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ABSTRACTS

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On the Numerical Solvability of the Cauchy Problem for Systems of Linear Ordinary Differential Equations

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There is investigated the numerical solvability question of the Cauchy problem for the system of ordinary differential equations

$$\frac{dx}{dt} = P(t)x + q(t), \tag{1}$$

$$x(t_0) = c_0, \tag{2}$$

where P and q are, respectively, real matrix- and vector-functions with the Lebesque integrable components defined on a closed interval $[a, b]$, $t_0 \in [a, b]$, $c_0 \in \mathbb{R}^n$ is a real vector.

Let the absolutely continuous vector function $x^0 : [a, b] \to \mathbb{R}^n$ be the unique solution of the problem $(1), (2)$.

Along with the problem $(1), (2)$ we consider the difference scheme

$$\Delta y(k-1) = \frac{1}{m}\big(G_{1m}(k)\,y(k) + G_{1m}(k)\,y(k-1) + g_m(k)\big) \ \ (k = 1, \dots, m), \tag{1_m}$$

$$y(k_m) = \gamma_m \tag{2_m}$$

$(m = 2, 3, \dots)$, where G_{jm} $(j = 1, 2)$ and q_m are, respectively, discrete real matrix- and vector-functions acting from the set $\{1, \dots, m\}$ into $\mathbb{R}^{n\times n}$, $k_m \in \{0, \dots, m\}$ and $\gamma_m \in \mathbb{R}^n$ for every $m \in \{2, 3, \dots\}$.

In the work, the necessary and sufficient and effective sufficient conditions are given for the convergence of the difference scheme $(1_m), (2_m)$ $(m = 2, 3, \dots)$ to the solution x^0 of the Cauchy problem $(1), (2)$.

The following notations and definitions will be used.

$\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ are, respectively, the sets of all natural, integer and real numbers. $\mathbb{R}_+ = [0, +\infty[$. $[a, b]$ is a closed interval.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$.

$O_{n\times m}$ is the zero $n \times m$-matrix. I_n is an identity $n \times n$ matrix.

O_n is the zero n-vector.

$\det(X)$ is the determinant of the $n \times n$-matrix X.

$\bigvee\limits_a^b(X)$ is the sum of total variations of the components x_{ij} $(i = 1, \dots, m;\ j = 1, \dots, m)$ of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$.

$\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$ is the set of all bounded variation matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$, i.e. such that $\bigvee\limits_a^b(X)<+\infty$.

$L([a,b];\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ whose components are Lebesgue integrable.

If $m\in\mathbb{N}$, then $\mathbb{N}_m=\{1,\dots,m\}$ and $\widetilde{\mathbb{N}}_m=\{0,1,\dots,m\}$.

If $J\subset\mathbb{Z}$, then $\mathrm{E}(J;\mathbb{R}^{n\times m})$ is the space of all matrix-functions $Y:J\to\mathbb{R}^{n\times m}$ with the norm

$$\|Y\|_J=\max\big\{\|Y(k)\|:\ k\in J\big\}.$$

Δ is the first order difference operator, i.e.

$$\Delta Y(i-1)=Y(i)-Y(i-1)\ \text{ for }\ Y\in\mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times m}),\ \ i\in\mathbb{N}_m.$$

Set

$$I_{10m}=\Big[a,a+\frac{\tau_m}{2}\Big[,\ \ I_{1km}=\Big[\tau_{km}-\frac{\tau_m}{2},\tau_{km}+\frac{\tau_m}{2}\Big[,\ \ I_{1mm}=\Big[b-\frac{\tau_m}{2},b\Big[,\ \ (k=1,\dots,m-1),$$
$$I_{20m}=\Big[a,a+\frac{\tau_m}{2}\Big],\ \ I_{1km}=\Big]\tau_{km}-\frac{\tau_m}{2},\tau_{km}+\frac{\tau_m}{2}\Big],\ \ I_{1mm}=\Big]b-\frac{\tau_m}{2},b\Big],\ \ (k=1,\dots,m-1),$$

where

$$\tau_{km}=a+k\tau_m\ \ (k=0,\dots,m),\quad \tau_m=\frac{b-a}{m}\,.$$

We introduce the operators $p_m:\mathrm{BV}([a,b];\mathbb{R}^n)\to\mathrm{E}(\widetilde{N}_m,\mathbb{R}^n)$ and $q_{jm}:\mathrm{E}(\widetilde{N}_m,\mathbb{R}^n)\to\mathrm{BV}([a,b];\mathbb{R}^n)$ $(j=1,2)$ defined by

$$p_m(x)(k)=x(\tau_{km})\ \text{ for }\ k\in\widetilde{N}_m$$

and

$$q_{jm}(y)(t)=y(k)\ \text{ for }\ t\in I_{jkm},\ \ k\in\widetilde{N}_m\ \ (j=1,2)$$

for every $m\in\{2,3,\dots\}$.

Definition. We say that a sequence $(G_{1m},G_{2m},g_m;k_m)$ $(m=2,3,\dots)$ belongs to the set $\mathcal{CS}(P,q,t_0)$ if for every $c_0\in\mathbb{R}^n$ and the sequence $\gamma_m\in\mathbb{R}^n$ $(m=2,3,\dots)$, satisfying the condition

$$\lim_{m\to+\infty}\gamma_m=c_0,$$

the difference problem $(1_m),(2_m)$ has a unique solution $y_m\in\mathrm{E}(\widetilde{N}_m;\mathbb{R}^n)$ for any sufficiently large m and the condition

$$\lim_{m\to+\infty}\big\|y_m-p_m(x^0)\big\|_{\widetilde{N}_m}=0$$

holds.

We assume that $P\in L([a,b];\mathbb{R}^{n\times n})$, $q\in L([a,b];\mathbb{R}^n)$, $G_{jm}\in\mathrm{E}(N_m;\mathbb{R}^{n\times n})$ $(j=1,2)$ and $g_m\in\mathrm{E}(N_m;\mathbb{R}^n)$ $(m=2,3,\dots)$. In addition, we define G_{jm} $(j=1,2)$ and g_m at the point zero by

$$G_{jm}(0)=O_{n\times n},\quad g_m(0)=0_n\ \ (j=1,2;\ \ m=2,3,\dots).$$

Theorem 1. *Let*

$$\lim_{m\to+\infty}\tau_{k_m m}=t_0. \tag{3}$$

Then

$$\big((G_{1m},G_{2m},g_m;k_m)\big)_{m=2}^{+\infty}\in\mathcal{CS}(P,q,t_0) \tag{4}$$

if and only if there exist matrix-functions $P_j \in L([a,b];\mathbb{R}^{n\times n})$ $(j=1,2)$ *and a sequence of matrix-functions* $H_m \in \mathrm{E}(\mathbb{N}_m;\mathbb{R}^{n\times n})$ $(m=2,3,\dots)$ *such that*

$$P_1(t)+P_2(t)=P(t) \ \textit{ for } \ t\in[a,b], \tag{5}$$

$$\lim_{m\to+\infty}\sup\Big(\frac{1}{m}\sum_{k=1}^{m}\|H_m(k)G_{jm}(k)\|\Big)<+\infty \ \ (j=1,2), \tag{6}$$

$$\lim_{k\to+\infty}\max\big\{\|H_m(k)-I_n\|:\ k\in\mathbb{N}_m\big\}=0, \tag{7}$$

$$\lim_{k\to+\infty}\max\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}H_m(l)G_{jm}(l)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}P_j(\tau)\,d\tau\bigg\|:\ \sigma<k;\ \sigma,k\in\widetilde{N}_m\bigg\}=0 \ \ (j=1,2) \tag{8}$$

and

$$\lim_{k\to+\infty}\max\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}H_m(l)g_m(l)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}q(\tau)\,d\tau\bigg\|:\ \sigma<k;\ \sigma,k\in\widetilde{N}_m\bigg\}=0. \tag{9}$$

Corollary 1. *Let the conditions* (3), (5)–(7) *hold and let*

$$\lim_{k\to+\infty}\max\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}H_m(l+i)G_{jm}(l)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}P_j(\tau)\,d\tau\bigg\|:\ \sigma<k;\ \sigma,\ k\in\widetilde{N}_m\bigg\}=0$$
$$(i=-1,1;\ \ j=1,2)$$

and

$$\lim_{k\to+\infty}\max\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}H_m(l+i)g_m(l)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}q(\tau)\,d\tau\bigg\|:\ \sigma<k;\ \sigma,k\in\widetilde{N}_m\bigg\}=0 \ \ (i=-1,1),$$

where $P_j\in L([a,b];\mathbb{R}^{n\times n})$ $(j=1,2)$, $H_m\in\mathrm{E}(\mathbb{N}_m;\mathbb{R}^{n\times n})$ $(m=2,3,\dots)$. *Let, moreover, either*

$$\lim_{k\to+\infty}\sup\bigg(\frac{1}{m}\sum_{k=1}^{m}\big(\|G_{1m}(k)\|+\|G_{2m}(k)\|+\|g_m(k)\|\big)\bigg)<+\infty$$

or

$$\lim_{k\to+\infty}\sup\bigg(\sum_{k=1}^{m}\big\|\Delta H_m(k-1)\big\|\bigg)<+\infty.$$

Then the inclusion (4) *holds.*

Theorem 2. *Let the conditions* (3), (5)–(7) *hold and let*

$$\lim_{k\to+\infty}\max\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}G_{jm}(l)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}P_j(\tau)\,d\tau\bigg\|:\ \sigma<k;\ \sigma,k\in\widetilde{N}_m\big\}=0 \ \ (j=1,2), \tag{10}$$

$$\lim_{k\to+\infty}\max\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}g_m(l)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}q(\tau)\,d\tau\bigg\|:\ \sigma<k;\ \sigma,k\in\widetilde{N}_m\bigg\}=0, \tag{11}$$

$$\lim_{k\to+\infty}\sup\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}\sum_{i=1}^{l}\Delta H(i)G_{jm}(i)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}P_{*j}(\tau)\,d\tau\bigg\|:\ \sigma<k;\ \sigma,k\in\widetilde{N}_m\bigg\}=0$$

and

$$\lim_{k\to+\infty}\sup\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}\sum_{i=1}^{l}\Delta H(i)g_m(i)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}q_*(\tau)\,d\tau\bigg\|:\ \sigma<k;\ \sigma,k\in\widetilde{N}_m\bigg\}=0,$$

where $P_j,P_{*j}\in L([a,b];\mathbb{R}^{n\times n})$ $(j=1,2)$, $q_*\in L([a,b];\mathbb{R}^n)$, $H_m\in \mathrm{E}(\mathbb{N}_m;\mathbb{R}^{n\times n})$ $(m=2,3,\dots)$. *Then*

$$\big((G_{1m},G_{2m},g_m;k_m)\big)_{m=2}^{+\infty}\in\mathcal{CS}(P-P_*,q-q_*,t_0),$$

where $P_*(t)\equiv P_{*1}(t)+P_{*2}(t)$.

Corollary 2. *Let the conditions* (3), (5) *hold and let there exist a natural* μ *and matrix-functions* $B_i\in\mathrm{E}(\mathbb{N}_m;\mathbb{R}^{n\times n})$ $(i=0,\dots,\mu-1)$ *such that*

$$\begin{aligned}
&\lim_{k\to+\infty}\sup\bigg(\frac{1}{m}\sum_{k=1}^{m}(\|G_{1m\mu}(k)\|+\|G_{2m\mu}(k)\|)\bigg)<+\infty,\\
&\lim_{k\to+\infty}\max\{\|H_{m\,\mu-1}(k)-I_n\|:k\in\mathbb{N}_m\}=0,\\
&\lim_{k\to+\infty}\max\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}G_{jm\mu}(l)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}P_j(\tau)d\tau\bigg\|:\sigma<k;\ \sigma,k\in\widetilde{N}_m\bigg\}=0\ \ (j=1,2)
\end{aligned}$$

and

$$\lim_{k\to+\infty}\max\bigg\{\bigg\|\frac{1}{m}\sum_{l=\sigma+1}^{k}g_{m\mu}(l)-\int\limits_{\tau_{\sigma m}}^{\tau_{km}}q_j(\tau)d\tau\bigg\|:\sigma<k;\ \sigma,k\in\widetilde{N}_m\bigg\}=0,$$

where $P_j\in L([a,b];\mathbb{R}^{n\times n})$ $(j=1,2)$,

$$\begin{gathered}
H_{m0}(k)\equiv I_n,\\
H_{m\,i+1}(k)\equiv\bigg(I_n-H_{mi}(k)-\frac{1}{m}\sum_{l=1}^{k}H_{mi}(l)G_{\sigma m}(l)-B_{i+1}(k)\bigg)H_{mi}(k),\\
G_{jm\,i+1}(k)\equiv H_{mi}(k)G_{jm}(k),\quad g_{m\,i+1}(k)\equiv H_{mi}(k)g_m(k)\\
(\sigma=1,2;\ \ i=1,\dots,\mu-1;\ \ m=2,3,\dots).
\end{gathered}$$

Then the inclusion (4) *holds.*

If $\mu=1$, then Corollary 2 has the following form.

Corollary 3. *Let the conditions* (3), (5), (10), (11) *and*

$$\lim_{k\to+\infty}\sup\bigg(\frac{1}{m}\sum_{k=1}^{m}\big(\|G_{1m}(k)\|+\|G_{2m}(k)\|\big)\bigg)<+\infty$$

hold, where $P_j\in L([a,b];\mathbb{R}^{n\times n})$ $(j=1,2)$. *Then the inclusion* (4) *holds.*

Corollary 4. *Let the conditions* (3), (5), (7)–(9) *hold and let there exist sequences of matrix-functions* $Q_{jm}\in\mathrm{E}(\widetilde{N}_m,\mathbb{R}^{n\times n})$ $(j=1,2;\ m=2,3,\dots)$ *such that*

$$\det\big(I_n+(-1)^jQ_{jm}(k)\big)\neq0\ \ for\ \ k\in\mathbb{N}_m\ \ (j=1,2;\ m=2,3,\dots)$$

and

$$\lim_{m\to+\infty}\sup\bigg(\frac{1}{m}\sum_{\sigma=1}^{2}\sum_{k=1}^{m}\|G_{\sigma m}(k)-Q_{\sigma m}(k)\|\bigg)<+\infty,$$

where

$$H_m(k)\equiv\prod_{l=1}^{k}(I_n+Q_{2m}(l))^{-1}(I_n-Q_{1m}(l)).$$

Then the inclusion (4) *holds.*

The question considered in the work is classical. There are a lot of papers where the problem has been investigated (see, for example, [5, 6] and the references therein). But we could not find the necessary and sufficient conditions for the convergence of the difference schemes. Note that there are obtained the necessary and sufficient conditions for stability of the difference schemes circumscribed above, as well.

The proofs of the results are based on the following concept. We rewrite the problems (1), (2) and $(1_m), (2_m)$ $(m=2,3,\dots)$ as the Cauchy problem for the systems of so called generalized ordinary differential equations in the sense of Kurzweil ([5, 7]). Therefore, the convergence of the differential scheme is equivalent to the well-posedness question for the Cauchy problem for the last systems. So, using the results of the papers [1–3] we established the present results.

The analogous results are valid for general linear boundary value problems, as well.

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On Exponential Stability of Invariant Tori of a Class of Nonlinear Systems

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One of the important issues in the qualitative theory of multifrequency oscillations is the question of the stability of invariant sets of dynamical systems defined in the direct product of the m-dimensional torus and the n-dimensional Euclidean space. The main results were obtained in the works of A. M. Samoĭlenko [3]. In this paper, we have established new conditions for the exponential stability of a trivial torus of nonlinear extensions of a dynamical system on a torus, which are formulated in terms of the properties of the right-hand sides of the system not on the whole torus, but only on the set of non-wandering points. The obtained results are applied to the investigation of the stability of toroidal sets of one class of impulsive dynamical systems [4]. Relevant studies for linear extensions of dynamical systems on the torus were used in [1, 5].

Consider the system of differential equations in the direct product m-dimensional torus $\mathcal{T}_m$ and n-dimensional Euclidean space $\mathbb{R}^n$

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = P(\varphi, x)x, \tag{1}$$

where $\varphi = (\varphi_1, \dots, \varphi_m)^T \in \mathcal{T}_m$, $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, the function P is continuous on $\mathcal{T}_m \times \mathbb{R}^n$ and for all $x \in \mathbb{R}^n$ $P(\,\cdot\,, x), a(\,\cdot\,) \in C(\mathcal{T}_m)$, $C(\mathcal{T}_m)$ – continuous space 2π-periodic for each component φ_v, $v = 1, \dots, m$, functions defined on $\mathcal{T}_m$.

Let the following conditions be fulfilled:

$$\exists\, M > 0 \ \text{ that is } \ \forall\,(\varphi, x) \in \mathcal{T}_m \times \mathbb{R}^n \quad \|P(\varphi, x)\| \le M, \tag{2}$$

$$\begin{gathered} \forall\, r > 0 \ \ \exists\, L = L(r) > 0 \ \text{ that is } \ \forall\, x', x'' \ \ \|x'\| \le r, \ \ \|x''\| \le r, \\ \forall\, \varphi \in \mathcal{T}_m \quad \big\|P(\varphi, x'') - P(\varphi, x')\big\| \le L\|x'' - x'\|, \end{gathered} \tag{3}$$

$$\exists\, A > 0 \ \ \forall\, \varphi', \varphi'' \in \mathcal{T}_m \quad \big\|a(\varphi'') - a(\varphi')\big\| \le A\|\varphi'' - \varphi'\|. \tag{4}$$

The condition (4) guarantees that the system

$$\frac{d\varphi}{dt} = a(\varphi) \tag{5}$$

generates a dynamic system on $\mathcal{T}_m$, which we will mark by $\varphi_t(\varphi)$.

Definition ([2])**.** Point $\varphi \in \mathcal{T}_m$ is called the wandering point of a dynamic system (5) if there is a neighborhood $U(\varphi)$ and moment of time $T = T(\varphi) > 0$ such that

$$U(\varphi) \cap \varphi_t(U(\varphi)) = \varnothing \ \ \forall\, t \ge T.$$

Denote by Ω the set of nonwandering points of the dynamic system (5). Since $\mathcal{T}_m$ is a compact, then Ω is a non-empty, invariant, compact subset $\mathcal{T}_m$. In addition, the following lemma is true.

Lemma ([2])**.** *For all $\varepsilon > 0$ there exist $T(\varepsilon) > 0$, $N(\varepsilon) > 0$ such that for all $\varphi \notin \Omega$ the corresponding trajectory $\varphi_t(\varphi)$ is only a finite period of time which does not exceed $T(\varepsilon)$, outside ε-neighborhood set Ω, leaving this neighborhood set no more $N(\varepsilon)$ times.*

The main purpose of the work is to establish the exponential stability of the trivial torus $x = 0$, $\varphi \in \mathcal{T}_m$ of the system (1) in terms of the properties of the function $\varphi \mapsto P(\varphi, 0)$ on the set of non-lattice points Ω of a dynamical system (5), as well as apply the obtained results to the study of the stability of toroidal sets of impulse dynamical systems generated by the problem (1).

Let's denote $\varphi \in \mathcal{T}_m$, $x \in \mathbb{R}^n$

$$\widehat{P}(\varphi, x) = \frac{1}{2}\big(P(\varphi, x) + P^T(\varphi, x)\big),$$
$$\lambda(\varphi, x) \ \text{– biggest eigenvalue } \widehat{P}(\varphi, x).$$

Theorem 1. *Let the condition be fulfilled*

$$\forall\, \varphi \in \Omega \quad \lambda(\varphi, 0) < 0. \tag{6}$$

Then the trivial torus of system (1) *is exponentially stable, i.e. there are constants* $K > 0$, $\gamma > 0$, $\delta > 0$ *such as for all* $\varphi \in \mathcal{T}_m$ *as* $x^0 \in \mathbb{R}^n$, $\|x^0\| \le \delta$ *fair inequality*

$$\forall\, t \ge 0 \quad \|x(t, \varphi, x^0)\| \le K\|x^0\|e^{-\gamma t},$$

where $x(t, \varphi, x^0)$ *– solution to the Cauchy problem*

$$\frac{dx}{dt} = P(\varphi_t(\varphi), x)x, \quad x(0) = x^0.$$

Remark. From the proof of the theorem it follows that for arbitrary $\varphi \in \mathcal{T}_m$ the following inequalities are performed

$$\forall\, t \ge 0 \quad \exp\bigg\{\int_0^t \delta(\varphi_s(\varphi), r)\, ds\bigg\} \le Ke^{-\gamma t},$$

where

$$\delta(\varphi, r) = \max_{\|x\| \le r} \lambda(\varphi, x).$$

As an example, consider system (on $\mathcal{T}_1 \times \mathbb{R}^2$)

$$\frac{d\varphi}{dt} = -\sin^2\Big(\frac{\varphi}{2}\Big), \tag{7}$$

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} -\cos(\varphi + x_1) & \sin(\varphi + x_2^2) \\ \sin(\varphi - x_2^3) & -\cos(\varphi + x_1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{8}$$

Dynamic system on $\mathcal{T}_1$, generated (7), has a set of nonwandering points

$$\Omega = \{\varphi = 0\}.$$

Symmetric matrix $P(0, \overline{0}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ has its own eigenvalues $\lambda_1 = \lambda_2 = -1$, so the condition (6) is fulfilled and by Theorem 1 trivial torus of system (7), (8) is exponentially stable.

As an application in the phase space $\mathcal{T}_m \times \mathbb{R}^n$ the impulsive system of differential equations is considered

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = P(\varphi, x)x, \tag{9}$$

$$\triangle x\big|_{\varphi \in \Gamma} = I(\varphi, x)x, \tag{10}$$

where are the functions a, P satisfie the conditions (2)–(5), I is continuous and limited to $\mathcal{T}_m \times \mathbb{R}^n$ and for all $x \in \mathbb{R}^n$ $I(\,\cdot\,,x) \in C(\mathcal{T}_m)$.

Impulse set Γ is given by equality

$$\Gamma = \big\{\varphi \in \mathcal{T}_m \mid \quad \Phi(\varphi) = 0\big\},$$

where $\Phi \in C(\mathcal{T}_m)$. Assume that $\forall\,\varphi \in \mathcal{T}_m$ there exist $\{t_i(\varphi)\}_{i=1}^{\infty} \subset (0,+\infty)$ – roots of an equation $\Phi(\varphi_t(\varphi)) = 0$, moreover,

$$\exists\,\theta > 0 \;\; \forall\,\varphi \in \mathcal{T}_m \;\; \forall\,i \geq 1 \quad t_{i+1}(\varphi) - t_i(\varphi) \geq \theta. \tag{11}$$

We will assume

$$\alpha = \max_{\varphi\in\Gamma} \|E + I(\varphi,0)\|,$$

where E – unit matrix.

Theorem 2. *Let the condition* (11) *be fulfilled and*

$$\forall\,\varphi \in \Omega \quad \frac{1}{\theta}\,\ln\alpha + \lambda(\varphi,0) < 0.$$

Then the trivial torus of system (8), (9) *is exponentially stable.*

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On Power-Law Asymptotic Behavior of Solutions to Weakly Superlinear Equations with Potential of General Form

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1 Introduction

Consider the equation

$$y^{(n)} = p(x, y, y', \dots, y^{(n-1)})|y|^k \operatorname{sign} y, \quad n > 4, \quad k > 1. \tag{1.1}$$

New results are proved on asymptotic behavior of blow-up and Kneser (see [7, Definition 13.1]) solutions to this equation. The same results concerning equation (1.1) with the constant potential $p = p_0 > 0$ are proved in [6]. In this paper one can also find the history of these problems. To prove the results, the equation is reduced to a dynamical system on an $(n-1)$-dimensional compact sphere (see [6]). We study the behavior of the trajectories of this system corresponding to constant-sign parts of solutions to (1.1). It is a modification of the method applied for the first time in [1] for the description of the asymptotic behavior of blow-up solutions to equation (1.1) with $n = 3, 4$. See also [2]. Later an asymptotic classification of solutions to (1.1) with $n = 3, 4$ was obtained by that method (see [3, 5] and the references here).

In particular, it was proved that for $n = 3, 4$ all blow-up and Kneser solutions to equation (1.1) have the power-law asymptotic behavior (see [2, 3]), namely, for blow-up at some point x^* solutions $y(x)$ it was obtained that

$$y(x) = C(x^* - x)^{-\alpha}(1 + o(1)) \tag{1.2}$$

with

$$\alpha = \frac{n}{k-1}\,, \quad C^{k-1} = \frac{1}{p_0} \prod_{j=0}^{n-1} (j + \alpha). \tag{1.3}$$

It was also proved for equation (1.1) with $(-1)^n p \equiv p_0 > 0$ for sufficiently large n (see [8]) and for $n = 12, 13, 14$ (see [4]) that there exists $k > 1$ such that equation (1.1) has a solution with non-power-law behavior, namely,

$$y(x) = (x^* - x)^{-\alpha}\ h(\log(x^* - x)),$$

where h is a positive periodic non-constant function on $\mathbf{R}$. We will discuss this problem for $n \geq 15$.

2 Main Results

Theorem 2.1. *Suppose $p \in C(\mathbf{R}^{n+1}) \cap \operatorname{Lip}_{y_0,\dots,y_{n-1}}(\mathbf{R}^n)$ and $p \to p_0 > 0$ as $x \to x^*$, $y_0 \to \infty, \dots, y_{n-1} \to \infty$. Then for any integer $n > 4$ there exists $K > 1$ such that for any real $k \in (1, K)$, any solution to equation* (1.1) *tending to $+\infty$ as $x \to x^* - 0$ has power-law asymptotic behavior* (1.2), (1.3).

Theorem 2.2. *Suppose* $p \in C(\mathbf{R}^{n+1}) \cap \mathrm{Lip}_{y_0,\dots,y_{n-1}}(\mathbf{R}^n)$ *and* $(-1)^n p \to p_0 > 0$ *as* $x \to \infty$, $y_0 \to 0, \dots, y_{n-1} \to 0$. *Then for any integer* $n > 4$ *there exists* $K > 1$ *such that all Kneser solutions to equation (1.1) with any real* $k \in (1, K)$ *tend to zero with power-law asymptotic behavior, namely,*

$$y(x) = C|x|^{-\alpha}(1 + o(1)), \quad x \to \infty,$$

with α *and* C *given by* (1.3).

3 Sketch of the Proof

Proof. To prove Theorem 2.1, as in the proof of Theorem 3.1 (see [6]), we put

$$\alpha = \frac{n}{k-1}\,, \quad \gamma = \frac{1}{\alpha}\,, \quad m = n - 1. \tag{3.1}$$

Consider equation (1.1) with $p = p_0 > 0$. Without loss of generality we can assume that $p_0 = 1$. To prove the theorem, an auxiliary dynamical system is investigated on the m-dimensional sphere. To define it note that if a function $y(x)$ is a solution to equation (1.1) with $p = p_0 > 0$, the same is true for the function

$$z(x) = Ay(A^\gamma x + B) \tag{3.2}$$

with any constants $A > 0$ and B.

Any non-trivial solution $y(x)$ of equation (1.1) with $p = p_0 > 0$ generates in $\mathbb{R}^n \setminus \{0\}$ the curve given parametrically by

$$\big(y(x), y'(x), y''(x), \dots, y^{(m)}(x)\big).$$

We can define an equivalence relation on $\mathbb{R}^n \setminus \{0\}$ such that all solutions obtained from $y(x)$ by (3.2) with $A > 0$ generate equivalent curves, i.e., curves passing through equivalent points (maybe for different x). We assume the points $(y_0, y_1, y_2, \dots, y_m)$ and $(z_0, z_1, z_2, \dots, z_m)$ in $\mathbb{R}^n \setminus \{0\}$ to be equivalent if and only if there exists a constant $\lambda > 0$ such that

$$z_j = \lambda^{n+j(k-1)} y_j, \quad j \in \{0, 1, \dots, m\}.$$

The obtained quotient space is homeomorphic to the m-dimensional sphere

$$S^m = \big\{y \in \mathbb{R}^n : \ y_0^2 + y_1^2 + y_2^2 + \dots + y_m^2 = 1\big\},$$

having exactly one representative of each equivalence class since the equation

$$\lambda^{2n} y_0^2 + \lambda^{2(n+2(k-1))} y_1^2 + \dots + \lambda^{2(n+m(k-1))} y_m^2 = 1$$

has exactly one positive root λ for any $(y_0, y_1, y_2, \dots, y_m) \in \mathbb{R}^n \setminus \{0\}$.

Equivalent curves in $\mathbb{R}^n \setminus \{0\}$ generate the same curves in the quotient space. The last ones are trajectories of an appropriate dynamical system, which can be described, in different charts covering the quotient space, by different formulae using different independent variables. A unique common independent variable can be obtained from those ones by using a partition of unity.

Within the chart that covers the points corresponding to positive values of solutions and has the coordinate functions

$$u_j = y^{(j)} y^{-1-\gamma j}, \quad j \in \{1, \dots, m\},$$

the dynamical system can be written as

$$\begin{cases} \dfrac{du_1}{dt} = u_2 - (1+\gamma)u_1^2, \\ \dfrac{du_j}{dt} = u_{j+1} - (1+\gamma j)u_1u_j, \quad j \in \{2, \dots, m-1\}, \\ \dfrac{du_m}{dt} = 1 - (1+\gamma m)u_1u_m \end{cases} \tag{3.3}$$

with the independent variable

$$t = \int\limits_{x_0}^{x} y(\xi)^\gamma \, d\xi.$$

The described dynamical system has some equilibrium points corresponding to the solutions to equation (1.1) with $p = p_0 > 0$ having the exact power-law behavior. One of them, which corresponds to the n-positive solutions with exact power-law behavior, can be found in terms of its u_j coordinates noted by $(a_1, \dots, a_m)$:

$$\begin{cases} a_{j+1} = (1+\gamma j)a_1a_j = a_1^{j+1} \prod\limits_{l=1}^{j} (1+\gamma l), \quad j \in \{1, \dots, m-1\}, \\ a_1 = \Big(\prod\limits_{l=1}^{m} (1+\gamma l) \Big)^{-1/n}. \end{cases} \tag{3.4}$$

Instead of system (3.3) it is more convenient for our current purposes to use another one obtained by the substitution $\tau = a_1 t$, $u_j = a_j v_j$, $j \in \{1, \dots, m\}$:

$$\begin{cases} \dfrac{dv_1}{d\tau} = (1+\gamma)(v_2 - v_1^2), \\ \dfrac{dv_j}{d\tau} = (1+\gamma j)(v_{j+1} - v_1v_j), \quad j \in \{2, \dots, m-1\}, \\ \dfrac{dv_m}{d\tau} = (1+\gamma m)(1 - v_1v_m). \end{cases}$$

The above equilibrium point has in the new chart all coordinates equal to 1.

Up to the moment, we actually considered, for each $\gamma > 0$, its own dynamical system defined on its own quotient space homeomorphic to the m-dimensional sphere. In what follows, we need one sphere with a γ-parameterized dynamical system having an equilibrium point common for all γ in consideration. Thus, the points $(y_0, y_1, \dots, y_m) \in \mathbb{R} \setminus \{0\}$ obtained while treating solutions to (1.1) with $p = p_0 > 0$ and different k will generate the same point on S^m if their corresponding coordinates have the same sign and the tuples

$$\bigg(|y| \ : \ \Big| \frac{y'}{a_1} \Big|^{\frac{1}{1+\gamma}} \ : \ \cdots \ : \ \Big| \frac{y^{(j)}}{a_j} \Big|^{\frac{1}{1+\gamma j}} \ : \cdots \ : \ \Big| \frac{y^{(m)}}{a_m} \Big|^{\frac{1}{1+\gamma m}} \bigg),$$

if considered as sets of projective coordinates, define the same point in the projective space $\mathbb{R}P^m$. In particular, for points corresponding to n-positive solutions this means that they have the same v_j coordinates in the related charts. Hereafter, the domain consisting of all points with positive v_j coordinates is denoted by S_+^m. The only equilibrium point in S_+^m, which has all v_j coordinates equal to 1, is denoted by v^*.

For further proof we need the following

Lemma 3.1 (see [6])**.** *There exist $\gamma_2 > 0$ and an open neighborhood U of the point v^* such that for any positive $\gamma < \gamma_2$, any trajectory of the global dynamical system passing through the closure $\overline{U}$ tends to v^*. If such a trajectory does not coincide with v^*, then it passes transversally, at some time, through the boundary ∂U.*

Now let us consider a solution $y(x)$ to equation (1.1), in suggestion that $P \to 1$ as $x \to x^*$, $y_0 \to \infty, \dots, y_{n-1} \to \infty$. This solution generates in S^m a curve described in the same chart by the system

$$\begin{cases} \dfrac{dv_1}{d\tau} = (1+\gamma)(v_2 - v_1^2), \\ \dfrac{dv_j}{d\tau} = (1+\gamma j)(v_{j+1} - v_1 v_j), \quad j \in \{2, \dots, m-1\}, \\ \dfrac{dv_m}{d\tau} = (1+\gamma m)(q(\tau) - v_1 v_m), \end{cases} \tag{3.5}$$

with the function $q(\tau)$ obtained by the correspondent substitution in P, and it tends to 1 as $\tau \to \infty$.

Lemma 3.2. *The set of all ω-limit points of the trajectory described by* (3.5) *with $q(\tau)$ tending to* 1 *as $\tau \to \infty$ is the union of some whole trajectories of system* (3.5)*.*

The proof of this lemma is almost the same as the proof of Lemma 5.6 in [3].

Since S^m is a compact set, any trajectory $s(\tau)$ on it has at least one ω-limit point. If this ω-limit point is unique, then it is the limit of the trajectory. So, if the trajectory does not tend to v^*, then it has at least one ω-limit point $w \neq v^*$. If the trajectory $s(\tau)$ is generated by a solution to equation (1.1) tending to $+\infty$ as $x \to x^* - 0$, then we can assume that $w \in S^m_+$. According to Lemma 3.1, the trajectory $s_1(\tau)$ of (3.5), passing through the point w, then it passes transversally, at some time, through the boundary ∂U for some $\gamma \in (0, \gamma_2)$. When the function $q(\tau)$ is sufficiently close to 1, then the trajectory $s(\tau)$ also passes transversally through ∂U. In this case it can enter U but cannot leave it. So, the points $s_1(\tau)$, outside of U, cannot be ω-limit points of $s(\tau)$. This contradiction to Lemma 3.2 shows that $s(\tau) \to v^*$ as $\tau \to \infty$. In particular,

$$v_1 = \Big(\frac{z_1}{z_0}\Big)^{1+\gamma} \longrightarrow 1 \ \text{ as } \ \tau \to \infty.$$

It means that the corresponding solution $y(x)$ to equation (1.1) satisfies the condition

$$\frac{y'}{a_1 y^{1+\gamma}} \longrightarrow 1 \ \text{ as } \ x \to x^* - 0.$$

So,

$$y' \sim a_1 y^{1+\gamma} \ \text{ as } \ x \to x^* - 0,$$
$$y \sim (a_1 \gamma)^{-\frac{1}{\gamma}} (x^* - x)^{-\frac{1}{\gamma}},$$

and from (3.1) and (3.4) we obtain

$$y \sim \big(\alpha(\alpha+1)\cdots(\alpha+n-1)\big)^{\frac{1}{k-1}} (x^* - x)^{-\alpha}, \ \ x \to x^* - 0. \tag{3.6}$$

It means that Theorem 2.1 for $p_0 = 1$ is proved.

If $y(x)$ is a solution to equation (1.1) with P tending to an arbitrary $p_0 > 0$, then $y p_0{}^{\frac{1}{k-1}}$ is a solution to equation (1.1) with a similar function P tending to 1. So, $y p_0{}^{\frac{1}{k-1}}$ satisfies (3.6), and,

$$y = \Big(\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{p_0}\Big)^{\frac{1}{k-1}} (x^* - x)^{-\alpha} (1 + o(1)) \ \text{ as } \ x \to x^* - 0.$$

Theorem 2.1 is proved. □

By similar considerations we can prove Theorem 2.2.

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Functions Defined by n-tuples of the Lyapunov Exponents of Linear Differential Systems Continuously Depending on the Parameter Uniformly on the Semiaxis

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1 Introduction

For a given positive integer number n we denote by $\mathcal{M}^n$ the vector space of linear differential systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+ \equiv [0, +\infty), \tag{1.1}$$

with continuous and bounded on the semiaxis $\mathbb{R}^+$ matrix functions $A\colon \mathbb{R}^+ \to \operatorname{End} \mathbb{R}^+$ (we identify systems (1.1) with their coefficient matrices) with usual operations of addition and multiplying by real numbers. Let us introduce the two most commonly used in the theory of Lyapunov exponents topologies in the vector space $\mathcal{M}^n$: the *uniform* one given by the norm

$$\|A\| = \sup_{t \in \mathbb{R}^+} |A(t)|, \quad A \in \mathcal{M}^n,$$

and the *compact-open* one given by the metric

$$\rho_C(A, B) = \sup_{t \in \mathbb{R}^+} \min \Big\{ |A(t) - B(t)|, \frac{1}{t} \Big\}, \quad A, B \in \mathcal{M}^n,$$

where $|A(t)| = \sup\limits_{|x|=1} |A(t)x|$. The resulting topological spaces we denote by $\mathcal{M}^n_U$ and $\mathcal{M}^n_C$, respectively.

The following definition of the Lyapunov exponents of system (1.1) is equivalent to the classical one [5, p. 34] and is more convenient for our purposes.

Definition 1.1. The Lyapunov exponents of system (1.1) are defined [2] by

$$\lambda_i(A) = \inf_{L \in G_i(S(A))} \sup_{x \in L \setminus \{0\}} \varlimsup_{t \to +\infty} \frac{1}{t} \ln |x(t)|, \quad i = \overline{1, n},$$

where $S(A)$ is the vector space of solutions of system (1.1) and $G_i(V)$ is the set of i-dimensional subspaces of a vector space V.

In our notation the Lyapunov exponents are numbered in non-decreasing order, unlike [7].

Let M be a metric space. Consider a family

$$\dot{x} = A(t,\mu)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}^+, \tag{1.2}$$

of linear differential systems depending on a parameter $\mu \in M$ and satisfying the property: for any fixed μ system (1.2) belongs to the space $\mathcal{M}^n$ (i.e. has a continuous and bounded on the semiaxis coefficient matrix). For any fixed $i \in \{1,\dots,n\}$ we put in correspondence with each $\mu \in M$ the i-th Lyapunov exponent of system (1.2) and as a result obtain the function $\Lambda_i^A \colon M \to \mathbb{R}$ called the i-th Lyapunov exponent of family (1.2). We identify families (1.2) with their coefficient matrices, the same as we do for systems (1.1).

Further we will consider families (1.2) with two different types of continuous dependence on a parameter $\mu \in M$. Matrix function A of family (1.2) represents a mapping $M \to \mathcal{M}^n$ defined by $\mu \mapsto A(\,\cdot\,,\mu)$. Therefore for families (1.2) definition of continuity in parameter depends on a topology in the space $\mathcal{M}^n$. Let $\mathcal{A}_C^n(M)$ denote the class of families (1.2) for which the mapping $M \to \mathcal{M}^n$ is continuous when $\mathcal{M}^n$ is endowed with the compact-open topology and let $\mathcal{A}_U^n(M)$ denote the class of families (1.2) for which the mapping $M \to \mathcal{M}^n$ is continuous when $\mathcal{M}^n$ is endowed with the uniform topology. In other words, the class $\mathcal{A}_C^n(M)$ consists of families (1.2) such that for any fixed $\mu \in M$ and any $T > 0$

$$\lim_{\mu\to\mu} \max_{t\in[0,T]} \big\|A(t,\nu) - A(t,\mu)\big\| = 0$$

holds, i.e. convergence is uniform on each line segment. The class $\mathcal{A}_U^n(M)$ consists of families (1.2) such that for any fixed $\mu \in M$

$$\lim_{\mu\to\mu} \big\|A(\,\cdot\,,\nu) - A(\,\cdot\,,\mu)\big\| = 0$$

holds, i.e. convergence is uniform on the whole semiaxis.

A natural problem stated by V. M. Millionshchikov [6] is to describe the Lyapunov exponents Λ_i^A of families (1.2) as functions on a metric space M. In a significant step towards its solution, V. M. Millionshchikov proved [6, 8] that for each $i \in \{1,\dots,n\}$ and any $A \in \mathcal{A}_C^n$ the function Λ_i^A can be represented as the limit of a decreasing sequence of functions of the first Baire class. In particular, this means that Λ_i^A belongs to the second Baire class. Simple examples show that for families from $\mathcal{A}_C^n$ the Lyapunov exponents Λ_i^A, $i = \overline{1,n}$, can be everywhere discontinuous even starting from $n = 1$. M. I. Rakhimberdiev proved [10] that in the Millionshchikov theorem the number of Baire class cannot be reduced. An exact characterization of Lyapunov exponents of families from $\mathcal{A}_C^n$ is given in paper [4]: a family $A \in \mathcal{A}_C^n(M)$ satisfying the equality $\Lambda_i^A = f$ exists if and only if the function f is upper-limit (the definition is given below) and has an upper semicontinuous minorant. Moreover, in paper [4] the author proved that an n-tuple $(f_1, f_2, \dots, f_n)$ of functions $M \to \mathbb{R}^n$ coincides with the n-tuple $(\Lambda_1^A, \dots, \Lambda_n^A)$ of the Lyapunov exponents of some family $A \in \mathcal{A}_C^n(M)$ if and only if each function f_i satisfies conditions above and the inequalities $f_1(\mu) \le \dots \le f_n(\mu)$ hold for all $\mu \in M$.

It is easy to see that for any space M and family $A \in \mathcal{A}_U^1(M)$ (the only) Lyapunov exponent of family (1.2) is continuous. O. Perron gave [9] (see also [3, 1.4]) an example of a mapping $A \in \mathcal{A}_U^2([0,1])$ such that the largest Lyapunov exponent of family (1.2) is not upper semicontinuous. For any metric space M, positive integers n and $i \in \{1,\dots,n\}$ the full description of the i-th Lyapunov exponent of family $A \in \mathcal{A}_U^n(M)$ is given in paper [1]: a family $A \in \mathcal{A}_U^n(M)$ satisfying the equality $\Lambda_i^A = f$ exists if and only if the function f is upper-limit and has continuous minorant and majorant.

The main purpose of this report is to describe the set of n-tuples $\{(\Lambda_1^A, \dots, \Lambda_n^A) :\ A \in \mathcal{A}_U^n(M)\}$ of the Lyapunov exponents for any given metric space M and positive integer n.

Definition 1.2. We call a function $f\colon M \to \mathbb{R}$ upper-limit if there exists a sequence of continuous functions $f_k\colon M \to \mathbb{R}$, $k \in \mathbb{N}$, such that

$$f(\mu) = \varlimsup_{k\to\infty} f_k(\mu), \quad \mu \in M.$$

Remark 1.1. The property of a function $f\colon M \to \mathbb{R}$ being upper-limit is equivalent to each of the next conditions:

(1) the function f can be represented as the pointwise limit of a decreasing sequence of functions of the first Baire class;

(2) pre-image of every semi-interval $[r, +\infty)$, $r \in \mathbb{R}$, under the mapping f is a G_δ-set.

In the notation of the monograph [2, § 37.1] functions satisfying this condition constitute class $({}^*, G_\delta)$. The equivalence of conditions (1) and (2) is established in [2, § 37.1] and that of condition (2) and Definition 1.2 is demonstrated in [4, Remark 3].

2 Main result

Theorem. *Consider an arbitrary metric space M, an integer number $n \ge 2$ and a set of functions $f_i\colon M \to \mathbb{R}$, $i = \overline{1,n}$. A family $A \in \mathcal{A}^n_U(M)$ satisfying equalities $\Lambda^A_i = f_i$, $i = \overline{1,n}$, exists if and only if* (1) *the inequalities $f_1(\mu) \le \dots \le f_n(\mu)$ hold for each $\mu \in M$ and* (2) *each function f_i, $i = \overline{1,n}$, is upper-limit and has continuous minorant and majorant. Moreover, if all functions f_i, $i = \overline{1,n}$, are bounded, then the coefficient matrix of family A can be chosen bounded.*

Remark 2.1. In the case $n = 1$ family (1.2) satisfying the required conditions exists if and only if the function f_1 is continuous.

Remark 2.2. It is easy to see that the conditions of the theorem above are stronger than those of an analogous theorem of paper [4]: our theorem requires the existence of continuous minorant and majorant for each of the given functions, while in [4] only the existence of an upper-semicontinuous minorant is required.

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Asymptotic Representations for Oscillatory Solutions of Higher Order Differential Equations

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We deal with an oscillation problem for the higher order nonlinear differential equation with a middle term

$$x^{(n)}(t)+q(t)x^{(n-2)}(t)+r(t)f(x(t))=0, \quad n\geq 3. \tag{0.1}$$

Precisely, we study the existence of oscillatory solutions of (0.1) which are bounded and not vanishing at infinity under the following assumptions:

(i) $q\in C^1[0,\infty)$, $q(t)\geq q_0>0$ for large t, and
$$\int_0^\infty |q'(t)|\,dt<\infty.$$

(ii) $r\in C[0,\infty)$.

(iii) $f\in C(\mathbb{R})$ such that $f(u)u>0$ for $u\neq 0$.

Note that the function r may change its sign.

By a solution of (0.1) we mean a continuously differentiable function x up to n order defined on $[T_x,\infty)$, $T_x\geq 0$, such that satisfies (0.1) on $[T_x,\infty)$ and $\sup\{|x(t)|:\ t\geq T\}>0$ for $T\geq T_x$. As usual, a solution x of (0.1) is said to be *oscillatory* if there exists a sequence $\{t_n\}$ tending to infinity such that $x(t_n)=0$.

The assumption (i) assures that the second order linear equation

$$h''(t)+q(t)h(t)=0 \tag{0.2}$$

is oscillatory. Moreover, since q is bounded and has bounded variation on $[0,\infty)$, all solutions of (0.2) are bounded together with their derivatives.

In our approach equation (0.1) is studied as a perturbation of the linear differential equation

$$y^{(n)}(t)+q(t)y^{(n-2)}(t)=0. \tag{0.3}$$

From this point of view, our results are mainly motivated by the previous ones obtained by I. Kiguradze [5] for the special case $q(t)\equiv 1$, namely for the equation

$$x^{(n)}(t)+x^{(n-2)}(t)+r(t)f(x(t))=0. \tag{0.4}$$

It was shown in [5] that, if r is positive and sufficient large in some sense, then for n even every solution of (0.4) is oscillatory and for n odd every proper solution of (0.4) is oscillatory, or is vanishing at infinity together with its derivatives, or admits the asymptotic representation

$$x(t) = c(1 + \sin(t - \varphi)) + \varepsilon(t),$$

where c, φ are suitable constants and ε is a continuous function for $t \geq 0$ which vanishes at infinity. The existence of bounded oscillatory solutions for equations of type (0.1) has attracted the attention of many authors, see, e.g., the monograph [6], the papers [1–3] and references therein. Observe that if q is a positive constant, then (0.3) has oscillatory, bounded and not vanishing at infinity solutions. If q is not constant and (i) is satisfied, then, as already claimed, these properties remain to hold for the second order equation (0.2). Thus, it is natural to ask under which assumptions these properties are valid also for (0.3) and the more general case (0.1). Here, we give a positive answer to both these questions. In particular, our main results yield the existence of oscillatory solutions of (0.1), which are bounded and not vanishing at infinity. These results complete recent ones in [2] and extend similar ones in [5, Theorem 1.4], which are proved for equation (0.4). An application that concerns the influence of the perturbing term r on the change of the oscillatory character passing from (0.3) to the linear equation

$$x^{(n)}(t) + q(t)x^{(n-2)}(t) + r(t)x(t) = 0, \quad n \geq 3, \tag{0.5}$$

is given.

Below we use the following notation for the growth of unbounded solutions.

The symbol $g_1 = O(g_2)$ as $t \to \infty$ means, as usual, that there exists a constant M such that $|g_1(t)| \leq M|g_2(t)|$ for large t.

1 Oscillatory solutions in the linear case

Equations (0.2) and (0.3) are strictly related. When $q(t) \equiv 1$, a basis of the space of solutions of (0.3) is given by

$$t^j, \quad j = 0, 1, \dots, n-3, \quad \sin t, \quad \cos t. \tag{1.1}$$

In the general case, that is when q is not constant, it is easy to see that a basis of the space of solutions of (0.3) is given by

$$t^j, \quad j = 0, 1, \dots, n-3, \quad \Gamma_u, \quad \Gamma_v, \tag{1.2}$$

where

$$\Gamma_u = \int_0^t (t-s)^{n-3} u(s)\, ds, \quad \Gamma_v = \int_0^t (t-s)^{n-3} v(s)\, ds \tag{1.3}$$

and u, v are two independent solutions of (0.2).

The following existence result for oscillatory solutions of (0.2), which are bounded and not vanishing at infinity, holds.

Theorem 1.1 ([3, Theorem 2])**.** *Let $n \geq 3$, u be a nontrivial solution of* (0.2) *and*

$$\int_0^\infty s^{n-3} |q'(s)|\, ds < \infty. \tag{1.4}$$

Then (0.3) *has an oscillatory solution* ϕ *such that*

$$\phi(t)=\begin{cases} u'(t)+\varepsilon(t) & \text{for } n \text{ odd},\\ u(t)+\varepsilon(t) & \text{for } n \text{ even},\end{cases}$$

where ε *is a continuous function on* $[0,\infty)$ *and* $\lim\limits_{t\to\infty}\varepsilon(t)=0$. *In particular,*

$$0<\limsup_{t\to\infty}|\phi(t)|<\infty.$$

The following asymptotic expressions of the integrals in (1.3) is needed for proving Theorem 1.1.

Lemma 1.1 ([3, Lemma 5])**.** *Let* $n\geq 3$ *and* (1.4) *hold. If* u *is a nontrivial* (*oscillatory*) *solution of* (0.2), *then there exist constants* c_i, $i=0,1\dots,n-2$, $c_{n-2}\neq 0$, *and a function* ε *such that*

$$\Gamma_u(t)=\begin{cases} \displaystyle\sum_{i=0}^{n-3}c_it^i+c_{n-2}u'(t)+\varepsilon(t), & \text{for } n \text{ odd},\\ \displaystyle\sum_{i=0}^{n-3}c_it^i+c_{n-2}u(t)+\varepsilon(t), & \text{for } n \text{ even},\end{cases}$$

where $\lim\limits_{t\to\infty}\varepsilon(t)=0$.

2 Oscillatory solutions in the nonlinear case

Let

$$F(u)=\max\big\{|f(v)|:\ -u\leq v\leq u\big\}.$$

The following criterion concerns the nonexistence of solutions of (0.1) vanishing at infinity.

Theorem 2.1 ([3, Theorem 1])**.** *Let* $n\geq 3$, $f\in C^1(\mathbb{R})$ *and*

$$\int\limits_0^\infty t^{n-3}|r(t)|\,dt<\infty. \tag{2.1}$$

Then (0.1) *does not have nontrivial solutions* x (*oscillatory or nonoscillatory*) *satisfying* $\lim\limits_{t\to\infty}x(t)=0$.

The following existence theorems hold.

Theorem 2.2 ([2, Theorem 1])**.** *Assume* $n\geq 3$. *Let for any positive constant* λ *and for some* $j=0,\dots,n-3$

$$\int\limits_0^\infty t^{n-3}F(\lambda t^j)|r(t)|\,dt<\infty.$$

Then for any solution y *of* (0.3) *such that* $y(t)=O(t^j)$ *as* $t\to\infty$, *there exists a solution* x *of* (0.1) *such that for large* t

$$x^{(i)}(t)=y^{(i)}(t)+\varepsilon_i(t),\ \ i=0,\dots,n-1,$$

where ε_i *are functions of bounded variation for large* t *and* $\lim\limits_{t\to\infty}\varepsilon_i(t)=0$, $i=0,\dots,n-1$.

Using Theorem 2.2 and Lemma 1.1 we get the asymptotic representations for solutions of (0.1).

Theorem 2.3 ([3, Theorem 4]). *Let $n \geq 3$ and u, v be two linearly independent solutions of* (0.2). *Assume* (1.4) *and for any positive constant λ*

$$\int_0^\infty t^{n-3}(\lambda t^{n-3})|r(t)|\,dt < \infty. \tag{2.2}$$

Then for any vector $(c_0, c_1, \dots, c_{n-1}) \in \mathbb{R}^n$ there exists a solution x of (0.1) *such that*

$$x(t) = \begin{cases} \displaystyle\sum_{i=0}^{n-3} c_i t^i + c_{n-2}u'(t) + c_{n-1}v'(t) + \varepsilon(t) & \text{for } n \text{ odd}, \\ \displaystyle\sum_{i=0}^{n-3} c_i t^i + c_{n-2}u(t) + c_{n-1}v(t) + \varepsilon(t) & \text{for } n \text{ even}, \end{cases} \tag{2.3}$$

where $\lim\limits_{t\to\infty} \varepsilon(t) = 0$. If, in addition, $f \in C^1(\mathbb{R})$ and there exists $M > 0$ such that

$$|f'(u)| \leq MF(u) \ \text{ for large } |u|, \tag{2.4}$$

then the solution x given by (2.3) *is unique.*

Theorem 2.3 extends [5, Theorem 1.4] stated for (0.4) with $r(t) > 0$.

The argument for proving Theorems 2.2 and 2.3 is based on the Ascoli theorem and an iterative method, which can be also useful for a numerical estimation of solutions. Moreover, in [2] the cases $n = 3$ and $n = 4$ are studied in details.

As application, consider the Emden–Fowler type equation

$$x^{(n)}(t) + q(t)x^{(n-2)}(t) + r(t)|x(t)|^\lambda \operatorname{sgn} x(t) = 0, \ \ \lambda > 0. \tag{2.5}$$

Then (2.4) is satisfied for any $\lambda > 0$ and (2.2) reads as

$$\int_0^\infty t^{(n-3)(\lambda+1)}|r(t)|\,dt < \infty.$$

Thus, according to Theorem 2.3, for a fixed vector $(c_0, c_1, \dots, c_{n-1})$ there exists a unique solution of (2.5) which has the asymptotic representation (2.3).

Another consequence of our results is the following.

Denote by S_y and S_x the solution space of (0.3) and (0.5), respectively. We say that (0.3) and (0.5) are *asymptotically equivalent*, if there exists a $1-1$ map $T : S_y \to S_x$ such that for every $y \in S_y$ there exists a unique $x \in S_x$ such that $T(y) = x$ and

$$\lim_{t\to\infty} (x(t) - y(t)) = 0.$$

Applying Theorems 2.1 and 2.2 we get the following.

Theorem 2.4 ([3, Theorem 5]). *Assume $n \geq 3$ and*

$$\int_0^\infty t^{2n-6}|r(t)|\,dt < \infty.$$

Then linear equations (0.3) *and* (0.5) *are asymptotically equivalent.*

The following example illustrates Theorem 2.3 and it is inspired from [4, page 113].

Example. Consider the equation

$$x^{(5)}(t)+q(t)x^{(3)}(t)+r(t)x^3(t)=0. \tag{2.6}$$

where

$$q(t)=1+\Big(t+\frac{1}{2}\Big)^{-3}\sin t+\frac{2}{3}\Big(t+\frac{1}{2}\Big)^{-4}\cos t-\frac{1}{9}\Big(t+\frac{1}{2}\Big)^{-5}\cos^2 t$$

and $r\in C[0,\infty)$ and $t^8r(t)\in L^1[0,\infty)$. A standard calculation shows that $q(t)>1/2$ for large t and $q'\in L^1[0,\infty)$. Thus, assumption (i) is satisfied. Moreover, also (1.4) and (2.2) are verified. Since the function

$$u(t)=(\cos t)\bigg[\exp\bigg(8\int\limits_0^t\frac{1}{(2s+1)^3}\,\cos s\,ds\bigg)\bigg]$$

is a solution of (0.2), see [4, page 113] with minor changes, in view of Theorem 2.3, for any vector $(c_0,\dots,c_3)$, equation (2.6) has the solution x given by

$$x(t)=c_0+c_1t+c_2t^2+c_3u'(t)+\varepsilon(t)$$

where $\lim\limits_{t\to\infty}\varepsilon(t)=0$.

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Construction of Partially Irregular Solutions of Linear Differential Systems in Critical Resonant Case

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Let D be a compact subset of $\mathbb{R}^n$ and $AP(\mathbb{R}\times D,\mathbb{R}^n)$ be a function space $f:\mathbb{R}\times D\to\mathbb{R}^n$. Each function $f(t,x)\in AP(\mathbb{R}\times D,\mathbb{R}^n)$ is continuous in the collection of variables and almost periodic in the t uniformly with respect to $x\in D$. According to [5, p. 60] we denote frequency modulus of function f as $\mathrm{Mod}(f)$, i.e. it is the smallest additive group of real numbers containing a set of Fourier exponents (frequencies) of function f. Throughout the paper we consider only systems written in the normal form. By the frequency modulus of a system of almost periodic equations we mean modulus of frequencies of its right-hand side. J. Kurzweil, O. Vejvoda in [6] showed that systems of ordinary almost periodic differential equations can have strongly irregular almost periodic solutions, i.e. intersection of frequency modulus this solutions with modulus of frequencies of system is trivially. Almost periodic solutions, frequency modulus of which contains only some frequencies of the system, were studied by A. K. Demenchuk in the articles [2–4] etc. This solutions are called partially irregular [4].

In this paper we investigate an existence problem for partially irregular almost periodic solutions of linear almost periodic system in the critical resonant case, where are purely imaginary eigenvalues with not simple elementary divisors of averaging of the coefficient matrix. The case of purely imaginary eigenvalues with simple elementary divisors was investigated in [4] and [1].

Let us consider the linear system

$$\frac{dx}{dt}=A(t)x+\varphi(t),\quad \mathrm{Mod}(A)\cap\mathrm{Mod}(\varphi)=\{0\},\ \ x\in\mathbb{R}^n, \tag{1}$$

and assume that $A(t)$ and $\varphi(t)$ are almost periodical such that intersection of frequency modules of the coefficient matrix $A(t)$ and driving forcing force $\varphi(t)$ is trivially. Almost periodic solutions $x(t)$, $\mathrm{Mod}(x)=\mathrm{Mod}(\varphi)$ of system (1) are called irregular forced [4]. Let us explore a existence problem of irregular with respect to $\mathrm{Mod}(A)$ almost periodic solutions $x(t)$ of system (1), i.e. such solutions that $(\mathrm{Mod}(x)+\mathrm{Mod}(\varphi))\cap\mathrm{Mod}(A)=\{0\}$, in critical resonant case.

Denote $\widehat{A}=\lim\limits_{T\to\infty}\frac{1}{T}\int\limits_0^T A(t)\,dt$, and $A_*(t)=A(t)-\widehat{A}$. Let Q_{A_*} be a constant nonsingular $n\times n$-matrix such that the first $n-d=s$ columns of matrix $A_*(t)Q_{A_*}$ are zero and the remaining columns are linearly independent. Let us consider the change of variables $x=Q_{A_*}y$, where $y=\mathrm{col}\,(y^{[s]},y_{[n-s]})$, $y^{[s]}=\mathrm{col}\,(y_1,\dots,y_s)$, $y_{[n-s]}=\mathrm{col}\,(y_{s+1},\dots,y_n)$. By $B^{[s,s]}$ and $B_{[n-s,s]}$ we denote respectively upper and lower blocks of $n\times s$-matrix that obtained from the matrix $B=Q_{A_*}^{-1}\widehat{A}Q_{A_*}$ deleting the last d columns (upper and lower indices indicate the dimensions of blocks). By $\psi(t)=Q_{A_*}^{-1}\varphi(t)$, $\psi(t)=\mathrm{col}\,(\psi^{[s]}(t),\psi_{[n-s]}(t))$ we denote the transformed driving force.

Lemma ([4])**.** *System* (1) *has almost periodic irregular solution $x(t)$ with respect to $Mod(A)$ if and only if:*

- *column rank of the matrix $A(t)-\widehat{A}(t)$ satisfies the inequality*
$$\operatorname{rank}_{\rm col} A_* = d < n; \tag{2}$$
- *the system*
$$\frac{dy^{[s]}}{dt} = B^{[s,s]}y^{[s]} + \psi^{[s]}(t) \tag{3}$$
has almost periodic solution $y^{[s]}(t)$ such that
$$\big(\operatorname{Mod}(y^{[s]}) + \operatorname{Mod}(\varphi)\big) \cap \operatorname{Mod}(A) = \{0\};$$
- *the following identity holds*
$$B_{[n-s,s]}y^{[s]}(t) + \psi_{[n-s]}(t) \equiv 0, \tag{4}$$
and $x(t) = Q_{A_*}\operatorname{col}(y^{[s]}(t),0,\dots,0)$.

Let $\alpha_k \pm i\beta_k$ ($k=1,\dots,k'$; $k' \leqslant n$; $i^2=-1$) be a eigenvalues of the matrix of coefficients $B^{[s,s]}$ of the reduced system (3). As noted above, in the article [1] a case of purely imaginary eigenvalues of matrix $B^{[s,s]}$ with simple elementary divisors was studied.

Suppose that there is a critical resonant case, when there is a pair of purely imaginary eigenvalues of matrix $B^{[s,s]}$ with multiplicity of two with not simple elementary divisors such that
$$\alpha_1 = \alpha_2 = 0, \quad \beta_2 = \beta_1 \in Mod(\varphi), \quad \alpha_q \neq 0 \quad (q = 3,\dots,k'). \tag{5}$$

Denote
$$G(t) = S_2^{-1}(t)\big(J_{B^{[s,s]}}S_2(t) - \dot{S}_2(t)\big), \quad S(t) = S_2^{-1}(t)S_1^{-1},$$
where
$$S_2(t) = \operatorname{diag}\big[e^{i\beta_1 t}, e^{i\beta_1 t}, e^{-i\beta_1 t}, e^{-i\beta_1 t}, 1,\dots,1\big],$$
$\dot{S}_2(t)$ is a derivative of matrix $S_2(t)$, and matrix S_1 transforms matrix $B^{[s,s]}$ to the Jordan normal form, i.e.,
$$S_1^{-1}B^{[s,s]}S_1 = J_{B^{[s,s]}} = \operatorname{diag}\big[J_1, J_2, J_3,\dots,J_{k'}\big] = \operatorname{diag}\big[J_1,J_2,J\big],$$
$$J_1 = \begin{pmatrix} i\beta_1 & 1 \\ 0 & i\beta_1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} -i\beta_1 & 1 \\ 0 & -i\beta_1 \end{pmatrix},$$
where J is a Jordan form, corresponding to the other eigenvalues of matrix $B^{[s,s]}$. Denote j-th row of matrix $g(t) = S(t)\psi^{[s]}(t)$ as $g_{(j)}(t)$ and j-th row of matrix $S(t)$ as $S_{(j)}(t)$.

Theorem. *Let coefficient matrix $A(t)$ and driving force $\varphi(t)$ of system* (1) *be almost periodic with trivial intersection of their frequency modules, and there be a critical resonant case* (5) *of the reduced system* (3)*. Then:*

- *If system* (1) *has almost periodic irregular solution $x(t)$ with respect to* $\operatorname{Mod}(A)$*, then this solution is irregular forced, i.e.* $\operatorname{Mod}(x) \subseteq \operatorname{Mod}(\varphi)$*.*

- *System* (1) *has an irregular forced almost periodic solution if and only if condition* (2) *and the estimates*

$$\sup_t \left| \int_0^t S_{(2)}(\tau)\psi^{[s]}(\tau)\,d\tau \right| < \infty, \quad \sup_t \left| \int_0^t \left(\int_0^\tau S_{(2)}(\sigma)\psi^{[s]}(\sigma)\,d\sigma + S_{(1)}\psi^{[s]}(\tau) \right) d\tau \right| < \infty \quad (6)$$

hold and almost periodic solution $y^{[s]}(t)$ *of reduced system* (3) *satisfy the identity* (4)*.*

The lemma and the theorem allow us to find partially irregular solutions of linear differential systems. For example, consider the quasi-periodic differential system

$$\begin{aligned}
\frac{dx_1}{dt} &= -x_1 + x_4 + x_5, \\
\frac{dx_2}{dt} &= x_1 \sin\sqrt{5}\,t + \big(1 + \sin\sqrt{5}\,t\big)x_2 - \big(1 + \sin\sqrt{5}\,t\big)x_4, \\
\frac{dx_3}{dt} &= x_1 \cos\sqrt{5}\,t + x_2 \cos\sqrt{5}\,t - x_4 \cos\sqrt{5}\,t + x_5 + \cos t, \\
\frac{dx_4}{dt} &= -2x_1 + x_4 + x_5, \\
\frac{dx_5}{dt} &= -x_1 \cos\sqrt{5}\,t - x_2 \cos\sqrt{5}\,t - x_3 + x_4 \cos\sqrt{5}\,t + \sin t,
\end{aligned} \quad (7)$$

wherein intersection of modules of frequencies of coefficients and driving force is trivially. System (7) has the solution

$$x = Q_{A_*}y = \mathrm{col}\,\big(a\sin t - b\cos t, a\cos t + b\sin t, \sin t, (a+b)\sin t + (a-b)\cos t, 0\big), \quad (8)$$

where a, b are arbitrary real constants. The frequency of solution (8) coincide with the frequency of driving force and incommensurable with the frequency of the coefficients of system (7), therefore this solution is irregular forced.

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Three-Layer Fully Linearized Difference Scheme for Symmetric Regularized Long Wave Equations

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The symmetric regularized-long-wave (SRLW) equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\,\frac{\partial^2 (u)^2}{\partial x\partial t}-\frac{\partial^4 u}{\partial x^2\partial t^2}=0 \tag{1}$$

was first derived in [7]. Such equation arises in different physical applications, including ion sound waves in plasma. The solvability and uniqueness of the solution of SRLW equation were studied in works [4–6].

Equation (1) can be rewritten in the form of equivalent first order system:

$$\begin{aligned}\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial x^2\partial t}+\frac{\partial \rho}{\partial x}+u\,\frac{\partial u}{\partial x}&=0,\\ \frac{\partial \rho}{\partial t}+\frac{\partial u}{\partial x}&=0.\end{aligned} \tag{2}$$

In the domain $x\in[a,b],\ t\in[0,T]$, let us define boundary and initial conditions for system (2) as follows:

$$u(a,t)=u(b,t)=0,\quad \rho(a,t)=\rho(b,t)=0,\ \ t\in[0,T], \tag{3}$$

$$u(x,0)=u_0(x),\quad \rho(x,0)=\rho_0(x),\ \ x\in[a,b]. \tag{4}$$

The domain $\overline{Q}$ is divided into rectangular grid by the points $(x_i,t_j)=(a+ih,j\tau)$, $i=0,1,2,\dots,$ $n, j=0,1,\dots,J$, where $h=(b-a)/n$ and $\tau=T/J$ denote the spatial and temporal mesh sizes, respectively.

The value of mesh function U at the node (x_i,t_j) is denoted by U_i^j, that is $U_i^j=U(x_i,t_j)$.

We define the difference quotients (forward, backward, and central, respectively) in x and t directions as follows:

$$\begin{gathered}(U_i^j)_x:=\frac{U_{i+1}^j-U_i^j}{h}\,,\quad (U_i^j)_{\overline{x}}:=\frac{U_i^j-U_{i-1}^j}{h}\,,\quad (U_i^j)_{\overset{\circ}{x}}:=\frac{1}{2}\big((U_i^j)_x+(U_i^j)_{\overline{x}}\big),\\ (U_i^j)_t:=\frac{U_i^{j+1}-U_i^j}{\tau}\,,\quad (U_i^j)_{\overline{t}}:=\frac{U_i^j-U_i^{j-1}}{\tau}\,,\quad (U_i^j)_{\overset{\circ}{t}}:=\frac{1}{2}\big((U_i^j)_t+(U_i^j)_{\overline{t}}\big).\end{gathered}$$

We approximate the problem (2)-(4) by the difference scheme

$$(U_i^j)_{\overset{\circ}{t}}-(U_i^j)_{\overline{x}x\overset{\circ}{t}}+\frac{1}{2}\,(\Phi_i^{j+1}+\Phi_i^{j-1})_{\overset{\circ}{x}}+\frac{1}{6}\,(\Lambda U)_i^j=0, \tag{5}$$

$$(\Phi_i^j)_{\overset{\circ}{t}}+\frac{1}{2}\,(U_i^{j+1}+U_i^{j-1})_{\overset{\circ}{x}}=0 \tag{6}$$

for $i=1,2,\dots,n-1$, $j=1,2,\dots J-1$, and

$$(U_i^0)_t-(U_i^0)_{\overline{x}xt}+\frac{1}{2}\,(\Phi_i^1+\Phi_i^0)_{\overset{\circ}{x}}+\frac{1}{6}\,(\Lambda U)_i^0=0, \tag{7}$$

$$(\Phi_i^0)_t+\frac{1}{2}\,(U_i^1+U_i^0)_{\overset{\circ}{x}}=0 \tag{8}$$

for $i=1,2,\dots,n-1$, $j=0$, with

$$U_i^0=u_0(x_i),\quad \Phi_i^0=\rho_0(x_i),\quad U_0^j=U_n^j=\Phi_0^j=\Phi_n^j=0. \tag{9}$$

Here

$$(\Lambda U)_i^j:=U_i^j(U_i^{j+1}+U_i^{j-1})_{\overset{\circ}{x}}+\big(U_i^j(U_i^{j+1}+U_i^{j-1})\big)_{\overset{\circ}{x}},\ \ j=1,2,\dots,J-1,$$
$$(\Lambda U)_i^0:=(U_i^0)(U_i^1+U_i^0)_{\overset{\circ}{x}}+\big(U_i^0(U_i^1+U_i^0)\big)_{\overset{\circ}{x}}.$$

Let H_0 be the set of functions defined on the mesh $\overline{\omega}=\{x_0,x_1,\dots,x_n\}$ and equal to zero at x_0, x_n. On H_0 we define the following inner products and norms

$$(U^j,V^j):=\sum_{i=1}^{n-1}hU_i^jV_i^j,\quad (U^j,V^j]:=\sum_{i=1}^{n}hU_i^jV_i^j,$$
$$\|U^j\|^2:=(U^j,U^j),\quad \|U^j]|^2:=(U^j,U^j],\quad \|U^j\|_\infty:=\max_{0\le i\le n}|U_i^j|.$$

Theorem. *The finite difference scheme* (5)–(9) *is uniquely solvable and possesses the following invariant*

$$E^j:=\|U^j\|^2+\|U_{\overline{x}}^j]|^2+\|\Phi^j\|^2=\|u_0\|^2+\|u_{0,\overline{x}}]|^2+\|\rho_0\|^2:=E^0,\ \ j=1,2,\dots. \tag{10}$$

Proof. Multiplying (5) by $\tau(U_i^{j+1}+U_i^{j-1})$ and summing over i, we obtain

$$A-B+\frac{1}{2}\,C+\frac{\tau}{6}\,D=0, \tag{11}$$

where

$$\begin{aligned}
A&:=\tau(U_{\overset{\circ}{t}}^j,U^{j+1}+U^{j-1})=\frac{1}{2}\,\big(\|U^{j+1}\|^2-\|U^{j-1}\|^2\big),\\
B&:=\tau(U_{\overline{x}x\overset{\circ}{t}}^j,U^{j+1}+U^{j-1})=-\frac{1}{2}\,\big(\|U_{\overline{x}}^{j+1}]|^2-\|U_{\overline{x}}^{j-1}]|^2\big),\\
C&:=\tau\big((\Phi^{j+1}+\Phi^{j-1})_{\overset{\circ}{x}},U^{j+1}+U^{j-1}\big)\\
&\quad=-\tau\big(\Phi^{j+1}+\Phi^{j-1},(U^{j+1}+U^{j-1})_{\overset{\circ}{x}}\big)=2\tau\big(\Phi^{j+1}+\Phi^{j-1},\Phi_{\overset{\circ}{t}}^j\big)=\|\Phi^{j+1}\|^2-\|\Phi^{j-1}\|^2,\\
D&:=(\Lambda U^j,U^{j+1}+U^{j-1})=0.
\end{aligned}$$

Thus, from (11) we have

$$\|U^{j+1}\|^2+\|U_{\overline{x}}^{j+1}]|^2+\|\Phi^{j+1}\|^2=\|U^{j-1}\|^2+\|U_{\overline{x}}^{j-1}]|^2+\|\Phi^{j-1}\|^2,\ \ j=1,2,\dots. \tag{12}$$

Multiplying (7) by $\tau(U_i^1+U_i^0)$ and summing over i, we obtain

$$A_0-B_0+\frac{1}{2}\,C_0+\frac{\tau}{6}\,D_0=0, \tag{13}$$

where

$$\begin{aligned}
A_0 &:= \tau(U_t^0, U^1 + U^0) = \big(\|U^1\|^2 - \|U^0\|^2\big),\\
B_0 &:= \tau(U_{\overline{x}xt}^0, U^1 + U^0) = \big((U^1 - U^0)_{\overline{x}x}, U^1 + U^0\big) = -\|U_{\overline{x}}^1]|^2 + \|U_{\overline{x}}^0]|^2,\\
C_0 &:= \tau\big((\Phi^1 + \Phi^0)_{\overset{\circ}{x}}, U^1 + U^0\big) = -\tau\big(\Phi^1 + \Phi^0, (U^1 + U^0)_{\overset{\circ}{x}}\big)\\
&\qquad = 2\tau(\Phi^1 + \Phi^0, \Phi_t^0) = 2\big(\|\Phi^1\|^2 - \|\Phi^0\|^2\big),\\
D_0 &:= (\Lambda U^0, U^1 + U^0) = 0.
\end{aligned}$$

Thus, from (13) we have

$$\|U^1\|^2 + \|U_{\overline{x}}^1]|^2 + \|\Phi^1\|^2 = \|U^0\|^2 + \|U_{\overline{x}}^0]|^2 + \|\Phi^0\|^2,$$

which together with (12) confirms the validity of (10).

Because the difference scheme is linear on each new level with respect to the unknown values, its unique solvability follows from (10). □

Theorem. *Difference scheme* (5)–(9) *is absolutely stable with respect to initial data.*

Theorem. *If the solution of problem* (2)–(4) *belongs to W_2^3 Sobolev space, then the order of convergence of the difference scheme equals $O(\tau^2 + h^2)$.*

Remark 1. Note that scheme (5), (6) is studied by Wang, Zhang, Chen in [8]. But there, for obtaining additional initial conditions on the first layer, they offer nonlinear two-layer scheme, requiring additional iterations, and which essentially worsens the result. Our approach uses an idea developed in [1–3], by which we obtain approximations (7), (8).

Remark 2. Note that equation (10) represents an perfect analogy of the well-known conservation law for SRLW equation

$$E(t) = \int\limits_a^b \Big(|u|^2 + \Big|\frac{\partial u}{\partial x}\Big|^2 + |\rho|^2\Big)\,dx = \|u_0\|_{L_2}^2 + \Big\|\frac{\partial u_0}{\partial x}\Big\|_{L_2}^2 + \|\rho_0\|_{L_2}^2 = E(0).$$

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Asymptotic Properties of Special Classes of Solutions of Second-Order Differential Equations with Nonlinearities in Some Sense Near to Regularly Varying

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The following differential equation is considered in the work

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')\exp\big(R(|\ln|yy'||)\big). \tag{1}$$

Here $\alpha_0 \in \{-1,1\}$, $p : [a;\omega[\,\to\,]0;+\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to\,]0;+\infty[$ are continuous functions, $Y_i \in \{0,\pm\infty\}$ $(i=0,1)$, Δ_{Y_i} is a one-sided neighborhood of Y_i, every function $\varphi_i(z)$ $(i=0,1)$ is a regularly varying function as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of order σ_i, $\sigma_0+\sigma_1 \ne 1$, $\sigma_1 \ne 0$, the function $R :]0;+\infty[\,\to\,]0;+\infty[$ is continuously differentiable and regularly varying on infinity of the order μ, $0<\mu<1$, the derivative function of the function R is monotone.

Definition. A solution y of equation (1) is called $P_\omega(Y_0,Y_1,\lambda_0)$-solution if it is defined on $[t_0,\omega[\,\subset [a,\omega[$ and

$$\lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i=0,1), \quad \lim_{t\uparrow\omega}\frac{(y'(t))^2}{y(t)y''(t)} = \lambda_0.$$

A lot of works (see, for example, [2, 3]) have been devoted to the establishing asymptotic representations of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equations of the form (1), in which $R \equiv 0$. The $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) are regularly varying functions as $t \uparrow \omega$ of index $\frac{\lambda_0}{\lambda_0-1}$ if $\lambda_0 \in R \setminus \{0,1t\}$. The asymptotic properties and necessary and sufficient conditions of existence of such solutions of equation (1) have been obtained in [1].

The cases $\lambda_0 \in \{0,1\}$ and $\lambda_0 = \infty$ are special. $P_\omega(Y_0,Y_1,1)$-solutions of equation (1) are rapidly varying functions as $t \uparrow \omega$. The cases $\lambda_0 = 0$ and $\lambda_0 = \infty$ are cases of the most difficulty because in this cases such solutions or their derivatives are slowly varying functions as $t \uparrow \omega$. Some results about asymptotic properties and existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) in these special cases are presented in the work.

We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $\theta : \Delta_Y \to\,]0;+\infty[$ satisfies the condition S if for any continuous differentiable function $L : \Delta_{Y_i} \to\,]0;+\infty[$ such that

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}} \frac{zL'(z)}{L(z)} = 0,$$

the following equality

$$\Theta(zL(z)) = \Theta(z)(1+o(1)) \ \text{ is true as } \ z \to Y \ \ (z \in \Delta_Y).$$

Let us introduce the following notations.

$$\pi_\omega(t) = \begin{cases} t & \text{if } \omega = +\infty,\\ t-\omega & \text{if } \omega < +\infty, \end{cases} \qquad \Theta_i(z) = \varphi_i(z)|z|^{-\sigma_i} \ \ (i=0,1),$$

$$I(t)=\alpha_0\int\limits_{A_\omega}^{t} p(\tau)\,d\tau,\qquad A_\omega=\begin{cases} a & \text{if } \int\limits_a^\omega p(\tau)\,d\tau=+\infty,\\ \omega & \text{if } \int\limits_a^\omega p(\tau)\,d\tau<+\infty.\end{cases}$$

In case $\lim\limits_{t\uparrow\omega}\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}=Y_1$, we put

$$J(t)=\int\limits_{B_\omega}^{t}\Big|I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau,$$

$$B_\omega=\begin{cases} b_1 & \text{if } \int\limits_{b_1}^{\omega}\Big|I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau=+\infty,\\ \omega & \text{if } \int\limits_{b_1}^{\omega}\Big|I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau<+\infty,\end{cases}$$

$$N_1(t)=\frac{(1-\sigma_1)I(t)\big|(1-\sigma_1)I(t)\Theta_1\big(\frac{y_1^0}{|\pi_\omega(t)|}\big)\big|^{\frac{1}{\sigma_1-1}}}{I'(t)R'(|\ln|\pi_\omega(t)||)}\,,$$

and in case $\lim\limits_{t\uparrow\omega}|\pi_\omega(\tau)|\operatorname{sign} y_0^0=Y_0$, we put

$$I_0(t)=\alpha_0\int\limits_{A_\omega^0}^{t} p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\Theta_0\big(|\pi_\omega(\tau)|\operatorname{sign} y_0^0\big)\,d\tau,$$

$$A_\omega^0=\begin{cases} b_2 & \text{if } \int\limits_{b_2}^{\omega} p(t)|\pi_\omega(t)|^{\sigma_0}\Theta_0\big(|\pi_\omega(t)|\operatorname{sign} y_0^0\big)\,dt=+\infty,\\ \omega & \text{if } \int\limits_{b_2}^{\omega} p(t)|\pi_\omega(t)|^{\sigma_0}\Theta_0\big(|\pi_\omega(t)|y_0^0\big)\,dt<+\infty,\end{cases}$$

$$N_2(t)=\alpha_0p(t)|\pi_\omega(t)|^{\sigma_0+1}\Theta_0\big(|\pi_\omega(t)|\operatorname{sign} y_0^0\big).$$

Here $b_1,b_2\in[a;\omega[$ are chosen in such a way that $\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\in\Delta_{Y_1}$ as $t\in[b_1;\omega]$ and $|\pi_\omega(\tau)|\operatorname{sign} y_0^0\in\Delta_{Y_0}$ as $t\in[b_2;\omega]$.

The first two theorems are devoted to the existence $P_\omega(Y_0,Y_1,0)$-solutions of equation (1). Such solutions are slowly varying functions as $t\uparrow\omega$, that makes difficulties in their investigations.

Theorem 1. *Let in equation* (1) *the function* φ_1 *satisfy the condition* S *and the following condition take place*

$$\lim_{t\uparrow\omega}\frac{R(|\ln|\pi_\omega(t)||)J(t)}{\pi_\omega(t)\ln|\pi_\omega(t)|J'(t)}=0. \tag{2}$$

Then for the existence of $P_\omega(Y_0,Y_1,0)$*-solutions of equation* (1) *the following conditions are necessary and sufficient*

$$\lim_{t\uparrow\omega}y_0^0|J(t)|^{\frac{1-\sigma_1}{1-\sigma_0-\sigma_1}}=Y_0,\quad \lim_{t\uparrow\omega}\frac{J'(t)}{y_1^0|J(t)|}=Y_1,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I'(t)}{I(t)}=\sigma_1-1,$$

$$\frac{I(t)}{y_1^0(1-\sigma_1)} > 0, \quad \frac{y_0^0 y_1^0 (1-\sigma_1)J(t)}{1-\sigma_0-\sigma_1} > 0 \;\; as \;\; t \in [b_1, \omega[\,.$$

For such solutions the following asymptotic representations take place as $t \uparrow \omega$

$$\frac{y(t)}{\big|\exp(R(|\ln|y(t)y'(t)||))\varphi_0(y(t))\big|^{\frac{1}{1-\sigma_1}}} = \frac{1-\sigma_0-\sigma_1}{1-\sigma_1}\,|1-\sigma_1|^{\frac{1}{1-\sigma_1}} J(t)[1+o(1)],$$

$$\frac{y'(t)}{y(t)} = \frac{(1-\sigma_1)J'(t)}{(1-\sigma_0-\sigma_1)J(t)}\,[1+o(1)].$$

Theorem 2. *Let condition* (2) *of Theorem* 1 *be not satisfied,* p *be a twice continuously differentiable function, function* φ_1 *satisfy the condition* S *and the following condition take place*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)N_1'(t)}{R'(|\ln|\pi_\omega(t)||)N_1(t)} = 0.$$

For the existence of such $P_\omega(Y_0, Y_1, 0)$*-solutions of equation* (1)*, that finite or infinite limit* $\lim\limits_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)}$ *exists, the following conditions are necessary and sufficient*

$$\lim_{t\uparrow\omega} y_0^0\Big(\exp\big(R(|\ln|\pi_\omega(t)||)\big)^{\frac{\sigma_1-1}{1-\sigma_0-\sigma_1}}\Big) = Y_0, \quad \lim_{t\uparrow\omega}\frac{-\alpha_0}{\pi_\omega(t)} = Y_1, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I'(t)}{I(t)} = \frac{\sigma_1-1}{\alpha_0}\,,$$

$$\alpha_0 y_1^0 \pi_\omega(t) < 0, \quad \alpha_0(1-\sigma_1)(1-\sigma_0-\sigma_1)y_0^0 R'\big(|\ln|\pi_\omega(t)||\big) > 0 \;\; as \;\; t \in [a, \omega[\,.$$

For such solutions the following asymptotic representations take place as $t \uparrow \omega$

$$\frac{y(t)}{|\varphi_0(y(t))\exp(R(|\ln|y(t)y'(t)||))\big|^{\frac{1}{1-\sigma_1}}} = (1-\sigma_0-\sigma_1)N_1(t)[1+o(1)],$$

$$\frac{y'(t)}{y(t)} = \frac{I'(t)R'(|\ln|\pi_\omega(t)||)}{(1-\sigma_0-\sigma_1)(1-\sigma_1)I(t)}\,[1+o(1)].$$

The next two theorems are devoted to the existence of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation (1). The first derivatives of such solutions are slowly varying functions as $t \uparrow \omega$, the fact creates difficulties in the investigation of such solutions.

Theorem 3. *For the existence of* $P_\omega(Y_0, Y_1, \pm\infty)$*-solutions of equation* (1) *the following conditions are necessary*

$$Y_0 = \begin{cases} \pm\infty, & if \;\; \omega = +\infty, \\ 0, & if \;\; \omega < +\infty, \end{cases} \quad \pi_\omega(t)y_0^0 y_1^0 > 0 \;\; as \;\; t \in [a, \omega[\,.$$

If the function φ_0 *satisfies the condition* S *and*

$$\lim_{t\uparrow\omega} \frac{R'(|\ln|\pi_\omega(t)||)I_0(t)}{\pi_\omega(t)I_0'(t)} = 0, \tag{3}$$

then (3) *together with the following conditions are necessary and sufficient for the existence of* $P_\omega(Y_0, Y_1, \pm\infty)$*-solutions of equation* (1)

$$\lim_{t\uparrow\omega} y_1^0 |I_0(t)|^{\frac{1}{1-\sigma_0-\sigma_1}} = Y_1, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)I_0'(t)}{I_0(t)} = 0, \quad y_1^0(1-\sigma_0-\sigma_1)I_0(t) > 0 \;\; as \;\; t \in [b_2, \omega[\,.$$

For such solutions the following asymptotic representations take place as $t \uparrow \omega$

$$\frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)||))} = (1-\sigma_0-\sigma_1)I_0(t)[1+o(1)], \quad \frac{y'(t)}{y(t)} = \frac{1}{\pi_\omega(t)}\,[1+o(1)].$$

Theorem 4. *If in* (1) *the function* p *is a continuously differentiable, the function* φ_0 *satisfies the condition* S *and*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)N'(t)}{R'(|\ln|\pi_\omega(t)||)N(t)}=0,$$

then with (3) *the following conditions are necessary and sufficient for the existence of* $P_\omega(Y_0,Y_1,\pm\infty)$*-solutions of equation* (1)

$$\lim_{t\uparrow\omega}y_1^0\exp\Big(\frac{1}{1-\sigma_0-\sigma_1}\,R(|\ln|\pi_\omega(t)||)\Big)=Y_1,\quad \alpha_0y_1^0(1-\sigma_0-\sigma_1)\ln|\pi_\omega(t)|>0\ \ as\ \ t\in[a,\omega[\,.$$

For such solutions the following asymptotic representations take place as $t\uparrow\omega$

$$\frac{|y'(t)|^{1-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)y'(t)||))}=\frac{|1-\sigma_0-\sigma_1|N(t)}{R'(|\ln|\pi_\omega(t)||)}\,[1+o(1)],\quad \frac{y'(t)}{y(t)}=\frac{1}{\pi_\omega(t)}\,[1+o(1)].$$

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On the Solvability of a Boundary Value Problem for Fourth Order Linear Functional Differential Equations

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Motivated by article [7], in this paper, we consider a boundary value problem for functional differential equation of the fourth order. We obtain sharp sufficient conditions for the existence and uniqueness of solutions.

Boundary value problems for fourth order functional differential equations are considered in [2–6, 8].

Definition 1. A linear operator T from the space of all continuous real functions $\mathbf{C}[0,1]$ into the space of all integrable functions $\mathbf{L}[0,1]$ is called positive if it maps every nonnegative continuous function into an almost everywhere nonnegative integrable function.

Consider the boundary value problem for a fourth order functional differential equation:

$$\begin{cases} x^{(4)}(t) = -(Tx)(t) + f(t), \ \ t \in [0,1], \\ x(0) = c_1, \quad \dot{x}(0) = c_2, \quad x(1) = c_3, \quad \dot{x}(1) = c_4, \end{cases} \tag{1}$$

where $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ is a linear bounded operator, $f \in \mathbf{L}[0,1]$, c_i, $i = 1, \dots, 4$, are real constants.

This problem possesses the Fredholm property (see, for example, [7]). Therefore, this problem is uniquely solvable if and only if the homogeneous problem

$$\begin{cases} x^{(4)}(t) = -(Tx)(t), \ \ t \in [0,1], \\ x(0) = 0, \quad \dot{x}(0) = 0, \quad x(1) = 0, \quad \dot{x}(1) = 0, \end{cases} \tag{2}$$

has only the trivial solution.

The Green function $G(t,s)$ of problem (2) is defined by the equality

$$G(t,s) = \begin{cases} \dfrac{t^2(1-s)^2(3s-t-2st)}{6} & \text{if } 0 \le t \le s \le 1, \\[2ex] \dfrac{(1-t)^2 s^2(3t-s-2st)}{6} & \text{if } 0 \le s < t \le 1. \end{cases}$$

So, problem (2) is equivalent to the equation

$$x = -GTx,$$

where $(Gz)(t) = \int\limits_0^1 G(t,s) z(s)\, ds$, $t \in [0,1]$, is the Green operator.

By using the principle of contraction mappings, we get that problem (1) has a unique solution if at least one from the following inequalities is fulfilled:

$$\|T\|_{\mathbf{C}[0,1] \to \mathbf{L}[0,1]} < 192, \quad \|T\|_{\mathbf{C}[0,1] \to \mathbf{L}_\infty[0,1]} < 384.$$

Let $p \in \mathbf{L}[0,1]$ be non-negative function.

Definition 2. $\mathbb{S}(p)$ is a set of all linear positive operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ satisfying the condition
$$T\mathbf{1} = p,$$
where $\mathbf{1}$ is the unit function.

Theorem 1. *Boundary value problem* (1) *has a unique solution for every operator $T \in \mathbb{S}(p)$ if and only if the inequality*
$$\begin{vmatrix} 1 + \int\limits_{t_0}^{1} G(t_1,s)p(s)\,ds & 1 + \int\limits_{0}^{1} G(t_1,s)p(s)\,ds \\ \int\limits_{t_0}^{1} G(t_2,s)p(s)\,ds & 1 + \int\limits_{0}^{1} G(t_2,s)p(s)\,ds \end{vmatrix} > 0$$
holds for all $0 \le t_1 \le t_2 \le 1$ and all $t_0 \in [0,1]$.

The base of the proof of Theorem 1 is the following lemma.

Lemma 1. *Let $p \in \mathbf{L}[0,1]$ be a non-negative function. Then the boundary value problem* (1) *has a unique solution for every operator $T \in \mathbb{S}(p)$ if and only if the problem*
$$\begin{cases} x^{(4)}(t) = -p_1(t)x(t_1) - p_2(t)x(t_2), \;\; t \in [0,1], \\ x(0) = 0, \quad \dot{x}(0) = 0, \quad x(1) = 0, \quad \dot{x}(1) = 0, \end{cases}$$
has only the trivial solution for all $0 \le t_1 \le t_2 \le 1$ and for all functions p_1, $p_2 \in \mathbf{L}[0,1]$ such that
$$p_1(t) + p_2(t) = p, \;\; 0 \le p_i(t) \le p(t), \;\; t \in [0,1], \;\; i = 1,2.$$

Consider the case where $p(t) \equiv P > 0$ is a constant.

Lemma 2. *Let $p(t) \equiv P > 0$ be a constant. If for some $T \in \mathbb{S}(P)$ problem* (2) *has a non-trivial solution, then for some $T \in \mathbb{S}(P)$ problem* (2) *has a symmetric non-trivial solution x such that $x(t) = -x(1-t)$ for all $t \in [0,1]$.*

By Lemma 2, we can put $t_0 = 1/2$ and $t_2 = 1 - t_1$, $t_1 \in [0,1/2]$ in Theorem 2 if $p(t) \equiv P$. So, by Theorem 1, in this case problem (1) is uniquely solvable for all operators $T \in \mathbb{S}(P)$ if and only if
$$P < \frac{1}{\max\limits_{t_1 \in [0,1/2]} \Big(\int\limits_{1/2}^{1} G(1-t_1,s)\,ds - \int\limits_{1/2}^{1} G(t_1,s)\,ds \Big)} \\ = \frac{192}{\max\limits_{t1 \in [0,1/2]} t_1^2(1-2t_1)(3-4t_1)} = \frac{1760\sqrt{33}}{3} - 416 \approx 2954.$$

Corollary 1. *Let $p \in \mathbf{L}[0,1]$ be a non-negative function such that*
$$\operatorname*{vrai\,sup}_{t \in [0,1]} p(t) \le \frac{1760\sqrt{33}}{3} - 416, \quad p(t) \not\equiv \frac{1760\sqrt{33}}{3} - 416.$$
Then boundary value problem (1) *is uniquely solvable for all operators $T \in \mathbb{S}(p)$.*

The constant in the Corollary 1 is sharp.

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Asymptotic Properties of Some Class of Solutions of Second Order Differential Equations with Rapidly and Regularly Varying Nonlinearities

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Let us consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y'). \tag{1}$$

In this equation $\alpha_0 \in \{-1;1\}$, functions $p : [a,\omega[\, \to\,]0,+\infty[$ $(-\infty < a < \omega \le +\infty)$, and $\varphi_i : \Delta_{Y_i} \to\,]0,+\infty[$ $(i \in \{0,1\})$ are continuous, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either an interval $[y_i^0, Y_i[$ [1] or an interval $]Y_i, y_i^0]$.

We also suppose that the function φ_1 is a regularly varying function of index σ_1 as $y \to Y_1$ $(y \in \Delta_{Y_1})$ ([3, pp. 10–15]), the function φ_0 is twice continuously differentiable on Δ_{Y_0} and satisfies the conditions

$$\varphi_0'(y) \neq 0 \ \text{ as } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi_0(y) \in \{0,+\infty\}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi_0(y)\varphi_0''(y)}{(\varphi_0'(y))^2} = 1.$$

The solution y of the equation (1), that is defined on the interval $[t_0,\omega[\, \subset [a,\omega[$, is called $P_\omega(Y_0,Y_1,\lambda_0)$-solution $(-\infty \le \lambda_0 \le +\infty)$ if the following conditions take place

$$y^{(i)} : [t_0,\omega[\, \to \Delta_{Y_i}, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i=0,1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0.$$

The aim of the work is to find the necessary and sufficient conditions for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of the equation (1) if $\lambda_0 \in R \setminus \{0;1\}$, to find asymptotic representations of such solutions and its first order derivatives as $t \uparrow \omega$.

Definition 1. We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $\theta : \Delta_Y \to\,]0;+\infty[$ satisfies the condition S as $z \to Y$ if for any continuous differentiable normalized slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $L : \Delta_{Y_i} \to\,]0;+\infty[$ the following relation is valid

$$\theta(zL(z)) = \theta(z)(1+o(1)) \ \text{ as } \ z \to Y \ \ (z \in \Delta_Y).$$

Definition 2. We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $L : \Delta_Y \to\,]0;+\infty[$ satisfies the condition S_1 as $z \to Y$ if for any finite segment $[a;b] \subset\,]0;+\infty[$

$$\limsup_{\substack{z\to Y\\ z\in\Delta_Y}} \left| \ln|z| \cdot \Big(\frac{L(\lambda z)}{L(z)} - 1\Big) \right| < +\infty \ \text{ for all } \ \lambda \in [a;b].$$

[1] If $Y_i = +\infty$ $(Y_i = -\infty)$, we will take $y_i^0 > 0$ or $y_i^0 < 0$, respectively.

Conditions S and S_1 are satisfied, for example, for the functions: $\ln|y|$, $|\ln|y||^{\mu}$ $(\mu\in R)$, $\ln\ln|y|$.

Let us introduce the following notations.

$$\pi_\omega(t)=\begin{cases} t & \text{if } \omega=+\infty,\\ t-\omega & \text{if } \omega<+\infty,\end{cases}\quad \theta_1(y)=\varphi_1(y)|y|^{-\sigma_1},$$

$$\Phi_0(y)=\int\limits_{A_\omega}^{y}|\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\,dz,\quad A_\omega=\begin{cases} y_0^0 & \text{if } \int\limits_{y_0^0}^{Y_0}|\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\,dz=\pm\infty,\\ Y_0 & \text{if } \int\limits_{y_0^0}^{Y_0}|\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\,dz=const,\end{cases}$$

$$\Phi_1(y)=\int\limits_{A_\omega}^{y}\frac{\Phi_0(\tau)}{\tau}\,d\tau,\quad Z_1=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi_1(y),\quad F(t)=\frac{\Phi_1^{-1}(I_1(t))\Phi_1'(\Phi_1^{-1}(I_1(t)))}{\pi_\omega(t)I_1'(t)}\,.$$

If $y_1^0\cdot\lim\limits_{t\uparrow\omega}|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}=Y_1$, we put

$$I(t)=|\lambda_0-1|^{\frac{1}{1-\sigma_1}}\cdot y_1^0\cdot\int\limits_{B_\omega^0}^{t}\Big|\pi_\omega(\tau)p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}y_1^0\big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau,$$

$$B_\omega^0=\begin{cases} b & \text{if } \int\limits_{b}^{\omega}\Big|\pi_\omega(\tau)p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}y_1^0\big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau=+\infty,\\ \omega & \text{if } \int\limits_{b}^{\omega}\Big|\pi_\omega(\tau)p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}y_1^0\big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau<+\infty,\end{cases}$$

$$I_1(t)=\int\limits_{B_\omega^1}^{t}\frac{\lambda_0 I(\tau)}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau,\quad B_\omega^1=\begin{cases} b & \text{if } \int\limits_{b}^{\omega}\frac{\lambda_0 I(\tau)}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau=\pm\infty,\\ \omega & \text{if } \int\limits_{b}^{\omega}\frac{\lambda_0|I(\tau)|}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau=const,\end{cases}$$

where $b\in[a;\omega[$ is chosen in such a way that $y_1^0|\pi_\omega(t))|^{\frac{1}{\lambda_0-1}}\in\Delta_{Y_1}$ as $t\in[b;\omega]$.

The following conclusions take place for the equation (1).

Theorem 1. *Let $\sigma_1\neq 1$, the function φ_1 satisfy the condition S, and the following limit relation be true*

$$\lim_{\substack{z\to Y_0\\ z\in\Delta_{Y_0}}}\frac{\big(\frac{\Phi_1'(z)}{\Phi_1(z)}\big)''\big(\frac{\Phi_1'(z)}{\Phi_1(z)}\big)}{\big(\big(\frac{\Phi_1'(z)}{\Phi_1(z)}\big)'\big)^2}=\gamma_0,\ \ \gamma_0\in R\setminus\{1,0\}. \tag{2}$$

The next conditions are necessary for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of the equation (1), if $\lambda_0\in\mathbb{R}\setminus\{0,1\}$:

$$\pi_\omega(t)y_1^0y_0^0\lambda_0(\lambda_0-1)>0,\quad \pi_\omega(t)y_1^0\alpha_0(\lambda_0-1)>0,\quad y_1^0\cdot\lim_{t\uparrow\omega}|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}=Y_1,$$

$$\lim_{t\uparrow\omega}\Phi_1^{-1}(I_1(t))=Y_0,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}=\infty,\quad \lim_{t\uparrow\omega}\frac{I_1'(t)\pi_\omega(t)}{\Phi_1'(\Phi_1^{-1}(I_1(t)))\Phi_1^{-1}(I_1(t))}=\frac{\lambda_0}{\lambda_0-1}\,.$$

These conditions are also sufficient for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the equation (1) *if*

$$I(t)I_1(t)\lambda_0(\sigma_1 - 1) > 0 \ \ as \ \ t \in [a;\omega[$$

and the function $\frac{|\pi_\omega(t)|^{1-\frac{(2-\gamma_0)\lambda_0}{(1-\gamma_0)(\lambda_0-1)}} I_1'(t)}{I_1(t)}$ *is a normalized slowly varying function as* $t \uparrow \omega$.

Moreover, for each such solution the following asymptotic representations take place as $t \uparrow \omega$

$$\Phi_1(y(t)) = I_1(t)[1 + o(1)], \quad \frac{y'(t)\Phi_1'(y(t))}{\Phi_1(y(t))} = \frac{I_1'(t)}{I_1(t)}\,[1 + o(1)].$$

Let us notice that the function $\Phi^{-1}(z) \cdot \frac{\Phi_1'(\Phi_1^{-1}(z))}{z}$ is a slowly varying function as $z \to Z_1$.

If the condition (2) is not true, the following theorem takes place.

Theorem 2. *Let for equation* (1) $\sigma_1 \neq 1$, *the function* $\theta_1(z)$ *satisfy the condition* S *as* $z \to Y_1$ $(z \in \Delta_{Y_1})$, *the function* $\Phi^{-1}(z) \cdot \frac{\Phi_1'(\Phi^{-1}(z))}{z}$ *satisfy the condition* S_1 *as* $z \to Z_1$. *Then for existence of the equation's* (1) $P_\omega(Y_0, Y_1, \lambda_0)$*-solutions, where* $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$, *it is necessary, and, if*

$$I(t)I_1(t)\lambda_0(\sigma_1 - 1) > 0 \ \ for \ \ t \in \,]b, \omega[\,,$$

and finite or infinite limits

$$\lim_{t\uparrow\omega} \pi_\omega(t)F'(t) \ \ and \ \ \lim_{t\uparrow\omega} \frac{\sqrt{|\frac{\pi_\omega(t)I_1'(t)}{} I_1(t)|}}{\ln|I_1(t)|}$$

exist, sufficient the fulfilment of the following conditions

$$\pi_\omega(t)y_1^0 y_0^0 \lambda_0(\lambda_0 - 1) > 0, \quad \pi_\omega(t)y_1^0 \alpha_0(\lambda_0 - 1) > 0 \ \ as \ \ t \in [a;\omega[\,,$$

$$y_1^0 \cdot \lim_{t\uparrow\omega} |\pi_\omega(t)|^{\frac{1}{\lambda_0-1}} = Y_1, \quad \lim_{t\uparrow\omega} I_1(t) = Z_1, \quad \lim_{t\uparrow\omega} \frac{I_1''(t)I_1(t)}{(I_1'(t))^2} = 1, \quad \lim_{t\uparrow\omega} F(t) = \frac{\lambda_0 - 1}{\lambda_0}.$$

Moreover, for every such solution the following asymptotic representations as $t \uparrow \omega$ *take place*

$$\Phi_1(y(t)) = I_1(t)[1 + o(1)], \quad \frac{\pi_\omega(t)y'(t)}{y(t)} = \frac{\lambda_0}{\lambda_0 - 1}\,[1 + o(1)].$$

Note that if in the limit relation (2) $\gamma_0 = 1$, the function $\Phi^{-1}(z) \cdot \frac{\Phi_1'(\Phi^{-1}(z))}{z}$ satisfies the condition S_1 as $z \to Z_1$.

The next example illustrates the obtained results of Theorem 1.

Let's consider the following differential equation

$$y'' = \frac{1}{4}\,t^{-3}L(t)e^{|y|^4 - t^8}|y'|^3, \tag{3}$$

where $L : [2, +\infty[\, \to \,]0, +\infty[\,$, $\frac{2-\sigma_1}{\beta+a} > 0$, $\beta > 0$, $\beta \neq 1$ as $t \in [2, +\infty[\,$.

This is an equation of the form (1), where $a = 2$, $\alpha_0 = 1$, $p(t) = \frac{1}{4}\,t^{-3}L(t)e^{-t^8}$, $\varphi_0(y) = e^{|y|^4}$, $\varphi_1(y) = |y|^3$.

Theorem 1 implies that the equation (3) has a one-parameter family of $P_{+\infty}(+\infty, +\infty, 2)$-solutions, and every such solution and the derivative of such solution satisfy the following asymptotic representations

$$\frac{1}{y^7(t)}\,e^{\frac{1}{2}\,y^4(t)} = t^{-14}(L(t))^{-\frac{1}{2}}\,e^{\frac{1}{2}\,t^8}[1 + o(1)], \quad y'(t)y^3(t) = 2t^7[1 + o(1)] \ \text{ as } \ t \uparrow \omega.$$

To illustrate the results of Theorem 2, we consider the differential equation

$$y'' = \psi(t)\exp\big(\exp(|y|^a) - \exp(t^d)\big)|y|^{\sigma_0}|y'|^{\sigma_1} \ \text{ as } \ t \in [2, +\infty[\,. \tag{4}$$

Here, $\sigma_0, \sigma_1 \in R$, $\sigma_1 > 1$, $a, d \in]0, +\infty[$, function $\psi : [2, +\infty[\to]0, +\infty[$ is continuous regularly varying on an infinity function of index γ, $\gamma \in R$.

This equation is an equation of the type (1), where

$$\alpha_0 = 1, \quad p(t) = \psi(t)\exp(-\exp(t^d)), \quad \varphi_0(y) = |y|^{\sigma_0}\exp(\exp(|y|^a)), \quad \varphi_1(y') = |y'|^{\sigma_1}.$$

We investigate the question of the existence and asymptotic behavior as $t \to +\infty$ of $P_{+\infty}(\infty, Y_1, \lambda_0)$-solutions of the equation (4) for which $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$.

In this case

$$\pi_\omega(t) = t, \quad \theta_1(y) = 1.$$

So, the function θ_1 satisfies the condition S.

Taking into account the choice of $B^0_{+\infty}$ as $t \to +\infty$, we have

$$I(t) = |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \cdot y_0^1 \cdot \frac{\sigma_1 - 1}{d} \cdot t^{1-d+\frac{1}{1-\sigma_1}} \cdot |\psi(t)|^{\frac{1}{1-\sigma_1}} \cdot \exp\Big(\frac{\exp(t^d)}{\sigma_1 - 1} - t^d\Big)[1 + o(1)].$$

At the same way as $t \to +\infty$ we have

$$I_1(t) = |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \cdot y_0^1 \cdot \Big(\frac{\sigma_1 - 1}{d}\Big)^2 \cdot t^{1-2d+\frac{1}{1-\sigma_1}} \cdot |\psi(t)|^{\frac{1}{1-\sigma_1}} \cdot \exp\Big(\frac{\exp(t^d)}{\sigma_1 - 1} - 2t^d\Big)[1 + o(1)].$$

In addition, in this case, since $Y_0 = \infty$, taking into account the choice of A^0_∞, we obtain

$$\Phi_0(y) = \frac{\sigma_1 - 1}{a} \cdot y^{\frac{\sigma_0}{\sigma_1 - 1} + 1 - a} \cdot \exp\Big(\frac{\exp(|y|^a)}{\sigma_1 - 1} - |y|^a\Big)[1 + o(1)] \ \text{ as } \ y \to \infty.$$

Similarly, we have

$$\Phi_1(y) = \Big(\frac{\sigma_1 - 1}{a}\Big)^2 \cdot y^{\frac{\sigma_0}{\sigma_1 - 1} + 1 - 2a} \cdot \exp\Big(\frac{\exp(|y|^a)}{\sigma_1 - 1} - 2|y|^a\Big)[1 + o(1)] \ \text{ as } \ y \to \infty.$$

At the same time,

$$\Phi_1^{-1}(y) \cdot \frac{\Phi_1'(\Phi_1^{-1}(y))}{y} = \frac{(\sigma_1 - 1)^2}{a} \ln y \cdot \big(\ln\big((\sigma_1 - 1)\ln y\big)\big)^{\frac{\frac{\sigma_0}{\sigma_1 - 1} - 2a + 1}{a}}[1 + o(1)] \ \text{ as } \ y \to \infty.$$

It means that condition S_1 is satisfied.

Thus, all the conditions of Theorem 2 are satisfied. By virtue of this theorem, the equation (4) can have only $P_{+\infty}(+\infty, +\infty, \frac{d}{d-a})$-solutions of the class of $P_{+\infty}(\infty, Y_1, \lambda_0)$-solutions. From Theorem 2 it also follows that the equation (4) has one-parameter family of $P_{+\infty}(+\infty, +\infty, \frac{d}{d-a})$-solutions.

Also, taking into account the known asymptotic behavior of the function Φ_1^{-1}, it is easy to obtain that every $P_{+\infty}(+\infty, +\infty, \frac{d}{d-a})$-solution of the equation (4) and the derivative of such solution satisfy the following asymptotic representations

$$(y(t))^{\frac{\sigma_0}{\sigma_1 - 1} + 1 - 2a} \cdot \exp\Big(\frac{\exp(|y(t)|^a)}{\sigma_1 - 1} - 2|y(t)|^a\Big)$$

$$= \Big|\frac{a}{d - a}\Big|^{\frac{1}{1-\sigma_1}} \cdot \Big(\frac{a}{d}\Big)^2 \cdot t^{1-2d+\frac{1}{1-\sigma_1}} \cdot \psi^{\frac{1}{1-\sigma_1}}(t) \cdot \exp\Big(\frac{\exp(t^d)}{\sigma_1 - 1} - 2t^d\Big)[1 + o(1)] \ \text{ as } \ t \to +\infty,$$

$$y'(t) = \frac{y(t)}{t}\,[1 + o(1)] \ \text{ as } \ t \to +\infty.$$

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On Relations Between Regularity Coefficients of Linear Difference Equations

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Denote by $\mathcal{M}d_s$ the class of difference equations

$$x(n+1)=A(n)x(n), \ \ x(n)\in\mathbb{R}^s, \ \ n\in\mathbb{N}_0 \stackrel{\text{def}}{=} \mathbb{N}\cup\{0\}, \tag{1}$$

of dimension $s\geq 2$ with matrix coefficient $A(\,\cdot\,)\colon \mathbb{N}_0\to \operatorname{End}\mathbb{R}^s$ such that

$$\sup\Big\{\max\big\{\|A(n)\|,\|A^{-1}(n)\|\big\}:\ n\in\mathbb{N}_0\Big\}<+\infty,$$

where $\|\cdot\|$ is the operator norm generated by the Euclidean norm in $\mathbb{R}^s$ (the Euclidean norm will be denoted by the same symbol). In our further consideration we will identify the system (1) with its coefficient matrix and we will write $A(\,\cdot\,)\in\mathcal{M}d_s$, or simply $A\in\mathcal{M}d_s$. The solution $x(\,\cdot\,)$ of the system (1) is a sequence $x(\,\cdot\,)=(x(n))_{n=0}^{+\infty}$ of vectors from $\mathbb{R}^s$ satisfying for all $n\in\mathbb{N}_0$ the equation (1). The set of all solution of a system $A(\,\cdot\,)\in\mathcal{M}d_s$ with standard operations of multiplication by scalars and vector addition forms a linear space over $\mathbb{R}$, which will be denoted by $\mathcal{X}_A$. A natural isomorphism between linear spaces $\mathbb{R}^s$ and $\mathcal{X}_A$ is given by a bijection $\xi\longleftrightarrow x(\,\cdot\,;\xi)$. For natural numbers $k\geq m$ denote by $\mathcal{A}(k,m)$ a matrix equal to

$$\prod_{i=k-1}^{m} A(i) \ \text{ if } \ k>m$$

and to identity $s\times s$ matrix I_s if $k=m$. With this notation we have $x(n;x_0)=\mathcal{A}(n,0)x_0$ and $\mathcal{A}(k,m)=\Phi(k)\Phi(m)^{-1}$, where $\Phi(\,\cdot\,)$ is any fundamental matrix of the system (1).

Together with the system (1) we will consider the adjoint system

$$y(n+1)=A^{-T}(n)y(n), \ \ y(n)\in\mathbb{R}^s, \ \ n\in\mathbb{N}_0, \ \ A^{-T}(\,\cdot\,)\stackrel{\text{def}}{=}(A^T(\,\cdot\,))^{-1}. \tag{2}$$

It is obvious that the system adjoint to the system (2) is the system (1), therefore the systems (1) and (2) are called mutually adjoint.

With each system $A\in\mathcal{M}d_s$ we associate the so-called: Lyapunov regularity coefficients $\sigma_L(A)$, Perron regularity coefficient $\sigma_P(A)$ and Grobman regularity coefficient $\sigma_G(A)$ [3, 4, 6, 8]. The role of these coefficients lies in the fact that they essentially characterize the response of the system (1) to linear exponentially decreasing and non-linear of higher order of smallness perturbations. In particular, the equality of at least one of them (and then any) to zero is equivalent to the regularity in Lyapunov sense of the system (1).

Now, we will present definitions of regularity coefficients of a system $A \in \mathcal{M}d_s$. Let $\lambda_1(A) \le \cdots \le \lambda_s(A)$ denote the Lyapunov exponents of the system (1) and $\mu_1(A) \ge \cdots \ge \mu_s(A)$ the Lyapunov exponents of the adjoint system (2) (the first ones are numbered in non-decreasing order and the second in non-increasing order). By $\Psi(A)$ we denote the set of all fundamental matrices of the system (1). For any sequence $(X(n))_{n=0}^{+\infty}$ of $s \times s$ matrices by $\lambda_i[X]$ we denote the Lyapunov exponent of its i-th column, $i = 1, \dots, s$. The Lyapunov, Perron and Grobman regularity coefficients are given by the following formulae:

$$\sigma_L(A) \stackrel{\text{def}}{=} \sum_{i=1}^{s} \lambda_i(A) - \varliminf_{n\to+\infty} n^{-1} \ln|\det \mathcal{A}(n,0)|, \tag{3}$$

$$\sigma_P(A) \stackrel{\text{def}}{=} \max_{1\le i\le s} \big\{\lambda_i(A) + \mu_i(A)\big\}, \tag{4}$$

$$\sigma_G(A) \stackrel{\text{def}}{=} \inf_{\Phi\in\Psi(A)} \max_{1\le i\le s} \big\{\lambda_i[\Phi] + \lambda_i[\Phi^{-T}]\big\}. \tag{5}$$

Let us notice that by the formula (3) the Lyapunov regularity coefficient of the adjoint system (2) is given by

$$\sigma_L(A^{-T}) = \sum_{i=1}^{s} \mu_i(A) + \varlimsup_{n\to+\infty} n^{-1} \ln|\det \mathcal{A}(n,0)|.$$

For continuous time system, in the monograph [2, pp. 55, 74], it has been shown that the regularity coefficients of any system $A \in \mathcal{M}_s$, $s \ge 2$ satisfy the following relations

$$0 \le \sigma_P(A) \le \sigma_G(A) \le s\sigma_P(A) \ \text{ and } \ 0 \le \sigma_G(A) \le \sigma_L(A) \le s\sigma_G(A),$$

where $\mathcal{M}_s$ denotes the set of liner differential systems with piecewise continuous coefficient $s \times s$-matrices uniformly bounded on the nonnegative half-line $[0, +\infty)$. In addition, in [2, p. 151] it has been shown that all these inequalities are achievable and there exists a system $A \in \mathcal{M}_s$ such that the Lyapunov, Perron and Grobman regularity coefficients are pairwise different. In the monograph [5, pp. 21, 22] the following improvement of the last inequality

$$0 \le \sigma_P(A) \le \sigma_G(A) \le \sigma_L(A) \le s\sigma_P(A) \tag{6}$$

has been proved for any system $A \in \mathcal{M}_s$, $s \ge 2$.

In the paper [9], it has been shown that the inequalities (6) describe all possible relations between the regularity coefficients of differential systems. In other words, it was shown that for any natural $s \ge 2$ and ordered triple of numbers (p, g, l) satisfying the inequalities $0 \le p \le g \le \ell \le sp$, there exists a system $A \in \mathcal{M}_s$, such that $\sigma_P(A) = p$, $\sigma_G(A) = g$ and $\sigma_L(A) = \ell$. For the difference systems an analogical result was presented in [1].

From the definitions (4) and (5) of the Perron $\sigma_P(A)$ and Grobman $\sigma_G(A)$ regularity coefficients it follows that $\sigma_P(A) = \sigma_P(A^{-T})$ and $\sigma_G(A) = \sigma_G(A^{-T})$. However, analogical equality for Lyapunov $\sigma_L(A)$ regularity coefficient does not hold in general. The example of systems $A \in \mathcal{M}_s$, such that $\sigma_L(A) \ne \sigma_L(A^{-T})$, is constructed in [2, p. 155]. Analogical example of system $A \in \mathcal{M}d_s$ can be constructed. The question about the relation between the Lyapunov regularity coefficients of mutually adjoint systems, i.e. the question about description of the set of pairs $(\sigma_L(A), \sigma_L(A^{-T}))$ was solved in [9]. In this paper it has been shown that for each natural $s \ge 2$ and each non-negative numbers ℓ and ℓ^* there exists a system $A \in \mathcal{M}_s$ such that $\ell = \sigma_L(A)$ and $\ell^* = \sigma_L(A^{-T})$ if and only if $s^{-1}\ell^* \le \ell \le s\ell^*$. Analogous result for the discrete-time systems has been proved in [7].

From (6) it is straightforward to obtain the following chain of the inequalities

$$0 \le \sigma_P(A) \le \sigma_G(A) \le \min\big(\sigma_L(A), \sigma_L(-A^T)\big) \le \max\big(\sigma_L(A), \sigma_L(-A^T)\big) \le s\sigma_P(A).$$

Consequently, it is important to know for which nonnegative numbers p, g, ℓ, and ℓ^* one can construct a system $A \in \mathcal{M}d_s$ satisfying the equalities $\sigma_p(A) = p$, $\sigma_G(A) = g$, $\sigma_L(A) = \ell$, and $\sigma_L(A^{-T}) = \ell^*$. The answer to the last question is given by the following theorem.

Theorem. *For any integer $s \geq 2$ and an ordered quadruple (p, g, ℓ, ℓ^*) of real numbers satisfying the inequalities $0 \leq p \leq g \leq \min(\ell, \ell^*) \leq \max(\ell, \ell^*) \leq sp$, there exists a system $A \in \mathcal{M}d_s$ such that $\sigma_P(A) = p$, $\sigma_G(A) = g$, $\sigma_L(A) = \ell$, and $\sigma_L(A^{-T}) = \ell^*$.*

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Asymptotics of Solutions of Second-Order Differential Equations with Regularly and Rapidly Varying Nonlinearities

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Consider the differential equation

$$y'' = \sum_{i=1}^{m} \alpha_i p_i(t) \varphi_i(y), \tag{1}$$

where $\alpha_i \in \{-1, 1\}$ $(i = \overline{1,m}\,)$, $p_i : [a, \omega[\, \to]0, +\infty[$ $(i = \overline{1,m}\,)$ are continuous functions, $-\infty < a < \omega \leq +\infty$, $\varphi_i : \Delta_{Y_0} \to]0, +\infty[$ $(i = \overline{1,m}\,)$, where Δ_{Y_0} is some one-sided neighborhood of the point Y_0, Y_0 is equal either to 0 or to $\pm\infty$, are continuous functions for $i = \overline{1,l}$ and twice continuously differentiable for $i = \overline{l+1,m}$, so that

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi_i(\lambda y)}{\varphi_i(y)} = \lambda^{\sigma_i} \;\; (i = \overline{1,l}\,) \;\; \text{for any} \;\; \lambda > 0, \tag{2}$$

$$\varphi_i'(y) \neq 0 \;\; \text{as} \;\; y \in \Delta_{Y_0}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \varphi_i(y) \in \{0, +\infty\}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi_i''(y)\varphi_i(y)}{\varphi_i'^2(y)} = 1 \;\; (i = \overline{l+1,m}\,). \tag{3}$$

It follows from the conditions (2) and (3) that φ_i $(i = \overline{1,l}\,)$ are regularly varying functions, as $y \to Y_0$, of orders σ_i and φ_i $(i = \overline{l+1,m}\,)$ are rapidly varying functions, as $y \to Y_0$ (see [5, Introduction, pp. 2, 4]).

Definition. A solution y of the differential equation (1) is called $P_\omega(Y_0, \lambda_0)$-solution, where $-\infty \leq \lambda_0 \leq +\infty$, if it is defined on some interval $[t_0, \omega[\, \subset [a, \omega[$ and satisfies the following conditions

$$\lim_{t \uparrow \omega} y(t) = Y_0, \quad \lim_{t \uparrow \omega} y'(t) = \begin{cases} \text{either} & 0, \\ \text{or} & \pm\infty, \end{cases} \quad \lim_{t \uparrow \omega} \frac{y'^2(t)}{y''(t) y(t)} = \lambda_0.$$

There have been known the results of the asymptotic behavior of $P_\omega(Y_0, \lambda_0)$-solutions of differential equation (1) in case when there is only one item with a regularly or rapidly varying nonlinearity on the right-hand side of the equation (1) (see [1–3]). The case $l = m$ has been also investigated when all nonlinearities on the right-hand side of differential equation (1) are regularly varying functions (see [4]). The general case, when, in addition to items with regularly varying nonlinearities there are items with rapidly varying nonlinearities on the right-hand side of the equation (1), has not been studied yet.

In this paper, for $\lambda_0 \in \mathbb{R} \setminus \{0; 1\}$ the existence conditions of $P_\omega(Y_0, \lambda_0)$-solutions of the differential equation (1) and asymptotic representations, as $t \uparrow \omega$, of such solutions and their first-order derivatives, are established in case when on each such solution the right-hand side of equation is equivalent, as $t \uparrow \omega$, to the s-th item, that is when

$$\lim_{t \uparrow \omega} \frac{p_i(t) \varphi_i(y(t))}{p_s(t) \varphi_s(y(t))} = 0 \;\; \text{for all} \;\; i \in \{1, \dots, m\} \setminus \{s\}. \tag{4}$$

Let

$$\Delta_{Y_0} = \Delta_{Y_0}(b), \ \ \text{where} \ \ \Delta_{Y_0}(b) = \begin{cases} [b, Y_0[& \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0, \\ ]Y_0, b] & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0, \end{cases}$$

and the number b satisfy the inequalities

$$|b| < 1 \ \text{ as } \ Y_0 = 0 \quad \text{and} \quad b > 1 \ \ (b < -1) \ \text{ as } \ Y_0 = +\infty \ \ (Y_0 = -\infty).$$

We set

$$\nu_0 = \operatorname{sign} b, \quad \nu_1 = \begin{cases} 1 & \text{if } \ \Delta_{Y_0}(b) = [b, Y_0[\,, \\ -1 & \text{if } \ \Delta_{Y_0}(b) =]Y_0, b], \end{cases} \quad \mu_i = \operatorname{sign} \varphi_i'(y) \ \ (i = \overline{l+1, m}\,),$$

$$\pi_\omega(t) = \begin{cases} t & \text{if } \ \omega = +\infty, \\ t - \omega & \text{if } \ \omega < +\infty, \end{cases} \quad J_i(t) = \int\limits_{A_i}^{t} \pi_\omega(\tau) p_i(\tau)\, d\tau,$$

$$H_i(y) = \int\limits_{B_i}^{y} \frac{ds}{\varphi_i(s)}\,, \quad Z_i = \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}(b)}} H_i(y) \ \ (i = \overline{1, m}\,),$$

where

$$A_i = \begin{cases} a & \text{if } \ \int\limits_a^\omega \pi_\omega(\tau) p_i(\tau)\, d\tau = \pm\infty, \\ \omega & \text{if } \ \int\limits_a^\omega \pi_\omega(\tau) p_i(\tau)\, d\tau = const, \end{cases} \qquad B_i = \begin{cases} b & \text{if } \ \int\limits_b^{Y_0} \dfrac{dy}{\varphi_i(y)} = \pm\infty, \\ Y_0 & \text{if } \ \int\limits_b^{Y_0} \dfrac{dy}{\varphi_i(y)} = const. \end{cases}$$

Theorem 1. *Let $\lambda_0 \in \mathbb{R} \setminus \{0; 1\}$ and $\sigma_s \neq 1$ for some $s \in \{1, \dots, l\}$. For the existence of $P_\omega(Y_0, \lambda_0)$-solutions of the equation* (1)*, satisfied the limit relations* (4)*, it is necessary that the inequalities*

$$\alpha_s \nu_0 \lambda_0 > 0, \quad \nu_0 \nu_1 \lambda_0 (\lambda_0 - 1) \pi_\omega(t) > 0 \ \ \textit{as} \ \ t \in]a, \omega[\tag{5}$$

and conditions

$$\alpha_s(\lambda_0 - 1) \lim_{t \uparrow \omega} J_s(t) = Z_s, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) J_s'(t)}{J_s(t)} = \frac{(1 - \sigma_s)\lambda_0}{\lambda_0 - 1}\,, \tag{6}$$

$$\lim_{t \uparrow \omega} \frac{p_i(t) \varphi_i(H_s^{-1}(\alpha_s(\lambda_0 - 1) J_s(t)))}{p_s(t) \varphi_s(H_s^{-1}(\alpha_s(\lambda_0 - 1) J_s(t)))} = 0 \ \ \textit{for all} \ \ i \in \{1, \dots, l\} \setminus \{s\}, \tag{7}$$

$$\lim_{t \uparrow \omega} \frac{p_i(t) \varphi_i(H_s^{-1}(\alpha_s(\lambda_0 - 1) J_s(t)(1 + \delta_i)))}{p_s(t) \varphi_s(H_s^{-1}(\alpha_s(\lambda_0 - 1) J_s(t)))} = 0 \ \ \textit{for all} \ \ i \in \{l+1, \dots, m\}$$

hold, where δ_i are arbitrary numbers of a one-sided neighborhood of zero. Moreover, for each of such solutions the following asymptotic representations hold

$$y(t) = H_s^{-1}\big(\alpha_s(\lambda_0 - 1) J_s(t)\big)[1 + o(1)] \ \ \textit{at} \ \ t \uparrow \omega, \tag{8}$$

$$y'(t) = \frac{\lambda_0 H_s^{-1}(\alpha_s(\lambda_0 - 1) J_s(t))}{(\lambda_0 - 1)\pi_\omega(t)}\, [1 + o(1)] \ \ \textit{at} \ \ t \uparrow \omega. \tag{9}$$

Theorem 2. *Let $\lambda_0 \in \mathbb{R} \setminus \{0;1\}$ and $\sigma_s \neq 1$ for some $s \in \{1, \dots, l\}$, the conditions* (5)–(7) *hold and*

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(H_s^{-1}(\alpha_s(\lambda_0-1)J_s(t)(1+u)))}{p_s(t)\varphi_s(H_s^{-1}(\alpha_s(\lambda_0-1)J_s(t)))} = 0 \ \text{ for all } \ i \in \{l+1, \dots, m\}$$

uniformly with respect to $u \in [-\delta, \delta]$ for any $0 < \delta < 1$. Let also one of the following two conditions hold

$$\text{or } \ \lambda_0 \neq -1, \quad \text{or } \ \lambda_0 = -1 \ \text{ and } \ \sigma_s < 1.$$

Then the differential equation (1) *has $P_\omega(Y_0, \lambda_0)$-solutions that admit the asymptotic representations* (8) *and* (9)*. Moreover, there is a one-parameter family of such solutions in case $\lambda_0(1-\sigma_s) < 0$ and two-parameter one in case $\lambda_0(1-\sigma_s) > 0$ and $\pi_\omega(t)(1-\lambda_0^2) < 0$ as $t \in]a, \omega[$.*

Besides the above-mentioned facts we also need the following auxiliary notations

$$J_{0i}(t) = \int_{A_i}^{t} \pi_\omega(\tau) p_{0i}(\tau)\, d\tau,$$

$$q_{0i}(t) = \frac{\alpha_i(\lambda_0-1)\pi_\omega^2(t)p_{0i}(t)\varphi_i(H_i^{-1}(\alpha_i(\lambda_0-1)J_{0i}(t)))}{H_i^{-1}(\alpha_i(\lambda_0-1)J_{0i}(t))},$$

$$G_{0i}(t) = \frac{y\varphi_i'(y)}{\varphi_i(y)}\bigg|_{y=H_i^{-1}(\alpha_i(\lambda_0-1)J_{0i}(t))}, \quad \psi_{0i}(t) = \int_{t_0}^{t} \frac{|G_{0i}(\tau)|^{\frac{1}{2}}\, d\tau}{\pi_\omega(\tau)},$$

$$\Phi_{0i}(t) = \frac{y\Big(\frac{\varphi_i'(y)}{\varphi_i(y)}\Big)'}{\frac{\varphi_i'(y)}{\varphi_i(y)}}\Bigg|_{y=H_i^{-1}(\alpha_i(\lambda_0-1)J_{0i}(t))} \qquad (i = \overline{l+1, m}\,),$$

where $p_{0i} : [a, \omega[\, \to]0, +\infty[$ are continuous functions so that $p_{0i}(t) \sim p_i(t)$ as $t \uparrow \omega$, t_0 is some number of $[a, \omega[$.

Theorem 3. *Let $\lambda_0 \in \mathbb{R} \setminus \{0;1\}$ and for some $s \in \{l+1, \dots, m\}$ the conditions*

$$\frac{\varphi_s(y)\varphi_i'(y)}{\varphi_s'(y)\varphi_i(y)} = O(1) \ \text{ as } \ y \to Y_0 \ \ (y \in \Delta_{Y_0}(b)) \ \text{ for all } \ i \in \{l+1, \dots, m\} \tag{10}$$

hold. For the existence of $P_\omega(Y_0, \lambda_0)$-solutions of the equation (1) *that admit the limit relations* (4)*, it is necessary that for some continuous function $p_{0s} : [a, \omega[\, \to]0, +\infty[$ such that $p_{0s}(t) \sim p_i(t)$ as $t \uparrow \omega$ the conditions*

$$\alpha_s\nu_0\lambda_0 > 0, \quad \alpha_s\mu_s(\lambda_0-1)J_{0s}(t) < 0 \ \text{ at } \ t \in]a, \omega[\,, \tag{11}$$

$$\alpha_s(\lambda_0-1)\lim_{t\uparrow\omega} J_{0s}(t) = Z_s, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_{0s}'(t)}{J_{0s}(t)} = \pm\infty, \quad \lim_{t\uparrow\omega} q_{0s}(t) = \frac{\lambda_0}{\lambda_0-1}, \tag{12}$$

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(H_s^{-1}(\alpha_s(\lambda_0-1)J_{0s}(t)))}{p_{0s}(t)\varphi_s(H_s^{-1}(\alpha_s(\lambda_0-1)J_{0s}(t)))} = 0 \ \text{ for all } \ i \in \{1, \dots, m\} \setminus \{s\} \tag{13}$$

hold. Moreover, for each of such solutions the following asymptotic representations hold

$$y(t) = H_s^{-1}(\alpha_s(\lambda_0-1)J_{0s}(t))\Big[1 + \frac{o(1)}{G_{0s}(t)}\Big] \ \text{ at } \ t \uparrow \omega,$$

$$y'(t) = \frac{\lambda_0 H_s^{-1}(\alpha_s(\lambda_0-1)J_{0s}(t))}{(\lambda_0-1)\pi_\omega(t)}\,[1+o(1)] \ \text{ at } \ t \uparrow \omega.$$

Theorem 4. *Let $\lambda_0 \in \mathbb{R} \setminus \{0;1\}$, for some $s \in \{l+1,\dots,m\}$ the function p_s might be represented in the form*

$$p_s(t) = p_{0s}(t)[1 + r_s(t)], \quad \textit{where} \ \lim_{t\uparrow\omega} r_s(t) = 0,$$

$p_{0s} : [a,\omega[\to]0,+\infty[$ *is a continuously differentiable function,* $r_s : [a,\omega[\to]-1,+\infty[$ *is a continuous function, the conditions* (10)–(13) *hold and there exist finite or equal to infinity limits*

$$\gamma_s = \lim_{t\uparrow\omega} \Phi_{0s}(t), \quad \lim_{t\uparrow\omega} \pi_\omega(t) q_{0s}'(t), \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}(b)}} \frac{\big(\frac{\varphi_s'(y)}{\varphi_s(y)}\big)'}{\big(\frac{\varphi_s'(y)}{\varphi_s(y)}\big)^2} \sqrt{\Big|\frac{y\varphi_s'(y)}{\varphi_s(y)}\Big|}, \quad \lim_{t\uparrow\omega} \frac{\psi_{0s}(t)\psi_{0s}''(t)}{\psi_{0s}'^2(t)}\,.$$

Then

1) *if $\alpha_s\mu_s = 1$, the differential equation* (1) *has a one-parameter family of $P_\omega(Y_0,\lambda_0)$-solutions with asymptotic representations*

$$y(t) = H_s^{-1}(\alpha_s(\lambda_0 - 1)J_{0s}(t))\Big[1 + \frac{o(1)}{G_{0s}(t)}\Big] \ \ \textit{at} \ t\uparrow\omega,$$
$$y'(t) = \frac{\lambda_0 H_s^{-1}(\alpha_s(\lambda_0-1)J_{0s}(t))}{(\lambda_0-1)\pi_\omega(t)}\Big[\frac{\lambda_0-1}{\lambda_0} q_{0s}(t) + |G_{0s}(t)|^{-\frac12} o(1)\Big] \ \ \textit{at} \ t\uparrow\omega;$$

2) *if $\alpha_s\mu_s = -1$ and*

$$\gamma_s \neq \lim_{\lambda\to\lambda_0} \frac{(\lambda-1)(2-3\lambda)}{\lambda(5\lambda-4)}, \quad \lim_{t\uparrow\omega} \psi_{0s}(t)\Big[q_{0s}(t)[1+r_s(t)] - \frac{\lambda_0}{\lambda_0-1}\Big] = 0,$$
$$\lim_{t\uparrow\omega} \psi_{0s}^2(t)\Big[\Big(\frac{\lambda_0}{\lambda_0-1} - q_{0s}(t)\Big)q_{0s}(t) + \frac{q_{0s}(t)r_s(t)}{\lambda_0-1} - \pi_\omega(t)q_{0s}'(t)\Big] = 0,$$
$$\lim_{t\uparrow\omega} \psi_{0s}^2(t) \sum_{\substack{i=1\\ i\neq s}}^{m} \frac{p_i(t)\varphi_i(H_s^{-1}(\alpha_s(\lambda_0-1)J_{0s}(t)))}{p_{0s}(t)\varphi_s(H_s^{-1}(\alpha_s(\lambda_0-1)J_{0s}(t)))} = 0,$$

the differential equation (1) *has a $P_\omega(Y_0,\lambda_0)$-solution with asymptotic at $t\uparrow\omega$ representations*

$$y(t) = H_s^{-1}(\alpha_s(\lambda_0-1)J_{0s}(t))\Big[1 + \frac{o(1)}{\psi_{0s}(t)G_{0s}(t)}\Big],$$
$$y'(t) = \frac{\lambda_0 H_s^{-1}(\alpha_s(\lambda_0-1)J_{0s}(t))}{(\lambda_0-1)\pi_\omega(t)}\Big[\frac{\lambda_0-1}{\lambda_0} q_{0s}(t) + \frac{o(1)}{\psi_{0s}(t)|G_{0s}(t)|^{\frac12}}\Big].$$

Moreover, there exists a two-parameter family of such solutions in case when

$$\beta\big(\lambda_0^2(5\gamma_s+3) + \lambda_0(-4\gamma_s-5)+2\big) < 0 \ \ \textit{as} \ \gamma_s = \textit{const}, \quad \frac45 < \lambda_0 < 1 \ \ \textit{as} \ \gamma_s = \pm\infty.$$

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Global Bifurcation of a Unique Limit Cycle in Some Class of Planar Systems

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1 Introduction

We consider the planar autonomous differential systems

$$\frac{dx}{dt} = P(x,y,\lambda), \quad \frac{dy}{dt} = Q(x,y,\lambda) \tag{1.1}$$

depending on a scalar parameter $\lambda \in \mathbb{R}$. Our goal is to derive conditions on P and Q such that there is a $\lambda_0 \in \mathbb{R}$ with the property that for all $\lambda > \lambda_0$ system (1.1) has a unique limit cycle in the phase plane which is hyperbolic and stable. Our approach to treat this problem is based on the bifurcation theory of planar autonomous systems. The underlying idea of our approach can be formulated as follows: We assume that $\lambda = \lambda_0$ and $\lambda = +\infty$ are bifurcation points of system (1.1) connected with the appearance of a limit cycle which is hyperbolic and stable, and we suppose that the interval $(\lambda_0, +\infty)$ does not contain any bifurcation point of system (1.1). The class of Dulac–Cherkas functions, the theory of one-parameter families of rotated vector fields and singularly perturbed systems are key ingredients in our approach [1, 3–5, 8–10]. In the Appendix their basic properties are summarized. We illustrate our approach by an example.

2 Assumptions. Main result

Consider system (1.1) under the following assumptions:

(A_1) *$P, Q : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are sufficiently smooth.*

(A_2) *System* (1.1) *has $\forall\, \lambda \in \mathbb{R}$ a unique equilibrium $E(\lambda)$ in the finite part of the phase plane.*

Without loss of generality we may suppose that $E(\lambda)$ is located at the origin $\forall\, \lambda$.

(A_3) *The origin changes its stability at $\lambda = \lambda_0$ and is unstable for $\lambda > \lambda_0$.*

(A_4) *There exists for $\lambda > \lambda_0$ a Dulac–Cherkas function $\Psi(x,y,\lambda)$ of system* (1.1) *in the phase plane such that the set $\mathcal{W}_\lambda := \{(x,y) \in \mathbb{R}^2 :\ \Psi(x,y,\lambda) = 0\}$ consists of a unique oval surrounding the origin.*

(A_5) *For $\lambda > \lambda_0$ there is a one-to-one mapping*

$$\overline{x} = \varphi_1(x, y, \lambda), \quad \overline{y} = \psi_1(x, y, \lambda)$$

such that system (1.1) *will be transformed into the system*

$$\frac{d\overline{x}}{dt} = \overline{P}(\overline{x}, \overline{y}, \lambda), \quad \frac{d\overline{y}}{dt} = \overline{Q}(\overline{x}, \overline{y}, \lambda) \tag{2.1}$$

with the following properties:

(i) *The functions $\overline{P}$ and $\overline{Q}$ have for $\lambda > \lambda_0$ the same smoothness as the functions P and Q.*

(ii) *The origin is the unique equilibrium of system* (2.1) $\forall\, \lambda > \lambda_0$.

(iii) λ_0 *is a Hopf bifurcation point for system* (2.1) *connected with the bifurcation of a stable limit cycle $\overline{\Gamma}_\lambda$ from the origin for increasing λ which is positively* (*that is anti-clockwise*) *oriented.*

(iv) *System* (2.1) *represents for $\lambda > \lambda_0$ a one-parameter family of positively rotated vector fields.*

(A_6) *For $\lambda > \lambda_0$ there is a one-to-one mapping*

$$\widetilde{x} = \varphi_2(x, y, \lambda), \quad \widetilde{y} = \psi_2(x, y, \lambda), \quad \tau = \chi(t, \lambda),$$

where τ increases with t for any $\lambda > \lambda_0$, such that system (1.1) *will be transformed into the system*

$$\frac{d\widetilde{x}}{d\tau} = \widetilde{P}(\widetilde{x}, \widetilde{y}, \varepsilon), \quad \varepsilon\,\frac{d\widetilde{y}}{d\tau} = \widetilde{Q}(\widetilde{x}, \widetilde{y}, \varepsilon) \tag{2.2}$$

with the following properties:

(i) *There is a smooth function $\zeta : (\lambda_0, +\infty) \to \mathbb{R}^+$ with $\zeta(\lambda) \to 0$ as $\lambda \to +\infty$ such that $\varepsilon = \zeta(\lambda)$.*

(ii) *The functions $\widetilde{P}$ and $\widetilde{Q}$ have for $\varepsilon > 0$ the same smoothness as the functions P and Q.*

(iii) *There is a sufficiently small positive number δ such that for $\varepsilon \in (0, \delta)$ system* (2.2) *has a family $\{\widetilde{\Gamma}_\varepsilon\}$ of uniformly bounded hyperbolic stable limit cycles which surround the origin and are positively oriented.*

The following theorem is our main result.

Theorem 2.1. *Under the assumptions (A_1)–(A_6) system* (1.1) *has for $\lambda > \lambda_0$ a unique family $\{\Gamma_\lambda\}$ of limit cycles which are hyperbolic, stable and positively oriented, and whose amplitudes are bounded on any bounded λ-interval.*

3 Example

We present an application of Theorem 2.1 for the Liénard system

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x - \lambda(x^{2q} - 1)y \tag{3.1}$$

with $q \in \mathbb{N}$. For $q = 1$, system (3.1) represents the famous van der Pol system. We show that system (3.1) has the same properties as the van der Pol system. For this purpose we prove that the assumptions (A_1)–(A_6) are fulfilled for system (3.1). In particular, we get the following results

Lemma 3.1. *The function*

$$\Psi(x,y,\lambda) \equiv x^2 + y^2 - 1$$

is a Dulac–Cherkas function for system (3.1) *in the phase plane for* $\lambda > 0$.

Finally, we apply Theorem 2.1 and get the result

Theorem 3.2. *System* (3.1) *has for all* $\lambda > 0$ $(\lambda < 0)$ *a unique limit cycle which is hyperbolic stable* (*unstable*) *and positively oriented.*

Full version of the derived results as a corresponding paper has been submitted for publication.

4 Appendix

Suppose that P, Q satisfy assumption (A_1). We denote by $X(\lambda)$ the vector field defined by system (1.1), by Λ some λ-interval and by Ω some region in $\mathbb{R}^2$.

Definition 4.1. A function $\Psi : \Omega \times \Lambda \to \mathbb{R}$ with the same smoothness as P, Q is called a Dulac–Cherkas function of system (1.1) in Ω for $\lambda \in \Lambda$ if there exists a real number $\kappa \neq 0$ such that

$$\Phi := (\operatorname{grad}\Psi, X(\lambda)) + \kappa\Psi \operatorname{div} X(\lambda) > 0 \ \ (< 0) \quad \text{for} \ \ (x,y,\lambda) \in \Omega \times \Lambda. \tag{4.1}$$

Remark 4.2. Condition (4.1) can be relaxed by assuming that Φ may vanish in Ω on a set of measure zero, and that no closed curve of this set is a limit cycle of (1.1).

The following two theorems can be found in [2].

Theorem 4.3. *Let* Ψ *be a Dulac–Cherkas function of* (1.1) *in* Ω *for* $\lambda \in \Lambda$. *Then any limit cycle* Γ_λ *of* (1.1) *in* Ω *is hyperbolic and its stability is determined by the sign of the expression* $\kappa\Phi\Psi$ *on* Γ_λ.

Theorem 4.4. *Let* Ω *be a* p*-connected region, let* Ψ *be a Dulac–Cherkas function of* (1.1) *in* Ω *such that the set* $\mathcal{W}_\lambda := \{(x,y) \in \Omega : \ \Psi(x,y,\lambda) = 0\}$ *consists of* s *ovals in* Ω. *Then system* (1.1) *has at most* $p - 1 + s$ *limit cycles in* Ω.

The following facts can be found in [7].

Definition 4.5. Let the assumption (A_1) be satisfied. System (1.1) is said to define a one-parameter family of negatively (positively) rotated vector fields for $\lambda \in \Lambda$ if for $\lambda \in \Lambda$ the equilibria of system (1.1) are isolated and at all ordinary points it holds

$$\Delta(x,y,\lambda) := P(x,y,\lambda)\,\frac{\partial Q(x,y,\lambda)}{\partial\lambda} - Q(x,y,\lambda)\,\frac{\partial P(x,y,\lambda)}{\partial\lambda} < 0 \ \ (> 0).$$

Remark 4.6. This condition can be relaxed by assuming that Δ vanishes on a set of measure zero and that no closed curve of this set is a limit cycle of (1.1).

Theorem 4.7. *Suppose that the assumptions* (A_1) *and* (A_2) *are satisfied and that system* (1.1) *represents a one-parameter family of negatively* (*positively*) *rotated vector fields. Let* $\{\Gamma_\lambda\}$ *be a family of hyperbolic stable limit cycles of system* (1.1) *with positive orientation. Then the amplitude of* Γ_λ *decreases monotonically with decreasing* (*increasing*) λ, *and the family terminates at* $\lambda = \lambda_*$ *when* Γ_{λ_*} *represents an equilibrium.*

Consider the singularly perturbed system

$$\frac{dx}{dt} = f(x,y), \quad \varepsilon \frac{dy}{dt} = g(x,y) \tag{4.2}$$

under the following assumptions

(C_1) *$f, g : \mathbb{R}^2 \to \mathbb{R}$ are sufficiently smooth, ε is a small positive parameter.*

(C_2) *The origin is the unique equilibrium of system* (4.2) *in the finite part of the phase plane. It is unstable for $\varepsilon > 0$. The trajectories are positively oriented near the origin.*

(C_3) *$g(x,y) = 0$ has the unique simple solution $x = \varphi(y)$, where φ is sufficiently smooth and satisfies*

$$\varphi(0) = 0, \quad \varphi'(0) < 0.$$

$\varphi'(y) = 0$ has exactly two real roots y_- and y_+ satisfying

$$y_- < 0, \ \ \varphi''(y_-) < 0, \ \ y_+ > 0, \ \ \varphi''(y_+) > 0.$$

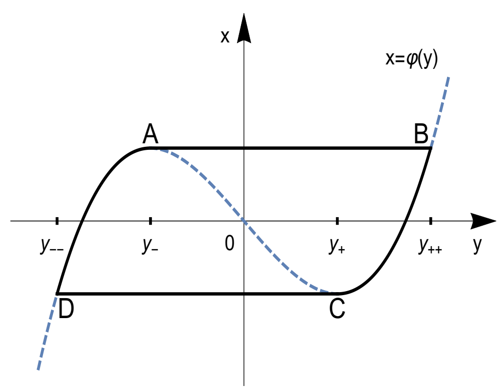

Figure 1. Closed curve $\mathcal{Z}_0$.

Using assumption (C_3) we can define a closed curve $\mathcal{Z}_0$ in the phase plane consisting of two finite segments of the curve $x = \varphi(y)$ bounded by the points $D = (y_{-}, \varphi(y_+))$, $A = (y_-, \varphi(y_-))$ and $C = (y_+, \varphi(y_+))$, $B = (y_{++}, \varphi(y_-))$ and of two finite segments of the straight lines $x = \varphi(y_-)$ and $x = \varphi(y_+)$ bounded by the points A, B and D, C, respectively (see Figure 1).

The following theorem is a special case of a more general theorem by E. F. Mishchenko and N. Kh. Rozov in [6].

Theorem 4.8. *Under the assumptions (C_1)–(C_3), system* (4.2) *has for sufficiently small ε a unique limit cycle Γ_ε in a small neighborhood of $\mathcal{Z}_0$ which is stable and positively oriented*

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Existence and Multiplicity of Periodic Solutions to Indefinite Singular Equations with the Phase Singularities

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The existence of a T-periodic solution to the second-order differential equation

$$u'' = h(t)g(u) \tag{1}$$

is studied in the first part. Here, $h \in L(\mathbb{R}/T\mathbb{Z})$ and $g \in C^1(\mathbb{R}_+;\mathbb{R}_+)$ ($\mathbb{R}_+$ stands for positive real numbers) is a nonincreasing function with a strong singularity at zero, i.e.,

$$\lim_{x\to0^+} \int\limits_x^1 g(s)\,ds = +\infty. \tag{2}$$

By a T-periodic solution to (1) we understand a T-periodic positive function $u : \mathbb{R} \to \mathbb{R}_+$ which is absolutely continuous together with its first derivative on $[0,T]$ and satisfies the equality (1) almost everywhere on $[0,T]$.

In addition to the assumptions imposed on g previously, we will need to assume the following technical condition hold:

$$\text{there exists } \ \gamma > 0 \ \text{ such that } \ \liminf_{x\to+\infty} \frac{g((1+\gamma)x)}{g(x)}\, H_- > H_+, \tag{3}$$

where

$$H_+ = \int\limits_0^T [h(s)]_+\,ds, \quad H_- = \int\limits_0^T [h(s)]_-\,ds,$$

denoting by $[a]_+ = \frac{1}{2}\,(|a|+a)$, $[a]_- = \frac{1}{2}\,(|a|-a)$ for any real number a. Obviously, the condition (3) implies that $\overline{h} \overset{def}{=} \frac{1}{T}\int\limits_0^T h(s)\,ds < 0$. However, this is not restrictive because $\overline{h} < 0$ is also a necessary condition for the existence of a T-periodic solution in the case when g is strictly decreasing (see Remark 2 below). For example, the condition (3) is satisfied when $g(x) = 1/x^\lambda$ (the nonlinearity in the model equation) provided $\overline{h} < 0$.

Remark 1. Without loss of generality we can and we will assume that

$$\lim_{x\to+\infty} g(x) < 1. \tag{4}$$

Indeed, if this is not the case, we can pass to the equation

$$u'' = \widetilde{h}(t)\widetilde{g}(u),$$

where $\widetilde{h}(t) = (g_\infty + 1)h(t)$ for $t \in \mathbb{R}$, $\widetilde{g}(x) = g(x)/(g_\infty + 1)$ for $x \in \mathbb{R}_+$, and $g_\infty = \lim\limits_{x\to+\infty} g(x)$.

Theorem 1. *Let* $\overline{h} < 0$, g *satisfy* (2), (3), *and* (4), *and let there exist pairwise disjoint intervals* $[a_k, b_k] \subset [0, T]$ $(k = 1, \dots, n)$ *such that*

$$h(t) \geq 0 \ \ \text{for a.e.} \ \ t \in \bigcup_{k=1}^{n} [a_k, b_k],$$
$$h(t) \leq 0 \ \ \text{for a.e.} \ \ t \in [0, T] \setminus \bigcup_{k=1}^{n} [a_k, b_k].$$

Let, moreover, there exist $c_k \in (a_k, b_k)$ $(k = 1, \dots, n)$ *such that*

$$\lim_{t \to t_0^+} \int_t^{b_k} h(s) g(C_k(s - t_0))\, ds = +\infty \ \ \text{for every} \ \ t_0 \in [a_k, c_k] \quad (k = 1, \dots, n),$$
$$\lim_{t \to t_0^-} \int_{a_k}^{t} h(s) g(D_k(t_0 - s))\, ds = +\infty \ \ \text{for every} \ \ t_0 \in [c_k, b_k] \quad (k = 1, \dots, n),$$

where

$$C_k = \frac{\Gamma}{b_k - c_k} + \frac{|\overline{h}|(b_k - c_k)}{4}\,, \quad D_k = \frac{\Gamma}{c_k - a_k} + \frac{|\overline{h}|(c_k - a_k)}{4}\,,$$
$$\Gamma = g^{-1}(1) + \frac{T}{4}\, \|h\|_1.$$

Then the equation (1) *has at least one* T*-periodic solution.*

Remark 2. Note that the condition $\overline{h} < 0$ is necessary for the existence of a T-periodic solution to (1) in the case when g is a strictly decreasing function. Indeed, if u is a T-periodic solution to (1), then dividing both sides of (1) by $g(u)$ and integrating it over $[0, T]$ we arrive at

$$0 > \int_0^T \frac{u'^2(s) g'(u(s))}{g^2(u(s))}\, ds = \int_0^T \frac{u''(s)}{g(u(s))}\, ds = \int_0^T h(s)\, ds,$$

provided $h(t) \not\equiv 0$.

The equation

$$u'' = \frac{h(t)}{u^{\lambda}}\,, \tag{5}$$

with $\lambda > 0$, can be viewed as a particular case of (1). Thus from Theorem 1 we obtain the following assertion.

Corollary 1. *Let* $\lambda \geq 1$ *and let there exist pairwise disjoint intervals* $[a_k, b_k] \subset [0, T]$ $(k = 1, \dots, n)$ *and* $\alpha > 0$ *such that*

$$h(t) \geq \alpha \big[(b_k - t)(t - a_k)\big]^{\lambda - 1} \ \ \text{for a.e.} \ \ t \in [a_k, b_k] \ \ (k = 1, \dots, n),$$
$$h(t) \leq 0 \ \ \text{for a.e.} \ \ t \in [0, T] \setminus \bigcup_{k=1}^{n} [a_k, b_k].$$

Then the equation (5) *has a* T*-periodic solution if and only if* $\overline{h} < 0$.

Slightly different result can be obtained in the case where the function g possesses two singularities. Therefore, we will consider the equation of the form

$$u'' = \sigma h(t) g(u) \tag{6}$$

in the second part. Here again, $h \in L(\mathbb{R}/T\mathbb{Z})$, $\sigma > 0$ is a parameter, and $g : (A, B) \to \mathbb{R}_+$ is a continuous function with $-\infty < A < B < +\infty$. Moreover, we assume that g is continuously differentiable, and there exists $P \in (A, B)$ such that

$$g'(x) \leq 0 \ \text{ for } \ x \in (A, P), \quad g'(x) \geq 0 \ \text{ for } \ x \in (P, B), \tag{7}$$

$$\lim_{x \to A^+} \int\limits_x^P g(s)\, ds = +\infty, \quad \lim_{x \to B^-} \int\limits_P^x g(s)\, ds = +\infty. \tag{8}$$

In this case, by a T-periodic solution to (6) we understand a T-periodic function $u : \mathbb{R} \to (A, B)$ which is absolutely continuous together with its first derivative on $[0, T]$ and satisfies the equality (6) almost everywhere on $[0, T]$.

Obviously, $H_+ H_- \neq 0$ is a necessary condition for solvability of a periodic problem for (6).

Theorem 2. *Let $\overline{h} \neq 0$, g satisfy (7), (8), and let there exist pairwise disjoint intervals $(a_k, b_k) \subset [0, T]$, $(x_i, y_i) \subset [0, T]$ ($k = 1, \dots, n$; $i = 1, \dots, m$) such that*

$$\bigcup_{k=1}^n [a_k, b_k] \cup \bigcup_{i=1}^m [x_i, y_i] = [0, T], \tag{9}$$

$$h(t) \geq 0 \ \text{ for a.e. } \ t \in \bigcup_{k=1}^n (a_k, b_k),$$

$$h(t) \leq 0 \ \text{ for a.e. } \ t \in \bigcup_{i=1}^m (x_i, y_i).$$

Let, moreover, there exist $c_k \in (a_k, b_k)$ and $z_i \in (x_i, y_i)$ ($k = 1, \dots, n$; $i = 1, \dots, m$) such that

$$\lim_{t \to t_0^+} \int\limits_t^{t_0 + \frac{P-A}{C_k}} h(s) g(A + C_k(s - t_0))\, ds = +\infty \ \text{ for every } \ t_0 \in [a_k, c_k] \ \ (k = 1, \dots, n),$$

$$\lim_{t \to t_0^-} \int\limits_{t_0 - \frac{P-A}{D_k}}^t h(s) g(A + D_k(t_0 - s))\, ds = +\infty \ \text{ for every } \ t_0 \in [c_k, b_k] \ \ (k = 1, \dots, n),$$

$$\lim_{t \to t_0^+} \int\limits_t^{t_0 + \frac{B-P}{K_i}} |h(s)| g(B - K_i(s - t_0))\, ds = +\infty \ \text{ for every } \ t_0 \in [x_i, z_i] \ \ (i = 1, \dots, m),$$

$$\lim_{t \to t_0^-} \int\limits_{t_0 - \frac{B-P}{L_i}}^t |h(s)| g(B - L_i(t_0 - s))\, ds = +\infty \ \text{ for every } \ t_0 \in [z_i, y_i] \ \ (i = 1, \dots, m).$$

where

$$C_k = \frac{B - A}{b_k - c_k}\,, \quad D_k = \frac{B - A}{c_k - a_k}\,, \quad K_i = \frac{B - A}{y_i - z_i}\,, \quad L_i = \frac{B - A}{z_i - x_i}\,.$$

Then there exists $\sigma_* > 0$ such that the equation (6) *has at least two T-periodic solutions for every $0 < \sigma < \sigma_*$ and at least one T-periodic solution for $\sigma = \sigma_*$. Moreover, there exists $\sigma^* \geq \sigma_*$ such that the equation* (6) *has no T-periodic solution for every $\sigma > \sigma^*$.*

The equation

$$u'' = \frac{\sigma h(t)}{u^\lambda (1-u)^\mu} \tag{10}$$

with $\lambda > 0$, $\mu > 0$, can be viewed as a particular case of (6). Thus from Theorem 2 we obtain the following assertion.

Corollary 2. *Let $\overline{h} \neq 0$, $\lambda \geq 1$, $\mu \geq 1$ and let there exist pairwise disjoint intervals $(a_k, b_k) \subset [0,T]$, $(x_i, y_i) \subset [0,T]$ $(k = 1, \dots, n;\ i = 1, \dots, m)$ such that* (9) *holds. Furthermore, let there exist $\alpha > 0$ such that*

$$\begin{aligned} h(t) &\geq \alpha\big[(b_k - t)(t - a_k)\big]^{\lambda-1} \ \ \text{for a.e. } t \in [a_k, b_k] \ \ (k = 1, \dots, n), \\ h(t) &\leq -\alpha\big[(y_i - t)(t - x_i)\big]^{\mu-1} \ \ \text{for a.e. } t \in [x_i, y_i] \ \ (i = 1, \dots, m). \end{aligned}$$

Then there exists $\sigma_* > 0$ such that the equation (10) *has at least two T-periodic solutions for every $0 < \sigma < \sigma_*$ and at least one T-periodic solution for $\sigma = \sigma_*$. Moreover, there exists $\sigma^* \geq \sigma_*$ such that the equation* (10) *has no T-periodic solution for every $\sigma > \sigma^*$.*

The proofs of the above-presented results can be found in the papers [1, 2].

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On the Relationships Between Stieltjes Type Integrals

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Integral equations of the form

$$x(t)=x(t_0)+\int_{t_0}^{t} \mathrm{d}[A]\,x=f(t)-f(t_0)$$

are natural generalizations of systems of linear differential equations. Their main goal is that they admit solutions which need not be absolutely continuous. Up to now such equations have been considered by several authors starting with J. Kurzweil [6] and T. H. Hildebrandt [3]. For further contributions see e.g. [1,5,8,9,11–14] and the references therein. These papers worked with several different concepts of the Stieltjes type integral like Young's (Hildebrandt), Kurzweil's (Kurzweil, Schwabik and Tvrdý), Dushnik's (Hönig) or Lebesgue's (Ashordia, Meng and Zhang). Thus an interesting question arises: what are the relationships between all these concepts?

It is known that (cf. [6, Theorem 1.2.1]) the Kurzweil–Stieltjes integral is in finite dimensional setting equivalent with the Perron–Stieltjes, while the relationship between the Perron–Stieltjes and the Lebesgue–Stieltjes integrals has been described already in [10, Theorem VI.8.1]. Furthermore, the relationship between the Young–Stieltjes and the Dushnik–Stieltjes integrals (DS) follows from [7, Theorem B]. Finally, the relationship between the Young–Stieltjes (YS) integral and the Kurzweil–Stieltjes (KS) one has been described in [11] and [12].

In this paper the symbols like $\mathbb{R}$, $\mathbb{N}$, $[a,b]$, (a,b), $\operatorname{var}_a^b f$ and $\|f\|_\infty$ have their usual and traditional meaning. For more details we refer to the preliminary version of the monograph [9]. In addition, recall that a finite ordered set $\boldsymbol{\alpha}=\{\alpha_0,\dots,\alpha_{\nu(P)}\}$ of points from $[a,b]$ is a division of $[a,b]$ if $a=\alpha_0<\dots<\alpha_{\nu(P)}=b$. The couple of ordered sets $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$ is a partition of $[a,b]$ if $\boldsymbol{\alpha}$ is a division of $[a,b]$ and $\boldsymbol{\xi}=\{\xi_1,\dots,\xi_{\nu(P)}\}$ is such that $\xi_j\in[\alpha_{j-1},\alpha_j]$ for all j. If $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$ is a division of $[a,b]$, the elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\xi}$ are always denoted respectively as α_j and ξ_j. At the same time the number of elements of $\boldsymbol{\xi}$ is always denoted by $\nu(P)$. For functions $f,g\colon[a,b]\to\mathbb{R}$ and a partition $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$ of $[a,b]$ we set

$$S(P)=\sum_{j=1}^{\nu(P)} f(\xi_j)\big[g(\alpha_j)-g(\alpha_{j-1})\big]$$

and, if g is regulated,

$$S_Y(P)=\sum_{j=1}^{\nu(P)}\Big(f(\alpha_{j-1})\Delta^+g(\alpha_{j-1})+f(\xi_j)\big[g(\alpha_j-)-g(\alpha_{j-1}+)\big]+f(\alpha_j)\Delta^-g(\alpha_j)\Big).$$

- *YS integral* (Y) $\int_a^b f\ \mathrm{d}g$ $\big($*DS integral* (D) $\int_a^b f\ \mathrm{d}g\big)$ exists and equals $I \in \mathbb{R}$ if

$$\begin{gathered}\text{for every } \varepsilon > 0 \text{ there is a division } \boldsymbol{\alpha}_\varepsilon \text{ of } [a,b] \text{ such that}\\ |S_Y(P) - I| < \varepsilon \quad \big(\text{or } |S(P) - I| < \varepsilon\big)\\ \text{holds for all partitions } P = (\boldsymbol{\alpha}, \boldsymbol{\xi}) \text{ of } [a,b] \text{ such that } \boldsymbol{\alpha} \supset \boldsymbol{\alpha}_\varepsilon \text{ and}\\ \alpha_{j-1} < \xi_j < \alpha_j \ \text{ for all } \ j \in \{1, \dots, \nu(\boldsymbol{\alpha})\}.\end{gathered}$$

- *KS integral* (K) $\int_a^b f\ \mathrm{d}g$ exists and has a value $I \in \mathbb{R}$ if

$$\begin{gathered}\text{for every } \varepsilon > 0 \text{ there exists a function } \delta_\varepsilon : [a,b] \to (0,1) \text{ such that}\\ |I - S(P)| < \varepsilon\\ \text{holds for all partitions } P = (\boldsymbol{\alpha}, \boldsymbol{\xi}) \text{ of } [a,b] \text{ such that}\\ [\alpha_{j-1}, \alpha_j] \subset [\xi_j - \delta_\varepsilon(\xi_j), \xi_j + \delta_\varepsilon(\xi_j)].\end{gathered}$$

It is not difficult to see that for all the three integrals under consideration the estimates

$$\bigg|\int_a^b f\ \mathrm{d}g\bigg| \le \|f\|_\infty \operatorname{var}_a^b g \ \text{ and } \ \bigg|\int_a^b f\ \mathrm{d}g\bigg| \le \big(|g(a)| + |g(b)| + \operatorname{var}_a^b g\big)\|f\|_\infty$$

are true whenever the corresponding integral exists. Indeed, it is enough to show that analogous inequalities are satisfied by the sums $|S(P)|$ and $|S_Y(P)|$ for arbitrary compatible partitions. To see how to prove the latter inequality for the YS integral, it helps to observe that the relation

$$\begin{aligned}f(\alpha)\big[g(\alpha+) - g(\alpha)\big] + f(\xi)\big[g(\beta-) - g(\alpha+)\big] + f(\beta)\big[g(\beta) - g(\beta-)\big]&\\ = \big[f(\alpha) - f(\xi)\big]g(\alpha+) + \big[f(\xi) - f(\beta)\big]g(\beta-) + f(\beta)g(\beta) - f(\alpha)g(\alpha)&\end{aligned}$$

is true if $f : [a,b] \to \mathbb{R}$, g is regulated on $[a,b]$, and $a \le \alpha \le \xi \le \beta \le b$.

Next convergence results are also true for all the three integrals under consideration.

Proposition. *Let $f : [a,b] \to \mathbb{R}$, $g \in \mathrm{BV}([a,b])$ and let the sequence $\{f_n\}$ tend uniformly to f on $[a,b]$. Then:*

- *If all the integrals $\int_a^b f_n\ \mathrm{d}g$, $n \in \mathbb{N}$, exist, then the integral $\int_a^b f\ \mathrm{d}g$ exists, too, and*

$$\lim_{n\to\infty} \int_a^b f_n\ \mathrm{d}g = \int_a^b f\ \mathrm{d}g.$$

- *If all the integrals $\int_a^b f\ \mathrm{d}g_n$, $n \in \mathbb{N}$, exist, then the integral $\int_a^b f\ \mathrm{d}g$ exists, too, and*

$$\lim_{n\to\infty} \int_a^b f\ \mathrm{d}g_n = \int_a^b f\ \mathrm{d}g.$$

For KS integrals the proofs are available in Section 6.3 *of* [9]*. Their ideas are pretty transparent and applicable also to YS and DS integrals: First, we notice that in both situations the sequences of integrals depending on* n *are Cauchy sequences in* $\mathbb{R}$ *and therefore they have a limit* $I \in \mathbb{R}$*. Further, uniform convergence and the above estimates implies that the limit integrals exist and equals* I*.*

Now we can formulate and justify our main result.

Theorem. *Suppose* f *and* g *are regulated on* $[a,b]$ *and at least one of them has a bounded variation on* $[a,b]$*. Then all the integrals* $(\mathrm{K})\int\limits_a^b f\,\mathrm{d}g$*,* $(\mathrm{Y})\int\limits_a^b f\,\mathrm{d}g$ *and* $(\mathrm{D})\int\limits_a^b f\,\mathrm{d}g$ *exist and*

$$(\mathrm{K})\int\limits_a^b f\,\mathrm{d}g = (\mathrm{Y})\int\limits_a^b f\,\mathrm{d}g = f(b)g(b) - f(a)g(a) - (\mathrm{D})\int\limits_a^b g\,\mathrm{d}f. \tag{1}$$

Sketch of the proof:

- It is not difficult to verify that the equalities (1) hold for every $f : [a,b] \to \mathbb{R}$ whenever g is a finite step function and, similarly, they are also true whenever g is regulated and f is a finite step function. (For analogous arguments see Examples 6.3.1 in [9].)
- Approximate uniformly regulated functions by sequences of finite step functions.
- Applying convergence results stated in Proposition, it is easy to complete the proof.

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The Version of Perron's Effect of Replacing the Values of Characteristic Exponents of Differential System by a Set of Positive Measure

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Consider the linear differential system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^2, \ \ t \geqslant t_0, \tag{1}$$

with bounded continuously differentiable on the semi-axis $[t_0, +\infty)$ coefficients and with negative characteristic exponents $\lambda_1(A) \leqslant \lambda_2(A) < 0$, which is a linear approximation for a nonlinear perturbed differential system

$$\dot{y} = A(t)y + f(t, y), \ \ y \in \mathbb{R}^2, \ \ t \geqslant t_0. \tag{2}$$

In this system, the so-called m-perturbation $f : [t_0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ is likewise continuously differentiable in its arguments $t \geqslant t_0$ and $y_1, y_2 \in \mathbb{R}$, of order $m > 1$ of smallness in the neighbourhood of the origin and admissible growth outside of it:

$$\|f(t, y)\| \leqslant C_f \|y\|^m, \ \ C_f = const, \ \ y \in \mathbb{R}^2, \ \ t \geqslant t_0. \tag{3}$$

Perron's effect [17], [15, pp. 50–51; 3–11] (see also [13,14]) of replacing the values of characteristic exponents establishes the existence both of the linear system (1) with fixed characteristic exponents $\lambda_1 \leqslant \lambda_2 < 0$ and of the nonlinear system (2) with perturbation (3) of order $m = 2$ of smallness and with all infinitely extendable to the right nontrivial solutions $y(t, c)$ with initial vectors $y(t_0, c) = c = (c_1, c_2) \neq 0$. In addition, all such solutions starting at the time moment $t = t_0$ on the axis $c_1 = 0$ have exponents, equal to the higher characteristic exponent λ_2 of the initial system (1) (that allows one to consider this effect partial), and the exponents of all the rest nontrivial solutions of system (2) coincide with some $\lambda_0 > 0$ (calculated incidentally in [5, pp. 13–15]). Generalizations of that effect in various directions have been obtained in [2, 3, 6–8, 10, 11].

The question on the realization of such a (continual) version of Perron's effect, when the set $\lambda(A, f)$ of Lyapunov's exponents of all nontrivial solutions (necessarily infinitely extendable to the right) of the corresponding system (2) with perturbation (3) would have been measurable, fully belonged to the positive semi-axis, had continuum power and even positive Lebesque measure, remained open. A positive answer to this question is contained, particularly, in the theorem below which defins in a general case an explicit representation of Lyapunov's exponents of all nontrivial solutions $y(t, c)$ of the needed nonlinear system (2) through their initial values $c = (c_1, c_2) \in \mathbb{R}^2$.

Note that earlier we have constructed the perturbed differential systems (2) with an exponentially stable zero solution whose set of characteristic exponents of solutions from a sufficiently small neighbourhood of a zero solution belongs fully to the negative semi-axis and: (1) is [4] of positive measure (a positive length segment); (2) consists [18] of a countable set of nonintersecting connectivity components. Further, these sets of exponents were fully described in [1]. Obviously, the results obtained there are not connected with the realization of Perron's effect (and, all the more, with its continual analogue) of replacing negative values of characteristic exponents of the system of linear approximation (1) by positive ones for all nontrivial solutions of the nonlinear system (2) with perturbations (3) of order $m > 1$ of smallness.

The following theorem is valid [9].

Theorem. *For any parameters $m > 1$, $\lambda_1 \leqslant \lambda_2 < 0$ and bounded continuously differentiable on the axis $\mathbb{R}_0 \equiv \mathbb{R} \setminus \{0\}$ functions*

$$\psi_i : \mathbb{R}_0 \xrightarrow{on} |\beta_i, b_i| \subset [\lambda_2, +\infty), \;\; b_1 \leqslant \beta_2, \;\; i = 1, 2, \tag{4}$$

there exist the linear system (1) with characteristic exponents $\lambda_1(A) = \lambda_1 \leqslant \lambda_2 = \lambda_2(A)$ and continuously differentiable in its arguments $t \geqslant t_0$ and $y_1, y_2 \in \mathbb{R}$, the m-perturbation $f(t,y)$ such that all nontrivial solutions $y(t,c)$ of the nonlinear perturbed system (2) are infinitely extendable to the right and have characteristic exponents

$$\lambda[y(\,\cdot\,, c)] = \begin{cases} \psi_1(c_1), & c_1 \neq 0, \;\; c_2 = 0, \\ \psi_2(c_2), & c_2 \neq 0, \;\; c = (c_1, c_2) \in \mathbb{R}^2. \end{cases}$$

Corollary. *A continual version of Perron's effect of replacing the negative values of characteristic exponents of the system of linear approximation (1) by positive ones of all nontrivial solutions of the nonlinear system (2) with perturbation (3) of order $m > 1$ is realized through the functions ψ_i with sets of values, that is, by the intervals $|\beta_i, b_i| \subset (0, +\infty)$ of positive length (and of positive Lebesgue measure).*

Remark. A set of values of each of the bounded, continuously differentiable on the axis $\mathbb{R} \setminus \{0\}$ functions ψ_1 and ψ_2 may consist of two connectivity components. The corresponding analogue of the above-formulated theorem establishing also a continual version of Perron's effect holds in this case, as well. The above theorem admits generalization with replacing the inclusion $|\beta_i, b_i| \subset [\lambda_2, +\infty)$ in condition (4) by a weaker inclusion $|\beta_i, b_i| \subset [\lambda_i, +\infty)$. Moreover, by sufficiently obvious changes in the proof of theorem cited in [9], one can prove an analogous statement for the bounded functions ψ_1 and ψ_2 from the Naire first class and their Suslin sets of values.

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Correctness and Additive Averaged Semi-Discrete Scheme for Two Nonlinear Multi-Dimensional Integro-Differential Parabolic Problems

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The note is devoted to the correctness of the initial-boundary value problems for two nonlinear multi-dimensional integro-differential equations of parabolic type. Construction and study of the additive averaged semi-discrete schemes with respect to time variable are also given. These type of equations are natural generalizations, on the one hand, of equations describing applied problems of mathematical physics and, on the other hand, of nonlinear parabolic equations in problems considered in [12] and [16]. The studied equations are based on well-known Maxwell's system arising in mathematical simulation of electromagnetic field penetration into a substance [9].

Maxwell's system is complex and its investigation and numerical resolution still yield for special cases (see, for example, [7] and references therein).

In [3] this system were proposed to the following integro-differential form

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\left[a\left(\int\limits_0^t |\operatorname{rot} H|^2\,d\tau\right)\operatorname{rot} H\right], \tag{1}$$

where $a = a(S)$ is defined for $S \in [0, \infty)$.

Making certain physical assumptions in mathematical description of the above-mentioned process G. I. Laptev is constructed a new integro-differential model, which represents a generalization of the system (1)

$$\frac{\partial H}{\partial t} = a\left(\int\limits_\Omega \int\limits_0^t |\operatorname{rot} H|^2\,dx\,d\tau\right)\Delta H. \tag{2}$$

Principal characteristic peculiarities of the systems (1) and (2) are connected with the appearance in the coefficients with derivatives of higher order nonlinear terms depended on the integral of time and space variables. These circumstances requires different discussions, than it is usually necessary for the solution of local differential problems.

The literature on the questions of existence, uniqueness, and regularity of the solutions to the models of above types is very rich. In [3–7, 10, 11] and in a number of other works as well the solvability of the initial-boundary value problems for (1) type models in scalar cases are studied. The correctness of these problems in [3–7] are proved using a modified version of the Galerkin's method and compactness arguments that are used in [12, 16] for investigation the nonlinear elliptic and parabolic equations.

Let us note that the unique solvability and large time behavior of initial-boundary value problems for (2) type equations at first are given in [4].

These questions and numerical resolution of initial-boundary value problems for (1) and (2) type models are discussed in many works as well (see, for example, [7, 14, 15] and references therein).

Many authors study the Rothe's type schemes, semi-discrete schemes with space variable, finite element and finite difference approximations for a integro-differential models (see, for example, [5–8, 12, 13] and references therein).

It is very important to study decomposition analogs for the above-mentioned multi-dimensional integro-differential models as well. At present there are some effective algorithms for solving the multi-dimensional problems (see, for example, [12, 13] and references therein).

In this paper the existence and uniqueness of solutions of initial-boundary value problems are fixed. Main attention is paid to investigation of Rothe's type semi-discrete additive averaged schemes. We shall focus our attention to the (1) and (2) type multi-dimensional integro-differential scalar equations.

Let Ω is bounded domain in the n-dimensional Euclidean space R^n with sufficiently smooth boundary $\partial\Omega$. In the domain $Q = \Omega \times (0, T)$ of the variables $(x, t) = (x_1, x_2, \dots, x_n, t)$ let us consider the following equations

$$\frac{\partial U}{\partial t} - \sum_{i=1}^{n} \left[\frac{\partial}{\partial x_i} a\left(\int_0^t \sum_{\ell=1}^{n} \left| \frac{\partial U}{\partial x_\ell} \right|^2 d\tau \right) \frac{\partial U}{\partial x_i} \right] = f(x,t), \quad (x,t) \in Q, \tag{3}$$

or

$$\frac{\partial U}{\partial t} - a\left(\int_\Omega \int_0^t \sum_{\ell=1}^{n} \left| \frac{\partial U}{\partial x_\ell} \right|^2 dx\, d\tau \right) \sum_{i=1}^{n} \frac{\partial^2 U}{\partial x_i^2} = f(x,t), \quad (x,t) \in Q, \tag{4}$$

with the first type initial-boundary value homogeneous conditions

$$U(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{5}$$

$$U(x,0) = 0, \quad x \in \overline{\Omega}, \tag{6}$$

where T is a fixed positive constant, f is a given function of its arguments.

The problems (3), (5), (6) and (4)–(6) are similar to problems considered in [2] and [4]. Using modified version of the Galerkin's method and compactness arguments [12, 16] it is not difficult to prove the following statement.

Theorem 1. *If $a(S) = 1 + S$, $f \in W_2^1(Q)$, $f(x,0) = 0$, then the problems* (3), (5), (6) *and* (4)–(6) *have solutions with the properties:*

$$U \in L_4(0, T; \overset{\circ}{W}{}_4^1(\Omega)), \quad \frac{\partial U}{\partial t} \in L_2(Q),$$

$$\sqrt{T-t}\, \frac{\partial^2 U}{\partial t \partial x_i} \in L_2(Q), \quad \sqrt{\psi}\, \frac{\partial^2 U}{\partial x_i \partial x_j} \in L_2(Q), \quad i, j = 1, \dots, n,$$

where $\psi \in C^\infty(\overline{\Omega})$, $\psi(x) > 0$, for $x \in \Omega$; $\frac{\partial \psi}{\partial \nu} = 0$, for $x \in \partial\Omega$ and ν is the outer normal of $\partial\Omega$.

Here we used usual L_p and W_p^k, $\overset{\circ}{W}{}_p^k$ Sobolev spaces.

The proof of the formulated theorem is divided into several steps. One of the basic step is to obtain necessary a priori estimates.

Using the scheme of investigation as in [4] it is not difficult to get the results of exponentially asymptotic behavior of solution as $t \to \infty$ of the initial-boundary value problems for the equations (3) and (4) with nonhomogeneous initial conditions.

On $[0, T]$ let us introduce a net with mesh points denoted by $t_j = j\tau$, $j = 0, 1, \dots, J$, with $\tau = T/J$.

Coming back to the problems (3), (5), (6) and (4)–(6) let us construct additive averaged Rothe's type schemes

$$\eta_i \frac{u_i^{j+1} - u^j}{\tau} = \frac{\partial}{\partial x_i} \left[\left(1 + \tau \sum_{k=1}^{j+1} \sum_{\ell=1}^{n} \left| \frac{\partial u_i^k}{\partial x_\ell} \right|^2 \right) \frac{\partial u_i^{j+1}}{\partial x_i} \right] + f_i^{j+1} \tag{7}$$

and

$$\eta_i \frac{u_i^{j+1} - u^j}{\tau} = \left(1 + \tau \sum_{k=1}^{j+1} \sum_{\ell=1}^{n} \int_\Omega \left| \frac{\partial u_i^k}{\partial x_\ell} \right|^2 dx \right) \frac{\partial^2 u_i^{j+1}}{\partial x_i^2} + f_i^{j+1}, \tag{8}$$

with homogeneous boundary and initial $u_i^0 = u^0 = 0$ conditions, where $u_i^j(x)$, $i = 1, \dots, n$, $j = 0, 1, \dots, J-1$ are solutions of the problems (7) and (8) consequently, and the following notations are introduced:

$$u^j(x) = \sum_{i=1}^{n} \eta_i u_i^j(x), \quad \sum_{i=1}^{n} \eta_i = 1, \ \ \eta_i > 0, \quad \sum_{i=1}^{n} f_i^{j+1}(x) = f^{j+1}(x) = f(x, t_{j+1}),$$

where u^j denotes approximation of exact solution U of the problem (4)–(6) at t_j. We use usual norm $\| \cdot \|$ of the space $L_2(\Omega)$.

Theorem 2. *If the problems* (3), (5), (6) *and* (4)–(6) *have sufficiently smooth solutions, then the solutions of problems* (7) *and* (8) *with homogeneous initial and boundary conditions converge to the solutions of the problems* (3), (5), (6) *and* (4)–(6) *and the following estimate is true*

$$\|U^j - u^j\| = O(\tau^{1/2}), \ \ j = 1, \dots, J.$$

Let us note that results analogical to Theorem 2 for the following variants of (3) and (4) integro-differential systems are obtained in the works [6] and [5], respectively:

$$\frac{\partial U}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\left(1 + \int_0^t \left| \frac{\partial U}{\partial x_i} \right|^2 dx\, d\tau \right) \frac{\partial U}{\partial x_i} \right] = f(x, t),$$

and

$$\frac{\partial U}{\partial t} - \sum_{i=1}^{n} \left(1 + \int_\Omega \int_0^t \left| \frac{\partial U}{\partial x_i} \right|^2 dx\, d\tau \right) \frac{\partial^2 U}{\partial x_i^2} = f(x, t).$$

It is very important to construct and investigate studied in this note type models for more general type nonlinearities and for (3) and (4) type multi-dimensional systems as well.

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Structure of Nonoscillatory Solutions of First Order Nonlinear Differential Systems of Emden–Fowler Type

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Section 1

We consider first order nonlinear cyclic differential systems of the form

$$x' - p(t)\varphi_\alpha(y) = 0, \quad y' + q(t)\varphi_\beta(x) = 0, \tag{A}$$

where α and β are positive constants, p and q are positive continuous functions on $[0,\infty)$ and φ_γ, $\gamma > 0$, denotes the function

$$\varphi_\gamma(u) = |u|^\gamma \operatorname{sgn} u, \quad u \in \mathbf{R}.$$

We are concerned exclusively with nonoscillatory solutions of (A), by which we mean those solutions (x,y) whose components x and y are nonoscillatory in the usual sense.

Oscillation theory of systems of the form (A) was created by Mirzov, whose achievements of great theoretical interest are summarized in the monograph [2]. There are some topics untouched in [2], one of which is the systematic study of nonoscillatory solutions of (A). The present work, motivated by this observation, aims to depict a clear picture of the overall structure of nonoscillatory solutions of (A) by analyzing their asymptotic behavior at infinity as precisely as possible.

Let (x,y) be a nonoscillatory solution of (A) on $[t_0,\infty)$. Since both x and y are eventually one-signed, they are monotone for all large t so that the limits $x(\infty) = \lim\limits_{t\to\infty} x(t)$ and $y(\infty) = \lim\limits_{t\to\infty} y(t)$ exist in the extended real numbers. Thus, $x(t)y(t) \neq 0$ on $[T,\infty)$ for some $T \geq t_0$. We say that (x,y) is a solution *of the first kind* (resp. *of the second kind*) if $x(t)y(t) > 0$ (resp. $x(t)y(t) < 0$) for $t \geq T$.

We use the notation

$$I_p = \int\limits_0^\infty p(t)\,dt, \quad I_q = \int\limits_0^\infty q(t)\,dt,$$

and examine the existence of nonoscillatory solutions of (A) by distinguishing the following four cases:

$$I_p = \infty \wedge I_q = \infty, \quad I_p = \infty \wedge I_q < \infty, \quad I_p < \infty \wedge I_q = \infty, \quad I_p < \infty \wedge I_q < \infty.$$

(The case $I_p = \infty \wedge I_q = \infty$) In this case, as is shown by Mirzov [2], all solutions of (A) are oscillatory, so that (A) has no nonoscillatory solutions.

(The case $I_p = \infty \wedge I_q < \infty$) In this case it can be shown that all nonoscillatory solutions (x, y) are of the first kind, and that $|x|$ are eventually increasing and $|y|$ are eventually decreasing. Thus their asymptotic behavior as $t \to \infty$ can be classified into the three types

I(i): $|x(\infty)| = \infty$, $0 < |y(\infty)| < \infty$,

I(ii): $|x(\infty)| = \infty$, $|y(\infty)| = 0$,

I(iii): $0 < |x(\infty)| < \infty$, $|y(\infty)| = 0$.

Nonoscillatory solutions of the types I(i) and I(iii) are termed, respectively, *maximal solutions* and *minimal solutions* of the first kind of (A), and their existence can be characterized as the following theorem shows.

Theorem 1.1. *Let α and β be any given positive constants.*

(i) *System* (A) *has solutions of the type* I(i) *if and only if*

$$\int_0^\infty q(t)P(t)^\beta\,dt < \infty, \quad \text{where } P(t) = \int_0^t p(s)\,ds.$$

(ii) *System* (A) *has solutions of the type* I(iii) *if and only if*

$$\int_0^\infty p(t)\rho(t)^\alpha\,dt < \infty, \quad \text{where } \rho(t) = \int_t^\infty q(s)\,ds.$$

Solutions of the type I(ii), which may be termed *intermediate solutions* of the first kind of (A), are so difficult to analyze that we have so far been able to handle only the system (A) whose nonlinearity is referred to as *sub-half-linear*.

Theorem 1.2. *Let $\alpha\beta < 1$. System* (A) *possesses a solution of the type* I(ii) *if and only if*

$$\int_0^\infty p(t)\rho(t)^\alpha\,dt < \infty \quad \text{and} \quad \int_0^\infty q(t)P(t)^\beta\,dt = \infty.$$

(The case $I_p < \infty \wedge I_q = \infty$) In this case it is shown that all nonoscillatory solutions (x, y) are of the second kind, and that $|x|$ are eventually decreasing and $|y|$ are eventually increasing. Thus their asymptotic behavior as $t \to \infty$ can be classified into the three types

II(i): $0 < |x(\infty)| < \infty$, $|y(\infty)| = \infty$,

II(ii): $|x(\infty)| = 0$, $|y(\infty)| = \infty$,

II(iii): $|x(\infty)| = 0$, $0 < |y(\infty)| < \infty$.

Solutions of the types II(i) and II(iii) are called, respectively, *maximal solutions* and *minimal solutions* of the second kind of (A), while those of the type II(ii) are called *intermediate solutions* of the second kind of (A). As for the solutions of this kind we have the following results which are in parallel with Theorems 1.1 and 1.2 regarding the solutions of the first kind.

Theorem 1.3. *Let α and β be any given positive constants.*

(i) *System* (A) *has solutions of the type* II(i) *if and only if*

$$\int\limits_0^\infty p(t)Q(t)^\alpha\,dt < \infty, \ \ \textit{where}\ \ Q(t) = \int\limits_0^t q(s)\,ds.$$

(ii) *System* (A) *has solutions of the type* II(iii) *if and only if*

$$\int\limits_0^\infty q(t)\pi(t)^\beta\,dt < \infty, \ \ \textit{where}\ \ \pi(t) = \int\limits_t^\infty p(s)\,ds.$$

Theorem 1.4. *Let $\alpha\beta < 1$. System* (A) *possesses a solution of the type* II(ii) *if and only if*

$$\int\limits_0^\infty q(t)\pi(t)^\beta\,dt < \infty \ \ \textit{and} \ \ \int\limits_0^\infty p(t)Q(t)^\alpha\,dt = \infty.$$

(The case $I_p < \infty \wedge I_q < \infty$) In this case it is shown without difficulty that all nonoscillatory solutions (x,y) of (A) are bounded and that both x and y have finite limits as $t \to \infty$. As a matter of fact, for any given constants c and d, zero or nonzero, there exists a nonoscillatory solution (x,y) of (A) such that $x(\infty) = c$ and $y(\infty) = d$. Thus, (A) possesses solutions of both the first and second kinds.

Section 2

We now turn our attention to scalar second order nonlinear differential equations of the form

$$(p(t)\varphi_\alpha(x'))' + q(t)\varphi_\beta(x) = 0, \tag{E}$$

where α and β are positive constants, and p and q are positive continuous functions on $[0,\infty)$. Equation (E) is often called a *generalized Emden–Fowler equation* and has been the object of intensive investigation in its own right (see e.g., [1,3]).

We are interested in the structure of the totality of nonoscillatory solutions of equation (E). Let x be a nonoscillatory solution of (E). We put $Dx = p(t)\varphi_\alpha(x')$ and call it the *quasi-derivative* of x. Worthy of note is the fact that by the substitution $y = Dx$ equation (E) is split into the differential system

$$x' - p(t)^{-\frac{1}{\alpha}}\varphi_{\frac{1}{\alpha}}(y) = 0, \quad y' + q(t)\varphi_\beta(x) = 0, \tag{B}$$

which can be regarded as a special case of system (A).

We say that a nonoscillatory solution x of (E) is of the first kind or of the second kind if $x(t)Dx(t) > 0$ or $x(t)Dx(t) < 0$ for all large t, respectively.

If $p^{-\frac{1}{\alpha}}$ and q satisfy

$$\int\limits_0^\infty p(t)^{-\frac{1}{\alpha}}\,dt = \infty \ \ \text{and} \ \ \int\limits_0^\infty q(t)\,dt < \infty, \tag{2.1}$$

then all nonoscillatory solutions x of (E) are of the first kind, and there are three possibilities for their asymptotic behavior at infinity

I(i): $|x(\infty)| = \infty$, $0 < |Dx(\infty)| < \infty$, I(ii): $|x(\infty)| = \infty$, $|Dx(\infty)| = 0$,

I(iii): $0 < |x(\infty)| < \infty$, $|Dx(\infty)| = 0$.

If $p^{-\frac{1}{\alpha}}$ and q satisfy

$$\int_0^\infty p(t)^{-\frac{1}{\alpha}}\,dt < \infty \ \text{ and } \int_0^\infty q(t)dt = \infty, \tag{2.2}$$

then all nonoscillatory solutions x of (E) are of the second kind, and there are three patterns of their asymptotic behavior at infinity

II(i): $0 < |x(\infty)| < \infty$, $|Dx(\infty)| = \infty$,

II(ii): $|x(\infty)| = 0$, $|Dx(\infty)| = \infty$,

II(iii): $|x(\infty)| = 0$, $0 < |Dx(\infty)| < \infty$.

In order to characterize the existence of solutions of these six types of (E) it suffices to specialize the results of Section 1 to system (B). For example, Theorems 1.2 and 1.4 applied to (B) which has to be sub-half-linear give rise to the following results on solutions of the types I(ii) and II(ii) which may well be termed *intermediate solutions* of equation (E) with α and β satisfying $\alpha > \beta$.

Theorem 2.5. *Let $\alpha > \beta$ and suppose that* (2.1) *holds. Then, equation* (E) *possesses a solution of the type* I(ii) *if and only if*

$$\int_0^\infty p(t)^{-\frac{1}{\alpha}}\bigg(\int_t^\infty q(s)\,ds\bigg)^\alpha dt < \infty \ \text{ and } \int_0^\infty q(t)\bigg(\int_0^t p(s)^{-\frac{1}{\alpha}}\,ds\bigg)^\beta dt = \infty.$$

Theorem 2.6. *Let $\alpha > \beta$ and suppose that* (2.2) *holds. Then, equation (E) possesses a solution of the type type* II(ii) *if and only if*

$$\int_0^\infty q(t)\bigg(\int_t^\infty p(s)^{-\frac{1}{\alpha}}\,ds\bigg)^\beta < \infty \ \text{ and } \int_0^\infty p(t)^{-\frac{1}{\alpha}}\bigg(\int_0^t q(s)\,ds\bigg)^\alpha dt = \infty.$$

Finally, in the case where

$$\int_0^\infty p(t)^{-\frac{1}{\alpha}}\,dt < \infty \ \text{ and } \int_0^\infty q(t)\,dt < \infty, \tag{2.3}$$

all nonoscillatory solutions x of (E) are bounded together with Dx on their intervals of existence, and for any given constants c and d, zero or non-zero, (E) has a nonoscillatory solution x such that $x(\infty) = c$ and $Dx(\infty) = d$.

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On a Mixed Nonlinear Hyperbolic Problem

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In the plane of independent variables x and t, in the domain $D_T: 0 < x < l,\ 0 < t < T$, consider a mixed problem of finding a solution $u(x,t)$ for semilinear wave equation of the form

$$u_{tt} - u_{xx} + g(u) = f(x,t),\ \ (x,t) \in D_T, \tag{1}$$

satisfying the following initial

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x),\ \ 0 \le x \le l, \tag{2}$$

and boundary value conditions

$$u_x(0,t) = F[u_t(0,t)], \quad u(l,t) = 0,\ \ 0 \le t \le T, \tag{3}$$

where f, φ, ψ, g and F are given, while u is unknown real functions.

Let the following conditions of smoothness

$$f \in C^1(\overline{D}_T),\ \ g, F \in C^1(\mathbb{R}),\ \ \varphi \in C^2([0,l]),\ \ \psi \in C^1([0,l]) \tag{4}$$

be fulfilled. We assume that at points $(0,0)$ and $(l,0)$ the following conditions of agreement

$$\begin{gathered} \varphi'(0) = F[\psi(0)],\ \ \psi'(0) = F'[\psi(0)]\big\{\varphi''(0) - g[\varphi(0)] + f(0,0)\big\}, \\ \varphi(l) = 0, \quad \psi(l) = 0, \quad g(0) - \varphi''(l) = f(l,0) \end{gathered} \tag{5}$$

are also fulfilled. Let

$$\int\limits_0^s g(s_1)\,ds_1 \ge -M_1 s^2 - M_2,\ \ sF(s) \ge -M_3\ \ \forall\, s \in \mathbb{R}, \tag{6}$$

where $M_i := const \ge 0$, $1 \le i \le 3$. When conditions (4)–(6) are fulfilled, for the solution $u \in C^2(\overline{D}_T)$ of the problem (1)–(3) it is valid the following a priori estimate

$$\|u\|_{C(\overline{D}_T)} \le c_1 \|f\|_{C(\overline{D}_T)} + c_2 \|\varphi\|_{C^1([0,l])} + c_3 \|\psi\|_{C([0,l])} + c_4 \|g\|_{C([-\|\varphi\|_{C([0,l])}, \|\varphi\|_{C([0,l])}])} + c_5 \tag{7}$$

with positive constants $c_i = c_i(M_1, M_2, M_3, l, T)$, $1 \le i \le 5$, independent on functions u, f, φ and ψ.

The problem (1)–(3) can be reduced to the system of Volterra type nonlinear integral equations, which has a continuous solution due to a priori estimate (7), Leray–Schauder's theorem and additional condition $F'(s) \ne -1$, $s \in \mathbb{R}$. In view of the conditions of smoothness (4) and agreement (5) this solution is a classical solution of the original problem. It is proved that the solution is unique.

Remark. When the conditions (6) are violated, the problem (1)–(3) may turn out to be insolvable even locally, or locally solvable with a blow-up solution. For example, when $g = 0$, $f = 0$, $F(s) = \arctan s - s$, $s \in \mathbb{R}$, and $|\varphi'(t) + \psi(t)| > \frac{\pi}{2}$, $t \in [0,l]$, then the problem (1)–(3) is insolvable even locally. Besides, if $|\varphi'(t) + \psi(t)| < \frac{\pi}{2}$ for $0 \le t < t_0 \le l$ and $|\varphi'(t_0) + \psi(t_0)| = \frac{\pi}{2}$, then a solution of this problem exists in $[0, t_0)$, and

$$\lim_{t \to t_0 - 0} \|u\|_{C^1(\overline{D}_t)} = \infty.$$

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Positive Invertible Matrices and Stability of Nonlinear Itô Equations

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Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},\mathbb{P})$ be a stochastic basis consisting of a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and an increasing, right-continuous family (a filtration) $(\mathcal{F}_t)_{t\geq0}$ of complete σ-subalgebras of $\mathcal{F}$. By E we denote the expectation on this probability space. The scalar stochastic processes $\mathcal{B}_i$, $i=2,\dots,m$ are scalar, independent Brownian motions on $(\mathcal{F}_t)_{t\geq0}$ (see e.g. [6]).

The following inequality holds true for any Brownian motion $\mathcal{B}(s)$ and any scalar stochastic process $f(s)$, which is integrable with respect to $\mathcal{B}(s)$ on $[0,t]$:

$$\bigg(E\bigg|\int\limits_0^t f(s)\,d\mathcal{B}(s)\bigg|^{2p}\bigg)^{1/2p}\leq c_p\bigg(E\bigg(\int\limits_0^t |f(s)|^2\,ds\bigg)^p\bigg)^{1/2p}.$$

Here c_p is some number depending on p. Some estimates on this number can be found e.g. in [6].

We consider the following system of Itô equations with delay:

$$\begin{aligned}dx_i(t)=\Big[-a_i(t)x_i(h_i(t))+\sum_{j=1}^n F_{ij}(t,x_j(h_{ij}(t)))\Big]\,dt\\ +\sum_{l=1}^m\Big[\sum_{j=1}^n G^l_{ij}\big(t,x_j(h^l_{ij}(t))\big)\Big]\,d\mathcal{B}_l(t)\ \ (t\geq0),\ \ i=1,\dots,n\end{aligned}\tag{1}$$

with the initial conditions

$$x_i(t)=\varphi_i(t)\ \ (t<0),\ \ i=1,\dots,n,,\tag{1_a}$$

$$x_i(t)=b_i,\ \ i=1,\dots,n,\tag{1_b}$$

where

1) a_i are Lebesgue measurable functions, which are defined on $[0,\infty)$ and satisfy $0<\overline{a}_i\leq a_i\leq A_i$ $(t\in[0,\infty))$ μ-everywhere for some positive numbers $\overline{a}_i$, A_i $(i=1,\dots,n)$;

2) $F_{ij}(\,\cdot\,,u)$ are Lebesgue measurable functions defined on $[0,\infty)$, $F_{ij}(t,\,\cdot\,)$ are continuous functions, which are defined on R^1 and satisfy $|F_{ij}(t,u)|\leq\overline{F}_{ij}|u|$ $(t\in[0,\infty))$ μ-everywhere for some positive numbers $\overline{F}_{ij}$ $(i,j=1,\dots,n)$;

3) $G^l_{ij}(\,\cdot\,,u)$ are Lebesgue measurable functions defined on $[0,\infty)$, $G^l_{ij}(t,\,\cdot\,)$ are continuous functions, which are defined on R^1 and satisfy $|G^l_{ij}(t,u)|\leq\overline{G}^l_{ij}|u|$ $(t\in[0,\infty))$ μ-everywhere for some positive numbers $\overline{G}^l_{ij}$ $(l=1,\dots,m;\ i,j=1,\dots,n)$;

4) h_i, h_{ij}, h_{ij}^l are Borel measurable functions defined on $[0,\infty)$ and satisfy $0 \le t - h_i(t) \le \tau_i$, $0 \le t - h_{ij}(t) \le \tau_{ij}$, $0 \le t - h_{ij}^l(t) \le \tau_{ij}^l$ $(t \in [0,\infty))$ μ-everywhere for some positive numbers τ_i, τ_{ij}, τ_{ij}^l for $l = 1,\dots,m$; $i,j = 1,\dots,n$;

5) φ_i are $\mathcal{F}_0$-measurable scalar stochastic processes defined on $[\sigma_i, 0)$, where $\sigma_i = \max\{\tau_i, \tau_{ij}, \tau_{ij}^l, l = 1,\dots,m;\ j = 1,\dots,n\}$;

6) b_i are $\mathcal{F}_0$-measurable scalar random values $(i = 1,\dots,n)$.

We remark that the initial value problem $(1), (1_a), (1_b)$ has a unique solution if the functions $F_{ij}(t,u)$, $G_{ij}^l t,u)$ are Lipschits with respect to u for all $l = 1,\dots,m$, $i,j = 1,\dots,n$ (see e. g. [3]). In what follows, we assume that this is the case and denote by $x(t,b,\varphi)$ the solution of (1) satisfying (1_a) and (1_b), so that $x(s,b,\varphi) = \varphi$ for $s < 0$ and $x(0,b,\varphi) = b$.

Definition 1. For a given real number p $(1 \le p < \infty)$ we say that system (1) is globally exponentially p-stable (w.r.t. the initial data) if there exist positive constants $\overline{c}$, β such that the inequality

$$E|x(t,x_0,\varphi)|^p \le \overline{c}\,\Big(E|x_0|^p + \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^p\Big)\exp\{-\beta s\}$$

holds true for all $t \ge 0$ and all φ, x_0.

An $n \times n$-matrix $\Gamma = (\gamma_{ij})_{i,j=1}^n$ is called nonnegative if $\gamma_{ij} \ge 0$, $i,j = 1,\dots,n$, and positive if $\gamma_{ij} > 0$, $i,j = 1,\dots,n$.

Definition 2. A matrix $\Gamma = (\gamma_{ij})_{i,j=1}^n$ is called an $\mathcal{M}$-matrix if $\gamma_{ij} \le 0$ for $i,j = 1,\dots,n$, $i \ne j$ and one of the following conditions is satisfied:

- Γ has a positive inverse matrix Γ^{-1};
- the principal minors of the matrix Γ are positive.

Below we define the $n \times n$-matrix Γ in the following way

$$\gamma_{ii} = 1 - \frac{A_i^2\tau_i^2 + A_i\overline{F}_{ii}\tau_i + c_pA_i\sqrt{\tau_i}\sum\limits_{i=1}^m \overline{G}_{ii}^l + \overline{F}_{ii}}{\overline{a}_i} - \frac{c_p\sum\limits_{l=1}^m \overline{G}_{ii}^l}{\sqrt{2\overline{a}_i}}\,, \quad i = 1,\dots,n,$$

$$\gamma_{ij} = -\frac{A_i\overline{F}_{ij}\tau_i + c_pA_i\sqrt{\tau_i}\sum\limits_{i=1}^m \overline{G}_{ij}^l + \overline{F}_{ij}}{\overline{a}_i} - \frac{c_p\sum\limits_{l=1}^m \overline{G}_{ij}^l}{\sqrt{2\overline{a}_i}}\,, \quad i,j = 1,\dots,n, \ \ i \ne j.$$

Theorem. *If the matrix Γ defined above is an $\mathcal{M}$-matrix, then system* (1) *is globally exponentially $2p$-stable.*

Outline of the proof (see [5] for the details).

The main idea is to use the W-method (see $[1,3,4]$ and the references therein) to regularize system (1) to obtain a certain integral operator in a suitable space of stochastic processes. This operator can be constructed with the help of an auxiliary linear equation, which is similar to the equation (1):

$$dx(t) = \big[(Qx)(t) + g(t)\big]\,dZ(t), \quad t \ge 0,$$

The solutions of this equation has the Cauchy representation

$$x(t) = U(t)x_0 + (Wg)(t), \quad t \ge 0,$$

where $U(t)$ is the fundamental matrix of the associated homogeneous equation, and W is the corresponding Cauchy operator.

Assuming for the sake of simplicity that system (1) is also linear, rewriting it in the operator form

$$dx(t) = \big[(Vx)(t) + f(t)\big]\,dZ(t), \;\; t \geq 0$$

and substituting the above Cauchy representation formula into this equation result in

$$dx(t) = \big[(Qx)(t) + ((V-Q)x)(t) + f(t)\big]\,dZ(t), \;\; t \geq 0,$$

or

$$x(t) = U(t)x(0) + (W(V-Q)x)(t) + (Wf)(t), \;\; t \geq 0.$$

Denoting $W(V-Q) = \Theta$, we obtain the operator equation $((I-\Theta)x)(t) = U(t)x(0) + (Wf)(t)$. If now the operator $I - \Theta$ is invertible in a suitable space of stochastic processes, then system (1) is globally asymptotically $2p$-stable.

In most implementations of this scheme, one tries to prove that the norm of the operator Θ is less than 1. Then $I - \Theta$ becomes invertible.

However, this approach may lead to too rough estimates. A more careful approach, based on the theory of positive matrices, was suggested in [2], where straight invertibility in norm is replaced by matrix inequalities. In particular, if the corresponding matrix is an $\mathcal{M}$-matrix, then we still can prove the global asymptotic $2p$-stability of system (1).

This approach is utilized in the paper [5] as well as in this presentation in the case of stochastic functional differential equations.

Let us now study system (1) in two dimensions.

Corollary 1. *Let $n = 2$ in system* (1) *and*

$$\sqrt{2}\,\Big(A_1^2\tau_1^2 + A_1\overline{F}_{11}\tau_1 + c_pA_1\sqrt{\tau_1}\sum_{i=1}^m \overline{G}_{11}^l + \overline{F}_{11}\Big) + \sqrt{\overline{a}_1}\,c_p\sum_{l=1}^m \overline{G}_{11}^l < \sqrt{2}\,\overline{a}_1,$$

$$\begin{aligned}
&\Big(\sqrt{2}\overline{a}_1 - \sqrt{2}\,\Big(A_1^2\tau_1^2 + A_1\overline{F}_{11}\tau_1 + c_pA_1\sqrt{\tau_1}\sum_{i=1}^m \overline{G}_{11}^l + \overline{F}_{11}\Big) - \sqrt{\overline{a}_1}c_p\sum_{l=1}^m \overline{G}_{11}^l\Big)\\
&\qquad\times\Big(\sqrt{2}\overline{a}_2 - \sqrt{2}\,\Big(A_2^2\tau_2^2 + A_2\overline{F}_{22}\tau_2 + c_pA_2\sqrt{\tau_2}\sum_{i=1}^m \overline{G}_{22}^l + \overline{F}_{22}\Big) - \sqrt{\overline{a}_2}c_p\sum_{l=1}^m \overline{G}_{22}^l\Big)\\
&> \Big(\sqrt{2}\,\Big(A_1\overline{F}_{12}\tau_1 + c_pA_1\sqrt{\tau_1}\sum_{i=1}^m \overline{G}_{12}^l + \overline{F}_{12}\Big) + \sqrt{\overline{a}_1}c_p\sum_{l=1}^m \overline{G}_{12}^l\Big)\\
&\qquad\times\Big(\sqrt{2}\Big(A_2\overline{F}_{21}\tau_2 + c_pA_2\sqrt{\tau_2}\sum_{i=1}^m \overline{G}_{21}^l + \overline{F}_{21}\Big) + \sqrt{\overline{a}_2}c_p\sum_{l=1}^m \overline{G}_{21}^l\Big).
\end{aligned}$$

Then system (1) *is globally exponentially* $2p$*-stable.*

Proof. We exploit the main theorem. Under the assumptions of Corollary 1 the matrix Γ becomes 2×2 with nonnegative off-diagonal entries. Thus, it will become an $\mathcal{M}$-matrix, if its principal minors γ_{11} and $\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}$ are positive. Straightforward calculations show that the first inequality of Corollary 1 yields $\gamma_{11} > 0$, while the second inequality of Corollary 1 yields $\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21} > 0$.

The corollaries below can be proven in a similar way. □

Corollary 2. *Consider the system*

$$dx_i(t) = \Big[-a_i(t)x_i(t) + \sum_{j=1}^{n} F_{ij}(t)x_j(h_{ij}(t))\Big]\,dt \\ + \sum_{l=1}^{m}\Big[\sum_{j=1}^{n} G^l_{ij}(t)x_j(h^l_{ij}(t))\Big]\,d\mathcal{B}_l(t) \ \ (t \geq 0), \ \ i = 1,\dots,n. \tag{2}$$

with $n = 2$ assuming that

$$\sqrt{2}\,\overline{F}_{11} + \sqrt{\overline{a}_1}\,c_p \sum_{l=1}^{m} \overline{G}^l_{11} < \sqrt{2}\,\overline{a}_1,$$

$$\Big(\sqrt{2}\,\overline{a}_1 - \sqrt{2}\,\overline{F}_{11} - \sqrt{\overline{a}_1}\,c_p \sum_{l=1}^{m} \overline{G}^l_{11}\Big)\Big(\sqrt{2}\,\overline{a}_2 - \sqrt{2}\,\overline{F}_{22} - \sqrt{\overline{a}_2}\,c_p \sum_{l=1}^{m} \overline{G}^l_{22})$$
$$> \Big(\sqrt{2}\,\overline{F}_{12} + \sqrt{\overline{a}_1}\,c_p \sum_{l=1}^{m} \overline{G}^l_{12}\Big)\Big(\sqrt{2}\,F_{21} + \sqrt{\overline{a}_2}\,c_p \sum_{l=1}^{m} \overline{G}^l_{21}\Big).$$

Then system (2) *is globally asymptotically* $2p$*-stable.*

Corollary 3. *Consider the system*

$$dx_i(t) = \Big[-a_i(t)x_i(t) + \sum_{j=1,\, i\neq j}^{n} F_{ij}(t)x_j(h_{ij}(t))\Big]\,dt \\ + \sum_{l=1}^{m}\Big[\sum_{j=1,\, i\neq j}^{n} G^l_{ij}(t)x_j(h^l_{ij}(t))\Big]\,d\mathcal{B}_l(t) \ \ (t \geq 0), \ \ i = 1,\dots,n. \tag{3}$$

with $n = 2$ assuming that

$$\Big(\sqrt{2}\,\overline{F}_{12} + \sqrt{\overline{a}_1}\,c_p \sum_{l=1}^{m} \overline{G}^l_{12}\Big)\Big(\sqrt{2}\,F_{21} + \sqrt{\overline{a}_2}\,c_p \sum_{l=1}^{m} \overline{G}^l_{21}\Big) < 2\overline{a}_1\overline{a}_2.$$

Then system (3) *is globally asymptotically* $2p$*-stable.*

Example 1. Consider the system

$$\begin{aligned} dx_1(t) &= \Big[-a_1x_1(t-h_1) + a_{11}F_{11}(x_1(t-h_{11})) + a_{12}F_{12}(x_2(t-h_{12}))\Big]\,dt \\ &\quad + \Big[b_{11}G_{11}(x_1(t-\tau_{11})) + b_{12}G_{12}(x_2(t-\tau_{12}))\Big]\,d\mathcal{B}(t) \ \ (t \geq 0), \\ dx_2(t) &= \Big[-a_2x_1(t-h_2) + a_{21}F_{21}(x_1(t-h_{21})) + a_{22}F_{22}(x_2(t-h_{22}))\Big]\,dt \\ &\quad + \Big[b_{21}G_{21}(x_1(t-\tau_{21})) + b_{22}G_{22}(x_2(t-\tau_{22}))\Big]\,d\mathcal{B}(t) \ \ (t \geq 0), \end{aligned} \tag{4}$$

where a_1, a_2, h_{ij}, τ_{ij}, a_{ij}, b_{ij}, $i,j = 1,2$ are positive numbers, F_{ij}, G_{ij}, $i,j = 1,2$ are continuous scalar functions on $(-\infty,+\infty)$ such that $|F_{ij}(u)| \leq |u|$, $|G_{ij}(u)| \leq |u|$, $i,j = 1,2$, and $\mathcal{B}$ is the standard scalar Brownian motion.

Then from Corollary 1 we deduce that the conditions

$$\sqrt{2}\,\Big(a_1^2h_1^2 + a_1a_{11}h_1 + c_pa_1\sqrt{h_1}\,b_{11} + a_{11}\Big) + \sqrt{a_1}\,c_pb_{11} < \sqrt{2}\,a_1,$$

$$\begin{aligned}\Big(\sqrt{2}\,a_1-\sqrt{2}\,\Big(a_1^2h_1^2+a_1a_{11}h_1+c_pa_1\sqrt{h_1}\,b_{11}+a_{11}\Big)-\sqrt{a_1}\,c_pb_{11}\Big)&\\ \times\Big(\sqrt{2}\,a_2-\sqrt{2}\,\Big(a_2^2h_2^2+a_2a_{22}h_2+c_pa_2\sqrt{h_2}\,b_{22}+a_{22}\Big)-\sqrt{a_2}\,c_pb_{22}\Big)&\\ >\Big(\sqrt{2}\,\big(a_1a_{12}h_1+c_pa_1\sqrt{h_1}\,b_{12}+a_{12}\big)+\sqrt{a_1}\,c_pb_{12}\Big)&\\ \times\Big(\sqrt{2}\,\big(a_2a_{21}h_2+c_pa_2\sqrt{h_2}\,b_{21}+a_{21}\big)+\sqrt{a_2}\,c_pb_{21}\Big)&\end{aligned}$$

imply the global asymptotic $2p$-stability of system (4).

Assume further that $a_{ii}=b_{ii}=0$, $i=1,2$ in system (4). In this case, the conditions

$$a_1h_1^2<1,$$

$$\begin{aligned}\Big(\sqrt{2}\,a_1-\sqrt{2}\,a_1^2h_1^2\Big)\Big(\sqrt{2}\,a_2-\sqrt{2}\,a_2^2h_2^2\Big)>\Big(\sqrt{2}\,\big(a_1a_{12}h_1+c_pa_1\sqrt{h_1}\,b_{12}+a_{12}\big)+\sqrt{a_1}\,c_pb_{12}\Big)&\\ \times\Big(\sqrt{2}\,\big(a_2a_{21}h_2+c_pa_2\sqrt{h_2}\,b_{21}+a_{21}\big)+\sqrt{a_2}\,c_pb_{21}\Big)&\end{aligned}$$

imply the global asymptotic $2p$-stability of system (4).

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The Boundary Value Problem for One Class of Semilinear Partial Differential Equations

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In the Euclidean space $\mathbb{R}^{n+1}$ of the variables $x=(x_1,x_2,\dots,x_n)$ and t we consider the semilinear equation of the type

$$\frac{\partial^{4k}u}{\partial t^{4k}}-\Delta u+f(u)=F, \tag{1}$$

where $f:\mathbb{R}\to\mathbb{R}$ is a given continuous function, $F=F(x,t)$ is a given, and $u=u(x,t)$ is an unknown real functions, k is a natural number, $\Delta:=\sum\limits_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$, $n\geq 2$.

For the equation (1) we consider the boundary value problem: find in the cylindrical domain $D_T=\Omega\times(0,T)$, where Ω is an open Lipschitz domain in $\mathbb{R}^n$, a solution $u(x,t)$ of that equation according to the boundary conditions

$$u\big|_{\Gamma}=0, \tag{2}$$

$$\frac{\partial^i u}{\partial t^i}\Big|_{\Omega_0\cup\Omega_T}=0,\ \ i=1,\dots,2k-1, \tag{3}$$

where $\Gamma:=\partial\Omega\times(0,T)$ is the lateral face of the cylinder D_T, $\Omega_0:\ x\in\Omega,\ t=0$ and $\Omega_T:\ x\in\Omega,\ t=T$ are the lower and upper bases of this cylinder, respectively.

Denote by $C^{2,4k}(\overline{D_T},\partial D_T)$ the space of functions u continuous in $\overline{D_T}$ and having continuous partial derivatives $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_i\partial x_j}$, $\frac{\partial^l u}{\partial t^l}$ in $\overline{D_T}$, $i,j=1,\dots,n$; $l=1,\dots,4k$. Let

$$C_0^{2,4k}(\overline{D_T},\partial D_T):=\Big\{u\in C^{2,4k}(\overline{D_T},\partial D_T):\ u\big|_{\Gamma}=0,\ \frac{\partial^i u}{\partial t^i}\Big|_{\Omega_0\cup\Omega_T}=0\ \ i=1,\dots,2k-1\Big\}.$$

Let $u\in C_0^{2,4k}(\overline{D_T},\partial D_T)$ be a classical solution of the problem $(1),(2),(3)$. Multiplying both parts of the equation (1) by an arbitrary function $\varphi\in C_0^{2,4k}(\overline{D_T},\partial D_T)$ and integrating the obtained equation by parts over the domain D_T, we obtain

$$\int\limits_{D_T}\bigg[\frac{\partial^{2k}u}{\partial t^{2k}}\,\frac{\partial^{2k}\varphi}{\partial t^{2k}}+\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\,\frac{\partial\varphi}{\partial x_i}\bigg]\,dx\,dt+\int\limits_{D_T}f(u)\varphi\,dx\,dt=\int\limits_{D_T}F\varphi\,dx\,dt. \tag{4}$$

Introduce the Hilbert space $W_0^{1,2k}(D_T)$ as a completion with respect to the norm

$$\|u\|^2_{W_0^{1,2k}(D_T)}=\int\limits_{D_T}\bigg[u^2+\sum_{i=1}^{2k}\Big(\frac{\partial^i u}{\partial t^i}\Big)^2+\sum_{i=1}^{n}\Big(\frac{\partial u}{\partial x_i}\Big)^2\bigg]\,dx\,dt$$

of the classical space $C_0^{2,4k}(\overline{D_T},\partial D_T)$.

We take the equality (4) as a basis for our definition of the weak generalized solution u of the problem $(1),(2),(3)$: the function $u\in W_0^{1,2k}(D_T)$ is said to be a weak generalized solution of the problem $(1),(2),(3)$ if for any function $\varphi\in W_0^{1,2k}(D_T)$ the integral equality (4) is valid.

It is not difficult to verify that if the weak generalized solution of the problem (1), (2), (3) belongs to the class $C_0^{2,4k}(\overline{D_T}, \partial D_T)$, then it will also be a classical solution of this problem.

Below, on the function $f = f(u)$ we impose the following requirements

$$f \in C(\mathbb{R}), \quad |f(u)| \leq M_1 + M_2|u|^{\alpha}, \quad u \in \mathbb{R}, \tag{5}$$

where

$$0 \leq \alpha = const < \frac{n+1}{n-1}\,, \tag{6}$$

and

$$uf(u) \geq 0 \quad \forall\, u \in \mathbb{R}. \tag{7}$$

Theorem. *Let the conditions* (5)–(7) *be fulfilled. Then for any* $F \in L_2(D)$ *the problem* (1), (2), (3) *has at least one weak generalized solution* $u \in W_0^{1,2k}(D_T)$.

Remark. Let us note that if along with the conditions (5)–(7) imposed on function f to demand that it is monotonous, then the solution $u \in W_0^{1,2k}(D_T)$ of the problem (1), (2), (3), the existence of which is stated in the theorem, is unique. As show the examples, when the conditions imposed on nonlinear function f are violated, then the problem (1), (2), (3) may not have a solution.

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Existence of Optimal Controls for Some Classes of Functional-Differential Equations

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Sufficient conditions for the existence of optimal controls for system of functional-differential equations which is nonlinear by phase variables and linear by control function are given. These conditions are obtained in terms of right-hand sides of the system and of the quality criterion function, which makes them convenient for verification.

Let $h>0$ be a value of delay, $|\cdot|$ denotes the norm of the vector in the space $\mathbb{R}^d$, $\|\cdot\|$ be the norm of $d\times m$-dimensional matrix which is consistent with the norm of the vector. Let us denote as $C=C([-h,0];\mathbb{R}^d)$ the Banach space of continuous maps of $[-h,0]$ into $\mathbb{R}^d$ with the uniform norm $\|\varphi\|_C=\max\limits_{\theta\in[-h,0]}|\varphi(\theta)|$. Also denote as $L_p=L_p([-h,0];\mathbb{R}^m)$, $p>1$, the Banach space of p-integrable m-dimensional vector-functions with standard norm

$$\|\varphi\|_{L_p}=\bigg(\int\limits_{-h}^{0}|\varphi(\tau)|^p\,d\tau\bigg)^{\frac{1}{p}}.$$

Let $x\in C([0,T];R^d)$, $\varphi\in C$. If $x(0)=\varphi(0)$, then the function

$$x(t,\varphi)=\begin{cases}\varphi(t), & t\in[-h,0],\\ x(t), & t\geq 0\end{cases}$$

is continuous on $[-h,T]$. For each $t\in[0,T]$ in the standard way by $\theta\in[-h,0]$ we put an element $x_t(\varphi)\in C$ as $x_t(\varphi)=x(t+\theta,\varphi)$. In what follows we shall write x_t instead of $x_t(\varphi)$. Let $t\in[0,T]$, D is some domain in $[0,T]\times C$, ∂D is its boundary and $\overline{D}=D\cup\partial D$.

Now we consider the optimal control problems for systems of functional differential equations:

$$\dot{x}=f_1(t,x_t)+\int\limits_{-h}^{0}f_2(t,x_t,y)u(t,y)\,dy,\ \ t\in[0,\tau], \tag{1}$$

$$x(t)=\varphi_0(t),\ \ t\in[-h,0],$$

with the quality criterion

$$J[u]=\int\limits_{0}^{\tau}L(t,x_t,u(t,\cdot\,))\,dt\longrightarrow\inf \tag{2}$$

on $[0,T]$, where $\varphi_0\in C$ is a fixed element such that $(0,\varphi_0)\in D$, $x(t)$ is the phase vector in $\mathbb{R}^d$, x_t is the phase vector in C, τ is the moment of the first exit (t,x_t) on the boundary ∂D, $f_1:D\to\mathbb{R}^d$, $f_2:D\times[-h,0]\to M^{d\times m}$ are $d\times m$-dimensional matrices, and for each $(t,\varphi)\in D$, $f_2(t,\varphi,\cdot\,)\in L_q([-h,0];M^{d\times m})$ with the norm

$$\|f_2(t,\varphi,\cdot\,)\|_{L_q}=\bigg(\int\limits_{-h}^{0}\|f_2(t,\varphi,y)\|^q\,dy\bigg)^{\frac{1}{q}},$$

$\frac{1}{q}+\frac{1}{p}=1$, $L: D\times L_p\to\mathbb{R}^1$. The control parameter $u\in L_p([0,T]\times[-h,0])$ is such that $u(t,y)\in U$, and U is convex and closed set in $\mathbb{R}^m$ for almost all t ,y.

Many works are devoted to the optimal control problems for functional-differential equations systems: the monograph [8] is devoted to the application of the method of dynamic programming and the principle of maximum to problems with aftereffect. In the case of compactness of the set of admissible controls in work [1] it was obtained an analogue of Filippov theorem of optimality control existence for ordinary differential equations. For noncompact set of admissible controls an analogue of the Cessari theorem is obtained in [5]. In the mentioned work the condition of compactness is imposed on a set of constraints and a certain condition of growth which connects the right sides of the system and the quality criterion. In [2] under the condition of compactness of the set of admissible controls value sufficient conditions for optimality on a fixed interval $[t_0,t_1]$ for neutral-type equations are obtained. In [3] the problem of optimal control of a delayed linear system is rewritten in a form that does not depend on the delay, which is studied by methods of ordinary differential equations. In the works [4, 6, 7] the optimal control problem of the system

$$\dot{x}(t)=rx(t)+f_0\bigg(x(t),\int\limits_{-T}^{0}a(\varsigma)x(t+\varsigma)\,d\varsigma\bigg)-u(t)$$

is considered. In [4] certain Hamilton–Jacobi–Bellman equations are obtained for certain quality functionals and, in terms of their solutions, sufficient conditions for optimality in the form of a reverse link are obtained. In [6] similar questions are considered for problems with phase restriction. In [7] for such problems it was obtained sufficient conditions for optimality under the condition of nondecreasing function $rx+f_0(x,y)$ in both the variables.

The main goal of our paper is to obtain the theorem on the existence of optimal controls for a wider class of problems under weaker conditions as compared with above mentioned works [1–8]. The following are the main conditions for the problem (1), (2) assumed in the manuscript.

Assumption 1. *Admissible controls are m-dimensional vector functions $u\in L_p([0,T][-h,0],\mathbb{R}^m)$, such that $u(t,y)\in U$ for almost all $t\in[0,T]$ and $y\in[-h,0]$.*

The set of admissible controls we denote as $\mathcal{U}$.

Assumption 2. *Maps $f_1(t,\varphi): D\to\mathbb{R}^d$ and $f_2(t,\varphi,y): D\times[-h,0]\to M^{d\times m}$ are defined and measurable with respect to all its arguments in the domains D and $D_1=\{(t,\varphi)\in D,\ y\in[-h,0]\}$ respectively, satisfy the linear growth condition and the Lipschitz condition with respect to φ, i.e. there exists the constant $K>0$ such that*

$$|f_1(t,\varphi)|+\|f_2(t,\varphi,y)\|\le K(1+\|\varphi\|_C) \tag{3}$$

for any $(t,\varphi)\in D$, $y\in[-h,0]$,

$$\big|f_1(t,\varphi_1)-f_1(t,\varphi_2)\big|+\big\|f_2(t,\varphi_1,y)-f_2(t,\varphi_2,y)\big\|\le K\|\varphi_1-\varphi_2\|_C \tag{4}$$

for all $(t,\varphi_1),(t,\varphi_2)\in D$, $y\in[-h,0]$.

Assumption 3. *Conditions for the criterion function:*

1) *the map $L(t,\varphi,z): D\times L_p\to\mathbb{R}^1$ is defined and continuous with respect to all its arguments in the domain $D_2=\{(t,\varphi)\in D,\ z\in L_p\}$;*

2) *there exists $a>0$ such that*

$$\big|L(t,\varphi_1,z)-L(t,\varphi_2,z)\big|\le a\|\varphi_1-\varphi_2\|_C$$

for all $(t,\varphi_1,z),(t,\varphi_2,z)\in D_2$;

3) *Frechet derivative L_u of the map L is continuous with respect to all its arguments in the domain D_2, and there exist constants $C_1 > 0$, $\alpha > 0$ such that for all $(t,\varphi,z) \in D_2$ the following inequality holds:*

$$\|L_u(t,\varphi,z)\|_{L_q} \le C_1\big(1 + \|\varphi\|_C^\alpha + \|z\|_{L_p}^{p-1}\big);$$

4) *there exists the constant $C > 0$ such that $L(t,\varphi,z) \ge C\|z\|_{L_p}^p$ for all $(t,\varphi,z) \in D_2$;*

5) *$L(t,\varphi,z)$ convex with respect to z for any fixed t, φ;*

Our first result concerns the existence, uniqueness and extension of the solution of the original problem (1) to the boundary ∂D of the domain D. It is some analogue of the Carathéodory theorem for ordinary differential equations.

Definition. The solution of the initial problem (1) on the segment $[-h, A]$, $A > 0$ is called a continuous on the segment $[-h, A]$ function $x(t)$ such that

1) $x(t) = \varphi_0(t)$, $t \in [-h, 0]$;

2) $(t, x_t) \in D$ on $t \in [0, A]$;

3) for $t \in [0, A]$ function $x(t)$ satisfies the integral equation

$$x(t) = \varphi_0(0) + \int\limits_0^t \bigg[f_1(s, x_s) + \int\limits_{-h}^0 f_2(s, x_s, y)u(s, y)\,dy\bigg]\,ds. \tag{5}$$

Remark. It is obvious that for $t \in [0, A]$ solution $x(t)$ is an absolutely continuous function and satisfies the equation (1) for almost all t on $[0, A]$.

Theorem 1. *Suppose that Assumptions* 1, 2 *are satisfied. Then there exists the solution of the initial problem* (5) *on the maximal segment* $[-h, \tau]$, $\tau > 0$ *and* $(\tau, x_\tau) \in \partial D$.

The following theorem gives for the problem (1), (2) the existence conditions of the optimal pair $x^*(t)$, $u^*(t,\theta)$), which provides the minimum of the quality criterion (2). In this case $u^* \in \mathcal{U}$ is called optimal control and the corresponding trajectory $x^*(t)$ (1) is called the optimal trajectory.

Theorem 2. *Suppose that Assumptions* 1–3 *are satisfied. Then there exists the solution of the optimal control problem* (1), (2).

As an application of the obtained results, we consider some particular cases of problem (1), (2). If $u = u(t)$ and does not depend on the value y, then the problem (1), (2) reduces to the "ordinary" optimal control problem for functional-differential equations. A particular case of the problem (1), (2) is the optimal control problem with maximum on the interval $[-h, T]$, $h > 0$.

$$\dot{x}(t) = f_1\big(t, x_t, \max_{s\in I(t)} x(s)\big) + f_2\big(t, x_t, \max_{s\in I(t)} x(s)\big)u(t), \tag{6}$$

$$x(t) = \varphi(t), \ \ t \in [-h, 0],$$

$$J[u] = \int\limits_0^\tau L(t, x(t), u(t))\,dt \longrightarrow \inf, \tag{7}$$

where $I(t) = [\beta(t), \alpha(t)]$, $\max x(s) = (\max x_1(s), \dots, \max x_d(s))$, $\beta(t)$, $\alpha(t)$ are continuous on $[0, T]$ functions such that $\beta(t) \le \alpha(t) \le t$ and $\min\limits_{t\in[0,T]} (\beta(t) - t) = -h$, $f(t, x, y) : [0, T] \times G \times G \to M^{d\times m}$, G is a domain in $\mathbb{R}^d$, $u \in U \subset \mathbb{R}^m$, $L(t, x, u) : [0, T] \times G \times U \to \mathbb{R}^1$.

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On the Cauchy Weighted Problem for Higher Order Singular Ordinary Differential Equations

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On a finite semi-open interval $]a,b]$ we consider the differential equation

$$u^{(n)} = f(t, u, \dots, u^{(n-1)}) \tag{1}$$

with the weighted initial conditions

$$\lim_{t\to a} u^{(i-1)}(t) = 0 \ \ (i = 1, \dots, n-1), \quad \limsup_{t\to a} \frac{|u^{(n-1)}(t)|}{\delta(t)} < +\infty, \tag{2}$$

where $n \geq 2$, the functions $f :]a,b] \times \mathbb{R}^n \to \mathbb{R}$ and $\delta :]a,b] \to]0,+\infty[$ are continuous and

$$\int\limits_a^b \delta(t)\, dt < +\infty.$$

Equation (1) is said to be singular in the time variable if

$$\int\limits_a^b f^*(t,x)\, dt = +\infty \ \text{ for } \ x > 0,$$

where

$$f^*(t,x) = \max\Big\{ |f(t, x_1, \dots, x_n)| : \ \sum_{i=1}^n |x_i| \leq x \Big\}.$$

For such equations, the weighted initial problem has been investigated earlier only in the case, when

$$\int\limits_a^b |f(t,0,\dots,0)|\, dt < +\infty, \quad \lim_{t\to a} \delta(t) = 0.$$

We have established unimprovable in a certain sense sufficient conditions for solvability and unique solvability of problem (1), (2) covering the cases, where

$$\int\limits_a^b (t-a)^\mu |f(t,0,\dots,0)|\, dt = +\infty \ \text{ for any } \ \mu > 0,$$

or the weighted function δ has no finite limit at the point a.

In Theorems 1–3 formulated below and dealing with the solvability, unsolvability, and unique solvability of problem (1), (2), respectively, it is assumed that the function f on the set $]a,b] \times \mathbb{R}^n$ satisfies one of the following three conditions:

$$f(t, x_1, \dots, x_n) \operatorname{sgn}(x_n) \leq \sum_{i=1}^n h_i(t)|x_i| + h_0(t), \tag{3}$$

$$f(t,x_1,\dots,x_n)-h_n(t)x_n \geq \sum_{i=1}^{n-1} h_i(t)|x_i|+h_0(t), \tag{4}$$

$$\big[f(t,x_1,\dots,x_n)-f(t,y_1,\dots,y_n)\big]\operatorname{sgn}(x_n-y_n) \leq \sum_{i=1}^{n} h_i(t)|x_i-y_i|, \tag{5}$$

where $h_i :]a,b] \to [0,+\infty[$ $(i=0,\dots,n-1)$ and $h_n :]a,b] \to \mathbb{R}$ are continuous functions. Note that unlike h_i $(i=0,\dots,n-1)$, the function h_n may be negative or with alternating sign.

Suppose

$$\delta_i(t)=\frac{1}{(n-1-i)!}\int\limits_a^t (t-s)^{n-1-i}\delta(s)\,ds \ \ (i=1,\dots,n-1).$$

Theorem 1. *If along with* (3) *the inequalities*

$$\limsup_{t\to a}\bigg[\frac{1}{\delta(t)}\sum_{i=1}^{n-1}\int\limits_a^t \exp\bigg(\int\limits_s^t h_n(\tau)\,d\tau\bigg)\delta_i(s)h_i(s)\,ds\bigg]<1, \tag{6}$$

$$\limsup_{t\to a}\bigg[\frac{1}{\delta(t)}\int\limits_a^t \exp\bigg(\int\limits_s^t h_n(\tau)\,d\tau\bigg)h_0(s)\,ds\bigg]<+\infty \tag{7}$$

are fulfilled, then problem (1), (2) *has at least one solution.*

Theorem 2. *Let along with* (4) *the conditions*

$$\liminf_{t\to a}\bigg[\delta(t)\exp\bigg(\int\limits_t^b h_n(s)\,ds\bigg)\bigg]=0, \tag{8}$$

$$\liminf_{t\to a}\bigg[\frac{1}{\delta(t)}\int\limits_a^t \exp\bigg(\int\limits_s^t h_n(\tau)\,d\tau\bigg)h_0(s)\,ds\bigg]>0$$

be fulfilled and there exist $b_0\in\,]a,b[$ *such that*

$$\sum_{i=1}^{n-1}\int\limits_a^t \exp\bigg(\int\limits_s^t h_n(\tau)\,d\tau\bigg)\delta_i(s)h_i(s)\,ds \geq \delta(t) \ \ \textit{for} \ \ a<t\leq b_0.$$

Then problem (1), (2) *has no solution.*

Assume now that the function δ is continuously differentiable on $]a,b]$ and, as an example, consider the differential equation

$$u^{(n)}=h_n(t)u^{(n-1)}+\sum_{i=1}^{n} h_i(t)|u^{(i-1)}|+f_0(t,u,\dots,u^{(n-1)}), \tag{9}$$

where

$$h_n(t)=\frac{\delta'(t)}{\delta(t)}-h(t), \ \ h_i(t)=\big[\alpha_i h(t)+\ell_i(t)\big]\,\frac{\delta(t)}{\delta_i(t)} \ \ (i=1,\dots,n-1),$$

α_i $(i = 1,\dots,n-1)$ are nonnegative constants, while $h :]a,b] \to [0,+\infty[$, $\ell_i :]a,b] \to [0,+\infty[$ $(i = 1,\dots,n-1)$ and $f_0 :]a,b] \times \mathbb{R}^n \to [0,+\infty[$ are continuous functions such that

$$\int_a^b h(t)\,dt = +\infty, \quad \int_a^b \ell_i(t)\,dt < +\infty \ \ (i = 1,\dots,n-1),$$
$$\alpha_0 h(t)\delta(t) \le f_0(t,x_1,\dots,x_n) \le \alpha h(t)\delta(t), \ \ \alpha > \alpha_0 > 0.$$

From Theorems 1 and 2 it follows

Corollary 1. *Problem* (9), (2) *is solvable if and only if*

$$\sum_{i=1}^{n-1} \alpha_i < 1.$$

Consequently, inequality (6) in Theorem 1 is unimprovable and it cannot be replaced by the nonstrict inequality

$$\limsup_{t\to a} \bigg[\frac{1}{\delta(t)} \sum_{i=1}^{n-1} \int_a^t \exp\bigg(\int_s^t h_n(\tau)\,d\tau\bigg)\delta_i(s)h_i(s)\,ds\bigg] \le 1.$$

Theorem 3. *If along with* (5) *conditions* (6)–(8) *are fulfilled, where* $h_0(t) = |f(t,0,\dots,0)|$, *then problem* (1), (2) *has one and only one solution.*

Conditions (6)–(8) are satisfied, for example, if

$$\delta(t) = (t-a)^\lambda, \ \ \lambda \in]-1,+\infty[\,, \quad h_n(t) = \frac{\lambda}{t-a} - \exp\Big(\frac{1}{t-a}\Big),$$
$$h_i(t) = \alpha_i (t-a)^{i+1-n} \exp\Big(\frac{1}{t-a}\Big) \ \ (i = 1,\dots,n-1), \quad f(t,0,\dots,0) = \alpha_0 (t-a)^\lambda \exp\Big(\frac{1}{t-a}\Big).$$

Consequently, Theorem 3 covers the case, where the functions $t \mapsto h_i(t)$ $(i = 1,\dots,n)$ and $t \mapsto f(t,0,\dots,0)$ have singularities of arbitrary order for $t = a$.

The following theorem contains the conditions guaranteeing the existence of an infinite set of solutions of problem (1), (2). It concerns the case, when on the set $]a,b] \times \mathbb{R}^n$ the inequality

$$-\sum_{i=1}^{n-1} h_i(t)|x_i| - h_0(t) \le \big(f(t,x_1,\dots,x_n) - h_n(t)x_n\big)\operatorname{sgn}(x_n) \le \sum_{i=1}^{n-1} \overline{h}_i(t)|x_i| + \overline{h}_0(t) \tag{10}$$

is satisfied, where $h_i :]a,b] \to \mathbb{R}_+$, $\overline{h}_i :]a,b] \to \mathbb{R}_+$ $(i = 0,\dots,n-1)$ and $h_n :]a,b] \to \mathbb{R}$ are continuous functions.

Theorem 4. *If along with* (10) *the conditions*

$$\int_a^b \exp\bigg(\int_s^b h_n(\tau)\,d\tau\bigg)\delta_i(s)h_i(s)\,ds < +\infty \ \ (i = 1,\dots,n-1),$$
$$\int_a^b \exp\bigg(\int_s^b h_n(\tau)\,d\tau\bigg)h_0(s)\,ds < +\infty, \quad \liminf_{t\to a} \bigg[\delta(t)\exp\bigg(\int_t^b h_n(\tau)\,d\tau\bigg)\bigg] > 0$$

are fulfilled, then problem (1), (2) *has an infinite set of solutions.*

Finally, let us consider the linear differential equation

$$u^{(n)} = \sum_{i=1}^{n} p_i(t)u^{(i-1)} + p_0(t) \tag{11}$$

with continuous coefficients $p_i :]a,b] \to \mathbb{R}$ $(i = 0, 1, \dots, n)$.

From Theorems 1, 3, 4 we have the following corollaries.

Corollary 2. *If*

$$\limsup_{t\to a} \bigg[\frac{1}{\delta(t)} \sum_{i=1}^{n-1} \int_a^t \exp\bigg(\int_s^t p_n(\tau)\,d\tau\bigg)\delta_i(s)|p_i(s)|\,ds\bigg] < 1, \tag{12}$$

$$\limsup_{t\to a} \bigg[\frac{1}{\delta(t)} \int_a^t \exp\bigg(\int_s^t p_n(\tau)\,d\tau\bigg)|p_0(s)|\,ds\bigg] < +\infty, \tag{13}$$

then problem (11), (2) *has at least one solution. If, however, along with* (12) *and* (13) *the condition*

$$\liminf_{t\to a} \bigg[\delta(t)\exp\bigg(\int_t^b h_n(s)\,ds\bigg)\bigg] = 0$$

is fulfilled, then this problem has a unique solution.

Corollary 3. *Let the function* $t \mapsto \delta(t)\exp\big(\int_t^b p_n(s)\,ds\big)$ *be nondecreasing and*

$$\int_a^b \frac{\delta_i(s)}{\delta(s)}\,|p_i(s)|\,ds < +\infty \;\; (i = 1, \dots, n-1), \quad \int_a^b \frac{|p_0(s)|}{\delta(s)}\,ds < +\infty.$$

Then problem (11), (2) *is uniquely solvable if and only if*

$$\lim_{t\to a} \bigg[\delta(t)\exp\bigg(\int_t^b p_n(s)\,ds\bigg)\bigg] = 0.$$

If, however,

$$\lim_{t\to a} \bigg[\delta(t)\exp\bigg(\int_t^b p_n(s)\,ds\bigg)\bigg] > 0,$$

then this problem has an infinite set of solutions.

The strict inequality (12) in Corollary 2 is unimprovable and it cannot be replaced by the nonstrict one.

A particular case of (2) is the condition

$$\lim_{t\to a} u^{(i-1)}(t) = 0 \;\; (i = 1, \dots, n-1), \quad \limsup_{t\to a} \frac{|u^{(n-1)}(t)|}{(t-a)^{\lambda}} < +\infty, \tag{14}$$

where $\lambda \in]-1, +\infty[$.

Corollary 4. *Let*

$$p(t) \overset{def}{=} \frac{\lambda}{t-a} - p_n(t) > 0 \ \ for \ \ a < t \leq b, \tag{15}$$

$$\lim_{t\to a} \frac{(t-a)^{n-i}p_i(t)}{p(t)} = 0, \ \ (i=1,\dots,n-1), \quad \limsup_{t\to a} \frac{|p_0(t)|}{(t-a)^{\lambda}p(t)} < +\infty. \tag{16}$$

Then problem (11), (14) *is uniquely solvable if and only if*

$$\int_a^b p(t)\,dt = +\infty. \tag{17}$$

Remark. If $\lambda > 0$, then, obviously, the conditions of Corollary 4 guarantee the existence of a solution of equation (11) satisfying the initial conditions

$$\lim_{t\to a} u^{(i-1)}(t) = 0 \ \ (i=1,\dots,n). \tag{18}$$

On the other hand, if $\lambda \in]-1,0]$ and along with (16)–(18) the condition

$$\liminf_{t\to a} \frac{|p_0(t)|}{(t-a)^{\lambda}p(t)} > 0$$

is fulfilled, then problem (11), (18) has no solution, whereas problem (11), (14) is uniquely solvable.

Ill-Posed Multi Dimensional Boundary Value Problems for Linear Hyperbolic Equations of Higher Order

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Let $m_1, \dots, m_n$ be positive integers. In the n-dimensional box $\Omega = [0, \omega_1] \times \cdots \times [0, \omega_n]$ for the linear hyperbolic equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + q(\mathbf{x}) \tag{1}$$

consider the boundary conditions

$$h_{ik}\big(u(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n)\big) = \varphi_{ik}(\widehat{\mathbf{x}}_i) \;\text{ for }\; \widehat{\mathbf{x}}_i \in \Omega_i \;\; (k = 1, \dots, m_i; \;\; i = 1, \dots, n). \tag{2}$$

Here $\mathbf{x} = (x_1, \dots, x_n)$, $\widehat{\mathbf{x}}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, $\Omega_i = [0, \omega_1] \times \cdots \times [0, \omega_{i-1}] \times [0, \omega_{i+1}] \times \cdots \times [0, \omega_n]$, $\mathbf{m} = (m_1, \dots, m_n)$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\widehat{\mathbf{m}}_i = \mathbf{m} - \mathbf{m}_i$ and $\mathbf{m}_i = (0, \dots, m_i, \dots, 0)$ are multi-indices,

$$u^{(\boldsymbol{\alpha})}(\mathbf{x}) = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} u(\mathbf{x})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\,,$$

$p_{\boldsymbol{\alpha}} \in C(\Omega)$ $(\boldsymbol{\alpha} < \mathbf{m})$, $q \in C(\Omega)$, $h_{ik} : C^{m_i - 1}([0, \omega_i]) \to \mathbb{R}$ $(k = 1, \dots, m_i;\ i = 1, \dots, n)$ are bounded linear functionals, and $\varphi_{ik} \in C^{\widehat{\mathbf{m}}_i}(\Omega_i)$ $(k = 1, \dots, m_i;\ i = 1, \dots, n)$. Furthermore, it is assumed that the functions φ_{ik} satisfy the following consistency conditions:

$$h_{ik}(\varphi_{jl})(\widehat{\mathbf{x}}_{ij}) \equiv h_{jl}(\varphi_{ik})(\widehat{\mathbf{x}}_{ij}) \;\; (k = 1, \dots, m_i; \;\; l = 1, \dots, m_j; \;\; i, j = 1, \dots, n),$$

where $\widehat{\mathbf{x}}_{ij} = \mathbf{x} - \widehat{\mathbf{x}}_i - \widehat{\mathbf{x}}_j$.

By a solution of problem (1), (2) we understand a classical solution, i.e., a function $u \in C^{\mathbf{m}}(\Omega)$ satisfying equation (1) and boundary conditions (2).

Along with problem (1), (2) consider its corresponding homogeneous problem

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}, \tag{1_0}$$

$$h_{ik}\big(u(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n)\big) = 0 \;\text{ for }\; \widehat{\mathbf{x}}_i \in \Omega_i \;\; (k = 1, \dots, m_i; \;\; i = 1, \dots, n). \tag{2_0}$$

We make use of following notations and definitions.

- $\operatorname{supp} \boldsymbol{\alpha} = \{i \mid \ \alpha_i > 0\}$, $\|\boldsymbol{\alpha}\| = |\alpha_1| + \cdots + |\alpha_n|$.
- $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) < \boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \iff \alpha_i \le \beta_i$ $(i = 1, \dots, n)$ and $\boldsymbol{\alpha} \ne \boldsymbol{\beta}$.
- $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \le \boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \iff \boldsymbol{\alpha} < \boldsymbol{\beta}$, or $\boldsymbol{\alpha} = \boldsymbol{\beta}$.
- $\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$, $\mathbf{1}_i = (0, \dots, 0, 1, 0, \dots, 0)$.
- $\boldsymbol{\Xi} = \{\boldsymbol{\sigma} \mid \ \mathbf{0} < \boldsymbol{\sigma} < \mathbf{1}\}$.
- $\widehat{\boldsymbol{\alpha}} = \mathbf{m} - \boldsymbol{\alpha}$. If $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$, then $\widehat{\boldsymbol{\sigma}} = \mathbf{1} - \boldsymbol{\sigma}$.

- $\mathbf{m}_{\boldsymbol\sigma} = (\sigma_1 m_1, \dots, \sigma_n m_n)$. It is clear that $\widehat{\mathbf{m}}_{\boldsymbol\sigma} = \mathbf{m} - \mathbf{m}_{\boldsymbol\sigma} = \mathbf{m}_{\widehat{\boldsymbol\sigma}}$.

- $\mathbf{x}_{\boldsymbol\sigma} = (\sigma_1 x_1, \dots, \sigma_n x_n)$. $\mathbf{x}_{\boldsymbol\sigma}$ will be identified with $(x_{i_1}, \dots, x_{i_l})$, as well as the set $\Omega_{\boldsymbol\sigma} = [0, \sigma_1\omega_1] \times \dots \times [0, \sigma_n\omega_n]$ will be identified with the set $[0, \omega_{i_1}] \times \dots \times [0, \omega_{i_l}]$, where $\{i_1, \dots, i_l\} = \operatorname{supp} \boldsymbol\sigma$.

- $C^{\mathbf{m}}(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{R}$, having continuous partial derivatives $u^{(\boldsymbol\alpha)}$, $\boldsymbol\alpha \le \mathbf{m}$, with the norm
$$\|u\|_{C^{\mathbf{m}}(\Omega)} = \sum_{\boldsymbol\alpha \le \mathbf{m}} \|u^{(\boldsymbol\alpha)}\|_{C(\Omega)}.$$

- $\widehat{C}^{\mathbf{m}}(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{R}$, having continuous partial derivatives $u^{(\boldsymbol\alpha)}$, $\boldsymbol\alpha < \mathbf{m}$, with the norm
$$\|u\|_{C^{\mathbf{m}}(\Omega)} = \sum_{\boldsymbol\alpha < \mathbf{m}} \|u^{(\boldsymbol\alpha)}\|_{C(\Omega)}.$$

Let $\boldsymbol\sigma \in \boldsymbol\Xi$. In the domain $\Omega_{\boldsymbol\sigma}$ consider the homogeneous boundary value problem depending on the parameter $\widehat{\mathbf{x}}_{\boldsymbol\sigma} \in \Omega_{\widehat{\boldsymbol\sigma}}$
$$v^{(\mathbf{m}_{\boldsymbol\sigma})} = \sum_{\boldsymbol\alpha < \mathbf{m}_{\boldsymbol\sigma}} p_{\boldsymbol\alpha + \widehat{\mathbf{m}}_{\boldsymbol\sigma}}(\mathbf{x}) v^{(\boldsymbol\alpha)}, \tag{$3_{\boldsymbol\sigma}$}$$
$$h_{ik}\big(v(x_1, \dots, x_{i_1}, \bullet, x_{i+1}, \dots, x_n)\big) = 0 \ \ (k = 1, \dots, m_i; \ \ i \in \operatorname{supp} \boldsymbol\sigma). \tag{$4_{\boldsymbol\sigma}$}$$
Problem $(3_{\boldsymbol\sigma}), (4_{\boldsymbol\sigma})$ is called an associated problem of level $l = \|\boldsymbol\sigma\|$.

Theorem 1. *Let all of the coefficients of equation* (1) *be constants, and let for some $\boldsymbol\sigma \in \boldsymbol\Xi$ associated problem $(3_{\boldsymbol\sigma}), (4_{\boldsymbol\sigma})$ be ill-posed. Furthermore, let*
$$p_{\boldsymbol\alpha+\boldsymbol\beta} + p_{a+\widehat{\mathbf{m}}_{\boldsymbol\sigma}} p_{\mathbf{m}_{\boldsymbol\sigma}+\boldsymbol\beta} = 0 \ \ \textit{for} \ \ \mathbf{0} < \boldsymbol\alpha < \mathbf{m}_{\boldsymbol\sigma}, \ \mathbf{0} < \boldsymbol\beta < \widehat{\mathbf{m}}_{\boldsymbol\sigma}.$$
Then for solvability of problem (1), (2) *it is necessary that for every $l \in \operatorname{supp} \widehat{\boldsymbol\sigma}$ and $j \in \{1, \dots, m_l\}$ the problem*
$$v^{(\mathbf{m}_{\boldsymbol\sigma})} = \sum_{\boldsymbol\alpha < \mathbf{m}_{\boldsymbol\sigma}} p_{\boldsymbol\alpha + \widehat{\mathbf{m}}_{\boldsymbol\sigma}} v^{(\boldsymbol\alpha)} + Q_{lj}(\widehat{\mathbf{x}}_l), \tag{5_{lj}}$$
$$h_{ik}\big(v(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n)\big) = \Psi^{lj}_{ik}(\widehat{\mathbf{x}}_{il}) \quad (k = 1, \dots, m_i; \ \ i \in \operatorname{supp} \boldsymbol\sigma), \tag{6_{lj}}$$
where
$$Q_{lj}(\widehat{\mathbf{x}}_l) = (p_{\mathbf{0}} + p_{\widehat{\mathbf{m}}_{\boldsymbol\sigma}} p_{\mathbf{m}_{\boldsymbol\sigma}}) \varphi_{lj}^{(\widehat{\mathbf{m}}_{\boldsymbol\sigma})}(\widehat{\mathbf{x}}_l) + h_{lj}(q)(\widehat{\mathbf{x}}_l)$$
and
$$\Psi^{lj}_{ik}(\widehat{\mathbf{x}}_{il}) = h_{lj}\big(\varphi_{ik}^{(\widehat{\mathbf{m}}_{\boldsymbol\sigma})}\big)(\widehat{\mathbf{x}}_{il}) - \sum_{\boldsymbol\beta < \widehat{\mathbf{m}}_{\boldsymbol\sigma}} p_{\mathbf{m}_{\boldsymbol\sigma}+\boldsymbol\beta} \varphi_{lj}^{(\boldsymbol\beta)}(\widehat{\mathbf{x}}_l)$$
is solvable.

Remark 1. Solvability of ill-posed nonhomogenous associated problem $(5_{lj}), (6_{lj})$ is in fact additional consistency condition between the boundary values φ_{ik}, the coefficients $p_{\boldsymbol\alpha}$ and the free term q. These are necessary conditions of solvability and they do not guarantee solvability of problem (1), (2) even if the homogeneous problem $(1_0), (2_0)$ has only the trivial solution.

Indeed, consider the periodic problem

$$u^{(1,1,1)} = \cos^2 x_1 u - q(x_1), \tag{7}$$

$$u(\pi, x_2, x_3) = u(0, x_2, x_3), \quad u(x_1, \pi, x_3) = u(x_1, 0, x_3), \quad u(x_1, x_2, \pi) = u(x_1, x_2, 0), \tag{8}$$

where q is a continuous function such that $q(\pi) = q(0)$. Problem (7), (8) is ill-posed, and its corresponding homogeneous problem has only the trivial solution. Furthermore, for problem (7), (8) all consistency conditions hold, Therefore, due to uniqueness, the only possible solution of problem (7), (8) should be

$$u(x_1) = \frac{q(x_1)}{\cos^2 x_1}\,.$$

On the other hand, it is clear that problem (7), (8) has a solution if and only if

$$q(x_1) = \cos^2 x_1 \widetilde{q}(x_1),$$

where $\widetilde{q} \in C^1([0,\pi])$. In particular, if $q(x_1) \equiv 1$, then problem (7), (8) has no solution despite the fact that all coefficients of equation (7) and boundary data are analytic functions.

In the rectangle $\Omega = [0,\omega_1] \times [0,\omega_2]$ consider the problem

$$u^{(2m_1,2m_2)} = p(x_1,x_2)\, u + q(x_1,x_2), \tag{9}$$

$$\begin{gathered} u^{(j-1,0)}(\omega_1,x_2) - u^{(j-1,0)}(0,x_2) = \varphi_j(x_2) \ \ (j = 1,\dots,2m_1), \\ u^{(0,k-1)}(x_1,\omega_2) - u^{(0,k-1)}(x_1,0) = \psi_k(x_1) \ \ (k = 1,\dots,2m_2). \end{gathered} \tag{10}$$

Theorem 2. *Let* $p,\ q \in \widehat{C}^{2m_1,2m_2}(\Omega)$, $\varphi_j \in C^{2m_2}([0,\omega_2])$ $(j = 1,\dots,2m_1-1)$, $\varphi_{2m_1} \in C^{4m_2}([0,\omega_2])$, $\psi_k \in C^{2m_1}([0,\omega_1])$ $(k = 1,\dots,2m_2-1)$, $\psi_{2m_2} \in C^{4m_1}([0,\omega_1])$,

$$\begin{gathered} p^{(j-1,0)}(\omega_1,x_2) - p^{(j-1,0)}(0,x_2) = 0 \ \ (j = 1,\dots,2m_1), \\ p^{(0,k-1)}(x_1,\omega_2) - p^{(0,k-1)}(x_1,0) = 0 \ \ (k = 1,\dots,2m_2), \end{gathered} \tag{11}$$

and let

$$(-1)^{m_1+m_2} \int\limits_0^{\omega_1} p(s,x_2)\,ds < 0, \quad (-1)^{m_1+m_2} \int\limits_0^{\omega_2} p(x_1,t)\,dt < 0 \ \ \textit{for} \ \ (x_1,x_2) \in \Omega. \tag{12}$$

Then problem (9), (10) *is solvable* ***if and only if***

$$\begin{aligned} \int\limits_0^{\omega_1} \bigg[\sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!}\, p^{(0,k-i)}(s,0)\psi_{i+1}(s) + q^{(0,k)}(s,\omega_2) - q^{(0,k)}(s,0) \bigg]\, ds& \\ = \varphi_{2m_1}^{(2m_2+k)}(\omega_2) - \varphi_{2m_1}^{(2m_2+k)}(0) \ \ (k = 0,\dots,2m_2-1),& \end{aligned} \tag{13}$$

$$\begin{aligned} \int\limits_0^{\omega_2} \bigg[\sum_{i=0}^{j} \frac{j!}{i!\,(j-i)!}\, p^{(j-i,0)}(0,t)\varphi_{i+1}(t) + q^{(j,0)}(\omega_1,t) - q^{(j,0)}(0,t) \bigg]\, dt& \\ = \psi_{2m_2}^{(2m_1+j)}(\omega_1) - \psi_{2m_2}^{(2m_1+j)}(0) \ \ \ (j = 0,\dots,2m_1-1).& \end{aligned} \tag{14}$$

Moreover, if the equalities (13) *and* (14) *hold, problem* (9), (10) *has a unique solution u admitting the estimate*

$$\|u\|_{C^{2m_1,2m_2}(\Omega)} \leq M\bigg(\sum_{j=1}^{2m_1-1} \|\varphi_j\|_{C^{2m_2}([0,\omega_2])} + \|\varphi_{2m_2}\|_{C^{4m_2}([0,\omega_2])} + \sum_{k=1}^{2m_2-1} \|\psi_k\|_{C^{2m_1}([0,\omega_1])} + \|\psi_{2m_2}\|_{C^{4m_1}([0,\omega_1])} + \|q\|_{\widehat{C}^{2m_1,2m_2}(\Omega)}\bigg), \quad (15)$$

where M is a positive constant independent of q.

Remark 2. Estimate (15) for a solution of problem (9), (10) is sharp, and regularity requirements on functions φ_k, ψ_j, p and q cannot be relaxed. For the sake of comparison, for equation (9) consider the Dirichlet boundary conditions

$$\begin{aligned} &u^{(j-1,0)}(\omega_1, x_2) = 0, \quad u^{(j-1,0)}(0, x_2) = 0 \ \ (j = 1, \dots, m_1), \\ &u^{(0,k-1)}(x_1, \omega_2) = 0, \quad u^{(0,k-1)}(x_1, 0) = 0 \ \ (k = 1, \dots, m_2). \end{aligned} \quad (16)$$

Unlike to problem (9), (10), by Theorem 2 from [1], problem (9), (16) is well-posed and its solution u admits the estimate

$$\|u\|_{C^{2m_1,2m_2}(\Omega)} \leq M\|q\|_{C(\Omega)},$$

where M is a positive constant independent of q.

This clearly demonstrates that ill-posedness of associated problems $(3_{\boldsymbol{\sigma}}), (4_{\boldsymbol{\sigma}})$ not only creates additional consistency conditions, but also increases regularity requirements on coefficients of the equation and the boundary data.

In order to better illustrate the affect of ill-posedness of associated problems on the regularity of solutions to problem (1), (2), consider the following examples.

Example 1. Let $\mathcal{E} = \{\mathbf{m}_{\boldsymbol{\sigma}} \mid \ \boldsymbol{\sigma} \in \boldsymbol{\Xi}\}$, $p_{\boldsymbol{\alpha}}$ be constants ($\boldsymbol{\alpha} \in \mathcal{E}$), and let p_0 and q be continuous functions such that

$$p_0(x_1, \dots, x_i + \omega_i, \dots, x_n) \equiv p_0(x_1, \dots, x_i, \dots, x_n) \ \ (i = 1, \dots, n), \quad (17)$$

$$q(x_1, \dots, x_i + \omega_i, \dots, x_n) \equiv q(x_1, \dots, x_i, \dots, x_n) \ \ (i = 1, \dots, n). \quad (18)$$

For the equation

$$u^{(\mathbf{2m})} = \sum_{\boldsymbol{\alpha} \in \mathcal{E}} p_{\boldsymbol{\alpha}}\, u^{(2\boldsymbol{\alpha})} + p_0(\mathbf{x})u + q(\mathbf{x}) \quad (19)$$

consider the periodic problem

$$u^{(k\mathbf{1}_i)}(x_1, \dots, 0, \dots, x_n) = u^{(k\mathbf{1}_i)}(x_1, \dots, \omega_i, \dots, x_n) \ \ (k = 0, \dots, 2m_i - 1; \ \ i = 1, \dots, n). \quad (20)$$

Assume that $(-1)^{\|\mathbf{m}\|+\|\boldsymbol{\alpha}\|} p_{\boldsymbol{\alpha}} < 0$ for $\boldsymbol{\alpha} \neq \mathbf{m}_n$, $p_{\mathbf{m}_n} = 0$, and

$$(-1)^{\|\mathbf{m}\|} p_0(\mathbf{x}) < 0, \ \ \mathbf{x} \in \Omega. \quad (21)$$

Then for $\boldsymbol{\sigma} = \mathbf{1}_n$ the associated problem $(3_{\boldsymbol{\sigma}}), (4_{\boldsymbol{\sigma}})$ (problem of level $n-1$) has a nontrivial solution. As a result, problem (19), (20) is ill-posed. It is solvable if $p_0, q \in C^{\mathbf{m}_n}(\Omega)$ and its unique solution u admits the estimates

$$\|u\|_{C^{\widehat{\mathbf{2m}_n}}(\Omega)} \leq M\|q\|_{C(\Omega)}$$

and

$$\|u\|_{C^{\mathbf{2m}}(\Omega)} \leq M\|q\|_{C^{\mathbf{2m}_n}}.$$

Example 2. Let $m_1 = m_2 = \cdots = m_n = m$, p_0 and q satisfy (17), (18) and (21). The problem

$$u^{(2\mathbf{m})} = (-1)^{mn+m-1} \sum_{i=1}^{n} u^{(2\mathbf{m}_i)} + p_0(\mathbf{x})u + q(\mathbf{x}), \tag{22}$$

$$u^{(k\mathbf{1}_i)}(x_1, \dots, 0, \dots, x_n) = u^{(k\mathbf{1}_i)}(x_1, \dots, \omega_i, \dots, x_n) \;\; (k = 0, \dots, 2m-1; \;\; i = 1, \dots, n). \tag{23}$$

is ill-posed because all of its associated problems are ill-posed. As a result, problem (22), (23) is solvable if $p_0, q \in C^{2\mathbf{m}}(\Omega)$ and its unique solution u admits the estimates

$$\|u\|_{C^{2\mathbf{m}}(\Omega)} \leq M \|q\|_{C^{2\mathbf{m}}},$$

and

$$\|u\|_{C^{m,\gamma}(\Omega)} \leq M \|q\|_{C^{\gamma}(\Omega)},$$

where $\gamma \in (0,1)$, and $C^{k,\gamma}(\Omega)$ is the space of k times continuously differentiable functions whose kth derivative is Hölder continuous with the exponent γ.

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Local Solvability of Multi Dimensional Initial-Boundary Value Problems for Higher Order Nonlinear Hyperbolic Equations

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Let $m_1, \dots, m_n$ be positive integers. In the n-dimensional box $\Omega = [0, \omega_1] \times \cdots \times [0, \omega_n]$ for the nonlinear hyperbolic equation

$$u^{(\mathbf{m})} = f(\mathbf{x}, \widehat{\mathcal{D}}^{\mathbf{m}}[u]) \tag{1}$$

consider the initial-boundary conditions

$$\begin{gathered} h_{ik}(u(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n)) = \varphi_{ik}(\widehat{\mathbf{x}}_i) \ \ (k = 1, \dots, m_i, \ \ i = 1, \dots, n-1), \\ u^{(0,\dots,0,k-1)}(x_1, \dots, x_{n-1}, 0) = \varphi_{nk}(\widehat{\mathbf{x}}_n) \ \ (k = 1, \dots, m_n). \end{gathered} \tag{2}$$

Here $\mathbf{x} = (x_1, \dots, x_n)$, $\widehat{\mathbf{x}}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\widehat{\mathbf{m}}_i = \mathbf{m} - \mathbf{m}_i$ and $\mathbf{m}_i = (0, \dots, m_i, \dots, 0)$ are multi-indices,

$$u^{(\alpha)}(\mathbf{x}) = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} u(\mathbf{x})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\,,$$

$\mathcal{D}^{\mathbf{m}}[u] = (u^{(\alpha)})_{\alpha \le \mathbf{m}}$, $\widehat{\mathcal{D}}^{\mathbf{m}}[u] = (u^{(\alpha)})_{\alpha < \mathbf{m}}$, $\Omega_i = [0, \omega_1] \times \cdots \times [0, \omega_{i-1}] \times [0, \omega_{i+1}] \times \cdots \times [0, \omega_n]$, $f \in C(\Omega \times \mathbb{R}^{m_1 \times \cdots \times m_n})$, $h_{ik} : C^{m_i - 1}([0, \omega_i]) \to \mathbb{R}$ $(k = 1, \dots, m_i;\ i = 1, \dots, n-1)$ are bounded linear functionals, and $\varphi_{ik} \in C^{\widehat{\mathbf{m}}_i}(\Omega_i)$ $(k = 1, \dots, m_i;\ i = 1, \dots, n)$. Furthermore, it is assumed that the functions φ_{ik} satisfy the following consistency conditions:

$$h_{ik}(\varphi_{jl})(\widehat{\mathbf{x}}_{ij}) \equiv h_{jl}(\varphi_{ik})(\widehat{\mathbf{x}}_{ij}) \ \ (k = 1, \dots, m_i; \ \ l = 1, \dots, m_j; \ \ i, j = 1, \dots, n),$$

where $\widehat{\mathbf{x}}_{ij} = \mathbf{x} - \widehat{\mathbf{x}}_i - \widehat{\mathbf{x}}_j$.

Set:

$$\mathbf{Z} = (z_\alpha)_{\alpha < \mathbf{m}}; \quad f_\alpha(\mathbf{x}, \mathbf{Z}) = \frac{\partial f(\mathbf{x}, \mathbf{Z})}{\partial z_\alpha}\,.$$

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \Upsilon_{\mathbf{m}} \iff \alpha_i = m_i \ \text{ for some } \ (i = 1, \dots, n).$$

The variables z_α $(\alpha \in \Upsilon_{\mathbf{m}})$ are called ***principal phase variables*** of the function $f(\mathbf{x}, \mathbf{Z})$.

By a solution of problem (1), (2) we understand a classical solution, i.e., a function $u \in C^{\mathbf{m}}(\Omega)$ satisfying equation (1) and boundary conditions (2).

Two-dimensional initial-boundary value problems were studied in [4, 5].

Definition. Let $\mathbf{n} = (n_1, \dots, n_r)$, $\Omega = [0, \omega_1] \times [0, \dots, \omega_r]$, $\mathbf{y} = (y_1, \dots, y_r)$, and let the function $g : C(\Omega \times \mathbb{R}^{n_1 \times \cdots \times n_r})$ be continuously differentiable with respect to the phase variables. A solution $v_0 \in C^n(\Omega)$ of the problem

$$v^{(\mathbf{n})} = g(\mathbf{y}, \widehat{\mathcal{D}}^{\mathbf{n}}[v]), \tag{3}$$

$$h_{ij}\big(v(y_1, \dots, y_{i-1}, \bullet, y_{i+1}, \dots, y_r)\big) = \psi_{ij}(\widehat{\mathbf{y}}_i) \ \ (j = 1, \dots, n_i; \ \ i = 1, \dots, r) \tag{4}$$

is called ***strongly isolated*** if the linearized problem

$$v^{(\mathbf{n})}=\sum_{\alpha<\mathbf{n}}p_\alpha(\mathbf{y})v^{(\alpha)},$$
$$h_{ij}\big(v(y_1,\dots,y_{i-1},\bullet,y_{i+1},\dots,y_r)\big)=0\ \ (j=1,\dots,n_i;\ \ i=1,\dots,r),$$

where $p_\alpha(\mathbf{y})=g_\alpha(\mathbf{y},\widehat{\mathcal{D}}^{\mathbf{k}}[v_0(\mathbf{y})])$, is well-posed.

Well-posed multi-dimensional boundary value problems for higher order linear hyperbolic equations were studied in [2].

The concept of a strongly isolated solution is closely related to the concept of strongly well-posedness. Strong well-posedness of two-dimensional boundary value problems for higher order nonlinear hyperbolic equations were introduced in [3].

Theorem 1. *Let the function f be continuously differentiable with respect to the phase variables, and let v_0 be a strongly isolated solution of the problem*

$$v^{(\widehat{\mathbf{m}}_n)}=p\big(\widehat{\mathbf{x}}_\mathbf{n},\widehat{\mathcal{D}}^{\widehat{\mathbf{m}}_n}[v]\big),\tag{5}$$
$$h_{ik}\big(u(x_1,\dots,x_{i-1},\bullet,x_{i+1},\dots,x_{n-1})\big)=\varphi_{ik}^{(m_n)}(\widehat{\mathbf{x}}_{ni})\ \ (k=1,\dots,m_i;\ \ i=1,\dots,n-1),\tag{6}$$

where

$$p\big(\widehat{\mathbf{x}}_\mathbf{n},\widehat{\mathcal{D}}^{\widehat{\mathbf{m}}_n}[v]\big)=f\Big(x_1,\dots,x_{n-1},0,\mathcal{D}^{\mathbf{m}-\mathbf{1}_n}[u_0](x_1,\dots,x_{n-1},0),\widehat{\mathcal{D}}^{\widehat{\mathbf{m}}_n}[v]\Big),$$

$\widehat{\mathbf{x}}_{ni}=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n-1})$ *and* $\mathbf{1}_n=(0,\dots,0,1)$*. Then there exists $\delta\in(0,\omega_n]$ such that in the set $\Omega_\delta=[0,\omega_1]\times\cdots\times[0,\omega_{n-1}]\times[0,\delta]$ problem* (1), (2) *has a unique solution u satisfying the condition*

$$u^{(\mathbf{m}_n)}(x_1,\dots,x_{n-1},0)=v_0(x_1,\dots,x_{n-1}).\tag{7}$$

Consider the "perturbed" equation

$$u^{(\mathbf{m})}=f\big(\mathbf{x},\widehat{\mathcal{D}}^{\mathbf{m}}[u]\big)+x_nq\big(\mathbf{x},\widehat{\mathcal{D}}^{\mathbf{m}}[u]\big).\tag{8}$$

Theorem 2. *Let the conditions of Theorem 1 hold, and let the function $q(\mathbf{x},\mathbf{Z})$ be continuously differentiable with respect to the principal phase variables z_α ($\alpha\in\Upsilon_\mathbf{m}$). Then there exists $\delta\in(0,\omega_n]$ such that in the set $\Omega_\delta=[0,\omega_1]\times\cdots\times[0,\omega_{n-1}]\times[0,\delta]$ problem* (8), (2) *has a at least one solution u satisfying condition* (7)*. Moreover, if the function q is locally Lipschitz continuous with respect to the rest of the phase variables, then such solution is unique.*

The following is a particular case of conditions (2):

$$h_{1k}\big(u(x_1,\dots,x_{i-1},\bullet,x_{i+1},\dots,x_n)\big)=\varphi_{1k}(\widehat{\mathbf{x}}_i)\ \ (k=1,\dots,m_i),$$
$$u^{(0,\dots,k_i-1,\dots,0)}(x_1,\dots,0,\dots,x_n)=\varphi_{ik_i}(\widehat{\mathbf{x}}_i)\ \ (k_i=1,\dots,m_i;\ \ i=2,\dots,n).\tag{9}$$

Corollary. *Let the function f be continuously differentiable with respect to the phase variables, and let v_0 be a strongly isolated solution of the problem*

$$v^{(m_1)}=p(x_1,v,v',\dots,v^{m_1-1}),$$
$$h_{1k}(v)=\varphi_{1k}^{(\widehat{\mathbf{m}}_1)}(\mathbf{0})\ \ (k=1,\dots,m_1),$$

where

$$p(x_1,v,v',\dots,v^{m_1-1})=f\Big(x_1,0,\dots,0,\widehat{\mathcal{D}}^{\widehat{\mathbf{m}}_1}\big[\mathcal{D}^{\mathbf{m}_1}[u_0]\big](x_1,0,\dots,0),v,v',\dots,v^{m_1-1}\Big).$$

Then there exist $\delta_i\in(0,\omega_i]$ ($i=2,\dots,n$) such that in the set $\Omega_{\delta_2\cdots\delta_n}=[0,\omega_1]\times[0,\delta_2]\times\cdots\times[0,\delta_n]$ problem (1), (7) *has a unique solution u satisfying the condition*

$$u^{(\widehat{\mathbf{m}}_1)}(x_1,0,\dots,0)=v_0(x_1).$$

Remark. In Theorem 1 the requirement of strong isolation of the solution v_0 cannot be replaced by well-posedness of problem (5), (6). In order to illustrate this, consider the problem

$$u^{(1,1)} = (u^{(0,1)})^3 - y^2 u^{(0,1)}, \tag{10}$$

$$u(\omega_1, y) - u(0, y) = \int\limits_0^y t \sin\frac{1}{t}\, dt, \quad u(x, 0) = 0. \tag{11}$$

For this case problem (5), (6) is the following one:

$$v' = v^3, \quad v(\omega_1) - v(0) = 0. \tag{12}$$

By Corollary 4.2 and Theorem 4.4 from [1], problem (12) has a unique solution $v_0(y) \equiv 0$ and is well-posed. On the other hand, it is clear, that $v_0(y) \equiv 0$ is not strongly isolated.

Our goal is to show that problem (10), (11) has no solution in the rectangle $\Omega_\delta = [0, \omega_1] \times [0, \delta]$ no matter how small $\delta > 0$ is.

Assume the contrary that problem (10), (11) has a solution u in Ω_δ for some $\delta > 0$. Then for an arbitrarily fixed $y \in (0, \delta]$, the function $v(\,\cdot\,) = u^{(0,1)}(\,\cdot\,, y)$ is a solution of the problem

$$v' = v^3 - y^2 v, \tag{13}$$

$$v(\omega_1) - v(0) = y \sin\frac{1}{y}\,. \tag{14}$$

containing the parameter $y \in [0, \omega_2]$. Moreover, if problem (10), (11) has a solution, then v is a solution (13), (14) depending continuously on the parameter y.

For every fixed $y \in (0, \delta]$ equation (13) has three constant solutions: $v_0(x) = 0$, $v_1(x) = y$ and $v_2(x) = -y$. Due to the existence and uniqueness theorem, a nonconstant solution v of equation (13) intersects v_0, v_1 or v_2, and thus $v'(x) \neq 0$ for $x \in [0, \omega_1]$. Let

$$k > \frac{1}{2\pi\delta} \ \text{ and } \ x \in \Big(\frac{1}{\pi + 2\pi k}, \frac{1}{2\pi k}\Big).$$

Then $v(\omega_1) > v(0)$ and $v'(x) > 0$ for $x \in [0, \omega_1]$. Therefore, either

$$v(x) > y \ \text{ for } \ x \in [0, \omega_1],$$

or

$$v(x) \in (-y, 0) \ \text{ for } \ x \in [0, \omega_1].$$

If $y = \frac{1}{\frac{\pi}{2} + 2\pi k}$, then $v(\omega_1) - v(0) = y$, and consequently,

$$v(x) \notin (-y, 0) \ \text{ for } \ x \in [0, \omega_1].$$

From the aforesaid, in view of continuity of $u^{(0,1)}$ in Ω_δ, it follows that

$$u^{(0,1)}(x, y) > y \ \text{ for } \ y \in \Big(\frac{1}{\pi + 2\pi k}, \frac{1}{2\pi k}\Big).$$

Similarly, one can show that

$$u^{(0,1)}(x, y) < -y \ \text{ for } \ y \in \Big(\frac{1}{2\pi(k+1)}, \frac{1}{\pi + 2\pi k}\Big).$$

However, the latter two inequalities imply that $u^{(0,1)}(x, y)$ is discontinuous along the lines $y = \frac{1}{\pi k}$ $(k = 1, 2, \dots)$. Thus we have proved that problem (10), (11) has no solution in Ω_δ for any $\delta > 0$.

In conclusion, as examples, consider the following initial-boundary value problems.

Example 1.

$$u^{(2,2,1)} = u^2 u^{(2,0,1)} + (u^{(1,1,0)})^4 u^{(0,2,1)} - (u^{(0,0,1)})^6 + q\big(x_1, x_2, x_3, u, u^{(1,0,0)}, u^{(0,1,0)}, u^{(1,1,0)}\big), \tag{15}$$

$$u(0, x_2, x_3) = 0, \quad u(\omega_1, x_2, x_3) = 0; \quad u(x_1, 0, x_3) = 0, \quad u(x_1, \omega_2, x_3) = 0;$$
$$u(x_1, x_2, 0) = \psi(x_1, x_2). \tag{16}$$

Let the function q be continuous. Then, by Corollary 4 from [2] and Theorem 2, there exists $\delta \in (0, \omega_3]$ such that in the set $\Omega_\delta = [0, \omega_1] \times [0, \omega_2] \times [0, \delta]$ problem $(15), (16)$ has a at least one solution. Moreover, if the function q is locally Lipschitz continuous with respect to the phase variables, then problem $(15), (16)$ is uniquely solvable.

Example 2.

$$u^{(2,2,1)} = u^{(2,0,1)} + (u^{(2,0,1)})^5 + u^{(0,2,1)} - u^{(0,0,1)} + q\big(x_1, x_2, x_3, u, u^{(1,0,0)}, u^{(0,1,0)}, u^{(1,1,0)}\big), \tag{17}$$

$$u^{(i,0,0)}(0, x_2, x_3) = u^{(i,0,0)}(\omega_1, x_2, x_3); \quad u^{(0,i,0)}(x_1, 0, x_3) = u^{(0,i,0)}(x_1, \omega_2, x_3) \ \ (i = 0, 1);$$
$$u(x_1, x_2, 0) = \psi(x_1, x_2). \tag{18}$$

Let the function q be continuous. Then, by Corollary 5 from [2] and Theorem 2, there exists $\delta \in (0, \omega_3]$ such that in the set $\Omega_\delta = [0, \omega_1] \times [0, \omega_2] \times [0, \delta]$ problem $(17), (18)$ has a at least one solution. Moreover, if the function q is locally Lipschitz continuous with respect to the phase variables, then problem $(17), (18)$ is uniquely solvable.

Example 3. Let p_α be smooth functions, q be a continuous function, $\mathbf{m} = (m_1, \dots, m_n, 0)$ and $\mathbf{1}_{n+1}) = (0, \dots, 0)$. For the equation

$$u^{(2\mathbf{m}+\mathbf{1}_{n+1})} = \sum_{\alpha<\mathbf{m}} \big(p_\alpha(x, u) u^{(\alpha+\mathbf{1}_{n+1})}\big)^{(\alpha)} + q\big(\mathbf{x}, \mathcal{D}^{\mathbf{2m}\text{ - }\mathbf{1}}[u]\big). \tag{19}$$

consider the initial-boundary value problems with the Dirichlet and periodic boundary conditions

$$u^{(k\mathbf{1}_i)}(x_1, \dots, 0, \dots, x_{n+1}) = 0, \quad u^{(k\mathbf{1}_i)}(x_1, \dots, \omega_i, \dots, x_{n+1}) = 0$$
$$(k = 0, \dots, m_i - 1;\ i = 1, \dots, n); \quad u(x_1, \dots, x_n, 0) = \varphi(x), \tag{20}$$

and

$$u^{(k\mathbf{1}_i)}(x_1, \dots, 0, \dots, x_{n+1}) = u^{(k\mathbf{1}_i)}(x_1, \dots, \omega_i, \dots, x_{n+1})$$
$$(k = 0, \dots, 2m_i - 1;\ \ i = 1, \dots, n); \quad u(x_1, \dots, x_n, 0) = \varphi(x). \tag{21}$$

Let $(-1)^{\|\mathbf{m}\|+\|\alpha\|} p_\alpha \le 0$ $((-1)^{\|\mathbf{m}\|+\|\alpha\|} p_\alpha < 0)$ for $\alpha < \mathbf{m}$. Then, by Theorem 2 from [2] and Theorem 2, there exists $\delta \in (0, \omega_{n+1}]$ such that in the set $\Omega_\delta = [0, \omega_1] \times \cdots \times [0, \omega_n] \times [0, \delta]$ problem $(19), (20)$ (problem $(19), (21)$) has a at least one solution. Moreover, if the function q is locally Lipschitz continuous with respect to the phase variables, then problem $(19), (20)$ (problem $(19), (21)$) is uniquely solvable.

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Convergence of Finite Difference Scheme and Uniqueness of a Solution for One System of Nonlinear Integro-Differential Equations with Source Terms

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Mathematical models of diffusive processes lead to nonstationary partial integro-differential equations and systems of those equations. Most of those problems, as a rule, are nonlinear. This moment significantly complicates the investigation of such models.

Our goal is to investigate and study numerical solutions of nonlinear integro-differential diffusion system, which appears at mathematical modeling of process of electromagnetic field propagation into a substance. The main characteristic of the corresponding system of Maxwell's equations is that it contains equations, which are strongly connected to each other. This circumstance dictates to use the corresponding investigation methods for each concrete model, as the general theory even for such linear systems is not yet fully developed. Naturally, the questions of numerical solution of these problems, which also are connected with serious complexities, arise as well.

In particular, our purpose is to study the following system of nonlinear integro-differential equations with source terms:

$$\begin{gathered}
\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\bigg[\bigg(1+\int\limits_0^t\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,d\tau\bigg)^p\,\frac{\partial U}{\partial x}\bigg]+|U|^{q-2}U=0,\\
\frac{\partial V}{\partial t}-\frac{\partial}{\partial x}\bigg[\bigg(1+\int\limits_0^t\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,d\tau\bigg)^p\,\frac{\partial V}{\partial x}\bigg]+|V|^{q-2}V=0,
\end{gathered}\tag{1}$$

where $0<p\le 1$, $q\ge 2$.

System above is obtained by adding the source terms to the resulting model which is derived after reduction of well-known Maxwell's equations to the system of nonlinear integro-differential equations. Such a reduction at first was made in [4].

The system of Maxwell's equation can be done in the following form [12]:

$$\frac{\partial H}{\partial t}=-\operatorname{rot}(\nu_m\operatorname{rot}H),\tag{2}$$

$$c_\nu\,\frac{\partial\theta}{\partial t}=\nu_m(\operatorname{rot}H)^2,\tag{3}$$

where $H=(H_1,H_2,H_3)$ is a vector of the magnetic field, θ is temperature, c_ν and ν_m characterizes the thermal heat capacity and electro-conductivity of the substance, respectively.

While propagating in the medium, variable magnetic field induces a variable electric field that generates a current. The current causes increase the medium's temperature, which should be taken into account for further investigations. Thus, we can say that coefficients $c_\nu=c_\nu(\theta)$ and

$\nu_m = \nu_m(\theta)$ are functions of temperature. After integration of (3) by time and substituting it in (2) the corresponding system of Maxwell's equations can be reduced to the following integro-differential form [4]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\left[a\left(\int_0^t |\operatorname{rot} H|^2\,d\tau\right)\operatorname{rot} H\right]. \tag{4}$$

The literature on the questions of existence, uniqueness, and regularity of solutions to the equations of the above types is very rich. In [1,2,4] the solvability of the first boundary value problem for scalar cases is studied using a modified version of the Galerkin's method and compactness arguments that are used in [13,14] for investigation of elliptic and parabolic models. The uniqueness of the solutions is investigated also in [1, 2, 4]. The asymptotic behavior of solutions is discussed in [3, 6–9, 11] and in a number of other works as well. Note also that to numerical resolution of (4) type one-dimensional equations were devoted many works as well, see, e.g., [5, 9–11] and references therein.

If we consider two component magnetic field $H = (0, U, V)$, where $U = U(x, t)$ and $V = V(x, t)$, then from (3) we get system of nonlinear integro-differential equations (1) with source terms. In the domain $[0, 1] \times [0, \infty)$ for system (1) let us consider the following initial-boundary value problem:

$$U(0,t) = U(1,t) = V(0,t) = V(1,t) = 0, \quad t \ge 0, \tag{5}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad x \in [0,1], \tag{6}$$

where U_0 and V_0 are given functions.

Let us consider the semi-discrete scheme for problem (1), (5), (6). On $[0, 1]$ let us introduce a net with mesh points denoted by $x_i = ih$, $i = 0, 1, \dots, M$, with $h = 1/M$. The boundaries are specified by $i = 0$ and $i = M$. The semi-discrete approximation at (x_i, t) are designed by $u_i = u_i(t)$ and $v_i = v_i(t)$. The exact solution to the problem at (x_i, t) is denoted by $U_i = U_i(t)$ and $V_i = V_i(t)$. At points $i = 1, 2, \dots, M-1$, the integro-differential equation will be replaced by approximation of the space derivatives by a forward and backward differences.

Let us correspond to (1), (5), (6) problem the following semi-discrete scheme:

$$\begin{gathered}
\left.\begin{aligned}
&\frac{du_i}{dt} - \left[\left(1 + \int_0^t \left[(u_{\overline{x},i})^2 + (v_{\overline{x},i})^2\right] d\tau\right)^p u_{\overline{x},i}\right]_x + |u_i|^{q-2}u_i = 0,\\
&\frac{dv_i}{dt} - \left[\left(1 + \int_0^t \left[(u_{\overline{x},i})^2 + (v_{\overline{x},i})^2\right] d\tau\right)^p v_{\overline{x},i}\right]_x + |v_i|^{q-2}v_i = 0,
\end{aligned}\right\} \quad i = 1, 2, \dots, M-1,\\
u_0(t) = u_M(t) = v_0(t) = v_M(t) = 0,\\
u_i(0) = U_{0,i}, \qquad v_i(0) = V_{0,i}, \quad i = 0, 1, \dots, M,
\end{gathered} \tag{7}$$

where

$$r_x = \frac{r_{i+1} - r_i}{h}, \quad r_{\overline{x}} = \frac{r_i - r_{i-1}}{h}.$$

So, we obtained the Cauchy problem (7) for nonlinear system of ordinary integro-differential equations.

It is not difficult to obtain the following estimates:

$$\|u(t)\|^2 + \int_0^t \|u_{\overline{x}}]|^2\,d\tau \le C, \quad \|v(t)\|^2 + \int_0^t \|v_{\overline{x}}]|^2\,d\tau \le C, \tag{8}$$

where

$$\|w(t)\|^2 = \sum_{i=1}^{M-1} w_i^2(t)h, \quad \|w_{\overline{x}}]|^2 = \sum_{i=1}^{M} w_{\overline{x},i}^2(t)h.$$

The a priori estimates (8) guarantee the global solvability of problem (7).

The following statement is true.

Theorem 1. *If* $0 < p \le 1$, $q \ge 2$ *and problem* $(1),(5),(6)$ *have a sufficiently smooth solution* $U(x,t)$, $V(x,t)$, *then the solution of problem* (7)

$$u = u(t) = (u_1(t), u_2(t), \dots, u_{M-1}(t)), \quad v = v(t) = (v_1(t), v_2(t), \dots, v_{M-1}(t))$$

tends to

$$U = U(t) = (U_1(t), U_2(t), \dots, U_{M-1}(t)), \quad V = V(t) = (V_1(t), V_2(t), \dots, V_{M-1}(t))$$

as $h \to 0$ *and the following estimates are true:*

$$\|u(t) - U(t)\| \le Ch, \quad \|v(t) - V(t)\| \le Ch.$$

Now let us consider the fully discrete scheme for problem $(1),(5),6$. On $[0,1] \times [0,T]$ let us introduce a net with mesh points denoted by $(x_i, t_j) = (ih, j\tau)$, where $i = 0,1,\dots,M$; $j = 0,1,\dots,N$ with $h = 1/M$, $\tau = T/N$. The initial line is denoted by $j = 0$. The discrete approximation at (x_i, t_j) is designed by (u_i^j, v_i^j) and the exact solution to problem $(1),(5),(6)$ by (U_i^j, V_i^j).

For problem $(1),(5),(6)$ let us consider the following finite difference scheme:

$$\begin{gathered}
\frac{u_i^{j+1} - u_i^j}{\tau} - \bigg\{ \Big(1 + \tau \sum_{k=1}^{j+1} \big[(u_{\overline{x},i}^k)^2 + (v_{\overline{x},i}^k)^2\big]\Big)^p u_{\overline{x},i}^{j+1} \bigg\}_x + |u_i^{j+1}|^{q-2} u_i^{j+1} = f_{1,i}^j, \\
\frac{v_i^{j+1} - v_i^j}{\tau} - \bigg\{ \Big(1 + \tau \sum_{k=1}^{j+1} \big[(u_{\overline{x},i}^k)^2 + (v_{\overline{x},i}^k)^2\big]\Big)^p v_{\overline{x},i}^{j+1} \bigg\}_x + |v_i^{j+1}|^{q-2} v_i^{j+1} = f_{2,i}^j, \\
i = 1,2,\dots,M-1; \;\; j = 0,1,\dots,N-1, \\
u_0^j = u_M^j = v_0^j = v_M^j = 0, \;\; j = 0,1,\dots,N, \\
u_i^0 = U_{0,i}, \quad v_i^0 = V_{0,i}, \;\; i = 0,1,\dots,M.
\end{gathered} \tag{9}$$

Multiplying equations in (9) scalarly by u_i^{j+1} and v_i^{j+1} respectively, it is not difficult to get the inequalities:

$$\|u^n\|^2 + \sum_{j=1}^{n} \|u_{\overline{x}}^j]|^2 \tau < C, \quad \|v^n\|^2 + \sum_{j=1}^{n} \|v_{\overline{x}}^j]|^2 \tau < C, \;\; n = 1,2,\dots,N, \tag{10}$$

where here and below C is a positive constant independent from τ and h.

The a priori estimates (10) guarantee the stability of scheme (9). The main statement of this note can be stated as follows.

Theorem 2. *If* $0 < p \le 1$, $q \ge 2$ *and problem* $(1),(5),(6)$ *has a sufficiently smooth solution* $(U(x,t), V(x,t))$, *then the solution* $u^j = (u_1^j, u_2^j, \dots, u_M^j)$, $v^j = (v_1^j, v_2^j, \dots, v_M^j)$, $j = 1,2,\dots,N$ *of the difference scheme* (9) *tends to the solution of the continuous problem* $(1),(5),(6)$ $U^j = (U_1^j, U_2^j, \dots, U_M^j)$, $V^j = (V_1^j, V_2^j, \dots, V_M^j)$, $j = 1,2,\dots,N$ *as* $\tau \to 0$, $h \to 0$ *and the following estimates are true:*

$$\|u^j - U^j\| \le C(\tau + h), \quad \|v^j - V^j\| \le C(\tau + h).$$

We have carried out numerous numerical experiments for problem $(1),(5),(6)$ with different kind of right hand sides and initial-boundary conditions.

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Asymptotics of Solutions of One Class of n-th Order Differential Equations with Regularly Varying Nonlinearities

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Consider the differential equation

$$y^{(n)} = \alpha p(t) \prod_{j=0}^{n-1} \varphi_j(y^{(j)}), \tag{1}$$

where $n \geq 2$, $\alpha \in \{-1, 1\}$, $p : [a, +\infty[\to]0, +\infty[$ is a continuous function, $a \in \mathbb{R}$, $\varphi_j : \Delta Y_j \to]0; +\infty[$ are continuous functions regularly varying, as $y^{(j)} \to Y_j$, of order σ_j, $j = \overline{0, n-1}$, ΔY_j is a one-sided neighborhood of the point Y_j, $Y_j \in \{0, \pm\infty\}$[1].

Among the set of monotone solutions of equation (1), defined in some neighborhood of $+\infty$, there might also be solutions for each of which there exists a number $k \in \{1, \dots, n\}$ such that

$$y^{(n-k)}(t) = c + o(1) \ \ (c \neq 0) \ \text{ as } \ t \to +\infty. \tag{2}$$

There have been obtained some results concerning the existence of solutions of type (2) in Corollaries 8.2, 8.6, 8.12 [4, Ch. II, § 8, pp. 207, 214, 223] and Corollaries 9.3, 9.7 [4, Ch. II, § 9, pp. 230, 233] of the monograph by I. T. Kiguradze and T. A. Chanturiya for the equations of general type, in Theorem 16.9 [4, Ch. IV, § 16, p. 321] for the differential equations of Emden–Fauler type. However, these results provide for a considerably strict restriction to the $(n-k+1)$-st derivative of a solution.

In the present paper, a question of performance of new results with less strict restrictions is investigated. When $k = 1, 2$, or the functions $\varphi_i(y^{(i)})$ $(i = \overline{n-k+1, n-2}\,)$ tend to the positive constants, as $y^{(i)} \to Y_i$, in the works [2] and [5] the necessary and sufficient existence conditions of solutions of type (2) of equation (1) and their asymptotic behaviour were obtained without any additional assumptions for these solutions. Otherwise, the new rather wide class of so-called $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions, $-\infty \leq \lambda_0 \leq +\infty$, of equation (1) has been assigned in the paper [3] as follows.

Definition. A solution y of the differential equation (1) is called (for $k \in \{3, \dots, n\}$) a $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution, where $-\infty \leq \lambda_0 \leq +\infty$, if it is defined on the interval $[t_0, +\infty[\subset [a, +\infty[$ and satisfies the conditions

$$\lim_{t\to+\infty} y^{(n-k)}(t) = c \ \ (c \neq 0), \quad \lim_{t\to+\infty} \frac{[y^{(n-1)}(t)]^2}{y^{(n-2)}(t) y^{(n)}(t)} = \lambda_0. \tag{3}$$

In accordance with its asymptotic properties the set of all $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions of equation (1) breaks up to the $k+1$ $(k \in \{3, \dots, n\}\,)$ disjoint subsets (see [1]) that correspond to the subsequent values of the parameter λ_0:

$$\lambda_0 \in \mathbb{R} \setminus \Big\{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\Big\}, \quad \lambda_0 = \pm\infty, \quad \lambda_0 = 1,$$
$$\lambda_0 = \frac{n-j-1}{n-j}\,, \ \ j \in \{n-k+2, \dots, n-1\}.$$

[1]For $Y_j = \pm\infty$ here and in the sequel, all numbers in the neighborhood of ΔY_j are assumed to have constant sign.

The case $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\}$ has been studied in the work [3]. The aim of the present paper is to investigate the question of existence and asymptotic behaviour of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions ($k \in \{3, \dots, n\}$) of equation (1) in special case $\lambda_0 \in \{1, \pm\infty\}$. The asymptotic, as $t \to +\infty$, formulas of their derivatives of order up to $n-1$ will be obtained too. Moreover, a question on the quantity of the studied solutions will be solved.

It is significant to note that by virtue of the results obtained by V. M. Evtukhov [1], the solutions of equation (1) satisfy the following a priori asymptotic conditions.

Lemma. *Let $k \in \{3, \dots, n\}$ and $y : [t_{0k}, +\infty[\to \mathbb{R}$ be an arbitrary $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution of equation* (1)*. Then the following, as $t \to +\infty$, assertions hold:*

- *if $\lambda_0 = \pm\infty$, then*

$$y^{(l-1)}(t) \sim \frac{t^{n-l}}{(n-l)!} y^{(n-1)}(t) \ \ (l = \overline{n-k+2, n-1}\,), \quad y^{(n)}(t) = o\Big(\frac{y^{(n-1)}(t)}{t}\Big);$$

- *if $\lambda_0 = 1$, then*

$$\frac{y^{(n-k+2)}(t)}{y^{(n-k+1)}(t)} \sim \frac{y^{(n-k+3)}(t)}{y^{(n-k+2)}(t)} \sim \dots \sim \frac{y^{(n)}(t)}{y^{(n-1)}(t)} \ \ \textit{and} \ \ \lim_{t\to+\infty} \frac{t y^{(n-k+2)}(t)}{y^{(n-k+1)}(t)} = +\infty.$$

It readily follows from the form of equation (1) that $y^{(n)}(t)$ has a constant sign in some neighborhood of $+\infty$. Then $y^{(n-l)}(t)$ $(l = \overline{1, k-1}\,)$ are strictly monotone functions in the neighborhood of $+\infty$ and, by virtue of (2), can tend only to zero, as $t \to +\infty$. Therefore, it is necessary that

$$Y_{j-1} = 0 \ \text{ for } \ j = \overline{n-k+2, n}. \tag{4}$$

Let us assume here and in the sequel that the numbers μ_j $(j = \overline{0, n-1}\,)$, defined in the following way

$$\mu_j = \begin{cases} 1 & \text{if } Y_j = +\infty, \text{ or } Y_j = 0 \text{ and } \Delta Y_j \text{ is a right neighborhood of the point } 0, \\ -1 & \text{if } Y_j = -\infty, \text{ or } Y_j = 0 \text{ and } \Delta Y_j \text{ is a left neighborhood of the point } 0 \end{cases}$$

are such that

$$\mu_j \mu_{j+1} > 0 \ \text{ for } \ j = \overline{0, n-k-1}, \quad \mu_j \mu_{j+1} < 0 \ \text{ for } \ j = \overline{n-k+1, n-2}, \tag{5}$$

$$\alpha \mu_{n-1} < 0. \tag{6}$$

These conditions on μ_j $(j = \overline{0, n-1}\,)$ and α are necessary for the existence of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions of equation (1) as long as for each of them in some neighborhood of $+\infty$

$$\operatorname{sign} y^{(j)}(t) = \mu_j \ \ (j = \overline{0, n-1}\,), \ \ \operatorname{sign} y^{(n)}(t) = \alpha.$$

It is obvious that by virtue of the first relative (3), for these solutions the following representations

$$y^{(l-1)}(t) = \frac{c t^{n-l-k+1}}{(n-l-k+1)!} \, [1 + o(1)] \ \ (l = \overline{1, n-k}\,) \ \text{ as } \ t \to +\infty \tag{7}$$

hold, $c \in \Delta Y_{n-k}$ and then

$$Y_{j-1} = \begin{cases} +\infty & \text{if } \ \mu_{n-k} > 0, \\ -\infty & \text{if } \ \mu_{n-k} < 0, \end{cases} \quad \text{for } \ j = \overline{1, n-k}. \tag{8}$$

We say that a continuous function $L : \Delta Y_0 \to]0,+\infty[$, slowly varying as $y \to Y_0$, satisfies the condition S_0 if

$$L\big(\mu e^{[1+o(1)]\ln|y|}\big) = L(y)[1+o(1)] \ \text{ as } \ y \to Y_0 \ \ (y \in \Delta Y_0),$$

where $\mu = \operatorname{sign} y$.

The condition S_0 is necessarily satisfied for functions L that have a nonzero finite limit, as $y \to Y_0$, for functions of the form

$$L(y) = |\ln|y||^{\gamma_1}, \quad L(y) = |\ln|y||^{\gamma_1}\big|\ln|\ln|y||\big|^{\gamma_2},$$

where $\gamma_1, \gamma_2 \neq 0$, and for many other functions.

Consider the case $\lambda_0 = \pm\infty$. The following statement holds for equation (1).

Theorem 1. *For $k \in \{3,\dots,n\}$ equation* (1) *doesn't have $\mathcal{P}^k_{+\infty}(\pm\infty)$-solutions.*

To investigate the case $\lambda_0 = 1$, besides the above-mentioned facts about the functions, regularly and slowly varying as $y^{(j)} \to Y_j$ $(j = \overline{0,n-1}\,)$, we need the following auxiliary notations:

$$\gamma_k = 1 - \sum_{j=n-k+1}^{n-1} \sigma_j, \quad \nu_k = \sum_{j=n-k+1}^{n-2} \sigma_j(n-j-1), \quad M_k(c) = \prod_{j=1}^{n-k} \Big|\frac{c}{(n-j-k+1)!}\Big|^{\sigma_{j-1}},$$

$$I_k(t) = \varphi_{n-k}(c)M_k(c)\int_{A_{0k}}^{t} p(\tau)\prod_{j=0}^{n-k-1}\varphi_j(\mu_j\tau^{n-k-j})\,d\tau, \quad I_{1k}(t) = \int_{A_{1k}}^{t} I_k(\tau)\,d\tau,$$

where A_{0k} (A_{1k}) is chosen equal either to $a_{0k} \geq a$ $(a_{1k} \geq a_{0k})$ or to $+\infty$ so as to ensure that the integral tends either to zero or to $+\infty$ as $t \to +\infty$.

Theorem 2. *Let $k \in \{3,\dots,n\}$ and $\gamma_k \neq 0$. Then, for existence of $\mathcal{P}^k_{+\infty}(1)$-solutions of equation* (1), *it is necessary that $c \in \Delta Y_{n-k}$, along with* (4)–(6), (8) *the following conditions*

$$\frac{I_k'(t)}{I_k(t)} \sim \frac{I_k(t)}{I_{1k}(t)} \ \ \text{as } t \to +\infty, \quad \lim_{t\to+\infty} |I_k(t)|^{\frac{1}{\gamma_k}} = 0 \ \ (j = \overline{n-k+1,n-1}\,) \tag{9}$$

and the inequalities, as $t \in]a,+\infty[$,

$$\gamma_k I_k(t) < 0, \quad I_{1k}(t) > 0, \quad (-1)^{n-j-1}\mu_j\mu_{n-1} > 0 \ \ (j = \overline{n-k+1,n-3}\,) \tag{10}$$

hold. Moreover, each solution of that kind admits along with (2) *and* (7) *the asymptotic, as $t \to +\infty$, representations*

$$y^{(j)}(t) = \Big(\frac{\gamma_k I_{1k}(t)}{I_k(t)}\Big)^{n-j-1} y^{(n-1)}(t)[1+o(1)] \ \ (j = \overline{n-k+1,n-2}\,),$$

$$\frac{|y^{(n-1)}(t)|^{\gamma_k}}{\prod\limits_{j=n-k+1}^{n-1} L_j\Big(\big(\frac{\gamma_k I_{1k}(t)}{I_k(t)}\big)^{n-j-1} y^{(n-1)}(t)\Big)} = \alpha\mu_{n-1}\gamma_k I_k(t)\Big|\frac{\gamma_k I_{1k}(t)}{I_k(t)}\Big|^{\nu_k}[1+o(1)].$$

Theorem 3. *Let $k \in \{3,\dots,n\}$, $\gamma_k \neq 0$ and functions L_j $(j = \overline{n-k+1,n-1}\,)$, slowly varying as $y^{(j)} \to Y_j$, satisfy the condition S_0. Then, in case of existence of $\mathcal{P}^k_{+\infty}(1)$-solutions of equation* (1), *the following condition*

$$\int_{a_{2k}}^{+\infty} \Big(\frac{I_{1k}(\tau)}{I_k(\tau)}\Big)^{k-2} \Big|\gamma_k I_k(\tau)\Big|\frac{\gamma_k I_{1k}(\tau)}{I_k(\tau)}\Big|^{\nu_k} \prod_{j=n-k+1}^{n-1} L_j\big(\mu_j|I_k(\tau)|^{\frac{1}{\gamma_k}}\big)\Big|^{\frac{1}{\gamma_k}} d\tau < +\infty \tag{11}$$

holds, where $a_{2k} \ge a_{1k}$ *such that* $\mu_{j-1}|I_k(t)|^{\frac{1}{\gamma_k}} \in \Delta Y_{j-1}$ $(j = \overline{n-k+2, n}\,)$ *for* $t \ge a_{2k}$, *and each solution of that kind admits along with* (7) *the following asymptotic, as* $t \to +\infty$, *representations*

$$y^{(n-k)}(t) = c + \mu_{n-1}\gamma_k^{k-2} W_k(t)[1 + o(1)], \tag{12}$$

$$y^{(l-1)}(t) = \mu_{n-1}\gamma_k^{n-l}\Big(\frac{I_{1k}(t)}{I_k(t)}\Big)^{n-l-k+2} W_k'(t)[1 + o(1)] \ \ (l = \overline{n-k+2, n}\,), \tag{13}$$

where

$$W_k(t) = \int\limits_{+\infty}^{t} \Big(\frac{I_{1k}(\tau)}{I_k(\tau)}\Big)^{k-2} \bigg|\gamma_k I_k(\tau)\bigg|\frac{\gamma_k I_{1k}(\tau)}{I_k(\tau)}\bigg|^{\nu_k} \prod_{j=n-k+1}^{n-1} L_j\big(\mu_j |I_k(\tau)|^{\frac{1}{\gamma_k}}\big)\bigg|^{\frac{1}{\gamma_k}} d\tau.$$

In the next theorem the sufficient existence conditions of $\mathcal{P}_{+\infty}^k(1)$-solutions of equation (1) with mentioned in Theorem 3 asymptotic representations are presented.

Theorem 4. *Let* $k \in \{3, \dots, n\}$, $\gamma_k \ne 0$, $c \in \Delta Y_{n-k}$, *the conditions* (4)–(6), (8)–(10), (11) *hold and the functions* L_j $(j = \overline{n-k+1, n-1}\,)$, *slowly varying as* $y^{(j)} \to Y_j$, *satisfy the condition* S_0. *In addition, let the inequality* $\sigma_{n-1} \ne 1$ *holds and the algebraic relative to* ρ *equation*

$$\sum_{l=2}^{k-1} \sigma_{n-l}(\rho + 1)^{k-l-1} - (1 - \sigma_{n-1} + \rho)(\rho + 1)^{k-2} = 0 \tag{14}$$

has no roots with zero real part. Then equation (1) *has a* $(n - k + m)$*-parameter family of* $\mathcal{P}_{+\infty}^k(1)$*-solutions that admit the asymptotic, as* $t \to +\infty$, *representations* (7), (12), (13), *where* m *is a number of roots* (*taking into account divisible*) *with positive real part of the algebraic equation* (14).

Remark. In fact, the algebraic equation (14) has no roots with zero real part if

$$\sum_{l=2}^{k-1} |\sigma_{n-l}| < |1 - \sigma_{n-1}|.$$

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On the Existence of Some Solutions of Systems of Ordinary Differential Equations which is Partially Resolved Relatively to the Derivatives in the Case of Fixed Singularity

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Let us consider the system of ordinary differential equations:

$$A(z)Y' = B(z)Y + f(z, Y, Y'), \tag{0.1}$$

where matrices $A, B : D_1 \to \mathbb{C}^{p\times n}$, $D_1 = \{z : \ |z| < R_1, \ R_1 > 0\} \subset \mathbb{C}$, matrices $A(z)$, $B(z)$ are analytic in the domain D_{10}, $D_{10} = D_1 \setminus \{0\}$, the pencil of matrices $A(z)\lambda - B(z)$ is singular on the condition that $z \to 0$, function $f : D_1 \times G_1 \times G_2 \to \mathbb{C}^p$, where domains $G_k \subset \mathbb{C}^n$, $0 \in G_k$, $k = 1, 2$, function $f(z, Y, Y')$ is analytic in $D_{10} \times G_{10} \times G_{20}$, $G_{k0} = G_k \setminus \{0\}$, $k = 1, 2$.

The system of ordinary differential equations (0.1) that satisfies conditions $p < n$, $A(z)$ is analytic matrix in the domain D_1 and $\operatorname{rang} A(z) = p$ on condition that $z \in D_1$.

Let us consider the function

$$Y = \operatorname{col}\begin{pmatrix} Y_1 & Y_2 \end{pmatrix}, \quad Y_1 : D_1 \to \mathbb{C}^p, \quad Y_2 : D_1 \to \mathbb{C}^{n-p}, \quad Y_1 = \operatorname{col}\big(Y_{11}(z), \dots, Y_{1p}(z)\big),$$
$$Y_2 = \operatorname{col}\big(Y_{21}(z), \dots, Y_{2n-p})(z)\big).$$

Without restricting the generality, assume that matrices $A(z)$, $B(z)$ and vector-function $f(z, Y, Y')$ take the forms:

$$A(z) = \begin{pmatrix} A_1(z) & A_2(z) \end{pmatrix}, \quad B(z) = \begin{pmatrix} B_1(z) & B_2(z) \end{pmatrix}, \quad f(z, Y, Y') = f^*(z, Y_1, Y_2, Y_1', Y_2'),$$

$A_1 : D_1 \to \mathbb{C}^{p\times p}$, $A_2 : D_1 \to \mathbb{C}^{p\times(n-p)}$, $B_1 : D_1 \to \mathbb{C}^{p\times p}$, $B_2 : D_1 \to \mathbb{C}^{p\times(n-p)}$, $\det A_1(z) \neq 0$ on the condition that $z \in D_1$, $f^* : D_1 \times G_{11} \times G_{12} \times G_{21} \times G_{22} \to C^p$, $G_{j1} \times G_{j2} = G_j$, $G_{j1} \subset C^p$, $G_{j2} \subset C^{n-p}$, $j = 1, 2$.

In this view the system (0.1) may be written as:

$$Y_1' = A_1^{-1}(z)B_1(z)Y_1 + A_1^{-1}(z)B_2(z)Y_2 - A_1^{-1}(z)A_2(z)Y_2' + A_1^{-1}(z)f^*(z, Y_1, Y_2, Y_1', Y_2'). \tag{0.2}$$

Let us suppose that matrices $A_1^{-1}(z)B_1(z)$, $A_1^{-1}(z)A_2(z)$, $A_1^{-1}(z)B_2(z)$ are analytic in the domain D_{10} and have removable singularity in the point $z = 0$.

Let us introduce the following notation:

$$P(z) = A_1^{-1}(z)B_1(z),$$
$$F^*(z, Y_1, Y_2, Y_1', Y_2') = A_1^{-1}(z)B_2(z)Y_2 - A_1^{-1}(z)A_2(z)Y_2' + A_1^{-1}f^*(z, Y_1, Y_2, Y_1', Y_2'),$$

then the system (0.2) may be written as

$$Y_1' = P(z)Y_1 + F^*(z, Y_1, Y_2, Y_1', Y_2'), \tag{0.3}$$

where $P(z)$ is analytic matrix in the domain D_{10} and has removable singularity in the point $z = 0$, $F^*(z, Y_1, Y_2, Y_1', Y_2')$ is analytic vector-function in the domain $D_{10} \times G_{110} \times G_{120} \times G_{210} \times G_{220}$, $G_{jk0} = G_{jk} \setminus 0$, $j, k = 1, 2$.

Let us introduce the following classes of functions:

- By H_0^{n-p} we basically mean class of $(n-p)$-dimensional analytic in the domain D_{10} functions that have removable singularity in the point $z=0$.
- By H_r^{n-p} we basically mean class of $(n-p)$-dimensional analytic in the domain D_{10} functions that have pole of r-order in the point $z=0$.

We study the system (0.3) that satisfies the hypothesis that $Y_2(z)$ is arbitrary state function from given class of function.

Let us consider the following two cases:

- vector-function Y_2 appertain to class of functions H_0^{n-p},
- vector-function Y_2 appertain to class of functions H_r^{n-p}.

1 Case when the function Y_2 has removable singularity at the point $z=0$

In the case $Y_2 \in H_0^{n-p}$, let us study question on the existence of the analytic solutions of Cauchy's problem

$$\begin{cases} Y_1' = P(z)Y_1 + F^*(z, Y_1, Y_2, Y_1', Y_2'), \\ Y_1(z) \to 0 \ \text{ on the condition that } \ z \to 0, \ \ z \in D_{10}, \end{cases} \tag{1.1}$$

that satisfies the additional condition

$$Y_1'(z) \to 0 \ \text{ on the condition that } \ z \to 0, \ \ z \in D_{10}. \tag{1.2}$$

Let us choose such vector-function $Y_2 \in H_0^{n-p}$ that after regularization in the point $z=0$, becomes analytic function in the domain D_1 and $Y_2(0)=0$.

In this case, the function F^* may be written as

$$F^*(z, Y_1, Y_2, Y_1', Y_2') = F^*\Big(z, Y_1, \sum_{k=1}^{\infty} A_k z^k, Y_1', \sum_{k=1}^{\infty} k \cdot A_k z^{k-1}\Big) = F(z, Y_1, Y_1'),$$

where $F : D_1 \times G_{11} \times G_{21} \to \mathbb{C}^p$.

Thus the problem (1.1) could be reduce to Cauchy's problem:

$$\begin{cases} Y_1' = P(z)Y_1 + F(z, Y_1, Y_1'), \\ Y_1(z) \to 0 \ \text{ on the condition that } \ z \to 0, z \in D_{10}. \end{cases} \tag{1.3}$$

The sufficient conditions were found in which for each arbitrary fixed function $Y_2 \in H_0^{n-p}$, $Y_2(0)=0$, there exists at least one analytic solution of Cauchy's problem (1.3) with the additional condition (1.2) in some subdomain of the domain D_{10} with point $z=0$ at the domain boundary.

2 Case when the function Y_2 has the pole of r-order at the point $z=0$

In this case, let us study question on existence of the analytic solutions of Cauchy's problem (1.1) satisfying the additional condition (1.2) for each arbitrary fixed function $Y_2 \in H_r^{n-p}$.

By condition, the function $Y_2 \in H_r^{n-p}$ may be written as

$$Y_2(z) = z^{-r} Y_2^*(z),$$

where $Y_2^*(z)$ is a analytic function in the domain D_1, and $Y_2^*(0) \neq 0$, moreover, function $Y_2^*(z)$ may be submitted in convergent power series on the condition that $z \in D_1$.

Let us suppose that the power series expansion of function F^* in the domain of point $(0,0,0,0,0)$ has finite number of summand containing vector-functions Y_2 and Y_2'.

Then vector-function $F^*(z, Y_1, Y_2, Y_1', Y_2')$ may be written as

$$F^*(z, Y_1, Y_2, Y_1', Y_2') = z^{-l} \cdot F(z, Y_1, Y_2^*, Y_1', Y_2^{*\prime}),$$

where vector-function $F(z, Y_1, Y_2^*, Y_1', Y_2^{*\prime})$ is analytic function in the domain $D_1 \times G_{11} \times G_{12} \times G_{21} \times G_{22}$, $l \in \mathbb{N}$, $l \geq r+1$.

The system (0.3) may be written as

$$\begin{aligned} z^l Y_1' = z^l A_1^{-1}(z) B_1(z) Y_1 - z^{l-r-1} A_1^{-1}(z) A_2(z) Y_2^{*\prime} \\ + z^{l-r} A_1^{-1}(z) B_2(z) Y_2^* + F(z, Y_1, Y_2^*, Y_1', Y_2^{*\prime}). \end{aligned} \tag{2.1}$$

Let us introduce the following notation

$$P(z) = A_1^{-1}(z) B_1(z), \quad R(z) = A_1^{-1}(z) A_2(z), \quad C(z) = A_1^{-1}(z) B_2(z).$$

Then the system (2.1) may be written

$$z^l Y_1' = z^l P(z) Y_1 - z^{l-r-1} R(z) Y_2^{*\prime} + z^{l-r} C(z) Y_2^* + F(z, Y_1, Y_2^*, Y_1', Y_2^{*\prime}), \tag{2.2}$$

where $P(z)$, $R(z)$, $C(z)$ are analytic matrices in the domain D_1.

The questions on the analytic solutions of Cauchy's problem existence (2.2) that satisfy the initial condition

$$Y_1(z) \to 0 \ \text{ on the condition that } \ z \to 0, \quad z \in D_{10}, \tag{2.3}$$

and the additional condition:

$$Y_1'(z) \to 0 \ \text{ on the condition that } \ z \to 0, \quad z \in D_{10}, \tag{2.4}$$

are considered.

The sufficient conditions were found on which for each arbitrary fixed function $Y_2 \in H_r^{n-p}$, there exists at least one analytic solution of Cauchy's problem (2.2), (2.3) with the additional condition (2.4) in some subdomain of the domain D_{10} with point $z = 0$ at the domain boundary.

For each of these cases we researched the properties of the relevant solutions of the system (0.1).

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Studying Integro-Differential CNN Model with Applications in Nano-Technology

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1 Introduction

The demand for smaller and faster devices has encouraged technological advances resulting in the ability to manipulate matter at nanoscales that have enabled the fabrication of nanoscale electromechanical systems. With the advances in materials synthesis and device processing capabilities, the importance of developing and understanding nanoscale engineering devices has dramatically increased over the past decade. For this purpose we shall study integro-differential equations for solution of dynamic coupled problems in multifunctional nano-heterogeneous piezoelectric composites.

Let $G \in R^2$ is a bounded piezoelectric domain with a set of inhomogeneities $I = \cup I_k \in G$ (holes, inclusions, nano-holes, nano-inclusions, which means that their diameter is less than $10^{-7}m$, see Figure 1.

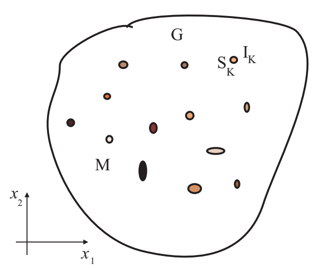

Figure 1. The geometry: PEM inclusions in a bounded PEM matrix.

The aim is to find the field in every point of $M = G \setminus I$, I and to evaluate stress concentration around the inhomogeneities. For this purpose we shall consider the case, when I is a nano-inclusion and we shall assume the following boundary conditions on S:

$$t_p^M = \frac{\partial \sigma_{lp}^S}{\partial l} \ \text{ on } S, \ \text{ or } \ \tau_3^I + t_3^M = \frac{\partial \sigma_{l3}^S}{\partial l}\,, \quad \tau_4^I + t_4^M = \frac{\partial \sigma_{l4}^S}{\partial l}\,, \tag{1.1}$$

where σ_{lp}^S is generalized stress [3], $p = 3, 4$, l is the tangential vector. In [3] the above formulated task is reduced to integro-differential equation. In this paper we shall consider CNN integro-differential model of the problem under consideration and we shall study its dynamics. We shall provide computer simulations for the evaluation of dynamic SCF (stress concentration factor). This characteristic is of interest in nano-mechanics and it is denoted by $|\sigma_{\varphi 3}/\sigma_0|$. Another characteristics of importance in nano-technology is the normalized dynamic Electric Field Concentration Field (EFCF) $|e_{15}^M E_\varphi/\sigma_0|$ along the perimeter of the inhomogeneity. Here φ is the polar angle of the observer point.

2 Dynamics of CNN integro-differential model

In [3] a system of integro-differential equations (IDE) is obtained for the unknowns u (displacement vectors) and τ (traction). The procedure is based on Gauss theorem [7] after finding the fundamental solutions of the boundary value problem formulated in the introduction.

In this section we shall consider the following system of integro-differential equations which is more general from the point of view of the applications in nano-technology:

$$u_t - u_{xx} = F(u,\tau) - b\int\limits_0^t u(s,x)\,ds, \tag{2.1}$$

$u(x,t)$, $0 < x,t < 1$, $b = const$. The proposed system (2.1) is a system of nonlinear integro-differential equation, in which $F(u)$ is a function of displacement vectors and the traction (u,τ) [3].

We shall construct CNN architecture of the above IDE (2.1). First, we map $u(x,t)$ into a CNN layer such that the state voltage of a CNN cell $v_{xij}(t)$ at a grid point (i,j) is associated with $u(ih,jh,t)$, $h = \Delta x$ and using the two-dimensional discretized Laplacian template $A_2 = \begin{pmatrix} 0 & 1 & 0\\ 1 & -4 & 1\\ 0 & 1 & 0\end{pmatrix}$, it is easy to design the CNN model:

(1) CNN cell dynamics:

$$\frac{du_{ij}}{dt} - I^s_{ij} = F(u_{ij},\tau) - b\int\limits_0^t u_{ij}(s)\,ds. \tag{2.2}$$

(2) CNN synaptic law:

$$I^s_{ij} = \frac{1}{h^2}\,(u_{ij-1} + u_{ij+1} - 4u_{ij} + u_{i-1j} + u_{i+1j}). \tag{2.3}$$

Let us assume for simplicity that the grid size of our CNN model is $h = 1$. Substituting (2.3) into (2.2) we obtain:

$$\frac{du_{ij}}{dt} - A_2 * u_{ij} = F(u_{ij}) - b\int\limits_0^t u_{ij}(s)\,ds, \ \ 1 \le i,j \le N. \tag{2.4}$$

The obtained CNN model (2.4) is actually a system of IDE which is identified as the state equation of an autonomous CNN made of $N \times N$ cells [1,5].

We shall study the dynamics of the CNN integro-differential model (2.4) by means of the theory of local activity [2,4]. The theory which will be presented below offers a constructive analytical method for uncovering local activity. One can determine the domain of the cell parameters in order for the cells to be locally active, and thus potentially capable of exhibiting complexity. This precisely defined parameter domain is called the edge of chaos.

Following the theory of local activity we shall find the equilibrium points E of (2.4) [6]. In general, the equilibrium points are functions of the cell parameters. We shall consider the equilibrium point $E^0 = (0,0)$. We calculate the four cell coefficients a_{11}, a_{12}, a_{21}, a_{22} of the Jacobian matrix at equilibrium point $E^0 = (0,0)$ and as well the trace $\mathrm{Tr}(E^0)$ and determinant $\Delta(E^0)$.Then we define stable and locally active region for the CNN integro-differential model (2.4).

Definition 2.1. We say that the cell is both stable and locally active at the equilibrium point E^0 for the CNN integro-differential model (2.4) if

$$a_{22} > 0 \ \text{ or } \ 4a_{11}a_{22} < (a_{12} + a_{21})^2$$

and

$$\mathrm{Tr}(E^0) < 0, \quad \Delta(E^0) > 0.$$

This region in the parameter space is called $\mathrm{SLAR}(E^0)$.

According to [2,4] edge of chaos (EC) is a region in the parameter space of a dynamical system in which emergence of complex phenomena and information processing is possible. Until now the definition of this phenomena is known only via empirical examples. Below we give more precise mathematical definition of EC.

Definition 2.2. CNN integro-differential model (2.4) operates in edge of chaos regime if and only if there is least one equilibrium point such that the cell is both locally active and stable.

Then the following theorem holds:

Theorem. *CNN integro-differential model* (2.4) *operates in edge of chaos if and only if the following conditions are satisfied:* $-1 < b < 1$, $F(0) = 0$, $F < 0 \in (0,b)$, $F > 0 \in (b,1)$, $F'(0) < 0$, $F'(1) < 0$. *This means that there is at least one equilibrium point which is both locally active and stable.*

Remark. It is very important to have circuit model for the physical implementation. Then we can apply results from the classical circuit theory in order to justify the cells local activity. If the cell acts like a source of small signal for at least one equilibrium point then we can say that it is locally active. In this case the cell can inject a net small-signal average power into the passive resistive grids [2,4].

3 Simulations

In this section we shall consider an illustrative example. Let us consider the domain $G_1G_2G_3G_4$ in Figure 2, which is a square elastic isotropic plate under uniform uni-axial time-harmonic traction of magnitude σ_0 applied to the vertical boundaries.

The heterogeneity is presented by a circular nano-inclusion with radius a. The size of the square plate is $10d$, where $d = 2a$. A dimensionless parameter is introduced and it is defined as $s = C_S/2\mu^M a$, where μ^M is the shear modulus of the plate material, $C_S = \lambda^S + 2\mu^S$. When the heterogeneity is presented by the inclusion the stiffness ratio of both phases is $\mu^I/\mu^M = 0.2$ and the densities correspond to frequency ratio of $\Omega^I/\Omega^M = 3.0$, where $\Omega^J = \omega a\sqrt{\rho^J/\mu^J}$, $J = I, M$. In all simulations the material damping is set to 5% and Poisson's ratio is 0.26 for both matrix and inclusion. The normalized hoop stresses spectra for representative point with polar angle $\phi = \pi/2$ of the heterogeneity interface versus normalized frequency for a single hole and inclusion cases are plotted in Figure 3. The dynamic SCF is defined as $|\sigma_{\phi\phi}/\sigma_0|$. Four different values of the surface parameter are considered namely $s = 0; 0.1; 0.5; 1.0$. The problem is solved for frequency range up to $\Omega^M = 0.8$.

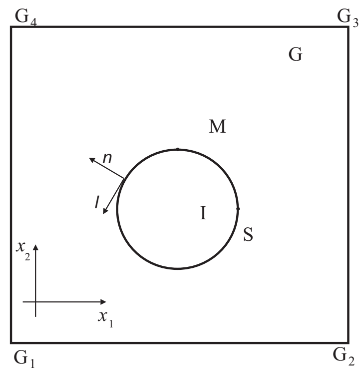

Figure 2. Rectangular PEM matrix with circle heterogeneity.

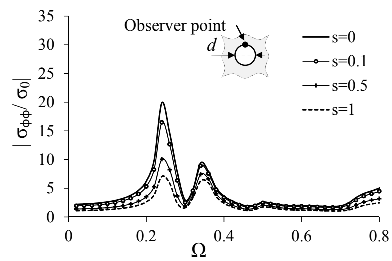

Figure 3. SCF versus frequency at observer point $\phi = \pi/2$ along interface between finite elastic isotropic matrix for nano-inclusion.

4 Conclusion

Time-harmonic elastodynamic analysis of anisotropic finite solids with defects such as nano-sized inclusions is presented in this work. The mathematical model combines classical 2D elastodynamic theory and surface elasticity model [3] allowing in such way to treat heterogeneities at nano-level. The analysis is carried out on IDE that employed the appropriate frequency-dependent fundamental solution, obtained with Radon transform [7]. The CNN architecture is implemented numerically by discretization of the IDE under consideration (2.1). Finally, numerical simulations show that the stress concentration field near defects is strongly influenced by the type and the size of the inclusion, the material anisotropy, the defect location and geometry, the dynamic load characteristics and the mutual interactions between defects and between them and the solid's boundary. The results of the present methodology are with application in the fields of computational fracture mechanics, geotechnical engineering and non-destructive testing evaluation of anisotropic composite materials.

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A Theorem on Differential Inequalities for Linear Functional Differential Equations in Abstract Spaces

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On the interval $[a,b]$, we consider the functional differential equation

$$\boxed{u'(t)=\ell(u)(t)+q(t)} \tag{1}$$

in a Banach space $(\mathbb{X},\|\cdot\|_{\mathbb{X}})$, where $\ell\colon \mathcal{C}([a,b];\mathbb{X})\to\mathcal{B}([a,b];\mathbb{X})$ is a **linear** operator and $q\in\mathcal{B}([a,b];\mathbb{X})$. Here, $\mathcal{C}([a,b];\mathbb{X})$, resp. $\mathcal{B}([a,b];\mathbb{X})$, denotes the Banach space of continuous, resp. Bochner integrable, abstract functions $f\colon [a,b]\to\mathbb{X}$ endowed with the standard norm.

Definition 1. By a solution of equation (1) we understand an abstract function $u\colon [a,b]\to\mathbb{X}$ which is strongly absolutely continuous on $[a,b]$, differentiable a.e. on $[a,b]$, and satisfies equality (1) a.e. on $[a,b]$.

Remark 2. Recall notions of strong absolute continuity and differentiability of abstract functions:

A function $u\colon [a,b]\to\mathbb{X}$ is said to be strongly absolutely continuous, if for each $\varepsilon>0$ there exists $\delta>0$ such that $\sum\limits_i \|u(b_i)-u(a_i)\|_{\mathbb{X}}<\varepsilon$ whenever $\{[a_i,b_i]\}$ is a finite system of mutually non-overlapping subintervals of $[a,b]$ that satisfies $\sum\limits_i (b_i-a_i)<\delta$.

We say that a function $u\colon [a,b]\to\mathbb{X}$ is differentiable at the point $t\in[a,b]$, if there is $\chi\in\mathbb{X}$ such that

$$\lim_{\delta\to 0}\Big\|\frac{u(t+\delta)-u(t)}{\delta}-\chi\Big\|_{\mathbb{X}}=0.$$

We denote $\chi=u'(t)$ the derivative of u at t. If u is differentiable at every point $t\in E\subseteq[a,b]$ with $\operatorname{meas}E=b-a$ (in the sense of Lebesgue measure), then u is called differentiable almost everywhere (a.e.) on $[a,b]$.

Remark 3. Differentiability a.e. on $[a,b]$ has to be assumed in Definition 1, because it, generally speaking, does not follow from the strong absolute continuity. Indeed, let $\mathbb{X}=\mathcal{L}([0,1];\mathbb{R})$ and

$$u(t)(x)=\begin{cases}1 & \text{if } 0\le x\le t\le 1,\\ 0 & \text{if } 0\le t<x\le 1.\end{cases}$$

Then u is strongly absolutely continuous on $[0,1]$, but not differentiable a.e. on $[0,1]$ (see [5, Example 7.3.9]).

In what follows, we assume that the Banach space $\mathbb{X}$ is equipped with the **preordering** $\le_K$ generated by a certain **wedge** K. It means that the elements $x_1,x_2\in\mathbb{X}$, by definition, satisfy the relation $x_1\le_K x_2$ if and only if $x_2-x_1\in K$ (we also write $x_2\ge_K x_1$). Recall that, by a wedge (see, e.g., [2]), a closed set $K\subseteq\mathbb{X}$ is understood such that $\alpha_1x_1+\alpha_2x_2\in K$ for arbitrary $\alpha_1,\alpha_2\in[0,+\infty[$ and $x_1,x_2\in K$. It should be noted that the fulfilment of both the relations $x_1\le_K x_2$ and $x_1\ge_K x_2$, generally speaking, does not imply that $x_1=x_2$.

The preordering $\leq_K$ in $\mathbb{X}$ allows one to define a preordering in the space $\mathcal{C}([a,b];\mathbb{X})$ in the following natural way. We say that for abstract functions $f_1, f_2 \in \mathcal{C}([a,b];\mathbb{X})$, the relation $f_1 \leqslant f_2$ holds if $f_1(t) \leq_K f_2(t)$ for every $t \in [a,b]$. However, in order to formulate a main result of this contribution (namely, Theorem 10), we need to introduce a certain strict type inequality in the space $\mathcal{C}([a,b];\mathbb{X})$.

Definition 4. We say that an element $f \in \mathcal{C}([a,b];\mathbb{X})$ is positive and we write $f \blacktriangleright 0$, if for any abstract function $g \in \mathcal{C}([a,b];\mathbb{X})$ there exists a number $\varepsilon > 0$ such that $\varepsilon g \leqslant f$, i.e.,

$$\varepsilon g(t) \leq_K f(t) \ \text{ for } \ t \in [a,b].$$

Remark 5. It is easy to see that, in the case $\mathbb{X} = \mathbb{R}$ and $K = [0, +\infty[$, the function $f \in \mathcal{C}([a,b];\mathbb{R})$ satisfies $f \blacktriangleright 0$ if and only if $f(t) > 0$ for $t \in [a,b]$.

Moreover, we assume in Theorem 10 that the operator ℓ in (1) is $\mathcal{B}$-positive in the sense of the following definition.

Definition 6. We say that a linear operator $\ell \colon \mathcal{C}([a,b];\mathbb{X}) \to \mathcal{B}([a,b];\mathbb{X})$ is $\mathcal{B}$-positive if the relation

$$\int\limits_a^t \ell(u)(s)\,\mathrm{d}s \geq_K 0 \ \text{ for } \ t \in [a,b]$$

holds for every $u \in \mathcal{C}([a,b];\mathbb{X})$ satisfying

$$u(t) \geq_K 0 \ \text{ for } \ t \in [a,b]. \tag{2}$$

Remark 7. It follows from [5, Proposition 5.1.2, Definition 3.2.1] (see also [4, Theorem 4.6]) that for any $g \in \mathcal{B}([a,b];\mathbb{X})$, the implication

$$g(t) \geq_K 0 \ \text{ for a.e. } \ t \in [a,b] \implies \int\limits_a^b g(s)\mathrm{d}s \geq_K 0$$

is true. Therefore, a linear operator $\ell \colon \mathcal{C}([a,b];\mathbb{X}) \to \mathcal{B}([a,b];\mathbb{X})$ is $\mathcal{B}$-positive provided that it is positive (increasing), i.e., the relation

$$\ell(u)(t) \geq_K 0 \ \text{ for a.e. } \ t \in [a,b]$$

holds for every $u \in \mathcal{C}([a,b];\mathbb{X})$ satisfying (2).

The problem whether the positivity of ℓ is also necessary for its $\mathcal{B}$-positivity is an open question for us.

It is well known that theorems on differential inequalities (maximum principles in other terminology) are powerful tool in the theory of both ordinary and partial differential equations. For abstract differential equation (1), one of possible theorems on differential inequalities can be formulated as follows.

Definition 8. We say that a theorem on differential inequalities holds for equation (1) if the implication

$$\left.\begin{array}{l} u \colon [a,b] \to \mathbb{X} \ \text{ is strongly absolutely continuous,} \\ u \text{ is differentiable a.e. on } [a,b], b \\ u'(t) \geq_K \ell(u)(t) \ \text{ for a.e. } \ t \in [a,b], \\ u(a) \geq_K 0 \end{array}\right\} \implies u(t) \geq_K 0 \ \text{ for } \ t \in [a,b] \tag{3}$$

is true.

Remark 9. The theorem on differential inequalities formulated in the form implication (3) is connected with the question on the existence, uniqueness, and "sign" of a solution to the Cauchy problem for equation (1), i.e., to the problem

$$u'(t) = \ell(u)(t) + q(t), \quad u(a) = c, \tag{4}$$

where ℓ, q are as in (1) and $c \in \mathbb{X}$.

Assume that $\mathbb{X} = \mathbb{R}$, $K = [0, +\infty[$, and that the theorem on differential inequalities holds for equation (1). Then the homogeneous problem

$$u'(t) = \ell(u)(t), \quad u(a) = 0 \tag{4_0}$$

has only the trivial solution. Indeed, let u be a solution of problem (4_0). Then, by virtue of (3), the inequality $u(t) \geq 0$ holds for $t \in [a, b]$. However, the function $-u$ is also a solution of problem (4_0) and thus, we get $u(t) \leq 0$ for $t \in [a, b]$. Consequently, we have $u \equiv 0$ because $K \cap (-K) = \{0\}$ in the considered particular case. Therefore, assuming (in addition) continuity of ℓ, we derive from the Fredholm alternative (see [1, Theorem 2.1]) that the Cauchy problem (4) is uniquely solvable for any $q \in \mathcal{B}([a, b]; \mathbb{X}) = \mathcal{L}([a, b]; \mathbb{R})$ and $c \in \mathbb{R}$ and, moreover, implication (3) yields that the corresponding Cauchy operator is positive.

In the case of general $\mathbb{X}$, the situation is much more complicated and needs a further investigation.

Theorem 10. *Let $\ell \colon \mathcal{C}([a, b]; \mathbb{X}) \to \mathcal{B}([a, b]; \mathbb{X})$ be a linear $\mathcal{B}$-positive operator and there exist a strongly absolutely continuous function $\gamma \colon [a, b] \to \mathbb{X}$, which is differentiable a.e. on $[a, b]$ and satisfies*

$$\gamma \blacktriangleright 0,$$
$$\gamma'(t) \geq_K \ell(\gamma)(t) \;\; \text{for a.e.} \;\; t \in [a, b].$$

Then the theorem on differential inequalities holds for equation (1)*.*

Remark 11. If $\mathbb{X} = \mathbb{R}^n$, $\sigma_1, \dots, \sigma_n \in \{-1, 1\}$,

$$K = \big\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \; \sigma_k x_k \geq 0 \big\},$$

and ℓ is positive (increasing) in the sense of Remark 7 then, in view of Remark 5, Theorem 10 coincides with one part of [6, Theorem 3.2]. The necessity of the existence of a function γ for the validity of a theorem on differential inequalities in Theorem 10 is still an open question. It is worth mentioning here that, in the case of general $\mathbb{X}$, the necessity indicated cannot be proved so easily as in [6, Theorem 3.2] because neither $K \cap (-K) = \{0\}$ nor the Fredholm alternative holds (some additional assumptions are needed).

Remark 12. One can show (see [7]) that the hyperbolic partial differential equation

$$\frac{\partial y^2(t, x)}{\partial t \, \partial x} = T(y)(t, x) + f(t, x), \tag{5}$$

where $T \colon \mathcal{C}([a, b] \times [c, d]; \mathbb{R}) \to \mathcal{L}([a, b] \times [c, d]; \mathbb{R})$ is a linear bounded operator[1] and $f \in \mathcal{L}([a, b] \times [c, d]; \mathbb{R})$, can be regarded as a particular case of abstract equation (1) in the space $\mathbb{X} = \mathcal{C}([c, d]; \mathbb{R})$. Therefore, from Theorem 10 we can derive a result concerning a theorem on differential inequalities for equation (5), which is in a compliance with [3, Theorem 3.1].

[1] $\mathcal{C}([a, b] \times [c, d]; \mathbb{R})$, resp. $\mathcal{L}([a, b] \times [c, d]; \mathbb{R})$, denotes the Banach space of continuous, resp. Lebesgue integrable, functions $y \colon [a, b] \times [c, d] \to \mathbb{R}$ endowed with the standard norm.

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Positive Solutions of a One-Dimensional Superlinear Indefinite Capillarity-Type Problem

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In this paper we are interested in the existence of positive solutions of the quasilinear Neumann problem

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1+(u')^2}}\Big)' = a(x)f(u) \ \text{ in } (0,1), \\ u'(0)=0, \quad u'(1)=0, \end{cases} \tag{1}$$

where $a \in L^1(0,1)$ changes sign and $f : [0,+\infty) \to [0,+\infty)$ is a continuous function having superlinear growth.

Problem (1) is a particular, one-dimensional, version of the elliptic problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = g(x,u) & \text{in } \ \Omega, \\ -\dfrac{\nabla u\cdot \nu}{\sqrt{1+|\nabla u|^2}} = \sigma & \text{on } \ \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, with outward pointing normal ν, and $g : \Omega\times\mathbb{R} \to \mathbb{R}$ and $\sigma : \partial\Omega \to \mathbb{R}$ are given functions. This problem plays a relevant role in the mathematical analysis of a number of physical or geometrical issues such as capillarity phenomena for incompressible fluids, reaction-diffusion processes where the flux features saturation at high regimes, or prescribed mean curvature problems for cartesian surfaces in the Euclidean space.

Although there is a large amount of literature devoted to the existence of positive solutions for semilinear elliptic problems with superlinear indefinite nonlinearities, no result is available for the problem (2), even in the one-dimensional case (1), in spite of the interest that this topic may have both mathematically and from the point of view of the applications.

As it will become clear later, according to Proposition below, the existence of a positive solution for the homogeneous Neumann problem (1) forces the right hand side of the equation to change sign, thus ruling out the possibility, if f is non-negative, that the sign of the weight function a be constant. Hence, the absence of any previous result in the existing literature might be attributable to the fact that superlinear indefinite weighted problems are fraught with a number of technical difficulties which do not arise in dealing with purely sublinear or superlinear problems, even in the most classical semilinear case, not to talk about the degenerate quasilinear problem dealt with in this paper. In addition, as an effect of the spatial heterogeneities incorporated into the formulation of the problem the complexity of the structure of the solution sets might be quite intricate, even in the semilinear case.

When the homogeneous Neumann boundary conditions are replaced in (1) by Dirichlet conditions, the existence of positive solutions is compatible with the right hand side of the equation having constant sign. As in this case technicalities are partially reduced, there are various results about existence, non-existence and multiplicity of positive solutions, even in higher dimension, assuming that both the functions a and f are non-negative.

Our aim here is therefore to begin the analysis of the effects of spatial heterogeneities in the simplest one-dimensional prototype problem (1). Although part of our discussion has slightly been inspired by some available results in the context of semilinear elliptic problems, it must be stressed that the specific structure of the mean curvature operator,

$$u \longmapsto \Big(\frac{u'}{\sqrt{1+(u')^2}}\Big)',$$

makes the analysis much more delicate and sophisticated, as it may determine the occurrence of discontinuous solutions.

Since problem (1) has a variational structure, it is natural to look for its solutions as critical points of an associated action functional such as

$$\mathcal{H}(v)=\int\limits_0^1\Big(\sqrt{1+(v')^2}-1\Big)\,dx-\int\limits_0^1 a\,F(v)\,dx,$$

with

$$F(s)=\int\limits_0^s f(\xi)\,d\xi. \tag{3}$$

As the functional $\mathcal{H}$ grows linearly with respect to the gradient v', it is well-defined in the Sobolev space $W^{1,1}(0,1)$ of all absolutely continuous functions in $(0,1)$. Yet, this space, which might be an obvious candidate where to settle the study of $\mathcal{H}$, is not a favorable framework to deal with critical point theory. Therefore, we replace the space $W^{1,1}(0,1)$ with the space $\mathrm{BV}(0,1)$ of all bounded variation functions in $(0,1)$, and the functional $\mathcal{H}$ with its relaxation $\mathcal{I}$ to $\mathrm{BV}(0,1)$. Namely, we introduce the functional $\mathcal{J}:\mathrm{BV}(0,1)\to\mathbb{R}$ defined by

$$\mathcal{J}(v)=\int\limits_0^1\Big(\sqrt{1+|Dv|^2}-1\Big)\,dx, \tag{4}$$

where, for $v\in\mathrm{BV}(0,1)$,

$$\int\limits_0^1\sqrt{1+|Dv|^2}\,dx=\sup_{\substack{w_1,w_2\in C_0^1(0,1)\\ \|w_1^2+w_2^2\|_{L^\infty}\le 1}}\int\limits_0^1(vw_1+w_2)\,dx,$$

Then, we denote by $\mathcal{I}:\mathrm{BV}(0,1)\to\mathbb{R}$ the functional defined by

$$\mathcal{I}(v)=\mathcal{J}(v)-\mathcal{F}(v), \tag{5}$$

where, for $v\in\mathrm{BV}(0,1)$,

$$\mathcal{F}(v)=\int\limits_0^1 a\,F(v)\,dx.$$

The relaxed functional $\mathcal{I}$ is not differentiable in $\mathrm{BV}(0,1)$, at least in the usual sense, yet it is the sum of the convex (Lipschitz) continuous functional $\mathcal{J}$ and of the continuously differentiable functional $\mathcal{F}$. Hence we say that a critical point of $\mathcal{I}$ is a function $u \in \mathrm{BV}(0,1)$ such that

$$\mathcal{F}'(u) \in \partial\mathcal{J}(u),$$

where $\partial\mathcal{J}(u)$ denotes the subdifferential of $\mathcal{J}$ at the point u in the sense of convex analysis, or, equivalently, such that the variational inequality

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \int\limits_0^1 af(u)(v-u)\,dx \tag{6}$$

holds for all $v \in \mathrm{BV}(0,1)$. Accordingly, the concept of solution used in this work is fixed by the next definition.

Definition 1. A solution of problem (1) is a function $u \in \mathrm{BV}(0,1)$ such that (6) holds for all $v \in \mathrm{BV}(0,1)$. In addition, a solution u of (1) is said to be positive if $\operatorname{ess\,inf} u \geq 0$ and $\operatorname{ess\,sup} u > 0$, and strictly positive if $\operatorname{ess\,inf} u > 0$.

The notion of solution for problem (1) introduced by Definition 1 has already been used and discussed in various papers. We just stress here its relevance because it allows to consider bounded variation solutions which arise as critical points of a different nature than minimizers of the associated action functional. However, here we will go further in the investigation of the regularity properties of the bounded variation solutions we will find, by proving that they are actually $W^{2,1}_{loc}$, and therefore classically satisfy the equation, on each open interval where the weight function a has a constant sign. Consequently, the discontinuities of the solutions that we construct may occur only in the nodal set of a, and we show that such discontinuity points must be 'vertical'ones.

In order to better motivate the hypotheses we are going to impose on the coefficients a and f, we first observe that if a positive solution u of (1) exists, then the function $a\,f(u)$ must change sign, unless it vanishes a.e. in $[0,1]$. Indeed, by choosing $v = u \pm 1$ as test functions in (6), we get

$$\int\limits_0^1 a\,f(u)\,dx = 0. \tag{7}$$

Thus, if f has a constant sign, the function $a(x)$ must change sign in $[0,1]$. However, in the frame of (1) a stronger property holds if f is assumed to be increasing, as expressed by the following result. As usual, we write

$$a^+ = \max\{a,0\} \ \text{ and } \ a^- = -\min\{a,0\}.$$

Proposition. *Assume that*

(a_1) $a \in L^1(0,1)$ *and* $a \neq 0$,

and

(f_1) $f \in C^1[0,+\infty)$ *is such that* $f(0) \geq 0$ *and* $f'(s) > 0$ *for all* $s > 0$.

Suppose that problem (1) *has a strictly positive solution. Then the following holds*

(a_2) $a^+ \neq 0$ *and* $\int\limits_0^1 a\,dx < 0$.

Remark. Even when $a \in L^1(0,1)$ satisfies (a_2), the condition (f_1) is not in general sufficient for guaranteeing the existence of a positive solution of (1). Indeed, suppose that there is an interval $[x_1, x_2] \subset (0,1)$ such that $a(x) > 0$ a.e. in $[x_1, x_2]$. Let ϕ_1 be a positive eigenfunction associated with the principal eigenvalue of $-d^2/dx^2$ in $H_0^1(x_1, x_2)$ and define

$$\phi(x) = \begin{cases} \phi_1(x) & \text{if } x \in [x_1, x_2], \\ 0 & \text{if } x \in [0,1] \setminus (x_1, x_2). \end{cases}$$

Suppose that (1) admits a positive solution u. Then, taking $u + \phi$ as a test function in (6) and using (f_1), we are driven to

$$\|\phi_1'\|_{L^1} \geq \int\limits_{x_1}^{x_2} a\, f(u)\, \phi_1\, dx \geq f(\operatorname{ess\,inf} u) \int\limits_{x_1}^{x_2} a\, \phi_1\, dx,$$

which clearly imposes a restriction on the size of f on the range of u, or on the amplitude of a in (x_1, x_2). This shows that some additional control on f, or on a, is needed.

Based on the observation that the mean curvature operator $(u'/\sqrt{1+(u')^2}\,)'$ behaves like the Laplace operator u'' at 0 and like the 1-Laplace operator $(u'/|u'|)'$ at infinity, and hence the functional $\mathcal{J}(u)$, defined in (4), behaves like $\frac{1}{2}\int\limits_0^1 |u'|^2\, dx$ at 0 and like $\int\limits_0^1 |u'|\, dx$ at infinity, we are led to impose on the potential F, defined in (3), some superquadraticity conditions at 0 and some superlinearity conditions at $+\infty$.

Theorem 1. *Assume that*

(a_3) $a \in L^1(0,1)$ *is such that* $\int\limits_0^1 a\, dx < 0$ *and* $a(x) > 0$ *a.e. on an interval* $K \subset [0,1]$,

(f_2) $f \in C^0[0, +\infty)$ *is such that* $f(s) \geq 0$ *for* $s \geq 0$,

(f_3) *there exist* $p > 2$ *and* $L > 0$ *such that*

$$\lim_{s \to 0^+} \frac{F(s)}{s^p} = L,$$

(f_4) *there exist* $q > 1$ *and* $M > 0$ *such that*

$$\lim_{s \to +\infty} \frac{F(s)}{s^q} = M,$$

(f_5) *there exists* $\vartheta > 1$ *such that*

$$\lim_{s \to +\infty} \frac{\vartheta F(s) - f(s)\, s}{s} = 0,$$

with F *defined in* (3)*. Then problem* (1) *has at least one positive solution* u*, with* $\mathcal{I}(u) > 0$*. In addition,*

$$u \in W^{2,1}_{loc}(\alpha, \beta) \cap W^{1,1}(\alpha, \beta)$$

for each interval $(\alpha, \beta) \subset (0,1)$ *such that* $a(x) \geq 0$ *a.e. in* (α, β)*, or* $a(x) \leq 0$ *a.e. in* (α, β)*. Moreover,* $u \in W^{2,1}_{loc}[0, \beta)$*, with* $u'(0) = 0$*, if* $\alpha = 0$*, while* $u \in W^{2,1}_{loc}(\alpha, 1]$*, with* $u'(1) = 0$*, if* $\beta = 1$*. Finally,* u *satisfies the equation*

$$-\bigg(\frac{u'}{\sqrt{1+(u')^2}}\bigg)' = a(x) f(u), \tag{8}$$

a.e. in each of such intervals.

Suppose further that

(f_6) *f is locally Lipschitz in $[0,+\infty)$.*

Then for every pair of adjacent intervals, (α,β), $(\beta,\gamma)\subset(0,1)$ with $a(x)\geq 0$ a.e. in (α,β) and $a(x)\leq 0$ a.e. in (β,γ) (respectively, $a(x)\leq 0$ a.e. in (α,β) and $a(x)\geq 0$ a.e. in (β,γ)), either

$$u\in W^{2,1}_{loc}(\alpha,\gamma),$$

or

$$u(\beta^-)\geq u(\beta^+)\ \text{ and }\ u'(\beta^-)=-\infty=u'(\beta^+)$$
$$\big(\text{respectively, }\ u(\beta^-)\leq u(\beta^+)\ \text{ and }\ u'(\beta^-)=+\infty=u'(\beta^+)\big),$$

where $u'(\beta^-)$, $u'(\beta^+)$ are, respectively, the left and the right Dini derivatives at β.

Assume further that

(a_4) *the function a changes sign finitely many times in $(0,1)$, in the sense that there is a decomposition*

$$[0,1]=\bigcup_{i=1}^{k}[\alpha_i,\beta_i],\ \text{ with }\ \alpha_i<\beta_i=\alpha_{i+1}<\beta_{i+1},\ \text{ for }\ i=1,\dots,k-1,$$

such that

$$(-1)^i a(x)\geq 0\ \text{ a.e. in }\ (\alpha_i,\beta_i),\ \text{ for }\ i=1,\dots,k,$$

or

$$(-1)^i a(x)\leq 0\ \text{ a.e. in }\ (\alpha_i,\beta_i),\ \text{ for }\ i=1,\dots,k.$$

Then u is a strictly positive (special) function of bounded variation.

The proof Theorem 1 relies on a perturbation argument. The approximating problems are solved by using a minimax technique. Then, the obtention of $W^{1,1}$-estimates allow us to pass to the limit in the approximation scheme to get a bounded variation solution of the original problem. A further concavity/convexity argument, combined with ordinary differential equations techniques, finally permits to conclude the partial regularity of the obtained bounded variation solutions. We refer to [1,2] for further results and complete proofs.

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On Coefficient Perturbation Classes with Degeneracies

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Consider the linear differential system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \tag{1}$$

with a piecewise continuous bounded coefficient matrix A such that $\|A(t)\| \leq a < +\infty$ for all $t \geq 0$. Together with system (1), consider the perturbed system

$$\dot{y} = A(t)y + Q(t)y, \ \ y \in \mathbb{R}^n, \ \ t \geq 0, \tag{2}$$

with a piecewise continuous bounded perturbation matrix Q. For the higher exponent of system (2), we use the notation $\lambda_n(A+Q)$. By $\mathbb{R}^{n\times n}$ we denote the set of all real $n \times n$-matrices with the spectral norm $\|\cdot\|$.

Let $\mathfrak{M}$ be a class of perturbations. The number $\Lambda(\mathfrak{M}) := \sup\{\lambda_n(A+Q) : \ Q \in \mathfrak{M}\}$ is an important asymptotic characteristics for system (1). Many authors investigated how to find $\Lambda(\mathfrak{M})$ for various $\mathfrak{M}$, see, e.g., the monographs [3, p. 157], [7, p. 39], the review [5], and the papers [1, 2, 4, 6, 8–16], where the following $\mathfrak{M}$ are considered:

- vanishing at infinity perturbations [15]

$$Q(t) \to 0, \ \ t \to +\infty;$$

- exponentially small perturbations [6]

$$\|Q(t)\| \leq N_Q \exp(-\sigma_Q t), \ \ \sigma_Q > 0, \ \ t \geq 0;$$

- σ-perturbations [4] (with fixed $\sigma > 0$)

$$\|Q(t)\| \leq N_Q \exp(-\sigma t), \ \ t \geq 0;$$

- power perturbations [1] with arbitrary $\gamma > 0$

$$\|Q(t)\| \leq N_Q t^{-\gamma}, \ \ t \geq 1;$$

- some generalized classes of perturbations [1, 9] similar to previous ones;
- classes defined by various integral conditions [2, 8, 10–13, 16].

Note that everywhere in the above formulas, $N_Q > 0$ is some number depending on Q.

In [1] sharp upper estimates for higher exponent of system (2) with perturbations of the class $\mathfrak{B}[\beta]$ defined by the condition

$$\|Q(t)\| \le N_Q \beta(t), \quad t \ge 0, \tag{3}$$

are obtained when β is some fixed positive piecewise continuous bounded function defined for all $t \ge 0$ and monotone decreasing to 0 with the rate of decrease less than exponential.

Non-monotonic case is partially considered in [11], where β instead of monotonicity obeys the following conditions:

(i) there exists $0<\varepsilon_0<1$ such that for each $\varepsilon \in]0,\varepsilon_0[$ the equality $\lim\limits_{m\to\infty} m^{-1} \sum\limits_{k=0}^{m-1} \beta_k^{\varepsilon} = 0$ is valid;

(ii) there exists $\rho > 0$ such that for any $k \in \mathbb{N}$ the inequality $\beta_k \le \rho\beta(t)$ holds for each $t \in [k-1,k]$ with the possible exception of a finite number of points.

In these conditions we use the notation $\beta_k = \int\limits_k^{k+1} \beta(t)\,dt$.

It should be stressed that in [1] as well as in [11] the algorithm for evaluation of $\Lambda(\mathfrak{M})$ is similar to the algorithm for evaluation of sigma-exponent due to N. A. Izobov [4].

All the above listed perturbation classes are nondegenerate in the sense that their definitions do not contain any restrictions on the sets $\mathfrak{M}(t) := \{Q(t) : \ Q \in \mathfrak{M}\}$, $t \ge 0$. Indeed, for each of them we have $\mathfrak{M}(t) = \mathbb{R}^{n\times n}$ for all $t \ge 0$. In this report we consider perturbations satisfying the condition (3) with non-negative β. It can be easily seen that $\mathfrak{B}[\beta](t) = 0 \in \mathbb{R}^{n\times n}$ for all t such that $\beta(t) = 0$. Hence, we can assume that $\mathfrak{B}[\beta]$ is to be be considered as one of the simplest examples of perturbation classes with degeneracies. In the future we plan to give a comprehencive consideration of such classes and as a first step in this direction we provide here an estimation of $\Lambda(\mathfrak{B}[\beta])$ for the functions β subject to the natural condition

$$\lim_{m\to\infty} \frac{1}{t} \int\limits_0^t \beta(s)\,ds = 0. \tag{4}$$

We show that N. A. Izobov's algorithm is also applicable in this case.

To obtain the required estimation we use the approach developed in [8, 11–13]. Let $X(t,\tau)$ and $Y(t,\tau)$ be the Cauchy matrices for systems (1) and (2) respectively. Denote $X_k := X(k+1,k)$, $Y_k := Y(k+1,k)$ for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Take some non-negative piecewise continuous function β defined for all $t \ge 0$ and satisfying condition (4). Put $\beta_k := \int\limits_k^{k+1} \beta(\tau)\,d\tau$, $k \in \mathbb{N}_0$, $b := \sup\limits_{t\ge 0} \beta(t)$. Obviously, $b \ge 0$ and $\beta_k \le b$ for all $k \in \mathbb{N}_0$. Now choose arbitrary perturbation $Q \in \mathfrak{B}[\beta]$ satisfying the inequality $\|Q(t)\| \le N_Q\beta(t)$ for all $t \ge 0$ with some $N_Q > 0$.

Lemma 1. *For each $k \in \mathbb{N}_0$ the matrix Y_k can be represented in the form $Y_k = X_k(E+V_k)$ where $V_k \in \mathbb{R}^{n\times n}$ is such that $\|V_k\| \le M\beta_k \le Mb$ and $M := N_Q e^{2a+N_Q b}$.*

Note that unlike [8, 11–13], there is an opportunity for some V_k to be zero for any perturbations $Q \in \mathfrak{B}[\beta]$. Indeed, we have $V_k = 0$ for each $k \notin \mathbb{N}_0^{\beta} := \{k \in \mathbb{N}_0 : \ \beta_k \ne 0\}$.

Denote $\langle m\rangle = \{0,1,\dots,m-1\}$ for $m \in \mathbb{N}$. Let d be any subset of $\langle m\rangle$. Further we assume that for $d \ne \varnothing$ the elements of d are arranged in the increasing order, so that $d_1 < d_2 < \dots < d_{|d|} =: H(d)$, where $|d|$ is the number of elements of the set d. Thus, $d = \{d_1, d_2, \dots, H(d)\}$.

Define the multipliers V_k, $k \in \mathbb{N}_0$, corresponding to the given perturbation Q by Lemma 1. Consider matrices $S_d^m := \prod\limits_{k=0}^{m-1} X_k W_k(d)$, $m \in \mathbb{N}$, where $W_k(d) = V_k$ if $k \in d$ and $W_k(d) = E$ if $i \notin d$.

Hereinafter we suppose that $\prod$ denotes the product of the factors arranged in descending order of indices. Since $X_{k+s}\cdots X_{k+1}X_k = X(k+s+1,k)$ for any $k,s\in\mathbb{N}_0$, multiplying all X_k with no intermediate multipliers V_k we get

$$S_d^m = X(m,H(d))V_{H(d)}\cdots X(d_2,d_1)V_{d_1}X(d_1,0).$$

Unlike [8, 11–13], here some V_k can be zero and, therefore, S_d^m is nonzero only when $d\subset\mathbb{N}_0^V := \{k\in\mathbb{N}_0:\ V_k\neq 0\}$. Nevertheless, the inequality

$$\|S_d^m\| \le \|X(m,H(d))\|\,\|V_{H(d)}\|\cdots\|X(d_2,d_1)\|\,\|V_{d_1}\|\,\|X(d_1,0)\| =: Z_d(m)$$

remains valid. Since

$$Y(m,0)=\prod_{i=0}^{m-1}X_i(E+V_i)=\sum_{d\subset\langle m\rangle}S_d^m,$$

we can estimate the value of $\|Y(m,0)\|$ by means of $Z_d(m)$.

Theorem 1. *Let h_i, $i\in\mathbb{N}_0$, be a sequence of non-negative numbers such that $h_i>0$ for $i\in\mathbb{N}_0^V$. Then the Cauchy matrix Y of system (2) satisfies the inequality*

$$\|Y(m,0)\|\le e^{K(m)}\max_{d\subset\langle m\rangle}R(d)Z_d(m),\ \ m\in\mathbb{N},$$

where $R(d)=\prod\limits_{i\in d}h_i$, $K(m)=\sum\limits_{i\in\langle m,V\rangle}h_i^{-1}$, $\langle m,V\rangle := \langle m\rangle\cap\mathbb{N}_0^V$.

The following Lemma is a necessary tool to remove condition (i) posed on β in [11].

Lemma 2. *If a sequence of non-negative numbers u_k, $k\in\mathbb{N}_0$, satisfies the condition*

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1}u_k=0, \tag{5}$$

then for any $\varepsilon\in\,]0,1]$ the sequence u_k^{ε}, $k\in\mathbb{N}_0$ satisfies condition (5) too.

As in [12], put

$$\Gamma_d^{\varphi}(m)=\|X(m,H(d))\|\varphi(H(d))\cdots\|X(d_2,d_1)\|\,\varphi(d_1)\,\|X(d_1,0)\|,$$

where $\varphi:\mathbb{N}_0\to[0,+\infty[\,$, $d\subset\langle m\rangle$, $m\in\mathbb{N}$. The main result of our work is given by the following statement.

Theorem 2. *The inequality*

$$\Lambda(\mathfrak{B}[\beta])\le\varlimsup_{m\to\infty}m^{-1}\ln\max_{d\subset\langle m,\beta\rangle}\Gamma_d^{\beta}(m) \tag{6}$$

holds for any non-negative piecewise continuous function β defined for all $t\ge 0$ and satisfying condition (4).

Attainability of the above estimation (6) is a separate problem to be solved by a special version of Millionshchikov's rotation method [14].

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On a Class of Linear Functional Differential Systems Under Integral Control

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1 Introduction

In the classical control problem for the differential system

$$(\mathcal{L}x)(t) \equiv \dot{x}(t) + A(t)x(t) = B(t)u(t) + f(t), \quad t \in [0, T],$$

one needs to find a control u taking the system from a given initial state $x(0) = \alpha$ to a prescribed terminal position $x(T) = \beta$. In the case with no constraints according to control, any terminal state β is attainable if the matrix

$$V = \int_0^T Y(t)B(t)B^\top(t)Y^\top(t)\,dt$$

is nonsingular, where $Y(t)$ is the inverse to the fundamental matrix of the system, $\cdot^\top$ stands for transposition. If the control $u(t)$ is constrained, say by $v(t) \le u(t) \le V(t)$, $t \in [0, T]$, there arises the question about the attainability set, i.e. the set of all terminal positions that are attainable by the use of controls such that the constraints are fulfilled. After the fundamental work by N. N. Krasovskiĭ [6] the questions of attainability are studied systematically for various classes of systems with continuous and discrete times (see, for instance, [1, 5, 8] and the references therein).

We consider a quite broad class of functional differential systems under control implemented by an integral operator in the case that the goal of control takes into account a collection of terminal and previous states of the system under control. Such problems find practical use, in particular, in economic dynamics [10, 12].

First we descript in detail a class of functional differential equations with linear Volterra operators and appropriate spaces where those are considered. Next the setting of the control problem (CP) is given and discussed, and some conditions for the solvability of CP are recalled. Those are obtained for the case of unconstrained control as applied to various classes of control (see, for instance, in [10, 11]). A theorem is formulated which gives a description of the attainability set for CP under consideration. Two illustrative example of application of the theorem are presented.

2 Control problem

We follow the notation and basic statements of the general functional differential theory in the part concerning linear systems with aftereffect [2, 4, 9].

Let $L^n = L^n[0, T]$ be the Lebesgue space of all summable functions $z : [0, T] \to R^n$ defined on a finite segment $[0, T]$ with the norm $\|z\|_{L^n} = \int_0^T |z(t)|\,dt$, where $|\cdot|$ is a norm in R^n. Denote by

$AC^n = AC^n[0,T]$ the space of absolutely continuous functions $x : [0,T] \to R^n$ with the norm

$$\|x\|_{AC^n} = |x(0)| + \|\dot{x}\|_{L^n}.$$

In what follows we will use some results from [2, 4, 9].

To give a description of the controlled functional differential system with aftereffect, introduce the linear operator $\mathcal{L}$:

$$(\mathcal{L}x)(t) = \dot{x}(t) - \int\limits_0^t K(t,s)\dot{x}(s)\,ds - A(t)x(0), \quad t \in [0,T]. \tag{2.1}$$

Here the elements $k_{ij}(t,s)$ of the kernel $K(t,s)$ are measurable on the set $\Delta = \{(t,s) :\ 0 \leqslant s \leqslant t \leqslant T\}$ and such that the estimates $|k_{ij}(t,s)| \leqslant \kappa(t)$, $i,j = 1,\dots,n$, hold on Δ with a function κ summable on $[0,T]$. The elements of $(n \times n)$-matrix $A(t)$ are summable on $[0,T]$. The operator $\mathcal{L} : AC^n \to L^n$ is bounded. The functional differential system $\mathcal{L}x = f$ covers linear differential equations with concentrated and/or distributed delay and Volterra integro-differential systems (see, for instance, [9]). For a particular case of $(\mathcal{L}x)(t) = \dot{x}(t) - \int\limits_0^t d_s\,R(t,s)x(s)$, where without loss of generality we can put $R(t,t) = 0$, we have $K(t,s) = R(t,s)$, $A(t) = R(t,0)$.

Under the listed conditions the linear operator $Q : L^n \to L^n$, $(Qz)(t) = z(t) - \int\limits_0^t K(t,s)z(s)(s)\,ds$ has the bounded inverse operator $(Q^{-1}f)(t) = f(t) + \int\limits_0^t \mathcal{R}(t,s)f(s)(s)\,ds$, where $\mathcal{R}(t,s)$ is the resolvent kernel with respect to $K(t,s)$. The matrix $C(t,s) = E + \int\limits_s^t \mathcal{R}(\xi,s)\,d\xi$, where E is the identity $(n \times n)$-matrix, is called the Cauchy matrix. The general solution of the equation $\mathcal{L}x = f$ has the form

$$x(t) = X(t)\alpha + \int\limits_0^t C(t,s)f(s)\,ds,$$

where $X(t)$ is the fundamental matrix to the homogeneous equation $\mathcal{L}x = 0$. The properties of the Cauchy matrix used below are studied in detail in [9].

The system under control is described by the equation

$$(\mathcal{L}x)(t) = (Bu)(t), \quad t \in [0,T]. \tag{2.2}$$

Here

$$(Bu)(t) = \int\limits_0^t B(t,s)u(s)\,ds,$$

the elements $b_{ij}(t,s)$ of the $(n \times r)$-matrix $B(t,s)$ are measurable on the set $\Delta = \{(t,s) :\ 0 \leqslant s \leqslant t \leqslant T\}$ and such that the estimates $|b_{ij}(t,s)| \leqslant b(t)$, $i,j = 1,\dots,n$, hold on Δ with a function b summable on $[0,T]$.

The initial state of system (2.2) is fixed:

$$x(0) = 0. \tag{2.3}$$

To define the on-target vector-functional, let us fix a collection of points $\{t_i\}$ in $[0,T]$: $0 < t_1 \leqslant \cdots \leqslant t_{m-1} \leqslant t_m = T$. The aim of control is prescribed with a given linear bounded vector-functional $\ell : AC^n \to R^N$:

$$\ell x \equiv \sum_{i=1}^m P_i x(t_i) = \beta, \tag{2.4}$$

where P_i, $i = 1, \dots, m$, are given $(N \times n)$-matrices. For the case of unconstrained control from the space of square summable functions $u : [0, T] \to R^r$, a condition of the solvability of (2.2)–(2.4) is given by the theorem appearing below, which is a corollary from Theorem 3.1 [10]. To formulate it, we introduce $(N \times n)$-matrix Φ:

$$\Phi(s) = \sum_{i=1}^{m} P_i \chi_i(s) C(t_i, s),$$

where $\chi_i(s)$ is the characteristic function of the segment $[0, t_i]$.

Theorem 2.1. *Problem* (2.2)–(2.4) *is solvable iff the* $(N \times N)$*-matrix*

$$W = \int_0^T \bigg\{ \int_\tau^T \Phi(s) B(s, \tau)\, ds \cdot \int_\tau^T B^\top(s, \tau) \Phi^\top(s)\, ds \bigg\}\, d\tau$$

is nonsingular.

Now we introduce the constraints with respect to the control $u(t)$:

$$Gu(t) \leqslant g, \quad t \in [0, T], \tag{2.5}$$

where G is a given $(N_1 \times r)$-matrix, $g \in R^{N_1}$. It is assumed that the set of all solutions to $Gv \leqslant g$ (i.e. the set $\mathcal{V}$ of admissible values of control $u(t)$) is nonempty and bounded in R^r.

Definition. We say that a set $\Xi \subset R^N$ is the ℓ-attainability set of (2.2) with (2.3) under constraints (2.5) iff problem (2.2)–(2.4) is solvable for any $\beta \in \Xi$.

Define $(N \times r)$-matrix $M(s)$ by the equality

$$M(s) = \int_s^T \Phi(\tau) B(\tau, s)\, d\tau.$$

Due to the Cauchy matrix we reduce problem (2.2)–(2.5) to the moment problem [6]

$$\int_0^T M(s) u(s)\, ds = \beta, \quad Gu(t) \leqslant g, \quad t \in [0, T],$$

and, employing Theorem 7.1 [7], obtain

Theorem 2.2. *Let* $B(\tau, \cdot)$ *be continuous on* $[0, \tau]$ *for almost all* $\tau \in [0, T]$*, and for any fixed* $\lambda \in R^N$ *the linear programming problem*

$$z = \lambda^\top M(s) v \longrightarrow \max, \quad Gv \leqslant g \tag{2.6}$$

be uniquely solvable for almost all $s \in [0, T]$*. Then problem* (2.2)–(2.5) *is solvable iff for any fixed* $\lambda \in R^N$ *the inequality*

$$\lambda^\top \beta \leqslant \int_0^T \lambda^\top M(s) u(s, \lambda)\, ds$$

holds, where $u(s, \lambda)$ *is a solution to* (2.6).

Example 1. Let us consider the system under control

$$\begin{aligned} \dot{x}_1(t) &= x_2(t-1) + \int\limits_0^t u_1(s)\,ds, \\ \dot{x}_2(t) &= -x_2(t) + \int\limits_0^t u_2(s)\,ds, \end{aligned} \qquad t \in [0,3], \tag{2.7}$$

where $x_2(s) = 0$ if $s < 0$, with the initial conditions

$$x_1(0) = 0, \quad x_2(0) = 0, \tag{2.8}$$

and on-target conditions as follows:

$$x_1(3) = \beta_1, \quad x_2(3) = \beta_2, \quad x_2(2) = \beta_3, \tag{2.9}$$

under the control constrained by the inequalities

$$0 \leqslant u_i(t) \leq 1, \ \ i = 1,2. \tag{2.10}$$

Here we have

$$\begin{aligned} (Bu)(t) &= \mathrm{col}\bigg(\int\limits_0^t u_1(s)\,ds, \int\limits_0^t u_2(s)\,ds\bigg), \\ C(t,s) &= \begin{pmatrix} 1 & \displaystyle\int\limits_s^t \chi_{[1,3]}(\tau)\chi_{[0,\tau-1]}(s)\exp(1-\tau+s)\,d\tau \\ 0 & \exp(s-t) \end{pmatrix}, \\ \ell x &= \mathrm{col}\big(x_1(3), x_2(3), x_2(2)\big). \end{aligned}$$

Now, to study the ℓ-attainability set, after calculations we obtain the system $\int\limits_0^3 M(s)u(s)\,ds = \beta$, where $\beta = \mathrm{col}(\beta_1,\beta_2,\beta_3)$, $u(s) = \mathrm{col}(u_1(s),u_2(s))$, $M(s) = \{\mu_{ij}(s)\}_{i=1,2,3;\,j=1,2}$,

$$\mu_{11}(s) = 3-s, \quad \mu_{12}(s) = \begin{cases} 1-s+e^{s-2} & \text{if } \ s \in [0,2], \\ 0 & \text{otherwise}, \end{cases} \quad \mu_{21}(s) = 0,$$

$$\mu_{22}(s) = 1-e^{s-3}, \quad \mu_{31}(s) = 0, \quad \mu_{32}(s) = \begin{cases} 1-e^{s-2} & \text{if } \ s \in [0,2], \\ 0 & \text{otherwise}. \end{cases}$$

By Theorem 2.2 it can be shown that the set $\mathcal{A} = \{(\beta_1,\beta_2,\beta_3) : \ \beta_1 \in [0,1.5], \ \beta_2 \in [0,0.35], \ 0 \leq \beta_3 \leq \beta_2\}$ is a subset of the ℓ-attainability set to problem (2.7)–(2.10).

Example 2. Consider system (2.7) with the initial conditions (2.8) and the on-target conditions

$$x_1(3) = \beta_1, \quad x_2(3) + x_2(2) = \beta_2, \tag{2.11}$$

under the control constrained by the inequalities

$$u_i(t) \geqslant 0, \ \ i = 1,2; \ \ u_1 + u_2 \leqslant 1. \tag{2.12}$$

For this case, we have $\ell x = \mathrm{col}(x_1(3), x_2(3)+x_2(2))$. By Theorem 2.2 it can be shown that the union of the triangle with corner points $(0;0)$, $(4.0;0)$, $(0.9;2.7)$ and the rectangle with corner points $(0;0)$, $(4.0;0)$, $(0;-0.5)$, $(4;-0.5)$ is a subset of the ℓ-attainability set to problem (2.7), (2.8), (2.11), (2.12).

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All Solutions of the Linear Periodic System with the Bounded Reflecting Matrix are Bounded

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We consider the system

$$\frac{dx}{dt} = P(t)x, \quad P(t+2\omega) \equiv P(t), \ \ x \in R^n \tag{1}$$

with the continuous matrix $P(t)$.

The reflecting function of this system has the form $F(t,x) := F(t)x$, where the reflecting matrix $F(t)$ is the solution of the Cauchy problem

$$\frac{\partial F}{\partial t} + F(t)P(t) + P(-t)F = 0, \quad F(0) = E.$$

About that see [1, p. 31]. So if $P(t) + P(-t) \equiv 0$, then $F(t) \equiv E$, where E is identity matrix.

Theorem. *If reflecting matrix $F(t)$ of the periodic system* (1) *is bounded on R, then all solutions of the system* (1) *are bounded on R.*

Example. The equation $\frac{dx}{dt} = x(1-\cos t)\operatorname{sign} t$ has bounded reflecting matrix $F(t) = 1$. The non-zero solutions of equations are not bounded. But this equation is non-periodic.

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The Periodic Type Problem for the Second Order Integro Differential Equations

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Consider on the interval $I=[0,\omega]$ the second order linear integro differential equation

$$u''(t)=\int\limits_0^\omega p(t,s)u(\tau(s))\,ds+q(t), \tag{1}$$

and the nonlinear functional differential equation

$$u''(t)=F(u)(t)+f(t), \tag{2}$$

with the periodic type two-point boundary value conditions

$$u^{(i-1)}(\omega)-u^{(i-1)}(0)=c_i \ \ (i=1,2), \tag{3}$$

where $c_1,c_2\in R$, $p\in L_\infty(I^2,R)$, $F:C(I,R)\to L(I,R)$ is a continuous operator, $\tau:I\to I$ is a measurable function, and $q\in L(I,R)$.

By a solution of problem (1), (3) we understand a function $u\in\widetilde{C}\,'(I,R)$ which satisfies equation (1) almost everywhere on I and satisfies conditions (3).

Throughout the paper we use the following notations.

R is the set of all real numbers, $R_+=[0,+\infty[$;

$C(I;R)$ is the Banach space of continuous functions $u:I\to R$ with the norm $\|u\|_C=\max\{|u(t)|:t\in I\}$;

$C\,'(I;R)$ is the Banach space of functions $u:I\to R$ which are continuous together with their first derivatives with the norm $\|u\|_{C'}=\max\{|u(t)|+|u'(t)|:\ t\in I\}$;

$\widetilde{C}\,'(I;R)$ is the set of functions $u:I\to R$ which are absolutely continuous together with their first derivatives;

$L(I;R)$ is the Banach space of the Lebesgue integrable functions $p:I\to R$ with the norm $\|p\|_L=\int\limits_a^b|p(s)|ds$;

$L_\infty(I,R)$ is the space of essentially bounded functions $p:I\to R$ with the norm $\|p\|_\infty=\operatorname{ess\,sup}\{|p(t)|:\ t\in I\}$;

$L_\infty(I^2,R)$ is the set of such functions $p:I^2\to R$ that for arbitrary $y\in L_\infty(I,R)$ and fixed $t\in I$, $p(t,\,\cdot\,),y(\,\cdot\,)\in L(I,\,R)$, and

$$\int\limits_0^\omega p(\,\cdot\,,s)y(s)\,ds\in L_\infty(I,\,R);$$

For arbitrary $p\in L_\infty(I^2,R)$ and measurable $\tau:I\to I$ we will use the notation:

$$\ell(p,\tau)=\frac{2\pi}{\omega}\bigg(\int\limits_0^\omega\int\limits_0^\omega|p(\xi,\,s)|\,|\tau(s)-\xi|\,ds\,d\xi\bigg)^{1/2}.$$

Definition 1. Let $\sigma \in \{-1, 1\}$. We say that the function $h \in L_\infty(I^2, R)$ belongs to the set $K^\sigma_{I,\tau}$ if $h(t,\, s) \geq 0$ and for an arbitrary function $p \in L_\infty(I^2, R)$ such that

$$0 \leq \sigma p(t,s) \leq h(t,s), \quad \int\limits_0^\omega p(t,\xi)\, d\xi \not\equiv 0 \ \text{ for } \ (t,s) \in I^2,$$

the homogeneous problem

$$v''(t) = \int\limits_0^\omega p(t,s) v(\tau(s))\, ds,$$
$$v^{(i-1)}(\omega) - v^{(i-1)}(0) = 0 \ \ (i = 1,2),$$

has no nontrivial solution.

Main results

Proposition 1. *Let* $\sigma \in \{-1,\, 1\}$,

$$h(t,s) \geq 0, \quad \int\limits_0^\omega h(t,\xi)\, d\xi \not\equiv 0 \ \ \textit{for} \ \ (t,\, s) \in I^2,$$

and for almost all $t \in I$ *the inequality*

$$\frac{(1-\sigma)}{2} \int\limits_0^\omega h(t,s)\, ds + \ell(h,\tau) \Big(\int\limits_0^\omega h(t,s)\, ds \Big)^{1/2} < \Big(\frac{2\pi}{\omega} \Big)^2$$

hold. Then

$$h \in K^\sigma_{I,\tau}.$$

Theorem 1. *Let* $\sigma \in \{-1,\, 1\}$,

$$\sigma p(t,s) \geq 0, \quad \int\limits_0^\omega p(t,\xi)\, d\xi \not\equiv 0 \ \ \textit{for} \ \ (t,s) \in I^2,$$

and for almost all $t \in I$ *the inequality*

$$\frac{(1-\sigma)}{2} \int\limits_0^\omega p(t,s)\, ds + \ell(p,\tau) \Big(\int\limits_0^\omega p(t,s)\, ds \Big)^{1/2} < \Big(\frac{2\pi}{\omega} \Big)^2$$

hold. Then problem $(1), (3)$ *is uniquely solvable.*

On the basis of Proposition 1 we can prove the necessary conditions for solvability and unique solvability of nonlinear problem $(2), (3)$. First for arbitrary $h \in L_\infty(I^2, R)$ and $r > 0$ introduce the set $U_{p,r}$ and the operator $\psi_{h,\tau}(x) : C(I, R) \to L(I, R)$ by the equalities

$$U_{I,r} = \bigg\{ x \in C'(I,R) : \ x^{(i-1)}(\omega) - x^{(i-1)}(\omega) = c_i \ (i = 1,2), \ \bigg| \int\limits_0^\omega \int\limits_0^\omega h(\xi,s) x(\tau(s))\, ds\, d\xi \bigg| \geq r \bigg\},$$

$$\psi_{p,\tau}(x)(t)=\begin{cases}1 & \text{for } \int\limits_0^\omega h(t,s)x(\tau(s))\,ds>0,\\ 0 & \text{for } \int\limits_0^\omega h(t,s)x(\tau(s))\,ds=0,\\ -1 & \text{for } \int\limits_0^\omega h(t,s)x(\tau(s))\,ds<0,\end{cases}$$

and the class $K_{p,\tau}(C,L)$ of the operators by the next definition.

Definition 2. Let $p\in L_\infty(I^2,R)$ and $\tau:I\to I$ be measurable function, then we say that $F\in K_{p,\tau}(C,L)$ if $F:C(I,R)\to L(I,R)$ is continuous operator and for arbitrary $r>0$

$$\sup\bigg\{|F(u)(t)|:\ \bigg|\int\limits_0^\omega\int\limits_0^\omega h(\xi,s)u(\tau(s))\,ds\,d\xi\bigg|\le r\bigg\}\in L(I,\,R).$$

Then the next theorem is true.

Theorem 2. *Let $F\in K_{h,\tau}(C,L)$, there exist numbers $\sigma\in\{-1,1\}$, $r_0>0$, measurable $\tau:I\to I$ and functions $g,g_0\in L(I,R_+)$, and $h\in K^\sigma_{I,\tau}$ such that for arbitrary $x\in U_{h,r_0}$ on I the next conditions hold*

$$g_0(t)\le\sigma F(x)(t)\psi_{h,\tau}(x)(t)\le\bigg|\int\limits_0^\omega h(t,s)x(\tau(s))\,ds\bigg|+g(t)$$

if $\psi_{h,\tau}(x)(t)\neq0$, and

$$F(x)(t)=0\ \text{ if }\ \psi_{h,\tau}(x)(t)=0.$$

Let, moreover,

$$\bigg|\int\limits_0^\omega f(s)\,ds\bigg|\le\int\limits_0^\omega g_0(s)\,ds-|c_2|. \tag{4}$$

Then problem (2), (3) *has at least one solution.*

From this theorem for the equation

$$u''(t)=f_0\bigg(t,\int\limits_0^\omega p(t,s)u(\tau(s))\,ds\bigg)+f(t), \tag{5}$$

where the function $f_0:I^2\to R$ is from the Carathéeodory's class, $f_0(t,0)\equiv0$, $p\in L_\infty(I^2,R)$, and $\tau:I\to I$ is a measurable function, it follows

Corollary 1. *Let there exist numbers $\sigma\in\{-1,1\}$, $r_0>0$, and functions $w,g,g_0\in L(I,\,R_+)$ such that $w(t)>0$, $\sigma,w,p\in K^\sigma_{I,\tau}$,*

$$g_0(t)\le f_0(t,x)\operatorname{sgn}x\le w(t)|x|+g(t)\ \text{ for }\ |x|>r_0,\ \ t\in I,$$

and inequality (4) *holds. Then problem* (5), (3) *has at least one solution.*

Remark. Inequality (4) in Theorem 2 (Corollary 1) cannot be replaced by the inequality

$$\Big|\int_0^\omega f(s)\,ds\Big| \le \int_0^\omega g_0(s)\,ds - |c_2| + \varepsilon \tag{6}$$

no matter how small $\varepsilon > 0$ would be. Indeed, if $F \equiv 0$, $f(t) \equiv \frac{\varepsilon}{\omega}$, $g_0 \equiv g \equiv 0$, $c_2 = 0$, and

$$h(t,s) \equiv \Big(\frac{2\pi}{\omega}\Big)^2 \frac{1-\varepsilon}{\ell(1,\tau)\sqrt{\omega}},$$

then instead of (4) inequality (6) holds and all other conditions of Theorem 2 (Corollary 1) are fulfilled. Nevertheless, in that case problem (2), (3) is not solvable.

Theorem 3. *Let $F \in K_{h,\tau}(C,L)$, there exist a number $\sigma \in \{-1,1\}$, a continuous functional $\eta : C(I,R) \to R_+$, measurable $\tau : I \to I$, and functions $h_0 \in L(I,R_+)$, $h \in K^\sigma_{I,\tau}$ such that for arbitrary $x, y \in U_{h,0}$ on I the next conditions hold*

$$h_0(t)\eta(x-y) \le \sigma(F(x)(t) - F(y)(t))\psi_{h,\tau}(x-g)(t) \le \Big|\int_0^\omega h(t,s)\big(x(\tau(s)) - y(\tau(s))\big)\,ds\Big|$$

if $\psi_{h,\tau}(x-y)(t) \ne 0$, and

$$F(x-y)(t) = F(x)(t) - F(y)(t) = 0$$

if $\psi_{h,\tau}(x-y)(t) = 0$, where

$$h_0(t) \gneqq 0 \ \text{ for } \ t \in I, \ \ \eta(z) > 0 \ \text{ if } \ \min_{t\in I}\{|z(t)|\} > 0.$$

Let, moreover, there exist a number $r_0 > 0$ such that condition (4) holds, where

$$g_0(t) = h_0(t)\min\left\{\eta(z) : \ \|z\|_C \ge \frac{r_0}{\int_0^\omega\int_0^\omega h(\xi,s)\,ds\,d\xi}\right\}.$$

Then problem (2), (3) is uniquely solvable.

As examples we consider the differential equations

$$u''(t) = \frac{\sigma(1 + |\sin u(\tau_0(t))|)}{2} \int_0^\omega h(t,s)u(\tau(s))\,ds + f_0(t) \tag{7}$$

and

$$u''(t) = \frac{\sigma\int_0^\omega h(t,s)u(\tau(s))\,ds}{(1 + \big|\int_0^\omega h(t,s)u(\tau(s))\,ds\big|)^\alpha} + f_0(t), \tag{8}$$

where $\sigma \in \{-1,1\}$, $\alpha \in (0,1]$, $h \in K^\sigma_{I,\tau}$, and $\int_0^\omega f_0(s)\,ds = 0$. Then in view of Theorem 2 with $r_0 > |c_2|\,\big|\int_0^\omega h(t,s)u(\tau(s))\,ds\big|^{-1}$ (Theorem 3) problem (7), (3) (problem (8), (3)) is solvable (uniquely solvable).

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Oscillation Criteria for Second-Order Linear Advanced Differential Equations

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On the half-line $\mathbb{R}_+ = [0, +\infty[$, we consider the second-order linear differential equation with argument deviation

$$u''(t) + p(t)u(\sigma(t)) = 0, \tag{1}$$

where $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a locally Lebesgue integrable function and $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that

$$\sigma(t) \geq t \ \text{ for } \ t \geq 0.$$

Oscillation theory for linear ordinary differential equations is a widely studied and well-developed topic of the general theory of differential equations. We mention some results which are closely related to those of this paper, in particular, works of E. Hille, E. Müller-Pfeiffer, and A. Wintner (see, e.g., [1–3, 6]). We should note that oscillation properties for the linear differential equation with deviating argument (1), but in the case when $\sigma(t)$ is a delay, were studied in [4, 5]

Solutions to equation (1) can be defined in various ways. Since we are interested in properties of solutions in a neighbourhood of $+\infty$, we introduce the following commonly used definitions.

Definition 1. Let $t_0 \in \mathbb{R}_+$. A continuous function $u\colon [t_0, +\infty[\to \mathbb{R}$ is said to be a *solution to equation* (1) *on the interval* $[t_0, +\infty[$ if it is absolutely continuous together with its first derivative on every compact interval contained in $[t_0, +\infty[$ and satisfies equality (1) almost everywhere in $[t_0, +\infty[$.

Definition 2. A solution to equation (1) is said to be *oscillatory* if it has a zero in any neighbourhood of infinity, and *non-oscillatory* otherwise.

Firstly, we remind that if $\int\limits_0^{+\infty} sp(s)\,\mathrm{d}s < +\infty$, then (1) has a proper non-oscillatory solution (see [4, Proposition 2.1]). Therefore, we assume throughout the paper that

$$\int\limits_0^{+\infty} sp(s)\,\mathrm{d}s = +\infty.$$

Let us put

$$F_* = \liminf_{t\to+\infty} t \int\limits_t^{+\infty} p(s)\,\mathrm{d}s, \quad F^* = \limsup_{t\to+\infty} t \int\limits_t^{+\infty} p(s)\,\mathrm{d}s. \tag{2}$$

We prove our main results by using lemma on a priori estimate of non-oscillatory solutions. If we have non-oscillatory solution, then we need to find a suitable a priori lower bound of the quantity $u(\sigma(t))/u(t)$. It is not difficult to verify that

$$1 \leq \frac{u(\sigma(t))}{u(t)} \ \text{ for large } t.$$

However, we succeeded in finding a more precise estimate in Lemma 1, which allow us to establish more efficient results.

Lemma 1. *Let u be a solution to equation* (1) *on the interval* $[t_u,+\infty[$ *satisfying the inequality*

$$u(t)>0 \ \textit{ for } \ t\geq t_u.$$

Then

$$F^*\leq 1$$

and, moreover, for any $\varepsilon\in[0,1[\,$, *there exists* $t_0(\varepsilon)\geq t_u$ *such that*

$$\Big(\frac{\sigma(t)}{t}\Big)^{\varepsilon F_*}\leq\frac{u(\sigma(t))}{u(t)} \ \textit{ for } \ \sigma(t)\geq t\geq t_0(\varepsilon),$$

where the numbers F_* *and* F^* *are given by relations* (2).

One can see that from Lemma 1 we obtain the following proposition.

Proposition. *Let*

$$F^*>1.$$

Then every proper solution to equation (1) *is oscillatory.*

Hence, it is natural to suppose that

$$F_*\leq 1. \tag{3}$$

Now we formulate main results. The first one contains Wintner type oscillation criterion.

Theorem 1. *Let condition* (3) *be fulfilled and let there exist* $\lambda\in[0,1[$ *and* $\varepsilon\in[0,1[$ *such that*

$$\int\limits_0^{+\infty}s^\lambda\Big(\frac{\sigma(s)}{s}\Big)^{\varepsilon F_*}p(s)\,\mathrm{d}s=+\infty. \tag{4}$$

Then every proper solution to equation (1) *is oscillatory.*

Next criterion generalizes a result of E. Müller-Pfeiffer proved for ordinary differential equations in [3].

Theorem 2. *Let conditions* (3) *hold and there exist* $\varepsilon\in[0,1[$ *such that*

$$\limsup_{t\to+\infty}\frac{1}{\ln t}\int\limits_0^t\Big(\frac{\sigma(s)}{s}\Big)^{\varepsilon F_*}p(s)\,\mathrm{d}s>\frac14\,.$$

Then every proper solution to equation (1) *is oscillatory.*

In view of Theorem 1, we can assume that

$$\int\limits_0^{+\infty}s^\lambda\Big(\frac{\sigma(s)}{s}\Big)^{\varepsilon F_*}p(s)\,\mathrm{d}s<+\infty \ \text{ for all } \ \lambda\in[0,1[\,, \ \ \varepsilon\in[0,1[\,.$$

It allows one to define, for any $\varepsilon\in[0,1[\,$, the function

$$Q(t;\varepsilon):=t\int\limits_t^{+\infty}\Big(\frac{\sigma(s)}{s}\Big)^{\varepsilon F_*}p(s)\,\mathrm{d}s \ \text{ for } \ t>0. \tag{5}$$

By using the lower and upper limits

$$Q_*(\varepsilon)=\liminf_{t\to+\infty} Q(t;\varepsilon), \quad Q^*(\varepsilon)=\limsup_{t\to+\infty} Q(t;\varepsilon), \tag{6}$$

we establish new Hille type oscillation criteria, which coincide with some well-known results in the case of ordinary differential equations (see, [2]).

Theorem 3. *Let conditions* (3) *hold and there exist* $\varepsilon\in[0,1[$ *such that*

$$Q^*(\varepsilon)>1.$$

Then every proper solution to equation (1) *is oscillatory.*

Theorem 4. *Let conditions* (3) *hold and there exist* $\varepsilon\in[0,1[$ *such that*

$$Q_*(\varepsilon)>\frac{1}{4}. \tag{7}$$

Then every proper solution to equation (1) *is oscillatory.*

Finally, we show two examples, where we can apply oscillatory criteria from Theorems 1 and 3 succesfully.

Example 1. Let us consider the following equation

$$u''(t)+\frac{1}{(t+1)^2}\,u\big((t+1)^2\big)=0 \ \text{ for } \ t\geq 0. \tag{8}$$

One can see that

$$F_*=\liminf_{t\to+\infty} t\int\limits_t^{+\infty}\frac{1}{(s+1)^2}\,\mathrm{d}s=\liminf_{t\to+\infty}\frac{t}{t+1}=1,$$

i.e. condition (3) is fulfilled.

On the other hand, if we put $\lambda=\varepsilon=\frac{1}{2}$, then we obtain

$$\int\limits_0^{+\infty} s^\lambda\Big(\frac{\sigma(s)}{s}\Big)^{\varepsilon F_*}p(s)\,\mathrm{d}s=\int\limits_0^{+\infty}\frac{1}{s+1}\,\mathrm{d}s=+\infty.$$

Consequently, condition (4) is satisfied and according to Theorem 1 every proper solution to equation (8) is oscillatory.

Example 2. Let us consider the equation

$$u''(t)+\frac{2+\sin(\ln t)+\cos(\ln t)}{t^2}\,u(4t)=0 \ \text{ for } \ t>0. \tag{9}$$

One can show that

$$F_*=\liminf_{t\to+\infty} t\int\limits_t^{+\infty}\frac{2+\sin(\ln s)+\cos(\ln s)}{s^2}\,\mathrm{d}s=\liminf_{t\to+\infty}(2+\cos(\ln t))=1,$$

i.e. condition (3) is fulfilled.

On the other hand, if we put $\varepsilon = \frac{1}{2}$, then from notation (5) and (6) we obtain

$$Q_*\Big(\frac{1}{2}\Big) = \liminf_{t\to+\infty} Q\Big(t;\frac{1}{2}\Big)$$
$$= \liminf_{t\to+\infty} t\int\limits_t^{+\infty} 2\,\frac{2+\sin(\ln s)+\cos(\ln s)}{s^2}\,\mathrm{d}s = \liminf_{t\to+\infty}(4+2\cos(\ln t)) = 2.$$

Consequently, condition (7) is satisfied and according to Theorem 3 every proper solution to equation (9) is oscillatory.

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On the Neumann Problem for Second Order Differential Equations with a Deviating Argument

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We consider the differential equation

$$u''(t) = f(t, u(\tau(t))) \tag{1}$$

with the Neumann boundary conditions

$$u'(a) = c_1, \quad u'(b) = c_2, \tag{2}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a function satisfying the local Carathèodory conditions, and $\tau : [a, b] \to [a, b]$ is a measurable function.

For $\tau(t) \equiv t$ problem (1), (2) is investigated in detail (see, e.g., [2–7] and the references therein). However, for $\tau(t) \not\equiv t$ that problem remains practically unstudied. The exception is only the case where equation (1) is linear (see [1]).

Theorems below on the solvability and unique solvability of problem (1), (2) are analogues of the theorem by I. Kiguradze [5] for differential equations with a deviating argument.

We use the following notation.

$$\mu(t) = \bigg(\frac{b-a}{2} + \Big|\frac{b+a}{2} - t\Big|\bigg)^{\frac{1}{2}}, \quad \mu_\tau(t) = \operatorname{ess\,sup}\Big\{|\tau(t) - \tau(t_0)|^{\frac{1}{2}} : \ a \le t_0 \le b\Big\},$$

$$\chi_\tau(t) = \begin{cases} 1 & \text{if } \tau(t) \neq t, \\ 0 & \text{if } \tau(t) = t, \end{cases}$$

$$f^*(t, x) = \max\Big\{|f(t, y)| : \ |y| \le x\Big\} \ \text{ for } \ t \in [a, b], \ \ x \ge 0.$$

Theorems 1 and 2 concern the cases where on the set $[a, b] \times \mathbb{R}$ either the inequality

$$f(t, x)\operatorname{sgn}(x) \ge \varphi(t, x), \tag{3}$$

or the inequality

$$f(t, x)\operatorname{sgn}(x) \le -\varphi(t, x) \tag{4}$$

is satisfied. Here the function $\varphi : [a, b] \times \mathbb{R} \to \mathbb{R}$ is such that $\varphi(\cdot, x)$ is Lebesgue integrable in the interval $[a, b]$,

$$\varphi(t, x) \ge \varphi(t, y) \ \text{ for } \ xy \ge 0, \ \ |x| \ge |y|, \tag{5}$$

and

$$\lim_{|x| \to +\infty} \int_a^b \varphi(t, x)\, dt > |c_1 - c_2|. \tag{6}$$

Theorem 1. *If conditions* (3), (5), (6), *and*

$$\lim_{|x| \to +\infty} \bigg(\frac{1}{|x|} \int_a^b |t - \tau(t)|^{\frac{1}{2}} f^*(t, \mu_\tau(t)x)\, dt\bigg) < 1$$

are fulfilled, then problem (1), (2) *has at least one solution.*

Theorem 2. *If conditions* (4)–(6)*, and*

$$\limsup_{|x|\to+\infty}\bigg(\frac{1}{|x|}\int\limits_a^b \mu(t)f^*(t,\mu_\tau(t)x)\,dt\bigg)<1$$

are fulfilled, then problem (1), (2) *has at least one solution.*

Example 1. Consider the differential equation

$$u''(t)=p(t)\frac{u(\tau(t))}{1+|u(\tau(t))|} \tag{7}$$

with the Lebesgue integrable coefficient $p:[a,b]\to\mathbb{R}$. It is clear that if $c_1\neq c_2$ and

$$\int\limits_a^b |p(t)|\,dt\le |c_1-c_2|,$$

then problem (7), (2) has no solution. Thus from Theorem 1 (from Theorem 2) it follows that if $c_1\neq c_2$ and

$$p(t)\ge 0 \ \text{ for } \ a\le t\le b \quad (p(t)\le 0 \ \text{ for } \ a\le t\le b),$$

then problem (7), (2) is solvable if and only if

$$\int\limits_a^b p(t)\,dt>|c_1-c_2| \qquad \bigg(\int\limits_a^b p(t)\,dt<-|c_1-c_2|\bigg).$$

The above example shows that condition (6) in Theorems 1 and 2 is unimprovable and it cannot be replaced by the condition

$$\lim_{|x|\to+\infty}\int\limits_a^b \varphi(t,x)\,dt\ge |c_1-c_2|.$$

Theorem 3. *Let on the set* $[a,b]\times\mathbb{R}$ *the conditions*

$$\big(f(t,x)-f(t,y)\big)\operatorname{sgn}(x-y)\ge p_1(t)|x-y|,$$
$$\chi_\tau(t)\big|f(t,x)-f(t,y)\big|\le p_2(t)|x-y|$$

hold, where $p_i:[a,b]\to\mathbb{R}_+$ $(i=1,2)$ *are integrable functions such that*

$$\int\limits_a^b p_1(t)\,dt>0,\quad \int\limits_a^b |t-\tau(t)|^{\frac12}\mu_\tau(t)p_2(t)\,dt<1.$$

Then problem (1), (2) *has one and only one solution.*

Theorem 4. *Let on the set* $[a,b]\times\mathbb{R}$ *the condition*

$$-p_2(t)|x-y|\le \big(f(t,x)-f(t,y)\big)\operatorname{sgn}(x-y)\le -p_1(t)|x-y|$$

be satisfied, where $p_i:[a,b]\to\mathbb{R}_+$ $(i=1,2)$ *are integrable functions such that*

$$\int\limits_a^b p_1(t)\,dt>0,\quad \int\limits_a^b \mu(t)\mu_\tau(t)p_2(t)\,dt<1.$$

Then problem (1), (2) *has one and only one solution.*

Example 2. Let $I \subset [a,b]$ and $[a,b] \setminus I$ be the sets of positive measure, and $\tau : [a,b] \to \mathbb{R}$ be the measurable function such that

$$\tau(t) = t \ \text{ for } \ t \in I, \quad \tau(t) \neq t \ \text{ for } \ t \notin I.$$

Let, moreover,

$$f(t,x) = \begin{cases} p(t)\big(\exp(|x|) - 1\big)\operatorname{sgn}(x) + q(t) & \text{if } \ t \in I, \\ p(t)x + q(t) & \text{if } \ t \in [a,b] \setminus I, \end{cases}$$

where $p : [a,b] \to (0,+\infty)$ and $q : [a,b] \to \mathbb{R}$ are integrable functions, and

$$\int\limits_a^b |t - \tau(t)|^{\frac{1}{2}} \mu_\tau(t) p(t)\, dt < 1. \tag{8}$$

Then by Theorem 3 problem (1), (2) has one and only one solution.

Consequently, Theorem 3 covers the case, where $\tau(t) \neq t$ and for any t from some set of positive measure the function f is rapidly increasing in the phase variable, i.e.,

$$\lim_{|x| \to +\infty} \frac{f(t,x)}{x} = +\infty.$$

At the end, consider the linear differential equation

$$u''(t) = p(t)u(\tau(t)) + q(t) \tag{9}$$

with integrable coefficients $p : [a,b] \to \mathbb{R}$ and $q : [a,b] \to \mathbb{R}$.

Theorem 3 yields the following statement.

Corollary 1. *If*

$$p(t) \geq 0 \ \textit{ for } \ a < t < b, \quad \int\limits_a^b p(t)\, dt > 0, \tag{10}$$

and inequality (8) *holds, then problem* (9), (2) *has one and only one solution.*

If $\tau(t) \equiv t$, then condition (10) guarantees the unique solvability of problem (9), (2). And if $\tau(t) \not\equiv t$, then this is not so. Indeed, if, for example, $a = 0$, $b = \pi$, $\tau(t) = \pi - t$, $p(t) = 1$ for $a \leq t \leq b$, and the function q satisfies the inequality

$$\int\limits_a^b q(t)\cos(t)\, dt \neq -c_1 - c_2, \tag{11}$$

then problem (9), (2) has no solution.

Therefore, condition (8) in Corollary 1 is essential and it cannot be omitted. However, the question on the unimprovability of that condition remains open.

Theorem 4 yields the following statement.

Corollary 2. *If*

$$p(t) \leq 0 \ \textit{ for } \ a < t < b, \quad 0 < \int\limits_a^b \mu(t)\mu_\tau(t)|p(t)|\, dt < 1, \tag{12}$$

then problem (9), (2) *has one and only one solution.*

Note that problem (9), (2) may be uniquely solvable also in the case where the differential equation
$$u''(t) = p(t)u(t) + q(t) \tag{13}$$
has no solution satisfying the boundary conditions (2). For example, if
$$p(t) = -1 \text{ for } a \le t \le b, \quad a = 0, \quad b = \pi, \tag{14}$$
and condition (11) holds, then problem (13), (2) has no solution. On the other hand, if along with (14) we have
$$\tau(t) = \begin{cases} t^3 & \text{for } 0 \le t \le \pi^{-1}, \\ \pi^{-3} & \text{for } \pi^{-1} \le t \le \pi, \end{cases}$$
then condition (12) is satisfied and, according to Corollary 2, problem (9), (2) is uniquely solvable for any c_1 and c_2.

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On Stability Properties of Global Attractors of Impulsive Infinite-Dimensional Systems

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One of the most popular mathematical approaches to describe the evolutionary processes with instantaneous changes is the theory of impulsive differential equations. Today due to the works of A. M. Samoĭlenko, M. O. Perestyuk [11], R. Lakshmikantham, D. Bainov [7] and many other mathematicians the theory of impulsive systems became a separate branch of the general theory of differential equations. An important subclass of systems with impulsive perturbations is impulsive (or discontinuous) dynamical systems, which are described by an autonomous evolutionary system, which trajectories has impulsive perturbations at moments of intersection with fixed subset of the phase space (impulsive set). Unlike systems with impulsive perturbations at fixed moments of time, the construction of a qualitative theory for impulsive dynamic systems is far from complete understanding. Various aspects of qualitative theory of such systems in the finite-dimensional case was studied in [1, 3, 6, 8, 9]. For infinite-dimensional dissipative systems one of the most powerful tools of investigation of the qualitative behavior of solutions is the theory of global attractors [12]. In [10] the theory of global attractors was used for investigation of impulsive systems with fixed moments of impulsive perturbations. However, the transferring of the basic constructions of this theory to impulsive dynamical systems has a fundamental problem – the absence of continuous dependence of the initial data. This requires a new concept for the global attractor, and for its main characteristics (invariance, stability, robustness). One of possible approaches was proposed in [5] and was based on the concept of a uniform attractor for non-autonomous systems – a compact minimal uniformly-attracting set. The absence an invariance condition in such definition allowed not to use strict conditions for the behavior of trajectories in the neighborhood of the impulsive set (see [2]) and obtain results about the existence and properties of the attractor for weakly nonlinear impulse-perturbed equations. Further in [4], this approach was extended to other classes of impulsive systems, in particular those for which the uniqueness of the solution of Cauchy problem is not fulfilled. The aim of this work is to study stability concept of the global attractor of the impulsive dynamical system and apply obtained results to a weakly nonlinear impulsive system.

Let $G:\mathbb{R}_+\times X\to R$ be a semigroup, which is given in the normed space X (it is not necessarily continuous), $\beta(X)$ be a set of all nonempty bounded subsets of X.

Definition ([5])**.** A compact set $\Theta\subset X$ is called a global attractor of G if

1) Θ is uniformly attracting set, i.e.,

$$\forall\, B\in\beta(X)\quad \operatorname{dist}(G(t,B),\Theta)\longrightarrow 0,\ \ t\to\infty;$$

2) Θ is minimal among closed uniformly attracting sets.

If global attractor exists, then it is unique. If G has a global attractor in a classical sense [12], i.e. if there exists a compact set $A\subset X$, which satisfies 1) and it is invariant ($G(t,A)=A\ \forall\, t\geq 0$), then A satisfies this definition. On the contrary, it is true that $G(t,\,\cdot\,)$ is continuous. It means that if Θ is global attractor of G and the following condition is fulfilled:

$$\forall\, t\geq 0 \text{ a map } x\to G(t,x) \text{ is continuous,} \tag{1}$$

then Θ is a global attractor of G in a classical sense, in particular

$$\forall\, t \geq 0 \quad \Theta = G(t, \Theta).$$

That is why, in impulsive (discontinuous) dynamical systems, where the semigroup G does not usually satisfy the condition (1), it's better to use this definition.

Another advantage of this definition is the following criterion:

Lemma 1 ([10])**.** *Assume that G satisfies dissipativity condition:*

$$\exists\, B_0 \in \beta(X) \ \ \forall\, B \in \beta(X) \ \ \exists\, T = T(B) \ \ \forall\, t \geq T \quad G(t, B) \subset B_0.$$

Then G has global attractor Θ if and only if, when G is asymtotically compact, i.e., $\forall\,\{x_n\} \in \beta(X)$ $\forall\,\{t_n \nearrow \infty\}$ sequence $\{G(t_n, x_n)\}$ is precompact. Moreover,

$$\Theta = \omega(B_0) := \bigcap_{\tau>0} \overline{\bigcup_{t\geq\tau} G(t, B_0)}.$$

Using this criterion, in [5], classes of impulsive infinite-dimensional dissipative problems, which have a global attractor were identified. In particular, it was shown that for sufficiently small $\varepsilon > 0$, solutions of the impulsive problem

$$\begin{aligned} &\frac{\partial y}{\partial t} = \Delta y - \varepsilon f(y), \ \ t > 0, \ \ x \in \Omega, \\ &y\big|_{\partial\Omega} = 0, \end{aligned} \tag{2}$$

where $\Omega \subset \mathbb{R}^p$ is bounded domain, f – continuously-differentiable function, $\forall\, y \in R$ $f'(y) \geq -C$, $|f(y)| \leq C$, in phase space $X = L^2(\Omega)$ with impulsive set

$$M = \big\{ y \in X \mid \ \ (y, \psi_1) = a \big\} \tag{3}$$

and impulsive map $I : M \mapsto X$

$$\text{for} \ \ y = a\psi_1 + \sum_{i=2}^{\infty} c_i\psi_i, \quad Iy = (1+\mu)a\psi_1 + \sum_{i=2}^{\infty} c_i\psi_i, \tag{4}$$

generate the semigroup G_ε, which has a global attractor Θ_ε, moreover,

$$\operatorname{dist}(\Theta_\varepsilon, \Theta_0) \longrightarrow 0, \ \ \varepsilon \to 0,$$

where

$$\Theta_0 = \bigcup_{t\in[0,\ln(1+\mu)]} \big\{ (1+\mu)ae^{-t}\psi_1 \big\} \cup \{0\} \tag{5}$$

is a global attractor of G_0, generated by (2)–(4), where $\varepsilon = 0$.

In case $\varepsilon = 0$ the set Θ_0 has a non-empty intersection with an impulsive set M is not invariant regarding the G_0. However, for $\Theta_0 \setminus M$ we obtain the next equality:

$$\forall\, t \geq 0 \quad G_0(t, \Theta_0 \setminus M) = \Theta_0 \setminus M. \tag{6}$$

This provides the basis to analyze the invariance and stability of the set $\Theta_\varepsilon \setminus M$ for the impulsive dynamical system G_ε with the global attractor Θ_ε. The next well-known result establishes the connection between different definitions of the stability of compact sets regarding the semigroup G.

Lemma 2. *If $A \subset X$ is compact and the condition is fulfilled:*

$$\forall\, x_n \to x \in A \;\; \forall\, t_n \geq 0 \quad \{G(t_n, x_n)\}\text{-precompact}, \tag{7}$$

then the following properties are equivalent:

1) $\forall\,\varepsilon > 0 \;\; \forall\, x \in A \;\; \exists\,\delta > 0 \;\; \forall\, t \geq 0$
$$G(t, O_\delta(x)) \subset O_\varepsilon(A);$$

2) $\forall\,\varepsilon > 0 \;\; \exists\,\delta > 0 \;\; \forall\, t \geq 0$
$$G(t, O_\delta(A)) \subset O_\varepsilon(A);$$

3) $\forall\, x \in A \;\; \forall\, y \notin A \;\; \exists\,\delta > 0 \;\; \forall\, t \geq 0$
$$G(t, O_\delta(x)) \cap O_\delta(y) = \varnothing;$$

4) $A = D^+(A) := \bigcup\limits_{x \in A} \big\{ y \mid \;\; y = \lim G(t_n, x_n), \;\; x_n \to x, \;\; t_n \geq 0 \big\}.$

Remark. Because of the construction $A \subset D^+(A)$, property 4) is equivalent to an embedding $D^+(A) \subset A$.

The following result is easily obtained by contradiction.

Lemma 3. *If Θ is global attractor of G and the condition is fulfilled*

$$\forall\, x_n \to x \in \Theta \;\; \forall\, t_n \to t \geq 0 \quad G(t_n, x_n) \longrightarrow G(t, x), \tag{8}$$

then Θ is stable in the sense of 1)–4).

The condition (8) is crucial. Its failure leads to the fact that the attractor Θ may not be stable in any sense of 1)–4). For example, for a semigroup G_0 attractor Θ_0, which is given by (5), does not satisfy 1)–4). Unfortunately, the same thing holds for an invariant set $\Theta_0 \setminus M$. However, it is easy to see that for G_0

$$D^+(\Theta_0 \setminus M) \subset \overline{\Theta_0 \setminus M}. \tag{9}$$

The main aim of this work is to prove that the properties (6), (9) are also fulfilled for a weakly nonlinear case.

Theorem. *For sufficiently small $\varepsilon > 0$ global attractor Θ_ε of impulsive dynamical system G_ε, which is generated by problem* (2)–(4)*, is invariant and stable in the following sense:*

$$\forall\, t \geq 0 \quad \Theta_\varepsilon \setminus M = G_\varepsilon(t, \Theta_\varepsilon \setminus M),$$
$$\Theta_\varepsilon = \overline{\Theta_\varepsilon \setminus M},$$
$$D^+(\Theta_\varepsilon \setminus M) \subset \overline{\Theta_\varepsilon \setminus M}.$$

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Antiperiodic Impulsive Problem Via Distributional Differential Equation

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1 Introduction

Analytical results presented here are based on a common research with Jan Tomeček. We focus our attention to problems where the evolution of systems is affected by rapid changes which is modelled by means of differential equations with impulses. Let us stress that abrupt changes of solutions of impulsive problems imply that such solutions do not preserve the basic properties which are associated with non-impulsive problems.

We work with a finite number $m \in \mathbb{N}$ of impulses on the compact interval $[0,T] \subset \mathbb{R}$. Most papers deal with *fixed-time* impulses where the moments of impulses

$$0 < t_1 < t_2 < \cdots < t_m < T$$

are fixed and known before. This is a special case of so called *state-dependent* impulses where the impulse moments depend on a solution of a differential equations and different solutions can have different moments of jumps. We present two ways of determining the impulse dependence on the solution:

- Let $\tau_1, \dots, \tau_m$ be functionals defined on a suitable functional space X and having values in $(0,T)$. Then the impulse moments $t_1, \dots, t_m$ are given as

  $$t_i = \tau_i(x) \in (0,T), \ \ x \in X, \ \ i = 1, \dots, m.$$

- Let $\gamma_1, \dots, \gamma_m$ be functions (barriers) defined on a suitable interval $[a,b] \subset \mathbb{R}$ and having values in $(0,T)$. Then the impulse moments $t_1, \dots, t_m$ are given as

  $$t_i = \gamma_i(x(t_i)) \in (0,T), \ \ x \in X, \ \ i = 1, \dots, m.$$

  In order to get the desired number of impulse points in this case it is necessary to impose additional conditions (transversality conditions) on $\gamma_1, \dots, \gamma_m$.

2 Periodic problems

A lot of papers studying impulsive periodic problems are population or epidemic models. Differential equations in these models have mostly a form of *autonomous* planar differential systems. However, there are a few existence results for non-autonomous periodic problems with state-dependent impulses:

- The first attempts can be seen in the monographs [1] and [11] investigating periodic solutions of quasilinear systems with state-dependent impulses.

- One of the first results that are trackable via Scopus is obtained by Bajo and Liz [2], where a scalar nonlinear first order differential equation is studied under the assumptions of the existence and uniqueness of a solution of the corresponding initial value problem with state-dependent impulses, and of the existence of lower and upper solutions of the periodic problem with state-dependent impulses. Their method of proof is based on a fixed point theorem for a Poincaré operator.
- A generalization to a system is done by Frigon and O'Regan [8] under the assumption that there exists a solution tube to the problem. They applied a fixed point theorem to a multi-valued Poincaré operator.
- Further interesting result is reached by Domoshnitsky, Drakhlin and Litsyn in [7]. They transformed a linear system with delay and state-dependent impulses to a system with fixed-time impulses and then they proved the existence of positive periodic solutions.
- Recently, Tomeček [12] proved the existence of a periodic solution to a nonlinear second order differential equation with ϕ-Laplacian and state-dependent impulses via lower and upper solutions method.

All the above problems have a *"classical"* formulation in which impulse conditions are given out of a differential equation. Let us demonstrate it on the van der Pol equation

$$x'(t) = y(t), \quad y'(t) = \mu\Big(x(t) - \frac{x^3(t)}{3}\Big)' - x(t) + f(t) \ \text{ for a.e. } \ t \in [0,T], \tag{1}$$

with the state-dependent impulse conditions

$$\triangle y(\tau_i(x)) = \mathcal{J}_i(x), \ \ i = 1, \dots, m, \tag{2}$$

where T, $\mu > 0$, $m \in \mathbb{N}$, τ_i, $\mathcal{J}_i$, $i = 1, \dots, m$, are functionals defined on the set of T-periodic functions of bounded variation and f is T-periodic Lebesgue integrable on $[0,T]$. Here x' and y' denote the classical derivatives of the functions x and y, respectively, $\triangle y(t) = y(t+) - y(t-)$.

Another possible formulation of the T-periodic problem with state-dependent impulses at the points $\tau_i(x) \in (0,T)$ can be written in the form of the *distributional* differential equation

$$D^2 z = \mu D\Big(z - \frac{z^3}{3}\Big) - z + f + \frac{1}{T}\sum_{i=1}^{m} \mathcal{J}_i(z)\delta_{\tau_i(z)}, \tag{3}$$

where Dz denotes the distributional derivative of a T-periodic function z of bounded variation and $\delta_{\tau_i(z)}$, $i = 1, \dots, m$, are the Dirac T-periodic distributions which involve impulses at the state-dependent moments $\tau_i(z)$, $i = 1, \dots, m$.

Results on the existence of periodic solutions to distributional equations of the type (3) have been reached by Belley, Virgilio and Guen in [4–6].

3 Antiperiodic problems

The study of antiperiodic solutions is closely related to the study of periodic solutions and their existence plays an important role in characterizing the behaviour of nonlinear differential equations. First order differential systems with antiperiodic conditions can describe neural networks and second order differential equations can serve as physical models, for example: Rayleigh equation (acoustics), Duffing, Liénard or van der Pol equations (oscillation theory).

In the study of T-antiperiodic solutions we work with functional spaces defined below which consist of *real-valued $2T$-periodic functions*: NBV is the space of functions of bounded variation normalized in the sense that $x(t)=\frac{1}{2}\,(x(t+)+x(t-))$, $\widetilde{\mathrm{NBV}}$ represents the Banach space of functions $x\in\mathrm{NBV}$ having zero mean value, which is equipped with the norm equal to the total variation $\mathrm{var}(x)$, C^∞ is the classical Fréchet space of functions having derivative of an arbitrary order, for finite $\Sigma\subset[0,2T)$ we denote by PAC_Σ the set of all functions $x\in\mathrm{NBV}$ such that $x\in\mathrm{AC}(J)$ for each interval $J\subset[0,2T]$ for which $\Sigma\cap J=\varnothing$, $\widetilde{\mathrm{AC}}=\mathrm{AC}\cap\widetilde{\mathrm{NBV}}$; for finite $\Sigma\subset[0,2T)$ we denote $\widetilde{\mathrm{PAC}}_\Sigma=\mathrm{PAC}_\Sigma\cap\widetilde{\mathrm{NBV}}$.

The first result about the existence and uniqueness of antiperiodic solutions of the distributional Liénard equation with state-dependent imupulses has been reached by Belley and Bondo in [3].

- In order to study T-antiperiodic solutions for the classical differential Liénard equation (1) with state-dependent impulses (2) we assume that f in (1) is T-antiperiodic and that the condition
$$\tau_i(x)\neq\tau_j(x)\ \text{ for all }\ i,j=1,\dots,m,\ \ i\neq j,\ \ x\in\widetilde{\mathrm{AC}}$$
is fulfilled.

- For $x\in\widetilde{\mathrm{AC}}$ we denote the set
$$\Sigma_x:=\big\{\tau_1(x),\dots,\tau_m(x),\tau_1(x)+T,\dots,\tau_m(x)+T\big\},$$
and say that the couple $(x,y)\in\widetilde{\mathrm{AC}}\times\widetilde{\mathrm{PAC}}_{\Sigma_x}$ is a *solution* of the impulsive problem (1), (2) if it satisfies the differential equation (1) and the impulse conditions (2). Such solution (x,y) is called *antiperiodic* if
$$x(0)=-x(T),\quad y(0)=-y(T).$$

- Motivated by [3] we construct the following distributional van der Pol equation
$$D^2z=\mu D\Big(z-\frac{z^3}{3}\Big)-z+f+\frac{1}{2T}\sum_{i=1}^{m}\mathcal{J}_i(z)\varepsilon_{\tau_i(z)}. \tag{4}$$
The Dirac T-periodic distribution δ_τ from (3) is replaced by the T-antiperiodic distribution $\varepsilon_\tau:=\delta_\tau-\mathbb{T}_T\delta_\tau$ in (4). Here $\mathbb{T}_T$ means the translation operator.

- We say that a function $z\in\widetilde{\mathrm{NBV}}$ is a *solution* of the distributional equation (4) if
$$\langle D^2z,\varphi\rangle=\Big\langle\mu D\Big(z-\frac{z^3}{3}\Big)-z+f+\frac{1}{2T}\sum_{i=1}^{m}\mathcal{J}_i(z)\varepsilon_{\tau_i(z)},\varphi\Big\rangle\ \text{ for every }\ \varphi\in\mathrm{C}^\infty. \tag{5}$$

By means of the method of apriori estimates and the Schauder fixed point theorem we get a new existence result for (4). Then, using an equivalence between classical and distributional problems which we proved in [9], we get the first result about the existence of a T-antiperiodic solution of equation (1) with state-dependent impulses (2):

Theorem 3.1. *Assume $T\in(0,\sqrt{3})$ and*

1. *f is T-antiperiodic and Lebesgue integrable on $[0,T]$;*
2. *$\tau_1,\dots,\tau_m$ are continuous with values in $(0,T)$;*
3. *if $i\neq j$, then $\tau_i(x)\neq\tau_j(x)$ for each $x\in\widetilde{\mathrm{AC}}$;*

4. $\mathcal{J}_1, \dots, \mathcal{J}_m$ *are continuous and bounded.*

Then there exists $\mu_0 > 0$ *such that for each* $\mu \in (0, \mu_0]$ *the problem* (1)*,* (2) *has a* T*-antiperiodic solution.*

Theorem 3.1 and its generalizations are published in [10] where the optimal value of μ_0 is also specified.

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Investigation of Transcendental Boundary Value Problems Using Lagrange Interpolation

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We study the non-local problem

$$u'(t) = f(t, u(t)), \ \ t \in [a, b]; \quad \phi(u) = d, \tag{1}$$

where $\phi : C([a, b], \mathbb{R}^n)$ is a vector functional (possibly non-linear), $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function defined on a bounded set and d is a given vector.

In [9], we have suggested an approach to this problem which involves a kind of reduction to a parametrized family of problems with separated conditions

$$u'(t) = f(t, u(t)), \ \ t \in [a, b], \tag{2}$$

$$u(a) = \xi, \quad u(b) = \eta, \tag{3}$$

where $z := \operatorname{col}(z_1, \dots, z_n)$, $\eta := \operatorname{col}(\eta_1, \dots, \eta_n)$ are unknown parameters. The techniques of [9] are based on properties of the iteration sequence $\{u_m(\,\cdot\,, \xi, \eta) : m \geq 0\}$,

$$u_0(t, \xi, \eta) := \Big(1 - \frac{t-a}{b-a}\Big) z + \frac{t-a}{b-a}\, \eta, \tag{4}$$

$$\begin{aligned} u_m(t, \xi, \eta) := u_0(t, \xi, \eta) &+ \int_a^t f\big(s, u_{m-1}(s, \xi, \eta)\big)\, ds \\ &- \frac{t-a}{b-a} \int_a^b f\big(s, u_{m-1}(s, \xi, \eta)\big)\, ds, \ \ t \in [a, b], \ \ m = 1, 2, \dots . \end{aligned} \tag{5}$$

Formulas (4) and (5) are used to compute the corresponding functions explicitly for certain values of m, which, under additional conditions, allows one to prove the solvability of the problem and construct approximate solutions.

The efficiency of application of this approach depends on the complexity of the non-linear terms appearing in (1). If the function f involves transcendental non-linearities with respect the second variable, the explicit computation according to (5) in the general case cannot be carried out due to the impossibility to find the exact values of the corresponding integrals. Here, we show how this difficulty can be overcome using the polynomial interpolation.

At first, we recall some results of the theory of approximations [1, 2, 4]. In a similar situation, we have used these facts in [7].

Denote by P_q the set of all polynomials of degree not higher than q on $[a,b]$. For any continuous $y:[a,b]\to\mathbb{R}$, there exists [2, 7] a unique polynomial $y^q\in H_q$ for which $\|y-y^q\|=\inf\limits_{p\in H_q}\|y-p\|=:E_q(y)$, where $\|\cdot\|$ is the uniform norm in $C([a,b])$. Then y^q is the polynomial of the *best uniform approximation* of y in H_q and the number $E_q(y)$ is called the *error* of the best uniform approximation.

For given continuous function $y:[a,b]\to\mathbb{R}$ and a natural number q, denote by T_qy the Lagrange interpolation polynomial of degree q such that $(T_qy)(t_i)=y(t_i)$, $i=1,2,\dots,q+1$, where

$$t_i=\frac{b-a}{2}\cos\frac{(2i-1)\pi}{2(q+1)}+\frac{a+b}{2}\,,\quad i=1,2,\dots,q+1, \tag{6}$$

are the Chebyshev nodes translated from $(-1,1)$ to the interval (a,b).

Lemma 1 ([5, p. 18])**.** *For any $q\geq 1$ and a continuous $y:[a,b]\to\mathbb{R}$, the corresponding interpolation polynomial constructed with the Chebyshev nodes admits the estimate*

$$|y(t)-(T_qy)(t)|\leq\Big(\frac{2}{\pi}\ln q+1\Big)E_q(y),\quad t\in[a,b]. \tag{7}$$

Definition 1. Let $y:[a,b]\to\mathbb{R}$ be continuous. The function

$$\delta\longmapsto\omega(y;\delta):=\sup_{t,s\in[a,b]:\ |t-s|\leq\delta}|y(t)-y(s)|$$

is called its *modulus of continuity.*

Note that $\delta\mapsto\omega(y;\delta)$ is a continuous non-decreasing function. The function y is uniformly continuous if and only if $\lim\limits_{\delta\to 0}\omega(y;\delta)=0$ [3, p. 131].

Lemma 2 (Jackson's theorem; [4, p. 22])**.** *If $y\in C([a,b],\mathbb{R})$, $q\geq 1$, then*

$$E_q(y)\leq 6\omega\Big(y;\frac{b-a}{2q}\Big). \tag{8}$$

Definition 2. A function $y:[a,b]\to\mathbb{R}$ satisfies the *Dini–Lipschitz condition* [2, p. 50] if

$$\lim_{\delta\to 0}\omega(y;\delta)\ln\delta=0.$$

It follows from (8) that

$$\lim_{q\to\infty}E_q(y)\ln q=0 \tag{9}$$

for any y satisfying the Dini–Lipschitz condition. In view of (7), equality (9) ensures the uniform convergence of interpolation polynomials at Chebyshev nodes for this class of functions. In particular, every α-Hölder continuous function $y:[a,b]\to\mathbb{R}$ with $0<\alpha\leq 1$ satisfies the Dini–Lipschitz condition.

Here, we need to construct interpolation polynomials for functions obtained as a result of application of the Nemytskii operator generated by the non-linearity from (1) and defined by the formula

$$(Ny)(t):=f(t,y(t)),\quad t\in[a,b], \tag{10}$$

for any y from $C([a,b],\mathbb{R}^n)$.

Lemma 3. *Let the function $f : [a,b] \times \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, satisfy the condition*

$$|f(t,x) - f(s,y)| \le k|t-s|^\alpha + K|x-y| \tag{11}$$

for $\{t,s\} \subset [a,b]$, $\{x,y\} \subset \Omega$, where $\alpha \in (0,1]$, $k \in \mathbb{R}^n_+$ and K is an $n \times n$ matrix with non-negative entries. Then, for any Hölder-continuous function $u : [a,b] \to \Omega$, the corresponding function Nu also has this property.

Here and below, the absolute value sign and inequalities between vectors are understood componentwise.

Rewrite (5) in the form

$$u_m(t,\xi,\eta) = u_0(t,\xi,\eta) + \big(\Lambda N u_{m-1}(\,\cdot\,,\xi,\eta)\big)(t), \ \ t \in [a,b], \ \ m = 1,2,\dots, \tag{12}$$

where N is the Nemytskii operator (10) and

$$(\Lambda y)(t) := \int_a^t y(s)\,ds - \frac{t-a}{b-a}\int_a^b y(s)\,ds, \ \ t \in [a,b], \tag{13}$$

for any y from $C([a,b],\mathbb{R}^n)$.

Fix a natural number q and extend the notation $T_q y$ for vector-functions by putting $T_q y := \operatorname{col}(T_q y_1, T_q T_q y_2, \dots, T_q y_n)$ for any $y = (y_i)_{i=1}^n$ from $C([a,b],\mathbb{R}^n)$, where $T_q y_i$ is the qth order interpolation polynomial for y_i constructed with the Chebyshev nodes (6).

Introduce now a modified iteration process keeping formula (4) for $u_0(\,\cdot\,,\xi,\eta)$:

$$v_0^q(\,\cdot\,,\xi,\eta) := u_0(\,\cdot\,,\xi,\eta) \tag{14}$$

and replacing (12) by the formula

$$v_m^q(t,\xi,\eta) := u_0(t,\xi,\eta) + \big(\Lambda T_q N v_{m-1}^q(\,\cdot\,,\xi,\eta)\big)(t), \ \ t \in [a,b], \ \ m \ge 1. \tag{15}$$

For any q, formula (15) defines a vector polynomial $v_m^q(\,\cdot\,,\xi,\eta)$ of degree $q+1$ (in particular, all these functions are continuously differentiable), which, moreover, satisfies the two-point boundary conditions (3). The coefficients of the interpolation polynomials depend on the parameters ξ and η.

Note that, under condition (11), in view of Lemma 3, the function $Nv_{m-1}^q(\,\cdot\,,\xi,\eta))$ appearing in (15) always satisfies the Dini–Lipschitz condition and, therefore, the corresponding interpolation polynomials at Chebyshev nodes uniformly converge to it.

Similarly to (12), functions (15) can be used to study the auxiliary problems (2).

In order to proceed, we introduce some notation. The symbol I_n stands for the unit matrix of dimension n, $r(K)$ denotes a spectral radius of a square matrix K. If $z \in \mathbb{R}^n$ and $\varrho \in \mathbb{R}^n_+$, the componentwise ϱ-neighbourhood of z is defined as $O_\varrho(\xi) := \{\xi \in \mathbb{R}^n : \ |\xi - z| \le \varrho\}$ and, similarly, we put $O_\varrho(\Omega) := \bigcup\limits_{z \in \Omega} O_\varrho(z)$ for any bounded $\Omega \subset \mathbb{R}^n$.

Let $\delta_\Omega(f) := \max\limits_{(t,z) \in [a,b]\times\Omega} f(t,z) - \min\limits_{(t,z)\in[a,b]\times\Omega} f(t,z)$, with the componentwise computation of maxima and minima for vector functions. For $\Omega \subset \mathbb{R}^n$, put $P_{q,\Omega} := \{u : \ \ u = (u_i)_{i=1}^n, \ u_i \in P_q, \ u([a,b]) \subset \Omega\}$. For a given continuous function $f : [a,b] \times \Omega \to \mathbb{R}^n$, set

$$l_{q,\Omega}(f) := \Big(\frac{2}{\pi}\ln q + 1\Big) \sup_{p \in P_{q,\Omega}} E_q(Np), \tag{16}$$

where N is given by (10) and E_q is the error of the best minimax approximation in P_q. In (16), we use the notation $E_q(u) = \operatorname{col}(E_q(u_1),\dots,E_q(u_n))$ for $u = (u_i)_{i=1}^n$, the least upper bound in (16) is understood componentwise.

Fix certain closed bounded sets D_a, D_b in $\mathbb{R}^n$ and assume that we are looking for solutions u of problem (1) with $u(a) \in D_a$ and $u(b) \in D_b$. Put

$$\Omega := \big\{(1-\theta)\xi + \theta\eta : \ \xi \in D_a, \ \eta \in D_b, \ \theta \in [0,1]\big\}. \tag{17}$$

Theorem 1. *Let there exist a non-negative vector ϱ such that*

$$\varrho \geq \frac{b-a}{4}\,\big(\delta_{O_\varrho(\Omega)}(f) + 2l_{q,O_\varrho(\Omega)}(f)\big). \tag{18}$$

Assume, in addition, that f in (1) *satisfies condition* (11) *on the set $[a,b] \times O_\varrho(\Omega)$ with some k and K and the maximal eigenvalue of K satisfies the inequality*

$$r(K) < \frac{10}{3(b-a)}\,. \tag{19}$$

Then, for all fixed $(\xi,\eta) \in D_a \times D_b$:

1) *For any $m \geq 0$, $q \geq 1$, the function $v_m^q(\,\cdot\,,\xi,\eta)$ is a vector polynomial of degree $q+1$ having values in $O_\varrho(\Omega)$ and satisfying the two-point conditions* (3)*.*

2) *The limits*

$$v_\infty^q(\,\cdot\,,\xi,\eta) := \lim_{m\to\infty} v_m^q(\,\cdot\,,\xi,\eta), \ \ q \geq 1; \quad v_\infty(\,\cdot\,,\xi,\eta) := \lim_{q\to\infty} v_\infty^q(\,\cdot\,,\xi,\eta) \tag{20}$$

exist uniformly on $[a,b]$. Functions (20) *satisfy conditions* (3)*.*

3) *The estimate*

$$\big|v_\infty(t,\xi,\eta) - v_m^q(t,\xi,\eta)\big| \leq \frac{5}{9}\,\alpha_1(t)K_*^m(1_n - K_*)^{-1}\big(\delta_{O_\varrho(\Omega)}(f) + 2l_{q,O_\varrho(\Omega)}(f)\big)$$

holds for any $t \in [a,b]$, $m \geq 0$, where $K_* := 3K(b-a)/10$ and

$$\alpha_1(t) = 2(t-a)\Big(1 - \frac{t-a}{b-a}\Big), \ \ t \in [a,b].$$

As follows from [9], the assumptions of Theorem 1, in particular, ensure the uniform convergence of sequence (4), (5) and its limit coincides with $v_\infty(\,\cdot\,,\xi,\eta)$. It is important to point out that, in contrast to formula (12), every component of $v_m^q(\,\cdot\,,\xi,\eta)$, $m \geq 0$, is a *polynomial* of degree $q+1$.

Theorem 2. *Let $(\xi,\eta) \in \Omega$. Under the assumptions of Theorem* 1*, the following two conditions are equivalent:*

1) *The function $u := v_\infty(\,\cdot\,,\xi,\eta) : [a,b] \to \mathbb{R}^n$ is a continuously differentiable solution of problem* (1) *such that $u(a) \in D_a$, $u(b) \in D_b$, and $u([a,b]) \subset O_\varrho(\Omega)$.*

2) *The pair (ξ,η) satisfies the system of $2n$ determining equations*

$$\eta - \xi = \int_a^b f\big(s, v_\infty(s,\xi,\eta)\big)\,ds, \quad \phi(v_\infty(\,\cdot\,,\xi,\eta)) = d. \tag{21}$$

The determining system (21) can be investigated by using properties of its approximate version

$$\eta - \xi = \int_a^b f\big(s, v_m^q(s,\xi,\eta)\big)\,ds, \quad \phi(v_m^q(\,\cdot\,,\xi,\eta)) = d, \tag{22}$$

where m and q are fixed. The solvability analysis based on properties of equations (22) can be carried out by analogy to [6, 10].

Although (18) is more restrictive than the corresponding condition from [9]

$$\varrho \geq \frac{b-a}{4}\,\delta_{O_\varrho(\Omega)}(f), \tag{23}$$

one can note that, by virtue of (16) and Lemmas 1–3, $\lim\limits_{q\to\infty} l_{q,O_\varrho(\Omega)}(f) = 0$. Furthermore, both (18) and (23) can be relaxed (and, in fact, the resulting conditions eventually fulfilled) by using interval divisions similarly to [8]. The same observation can be made on condition (19), which, in particular, after one division in the ratio $1:2$, is replaced by the condition

$$r(K) < \frac{20}{3(b-a)}\,.$$

As an example of application of the approach based on the polynomial approximations (14), (15), consider the system of differential equations

$$\begin{aligned} u_1'(t) &= u_1(t)u_2(t),\\ u_2'(t) &= -\ln(2u_1(t)), \ \ t \in \Big[0, \frac{\pi}{4}\Big], \end{aligned} \tag{24}$$

with the non-linear two-point boundary conditions

$$u_1(0) - \Big(u_2\Big(\frac{\pi}{4}\Big)\Big)^2 = \frac{3}{8}, \quad u_1(0)u_2\Big(\frac{\pi}{4}\Big) = \frac{\sqrt{2}}{8}\,. \tag{25}$$

Choose the subsets D_a and D_b, where one looks for the values $u(a)$ and $u(b)$, e.g., as follows:

$$D_a = \big\{(u_1,u_2):\ 0.45 \leq u_1 \leq 0.75,\ 0.4 \leq u_2 \leq 0.55\big\}, \ \ D_b = D_a.$$

In this case, set (17) has the form $\Omega = D_a = D_b$. Putting $\varrho = \mathrm{col}\,(0.2, 0.35)$, we get

$$O_\varrho(\Omega) = \big\{(u_1,u_2):\ 0.25 \leq u_1 \leq 0.95,\ 0.05 \leq u_2 \leq 0.9\big\}.$$

A direct computation shows that the conditions of Theorem 1 are satisfied for q large enough. By solving the polynomial approximate determining equations (22), we obtain the numerical values of parameters ξ_1, ξ_2, η_1, η_2, which determine the polynomial approximate solution of the given problem (24), (25). In particular, for $q=4$ and $m=7$, the approximate solution $u_7^4 = \mathrm{col}\,(u_{71}^4, u_{72}^4)$ is a vector polynomial of degree 5,

$$\begin{aligned} u_{71}^4(t) &\approx 0.00456\,t^5 - 0.02668\,t^4 - 0.02838\,t^3 + 0.06195\,t^2 + 0.24987\,t + 0.5,\\ u_{72}^4(t) &\approx 0.49982 - 0.0017\,t^5 + 0.02231\,t^4 - 0.00062\,t^3 - 0.24956\,t^2 + 0.49982. \end{aligned}$$

A comparison with the exact solution

$$u_1(t) = \frac{1}{2}\,\exp\Big(\frac{1}{2}\,\sin t\Big), \quad u_2(t) = \frac{1}{2}\,\cos t$$

shows a high degree of accuracy of the approximate polynomial solution.

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The Vinograd–Millionshchikov Central Exponents and their Simplified Variants

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1 The original central exponents

For the Euclidean space $\mathbb{R}^n$, $n>1$, we denote by $\mathcal{M}^n$ the set of bounded and piecewise continuous operator-functions $A:\mathbb{R}^+\to\operatorname{End}\mathbb{R}^n$ generating systems of the form

$$\dot{x}=A(t)x,\ \ x\in\mathbb{R}^n,\ \ t\in\mathbb{R}^+\equiv[0,\infty).$$

The *Lyapunov* and *Perron exponents* play an important role in the investigation of solutions of differential equations and systems for Lyapunov and Poisson stability. From this point of view, the so-called *central exponents* are no less useful, being responsible for the stability of all slightly perturbed systems with any sufficiently close linearization (see, e.g., the monograph [1]).

Definition 1.1. The *Vinograd–Millionshchikov central* exponents of a system $A\in\mathcal{M}^n$ are given by the formulas: two *upper* (*up-limit* with a hat, and *low-limit* with a tick) ones

$$\widehat{\Omega}(A)\equiv\inf_{T>0}\varlimsup_{m\to\infty}\frac{1}{Tm}\sum_{i=1}^{m}\ln\big\|X_A(Ti,T(i-1))\big\|,$$
$$\check{\Omega}(A)\equiv\inf_{T>0}\varliminf_{m\to\infty}\frac{1}{Tm}\sum_{i=1}^{m}\ln\big\|X_A(Ti,T(i-1))\big\|,$$

and, respectively, two *lower* ones

$$\widehat{\omega}(A)\equiv\sup_{T>0}\varlimsup_{m\to\infty}\frac{1}{Tm}\sum_{i=1}^{m}\ln\big|X_A(Ti,T(i-1))\big|,$$
$$\check{\omega}(A)\equiv\sup_{T>0}\varliminf_{m\to\infty}\frac{1}{Tm}\sum_{i=1}^{m}\ln\big|X_A(Ti,T(i-1))\big|,$$

where $\|X\|\equiv\sup\limits_{|x|=1}|Xx|$, $|X|\equiv\|X^{-1}\|^{-1}$, and X_A is the Cauchy operator of the system A.

The first and the last of these four exponents are introduced by R. E. Vinograd in 1957, and the penultimate one is suggested by V. M. Millionshchikov in 1969. The upper (lower) central exponents coincide with the upper (lower) limits of the highest (lowest) Lyapunov and Perron exponents at the point A in the *uniform topology* of the space $\mathcal{M}^n$.

2 The various time scales

Non-uniform scales were employed for the first time by N. A. Izobov in 1982.

Definition 2.1. Let $\mathcal{T}$ be the set of all time *scales*, i.e. strictly increasing unbounded sequences of the form

$$\tau \equiv (\tau_k)_{k\in\mathbb{N}}, \quad \tau_k \in \mathbb{R}^+ \quad (\tau_0 \equiv 0).$$

We say that the scale $\tau \in \mathcal{T}$ is:

1) *uniform* (with the *difference* $T > 0$), if $\tau = \tau(T) \equiv (Tk)_{k\in\mathbb{N}}$;
2) *slowly increasing* (denoted by $\tau \in \mathcal{T}^1$), if $\overline{\lim\limits_{k\to\infty}}\, \tau_k/\tau_{k-1} = 1$.
3) *dense* (denoted by $\tau \in \mathcal{T}^0$), if $\|\tau\| \equiv \overline{\lim\limits_{k\to\infty}}\,(\tau_k - \tau_{k-1}) < \infty$;
4) *expanding* (denoted by $\tau \in \mathcal{T}_\infty$), if $|\tau| \equiv \underline{\lim\limits_{k\to\infty}}\,(\tau_k - \tau_{k-1}) = \infty$;
5) *rarefied* (denoted by $\tau \in \mathcal{T}^\infty$), if $|\tau| < \infty = \|\tau\|$;
6) *slowly expanding* (denoted by $\tau \in \mathcal{T}^1_\infty \equiv \mathcal{T}^1 \cap \mathcal{T}_\infty$), if it is both slowly increasing and expanding.

The distribution of scales in three types (dense, expanding and rarefied ones) defines their complete classification, i.e. the representation $\mathcal{T} = \mathcal{T}^0 \sqcup \mathcal{T}^\infty \sqcup \mathcal{T}_\infty$ holds.

3 The simplified central exponents

As we saw above, the formulas for calculating the central exponents in terms of the Cauchy operators of the original linear system are objectively rather complex. We try to simplify those formulas by an appropriate choice of a non-uniform scale.

Definition 3.1. For each scale $\tau \in \mathcal{T}$ we define four *simplified central* exponents of a system $A \in \mathcal{M}^n$: two *upper* (*up-limit* and *low-limit*) ones

$$\widehat{\Delta}_\tau(A) \equiv \overline{\lim_{m\to\infty}} \frac{1}{\tau_m} \sum_{i=1}^m \ln \|X_A(\tau_i, \tau_{i-1})\|, \quad \check{\Delta}_\tau(A) \equiv \underline{\lim_{m\to\infty}} \frac{1}{\tau_m} \sum_{i=1}^m \ln \|X_A(\tau_i, \tau_{i-1})\|, \tag{3.1}$$

and, respectively, two *lower* ones

$$\widehat{\delta}_\tau(A) \equiv \overline{\lim_{m\to\infty}} \frac{1}{\tau_m} \sum_{i=1}^m \ln |X_A(\tau_i, \tau_{i-1})|, \quad \check{\delta}_\tau(A) \equiv \underline{\lim_{m\to\infty}} \frac{1}{\tau_m} \sum_{i=1}^m \ln |X_A(\tau_i, \tau_{i-1})|. \tag{3.2}$$

The Vinograd–Millionshchikov central exponents (non-simplified) of a system $A \in \mathcal{M}^n$ can now be given by the following formulas (with an additional exact bound): *upper* ones

$$\widehat{\Omega}(A) \equiv \inf_{T>0} \widehat{\Delta}_{\tau(T)}(A), \quad \check{\Omega}(A) \equiv \inf_{T>0} \check{\Delta}_{\tau(T)}(A), \tag{3.3}$$

and, respectively, *lower* ones

$$\widehat{\omega}(A) \equiv \sup_{T>0} \widehat{\delta}_{\tau(T)}(A), \quad \check{\omega}(A) \equiv \sup_{T>0} \check{\delta}_{\tau(T)}(A). \tag{3.4}$$

4 The central exponents in dense scales

The properties of simplified central exponents in dense scales are similar to those of the corresponding ones in uniform scales. Thus, estimates of the central exponents, which in the special case of a uniform scale follow directly from Definition 1.1, are also valid for all dense scales.

Theorem 4.1. *For any system $A \in \mathcal{M}^n$ and any dense scale $\tau \in \mathcal{T}^0$ the following inequalities hold:*

$$\widehat{\Delta}_\tau(A) \geqslant \widehat{\Omega}(A), \quad \check{\Delta}_\tau(A) \geqslant \check{\Omega}(A), \quad \widehat{\delta}_\tau(A) \leqslant \widehat{\omega}(A), \quad \check{\delta}_\tau(A) \leqslant \check{\omega}(A). \tag{4.1}$$

The estimates (4.1) cannot be improved even for uniform scales, let alone for all dense ones.

Theorem 4.2. *For any system $A \in \mathcal{M}^n$ the equalities hold*

$$\widehat{\Omega}(A) = \inf_{\tau\in\mathcal{T}^0} \widehat{\Delta}_\tau(A), \quad \check{\Omega}(A) = \inf_{\tau\in\mathcal{T}^0} \check{\Delta}_\tau(A), \quad \widehat{\omega}(A) = \sup_{\tau\in\mathcal{T}^0} \widehat{\delta}_\tau(A), \quad \check{\omega}(A) = \sup_{\tau\in\mathcal{T}^0} \check{\delta}_\tau(A). \tag{4.2}$$

The exact bounds in the equalities (4.2) (just as in the equalities (3.3) and (3.4)) can be replaced by the limits as $|\tau| \to \infty$.

Theorem 4.3. *For any system $A \in \mathcal{M}^n$ the equalities hold*

$$\widehat{\Omega}(A) = \lim_{\tau\in\mathcal{T}^0,\, |\tau|\to\infty} \widehat{\Delta}_\tau(A), \quad \check{\Omega}(A) = \lim_{\tau\in\mathcal{T}^0,\, |\tau|\to\infty} \check{\Delta}_\tau(A),$$
$$\widehat{\omega}(A) = \lim_{\tau\in\mathcal{T}^0,\, |\tau|\to\infty} \widehat{\delta}_\tau(A), \quad \check{\omega}(A) = \lim_{\tau\in\mathcal{T}^0,\, |\tau|\to\infty} \check{\delta}_\tau(A).$$

The exact bounds in the equalities (4.2), generally speaking, are not attained and, moreover, for some system all four bounds are not attained at once.

Theorem 4.4. *There exists a diagonal system $A \in \mathcal{M}^2$ such that for each dense scale $\tau \in \mathcal{T}^0$ all the inequalities* (4.1) *are strict.*

The assertion of Theorem 4.4 extends to systems of an arbitrary order $n > 1$. Theorems 5.2 and 5.5–6.2 below admit a similar generalization.

5 The central exponents in expanding scales

Simplified central exponents in slowly expanding scales are also estimated by the corresponding central ones, but from the other side, opposite to the estimates (4.1).

Theorem 5.1. *For any system $A \in \mathcal{M}^n$ and any expanding scale $\tau \in \mathcal{T}_\infty$ the inequalities*

$$\widehat{\Delta}_\tau(A) \leqslant \widehat{\Omega}(A), \quad \check{\delta}_\tau(A) \geqslant \check{\omega}(A) \tag{5.1}$$

hold and for any slowly expanding scale $\tau \in \mathcal{T}^1_\infty$ additional inequalities hold:

$$\check{\Delta}_\tau(A) \leqslant \check{\Omega}(A), \quad \widehat{\delta}_\tau(A) \geqslant \widehat{\omega}(A). \tag{5.2}$$

The requirement that an expanding scale be slowly increasing is not superfluous for the validity of the inequalities (5.2) in Theorem 5.1.

Theorem 5.2. *For any not slowly increasing scale $\tau \in \mathcal{T} \setminus \mathcal{T}^1$ there exists a diagonal system $A \in \mathcal{M}^2$ such that both the inequalities* (5.2) *are not true.*

The estimates (5.1) and (5.2) in Theorem 5.1 cannot be improved. Moreover, for none of them there is an assertion analogous to Theorem 4.4 (for dense scales).

Theorem 5.3. *For any system $A \in \mathcal{M}^n$ there exists a slowly expanding scale $\tau \in \mathcal{T}_\infty^1$ such that the following equalities hold:*

$$\widehat{\Delta}_\tau(A) = \widehat{\Omega}(A), \quad \check{\Delta}_\tau(A) = \check{\Omega}(A), \quad \widehat{\delta}_\tau(A) = \widehat{\omega}(A), \quad \check{\delta}_\tau(A) = \check{\omega}(A). \tag{5.3}$$

Thus, one more view is possible on the central exponents (3.3) and (3.4).

Theorem 5.4. *For any system $A \in \mathcal{M}^n$ the following equalities hold:*

$$\max_{\tau\in\mathcal{T}_\infty} \widehat{\Delta}_\tau(A) = \widehat{\Omega}(A) = \max_{\tau\in\mathcal{T}_\infty^1} \widehat{\Delta}_\tau(A), \quad \check{\Omega}(A) = \max_{\tau\in\mathcal{T}_\infty^1} \check{\Delta}_\tau(A),$$
$$\widehat{\omega}(A) = \min_{\tau\in\mathcal{T}_\infty^1} \widehat{\delta}_\tau(A), \quad \min_{\tau\in\mathcal{T}_\infty} \check{\delta}_\tau(A) = \check{\omega}(A) = \min_{\tau\in\mathcal{T}_\infty^1} \check{\delta}_\tau(A).$$

The inequalities (5.1) and (5.2) under the conditions of Theorem 5.1 do not, in general, turn into equalities for any fixed expanding scale.

Theorem 5.5. *For any expanding scale $\tau \in \mathcal{T}_\infty$ there exists a diagonal system $A \in \mathcal{M}^2$ such that all the inequalities* (5.1) *and* (5.2) *are strict.*

6 The central exponents in rarefied scales

Rarefied scales, occupying, by definition, an intermediate position between dense and expanding ones, may possess the properties of both the types of scales. In particular, none of Theorems 4.1 and 5.1 applies to all rarefied scales.

Theorem 6.1. *For some rarefied scale $\tau \in \mathcal{T}^\infty$ there exist two diagonal systems $A', A'' \in \mathcal{M}^2$ such that for $A = A'$ all the inequalities* (4.1) *are strict and for $A = A''$ all the inequalities* (5.1) *and* (5.2) *are strict.*

No rarefied scale ensures any of the equalities (5.3) for all systems at once.

Theorem 6.2. *For any rarefied scale $\tau \in \mathcal{T}^\infty$ there exists a diagonal system $A \in \mathcal{M}^2$ such that either all the inequalities* (4.1) *are strict or all the inequalities* (5.1) *and* (5.2) *are strict.*

7 Universal scales on a subset

We consider the possibility of completely removing exact bounds or limits appearing explicitly in the formulas (3.3), (3.4) for central exponents, by replacing them with simplified ones (3.1), (3.2) with a well-chosen scale.

Definition 7.1. The scale $\tau \in \mathcal{T}$ will be called *universal on a subset* $\mathcal{M} \subset \mathcal{M}^n$ if for any system $A \in \mathcal{M}$ it satisfies all the equalities (5.3).

On the one hand, Theorems 4.4, 5.5 and 6.2 imply that there is no universal scale on the entire set $\mathcal{M}^n$. On the other hand, such scales still exist on certain standard subsets of it, as the following three theorems say.

Theorem 7.1. *Each scale $\tau \in \mathcal{T}$ is universal on the subset of autonomous diagonal systems $A \in \mathcal{M}^n$.*

Theorem 7.2. *Each expanding scale $\tau \in \mathcal{T}_\infty$ is universal on the subset of autonomous systems $A \in \mathcal{M}^n$.*

Theorem 7.3. *Each slowly expanding scale $\tau \in \mathcal{T}^1_\infty$ is universal on the subset of exponentially separated systems $A \in \mathcal{M}^n$.*

We note that the subset of exponentially separated systems is open and everywhere dense in the topological space $\mathcal{M}^n$ (the last assertion was proved by V. M. Millionshchikov in 1969).

Finally, according to Theorem 5.3, on each *one-point* subset, regardless of the properties of the corresponding system, there is an *individual* universal slowly expanding scale. This statement extends to arbitrary *compact* subsets.

Theorem 7.4. *For any compact set $\mathcal{K} \subset \mathcal{M}^n$ there is slowly expanding scale $\tau \in \mathcal{T}^1_\infty$ that is universal on $\mathcal{K}$.*

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Variation Formulas of Solutions for Nonlinear Controlled Functional Differential Equations with Constant Delay and the Discontinuous Initial Condition

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Let $O \subset \mathbb{R}^n$ and $U_0 \subset \mathbb{R}^r$ be open sets. Let $\theta_2 > \theta_1 > 0$ be given numbers and n-dimensional function $f(t,x,y,u)$ satisfy the following conditions: for almost all fixed $t \in I = [a,b]$ the function $f(t,\,\cdot\,) : O^2 \times U_0 \to \mathbb{R}^n$ is continuously differentiable; for each fixed $(x,y,u) \in O^2 \times U_0$ the functions $f(t,x,y,u)$, $f_x(t,\,\cdot\,)$, $f_y(t,\,\cdot\,)$ and $f_u(t,\,\cdot\,)$ are measurable on I; for compact sets $K \subset O$ and $U \subset U_0$ there exists a function $m_{K,U}(t) \in L_1(I,[0,\infty))$ such that

$$|f(t,x,y,u)| + |f_x(t,\,\cdot\,)| + |f_y(t,\,\cdot\,) + |f_u(t,\,\cdot\,)| \le m_{K,U}(t)$$

for all $(x,y,u) \in K^2 \times U$ and for almost all $t \in I$. Furthermore, Φ is the set of continuous initial functions $\varphi : I_1 = [\widehat{\tau}, b] \to O$, $\widehat{\tau} = a - \theta_2$ and Ω is the set of measurable control functions $u : I \to U$ with $\operatorname{cl} u(I)$ is a compact set and $\operatorname{cl} u(I) \subset U$.

To each element $\mu = (t_0, \tau, x_0, \varphi(t), u(t)) \in \Lambda = [a,b) \times [\theta_1, \theta_2] \times O \times \Phi \times \Omega$ we assign the delay controlled functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau), u(t)\big) \tag{1}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \;\; t \in [\widehat{\tau}, t_0), \quad x(t_0) = x_0. \tag{2}$$

Condition (2) is said to be the discontinuous initial condition because, in general, $x(t_0) \neq \varphi(t_0)$.

Definition. Let $\mu = (t_0, \tau, x_0, \varphi(t), u(t)) \in \Lambda$. A function $x(t) = x(t;\mu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$ is called a solution of equation (1) with the initial condition (2) or the solution corresponding to μ and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Let us introduce the set of variation:

$$V = \Big\{ \delta\mu = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi, \delta u) : \; \delta t_0| \le \alpha, \; |\delta\tau| \le \alpha, \; |\delta x_0| \le \alpha,$$
$$\delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i, \; \delta u = \sum_{i=1}^{k} \lambda_i \delta u_i, \; |\lambda_i| \le \alpha, \; i = \overline{1,k} \Big\},$$

where $\delta\varphi_i \in \Phi - \varphi_0$, $\delta u_i \in \Omega - u_0$, $i = \overline{1,k}$. Here $\varphi_0 \in \Phi, u_0 \in \Omega$ are fixed functions and $\alpha > 0$ is a fixed number.

Let $\mu_0 = (t_{00}, \tau_0, x_{00}, \varphi_0(t), u_0(t)) \in \Lambda$ be a fixed element, where $t_{00}, t_{10} \in (a,b)$, $t_{00} < t_{10}$ and $\tau_0 \in (\theta_1, \theta_2)$. Let $x_0(t)$ be the solution corresponding to μ_0. There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$

such that for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_1)\times V$, we have $\mu_0+\varepsilon\delta\mu\in\Lambda$, and the solution $x(t;\mu_0+\varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau},t_{10}+\delta_1]\subset I_1$ corresponds to it (see, [2, Theorem 1.4]).

By the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau},t_{10}+\delta_1]$. Therefore, we can assume that the solution $x_0(t)$ is defined on the whole interval $[\widehat{\tau},t_{10}+\delta_1]$. Now we introduce the increment of the solution $x_0(t)=x(t;\mu_0)$:

$$\Delta x(t;\varepsilon\delta\mu)=x(t;\mu_0+\varepsilon\delta\mu)-x_0(t),\ \ (t,\varepsilon,\delta\mu)\in[\widehat{\tau},t_{10}+\delta_1]\times(0,\varepsilon_1)\times V.$$

Theorem 1. *Let the following conditions hold:*

1) $t_{00}+\tau_0<t_{10}$;
2) *the function $\varphi_0(t)$ is absolutely continuous and $\dot\varphi_0(t),t\in I_1$ is bounded;*
3) *the function $f(w,u)$, where $w=(t,x,y)\in I\times O^2$ is bounded on $I\times O^2\times U_0$;*
4) *there exists the finite limit*
$$\lim_{w\to w_0}f(w,u_0(t))=f^-,\ \ w\in(a,t_{00}]\times O^2,$$
where $w_0=(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_0))$.
5) *there exist the finite limits*
$$\lim_{(w_1,w_2)\to(w_1^0,w_2^0)}\big[f(w_1,u_0(t))-f(w_2,u_0(t))\big]=f_1,\ \ w_1,w_2\in(a,b)\times O^2,$$
where
$$w_1^0=(t_{00},x_0(t_{00}+\tau_0),x_{00}),\ \ w_2^0=(t_{00},x_0(t_{00}+\tau_0),\varphi_0(t_{00})).$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ with $t_{10}-\delta_2>t_{00}+\tau_0$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^-$, where $V^-=\{\delta\mu\in V:\delta t_0\le 0\}$, we have

$$\Delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x(t;\delta\mu)+o(t;\varepsilon\delta\mu). \tag{3}$$

Here

$$\delta x(t;\delta\mu)=-Y(t_{00};t)f^-\delta t_0+\beta(t;\delta\mu), \tag{4}$$
$$\beta(t;\delta\mu)=Y(t_{00};t)\delta x_0-Y(t_{00}+\tau_0;t)f_1\delta t_0-\bigg[Y(t_{00}+\tau_0;t)f_1+\int_{t_{00}}^{t}Y(\xi;t)f_y[\xi]\dot x_0(\xi-\tau_0)\,d\xi\bigg]\delta\tau$$
$$+\int_{t_{00}-\tau_0}^{t}Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)\,d\xi+\int_{t_{00}}^{t}Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi,$$

where it is assumed that

$$\int_{t_{00}}^{t}Y(\xi;t)f_y[\xi]\dot x_0(\xi-\tau_0)\,d\xi=\int_{t_{00}}^{t_{00}+\tau_0}Y(\xi;t)f_y[\xi]\dot\varphi_0(\xi-\tau_0)\,d\xi+\int_{t_{00}+\tau_0}^{t}Y(\xi;t)f_y[y]\dot x_0(\xi-\tau_0)\,d\xi.$$

Next, $Y(\xi;t)$ is the $n\times n$-matrix function satisfying the equation

$$Y_\xi(\xi;t)=-Y(\xi;t)f_x[\xi]-Y(\xi+\tau_0;t)f_y[\xi+\tau_0],\ \ \xi\in[t_{00},t]$$

and the condition

$$Y(\xi;t)=\begin{cases} H & \text{for } \xi=t,\\ \Theta & \text{for } \xi>t,\end{cases}$$

H is the identity matrix and Θ is the zero matrix; $f_y=\frac{\partial}{\partial y}f$, $f_y[\xi]=f_y(\xi,x_0(\xi),x_0(\xi-\tau_0),u_0(\xi))$.

Some comments. The expression (3) is called the variation formula of a solution.

The addend $-Y(t_{00};t)f^-\delta t_0-Y(t_{00}+\tau_0;t)f_1\delta t_0$ in the formula (4) is the effect of the discontinuous initial condition (2) and perturbation of the initial moment t_{00}.

The addend

$$-\Big[Y(t_{00}+\tau_0;t)f_1+\int\limits_{t_{00}}^{t}Y(\xi;t)f_{x_1}[\xi]\dot x_0(\xi-\tau_0)\,d\xi\Big]\delta\tau$$

in formula (4) is the effect of the discontinuous initial condition (2) and perturbation of the delay parameter τ_0.

The expression

$$Y(t_{00};t)\delta x_0+\int\limits_{t_{00}-\tau_0}^{t}Y(\xi+\tau_0;t)f_{x_1}[\xi+\tau_0]\delta\varphi(\xi)\,d\xi$$

in formula (4) is the effect of perturbations of the initial vector x_{00} and the initial function $\varphi_0(t)$.

The expression

$$\int\limits_{t_{00}}^{t}Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi$$

is the effect of perturbation of the control function $u_0(t)$. Finally, we note that in [4] variation formulas of solutions were proved for equation (1) with the discontinuous initial condition (2) in the case when the initial moment and delay variations have the same signs. In the present paper variation formulas of solutions are obtained with respect to wide classes of variations (see, V^- and V^+). The variation formulas of solutions for various classes of controlled delay functional differential equations, without perturbations of delay, are proved in [1,3].

Theorem 2. *Let the conditions* 1)–3) *and* 5) *of Theorem* 1 *hold. Moreover, there exists the finite limit*

$$\lim_{w\to w_0}f(w,u_0(t))=f^+,\ \ w\in[t_{00},b)\times O^2.$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$, with $t_{10}-\delta_2>t_{00}+\tau_0$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^+$, where $V^+=\{\delta\mu\in V:\delta t_0\geq 0\}$, formula (4) *holds, where*

$$\delta x(t;\delta\mu)=-Y(t_{00};t)f^+\delta t_0+\beta(t;\delta\mu).$$

Theorem 3. *Let the conditions* 1)–5) *of Theorem* 1 *and the condition* 6) *hold. Moreover, $f^-=f^+:=\widehat f$. Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$, with $t_{10}-\delta_2>t_{00}+\tau_0$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V$, formula* (4) *holds, where*

$$\delta x(t;\delta\mu)=-Y(t_{00};t)\widehat f\delta t_0+\beta(t;\delta\mu).$$

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The Analogue of the Floquet's–Lyapunov's Theorem for Linear Differential Systems with Slowly Varying Parameters

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Let

$$G(\varepsilon_0)=\big\{t,\varepsilon:\ 0<\varepsilon<\varepsilon_0,\ -L\varepsilon^{-1}\le t\le L\varepsilon^{-1},\ 0<L<+\infty\big\}.$$

Definition 1. We say that a function $p(t,\varepsilon)$ belongs to the class $S(m;\varepsilon_0)$ $(m\in\mathbf{N}\cup\{0\})$ if

- $p:G(\varepsilon_0)\to\mathbf{C}$;
- $p(t,\varepsilon)\in C^m(G(\varepsilon_0))$ with respect to t;
- $d^kp(t,\varepsilon)/dt^k=\varepsilon^kp_k^*(t,\varepsilon)$ $(0\le k\le m)$,

$$\|p\|_{S(m;\varepsilon_0)}\overset{def}{=}\sum_{k=0}^{m}\sup_{G(\varepsilon_0)}|p_k^*(t,\varepsilon)|<+\infty.$$

Under a slowly varying function we mean a function of class $S(m;\varepsilon_0)$.

Definition 2. We say that a function $f(t,\varepsilon,\theta(t,\varepsilon))$ belongs to the class $F(m;\varepsilon_0;\theta)$ $(m\in\mathbf{N}\cup\{0\})$ if this function can be represented as

$$f(t,\varepsilon,\theta(t,\varepsilon))=\sum_{n=-\infty}^{\infty}f_n(t,\varepsilon)\exp(in\,\theta(t,\varepsilon)),$$

and

- $f_n(t,\varepsilon)\in S(m;\varepsilon_0)$;
- $$\|f\|_{F(m;\varepsilon_0;\theta)}\overset{def}{=}\sum_{n=-\infty}^{\infty}\|f_n\|_{S(m;\varepsilon_0)}<+\infty,$$
- $\theta(t,\varepsilon)=\int\limits_0^t\varphi(\tau,\varepsilon)\,d\tau$, $\varphi(t,\varepsilon)\in\mathbf{R}^+$, $\varphi(t,\varepsilon)\in S(m;\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)}\varphi(t,\varepsilon)=\varphi_0>0$.

We consider the next system of differential equations

$$\frac{dx_j}{dt}=\lambda_j(t,\varepsilon)x_j+\mu\sum_{k=1}^{N}p_{jk}(t,\varepsilon,\theta)x_k,\ \ j=\overline{1,N},\tag{1}$$

where $\lambda_j(t,\varepsilon)\in S(m;\varepsilon_0)$, $p_{jk}(t,\varepsilon,\theta)\in F(m;\varepsilon_0;\theta)$ $(j,k=\overline{1,N})$, $\mu\in(0,\mu_0)\subset\mathbf{R}^+$.

We study the problem about the structure of fundamental system of solutions $x_{jk}(t,\varepsilon,\mu)$ $(j,k=\overline{1,N})$ of system (1).

Lemma 1. *Let the function*

$$f(t,\varepsilon,\theta(t,\varepsilon)) = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} f_n(t,\varepsilon)\exp(in\,\theta(t,\varepsilon))$$

belong to the class $F(m-1;\varepsilon_0;\theta)$. Then the function

$$x(t,\varepsilon,\theta(t,\varepsilon)) = \varepsilon\int_0^t f(\tau,\varepsilon,\theta(\tau,\varepsilon))\,d\tau$$

belongs to the class $F(m-1;\varepsilon_0;\theta)$ also, and there exists $K_1 \in (0,+\infty)$, that does not depend on the function f such that

$$\|x(t,\varepsilon,\theta)\|_{F(m-1;\varepsilon_0;\theta)} \le K_1\|f(t,\varepsilon,\theta)\|_{F(m-1;\varepsilon_0;\theta)}.$$

Lemma 2. *Let we have the linear nonhomogeneous first-order differential equation:*

$$\frac{dx}{dt} = \lambda(t,\varepsilon)x + \varepsilon u(t,\varepsilon,\theta(t,\varepsilon)), \tag{2}$$

where $\lambda(t,\varepsilon) \in S(m;\varepsilon_0)$, $u(t,\varepsilon,\theta) \in F(m-1;\varepsilon_0;\theta)$. Let the condition $|\operatorname{Re}\lambda(t,\varepsilon)| \ge \gamma_0 > 0$ hold. Then equation (2) has a particularly solution $x(t,\varepsilon,\theta(t,\varepsilon)) \in F(m-1;\varepsilon_0;\theta)$, and there exists $K_2 \in (0,+\infty)$, that does not depend on the function $u(t,\varepsilon,\theta)$ such that

$$\|x(t,\varepsilon,\theta)\|_{F(m-1;\varepsilon_0;\theta)} \le \frac{K_2}{\gamma_0}\,\|u(t,\varepsilon,\theta)\|_{F(m-1;\varepsilon_0;\theta)}. \tag{3}$$

Lemma 3. *Let the system* (1) *be such that*

$$\big|\operatorname{Re}(\lambda_j(t,\varepsilon) - \lambda_k(t,\varepsilon))\big| \ge \gamma_1 > 0 \ \ (j \neq k). \tag{4}$$

Then there exists $\mu_1 \in (0,\mu_0)$ such that for all $\mu \in (0,\mu_1)$ there exists the Lyapunov's transformation of kind

$$x_j = y_j + \mu\sum_{k=1}^{N}\psi_{jk}(t,\varepsilon,\theta,\mu)y_k, \ \ j=\overline{1,N}, \tag{5}$$

where $\psi_{jk} \in F(m-1;\varepsilon_0;\theta)$, reducing the system (1) *to*

$$\frac{dy_j}{dt} = \big(\lambda_j(t,\varepsilon) + \mu u_j(t,\varepsilon,\mu)\big)y_j + \mu\varepsilon\sum_{k=1}^{N} v_{jk}(t,\varepsilon,\theta,\mu)y_k, \ \ j=\overline{1,N}, \tag{6}$$

where $u_j \in S(m;\varepsilon_0)$, $v_{jk} \in F(m-1;\varepsilon_0;\theta)$ $(j,k=\overline{1,N})$.

Lemma 4. *Let the condition* (4) *hold. Then there exists $\mu_2 \in (0,\mu_1)$ (μ_i are defined in Lemma* 3*) such that for all $\mu \in (0,\mu_2)$ there exists the Lyapunov's transformation of kind*

$$y_j = z_j + \mu\sum_{k=1}^{N} q_{jk}(t,\varepsilon,\theta,\mu)z_k, \ \ j=\overline{1,N}, \tag{7}$$

where $q_{jk} \in F(m-1;\varepsilon_0;\theta)$, reducing the system (6) *to the pure diagonal form*

$$\frac{dz_j}{dt} = d_j(t,\varepsilon,\theta,\mu)z_j, \ \ j=\overline{1,N}, \tag{8}$$

where

$$d_j = \lambda_j(t,\varepsilon) + \mu u_j(t,\varepsilon,\mu) + \mu\varepsilon v_{jj}(t,\varepsilon,\theta,\mu) + \mu\varepsilon \sum_{\substack{k=1\\(k\neq j)}}^{N} v_{jk}(t,\varepsilon,\theta,\mu) q_{kj}(t,\varepsilon,\theta,\mu), \ \ j=\overline{1,N}. \tag{9}$$

Theorem. *Let for the system* (1) *the condition* (4) *holds. Then there exists* $\mu_3 \in (0,\mu_0)$ *such that for all* $\mu \in (0,\mu_3)$ *the system* (3) *has a fundamental system of solutions of kind:*

$$x_{jk} = r_{jk}(t,\varepsilon,\theta,\mu) \exp\Big(\int\limits_0^t \sigma_j(s,\varepsilon,\mu)\,ds\Big), \ \ j,k=\overline{1,N} \tag{10}$$

j – the number of solution, k – the number of component, where $r_{jk}(t,\varepsilon,\theta,\mu) \in F(m-1;\varepsilon_0;\theta)$, $\sigma_j(t,\varepsilon,\mu) \in S(m-1;\varepsilon_0)$.

Proof. The fundamental system of solutions (FSS) of the system (8) has a kind:

$$z_{jk} = \delta_j^k \exp\Big(\int\limits_0^t d_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds\Big), \ \ j,k=\overline{1,N}$$

j – the number of solution, k – the number of component, δ_j^k – the symbol of Kronecker. By virtue (7) FSS of system (6) has a kind:

$$y_{jk} = \widetilde{q}_{jk}(t,\varepsilon,\theta,\mu) \exp\Big(\int\limits_0^t d_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds\Big), \ \ k=\overline{1,N},$$

where $\widetilde{q}_{jk} = \delta_j^k + (1-\delta_j^k)\mu q_{jk}$ (j – the number of solution, k – the number of component). By virtue (5) FSS of system (1) has a kind:

$$x_{jk} = \Big(\sum_{l=1}^{N} \widetilde{\psi}_{kl}(t,\varepsilon,\theta,\mu)\widetilde{q}_{lj}(t,\varepsilon,\theta,\mu)\Big) \exp\Big(\int\limits_0^t d_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds\Big), \ \ j,k=\overline{1,N}, \tag{11}$$

where $\widetilde{\psi}_{jk} = \delta_j^k + \mu\psi_{jk}$($\psi_{jk}$ are defined in Lemma 3).

Consider

$$\int\limits_0^t d_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds = \int\limits_0^t \big(\lambda_j(s,\varepsilon) + \mu_j(s,\varepsilon,\mu)\big)\,ds + \mu\varepsilon \int\limits_0^t w_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds,$$

where $w_j = v_{jj} + \sum\limits_{k=1}^{N} \psi_{jk} q_{kj} \in F(m-1;\varepsilon_0;\theta)$. We represent the functions w_j as $w_j = w_j^*(t,\varepsilon,\mu) + \widetilde{w}_j(t,\varepsilon,\theta,\mu)$, where

$$w_j^*(t,\varepsilon,\mu) = \overline{w_j(t,\varepsilon,\theta,\mu)} = \frac{1}{2\pi}\int\limits_0^{2\pi} w_j(t,\varepsilon,\theta,\mu)\,d\theta \in S(m-1;\varepsilon_0).$$

Accordingly, $\widetilde{w}_j \in F(m-1;\varepsilon_0;\theta)$, and $\overline{\widetilde{w}_j(t,\varepsilon,\theta,\mu)} \equiv 0$. Then

$$\exp\bigg(\int\limits_0^t d_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds\bigg) \\ = \exp\bigg(\int\limits_0^t \big(\lambda_j(s,\varepsilon)+\mu u_j(s,\varepsilon,\mu)+\mu\varepsilon w_j^*(s,\varepsilon,\mu)\big)\bigg)\exp\bigg(\mu\varepsilon\int\limits_0^t \widetilde{w}_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds\bigg). \tag{12}$$

By virtue Lemma 1, we conclude

$$\varepsilon\int\limits_0^t \widetilde{w}_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds \in F(m-1;\varepsilon_0;\theta) \ \ (j=\overline{1,N}).$$

It follows by virtue of the properties of functions from class $F(m;\varepsilon_0;\theta)$ that

$$g_j(t,\varepsilon,\theta,\mu) = \exp\bigg(\mu\varepsilon\int\limits_0^t \widetilde{w}_j(s,\varepsilon,\theta(s,\varepsilon),\mu)\,ds\bigg) \in F(m-1;\varepsilon_0;\theta) \ \ (j=\overline{1,N}). \tag{13}$$

By virtue of (11)–(13) we obtain the statement of the theorem. □

Obviously, the formula (10) is an analogue of Floquet's–Lyapunov's theorem for the systems of kind (1).

Solutions of Two-Term Fractional Differential Equations on the Half-Line Via Lower and Upper Solutions Method

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We are interested in the fractional initial value problem

$$ {}^c\mathcal{D}^\alpha u(t) = q(t,u(t))\,{}^c\mathcal{D}^\beta u(t) + f(t,u(t)), \tag{1}$$

$$ u(0) = a, \tag{2}$$

where $0 < \beta < \alpha \le 1$, $q, f \in C(\mathbb{R}_0 \times \mathbb{R})$, $\mathbb{R}_0 = [0,\infty)$, $a \in \mathbb{R}$ and ${}^c\mathcal{D}$ denotes the Caputo fractional derivative.

We recall that the Caputo fractional derivative ${}^c\mathcal{D}^\gamma x$ of order $\gamma > 0$, $\gamma \notin \mathbb{N}$, of a function $x\colon \mathbb{R}_0 \to \mathbb{R}$ is given as [1,2]

$$ {}^c\mathcal{D}^\gamma x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)} \Big(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!}\, s^k \Big)\, \mathrm{d}s, $$

where Γ is the Euler gamma function, $n = [\gamma] + 1$, $[\gamma]$ means the integral part of the fractional number γ. If $\gamma \in \mathbb{N}$, then ${}^c\mathcal{D}^\gamma x(t) = x^{(\gamma)}(t)$.

In particular,

$$ {}^c\mathcal{D}^\gamma x(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\, (x(s) - x(0))\, \mathrm{d}s, \quad \gamma \in (0,1). $$

If $\alpha = 1$, then (1) has the form

$$ u'(t) = q(t,u(t))\,{}^c\mathcal{D}^\beta u(t) + f(t,u(t)). $$

This equation is called the generalized Basset fractional differential equation [1–4].

Definition 1. We say that $u : \mathbb{R}_0 \to \mathbb{R}$ is *a solution of equation* (1) if u, ${}^c\mathcal{D}^\alpha u \in C(\mathbb{R}_0)$ and (1) holds for $t \in \mathbb{R}_0$. A solution u of (1) satisfying the initial condition (2) is called *a solution of problem* (1), (2).

The existence, uniqueness and the structure of solutions to problem (1), (2) is proved by the lower and upper solutions method combined with the extension method, the Schauder fixed point theorem and the maximum principle for the Caputo fractional derivative [5].

Definition 2. A function $\sigma : \mathbb{R}_0 \to \mathbb{R}$ is called *a lower solution of* (1) if $\sigma, {}^c\mathcal{D}^\alpha \sigma \in C(\mathbb{R}_0)$ and

$$ {}^c\mathcal{D}^\alpha \sigma(t) \le q(t,\sigma(t))\,{}^c\mathcal{D}^\beta \sigma(t) + f(t,\sigma(t)) \ \text{ for } \ t \in \mathbb{R}_0. $$

If the inequality is reversed, then σ is called *an upper solution of* (1).

Theorem 1. *Let*

(H_1) *$q(t,x) \le 0$ for $(t,x) \in \mathbb{R}_0 \times \mathbb{R}$ if $\alpha < 1$ and $q(t,x) < 0$ for $(t,x) \in \mathbb{R}_0 \times \mathbb{R}$ if $\alpha = 1$;*

(H_2) *there are a lower solution φ and an upper solution ρ of (1) such that*

$$\varphi(t) < \rho(t) \ \text{ for } \ t \in \mathbb{R}_0.$$

Then for $a \in (\varphi(0), \rho(0))$ there exists at least one solution of problem $(1),(2)$ and its solutions u satisfy

$$\varphi(t) < u(t) < \rho(t) \ \text{ for } \ t \in \mathbb{R}_0.$$

Example 1. Let $\mu > 0$, $q \in C(\mathbb{R}_0 \times \mathbb{R})$ satisfy (H_1) and $r \in C(\mathbb{R}_0)$, $0 \le r(t) \le (1+t)^\mu$ for $t \in \mathbb{R}_0$. Then $\varphi(t) = 0$ and $\rho(t) = 1+t$ for $t \in \mathbb{R}_0$ are lower and upper solutions of the equation

$$^c\!D^\alpha u(t) = q(t,u(t))\,^c\!D^\beta u(t) + r(t) - |u(t)|^\mu. \tag{3}$$

Theorem 1 ensures that for $a \in (0,1)$ solutions u of problem $(3),(2)$ satisfy $0 < u(t) < 1+t$ for $t \in \mathbb{R}_0$.

Corollary 1. *Let (H_1) hold and let there exist $A, B \in \mathbb{R}$, $A < B$, such that*

$$f(t,A) \ge 0, \ \ f(t,B) \le 0 \ \text{ for } \ t \in \mathbb{R}_0.$$

Then for $a \in (A,B)$ there exists at least one solution of problem $(1),(2)$ and its solutions u satisfies $A < u(t) < B$ for $t \in \mathbb{R}_0$.

Corollary 2. *Let (H_1) and (H_2) hold and let $\lim\limits_{t\to\infty} \varphi(t) = 0$, $\lim\limits_{t\to\infty} \rho(t) = 0$. Then for $a \in (\varphi(0), \rho(0))$ there exists at least one solution of problem $(1),(2)$ and its solutions u satisfy $\lim\limits_{t\to\infty} u(t) = 0$ and $\varphi(t) < u(t) < \rho(t)$ for $t \in \mathbb{R}_0$.*

Example 2. Let $r, w \in C(\mathbb{R}_0)$ and $0 < r(t) \le M$, $|w(t)| < 2 + M$ for $t \in \mathbb{R}_0$, where M is a positive constant. Let $\mu > 0$ and $S \ge 4(2+M)$. Then $\varphi(t) = -e^{-t}$ and $\rho(t) = e^t$ are lower and upper solutions of the equation

$$^c\!D^\alpha u(t) = -\big(r(t) + |u(t)|^\mu\big)\,^c\!D^\beta u(t) + w(t) - Se^t u(t). \tag{4}$$

By Corollary 2, for $a \in (-1,1)$ solutions u of problem $(4),(2)$ satisfy $\lim\limits_{t\to\infty} u(t) = 0$ and $-e^{-t} < u(t) < e^{-t}$ for $t \in \mathbb{R}_0$.

Now, we consider the equation

$$^c\!D^\alpha u(t) = b(t)\,^c\!D^\beta u(t) + f(t,u(t)), \tag{5}$$

that is the special case of (1). The separation property of solutions to equation (5) is given in the following theorem.

Theorem 2. *Let (H_2) and*

(H_1^*) *$b \in C(\mathbb{R}_0)$, $b \le 0$ on $\mathbb{R}_0$ if $\alpha < 1$ and $b < 0$ on $\mathbb{R}_0$ if $\alpha = 1$*

hold. Let u, v be solutions of equation (5) such that $\varphi(0) < u(0) < v(0) < \rho(0)$. Then

$$\varphi(t) < u(t) < v(t) < \rho(t) \ \text{ for } \ t \in \mathbb{R}_0.$$

The following result gives the existence of a unique solution of problem $(5),(2)$.

Theorem 3. *Let (H_1^*) and (H_2) hold. Let for each $t \in \mathbb{R}_0$ the function $f(t,x)$ is decreasing in the variable x on the interval $[\varphi(t),\rho(t)]$. Then for $a \in (\varphi(0),\rho(0))$ there exists a unique solution u of problem $(5),(2)$ and $\varphi(t) < u(t) < \rho(t)$ for $\mathbb{R}_0$.*

Under the conditions of Theorem 3, for each $a \in (\varphi(0),\rho(0))$ there exists a unique solution of problem $(5),(2)$. We denote u_a this solution. By Theorem 2,

$$\varphi(t) < u_{a_1}(t) < u_{a_2}(t) < \rho(t) \ \text{ for } \ t \in \mathbb{R}_0, \ \ \varphi(0) < a_1 < a_2 < \rho(0).$$

For $a \in \{\varphi(0),\rho(0)\}$, we have the following result.

Lemma 1. *Let the conditions of Theorem 3 be satisfied. Then for $a \in \{\varphi(0),\rho(0)\}$ there exists a unique solution u_a of problem $(5),(2)$ satisfying $\varphi(t) \le u_a(t) \le \rho(t)$ on $\mathbb{R}_0$.*

We denote by $u_{\varphi(0)}$ and $u_{\rho(0}$ the unique solution of $(5),(2)$ for $a = \varphi(0)$ and $a = \rho(0)$, respectively.

The following result says that the set $\mathcal{Z} = \{(t,x) \in \mathbb{R}^2 \colon t \in \mathbb{R}_0,\ u_{\varphi(0)}(t) \le x \le u_{\rho(0)}(t)\}$ is covered by graphs of solutions u of equation (5) with $u(0) \in [\varphi(0),\rho(0)]$.

Theorem 4. *Let the conditions of Theorem 3 hold. Then for $(T,x_0) \in \mathbb{R}_0 \times [u_{\varphi(0)}(T), u_{\rho(0)}(T)]$ there exists a unique solution u of equation (5) such that $u(T) = x_0$ and $\varphi(t) \le u(t) \le \rho(t)$ on $\mathbb{R}_0$.*

Example 3. Let $\mu > 0$, $w \in C(\mathbb{R}_0)$, $0 \le w \le M$ on $\mathbb{R}_0$, where M is a positive constant, and let b satisfy (H_1^*). Then the constant functions $\varphi = 0$ and $\rho = \sqrt[\mu]{M}$ are lower and upper solutions of the fractional differential equation

$$^{c}\!D^{\alpha}u(t) = b(t)\,^{c}\!D^{\beta}u(t) + w(t) - |u(t)|^{\mu}. \tag{6}$$

By Theorem 3 and Lemma 1, for $a \in [0, \sqrt[\mu]{M}]$ there exists a unique solution of problem $(6),(2)$. Theorem 4 guarantees that for $T \in \mathbb{R}_0$ and $x_0 \in [0, \sqrt[\mu]{M}]$ there exists a unique solution u of (6) satisfying $u(T) = x_0$ and $0 \le u(t) \le \sqrt[\mu]{M}$ on $\mathbb{R}_0$.

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On Comparison Theorem for Neutral Stochastic Differential Equations of Reaction-Diffusion Type

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Abstract We prove a comparison theorem for the mild solutions (solutions) of Cauchy problems for two stochastic integro-differential equations of reaction-diffusion type with delay. According to our result, if $f_1 \geq f_2$, then $u_1 \geq u_2$ with probability one.

1 Introduction

We study the following Cauchy problems for two neutral partial stochastic integro-differential equations of reaction-diffusion type

$$d\bigg(u_i(t,x) + \int\limits_{\mathbb{R}^d} b_i(t,x,u_i(t-r,\xi),\xi)\,d\xi\bigg) = \big(\Delta_x u_i(t,x) + f_i(t,u_i(t-r,x),x)\big)\,dt$$

$$+\sigma(t,x)\,dW(t,x), \ \ 0 < t \leq T, \ \ x \in \mathbb{R}^d, \ \ i \in \{1,2\}, \tag{1.1}$$

$$u_i(t,x) = \phi_i(t,x), \ \ -r \leq t \leq 0, \ \ x \in \mathbb{R}^d, \ \ r > 0, \ \ i \in \{1,2\}, \tag{1.1*}$$

where $\Delta_x \equiv \sum\limits_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$, $i \in \{1,\dots,d\}$, W is $L_2(\mathbb{R}^d)$-valued Q-Wiener process, $f_i : [0,T]\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$, $i \in \{1,2\}$, $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ and $b_i : [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$, $i \in \{1,2\}$, are some given functions to be specified later, and $\phi_i : [-r,0]\times\mathbb{R}^d \to \mathbb{R}$, $i \in \{1,2\}$, are initial-datum functions.

A problem of comparison of solutions to stochastic differential equations in finite-dimensional case has firstly arised in [8]. It has been proved in this work that under certain assumptions the solution is monotonously non-decreasing function from "drift"-coefficient f. Variations of this result have been proposed in the works [1–4, 6, 7]. In [5] the proof of comparison theorem for solutions to the Cauchy problem for stochastic differential equations with multidimensional Wiener processes in Hilbert space is given. Our aim was to prove the comparison theorem for solutions of problem

(1.1), (1.1*) using the idea from this work. This result plays an important role when studying the existence of solutions to the Cauchy problem for stochastic differential equations with non-Lipschitz conditions on "drift"-coefficients.

2 Formulation of the problem

Throughout the article $(\Omega, \mathcal{F}, \mathbf{P})$ will note a complete probability space. Let $\{e_n(x),\ n \in \{1,2,\dots\}\}$ be an orthonormal basis on $L_2(\mathbb{R}^d)$ such that $\sup\limits_{n\in\{1,2,\dots\}} \operatorname*{ess\,sup}\limits_{x\in\mathbb{R}^d} |e_n(x)| \le 1$. We define a Q-Wiener $L_2(\mathbb{R}^d)$-valued process

$$W(t,x) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n(x) \beta_n(t), \quad t \ge 0, \quad x \in \mathbb{R}^d,$$

where $\{\beta_n(t),\ n \in \{1,2,\dots\}\} \subset \mathbb{R}$ are independent standard one-dimensional Brownian motions on $t \ge 0$, $\{\lambda_n,\ n \in \{1,2,\dots\}\}$ is a sequence of positive numbers such that $\lambda = \sum\limits_{n=1}^{\infty} \lambda_n < \infty$. Let $\{\mathcal{F}_t,\ t \ge 0\}$ be a normal filtration on $\mathcal{F}$. We assume that $W(t,\,\cdot\,)$, $t \ge 0$, is $\mathcal{F}_t$-measurable and the increments $W(t+h,\,\cdot\,) - W(t,\,\cdot\,)$ are independent of $\mathcal{F}_t$ for all $h > 0$ and $t \ge 0$. Throughout the article $L_2(\mathbb{R}^d)$ will note real Hilbert space with the norm

$$\|f\|_{L_2(\mathbb{R}^d)} = \bigg(\int\limits_{\mathbb{R}^d} f^2(x)\,dx \bigg)^{\frac{1}{2}}.$$

Let the following seven assumptions be true.

(1) $f_i : [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $i \in \{1,2\}$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $b_i : [0,T] \times \mathbb{R}^d \times \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $i \in \{1,2\}$, are measurable with respect to all of their variables functions, b_i, $i \in \{1,2\}$, are continuous in the first argument;

(2) $\phi_i(t,x,\omega) : [-r,0] \times \mathbb{R}^d \times \Omega \to L_2(\mathbb{R}^d)$, $i \in \{1,2\}$, are $\mathcal{F}_0$-measurable random variables with almost surely continuous paths and such that

$$\sup_{-r \le t \le 0} \mathbf{E}\|\phi_i(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} < \infty, \quad i \in \{1,2\};$$

(3) b_i, $i \in \{1,2\}$, are uniformly continuous in the first argument and satisfy the Lipshitz condition in the third argument of the form

$$\big|b_i(t,x,u,\xi) - b_i(t,x,v,\xi)\big| \le l(t,x,\xi)|u-v|, \quad 0 \le t \le T, \quad \{x,\xi\} \subset \mathbb{R}^d, \quad \{u,v\} \subset \mathbb{R}, \quad i \in \{1,2\},$$

where $l : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is such that

$$\sup_{0 \le t \le T} \int\limits_{\mathbb{R}^d} \sqrt{\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)\,d\xi}\,dx < \infty, \quad \sup_{0 \le t \le T} \int\limits_{\mathbb{R}^d} \bigg(\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)\,d\xi \bigg)\,dx < \frac{1}{4};$$

(4) there exists a function $\chi : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$, satisfying the conditions

$$\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \chi(x,\xi)\,d\xi\,dx < \infty, \quad \int\limits_{\mathbb{R}^d} \bigg(\int\limits_{\mathbb{R}^d} \chi(x,\xi)\,d\xi \bigg)^2 dx < \infty,$$

such that

$$\sup_{0 \le t \le T} |b_i(t,x,0,\xi)| \le \chi(x,\xi), \quad 0 \le t \le T, \quad x \in \mathbb{R}^d, \quad \xi \in \mathbb{R}^d, \quad i \in \{1,2\};$$

(5) there exists a function $\eta : [0,T] \times \mathbb{R}^d \to [0,\infty)$ with

$$\sup_{0\le t\le T} \int_{\mathbb{R}^d} \eta^2(t,\xi)\,d\xi < \infty,$$

such that the following linear-growth and Lipschitz conditions are valid

$$|f_i(t,u,x)| \le \eta(t,x) + L|u|, \ \ 0\le t\le T, \ \ u\in\mathbb{R}, \ \ x\in\mathbb{R}^d \ \ i\in\{1,2\},$$
$$|f_i(t,u,x) - f_i(t,v,x)| \le L|u-v|, \ \ 0\le t\le T, \ \ \{u,v\}\subset\mathbb{R}, \ \ x\in\mathbb{R}^d, \ \ i\in\{1,2\};$$

(6) the following condition is valid for σ

$$\sup_{0\le t\le T} \|\sigma(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} < \infty;$$

(7) for any $x\in\mathbb{R}^d$ there exist $\partial_x b$, $D_x^2 b$, and for $\nabla_x b$ and $D_x^2 b$ the following linear-growth condition with respect to the third argument is true

$$|\nabla_x b(t,x,u,\xi)| + \|D_x^2 b(t,x,u,\xi)\|_d \le \psi(t,x,\xi)\big(1+|u|\big), \ \ 0\le t\le T, \ \ \{x,\xi\}\subset\mathbb{R}^d, \ \ u\in\mathbb{R},$$

for $D_x^2 b$ – the following Lipschitz condition

$$\big\|D_x^2 b(t,x,u,\xi) - D_x^2 b(t,x,v,\xi)\big\|_d \le \psi(t,x,\xi)|u-v|, \ \ 0\le t\le T, \ \ \{x,\xi\}\subset\mathbb{R}^d, \ \ \{u,v\}\subset\mathbb{R},$$

where $\psi : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to [0,\infty)$ is such that

$$\sup_{0\le t\le T} \int_{\mathbb{R}^d} \bigg(\int_{\mathbb{R}^d} \psi(t,x,\xi)\,d\xi \bigg)^2 dx < \infty, \quad \sup_{0\le t\le T} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi^2(t,x,\xi)\,d\xi\,dx < \infty,$$

and besides for any point $x_0\in\mathbb{R}^d$ there is its neighborhood $B_\delta(x_0)$ and a nonnegative function φ such that

$$\sup_{0\le t\le T} \varphi(t,\,\cdot\,,x_0,\delta) \in L_2(\mathbb{R}^d)\cap L_1(\mathbb{R}^d), \ \ \delta>0,$$
$$\big|\psi(t,x,\xi) - \psi(t,x_0,\xi)\big| \le \varphi(t,\xi,x_0,\delta)|x-x_0|, \ \ 0\le t\le T, \ \ |x-x_0|<\delta, \ \ \xi\in\mathbb{R}^d.$$

Our main result is the following comparison theorem.

Theorem. *Suppose assumptions* (1)–(7) *are satisfied and*

(1) *the initial-datum functions* ϕ_i, $i\in\{1,2\}$, *satisfy the condition*

$$\phi_1(t,x) \ge \phi_2(t,x), \ \ 0\le t\le T, \ \ x\in\mathbb{R}^d;$$

(2) *the functions* b_i, $i\in\{1,2\}$, *satisfy the conditions*

$$b_1(0,x,\phi_2(-r,\xi),\xi) = b_2(0,x,\phi_2(-r,\xi),\xi), \ \ \{x,\xi\}\subset\mathbb{R}^d,$$
$$b_1(0,x,\phi_1(-r,\xi),\xi) = b_2(0,x,\phi_1(-r,\xi),\xi), \ \ \{x,\xi\}\subset\mathbb{R}^d,$$
$$b_1(0,x,\phi_1(-r,\xi),\xi) = b_1(0,x,\phi_2(-r,\xi),\xi), \ \ \{x,\xi\}\subset\mathbb{R}^d,$$
$$b_1(t,x,u,\xi) \le b_2(t,x,u,\xi), \ \ 0\le t\le T, \ \ \{x,\xi\}\subset\mathbb{R}^d, \ \ u\in\mathbb{R};$$

(3) *for the "drift"-functions* $f_1(t,u,x) \geq f_2(t,u,x)$, $0 \leq t \leq T$, $u \in \mathbb{R}$, $x \in \mathbb{R}^d$.

Let one of the following conditions be true:

(M1) b_1 *is monotonously non-increasing,* f_1 *is monotonously non-decreasing with respect to* u*;*

(M2) b_2 *is monotonously non-increasing,* f_2 *is monotonously non-decreasing with respect to* u*.*

Then with probability one the solutions of (1.1), (1.1*) *satisfy*

$$u_1(t,x) \geq u_2(t,x) \ \ x \in \mathbb{R}^d, \ \ 0 \leq t \leq T.$$

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On a Delay Parameter Optimization Problem: Existence, Sensitivity of a Functional Minimum, Necessary Optimality Conditions

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Let $t_1 > t_0$ and $\theta_2 > \theta_1 > 0$ be given numbers with $t_0 + \theta_2 < t_1$; let $O \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^r$ be open and compact sets. Let us consider the function $F(t,x,y,u) = (f^0, f)$, $(t,x,y,u) \in I \times O^2 \times U$ satisfying the standard conditions: for almost all fixed $t \in I = [t_0, t_1]$ the function $F(t,\,\cdot\,) : O^2 \times U \to \mathbb{R}^{1+n}$ is continuous in $(x,y,u) \in O^2 \times U$ and continuously differentiable in $(x,y) \in O^2$; for each fixed $(x,y,u) \in O^2 \times U$ the functions $F(t,x,y,u)$, $F_x(t,\,\cdot\,)$ and $F_y(t,\,\cdot\,)$ are measurable on I; for any compact set $K \subset O$ there exists a function $m_K(t) \in L_1(I, \mathbb{R}_+)$, $\mathbb{R}_+ = [0,\infty)$ such that

$$|F(t,x,y,u)| + |F_x(t,\,\cdot\,)| + |F_y(t,\,\cdot\,)| \le m_K(t)$$

for all $(x,y,u) \in K^2 \times U$ and for almost all $t \in I$.

Furthermore, by Ω we denote the set of measurable control functions $u : I \to U$; the initial function $\varphi_0(t) \in C(I_1, O)$, where $I_1 = [\widehat{\tau}, t_1]$, $\widehat{\tau} = t_0 - \theta_2$; the initial vector $x_{00} \in O$, the function $q^0(\tau, x) \in C(I_2 \times O, \mathbb{R})$, where $I_2 = [\theta_1, \theta_2]$.

To each element $w = (\tau, u(t)) \in W = I_2 \times \Omega$ we set in correspondence the controlled delay functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau), u(t)\big) \tag{1}$$

with the initial condition

$$x(t) = \varphi_0(t), \ \ t \in [\widehat{\tau}, t_0), \quad x(t_0) = x_{00}. \tag{2}$$

Definition 1. Let $w = (\tau, u(t)) \in W$. A function $x(t) = x(t;w) \in O$, $t \in [\widehat{\tau}, t_1]$, is called a solution of the equation (1) with the initial condition (2) or a solution corresponding to the element w and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies the condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (1) almost everywhere on $[t_0, t_1]$.

By W_0 we denote the set of all $w \in W$ elements for which there exist solutions $x(t;w)$ defined on the interval I. In the sequel it is assumed that $W_0 \ne \varnothing$.

Definition 2. An element $w_0 = (\tau_0, u_0(t)) \in W_0$ is said to be optimal or a solution of the problem (1)–(3) if for an arbitrary element $w \in W_0$ the inequality

$$\begin{aligned} J(w_0) &= q^0(\tau_0, x_0(t_1)) + \int\limits_{t_0}^{t_1} f^0\big(t, x_0(t), x_0(t-\tau_0), u_0(t)\big)\,dt \\ &\le J(w) = q^0(\tau, x(t_1)) + \int\limits_{t_0}^{t_1} f^0\big(t, x(t), x(t-\tau), u(t)\big)\,dt \end{aligned} \tag{3}$$

holds, where $x_0(t) = x(t;w_0)$, $x(t) = x(t;w)$.

The problem (1)–(3) is called the optimization problem of delay parameter.

Theorem 1. *There exists an optimal element w_0 if the following conditions hold:*

1.1 *there exists a compact set $K_0 \subset O$ such that for an arbitrary $w \in W_0$*

$$x(t;w) \in K_0, \ \ t \in I;$$

1.2 *the set*

$$P_F(t,x,y) = \big\{(p^0,p)^T \in \mathbb{R}^{1+n} : \ \exists\, u \in U, \ p^0 \geq f^0(t,x,y,u), \ p = f(t,x,y,u)\big\}$$

is convex for all fixed $(t,x,y) \in I \times K_0^2$.

Remark 1. Let U be the convex set. Let $f(t,x,y,u) = A(t,x,y) + B(t,x,y)u$ and let the function $f^0(t,x,y,u)$ be convex in $u \in U$, then the condition 1.2 of the Theorem 1 holds.

Theorem 2. *Let the conditions* 1.1 *and* 1.2 *of Theorem* 1 *hold. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every $(x_{0\delta}, \varphi_\delta, q_\delta, G_\delta)$, where $x_{0\delta} \in O$, $\varphi_\delta \in C(I_1,O)$, $q_\delta^0 \in C(I_2 \times O, \mathbb{R})$, $G_\delta = (g_\delta^0, g_\delta)$ satisfying the condition*

$$|x_{00} - x_{0\delta}| + \|\varphi_0 - \varphi_\delta\| + \|q^0 - q_\delta^0\| + \|G_\delta\|_{K_1} \leq \delta,$$

there exists a solution $w_\delta = (\tau_\delta, u_\delta(t))$ of the perturbed optimal problem

$$\begin{gathered}
\dot{x}(t) = f\big(t, x(t), x(t-\tau), u(t)\big) + g_\delta\big(t, x(t), x(t-\tau)\big), \\
x(t) = \varphi_\delta(t), \ \ t \in [\widehat{\tau}, t_0), \quad x(t_0) = x_{0\delta}, \\
J(w;\delta) = q_\delta^0(\tau, x(t_1)) + \int\limits_{t_0}^{t_1} \Big[f^0\big(t, x(t), x(t-\tau), u(t)\big) + g_\delta^0\big(t, x(t), x(t-\tau)\big)\Big]\, dt \longrightarrow \min
\end{gathered}$$

and $|J(w_0) - J(w_\delta;\delta)| < \varepsilon$. Here, the functions $G_\delta(t,x,y)$ satisfy the standard conditions on the set $I \times O^2$ and

$$\int\limits_I \sup\Big\{|G_\delta(t,x,y)| + |G_{\delta x}(t,\,\cdot\,)| + |G_{\delta y}(t,\,\cdot\,)| : \ (x,y) \in K_1^2\Big\}\, dt \leq const,$$

where $K_1 \subset O$ is a compact set containing a neighborhood of K_0;

$$\begin{gathered}
G_{\delta x} = \frac{\partial}{\partial x}\, G_\delta, \quad \|\varphi_0 - \varphi_\delta\| = \sup\big\{|\varphi_0(t) - \varphi_\delta(t)| : \ t \in I_1\big\}, \\
\|q_0 - q_\delta\| = \sup\Big\{|q_0(\tau,x) - q_\delta(\tau,x)| : \ (\tau,x) \in I_2 \times K_1\Big\}, \\
\|G_\delta\|_{K_1} = \sup\bigg\{\bigg|\int\limits_{s_1}^{s_2} G_\delta(t,x,y)\, dt\bigg| : \ (s_1,s_2,x,y) \in I^2 \times K_1^2\bigg\}.
\end{gathered}$$

Theorem 3. *Let the conditions* 1.1 *and* 1.2 *of Theorem* 1 *hold. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every $(x_{0\delta}, \varphi_\delta, q_\delta^0, G_\delta)$, where $x_{0\delta} \in O$, $\varphi_\delta \in C(I_1,O)$, $q_\delta^0 \in C(I_2 \times O, \mathbb{R})$, $G_\delta(t,x,y,u) = (g_\delta^0, g_\delta)$ satisfying the conditions*

$$|x_{00} - x_{0\delta}| + \|\varphi_0 - \varphi_\delta\| + \|q^0 - q_\delta^0\| + \|G_\delta\|_1 \leq \delta$$

and the set $P_{F+G_\delta}(t,x,y)$ *is convex, there exists a solution* $w_\delta=(\tau_\delta,u_\delta(t))$ *of the perturbed optimal problem*

$$\dot x(t)=f\big(t,x(t),x(t-\tau),u(t)\big)+g_\delta\big(t,x(t),x(t-\tau),u(t)\big),$$
$$x(t)=\varphi_\delta(t),\ \ t\in[\widehat\tau,t_0),\quad x(t_0)=x_{0\delta},$$
$$J(w;\delta)=q_\delta^0(\tau,x(t_1))+\int\limits_{t_0}^{t_1}\Big[f^0\big(t,x(t),x(t-\tau),u(t)\big)+g_\delta^0\big(t,x(t),x(t-\tau),u(t)\big)\Big]\,dt\longrightarrow\min$$

and $|J(w_0)-J(w_\delta;\delta)|<\varepsilon$. *Here, the functions* $G_\delta(t,x,y,u)$ *satisfy the standard conditions on the set* $I\times O^2\times U$ *and*

$$\int\limits_I \sup\Big\{|G_{\delta x}(t,x,y,u)|+|G_{\delta y}(t,\,\cdot\,)|:\ (x,y,u)\in K_1^2\times U\Big\}\,dt\le const;$$
$$\|G_\delta\|_1=\int\limits_I \sup\big\{|G_\delta(t,x,y,u)|:\ (x,y,u)\in K_1^2\times U\big\}\,dt.$$

Theorem 4. *Let* $w_0=(\tau_0,u_0(t)),\tau_0\in(\theta_1,\theta_2)$, *be an optimal element and the following conditions hold:*

4.1 *the initial function* $\varphi_0(t)$ *is absolutely continuous and* $\dot\varphi_0(t)$ *is bounded;*

4.2 *the function* $F(t,x,y,u_0(t)),(t,x,y)\in I\times O^2$ *is bounded;*

4.3 *there exist the finite limit*

$$\lim_{(v_1,v_2)\to(v_{10},v_{20})}\big[F(v_1,u_0(t))-F(v_2,u_0(t))\big]=F_0=(f_0^0,f_0)^T,$$

where $v_1,v_2\in I\times O^2$,

$$v_{10}=\big(t_0+\tau_0,x_0(t_0+\tau_0),x_0\big),\quad v_{20}=\big(t_0+\tau_0,x_0(t_0+\tau_0),\varphi_0(t_0)\big).$$

Then the following conditions hold:

4.4 *the condition for the optimal delay parameter* τ_0

$$-\frac{\partial}{\partial\tau}\,q_0(\tau_0,x_0(t_1))=-f_0^0+\psi(t_0+\tau_0)f_0$$
$$+\int\limits_{t_0}^{t_0+\tau_0}\big\{-f_y^0[t+\tau_0]+\psi(t)f_y[t+\tau_0]\big\}\dot\varphi_0(t-\tau_0)\,dt$$
$$+\int\limits_{t_0+\tau_0}^{t_1}\big\{-f_y^0[t+\tau_0]+\psi(t)f_y[t+\tau_0]\big\}\dot x_0(t-\tau_0)\,dt;$$

4.5 *the condition for the optimal control* $u_0(t)$

$$-f^0\big(t,x_0(t),x_0(t-\tau_0),u_0(t)\big)+\psi(t)f\big(t,x_0(t),x_0(t-\tau_0),u_0(t)\big)$$
$$=\max_{u\in U}\Big[-f^0(t,x_0(t),x_0(t-\tau_0),u)+\psi(t)f(t,x_0(t),x_0(t-\tau_0),u)\Big].$$

Here $\psi(t)$ is the solution of the equation

$$\psi(t) = f_x^0[t] - \psi(t)f_x[t] + \chi(t+\tau_0)\big\{f_y^0[t+\tau_0] - \psi(t+\tau_0)f_y[t+\tau_0]\big\}, \ \ t \in [t_0, t_1]$$

with the initial condition

$$\psi(t_1) = -q_x^0(\tau_0, x_0(t_1));$$
$$f_x^0[t] = f_x^0\big(t, x_0(t), x_0(t-\tau_0), u_0(t)\big).$$

Some comments. The theorems of existence and sensitivity of the functional minimum for optimal problems involving various functional differential equations with fixed delay are given in [1–3]. Theorems 1–3 and Theorem 4 are proved by the scheme given in [3] and [4], respectively.

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Existence of Rapidly Varying Solutions of Second Order Half-Linear Differential Equations

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1 Introduction

This paper is concerned with positive solutions of generalized Thomas–Fermi equations of the form

$$(p(t)\varphi_\alpha(x'))' = q(t)\varphi_\alpha(x), \ \ t \geqq a \quad \big(\varphi_\gamma(\xi) = |\xi|^{\gamma-1}\xi = |\xi|^\gamma \operatorname{sgn}\xi, \ \ \xi \in \mathbb{R}, \ \ \gamma > 0\big) \tag{A}$$

for which the following conditions are always assumed to hold:

(a) α is a positive constant;

(b) $p, q : [a, \infty) \to (0, \infty)$, $a \geqq 0$ are continuous functions;

(c) $p(t)$ satisfies that either

$$\int_a^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} = \infty \quad \bigg(P(t) = \int_a^t \frac{ds}{p(s)^{\frac{1}{\alpha}}}\bigg) \tag{1.1}$$

or

$$\int_a^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} < \infty \quad \bigg(\pi(t) = \int_t^\infty \frac{ds}{p(s)^{\frac{1}{\alpha}}}\bigg). \tag{1.2}$$

By a positive solution on an interval J of the differential equation (A) we mean a function $x : J \to (0, \infty)$ which is continuously differentiable on J together with $p(t)\varphi_\alpha(x'(t))$ and satisfies (A) there.

Since the publication of the book [11] of Marić in the year 2000, the class of rapidly varying functions in the sense of Karamata [7] is a well-suited framework for the asymptotic analysis of nonoscillatory solutions of second order linear differential equation of the form

$$x''(t) = q(t)x(t), \ \ q(t) > 0. \tag{B}$$

2 Definitions of rapidly varying functions

Definition 2.1. A measurable function $f : [a, \infty) \to (0, \infty)$ is said to be a rapidly varying of index ∞ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \begin{cases} \infty & \text{for } \ \lambda > 1, \\ 0 & \text{for } \ 0 < \lambda < 1. \end{cases}$$

Moreover, a measurable function $f : [a, \infty) \to (0, \infty)$ is said to be a rapidly varying of index $-\infty$ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \begin{cases} 0 & \text{for } \ \lambda > 1, \\ \infty & \text{for } \ 0 < \lambda < 1. \end{cases}$$

The totality of rapidly varying functions of index ∞ (or $-\infty$) is denoted by $\mathrm{RPV}(\infty)$ (or $\mathrm{RPV}(-\infty)$). The functions

$$f(t)=e^{kt},\quad f(t)=\exp\{t^k\},\quad f(t)=\exp\{e^t\},$$

and

$$f(t)=e^{-kt},\quad f(t)=\exp\{-t^k\},\quad f(t)=\exp\{-e^t\}$$

belong to the $\mathrm{RPV}(\infty)$ and $\mathrm{RPV}(-\infty)$, respectively, for all $k>0$. For the rapidly varying functions the reader referred to the book of Bingham, Goldie and Teugels [1].

3 Definitions of generalized rapidly varying functions

In 2004, Jaroš and Kusano [6] set up the framework of positive solutions which is suitable for the asymptotic analysis of the self-adjoint differential equation

$$(p(t)x'(t))'=q(t)x(t), \tag{C}$$

because the asymptotic behavior of solutions of (C) depends heavily on the function $P(t)$ or $\pi(t)$ given by (1.1) or (1.2), respectively. Therefore, they needed to make the properly generalizing the class of rapidly varying functions in the sense of Karamata. In the generalization use is made of a positive function $R(t)$ which is continuously differentiable on $[a,\infty)$ and satisfies

$$R'(t)>0 \ \text{ for } \ t\ge t_0 \ \text{ and } \ \lim_{t\to\infty}R(t)=\infty.$$

The inverse function of $R(t)$ is denoted by $R^{-1}(t)$.

Definition 3.1. A measurable function $f:[a,\infty)\to(0,\infty)$ is said to be rapidly varying with index ∞ (or $-\infty$) with respect to $R(t)$ if the function $f\circ R^{-1}(t)$ is rapidly varying with index ∞ (or $-\infty$) in the sense of Definition 2.1. The set of all rapidly varying functions with index ∞ (or $-\infty$) with respect to $R(t)$ is denoted by $\mathrm{RPV}_R(\infty)$ (or $RPV_R(-\infty)$), that is, $f\in\mathrm{RPV}_R(\infty)$ satisfies

$$\lim_{t\to\infty}\frac{f(R^{-1}(\lambda t))}{f(R^{-1}(t))}=\begin{cases}\infty & \text{for } \ \lambda>1,\\ 0 & \text{for } \ 0<\lambda<1,\end{cases}$$

and that $f\in\mathrm{RPV}_R(-\infty)$ satisfies

$$\lim_{t\to\infty}\frac{f(R^{-1}(\lambda t))}{f(R^{-1}(t))}=\begin{cases}0 & \text{for } \ \lambda>1,\\ \infty & \text{for } \ 0<\lambda<1.\end{cases}$$

4 Main result

The equation (A_0) $(\varphi_\alpha(x'))'+q(t)\varphi_\alpha(x)=0$ has been the object of intensive investigations from the late 20th century because of the fact that it has many fundamental qualitative properties in common with those of the linear equation

$$x''(t)+q(t)x(t)=0. \tag{D}$$

This fact was first discovered by Elbert [4] and Mirzov [12], who showed in particular that the classical Sturmian comparison and separation theorems for (D) can be carried over to (A_0) almost verbatim and literatim. Since the pioneering work of Elbert and Mirzov much efforts have been directed towards an in-depth analysis of the similarity existing between (A_0) and (D). The reader is referred to the papers [2,3,5,8–10] for typical results on oscillation and/or nonoscillation of (A_0) which are derived in this way on the basis of the well-developed linear oscillation theory of (D).

Theorem A (V. Marić [11])**.** *The equation* (B) *has a fundamental set of solutions* $\{x_1(t), x_2(t)\}$ *such that*

$$x_1 \in \mathrm{RPV}(-\infty) \ \text{ and } \ x_2 \in \mathrm{RPV}(\infty)$$

if and only if

$$\lim_{t\to\infty} t \int\limits_t^{\lambda t} q(s)\,ds = \infty \ \text{ for all } \ \lambda > 1.$$

The purpose of this paper is to obtain the conditions of the existence of rapidly varying solutions of (A) based on the above result of Marić. It will turn out that $R(t) = P(t)$ or $R(t) = 1/\pi(t)$ is the best choice of $R(t)$ for the analysis of the equation (A) with $p(t)$ subject to (1.1) or (1.2), respectively.

We will establish the conditions for the existence of rapdily varying solutions of the equation (A) with the case where the function $p(t)$ satisfies the condition (1.1) or (1.2), respectively.

Theorem 4.1. *Suppose that the function* $p(t)$ *satisfies the condition* (1.1) *and*

$$\lim_{t\to\infty} P(t)^{\alpha} \int\limits_t^{P^{-1}(\lambda P)(t)} q(s)\,ds = \infty$$

are satisfied for all $\lambda > 1$*. Then, the equation* (A) *possesses*

$$x_1 \in \mathrm{RPV}_P(-\infty) \ \text{ and } \ x_2 \in \mathrm{RPV}_P(\infty).$$

Theorem 4.2. *Suppose that the function* $p(t)$ *satisfies the condition* (1.2) *and*

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \int\limits_t^{(\frac{1}{\pi})^{-1}(\frac{\lambda}{\pi})(t)} \pi(s)^{\alpha+1} q(s)\,ds = \infty$$

are satisfied for all $\lambda > 1$*. Then, the equation* (A) *possesses*

$$x_1 \in \mathrm{RPV}_{\frac{1}{\pi}}(-\infty) \ \text{ and } \ x_2 \in \mathrm{RPV}_{\frac{1}{\pi}}(\infty).$$

5 Examples

We here present two examples illustrating the results developed in the preceding Theorems 4.1 and 4.2.

Example 5.1. Consider the differential equation

$$(e^{-\alpha t}\varphi_\alpha(x'))' = \alpha e^t \varphi_\alpha(x), \ \ t \geqq 0, \tag{5.1}$$

where α is as in (A). The function $P(t)$ defined by (1) can be taken to be $P(t) = e^t$. Since, for all $\lambda > 1$,

$$\lim_{t\to\infty} P(t)^{\alpha} \int\limits_t^{P^{-1}(\lambda P)(t)} q(s)\,ds = \alpha(\lambda - 1)e^{(\alpha+1)t} \longrightarrow \infty \ \text{ as } \ t \to \infty,$$

we see from Theorem 4.1 that the equation (5.1) possesses rapidly varying solutions such that

$$x_1 \in \mathrm{RPV}_{e^t}(-\infty) \ \text{ and } \ x_2 \in \mathrm{RPV}_{e^t}(\infty).$$

It is easy to check that $x(t) = \exp\{-e^t\} \in \mathrm{RPV}_{e^t}(-\infty)$ is one such solution of (5.1).

Example 5.2. Consider the differential equation

$$\big(t^\alpha (\log t)^{2\alpha} \varphi_\alpha(x')\big)' = q(t)\varphi_\alpha(x), \ \ t \geqq e, \ \ q(t) \sim \frac{\alpha(\log t)^{2\alpha}}{t} \ \text{ as } \ t \to \infty, \tag{5.2}$$

where α is as in (A) and the symbol $\sim$ is used to denote the asymptotic equivalence

$$f(t) \sim g(t) \ \text{ as } \ t \to \infty \iff \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

Since $\pi(t) \sim (\log t)^{-1}$, $t \to \infty$, we have, for all $\lambda > 1$,

$$\frac{1}{\pi(t)} \int\limits_t^{(\frac{1}{\pi})^{-1}(\frac{\lambda}{\pi})(t)} \pi(s)^{\alpha+1} q(s)\,ds \sim \alpha\lambda \log t \longrightarrow \infty \ \text{ as } \ t \to \infty,$$

from which it follows from Theorem 4.2 that the equation (5.2) possesses rapidly varying solutions such that

$$x_1 \in \mathrm{RPV}_{\log t}(-\infty) \ \text{ and } \ x_2 \in \mathrm{RPV}_{\log t}(\infty).$$

$x(t) = t \in \mathrm{RPV}_{\log t}(\infty)$ is one such solution of (5.2).

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Analytic Representation and Some Properties of "Bulky" Links, Generated by Generalised Möbius–Listing's Bodies

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Natural forms affect all of us, not only for their beauty, but also for their diversity. It is still not known whether forms define the essence of the phenomena associated with them, or vice versa that is, forms are natural consequences of the phenomena. The essence of one "unexpected" phenomenon is as follows: Usually after one "full cutting", an object is split into two parts. The Möbius strip is a well-known exception, however, which still remains whole after cutting. In 2008–2009 author discovered a class of surfaces, which have following properties – after full cutting more than two surfaces appear, but this is a result for specific class of pure mathematical surfaces [2]. It turns out that three-dimensional Möbius Listing bodies, GML_m^n, (m is number of symmetry of the radial cross section and n is a number of twisting) which is a wide subclass of the Generalized Twisting and Rotated figures – shortly GTR_m^n – which, through their analytic representation, could yield more than two objects after only single cutting ([3] or [5]). These are not only theoretical results, as can be proved by real-life examples. Many classical objects (torus with different forms of radial cross sections, helicoid, helix, Möbius strip, ... etc.) are elements of this wide class of GTR_m^n figures, so it is important to study the similarity and difference between these figures and surfaces. One possible application of these results is assumed in the description of the properties of the middle surfaces in the theory of elastic shells [6].

Based on the one form of analytical representation

$$\begin{cases} X(\tau,\theta) = \Big(R + r(\tau,\theta)\cos\Big(\psi + \dfrac{n\theta}{m}\Big)\Big)\cos(\theta), \\ Y(\tau,\theta) = \Big(R + r(\tau,\theta)\cos\Big(\psi + \dfrac{n\theta}{m}\Big)\Big)\sin(\theta), \\ Z(\tau,\theta) = r(\tau,\theta)\sin\Big(\psi + \dfrac{n\theta}{m}\Big), \end{cases} \tag{1}$$

and on the definition of operation of cutting defined earlier [2, 4], some basic questions to be answered appear, for example:

1. How many objects appear after cutting of the GML_m^n surfaces or bodies?
2. What type of GML_m^n surfaces or bodies appear after cutting (this question for Möbius strip was formulated for the first time by Sosinski see e.g. [1])?
3. What is a link-structure of the surfaces or bodies, which appear after cutting?
4. What are shapes of radial cross sections of the geometric objects which appear after cutting of GML_m^n surfaces or bodies?
5. How many different combinations of geometric objects (in the sense of shapes of the radial cross sections) appear after cutting for arbitrary number m in GML_m^n?

6. What are differential geometric characteristics of GML_m^n surfaces or bodies?

At this stage, we unfortunately do not have answers to all of these questions raised in the case of arbitrary values of m, but some particular cases were reported by the author and his colleagues [2, 3, 5].

In this report we give some general results.

Remark.

- **A.** If m is an even number, then for different n (more precisely, if $\gcd(m,n) = 1$) – after one full cutting of GML_m^n bodies, maximum $m/2+1$ independent geometric objects appear (this number depends also on the geometric place of the cutting line in the cross section of body), i.e. link-$(m/2+1)$ appear and only one element has structure similar to figure before cutting;

- **B.** If m is an odd number, then for different n (more precisely, if $\gcd(m,n) = 1$) – after one full cutting of GML_m^n bodies, maximum $[m/2]+2$ independent geometric objects appear (in the cross section of body), i.e. link-$([m/2]+2)$ appear and only one element has structure similar to figure before cutting; $[m/2]$ is the integer part of number $m/2$; (see example for $m = 5$ and $n = 1$ in Figure 1).

- **C.** If m is even number, then always exist some values of n, such that after one full cutting of GML_m^n bodies only 1 independent geometric object appears (for this, the cutting line should include the center of symmetry of the radial cross section of the body), i.e. knot (link-1 appears, whose index is defined by $\gcd(m,n) = 1$;

  ⋆ If m is an even number, then always some values of n exist, and the phenomenon of the Möbius strip is realized (see example for $m = 6$ and $n = 1$ in Figure 2)!

- **D.** If m is odd number, then there exist some values of n, such that after one cutting of GML_m^n at least 2 independent geometric objects appear (for this the cutting line should include the center of symmetry of the radial cross section of body), i.e. (link-2 appears), whose index is defined by $\gcd(m,n) = 1$;

  ⋆ If m is an odd number, then for any value of the parameter n, the phenomenon of the Möbius band is never realized!

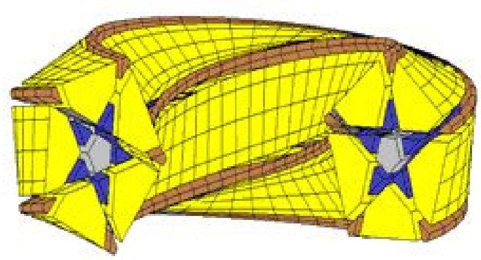

Figure 1.

Acknowledgements

This report is a collection of results obtained at different times since 2006. Results were discussed with many colleagues in different countries and in different institutions. For the authors were very

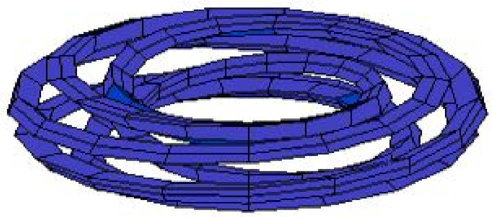

Figure 2.

valuable discussions with professors P. V. Checcerini and M. Tallini in 2008 in Rome. A special role is played by creative contact with Paolo Emilio Ricci and Johan Gielis, with which some important results have been obtained. At different times and different results were obtained with C. Cassisa, D. Caratelli, M. Rogava and M. Transirico; For valuable discussions, the authors thank colleagues N. Avazashvili, M. Nakashidze, P. P. Natalini, S. Camiz, Antonio di Nola. Authors are grateful to Ph.D. Student of St. Andrew the First-Called Georgian University of the Patriarchate of Georgia Levan Roinishvili for creating the convenient program for demonstration of the obtained mathematical results. Some part of results has been fulfilled by a financial support of Shota Rustaveli National Science Foundation (Grant # SRNSF/FR/358/5-109/14).

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The Shift Invariance of Time Scales and Applications

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Abstract Recently Wang, Agarwal and O'Regan studied the problem of the shift invariance of time scales. This contributes to investigating functions defined by shifts, for example periodic functions, almost periodic functions and almost automorphic functions, etc. Moreover the related theory of dynamic equations was established. In addition, the shift invariance of time scales was employed to construct delay functions effectively in delay dynamic equations on time scales. As an extension, the almost translation invariance of time scales has been presented.

1 A matched space and the shift invariance of time scales

Recently, to consider the problem of the shift invariance of time scales, the authors introduced the concept of matched space for time scales and applied it to propose new concepts of almost periodic functions and almost automorphic functions (see [7]). These concepts are effective not only on periodic time scales but also are valid on irregular time scales such as quantum-like time scales, e.g. $\pm\overline{q^{\mathbb{Z}}}$, $\overline{(-q)^{\mathbb{Z}}}$, $\overline{-q^{\mathbb{Z}}\cup q^{\mathbb{Z}}}$ and others such as $\pm\mathbb{N}^{\frac{1}{2}}$ and $\pm\mathbb{N}^2$, etc. Under a matched space for the time scale, a new almost periodic theory for dynamic equations was established (see [10]). To investigate periodic solutions to dynamic equations on a quantum time scale, Adıvar proposed a new concept of periodic time scales under shift operators in 2013, and this approach enables one to investigate this periodic notion on time scales (see [1]). In 2017, using the idea of shift operators established in [1], the authors introduced the concept of piecewise almost periodic functions which includes q-difference equations on a quantum-like time scale under a stochastic background and they used it to study almost periodic solutions to stochastic impulsive dynamic delay models (see [8, 11]).

To develop an effective tool to consider the shift invariance of time scales the authors in [7] used the algebraic structure of an Abelian group and introduced the concept of a matched space for time scales and they constructed the algebraic structure of matched spaces to solve the problem of closedness of time scales under shifts (including translational and non-translational shifts). The concept of periodic time scales under matched spaces includes the definition of periodic time scales proposed by Adıvar. Also functions defined by shifts can be introduced on a larger group of time scales and the shift invariance of time scales can be guaranteed under the matched space.

To introduce the concept of matched space for time scales, the following algebraic structure for a pair $(\Pi^*, \widetilde{\delta})$ is needed.

Definition 1.1 ([7])**.** Let Π^* be a subset of $\mathbb{R}$ together with an operation $\widetilde{\delta}$ and a pair $(\Pi^*, \widetilde{\delta})$ be an Abelian group and $\widetilde{\delta}$ be increasing with respect to its second argument, i.e., Π^* and $\widetilde{\delta}$ satisfy the following conditions:

(1) Π^* is closed with respect to an operation $\widetilde{\delta}$, i.e., for any $\tau_1, \tau_2 \in \Pi^*$, we have $\widetilde{\delta}(\tau_1, \tau_2) \in \Pi^*$.

(2) For any $\tau \in \Pi^*$, there exists an identity element $e_{\Pi^*} \in \Pi^*$ such that $\widetilde{\delta}(e_{\Pi^*}, \tau) = \tau$.

(3) For all $\tau_1, \tau_2, \tau_3 \in \Pi^*$, $\widetilde{\delta}(\tau_1, \widetilde{\delta}(\tau_2, \tau_3)) = \widetilde{\delta}(\widetilde{\delta}(\tau_1, \tau_2), \tau_3)$ and $\widetilde{\delta}(\tau_1, \tau_2) = \widetilde{\delta}(\tau_2, \tau_1)$.

(4) For each $\tau \in \Pi^*$, there exists an element $\tau^{-1} \in \Pi^*$ such that $\widetilde{\delta}(\tau, \tau^{-1}) = \widetilde{\delta}(\tau^{-1}, \tau) = e_{\Pi^*}$, where e_{Π^*} is the identity element in Π^*.

(5) If $\tau_1 > \tau_2$, then $\widetilde{\delta}(\,\cdot\,, \tau_1) > \widetilde{\delta}(\,\cdot\,, \tau_2)$.

A subset S of $\mathbb{R}$ is called relatively dense with respect to the pair $(\Pi^*, \widetilde{\delta})$ if there exists a number $L \in \Pi^*$ and $L > e_{\Pi^*}$ such that $[a, \widetilde{\delta}(a, L)]_{\Pi^*} \cap S \neq \varnothing$ for all $a \in \Pi^*$. The number $|L|$ is called the inclusion length with respect to the group $(\Pi^*, \widetilde{\delta})$.

According to Definition 1.1, one can introduce the definition of a relatively dense set with respect to the group $(\Pi^*, \widetilde{\delta})$, where Π^* is a subset of $\mathbb{R}$ together with an operation $\widetilde{\delta}$.

Definition 1.2 ([7])**.** A subset S of $\mathbb{R}$ is called relatively dense with respect to the pair $(\Pi^*, \widetilde{\delta})$ if there exists a number $L \in \Pi^*$ and $L > e_{\Pi^*}$ such that $[a, \widetilde{\delta}(a, L)]_{\Pi^*} \cap S \neq \varnothing$ for all $a \in \Pi^*$. The number $|L|$ is called the inclusion length with respect to the group $(\Pi^*, \widetilde{\delta})$.

Definition 1.3 ([7])**.** Let the pair $(\Pi^*, \widetilde{\delta})$ be an Abelian group and Π^*, $\mathbb{T}^*$ be the largest subsets of the time scales Π and $\mathbb{T}$, respectively. Further, let Π be the adjoint set of $\mathbb{T}$ and F the adjoint mapping between $\mathbb{T}$ and Π. The operator $\delta : \Pi^* \times \mathbb{T}^* \to \mathbb{T}^*$ satisfies the following properties:

(P_1) (Monotonicity) The function δ is strictly increasing with respect to its all arguments, i.e., if

$$(T_0, t),\ (T_0, u) \in \mathcal{D}_\delta := \big\{(s,t) \in \Pi^* \times \mathbb{T}^* :\ \delta(s,t) \in \mathbb{T}^*\big\},$$

then $t < u$ implies $\delta(T_0, t) < \delta(T_0, u)$; if (T_1, u), $(T_2, u) \in \mathcal{D}_\delta$ with $T_1 < T_2$, then $\delta(T_1, u) < \delta(T_2, u)$.

(P_2) (Existence of inverse elements) The operator δ has the inverse operator $\delta^{-1} : \Pi^* \times \mathbb{T}^* \to \mathbb{T}^*$ and $\delta^{-1}(\tau, t) = \delta(\tau^{-1}, t)$, where $\tau^{-1} \in \Pi^*$ is the inverse element of τ.

(P_3) (Existence of identity element) $e_{\Pi^*} \in \Pi^*$ and $\delta(e_{\Pi^*}, t) = t$ for any $t \in \mathbb{T}^*$, where e_{Π^*} is the identity element in Π^*.

(P_4) (Bridge condition) For any $\tau_1, \tau_2 \in \Pi^*$ and $t \in \mathbb{T}^*$, $\delta(\widetilde{\delta}(\tau_1, \tau_2), t) = \delta(\tau_1, \delta(\tau_2, t)) = \delta(\tau_2, \delta(\tau_1, t))$.

Then the operator $\delta(s,t)$ associated with $e_{\Pi^*} \in \Pi^*$ is said to be a shift operator on the set $\mathbb{T}^*$. The variable $s \in \Pi^*$ in δ is called the shift size. The value $\delta(s,t)$ in $\mathbb{T}^*$ indicates s units shift of the term $t \in \mathbb{T}^*$. The set $\mathcal{D}_\delta$ is the domain of the shift operator δ.

Definition 1.4. Let the pair $(\Pi^*, \widetilde{\delta})$ be an Abelian group, and Π^*, $\mathbb{T}^*$ be the largest subsets of the time scales Π and $\mathbb{T}$, respectively. Further, let Π be an adjoint set of $\mathbb{T}$ and F the adjoint mapping between $\mathbb{T}$ and Π. If there exists the shift operator δ satisfying Definition 1.3, then we say the group $(\mathbb{T}, \Pi, F, \delta)$ is a matched space for the time scale $\mathbb{T}$.

Using the algebraic structure of matched spaces, we introduce the following new concept of periodic time scales.

Definition 1.5. A time scale $\mathbb{T}$ is called a periodic time scale (or bi-direction shift invariant time scale) under a matched space $(\mathbb{T}, \Pi, F, \delta)$ if

$$\widetilde{\Pi} := \big\{\tau \in \Pi^* :\ (\tau^{\pm 1}, t) \in \mathcal{D}_\delta,\ \forall\, t \in \mathbb{T}^*\big\} \notin \big\{\{e_{\Pi^*}\}, \varnothing\big\}. \tag{1.1}$$

The shift invariance of time scales attached with shift directions is also considered in [7]. In this case, the pair $(\Pi^*, \widetilde{\delta})$ in Definition 1.1 is not an Abelian group but a semigroup with a direction, which indicates that the shift direction has an impact on the shift closedness of time scales (see Definition 2.8, Definition 2.10 from [7]). In [6] the authors noted that that the translation direction should be taken into account when one considers the translation invariance of time scales and the authors introduced the concept of periodic time scales attached with translation direction (i.e., oriented-direction translation invariant time scales). During 2015–2017, Wang, Agarwal and O'Regan considered the local translation invariance of time scales attached with translation direction and introduced the concept of changing-periodic time scales and established a composition theorem of time scales to divide an arbitrary time scale into a countable union of translation invariant time scales which are with translation directions (see [2, 4]). Therefore, shift direction is also another factor that should be considered when discussing the shift invariance of time scales.

2 Almost periodic and almost automorphic functions

Under the matched space of time scales, some new concepts of almost periodic functions and almost automorphic functions were introduced and a related theory with applications to dynamic equations was established in the literature (see [7, 10]).

Now, we assume that $(\mathbb{T}, \Pi, F, \delta)$ is a bi-direction matched space, then all the elements from Π^* have the corresponding inverse elements in Π^*.

Definition 2.1. Let $\mathbb{T}$ be a periodic time scale under the matched space $(\mathbb{T}, \Pi, F, \delta)$. A function $f \in C(\mathbb{T} \times D, \mathbb{X})$ is called an almost periodic function with shift operators in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-shift set of f

$$E\{\varepsilon, f, S\} = \Big\{\tau \in \widetilde{\Pi} :\ \big\|f(\delta_{\tau^{\pm 1}}(t), x) - f(t, x)\big\| < \varepsilon \ \text{ for all } t \in \mathbb{T}^* \text{ and } x \in S\Big\}$$

is a relatively dense set with respect to the pair $(\Pi^*, \widetilde{\delta})$ for all $\varepsilon > 0$ and for each compact subset S of D; that is, for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$\big\|f(\delta_{\tau^{\pm 1}}(t), x) - f(t, x)\big\| < \varepsilon \ \text{ for all } t \in \mathbb{T}^* \text{ and } x \in S.$$

Now τ is called the ε-shift number of f and $l(\varepsilon, S)$ is called the inclusion length of $E\{\varepsilon, f, S\}$.

Definition 2.2. Let $\mathbb{T}$ be a periodic time scale under the matched space $(\mathbb{T}, \Pi, F, \delta)$, the shift $\delta_{\tau^{\pm 1}}(t)$ is Δ-differentiable with rd-continuous bounded derivatives $\delta^{\Delta}_{\tau^{\pm 1}}(t) := \delta^{\Delta}(\tau^{\pm 1}, t)$ for all

$t \in \mathbb{T}^*$. A function $f \in C(\mathbb{T} \times D, \mathbb{X})$ is called an almost periodic function with shift operators in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-shift set of f

$$E\{\varepsilon, f, S\} = \Big\{\tau \in \widetilde{\Pi} : \ \big\|f(\delta_{\tau^{\pm 1}}(t), x)\delta^{\Delta}_{\tau^{\pm 1}}(t) - f(t, x)\big\| < \varepsilon \ \text{ for all } t \in \mathbb{T}^* \text{ and } x \in S\Big\}$$

is a relatively dense set with respect to the pair $(\Pi^*, \widetilde{\delta})$ for all $\varepsilon > 0$ and for each compact subset S of D; that is, for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$\big\|f(\delta_{\tau^{\pm 1}}(t), x)\delta^{\Delta}_{\tau^{\pm 1}}(t) - f(t, x)\big\| < \varepsilon \ \text{ for all } t \in \mathbb{T}^* \text{ and } x \in S.$$

Now τ is called the ε-shift number of f and $l(\varepsilon, S)$ is called the inclusion length of $E\{\varepsilon, f, S\}$.

Definition 2.3. Let $\mathbb{T}$ be a periodic time scale under the matched space $(\mathbb{T}, \Pi, F, \delta)$.

(i) Let $f : \mathbb{T} \to \mathbb{X}$ be a bounded continuous function. We say that f is almost automorphic if from every sequence $\{s_n\} \subset \widetilde{\Pi}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$g(t) = \lim_{n\to\infty} f(\delta_{\tau_n}(t)),$$

is well defined for each $t \in \mathbb{T}^*$ and

$$\lim_{n\to\infty} g(\delta_{\tau_n^{-1}}(t)) = \lim_{n\to\infty} g(\delta^{-1}_{\tau_n}(t)) = f(t)$$

for each $t \in \mathbb{T}^*$. Denote by $AA_\delta(\mathbb{T}, \mathbb{X})$ the set of all such functions.

(ii) A continuous function $f : \mathbb{T} \times \mathbb{X} \to \mathbb{X}$ is said to be almost automorphic if $f(t, x)$ is almost automorphic in $t \in \mathbb{T}^*$ uniformly for $x \in B$, where B is any bounded subset of $\mathbb{X}$. Denote by $AA_\delta(\mathbb{T} \times \mathbb{X}, \mathbb{X})$ the set of all such functions.

Definition 2.4. Let $\mathbb{T}$ be a periodic time scale under the matched space $(\mathbb{T}, \Pi, F, \delta)$.

(i) Let $f : \mathbb{T} \to \mathbb{X}$ be a bounded continuous function and the shift $\delta_\tau(t)$ is Δ-differentiable with rd-continuous bounded derivatives $\delta^{\Delta}_{\tau}(t) := \delta^{\Delta}(\tau, t)$ for all $t \in \mathbb{T}^*$. We say that f is Δ-almost automorphic if from every sequence $\{s_n\} \subset \widetilde{\Pi}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$g(t) = \lim_{n\to\infty} f(\delta_{\tau_n}(t))\delta^{\Delta}_{\tau_n}(t),$$

is well defined for each $t \in \mathbb{T}^*$ and

$$\lim_{n\to\infty} g(\delta_{\tau_n^{-1}}(t))\delta^{\Delta}_{\tau_n^{-1}}(t) = f(t)$$

for each $t \in \mathbb{T}^*$. Denote by $AA_\delta(\mathbb{T}, \mathbb{X})$ the set of all such functions.

(ii) A continuous function $f : \mathbb{T} \times \mathbb{X} \to \mathbb{X}$ is said to be Δ-almost automorphic if $f(t, x)$ is Δ-almost automorphic in $t \in \mathbb{T}^*$ uniformly for $x \in B$, where B is any bounded subset of $\mathbb{X}$. Denote by $AA_\delta(\mathbb{T} \times \mathbb{X}, \mathbb{X})$ the set of all such functions.

Note the above concepts under oriented-direction matched space can be found in [7].

3 Delay dynamic equations and models

In [5], the authors proposed some types of delay dynamic equations on time scales. The range of the delay functions should be a subset of the shift invariant number set, for example, when a time scale is a bi-direction shift invariant time scale (i.e., a periodic time scale in the sense of Definition 1.5), the shift invariant number set is the set formed by all the periods of $\mathbb{T}$. In [10], an n_0-order Δ-almost periodic theory of dynamic equations was established, and we considered the following almost periodic dynamic equation with variable delays under the matched space $(\mathbb{T}, F, \Pi, \delta)$:

$$x^{\Delta}(t) = S_A^{n_0}(t)x(t) + \sum_{i=1}^{n} S_f^{n_0}\big(t, x(\delta(\tau_i(t), t))\big),$$

where $A(t)$ is an $\Delta_{n_0}^{\delta}$-almost periodic matrix function on $\mathbb{T}$, $\tau_i(t) : \mathbb{T}^* \to \Pi^*$ is $\Delta_{n_0}^{\delta}$-almost periodic on $\mathbb{T}$ for every $i = 1, 2, \dots, n$, and $f \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ is $\Delta_{n_0}^{\delta}$-almost periodic uniformly in t for $x \in \mathbb{R}^n$.

In [8], the following almost periodic impulsive stochastic Lasota–Wazewska timescale model was considered.

$$\begin{cases} \Delta\big(x_i(t) + c_i(t)x_i(\delta_-(\tau_i, t))\big) = \Big[-\alpha_i(t)x_i(t) + \displaystyle\sum_{j=1}^{m} \beta_{ij}(t)e^{-\gamma_{ij}(t)x_j(\delta_-(\tau_{ij}, t))}\Big]\Delta t & \\ \qquad + \displaystyle\sum_{j=1}^{m} H_{ij}\big(t, x_j(\delta_-(\sigma_{ij}, t))\big)\Delta\omega_j(t), & t \neq t_k, \\ \widetilde{\Delta} x_i(t_k) = x_i(t_k^+) - x_i(t_k^+) = I_{ik}(x_i(t_k)) + \alpha_{ik}x_i(t_k) + \nu_{ik}, & t = t_k; \end{cases}$$

see [8] for the biological background of the above model. Therefore, the shift invariance of time scales contributes to constructing the delay functions effectively in delay dynamic equations on time scales.

4 Almost translation invariance of time scales

In the literature [9], we employ an approximation method to introduce the concept of almost translation invariance of time scales (i.e., almost-complete closedness time scale).

Definition 4.1. We say $\mathbb{T}$ is an almost-complete closedness time scale (ACCTS) if for any given $\varepsilon_1 > 0$, there exist a constant $l(\varepsilon_1) > 0$ such that each interval of length $l(\varepsilon_1)$ contains a $\tau(\varepsilon_1)$ and sets $A_\tau^{\varepsilon_1}$ such that

$$d(\overline{\mathbb{T} \setminus A_\tau^{\varepsilon_1}}, \mathbb{T}^\tau) < \varepsilon_1$$

i.e., for any $\varepsilon_1 > 0$, the following set

$$\mathrm{E}\{\mathbb{T}, \varepsilon_1\} = \big\{\tau \in \Pi :\ d(\overline{\mathbb{T} \setminus A_\tau^{\varepsilon_1}}, \mathbb{T}^\tau) < \varepsilon_1\big\} := \Pi_{\varepsilon_1}$$

is relatively dense in Π. Here, τ is called the ε_1-translation number of $\mathbb{T}$, $l(\varepsilon_1)$ is called the inclusion length of $\mathrm{E}\{\mathbb{T}, \varepsilon_1\}$, and $\mathrm{E}\{\mathbb{T}, \varepsilon_1\}$ the ε_1-translation set of $\mathbb{T}$, $A_\tau^{\varepsilon_1}$ is called the ε_1-improper set of $\mathbb{T}$,

$$\mathscr{R}_{\mathbb{T}}(\tau, \varepsilon_1) := \mathbb{T} \cap \Big(\bigcup_{\tau \in \Pi_{\varepsilon_1}} \overline{\mathbb{T}^{-\tau} \setminus A_{-\tau}^{\varepsilon_1}} \Big)$$

the ε_1-main region of $\mathbb{T}$, where

$$A_{-\tau}^{\varepsilon_1} = (A_\tau^{\varepsilon_1})^{-\tau} := \big\{a - \tau :\ a \in A_\tau^{\varepsilon_1}\big\}.$$

Note that Definition 4.1 will include the concept of almost periodic time scales proposed in [3], which was developed in [5, 6]. This concept was applied to study double-almost periodic functions and solutions with double almost periodicity to dynamic equations on time scales (see [9]).

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