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ON FIBER STRONG SHAPE THEORY

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Contents

Introduction

Shape Theory is an important and rich branch of Geometric Topology. Its methods can be successfully applied to the study of problems of Topology as well as other branches of Mathematics. Shape theory is meaningful extension of homotopy theory of spaces having homotopy type of ANR-spaces, polyhedras and simplicial complexes to the categories of more general spaces.

The shape theories that satisfy the main results of classical shape theory play essential role in modern topology. Their quantity and importance are systematically growing in the process of the research of the various problems of topology (Homology theory, Homotopy theory, Retracts theory, Shape Theory, Dynamical Systems, C*-algebra and others).

At the begining shape theory was constructed by K.Borsuk ([Bo_2]- [Bo_4]]) for the category of compact metric spaces. S.Mardešić and J.Segal extended Borsuk's shape theory to the category of compact Hausdorff spaces ([M-S_1]- [M-S_3]]). After that R.H.Fox spread Borsuk's theory on the category of metrizable spaces [Fo]. The another generalization of shape theory was described by B.J.Ball and R.B.Sher in [Ba-Sh], where they constructed proper shape theory for the category of locally compact separable spaces and proper maps. Besides, B.J.Ball investigated the proper shape theory for the category of locally compact metrizable spaces and proper maps [Ba]. The proper shape theory for the category of locally compact paracompact spaces and proper maps was developed by V.Baladze [B_7]. Shape classifications of paracompact and p-paracompact spaces was described by A.Šostak [Š] and S. Mardešić and A.Šostak [M-Š]. Shape theory for the category of arbitrary topological spaces was developed by K.Morita [Mor] and S.Mardešić [M_1].

The categorical aspects of shape theory were studied by J.M. Cordier and T.Porter [Co-P]

The shape type extensions of uniform homotopy theory of absolute neighbourhood uniform retracts, equivariant homotopy theory of equivariant absolute neighbourhood retracts and n-homotopy theory of absolute neighbourhood retracts were constructed and investigated by several authors.

Uniform shape theory for the category of uniform spaces introduced by Agaronian and Smirnov [A-S], V.Baladze ([B_8], [B_9], [B_{11}]), V.Baladze and L. Turmanidze ([B-Tu_1], [B-Tu_2]), D. Doičinov ([Do_1]- [Do_3]), Nguen Anh Kiet [Ki], T.Miyata ([Mi_1]- [Mi_2]), T. Miyata and J. Segal [Mi-S], T.Miyata and T. Watanabe [Mi-W], Nguen To Nhu [Nh].

The origins of equivariant shape theory of spaces with action topological group can be traced back to papers by S. A. Antonyan, R. Jimenez and S. de Neymet [An-J-N],S.A. Antonian and S.Mardešić [An-M], Z. Čerin [$Č_3$], P.S. Gevorgian ([G_1]- [G_3]) and Yu.M.Smirnov ([Sm_1]- [Sm_3]). In the solution of problems of equivariant shape theory important role plaid the methods and results of papers ([An_1]- [An_4], [An-J-N], [An-M]).

The n-shape theory was constructed by A.Chigogidze ([Ch_1], [Ch_2]) for the category of compact metric spaces. His results have been expanded on the category locally compact separable spaces and proper maps by Y. Akaike ([Ak_1], [Ak_2]), Y.Akaike and K.Sakai [Ak-Sa] and K. Sakai [Sa]. The n-shape theory for the category of arbitrary compact Hausdorff spaces was investigated by R. Jimenez and L.R.Rubin [Ji-R].

There are several approaches to the fiber shape theory for spaces over a fixed space B and continuous maps. The fiber shape theory is extension of fiber of homotopy theory of ANR_{B_0}-spaces ([Dol_2], [Y_2]) and ANR-maps ([U], [N-S]).

Fiber shape theory for the category of compact metric spaces over fixed space B_0 and fiberpreserving maps where introduced by H.Kato ([K_1]- [K_4]) and M.Clap and L.Montejano [Cl-Mo]. In papers ([Y_1]- [Y_4]) T. Yagasaki considered and investigated fiber shape theory of category metric spaces over B_0 and fiber preserving maps.Fiber shape theories for arbitrary spaces over B_0, maps of metric spaces and maps of topological spaces developed in papers V.Baladze ([B_2]- [B_6], [B_{11}]), Z.Čerin [$Č_2$] and D. A. Edwards and P. T. Mc. Auley [E-A].

Together with the classical shape theory and its variants there exists an important branch of modern geometric topology, so called strong shape theory, which besides the applications in topology (general topology, algebraic topology, geometric topology) ([M_3],[Md]), has also applications in other branches of mathematics (dynamical systems, C^*-algebras)([H], [D]).

Strong shape theory for different categories of spaces was investigated by several authors. For the category of compact metric spaces equivalent strong shape theories were introduced by F.W.Bauer [Bau], A. Calder and H.M.Hastings [Ca-H], F.W.Cathey [C_1], J.Dydak and J.Segal [Dy-S], D.A.Edwards and H.M.Hastings [E-H], Y.Kodama and J.Ono [Ko-O], Yu.T.Lisica [L_4] and J.B.Quigly [Q].

Strong shape theory for the category of general topological spaces and arbitrary categories was constructed by M. Batanin [Bat], F.W. Bauer [Bau], J.Dydak and S.Nowak ([Dy-N_1], [Dy-N_2]), Yu.T.Lisica [L_3], Yu.T.Lisica and S.Mardešić [L-M], Z.Miminoshvili [Mim] and L. Stramaccia [St]. In the papers are solved several serious problems of Topology [M_3].

For the present period of the shape theory development it is characteristic to design and research different versions of strong shape theory.

Strong shape theory based on the notion of equivariant homotopy constructed by V.Baladze [B_1] for metric G-spaces and A.Bykov and M.Texis for compact metric G-spaces [By-Te_2].

Strong shape theory based on the notion of n-homotopy was developed by Y.Iwamoto and K.Sakai [I-Sa].

Fiberwise topology is a new direction of topology developed on the basis of General Topology, Algebraic Topology and Geometric Topology. Fiberwise topology occupies a central place in topology today. It's methods were played important role in the solutions some problems of Differential Geometry, Lie Groups and Dynamical Systems, so establishment of new properties and characteristics of fiber spaces has more important significance.

The aspects of Algebraic topology and Homotopy topology for fiberwise topology were studied by James [J_2] and James and Crabb [Cr-J]. The investigation of fiberwise topology in the view of general topology was developed by F.Cammaroto, B.Pasymkov, D.Buhagiar and T.Miwa ([Bu-Miw-Pa], [Ca-Pa]).

The problem of construction of strong shape theory for fiberwise topology is one of interesting problems. As the strong shape theory arises from homotopy theory, so fiber strong shape theory arises from fiberwise homotopy theory. To develop of the fiber strong shape theory is natural. It is hoped that this may stimulate further research of fiberwise topology, in particular, of fiberwise homotopy theory.

The main goal of the dissertation work is to develop the strong shape theory of fiber topology, the investigation of the problem of construction of fiber strong shape classification for compact metric spaces and general topological spaces, the search of necessary and sufficient conditions the fulfillment of which will imply that the shape morphisms will be fiber strong shape equivalences. The aim of the thesis is also to transfer main focus on the geometric interpretation of fiber strong shape theory.

We begin with a short description of results of the thesis by chapters. The dissertation work consists of Introduction, three Chapters and Bibliography.

Introduction shortly describes the history of strong shape theory and the results obtained in dissertation work. Chapter 1 provides a survey of fiberwise topology, begining with basis theory and proceeding to a selection and specialized topics of fiberwise homotopy theory and fiberwise retracts theory. Chapter1 also include the fiber Borsuk's pairs, strong shape deformation maps and fibrant spaces. In chapter 2 are defined fiber cotelescopes, constructed fiber strong shape category and established the characterizations of fiber strong shape equivalences of compact metrizable spaces. In Chapter 3 studied fiber strong $\mathrm{ANR}_{\mathrm{B}_0}$-expansion and investigated the fiber strong shape theory for arbitrary spaces. At the end of dissertation is given the bibliography of used references.

Now we give a short survey of obtained results.

In the section 1.1 of Chapter 1 are obtained some results concerning the characterizations of Borsuk's pairs over B_0 used in other sections of work.

The properties of fiberwise Borsuk's pairs are described in the following propositions.

Theorem 1.1.1. *A map* $i : (A, \pi_A) \to (X, \pi_X)$ *over* B_0 *is a cofibration over* B_0 *if and only if the map* $j : (\mathrm{Cyl}(i), \pi_{\mathrm{Cyl}(i)}) \to (X \times I, \pi_{X\times I})$ *over* B_0 *is retractible.*

Corollary 1.1.2. *A closed pair* (X, A) *of space* X *over* B_0 *and its closed subspace* A *is a Borsuk pair over* B_0 *if and only if the subspace* $(X \times \{0\}) \cup (A \times I) \subset X \times I$ *is a retract over* B_0 *of* $X \times I$.

Corollary 1.1.3. *For each closed Borsuk's pair* (X, A) *over* B_0 *and for every space* Y *over* B_0 *the pair* $(X \times Y, A \times Y)$ *over* B_0 *is a closed Borsuk's pair over* B_0.

Corollary 1.1.4. *If* (X, A) *is a Borsuk's pair over* B_0 *and* A *is a closed subspace of locally compact Hausdorff space* X *then for each space* Y *over* B_0 *the map* $i^* : Y^X \to Y^A$ *is a cofibration over* B_0.

Theorem 1.1.5. *A pair* (X, A) *of space* (X, π_X) *over* B_0 *and its closed subspace* $(A, \pi_{X|A})$ *is a Borsuk pair over* B_0 *if and only if there exist a map* $\psi : X \to I$ *and a fiber homotopy* $G : (X \times I, \pi_{X\times I}) \to (X, \pi_X)$ *with respect* A *such that* $A = \psi^{-1}(0)$, $G(x, 0) = x$ *and* $G(x, t) \in A$ *when* $\psi(x) < t$.

Theorem 1.1.6. *Let* (X, A) *be a Borsuk pair over* B_0. *Then* $(X \times I, (X \times \{0\}) \cup (A \times I) \cup X \times \{1\})$ *is the Borsuk pair over* B_0.

Theorem 1.1.7. *Let* (X, A) *be a Borsuk pair over* B_0. *Then each deformation retraction* $r : (X, \pi_X) \to (A, \pi_{X|A})$ *over* B_0 *is a strong deformation retraction over* B_0.

Theorem 1.1.8. *A closed pair* (X, A) *of spaces over* B_0 *is a Borsuk pair over* B_0 *if and only if* $\tilde{A} = (X \times \{0\}) \cup (A \times I)$ *is a strong deformation retract over* B_0 *of* $(X \times I, \pi_{X\times I})$.

Corollary 1.1.9. *Let* (X, A) *be a closed Borsuk pair over* B_0. *Then the subspace* (A, π_A) *is a strong deformation retraction over* B_0 *of* (X, π_X) *if and only if the inclusion* $i : (A, \pi_A) \to (X, \pi_X)$ *is a fiber homotopy equivalence.*

In section 1.2 of Chapter 1 are given definitions and various concepts associated to fiber SSDR-maps and fibrant spaces and established their properties.

All spaces in Section 1.2 are metrizable. Here the basic definition is the following

Definition 1.2.1. *Let* $(X, \pi_X) \in ob(\mathbf{M_{B_0}})$ *and let* A *be a closed subspace of* X. *The subspace* $(A, \pi_{X|A})$ *over* B_0 *is called a shape strong deformation retract over* B_0 *of* (X, π_X) *if there exists an embedding* $\alpha : (X, \pi_X) \hookrightarrow (Y, \pi_Y) \in \mathrm{AR}_{\mathrm{B}_0}$ *over* B_0 *satisfying the following condition:*

for any pair of neighbourhoods U *and* V *of* $\alpha(X)$ *and* $\alpha(A)$ *respectively in* (Y, π_Y), *there is a homotopy* $H : (X \times I, \pi_{X\times I}) \to (U, \pi_{Y|U})\mathrm{rel}A$ *over* B_0 *such that* $H(x,0) = \alpha(x)$ *and* $H(x,1) \in V$ *for each* $x \in X$.

This definition involves that if an embedding $\alpha : (X, \pi_X) \to (M, \pi_M)$ over B_0 satisfies the conditions of definition 1.2.1, then these conditions hold for any closed embedding $\beta : (X, \pi_X) \to (Z, \pi_Z) \in \mathrm{AR}_{\mathrm{B}_0}$.

A closed embedding $i : (A, \pi_A) \to (X, \pi_X)$ over B_0 is called $\mathrm{SSDR}_{\mathrm{B}_0}$-map if i embeds (A, π_A) in (X, π_X) as a shape strong deformation retract over B_0 of (X, π_X).

The notion of $\mathrm{SSDR}_{\mathrm{B}_0}$-map generalizes the notion of $\mathrm{SDR}_{\mathrm{B}_0}$-map.

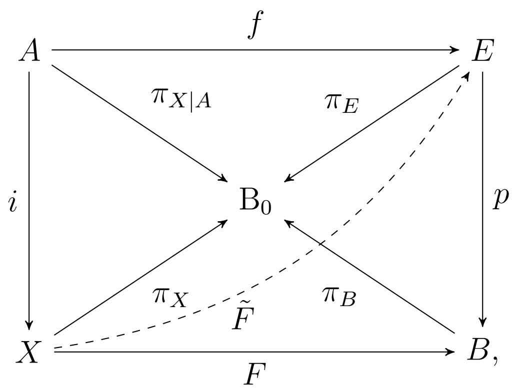
One of main results of section 2.1 of Chapter 1 is the following

Theorem 1.2.2. *Let* $(X, \pi_X) \in \mathbf{M}_{\mathbf{B_0}}$ *and* A *be a closed subspace of* X. *Then the following conditions are equivalent:*

a) $i : (A, \pi_{X|A}) \hookrightarrow (X, \pi_X)$ *is an* SSDR-*map over* B_0;

b) *for any map* $f : (A, \pi_{X|A}) \to (Y, \pi_Y) \in \mathrm{ANR}_{\mathrm{B}_0}$ *over* B_0, *there is an extension* $\tilde{f} : (X, \pi_X) \to (Y, \pi_Y)$ *over* B_0 *such that* $\tilde{f} \cdot i = f$ *and any two such extensions over* B_0 *are fiber homotopic with respect* iA;

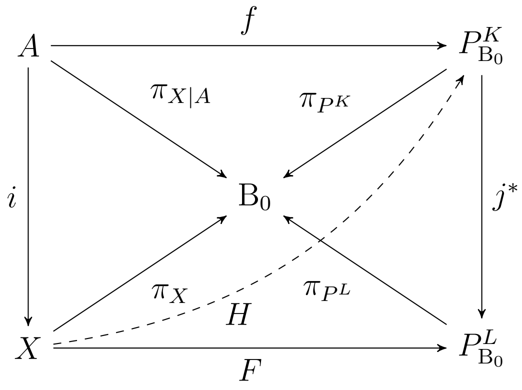
c) *for any commutative diagram*



where $p : (E, \pi_E) \to (B, \pi_B)$ *is a fibration over* B_0 *and* (E, π_E) *and* (B, π_B) *are* $\mathrm{ANR}_{\mathrm{B}_0}$*-spaces, there exists a map* $\tilde{F} : (X, \pi_X) \to (E, \pi_E)$ *over* B_0 *such that* $\tilde{F} \cdot i = f$

and $p \cdot \tilde{F} = F$.

d) *for any commutative diagram of maps over* B_0

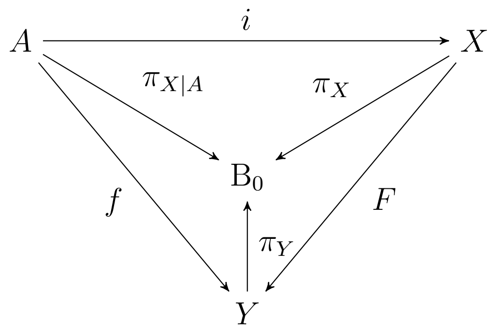


there exists a filler $H : (X, \pi_X) \to (P^K, \pi_{P^K})$ *over* B_0 *provided* $P \in \mathrm{ANR}_{\mathrm{B}_0}$ *and* L *is a subcomplex of a finite* CW-*complex* K *with an inclusion map* $j : L \hookrightarrow K$.

This result is plaining assertional role in whole work.

In Chapter 1 also introduced definition and investigation of fibrant spaces over B_0.

Definition 1.2.3. *A space* (Y, π_Y) *over* B_0 *is called a fibrant space over* B_0 *if for every* SSDR-*map* $i : (A, \pi_{X|A}) \to (X, \pi_X)$ *over* B_0 *and every map* $f : (A, \pi_{X|A}) \to (Y, \pi_Y)$ *over* B_0, *there is a map* $F : (X, \pi_X) \to (Y, \pi_Y)$ *over* B_0 *such that* $F \cdot i = f$, *i.e. the following diagram commutes*:



The class of fibrant spaces over B_0 is sufficiently large. It contains the class of absolute neighbourhood retracts over B_0 (Theorem 1.2.4).

Apart from this result here is proved that, if (Y, π_Y) is a fibrant space over B_0 and K is a compact metric space, then $(Y^K_{\mathrm{B}_0}, \pi_{Y^K_{\mathrm{B}_0}})$ also is a fibrant space over B_0(Theorem 1.2.3).

The result of Chapter 1 are summarized in the propositions which systematically are used in next parts of work.

Theorem 1.2.6. *Let* $\mathbf{Y} = ((Y_n, \pi_{Y_n}), p_{n,n+1}, N^+)$ *be an inverse system of fibrant spaces over* B_0 *and fibrations over* B_0. *Then the fiber limit space* $Y = \varprojlim \mathbf{Y}$ *is a fibrant space over* B_0 *and the natural projections* $p_n : (Y, \pi_Y) \to (Y_n, \pi_{Y_n})$ *are fibrations over* B_0.

Theorem 1.2.7. *Let* $f : (X, \pi_X) \to (Y, \pi_Y)$ *be a map over* B_0. *If* $(X, \pi_X), (Y, \pi_Y) \in \mathrm{ANR}_{\mathrm{B}_0}$, *then* $\mathrm{coCyl}_{\mathrm{B}_0}(f) \in \mathrm{ANR}_{\mathrm{B}_0}$.

Theorem 1.2.8. *Let* $f : (X, \pi_X) \to (Y, \pi_Y)$ *be a map over* B_0 *of fibrant spaces over* B_0. *Then the* $\mathrm{coCyl}_{\mathrm{B}_0}(f)$ *over* B_0 *is a fibrant space over* B_0.

The section 2.1 of Chapter 2 are begined to study of fiber cotelescope $\mathrm{coTel}(\mathbf{X})$ of inverse sequence $\mathbf{X} = \{(X_n, \pi_{X_n}, q_n^{n+1}, N^+)$ over B_0.

The detailed descriptions of constructions given here allows us to prove the following main

Theorem 2.1.1. *Let* $\mathbf{X} = ((X_n, \pi_{X_n}), q_n^{n+1}, N^+)$ *be an inverse sequence consisting of fibrant spaces over* B_0 *and maps over* B_0. *Then the cotelescope* $\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X})$ *is a fibrant space over* B_0. *If all* (X_n, π_{X_n}) *members of the inverse system* $\mathbf{X}$ *are* $\mathrm{ANR}_{\mathrm{B}_0}$-*spaces, then* $\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X})$ *is a fibrant space over* B_0 *too.*

There exists the unique natural embedding $i_{\mathbf{q}} : (X, \pi_X) \to (\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X}), \pi_{\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X})})$ over B_0 such that $\tilde{q}_n \cdot i_{\mathbf{q}} = i_n \cdot q_n$ for each $n \geq 0$.

In order to define the fiber strong shape classification in dissertation work are offered the notion of fiber resolution, which is a special case of the definition of resolution over B_0 given in [B_4] by V.Baladze.

Definition 2.1.2. *An inverse sequence* $\mathbf{X} = ((X_n, \pi_{X_n}), q_n^{n+1}, N^+)$ *is called resolution over* B_0 *of compact space* (X, π_X) *over* B_0 *if*

a)$(X, \pi_X) = \varprojlim \mathbf{X}$;

b)*the family* $\mathbf{q} = \{q_n : (X, \pi_X) \to (X_n, \pi_{X_n})\}_{n \in N^+}$ *satisfies the following condition: for each* $n \in N^+$ *and open neighbourhood* U *of* $q_n(X)$ *in* (X,π_{X_n}) *there exists* $m \geq n$ *such that* $q_n^m(X_m) \subseteq U$.

If all the $(X_n, \pi_{X_n}) \in \mathrm{ANR}_{B_0}$, *then* $\mathbf{q}$ *is called an* ANR_{B_0}*-resolution over* B_0.

One of the crucial point of the methods developed in Chapter 2 is the theorem of existence of fiber resolution.

Theorem. 2.1.3. *For each compact metrizable space* (X, π_X) *over* B_0 *there exists an* ANR_{B_0}*-resolution* $\mathbf{q} : (X, \pi_X) \to \mathbf{X}$ *over* B_0.

From this results follows that for every fiber resolution of compact metric space (X, π_X) over B_0 there corresponds an fiber fibrant estension of (X, π_X), namely the fiber cotelescope of this fiber resolution. The following result plays essential role in constructions given in work.

Theorem 2.1.4. *Let* (X, π_X) *be a compact metrizable space over* B_0. *If* $\mathbf{q} : (X, \pi_X) \to \mathbf{X} = ((X_n, \pi_{X_n}), q_n^{n+1}, N^+)$ *is a resolution over* B_0 *of* (X, π_X), *then there exists an infinite strong deformation*

$$D : \mathrm{coTel}_{B_0}(\mathbf{X}) \times [0, \infty) \to \mathrm{coTel}_{B_0}(\mathbf{X})$$

of $\mathrm{coTel}_{B_0}(\mathbf{X})$ *over* B_0 *onto* $i_{\mathbf{q}}(X)$. *In particular, the map* $i_{\mathbf{q}} : (X, \pi_X) \to \mathrm{coTel}_{B_0}(\mathbf{X})$ *is an* SSDR*-map over* B_0.

The effect of Theorem 2.1.1, Theorem 2.1.3 and Theorem 2.1.4 is given by the following result. Let $\tilde{X} = \mathrm{coTel}_{B_0}(\mathbf{X})$.

Theorem 2.1.5. *For each compact metrizable space* (X, π_X) *over* B_0 *there is a fibrant extension* $i_X : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})$ *over* B_0. *In particular, if* $\mathbf{q} : (X, \pi_X) \to \mathbf{X} = ((X_n, \pi_{X_n}), q_n^{n+1}, N^+)$ *is an* $\mathrm{ANR}_{\mathrm{B}_0}$*-resolution over* B_0, *then the embedding* $i_{\mathbf{q}}$: $(X, \pi_X) \to (\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X}), \pi_{\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X})})$ *is a fibrant extension over* B_0.

The results obtained in dissertation work yield that the fiber strong shape theory is coarser than the fiber homotopy theory, but is finer that the fiber shape theory.

The main aim of Chapter 2 is the construction of fiber strong shape theory for compact metrizable spaces over a fixed base space B_0, using the fiber versions of cotelescop and fibrant space.

The fiber strong shape category here constructed is the full image of functor reflector from the fiber homotopy category $\mathbf{H}(\mathbf{CM}_{\mathbf{B_0}})$ of compact metrizable spaces over B_0 in the fiber homotopy category $\mathbf{H}(\mathbf{F}_{\mathbf{B_0}})$ of fiber fibrant spaces.

The Theorems 2.1.1, 2.1.3, 2.1.4 and 2.1.5 and rountine diagram-choicing, as in the analogous situation in category theory, yield the following

Theorem 2.1.6. *Let* $i_X : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})$ *be a fibrant extension over* B_0 *of space* $(X, \pi_X) \in \mathbf{CM}_{\mathbf{B_0}}$. *Then the morphism* $[i_X]_{\mathrm{B}_0} : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})$ *of category* $\mathbf{H}(\mathbf{CM}_{\mathbf{B_0}})$ *is an* $\mathbf{H}(\mathbf{F}_{\mathbf{B_0}})$*-reflection.*

The family $\{i_X : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})\}_{(X,\pi_X) \in ob(\mathbf{H}(\mathbf{CM}_{\mathbf{B_0}}))}$ induces the $\mathbf{H}(\mathbf{F}_{\mathbf{B_0}})$-reflector

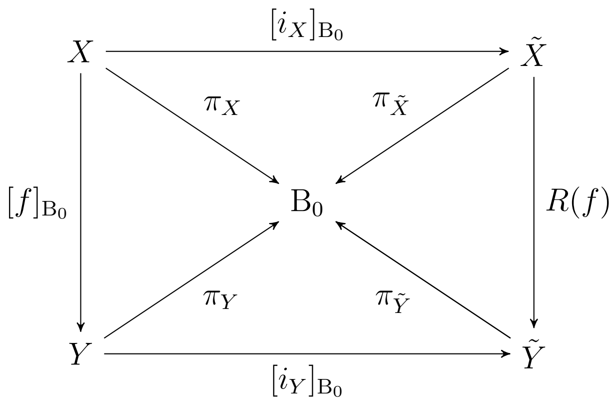
$$\mathrm{R} : \mathbf{H}(\mathbf{CM}_{\mathbf{B_0}}) \to \mathbf{H}(\mathbf{F}_{\mathbf{B_0}})$$

that is a functor given by formula

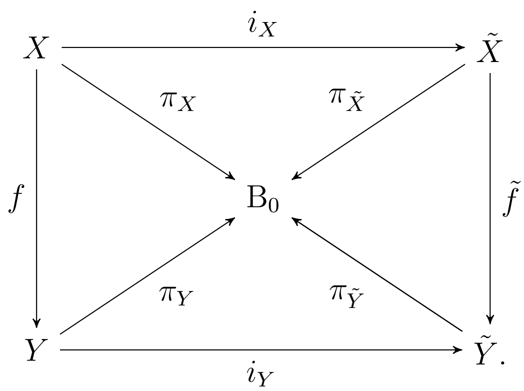
$$\mathrm{R}((X, \pi_X)) = (\tilde{X}, \pi_{\tilde{X}}), (X, \pi_X) \in ob(\mathbf{H}(\mathbf{CM}_{\mathbf{B_0}}))$$

and satisfying the condition:

for each map $f : (X, \pi_X) \to (Y, \pi_Y)$ over B_0 of compact metrizable spaces the diagram



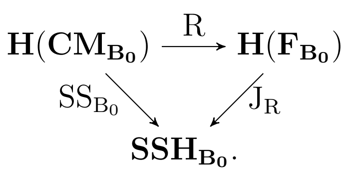
is commutative.For the map f over B_0 there exists a unique up to fiber homotopy map $\tilde{f} : (\tilde{X}, \pi_{\tilde{X}}) \to (\tilde{Y}, \pi_{\tilde{Y}})$ over B_0 such that the following diagram commutes



In this case the pair $(i_X, i_Y) : f \to \tilde{f}$ is called a fibrant extension over B_0 of map f.

Definition 2.1.7. *The fiber strong shape category* $\mathbf{SSH_{B_0}}$ *of compact metrizable spaces over* B_0 *is full image of the reflector* $R : \mathbf{H}(\mathbf{CM_{B_0}}) \to \mathbf{H}(\mathbf{F_{B_0}})$.

There is a commutative diagram



Note that, for each

$$(X,\pi_X),(Y,\pi_Y)\in ob(\mathbf{H}(\mathbf{CM_{B_0}}))$$

$$ob(\mathbf{SSH_{B_0}})=ob(\mathbf{H}(\mathbf{CM_{B_0}}))$$

$$\mathrm{Mor}_{\mathbf{SSH_{B_0}}}((X,\pi_X),(Y,\pi_Y))=[(\tilde{X},\pi_{\tilde{X}}),(\tilde{Y},\pi_{\tilde{Y}})]_{\mathrm{B}_0},$$

$$\mathrm{SS}_{\mathrm{B}_0}((X,\pi_X))=(X,\pi_X)$$

and for a fibrant extension $(i_X,i_Y):f\to\tilde{f}:(\tilde{X},\pi_{\tilde{X}})\to(\tilde{Y},\pi_{\tilde{Y}})$ over B_0 of each map $f:(X,\pi_X)\to(Y,\pi_Y)$ over B_0

$$\mathrm{SS}_{\mathrm{B}_0}([f]_{\mathrm{B}_0})=\mathrm{R}([f]_{\mathrm{B}_0})=[\tilde{f}]_{\mathrm{B}_0}.$$

According to J.Dydak and S.Novak [Dy-N$_1$] in section 2.2 of Chapter 2 defined fiber strong shape equivalence.

Definition 2.2.1. *A map $f:(X,\pi_X)\to(Y,\pi_Y)$ over B_0 is a fiber shape equivalence if for each $\mathrm{ANR}_{\mathrm{B}_0}$-space (P,π_P) induces a bijection $f^*:[Y,P]_{\mathrm{B}_0}\to[X,P]_{\mathrm{B}_0}$. A fiber shape equivalence f is called a fiber strong shape equivalence if for any two maps $g,h:(Y,\pi_Y)\to(P,\pi_P)\in\mathrm{ANR}_{\mathrm{B}_0}$ over B_0 and a fiber homotopy $H:(X\times I,\pi_{X\times I})\to(P,\pi_P)$ over B_0 joining $g\,f$ and $h\,f$, H is fiber homotopic rel $X\times\{0,1\}$ to $H'\,(f\times 1_I)$, where $H':(Y\times I,\pi_{Y\times I})\to(P,\pi_P)$ is a fiber homotopy between g and h.*

The notion of fiber double mapping cylinder is very useful and simple geometric object. It is a comfortable tool for investigation of fiber strong shape theory.

The double mapping cylinder $\mathrm{dCyl}_{\mathrm{B}_0}(f)$ over B_0 of map $f:(X,\pi_X)\to(Y,\pi_Y)$ over B_0 is the subspace $X\times I\cup\mathrm{Cyl}_{\mathrm{B}_0}(f)\times\{0,1\}$ of space $\mathrm{Cyl}_{\mathrm{B}_0}(f)\times I$ over B_0.

Using the notion of fiber double mapping cylinder are given the characterizations of fiber strong shape morphisms. Here are found necessary and sufficient conditions under which a map over B_0 is a fiber strong shape equivalence. Using the properties of fiber function spaces here are proved the following results.

One of main results is the following

Theorem 2.2.3. *Let* $f : (X, \pi_X) \to (Y, \pi_Y)$ *be a map over* B_0. *The following conditions are equivalent:*

1). f is a fiber strong shape equivalences;

2). for a given space (Z, π_Z) over B_0 containing (X, π_X) as a closed subspace over B_0, every map $g : (Z, \pi_Z) \to (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$ over B_0 extends to $(Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)})$ and every map

$$H : (Z \times I \cup \mathrm{dCyl}_{\mathrm{B}_0}(f), \pi_{Z \times I \cup \mathrm{dCyl}_{\mathrm{B}_0}(f)}) \to (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$$

over B_0 extends to $((Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)) \times I, \pi_{(Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)) \times I})$;

3). if (X, π_X) is a closed subspace of (Z, π_Z), then the fiber inclusions

$$i : (Z, \pi_Z) \to (Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)})$$

and

$$j : (Z \times I \cup \mathrm{dCyl}_{\mathrm{B}_0}(f), \pi_{Z \times I \cup \mathrm{dCyl}_{\mathrm{B}_0}(f)}) \to ((Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)) \times I, \pi_{(Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)) \times I})$$

are fiber shape equivalences;

4). if (X, π_X) is a closed subspace of (Z, π_Z), then the fiber inclusion

$$i : (Z, \pi_Z) \to (Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)})$$

is a fiber strong shape equivalence;

5). if (X, π_X) is a closed subspace of (Z, π_Z), then the fiber inclusion

$$i : (Z, \pi_Z) \to (Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)})$$

is a fiber shape equivalence;

6). the fiber inclusions

$$k : (X, \pi_X) \to (\mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(f)})$$

and

$$l : (\mathrm{dCyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{dCyl}_{\mathrm{B}_0}(f)}) \to (\mathrm{Cyl}_{\mathrm{B}_0}(f) \times I, \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(f) \times I})$$

are fiber shape equivalences;

7). every map $g : (X, \pi_X) \to (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$ over B_0 extends to $(\mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(f)})$ and every map $H : (\mathrm{dCyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{dCyl}_{\mathrm{B}_0}(f)}) \to (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$ over B_0 extends to $(\mathrm{Cyl}_{\mathrm{B}_0}(f) \times I, \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(f) \times I})$.

The consequences of theorem 2.2.3 are the following propositions.

Corollary 2.2.4. *Let (X, π_X) be a space over B_0 and $A \subset X$. The fiber inclusion $i : (A, \pi_{X|A}) \to (X, \pi_X)$ is a fiber strong shape equivalence if and only if i and $j : (X \times \{0\} \cup A \times I \cup X \times \{1\}, \pi_{X \times \{0\} \cup A \times I \cup X \times \{1\}}) \to (X \times I, \pi_{X \times I})$ are fiber shape equivalences.*

Corollary 2.2.5. *Let $f : (X, \pi_X) \to (Y, \pi_Y)$ be a fiber homotopy equivalence. Then f is a fiber strong shape equivalence.*

Corollary 2.2.6. *If $g : (X, \pi_X) \to (Y, \pi_Y)$ is fiber homotopic to a fiber strong shape equivalence $f : (X, \pi_X) \to (Y, \pi_Y)$, then g is a fiber strong shape equivalence.*

Theorem 2.2.7. *Let $f : (X, \pi_X) \to (Y, \pi_Y)$ and $g : (Y, \pi_Y) \to (Z, \pi_Z)$ be fiber* *strong shape equivalences. Then the composition* $g\,f : (X,\pi_X) \to (Z,\pi_Z)$ *is a fiber strong shape equivalence.*

Theorem 2.2.8. *Let* $f : (X,\pi_X) \to (Y,\pi_Y)$ *and* $g : (Y,\pi_Y) \to (Z,\pi_Z)$ *be maps over* B_0 *such that* $g\,f$ *is a fiber strong shape equivalence. If one of* f *and* g *is a fiber strong equivalence, then both* f *and* g *are fiber strong shape equivalences.*

Corollary 2.2.9. *Let* $f : (X,\pi_X) \to (Y,\pi_Y)$ *be a fiber shape equivalence. If* (X,π_X) *has the fiber homotopy type of an* $\mathrm{ANR}_{\mathrm{B}_0}$*, then* f *is a fiber strong shape equivalence.*

The next theorem show that in terms of fiber double cylinders it is possible to describe fiber strong shape isomorphisms of category $\mathrm{SSH}_{\mathrm{B}_0}$.

Theorem 2.2.10. *A closed fiber embedding* $i : (A,\pi_{X|A}) \to (X,\pi_X)$ *is a fiber strong shape equivalence if and only if* i *is a* SSDR*-map over* B_0.

Theorem 2.2.11. *Let* $f : (X,\pi_X) \to (Y,\pi_Y)$ *be a map over* B_0 *of compact metrizable spaces over* B_0 *and* $(i_X, i_Y) : f \to \tilde{f}$ *a fibrant extension over* B_0 *of* f*. Then* f *is a fiber strong shape equivalence if and only if* $\tilde{f}$ *is a fiber homotopy equivalence.*

Corollary 2.2.12. *A mapf over* B_0 *of compact metrizable spaces over* B_0 *is a fiber strong shape equivalence in the sense of Definition 2.2.1 if and only if* $\mathrm{SS}_{\mathrm{B}_0}([\mathrm{f}]_{\mathrm{B}_o})$ *is an isomorphism of the category* $\mathbf{SSH}_{\mathbf{B}_0}$.

In the Chapter 3 is constructed and developed a fiber strong shape theory for arbitrary spaces over fixed metrizable space B_0. The approach given here is based on the method of Mardešić-Lisica and instead of resolutions, introduced by Mardešić, their fiber preserving analogues are used. The fiber strong shape theory yields the classification of spaces over B_0 which is coarser than the classification of spaces over B_0 induced by fiber homotopy theory, but is finer than the classification of spaces over B_0 given by usual fiber shape theory.

The construction of fiber strong shape category uses the notion of fiber strong $\mathrm{ANR}_{\mathrm{B}_0}$-expansion of space over B_0. Fiber strong expansions of spaces over B_0 are morphisms of category $\mathbf{pro}-\mathbf{Top_{B_0}}$ from spaces over B_0 to inverse systems of spaces over B_0, which satisfy a stronger version of fiber homotopy conditions of $\mathrm{ANR}_{\mathrm{B}_0}$-expansion defined by V.Baladze ([B_4], [B_{10}]).

In the section 3.1 it is proved that fiber resolutions of spaces over B_0 induce fiber strong expansions of spaces over B_0. In order to construct the fiber strong shape category $\mathbf{SSH_{B_0}}$ is used this result.

The essential role in section 3.1 play the following notions and results.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathscr{A}}$ be a covering of a space (Y,π_Y) over B_0. We say that the fiber preserving maps $f,g:(X,\pi_X)\to(Y,\pi_Y)$ are $\mathcal{U}$-near, if for every $x\in X$ there exists a $U_\alpha\in\mathcal{U}$ such that, $f(x),g(x)\in U_\alpha$. We say that a fiber preserving homotopy $H:(X\times I,\pi_{X\times I})\to(Y,\pi_Y)$ which connects f and g, is a $\mathcal{U}$-homotopy if for every $x\in X$ there exists a $U_\alpha\in\mathcal{U}$ such that $H(x,t)\subseteq U_\alpha$ for all $t\in I$.

Proposition 3.1.1 (Comp. [B_5],Proposition 7) Let (Y,π_Y) be an $\mathrm{ANR}_{\mathrm{B}_0}$-space.Then every open covering $\mathcal{U}$ of (Y,π_Y) admits an open covering $\mathcal{V}$ of (Y,π_Y) such that, whenever any two f.p. maps $f,g:(X,\pi_X)\to(Y,\pi_Y)$ from an arbitrary space (X,π_X) over B_0 into the space (Y,π_Y) over B_0 are $\mathcal{V}$-near, then there exists f.p. $\mathcal{U}$-homotopy $H:(X\times I,\pi_{X\times I})\to(Y,\pi_Y)$ which connects f and g. Moreover, if for a subset $A\subseteq X$, $f_{|A}=g_{|A}$, then H is f.p. homotopy rel$A$.

Definition 3.1.4. (V.Baladze, see [B_4]- [B_6], [B_{10}]) *Let (X,π_X) be a topological space over B_0, $\mathbf{X}=((X_\alpha,\pi_{X_\alpha}),p_{\alpha\alpha'},\mathscr{A})$ an inverse system in $\mathbf{Top_{B_0}}$ and $\mathbf{p}=(p_\alpha):(X,\pi_X)\to\mathbf{X}$ a morphism of $\mathbf{pro}-\mathbf{Top_{B_0}}$. We call $\mathbf{p}$ an expansion over B_0 of the space (X,π_X) over B_0 provided it has the following properties:*

$\mathrm{E}_{\mathrm{B}_0}1$). *For every $\mathrm{ANR}_{\mathrm{B}_0}$-space (P,π_P) over B_0 and f.p. map $f:(X,\pi_X)\to(P,\pi_P)$ there is an index $\alpha\in\mathscr{A}$ and a f. p. map $h:(X_\alpha,\pi_{X_\alpha})\to(P,\pi_P)$ such that $h\,p_\alpha\underset{\mathrm{B}_0}{\simeq}f$.*

$\mathrm{E}_{\mathrm{B}_0}2$). *If $f,f':(X_\alpha,\pi_{X_\alpha})\to(P,\pi_P)$ are f. p. maps, $(P,\pi_P)\in\mathrm{ANR}_{\mathrm{B}_0}$ and $f\,p_\alpha\underset{\mathrm{B}_0}{\simeq}f'\,p_\alpha$, then there is an index $\alpha'\geq\alpha$ such that $f\,p_{\alpha\alpha'}\underset{\mathrm{B}_0}{\simeq}f'\,p_{\alpha\alpha'}$.*

Definition 3.1.5. *A morphism* $\mathbf{p} : (X, \pi_X) \to ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathscr{A})$ *is called a strong expansion over* B_0 *provided it satisfies condition* $\mathrm{E}_{\mathrm{B}_0}1)$ *and the following condition:*

$\mathrm{SE}_{\mathrm{B}_0}2)$. *Let* (P, π_P) *be an* $\mathrm{ANR}_{\mathrm{B}_0}$*-space, let* $f_0, f_1 : (X_\alpha, \pi_{X_\alpha}) \to (P, \pi_P)$, $\alpha \in \mathscr{A}$ *be f.p. maps and let* $S : (X \times I, \pi_{X\times I}) \to (P, \pi_P)$ *be a f.p. homotopy such that*

$$S(x, 0) = f_0 p_\alpha(x), \qquad x \in X,$$

and

$$S(x, 1) = f_1 p_\alpha(x), \qquad x \in X.$$

Then there exists a $\alpha^{'} \geq \alpha$ *and a f.p. homotopy* $H : (X_{\alpha'} \times I, \pi_{X_{\alpha'} \times I}) \to (P, \pi_P)$, *such that*

$$H(x, 0) = f_0 p_{\alpha\alpha'}(z), \qquad z \in X_{\alpha'},$$

$$H(x, 1) = f_1 p_{\alpha\alpha'}(z), \qquad z \in X_{\alpha'},$$

$$H(p_{\alpha'} \times 1_I) \underset{\mathrm{B}_0}{\simeq} S(\mathrm{rel}(X \times \partial I)).$$

Every strong expansion over B_0 is an expansion over B_0.

If all $(X_\alpha, \pi_{X_\alpha}) \in \mathrm{ANR}_{\mathrm{B}_0}$, then $\mathbf{p}$ is called an $\mathrm{ANR}_{\mathrm{B}_0}$-expansion and strong $\mathrm{ANR}_{\mathrm{B}_0}$-expansion, respectively.

The main results of section 3.1 is the following theorem.

Theorem 3.1.6. *Let* (X, π_X) *be a topological space over* B_0. *Then every resolution* $\mathbf{p} : (X, \pi_X) \to \mathbf{X}$ *over* B_0 *induces a strong* $\mathrm{ANR}_{\mathrm{B}_0}$*-expansion over* B_0.

Corollary 3.1.7 *Every* $\mathrm{ANR}_{\mathrm{B}_0}$*-resolution over* B_0 *induces* $\mathrm{ANR}_{\mathrm{B}_0}$*-expansion over* B_0.

Corollary 3.1.8 *Every space* (X, π_X) *over* B_0 *admits a cofinite strong* $\mathrm{ANR}_{\mathrm{B}_0}$-

expansion over B_0.

In the proof of Theorem Theorem 3.1.6 are used the following lemmas.

Lemma 3.1.9. *Let* (X, π_X) *be a topological space over metrizamble space* B_0, *let* $(P, \pi_P), (P^{'}, \pi_{P^{'}})$ *be* $\mathrm{ANR}_{\mathrm{B}_0}$*-spaces, let* $f : (X, \pi_X) \to (P^{'}, \pi_{P^{'}})$, $h_0, h_1 : (P^{'}, \pi_{P^{'}}) \to (P, \pi_P)$ *be f.p. maps and let* $S : (X \times I, \pi_{X\times I}) \to (P, \pi_P)$ *be a f.p. homotopy such that*

$$S(x, 0) = h_0 f(x), \qquad x \in X,$$

$$S(x, 1) = h_1 f(x), \qquad x \in X.$$

Then there exists an $\mathrm{ANR}_{\mathrm{B}_0}$*-space* $(P^{''}, \pi_{P^{''}})$, *f.p. maps* $f^{'} : (X, \pi_X) \to (P^{''}, \pi_{P^{''}})$, $h : (P^{''}, \pi_{P^{''}}) \to (P^{'}, \pi_{P^{'}})$ *and a f.p. homotopy* $K : (P^{''} \times I, \pi_{P^{''}\times I}) \to (P, \pi_P)$ *such that*

$$hf^{'} = f,$$

$$K(z, 0) = h_0 h(z), \qquad z \in P^{''}$$

$$K(z, 1) = h_1 h(z), \qquad z \in P^{''}$$

$$K(f^{'} \times 1_I) = S.$$

Lemma 3.1.10. *Let* $\mathbf{p} : (X, \pi_X) \to \mathbf{X}$ *be a resolution over* B_0 *and let* $\alpha, (P, \pi_P), f_0, f_1$ *and* (F, π_F) *be as in* $\mathrm{SE}_{\mathrm{B}_0}2)$. *Then for every open covering* $\mathcal{U}$ *of* (P, π_P), *there exist a* $\alpha^{'} \geq \alpha$ *and a f.p. homotopy* $H : (X_{\alpha^{'}} \times I, \pi_{X_{\alpha^{'}}\times I}) \to (P, \pi_P)$ *such that*

$$H(y, 0) = f_0\ p_{\alpha\alpha^{'}}(y), \qquad y \in X_{\alpha^{'}}$$

$$H(y, 1) = f_1\ p_{\alpha\alpha^{'}}(y), \qquad y \in X_{\alpha^{'}}$$

$$(S, H(1 \times p_{\alpha^{'}})) \leq \mathcal{U}.$$

In the section 3.2 of Chapter 3 is constructed fiber coherent prohomotopy category $\mathbf{CPHTop_{B_0}}$. The fiber coherent prohomotopy category $\mathbf{CPHTop_{B_0}}$ has as objects inverse systems $\mathbf{X} = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathscr{A})$ of topological spaces over B_0 and f.p. maps over directed cofinite index sets. The morphisms are f.p. coherent homotopy classes $[f] : \mathbf{X} \to \mathbf{Y}$ of f.p. coherent maps $f : \mathbf{X} \to \mathbf{Y}$ of such systems. Composition is defined by composing representatives, which are special f.p. coherent maps.

There exist the functors $\mathrm{C} : \mathbf{pro-Top_{B_0}} \to \mathbf{CPHTop_{B_0}}$ and $\mathrm{E} : \mathbf{CPHTop_{B_0}} \to \mathbf{pro-HTop_{B_0}}$. The composition $\mathrm{E} \circ \mathrm{C} : \mathbf{pro-Top_{B_0}} \to \mathbf{pro-HTop_{B_0}}$ is the functor induced by the f.p. homotopy functor $H : \mathbf{Top_{B_0}} \to \mathbf{HTop_{B_0}}$.

The objects of fiber strong shape category $\mathbf{SSH_{B_0}}$ are all topological spaces over B_0. The morphisms of category $\mathbf{SSH_{B_0}}$ are defined by the following way.

Let $\mathbf{p} : (X, \pi_X) \to \mathbf{X}$ and $\mathbf{q} : (Y, \pi_Y) \to \mathbf{Y}$ be an $\mathrm{ANR_{B_0}}$-resolutions of (X, π_X) and (Y, π_Y), respectively. Let $[f] : \mathbf{X} \to \mathbf{Y}$ be a some morphism of category $\mathbf{CPHTop_{B_0}}$. Let $\mathbf{p}' : (X, \pi_X) \to \mathbf{X}'$, $\mathbf{q}' : (Y, \pi_Y) \to \mathbf{Y}'$,$[f'] : \mathbf{X}' \to \mathbf{Y}'$ be another triple of fiber resolutions of spaces (X, π_X) and (Y, π_Y) over B_0 and morphism of category $\mathbf{CPHTop_{B_0}}$.

The triples $(\mathbf{p}, \mathbf{q}, [f])$ and $(\mathbf{p}', \mathbf{q}', [f'])$ are called equivalent if $[f']\ [i] = [j]\ [f]$, where $[i] : \mathbf{X} \to \mathbf{X}'$ and $[j] : \mathbf{Y} \to \mathbf{Y}'$ are isomorphisms of category $\mathbf{CPHTop_{B_0}}$.

The fiber strong shape morphisms $\mathrm{F} : (X, \pi_X) \to (Y, \pi_Y)$ are the equivalence classes of triples $(\mathbf{p}, \mathbf{q}, [f])$ with respect to the defined equivalence relation.

By symbol $\mathrm{ssh_{B_0}}((X, \pi_X))$ is denoted the equivalence class of topological space (X, π_X) and call the fiber strong shape of (X, π_X).

In the sections 3.2 are constructed a fiber strong shape functor $\mathrm{SS_{B_0}} : \mathbf{HTop_{B_0}} \to \mathbf{SSH_{B_0}}$ and a functor $\mathrm{S} : \mathbf{SSH_{B_0}} \to \mathbf{SH_{B_0}}$ into V.Baladze fiber shape category [B_4].

One of main results of this section is the following.

Theorem 3.2.5 *There exists the following commutative diagram*

$$
\begin{array}{ccc}
 & \xrightarrow{\mathrm{S}_{\mathrm{B}_0}} & \mathbf{SH_{B_0}} \\
\mathbf{HTop_{B_0}} & & \uparrow \mathrm{S} \\
 & \xrightarrow[\mathrm{SS}_{\mathrm{B}_0}]{} & \mathbf{SSH_{B_0}},
\end{array}
$$

where $\mathrm{S}_{\mathrm{B}_0}$ *is V.Baladze fiber shape functor* [B_4].

Corollary 3.2.6. *Let* (X, π_X) *and* (Y, π_Y) *be topological spaces over* B_0. *If* $\mathrm{ssh}_{\mathrm{B}_0}((X, \pi_X)) = \mathrm{ssh}_{\mathrm{B}_0}((Y, \pi_Y))$, *then* $\mathrm{sh}_{\mathrm{B}_0}((X, \pi_X)) = \mathrm{sh}_{\mathrm{B}_0}((Y, \pi_Y))$.

Chapter 1

Fiber Strong Shape Deformation Retractions and Fibrant Spaces

Chapter 1 provides a survey of fiberwise topology, begining with basis theory and proceeding to a selection and specialized topics of fiberwise homotopy theory and fiberwise retracts theory. Chapter 1 also include investigation of fiber Borsuk's pairs, strong shape deformation maps and fibrant spaces.

1.0 On Fiberwise Topological Preliminaries and Auxiliary Facts

In this section we introduce the basic notations and results which we use in the next.

Let $\mathrm{R} : \mathscr{K} \longrightarrow \mathscr{L}$ be a functor. The full image of functor R is a category **fimR** and a factorization of R

$$\begin{array}{ccc} \mathscr{K} & \xrightarrow{\ \mathrm{R}\ } & \mathscr{L} \\ & {}_{\mathrm{E}}\searrow \quad \nearrow_{\mathrm{J}} & \\ & \mathbf{fimR}, & \end{array}$$

where E is the identity on objects of category $\mathscr{K}$ and J is fully faithful.

Let $\mathscr{L}$ is a full subcategory of $\mathscr{P}$. Then an element $\tau : X \to Y$ of set $\mathrm{Mor}_{\mathscr{P}}(X, Y)$ with $Y \in ob(\mathscr{L})$ is called $\mathscr{L}$-reflection of X if the function $\tau^{\#} : \mathscr{L}(Y, L) \to \mathscr{P}(X, L)$ is bijective for each $L \in ob(\mathscr{L})$. Let $\mathscr{K} \subseteq \mathscr{P}$ be a subcategory of $\mathscr{P}$ and let $\{\tau_X : X \to \mathrm{R}X\}_{X \in ob(\mathscr{K})}$ be a family of $\mathscr{L}$-reflections, where R is a function mapping objects of $\mathscr{K}$ to objects of $\mathscr{L}$. It is clear that the function R extends to a functor $\mathrm{R} : \mathscr{K} \to \mathscr{L}$. By definition, for each element $f : A \to X$ of $\mathrm{Mor}_{\mathscr{K}}(A, X)$, $\mathrm{R}f$ is a morphism $\mathrm{R}f : \mathrm{R}A \to \mathrm{R}X$ for which $(\mathrm{R}f) \cdot \tau_A = \tau_X \cdot f$. Defined functor is called a reflection or reflector of $\mathscr{K}$ in $\mathscr{L}$.

For a given fixed object B_0 of category $\mathscr{K}$ by $\mathscr{K}_{\mathbf{B_0}}$ denote the following category. The objects of $\mathscr{K}_{\mathbf{B_0}}$ are pairs (X, π_X) consisting of object $X \in ob(\mathscr{K})$ and morphism $\pi_X : X \to B$ from $\mathrm{Mor}_{\mathscr{K}}(X, Y)$, called the projection.

The morphisms of $\mathscr{K}_{\mathbf{B_0}}$ are morphisms $f : X \to Y$ of $\mathscr{K}$ with property $\pi_X = \pi_Y \cdot f$. These morphisms are called morphisms over B_0.

We will denote by **Top**, **M** and **CM** the categories of topological spaces, metrizable spaces and compact metrizamble spaces, respectively. Consequently, for fixed objects B_0 of given categories there exist the categories $\mathbf{Top_{B_0}}$, $\mathbf{M_{B_0}}$ and $\mathbf{CM_{B_0}}$. In this categories the notion of fiber homotopy is defined.

For each object (X, π_X) of some category of spaces over B_0 the pair $(X \times Z, \pi_{X \times Z})$, where Z is a space and $\pi_{X \times Z}$ is the projection given by formula

$$\pi_{X \times Z}(x, z) = \pi_X(x), \ (x, z) \in X \times Z,$$

is the space over B_0. Note that the natural projection $p_X : X \times Z \to X$ is the map over B_0.

Let Y^Z be the function space with compact-open topology. Consider the subspace

$Y^Z_{\mathrm{B}_0}$ of the space Y^Z:

$$Y^Z_{\mathrm{B}_0} = \{f \in Y^Z : \pi_Y \cdot f = const\}.$$

Let $\pi_{Y^Z_{\mathrm{B}_0}} : Y^Z_{\mathrm{B}_0} \to \mathrm{B}_0$ be a map given by

$$\pi_{Y^Z_{\mathrm{B}_0}}(f) = \pi_Y(f(z)), z \in Z.$$

Consequently, the pair $(Y^Z_{\mathrm{B}_0}, \pi_{Y^Z_{\mathrm{B}_0}})$ is a space over B_0.

By exponential law there exists a homeomorphism map over B_0

$$E : (Y, \pi_Y)^{(X\times Z, \pi_{X\times Z})} \to (Y^Z_{\mathrm{B}_0}, \pi_{Y^Z_{\mathrm{B}_0}})^{(X,\pi_X)}$$

given by formula

$$(E(H)(x))(z) = H(x, z), H : (X \times Z, \pi_{X\times Z}) \to (Y, \pi_Y), x \in X, z \in Z.$$

Let $f, g : (X, \pi_X) \to (Y, \pi_Y)$ be maps over B_0 and $I = [0, 1]$. A fiber homotopy from f to g is called a map $H : (X \times I, \pi_{X\times I}) \to (Y, \pi_Y)$ over B_0 such that $H_0 = f$ and $H_1 = g$.

A fiber homotopy H from f to g, $H : f \underset{\mathrm{B}_0}{\simeq} g$, we also call a homotopy over B_0. The fiber homotopy class of fiber map f is denoted by $[f]_{\mathrm{B}_0}$. We write $[X, Y]_{\mathrm{B}_0}$ for the set of all fiber homotopy classes. By $\mathbf{H}(\mathbf{Top}_{\mathbf{B_0}})$, $\mathbf{H}(\mathbf{M}_{\mathbf{B_0}})$ and $\mathbf{H}(\mathbf{CM}_{\mathbf{B_0}})$ we denote the fiber homotopy categories of categories $\mathbf{Top}_{\mathbf{B_0}}$, $\mathbf{M}_{\mathbf{B_0}}$ and $\mathbf{CM}_{\mathbf{B_0}}$, respectively.

By exponential law a homotopy $H : (X \times I, \pi_{X\times I}) \to (Y, \pi_Y)$ over B_0 induces a map $E(H) : (X, \pi_X) \to (Y^I_{\mathrm{B}_0}, \pi_{Y^I_{\mathrm{B}_0}})$ over B_0, where $Y^I_{\mathrm{B}_0} = \{f : I \to Y | \pi_Y \cdot f = const\}$ and $\pi_{Y^I_{\mathrm{B}_0}}$ is a map defined by formula

$$\pi_{Y^I_{\mathrm{B}_0}}(f) = \pi_Y(f(t)), t \in I, f \in Y^I_{\mathrm{B}_0}.$$

Now we give definitions of some maps which are used in the next.

By $\omega_0 : Y^I_{\mathrm{B}_0} \to (Y, \pi_Y)$ and $\omega_1 : Y^I_{\mathrm{B}_0} \to (Y, \pi_Y)$ we denote maps given by formulas

$$\omega_0(\varphi) = \varphi(0), \varphi \in Y^I_{\mathrm{B}_0},$$

$$\omega_1(\varphi) = \varphi(1), \varphi \in Y^I_{\mathrm{B}_0},$$

respectively.

For each $t \in I$ there exist the embedding maps $\sigma_t : (X, \pi_X) \to (X \times I, \pi_{X\times I})$ over B_0 given by formula

$$\sigma_t(x) = (x, t), x \in X.$$

Let $j : L \to K$ be a map. By $j^* : P^K_{\mathrm{B}_0} \to P^L_{\mathrm{B}_0}$ we denote a map over B_0 given by formula

$$j^*(u) = u \cdot j, u \in P^K_{\mathrm{B}_0}.$$

There exists a fiber homotopy functor $\mathrm{H} : \mathbf{Top_{B_0}} \to \mathbf{H}(\mathbf{Top}_{\mathrm{B}_0})$ given by formulas

$$\mathrm{H}(f) = [f]_{\mathrm{B}_0}, f \in \mathrm{Mor}_{\mathbf{Top_{B_0}}}(X, Y)$$

and

$$\mathrm{H}((X, \pi_X)) = (X, \pi_X), (X, \pi_X) \in ob(\mathbf{Top_{B_0}}).$$

Let $A \subset X$ and $\pi_A = \pi_{X|_A}$. A map $r : (X, \pi_X) \to (A, \pi_A)$, over B_0 is a fibrewize retraction over B_0, if $r \cdot i = 1_A$ and, in addition, $i \cdot r \simeq_{\mathrm{B}_0} 1_A$, then r is called a fibrewise deformation retraction, or deformation retraction over B_0.

A subspace A of metrizable space X over B_0 is called a fibrewize neighborhood retract of X if there exist an open neighborhood U of A in X and a fibrewize retraction $r : U \to A$.

A deformation retraction $r : (X, \pi_X) \to (A, \pi_A)$ over B_0 is called a strong deformation retraction over B_0, or $\mathrm{SDR}_{\mathrm{B}_0}$-map over B_0, if $i \cdot r \underset{\mathrm{B}_0}{\simeq} 1_X \mathrm{rel} A$.

Note that for each fiber homotopy equivalence $f : (X, \pi_X) \to (Y, \pi_Y)$ the subspace $(X, \pi_X) \subset (\mathrm{Cyl}(f), \pi_{\mathrm{Cyl}(f)})$ over B_0 is a strong deformation retract over B_0 of $\mathrm{Cyl}(f)$.

Let A be a closed subset of a space $(X, \pi_X) \in \mathbf{M_{B_0}}$ over B_0. We say that the map $D : (X \times [0, +\infty), \pi_{X \times [0,+\infty)}) \to (X, \pi_X)$ over B_0 is an infinite strong deformation of (X, π_X) onto $(A, \pi_{X|A})$ if $D(x, 0) = x$ for all $x \in X$, $D(a, t) = a$ for all $a \in A, t \in [0, +\infty)$ and for any open neighbourhood U of A in X there exists a $\lambda \in [0, +\infty)$ such that $D(X \times [\lambda, \infty)) \subseteq U$.

We also use the following notions. Let B_0 be a fixed metrizable space. A space $(Y, \pi_Y) \in ob(\mathbf{M_{B_0}})$ is an absolute retract over B_0, $(Y, \pi_Y) \in \mathrm{AR}_{\mathrm{B}_0}$ (an absolute neighbourhood retract over B_0, $(Y, \pi_Y) \in \mathrm{ANR}_{\mathrm{B}_0}$), if (Y, π_Y) has the following property: for any closed embedding $i : (Y, \pi_Y) \to (X, \pi_X) \in ob(\mathbf{M_{B_0}})$ over B_0 there exists a fiberwise retraction $r : (X, \pi_X) \to (i(Y), \pi_{X|_{i(Y)}})$ (an open neighbourhood U of $i(Y)$ in X and a fiberwise retraction $r : (U, \pi_{X|U}) \to (i(Y), \pi_{X|i(Y)})$).

The space $(Y, \pi_Y) \in ob(\mathbf{M_{B_0}})$ is an absolute extensor over B_0, $Y \in \mathrm{AE}_{\mathrm{B}_0}$ (an absolute neighbourhood extensor over B_0, $(Y, \pi_Y) \in \mathrm{ANE}_{\mathrm{B}_0}$), if it has the following property: for any space $(X, \pi_X) \in ob(\mathbf{M_{B_0}})$ over B_0 and any closed subspace $A \subseteq X$, every map $f : (A, \pi_{X|A}) \to (Y, \pi_Y)$ over B_0 has an extension $\tilde{f} : (X, \pi_X) \to (Y, \pi_Y)$ over B_0 ($\tilde{f} : (U, \pi_{X|U}) \to (Y, \pi_Y)$, where U is an open neighbourhood of A in X).

The next results are routine generalizations of the results of the retracts theory.

A metrizable space over B_0 is an $\mathrm{A(N)R}_{\mathrm{B}_0}$-space if and only if it is an $\mathrm{A(N)E}_{\mathrm{B}_0}$-space [$\mathrm{Y}_2$].

For every metric space (X, π_{B_0}) over B_0 there exist fibrepreserving closed embedding into $\mathrm{ANR}_{\mathrm{B}_0}$-space (M, π_M) with weight $w(M) \leqslant max(w(X), w(\mathrm{B}_0), \aleph_0)$ [B_4].

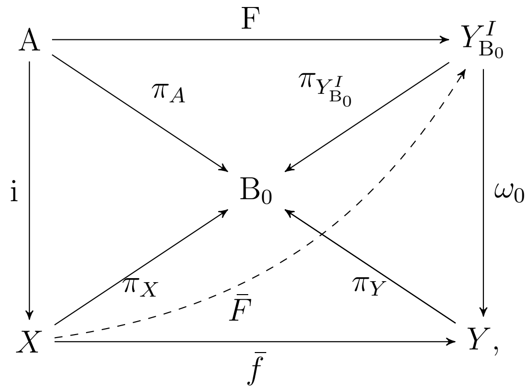
The space $Y^Z_{\mathrm{B}_0}$ of maps $\varphi : Z \to Y$ from compact metrizable space Z into $\mathrm{ANR}_{\mathrm{B}_0}$-space Y, with compact-open topology and property $\pi_Y \cdot \varphi = const$ is $\mathrm{ANR}_{\mathrm{B}_0}$-space [B₄].

Let $(Y, \pi_Y) \in \mathrm{ANR}_{\mathrm{B}_0}$ and $A \subset X$ be a closed subspace of $(X, \pi_X) \in ob(\mathbf{M_{B_0}})$. Let $f : (X, \pi_X) \to (Y, \pi_Y)$ be a map over B_0, and let $H : (A \times I, \pi_{X \times I|_{A \times I}}) \to (Y, \pi_Y)$ be a homotopy over B_0 of map $f_{|A}$. Then there exists an extension of $\tilde{H}$ to a homotopy over B_0 of f itself [Y_2].

Apart from this result in [Y_2] the following proposition is shown:

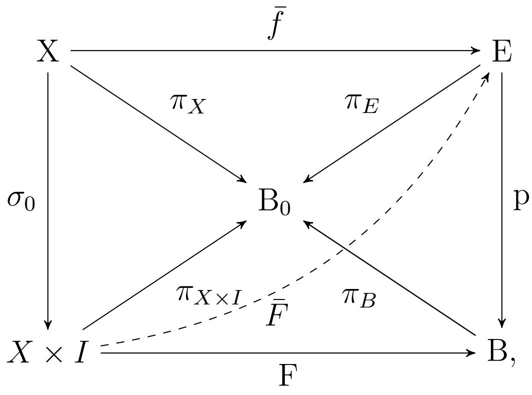
Let $f, g : (X, \pi_X) \to (Y, \pi_Y) \in \mathrm{ANR}_{\mathrm{B}_0}$ be maps over B_0 from metric space over B_0 and let $H : (A \times I, \pi_{A \times I}) \to (Y, \pi_Y)$ be a homotopy over B_0 between restrictions on a closed subspace A of X of maps f and g. Then there exist an open neighborhood U of A in X and a homotopy $\tilde{H} : (U \times I, \pi_{X \times I|U \times I}) \to (Y, \pi_Y)$ over B_0 between restrictions $f_{|U}$ and $g_{|U}$ such that $\tilde{H}_{|A \times I} = H$.

A map $i : (A, \pi_A) \to (X, \pi_X)$ over B_0 is called a cofibration over B_0 if for each commutative diagram



where all maps are maps over B_0 and $\omega_0 \cdot F = \bar{f} \cdot i$, there exists a map $\bar{F} : (X : \pi_X) \to (Y^I_{\mathrm{B}_0}, \pi_{Y^I_{\mathrm{B}_0}})$ over B_0 such that $F = \bar{F} \cdot i$ and $\omega_0 \cdot \bar{F} = \bar{f}$.

The map $p : (E, \pi_E) \to (B, \pi_B)$ over B_0 is called a fibration over B_0 if for each commutative diagram



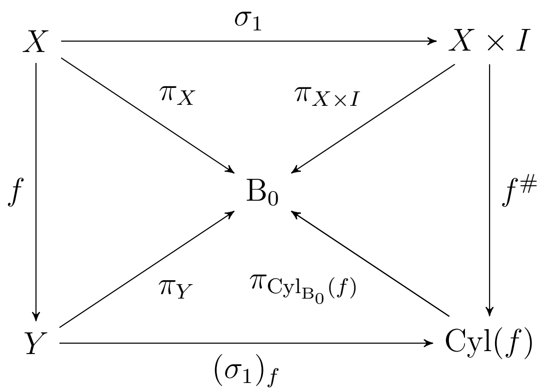
where all maps are maps over B_0 and $p \cdot \bar{f} = F \cdot \sigma_0$, there exists a map $\bar{F} : (X \times I, \pi_{X\times I}) \to (E, \pi_E)$ over B_0 such that $F = p \cdot \bar{F}$ and $\bar{F} \cdot \sigma_0 = \bar{f}$.

The cylinder $\mathrm{Cyl}_{\mathrm{B}_0}(f)$ of a map $f : (X, \pi_X) \to (Y, \pi_Y)$ over B_0 is the pair consisting of cylinder $\mathrm{Cyl}(f)$ of map $f : X \to Y$ and projection $\pi_{\mathrm{Cyl}(f)} : \mathrm{Cyl}(f) \to B$ given by formulas

$$\pi_{\mathrm{Cyl}(f)}([x,t]) = \pi_X(x), [x,t] \in \mathrm{Cyl}(f),$$

$$\pi_{\mathrm{Cyl}(f)}(y) = \pi_Y(y), y \in Y \subset \mathrm{Cyl}(f).$$

There exists a commutative diagram



where $\sigma_1, (\sigma_1)_f$ and $f^{\#}$ are maps given by formulas

$$\sigma_1(x) = (x, 1), \qquad x \in X,$$

$$(\sigma_1)_f(y) = [y], \qquad y \in Y,$$

$$f^{\#}((x,t)) = [x,t], \qquad (x,t) \in X \times I.$$

Let $j : (\mathrm{Cyl}(f), \pi_{\mathrm{Cyl}(f)}) \to (Y \times I, \pi_{Y\times I})$ be a map over B_0 defined by formulas

$$j[(x,t)] = (f(x), t), \qquad (x,t) \in X \times I,$$

$$j(y) = (y, 0), \qquad y \in Y.$$

It is easy to see that the map $i : (X, \pi_X) \to (\mathrm{Cyl}(f), \pi_{\mathrm{Cyl}(f)})$ over B_0 is a cofibration over B_0 and the retraction map $r : (\mathrm{Cyl}(f), \pi_{\mathrm{Cyl}(f)}) \to (Y, \pi_Y)$ over B_0 given by formulas

$$r([x,t]) = [x, 1], \qquad [x,t] \in \mathrm{Cyl}(f),$$

$$r(y) = y, \qquad y \in Y \subset \mathrm{Cyl}(f).$$

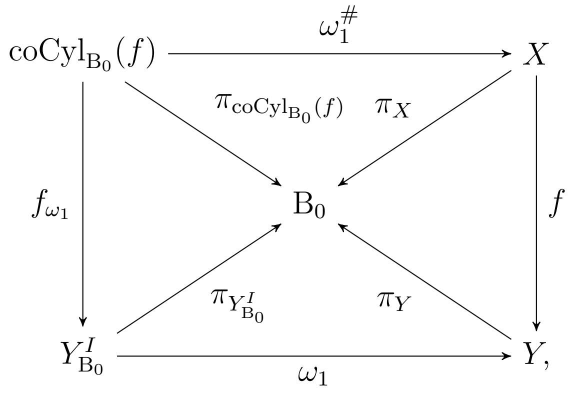
is a fiber homotopy equivalence.

A map $f : (X, \pi_X) \to (Y, \pi_Y)$ over B_0 is a cofibration over B_0 if and only if the map j over B_0 is a retractionable map, i.e. there exists a retraction $r : (X \times I, \pi_{X\times I}) \to (\mathrm{Cyl}(f), \pi_{\mathrm{Cyl}(f)})$ over B_0.

The cocylinder over B_0 of map $f : (X, \pi_X) \to (Y, \pi_Y)$ over B_0, denoted by $\mathrm{coCyl}_{\mathrm{B}_0}(f)$, is the subspace of cocylinder $\mathrm{coCyl}(f)$ consisting of pairs (u, x), where $u \in Y^I_{\mathrm{B}_0}$, $x \in X$ and $u(1) = f(x)$, i.e. $\pi_Y \cdot u = const$. The subspace $\mathrm{coCyl}_{\mathrm{B}_0}(f)$ is a space over B_0 with the projection $\pi_{\mathrm{coCyl}_{\mathrm{B}_0}(f)} : \mathrm{coCyl}_{\mathrm{B}_0}(f) \to \mathrm{B}_0$ given by formula

$$\pi_{\mathrm{coCyl}_{\mathrm{B}_0}(f)}(u, x) = \pi_Y(f(x)), (u, x) \in \mathrm{coCyl}_{\mathrm{B}_0}(f).$$

There exists a commutative diagram



where $\omega_1^{\#}$, f_{ω_1} and ω_1 are maps given by formulas

$$\omega_1^{\#}(u,x) = x, \qquad (u,x) \in \mathrm{coCyl}_{\mathrm{B}_0}(f),$$

$$f_{\omega_1}(u,x) = u, \qquad (u,x) \in \mathrm{coCyl}_{\mathrm{B}_0}(f),$$

$$\omega_1(u) = u(1), \qquad u \in Y^I_{\mathrm{B}_0}.$$

Let $p : \mathrm{coCyl}_{\mathrm{B}_0}(f) \to Y$ be a map defined as follows:

$$p(u,x) = u(0), (u,x) \in \mathrm{coCyl}_{\mathrm{B}_0}(f).$$

It is clear that p is a map over B_0. Observe that $p = \omega_0 \cdot f_{\omega_1}$. Note that the map $p : (\mathrm{coCyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{coCyl}_{\mathrm{B}_0}(f)}) \to (Y, \pi_Y)$ is a fibration over B_0.

Let $0_y : I \to Y$ be the constant path in point $y \in Y$. The pair $(0_{f(x)}, x)$ belongs to $\mathrm{coCyl}_{\mathrm{B}_0}(f)$ because $0_{f(x)}(1) = f(x)$.

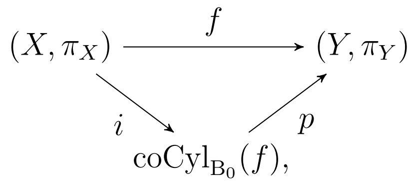
Let $i : X \to \mathrm{coCyl}_{\mathrm{B}_0}(f)$ be a map defined by formula

$$i(x) = (0_{f(x)}, x), x \in X.$$

Now we define a map $r : \mathrm{coCyl}_{\mathrm{B}_0}(f) \to X$ by formula

$$r(u, x) = x, (u, x) \in \mathrm{coCyl}_{\mathrm{B}_0}(f).$$

Then $r \cdot i = 1_X$ and $i \cdot r \underset{\mathrm{B}_0}{\simeq} 1_X$. Hence, X is embeddable in $\mathrm{coCyl}_{\mathrm{B}_0}(f)$ and it is strong deformation retract over B_0 of $\mathrm{coCyl}_{\mathrm{B}_0}(f)$. Thus, i is a homotopy equivalence over B_0 and there exists a factorization

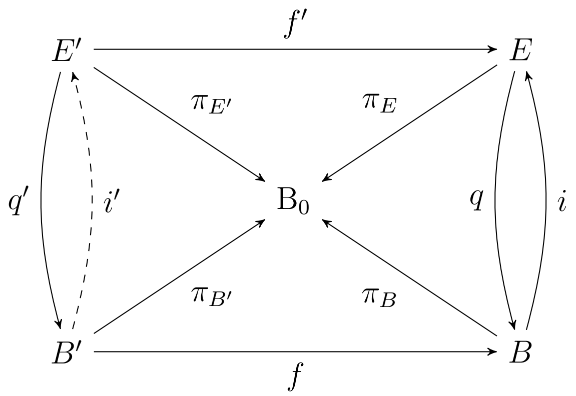


i.e. $f = p \cdot i$. Indeed,

$$f(x) = 0_{f(x)}(0) = p(0_{f(x)}, x) = (p \cdot i)(x).$$

The map $r : (\mathrm{coCyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{coCyl}_{\mathrm{B}_0}(f)}) \to (X, \pi_X)$ over B_0 is a shrinkable fibration over B_0 with respect to $i : (X, \pi_X) \to (\mathrm{coCyl}_{\mathrm{B}_0}(f),\ \pi_{\mathrm{coCyl}_{\mathrm{B}_0}(f)})$, if $r \cdot i = 1_X$ and $i \cdot r \underset{\mathrm{B}_0}{\simeq} 1_{\mathrm{coCyl}_{\mathrm{B}_0}(f)} \mathrm{rel}\, i(X)$.

It is easy to see that if in the pull-back diagram of maps over B_0



q is a shrinkable fibration over B_0 with respect to $i : (B, \pi_B) \to (E, \pi_E)$, then q' is also a shrinkable fibration over B_0 with respect to a uniquely defined embedding $i' : (B', \pi_{B'}) \to (E', \pi_{E'})$ over B_0 such that $f' \cdot i' = i \cdot f$.

1.1 On fiber Borsuk pairs

A pair (X, A) consisting of a space (X, π_X) over B_0 and subspace $A \subset X$ is a pair of Borsuk over B_0 or fiber Borsuk pair, if the inclusion $i : (A, \pi_{X|_A}) \to (X, \pi_X)$ over B_0 is a cofibration over B_0. Note that a closed pair (X, A) is a pair of Borsuk over B_0, if $X \times 0 \cup A \times I$ is a retract over B_0 of $X \times I$.

First we prove some propositions about cofibrations over B_0 and Borsuk's pairs over B_0.

Theorem 1.1.1. *A map $i : (A, \pi_A) \to (X, \pi_X)$ over B_0 is a cofibration over B_0 if and only if the map $j : (\mathrm{Cyl}(i), \pi_{\mathrm{Cyl}(i)}) \to (X \times I, \pi_{X\times I})$ over B_0 is fiberwise retractible.*

Proof. Let $F : (A \times I, \pi_{A\times I}) \to (Y, \pi_Y)$ be a homotopy over B_0 and let $f_0 = F \cdot \sigma_0$. Consider an extension map $\bar{f} : (X, \pi_X) \to (Y, \pi_Y)$ over B_0 of f_0. The maps F and $\bar{f}$ over B_0 induce a map $g : \mathrm{Cyl}_{\mathrm{B}_0}(i) \to Y$ over B_0 such that

$$\begin{aligned} g([a,t]) &= F(a,t), && [(a,t)] \in \mathrm{Cyl}_{\mathrm{B}_0}(i), \\ g(x) &= \bar{f}(x), && x \in \mathrm{Cyl}_{\mathrm{B}_0}(f). \end{aligned}$$

The pair $(F, \bar{f})$ is cone over (σ_0, i). It is clear that g is a morphism of the cone $(i_\#, (\sigma_0)_i)$ into the cone $(F, \bar{f})$. Consequently, if there is a retraction $r : (X \times I, \pi_{X\times I}) \to (\mathrm{Cyl}_{\mathrm{B}_0}(i), \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(i)})$ over B_0, then the composition $\bar{F} = g \cdot r : (X \times I, \pi_{X\times I}) \to (Y, \pi_Y)$

is a homotopy over B_0 of map f over B_0 because $\bar{F} \cdot j = g$ and, hence,

$$\bar{F}(x,0) = (\bar{F} \cdot j)(x) = g(x) = \bar{f}(x), \qquad x \in X.$$

Note that

$$\bar{F}((i(a),t)) = (\bar{F} \cdot j)([(a,t)]) = g[(a,t)] = F((a,t))$$

for each pair $(a,t) \in (A \times I, \pi_{A\times I})$.

Thus, if the map j over B_0 is retractible, then the map i over B_0 is a cofibration.

Now conversely assume that the map $i : (A, \pi_A) \to (X, \pi_X)$ over B_0 is a cofibration over B_0. Then there exists a retraction $r : (X \times I, \pi_{X\times I}) \to (\mathrm{Cyl}_{\mathrm{B}_0}(i), \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(i)})$ over B_0 such that

$$r((x,0)) = x, \qquad (x,0) \in X \times I,$$

$$r(i(a),t) = [(a,t)], \qquad a \in A, t \in I.$$

Thus, r is a retraction of j. □

Corollary 1.1.2. *A closed pair (X,A) of space X over B_0 and its closed subspace A is a Borsuk pair if and only if the subspace $(X \times \{0\}) \cup (A \times I) \subset X \times I$ is a retract over B_0 of $X \times I$.* □

Corollary 1.1.3. *For each closed Borsuk's pair (X,A) over B_0 and for every space Y over B_0 the pair $(X \times Y, A \times Y)$ over B_0 is a closed Borsuk's pair over B_0.* □

Corollary 1.1.4. *If (X,A) is a Borsuk's pair over B_0 and A is a closed subspace of locally compact Hausdorff space X then for each space Y over B_0 the map $i^* : Y^X \to Y^A$ is a cofibration over B_0.* □

Theorem 1.1.5. *A pair (X,A) of space (X,π_X) over B_0 and its closed subspace $(A, \pi_{X|A})$ is a Borsuk pair over B_0 if and only if there exist a map $\psi : X \to I$ and* *a fiber homotopy* $G : (X \times I, \pi_{X\times I}) \to (X, \pi_X)$ *with respect* A *such that* $A = \psi^{-1}(0)$, $G(x, 0) = x$ *and* $G(x, t) \in A$ *when* $\psi(x) < t$.

Proof. Let (X, A) be a Borsuk pair over B_0. By Corollary 1.1.2 there exists a retraction $r : X \times I \to \tilde{A} = (X \times \{0\}) \cup (A \times I)$ over Bo. Let $r((x, t)) = (\bar{r}(x, t), \rho(x, t))$, where $\bar{r}(x, t) \in X$ is first coordinate of $r(x, t)$ and $\rho(x, t) \in I$ is the projection in I of point $r(x, t)$.

Let $\psi : X \to I$ be a function given by

$$\psi(x) = \max\{t - \rho(x, t) | x \in X\}.$$

Note that $A = \psi^{-1}(0)$. Besides, if $\psi(x) < t$, then $\rho(x, t) > 0$ and consequently, $\bar{r}(x, t) \in A$.

Let $G = \bar{r} : (X \times I, \pi_{X\times I}) \to (X, \pi_X)$. It is clear that $G(x, 0) = \bar{r}(x, 0) = x$, $G(x, t) \in A$ for $\psi(x) < t$ and

$$\pi_X(x) = \pi_{X\times I}(x, t) = \pi_{\tilde{A}}(\bar{r}(x, t), \rho(x, t)) = \pi_X(\bar{r}(x, t)),$$

i.e. $\bar{r}$ is a map over B_0.

Now assume that hold the conditions of theorem. Then the map $r : (X\times I, \pi_{X\times I}) \to (X, \pi_X)$ given by

$$r(x, t) = \begin{cases} (G(x, t), 0), & t \leq \psi(x) \\ (G(x, t), t - \psi(x)), & t \geq \psi(x). \end{cases}$$

is a retraction over B_0. Consequently, (X, A) is a Borsuk pair over B_0. □

Theorem 1.1.6. *Let* (X, A) *be a Borsuk pair over* B_0. *Then* $(X \times I, (X \times \{0\}) \cup (A \times I) \cup X \times \{1\})$ *is the Borsuk pair over* B_0.

Proof. For simplicity by X_A denote the set $(X\times\{0\})\cup(A\times I)\cup X\times\{1\})$. By Theorem 1.1.5 there exist a function $\varphi : X \to I$ and a fiber homotopy $g_t : U \to X \mathrm{rel} A$ from $U = X \setminus \varphi^{-1}(0)$ to X such that $\varphi^{-1}(0) = A$, $g_0(x) = x$ and $g_1(x) \in A$ for each $x \in U$.

The function $\psi : X \times I \to I$ defined by

$$\psi(x,t) = 2\min(2\varphi(x), \tau, 1-\tau), \qquad (x,\tau) \in X \times I$$

has property $\psi^{-1}(0) = X_A$.

Let $V = X \times I \setminus \psi^{-1}(1)$ be a set consisting of points $(x,\tau) \in X \times I$ for which $\tau \neq \frac{1}{2}$ or $\psi(x) = \frac{1}{4}$.

The maps $h_t : V \to V \times I$ given by formulas

$$h_t(x,\tau) = \begin{cases} (x, \tau(1-t)), & 2\tau \leq \varphi(x); \\ (g(x, (\frac{2t}{\varphi(x)} - 1)t), \tau(1-t)), & \varphi(x) \leq 2\tau \leq \min(2\varphi(x), 1); \\ (g(x,t), (\tau - 2\varphi(x))t + \tau), & \varphi(x) \leq \tau \leq \min(2\varphi(x), \frac{1}{2}); \\ (g(x,t), \tau), & 2\varphi(x) \leq \tau \leq 1 - 2\varphi(x); \\ (g(x,t), \tau + (2\varphi(x) + \tau - 1)t), & \max(1 - 2\varphi(x), \frac{1}{2}) \leq \tau \leq 1 - \varphi(x); \\ (g(x, (\frac{2(1-\tau)}{\varphi(x)} - 1)t), \tau + t - \tau t), & \max(2(1 - \varphi(x)), 1) \leq 2\tau \leq 2 - \varphi(x); \\ (x, \tau + t - \tau t), & 2 - \varphi(x) \leq 2\tau \end{cases}$$

have properties

$$h_0(x,\tau) = (x,\tau), h_1(x,\tau) \in X_A, (x,\tau) \in X \times I$$

and

$$h_t(x,\tau) = (x,\tau), (x,\tau) \in X_A.$$

Thus, the pair $(X \times I, (X \times \{0\}) \cup (A \times I) \cup (X \times \{1\}))$ is the Borsuk pair over

B_0. □

Theorem 1.1.7. *Let* (X, A) *be a Borsuk pair over* B_0. *Then each deformation retraction* $r : (X, \pi_X) \to (A, \pi_{X|A})$ *over* B_0 *is a strong deformation retraction over* B_0.

Proof. Let $F : (X \times I, \pi_{X\times I}) \to (X, \pi_X)$ be a fiber homotopy between $i \cdot r : (X, \pi_X) \to (X, \pi_X)$ and $1_X : (X, \pi_X) \to (X, \pi_X)$. By Theorem 1.1.6 the pair $(X \times I, X_A)$ is the Borsuk pair over B_0.

Hence, there exists a fiber homotopy given by

$$F_\tau(x,t) = \begin{cases} F((i\cdot r)(x), \tau), & t = 0, \\ F(x, t + (1-t)\tau), & x \in A, t \in I, \\ x, & t = 1. \end{cases}$$

Note that $F_0(x,t) = F(x,t)$ and $F_1 : X \times I \to X \mathrm{rel} A$ is a fiber homotopy between 1_X and $i \cdot r$. □

Theorem 1.1.8. *A closed pair* (X, A) *of spaces over* B_0 *is a Borsuk pair over* B_0 *if and only if* $\tilde{A} = (X \times \{0\}) \cup (A \times I)$ *is a strong deformation retract over* B_0 *of* $(X \times I, \pi_{X\times I})$.

Proof. Let $(\tilde{A}, \pi_{\tilde{A}})$ be a strong deformation retract over B_0 of $(X \times I, \pi_{X\times I})$. By Corollary 1.1.2 the pair (X, A) is a Borsuk pair over B_0.

Consequently, as the product $(X \times I, \pi_{X\times I})$ is deformable in $X \times \{0\}$ and hence, in $\tilde{A}$, by Corollary 1.1.2 there exists a retraction $r : (X \times I, \pi_{X\times I}) \to (\tilde{A}, \pi_{\tilde{A}})$ over B_0. This retraction is deformation retraction over B_0. By Theorem 1.1.7 r is a strong deformation retraction over B_0. Let $r(x,t) = (\bar{r}(x,t), \rho(x,t))$, where $x \in X$, $t \in I$ and $\bar{r}(x,t) \in X$, $\rho(x,t) \in I$.

The deformation $g_\tau : X \times I \to X \times I$ defined by formula

$$g_\tau(x,t) = (\bar{r}(x,(1-\tau)t),(1-\tau)\rho(x,t)+\tau t), x \in X, t \in I$$

is deformation over B_0 and it satisfies the following conditions:

$$g_0 = i \cdot r,$$

$$g_1 = 1_X,$$

$$g_\tau(x,t) = (x,t), \qquad (x,t) \in \tilde{A}.$$

□

Corollary 1.1.9. *Let (X,A) be a closed Borsuk pair over B_0. Then the subspace (A,π_A) is a strong deformation retraction over B_0 of (X,π_X) if and only if the inclusion $i:(A,\pi_A)\to(X,\pi_X)$ is a fiber homotopy equivalence.* □

1.2 On Fiber SSDR-maps and Fibrant Spaces

In this section we give the definition and discuss various concepts which are associated to SSDR-maps over B_0. The following provides a shape version of SDR-map over B_0.

All spaces in Section 1.2 are metrizable.

Here the basic definition is the following

Definition 1.2.1. *Let $(X,\pi_X) \in ob(\mathbf{M_{B_0}})$ and let A be a closed subspace of X. The subspace $(A,\pi_{X|A})$ over B_0 is called a shape strong deformation retract over* B_0 of *(X,π_X) if there exists an embedding $\alpha:(X,\pi_X)\hookrightarrow(Y,\pi_Y)\in \mathrm{AR}_{\mathrm{B}_0}$ over B_0 satisfying the following condition*:

for any pair of neighbourhoods U and V of $\alpha(X)$ and $\alpha(A)$ respectively in (Y,π_Y), *there is a homotopy* $H : (X \times I, \pi_{X\times I}) \to (U, \pi_{Y|U})\mathrm{rel}A$ *over* B_0 *such that* $H(x, 0) = \alpha(x)$ *and* $H(x, 1) \in V$ *for each* $x \in X$.

It is clear that if an embedding $\alpha : (X, \pi_X) \to (M, \pi_M)$ over B_0 satisfies the conditions of definition 1.2.1, then these conditions hold for any closed embedding $\beta : (X, \pi_X) \to (Z, \pi_Z) \in \mathrm{AR}_{\mathrm{B}_0}$.

A closed embedding $i : (A, \pi_A) \to (X, \pi_X)$ over B_0 is called $\mathrm{SSDR}_{\mathrm{B}_0}$-map if i embeds (A, π_A) in (X, π_X) as a shape strong deformation retract over B_0 of (X, π_X).

Note that the notion of $\mathrm{SSDR}_{\mathrm{B}_0}$-map generalizes the notion of $\mathrm{SDR}_{\mathrm{B}_0}$-map.

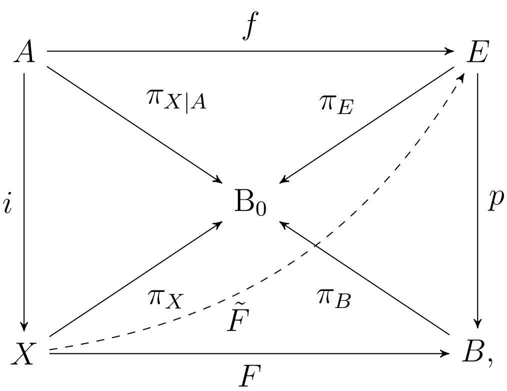
We get the following theorem which is a fiber version of Theorem 1.2 of ([C_1], [C_2]).

Theorem 1.2.2. *Let* $(X, \pi_X) \in \mathbf{M}_{\mathbf{B_0}}$ *and* A *be a closed subspace of* X. *Then the following conditions are equivalent:*

a) $i : (A, \pi_{X|A}) \hookrightarrow (X, \pi_X)$ *is an* SSDR-*map over* B_0;

b) for any map $f : (A, \pi_{X|A}) \to (Y, \pi_Y) \in \mathrm{ANR}_{\mathrm{B}_0}$ *over* B_0, *there is an extension* $\tilde{f} : (X, \pi_X) \to (Y, \pi_Y)$ *over* B_0 *such that* $\tilde{f} \cdot i = f$ *and any two such extensions over* B_0 *are fiber homotopic with respect* iA;

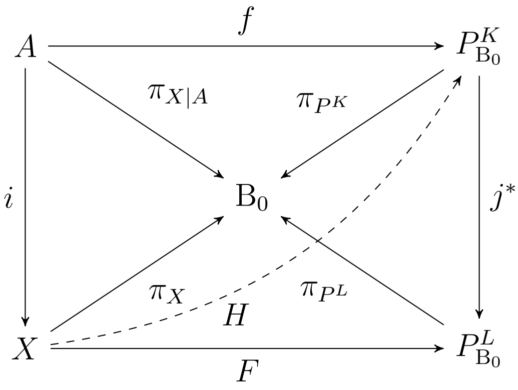
c) for any commutative diagram



where $p : (E, \pi_E) \to (B, \pi_B)$ *is a fibration over* B_0 *and* (E, π_E) *and* (B, π_B) *are* $\mathrm{ANR}_{\mathrm{B}_0}$-*spaces, there exists a map* $\tilde{F} : (X, \pi_X) \to (E, \pi_E)$ *over* B_0 *such that* $\tilde{F} \cdot i = f$ *and*

$p \cdot \tilde{F} = F$.

d) for any commutative diagram of maps over B_0



there exists a filler $H : (X, \pi_X) \to (P^K, \pi_{PK})$ *over* B_0 *provided* $P \in \mathrm{ANR}_{\mathrm{B}_0}$ *and* L *is a subcomplex of a finite* CW*-complex* K *with an inclusion map* $j : L \hookrightarrow K$.

Proof. We check up the following implications a)⇒b)⇒c)⇒d)⇒a).

a)⇒b). As in the proof of Proposition 2 of [B4] we can show that (X, π_X) is a closed subspace of $\mathrm{AE}_{\mathrm{B}_0}$-space for metric spaces (M, π_M) over B_0 with weight $w(M) \leq max(w(X), w(\mathrm{B}_0), \aleph_0)$. Here $M = B \times K$, where K is a convex hull of X in a normed vector space L. Since (Y, π_Y) is $\mathrm{ANE}_{\mathrm{B}_0}$-space there exist an open neighbourhood V of A in M and extension $\hat{f} : (V.\pi_{M|A}) \to (Y, \pi_Y)$ over B_0 of map $f : (A, \pi_{X|A}) \to (Y, \pi_Y)$. By condition a) there exists a homotopy $H : (X \times I, \pi_{X \times I}) \to (M, \pi_M)$ over B_0 such that

$$H(x,0) = x, \qquad x \in X,$$

$$H(x,1) \in V, \qquad x \in X,$$

$$H(a,t) = a, \qquad a \in A, t \in I.$$

Let $\tilde{f} : (X, \pi_X) \to (Y, \pi_Y)$ be a map given by the following formula

$$\tilde{f} = \hat{f}(H(x,1)), \qquad x \in X.$$

Note that

$$(\pi_Y \cdot \tilde{f})(x) = \pi_Y(\tilde{f}(x)) = \pi_Y(\hat{f}(H(x,1))) = (\pi_Y \cdot \hat{f})(H(x,1)) =$$
$$= \pi_{M|A}(H(x,1)) = \pi_M(H(x,1)) = \pi_{X\times I}(x,1) = \pi_X(x).$$

Thus, $\pi_Y \cdot \tilde{f} = \pi_X$ and hence, $\tilde{f}$ is a map over B_0. Now show that any two such type extensions are fiber homotopic with respect A. Let $\tilde{f}_1, \tilde{f}_2 : (X, \pi_X) \to (Y, \pi_Y)$ be extensions over B_0 of map f. Consider a subspace $N = X \times \{0\} \cup A \times I \cup X \times \{1\}$ of space $(M \times I,\ \pi_{M\times I})$. Define a map $F : N \to Y$ over B_0 by

$$F(x,0) = \tilde{f}_1(x), \qquad x \in X,$$
$$F(x,1) = \tilde{f}_2(x), \qquad x \in X,$$
$$F(x,a) = f(a), \qquad a \in A, t \in I.$$

It is clear that (see Proposition 1.1 of [Y_2]) $M \times I = (B \times K) \times I \approx B \times (K \times I) \in \mathrm{AE}_{\mathrm{B}_0}$, because $K \times I \in \mathrm{AE}(M)$. There exists an extension $\bar{F} : (W, \pi_{M\times I|W}) \to (Y, \pi_Y)$ over B_0 of map $F : (N, \pi_{M\times I|N}) \to (Y, \pi_Y)$ over B_0 on some open neighbourhood W of N in $M \times I$.

Let U be an open neighbourhood of X in M such that $U \times \{0\} \subset W$ and $U \times \{1\} \subset W$. Besides, consider an open neighbourhood V of A in M such that $V \times I \subset W$. By condition a) it follows the existence of homotopy $D : (X \times I, \pi_{X\times I}) \to (U, \pi_{M|U})$ over

B_0 with properties

$$D(x,0) = x, \qquad x \in X,$$

$$D(x,1) \in V, \qquad x \in X,$$

$$D(a,t) = a, \qquad a \in A, t \in I.$$

Let $F'(x,t) = \bar{F}(D(x,t),0)$, $F''(x,t) = \bar{F}(D(x,t),1)$ and $H(x,t) = \bar{F}(D(x,1),t)$. Note that F', F'' and H induce the fiber homotopies:

$$F' : \tilde{f}_1 \simeq_{\mathrm{B}_0} h_1 \mathrm{rel} A,$$

$$F'' : \tilde{f}_2 \simeq_{\mathrm{B}_0} h_2 \mathrm{rel} A,$$

$$H : h_1 \simeq_{\mathrm{B}_0} h_2 \mathrm{rel} A.$$

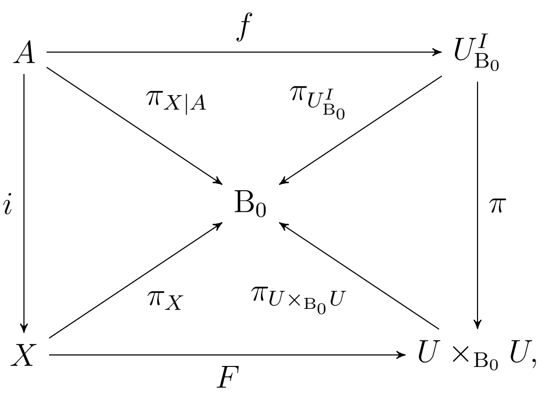
Therefore, $\tilde{f}_1 \simeq_{\mathrm{B}_0} \tilde{f}_2 \mathrm{rel} A$.

b)⇒c). By condition b) for a space $(E, \pi_E) \in \mathrm{ANE}_{\mathrm{B}_0}$ over B_0 there is a map $\bar{F} : (X, \pi_X) \to (E, \pi_E)$ over B_0 such that $\bar{F} \cdot i = f$. Note that $F \cdot i = p \cdot f = p \cdot \bar{F} \cdot i$. From condition b) also follows the existence of homotopy $H : F \simeq_{\mathrm{B}_0} p \cdot \bar{F} \mathrm{rel} i(A)$ over B_0. Thus there is a fiber homotopy $\tilde{H} : (X \times I, \pi_{X \times I}) \to (E, \pi_E)$ such that $p \cdot \tilde{H} = H$. The fiber homotopy $\tilde{H}$ induces a map $\tilde{F} : (X, \pi_X) \to (E, \pi_E)$ over B_0 with properties $\tilde{F} \cdot i = f$ and $p \cdot \tilde{F} = F$.

c)⇒d).By proposition 9 of [B4] the space $P^K_{\mathrm{B}_0}$ and $P^L_{\mathrm{B}_0}$ are $\mathrm{ANR}_{\mathrm{B}_0}$ -spaces. Also note that $j^* : P^K_{\mathrm{B}_0} \to P^L_{\mathrm{B}_0}$ is a fibration over B_0. Hence, there exists a filler $H : (X, \pi_X) \to (P^K_{\mathrm{B}_0}, \pi_{P^K_{\mathrm{B}_0}})$ over B_0.

d)⇒a). Let (X, π_X) be a closed subspace over B_0 of $\mathrm{AR}_{\mathrm{B}_0}$-space (M, π_M) . Let $i : (A, \pi_{X|A}) \to (X, \pi_X)$ be the inclusion over B_0 of closed set A of X given by $i(a) = a$ for each $a \in A$.

Consider open neighbourhoods U and V of X and of A, respectively, in M such that $V \subseteq U$. Note that $(U, \pi_{M|U})$,$(V, \pi_{M|V}) \in \mathrm{ANR}_{\mathrm{B}_0}$. Let $P = V$, $K = \{*\}$, $L = \emptyset$ and let $f : (A, \pi_{X|A}) \to (V, \pi_{M|V})$ be the inclusion map over B_0. By condition d) there exists a map $r : (X, \pi_X) \to (V, \pi_{M|V})$ over B_0 such that $r \cdot i = f$. Now assume that $P = U$, $K = I$ and $L = \{0, 1\}$. Consider a commutative diagram



where

$$\pi(\omega) = (\omega(0), \omega(1)), \qquad \omega \in U^I_{\mathrm{B}_0},$$

$$f(a)(t) = a, a \in A, \qquad t \in I,$$

$$F(x) = (x, r(x)), \qquad x \in X.$$

It is clear that $\pi_{X|A} = \pi_{U^I_{\mathrm{B}_0}} \cdot f$, $\pi_{U^I_{\mathrm{B}_0}} = \pi_{U \times_{\mathrm{B}_0} U} \cdot \pi$ and $\pi_X = \pi_{U \times_{\mathrm{B}_0} U} \cdot F$. Also note that $U \times_{\mathrm{B}_0} U$ and U_{B_0} are $\mathrm{ANR}_{\mathrm{B}_0}$-spaces.

By condition d) there exists a map $H : (X, \pi_X) \to (U^I_{\mathrm{B}_0}, \pi_{U^I_{\mathrm{B}_0}})$ over B_0 such that $H \cdot i = f$ and $\pi \cdot H = F$.

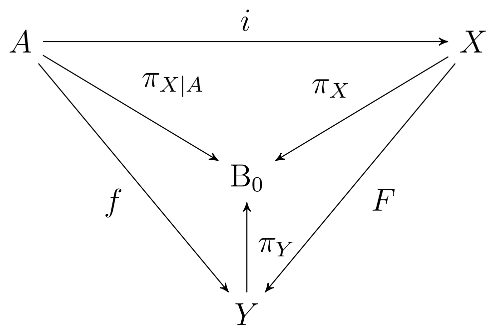
Let $D : (X \times I, \pi_{X \times I}) \to (U, \pi_{M|U})$ be a map over B_0 given by formula

$$D(x, t) = H(x, t), \qquad (x, t) \in X \times I.$$

The map D satisfies the conditions of the definition of SSDR-map over B_0. □

Now we need to introduce definition and investigation of fibrant spaces over B_0.

Definition 1.2.3. *A space* (Y, π_Y) *over* B_0 *is called a fibrant space over* B_0 *if for every* SSDR-*map* $i : (A, \pi_{X|A}) \to (X, \pi_X)$ *over* B_0 *and every map* $f : (A, \pi_{X|A}) \to (Y, \pi_Y)$ *over* B_0, *there is a map* $F : (X, \pi_X) \to (Y, \pi_Y)$ *over* B_0 *such that* $F \cdot i = f$, *i.e. the following diagram commutes*:



We have the following proposition.

Theorem 1.2.4. *If* Y *is an* $\mathrm{ANR}_{\mathrm{B}_0}$-*space, then* Y *is a fibrant space over* B_0.

Proof. Let $i : (A, \pi_{X|A}) \to (X, \pi_X)$ be a $\mathrm{SSDR}_{\mathrm{B}_0}$-map and $(Y, \pi_Y) \in \mathrm{ANR}_{\mathrm{B}_0}$. By implication a⇒b) for each map $f : (A, \pi_{X|A}) \to (Y, \pi_Y)$ over B_0 there exists an extension $\tilde{f} : (X, \pi_X) \to (Y, \pi_Y)$ over B_0 with property $\tilde{f} \cdot i = f$. Thus Y is an fibrant space over B_0. □

Theorem 1.2.5. *If* (Y, π_Y) *is a fibrant space over* B_0 *and* Z *is a compact space, then* $(Y^Z_{\mathrm{B}_0}, \pi_{Y^Z_{\mathrm{B}_0}})$ *also is a fibrant space over* B_0.

Proof. Let (X, π_X) be a metric space over B_0, A a closed subset of X, $i : (A, \pi_{X|A}) \to (X, \pi_X)$ a $\mathrm{SSDR}_{\mathrm{B}_0}$-map and $f : (A, \pi_{X|A}) \to (Y^Z_{\mathrm{B}_0}, \pi_{Y^Z_{\mathrm{B}_0}})$ a map over B_0. The map $F : (A \times Z, \pi_{A\times Z}) \to (Y, \pi_Y)$ given by formula

$$F(a, z) = (f(a))(z), (a, z) \in (A \times Z, \pi_{A\times Z})$$

is a map over B_0. Indeed, for every $(a,z) \in A \times Z$ we have

$$\begin{aligned}(\pi_Y \cdot F) &= \pi_Y(F(a,z)) = \pi_Y(f(a)(z)) = \pi_{Y^Z_{B_0}}(f(a)) = \\ &= \pi_{X|A}(a) = \pi_{A\times Z}(a,z).\end{aligned}$$

Observe that if $i : (A, \pi_{X|A}) \to (X, \pi_X)$ is a SSDR-map over B_0, then $i \times 1_Z : (A \times Z, \pi_{A\times Z}) \to (X \times Z, \pi_{X\times Z})$ is a SSDR-map over B_0. Indeed, we can assume the pair $(X \times Z, \pi_{X\times Z})$, where $\pi_{X\times Z}((x,z)) = \pi_X(x)$, is embeddable in some AR_{B_0}-space $(M \times N, \pi_{M\times N})$ such that (X, π_X) and Z are embeddable in $(M, \pi_M) \in AR_{B_0}$ and $N \in AR$, respectively. Let W and Q be open neighbourhoods of $X \times Z$ and $A \times Z$ in an AR_{B_0}-space $M \times N$, respectively. There exist open neighbourhoods U and V of X and A respectively in M such that $U \times Z \subset W$ and $V \times Z \subset Q$. Since i is a SSDR-map over B_0 there exists a homotopy $H : (X, \pi_X) \times I \to (U, \pi_{M|U})$ over B_0 with properties $H(x,0) = i(x)$ and $H(x,1) \in V$.

Let $\tilde{H} : (X \times Z \times I, \pi_{X\times Z\times I}) \to (U \times Z, \pi_{U\times Z})$ be a map given by formula

$$\tilde{H}(x,z,t) = (H(x,t),z), (x,z) \in X \times Z, t \in I.$$

Note that $\tilde{H}$ is a map over B_0 satisfying the following conditions

$$\tilde{H}(x,z,0) \;= (H(x,0),z) = (i(x),z) = (i \times 1_Z)(x,z)$$

and

$$\tilde{H}(x,z,1) \;= (H(x,1),z) \in V \times Z \subset Q.$$

Since (Y, π_Y) is an fibrant space over B_0 there is a map $\bar{F} : (X \times Z, \pi_{X\times Z}) \to (Y, \pi_Y)$

over B_0 such that $\bar{F} \cdot (i \times 1_Z) = \tilde{f}$, where $\tilde{f} : (A \times Z, \pi_{A \times Z}) \to (Y, \pi_Y)$ be a map over B_0 given by formula

$$\tilde{f}(a, z) = (f(a))(z), (a, z) \in A \times Z.$$

Let $\tilde{F} : (X, \pi_X) \to (\mathrm{Y}^Z_{\mathrm{B}_0}, \pi_{\mathrm{Y}^Z_{\mathrm{B}_0}})$ be a map given by

$$(\tilde{F}(x))(z) = \bar{F}(x, z), x \in X, z \in Z.$$

It is clear that $\tilde{F} \cdot i = f$. □

Theorem 1.2.6. *Let* $\mathbf{Y} = ((Y_n, \pi_{Y_n}), p_{n,n+1}, N^+)$ *be an inverse system of fibrant spaces over* B_0 *and fibrations over* B_0. *Then the fiber limit space* $Y = \varprojlim \mathbf{Y}$ *is a fibrant space over* B_0 *and the natural projections* $p_n : (Y, \pi_Y) \to (Y_n, \pi_{Y_n})$ *are fibrations over* B_0.

Proof. Let $(y_n) \in Y = \varprojlim \mathbf{Y}$. It is clear that for each $n < n + 1$

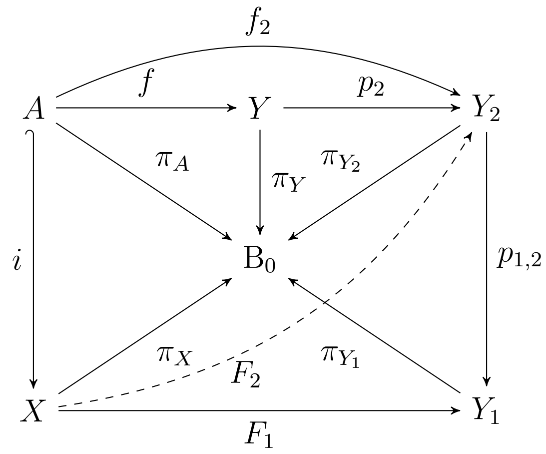
$$\pi_n(y_n) = (\pi_n \cdot p_{n,n+1})(y_{n+1}) = \pi_{n+1}(y_{n+1}).$$

Assume that

$$\pi_Y((y_n)) = \pi_n(y_n), (y_n) \in Y.$$

Note that $\pi_{Y_n} \cdot p_n = \pi_Y$. Consequently, (Y, π_Y) is a space over B_0 and $p_n : Y \to Y_n$ is a map over B_0.

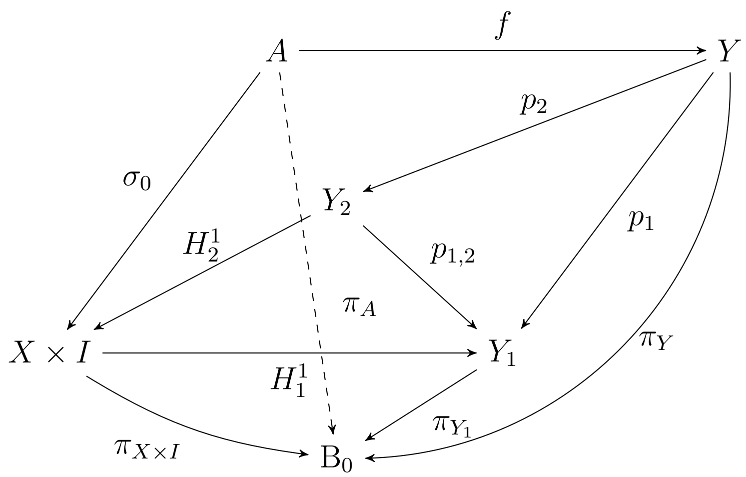
Let $f_n = p_n \cdot f, n \in N$. It is clear that there exists a map $F_1 : X \to Y_1$ over B_0 such that $F_1 \cdot i = p_1 \cdot f$. For the commutative diagram



there is a map $F_2 : (X, \pi_X) \to (Y_2, \pi_{Y_2})$ over B_0 with properties $F_2 \cdot i = f_2$ and $p_{1,2} \cdot F_2 = F_1$. Inductively we can construct the sequence $\{F_n\}_{n \in N^+}$ of maps $F_n : X \to Y_n$ over B_0 for which $p_{n,n+1} \cdot F_{n+1} = F_n$ and $F_n \cdot i = f_n$.

Let $F = \underset{n \in N^+}{\Delta} F_n : X \to \prod_{n \in N^+} Y_n$ be the diagonal product over B_0 of maps $F_n : (X, \pi_X) \to (Y_n, \pi_{Y_n}), n \in N^+$. The map F induces a map over B_0 which we again denote by $F : (X, \pi_X) \to (Y, \pi_Y)$. It is clear that $F \cdot i = f$.

Now show that $p_1 : (Y, \pi_Y) \to (Y_1, \pi_{Y_1})$ is fibration over B_0. Consider the diagram



There exists a map $H_1^1 : (X \times I, \pi_{X \times I}) \to (Y_2, \pi_{Y_2})$ over B_0 such that $H_1^1 \cdot \sigma_0 = p_1 \cdot f = p_{1,2} \cdot (p_2 \cdot f)$. Hence, we can choose a map $H_2^1 : (X \times I, \pi_{X \times I}) \to (Y_2, \pi_{Y_2})$ over B_0 for which $H_2^1 \cdot \sigma_0 = p_2 \cdot f$ and $p_{1,2} \cdot H_2^1 = H_1^1$. Thus, inductively we can construct a sequence $H_1^1, H_2^1, \cdots H_n^1, \cdots$ of maps $H_n^1 : (X \times I, \pi_{X\times I}) \to (Y_n, \pi_{Y_n})$ over B_0 such that $H_n^1 = p_{n,n+1} \cdot H_{n+1}^1, n \in N^+$. Let $H_1 : (X \times I, \pi_{X\times I}) \to (Y, \pi_Y)$ be a map given by

$$H_1 = \underset{n\in N}{\Delta} H_n^1 : (X \times I, \pi_{X\times I}) \to (Y, \pi_Y).$$

Finally, we observe that $H^1 \cdot \sigma_0 = f$ and $p_1 \cdot H_1 = H_1^1$.

Analogously, we can prove that $p_2, p_3, \cdots$ maps over B_0 are fibration over B_0. □

Theorem 1.2.7. *Let* $f : (X, \pi_X) \to (Y, \pi_Y)$ *be a map over* B_0*. If* $(X, \pi_X), (Y, \pi_Y) \in \mathrm{ANR}_{\mathrm{B}_0}$*, then* $\mathrm{coCyl}_{\mathrm{B}_0}(f) \in \mathrm{ANR}_{\mathrm{B}_0}$.

Proof. Let $(Z, \pi_Z) \in ob(\mathbf{M_{B_0}})$ and A be a closed subspace of Z and let $g : (A, \pi_{Z|A}) \to \mathrm{coCyl}_{\mathrm{B}_0}(f)$ be a map over B_0. For the composition $g_2 = \omega_1^{\#} \cdot g : (A, \pi_{Z|A}) \to (X, \pi_X)$ there exist a neighbourhood U of A in Z and an extension $\tilde{g}_2 : (U, \pi_{Z|U}) \to (X, \pi_X)$ over B_0 of map $\omega_1^{\#} \cdot g$ over B_0. Note that

$$f \cdot \tilde{g}_2(a) = f(\tilde{g}_2(a)) = f(g_2(a)) = f \cdot \omega_1^{\#} \cdot g(a) = \omega_1 \cdot f_{\omega_1} \cdot g(a) = (f_{\omega_1} \cdot g(a))(1).$$

The composition $f_{\omega_1} \cdot g : (A, \pi_{Z|A}) \to (Y^I_{\mathrm{B}_0}, \pi_{Y^I_{\mathrm{B}_0}})$ induces the map $H : A \times I \to Y$ over B_0 given by

$$H(a,t) = ((f_{\omega_1} \cdot g)(a))(t), (a,t) \in A \times I.$$

It is clear that for each $a \in A$ and $t \in I$

$$\begin{aligned} H(a,1) &= ((f_{\omega_1} \cdot g)(a))(1) = (\omega_1 \cdot f_{\omega_1} \cdot g)(a) = (f \cdot \omega_1^{\#} \cdot g)(a) = \\ &= f \cdot ((\omega_1^{\#} \cdot g)(a)) = f(\tilde{g}_2(a)) = (f \cdot \tilde{g}_2)(a) = (f \cdot \tilde{g}_{2|A})(a). \end{aligned}$$

Let $G : \big((U \times \{0\}) \cup A \times I, \pi_{U\times\{0\}\cup A\times I}\big) \to (Y, \pi_Y)$ be a map defined by formula

$$\begin{aligned} G(u,1) &= f\tilde{g}_2(u), u \in U, \\ G(a,t) &= H(a,t), (a,t) \in A \times I. \end{aligned}$$

There exists an extension $\tilde{G} : (U \times I, \pi_{U\times I}) \to (Y, \pi_Y)$ over B_0 such that

$$\tilde{G}_{|U\times\{\mathbf{1}\}} = f \cdot \tilde{g}_2$$

and

$$\tilde{G}_{|A\times I} = H.$$

The map $\tilde{G}$ induces a map $\tilde{g}_1 : (U, \pi_{Z|_U}) \to (Y^I_{\mathrm{B}_0}, \pi_{Y^I_{\mathrm{B}_0}})$ for which

$$(\tilde{g}_1(u))(t) = \tilde{G}(u,t), u \in U, t \in I.$$

Let $\tilde{g} = \tilde{g}_1 \Delta \tilde{g}_2 : (U, \pi_{Z|_U}) \to (\mathrm{coCyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{coCyl}_{\mathrm{B}_0}(f)})$.

Also note that for each pair $\hat{g}(u) = (\tilde{g}_1(u), \tilde{g}_2(u))$ holds the condition

$$\tilde{g}_1(u)(1) = \tilde{G}(u,1) = f\tilde{g}_2(u), u \in U,$$

i.e. $\tilde{g}(u) \in \mathrm{coCyl}_{\mathrm{B}_0}(f)$. Besides,

$$\tilde{g}(a) = (\tilde{g}_1(a), \tilde{g}_2(a)) = (\tilde{g}_1(a), g_2(a)) = (\tilde{g}_1(a), \omega_1^{\#} g(a)), a \in A.$$

Note that $\tilde{g}_1(a)$ is a map $\tilde{g}_1(a) : I \to Y$ such that

$$\tilde{g}_1(a)(t) = \tilde{G}(a,t) = H(a,t) = ((f_{\omega_1} \cdot g)(a))(t),$$

i.e. $\tilde{g}_1(a) = f_{\omega_1} \cdot g(a)$.

Hence, for each $a \in A$ we have

$$\tilde{g}(a) = (\tilde{g}_1(a), \tilde{g}_2) = (f_{\omega_1} \cdot g(a), \omega_1^{\#} \cdot g(a)) = g(a).$$

$\square$

Theorem 1.2.8. *Let* $f : (X, \pi_X) \to (Y, \pi_Y)$ *be a map over* B_0 *of fibrant spaces over* B_0*. Then the* $\mathrm{coCyl}_{\mathrm{B}_0}(f)$ *over* B_0 *is a fibrant space over* B_0*.*

Proof. Let $g : (A, \pi_{Z|_A}) \to (\mathrm{coCyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{coCyl}_{\mathrm{B}_0}(f)})$ be a map over B_0 from a closed subspace $(A, \pi_{Z|A}$ of $(Z, \pi_Z) \in ob(\mathbf{M_{B_0}})$ to the $(\mathrm{coCyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{coCyl}_{\mathrm{B}_0}(f)}$. There exists an extension $\tilde{g}_2 : (Z, \pi_Z) \to (X, \pi_X)$ over B_0 of map $g_2 = \omega_1^{\#} \cdot g : (A, \pi_{Z|_A}) \to (X, \pi_X)$ over B_0. Note that from the equivalence a)⇔b) of Theorem 1.2.2 it follows that the inclusion $(X \times \{0\} \cup A \times I, \pi_{X \times \{0\} \cup A \times I}) \to (X \times I, \pi_{X \times I})$ over B_0 is an SSDR-map over B_0.

Let

$$G : (Z \times \{0\} \cup A \times I, \pi_{Z \times \{0\} \cup A \times I}) \to (Y, \pi_Y)$$

be a map given by formulas

$$G(z, 1) = f\tilde{g}_2(z), z \in X$$

and

$$G(a, t) = H(a, t), (a, t) \in A \times I,$$

where $H : (A \times I, \pi_{A \times I}) \to (Y, \pi_Y)$ is a map over B_0 given by $H(a, t) = ((f_{\omega_1} \cdot g)(a))(t)$.

As in the proof of Proposition 1.2.7 we can check up that there exists map $\tilde{g}_1$:

$(Z, \pi_Z) \to (Y^I, \pi_{Y^I_{\mathrm{B}_0}})$ over B_0 such that

$$\tilde{g}_1(z)(1) = f\tilde{g}_2(z), z \in Z.$$

Let $\tilde{g} = \tilde{g}_1 \Delta \tilde{g}_2 : (Z, \pi_Z) \to (\mathrm{coCyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{coCyl}_{\mathrm{B}_0}(f)})$. It is clear that $\tilde{g}_{|A} = g$.

Thus, the pair $(\mathrm{coCyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{coCyl}_{\mathrm{B}_0}(f)})$ is a fibrant space over B_0. □

Chapter 2

Fiber Strong Shape Classifications of Compact Metrizable Spaces

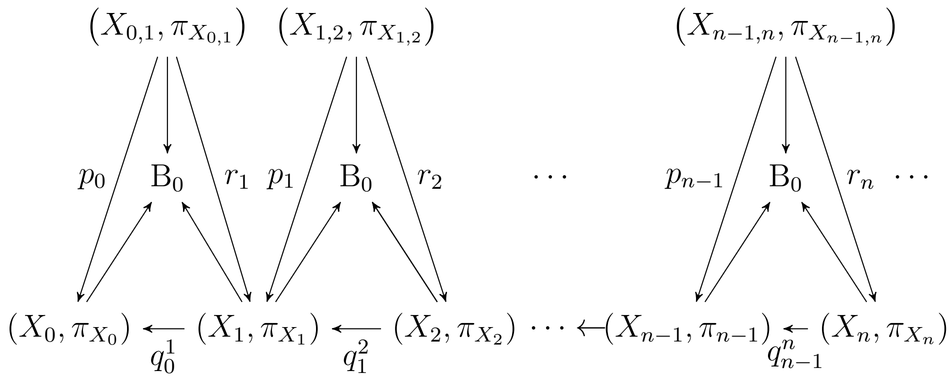
In chapter 2 are defined and studied fiber cotelescopes and $\mathrm{ANR}_{\mathrm{B}_0}$-resolutions, proved the theorem of existence of $\mathrm{ANR}_{\mathrm{B}_0}$-resolution, constructed fiber strong shape category of compact metrizable spaces and established the characterizations of fiber strong shape equivalences based on the notion of the double mapping cylinder over B_0.The constructed fiber strong shape category is the full image of functor reflector from the fiber homotopy category of compact metrizable spaces over B_0 in the fiber homotopy category of fiber fibrant spaces.

2.1 On Fiber Strong Shape Category of Compact Metrizable Spaces

First we consider cotelescopes of inverse sequences over B_0. Let $\mathbf{X} = \{(X_n, \pi_{X_n}), q_n^{n+1}, N^+\}$ be an inverse sequence over B_0. For each bonding map $q_n^{n+1} : (X_{n+1}, \pi_{X_{n+1}}) \to (X_n, \pi_{X_n})$ over B_0 consider the cocylinder $X_{n,n+1} = \mathrm{coCyl}_{\mathrm{B}_0}(q_n^{n+1})$ over B_0 of map

$q_n^{n+1} : (X_{n+1}, \pi_{X_{n+1}}) \to (X_n, \pi_{X_n})$ over B_0, fibration $p_n = \omega_1 \cdot f_{\omega_1} : X_{n,n+1} \to X_n$ over B_0 and the shrinkable fibration $r_{n+1} = \omega_1^{\#} : X_{n,n+1} \to X_{n+1}$ over B_0 with respect to the $\mathrm{SDR}_{\mathrm{B}_0}$-map $i_{n+1} : X_{n+1} \to X_{n,n+1}$.

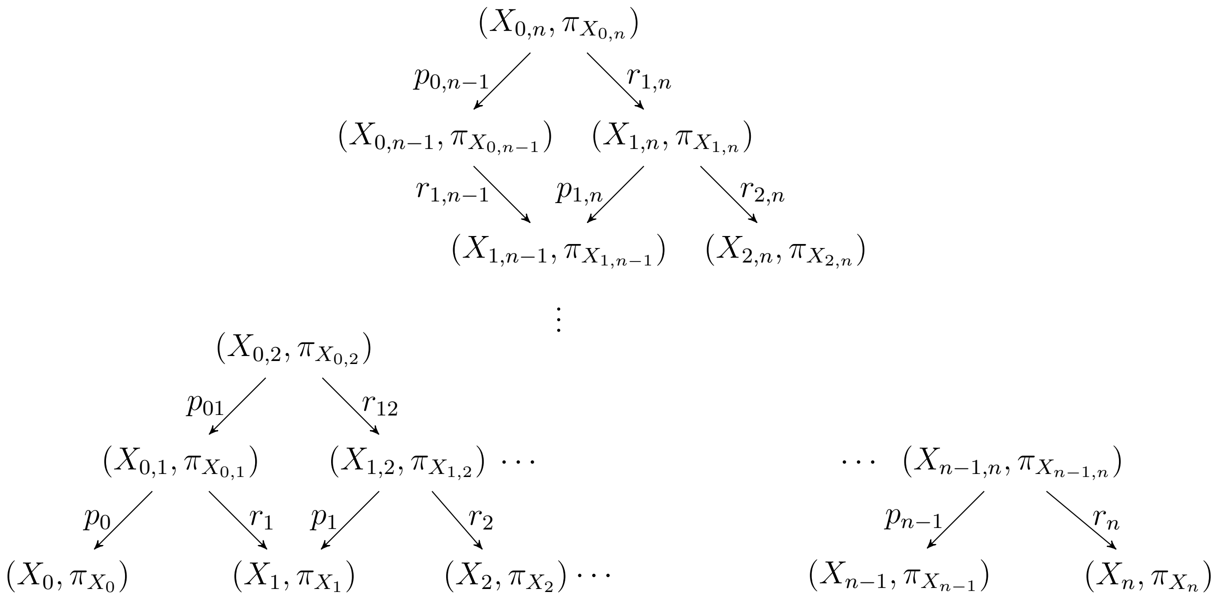
The cotelescope over B_0 of the inverse sequence $\mathbf{X}$, denoted by $\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X})$, is defined as the inverse limit of the diagram $T(\mathbf{X})$



By definition of cotelescope over B_0, $\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X}) = \varprojlim T(\mathbf{X})$ is a space over B_0 of points $(x_0, \omega_0, x_1, \omega_1, x_2, \omega_2, \cdots) \in \prod_{i=0}^{\infty} \left(X_i \times_{\mathrm{B}_0} X_i^I\right)$ for which

$$\omega_0(0) = x_0, \omega_0(1) = q_0^1(x_1), \omega_1(1) = q_1^2(x_2), \cdots .$$

Let $T_n(\mathbf{X})$ be a finite subdiagram consisting of first n numbers of diagram $T(\mathbf{X})$ and $\left(X_{0,n}, \pi_{X_{0,n}}\right) = \varprojlim T_n(\mathbf{X})$. Now consider the following diagram



Note that in this diagram $p_1, p_2, ..., p_n$ are fibrations over B_0. Hence, the maps $p_{n,m}$ also are fibrations over B_0 and $r_{n,m}$ maps are shrinkable fibrations with respect to maps $i_{n,m}$ since each r_n is a shrinkable fibration with respect to i_n. Changing r_n by i_n, $r_{n,m}$ by $i_{n,m}$ and putting $(\tilde{X}_0, \pi_{\tilde{X}_0}) = (X_0, \pi_{X_0})$, $\tilde{q}_0^1 = p_0, \tilde{i}_0 = 1_{X_0}, \tilde{i}_1 = i_1$, $(\tilde{X}_n, \pi_{\tilde{X}_n}) = (X_{0,n}, \pi_{X_{0,n}})$, $\tilde{q}_{n-1}^n = p_{0,n-1}$, $\tilde{i}_n = i_{1,n} \cdot \cdots \cdot i_{n-1,n} \cdot i_n$ for $n > 1$ we obtain the following inverse system $\tilde{\mathbf{X}} = ((\tilde{X}_n, \pi_{\tilde{X}_n}), \tilde{q}_n^{n+1}, N^+)$ and commutative diagram

$$\begin{array}{cccccccccc}
(\tilde{X}_0, \pi_{\tilde{X}_0}) & \xleftarrow{\tilde{q}_0^1} & (\tilde{X}_1, \pi_{\tilde{X}_1}) & \xleftarrow{\tilde{q}_1^2} & (\tilde{X}_2, \pi_{\tilde{X}_2}) & \leftarrow \cdots \leftarrow & (\tilde{X}_n, \pi_{\tilde{X}_n}) & \xleftarrow{\tilde{q}_n^{n+1}} & (\tilde{X}_{n+1}, \pi_{\tilde{X}_{n+1}}) & \longleftarrow \\
1_{\tilde{X}_0} = \tilde{i}_0 \uparrow & & \tilde{i}_1 \uparrow & & \tilde{i}_2 \uparrow & & \tilde{i}_n \uparrow & & \tilde{i}_{n+1} \uparrow & \\
(X_0, \pi_{X_0}) & \xleftarrow[q_0^1]{} & (X_1, \pi_{X_1}) & \xleftarrow[q_1^2]{} & (X_2, \pi_{X_2}) & \leftarrow \cdots \leftarrow & (X_n, \pi_{X_n}) & \xleftarrow[q_n^{n+1}]{} & (X_{n+1}, \pi_{X_{n+1}}) & \longleftarrow
\end{array}$$

Note that $\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X}) = \varprojlim \tilde{\mathbf{X}}$, $\tilde{q}_n^{n+1} : (\tilde{X}_{n+1}, \pi_{\tilde{X}_{n+1}}) \to (\tilde{X}, \pi_{\tilde{X}})$ is a fibration over B_0 and $\tilde{i}_n : (X_n, \pi_{X_n}) \to (\tilde{X}_n, \pi_{\tilde{X}_n})$ is $\mathrm{SDR}_{\mathrm{B}_0}$-map over B_0 for each $n \geqslant 0$. Also note that if all (X_n, π_{X_n}) are $\mathrm{ANR}_{\mathrm{B}_0}$-spaces (fibrant spaces over B_0), then all $(\tilde{X}_n, \pi_{\tilde{X}_n})$ are $\mathrm{ANR}_{\mathrm{B}_0}$-spaces (fibrant spaces over B_0). In particular, we have obtained the following theorem.

Theorem 2.1.1. *Let* $\mathbf{X} = ((X_n, \pi_{X_n}), q_n^{n+1}, N^+)$ *be an inverse sequence consisting of fibrant spaces over* B_0 *and maps over* B_0. *Then the cotelescope* $\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X})$ *is a fibrant space over* B_0. *If all* (X_n, π_{X_n}) *members of the inverse system* $\mathbf{X}$*are* $\mathrm{ANR}_{\mathrm{B}_0}$*-spaces, then* $\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X})$ *is a fibrant space over* B_0 *too.* □

Let $X = \varprojlim \mathbf{X}$ and $\mathbf{q} = \{q_n\}_{n \in N^+}$, where $q_n : X \to X_n$ are the natural projections over B_0. Then SSDR-maps $\tilde{i}_n$ over B_0 from the above given diagram induce the unique natural embedding $i_{\mathbf{q}} : (X, \pi_X) \to (\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X}), \pi_{\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X})})$ over B_0 such that $\tilde{q}_n \cdot i_{\mathbf{q}} = i_n \cdot q_n$ for each $n \geq 0$.

Definition 2.1.2. An inverse sequence $\mathbf{X} = ((X_n, \pi_{X_n}), q_n^{n+1}, N^+)$ is called resolution over B_0 of compact space (X, π_X) over B_0 if

a)$(X, \pi_X) = \varprojlim \mathbf{X}$;

b)the family $\mathbf{q} = \{q_n : (X, \pi_X) \to (X_n, \pi_{X_n})\}_{n \in N^+}$ satisfies the following condition: for each $n \in N^+$ and open neighbourhood U of $q_n(X)$ in (X, π_{X_n}) there exists $m \geq n$ such that $q_n^m(X_m) \subseteq U$.

If all the $(X_n, \pi_{X_n}) \in \mathrm{ANR}_{\mathrm{B}_0}$, then $\mathbf{q}$ is called an $\mathrm{ANR}_{\mathrm{B}_0}$-resolution over B_0.

Note that this definition of resolution over B_0 is a special case of the definition of resolution over B_0 given in [B4].

Now prove the theorem of existence of resolution over B_0 of compact metrizable spaces over B_0.

Theorem 2.1.3. *For each compact metrizable space* (X, π_X) *over* B_0 *there exists an* $\mathrm{ANR}_{\mathrm{B}_0}$*-resolution* $\mathbf{q} : (X, \pi_X) \to \mathbf{X}$ *over* B_0.

Proof. We can assume that (X, π_X) is a closed subspace of some $\mathrm{AR}_{\mathrm{B}_0}$-space (M, π_M). Indeed, there exists a closed embedding $j = i \triangle \pi_X : (X, \pi_X) \to (M, \pi_M) = (N \times \mathrm{B}_0, \pi_{N \times \mathrm{B}_0})$, where $i : X \to N$ is an closed inclusion of X into AR-space N. Let

X_n be the union $\bigcup_{x\in X} B(x, \frac{1}{n})$, where $B(x, \frac{1}{n})$ is the open ball in M with center x and radious $\varepsilon = \frac{1}{n}$. For any neighbourhood U of X in M and $x \in X$ there exists ε_x such that $B(x, \varepsilon_x) \subset U$. There exists a finite set $\{x_1, x_2, ..., x_k\} \subset X$ such that $X \subseteq \bigcup_{i=1}^{k} B(x_i, \varepsilon_{x_i})$. Let $\varepsilon = \frac{1}{n} \leq min\{\varepsilon_{x_1}, \varepsilon_{x_2}, ..., \varepsilon_{x_k}\}$. It is clear that $X_n = \bigcup_{x\in X} B(x, \frac{1}{n})$ has the property $X_n \subseteq U$. Note that obtained family of neighbourhoods of X in M form an inverse sequence $\mathbf{X} = (X_n, q_n^{n+1}, N^+)$ of $\mathrm{ANR}_{\mathrm{B}_0}$-spaces, where q_n^{n+1} is the inclusion of X_{n+1} into X_n. Since $X = \bigcap_{n=1}^{\infty} X_n$, we can conclude $(X, \pi_X) = \varprojlim \mathbf{X}$.

Therefore, the family $\mathbf{q} = \{q_n\}_{n\in N^+}$ of inclusions $q_n : (X, \pi_X) \to (X_n, \pi_{X_n})$ over B_0 form a resolution $\mathbf{q} : (X, \pi_X) \to \mathbf{X}$ over B_0 of space (X, π_X) over B_0. □

Theorem 2.1.4. *Let (X, π_X) be a compact metrizable space over* B_0. *If $\mathbf{q} : (X, \pi_X) \to \mathbf{X} = ((X_n, \pi_{X_n}), q_n^{n+1}, N^+)$ is a resolution over* B_0 *of (X, π_X), then there exists an infinite strong deformation*

$$D : \mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X}) \times [0, \infty) \to \mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X})$$

of $\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X})$ *over* B_0 *onto $i_{\mathbf{q}}(X)$. In particular, the map $i_{\mathbf{q}} : (X, \pi_X) \to$* $\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X})$ *is an* SSDR*-map over* B_0.

Proof. Let $\tilde{X} = \mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X})$. The projections $\tilde{q}_i : (\tilde{X}, \pi_{\tilde{X}}) \to (\tilde{X}_i, \pi_{\tilde{X}_i})$ over B_0 are fibrations over B_0 and they have fiber homotopy lifting property.

Hence, there are deformations $\tilde{D}_n : (\tilde{X} \times I, \pi_{\tilde{X}\times I}) \to (\tilde{X}_n, \pi_{\tilde{X}_n})$ over B_0 of $\tilde{X}$ onto $F_n = \tilde{q}_n^{-1} i_n(X_n)$. The family $\{F_n\}$ is a decreasing family of closed subsets of $\tilde{X}$, i.e. for each $n \geq 0$

$$\tilde{X} = F_0 \supset F_1 \supset ... \supset F_n \supset F_{n+1} \supset i_{\mathbf{q}}(X).$$

Since $\mathbf{q}$ is a resolution over B_0, then for each neighborhood $\tilde{U}$ of $i_{\mathbf{q}}(X)$ in $(\tilde{X}, \pi_{\tilde{X}})$ there exists an index m such that $F_m \subset \tilde{U}$. There are an index n and neighborhood $\tilde{V}$ of $q_n(i_{\mathbf{q}}(X))$ in $(\tilde{X}_n, \pi_{\tilde{X}_n})$ such that $\tilde{q}_n^{-1}(\tilde{V}) \subset \tilde{U}$. Let $V = \tilde{i}_n^{-1}(\tilde{U})$ and $q_n(X) \subset V \subset X_n$.

There is an index $m \geq n$ for which $q_n^m(X_m) \subset V$ and $\tilde{q}_n^m(i_m(X_m)) \subset \tilde{V}$. Note that

$$F_m = \tilde{q}_m^{-1}(\tilde{i}_m(X_m)) \subseteq \tilde{q}_n^{-1}(\tilde{q}_n^m(\tilde{i}_m(X_m))) \subseteq \tilde{q}_n^{-1}(\tilde{V}) \subset \tilde{U}.$$

The strong deformations D_i over B_0 induce the required infinite deformation D : $\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X}) \times [0, +\infty) \to \mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X})$ over B_0. □

The next theorem follows directly from Theorems 2.1.1, 2.1.3 and 2.1.4.

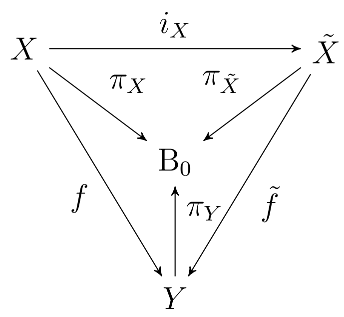
Theorem 2.1.5. *For each compact metrizable space (X, π_X) over B_0 there is a fibrant extension $i_X : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})$ over B_0. In particular, if $\mathbf{q} : (X, \pi_X) \to \mathbf{X} = ((X_n, \pi_{X_n}), q_n^{n+1}, N^+)$ is an $\mathrm{ANR}_{\mathrm{B}_0}$-resolution over B_0, then the embedding $i_{\mathbf{q}}$: $(X, \pi_X) \to (\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X}), \pi_{\mathrm{coTel}_{\mathrm{B}_0}(\mathbf{X})})$ is a fibrant extension over B_0.* □

The purpose of this section is to construct of fiber strong shape theory for compact metrizable spaces over a fixed base space B_0, using the fiber versions of cotelescop and fibrant space.

The constructed fiber strong shape category is the full image of functor reflector from the fiber homotopy category of compact metrizable spaces over B_0 in the fiber homotopy category of fiber fibrant spaces.

The obtained classification of spaces over B_0 demonstrates the advantage of fiber strong shape theory over fiber shape theory. Now define the fiber strong shape category $\mathbf{SSH_{B_0}}$ for compact metrizable spaces over B_0 in a quite usual way as the full image of some functor-reflector. Here we consider the reflector of the fiber homotopy category $\mathbf{H(CM_{B_0})}$ of compact metrizable spaces over B_0 in the fiber homotopy category $\mathbf{H(F_{B_0})}$ of fibrant spaces over B_0.

Let $(X, \pi_X) \in ob(\mathbf{CM_{B_0}})$ and $i_X : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})$ be a fibrant extension over B_0. For each map $f : (X, \pi_X) \to (Y, \pi_Y)$ over B_0, where (Y, π_Y) is a fibrant space over B_0, there exists a map $\tilde{f} : (\tilde{X}, \pi_{\tilde{X}}) \to (Y, \pi_Y)$ over B_0 such that the following diagram



commutes, i.e. $f = \tilde{f} \cdot i_X$. From Theorem 1.1.2 follows that if $f \underset{\mathrm{B}_0}{\simeq} f' : (\tilde{X}, \pi_{\tilde{X}}) \to (Y, \pi_Y)$ and $\tilde{f}' \cdot i_X = f'$, then $\tilde{f} \underset{\mathrm{B}_0}{\simeq} \tilde{f}'$. Hence, the map

$$[i_X]^{\#}_{\mathrm{B}_0} : [\tilde{X}, Y]_{\mathrm{B}_0} \to [X, Y]_{\mathrm{B}_0}$$

given by formula

$$[i_X]^{\#}_{\mathrm{B}_0}([\tilde{f}]_{\mathrm{B}_0}) = [\tilde{f} \cdot i_X]_{\mathrm{B}_0}$$

is bijective. Thus, we have the following.

Theorem 2.1.6. *Let* $i_X : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})$ *be a fibrant extension over* B_0 *of space* $(X, \pi_X) \in \mathbf{CM_{B_0}}$. *Then the morphism* $[i_X]_{\mathrm{B}_0} : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})$ *of category* $\mathbf{H}(\mathbf{CM_{B_0}})$ *is an* $\mathbf{H}(\mathbf{F_{B_0}})$*-reflection.* □

It is clear that the family $\{i_X : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})\}_{(X,\pi_X) \in ob(\mathbf{H}(\mathbf{CM_{B_0}}))}$ induces the $\mathbf{H}(\mathbf{F_{B_0}})$-reflector

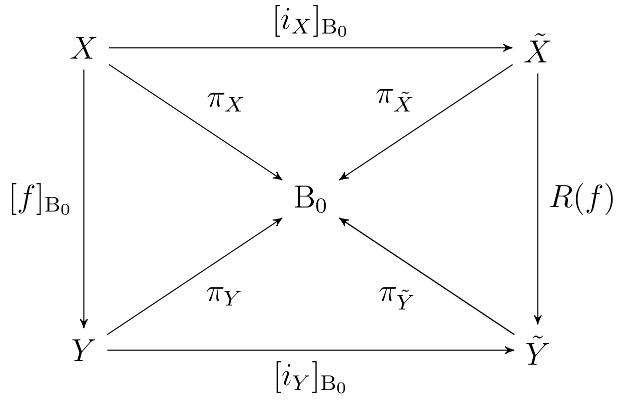
$$\mathrm{R} : \mathbf{H}(\mathbf{CM_{B_0}}) \to \mathbf{H}(\mathbf{F_{B_0}})$$

that is a functor given by formula

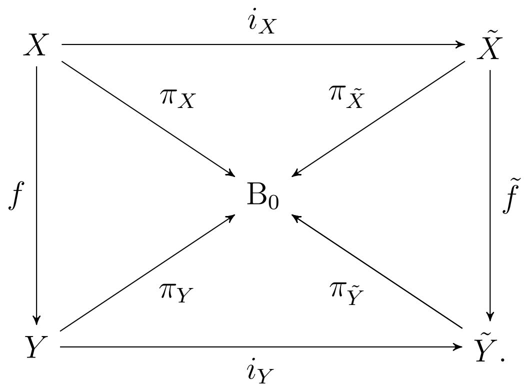
$$\mathrm{R}((X, \pi_X)) = (\tilde{X}, \pi_{\tilde{X}}), (X, \pi_X) \in ob(\mathbf{H}(\mathbf{CM_{B_0}}))$$

and satisfying the condition:

for each map $f : (X, \pi_X) \to (Y, \pi_Y)$ over B_0 of compact metrizable spaces the diagram



is commutative. Indeed, for the map there exists a unique up to fiber homotopy map $\tilde{f} : (\tilde{X}, \pi_{\tilde{X}}) \to (\tilde{Y}, \pi_{\tilde{Y}})$ such that the following diagram commutes



In this case the pair $(i_X, i_Y) : f \to \tilde{f}$ is called a fibrant extension over B_0 of map f.

Definition 2.1.7. The fiber strong shape category $\mathbf{SSH_{B_0}}$ of compact metrizable spaces over B_0 is full image of the reflector $R : \mathbf{H}(\mathbf{CM_{B_0}}) \to \mathbf{H}(\mathbf{F_{B_0}})$.

Note that

$$ob(\mathbf{SSH_{B_0}}) = ob(\mathbf{H}(\mathbf{CM_{B_0}}))$$

and

$$\mathrm{Mor}_{\mathbf{SSH_{B_0}}}((X, \pi_X), (Y, \pi_Y) = [(\tilde{X}, \pi_{\tilde{X}}), (\tilde{Y}, \pi_{\tilde{Y}})]_{B_0}, (X, \pi_X), (Y, \pi_Y) \in ob(\mathbf{HCM_{B_0}})).$$

Besides,

$$\mathrm{SS}_{\mathrm{B}_0}((X, \pi_X)) = (X, \pi_X)$$

for each $(X, \pi_X) \in ob(\mathbf{HCM_{B_0}})$ and

$$\mathrm{SS}_{\mathrm{B}_0}([f]_{\mathrm{B}_0}) = R([f]_{\mathrm{B}_0}) = [\tilde{f}]_{\mathrm{B}_0}$$

for a fibrant extension $(i_X, i_Y) : f \to \tilde{f} : (\tilde{X}, \pi_{\tilde{X}}) \to (\tilde{Y}, \pi_{\tilde{Y}})$ over B_0 of map $f : (X, \pi_X) \to (Y, \pi_Y)$ over B_0.

There is a commutative diagram

$$\begin{array}{ccc} \mathbf{H}(\mathbf{CM_{B_0}}) & \xrightarrow{\mathrm{R}} & \mathbf{H}(\mathbf{F_{B_0}}) \\ & {\scriptstyle \mathrm{SS}_{\mathrm{B}_0}} \searrow \quad \swarrow {\scriptstyle \mathrm{J}_{\mathrm{R}}} & \\ & \mathbf{SSH_{B_0}}. & \end{array}$$

2.2 On Fiber Strong Shape Equivalences of Compact Metrizable Spaces

The double mapping cylinder $\mathrm{dCyl}_{\mathrm{B}_0}(f)$ over B_0 of map $f : (X, \pi_X) \to (Y, \pi_Y)$ over B_0 is the subspace $X \times I \cup \mathrm{Cyl}_{\mathrm{B}_0}(f) \times \{0, 1\}$ of space $\mathrm{Cyl}_{\mathrm{B}_0}(f) \times I$ over B_0.

By J. Dydak and S.Nowak in ([Dy-N_1], [Dy-N_2]) were defined a strong shape equivalence. We give the definition of fiber version of strong shape equivalence.

Definition 2.2.1. A map $f : (X, \pi_X) \to (Y, \pi_Y)$ over B_0 is a shape equivalence if for each $\mathrm{ANR}_{\mathrm{B}_0}$-space (P, π_P) induces a bijection $f^* : [Y, P]_{\mathrm{B}_0} \to [X, P]_{\mathrm{B}_0}$. A fiber shape equivalence f is called a fiber strong shape equivalence if for any two maps $g, h : (Y, \pi_Y) \to (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$ over B_0 and a fiber homotopy $H : (X \times I, \pi_{X \times I}) \to (P, \pi_P)$ over B_0 joining $g\, f$ and $h\, g$, H is fiber homotopic rel $X \times \{0, 1\}$ to $H' \, (f \times 1_I)$, where $H' : (Y \times I, \pi_{Y \times I}) \to (P, \pi_P)$ is a fiber homotopy between g and h.

Theorem 2.2.2. *Let* $f : (X, \pi_X) \to (Y, \pi_Y)$ *be a fiber shape equivalence and let* $g : (\partial I^n \times Y, \pi_{\partial I^n \times Y}) \to (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$ *be a map over* B_0 *such that the composition* $g\,(1_{I^n} \times f) : (I^n \times X, \pi_{I^n \times X}) \to (P, \pi_P)$ *has an extension onto* $(I^n \times X, \pi_{I^n \times X})$. *Then* g *has an extension onto* $(I^n \times X, \pi_{I^n \times X})$.

Proof. The map $g : (\partial I^n \times Y, \pi_{\partial I^n \times Y}) \to (P, \pi_P)$ induce the map over B_0 from (Y, π_Y) into $(P^{\partial I^n}, \pi_{P^{\partial I^n}})$ which we also denoted by $g : (Y, \pi_Y) \to (P^{\partial I^n}, \pi_{P^{\partial I^n}})$.

Let $h : (X, \pi_X) \to (P^{\partial I^n}, \pi_{P^{\partial I^n}})$ be a fiber extension of $g\,f$. By condition of theorem f is a fiber shape equivalence. Hence, there exists a map $h' : (Y, \pi_Y) \to (P^{\partial I^n}, \pi_{P^{\partial I^n}})$ over B_0 such that $h'\,f \underset{\mathrm{B}_0}{\simeq} h$. By h' again denote map $h' : (Y^n \times I, \pi_{Y^n \times I}) \to (P, \pi_P)$ over B_0 induced by h'. From the relation $h'\,f \underset{\mathrm{B}_0}{\simeq} h$ and the equality $h = g\,f$ it follows that $h'f \underset{\mathrm{B}_0}{\simeq} g \cdot f$. Hence, $h'_{|\partial I^n \times Y} \underset{\mathrm{B}_0}{\simeq} g$. Since the pair $(I^n \times Y, \partial I^n \times Y)$ has the fiber homotopy extension property g extends onto $I^n \times Y$. □

Theorem 2.2.3. *Let* $f : (X, \pi_X) \to (Y, \pi_Y)$ *be a map over* B_0. *The following conditions are equivalent:*

1). f is a fiber strong shape equivalence;

2). for a given space (Z, π_Z) *over* B_0 *containing* (X, π_X) *as a closed subspace over* B_0, *every map* $g : (Z, \pi_Z) \to (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$ *over* B_0 *extends to* $(Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)})$ *and every map*

$$H : (Z \times I \cup \mathrm{dCyl}_{\mathrm{B}_0}(f), \pi_{Z \times I \cup \mathrm{dCyl}_{\mathrm{B}_0}(f)}) \to (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$$

over B_0 *extends to* $((Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)) \times I, \pi_{(Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)) \times I})$;

3). if (X, π_X) *is a closed subspace of* (Z, π_Z), *then the fiber inclusions*

$$i : (Z, \pi_Z) \to (Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)})$$

and

$$j : (Z \times I \cup \mathrm{dCyl}_{\mathrm{B}_0}(f), \pi_{Z\times I\cup \mathrm{dCyl}_{\mathrm{B}_0}(f)}) \to ((Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)) \times I, \pi_{(Z\cup \mathrm{Cyl}_{\mathrm{B}_0}(f))\times I})$$

are fiber shape equivalence;

4). if (X, π_X) is a closed subspace of (Z, π_Z), then the fiber inclusion

$$i : (Z, \pi_Z) \to (Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{Z\cup \mathrm{Cyl}_{\mathrm{B}_0}(f)})$$

is a fiber strong shape equivalences;

5). if (X, π_X) is a closed subspace of (Z, π_Z), then the fiber inclusion

$$i : (Z, \pi_Z) \to (Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{Z\cup \mathrm{Cyl}_{\mathrm{B}_0}(f)})$$

is a fiber shape equivalence;

6). the fiber inclusions

$$k : (X, \pi_X) \to (\mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(f)})$$

and

$$l : (\mathrm{dCyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{dCyl}_{\mathrm{B}_0}(f)}) \to (\mathrm{Cyl}_{\mathrm{B}_0}(f) \times I, \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(f)\times I})$$

are fiber shape equivalences;

7). every map $g : (X, \pi_X) \to (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$ over B_0 extends to $(\mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(f)})$ and every map $H : (\mathrm{dCyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{dCyl}_{\mathrm{B}_0}(f)}) \to (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$ over B_0 extends to $(\mathrm{Cyl}_{\mathrm{B}_0}(f) \times I, \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(f)\times I})$.

Proof. 1)⇒2). Let $g : (Z, \pi_Z) \to (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$ be a map over B_0. Consider the fiberpreserving restriction $g_{|X} : (X, \pi_X) \to (P, \pi_P)$. This map has a fiber extension

$g' : (\mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(f)}) \to (P, \pi_P)$. The maps g' and g induce a map $g'' : (Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{Z\cup\mathrm{Cyl}_{\mathrm{B}_0}(f)}) \to (P, \pi_P)$ over B_0 which is fiber extension of g.

Let $q : (\mathrm{dCyl}_{\mathrm{B}_0}(1_X), \pi_{\mathrm{dCyl}_{\mathrm{B}_0}(1_X)}) \to (\mathrm{dCyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{dCyl}_{\mathrm{B}_0}(f)})$ be the fiber natural projection and let $f' : (\mathrm{dCyl}_{\mathrm{B}_0}(1_X), \pi_{\mathrm{dCyl}_{\mathrm{B}_0}(1_X)}) \to (\mathrm{dCyl}_{\mathrm{B}_0}(1_Y), \pi_{\mathrm{dCyl}_{\mathrm{B}_0}(1_Y)})$ be a map over B_0 induced by f. Note that

$$H\,q \underset{\mathrm{B}_0}{\simeq} H'\,f' \,\mathrm{rel}\, X \times \{1\} \times \{0,1\},$$

where $H' : H_{|Y\times\{1\}\times\{0\}} \underset{\mathrm{B}_0}{\simeq} H_{|Y\times\{1\}\times\{1\}}$ is a homotopy over B_0.Consequently, the map H has a fiber extension onto $(Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f) \times I)$.

2)$\Rightarrow$3). Note that $i_* : [Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f), P]_{\mathrm{B}_0} \to [Z, P]_{\mathrm{B}_0}$ is the surjection for each $P \in \mathrm{ANR}_{\mathrm{B}_0}$. Prove that i_* is an injective map.

Let $g, h : (Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{Z\cup\mathrm{Cyl}_{\mathrm{B}_0}(f)}) \to (P, \pi_P)$ be maps over B_0 with some fiber homotopy

$$H : g_{|Z} \underset{\mathrm{B}_0}{\simeq} h_{|Z}.$$

There exists a map $G : (Z \times I \cup \mathrm{dCyl}_{\mathrm{B}_0}(f), \pi_{Z\times I\cup\mathrm{dCyl}_{\mathrm{B}_0}(f)}) \to (P, \pi_P)$ over B_0 such that

$$G_{|\mathrm{Cyl}_{\mathrm{B}_0}(f)\times\{0\}} = g,$$

and

$$G_{|\mathrm{Cyl}_{\mathrm{B}_0}(f)\times\{1\}} = h.$$

Let $G' : (\mathrm{Cyl}_{\mathrm{B}_0}(f) \times I, \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(f)\times I}) \to (P, \pi_P)$ be a fiber extension of G. Then $G' : g \underset{\mathrm{B}_0}{\simeq} h$. Now show that $j_* : [(Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)) \times I, P]_{\mathrm{B}_0} \to [Z \times I \cup \mathrm{dCyl}_{\mathrm{B}_0}(f), P]_{\mathrm{B}_0}$ is a bijection for each $(P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$. Let $G, H : ((Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)) \times I, \pi_{(Z\cup\mathrm{Cyl}_{\mathrm{B}_0}(f))\times I}) \to (P, \pi_P)$ be maps over B_0 whose restrictions on subspace $Z \times I \times \mathrm{dCyl}_{\mathrm{B}_0}(f)$ are fiber homotopic. Notice that

$$G_{|(Z\cup \mathrm{dCyl}_{\mathrm{B}_0}(f))\times\{0\}} \underset{\mathrm{B}_0}{\simeq} H_{|(Z\cup \mathrm{dCyl}_{\mathrm{B}_0}(f))\times\{0\}}.$$

Since the inclusion $(Z\cup \mathrm{Cyl}_{\mathrm{B}_0}(f))\times\{0\}\to(Z\cup \mathrm{Cyl}_{\mathrm{B}_0}(f))\times I$ is the fiber inclusion the maps G and H over B_0 are homotopic over B_0.

3)⇒4) Let $H:(Z\times I,\pi_{Z\times I})\to(P,\pi_P)\in \mathrm{ANR}_{\mathrm{B}_0}$ be a fiber homotopy between restrictions $g_{|Z}$ and $h_{|Z}$ of maps $g,h:(Z\cup \mathrm{Cyl}_{\mathrm{B}_0}(f),\pi_{Z\cup \mathrm{Cyl}_{\mathrm{B}_0}(f)})\to(P,\pi_P)$ over B_0. There exists an extension map $G:(Z\times I\cup \mathrm{dCyl}_{\mathrm{B}_0}(f),\pi_{Z\times I\cup \mathrm{dCyl}_{\mathrm{B}_0}(f)})\to(P,\pi_P)$ over B_0 of H such that $G_{|\mathrm{Cyl}_{\mathrm{B}_0}(f)\times\{0\}}=g$ and $G_{|\mathrm{Cyl}_{\mathrm{B}_0}(f)\times\{1\}}=h$. By condition iii) there exists a fiber homotopy extension $G':((Z\cup \mathrm{Cyl}_{\mathrm{B}_0}(f))\times I,\pi_{(Z\cup \mathrm{Cyl}_{\mathrm{B}_0}(f))\times I})\to(P,\pi_P)$of G. The pair $((Z\cup \mathrm{Cyl}_{\mathrm{B}_0}(f))\times I, Z\times I\cup \mathrm{dCyl}_{\mathrm{B}_0}(f))$ has the fiber homotopy extension property with respect to any space over B_0 because

$$(Z\times I\cup \mathrm{dCyl}_{\mathrm{B}_0}(f))\times I\cup \mathrm{Cyl}_{\mathrm{B}_0}(f)\times I\times\{0\}$$

is a fiber retract of

$$(Z\times I\cup \mathrm{dCyl}_{\mathrm{B}_0}(f)\cup Y\times I)\times I\cup \mathrm{Cyl}_{\mathrm{B}_0}(f)\times I\times\{0\}$$

and

$$(Z\times I\cup \mathrm{dCyl}_{\mathrm{B}_0}(f)\cup Y\times I)\times I\cup \mathrm{Cyl}_{\mathrm{B}_0}(f)\times I\times\{0\}$$

is a fiber retract of $(Z\cup \mathrm{Cyl}_{\mathrm{B}_0}(f))\times I\times I$. Consequently, G' is a fiber homotopy between g and h and the restriction of G' on $Z\times I$ is equal to G.

4)⇒5). The verification of this implications is trivial.

5)⇒6). Let $Z=X\times I\cup \mathrm{Cyl}_{\mathrm{B}_0}(f)\times\{0\}$. By condition v) we infer that the fiber inclusion $X\times I\cup \mathrm{Cyl}_{\mathrm{B}_0}(f)\times\{0\}\to \mathrm{dCyl}_{\mathrm{B}_0}(f)$ is a fiber shape equivalence. Consequently, the fiber inclusion $\mathrm{dCyl}_{\mathrm{B}_0}(f) \to \mathrm{Cyl}_{\mathrm{B}_0} \times I$ is shape equivalence. Besides, for $Z = X$ we get that $X \to \mathrm{Cyl}_{\mathrm{B}_0}(f)$ is a fiber shape equivalence over B_0.

6)⇒7). This implication is obvious because $(\mathrm{Cyl}_{\mathrm{B}_0}(f), X)$ and $(\mathrm{Cyl}_{\mathrm{B}_0}(f)\times I, \mathrm{dCyl}_{\mathrm{B}_0}(f))$ have the fiber homotopy extension property with respect to any space over B_0.

7)⇒1). Let $H : (\mathrm{dCyl}_{\mathrm{B}_0}(1_X), \pi_{\mathrm{dCyl}_{\mathrm{B}_0}(1_X)}) \to (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$ be a fiber homotopy between gf and hf, where $g, h : (Y, \pi_Y) \to (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$ are maps over B_0. There exists a map $G : (\mathrm{dCyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{dCyl}_{\mathrm{B}_0}(f)}) \to (P, \pi_P)$ over B_0 such that $G_{Y\times\{0\}} = g$ and $G_{Y\times\{1\}} = h$. Let $G' : (\mathrm{Cyl}_{\mathrm{B}_0}(f) \times I, \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(f)\times I}) \to (P, \pi_P)$ be an extension over B_0 of G. Using the fiber projection $\pi : X \times I \times I \to \mathrm{Cyl}_{\mathrm{B}_0}(f) \times I$ and strong fiber deformation retraction of $X \times I \times I$ onto $X \times \{1\} \times I$ we infer that H is fiber homotopic rel$X \times \{1\} \times \{0, 1\}$ to $H' \times (f \times 1_I)$, where $H' : (\mathrm{dCyl}_{\mathrm{B}_0}(1_Y), \pi_{\mathrm{dCyl}_{\mathrm{B}_0}(1_Y)}) \to (P, \pi_P)$ is a fiber homotopy between g and h. Hence, f is a fiber strong shape equivalence. □

Corollary 2.2.4. *Let (X, π_X) be a space over B_0 and $A \subset X$. The fiber inclusion $i : (A, \pi_{X|A}) \to (X, \pi_X)$ is a fiber strong shape equivalence if and only if i and j : $(X \times \{0\} \cup A \times I \cup X \times \{1\}, \pi_{X\times\{0\}\cup A\times I\cup X\times\{1\}}) \to (X \times I, \pi_{X\times I})$ are fiber shape equivalences.*

Proof. Let $f = i$. This corollary is straight consequence of equivalence of conditions 1) and 6). □

Corollary 2.2.5. *Let $f : (X, \pi_X) \to (Y, \pi_Y)$ be a fiber homotopy equivalence. Then f is a fiber strong shape equivalence.*

Proof. The space (X, π_X) is a strong deformation retract over B_0 of $\mathrm{Cyl}_{\mathrm{B}_0}(f)$. Hence, $(Y \times \{0\}, \pi_{Y\times\{0\}})$ is a strong deformation retract of $\mathrm{dCyl}_{\mathrm{B}_0}(f)$. Thus, the fiber inclusions of (X, π_X) into $\mathrm{Cyl}_{\mathrm{B}_0}(f)$ and $(\mathrm{dCyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{dCyl}_{\mathrm{B}_0}(f)})$ into $\mathrm{Cyl}_{\mathrm{B}_0}(f)\times I$ are fiber homotopy equivalences. □

Corollary 2.2.6. *If $g : (X, \pi_X) \to (Y, \pi_Y)$ is fiber homotopic to a fiber strong shape equivalence $f : (X, \pi_X) \to (Y, \pi_Y)$, then g is a fiber strong shape equivalence.*

Proof. The cylinder $\mathrm{Cyl}_{\mathrm{B}_0}(g)$ over B_0 is fiber homotopy equivalence to cylinder $\mathrm{Cyl}_{\mathrm{B}_0}(f)\mathrm{rel}X$. Hence, for every space (M, π_M) over B_0 containing X as a closed set the spaces $M \cup \mathrm{Cyl}_{\mathrm{B}_0}(f)$ and $Z \cup \mathrm{Cyl}_{\mathrm{B}_0}(g)$ over B_0 are fiber homotopy equivalent with respect M. By equivalence of conditions 1) and 5) of Theorem 3 g is a fiber strong shape equivalence. □

Now prove the following

Theorem 2.2.7. *Let $f : (X, \pi_X) \to (Y, \pi_Y)$ and $g : (Y, \pi_Y) \to (Z, \pi_Z)$ be fiber strong shape equivalences. Then $g\ f : (X, \pi_X) \to (Z, \pi_Z)$ fiberpreserving map is a fiber strong shape equivalence.*

Proof. It is clear that the composition $g\ f$ is a fiber shape equivalence. Let $\varphi, \psi : (Z, \pi_Z) \to (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$ be fiberpreserving maps and $H : (X \times I, \pi_{X \times I}) \to (P, \pi_P)$ be a fiber homotopy $H : \varphi g f \underset{\mathrm{B}_0}{\simeq} \psi g f$. By condition of theorem there exists a fiberpreserving homotopy $H' : (Y \times I, \pi_{Y \times I}) \to (P, \pi_P)$ between fiberpreserving maps $\varphi\, g$ and $\psi\, g$ such that

$$H \underset{\mathrm{B}_0}{\simeq} H'\,(f \times 1_I)\mathrm{rel}\, X \times \{0, 1\}.$$

Besides, there is a fiber homotopy $H'' : (Z \times I, \pi_{Z \times I}) \to (P, \pi_P)$ between φ and ψ such that

$$H' \underset{\mathrm{B}_0}{\simeq} H''\,(g \times 1_I)\mathrm{rel}\, Y \times \{0, 1\}.$$

Consequently, we have the following fiber homotopy

$$H \underset{\mathrm{B}_0}{\simeq} H''\,(g\ f \times 1_I)\mathrm{rel}\, X \times \{0.1\}.$$

□

Theorem 2.2.8. *Let $f : (X, \pi_X) \to (Y, \pi_Y)$ and $g : (Y, \pi_Y) \to (Z, \pi_Z)$ be maps over* B_0 *such that $g\,f$ is a fiber strong shape equivalence. If one of f and g is a fiber strong equivalence, then both f and g are fiber strong shape equivalences.*

Proof. By condition of theorem f and g are fiber shape equivalences. Let H be some map over B_0 from $(\mathrm{dCyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{dCyl}_{\mathrm{B}_0}(f)})$ into $(P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$. There is an fiber extension $H' : (X \times I \cup \mathrm{Cyl}_{\mathrm{B}_0}(f) \times \{0,1\} \cup \mathrm{Cyl}_{\mathrm{B}_0}(g) \times \{0,1\}, \pi_{X \times I \cup \mathrm{Cyl}_{\mathrm{B}_0}(f) \times \{0,1\} \cup \mathrm{Cyl}_{\mathrm{B}_0}(g) \times \{0,1\}}) \to (P, \pi_P)$ of H because the fiber inclusion $(Y, \pi_Y) \to (\mathrm{Cyl}_{\mathrm{B}_0}(g), \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(g)})$ is a fiber shape equivalence. By Corollary 2.5 of [F]$(\mathrm{Cyl}_{\mathrm{B}_0}(f) \cup \mathrm{Cyl}_{\mathrm{B}_0}(g), \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(f) \cup \mathrm{Cyl}_{\mathrm{B}_0}(g)})$ is fiber homotopy equivalent of $(\mathrm{Cyl}_{\mathrm{B}_0}(g\,f), \pi_{(\mathrm{Cyl}_{\mathrm{B}_0}(g\,f))})$. Besides, by condition gf is a fiber strong shape equivalence. Consequently, H' extends onto $(\mathrm{Cyl}_{\mathrm{B}_0}(f) \cup \mathrm{Cyl}_{\mathrm{B}_0}(g)) \times I$, and hence, on $\mathrm{Cyl}_{\mathrm{B}_0}(f) \times I$. Thus, by equivalence 1)⇔7) f is a fiber strong shape equivalence.

Let $H : (Y \times I, \pi_{Y \times I}) \to (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$ be a fiber homotopy berween $g\,\varphi$ and $g\,\psi$, where $\varphi, \psi : (Z, \pi_Z) \to (P, \pi_P)$. There is a fiber homotopy $H'' : (Z \times I, \pi_{Z \times I}) \to (P, \pi_P)$ such that $H'' : \varphi \underset{\mathrm{B}_0}{\simeq} \psi$, $H''\,(gf \times 1_I) \underset{\mathrm{B}_0}{\simeq} H\,(f \times 1_I) \mathrm{rel}\, X \times \{0,1\}$. Let $G : (Y \times \partial I^2, \pi_{Y \times \partial I^2}) \to (P, \pi_P)$ be a map over B_0 given by

$$G(y, 0, t) = H(y, t), y \in Y, t \in I,$$

$$G(y, 1, t) = H''(g(y), t), y \in Y, t \in I,$$

$$G(y, t, 0) = \varphi g(y), y \in Y, t \in I,$$

$$G(y, t, 1) = \psi g(y), y \in Y, t \in I.$$

Then $G(f \times 1_I) : (X \times (\partial I^2), \pi_{X \times (\partial I^2)}) \to (P, \pi_P)$ extends onto $X \times I^2$. By Theorem

2.2.2 the map G extends onto $Y \times I^2$. Hence, we have

$$H \underset{\mathrm{B}_0}{\simeq} H''(g \times 1_I) \mathrm{rel}\, Y \times \{0,1\}.$$

□

Corollary 2.2.9. *Let $f : (X, \pi_X) \to (Y, \pi_Y)$ be a fiber shape equivalence. If (X, π_X) has the fiber homotopy type of an $\mathrm{ANR}_{\mathrm{B}_0}$, then f is a fiber strong shape equivalence.*

Proof. Note that there is a map $g : (Y, \pi_Y) \to (X, \pi_X)$ over B_0 such that $g\,f \underset{\mathrm{B}_0}{\simeq} 1_X$. By Theorem 2.2.8, $g\,f$ is a fiber strong shape equivalence. Since $g\,f$ is fiber strong shape equivalence and f is fiber shape equivalences, then f and g are fiber strong shape equivalences. □

The next Theorem 2.2.10 and Theorem 2.2.11 show that in terms of fiber double cylinders it is possible to describe fiber strong shape isomorphisms of category $\mathbf{SSH_{B_0}}$.

Theorem 2.2.10. *A closed fiber embedding $i : (A, \pi_{X|A}) \to (X, \pi_X)$ is a fiber strong shape equivalence if and only if i is a* SSDR*-map over* B_0.

Proof. Let i is a SSDR-map over B_0. First show that the function $i_* : [X, P]_{\mathrm{B}_0} \to [A, P]_{\mathrm{B}_0}, (P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$ is a bijection. From the equivalence a)⇒ b) of Theorem 2.2.2 follows that i_* is a surjection because for each map $f : (A, \pi_{X|A}) \to (P, \pi_P)$ over B_0 there is a map $\tilde{f} : (X, \pi_X) \to (P, \pi_P)$ over B_0 such that $\tilde{f}\,i = f$ and $i_*(\tilde{f}) = f$. The map i_* over B_0 also is an injection. Indeed, let $g, h : (X, \pi_X) \to (P, \pi_P)$ be two maps over B_0 such that $i_*(h) = g = i_*(f)$, i.e. $h\,i \underset{\mathrm{B}_0}{\simeq} f \underset{\mathrm{B}_0}{\simeq} g\,i$. By fiber version of Borsuks homotopy extension theorem [Y$_2$] there exists maps $\tilde{f}_1, \tilde{f}_2 : (X, \pi_X) \to (P, \pi_P)$ over B_0 such that $\tilde{f}_{1|A} = f = \tilde{f}_{2|A}$, $\tilde{f}_1 \underset{\mathrm{B}_0}{\simeq} g$ and $\tilde{f}_2 \underset{\mathrm{B}_0}{\simeq} h$. By the implication a)⇒b) we have $\tilde{f}_1 \underset{\mathrm{B}_0}{\simeq} \tilde{f}_2 \mathrm{rel}\, i(A)$. Hence, $g \underset{\mathrm{B}_0}{\simeq} h$. One easily sees that $[q]_{\mathrm{B}_0} = [h]_{\mathrm{B}_0}$.

Let now $H : (A \times I, \pi_{A\times I}) \to (P, \pi_P)$ be a fiber homotopy between $g\,i$ and $h\,i$. Since $(P^I, \pi_{P^I}) \in \mathrm{ANR}_{\mathrm{B}_0}$, the function $i_* : [X, P^I]_{\mathrm{B}_0} \to [A, P^I]_{\mathrm{B}_0}$ is a bijection. Consequently, the function $(i \times 1_I)_* : [X \times I, P]_{\mathrm{B}_0} \to [A \times I, P]_{\mathrm{B}_0}$ is a bijection too. Hence, there exists a map $F : (X \times I, \pi_{X\times I}) \to (P, \pi_P)$ over B_0 and a fiber homotopy $S : (i \times 1_I)_*(F) = F\,(i \times 1_I) \underset{\mathrm{B}_0}{\simeq} H$. Let $G = S \cup F : X \times I \times \{0\} \cup (A \times I) \times I \to (P, \pi_P)$ be a map over B_0 given by formulas

$$G_{|X\times I\times\{0\}} = F,$$

$$G_{A\times I\times I} = S.$$

By Borsuk's fiber homotopy extension theorem there exists a map $\tilde{G} : (X \times I \times I, \pi_{X\times I}) \to (P, \pi_Y)$ over B_0 such that $\tilde{G}_{|X\times I\times\{1\}}$ is a fiber homotopy between fiber maps $\tilde{g} : (X, \pi_X) \to (P, \pi_P)$ and $\tilde{h} : (X, \pi_X) \to (P, \pi_P)$ given by formulas

$$\tilde{g}(x) = G(x, 1, 0), x \in X,$$

$$\tilde{h}(x) = \tilde{G}(x, 1, 1), x \in X,$$

$$\tilde{g}_{|A} = g\,i,$$

$$\tilde{h}_{|A} = h\,i.$$

By the Theorem 3 of [B,T$_1$], there exist fiber homotopies $T : g \underset{\mathrm{B}_0}{\simeq} \tilde{g}$ and $Q : \tilde{h} \underset{\mathrm{B}_0}{\simeq} h$. The combination of given fiber homotopies

$$L = T \cup \tilde{G}_{X\times I\times\{1\}} \cup Q : X \times I \times \{1\} \to (P, \pi_P)$$

is fiber homotopy between g and h. Note that

$$L\,(i \times 1_I) \underset{\mathrm{B}_0}{\simeq} H \,\mathrm{rel}\, A \times \{0,1\},$$

i.e. i is fiber strong shape equivalence.

Now prove inverse fact. Let i be a fiber strong shape equivalence. Then i_* is a bijection. Consequently, for each map $f : (A, \pi_{X|A}) \to (P, \pi_P)$ over B_0 there is a map $\tilde{F} : (X, \pi_X) \to (P, \pi_P)$ over B_0 such that $i_*(\tilde{F}) = \tilde{F}\, i \underset{\mathrm{B}_0}{\simeq} f$.

Using Borsuk's fiber homotopy extension theorem we can conclude that there exists a fiber extension $\tilde{f} : (X, \pi_X) \to (P, \pi_P)$ for which $\tilde{f} \underset{\mathrm{B}_0}{\simeq} F$.

Let $\tilde{f}_1, \tilde{f}_2 : (X, \pi_X) \to (P, \pi_P)$ be two such fiber extensions of f. Then there is a fiber homotopy $H' : \tilde{f}_1 \underset{\mathrm{B}_0}{\simeq} \tilde{f}_2$, for which

$$(i \times 1_Y)\, H' \underset{\mathrm{B}_0}{\simeq} H : f \underset{\mathrm{B}_0}{\simeq} f \,\mathrm{rel}\, A \times \{0,1\}.$$

Hence, by implication b)⇒a) of Theorem 2, i is an SSDR-map over B_0. □

Theorem 2.2.11. *Let $f : (X, \pi_X) \to (Y, \pi_Y)$ be a map over B_0 of compact metrizable spaces over B_0 and $(i_X, i_Y) : f \to \tilde{f}$ a fibrant extension over B_0 of f. Then f is a fiber strong shape equivalence if and only if $\tilde{f}$ is a fiber homotopy equivalence.*

Proof. It is known that $f = p\, i$, where $i : (X, \pi_X) \to (\mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(f)})$ and $p : (\mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(f)}) \to (Y, \pi_Y)$ are cofibration and fiber homotopy equivalence over B_0, respectively. Let $i_{\mathrm{Cyl}_{\mathrm{B}_0}(\mathrm{f})} : (\mathrm{Cyl}_{\mathrm{B}_0}(f), \pi_{\mathrm{Cyl}_{\mathrm{B}_0}(f)}) \to (\tilde{Z}, \pi_{\tilde{Z}})$ be a fibrant extension over B_0 of the mapping cylinder of$f$. There exist maps $\tilde{i} : (\tilde{X}, \pi_{\tilde{X}}) \to (\tilde{Z}, \pi_{\tilde{Z}})$ and $\tilde{p} : (\tilde{Z}, \pi_{\tilde{Z}}) \to (\tilde{Y}, \pi_{\tilde{Y}})$ over B_0 such that

$$i_{\mathrm{Cyl}_{\mathrm{B}_0}(f)}\, i = \tilde{i}\, i_X,$$

$$i_Y\, p = \tilde{p}\, i_{\mathrm{Cyl}_{\mathrm{B}_0}(f)}.$$

Let f be a fiber strong shape equivalence, in the sense of Definition 2.2.1. Since p is a fiber homotopy equivalence it is strong shape equivalence. Thus from the equality $f = p\, i$ it follows that i is a fiber strong shape equivalence. By Theorem 2.2.10 i is SSDR-map over B_0. Consequently, $\tilde{i}$ is a fiber homotopy equivalence. Hence, the composition $\tilde{p}\,\tilde{i}$ is a fiber homotopy equivalence. Note that $\tilde{p}\,\tilde{i}$ and $\tilde{f}$ are fiber extensions over B_0 of map f. Therefore $\tilde{p}\,\tilde{i} \underset{\mathrm{B}_0}{\simeq} \tilde{f}$. It follows that $\tilde{f}$ is fiber homotopy equivalence over B_0.

Now prove that if $\tilde{f} : (\tilde{X}, \pi_{\tilde{X}}) \to (\tilde{Y}, \pi_{\tilde{Y}})$ is a fiber homotopy equivalence then f is a fiber strong shape equivalence. Note that for each $P \in \mathrm{ANR}_{\mathrm{B}_0}$ the functions $\tilde{f}_* : [\tilde{Y}, P]_{\mathrm{B}_0} \to [\tilde{X}, P]_{\mathrm{B}_0}, (i_X)_* : [\tilde{X}, P]_{\mathrm{B}_0} \to [X, P]_{\mathrm{B}_0}$ and $(i_{\tilde{Y}})_* : [\tilde{Y}, P]_{\mathrm{B}_0} \to [Y, P]_{\mathrm{B}_0}$ are bijections. Since $(f)_*\,(i_Y)_* = (i_X)_*\,\tilde{f}_*$, we conclude f_* is a bijection too. The space $P^I_{\mathrm{B}_0}$ over B_0 is an $\mathrm{ANR}_{\mathrm{B}_0}$-space. Hence, $f_* : [Y, P^I_{\mathrm{B}_0}]_{\mathrm{B}_0} \to [X, P^I_{\mathrm{B}_0}]_{\mathrm{B}_0}$ is a bijection.

Let $H : g\, f \underset{\mathrm{B}_0}{\simeq} h\, f$ be a fiber homotopy, where $f, g : (Y, \pi_Y) \to (P, \pi_P)$ are maps over B_0. Then there exists a map $H' : (Y \times I, \pi_{Y \times I}) \to (P, \pi_P)$ over B_0 such that

$$H'\,(f \times 1_I) \underset{\mathrm{B}_0}{\simeq} H.$$

Using the argument of proof of Theorem 2.2.3 for fiber inclusion $i : (f(X), \pi_{Y|f(X)}) \to (Y, \pi_Y)$ we can construct a fiber homotopy $\tilde{H} : g \underset{\mathrm{B}_0}{\simeq} h$ for which $\tilde{H}\,(f \times 1_I) \underset{\mathrm{B}_0}{\simeq} H$. Thus, f is a fiber strong shape equivalence in the sense of Definition 2.2.1. □

Corollary 2.2.12. *A mapf over* B_0 *of compact metrizable spaces over* B_0 *is a fiber strong shape equivalence in the sense of Definition 2.2.1 if and only if* $\mathrm{SS}_{\mathrm{B}_0}([\mathrm{f}]_{\mathrm{B}_o})$ *is an isomorphism of the category* $\mathbf{SSH}_{\mathbf{B_0}}$. □

Chapter 3

Fiber Strong Shape Theory of Arbitrary Topological Spaces

In the Chapter 3 we construct and develop a fiber strong shape theory for arbitrary spaces over fixed metrizable space B_0. Our approach is based on the method of Mardešić-Lisica and instead of resolutions, introduced by Mardešić, their fiber preserving analogues are used. The fiber strong shape theory yields the classification of spaces over B_0 which is coarser than the classification of spaces over B_0 induced by fiber homotopy theory, but is finer than the classification of spaces over B_0 given by usual fiber shape theory.

3.1 Resolution and Strong Expansions of Spaces over B_0

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathscr{A}}$ be a covering of a space Y. We say that the maps $f, g : X \to Y$ are $\mathcal{U}$-near, if for every $x \in X$ there exists a $U_\alpha \in \mathcal{U}$ such that, $f(x), g(x) \in U_\alpha$. We say that a homotopy $H : X \times I \to Y$ which connects f and g, is a $\mathcal{U}$-homotopy if for every

$x \in X$ there exists a $U_\alpha \in \mathcal{U}$ such that $H(x,t) \subseteq U_\alpha$ for all $t \in I$.

Proposition 3.1.1. (Comp. [B_5],Proposition 7) Let (Y, π_Y) be a $\mathrm{ANR}_{\mathrm{B}_0}$.Then every open covering $\mathcal{U}$ of (Y, π_Y) admits an open covering $\mathcal{V}$ of Y such that, whenever any two f.p. maps $f, g : (X, \pi_X) \to (Y, \pi_Y)$ from an arbitrary space (X, π_X) over B_0 into the space (Y, π_Y) over B_0 are $\mathcal{V}$-near, then there exists f.p. $\mathcal{U}$-homotopy $H : (X \times I, \pi_{X\times I}) \to (Y, \pi_Y)$ which connects f and g. Moreover, if for a subset $A \subseteq X$, $f_{|A} = g_{|A}$, then H is f.p. homotopy rel$A$.

Proof. We may assume that (Y, π_Y) is a closed subset of space $\mathrm{B}_0 \times K$, where K is a convex set of normed vector space L. Let $\pi : \mathrm{B}_0 \times K \to K$ be the map given by the formula $\pi(b,k) = k$ for every $(b,k) \in \mathrm{B}_0 \times K$. Since (Y, π_Y) is an $\mathrm{ANR}_{\mathrm{B}_0}$, there is an open neighbourhood (G, π_G) of (Y, π_Y) in $\mathrm{B}_0 \times K$ together with a fibrewise retraction $r : (G, \pi_G) \to (Y, \pi_Y)$. Let $\{O_\alpha \times Q_\alpha\}_{\alpha\in\mathscr{A}}$ be a refinement of $r^{-1}(\mathcal{U})$, where Q_α is convex for every $\alpha \in \mathscr{A}$. Then $\mathcal{V} = \{(O_\alpha \times Q_\alpha) \cap Y\}_{\alpha\in\mathscr{A}}$ is an open refinement of the covering $\mathcal{U}$. For any two $\mathcal{V}$-near f.p. maps $f, g : (X, \pi_X) \to (Y, \pi_Y) \subseteq \mathrm{B}_0 \times K$ we can define a f.p. homotopy $H : (X \times I, \pi_{X\times I}) \to K$ by formula

$$H^{'}(x,t) = (\pi_X(x), (1-t)\pi(f(x)) + t\pi(g(x)), \qquad (x,t) \in X \times I.$$

Define a f.p. map $H : (X \times I, \pi_{X\times I}) \to (Y, \pi_Y)$ by taking

$$H(x,t) = r(H'(x,t)), \qquad (x,t) \in X \times I.$$

Clearly, we have $H_0 = f$, $H_1 = g$ and H is a $\mathcal{U}$-homotopy. Obviously, if $f(x) = g(x)$, for each $x \in A$, then $H(x,t) = f(x) = g(x)$ for every $t \in I$. □

An inverse system of the category $\mathbf{Top_{B_0}}$ is a collection $\mathbf{X} = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathscr{A})$ of space $(X_\alpha, \pi_{X_\alpha})$ over B_0 indexed by a directed set $\mathscr{A}$ and f.p. maps $p_{\alpha\alpha'} : (X_{\alpha'}, \pi_{X_{\alpha'}}) \to$

$(X_\alpha, \pi_{X_\alpha})$, $\alpha \le \alpha'$, such that $p_{\alpha\alpha'}\ p_{\alpha'\alpha''} = p_{\alpha\alpha''}$ and $p_{\alpha\alpha} = 1_{X_\alpha}$, $\alpha \in \mathscr{A}$.

A morphism $(f_\beta, \varphi) : \mathbf{X} \to \mathbf{Y} = ((Y_\beta, \pi_{Y_\beta}), q_{\beta\beta'}, \mathscr{B})$ of inverse systems of the category $\mathbf{Top_{B_0}}$ consists of a function $\varphi : \mathscr{B} \to \mathscr{A}$ and of f.p. maps $f_\beta : (X_{\varphi(\beta)}, \pi_{X_{\varphi(\beta)}}) \to (Y_\beta, \pi_{Y_\beta})$, $\beta \in \mathscr{B}$, such that whenever $\beta \le \beta'$, then there is an index $\alpha \ge \varphi(\beta), \varphi(\beta')$ for which $f_\beta\ p_{\varphi(\beta)} = q_{\beta\beta'}\ f_{\beta'}\ p_{\varphi(\beta')\alpha}$.

Two morphisms $(f_\beta, \varphi), (g_\beta, \psi) : \mathbf{X} \to \mathbf{Y}$ are said to be equivalent, $f \underset{\mathrm{B_0}}{\simeq} g$, provided for each $\beta \in \mathscr{B}$ there is an $\alpha \in \mathscr{A}$, $\alpha \ge \varphi(\beta), \psi(\beta)$, such that $f_\beta\ p_{\varphi(\beta)\alpha} = g_\beta\ p_{\psi(\beta)\alpha}$.

Let $\mathbf{pro} - \mathbf{Top_{B_0}}$ be a category, whose objects are the inverse systems $\mathbf{X}$ of the category $\mathbf{Top_{B_0}}$ and whose morphisms are the equivalence classes $\mathbf{f}$ of morphisms $(f_{\beta,\varphi}) : \mathbf{X} \to \mathbf{Y}$ with respect to relation $\underset{\mathrm{B_0}}{\simeq}$.

A morphism $\mathbf{p} = (p_\alpha) : (X, \pi_X) \to \mathbf{X} = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathscr{A})$ from a rudimentary system $((X, \pi_X))$ to an inverse system $\mathbf{X}$ consists of the f.p. maps $p_\alpha : (X, \pi_X) \to (X_\alpha, \pi_{X_\alpha})$,$\alpha \in \mathscr{A}$, such that $p_\alpha = p_{\alpha\alpha'}\ p_{\alpha'}$, $\alpha \le \alpha'$.

Definition 3.1.2 (V.Baladze, see [B$_4$]- [B$_6$])**.** *Let (X, π_X) be a space over* $\mathrm{B_0}$ *and let* $\mathbf{X} = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathscr{A})$ *be an inverse system of the category* $\mathbf{Top_{B_0}}$. *We say that* $\mathbf{p} : (X, \pi_X) \to \mathbf{X}$ *is a resolution over* $\mathrm{B_0}$ *or fiber resolution of the space (X, π_X) over* $\mathrm{B_0}$ *provided it satisfies the following two conditions:*

$\mathrm{R_{B_0}1}$). *Let $(P, \pi_P) \in \mathrm{ANR_{B_0}}$, let $\mathfrak{U}$ be an open covering of (P, π_P) and let $h : (X, \pi_X) \to (P, \pi_P)$ be a f.p. map. Then there exist an index $\alpha \in \mathscr{A}$ and a f.p. map $f : (X_\alpha, \pi_{(P,\pi_P)}) \to (P, \pi_P)$ such that h and $f\ p_\alpha$ are $\mathfrak{U}$-near.*

$\mathrm{R_{B_0}2}$). *Let $(P, \pi_P) \in \mathrm{ANR_{B_0}}$ and let $\mathfrak{U}$ be an open covering of (P, π_P). Then there is an open cover $\mathfrak{U}'$ of (P, π_P) with the following property: if $\alpha \in \mathscr{A}$ and $f, f' : (X, \pi_X) \to (P, \pi_P)$ are f.p. maps such that the f.p. maps $f\ p_\alpha$ and $f'\ p_\alpha$ are $\mathfrak{U}'$-near, then there is an index $\alpha' \ge \alpha$ such that the f.p. maps $f\ p_{\alpha\alpha'}$ and $f'\ p_{\alpha\alpha'}$ are $\mathfrak{U}$-near.*

If in a fiber resolution $\mathbf{p} : (X, \pi_X) \to \mathbf{X} = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathscr{A})$ *of the space* (X, π_X) *over* B_0 *each* $(X_\alpha, \pi_{X_\alpha})$ *is an* $\mathrm{ANR}_{\mathrm{B}_0}$*, then we say that* $\mathbf{p}$ *is a fiber* $\mathrm{ANR}_{\mathrm{B}_0}$*-resolution.*

The next theorem of V.Baladze ([B$_4$]- [B$_6$]) is essential in the construction of the fiber shape category for arbitrary spaces over B_0.

Theorem 3.1.3. *Every space* (X, π_X) *over a metrizable space* B_0 *admits an* $\mathrm{ANR}_{\mathrm{B}_0}$*-resolution over* B_0. □

Definition 3.1.4 (V.Baladze, see [B$_4$]- [B$_6$], [B$_{10}$])**.** *Let* (X, π_X) *be a topological space over* B_0*,* $\mathbf{X} = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathscr{A})$ *an inverse system in* $\mathbf{Top_{B_0}}$ *and* $\mathbf{p} = (p_\alpha) : (X, \pi_X) \to \mathbf{X}$ *a morphism of* $\mathbf{pro} - \mathbf{Top_{B_0}}$*. We call* $\mathbf{p}$ *an expansion over* B_0 *of the space* (X, π_X) *over* B_0 *provided it has the following properties:*

$\mathrm{E}_{\mathrm{B}_0}1)$. *For every* $\mathrm{ANR}_{\mathrm{B}_0}$*-space* (P, π_P) *over* B_0 *and f.p. map* $f : (X, \pi_X) \to (P, \pi_P)$ *there is an index* $\alpha \in \mathscr{A}$ *and a f. p. map* $h : (X_\alpha, \pi_{X_\alpha}) \to (P, \pi_P)$ *such that* $h\ p_\alpha \underset{\mathrm{B}_0}{\simeq} f$.

$\mathrm{E}_{\mathrm{B}_0}2)$. *If* $f, f' : (X_\alpha, \pi_{X_\alpha}) \to (P, \pi_P)$ *are f. p. maps,* $(P, \pi_P) \in \mathrm{ANR}_{\mathrm{B}_0}$ *and* $f\ p_\alpha \underset{\mathrm{B}_0}{\simeq} f'\ p_\alpha$*, then there is an index* $\alpha' \geq \alpha$ *such that* $f\ p_{\alpha\alpha'} \underset{\mathrm{B}_0}{\simeq} f'\ p_{\alpha\alpha'}$.

Definition 3.1.5. *A morphism* $\mathbf{p} : (X, \pi_X) \to ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathscr{A})$ *is called a strong expansion over* B_0 *provided it satisfies condition* $E_{\mathrm{B}_0}1)$ *and the following condition:*

$\mathrm{SE}_{\mathrm{B}_0}2)$. *Let* (P, π_P) *be an* $\mathrm{ANR}_{\mathrm{B}_0}$*-space, let* $f_0, f_1 : (X_\alpha, \pi_{X_\alpha}) \to (P, \pi_P)$, $\alpha \in \mathscr{A}$ *be f.p. maps and let* $F : (X \times I, \pi_{X\times I}) \to (P, \pi_P)$ *be a f.p. homotopy such that*

$$S(x, 0) = f_0 p_\alpha(x), \qquad x \in X,$$

$$S(x, 1) = f_1 p_\alpha(x), \qquad x \in X.$$

Then there exists a $\alpha' \geq \alpha$ *and a f.p. homotopy* $H : (X_{\alpha'} \times I, \pi_{X_{\alpha'}\times I}) \to (P, \pi_P)$*, such that*

$$H(x, 0) = f_0 p_{\alpha\alpha'}(z), \qquad z \in X_{\alpha'},$$

$$H(x,1) = f_1 p_{\alpha\alpha'}(z), \qquad z \in X_{\alpha'},$$

$$H(p_{\alpha'} \times 1_I) \underset{\mathrm{B}_0}{\simeq} S(\mathrm{rel}(X \times \partial I)).$$

It is clear that, every strong expansion over B_0 is an expansion over B_0.

If all $(X_\alpha, \pi_{X_\alpha}) \in \mathrm{ANR}_{\mathrm{B}_0}$, then $\mathbf{p}$ is called an $\mathrm{ANR}_{\mathrm{B}_0}$-expansion and strong $\mathrm{ANR}_{\mathrm{B}_0}$-expansion, respectively.

The main result of section 4.1 is the following theorem.

Theorem 3.1.6. *Let* (X, π_X) *be a topological space over* B_0*. Then every resolution* $\mathbf{p} : (X, \pi_X) \to \mathbf{X}$ *over* B_0 *induces a strong* $\mathrm{ANR}_{\mathrm{B}_0}$*-expansion.* □

Corollary 3.1.7. *Every* $\mathrm{ANR}_{\mathrm{B}_0}$*-resolution over* B_0 *induces* $\mathrm{ANR}_{\mathrm{B}_0}$*-expansion over* B_0. □

Corollary 3.1.8. *Every space* (X, π_X) *over* B_0 *admits a cofinite strong* $\mathrm{ANR}_{\mathrm{B}_0}$*-expansion over* B_0. □

In the proof of Theorem 3.1.6 we need the following lemmas.

Lemma 3.1.9. *Let* (X, π_X) *be a topological space over metrizable space* B_0*, let* $(P, \pi_P), (P', \pi_{P'})$ *be* $\mathrm{ANR}_{\mathrm{B}_0}$*-spaces, let* $f : (X, \pi_X) \to (P', \pi_{P'})$*,* $h_0, h_1 : (P', \pi_{P'}) \to (P, \pi_P)$ *be f.p. maps and let* $S : (X \times I, \pi_{X \times I}) \to (P, \pi_P)$ *be a f.p. homotopy such that*

$$S(x,0) = h_0 f(x), \qquad x \in X,$$

$$S(x,1) = h_1 f(x), \qquad x \in X.$$

Then there exists an $\mathrm{ANR}_{\mathrm{B}_0}$*-space* $(P'', \pi_{P''})$*, f.p. maps* $f' : (X, \pi_X) \to (P'', \pi_{P''})$*,*

$h : (P^{''}, \pi_{P^{''}}) \to (P^{'}, \pi_{P^{'}})$ *and a f.p. homotopy* $K : (P^{''} \times I, \pi_{P^{''} \times I}) \to (P, \pi_P)$ *such that*

$$
\begin{aligned}
hf^{'} &= f,\\
K(z,0) &= h_0 h(z), \qquad z \in P^{''}\\
K(z,1) &= h_1 h(z), \qquad z \in P^{''}\\
K(f^{'} \times 1_I) &= S.
\end{aligned}
$$

Proof. Let $S : (X \times I, \pi_{X\times I}) \to (P, \pi_P)$ be a map such that $S(x,0) = (h_0\ f)(x)$, $S(x,1) = (h_1\ f)(x)$ and $\pi_P\ S = \pi_{X\times I}$. Consider the subspace $C_{\mathrm{B}_0}(I,P)$ of the space $C(I,P)$. Let $\pi_{C_{\mathrm{B}_0}(I,P)} : C_{\mathrm{B}_0}(I,P) \to \mathrm{B}_0$ be the map given by $\pi_{C_{\mathrm{B}_0}(I,P)}(\varphi) = \pi_P(\varphi(t))$.

Consequently, $C_{\mathrm{B}_0}(I,P)$ is a space over B_0. The f.p. map $S : (X \times I, \pi_{X\times I}) \to (P, \pi_P)$ defines the map $s : (X, \pi_X) \to C_{\mathrm{B}_0}(I,P)$ such that $(s(x))(t) = S(x,t)$, $x \in X$, $t \in I$. The image of the point $x \in X$, $s(x) \in C_{\mathrm{B}_0}(I,P)$, because $\pi_P\ s(x) : I \to \mathrm{B}_0$ is a constant map. Indeed,

$$(\pi_P\ s(x))(t) = \pi_P(s(x))(t) = \pi_P(S(x,t)) = \pi_{X\times I}(x,t) = \pi_X(x)$$

for every $t \in I$.

For each $x \in X$ we have

$$(\pi_{C_{\mathrm{B}_0}(I,P)}\ s)(x) = (\pi_{C_{\mathrm{B}_0}(I,P)}(s(x)) = \pi_P(s(x))(t) =$$

$$= \pi_P(S(x,t)) = \pi_{X\times I}(x,t) = \pi_X(x).$$

Thus, $\pi_{C_{\mathrm{B}_0}(I,P)}\ s = \pi_X$. Hence, $s : (X, \pi_X) \to C_{\mathrm{B}_0}(I,P)$ is a f.p. map. For all $x \in X$ we have

$$(s(x))(0) = S(x,0) = (h_0\ f)(x)$$

and

$$(s(x))(1) = S(x,1) = (h_1\ f)(x).$$

Let $P^{'} \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I,P) = \{(y,\varphi)|\pi_{P'}(y) = \pi_{C_{\mathrm{B}_0}(I,P)}(\varphi)\}$. The map $f^{'} : (X,\pi_X) \to P^{'} \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I,P)$, given by $f^{'}(x) = (f(x),s(x))$, is a f.p. map. Let $\pi_{P' \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I,P)} : P^{'} \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I,P) \to \mathrm{B}_0$ be a map defined by

$$\pi_{P' \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I,P)}(y,\varphi) = \pi_{P'}(y) = \pi_{C_{\mathrm{B}_0}(I,P)}(y).$$

Then we have

$$\pi_{P' \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I,P)}\ f^{'} = \pi_{P' \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I,P)}(f(x),s(x)) = \pi_{P'}(f(x)) = \pi_X(x).$$

Thus, $\pi_X = \pi_{P' \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I,P)}\ f^{'}$.

It is clear that the first projection $h : P^{'} \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I,P) \to (P^{'},\pi_{P'})$ is a f.p. map and $h\ f^{'} = f$.

We define the subset $(P^{''},\pi_{P''})$ of $P^{'} \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I,P)$ be the following way:

$$P^{''} = \{(y,\varphi) \in P^{'} \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I,P)|\varphi(0) = h_0(y), h_1(y) = \varphi(1)\}.$$

Let $K : P^{'} \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I,P) \times I \to P$ be a map given by formula

$$K((y,\varphi),t) = \varphi(t), y \in P^{'}, \varphi \in C_{\mathrm{B}_0}(I,P), t \in P.$$

The restriction of K on $(P^{''} \times I, \pi_{P'' \times I})$ again denote by $K : (P^{''} \times I, \pi_{P'' \times I}) \to (P,\pi_P)$. This map is a f.p. homotopy between $h_0\ h_{|P''}$ and $h_1\ h_{|P''}$.

Indeed, for every $(y,\varphi) \in P^{''}$ and $t \in I$ we have

$$K((y,\varphi),0) = \varphi(0) = h_0(y) = h_0\ h(y,\varphi),$$

$$K((y,\varphi),1) = \varphi(1) = h_1(y) = h_1\ h(y,\varphi),$$

$$\pi_{P' \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I,P) \times I}((y,\varphi),t) = \pi_{P' \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I,P)}(y,\varphi) =$$

$$= \pi_{P'}(y) = \pi_{C_{\mathrm{B}_0}(I,P)}(\varphi) = \pi_P(\varphi(t)) = \pi_P(K(y,\varphi),t).$$

Note that for each $x \in X$ and $t \in I$

$$K(f^{'} \times 1_I)(x,t) = K((f(x),s(x)),t) = (s(x))(t) = (S(x,t)).$$

Hence, $K(f^{'} \times 1_I) = S$.

We shall prove that $(P^{''},\pi_{P^{''}}) \in \mathrm{ANR}_{\mathrm{B}_0}$. Now suppose that A is a closed subspace of a space Z over B_0 and $l : A \to P^{''}$ is a map such that $\pi_A = \pi_{Z|A} = \pi_P\ l$.

Denote by $L : (A \times I, \pi_{A\times I}) \to (P,\pi_P)$ the map defined by

$$L(a,t) = (h^{'}\ l(a))(t), (a,t) \in A \times I,$$

where $h^{'}$ is the second projection $P^{'} \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I,P) \to C_{\mathrm{B}_0}(I,P)$. It is clear that L is a f.p. map. Indeed,

$$(\pi_P\ L)(a,t) = \pi_P(L(a,t)) = \pi_P((h^{'}\ l(a))(t) =$$

$$= \pi_{C_{\mathrm{B}_0}(I,P)}(h^{'}(l(a))) = \pi_A(a) = \pi_{A\times I}(a,t).$$

The map L is a f.p. homotopy between $h_0\ h\ l$ and $h_1\ h\ l$. Indeed,

$$L(a,0) = (h^{'}\ l(a))(0) = h_0\ h\ l(a), \quad a \in A$$

and

$$L(a,1) = (h^{'}\ l(a))(1) = h_1\ h\ l(a), \quad a \in A.$$

Observe that, since $(P^{'}, \pi_{P'}) \in \mathrm{ANR}_{\mathrm{B}_0}$ and $h\ l : (A, \pi_A) \to (P^{'}, \pi_{P'})$ is a f.p. map, there is a neighbourhood U of A in Z and there exists a f.p. map $\tilde{l}^{'} : (U, \pi_U) \to (P^{'}, \pi_{P'})$ such that $\tilde{l}^{'}_{|A} = h\ l$.

There exist a neighbourhood V of A in U and a f.p. homotopy $\tilde{L} : (V \times I, \pi_{V\times I}) \to (P, \pi_P)$ between $h_0\ \tilde{l}^{'}_{|V}$ and $h_1\ \tilde{l}^{'}_{|V}$. Also note that $\tilde{L}(a,t) = L(a,t)$ for each $a \in A$ and $t \in I$. Let $\tilde{l}^{''}$ be a f.p. map $\tilde{l}^{''} : (V, \pi_V) \to C_{\mathrm{B}_0}(I, P)$, given by $(\tilde{l}^{''}(z))(t) = \tilde{L}(z,t)$, $z \in V, t \in I$. For every $a \in A$ we have

$$(\tilde{l}^{''}(a))(t) = \tilde{L}(a,t) = L(a,t) = (h^{'}\ l(a))(t).$$

Consequently, $\tilde{l}^{''}_{|A} = h^{'}\ l$. Now define the f.p. map $\tilde{l} : (V, \pi_V) \to P^{'} \times_{\mathrm{B}_0} C_{\mathrm{B}_0}(I, P)$ by the formula

$$\tilde{l}(z) = (\tilde{l}^{'}, \tilde{l}^{''}), \quad z \in V.$$

For each $z \in V$ we have

$$(\tilde{l}^{''}(z))(0) = \tilde{L}(z,0) = h_0\ \tilde{l}^{'}(z),$$

$$(\tilde{l}^{''}(z))(1) = \tilde{L}(z,1) = h_1\ \tilde{l}^{'}(z).$$

Consequently, $\tilde{l} : (V, \pi_V) \to (P^{''}, \pi_{P''})$ is an extension of the f.p. map $l : (A, \pi_A) \to (P^{''}, \pi_{P''})$. This fact completes the proof of lemma 3.1.9. □

Lemma 3.1.10. *Let* $\mathbf{p} : (X, \pi_X) \to \mathbf{X}$ *be a resolution over* B_0 *and let* α, P, f_0, f_1 *and* F *be as in* $\mathrm{SE}_{\mathrm{B}_0}2)$*. Then for every open covering* $\mathcal{U}$ *of* (P, π_P)*, there exist a* $\alpha' \geq \alpha$ *and a f.p. homotopy* $H : (X_{\alpha'} \times I, \pi_{X_{\alpha'} \times I}) \to (P, \pi_P)$ *such that*

$$H(y,0) = f_0\ p_{\alpha\alpha'}(y), \qquad y \in X_{\alpha'}$$

$$H(y,1) = f_1\ p_{\alpha\alpha'}(y), \qquad y \in X_{\alpha'}$$

$$(S, H(1 \times p_{\alpha'})) \leq \mathcal{U}.$$

Proof. Let $\mathcal{U}$ be an open covering of (P, π_P). There exists an open star-refinement $\mathcal{U}'$ of $\mathcal{U}$. Now we choose an open covering $\mathcal{V}$ of (P, π_P) such that the assertions of Proposition 3.1.1 hold for $\mathcal{U}'$. We can assume that $\mathcal{V}$ is a star-refinement of $\mathcal{U}'$. We choose $\mathcal{V}'$ so that $\mathcal{V}'$ is a star-refinement of $\mathcal{V}$ and $\mathrm{R}_{\mathrm{B}_0}2)$ holds for (P, π_P), $\mathcal{V}$ and $\mathcal{V}'$.

Let $P' = P \times_{\mathrm{B}_0} P$. By $g_0, g_1 : (P', \pi_{P'}) \to (P, \pi_P)$ denote the two projections. Let $f : (X, \pi_X) \to (P', \pi_{P'})$ be the diagonal product of f.p. maps $f_0\ p_\alpha : (X, \pi_X) \to (P, \pi_P)$ and $f_1\ p_\alpha : (X, \pi_X) \to (P, \pi_P)$. It is clear that $g_0\ f = f_0\ p_\alpha$, $g_1\ f = f_1\ p_\alpha$, $F_0 = g_0\ f$ and $F_1 = g_1\ f$.

By the lemma 3.1.9 there exists an $\mathrm{ANR}_{\mathrm{B}_0}$-space $(P'', \pi_{P''})$, f.p. maps $f' : (X, \pi_X) \to (P'', \pi_{P''})$, $g : (P'', \pi_{P''}) \to (P', \pi_{P'})$ and a f.p. homotopy $G : (P'' \times I, \pi_{P'' \times I}) \to (P, \pi_P)$ such that

$$g\ f' = f,$$

$$G_0 = g_0\ g, G_1 = g_1\ g,$$

$$G(f' \times 1) = F.$$

We choose for the open covering $G^{-1}(\mathcal{V}')$ of $(P'' \times I, \pi_{P'' \times I})$ a refinement, which is a stacked covering $\mathscr{V}$ of $(P'' \times I, \pi_{P'' \times I})$, given by a locally finite open covering $\mathcal{W}$ of $(P'', \pi_{P''})$ and by finite open coverings $\mathcal{J}_W, W \in \mathscr{W}$ of I.

By condition $R_{B_0}1)$ there exists a $\alpha^{''} \geq \alpha$ and f.p. mapping $h : (X_{\alpha''}, \pi_{X_{\alpha''}}) \to (P^{''}, \pi_{P''})$ such that

$$(f^{'}, h\ p_{\alpha''}) \leq \mathscr{W}$$

It is clear that for any $W \in \mathcal{W}$, $W \times 0 \subseteq W \times J$, where $J \in \mathcal{J}_{\mathcal{W}}$ and $W \times J \subset G^{-1}(V^{'})$ for some $V^{'} \in \mathcal{V}^{'}$.

Note that

$$g_0 g(W) = G_0(W) = G(W \times 0) \subseteq G(W \times J) \subseteq V^{'}.$$

Hence, $g_0 g(\mathscr{W})$ refines $\mathscr{V}^{'}$ and $(g_0\ g\ f^{'}, g_0\ g\ h\ p_{\alpha''}) \leq \mathscr{V}^{'}$.

From the equalities

$$g_0\ g f^{'} = g_0 f = f_0 p_\lambda = f_0 p_{\alpha\alpha'} p_{\alpha''}$$

it follows that

$$(g_0 g h p_{\alpha''}, f_0\ p_{\alpha\alpha''}\ p_{\alpha''}) \leq \mathcal{V}^{'}.$$

We also can claim that

$$(g_1 g h p_{\alpha''}, f_1\ p_{\alpha\alpha''}\ p_{\alpha''}) \leq \mathcal{V}^{'}.$$

By condition $R_{B_0}2)$ there is a $\alpha^{'} \geq \alpha^{''}$ such that

$$(g_0 g h p_{\alpha''\alpha'}, f_0\ p_{\alpha\alpha'}) \leq \mathcal{V}$$

and

$$(g_1 g h p_{\alpha''\alpha'}, f_1\ p_{\alpha\alpha'}) \leq \mathcal{V}.$$

Besides, there exist $\mathcal{U}'$-f.p. homotopies $K, L : (X_{\alpha'} \times I, \pi_{X_{\alpha'} \times I}) \to (P, \pi_P)$ such that $K_0 = f_0 p_{\alpha\alpha'}, K_1 = g_0 g h p_{\alpha''\alpha'}$, $L_0 = f_1 p_{\alpha\alpha'}$ and $L_1 = g_1 g h p_{\alpha''\alpha'}$.

Note that for any $t \in I$ the pairs $(f'(x), t)$ and $(hp_{\alpha''}(x), t)$ belong to some elements of $\mathscr{V}$ and consequently to $G^{-1}(V')$ for some $V' \in \mathcal{V}'$. Thus $G(f' \times 1_I)$ and $G(hp_{\alpha''} \times 1_I)$ are $\mathcal{V}'$-near. Hence,

$$(G(f' \times 1_I), G(hp_{\alpha''} \times 1_I)) \leq \mathcal{V}.$$

Now we define f.p. homotopy $H : (X_{\alpha'} \times I, \pi_{X_{\alpha'} \times I}) \to (P, \pi_P)$ by formulas

$$H(y,t) = \begin{cases} K(y, \frac{t}{\varphi(z)}), & 0 \leq t \leq \varphi(z), \\ G(z, \frac{t-\varphi(z)}{1-2\varphi(z)}), & \varphi \leq t \leq 1 - \varphi(z), \\ L(y, \frac{1-t}{\varphi(z)}), & 1 - \varphi(z) \leq t \leq 1, \end{cases}$$

where $z = hp_{\alpha''\alpha'}(y)$ and $\varphi : (P'', \pi_{P''}) \to I$ is a continuous map defined in [M$_2$].

As in [M$_2$] we can prove that for every $(x,t) \in X \times I$, there is a $U \in \mathscr{U}$ such that

$$F(x,t), H(p_\alpha(x), t) \in U.$$

$\square$

Proof of Theorem 3.1.6. . First prove the following condition.

$\mathrm{E_{B_0}}1)$. Let $\mathcal{U}$ be a open covering of (P, π_P). Consider open covering $\mathcal{V}$ as in Proposition 3.1.1. By $\mathrm{R_{B_0}}1)$ there exist an index $\alpha \in \mathscr{A}$ and a f.p. mapping $h : (X_\alpha, \pi_{X_\alpha}) \to (P, \pi_P)$ which satisfies condition $(hp_\alpha, f) \leq \mathcal{V}$. Thus, by the choice of $\mathcal{V}$, $f \underset{\mathrm{B_0}}{\simeq} h\ p_\alpha$.

$\mathrm{S_{B_0}}2)$. Let $\mathcal{U}$ be a open covering $\mathcal{U}$ of. Consider a covering $\mathcal{V}$ as in Proposition 3.1.1. By Lemma 3.1.9 there exist a $\alpha' \geq \alpha$ and f.p. homotopy $H : (X_{\alpha'} \times I, \pi_{X_{\alpha'} \times I}) \to (P, \pi_P)$

which satisfies

$$H(z,0) = f_0 p_{\alpha\alpha'}(z), \qquad z \in X_{\lambda'},$$

$$H(z,1) = f_1 p_{\alpha\alpha'}(z), \qquad z \in X_{\lambda'},$$

$$(S, H(1 \times p_{\alpha'})) \leq \mathcal{V}.$$

Consider the spaces $Z = X \times I$ and $A = X \times \partial I$ over B_0 and f.p. mappings $h_0 = F$ and $h_1 = H(p_{\alpha'} \times 1)$.

Note that $h_{0_{|A}} = h_{1_{|A}}$. Indeed, for each $x \in X$

$$h_0(x,0) = F(x,0) = f_0 p_\alpha(x) = f_0 p_{\alpha\alpha'} p_{\alpha'}(x) = H(p_{\alpha'}(x), 0) = h_1(x,0).$$

Analogously, for each $x \in X$ we have

$$h_0(x,1) = F(x,1) = f_1 p_\alpha(x) = f_1 p_{\alpha\alpha'} p_{\alpha'}(x) = H(p_{\alpha'}(x), 1) = h_1(x,0).$$

Consequently, $(h_0, h_1) \leq \mathcal{V}$. By Preposition 3.1.1 there exists a f.p. homotopy rel$(X \times \partial I)$, which connects F and $H(p_{\alpha'} \times 1_I)$. $\square$

3.2 On Fiber Strong Shape Category for Arbitrary Topological Spaces

Let Δ^n be the standard n-simplex, i.e. the set of all points $t = \{t = (t_0, t_1, \cdots, t_n) \in R^{n+1}\}$, where $t_0 \geq 0, \cdots, t_n \geq 0$ and $t_0 + \cdots + t_n = 1$.

For $n > 0$ and $0 \leq j \leq n$ there exist $\partial_j^n : \Delta^{n-1} \to \Delta^n$ j-th face operators and for $n \geq 0$ and $0 \leq j \leq n$ there exist $\sigma_j^n : \Delta^{n+1} \to \Delta^n$ j-th degeneracy operators given by formulas

$$\partial_j^n(t_0, \cdots, t_{n-1}) = (t_0, \cdots, t_{j-1}, 0, t_j, \cdots, t_{n-1}),$$

$$\sigma_j^n(t_0, \cdots, t_{n+1}) = (t_0, \cdots, t_{j-1}, t_j + t_{j+1}, t_{j+2}, \cdots, t_{n+1}).$$

Let $\mathscr{B}$ be a directed set. By $\mathscr{B}^n$ denote the set of all sequences $\boldsymbol{\beta} = (\beta_0, \cdots, \beta_n)$, $\beta_0 \le \cdots \le \beta_n$ of elements of $\mathscr{B}$.

For $n > 0$ and $0 \le j \le n$ we consider the j-th face operator $d_j^n : \mathscr{B}^n \to \mathscr{B}^{n-1}$ given by formula

$$d_j^n(\beta_0, \cdots, \beta_n) = (\beta_0, \cdots, \beta_{j-1}, \beta_{j+1}, \cdots, \beta_n)$$

and for $n \ge 0$ and $0 \le j \le n$ by s_j^n we denote j-th degeneracy operator $s_j^n : \mathscr{B}^n \to \mathscr{B}^{n+1}$ given by formula

$$s_j^n(\beta_0, \cdots, \beta_n) = (\beta_0, \cdots, \beta_j, \beta_j, \cdots, \beta_n).$$

For simplicity the images $d_j^n(\boldsymbol{\beta})$ and $s_j^n(\boldsymbol{\beta})$ we denote by $\boldsymbol{\beta}_j$ and $\boldsymbol{\beta}^j$, respectively.

Let $\mathbf{X} = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathscr{A})$ and $\mathbf{Y} = ((Y_\beta, \pi_{Y_\beta}), p_{\beta\beta'}, \mathscr{B})$ be the objects of category $\mathbf{pro} - \mathbf{Top_{B_0}}$.

A coherent map $f : \mathbf{X} \to \mathbf{Y}$ over B_0 or fiber preserving (f.p) coherent map consists of function $\varphi : \mathscr{B}^n \to \mathscr{A}$ and fiber preserving maps $f_{\boldsymbol{\beta}} : X_{\varphi(\boldsymbol{\beta})} \times \Delta^n \to Y_{\beta_0}$, $\boldsymbol{\beta} = (\beta_0, \cdots, \beta_n) \in \mathscr{B}^n$, $n \ge 0$ having the following properties:

i). The function φ, which assigns to every $n \ge 0$ and $\boldsymbol{\beta} = (\beta_0, \cdots, \beta_n) \in \mathscr{B}^n$ an element $\varphi(\boldsymbol{\beta}) = \varphi(\beta_0, \cdots, \beta_n) \in \mathscr{A}$, satisfies condition:

$$\varphi(\boldsymbol{\beta}) \ge \varphi(\boldsymbol{\beta}_j), \qquad 0 \le j \le n, n > 0.$$

ii). For every $n \ge 0$ and every $\boldsymbol{\beta} = (\beta_0, \cdots, \beta_n) \in \mathscr{B}^n$ the fiber preserving maps

$f_{\boldsymbol{\beta}} : (X_{\varphi(\boldsymbol{\beta})} \times \Delta^n, \pi_{X_{\varphi(\boldsymbol{\beta})}\times\Delta^n}) \to (Y_{\beta_0}, \pi_{Y_{\beta_0}})$ satisfies condition:

$$f_{\boldsymbol{\beta}}(x, \partial_j^n t) = \begin{cases} q_{\beta_0\beta_1} f_{\boldsymbol{\beta_0}}(p_{\varphi(\boldsymbol{\beta_0})} p_{\varphi(\boldsymbol{\beta})}(x), t), & j = 0 \\ f_{\boldsymbol{\beta_j}}(p_{\varphi(\boldsymbol{\beta_j})} p_{\varphi(\boldsymbol{\beta})}(x), t), & 0 \le j \le n, \end{cases}$$

where $x \in X_{\varphi(\boldsymbol{\beta})}$, $t \in \Delta^{n-1}$, $n \ge 0$, $X_{\varphi(\boldsymbol{\beta})} \times \Delta^n$ is the space over B_0 with projection $\pi_{X_{\varphi(\boldsymbol{\beta})}\times\Delta^n} : X_{\varphi(\boldsymbol{\beta})} \times \Delta^n \to \mathrm{B}_0$ given by formula

$$\pi_{X_{\varphi(\boldsymbol{\beta})}\times\Delta^n}(x, t) = \pi_{X_{\varphi(\boldsymbol{\beta})}}(x), \qquad x \in X_{\varphi(\boldsymbol{\beta})}, t \in \Delta^n$$

and

$$f_{\boldsymbol{\beta}}(p_{\varphi(\boldsymbol{\beta})\varphi(\boldsymbol{\beta_j})}(x), \sigma_j^n(t)) = f_{\boldsymbol{\beta^j}}(x, t), 0 \le j \le n, x \in X_{\varphi(\boldsymbol{\beta^j})}, t \in \Delta^{n+1}, n \ge 0.$$

The identity coherent map $1_{\mathbf{X}} : \mathbf{X} \to \mathbf{X}$ over B_0 is given by formulas:

$$\varphi(\boldsymbol{\alpha}) = \alpha_n, \boldsymbol{\alpha} = (\alpha_0, \cdots, \alpha_n) \in \mathscr{A}^n,$$

$$1_{\boldsymbol{\alpha}}(x, t) = p_{\alpha_0\alpha_n}(x), x \in X_{\alpha_n}, t \in \Delta^n, n \ge 0.$$

A coherent homotopy over B_0 or fiber preserwing (f.p.) homotopy $F : \mathbf{X} \times I \to \mathbf{Y}$ connecting f.p. coherent maps $f, f' : \mathbf{X} \to \mathbf{Y}$, is a f.p. coherent map of $\mathbf{X} \times I = ((X_\alpha \times I, \pi_{X_\alpha \times I}), p_{\alpha\alpha'} \times 1_I, \mathscr{A})$ to $\mathbf{Y}$, given by a function Φ and by f.p. maps $F_{\boldsymbol{\beta}} : (X_{\varphi(\boldsymbol{\beta})} \times I \times \Delta_n, \pi_{X_{\varphi(\boldsymbol{\beta^j})}\times I\times\Delta_n}) \to (Y_{\beta_0}, \pi_{Y_{\beta_0}})$, witch have i) and ii) properties and satisfy the conditions

$$\Phi(\boldsymbol{\beta}) \ge \varphi(\boldsymbol{\beta}), \varphi'(\boldsymbol{\beta}),$$

$$F_{\boldsymbol{\beta}}(x, 0, t) = f_{\boldsymbol{\beta}}(p_{\varphi(\boldsymbol{\beta})\Phi(\boldsymbol{\beta})}(x), t),$$

$$F_{\boldsymbol{\beta}}(x,1,t) = f'_{\boldsymbol{\beta}}(p_{\varphi'(\boldsymbol{\beta})\Phi(\boldsymbol{\beta})}(x),t),$$

where $x \in X_{\varphi(\boldsymbol{\beta})}, t \in \Delta^n, n \geq 0$.

As in [L-M] we can prove the following

Proposition 3.2.1. *The f.p. coherent homotopy relation of f.p. coherent maps is an equivalence relation.* □

A f.p. coherent map $f : \mathbf{X} \to \mathbf{Y}$ is called a special f.p. coherent map or a special coherent map over B_0 if $\varphi(\boldsymbol{\beta}) = \varphi(\beta_n)$ for each $\boldsymbol{\beta} \in \mathscr{B}^n$ and $\varphi_{|\mathscr{B}} : \mathscr{B} \to \mathscr{A}$ is an increasing function.

The composition $h = g\ f$ of special f.p. coherent maps over B_0 is defined as in [L-M].

A special f.p. coherent homotopy connecting two special f.p. coherent maps $f, f' :$ $\mathbf{X} \to \mathbf{Y}$ is a f.p. coherent homotopy $F : \mathbf{X} \times I \to \mathbf{Y}$ between f and f' and at the same time it is a special f.p. coherent map.

Note that if the index set $\mathscr{B}$ of $\mathbf{Y}$ is cofinite, then special f.p. coherent homotopy relation of special f.p. coherent maps is an equivalence relation.

The proofs of the following proposition pass as in [L-M].

Proposition 3.2.2. *Let $f, f' : \mathbf{X} \to \mathbf{Y}$, $g, g' : \mathbf{Y} \to \mathbf{Z} = ((Z_\gamma, \pi_{Z_\gamma}), r_{\gamma\gamma'}, \mathscr{C})$ be special f.p. coherent maps and let F, G be special f.p. coherent homotopies connecting f with f' and g with g', respectively. If the index set $\mathscr{C}$ is cofinite, then there is a special f.p. coherent homotopy connecting $g\ f$ and $g'\ f'$.* □

Proposition 3.2.3. *If $f : \mathbf{X} \to \mathbf{Y}$, $g : \mathbf{Y} \to \mathbf{Z}$ and $h : \mathbf{Z} \to \mathbf{W}$ are special f.p. coherent maps of inverse systems of $\mathbf{Top_{B_0}}$ over cofinite index sets, then there is a special f.p. coherent homotopy connecting $h(gf)$ with $(hg)f$.* □

Proposition 3.2.4. *If $f : \mathbf{X} \to \mathbf{Y}$ is a special f.p. coherent map of inverse systems of $\mathbf{Top_{B_0}}$ over cofinite index sets and $1_{\mathbf{X}}$ and $1_{\mathbf{Y}}$ are the f.p. coherent identity maps, then there exist special f.p. coherent homotopies connecting $f\ 1_{\mathbf{X}}$ with f and $1_{\mathbf{Y}}\ f$ with f.* □

As in [L-M] we can show that whenever the index set $\mathscr{B}$ of $\mathbf{Y}$ is cofinite, then every f.p. coherent homotopy class $[f] : \mathbf{X} \to \mathbf{Y}$ of f.p. coherent maps $f : \mathbf{X} \to \mathbf{Y}$ contains a unique f.p. coherent homotopy class of special f.p. coherent maps. Consequently, in the cofinite case one can define composition of f.p. coherent homotopy classes by composing their special representatives.

Now define the following category. The f.p. coherent prohomotopy category $\mathbf{CPHTop_{B_0}}$ has as objects inverse systems $\mathbf{X} = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathscr{A})$ of topological spaces over B_0 and f.p. maps over directed cofinite index sets. The morphisms are f.p. coherent homotopy classes $[f] : \mathbf{X} \to \mathbf{Y}$ of f.p. coherent maps $f : \mathbf{X} \to \mathbf{Y}$ of such systems. Composition is defined by composing representatives, which are special f.p. coherent maps. Identity morphism of $\mathbf{X}$ is the class, containing the coherent map $1_{\mathbf{X}} : \mathbf{X} \to \mathbf{X}$.

Now define the functor $C : \mathbf{pro} - \mathbf{Top_{B_0}} \to \mathbf{CPHTop_{B_0}}$. Let $(f_\beta, \varphi) : \mathbf{X} \to \mathbf{Y}$ be a map of inverse systems. We associate with (f_β, φ) a f.p. coherent map $f : \mathbf{X} \to \mathbf{Y}$. For this aim we extend $\varphi : \mathscr{B} \to \mathscr{A}$ to a function φ defined for all $\boldsymbol{\beta} = (\beta_0, \cdots, \beta_n)$ in such a way that

$$\varphi(\boldsymbol{\beta}) \geq \varphi(\beta_j), 0 \leq j \leq n.$$

We use the method of induction. Let $n = 1$ and $\boldsymbol{\beta} = (\beta_0, \beta_1)$. Note that

$$f_{\beta_0}\ p_{\varphi(\beta_0)\varphi(\beta_0,\beta_1)} = q_{\beta_0\beta_1} f_{\beta_1}\ p_{\varphi(\beta_1)\varphi(\beta_0,\beta_1)}.$$

Let $f_{\boldsymbol{\beta}} : (X_{\varphi(\boldsymbol{\beta})} \times \Delta^n, \pi_{X_{\varphi(\boldsymbol{\beta})} \times \Delta^n}) \to (Y_{\beta_0}, \pi_{Y_{\beta_0}})$ a f.p. mapping defined by

$$f_{\boldsymbol{\beta}}(x,t) = f_{\beta_0} p_{\varphi(\beta_0)\varphi(\boldsymbol{\beta})}(x), \qquad x \in X_{\varphi(\boldsymbol{\beta})}, t \in \Delta^n.$$

Also note that

$$f_{\boldsymbol{\beta}}(x, \partial_0^n t) = f_{\beta_0} p_{\varphi(\beta_0)\varphi(\boldsymbol{\beta})}(x) = q_{\beta_0\beta_1} f_{\beta_1} p_{\varphi(\beta_1)\varphi(\boldsymbol{\beta})}(x) = q_{\beta_0\beta_1} f_{\boldsymbol{\beta_0}}(p_{\varphi(\boldsymbol{\beta_0})\varphi(\boldsymbol{\beta})}(x), t)$$

and

$$f_{\boldsymbol{\beta}}(x, \partial_j^n t) = f_{\beta_0} p_{\varphi(\beta_0)\varphi(\boldsymbol{\beta})}(x) = f_{\boldsymbol{\beta_j}}(p_{\varphi(\boldsymbol{\beta_j})\varphi(\boldsymbol{\beta})}(x), t), \quad 0 < j \leq n,$$

$$f_{\boldsymbol{\beta}}(p_{\varphi(\boldsymbol{\beta})\varphi(\boldsymbol{\beta}^j)}(x), \sigma_j^n t) = f_{\beta_0} p_{\varphi(\beta_0)\varphi(\boldsymbol{\beta}^j)}(x) = f_{\boldsymbol{\beta}^j}(x,t), \quad 0 \leq j \leq n.$$

Let $\varphi^{'}$ be another extension of φ. We obtain another f.p. coherent map $f^{'}$. Note that f and $f^{'}$ are f.p. coherently homotopic.

Let $(f_\beta, \varphi), (f_\beta^{'}, \varphi^{'}) : \mathbf{X} \to \mathbf{Y}$ are equivalent morphisms. As in [L-M] we can show that the associated f.p. coherent maps f and $f^{'}$ are connected by some f.p. coherent homotopy $F : \mathbf{X} \times I \to \mathbf{Y}$.

Thus, to every morphism of $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ of $\mathbf{pro}-\mathbf{Top}_{\mathbf{B_0}}$ we can associate a morphism $[f] = \mathrm{C}(\mathbf{f})$ of $\mathbf{CPHTop}_{\mathbf{B_0}}$. If we restrict $\mathbf{pro} - \mathbf{Top}_{\mathbf{B_0}}$ to inverse systems over cofinite index sets, then we have defined a functor $\mathrm{C} : \mathbf{pro} - \mathbf{Top}_{\mathbf{B_0}} \to \mathbf{CPHTop}_{\mathbf{B_0}}$.

By definition,

$$\mathrm{C}(\mathbf{f}) = [f], \qquad \mathbf{f} \in \mathrm{Mor}_{\mathbf{pro}-\mathbf{Top}_{\mathbf{B_0}}}(\mathbf{X}, \mathbf{Y}),$$

$$\mathrm{C}(\mathbf{X}) = \mathbf{X}, \qquad \mathbf{X} \in ob(\mathbf{pro} - \mathbf{Top}_{\mathbf{B_0}}).$$

$\mathrm{C}(1_{\mathbf{Y}})$ is the f.p. coherent homotopy class of $1_{\mathbf{Y}}$. Let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ and $\mathbf{g} : \mathbf{Y} \to \mathbf{Z}$ be morphism of $\mathbf{pro} - \mathbf{Top}_{\mathbf{B_0}}$. As in [L-M] we can prove that $\mathrm{C}(\mathbf{g}\ \mathbf{f}) = \mathrm{C}(\mathbf{g})\ \mathrm{C}(\mathbf{f})$.

Besides, there exists a functor $\mathrm{E} : \mathbf{CPHTop}_{\mathbf{B_0}} \to \mathbf{pro} - \mathbf{HTop}_{\mathbf{B_0}}$. Assume that for each inverse system $\mathbf{X} = (X_\alpha, p_{\alpha\alpha'}, \mathscr{A})$ in $\mathbf{Top_{B_0}}$, $\mathrm{E}\mathbf{X} = (X_\alpha, [p_{\alpha\alpha'}]_{\mathrm{B}_0}, \mathscr{A})$.

Let $f : \mathbf{X} \to \mathbf{Y}$ be a f.p. coherent map given by $f_{\boldsymbol{\beta}}$ and φ. We associate with f the morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ of $\mathbf{pro} - \mathbf{HTop_{B_0}}$, given by function $\varphi_{|\mathscr{B}} : \mathscr{B} \to \mathscr{A}$ and the fiber homotopy classes over B_0,$[f_{\beta_0}]_{\mathrm{B}_0} : X_{\varphi(\beta_0)} \to Y_{\beta_0}$.

Note that $\mathbf{f}$ is a morphism of $\mathbf{pro}-\mathbf{HTop_{B_0}}$. Indeed, for $\beta_0 \leq \beta_1$ and $\alpha = \varphi(\beta_0, \beta_1)$ we have $\alpha \geq \varphi(\beta_0), \varphi(\beta_1)$. Besides, the f.p. map $f_{\beta_0\beta_1} : (X_\alpha \times \Delta^1, \pi_{X_\alpha \times \Delta^1}) \to (Y_{\beta_0}, \pi_{Y_{\beta_0}})$ satisfies the conditions

$$f_{\beta_0\beta_1}(x, \partial_0^1(1)) = q_{\beta_0\beta_1} f_{\beta_1}(p_{\varphi(\beta_1)\alpha}(x), 1)$$

and

$$f_{\beta_0\beta_1}(x, \partial_1^1(1)) = f_{\beta_0}(p_{\varphi(\beta_0)\alpha}(x), 1).$$

Thus,

$$[f_{\beta_0}]_{\mathrm{B}_0}\ [p_{\varphi(\beta_0)\alpha}]_{\mathrm{B}_0} = [q_{\beta_0\beta_1}]_{\mathrm{B}_0}\ [f_{\beta_1}]_{\mathrm{B}_0}\ [p_{\varphi(\beta_1)\alpha}]_{\mathrm{B}_0}.$$

Let $f, f' : \mathbf{X} \to \mathbf{Y}$ be f.p. coherent homotopic maps. Let $F : \mathbf{X} \times I \to \mathbf{Y}$ be a f.p. coherent homotopy between f and f', given by Φ and $F_{\boldsymbol{\beta}}$. Note that $\Phi(\beta_0) \geq \varphi(\beta_0), \varphi'(\beta_0)$ and $F_{\beta_0} : X_{\Phi(\beta_0) \times I \times \Delta^0} \to Y_{\beta_0}$ is a f.p. map satisfying conditions

$$F_{\beta_0}(x, 0, 1) = f_{\beta_0}(p_{\varphi(\beta_0)\Phi(\beta_0)}(x), 1)$$

and

$$F_{\beta_0}(x, 1, 1) = f'_{\beta_0}(p_{\varphi'(\beta_0)\Phi(\beta_0)}(x), 1).$$

Consequently,

$$[f_{\beta_0}]_{\mathrm{B}_0}\ [p_{\varphi(\beta_0)\Phi(\beta_0)}]_{\mathrm{B}_0} = [f'_{\beta_0}]_{\mathrm{B}_0}\ [p_{\varphi'(\beta_0)\Phi(\beta_0)}]_{\mathrm{B}_0}.$$

Thus, with f and with $f^{'}$ is associated the same morphism of $\mathbf{pro-HTop_{B_0}}$. Consequently, it is possible to define a functor $\mathrm{E} : \mathbf{CPHTop_{B_0}} \to \mathbf{pro-HTop_{B_0}}$.

The composition $\mathrm{E} \circ \mathrm{C} : \mathbf{pro-Top_{B_0}} \to \mathbf{pro-HTop_{B_0}}$ is the functor induced by the f.p. homotopy functor $H : \mathbf{Top_{B_0}} \to \mathbf{HTop_{B_0}}$.

A f.p. coherent map $f : X \to \mathbf{Y}$ consists of f.p. maps $f_{\boldsymbol{\beta}} : (X \times \Delta^n, \pi_{X\times\Delta^n}) \to (Y_{\beta_0}, \pi_{Y_{\beta_0}}), \boldsymbol{\beta} = (\beta_0, \cdots, \beta_n) \in \mathscr{B}$, $n \geq 0$, satisfying the following conditions: for each $x \in X$, $t \in \Delta^{n-1}, n > 0$

$$f_{\boldsymbol{\beta}}(x, \partial_j^n t) = \begin{cases} q_{\beta_0\beta_1} f_{\boldsymbol{\beta_0}}(x,t), & j = 0, \\ f_{\boldsymbol{\beta_j}}(x,t), & 0 < j \leq n \end{cases}$$

and for each $x \in X$, $t \in \Delta^{n+1}, n \geq 0$

$$f_{\boldsymbol{\beta}}(x, \sigma_j^n t) = f_{\boldsymbol{\beta}^j}(x,t), \quad 0 \leq j \leq n.$$

Note that a f.p. coherent map $f : X \to \mathbf{Y}$ is always a special f.p. coherent map.

A f.p. coherent homotopy $F : X \times I \to \mathbf{Y}$, connecting f and $f^{'}$, is a f.p. coherent map given by $F_{\boldsymbol{\beta}}$ and satisfying the conditions: for each $x \in X, t \in \Delta^n$

$$F_{\boldsymbol{\beta}}(x, 0, t) = f_{\boldsymbol{\beta}}(x,t)$$

and

$$F_{\boldsymbol{\beta}}(x, 1, t) = f^{'}_{\boldsymbol{\beta}}(x,t).$$

Let $\mathbf{p} = (p_\alpha) : X \to \mathbf{X}$ be a morphism of $\mathbf{pro-Top_{B_0}}$. It is clear that with $\mathbf{p}$ is associated a unique f.p. coherent map $p : X \to \mathbf{X}$ given by formula

$$p_{\boldsymbol{\alpha}}(x,t) = p_{\alpha_0}(x),$$

where $\boldsymbol{\alpha} = (\alpha_0, \cdots, \alpha_n) \in \mathscr{A}^n, x \in X, t \in \Delta^n$.

The objects of category $\mathbf{SSH_{B_0}}$ are all topological spaces over B_0. The morphisms of category $\mathbf{SSH_{B_0}}$ are defined by the following way.

Let $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$ be an $\mathrm{ANR}_{\mathrm{B}_0}$-resolutions of X and Y, respectively. Let $[f] : \mathbf{X} \to \mathbf{Y}$ be a some morphism of category $\mathbf{CPHTop_{B_0}}$. Let $\mathbf{p}^{'} : X \to \mathbf{X}'$, $\mathbf{q}^{'} : Y \to \mathbf{Y}'$,$[f'] : \mathbf{X}^{'} \to \mathbf{Y}^{'}$ be another triple of fiber resolutions of spaces X and Y over B_0 and morphism of category $\mathbf{CPHTop_{B_0}}$.

Now define the following equivalence relation. We say the triples $(\mathbf{p}, \mathbf{q}, [f])$ and $(\mathbf{p}^{'}, \mathbf{q}^{'}, [f^{'}])$ are equivalent if

$$[f^{'}]\ [i] = [j]\ [f],$$

where $[i] : \mathbf{X} \to \mathbf{X}^{'}$ and $[j] : \mathbf{Y} \to \mathbf{Y}^{'}$ are isomorphisms of category $\mathbf{CPHTop_{B_0}}$.

The fiber strong shape morphisms $\mathrm{F} : (X, \pi_X) \to (Y, \pi_Y)$ are the equivalence classes of triples $(\mathbf{p}, \mathbf{q}, [f])$ with respect to the above defined relation $\sim$.

Let $F : (X, \pi_X) \to (Y, \pi_Y)$ and $G : (Y, \pi_Y) \to (Z, \pi_Z)$ be the fiber strong shape morphisms, defined by triples $(\mathbf{p}, \mathbf{q}, [f])$ and $(\mathbf{p}^{'}, \mathbf{q}^{'}, [g])$, where $\mathbf{p}^{'} : (Y, \pi_Y) \to \mathbf{Y}'$, $\mathbf{q}^{'} : (Z, \pi_Z) \to \mathbf{Z}$ and $[g] : \mathbf{Y}^{'} \to \mathbf{Z}$.

As we know there exists an unique morphism $[h] : \mathbf{Y} \to \mathbf{Y}^{'}$ of category $\mathbf{CPHTop_{B_0}}$ such that $[h]\ [q] = [q^{'}]$. Note that

$$[j][q] = [q^{'}] = [h]\ [q].$$

Hence, $[j] = [h]$. Besides, $[g]\ [j] = [g]\ [h]\ [1_{\mathbf{Z}}]$.

Thus, we can assume that the morphisms F and G are given by triples $(\mathbf{p}, \mathbf{q}, [f])$ and $(\mathbf{q}, \mathbf{r}, [g])$.

Consequently, we can define the composition $G\ F : X \to Z$ as the morphism given by triple $(\mathbf{p}, \mathbf{r}, [g]\ [f])$.

In the role an identity morphism $\mathscr{I} : X \to X$ we can take the morphism defined by triple $(\mathbf{p}, \mathbf{p}, [1_X])$.

The obtained category $\mathbf{SSH_{B_0}}$ call the fiber strong shape category.

Let $X \in ob(\mathbf{SSH_{B_0}})$. By symbol $\mathrm{ssh}_{\mathrm{B}_0}(X)$ denote the equivalence class of topological space (X, π_X) and call the fiber strong shape of (X, π_X).

For each f.p. map $\varphi : (X, \pi_X) \to (Y, \pi_Y)$ choose $\mathrm{ANR}_{\mathrm{B}_0}$-resolutions $\mathbf{p} : (X, \pi_X) \to \mathbf{X}$ and $\mathbf{q} : (Y, \pi_Y) \to \mathbf{Y}$. There exists a unique morphism $[f] : \mathbf{X} \to \mathbf{Y}$ of category $\mathbf{CPHTop_{B_0}}$ such that $[q]\ [\varphi] = [f]\ [p]$.

We can define a functor $\mathrm{SS}'_{\mathrm{B}_0} : \mathbf{Top_{B_0}} \to \mathbf{SSH_{B_0}}$. By definition,

$$\mathrm{SS}'(X) = X, \qquad X \in ob(\mathbf{Top_{B_0}})$$

and

$$\mathrm{SS}'(\varphi) = \Phi, \qquad \varphi \in \mathrm{Mor}_{\mathbf{Top_{B_0}}}(X, Y).$$

Here Φ is a fiber strong shape morphism defined by triple $(\mathbf{p}, \mathbf{q}, [f])$.

As in [L-M] we can prove that functor $\mathrm{SS}'_{\mathrm{B}_0}$ induces a functor $\mathrm{SS}_{\mathrm{B}_0} : \mathbf{HTop_{B_0}} \to \mathbf{SSH_{B_0}}$, which we call the fiber strong shape functor. By definition,

$$\mathrm{SS}_{\mathrm{B}_0}(X) = X, X \in ob(\mathbf{HTop_{B_0}})$$

and

$$\mathrm{SS}_{\mathrm{B}_0}([\varphi]_{\mathrm{B}_0}) = \mathrm{SS}'(\varphi), [\varphi]_{\mathrm{B}_0} \in \mathrm{Mor}_{\mathbf{HTop_{B_0}}}(X, Y).$$

Let us define a functor $\mathrm{S} : \mathbf{SSH_{B_0}} \to \mathbf{SH_{B_0}}$. Assume that $\mathrm{S}(X) = X$ for each object $X \in ob(\mathbf{SSH_{B_0}})$. Let $F : (X, \pi_X) \to (Y, \pi_Y)$ be a fiber strong shape morphism given by a triple $(\mathbf{p}, \mathbf{q}, [f])$.

Consider the morphism $\mathrm{E}([f])$ as an image of $[f]$ with respect the functor E : $\mathbf{CPHTop_{B_0}} \to \mathbf{pro} - \mathbf{HTop_{B_0}}$. The triple $(\mathrm{H}\mathbf{p}, \mathrm{H}\mathbf{q}, \mathrm{E}[f])$ generates a fiber shape morphism, which we denote by $\mathrm{S}(F) : (X, \pi_X) \to (Y, \pi_Y)$.

Now we can formulate the following

Theorem 3.2.5. *There exists a commutative diagram*

$$\begin{array}{ccc} & \overset{\mathrm{S_{B_0}}}{\nearrow} & \mathbf{SH_{B_0}} \\ \mathbf{HTop_{B_0}} & & \uparrow \mathrm{S} \\ & \underset{\mathrm{SS_{B_0}}}{\searrow} & \mathbf{SSH_{B_0}}, \end{array}$$

where $\mathrm{S_{B_0}}$ *is V.Baladze fiber shape functor* $[\mathrm{B}_4]$. □

Corollary 3.2.6. *Let* (X, π_X) *and* (Y, π_Y) *be topological spaces over* B_0*. If* $\mathrm{ssh_{B_0}}(X) = \mathrm{ssh_{B_0}}(Y)$*, then* $\mathrm{sh_{B_0}}(X) = \mathrm{sh_{B_0}}(Y)$*.* □

Remark 3.2.7. Using the methods developed in this paper and papers ($[\mathrm{B}_{10}]$, [L-M], $[\mathrm{M}_2]$, $[\mathrm{M}_3]$) it is possible to construct fiber strong shape theory for category of arbitrary continuous maps.

Conclusion

The basic achievements made in the thesis are as follows:

1. The study of Borsuks fiber pairs and investigation of their properties.

2. The definition of fiber strong shape deformation retracts, so called SSDR_{B_0}-maps and investigation of their properties.

3.The definition of fibrant spaces over B_0 and establishment of their properties.

4. The construction of fiber cotelescope of inverse sequence of spaces over B_0 and study of their properties.

5. The construction of fiber strong shape classification of compact metric spaces by means of fiber cotelescope, fibrant spaces over B_0 and fiber resolutions.

6. The characterization of fiber strong shape equivalences by means of double map cylinder.

7. The introduction of a concept of fiber strong $\mathrm{ANR}_{\mathrm{B}_0}$-extension and proof of its existence theorem.

8. The constructions of fiber strong shape category $\mathbf{SSH_{B_0}}$ of general topological spaces, the fiber strong shape functor $\mathrm{SS}_{\mathrm{B}_0} : \mathbf{HTop_{B_0}} \to \mathbf{SSH_{B_0}}$, the functor S:$\mathbf{SSH_{B_0}} \to \mathbf{SH_{B_0}}$ with values in V. Baladzes fiber shape category $\mathbf{SH_{B_0}}$ and proof of the equality $\mathrm{S} \cdot \mathrm{SS}_{\mathrm{B}_0} = \mathrm{S}_{\mathrm{B}_0}$, where $\mathrm{S}_{\mathrm{B}_0} : \mathbf{HTop_{B_0}} \to \mathbf{SH_{B_0}}$ is V.Baladze fiber shape functor [B_4].

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