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ON THE OCCASION OF BORIS KHVEDELIDZE'S 100TH BIRTHDAY ANNIVERSARY

This year we mark the centenary of Boris Khvedelidze, one of the brilliant representatives of the Georgian mathematical school, outstanding scientist, academician of the Georgian National Academy of Sciences.

Boris Khvedelidze was born on November 7, 1915 in the town of Chiatura (Georgia). His father Vladimir Khvedelidze and mother Olga Berishvili were doctors.

In 1931, upon graduation from the Tbilisi pedagogical technical college he worked in the fundamental library of the Georgian Polytechnical Institute as librarian. In 1933, he continued his education at the faculty of physics and mathematics of the Tbilisi State University. During his studies at this faculty he was deeply impressed by the lectures delivered by professors Levan Gokieli, Archil Kharadze, Niko Muskhelishvili, Ilya Vekua and Levan Magnaradze whom he always recalled with a great warmth.

Having graduated with honours from the University, in 1938 Boris Khvedelidze was successfully enrolled in the post-graduate course at the Institute of Mathematics of the Georgian branch of Academy of Sciences of the USSR. His supervisor was Ilya Vekua. Under his guidance B. Khvedelidze set about investigation of the Poincaré boundary value problem for the second order differential equation of elliptic type. His first research work in this subject has been presented by N. Muskhelishvili for publication in "Transactions of the Academy of Sciences of the USSR".

The years of B. Khvedelidze's post-graduate studies coincided with the period when mathematical research work in Georgia were effectively developing. Under the N. Muskhelishvili's supervision the seminar in the theory of Cauchy integrals and their applications to the boundary value problems of analytic and harmonic functions was working intensively. This seminar has played an important role in the formation of many Georgian mathematicians who have in the sequel carried out the well-known investigations dealt with the boundary value problems of the function theory of a complex variable and with singular integral equations. At one of the seminar sessions the supervisor put the question on the extension of the known results obtained for the Riemann problem in a simply-connected domain to a multiply connected domain. This problem was successfully solved by Boris Khvedelidze.

In 1942 B. Khvedelidze defended his Candidate's thesis. About the results obtained in this dissertation N. Muskhelishvili in his monograph "Singular Integral Equations" (M., 1968) wrote: "The first complete solution of the problem

$$A(s)\frac{\partial u}{\partial n} + B(s)\frac{\partial u}{\partial s} + c(s)u = f(s)$$

has been given by B. Khvedelidze". The results of this work and those obtained by I. Vekua were applied in Holland to study mathematical problems arisen after the destructive flood in 1953. On the mathematical method employed to this event, professor Dantzig at the International Congress of Mathematicians in Amsterdam declared: "These methods have been discussed in detal at the Tbilisi School under the supervision of N. Muskhelishvili. For our aims, the results obtained by I. Vekua and B. Khvedelidze are of particular importance".

In 1957 B. Khvedelidze defended his Doctoral dissertation under the title "Linear Boundary Value Problems of the Function Theory, Singular Integral Equations and Some Their Applications". Its content is presented in detail in his monograph under the same title (Proceedings of Tbilisi A. Razmadze Mathematical Institute, vol. 23, 1956). In this work B. Khvedelidze developed the method of Cauchy type integrals with density from $L^p(\Gamma)$, $p > 1$ to solve that group of boundary value problems which he called discontinuous (i.e., the problems when an unknown function may have on the boundary an infinite set of singularities). Towards this end, it was, first of all, necessary to study the problem of continuity of the operator generated by a singular Cauchy integral in Lebesgue spaces. In the case, where the line of integration is a Lyapunov's curve, he proved that the operator in the weighted space $L^p(\Gamma,\rho)$, $p > 1$ is continuous when ρ is the power function

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\alpha_k}, \quad t_k \in \Gamma, \quad t_i \neq t_j, \quad \text{when} \quad i \neq j, \quad -\frac{1}{p} < \alpha_k < \frac{1}{p'}.$$

Such a result for conjugate functions and Hilbert transformation (which are the Cauchy singular operators for a circumference and a straight line, respectively) was, for the first time, stated by Hardy and Littlewood. This result in the case of general rectifiable curves has found wide applications and, seemingly, therefore the above weighted function is frequently called in literature as Khvedelidze's weight.

Further, the method of Cauchy type integrals has been effectively used by B. Khvedelidze for inversion of a singular integral and for solution of boundary value problems of the theory of analytic functions. This is, first of all, concerns with the problem of linear conjugation in Privalov's statement, that is, with the solution of the problem in the class of Cauchy type integrals with density from $L^p(\Gamma)$. He has achieved "essential progress in a class of free terms of a boundary condition and in a class of admissible solutions" (F. D. Gakhov, in: "Investigation of Modern Problems of the Theory of Functions of a Complex Variable", M., 1964, p. 361).

With the same success B. Khvedelidze studied singular integral equations with the Cauchy kernel in Lebesgue spaces. In his book "Singular Integral Equations" (M., 1960, 404-405) N. Muskhelishvili writes: "At last we note one more of the results of great interest from the point of view of the questions dealt with in the present book". B. Khvedelidze has shown that singular integral equations considered in this chapter have the same solutions in the class H^*, as well as in classes $L^p(\Gamma)$, $p > 1$.

The above-mentioned results obtained by B. Khvedelidze were later on developed by him, his pupils and collaborators and also by many followers beyond Georgia. A part of the results obtained in this direction have been skillfully expounded in his paper "The Method of Cauchy Type Integrals in Discontinuous Boundary Value Problems of the Theory of Holomorphic Functions of One Complex Variable" (Modern Problems of Mathematics, vol. 7, M., 1975, 5-162; English translation in: "J. Sov. Math.", 7(1977), 309-414). This and the above-mentioned work published in "Proceedings of A. Razmadze Mathematical Institute" are up to the present days the handbooks of many specialists engaged in this area. It is difficult to find research works in the boundary value problems of the function theory and singular integral equations lack of references to the works of Boris Khvedelidze. Besides numerous works in this subject, there are more than ten monographs.

In 1967 B. Khvedelidze was elected Corresponding Member and in 1983 Full Member of the Georgian Academy of Sciences.

In is not easy to list his vast scientific, pedagogical and public activity he led for many dozens of years. From 1957 to 1986 he headed department of Function Theory and Functional Analysis at A. Razmadze Mathematical Institute, and from 1986 to the end of his life he was the head of organized by his initiative department of the Methods of Complex Analysis.

B. Khvedelidze was the head of Organizing Committee of the Georgian Mathematical Society and repeatedly he was elected its Vice-President. For many years he was at the head of the Chair of Higher Mathematics at Georgian Polytechnical Institute. B. Khvedelidze made an important contribution to the formation of Abkhazian State University and Sukhumi branch of the Tbilisi State University, being one of the leading professors from the day of its foundation.

Together with A. Kharadze, V. Chelidze and I. Kartsivadze, he was the author of the course in mathematical analysis in Georgian language which has played an important role in formation of many generations of Georgian mathematicians. It is no less important to mention B. Khvedelidze's remarkable human gualities-tenderness, benevolence, willingness to render assistance. B. Khvedelide was always strong-willed, with fortitude he endured vital confusions. In the years of Soviet repressions his family was deported to the South Kazakhstan (his nephew after the war has stayed in France). For a year and a half he was a teacher at zooveterinary technical school in the town of Kaplanbec. Thanks to his friends, who supplied him with the needed literature, he was able to continue scientific work even in exile. With a great gratitude he recalled the fact that his own library gifted by him before exile to the Tbilisi State University has been given him back after exile.

It should be noted that in days when scientific circles mark Boris Khvedelidze's centenary, officials of A. Razmadze Mathematical Institute intensively continue research work in new aspects of boundary value problems of the function theory, theory of operators and singular integral equations. Recently, several monographs in this direction have been published by the research workers of the institute.

Boris Khvedelidze passed away on March 27, 1993.

His name once again reminds us of his veritable professionalism, rear pedagogical talent, devotion to his people and work.

Blessed memory on the well-known scientist and wonderful person will for a long time remain in the hearts.

V. Kokilashvili and V. Paatashvili

List of Publications of B. Khvedelidze

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ON ONE NONLINEAR ANALOGUE OF THE DARBOUX PROBLEM

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Abstract. For one nonlinear oscillation equation, we consider a problem which is a nonlinear analogue of the Darboux problem and consists in the simultaneous determination of a regular solution and its domain of definition. The question of solvability of the formulated problem is solved by the method of characteristics.

რეზიუმე. არაწრფივი რხევების ერთი განტოლებისათვის განხილულია ამოცანა, რომელიც წარმოადგენს დარბუს ამოცანის არაწრფივ ანალოგს და მოითხოვს რეგულარული ამოხსნისა და მისი განსაზღვრის არის ერთდროულად დადგენას. განხილული ამოცანის ამოხსნადობის პრობლემა გადაჭრილია მახასიათებელთა მეთოდით.

As is known, the carrier of the initial characteristic Darboux problem for linear equations consists of two curves drawn from the common point of these curves [1]. One of these curves is characteristic, and the other has nowhere characteristic direction.

The characteristics of linear hyperbolic equations are completely defined by means of principal coefficients. In nonlinear cases these coefficients already depend on a sought solution and its lower derivatives. Since the characteristics, too, depend on them, the linear formulation of a Darboux characteristic problem cannot be automatically extended to the case of nonlinear equations which are of particular interest from the standpoint of application [2]. Therefore the formulations of Darboux problems for such equations should be revised with regard for general characteristic invariants [3]–[7].

In this paper, an attempt is made to formulate correctly a partially characteristic problem for a quasilinear equation, which arises when studying

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nonlinear oscillations

$$x^2(u_y^4 u_{xx} - u_{yy}) = cuu_y^4, \quad c = \text{const}\,. \tag{1}$$

The particular case of equation (1) for the purpose of outfitting warships by order of Pentagon has been investigated in [8], [9]. The general solution of the equation has been constructed for $c = 0$.

Equation (1) is interesting by the degeneracy of its order and, perhaps, by the hyperbolicity, too. The former is completely defined and occurs on the coordinate axis. The parabolic degeneracy [10] depends on the behavior of the derivative u_y of an unknown solution $u(x, y)$. Hence the set of points of this degeneracy is not a priori prescribed in this case and has to be defined simultaneously with a solution.

Since the set of points of parabolic degeneracy and the characteristics are not defined by the equation, they have to be defined by the conditions of the problem. For this we need all characteristic rules of equation (1).

The characteristic roots of equation (1)

$$\lambda_1 = u_y^{-2}, \quad \lambda_2 = -u_y^{-2}$$

provide differential relations of characteristic directions

$$u_y^2 dy - dx = 0, \quad u_y^2 dy + dx = 0. \tag{2}$$

If, taking into account (2), we consider equation (1), we come to the differential characteristic relations

$$x^2 u_y^4\, du_x - x^2 u_y^2\, du_y - cu_y^4 u\, dx = 0, \quad x^2 u_y^4\, du_x + x^2 u_y^2\, du_y - cu_y^4 u\, dx = 0.$$

The following theorem [11] is true.

Theorem 1. *Assuming $c > -\frac{1}{4}$, each of the characteristic systems of equation* (1) *admits exactly two first integrals, and they are represented explicitly as*

$$\begin{cases} \xi \equiv (u_y^{-1} + u_x)x^{\alpha} - \alpha u x^{\alpha-1}, \\ \xi_1 \equiv (u_y^{-1} + u_x)x^{1-\alpha} - (1-\alpha)ux^{-\alpha}, \end{cases} \tag{3}$$

for the family of the root λ_1, and as

$$\begin{cases} \eta \equiv (u_y^{-1} - u_x)x^{\alpha} + \alpha u x^{\alpha-1}, \\ \eta_1 \equiv (u_y^{-1} - u_x)x^{1-\alpha} + (1-\alpha)ux^{-\alpha}, \quad \alpha = \dfrac{1}{2}\left(1 + \sqrt{4c+1}\right), \end{cases} \tag{4}$$

for the family of the root λ_2.

By virtue of these two pairs of first integrals (ξ, ξ_1) and (η, η_1), which are actually characteristic invariants, it follows that in the class of hyperbolic solutions we can construct two intermediate integrals

$$\xi_1 = \varphi'(\xi), \quad \eta_1 = \psi'(\eta)$$

of equation (1) [12]. In these integrals, φ, ψ are arbitrary smooth functions such that they ensure the differentiability of the sought solution up to the second order.

Theorem 2. *If $\varphi, \psi \in C^3(R)$, then equation* (1) *is equivalent to the triple of the following relations* [11]

$$x = \Big(\frac{\varphi'(\xi) + \psi'(\eta)}{\xi + \eta}\Big)^{\frac{1}{1-2\alpha}}, \tag{5}$$

$$y = \frac{1}{4(1-2\alpha)}\Big[(\xi+\eta)\big(\psi'(\eta) - \varphi'(\xi)\big) + 2\big(\varphi(\xi) - \psi(\eta)\big)\Big], \tag{6}$$

$$u = \frac{1}{1-2\alpha}\bigg[\xi\Big(\frac{\varphi'(\xi) + \psi'(\eta)}{\xi + \eta}\Big)^{\frac{1-\alpha}{1-2\alpha}} - \varphi'(\xi)\Big(\frac{\varphi'(\xi) + \psi'(\eta)}{\xi + \eta}\Big)^{\frac{\alpha}{1-2\alpha}}\bigg]. \tag{7}$$

To relations (5)–(7) we come from equation (1) without any additional conditions. By removing arbitrary parameters φ, ψ, from these relations we return to equation (1). Hence this triple of relations can be taken as a general integral of equation (1), and the invariants ξ, η as characteristic variables.

However, the above-constructed general integral (5)–(7) does not define in any way at least one characteristic of either of the families in order to take it as a data carrier of a mixed characteristic problem. Hence we have to choose such a characteristic arbitrarily, at our discretion. Suppose it is some arc γ given in explicit form

$$\gamma: \ y = g(x), \quad 0 < a \le x \le b, \quad g \in C^3(R). \tag{8}$$

The function g is assumed to be strictly monotonic, and the arc γ to be ascending. Without loss of generality, it can be assumed that

$$g(a) = 0.$$

Let the function h given on some segment $[a, d]$ be twice continuously differentiable and contracting this segment to the segment $[a, b]$. It is assumed that h satisfies the conditions

$$h(a) = a, \quad h(d) = b, \ \ h' < 0.$$

The Problem. Find a regular hyperbolic solution $u(x, y)$ of equation (1) and define simultaneously its domain of definition if along this solution the curve γ is characteristic, the solution itself satisfies the conditions

$$u(a, 0) = \mu, \quad u_x(a, 0) = \theta, \tag{9}$$

and each pair of points $(x, 0)$, $(h(x), g(h(x)))$ connected with the mapping of h belongs to the respective general characteristic of the family of the root λ_2.

According to the formulation of the problem, the curve γ is actually attributed to the family of characteristics of the root λ_1. This is equivalent to the equality

$$g'(x) = u_y^{-2}(x, g(x)). \tag{10}$$

Thus we can define two variants of values of the derivative u_y along the curve γ:

$$u_y = \frac{1}{\pm\sqrt{g'(x)}}\,. \tag{11}$$

Of them we choose the arithmetic value of the root. The reasoning for the other root is analogous. To solve the problem, along with (11) we also need to define on the arc γ the values of a solution u and its derivative u_x. To this end, we have to use the characteristic invariants ξ, ξ_1 of the family λ_1. The values of u and u_x at the initial point $(a, 0)$ of the curve γ are known. Using (9), (11), we calculate the characteristic invariants ξ, ξ_1 at the point $(a, 0)$ for which we introduce the notation

$$\begin{aligned} \xi\big|_{(a,0)} &= \big(\sqrt{g'(a)} + \theta\big)a^{\alpha} - \alpha\mu a^{\alpha-1} \equiv [\xi]_a, \\ \xi_1\big|_{(a,0)} &= \big(\sqrt{g'(a)} + \theta\big)a^{1-\alpha} - (1-\alpha)\mu a^{-\alpha} \equiv [\xi_1]_a. \end{aligned} \tag{12}$$

Since the characteristic invariants ξ, ξ_1 take constant values along γ, we have

$$\begin{aligned} \big[(u_y^{-1} + u_x)x^{\alpha} - \alpha u x^{\alpha-1}\big]\Big|_{\gamma} &= [\xi]_a, \\ \big[(u_y^{-1} + u_x)x^{1-\alpha} - (1-\alpha) u x^{-\alpha}\big]\Big|_{\gamma} &= [\xi_1]_a. \end{aligned}$$

Considering these two relations as a system relative to u and u_x, we define their values on γ as follows:

$$\begin{aligned} u\big|_{\gamma} &= \frac{1}{2-\alpha}\Big[x^{1-\alpha}[\xi]_a - [\xi_1]_a x^{\alpha}\Big], \\ u_x\big|_{\gamma} &= \frac{1-\alpha}{2-\alpha}\,[\xi]_a x^{-\alpha} - \frac{\alpha}{1-2\alpha}\,[\xi_1]_a x^{\alpha-1} - \sqrt{g'(x)}\,. \end{aligned}$$

Thus we have succeeded in defining the values of the sought solution and its first order derivatives all over the characteristic γ. Using these values, we can define the solution and its first order derivatives outside γ and establish the limits of their propagation.

To define the values of $u(x, 0)$, $u_x(x, 0)$ and $u_y(x, 0)$, from an arbitrary point $P(x, 0)$, $a < x \le d$, we draw the characteristic Γ of the family of the root λ_2, which by the conditions of problem (1), (9) intersects the characteristic γ at the point $N(h(x), g(h(x)))$. The invariants η and η_1 must be constant along the characteristic Γ.

Since the values of the invariants η, η_1 at the point N

$$\eta\big|_N = 2\sqrt{g'(h(x))}\, h^{\alpha}(x) - [\xi]_a,$$
$$\eta_1\big|_N = 2\sqrt{g'(h(x))}\, h^{1-\alpha}(x) - [\xi_1]_a,$$

remain unchanged all over the characteristic Γ, the point $(x,0)$ inclusive, the equalities

$$\eta(x,0) = \eta\big|_N, \quad \eta_1(x,0) = \eta_1\big|_N$$

will be fulfilled. These invariants can be written in the explicit form

$$\begin{aligned}[\eta]_x \equiv & \big(u_y^{-1}(x,0) - u_x(x,0)\big)x^{\alpha} + \alpha u(x,0)x^{\alpha-1} = \\ & = 2\sqrt{g'(h(x))}\, h^{\alpha}(x) - [\xi]_a, \end{aligned} \tag{13}$$

$$\begin{aligned}[\eta_1]_x \equiv & \big(u_y^{-1}(x,0) - u_x(x,0)\big)x^{1-\alpha} + (1-\alpha) u(x,0)x^{-\alpha} = \\ & = 2\sqrt{g'(h(x))}\, h^{1-\alpha}(x) - [\xi_1]_a. \end{aligned} \tag{14}$$

We take these equalities as a linear algebraic system and define the sought solution at an arbitrary point $(x,0)$ of the segment $[a,d]$

$$\begin{aligned} u(x,0) = & \frac{2}{2\alpha-1}\, \sqrt{g'(h(x))}\, \big(h^{\alpha}(x)x^{1-\alpha} - h^{1-\alpha}(x)x^{\alpha}\big) - \\ & - \frac{1}{2\alpha-1}\, [\xi]_a x^{1-\alpha} + \frac{1}{2\alpha-1}\, [\xi_1]_a x^{\alpha}. \end{aligned} \tag{15}$$

This is quite sufficient in order to define at the same points the first order derivatives $u_x(x,0)$ and $u_y(x,0)$ of the sought solution u. The derivative u_x is obtained by direct differentiation of (15)

$$\begin{aligned} u_x(x,0) = & \frac{1}{2\alpha-1}\, \frac{g''(h(x))\cdot h'(x)}{\sqrt{g'(h(x))}}\, \big(h^{\alpha}(x)x^{1-\alpha} - h^{1-\alpha}(x)x^{\alpha}\big) + \\ & + \frac{2}{2\alpha-1}\, \sqrt{g'(h(x))}\, \Big(\alpha h^{\alpha-1}(x)h'(x)x^{1-\alpha} + (1-\alpha)h^{\alpha}(x)x^{-\alpha} - \\ & \qquad - (1-\alpha)h^{-\alpha}(x)h'(x)x^{\alpha} - \alpha h^{1-\alpha}(x)x^{\alpha-1}\Big) - \\ & - \frac{1-\alpha}{2\alpha-1}\, [\xi]_a x^{-\alpha} + \frac{\alpha}{2\alpha-1}\, [\xi_1]_a x^{\alpha-1}. \end{aligned} \tag{16}$$

The other derivative u_y is defined by substituting (15), (16) into (13) or (14)

$$u_y(x,0)=\left\{\frac{2\sqrt{g'(h(x))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(x)x^{-\alpha}+(1-\alpha)h^{1-\alpha}(x)x^{\alpha-1}+ \right.$$
$$+\alpha h^{\alpha-1}(x)h'(x)x^{1-\alpha}-(1-\alpha)h^{-\alpha}(x)h'(x)x^{\alpha}\Big]+\frac{1}{2\alpha-1}\times$$
$$\left.\times\frac{g''(h(x))\cdot h'(x)}{\sqrt{g'(h(x))}}\left(h^{\alpha}(x)x^{1-\alpha}-h^{1-\alpha}(x)x^{\alpha}\right)+\frac{1-\alpha}{2\alpha-1}[\xi]_a x^{\alpha}\right\}^{-1}. \quad (17)$$

Because of the nonlinearity of equation (1) and depending on the $u_x(x,0)$, $u_y(x,0)$, the segment $[a,d]$ may turn out to be the characteristic of either of the families. This is the cause for which the problem under consideration may be ill-posed or even unsolvable.

In order to avoid transformation of the segment $[a,d]$ to the characteristic, we should find the conditions ensuring an a priori estimate

$$0<|u_y(x,0)|<\infty.$$

It is understood that these conditions should be expressed in terms g, h.

The above estimate excludes for $y=0$ not only the characteristic direction of the carrier, but also the parabolic degeneracy of equation (1).

The assumptions $h\in C^1(\overline{J})$, $g\in C^2(\overline{I})$, $J\equiv(a,d)$, $I\equiv(a,b)$ ensure the fulfillment of the condition

$$u_y(x,0)\neq 0,\quad x\in\overline{J}$$

and the existence of minimal and maximal values of the functions g', $|g''|$ on $\overline{I}$ and of $|h|$ on $\overline{J}$. We denote by n the smallest of minimal values and by N the largest of maximal values. We obtain the estimate

$$|u_y(x,0)|<+\infty,\quad x\in\overline{J}$$

if one of the following conditions

$$2\sqrt{N}\left[(\alpha-1)\Big(\frac{a}{d}\Big)^{\alpha}-(\alpha-1)\Big(\frac{d}{a}\Big)^{\alpha-1}-\alpha\Big(\frac{b}{a}\Big)^{\alpha-1}N-(\alpha-1)\Big(\frac{d}{a}\Big)^{\alpha}N\right]-$$
$$-\frac{N^2}{\sqrt{n}}\left(d^{\alpha}a^{1-\alpha}-d^{1-\alpha}a^{\alpha}\right)-(\alpha-1)[\xi]_a a^{\alpha}>0, \quad (18)$$

is fulfilled, where the value $[\xi]_a$ is given by formula (12), and $\alpha>1$;

$$\sqrt{N}\,\eta_*(1+\operatorname{sgn}\eta_*)+\sqrt{n}\,\eta_*(1-\operatorname{sgn}\eta_*)+\frac{N^2}{\sqrt{n}}\left(d^{\alpha}a^{1-\alpha}-d^{1-\alpha}a^{\alpha}\right)-$$
$$-(\alpha-1)\xi^{[a]}\Big(a^{\alpha}\,\frac{1+\operatorname{sgn}[\xi]_a}{2}+d^{\alpha}\,\frac{1-\operatorname{sgn}[\xi]_a}{2}\Big)<0, \quad (19)$$

$$\eta_* \equiv (\alpha-1)\Big(\frac{b}{a}\Big)^{\alpha} - (\alpha-1)\Big(\frac{a}{b}\Big)^{\alpha-1} - \alpha\Big(\frac{a}{d}\Big)^{\alpha-1} n - (\alpha-1)\Big(\frac{a}{b}\Big)^{\alpha} n, \quad \alpha>1,$$

$$g''(x) \le 0, \quad x \in \overline{I}, \quad \alpha = 1; \tag{20}$$

$$g''(x) > 0, \quad -2n^2 + N^2(d-a) < 0, \quad x \in \overline{I}, \quad \alpha = 1; \tag{21}$$

$$\sqrt{N}\,\eta^*(1-\operatorname{sgn}\eta^*) + \sqrt{n}\,\eta^*(1+\operatorname{sgn}\eta^*) - \frac{N^2}{\sqrt{n}}(d-a) + \\ +(1-\alpha)\xi^{[a]}\Big[a^{\alpha}\,\frac{1+\operatorname{sgn}[\xi]_a}{2} + d^{\alpha}\,\frac{1-\operatorname{sgn}[\xi]_a}{2}\Big] > 0, \tag{22}$$

$$\eta^* \equiv (\alpha-1)\Big(\frac{b}{a}\Big)^{\alpha} + (1-\alpha)\Big(\frac{a}{d}\Big)^{1-\alpha} - \alpha\Big(\frac{d}{a}\Big)^{1-\alpha} N + \\ + (1-\alpha)\Big(\frac{a}{d}\Big)^{\alpha} n, \quad \frac{1}{2} < \alpha < 1;$$

$$\eta_0\sqrt{n}\,(1-\operatorname{sgn}\eta_0) + \eta_0\sqrt{N}\,(1+\operatorname{sgn}\eta_0) + \frac{n^2}{\sqrt{n}}(d-a) + \\ + (1-\alpha)[\xi]_a\Big(d^{\alpha}\,\frac{1+\operatorname{sgn}[\xi]_a}{2} + a^{\alpha}\,\frac{1-\operatorname{sgn}[\xi]_a}{2}\Big) < 0, \tag{23}$$

$$\eta_0 \equiv (\alpha-1)\Big(\frac{a}{d}\Big)^{\alpha} + (1-\alpha)\Big(\frac{b}{a}\Big)^{1-\alpha} - \alpha\Big(\frac{a}{b}\Big)^{1-\alpha} n + \\ +(1-\alpha)\Big(\frac{d}{a}\Big)^{\alpha} N, \quad \frac{1}{2} < \alpha < 1.$$

Let $(\rho,0)$ and $(\sigma,0)$ be arbitrarily chosen points from the segment $[a,d]$.

Using the values of $u(\sigma,0)$, $u_x(\sigma,0)$, $u_y(\sigma,0)$, we define the constants $[\xi]_\sigma$, $[\xi_1]_\sigma$, whose values must coincide with the invariants ξ, ξ_1 on the unknown yet characteristic Γ_1 of the family of the root λ_1 drawn from the point $A(\sigma,0)$. Assume that this characteristic is given by the formula $y=m(x)$, where the function m is to be defined. Then on this curve, we have

$$\xi\big|_{\Gamma_1} = \big(u_y^{-1}(x,m(x)) + u_x(x,m(x))\big)x^{\alpha} - \alpha u(x,m(x))x^{\alpha-1} = [\xi]_\sigma, \tag{24}$$

$$\xi_1\big|_{\Gamma_1} = \big(u_y^{-1}(x,m(x)) + u_x(x,m(x))\big)x^{1-\alpha} - \\ -(1-\alpha)u(x,m(x))x^{-\alpha} = [\xi_1]_\sigma, \tag{25}$$

where

$$[\xi]_\sigma = \xi\big|_A = \frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(\sigma) + (1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-1} - \\ - \alpha h^{\alpha-1}(\sigma)h'(\sigma)\sigma^{1-\alpha} - (1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha}\Big] + \\ + \frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\big(h^{\alpha}(\sigma)\sigma - h^{1-\alpha}\sigma^{2\alpha}\big) +$$

$$
\begin{aligned}
&+\frac{2-2\alpha}{2\alpha-1}\,[\xi]_a\sigma^{2\alpha}-2\sqrt{g'(h(\sigma))}\,h^{\alpha}(\sigma)+[\xi]_a,\\
[\xi_1]_\sigma=\xi_1\big|_A=&\ \frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(\sigma)\sigma^{-1}+(1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-2}+\\
&+\alpha h^{\alpha-1}(\sigma)h'(\sigma)-(1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha-1}\Big]+\\
&+\frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\big(h^{\alpha}(\sigma)-h^{1-\alpha}(\sigma)\sigma^{2\alpha-1}\big)+\\
&+\frac{2-2\alpha}{2\alpha-1}\,[\xi]_a\sigma^{2\alpha-1}-2\sqrt{g(h(\sigma))}+[\xi_1]_a.
\end{aligned}
$$

In an absolutely analogous manner we define the invariants η, η_1 on the characteristic Γ_2 of the other family drawn from the point $B(\rho,0)$. Assume that this characteristic is given by the equation $y=\ell(x)$, where ℓ is an unknown yet function. Thus we have

$$
\eta\big|_{\Gamma_2}=\big(u_y^{-1}(x,\ell(x))-u_x(x,\ell(x))\big)x^{\alpha}+\alpha u(x,\ell(x))x^{\alpha-1}=[\eta]_\rho, \tag{26}
$$

$$
\begin{aligned}
\eta_1\big|_{\Gamma_2}=&\big(u_y^{-1}(x,\ell(x))-u_x(x,\ell(x))\big)x^{1-\alpha}+\\
&+(1-\alpha)u(x,\ell(x))x^{-\alpha}=[\eta_1]_\rho,
\end{aligned} \tag{27}
$$

where

$$
\begin{aligned}
[\eta]_\rho&=2\sqrt{g'(h(\rho))}\,h^{\alpha}(\rho)-[\xi]_a,\\
[\eta_1]_\rho&=2\sqrt{g'(h(\rho))}\,h^{1-\alpha}(\rho)-[\xi_1]_a.
\end{aligned}
$$

At the intersection point (x_1,y_1) of these characteristics, if such a point exists, conditions (24)–(27) and $\ell(x_1)=m(x_1)$ must be fulfilled simultaneously. Therefore in the left-hand parts of (24), (25) we can replace $m(x_1)$ by $\ell(x_1)$. As a result, we obtain the following system for defining the values of x, u, u_x, u_y at the point $C(x_1,\ell(x_1))$

$$
\big(u_y^{-1}(x_1,\ell(x_1))+u_x(x_1,\ell(x_1))\big)x_1^{\alpha}-\alpha u(x_1,\ell(x_1))x_1^{\alpha-1}=[\xi]_\sigma, \tag{28}
$$

$$
\big(u_y^{-1}(x_1,\ell(x_1))+u_x(x_1,\ell(x_1))\big)x_1^{1-\alpha}-(1-\alpha)u(x_1,\ell(x_1))x_1^{-\alpha}=[\xi_1]_\sigma, \tag{29}
$$

$$
\big(u_y^{-1}(x_1,\ell(x_1))-u_x(x_1,\ell(x_1))\big)x_1^{\alpha}+\alpha u(x_1,\ell(x_1))x_1^{\alpha-1}=[\eta]_\rho, \tag{30}
$$

$$
\big(u_y^{-1}(x_1,\ell(x_1))-u_x(x_1,\ell(x_1))\big)x_1^{1-\alpha}+(1-\alpha)u(x_1,\ell(x_1))x_1^{-\alpha}=[\eta_1]_\rho. \tag{31}
$$

Taking these equalities as a linear algebraic system, we define the values of the abscissa x_1 of the intersection point C of the characteristics Γ_1 and Γ_2, and also of the sought solution $u(x_1,\ell(x_1))$ together with its first order derivatives $u_x(x_1,\ell(x_1))$ and $u_y(x_1,\ell(x_1))$.

So far ρ, σ have been chosen arbitrarily on the segment $[a,d]$ and it has been through them that we have defined the coordinates (x_1,y_1) of the intersection point of the characteristics. Now, if we assume that they run through this segment, we obtain the set of intersection points of the characteristics drawn from all possible pairs of points $(\rho, 0)$, $(\sigma, 0)$. That is why in the notations of solutions of the algebraic system (28)–(31) we omit the indexes

$$x = X(\rho, \sigma), \tag{32}$$

$$u = U(\rho, \sigma), \tag{33}$$

$$u_x = P(\rho, \sigma), \tag{34}$$

$$u_y = Q(\rho, \sigma), \tag{35}$$

where

$$\begin{aligned}
X(\rho,\sigma) =&\Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{\frac{1}{2\alpha-1}},\\
U(\rho,\sigma) =&\frac{1}{1-2\alpha}\left[\Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{\frac{1-\alpha}{2\alpha-1}}[\xi]_\sigma - [\xi_1]_\sigma\Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{\frac{\alpha}{2\alpha-1}}\right],\\
P(\rho,\sigma) =&\Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{-\frac{\alpha}{2\alpha-1}}\Big(\frac{1}{2-4\alpha}[\xi]_\sigma -\\
&-\frac{1}{2}[\eta]_\rho\Big) - \frac{\alpha}{1-2\alpha}[\xi_1]_\sigma\Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{\frac{\alpha-1}{2\alpha-1}},\\
Q(\rho,\sigma) =&2\Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{\frac{\alpha}{2\alpha-1}}\big([\xi]_\sigma + [\eta]_\rho\big)^{-1}.
\end{aligned}$$

To describe the structure of this set of points, we must express the ordinate y as a function of arguments ρ, σ, in the same way as all other were represented by formulas (32)–(35). To construct the function $y = Y(\rho, \sigma)$, we need the explicit representations of X and Q in the form

$$\begin{aligned}
X(\rho,\sigma) = &\left\{\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^\alpha(\sigma) + (1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-1} -\right.\\
&-\alpha h^{\alpha-1}(\sigma)h'(\sigma)\sigma^{1-\alpha} - (1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha}\Big] +\\
&+\frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\big(h^\alpha(\sigma)\sigma - h^{1-\alpha}(\sigma)\sigma^{2\alpha}\big) + \frac{2-2\alpha}{2\alpha-1}[\xi]_a\sigma^{2\alpha} -\\
&\left.-2\sqrt{g'(h(\sigma))}\,h^\alpha(\sigma) + 2\sqrt{g'(h(\rho))}\,h^\alpha(\rho)\right\}^{\frac{1}{2\alpha-1}}\times\\
&\times\left\{\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^\alpha(\sigma) + (1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-1} -\right.\\
&-\alpha h^{\alpha-1}(\sigma)h'(\sigma)\sigma^{1-\alpha} - (1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha}\Big] +\\
&+\frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\big(h^\alpha(\sigma)\sigma - h^{1-\alpha}(\sigma)\sigma^{2\alpha}\big) + \frac{2-2\alpha}{2\alpha-1}[\xi]_a\sigma^{2\alpha} -
\end{aligned}$$

$$
-2\sqrt{g'(h(\sigma))}\,h^{1-\alpha}(\sigma)+2\sqrt{g'(h(\rho))}\,h^{1-\alpha}(\rho)\Bigg\}^{\frac{1}{1-2\alpha}}, \tag{36}
$$

$$
\begin{aligned}
Q(\rho,\sigma)=2\Bigg\{&\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(\sigma)+(1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-1}-\\
&-\alpha h^{\alpha-1}(\sigma)h'(\sigma)\sigma^{1-\alpha}-(1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha}\Big]+\\
&+\frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\left(h^{\alpha}(\sigma)\sigma-h^{1-\alpha}(\sigma)\sigma^{2\alpha}\right)+\frac{2-2\alpha}{2\alpha-1}\,[\xi]_a\sigma^{2\alpha}-\\
&-2\sqrt{g'(h(\sigma))}\,h^{\alpha}(\sigma)+2\sqrt{g'(h(\rho))}\,h^{\alpha}(\rho)\Bigg\}^{\frac{\alpha}{2\alpha-1}}\times\\
\times\Bigg\{&\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(\sigma)+(1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-1}-\\
&-\alpha h^{\alpha-1}(\sigma)h'(\sigma)\sigma^{1-\alpha}-(1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha}\Big]+\\
&+\frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\left(h^{\alpha}(\sigma)\sigma-h^{1-\alpha}(\sigma)\sigma^{2\alpha}\right)+\frac{2-2\alpha}{2\alpha-1}\,[\xi]_a\sigma^{2\alpha}-\\
&-2\sqrt{g'(h(\sigma))}\,h^{1-\alpha}(\sigma)+2\sqrt{g'(h(\rho))}\,h^{1-\alpha}(\rho)\Bigg\}^{\frac{\alpha}{1-2\alpha}}\times\\
\times\Bigg\{&\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(\sigma)\sigma^{-1}+(1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-2}+\\
&+\alpha h^{\alpha-1}(\sigma)h'(\sigma)-(1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha-1}\Big]+\\
&+\frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\left(h^{\alpha}(\sigma)-h^{1-\alpha}(\sigma)\sigma^{2\alpha-1}\right)+\frac{2-2\alpha}{2\alpha-1}\,[\xi]_a\sigma^{2\alpha-1}-\\
&-2\sqrt{g'(h(\sigma))}\,h^{1-\alpha}(\sigma)+2\sqrt{g'(h(\rho))}\,h^{\alpha}(\rho)\Bigg\}^{-1}.
\end{aligned} \tag{37}
$$

To define the function m, the equation of the characteristic Γ_1 is formally written in the form

$$
y=m(x)=m[X(\rho,\overline{\sigma})]\equiv M(\rho,\overline{\sigma}),
$$

where the little line over the letter means this value is constant. The direction of this characteristic is defined by the root λ_1 or, in other words, by the values of the derivative $u_y=Q(\overline{\sigma},\rho)$. Therefore we have

$$
\frac{dm(X(\rho,\overline{\sigma}))}{dX(\rho,\overline{\sigma})}=\frac{dm(X(\rho,\overline{\sigma}))}{X'_{\rho}(\rho,\overline{\sigma})\,d\rho}=\frac{1}{Q^2(\rho,\overline{\sigma})}
$$

or, which is the same,

$$\frac{dM(\rho,\overline{\sigma})}{d\rho}=\frac{X'_\rho(\rho,\overline{\sigma})}{Q^2(\rho,\overline{\sigma})}\,. \tag{38}$$

Hence by integration we obtain

$$M(\rho,\overline{\sigma})=\int\limits_a^\rho \frac{X'_t(t,\overline{\sigma})}{Q^2(t,\overline{\sigma})}\,dt+M(a,\overline{\sigma}),\quad \rho\in[a,d],$$

where the value $M(a,\overline{\sigma})$ is unknown yet.

By an analogous reasoning, using the notation $y=\ell(x)=\ell\big[X(\overline{\rho},\sigma)\big]\equiv L(\overline{\rho},\sigma)$ and taking into account the direction of the characteristic Γ_2 defined by the root λ_2, we obtain

$$\frac{dL(\overline{\rho},\sigma)}{d\sigma}=-\frac{X'_\sigma(\overline{\rho},\sigma)}{Q^2(\overline{\rho},\sigma)}$$

and

$$L(\overline{\rho},\sigma)=-\int\limits_a^\sigma \frac{X_z(\overline{\rho},z)}{Q^2(\overline{\rho},z)}\,dz+L(\overline{\rho},a),\quad \sigma\in[a,d],$$

where $L(\overline{\rho},a)$ is not known either and has to be defined.

To define the unknown values, note that $L(\overline{\rho},a)$ is the value of L at the intersection point of the characteristics γ and Γ_2. Therefore

$$L(\overline{\rho},a)=g(h(\overline{\rho})),$$

and

$$M(a,\overline{\sigma})=L(a,\overline{\sigma})=-\int\limits_a^{\overline{\sigma}} \frac{X'_z(a,z)}{Q^2(a,z)}\,dz+g(h(a)),$$

where $g(h(a))=g(a)=0$.

In defining the characteristics of the families of the roots λ_1 and λ_2, the functions $M(\rho,\overline{\sigma})$ and $L(\overline{\rho},\sigma)$ are given by the equalities

$$M(\rho,\overline{\sigma})=\int\limits_a^\rho \frac{X'_t(t,\overline{\sigma})}{Q^2(t,\overline{\sigma})}\,dt-\int\limits_a^{\overline{\sigma}} \frac{X'_z(a,z)}{Q^2(a,z)}\,dz \tag{39}$$

with an argument $\rho\in[a,d]$ and a parameter $\overline{\sigma}\in[a,d]$, and

$$L(\overline{\rho},\sigma)=-\int\limits_a^\sigma \frac{X'_z(\overline{\rho},z)}{Q^2(\overline{\rho},z)}\,dz+g(h(\overline{\rho})), \tag{40}$$

with a variable $\sigma\in[a,d]$ and a parameter $\overline{\rho}\in[a,d]$.

Thus the integral of problem (1), (9) is given by formulas (32), (33) and

$$y=Y(\rho,\sigma), \tag{41}$$

where

$$Y(\rho,\sigma) = -\int\limits_a^\sigma \frac{X_z'(\rho,z)}{Q^2(\rho,z)}\,dz + g(h(\rho)),$$

and the variables $\rho,\sigma \in [a,d]$.

The domain of definition D of the solution of problem (1), (9) is completely defined by relations (32), (41), where expressions of x, y depend on ρ, σ. The values of these functions are treated as the current coordinates describing the domain D.

The domain of definition of the solution of the problem under consideration is bounded by four characteristics. The first of them which is an arch of the curve γ is given by the condition of the problem. The other characteristics are represented parametrically. In our representations we take as parameters the values ρ, σ of the abscissa of the intersection points through which these characteristics pass:

$$\Gamma_3: \quad x = X(d,\sigma), \quad y = L(d,\sigma), \tag{42}$$

$$\Gamma_4: \quad x = X(\rho,d), \quad y = M(\rho,d), \tag{43}$$

$$\Gamma_5: \quad x = X(a,\sigma), \quad y = L(a,\sigma), \tag{44}$$

where the functions X, M, L are given by (36), (39), (40).

Such is the structure of the domain of definition of the solution of the problem when the values of the derivative u_y on the arc γ in formula (11) are defined by the positive root. The domain has the same kind of structure when the root in (11) is negative. The latter case is investigated by analogy with the preceding case.

Thus the following theorem is valid.

Theorem 3. *If along the curve γ it occurs that $u_y > 0$, then under the conditions* (18)–(23) *there exists the solution of problem* (1), (9) *given by the formulas* (32), (33), (41). *The domain D of the solution is bounded by the arcs of characteristic curves* (8), (42), (43) *and* (44).

The case $u_y > 0$ along the curve γ can be studied in a similar way.

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OPTIMIZATION OF A STATE FINANCING MODEL OF VOCATIONAL COLLEGES

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Abstract. The paper presents the results of the quantitative sociological research. The basic factors which essentially determine the attitude of students to vocational education are identified and analyzed, which on its part is one of the important prerequisites for the formation of a rating system of vocational colleges and for the optimization of the model of their financial support by the state.

რეზიუმე. სტატიაში წარმოდგენილია რაოდენობრივი სოციოლოგიური კვლევის შედეგები. მათ შორის, იდენტიფიცირებული და გაანალიზებულია ძირითადი ფაქტორები რომლებიც არსებითად განსაზღვრავს სტუდენტთა დამოკიდებულებას პროფესიული სასწავლებლის მიმართ, რაც, თავის მხრივ, პროფესიული სასწავლებლების სარეიტინგო სისტემის ფორმირებისა და სახელმწიფო დაფინანსების მოდელის ოპტიმიზების ერთ-ერთ მნიშვნელოვან წინაპირობას წარმოადგენს.

Post-Soviet Georgia is faced with complex and multivariate challenges of modernization. In the process of integration into the modern democratic world and formation of an independent state one of the most urgent tasks is to make the education system match the up-to-date international standards. Proceeding from the fact that the reform of the education system takes place against the background of acute economic problems, it is very important to use such methods that are directed not to additional investments but to an optimal distribution of the existing resources, which will make it possible to get a maximal effect at minimal expenditures.

Resolution of the Government of Georgia, No. 244 [1] dated September 19, 1913 (On the Determination of the Order and Terms of the Financing of Vocational Education and Confirmation of a Maximal Payment for the Study in Vocational Education Programs) established the rules and terms

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of a financial support given by the Government to the vocational education in Georgia.

According to these rules, every student receives–according to his profession–a voucher of financial support that exceeds a minimal payment amount needed for his training. As a result, after students finish their training course, a certain amount of money ("surplus") remains in the possession of the college which the latter can spend at its discretion for the purpose of development of the training process. If the students of the college under investigation are regarded as a united set, then their financial "surplus" at the current moment of time is "scattered" as a non-uniform amount among various educational institutions–so, there does not exist a unified mechanism of surplus calculation and expenditure.

The training of a student in one and the same profession in various colleges may involve different cost sums and, accordingly, the surplus amount that remains can be different. Moreover, there does not exist a monitoring mechanism by means of which we could evaluate how effectively the optimal sum was spent. It is not excluded that a minimal expenditure will negatively affect the quality of training or a maximal expenditure does not at all mean a better quality of training. Therefore, based only on the expenditure and surplus amounts, no conclusion can be made as to how purposefully the state resource was spent, i.e. to what an extent each college acquires a "free" surplus according to its individual rule and ensures, also according to its individual rule, a rise in the competitiveness of a graduate student at the job market.

We have to deal with yet another problematic fact that the surplus of each college and the sum of surpluses of the same college are qualitatively different sums: in individual colleges this surplus cannot produce any significant influence on the resolution of their own problems (this especially concerns a college with a minimal budget), whereas the concentration and purposeful resolution enables a college to solve important and large-scale problems.

The purpose of the statistical survey results presented in the present paper is to make a contribution to the optimization of the process of financing vocational college students by the state, namely: to contribute to the elaboration of an optimal model of surplus distribution and monitoring.

To accomplish this aim, it is necessary, on the one hand, to carry out an analysis of the needs of vocational colleges and, on the other hand, to range the existing colleges, i.e. to work out the national rating system, on the basis of which a model of concentration and maximally effective use of surpluses can be elaborated.

In the initial stage, to solve the above-mentioned problems we used the method of quantitative sociological investigation to identify those basic factors which essentially determine the attitude of students to vocational training, which, on its part, is one of the most important components of vocational training estimation.

The general parameters of the research are as follows: a general set consisting of students of the existing state-founded vocational colleges in Georgia and the selected quantity of students equal to 1036. In selecting students, a simple random method was used. The method of questioning is a face-to-face interview. An average interviewing time is 40–50 minutes.

The investigation techniques were prepared on the basis of consultations with education and professional experts by using the approved measurement methods. Field work was carried out observing the ethics investigation standards (see [2–6]). Data analysis was carried out by the methods of descriptive statistics analysis and factor analysis (SPSS software, version 20.0).

Basic Results

The frequency indices of sex values–52% for female sex and 48% for male sex give the grounds to make a conclusion that the choice to receive education at a vocational college does not depend on sex (analogous indices of Georgia's population are 56% for female sex and 44% for male sex. These are the data of the poll of 2014 [7]).

The distribution of students by age groups is as follows:

15-19 yrs	47.3%
20-24 yrs	26.4%
25-29 yrs	8.3%
30-34 yrs	4.2%
35-39 yrs	2.3%
40-44 yrs	2.2%
45+	9.3%

Approximately three-thirds of students are collected in 15-24 years age groups. Attention is given to the fact that the specific fraction of students aged 45 years and more noticeably exceeds an analogous index of the age group of first-year students aged 30-44, which, in view of demands of the job market, can be associated with the necessity to change the professional qualification or to receive a new qualification.

The majority of students (71%) have the base or secondary education.

Students having a higher degree of education are represented by a much lower specific fraction. It should be noted that the specific fraction of bachelors who finished the course at technical secondary schools, colleges, special secondary schools exceeds the index. This result is quite important from the viewpoint of investigation of the mutual relationship between these two steps of education, though in order to make a concrete conclusion this question demands a deeper study.

Education	
Base	30%
Secondary	41%
secondary school, college, special secondary school,	10%
Bachelor	13%
Master	3%
Other	3%

To estimate the attitude of students to vocational colleges we chose the following three general parameters: estimation of services rendered to students and the related activities of the administration and auxiliary personnel; estimation of the teaching quality, which includes estimation of the pedagogical resource and curricula; estimation of the logistics (material-and-technical base). Each of these parameters is, in turn, subdivided into concrete indicators. As a result, measurements were done by means of 48 variables. Factor analysis or, more specifically, the method of selection of concrete components was applied. From 48 variables we chose 8 general factor variables of latent character which essentially influence the attitude of students to a vocational school.

Factor 1 includes 11 variables. This factor is conditionally called "servicing of students" since it contains all the variables which were used to estimate the services rendered directly to students and also some of those variables which are related to the administrative and auxiliary personnel who render these services.

Measurements were done by the ten-point scale, the minimal and maximal values being 1 and 10, respectively.

Factor No. 1. Services Rendered to Students

Variable description	Average	Standard deviation	Variation coefficient
Information is timely provided by college administration	9.00	1.881	20.9%
Administration is staffed with professional specialists	9.14	1.756	19.2%
Administration resolves problems in proper time	8.97	1.998	22.3%
Information on the college web-page is regularly up-graded	8.71	2.261	25.9%
College web-page performs information and communication function	8.73	2.223	25.5%
Health of college students and personnel is properly protected	8.98	1.957	21.8%
Security of college students and personnel is properly provided	9.02	1.945	21.6%
Students have access to internet	8.34	2.731	32.8%
Time-tables are timely prepared	9.09	1.872	20.6%
Registration of students takes place	9.32	1.569	16.8%
in connection with additional exams and training process	9.01	2.000	22.2%

It should be noted that an individual factor unites variables which are used to for estimating the personal mutual relations of a student with the administrative or auxiliary personnel.

Factor No. 2. Personal Communication with the Administration

Variable description	Average	Standard deviation	Variation coefficient
Students' files connected with administrative matters are organized	9.11	1.741	19.1%
Student has support on the part of administration	9.02	1.892	21.0%
Communication with administrative personnel is simple	9.09	1.747	19.2%

Thus, on the part of administration the resolution of general problems and the communication with individual students are estimated differentially, which in a vocational college even more clearly reveals the relevance of individual contacts with every student as they depend on his concrete demands.

Factor No. 3. Teaching Quality

The qualification of the pedagogical resource and the quality of training programs are apprehended by students from the viewpoint of indivisible integrity–all variables connected with these two aspects of the teaching process are united into a single factor. This logically expected result once more emphasizes the fact that both aspects should continue their development in this harmony. Otherwise the effect of failure of one of them will automatically affect the other.

Variable description	Average	Standard deviation	Variation coefficient
General estimation of the general education program	9.16	1.459	15.9%
Estimation of teaching personnel professionalism	9.54	1.233	12.9%
Estimation of assessment system existing in college	9.14	1.569	17.2%
Estimation of theoretical lectures	9.41	1,261	13.4%
Estimation of work in group	9.17	1.546	16.9%
Estimation of training practical work	9.14	1.719	18.Ge8%
Estimation of examination process	9.29	1.469	15.8%
Modern methods are actively used in delivering lectures	9.03	1.729	19.2%
Teacher is always well prepared for lecture	9.51	1.311	13.8%

Master-classes which are not directly integrated into the teaching process, buffet and various events outside the educational process (competitions, sports contests and so on) are in fact regarded on the part of students as a single factor. It is of interest to note that as compared with other variables the average estimates included in this factor have low indices. We call conditionally this factor the student medium.

From the picture presented above it is obvious that: a) as different from the teaching process, the student medium outside the teaching process is problematic and needs improvement; b) the integration of master-classes into the non-teaching medium shows that a master-class is regarded not as an improving phenomenon of professional competence, but as an unimportant addition to the education process, which indicates the necessity of a further investigation of this issue and the obligatory integration of master-classes into the education process.

Factor No. 4. Student Medium Outside the Education Process

Variable description	Average	Standard deviation	Variation coefficient
Education process includes master-classes	7.21	3.194	44.3%
Functioning of buffet in educational establishments	4.76	4.275	89.8%
Various events such as competitions, sports contests and so on are organized	7.27	3.363	46.2%

Resources that are connected with information receiving and communication are grouped in a separate factor.

Factor No. 5. Information and Communication Resources

Variable description	Average	Standard deviation	Variation coefficient
Education process is provided with computer facilities	8.58	2.367	27.6%
Education process is provided with library	8.47	2.437	28.8%
Education process is provided with internet	8.39	2.589	30.9%
Use of computer class is accessible	8.42	2.572	30.5%
Assessment of logistics available in library	8.18	2.447	29.9%
Quantity of computers available for students is satisfactory	8.49	2.517	29.7%
Computer software is satisfactory	8.43	2.386	28.3%
Monitors mounted in lecture-rooms function properly	7.85	3.029	38.6%
Xerox is available for students	7.22	3.381	46.8%

General infrastructure, equipment of students' practical work and the state of lecture-rooms are estimated by students differentially and regarded as independent factors.

Factor No. 6. General Infrastructure

Variable description	Average	Standard deviation	Variation coefficient
Floors are in good condition and do not hamper normal conducting of training process	9.12	1.984	21.7%
Walls, windows, doors are in good condition	9.06	2.035	22.5%
Electric power supply system functions properly	8,85	2.224	25.1%
Heating and air conditioning system functions properly	8.20	2.777	33.9%
Running water supply system functions properly	8.50	2.557	30.1%
Educational institutions are equipped in conformity with modern standards	8.30	2.561	30.9%
Training practical work is carried out in college on permanent basis	8.12	2.695	33.2%
Wet points are in good order	8.23	2.737	33.3%
Sanitary conditions are satisfactory	8.64	2.421	28.0%

Factor No. 7. Training Practice – Equipment

Variable description	Average	Standard deviation	Variation coefficient
Lecture-rooms are provided with required hardware	8.48	2.400	28.3%
Training practice laboratories are provided with required equipment/tools	8.30	2.479	29.9%
Training practice is provided with required materials	8.36	2.405	28.7%

Factor No. 8. Lecture Rooms

Variable description	Average	Standard deviation	Variation coefficient
Sufficient quantity of desks and chairs are available in lecture-rooms	9.41	1.589	16.9%
Desks used by students are convenient	9.00	2.043	22.7%
All lecture-rooms have blackboards	9.48	1.499	15.8%

To conclude, it can be said that students estimate the main components of the educational process with sufficient conscientiousness. Based on students' estimates, the selected factors, i.e. variables of general character by means of which students form their attitude to vocational education, characterize and the position and functioning of an educational institution and successfully perform the function of an essential parameter for establishing the rating of vocational institutions.

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THE JACOBI TRANSFORM METHOD IN APPROXIMATION THEORY

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Abstract. In this paper, the behavior of Fourier coefficients of some classes of functions on an arbitrary orthogonal system is studied. The estimations of order of convergence to zero of Fourier–Jacobi coefficients are found. These estimations are precise and of terminal character. The obtained results are used for the convergence of Fourier–Jacobi series.

რეზიუმე. ნაშრომში შესწავლილია ნებისმიერი ორთოგონალური სისტემის ზოგიერთი კლასის ფუნქციების ფურიეს კოეფიციენტების ყოფაქცევა. ნაპოვნია ფურიე-იაკობის კოეფიციენტების ნულთან კრებადობის რიგის შეფასებები. ეს შეფასებები ზუსტია და აქვთ ზღვრული ხასიათი. მიღებული შედეგები გამოიყენება ფურიე-იაკობის მწკრივების კრებადობისათვის.

0. Introduction

The estimations of Fourier-Legendre coefficients of functions belonging to one of the classes $C\,[-1,\,1]$, $L\,[-1,\,1]$ or $L^2\,[-1,\,1]$ were given in [1]. The obtained inequalities were applied to the problems of convergence of Fourier-Legendre series. In [2], these results were generalized to ultraspherical series for $f \in L_{p,\,\mu}\,[-1,\,1]$, $1 \le p \le \infty$. In [3], the author obtained the estimations of the Fourier–Jacobi coefficient of smooth functions of bounded variation.

Unlike the above-indicated papers, in this paper we study the behavior of Fourier coefficients of some classes of functions on an arbitrary orthogonal system.

Suppose μ be a measure on $[a,b]$, such that $\mu[a,b]=1$.

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Let $\varphi_n(x)$, $n = 0, 1, \dots$ be a system of orthogonal functions with respect to μ on the segment $[a, b]$ and let

$$\hat{f}(n) = \int_a^b f(x)\,\varphi_n(x)\,d\mu(x) \tag{0.1}$$

be the n-th Fourier coefficient of the functions f, belonging to one of the classes $L_{p,\mu}[a, b]$, $(1 \le p < \infty)$, that is to a class of summable functions of p-th degree, with respect to the measure μ. L' is a class of functions with an integrable derivative on $[a, b]$.

Denote by X one of the linear spaces $L_{p,\mu}$ or L' and by $L = L(X, X)$ the space of linear operators acting from X to X, for which the equality

$$\int_a^b (A^r f)(x)\,g(x)\,d\mu(x) = \int_a^b f(x)\,(A^r g)(x)\,d\mu(x), \quad r = 1, 2, \dots . \tag{0.2}$$

is fulfilled.

In the case $f \in L_{p,\mu}$ we assume $g \in L_{q,\mu}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

We'll say that $f \in W_X^r$, if $\exists g \in X$ such that

$$f(x) = (A^r g)(x) + c, \quad r = 1, 2, \dots, \tag{0.3}$$

where $A \in L(X, X)$, $A^0 f = f$, $A^r f = A\left(A^{r-1} f\right)$, $r = 1, 2, \dots$ and c is some constant.

Define the norm $f \in L_{p,\mu}$ by

$$\|f\|_{L_{p,\mu}} \equiv \|f\|_{p,\mu} = \left(\int_a^b |f(x)|^p\,d\mu(x)\right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty,$$

and the norm of $f \in L'$ by $\|f\|_{L'} \equiv \|f\|_C = \sup\limits_{a \le x \le b} |f(x)|$.

Hereafter the operator satisfying the condition (0.2) for which presentation (0.3) is true will be constructed.

In Section 1 we prove general theorems on the convergence to zero of Fourie coefficients of the functions from X on an arbitrary orthogonal system. In Section 2 we study basic properties of the Jacobi transform of the functions in X. The operator satisfying the conditions (0.2) and (0.3) is constructed in Section 3. Here we establish integral estimates for Jacobi polynomials. The results of Section 4 have auxiliary character. The results of Section 5 are realization of generalized theorems of Section 1. The order of convergence to zero of Fourier–Jacobi coefficients of the functions from X are found. In Section 6 we prove the asymptotics of theorems on the order of convergence for particular sums of Fourier–Jacobi series.

1. On Fourier Coefficients of Classes X

In this section we prove the generalized theorems on the convergence to zero of Fourier coefficients from X.

Theorem 1.1. *Let* $f \in W_X^r\,(X = L_{p,\mu})$, $(1 < p < \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$. *If*
1^0. $\|A^r \varphi_n\|_{q,\,\mu} \le M - \text{const}$, $q > 1$, $r = 0, 1, \dots$;
2^0. $\lim\limits_{n\to\infty} \int\limits_{\alpha}^{\beta} (A^r \varphi_n)\,(x)\,d\mu\,(x) = 0$, $a \le \alpha < \beta \le b$, $r = 0, 1, \dots$, *where* α *and* β *are arbitrary numbers, then*

$$\lim_{n\to\infty} \hat{f}(n) = 0.$$

Proof. According to (0.2), we can write

$$\hat{f}(n) = \int\limits_a^b f\,(x)\;\varphi_n\,(x)\,d\mu\,(x) = \int\limits_a^b ((A^r g)\,(x) + c)\,\varphi_n\,(x)\,d\mu\,(x) =$$

$$= \int\limits_a^b (A^r g)\,(x)\,\varphi_n\,(x)\,d\mu\,(x) + c\int\limits_a^b \varphi_n\,(x)\,d\mu\,(x) =$$

$$= \int\limits_a^b g\,(x)\,(A^r \varphi_n)\,(x)\,d\mu\,(x) + c\int\limits_a^b \varphi_n\,(x)\,d\mu\,(x) = A_n + B_n. \tag{1.1}$$

By condition 2^0 of the theorem,

$$\lim_{n\to\infty} B_n = 0, \tag{1.2}$$

since $A^0 \varphi_n = \varphi_n$.

Let's turn to A_n. Let $g \in L_{p,\mu}\,[a, b]$. By density of C in $L_{p,\mu}$, $\exists h \in C$, such that

$$\|h - g\|_{p,\,\mu} < \frac{\varepsilon}{M}. \tag{1.3}$$

Further,

$$|A_n| \le \left| \int\limits_a^b (g\,(x) - h\,(x))\,(A^r \varphi_n)\,(x)\,d\mu\,(x) \right| +$$

$$+ \left| \int\limits_a^b h\,(x)\,(A^r \varphi_n\,)\,(x)\,d\mu\,(x) \right| = |A_{n.1}| + |A_{n.2}|. \tag{1.4}$$

By condition 1^0 of the theorem, inequality (1.3) and Hőlder's inequality

$$|A_{n.1}| \leq \int_a^b |g(x) - h(x)| \; |(A^r\varphi_n)(x)| \, d\mu(x) \leq$$

$$\leq \left(\int_a^b |g(x) - h(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \left(\int_a^b |(A^r\varphi_n)(x)|^q \, d\mu(x) \right)^{\frac{1}{q}} =$$

$$= \|A^r\varphi_n\|_{q,\,\mu} \, \|f - g\|_{p,\,\mu} < \varepsilon. \tag{1.5}$$

It remains to consider $A_{n,2}$. According to the Cantour theorem, we partition the segment $[a, b]$ by the points $a = x_0 < x_1 < \cdots < x_m = b$ so that at each partial interval $[x_k, \, x_{k+1}]$, $k = 0, 1, \ldots, m-1$, the vibrations of the function h couldn't exceed the given $\varepsilon > 0$.

Then

$$|A_{n,2}| = \left| \int_a^b h(x) \; (A^r\varphi_n)(x) \, d\mu(x) \right| \leq$$

$$\leq \left| \sum_{k=1}^m \int_{x_{k-1}}^{x_k} (h(x) - h(x_{k-1})) \, (A^r\varphi_n)(x) \, d\mu(x) \right| +$$

$$+ \left| \sum_{k=1}^m h(x_{k-1}) \int_{x_{k-1}}^{x_k} (A^r\varphi_n)(x) \, d\mu(x) \right| = A^{(1)}_{n\cdot 2} + A^{(2)}_{n\cdot 2}.$$

But

$$A^{(1)}_{n\cdot 2} \leq \sum_{k=1}^m \int_{x_{k-1}}^{x_k} |h(x) - h(x_{k-1})| \, |(A^r\varphi_n)(x)| \, d\mu(x) <$$

$$< \varepsilon \int_a^b |(A^r\varphi_n)(x)| \, d\mu(x) < \varepsilon \|A^r\varphi_n\|_{q,\mu} < \varepsilon \cdot M. \tag{1.6}$$

And the sum $A^{(2)}_{n\cdot 2}$ by condition 2^0 tends to zero as $n \to \infty$ and therefore for the great enough numbers $n > n_0(\varepsilon)$ turns out lesser than $\varepsilon > 0$, i.e.,

$$\left| A^{(2)}_{n\cdot 2} \right| < \varepsilon, \quad \text{for} \quad n > n_0$$

This and (1.6) imply that

$$|A_{n\cdot 2}| < \varepsilon(M + 1). \tag{1.7}$$

Taking into account (1.5) and (1.7) in (1.4), we get

$$|A_n| < \varepsilon(M + 2). \tag{1.8}$$

Using (1.2) and (1.8) on (1.1), we get the assertion of the above theorem. $\square$

Theorem 1.2. *Let* $f \in W_X^r\,(X = L_{1,\mu})$. *If*

1^0. $|(A^r\varphi_n)(x)| \le M - \text{const},\ r = 0, 1, \dots,\ x \in [a, b]$;

2^0. $\lim\limits_{n\to\infty} \int\limits_{\alpha}^{\beta} (A^r\varphi_n)(x)\, d\mu(x) = 0$, $a \le \alpha < \beta \le b$, *where* α *and* β *are arbitrary numbers, then*

$$\lim_{n\to\infty} \hat{f}(n) = 0.$$

Proof. As in Theorem 1.1, the proof is reduced to the study of the integral A_n. Let first $g \in C[a, b]$, then by the Cantour theorem we partition the segment $[a, b]$, $a = x_0 < \cdots < x_m = b$ so that $\forall\varepsilon > 0$, $\exists\delta(\varepsilon) > 0$, such that $\forall x \in [x_k, x_{k+1}]$, $\max\limits_k |x_k - x_{k+1}| < \delta$,

$$|f(g) - g(x_k)| < \varepsilon. \tag{1.9}$$

Therefore

$$\begin{aligned} |A_n| &= \left| \int_a^b (g(x) - g(x_k))\,(A^r\varphi_n)(x)\, d\mu(x) + \right. \\ &\quad \left. + \int_a^b g(x_k)\,(A^r\varphi_n)(x)\, d\mu(x) \right| \le \\ &\le \int_a^b |g(x) - g(x_k)|\, |(A^r\varphi_n)(x)|\, d\mu(x) + \left| \int_a^b g(x_k)\,(A^r\varphi_n)(x)\, d\mu(x) \right| = \\ &= \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} |g(x) - g(x_k)|\, |(A^r\varphi_n)(x)|\, d\mu(x) + \\ &\quad + \left| \sum_{k=1}^{m} g(x_k) \int_{x_{k-1}}^{x_k} (A^r\varphi_n)(x)\, d\mu(x) \right| = A_{n.1} + A_{n.2}. \end{aligned} \tag{1.10}$$

From (1.9) and condition 1^0 of the theorem, we have

$$A_{n.1} < \varepsilon \cdot M \int_a^b d\mu(x) = \varepsilon \cdot \mu. \tag{1.11}$$

From condition 2^0 of the theorem,

$$\lim_{n\to\infty} A_{n.2} = 0. \tag{1.12}$$

Taking into account (1.10) and (1.11) in (1.9), we get

$$\lim_{n\to\infty} A_n = 0. \tag{1.13}$$

Now let g be a measurable bounded function

$$|g(x)| \le M_1 - \text{const}, \; x \in [a,b]. \tag{1.14}$$

By N. Lusin's theorem (see [11], p. 118), $\varepsilon > 0 \; \exists \nu(x) \in C[a,b]$ such that

$$mE(g \ne \nu) < \varepsilon, \;\; |\nu(x)| \le M_1. \tag{1.15}$$

Then

$$|A_n| = \left| \int_b^a g(x)(A^r\varphi_n)(x)\, d\mu(x) \right| \le$$

$$\le \left| \int_a^b [g(x) - \nu(x)](A^r\varphi_n)(x)\, d\mu(x) \right| +$$

$$+ \left| \int_a^b \nu(x)(A^r\varphi_n)(x)\, d\mu(x) \right|.$$

But by (1.15) and condition 1^0 of the theorem,

$$\left| \int_a^b [g(x) - \nu(x)](A^r\varphi_n)(x)\, d\mu(x) \right| =$$

$$= \left| \int_{E(f\ne\nu)} [g(x) - \nu(x)](A^r\varphi_n)(x)\, d\mu(x) \right| < 2MM_1\varepsilon.$$

On the other hand, by (1.13) for great enough n one has $\beta_n < \varepsilon$. Thus we have

$$|A_n| < (2MM_1 + 1)\,\varepsilon. \tag{1.16}$$

From this follows the assertion of the theorem for measurable bounded function.

Finally, let $g \in L_{1,\mu}$. Taking $\varepsilon > 0$ and using the absolute continuity of the integral, we find $\delta > 0$ such that for any measurable set $e \in [a,b]$ with measure $me < \delta$ (see [11], p. 165),

$$\int_e |g(x)|\, d\mu(x) < \varepsilon. \tag{1.17}$$

We find a bounded measurable function $\nu(x)$, so that (see [11], p. 113)

$$mE(g \ne \nu) < \delta, \quad |\nu(x)| \le M_2 - \text{const}. \tag{1.18}$$

Then by (1.15)–(1.18) and condition 1^0 of the theorem,

$$|A_n| \leq \left| \int_a^b (g-\nu)(x)\,(A^r\varphi_n)(x)\,d\mu(x) \right| +$$

$$+\left| \int_a^b \nu(x)\,(A^r\varphi_n)(x)\,d\mu(x) \right| \leq \left| \int\limits_{E(g\neq\nu)} (g-\nu)(x)\,(A^r\varphi_n)(x)\,d\mu(x) \right| + \varepsilon <$$

$$< M\varepsilon + \varepsilon = (M+1)\,\varepsilon,$$

Thus the proof of the theorem is complete. □

Note that to essence for $\mu(x) \equiv 1$ this theorem was proved by Henri Lebesgue (see [11], p. 300). We present the proof for completeness of explanation.

Theorem 1.3. *Let* $f \in W_X^r$ $(X = L')$. *If*

1^0. $\left| \int_a^x (A^r\varphi_n)(t)\,d\mu(t) \right| \leq M,\ r = 0, 1, \ldots,\ x \in [a,b]$;

2^0. $\lim\limits_{n\to\infty} \int_\alpha^\beta (A^r\varphi_n)(x)\,d\mu(x) = 0,\ a \leq \alpha < \beta \leq b,$ *then almost everywhere*

$$\lim_{n\to\infty} \hat{f}(n) = 0.$$

Proof. Since $f \in L'$, then ([11], p. 292)

$$f(x) = f(a) + \int_a^x f'(t)\,dt.$$

Then

$$\int_a^b f(x)\,(A^r\varphi_n)(x)\,d\mu(x) =$$

$$= \int_a^b \left(f(a) + \int_a^x f'(t)\,dt \right) (A^r\varphi_n)(x)\,d\mu(x) =$$

$$= f(a) \int_a^b (A^r\varphi_n)(x)\,d\mu(x) +$$

$$+ \int_a^b \int_a^x f'(t)\,dt\,(A^r\varphi_n)(x)\,d\mu(x) = A_{n.1} + A_{n.2}. \tag{1.19}$$

By condition 2^0 of the theorem,

$$\lim_{n\to\infty} A_{n.1} = 0. \tag{1.20}$$

By 1^0 ([11], p. 113), almost everywhere

$$A_{n.2} = \int_a^b \left(\int_a^x f'(t)\, dt \right) d \int_a^x (A^r \varphi_n)(t)\, d\mu(t) =$$

$$= \int_a^x f'(t)\, dt \int_a^x (A^r \varphi_n)(t)\, d\mu(t) \Big|_a^b -$$

$$- \int_a^b \left(\int_a^x (A^r \varphi_n)(t)\, d\mu(t) \right) f'(x)\, dx = \int_a^b f'(t)\, dt \int_a^b (A^r \varphi_n)(t)\, d\mu(t) -$$

$$- \int_a^b \left(\int_a^x (A^r \varphi_n)(t)\, d\mu(t) \right) f'(x)\, dx =$$

$$= (f(b) - f(a)) \int_a^b (A^r \varphi_n)(t)\, d\mu(t) -$$

$$- \int_a^b \left(\int_a^x (A^r \varphi_n)(t)\, d\mu(t) \right) f'(x)\, dx = A'_{n.2} + A''_{n.2}. \tag{1.21}$$

By condition 2^0 of the theorem,

$$\lim_{n\to\infty} A'_{n.2} = 0.$$

And by the conditions of the theorem and Lebesgue theorem ([11] p. 139),

$$\lim_{n\to\infty} A''_{n.2} = 0.$$

Taking into account (1.22) and (1.23) in (1.21), we get

$$\lim_{n\to\infty} A_{n.2} = 0.$$

Using (1.20) and (1.24) in (1.19), we get the assertion of the theorem. $\square$

Remark. Theorems 1.1–1.3 are just for arbitrary linear operator satisfying the condition (0.3).

2. Basic Properties of the Jacobi Transform

In this section, we study the properties of Jacobi's transform of some classes of functions. We introduce the concept of a strong derivative and of the Jacobi integral. The connection between them is established. Owing to this concept, becomes clear structural description of classes of functions. The obtained results are analogues to some theorems proved in [4] for Legendre transform.

Next, let X be one of the spaces $L_{p,\alpha}[-1, 1]$, $1 \le p < \infty$ or $C[-1, 1]$ endowed with the norms

$$\|f\|_{L_{p,\alpha}} \equiv \|f\|_{p,\alpha} = \left(\int_{-1}^{1} |f(x)|^p \, d\mu_\alpha(x)\right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty,$$

$$\|f\|_{L'} \equiv \|f\|_C = \sup_{-1 \le x \le 1} |f(x)|,$$

where $d\mu_\alpha(x) = c_1(\alpha)(1-x)^\alpha (1+x)^{-\frac{1}{2}} \, dx$, $-\frac{1}{2} < \alpha < \frac{1}{2}$,

$$c_1(\alpha) = \frac{\Gamma\left(\alpha + \frac{3}{2}\right)}{2^{\alpha+\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha+1)} = \left(\int_{-1}^{1} (1-x)^\alpha (1+x)^{-\frac{1}{2}} \, dx\right)^{-1}.$$

We consider the Jacobi polynomials $P_n^{\left(\alpha,-\frac{1}{2}\right)}(x)$, for $-\frac{1}{2} < \alpha < \frac{1}{2}$, $n = 0, 1, \ldots$, which form the orthogonal system of functions on the segment $[-1, 1]$ with weight $(1-x)^\alpha (1+x)^{-\frac{1}{2}}$, that is (see [5], p. 80)

$$\int_{-1}^{1} P_n^{\left(\alpha,-\frac{1}{2}\right)}(x) P_k^{\left(\alpha,-\frac{1}{2}\right)}(x)(1-x)^\alpha (1+x)^{-\frac{1}{2}} \, dx =$$

$$= \begin{cases} 0, & k \ne n, \\ h_n(\alpha), & k = n, \end{cases} \tag{2.1}$$

where

$$h_n(\alpha) = \frac{2^{\alpha+\frac{1}{2}}\Gamma(n+\alpha+1)\Gamma\left(n+\frac{1}{2}\right)}{\left(\alpha+\frac{1}{2}+2n\right)\Gamma(n+1)\Gamma\left(n+\alpha+\frac{1}{2}\right)}.$$

Further (see [6], p. 250),

$$\max_{|x|\le 1}\left|P_n^{\left(\alpha,-\frac{1}{2}\right)}(x)\right| = P_n^{\left(\alpha,-\frac{1}{2}\right)}(1) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}, \tag{2.2}$$

$$\frac{d}{dx}P_n^{\left(\alpha,-\frac{1}{2}\right)}(x) = \frac{1}{2}\left(n+\alpha+\frac{1}{2}\right)P_{n-1}^{\left(\alpha+1,\frac{1}{2}\right)}(x). \tag{2.3}$$

Assume $R_n^{\left(\alpha,-\frac{1}{2}\right)}(x) = P_n^{\left(\alpha,-\frac{1}{2}\right)}(x)/P_n^{\left(\alpha,-\frac{1}{2}\right)}(1)$.

According to (1.2),

$$\max_{|x|\le 1}\left|R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\right| = 1. \tag{2.4}$$

The Jacobi transform (the Fourier–Jacobi coefficients) is defined for $f\in X$ by

$$\hat{f}_{\left(\alpha,-\frac{1}{2}\right)}(n) = \hat{f}(n) = \int_{-1}^{1} f(x)\, R_n^{\left(\alpha,-\frac{1}{2}\right)}(x)\, d\mu_\alpha(x)\,.$$

Lemma 2.1. *Assuming $f,\, g \in X$ and $c \in R = (-\infty,\,\infty)$, we have*

(a) $\left|\hat{f}(n)\right| \le \|f\|_X\,,\quad n \in P := \{0,\,1,\,2,\,\dots\}\,;$

(b) $(f+g)\wedge(n) = \hat{f}(n) + \hat{g}(n),\quad (cf)\wedge(n) = cf\wedge(n)\,;$

(c) $\left(R_n^{\left(\alpha,\,-\frac{1}{2}\right)}\right)\wedge(n) = \begin{cases} 0, & k\ne n,\\ \frac{2^{\alpha+\frac{1}{2}}\Gamma^2(\alpha+1)\Gamma(n+\frac{1}{2})\Gamma(n+1)}{(\alpha+\frac{1}{2}+2n)\Gamma(n+\alpha+\frac{1}{2})\Gamma(n+\alpha+1)}, & k=n; \end{cases}$

(d) *for all $n \in P$, the relation*

$$\hat{f}(n) = 0 \Leftrightarrow f(x) = 0 \quad (a.e),$$

is true and means that the assertion holds for all $x \in [-1,\,1]$ if $X = C\,[-1,\,1]$, and for almost all $x \in [-1,\,1]$ if $X = L_{p,\,\alpha}\,[-1,\,1]$, $1 \le p < \infty$.

Proof. We prove (a). Let $f \in L'\,[-1,1]$, then by (1.4),

$$\left|\hat{f}(n)\right| \le \sup_{|x\le 1|}|f(x)|\int_{-1}^{1}\left|R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\right| d\mu_\alpha(x) \le \|f\|_C\,. \tag{2.5}$$

Now let $f \in L_{p,\,\alpha}$. For $p=1$, by (2.4), we have

$$\left|\hat{f}(n)\right| \le \int_{-1}^{1}|f(x)|\left|R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\right| d\mu_\alpha(x) \le \|f\|_{1,\,\alpha}\,, \tag{2.6}$$

and for $p>1$, by Hölder's inequality

$$\left|\hat{f}(n)\right| \le \|f\|_{p,\,\alpha}\left\|R_n^{\left(\alpha,\,-\frac{1}{2}\right)}\right\|_{q,\,\alpha} \le \|f\|_{p,\,\alpha}\,. \tag{2.7}$$

Thus from (2.5)–(2.7) it follows that for all $f \in X$

$$\left|\hat{f}(n)\right| \le \|f\|_X\,.$$

The properties (b) are obvious and (c) follows by (2.1).

We prove (d). The direct assertion is obvious. The conserve assertion follows from the uniquesness theorem for the Jacobi transform.

In this way, for all $n \in P$, we have

$$f^{\wedge}(n) = g^{\wedge}(n) \Leftrightarrow f(x) = g(x) \quad (a.e.).$$

Thus Lemma is proved. □

Corollary 2.1. *For all $n \in N$*

$$\overset{\wedge}{f(n)}(n) = 0 \Rightarrow f(x) = constant \quad (a.e.).$$

Proof. Really, let $n \in N$. From property (c) it follows that $c^{\wedge}(n) = 0$ for $n \in N$, where (c) is an arbitrary constant. But then we have

$$f^{\wedge}(n) = f^{\wedge}(n) - c^{\wedge}(n) = (f - c)^{\wedge}(n) =$$

$$= \int_{-1}^{1} (f(x) - c)\, R_n^{\left(\alpha, -\frac{1}{2}\right)}(x)\, d\mu_\alpha(x).$$

This implies that

$$f^{\wedge}(n) = 0 \Rightarrow f(x) - c = 0 \quad \text{(a.e.)}.$$

Thus Corollary 2.1 is proved. □

For $\alpha > \beta = -\frac{1}{2}$, the generalized Jacobi shift operator is of the form (see [7])

$$(\tau_t f)(x) = \int_{-1}^{1} f(x, t, r)\, dm_\alpha(r), \tag{2.8}$$

where

$$f(x, t, r) = f\Big(xt + r\sqrt{1-x^2}\sqrt{1-t^2} - \frac{1}{2}\left(1-r^2\right)(1-x)(1-t)\Big),$$

$$dm_\alpha(r) = c_2(\alpha)\left(1-r^2\right)^{a-\frac{1}{2}} dr \quad \text{and}$$

$$c_2(\alpha) = \frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)} = \Bigg(\int_{-1}^{1} \left(1-r^2\right)^{\alpha-\frac{1}{2}} dr\Bigg)^{-1}.$$

The following important equality

$$\left(\tau_t R_n^{\left(\alpha, -\frac{1}{2}\right)}\right)(x) = R_n^{\left(\alpha, -\frac{1}{2}\right)}(x)\, R_n^{\left(\alpha. -\frac{1}{2}\right)}(t) \tag{2.9}$$

follows from the “multiplication theorem” for Jacobi polynomials (see [8], p. 130):

$$P_n^{\left(\alpha, -\frac{1}{2}\right)}(x)\, P_n^{\left(\alpha, -\frac{1}{2}\right)}(t) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)} \int_{-1}^{1} P_n^{\left(\alpha, -\frac{1}{2}\right)}(x, t, r)\, dm_\alpha(r).$$

Lemma 2.2. *The operator τ_t is linear from X into itself, satisfying*
(a) $\|\tau_t 1\|_{[X,\,X]} = 1,\ (t \in [-1,\,1])$;

(b) $\lim\limits_{t\to 1-0} \|\tau_t f - f\|_X = 0,\ (f \in X)$;

(c) $(\tau_t f)^{\wedge}(n) = \hat{f}(n)\, R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(t),\ (f \in X,\ \ t \in [-1,\,1]\,,\ n \in P)$;
(d) $(\tau_t f)(x) = (\tau_x f)(t),\ (f \in X,\ \ x,\, t \in [-1,\,1])$.

(e) $\lim_{n\to\infty} \hat{f}(n) = 0$.

Proof. We prove (a). First, we'll show that

$$\|\tau_t f\|_{p,\,\alpha} \le \|f\|_{p,\,\alpha}\,,\ \ 1 \le p < \infty. \tag{2.10}$$

By (2.8) and Hölder's inequality, we have

$$|(\tau_t f)(x)|^p = \left(\left|\int_{-1}^{1} f(x,\,t,\,r)\, dm_\alpha(r)\right|\right)^p \le$$

$$\le \left(\int_{-1}^{1} dm_\alpha(r)\right)^{p-1} \int_{-1}^{1} |f(x,\,t,\,r)|^p\, dm_\alpha(r) = \int_{-1}^{1} |f(x,\,t,\,r)|^p\, dm_\alpha(r)\,.$$

Hence we have

$$\|\tau_t f\|_{p,\,\alpha}^p = \int_{-1}^{1} |(\tau_t f)(x)|^p\, d\mu_\alpha(x) \le$$

$$\le c_1(\alpha)\, c_2(\alpha) \int_{-1}^{1}\int_{-1}^{1} (1-x)^\alpha (1+x)^{-\frac{1}{2}} \left(1-r^2\right)^{\alpha-\frac{1}{2}} \times$$

$$\times \left| f\left(xt + r\sqrt{1-x^2}\sqrt{1-t^2} - \frac{1}{2}\left(1-r^2\right)(1-x)(1-t)\right)\right|^p dxdr.$$

Assuming $t = \cos u$ and $y = \cos\frac{u}{2}$, we obtain

$$\cos u = 2y^2 - 1, \quad \sin u = 2y\sqrt{1-y^2},$$

then

$$\|\tau_t f\|_{p,\,\alpha} \le c_1(\alpha)\, c_2(\alpha) \int_{-1}^{1}\int_{-1}^{1} (1-x)^\alpha (1+x)^{-\frac{1}{2}} \left(1-r^2\right)^{\alpha-\frac{1}{2}} \times$$

$$\times \left| f\left[x\left(2y^2-1\right) + 2ry\sqrt{1-y^2}\sqrt{1-x^2} - \left(1-r^2\right)(1-x)\left(1-y^2\right)\right]\right|^p drdx.$$

Making substitution $x=\cos\theta$ and denoting $z=\cos\frac{\theta}{2}$ we get

$$\|\tau_t f\|_{p,\alpha} \le c_1(\alpha)\, c_2(\alpha)\, 2^{\alpha+\frac{3}{2}} \int\limits_0^1 \int\limits_{-1}^1 \left(1-z^2\right)^{\alpha} \left(1-r^2\right)^{\alpha-\frac{1}{2}} \times$$
$$\times \left| f\left[\left(2z^2-1\right)\left(2y^2-1\right)+\right.\right.$$
$$\left.\left.+4ryz\sqrt{1-y^2}\sqrt{1-z^2}-2\left(1-r^2\right)\left(1-y^2\right)\left(1-z^2\right)\right]\right|^p drdz=$$
$$=2^{\alpha+\frac{3}{2}} c_1(\alpha)\, c_2(\alpha) \int\limits_0^1 \int\limits_{-1}^1 \left(1-z^2\right)^{\alpha}\left(1-r^2\right)^{\alpha-\frac{1}{2}} \left| f\Big[2\,(yz+\right.$$
$$\left.+r\sqrt{1-y^2}\sqrt{1-z^2}\Big)^2-1\Big]\right|^p drdz.$$

Substituting the inner integral, putting $v=yz+r\sqrt{1-y^2}\sqrt{1-z^2}$ and taking into account that

$$r=(v-yz)\left(1-y^2\right)^{-\frac{1}{2}}\left(1-z^2\right)^{-\frac{1}{2}},\quad dr=\left(1-y^2\right)^{-\frac{1}{2}}\left(1-z^2\right)^{-\frac{1}{2}}dv,$$

we obtain

$$\|\tau_t f\|_{p,\alpha} \le 2^{\alpha+\frac{3}{2}} c_1(\alpha)\, c_2(\alpha)\left(1-y^2\right)^{-\alpha} \int\limits_0^1 dz\times$$
$$\times \int\limits_{yz-\sqrt{1-y^2}\sqrt{1-z^2}}^{yz+\sqrt{1-y^2}\sqrt{1-z^2}} \left|f\left(2v^2-1\right)\right|^p \left(1-y^2-z^2-v^2+2yzv\right)^{\alpha-\frac{1}{2}} dv.$$

Changing the order of integration and using the formula (see [9], p. 298)

$$\int\limits_a^b (b-x)^{\mu-1}(x-a)^{\nu-1}\,dx=(b-a)^{\mu+\nu-1}\frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)},$$

we obtain

$$\|\tau_t f\|_{p,\alpha} \le 2^{\alpha+\frac{3}{2}} c_1(\alpha)\, c_2(\alpha)\left(1-y^2\right)^{-\alpha} \int\limits_0^1 \left|f\left(2v^2-1\right)\right|^p dv\times$$
$$\times \int\limits_{yv-\sqrt{1-y^2}\sqrt{1-v^2}}^{yv+\sqrt{1-y^2}\sqrt{1-v^2}} \left(1-y^2-z^2-v^2+2yzv\right)^{\alpha-\frac{1}{2}} dz=$$

$$= 2^{3\alpha+\frac{3}{2}} c_1(\alpha)\, c_2(\alpha)\, \frac{\Gamma^2\left(\alpha+\frac{1}{2}\right)}{\Gamma(2\alpha+1)} \int\limits_0^1 \left(1-v^2\right)^{\alpha} \left|f\left(2v^2-1\right)\right|^p dv.$$

Substituting $v=\sqrt{\frac{1+u}{2}}$ and the equality (see [12], p. 760)

$$\frac{2^{2\alpha}\Gamma^2\left(\alpha+\frac{1}{2}\right)}{\Gamma(2\alpha+1)} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha+1)} = \frac{1}{c_2(\alpha)},$$

we obtain

$$\|\tau_t f\|_{p,\,\alpha} \leq \left(\int\limits_{-1}^{1} |f(u)|^p\, dm_\alpha(u)\right)^{\frac{1}{p}} = \|f\|_{p,\,\alpha},$$

from which follows (2.10).

On the other hand,

$$\|\tau_t 1\|_{p,\,\alpha} = \left(\int\limits_{-1}^{1}\left|\int\limits_{-1}^{1} dm_\alpha(r)\right|^p d\mu_\alpha(x)\right)^{\frac{1}{p}} = \|1\|_{p,\,\alpha} = 1.$$

From this and (2.10) it follows that in the case $X = L_{p,\,\alpha}$, $p \geq 1$,

$$\|\tau_t 1\|_{p,\,\alpha} = 1, \quad (t \in [-1,\,1]). \tag{2.11}$$

But the case $X = L'$ is elementary.

Really, assuming

$$z = xt + r\sqrt{1-x^2}\sqrt{1-t^2} - \frac{1}{2}\left(1-r^2\right)(1-x)(1-t) \leq$$
$$\leq xt + r\sqrt{1-x^2}\sqrt{1-t^2} \leq xt + \sqrt{1-x^2}\sqrt{1-t^2}$$

and taking $x=\cos u$, $t=\cos v$, we obtain

$$z \leq \cos x\ \cos t + \sin x\ \sin t = \cos(x-t) \leq 1.$$

On the other hand,

$$z = xt + r\sqrt{1-x^2}\sqrt{1-t^2} - \frac{1}{2}\left(1-r^2\right)(1-x)(1-t) \geq -1 \Leftrightarrow$$
$$\Leftrightarrow 2xt + 2r\sqrt{\left(1-x^2\right)\left(1-t^2\right)} + r^2(1-x)(1-t) - 1 - xt + x + t \geq -2 \Leftrightarrow$$
$$\Leftrightarrow r^2(1-x)(1-t) + 2r\sqrt{\left(1-x^2\right)\left(1-t^2\right)} + xt + x + t + 1 \geq 0 \Leftrightarrow$$
$$\Leftrightarrow r^2(1-x)(1-t) + 2r\sqrt{\left(1-x^2\right)\left(1-t^2\right)} + (1+x)(1+t) \geq 0 \Leftrightarrow$$
$$\Leftrightarrow \left(r\sqrt{(1-x)(1-t)} + \sqrt{(1+x)(1+t)}\right)^2 \geq 0.$$

Thus $-1 \leq z \leq 1$, then by (2.8),

$$\|\tau_t f\|_C \leq \sup_{-1\leq z\leq 1} |f(z)| = \|f\|_C\,.$$

This implies that

$$\|\tau_t 1\|_C = 1. \tag{2.12}$$

Property (a) follows from (2.11) and (2.12).

We prove (b). Let $f \in L_{p,\,\alpha}$ $(p \geq 1)$. Then by density of C in $L_{p,\,\alpha}$ for an arbitrary number $\varepsilon > 0$ there exists $\psi \in C\,[-1,\,1]$ such that

$$\|f - \psi\|_{p,\,\alpha} < \frac{\varepsilon}{3}. \tag{2.13}$$

If $\psi \in C\,[-1,\,1]$, then $\forall \varepsilon > 0$, $\exists \delta\,(\varepsilon) > 0$.

$$|\psi\,(x,\,t,\,r) - \psi\,(x)| < \frac{\varepsilon}{3} \tag{2.14}$$

for all $t \in (1-\delta;\ 1)$. Then for any $t \in (1-\delta;\ 1)$ one has

$$|(\tau_t \psi)\,(x) - \psi\,(x)| < \frac{\varepsilon}{3}, \tag{2.15}$$

from which follows

$$\|\tau_t \psi - \psi\|_{p,\,\alpha} = \left(\int_{-1}^{1} |(\tau_t \psi\,(x) - \psi\,(x))|^p \, d\mu_\alpha\,(x)\right)^{\frac{1}{p}} < \frac{\varepsilon}{3}. \tag{2.16}$$

Now taking into account (2.10), (2.13) and (2.16), for any $t \in (1-\delta,\,1)$, we obtain

$$\|\tau_t f - f\|_{p,\,\alpha} \leq \|\tau_t \psi - f\|_{p,\,\alpha} + \|\psi - \tau_t \psi\|_{p,\,\alpha} + \|\tau_t \psi - \tau_t f\|_{p,\,\alpha} < \varepsilon,$$

equivalent to

$$\lim_{t \to 1-0} \|\tau_t f - f\|_{p,\,\alpha} = 0, \quad p \geq 1. \tag{2.17}$$

Now, from (2.14) and (2.8) follows

$$\lim_{t \to 1-0} \|\tau_t f - f\|_C = 0. \tag{2.18}$$

The validity of assertion (b) follows from (2.17) and (2.18).

We prove (c). Doing as in proving property (a), we obtain

$$(\tau_t f)^\wedge\,(n) = \int_{-1}^{1} R_n^{\left(\alpha,\,-\frac{1}{2}\right)}\,(x)\,(\tau_t f)\,(x)\,d\mu_\alpha\,(x) =$$

$$= 2^{\alpha+\frac{1}{2}} c_1\,(\alpha)\,c_2\,(\alpha)\left(1-y^2\right)^{-\alpha} \int_0^1 R_n^{\left(\alpha,\,-\frac{1}{2}\right)}\left(2z^2-1\right) \times$$

$$\times \int_{yz-\sqrt{1-y^2}\sqrt{1-z^2}}^{yz+\sqrt{1-y^2}\sqrt{1-z^2}} f\left(2v^2-1\right)\left(1-y^2-z^2-v^2+2yzv\right)^{\alpha-\frac{1}{2}} dz dv.$$

It is known (see [5], p. 71) that

$$P_n^{(\alpha,\,-\frac{1}{2})}\left(2z^2-1\right)=\frac{\Gamma\left(n+\alpha+1\right)\Gamma\left(2n+1\right)}{\Gamma\left(2n+\alpha+1\right)\Gamma\left(n+1\right)}P_{2n}^{(\alpha,\,\alpha)}\left(z\right), \tag{2.19}$$

where $P_n^{(\alpha,\,\alpha)}(z)$ are ultraspherical polynomials which form the orthogonal system of functions on the segment $[-1,\,1]$ with weight $\left(1-z^2\right)^{\alpha}$. Then according (1.2), we have

$$R_n^{(\alpha,\,-\frac{1}{2})}\left(2z^2-1\right)=\frac{\Gamma\left(\alpha+1\right)\Gamma\left(2n+1\right)}{\Gamma\left(2n+\alpha+1\right)}P_{2n}^{(\alpha,\,\alpha)}\left(z\right). \tag{2.20}$$

Taking into account (2.20) and changing the order of integration, we obtain

$$(\tau_t f)\wedge(n)=2^{\alpha+\frac{1}{2}}c_1\left(\alpha\right)c_2\left(\alpha\right)\frac{\Gamma\left(\alpha+1\right)\Gamma\left(2n+1\right)}{\Gamma\left(2n+\alpha+1\right)}\left(1-y^2\right)^{-\alpha}\int\limits_0^1 f\left(2v^2-1\right)\times$$

$$\times\int\limits_{yv-\sqrt{1-y^2}\sqrt{1-v^2}}^{yv+\sqrt{1-y^2}\sqrt{1-v^2}}P_{2n}^{(\alpha,\,\alpha)}\left(z\right)\left(1-y^2-z^2-v^2+2yzv\right)^{\alpha-\frac{1}{2}}dzdv.$$

Substituting the inner integral

$$z=yv+r\sqrt{1-y^2}\sqrt{1-v^2},$$

we obtain

$$(\tau_t f)\wedge(n)=2^{\alpha+\frac{1}{2}}c_1\left(\alpha\right)c_2\left(\alpha\right)\frac{\Gamma\left(\alpha+1\right)\Gamma\left(2n+1\right)}{\Gamma\left(2n+\alpha+1\right)}\int\limits_0^1\left(1-v^2\right)^{\alpha}f\left(2v^2-1\right)\times$$

$$\times\int\limits_{-1}^1 P_{2n}^{(\alpha,\,\alpha)}\left(yv+r\sqrt{1-y^2}\sqrt{1-v^2}\right)\left(1-r^2\right)^{\alpha-\frac{1}{2}}drdv.$$

By the “multiplication theorem”, for ultraspherical polynomials

$$P_{2n}^{(\alpha,\,\alpha)}\left(y\right)P_{2n}^{(\alpha,\,\alpha)}\left(v\right)=$$

$$=\frac{\Gamma\left(2n+\alpha+1\right)}{\Gamma\left(2n+1\right)\Gamma\left(\alpha+1\right)}\int\limits_{-1}^1 P_{2n}^{(\alpha,\,\alpha)}\left(yv+r\sqrt{1-y^2}\sqrt{1-v^2}\right)dm_{\alpha}\left(r\right),$$

we have

$$(\tau_t f)\wedge(n)=\frac{\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma^2\left(2n+1\right)\Gamma\left(\alpha+1\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma^2\left(2n+\alpha+1\right)}P_{2n}^{(\alpha,\,\alpha)}\left(y\right)\times$$

$$\times\int\limits_0^1\left(1-v^2\right)^{\alpha}f\left(2v^2-1\right)P_{2n}^{(\alpha,\,\alpha)}\left(v\right)dv.$$

And by formula (2.20),

$$(\tau_t f)^\wedge(n) = \frac{\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha+1)} R_n^{\left(\alpha,-\frac{1}{2}\right)}\left(2y^2-1\right) \times$$

$$\times \int_0^1 \left(1-v^2\right)^\alpha f\left(2v^2-1\right) R_n^{\left(\alpha,-\frac{1}{2}\right)}\left(2v^2-1\right) dv.$$

Since $2y^2-1=t$, making the change of variables $v=\sqrt{\frac{1+u}{2}}$, we obtain

$$(\tau_t f)^\wedge(n) = R_n^{\left(\alpha,-\frac{1}{2}\right)}(t)\int_{-1}^{1} f(u)\, R_n^{\left(\alpha,-\frac{1}{2}\right)}(u)\, d\mu_\alpha(u) = R_n^{\left(\alpha,-\frac{1}{2}\right)}(t)\, \hat{f}(n).$$

Property (d) is obvious by the definition.

It remains to prove (e). Let x_n be the greater root of $P_n^{\left(\alpha,-\frac{1}{2}\right)}(x)$. From Lemma 2.1 $(a,\ b)$ and Lemma 2.2 (c) for $n\in N$ one can deduce that

$$\left|\hat{f}(n)\right| = \left|\left(f-\tau_{x_n f}\right)^\wedge(n)\right| \le \left\|f-f_{x_n}\right\|_X. \tag{2.21}$$

By Stieltjes inequality (see [5], p. 131, formula (6.21.5)),

$$\frac{2n-1}{2n+1}\pi \le x_n \le \frac{2n}{2n+1}\pi \Rightarrow \lim_{n\to\infty} x_n = 1. \tag{2.22}$$

Assertion (e) follows from Lemma 2.2 (b), (2.21) and (2.22). Part (e) is a Riemann–Lebesgue type result.

For the functions f, g defined on $[-1,\ 1]$, the Jacobi convolution is given by

$$(f*g)(x) = \int_{-1}^{1} g(u)\,(\tau_u f)(x)\, d\mu_\alpha(u),$$

whenever the integral exists. □

Lemma 2.3. *If $f\in X$, $g\in L_{1,\alpha}$, then $f*g$ exists (a.e.) and belongs to X. Furthermore, one has:*

(a) $(f*g)(x) = (g*f)(x)$

(b) $\|f*g\|_X \le \|f\|_X\, \|g\|_{1,\alpha}$ $(X=L_{p,\alpha},\ 1\le p<\infty)$,

(c) $(f*g)^\wedge(n) = \hat{f}(n)\, \hat{g}(n)$, $n\in P$.

Proof. We prove (a). By definition (see the prove of Lemma 2.2 (a)),

$$(f*g)(x) = \int_{-1}^{1} g(u)\left(\int_{-1}^{1} f(x,\, t,\, r)\, dm_\alpha(r)\right) d\mu_\alpha(u) =$$

$$=2^{\alpha+\frac{1}{2}}c_1(\alpha)\,c_2(\alpha)\int_0^1\int_{-1}^1 g\left(2u^2-1\right) f\left[2\left(uy+r\sqrt{1-u^2}\sqrt{1-y^2}\right)^2-1\right]\times$$

$$\times\left(1-r^2\right)^{\alpha-\frac{1}{2}}\left(1-u^2\right)^{\alpha} drdu=$$

$$=2^{\alpha+\frac{1}{2}}c_1(\alpha)\,c_2(\alpha)\left(1-y^2\right)^{-\alpha}\int_0^1 g\left(2u^2-1\right)\int_{uy=\sqrt{1-u^2}\sqrt{1-y^2}}^{uy+\sqrt{1-u^2}\sqrt{1-y^2}} f\left(2v^2-1\right)\times$$

$$\times\left(1-u^2-y^2-v^2+2uyv\right)^{\alpha-\frac{1}{2}} dvdu=$$

$$=2^{\alpha+\frac{1}{2}}c_1(\alpha)\,c_2(\alpha)\left(1-y^2\right)^{-\alpha}\int_0^1 f\left(2v^2-1\right)\times$$

$$\times\int_{vy-\sqrt{1-v^2}\sqrt{1-y^2}}^{vy+\sqrt{1-v^2}\sqrt{1-y^2}} g\left(2u^2-1\right)\left(1-u^2-y^2-v^2+2uyv\right)^{\alpha-\frac{1}{2}} dudv=$$

$$=2^{\alpha+\frac{1}{2}}c_1(\alpha)\,c_2(\alpha)\int_0^1\left(1-v^2\right) f\left(2v^2-1\right)\times$$

$$\times\int_{-1}^1 g\left[2\left(vy+r\sqrt{1-v^2}\sqrt{1-y^2}\right)^2-1\right]\left(1-r^2\right)^{\alpha-\frac{1}{2}} drdv=$$

$$=\int_{-1}^1 f(u)\,(\tau_u g)(x)\,d\mu_\alpha(u)=(g*f)(x)\,.$$

We prove (b). Let $X=L_{p,\,\alpha}$, $1\le p<\infty$.
By Minkovski's inequality (see [10], p. 179) and (2.10), we obtain

$$\|f*g\|_{p,\,\alpha}=\left(\int_{-1}^1\left|\int_{-1}^1 g(u)\,(\tau_x f)(u)\,d\mu_\alpha(u)\right|^p d\mu_\alpha(x)\right)^{\frac{1}{p}}\le$$

$$\le\int_{-1}^1|g(u)|\left(\int_{-1}^1|\tau_x f(u)|^p\,d\mu_\alpha(x)\right)^{\frac{1}{p}} d\mu_\alpha(u)=$$

$$=\int_{-1}^1\|\tau_u f\|_{p,\,\alpha}\,|g(u)|\,d\mu_\alpha(u)\le\|f\|_{p,\,\alpha}\,\|g\|_{1,\,\alpha}\,. \tag{2.23}$$

Let now $X = L'$. Then

$$\|f * g\|_{L'} = \sup_{|x|\le 1} \left| \int_{-1}^{1} g(u)(\tau_u f)(x)\, d\mu_\alpha(u) \right| \le$$

$$\le \int_{-1}^{1} \|\tau_u f\|_C \, |g(u)| \, d\mu_\alpha(u) \le \|f\|_C \, \|g\|_{1,\alpha} . \tag{2.24}$$

Property (b) follows from (2.23) and (2.24).

It remains to prove (c). By Fubini's theorem (see [11], p. 379) and Lemma 2.2 (c), we obtain

$$(f * g)^\wedge(n) = \int_{-1}^{1} R_n^{\left(\alpha,-\frac{1}{2}\right)}(x) \left(\int_{-1}^{1} g(u)(\tau_u f)(x)\, d\mu_\alpha(u) \right) d\mu_\alpha(x) =$$

$$= \int_{-1}^{1} R_n^{\left(\alpha,-\frac{1}{2}\right)}(x)(\tau_u f)(x)\, d\mu_\alpha(x) \int_{-1}^{1} g(u)\, d\mu_\alpha(u) =$$

$$= \int_{-1}^{1} g(u)(\tau_u f)^\wedge(n)\, d\mu_\alpha(u) =$$

$$= f\overset{\wedge}{(n)} \int_{-1}^{1} g(u)\, R_n^{\left(\alpha,-\frac{1}{2}\right)}(u)\, d\mu_\alpha(u) = f\overset{\wedge}{(n)}\, g\overset{\wedge}{(n)} .$$

Thus the lemma is proved. □

3. The Jacobi Derivative and Integral

We start with the definition of a strong (or norm) derivative.

Definition 3.1. If for $f \in X$ there exists $g \in X$ such that

$$\lim_{t\to 1-0} \left\| \frac{f - \tau_t f}{1-t} - g \right\|_X = 0,$$

then g is called a strong Jacobi derivative of f which we denote by Df. For any $r \in N$, the r-th strong derivative of f is defined with $D^0 f = f$ by $D^r f = D\left(D^{r-1} f\right)$, whenever this is meaningful. The set of all $f \in X$ for which $D^r f$ exists as an element of X, we denote by W_X^r.

Lemma 3.1. *If $f \in W_X^r$, $r \in N$, then*

$$(D^r f)^\wedge(n) = \left(\frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} \right)^r f\overset{\wedge}{(n)} , \quad n \in P. \tag{3.1}$$

Proof. Let $r=1$. Using Lemma 2.2 (c) and Lemma 2.1 (a, b), we obtain

$$\left|\frac{1-R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(t)}{1-t}\,\hat{f}(n)-(D\hat{f})\,(n)\right|=$$

$$=\left|\left(\frac{f-\tau_t f}{1-t}-Df\right)^{\wedge}(n)\right|\le\left\|\frac{f-\tau_t f}{1-t}-Df\right\|_X .$$

Since the right-hand side tends to zero as $t\to 1-0$, it follows that

$$\lim_{t\to 1-0}\frac{1-R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(t)}{1-t}\,\hat{f}(n)=(Df)^{\wedge}(n)\,.$$

Taking into account (2.2) and (2.3), we obtain

$$\lim_{t\to 1-0}\frac{1-R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(t)}{1-t}=\left(R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(t)\right)'_{t=1}=\frac{\left(n+\alpha+\frac{1}{2}\right)P_{n-1}^{\left(\alpha+1,\,\frac{1}{2}\right)}(1)}{2P_n^{\left(\alpha,\,-\frac{1}{2}\right)}(1)}=$$

$$=\frac{\left(n+\alpha+\frac{1}{2}\right)\Gamma(\alpha+1)\,\Gamma(n+1)}{2\Gamma(\alpha+2)\,\Gamma(n)}=\frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)}, \tag{3.2}$$

and then for $r=1$,

$$(Df)^{\wedge}(n)=\frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)}\,\hat{f}(n)\,.$$

The result for $r\ge 2$ follows by induction.

Lemma is proved. □

A simple consequence of this results is

Corollary 3.1. *$f\in W_X^r$ and $D^r f=0$ (a.e.) for some $r\in N$ holds if and only if $f=\mathrm{const}$ (a.e.).*

Proof. Direct assertion follows from Corollary 2.1 of Lemma 2.1. The converse follows from the definition of $D^r f$, since $(\tau_t f)(x)=f(x)$ (a.e.) if $f=\mathrm{const}$ (a.e.).

In order to define an inverse operator D^r, one has to look for a function $\psi_r\in L_{1,\,\alpha}(-1,\,1)$ whose Jacobi transform is given by

$$\psi_r{}^{\wedge}(n)=\left(\frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\right)^r,\ \ r\in N. \tag{3.3}$$

Thus Corollary is proved. □

Proposition 3.1. *The functions*

$$\psi_1(u)=2(\alpha+1)\int\limits_{-1}^{u}\left[(1-x)^{-1-\alpha}(1+x)^{-\frac{1}{2}}\int\limits_{-1}^{x}(1-t)^{\alpha}(1+t)^{-\frac{1}{2}}\,dt\right]dx,$$

$$u\in(-1,\,1)\quad \psi_r(u)=(\psi_1*\psi_{r-1})(u)\,,\ \ r=2,\,3,\ldots$$

belong to $L_{1,\,\alpha}\,(-1,\,1)$ *for each* $r \in N$*, and their Jacobi coefficients are given by* (3.3)*.*

Proof. First we'll show that $\psi_1 \in L_{1,\,\alpha}\,(-1,\,1)$.
Really

$$\int\limits_{-1}^{1} \psi_1\,(u)\,d\mu_\alpha\,(2\alpha+2)\times$$

$$\times \int\limits_{-1}^{1} \left\{ \int\limits_{-1}^{u} \left[(1-x)^{-1-\alpha}\,(1+x)^{-\frac{1}{2}} \int\limits_{-1}^{x} \frac{(1-t)^{\alpha+\frac{1}{2}}\,dt}{(1-t)^{\frac{1}{2}}\,(1+t)^{\frac{1}{2}}} \right] dx \right\} d\mu_\alpha\,(u) \le$$

$$\le 2^{\alpha+\frac{1}{2}}\,(\alpha+1) \int\limits_{-1}^{1} \left\{ \int\limits_{-1}^{u} \left[\frac{(1-x)^{-\alpha-\frac{3}{2}}}{(1+x)^{\frac{1}{2}}} \int\limits_{-1}^{x} (1+t)^{-\frac{1}{2}}\,dt \right] dx \right\} d\mu_\alpha\,(u) =$$

$$= 2^{\alpha+\frac{5}{2}}\,(\alpha+1) \int\limits_{-1}^{1} \left[\int\limits_{-1-1}^{u} (1-x)^{-\alpha-\frac{3}{2}}\,(1+x)^{-\frac{1}{2}}\,(1+t)^{\frac{1}{2}}\,|_{-1}^{x} dx \right] d\mu_\alpha\,(u) =$$

$$= 2^{\alpha+\frac{5}{2}}\,\frac{\alpha+1}{\alpha+\frac{1}{2}} \int\limits_{-1}^{1} (1-u)^{\alpha}\,(1+u)^{-\frac{1}{2}} \left((1-u)^{-\alpha-\frac{1}{2}} - 2^{-\alpha-\frac{1}{2}} \right) du =$$

$$= 2^{\alpha+\frac{7}{2}}\,\frac{\alpha+1}{2\alpha+1} \left(\int\limits_{-1}^{1} \left(1-u^2\right)^{-\frac{1}{2}} du - 2^{-\alpha-\frac{1}{2}} \int\limits_{-1}^{1} (1-u)^{\alpha}\,(1+u)^{-\frac{1}{2}}\,du \right) =$$

$$= 2^{\alpha+\frac{7}{2}}\,\frac{\alpha+1}{2\alpha+1} \arcsin u|_{-1}^{1} - \frac{8\,(\alpha+1)}{2\alpha+1}\,\frac{2^{\alpha+\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)\Gamma\,(\alpha+1)}{\Gamma\left(\alpha+\frac{3}{2}\right)} =$$

$$= 2^{\alpha+\frac{7}{2}}\,\frac{(\alpha+1)\,\pi}{2\alpha+1} - \frac{8\,(\alpha+1)}{2\alpha+1}\,\frac{2^{\alpha+\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)\Gamma\,(\alpha+1)}{\Gamma\left(\alpha+\frac{3}{2}\right)} =$$

$$= 2^{\alpha+\frac{7}{2}}\,\frac{\alpha+1}{2\alpha+1}\Gamma\left(\frac{1}{2}\right) \left(\Gamma\left(\frac{1}{2}\right) - \frac{\Gamma\,(\alpha+1)}{\Gamma\left(\alpha+\frac{3}{2}\right)} \right) = c_\alpha.$$

This implies that

$$\|\psi_1\|_{1,\,\alpha} \le c_\alpha.$$

Now we'll show that

$$\psi_r \in L_{1,\,\alpha}\,(-1,\,1)\,, \quad r = 2, 3, \dots\,.$$

Assume $r = 2$. Using Lemma 2.3 (b), we can write

$$\|\psi_2\|_{1,\,\alpha} = \|\psi_1 * \psi_1\|_{1,\,\alpha} \le \|\psi_1\|_{1,\,\alpha}^2 \le c_\alpha^2.$$

The result for $r \geq 3$ follows by induction

$$\|\psi_r\|_{1,\,\alpha} = \|\psi_1 * \psi_{r-1}\|_{1,\,\alpha} \leq \|\psi_1\|_{1,\,\alpha} \|\psi_{r-1}\|_{1,\,\alpha} \leq c_\alpha^2.$$

Now we show that for $\psi_r(u)$, $r \in N$ the equality (3.3) holds. According to Lemma 2.3, it suffices to have the differential equation (see [5], p. 73)

$$P_n^{\left(\alpha,-\frac{1}{2}\right)}(u) = -\frac{(1-u)^{-\alpha}(1+u)^{\frac{1}{2}}}{n\left(n+\alpha+\frac{1}{2}\right)} \frac{d}{du}\left[(1-u)^{\alpha+1}(1+u)^{\frac{1}{2}} \frac{d}{du} P_n^{\left(\alpha,-\frac{1}{2}\right)}(u)\right].$$

Integrating by parts, we find

$$\overset{\wedge}{\psi_1}(n) = 2(\alpha+1)\int_{-1}^{1}\left\{\int_{-1}^{u}\left[(1-x)^{-\alpha-1}(1+x)^{-\frac{1}{2}}\int_{-1}^{x} d\mu_\alpha(t)\right]dx\right\}\times$$
$$\times R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(u)\, d\mu_\alpha(u) =$$
$$= -\frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\int_{-1}^{1}\left\{\int_{-1}^{u}\left[(1-x)^{-\alpha-1}(1+x)^{-\frac{1}{2}}\int_{-1}^{x} d\mu_\alpha(t)\right]dx\right\}\times$$
$$\times d\left[(1-u)^{\alpha+1}(1+u)^{\frac{1}{2}} \frac{d}{du} R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(u)\right] =$$
$$= \frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\int_{-1}^{1} \frac{d}{du} R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(u)\int_{-1}^{u} d\mu_\alpha(t) =$$
$$= \frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\int_{-1}^{1}\left(\int_{-1}^{u} d\mu_\alpha(t)\right) dR_n^{\left(\alpha,\,-\frac{1}{2}\right)}(u) =$$
$$= \frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\left[R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(1)\int_{-1}^{1} d\mu_\alpha(t) - \int_{-1}^{1} R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(u)\, d\mu_\alpha(u)\right] =$$
$$= \frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)},$$

where we have also used Lemma 2.1 (c) and (2.4).
Thus Proposition 3.1 is proved. □

For $r \in N$, the Jacobi integral I^r can now be defined as follows:

$$(I^r f)(x) := (f * \psi_r)(x), \quad (x \in [-1,\, 1];\ f \in X). \tag{3.4}$$

Proposition 3.2. *The integral I^r is the bounded linear operator from X into itself, which satisfies for each $r \in N$ and $f \in X$, $r, s \in N$:*
(a) $(I^r I^s f)(x) = (I^s I^r f)(x) = (I^{r+s} f)(x)$, (*a.e.*)

(b) $(I^r f)^\wedge(n) = \left(\frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\right)^r \hat{f}(n)\ , \qquad n \in N;$

(c) *for* $f \in W_X^r$, *one has*

$$(I^r D^r f)(x) = f(x) - c, \qquad (a.e.)\,; \tag{3.5}$$

(d) *for any* $f,\, g \in X$, *the equality*

$$\int_{-1}^{1} g(x)\,(I^r f)(x)\, d\mu_\alpha(x) = \int_{-1}^{1} f(x)\,(I^r g)(x)\, d\mu_\alpha(x)$$

is valid.

Proof. The linearity of the operator is obvious, and the boundedness follows from the inequality (see Lemma 2.3 (b))

$$\|I^r f\|_X = \|f * \psi_r\|_X \le \|\psi_r\|_{1,\,\alpha}\,\|f\|_X \le c_\alpha^r\,\|f\|_X\,.$$

We prove (a). By definition,

$$\psi_{r+s} = \psi_1 * \psi_{r+s-1} = \psi_1 * (\psi_1 * \psi_{r+s-2}) =$$
$$= (\psi_1 * \psi_1) * \psi_{r+s-2} = \psi_2 * \psi_{r+s-2} = \cdots = \psi_r * \psi_s$$

from which we have

$$(I^r I^s f)(x) = (I^r f * \psi_s)(x) = ((f * \psi_r) * \psi_s)(x) =$$
$$= (f * (\psi_r * \psi_s))(x) = (f * \psi_{r+s})(x) = \left(I^{r+s} f\right)(x)\,.$$

We'll prove (b). By Proposition 3.1 and Lemma 2.3 (c), we obtain

$$(I^r f)^\wedge(n) = (f * \psi_r)^\wedge(n) = \hat{f}(n)\,\overset{\wedge}{\psi_r}(n) = \left(\frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\right)^r \hat{f}(n)\,.$$

We'll prove (c). By Proposition 3.2 (b) and (3.1), we have

$$(I^r D^r f)^\wedge(n) = \left(\frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\right)^r (D^r f)^\wedge(n) = \hat{f}(n), \quad n \in N. \tag{3.6}$$

Since $R_0^{\left(\alpha,\,-\frac{1}{2}\right)}(x) = 1$ (see [5], p.82), by the orthogonality of polynomials $R_0^{\left(\alpha,\,-\frac{1}{2}\right)}(x)$, we have

$$\hat{c}(n) = c\int_{-1}^{1} R_0^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\,R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\,d\mu_\alpha(x) = \begin{cases} 0, & n \in N, \\ c, & n = 0. \end{cases}$$

Then from (3.6) we find that

$$(I^r D^r f)^\wedge(n) = \hat{f}(n) - \hat{c}(n) = (f(x) - c)^\wedge(n)\,, \quad n \in N,$$

from which it follows that

$$(I^r D^r f)(n) = f(x) - c, \quad (a.e.)\,.$$

We'll prove (d).

$$\int_{-1}^{1} g(x)\left(I^{r} f\right)(x)\, d\mu_{\alpha}(x)=\int_{-1}^{1} g(x)\left(f * \psi_{r}\right)(x)\, d\mu_{\alpha}(x)=$$

$$=\int_{-1}^{1} g(x)\left\{\int_{-1}^{1} f(t)\left[\int_{-1}^{1} \psi_{r}(x, t, r)\, dm_{\alpha}(r)\right] d\mu_{\alpha}(t)\right\} d\mu_{\alpha}(x)=$$

$$=\int_{-1}^{1} f(t)\left\{\int_{-1}^{1} g(x)\left[\int_{-1}^{1} \psi_{r}(x, t, r)\, dm_{\alpha}(r)\right] d\mu_{\alpha}(x)\right\} d\mu_{\alpha}(t)=$$

$$=\int_{-1}^{1} f(t)\left[\int_{-1}^{1} g(x)\left(\tau_{t} \psi_{r}\right)(x)\, d\mu_{\alpha}(x)\right] d\mu_{\alpha}(t)=$$

$$=\int_{-1}^{1} f(t)\left(g * \psi_{r}\right)(t)\, d\mu_{\alpha}(t)=\int_{-1}^{1} f(t)\left(I^{r} g\right)(t)\, d\mu_{\alpha}(t).$$

The proposition is proved. $\square$

The analogue of (3.5) for $r=1$ in the classical analysis is

$$\int_{-1}^{x} f'(t) dt=f(x)-f(-1), \tag{3.7}$$

which holds for each continuous function f integrable derivative. Making interchange of order of integration and performing differentiation in (3.7), one can get an equation which will be valid for each continuous function f, namely,

$$\frac{d}{dx} \int_{-1}^{x} f(t)\, dt=f(x).$$

Therefore there arises the question whether the equation

$$\left(D^{r} I^{r} f\right)(x)=f(x)-c \qquad (a.e.) \tag{3.8}$$

is likewise true for each $f \in X$.

Thus we consider the function

$$(x;\, t)=\begin{cases}\frac{\alpha+1}{c_{1}(\alpha)}\left(\log \frac{2}{1+t}\right)^{-1} \int_{t}^{x}(1-u)^{-\alpha-1}(1+u)^{-\frac{1}{2}}\, du, & -1<t \leq x<1; \\ 0, & \text{otherwise },\end{cases}$$

and put $A_{t} f=f *(\cdot\,;\, t)$, $t \in(-1,\, 1)$.

Lemma 3.2. *For each $t \in (-1, 1)$ and $x \in [-1, 1]$, the function $(\cdot\,;t)$ belongs to $L_{1,\alpha}(-1, 1)$, it is nonnegative and satisfies:*

(a) $$((\cdot\,;t))^\wedge(n) = \frac{(\alpha+1)\left(1-R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(t)\right)}{n\left(n+\alpha+\frac{1}{2}\right)}\left(\log\frac{2}{1+t}\right)^{-1}, \quad n \in N;$$

(b) *For each $t \in (-1, 1)$, the A_t is the positive linear bounde operator from X into itself and*

$$\lim_{t\to 1-0}\|A_t f - f\|_X = 0 \quad (f \in X). \tag{3.9}$$

Let $n \in N$. Then by the partial integration (see the proof of Proposition 3.1.), we have

$$((\cdot\,;t))^\wedge(n) = \frac{\alpha+1}{c_1(\alpha)}\left(\log\frac{2}{1+t}\right)^{-1}\int_t^1\left[\int_t^x(1-u)^{-\alpha-1}(1+u)^{-\frac{1}{2}}\,du\right]\times$$

$$\times(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\,dx = -\frac{(\alpha+1)\left(\log\frac{2}{1+t}\right)^{-1}}{c_1(\alpha)\,n\left(n+\alpha+\frac{1}{2}\right)}\times$$

$$\times\int_t^1\left[\int_t^x(1-u)^{-\alpha-1}(1+u)^{-\frac{1}{2}}\,du\right]d\left[\frac{(1-u)^{\alpha+1}}{(1+u)^{-\frac{1}{2}}}\frac{d}{du}R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(u)\right] =$$

$$= \frac{(\alpha+1)\left(\log\frac{2}{1+t}\right)^{-1}}{n\left(n+\alpha+\frac{1}{2}\right)}\int_t^1\frac{d}{dx}R_n^{\left(\alpha,-\frac{1}{2}\right)}(x)\,dx = \frac{\alpha+1}{n\left(n+\alpha+\frac{1}{2}\right)}\frac{1-R_n^{\left(\alpha,-\frac{1}{2}\right)}(t)}{\log\frac{2}{1+t}}.$$

We can show that $X(x;t)$ belongs to $L_{1,\alpha}(-1, 1)$.

$$\|(\cdot\,;t)\|_{1,\alpha} = \frac{\alpha+1}{\log\frac{2}{1+t}}\int_t^1(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}\int_t^x(1-u)^{-\alpha-1}(1+u)^{-\frac{1}{2}}\,dudx =$$

$$= -\left(\log\frac{2}{1+t}\right)^{-1}\int_t^1(1+x)^{-\frac{1}{2}}\left[\int_t^x(1-u)^{-\alpha-1}(1+u)^{-\frac{1}{2}}\,du\right]d(1-x)^{\alpha+1}.$$

By the partial integration, we obtain

$$\|(\cdot\,;t)\|_{1,\alpha} = \left(\log\frac{2}{1+t}\right)^{-1}\times$$

$$\times\int_t^1\left[\frac{(1-x)^{-\alpha-1}}{1+x} - \frac{1}{2}(1+x)^{-\frac{3}{2}}\int_t^x(1-u)^{-\alpha-1}(1+u)^{-\frac{1}{2}}\,du\right](1-x)^{\alpha+1}\,dx =$$

$$= \left(\log\frac{2}{1+t}\right)^{-1}\times$$

$$\times\int\limits_t^1\left[\frac{1}{1+x}-\frac{1}{2}\left(1-x\right)^{\alpha+1}\left(1+x\right)^{-\frac{3}{2}}\int\limits_t^x\left(1-u\right)^{-\alpha-1}(1+u)^{-\frac{1}{2}}\,du\right]dx=$$

$$=1-\frac{1}{2\log\frac{2}{1+t}}\int\limits_t^1\left(1-x\right)^{\alpha+1}\left(1+x\right)^{-\frac{3}{2}}\int\limits_t^x\left(1-u\right)^{-\alpha-1}\left(1+u\right)^{-\frac{1}{2}}\,dudx.$$

Then we have

$$\|\left(\cdot\,;\,t\right)\|_{1,\,\alpha}\leq 1.$$

From this, taking into account Lemma 2.3 (b), we obtain

$$\|A_tf\|_X\leq\|\left(\cdot\,;\,;\,t\right)\|_{1,\,\alpha}\,\|f\|_X\leq\|f\|_X\,. \tag{3.10}$$

Further, by Lemma 2.3 (c), we have

$$\left(A_tR_n^{\left(\alpha,\,-\frac{1}{2}\right)}\right)\wedge(n)=\left(R_n^{\left(\alpha,\,-\frac{1}{2}\right)}*\left(\cdot\,;\,t\right)\right)\wedge(n)=$$

$$=\left(R_n^{\left(\alpha,\,-\frac{1}{2}\right)}\right)\wedge(n)\left(\left(\cdot\,;\,t\right)\right)\wedge(n)=$$

$$=\frac{(\alpha+1)\left(1-R_n^{\alpha,\,-\frac{1}{2}}(t)\right)}{n\left(n+\alpha+\frac{1}{2}\right)\log\frac{2}{1+t}}\left(R_n^{\left(\alpha,\,-\frac{1}{2}\right)}\right)\wedge(n)\,.$$

This implies that

$$A_tR_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)=\frac{2\left(\alpha+1\right)\left(1-R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(t)\right)}{n\left(n+\alpha+\frac{1}{2}\right)2\log\frac{2}{1+t}}R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\ \ (a.e.) \tag{3.11}$$

Using (3.2) in (3.11), by de L'Hospital's rule, we obtain

$$\lim_{t\to1-0}A_tR_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)=\frac{2\left(\alpha+1\right)}{n\left(n+\alpha+\frac{1}{2}\right)}R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\lim_{t\to1-0}\frac{1-R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(t)}{1-t}\times$$

$$\times\lim_{t\to-0}\frac{1-t}{2\log\frac{2}{1+t}}=R_n^{\left(\alpha,\,-\frac{1}{2}\right)}(x)\,,$$

from which it follows that

$$\lim_{t\to1-0}\left\|A_tR_n^{\left(\alpha,\,-\frac{1}{2}\right)}-R_n^{\left(\alpha,\,-\frac{1}{2}\right)}\right\|_X=0. \tag{3.12}$$

Using the density of the space of polynomials in the space X for any $f\in X$, there exists the polynomial $Q_n(x)$ such that for every $\varepsilon>0$ and for sufficient n,

$$\|f-Q_n\|_X<\frac{\varepsilon}{3}.$$

On the other hand (see [12], p. 334), there can appear $Q_n(x)$ in the form

$$Q_n(x) = \sum_{k=0}^{n} \alpha_k R_k^{\left(\alpha,\, -\frac{1}{2}\right)}(x),$$

where α_k is same number.

But then by (3.12) for the chosen ε, there exists δ $(0 < \delta < 1)$ such that for $t \in (1-\delta;\, 1)$, we have

$$\|A_t Q_n - Q_n\|_X \leq \sum_{k=0}^{n} \alpha_k \left\| A_t R_k^{\left(\alpha,\, -\frac{1}{2}\right)} - R_k^{\left(\alpha,\, -\frac{1}{2}\right)} \right\|_X < \frac{\varepsilon}{3} \tag{3.13}$$

Taking now into account (3.10) and (3.13), from the inequality

$$\begin{gathered} \|A_t f - f\|_X \leq \|A_t (f - Q_n)\|_X + \|A_t Q_n - Q_n\|_X + \\ + \|f - Q_n\|_X \leq 2\,\|f - Q_n\| + \|A_t Q_n - Q_n\|_X < \varepsilon \end{gathered}$$

we obtain the approval (3.9).

The following theorem is the analogue, suitable for Theorem 1 from [4], obtained for the Legendre transform.

Theorem 3.3. *The following statements are equivalent to $f \in X,\, r \in N$:*
(a) $f \in W_X^r = \{f \in X;\ D^r f \in X\}$
(b) *there exists $g_1 \in X$ such that*

$$\overset{\wedge}{g_1}(n) = \left(\frac{n\left(n + \alpha + \frac{1}{2}\right)}{2(\alpha+1)}\right)^r \hat{f}(n),$$

for any $n \in N$.

(c) *there exists $g_2 \in X$ such that*

$$f(x) = (I^r g_2)(x) + \text{const} \qquad (a.e.). \tag{3.14}$$

The functions g_1, g_2 are uniquely determined $(a.e.)$ *a part form additive constant and one has*

$$(D^r f)(x) = g_1(x) - \overset{\wedge}{g_1}(0) = g_2(x) - \overset{\wedge}{g_2}(0) \qquad (a.e.) \tag{3.15}$$

Proof. Let $f \in W_X^r$. By Lemma 3.1, one has

$$(D^r f)^\wedge(n) = \left(\frac{n\left(n + \alpha + \frac{1}{2}\right)}{2(\alpha+1)}\right)^r \hat{f}(n)$$

i.e., (b) is valid with $g_1 = D^r f$.

Let now $g_1 \in X$ and

$$g_1{}^\wedge(n) = \left(\frac{n\left(n + \alpha + \frac{1}{2}\right)}{2(\alpha+1)}\right)^r \hat{f}(n), \quad (n \in N),$$

then by Lemma 3.1, $\overset{\wedge}{g_1}(n) = (D^r f)^\wedge(n)$, from which it follows that $g_1(x) = (D^r f)(x)$ (a.e.), i.e., $D^r f \in X$. On the other hand, if (b) holds, then by Proposition 3.2 (b),

$$\hat{f}(n) = \left(\frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\right)^r g_1{}^\wedge(n) = (I^r g_1)^\wedge(n), \quad (n \in N).$$

This implies that $f(x) = (I^r g_1)(x)$ (a.e.). But since $g_1 \in X$ and $I^r : X \to X$, therefore $f \in X$.

We show that (a) is equivalent to (c). If $f \in W_X^r$, then by Proposition 3.2 (c),

$$f(x) = (I^r D^r f)(x) + c \quad (a.e.)$$

and it suffices to put $g_2 = D^r f$.

Now let (c) be satisfied with $r = 1$. We show that

$$\frac{f(x) - (\tau_t f)(x)}{1-t} = \frac{2\log\frac{2}{1+t}}{1-t}\left[(A_t g_2)(x) - \overset{\wedge}{g_2}(0)\right] \quad (a.e.). \tag{3.16}$$

On the one hand,

$$f(x) - (\tau_t f)(x) = (I g_2)(x) - (\tau_t I g_2)(x) =$$
$$= (g_2 * \psi_1)(x) - \tau_t(g_2 * \psi_1)(x).$$

By Lemmas 2.2 (c), 2.3 (c) and (3.3), we have

$$((g_2 * \psi_1) - \tau_t(g_2 * \psi)(x))^\wedge(n) = \overset{\wedge}{g_2}(n)\,\overset{\wedge}{\psi_1}(n)\left(1 - R_n^{\left(\alpha,-\frac{1}{2}\right)}(t)\right)$$
$$\frac{2(\alpha+1)}{n\left(n+\alpha+\frac{1}{2}\right)}\left(1 - R_n^{\left(\alpha,-\frac{1}{2}\right)}(t)\right)\overset{\wedge}{g_2}(n). \tag{3.17}$$

On the other hand, by Lemmas 2.3 (c), 3.2 (b) and (3.3),

$$\left(A_t g_2 - \overset{\wedge}{g_2}(0)\right)^\wedge(n) = (g_2 * X(\cdot;t))^\wedge(n) - \left(\overset{\wedge}{g_2}(0)\right)^\wedge(n) =$$
$$= \overset{\wedge}{g_2}(n)\,(X(\cdot\,;t))^\wedge(n) = \frac{\alpha+1}{n\left(n+\alpha+\frac{1}{2}\right)}\,\overset{\wedge}{g_2}(n)\,\frac{1 - R_n^{\left(\alpha,-\frac{1}{2}\right)}(t)}{\log\frac{2}{1+t}}. \tag{3.18}$$

Now (3.16) follows from (3.17) and (3.18).

Taking into account (3.9) and (3.16), we obtain

$$\left\|\frac{f - \tau_t f}{1-t} - g_2 + \overset{\wedge}{g_2}(0)\right\|_X = \left\|\frac{2\log\frac{2}{1+t}}{1-t}\left(A_t g_2 - \overset{\wedge}{g_2}(0)\right) - g_2 + \overset{\wedge}{g_2}(0)\right\|_X \le$$
$$\le \left|1 - \frac{2\log\frac{2}{1+t}}{1-t}\right|\left(\|A_t g_2\|_X + \left|\overset{\wedge}{g_2}(0)\right|\right) + \|A_t g_2 - g_2\|_X = 0(1), \ \ t \to 1-0,$$

since by de L'Hospital's rule,

$$\lim_{t\to 1-0}\frac{2\log\frac{2}{1+t}}{1-t}=\lim_{t\to 1-0}\frac{2}{1+t}=1,$$

from which it follows that (3.16) holds for $r=1$, i.e., $Df\in X$. The general case follows by induction.

We show that the presentation (3.15) is unique. We suppose that there exists $g_1\in X$ such that

$$f\left(x\right)=\left(I^r g_1\right)\left(x\right)+\text{const},$$

then

$$\left(D^r f\right)\left(x\right)=g_1\left(x\right)-\overset{\wedge}{g_1}\left(0\right). \tag{3.19}$$

From (3.15) and (3.19) it follows that for $n\in N$,

$\overset{\wedge}{g_2}\left(n\right)=\overset{\wedge}{g_1}\left(n\right)\Rightarrow g_2\left(x\right)=g_1\left(x\right)$, (a.e.)

i.e., the presentation (3.15) is unique.

Thus the theorem is proved. □

Corollary 3.2. *If $f\in W_X^r$, $r\in N$ and $g\in L_{1,\,\alpha}$, then $f*g\in W_X^r$ and*

$$\text{(a)}\quad \left(D^r\left(f*g\right)\right)\left(x\right)=\left(\left(D^r f\right)*g\right)\left(x\right)\quad (a.e.), \tag{3.20}$$

$$\text{(b)}\quad \left(D^r R_n^{\left(\alpha,\,-\frac{1}{2}\right)}\right)\left(x\right)=\left(\frac{n\left(n+\alpha+\frac{1}{2}\right)}{2\left(\alpha+1\right)}\right)^r R_n^{\left(\alpha,\,-\frac{1}{2}\right)}\left(x\right),$$

$$\left(x\in\left[-1,1\right],\, r\in N,\, n\in P\right).$$

(c) *The Jacobi integral $I^r f$ of $f\in X$ belongs to W_X^r for each $r\in N$, and one has*

$$\left(D^r I^r f\right)\left(x\right)=f\left(x\right)-\hat{f}\left(0\right)\quad (a.e.) \tag{3.21}$$

(d) *$A_t f\in W_X^r$ of $f\in X$ for each $t\in(-1,\,1)$ and*

$$\left(DA_t f\right)\left(x\right)=\frac{f\left(x\right)-\left(\tau_t f\right)\left(x\right)}{2\log\frac{2}{1+t}}\quad (a.e.),$$

moreover,

$$\left(DA_t f\right)\left(x\right)=\left(A_t Df\right)\left(x\right)\quad (a.e.).$$

(e) *The operator $D^r: W_X^r\to X$ is closed, i.e., if*

$$\lim_{n\to\infty}\left\|f_n-f\right\|_X=\lim_{n\to\infty}\left\|D^r f_n-g\right\|_X=0$$

for a sequence $\{f_n\}_{n=1}^{\infty}\in W_X^r$ and $f,\,g\in X$, then $f\in W_X^r$ and $D^r f=g$ (a.e.).

We prove (a). Let $f \in W_X^r$ and $g \in L_{1,\alpha}(-1, 1)$, then $D^r f \in X$, and by Lemma 2.3, (a) $(D^r f) * g \in X$. But by Lemma 2.3 (c) and Lemma 3.1, we have

$$((D^r f) * g)^\wedge(n) = (D^r f)^\wedge(n)\, \hat{g}(n) = \left(\frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)}\right)^r \hat{f}(n)\, \hat{g}(n)$$

$$\left(\frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)}\right)^r (f * g)^\wedge(n) = (D^r(f * g))^\wedge(n),$$

whence follows (3.20), and $f * g \in W_X^r$.

We prove (b). First, let $r = 1$. By (2.9) and (3.2), we have

$$\lim_{t\to 1-0} \frac{R_n^{\left(\alpha,-\frac{1}{2}\right)}(x) - \left(\tau_t R_n^{\left(\alpha,-\frac{1}{2}\right)}\right)(x)}{1-t} =$$

$$= \lim_{t\to 1-0} \frac{1 - R_n^{\left(\alpha,-\frac{1}{2}\right)}}{1-t} R_n^{\left(\alpha,-\frac{1}{2}\right)}(x) = \frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} R_n^{\left(\alpha,-\frac{1}{2}\right)}(x).$$

The last means that $\forall \varepsilon > 0$ $\exists \delta(\varepsilon) > 0$ such that $\forall t \in (1-\delta, 1)$ and $\forall x \in [-1, 1]$

$$\left| \frac{R_n^{\left(\alpha,-\frac{1}{2}\right)}(x) - \left(\tau_t R_n^{\left(\alpha,-\frac{1}{2}\right)}\right)(x)}{1-t} - \frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} R_n^{\left(\alpha,-\frac{1}{2}\right)}(x) \right| < \varepsilon,$$

whence it follows that for $\forall t \in (1-\delta, 1)$,

$$\left\| \frac{R_n^{\left(\alpha,-\frac{1}{2}\right)} - \left(\tau_t R_n^{\left(\alpha,-\frac{1}{2}\right)}\right)}{1-t} - \frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} R_n^{\left(\alpha,-\frac{1}{2}\right)} \right\| < \varepsilon,$$

i.e., $\left(DR_n^{\left(\alpha,-\frac{1}{2}\right)}\right)(x) = \frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} R_n^{\left(\alpha,-\frac{1}{2}\right)}(x)$.

The general case for $r \geq 2$ follows by induction. If in Theorem 3.1 (c) we put $g_2 = f$, then we obtain (3.21) and the assertion $I^r f \in W_X^r$.

We prove (d). By Lemmas 3.1 and 2.3, we have

$$(DA_t f)^\wedge(n) = \frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} (A_t f)^\wedge(n)$$

$$\frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} (f * (\cdot\,; t))^\wedge(n) = \frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} \hat{f}(n)\, \overset{\wedge}{(n)}. \tag{3.22}$$

From (3.22), according to Lemma 3.2 (a), we obtain

$$(DA_t f)^\wedge(n) = \frac{1 - R_n^{\left(\alpha,-\frac{1}{2}\right)}(t)}{2\log\frac{2}{1+t}} \hat{f}(n) = \left(\frac{f - \tau_t f}{2\log\frac{2}{1+t}}\right)^\wedge(n),$$

from which it follows that

$$(DA_t f)(x) = \frac{f(x) - (\tau_t f)(x)}{2\log\frac{2}{1+t}} \quad (a.e.).$$

Further, if $f \in W_X^r$, then

$$(A_t Df)\wedge(n) = (Df * (\cdot;\, t))\wedge(n) = (Df)\wedge(n)\,((\cdot;\, t))\wedge(n) =$$
$$= \frac{n\left(n+\alpha+\frac{1}{2}\right)}{2(\alpha+1)} f\wedge(n)\ \overset{\wedge}{(n)}\,. \tag{3.23}$$

From (3.22) and (3.23) follows

$$(DA_t f)(x) = (A_t Df)(x) \quad (a.e.).$$

Finally, we prove (e). From (3.15), we have

$$(D^r f)(x) = g_1(x) - g_1\wedge(0)\,, \ \ g_1 \in X$$

and

$$(D^r f_n)(x) = g_{1n}(x) - g_{1n}\wedge(0)\,, \ \ g_{1n} \in X, \ n = 1, 2, \ldots\,.$$

By supposition

$$\lim_{n\to\infty} \|g_1 - g_{1n}\|_X = 0.$$

But then along with Lemma 2.1 (a), we have

$$|\,(D^r f - g)\wedge(n)\,| \le \|D^r f - g\|_X \le$$
$$\le \|D^r f - D^r f_n\|_X + \|D^r f_n - g\|_X \le$$
$$\le \|D^r f_n - g\|_X + \|g_1 - g_{1n}\|_X + |g_1\wedge(0) - g_{1n}\wedge(0)\,| \le \|D^r f_n - g\|_X +$$
$$+2\|g_1 - g_{1n}\|_X \to 0 \quad \text{as} \quad n \to \infty,$$

which implies that $(D^r f)(x) = g(x)$ (a.e.), but $g \in X \Rightarrow f \in W_X^r$.

Thus Corollary is proved.

4. Auxiliary Assertions

In this section we prove some facts that we'll need later on, though they are of independent interest.

In what follows, by M we denote positive constants, independent of x and n.

Lemma 4.1. *Let* $-\frac{1}{2} < \alpha < \frac{1}{2}$. *For* $n = 2, 3, \ldots$, *the estimations*

$$\left|\int_{-1}^{t} (1-x)^{\alpha}(1+x)^{-\frac{1}{2}} P_n^{(\alpha,-\frac{1}{2})}(x)dx\right| \le$$
$$\le M(1-t)^{\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{1}{2}} n^{-\frac{3}{2}}, t \in (-1, 1)\,, \tag{4.1}$$

$$\left|\int_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx\right| \le Mn^{\frac{2}{2\alpha+3}-\frac{3}{2}}, t\in[-1,1] \tag{4.2}$$

are valid.

Here and in the sequel, M will denote different constants.

Proof. Using the formula ([5], p. 73)

$$P_n^{(\alpha,-\frac{1}{2})}(x) = -\frac{1}{n(n+\alpha+\frac{1}{2})}\Big\{(1-x^2)\frac{d^2}{dx^2}P_n^{(\alpha,-\frac{1}{2})}(x)-$$
$$-\left[\frac{1}{2}+\alpha+\Big(\alpha+\frac{3}{2}\Big)x\right]\frac{d}{dx}P_n^{(\alpha,-\frac{1}{2})}(x)\Big\},$$

we obtain

$$\int_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx =$$
$$= -\frac{1}{n(n+\alpha+\frac{1}{2})}\left\{\int_{-1}^{t}(1-x)^{\alpha+1}(1+x)^{\frac{1}{2}}\frac{d^2}{dx^2}P_n^{(\alpha,-\frac{1}{2})}(x)-\right.$$
$$\left.-\int_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}\left[\frac{1}{2}+\alpha+\left(\alpha+\frac{3}{2}\right)x\right]\frac{d}{dx}P_n^{(\alpha,-\frac{1}{2})}(x)dx\right\} =$$
$$= -\frac{1}{n(n+\alpha+1/2)}\left\{(1-x)^{\alpha+1}(1+x)^{\frac{1}{2}}\frac{d}{dx}P_n^{(\alpha,-\frac{1}{2})}(x)\Big|_{-1}^{t}+\right.$$
$$+\int_{-1}^{\mathrm{t}}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}\left[\frac{1}{2}+\alpha+\left(\alpha+\frac{3}{2}\right)x\right]\frac{d}{dx}P_n^{(\alpha,-\frac{1}{2})}(x)dx-$$
$$\left.-\int_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}\left[\frac{1}{2}+\alpha+\left(\alpha+\frac{3}{2}\right)x\right]\frac{d}{dx}P_n^{(\alpha,-\frac{1}{2})}(x)dx\right\} =$$
$$= -\frac{1}{n(n+\alpha+1/2)}(1-t)^{\alpha+1}(1+t)^{\frac{1}{2}}\frac{d}{dt}P_n^{(\alpha,-\frac{1}{2})}(t). \tag{4.3}$$

Since ([5], pp. 84, 82)

$$(1-x^2)\frac{d}{dx}P_n^{(\alpha,-\frac{1}{2})}(x) = \frac{(n+\alpha+\frac{1}{2})[(2n+\alpha+\frac{3}{2})x+\alpha+\frac{1}{2}]P_n^{(\alpha,-\frac{1}{2})}(x)}{2n+\alpha+\frac{3}{2}}-$$
$$-\frac{2(n+1)(n+\alpha+\frac{1}{2})}{2n+\alpha+\frac{3}{2}}P_{n+1}^{(\alpha,-\frac{1}{2})}(x) = \left(n+\alpha+\frac{1}{2}\right)xP_{n+1}^{(\alpha,-\frac{1}{2})}(x)+$$

$$+\frac{(\alpha+\frac{1}{2})(n+a+\frac{1}{2})}{2n+\alpha+\frac{3}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)-\frac{2(n+1)(n+\alpha+\frac{1}{2})}{2n+\alpha+\frac{3}{2}}P_{n+1}^{(\alpha,-\frac{1}{2})}(x) \quad (4.4)$$

and

$$2(n+1)\left(n+\alpha+\frac{1}{2}\right)\left(2n+\alpha-\frac{1}{2}\right)P_{n+1}^{(\alpha,-\frac{1}{2})}(x)=$$
$$=\left(2n+\alpha+\frac{1}{2}\right)\left[\left(2n+\alpha+\frac{3}{2}\right)\left(2n+\alpha-\frac{1}{2}\right)x+\alpha^2-\frac{1}{4}\right]P_n^{(\alpha,-\frac{1}{2})}(x)-$$
$$-2(n+\alpha)\left(n-\frac{1}{2}\right)\left(2n+\alpha+\frac{3}{2}\right)P_{n+1}^{(\alpha,-\frac{1}{2})}(x),$$

therefore

$$\left(2n+\alpha+\frac{1}{2}\right)\left(2n+\alpha+\frac{3}{2}\right)\left(2n+\alpha-\frac{1}{2}\right)xP_n^{(\alpha,-\frac{1}{2})}(x)=$$
$$=2(n+1)\left(n+\alpha+\frac{1}{2}\right)\left(2n+\alpha-\frac{1}{2}\right)P_{n+1}^{(\alpha,-\frac{1}{2})}(x)+$$
$$+2(n+\alpha)\left(n-\frac{1}{2}\right)\left(2n+\alpha+\frac{3}{2}\right)P_{n+1}^{(\alpha,-\frac{1}{2})}(x)-$$
$$-\left(2n+\alpha+\frac{1}{2}\right)\left(\alpha^2-\frac{1}{4}\right)P_n^{(\alpha-\frac{1}{2})}(x).$$

Further,

$$xP_n^{(\alpha,-\frac{1}{2})}(x)=\frac{2(n+1)(n+\alpha+\frac{1}{2})}{(2n+\alpha+\frac{1}{2})(2n+\alpha+\frac{3}{2})}P_{n+1}^{(\alpha,-\frac{1}{2})}(x)+$$
$$+\frac{2(n+\alpha)(n-\frac{1}{2})}{(2n+\alpha+\frac{1}{2})(2n+\alpha-\frac{1}{2})}P_{n+1}^{(\alpha,-\frac{1}{2})}(x)+$$
$$+\frac{\frac{1}{4}-\alpha^2}{(2n+\alpha+\frac{3}{2})(2n+\alpha-\frac{1}{2})}P_n^{(\alpha,-\frac{1}{2})}(x). \quad (4.5)$$

Using (4.5) in (4.4) and (4.3), we obtain

$$\int_{-1}^{t}(1-x)^\alpha(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx=\frac{(1-t)^\alpha(1+t)^{-\frac{1}{2}}}{n(n+\alpha+\frac{1}{2})}\times$$
$$\times\Bigg\{\frac{2(n+1)(n+\alpha+\frac{1}{2})}{2n+\alpha+\frac{3}{2}}P_{n+1}^{(\alpha,-\frac{1}{2})}(x)-\frac{(\alpha+\frac{1}{2})(n+\alpha+\frac{1}{2})}{2n+\alpha+\frac{3}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)-$$
$$-\frac{2(n+1)(n+\alpha+\frac{1}{2})^2}{(2n+\alpha+\frac{1}{2})(2n+\alpha+\frac{3}{2})}P_{n+1}^{(\alpha,-\frac{1}{2})}(x)-$$
$$-\frac{2(n+\alpha)(n-\frac{1}{2})(n+\alpha+\frac{1}{2})}{(2n+\alpha+\frac{3}{2})(2n+\alpha-\frac{1}{2})}P_{n+1}^{(\alpha,-\frac{1}{2})}(x)-$$

$$-\frac{(\frac{1}{4}-\alpha^2)(n+\alpha+\frac{1}{2})}{(2n+\alpha+\frac{3}{2})(2n+\alpha-\frac{1}{2})}P_{n+1}^{(\alpha,-\frac{1}{2})}(x)\Big\},$$

which implies that

$$\int_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx=O(\frac{1}{n})(1-t)^{\alpha}(1+t)^{-\frac{1}{2}}\left|P_n^{(\alpha,-\frac{1}{2})}(t)\right|.$$

Using estimation of the latter correlation ([6], p. 265)

$$(1-x)^{\frac{\alpha}{2}+\frac{1}{4}}\left|P_n^{(\alpha,-\frac{1}{2})}(x)\right|\le Mn^{-\frac{1}{2}},\ \ \alpha\ge-\frac{1}{2},\ \ -1\le x\le 1, \tag{4.6}$$

we obtain (4.1).

Now we prove inequality (4.2). Let $-1\le t\le -1+n^{-4/(3+2\alpha)}$.

Then using (4.6), we obtain

$$\left|\int_{-1}^{\mathrm{t}}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx\right|\le$$

$$\le\int_{-1}^{-1+n^{-4/(3+2\alpha)}}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}\left|P_n^{(\alpha,-\frac{1}{2})}(x)\right|dx\le$$

$$\le Mn^{-\frac{1}{2}}\int_{-1}^{-1+n^{-4/(3+2\alpha)}}(1-x)^{\frac{\alpha}{2}-\frac{1}{4}}(1+x)^{-\frac{1}{2}}dx\le$$

$$\le Mn^{-\frac{1}{2}}\int_{-1}^{-1+n^{-4/(3+2\alpha)}}(1+x)^{-\frac{1}{2}}dx\le Mn^{-\frac{1}{2}}n^{-\frac{2}{3+2a}}\le Mn^{\frac{2}{2\alpha+3}-\frac{3}{2}}. \tag{4.7}$$

Let now $-1+n^{-4/(3+2\alpha)}\le t\le\delta<1$.

Then from (4.1) we obtain

$$\left|\int_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx\right|\le Mn^{\frac{2}{2\alpha+3}-\frac{3}{2}}. \tag{4.8}$$

Let $\delta\le t\le 1-n^{-\frac{4}{3+2\alpha}}$. Again, from (4.1), we have

$$\left|\int_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx\right|\le Mn^{(\frac{1}{4}-\frac{\alpha}{2})\frac{4}{2\alpha+3}-\frac{3}{2}}=$$

$$=Mn^{\frac{1-2\alpha}{3+2\alpha}-\frac{3}{2}}\le M\cdot n^{\frac{2}{2\alpha+3}-\frac{3}{2}}. \tag{4.9}$$

At last, let

$$1-n^{-\frac{4}{3+2\alpha}}\le t\le 1.$$

Then we have

$$\int_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx =$$

$$= \int_{-1}^{1-n^{-\frac{4}{3+2\alpha}}}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx +$$

$$+\int_{1-n^{-\frac{4}{3+2\alpha}}}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx = J_1 + J_2. \tag{4.10}$$

Applying the estimates (4.7)–(4.9) for J_1, we obtain the estimate (4.11)

$$J_1 \le Mn^{\frac{2}{2a+3}-\frac{3}{2}}. \tag{4.11}$$

It remains to estimate J_2. Using inequality (4.6), we obtain

$$J_2 \le \int_{1-n^{-\frac{4}{3+2\alpha}}}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}\left|P_n^{(\alpha,-\frac{1}{2})}(x)\right|dx \le$$

$$\le Mn^{-\frac{1}{2}}\int_{1-n^{-\frac{4}{3+2\alpha}}}^{t}(1-x)^{\frac{\alpha}{2}-\frac{1}{4}}(1+x)^{-\frac{1}{2}}dx \le$$

$$\le Mn^{-\frac{1}{2}}\int_{1-n^{-\frac{4}{3+2\alpha}}}^{t}(1-x)^{\frac{\alpha}{2}-\frac{1}{4}}dx \le Mn^{-\frac{1}{2}}n^{-\left(\frac{\alpha}{2}+\frac{3}{4}\right)\frac{4}{3+2\alpha}} = Mn^{-\frac{3}{2}}. \tag{4.12}$$

Using (4.11) and (4.12) in (4.10), we obtain for which $1-n^{-4/(3+2a)} \le t \le 1$ the estimate

$$\left|\int_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(x)dx\right| \le Mn^{\frac{2}{2\alpha+3}-\frac{3}{2}}. \tag{4.13}$$

is valid. Now, using (4.7), (4.8), (4.9) and (4.13), we obtain inequality (4.2). Thus Lemma 4.1 is proved. □

Since ([5], p. 70 and [11], p. 131)

$$P_n^{(\alpha,-\frac{1}{2})}(1) = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)} = \frac{n^{\alpha}}{\Gamma(\alpha+1)}\left\{1+O\Big(\frac{1}{n}\Big)\right\}, \tag{4.14}$$

for Lemma 4.1 we get the following

Corollary 4.1. *For $n=2,3,\ldots;\ -\frac{1}{2}<\alpha<\frac{1}{2}$, the estimates*

$$\left|\int_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}R_n^{(a,-\frac{1}{2})}(x)dx\right|\leq$$

$$\leq M\begin{cases}(1-t)^{\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{1}{2}}n^{-\alpha-\frac{3}{2}}, & t\in(-1,1)\,,\\ n^{\frac{2}{2\alpha+3}-\frac{3}{2}-\alpha}, & t\in[-1,1]\end{cases}$$

are true.

Lemma 4.2. *Let $-\frac{1}{2}<\alpha<\frac{1}{2}$. Then for $n=2,3,\ldots$, the estimates*

$$\left|\int_{-1}^{t}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}(I^rR_n^{(\alpha,-\frac{1}{2})})(x)dx\right|\leq$$

$$\leq M\begin{cases}(1-t)^{\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{1}{2}}n^{-2r-\alpha-\frac{3}{2}}, & t\in(-1,1)\,,\\ n^{\frac{2}{2\alpha+3}-\frac{3+2\alpha}{2}-2r}, & t\in[-1,1]\end{cases}$$

are true.

Proof. From (2.4) and (1.9), we have

$$\int_{-1}^{t}\left(I^rR_n^{(\alpha,-\frac{1}{2})}\right)(x)d\mu_\alpha(x)=\int_{-1}^{t}\left(R_n^{(\alpha,-\frac{1}{2})}*\psi_r\right)(x)d\mu_\alpha(x)=$$

$$=\int_{-1}^{t}\int_{-1}^{1}\left(\tau_uR_n^{(\alpha,-\frac{1}{2})}\right)(x)\psi_r(u)\,d\mu_\alpha(u)\,d\mu_\alpha(x)=$$

$$=\int_{-1}^{1}\psi_r(u)R_n^{(\alpha,-\frac{1}{2})}(u)d\mu_\alpha(u)\int_{-1}^{t}R_n^{(\alpha,-\frac{1}{2})}(x)d\mu_x=$$

$$=\overset{\wedge}{\psi_r}(n)\int_{-1}^{t}R_n^{(\alpha,-\frac{1}{2})}(x)d\mu_\alpha(x)=$$

$$=\overset{\wedge}{\psi_r}(n)\int_{-1}^{t}R_n^{(\alpha,\,-\frac{1}{2})}(x)d\mu_\alpha(x).\tag{4.15}$$

Using (2.3), Corollary 4.1 in (4.15), we obtain assertions of Lemma 4.2. □

Lemma 4.3. *For $n = 1, 2, \ldots$, the estimates*

$$\left\| R_n^{(\alpha,-\frac{1}{2})} \right\|_{q,\alpha} \le M \begin{cases} n^{-\frac{1}{q}}, & (\alpha+\frac{1}{2})q > 1, \\ n^{-(\alpha+\frac{1}{2})}\sqrt{\log n}, & (\alpha+\frac{1}{2})q = 1, \\ n^{-(\alpha+\frac{1}{2})}, & (\alpha+\frac{1}{2})q < 1. \end{cases}$$

are true.

$$\left\| R_n^{(\alpha,-\frac{1}{2})} \right\|_{q,\alpha} = C_1(\alpha) \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{-\frac{1}{2}} \left| R_n^{(\alpha,-\frac{1}{2})}(x) \right| dx =$$

$$= 2^{a+\frac{3}{2}} C_1(\alpha) \int_0^1 (1-x^2)^{\alpha} \left| R_n^{(\alpha,-\frac{1}{2})}(2x^2-1) \right| dx. \tag{4.16}$$

Using the equality ([5], p. 71)

$$P_n^{(\alpha,-\frac{1}{2})}(2x^2-1) = \frac{\Gamma(n+1+\alpha)\Gamma(2n+1)}{\Gamma(n+1)\Gamma(2n+\alpha+1)} P_{2n}^{(\alpha,\alpha)}(x)$$

and (4.14), we find

$$R_n^{(\alpha,-\frac{1}{2})}(2x^2-1) = \frac{\Gamma(1+\alpha)\Gamma(2n+1)}{\Gamma(2n+\alpha+1)} P_{2n}^{(\alpha,\alpha)}(x). \tag{4.17}$$

Using (4.17) in (4.16), we obtain

$$\left\| R_n^{(\alpha,-\frac{1}{2})} \right\|_{q,\mu}^q =$$

$$= 2^{\alpha+\frac{3}{2}} C_1(\alpha) \left\{ \frac{\Gamma(1+\alpha)\Gamma(2n+1)}{\Gamma(2n+\alpha+1)} \right\}^q \int_0^1 (1-x^2)^{\alpha} \left| P_{2n}^{(\alpha,\alpha)}(x) \right|^q dx.$$

by the Cauchy-Buniakowsky's inequality

$$\left\| R_n^{(\alpha,-\frac{1}{2})} \right\|_{q,\alpha}^q \le$$

$$\le 2^{\alpha+\frac{3}{2}} C_1(\alpha) \left\{ \frac{\Gamma(\alpha+1)\Gamma(2n+1)}{\Gamma(2n+\alpha+1)} \right\}^q \left(\int_0^1 (1-x^2)^{2\alpha} dx \right)^{\frac{1}{2}} \le$$

$$\le \left(\int_0^1 |P_{2n}^{(\alpha,\alpha)}(x)|^{2q} dx \right)^{\frac{1}{2}} \le$$

$$\le C_3(\alpha) \left(\int_0^1 |P_{2n}^{(\alpha,\alpha)}(x)|^{2q} dx \right)^{\frac{1}{2}} \tag{4.18}$$

where

$$C_3(\alpha)=2^{\alpha+1}C_1(\alpha)\left\{\frac{\Gamma(2\alpha+1)\Gamma(\frac{1}{2})}{\Gamma(2\alpha+\frac{3}{2})}\right\}^{\frac{1}{2}}\left\{\frac{\Gamma(\alpha+1)\Gamma(2n+1)}{\Gamma(2n+\alpha+1)}\right\}^{q}\leq Mn^{-\alpha q}$$

From the inequality ([5], p. 177)

$$\left|P_{2n}^{(\alpha,\alpha)}(x)\right| \leq Mn^{-\frac{1}{2}}(1-x+n^{-2})^{-\frac{\alpha}{2}-\frac{1}{4}},\ 0\leq x\leq 1$$

it follows that

$$\left(\int\limits_0^1 \left|P_{2n}^{(\alpha,\alpha)}(x)\right|^{2q} dx\right)^{\frac{1}{2}} \leq Mn^{-\frac{q}{2}}\begin{cases} n^{\left(\alpha+\frac{1}{2}\right)q-1}, & (\alpha+\frac{1}{2})q>1,\\ \sqrt{\log n}, & (\alpha+\frac{1}{2})q=1,\\ 1, & (\alpha+\frac{1}{2})q<1,\end{cases} \leq$$

$$\leq M\begin{cases} n^{q\alpha-1}, & (\alpha+\frac{1}{2})q>1,\\ n^{-q/2}\sqrt{\log n}, & (\alpha+\frac{1}{2})q=1,\\ n^{-q/2}, & (\alpha+\frac{1}{2})q<1.\end{cases} \tag{4.19}$$

are true.

Using (4.14) and (4.19) in (4.15), we obtain the assertion of Lemma 4.3.

Lemma 4.4. *For* $n=1,2,\ldots$ *the estimates*

$$\left\|I^r R_n^{(\alpha,-\frac{1}{2})}\right\|_{q,\mu}\leq M\begin{cases} n^{-2r-\alpha-1/2}, & (\alpha+\frac{1}{2})q<1,\\ n^{-2r-\alpha-1/2}\sqrt{\log n}\frac{1}{2q}, & (\alpha+\frac{1}{2})\sqrt{\log n},\\ n^{-2r-1/q}, & (\alpha+\frac{1}{2})q>1\end{cases}$$

are true.

Proof. By (2.4) and (1.9), we obtain

$$\left\|I^r R_n^{(\alpha,-\frac{1}{2})}\right\|_{q,\alpha}=\left\|R_n^{(\alpha,-\frac{1}{2})}*\psi_r\right\|_{q,\alpha}=$$

$$=\left\|\int\limits_{-1}^{1}\psi_r(u)\left(\tau_u R_n^{(\alpha,-\frac{1}{2})}(\cdot)\,d\mu_\alpha(u)\right)\right\|_{q,\alpha}=$$

$$=\left\|\int\limits_{-1}^{1}\psi_r(u)R_n^{(\alpha,-\frac{1}{2})}(u)R_n^{(\alpha,-\frac{1}{2})}(\cdot)\,d\mu_\alpha(u)\right\|_{q,\alpha}=$$

$$=\left\|\overset{\wedge}{\psi_r}(n)\,R_n^{(\alpha,-\frac{1}{2})}\right\|_{q,\alpha}=\overset{\wedge}{\psi_r}(n)\left\|R_n^{(\alpha,-\frac{1}{2})}\right\|_{q,\alpha},$$

from which using (3.3) and Lemma 4.3, we obtain the statement of Lemma 4.4. $\square$

Lemma 4.5. *Let* $-\dfrac{1}{2} < \alpha < \dfrac{1}{2}$. *For* $r = 0, 1, \dots$ *and* $n = 1, 2, \dots$ *the equality*

$$\left\| I^r R_n^{(\alpha,-\frac{1}{2})} \right\|_{C[-1,1]} = \overset{\wedge}{\psi_r}(n) = \left\{ \frac{2(\alpha+1)}{n(n+\alpha+\frac{1}{2})} \right\}^r$$

is valid.

Proof. By (2.7) and (3.3), we obtain

$$\left\| I^r R_n^{(\alpha,-\frac{1}{2})} \right\|_{C[-1,1]} = \left\| R_n^{(\alpha,-\frac{1}{2})} * \psi \right\|_{C[-1,1]} =$$

$$= \left\| \int_{-1}^{1} \psi_r(u) \left(\tau_u R_n^{(\alpha,-\frac{1}{2})} \right)(\cdot) d\mu_\alpha(u) \right\|_{C[-1,1]} =$$

$$= \left\| \int_{-1}^{1} \psi_r(u) R_n^{(\alpha,-\frac{1}{2})}(u) d\mu_\alpha(u) R_n^{(\alpha,-\frac{1}{2})}(\cdot) \right\|_{C[-1,1]} =$$

$$= \overset{\wedge}{\psi_r}(n) \left\| R_n^{(\alpha,-\frac{1}{2})}(\cdot) \right\|_{C[-1,1]} = \overset{\wedge}{\psi_r}(n) = \left\{ \frac{2(\alpha+1)}{n(n+\alpha+\frac{1}{2}} \right\}^r.$$

Thus Lemma 4.5 is proved. □

Lemma 4.6. *Let* $-\dfrac{1}{2} < \alpha < \dfrac{1}{2}$. *For* $n = 2, 3, \dots$ *and* $r = 0, 1, \dots,$ *the estimate*

$$\left\| I^r R_n^{(\alpha,-\frac{1}{2})} \right\|_{1,\alpha} \le M n^{-2r-\alpha-\frac{1}{2}}$$

is valid.

Proof. Also, as when proving Lemma 4.4, we find

$$\left\| \left(I^r R_n^{(\alpha,-\frac{1}{2})} \right) \right\|_{1,\alpha} = \overset{\wedge}{\psi_r}(n) \left\| R_n^{(\alpha,-\frac{1}{2})} \right\|_{1,\alpha}. \tag{4.20}$$

Since (see proof of Lemma 4.3)

$$\|R_n^{(\alpha,-\frac{1}{2})}\|_{1,\alpha} = 2^{\alpha+\frac{3}{2}} C_1(\alpha) \frac{\Gamma(\alpha+1)\Gamma(2n+1)}{\Gamma(2n+\alpha+1)} \int_0^1 (1-x^2)^\alpha |P_{2n}^{(\alpha,\alpha)}(x)| dx,$$

using the estimate (4.6) and the relation (4.14), we obtain

$$\|R_n^{(\alpha,-\frac{1}{2})}\|_{1\alpha} \le M n^{-\alpha-1/2} \int_0^1 (1-x^2)^{\frac{\alpha}{2}-\frac{1}{4}} dx \le$$

$$\le M n^{-\alpha-1/2} \int_0^1 (1-x)^{\frac{\alpha}{2}-\frac{1}{4}} x^{-\frac{1}{2}} dx = M \frac{\Gamma\left(\frac{2\alpha+3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2\alpha+5}{4}\right)} n^{-\alpha-\frac{1}{2}}. \tag{4.21}$$

Using Lemma 4.5 and (4.12) in (4.20), we obtain the assertion of Lemma 4.6. □

5. On Estimations of Coefficients of Fourier–Jacobi Functions from W_X^r

In this section we give applications of general theorems from Section 1. We formulate them in conformity with our case.

Theorem A. Let $f \in W_X^r \left(X = L_{p,\alpha},\ \ 1 < p < \infty\right)$, $\frac{1}{p} + \frac{1}{q} = 1$. If

1. $\|I^r \varphi_n\|_{q,\alpha} \leq M -$ const., $q > 1$, $r = 0, 1, \ldots$;

2. $\lim_{n\to\infty} \int_{-1}^{t} \left(I^r \varphi_n\right)(x) d\mu_\alpha(x) = 0$, $t \in [-1, 1]$,

then

$$\lim_{n\to\infty} \int_{-1}^{1} f(x)\varphi_n(x) d\mu_\alpha(x) = 0,$$

where $\varphi_n(x)$, $n = 0, 1, \ldots$ is the system of orthogonal functions with the weight $\mu_\alpha(x) = C_1(\alpha)(1-x)^\alpha(1+x)^{-\frac{1}{2}}$ on the segment $[-1, 1]$.

Theorem B. Let $f \in W_X^r \left(X = L_{1,\alpha}\right)$. If

1. $|\left(I^r \varphi_n(x)\right)| \leq M -$ const, $r = 0, 1, \ldots$, $x \in [-1, 1]$;

2. $\lim_{n\to\infty} \int_{-1}^{t} \left(I^r \varphi_n\right)(x) d\mu_\alpha(x) = 0,\ t \in [-1, 1]$,

then

$$\lim_{n\to\infty} \int_{-1}^{1} f(x)\varphi_n(x) d\mu_\alpha(x) = 0.$$

Theorem C. Let $f \in W_X^r \big(X = L'\big)$. If

1. $\left|\int_{-1}^{x} \left(I^r \varphi_n\right)(t) d\mu_\alpha(t)\right| \leq M\ \ -$ const, $r = 0, 1, \ldots$, $x \in [-1, 1]$,

2. $\lim_{n\to\infty} \int_{-1}^{t} \left(I^r \varphi_n\right)(x) d\mu_\alpha(x) = 0,\ t \in [-1, 1]$,

then

$$\lim_{n\to\infty} \int_{-1}^{1} f(x)\varphi_n(x) d\mu_\alpha(x) = 0.$$

To each $f \in X$ let us now associate its Fourier–Jacobi series

$$f(x) \sim \sum_{n=0}^{\infty} \frac{(2n+\alpha+\frac{1}{2})\Gamma(n+\alpha+\frac{1}{2})\Gamma(n+\alpha+1)}{2^{\alpha+\frac{1}{2}}\Gamma^2(\alpha+1)\Gamma(n+\frac{1}{2})\Gamma(n+1)} \hat{f}(n)\, R_n^{(\alpha,-\frac{1}{2})}(x),$$

where

$$\hat{f}(n) = \int_{-1}^{1} f(t) R_n^{(\alpha,-\frac{1}{2})}(t) d\mu_\alpha(t).$$

Theorem 5.1. *Let* $f \in W_X^r$ $(X = L_{p,\alpha}, 1 < p < \infty)$, *then*

$$\lim_{n\to\infty} n^{2r+\alpha+\frac{1}{2}} \hat{f}(n) = 0, \;\; 0 < \Big(\alpha + \frac{1}{2}\Big) q < 1, \;\; r = 0, 1, \dots .$$

Proof. Proof of the theorem is reduced to the verification of the condition of Theorem A. According to (5.2),

$$n^{2r+\alpha+\frac{1}{2}} \hat{f}(n) = \int_{-1}^{1} \varphi_n(x) d\mu_\alpha(x),$$

where

$$\varphi_n^{(\alpha,-\frac{1}{2})} \equiv \varphi_n(x) = n^{2r+\alpha+\frac{1}{2}}(x) R_n^{(\alpha,-\frac{1}{2})}(x).$$

The first condition of Theorem A follows easily from Lemma 4.4.

$$\|I^r \varphi_n\|_{q,\alpha} = n^{2r+\alpha+\frac{1}{2}} \|R_n^{(\alpha,-\frac{1}{2})}\|_{q,\alpha} \le M,$$

and by Lemma 4.2, we have

$$\left| \int_{-1}^{t} (I^r \varphi_n)(x)\, d\mu_\alpha(x) \right| = n^{2r+\alpha+\frac{1}{2}} \left| \int_{-1}^{t} \Big(I^r R_n^{\alpha,-\frac{1}{2}} \Big)(x)\, d\mu_\alpha(x) \right| \le$$

$$\le M \begin{cases} (1-t)^{\frac{\alpha}{2}-\frac{1}{4}} (1+t)^{\frac{-1}{2}} \frac{1}{n}, & t \in (-1,\, 1), \\ n^{-\frac{2\alpha+1}{2\alpha+3}}, & t \in [-1,\, 1]. \end{cases}$$

From this we obtain condition 2 of Theorem A.

Thus Theorem 5.1 is proved. □

Remark. If $f \in W_X^r$ $(X = L_{p,\alpha})$, then for each sequence of real numbers, γ_n tends to infinity

$$\lim_{n\to\infty} n^{2r+\alpha+\frac{1}{2}} \hat{f}(n)\, \gamma_n,$$

not approaching zero.

Proof. Just as in proving Lemma 4.2, we obtain

$$\|\gamma_n (I^r \varphi_n)\|_{2,\mu} = n^{2r+\alpha+\frac{1}{2}} \overset{\wedge}{\psi_r}(n) \left\| R_n^{(\alpha,-\frac{1}{2})} \right\|_{2,\mu} \gamma_n.$$

Using (1.1) and (1.2), we get

$$\|R_n^{(\alpha,-1/2)}\|_{2,\mu} = \left\{ \int_{-1}^{1} \Big[R_n^{(\alpha,-\frac{1}{2})}(x) \Big]^2 d\mu_\alpha(x) \right\}^{\frac{1}{2}} =$$

$$= \left\{ C_1(\alpha) \frac{2^{\alpha+\frac{1}{2}} \Gamma(n+\alpha+1) \Gamma(n+\frac{1}{2})}{(2n+\alpha+\frac{1}{2}) \Gamma(n+1) \Gamma(n+\alpha+\frac{1}{2})} \left(\frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \right)^2 \right\}^{\frac{1}{2}} =$$

$$= \left\{ C_1(\alpha) \frac{2^{\alpha+\frac{1}{2}}\Gamma(n+\frac{1}{2})\Gamma(n+1)\Gamma^2(\alpha+1)}{(2n+\alpha+\frac{1}{2})\Gamma(n+\alpha+1)\Gamma(n+\alpha+\frac{1}{2})} \right\}^{\frac{1}{2}} \sim$$

$$\sim \left(2^{\alpha+\frac{1}{2}}\Gamma^2(\alpha+1)C_1(\alpha)\frac{1}{n^{2\alpha+1}} \right)^{\frac{1}{2}} = 2^{\frac{\alpha}{2}+\frac{1}{4}}\Gamma(\alpha+1)\sqrt{C_1(\alpha)}n^{-\alpha-\frac{1}{2}}.$$

Then

$$\|\gamma_n(I^r\varphi_n)(\cdot)\|_{2,\mu} \sim$$

$$\sim 2^{\frac{\alpha}{2}+\frac{1}{4}}\Gamma(\alpha+1)C_1^{\frac{3}{2}}(\alpha)(2(\alpha+1))^r\gamma_n \frac{n^{2r+\alpha+\frac{1}{2}}n^{-\alpha-\frac{1}{2}}}{\left(n(n+\alpha+\frac{1}{2})\right)^r} \sim$$

$$\sim 2^{r+\frac{\alpha}{2}+\frac{1}{4}}(\alpha+1)^r(C_1(\alpha))^{3/2}\gamma_n,$$

since $\frac{\Gamma(\alpha+\lambda)}{\Gamma(\lambda)} \sim \alpha^\lambda, \alpha \to \infty$, ([9], p. 951).

Hence we find that

$$\lim_{n\to\infty} \|\gamma_n(I^r\varphi_n)\|_{2,\mu} = \infty,$$

i.e., not fluttering the first condition of Theorem A. Consequently, if $f \in W_X^r(L_{2,\alpha})$, then the order is final in Theorem 5.1. ☐

Theorem 5.2. *Let* $f \in W_X^r\,(X = L_{1,\alpha})$. *Then for* $-\frac{1}{2} < \alpha < \frac{1}{2}$ *and* $r = 0, 1, \ldots,$

$$\lim_{\gamma\mapsto\infty} n^{2r}\,\hat{f}(n) = 0, \ \ r = 0, 1, \ldots,$$

but

$$\lim_{n\to\infty} \gamma_n n^{2r}\,\hat{f}(n)$$

not approaching zero, as γ_n*-tends to infinity.*

Proof. According to (5.2),

$$n^{2r}\,\hat{f}(n) = \int_{-1}^{1} f(x)\varphi_n(x)d\mu_\alpha(x),$$

where

$$\varphi_n(x) = n^{2r}R_n^{(\alpha,-\frac{1}{2})}(x).$$

The first condition of Theorem B follows by Lemma 4.5.

Using Lemma 4.2, we have

$$\left|\int_{-1}^{t}(I^r\varphi_n)\,d\mu_\alpha(x)\right| = n^{2r}\left|\int_{-1}^{t}\left(I^rR_n^{(\alpha,\,-\frac{1}{2})}\right)(x)d\mu_\alpha(x)\right| \le$$

$$\leq M \begin{cases} (1-t)^{\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{\frac{-1}{2}} n^{-n-\frac{3}{2}}, & t \in (-1,\,1), \\ n^{\frac{2}{2\alpha+3}-\frac{2\alpha+3}{2}}, & t \in [-1,\,1], \end{cases}$$

from which we get the second condition of Theorem B. Thus the first assertion of Theorem 5.2 is proved.

We prove the second assertion of the theorem.

From the proof of Lemma 4.5, it follows that

$$\|\gamma_n \left(I^r \varphi_n\right)\|_{C[-1,1]} = \gamma_n n^{2r} \overset{\wedge}{\psi_r}(n) \, \||R_n^{(\alpha,-\frac{1}{2})}|\| =$$

$$= \gamma_n n^{2r} \overset{\wedge}{\psi_r}(n) = (2(\alpha+1))^r \frac{n^{2r}\gamma_n}{(n(n+\alpha+\frac{1}{2}))^r},$$

whence it follows that

$$\lim_{n\to\infty} \|\gamma_n \left(J^r \varphi_n\right)\|_{C[-1,1]} = \infty,$$

i.e., the first condition of Theorem B is not fulfill.

Thus Theorem 5.2 is proved. □

Theorem 5.3. *Let* $f \in W_X^r \left(X = L'\right)$. *Then for* $-\frac{1}{2} < \alpha < \frac{1}{2}$, *the equality*

$$\lim_{n\to\infty} n^{2r+\frac{2\alpha+3}{2}-\frac{2}{2\alpha+3}} \hat{f}(n) = 0, \quad (r = 0, 1, \dots)$$

is valid.

Proof. By (5.2), we have

$$n^{2r+\frac{2\alpha+3}{2}-\frac{2}{2\alpha+3}} \hat{f}(n) = \int_{-1}^{1} f(x)\varphi_n(x) d\mu_\alpha(x),$$

where

$$\varphi_n(x) = n^{2r+\frac{2\alpha+3}{2}-\frac{2}{2\alpha+3}} R_n^{(\alpha,-\frac{1}{2})}(x). \tag{5.1}$$

The first condition of Theorem C for (5.4) follows from Lemma 4.2. Further, just as in proving Lemma 4.5, we have

$$\int_{-1}^{1} (I^r\varphi_n)(x) d\mu_\alpha(x) = n^{2r+\frac{2\alpha+3}{2}-\frac{2}{2\alpha+3}} \int_{-1}^{t} (I^r\varphi_n)(x) d\mu_\alpha(x) =$$

$$= n^{2r+\frac{2\alpha+3}{2}-\frac{2}{2\alpha+3}} \left\{ \frac{2(\alpha+1)}{n(n+\alpha+\frac{1}{2})} \right\}^r \int_{-1}^{t} R_n^{(\alpha,-\frac{1}{2})}(x) d\mu_\alpha(x) \leq$$

$$\leq (2(\alpha+1))^r \begin{cases} 0, & n = 1, 2, \dots; \qquad t = \pm 1, \\ n^{\frac{2\alpha+3}{2}-\frac{2}{2\alpha+3}} \int_{-1}^{t} R_n^{(\alpha,-\frac{1}{2})}(x) d\mu_\alpha(x), & t \neq \pm 1. \end{cases}$$

Taking into account Corollary 4.1, we obtain

$$\left|\int_{-1}^{t}(I^r\varphi_n)(x)d\mu_\alpha(x)\right|\le M(1-t)^{\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{1}{2}}n^{-\frac{2}{2\alpha+3}}.$$

This implies that the second condition of Theorem C for (5.4) is correct.

Thus Theorem 5.3 is proved. □

6. On the Convergence of Fourier–Jacobi Series

In this section, using the results of Section 5 dealt with the convergence of Fourier–Jacobi series,

$$\sum_{n=0}^{\infty}\frac{\left(2n+\alpha+\frac{1}{2}\right)\Gamma(n+1)\Gamma\left(n+\alpha+\frac{1}{2}\right)}{2^{\alpha+\frac{1}{2}}\Gamma(n+\alpha+1)\Gamma\left(n+\frac{1}{2}\right)}\times$$
$$\times\left(\int_{-1}^{1}P_n^{(\alpha,-\frac{1}{2})}(t)d\mu_\alpha(t)\right)P_n^{(\alpha,-\frac{1}{2})}(x). \tag{6.1}$$

Let us consider the $n-$th partial sum of series (6.1)

$$S_n(f;x)=\int_{-1}^{1}f(t)K_n(t,x)d\mu_\alpha(t), \tag{6.2}$$

where

$$K_n(t,x)=\sum_{k=0}^{n}\frac{\left(2k+\alpha+\frac{1}{2}\right)\Gamma(k+1)\Gamma\left(k+\alpha+\frac{1}{2}\right)}{2^{\alpha+\frac{1}{2}}\Gamma(k+\alpha+1)\Gamma\left(k+\frac{1}{2}\right)}P_k^{(\alpha,-\frac{1}{2})}(t)P_k^{(\alpha,-\frac{1}{2})}(x).$$

Applying multiplication theorem to the Jacobi polynomials [8]

$$\frac{\Gamma(\alpha+1)\Gamma(n+1)}{C_2(\alpha)\Gamma(n+\alpha+1)}P_n^{(\alpha,-\frac{1}{2})}(t)P_n^{(\alpha,-\frac{1}{2})}(x)=$$
$$=\int_{-1}^{1}P_n^{(\alpha,-\frac{1}{2})}\Big(xt+r\sqrt{1-x^2}\sqrt{1-t^2}-$$
$$-\frac{1}{2}(1-r^2)(1-x)(1-t)\Big)(1-r^2)^{\alpha-\frac{1}{2}}dr$$

in (6.2), we obtain

$$S_n(f;x)=\frac{C_2(\alpha)}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)}\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{-\frac{1}{2}}f(t)\times$$

$$\times\left\{\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\int_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}\biggl(xt+\right.$$

$$\left.+r\sqrt{1-x^2}\sqrt{1-t^2}-\frac{1}{2}(1-r^2)(1-x)(1-t)\biggr)dr\right\}dt.$$

Substituting $t=\cos u$ and $y=\cos\frac{u}{2}$, we obtain

$$S_n(f;x)=\frac{C_2(\alpha)}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)}\int_{0}^{\pi}(1-\cos u)^{\alpha}(1+\cos u)^{-\frac{1}{2}}f(\cos u)\times$$

$$\times\left\{\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\int_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}\left(x\cos u+\right.\right.$$

$$\left.\left.+r\sqrt{1-x^2}\sin u-(1-r^2)(1-x)\sin^2\frac{u}{2}\right)dr\right\}\sin u du=$$

$$=\frac{C_2(\alpha)}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)}\int_{0}^{\pi}\left(2\sin^2\frac{u}{2}\right)^{\alpha}\left(2\cos^2\frac{u}{2}\right)^{-\frac{1}{2}}f\left(2\cos^2\frac{u}{2}-1\right)\times$$

$$\times\left\{\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\int_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}\left[x\left(2\cos^2\frac{u}{2}-1\right)+\right.\right.$$

$$\left.\left.+2r\sqrt{1-x^2}\sqrt{1-\cos^2\frac{u}{2}}\cos\frac{u}{2}-(1-r^2)(1-x)sin^2\frac{u}{2}\right]dr\right\}\times$$

$$\times 2\sqrt{1-\cos^2\frac{u}{2}}\cos\frac{u}{2}du=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\int_{0}^{1}(1-y^2)^{\alpha}f(2y^2-1)\times$$

$$\times\left\{\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\int_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}\left[x\left(2y^2-1\right)+\right.\right.$$

$$\left.\left.+2ry\sqrt{1-x^2}\sqrt{1-y^2}-\frac{1}{2}(1-r^2)(1-x)(1-y^2)\right]dr\right\}dy.$$

Assuming that in the inside integral $x=\cos\theta$ and $z=\cos\frac{\theta}{2}$, we obtain

$$S_n(f;x)=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\times$$

$$\times\int_{-1}^{1}\left(1-y^2\right)^{\alpha}f(2y^2-1)\left\{\int_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}\Bigl[(2z^2-1)(2y^2-1)+\right.$$

$$+4ryz\sqrt{1-z^2}\sqrt{1-y^2}-2(1-r^2)(1-z^2)(1-y^2)\Big]dr\Big\}dy=$$

$$=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\int\limits_0^1(1-y^2)^{\alpha}f(2y^2-1)\times$$

$$\times\Big\{\int\limits_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}\left[2\left(zy+r\sqrt{1-z^2}\sqrt{1-y^2}\right)^2-1\right]dr\Big\}dy.$$

Making substitution in the inside integral

$$v=zy+r\sqrt{1-z^2}\sqrt{1-y^2}$$

and taking into account that

$$r=(v-zy)(1-y^2)^{-\frac{1}{2}}(1-z^2)^{-\frac{1}{2}},\ \ dr=(1-y^2)^{-\frac{1}{2}}(1-z^2)^{-\frac{1}{2}}dv,$$

we obtain

$$S_n(f;x)=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma\left(k+\alpha+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}\times$$

$$\times\int\limits_0^1(1-y^2)^{\alpha}f(2y^2-1)(1-y^2)^{-\frac{1}{2}}(1-z^2)^{-\frac{1}{2}}\times$$

$$\times\int\limits_{zy-\sqrt{1-z^2}\sqrt{1-y^2}}^{zy+\sqrt{1-z^2}\sqrt{1-y^2}}\left[1-\left(\frac{v-zy}{\sqrt{1-z^2}\sqrt{1-y^2}}\right)^2\right]^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(2v^2-1)dvdy=$$

$$=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma\left(k+\alpha+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}(1-z^2)^{-\alpha}\int\limits_0^1 f(2y^2-1)\times$$

$$\times\left\{\int\limits_{zy-\sqrt{1-z^2}\sqrt{1-y^2}}^{zy+\sqrt{1-z^2}\sqrt{1-y^2}}(1-z^2-y^2-v^2+2zyv)^{\alpha-\frac{1}{2}}P_n^{(\alpha,-\frac{1}{2})}(2v^2-1)dv\right\}dy.$$

By the change of order of integration, we obtain

$$S_n(f;x)=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\times$$

$$\times\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma\left(k+\alpha+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}(1-z^2)^{-\alpha}\int\limits_0^1 P_n^{(\alpha,-\frac{1}{2})}(2v^2-1)\times$$

$$\times\left\{\int\limits_{vz-\sqrt{1-v^2}\sqrt{1-z^2}}^{vz+\sqrt{1-v^2}\sqrt{1-z^2}}(1-z^2-y^2-v^2+2zyv)^{\alpha-\frac{1}{2}}f(2y^2-1)dy\right\}dv.$$

Now, making substitution

$$y=vz+r\sqrt{1-v^2}\sqrt{1-z^2},$$

we obtain

$$S_n(f;x)=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\times$$

$$\times\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma\left(k+\alpha+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}\int\limits_0^1(1-v^2)P_n^{(\alpha,-\frac{1}{2})}(2v^2-1)\times$$

$$\times\left\{\int\limits_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}f\left[2\left(vz+r\sqrt{1-v^2}\sqrt{1-z^2}\right)^2-1\right]dr\right\}dv=$$

$$=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma\left(k+\alpha+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}\times$$

$$\times\int\limits_0^1(1-v^2)^{\alpha}P_n^{(\alpha,-\frac{1}{2})}(2v^2-1)\left\{\int\limits_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}f\left[2v^2z^2+\right.\right.$$

$$\left.\left.+4rzv\sqrt{1-v^2}\sqrt{1-z^2}+r^2(1-v^2)(1-z^2)-1\right]dr\right\}dv=$$

$$=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\int\limits_0^1\left(1-v^2\right)^{\alpha}P_n^{(\alpha,-\frac{1}{2})}(2v^2-1)\times$$

$$\times\left\{\int\limits_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}f\left[(2z^2-1)(2v^2-1)+\right.\right.$$

$$+4rzv\sqrt{1-v^2}\sqrt{1-z^2}+r^{2(1-v^2)}(1-z^2)-$$

$$\left.\left.-2(1-r^2)(1-z^2)(1-v^2)\right]dr\right\}dv=\left|z=\cos\frac{\theta}{2};x=\cos\theta\right|=$$

$$=\frac{2C_2(\alpha)}{\Gamma(\alpha+1)}\sum_{k=0}^{n}\left(2k+\alpha+\frac{1}{2}\right)\frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\times$$

$$\times\int\limits_0^1\left(1-v^2\right)^{\alpha}P_n^{(\alpha,-\frac{1}{2})}(2v^2-1)\left\{\int\limits_{-1}^{1}(1-r^2)^{\alpha-\frac{1}{2}}f\left[x(2v^2-1)+\right.\right.$$

$$\left.\left.+2rv\sqrt{1-x^2}\sqrt{1-v^2}-(1-r^2)(1-x)(1-v^2)\right]dr\right\}dv=$$

$$= \left| 2v^2 - 1 = u; dv = 2^{-\frac{3}{2}}(1+u)^{-\frac{1}{2}} du \right| =$$

$$= \frac{C_2(\alpha)}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)} \sum_{k=0}^{n} \left(2k+\alpha+\frac{1}{2}\right) \frac{\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})} \times$$

$$\times \int_{-1}^{1} (1-u)^{\alpha}(1+u)^{-\frac{1}{2}} P_n^{(\alpha,-\frac{1}{2})}(u) \left\{ \int_{-1}^{1} (1-r^2)^{\alpha-\frac{1}{2}} f\left[xu + r\sqrt{1-u^2}\sqrt{1-x^2} - \right.\right.$$

$$\left.\left. -\frac{1}{2}(1-r^2)(1-x)(1-u) \right] dr \right\} = C_2(\alpha) \int_{-1}^{1} (1-u)^{\alpha}(1+u)^{-\frac{1}{2}} (\tau_u f)(x) K_n(u) du,$$

where

$$K_n(u) = \sum_{k=0}^{n} \frac{(2k+\alpha+\frac{1}{2})\Gamma(2k+\alpha+\frac{1}{2})}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)\Gamma(k+\frac{1}{2})} P_k^{(\alpha,-\frac{1}{2})}(u).$$

Applying Cristoffel-Darbu formula

$$\sum_{\nu=0}^{n} \frac{(2\nu+\alpha+\frac{1}{2})\Gamma(\nu+\alpha+\frac{1}{2})}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)\Gamma(\nu+\frac{1}{2})} P_\nu^{(\alpha,-\frac{1}{2})}(u) =$$

$$= \frac{2^{\frac{1}{2}-\alpha}\Gamma(n+\alpha+\frac{3}{2})}{(2n+\alpha+\frac{3}{2})\Gamma(\alpha+\frac{1}{2})\Gamma(n+\frac{1}{2})} \times$$

$$\times \frac{(n+\alpha+1)P_n^{(\alpha,-\frac{1}{2})}(x) - (n+1)P_{n+1}^{(\alpha,-\frac{1}{2})}(x)}{1-x}$$

and using $S_n(1;x) \equiv 1$, we obtain

$$f(x) - S_n(f;x) =$$

$$= \frac{C_2(\alpha)2^{\frac{1}{2}-\alpha}\Gamma(n+\alpha+\frac{3}{2})(n+1+\alpha)}{\Gamma(\alpha+1)(2n+\alpha+\frac{3}{2})\Gamma(n+\frac{1}{2})} \int_{-1}^{1} \mu(u)\varphi_u(x) P_n^{(\alpha,-\frac{1}{2})}(u) du$$

$$\frac{C_2(\alpha)2^{\frac{1}{2}-\alpha}\Gamma(n+\alpha+\frac{3}{2})(n+1)}{\Gamma(\alpha+1)(2n+\alpha+\frac{3}{2})\Gamma(n+\frac{1}{2})} \int_{-1}^{1} \mu(u)\varphi_u(x) P_{n+1}^{(\alpha,-\frac{1}{2})}(u) du,$$

where

$$\varphi_u(x) = \frac{f(x) - (\tau_u f)(x)}{1-u}.$$

And since

$$P_n^{(\alpha,-\frac{1}{2})}(x) = P_n^{(\alpha,-\frac{1}{2})}(1) R_n^{(\alpha,-\frac{1}{2})}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)} R_n^{(\alpha,-\frac{1}{2})}(x),$$

therefore

$$f(x) - S_n(f;x) =$$

$$= \frac{C_2(\alpha)2^{\frac{1}{2}-\alpha}\Gamma(n+\alpha+\frac{3}{2})\Gamma(n+\alpha+2)}{\Gamma^2(\alpha+1)\Gamma(n+1)(2n+\alpha+\frac{3}{2})\Gamma(n+\frac{1}{2})}\int\limits_{-1}^{1}\mu(u)\varphi_u(x)R_n^{(\alpha,-\frac{1}{2})}(u)du-$$

$$-\frac{C_2(\alpha)2^{\frac{1}{2}-\alpha}\Gamma(n+\alpha+\frac{3}{2})\Gamma(n+\alpha+2)}{\Gamma^2(\alpha+1)\Gamma(n+1)(2n+\alpha+\frac{3}{2})\Gamma(n+\frac{1}{2})}\int\limits_{-1}^{1}\mu(u)\varphi_u(x)R_{n+1}^{(\alpha,-\frac{1}{2})}(u)du =$$

$$= \frac{C_2(\alpha)2^{\frac{1}{2}-\alpha}\Gamma(n+\alpha+\frac{3}{2})\Gamma(n+\alpha+2)((\varphi_x\overset{\wedge}{(n)}) - (\varphi_x\,(\hat{n}+1)))}{\Gamma^2(\alpha+1)C_1(\alpha)\Gamma(n+1)(2n+\alpha+\frac{3}{2})\Gamma(n+\frac{1}{2})} \sim$$

$$\sim \frac{C_2(\alpha)2^{\frac{1}{2}-\alpha}}{C_1(\alpha)\Gamma^2(\alpha+1)}((\varphi_x\overset{\wedge}{(n)}) - (\varphi_x\,(\hat{n}+1)))n^{2\alpha+1} =$$

$$= \frac{2}{\Gamma(\alpha+\frac{3}{2})}((\varphi_x\overset{\wedge}{(n)}) - (\varphi_x\,(\hat{n}+1)))n^{2\alpha+1}.$$

For the theorems in Section 5 we have obtained the following theorems on the order of pointwise convergence of Fourier–Jacobi series.

Theorem 6.1. *For each point $x \in [-1,1|$, $\varphi_{\mathrm{u}}(x) \in W_X^r$ ($X = L_{p,\alpha}$, $1 < p < \infty$), for $r = 0,1,\ldots,$ and $0 < (\alpha+\frac{1}{2})\ q < 1$, the equality*

$$\lim_{n\to\infty} n^{2r-\alpha-\frac{1}{2}}\{f(x) - S_n(f;x)\} = 0$$

is valid, moreover, for $p = 2$, the order here is final.

Theorem 6.2. *For each point $x \in [-1,1|$, $\varphi_u(x) \in W_X^r$ ($X = L_{1,\alpha}$), for $r = 0,1,\ldots,$ and $-\frac{1}{2} < \alpha < \frac{1}{2}$, the equality*

$$\lim_{n\to\infty} n^{2r-2\alpha-1}\{f(x) - S_n(f;x)\} = 0,$$

is valid, moreover, the order here is final.

Theorem 6.3. *For each point $x \in [-1,1|$, $\varphi_u(x) \in W_X^r$ ($X = L'_{1,\alpha}$), for $r = 0,1,\ldots,$ and $-\frac{1}{2} < \alpha < \frac{1}{2}$, the equality*

$$\lim_{n\to\infty} n^{2r+\frac{2\alpha-1}{4\alpha+6}-\alpha}\{f(x) - S_n(f;x)\} = 0 \quad \textit{is valid.}$$

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CYCLIC REFINEMENT OF BECK'S INEQUALITIES

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Abstract. In this paper, we refine the discrete Jensen's inequality for vectors by the idea recently given in [2]. As a consequence, we are able to refine the inequality of E. Beck [1] with the help of cyclic generalized mixed symmetric means. This leads to the refinements of the classical Hölder and Minkowski's inequalities.

რეზიუმე. [2] შრომაში განვითარებულ იდეაზე დაყრდნობით დაზუსტებულია იენსენის უტოლობა ვექტორებისათვის. ამ შედეგზე დაყრდნობით ზუსტდება ე. ბეკის [1] უტოლობა ციკლური განზოგადებული შერეული საშუალოების გამოყენებით. ამ უკანსაკნელ უტოლობას მოსდევს ჰელდერისა და მინკოვსკის კლასიკური უტოლობების დაზუსტება.

1. Introduction and Preliminary Results

Let U be a convex subset of a real linear space, and let $f : U \to \mathbb{R}$ be a convex function. If $x_i \in U$ $(1 \leq i \leq n)$ and $p_i \geq 0$ $(1 \leq i \leq n)$ such that $\sum\limits_{i=1}^{n} p_i = 1$, then the discrete Jensen's inequality

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

holds. Particularly, we have

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} f(x_i). \tag{2}$$

Let $I \subset \mathbb{R}$ be an interval, $h : I \to \mathbb{R}$ be a continuous and strictly monotone function, and let $\mathbf{a} = (a_1, \ldots, a_n) \in I^n$. Then the quasi-arithmetic

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h-mean of vector $\mathbf{a}$ is defined by

$$h_n(\mathbf{a}) = h_n(a_i; 1 \leq i \leq n) = h(\mathbf{a}; n) := h^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} h(a_i)\right).$$

First, we extend Beck's results (see [1]). The use will be made of the following hypothesis:

(A_1) Let $L_t : I_t \to \mathbb{R}$ $(t = 1, \ldots, m)$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$, and let $f : I_1 \times \cdots \times I_m \to I_N$ be a continuous function. Let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)} \in \mathbb{R}^n$ $(n \geq 2)$ such that $\mathbf{x}^{(t)} \in I_t^n$ for each $t = 1, \ldots, m$.

The following result is a simple consequence of the discrete Jensen's inequality (2).

Theorem 1.1. *Assume (A_1). If N is an increasing function, then the inequality*

$$f\left(L_1(\mathbf{x}^{(1)}; n), \ldots, L_m(\mathbf{x}^{(m)}; n)\right) \geq$$
$$\geq N^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} N(f(x_i^{(1)}, \ldots, x_i^{(m)}))\right), \tag{3}$$

holds for all possible $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$, if and only if the function H defined on $L_1(I_1) \times \cdots \times L_m(I_m)$ by

$$H(t_1, \ldots, t_m) := N\left(f\left(L_1^{-1}(t_1), \ldots, L_m^{-1}(t_m)\right)\right) \tag{4}$$

is concave. The inequality in (3) is reversed for all possible $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$, if and only if H is convex.

Proof. We replace the convex function f by $-H$ or H, and x_i by $L_t(x_i^{(t)})$ in (2) and then, applying the increasing function N^{-1}, we get the required results. □

Beck's original result (see [4], p. 249 or [3], p. 300) was the weighted form of Theorem 1.1 (see in [10], p. 157), but with $m = 2$ and $I_1 = [k_1, k_2]$, $I_2 = [l_1, l_2]$ and $I_N = [n_1, n_2]$.

For the simplicity, in case $m = 2$ we use the following form of (A_1):

(A_2) Let $K : I_K \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$, and let $f : I_K \times I_L \to I_N$ be a continuous function. Let $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ $(n \geq 2)$ such that $\mathbf{a} \in I_K^n$ and $\mathbf{b} \in I_L^n$.

Then (3) has the form

$$f\left(K_n(\mathbf{a}), L_n(\mathbf{b})\right) \geq N_n\left(f(\mathbf{a}, \mathbf{b})\right), \tag{5}$$

where $f(\mathbf{a}, \mathbf{b})$ means $(f(a_1, b_1), \ldots, f(a_n, b_n))$.

The following results are the important special cases of Theorem 1.1 and generalize the corresponding results of Beck. We use the following hypothesis:

(A_3) Let $K : I_K \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$ such that either $I_K + I_L \subset I_N$ and $f(x,y) = x+y$ $((x,y) \in I_K \times I_L)$, or I_K, $I_L \subset]0,\infty[$, $I_K \cdot I_L \subset I_N$ and $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$. Assume further that the functions K, L and N are twice continuously differentiable on the interior of their domains, respectively. Let $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ $(n \geq 2)$ such that $\mathbf{a} \in I_K^n$ and $\mathbf{b} \in I_L^n$.

The interior of a subset A of $\mathbb{R}$ is denoted by A°.

Corollary 1.2. *Assume* (A_3) *with* $f(x,y) = x + y$ $((x,y) \in I_K \times I_L)$, *and assume that* K', L', N', K'', L'' *and* N'' *are all positive. Introducing* $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$, (5) *holds for all possible tuples* $\mathbf{a}$ *and* $\mathbf{b}$, *if and only if*

$$E(x) + F(y) \leq G(x+y), \quad (x,y) \in I_K^\circ \times I_L^\circ. \tag{6}$$

Corollary 1.3. *Assume* (A_3) *with* $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$. *Suppose the functions* $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* I_K°, I_L° *and* I_N°, *respectively. Assume further that* K', L', N', A, B *and* C *are all positive. Then* (5) *holds for all possible tuples* $\mathbf{a}$ *and* $\mathbf{b}$, *if and only if*

$$A(x) + B(y) \leq C(xy), \quad (x,y) \in I_K^\circ \times I_L^\circ.$$

To prove these corollaries, similar arguments can be applied as in the analogous results of Beck. We just sketch the proof of Corollary 1.2.

Proof. By Theorem 1.1, it is enough to prove that the function

$$H : K(I_K) \times L(I_L) \to \mathbb{R}, \quad H(t,s) := N\left(K^{-1}(t) + L^{-1}(s)\right)$$

is concave. Since H is continuous, and twice continuously differentiable on the interior $K(I_K^\circ) \times L(I_L^\circ)$ of its domain, we have to show that

$$h_1^2 \frac{\partial^2 H(t,s)}{\partial t^2} + 2h_1 h_2 \frac{\partial^2 H(t,s)}{\partial t \partial s} + h_2^2 \frac{\partial^2 H(t,s)}{\partial s^2} \leq 0$$

for all $(t,s) \in K(I_K^\circ) \times L(I_L^\circ)$ and $(h_1,h_2) \in \mathbb{R}^2$. By computing the partial derivatives of H of order 2 at the points of $K(I_K^\circ) \times L(I_L^\circ)$, we have the condition (6) (see [3] p. 303). $\square$

The interpolations of the discrete Jensen's inequality (2) given in [13] are used in [11] (see also [12], p.195) to refine the inequality of E. Beck for a function of two variables. The similar idea is utilized in [8, 9] (see also [10], Chapter 7) for the refinements of weighted discrete Jensen's inequality (1) appeared in [5, 6, 7]. Analogously, in this paper we work out the new refinement of Beck's inequality (3) by cyclic mixed symmetric means as a consequence of the new refinement of the discrete Jensen's inequality (2) constructed in [2]. This, obviously, leads to some new refinements of the classical Hölder and Minkowski's inequalities.

We need another hypothesis:

(H_2) Let U be a convex set in $\mathbb{R}^m$, $\mathbf{x}_1, \dots, \mathbf{x}_n \in U$, such that $\mathbf{x}_{i+n} = \mathbf{x}_i$, and $\lambda := (\lambda_1, \dots, \lambda_n)$ be a positive n-tuple such that $\sum_{i=1}^{k} \lambda_i = 1$ for $2 \leq k \leq n$. Further, let $f : U \to \mathbb{R}$ be a convex function.

The following refinement of the discrete Jensen's inequality for functions of several variables is analogous to the refinement given in [2] for the function of one variable:

Theorem A. *Assume (H_2), and consider the following sum*

$$S = \frac{1}{n} \sum_{i=1}^{n} f\left(\sum_{j=0}^{k-1} \lambda_{j+1} \mathbf{x}_{i+j} \right). \tag{7}$$

Then

$$f\left(\frac{\sum_{i=1}^{n} \mathbf{x}_i}{n} \right) \leq \frac{1}{n} \sum_{i=1}^{n} f\left(\sum_{j=0}^{k-1} \lambda_{j+1} \mathbf{x}_{i+j} \right) \leq \frac{\sum_{i=1}^{n} f(\mathbf{x}_i)}{n}. \tag{8}$$

Proof. The idea of proof is the same as that given in [2].

First, we prove the second inequality in (8). Since f is convex, by Jensen's inequality, we have

$$\sum_{i=1}^{n} f\left(\sum_{j=0}^{k-1} \lambda_{j+1} \mathbf{x}_{i+j} \right) \leq \sum_{i=1}^{n} \sum_{j=0}^{k-1} \lambda_{j+1} f(\mathbf{x}_{i+j}) =$$
$$= \sum_{i=1}^{n} f(\mathbf{x}_i) \sum_{j=1}^{k} \lambda_j = \sum_{i=1}^{n} f(\mathbf{x}_i)$$

Now we prove the first inequality in (8). Since f is convex, by Jensen's inequality, we have

$$\frac{1}{n} \sum_{i=1}^{n} f\left(\sum_{j=0}^{k-1} \lambda_{j+1} \mathbf{x}_{i+j} \right) \geq f\left(\frac{\sum_{i=1}^{n} \sum_{j=0}^{k-1} \lambda_{j+1} \mathbf{x}_{i+j}}{n} \right) =$$
$$= f\left(\frac{\sum_{i=1}^{n} \mathbf{x}_i \sum_{j=1}^{k} \lambda_j}{n} \right) = f\left(\frac{\sum_{i=1}^{n} \mathbf{x}_i}{n} \right) \qquad \square$$

2. Refinement of Beck's Inequality

In what follows, we assume (A_1) such that $x_{i+n}^{(t)} = x_i^{(t)}$ $(t = 1, \dots, m)$ and $(\lambda_1, \dots, \lambda_n)$ is a positive n-tuple in the way that $\sum_{i=1}^{k} \lambda_i = 1$ for $2 \leq k \leq n$.

The cyclic mixed symmetric means relative to (7) are defined by

$$M(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)}) := \\ = N^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} N\left(f\left(L_1(\mathbf{x}^{(1)};k),\ldots,L_m(\mathbf{x}^{(m)};k)\right)\right)\right) \tag{9}$$

$$L_t(\mathbf{x}^{(t)};k) = L_t^{-1}\left(\sum_{j=0}^{k-1}\lambda_{j+1}L_t(x_{i+j}^{(t)})\right);\quad t=1,\ldots,m.$$

Now, we get an interpolation of (3) by the direct application of Theorem A as follows.

Theorem 2.1. *Assume* (A_1) *such that* $x_{i+n}^{(t)} = x_i^{(t)}$ $(t=1,\ldots,m)$, *and* $(\lambda_1,\ldots,\lambda_n)$ *is a positive* n*-tuple such that* $\sum_{i=1}^{k}\lambda_i = 1$ *for* $2\le k\le n$. *If* N *is an increasing* (*decreasing*) *function, then the inequalities*

$$f\left(L_1(\mathbf{x}^{(1)};n),\ldots,L_m(\mathbf{x}^{(m)};n)\right) \le M(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)}) \le \\ \le N^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} N(f(x_i^{(1)},\ldots,x_i^{(m)}))\right), \tag{10}$$

hold for all possible $\mathbf{x}^{(t)}$ $(t=1,\ldots,m)$, *if and only if the function* H *defined in Theorem* 1.1 *is convex* (*concave*). *If* N *is an increasing* (*decreasing*) *function, then the inequalities in* (10) *are reversed for all possible* $\mathbf{x}^{(t)}$ $(t = 1,\ldots,m)$, *if and only if* H *is concave* (*convex*).

Proof. Suppose N is increasing and the function $H : L_1(I_1)\times\cdots\times L_m(I_m) \to \mathbb{R}$,

$$H(t_1,\ldots,t_m) = N\left(f\left(L_1^{-1}(t_1),\ldots,L_m^{-1}(t_m)\right)\right)$$

is convex. We apply Theorem A to the function H and to the vectors $(L_1(x_i^{(1)}),\ldots,L_m(x_i^{(m)}))$, $i=1,\ldots,n$. Then the first term in (8) gives

$$H\left(\frac{1}{n}\sum_{i=1}^{n}\left(L_1(x_i^{(1)}),\ldots,L_m(x_i^{(m)})\right)\right) = \\ = H\left(\frac{1}{n}\sum_{i=1}^{n}L_1(x_i^{(1)}),\ldots,\frac{1}{n}\sum_{i=1}^{n}L_m(x_i^{(m)})\right) = \\ = N\left(f\left(L_1^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}L_1(x_i^{(1)})\right),\ldots,L_m^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}L_m(x_i^{(m)})\right)\right)\right) = \\ = N\left(f\left(L_1(\mathbf{x}^{(1)};n),\ldots,L_m(\mathbf{x}^{(m)};n)\right)\right).$$

The last term in (8) is

$$\frac{1}{n}\sum_{i=1}^{n} H(L_1(x_i^{(1)}),\dots,L_m(x_i^{(m)})) = \frac{1}{n}\sum_{i=1}^{n} N\left(f\left(x_i^{(1)},\dots,x_i^{(m)}\right)\right),$$

and the middle term in (8) has the form

$$\frac{1}{n}\sum_{i=1}^{n} H\left(\sum_{j=0}^{k-1}\lambda_{j+1}\left(L_1(x_{i+j}^{(1)}),\dots,L_m(x_{i+j}^{(m)})\right)\right) =$$

$$= \frac{1}{n}\sum_{i=1}^{n} H\left(\sum_{j=0}^{k-1}\lambda_{j+1}L_1(x_{i+j}^{(1)}),\dots,\sum_{j=0}^{k-1}\lambda_{j+1}L_m(x_{i+j}^{(m)})\right) =$$

$$= \frac{1}{n}\sum_{i=1}^{n} N\left(f\left(L_1^{-1}\left(\sum_{j=0}^{k-1}\lambda_{j+1}L_1(x_{i+j}^{(1)})\right),\dots,L_m^{-1}\left(\sum_{j=0}^{k-1}\lambda_{j+1}L_m(x_{i+j}^{(m)})\right)\right)\right) =$$

$$= \frac{1}{n}\sum_{i=1}^{n} N\left(f\left(L_1(\mathbf{x}^{(1)};k),\dots,L_m(\mathbf{x}^{(m)};k)\right)\right).$$

The inequalities (10) follow from these observations and Theorem A since N^{-1} is increasing.

The converse is obtained by Theorem 1.1. □

Assume (A_2) such that $a_{i+n} = a_i$, $b_{i+n} = b_i$, and $(\lambda_1,\dots,\lambda_n)$ is a positive n-tuple such that $\sum_{i=1}^{k}\lambda_i = 1$ for $2 \le k \le n$. Then, for $m = 2$, the reverse of (10) can be written as

$$f\left(K_n(\mathbf{a}), L_n(\mathbf{b})\right) \ge M(K, L; \mathbf{a}, \mathbf{b}) \ge N^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} N(f(a_i, b_i)).\right) \tag{11}$$

Example 2.2. Let $f(x)=xy$ and $N(x)=x$, then $H(s,t)=K^{-1}(s)L^{-1}(t)$. If H is concave, then (11) gives the following refinement of Hölder's inequality,

$$\frac{1}{n}\sum_{i=1}^{n} a_i b_i \le \frac{1}{n}\sum_{i=1}^{n} K(\mathbf{a};k)L(\mathbf{b};k) \le K_n(\mathbf{a})L_n(\mathbf{b}). \tag{12}$$

In particular, if $H(s,t) = s^{1/q}t^{1/r}$, then H is concave for $q, r > 1$ and $q^{-1} + r^{-1} = 1$; we get the following refinement of the classical Hölder's inequality for positive n-tuples $\mathbf{a}$ and $\mathbf{b}$.

$$\sum_{i=1}^{n} a_i b_i \le \sum_{i=1}^{n}\left(\sum_{j=0}^{k-1}\lambda_{j+1}a_{i+j}^{q}\right)^{\frac{1}{q}}\left(\sum_{j=0}^{k-1}\lambda_{j+1}b_{i+j}^{r}\right)^{\frac{1}{r}} \le \left(\sum_{i=1}^{n} a_i^q\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n} b_i^r\right)^{\frac{1}{r}}.$$

Example 2.3. If $H(s,t) = (s^{1/p} + t^{1/p})^p$, then H is concave for $p > 1$, and (11) reduces to the following refinement of the classical Minkowski's inequality for positive n-tuples $\mathbf{a}$ and $\mathbf{b}$.

$$\left(\sum_{i=1}^{n}(a_i+b_i)^p\right)^{\frac{1}{p}} \le$$
$$\le \left(\sum_{i=1}^{n}\left(\left(\sum_{j=0}^{k-1}\lambda_{j+1}a_{i+j}^p\right)^{\frac{1}{p}}+\left(\sum_{j=0}^{k-1}\lambda_{j+1}b_{i+j}^p\right)^{\frac{1}{p}}\right)^p\right)^{\frac{1}{p}} \le$$
$$\le \left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}b_i^p\right)^{\frac{1}{p}}.$$

On the analogy of Corollary 1.2 and Corollary 1.3, we have the following consequences of Theorem 2.1.

Corollary 2.4. *Assume* (A_3) *such that* $a_{i+n}=a_i$, $b_{i+n}=b_i$, *and* $(\lambda_1,\ldots,\lambda_n)$ *is a positive* n*-tuple such that* $\sum_{i=1}^{k}\lambda_i=1$ *for* $2\le k\le n$. *Suppose* $f(x,y)=x+y$ $((x,y)\in I_K\times I_L)$, *and assume that* K', L', N', K'', L'' *and* N'' *are all positive. Introducing* $E:=\frac{K'}{K''}$, $F:=\frac{L'}{L''}$, $G:=\frac{N'}{N''}$, (11) *holds for all possible* $\mathbf{a}$ *and* $\mathbf{b}$, *if and only if*

$$E(x)+F(y)\le G(x+y),\quad (x,y)\in I_K^\circ\times I_L^\circ.$$

In this case,

$$M(K,L;\mathbf{a},\mathbf{b})=N^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}N\left(K(\mathbf{a};k)+L(\mathbf{b};k)\right)\right). \tag{13}$$

Corollary 2.5. *Assume* (A_3) *such that* $a_{i+n}=a_i$, $b_{i+n}=b_i$, $(\lambda_1,\ldots,\lambda_n)$ *is a positive* n*-tuple such that* $\sum_{i=1}^{k}\lambda_i=1$ *for* $2\le k\le n$, *and* $f(x,y)=xy$ $((x,y)\in I_K\times I_L)$. *Suppose the functions* $A(x):=\frac{K'(x)}{K'(x)+xK''(x)}$, $B(x):=\frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x):=\frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* I_K°, I_L° *and* I_N°, *respectively. Assume further that* K', L', M', A, B *and* C *are all positive. Then* (11) *holds for all possible* $\mathbf{a}$ *and* $\mathbf{b}$, *if and only if*

$$A(x)+B(y)\le C(xy),\quad (x,y)\in I_K^\circ\times I_L^\circ.$$

In this case,

$$M(K,L;\mathbf{a},\mathbf{b})=N^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}N\left(K(\mathbf{a};k)L(\mathbf{b};k)\right)\right). \tag{14}$$

3. Refinement of Minkowski's Inequality

(A_4) Let I be an interval in $\mathbb{R}$, and let $M: I\to\mathbb{R}$ be a continuous and strictly monotone function. Let $\mathbf{x}_i\in I^m$ be such that $\mathbf{x}_{i+n}=\mathbf{x}_i$ $(i=1,\ldots,n)$, $(\lambda_1,\ldots,\lambda_n)$ be a positive n-tuple such that $\sum_{i=1}^{k}\lambda_i=1$ for $2\le k\le n$, and let $\mathbf{w}=(w_1,\ldots,w_m)$ be a nonnegative m-tuple such that $\sum\limits_{i=1}^{m}w_i=1$.

We give a refinement of Minkowski's inequality by using Theorem A.

Theorem 3.1. *Assume* (A_4)*, and let the quasi-arithmetic mean function*

$$\mathbf{x} \to M_m(\mathbf{x};\mathbf{w}) := M^{-1}\left(\sum_{i=1}^{m} w_i M(x_i)\right), \quad \mathbf{x} \in I^m$$

be convex. Then

$$M_m\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i;\mathbf{w}\right) \leq$$

$$\leq \frac{1}{n}\sum_{i=1}^{n} M_m\left(\sum_{j=0}^{k-1}\lambda_{j+1}\mathbf{x}_{i+j};\mathbf{w}\right) \leq \frac{1}{n}\sum_{r=1}^{n} M_m(\mathbf{x}_r;\mathbf{w}). \tag{15}$$

Proof. This is obtained by applying Theorem A to the function $M_m(\cdot;\mathbf{w})$ and to the vectors $\mathbf{x}_i$ $(i = 1, \ldots, n)$. □

The following necessary and sufficient condition for the quasi-arithmetic mean function to be convex is given in [12], p. 197:

Theorem B. *If* $M : [m_1, m_2] \to R$ *has continuous derivatives of second order and it is strictly increasing and strictly convex, then the quasi-arithmetic mean function* $M_m(\cdot; w)$ *is convex, if and only if* M'/M'' *is a concave function.*

(A_5) Let $M :]0,\infty[\to]0,\infty[$ be a continuous and strictly monotone function such that $\lim_{x\to 0} M(x) = \infty$ or $\lim_{x\to\infty} M(x) = \infty$. Let $\mathbf{x}_i \in I^m$ be such that $\mathbf{x}_{i+n} = \mathbf{x}_i$ $(i = 1, \ldots, n)$, $(\lambda_1, \ldots, \lambda_n)$ be a positive n-tuple such that $\sum_{i=1}^{k}\lambda_i = 1$ for $2 \leq k \leq n$. Let $\mathbf{w} = (w_1, \ldots, w_m)$ be positive m-tuple such that $w_i \geq 1$ $(i = 1, \ldots, m)$.

Then we define

$$\widetilde{M}_m(\mathbf{x};\mathbf{w}) = M^{-1}\left(\sum_{i=1}^{m} w_i M(x_i)\right). \tag{16}$$

The following result is also given in ([12], page 197):

Theorem C. *If* $M :]0,\infty[\to]0,\infty[$ *has continuous derivatives of second order and it is strictly increasing and strictly convex, then* $\widetilde{M}_m(\cdot; w)$ *is a convex function if* M/M' *is a convex function.*

By using (16), we have

Theorem 3.2. *Assume* (A_5)*. If the function*

$$\mathbf{x} \to \widetilde{M}_m(\mathbf{x};\mathbf{w}), \quad \mathbf{x} \in]0,\infty[^m$$

is convex, then Theorem 3.1 *remains valid for* $\widetilde{M}_m(\mathbf{x};\mathbf{w})$ *instead of* $M_m(\mathbf{x};\mathbf{w})$.

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ON VARIABLE EXPONENT HARDY CLASSES OF ANALYTIC FUNCTIONS

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Abstract. The paper studies the Hardy type classes $H^{p(t)}$ and $h^{p(t)}$ of analytic and harmonic functions respectively when a variable exponent $p(t)$ satisfies the log-continuity condition and its least value equals to one. Generalizations of the Fichtenholz, Smirnov and Tumarkin's theorems known for the classical Hardy classes are given. The Dirichlet problem is solved in the framework of spaces $H^{p(t)}$ in two different statements.

რეზიუმე. ნაშრომში შესწავლილია ანალიზურ და ჰარმონიულ ფუნქციათა ჰარდის ტიპის $H^{p(t)}$ და $h^{p(t)}$ კლასები, როდესაც ცვლადი მაჩვენებელი $p(t)$ აკმაყოფილებს ლოგარითმული უწყვეტობის პირობას და მისი უმცირესი მნიშვნელობა ტოლია ერთის. განზოგადებულია ფიხტენგოლცის, სმირნოვის და ტუმარკინის თეორემები, რომლებიც კარგადაა ცნობილი ჰარდის კლასიკური სივრცეების შემთხვევაში. ამოხსნილია დირიხლეს ამოცანა ორი სხვადასხვა დასმით ჰარმონიულ ფუნქციათა ჰარდის ცვლადმაჩვენებლიან კლასებში.

The interest in new functional spaces including those which involve Lebesgue integration with a variable exponent $p(t)$ has appreciably increased in the last two decades, and these spaces have become the subject of study by many mathematicians. This was motivated by the fact that investigation of applied problems in such classes allows one to consider local singularities of the given and unknown functions in more detail (see, e.g., [1]–[7] et al.)

In studying boundary value problems of the theory of analytic functions and certain problems for harmonic functions, the more fruitful turned out to be the notion of variable exponent Hardy classes suggested in [8].

Here we introduce some definitions.

Let $U = \{w : |w| < 1\}$ be a circle with the boundary $\gamma = \{t : |t| = 1\}$ and $p(t) = p(e^{i\sigma}) \equiv p(\sigma)$, $0 \le \sigma \le 2\pi$ be the given on γ positive measurable function.

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We say that an analytic in U function $\Phi(w)$ belongs to the class $H^{p(\cdot)}$, if

$$\sup_{0<r<1} \int_0^{2\pi} \left|\Phi(re^{i\sigma})\right|^{p(\sigma)} d\sigma < \infty; \tag{1}$$

analogously, a harmonic function $u(w)$ belongs to the class $h^{p(\cdot)}$, if

$$\sup_{0<r<1} \int_0^{2\pi} \left|u(re^{i\sigma})\right|^{p(\sigma)} d\sigma < \infty. \tag{2}$$

Assume

$$\widetilde{h}^{p(\cdot)} = \left\{u : \exists\, \Phi \in H^{p(\cdot)}\ u(w) = \operatorname{Re}\Phi(w),\ w \in U\right\}.$$

In the most of the above-mentioned works it is assumed that $p(t)$ satisfies the following conditions:

(1) there exists the constant $C(p)$ such that for any $t_1, t_2 \in \gamma$,

$$\left|p(t_1) - p(t_2)\right| < C(p)\left|\ln(t_1 - t_2)\right|^{-1}; \tag{3}$$

(2) $\min\limits_{t\in\gamma} p(t) = \underline{p} > 1.$

A set of such functions we denote by $\mathcal{P}(\gamma)$.

The class of functions $p(t)$ for which (3) holds and

(2′) $\min\limits_{t\in\gamma} p(t) = \underline{p} = 1,$ (4)

we denote by $\mathcal{P}_1(\gamma)$.

The classes indicated in [8]–[11] have been investigated under the assumption that $p \in \mathcal{P}(\gamma)$. However, from the point of view of applications, it is desirable to maintain the case $\underline{p} = 1$.

In the present paper we present some properties of functions from the classes $H^{p(\cdot)}$, $h^{p(\cdot)}$ and $\widetilde{h}^{p(\cdot)}$ for $p \in \mathcal{P}_1(\gamma)$. The classes Hardy are considered in the domain $U^- = \{w : |w| > 1\}$, as well. It turns out that for $p \in \mathcal{P}(\gamma)$ the equality

$$h^{p(\cdot)} \approx \widetilde{h}^{p(\cdot)}$$

holds. For $p \in \mathcal{P}_1(\gamma)$, this is, generally speaking, false (the corresponding example can be found in item 4.2).

In the final part of the present work we consider the Dirichlet problem in two different statements:

I. Find a harmonic function $u(w)$ of the class $\widetilde{h}^{p(\cdot)}$ such that almost everywhere on γ we have

$$u^+(t) = b(t). \tag{5}$$

II. The certain new class of functions $V \subset L^{p(\cdot)}(\gamma)$ which is invariant with respect to the Cauchy singular operator

$$S : b \to Sb, \quad (Sb)(t) = \frac{1}{\pi i} \int_{\gamma} \frac{b(\tau)d\tau}{\tau - t},$$

i.e.,

$$S(V) = V$$

has been introduced in [12].

We consider the problem: find the function u from the set

$$\widetilde{h}^p(\gamma; U) = \Big\{ u : \exists\, \Phi \in H^{p(\cdot)},$$
$$\Phi(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{\Phi^+(\tau)d\tau}{\tau - w}, \ \Phi^+ \in L^{p(\cdot)}(\gamma), \ u = \operatorname{Re}\Phi \Big\}$$

for which equality (5) holds.

We prove that for problem (5) to be solvable in the first statement, it is necessary and sufficient that

$$b(t) \in L^{p(\cdot)}(\gamma), \quad (Sb)(t) \in L^{p(\cdot)}(\gamma). \tag{6}$$

The problem in the second statement is solvable for any $b \in V$.

In both cases we have a unique solution.

2. Preliminaries

2.1. **The Class $L^{p(\cdot)}(\gamma)$.** Let $p(t)$ be a positive measurable function on γ. For the measurable on γ function $f(\tau) = f(e^{i\sigma})$, $0 \le \sigma \le 2\pi$ we put

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_0^{2\pi} \left| \frac{f(e^{i\sigma})}{\lambda} \right|^{p(\sigma)} d\sigma \le 1,\, p(\sigma) = p(e^{i\sigma}) \right\}.$$

Let

$$L^{p(\cdot)}(\gamma) = \big\{ f : \|f\|_{p(\cdot)} < \infty \big\}.$$

2.2. **The Hardy Classes $H^{p(\cdot)}(U^-)$.**

Definition. We say that the function $\Phi(w)$, analytic in the domain $U^- = \{w : |w| > 1\}$, belongs to the class $H^{p(\cdot)}(U^-)$, if

$$\sup_{R>1} \int_0^{2\pi} \big|\Phi(\mathrm{R}e^{i\sigma})\big|^{p(\sigma)} R\, d\sigma < \infty.$$

For $p \equiv 1$, we write $H^1(U^-)$.

2.3. **Classes of Functions Representable by the Cauchy Type Integral.** By $K^{p(\cdot)}(\gamma)$ we denote a set of functions $\Phi(w)$, analytic in the plane, cut along γ, and representable in the form of the integral

$$\Phi(w)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\varphi(\tau)}{\tau-w}\,d\tau=(K_\gamma\varphi)(w),\quad w\overline{\in}\gamma,\quad \varphi\in L^{p(t)}(\Gamma). \tag{7}$$

3. Some Properties of Hardy Class Functions

3.1. **The existence of boundary values.** Relying on the Fatou's lemma, it is not difficult to prove that functions of classes $h^{p(\cdot)}$ and $H^{p(\cdot)}$ for almost all points $t\in\gamma$ possess an angular boundary value, and the boundary functions belong to $L^{p(\cdot)}(\gamma)$.

3.2. **The condition for belonging of analytic function to the class $H^{p(\cdot)}$.**

Theorem 1. *Let $p\in\mathcal{P}_1(\gamma)$. If the analytic in U function $\Phi(w)$ is representable by one of the formulas*

$$\Phi(w)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\Phi^+(\tau)}{\tau-w}\,d\tau,\quad w\in U,\quad (\textit{the Cauchy formula}), \tag{8}$$

or

$$\Phi(w)=\Phi(r\,e^{i\vartheta})=\frac{1}{2\pi}\int\limits_0^{2\pi}\Phi^+(e^{i\sigma})\frac{1-r^2}{1+r^2-2r\cos(\sigma-\vartheta)}\,d\sigma, \tag{9}$$

$$(\textit{the Poisson formula}),$$

where $\Phi^+\in L^{p(\cdot)}(\gamma)$, then it is representable by another formula, as well.

A set of such functions coincides with the class $H^{p(\cdot)}$.

Proof. We make use of the following result from [6].

If f is 2π-periodic function from $L^{p(\cdot)}(T)$, $T=[0,2\pi]$, $p\in\mathcal{P}_0(\gamma)$, then for the Poisson integral

$$u_f(r,\vartheta)=\frac{1}{2\pi i}\int\limits_0^{2\pi}f(e^{i\sigma})\frac{1-r^2}{1+r^2-2r\cos(\sigma-\vartheta)}\,d\sigma$$

the estimate

$$\big\|u_f(r,\vartheta)\big\|_{p(\cdot)}\le M\|f\|_{p(\cdot)} \tag{10}$$

is valid, where M does not depend on f.

This implies that for almost all $t\in\gamma$ there exists an angular limit $u^+(t)$ which is equal to $t(e^{i\vartheta})$, and

$$\lim_{r\to1}\big\|u_f(re^{i\vartheta})-f(e^{i\vartheta})\big\|_{p(\cdot)}=0,$$

hence,

$$u_+(re^{i\vartheta}) \in h^{p(\cdot)}, \quad p \in \mathcal{P}_0(\gamma). \tag{11}$$

Let now (9) hold, where $\Phi = u + iv$ and $\Phi^+ = (u^+ + iv^+) \in L^{p(\cdot)}(\gamma)$, then $u(r,\vartheta) = u_{\mathrm{Re}\,\Phi^+}(r,\vartheta)$, $v(r,\vartheta) = u_{\mathrm{Im}\,\Phi^+}(r,\vartheta)$ and by virtue of equality (11), we have $u \in h^{p(\cdot)}$, $v \in h^{p(\cdot)}$. Thus $\Phi \in H^{p(\cdot)} \subset H^1$, and according to the Fichtenholz theorem, Φ is representable by formula (8), where $\Phi^+ \in L^{p(\cdot)}(\gamma)$.

If (8) is valid, then $\Phi \in H^1$ and $\Phi^+ \in L^{p(\cdot)}(\gamma)$. Again, by virtue of Fichtenholz theorem, formula (9), where $\Phi^+ \in L^{p(\cdot)}(\gamma)$, is valid according to the assumption, and $\Phi \in H^{p(\cdot)}$, by the above proven.

If $\Phi \in H^{p(\cdot)}$, then it belongs to H^1 and $\Phi^+ \in L^{p(\cdot)}(\gamma)$ (see item 3.1). This implies that both equalities (8) and (9) are valid. □

3.3. **On the functions of the class $H^1(U^-)$.** (a) If the analytic in U^- function $\Phi(w)$ belongs to $H^1(U^-)$, then the function $F(\zeta) = \Phi(\frac{1}{\zeta})$, $\zeta \in U$ belongs to H^1, and $F(0) = 0$.

Conversely, if $F(0) = 0$ and $F(\zeta) \in H^1$, then $\Phi(\zeta) = F(\frac{1}{\zeta})$, $\zeta \in U^-$ belongs to $H^1(U^-)$.

(b) For the analytic in U^- function $\Phi(w)$ to belong to $H^1(U^-)$, it is necessary and sufficient that it be representable by the Cauchy integral

$$\Phi(w) = -\frac{1}{2\pi i}\int\limits_\gamma \frac{\Phi^-(\tau)d\tau}{\tau - w}, \quad w \in U^-. \tag{12}$$

3.4. **On the representability of a pair of functions given on U and U^- by the Cauchy type integral.**

(a) If $\Phi_1 \in H^1$, $\Phi_2 \in H^1(U^-)$, then the function

$$F(w) = \frac{1}{2\pi i}\int\limits_\gamma \frac{\Phi_1^+(\tau) - \Phi_2^-(\tau)}{\tau - w}\, d\tau, \quad w \overline{\in} \gamma$$

coincides both with $\Phi_1(w)$ for $w \in U$ and with $\Phi_2(w)$ for $w \in U^-$. If, however, $\Phi_1 \in H^{p(\cdot)}$, $\Phi_2 \in H^{p(\cdot)}(U^-)$, then $F \in K^{p(\cdot)}(\gamma)$.

(b) If $\Phi_1 \in H^1$, $\Phi_2 \in H^1(U^-)$ and almost for all $t \in \gamma$ we have

$$\Phi_1^+(t) = \Phi_2^-(t),$$

then $\Phi_1(w) = 0$, $w \in U$, $\Phi_2(w) = 0$, $w \in U^-$.

3.5. **On the classes $h^{p(\cdot)}$ and $\widetilde{h}^p$.**

Theorem 2. *If $p \in \mathcal{P}(\gamma)$, then*

$$h^{p(\cdot)} = \widetilde{h}^{p(\cdot)}. \tag{13}$$

Proof. The fact that $\widetilde{h}^{p(\cdot)} \subset h^{p(\cdot)}$ is obvious. Let us prove that $h^{p(\cdot)} \subset \widetilde{h}^{p(\cdot)}$.

Let $u \in h^{p(\cdot)}$, then $u \in h^{\underline{p}}$, $\underline{p} > 1$; by the known Riesz theorem, the function v, harmonically conjugate to u, likewise belongs to $h^{\underline{p}}$, hence

$$\Phi(w) = [u(w) + iv(w)] \in H^{\underline{p}} \subset H^1.$$

By the Fichtenholz theorem,

$$\Phi(w) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{u(\tau) + iv(\tau)}{\tau - w}\, d\tau,$$

where $(u + iv) \in L^{p(\cdot)}(\gamma)$, $p \in \mathcal{P}(\gamma)$. But the Cauchy type integral $\frac{1}{2\pi i} \int_\gamma \frac{f(\tau)d\tau}{\tau - w}$, $w \in U$ for $f \in L^{p(\cdot)}(\gamma)$ belongs to $H^{p(\cdot)}$ (see [11], p. 76). Consequently, $\Phi \in H^{p(\cdot)}$, and $u = \operatorname{Re}\Phi$, i.e., $u \in \widetilde{h}^{p(\cdot)}$. □

3.6. **Generalization of one Smirnov's theorem.** The following Smirnov's theorem is well known [13].

Theorem. *If $\Phi \in H^p$, $\Phi^+ \in L^{p_1}(\gamma)$, $p_1 > p$, then $\Phi \in H^{p_1}$.*

For the variable p, the theorem below is valid.

Theorem 3. *If $\Phi \in H^{p(\cdot)}$, $\underline{p} > 0$ and $\Phi^+ \in L^{\mu(\cdot)}(\gamma)$, $\mu \in \mathcal{P}_1(\gamma)$, then $\Phi \in H^{\lambda(\cdot)}$, where $\lambda(t) = \max(p(t), \mu(t))$.*

In [11] (p. 76), this theorem has been proved under the assumptions $\underline{p} > 0$, $\mu \in \mathcal{P}(\gamma)$.

Proof. Let $\Phi(z) \in H^{p(\cdot)}$, $\underline{p} > 0$ and $\Phi \in L^{\mu(\cdot)}(\gamma)$. This implies that $\Phi^+ \in L^1(\gamma)$; consequently, $\Phi(z) \in H^1$. Then $\Phi(z) = \frac{1}{2\pi i} \int_\gamma \frac{\Phi^+(\tau)d\tau}{\tau - z}$. Here $\Phi^+(t) \in L^{\mu(t)}(\gamma)$, where $\mu(t) \in \mathcal{P}_1(\gamma)$. By Theorem 1, we conclude that $\Phi \in H^{\mu(t)}$. Thus $\Phi \in H^{p(\cdot)}$ (by the assumption) and $\Phi \in H^{\mu(\cdot)}$ (by the above proven), hence $\Phi \in H^{\mu(t)}$. □

3.7. **On the convergence of a function sequence from $H^{p(\cdot)}$, $p \in \mathcal{P}_0(\gamma)$.**

Theorem 4. *Let $\{\Phi_n(\zeta)\}$ be a sequence of boundary values of functions $\Phi_n(z) \in H^{p(\cdot)}$, $p \in \mathcal{P}_1(\gamma)$ and*

$$\int\limits_{\gamma} \left|\Phi_n(\zeta)\right|^{p(\zeta)} |d\zeta| = \int\limits_{0}^{2\pi} \left|\Phi_n(\zeta^{i\vartheta})\right|^{p(\vartheta)} d\vartheta < C, \quad p(\vartheta) = p(e^{i\vartheta}),$$

where ζ is independent of n.

If $\{\Phi_n(\zeta)\}$ converges in measure on γ, then the sequence $\{\Phi_n(\zeta)\}$ converges uniformly in U to some function $\Phi(z)$ of the class $H^{p(\cdot)}$, and $\{\Phi_n(\zeta)\}$ converges in measure on γ to the function $\Phi^+(\zeta)$.

Proof. We have

$$\int_\gamma \big|\Phi_n(\zeta)\big||d\zeta| = \int_{\{\zeta:|\Phi_n(\zeta)|\le 1\}} \big|\Phi_n(\zeta)\big||d\zeta| + \int_{\{\zeta:|\Phi_n(\zeta)|>1\}} \big|\Phi_n(\zeta)\big||d\zeta| \le$$
$$\le 2\pi + \int_\gamma \big|\Phi_n(\zeta)\big|^{p(\zeta)}|d\zeta| \le 2\pi + C.$$

Using Tumarkin's theorem ([14], p. 263-9) (in which it is stated that the provable theorem is valid for $p = \text{const}$), we conclude that $\{\Phi_n(\zeta)\}$ converges in U to some function $\Phi \in H^1$. Let us show that $\Phi^+ \in L^{p(\cdot)}(\gamma)$.

From the converging in measure on γ sequence $\{\Phi_n(\zeta)\}$ we select the subsequence $\{\Phi_n(z)\}$, converging almost everywhere on γ. Then $|\Phi_{n_k}(e^{i\vartheta})|^{p(\vartheta)}$ converges almost everywhere on γ to the function $|\Phi(e^{i\vartheta})|^{p(\vartheta)}$. By the Fatou's lemma, we obtain

$$\int_\gamma \big|\Phi^+(\zeta)\big|^{p(\zeta)}|d\zeta| = \int_\gamma \lim_{k\to\infty} \big|\Phi_{n_k}(\zeta)\big|^{p(\zeta)}|d\zeta| \le \int_\gamma \big|\Phi_{n_k}(\zeta)\big|^{p(\zeta)}|d\zeta| < C.$$

Thus $\Phi \in H^1$, $\Phi^+ \in L^{p(\cdot)}(\gamma)$. Hence

$$\Phi(z) = \frac{1}{2\pi i}\int_\gamma \frac{\Phi^+(t)dt}{t-z}, \quad \Phi^+ \in L^{p(\cdot)}(\gamma).$$

By Theorem 1, $\Phi(z) \in H^{p(\cdot)}$. □

Theorem 4 is a partial generalization of G. Tumarkin's theorem (see [14], p. 268–269).

4. The Dirichlet Problem in the Class $\widetilde{h}^{p(\cdot)}$

4.1. For $p \in \mathcal{P}(\gamma)$, the Dirichlet problem is solved in the class $h^{p(\cdot)}$ for $b \in L^{p(\cdot)}(\gamma)$ (see, e.g., [11], p. 219). The solution is unique and representable by the Poisson integral.

When $p \in \mathcal{P}_1(\gamma)$, situation changes in the main. Here we have the following

Theorem 5. *Let $p \in \mathcal{P}_1(\gamma)$; for the solvability of the Dirichlet problem in the class $\widetilde{h}^{p(\cdot)}$, that is, for the existence of the function $u(w)$ which is the real part of some function from $H^{p(\cdot)}$ and*

$$u^+(t) = b(t) \tag{14}$$

it is necessary and sufficient that the conditions

$$b(t) \in L^{p(\cdot)}(\gamma), \quad (Sb)(t) \in L^{p(t)}(\gamma). \tag{15}$$

be fulfilled. If these conditions are fulfilled, then the Dirichlet problem in the class $\widetilde{h}^{p(\cdot)}(\gamma)$ is uniquely solvable and the solution $u(w)$ is given by the equality

$$u(w) = \Re \frac{1}{2\pi} \int_{\gamma} b(\tau) \frac{\tau + w}{\tau - w} \frac{d\tau}{\tau}\,, \tag{16}$$

or what us the same,

$$u(w) = \frac{1}{2\pi} \int_{\gamma} b(e^{i\sigma}) \frac{1 - r^2}{1 + r^2 - 2r\cos(\sigma - \vartheta)}\, d\sigma, \quad w = re^{i\vartheta}. \tag{17}$$

Proof. The necessity. We use the following result.

If $\Phi(w) \in H^1$, then it is representable in the form

$$\Phi(w) = \frac{1}{2\pi} \int_{\gamma} \operatorname{Re} \Phi^+(\tau) \frac{\tau + w}{\tau - w} \frac{d\tau}{\tau} + i \operatorname{Im} \Phi(0). \tag{18}$$

(This statement is well-known for the functions Φ, analytic in U and continuous in $\overline{U}$. In the above formulation, this statement can be found in [15] (see also [11], p. 11)).

Thus, let $u(w) \in \widetilde{h}^{p(\cdot)}$ and satisfy the condition (14), then there exists the function $\Phi(w) \in H^{p(\cdot)} \subset H^1$ such that $u(w) = \operatorname{Re} \Phi(w)$.

By virtue of the statement from item 3.2, a solution $u(w)$ may be only the function given by equality (16). For this function to be a solution, it is necessary that the function

$$\Phi_b(w) = \frac{1}{2\pi} \int_{\gamma} b(\tau) \frac{\tau + w}{\tau - w} \frac{d\tau}{\tau}$$

belongs to H^1, i.e., the equality

$$\Phi_b(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{\Phi_b^+(\tau) d\tau}{\tau - w}$$

be valid.

Since

$$\Phi_b(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{2b(\tau) d\tau}{\tau - w} - \frac{1}{2\pi i} \int_{\gamma} \frac{b(\tau)}{\tau}\, d\tau, \tag{19}$$

by virtue of Sokhotskii-Plemelj formula we, find

$$\Phi_b^+(t) = b(t) + (Sb)(t) + \text{const}\,. \tag{20}$$

By the statement from item 3.1, we should have $[b(t) + (Sb)(t)] \in L^{p(\cdot)}(\gamma)$. Since $b(t) \in L^{p(\cdot)}(\gamma)$, we should have $(Sb)(t) \in L^{p(\cdot)}(\gamma)$. Hence conditions (15) are fulfilled.

The sufficiency. Let the conditions (15) be fulfilled. Let us prove that $\Phi_b(w) \in H^{p(\cdot)}$. It is seen from (19) that $\Phi_b(w)$, as the Cauchy type integral, belongs to $\bigcap\limits_{\delta<1} H^\delta$ (see [14]. p. 96). It follows from (15) that $\Phi_b(w) \in H^1$, and hence,

$$\Phi_b(w) = \frac{1}{2\pi i} \int\limits_\gamma \frac{\Phi_b^+(\tau)d\tau}{\tau - w}.$$

According to (20) and (15), we find that $\Phi_b(w)$ is representable by the Cauchy integral with density $\Phi_b^+ \in L^{p(\cdot)}(\gamma)$. By virtue of Theorem 1, we conclude that $\Phi_b \in H^{p(\cdot)}$. Consequently, $u = \operatorname{Re} \Phi_b$ is the solution of problem (14) of the class $\widetilde{h}^{p(t)}$. □

Remark. The fact that $\Phi_b(w)$ belongs to the class $H^{p(\cdot)}$ can be also proved as follows.

As is mentioned above, $\Phi_b \in \bigcap\limits_{\delta<1} H^\delta$; assume $\Phi_b \in H^{1/2}$. Next, owing to (15), the function $\Phi_b^+ \in L^{p(t)}(\gamma)$.

Using Theorem 3, we find that $\Phi_b \in H^{\lambda(t)}$, where $\lambda(t) = \max(\frac{1}{2}; p(t)) = p(t)$.

4.2. On the functions $b(t)$ for which problem (14) is unsolvable. If $p \in \mathcal{P}(\gamma)$, then for any function $b \in L^{p(\cdot)}(\gamma)$ we have $Sb \in L^{p(\cdot)}(\gamma)$ (see [15] and also [11], p. 44). But when $p \in \mathcal{P}_1(\gamma)$, then this is, generally speaking, impossible at least for such $p(t)$ which admit value 1 on some arc $\gamma_0 \subset \gamma$. Indeed, were Sb for any b from $L^{p(\cdot)}(\gamma)$ belong to $L^{p(\cdot)}(\gamma)$, the Cauchy operator $S : b \to Sb$ would be continuous in $L^{p(\cdot)}(\gamma)$ (see [16], and also [11], p. 101). But this is impossible, since there exist the functions $\widetilde{b} \in L^1(\gamma_0)$ for which $S\widetilde{b} \,\overline{\in}\, L^1(\gamma_0)$; taking as $b(t)$ the function b_1 from $L^{p(\cdot)}(\gamma)$ which equals $\widetilde{b}$ on γ_0, we have $b_1 \,\overline{\in}\, L^1(\gamma_0)$, and hence, $Sb_1 \,\overline{\in}\, L^{p(\cdot)}(\gamma)$.

Obviously, in the case under consideration there exist linearly independent functions $b_1, b_2, \dots, b_n, \dots$ for which problem (14) is unsolvable in the class $\widetilde{h}^{p(\cdot)}$.

4.3. Certain subsets of functions from $L^{p(\cdot)}(\gamma)$, $\min\limits_{t\in\gamma} p(t) = 1$, for which conditions (15) are fulfilled. In [12], in connection with the investigation of problems dealing with the approximation of functions from $L^{p(\cdot)}(\gamma)$, it was considered the sets V_r, $r \in N_0 = \{0, 1, 2, \dots\}$ of those function f from $L^{p(\cdot)}(\gamma)$ for which

$$\int\limits_0^{\delta_0} \frac{\Omega(t,\delta)}{\delta} \Big(\ln \frac{1}{\delta}\Big)^r d\delta < \infty, \quad \text{where} \quad \delta_0 > 0,$$

and

$$\Omega(f,\delta)=\sup_{h\le\delta}\Bigg\|\int\limits_{s-h}^{s+h} f(e^{i\sigma})d\sigma - f(s)\Bigg\|_{p(\cdot)}.$$

It has been proved that the Cauchy operator $S: b\to Sb$ transfers V_{r+1} into V_r, and $S(V_0)\subset L^{p(\cdot)}(\gamma)$.

Consequently, the following theorem is valid.

Theorem 6. *If $b\in V_0$, then problem* (14) *is solvable in the class $\widetilde{h}^{p(\cdot)}$.*

4.4. On classes of functions V and $\widetilde{h}^{p(\cdot)}(\gamma;V)$. The Dirichlet problem in the class $\widetilde{h}^{p(\cdot)}(\gamma;V)$. The above-mentioned work [12] considers also the set

$$V=\bigcap_{z\in N_0} V_r.$$

which is invariant with respect to the operator S, i.e.,

$$S(V)=V_l.$$

Let us consider the Dirichlet problem in the following statement: find the function $u(w)$ from the set

$$\widetilde{h}^{p(\cdot)}(\gamma;V)=\Bigg\{u:\exists\,\Phi\in H^{p(\cdot)}\Phi(w)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\Phi^+(\tau)d\tau}{\tau-w},$$
$$\Phi^+\in V,\;\; u(w)=\operatorname{Re}\Phi(w)\Bigg\},$$

which satisfies the boundary condition

$$u^+(t)=b(t).$$

In this case the theorem below is valid.

Theorem 7. *If $b\in V$, then the Dirichlet problem is solvable in the class $\widetilde{h}^{p(\cdot)}(\gamma;U)$.*

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THE RIEMANN BOUNDARY VALUE PROBLEM IN VARIABLE EXPONENT SMIRNOV CLASS OF GENERALIZED ANALYTIC FUNCTIONS

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Abstract. The present paper studies the Riemann boundary value problem for generalized analytic in I. Vekua sense functions. The problem is formulated as follows: on the plane, cut along a simple, closed, rectifiable curve Γ, find the generalized analytic function $W(z)$ which in the domains G^+ and G^-, bounded by the curve Γ, belongs to the Smirnov classes with a variable exponent and $W^{\pm}(t)$ its boundary values almost for all $t \in \Gamma$ satisfy the condition

$$W^+(t) = a(t)W^-(t) + b(t),$$

where $a(t)$ and $b(t)$ are the given on Γ functions.

Various conditions of solvability are revealed and solutions (if any) are constructed.

რეზიუმე. ნაშრომში ი. ვეკუას აზრით განზოგადებული ანალიზური ფუნქციებისათვის გამოკვლეულია რიმანის სასაზღვრო ამოცანა მარტივ შეკრულ Γ წირზე გაჭრილ სიბრტყეში. ვიპოვოთ ისეთი განზოგადებული ანალიზური ფუნქცია $W(z)$, რომელიც Γ-თი შემოსაზღვრულ არეებში მიეკუთვნება სმირნოვის ცვლად $p(t)$ მაჩვენებლიან კლასებს და რომლის სასაზღვრო მნიშვნელობები და $W^{\pm}(t)$ Γ წირის თითქმის ყველა წერტილში აკმაყოფილებენ ტოლობას

$$W^+(t) = a(t)W^-(t) + b(t),$$

სადაც $a(t)$ და $b(t)$-ზე მოცემული ფუნქციებია.

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1. Introduction

In the boundary value problems appearing in various fields of mathematics it is frequently required of the solution that unknown functions would belong to a certain Lebesque class (see, e.g., [2], [6], [7], [26], etc.)

Recently, the problems of pseudo-differential equations are being intensively studied in nonstandard Banach functional spaces, in particular, in the framework of variable exponent Lebesgue spaces. Such a statement of the problem is motivated by the fact that the classes of functions in definition of which the integration exponent is, generally speaking, a function, more precisely take into account local singularities of the given functions. Such spaces are natural ones in which we seek for solutions.

There is a vast literature devoted to the investigation of variable exponent Lebesgue spaces. It suffices to mention monographs [1], [3], [12] and references therein.

The works [3], [9], [11], [12], [22], [24], [25], etc. dealing with the boundary value problems for analytic and harmonic functions and related singular integral equations have been studied in the framework of variable exponent Lebesgue spaces.

In these problems regarding $p(t)$ it is more frequently assumed that $p(t) \in P(\Gamma)$, i.e., the conditions:

(a) there exists the number M such that for any t_1, $t_2 \in \Gamma$ we have

$$|p(t_1) - p(t_2)| < M|\ln|t_1 - t_2||^{-1}; \tag{1}$$

(b)

$$\min_{t\in\Gamma} p(t) = \underline{p} > 1, \tag{2}$$

are fulfilled.

Further, the generalized Cauchy type integral and the generalized singular integral have been investigated in [13], and Smirnov classes with a variable exponent $p(t)$ for generalized analytic functions have been introduced and studied in [21]. The results obtained in these works give every reason to investigate boundary value problems for generalized analytic functions when boundary values of unknown functions and those prescribed in the boundary conditions belong to $L^{p(t)}(\Gamma)$.

The Riemann problem for continuous statement has been considered in [17]. Smirnov classes for a constant p are studied in [18]. A number of problems in these classes have been investigeted in [14], [5], [7], [8], [15], [16], [18], [19].

In the present paper we investigate the Riemann problem which is formulated as follows.

Let Γ be a simple, closed, rectifiable curve dividing the plane $\mathbb{C}$ into two domains G^+ and G^-. Find such a generalized analytic in I. Vekua sense function W which

1) is a regular solution of the class $U^{s,2}(A,B,\mathbb{C})$ $s>2$ of equation;

$$LW=\partial_{\overline{z}}W+A(z)W(z)+B(z)\overline{W(z)}=0; \tag{3}$$

2) belongs to the class $K^{p(t)}(A;B;\Gamma)$, i.e., is representable by the generalized Cauchy type integral

$$W(z)=(\widetilde{K}_{\Gamma}\varphi)(z)=$$
$$=\frac{1}{2\pi}\int_{\Gamma}\Omega_1(z,\tau)\varphi(\tau)d\tau-\Omega_2(z,\tau)\overline{\varphi}(\tau)d\overline{\tau},\quad \varphi\in L^{p(\cdot)}(\Gamma),\quad z\overline{\in}\Gamma; \tag{4}$$

3) the boundary functions $W^+(t)$ and $W^-(t)$ almost everywhere on Γ satisfy the condition

$$W^+(t)=a(t)W^-(t)+b(t), \tag{5}$$

where $b(t)\in L^{p(t)}(\Gamma)$.

Regarding Γ, $p(t)$ and $a(t)$, it is assumed that

(a) Γ is a curve of the class I^* containing, in particular, piece-wise smooth and Radon's curves without external peaks;

(b) $p(t)$ is the function of the class $\mathcal{P}(\Gamma)$;

(c) $a(t)$ belongs to the $A(p(t),\Gamma)$ class of measurable functions on Γ which is a natural generalization of I. Simonenko's class (see [26]).

Under the adopted assumptions, the generalized Cauchy type integrals (2) on the domains G^+ and G^- belong to the Smirnov classes $E^{p(\cdot)}(A;B;G^+)$ and $E^{p(\cdot)}(A;B;G^-)$, respectively [21].

A set of generalized analytic functions in the plane, cut along the closed curve Γ such that in the domains G^+ and G^- bounded by Γ they belong to the classes $E^{p(\cdot)}(A;B;G^{\pm})$, we denote by $PE^{p(\cdot)}(A;B;\Gamma)$. Such functions in the conditions (1), (a) and (b) are representable by the generalized Cauchy type integral in the domains G^+ and G^-, and therefore are representable by the Cauchy integral with density from $L^{p(\cdot)}(\Gamma)$ [21]. By virtue of the abovesaid, a picture of solvability of the Riemann problem in classes $K^{p(\cdot)}(A;B;\Gamma)$ and $PE^{p(\cdot)}(A;B;\Gamma)$ is the same.

2. Preliminaries

2.1. **The function of the class $L^{s,\nu}(G)$.** Let G be the domain in the plane $\mathbb{C}$, and $f(z)$ be the function of the class $L^s(G)$, $s>0$. We continue it on $\mathbb{C}\backslash G$ by zero and for the obtained function we preserve the notation $f(z)$. Assume $f_\nu(z)=z^\nu f\left(\frac{1}{z}\right)$.

A set of functions f for which

$$f\in L^s(\mathbb{C}),\quad f_\nu(z)\in L^s(U),\quad U=\{z:|z|<1\}. \tag{6}$$

we denote by $L^{s,\nu}(\mathbb{C})$ [27, p. 29].

2.2. **Regular solutions of equation (3).** We say that the function $W = W(z)$ is a regular solution in G of equation (3), if every point $z_0 \in G$ possesses a neighbourhood G_0 in which W has a generalized in Sobolev sense derivative $\partial_{\overline{z}} W = \frac{1}{2}\left(\frac{\partial W}{\partial x} + i\frac{\partial W}{\partial y}\right)$, $(z = x+iy)$ and almost everywhere in $G_0 - LW = 0$.

A set of regular solutions of equation (3), when $A, B \in L^{s,2}(G)$, we denote by $U^{s,2}(A;B;G)$.

For $s > 2$, every function $W \in U^{s,2}(A;B;G)$ is representable in the form

$$W = \Phi_W \exp \omega_W, \quad \omega_W(z) = \iint\limits_G \left(A + B\frac{\overline{W}}{W}\right)\frac{d\zeta d\eta}{\zeta - z}, \quad \zeta = \xi + i\eta, \tag{7}$$

where Φ_W is holomorphic in G, ω_W belongs to the Hölder class $H_{\frac{s-2}{s}}(\mathbb{C})$, and $\omega_W(\infty) = 0$ [27, pp. 160–162].

The function Φ_W is called an analytic divisor and ω_W is a logarithmic difference of the generalized analytic function $W(z)$.

2.3. **The principal kernels.** Let $A, B \in L^{s,2}(G)$, $s > 2$, Φ be an analytic function in G and t be a fixed point from $\mathbb{C}$. It is proved in [27] (p. 175-7) that there exists a regular solution $W(z;t)$ of equation (3) such that: 1) $W_0 = \frac{W(z,t)}{\Phi(z)}$ is continuous in G and continuously extendable on $\mathbb{C}$; 2) $W_0(z) \neq 0$; 3) $W(t) = 1$; 4) $W_0(z)$ is holomorphic outside of G.

The operator which assigns to each pair Φ and t the function $W(z;t)$ we denote by $R_t^{A;B}(\Phi(z))$.

If $\Phi_1(z) = \frac{1}{2(t-z)}$, $\Phi_2(z) = \frac{1}{2i(t-z)}$ and $X_j(z,t) = R_t^{A,B}(\Phi_j(z))$, $j = 1, 2$ are regular solutions of equation (3) in $\mathbb{C}\{t\}$, then the functions

$$\Omega_1(z,t) = X_1(z,t) + iX_2(z,t), \quad \Omega_2(z,t) = X_1(z,t) - iX_2(z,t)$$

are called the principal kernels of the class $U^{s,2}(A;B;G)$.

2.4. **Generalized polynomials.** A generalized polynomial of order n of the class $U^{s,2}(A;B;\mathbb{C})$ is called that regular solution of equation (3) whose analytic divisor is a classical polynomial of order n [27, p. 167].

Suppose

$$\nu'_{2k} = R_\infty^{-A,-\overline{B}}(z^k), \quad \nu'_{2k+1}(z) = R_\infty^{-A,-\overline{B}}(iz^k).$$

2.5. **The space $L^{p(\cdot)}(\Gamma)$.** For the measurable on Γ function $f(t)$ we put

$$\|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \int\limits_0^b \Big|\frac{f(t(\sigma))}{\lambda}\Big|^{p(t(\sigma))} d\sigma \leq 1\},$$

where $t = t(\sigma)$, $0 \le \sigma \le l$ is the equation of the curve Γ with respect to the arc abscissa σ. And let

$$L^{p(\cdot)}(\Gamma) = \{f : \|f\|_{p(\cdot)} < \infty\}.$$

For $p \in P(\Gamma)$, the set $L^{p(\sigma)}(\Gamma)$ with the norm $\|\cdot\|_{p(\cdot)}$ is the Banach space.

3. The Variable Exponent Smirnov Class

3.1. **Definition.** We say that the generalized analytic function W belongs to the class $E^{p(\cdot)}(A;B;G)$, if $W \in U^{s,2}(A;B;G)$, $s > 2$ and

$$\sup_{0<\rho<1} \int_0^{2\pi} |W(z(\rho e^{i\theta}))|^{p(\theta)} |z'(\rho e^{i\theta})| d\theta < \infty, \quad p(\theta) \equiv p(z(e^{i\theta})), \tag{8}$$

where $z = z(\rho e^{i\theta})$ is the comformal mapping of U onto G (for details on those classes, see [21]).

If $W \in E^{p(\cdot)}(A;B;G)$, $p \in P(\Gamma)$ then almost for all $t \in \Gamma$, there exists an angular boundary value $W^+(t)$, and the function $t \to W^+(t)$ belongs to $L^{p(\cdot)}(\Gamma)$.

It follows from the representation (7) that the belonging of W to the class $E^{p(\cdot)}(A;B;G)$ is equivalent to the fact that the function Φ_W belongs to the class $E^{p(\cdot)}(G)$, i.e.,

$$\sup \int_0^{2\pi} |\Phi_W(z(\rho e^{i\theta}))|^{p(\theta)} |z'(\rho e^{i\theta})| d\theta < \infty.$$

If G is an unbounded domain and there is the polynomial $Q(z)$ such that $[\Phi(z) - Q(z)] \in E^{p(\cdot)}(G)$, we write $\Phi \in \widetilde{E}^{p(t)}(G)$.

3.2. **Classes of functions representable by the generalized Cauchy type integral.** Let Γ be a simple rectifiable curve bounding the domains G^+ and G^-, $\Omega_1(z,t)$ and let $\Omega_2(z,t)$ be the kernels of the class $U^{s,2}(A;B;\mathbb{C})$, $f \in L(\Gamma)$. The function

$$(\widetilde{K}_\Gamma f)(z) = \int_\Gamma \Omega_1(z,t) f(t) dt - \Omega_2(z,t)\overline{f}(t) d\bar{t}, \quad z\overline{\in}\Gamma$$

is called the generalized Cauchy type integral [27, p. 198].

This function is a regular solution of (3) of the class $U^{s,2}(A;B;\mathbb{C})$.

Assume

$$K^{p(t)}(A;B;\Gamma) = \{W : W(z) = (\widetilde{K}_\Gamma f)(z), \ \ f \in L^{p(\cdot)}(\Gamma)\};$$
$$K^{p(t)}(\Gamma) = K^{p(t)}(0;0;\Gamma).$$

$\widetilde{K}^{p(t)}(A;B;\Gamma) = \{W : \exists \text{ polynomial } p_{_W} : W(z) = W_0(z) + p_{_W}(z),\ W_0 \in K^{p(t)}(A;B;\Gamma)\}$.

4. Classes of Curves

4.1. **Lavrent'ev's curves (of the class Λ).** The curve Γ belongs to the class Λ, if $\sup_{t_1,t_2\in\Gamma} s(t_1,t_2)[|t_1-t_2|^{-1}]<\infty$, where $s(t_1,t_2)$ is the length of the least of two arcs lying on Γ and joining t_1 and t_2.

4.2. **The class I_0.** I_0 is a set of curves Γ with the equation $t=t(\sigma)$, $0\le\sigma\le l$ (with respect to the arc abscissa) for which there exists a smooth curve with the equation $\mu=\mu(\sigma)$, $0\le\sigma\le l$ such that

$$\operatorname*{ess\,sup}_{0\le\sigma_0\le l}\int_0^l\Big|\frac{t'(\sigma)}{t(\sigma)-t(\sigma_0)}-\frac{\mu'(\sigma)}{\mu(\sigma)-\mu(\sigma_0)}\Big|d\sigma<\infty.$$

4.3. **The class I^*.** The simple curve Γ belongs to the class I^*, if $\Gamma\in\Lambda$ and it can be represented as a finite union of arcs of the class I_0, having tangents at the ends.

4.4. **Examples.** I^* contains piecewis-smooth and piecewise-Radonean curves without cusps (see [4], pp. 23-30, [1], pp. 146-7).

5. The class of functions $A(p(t),\Gamma)$.

A measurable function $a(t)$ belongs to the class $A(p(t),\Gamma)$, if

1) $0<m=\operatorname*{ess\,inf}_{t\in\Gamma}|a(t)|\le\operatorname*{ess\,sup}_{t\in\Gamma}|a(t)|=M<\infty$;

2) for every point $\tau\in\Gamma$, there exists the arc $\Gamma_\tau\subset\Gamma$ containing τ on which almost all values $a(t)$ lie inside of the angle with vertex at the origin, of size less than

$$\alpha_\tau=2\pi\Big[\sup_{t\in\Gamma_\tau}\max(p(t),q(t))\Big]^{-1},\quad q(\tau)=\frac{p(\tau)}{p(\tau)-1}.$$

For the function $a(t)$ from $A(p(t),\Gamma)$, following [26], we define a branch of the function $\arg a(t)$. We select a finite covering of Γ by the arcs $\Gamma_k=\Gamma_{\tau_k}$.

Let c be the point on Γ_{τ_1} at which there exists the tangent and the point $a(\sigma)$ lies inside of the angle of size α_{r_1}. We fix $(\arg a(c))^-\in[0,2\pi)$. Moving along γ, we define the value $\arg a(t)$ so as for t_1,t_2, lying on one of the arcs Γ_{τ_k}, to have $|\arg a(t_1)-\arg a(t_2)|<\alpha_{\tau_k}$. Going around Γ, the point c falls into Γ_{τ_1} with a new value $(\arg(c))^+$.

The number

$$\varkappa=\frac{1}{2\pi}\Big[(\arg a(c))^+-(\arg a(c))^-\Big]\tag{9}$$

is the integer, independent of the covering of Γ by the arcs Γ_k, and the choice of c. We call this number an index of the function $a(t)$ and write $\varkappa=\operatorname{ind}a(t)$.

For $p=\mathrm{const}$, the class $A(p,\Gamma)$ coincides with the known I. Simonenko's class [26].

6. Statement of the Riemann problem.

When Γ is the Carleson curve bounding the domains G^+ and G^-, and

$$A, B \in L^{s,2}(G^+), \quad s > 2, \quad p \in P(\Gamma), \quad \overline{p} = \sup_{t\in\Gamma} p(t),$$

$$\overline{p}' = \frac{\overline{p}}{\overline{p}-1}, \quad \frac{s}{2} > \overline{p}' \tag{10}$$

then as is proved in [21], the equality

$$K^{p(\cdot)}(A; B; G^+) = E^{p(\cdot)}(A; B; G^+) \tag{11}$$

holds.

In particular, inclusion (11) holds if

$$A, B \in L^{\infty}(G^+), \quad p \in P(\Gamma). \tag{12}$$

When

$$\Gamma \in I^*, \quad p \in P(\Gamma), \quad a \in A(p(t), \Gamma), \quad b \in L^{p(t)}(\Gamma) \tag{13}$$

problem (5) in the class $K^{p(t)}(\Gamma)$ has been investigated in [22].

Since when solving problem (5) in the class $K^{p(t)}(A; B; \Gamma)$, of importance for us is equality (11) and knowledge of a picture of its solvability in $K^{p(t)}(A; B; \Gamma)$, we will assume that the condition

$$\Gamma \in I^*, \quad A, B \in L^{s,2}(G), \quad s > 2, \quad p \in P(\Gamma), \quad \frac{s}{2} > \overline{p}', \tag{14}$$

or

$$\Gamma \in I^*, \quad A, B \in L^{\infty}(G), \quad p \in P(\Gamma), \tag{15}$$

is fulfilled.

In the first case, the choice for A, B is wide, but the set of admissible $p(t)$ is bounded by the condition $\frac{s}{2} > \overline{p}'$. In the second case, the set of A and B contracts, but now $p(t)$ is arbitrary from $P(\Gamma)$.

Thus, let condition (14) or (15) be fulfilled and we are required to find a generalized analytic function W which is a regular solution of equation (3), representable by the generalized Cauchy type integral with density $L^{p(\cdot)}(\Gamma)$ and almost everywhere on Γ equality (5) is valid.

When we say that W is a regular solution of problem (5), we regard that all the conditions adopted in this section for W are satisfied.

7. Solution of the Problem

7.1. One necessary condition of solvability. If

$$\Gamma \in I^*, \quad a \in A(p(t), \Gamma) \quad p \in P(\Gamma). \tag{16}$$

then the function

$$X(z) = \begin{cases} \exp h(z), \quad z \in G^+, \\ (z - z_0)^{-\varkappa} \exp h(z), \quad z_0 \in G^+, \quad z \in G^- \end{cases} \tag{17}$$

satisfies the following conditions: there exists $\delta > 0$ such that

$$X(z) \in \widetilde{E}^{p(t)+\delta}(G^{\pm}), \tag{18}$$

$$[X(z)]^{-1} \in \widetilde{E}^{q(t)+\delta}(G^{\pm}) \tag{19}$$

$$a(t) = X^{+}(t)[X^{-}(t)]^{-1} \tag{20}$$

(see [22]).

We write condition (5) in the form

$$W^{+}(X^{+})^{-1} - W^{-}(X^{-})^{-1} = b(X^{+})^{-1}, \tag{21}$$

and assume

$$V = W(X)^{-1}. \tag{22}$$

Lemma 1. *Let*

$$LW = \partial_{\overline{z}}W + AW + B\overline{W}, \quad L_1V = \partial_{\overline{z}}V + AV + B\frac{\overline{X}}{X}\overline{V}.$$

If $LW = 0$, then $L_1V = 0$, where V is given by equality (22)*. Conversely, if $L_1V = 0$ and $W = VX$, then $LW = 0$.*

Proof. Since $X(z)$ and $(X(z))^{-1}$ are the functions, analytic in G, it can be easily verified that

$$L_1V = L_1\frac{W}{X} = \frac{1}{X}(\partial_{\overline{z}}W + AW + B\overline{W}) = \frac{1}{X}LW.$$

From the above equality follow two statements of the lemma. □

Corollary 1. *If $W \in U^{s,2}(A;B;G)$, $s > 2$ then $V \in U^{s,2}(A;B\frac{\overline{X}}{X};G)$.*

Corollary 2. *If V is the function given by equality* (22)*, then*

$$V = \Phi_V \exp \omega_{V}, \tag{23}$$

where

$$\Phi_V = \frac{\Phi_W}{X}, \quad \omega_{V} = \omega_{W}. \tag{24}$$

Proof. We have

$$V = \frac{W}{X} = \frac{\Phi_W}{X}\exp\omega_W,$$

$$\omega_{W} = \iint\limits_G \Big(A + B\frac{\overline{W}}{W}\Big)\frac{d\xi d\eta}{\zeta - z} = \iint\limits_G \Big(A + B\frac{\overline{XV}}{XV}\Big)\frac{d\xi d\eta}{\zeta - z} =$$

$$= \iint\limits_G \Big(A + B\frac{\overline{X}}{X}\frac{\overline{V}}{V}\Big)\frac{d\xi d\eta}{\zeta - z} = \omega_{V}; \tag{25}$$

(We have used here the equality $W = VX$ and Corollary 1).

Thus,

$$V=\Phi_V\exp\omega_{_V}=\frac{\Phi_W}{X}\exp\omega_{_V}$$

and hence,

$$\Phi_V=\frac{\Phi_W}{X}. \tag{26}$$

Equalities (25) and (26) are just the provable by us equalities (24). $\square$

Since $\Phi_W\in E^{p(\cdot)}(G^+)$ and $\frac{1}{X}\in\widetilde{E}^{q(\cdot)+\delta}(G^\pm)$ (see (19)), it follows from (26) that $\Phi_V\in E^{1+\varepsilon}(G^+)$, $\varepsilon>0$ and hence $V\in E^{1+\varepsilon}\Big(A;B\frac{\overline{X}}{X};G^+\Big)$. Behavior of the function $\phi_{_V}$ in the domain G^- depends on $\frac{1}{X}$.

If $\varkappa=\operatorname{ind}a\geq 0$, then it is easily seen from (17) that $\lim_{z\to\infty}V(z)=0$ for $\varkappa=0$ and $\lim_{z\to\infty}V(z)=\text{const}$ for $\varkappa=1$.

For $\varkappa>1$, the function V at the point $z=\infty$ admits the pole of order $\varkappa-1$. Therefore there exist $\widetilde{\Phi}\in E^{1+\varepsilon}(G^-)$, $\varepsilon>0$ and the polynomial $Q_{\varkappa-1}$ of order $\varkappa-1$ such that

$$\Phi_V=\widetilde{\Phi}+Q_{\varkappa-1}.$$

By virtue of (21)–(22), we have

$$V^+-V^-=\frac{b}{X^+}.$$

Since Φ_V and $\widetilde{\Phi}_V$ belong to $E^{1+\varepsilon}(G^\pm)$, then W belongs to the class $E^{1+\varepsilon}\Big(A;B\frac{\overline{X}}{X};G^\pm\Big)$.

Let $\Omega_{1,1}(z,t)$ and $\Omega_{2,1}(z,t)$ be the principal kernels of the class $U^{s,2}\Big(A;B\frac{\overline{X}}{X};G^\pm\Big)$ $s>2$. Then

$$\frac{W(z)}{X(z)}=V(z)=\widetilde{K}_{\Gamma,1}\Big(\frac{b}{X^+}\Big)+V_{\varkappa-1}(z),$$

where

$$\widehat{K}_{\Gamma,1}\Big(\frac{b}{X^+}\Big)=\int\limits_\Gamma\Omega_{1,1}(z,t)\frac{b(t)}{X^+(t)}-\Omega_{2,1}(z,t)\Big(\overline{\frac{b(t)}{X^+(t)}}\Big)dt$$

where $\widehat{V}_{\varkappa-1}(z)$ is the generalized polynomial of order $\varkappa-1$. This implies that one possible solution of problem (5) will be

$$W(z)=X(z)W_b(z)+X(z)\widehat{V}_{\chi-1}(z), \tag{27}$$

where

$$W_b(z)=\widetilde{K}_{\Gamma,1}\Big(\frac{b}{X^+}\Big).$$

Since

$$\frac{b}{X^+}\in L^{1+\eta}(\Gamma),\quad \eta>0$$

we have

$$W = X\Big(\widetilde{K}_{\Gamma,1}\frac{b}{X^+}\Big) = X\Phi_{W_b}\exp\omega_{W_b} \in E^{\eta}\Big(A; B\frac{\overline{X}}{X}; G^{\pm}\Big). \tag{28}$$

Next,

$$W_b^+ = \frac{1}{2}\Big(b + X^+\widetilde{S}_{\Gamma,1}\Big(\frac{b}{X^+}\Big)\Big), \quad W_b^- = \frac{1}{2a}\Big(-b + X^+\widetilde{S}_{\Gamma,1}\Big(\frac{b}{X^+}\Big)\Big). \tag{29}$$

This implies that for the inclusion $W_b \in K^{p(\cdot)}(A;B;\Gamma)$ it is necessary that

$$(Tb)(t) = X^+(t)\widetilde{S}_{\Gamma,1}\Big(\frac{b}{X^+}\Big)(t) \in L^{p(t)}(\Gamma). \tag{30}$$

Conversely, if (30) holds, then $W_b \in E^{\eta}\Big(A; B\frac{\overline{X}}{X}; G^{\pm}\Big)$ and $(W_b)^+ \in L^{p(t)}(\Gamma)$. According to the generalized Smirnov's theorem (see [17]), we will have $W_b \in E^{\widetilde{p}(t)}(A;B;G)$, where $\widetilde{p}(t) = \max(p(t),\eta) = p(t)$, i.e.,

$$W_b(z) \in E^{p(t)}(A;B;G^{\pm}). \tag{31}$$

Thus the following lemma is valid.

Lemma 2. *For problem* (5) *to be solvable for* $\varkappa \geq 0$*, it is necerssary and sufficient that inclusions* (30) *be fulfilled.*

Let condition (30) be fulfilled. Find out under what additional conditions W_b is a particular solution of problem (5) and construct its general solution. We consider separately the cases $\varkappa \geq 0$ and $\varkappa < 0$.

7.2. **The case $\varkappa \geq 0$.** By virtue of (31),

$$W_b(z) = \widetilde{K}_{\Gamma,1}(W_b^+ - W_b^-) = (\widetilde{K}_{\Gamma,1}f)(z), \quad f(t) = (W_b^+(t) - W_b^-(t)) \in L^{p(t)}(\Gamma).$$

Since $\Gamma \in I^*$; $p \in H(\Gamma)$ and $W_b(\infty) = 0$, we have $W_b(z) \in K^{p(\cdot)}\Big(A; B\frac{\overline{X}}{X}; \Gamma\Big)$ (see Corollary 1).

Therefore $(\widetilde{K}_{\Gamma,1}f)(z) \in E^{p(\cdot)}\Big(A; B\frac{\overline{X}}{X}; G^{\pm}\Big)$, and hence, W_b is the solution of problem (5). Now, to find its general solution, we have to solve the problem

$$V^+ - V^- = 0 \tag{32}$$

in the class of functions whose analytic divisor admits the representation $\Phi_v = \widetilde{\Phi}_v + Q_{\varkappa-1}$, $\widetilde{\Phi} \in E^{p(t)}(G^{\pm})$.

It follows from (32) that $\widetilde{\Phi}_v^+ - \widetilde{\Phi}_v = 0$, and since $W_b \in K^{p(\cdot)}\Big(A, \frac{B\overline{X}}{X}, \Gamma\Big)$, therefore $\widetilde{\Phi}_v = 0$. Consequently, the solutions of (32) are the functions V for which analytic divisor is the polynomial $Q_{\varkappa-1}$.

We denote such a function by $\widehat{V}_{\varkappa-1}$. Then if condition (30) is fulfilled, a general solution of the problem is

$$W(z) = X(z)\widetilde{K}_{\Gamma,1}\Big(\frac{b}{X^+}\Big)(z) + X(z)\widetilde{V}_{\varkappa-1}(z). \tag{33}$$

7.3. **The case $\varkappa < 0$.** In this case the only one possible solution of the problem may be only the function $W_b(z)$; however, for this function to be of the class $E^{p(\cdot)}(A;B;G^-)$, it is necessary and sufficient that the function $\widetilde{K}_{\Gamma,1}\Big(\frac{b}{X^+}\Big)(z)$ at the point $z=\infty$ have zero of order $|\varkappa|$. For this to be so, it is necessary and sufficient that

$$\operatorname{Im}\int\limits_{\Gamma} u_k(t)b(t)dt = 0, \quad k = 0,1,\dots,2(1+|\varkappa|)-3, \tag{34}$$

where u_k are linearly independent solutions of the homogeneous problem

$$u^+(t) = \frac{1}{a(t)}u^-(t) \tag{35}$$

(see [4], p. 53).

Let us show that u_k belongs to $E^{q(t)}(G^\pm)$.

Since $\frac{1}{a(t)} \in A(q(t),\Gamma)$ and $\operatorname{ind}\frac{1}{a(t)} = -\varkappa > 0$, according to the result obtained in item 7.2, the solutions of problem (35) are given by the equality

$$u(z) = \frac{1}{X(z)}\widetilde{u}_{|\varkappa|-1},$$

where $\widehat{u}_{|\varkappa|-1}$ is the generalized polynomial of order $|\varkappa|-1$.

Consequently, the analytic divisor of the generalized analytic function $u(z)$ is $\frac{Q_{|\varkappa|-1}(z)}{X(z)}$.

By virtue of the fact that we have inclusion (19) and $\Phi_u(\infty)=0$, we can conclude that $\Phi_W(z) \in E^{q(t)}(G^\pm)$, and hence, $u(z) \in E^{q(\cdot)}(A;B;G^\pm)$. This implies that the function W_b under conditions (30) and (34) is the solution of problem (5).

7.4. **The main theorem.** From the results obtained in items 6.2 and 6.3 it follows that if condition (14) or (15) with respect to Γ, $p(t)$, $a(t)$, $b(t)$ are fulfilled, then for the Riemann problem considered in the class $K^{p(t)}(A;B;\Gamma)$ (or in $PE^{p(t)}(A;B;\Gamma)$), the theorem, analogous to that appearing in the classical assumptions and in the class $K^{p(t)}(\Gamma)$, is valid.

Theorem. *Let Γ be the simple closed curve bounding the domains G^+ and G^- and let the condition* (14) *or* (15) *be fulfilled. If, moreover, $a(t) \in \Lambda(p(t),\Gamma)$, $b(t) \in L^{p(\cdot)}(\Gamma)$ and $\varkappa = \operatorname{ind} a(t)$, then for problem* (5) *to be solvable in the class $K^{p(t)}(A;B;\Gamma)$, it is necessary and sufficient that the condition*

$$(\widetilde{T}b)(t) = X^+(t)\Big(\widetilde{S}_{\Gamma,1}\frac{b}{X^+}\Big)(t) \in L^{p(t)}(\Gamma)$$

be fulfilled, where $\widetilde{S}_{\Gamma,1}$ is the generalized Cauchy singular integral with principal kernels $\Omega_{1,1}$ and $\Omega_{1,2}$ of the class $U^{s,2}(A;B\frac{\overline{X}}{X};G^\pm)$, and $X(z)$ is the function given by equality (17).

If this condition is fulfilled, then:

(i) when $\varkappa = \operatorname{ind} a \geq 0$, the problem is solvable and its general solution is given by the equality

$$W(z) = X(z)\widetilde{K}\Big(\frac{b}{X^+}\Big)(z) + X(z)\widehat{V}_{\varkappa-1}(z),$$

where $\widetilde{V}_{\varkappa-1}(z)$ is an arbitrary generalized polynomial of order $(V_{\varkappa-1}(z) = 0)$;

(ii) when $\varkappa < 0$, then for the solvability of the problem it is necessary and sufficient that the condition $\widetilde{T}b \in L^{p(t)}(\Gamma)$ and

$$\operatorname{Im} \int\limits_{\Gamma} u_k(t)b(t)dt = 0, \quad k = 0, 1, \dots, 2(1 + |x|) - 3$$

be fulfilled, where u_k are linearly independent solutions of the class $K^{p(t)}(-A; -B\frac{\overline{X}}{X}; \Gamma)$ of the problem

$$u^+(t) = \frac{1}{a(t)} u^-(t).$$

Remark. If $b \in L^{p(t)+\delta}(\delta)$, $\delta > 0$ then $\widetilde{T}_b \in L^{p(t)}(\Gamma)$.

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CRITERIA FOR THE BOUNDEDNESS OF POTENTIAL OPERATORS IN GRAND LEBESGUE SPACES

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Abstract. It is shown that the fractional integral operators with the parameter α, $0 < \alpha < 1$, are not bounded between the generalized grand Lebesgue spaces $L^{p),\theta_1}$ and $L^{q),\theta_2}$ for $\theta_2 < \theta_1 q/p$, where $1 < p < 1/\alpha$ and $q = \frac{p}{1-\alpha p}$. It is proved that the one–weight inequality

$$\|I_\alpha(fw^\alpha)\|_{L^{q),\,\theta q/p}_w} \le c\|f\|_{L^{p),\theta}_w},$$

where I_α is the potential operator on the interval $[0,1]$, holds if and only if $w \in A_{1+q/p'}([0,1])$.

რეზიუმე. ნაჩვენებია, რომ წილადური ინტეგრალური ოპერატორი α პარამეტრით, $0 < \alpha < 1$, არ არის შემოსაზღვრული $L^{p),\theta_1}$ და $L^{q),\theta_2}$ გრანდ ლებეგის სივრცეებს შორის, სადაც $\theta_2 < \theta_1 q/p$, $1 < p < 1/\alpha$ და $q = \frac{p}{1-\alpha p}$. ნაჩვენებია, რომ ერთწონიანი უტოლობა

$$\|I_\alpha(fw^\alpha)\|_{L^{q),\,\theta q/p}_w} \le c\|f\|_{L^{p),\theta}_w},$$

სადაც I_α პოტენციალის ოპერატორია $[0,1]$ ინტერვალზე, ძალაშია მაშინ და მხოლოდ მაშინ, როცა $w \in A_{1+q/p'}([0,1])$.

Introduction

In this paper we prove that potential operators with the parameter α, $0 < \alpha < 1$, are not bounded from $L^{p)}$ to $L^{q)}$, where $1 < p < \infty$ and q is the Hardy–Littlewood–Sobolev exponent of p: $q = \frac{p}{1-\alpha p}$. This phenomena motivates us to investigate the boundedness problem for the Riesz potential operator I_α in the generalized grand Lebesgue spaces. In particular, we study this problem in weighted $L^{p),\theta}_w$ spaces and prove that the one–weight

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inequality

$$\|I_\alpha(fw^\alpha)\|_{L^{q),\,\theta q/p}_w([0,1])} \le c\|f\|_{L^{p),\theta}_w([0,1])}$$

holds if and only if w belongs to the Muckenhoupt's class $A_{1+q/p'}$.

The unweight spaces $L^{p),\theta}$ (i.e. $L^{p),\theta}_w$ for $w \equiv \text{const}$) were introduced by E. Greco, T. Iwaniec and C. Sbordone [6] when they studied existence and uniqueness of the nonhomogeneous $n-$ harmonic equation $divA(x,\nabla u) = \mu$.

The grand Lebesgue spaces $L^{p)} = L^{p),1}$ first appeared in the paper by T. Iwaniec and C. Sbordone [7]. In that paper the authors showed that if $f = (f_1,\dots,f_n) : \Omega \to \mathbb{R}^n$ belongs to the Sobolev class $W^{1,1}$, where Ω is an open subset in $\mathbb{R}^n$, $n \ge 2$, then the Jacobian determinant $J = J(f,x) = det\ Df(x)$ ($J(x,f) \ge 0$ a.e.) of f belongs to the class $L^1_{\text{loc}}(\Omega)$ provided that $g \in L^{n)}$, where

$$g(x) := |Df(x)| = \{\sup|Df(x)y| : y \in S^{n-1}\}.$$

Recently necessary and sufficient conditions guaranteeing the one–weight inequality for the Hardy–Littlewood maximal operator in $L^{p)}_w(I)$, where $I = [0,1]$, were established by A. Fiorenza, B. Gupra and P. Jain [4], while the same problem for the Hilbert transform was studied in the paper [9]. In particular, it turned out that the Hardy–Littlewood maximal operator (resp. the Hilbert transform) is bounded in $L^{p)}_w(I)$ if and only if the weight w belongs to the Muckenhoupt class $A_p(I)$.

1. Preliminaries

Let Ω be a bounded subset of $\mathbb{R}^n$ and let w be an a.e. positive, integrable function on Ω (i.e. a weight). The weighted generalized grand Lebesgue space $L^{p),\theta}(\Omega)$ $(1 < p < \infty)$ is the class of those $f : \Omega \to \mathbb{R}$ for which the norm

$$\|f\|_{L^{p),\theta}_w(\Omega)} = \sup_{0<\varepsilon\le p-1}\left(\frac{\varepsilon^\theta}{|\Omega|}\int\limits_\Omega |f(t)|^{p-\varepsilon}w(t)dt\right)^{1/(p-\varepsilon)}$$

is finite.

If $w \equiv$ const 1, then we denote $L^{p),\theta}(\Omega) := L^{p),\theta}_w(\Omega)$. The space $L^{p),\theta}_w(\Omega)$ is not rearrangement invariant unless $w \equiv$ const.

Hölder's inequality and simple estimates yield the following embeddings (see also [6], [4]):

$$L^p_w(\Omega) \subset L^{p),\theta_1}_w(\Omega) \subset L^{p),\theta_2}_w(\Omega) \subset L^{p-\varepsilon}_w(\Omega), \tag{1.1}$$

where $0 < \varepsilon < p-1$ and $\theta_1 < \theta_2$.

In the classical weighted Lebesgue spaces L^p_w the equality

$$\|f\|_{L^p_w} = \|w^{1/p}f\|_{L^p}$$

holds but this property fails in the case of grand Lebesgue spaces. In particular, there is $f \in L^{p)}_w$ such that $w^{1/p} f \notin L^{p)}$ (see also [4] for the details).

Let φ be positive increasing function on $(0, p-1)$ satisfying the condition $\varphi(0+) = 0$, where $1 < p < \infty$. We will also need the following auxiliary class of functions defined on Ω and associated with φ:

$$L^{p),\varphi(x)}_w(\Omega) := \left\{ f : \sup_{0<\varepsilon\le p-1} \left(\varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}_w} \right) < \infty \right\}.$$

The space $L^{p),\theta}_w(\Omega)$, $\theta > 0$, is the special case of $L^{p),\varphi(x)}_w(\Omega)$ taking $\varphi(x) = \frac{x^\theta}{|\Omega|}$.

Throughout the paper the symbol $\varphi(t) \approx \psi(t)$ means that there exist positive constants c_1 and c_2 such that $c_1\varphi(t) \le \psi(t) \le c_2\psi(t)$. Constants (often different constants in the same series of inequalities) will generally be denoted by c or C. By the symbol p' we denote the conjugate number of p, i.e. $p' := \frac{p}{p-1}$, $1 < p < \infty$.

2. Fractional Integrals and Fractional Maximal Functions in Unweighted Grand Lebesgue Spaces

Let

$$(I_\alpha f)(x) = \int_0^1 \frac{f(y)}{|x-y|^{1-\alpha}}\, dy, \quad 0 < \alpha < 1$$

be the Riesz potential operator defined on $[0,1]$. We begin this section with the following result:

Theorem 2.1. *Let $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, θ_1 and θ_2 be positive numbers such that $\theta_2 < \theta_1 q/p$, where $q = \frac{p}{1-\alpha p}$. Then the operator I_α is not bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$.*

Proof. First observe that $q/p = 1+\alpha q$. Suppose the contrary: I_α is bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$ i. e. the inequality

$$\|I_\alpha f\|_{L^{q),\theta_2}} \le c\|f\|_{L^{p),\theta_1}} \tag{2.1}$$

holds, where the positive constant c does not depend on f. Taking $f = \chi_J$ in (2.1), where J is an interval in $[0,1]$, we have

$$(I_\alpha f)(x) = \int_J \frac{dy}{|x-y|^{1-\alpha}} \ge |J|^\alpha, \quad x \in J.$$

Consequently,

$$\|I_\alpha f\|_{L^{q),\theta_2}} \ge |J|^\alpha \|\chi_J\|_{L^{q),\theta_2}}.$$

Taking inequality (2.1) into account we have that

$$|J|^\alpha \|\chi_J\|_{L^{q),\theta_2}} \le c\|\chi_J\|_{L^{p),\theta_1}}, \tag{2.2}$$

where the positive constant c does not depend on J.

Let us define the number $\varepsilon_J \in (0, p-1]$ satisfying the condition

$$\sup_{0<\varepsilon\le p-1} \left(\varepsilon^{\theta_1}|J|\right)^{\frac{1}{p-\varepsilon}} = \left(\varepsilon_J^{\theta_1}|J|\right)^{\frac{1}{p-\varepsilon_J}}. \tag{2.3}$$

Now we claim that $\lim\limits_{|J|\to 0} \varepsilon_J = 0$. Indeed, suppose the contrary: that there is a sequence of intervals J_n and a positive number λ such that $|J_n| \to 0$ and $\varepsilon_{J_n} \ge \lambda > 0$ for all $n \in N$. It is obvious that we can choose J_{n_0} so that

$$\frac{|J_{n_0}|^{\frac{1}{\theta_1}}(p-1)}{e} < e^{-\frac{p}{\lambda/2}}.$$

Now we claim that $g'(x) < 0$ for x satisfying the condition $x \in [\lambda/2, p-1]$, where $g(x) = \left(x^{\theta_1}|J_{n_0}|\right)^{\frac{1}{p-x}}$. Indeed, it is easy to see that for such an x,

$$\frac{|J_{n_0}|^{\frac{1}{\theta_1}}x}{e} \le \frac{|J_{n_0}|^{\frac{1}{\theta_1}}(p-1)}{e} < e^{-\frac{p}{\lambda/2}} \le e^{-\frac{p}{x}}$$

hold. Hence, using the formula

$$g'(t) = g(t)\cdot\frac{1}{p-t}\left[\frac{\ln\left(t^{\theta_1}|J_{n_0}|\right)}{p-t} + \frac{\theta_1}{t}\right]$$

and the fact that

$$g'(t) < 0 \Longleftrightarrow \frac{t|J_{n_0}|^{\frac{1}{\theta_1}}}{e} < e^{-\frac{p}{t}}$$

we conclude that $g'(x) < 0$ for $x \in [\lambda/2, p-1]$. This observation together with the equality $\lim\limits_{x\to 0} g(x) = 0$ gives that $0 < \varepsilon_{J_{n_0}} < \lambda$, where $\varepsilon_{J_{n_0}}$ is defined by

$$\sup_{0<\varepsilon\le p-1} \left(\varepsilon^{\theta_1}|J_{n_0}|\right)^{\frac{1}{p-\varepsilon}} = \left(\varepsilon_{J_{n_0}}^{\theta_1}|J_{n_0}|\right)^{1/(p-\varepsilon_{J_{n_0}})}.$$

This contradicts the assumption that $\varepsilon_{J_n} \ge \lambda > 0$ for all n.

Further, we choose η_J so that

$$\alpha = \frac{1}{p} - \frac{1}{q} = \frac{1}{p-\varepsilon_J} - \frac{1}{q-\eta_J}.$$

This is equivalent to say that

$$\eta_J = q - \frac{p-\varepsilon_J}{1-\alpha(p-\varepsilon_J)}. \tag{2.4}$$

By (2.2) and (2.3) we have that

$$|J|^{\alpha}\eta_J^{\frac{\theta_2}{q-\eta_J}}|J|^{\frac{1}{q-\eta_J}} \le c\varepsilon_J^{\frac{\theta_1}{p-\varepsilon_J}}|J|^{\frac{1}{p-\varepsilon_J}}. \tag{2.5}$$

(here we used the fact that if ε_J is small, then $0 < \eta_J < q - 1$). Now (2.5) yields:

$$\eta_J^{\frac{\theta_2}{q-\eta_J}} \varepsilon_J^{-\frac{\theta_1}{p-\varepsilon_J}} \le c. \tag{2.6}$$

Further, (2.4) and (2.6) imply

$$\left(\frac{q - \frac{p-\varepsilon_J}{1-\alpha(p-\varepsilon_J)}}{\varepsilon_J} \right)^{\frac{\theta_2}{p-\varepsilon_J} - \alpha\theta_2} \varepsilon_J^{-\frac{\theta_1}{p-\varepsilon_J} + \frac{\theta_2}{p-\varepsilon_J} - \alpha\theta_2} \le c. \tag{2.7}$$

Passing now to the limit as $|J| \to 0$ we see that the left-hand side of (2.7) tends to $+\infty$ because the limit of the first factor is $\left[\frac{1}{(1-\alpha p)^2}\right]^{\frac{\theta_2}{p} - \alpha\theta_2}$, and

$$\lim_{|J|\to 0} \varepsilon_J^{\frac{\theta_2-\theta_1}{p-\varepsilon_J} - \alpha\theta_2} = \lim_{|J|\to 0} \varepsilon_J^{\frac{\theta_2-\theta_1}{p} - \alpha\theta_2} = \infty$$

(Here we used the observation $\frac{\theta_2}{\theta_1} < 1 + \alpha q \Longleftrightarrow \frac{\theta_2-\theta_1}{p} - \alpha\theta_2 < 0$). □

Analysing the proof of Theorem 2.1 we have the result similar to that of the previous statement for the fractional maximal operator

$$M_\alpha f(x) = \sup_{\substack{J \ni x \\ J \subset [0,1]}} \frac{1}{|J|^{1-\alpha}} \int_J |f|, \quad x \in [0,1].$$

Theorem 2.2. *Let the conditions of Theorem* 2.1 *be satisfied. Then the operator M_α is not bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$.*

Proof. Proof is the same as in the case of Theorem 2.1. We only need to observe that the inequality

$$M_\alpha f(x) \ge \frac{1}{|J|^{1-\alpha}} \int_J dx = |J|^\alpha, \quad x \in J,$$

holds for $f(x) = \chi_J(x)$, where J is a subinterval of $[0,1]$. Details are omitted. □

3. Sobolev's Embedding in Weighted Generalized Grand Lebesgue Spaces

This section is devoted to the investigation of the one–weight inequality for the operator I_α in $L_w^{p),\theta}$ spaces.

First we introduce the function

$$\varphi(u) = \left[\frac{u-q}{1-\alpha(u-q)} + p \right]^{1-(u-q)\alpha} \tag{3.1}$$

where $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $q = \frac{p}{1-\alpha p}$.

To prove the main results we need some auxiliary statements.

Lemma 3.1. *$\varphi(x) \approx x^{1+\alpha q}$ near 0.*

The proof is straightforward and therefore is omitted.

Lemma 3.2. *Let $1 < q < \infty$ and let w be a weight. Then*

$$\|f\|_{L_w^{q),\varphi(x)}([0,1])} \approx \|f\|_{L_w^{q),1+\alpha q}([0,1])}$$

where φ is defined by (3.1).

Proof. Follows immediately from Lemma 3.1. □

Lemma 3.3. *Let $1 < q < \infty$ and let $\theta > 0$. Then*

$$\|f\|_{L_w^{q),\varphi(x^\theta)}([0,1])} \approx \|f\|_{L_w^{q),\theta(1+\alpha q)}([0,1])},$$

where φ is defined by (3.1).

The proof follows immediately from Lemma 3.1.

Lemma 3.4. *Let $1 < p < \infty$ and let Φ be a positive increasing function on $(0, p-1)$ satisfying $\Phi(0+) = 0$. Then there is a positive constant c such that for all intervals $J \subset [0,1]$ and $f \in L_w^{p,\Phi(x)}$ the inequality*

$$\|f\|_{L_w^{p),\Phi(x)}(J)} \le c(w(J))^{-\frac{1}{p}} \left(\int_J |f(t)|^p w(t)dt \right)^{\frac{1}{p}} \|\chi_J\|_{L_w^{p),\Phi(x)}}$$

holds.

Proof. We have

$$\|f\|_{L_w^{p),\Phi(x)}(J)} = \sup_{0<\varepsilon\le p-1} \left(\Phi(\varepsilon) \int_J |f(x)|^{p-\varepsilon} w(x)dx \right)^{\frac{1}{p-\varepsilon}} =$$

$$= \sup_{0<\varepsilon\le p-1} \left(\Phi(\varepsilon) \int_J |f(x)|^{p-\varepsilon} w(x)^{\frac{p-\varepsilon}{p}} w(x)^{\frac{\varepsilon}{p}} dx \right)^{\frac{1}{p-\varepsilon}} \le$$

$$\le \sup_{0<\varepsilon\le p-1} \Phi(\varepsilon)^{\frac{1}{p-\varepsilon}} \left(\int_J \left(|f(x)|^{p-\varepsilon} w(x)^{\frac{p-\varepsilon}{p}} \right)^{\frac{p}{p-\varepsilon}} dx \right)^{\frac{1}{p}} \times$$

$$\times \left(\int_J \left[w^{\frac{\varepsilon}{p}}(x) \right]^{\frac{p}{\varepsilon}} dx \right)^{\frac{\varepsilon}{p(p-\varepsilon)}} =$$

$$= \sup_{0<\varepsilon\le p-1} \Phi(\varepsilon)^{\frac{1}{p-\varepsilon}} \left(\int_J |f(x)|^p w(x)dx \right)^{\frac{1}{p}} \left(\int_J w(x)dx \right)^{\frac{\varepsilon}{p(p-\varepsilon)}} =$$

$$= \left(\int_J |f(x)|^p w(x)dx \right)^{\frac{1}{p}} \left(\int_J w(x)dx \right)^{-\frac{1}{p}} \sup_{0<\varepsilon\le p-1} \left(\Phi(\varepsilon) \int_J w(x)dx \right)^{\frac{1}{p-\varepsilon}} =$$

$$= \left(\int\limits_J |f(x)|^p w(x)dx \right)^{\frac{1}{p}} (w(J))^{-\frac{1}{p}} \|\chi_J\|_{L_w^{p),\Phi(x)}(J)}. \qquad \square$$

Lemma 3.5. *Let $\theta > 0$, $1 < p < \infty$, $0 < \alpha < 1/p$ and let $q = \frac{p}{1-\alpha p}$. Let ψ be a positive increasing function on $(0, q-1)$ satisfying the condition $\psi(0+) = 0$. Suppose that the inequality*

$$\|I_\alpha(fw^\alpha)\|_{L_w^{q),\psi(x)}([0,1])} \leq c\|f\|_{L_w^{p),\theta}([0,1])} \tag{3.2}$$

holds. Then

$$\int\limits_0^1 w^{-p'/q}(x)dx < \infty.$$

Proof. Suppose the contrary: $\int\limits_0^1 w^{-p'/q}(x)dx = \|w^{\alpha-1}\|_{L_w^{p'}} = \infty$. This means that there is a function $g \in L_w^p$ such that $\int\limits_0^1 gw^\alpha = \infty$.

On the other hand,

$$I_\alpha(gw^\alpha)(x) = \int\limits_0^1 \frac{g(t)w^\alpha(t)}{|x-t|^{1-\alpha}}dt \geq \int\limits_0^1 g(t)w^\alpha(t)dt = \infty, \quad x \in [0,1].$$

Further, Lemma A implies that $g \in L_w^{p),\theta}([0,1])$. This contradicts (i). $\square$

Definition 3.1. Let $1 < r < \infty$. We say that a weight function w belongs to the Muckenhoupt's class $A_r([0,1])$ ($w \in A_r([0,1])$) if

$$A_r(w) := \sup_{J \subset [0,1]} \left(\frac{1}{|J|} \int\limits_J w(t)dt \right)^{1/r} \left(\frac{1}{|J|} \int\limits_J w^{1-r'}(t)dt \right)^{1/r'} < \infty,$$

where the supremum is taken over all subintervals J of $[0,1]$.

Lemma 3.6. *Let $0 < \alpha < 1$, $1 < p < 1/\alpha$. We set $q = \frac{p}{1-\alpha p}$. Suppose that $w \in A_{1+q/p'}([0,1])$, i.e.,*

$$\sup_{J \subset [0,1]} \left(\frac{1}{|J|} \int\limits_J w(t)dt \right)^{1/q} \left(\frac{1}{|J|} \int\limits_J w^{-p'/q}(t)dt \right)^{1/p'} < \infty.$$

Then there are positive constants σ_1, σ_2 and L satisfying the conditions:

$$\frac{1}{p-\sigma_2} - \frac{1}{q-\sigma_1} = \alpha, \quad w \in A_{1+\frac{q-\sigma_1}{(p-\sigma_2)'}},$$
$$\|K_\alpha\|_{L_w^{p-\eta} \to L_w^{q-\varepsilon}} \leq L$$

for all $0 \leq \varepsilon \leq \sigma_1$, $0 \leq \eta \leq \sigma_2$ with $\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \alpha$, where K_α is the operator defined as follows $K_\alpha f = I_\alpha(fw^\alpha)$.

Proof. Since $w \in A_{1+q/p'}$ by the openness property of Muckenhoupt's classes (see [11]) we have that there are small positive numbers σ_1 and σ_2 such that $\frac{1}{p-\sigma_2} - \frac{1}{q-\sigma_1} = \alpha$ and $w \in A_{1+(q-\sigma_1)/(p-\sigma_2)'}$.

Now we use the idea from [8]. By the result of B. Muckenhoupt and R. L. Wheeden [12] we have that the operator K_α is bounded from L^p_w to L^q_w and from $L^{p-\sigma_2}_w$ to $L^{q-\sigma_1}_w$. Let $0 < t < 1$ and let us define positive numbers η and ε so that

$$\frac{1}{p-\eta} = \frac{t}{p} + \frac{1-t}{p-\sigma_2}, \quad \frac{1}{q-\varepsilon} = \frac{t}{q} + \frac{1-t}{q-\sigma_1}.$$

Then by applying the Riesz–Thorin theorem (see e.g. [2], p. 16) we have that K_α is bounded from $L^{p-\eta}_w$ to $L^{q-\varepsilon}_w$ and moreover,

$$\|K_\alpha\|_{L^{p-\eta}_w \to L^{q-\varepsilon}_w} \le \|K_\alpha\|^t_{L^p_w \to L^q_w} \|K_\alpha\|^{1-t}_{L^{p-\sigma_2}_w \to L^{q-\sigma_1}_w}.$$

Observe now that

$$\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \frac{t}{p} - \frac{t}{q} + \frac{1-t}{p-\sigma_2} - \frac{1-t}{q-\sigma_1} =$$
$$= t\Big(\frac{1}{p} - \frac{1}{q}\Big) + (1-t)\Big(\frac{1}{p-\sigma_2} - \frac{1}{q-\sigma_1}\Big) = t\alpha + (1-t)\alpha = \alpha.$$

The lemma is proved since we can take $L = \|K_\alpha\|_{L^p_w \to L^q_w} \|K_\alpha\|_{L^{p-\sigma_2}_w \to L^{q-\sigma_1}_w}$ (since without loss of generality we can assume that each factor in the latter expression is greater or equal to 1). □

Theorem 3.1. *Let $1 < p < \infty$ and let $0 < \alpha < 1/p$. Suppose that $\theta > 0$. We set $q = \frac{p}{1-\alpha p}$. Then the inequality*

$$\|I_\alpha(fw^\alpha)\|_{L^{q),\theta q/p}_w([0,1])} \le c\|f\|_{L^{p),\theta}_w([0,1])} \tag{3.3}$$

holds if and only if $w \in A_{1+q/p'}([0,1])$.

Proof. By Lemma 3.1 we have that (3.3) is equivalent to the inequality

$$\|I_\alpha(fw^\alpha)\|_{L^{q),\psi(x)}_w([0,1])} \le c\|f\|_{L^{p),\theta}_w([0,1])}, \tag{3.4}$$

where

$$\psi(x) = \varphi(x^\theta), \quad \varphi(x) = \bigg[\frac{x-q}{1-\alpha(x-q)} + p\bigg]^{1-(x-q)\alpha}. \tag{3.5}$$

Necessity. Let (3.3) and hence (3.4) hold. By Lemma 3.5 we have that $\int\limits_0^1 w^{-p'/q} < \infty$. Let us take $f = \chi_J w^{-\alpha-p'/q}$. Then for $x \in J$, we get that

$$I_\alpha(w^\alpha f)(x) \ge \frac{1}{|J|^{1-\alpha}} \int\limits_J w^\alpha f = \frac{1}{|J|^{1-\alpha}} \int\limits_J w^{-p'/q}.$$

Hence,

$$\|I_\alpha(w^\alpha f)\|_{L_w^{q),\psi(x)}([0,1])} \geq |J|^{\alpha-1}\Big(\int\limits_J w^{-p'/q}\Big)\|\chi_J\|_{L_w^{q),\psi(x)}([0,1])}.$$

Further, by Lemma 3.4 with $\Phi(x) = x^\theta$ we find that

$$|J|^{\alpha-1}\Big(\int\limits_J w^{-p'/q}\Big)\|\chi_J\|_{L_w^{q),\psi(x)}([0,1])} \leq$$

$$\leq c\|f\|_{L^{p),\theta}([0,1])} \leq c(w(J))^{-\frac{1}{p}}\Big(\int\limits_J |f(t)|^p w(t)dt\Big)^{\frac{1}{p}}\|\chi_J\|_{L_w^{p),\theta}([0,1])} =$$

$$= cw(J)^{-\frac{1}{p}}\Big(\int\limits_J w^{-p'/q}\Big)^{1/p}\|\chi_J\|_{L_w^{p),\theta}([0,1])}.$$

It is easy to see that there is a number η_J depending on J such that $0 < \eta_J \leq p-1$ and

$$|J|^{\alpha-1}w(J)^{\frac{1}{p}}\Big(\int\limits_J w^{-p'/q}\Big)^{\frac{1}{p'}}\|\chi_J\|_{L_w^{q),\psi(x)}([0,1])} \leq c\left(\eta_J w(J)\right)^{\frac{1}{p-\eta_J}}.$$

For such an η_J we choose ε_J so that

$$\frac{1}{p-\eta_J} - \frac{1}{q-\varepsilon_J} = \alpha.$$

Then $0 < \varepsilon_J \leq q-1$ and

$$|J|^{\alpha-1}w(J)^{\frac{1}{p}-\frac{1}{p-\eta_J}}\eta_J^{-\frac{\theta}{p-\eta_J}}\psi(\varepsilon_J)^{\frac{1}{q-\varepsilon_J}}w(J)^{\frac{1}{q-\varepsilon_J}}\Big(\int\limits_J w^{-p'/q}\Big)^{\frac{1}{p'}} \leq c.$$

Observe that by Lemma 3.1 we have that

$$\eta_J^{-\frac{\theta}{p-\eta_J}}\psi(\varepsilon_J)^{\frac{1}{q-\varepsilon_J}} = \eta_J^{-\frac{\theta}{p-\eta_J}}\varphi\left(\varepsilon_J^\theta\right)^{\frac{1}{q-\varepsilon_J}} \approx \eta_J^{-\frac{\theta}{p-\eta_J}}\varepsilon_J^{\frac{\theta(1+\alpha q)}{q-\varepsilon_J}} =$$

$$= \Big(\eta_J^{-\frac{1}{p-\eta_J}}\varepsilon_J^{\frac{1+\alpha q}{q-\varepsilon_J}}\Big)^\theta \approx \Big(\eta_J^{-\frac{1}{p-\eta_J}}\varphi(\varepsilon_J)^{\frac{1}{q-\varepsilon_J}}\Big)^\theta = 1$$

and also,

$$\frac{1}{p} - \frac{1}{p-\eta_J} + \frac{1}{q-\varepsilon_J} = \frac{1}{p} - \alpha = \frac{1}{q}.$$

Finally, we have that

$$|J|^{\alpha-1}w(J)^{\frac{1}{q}}\Big(\int\limits_J w^{-p'/q}\Big)^{1/p'} \leq c.$$

Necessity is proved.

Sufficiency. Using Lemma 3.6 we have that there are positive constants σ_1, σ_2 and L satisfying the conditions: $\frac{1}{p-\sigma_2}-\frac{1}{q-\sigma_1}=\alpha$, $w\in A_{1+\frac{q-\sigma_1}{(p-\sigma_2)'}}$, $\|K_\alpha\|_{L^{p-\eta}_w\to L^{q-\varepsilon}_w}\le L$ for all $0\le\varepsilon\le\sigma_1$, $0\le\eta\le\sigma_2$ with $\frac{1}{p-\eta}-\frac{1}{q-\varepsilon}=\alpha$, where K_α is the operator defined by $K_\alpha f=I_\alpha(fw^\alpha)$.

Let σ be a small positive number such that $\sigma<\sigma_1<q-1$ and let us fix $\varepsilon\in(\sigma,q-1]$. Then $\frac{q-\sigma}{q-\varepsilon}>1$. By Hölder's inequality we have that

$$\|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])}\le\Bigg(\int\limits_0^1|I_\alpha(fw^\alpha)(x)|^{q-\sigma}w(x)dx\Bigg)^{\frac{1}{q-\sigma}}w([0,1])^{\frac{\varepsilon-\sigma}{(q-\sigma)(q-\varepsilon)}}$$

because $\left(\frac{q-\sigma}{q-\varepsilon}\right)'=\frac{q-\sigma}{\varepsilon-\sigma}$.

Further, the conditions $\sigma<q-1$ and $\sigma<\varepsilon<q-1$ yield

$$0<\frac{\varepsilon-\sigma}{(q-\sigma)(q-\varepsilon)}<\frac{q-1-\sigma}{q-\sigma}.$$

Consequently, using the well–known result by B. Muckenhoupt and R. L. Wheeden [12] for the classical weighted Lebesgue spaces:

$$\|I_\alpha(fw^\alpha)\|_{L^q_w([0,1])}\le c\|f\|_{L^p_w([0,1])}\Longleftrightarrow w\in A_{1+q/p'}([0,1]),\ \ q=\frac{p}{1-\alpha p},$$

we find that

$$\begin{aligned}
\|I_\alpha(fw^\alpha)\|_{L^{q),\psi(x)}_w([0,1])}&=\max\Bigg\{\sup_{0<\varepsilon\le\sigma}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])},\\
&\qquad\sup_{\sigma<\varepsilon\le q-1}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])}\Bigg\}\le\\
&\le\max\Bigg\{\sup_{0<\varepsilon\le\sigma}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])},\\
&\qquad\|I_\alpha(fw^\alpha)\|_{L^{q-\sigma}_w}\sup_{\sigma<\varepsilon\le q-1}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}w([0,1])^{\frac{\varepsilon-\sigma}{(q-\sigma)(q-\varepsilon)}}\Bigg\}\le\\
&\le\max\Bigg\{1,\sup_{\sigma<\varepsilon\le q-1}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\psi(\sigma)^{-\frac{1}{q-\sigma}}w([0,1])^{\frac{\varepsilon-\sigma}{(q-\sigma)(q-\varepsilon)}}\Bigg\}\times\\
&\qquad\times\sup_{0<\varepsilon\le\sigma}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\|I_\alpha(fw^\alpha)\|_{L^{q-\varepsilon}_w([0,1])}\le\\
&\le c\max\Bigg\{1,\Bigg[\sup_{\sigma<\varepsilon\le q-1}(\psi(\varepsilon))^{\frac{1}{q-\varepsilon}}\Bigg]\psi(\sigma)^{-\frac{1}{q-\sigma}}(1+w([0,1])^{\frac{q-1-\sigma}{q-\sigma}}\Bigg\}\times\\
&\qquad\times\sup_{0<\eta\le\sigma_0}\eta^{\frac{\theta}{p-\eta}}\|f\|_{L^{p-\eta}_w([0,1])}\le\\
&\le c\Bigg(\sup_{\sigma<\varepsilon\le q-1}\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}\Bigg)\psi(\sigma)^{-\frac{1}{q-\sigma}}(1+w([0,1]))^{\frac{q-1-\sigma}{q-\sigma}}\|f\|_{L^{p),\theta}_w([0,1])}.
\end{aligned}$$

Here σ_0 is small positive number chosen so that if $0 < \varepsilon \le \sigma$, then $0 < \eta \le \sigma_0 < \sigma_1 < p-1$. Also, we used the estimates:

$$\psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \approx \varepsilon^{\frac{\theta(1+\alpha q)}{q-\varepsilon}} \approx \varphi(\varepsilon)^{\frac{\theta}{q-\varepsilon}} = \eta^{\frac{\theta}{p-\eta}}, \quad \text{as} \quad \varepsilon \to 0,$$

where $\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \alpha$. □

Remark 3.1. Theorem 3.1 implies that if $1 < p < \infty$, $0 < \alpha < 1/p$, $q = \frac{p}{1-\alpha}$ and $\mu > 0$, then the one-weight inequality

$$\sup_{0<\varepsilon<q-1} \varepsilon^{\mu} \Big(\int_0^1 |I_\alpha(f w^\alpha)(x)|^{q-\varepsilon} w(x)dx \Big)^{\frac{1}{q-\varepsilon}} \le$$

$$\le C \sup_{0<\eta<p-1} \eta^{\mu} \Big(\int_0^1 |f(x)|^{p-\eta} w(x)dx \Big)^{\frac{1}{p-\eta}}$$

with the positive constant C independent of f holds if and only if $w \in A_{1+q/p'}([0,1])$.

This follows from the following easily verifiable relation

$$\|g\|_{L_w^{r),\theta}([0,1])} \approx \sup_{0<\varepsilon<r-1} \varepsilon^{\frac{\theta}{r}} \Big(\int_0^1 |g(x)|^{r-\varepsilon} w(x)dx \Big)^{\frac{1}{r-\varepsilon}},$$

which holds for weighted grand Lebesgue space $L_w^{r),\theta}([0,1])$, where $1<r<\infty$ and $\theta > 0$.

Corollary 3.1. *Let $\theta > 0$ and let $1 < p < \infty$. Suppose that $0 < \alpha < 1/p$. We set $q = \frac{p}{1-\alpha p}$. Then I_α is bounded from $L^{p),\theta_1}([0,1])$ to $L^{q),\theta_2}([0,1])$ provided that $\theta_2 > \theta_1 q/p$.*

Proof. follows immediately from Theorem 3.1 (in the unweighted case $w(x) \equiv$ const) and (1.1). □

4. One-sided potentials

In this section we show that the unboudedness result in grand Lebesgue spaces is also true for the one–sided potentials:

$$(R_\alpha f)(x) = \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x \in (0,1);$$

and

$$(W_\alpha f)(x) = \int_x^1 \frac{f(t)}{(t-x)^{1-\alpha}} dt, \quad x \in (0,1),$$

where $0<\alpha<1$. In particular, we claim that R_α and W_α are not bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$, where $q=\frac{p}{1-\alpha p}$, $1<p<\infty$, θ_1, $\theta_2>0$, $\theta_2<\frac{\theta_1 q}{p}$. Indeed, let us show the result first for R_α.

Suppose the contrary:

$$\|R_\alpha f\|_{L^{q),\theta_2}([0,1])} \le c\|f\|_{L^{p),\theta_1}([0,1])}, \qquad \theta_2<\frac{\theta_1 q}{p}, \tag{4.1}$$

where c does not depend on f. Let $f_n(x)=\chi_{(0,1/2n)}(x)$ in (4.1). Then taking the following inequality

$$(R_\alpha f_n)(x) \ge \int_0^{\frac{1}{2n}} \frac{1}{(x-t)^{1-\alpha}}dt \ge \left(\frac{1}{2n}\right)^\alpha, \quad x\in\left(\frac{1}{2n},\frac{1}{n}\right), \tag{4.2}$$

into account, (4.1) yields that

$$(2n)^{-\alpha}\left\|\chi_{\left(\frac{1}{2n},\frac{1}{n}\right)}\right\|_{L^{q),\theta_2}([0,1])} \le c\left\|\chi_{(0,1/2n)}\right\|_{L^{p),\theta_1}([0,1])}. \tag{4.3}$$

Now we choose ε_n positive number so that

$$\sup_{0<\varepsilon\le p-1}\left(\varepsilon^{\theta_1}\frac{1}{2n}\right)^{\frac{1}{p-\varepsilon}} = \left(\varepsilon_n^{\theta_1}\frac{1}{2n}\right)^{\frac{1}{p-\varepsilon_n}}. \tag{4.4}$$

We now observe that $\lim\limits_{n\to 0}\varepsilon_n=0$ (see the proof of Theorem 2.1 for the similar arguments). Choose now η_n so that

$$\alpha=\frac{1}{p}-\frac{1}{q}=\frac{1}{p-\varepsilon_n}-\frac{1}{q-\eta_n}.$$

Hence,

$$\eta_n = q-\frac{p-\varepsilon_n}{1-\alpha(p-\varepsilon_n)}. \tag{4.5}$$

By (4.3)–(4.5) we conclude that

$$(2n)^{-\alpha}\eta_n^{\frac{\theta_2}{q-\eta_n}}\left(\frac{1}{2n}\right)^{\frac{1}{q-\eta_n}} \le c\varepsilon_n^{\frac{\theta_1}{p-\varepsilon_n}}(2n)^{-1/(p-\varepsilon_n)}. \tag{4.6}$$

From (4.6) we have that

$$\eta_n^{\frac{\theta_2}{q-\eta_n}}\varepsilon_n^{-\frac{\theta_1}{p-\varepsilon_n}} \le c_p, \ \text{ for all } n\in N \tag{4.7}$$

because

$$\frac{1}{2}\le\left(\frac{1}{2}\right)^{\frac{1}{p-\varepsilon_n}}\le\left(\frac{1}{2}\right)^{\frac{1}{p}},$$
$$\frac{1}{2}\le\left(\frac{1}{2}\right)^{\frac{1}{q-\eta_n}}\le\left(\frac{1}{2}\right)^{\frac{1}{q}}.$$

Now (4.5) yields

$$\left[\frac{q-\frac{p-\varepsilon_n}{1-\alpha(p-\varepsilon_n)}}{\varepsilon_n}\right]^{\frac{\theta_2}{p-\varepsilon_n}-\alpha\theta_2} \varepsilon_n^{-\frac{\theta_1}{p-\varepsilon_n}+\frac{\theta_2}{p-\varepsilon_n}-\alpha\theta_2} \le c_p.$$

Hence,

$$\left[\frac{q-\frac{p-\varepsilon_n}{1-\alpha(p-\varepsilon_n)}}{\varepsilon_n}\right]^{\frac{\theta_2}{p-\varepsilon_n}-\alpha\theta_2} \varepsilon_n^{\frac{\theta_2-\theta_1}{p-\varepsilon_n}-\alpha\theta_2} \le c_p,$$

which is impossible, because $\lim\limits_{n\to\infty} \varepsilon_n^{\frac{\theta_2-\theta_1}{p-\varepsilon_n}-\alpha\theta_2} = \infty$ (recall that $\frac{\theta_2-\theta_1}{p}-\alpha\theta_2 = \frac{\theta_2}{q}-\frac{\theta_1}{p} < 0$).

Analogously, we have that W_α is not bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$. This follows from the inequalities

$$(W_\alpha)(x) \ge \int\limits_x^{1-\frac{1}{3n}} \frac{f(t)}{(t-x)^{1-\alpha}}dt \ge \left(\frac{2}{3n}\right)^{\alpha-1} \cdot \frac{1}{6n} = c_\alpha n^{-\alpha}, \quad x \in \left(1-\frac{1}{n}, 1-\frac{1}{2n}\right),$$

where $f(t) = \chi_{(1-\frac{1}{2n},1-\frac{1}{3n})}(t)$. Hence,

$$c_\alpha n^{-\alpha} \left\|\chi_{\left(1-\frac{1}{n},1-\frac{1}{2n}\right)}\right\|_{L^{q),\theta_2}([0,1])} \le c \left\|\chi_{\left(1-\frac{1}{2n},1-\frac{1}{3n}\right)}\right\|_{L^{p),\theta_1}([0,1])}.$$

Choosing now ε_n so that

$$\left[\varepsilon_n^{\theta_1}\frac{1}{6n}\right]^{\frac{1}{p-\varepsilon_n}} = \sup_{0<\varepsilon_n\le p-1} \left[\varepsilon_n^{\theta_1}\frac{1}{6n}\right]^{\frac{1}{p-\varepsilon}}, \quad 0<\varepsilon_n \le p-1,$$

and observing that $\lim\limits_{n\to\infty} \varepsilon_n = 0$ (see the proof of Theorem 2.1 for the similar arguments) we find that the conclusion similar to the case of R_α is valid.

4.1. **Conclusions and Remarks.**

Let $0<\alpha<1$ and let I_α, R_α, W_α be potential operators defined above. In the sequel we denote by T_α one of these operators.

Corollary 4.1. *Let $1<p<\infty$ and let $0<\alpha<1/p$. We set $q=\frac{p}{1-\alpha p}$. Suppose that θ_1 and θ_2 be positive numbers. Then:*

(i) *If $\theta_2 < \theta_1 q/p$, then T_α is not bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$.*

(ii) *If $\theta_2 \ge \theta_1 q/p$, then T_α is bounded from $L^{p),\theta_1}$ to $L^{q),\theta_2}$.*

Remark 4.1. There is a function f from $L^{p)}\backslash L^p$ such that $T_\alpha f \in L^{q)}\backslash L^q$.

Indeed, let $f(t)=t^{-\frac{1}{p}}$, $t\in(0,1)$. Then $f\in L^{p)}\backslash L^p$. On the other hand, (see e. g. [13]), $T_\alpha f \approx t^{-\frac{1}{q}}$. Hence $T_\alpha f \in L^{q)}\backslash L^q$.

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ON SLIGHTLY CONTINUOUS MULTIFUNCTIONS VIA GENERALIZED TOPOLOGY

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Abstract. In this paper, the notion of upper (lower) slightly (μ,σ)-continuous multifunctions has been introduced. Some characterizations of these types of multifunctions have been given. Several properties of such multifunctions are also obtained.

რეზიუმე. სტატიაში შემოღებულია ზედა (ქვედა) მცირედ (μ,σ)-უწყვეტი მულტიფუნქციები. მიღებულია ასეთი ტიპის ფუნქციების გარკვეული დახასიათება და დადგენილია მათი ზოგიერთი თვისება.

1. Introduction

One of the most important area in the theory of classical point set topology is continuity of functions and multifunctions as they are important tools for studying properties of spaces and for constructing new spaces from previously existing ones. Several weaker forms of continuous functions have been introduced and studied by different mathematicians. In [7] A. Kanibir and I. L. Reilly introduced upper (lower) semi generalized continuous multifunctions by using the concept of generalized topology. Similar types of functions have also been studied by C. Boonpok [2]. Such a generalized topology was first introduced by A. Cs*ászár*. We first recall some notions defined in [3]. Let X be a non-empty set, $\exp X$ denotes the power set of X. We call a class $\mu \subsetneqq \exp X$ a generalized topology [3], (briefly, GT) if $\varnothing \in \mu$ and union of elements of μ belongs to μ. A set X, with a GT μ on it is said to be a generalized topological space (briefly, GTS) and is denoted by (X,μ). For a GTS (X,μ), the elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets. For $A \subsetneqq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A; and by $i_\mu(A)$ the union of all μ-open

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sets contained in A, i.e., the largest μ-open set contained in A (see [3, 4] for details).

It is easy to observe that i_μ and c_μ are idempotent and monotonic, where $\gamma : \exp X \to \exp X$ is said to be idempotent iff $A \subseteqq B \subseteqq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic iff $\gamma(A) \subseteqq \gamma(B)$. It is also well known from [4, 5] that if μ is a GT on X and $A \subseteqq X$, $x \in X$, then $x \in c_\mu(A)$ iff $x \in M \in \mu \Rightarrow M \cap A \neq \varnothing$ and $c_\mu(X \setminus A) = X \setminus i_\mu(A)$.

Throughout the paper, we shall use (X, μ) to mean a generalized topological space and (Y, σ) will denote a topological space. For a subset A, $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote the closure and interior of A respectively. For a multifunction $F : (X, \mu) \to (Y, \sigma)$, the upper and lower inverse of a set A of Y are denoted by $F^+(A)$ and $F^-(A)$ and defined by $F^+(A) = \{x \in X : F(x) \subseteqq A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \varnothing\}$. Also here $\mu(x) = \{U \in \mu : x \in U\}$.

2. Slightly (μ, σ)-Continuous Multifunctions

Definition 2.1. A multifunction $F : (X, \mu) \to (Y, \sigma)$ is said to be

(a) upper slightly (μ, σ)-continuous if for each point $x \in X$ and each clopen set V in Y containing $F(x)$, there exists $U \in \mu(x)$ in X such that $F(U) \subseteqq V$.

(b) lower slightly (μ, σ)-continuous if for each point $x \in X$ and each clopen set V in Y with $F(x) \cap V \neq \varnothing$, there exists $U \in \mu(x)$ in X such that $F(u) \cap V \neq \varnothing$ for each $u \in U$.

Theorem 2.2. *For a multifunction $F : (X, \mu) \to (Y, \sigma)$ the followings are equivalent:*

(a) *F is upper slightly (μ, σ)-continuous;*

(b) *$F^+(V)$ is μ-open for each clopen set V of Y;*

(c) *$F^-(V)$ is μ-closed for each clopen set V of Y.*

Proof. (a) $\Rightarrow$ (b): Let V be any clopen set in Y and $x \in F^+(V)$. Then $F(x) \subseteqq V$. Thus by (a), there exists $U \in \mu(x)$ in X such that $F(U) \subseteqq V$. Thus $x \in U \subseteqq F^+(V)$ and hence $x \in i_\mu(F^+(V))$. Therefore, $F^+(V) \subseteqq i_\mu(F^+(V))$ i.e., $F^+(V)$ is μ-open.

(b) $\Rightarrow$ (c): Let V be a clopen set of Y. Then $Y \setminus V$ is clopen in Y. Then by (b), $X \setminus F^-(V) = F^+(Y \setminus V) = i_\mu(F^+(Y \setminus V)) = X \setminus c_\mu(F^-(V))$. Thus $F^-(V) = c_\mu(F^-(V))$ i.e., $F^-(V)$ is μ-closed.

(c) $\Rightarrow$ (b): This follows from the fact that $F^-(Y \setminus B) = X \setminus F^+(B)$ for any subset B of Y.

(b) $\Rightarrow$ (a): Let $x \in X$ and V be any clopen set of Y containing $F(x)$. Then $x \in F^+(V) = i_\mu(F^+(V))$. Thus there exists $U \in \mu(x)$ such that $U \subseteqq F^+(V)$. Therefore, there exists a μ-open set U in X containing x such that $F(U) \subseteqq V$. Thus F is upper slightly (μ, σ)-continuous. $\square$

Theorem 2.3. *For a multifunction $F : (X, \mu) \to (Y, \sigma)$ the followings are equivalent:*
(a) *F is lower slightly (μ, σ)-continuous;*
(b) *$F^-(V)$ is μ-open for each clopen set V of Y;*
(c) *$F^+(V)$ is μ-closed for each clopen set V of Y.*

Proof. (a) $\Rightarrow$ (b): Let V be a clopen set of Y and $x \in F^-(V)$. Then $F(x) \cap V \neq \varnothing$ and hence by (a) there exists $U \in \mu(x)$ such that $F(u) \cap V \neq \varnothing$ for each $u \in U$. Therefore, we have $U \subseteq F^-(V)$ and hence $x \in U \subseteq i_\mu(F^-(V))$. Thus $F^-(V) \subseteq i_\mu(F^-(V))$ i.e., $F^-(V)$ is μ-open.

(b) $\Rightarrow$ (c): Let V be a clopen set in Y. Then $Y \setminus V$ is clopen in Y and by (b), we have $X \setminus F^+(V) = F^-(Y \setminus V) = i_\mu(F^-(Y \setminus V)) = X \setminus c_\mu(F^+(V))$. Thus $F^+(V)$ is μ-closed.

(c) $\Rightarrow$ (a): Let x be any point of X and V be any clopen set in Y such that $F(x) \cap V \neq \varnothing$. Then $x \in F^-(V)$ and hence $x \notin X \setminus F^-(V) = F^+(Y \setminus V)$. As $Y \setminus V$ is clopen in Y, by (c) we have $x \notin c_\mu(F^+(Y \setminus V))$. Thus there exists $U \in \mu(x)$ such that $U \cap F^+(Y \setminus V) = \varnothing$; hence $U \subseteq F^-(V)$. Thus $F(u) \cap V \neq \varnothing$ for each $u \in U$. Therefore F is lower slightly (μ, σ)-continuous. $\square$

Definition 2.4. A topological space (X, τ) is said to be extremally disconnected (in short, E.D.) if closure of each open set in X is open in X.

Theorem 2.5. *Let (Y, σ) be an extremally disconnected space. For a multifunction $F : (X, \mu) \to (Y, \sigma)$ the followings are equivalent:*
(a) *F is upper slightly (μ, σ)-continuous;*
(b) *$c_\mu(F^-(V)) \subseteq F^-(\mathrm{cl}(V))$ for each open set V in Y;*
(c) *$F^+(\mathrm{int}(K)) \subseteq i_\mu(F^+(K))$ for each closed set K in Y.*

Proof. (a) $\Rightarrow$ (b): Let V be an open set in Y. Then $\mathrm{cl}(V)$ is clopen in Y (as Y is E.D.). By Theorem 2.2, $F^-(\mathrm{cl}(V)) = c_\mu(F^-(\mathrm{cl}(V)))$ and $F^-(V) \subseteq F^-(\mathrm{cl}(V))$. Thus $c_\mu(F^-(V)) \subseteq c_\mu(F^-(\mathrm{cl}(V))) = F^-(\mathrm{cl}(V))$. So $c_\mu(F^-(V)) \subseteq F^-(\mathrm{cl}(V))$.

(b) $\Rightarrow$ (c): Let K be any closed set in Y. Put $V = Y \setminus K$. Then V is an open set in Y. Then $X \setminus i_\mu(F^+(K)) = c_\mu(X \setminus F^+(K)) = c_\mu(F^-(V)) \subseteq F^-(\mathrm{cl}(V))$ (by (b)) $= F^-(Y \setminus \mathrm{int}(K)) = X \setminus F^+(\mathrm{int}(K))$. Thus we have $F^+(\mathrm{int}(K)) \subseteq i_\mu(F^+(K))$.

(c) $\Rightarrow$ (a): Let x be any point of X and V be a clopen set in Y containing $F(x)$. Then by (c) we have $x \in F^+(\mathrm{int}(V)) = F^+(V) \subseteq i_\mu(F^+(V))$. Therefore, there exists $U \in \mu(x)$ such that $U \subseteq F^+(V)$. Thus there exists a μ-open set U in X containing x such that $F(U) \subseteq V$. Therefore, F is upper slightly (μ, σ)-continuous. $\square$

Theorem 2.6. *Let (Y, σ) be an extremally disconnected space. For a multifunction $F : (X, \mu) \to (Y, \sigma)$ the followings are equivalent:*
(a) *F is lower slightly (μ, σ)-continuous;*

(b) $c_{\mu}(F^{+}(V)) \subseteqq F^{+}(\mathrm{cl}(V))$ *for each open set* V *in* Y*;*
(c) $F^{-}(\mathrm{int}(K)) \subseteqq i_{\mu}(F^{-}(K))$ *for every closed set* K *in* Y*.*

Proof. The proof is similar to that of Theorem 2.5. ☐

Lemma 2.7 ([8])**.** *For a topological space* (Y, σ)*, the followings are equivalent:*
(a) (Y, σ) *is extremally disconnected;*
(b) *The closure of every semi-open set of* (Y, σ) *is open;*
(c) *The closure of every pre-open set of* (Y, σ) *is open;*
(d) *The closure of every* β*-open set of* (Y, σ) *is open.*

Lemma 2.8. *Let* (Y, σ) *be an extremally disconnected space. For a multifunction* $F : (X, \mu) \to (Y, \sigma)$ *the followings are equivalent:*
(a) F *is upper slightly* (μ, σ)*-continuous;*
(b) $c_{\mu}(F^{-}(V)) \subseteqq F^{-}(\mathrm{cl}(V))$ *for each semi-open* (*resp. pre-open,* β*-open*) *set* V *in* Y*;*
(c) $F^{+}(\mathrm{int}(K)) \subseteqq i_{\mu}(F^{+}(K))$ *for each semi-closed* (*resp. pre-closed,* β*-closed*) *set* K *in* Y*.*

Proof. The proof is similar to that of Theorem 2.5 and it follows from Theorem 2.2 and Lemma 2.7. ☐

Theorem 2.9. *Let* (Y, σ) *be an extremally disconnected space. For a multifunction* $F : (X, \mu) \to (Y, \sigma)$ *the followings are equivalent:*
(a) F *is lower slightly* (μ, σ)*-continuous;*
(b) $c_{\mu}(F^{+}(V)) \subseteqq F^{+}(\mathrm{cl}(V))$ *for each semi-open* (*resp. pre-open,* β*-open*) *set* V *in* Y*;*
(c) $F^{-}(\mathrm{int}(K)) \subseteqq i_{\mu}(F^{-}(K))$ *for each semi-closed* (*resp. pre-closed,* β*-closed*) *set* K *in* Y*.*

Proof. The proof is similar to that of Theorem 2.6 and it follows from Theorem 2.3 and Lemma 2.7. ☐

Definition 2.10. A multifunction $F : (X, \mu) \to (Y, \sigma)$ is said to be
(a) upper (μ, σ)-continuous (resp. upper almost (μ, σ)-continuous, upper weakly (μ, σ)-continuous) if for each point $x \in X$ and each open set V of Y containing $F(x)$, there exists a μ-open set U in X containing x such that $F(U) \subseteqq V$ (resp. $F(U) \subseteqq \mathrm{int}(\mathrm{cl}(V))$, $F(U) \subseteqq \mathrm{cl}(V)$).
(b) lower (μ, σ)-continuous (resp. lower almost (μ, σ)-continuous, lower weakly (μ, σ)-continuous) if for each point $x \in X$ and each open set V of Y with $F(x) \cap V \neq \varnothing$, there exists a μ-open set U in X containing x such that $F(u) \cap V \neq \varnothing$ (resp. $F(u) \cap \mathrm{int}(\mathrm{cl}(V)) \neq \varnothing$, $F(u) \cap \mathrm{cl}(V) \neq \varnothing$) for each $u \in U$.

Theorem 2.11. *If a multifunction* $F : (X, \mu) \to (Y, \sigma)$ *is upper weakly* (μ, σ)*-continuous, then it is upper slightly* (μ, σ)*-continuous.*

Proof. Let $x \in X$ and V be a clopen set in Y containing $F(x)$. Since F is upper weakly (μ, σ)-continuous, there exists a μ-open set U containing x such that $F(U) \subseteqq \mathrm{cl}(V) = V$. So F is upper slightly (μ, σ)-continuous. $\square$

Theorem 2.12. *If a multifunction $F : (X, \mu) \to (Y, \sigma)$ is lower weakly (μ, σ)-continuous, then it is lower slightly (μ, σ)-continuous.*

Proof. Similar to that of Theorem 2.11. $\square$

Example 2.13. Let $X = Y = \{a, b, c\}$, $\mu = \{\varnothing, \{a\}, \{a, b\}, X\}$, $\sigma = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$. It can be shown that the function $F : (X, \mu) \to (Y, \sigma)$ defined by $F(a) = \{b, c\}$, $F(b) = F(c) = \{a\}$ is upper slightly (μ, σ)-continuous, but not upper weakly (μ, σ)-continuous.

Lemma 2.14. *A multifunction $F : (X, \mu) \to (Y, \sigma)$ is upper almost (μ, σ)-continuous (resp. lower almost (μ, σ)-continuous) if and only if for each regular open set V containing $F(x)$ (resp. $V \cap F(x) \neq \varnothing$) there exists a μ-open set U containing x such that $F(U) \subseteqq V$ (resp. $F(u) \cap V \neq \varnothing$ for each $u \in U$).*

Theorem 2.15. *If a multifunction $F : (X, \mu) \to (Y, \sigma)$ is upper slightly (μ, σ)-continuous and (Y, σ) is extremally disconnected then F is upper almost (μ, σ)-continuous.*

Proof. Let $x \in X$ and V be any regular open set of Y containing $F(x)$. Then V is a clopen set (as Y is extremally disconnected [see [10], Lemma 5.6]). Since F is upper slightly (μ, σ)-continuous, there exists a μ-open set U in X containing x such that $F(U) \subseteqq V$. Thus by Lemma 2.14, F is upper almost (μ, σ)-continuous. $\square$

Theorem 2.16. *If a multifunction $F : (X, \mu) \to (Y, \sigma)$ is lower slightly (μ, σ)-continuous and (Y, σ) is extremally disconnected, then F is lower almost (μ, σ)-continuous.*

Proof. Similar to that of Theorem 2.15. $\square$

Definition 2.17. A topological space (X, τ) is said to be 0-dimensional [14] if each point of X has a base consisting of clopen sets.

Definition 2.18. A topological space (X, τ) is said to be mildly compact [13] or slightly compact [10] if every clopen cover of X admits a finite subcover. A subset A of X is called mildly compact relative to X if every cover of A by clopen subsets of X has a finite subcover.

Theorem 2.19. *If a multifunction $F : (X, \mu) \to (Y, \sigma)$ is upper slightly (μ, σ)-continuous and (Y, σ) is 0-dimensional and $F(x)$ is mildly compact relative to Y for each $x \in X$, then F is upper (μ, σ)-continuous.*

Proof. Let $x \in X$ and V be any open set of (Y, σ) containing $F(x)$. Then by 0-dimensionality of (Y, σ), for each $y \in F(x)$ there exists a clopen set G_y such that $y \in G_y \subsetneqq V$. Since $F(x)$ is mildly compact relative to Y, there exists a finite number of points $y_1, y_2, \ldots, y_n \in F(x)$ such that G_{y_i} is clopen in Y for each i and $F(x) \subsetneqq \cup\{G_{y_i} : i = 1, 2, \ldots, n\} \subsetneqq V$. Let $G = \cup\{G_{y_i} : i = 1, 2, \ldots, n\}$. Then G is clopen in Y and $F(x) \subsetneqq G \subsetneqq V$. Since F is upper slightly (μ, σ)-continuous, there exists a μ-open set U with $x \in U$ such that $F(U) \subsetneqq G \subsetneqq V$. Thus F is upper (μ, σ)-continuous. □

Theorem 2.20. *If a multifunction $F : (X, \mu) \to (Y, \sigma)$ is lower slightly (μ, σ)-continuous and (Y, σ) is 0-dimensional, then F is lower (μ, σ)-continuous.*

Proof. Let $x \in X$ and V be an open set in Y such that $F(x) \cap V \neq \varnothing$. Then there exists a clopen set V_x such that $V_x \cap F(x) \neq \varnothing$ and $V_x \subsetneqq V$. Since F is lower slightly (μ, σ)-continuous and $V_x \cap F(x) \neq \varnothing$ there exists $U \in \mu(x)$ such that $F(u) \cap V_x \neq \varnothing$ for each $u \in U$. Thus there exists $U \in \mu(x)$ such that $F(u) \cap V \neq \varnothing$ for each $u \in U$ (as $V_x \subsetneqq V$). Therefore F is lower (μ, σ)-continuous. □

The clopen subsets of a topological space (X, τ) forms a base for a topology on X. This topology is called ultra-regularization [9] of τ and is denoted by τ_u. A topological space (X, τ) is said to be ultra-regular [6] if $\tau = \tau_u$.

Theorem 2.21. *If a multifunction $F : (X, \mu) \to (Y, \sigma)$ is upper slightly (μ, σ)-continuous and (Y, σ) is ultra-regular and $F(x)$ is mildly compact relative to Y for each $x \in X$, then F is upper (μ, σ)-continuous.*

Proof. Similar to that of Theorem 2.19. □

Theorem 2.22. *If a multifunction $F : (X, \mu) \to (Y, \sigma)$ is lower slightly (μ, σ)-continuous and (Y, σ) is ultra-regular, then F is lower (μ, σ)-continuous.*

Proof. Similar to that of Theorem 2.20. □

3. Properties of Upper (Lower) Slightly (μ, σ)-Continuous Multifunctions

Definition 3.1. A GTS (X, μ) is said to be μ-connected [12] if X can not be written as union of two non-empty μ-open sets.

Theorem 3.2. *Let $F : (X, \mu) \to (Y, \sigma)$ be an upper (lower) slightly (μ, σ)-continuous surjection. If (X, μ) is μ-connected and $F(x)$ is connected for each $x \in X$, then (Y, σ) is connected.*

Proof. If possible let (Y, σ) be not connected. Then there exists a pair of disjoint open sets U and V such that $Y = U \cup V$. Since $F(x)$ is connected, for each $x \in X$ either $F(x) \subseteqq U$ or $F(x) \subseteqq V$. If $x \in F^+(U \cup V)$, then $F(x) \in U \cup V$ and hence $x \in F^+(U) \cup F^+(V)$. Also, as F is surjective there exist $x, y \in X$ such that $F(x) \subseteqq U$, $F(y) \subseteqq V$ hence $x \in F^+(U)$ and $y \in F^+(V)$. Thus $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$, $F^+(U) \cap F^+(V) = F^+(U \cap V) = \varnothing$ and $F^+(U) \neq \varnothing \neq F^+(V)$.

If F is upper slightly (μ, σ)-continuous then since U and V are clopen, by Theorem 2.2 $F^+(U)$ and $F^+(V)$ are μ-clopen in X- a contradiction to the fact that X is μ-connected. If F is lower slightly (μ, σ)-continuous then by Theorem 2.3 we can have a similar contradiction. □

Definition 3.3. A multifunction $F : (X, \mu) \to (Y, \sigma)$ is said to have a (μ, σ)-clopen graph if for each $(x, y) \in X \times Y \setminus G(F)$, there exist a μ-open set U in X containing x and a clopen set V in Y containing y such that $(U \times V) \cap G(F) = \varnothing$.

Lemma 3.4. *A multifunction $F : (X, \mu) \to (Y, \sigma)$ has a (μ, σ)-clopen graph if and only if for each $(x, y) \in X \times Y \setminus G(F)$, there exist a μ-open set U in X containing x and a clopen set V in Y containing y such that $F(U) \cap V = \varnothing$.*

Definition 3.5. A topological space (X, τ) is said to be ultra-Hausdorff [13] if for each pair of distinct points $x, y \in X$, there exist disjoint pair of clopen sets U and V such that $x \in U$ and $y \in V$.

Theorem 3.6. *If $F : (X, \mu) \to (Y, \sigma)$ is an upper slightly (μ, σ)-continuous multifunction such that $F(x)$ is mildly compact relative to Y for each $x \in X$ and (Y, σ) is ultra-Hausdorff, then $G(F)$ is (μ, σ)-clopen.*

Proof. Suppose that $(x_0, y_0) \in X \times Y \setminus G(F)$. Then $y_0 \notin F(x_0)$. Since Y is ultra-Hausdorff, for each $y \in F(x_0)$ there exist clopen sets G_y and H_y in Y containing y and y_0 respectively, such that $G_y \cap H_y = \varnothing$. Then the family $\{G_y : y \in F(x_0)\}$ is a clopen cover of $F(x_0)$. Since $F(x_0)$ is mildly compact relative to Y, there exists a finite number of points $y_1, y_2, \ldots, y_n$ in $F(x_0)$ such that $F(x_0) \subseteqq \cup\{G_{y_i} : i = 1, 2, \ldots, n\} = G$ (say). Let $H = \cap\{H_{y_i} : i = 1, 2, \ldots, n\}$. Then G and H both are clopen in Y such that $F(x_0) \subseteqq G$, $y_0 \in H$ and $G \cap H = \varnothing$. Since F is upper slightly (μ, σ)-continuous, there exists a μ-open set U in X containing x_0 such that $F(U) \subseteqq G$. Thus $F(U) \cap H = \varnothing$. Hence by Lemma 3.4, $G(F)$ is (μ, σ)-clopen. □

Definition 3.7. For any subset A of a GTS (X, μ), the μ-frontier [11] of A is denoted by $Fr_\mu(A)$ and defined by $Fr_\mu(A) = c_\mu(A) \cap c_\mu(X \setminus A)$.

Theorem 3.8. *The set of all points $x \in X$ at which a multifunction $F : (X, \mu) \to (Y, \sigma)$ is not upper (lower) slightly (μ, σ)-continuous is identical with the union of μ-frontier of the upper (resp. lower) inverse image of clopen sets containing (resp. meeting) $F(x)$.*

Proof. We shall prove the theorem when F is upper slightly (μ,σ)-continuous. The case for lower slightly (μ,σ)-continuous can be shown in a similar fashion. Suppose that F is not upper slightly (μ,σ)-continuous at $x \in X$. Then there exists a clopen set V in Y containing $F(x)$ such that $U \cap (X \setminus F^+(V)) \neq \varnothing$ for each μ-open set U containing x. Then $x \in c_\mu(X \setminus F^+(V))$. On the other hand, we have $x \in F^+(V) \subseteqq c_\mu(F^+(V))$. Hence $x \in Fr_\mu(F^+(V))$.

Conversely, suppose that F is upper slightly (μ,σ)-continuous at $x \in X$. Let V be any clopen set in Y containing $F(x)$. Then there exists a μ-open set U in X containing x such that $U \subseteqq F^+(V)$; hence $x \in i_\mu(F^+(V))$. Therefore, $x \notin Fr_\mu(F^+(V))$ for each clopen set V in Y containing $F(x)$. $\square$

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MULTILINEAR FRACTIONAL INTEGRALS IN WEIGHTED GRAND LEBESGUE SPACES

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Our goal is to present weighted inequalities for multilinear fractional integral operators in grand Lebesgue spaces. The theory of grand Lebesgue spaces introduced by T. Iwaniec and C. Sbordone [14] is one of the intensively developing directions of the modern analysis. It was realized the necessity for the study of these spaces because of their rather essential role and applications in various fields. These spaces naturally arise, for example, in the integrability problems of the Jacobian under minimal hypothesis (see [14] for the details).

Structural properties of grand Lebesgue spaces were investigated in the papers [4], [2]. In [5] the authors proved that for the boundedness of the Hardy–Littlewood maximal operator defined on $[0,1]$ in weighted grand Lebesgue spaces $L^{p)}_w([0,1])$ it is necessary and sufficient that the weight w satisfies the Muckenhoupt's A_p condition on the interval $[0,1]$.

The same phenomenon was noticed for the Hilbert transform in [22]. We refer also to [17], [16], [28] for one–weight results regarding maximal and singular integrals of various type in these spaces.

In [25] the boundedness criteria for fractional integral operators in weighted grand Lebesgue spaces from the one–weight viewpoint were established. In particular, in that paper the author determined values of the second parameter for grand Lebesgue spaces governing the boundedness of fractional integral operator in these spaces, and established criterion for which inequality (8) (see below) holds in the linear case (see also [23] for related topics).

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In [21] trace inequality criteria for fractional integrals in grand Lebesgue spaces defined on metric measure spaces were derived.

Multilinear fractional integrals were introduced and studied in the papers by L. Grafakos [6], C. Kenig and E. Stein [15], L. Grafakos and N. Kalton [8].

For the boundedness and other properties of multi(sub)linear fractional integrals in (weighted) Lebesgue spaces we refer, e.g., to [26], [27], [3], [29], [18], [19].

Recently, in [20] the authors of this paper presented the one-weight inequality for the multi(sub)linear Hardy–Littlewood maximal and Calderón-Zygmund operators defined on an SHT.

It should be stressed that the results of this paper are new even for Euclidean spaces. In the most cases the derived conditions are simultaneously necessary and sufficient for appropriate one-weight inequality in the linear case (see, e.g., [23], [22], [21], [25]).

In the sequel the following notation will be used:

$$\overrightarrow{p} := (p_1, \dots, p_m),$$

where $p_i \in (0, \infty)$ for each $1 \le i \le m$;

$$\overrightarrow{f} = (f_1, \dots, f_m),$$

where f_i are $\mu-$ measurable functions defined on X;

$$\frac{1}{p} := \sum_{i=1}^{m} \frac{1}{p_i} d\mu(\overrightarrow{y}), \quad d\overrightarrow{y} := d\mu(y_1) \cdots d\mu(y_m);$$

$$\nu_{\overrightarrow{w}} := \prod_{j=1}^{m} w_j^{p/p_j}, \quad \widetilde{\nu}_{\overrightarrow{w}} := \prod_{j=1}^{m} w_j^{q/q_j};$$

$$B_{xy} := \mu\big(B(x, d(x,y))\big).$$

Let $s \in [1, \infty]$. As usual we put $s' := \frac{s}{s-1}$ if $s \in (1, \infty)$ and $s' := \infty$ for $s = 1$ and $s' := 1$ for $s = \infty$;

$$A_{p,q,\alpha} := q(1/p - \alpha)$$

for $1 < p < q < \infty$ and $0 < \alpha < 1/p$;

Let (X, d, μ) be a quasi-metric measure space with quasi-metric d and measure μ. If μ satisfies the well-known doubling condition, then (X, d, μ) is called space of homogeneous type (SHT).

Let $1 \le r < \infty$. We denote by $L^r(X, \mu)$ the Lebesgue space on X with an exponent r.

If w is a weight (locally integrable, μ-a.e. positive function on X), then we denote the weighted Lebesgue spaces by $L^r_w(X, \mu)$, i.e., $f \in L^r_w(X, \mu)$ if $\|f\|_{L^r_w(X,\mu)} = \|f\|_{L^r(X, w d\mu)} < \infty$.

Let $\mu(X) < \infty$, $1 < p < \infty$ and let φ be a continuous positive function on $(0, p-1)$ such that it is non-decreasing on $(0, \sigma)$ for some small positive σ

and satisfies the condition $\lim\limits_{x\to 0+}\varphi(x)=0$. The generalized grand Lebesgue space $L^{p),\varphi}(X,\mu)$ is the class of those $f:X\to\mathbb{R}$ for which the norm

$$\|f\|_{L^{p),\varphi}(X,\mu)}=\sup_{0<\varepsilon<p-1}\left(\varphi(\varepsilon)\int\limits_X |f(x)|^{p-\varepsilon}\,d\mu(x)\right)^{1/(p-\varepsilon)}$$

is finite. If w is a weight on X, then the weighted grand Lebesgue space with weight w is denoted by $L^{p),\varphi}_w(X,\mu)$ and coincides with the class $L^{p),\varphi}(X,wd\mu)$. In this case we assume that $\|f\|_{L^{p),\varphi}_w(X,\mu)}=\|f\|_{L^{p),\varphi}(X,wd\mu)}$.

If $\varphi(x)=x^\theta$, where θ is a positive number, then we denote $L^{p),\varphi}(X,\mu)$ (resp., $L^{p),\varphi}_w(X,\mu)$) by $L^{p),\theta}(X,\mu)$ (resp. by $L^{p),\theta}_w(X,\mu)$).

The space $L^{p),\theta}(X,\mu)$ is a Banach space.

It is easy to check that the following continuous embeddings hold:

$$L^p(X,\mu)\hookrightarrow L^{p),\theta_1}(X,\mu)\hookrightarrow L^{p),\theta_2}(X,\mu)\hookrightarrow L^{p-\varepsilon}(X,\mu),$$

where $0<\varepsilon\le p-1$ and $\theta_1<\theta_2$.

It turns out that in the theory of PDEs the generalized grand Lebesgue spaces are appropriate to the solutions of existence and uniqueness, and, also, the regularity problems for various nonlinear differential equations. The space $L^{p),\theta}$ (defined on bounded domains in $\mathbb{R}^n$) for arbitrary positive θ was introduced in the paper [12], where the authors studied the nonhomogeneous n-harmonic equation div $A(x,\nabla u)=\mu$. If $\theta=1$, then $L^{p),\theta}(X,\mu)$ coincides with the Iwaniec–Sbordone space, which we denote by $L^{p)}(X,\mu)$. The grand Lebesgue space is non-reflexive, non-separable and, in general, is non-rearrangement invariant (see, e.g., [4]).

We define the class $\prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j)$ of vector functions $\overrightarrow{f}$ as follows: $\overrightarrow{f}\in\prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j)$ if

$$\|\overrightarrow{f}\|_{\prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j)}=\sup_{1<r<p}\left\{\varphi\Big(\frac{p}{r'}\Big)^{\frac{r}{p}}\prod_{j=1}^m\|f_j\|_{L^{p_j/r}(X,\mu_j)}\right\}=$$
$$=\sup_{1<r<p}\left\{\prod_{j=1}^m\varphi\Big(\frac{p}{r'}\Big)^{\frac{r}{p_j}}\|f_j\|_{L^{p_j/r}(X,\mu_j)}\right\}<\infty.$$

The expression $\|\overrightarrow{f}\|_{\prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j)}$ can be rewritten as follows:

$$\|\overrightarrow{f}\|_{\prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j)}=$$
$$=\sup_{0<\eta<p-1}\left\{\prod_{j=1}^m\varphi(\eta)^{\frac{1}{p_j-\eta_j}}\|f_j\|_{L^{p_j-\eta_j}(X,\mu_j)}:\frac{p}{p-\eta}=\frac{p_j}{p_j-\eta_j},\ \ j=1,\dots,m\right\}.$$

It is easy to check that

$$\prod_{j=1}^{m} L^{p_j),\varphi}(X,\mu_j) \hookrightarrow \prod_{j=1}^{m} \mathcal{L}^{p_j),\varphi}(X,\mu_j),$$

in particular,

$$\|\overrightarrow{f}\|_{\prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j)} \le \|\overrightarrow{f}\|_{\prod_{j=1}^m L^{p_j),\varphi}(X,\mu_j)}.$$

This follows from the fact that if $\frac{p}{p-\eta} = \frac{p_j}{p_j-\eta_j}$, $j = 1,\dots,m$, then $\eta \le \eta_j$ because $\frac{1}{\eta} = \sum_{j=1}^m \frac{1}{\eta_j}$.

When we deal with grand Lebesgue spaces we assume that $\mu(X) < \infty$.

Let $1 < r < \infty$. We say that a weight function w belongs to the Muckenhoupt class $A_r(X)$ if

$$\|w\|_{A_r} := \sup_B \bigg(\frac{1}{\mu(B)}\int_B w\,d\mu\bigg)\bigg(\frac{1}{\mu(B)}\int_B w^{1-r'}\,d\mu\bigg)^{r-1} < \infty.$$

Let us recall the definition of the vector Muckenhoupt condition (see [24] for Euclidean spaces and [9] for metric measure spaces).

Definition A. Let $1 \le p_j < \infty$ for each $1 \le j \le m$, and $0 < p < \infty$. We say that $\overrightarrow{w}$ satisfies the $A_{\overrightarrow{p}}(X)$ condition ($\overrightarrow{w} \in A_{\overrightarrow{p}}$) if

$$\|\overrightarrow{w}\|_{A_{\overrightarrow{p}}} := \sup_{B\subset X} \bigg(\frac{1}{\mu(B)}\int_B \nu_{\overrightarrow{w}}(x)d\mu(x)\bigg)\times$$
$$\times \prod_{j=1}^{m} \bigg(\frac{1}{\mu(B)}\int_B w_j^{1-p'}(x)\,d\mu(x)\bigg)^{p/p_j'} < \infty,$$

where the supremum is taken over all balls B in X. For $p_j = 1$, the expression $\big(\frac{1}{\mu(B)}\int_B w_j^{1-p'}(x)d\mu(x)\big)^{1/p_j'}$ is understood as $(\inf_B w_j)^{-1}$.

The expression $\|\overrightarrow{w}\|_{A_{\overrightarrow{p}}}$ is called $A_{\overrightarrow{p}}$ characteristic of $\overrightarrow{w}$.

It is known (see [24]) that if $\overrightarrow{w}$ satisfies the condition $A_{\overrightarrow{p}}(\mathbb{R}^n)$, then the boundedness of the multi(sub)linear Hardy–Littlewood and Calderón-Zygmund operators defined on $\mathbb{R}^n$ from $\prod_{j=1}^m L^{p_j}_{w_j}(\mathbb{R}^n)$ to $L^p_{\nu_{\overrightarrow{w}}}(\mathbb{R}^n)$ holds, where $1 < p_j < \infty$ for each $1 \le j \le m$, and $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$.

In the linear case ($m = 1$) the class $A_{\overrightarrow{p}}$ coincides with the well-known Muckenhoupt class A_p.

Definition B (vector Muckenhoupt-Wheeden condition). Let (X,d,μ) be a metric measure space, $1 \le p_i < \infty$ for $i = 1,\dots,m$. Suppose that $p < q < \infty$. Let $w_1,\dots,w_m$ be a weight functions on X. We say that

$\overrightarrow{w} = (w_1, \dots, w_m)$ satisfies $A_{\overrightarrow{p},q}(X)$ condition ($\overrightarrow{w} \in A_{\overrightarrow{p},q}(X)$) if

$$\sup_B \left(\frac{1}{\mu B} \int\limits_B \Big(\prod_{i=1}^m w_i \Big)^q d\mu \right)^{1/q} \prod_{i=1}^m \left(\frac{1}{\mu B} \int\limits_B w_i^{-p_i'} \, d\mu \right)^{1/p_i'} < \infty,$$

where the supremum is taken over all balls B in X. For $p_j = 1$, the expression $\big(\frac{1}{\mu(B)} \int\limits_B w_j^{1-p'}(x) d\mu(x) \big)^{1/p_j'}$ is understood as $(\inf_B w_j)^{-1}$.

Theorem A ([26]). *Let $1 < p_1, \dots, p_m < \infty$, $0 < \alpha < mn$, $\frac{1}{m} < p < \frac{n}{\alpha}$. Suppose that q is an exponent satisfying the condition $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose that w_i are a.e. positive functions on $\mathbb{R}^n$ such that $w_i^{p_i}$ are weights. Then the inequality*

$$\left(\int\limits_{\mathbb{R}^n} \Big(|\mathcal{J}_\alpha(\overrightarrow{f})(x)| \prod_{i=1}^m w_i(x) \Big)^q dx \right)^{1/q} \le C \prod_{i=1}^m \left(\int\limits_{\mathbb{R}^n} \big(|f_i(y_i)| w_i \big)^{p_i} dy_i \right)^{1/p_i},$$

holds, where

$$\mathcal{J}_\alpha(\overrightarrow{f})(x) = \int\limits_{(\mathbb{R}^n)^m} \frac{f_1(y_1) \cdots f_m(y_m)}{(|x-y_1| + \dots + |x-y_m|)^{mn-\alpha}} \, d\overrightarrow{y},$$

holds, if and only if $\overrightarrow{w} \in A_{\overrightarrow{p},q}(\mathbb{R}^n)$.

The next statement characterizes those weights v on $\mathbb{R}^n$ for which the $\mathcal{I}_\alpha : \prod_{j=1}^m L^{p_j}(\mathbb{R}^n) \to L^q_v(\mathbb{R}^n)$ holds, where $p < q < \infty$.

Theorem B ([18]). *Let $1 < p_i < \infty$ for each $1 \le i \le m$. Let $p < q$. Then $\mathcal{J}_\alpha$ is bounded from $\prod_{j=1}^m L^{p_j}(\mathbb{R}^n)$ to $L^q_v(\mathbb{R}^n)$ if and only if*

$$\sup_Q \left(\int\limits_Q v(x)(x) \, dx \right)^{1/q} |Q|^{\alpha - n/p} < \infty$$

is satisfied, where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

This statement remains valid if we replace $\mathbb{R}^n$ by an interval in $\mathbb{R}$, and $\mathcal{J}_\alpha$ by potential operator on an interval:

$$(J_\alpha f)(x) = \int\limits_0^1 \frac{f(t)}{|x-t|^{1-\alpha}} \, dt, \quad 0 < \alpha < 1, \quad x \in [0,1]. \tag{1}$$

Let $1 < p < \infty$, $0 < \alpha < 1/p$ and q be the Hardy–Littlewood–Sobolev exponent, i.e., $q = \frac{p}{1-\alpha p}$. It is known (see [25]) that the operator J_α and, consequently, appropriate fractional maximal operator

$$(M_\alpha f)(x) = \sup_{I \ni x} \frac{1}{|I|^{1-\alpha}} \int\limits_I |f(t)| \, dt, \quad 0 < \alpha < 1, \quad x \in [0,1]. \tag{2}$$

is bounded from $L^{p),\theta_1}([0,1])$ to $L^{q),\theta_2}([0,1])$ if $\theta_2 \geq \frac{q\theta_1}{p}$. However, this boundedness fails if $\theta_2 < \frac{q\theta_1}{p}$. Moreover, it was shown that the one-weight inequality

$$\big\|T_\alpha(fw^\alpha)\big\|_{L_w^{q),\theta q/p}([0,1])} \leq C\|f\|_{L_w^{p),\theta}([0,1])},$$

where T_α is J_α or M_α, $1 < p < \frac{1}{\alpha}$, $q = \frac{p}{1-\alpha p}$, $\theta > 0$, holds if and only if $w \in A_{1+q/p'}([0,1])$.

The next statement gives D. Adams type (see [1]) trace inequality characterization for the fractional integrals and corresponding fractional maximal functions defined by

$$(\mathcal{T}_\alpha f)(x) = \int\limits_X \frac{f(y)}{\mu(B_{xy})^{1-\alpha}}\,d\mu(y), \quad x \in X, \quad 0 < \alpha < 1,$$

$$(\mathcal{M}_\alpha f)(x) = \sup_{B \ni x} \frac{1}{\mu(B)^{1-\alpha}} \int\limits_B |f(y)|\,d\mu(y), \quad 0 < \alpha < 1,$$

where the supremum is taken over all balls $B \subset X$ containing x.

Theorem C ([21]). *Let $1 < p < q < \infty$ and let $0 < \alpha < 1/p$. Suppose that (X, d, μ) is an SHT and ν is an another finite measure on X. Let $\theta > 0$. Then the following conditions are equivalent*:

(i) *the operator $\mathcal{T}_\alpha$ is bounded from $L^{p),\theta}(X,\mu)$ to $L^{q),q\theta/p}(X,\nu)$;*

(ii) *the operator $\mathcal{M}_\alpha$ is bounded from $L^{p),\theta}(X,\mu)$ to $L^{q),q\theta/p}(X,\nu)$;*

(iii) *there is a positive constant C such that for all balls B in X the inequality*

$$\nu(B) \leq C\big(\mu(B)\big)^{A_{p,q,\alpha}} \tag{3}$$

holds, where

$$A_{p,q,\alpha} := q\Big(\frac{1}{p} - \alpha\Big). \tag{4}$$

1. The Main Results

In this section we the main results.

1.1. **Unboundedness of Multilinear Fractional Integrals.** Let (X, d, μ) be an SHT and let

$$\mathcal{M}_\alpha(\overrightarrow{f})(x) = \sup_{B \ni x} \prod_{i=1}^m \frac{1}{\mu(B)^{1-\alpha/m}} \int\limits_B |f_i(y_i)|\,d\mu(y_i), \quad 0 \leq \alpha < m;$$

$$\mathcal{I}_\alpha(\overrightarrow{f})(x) = \int\limits_{X^m} \frac{f_1(y_1)\cdots f_m(y_m)}{\big(B_{xy_1} + \cdots + B_{xy_m}\big)^{m-\alpha}}\,d\mu(\overrightarrow{y}),$$

defined, generally speaking, on an STH in the classical Lebesgue spaces.

The following statement shows the range of the second parameter for which the boundedness of the operator $\mathcal{M}_\alpha$ (resp. $\mathcal{I}_\alpha$) from the product space to grand Lebesgue space fails (for linear fractional integrals on an interval see [25] and linear potentials on an SHT we refer to [21]).

Proposition 1. *Let (X,d,μ) be an SHT with $\mu(X)<\infty$. Suppose that $1<p_j<\infty$, $\frac{1}{p}=\sum_{j=1}^m \frac{1}{p_j}$ and $1<p<q<\infty$. Let*

$$\liminf_{\mu(B)\to 0} \nu(B)\mu(B)^{A_{p,q,\alpha}} \neq 0,$$

where $A_{p,q,\alpha}$ is defined by (4). If $0<\theta_2<\frac{\theta_1 q}{p}$, then the operator $\mathcal{N}_\alpha$, where $\mathcal{N}_\alpha$ is either $\mathcal{M}_\alpha$ or $\mathcal{I}_\alpha$, is not bounded from $\prod_{j=1}^m \mathcal{L}_{L^{p_j),\theta_1}}(X,\mu)$ to $L^{q),\theta_2}(X,\nu)$.

Corollary 1. *Let (X,d,μ) be an SHT. Suppose that $0<\alpha<1$, $1<p_j<\infty$ for each $1\le j\le m$, $\frac{1}{p}=\sum_{j=1}\frac{1}{p_j}$ and $1/m<p<1/\alpha$. We set $q=\frac{p}{1-\alpha p}$. Suppose that $0<\theta_2<\frac{\theta_1 q}{p}$. Then the operator $\mathcal{N}_\alpha$ is not bounded from $\prod_{j=1}^m \mathcal{L}^{p_j),\theta_1}(X,\mu)$ to $L^{q),\theta_2}(X,\mu)$, where $\mathcal{N}_\alpha$ is $\mathcal{M}_\alpha$ or $\mathcal{I}_\alpha$.*

Let (X,d,μ) be an SHT. To formulate the next statement we need to introduce the class $M_{\overrightarrow{p},q}(X,\nu,\mu_1,\dots,\mu_m)$ ($p_j,q>1$, $1\le j\le m$) of $m+1$-tuple of finite measures $(\nu,\mu_1,\dots,\mu_m)$ defined on X.

Definition C. Let (X,d,μ) be an SHT and let $\mu_1,\dots,\mu_m,\nu$ be measures on X. A multilinear operator T belongs to the class $M_{\overrightarrow{p},q}(X,\mu_1,\dots,\mu_m,\nu)$ if T is bounded from $\prod_{j=1}^m L^{p_j}(X,\mu_j)$ to $L^q(X,\nu)$.

If $d\mu_j=w_j d\mu$ for every $1\le j\le m$, $d\nu=v d\mu$ for some weight functions $w_1,\dots,w_m,v$ then we denote $M_{\overrightarrow{p},q}(X,\mu_1,\dots,\mu_m,\nu)$ by $M_{\overrightarrow{p},q}(X,w_1,\dots,w_m,v)$.

Let $1<q<\infty$, $\varepsilon_0\in(0,q-1)$ and $\eta_0\in(0,a)$, where a is sufficiently small positive number. Ne denote

$$g(x):=\frac{q\varepsilon_0(p-\eta_0)x}{\eta_0(q-\varepsilon_0)(p-x)+x\varepsilon_0(p-x)}, \tag{5}$$

$$\Psi(x):=\Phi(g(x))^{\frac{p-x}{q-g(x)}}, \tag{6}$$

with $\Phi\in R(0,\sigma)$, where $R(0,\sigma)$ is the class of those increasing functions ϕ an interval $(0,\sigma)$, with small positive σ, such that $\lim_{x\to 0}\phi(x)=0$.

Theorem 1. *Let (X,d,μ) be an SHT. Let $1<p_j<\infty$ for each $1\le j\le m$. Let $1<q<\infty$. We set $\frac{1}{p}=\sum_{j=1}^m\frac{1}{p_j}$. Let $\Psi\in R(0,\sigma)$ and Ψ is defined by (6). Suppose that a multilinear operator T satisfies the condition $T\in M_{\overrightarrow{p},q}(X,\mu_1,\dots,\mu_m,\nu)\cap M_{\overrightarrow{p}/r,q/s}(X,\mu_1,\dots,\mu_m,\nu)$ for some $r,s>1$. Then T is bounded from $\prod_{j=1}^m \mathcal{L}^{p_j),\Psi}(X,\mu_j)$ to $L^{q),\Phi}(X,\nu)$.*

Now we reformulate the one-weight result for $\mathcal{I}_\alpha$ and $\mathcal{M}_\alpha$, where $X = [0,1]$ and $d\mu = dx$ is the Lebesgue measure (cf. Theorem A).

Let w_j are weights on $[0,1]$ for $1 \le j \le m$. In what follows we assume that

$$\widetilde{\mathcal{N}}_{\alpha,\overrightarrow{w}}\overrightarrow{f} := \mathcal{N}_\alpha(f_1 w_1^{\alpha_1}, \dots, f_m w_m^{\alpha_m}),$$

where $\mathcal{N}_\alpha$ is $\mathcal{I}_\alpha$ or $\mathcal{M}_\alpha$. We put $\alpha_j = \frac{1}{p_j} - \frac{1}{q_j}$ for each $1 \le j \le m$ and

$$\alpha = \frac{1}{p} - \frac{1}{q}, \quad \frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}, \quad \frac{1}{q} = \sum_{j=1}^m \frac{1}{q_j}. \tag{7}$$

Taking the version of Theorem B for bounded interval into account we find that the next statement holds:

Proposition B. *Let $1 < p_j < \infty$ for each $1 \le j \le m$, and $\frac{1}{m} < p < \frac{1}{\alpha}$, where $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$. We set $q = \frac{p}{1-\alpha p}$. Let for weight functions w_j, $1 \le j \le m$,*

$$\widetilde{\nu}_{\overrightarrow{w}} := \prod_{j=1}^m w_j^{q/q_j}.$$

Then the inequality

$$\|\widetilde{\mathcal{N}}_{\alpha,\overrightarrow{w}}\overrightarrow{f}\|_{L^q_{\widetilde{\nu}_{\overrightarrow{w}}}([0,1])} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}([0,1])}$$

holds if and only if $\overrightarrow{w} \in A_{\overrightarrow{l(p,q)}}([0,1])$, where $\overrightarrow{l(p,q)} := (1 + q_1/p_1', \dots, 1 + q_m/p_m')$, i.e.

$$\sup_I \left(\frac{1}{|I|}\int_I \widetilde{\nu}_{\overrightarrow{w}}(x)\,dx\right)^{1/q} \prod_{j=1}^m \left(\frac{1}{|I|}\int_I w_j^{-p_j'/q}(x)\,dx\right)^{1/p_j'} < \infty,$$

where the supremum is taken over all subintervals I of $[0,1]$.

Theorem 2. *Let $1/m < p < \infty$, $p_i = mp$ for each $1 \le i \le m$. We set $q = \frac{p}{1-\alpha p}$. Let $\frac{1}{q_j} = \frac{1}{p_j} - \frac{\alpha}{m} \ge 0$. Suppose that $\theta > 0$. Then the condition $\overrightarrow{w} \in A_{\overrightarrow{l(p,q)}}([0,1])$, where $\overrightarrow{l(p,q)} := (1 + q_1/p_1', \dots, 1 + q_m/p_m')$ guarantees the following one-weight inequality*

$$\big\|\mathcal{N}_\alpha(f_1 w_j^{\alpha_j}, \dots, f_m w_m^{\alpha_m})\big\|_{L^{q),\theta q/p}_{\widetilde{\nu}_{\overrightarrow{w}}}([0,1])} \le C \|\overrightarrow{f}\|_{\prod_{j=1}^m \mathcal{L}^{p_j),\theta}_{w_j}([0,1])}, \tag{8}$$

where $\mathcal{N}_\alpha$ is $\mathcal{I}_\alpha$ or $\mathcal{M}_\alpha$, and α_j are defined by (7), $j = 1, \dots, m$.

Now we formulate another type of one-weight inequality which is new even in the linear ($m = 1$) case.

Theorem 3. *Let* $1/m < p < \infty$, $p_j = mp$ *for each* $1 \le i \le m$. *We set* $q = \frac{p}{1-\alpha p}$. *Let* $\frac{1}{q_j} = \frac{1}{p_j} - \frac{\alpha}{m} > 0$. *Suppose that* $\theta > 0$. *Let for weight functions* w_j, $1 \le j \le m$,

$$\widetilde{\nu}_{\vec{w}} := \prod_{j=1}^{m} w_j^{q/q_j}.$$

Then the condition $\overrightarrow{w} \in A_{\vec{p},q}([0,1])$ *implies the one-weight inequality*

$$\big\|(\mathcal{N}_\alpha \overrightarrow{f})\widetilde{\nu}_{\vec{w}}\big\|_{L^{q),\theta q/p}([0,1])} \le C \prod_{j=1}^{m} \big\|f_j w_j\big\|_{L^{p_j),\theta}([0,1])},$$

where $\mathcal{N}_\alpha$ *is* $\mathcal{I}_\alpha$ *or* $\mathcal{M}_\alpha$ *and the positive constant* C *is independent of* f_j, $1 \le j \le m$.

In the linear case the latter statement is formulated as follows:

Corollary 2. *Let* $m = 1$ *and let* $1 < p < \infty$. *We set* $q = \frac{p}{1-\alpha p}$. *Suppose that* $\theta > 0$. *Let* J_α *be the fractional integral operator defined by* (1). *If the condition* $w \in A_{\vec{p},q}([0,1])$ *is satisfied, then the following one-weight inequality holds*

$$\big\|(J_\alpha f)w\big\|_{L^{q),\theta q/p}([0,1])} \le C\big\|fw\big\|_{L^{p),\theta}([0,1])}$$

with the positive constant C *independent of* f.

1.2. **Trace type inequality.** Now we give necessary and sufficient condition governing the boundedness of $\mathcal{N}_\alpha$ from $\prod_{j=1}^{m} \mathcal{L}^{p_j),\theta}([0,1])$ to $L^{q),\theta q/p}([0,1],\nu)$, where T_α is J_α or M_α and ν is another measure on $[0,1]$. Here J_α or M_α are defined by (1), (2) respectively.

Theorem 4. *Let* $1 < p_j < \infty$ *for every* $1 \le j \le m$ *and let* $\theta > 0$. *Let* $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$. *Suppose that* $0 < \alpha < \frac{1}{p}$ *and* $p < q < \infty$. *Then the following conditions are equivalent*:

(i) the operator J_α is bounded from $\prod_{j=1}^{m} \mathcal{L}^{p_j),\theta}([0,1])$ to $L_v^{q),\theta q/p}([0,1])$;

(ii) the operator J_α is bounded from $\prod_{j=1}^{m} \mathcal{L}^{p_j),\theta}([0,1])$ to $L_v^{q),\theta q/p}([0,1])$;

(iii) there is a positive constant C such that

$$v(I) \le C|I|^{A_{\alpha,p,q}}. \tag{9}$$

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The editors express their regret regarding the following matter.

The paper "Strong Homology Groups of Continuous Map" by A. Beridze was published in "Proceedings of A. Razmadze Mathematical Institute" (PRMI) **162** (2013). Later on, we were informed that essentially the same paper in Russian was published earlier in "Sovr. Mat. Probl." **84** (2012), Part 3; its English version has appeared in "J. Math. Sci." (N.Y.), **197** (2014), No. 6. The reviewers of PRMI claimed that they had no access to the Russian Journal "Sovr. Mat. Probl."

On the other hand, at the same time the author submitted his paper to two different journals.

We confirm that the aforementioned fact does not meet publication ethics.