omega X; \Bbbk) \to H_{*+1}(X; \Bbbk)$ be the suspension map. If $y \notin \operatorname{Ker} \sigma_*$ and $y^2 \neq 0$, then $y^n \neq 0$ for all $n \geq 2$.*

Given an odd prime p, consider the Hirsch algebra $A \otimes \mathbb{Z}_p$, let $x \in H^{2m+1}(A \otimes \mathbb{Z}_p)$, and let β be the Bockstein operator. We obtain the formula

$$\langle x\rangle^p = -\beta \mathcal{P}_1(x), \tag{1.1}$$

which has the same form as Kraines's formula in [23], however, the cohomology operation $\mathcal{P}_1 : H^{2m+1}(A \otimes \mathbb{Z}_p) \to H^{2mp+1}(A \otimes \mathbb{Z}_p)$ in (1.1) is canonically determined by the iteration of the $\smile_1$-product on $A \otimes \mathbb{Z}_p$ (Theorem 3). Dually, if A is the singular chains on the triple loop space $\Omega^3 X$, we can identify $\mathcal{P}_1$ with the Dyer–Lashof operation (see [22]). In fact the validity of (1.1) in a general algebraic framework is conjectured by May [25, Section 6]. Furthermore, when $X = BF_4$, the classifying space of the exceptional group F_4, we exhibit explicit perturbations in the filtered model of X and recover formula (1.1) in $H^*(X; \mathbb{Z}_3)$.

Although Theorem 1 provides a theoretical model of a Hirsch algebra A endowed with higher order operations $E_{p,q}$, in practice one can construct a small *multiplicative* model for recognizing $H^*(BA)$ as an algebra in which the product is determined only by the binary operation $E_{1,1} = \smile_1$. Thus, a (minimal) multiplicative resolution of $H^*(A)$ endowed with a $\smile_1$-product provides an economical way to calculate the algebra $H^*(BA)$. We apply this technique to the Hochschild cochain complex $A = C^\bullet(P; P)$ of an associative algebra P over a field $\Bbbk$ of characteristic zero to establish the following.

Theorem B. *If the Hochschild cohomology $H^* = H(C^\bullet(P; P))$ is a free algebra, then the Lie algebra structure on $Tor_*^A(\Bbbk, \Bbbk)$ is completely determined by that of the G-algebra H^*. Consequently, the product μ^* on $Tor_*^A(\Bbbk, \Bbbk)$ is commutative if and only if the G-product on H^* is trivial.*

Some applications of filtered Hirsch algebras considered in an earlier version of this paper are also considered in [31,32] (see also [29,33]).

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2. The category of Hirsch algebras

This section defines the generalized notion of a Hirsch algebra applied here, the morphisms between them, and the notion of a Hirsch resolution.

Let $\Bbbk$ be a commutative ring with unity 1 and characteristic ν; in the applications, $\Bbbk$ will be the integers $\mathbb{Z}$, a finite field $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ with p prime, or a field of characteristic zero. Graded $\Bbbk$-modules A^* are assumed to be graded over $\mathbb{Z}$. A module A^* is connected if $A^0 = \Bbbk$, and a non-negatively graded, connected module A^* is 1-*reduced* if $A^1 = 0$.

For a module A, let $T(A)=\bigoplus_{i=0}^{\infty} A^{\otimes i}$, where $A^0=\Bbbk$, be the tensor module of A. An element $a_1\otimes\cdots\otimes a_n\in A^{\otimes n}$ is denoted by $[a_1|\cdots|a_n]$ when $T(A)$ is viewed as the tensor coalgebra or by $a_1\cdots a_n$ when $T(A)$ is viewed as the tensor algebra. We denote by $s^{-1}A$ the desuspension of A, i.e., $(s^{-1}A)^i=A^{i+1}$.

A dga (A,d_A) is assumed to be supplemented; in particular, it has the form $A=\tilde{A}\oplus\Bbbk$. The (reduced) bar construction BA on A is the tensor coalgebra $T(\bar{A})$, $\bar{A}=s^{-1}\tilde{A}$, with differential $d=d_1+d_2$ given for $[\bar{a}_1|\cdots|\bar{a}_n]\in T^n(\bar{A})$ by

$$d_1[\bar{a}_1|\cdots|\bar{a}_n]=-\sum_{1\le i\le n}(-1)^{\epsilon^a_{i-1}}[\bar{a}_1|\cdots|\overline{d_A(a_i)}|\cdots|\bar{a}_n]$$

and

$$d_2[\bar{a}_1|\cdots|\bar{a}_n]=-\sum_{1\le i<n}(-1)^{\epsilon^a_{i}}[\bar{a}_1|\cdots|\overline{a_ia_{i+1}}|\cdots|\bar{a}_n],$$

where $\epsilon^x_i=|x_1|+\cdots+|x_i|+i$.

Let us generalize (slightly) the definition of a Hirsch algebra [20]. Let A be a dga and consider the dg module $(Hom(BA\otimes BA,A),\nabla)$, where ∇ is the canonical Hom differential. Since the tensor product $BA\otimes BA$ is a dgc with the standard coalgebra structure, the $\smile$-product induces a dga structure on $(Hom(BA\otimes BA,A),\nabla,\smile)$.

Definition 1. A **Hirsch algebra** is an associative dga A equipped with multilinear maps

$$E_{p,q}:A^{\otimes p}\otimes A^{\otimes q}\to A,\quad p,q\ge 0,\ p+q>0,$$

satisfying the following conditions:

(i) $\deg E_{p,q}=1-p-q$;
(ii) $E_{1,0}=Id=E_{0,1}$ and $E_{p>1,0}=0=E_{0,q>1}$;
(iii) The homomorphism $E:BA\otimes BA\to A$ defined by

$$E([\bar{a}_1|\cdots|\bar{a}_p]\otimes[\bar{b}_1|\cdots|\bar{b}_q])=E_{p,q}(a_1,\ldots,a_p;b_1,\ldots,b_q)\tag{2.1}$$

is a twisting cochain in the dga $(Hom(BA\otimes BA,A),\nabla,\smile)$, i.e., $\nabla E=-E\smile E$.

A *morphism* $f:A\to B$ between two Hirsch algebras is a dga map f that commutes with $E_{p,q}$ for all p,q.

Condition (iii) implies that $\mu_E:BA\otimes BA\to BA$ is a chain map; thus BA is a dg bialgebra whose multiplication μ_E is not necessarily associative (compare [8,37,5,21,26]); in particular, $\mu_{E_{10}+E_{01}}$ is the shuffle product on BA, and a Hirsch algebra with $E_{p,q}\equiv 0$ for all $p,q\ge 1$ is just a cdga (cf. (2.3)). It is useful to express Eq. (2.1) componentwise:

$$\begin{aligned}
&dE_{p,q}(a_1,\ldots,a_p;b_1,\ldots,b_q)\\
&\quad=\sum_{1\le i\le p}(-1)^{\epsilon^a_{i-1}}E_{p,q}(a_1,\ldots,da_i,\ldots,a_p;b_1,\ldots,b_q)\\
&\qquad+\sum_{1\le j\le q}(-1)^{\epsilon^a_p+\epsilon^b_{j-1}}E_{p,q}(a_1,\ldots,a_p;b_1,\ldots,db_j,\ldots,b_q)\\
&\qquad+\sum_{1\le i<p}(-1)^{\epsilon^a_i}E_{p-1,q}(a_1,\ldots,a_ia_{i+1},\ldots,a_p;b_1,\ldots,b_q)\\
&\qquad+\sum_{1\le j<q}(-1)^{\epsilon^a_p+\epsilon^b_j}E_{p,q-1}(a_1,\ldots,a_p;b_1,\ldots,b_jb_{j+1},\ldots,b_q)\\
&\qquad+\sum_{\substack{0\le i\le p\\ 0\le j\le q\\ (i,j)\ne(0,0)}}(-1)^{\epsilon_{i,j}}E_{i,j}(a_1,\ldots,a_i;b_1,\ldots,b_j)\cdot E_{p-i,q-j}(a_{i+1},\ldots,a_p;b_{j+1},\ldots,b_q),
\end{aligned}\tag{2.2}$$

$$\epsilon_{i,j}=\epsilon^a_i+\epsilon^b_j+(\epsilon^a_i+\epsilon^a_p)\epsilon^b_j+1.$$

In particular, the operation $E_{1,1}$ satisfies conditions similar to Steenrod's cochain $\smile_1$-product:

$$dE_{1,1}(a;b) - E_{1,1}(da;b) + (-1)^{|a|}E_{1,1}(a;db) = (-1)^{|a|}ab - (-1)^{|a|(|b|+1)}ba; \tag{2.3}$$

consequently, $E_{1,1}$ measures the non-commutativity of the product $\cdot$ on A. We shall use the notation $a \smile_1 b = E_{1,1}(a;b)$ interchangeably. The following special cases will also be important for us, so we write them explicitly:

The Hirsch formulas up to homotopy

$$\begin{aligned} dE_{2,1}(a,b;c) = {} & E_{2,1}(da,b;c) - (-1)^{|a|}E_{2,1}(a,db;c) + (-1)^{|a|+|b|}E_{2,1}(a,b;dc) \\ & - (-1)^{|a|}(ab)\smile_1 c + (-1)^{|a|+|b|+|b||c|}(a\smile_1 c)b + (-1)^{|a|}a(b\smile_1 c) \end{aligned}$$

and

$$\begin{aligned} dE_{1,2}(a;b,c) = {} & E_{1,2}(da;b,c) - (-1)^{|a|}E_{1,2}(a;db,c) + (-1)^{|a|+|b|}E_{1,2}(a;b,dc) \\ & + (-1)^{|a|+|b|}a\smile_1(bc) - (-1)^{|a|+|b|}(a\smile_1 b)c - (-1)^{|a|(|b|-1)}b(a\smile_1 c) \end{aligned}$$

tell us that the deviations of the binary operation $\smile_1$ from left and right derivation of the $\cdot$ product are measured by the respective boundaries of the operations $E_{1,2}$ and $E_{2,1}$ on three variables.

The following definition describes a class of Hirsch algebras in which the $\smile_1$-product itself is homotopy commutative (cf. (2.5)).

Definition 2. A **quasi-homotopy commutative Hirsch algebra** (QHHA) is a Hirsch algebra A equipped with a binary product $\cup_2 : A\otimes A \to A$ such that

$$d(a\cup_2 b) = da\cup_2 b + (-1)^{|a|}a\cup_2 db + (-1)^{|a|}a\smile_1 b + (-1)^{(|a|+1)|b|}b\smile_1 a - q(a;b), \tag{2.4}$$

where $q(a;b)$ satisfies:

$(2.4)_1$ Leibniz rule: $dq(a;b) = -q(da;b) - (-1)^{|a|}q(a;db)$;
$(2.4)_2$ Acyclicity: $[q(a,b)] = 0 \in H(A,d)$ for $da = db = 0$.

Note that $(2.4)_1$ follows from the equalities (2.2) and $d^2 = 0$. Obviously, discarding the parameter $q(a;b)$, the above formula just becomes the Steenrod formula for the $\smile_2$-cochain product:

$$d(a\smile_2 b) = da\smile_2 b + (-1)^{|a|}a\smile_2 db + (-1)^{|a|}a\smile_1 b + (-1)^{(|a|+1)|b|}b\smile_1 a. \tag{2.5}$$

However, $q(-;-)$ may be non-zero when passing to models constructed via cohomology as below. In the following four examples, the first is a naturally occurring example of a *cochain* Hirsch algebra (compare Example 5); in the second example QHHA structures are considered for certain Hirsch algebras; in the third and fourth examples a Hirsch algebra structure is lifted to the cohomology level. In fact, the fourth example was the original motivation for this paper.

Example 1. The primary examples of Hirsch algebras for topological spaces X are their cubical or simplicial cochain complexes [20,19,21]. In the simplicial case one can choose $E_{p,q} = 0$ for $q \geq 2$ and obtain an HGA structure on the simplicial cochains $C^*(X;\Bbbk)$ [2] (see also [19]). Furthermore, the product μ_E on $BC^*(X;\Bbbk)$ gives the multiplicative structure of the loop space cohomology $H^*(\Omega X;\Bbbk)$.

Here the cochain complex $C^*(X;\Bbbk)$ of a space X is 1-reduced, since by definition $C^*(X;\Bbbk) = C^*(\mathrm{Sing}^1 X;\Bbbk)/C^{>0}(\mathrm{Sing}\, x;\Bbbk)$ where $\mathrm{Sing}^1 X \subset \mathrm{Sing}\, X$ is the Eilenberg 1-subcomplex generated by the singular simplices that send the 1-skeleton of the standard n-simplex Δ^n to the base point x of X. Unlike the cubical cochains, the Hirsch algebra structure of the simplicial cochains is *associative*, i.e., the above product μ_E is associative.

Example 2. First, note that the Hirsch algebras from the previous example are also QHHA's by setting $\cup_2 = \smile_2$ and $q(-;-) = 0$. Let A be a *special Hirsch algebra*, i.e., A is an associative Hirsch algebra and BA also admits a Hirsch algebra structure. Then A is a QHHA since it admits a $\cup_2$-product satisfying (2.5) (cf. [18]). An important example of a special Hirsch algebra is $A = C^*(X;\Bbbk)$ from the previous example (cf. [20,34]). Finally, for a QHHA A with ν

to be zero or odd and $\smile_2$-product satisfying (2.5), define the *divided* $\smile_2$-operation $\cup_2$ as

$$a \cup_2 b = \begin{cases} \frac{1}{2}\, a \smile_2 a, & a = b \\ a \smile_2 b, & \text{otherwise.} \end{cases}$$

Then A with this $\cup_2$-operation is again a QHHA.

Example 3. Let $(H, d = 0)$ be a free cga $H = S\langle \mathcal{H}^* \rangle$ generated by a graded set $\mathcal{H}^*$. Then any map of sets $\tilde{E}_{p,q} : \mathcal{H}^{\times p} \times \mathcal{H}^{\times q} \to H$ of degree $1 - p - q$ extends to a Hirsch algebra structure $E_{p,q} : H^{\otimes p} \otimes H^{\otimes q} \to H$ on H. Indeed, using formula (2.2) the construction goes by induction on the sum $p + q$. In particular, if only $\tilde{E}_{1,1}$ is non-zero then the image of $E_{p,q}$ for $p + q \geq 3$ is into the submodule of H spanned by the monomials of the form $\tilde{E}_{1,1}(a_1; b_1) \cdots \tilde{E}_{1,1}(a_k; b_k) \cdot x$ for $a_i, b_i \in \mathcal{H}$, $x \in H$, and $k \geq 1$.

Example 4. The argument in Example 3 suggests how to lift a Hirsch $\mathbb{Z}_2$-algebra structure from the cochain level to cohomology. Given a Hirsch algebra A, let $H = H^*(A)$. For a cocycle $a \in A^m$, one has $d_A E_{1,1}(a, a) = 0$ and $Sq_1 : H^m \to H^{2m-1}$ is defined by

$$[a] \to [E_{1,1}(a, a)].$$

The trick here is to convert the Hirsch formulas up to homotopy on A to the Cartan formula $Sq_1(ab) = Sq_1 a \cdot Sq_0 b + Sq_0 a \cdot Sq_1 b$ on H by fixing a set of multiplicative generators $\mathcal{H} \subset H$. Define the map $\tilde{Sq}_{1,1} : \mathcal{H} \times \mathcal{H} \to H$ for $a, b \in \mathcal{H}$ by

$$\tilde{Sq}_{1,1}(a; b) = \begin{cases} Sq_1 a, & a = b, \\ 0, & \text{otherwise} \end{cases}$$

and extend to the operation $Sq_{1,1} : H \otimes H \to H$ as a (two-sided) derivation with respect to the $\cdot$ product; then in particular, $Sq_{1,1}(u; u) = Sq_1 u$ for all $u \in H$. Define $Sq_{p,q} = E_{p,q} : H^{\otimes p} \otimes H^{\otimes q} \to H$ for $p + q \geq 3$ by means of (2.2). Note that if the multiplicative structure on H is not free, such an extension might not exist. This procedure gives a Hirsch algebra structure $\{Sq_{p,q}\}$ on the cohomology algebra H in the following situations:

(i) H has trivial multiplication (e.g. the cohomology of a suspension).
(ii) H is a polynomial algebra.
(iii) H has the following property: If $a \cdot b = 0$, then $Sq_1 a \cdot b = 0 = Sq_1 a \cdot Sq_1 b$ for all $a, b \in H$.

Obviously we have the following proposition:

Proposition 1. *A morphism $f : A \to A'$ of Hirsch algebras induces a Hopf dga map of the bar constructions*

$$Bf : BA \to BA'.$$

If the modules A, A' are $\Bbbk$-free and f is a homology isomorphism, so is Bf.

This proposition is useful when applying special models for a Hirsch algebra A to calculate the cohomology algebra $H^*(BA) = Tor^A(\Bbbk, \Bbbk)$ (see Section 3.4), and consequently, the loop space cohomology $H^*(\Omega X; \Bbbk)$ when $A = C^*(X; \Bbbk)$ (see, for example, [31]).

Given a Hirsch algebra A with cohomology $H = H(A)$, let us construct a Hirsch algebra model of A. The commutative algebra H admits a special *multiplicative* resolution (RH, d), which is endowed with the Hirsch algebra structure $\{E_{p,q}\}$. The perturbed differential d_h on RH gives the desired Hirsch algebra model (RH, d_h) of A.

2.1. Hirsch resolution

Let H^* be a graded algebra and recall that a multiplicative resolution (R^*H^*, d) of H^* is the bigraded tensor algebra $T(V)$ generated by the bigraded free $\Bbbk$-module

$$V = \bigoplus_{j,m \geq 0} V^{-j,m},$$

where $V^{-j,m} \subset R^{-j}H^m$. The total degree of $R^{-j}H^m$ is the sum $-j+m$, d is of bidegree (1, 0) and $\rho : (RH, d) \to H$ is a map of bigraded algebras inducing an isomorphism $\rho^* : H^*(RH, d) \xrightarrow{\approx} H^*$ where H^* is bigraded via $H^{0,*} = H^*$ and $H^{<0,*} = 0$ ([27]; compare [11,13]). In other words,

$$\left((R^*H^m, d) \xrightarrow{\rho} H^m\right) = (\cdots \xrightarrow{d} R^{-2}H^m \xrightarrow{d} R^{-1}H^m \xrightarrow{d} R^0H^m \xrightarrow{\rho} H^m)$$

is a usual free (additive) resolution of the $\Bbbk$-module H^m for each m, and there is a multiplication on the family $\{R^*H^m\}_{m\in\mathbb{Z}}$, which is compatible with both d and the bidegree. When each H^m is $\Bbbk$-free, ΩBH (the cobar–bar construction of H) is an example of RH with $V = BH$. In general, the multiplicative structure of H^* gives rise to (additively) *non-minimal* submodules (R^*H^m, d) even for H^m to be $\Bbbk$-free or $H^m = 0$. The reason for this is that a (multiplicative) relation in H involving elements of degree $<m$ can produce an element $a \in R^{-1}H^k$ with $k < m$, say $m = kn$, some $n \geq 2$, and since the multiplication on R^*H^* respects the bidegree, the non-zero element a^n, the nth power of a, ultimately belongs to $R^{-n}H^m$, the nth component of a $\Bbbk$-module resolution of H^m (see the proof of Proposition 3). Furthermore, even for H to be a free cga over a field $\Bbbk$, the non-commutative nature of RH fails to imply R^*H^m to be a minimal $\Bbbk$-module resolution of H^m, i.e.,

$$R^0H^m = H^m \quad \text{and} \quad R^{-i}H^m = 0, \quad i > 0;$$

this is quite different from the situation in [11].

For example, consider the polynomial algebra $H = \mathbb{Z}_2[x, y]$ with $x, y \in H^2$ and $x_0, y_0 \in R^0H^2$ satisfying $\rho x_0 = x$ and $\rho y_0 = y$. Then $R^{-1}H^4 \neq 0$ since there is an element $a \in R^{-1}H^4$ such that $da = x_0y_0 + y_0x_0$. In particular, if H is the cohomology of a dga A with a *non-commutative* $\smile_1$-product (and perhaps higher order operations $E_{p,q}$; cf. Examples 1 and 5), then the construction of a *Hirsch algebra* model of A using RH requires to add another element b in $R^{-1}H^4$ with $db = x_0y_0 + y_0x_0$. Then denote $a = x_0 \smile_1 y_0$ and $b = y_0 \smile_1 x_0$ respectively (see Theorem 1). Furthermore, if H^* is 1-reduced and we wish to have a 1-reduced multiplicative resolution RH, we must restrict the resolution length of R^*H^m so that $R^{-i}H^m = 0$ for $i \geq m-1$ (e.g. H^m is $\Bbbk$-free for all m or H^2 is $\Bbbk$-free and $\Bbbk$ is a principal ideal domain). This motivates the following definition:

Definition 3. Let H^* be a cga. An absolute Hirsch resolution of H is a multiplicative resolution

$$\rho : R^*H^* \to H^*, \qquad RH = T(V), \quad V = \langle \mathcal{V} \rangle,$$

endowed with the Hirsch algebra structural operations

$$E_{p,q} : RH^{\otimes p} \otimes RH^{\otimes q} \to V \subset RH$$

such that V is decomposed as $V^{*,*} = \mathcal{E}^{*,*} \oplus U^{*,*}$ in which $\mathcal{E}^{0,*} = 0$, $U^{0,*} = V^{0,*}$ and $\mathcal{E}^{*,*} = \bigoplus_{p,q\geq 1} \mathcal{E}^{<0,*}_{p,q}$ is distinguished by an isomorphism of modules

$$E_{p,q} : \bigoplus_{\substack{i_{(p)}+j_{(q)}=s \\ k_{(p)}+\ell_{(q)}=t}} \left(\underset{1\leq r\leq p}{\otimes} R^{i_r}H^{k_r} \bigotimes \underset{1\leq n\leq q}{\otimes} R^{j_n}H^{\ell_n} \right) \xrightarrow{\approx} \mathcal{E}^{s-p-q+1\,,\,t}_{p,q} \subset V^{*,*}$$

where $x_{(r)} = x_1 + \cdots + x_r$.

Given a Hirsch algebra $(A, \{E_{p,q}\}, d)$, a submodule $J \subset A$ is a *Hirsch ideal* of A if it is an ideal with $E_{p,q}(a_1, \ldots, a_p; a_{p+1}, \ldots, a_{p+q}) \in J$ whenever $a_i \in J$ for some i.

Definition 4. Let $\rho_a : (R_aH, d) \to H$ be an absolute Hirsch resolution and $J \subset R_aH$ be a Hirsch ideal such that $d : J \to J$ and the quotient map $g : R_aH \to R_aH/J$ is a homology isomorphism. A Hirsch resolution of H is the Hirsch algebra $RH = R_aH/J$ with a map $\rho : RH \to H$ such that $\rho_a = \rho \circ g$.

Thus an absolute Hirsch resolution is a Hirsch resolution by taking $J = 0$.

Proposition 2. *Every cga H^* has an (absolute) Hirsch resolution $\rho : R^*H^* \to H^*$.*

Proof. We build a Hirsch resolution of H^* by induction on the resolution degree. Let $\mathcal{H}^* \subset H^*$ be a set of multiplicative generators. Denote $\mathcal{V}^{0,*} = \mathcal{H}^*$; let $V^{0,*} = \langle \mathcal{V}^{0,*} \rangle$ be the free $\Bbbk$-module span of $\mathcal{V}^{0,*}$ and form the free (tensor) graded algebra $R^0H^* = T(V^{0,*})$. Obviously, there is a dga epimorphism $\rho^0 : (R^0H^*, 0) \to H^*$. Inductively, given $n \geq 0$, assume we have constructed a $\Bbbk$-module $R^{(-n)}H^* = \oplus_{0\leq r\leq n} R^{-r}H^*$ with a map $\rho^{(n)} : (R^{(-n)}H^*, d) \to H^*$ with $\rho^r(R^{-r}H^*) = 0$ for $1 \leq r \leq n$, where $d : R^{-r}H^* \to R^{-r+1}H^*$ is a differential of bidegree (1, 0) defined for $1 \leq r \leq n$ and acyclic in resolution degrees $-r$ for $1 \leq r < n$; $R^{-r}H^*$ is a component of bidegree $(-r, *)$ of $T(V^{(-r),*})$ for $V^{(-r),*} = V^{0,*} \oplus \cdots \oplus V^{-r,*}$, so that

$$R^{-r}H^* = V^{-r,*} \oplus \mathcal{D}^{-r,*} = \mathcal{E}^{-r,*} \oplus U^{-r,*} \oplus \mathcal{D}^{-r,*}$$

where $\mathcal{E}^{-r,*} = \bigoplus_{p,q\geq 1} \mathcal{E}^{-r,*}_{p,q}$ and $\mathcal{E}^{-r,*}_{p,q}$ spans the set of (formal) expressions $E_{p,q}(a_1, \dots, a_p; b_1, \dots, b_q)$, $a_j \in R^{-i_k}H^*$, $b_\ell \in R^{-j_\ell}H^*$, $r = i_{(p)} + j_{(q)} + p + q - 1$, while $\mathcal{D}^{-r,*}$ is the module of decomposables of bidegree $(-r, *)$ in $T(V^{(-r),*})$; d is given by formula (2.2) on $\mathcal{E}^{-r,*}$, while acts as a derivation on $\mathcal{D}^{-r,*}$.

Let $\mathcal{E}^{-n-1,*} = \bigoplus_{p,q\geq 1} \mathcal{E}^{-n-1,*}_{p,q}$ where $\mathcal{E}^{-n-1,*}_{p,q}$ spans the set of expressions $E_{p,q}(a_1, \dots, a_p; b_1, \dots, b_q)$, $a_k \in R^{-i_k}H^*$, $b_\ell \in R^{-j_\ell}H^*$, $n+1 = i_{(p)} + j_{(q)} + p + q - 1$, and let $\mathcal{D}^{-n-1,*}$ be the module of decomposables of bidegree $(-n-1, *)$ in $T\left(V^{(-n),*} \oplus \mathcal{E}^{-n-1,*}\right)$; define d by formula (2.2) on $\mathcal{E}^{-n-1,*}$ and as a derivation on $\mathcal{D}^{-n-1,*}$ so that

$$\mathcal{E}^{-n-1,*} \oplus \mathcal{D}^{-n-1,*} \xrightarrow{d} R^{-n}H^* \xrightarrow{d} R^{-n+1}H^*.$$

Define a free $\Bbbk$-module $U^{-n-1,*}$ and d on it to achieve acyclicity in resolution degree $-n$, i.e, denoting $V^{-n-1,*} = \mathcal{E}^{-n-1,*} \oplus U^{-n-1,*}$, we obtain a partial resolution for each $m \in \mathbb{Z}$

$$V^{-n-1,m} \oplus \mathcal{D}^{-n-1,m} \xrightarrow{d} R^{-n}H^m \xrightarrow{d} R^{-n+1}H^m \xrightarrow{d} \cdots \xrightarrow{d} R^{-1}H^m \xrightarrow{d} R^0H^m \xrightarrow{\rho} H^m.$$

Define $R^{-n-1}H^* = V^{-n-1,*} \oplus \mathcal{D}^{-n-1,*}$ and $\rho^{n+1} : R^{-n-1}H^* \to H^*$ to be trivial. This completes the inductive step.

Finally, set $R^*H^* = \oplus_n R^{(-n)}H^*$ with $V^{*,*} = \langle \mathcal{V}^{*,*} \rangle$, $\mathcal{E}^{*,*} = \oplus_n \mathcal{E}^{-n,*}$, $U^{*,*} = \oplus_n U^{-n,*}$, $\rho|_{R^0H^*} = \rho^0$ and $\rho|_{R^{-n}H^*} = 0$ for $n > 0$ to obtain the desired resolution map $\rho : RH \to H$. $\square$

Note that in a Hirsch resolution $(RH, \{E_{p,q}\}, d)$, we may have relations among $E_{p,q}$'s (e.g. $E_{p,q} = 0$ for some $p, q \geq 1$; cf. Section 2.6). For example, the Hirsch structure of RH is *associative* if the product μ_E on the bar construction $B(RH)$ is associative and is equivalent to the equalities among $E_{p,q}$'s as follows.

Given a Hirsch algebra A and an arbitrary triple

$$(\mathbf{a}; \mathbf{b}; \mathbf{c}) = (a_1, \dots, a_k\, ; b_1, \dots, b_\ell\, ; c_1, \dots, c_r), \quad a_i, b_j, c_s \in A,$$

denote

$$\mathcal{R}_{k,\ell,r}((\mathbf{a}; \mathbf{b}); \mathbf{c}) = \sum_{\substack{k_{(p)}=k;\ell_{(p)}=\ell \\ 1\leq p\leq k+\ell}} (-1)^{\varepsilon} E_{p,r}(E_{k_1,\ell_1}(a_1, \dots, a_{k_1}; b_1, \dots, b_{\ell_1}),$$
$$\dots, E_{k_p,\ell_p}(a_{k-k_p+1}, \dots, a_k; b_{\ell-\ell_p+1}, \dots, b_p)\, ; c_1, \dots, c_r)$$

and

$$\mathcal{R}_{k,\ell,r}(\mathbf{a}; (\mathbf{b}; \mathbf{c})) = \sum_{\substack{\ell_{(q)}=\ell;r_{(q)}=r \\ 1\leq q\leq \ell+r}} (-1)^{\delta} E_{k,q}(a_1, \dots, a_k; E_{\ell_1,r_1}(b_1, \dots, b_{\ell_1}; c_1, \dots, c_{r_1}),$$
$$\dots, E_{\ell_q,r_q}(b_{\ell-\ell_q+1}, \dots, b_\ell; c_{r-r_q+1}, \dots, c_q)),$$

where we use the convention that $E_{0,1}(-; a) = E_{1,0}(a; -) = a$, $E_{0,m}(-; a_1, \dots, a_m) = E_{m,0}(a_1, \dots, a_m; -) = 0$, $m \geq 2$, and $x_{(n)} = x_1 + \cdots + x_n$, while the signs ε and δ are induced by permutations of symbols a_i, b_j, c_s (cf. [37]). Then the associativity of A is equivalent to the equalities

$$\mathcal{R}_{k,\ell,r}((\mathbf{a}; \mathbf{b}); \mathbf{c}) = \mathcal{R}_{k,\ell,r}(\mathbf{a}; (\mathbf{b}; \mathbf{c})), \quad k, \ell, r \geq 1.$$

Now consider the expression

$$\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c}))-\mathcal{R}_{k,\ell,r}((\mathbf{a};\mathbf{b});\mathbf{c})\in\mathcal{E}^{1-k-\ell-r,*}$$

in an absolute Hirsch resolution RH. We have that this expression belongs to $\mathcal{E}^{-2,*}$ and is a cocycle for $(\mathbf{a};\mathbf{b};\mathbf{c})=(a;b;c)$, $a,b,c\in R^0H$ (see (2.6) and Fig. 1 in which the boundaries of both hexagons are labeled by the 6 components of $d\mathcal{R}_{1,1,1}(a;(b;c))=d\mathcal{R}_{1,1,1}((a;b);c)$). So there is an element, denoted by $s(\mathcal{R}_{1,1,1}(a;(b;c)))\in V^{-3,*}$ such that $ds(\mathcal{R}_{1,1,1}(a;(b;c)))=\mathcal{R}_{1,1,1}(a;(b;c))-\mathcal{R}_{1,1,1}((a;b);c)$. In general, define elements $s(\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c})))\in V$ such that

$$\begin{aligned}&ds(\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c})))+s(\mathcal{R}_{k,\ell,r}(d\mathbf{a};(\mathbf{b};\mathbf{c})))+(-1)^{\varepsilon_1}s(\mathcal{R}_{k,\ell,r}(\mathbf{a};(d\mathbf{b};\mathbf{c})))\\&\quad+(-1)^{\varepsilon_2}s(\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};d\mathbf{c})))=\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c}))-\mathcal{R}_{k,\ell,r}((\mathbf{a};\mathbf{b});\mathbf{c})\\&\varepsilon_1=|\mathbf{a}|+k,\ \varepsilon_2=|\mathbf{a}|+|\mathbf{b}|+k+\ell.\end{aligned}$$

Consequently, $RH=R_aH/J_{ass}$ is an associative Hirsch resolution, where $J_{ass}\subset R_aH$ is a Hirsch ideal generated by

$$\{\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c}))-\mathcal{R}_{k,\ell,r}((\mathbf{a};\mathbf{b});\mathbf{c}),\ s(\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c})))\}.$$

In particular, for $(\mathbf{a};\mathbf{b};\mathbf{c})=(a;b;c)$ the associativity of a Hirsch resolution implies the following.

Proposition 3. *For $a,b,c\in RH$, there is the equality*

$$\begin{aligned}&(a\smile_1 b)\smile_1 c+E_{2,1}(a,b;c)+(-1)^{(|a|+1)(|b|+1)}E_{2,1}(b,a;c)\\&\quad=a\smile_1(b\smile_1 c)+E_{1,2}(a;b,c)+(-1)^{(|b|+1)(|c|+1)}E_{1,2}(a;c,b).\end{aligned}\tag{2.6}$$

A Hirsch resolution (RH,d) is *minimal* if

$$d(u)\in\mathcal{E}+\mathcal{D}+\kappa_u\cdot V\quad\text{for all } u\in U,$$

where $\mathcal{D}^{*,*}\subset R^*H^*$ denotes the submodule of decomposables $RH^+\cdot RH^+$ (RH^+ denotes RH modulo the unital component) and $\kappa_u\in\Bbbk$ is non-invertible. For example, when $\Bbbk=\mathbb{Z}$ we have $\kappa_u\in\mathbb{Z}\setminus\{-1,1\}$; when $\Bbbk$ is a field we have $\kappa_u=0$ for all u. Note that a minimal Hirsch resolution is *not* minimal in the category of dgas since the resolution differential does not send multiplicative generators into $\mathcal{D}$ even when $\Bbbk$ is a field. Furthermore, the notion of minimality of RH does not depend upon whether some operation $E_{p,q}$ is zero (cf. Section 2.6). On the other hand, in order to define a $\smile_2$-operation in a simple way on RH we have to consider a non-minimal Hirsch resolution in the next subsection.

Such a flexibility of choice of RH is due to the trivial Hirsch structure of H, and, in practice, the choice is suggested by a Hirsch algebra A that realizes H as the cohomology algebra.

2.2. QHHA structures on Hirsch algebras

First, note that one can introduce a $\smile_2$-product on a Hirsch resolution that satisfies (2.5). However, such a QHHA structure on RH in not always satisfactory, and we shall consider a $\cup_2$-operation simultaneously for the reasons explained below. For an even dimensional a, or for any a whenever $\nu=2$, we have that $a\smile_1 a$ is cocycle for $da=0$; hence, there is an element $x\in RH$ with $dx=a\smile_1 a$. But we cannot identify x with $a\smile_2 a$ because $d(a\smile_2 a)=0$ according to (2.5). On the other hand, it is helpful to denote $x:=a\cup_2 a$ since certain formulas are conveniently expressed in terms of the binary operation $\cup_2$ (see, for example, Proposition 5 or Remark 7). Furthermore, we can identify $a\cup_2 a$ with $\frac{1}{2}a\smile_2 a$ for $|a|$ even and 2 invertible in $\Bbbk$.

By construction of a Hirsch resolution in Proposition 2, the definition of $\smile_2$ mimics that of $\smile_1$. We start with the consideration of the expression

$$(-1)^a a\smile_1 b+(-1)^{(|a|+1)|b|}b\smile_1 a\ \in\mathcal{E}^{-1,*}\quad\text{for } a,b\in\mathcal{V}^{0,*}.$$

It is a cocycle in (RH, d), and hence, must be killed by a multiplicative generator; denote this generator by $a \smile_2 b \in U^{-2,*}$. Inductively, assume that the right-hand side of (2.5) has been defined as an element of $U^{-n+1,*}$. Then it is bounded by a multiplicative generator $a \smile_2 b \in U^{-n,*}$. Thus, $a \smile_2 b \in U$ for all $a, b \in RH$. In particular, if $dx = 0$, then $d(x \smile_2 x) = 0$ or $d(\frac{\nu}{2} x \smile_2 x) = 0$ for $|x|$ to be odd or for both $|x|$ and ν to be even respectively in which case a multiplicative generator $y \in U$ with $dy = x \smile_2 x$ is denoted by $x \cup_3 x$.

Now define a $\cup_2$-operation by

$$a \cup_2 b = \begin{cases} a \smile_2 b, & a \neq b,\ a, b \text{ are in a basis of } RH \\ 0, & a = b,\ |a| \text{ and } \nu \text{ are odd,} \end{cases} \tag{2.7}$$

while, otherwise, define $a \cup_2 a \in U$ by

$$d(a \cup_2 a) = \begin{cases} a \smile_1 a + a \smile_2 da + da \cup_3 da, & |a| \text{ is even} \\ \dfrac{\nu}{2}(a \smile_1 a + a \smile_2 da) + da \cup_3 da, & |a| \text{ is odd, } \nu \text{ is even.} \end{cases}$$

Hence, $a \cup_2 b \in U$ for any $a, b \in RH$, and let

$$\mathcal{T} = \{a \cup_2 b \in U \mid a, b \in RH\}.$$

Thus, we obtain the decomposition $U = \mathcal{T} \oplus \mathcal{M}$, some $\mathcal{M}$, and, hence, the decomposition

$$V = \mathcal{E} \oplus U = \mathcal{E} \oplus \mathcal{T} \oplus \mathcal{M}.$$

In particular, $\mathcal{T}$ contains elements of the form $a_1 \cup_2 \cdots \cup_2 a_n$, $a_i \in RH$, obtained by the iteration of the $\cup_2$-product for $n \geq 2$. In particular, for $a_i \in V^{0,2r}$ we have the following equality

$$d(a_1 \cup_2 \cdots \cup_2 a_n) = \sum_{(\mathbf{i};\mathbf{j})} sgn(\mathbf{i};\mathbf{j})(a_{i_1} \cup_2 \cdots \cup_2 a_{i_k}) \smile_1 (a_{j_1} \cup_2 \cdots \cup_2 a_{j_\ell}),$$

where the summation is over unshuffles $(\mathbf{i};\mathbf{j}) = (i_1 < \cdots < i_k; j_1 < \cdots < j_\ell)$ of $\underline{n}$ with $(a_{i_1}, \ldots, a_{i_k}) = (a_{i'_1}, \ldots, a_{i'_k})$ if and only if $\mathbf{i} = \mathbf{i}'$ and $sgn(\mathbf{i};\mathbf{j})$ is induced by the permutation sign $a_i \cup_2 a_j = (-1)^{|a_i||a_j|} a_j \cup_2 a_i$ (see also Fig. 1 for $n = 3$); consequently, for $a_1 = \cdots = a_n = a$ and $a^{\cup_2 n} := a \cup_2 \cdots \cup_2 a$, we get

$$da^{\cup_2 n} = \sum_{k+\ell=n} a^{\cup_2 k} \smile_1 a^{\cup_2 \ell}, \quad k, \ell \geq 1. \tag{2.8}$$

Note that the above equalities do not depend on the parity of a_i's when $\nu = 2$.

Remark 1. 1. The definition of $\mathcal{T}$ does not depend on the (Hirsch) associativity of RH.

2. In a minimal Hirsch resolution one can also minimize the module $\mathcal{T}$ as

$$\mathcal{T} = \{a \cup_2 b \in U \mid a, b \in \mathcal{M}\},$$

while $a \cup_2 b$ for $a, b \in RH$ is extended by certain *derivation* formulas. These formulas are rather complicated, but they could be written down if necessary.

3. The module $\mathcal{M}$ reflects the complexity of the multiplicative relations of the commutative algebra H.

For example, if H is a polynomial algebra and RH is a minimal Hirsch resolution, then $\mathcal{M} = \mathcal{M}^{0,*} = V^{0,*}$ and, consequently, RH is completely determined by the $\smile_1$- and $\cup_2$-operations [31] (see also Theorem 4).

2.3. Some canonical syzygies in the Hirsch resolution

Below we give topological interpretation of some canonical syzygies in the Hirsch resolution RH. In particular these syzygies reflect the non-associativity of the $\smile_1$-product. Remark that higher order canonical syzygies should be also related with the combinatorics of permutahedra. In practice, such relations are helpful to construct small Hirsch resolutions RH (cf. [31], see also Remark 1).

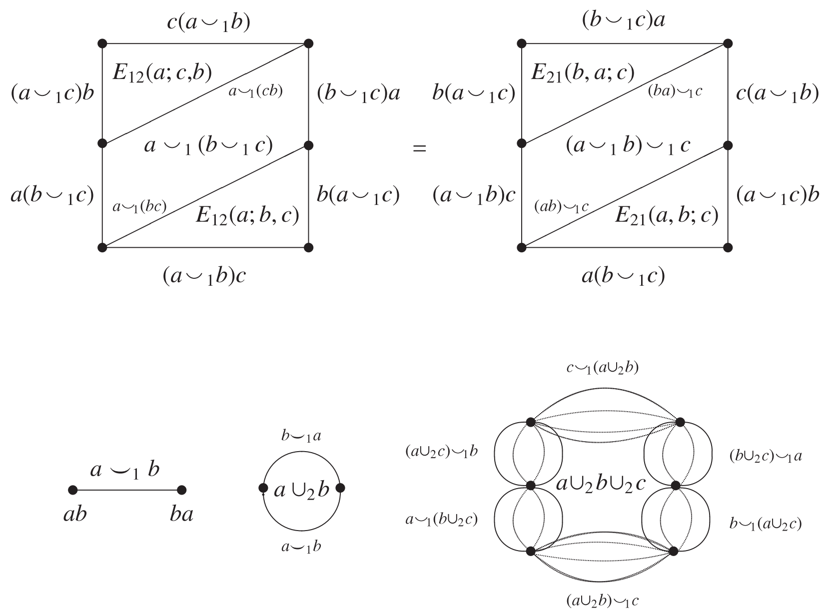

Fig. 1. Topological interpretation of some canonical syzygies in the Hirsch resolution RH.

The symbol "=" in the figure above assumes equality (2.6); the picture for $a\cup_2 b\cup_2 c$ is in fact 4-dimensional and must be understood as follows: Whence $a\cup_2 b$ corresponds to the 2-ball, the boundary of $a\cup_2 b\cup_2 c$ consists of the six 3-balls each of which is subdivided into four 3-cells by fixing two equators (these cells just correspond to the four summand components of the differential evaluated on the compositions of the $\smile_1$- and $\cup_2$-products). Then given a 3-ball, two cells from these four cells are glued to the ones of the boundary of the (diagonally) opposite 3-ball, and the other cells are glued to the ones of the boundaries of the neighboring 3-balls according to the relation

$$x\smile_1(y\smile_1 z)+(x\smile_1 y)\smile_1 z=y\smile_1(x\smile_1 z)+(y\smile_1 x)\smile_1 z.$$

2.4. Filtered Hirsch model

Recall that a dga (A^*,d) is *multialgebra* if it is bigraded $A^n=\bigoplus_{n=i+j}A^{i,j}$, $i\leq 0$, $j\geq 0$, and $d=d^0+d^1+\cdots+d^n+\cdots$ with $d^n:A^{p,q}\to A^{p+n,q-n+1}$ [12]. A dga A is bigraded via $A^{0,*}=A^*$ and $A^{i,*}=0$ for $i\neq 0$; consequently, A is a multialgebra. A multialgebra A is *homological* if $d^0=0$ (hence $d^1d^1=0$) and

$$H^i(\cdots\xrightarrow{d^1}A^{i,*}\xrightarrow{d^1}A^{i+1,*}\xrightarrow{d^1}\cdots\xrightarrow{d^1}A^{0,*})=0,\quad i<0.$$

For a homological multialgebra the sum $d^2+d^3+\cdots+d^n+\cdots$ is called a *perturbation* of d^1. In the sequel we always consider homological multialgebras, d^1 is denoted by d, d^r is denoted by h^r, and the sum $h^2+h^3+\cdots+h^n+\cdots$ is denoted by h. We sometimes denote $d+h$ by d_h.

A *multialgebra morphism* $\zeta:A\to B$ between two multialgebras A and B is a dga map of total degree zero that preserves the resolution (column) filtration, so that ζ has the components $\zeta=\zeta^0+\cdots+\zeta^i+\cdots$, $\zeta^i:A^{s,t}\to B^{s+i,t-i}$. A chain homotopy $s:A\to B$ between two multiplicative maps $f,g:A\to B$ is an (f,g)-*derivation* homotopy if $s(ab)=s(a)g(b)+(-1)^{|a|}f(a)s(b)$. A *homotopy* between two morphisms $f,g:A\to B$ of multialgebras is an (f,g)-derivation homotopy $s:A\to B$ of total degree -1 that lowers the column filtration by 1.

A multialgebra is *quasi-free* if it is a tensor algebra over a bigraded $\Bbbk$-module. Given $m\geq 2$, the map $h^m|_{A^{-m,*}}:A^{-m,*}\to A^{0,*}$ is referred to as the *transgressive* component of h and is denoted by h^{tr}. A multialgebra A with a Hirsch algebra structure

$$E_{p,q}:\otimes_{r=1}^{p}A^{i_r,k_r}\bigotimes\otimes_{n=1}^{q}A^{j_k,\ell_n}\longrightarrow A^{s-p-q+1\,,\,t}$$

with $(s,t) = \left(i_{(p)} + j_{(q)}\,,\, k_{(p)} + \ell_{(q)}\right),\ p,q \geq 1$, is called *Hirsch multialgebra*. A *homotopy* between two morphisms $f,g : A \to A'$ of Hirsch (multi)algebras is a homotopy $s : A \to A'$ of underlying (multi)algebras and

$$\begin{aligned}
&s(E_{p,q}(a_1,\dots,a_p\,;b_1,\dots,b_q))\\
&\quad= \sum_{1\leq \ell\leq q} (-1)^{\epsilon^a_p+\epsilon^b_{\ell-1}} E_{p,q}(fa_1,\dots,fa_p\,;fb_1,\dots,fb_{\ell-1},sb_\ell,gb_{\ell+1},\dots,gb_q)\\
&\qquad+ \sum_{1\leq k\leq p} (-1)^{\epsilon^a_{k-1}} E_{p,q}(fa_1,\dots,fa_{k-1},sa_k,ga_{k+1},\dots,ga_p\,;gb_1,\dots,gb_q)\\
&\qquad- \sum_{\substack{1\leq i\leq p\\ 1<\ell\leq j\leq q}} (-1)^{\epsilon_{i,j,\ell}} E_{i,j}(fa_1,\dots,fa_i\,;fb_1,\dots,fb_{\ell-1},sb_\ell,gb_{\ell+1},\dots,gb_j)\\
&\qquad\times E_{p-i,q-j}(fa_{i+1},\dots,fa_{p-1},sa_p\,;gb_{j+1},\dots,gb_q)\\
&\qquad- \sum_{\substack{0\leq i<k\leq p\\ 1\leq j\leq q}} (-1)^{\epsilon_{i,j,k}} E_{i,j}(fa_1,\dots,fa_i\,;sb_1,gb_2,\dots,gb_j)\\
&\qquad\times E_{p-i,q-j}(fa_{i+1},\dots,fa_{k-1},sa_k,ga_{k+1},\dots,ga_p\,;gb_{j+1},\dots,gb_q),
\end{aligned}\tag{2.9}$$

$$\epsilon_{i,j,m} = \epsilon^a_{p-1} + \epsilon^b_{m-1} + (\epsilon^a_p + \epsilon^a_i)\epsilon^b_j,\quad p,q\geq 1,$$

in which the first equality is

$$s(a \smile_1 b) = (-1)^{|a|+1} fa \smile_1 sb + sa \smile_1 gb - (-1)^{(|a|+1)(|b|+1)} sb\cdot sa.$$

Denote the homotopy classes of morphisms between two Hirsch (multi)algebras by $[-,-]$.

Definition 5. A quasi-free Hirsch homological multialgebra $(A,\{E_{p,q}\},d+h)$ is a **filtered Hirsch algebra** if it has the following additional properties:

(i) In $A = T(V)$ a decomposition

$$V^{*,*} = \mathcal{E}^{*,*} \oplus U^{*,*}$$

is fixed where $\mathcal{E}^{*,*} = \bigoplus_{p,q\geq 1} \mathcal{E}^{<0,*}_{p,q}$ is distinguished by an isomorphism of modules

$$E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \xrightarrow{\approx} \mathcal{E}_{p,q} \subset V,\quad p,q\geq 1;$$

(ii) The restriction of the perturbation h to $\mathcal{E}$ has no transgressive components h^{tr}, i.e., $h^{tr}|_{\mathcal{E}} = 0$.

Given a Hirsch algebra B, a *filtered Hirsch model* for B is a filtered Hirsch algebra A together with a Hirsch algebra map $A \to B$ that induces an isomorphism on cohomology. Our next proposition, which is a Adams–Hilton type of statement, exhibits a basic property of filtered Hirsch algebras:

Proposition 4. *Let $\zeta : B \to C$ be a map of (filtered) Hirsch algebras that induces an isomorphism on cohomology. If A is a filtered Hirsch algebra, there is a bijection of sets of homotopy classes of (filtered) Hirsch algebra maps*

$$\zeta_\# : [A,B] \xrightarrow{\approx} [A,C].$$

Proof. Discarding Hirsch algebra structures, the proof goes by induction on the resolution grading and is similar to that of Theorem 2.5 in [12] (see also [28]). The Hirsch algebra structure serves to specify a choice of homotopy s on the multiplicative generators $\mathcal{E} \subset V$. When constructing a chain homotopy $s : A \to C$ between two multiplicative maps $f,g : A \to C$, we can choose an s on $\mathcal{E}^{i,*}$ that satisfies formula (2.9) in each step of the induction. □

The basic examples of a filtered Hirsch algebra are provided by the following theorem, which states our main result on Hirsch algebras:

Theorem 1. *Let H be a cga and let $\rho : (RH, d) \to H$ be an absolute Hirsch resolution. Given a Hirsch algebra A, assume there exists an isomorphism $i_A : H \approx H(A, d)$. Then*

(i) *Existence. There is a pair (h, f) where $h : RH \to RH$ is a perturbation of the resolution differential d on RH and*

$$f : (RH, d + h) \to A$$

is a filtered Hirsch model of A such that $(f|_{R^0H})^* = i_A\rho|_{R^0H} : R^0H \to H(A)$.

(ii) *Uniqueness. If $(\bar{h}, \bar{f})$ and $\bar{f} : (RH, d + \bar{h}) \to A$ satisfy the conditions of* (i)*, there is an isomorphism of filtered Hirsch models*

$$\zeta : (RH, d + h) \xrightarrow{\approx} (RH, d + \bar{h})$$

of the form $\zeta = Id + \zeta^1 + \cdots + \zeta^r + \cdots$ with $\zeta^r : R^{-s}H^t \to R^{-s+r}H^{t-r}$ such that f is homotopic to $\bar{f} \circ \zeta$.

Note that the proof of the theorem uses an induction on resolution grading as it is used by the construction of filtered model due to Halperin–Stasheff [11] (compare also [27,28]); although in the rational case for the existence and the uniqueness of a pair (h, f) the zero characteristic of $\Bbbk$ is essentially involved, the proof below shows that such a restriction can be simply avoided. Here a technical subtlety is that we have certain canonically chosen multiplicative generators on which (h, f) must act by a canonical rule.

Proof. *Existence.* Let $RH = T(V)$ with $V = \mathcal{E} \oplus U$. We define a perturbation h and a Hirsch algebra map $f : (RH, d+h) \to (A, d)$ by induction on resolution (column) grading. First consider $R^0H = T(V^{0,*})\ (=T(U^{0,*}))$. Define a chain map $\mathfrak{f}^0 : (V^{0,*}, 0) \to (A, d)$ by $(\mathfrak{f}^0)^* = i_A\rho|_{V^{0,*}} : V^{0,*} \to H(A)$. Extend $\mathfrak{f}^0$ multiplicatively to obtain a dga map $f^0 : R^0H \to A$. There is a map $\mathfrak{f}^1 : V^{-1,*} \to A^{*-1}$ with $f^0 d|_{V^{-1,*}} = d\mathfrak{f}^1$; in particular, choose $\mathfrak{f}^1$ on $\mathcal{E}^{-1,*}\ (=\mathcal{E}^{-1,*}_{1,1})$ defined by the formula $\mathfrak{f}^1(a \smile_1 b) = f^0a \smile_1 f^0b$ for $a, b \in R^0H$. Then extend $\mathfrak{f}^0 + \mathfrak{f}^1$ multiplicatively to obtain a dga map $\mathfrak{f}^{(1)}_{\#} : T(V^{(-1),*}) \to (A, d)$; then denote the restriction of $\mathfrak{f}^{(1)}_{\#}$ to $R^{(-1)}H$ by $f^{(1)} : (R^{(-1)}H, d) \to (A, d)$.

Inductively, assume that a pair $(h^{(n)}, f^{(n)})$ has been constructed that satisfies the following conditions:

(1) $h^{(n)} = h^2 + \cdots + h^n$ is a derivation on RH,

(2) Equality (2.2) holds on $R^{(-n)}H$ for $d + h^{(n)}$ in which

$$\begin{aligned} h^r E_{p,q}(a_1, \ldots, a_p\, ; b_1, \ldots, b_q) = {} & \sum_{i=1}^{p} (-1)^{\epsilon^a_{i-1}} E_{p,q}(a_1, \ldots, h^r a_i, \ldots, a_q\, ; b_1, \ldots, b_q) \\ & + \sum_{j=1}^{q} (-1)^{\epsilon^a_p + \epsilon^b_{j-1}} E_{p,q}(a_1, \ldots, a_p\, ; b_1, \ldots, h^r b_j, \ldots, b_q), \\ & 2 \le r \le n, \end{aligned}$$

(3) $dh^n + h^n d + \sum_{i+j=n+1} h^i h^j = 0$,

(4) $f^{(n)} : R^{(-n)}H \to A$ is the restriction of a dga map $\mathfrak{f}^{(n)}_{\#} : T\big(V^{(-n),*}\big) \to A$ to $R^{(-n)}H$ for $f^{(n)} = f^0 + \cdots + f^n$;

(5) $f^{(n)}(d + h^{(n)}) = df^{(n)}$ on $R^{(-n)}H$, and

(6) $f^{(n)}$ is compatible with the maps $E_{p,q}$ on $\mathcal{E}^{(-n),*}$.

Consider

$$f^{(n)}(d + h^{(n)})|_{V^{-n-1,*}} : V^{-n-1,*} \to A^{*-n-1};$$

clearly $df^{(n)}(d + h^{(n)}) = 0$. Define a linear map $h^{n+1} : U^{-n-1,*} \to R^0H^{*-n}$ with $\rho h^{n+1} = i_A^{-1}[f^{(n)}(d + h^{(n)})]$ and extend h^{n+1} on RH as a derivation (denoting by the same symbol) with

$$dh^{n+1} + h^{n+1}d + \sum_{i+j=n+2} h^i h^j = 0$$

and

$$h^{n+1}E_{p,q}(a_1,\ldots,a_p\,;b_1,\ldots,b_q)=\sum_{i=1}^{p}(-1)^{\epsilon^a_{i-1}}E_{p,q}(a_1,\ldots,h^{n+1}a_i,\ldots,a_q\,;b_1,\ldots,b_q)$$
$$+\sum_{j=1}^{q}(-1)^{\epsilon^a_p+\epsilon^b_{j-1}}E_{p,q}(a_1,\ldots,a_p\,;b_1,\ldots,h^{n+1}b_j,\ldots,b_q).$$

Then there is a map $\mathfrak{f}^{n+1}:V^{-n-1,*}\to A^{*-n-1}$ such that it is compatible with $E_{p,q}$ on $\mathcal{E}^{-n-1,*}$ and

$$f^{(n)}(d+h^{(n+1)})|_{V^{-n-1,*}}=d\mathfrak{f}^{n+1}.$$

Extend $\mathfrak{f}^{(n+1)}:=\mathfrak{f}^0+\cdots+\mathfrak{f}^{n+1}$ multiplicatively to obtain a dga map $\mathfrak{f}^{(n+1)}_{\#}:T(V^{(-n-1),*})\to A$; the restriction of $\mathfrak{f}^{(n+1)}_{\#}$ to $R^{(-n-1)}H$ is denoted by

$$f^{(n+1)}:R^{(-n-1)}H\to A.$$

Thus the construction of the pair $\left(h^{(n+1)},f^{(n+1)}\right)$ completes the inductive step. Finally, a perturbation $h=h^2+\cdots+h^n+\cdots$ and a Hirsch algebra map f such that $f=f^0+\cdots+f^n+\cdots$ are obtained as desired.

Uniqueness. Using Proposition 4 we construct a multialgebra morphism

$$\zeta:(RH,d+h)\to(RH,d+\bar{h}),$$

$\zeta=\zeta^0+\zeta^1+\cdots$, with $\bar{f}\circ\zeta\simeq f$; in addition, it is easy to choose ζ with $\zeta^0=Id$. □

2.5. Filtered model for a QHHA

Referring to Section 2.2, this section considers the compatibility of the perturbation h and the Hirsch map f with the $\cup_2$-product of RH in Theorem 1. Even if A is a QHHA in the theorem, it is impossible to obtain a *QHHA map f* which commutes with $\cup_2$-products because the compatibility of parameters $q(-;-)$ under f is obstructed. When A is a $\mathbb{Z}_2$-algebra, for example, the obstruction is caused by the non-free action of Sq_1 on H. However, when $q(-;-)=0$ for the $\cup_2$-operation in A (cf. Example 2), one can refine the perturbation h in Theorem 1 as it is stated in Proposition 5 (in particular, item (i) of this proposition is an essential detail of the proof of the main result in [33]).

Let $\mathtt{T}\subset\mathcal{T}$ be a submodule defined by

$$\mathtt{T}=\langle a\cup_2 b\in\mathcal{T}\mid a\neq b\text{ in a basis of }\mathcal{M}\rangle.$$

For $\nu=2$, let $Sq_1:H^m(A)\to H^{2m-1}(A)$ be the map from Example 4.

Proposition 5. *Let A be a QHHA with $\cup_2$-operation satisfying* (2.5) *(e.g. A is a special Hirsch algebra from Example* 2*). Then in the filtered Hirsch model $f:(RH,d_h)\to A$ given by Theorem* 1*, the perturbation h can be chosen such that*

(i) $h^{tr}|_{\mathtt{T}}=0$;
(ii) *Let $\nu=2$. Then for $z_i=h^{tr}(a^{\cup_2 2^i})$ with $a\in R^0H$,*

$$\rho z_1=Sq_1(\rho a)\quad\textit{and}\quad h(a^{\cup_2 2^n})=\sum_{1\le i<n}z_i\cup_2 a^{\cup_2(2^n-2i)}+z_n.$$

Proof. (i) First, remark that any element of $\mathtt{T}$ satisfies (2.5) (cf. (2.7)). Following the construction of a pair (h,f) in the proof of Theorem 1, define f for $a\cup_2 b\in\mathtt{T}^{-2,*}$ with $a,b\in\mathcal{V}^{0,*}$ by the formula

$$f(a\cup_2 b)=fa\cup_2 fb. \tag{2.10}$$

Since (2.5), f is chain with respect to the resolution differential d of RH, so we can set $h^2(a\cup_2 b)=0$. Inductively, assume that for $a\cup_2 b\in\mathtt{T}^{-r,*}$, $2\le r<n$, the map f is defined by (2.10), while h is defined by

$$h(a\cup_2 b)=ha\cup_2 b+(-1)^{|a|}a\cup_2 hb. \tag{2.11}$$

Then for $a \cup_2 b \in \mathtt{T}^{-n,*}$ define h again by (2.11). Clearly, $fd_h(a \cup_2 b)$ is a cocycle in A and is bounded by $fa \cup_2 fb$. Therefore, we can define f on $a \cup_2 b$ by (2.10). Consequently, we set $h^{tr}(a \cup_2 b) = 0$ as required.

(ii) Since f is a Hirsch map, it commutes with $\smile_1$-products and the first equality follows from the definition of Sq_1. The verification of the second equality follows immediately from (2.8). $\square$

Remark 2. Whereas Sq_1 induces the product on $H(BA)$, the transgressive values z_i in item (ii) of Proposition 5 are closely related with the existence of the symmetric Massey products of the element $\sigma^*(\rho a) \in H(BA)$ for the suspension map $\sigma^* : H^*(A) \to H^{*-1}(BA)$ (compare Theorem 3 and Remark 7): When $\sigma^*(\rho z_k) = 0$ for $k < i$ (e.g. $z_k \in \mathcal{D}^{0,*}$), the cohomology class $\sigma^*(\rho z_i)$ is automatically identified with the symmetric Massey product $\langle \sigma^*(\rho a)\rangle^{2^i}$.

Unlike Example 1, the Hirsch algebra A provided by the following example does not have a $\smile_2$-product. This fact allows us to lift a combination $a \smile_1 b \pm b \smile_1 a$ for cocycles $a, b \in A$ to the cohomology level as a non-trivial (binary) product (see also Section 3.4).

Example 5. It is known that the Hochschild cochain complex $C^\bullet(P; P)$ of an associative algebra P admits an HGA structure [17,8], which is a particular Hirsch algebra. Furthermore, whereas the Hochschild cohomology $H = H(C^\bullet(P; P))$ is a cga, H is also endowed with the binary operation $x * y$ defined for $x = [a]$ and $y = [b]$ by $x * y = [a \circ b - (-1)^{(|a|+1)(|b|+1)} b \circ a]$, where $\circ (= \smile_1)$ is Gerstenhaber's operation on the Hochschild cochain complex. The $*$ product on the Hochschild cohomology is referred to as the G-algebra structure. Since H is a cga, we can apply Theorem 1 for $A = C^\bullet(P; P)$ and obtain the filtered Hirsch model $f : (RH, d + h) \to C^\bullet(P; P)$. Given $a, b \in V^{0,*}$, obviously we have $\rho h^2(a \cup_2 b) = \rho a * \rho b$ (since $f^1(a \smile_1 b) = f^0 a \circ f^0 b$). In other words, the non-triviality of the G-algebra structure on H implies the non-triviality of perturbation h^2 restricted to the submodule $\mathcal{T} \subset V$. Consequently, the operation $a \cup_2 b$ with $q(a, b)$ satisfying item $(2.4)_2$ does not exist on the filtered Hirsch model of $C^\bullet(P; P)$ in general.

2.6. A small Hirsch resolution $R_\varsigma H$

Let A be a Hirsch algebra over $\Bbbk$. Whereas $(RH, d_h) = (T(V), d_h)$ in a filtered Hirsch model $f : (RH, d_h) \to A$, the calculation of $H(BA)$ can be carried out in terms of V as follows. Denote $\bar V = s^{-1}(V^{>0}) \oplus \Bbbk$ and define the differential $\bar d_h$ on $\bar V$ by the restriction of $d + h$ to V to obtain the cochain complex $(\bar V, \bar d_h)$. There are isomorphisms

$$H^*(\bar V, \bar d_h) \approx H^*(B(RH), d_{B(RH)}) \overset{Bf^*}{\approx} H^*(BA, d_{BA}) \approx Tor^A(\Bbbk; \Bbbk). \tag{2.12}$$

In particular, for $A = C^*(X; \Bbbk)$ with X simply connected (cf. Example 1),

$$H^*(\bar V, \bar d_h) \approx H^*(BC^*(X; \Bbbk), d_{BC}) \approx H^*(\Omega X; \Bbbk).$$

Remark 3. Note that the first isomorphism of (2.12) is a consequence of a general fact about tensor algebras [6], while the second follows from Proposition 1.

Furthermore, to conveniently involve the multiplicative structure of (2.12), one can reduce V at the cost of $\mathcal{E} \subset V$ in the manner we shall describe. Let $J_\varsigma \subset R_a H$ be the Hirsch ideal of an absolute Hirsch resolution $R_a H$ generated by

$$\{E_{p,q}(a_1, \ldots, a_p; a_{p+1}, \ldots, a_{p+q}),\ dE_{p,q}(a_1, \ldots, a_p; a_{p+1}, \ldots, a_{p+q}) \mid p + q \ge 3\}$$

with

$$\begin{array}{ll} a_1, \ldots, a_p \in R_a H,\ a_{p+1} \in V, & p \ge 1,\ q = 1 \\ a_1, \ldots, a_{p+q} \in R_a H, & p \ge 1,\ q > 1. \end{array}$$

Then

$$R_\varsigma H = R_a H / J_\varsigma$$

is a Hirsch resolution of H. Indeed, using (2.2) we see that $d : J_\varsigma \to J_\varsigma$ and $H(J_\varsigma, d) = 0$. Thus $g_\varsigma : (R_a H, d) \to (R_\varsigma H, d)$ is a homology isomorphism. We have an obvious projection $\rho_\varsigma : (R_\varsigma H, d) \to H$ such that $\rho = \rho_\varsigma \circ g_\varsigma$.

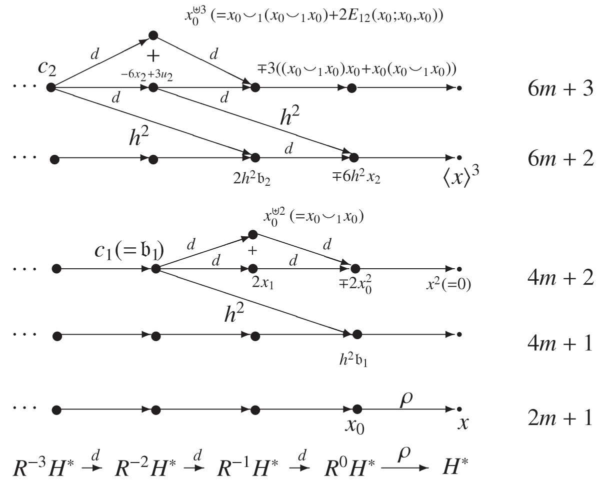

Fig. 2. A fragment of the filtered Hirsch $\mathbb{Z}$-algebra obtained as a perturbed resolution $(RH, d+h)$ of a cga H.

Consequently, ρ_ς is also a resolution map. Furthermore, we have $h : J_\varsigma \to J_\varsigma$ so that $(R_\varsigma H, d_h)$ is a Hirsch algebra (in fact an HGA) and g_ς extends to a quasi-isomorphism of filtered Hirsch algebras

$$g_\varsigma : (R_a H, d_h) \to (R_\varsigma H, d_h). \tag{2.13}$$

Thus, the Hirsch (HGA) structure of $R_\varsigma H = T(V_\varsigma)$ is generated by the $\smile_1$-product and (2.2) is equivalent to the following two equalities:

1. *The (left) Hirsch formula.* For $a, b, c \in R_\varsigma H$:

$$c \smile_1 ab = (c \smile_1 a)b + (-1)^{(|c|+1)|a|} a(c \smile_1 b).$$

2. *The (right) generalized Hirsch formula.* For $a, b \in R_\varsigma H$ and $c \in V_\varsigma$ with $d_h(c) = \sum c_1 \cdots c_q,\ c_i \in V_\varsigma$:

$$ab \smile_1 c = \begin{cases} a(b \smile_1 c) + (-1)^{|b|(|c|+1)}(a \smile_1 c)\, b, & q = 1, \\ a(b \smile_1 c) + (-1)^{|b|(|c|+1)}(a \smile_1 c)\, b & \\ \quad + \sum\limits_{1 \le i < j \le q} (-1)^{\varepsilon}\, c_1 \cdots c_{i-1}(a \smile_1 c_i)\, c_{i+1} \cdots c_{j-1}(b \smile_1 c_j)\, c_{j+1} \cdots c_q, & q \ge 2, \end{cases} \tag{2.14}$$

where $\varepsilon = (|a|+1)\left(\epsilon^c_{i-1} + i + 1\right) - (|b|+1)\left(\epsilon^c_{j-1} + j\right)$.

Remark 4. First, Formula (2.14) can be thought of as a generalization of Adams' formula for the $\smile_1$-product in the cobar construction [1, p. 36] from $q = 2$ to any $q \ge 2$. Second, the usage of $R_\varsigma H$ shows that the multiplication μ^*_E on $H^*(BA) \approx H^*(\bar{V}_\varsigma, \bar{d}_h)$ is in fact determined only by the $\smile_1$-product on V_ς.

Note that for any Hirsch resolution of H considered here, and consequently for any filtered Hirsch model, the first two columns in Fig. 2 are the same.

3. Some examples and applications

In the discussion that follows we sometimes abuse notation and denote $R_\varsigma H$ by RH. As we mentioned in the introduction, certain applications of the above material are given in [31,32]. The applications that appear here are new.

3.1. Symmetric Massey products

Recall the definition of the n-fold symmetric Massey product $\langle x \rangle^n$ (cf. [23,25]). Let $x \in H(A)$ be an element for a dga A, and $x_0 \in A$ be a cocycle with $x = [x_0]$. Given $n \ge 3$, consider a sequence $(x_0, x_1, \ldots, x_{n-2})$ in A such that

$$dx_k = \sum_{i+j=k-1} (-1)^{|x_i|+1} x_i x_j, \quad 1 \le k \le n-2; \tag{3.1}$$

in particular, $dx_1 = -(-1)^{|x_0|} x_0^2$, i.e., $x^2 = 0$. Then $\sum_{i+j=n-2}(-1)^{|x_i|+1}x_i x_j$ is a cocycle, and a subset of $H(A)$ formed by the classes of all such cocycles is denoted by $\langle x\rangle^n$. (In other words, the existence of a sequence $(x_0, x_1, \ldots, x_k, \ldots)$ satisfying (3.1) for all k implies that $c := \sum_{k\ge 0} x_k$ is a *twisting* element in A whenever this sum (possibly infinite) has a sense; an element $c \in A$ is twisting if $dc = \pm c \cdot c$; cf. [3].)

When $A = C^*(X; \mathbb{Z}_p)$ for p to be an odd prime, and $x \in H^{2m+1}(X; \mathbb{Z}_p)$ is odd dimensional, the following formula is established in [23] (for the dual case see [22]):

$$\langle x\rangle^p = -\beta\mathcal{P}_1(x) \tag{3.2}$$

where $\mathcal{P}_1 : H^{2m+1}(X; \mathbb{Z}_p) \to H^{2mp+1}(X; \mathbb{Z}_p)$ is the Steenrod cohomology operation. Thus, the formulas in [23] and [22] involve the connection of the symmetric Massey products with the Steenrod and Dyer–Lashof (co)homology operations in their respective topological settings (cf. [25]). Below Theorem 3 emphasizes the algebraic content of these formulas and generalizes them using a filtered Hirsch model over the integers.

3.2. Massey syzygies in the Hirsch resolution

Let (RH, d) be a Hirsch resolution of H. Given a sequence of relations of the form $da_i = \lambda b_i$ and

$$\begin{aligned} &du_i = (-1)^{|a_i|+1} a_i a_{i+1} + \lambda v_i, \qquad dv_i = (-1)^{|a_i|} b_i a_{i+1} + a_i b_{i+1},\\ &a_i, u_i, v_i \in RH,\ \lambda \in \mathbb{Z}\setminus\{-1, 1\},\ 1 \le i < n, \end{aligned} \tag{3.3}$$

in (RH, d), there are elements $u_{a_{i_1},\ldots,a_{i_k}} \in RH$, $3 \le k \le n$, defined in terms of syzygies that mimic the definition of k-fold Massey products arising from k-tuples $(a_{i_1}, \ldots, a_{i_k})$ [23]. Precisely, $u_{a_1,\ldots,a_n}$ is defined by

$$\begin{aligned} du_{a_1,\ldots,a_n} &= \sum_{0\le i<n} (-1)^{\epsilon_i^a} u_{a_1,\ldots,a_i} u_{a_{i+1},\ldots,a_n} + \lambda v_{a_1,\ldots,a_n},\\ dv_{a_1,\ldots,a_n} &= \sum_{0\le i<n} ((-1)^{\epsilon_i^a+1} v_{a_1,\ldots,a_i} u_{a_{i+1},\ldots,a_n} + u_{a_1,\ldots,a_i} v_{a_{i+1},\ldots,a_n}), \end{aligned} \tag{3.4}$$

with the convention that $u_{a_i} = a_i$, $u_{a_i,a_{i+1}} = u_i$ and $v_{a_i} = b_i$, $v_{a_i,a_{i+1}} = v_i$. When $b_i = 0$, Eq. (3.4) reduces to

$$du_{a_1,\ldots,a_n} = \sum_{0\le i<n} (-1)^{\epsilon_i^a} u_{a_1,\ldots,a_i} u_{a_{i+1},\ldots,a_n}.$$

We are interested in the special case of (3.3) obtained by setting $a_1 = \cdots = a_n$. More precisely, we consider the following situation (see also Example 6).

Let A be a torsion free Hirsch algebra over $\mathbb{Z}$ and fix a filtered model $f : (RH, d_h) \to A$. For a module C over $\mathbb{Z}$, let $C_{\Bbbk} := C \otimes_{\mathbb{Z}} \Bbbk$ and let $t_{\Bbbk} : C \to C_{\Bbbk}$ be the standard map; then $A_{\Bbbk} = A \otimes_{\mathbb{Z}} \Bbbk$ and $RH_{\Bbbk} = RH \otimes_{\mathbb{Z}} \Bbbk$. Also let $H_{\Bbbk} := H(A_{\Bbbk})$. There is the Hirsch model of $(A_{\Bbbk}, d_{A_{\Bbbk}})$ given by

$$f_{\Bbbk} = f \otimes 1 : (RH_{\Bbbk}, d_h \otimes 1) \to (A_{\Bbbk}, d_{A_{\Bbbk}}).$$

Given an element $x \in H_{\Bbbk}$, let x_0 be a *representative* of x in RH so that $[t_{\Bbbk} f(x_0)] = x$. In particular, $x_0 \in R^0H^*$ for $\beta(x) = 0, k \ge 1$, and $x_0 \in R^{-1}H^*$ with $dx_0 = \lambda x_0'$, $x_0' \in R^0H^*$, for $\beta(x) \ne 0$, where β denotes the Bockstein cohomology homomorphism associated with the sequence

$$0 \to \mathbb{Z}_\lambda \to \mathbb{Z}_{\lambda^2} \to \mathbb{Z}_\lambda \to 0.$$

If $x \in H = H^*(A)$, then obviously $x_0 \in R^0H^*$. In any case, assuming $x^2 = 0$ we have the corresponding relation in (RH, d):

$$dx_1 = (-1)^{|x_0|+1} x_0^2 + \lambda x_1'$$

with the convention that $x'_1 = 0$ whenever $x_0 \in R^0H^*$. This equality is a special case of (3.3), so (3.4) gives the following sequence of relations in (RH, d):

$$dx_n = \sum_{\substack{i+j=n-1\\ i,j\geq 0}} (-1)^{|x_i|+1}x_ix_j + \lambda x'_n, \quad n \geq 1, \tag{3.5}$$

where $x'_n = 0$ for $x_0 \in R^0H$.

We have the following description of Massey symmetric products in terms of the sequence $\mathbf{x} = \{x_n\}_{n\geq 0}$ in (RH, d_h). Denote $y_i = t_{\Bbbk}x_i$ in $(RH_{\Bbbk}, d_h)$. If $hy_i = 0$ for $0 \leq i < n$, then (3.5) implies $d_hd(y_n) = dd(y_n) = 0$, and consequently, $[dy_n] = -[hy_n]$. Therefore

$$f^*_{\Bbbk}[dy_n] = -f^*_{\Bbbk}[hy_n] \in \langle x\rangle^{n+1}. \tag{3.6}$$

Furthermore, the elements x_n appear in a family of relations in (RH, d). For example, these relations can be deduced from the following observation. For $x \in H$ with $x^2 = 0$, let $\iota : BH \to B(RH, d)$ be a chain map such that $\iota([\bar{x}|\cdots|\bar{x}]) = (-1)^n[\overline{x_n}]$ for $[\bar{x}|\ldots|\bar{x}] \in B^{n+1}H$, $n \geq 0$. Assuming BH is endowed with the shuffle product sh_H, the map ι will be multiplicative up to a chain homotopy $\mathfrak{b}$. Since $B(RH)$ is cofree, we can choose $\mathfrak{b}$ to be $(\mu_E \circ (\iota \otimes \iota), \iota \circ sh_H)$-coderivation. This observation easily extends to the mod λ case when $x_0 \in R^{-1}H$ with $dx_0 = \lambda x'_0$. Now let

$$\bar{\mathfrak{b}}_{k,\ell} := \mathfrak{b}(\overbrace{[\bar{x}|\cdots|\bar{x}]}^{k}\otimes\overbrace{[\bar{x}|\cdots|\bar{x}]}^{\ell})\,|_{\overline{RH}} \quad \text{and} \quad i_{[n]} := i_1 + \cdots + i_n + n;$$

then the equality $\mu_E(\iota \otimes \iota) - \iota \circ sh_H = d_{B(RH)}\mathfrak{b} + \mathfrak{b}d_{BH\otimes BH}$ implies in (RH, d):

For $|x_0|$ odd:

$$\begin{aligned} d\mathfrak{b}_{k,\ell} = {} & (-1)^{k+\ell}\binom{k+\ell}{k}x_{k+\ell-1} \\ & + \sum_{i_{[p]}=k,\ j_{[q]}=\ell} (-1)^{k+\ell+p+q}E_{p,q}(x_{i_1},\ldots,x_{i_p};x_{j_1},\ldots,x_{j_q}) \\ & - \sum_{\substack{0\leq r<k,0\leq m<\ell\\ i_{[s]}=r,\ j_{[t]}=m}} (-1)^{r+m}\Big((-1)^{s+t}E_{s,t}(x_{i_1},\ldots,x_{i_s};x_{j_1},\ldots,x_{j_t})\mathfrak{b}_{k-r,\ell-m} \\ & + \binom{r+m}{r}\mathfrak{b}_{k-r,\ell-m}\,x_{r+m-1}\Big) + \lambda\mathfrak{b}'_{k,\ell} \end{aligned} \tag{3.7}$$

in which $\mathfrak{b}'_{k,\ell} = 0$ for $x_0 \in R^0H$, and the first equalities are:

$$\begin{aligned} &d\mathfrak{b}_{1,1} = 2x_1 + x_0 \smile_1 x_0 + \lambda\mathfrak{b}'_{1,1},\\ &d\mathfrak{b}_{2,1} = -3x_2 + E_{2,1}(x_0,x_0;x_0) - x_1 \smile_1 x_0 - x_0\mathfrak{b}_{1,1} + \mathfrak{b}_{1,1}x_0 + \lambda\mathfrak{b}'_{2,1},\\ &d\mathfrak{b}_{1,2} = -3x_2 + E_{1,2}(x_0;x_0,x_0) - x_0 \smile_1 x_1 - x_0\mathfrak{b}_{1,1} + \mathfrak{b}_{1,1}x_0 + \lambda\mathfrak{b}'_{1,2}. \end{aligned}$$

For $|x_0|$ even:

$$\begin{aligned} d\mathfrak{b}_{k,\ell} = {} & (-1)^{k+\ell}\alpha_{k,\ell}\,x_{k+\ell-1} \\ & + \sum_{i_{[p]}=k,\ j_{[q]}=\ell} (-1)^{k+\ell+p+q}E_{p,q}(x_{i_1},\ldots,x_{i_p};x_{j_1},\ldots,x_{j_q}) \\ & - \sum_{\substack{0\leq r<k,0\leq m<\ell\\ i_{[s]}=r,\ j_{[t]}=m}} \Big((-1)^{(k+r+1)m+s+r+t}E_{s,t}(x_{i_1},\ldots,x_{i_s};x_{j_1},\ldots,x_{j_t})\mathfrak{b}_{k-r,\ell-m} \\ & + (-1)^{k+\ell+r(\ell+m)}\alpha_{r,m}\mathfrak{b}_{k-r,\ell-m}\,x_{r+m-1}\Big) + \lambda\mathfrak{b}'_{k,\ell}, \end{aligned} \tag{3.8}$$

$$\alpha_{i,j}=\begin{cases}\binom{(i+j)/2}{i/2}, & i,j \text{ are even},\\ \binom{(i+j-1)/2}{i/2}, & i \text{ is even, } j \text{ is odd},\\ 0, & i,j \text{ are odd},\end{cases}$$

in which $\mathfrak{b}'_{k,\ell}=0$ for $x_0\in R^0H$, and the first equalities are:

$$d\mathfrak{b}_{1,1}=x_0\smile_1 x_0+\lambda\mathfrak{b}'_{1,1}\quad (\text{i.e., } \mathfrak{b}_{1,1}=x_0\cup_2 x_0 \text{ when } x_0\in R^0H^*),$$
$$d\mathfrak{b}_{2,1}=-x_2+E_{2,1}(x_0,x_0;x_0)-x_1\smile_1 x_0-x_0\mathfrak{b}_{1,1}-\mathfrak{b}_{1,1}x_0+\lambda\mathfrak{b}'_{2,1},$$
$$d\mathfrak{b}_{1,2}=-x_2+E_{1,2}(x_0;x_0,x_0)-x_0\smile_1 x_1+x_0\mathfrak{b}_{1,1}+\mathfrak{b}_{1,1}x_0+\lambda\mathfrak{b}'_{1,2}.$$

Of course, for the sake of minimality, one can choose only certain $\mathfrak{b}_{k,\ell}$ above to be nontrivial. For example, let $|x|$ be even, let $\mathfrak{b}_{2j+1}:=\mathfrak{b}_{1,2j+1}$, and set x_{2n} in (3.5) as

$$x_{2n}=-x_0\smile_1 x_{2n-1}+\sum_{i+j=n-1}(x_{2i}\mathfrak{b}_{2j+1}-\mathfrak{b}_{2j+1}x_{2i}).\tag{3.9}$$

Thus one can also set $\mathfrak{b}_{1,2n}=0$ and eliminate $\mathfrak{b}_{1,2n}$ from (3.8); in particular, $\mathfrak{b}_{2,1}$ can be identified with $x_0\smile_2 x_1$ for $n=1$.

Note that for an HGA A (e.g. $A=C^*(X;\mathbb{Z})$) we have that $E_{p,q}=0$ for all $q\ge 2$, that the second Hirsch formula up to homotopy from Section 2 becomes strict, and consequently, the formulas above are much simpler (see also Section 2.6).

Theorem 2. *Let A be a Hirsch algebra over $\mathbb{Z}$ and let $\Bbbk$ be a field of characteristic $p\ge 0$.*

(i) *Let $x\in H(A)$ with $x^2=0$. If $\langle x\rangle^n$ is defined for $n\ge 3$, it has a finite order.*
(ii) *Let $x\in H_{\Bbbk}$ with $x^2=0$ and $p>0$. Then $\langle x\rangle^n$ is defined for $3\le n\le p$ and vanishes whenever $3\le n<p$.*
(iii) *Let $x\in H_{\Bbbk}$ with $x^2=0$ and $p=0$. Then $\langle x\rangle^n$ is defined and vanishes for all n.*

Proof. (i) Observe that the inductive construction of the terms h^r, $r\ge 2$, of h in (RH,d_h) implies $hx_i=0$ for $0\le i\le n-2$ whenever $\langle x\rangle^n$ is defined. Apply formulas (3.7)–(3.8) to verify that $m\langle x\rangle^n=0$ with $m=n$ for $|x|$ odd (take $(k,\ell)=(1,n-1)$ in (3.7)), while $m=r-1$ or $m=r$ for $n=2r$ or $n=2r+1$ (take $(k,\ell)=(2,n-2)$ in (3.8)) for $|x|$ even.

(ii)–(iii) The proof follows an argument similar to that in (i). □

Remark 5. First, regarding Theorem 2, item (i), note that formula (3.9) implies that $\langle x\rangle^n=0$ whenever $|x|$ and n are even. Second, if $|x|$ is odd, formulas (3.7) –(3.8) imply that whenever defined, $\langle x\rangle^n$ consists of a single cohomology class independent of the parity of n (see [23,22]).

3.3. The Kraines formula

Let $p:=\lambda$ be an odd prime. Let $a\in A^{2m+1}$ be an element with $da=0$ or $da=pa'$ for some a'. Given $n\ge 2$, take (the right most) nth-power of $\bar a\in\bar A$ under the μ_E product on BA and consider its component in $\bar A$. Denote this component by $s^{-1}(a^{\uplus n})$ for $a^{\uplus n}\in A^{2mn+1}$. The element $a^{\uplus n}$ has the form

$$a^{\uplus n}=a^{\smile_1 n}+Q_n(a),$$

where $Q_n(a)$ is expressed in terms of $E_{1,k}$ for $1<k<n$ (for the relations of small degrees involving this power, see also Fig. 2). For example, $Q_2(a)=0$ since $a^{\uplus 2}=a^{\smile_1 2}$ and $Q_3(a)=2E_{1,2}(a;a,a)$. In particular, if A is an HGA, then obviously $a^{\uplus n}=a^{\smile_1 n}$. Thus $da^{\uplus n}$ is divided by an integer $p\ge 2$ if and only if p is a prime and $n=p^i$, some $i\ge 1$. Consequently, the homomorphism

$$\mathcal{P}_1: H^{2m+1}_{\mathbb{Z}_p}\to H^{2mp+1}_{\mathbb{Z}_p},\qquad [t_{\mathbb{Z}_p}(a)]\to[t_{\mathbb{Z}_p}(a^{\uplus p})],\quad a\in A,\ d(t_{\mathbb{Z}_p}(a))=0,\tag{3.10}$$

is well defined.

Theorem 3. *Let A be a Hirsch algebra as in Proposition 5. Let A be torsion free and p be an odd prime. Then formula (3.2) holds in $H_{\mathbb{Z}_p}$ for $\mathcal{P}_1$ given by (3.10).*

Proof. Given $n \geq 1$, let $\mathfrak{b}_n := \mathfrak{b}_{1,n}$ and set $(k, \ell) = (1, n)$ in (3.7) to obtain

$$d\mathfrak{b}_n = (-1)^{n+1}\left((n+1)x_n - \sum_{\substack{j_{[q]}=n\\ 1\leq q\leq n}} (-1)^q E_{1,q}(x_0; x_{j_1}, \ldots, x_{j_q})\right) + \sum_{i+j=n-1} (-1)^i \left(\mathfrak{b}_j x_i - x_i \mathfrak{b}_j\right) + p\mathfrak{b}'_n. \quad (3.11)$$

By means of the element x_0 and the sequence $\{\mathfrak{b}_n\}_{n\geq 1}$, form the sequence $\{c_n\}_{n\geq 1}$ in RH as follows:

$$c_1 = \mathfrak{b}_1 \quad \text{and} \quad c_n = n!\,\mathfrak{b}_n + x_0 \smile_1 c_{n-1}, \quad n \geq 2.$$

For $n = p-1$, relation (3.11) implies a relation of the form

$$dc_{p-1} = -p!\,x_{p-1} + x_0^{\uplus p} + pu_{p-1}, \quad (3.12)$$

where $u_{p-1} \in RH^+ \cdot RH^+$ for $\beta(x) = 0$, while $u_{p-1} = w_{p-1} + (p-1)!\,\mathfrak{b}'_{p-1}$ with $w_{p-1} \in RH^+ \cdot RH^+$ for $\beta(x) \neq 0$. Hence, from $d^2(c_{p-1}) = 0$ we get

$$d(x_0^{\uplus p}) = p!\,dx_{p-1} - p\,du_{p-1} = p((p-1)!\,dx_{p-1} - du_{p-1}).$$

Obviously, $h(x_0^{\uplus p}) = 0$ because $h(x_0) = 0$ (recall that a perturbation h annihilates $R^{(-1)}H$ and is a derivation on $\mathcal{E}$). Consequently,

$$d_h(x_0^{\uplus p}) = p((p-1)!\,dx_{p-1} - du_{p-1}).$$

Taking into account $(p-1)! = -1 \bmod p$, and passing to $H_{\mathbb{Z}_p}$ we obtain

$$\beta\mathcal{P}_1(x) = f^*_{\mathbb{Z}_p}[-dy_{p-1} - dv_{p-1}] = -f^*_{\mathbb{Z}_p}[dy_{p-1}] - f^*_{\mathbb{Z}_p}[dv_{p-1}] \quad \text{for } v_{p-1} := t_{\mathbb{Z}_p}(u_{p-1}).$$

Since $f^*_{\mathbb{Z}_p}[dy_{p-1}] = \langle x\rangle^p$ by (3.6), it remains to show that $f^*_{\mathbb{Z}_p}[dv_{p-1}] = 0$. Indeed, if $\beta(x) = 0$, then $x_0 \in R^0H$, $u_{p-1} \in RH^+ \cdot RH^+$, and $hv_{p-1} = 0$ by the similar argument as in the proof of Theorem 2 (ii). Consequently, $0 = f^*_{\mathbb{Z}_p}[-hv_{p-1}] = f^*_{\mathbb{Z}_p}[dv_{p-1}]$. If $\beta(x) \neq 0$, then $x_0 \in R^{-1}H$, and let $dx_0 = px'_0$. We have that u_{p-1} contains $\mathfrak{b}'_{p-1}$ as a summand, and $hv_{p-1} = -h\mathfrak{b}'_{p-1}$. Denoting $z_0 = g_\varsigma(x_0)$ and $z'_0 = g_\varsigma(x'_0)$ in (R_ς, d_h) where g_ς is given by (2.13), we have that $g_\varsigma(x_0^{\uplus p}) = z_0^{\smile_1 p}$ and $g_\varsigma(h\mathfrak{b}'_{p-1})$ is mod p cohomologous to

$$\sum_{0\leq i<p} z_0^{\smile_1 i} \smile_1 z'_0 \smile_1 z_0^{\smile_1 p-i-1}, \text{ a summand component of } d(z_0^{\smile_1 p}).$$

But this component bounds $\sum_{0\leq i\leq p-2} z_0^{\smile_1 i} \smile_1 (z_0 \cup_2 z'_0) \smile_1 z_0^{\smile_1 p-i-2}$ mod p that finishes the proof. □

Remark 6. When $p = 2$ the relation $d(x_0 \smile_1 x_0) = -2x_0^2 + 2(x'_0 \smile_1 x_0 + x_0 \smile_1 x'_0)$ implies the Adem relation $Sq_0(a) = Sq^1 Sq_1(a)$ in $H_{\mathbb{Z}_2}$ thought of as the "Kraines formula" $\langle a\rangle^2 = a^2 = \beta Sq_1(a)$.

Example 6. Fix a Hirsch filtered model $f : (RH, d_h) \to A$ with $RH = T(V)$. Suppose that we are given a single relation

$$da = \lambda b, \quad a \in V^{-1,2k+1},\ b \in V^{0,2k+1},\ \lambda \geq 2,\ k \geq 1, \quad (3.13)$$

and deduce the following relations in (RH, d): First, define $c \in V$ by

$$dc = \begin{cases} ab + \dfrac{\lambda}{2} b \smile_1 b, & \lambda \ \text{ is even} \\ 2ab + \lambda b \smile_1 b, & \lambda \ \text{ is odd.} \end{cases} \quad (3.14)$$

When λ is odd, denote (cf. (3.3))

$$u_{2a,b} := -c, \qquad u_{b,2a} := c - 2a \smile_1 b \quad \text{and} \quad u_{2b,b} := 2ab + (\lambda - 1)b \smile_1 b$$

and obtain

$$
\begin{aligned}
&du_{a,a} = -a^2 + \lambda v_{a,a}, \quad v_{a,a} = c - a \smile_1 b,\\
&du_{a,2b,b} = -au_{2b,b} - u_{a,2b}b + \lambda v_{a,2b,b} = -2a^2b - (\lambda - 1)a(b \smile_1 b) + cb + \lambda u_{b,2b,b},\\
&du_{b,2a,b} = bu_{2a,b} - u_{b,2a}b + \lambda v_{b,2a,b} = bc - (c - 2a \smile_1 b)b + \lambda u_{b,2b,b},\\
&du_{a,2a,b} = -au_{2a,b} + u_{2a,a}b + \lambda v_{a,2a,b},
\end{aligned}
$$

where $v_{a,2b,b} = v_{b,2a,b} = u_{b,2b,b} = 2u_{b,b,b}$. Keeping in mind the fact that $d_h^2 = 0$, there is the following action of the perturbation h on the relations above:

$$
\begin{aligned}
&dh^2u_{a,a} = -\lambda h^2c,\\
&dh^2u_{a,2b,b} = -h^2c \cdot b - \lambda h^2u_{b,2b,b},\\
&dh^2u_{b,2a,b} = b \cdot h^2c + h^2c \cdot b - \lambda h^2u_{b,2b,b},\\
&dh^2u_{a,2a,b} = -a \cdot h^2c - 2h^2u_{a,a} \cdot b - \lambda h^2v_{a,2a,b},\\
&dh^3u_{a,2a,b} = -h^3u_{2a,a} \cdot b - \lambda h^3v_{a,2a,b} - h^2h^2u_{a,2a,b}.
\end{aligned}
$$

Below we shall exploit the third equality in list of relations above. First, we have

$$d\left(h^2u_{b,2a,b} + b \smile_1 h^2c\right) = -\lambda h^2u_{b,2b,b}.$$

Suppose that $\Bbbk$ is a ring such that ν divides λ and

$$[t_{\Bbbk}(a)][t_{\Bbbk}(b)] = 0. \tag{3.15}$$

By (3.14) one has $[t_{\Bbbk}(ab)] = -[t_{\Bbbk}h^2c]$, so that $h^2c = 0 \bmod \nu$ above. Denoting $[t_{\Bbbk}f(a)] := y$ and $[t_{\Bbbk}f(b)] := x$, we have $xy = 0$ by (3.15). Thus the triple Massey product $\langle x, y, x\rangle$ is defined in $H_{\Bbbk}$ and contains $[t_{\Bbbk}f(bu_{a,b} - u_{b,a}b)]$ $(= -[t_{\Bbbk}f(hu_{b,a,b})])$. Obviously, $\langle x\rangle^3$ is also defined and

$$\beta_\lambda\langle x, y, x\rangle = -\langle x\rangle^3$$

(here β_λ denotes the Bockstein map associated with $0 \to \mathbb{Z}_\nu \to \mathbb{Z}_{\nu\lambda} \to \mathbb{Z}_\lambda \to 0$). Now let $p = \lambda = 3$ and consider (3.12) for x. Then

$$c_2 = 2\mathfrak{b}_2 + x_0 \smile_1 \mathfrak{b}_1, \qquad x_0^{\uplus 3} = x_0^{\smile_1 3} + 2E_{1,2}(x_0; x_0, x_0), \qquad u_2 = \mathfrak{b}_1x_0 - x_0\mathfrak{b}_1$$

and

$$dc_2 = -6x_2 + x_0^{\smile_1 3} + 2E_{1,2}(x_0; x_0, x_0) + 3(\mathfrak{b}_1x_0 - x_0\mathfrak{b}_1).$$

Since $[x_0]^2 = 0$, one has $h^2\mathfrak{b}_1 = 0$ and hence

$$hc_2 = 2(h^2 + h^3)\mathfrak{b}_2$$

(for the relations above, see also Fig. 2). In particular, $dh^2c_2 = 6h^2x_2$. Let $a := y_0$, $b := x_0$, $u_{b,b} := x_1$ and $u_{b,b,b} := x_2$ and set $h^2c_2 = -2h^2u_{x_0,y_0,x_0}$. Furthermore, if we also have $h^3c_2 = h^3u_{x_0,y_0,x_0} \bmod 3$, then $[t_{\Bbbk}f(x_0^{\uplus 3})] = -[t_{\Bbbk}f(hc_2)] = -[t_{\Bbbk}f(hu_{x_0,y_0,x_0})]$ and, consequently,

$$\mathcal{P}_1(x) \in \langle x, y, x\rangle. \tag{3.16}$$

For example, let $A = C^*(BF_4; \mathbb{Z}_3)$, the cochain complex of the classifying space BF_4 of the exceptional group F_4. Then equality (3.15) together with (3.16) holds in $H(BF_4; \mathbb{Z}_3)$. More precisely, let $x_i \in H^i(BF_4; \mathbb{Z}_3)$ be multiplicative generators in notation of [36] and recall the following relations among them: $x_8x_9 = 0 = x_4x_{21}$, $\delta x_8 = x_9$, $\delta x_{25} = x_{26}$; also $\mathcal{P}^3(x_9) = x_{21}$ and $\mathcal{P}^1(x_{21}) = x_{25}$; thus $\mathcal{P}^1\mathcal{P}^3(x_9) = \mathcal{P}_1(x_9) = x_{25}$ by an application of the Adem relation. Thus the knowledge of both $H^*(BF_4; \mathbb{Z}_3)$ and $H^*(F_4; \mathbb{Z}_3)$ in low degrees enables us to use the filtered Hirsch model of BF_4 to deduce the following: Let a and b be defined in (3.13) by $[t_{\mathbb{Z}_3}f(a)] = x_8$ and $[t_{\mathbb{Z}_3}f(b)] = x_9$. Then $[t_{\mathbb{Z}_3}f(hc_2)] = [t_{\mathbb{Z}_3}f(hu_{b,a,b})] = -x_{25}$ and $[t_{\mathbb{Z}_3}f(h^2u_{b,b,b})] = x_{26}$ so that

$$\langle x_9\rangle^3 = -\beta\mathcal{P}_1(x_9) \quad \text{with } \mathcal{P}_1(x_9) = \langle x_9, x_8, x_9\rangle.$$

Finally, we remark that the both sides of this formula become trivial under the loop suspension map $\sigma^* : H^*(BF_4;\mathbb{Z}_3) \to H^{*-1}(F_4;\mathbb{Z}_3)$ by a general well-known fact about Massey products [23,24] (compare $\mathcal{P}_1(i_3)$ for $i_3 \in H^3(K(\mathbb{Z}_3;3);\mathbb{Z}_3)$).

3.4. Hochschild cohomology with the G-algebra structure

In this section we assume that $\Bbbk$ is a field of characteristic zero. Refer to Example 5 and recall that the HGA structure $E = \{E_{p,q}\}_{p\geq 0;q=0,1}$ on the Hochschild cochain complex $A = C^\bullet(P;P)$ induces an associative product μ_E on the bar construction BA and hence the product μ_E^* on $H^*(BA) = Tor_*^A(\Bbbk,\Bbbk)$. Since $Tor_*^A(\Bbbk,\Bbbk)$ is an associative algebra, it can be converted into a Lie algebra in the standard way.

Theorem 4. *If the Hochschild cohomology $H^* = H(C^\bullet(P;P))$ is a free algebra, then the Lie algebra structure on $Tor_*^A(\Bbbk,\Bbbk)$ is completely determined by that of the G-algebra H^*. Consequently, the product μ_E^* on $Tor_*^A(\Bbbk,\Bbbk)$ is commutative if and only if the G-product on H^* is trivial.*

Proof. For a free algebra H, the module $\mathcal{M} \subset V$ has simple form in the (minimal) Hirsch resolution (RH,d), i.e., $\mathcal{M}^{<0,*} = 0$. Indeed, given an odd dimensional multiplicative generator $x \in H$ and a representative $x_0 \in R^0H$ of x, the elements x_n in the sequence (3.5) can be defined as $x_n = \frac{(-1)^n}{(n+1)!}\, x_0^{\smile_1 n+1}$ and hence $x_n \in \mathcal{E}$ for $n \geq 1$. In particular, there is a map of dg algebras $(RH,d) \to A$ and hence an isomorphism of dg coalgebras $H^*(BA) \approx H^*(BH)$ for a dga A with $H = H^*(A)$ (a free $\Bbbk$-algebra H is intrinsically $\Bbbk$-formal). Regarding the filtered Hirsch model (RH,d_h), the perturbation h may be non-zero only on $\mathcal{T}$. More precisely, according to Example 5 the cohomology class $[h(a \cup_2 b)] \in H^*(RH,d_h)$ is defined by $\rho a * \rho b \in H$ for $a,b \in V^{0,*}$. Since $H^*(BH) \approx H^*(BA) \approx H^*(\bar V,\bar d_h)$ (cf. (2.12)), the multiplication μ_E^* on $H^*(BH)$ is induced by the $\smile_1$-product on V (cf. Remark 3). Therefore, the Lie bracket on $H^*(BH)$ is determined by the bracket

$$[a,b] = a \smile_1 b - (-1)^{(|a|+1)(|b|+1)} b \smile_1 a$$

on V. The observation that $s^{-1}[a,b]$ is cohomologous to $s^{-1}h(a \cup_2 b)$ in $\bar V$ for all $a,b \in V^{0,*}$ completes the proof. $\square$

Remark 7. Note that the transgressive component h^{tr} evaluated on the elements $a_1 \cup_2 \cdots \cup_2 a_n \in \mathcal{T}$ for $a_i \in V^{0,*}$, $n \geq 3$, determines higher order operations on $Tor^A(\Bbbk;\Bbbk)$ that extend the Lie algebra structure to an L_∞-algebra structure.

For example, a polynomial algebra $P = \Bbbk[x_1,\ldots,x_n]$ provides the case of H^* in the theorem. Indeed, in general, to calculate the Hochschild cohomology of an algebra P construct a small complex $(C_V^\bullet(P),\bar d)$, which is quasi-isomorphic to $C^\bullet(P;P)$ as follows (compare [15]): Fix an ordinary multiplicative resolution $\rho : RP \to P$ with $RP = T(V)$, view P as an RP-bimodule via ρ, and let $B(\rho)^\bullet : C^\bullet(P;P) \to C^\bullet(RP;P)$ be a quasi-isomorphism induced by $B(\rho) : B(RP) \to BP$. Set $(C_V^\bullet(P),\bar d) = (Hom(\bar V,P),\bar d)$ in which $\bar d$ is defined for $f \in C_V^\bullet(P)$ by $\bar d f = g$,

$$g(\bar x) = \sum_{1\leq i\leq k} (-1)^{\nu_i} \rho(v_1)\cdots f(\bar v_i)\cdots \rho(v_k), \qquad dx = \sum v_1\cdots v_k,\ v_i \in V, \quad k \geq 1,$$

$\nu_i = (|f|+1)(|v_1|+\cdots+|v_{i-1}|)$, and define a chain map $\chi : C_V^\bullet(P) \to C^\bullet(RP;P)$ by $\chi f = f'$,

$$f'(\bar x) = \begin{cases} f(\bar x), & x \in V, \\ \displaystyle\sum_{1\leq i\leq n} (-1)^{\nu_i} \rho(v_1)\cdots f(\bar v_i)\cdots \rho(v_n), & x = \sum v_1\cdots v_n,\ v_i \in V, n \geq 2. \end{cases}$$

Isomorphism (2.12) implies that χ is a homology isomorphism. On the other hand, the $\smile$-product on $C^\bullet(P;P)$ induces a $\smile$-product on $C_V^\bullet(P)$; more precisely, we have that $\bar V$ is a coalgebra with the coproduct $\bar\Delta : \bar V \to \bar V \otimes \bar V$ induced by the standard coproduct of BP and, consequently, $Hom(\bar V,P)$ is endowed with the standard $\smile$-product. When P is polynomial, the minimal V^* can be thought of as generated by the iterations of a (commutative) $\smile_1$-product [30]; consequently, $(\bar V^*,\bar\Delta)$ is an exterior coalgebra. Dually, $\bar V_*$ is an exterior algebra on generators $\bar x_1,\ldots,\bar x_n$. Furthermore, $\bar d = 0$ and hence $H(C_V^\bullet(P),\bar d) = C_V^\bullet(P)$. Thus the Hochschild cohomology H^* is isomorphic to the algebra $C^{\bullet}_V(P) \approx \bar{V}_{*-1} \otimes P^*$, which is the tensor product of an exterior algebra and a polynomial algebra, as required.

3.5. Symmetric Massey products in $C^*(X;\Bbbk)$ and powers in the loop homology $H_*(\Omega X;\Bbbk)$

Let A_* be a dg coalgebra over a field $\Bbbk$ and let $A^* = Hom(A_*,\Bbbk)$ be a dg algebra so that $H(A^*) = Hom(H(A_*),\Bbbk)$. Let

$$\iota : H(BA^*) \to Hom(H(\Omega A_*),\Bbbk)),$$

be the canonical map, where ΩA_* denotes the cobar construction of the coalgebra A_*. Given the suspension map $\sigma^* : H^*(A^*) \to H^{*-1}(BA^*)$, let $x \in H_*(A^*)$ and $y \in H_{*-1}(\Omega A_*)$, where y is a basis element with $\iota(\sigma^* x)(y) = 1 \in \Bbbk$, and $\iota(\sigma^* x)(y') = 0$ for any basis element $y' \neq y$.

Suppose that $\langle x\rangle^n$ is defined for x. Let $\{a_i\}_{0\leq i<n}$ be a defining system of $\langle x\rangle^n$ with $a_0 \in A^*$ a representative cocycle of x. Then $\bar{a}_0 \in BA^*$ is a cocycle with $[\bar{a}_0] = \sigma^* x$ and $\{a_i\}_{0\leq i<n}$ lifts to a cocycle $a \in BA^*$ so that the cohomology class $[a] \in H^*(BA^*)$ is represented by the y^n (the nth-power of y) in $H_*(\Omega A_*)$ via the map ι. Then Theorem 2 immediately implies the following:

Theorem 5. *Let X be a simply connected space, let $\Bbbk$ be a field of characteristic zero, and let $\sigma_* : H_*(\Omega X;\Bbbk) \to H_{*+1}(X;\Bbbk)$ be the suspension map. If $y \in H_*(\Omega X;\Bbbk)$ such that $y \notin \operatorname{Ker}\sigma_*$ and $y^2 \neq 0$, then $y^n \neq 0$ in $H_*(\Omega X;\Bbbk)$ for all $n \geq 2$.*

Finally, recalling the connection between symmetric Massey products and twisting elements in A^*, which arise from the sequences $\{a_i\}_{i\geq 0}$ above, we remark that the observation above relates the existence of twisting elements in A^* with the existence of polynomial generators in $H_*(\Omega A_*)$.

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Original article

Spectral estimates of the p-Laplace Neumann operator in conformal regular domains

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Abstract

In this paper we study spectral estimates of the p-Laplace Neumann operator in conformal regular domains $\Omega \subset \mathbb{R}^2$. This study is based on (weighted) Poincaré–Sobolev inequalities. The main technical tool is the theory of composition operators in relation with the Brennan's conjecture. We prove that if the Brennan's conjecture holds for any $p \in (4/3, 2)$ and $r \in (1, p/(2-p))$ then the weighted (r, p)-Poincare–Sobolev inequality holds with the constant depending on the conformal geometry of Ω. As a consequence we obtain classical Poincare–Sobolev inequalities and spectral estimates for the first nontrivial eigenvalue of the p-Laplace Neumann operator for conformal regular domains.

Keywords: Conformal mappings; Sobolev spaces; Elliptic equations

1. Introduction and methodology

Let $\Omega \subset \mathbb{R}^2$ be a simply connected planar domain with a smooth boundary $\partial\Omega$. We consider the Neumann eigenvalue problem for the p-Laplace operator $(1 < p < 2)$:

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \mu_p |u|^{p-2} u & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The weak statement of this spectral problem is as follows: a function u solves the previous problem if and only if $u \in W^{1,p}(\Omega)$ and

$$\iint_{\Omega} \left(|\nabla u(x,y)|^{p-2}\nabla u(x,y)\right) \cdot \nabla v(x,y)\, dxdy = \mu_p \iint_{\Omega} |u|^{p-2} u(x,y) v(x,y)\, dxdy$$

for all $v \in W^{1,p}(\Omega)$.

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The first nontrivial Neumann eigenvalue μ_p can be characterized as

$$\mu_p(\Omega) = \min\left\{ \frac{\iint_\Omega |\nabla u(x,y)|^p \, dxdy}{\iint_\Omega |u(x,y)|^p \, dxdy} : u \in W^{1,p}(\Omega) \setminus \{0\}, \iint_\Omega |u|^{p-2} u \, dxdy = 0 \right\}.$$

Moreover, $\mu_p(\Omega)^{-\frac{1}{p}}$ is the best constant $B_{p,p}(\Omega)$ (see, for example, [1,2]) in the following Poincaré–Sobolev inequality

$$\inf_{c\in\mathbb{R}} \|f - c \mid L^p(\Omega)\| \leq B_{p,p}(\Omega) \|\nabla f \mid L^p(\Omega)\|, \quad f \in W^{1,p}(\Omega).$$

We prove, that $\mu_p(\Omega)$ depends on the conformal geometry of Ω and can be estimated in terms of Sobolev norms of a conformal mapping of the unit disc $\mathbb{D}$ onto Ω (Theorem A).

The main technical tool is existence of universal weighted Poincaré–Sobolev inequalities

$$\inf_{c\in\mathbb{R}} \left(\iint_\Omega |f(x,y) - c|^r h(x,y) \, dxdy \right)^{\frac{1}{r}} \leq B_{r,p}(\Omega, h) \left(\iint_\Omega |\nabla f(x,y)|^p \, dxdy \right)^{\frac{1}{p}}, \quad f \in W^{1,p}(\Omega), \tag{1.2}$$

in any simply connected planar domain $\Omega \neq \mathbb{R}^2$ for conformal weights $h(x,y) := J_\varphi(x,y) = |\varphi'(x,y)|^2$ induced by conformal homeomorphisms $\varphi : \Omega \to \mathbb{D}$.

Main results of this article can be divided onto two parts. The first part is the technical one and concerns weighted Poincaré–Sobolev inequalities in arbitrary simply connected planar domains with nonempty boundaries (Theorem C *and its consequences). Results of the first part will be used for (non weighted) Poincaré–Sobolev inequalities in so-called conformal regular domains (Theorem* B*) that lead to lower estimates for the first nontrivial eigenvalue* μ_p *(Theorem* A*). To the best of our knowledge lower estimates were known before for convex domains only. The class of conformal regular domains is much larger. It includes, for example, bounded domains with Lipschitz boundaries and quasidiscs, i.e images of discs under quasiconformal homeomorphisms of whole plane.*

Brennan's conjecture [3] is that for a conformal mapping $\varphi : \Omega \to \mathbb{D}$

$$\iint_\Omega |\varphi'(x,y)|^\beta \, dxdy < +\infty, \quad \text{for all } \frac{4}{3} < \beta < 4. \tag{1.3}$$

For the inverse conformal mapping $\psi = \varphi^{-1} : \mathbb{D} \to \Omega$ Brennan's conjecture [3] states

$$\iint_{\mathbb{D}} |\psi'(u,v)|^\alpha \, dudv < +\infty, \quad \text{for all } -2 < \alpha < \frac{2}{3}. \tag{1.4}$$

A connection between Brennan's Conjecture and composition operators on Sobolev spaces was established in [4]:

Equivalence Theorem. *Brennan's Conjecture* (1.3) *holds for a number* $\beta \in (4/3; 4)$ *if and only if a conformal mapping* $\varphi : \Omega \to \mathbb{D}$ *induces a bounded composition operator*

$$\varphi^* : L^{1,p}(\mathbb{D}) \to L^{1,q(p,\beta)}(\Omega)$$

for any $p \in (2; +\infty)$ *and* $q(p,\beta) = p\beta/(p + \beta - 2)$.

The inverse Brennan's Conjecture states that for any conformal mapping $\psi : \mathbb{D} \to \Omega$, the derivative ψ' belongs to the Lebesgue space $L^\alpha(\mathbb{D})$, for $-2 < \alpha < 2/3$. The integrability of the derivative in the power greater than $2/3$ requires some restrictions on the geometry of Ω. If $\Omega \subset \mathbb{R}^2$ is a simply connected planar domain of finite area, then

$$\iint_{\mathbb{D}} |\psi'(u,v)|^2 \, dudv = \iint_{\mathbb{D}} J_\psi(u,v) \, dudv = |\Omega| < \infty.$$

Integrability of the derivative in the power $\alpha > 2$ is impossible without additional assumptions on the geometry of Ω. For example, for any $\alpha > 2$ the domain Ω necessarily has a finite geodesic diameter [5].

Let $\Omega \subset \mathbb{R}^2$ be a simply connected planar domain. Then Ω is called a conformal α-regular domain if there exists a conformal mapping $\varphi : \Omega \to \mathbb{D}$ such that

$$\iint_{\mathbb{D}} \left|(\varphi^{-1})'(u,v)\right|^{\alpha} \, dudv < \infty \quad for\ some\ \alpha > 2. \tag{1.5}$$

If Ω is a conformal α-regular domain for some $\alpha > 2$ we call Ω a conformal regular domain.

The property of α-regularity does not depend on choice of a conformal mapping φ and depends on the hyperbolic geometry of Ω only [6]. For connection between conformal mapping and hyperbolic geometry see, for example, [7].

Note that a boundary $\partial\Omega$ of a conformal regular domain can have any Hausdorff dimension between one and two, but cannot be equal two [8].

The next theorem gives lower estimates of the first nontrivial p-Laplace Neumann eigenvalue:

Theorem A. *Let $\varphi : \Omega \to \mathbb{D}$ be a conformal homeomorphism from a conformal α-regular domain Ω to the unit disc $\mathbb{D}$ and Brennan's Conjecture holds. Then for every $p \in (\max\{4/3, (\alpha+2)/\alpha\}, 2)$ the following inequality holds*

$$\frac{1}{\mu_p(\Omega)} \leq \inf_{q\in[1,2p/(4-p))} \left\{ \|(\varphi^{-1})' \mid L^{\alpha}(\mathbb{D})\|^2 \left(\iint_{\mathbb{D}} \left|\left(\varphi^{-1}\right)'\right|^{\frac{(p-2)q}{p-q}} \, dudv \right)^{\frac{p-q}{q}} \cdot B^p_{\frac{\alpha p}{\alpha-2},q}(\mathbb{D}) \right\}.$$

Here $B_{r,q}(\mathbb{D})$ is the best constant in the corresponding (r,q)-Poincare–Sobolev inequality in the unit disc $\mathbb{D}$ for $r = \alpha p/(\alpha-2)$.

In the limit case $\alpha = \infty$ and $p = q$ we have

Corollary A. *Suppose that Ω is a smooth bounded Jordan domain with a boundary $\partial\Omega$ of a class C^1 with a Dini continuous normal. Let $\varphi : \Omega \to \mathbb{D}$ be a conformal homeomorphism from Ω onto the unit disc $\mathbb{D}$. Then for every $p \in (1,2)$ the following inequality holds*

$$\frac{1}{\mu_p(\Omega)} \leq \|(\varphi^{-1})' \mid L^{\infty}(\mathbb{D})\|^p \frac{1}{\mu_p(\mathbb{D})}.$$

Remark 1.1. The constant $B_{r,q}(\mathbb{D})$ satisfies [9,10]:

$$B_{r,q}(\mathbb{D}) \leq \frac{2}{\pi^{\delta}} \left(\frac{1-\delta}{1/2-\delta} \right)^{1-\delta}, \quad \delta = 1/q - 1/r.$$

Remark 1.2. The Brennan's conjecture was proved for $\alpha \in [\alpha_0, 2/3)$ when $\alpha_0 = -1.752$ [11].

In the Introduction we formulate main results under the assumptions that the Brennan's conjecture holds true for all $-2 < \alpha < 2/3$. In the main part of the paper we prove main results for $\alpha_0 = -1.752 < \alpha < 2/3$ that was proved recently.

This difference is related to our belief that the Brennan's conjecture is correct.

Remark 1.3. The estimates for the $\mu_p(\Omega)$ were known before only for convex domains. For example, in [12] it was proved that

$$\mu_p(\Omega) \geq \left(\frac{\pi_p}{d(\Omega)} \right)^p$$

where

$$\pi_p = 2\int_0^{(p-1)^{\frac{1}{p}}} \frac{dt}{(1 - t^p/(p-1))^{\frac{1}{p}}} = 2\pi \frac{(p-1)^{\frac{1}{p}}}{p(\sin(\pi/p))}.$$

Theorem A has a direct connection with the spectral stability problem for the p-Laplace operator. See, the recent papers, [13–16], where one can found the history of the problem, main results in this area and appropriate references.

Theorem A is a corollary (after simple calculations) of the following version of the Poincaré–Sobolev inequality.

Theorem B. *Suppose that $\Omega \subset \mathbb{R}^2$ is a conformal α-regular domain and Brennan's Conjecture holds. Then for every $p \in (\max\{4/3, \alpha/(\alpha-1)\}, 2)$, every $s \in (1, \frac{\alpha-2}{\alpha}\frac{p}{2-p})$ and every function $f \in W^{1,p}(\Omega)$, the inequality*

$$\inf_{c\in\mathbb{R}}\left(\iint_\Omega |f(x,y)-c|^s\,dxdy\right)^{\frac{1}{s}} \le B_{s,p}(\Omega)\left(\iint_\Omega |\nabla f(x,y)|^p\,dxdy\right)^{\frac{1}{p}} \tag{1.6}$$

holds with the constant

$$B_{s,p}(\Omega) \le \|(\varphi^{-1})'|L^\alpha(\mathbb{D})\|^{\frac{2}{s}} B_{r,p}(\Omega,h) \le \inf_{q\in[1,2p/(4-p))}\left\{B_{\frac{\alpha s}{\alpha-2},q}(\mathbb{D})\cdot\|(\varphi^{-1})'|L^\alpha(\mathbb{D})\|^{\frac{2}{s}} K_{p,q}(\mathbb{D})\right\}.$$

Here $B_{r,p}(\Omega,h)$, $r=\alpha s/(\alpha-2)$, is the best constant of the following weighted Poincaré–Sobolev inequality:

Theorem C. *Suppose $\Omega \subset \mathbb{R}^2$ is a simply connected domain with non empty boundary, Brennan's Conjecture holds and $h(z)=J(z,\varphi)$ is the conformal weight defined by a conformal homeomorphism $\varphi:\Omega\to\mathbb{D}$. Then for every $p\in(4/3,2)$ and every function $f\in W^{1,p}(\Omega)$, the inequality*

$$\inf_{c\in\mathbb{R}}\left(\iint_\Omega |f(x,y)-c|^r h(x,y)\,dxdy\right)^{\frac{1}{r}} \le B_{r,p}(\Omega,h)\left(\iint_\Omega |\nabla f(x,y)|^p\,dxdy\right)^{\frac{1}{p}} \tag{1.7}$$

holds for any $r\in[1,p/(2-p))$ with the constant

$$B_{r,p}(\Omega,h) \le \inf_{q\in[1,2p/(4-p))}\left\{K_{p,q}(\mathbb{D})\cdot B_{r,q}(\mathbb{D})\right\}.$$

Here $B_{r,q}(\mathbb{D})$ is the best constant in the (non-weighted) $(r.q)$-Poincaré–Sobolev inequality in the unit disc $\mathbb{D}\subset\mathbb{R}^2$ and $K_{p,q}(\Omega)$ is the norm of composition operator

$$\left(\varphi^{-1}\right)^*: L^{1,p}(\Omega)\to L^{1,q}(\mathbb{D})$$

generated by the inverse conformal mapping $\varphi^{-1}:\mathbb{D}\to\Omega$:

$$K_{p,q}(\Omega) \le \left(\iint_{\mathbb{D}}\left|\left(\varphi^{-1}\right)'\right|^{\frac{(p-2)q}{p-q}}\,dudv\right)^{\frac{p-q}{pq}}.$$

Remark 1.4. Theorem C holds (without referring the Brennan's Conjecture) for

$$1\le r\le\frac{2p}{2-p}\cdot\frac{|\alpha_0|}{2+|\alpha_0|}<\frac{p}{2-p} \tag{1.8}$$

and $p\in((|\alpha_0|+2)/(|\alpha_0|+1),2)$, where $\alpha_0=-1.752$ represents the best result for which Brennan's conjecture was proved.

Remark 1.5. Let $\Omega\subset\mathbb{R}^2$ be a simply connected smooth domain. Then $\varphi^{-1}\in L^\alpha(\mathbb{D})$ for all $\alpha\in\mathbb{R}$ and we have the weighted Poincaré–Sobolev inequality (1.2) for all $p\in[1,2)$ and all $r\in[1,2p/(2-p)]$.

In the case, when we have an embedding of a weighted Lebesgue space into a non-weighted one, the weighted Poincaré–Sobolev inequality (1.7) implies the standard Poincaré-Sobolev inequality (1.6).

Let us give some historical remarks about the notion of conformal regular domains. This notion was introduced in [16] and was applied to the stability problem for eigenvalues of the Dirichlet–Laplace operator. In [10] the lower estimates for the first non-trivial eigenvalues of the Neumann–Laplace operator in conformal regular domains were obtained. In [5] we proved but did not formulated the following important fact about conformal regular domains and the Poincaré–Sobolev inequality:

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain of finite area such that the $(s, 2)$-Poincaré–Sobolev inequality*

$$\inf_{c\in\mathbb{R}}\left(\iint_{\Omega} |f(x, y) - c|^s \, dxdy\right)^{\frac{1}{s}} \leq B_{s,2}(\Omega)\left(\iint_{\Omega} |\nabla f(x, y)|^2 \, dxdy\right)^{\frac{1}{2}}$$

does not hold for some $s \geq 2$. Then Ω is not a conformal regular domain.

In the present work we suggest for the conformal regular domains a new method based on the theory of composition operators [17,18] and its applications to the Sobolev type embedding theorems [19,20].

The following diagram illustrates this idea:

$$\begin{array}{ccc} W^{1,p}(\Omega) & \xrightarrow{\varphi^*} & W^{1,q}(\mathbb{D}) \\ \downarrow & & \downarrow \\ L^s(\Omega) & \xleftarrow{(\varphi^{-1})^*} & L^r(\mathbb{D}). \end{array}$$

Here the operator φ^* defined by the composition rule $\varphi^*(f) = f \circ \varphi$ is a bounded composition operator on Sobolev spaces induced by a homeomorphism φ of Ω and $\mathbb{D}$ and the operator $(\varphi^{-1})^*$ defined by the composition rule $(\varphi^{-1})^*(f) = f \circ \varphi^{-1}$ is a bounded composition operator on Lebesgue spaces. This method allows to transfer Poincaré–Sobolev inequalities from regular domains (for example, from the unit disc $\mathbb{D}$) to Ω.

In recent works we studied composition operators on Sobolev spaces in connection with the conformal mappings theory [21]. This connection leads to weighted Sobolev embeddings [22,4] with the universal conformal weights. Another application of conformal composition operators was given in [16] where the spectral stability problem for conformal regular domains was considered.

2. Composition operators

Since all composition operators that will be used in this paper are induced by conformal homeomorphisms we formulate results about composition operators for diffeomorphisms only.

2.1. Composition operators on Lebesgue spaces

For any domain $\Omega \subset \mathbb{R}^2$ and any $1 \leq p < \infty$ we consider the Lebesgue space $L^p(\Omega)$ of measurable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid L^p(\Omega)\| := \left(\iint_{\Omega} |f(x, y)|^p \, dxdy\right)^{1/p} < \infty.$$

The following theorem about composition operators on Lebesgue spaces is well known (see, for example [18]):

Theorem 2.1. *Let $\varphi : \Omega \to \Omega'$ be a diffeomorphism between two planar domains Ω and Ω'. Then the composition operator*

$$\varphi^* : L^r(\Omega') \to L^s(\Omega), \quad 1 \leq s \leq r < \infty,$$

is bounded, if and only if

$$\left(\iint_{\Omega'} \left(J_{\varphi^{-1}}(u, v)\right)^{\frac{r}{r-s}} \, dudv\right)^{\frac{r-s}{rs}} = K < \infty, \quad 1 \leq s < r < \infty,$$

$$\sup_{(u,v)\in\Omega'} \left(J_{\varphi^{-1}}(u, v)\right)^{\frac{1}{s}} = K < \infty, \quad 1 \leq s = r < \infty.$$

The norm $\|\varphi^*\|$ of the composition operator φ^* equals K.

2.2. Composition operators on Sobolev spaces

We define the Sobolev space $W^{1,p}(\Omega)$, $1 \leq p < \infty$ as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W^{1,p}(\Omega)\| = \left(\iint_{\Omega} |f(x,y)|^p \, dxdy \right)^{\frac{1}{p}} + \left(\iint_{\Omega} |\nabla f(x,y)|^p \, dxdy \right)^{\frac{1}{p}}.$$

We define also the homogeneous seminormed Sobolev space $L^{1,p}(\Omega)$ of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L^{1,p}(\Omega)\| = \left(\iint_{\Omega} |\nabla f(x,y)|^p \, dxdy \right)^{\frac{1}{p}}.$$

Recall that the embedding operator $i : L^{1,p}(\Omega) \to L^1_{\mathrm{loc}}(\Omega)$ is bounded.

Let Ω and Ω' be domains in $\mathbb{R}^2$. We say that a diffeomorphism $\varphi : \Omega \to \Omega'$ induces a bounded composition operator

$$\varphi^* : L^{1,p}(\Omega') \to L^{1,q}(\Omega), \quad 1 \leq q \leq p \leq \infty,$$

by the composition rule $\varphi^*(f) = f \circ \varphi$ if $\varphi^*(f) \in L^{1,q}(\Omega)$ and there exists a constant $K < \infty$ such that

$$\|\varphi^*(f) \mid L^{1,q}(\Omega)\| \leq K \|f \mid L^{1,p}(\Omega')\| \quad \text{for all } f \in L^{1,p}(\Omega).$$

The main result of [17] gives the analytic description of composition operators on Sobolev spaces (see also [18]) and asserts (in the case of diffeomorphisms) that

Theorem 2.2 (*[17]*). *A diffeomorphism $\varphi : \Omega \to \Omega'$ between two domains Ω and Ω' induces a bounded composition operator*

$$\varphi^* : L^{1,p}(\Omega') \to L^{1,q}(\Omega), \quad 1 \leq q < p < \infty,$$

if and only if

$$K_{p,q}(\Omega) = \left(\iint_{\Omega} \left(\frac{|\varphi'(x,y)|^p}{|J_\varphi(x,y)|} \right)^{\frac{q}{p-q}} dxdy \right)^{\frac{p-q}{pq}} < \infty.$$

Definition 2.3. We call a bounded domain $\Omega \subset \mathbb{R}^2$ as a (r,q)-embedding domain, $1 \leq q, r \leq \infty$, if the embedding operator

$$i_\Omega : W^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$$

is bounded. The unit disc $\mathbb{D} \subset \mathbb{R}^2$ is an example of the $(r,2)$-embedding domain for all $r \geq 1$.

The following theorem gives a characterization of composition operators in the normed Sobolev spaces [10]. For readers convenience we reproduce here the proof of the theorem.

Theorem 2.4. *Let Ω be an (r,q)-embedding domain for some $1 \leq q \leq r < \infty$ and $|\Omega'| < \infty$. Suppose that a diffeomorphism $\varphi : \Omega \to \Omega'$ induces a bounded composition operator*

$$\varphi^* : L^{1,p}(\Omega') \to L^{1,q}(\Omega), \quad 1 \leq q \leq p < \infty,$$

and the inverse diffeomorphism $\varphi^{-1} : \Omega' \to \Omega$ induces a bounded composition operator

$$(\varphi^{-1})^* : L^r(\Omega) \to L^s(\Omega')$$

for some $p \leq s \leq r$.

Then $\varphi : \Omega \to \Omega'$ induces a bounded composition operator

$$\varphi^* : W^{1,p}(\Omega') \to W^{1,q}(\Omega), \quad 1 \leq q \leq p < \infty.$$

Proof. Because the composition operator $(\varphi^{-1})^* : L^r(\Omega) \to L^s(\Omega')$ is bounded, then the following inequality

$$\|(\varphi^{-1})^* g \mid L^s(\Omega')\| \leq A_{r,s}(\Omega)\|g \mid L^r(\Omega)\|$$

holds. Here $A_{r,s}(\Omega)$ is a positive constant.

If a domain Ω is an embedding domain and the composition operators

$$(\varphi^{-1})^* : L^r(\Omega) \to L^s(\Omega'), \qquad \varphi^* : L^{1,p}(\Omega') \to L^{1,q}(\Omega)$$

are bounded, then for a function $f = g \circ \varphi^{-1}$ the following inequalities

$$\begin{aligned}\inf_{c\in\mathbb{R}} \|f - c \mid L^s(\Omega')\| &\leq A_{r,s}(\Omega) \inf_{c\in\mathbb{R}} \|g - c \mid L^r(\Omega)\| \\ &\leq A_{r,s}(\Omega) M \|g \mid L^{1,q}(\Omega)\| \leq A_{r,s}(\Omega) K_{p,q}(\Omega) M \|f \mid L^{1,p}(\Omega')\|\end{aligned}$$

hold. Here M and $K_{p,q}(\Omega)$ are positive constants.

The Hölder inequality implies the following estimate

$$\begin{aligned}|c| &= |\Omega'|^{-\frac{1}{p}} \|c \mid L^p(\Omega')\| \leq |\Omega'|^{-\frac{1}{p}} \big(\|f \mid L^p(\Omega')\| + \|f - c \mid L^p(\Omega')\|\big) \\ &\leq |\Omega'|^{-\frac{1}{p}} \|f \mid L^p(\Omega')\| + |\Omega'|^{-\frac{1}{s}} \|f - c \mid L^s(\Omega')\|.\end{aligned}$$

Since $q \leq r$ we have

$$\begin{aligned}\|g \mid L^q(\Omega)\| &\leq \|c \mid L^q(\Omega)\| + \|g - c \mid L^q(\Omega)\| \leq |c||\Omega|^{\frac{1}{q}} + |\Omega|^{\frac{r-q}{r}} \|g - c \mid L^r(\Omega)\| \\ &\leq \left(|\Omega'|^{-\frac{1}{p}} \|f \mid L^p(\Omega')\| + |\Omega'|^{-\frac{1}{s}} \|f - c \mid L_s(\Omega')\|\right) |\Omega|^{\frac{1}{q}} + |\Omega|^{\frac{r-q}{r}} \|g - c \mid L^r(\Omega)\|.\end{aligned}$$

From previous inequalities we obtain for $\varphi^*(f) = g$ finally

$$\begin{aligned}\|g \mid L^q(\Omega)\| &\leq |\Omega|^{\frac{1}{q}} |\Omega'|^{-\frac{1}{p}} \|f \mid L^p(\Omega')\| + A_{r,s}(\Omega) K_{p,q}(\Omega) M |\Omega|^{\frac{1}{q}} |\Omega'|^{-\frac{1}{p}} \|f \mid L^{1,p}(\Omega')\| \\ &\quad + K_{p,q}(\Omega) M |\Omega|^{\frac{r-q}{r}} \|f \mid L^{1,p}(\Omega)\|.\end{aligned}$$

Therefore the composition operator

$$\varphi^* : W^{1,p}(\Omega') \to W^{1,q}(\Omega)$$

is bounded. □

3. Poincaré–Sobolev inequalities

3.1. Weighted Poincare–Sobolev inequalities

Let $\Omega \subset \mathbb{R}^2$ be a planar domain and let $v : \Omega \to \mathbb{R}$ be a smooth positive real valued function in Ω. For $1 \leq p < \infty$ consider the weighted Lebesgue space $L^p(\Omega, v)$ of measurable functions $f : \Omega \to \mathbb{R}$ with the finite norm

$$\|f \mid L^p(\Omega, v)\| := \left(\iint_\Omega |f(x, y)|^p v(x, y)\, dxdy\right)^{1/p} < \infty.$$

It is a Banach space for the norm $\|f \mid L^p(\Omega, v)\|$.

Applications of the conformal mappings theory to the Poincaré–Sobolev inequalities in planar domains are based on the following result (Theorem 3.3, Proposition 3.4 [4]) which connected the classical mappings theory and the Sobolev spaces theory.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain with non-empty boundary and $\varphi : \Omega \to \mathbb{D}$ be a conformal homeomorphism. Suppose that the (Inverse) Brennan's Conjecture holds for the interval $[\alpha_0, 2/3)$ where $\alpha_0 \in \big(-2, 0\big)$ and $p \in \big(\frac{|\alpha_0|+2}{|\alpha_0|+1}, 2\big)$.*

Then the inverse mapping φ^{-1} induces a bounded composition operator

$$(\varphi^{-1})^* : L^1_p(\Omega) \to L^1_q(\mathbb{D})$$

for any q such that

$$1 \leq q \leq \frac{p\,|\alpha_0|}{2+|\alpha_0|-p} < \frac{2p}{4-p}$$

and for any function $f \in L^1_p(\Omega)$ the inequality

$$\|(\varphi^{-1})^* f \mid L^{1,q}(\mathbb{D})\| \leq \left(\iint_{\mathbb{D}} |(\varphi^{-1})'|^{\frac{(p-2)q}{p-q}}\, dudv\right)^{\frac{p-q}{pq}} \|f \mid L^{1,p}(\Omega)\|$$

holds.

Remark 3.2. Let us remark that $\frac{|\alpha_0|+2}{|\alpha_0|+1} > \frac{4}{3}$ for any $\alpha_0 \in \big(-2, 0\big)$.

Using this theorem we prove

Theorem C′. *Suppose that $\Omega \subset \mathbb{C}$ is a simply connected domain with non empty boundary, the Brennan's Conjecture holds for the interval $[\alpha_0, 2/3)$, where $\alpha_0 \in \big(-2, 0\big)$ and $h(z) = J_\varphi(z)$ is the conformal weight defined by a conformal homeomorphism $\varphi : \Omega \to \mathbb{D}$. Then for every $p \in ((|\alpha_0|+2)/(|\alpha_0|+1), 2)$ and every function $f \in W^{1,p}(\Omega)$, the inequality*

$$\inf_{c\in\mathbb{R}} \left(\iint_{\Omega} |f(x,y)-c|^r h(z)\, dxdy\right)^{\frac{1}{r}} \leq B_{r,p}(\Omega, h)\left(\iint_{\Omega} |\nabla f(x,y)|^p\, dxdy\right)^{\frac{1}{p}}$$

holds for any r such that

$$1 \leq r \leq \frac{2p}{2-p} \cdot \frac{|\alpha_0|}{2+|\alpha_0|} < \frac{p}{2-p}$$

with the constant

$$B_{r,p}(\Omega, h) \leq \inf_{q\in[1,2p/(4-p))} \left\{K_{p,q}(\mathbb{D}) \cdot B_{r,q}(\mathbb{D})\right\}.$$

Here $B_{r,q}(\mathbb{D})$ is the best constant in the (non-weighted) Poincaré–Sobolev inequality in the unit disc $\mathbb{D} \subset \mathbb{C}$ and $K_{p,q}(\Omega)$ is the norm of composition operator

$$\left(\varphi^{-1}\right)^* : L^{1,p}(\Omega) \to L^{1,q}(\mathbb{D})$$

generated by the inverse conformal mapping $\varphi^{-1} : \mathbb{D} \to \Omega$:

$$K_{p,q}(\Omega) \leq \left(\iint_{\mathbb{D}} \left|\left(\varphi^{-1}\right)'\right|^{\frac{(p-2)q}{p-q}}\, dudv\right)^{\frac{p-q}{pq}}.$$

Proof. By the Riemann Mapping Theorem, there exists a conformal mapping $\varphi : \Omega \to \mathbb{D}$, and by the (Inverse) Brennan's Conjecture,

$$\int_{\mathbb{D}} |(\varphi^{-1})'(u,v)|^{\alpha}\, dudv < +\infty, \quad \text{for all } -2 < \alpha_0 < \alpha < 2/3.$$

Hence, by Theorem 3.1, the inequality

$$\|\nabla(f \circ \varphi^{-1}) \mid L^q(\mathbb{D})\| \leq K_{p.q}(\mathbb{D})\|\nabla f \mid L^p(\Omega)\|$$

holds for every function $f \in L^{1,p}(\Omega)$ and for any q such that

$$1 \leq q \leq \frac{p\,|\alpha_0|}{2+|\alpha_0|-p} < \frac{2p}{4-p}. \tag{3.1}$$

Choose arbitrarily $f \in C^1(\Omega)$. Then $g = f \circ \varphi^{-1} \in C^1(\mathbb{D})$ and, by the classical Poincaré–Sobolev inequality,

$$\inf_{c\in\mathbb{R}} \|f \circ \varphi^{-1} - c \mid L^r(\mathbb{D})\| \leq B_{q,r}(\mathbb{D})\|\nabla(f \circ \varphi^{-1}) \mid L^q(\mathbb{D})\| \tag{3.2}$$

for any r such that

$$1 \leq r \leq \frac{2q}{2-q}.$$

By elementary calculations from the inequality (3.1), it follows that

$$\frac{2q}{2-q} \leq \frac{2p}{2-p} \cdot \frac{|\alpha_0|}{2+|\alpha_0|} < \frac{p}{2-p}.$$

Combining inequalities for $2q/(2-q)$ and r we conclude that the inequality (3.2) holds for any r such that

$$1 \leq r \leq \frac{2p}{2-p} \cdot \frac{|\alpha_0|}{2+|\alpha_0|} < \frac{p}{2-p}.$$

Using the change of variable formula, the classical Poincaré–Sobolev inequality for the unit disc

$$\inf_{c\in\mathbb{R}}\left(\iint_{\mathbb{D}} |g(u,v)-c|^r\, dudv\right)^{\frac{1}{r}} \leq B_{r,q}(\mathbb{D})\left(\iint_{\mathbb{D}} |\nabla g(u,v)|^q\, dudv\right)^{\frac{1}{q}}$$

and Theorem 3.1, we finally infer

$$\begin{aligned}
&\inf_{c\in\mathbb{R}}\left(\iint_{\Omega} |f(x,y)-c|^r h(x,y)\, dxdy\right)^{\frac{1}{r}}\\
&\quad= \inf_{c\in\mathbb{R}}\left(\iint_{\Omega} |f(x,y)-c|^r J_\varphi(x,y)\, dxdy\right)^{\frac{1}{r}} = \inf_{c\in\mathbb{R}}\left(\iint_{\mathbb{D}} |g(u,v)-c|^r\, dudv\right)^{\frac{1}{r}}\\
&\quad\leq B_{r,q}(\mathbb{D})\left(\iint_{\mathbb{D}} |\nabla g(u,v)|^q\, dudv\right)^{\frac{1}{q}} \leq K_{p,q}(\mathbb{D}) \cdot B_{r,q}(\mathbb{D})\left(\iint_{\Omega} |\nabla f(x,y)|^p\, dxdy\right)^{\frac{1}{p}}.
\end{aligned}$$

Approximating an arbitrary function $f \in W^{1,p}(\Omega)$ by smooth functions we obtain

$$\inf_{c\in\mathbb{R}}\left(\iint_{\Omega} |f(x,y)-c|^r h(z)\, dxdy\right)^{\frac{1}{r}} \leq B_{r,p}(\Omega,h)\left(\iint_{\Omega} |\nabla f(x,y)|^p\, dxdy\right)^{\frac{1}{p}}$$

with the constant

$$B_{r,p}(\Omega,h) \leq \inf_{q:q\in[1,2p/(4-p))} \left\{K_{p,q}(\mathbb{D}) \cdot B_{r,q}(\mathbb{D})\right\}. \quad \square$$

The property of the conformal α-regularity implies the integrability of a Jacobian of conformal mappings (conformal weights) and therefore for any conformal α-regular domain we have the embedding of weighted Lebesgue spaces $L^r(\Omega,h)$ into non-weighted Lebesgue spaces $L^s(\Omega)$ for $s = \frac{\alpha-2}{\alpha}r$.

Lemma 3.3. *Let Ω be a conformal α-regular domain. Then for any function $f \in L^r(\Omega,h)$, $\alpha/(\alpha-2) \leq r < \infty$, the inequality*

$$\|f \mid L^s(\Omega)\| \leq \left(\iint_{\mathbb{D}} \left|(\varphi^{-1})'\right|^\alpha\, dudv\right)^{\frac{2}{\alpha}\cdot\frac{1}{s}} \|f \mid L^r(\Omega,h)\|$$

holds for $s = \frac{\alpha-2}{\alpha}r$.

Proof. Since Ω is a conformal α-regular domain then there exists a conformal mapping $\varphi : \Omega \to \mathbb{D}$ such that

$$\left(\iint_{\mathbb{D}} \left|J_{\varphi^{-1}}(u,v)\right|^{\frac{r}{r-s}}\, dudv\right)^{\frac{r-s}{rs}} = \left(\iint_{\mathbb{D}} \left|(\varphi^{-1})'(u,v)\right|^\alpha\, dudv\right)^{\frac{2}{\alpha}\cdot\frac{1}{s}} < \infty,$$

for $s=\frac{\alpha-2}{\alpha}r$. Then

$$
\begin{aligned}
\|f\mid L^s(\Omega)\| &= \left(\iint_\Omega |f(x,y)|^s\,dxdy\right)^{\frac{1}{s}} = \left(\iint_\Omega |f(x,y)|^s J_\varphi^{\frac{s}{r}}(x,y) J_\varphi^{-\frac{s}{r}}(x,y)\,dxdy\right)^{\frac{1}{s}}\\
&\le \left(\iint_\Omega |f(x,y)|^r J_\varphi(x,y)\,dxdy\right)^{\frac{1}{r}} \left(\iint_\Omega J_\varphi^{-\frac{s}{r-s}}(x,y)\,dxdy\right)^{\frac{r-s}{rs}}\\
&= \left(\iint_\Omega |f(x,y)|^r h(x,y)\,dxdy\right)^{\frac{1}{r}} \left(\iint_{\mathbb{D}} J_{\varphi^{-1}}^{\frac{r}{r-s}}(u,v)\,dudv\right)^{\frac{r-s}{rs}}\\
&= \left(\iint_\Omega |f(x,y)|^r h(x,y)\,dxdy\right)^{\frac{1}{r}} \left(\iint_{\mathbb{D}} \left|(\varphi^{-1})'(u,v)\right|^\alpha\,dudv\right)^{\frac{2}{\alpha}\cdot\frac{1}{s}}.
\end{aligned}
\tag{3.3}
$$

The Lemma is proved. □

From Theorem C′ and Lemma 3.3 follows Theorem B′:

Theorem B′. *Suppose that $\Omega\subset\mathbb{C}$ is a simply connected domain with non empty boundary, the Brennan's Conjecture holds for the interval $[\alpha_0, 2/3)$, where $\alpha_0\in(-2,0)$.*

Then for every

$$p\in\left(\max\left\{\frac{4}{3},\frac{2\alpha(|\alpha_0|+2)}{(2\alpha+3\alpha|\alpha_0|-4|\alpha_0|)},2\right\}\right),$$

every $s\in[1,\frac{\alpha-2}{\alpha}\frac{p}{2-p}\frac{|\alpha_0|}{2+|\alpha_0}]$ and every function $f\in W^{1,p}(\Omega)$, the inequality

$$\inf_{c\in\mathbb{R}}\left(\iint_\Omega |f(x,y)-c|^s\,dxdy\right)^{\frac{1}{s}} \le B_{s,p}(\Omega)\left(\iint_\Omega |\nabla f(x,y)|^p\,dxdy\right)^{\frac{1}{p}} \tag{3.4}$$

holds with the constant

$$B_{s,p}(\Omega)\le \|(\varphi^{-1})'|L^\alpha(\mathbb{D})\|^{\frac{2}{s}} B_{r,p}(\Omega,h)\le \inf_{q\in[1,2p/(4-p))}\left\{B_{\frac{\alpha s}{\alpha-2},q}(\mathbb{D})\cdot\|(\varphi^{-1})'|L^\alpha(\mathbb{D})\|^{\frac{2}{s}} K_{p,q}(\mathbb{D})\right\}.$$

Proof. The inequality (3.4) immediately follows from the main inequality of Theorem C′ and the main inequality of Lemma 3.3. The last part of this inequality used known estimates for the constant of the Poincaré–Sobolev inequality in the unit disc.

It is necessary to clarify the restrictions for parameters p, r, s, because these restrictions do not follow directly from Theorem A′ and Lemma 3.3.

By Lemma 3.3 $s=\frac{\alpha-2}{\alpha}r$. By Theorem C′

$$1\le r\le \frac{2p}{2-p}\cdot\frac{|\alpha_0|}{2+|\alpha_0|}<\frac{p}{2-p}.$$

Hence

$$1\le s\le \frac{\alpha-2}{\alpha}\cdot\frac{2p}{2-p}\cdot\frac{|\alpha_0|}{2+|\alpha_0|}<\frac{\alpha-2}{\alpha}\cdot\frac{p}{2-p}.$$

Since $1\le s$ we have from this inequality that

$$\frac{\alpha}{\alpha-2}\le\frac{2p}{2-p}\cdot\frac{|\alpha_0|}{2+|\alpha_0|}<\frac{p}{2-p}.$$

By elementary calculations

$$p \geq \frac{2\alpha(2+|\alpha_0|)}{2\alpha+3\alpha|\alpha_0|-4|\alpha_0|} > \frac{\alpha}{\alpha-1}.$$

The last inequality is correct by factor that Brennan's conjecture is correct for all $\alpha : -2 < \alpha < 2/3$. □

Theorem A′ follows from Theorem B′, using $s = p$, that is necessary for coincidence of the first nontrivial Neumann–Laplace eigenvalue and the constant in the Poincaré–Sobolev inequality of Theorem B′.

Theorem A′. *Let $\varphi : \Omega \to \mathbb{D}$ be a conformal homeomorphism from a conformal α-regular domain Ω to the unit disc $\mathbb{D}$ and the Brennan's Conjecture holds for the interval $[\alpha_0, 2/3)$, where $\alpha_0 \in \big(-2, 0\big)$.*

Then for every

$$p \in \left(\max\left\{4/3, \frac{4(\alpha+|\alpha_0|)}{\alpha(2+|\alpha_0|)}\right\}, 2\right)$$

the following inequality holds

$$\frac{1}{\mu_p(\Omega)} \leq \inf_{q\in[1,2p/(4-p))} \left\{ B^p_{\frac{\alpha p}{\alpha-2},q}(\mathbb{D}) \cdot \|(\varphi^{-1})'|L^\alpha(\mathbb{D})\|^2 \left(\iint_{\mathbb{D}} \left|\left(\varphi^{-1}\right)'\right|^{\frac{(p-2)q}{p-q}} dudv \right)^{\frac{p-q}{q}} \right\}.$$

Proof. By Lemma 3.3 $p = \frac{\alpha-2}{\alpha} r$. By Theorem C′

$$1 \leq r \leq \frac{2p}{2-p} \cdot \frac{|\alpha_0|}{2+|\alpha_0|} < \frac{p}{2-p}.$$

Hence

$$\frac{\alpha}{\alpha-2} \leq \frac{1}{2-p} \cdot \frac{2\,|\alpha_0|}{2+|\alpha_0|}.$$

By elementary calculations

$$p \geq 2 - \frac{2\,|\alpha_0|}{2+|\alpha_0|}\frac{\alpha-2}{\alpha} = \frac{4(\alpha+|\alpha_0|)}{\alpha(2+|\alpha_0|)} > \frac{\alpha+2}{\alpha}.$$

The last inequality holds provided that the Brennan's conjecture holds true all $\alpha : -2 < \alpha < 2/3$. □

Corollary A′. *Suppose that Ω is smoothly bounded Jordan domain with a boundary $\partial\Omega$ of a class C^1 with a Dini continuous normal. Let $\varphi : \Omega \to \mathbb{D}$ be a conformal homeomorphism from Ω onto the unit disc $\mathbb{D}$. Then for every $p \in (1, 2)$ the following inequality is correct*

$$\frac{1}{\mu_p(\Omega)} \leq \|(\varphi^{-1})'|L^\infty(\mathbb{D})\|^p \frac{1}{\mu_p(\mathbb{D})}.$$

Proof. If Ω is smoothly bounded Jordan domain with a boundary $\partial\Omega$ of a class C^1 with a Dini continuous normal, then for a conformal mapping $\varphi : \Omega \to \mathbb{D}$, the derivative φ' is bounded away from 0 and ∞ [23]. Hence. we can apply Theorem A′ in the limit case $\alpha = \infty$ and $p = q$. Then

$$\frac{1}{\mu_p(\Omega)} \leq B^p_{p,p}(\mathbb{D}) \cdot \|(\varphi^{-1})'|L^\infty(\mathbb{D})\|^2 \|(\varphi^{-1})'|L^\infty(\mathbb{D})\|^{p-2} = \|(\varphi^{-1})'|L^\infty(\mathbb{D})\|^p \frac{1}{\mu_p(\mathbb{D})}.$$

The corollary is proved. □

As an application we obtain the lower estimate of the first non-trivial eigenvalue on the Neumann eigenvalue problem for the p-Laplace operator in the interior of the cardioid (which is a non-convex domain with a non-smooth boundary).

Example 3.4. Let Ω_c be the interior of the cardioid $\rho = 2(1+\cos\theta)$. The diffeomorphism

$$z = \psi(w) = (w+1)^2, \quad w = u + iv,$$

is conformal and maps the unit disc $\mathbb{D}$ onto Ω_c. Then by Theorem A′:

$$\frac{1}{\mu_p(\Omega_c)} \le \inf_{1\le q\le \frac{2p}{4-p}} \left(\frac{2}{\pi^\delta}\left(\frac{1-\delta}{1/2-\delta}\right)^{1-\delta}\right)^p \left(\iint_{\mathbb{D}} (2|w+1|)^\alpha \, dudv\right)^{\frac{2}{\alpha}}$$
$$\times \left(\iint_{\mathbb{D}} (2|w+1|)^{\frac{(p-2)q}{p-q}} \, dudv\right)^{\frac{p-q}{q}}. \tag{3.5}$$

Here $\delta = 1/q - (\alpha-2)/\alpha p$.

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