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Mathematics

On the Cramer-Rao Inequality in an Infinite Dimensional Space

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ABSTRACT. The Cramer-Rao inequality is obtained in a Banach space by using the technique of smooth measures. The principle of maximum likelihood is formulated. The examples are considered.

Key words: *Cramer-Rao inequality, smooth measure, maximum likelihood principle.*

The current state of infinite dimensional analysis makes it possible to consider many fundamental problems of statistics in more general terms. The theory of smooth measures [1] provides vast opportunities in this direction. In the present paper, the realization of one of such opportunities is illustrated by an example of generalization of the Cramer-Rao (C-R) inequality and formulation of the maximum likelihood principle in the infinite dimensional case. This approach was actually suggested by E. Gobet in [2] and is based on the Malliavin calculus theory [3, 4]. In [5], the Malliavin calculus was used in the finite-dimensional (more precisely, one-dimensional) case. In this paper, we use the technique of smooth functions which enables us to formulate a general approach to problems for the Cramer-Rao inequality and tackle the questions associated with them. In [6], the theory of smooth measures was applied to the estimation of the logarithmic derivative of a measure.

1. The Logarithmic Derivative of a Measure.

Let $\{\Omega, \Im, P\}$ be a complete probability space. Consider a random element $X = X(\omega;\theta)$ with parameter $\theta \in \Theta$, where $\Theta \subset \Xi$ is a subset of a separable real Banach space Ξ with norm $\|\cdot\|_{\Xi}$. Let $X = X(\omega;\theta)$ have values in a linear space E.

The primary goal of statistics is to estimate an unknown parameter θ. The estimation is based on (iid) observations $X_1, X_2, \dots, X_n, \dots$ of the given random variable. We must construct a statistical function

$T = T(X_1, X_2, ..., X_n)$ such that estimates the parameter θ .

Usually, in this situation we obtain the sequence of statistical structures $\{\aleph, \Re, (P(\theta;\cdot), \theta \in \Theta)\}$, where $\aleph = E^n$ $(n = 1, 2, ..., \infty)$ is the linear space generated by the sequence of random variables $X_1, X_2, ..., X_n$, $\Re$ is the σ-algebra generated by observable sets and $\{P(\theta;\cdot), \theta \in \Theta\}$ is the family of probability measures (distributions) generated by the vector $Y = (X_1, X_2, ..., X_n)$ with the help of the relation $P(\theta; A) = P(Y^{-1}(A))$, $A \in \Re$. In classical statistics, the structure $\{\aleph, \Re, (P(\theta;\cdot), \theta \in \Theta)\}$ is the main object of investigation.

On the other hand, there are a wide range of problems in which it is more convenient to operate with the function $X = X(\omega; \theta)$ if we impose on it the condition of smoothness (regularity condition) of the parameter θ . Thus this is a good chance to apply the tool of stochastic calculus of variations.

We in fact use two calculi: one is based on studying the properties of the statistical structure $\{\aleph, \Re, (P(\theta;\cdot), \theta \in \Theta)\}$ where the family of measures $P(\theta, \cdot)$ is assumed to be smooth, and the other employs direct stochastic methods for which the object of investigation is the function $X(\omega, \theta)$.

Thus the family of distributions $\{P(\theta; A), \theta \in \Theta, A \in \Re\}$ is interesting for us in terms of smoothness imposed on the two parameters θ and A .

Let us assume that $\aleph$ is a separable real reflective Banach space. For every fixed $\theta \in \Theta$, $P(\theta, \cdot)$ is a positive measure. If $h \in \aleph$ is some vector, then we denote by $P_h(\theta; A)$ the measure obtained by the shift $P_h(\theta; A) = P(\theta; A + h)$. We say that the measure $P(\theta, \cdot)$ is differentiable along the vector h if there exists a bounded linear functional on $\aleph$ denoted by $d_h P(\theta;\cdot)$ such that for every $A \in \Re$ the following equality is true

$$P_h(\theta; A) - P(\theta; A) = d_h P(\theta; A) h + \alpha(\theta, A; h),$$

where $\alpha(\theta, A; h)$ is a function such that $\alpha(\theta, A; th) = o(t)$, $t \in R$.

In the case where $\aleph$ is a separable real Hilbert space with scalar product $(\ \cdot\ ,\ \cdot\ )_\aleph$ and norm $\|h\|_\aleph$, $h \in \aleph$, we write $P_h(\theta; A) - P(\theta; A) = (d_h P(\theta; A), h)_\aleph + \alpha(\theta, A; h)$ and sometimes (when it does not lead to confusion) under the derivative $d_h P(\theta;\cdot)$ we will understand an element of the Hilbert space. Clearly, the function $d_h P(\theta;\cdot) h$ is a σ-additive (alternating) measure on $\Re$.

The function $\psi_\theta(t) = P(\theta, A + th)$ is nonnegative and everywhere differentiable. If $P(\theta; A) = 0$, $A \in \Re$, then $t = 0$ is the point of a minimum for functions $\psi_\theta(t)$. Therefore $d_h P(\theta; A) = 0$. Thus, by the Radon-Nikodym theorem, there is a measurable function $\beta_\theta(x; h)$ such that $\dfrac{d_h P(\theta; dx)}{P(\theta; dx)} = \beta_\theta(x; h)$. This function is called the logarithmic derivative of the measure $P(\theta;\ \cdot\ )$ along a vector $h \in \aleph$. The logarithmic derivative $\beta_\theta(x; h)$ is linear on the second argument. A vector h is called an admissible direction for the measure

$P(\theta;\cdot)$. The set of all admissible directions is called an admissible subspace.

Example 1. Let $H_+ \subset H \subset H_-$ be three Hilbert spaces, the enclosure operator $i: H_+ \to H$ be the Hilbert-Schmidt operator. Such a triple is called the Hilbert-Schmidt structure. Let γ_θ be a Gaussian measure in γ_θ with the correlation operator equal to I in H and an average θ, $\theta \in H_-$. If $h \in H_+$ then the logarithmic derivative of the measure γ_θ along h is $(\theta - x, h)_H$. ■

In the theory of differentiable measures the fact that the integration by parts formula is fulfilled is very important. Let $\aleph$ be a separable real Hilbert space and $f(x)$ be a functional on it. Suppose that $f(x)$ has a derivative along a vector $h \in \aleph : d_h f(x) = \lim_{t \to 0} t^{-1}\left[f(x+th) - f(x)\right]$ and $d_h f(\cdot) \in L_1(P(\theta;\cdot))$ for fixed $\theta \in \Theta$. In that case, if the measure $P(\theta;\cdot)$ is differentiable along h, then

$$\int_{\aleph}(d_h f(x), h)_{\aleph} P(\theta; dx) = -\int_{\aleph} f(x) d_h P(\theta; dx) = -\int_{\aleph} f(x)\beta_\theta(x;h) P(\theta; dx). \tag{1}$$

We can define the logarithmic derivative along some changeable direction (the so-called logarithmic gradient). Equality (1) can be considered on the basis of such a definition or we can act as when defining the derivative measure along a constant direction.

Let $z(x): \aleph \to \aleph$ be a differentiable vector field with a bounded derivative $\sup_{x \in \aleph}\|z'(x)\| < \infty$. Denote the integral stream corresponding to $z(x)$ by S_t, $t \in R$. This means that $\frac{dS_t}{dt} = z(S_t)$, $S_0 = I$.

According to the transformation $P_t(\theta; A) = P(\theta; S_t^{-1}(A))$, $A \in \mathfrak{R}$, to the family of measures $(P(\theta;\ \cdot\ ), \theta \in \Theta)$ there corresponds a class of measures $(P_t(\theta;\cdot), \theta \in \Theta, t \in R)$. The measure $P(\theta;\cdot)$ is differentiable along the vector field $z(x)$ if there is a measure (necessarily alternating) $D_z P(\theta; A)$ such that for any bounded and differentiable function $\varphi: \aleph \to R$, $\varphi \in C^1(\aleph; R)$ we have

$$\int_{\aleph}\varphi(x) D_z P(\theta; dx) = -\int_{\aleph}\varphi'(x) z(x) P(\theta; dx).$$

If $D_z P(\theta;\cdot) \ll P(\theta;\cdot)$, then the Radon-Nikodym density is called the logarithmic derivative of $P(\theta;\cdot)$ along the vector field $z(x): \beta_\theta(x;z) = \frac{D_z P(\theta; dx)}{P(\theta; dx)}$.

Let H be embedded in the Hilbert space $\aleph$, where the embedment operator is the Hilbert-Schmidt operator. Then we can consider the Hilbert-Schmidt structure $\aleph^* \subset H \subset \aleph$. Let us choose a class of measures £ for which there exists a measurable, locally bounded function $\lambda: \aleph \to \aleph$ such that for each constant direction $h \in \aleph^*$ there exists a logarithmic derivative along h which has the form $\beta_\theta(x;h) = \lambda(\theta;x)h = (\lambda(\theta,x), h)_H$. In this case we say that the measure has the logarithmic gradient $\lambda(\theta;x)$. If $P(\theta) \in$ £ and the vector field $z: \aleph \to \aleph^*$ and its derivative is bounded, then the measure $P(\theta)$

has a logarithmic gradient and

$$\beta_\theta\left(x;z(x)\right)=\left\langle\lambda(\theta;x),z(x)\right\rangle+trz'(x).$$

By the principle of continuity, this functional can be extended to smooth vector fields $z(x):\aleph\to H$.

Example 2. In the conditions of Example 1, consider the vector field $z(x):H_-\to H_-$ with a bounded derivative $\sup_{x\in H_-}\|z'(x)\|<\infty$. If $z:H_-\to H$, then the logarithmic gradient exists and $\beta_\theta(x;z)=\left(\theta-x,z(x)\right)_H+trz'(x)$.

We need to show the smoothness of measures with respect to the parameter. Assume as above that we have $\left\{\aleph,\Re,\left(P(\theta;\cdot),\theta\in\Theta\right)\right\}$, where $\aleph$ is a separable real Banach space, and Θ is a smooth manifold embedded into another separable real Banach space Ξ. For every fixed $A\in\Re$ and a vector $\vartheta\in\Xi$, consider the derivative of the function $\tau(\theta)=P(\theta;A)$ at a point θ along ϑ. Denote this derivative by $d_\theta P(\theta;A)\vartheta$. For fixed θ and ϑ, it is an alternating measure. It is easy to see that $d_\theta P(\theta,\cdot)\vartheta\ll P(\theta,\cdot)$ and, by the Radon-Nikodym theorem, there exists a measurable function $l_\theta(x;\vartheta)=\dfrac{d_\theta P(\theta;dx)\vartheta}{P(\theta;dx)}$. The function $l_\theta(x;\vartheta)$ is called the logarithmic derivative of the measure $P(\theta,\cdot)$ with respect to the parameter.

When Ξ is a separable Hilbert space, we denote by $\mathcal{K}$ the space of measures, for which the logarithmic derivative with respect to the parameter is represented as the scalar product $l_\theta(x;\vartheta)=\left(\mathscr{k}(x,\theta),\vartheta\right)_\Xi$. Thus we call $\mathscr{k}(x,\theta)$ a vector logarithmic gradient. For Examples 1 and 2 we have $\lambda(x,\theta)=\theta-x$ and $\mathscr{k}(x,\theta)=x-\theta$.

For the family of measures $\left(P(\theta;\cdot),\theta\in\Theta\right)$ with the logarithmic derivative with respect to the parameter along ϑ there exists a measure ν dominating this family. As is known [7], all measures $P(\theta;\cdot)$ are equivalent to one another and $\dfrac{P(\theta_2;dx)}{P(\theta_1;dx)}=\exp\int_{\theta_1}^{\theta_2}l_\theta(x;\vartheta)\,d\theta$.

2. Regularity Conditions

We present the following regularity conditions.

Condition I. For $X(\theta)=X(\theta;\omega):\Theta\times\Omega\to\aleph$ there is a derivative $X'(\theta)$ on θ along $\vartheta\in\Xi_0$, where $\Xi_0\subset\Xi$ is a subspace of Ξ. It is a linear mapping $\Xi\to\aleph$ for every $\theta\in\Theta$. Thus for any $\vartheta\in\Xi_0$ and $\theta\in\Theta$ we have $\left\|X'(\theta)\vartheta\right\|_\aleph\in L_2(\Omega,P)$.

Condition II. $E\left\{X'(\theta)\vartheta\mid X(\theta)=x\right\}$ is a strongly continuous function of x for any $\vartheta\in\Xi_0$, $\theta\in\Theta$.

Condition III. The family of measures $\left(P(\theta;\cdot),\theta\in\Theta\right)$ has a logarithmic derivative with respect to the parameter along a constant direction from a subspace $\Xi_0\subset\Xi$ and

$$\beta_\theta(x;h) \in L_2(\aleph, P(\theta)), \quad \vartheta \in \Xi_0, \quad \theta \in \Theta .$$

Condition IV. The family of measures $(P(\theta;\cdot), \theta \in \Theta)$ has a logarithmic derivative with respect to the parameter along a constant direction of subspace $h \in \aleph_0$ and

$$\beta_\theta(x;h) \in L_2(\aleph, P(\theta)), \; h \in \aleph_0, \theta \in \Theta .$$

Condition V. For the statistic $T = T(x) : \aleph \to R$ the following equality is valid

$$d_\vartheta \int_\aleph T(x) P(\theta; dx) = \int_\aleph T(x) d_\vartheta P(\theta; dx).$$

Lemma. *Under the regularity conditions* I-IV *for the logarithmic derivatives* $\beta_\theta(x;h)$ *and* $l_\theta(x;\vartheta)$ *the following equality is true*

$$l_\theta(x;\vartheta) = -\beta_\theta(x; K_{\theta,\vartheta}(x)), \tag{2}$$

where

$$K_{\theta,\vartheta}(x) = E\left\{\frac{d}{d\theta} X(\theta)\vartheta \mid X(\theta) = x\right\}.$$

Proof. By definition, $P(\theta; A) = P(X^{-1}(\theta; A))$. Let $f(x)$ be a bounded, continuously differentiable along $h \in \aleph$ real function. Using the formula of the change of variables, we obtain

$$\int_\aleph f(x) P(\theta; dx) = Ef(X(\theta)).$$

Consider the derivatives on both sides of θ along ϑ. Then we obtain

$$\int_\aleph f(x) d_\theta P(\theta; dx)\vartheta = E\frac{d}{dx} f(X(\theta)) \frac{d}{d\theta} X(\theta)\vartheta$$

or

$$\int_\aleph f(x) l_\theta(x;\vartheta) P(\theta; dx) = \int_\aleph f'(x) E\{X'(\theta)\vartheta \mid X(\theta) = x\} P(\theta; dx).$$

Denote $K_{\theta,\vartheta}(x) = E\left\{\frac{d}{d\theta} X(\theta)\vartheta \mid X(\theta) = x\right\}$, then we write

$$\int_\aleph f'(x) K_{\theta,\vartheta}(x) P(\theta; dx) = -\int_\aleph f(x) \beta_\theta(x, K_{\theta,\vartheta}(x)) P(\theta; dx).$$

Since $f(x)$ is arbitrary, we obtain (2). ■

3. The Cramer-Rao Inequality

Let $\{\aleph, \Re, (P(\theta, \cdot), \theta \in \Theta)\}$ be the statistical structure corresponding to a random element $X(\omega) = X(\theta, \omega)$. Here $\aleph$ is a separable real reflective Banach space, $\Re$ is a σ-algebra of Borel sets, $\Theta \subset \Xi$ is an open subset of the separable real Banach space Ξ. Assume that regularity conditions I-IV are fulfilled.

Suppose that $g(\theta)=E_\theta\left(T(X)\right)$, where $T:\aleph\to R$ is a measurable mapping (a statistical function). For the statistical function we have one more regularity condition.

Theorem 1 (Cramer-Rao inequality). *Let regularity conditions I-V be fulfilled. Then*

$$Var\,T(X)\ge\frac{\left(g'_\vartheta(\theta)\right)^2}{E_\theta l_\theta^2(X;\vartheta)}. \tag{3}$$

Proof. Consider the derivative of $g(\theta)$ along $\vartheta\in\Xi$. Then

$$d_\vartheta E_\theta T(X)=d_\vartheta\int_\aleph T(x)P(\theta;dx)=\int_\aleph T(x)d_\vartheta P(\theta;dx)\vartheta=$$

$$=\int_\aleph T(x)l_\theta(x,\vartheta)P(\theta;dx)=E_\theta T(X)l_\theta(X;\vartheta).$$

Thus

$$d_\vartheta E_\theta T(X)=E_\theta T(X)l_\theta(X;\vartheta). \tag{4}$$

Put $T(x)=1$ in (4). We obtain $E_\theta l_\theta(X;\vartheta)=0$. Therefore

$$d_\vartheta E_\theta\left(T(X)\right)=E_\theta\left(\left(T(X)-g(\theta)\right)l_\theta(X;\vartheta)\right)$$

and

$$\left(d_\vartheta E_\theta\left(T(X)\right)\right)^2\le E_\theta\left(T(X)-g(\theta)\right)^2\cdot E_\theta l_\theta^2(X;\vartheta).$$

Hence

$$Var\,T(X)\ge\frac{\left(g'_\vartheta(\theta)\right)^2}{E_\theta\beta_\theta^2\left(X;E\left(X'(\theta)\vartheta\mid X\right)\right)}. \blacksquare$$

Therefore

4. Maximum Likelihood Principle

Let $\{\aleph,\Re,(P(\theta),\theta\in\Theta)\}$ be the statistical structure corresponding to a random element $X=X(\theta)=X(\theta,\omega)$, $\omega\in\Omega$, where $\aleph$ is a separable real Banach space, $\Re$ is a σ-algebra of Borel subsets, Θ is an open subset of another separable real Banach space Ξ. Assume that the family of measures $\left(P(\theta,\cdot),\theta\in\Theta\right)$ has a logarithmic derivative $l_\theta(x,\vartheta)$ with respect to the parameter along $\vartheta\in\Xi$. Then, according to Theorem 1, there is a logarithmic derivative with respect to the measure and

$$l_\theta(x,\vartheta)=-\beta_\theta\left(x,E\left\{X'(\theta)\vartheta\mid X(\theta)=x\right\}\right).$$

Consider the structure of the repeated sample

$$\{\aleph,\Re,(P(\theta),\theta\in\Theta)\}^n=\{\aleph^n,\Re^n,(P^n(\theta),\theta\in\Theta)\}.$$

Theorem 2. If the logarithmic derivative $l_\theta(x,\vartheta)$ with respect to the parameter exists in the statistical structure $\{\aleph,\Re,(P(\theta),\theta\in\Theta)\}$, then there also exists the logarithmic derivative $L_\theta((x_1,...,x_n),\vartheta^n)$ with respect to the parameter along $\vartheta^n \overset{def}{=} (\vartheta,...,\vartheta)$ for the structure $\{\aleph,\Re,(P(\theta),\theta\in\Theta)\}^n$ of the repeated sample and

$$L_\theta((x_1,...,x_n),\vartheta^n)=\sum_{k=1}^{n} l_\theta(x_k,\vartheta)=-\sum_{k=1}^{n}\beta_\theta(x_n,E\{X_k'(\theta)\vartheta\mid X_k(\theta)=x_k\}). \qquad (6)$$

Proof. Since there exists $d_\theta^\vartheta P(\theta)$ by condition, it is easy to show that there also exists

$$d_\theta^{\vartheta,...,\vartheta}P^n(\theta)=\sum_{k=1}^{n} d_\theta^\vartheta(x_k,\vartheta)\prod_{\substack{j=1\\ j\neq k}}^{n}P(\theta)$$

which is absolutely continuous with respect to $P^n(\theta)$. The validity of the theorem and formula (6) follow from the Radon-Nikodym theorem. ■

According to Theorem 2, we can formulate the maximum likelihood principle as follows.

Let $X_1,X_2,...,X_n$ be a sample of random variables $X(\theta)$. Here θ is the unknown parameter to be estimated using the sample. Assume that for a distribution $P(\theta)$ of $X(\theta)$, there is a logarithmic derivative $l_\theta(x,\vartheta)$ with respect to the parameter along any vector $\vartheta\in\Xi_0$, and $l_\theta(x,\vartheta)=\langle\lambda(x,\theta),\vartheta\rangle$. Here Ξ_0 is a dense subset of Ξ.

As is known, all measures $P(\theta)$ are equivalent to one another. Let $\theta_0\in\Xi_0$ be a fixed point. Consider the likelihood function $\dfrac{dP(\theta)}{dP(\theta_0)}(x)=\rho(x,\theta)$.

It is easy to see that if $P\in\mathcal{L}$, then

$$\frac{\rho_\theta'(x,\theta)\vartheta}{\rho(x,\theta)}=l_\theta(x,\vartheta).$$

For the sample $X_1,X_2,...,X_n$ the likelihood function is

$$L(X_1,...,X_n,\theta;\vartheta)=\prod_{k=1}^{n}\rho(X_k,\theta).$$

By the likelihood principle, a value $\theta=\hat{\theta}$ for which the likelihood function takes a maximum value (provided that the parameter θ has such a value) is called a maximum likelihood estimate. Since

$$\ln L(X_1,...,X_n,\theta;\vartheta)=\sum_{k=1}^{n}\ln\rho(X_k,\theta),$$

the condition for a maximum makes it possible to formulate this definition in terms of a logarithmic derivative with respect to the parameter.

A solution (if it exists) of the equation

$$\sum_{k=1}^{n} l_\theta\left(x_k, \vartheta\right) = 0, \qquad \forall \vartheta \in \Xi_0 \tag{8}$$

with respect to θ, is called a maximum likelihood estimate if $\frac{d}{d\theta} l\left(x,\theta\right)$ is negatively determined.

Example 3. Let us consider the sample $X_1, X_2, \ldots, X_n$ of a canonical Gaussian variable with an unknown average θ in the equipped Hilbert space $H_+ \subset H \subset H_-$. Then

$$\beta_\theta(x,h) = \left(\theta - x, h\right)_H, \; h \in H_+.$$

It is obvious that $X\left(\theta\right) = N + \theta$, where N is a canonical Gaussian variable with an average, is equal to 0. $X'\left(\theta\right) = I$, $X'\left(\theta\right)h = h$ and therefore

$$E\left\{X'_k\left(\theta\right)h \mid X_k\left(\theta\right) = x\right\} = h.$$

So (9) becomes

$$\sum_{k=1}^{n} \left(\theta - x_k, h\right)_H = 0.$$

From here

$$\left(\hat{\theta}, h\right)_H = \frac{1}{n}\sum_{k=1}^{n}\left(X_k, h\right)_H, \quad \hat{\theta} = \frac{1}{n}\sum_{k=1}^{n} X_k = \overline{X}$$

and

$$\sum_{k=1}^{n} \frac{d^h}{d\theta}\left(x - \theta, h\right)_H = -n\|h\|_H^2 \le 0.$$

As an application we consider a random process $x\left(t\right) = \varphi\left(t\right) + w\left(t\right)$ where $w\left(t\right)$ is a standard Wiener process, $\varphi \in C\left[0,\infty\right) = \Xi$ is an unknown component of the observable process. Clearly, $Ex(t) = \varphi(t)$. In this case $H_+ = C'\left[0,\infty\right)$, $H_- = L_2\left[0,\infty\right)$. If $x_1\left(t\right), x_2\left(t\right), \ldots, x_n\left(t\right)$ are observations, then

$$\hat{\varphi}\left(t\right) = \frac{1}{n}\sum_{k=1}^{n} x_k\left(t\right).$$

If $H = R^n$ is of finite dimension, then we obtain a maximum likelihood estimate along any vector $h = \left(h_1, \ldots, h_n\right)$:

$$\hat{\theta} = \left(\hat{\theta}_1, \ldots, \hat{\theta}_m\right) = \left(\frac{1}{n}\sum_{k=1}^{n} X_k^1, \ldots, \frac{1}{n}\sum_{k=1}^{n} X_k^m\right). \; \blacksquare$$

მათემატიკა

კრამერ-რაოს უტოლობის შესახებ უსასრულო განზომილებიან სივრცეში

ე. ნადარაია*, პ. ბაბილუა**, მ. ფაცაცია§, გ. სოხაძე**

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§ სოხუმის სახელმწიფო უნივერსიტეტი

გლუვი ზომების ტექნიკის გამოყენებით მიღებულია კრამერ-რაოს უტოლობა ბანახის სივრცეში. ჩამოყალიბებულია მაქსიმალური დასაჯერობის პრინციპი. განხილულია მაგალითები.

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Mathematics

The Cauchy-Nicoletti Multipoint Boundary Value Problem for Systems of Linear Generalized Differential Equations with Singularities

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ABSTRACT. The Cauchy-Nicoletti multipoint boundary value problem

$$dx(t) = dA(t)\cdot x(t) + df(t) \quad for \quad t\in[a,b],$$

$$x_i(t_i+) = 0, \quad x_i(t_i-) = 0, \ (i = 1,\dots,n),$$

is considered, where $x_1,\dots,x_n$ **are the components of the desired solution** x, $-\infty < a < t_i \le t_{i+1} < b < \infty$, $f = (f_l)_{l=1}^n : [a,b] \to R^n$ **is a vector-function the components of which are functions with bounded variations, and** $A = (a_{il})_{i,l=1}^n : [a,b] \to R^{n\times n}$ **is a matrix-function such that the functions** $a_{i1},\dots,a_{il}$ **have bounded variations on every interval from [*a*, *b*] which do not include the point** t_i **for every** $i\in\{1,\dots,n\}$.

The sufficient conditions are established for the unique solvability of this problem in the case when the considered system is singular, i. e., the components of the matrix-function *A* do not have bounded variation on the interval [*a*, *b*]. © *2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *systems of linear generalized ordinary differential equations, singularity, the Lebesgue-Stiltjes integral, a multipoint boundary value problem.*

1. Statement of the Problem and Basic Notation

In the paper for the system of linear singular generalized ordinary differential equations

$$dx(t) = dA(t)\cdot x(t) + df(t) \quad for \quad t\in[a,b], \tag{1}$$

we consider the Cauchy-Nicoletti multipoint boundary value problem

$$x_i(t_i+) = 0, \quad x_i(t_i-) = 0, \ (i = 1,\dots,n), \tag{2}$$

where $-\infty < a < t_i \le t_{i+1} < b < \infty$, $x_1,\dots,x_n$ are the components of the desired solution x,

$f=(f_l)_{l=1}^n:[a,b]\to R^n$ is a vector-function the components of which are functions with bounded variations on the interval $[a,b]$, and $A=(a_{il})_{i,l=1}^n:[a,b]\to R^{n\times n}$ is a matrix-function the components $a_{i1},\dots,a_{il}$ of which have bounded variations on every closed interval contained in $[a,t_i[\cup]t_i,b]$ for every $i\in\{1,\dots,n\}$. We have investigated the question of the unique solvability of the problem (1), (2) in the singular case, i.e., in the case when the components of the matrix-functions A may have unbounded variation on the closed interval $[a,b]$.

We give a general theorem for solvability of the problem (1), (2). On the basis of this theorem we have obtained effective criteria for the solvability of this problem.

Analogous and related questions are investigated in [1-7] (see also the references therein) for the singular boundary value problems for linear and nonlinear systems of ordinary differential equations, and in [8-14] for regular and singular multipoint boundary value problems for systems of linear and nonlinear generalized differential equations. As to multipoint singular boundary value problems for generalized differential systems, they have not been sufficiently studied yet, and, despite some results [13, 14], their theory is far from completion even in the linear case. Therefore, the problem considered in the paper is actual.

To a considerable extent, the interest in the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view [8-18] and the references therein).

Throughout the paper the following notation and definitions will be used. $R=]-\infty,\infty[$, $[a,b]$, $]a,b[$ and $[a,b[$, $]a,b]$ $(a,b\in R)$ are, respectively, closed, open and semi-open intervals. $R^{n\times m}$ is the space of all real $n\times m$-matrices $X=(x_{i,l=1})^{n,m}$ with the norm

$$\|X\|=\sum_{i,l=1}^{n,m}|x_{il}|;$$

$R_+^{n\times m}=\{X=(x_{il})_{i,l=1}^{n,m}:x_{il}\ge 0\ (i=1,\dots,n;\ l=1,\dots,m)\}$; $|X|=(|x_{il}|)_{i,l=1}^{n,m}$; $O_{n\times m}$ $(or\ O)$ is the zero $n\times m$ – matrix.

If $X=(x_{il})_{i,l=1}^n\in R^{n\times n}$, then X^{-1}, $\det(X)$ and $r(X)$ are, respectively, the matrix inverse to X the determinant of X and the spectral radius of X; I_n is the identity $n\times n$ – matrix.

$R^n=R^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$; $R_+^n=R_+^{n\times 1}$.

$V_c^d(X)$, where $a<c<d<b$, is the total variation of the matrix-function $X=(x_{il})_{i,l=1}^n:[a,b]\to R^{n\times n}$, i.e., the sum of total variations of the latter's components $x_{il}\ (i=1,\dots,n;\ l=1,\dots,m)$; if $d<c$, then $V_c^d(X)=-V_d^c(X)$; $V(X)(t)=(v(x_{il})(t))_{i,l=1}^{n,m}$, where $v(x_{il})(c_0)=0$, $v(x_{il})(t)=V_{c_0}^t(x_{il})$ for $a<t<b$, $c_0=(a+b)/2$; $X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X:]a,b[\to R^{n\times m}$ at the point $t\in]a,b[$ (we will assume $X(t)=X(a+)$ for $t\le a$ and $X(t)=X(b-)$ for $t\ge b$ if necessary); $d_1X(t)=x(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

$BV([a,b],R^{n\times m})$ is the set of all matrix-functions of bounded variation $X:[a,b]\to R^{n\times m}$ (i.e., such that $V_a^b(X)<\infty$;

$BV_{loc}(]a,b[,R^{n\times m})$ is the set of all matrix-functions $X:]a,b[\to R^{n\times m}$ such that $V_a^b(X)<\infty$ for every $a<c<d<b$;

If I is an arbitrary interval from R and $t_1,\dots,t_n\in I$, then $BV_{loc}(I,t_1,\dots,t_n;R^{n\times m})$ is the set of all matrix-functions $X:I\to R^{n\times m}$ the restrictions of which on every closed interval $[c,d]\subset I\setminus\{t_1,\dots,t_n\}$ belongs to $BV([a,b],R^{n\times m})$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If a function $\alpha\in BV([a,b],R)$ has no more than a finite number of points of discontinuity, and $m\in\{1,2\}$, then by $D_{\alpha m}=\{t_{\alpha m1},\dots,t_{\alpha mn_{\alpha m}}\}(t_{\alpha m1}<\dots<t_{\alpha mn_{\alpha m}})$ we denote the set of all points $t\in[a,b]$ for which $d_m\alpha(t)=0$; moreover, we put $\mu_{\alpha m}=\max\{d_m\alpha(t):t\in D_{\alpha m}\}$.

If $\beta\in BV([a,b],R)$, then

$$\nu_{\alpha m\beta j}=max\left\{d_j\beta(t_{\alpha ml})+\sum_{t_{\alpha ml+1-m}<\tau<t_{\alpha ml+2-m}}d_j\beta(\tau):l=1,\dots,n_{\alpha m}\right\}\ (j,m=1,2),$$

where $t_{\alpha 20}=a-1,\ t_{\alpha 2n_{\alpha 1}+1}=b+1$.

$s_j:BV([a,b],R)\to BV([a,b],R)\ (j=0,1,2)$ are the operators defined, respectively, by

$$s_1(x)(a)=s_2(x)(b)=0,$$

$$s_1(x)(t)=\sum_{a<\tau\le t}d_1x(\tau),\ s_2(x)(t)=\sum_{a\le\tau<t}d_2x(\tau)\quad for\ a<t\le b$$

and

$$s_0(x)(t)=x(t)-s_1(x)(t)-s_2(x)(t)\quad for\ a\le t\le b.$$

If $g:[a,b]\to R$ is a nondecreasing function, and $a\le s\le t\le b$, then

$$\int_s^t x(\tau)dg(\tau)=\int_{]s,t[}x(\tau)ds_0(g)(\tau)+\sum_{s<\tau\le t}x(\tau)d_1g(\tau)+\sum_{s\le\tau\le t}x(\tau)d_2g(\tau),$$

where $\int_{]s,t[}x(\tau)ds_0(g)(\tau)$ is the Lebesgue-Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu(s_0(g))$ corresponding to the function $s_0(g)$; moreover, we assume $\int_t^s x(\tau)dg(\tau)=-\int_s^t x(\tau)dg(\tau)$ and $\int_s^s x(\tau)dg(\tau)=0$;

$L([a,b],R;g)$ is the space of all functions $x:[a,b]\to R$, measurable and integrable with respect to the measure $\mu(g)$ with the norm

$$\|X\|_{L,g}=\int_a^b|x(t)|\,dg(t).$$

If $g(t)\equiv g_1(t)-g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)dg(\tau)=\int_s^t x(\tau)dg_1(\tau)-\int_s^t x(\tau)dg_2(\tau)\quad for\ s\le t.$$

If $G=(g_{ik})_{i,k=1}^{l,n}:[a,b]\to R^{l\times n}$ is a nondecreasing matrix-function and $D\subset R^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X=(x_{kj})_{k,j=1}^{n,m}:[a,b]\to D$ such that $x_{kj}\in L([a,b],R;g_{ik})(i=1,\dots,l;k=1,\dots,n;j=1,\dots,m)$;

$$\int_s^t dG(\tau)\cdot X(\tau)=\left(\sum_{k=1}^{n}\int_s^t x_{kj}(\tau)dg_{ik}(\tau)\right)_{i,j=1}^{l,n}\quad for\ \ a\le s\le t\le b,$$

$$S_j(G)(t)\equiv(s_j(g_{ik})(t))_{i,k=1}^{l,n}\ (j=0,1,2).$$

If $G(t)\equiv G_1(t)-G_2(t)$, where $G_1(t)$ and $G_2(t)$ are nondecreasing matrix-functions, then

$$\int_s^t dG(\tau)\cdot X(\tau)=\int_s^t dG_1(\tau)\cdot X(\tau)-\int_s^t dG_2(\tau)\cdot X(\tau)\ for\ s\le t,$$

$$S_k(G)=S_k(G_1)-S_k(G_2)\ (k=0,1,2),$$

$$L([a,b],D;G)=L([a,b],D;G_1)\cap L([a,b],D;G_2).$$

The inequalities between the matrices are understood component-wise.

A vector-function $x\in BV_{loc}([a,b],t_1,\dots,t_n;R^n)$ is said to be a solution of the system (1) if

$$x(t)=x(s)+\int_s^t dA(\tau)\cdot x(\tau)+f(t)-f(s)\ for\ s<t,\ [s,t]\subset[a,b]\setminus\{t_1,\dots,t_n\}.$$

By a solution of the problem (1), (2) we mean the solution $x=(x_i)_{i=1}^n$ of the system (1) such that the one-sided limits $x_i(t_i-),\ x_i(t_i+)\ (i=1,\dots,n)$ exist and the equalities (2) are valid.

A vector-function $x\in BV_{loc}([a,b],t_1,\dots,t_n;R^n)$ is said to be a solution of the system of generalized differential inequalities $dx(t)\le dB(t)\cdot x(t)+df(t)\ (\ge)\quad for\quad t\in[a,b]$, if

$$x(t)\le x(s)+\int_s^t dB(\tau)\cdot x(\tau)+f(t)-f(s)\ \ for\, s<t,\ [s,t]\subset[a,b]\setminus\{t_1,\dots,t_n\}.$$

Without loss of generality we assume that $A(a)=O_{n\times n},\ f(0)=0_n$. Let, moreover,

$$\det(I_n+(-1)^j dA(t))\ne 0\ \ for\ \ t\in[a,b]\setminus\{t_1,\dots,t_n\}\ (j=1,2).$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding system (1) (see [18, Theorem III.1.4]).

If $s\in]a,b[$ and $\alpha\in BV_{loc}(]a,b[,R)$ are such that

$$1+(-1)^j d_j\alpha(t)\ne 0\ \ for\ t\in]a,b[\ (j=1,2),$$

then by $\gamma_\alpha(.,s)$ we denote the solution of the Cauchy problem $d\gamma(t)=\gamma(t)d\alpha(t),\ \gamma(s)=1$.

It is known (see [15], [16]) that this problem has a unique solution and it is given by

$$\gamma_\alpha(t,s)=\begin{cases}\exp\big(s_0(\alpha)(t)-s_0(\alpha)(s)\big)\prod\limits_{s<\tau\le t}\big(1-d_1\alpha(\tau)\big)^{-1}\prod\limits_{s\le\tau<t}\big(1+d_2\alpha(\tau)\big) & for\ t>s,\\ \exp\big(s_0(\alpha)(t)-s_0(\alpha)(s)\big)\prod\limits_{s<\tau\le t}\big(1-d_1\alpha(\tau)\big)\prod\limits_{s\le\tau<t}\big(1+d_2\alpha(\tau)\big)^{-1} & for t<s,\\ 1 & for\ t=s.\end{cases}$$

Definition 1. We say that a matrix-function $C=\left(c_{il}\right)_{i,l=1}^{n}\in BV([a,b],R^{n\times m})$ belongs to the set $\mathrm{U}([a,b],t_1,\dots,t_n)$ if the functions $c_{il}\ (i\ne l;\ i,l=1,\dots,n)$ are nondecreasing on $[a,b]$ and the system

$$\operatorname{sgn}(t-t_i)\cdot dx_i(t)\le\sum_{l=1}^{n}x_l(t)dc_{il}(t)\ for\ t\in[a,b]\ (i=1,\dots,n)$$

has no nontrivial, nonnegative solution satisfying the condition (2).

A similar definition of set $U([a,b],t_1,\dots,t_n)$ has been introduced by I. Kiguradze for ordinary differential equations (see [4]).

We note that the problem (1),(2) under the condition $a\le t_i\le t_{i+1}\le b\ (i=1,\dots,n)$ is reduced to the case given above. Indeed, if $t_i=a\ (t_i=b)$ for some $i\in\{1,\dots,n\}$, then setting $A(t)\equiv A(a)$ and $f(t)\equiv f(a)$ for $t\le a\ (A(t)\equiv A(b)$ and $f(t)\equiv f(b)$ for $t\ge b)$, we can consider the problem on every interval $[a_0,b_0]$, where $a_0<a<b<b_0$. Moreover, without loss of generality we assume that $a<t_i-1/k<t_i+1/k<b$ for every natural k.

2. Formulation of Main Results

Theorem 1. Let the vector-function $f=(f_l)_{l=1}^n$ belong to $BV([a,b],R^n)$, and the matrix-function $A=(a_{il})_{i,l=1}^n\in BV_{loc}([a,b],t_1,\dots,t_n;R^{n\times n})$ be such that the conditions

$$\left(s_0(a_{ii})(t)-s_0(a_{ii})(s)\right)\operatorname{sgn}(t-t_i)\le s_0(c_{ii}-\alpha_i)(t)-s_0(c_{ii}-\alpha_i)(s)$$
$$for\ a\le s<t<t_i\ or\ t_i<s<t\le b\ (i=1,\dots,n), \quad (3)$$

$$(-1)^j\left(\left|1+(-1)^j d_ja_{ii}(t)\right|-1\right)\operatorname{sgn}(t-t_i)\le d_j\left(c_{ii}(t)-\alpha_i(t)\right)\ for\ t\in[a,t_i[\cup]t_i,b]\,(j=1,2;i=1,\dots,n), \quad (4)$$

$$\left|s_0(a_{il})(t)-s_0(a_{il})(s)\right|\le s_0(c_{il})(t)-s_0(c_{il})(s)\ for\ a\le s<t<t_i\ or\ t_i<s<t\le b\ (i\ne l;i,l=1,\dots,n) \quad (5)$$

and

$$\left|d_ja_{il}(t)\right|\le d_jc_{il}(t)\ for\ t\in[a,t_i[\cup]t_i,b]\ (j=1,2;i\ne l;i,l=1,\dots,n) \quad (6)$$

are fulfilled, where

$$C=(c_{il})_{i,l=1}^n\in U([a,b],t_1,\dots,t_n),\ \ \alpha_i:[a,t_i[\cup]t_i,b]\to R\ (l=1,\dots,n)$$

are functions, nondecreasing on every interval $[a,t_i[$ and $]t_i,b]$, having one-sided limits $\alpha_i(t_i-)$ and $\alpha_i(t_i-)$ and satisfying the conditions

$$\lim_{t\to t_i+}d_1\alpha_i(t)<1\ (i=1,\dots,n_0),\ \ \lim_{t\to t_i-}d_2\alpha_i(t)<1\ (i=n_0+1,\dots,n) \quad (7)$$

and

$$\lim_{t\to t_i+}\sup\left\{\gamma_{\alpha_i}(t,t_i+1/k):k=1,2,\dots\right\}=0\ (i=1,\dots,n),$$
$$\lim_{t\to t_i-}\sup\left\{\gamma_{\alpha_i}(t,t_i-1/k):k=1,2,\dots\right\}=0\ (i=1,\dots,n). \quad (8)$$

Then the problem (1), (2) has one and only one solution.

Corollary 1. Let the vector-function $f=(f_l)_{l=1}^n$ belong to $BV([a,b],R^n)$, and the matrix-function $A=(a_{il})_{i,l=1}^n \in BV_{loc}([a,b],t_1,\dots,t_n;R^{n\times n})$ be such that the conditions

$$(s_0(a_{ii})(t)-s_0(a_{ii})(s))\operatorname{sgn}(t-t_i) \le -(s_0(\alpha_i)(t)-s_0(\alpha_i)(s))+\int_s^t h_{ii}(\tau)ds_0(\beta_i)(\tau)$$
for $a \le s < t < t_i$ or $t_i < s < t \le b$ $(i=1,\dots,n)$,

$$(-1)^j\left(\left|1+(-1)^j d_j a_{ii}(t)\right|-1\right)\operatorname{sgn}(t-t_i) \le h_{ii}(t)d_j\beta_i(t)-d_j\alpha_i(t)$$
for $t\in [a,t_i[\cup]t_i,b]$ $(j=1,2;i=1,\dots,n)$,

$$|s_0(a_{il})(t)-s_0(a_{il})(s)| \le \int_s^t h_{il}(\tau)ds_0(\beta_l)(\tau) \text{ for } a\le s<t<t_i \text{ or } t_i<s<t\le b (i\ne l;i,l=1,\dots,n)$$

$$|d_j a_{il}(t)| \le h_{il}(t)d_j\beta_l(t) \text{ for } t\in [a,t_i[\cup]t_i,b] (j=1,2;i\ne l;i,l=1,\dots,n)$$

are fulfilled, where $\alpha_i:[a,t_i[\cup]t_i,b]\to R$ $(i=1,\dots,n)$ are functions, nondecreasing on every interval $[a,t_i[$ and $]t_i,b]$, having one-sided limits $\alpha_i(t_i-)$ and $\alpha_i(t_i+)$ and satisfying the conditions (3), (4); β_l $(l=1,\dots,n)$ are functions nondecreasing on $[a,b]$ and having not more than a finite numder of points of discontinuity; $h_{ii}\in L^\mu([a,b],R;\beta_i)$, $h_{il}\in L^\mu([a,b],R_+;\beta_l)$ $(i\ne l;i,l=1,\dots,n)$, $1\le\mu\le\infty$. Let, moreover,

$$r(H)<1,$$

where $3n\times 3n$-matrix $H=\left(H_{j+1m+1}\right)_{j,m=0}^2$ is defined by

$$H_{j+1m+1}=\left(\lambda_{kmij}\|h_{ik}\|_{\mu,s_m(\beta_i)}\right)_{i,k=1}^n \quad (j,m=0,1,2),$$

$$\xi_{ij}=\left(s_j(\beta_i)(b)-s_j(\beta_i)(a)\right)^{\frac{1}{\nu}} \quad (j=0,1,2;i=1,\dots,n);$$

$$\lambda_{k0i0}=\begin{cases}\left(\dfrac{4}{\pi^2}\right)^{\frac{1}{\nu}}\xi_{k0}^{\,2} & \text{if } s_0(\beta_i)(t)\equiv s_0(\beta_k)(t),\\ \xi_{k0}\xi_{i0} & \text{if } s_0(\beta_i)(t)\not\equiv s_0(\beta_k)(t)(i,k=1,\dots,n);\end{cases}$$

$$\lambda_{kmij}=\xi_{km}\xi_{ij} \text{ if } m^2+j^2>0, mj=0 \ (j,m=0,1,2;\ k=1,\dots,n),$$

$$\lambda_{kmij}=\left(\frac{1}{4}\mu_{\alpha_k m}\nu_{\alpha_k m\alpha_i j}\sin^{-2}\frac{\pi}{4n_{\alpha_k m}+2}\right)^{\frac{1}{\nu}} \quad (j,m=1,2;k=1,\dots,n),$$

and $\dfrac{1}{\mu}+\dfrac{2}{\nu}=1$. Then the problem (1), (2) has one and only one solution.

Remark 1. In Corollary 1, $3n\times 3n$-matrix H can be replaced by the $n\times n$-matrix

$$\left(\max\left\{\sum_{j=0}^2 \lambda_{kmij}\|h_{ik}\|_{\mu,s_m(\alpha_k)} : m=0,1,2\right\}\right)_{i,k=1}^n.$$

By Remark 1, Corollary 1 has the following form for $h_{il}(t) \equiv h_{il} = \text{const}(i,l=1,\dots,n)$, $\alpha_i(t) \equiv \alpha(t)\,(i=1,\dots,n)$, $\beta_i(t) \equiv \beta(t)\ (i=1,\dots,n)$ and $\mu = \infty$.

Corollary 2. Let the vector-function $f=(f_l)_{l=1}^n$ belong to $BV([a,b],R^n)$, and the matrix-function $A=\left(a_{il}\right)_{i,l=1}^n \in BV_{loc}([a,b],t_1,\dots,t_n;R^{n\times n})$ be such that the conditions

$$\left(s_0\left(a_{ii}\right)(t)-s_0\left(a_{ii}\right)(s)\right)\operatorname{sgn}\left(t-t_i\right) \le h_{ii}\left(s_0(\beta)(t)-s_0(\beta)(s)\right)-\left(s_0(\alpha)(t)-s_0(\alpha)(s)\right)$$
$$\text{for } a \le s < t < t_i \text{ or } t_i < s < t \le b \ (i=1,\dots,n),$$

$$(-1)^j\left(\left|1+(-1)^j d_j a_{ii}(t)\right|-1\right)\operatorname{sgn}\left(t-t_i\right) \le h_{ii} d_j\beta(t)-d_j\alpha(t) \text{ for } t \in [a,t_i[\cup]t_i,b]\,(j=1,2;i=1,\dots,n),$$

$$\left|s_0\left(a_{il}\right)(t)-s_0\left(a_{il}\right)(s)\right| \le h_{il}\left(s_0(\beta)(t)-s_0(\beta)(s)\right) \text{ for } a \le s < t < t_i \text{ or } t_i < s < t \le b\,(i \ne l;i,l=1,\dots,n),$$

$$\left|d_j a_{il}(t)\right| \le h_{il} d_j \beta_i(t) \text{ for } t \in [a,t_i[\cup]t_i,b] \ (j=1,2;i \ne l;i,l=1,\dots,n)$$

are fulfilled, where $\alpha_i:[a,t_i[\cup]t_i,b] \to R\ (i=1,\dots,n)$ are functions, nondecreasing on every interval $[a,t_i[$ and $]t_i,b]$, having one-sided limits $\alpha_i(t_i-)$ and $\alpha_i(t_i+)$ and satisfying the conditions

$$\lim_{t\to t_i+} d_1\alpha(t) < 1 \ (i=1,\dots,n_0),\ \lim_{t\to t_i-} d_2\alpha(t) < 1 \ \ (i=1,\dots,n)$$

and

$$\lim_{t\to t_i+} \sup\left\{\gamma_\alpha(t,t_i+1/k): k=1,2,\dots\right\}=0 \ \ (i=1,\dots,n),$$
$$\lim_{t\to t_i-} \sup\left\{\gamma_\alpha(t,t_i-1/k): k=1,2,\dots\right\}=0 \ (i=1,\dots,n);$$

β is a function nondecreasing on $[a,b]$ and having not more than a finite number of points of discontinuity; $h_{ii} \in R$, $h_{il} \in R\ (i \ne l;\ i,l=1,\dots,n)$. Let, moreover,

$$\rho_0 r(H) < 1,$$

where $H=\left(h_{ik}\right)_{i,k=1}^n$,

$$\rho_0=\max\left\{\left\|\sum_{j=0}^{2}\lambda_{mj}: m=0,1,2\right\|\right\},\ \lambda_{00}=\frac{2}{\pi}\left(s_0(\beta)(b)-s_0(\beta)(a)\right),$$

$$\lambda_{0j}=\lambda_{0j}=\left(s_0(\beta)(b)-s_0(\beta)(a)\right)^{\frac{1}{2}}\cdot\left(s_j(\beta)(b)-s_j(\beta)(a)\right)^{\frac{1}{2}}\ (j=1,2),$$

$$\lambda_{mj}=\left(\frac{1}{4}\mu_{\alpha m}\nu_{\alpha m \alpha j}\sin^{-2}\frac{\pi}{4n_{\alpha m}+2}\right)^{\frac{1}{\nu}}\ \ (m,j=1,2).$$

Then the problem (1), (2) has one and only one solution.

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მათემატიკა

კოში-ნიკოლეტის მრავალწერტილოვანი სასაზღვრო ამოცანა წრფივ განზოგადებულ დიფერენციალურ განტოლებათა სისტემებისათვის სინგულარობებით

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** *სოხუმის სახელმწიფო უნივერსიტეტი, თბილისი*

(წარმოდგენილია აკადემიკოს ი. კიღურაძის მიერ)

განხილულია კოში-ნიკოლეტის მრავალწერტილოვანი სასაზღვრო ამოცანა

$$dx(t) = dA(t)\cdot x(t) + df(t) \quad for \quad t \in [a,b],$$

$$x_i(t_i+) = 0, \quad x_i(t_i-) = 0, \ (i = 1,\dots,n),$$

სადაც $x_1,\dots,x_n$ საძიებელი x ამონახსნის კომპონენტებია, $-\infty < a < t_i \le t_{i+1} < b < \infty$, $f = (f_l)_{l=1}^n : [a,b] \to R^n$ არის ვექტორული ფუნქცია, რომლის კომპონენტები სასრული ვარიაციის მქონე ფუნქციებია, ხოლო მატრიცული ფუნქცია $A = (a_{il})_{i,l=1}^n : [a,b] \to R^{n\times n}$ ისეთია, რომ ყოველი $i \in \{1,\dots,n\}$-თვის $a_{i1},\dots,a_{il}$ ფუნქციებს გააჩნია სასრული ვარიაციები $[a, b]$-ში შემავალ ნებისმიერ შუალედზე, რომელიც არ შეიცავს t_i წერტილს.

დადგენილია აღნიშნული სასაზღვრო ამოცანის ცალსახად ამოხსნადობისათვის საკმარისი პირობები იმ შემთხვევაში, როცა განსახილველი დიფერენციალური სისტემა სინგულარულია, ანუ როცა A მატრიცული ფუნქციის კომპონენტებს არ გააჩნია სასრული ვარიაციები $[a, b]$.

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Mathematics

Partially Independent Random Variables

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ABSTRACT. In this paper the definition of *A*- independence of *X* and *Y* random variables is introduced and the example of *A*-independent random variables is constructed. Regression of *X* on *Y* and regression of *Y* on *X* are investigated. Also the joint characteristic function of this random variables is obtained.

Key words: *random variables, A-independence,regression, characteristic function.*

Introduction. One of the important and fundamental problems of probability theory is independence of random variables. In this paper we introduce the definition of partial independence of two random variables. Using the standard bivariate normal distribution density we construct a nontrivial example of joint probability distribution density for such partially independent random variables.We investigate the properties of this distribution, find the conditional probability distribution density and calculated regressions. Also we give the expression for characteristic function of this joint probability distribution.

1. *A*-independent random variables.

Definition. We say that real random variables X and Y on the probability space (Ω, F, P) are A-independent (A is the subset of R^2) if and only if $F_{XY}(x,y) = F_x(x)F_Y(y)$ for all $(x,y) \in A$, where $F_{XY}(x,y) = P(X \le x, Y \le y)$ is the joint probability distribution function of X and Y, $F_X(x)$ and $F_Y(y)$ are the probability distribution functions of X and Y respectively.

It is clear that independence in the usual sense (see definition, for examples, in [1]) of random variables X and Y coincides with $A = R^2$ -independence.

If there exists the joint probability distribution density $f_{XY}(x,y)$, then we say that X and Y are A-independent if and only if $f_{XY}(x,y) = f_X(x)f_y(y)$ for all $(x,y) \in A$, where $f_X(x)$ and $f_y(y)$ are the probability distribution densities of X and Y respectively.

We begin to construct a special example of A-independent random variables using the joint standard normal distribution density

$$f(x,y)=\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\{-\frac{(x^2+y^2-2xy\rho)}{2(1-\rho^2)}\},/\rho/<1.$$

It is known (see, for example [2]) that $f(x,y)=f(x)f(y/x)=f(y)f(x/y)$,
where $f(x)$ and $f(y)$ are the standard normal distribution and

$$f(x/y)=\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\{-\frac{(x-\rho y)^2}{2(1-\rho^2)}\},\ f(y/x)=\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\{-\frac{(y-\rho x)^2}{2(1-\rho^2)}\}.$$

Let

$$A_{++}=\{(x,y)\in R^2: x\geq 0, y\geq 0\}, A_{--}=\{(x,y)\in R^2: x<0, y<0\},$$
$$A_{+-}=\{(x,y)\in R^2: x\geq 0, y<0\}, A_{-+}=\{(x,y)\in R^2: x<0, y\geq 0\},$$

and

$$g(x,y)=C_+^2 I_{A_{++}}(x,y)f(x,y)+C_-^2 I_{A_{--}}(x,y)f(x,y)+I_{A_{+-}}(x,y)u_+(x)f(x)u_-(y)f(y)+ I_{A_{-+}}(x,y)u_-(x)f(x)u_+(y)f(y), \tag{1}$$

Where $I_A(x,y)$ is the indicator of A and

$$u_+(x)=\alpha^{-\frac{1}{2}}C_+\int_0^\infty f(x/y)dy, \text{ if } x\geq 0 \text{ and } u_+(x)=0, \text{ if } x<0, \tag{2}$$

$$u_-(x)=\alpha^{-\frac{1}{2}}C_-\int_{-\infty}^0 f(x/y)dy, \text{ if } x<0 \text{ and } u_+(x)=0, \text{ if } x\geq 0. \tag{3}$$

Here

$$\alpha=\int_0^\infty\int_0^\infty f(x,y)dxdy=\int_{-\infty}^0\int_{-\infty}^0 f(x,y)dxdy \tag{4}$$

(the values of $\alpha=\alpha(\rho)$ can be found from tables of standard bivariate normal distribution, see, for example, [3]-[5]);

C_+ and C_- are some constants.

Denote

$$A_+=\int_0^\infty u_+(x)f(x)dx, A_-=\int_{-\infty}^0 u_-(x)f(x)dx. \tag{5}$$

After substitution $u_+(x)$ and $u_-(x)$ from (2) and (3) in (5) we obtain

$$A_+=C_+\alpha^{\frac{1}{2}} \text{ and } A_-=C_-\alpha^{\frac{1}{2}}. \tag{6}$$

Let us choose C_+ and C_- in such a way that $C_+>0,\ C_->0$ and

$$C_+ + C_- = \alpha^{-\frac{1}{2}}$$

(for example, $C_+ = C_- = \frac{1}{2\alpha^{\frac{1}{2}}}$ or $C_+ = \frac{1}{3\alpha^{\frac{1}{2}}}$, $C_- = \frac{2}{3\alpha^{\frac{1}{2}}}$). Then from (6):

$$A_+ + A_- = 1 \tag{7}$$

and if

$$g(x) = u(x) f(x), \tag{8}$$

where

$$u(x) = \begin{cases} u_+(x) & if \quad x \geq 0, \\ u_-(x) & if \quad x < 0, \end{cases} \tag{9}$$

we have

$$\int_{-\infty}^{\infty} g(x)dx = \int_{-\infty}^{\infty} u(x)f(x)dx = \int_{-\infty}^{0} u(x)f(x)dx + \int_{0}^{\infty} u(x)f(x)dx = \int_{-\infty}^{0} u_-(x)f(x)dx + \int_{0}^{\infty} u_+(x)f(x)dx = A_- + A_+ = 1.$$

Therefore $g(x)$ is a probability distribution density.

Now we can verify that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)dxdy = 1,$$

where $g(x,y)$ is defined by (1).

Really, from (1)-(4)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)dxdy = (C_+^2 + C_-^2)\alpha + 2\alpha C_+ C_- = \alpha(C_+^2 + C_-^2 + 2C_+C_-) = \alpha(C_+ + C_-)^2 = 1.$$

We show that marginal distribution densities of $g(x,y)$ are $g(x) = u(x)f(x)$ and $g(y) = u(y)f(y)$:

$$\int_{-\infty}^{\infty} g(x,y)dy = I_{[0,\infty)}(x)\int_{-\infty}^{\infty} g(x,y)dy + I_{(-\infty,0)}(x)\int_{-\infty}^{\infty} g(x,y)dy =$$

$$I_{[0,\infty)}(x)[C_+^2 f(x)\int_{0}^{\infty} f(y/x)dy + u_+(x)f(x)A_-] + I_{(-\infty,0)}(x)[C_-^2 f(x)\int_{-\infty}^{0} f(y/x)dy + u_-(x)f(x)A_+] =$$

$$I_{[0,\infty)}(x)[C_+\alpha^{\frac{1}{2}} u_+(x)f(x) + u_+(x)f(x)A_-] + I_{(-\infty,0)}(x)[C_-\alpha^{\frac{1}{2}} u_-(x)f(x) + u_-(x)f(x)A_+] =$$

$$I_{[0,\infty)}(x)[u_+(x)f(x)(C_+\alpha^{\frac{1}{2}} + A_-)] + I_{(-\infty,0)}(x)[u_-(x)f(x)(C_-\alpha^{\frac{1}{2}} + A_+)] =$$

$$I_{[0,\infty)}(x)u_+(x)f(x) + I_{(-\infty,0)}(x)u_-(x)f(x) = g(x).$$

Similarly

$$\int_{-\infty}^{\infty} g(x,y)dx = g(y).$$

From (1)-(3), (8), (9) it is clear that $g(x,y) = g(x)g(y)$ on $A_{+-} \bigcup A_{-+}$ and really

$$g(x,y) = C_+^2 I_{A_{++}}(x,y) f(x,y) + C_-^2 I_{A_{--}}(x,y) f(x,y) + I_{A_{+-}}(x,y) g(x) g(y) + I_{A_{-+}}(x,y) g(x) g(y),$$

Thus we have proved the folloving

Theorem. *The real function* $g(x,y)$ *defined on* R^2 *by (1) is the probability distribution density with marginal distribution densities* $g(x)$ *and* $g(y)$ *of same form, defined from (8). The random variables X and Y with this joint distribution density* $f_{XY}(x,y) = g(x,y)$ *are* $A_{+-} \bigcup A_{-+}$ *– independent.*

3. Regression.

Suppose the random variables X and Y are $A_{+-} \bigcup A_{-+}$ – independent and have the joint probability distribution density $f_{XY}(x,y) = g(x,y)$, where $g(x,y)$ is defined by (1). It is not difficult to obtain the conditional density

$$f_X(x / Y = y) = \frac{g(x,y)}{g(y)} = C_+^2 I_{A_{++}}(x,y) \frac{f(x,y)}{u_+(y) f(y)} + C_-^2 I_{A_{--}}(x,y) \frac{f(x,y)}{u_-(y) f(y)} + I_{A_{+-}}(x,y) u_+(x) f(x) +$$

$$I_{A_{-+}}(x,y) u_-(x) f(x) = C_+^2 I_{A_{++}}(x,y) \frac{f(x/y)}{u_+(y)} + C_-^2 I_{A_{--}}(x,y) \frac{f(x,y)}{u_-(y)} + I_{A_{+-}}(x,y) u_+(x) f(x) +$$

$$I_{A_{-+}}(x,y) u_-(x) f(x) = I_{A_{++}}(x,y) C_+ \alpha^{\frac{1}{2}} \frac{f(x/y)}{\int_0^{\infty} f(u/y) du} + I_{A_{--}}(x,y) C_- \alpha^{\frac{1}{2}} \frac{f(x/y)}{\int_{-\infty}^{0} f(u/y) du} +$$

$$I_{A_{+-}}(x,y) C_+ \alpha^{-\frac{1}{2}} f(x) \int_0^{\infty} f(u/x) du + I_{A_{-+}}(x,y) C_- \alpha^{-\frac{1}{2}} f(x) \int_{-\infty}^{0} f(u/x) du.$$

So

$$f_X(x / Y = y) = I_{A_{++}}(x,y) C_+ \alpha^{\frac{1}{2}} \frac{f(x/y)}{\int_0^{\infty} f(u/y) du} + I_{A_{--}}(x,y) C_- \alpha^{\frac{1}{2}} \frac{f(x/y)}{\int_{-\infty}^{0} f(u/y) du} +$$

$$I_{A_{+-}}(x,y) C_+ \alpha^{-\frac{1}{2}} f(x) \int_0^{\infty} f(u/x) du + I_{A_{-+}}(x,y) C_- \alpha^{-\frac{1}{2}} f(x) \int_{-\infty}^{0} f(u/x) du \qquad (10)$$

and

$$f_Y(y / X = x) = I_{A_{++}}(x,y) C_+ \alpha^{\frac{1}{2}} \frac{f(y/x)}{\int_0^{\infty} f(u/x) du} + I_{A_{--}}(x,y) C_- \alpha^{\frac{1}{2}} \frac{f(y/x)}{\int_{-\infty}^{0} f(u/x) du} +$$

$$I_{A_{+-}}(x,y) C_+ \alpha^{-\frac{1}{2}} f(y) \int_0^{\infty} f(u/y) du + I_{A_{-+}}(x,y) C_- \alpha^{-\frac{1}{2}} f(y) \int_{-\infty}^{0} f(u/y) du. \qquad (11)$$

From these expressions we see that on the set $A_{+-}\bigcup A_{-+}$ the conditional distribution density $f_X(x/Y=y)$ is the function only of x and $f_Y(y/X=x)$ is the function only of y. It is natural because x and y are independent on this set .

Using (10) and (11) we find regression of X on Y and Y on X.

Regression of X on Y:

$$E(X/Y=y)=\int_{-\infty}^{\infty} xf_X(x/Y=y)dx = I_{[0,\infty)}(y)C_+\alpha^{\frac{1}{2}}\frac{\int_0^\infty xf(x/y)dx}{\int_0^\infty f(u/y)du}+$$

$$I_{(-\infty,0)}(y)C_-\alpha^{\frac{1}{2}}\frac{\int_{-\infty}^0 xf(x/y)dx}{\int_{-\infty}^0 f(u/y)du}+I_{[0,\infty)}(y)C_+\alpha^{-\frac{1}{2}}\int_0^\infty xf(x)(\int_0^\infty f(u/x)du)dx+$$

$$I_{(-\infty,0)}(y)C_-\alpha^{-\frac{1}{2}}\int_{-\infty}^0 xf(x)(\int_{-\infty}^0 f(u/x)du)dx. \qquad (12)$$

Note that here

$$\int_0^\infty f(u/y)du=\frac{1}{\sqrt{2\pi(1-\rho^2)}}\int_0^\infty e^{-\frac{(u-\rho y)^2}{2(1-\rho^2)}}du=\frac{1}{\sqrt{2\pi}}\int_{-\frac{\rho y}{\sqrt{1-\rho^2}}}^{\infty} e^{-\frac{u^2}{2}}du=1-\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}}),$$

$$\int_{-\infty}^0 f(u/y)du=\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}}),$$

$$\int_0^\infty uf(u/y)du=-(1-\rho^2)\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{\rho^2y^2}{2(1-\rho^2)}}+\rho y\left(1-\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}})\right), \qquad (13)$$

$$\int_{-\infty}^0 uf(u/y)du=(1-\rho^2)\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{\rho^2y^2}{2(1-\rho^2)}}+\rho y\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}}),$$

Denote

$$K_+=C_+\alpha^{-\frac{1}{2}}\int_0^\infty xf(x)(\int_0^\infty f(u/x)du)dx,$$

$$K_-=C_-\alpha^{-\frac{1}{2}}\int_{-\infty}^0 xf(x)(\int_{-\infty}^0 f(u/x)du)dx \qquad (14)$$

It is clear that

$$K_+ = K_+(\rho) = \int_0^\infty xg(x)dx, K_- = K_-(\rho) = \int_{-\infty}^0 xg(x)dx,$$

and

$$K_+ + K_- = \int_{-\infty}^\infty xg(x)dx = EX.$$

Using (13) and (14) from (12) we obtain:

$$E(X/Y=y) = I_{[0,\infty)}(y)C_+\alpha^{\frac{1}{2}}[(\rho^2-1))\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{\rho^2y^2}{2(1-\rho^2)}} + \rho y\left(1-\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}})\right)][1-\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}})]^{-1} + I_{(-\infty,0)}(y)C_-\alpha^{\frac{1}{2}}[(1-\rho^2))\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{\rho^2y^2}{2(1-\rho^2)}} + \rho y\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}})][\Phi(\frac{-\rho y}{\sqrt{1-\rho^2}})]^{-1} + I_{(-\infty,0)}(y)K_+ + I_{[0,\infty)}(y)K_-. \quad (15)$$

Similarly

$$E(Y/X=x) = I_{[0,\infty)}(x)C_+\alpha^{\frac{1}{2}}[(\rho^2-1))\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{\rho^2x^2}{2(1-\rho^2)}} + \rho y\left(1-\Phi(\frac{-\rho x}{\sqrt{1-\rho^2}})\right)][1-\Phi(\frac{-\rho x}{\sqrt{1-\rho^2}})]^{-1} + I_{(-\infty,0)}(x)C_-\alpha^{\frac{1}{2}}[(1-\rho^2))\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{\rho^2x^2}{2(1-\rho^2)}} + \rho y\Phi(\frac{-\rho x}{\sqrt{1-\rho^2}})][\Phi(\frac{-\rho x}{\sqrt{1-\rho^2}})]^{-1} + I_{(-\infty,0)}K_+ + I_{[0,\infty)}(x)K_-. \quad (16)$$

Representations (15) and (16) show that regression of X on Y and regression of Y on X are not linear.

Remark. Denote

$$f_+(x/y) = \begin{cases} \dfrac{f(x/y)}{\int_0^\infty f(u/y)du}, & x \geq 0, \\ 0, & x < 0 \end{cases} \quad (17)$$

and

$$f_-(x/y) = \begin{cases} \dfrac{f(x/y)}{\int_{-\infty}^0 f(u/y)du}, & x < 0, \\ 0, & x \geq 0. \end{cases} \quad (18)$$

Then we can rewrite (10) in the following form

$$f_X(x/Y=y)=I_{A_{++}}(x,y)C_+\alpha^{\frac{1}{2}}f_+(x/y)+I_{A_{--}}(x,y)C_-\alpha^{\frac{1}{2}}f_-(x/y)+$$
$$I_{A_{+-}}(x,y)C_+\alpha^{-\frac{1}{2}}f(x)\int_0^{\infty}f(u/x)du+I_{A_{-+}}(x,y)C_-\alpha^{-\frac{1}{2}}f(x)\int_{-\infty}^{0}f(u/x)du. \quad (19)$$

Note that $f_+(x/y)$ and $f_-(x/y)$ defined by (17) and (18) are conditional densities.

3. Joint characteristic function.

Let

$C_+=C_-=\dfrac{1}{2\alpha^{\frac{1}{2}}}$, then from (1):

$$g^*(x,y)=\frac{1}{4\alpha}I_{A_{++}}(x,y)f(x,y)+\frac{1}{4\alpha}I_{A_{--}}(x,y)f(x,y)+I_{A_{+-}}(x,y)g^*(x)g^*(y)+I_{A_{-+}}(x,y)g^*(x)g^*(y), \quad (20)$$

where $g^*(x)$ and $g^*(y)$ are defined by (8), (9), (2) and (3) with $C_+=C_-=\dfrac{1}{2\alpha^{\frac{1}{2}}}$.

Denote

$$\beta=\int_0^{\infty}g^*(x)dx,\gamma=\int_{-\infty}^{0}g^*(x)dx \quad (\beta+\gamma=1). \quad (21)$$

And rewrite (20) in the following form

$$g^*(x,y)=I_{A_{++}}(x,y)\frac{1}{4}\frac{f(x,y)}{\alpha}+I_{A_{--}}(x,y)\frac{1}{4}\frac{f(x,y)}{\alpha}+I_{A_{+-}}(x,y)\beta\gamma\frac{g^*(x)}{\beta}\frac{g^*(y)}{\gamma}+$$
$$I_{A_{-+}}(x,y)\beta\gamma\frac{g^*(x)}{\gamma}\frac{g^*(y)}{\beta}. \quad (22)$$

Note that here

$$\alpha=\int_0^{\infty}\int_0^{\infty}f(x,y)dxdy=\int_{-\infty}^{0}\int_{-\infty}^{0}f(x,y)dxdy.$$

Let the random variables U and V defined on (Ω,F,P) with values in $A_{++}\bigcup A_{--}$ have the joint probability distribution density

$$f_{UV}(x,y)=\frac{f(x,y)}{2\alpha},\quad (x,y)\subset A_{++}\bigcup A_{--},$$
$$f_{UV}(x,y)=0,\qquad (x,y)\subset A_{+-}\bigcup A_{-+}. \quad (23)$$

Assume that the random variables ξ and η with values in $[0,\infty)$ and in $(-\infty,0)$, respectively, have the probability distribution densities :

$$f_\xi(x)=\frac{g^*(x)}{\beta}, x\ge 0,$$
$$f_\xi(x)=0, x<0 \tag{24}$$

and

$$f_\eta(x)=\frac{g^*(x)}{\gamma}, \quad x<0,$$
$$f_\eta(x)=0, \quad x\ge 0. \tag{25}$$

Then, it is clear that the joint characteristic function corresponding to $g^*(x,y)$ has the form

$$\varphi(z_1,z_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(z_1x+z_2y)}g^*(x,y)dxdy=\frac{1}{2}\varphi_{UV}(z_1,z_2)+\beta\gamma[\varphi_\xi(z_1)\varphi_\eta(z_2)+\varphi_\xi(z_2)\varphi_\eta(z_1)], \tag{26}$$

where $\varphi_{UV}(z_1,z_2)$ is the characteristic function corresponding to joint density (23), $\varphi_\xi(z)$ and $\varphi_\eta(z)$ are the characteristic functions corresponding to densities (24) and (25) respectively.

Remark 2. If in (26) $z_1=0$ and $z_2=0$, we obtain $\beta\gamma=\frac{1}{4}$. Really in this case, when $C_+=C_-=\frac{1}{2\alpha^{\frac{1}{2}}}$ we have:

$$\beta=\int_0^\infty g^*(x)dx=\int_0^\infty u^*_+(x)f(x)dx=\frac{1}{2\alpha}\int_0^\infty(\int_0^\infty f(y/x))dy)f(x)dx=\frac{1}{2\alpha}\int_0^\infty\int_0^\infty f(x,y)dxdy=\frac{1}{2}$$

and

$$\gamma=\int_{-\infty}^0 g^*(x)dx=\int_{-\infty}^0 u^*_-(x)f(x)dx=\frac{1}{2\alpha}\int_{-\infty}^0(\int_{-\infty}^0 f(y/x))dy)f(x)dx=\frac{1}{2\alpha}\int_{-\infty}^0\int_{-\infty}^0 f(x,y)dxdy=\frac{1}{2}.$$

Therefore $\beta=\gamma=\frac{1}{2}$ and $\beta\gamma=\frac{1}{4}$.

მათემატიკა

ნაწილობრივ დამოუკიდებული შემთხვევითი სიდიდეები

ო. ღლონტი

ი. ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტი

(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

სტატიაში განსაზღვრულია X და Y შემთხვევითი სიდიდეების A-დამოუკიდებლობის ცნება და A-დამოუკიდებელი შემთხვევითი სიდიდეების მაგალითი არის აგებული. განხილულია რეგრესია X-სა Y-ზე და Y-სა X-ზე. ნაპოვნია ასეთი შემთხვევითი სიდიდეების მახასიათებელი ფუნქციის სახე.

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Mathematics

On Some Approximation Properties of a Generalized Fejér Integral

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ABSTRACT. Problems of approximation in spaces of p-integrable for some $p \geq 1$, as well as essentially bounded functions defined on a locally compact Abelian group are considered. Analogs of Fejér well-known positive operators are taken as approximate aggregates. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *locally compact Abelian group, positive operator, Fejér integral, approximation.*

In papers [1-5] problems of approximative nature are considered for some spaces of real or complex valued functions and also measures defined on a locally compact Abelian group. Let G be a locally compact Abelian Hausdorf group and $\hat{G}$ be the dual group, i.e. the set of all characters on G. $\hat{G}$ is also a locally compact Abelian group in the topology of uniform convergence of characters on compact subsets of G. $U_{\hat{G}}$ will stand for the collection of all symmetric compact sets from $\hat{G}$ which are closures of neighborhoods of the unity in $\hat{G}$. $KT = \{g : g = g_1 g_2, \quad g_1 \in K, \quad g_2 \in T\}$ will stand for the product of the sets K and T, while $(1)_K$ will denote the characteristic function of the set K. By $L^p(G) \equiv L^p(G,\mu), \quad 1 \leq p < \infty$, is denoted the space of p-th power integrable functions on G with respect to the Haar measure μ. $L^\infty(G) \equiv L^\infty(G,\mu)$ denotes the space of essentially bounded on G functions with respect to μ. For arbitrary K and T in $\hat{G}$ we consider the following functions defined on G ([6], Ch.5, §1)

$$V_{K,T}(g) = (\text{mes}T)^{-1}(\hat{1})_T(g)(\hat{1})_{KT}(g). \tag{1}$$

Here and in the sequel by $\hat{f}$ (resp. $\tilde{f}$) is denoted the Fourier (resp. the inverse) transform of $L^p(G,\mu), \quad 1 \leq p \leq 2$. Usually, the Haar measures on G and $\hat{G}$ are normalized so that the inversion formula $f = (\hat{f})^{\sim}$ holds for functions $f \in L^1(G), \quad \hat{f} \in L^1(\hat{G})$.

In [1] is introduced the set $W^p(K)$, $K \in U_{\hat{G}}$ of continuous functions $f \in L^p(G)$, such that

$$f(g) = (f * V_{K,T})(g) \equiv \int_G f(h) \ V_{K,T}(h^{-1}g)d\mu(h) \equiv \int_G f(h) \ V_{K,T}(h^{-1}g)dh,$$

for all $g \in G$, $T \in \hat{G}$.

If $G = \mathbb{R}^m$, $m \geq 1$, and K is a symmetric body of $\mathbb{R}^m$, the class $W^p(K)$, $1 \leq p \leq \infty$, coincides with the well-known class of entire functions of exponential type K whose traces on $\mathbb{R}^m$ belong to the space $L^p(\mathbb{R}^m)$ [7, 8]. For the case of compact G the functions from $W^p(K)$, $1 \leq p \leq \infty$, are finite linear combinations of characters of G.

In papers [1, 2] it is proved that $W^p(K)$, $1 \leq p \leq \infty$, is the shift invariant closed subspaces of $L^p(G)$. In the case when $1 \leq p \leq 2$, the set $W^p(K)$ coincides with the set $F^p(K)$, which consists of continuous functions on G, whose Fourier transform supports belong to K. Moreover, the set of functions $W^p(K)$ for all possible compact $K \in U_{\hat{G}}$ is dense in $L^p(G)$, $1 \leq p < \infty$. In connection with the density problem we remark the following.

Let K be a symmetric compact set from $\hat{G}$. In [4] we have introduced a definition of B-property, which is an analogy of the notion of convexity for locally compact Abelian groups. According to this definition, a set K possesses the B-property if the element $g \in \hat{G}$, which admits for a certain natural number n the representation $g^n = g_1^{n_1} \cdots g_k^{n_k}$, where $n_1, \cdots, n_k$ are natural numbers, $n = n_1 + \cdots + n_k$, while $g_1, \cdots, g_k \in K$, belongs to K. If T is a set from $U_{\hat{G}}$, without the B-property, it can be put in a $K \in U_{\hat{G}}$ which does have the B-property. To construct such a minimal set K, we must consider all elements $g \in \hat{G}$ which can be represented in a form $g^n = g_1^{n_1} \cdots g_k^{n_k}$, where $g_1, \cdots, g_k \in T$, $n_1, \cdots, n_k \in N$, $n = n_1 + \cdots + n_k$ and connect them to T such g, if $g \notin T$. If we connect to T all such g, then the obtained set K possesses the required property. Really, let us consider elements $g_1, \cdots, g_k \in K$ and suppose that $g^n = g_1^{n_1} \cdots g_k^{n_k}$ for some $g \in G$, $g_1, \cdots, g_k \in K$, $n_1, \cdots, n_k \in N$, $n = n_1 + \cdots + n_k$. It suffices to consider the case k=2. Let $g_1^{n_1} = g_{11}^{n_{11}} \cdots g_{1h}^{n_{1h}}$, $g_2^{n_2} = g_{21}^{n_{21}} \cdots g_{2l}^{n_{2l}}$, $n_1 = n_{11} + \cdots + n_{1h}$, $n_2 = n_{21} + \cdots + n_{2l}$, $g_{1i} \in T$, $i = 1, \cdots, h$, $g_{2j} \in T$, $j = 1, \cdots, l$. Let n be the least common multiple of the numbers n_1 and n_2, and $n = n_1 m_1 = n_2 m_2$. Let $g^s = g_1^{s_1} g_2^{s_2}$, $s = s_1 + s_2$ $(s_1, s_2 \in N)$. Let us prove that $g \in K$. We have $g^{ns} = (g^s)^n = g_1^{ns_1} g_2^{ns_2} =$ $= g_{11}^{n_{11} m_1 s_1} \cdots g_{1h}^{n_{1h} m_1 s_1} g_{21}^{n_{21} m_2 s_2} \cdots g_{2l}^{n_{2l} m_2 s_2}$. Here $n_{11} m_1 s_1 + \cdots + n_{1h} m_1 s_{12} + n_{21} m_2 s_2 + \cdots + n_{2l} m_2 s_2 = ns$, $g_{11}, \cdots, g_{2l} \in T$ and by construction of K, we have that $g \in K$. Thus K has the B-property. Inclusion $K \in U_{\hat{G}}$ is clear. Thus every $T \in U_{\hat{G}}$ may be put in some $K \in U_{\hat{G}}$, which has the B-property. It is shown in [1], that $W^p(T) \subset W^p(K)$, $1 \leq p \leq \infty$, for every $T \subset K$, and therefore we can take the sets $W^p(K)$ as a dense set in $L^p(G)$ for all $K \in U_{\hat{G}}$, having the B-property too.

Let I be an ordered unbounded set $I \subset \mathbb{R}^+$ and consider a generalized sequence of sets $K \in U_{\hat{G}}$, such that $K_\alpha \subset K_\beta$ if $\alpha < \beta$ $(\alpha,\ \beta \in I)$ and $\bigcup_{\alpha \in I} K_\alpha = \hat{G}$. For a such sequence $L^p(R^m)$, $1 \le p \le \infty$ we have studied in [5] the following sequence of positive operators

$$\sigma_{K_\alpha}(f)(g) \equiv (f * V_{K_\alpha})(g) = \int_G f(h)\ V_{K_\alpha}(h^{-1}g)dh, \tag{2}$$

where

$$V_{K_\alpha}(g) = (\text{mes}K_\alpha)^{-1}((\hat{1})_{K_\alpha}(g))^2, \qquad K_\alpha \in U_{\hat{G}}. \tag{3}$$

The kernels (3) are mentioned in [6], (Ch.5, §1) and they represent the limiting case of the kernel $V_{K,T}$ defined by (1), when K converges to the unity of $\hat{G}$ and T is replaced by K_α. In [5] it is proved that if $f \in L^p(G)$, $1 \le p \le \infty$ then $\sigma_{K_\alpha}(f) \in W^p(K_\alpha^2)$.

Some results concerning positive linear operators and the approximation of continuous functions on locally compact Abelian groups are given in [9]. First of all the positive operators (2) are of importance as in the case $G = \mathbb{R}^m$ they coincide with the well-known Fejér operator. For example, if $m = 1$, $I = (0, \infty)$, $\alpha \in I$ and $K_\alpha = [-\alpha, \alpha]$, we have the sequence of operators (see, for example, [10], 3.1.2, p. 122)

$$\sigma_{[-\alpha,\alpha]}(f)(x) \equiv \sigma_\alpha(f)(x) = \frac{2}{\pi\tau} \int_{-\infty}^{\infty} f(t)(x-t)^{-2} \sin^2 \frac{\tau}{2}(x-t)dt, \qquad \tau \in \mathbb{R}^+.$$

In the case when $G = \mathbb{R}^m$, $m \ge 1$ and K is the ball of a radius $\tau \in \mathbb{R}^+$, the kernel of the operator (2) takes the form

$$V_K(x) = (\omega_m)^{-1} |x|^{-m} J_{m/2}^2(\tau |x|/2), \qquad x \in \mathbb{R}^m,$$

where $J_{m/2}$ is the Bessel function of order $m/2$ and ω_m is the area of the m-dimensional unit sphere. In that case, the operator (2) is studied in [11] in connection with the saturation problem in $L^p(R^m)$, $1 \le p \le \infty$. In [5] it is proved that if $f \in L^p(G)$, $1 \le p < \infty$, and the sequence of sets $K_\alpha \in U_{\hat{G}}$ satisfies the condition

$$\lim_{\alpha \to \infty} \frac{\text{mes}\ \ (TK_\alpha)}{\text{mes}\ \ K_\alpha} = 1 \tag{4}$$

for all fixed $T \in U_{\hat{G}}$, then

$$\lim_{\alpha \to \infty} \| f - \sigma_{K_\alpha}(f) \|_{L^p(G)} = 0.$$

Under the condition (4) the sequence (2) converges to f also in the space $L^\infty(G)$, but in the weak* topology of $L^\infty(G)$. An analogous result is valid also in the space $M(G)$ of bounded regular complex valued Borel measures on G [5]. In addition to these results we state the following

Theorem 1. *Let $\{K_\alpha\}$ be a sequence in $U_{\hat{G}}$ satisfying (4) and $S \subset G$ is a compact set. If a function $f \in L^\infty(G)$ is continuous at a neighborhood of S, then $\sigma_{K_\alpha}(f)$ converges to f uniformly on S as $\alpha \to \infty$.*

Proof. Since f is uniformly continuous on S, given $\delta > 0$ find a neighborhood $V \in U_{\hat{G}}$ of the unity of

$\hat{G}$ such that

$$|f(hg)-f(g)|<\delta/2 \qquad \text{for all} \quad g\in T, \quad h\in V. \tag{5}$$

In [5] (Proposition 4) it is proved that under assumption (4) the sequence of kernels $\{V_{K_\alpha}\}$ represents an approximate unity. This means that for every fixed $U\in U_{\hat{G}}$ the following equalities hold

$$\lim_{\alpha\to\infty}\int_{G\setminus V} V_{K_\alpha}(g)dg=0, \qquad \int_G V_{K_\alpha}(g)dg=1.$$

Therefore there exists a $\alpha_0>0$ such that

$$\int_{G\setminus V} V_{K_\alpha}(g)dg < \frac{\delta}{4\|f\|_{L^\infty(G)}}, \quad \text{for all } \alpha>\alpha_0. \tag{6}$$

By use of the equality $\int_G V_{K_\alpha}(g)dg=1$ we have

Applying (5) and (6), and representing the integral of the right-hand as $\int_V + \int_{G\setminus V}$, we obtain

$$\sup_{g\in T}|f(g)-\sigma_{K_\alpha}(f)(g)|\le \delta/2+\delta/2=\delta, \qquad \text{when} \quad \alpha\to\infty.$$

Now we prove that the rate of convergence of the operators σ_{K_α} to the identical operator in general is not greater than $\text{mes}K_\alpha$.

Theorem 2 *. Let $f\in L^p(G)$, $1\le p\le 2$, and a sequence of sets $K_\alpha\in U_{\hat{G}}$ satisfy the condition*

$$\limsup_{\alpha\in I}\{\text{mes}\,K_\alpha\setminus \text{mes}\,(K_\alpha\cap(\chi K_\alpha))\}\ne 0 \tag{7}$$

for any fixed $\chi\in\hat{G}$. Then from the condition

$$\|f-\sigma_{K_\alpha}(f)\|_{L^p(G)}=o(\text{mes}\,K_\alpha), \qquad \alpha\to\infty, \tag{8}$$

it follows that $f(g)=0$ a.e. on G.

Proof. Since $f\in L^p(G)$, $1\le p\le 2$, and $V_{K_\alpha}\in L^1(G)$ for all $\alpha\in I$, we have for the Fourier transform of (2) ([12], Ch.8)

$$(\sigma_{K_\alpha}(f))^\wedge(\chi)=\hat{f}(\chi)(V_{K_\alpha})^\wedge(\chi), \tag{9}$$

and

$$(V_{K_\alpha})^\wedge(\chi)=(\text{mes}\,K_\alpha)^{-1}(((1)_{K_\alpha}*(1)_{K_\alpha})^\wedge)^\wedge(\chi).$$

It is clear that $(1)_{K_\alpha}*(1)_{K_\alpha}\in L^1(\hat{G})$ and $((1)_{K_\alpha}*(1)_{K_\alpha})^\wedge\in L^1(G)$. Therefore we have from (9)

$$(\sigma_{K_\alpha}(f))^\wedge(\chi)=\hat{f}(\chi)(\text{mes}\,K_\alpha)^{-1}((1)_{K_\alpha}*(1)_{K_\alpha})(\chi^{-1})$$

It follows from here

$$(\text{mes } K_\alpha)(f-\sigma_{K_\alpha}(f))^\wedge(\chi)=\hat{f}(\chi)(\text{mes } K_\alpha-\int_{K_\alpha}(1)_{K_\alpha}(h^{-1}\chi^{-1})dh). \quad (10)$$

The upper bound of the L^q norm $(p^{-1}+q^{-1}=1)$ on the left-hand is, according to the Hausdorf inequality ([12], §31.22), $(\text{mes } K_\alpha)\| f-\sigma_{K_\alpha}\|_{L^p(G)}$. Then it follows from (8) and (10)

$$\lim_{\alpha\to\infty}\|\hat{f}(\chi)(\text{mes } K_\alpha-\int_{K_\alpha}(1)_{K_\alpha}(h^{-1}\chi^{-1})dh)\|_{L^q(G)}=0.$$

According to (3), this means that $\|\hat{f}\|_{L^q(G)}=0$, i.e. $f(g)=0$ a.e. on G ([12], §31.31).

Now we give examples of operators σ_{K_α} for some groups G and sets K_α:

1. Let $G=\mathbb{R}^m$, $m\ge 1$ and consider the sets $K_\alpha=\{u:\ u\in\mathbb{R}^m,\ d(u)\le\alpha\}$, $\alpha>0$, where $d^2(u)=(u,Au)=$ $=\sum_{k,l=1}^{m}a_{k,l}u_k u_l$ and $A=(a_{k,l})_{k,l=1}^{m}$ is a positive definite matrix. It is clear that $K_\alpha\subset U_{\hat{G}}=U_{R^m}$. It is possible to calculate $(\hat{1})_{K_\alpha}$ by analogy to [13] (we must keep in mind that according to our agreement the Haar measure μ in $\mathbb{R}^m$ is normalized in such a way that $\mu(E)=(2\pi)^{m/2}l(E)$, $E\subset\mathbb{R}^m$, where $l(E)$ is the Lebesgue measure of E). Actually, we can obtain the following representation of σ_{K_α}

$$\sigma_{K_\alpha}(F)(x)=\frac{2^m\pi^{m/2}\Gamma(m/2+1)}{\sqrt{\det\Lambda}}\int_{\mathbb{R}^m}f(t+x)\frac{J^2_{m/2}((\alpha/2)\sqrt{(x,A^{-1}x)})}{(x,A^{-1}x)^{m/2}}dx, \quad (11)$$

where $J_{m/2}$ is the Bessel function of order $m/2$ and Γ is the Euler gamma function.

Let us consider the matrix B, whose columns are orthonormal eigenvectors of A. Let B' be the matrix conjugate to B and Λ be the diagonal matrix composed with the eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_m$ of A. It is well-known that $\Lambda=B'AB$, $\det A=\det\Lambda$, $\det B=1$. The change of variables $u=Bv$, $v_k=\lambda_k^{-1/2}z_k$ gives

$$(u,Au)=(Bv,ABv)=(v,B'ABv)=(v,\Lambda v)=|z|^2=z_1^2+z_2^2+\cdots+z_m^2.$$

Thus $d(u)\le\alpha\ \Leftrightarrow\ |z|\le\alpha$. This linear correspondence between K_α and $C_\alpha:=\{z\in R^m:\ |z|\le\alpha\}$ is $z=(B\Lambda)^{-1}u$. By this map the image of $C_\alpha+\mu$ for any $\mu\in K_\alpha$ is translation of C_α by the vector $\mu'=(B\Lambda)^{-1}u$. It is clear that $\text{mes } K_\alpha\setminus\text{mes }(K_\alpha+\mu)=\text{mes } C_\alpha\setminus\text{mes }(C_\alpha+\mu')$. It may be proved by calculation that $\text{mes } C_\alpha\setminus\text{mes }(C_\alpha+\mu')\ge C(\mu')\alpha^{m-1}$, where $C(\mu')$ depends only on μ' a positive number. Thus, in the considered example condition (7) is satisfied. Therefore Theorem 2 implies that in this example the approximation by operators (11) of order α of nontrivial functions from $L^p(\mathbb{R}^m)$, $1\le p\le 2$, is impossible.

2. If $G=\mathbb{Z}$ is the group of whole numbers, then $\hat{G}=E$ is the unit circumference from $\mathbb{R}^2$ up to an isomorphism. Characters of G have the form $\chi_t(n)=t^n$, $n\in Z$, $t\in E$. Let $K_\alpha=\{e^{i\theta}:-\pi+\frac{\pi}{\alpha}\le\theta\le\pi+\frac{\pi}{\alpha},\ \ 1<\alpha<\infty\}$. Take, as a dual measure on E, the arc length divided into $i\sqrt{2\pi}$. We have

$$(\hat{1})_{K_\alpha}(n)=\frac{1}{i\sqrt{2\pi}}\int_{K_\alpha}\xi^n d\xi=\begin{cases}\frac{\sqrt{2}}{\sqrt{\pi}\ (n+1)}\sin\frac{(\alpha-1)(n+1)\pi}{\alpha}, & n\neq -1,\\ \sqrt{2\pi}\ \frac{\alpha-1}{\alpha}, & n=-1,\end{cases}$$

and therefore, in this case

$$\sigma_{K_\alpha}(f)(n)=\frac{\alpha}{(\alpha-1)\pi^2}\sum_{k=-\infty,\ k\neq n+1}^{\infty} f(k)\frac{\sin^2(\alpha-1)(n-k+1)\pi/\alpha}{(n-k+1)^2}+\frac{\alpha}{\pi}f(n+1).$$

3. Let $G=\mathbb{R}^+$ be the multiplicative group of positive integers with the unit e=1. This group has a character $\chi(\xi)=\xi^{ix}$, where $x\in\mathbb{R}$ and $\hat{G}=\mathbb{R}$ up to an isomorphism. Take as K_α the interval $K_\alpha=[-\alpha,\ \alpha]$, $\alpha\in\mathbb{R}^+$. Then

$$(\hat{1})_{K_\alpha}(\xi)=\int_{-\alpha}^{\alpha}\xi^{ix}dx=\frac{2\sin(\alpha\ln\xi)}{\ln\xi}$$

And, therefore, in this case

$$\sigma_{K_\alpha}(f)(x)=\frac{1}{\alpha\pi}\int_0^\infty f(h)\frac{\sin^2(\alpha\ln\frac{x}{h})}{h\ln^2\frac{x}{h}}dh.$$

4. Let $G=\mathbb{Q}_p$ be the field of the p-adic numbers with a prime p. With respect to the additional operation of p-adic numbers, $\mathbb{Q}_p$ is a locally compact Abelian group [14]. Its dual group is isomorphic to the addition group $\mathbb{Q}_p$. The character χ_ξ, corresponding to a p-adic number ξ, has the form $\chi_\xi(x)=\exp(2\pi i\{\xi x\}_p)$. Here $\{x\}_p$ is defined by the p-adic expansion $x=\sum_{n\geq \operatorname{ord}_p(x)}a_n p^n$ as

$$\{x\}_p=\begin{cases}\sum_{n=\operatorname{ord}_p(x)}^{-1}a_n p^n, & \text{if}\quad \operatorname{ord}_p(x)<0,\\ 0, & \text{if}\quad \operatorname{ord}_p(x)\geq 0,\end{cases}$$

and ξx is the product of p-adic numbers ξ and x in the field $\mathbb{Q}_p$.

Let $n\in\mathbb{Z}$, $\alpha=p^n$ and K_α be the p-adic ball of the radius p^n with the center in zero, i.e. $K_\alpha=\{h\in\mathbb{Q}_p : |h|_p\leq p^n\}$, where $|h|_p$ is the p-adic norm of h. The Haar measure of this ball is mes $K_\alpha=p^n$ ([14], §4, (2.3)). By means of the formula ([14], §4, (3.1))

$$\int_{K_\alpha}\chi_\xi(x)dx=\begin{cases}p^n, & \text{if}\quad |\xi|_p\leq p^{-n},\\ 0, & \text{if}\quad |\xi|_p>p^{-n+1},\end{cases}$$

we obtain that in the considered case the operator σ_{K_α} has the form $\sigma_{K_\alpha}(x)=$

$$=p^{-n}\int_{\mathbb{Q}_p}f(\xi+x)d\xi\left(\int_{|t|_p\leq p^n}e^{2\pi i\{t\xi\}_p}dt\right)^2=p^n\int_{|\xi|_p\leq p^{-n}}f(\xi+x)d\xi\ .$$

მათემატიკა

ფეიერის განზოგადებული ინტეგრალის აპროქსიმაციული თვისებების შესახებ

დ. უგულავა

საქართველოს ტექნიკური უნივერსიტეტი, ნიკო მუსხელიშვილის გამოთვლითი მათემატიკის ინსტიტუტი, თბილისი

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შესწავლილია ლოკალურად კომპაქტურ აბელის ჯგუფზე ჰაარის ზომით რომელიღაც $p \geq 1$-სათვის p-ინტეგრებად და აგრეთვე არსებითად შემოსაზღვრულ ფუნქციათა აპროქსიმაციის საკითხები. მააპროქსიმირებელ აგრეგატებად აღებულია ფეიერის ცნობილი დადებითი ოპერატორების ანალოგები.

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Mathematics

On Reconstruction of Coefficients of a Multiple Trigonometric Series with Lebesgue Nonintegrable Sum

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(Presented by Academy Member Vakhtang Kokilashvili)

ABSTRACT. Cantor's functionals sequence notion for one-dimensional trigonometric series is introduced. Also, the possibility of reconstruction of coefficients of multiple trigonometric series with Lebesgue nonintegrable sum by iterated use of Cantor's functionals is established.

Key words: *uniqueness of trigonometric series, Cantor's and Valle-Poussin's theorems, Denjoy integral, T-integrals, Cantor's functionals sequence, multiple trigonometric series.*

We denote the trigonometric system defined on $[0,1]$ by $T^1 = \{t_i(\tau)\}_{i=0}^{\infty}$, where $t_0(\tau) \equiv 1$, $t_{2i-1}(\tau) = \sqrt{2}\cos 2\pi i\tau$ and $t_{2i}(\tau) = \sqrt{2}\sin 2\pi i\tau$, $i = 1,2,\ldots$.

Let consider a trigonometric series

$$\sum_{i=0}^{\infty} a_i t_i(\tau). \tag{1}$$

Partial sums of the series (1) will be denoted by

$$S_m(\tau) = \sum_{i=0}^{2m} a_i t_i(\tau).$$

Definition 1. A set $A \subset [0,1]$ belongs to the class $U(T^1)$ if the only series (1) that converges to zero on A is the series all of whose coeffitients are zero.

Definition 2. We say that a function $f(\tau)$ belongs to the class $J(A, T^1)$ if $A \in U(T^1)$ and there exists a series (1) such that equality

$$\sum_{i=0}^{\infty} a_i t_i(\tau) = f(\tau) \qquad (2)$$

holds true for any $\tau \in A$.

Note that if $f \in J(A,T^1)$, then the uniqueness of coefficients of series (2) follows from the definition of $U(T^1)$.

Definition 3. We say that a sequence of functionals $\{G_i^A(f(\tau))\}_{i=0}^{\infty}$ is Cantor's functionals sequence if for any function $f \in J(A,T^1)$ and for every $i = 0,1,2,\ldots$ the equality

$$a_i = G_i^A(f(\tau))$$

holds.

The following theorem, proved by Cantor [1] in 1872 is a fundamental result in the uniqueness theory for trigonometric series.

Theorem A. *If the series (1) is everywhere convergent to zero, then all coefficients of the series are zero.*

This theorem was generalized in various direction. In particular, in 1912 Valle-Poussin [3] proved the following statement.

Theorem B. *If the series* (1) *converges to a finite integrable in the Lebesgue sens, function f everywhere possibly exept at countably many points, then it is the Fourier-Lebesgue series of f.*

The author [2] proved some results concerning the uniqueness of certain multiple function series for Pringsheim convergence. In particular, these results imply the Theorems A and B remain valid for multiple trigonometric series.

It is well-known that there exsits such trigonometric series, that their sums are not integrable functions in the Lebesgue sens. An example of such series is given by the series

$$\sum_{n=2}^{\infty} \frac{\sin 2\pi n\tau}{\ln n}.$$

In the other hand, the uniqueness of coefficients of everywhere convergent trigonometric series follows from Cantor's theorem. This circumstance caused the necessity of such generalization of Lebesgue integral notion, that any everywhere convergent trigonometric series would be Fourier series in the generalized integral sens. This problem was solved by Denjoy. It is known other generalizations of Lebesgue integral notion, which also solve the problem. Examples of such generalization are: MZ Marcinkiewicz-Zygmund integral, P^2 James integral, SCP Burkill integral and other, so called T-integrals (see [4]). So the Fourier formulas for coefficients of everywhere convergent trigonometric series

$$a_i = \int_0^1 f(\tau) t_i(\tau)\, d\tau, \qquad i = 0,1,2,\ldots$$

in every above mentioned generalized integral sens give examples of Cantor's functionals sequences when A=[0,1].

Let $d \ge 2$ be an integer, R^d Euclidean space of dimension d, Z_0^d the set of all points with nonnegative integer coordinates in R^d. We denote points of the set Z_0^d by $m = (m_1,\ldots,m_d)$ and $n = (n_1,\ldots,n_d)$. We use

$x=(x_1,\dots,x_d)$ to denote points of the unit cube $[0,1]^d$.

We consider d-multiple trigonometric series

$$\sum_{n_1=0}^{\infty}\cdots\sum_{n_d=0}^{\infty} a_{n_1,\dots,n_d}\prod_{j=1}^{d} t_{n_j}(x_j). \tag{3}$$

Rectangular partial sums of the series (3) we denote by $S_m(x)$. i. e.

$$S_m(x)=\sum_{n_1=0}^{2m_1}\cdots\sum_{n_d=0}^{2m_d} a_{n_1,\dots,n_d}\prod_{j=1}^{d} t_{n_j}(x_j).$$

The convergence of the series(3) will be understood as Pringsheim convergence.

The Cartesian product for every $2\le p\le d$ and any set A will be denoted by

$$\underbrace{A\times\cdots\times A}_{p}=A^p.$$

Let consider a function $F(x_1,\dots,x_p)$, where $(x_1,\dots,x_p)\in A^p$.

We introduce a sumbol $G_i^{A,p}\left(F(x_1,\dots,x_p)\right)$ which generalize G_i^A for multivariable situations. Namely, for every $2\le p\le d$, the symbol $G_i^{A,p}\left(F(x_1,\dots,x_{p-1},x_p)\right)$ means that G_i^A acts on a function $F(x_1,\dots,x_{p-1},x_p)$, where only $x_p\in A$ is an independent variable and $(x_1,\dots,x_{p-1})$ is a fixed point of the set A^{p-1}. Also, $F(x_1,\dots,x_{p-1},x_p)\in J(A,T^1)$ for any $(x_1,\dots,x_{p-1})\in A^{p-1}$. If $p=1$, then $G_i^{A,1}\left(F(x_1)\right)=G_i^A\left(F(x_1)\right)$.

We established that it is possible to calculate coefficients of convergent series (3) by iterated using of Cantor's functionals. Namely, the following holds true.

Theorem. *Let* $A\in U(T^1)$ *and* $\left\{G_i^A\left(f(\tau)\right)\right\}_{i=0}^{\infty}$ *is Cantor's functionals sequence and for any* $(x_1,\dots,x_d)\in A^d$ *the series* (3) *converges to the function* $F(x_1,\dots,x_d)$, *then for every* $n=(n_1,\dots,n_d)\in Z_0^d$ *the equality*

$$a_{n_1,\dots,n_d}=G_{n_1}^{A,1}\left(G_{n_2}^{A,2}\left(\cdots\left(G_{n_d}^{A,d}\left(F(x_1,\dots,x_d)\right)\right)\cdots\right)\right)$$

holds.

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მათემატიკა

ლებეგის აზრით არაინტეგრებადი ფუნქციებისაკენ კრებადი ჯერადი ტრიგონომეტრიული მწკრივების კოეფიციენტთა აღდგენის შესახებ

შ. ტეტუნაშვილი

ი. ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტის ა. რაზმაძის მათემატიკის ინსტიტუტი; საქართველოს ტექნიკური უნივერსიტეტი, თბილისი

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ნაშრომში შემოტანილია კანტორის ფუნქციონალთა მიმდევრობის ცნება და მოყვანილია თეორემა, რომლის თანახმადაც კრებადი ჯერადი ტრიგონომეტრიული მწკრივების კოეფიციენტების გამოთვლა ხდება კანტორის ფუნქციონალთა განმეორებითი გამოყენებით. შევნიშნავთ, რომ კანტორის ფუნქციონალთა მიმდევრობის მაგალითებს წარმოადგენს დანჟუას, მარცინკევიჩ-ზიგმუნდის, ჯეიმსის, ბერკილის და სხვა ინტეგრალების საშუალებით ერთმაგი ტრიგონომეტრიული მწკრივების კოეფიციენტთა გამოსათვლელი ფურიეს ფორმულები.

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Mathematics

On the $\mathcal{D}$-equivalence Class of a Graph

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ABSTRACT. Let G be a simple graph of order n. The domination polynomial of G is the polynomial $D(G,x)=\sum_{i=\gamma(G)}^{n} d(G,i)x^i$, where $d(G, i)$ is the number of dominating sets of G of size i, and $\gamma(G)$ is the domination number of G. A dominating set with cardinality $\gamma(G)$ is called a γ-set. Two graphs G and H are said to be $\mathcal{D}$-equivalent, written $G \sim H$, if $D(G, x) = D(H, x)$. The $\mathcal{D}$-equivalence class of G is $[G]=\{H : H \sim G\}$. A graph G is said to be $\mathcal{D}$-unique, if $[G] = \{G\}$. In this paper we study the $\mathcal{D}$-equivalence of some graphs. Also, we obtain some properties of graphs with unique γ-set. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *domination polynomial, equivalence.*

1. Introduction

Let $G=(V,E)$ be a simple graph. The order of G is the number of vertices of G. For any vertex $v\in V$, the open neighborhood of v is the set $N(v)=\{u\in V \mid uv\in E\}$ and the closed neighborhood is the set $N[v]=N(v)\cup\{v\}$. For a set $S\subseteq V$, the open neighborhood is $N(S)=\bigcup_{v\in S} N(v)$ and the closed neighborhood is $N[S]=N(S)\cup S$. A set $S\subseteq V$ is a dominating set if $N[S]=V$, or equivalently, every vertex in $V\setminus S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set with cardinality $\gamma(G)$ is called a γ-set. For a detailed treatment of this parameter, the reader is referred to [2]. Let $\mathcal{D}(G, i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G, i)=|\mathcal{D}(G, i)|$. The domination polynomial $D(G, x)$ of G is defined as $D(G, x) =\sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$, [1].

The following theorem follows from the definitions of isomorphic graphs and domination polynomial.

Theorem 1. *If G and H are isomorphic, then $D(G, x) = D(H, x)$.*

The converse of the above theorem is not true. There are numerous graphs with the same domination polynomials. Two graphs G and H are said to be *dominating equivalence*, or simply $\mathcal{D}$-equivalent, written G

$\sim H$, if $D(G,x) = D(H,x)$. It is evident that the relation $\sim$ of being $\mathcal{D}$-equivalence is an equivalence relation on the family $\mathcal{D}$ of graphs, and thus G is partitioned into equivalence classes, called the *$\mathcal{D}$-equivalence classes*. Given $G \in \mathcal{G}$, let $[G] = \{H \in \mathcal{G} : H \sim G\}$. We call $[G]$ the equivalence class determined by G. A graph G is said to be *dominating unique*, or simply $\mathcal{D}$-unique, if $[G] = \{G\}$. Determining $\mathcal{D}$-equivalence class of graphs is one of the interesting problems on equivalence classes. Fig. 1 shows all connected graphs of order 5 with the same domination polynomials. Note that for disconnected graphs, we can use the following theorem:

Theorem 2. [1] *If a graph G has m components $G_1,\ldots,G_m$, then $D(G,x) = D(G_1,x) \ldots D(G_m,x)$.*

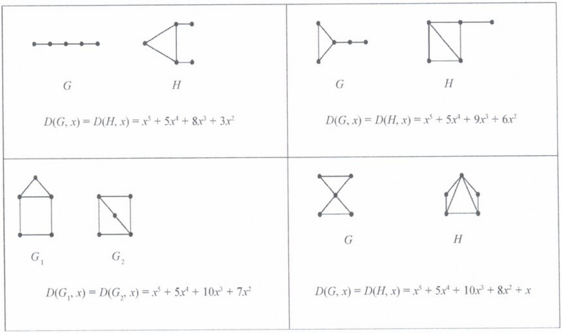

Fig. 1. Graphs of order 5 with identical domination polynomials.

The *join* of two graphs G_1 and G_2, denoted by $G_1 + G_2$ is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } V(G_2)\}$. As usual we denote the complete bipartite graph by $K_{m,n}$, the complete graph, path and cycle of order n by K_n, P_n and C_n, respectively. Also $K_{1,n}$ is the star graph with $n+1$ vertices.

In Section 2 we study the $\mathcal{D}$-equivalence classes of some specific graphs. In Section 3 we study some properties of graphs with unique γ-set.

2. $\mathcal{D}$-Equivalence classes of some graphs

In this section, we study the $\mathcal{D}$-equivalence class of K_n and $K_{1,n}$. First, we recall the following theorem which gives a formula for the computation of the domination polynomial of join of two graphs.

Theorem 3. [1] *Let G_1 and G_2 be graphs of orders n_1 and n_2, respectively. Then*

$$D(G_1 + G_2, x) = \left((1+x)^{n_1} - 1\right)\left((1+x)^{n_2} - 1\right) + D(G_1,x) + D(G_2,x).$$

Theorem 4. *Assume that G is a graph of order n and $v \in V(G)$. If $deg(v) = n-1$, then G is $\mathcal{D}$-unique, if and only if $G \backslash \{v\}$ is $\mathcal{D}$-unique. Hence K_n and $K_{1,n}$ are $\mathcal{D}$-unique for every natural number n.*

Proof. By Theorem 3, $D(G,x) = x\left((1+x)^{n-1}-1\right)+x+D(G\setminus\{v\},x)$. Thus G is $\mathcal{D}$-unique if and only if $G\setminus\{v\}$ is $\mathcal{D}$-unique.

Remark. The $\mathcal{D}$-equivalence class of $K_{n,n}$ is not unique. We can see this as follows. Consider two disjoint copies of complete graph K_n with vertex sets $\{v_1,\ldots,v_n\}$ and $\{v_1',\ldots,v_n'\}$, respectively. Join v_i to v_i' for every i, $1\le i\le n$. It can be easily seen that the domination polynomial of this graph is the same as the domination polynomial of $K_{n,n}$.

3. Some graphs with unique γ-set

Lemma 1. *Let G be a graph and u be a pendant vertex of G. Suppose that* $uv\in E(G)$, $deg(v)=2$ and $N(v) = \{u,w\}$. *If G has a unique* γ*-set, then* $G\setminus\{u,v,w\}$ *has a unique* γ*-set.*

Proof. Assume that $\gamma(G)=t$, and $D\subseteq V(G)$ is a unique γ-set of G with size t. Since u is a pendant vertex, either one of the vertices u or v should be contained in D. Obviously, $v\in D$, for otherwise $(D\setminus\{u\})\cup\{v\}$ is another γ-set for G, a contradiction. Clearly, $w\notin D$, for otherwise $(D\setminus\{v\})\cup\{u\}$ is another γ-set for G, a contradiction. Hence $|D\cap(G\setminus\{u,v,w\})|=t-1$. We claim that $\gamma(G\setminus\{u,v,w\})=t-1$. If $\gamma(G\setminus\{u,v,w\})<t-1$, then by adding v to a γ-set for $G\setminus\{u,v,w\}$ we obtain a γ-set for G with size at most $t-1$, a contradiction. Thus $\gamma(G\setminus\{u,v,w\})=t-1$. For every γ-set S of $G\setminus\{u,v,w\}$, $S\cup\{v\}$ is a γ-set of G. This implies that $G\setminus\{u,v,w\}$ has a unique γ-set and the proof is complete.

We have the following corollary for path P_n by Lemma 1:

Corollary 1. *If* $n\equiv 0 (\mathrm{mod}\ 3)$, *then* P_n *has a unique* γ*-set.*

By Lemma 1, we have the following corollaries for the graphs $C_n(m)$ and T_{n_1,n_2,n_3} (see Fig. 2):

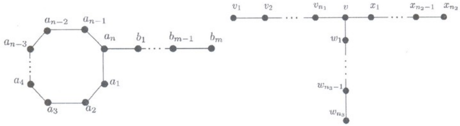

Fig. 2. The graphs $C_n(m)$ and T_{n_1,n_2,n_3}, respectively.

Corollary 2. *Suppose that m and n are natural numbers. The graph* $C_n(m)$ *has a unique* γ*-set, if and only if one of the following holds*:

(*i*) $n\equiv 0(\mathrm{mod}\ 3)$ *and* $n\equiv 1(\mathrm{mod}\ 3)$,

(*ii*) $n\equiv 1(\mathrm{mod}\ 3)$ *and* $n\equiv 2(\mathrm{mod}\ 3)$.

Corollary 3. *Suppose that* n_1; n_2 *and* n_3 *are natural numbers. If* $n_1\equiv n_2\equiv n_3\equiv 1(mod\ 3)$, *then the tree* T_{n_1,n_2,n_3} *has a unique* γ*-set.*

Lemma 2. *Let G be a graph,* $u,v\in V(G)$ *and* $deg(u)=deg(v)=1$. *If* $uw, vw'\in E(G)$ *and* $ww'\notin E(G)$, *then* $D(G,x)=D(G+ww',x)$, *and hence G is not* $\mathcal{D}$*-unique.*

Proof. Clearly, every dominating set for G is a dominating set for $G+ww'$. Now, let $S\subseteq V(G)$ be a dominating set for $G+ww'$. If both $w, w' \in S$ or both $w, w' \notin S$, then obviously S is also a dominating set for G. So suppose that $w \in S$ and $w' \notin S$ (or $w \notin S$ and $w' \in S$). Since S is a dominating set for $G+ww'$, we have v 2 S. This implies that S is a dominating set for G. Therefore we conclude that $D(G,x)=D(G+ww',x)$ and the proof is complete.

The following theorem is an immediate conclusion of Lemma 2:

Theorem 5. *For every $n \geq 5$, P_n is not $\mathcal{D}$-unique, and $[P_n]$ contains at least two following graphs:*

Fig. 3. Two graphs of $[P_n]$.

მათემატიკა

გრაფის $\mathcal{D}$-ეკვივალენტურობის კლასების შესახებ

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ვთქვათ G n-რიგის მარტივი გრაფია. G-ს დომინატური მრავალწევრი ეწოდება $D(G,x)=\sum_{i=\gamma(G)}^{n} d(G,i)x^i$ გამოსახულებას, სადაც $d(G,i)$ არის G-ს i სიგრძის დომინატური სიმრავლეების რიცხვი, ხოლო $\gamma(G)$ კი G-ს დომინირების რიცხვი. $\gamma(G)$-ს ტოლი დომინირების რიცხვის მქონე დომინატურ სიმრავლეს ეწოდება γ-სიმრავლე. ორ G და H გრაფს ეწოდება $\mathcal{D}$-ეკვივალენტური და იწერება $G \sim H$, თუ $D(G,x)=D(H,x)$. G-ს $\mathcal{D}$-ეკვივალენტურობის კლასია $[G]=\{H : H \sim G\}$. G გრაფს ეწოდება $\mathcal{D}$-ერთადერთი, თუ $[G]=\{G\}$. ნაშრომში შესწავლილია გარკვეული ტიპის გრაფების $\mathcal{D}$-ეკვივალენტურობა. აგრეთვე, მიღებულია ერთადერთი γ-სიმრავლის მქონე გრაფების ზოგიერთი თვისება.

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Mechanics

Characteristics of the Solution of the Consistently Linearized Eigenproblem for Lateral Torsional Buckling

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ABSTRACT. The consistently linearized eigenproblem has proved to be a powerful mathematical tool for classification of buckling, based on the percentage bending energy of the total strain energy. Of particular interest are prebuckling states with a constant percentage strain energy. The two limiting cases of such states are membrane stress states and states of pure bending. Buckling at pure bending, referred to as lateral torsional buckling, is the topic of this work. The transfer matrix method is used to derive a secant stiffness matrix in analytical form. Formulation of the consistently linearized eigenproblem by means of this matrix yields the same solution as would be obtained by a formulation based on the tangent stiffness matrix which is an essential ingredient of nonlinear Finite Element Analysis. This remarkable finding permits analytical verification of hypothesized subsidiary conditions for lateral torsional buckling. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *consistently linearized eigenproblem, lateral torsional buckling, transfer matrix method.*

1. Introduction

Recently, Mang et al. [1] reported on Finite Element Analysis (FEA) of elastic structures at special prebuckling states which were defined as states with a constant percentage bending energy of the strain energy in the prebuckling regime. This regime is characterized by a proportional increase of the reference load. One of the two limiting cases of such states is a membrane stress state. This case was treated in detail in a paper by Mang and Höfinger [2].

The present work deals with the second limiting case, which is buckling at pure bending, referred to as lateral torsional buckling. The purpose of the paper is to prove subsidiary conditions for this special case, in the context of the Finite Element Method (FEM), hypothesized in [1] on the basis of the consistently linearized eigenproblem which was introduced by Helnwein [3]. In order to free the proof from discretization errors, typical for results obtained by the Finite Element Method (FEM), these conditions are verified through numerical evaluation of an analytical solution obtained by means of the transfer matrix method [4].

The paper is organized as follows: In Chapter 2, the differential equation for the rotation of the cross-section of a beam subjected to pure, skew bending about the axis of the beam is presented. Chapter 3 is devoted to the analytical solution of this differential equation, followed by the formation of the transfer matrix. The mathematical expression for the rotation depends on a dimensionless load factor by which the reference bending moment is multiplied, and on the axial coordinate. Buckling at pure bending requires a constant reference bending moment. In Chapter 4, the secant stiffness matrix is derived analytically with the help of the transfer matrix method. In Chapter 5, the formulation of the consistently linearized eigenproblem on the basis of the secant stiffness matrix is shown to give the same result as would be obtained by means of the tangent stiffness matrix. Chapter 6 contains the numerical investigation. Conclusions from this work are drawn in Chapter 7.

2. Differential Equation

Fig. 1 shows a fork-supported beam of length l with a constant, doubly symmetric cross-section. At its ends, the beam is subjected to bending moments

$$M_{y,b} = M_{y,a} = M_y = \lambda \bar{M}_y, \tag{1}$$

$$M_{z,b} = M_{z,a} = M_z = \lambda \bar{M}_z, \tag{2}$$

where $\bar{M}_y$ and $\bar{M}_z$ are reference quantities and λ is a dimensionless load factor. Hence, the prebuckling state is one of pure bending.

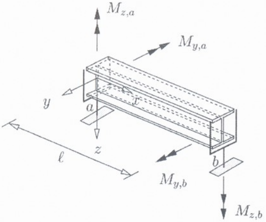

Fig. 1. Fork-supported beam subjected to pure, skew bending.

The differential equation for the rotation ϑ of the cross-section of the beam about the x-axis (Fig. 2) follows from [5], considering (1) and (2), as

$$EI_\omega \vartheta'''' - GI_T \vartheta'' - \frac{1}{EI_y}\left(\frac{I_y}{I_z}\bar{M}_y^2 + \bar{M}_z^2\right)\lambda^2\vartheta = \frac{\bar{M}_y \bar{M}_z}{EI_y}\left(1 - \frac{I_y}{I_z}\right)\lambda^2, \tag{3}$$

where I_y and I_z are the principal moments of inertia, I_T is St. Venant's torsion constant, I_ω is the warping constant, E is Young's modulus, and G is the shear modulus.

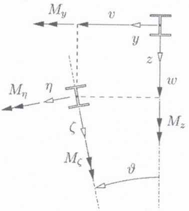

Fig. 2. Cross-section of the beam in the undeformed and the deformed position of the structure.

Eq. (3) is based on the assumption of small prebuckling rotations for which

$$\sin\vartheta \approx \vartheta, \quad \cos\vartheta \approx 1, \tag{4}$$

resulting in

$$M_\eta = M_y + M_z\vartheta, \quad M_\zeta = M_z - M_y\vartheta. \tag{5}$$

If $\bar{M}_y = 0$ or $\bar{M}_z = 0$, or $I_z = I_y$, (3) becomes a homogeneous differential equation, representing an eigenproblem. The smallest eigenvalue, $\lambda = \lambda_S$, defines the buckling moment. The corresponding eigenfunction, $\hat{\vartheta}(x)$, permits determination of the buckling mode. If $\bar{M}_y \neq 0$ and $\bar{M}_z \neq 0$, and $I_z \neq I_y$, $\vartheta(x,\lambda)$, $0 < x < l$, tends to infinity as λ approaches λ_S, which is in contradiction to (4). However, this well-known deficiency of second-order theory is no obstacle for reaching the goals of this work.

Introducing the abbreviations

$$b = \frac{GI_T}{EI_\omega},$$

$$c(\lambda) = \frac{GI_T}{E^2 I_z I_\omega}\left(\frac{I_y}{I_z}\bar{M}_y^2 + \bar{M}_z^2\right)\lambda^2,$$

$$d(\lambda) = \frac{\bar{M}_y \bar{M}_z}{EI_y}\left(1 - \frac{I_y}{I_z}\right)\lambda^2, \quad (6)$$

Eq. 3 is rewritten as

$$\vartheta''''(x,\lambda) - b\vartheta''(x,\lambda) - c(\lambda)\vartheta(x,\lambda) = d(\lambda). \quad (7)$$

3. Solution of the Differential Equation and Formation of the Transfer Matrix

According to Schneider and Rubin [6], the solution of the inhomogeneous, linear, ordinary differential equation of fourth order in ϑ (7) is obtained as

$$\vartheta(x,\lambda) = \vartheta_a(\lambda)\mu_1(x,\lambda) + \vartheta_a'(\lambda)\mu_2(x,\lambda) + \vartheta_a''(\lambda)\mu_3(x,\lambda) + \vartheta_a'''(\lambda)\mu_4(x,\lambda) + d(\lambda)\mu_5(x,\lambda), \qquad \lambda > 0, \quad (8)$$

where

$$\mu_1(x,\lambda) = \frac{f^2(\lambda)\cosh(f(\lambda)x) + g^2(\lambda)\cos(g(\lambda)x)}{2\sqrt{r(\lambda)}}, \quad (9)$$

$$\mu_2(x,\lambda) = \frac{f(\lambda)\sinh(f(\lambda)x) + g(\lambda)\sin(g(\lambda)x)}{2\sqrt{r(\lambda)}}, \quad (10)$$

$$\mu_3(x,\lambda) = \frac{\cosh(f(\lambda)x) - \cos(g(\lambda)x)}{2\sqrt{r(\lambda)}}, \quad (11)$$

$$\mu_4(x,\lambda) = \frac{\dfrac{\sinh(f(\lambda)x)}{f(\lambda)} - \dfrac{\sin(g(\lambda)x)}{g(\lambda)}}{2\sqrt{r(\lambda)}}, \quad (12)$$

$$\mu_5(x,\lambda) = \frac{1}{c(\lambda)}\left(\mu_1(x,\lambda) - 1 - b\mu_3(x,\lambda)\right), \quad (13)$$

with

$$r(\lambda) = \frac{1}{4}b^2 + c(\lambda), \quad (14)$$

$$f(\lambda) = \sqrt{\frac{b}{2} + \sqrt{r(\lambda)}}, \quad (15)$$

$$g(\lambda) = \sqrt{-\frac{b}{2} + \sqrt{r(\lambda)}}. \quad (16)$$

Formation of the transfer matrix begins with expressing the vector $\boldsymbol{\vartheta}_x$, the transpose of which is defined as

$$\boldsymbol{\vartheta}_x^T = \lfloor \vartheta(x,\lambda), \vartheta'(x,\lambda), \vartheta''(x,\lambda), \vartheta'''(x,\lambda) | 1 \rfloor, \quad (17)$$

in terms of the vector $\boldsymbol{\vartheta}_a = \boldsymbol{\vartheta}_x(x=0)$, the transpose of which is given as

$$\boldsymbol{\vartheta}_a^T = \lfloor \vartheta_a(\lambda),\ \vartheta_a'(\lambda),\ \vartheta_a''(\lambda),\ \vartheta_a'''(\lambda)\ |\ 1 \rfloor. \quad (18)$$

The purpose of the last coefficient in (17) and (18) is to render the matrix $\boldsymbol{F}_{xa}^*$ in the relation

$$\boldsymbol{\vartheta}_x = \boldsymbol{F}_{xa}^* \cdot \boldsymbol{\vartheta}_a \quad (19)$$

quadratic. Making use of (8) and its first three derivatives with respect to x, $\boldsymbol{F}_{xa}^*$ is obtained as follows:

$$\boldsymbol{F}_{xa}^* = \left[\begin{array}{cccc|c} \mu_1(x,\lambda) & \mu_2(x,\lambda) & \mu_3(x,\lambda) & \mu_4(x,\lambda) & d(\lambda)\mu_5(x,\lambda) \\ \mu_1'(x,\lambda) & \mu_2'(x,\lambda) & \mu_3'(x,\lambda) & \mu_4'(x,\lambda) & d(\lambda)\mu_5'(x,\lambda) \\ \mu_1''(x,\lambda) & \mu_2''(x,\lambda) & \mu_3''(x,\lambda) & \mu_4''(x,\lambda) & d(\lambda)\mu_5''(x,\lambda) \\ \mu_1'''(x,\lambda) & \mu_2'''(x,\lambda) & \mu_3'''(x,\lambda) & \mu_4'''(x,\lambda) & d(\lambda)\mu_5'''(x,\lambda) \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad (20)$$

The constitutive equations are cast in matrix form analogous to (19):

$$\boldsymbol{\vartheta}_x = \boldsymbol{P}_x \cdot \boldsymbol{Z}_x \quad (21)$$

with

$$\boldsymbol{Z}_x^T = \lfloor \vartheta(x,\lambda), M_{Tp}(x,\lambda), M_\omega(x,\lambda), M_{Ts}(x,\lambda) | 1 \rfloor, \quad (22)$$

where M_{Tp}, M_ω, and M_{Ts} denote the primary torque, the warping moment, and the secondary torque, respectively, and

$$\boldsymbol{P}_x = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{1}{GI_T} & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{1}{GI_\omega} & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{GI_\omega} & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad (23)$$

containing the compliances $1/GI_T$ and $1/GI_\omega$. The purpose of the last coefficient in (22) is to render the matrix $\boldsymbol{P}_x$ quadratic. Solving (21) for $\boldsymbol{Z}_x$, considering (19) and a relation for $\boldsymbol{\vartheta}_x$ analogous to (21), results in

$$\boldsymbol{Z}_x = \boldsymbol{P}_x^{-1} \cdot \boldsymbol{\vartheta}_x = \boldsymbol{P}_x^{-1} \cdot \boldsymbol{F}^*_{xa} \cdot \boldsymbol{\vartheta}_a =$$

$$\boldsymbol{P}_x^{-1} \cdot \boldsymbol{F}^*_{xa} \cdot \boldsymbol{P}_x \cdot \boldsymbol{Z}_a = \boldsymbol{F}_{xa} \cdot \boldsymbol{Z}_a , \qquad (24)$$

where

$$\boldsymbol{F}_{xa} = \boldsymbol{P}_x^{-1} \cdot \boldsymbol{F}^*_{xa} \cdot \boldsymbol{P}_x \quad (25)$$

is referred to as transfer matrix.

4. Secant Stiffness Matrix

Specialization of (19) for $x = l$ yields

$$\boldsymbol{\vartheta}_b = \boldsymbol{F}^*_{ba} \cdot \boldsymbol{\vartheta}_a. \qquad (26)$$

Exchanging the first two rows in (26) and replacing ϑ_a'' by $-\vartheta_a''$, ϑ_a''' by $-\vartheta_a'''$, ϑ_b' by $-\vartheta_b'$, and ϑ_b''' by $-\vartheta_b'''$ in order to comply with the sign convention of the displacement method, gives

$$\bar{\boldsymbol{\vartheta}}_a^T = \lfloor \vartheta_a'(\lambda), \vartheta_a(\lambda), -\vartheta_a''(\lambda), -\vartheta_a'''(\lambda) \mid 1 \rfloor \qquad (27)$$

$$\bar{\boldsymbol{\vartheta}}_b^T = \lfloor -\vartheta_b'(\lambda), \vartheta_b(\lambda), \vartheta_b''(\lambda), -\vartheta_b'''(\lambda) \mid 1 \rfloor. \qquad (28)$$

Accordingly, the matrix $\bar{\boldsymbol{F}}^*_{ba}$ in the relation

$$\bar{\boldsymbol{\vartheta}}_b = \bar{\boldsymbol{F}}^*_{ba} \cdot \bar{\boldsymbol{\vartheta}}_a \qquad (29)$$

is obtained as

$$\bar{\boldsymbol{F}}^*_{ba} = \left[\begin{array}{cccc|c} -\mu_2'(\lambda) & -\mu_1'(\lambda) & \mu_3'(\lambda) & \mu_4'(\lambda) & -d(\lambda)\mu_5'(\lambda) \\ \mu_2(\lambda) & \mu_1(\lambda) & -\mu_3(\lambda) & -\mu_4'(\lambda) & d(\lambda)\mu_5(\lambda) \\ -\mu_1''(\lambda) & -\mu_2''(\lambda) & \mu_3''(\lambda) & \mu_4''(\lambda) & -d(\lambda)\mu_5''(\lambda) \\ \mu_1'''(\lambda) & \mu_2'''(\lambda) & -\mu_3'''(\lambda) & -\mu_4'''(\lambda) & d(\lambda)\mu_5'''(\lambda) \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad (30)$$

In the terms in (30) with the letter symbol μ the argument $x=l$ was omitted.

The first two lines of (29) can be written as

$$\boldsymbol{q}_b = \boldsymbol{T}^{(1)}_{ba} \cdot \boldsymbol{q}_a + \boldsymbol{T}^{(2)}_{ba} \cdot \boldsymbol{s}_a + \boldsymbol{t}_{ba}, \qquad (31)$$

where

$$\boldsymbol{q}_a = \begin{Bmatrix} \vartheta_a'(\lambda) \\ \vartheta_a(\lambda) \end{Bmatrix}, \quad \boldsymbol{q}_b = \begin{Bmatrix} -\vartheta_b'(\lambda) \\ \vartheta_b(\lambda) \end{Bmatrix}, \qquad (32)$$

$$\boldsymbol{s}_a = \begin{Bmatrix} -\vartheta_a''(\lambda) \\ -\vartheta_a'''(\lambda) \end{Bmatrix}, \qquad (33)$$

$$\boldsymbol{T}^{(1)}_{ba} = \begin{bmatrix} -\mu_2'(\lambda) & -\mu_1'(\lambda) \\ \mu_2(\lambda) & \mu_1(\lambda) \end{bmatrix},$$

$$\boldsymbol{T}^{(2)}_{ba} = \begin{bmatrix} \mu_3'(\lambda) & \mu_4'(\lambda) \\ -\mu_3(\lambda) & -\mu_4(\lambda) \end{bmatrix} \qquad (34)$$

and

$$\boldsymbol{t}_{ba} = \begin{Bmatrix} -d(\lambda)\mu_5'(\lambda) \\ d(\lambda)\mu_5(\lambda) \end{Bmatrix}. \qquad (35)$$

Solving (31) for $\boldsymbol{s}_a$ gives

$$\boldsymbol{s}_a = \boldsymbol{K}_{as} \cdot \boldsymbol{q}_a + \boldsymbol{K}_{ab} \cdot \boldsymbol{q}_b + \boldsymbol{s}_{ab}, \qquad (36)$$

where

$$\boldsymbol{K}_{as} = -\left(\boldsymbol{T}^{(2)}_{ba}\right)^{-1} \cdot \boldsymbol{T}^{(1)}_{ba},$$

$$\boldsymbol{K}_{ab} = \left(\boldsymbol{T}^{(2)}_{ba}\right)^{-1}, \qquad (37)$$

and

$$\boldsymbol{s}_{ab} = -\left(\boldsymbol{T}^{(2)}_{ba}\right)^{-1} \cdot \boldsymbol{t}_{ba}. \qquad (38)$$

Inversion of (29) yields

$$\bar{\boldsymbol{\vartheta}}_a = (\bar{\boldsymbol{F}}^*_{ba})^{-1} \cdot \bar{\boldsymbol{\vartheta}}_b, \qquad (39)$$

where

$$(\bar{\boldsymbol{F}}^*_{ba})^{-1} = \left[\begin{array}{cccc|c} t_{11}(\lambda) & t_{12}(\lambda) & t_{13}(\lambda) & t_{14}(\lambda) & t_{15}(\lambda) \\ t_{21}(\lambda) & t_{22}(\lambda) & t_{23}(\lambda) & t_{24}(\lambda) & t_{25}(\lambda) \\ t_{31}(\lambda) & t_{32}(\lambda) & t_{33}(\lambda) & t_{34}(\lambda) & t_{35}(\lambda) \\ t_{41}(\lambda) & t_{42}(\lambda) & t_{43}(\lambda) & t_{44}(\lambda) & t_{45}(\lambda) \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]. \qquad (40)$$

Analogous to (31), the first two rows of (39) can be written as

$$\boldsymbol{q}_a = \boldsymbol{T}^{(1)}_{ab} \cdot \boldsymbol{q}_b + \boldsymbol{T}^{(2)}_{ab} \cdot \boldsymbol{s}_b + \boldsymbol{t}_{ab} \qquad (41)$$

where

$$\boldsymbol{s}_b = \begin{Bmatrix} \vartheta_b''(\lambda) \\ -\vartheta_b'''(\lambda) \end{Bmatrix}, \qquad (42)$$

$$T_{ab}^{(1)} = \begin{bmatrix} t_{11}(\lambda) & t_{12}(\lambda) \\ t_{21}(\lambda) & t_{22}(\lambda) \end{bmatrix},$$

$$T_{ab}^{(2)} = \begin{bmatrix} t_{13}(\lambda) & t_{14}(\lambda) \\ t_{23}(\lambda) & t_{24}(\lambda) \end{bmatrix}, \quad (43)$$

$$t_{ab} = \begin{Bmatrix} t_{15}(\lambda) \\ t_{25}(\lambda) \end{Bmatrix}. \quad (44)$$

Solving (42) for s_b gives

$$s_b = K_{bs} \cdot q_b + K_{ba} \cdot q_a + s_{ba}, \quad (45)$$

where

$$K_{bs} = -\left(T_{ab}^{(2)}\right)^{-1} \cdot T_{ab}^{(1)}, K_{ba} = \left(T_{ab}^{(2)}\right)^{-1}, \quad (46)$$

and

$$s_{ba} = -\left(T_{ab}^{(2)}\right)^{-1} \cdot t_{ab}. \quad (47)$$

Combining (36) and (45) results in

$$K_S \cdot q = P, \quad (48)$$

where, in the terminology of the FEM,

$$K_S = \left[\begin{array}{c|c} K_{as} & K_{ab} \\ \hline K_{ba} & K_{bs} \end{array}\right] \quad (49)$$

represents the secant stiffness matrix for the given system, considered as a single finite element,

$$q = \begin{Bmatrix} q_a \\ \hline q_b \end{Bmatrix} \quad (50)$$

is the vector of nodal "displacements", and

$$P = \begin{Bmatrix} s_a - s_{ab} \\ \hline s_b - s_{ba} \end{Bmatrix} \quad (51)$$

stands for a vector of nodal "forces". Hence (48) represents the equilibrium equation.

Lateral torsional buckling occurs for the smallest value of λ, λ_S, for which

$$Det\, K_S = 0. \quad (52)$$

The eigenvector v_1, corresponding to the eigenvalue $\lambda_S = \lambda_1$, follows from

$$K_S(\lambda_1) \cdot v_1 = 0. \quad (53)$$

For $\lambda \to \lambda_S$, $q(x,\lambda) \to \infty$, $0 < x < l$, which is in contradiction to the assumption of small prebuckling rotations (see (4)). An analogous contradiction occurs if a simply supported beam, subjected to eccentric compressive forces at both ends, is analyzed by means of second-order theory. As these forces approach the Euler load, the displacements tend to infinity.

5. Consistently Linearized Eigenproblem

The mathematical formulation of the consistently linearized eigenproblem reads [3]

$$[K_T + (\lambda^* - \lambda)\, K_{T,\lambda}] \cdot v^* = 0, \quad (54)$$

where K_T is the $[N \times N]$ tangent stiffness matrix in the frame of the FEM and $K_{T,\lambda}$ is its first derivative with respect to λ along a direction parallel to the primary path. $v_1^*, v_2^*, \ldots, v_N^*$ are the eigenvectors corresponding to the distinct eigenvalues $\lambda_1^* - \lambda$, $\lambda_2^* - \lambda, \ldots,$ $\lambda_N^* - \lambda$. Writing (3) for the first eigenpair $(\lambda_1^* - \lambda,\ v_1^*)$, gives

$$[K_T + (\lambda_1^* - \lambda)\, K_{T,\lambda}] \cdot v_1^* = 0. \quad (55)$$

Specialization of (55) for the stability limit $(\lambda_1^* - \lambda = 0,\ v_1^* = v_1)$, where $\lambda = \lambda_S$, yields

$$K_T \cdot v_1 = 0. \quad (56)$$

Originally, the consistently linearized eigenproblem was used as a tool for circumventing numerical problems in the vicinity of snap – through points and bifurcation points on nonlinear primary paths. More recently, this eigenproblem was employed for derivation of subsidiary conditions for buckling at special prebuckling states, such as, e.g. , membrane stress states.

In the following, it will be shown that for lateral torsional buckling the eigensolution obtained from

$$[K_S + (\tilde{\lambda}_1 - \lambda)\, K_{S,\lambda}] \cdot \tilde{v}_1 = 0 \quad (57)$$

is the same as the one resulting from (55), i.e.,

$$\lambda_1^*(\lambda) = \tilde{\lambda}_1(\lambda), \quad \boldsymbol{v}_1^*(\lambda) = \tilde{\boldsymbol{v}}_1(\lambda). \tag{58}$$

The reason for this remarkable result is that for lateral torsional buckling,

$$\boldsymbol{v}_1^*(\lambda) = \tilde{\boldsymbol{v}}_1(\lambda) = \boldsymbol{v}_1 = \textbf{const}\,. \tag{59}$$

which, e.g., is not the case for torsional flexural buckling. Consideration of (59) in (55) and (57), premultiplication of the resulting relations by $\boldsymbol{v}_1$, and combination of the so-obtained equations, considering (58.1), gives

$$\frac{\boldsymbol{v}_1 \cdot \boldsymbol{K}_S \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{K}_{S,\lambda} \cdot \boldsymbol{v}_1} = \frac{\boldsymbol{v}_1 \cdot \boldsymbol{K}_T \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{K}_{T,\lambda} \cdot \boldsymbol{v}_1}. \tag{60}$$

As follows from (53) and (56), (60) is satisfied at the stability limit. To check whether this relation is also fulfilled for λ=0, (48) is derived with respect to λ:

$$\boldsymbol{K}_{S,\lambda} \cdot \boldsymbol{q} + \boldsymbol{K}_S \cdot \boldsymbol{q}_{,\lambda} = \boldsymbol{P}_{,\lambda}\,, \tag{61}$$

where

$$\boldsymbol{P}_{,\lambda} = \boldsymbol{K}_T \cdot \boldsymbol{q}_{,\lambda} \tag{62}$$

represents the rate form of the equilibrium equations. Specialization of (61) for $\lambda = 0 \rightarrow \boldsymbol{q} = \boldsymbol{0}$ obviously yields

$$\boldsymbol{K}_S = \boldsymbol{K}_T\,. \tag{63}$$

Combination of (61) and (62), followed by derivation with respect to λ, results in

$$\boldsymbol{K}_{S,\lambda\lambda} \cdot \boldsymbol{q} + 2\boldsymbol{K}_{S,\lambda} \cdot \boldsymbol{q}_{,\lambda} + \boldsymbol{K}_S \cdot \boldsymbol{q}_{,\lambda\lambda} = \boldsymbol{K}_{T,\lambda} \cdot \boldsymbol{q}_{,\lambda} + \boldsymbol{K}_T \cdot \boldsymbol{q}_{,\lambda\lambda}\,. \tag{64}$$

Specialization of (64) for λ=0 including consideration of (63) gives

$$2\boldsymbol{K}_{S,\lambda} = \boldsymbol{K}_{T,\lambda}. \tag{65}$$

As follows from (63) and (65) and from the positive definiteness $\boldsymbol{K}_T(\lambda{=}0){=}\boldsymbol{K}_0$, where $\boldsymbol{K}_0$ is the constant small – displacement stiffness matrix [7], satisfaction of (60) requires

$$\boldsymbol{K}_{S,\lambda} \cdot \boldsymbol{v}_1 = \boldsymbol{0} \quad \Leftrightarrow \quad \boldsymbol{K}_{T,\lambda} \cdot \boldsymbol{v}_1 = \boldsymbol{0}\,. \tag{66}$$

Specialization of (55) and (57) for $\lambda = 0$ and consideration of (59) and (66) yields

$$\tilde{\lambda}_1(\lambda = 0) = \lambda_1^*(\lambda = 0) = \infty. \tag{67}$$

In Chapter 6, (67) will be verified numerically.

For lateral torsional buckling, the eigenproblem (57) which is based on an analytical result for the secant stiffness matrix $\boldsymbol{K}_S$ is superior to the eigenproblem (55) in the frame of the FEM. Hence, there is no need to use (55) for numerical verification of subsidiary conditions for lateral torsional buckling, hypothesized by means of this relation.

The basis for derivation of such conditions is the relation

$$\left([\boldsymbol{K}_T + (\lambda_1^* - \lambda)\,\boldsymbol{K}_{T,\lambda}] \cdot \boldsymbol{v}_1^*\right)_{,\lambda\lambda\lambda}\Big|_{\lambda=\lambda_S} = \boldsymbol{0}\,, \tag{68}$$

resulting in [1]

$$\lambda^*_{1,\lambda\lambda\lambda} = -3\lambda^{*2}_{1,\lambda\lambda} + 2\frac{\boldsymbol{v}_1 \cdot \boldsymbol{K}_{T,\lambda\lambda\lambda} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{K}_{T,\lambda} \cdot \boldsymbol{v}_1}. \tag{69}$$

For buckling at general stress states, characterized by a percentage bending energy of the strain energy that increases in the prebuckling regime with increasing λ,

$$\lambda^*_{1,\lambda\lambda\lambda} > 0. \tag{70}$$

For buckling at special stress states, characterized by a constant percentage strain energy in the prebuckling regime,

$$\lambda^*_{1,\lambda\lambda\lambda} = 0, \tag{71}$$

with the exception of buckling from a membrane stress state, for which

$$\lambda^*_{1,\lambda\lambda\lambda} \le 0\,. \tag{72}$$

Hence, according to the above hypotheses, for lateral torsional buckling, characterized by 100% of percentage bending energy of the strain energy in the prebuckling regime,

$$\lambda^{*2}_{1,\lambda\lambda} = \frac{2}{3}\frac{\boldsymbol{v}_1 \cdot \boldsymbol{K}_{T,\lambda\lambda\lambda} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{K}_{T,\lambda} \cdot \boldsymbol{v}_1}, \tag{73}$$

as follows from substitution of (71) into (69).

Lateral torsional buckling represents a limiting case of torsional flexural buckling for which (70) holds. By contrast, lateral buckling is no limiting case of buckling at a constant percentage bending energy of the total strain energy in the prebuckling regime. This fact is reflected by different signs of the curvatures of the eigenvalue curves $\lambda_1^*(\lambda)$ at the stability limit where $\lambda_{1,\lambda}^* = 0$ [1]. (For convenience, $\lambda_1^*(\lambda)$ is occasionally referred to as the eigenvalue curve, although $\lambda_1^* - \lambda$ is actually the eigenvalue.) Lateral torsional buckling is the only case of (71) for which

$$\lambda_{1,\lambda\lambda}^* > 0 \tag{74}$$

holds in (73). In Chapter 6, (74) will be verified numerically.

6. Numerical Investigation

6.1 Solution of the eigenproblem

The numerical investigation consists of stability analysis of a beam as shown in Fig. 1. The structural steel shape is a HE-A 200 [8]. The input data for the analysis are given as follows:

$l = 2\text{m}$,

$I_y = 3690 \cdot 10^{-8}\text{m}^4$, $I_z = 1340 \cdot 10^{-8}\text{m}^4$,

$I_T = 21.1 \cdot 10^{-8}\text{m}^4$, $I_\omega = 10.8 \cdot 10^{-8}\text{m}^6$,

$E = 21 \cdot 10^7\text{kN/m}^2$, $\nu = 0.3$,

$\bar{M}_y = 80$ kNm, $\bar{M}_z = 1$ kNm.

Because of $\bar{M}_z \neq 0$, the deformed axis of the beam is a space curve. Since, $\bar{M}_z/\bar{M}_y = 1/80$, the deviation of this curve from the plane curve that would be obtained for $\bar{M}_z = 0$ is small, representing an imperfection that is characterized by a prebuckling rotation ϑ of the cross-section of the beam about the x-axis (Fig. 2).

The boundary conditions of the fork-supported beam are given as

$$\vartheta_a = \vartheta_b = 0, \tag{75}$$

and

$$M_{\omega,a} = -EI_\omega \vartheta_a'' = 0 \;\Rightarrow\; \vartheta_a'' = 0,$$
$$M_{\omega,b} = -EI_\omega \vartheta_b'' = 0 \;\Rightarrow\; \vartheta_b'' = 0. \tag{76}$$

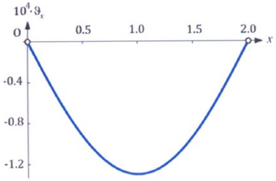

Fig. 3. $\vartheta(x,\lambda=1)$.

To compute $\boldsymbol{\vartheta}_x$, $\boldsymbol{Z}_x$ must be known (see (21)). Determination of $\boldsymbol{Z}_x$ requires knowledge of $\boldsymbol{Z}_a$ (see (24)). The transpose of $\boldsymbol{Z}_a$ follows from specialization of (22) for $x = 0$. Considering the boundary conditions at this point, $\boldsymbol{Z}_a^T$ is obtained as

$$\boldsymbol{Z}_a^T = \lfloor 0,\ M_{Tp,a},\ 0,\ M_{Ts,a} \mid 1 \rfloor. \tag{77}$$

Use of the first and the third line of the transfer matrix $\boldsymbol{F}_{ba}$ gives

$$\begin{bmatrix} F_{ba,12}(\lambda) & F_{ba,14}(\lambda) \\ F_{ba,32}(\lambda) & F_{ba,34}(\lambda) \end{bmatrix} \begin{Bmatrix} M_{Tp,a}(\lambda) \\ M_{Ts,a}(\lambda) \end{Bmatrix} = \begin{Bmatrix} -F_{ba,15}(\lambda) \\ -F_{ba,35}(\lambda) \end{Bmatrix} \tag{78}$$

which can be solved for $M_{Tp,a}$ and $M_{Ts,a}$.

Fig. 3 shows the function $\vartheta(x,\lambda)$, evaluated for $\lambda = 1$, i.e., for the reference moments $\bar{M}_y$ and $\bar{M}_z$ (see (1) and (2)).

Since $\vartheta(x,\lambda)$ is symmetric with respect to midspan, so are M_η and M_ζ (see (5)) and

$$M_\omega = -EI_\omega \vartheta''. \tag{79}$$

Table 1 contains the results for M_η and M_ζ at five points in the interval <0, 2.0>. The deviations of M_η from $\bar{M}_y = 80$ kNm and of M_ζ from $\bar{M}_z = 1$ kNm are small.

Symmetry of M_η and M_ζ entails symmetry of w and v.

Since

$$M_{Tp} = GI_T\vartheta',\; M_{Ts} = -EI_\omega \vartheta''' \tag{80}$$

are antisymmetric with respect to midspan, so is

$$M_T = M_{Tp} + M_{Ts}. \tag{81}$$

Table 1. $M_\eta(x, \lambda=1)$, $M_\zeta(x, \lambda=1)$

x [m]	M_η [kNm]	M_ζ [kNm]
0.0	80	1
0.5	79.99991	1.00737
1.0	79.99987	1.01032
1.5	79.99991	1.00737
2.0	80	1

Since $\vartheta_a'(\lambda)$ tends to infinity as λ approaches λ_S (see Fig.4), so does $M_{Tp,a}(\lambda)$. By analogy, in this case also $M_{Ts,a}(\lambda)$ tends to infinity. This characteristic feature of second-order theory has no influence on the following solution of the underlying eigenproblem.

The zeros of the determinant of the coefficient matrix

$$U(\lambda)=\begin{bmatrix} F_{ba,12}(\lambda) & F_{ba,14}(\lambda) \\ F_{ba,32}(\lambda) & F_{ba,34}(\lambda) \end{bmatrix} \tag{82}$$

in (78) are the eigenvalues of the underlying eigenproblem. The first two eigenvalues are obtained as λ_1=8.899 and λ_2=32.331. $\lambda_1=\lambda_S$ is the load factor that defines the buckling moment $\lambda_S \bar{M}_z$. Fig. 5 shows the corresponding eigenform, representing a spatial halfwave. To obtain this eigenform requires specification of the symmetry condition

$$\hat{\vartheta}_a' = -\hat{\vartheta}_b' . \tag{83}$$

Herein, $\hat{\vartheta}_a'$ is chosen as 1.

Fig. 6 shows the eigenform corresponding to λ_2, consisting of two spatial halfwaves. To obtain this eigenform requires specification of the antisymmetry condition

$$\hat{\vartheta}_a' = \hat{\vartheta}_b' . \tag{84}$$

To verify that the zeros of the determinant of the

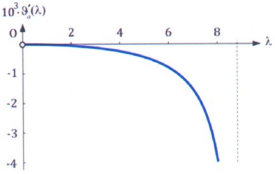

Fig. 4. $\vartheta_a'(\lambda)$

secant stiffness matrix $\boldsymbol{K}_S$ (see (49)) are identical with the zeros of the determinant of the matrix U (see (82)), the boundary conditions

$$\hat{\vartheta}_a = \hat{\vartheta}_b = 0 \tag{85}$$

for the eigenform must be considered. Accordingly, the number of the elements of the eigenvector is reduced from four to two, resulting in

$$\hat{\boldsymbol{v}} = \begin{Bmatrix} \hat{\vartheta}_a' \\ -\hat{\vartheta}_b' \end{Bmatrix}. \tag{86}$$

The minus sign in (86) correlates with the minus sign in the expression for $\boldsymbol{q}_b$ (see (32)). Because of the reduction of the eigenvector, the secant stiffness matrix $\boldsymbol{K}_S$ must be reduced from a [4 × 4] to a [2 × 2] submatrix:

$$\hat{\boldsymbol{K}}_S(\lambda)=\begin{bmatrix} K_{11}(\lambda) & K_{13}(\lambda) \\ K_{31}(\lambda) & K_{33}(\lambda) \end{bmatrix}. \tag{87}$$

Although $Det\,\hat{\boldsymbol{K}}_S(\lambda)$ is different from $Det\,\boldsymbol{U}(\lambda)$ (see Fig.7), the first two zeros of $Det\,\hat{\boldsymbol{K}}_S(\lambda)$ were found to agree with the corresponding zeros of $Det\,\boldsymbol{U}(\lambda)$.

The location of the vertical asymptote of the function $Det\,\hat{\boldsymbol{K}}_S(\lambda)$ between its first and second zero

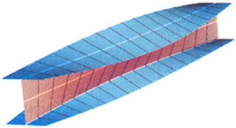

Fig. 5. First eigenform (buckling mode)

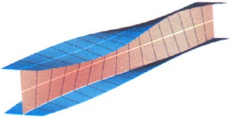

Fig. 6. Second eigenform

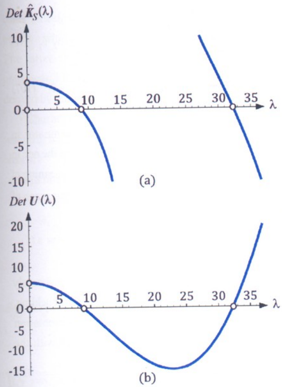

Fig. 7. (a) $Det\ \hat{\boldsymbol{K}}_S(\lambda)$, (b) $Det\ \boldsymbol{U}(\lambda)$

agrees with the location of the point of intersection of the mechanically insignificant second branch of the load-displacement curves, consisting of infinitely many branches, with the λ-axis. This follows from inversion of (48), resulting in

$$q = \hat{\boldsymbol{K}}_S^{-1} \cdot \boldsymbol{P}, \quad Det\ \hat{\boldsymbol{K}}_S \neq 0, \tag{88}$$

where

$$\hat{\boldsymbol{K}}_S^{-1} = \frac{\left(Adj\ \hat{\boldsymbol{K}}_S\right)^T}{Det\ \hat{\boldsymbol{K}}_S}. \tag{89}$$

For

$$Det\ \hat{\boldsymbol{K}}_S = \infty, \tag{90}$$

$$\hat{\boldsymbol{K}}_S^{-1} = \boldsymbol{0} \quad \Rightarrow \quad q = \boldsymbol{0}. \tag{91}$$

6.2. Characteristics of the consistently linearized eigenproblem

Formulation of the consistently linearized eigenproblem on the basis of $\hat{\boldsymbol{K}}_S(\lambda)$ and $\hat{\boldsymbol{v}}$ gives

$$[\hat{\boldsymbol{K}}_S + (\tilde{\lambda} - \lambda)\hat{\boldsymbol{K}}_{S,\lambda}] \cdot \hat{\boldsymbol{v}} = 0, \tag{92}$$

where

$$\hat{\boldsymbol{v}} = \begin{Bmatrix} \hat{\vartheta}'_a \\ -\hat{\vartheta}'_b \end{Bmatrix}. \tag{93}$$

Normalization of $\hat{\boldsymbol{v}}$ such that

$$\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{v}} = 1 \tag{94}$$

yields

$$|\hat{\boldsymbol{v}}| = \sqrt{\left(\hat{\vartheta}'_a\right)^2 + \left(-\hat{\vartheta}'_b\right)^2} = 1. \tag{95}$$

The solution of (92) consists of two eigenpairs:

$$\left(\tilde{\lambda}_1 - \lambda,\ \hat{\boldsymbol{v}}_1\right), \quad \left(\tilde{\lambda}_2 - \lambda,\ \hat{\boldsymbol{v}}_2\right). \tag{96}$$

The first eigenpair refers to symmetric eigenforms, for which

$$\hat{\vartheta}'_a = -\hat{\vartheta}'_b, \tag{97}$$

resulting in

$$\hat{\boldsymbol{v}}_1 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \textbf{const}. \tag{98}$$

The second eigenpair refers to antisymmetric eigenforms, for which

$$\hat{\vartheta}'_a = \hat{\vartheta}'_b, \tag{99}$$

resulting in

$$\hat{\boldsymbol{v}}_2 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \textbf{const}. \tag{100}$$

The independence of $\hat{\boldsymbol{v}}_1$ and $\hat{\boldsymbol{v}}_2$ of λ verifies (59). The two eigenvectors satisfy the orthogonality condition

$$\hat{\boldsymbol{v}}_1 \cdot \hat{\boldsymbol{v}}_2 = 0. \tag{101}$$

$\hat{\boldsymbol{K}}_S(\lambda)$ is an ingredient of an analytical solution of the underlying eigenproblem. Hence,

$$Det\ \hat{\boldsymbol{K}}_S(\lambda) = 0 \tag{102}$$

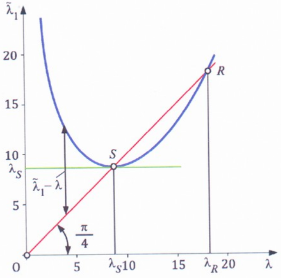

Fig. 8. $\tilde{\lambda}_1(\lambda)$

for countably infinitely many values of $\lambda > 0$, associated with zero eigenvalues

$$\tilde{\lambda}_1 - \lambda = 0, \tag{103}$$

and

$$\tilde{\lambda}_2 - \lambda = 0. \tag{104}$$

The smallest zero eigenvalue represents the stability limit $\lambda = \lambda_S = \tilde{\lambda}_1$ and, thus, defines the buckling moment $\lambda_S \bar{M}_y$. Fig. 8 illustrates the function $\tilde{\lambda}_1(\lambda)$.

Fig. 8 confirms $\tilde{\lambda}_{1,\lambda}(\lambda_S) = 0$ [1] and verifies the hypotheses $\tilde{\lambda}_{1,\lambda\lambda}(\lambda_S) > 0$ (see (74), recalling that $\lambda_1^* = \tilde{\lambda}_1$) and $\tilde{\lambda}_1(0) = \infty$ (see (67)). Closer inspection of $\tilde{\lambda}_1(\lambda)$ also seems to confirm the hypothesis that the curvature of this curve becomes a minimum at S, which implies $\tilde{\lambda}_{1,\lambda\lambda\lambda}(\lambda_S)=0$ (see (71)).

Continuation of the curve $\tilde{\lambda}_1(\lambda)$ beyond $\lambda = \lambda_S$ shows that (103) also holds at $\lambda = \lambda_R$, although

$$\tilde{\lambda}_{1,\lambda}(\lambda_R) \neq 0. \tag{105}$$

For $\lambda \to \lambda_R$,

$$Det\, \hat{\boldsymbol{K}}_S(\lambda) \to -\infty \tag{106}$$

(see Fig 9(a)). Specialization of (92) for $\lambda = \lambda_R$ and premultiplication of the obtained relation by $\hat{\boldsymbol{v}}_1$ gives

$$\lambda_R - \lambda = -\frac{\hat{\boldsymbol{v}}_1 \cdot \hat{\boldsymbol{K}}_S(\lambda_R) \cdot \hat{\boldsymbol{v}}_1}{\hat{\boldsymbol{v}}_1 \cdot \hat{\boldsymbol{K}}_{S,\lambda}(\lambda_R) \cdot \hat{\boldsymbol{v}}_1} = -\frac{(-\infty)}{(-\infty)} = -0, \tag{107}$$

because the quadratic form in the denominator in (107) is tending more strongly to $-\infty$ than the one in the numerator (Fig. 9). The mathematical meaning of λ_R ($\boldsymbol{q}(\lambda_R)$=0) was explained in the last paragraph of Subchapter 6.1.

7. Conclusions

• Remarkably, the solution of the consistently linearized eigenproblem for lateral torsional buckling, formulated by means of an analytical representation of the secant stiffness matrix, is identical with the

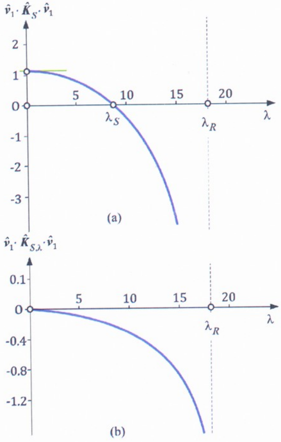

Fig. 9. (a) $\hat{\boldsymbol{v}}_1 \cdot \hat{\boldsymbol{K}}_S(\lambda) \cdot \hat{\boldsymbol{v}}_1$, (b) $\hat{\boldsymbol{v}}_1 \cdot \hat{\boldsymbol{K}}_{S,\lambda}(\lambda) \cdot \hat{\boldsymbol{v}}_1$.

one resulting from formulation of such an eigenproblem with the help of the tangent stiffness matrix. The latter represents an essential ingredient of nonlinear FEA. The secant stiffness matrix was derived by means of the transfer matrix method.

• Characteristics of the eigenvalue curve $\lambda_1^*(\lambda)$ resulting from the consistently linearized eigenproblem are

(a) a minimum of the curvature of this curve at the stability limit where $\lambda_1^* = \lambda_S$, and $\lambda_{1,\lambda}^*(\lambda_S) = 0$, $\lambda_{1,\lambda\lambda}^*(\lambda_S) > 0$, and

(b) a vertical asymptote of the curve at λ=0, i.e.,

$$\lambda_1^*(\lambda = 0) = \infty.$$

The characteristic feature of the eigenvector function $\boldsymbol{v}_1^*(\lambda)$ is its constancy. Hence, the solution of the consistently linearized eigenproblem applied to lateral torsional buckling takes up a position between the general case, where both the eigenvalue and the eigenvector function are variables, and the special case of linear stability problems, where both are constants.

• The presented solution closes a gap in a new concept of categorization of buckling by means of spherical geometry.

მექანიკა

საკუთრივი მნიშვნელობის ამოცანის თანმიმდევრობითი გაწრფივების დახასიათება ღუნვის გვერდითი გრეხვისათვის

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საკუთრივი მნიშვნელობის ამოცანის თანმიმდევრობით გაწრფივება მძლავრი მათემატიკური იარაღი აღმოჩნდა ღუნვების კლასიფიკაციისათვის. განსაკუთრებით საინტერესოა პრეღუნვის მდგომარეობები მუდმივი სრული ენერგიით. ასეთი მდგომარეობის ორი ზღვრული შემთხვევაა მებრანის დაძაბულობისა და წმინდა ღუნვების მდგომარეობები. კვლევის საგანია წმინდა ღუნვების მდგრადობა, რომელიც გვერდითი ღუნვის გრეხვად იწოდება. ტრანსფერ-მატრიცის მეთოდი გამოიყენება ანალიზური სახით სიმტკიცის მატრიცის მისაღებად. საკუთრივი ფუნქციის გაწრფივებული ამოცანისადმი ასეთი მიდგომით მიღებული ამონახსნი იგივეა, რაც მხები მატრიცის მეთოდით მიღებული, რომელიც თავის მხრივ არაწრფივი სასრული ელემენტის ანალიზის მნიშვნელოვანი ინგრედიენტია. მიღებული შედეგი საშუალებას იძლევა ანალიზურად შემოწმდეს ჰიპოთეზები გვერდით გრეხვაზე დამატებითი პირობების არსებობის შემთხვევაში.

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Mechanics

Nonlocal Refined Theory for Nanobeams with Surface Effects

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ABSTRACT. In this study we propose governing differential equations for beams taking into account shear deformation, rotary inertia, locality and surface stress effects. It is shown that the equation is both simpler and more consistent than the appropriate Bresse-Timoshenko equations extended to include locality and surface stress effects. Proposed equation contains 11 terms with respect to displacement versus 19 terms appearing in the equations that extend the Bresse-Thimoshenko equations to include nonlocality and surface effects. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *nanotube, surface effect, nonlocal.*

1. Introduction.

We consider the following Bresse-Timoshenko equations for a beam analyzed by local theory and without surface effects

$$EI\frac{\partial^2\phi}{\partial x^2}-\kappa AG\left(\phi-\frac{\partial v}{\partial x}\right)-\rho I\frac{\partial^2\phi}{\partial t^2}=0, \tag{1}$$

$$\rho A\frac{\partial^2 v}{\partial t^2}+\kappa AG\left(\frac{\partial\phi}{\partial x}-\frac{\partial^2 v}{\partial x^2}\right)=0, \tag{2}$$

where κ is the shear coefficient which depends on the shape of the cross-section, A is the area of the cross-section, G is the modulus of elasticity in shear, ϕ is the slope of the deflection curve when the shear force is neglected, v is the total deflection, x is the axial displacement, ρ is the mass density per unit volume, t is the time, E is the Young's modulus of elasticity and I is the area moment of inertia.

Equations (1) and (2) can be reduced to a single differential equation in terms of displacement $v(x,t)$:

$$EI\frac{\partial^4 v}{\partial x^4}+\rho A\frac{\partial^2 v}{\partial t^2}-\rho I\left(1+\frac{E}{\kappa G}\right)\frac{\partial^4 v}{\partial x^2\partial t^2}+\frac{\rho^2 I}{\kappa G}\frac{\partial^4 v}{\partial t^4}=0. \tag{3}$$

Timoshenko [1] evaluated the natural frequencies of the simply supported beam at both ends and arrived at the conclusion that the term associated with the last term in equation (3) " is small quantity of the second order compared with" other terms in the characteristic equation.

Let us look at the third term in dynamic equilibrium equation considered by Timoshenko

$$-V+\frac{\partial M}{\partial x}-\rho I\frac{\partial^2\phi}{\partial t^2}=0 \tag{4}$$

namely $-\rho I\partial^2\phi/\partial t^2$ which replaces its counterpart $-\rho I\partial^3 v/\partial x\partial t^2$ in equation associated with taking into account the rotary inertia alone

$$-V+\frac{\partial M}{\partial x}-\rho I\frac{\partial^3 v}{\partial x\partial t^2}=0\,. \tag{5}$$

Timoshenko's [1] purpose was, as the word "correction" in the title of his paper suggests, a correction of the original rotary inertia term by Bresse [2] and Rayleigh [3] by incorporating the shearing force. However, it appears that such a correction has a secondary effect, since it attempts to correct the correction due to the rotary inertia effect. Such a process of correction of correction apparently leads to secondary effect and results in an inconsistent theory including, as was shown by Timoshenko [1] himself, a second order term.

The situation was remedied by Elishakoff [4]. Namely, it was shown that simpler and more consistent set of equations is derivable by retaining the equation (2) whereas the equation (1) should be replaced by

$$EI\frac{\partial^2\phi}{\partial x^2}-\kappa AG\left(\phi-\frac{\partial v}{\partial x}\right)-\rho I\frac{\partial^3 v}{\partial x\partial t^2}=0\,. \tag{6}$$

Equations (2) and (6) can also be reduced to the single differential equation

$$EI\frac{\partial^4 v}{\partial x^4}+\rho A\frac{\partial^2 v}{\partial t^2}-\rho I\left(1+\frac{E}{\kappa G}\right)\frac{\partial^4 v}{\partial x^2\partial t^2}=0\,. \tag{7}$$

As is seen, the consistent equation (7) differs from the original Bresse-Timoshenko equation (3) by absence of the term containing the fourth derivative with respect to time.

It will be our goal to develop consistent equation also for the flexural beams within nonlocal theory with or without surface effects.

2. Analysis without surface effects.

Lu, Lee, Lu and Zhang [5] and Lu, Lee, Lu and Zhang [6] proposed a generalization of Bresse-Timoshenko equations for the flexural beams analyzed by the nonlocal theory without surface effects

$$EI\frac{\partial^2\phi}{\partial x^2}+\kappa GA\left(\frac{\partial v}{\partial x}-\phi\right)-\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\rho I\frac{\partial^2\phi}{\partial t^2}=0\,, \tag{8}$$

$$\kappa GA\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial\phi}{\partial x}\right)-\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\rho A\frac{\partial^2 v}{\partial t^2}=-\left[p-(e_0a)^2\frac{\partial^2 p}{\partial x^2}\right], \tag{9}$$

where $e_0 a$ is the scale coefficient that incorporates the nonlocality effect, a is an internal characteristic length, and e_0 is a constant for adjusting the model in matching some reliable results by experiments or other models. When $e_0 a = 0$, equations (8) and (9) are reduced to the equations of classical Bresse-Timoshenko beam.

We can extract $\partial\phi/\partial x$ from equation (9) to get

$$\frac{\partial\phi}{\partial x}=\frac{\partial^2 v}{\partial x^2}-\frac{1}{\kappa GA}\left(\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\rho A\frac{\partial^2 v}{\partial t^2}-\left[p-(e_0a)^2\frac{\partial^2 p}{\partial x^2}\right]\right). \tag{10}$$

Differentiation of equation (8) with respect to x yields

$$EI\frac{\partial^3\phi}{\partial x^3}+\kappa GA\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial\phi}{\partial x}\right)-\left[\frac{\partial}{\partial x}-(e_0a)^2\frac{\partial^3}{\partial x^3}\right]\rho I\frac{\partial^2\phi}{\partial t^2}=0. \tag{11}$$

In order to substitute $\partial^3\phi/\partial x^3$, $\partial^3\phi/\partial x\partial t^2$ and $\partial^5\phi/\partial x^3\partial t^2$ in equation (11), we first write the following derivatives using equation (10)

$$\frac{\partial^3\phi}{\partial x^3}=\frac{\partial^4 v}{\partial x^4}-\frac{1}{\kappa GA}\left(\left[\frac{\partial^2}{\partial x^2}-(e_0a)^2\frac{\partial^4}{\partial x^4}\right]\rho A\frac{\partial^2 v}{\partial t^2}-\left[\frac{\partial^2 p}{\partial x^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^4}\right]\right), \tag{12}$$

$$\frac{\partial^3\phi}{\partial x\,\partial t^2}=\frac{\partial^4 v}{\partial x^2\partial t^2}-\frac{1}{\kappa GA}\left(\left[\frac{\partial^2}{\partial t^2}-(e_0a)^2\frac{\partial^4}{\partial x^2\partial t^2}\right]\rho A\frac{\partial^2 v}{\partial t^2}-\left[\frac{\partial^2 p}{\partial t^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^2\partial t^2}\right]\right), \tag{13}$$

$$\frac{\partial^5\phi}{\partial x^3\partial t^2}=\frac{\partial^6 v}{\partial x^4\partial t^2}-\frac{1}{\kappa GA}\left(\left[\frac{\partial^4}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^2}{\partial x^4\partial t^2}\right]\rho A\frac{\partial^2 v}{\partial t^2}-\left[\frac{\partial^4 p}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^6 p}{\partial x^4\partial t^2}\right]\right). \tag{14}$$

Substituting equations (10), (12), (13) and (14) into equation (11), equations (8) and (9) are reduced to a single differential equation in terms of displacement $v(x,t)$:

$$\begin{aligned}
&EI\left(\frac{\partial^4 v}{\partial x^4}-\frac{\rho}{\kappa G}\left[\frac{\partial^4 v}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^6 v}{\partial x^4\partial t^2}\right]\right)+\rho A\left(\frac{\partial^2 v}{\partial t^2}-(e_0a)^2\frac{\partial^4 v}{\partial x^2\partial t^2}\right)-\\
&-\rho I\left(\frac{\partial^4 v}{\partial x^2\partial t^2}-\frac{\rho}{\kappa G}\left[\frac{\partial^4 v}{\partial t^4}-(e_0a)^2\frac{\partial^6 v}{\partial x^2\partial t^4}\right]\right)+\\
&+(e_0a)^2\rho I\left(\frac{\partial^6 v}{\partial x^4\partial t^2}-\frac{\rho}{\kappa G}\left[\frac{\partial^6 v}{\partial x^2\partial t^4}-(e_0a)^2\frac{\partial^8 v}{\partial x^4\partial t^4}\right]\right)=-\frac{EI}{\kappa GA}\left[\frac{\partial^2 p}{\partial x^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^4}\right]+\\
&+\left[p-(e_0a)^2\frac{\partial^2 p}{\partial x^2}\right]+\frac{1}{\kappa GA}\left[\frac{\partial^2 p}{\partial t^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^2\partial t^2}\right]+(e_0a)^2\rho I\left[\frac{\partial^4 p}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^6 p}{\partial x^4\partial t^2}\right].
\end{aligned} \tag{15}$$

We would like to compare the equation (15) with the equation that is derivable via consistent analysis. Following Ref. [4] a simpler and more consistent set of equations is constituted by retaining the equation (9) whereas the equation (8) should be replaced by

$$EI\frac{\partial^2\phi}{\partial x^2}+\kappa GA\left(\frac{\partial v}{\partial x}-\phi\right)-\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\rho I\frac{\partial^3 v}{\partial x\partial t^2}=0. \tag{16}$$

Then equation (10) should be retained. Differentiation of equation (16) with respect to x yields

$$EI\frac{\partial^3\phi}{\partial x^3}+\kappa GA\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial\phi}{\partial x}\right)-\left[\frac{\partial}{\partial x}-(e_0a)^2\frac{\partial^3}{\partial x^3}\right]\rho I\frac{\partial^3 v}{\partial x\partial t^2}=0. \tag{17}$$

Substituting equations (10) and (12) into equation (17), equations (9) and (16) we obtain a single differential equation in terms of displacement $v(x,t)$:

$$EI\left(\frac{\partial^4 v}{\partial x^4}-\frac{1}{\kappa GA}\left[\frac{\partial^2}{\partial x^2}\left(\rho A\frac{\partial^2 v}{\partial t^2}\right)-(e_0a)^2\frac{\partial^4}{\partial x^4}\left(\rho A\frac{\partial^2 v}{\partial t^2}\right)\right]\right)+\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\rho A\frac{\partial^2 v}{\partial t^2}-$$

$$-\rho I\left[\frac{\partial^4 v}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^6 v}{\partial x^4\partial t^2}\right]=-\frac{EI}{\kappa GA}\left[\frac{\partial^2 p}{\partial x^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^4}\right]+\left[p-(e_0a)^2\frac{\partial^2 p}{\partial x^2}\right]. \tag{18}$$

As is seen, the consistent set differs from the original Bresse-Timoshenko equations by absence of the following terms

$$-\frac{\rho A}{\kappa GA}\left[\frac{\partial^4 v}{\partial t^4}-(e_0a)^2\frac{\partial^6 v}{\partial x^2\partial t^4}\right]\rho I-\frac{(e_0a)^2\rho^2 I}{\kappa G}\left[\frac{\partial^6 v}{\partial x^2\partial t^4}-(e_0a)^2\frac{\partial^8 v}{\partial x^4\partial t^4}\right]$$

$$\frac{1}{\kappa GA}\left[\frac{\partial^2 p}{\partial t^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^2\partial t^2}\right](e_0a)^2\rho I\left[\frac{\partial^4 p}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^6 p}{\partial x^4\partial t^2}\right]. \tag{19}$$

Specifically, equation (15) consists of 11 terms containing $v(x,t)$ whereas equation (18) consists of 7 such terms. Moreover, equation (15) contains 8 terms containing $p(x,t)$, whereas equation (18) contains only 4 such terms.

As expected, the consistent set differs from the original Bresse-Timoshenko equations by absence of the terms containing the fourth derivative with respect to time. Moreover, four additional terms with respect to $v(x,t)$, and four additional terms with respect to loading $p(x,t)$ are absent in the consistent set.

3. Analysis with surface effects

Lee and Chang [7] derived the equations of motion for the nonlocal Bresse-Timoshenko beam model with surface effects as follows

$$(EI)^*\frac{\partial^2\phi}{\partial x^2}+\kappa GA\left(\frac{\partial v}{\partial x}-\phi\right)-\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\rho I\frac{\partial^2\phi}{\partial t^2}=0, \tag{20}$$

$$\kappa GA\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial\phi}{\partial x}\right)-\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2}-\frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]=-\left[p-(e_0a)^2\frac{\partial^2 p}{\partial x^2}\right], \tag{21}$$

where $(EI)^*$ is the effective flexural rigidity which includes the surface bending elasticity on the nanotube and its flexural rigidity and defined as $(EI)^* = \pi E^S\left(R_i^3 + R_o^3\right) + EI$, with E^S being the surface elasticity modulus, H is the constant parameter which is determined by the residual surface tension and the shape of cross section and defined as $H = 4\tau\left(R_i + R_o\right)$, with τ the residual surface tension per length on the nanotube.

We extract $\partial\phi/\partial x$ from equation (21) to get

$$\frac{\partial\phi}{\partial x} = \frac{\partial^2 v}{\partial x^2} - \frac{1}{\kappa GA}\left(\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right] - \left[p-(e_0a)^2\frac{\partial^2 p}{\partial x^2}\right]\right). \tag{22}$$

Differentiation of equation (20) with respect to x yields

$$(EI)^*\frac{\partial^3\phi}{\partial x^3} + \kappa GA\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial\phi}{\partial x}\right) - \left[\frac{\partial}{\partial x} - (e_0a)^2\frac{\partial^3}{\partial x^3}\right]\rho I\frac{\partial^2\phi}{\partial t^2} = 0. \tag{23}$$

In order to substitute $\partial^3\phi/\partial x^3$, $\partial^3\phi/\partial x\partial t^2$ and $\partial^5\phi/\partial x^3\partial t^2$ in equation (23), we obtain the following derivatives using equation (22)

$$\frac{\partial^3\phi}{\partial x^3} = \frac{\partial^4 v}{\partial x^4} - \frac{1}{\kappa GA}\left(\left[\frac{\partial^2}{\partial x^2} - (e_0a)^2\frac{\partial^4}{\partial x^4}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right] - \left[\frac{\partial^2 p}{\partial x^2} - (e_0a)^2\frac{\partial^4 p}{\partial x^4}\right]\right), \tag{24}$$

$$\frac{\partial^3\phi}{\partial x\,\partial t^2} = \frac{\partial^4 v}{\partial x^2\partial t^2} - \frac{1}{\kappa GA}\left(\left[\frac{\partial^2}{\partial t^2} - (e_0a)^2\frac{\partial^4}{\partial x^2\partial t^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right] - \left[\frac{\partial^2 p}{\partial t^2} - (e_0a)^2\frac{\partial^4 p}{\partial x^2\partial t^2}\right]\right), \tag{25}$$

$$\frac{\partial^5\phi}{\partial x^3\partial t^2} = \frac{\partial^6 v}{\partial x^4\partial t^2} - \frac{1}{\kappa GA}\left(\left[\frac{\partial^4}{\partial x^2\partial t^2} - (e_0a)^2\frac{\partial^6}{\partial x^4\partial t^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right] - \left[\frac{\partial^4 p}{\partial x^2\partial t^2} - (e_0a)^2\frac{\partial^6 p}{\partial x^4\partial t^2}\right]\right). \tag{26}$$

Substituting equations (22), (24), (25) and (26) into equation (23), equations (20) and (21) can be reduced to a single equation:

$$\begin{aligned}
&(EI)^*\left(\frac{\partial^4 v}{\partial x^4} - \frac{1}{\kappa GA}\left[\frac{\partial^2}{\partial x^2} - (e_0a)^2\frac{\partial^4}{\partial x^4}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]\right) + \left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]\\
&\quad -\rho I\left\{\frac{\partial^4 v}{\partial x^2\partial t^2} - \frac{1}{\kappa GA}\left[\frac{\partial^2}{\partial t^2} - (e_0a)^2\frac{\partial^4}{\partial x^2\partial t^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]\right.\\
&\quad \left. -(e_0a)^2\left(\frac{\partial^6 v}{\partial x^4\partial t^2} - \frac{1}{\kappa GA}\left[\frac{\partial^4}{\partial x^2\partial t^2} - (e_0a)^2\frac{\partial^6}{\partial x^4\partial t^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]\right)\right\}\\
&= -\frac{(EI)^*}{\kappa GA}\left[\frac{\partial^2 p}{\partial x^2} - (e_0a)^2\frac{\partial^4 p}{\partial x^4}\right] + \left[p-(e_0a)^2\frac{\partial^2 p}{\partial x^2}\right] + \frac{\rho I}{\kappa GA}\left\{\left[\frac{\partial^2 p}{\partial t^2} - (e_0a)^2\frac{\partial^4 p}{\partial x^2\partial t^2}\right]\right.\\
&\quad \left. -(e_0a)^2\left[\frac{\partial^4 p}{\partial x^2\partial t^2} - (e_0a)^2\frac{\partial^6 p}{\partial x^4\partial t^2}\right]\right\}.
\end{aligned} \tag{27}$$

A simpler and more consistent set of equations is constituted (see Ref. [4]) by retaining the equation (21) whereas the equation (20) should be replaced by

$$(EI)^{*}\frac{\partial^2\phi}{\partial x^2}+\kappa GA\left(\frac{\partial v}{\partial x}-\phi\right)-\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\rho I\frac{\partial^3 v}{\partial x\partial t^2}=0. \tag{28}$$

Then equation (22) remains and differentiation with respect to x yields

$$(EI)^{*}\frac{\partial^3\phi}{\partial x^3}+\kappa GA\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial\phi}{\partial x}\right)-\left[\frac{\partial}{\partial x}-(e_0a)^2\frac{\partial^3}{\partial x^3}\right]\rho I\frac{\partial^3 v}{\partial x\partial t^2}=0 \tag{29}$$

and substituting equations (23) and (24) into equation (29) gives

$$(EI)^{*}\left(\frac{\partial^4 v}{\partial x^4}-\frac{1}{\kappa GA}\left[\frac{\partial^2}{\partial x^2}-(e_0a)^2\frac{\partial^4}{\partial x^4}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2}-\frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]\right)+\left[1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2}-\frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]$$

$$-\left[\frac{\partial}{\partial x}-(e_0a)^2\frac{\partial^3}{\partial x^3}\right]\rho I\frac{\partial^3 v}{\partial x\partial t^2}=-\frac{(EI)^{*}}{\kappa GA}\left[\frac{\partial^2 p}{\partial x^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^4}\right]+\left[p-(e_0a)^2\frac{\partial^2 p}{\partial x^2}\right]. \tag{30}$$

As is seen, the consistent set differs from the original Lee and Chang [7] equations by absence of the following terms

$$\frac{\rho I}{\kappa GA}\left[\frac{\partial^2}{\partial t^2}-(e_0a)^2\frac{\partial^4}{\partial x^2\partial t^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2}-\frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]$$

$$\frac{1}{\kappa GA}(e_0a)^2\left[\frac{\partial^4}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^6}{\partial x^4\partial t^2}\right]\left[\rho A\frac{\partial^2 v}{\partial t^2}-\frac{\partial}{\partial x}\left(H\frac{\partial v}{\partial x}\right)\right]$$

$$\frac{\rho I}{\kappa GA}\left[\frac{\partial^2 p}{\partial t^2}-(e_0a)^2\frac{\partial^4 p}{\partial x^2\partial t^2}\right]-\frac{\rho I}{\kappa GA}(e_0a)^2\left[\frac{\partial^4 p}{\partial x^2\partial t^2}-(e_0a)^2\frac{\partial^6 p}{\partial x^4\partial t^2}\right]. \tag{31}$$

Note that equation (27) corresponding to the original Bresse-Timoshenko theory with nonlocality and surface effects incorporates 19 terms for displacement $v(x,t)$ whereas equation (30) contains 11 such terms. Likewise, equation (27) consists of 8 terms for the distributed load $p(x,t)$ whereas equation (30) contains 4 such terms.

As observed in the case without surface effects, comparison of the consistent set with the original Bresse-Timoshenko equations shows the absence of the terms containing the fourth derivative with respect to time and eight additional terms.

4. Free Vibrations of Nonlocal Bernoulli-Euler Beams

We study free vibrations of the Bresse-Timoshenko beams that are simply-supported at both ends. To do so, we postulate that harmonic vibrations with natural frequency ω are taking place. The boundary conditions are satisfied if we let

$$v(x,t)=V\sin\frac{m\pi x}{L}\sin\omega t, \tag{32}$$

where m is the number of half-waves in axial direction, L is the beam length. In equations (15), (18), (27) and (30) we let $p(x,t)\equiv 0$.

We would like to compare the solutions obtained for ω^2 from equations (15), (18), (27) and (30) with the solution derived from the nonlocal Bernoulli-Euler beam, namely

$$\omega_{BE}^2=\frac{EI}{\rho A\left[1+\left(e_0a\right)^2\left(\frac{m\pi}{L}\right)^2\right]}\left(\frac{m\pi}{L}\right)^4. \tag{33}$$

This equation corresponds to the equation discussed by Zhang, Liu and Xie [8] and Lu, Lee, Lu and Zhang [9]. As is seen the squared natural frequency of the nonlocal Bernoulli-Euler beam is smaller than its counterpart for the beam treated by local theory $EI\left(m\pi/L\right)^4/\rho A$.

5. Free Vibration Analysis of Bresse-Timoshenko Beams without Surface Effects

Substitution of equation (32) obtained from the original set into equation (15) leads to the following expression for ω_O^2

$$\begin{aligned}\omega_O^2=&\frac{1}{2}\frac{1}{\rho I\left(L^4+L^2\left(e_0a\right)^2\pi^2m^2+\left(e_0a\right)^4\pi^4m^4\right)}\Big[\Big(EIL^2m^2\pi^2+EI\,m^4\pi^4\left(e_0a\right)^2+A\kappa GL^4\\&+A\kappa GL^2\left(e_0a\right)^2\pi^2m^2+I\pi^2m^2\kappa GL^2+I\left(e_0a\right)^2\pi^4m^4\kappa G-\Big(A^2\kappa^2G^2L^8-2EI^2L^4\pi^4m^4\kappa G\\&-2EI^2\pi^8m^8\left(e_0a\right)^2\kappa G+2A^2\kappa^2G^2L^6\left(e_0a\right)^2\pi^2m^2+2AI\kappa^2G^2L^6\pi^2m^2+A^2\kappa^2G^2L^4\left(e_0a\right)^2\pi^4m^4\\&+2I^2m^6\pi^6\kappa^2G^2L^2\left(e_0a\right)^2+E^2I^2L^4m^4\pi^4+E^2I^2m^8\pi^8\left(e_0a\right)^4+2E^2I^2L^2m^6\pi^6\left(e_0a\right)^2\\&+I^2L^4m^4\pi^4\kappa^2G^2+I^2\left(e_0a\right)^4m^8\pi^8\kappa^2G^2+2EIL^6A\kappa Gm^2\pi^2+4EIL^4A\kappa Gm^4\pi^4\left(e_0a\right)^2\\&+2EIA\kappa GL^2m^6\pi^6\left(e_0a\right)^4+4IL^4A\kappa^2G^2m^4\pi^4+2AI\kappa^2G^2L^2m^6\pi^6\left(e_0a\right)^4\Big)^{1/2}\Big].\end{aligned} \tag{34}$$

Equation (18) associated with the consistent set yields the following equation for squared natural frequency

$$\omega_C^2=\frac{EI\pi^4m^4\kappa G}{\rho\left(EI\pi^2m^2L^2+EI\pi^4m^4\left(e_0a\right)^2+A\kappa GL^4+A\kappa GL^2m^2\pi^2\left(e_0a\right)^2+I\pi^2m^2\kappa GL^2+I\pi^4m^4\kappa G\left(e_0a\right)^2\right)}. \tag{35}$$

In order to compare the natural frequencies obtained by original and consistent set we form the ratio ω_C^2/ω_O^2. The first natural frequencies are $\omega_O^{(1)}=1.293109212\cdot 10^{12}$ rad, $\omega_C^{(1)}=1.285469485\cdot 10^{12}$ rad, respectively. As is seen the difference is less than 0.60%. The second frequencies equal $\omega_O^{(2)}=3.702494824\cdot 10^{12}$ rad and $\omega_C^{(2)}=3.600545631\cdot 10^{12}$ rad, respectively, the difference being less than 2.76%. For the third frequency ($\omega_O^{(3)}=5.891090015\cdot 10^{12}$ rad, $\omega_C^{(3)}=5.636053802\cdot 10^{12}$ rad) the difference constitutes less than 4.33%.

As is seen, the consistent set yields nearly the same fundamental natural frequency as the original set, but the equation is much simpler than the one associated with original Bresse-Timoshenko theory. Moreover, as shown in Ref. 4, the simpler set is also more consistent than the original one.

6. Free Vibrations Analysis of Bresse-Timoshenko Beams with Surface Effects

Substitution of equation (32) obtained from the original set into equation (27) leads to the following expression for ω_O^2

$$
\begin{aligned}
\omega_O^2 = \frac{1}{2\rho IAL^2\left(L^2+(e_0a)^2\pi^2m^2\right)}\Big[& A^2\kappa GL^4 + L^2 IH\pi^2m^2 + L^2(EI)^* A\pi^2m^2 + L^2 I\kappa GA\pi^2m^2 \\
& + IH(e_0a)^2 m^2\pi^2 - \Big(2L^2I^2H^2m^6\pi^6(e_0a)^2 + L^4I^2H^2\pi^4m^4 + L^4(EI)^{*2}\pi^4m^4A^2 \\
& + I^2H^2(e_0a)^4m^8\pi^8 + 2A^3\kappa GL^6(EI)^*\pi^2m^2 + 2A^3\kappa^2G^2L^6I\pi^2m^2 - 2L^4IH\pi^4m^4(\dot{E}I)\ A \\
& + L^4(EI)^{*2}\pi^4m^4G^2A^2 + A^4\kappa^2G^2L^2 - 2A^2\kappa GL^6IH\pi^2m^2 - 2A\kappa GL^4IH(e_0a)^2\pi^4m^4 \\
& + 2L^4I^2H\kappa GA\pi^4m^4 - 2L^4(EI)^*A^2I\kappa G + 2L^2(EI)^*\pi^6m^6AIH(e_0a)^2 \\
& + 2L^2I^2\pi^6m^6\kappa GAH(e_0a)^2 + 4IA^2L^2(EI)^*\pi^6m^6(e_0a)^4\rho H\Big)^{1/2}\Big].
\end{aligned}
\tag{36}
$$

Equation (30) obtained from the consistent set yields

$$
\omega_C^2 = \frac{\pi^2m^2\left[(EI)^*\pi^2m^2L^2\kappa GA+(EI)^*\pi^2m^2HL^2+(EI)^*H\pi^4m^4(e_0a)^2+HL^4\kappa GA+(e_0a)^2H\pi\right]m^2L^2\kappa GA}{\rho AL^2\left[(EI)^*\pi^2m^2L^2+(EI)^*\pi^4m^4(e_0a)^2+AL^4\kappa G+(e_0a)^2A\pi^2m^2L^2\kappa G+I\pi^2m^2L^2\kappa G+I\right]\pi^4m^4\kappa G(e_0a)^2}. \tag{37}
$$

The first natural frequencies are $\omega_O^{(1)} = 1.951736705\cdot10^{12}$ rad and $\omega_C^{(1)} = 1.941921730\cdot10^{12}$ rad, respectively. As is seen, the difference is less than 0.51%. The second frequencies equal, respectively, $\omega_O^{(2)} = 4.726408202\cdot10^{12}$ rad and $\omega_C^{(2)} = 4.644847128\cdot10^{12}$ rad, difference being 1.76%. For the third frequency ($\omega_O^{(3)} = 7.116491920\cdot10^{12}$ rad, $\omega_C^{(3)} = 7.009604822\cdot10^{12}$ rad) the percentagewise difference constitutes 1.53%.

As is seen, the consistent set yields nearly the same fundamental natural frequency as the original set, but the equation is much simpler than the one associated with original Bresse-Timoshenko theory. Moreover, as shown in Ref. [4], the simpler set is also more consistent than the original one.

7. Conclusion

In this paper we propose a consistent set of nonlocal Bresse-Timoshenko equations for nonlocal beams with or without surface stress effects. We demonstrate that for the lower end of natural frequencies the consistent set and original set yield close values for natural frequencies. Moreover, the suggested governing equation is simpler than that associated with original equations extended to include nonlocal and surface effects.

Our conclusions are compatible with the results of study by Sadeghian *et al* [10] who conclude that "surface stress can cause significant difference in resonant frequency compared to the value obtained by classical beam theory."

It is hoped that equations developed in this study will find wide application in engineering research and practice.

მექანიკა

ნანოღეროების არალოკალური დაზუსტებული თეორია ზედაპირული ეფექტებით

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მოცემულ გამოკვლევაში შემოთავაზებულია დიფერენციალური განტოლებები ღეროებისათვის რომლებიც ითვალისწინებენ განივ დეფორმაციებს, ბრუნვის ინერციას, ლოკალურობას და ზედაპირული ძაბვის გავლენას. ნაჩვენებია, რომ ჩვენს მიერ შემოთავაზებული განტოლება არის უფრო მარტივიც და თავსებადიც ვიდრე შესაბამისი ბრესსე-ტიმოშენკოს განტოლებები და მასში ჩართულია ლოკალურობის და ზედაპირული ძაბვის ეფექტი. შემოთავაზებული განტოლება შეიცავს 11 წევრს გადაადგილების ვექტორის მიმართ ნაცვლად 19 წევრისა, რომლებიც მონაწილეობენ ბრესსე-ტიმოშენკოს განტოლებების პირდაპირ განზოგადებაში, რაც ითვალისწინებს არალოკალურობას და ზედაპირულ ეფექტებს.

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Physics

Delta-Like Singularity in the Radial Laplace Operator and the Status of the Radial Schrödinger Equation

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ABSTRACT. By careful exploration of separation of variables into the Laplacian in spherical coordinates, we obtain the extra delta-like singularity, elimination of which restricts the radial wave function at the origin. This constraint has the form of boundary condition for the radial Schrödinger equation. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *Laplace operator, radial equation, boundary condition, singular potentials.*

1. Introduction.

It is well known that the Laplace operator appears in many physical as well as of mathematical problems. Especially in quantum mechanics the dynamics of any physical system is described by the three-dimensional Schrödinger equation [1,2]

$$\Delta\psi(\vec{r})+2m\left[E-V(r)\right]\psi(\vec{r})=0. \qquad (1)$$

The central potential $V(\vec{r})=V(|\vec{r}|)\equiv V(r)$ is frequently encountered in most interesting physical problems, therefore reduction to the one-dimensional (radial) equation is a widespread procedure.

The traditional way is the application of the substitution $\psi(\vec{r})=R(r)Y_l^m(\theta,\varphi)$, where $Y_l^m(\theta,\varphi)$ is the spherical harmonics and because of the continuity and uniqueness, orbital quantum numbers l are integers, $l=0,1,2,...,$ whereas $m=-|l|,...,l.$

After this substitution angular variables are separated and we are left with the equation for the full radial function $R(r)$

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+2m\left[E-V(r)\right]R-\frac{l(l+1)}{r^2}R=0. \qquad (2)$$

It is a traditional trick in quantum mechanics to avoid the first derivative term from this equation by substitution

$$R(r)=\frac{u(r)}{r} \qquad (3)$$

after which a naïve calculation gives the equation for the new radial wave function $u(r)$ in the form

$$\frac{d^2u(r)}{dr^2}-\frac{l(l+1)}{r^2}u(r)+2m\left[E-V(r)\right]u(r)=0. \qquad (4)$$

It is this equation that plays an important role in quantum mechanics since its birth. However, as has been clarified in recent years, there is an ambiguity in derivation of boundary condition for $u(r)$ at the origin r=0, especially in the case of singular potentials [3-5].

According to this reason many authors content themselves by consideration only of a square integrability of radial function and do not pay attention to its behavior at the origin. Of course, this is permissible mathematically and the strong theory of linear differential operators allows for such an approach [6-8]. There appears the so-called self-adjoint extended (SAE) physics [9], in the framework of which among physically reasonable solutions one encounters also many curious results, such as bound states in the case of repulsive potential [10] and so on. We think that these highly unphysical results are caused by the fact that without suitable boundary condition at the origin a functional domain for radial Schrödinger Hamiltonian is not restricted correctly..

Careful investigation, performed below, shows that the validity of radial equation (4) is not correctly established. Indeed, it is physically (and mathematically, of course) warranted that the equation, obtained after separation of variables, must be compatible with the primary equation. It is a necessary condition for the correctness of a separation procedure.

2. Rigorous derivation of radial equation.

In the case of reduction of Laplace operator the transition from Cartesian to spherical coordinates is not unambiguous, because the Jacobian of this transformation [11] $J = r^2 \sin\theta$ is singular at $r = 0$ and $\theta = n\pi\,(n = 0,1,2,...)$. The angular part is fixed by the requirement of continuity and uniqueness. This gives the unique spherical harmonics $Y_l^m(\theta,\varphi)$ mentioned above.

Note that in the reduction of Laplace operator it usually is pointed out that $r > 0$. However, $\vec{r} = 0$ is an ordinary point in the full Schrödinger equation (1), but it is a point of singularity in the reduction of variables. Thus, the knowledge of specific boundary behavior is necessary. We underline that the equation (2) is correct, but the substitution (3) enhances singularity at r=0 and may cause some misunderstandings.

Indeed, let us rewrite the full radial equation (2) after this substitution

$$\frac{1}{r}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)u(r)+u(r)\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)\left(\frac{1}{r}\right)+ +2\frac{du}{dr}\frac{d}{dr}\left(\frac{1}{r}\right)-\left[\frac{l(l+1)}{r^2}-2m\left(E-V(r)\right)\right]\frac{u}{r}=0. \quad (5)$$

We write the equation in this form deliberately, indicating the action of radial part of Laplacian on relevant factors explicitly. It seems that the first derivatives of $u(r)$ are cancelled and we are faced with the following equation

$$\frac{1}{r}\left(\frac{d^2u}{dr^2}\right)+u\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)\left(\frac{1}{r}\right)- -\frac{l(l+1)}{r^2}\frac{u}{r}+2m\left(E-V(r)\right)\frac{u}{r}=0. \quad (6)$$

Now if we differentiate the second term "naively", we'll derive zero. But it is true only in the case when $r \neq 0$. However, below we show that in general this term is proportional to the 3-dimensional delta function. Indeed, taking into account that

$$\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}=\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\equiv\Delta_r \quad (7)$$

is the radial part of the Laplace operator and therefore [12]

$$\Delta_r\left(\frac{1}{r}\right)=\Delta\left(\frac{1}{r}\right)=-4\pi\delta^{(3)}(\vec{r}) \quad (8)$$

we obtain the equation for $u(r)$

$$\frac{1}{r}\left[-\frac{d^2u(r)}{dr^2}+\frac{l(l+1)}{r^2}u(r)\right]+4\pi\delta^{(3)}(\vec{r})u(r)$$

$$-2m\left[E-V(r)\right]\frac{u(r)}{r}=0. \qquad (9)$$

We see that there appears the extra delta-function term. Its presence in the radial equation is physically nonsense and must be eliminated. Note that when $r \neq 0$, this extra term vanishes owing to the property of the delta function and if, in this case, we multiply this equation by r, we'll obtain an ordinary radial equation (4).

However if r=0, multiplication by r is not permissible and this extra term remains in Eq. (9). Therefore one has to investigate this term separately and find another ways to abandon it.

The term with 3-dimensional delta-function must be comprehended as being integrated over $d^3r = r^2 dr \sin\theta d\theta d\varphi$. On the other hand [12]

$$\delta^{(3)}(\vec{r})=\frac{1}{|J|}\delta(r)\delta(\theta)\delta(\varphi), \qquad (10)$$

where $J = r^2 \sin\theta$ is the Jacobian of transformation.

Taking into account all the above mentioned relations, one is convinced that the extra term still survives, but now in the one-dimensional form

$$u(r)\delta^{(3)}(\vec{r})d^3\vec{r} \to u(r)\delta(r)dr. \qquad (11)$$

Its appearance as a point-like source breaks many fundamental principles of physics, which is not desirable. The only reasonable way to remove this term without modifying Laplace operator or including compensating delta function term into the potential $V(r)$, is to impose the requirement

$$u(0)=0 \qquad (12)$$

(note that multiplication of Eq. (9) by r and then elimination of this extra term owing to the property $r\delta(r)=0$ is not a legitimate procedure, because effectively it is equivalent to multiplication by zero).

Therefore we conclude that the radial equation (4) for $u(r)$ is compatible with the full Schrödinger equation (1) if and only if the condition $u(0)=0$ is fulfilled. *The radial equation (4) supplemented by the condition (12) is equivalent to the full Schrödinger equation (1).* It is in accordance with the Dirac requirement [2], that the solutions of the radial equation must be compatible with the full Schrodinger equation. It is remarkable to see that the supplementary condition (12) has a form of boundary condition at the origin.

3. Comments, some applications and conclusions

Some comments are in order here: equation for $R(r)=\frac{u(r)}{r}$ has its usual form (2). Derivation of boundary behavior from this equation is as problematic as for $u(r)$ from Eq. (4). Problem with delta function arises only in the course of elimination of the first derivative. Now, after the condition (12) is established, it follows that the full wave function $R(r)$ is less singular at the origin than r^{-1}. However, this conclusion could be hasty because the transition to Eq. (4) for $R(r)$ is not necessary at all. It is also remarkable to note that the boundary condition (12) is valid whether potential is regular or singular. It is only the consequence of particular transformation of Laplacian. Different potentials can only the determined in the specific way of $u(r)$ tending to zero at the origin and the delta function arises in the reduction of the Laplace operator every time. All of these statements can easily be verified also by explicit integration of Eq. (9) over a small sphere with radius a tending to zero at the end of calculations.

It seems very curious that this fact has hitherto remained unnoticed by physicists in spite of numerous discussions [1, 2].

Apparently mathematicians knew about the singular behavior of Laplace operator for a long time. But their results did not find a relevant presentation in physical literature, while the delta function became popular only after Dirac. Therefore the fact, described above, seems to us as being very curious.

We discuss another important point with regard to radial Laplacian. It is well known from some books on special functions that there is the following operator relation [12]

$$\Delta_r = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\cdot\right) = \frac{1}{r}\frac{d^2}{dr^2}(r\cdot). \qquad (13)$$

Here the dot denotes the impact of this expression on some function. The validity of this relation is easily verified by direct calculation. But this equality fails at point $r=0$. Indeed, let us act by both sides on the full radial function $R(r)$:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{1}{r}\frac{d^2}{dr^2}(rR) = \frac{1}{r}\frac{d^2}{dr^2}u(r). \; (14)$$

It is this relation that is used in mathematical literature for special functions [13].

If it is true everywhere, then there does not appear any problem in the derivation of the radial equation. But now we know that after substitution of $R(r) = \frac{u(r)}{r}$ on the left-hand side it follows

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\frac{u}{r}\right) = \frac{1}{r}\frac{d^2u}{dr^2} - 4\pi\delta(\vec{r})u. \qquad (15)$$

Therefore previous operator equality must be modified perhaps as follows

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\cdot\right) = \frac{1}{r}\frac{d^2}{dr^2}(r\cdot) - 4\pi\delta^{(3)}(\vec{r})r\cdot. \; (16)$$

This relation is correct at every point including the origin. The validity of this relation may be checked by acting on $R(r)$, and then using equality $u = rR$.

The relation $u(0)=0$ is not only the boundary condition for the radial equation, but it is a relation which must be necessarily fulfilled in order to have the radial equation in its usual form compatible to the full Schrödinger equation. Accidentally it has a boundary condition form. Without this condition the radial equation is not valid.

Now, after this condition has been established, many problems can be considered rigorously by taking it into account. Remarkably, all the results obtained earlier for regular potentials with the boundary condition (12) remain unchanged. In most textbooks on quantum mechanics $r \to 0$ behavior is obtained from Eq. (4) in the case of regular potentials. When an equation like (4) is known, the derivation of boundary behavior from it is an almost trivial procedure. It depends on the behavior of the potential under consideration.

But we have shown that this equation takes place only together with a boundary condition (12). On the other hand, for *singular potentials* this condition will have far-reaching implications. Many authors neglected the boundary condition entirely and were satisfied only by square integrability. But in this treatment some parameters of wave functions go out of allowed regions and a self-adjoint extension procedure can yield unphysical results. Below we consider some simple consequences showing the differences which arise with and without the above mentioned boundary condition:

(i) Regular potentials at the origin:

$$\lim_{r\to 0} r^2V(r) = 0. \qquad (17)$$

In this case, after substitution at the origin of $u \sim r^s$, it follows from inicial equation that $s(s-1) = l(l+1)$, which gives two solutions $\underset{r\to 0}{u} \sim c_1 r^{l+1} + c_2 r^{-l}$ (see, any textbooks on quantum mechanics). For non-zero l -s the second solution is not square integrable and is usually ignored. But for $l=0$, many authors discuss how to deal with this solution [14], which is also square integrable near the origin. According to condition (12), this solution must be ignored. This result justifies the assumption made in the book of A.Messiah [15] about the behavior of the *s*-state wave function at the origin.

(ii) Transitive attractive singular potentials at the origin:

$$\lim_{r\to 0} r^2V(r) = -V_0 = const; \qquad V_0 > 0. \qquad (18)$$

In this case, the indicial equation takes the form $s(s-1) = l(l+1) - 2mV_0$, which has two solutions: $s = \frac{1}{2} \pm \sqrt{\left(l+\frac{1}{2}\right)^2 - 2mV_0}$. Therefore

$$\underset{r\to 0}{u} \sim c_1 r^{\frac{1}{2}+P} + c_2 r^{\frac{1}{2}-P}; \quad P = \sqrt{\left(l+\frac{1}{2}\right)^2 - 2mV_0}. \quad (19)$$

It seems that both solutions are square integrable at origin as long as $0 \le P < 1$. Exactly this range is studied in most papers (see for example [10]), whereas according to boundary condition (12) we have $0 \le P < \frac{1}{2}$. The difference is essential. Indeed, the radial equation has a form

$$u'' - \frac{P^2 - 1/4}{r^2} u + 2mEu = 0. \quad (20)$$

Depending on whether P exceeds 1/2 or not, the sign in front of the fraction changes and one can derive attraction in the case of repulsive potential and vice versa. Boundary condition (12) avoids this nonphysical region $\frac{1}{2} \le P < 1$.

Lastly, we note that the same holds for radial reduction of the Klein-Gordon equation, because in three dimensions it has the following form

$$\left(-\Delta + m^2\right)\psi(\vec{r}) = \left[E - V(r)\right]^2 \psi(\vec{r}) \quad (21)$$

and the reduction of variables in spherical coordinates will proceed in an absolutely same direction as in Schrödinger equation. Interestingly enough, something like that arises in classical electrodynamics [16] in calculations of electric dipole and magnetic fields, but cancels without any physical consequences. The situation in quantum mechanics differs because the extra delta term necessitates the restriction of the radial wave function.

ფიზიკა

დელტასმაგვარი სინგულარობა რადიალურ ლაპლასის ოპერატორში და შრედინგერის რადიალური განტოლების სტატუსი

ა. ხელაშვილი*, თ. ნადარეიშვილი**

** აკადემიის წევრი, ი.ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტის მაღალი ენერგიების ფიზიკის ინსტიტუტი; საქართველოს საპატრიარქოს წმინდა ანდრია პირველწოდებულის სახელობის ქართული უნივერსიტეტი, თბილისი*
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სფერულ კოორდინატებში ლაპლასის ოპერატორში ცვლადების განცალების პროცედურის კორექტულად ჩატარების შედეგად, მივიღეთ დამატებითი დელტასმაგვარი სინგულარობა, რომლის გამორიცხვაც ზღუდავს ტალღურ ფუნქციას სათავეში. ამ შეზღუდვას სასაზღვრო პირობის სახე აქვს შრედინგერის განტოლებისათვის.

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Physics

Fusion and Fission of Rare Radioactive Isotopes by Laser Driven Ions

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(Presented by Academy Member Nodar Tsintsadze)

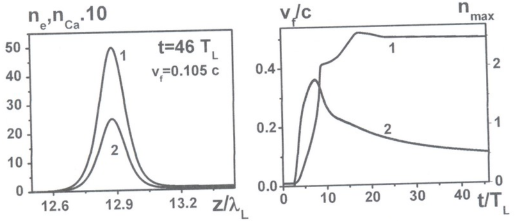
ABSTRACT. The dynamics of overdense ion bunches created by ultra-intense femtosecond laser pulses is studied in order to use them for the fusion of rare isotopes. It is possible to use them for creation of superheavy elements as well as for fission of radioactive nuclei. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *ion acceleration, laser-driven, nuclear fusion.*

Development of compact laser systems in recent years enabled generation of femtosecond pulses of $10^{22}\div10^{23}$ W/cm^2 intensities. Acceleration of ions up to relativistic velocities by the pressure of such pulses is considered as one of the most important use of these lasers [1-4]. Bunches of light ions obtained in this way can be accelerated up to very high (v~$0.4\div0.8$c) velocities. As opposed to the traditional accelerators, the time (t~$10\div20\ T_L$ where T_L is the period of the laser irradiation and is of femtosecond scale) and distance (l~$10\div15\lambda_L$, where T_L is the wavelength of the laser radiation and is of micron scale) for the acceleration in this case are very short. In addition, concentrations of bunches obtained in such way are much more than that in the ion fluxes generated in traditional accelerators. The incidence of ultraintense circularly polarized pulses on a thin solid foil compressed in advance at cryogenic temperatures (n_{i0}~$0.2\div4\cdot10^{23}$ cm^{-3}), leads to its further compression to very high concentrations (n_i~10^{25} cm^{-3}). The dimensions of these few GeV ion bunches are very small ($a_\perp$ ~$20\div50\ \lambda_L$, $a_\parallel$~ λ_L). The field of use of such bunches can be quite wide including creation of femtosecond sources of energetic neutrons, generation of fast ignition in laser confined fusion, proton imaging and oncology, formatting "flying" mirrors for generation monochromatic terahertz and x-rays generation, rare isotopes production and heavy ion collider [5-12].

In this work, the accelerating of bunches of different nuclei to the optimal energies for the fusion of expensive isotopes and decay of radioactive nuclei are considered. Possible application of laser-driven relativistic ion bunches for such purposes was mentioned in one of the pioneering works dedicated to acceleration of ions in this way [1].

The advantage of ion acceleration using pressure of laser radiation is well seen at obtaining superheavy



a) b)

Fig. 1. Acceleration and contraction of Ca^{48} plasma foil (thickness – $10\lambda_L$, n_e=18.6^2n_{cr}) by ultraintensive supergaussian (m=4) laser radiation (amplitude – 300, pulse duration – $4T_L$).
a) Concentrations of Ca^{48} isotope electrons (1) and ions (2) at t=46T_L;
b) Time dependences of Ca^{48} ions concentration maximum – n_{max} (1) and its velocity – v_f/c (2).

(Z>100) unstable elements. For fusion of such nuclei, thin foils of Pu^{242}, Pu^{244}, Am^{248}, Cm^{245}, Cm^{248}, Cf^{249}, Bk^{249} were radiated for 70 days by $7\cdot10^{12}$ s^{-1} flux of Ca^{48} [13-14]. The results of these experiments show that the number of generated superheavy nuclei is too small (for instance, only five Z=117 nuclei were observed after Bk^{249}+Ca^{48} reactions). This was caused by both low cross-section of the reactions (σ~1.3 ÷1.5 pb), and low concentration of the accelerated calcium ions. We have shown that better results can be expected at irradiation of thin foils (l_{Bk}~20μm) made of Ca^{48} with n_{i0}=1.9 · 10^{22} cm^{-3} concentration by ultraintense (e/mc^2, A_0=300) Nd (λ_L=1.06μm) short (t_L~10 T_L) circularly polarized narrow ($a_\perp$~40λ_L) pulses. In this case, formation of 252 MeV overdense n_i=0.95 · 10^{25} cm^{-3} bunch will take place (Fig.1) and the number of events of generation of superheavy nuclei at its interaction with thin (10 λ_L) Bk^{249} foil exceeds one.

$$N_R = n_R a_\perp^2 l_{Bk} = n_{Bk} a_\perp^2 l_{Bk} \int n_i \sigma v_i dt > 1 .$$

Obtaining of heavy nuclei in this way is not only much quicker, but also requires much less energy as well.

At the above evaluations, we used the experimental values of the nuclear reaction cross sections, which were obtained at colliding of charged nuclei of much less densities in traditional colliders. Our opinion is that it should be increased at collision of high concentration electrically neutral bunches, where the ions are enveloped by electronic coatings. Importance of such experiments is even greater as there is no information about the probability of many nuclear reactions due to the duration and expensiveness of the experiments needed for obtaining such information. Compact, fast and inexpensive mechanism of obtaining dense energetic ions accelerated by laser irradiation allows one to provide numerous experiments for establishing the cross sections of quite a wide spectrum of nuclear reactions. First of all, the probabilities of obtaining expensive and rare elements need to be defined and in some cases adjusted. The method of fusion is not in fact used currently for obtaining rare isotopes. Usually they are obtained in centrifuges by separation from the isotopes with similar atomic numbers. Often this is a rather time and energy consuming way, especially in the cases of heavy elements, which conditions their high prices. On the other hand, this was still more cost-effective than in nuclear fusion in traditional accelerators. Nuclear fusion by laser-operated relativistic ion bunches might be more convenient as well as more cost-effective. By the same token, in many cases, there is a possibility of chemical separation of heavy

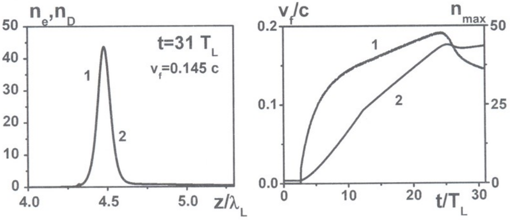


Fig. 2. Acceleration and contraction of deuterium plasma foil (thickness – $5\lambda_L$, n_e=18²n$_{cr}$) by ultraintensive supergaussian (m=20) laser radiation (amplitude – 60, pulse duration – $10T_L$).
a) Concentrations of deuterium electrons (1) and ions (2) at t=31T_L;
b) Time dependences of deuterium ions concentration maximum – n_{max} (1) and its velocity – v_f/c (2).

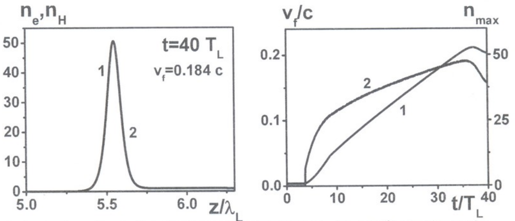


Fig. 3. Acceleration and contraction of hydrogen plasma foil (thickness – $7\lambda_L$, n_e=18²n$_{cr}$) by ultraintensive supergaussian (m=20) laser radiation (amplitude – 30, pulse duration – $30T_L$).
a) Concentrations of hydrogen electrons (1) and ions (2) at t=40T_L;
b) Time dependences of hydrogen ions concentration maximum – n_{max} (1) and its velocity – v_f/c (2).

nuclei obtained in such way from the others. As an example, we can consider the fusion of Os^{187} the most expensive isotope. If we bomb a thin (l_{Re}~20 μm) foil of Re^{187} by overdense (n_i=1.8 ·10^{25} cm^{-3}) bunch of deuterium that was accelerated to v_i~0.145c velocities by irradiation of 0.5 ·10^{22} W/cm^2 intensity (Fig.2) and t_L~14T_L duration (at these energies, the local maximum of the cross section is ~0.25b (Interpreted ENDF file "RE-187(D,2N)OS-187,SIG MAT=7531 MF=3 MT=16 Library: TENDL-2009), about 1% of it will be transferred in Os^{187}. If the width of the pulses is $40\lambda_1$, then the fusion of 1g Os^{187} would require about ten million pulses. The cost of Os^{187} produced in this way, will be much less than its current market price.

Similarly, at collision of deuterium and hydrogen nuclei with Np^{237}, there occur Pu^{238} and Pu^{236} fusion reactions. The cross sections are relatively high and reach 275 mb and 200 mb respectively, when the deuterium energy is 16.5 MeV, and that of hydrogen is 12.5 MeV (Fig.3) [15]. By adjusting the parameters of the pulses and foils, it is possible to create bunches which, after interaction with 10 μm Np^{237} target, will lead to transformation of about 2% of its nuclei to

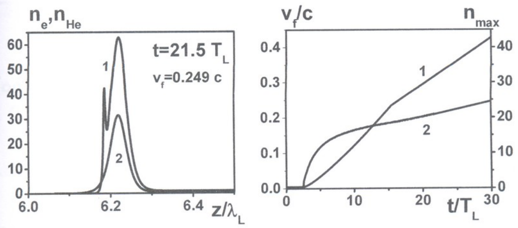


Fig. 4. Acceleration and contraction of He^4 plasma foil (thickness – $5\lambda_L$, $n_e=18^2 n_{cr}$) by ultraintensive supergaussian (m=20) laser radiation (amplitude – 85, pulse duration – $22T_L$).
a) Concentrations of He^4 isotope electrons (line 1) and ions (line 2) at $t=21.5T_L$;
b) Time dependences of He^4 ions concentration maximum – n_{max} (1) and their velocity – v_f/c (2).

plutonium.

Laser-driven ion bunches can be used also for decay of radioactive nuclei to neutralize them. This would allow security against radioactive dust. With this regard, let us first of all discuss the process of U^{235} and U^{238} decay to inactive nuclei. The cross-section of U^{235} decay reactions goes up to 2.7b at their interaction with α particles of 100 MeV energy, and up to 1.2b at their collision with protons of 100÷1000 MeV energy. Effective decay of U^{238} occurs at bombardment with 103 MeV energy α particles (σ=3.2b) [16]. We have found the conditions (Fig.4) when generation of superdense bunches of sufficient energies takes place. One such process leads to decay of 4÷5% of target nuclei.

As was shown in the above studies, at interaction of laser-driven overdense bunches with the target, effective nuclear reactions take place and the colliders based on such process can be used successfully for investigation of nuclear occurrences as well as for production of rare isotopes.

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ფიზიკა

ლაზერით აჩქარებული იონებით იშვიათი იზოტოპების სინთეზი და დაშლა

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ი. ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტის ე. ანდრონიკაშვილის ფიზიკის ინსტიტუტი

(წარმოდგენილია აკადემიის წევრის ნ. ცინცაძის მიერ)

შესწავლილია ულტრაინტენსიური ლაზერული გამოსხივების წნევის მოქმედებით ზემკვრივი პლაზმის იონების იშვიათი იზოტოპების მიღებისთვის საჭირო ოპტიმალურ ენერგიებამდე აჩქარების დინამიკა. ამ მეთოდით ზემძიმე ელემენტების სინთეზი მიიღწევა ბევრად უფრო მცირე დროში, ვიდრე ტრადიციულ ამაჩქარებლებში. რელატივისტური ზემკვრივი იონების გამოყენება შეიძლება რადიაქტიური ბირთვების დაშლისთვისაც.

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Physics

The Non-Perturbative Analytical Equation of State for the Gluon Matter

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(Presented by Academy Member Anzor Khelashvili)

ABSTRACT. In order to derive the equation of state for the pure $SU(3)$ Yang-Mills fields from first principles, it is proposed to generalize the effective potential approach for composite operators to non-zero temperature. It is essentially non-perturbative by construction, since it assumes the summation of an infinite number of the corresponding contributions. There is no dependence on the coupling constant, only a dependence on the mass gap, which is responsible for the large-scale structure of the QCD ground state. The equation of state generalizes the bag constant at non-zero temperature, while its nontrivial Yang-Mills part has been approximated by the generalization of the free gluon propagator to non-zero temperature, as a first necessary step. Even in this case we were able to show explicitly that the pressure may continuously change its regime at T* = 266.5 MeV. All the other thermodynamical quantities such as the energy density, entropy, etc. are to be understood to have drastic changes in their regimes in the close vicinity of T*. All this is in qualitative and quantitative agreement with thermal lattice QCD results for the pure Yang-Mills fields. We have firmly established the behaviour of all the thermodynamical quantities in the region of low temperatures, where thermal lattice QCD calculations suffer from big uncertainties. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *temperature, QCD, Yang-Mills, gluon, bag constant.*

I. Introduction

The equation of state (EoS) for the Quark-Gluon Plasma (QGP) has been derived analytically in QCD up to the order $g^6 \ln(1/g^2)$ by generalizing the perturbation theory (PT) method at non-zero temperature and density [1-4] and references therein). However, due to its non-analytical dependence on the QCD coupling constant g^2, nobody can trust its description of the QGP dynamics, apart from maybe at very high temperature. So there is an exact indication that the analytical EoS derived by thermal PT QCD is wrong.

At present, the only method to be used in order to investigate thermal QCD is the lattice QCD at finite temperature and baryon density which underwent a rapid recent progress [1,3,5-7] and references therein).

However, the lattice QCD, being a very specific regularization scheme, first of all is aimed at obtaining the well-defined corresponding expressions in order to get correct numbers and curves from them. So, one gets numbers and curves, but not understanding of what is the physics behind them. Such kind of understanding can only come from the dynamical theory which is continuous QCD. For example, any description of the QGP is to be formulated in the framework of the dynamical theory. The lattice thermal QCD is useless in this. The need in the analytical EoS remains, but, of course it should be essentially non-perturbative (NP), reproducing the thermal PT QCD results at a very high temperature only. Thus analytic NP QCD and lattice QCD approaches to finite-temperature QCD do not exclude each other, but on the contrary they should complement each other. Especially this is true for low temperatures where lattice QCD calculations suffer from big uncertainties [1,3,5-7]. There already exist interesting analytic approaches based on quasi-particle and liquid model pictures [8-17] to analyze the results of $SU(3)$ thermal lattice QCD calculations for the QGP EoS.

The formalism we are going to use in order to generalize it to non-zero temperature is the effective potential approach for composite operators [18-20]. It is essentially NP from the very beginning, since it is dealing with the expansion of the corresponding skeleton loop contributions (for a more detailed description see below). The main purpose of this paper is to derive EoS for the gluon matter by introducing the temperature dependence into the effective potential approach in a self-consistent way.

II. The Vacuum Energy Density

The quantum part of the vacuum energy density (VED) is determined by the effective potential approach for composite operators [18]. In the absence of external sources the effective potential is nothing but the VED. It is given in the form of the skeleton loop expansion, containing all the types of the QCD full propagators and vertices. So each vacuum skeleton loop itself is a sum of an infinite number of the corresponding PT vacuum loops, i.e., it contains the point-like vertices and free propagators (the figures of these expansions are explicitly shown in [20]). The number of the vacuum skeleton loops is equal to the power of the Planck constant $\hbar$.

Here we are going to formulate a general method of numerical calculation of the quantum part of the truly NP Yang-Mills (YM) VED in the covariant gauge QCD. The gluon part of the VED to leading order (the so-called log-loop level $\sim \hbar$) is given analytically by the effective potential for composite operators as follows [18]:

$$V(D) = \frac{i}{2}\int \frac{d^4q}{(2\pi)^4}\, Tr\{\ln(D_0^{-1}D) - D_0^{-1}D + 1\}, \qquad (1)$$

here $D(q)$ is the full gluon propagator and $D_0(q)$ is its free counterpart (see below). Traces over space-time and color group indices are assumed. Evidently, the effective potential is normalized to $V(D_0) = 0$. Next-to-leading and higher order contributions (two and more vacuum skeleton loops) are suppressed at least by one order of magnitude in powers of $\hbar$.

The two-point Green's function, describing the full gluon propagator, is $D_{\mu\nu}(q) = -i\{T_{\mu\nu}(q)d(-q^2,\xi) + \xi L_{\mu\nu}(q)\}(1/q^2)$, where ξ is the gauge-fixing parameter and $T_{\mu\nu}(q) = g_{\mu\nu} - (q_\mu q_\nu/q^2) = g_{\mu\nu} - L_{\mu\nu}(q)$. Its free counterpart $D_0 \equiv D^0_{\mu\nu}(q)$ is obtained by putting the full gluon form factor $d(-q^2,\xi)$ simply to one, i.e., $D^0_{\mu\nu}(q) = -i\{T_{\mu\nu}(q)d(-q^2,\xi) + \xi L_{\mu\nu}(q)\}(1/q^2)$.

In order to evaluate the effective potential (1), we use the well-known expression

$Tr\{\ln(D_0^{-1}D) = 8\times 4 \ln det(D_0^{-1}D) = 32 \ln[(3/4)\, d(-q^2,\xi) + (1/4)]$. Going over to four-dimensional Euclidean space in Eq. (1), one obtains ($\varepsilon_g = V(D)$)

$$\varepsilon_g = -16\int \frac{d^4q}{(2\pi)^4}\, \{\ln[1+3\, d(q^2,\xi)] - \frac{3}{4}\, d(q^2,\xi) + a\}, (2)$$

where the constant a = (3/4) – 2ln2 = -0.6363 and the integration over q^2 from zero to infinity is assumed. The VED ε_g derived in Eq. (2) is already a colorless quantity, since it has been already summed over color indices. Also, only the transversal ("physical") degrees of freedom of gauge bosons contribute to this equation (up to one skeleton loop order). So, there is no need for ghosts to cancel their longitudinal (unphysical) counterparts because of a normalization condition in this case.

However, the derived expression (2) remains rather formal, since it suffers from different types of PT contributions ("contaminations"). In order to define correctly the truly NP VED, let us make first the identical transformation of the full effective charge in Eq. (2) as follows: $d(q^2,\xi) = d(q^2,\xi) - d^{PT}(q^2,\xi) + d^{PT}(q^2,\xi) = d^{NP}(q^2) + d^{PT}(q^2,\xi)$, where $d^{PT}(q^2,\xi)$ correctly describes the PT structure of the full effective charge $d(q^2,\xi)$, including its behaviour in the ultra-violet (UV) limit compatible with asymptotic freedom (AF), otherwise remaining arbitrary. On the other hand, $d^{NP}(q^2)$ is assumed to reproduce correctly the NP structure of the full effective charge, including its asymptotic behaviour in the deep infrared (IR) limit. Evidently, both terms are valid in the whole energy/momentum range, i.e., they are not asymptotics. Let us also emphasize the principal difference between $d^{PT}(q^2,\xi)$ and $d^{NP}(q^2)$. The former is NP quantity "contaminated" by PT contributions, while the latter, being also NP, is, nevertheless, free of them. Thus the separation between the truly NP effective charge $d^{NP}(q^2)$ and its nontrivial PT counterpart $d^{PT}(q^2,\xi)$ is achieved. For example, if the full effective charge explicitly depends on the scale responsible for the truly NP dynamics in QCD, say Λ^2_{NP}, then one can define the subtraction $d^{NP}(q^2, \Lambda^2_{NP}) = d(q^2, \Lambda^2_{NP}) - d(q^2, \Lambda^2_{NP} = 0) = d(q^2, \Lambda^2_{NP}) - d^{PT}(q^2)$, which is obviously equivalent to the previous decomposition.

III. Generalization to Non-zero Temperature

Substituting the above-discussed exact decomposition into Eq. (2), introducing further the effective scale squared, separating the NP region from the PT one (soft momenta from hard momenta), and omitting some algebraic rearrangement (for details see [20]), one obtains:

$$\varepsilon_{YM} = - B_{YM} + B_{YM}(T) + P_{YM}(T) \tag{3}$$

Here evidently $\varepsilon_g \equiv \varepsilon_{YM}$ and B_{YM} is the bag constant at zero temperature [20,21]. Also, $B_{YM}(T)$ and $P_{YM}(T)$ are explicitly given by the following expressions:

$$B_{YM}(T) = 16 \int^{q_{eff}} \frac{d^4 q}{(2\pi)^4} \cdot \left\{ \ln\left[1 + 3\alpha_s^{NP}(q^2)\right] - \frac{3}{4}\alpha_s^{NP}(q^2) \right\} \tag{4}$$

and $P_{YM}(T)$ has more complicate form, namely

$$P_{YM}(T) = -16 \int \frac{d^4 q}{(2\pi)^4} \{ [\ln [1 + 3\alpha_s^{PT}(q^2) + 3\alpha_s^{NP}(q^2)] - \frac{3}{4} [\alpha_s^{NP}(q^2) + \alpha_s^{NP}(q^2)] + a\} \tag{5}$$

respectively, since it depends on both effective charges, $\alpha_s^{NP}(q^2) \equiv d^{NP}(q^2)$ and $\alpha_s^{PT}(q^2) \equiv d^{PT}(q^2)$. Precisely these expressions should be generalized to non-zero temperatures in order to get EoS for the pure YM fields. That is why we introduce the dependence on the temperature T in advance. Evidently, Eq. (4) will reproduce the temperature-dependent bag constant. In the expression for $P_{YM}(T)$ the integration is from zero to infinity, while in the integral for $B_{YM}(T)$ it is from zero to the effective scale squared q^2_{eff}, which is symbolically shown in Eq. (4). It is worth emphasizing that a so defined bag constant (4) is free of all types of PT contributions ("contaminations"), as it is required (this was a reason for the above-mentioned algebraic rearrangements and subtractions, see [20-22] and references therein).

The problem remaining to solve is to choose the truly NP effective charge $\alpha_s^{NP}(q^2)$. For the different truly NP effective charges we will get different analytical and numerical results. That is why the choice for its explicit expression should be physically and mathematically well justified. Let us choose the truly NP effective charge as follows:

$$\alpha_s^{NP}(q^2) = \frac{\Lambda_{NP}^2}{q^2}, \qquad (6)$$

where Λ_{NP}^2 is the mass scale parameter (the mass gap) responsible for the large-scale structure of the true QCD vacuum. It is well known that in continuous QCD it leads to a linear rising potential between heavy quarks, "seen" by lattice QCD [23] as well ($(q^2)^{-2}$-type behaviour for the full gluon propagator). Moreover, in [24-25] (see references therein as well) it has been explicitly shown that it is a direct nonlinear iteration solution of the transcendental equation for the full gluon propagator in the presence of a renormalized mass gap (see also Ref. [26]). The separation between the truly NP and the PT effective charges is both exact and unique, since the PT effective charge is always regular at zero, while the truly NP effective charge is singular at the origin. Let us also note that the chosen effective charge (6) does not depend explicitly on the gauge choice. It has been already used [20,21] in order to calculate the bag constant, which turned out to be in a very good agreement with such an important phenomenological parameter as the gluon condensate. It leads to many other desirable properties for the bag pressure at zero temperature [20]. Thus, our choice (6) is physically justified and mathematically confirmed.

In the imaginary time formalism [1,27] these expressions can be easily generalized to non-zero temperatures $T \equiv \beta^{-1}$ according to the prescription (let us recall that there is already Euclidean signature)

$$\int \frac{dq_0}{2\pi} \to T \sum_{n=-\infty}^{+\infty},$$

$$q^2 = \boldsymbol{q}^2 + q_0^2 = \boldsymbol{q}^2 + \omega_n^2 = \omega^2 + \omega_n^2, \quad \omega_n = 2\pi n T, \qquad (7)$$

i.e., each integral over q_0 of the loop momentum is to be replaced by the sum over Matsubara frequencies labeled by n, which obviously assumes the replacement $q_0 \to \omega_n = 2\pi n T$ for bosons (gluons). In frequency-momentum space the truly NP effective charge becomes

$$\alpha_s^{NP}(q^2) = \alpha_s^{NP}(\boldsymbol{q}^2, \omega_n^2) = \frac{\Lambda_{NP}^2}{\omega^2 + \omega_n^2},$$

$$\alpha_s^{PT}(q^2) = \alpha_s^{PT}(\boldsymbol{q}^2, \omega_n^2) = \alpha_s^{PT}(\omega^2, \omega_n^2). \qquad (8)$$

Here and everywhere below $\omega = (\boldsymbol{q}^2)^{1/2}$ and $\boldsymbol{q}^2$ is the three-dimensional loop momentum squared in complete agreement with the relations (7).

IV. The Derivitons of $B_{YM}(T)$ and $P_{YM}(T)$

In frequency-momentum space the bag pressure (4) after the substitution of the expressions (7) and (8) becomes:

$$B_{YM}(T) = 16 \int \frac{d^3q}{(2\pi)^3} T \sum_{n=-\infty}^{+\infty} \Big[\ln\left[3\Lambda_{NP}^2 + \boldsymbol{q}^2 + \omega_n^2\right] - \ln\left[\boldsymbol{q}^2 + \omega_n^2\right] - \frac{3}{4}\Lambda_{NP}^2 \left(\boldsymbol{q}^2 + \omega_n^2\right)^{-1} \Big], \qquad (9)$$

where the dependence on the effective scale ω_{eff} is omitted (see below), for simplicity. Here it is also convenient to introduce the following notation: $\omega' = (\boldsymbol{q}^2 + m'^2_{\text{eff}})^{1/2} = (\boldsymbol{q}^2 + 3\Lambda_{NP}^2)^{1/2} = (\omega^2 + 3\Lambda_{NP}^2)^{1/2}$. So it is possible to say that we have two sorts of gluons: massless ω and massive ω' with the effective mass $m'_{\text{eff}} = \sqrt{3}\Lambda_{NP}^2$. In this case the summation over the Matsubara frequencies ω_n can be easily done, as well as performing an almost trivial integration over angular variables.

Due to the above-mentioned normalization of the effective potential approach in Eq. (1), the investigation of the YM part (5) of the future gluon matter EoS makes sense to begin with putting first $\alpha_s^{PT}(q^2)=1$, i.e., approximating the nontrivial PT effective charge by its free PT counterpart. Then on account of the relations (7), the YM pressure (5) in frequency-momentum space becomes

$$P_{YM}(T) = -16 \int \frac{d^3q}{(2\pi)^3} T \sum_{n=-\infty}^{+\infty} \Big[\ln\left[\frac{3}{4}\Lambda_{NP}^2 + \boldsymbol{q}^2 + \omega_n^2\right] + \ln\left[\boldsymbol{q}^2 + \omega^2\right] - \frac{3}{4}\Lambda_{NP}^2 \left(\boldsymbol{q}^2 + \omega_n^2\right)^{-1} \Big]. \qquad (10)$$

Comparing Eqs. (9) and (10) one can write down the final result directly. For this purpose, in the final evaluation of Eq. (9) one must change the overall sign and replace ω' by $\bar{\omega} = (\omega^2 + (3/4)\Lambda_{NP}^2)^{1/2}$. We should also integrate from zero to infinity. All the aspects of these derivations can be found in [22] in detail.

V. The Gluon Matter EoS

Denoting further $\varepsilon_{YM}(T) + B_{YM} = P_{GM}(T)$ in the left-hand-side of our EoS (3) and summing up all the results of the summation over the Matsubara frequencies in the expressions (9) and (10), one obtains that EoS (3) finally becomes

$$P_{GM}(T) = \frac{6}{\pi^2}\Lambda_{NP}^2 P_1(T) +$$

$$\frac{16}{\pi^2} T\{P_2(T) + P_3(T) - P_4(T)\}, \qquad (11)$$

where the dependence on the thermodynamical variable T is only shown explicitly and

$$P_1(T) = \int_{\omega_{eff}}^{\infty} d\omega \frac{\omega}{e^{\beta\omega} - 1}$$

$$P_2(T) = \int_{\omega_{eff}}^{\infty} d\omega\ \omega^2 \ln[1 - e^{\beta\omega}]$$

$$P_3(T) = \int_{0}^{\omega_{eff}} d\omega\ \omega^2 \ln[1 - e^{\beta\omega'}]$$

$$P_4(T) = \int_{0}^{\infty} d\omega\ \omega^2 \ln[1 - e^{\beta\bar{\omega}}]. \qquad (12)$$

In the formal PT limit $\Lambda_{NP}^2 = 0$ it follows $\bar{\omega} = \omega' = \omega$ and thus the gluon matter pressure (11) in this limit vanishes, i.e., it is truly NP, indeed. The effective potential has been normalized to zero in the $D \to D_0$ limit, which reproduces the case of the so-called Stefan-Boltzmann (SB) non-interacting (ideal) gas of massless particles (gluons) at high temperatures [1]. So the SB limit $P_{SB}(T) = (8/45)\pi^2 T^4$ can be added (if necessary) to the truly NP pressure (11) in the $T \to \infty$ ($\beta \to 0$) limit only.

Other thermodynamical quantities. In quantum statistics the thermodynamical potential $\Omega(T)$ is nothing but the pressure $P(T)$ apart from the sign, so in our case we can put $\Omega(T) = -P_{GM}(T)$. In the quantum statistical theory all the important quantities such as energy density, entropy, etc. are to be expressed in terms of the thermodynamical potential. So the general formulae to be used are [1]: $\varepsilon(T) = -T\frac{\partial\Omega(T)}{\partial T} + \Omega(T)$, $s(T) = -\frac{\partial\Omega(T)}{\partial T}$, $c_V(T) = \frac{\partial\varepsilon(T)}{\partial T} = T\frac{\partial s(T)}{\partial T}$ for the pure YM fields, i.e., when the chemical potential is equal to zero. Evidently, here and everywhere below $\varepsilon(T)$ and $s(T)$ are the energy density and entropy, respectively, of the pure NP gluon matter, and one of the interesting thermodynamical characteristics of the QGP is the heat capacity $c_V(T)$. Their corresponding SB limits are: $\varepsilon_{SB}(T) = (24/45)\pi^2 T^4$, $s_{SB}(T) = (32/45)\pi^2 T^3$, $c_V^{SB}(T) = (96/45)\pi^2 T^3$ which should be added to our expressions in the high temperature $T \to \infty$ ($\beta \to 0$) limit only.

VI. The Scale-setting System

From the relations (7) it follows that in frequency-momentum space a possible free parameter of our approach is the effective scale $\omega_{eff} = \sqrt{q_{eff}^2 - \omega_c^2}$, where we introduced the constant Matsubara frequency ω_c, which is always positive. So ω_{eff} is always less than or equal to q_{eff} of the four-dimensional QCD, i.e., $\omega_{eff} \leq q_{eff}$. One then can conclude that q_{eff} is a very good upper limit for ω_{eff}. In this connection, let us recall now that the bag constant B_{YM} at zero temperatures has been successfully calculated at a scale q_{eff} = 1 GeV^2, in fair agreement with other phenomenological quantities such as gluon condensate [20]. So ω_{eff} is fixed as follows: $\omega_{eff} = q_{eff} = 1$ GeV^2. The mass gap squared

Λ_{NP}^2 calculated just at this scale is equal to Λ_{NP}^2 = 0.4564 GeV^2 [20]. Thus, we have no more free parameters in our approach. The confinement dynamics is nontrivially taken into account directly through the mass gap and the bag constant itself.

VII. Numerical Results And Discussion

All our numerical results are shown in Fig. 1. It is seen explicitly that the NP gluon pressure may continuously change its regime in the close neighborhood of a maximum at T^* = 266.5 MeV (which is obtained after the parameters ω_{eff} and Λ_{NP}^2 have been fixed) in order to achieve the thermodynamical SB limit at high temperatures. For the displayed quantities in Fig. 1 the SB limits are the corresponding constants. At the same time, for all the other thermodynamical quantities such as the energy density, entropy and heat capacity this is impossible (none of their power-type fall off at this point can be smoothly transformed into constant behaviour at high temperatures). In order to achieve the thermodynamical SB limits at high temperatures their full counterparts should undergo drastic changes in their regimes in the close neighborhood of this point. As we already know from thermodynamics of $SU(3)$ lattice QCD [1], [2], [28] the energy and entropy densities have a discontinuity at about T_c =260 MeV, while the pressure remains continuous. Our characteristic temperature T^*=266.5 MeV is, surprisingly, very close to the same value. A clear evidence that something nontrivial in the behaviour of the thermodynamical quantities in the vicinity of our characteristic temperature T^* = 266.5 MeV should actually take place follows from the fact that at this point $\varepsilon = 3P$, which should be valid at very high temperatures only (SB limit). In other words, in order to derive EoS valid above T^*, and thus to provide a correct picture of thermodynamics of the gluon matter in the whole range of temperature, one needs a nontrivial approximation of the YM part (5), compatible with the asymptotic freedom phenomenon in QCD. This will be the subject of the subsequent paper.

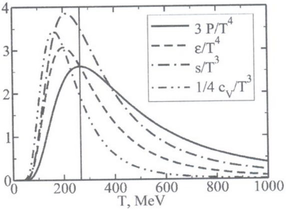

Fig. 1. The NP pressure P, energy density ε, entropy s and heat capacity c_V as functions of the temperature. The NP gluon pressure P has a maximum at T^* = 266.5 MeV.

It is worth emphasizing that we have no problems in describing and predicting the behaviour of all the important thermodynamical quantities at low temperatures below T^* (see Fig. 1). We do not expect any serious changes in the behaviour of the thermodynamical quantities in this region (exponential fall off or rise when the temperature goes down or up, respectively) even after taking into account the above-mentioned nontrivial approximation of the YM part (5), apart from the "non-physical" maximums which should disappear, of course. However, whatever changes may occur they will be under our control.

The confinement dynamics (6), generalized to non-zero temperatures in Eq. (8), is still important especially in the region of low temperatures even up to the temperature at which all the important thermodynamical quantities may undergo drastic changes in their behaviour (apart from pressure). From the structure of our EoS (see Eqs. (11)-(12)), it clearly follows that we have two massive gluonic excitations ω' and $\bar{\omega}$. The former can be interpreted as glueballs with masses $m'_{eff} = \sqrt{3}\ \Lambda_{NP}$ = 1.17 GeV, while the latter as gluons with effective masses $\bar{m}_{eff} = \frac{\sqrt{3}}{2}\Lambda_{NP}$ = 0.585 GeV. We have also two massless gluonic excitations propagating in accordance with the two first integrals in Eq.~(12). However, they are not free since in the formal PT $\Lambda^2_{NP} = 0$ limit they vanish. So all our massive and massless gluonic excitations are of the NP dynamical origin. At the same time, the generalization of our formalism to non-zero temperature in order to introduce into the consideration topological objects like instantons and related issues (for example, tunneling) would be of great interest [29,30] (work is in progress).

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ფიზიკა

გლუონის მატერიის მდგომარეობის არაპერტურბაციული ანალიზური განტოლება

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ეფექტური პოტენციალის მიახლოება შედგენილი ოპერატორებისთვის განზოგადებულია არანულოვანი ტემპერატურებისთვის და მისი მეშვეობით, პირველადი პრინციპებიდან გამომდინარე, მიღებულია მდგომარეობის განტოლება SU(3) იანგ-მილსის ველებისთვის. ის არსებითად არაპერტურბაციული ხასიათისაა, რადგან უსასრულო რაოდენობის წევრების აჯამვას გულისხმობს. იგი დამოკიდებულია არა ბმის მუდმივაზე, არამედ მასურ ღრეჩოზე, რომელიც პასუხისმგებელია კვანტური ქრომოდინამიკის ძირითადი მდგომარეობის სტრუქტურაზე დიდ მანძილებზე. მდგომარეობის განტოლება არის ჩანთის მოდელის მუდმივას განზოგადება არანულოვანი ტემპერატურებისთვის, მაშინ როდესაც მისი იანგ-მილსის არატრივიალური ნაწილი აპროქსიმირებულია გლუონის თავისუფალი პროპაგატორით არანულოვანი ტემპერატურებისთვის, როგორც პირველი აუცილებელი ნაბიჯი. ამ შემთხვევაშიც კი ჩვენ შევძელით ცხადად გვეჩვენებინა, რომ წნევა უწყვეტად იცვლის თავის რეჟიმს T^*=266.5MeV ტემპერატურაზე. ყველა სხვა თერმოდინამიკური სიდიდეები, როგორიცაა ენერგიის სიმკვრივე, ენტროპია და სხვა, მკვეთრად იცვლიან თავიანთ რეჟიმებს T* ტემპერატურის მახლობლობაში. ყოველივე ეს თვისებრივ და რაოდენობრივ თანხმობაშია თერმული კვანტური ქრომოდინამიკის მესერული მიახლოების შედეგებთან წმინდა იანგ-მილსის ველებისთვის. ჩვენ მკაფიოდ დავადგინეთ ყველა თერმოდინამიკური სიდიდე დაბალი ტემპერატურების არეში, სადაც მესერეული მიახლოების გამოთვლები აწყდებიან დიდ არაცალსახობებს.

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Physics

An Investigation of Bound *qqq*-Systems on the Basis of Salpeter Equation in the Framework of Simple Approach with Use of Expansion in Terms of Hyperspherical Harmonics

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ABSTRACT. The approach is developed to the solution of a problem of three bound constituent quarks (baryon) on the basis of Salpeter equation with two required 8-component spinors, having clear physical sense in the meaning of a particle-antiparticle (baryon-antibaryon), without so-called relativization of full wave function. The doubtful character of consideration of two-particle interaction under quark confinement conditions is stressed. It is proposed to use expansion in terms of hyperspherical harmonics for calculations of compact bound systems with three-particle interactions. Two elementary types of the central three-particle interaction - linear and oscillatory potentials - are considered. The approach proposed in this paper will be applied numerically to light baryon calculations. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *bound three-quarks systems, baryons, Salpeter equation, 8-component spinors, three-particle linear and oscillatory potentials, hyperspherical harmonics.*

Advances in nonrelativistic quark models for description of the bound states of *qqq*-systems are well-known [1]. However relatively large binding energies of light baryons (N, Σ, Λ, Ξ, Δ, Ω), consisting of constituent quarks (u, d, s), lead to the necessity of accounting for relativism. The relativistic-covariant approach is realised within the framework of Bethe-Salpeter equation [2] generalized to three particles [3]. The homogeneous integral B-S equation for the bound states of quarks is derived from the first principles – the 4-dimensional six-point Green's function has a pole at energy equal to bound state (baryon) mass [4-6]. Therefore quark confinement (lack of the free spectator quark, or lack of asymptotic states of the inhomogeneous B-S equation) does not interfere with the statement of the equation for three bound quarks, rather it specifies the basic role of three-particle interactions in baryons. However, problems related to probabilistic interpretation and normalization of a wave function arise. The absurdity of wave function dependence on relative time of particles is precisely enough and wittily described by the following expression: "Electron today and proton tomorrow do not form the bound state – a hydrogen atom" (see e.g. [7]). As with two particles [6], these problems are overcome at the instantaneous approximation excluding the B-S equation kernel dependence on relative energy variables in momentum space. It is very important that the obtained 3-dimensional Salpeter equation remains relativistic-invariant [8] and the wave function gains usual probabilistic sense. At the same time, in a coordinate space the instantaneous approximation arranges all three quarks on a spacelike hypersurface, interquark interaction takes a potential character, i.e. interaction propagates with infinite velocity and effects of retardation are formally missing. The conventional two-particle forces (considered for either reasons in the kernel of 4-dimensional B-S equation [8]) in Salpeter equation have the formal character, because of quark confinement they really are three-particle. At first it is necessary to make the instantaneous approximation in the kernel of 4-dimensional B-S equation, and then to talk (or not to talk!) about multiparticleness of interquark interactions, rather than the reverse. The principal virtues of Salpeter equation – relativistic invariance and simultaneous affinity of potential reviewing to a nonrelativistic picture, make this equation especially attractive for the description of baryons. At the same time the present state of QCD does not give the possibility of constructing the kernel of B-S equation and consequently we are forced to choose it phenomenologically.

The basic role of three-particle forces in calculations of baryons together with phenomenologycal choice of Salpeter equation's kernel pushes us to use expansions (for required wave functions) in terms of hyperspherical harmonics (HH) most natural for this case [9-12]. In particular, using solutions of a nonrelativistic Schrödinger equations with oscillatory three-particle potential along with simplification of analytical calculations we hope to achieve fast convergence in specific numerical calculations. In a sense it is possible to consider this paper as prolongation of early examinations for $q\bar{q}$ systems [13-18].

Proceeding from B-S equation for three bound quarks [2, 3] and using an average procedure over energy variables [19], at first we shall come to quasipotential equations and then in instantaneous approximation (in center-of-mass system: $\vec{K}=\vec{k}_1+\vec{k}_2+\vec{k}_3=0$; $\vec{k}_1$, $\vec{k}_2$ and $\vec{k}_3$ are quark 3-momenta) we shall obtain Salpeter equations:

$$\left[M+i\varepsilon-h_1(\vec{k}_1)-h_2(\vec{k}_2)-h_3(\vec{k}_3)\right]\Phi_M(\vec{\kappa})=I^{9/4}\hat{\Pi}(\vec{\kappa})\gamma_0^{(1)}\gamma_0^{(2)}\gamma_0^{(3)}\int V(M;\vec{\kappa},\vec{\kappa}')\left[d\vec{\kappa}'/(2\pi)^6\right]\Phi_M(\vec{\kappa}'). \quad (1)$$

In this equation the 6-dimensional vector $\vec{\kappa}=(\vec{q},\vec{p})$ with length $\kappa=\sqrt{q^2+p^2}$ is made up of Jacobi coordinates (in $\vec{\kappa}$-representation):

$$\left.\begin{aligned}\vec{q}&=\sqrt{(\mu_1+\mu_2)/(\mu_1\mu_2)}(\mu_2\vec{k}_1-\mu_1\vec{k}_2)/(\mu_1+\mu_2),\\ \vec{p}&=[\mu_3(\vec{k}_1+\vec{k}_2)-(\mu_1+\mu_2)\vec{k}_3]/\sqrt{\mu_3(\mu_1+\mu_2)},\end{aligned}\right\} \quad (2)$$

where $\mu_i=m_i/M_0$ $(M_0=m_1+m_2+m_3)$ is relative mass of constituent quark. The 6-volume element of momentum space is equal to

$$\left.\begin{aligned}&d\vec{\kappa}=d\vec{q}d\vec{p}=\kappa^5 d\kappa d\Omega_{\vec{\kappa}},\\ &d\Omega_{\vec{\kappa}}=\cos^2\alpha\sin^2\alpha\, d\alpha\, d\Omega_{\vec{q}}d\Omega_{\vec{p}},\\ &\int d\Omega_{\vec{\kappa}}=\pi^3,\ 0\le\alpha\le\pi/2.\end{aligned}\right\} \quad (3)$$

The projection operator

$$\hat{\Pi}(\vec{\kappa})=\Lambda_1^+(\vec{k}_1)\Lambda_2^+(\vec{k}_2)\Lambda_3^+(\vec{k}_3)+\Lambda_1^-(\vec{k}_1)\Lambda_2^-(\vec{k}_2)\Lambda_3^-(\vec{k}_3) \quad (4)$$

is expressed through usual one-particle projection operators:

$$\Lambda^\pm(\vec{k})=[\omega(\vec{k})\pm h(\vec{k})]/[2\omega(\vec{k})], \quad (5)$$

where $h(\vec{k})=\gamma_0\vec{\gamma}\cdot\vec{k}+\gamma_0 m$ is Dirac Hamiltonian of the free quark (γ_0, $\vec{\gamma}$ are Dirac matrices), $\omega(\vec{k})=\sqrt{m^2+\vec{k}^2}$. In the equation (1) $I=1/(\mu_1\mu_2\mu_3)^2$ represents the Jacobian of transformation from Cartesian 4-momenta k_1, k_2 and k_3 to Jacobi 4-momenta K, q and p. The relation between these 4-momenta is similar to 3-dimensional relations (2). Usually the kernel of interaction $V(M;\vec{\kappa},\vec{\kappa}')$ is chosen phenomenologically, mass-independent.

Acting on the equation (1) with operator $\left[M+i\varepsilon-h_1(\vec{k}_1)-h_2(\vec{k}_2)-h_3(\vec{k}_3)\right]^{-1}$ at the left, and using the relations

$$\left.\begin{aligned}&\gamma_0 h(\vec{k})=h(-\vec{k})\gamma_0, \qquad \gamma_0\Lambda^\pm(-\vec{k})=\Lambda^\pm(\vec{k})\gamma_0,\\ &h(\vec{k})\Lambda^\pm(\vec{k})=\pm\omega\Lambda^\pm(\vec{k}), \quad h^2(\vec{k})=\omega^2,\end{aligned}\right\} \quad (6)$$

we shall obtain another form of Salpeter equation:

$$\Phi_M(\vec{\kappa}) = I^{9/4}\{\Lambda_1^+(k_1)\Lambda_2^+(k_2)\Lambda_3^+(k_3)/(M+i\varepsilon-\omega_1-\omega_2-\omega_3)+$$
$$+\Lambda_1^-(k_1)\Lambda_2^-(k_2)\Lambda_3^-(k_3)/(M+\omega_1+\omega_2+\omega_3)\}\times\gamma_0^{(1)}\cdot\gamma_0^{(2)}\cdot\gamma_0^{(3)}\int V(M;\vec{\kappa},\vec{\kappa}')[d\vec{\kappa}'/(2\pi)^6]\Phi_M(\vec{\kappa}'). \qquad (7)$$

From here, taking into account properties of projection operators $\Lambda^{\pm}(\vec{k})$, it is easy to obtain very useful additional equations:

$$\Phi_M^{++-}(\vec{\kappa}) = \Phi_M^{+-+}(\vec{\kappa}) = \Phi_M^{-++}(\vec{\kappa}) = \Phi_M^{--+}(\vec{\kappa}) = \Phi_M^{-+-}(\vec{\kappa}) = \Phi_M^{--+}(\vec{\kappa}) = 0\,, \qquad (8)$$

$$\Phi_M^{+++}(\vec{\kappa}) = \Lambda_1^{\pm}(\vec{k}_1)\Lambda_2^{\pm}(\vec{k}_2)\Lambda_3^{\pm}(\vec{k}_3)\Phi_M(\vec{\kappa})\,. \qquad (9)$$

Further, representing 64-component spinor $\Phi_M(\vec{\kappa})$ in the form of the block column involving eight 8-component spinors $\Phi_1, \Phi_2, \ldots, \Phi_8$ and solving six (block) equations (8), it is possible to express the spinors Φ_2, Φ_3, Φ_4, Φ_5, Φ_6 and Φ_7 through two spinors Φ_1 and Φ_8:

$$\left.\begin{aligned}
\Phi_2 &= \{(1-\varepsilon_1\varepsilon_2)\lambda_3\Phi_1+(1+\varepsilon_3)\lambda_1\lambda_2\Phi_8\}/(1+\varepsilon_1\varepsilon_2\varepsilon_3),\\
\Phi_3 &= \{(1-\varepsilon_1\varepsilon_3)\lambda_2\Phi_1+(1+\varepsilon_2)\lambda_1\lambda_3\Phi_8\}/(1+\varepsilon_1\varepsilon_2\varepsilon_3),\\
\Phi_4 &= \{(1+\varepsilon_1)\lambda_2\lambda_3\Phi_1-(1-\varepsilon_2\varepsilon_3)\lambda_1\Phi_8\}/(1+\varepsilon_1\varepsilon_2\varepsilon_3),\\
\Phi_5 &= \{(1-\varepsilon_2\varepsilon_3)\lambda_1\Phi_1+(1+\varepsilon_1)\lambda_2\lambda_3\Phi_8\}/(1+\varepsilon_1\varepsilon_2\varepsilon_3),\\
\Phi_6 &= \{(1+\varepsilon_2)\lambda_1\lambda_3\Phi_1-(1-\varepsilon_1\varepsilon_3)\lambda_2\Phi_8\}/(1+\varepsilon_1\varepsilon_2\varepsilon_3),\\
\Phi_7 &= \{(1+\varepsilon_3)\lambda_1\lambda_2\Phi_1-(1-\varepsilon_1\varepsilon_2)\lambda_3\Phi_8\}/(1+\varepsilon_1\varepsilon_2\varepsilon_3),
\end{aligned}\right\} \qquad (10)$$

$$\lambda_i(\vec{k}_i) \equiv \vec{\sigma}_i\cdot\vec{k}_i/[\omega_i(\vec{k}_i)+m_i]\,, \quad \lambda_i\lambda_i = \varepsilon_i\,, \quad \varepsilon_i = (\omega_i - m_i)/(\omega_i + m_i)\,. \qquad (11)$$

Thus, determination of 64 components of spinor F_M is reduced to determination of 16 components of two 8-component spinors F_1 and F_8. This circumstance favourably distinguishes Salpeter equation from all other 3-dimensional relativistic equations. The normalizing condition of Salpeter amplitude F_M for the kernel $V(\vec{\kappa},\vec{\kappa}')$ (mass-independent) taking into account relations (10) has the following form:

$$\int[d\vec{\kappa}/(2\pi)^6]\mathcal{N}(\vec{\kappa})\left\{|\Phi_1(\vec{\kappa})|^2+|\Phi_8(\vec{\kappa})|^2\right\} = 2M_B, \qquad (12)$$

$$\mathcal{N}(\vec{\kappa}) = (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)/[I^{1/4}(1+\varepsilon_1\varepsilon_2\varepsilon_3)]. \qquad (13)$$

Let us pay attention to one interesting circumstance of sufficiently deep physical sense. As is known [20], the Feynman rule of round of poles $m \to m - i\varepsilon$ ($\varepsilon > 0$, $\varepsilon \to 0$)in the full propagator of the free quark leads to a possibility of a motion of particles in time both forward and back. The particle moving back in time is equivalent to an antiparticle moving forward. By the way, this rule maintains the covariance of theory as infinitesimal imaginary addition $-i\varepsilon$ is introduced into invariant mass m. In turn B-S amplitude dependence on relative time is caused by the existence of antiparticles. Motion forward-back in time makes essential configurations for which individual routes in time are different for the bound particles, and so the relative time is large [21, 8]. Projection operator $\hat{\Pi}(\vec{\kappa})$ in Salpeter equation (1) is that "relict" of an average of B-S equation, which corresponds to the contribution of these configurations to bound state amplitude. On the other hand, thanks to this operator in Salpeter equation it is possible to reduce the problem to determination only of two 8-component spinors. Below we shall make clear the physical sense of these spinors.

Charge conjugation $\mathcal{C}$, space parity $\mathcal{P}$ and time-reversal $\mathcal{T}$ operators for baryons are obtained by direct multiplication of corresponding one-particle quark operators and in $\vec{\kappa}$-representation operate as follows:

$$\mathscr{C}\Phi_M(\vec{\kappa}) = \gamma_0\gamma_2 \cdot \gamma_0\gamma_2 \cdot \gamma_0\gamma_2 \Phi_M^*(\vec{\kappa}) , \tag{14}$$

$$\mathscr{P}\Phi_M(\vec{\kappa}) = \gamma_0 \cdot \gamma_0 \cdot \gamma_0 \Phi_M(-\vec{\kappa}) , \tag{15}$$

$$\mathscr{T}\Phi_M(\vec{\kappa}) = -i\gamma_1\gamma_3 \cdot \gamma_1\gamma_3 \cdot \gamma_1\gamma_3 \Phi_M^*(-\vec{\kappa}) . \tag{16}$$

From here we obtain an action $\mathscr{C}\mathscr{P}\mathscr{T}$ -operator on $\Phi_M(\vec{\kappa})$:

$$\mathscr{C}\mathscr{P}\mathscr{T}\Phi_M(\vec{\kappa}) = -\gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \Phi_M(\vec{\kappa}) , \tag{17}$$

where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Therefore $\mathscr{C}\mathscr{P}\mathscr{T}$ -symmetry of a strong interaction will be expressed by the following commutation relation (see eq. (1)):

$$\left[\gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5 ,\, V(M;\vec{\kappa},\vec{\kappa}')\right] = 0 . \tag{18}$$

Besides, $\Phi_M(\vec{\kappa})$ solutions of Salpeter equation (1) have also a certain value of parity π since parity operator $\mathscr{P}$ commutes with interaction V:

$$\mathscr{P}\Phi_{M,\pi}(\vec{\kappa}) = \pi\, \Phi_{M,\pi}(\vec{\kappa}) . \tag{19}$$

Further in Salpeter equation (1) with negative value of mass $-M < 0$ we shall insert expressions $1 = (\gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5)(\gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5)$ before amplitudes $\Phi_{-M,\pi}(\vec{\kappa})$ and $\Phi_{-M,\pi}(\vec{\kappa}')$. Then we shall carry remaining expressions $\gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5$ up to the end to the left, using a commutator (18) and an anticommutator

$$\{\gamma_0\gamma_5, h(\vec{k})\} = 0 . \tag{20}$$

The "new" Salpeter equation (obtained as a result of these manipulations) already describes a baryon with the positive mass $M > 0$ and opposite parity $-\pi$:

$$\Phi'_{M,-\pi}(\vec{\kappa}) = \gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \Phi_{-M,\pi}(\vec{\kappa}). \tag{21}$$

Really, by means of relation $\gamma_0\gamma_5 = -\gamma_5\gamma_0$ and direct evaluations we shall obtain:

$$\mathscr{P}\Phi'_M(\vec{\kappa}) = -\pi\Phi'_M(\vec{\kappa}). \tag{22}$$

Thus, the solution (21) describes an antibaryon. Taking into account the zero anticommutator (20), from the relation (21) the following relations imply also:

$$\left.\begin{aligned} \Phi^{+++}_{M,-\pi}(\vec{\kappa}) &= \gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \Phi^{---}_{-M,\pi}(\vec{\kappa}), \\ \Phi^{---}_{M,-\pi}(\vec{\kappa}) &= \gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \Phi^{+++}_{-M,\pi}(\vec{\kappa}), \end{aligned}\right\} \tag{23}$$

where the summands of positive and negative energy ($\Phi^{+++}_M(\vec{\kappa})$ and $\Phi^{---}_M(\vec{\kappa})$) of the total Salpeter amplitude

$$\Phi_M(\vec{\kappa}) = \Phi^{+++}_M(\vec{\kappa}) + \Phi^{---}_M(\vec{\kappa}) \tag{24}$$

are definited according to relations (9), (8).

At the same time the component-wise structure of Salpeter amplitude $\Phi_M(\vec{\kappa})$ and relation (21) (with an explicit form of matrix $\gamma_0\gamma_5 \cdot \gamma_0\gamma_5 \cdot \gamma_0\gamma_5$) allow to explain the physical sense of 8-component spinors in the meaning of "particle-antiparticle":

$$\left.\begin{aligned}\Phi'_{M1,-\pi}(\vec{\kappa}) &= -\Phi_{-M8,\pi}(\vec{\kappa})\,,\\ \Phi'_{M8,-\pi}(\vec{\kappa}) &= \ \ \Phi_{-M1,\pi}(\vec{\kappa})\,.\end{aligned}\right\} \tag{25}$$

The baryon consisting of three constituent quarks is described by the total Salpeter amplitude $\Phi_M^{J^\pi M_J TM_T \mathscr{S}^*}$ having a quite definite quantum number set: JM_J – the total angular momentum and its projection, π – space parity, TM_T – the total isospin and its projection, $\mathscr{S}^*$ – strangeness. It is possible to become convinced that two 8-component spinors $\Phi_{M1,\pi}^{JM_JTM_T\mathscr{S}^*}$ and $\Phi_{M8,-\pi}^{JM_JTM_T\mathscr{S}^*}$ (which are components of the total amplitude) have almost the same quantum number set. Elimination is orbital parity. Here in the lower rows of indexes «π» and «$-\pi$» designate numerical values of orbital parity π'. In the former case numerically it is equal to the total space parity, and in the latter case it is opposite to it. All can be checked up acting with a parity operator on $\Phi_M(\vec{\kappa})$ (using an explicit form of matrix $\gamma_0\cdot\gamma_0\cdot\gamma_0$) and then line by line comparing the obtained expression with $\pi\cdot\Phi_M(\vec{\kappa})$.

Further we shall construct the required totally antisymmetric 8-component spinor $\Phi_{M1,\pi}^{JM_JTM_T\mathscr{S}^*}$ ($\Phi_{M8,-\pi}^{JM_JTM_T\mathscr{S}^*}$ can be constructed similarly; labels are borrowed from paper [8]):

$$\Phi_{M1,\pi}^{JM_JTM_T\mathscr{S}}(\vec{\kappa}) = \sum_{\mathscr{R}_L\mathscr{R}_S\mathscr{R}_F} \left\{\left\{\left\{\left[\psi^{\pi L}(\vec{\kappa})\right]_{\mathscr{R}_L}\cdot\left[\chi^S\right]_{\mathscr{R}_S}\right\}^{JM_J}\times\left[\varphi^{TM_T\mathscr{S}^*}\right]_{\mathscr{R}_F}\right\}_{\mathscr{S}}\cdot c_{\mathscr{A}}\right\}_{\mathscr{A}}. \tag{26}$$

Here $\left[\psi^{\pi L}(\vec{\kappa})\right]_{\mathscr{R}_L}$ is space wave function in $\vec{\kappa}$-representation with total orbital momentum L, orbital parity π (coinciding with baryon total parity) and symmetry $\mathscr{R}_L\in\{\mathscr{S},\mathscr{M}_{\mathscr{S}},\mathscr{M}_{\mathscr{A}}\}$ concerning permutations of quarks [22, 23]. $\left[\chi^S\right]_{\mathscr{R}_S}$ is 8-component (!) spin wave function (spinor) with total spin S and symmetry $\mathscr{R}_S\in\{\mathscr{S},\mathscr{M}_{\mathscr{S}},\mathscr{M}_{\mathscr{A}}\}$. $\left[\varphi^{TM_T\mathscr{S}^*}\right]_{\mathscr{R}_F}$ is flavour wave function with total isospin T and its projection M_T, with strangeness $\mathscr{S}^*$ and symmetry $\mathscr{R}_F\in\{\mathscr{S},\mathscr{M}_{\mathscr{S}},\mathscr{M}_{\mathscr{A}},\mathscr{A}\}$. $c_{\mathscr{A}}$ is a wave function describing a totally antisymmetric colour (colourless) singlet.

Let us pay attention to one circumstance. In the proposed approach there is no need for the so-called relativization of Salpeter amplitude $\Phi_M^{J^\pi M_JTM_T\mathscr{S}^*}(\vec{\kappa})$ [8] as its spinor (relativistic) structure is ensured with required spinors $\Phi_{M1,\pi}^{JM_JTM_T\mathscr{S}^*}(\vec{\kappa})$ and $\Phi_{M8,-\pi}^{JM_JTM_T\mathscr{S}^*}(\vec{\kappa})$, whose 8-component structure in turn is set by spin wave function $\left[\chi^S\right]_{\mathscr{R}_S}$. All "remaining" gives us a solution of Salpeter equation with required unknown quantities in space wave function $\left[\psi^{\pi L}(\vec{\kappa})\right]_{\mathscr{R}_L}$.

The most labour-consuming is construction of a space wave function with a given symmetry. It is obtained by means of the so-called Young's symmetrization operators [9] comprising quark permutations. We have HH $\phi_K^{l_{q_j}l_{p_j}LM_L}(\Omega_{\vec{\kappa}})$ depending on Jacobi coordinate set (2). They satisfy the equation:

$$\hat{K}^2\phi_K^{l_q l_p m_q m_p}(\Omega_{\vec{\kappa}}) = K(K+4)\phi_K^{l_q l_p m_q m_p}(\Omega_{\vec{\kappa}}),\ K=0,1,2,..., \tag{27}$$

$$\hat{K}^2 = -\partial^2/\partial\alpha^2 - 4ctg2\alpha(\partial/\partial\alpha) + \hat{l}_q^2/\cos^2\alpha + \hat{l}_p^2/\sin^2\alpha. \tag{28}$$

$\hat{K}^2$ is square of 6-dimensional orbital momentum $\hat{K}$ or an angular part of 6-dimensional Laplacian:

$$\Delta_{\vec{\kappa}} \equiv \partial^2/\partial\vec{\kappa}^2 = \partial^2/\partial\kappa^2 + (5/\kappa)(\partial/\partial\kappa) - \hat{K}^2/\kappa^2 . \tag{29}$$

HH are orthonormal functions:

$$\int\left(\phi_{K'}^{l'_q l'_p m'_q m'_p}(\Omega_{\vec{\kappa}})\right)^* \phi_K^{l_q l_p m_q m_p}(\Omega_{\vec{\kappa}})d\Omega_{\vec{\kappa}} = \delta_{K'K}\delta_{l'_q l_q}\delta_{l'_p l_p}\delta_{m'_q m_q}\delta_{m'_p m_p}. \tag{30}$$

Concrete expressions of HH can be found in [9]. However Young's symmetrization operators action on HH leads to need of having relation between HH depending on different Jacobi coordinate set obtained from starting set 1≡(123) by cyclical permutations of particles: 2≡(231), 3≡(312). Such relation is ensured with Reynal-Revai coefficients [24, 9]:

$$\phi_K^{l_{q_j} l_{p_j} LM_L}(\Omega_j) = \sum_{l_{q_i} l_{p_i}} \left\langle l_{q_i} l_{p_i} \middle| l_{q_j} l_{p_j} \right\rangle_{KL} \phi_K^{l_{q_i} l_{p_i} LM_L}(\Omega_i). \tag{31}$$

Because of invariancy of $\Delta_{\vec{\kappa}}$ concerning permutations of particles and rotations to Reynal-Revai coefficients conserved quantum numbers K and L are assigned. By means of relation (31) it is possible to express orbital wave function $\left[\psi^{\pi L}(\vec{\kappa})\right]_{\mathscr{R}_L}$ through a set of one type basis HH that considerably simplifies calculations. However per se HH $\phi_{KL}^{l_1 l_2}$ have no certain symmetry with respect to permutations group S_3, i.e. they are not basis functions of representation of this group. Thus there is a problem of constructing from $\phi_{KL}^{l_1 l_2}$ such complete and orthonormal basis $\phi_{KL}^{\{f\}\mu\sigma}$ which would be the basis of representation S_3. Such problem is solved by calculation of so-called symmetrization coefficients [9]. Here $\{f\}$ is Young's diagram: $\{3\}$ – symmetrical and $\{1^3\}$ – antisymmetric 1-dimensional representations, $\{21\}$ – 2-dimensional mixed symmetry representations; μ – Yamanouchi symbol for $\{21\}$ diagram; σ – the number of identical representations of S_3 groups.

For the elementary central three-partical interaction

$$\left.\begin{aligned}&\langle\vec{\kappa}|V|\vec{\kappa}'\rangle = \hat{\Pi}_0\mathscr{V}(\vec{\kappa},\vec{\kappa}'),\\ &\hat{\Pi}_0 = \lim_{\vec{\kappa}\to 0}\hat{\Pi}(\vec{\kappa}) = \{1\cdot 1\cdot 1 + \gamma_0\cdot\gamma_0\cdot 1 + \gamma_0\cdot 1\cdot\gamma_0 + 1\cdot\gamma_0\cdot\gamma_0\}/4.\end{aligned}\right\} \tag{32}$$

Salpeter equation (1) has the following form:

$$\left.\begin{aligned}&\{M - \Omega B/A\}\Phi_1(\vec{\kappa}) - [\Omega\hat{C}/A]\Phi_8(\vec{\kappa}) = \Pi(\vec{\kappa})\int\mathscr{V}(\vec{\kappa},\vec{\kappa}')\Phi_1(\vec{\kappa}')[d\vec{\kappa}'/(2\pi)^6],\\ &[\Omega\hat{C}/A]\Phi_1(\vec{\kappa}) - \{M + \Omega B/A\}\Phi_8(\vec{\kappa}) = \Pi(\vec{\kappa})\int\mathscr{V}(\vec{\kappa},\vec{\kappa}')\Phi_8(\vec{\kappa}')[d\vec{\kappa}'/(2\pi)^6].\end{aligned}\right\} \tag{33}$$

The scalars are defined as follows:

$$\left.\begin{aligned} &\Omega(\vec{\kappa}) = \omega_1 + \omega_2 + \omega_3, \\ &A(\vec{\kappa}) = m_1 m_2 \omega_3 + m_1 \omega_2 m_3 + \omega_1 m_2 m_3 + \omega_1 \omega_2 \omega_3, \\ &B(\vec{\kappa}) = \omega_1 \omega_2 m_3 + \omega_1 m_2 \omega_3 + m_1 \omega_2 \omega_3 + m_1 m_2 m_3, \\ &\Pi(\vec{\kappa}) = A(\vec{\kappa})/(4\omega_1\omega_2\omega_3) = 1/(I^{1/4}\mathcal{N}(\vec{\kappa})). \end{aligned}\right\} \tag{34}$$

A single matrix $\hat{C}$ (which defines the spinor character of a set of equations (33)) has the form:

$$\hat{C}(\vec{\kappa}) \equiv (\vec{\sigma}_1 \cdot \vec{k}_1)(\vec{\sigma}_2 \cdot \vec{k}_2)(\vec{\sigma}_3 \cdot \vec{k}_3), \tag{35}$$

where $\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ is a vector with components in the form of usual Pauli's matrixes. For variables A, B and $\hat{C}$ the relation $A^2 = B^2 + \hat{C}^2$ is valid.

The most popular (local) potentials used in calculations of two-quark bound systems [25] are easily generalized for the case of three-quark bound systems:

$$\langle \vec{\rho} | V_n | \vec{\rho}' \rangle = \hat{\Pi}_0 \mathcal{V}_n(\vec{\rho}, \vec{\rho}') = \hat{\Pi}_0 \delta(\vec{\rho} - \vec{\rho}')[\mathcal{V}_0 + \eta_n \rho^n e^{-\varepsilon\rho}], \tag{36}$$

where 6-vector $\vec{\rho} = (\vec{\eta}, \vec{\xi})$ with length $\rho = \sqrt{\eta^2 + \xi^2}$ (so-called collective variable) is made up of conjugate Jacobi coordinates

$$\left.\begin{aligned} &\vec{\eta} = \sqrt{\mu_1\mu_2/(\mu_1+\mu_2)}(\vec{x}_1 - \vec{x}_2), \\ &\vec{\xi} = -\sqrt{\mu_3(\mu_1+\mu_2)}[\vec{x}_3 - (\mu_1\vec{x}_1 + \mu_2\vec{x}_2)/(\mu_1+\mu_2)]. \end{aligned}\right\} \tag{37}$$

Here $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$ are usual Cartesian coordinates. $\vec{\kappa}$ - and $\vec{\rho}$ -vectors are costate variables. In relation (36) $\mathcal{V}_0$ and η_n are fitting parameters. Factor $e^{-\varepsilon}$ $(\varepsilon \to 0)$ is introduced for regularization of potentials in $\vec{\kappa}$ -representation. Values $n = -1, 1, 2$ correspond to Coulomb, linear and oscillatory potentials respectively. Omitting fairly long evaluations we will write the expression $\mathcal{V}(\vec{\kappa}, \vec{\kappa}')$ for the linear potential:

$$\mathcal{V}(\vec{\kappa}, \vec{\kappa}') = (2\pi)^6 \delta(\vec{\kappa} - \vec{\kappa}')\mathcal{V}_0 + [(2\pi)^6/(\kappa\kappa')^2] \times \sum_K v_K^1(\kappa, \kappa'; \varepsilon)\phi_K^*(\Omega_{\vec{\kappa}})\phi_K(\Omega_{\vec{\kappa}'}), \tag{38}$$

$$\begin{aligned} v_K^1(\kappa, \kappa'; \varepsilon) = {} & (2K+3)/[\pi(\kappa\kappa')^{3/2}(z^2-1)] \cdot \{(z - z_0)[(K + 5/2 - 2z^2/(z^2-1)) \times \\ & \times Q_{K+3/2}(z) + (2z/(z^2-1))Q_{K+1/2}(z)] + [zQ_{K+3/2}(z) - Q_{K+1/2}(z)]/2\}, \end{aligned} \tag{39}$$

$$z = z_0 + \varepsilon^2/(2\kappa\kappa'), \quad z_0 = (\kappa/\kappa' + \kappa'/\kappa)/2. \tag{40}$$

Here $Q_\nu(z)$ is Legendre function of the second kind.

However, oscillatory potential in momentum representation can be obtained in another more simple way. Fourier transform of potential (36) with $n = 2$ and $\eta_2 = M_0\Omega_0^2/2$ (Ω_0 - fitting parameter) in $\vec{\kappa}$ -representation has the form:

$$\begin{aligned} \mathcal{V}(\vec{\kappa}, \vec{\kappa}') &= (2\pi)^6 \delta(\vec{\kappa} - \vec{\kappa}')\mathcal{V}_0 - (2\pi)^6[M_0\Omega_0^2/2]\left[\Delta_{\vec{\kappa}'}\delta(\vec{\kappa} - \vec{\kappa}')\right] = \\ &= (2\pi)^6 \delta(\vec{\kappa} - \vec{\kappa}')\left[\mathcal{V}_0 - \vec{\kappa}^2/(2M_0)\right] + \sum_\nu \Omega_0(2N + K + 3)\langle \nu | \vec{\kappa}' \rangle \langle \vec{\kappa} | \nu \rangle. \end{aligned} \tag{41}$$

In deriving relation (41) we used a completeness condition

$$\sum_{\nu} \langle \vec{\kappa} | \nu \rangle \langle \nu | \vec{\kappa}' \rangle = (2\pi)^6 \delta(\vec{\kappa} - \vec{\kappa}') \qquad (42)$$

for 6-dimensional oscillatory basis functions $\langle \vec{\kappa} | \nu \rangle$ with quantum numbers $\nu \equiv_{NK}^{l_q l_p L M_L}$.

These functions satisfy the known oscillatory equation

$$\{-[M_0 \Omega_0^2 / 2] \Delta_{\vec{\kappa}} + \vec{\kappa}^2 / (2M_0) - (E - \mathscr{V}_0)\} \langle \vec{\kappa} | \Psi \rangle = 0 \qquad (43)$$

and have the following form:

$$\left. \begin{aligned} &\langle \vec{\kappa} | \nu \rangle = (2\pi)^3 a \left[\chi_{NK}(a\kappa) / \kappa^{5/2} \right] \phi_K^{l_q l_p L M_L}(\Omega_{\vec{\kappa}}), \\ &\chi_{NK}(x) = \sqrt{2\Gamma(N+1) / [a\Gamma(N+K+3)]}\; e^{-x^2/2} x^{K+5/2} L_N^{K+2}(x^2), \quad a = 1/\sqrt{M_0 \Omega_0} \end{aligned} \right\} \qquad (44)$$

$L_N^{K+2}(x^2)$ is Laguerre polynomial, N is positive integer, the energy eigenvalue is equal to

$$E_{NK} = \mathscr{V}_0 + \Omega_0 (2N + K + 3). \qquad (45)$$

It is easy to verify that the set of equations (33) has correct solutions in the case of the free motion: $\mathscr{V}(\vec{\kappa}, \vec{\kappa}') = 0$, and also has the correct nonrelativistic limit.

In summary it should be noted that having reduced the solution of Salpeter equation for three-particle bound systems to determination of two 8-component spinors without additional "relativization" of wave function, effectively using three-particle interquark interaction (in quark confinement conditions) and a powerful method of expansions in terms of hyperspherical harmonics (the most suitable to description of compact bound systems), we hope to appreciably simplify calculations of the basic characteristics of baryons.

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ფიზიკა

ბმული *qqq*-სისტემების შესწავლა სოლპიტერის განტოლების საფუძველზე უბრალო მიდგომის ფარგლებში ჰიპერსფერული ჰარმონიკების მწკრივად გაშლის მეთოდის გამოყენებით

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Astronomy

Electropolarimetric Study of Jupiter's Galilean Satellites

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ABSTRACT. The polarization properties of light, reflected from Jupiter's satellites, are studied. Maximum difference is noticeable between polarization degrees of light, reflected from the satellites' front and rear hemispheres. For the satellites located relatively close to Jupiter (Io, Europe, Ganymede), the magnitude of polarization degree of light, reflected from the front hemisphere, is comparatively less than the magnitude of polarization degree of light, reflected from the rear hemisphere, and *vice versa* for satellite Callisto. A hypothesis is presented in order to explain the mentioned differences. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *Galilean satellites, polarization degree, front and rear hemispheres.*

It is possible to name only several scientific works [1-3] in which polarization properties of Jupiter's Galilean satellites are studied .

Proceeding from the above stated I have set my mind since 1981 on the investigation of polarization properties of Jupiter's Galilean satellites in the alpha angle of each phase and in ten different areas of the visible spectrum . The observed material was obtained at the Abastumani Astrophysical Observatory, Georgian Academy of Sciences on both 40-cm refractor and 125-cm reflector, to which polarimeter ASEP-78 was attached during observations.

Due to the fact that consideration of the effect of Jupiter's surrounding background is rather complicated, we did not conduct Jupiter's observations from limb along 2-3 radii vision ray. It should also be mentioned here that a mean square error of one measurement during observation without filter does not exceed 0.05%. The observational method is described in detail in [4].

Taking all the above stated into account, a relatively reliable observation of Jupiter's Galilean satellites is possible when they are within the following orbital intervals:

Io	]30°; 150°[and]210°; 330°[
Europe	]19°; 161°[and [119°; 341°[
Ganymede	]12°; 168°[and]192°; 348°[
Callisto	]7°; 173°[and [187°; 353°[

In general the magnitude of polarization degree of light, reflected from a satellite's surface, must vary depending on the α-phase angle, satellite orbital longitude L, wave length λ and observation period t, or P = P (α, L, λ, t).

Observations. In the present work polarization properties of light, reflected from Jupiter's satellites, are studied. Maximum difference is noticeable between the polarization degrees of light, reflected from satellites' front and rear hemispheres. For satellites located relatively close to Jupiter (Io, Europe, Ganymede), the magnitude of polarization degree of light, reflected from front hemisphere, is comparatively less than the magnitude of polarization degree of light reflected from the rear hemisphere, and vice versa for satellite Callisto. In the present paper an acceptable hypothesis is proposed in order to explain the mentioned differences.

Based on the processing of obtained materials the author deduces that:

1. The magnitude of polarization degree of light reflected from the front side ($L \approx 90^{\circ}$) of satellite Io during observations without filter is in absolute magnitude by 0.15 – 0.20% less than the magnitude of polarization degree of light, reflected from the rear side ($L \approx 270^{\circ}$) when phase angle $\alpha \approx 5^{\circ}$, while the magnitude of polarization degree of light reflected from satellite Io's rear hemisphere is equal to $P(\alpha) = P(5^{\circ}) = -0.38\%$.

2. The magnitude of polarization degree of light, reflected from satellite Europe's front side during observations without filter is in absolute magnitude by 0.12% less than the magnitude of polarization degree of light, reflected from the rear side, when phase angle $\alpha \approx 3^{\circ}.5$, and the magnitude of polarization degree of light, reflected from satellite Europe's rear hemisphere, is equal to $P(\alpha) = P(3^{\circ}.5) = -0.25\%$.

The magnitude of polarization degree of light reflected from satellite Ganymede's front side during observations without filter in absolute magnitude is by 0.15 – 0.18% less than the magnitude of polarization degree of light, reflected from the rear, when phase angle $\alpha \approx 5^{\circ}$, while the magnitude of polarization degree of light, reflected from satellite Ganymede's rear hemisphere, constitutes $P(\alpha) = P(5^{\circ}) = -0.40\%$.

3. The magnitude of polarization degree of light reflected from satellite Callisto's front side during observations without filter in absolute magnitude is by 0.65% more than the magnitude of polarization degree of light reflected from the rear side, and constitutes 0.35%.

Analysis. It is evident that the magnitude of polarization degree of light, reflected from the front hemisphere of the first three satellites (Io, Europe, Ganymede), is less that the magnitude of polarization degree of light, reflected from the rear hemisphere, while in the case of satellite Callisto it is vice versa. One of the possible hypotheses for explaining this phenomenon is the following: as is known there is a shower of a multitude of meteorites, moving both on circular and elliptic orbits. Showers of meteors, moving on elliptic orbits in the direction, coinciding with the satellites' direction, must be the reason of the above mentioned exposed difference. These showers are falling asymmetrically upon the satellites' front and rear hemispheres.

In order to facilitate our calculations let us review meteor showers, the pericenter of which is $\sim 6R_J$ close to the satellites' (specifically Io's) orbit, located near the planet, and the apocenter $\approx 26R_J$ close to satellite Callisto's orbit.

In such case, as is well-known from celestial mechanics, the velocity of a body's movement in pericenter and apocenter is calculated using the following formulae:

$V^2 = V_c^2\,(1+e)/(1-e)$ (in pericenter), $V^2 = V_c^2 \times (1-e)/(1+e)$ (in apocenter), where V_c is the main velocity of an object moving on orbit, e is the orbit's excentricity.

On the one hand, it may be easily obtained that the velocity of meteoric bodies, having the above mentioned properties, will equal V = 22.50 km/sec. in pericenter and V = 5.04 km/sec in apocenter.

On the other hand, optimum velocities of Galilean satellites moving on circular orbits, are: for Io 16.94 km/sec., Europa 13.43 km/sec., Ganymede 10.63 km/sec. and Callisto 8.01 km/sec.

Evidently, the indicated meteoric bodies are falling upon Io from the rear side ($V_{Flow} > V_{Io}$), while in the case of Gallisto ($V_{Cal} > V_{Flow}$) we have the opposite picture. Callisto is gathering on and overtaking meteor showers, which bombard it from the front side due to the fact that the majority of meteoric bodies are dark (have less albedo and a high polarization degree). Consequently the light reflected from the satellite's indicated side corresponds to the higher polarization degree [4]. As the mentioned effect lasts for billions of years, the satellite's front and rear sides differ from each other.

ასტრონომია

იუპიტერის გალილეისეული თანამგზავრების ელექტროპოლარიმეტრიული შესწავლა

 რ. ჭიღლაძე

ილიას სახელმწიფო უნივერსიტეტი, ე.ხარაძის ეროვნული ასტროფიზიკური ობსერვატორია, თბილისი

(წარმოდგენილია აკადემიკოს ჯ. ლომინაძის მიერ)

ნაშრომში შესწავლილია იუპიტერის გალილეისეული თანამგზავრების ზედაპირებიდან არეკლილი სინათლის პოლარიზაციული თვისებები, რომლის ანალიზის საფუძველზე შეიმჩნევა მაქსიმალური განსხვავება თანამგზავრების წინა და უკანა ნახევარსფეროებიდან არეკლილი სინათლის პოლარიზაციის ხარისხთა შორის იუპიტერთან შედარებით ახლოს მყოფი თანამგზავრებისათვის (იო, ევროპა, განიმედე), წინა ნახევარსფეროდან (მოძრაობის მიმართულებით) არეკლილი სინათლის პოლარიზაციის ხარისხის სიდიდე გაცილებით ნაკლებია უკანა ნახევარსფეროდან არეკლილი სინათლის პოლარიზაციის ხარისხის სიდიდეზე, ხოლო თანამგზავრ კალისტოსათვის - პირიქით. ნაშრომში გამოთქმულია ჰიპოთეზა აღნიშნულ განსხვავებათა ასახსნელად.

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Materials Science

Corrosion of Chromium in Ambient $CO+CO_2$ Mixtures

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(Presented by Academy Member Irakli Zhordania)

ABSTRACT. The results of an investigation of high-temperature corrosion of unalloyed chromium and the binary alloy Cr+0.5%Ce in an ambient 75%CO+25%CO_2 gas mixture under a pressure of 0.01 atm are presented. It is shown that cerium additions have a beneficial effect on the oxidation resistance of chromium. © *2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *chromium, cerium, scale, gas mixture.*

Introduction

During the operation of aircraft engines and other installations used in various branches of industry (chemical, power) metallic materials come into contact with carbon- and oxygen containing gases and are subjected to their aggressive action. Researches aimed at creating corrosion-resistant protective outer zones able to protect the base metal against the chemical and mechanical action of corrosive environments are therefore essential.

The known method for preprocessing Cr_2O_3-forming alloys to improve their oxidation resistance at high temperatures in the ambient air [1] has been successfully used in the case of complex gas mixtures under low oxygen partial pressure.

In the presence of the $CO+CO_2$ gas mixture the reaction product resulting from its interaction with the metal can either be oxides or carbides. When chromium (Cr) or alloys based on chromium are subjected to such chemical action and oxygen activity (oxygen partial pressure P_{02}) in the system is below the range of stability (dissociation pressure) of Cr_2O_3 the occurrence of carbide phases [2] predominates. In contrast to chromium oxide (Cr_2O_3) the latter do not possess protective properties and they cannot serve as diffusion barriers.

Moreover, it is known that rare-earth metals, including cerium (Ce) intensify selective formation of Cr_2O_3 scale, slow down its growth and improve adherence [3]. Besides, the presence of these elements slows down the growth of grains in the scale and ultimately changes the diffusion direction of mass transfer from predominantly external diffusion of chromium (Cr) to internal diffusion of oxygen (O_2).

Furthermore, it is known that the grain boundaries of the material under investigation [3] play an essential role in the oxidizing process.

In this investigation the effect of graininess as well as cerium additions on the oxidizing rate of chromium in the $CO+CO_2$ environment has been studied, taking into consideration the foregoing.

Materials and methods

Sample processing procedures for the experiments are described in [3].

Thermally oxidized specimens were placed in the reaction zone of the installation after establishing the specified parameters for the experiment: temperature 1100°C, the pressure of gaseous atmosphere ($75\%CO+25\%CO_2$) 0.01 atm. Recording specific weight gain of test specimens was initiated ~ 30 seconds on, the reaction products were then removed from the reaction zone and the ratio of the initial components ($75\%CO+25\%CO_2$) as well as the pressure of gaseous atmosphere (0.01 atm) were re-established. The experimental conditions continued to be stabilized during the whole process.

Electrodeposited chromium with the grain size ~ 0.01 μm was used as the initial material; A coarse-grained metal with the grain size 20-30 μm was obtained through annealing of the initial material at 1100-1200°C.

Weight gains of the specimens (mg/cm^2) were determined by the continuous weighing method, while the microstructure of the scale was investigated with the help of the scanning electron microscope.

Experimental results and evaluation

1) Thermogravimetric investigation

Specimens of fine-grained and coarse-grained chromium as well as the binary alloy Cr+0.5%Ce were oxidized at 1100°C in gaseous atmosphere comprising $75\%CO+25\%CO_2$ under a pressure of 0.01 atm. Under these conditions (high temperature and low oxygen activity) no evaporation of chromium oxide was observed, as could be expected.

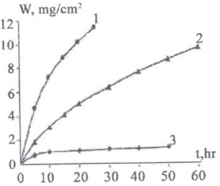

Fig. 1. The kinetics of oxidation of unalloyed chromium (Cr) and the binary alloy Cr+0.5%Ce ; 1 - fine-grained chromium; 2 - coarse-grained chromium; 3 - alloy Cr+0.5%Ce.

The results of the investigation are given in Figure 1.

Like the tests conducted in pure oxygen [3] the effect of cerium on the oxidizing rate was detectible in this particular case as well. Besides, it is demonstrated that fine-grained chromium oxidizes much faster than coarse-grained chromium.

The obtained results have corroborated the premise of previous investigations concerning the enhancement of the oxidation resistance of Cr_2O_3-forming alloys by virtue of rare-earth metal additions and are in agreement with the concept that a reduction in the dimensions of grain boundary surfaces or blocking them by diffusion barriers slows down mass transfer through the scale [4,5].

2) The characteristics of oxidized specimens

A scale formed during high temperature oxidation of unalloyed chromium bulges out heavily and deforms (Fig.2); it is characterized by poor adhesion and easily separates from the matrix. Fig. 3 shows the underside of the scale presented in Fig. 2. It is cracked and deformed. The oxide grain size in this section of the scale is small, which makes it qualitatively different from a similar scale formed under ambient oxygen [6].

Fig. 4 illustrates a scale formed on the surface of the alloy Cr+0.5%Ce specimen following its heating to 1100°C in ambient gas mixture ($75\%CO+25\%CO_2$). It can be seen that a part of it has broken away.

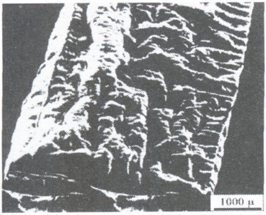

Fig. 2. SEM image of the oxidized unalloyed chromium surface.

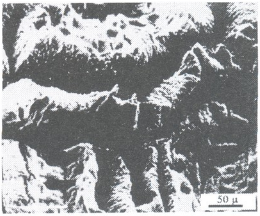

Fig. 3. SEM image of the underside of the scale shown in Fig. 2.

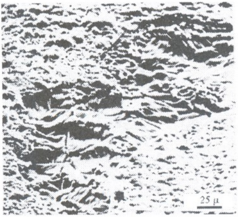

Fig. 4. SEM image of the oxidized Cr+0.5%Ce alloy surface

Moreover, during corrosion of unalloyed chromium and an alloy with cerium additions under the conditions described above no significant difference in the gross features of morphology of scales development are observed (Fig. 2 and Fig. 4). The difference is significant, however, in the case of using the medium of pure oxygen. However, the scale is significantly thinner on the alloy specimens (Fig. 1) pro rata with lower weight gains.

Conclusion

The results of the investigation are thus indicative of a positive effect of small additives of cerium on the oxidation resistance of chromium in the gas atmosphere (75%CO+25%CO_2) at high temperatures and under low oxygen activity. This is reflected in a significant slowdown of oxidation of unalloyed chromium and is accounted for by the formation of high density inner layer of the scale.

The impact of cerium on the morphology of the scale growth is less pronounced, although its thickness significantly decreases in the process.

მასალათმცოდნეობა

ქრომის კოროზია გარემომცველ $CO+CO_2$ ნარევებში

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ნაშრომში განხილულია სუფთა ქრომისა და ცერიუმთან მისი შენადნობის (Cr+0,5%Ce) მაღალტემპერატურული კოროზიის კვლევის მეთოდიკა და შედეგები (75% CO + 25% CO_2) შედგენილობის აირად ატმოსფეროში 0,01 ატმ წნევის პირობებში. ნაჩვენებია, რომ ქრომის ცერიუმით ლეგირება დადებითად მოქმედებს მის კოროზიულ მდგრადობაზე. ახსნილია მიღებული შედეგის მექანიზმი.

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Geology

Paleobiogeographic Zoning of the Basins of the Caucasus in the Early Jurassic-Bajocian by Ammonites

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ABSTRACT. On the basis of study of the ecology and evolution of Early Jurassic-Bajocian ammonites of the Caucasus, the routes of migration of these organisms and areas of their dispersal were established. The paleobiogeographic boundaries were specified and the existence of 4 palaeobiogeographic regions on the territory of the Caucasus in the Early Jurassic-Aalenian time was verified: 1. The Lesser Caucasian, in its southern part; 2. The Southern Caucasus intermountain area including the Dzirula massif; 3. The Southern slope of the Greater Caucasus including the territories of Georgia and Azerbaijan; 4. The Northern Caucasus. At the end of the Bajocian age considerable differentiation of ammonite fauna took place. It led to the appearance of new families and genera. Ranking of the earlier distinguished palaeobiogeographic areas to the subprovinces and also the existence of the Nakhichevan subprovince are justified.

Key words: *palaeobiogeography, Early Jurassic, Bajocian, Caucasus, ammonites.*

Over the past few decades our knowledge of Jurassic ammonites has increased considerably. Due to numerous studies of this important group of fauna, the possibility of solving the problems of biostratigraphy and paleobiogeography appeared. The changes in ammonite fauna in space and time were traced on the basis of extensive material.

Numerous researches were dedicated to Lower Jurassic-Bajocian palaeobiogeographical structures. However, information on the palaeobiogeography of this geological time interval in the Caucasus can be found only in [1, 2].

Recently new data on the Early and Middle Jurassic ammonites of the South Caucasus were received. They are of major importance in revealing general patterns of their geographic differentiation.

Taxa of palaeobiogeographical subdivisions used in this study are determined by the rank of groups of ammonites characteristic of them. In particular, biogeographic unit of the highest rank, i.e. an area covering an extensive territory of land or sea that differs from the adjacent one by the presence or disappearance of superfamilies and genera. Biogeographical unit of the second rank is a province, which is a part of the area characterized by species and subspecies and more fractionated subdivisions belonging to the provinces, such as subprovinces and districts.

Based on the study of ammonite fauna and biofacies analysis, in the Caucasus in the Early Jurassic and Aalenian 4 regions are distinguished: 1) the Lesser Caucasian, located in its southern part; 2) the South Caucasian intermountain area including the salient of the Dzirula crystalline basement; 3) the Southern slope of the Greater Caucasus including the territories of Georgia and Azerbaijan; 4) the Northern Caucasus. These paleobiogeographical units are considered in this paper (Fig. 1).

Geographic differentiation of ammonite fauna in the Early Jurassic and Aalenian was not sufficiently expressed and manifested mainly at the species level. At that time mainly the same genera were widespread. This, of course, complicates the distinguishing of a palaeobiogeographical unit that is larger than a province. The Caucasian marine basin in the Early Jurassic and Aalenian was a part of the Mediterranean province and was located in its northeastern part. Marine basins of North-Western Europe were identified as the Middle European province [2].

The complex of Early Jurassic and Aalenian ammonite fauna of the Caucasus is fairly abundant. Its members are distributed unevenly. A great number of ammonite genera of the Sinemurian age of the Early Jurassic epoch are recorded in the marine basin situated in the northern part of the Lesser Caucasus region. Here, in the Sinemurian age shallow marine conditions with normal water temperature and salinity favorable for intensive development of vital processes were established. In the organic world the ammonites of the following genera play an important role: the Mediterranean- *Partshiceras, Eoderoceras* and *Epideroceras*, the Middle European - *Canavarites, Arietites Coroniceras, Paracoroniceras* and *Metophioceras*, as well as *Vermiceras, Arnioceras, Oxynoticeras, Echioceras, Microderoceras*, found both in the Middle European and Mediterranean provinces and predominantly the Middle European - *Gleviceras* and *Paltechioceras* [3-5].

Ammonites of the Pliensbachian age are more exhausted and represented by the Mediterranean *Arieticeras, Liparoceras*, the Middle European *Tropidoceras*, mainly by the Mediterranean *Pleuroceras* and by widely spread *Amaltheus*.

The ammonite complexes of the Toarcian and Aalenian differ in a restricted amount of ammonites, belonging mainly to the Mediterranean *Callirhylloceras, Peronoceras, Harpoceras, Phymatoceras* and to the Middle European *Grammoceras, Pseudogrammoceras, Dumortieria, Leioceras, Costileioceras* and *Ludwigia*.

It is clear from the above list that the ratio of ammonite genera in some epochs changes. Apparently, in the Sinemurian and Toarcian ages, marine conditions on the territory of the Lesser Caucasus region were favorable for ammonite fauna habitat.

Species composition of genera is typical of the Mediterranean and Middle European provinces. The penetration of Middle European ammonites into the territory of the Lesser Caucasus probably took place from Central Europe along the northern margin of the Tethys, via North Anatolia.

In the Sinemurian age, the sea invading from the Lesser Caucasus basin through a wide strait on the territory of the South Caucasian intermountain area reached the Dzirula massif. During the Early Jurassic and Aalenian, the land on both sides of the strait was highly elevated [6].

The Sinemurian ammonite fauna of the Dzirula massif is considerably poorer than that of the Lesser Caucasus. It is represented by only three genera - *Vermiceras, Arnioceras* and *Microderoceras*.

The biocenosis of the Pliensbachian of the Dzirula massif is composed mainly of brachiopods and ammonites. An overwhelming majority of this fauna developed in the shallow part of the marine basin. As a part of taphocoenosis, the ammonites belong to the Mediterranean *Juraphyllites, Calliphylloceras, Arieticeras, Pleurocerass, Fuciniceras* and to the Middle European *Crucilobiceras, Phricodoceras, Uptonia, Polymorphites, Acanthopleuroceras, Tropidoceras* and *Pseudogrammoceras* [7, 8].

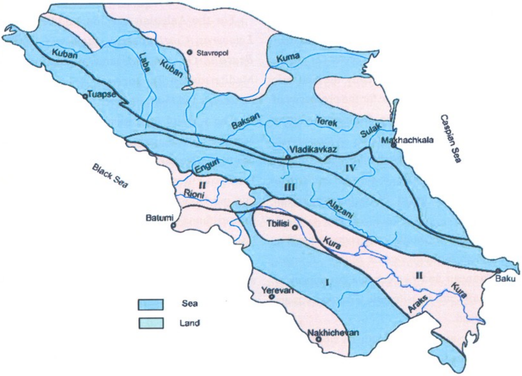

Fig. 1. Scheme of palaeobiogeographic zoning of the Caucasus in the Early Jurassic-Bajocian. The regions: I.The Lesser Caucasus; II.The Southern Caucasus intermountain area; III. The Southern slope of the Greater Caucasus; IV.The Northern Caucasus

In the Toarcian- Early Aalenian the sedimentary environments inherited from the Late Pliensbachian dominate. The abundance of fauna, represented by different groups, indicates optimal conditions for their existence.

The ammonite complex of these epochs in the South Caucasian intermountain area, within the limits of the Dzirula massif basin, is rather rich and diverse in genera and species composition. In the Toarcian age the Mediterranean genera *Calliphylloceras, Lytoceras, Harpoceras, Phymatoceras, Hammatoceras, Peronoceras, Catacoeloceras, Pseudolioceras, Polyplectus, Praehaploceras* prevail and the Middle European *Hildoceras, Grammoceras, Pleydellia* also occur.

The Aalenian age is represented predominantly by the Mediterranean *Tatrophylloceras, Lytoceras, Hudlestonia, Planammatoceras, Erycites* and the Middle European *Costileioceras* and *Leioceras*.

Thus, in the given ammonite complex the genera of the Middle European and Mediterranean provinces as well as the genera common for both biogeographic units were spread. Migration routes and dispersal of ammonites passed through Northern Anatolia by the strait located in the territory of the South Caucasian intermountain area.

In the Sinemurian, water penetrated into the territory of the region of the Southern slope of the Greater Caucasus. At that time a normal hydrochemical regime favorable for the dispersal of a rich complex of ammonites (42 species) was established in the basin. The complex is represented by the ammonite genera found in the Mediterranean and Middle European provinces - *Phylloceras, Partschiceras, Juaphyllites, Arietites, Coroniceras, Vermiceras, Arnioceras, Euasteroceras, Oxynoticeras, Gleviceras , Radstockiceras, Echioceras, Paltechioceras, Leptechioceras* [9, 3, 5, 10].

In the Pliensbachian age the Mediterranean *Partshiceras* and *Audaxlytoceras, Arieticeras,* the

Middle European *Zetoceras, Tropidoceras, Uptonia* and a cosmopolitan *Amaltheus* occur.

In the Toarcian age the ammonites reached their bloom. The number of genera and species (27 genera and 80 species) increased. Apparently, the bionomic conditions were most favorable for the prosperity of ammonite fauna. Among the discovered genera *Calliphylloceras, Partshiceras, Harpoceras, Hammatoceras, Planammatoceras* are Mediterranean and *Hildoceras, Polyplectus, Grammoceras, Pseudogrammoceras, Dumortieria, Pleydellia* are Central European.

In the Aalenian the number of genera reduced to 16. Among them *Tatrophylloceras, Brediya, Erycites* are Mediterranean and *Leioceras, Costileioceras, Staufenia, Ludwigia, Brasilia* and *Graphoceras* are Middle European.

Thus, in the area of the Southern slope of the Greater Caucasus both Mediterranean and Middle European ammonites were found. Apparently, the migration routes of the Mediterranean ammonite fauna passed across Italy and Anatolia, where analogous forms were recorded [1]. As to the Middle European representatives, they infiltrated into the region of the Southern slope of the Greater Caucasus through the Carpathians and the Crimea.

In the Northern Caucasus region, the complex of Sinemurian ammonites consists of the genera (*Arietites, Oxynoticeras Echioceras* and *Microderoceras)* characteristic of the Middle European and Mediterranean provinces [11].

In the Pliensbachian, the number of genera increased slightly. They are represented by *Pleuroceras, Arieticeras* and the Central European *Tragophylloceras, Androgunoceras* and a cosmopolitan *Amaltheus* [11].

The number of Toarcian genera and species increased compared to the Pliensbachian age. The Middle European *Hildoceras, Grammoceras, Pseudogrammoceras, Dumortieria, Pleydellia* and predominantly the Mediterranean *Peronoceras, Dactylioceras, Harpoceras, Phymatoceras, Haugia* and *Brodieia* were found.

For the Aalenian age the Middle European *Leioceras, Costileioceras, Staufenia, Ludwigia, Brasilia, Graphoceras* and predominantly the Mediterranean *Tmetoceras, Erycites, Planammatoceras* and *Hammatoceras* are characteristic [11].

The ways of migration and settlement of the Middle European Early Jurassic-Aalenian ammonoidea to the Northern Caucasus bypassed the sea of Southern Europe, directly across the Danish-Polish and pre-Dobrogea troughs [2]. And the Mediterranean fauna migrated through the Balkans, the Carpathians and the Crimea.

In addition, in the Early Jurassic-Aalenian time also a direct exchange with fauna between the seas of the Northern and Southern slopes of the Greater Caucasus took place.

In the Bajocian, differentiation of ammonite fauna significantly intensified. It led to the appearance of new families and genera that allowed to establish the Mediterranean and Middle European provinces and to raise the palaeobiogeographical areas, distinguished by us, to the rank of subprovinces.

The complex of ammonoidea of the Lesser Caucasian subprovince consists of the Mediterranean *Calliphylloceras, Phylloceras, Partschiceras, Pseudophylloceras, Thysanolytoceras, Nannolytoceras, Dinolutoceras, Eurystomiceras, Vermisphinctes* and the Middle European *Oppelia, Stephanoceras, Parkinsonia, Cadomites* [4, 12].

To the south, the Nakhichevan subprovince is distinguished [2]. It contains a rich ammonite fauna [13]. Here, together with Phylloceras and Lytoceras the Mediterranean *Spiroceras, Dorsetensia, Lissoceras, Otoites, Sphaeroceras, Leptosphinctes* and the Middle European *Chondroceras, Strigoceras, Oppelia, Oecotraustes, Cadomites, Stephanoceras, Strenoceras, Garantiana, Pseudogarantiana* and *Parkinsonia* are recorded. Apparently, the penetration and settlement of ammonite fauna in the Nakhichevan subprovince went on across the system of depressions of the southern branch of the Tethys, via Iran [2].

In the Bajocian age, the subprovince of the South Caucasian intermountain area was involved in the downward movements. The width of the strait located here since the Early Jurassic and Aalenian considerably reduced. The Bajocian transgression completely blocked the southern periphery of the Dzirula massif, although with some delay, as the base of the Bajocian verified with ammonite fauna starts here from its second zone [9]. Westward, in Okriba and Khreiti, marine regime was established. In spite of the intense manifestation of volcanic activity in this area, from time to time the ecological environment was favorable for the development of numerous mollusk fauna. The presence of cephalopods with prevalent representatives of *Lytoceras* and *Phylloceras* shows the close connection between the subprovince of the South Caucasian intermountain area and the Tethys Ocean.

In the Bajocian ammonite complexes of Okriba, Khreiti and of the Dzirula massif *Phylloceras, Calliphylloceras, Thysanoceras, Thysanolytoceras, Eurystomiceras, Nannolytoceras, Okribites, Oppelia, Sphaeroceras, Stephanoceras, Strenoceras, Parkinsonia, Emileia, Garantiana, Vermisphinctes* and the Middle European *Orthogarantiana* were recorded [12,14,15].

Further to the north, in the subprovince of the Southern slope of the Greater Caucasus in the ammonite fauna, together with phylloceras and lytoceras (*Pseudophylloceras, Thysanolytoceras, Eurystomiceras, Nannolytoceras, Dinolytoceras* and *Lytoceras*) *Hyperlioceras, Toxolioceras, Sonninia, Otoites, Spaeroceras, Garantiana, Parkinsonia* were discovered [12].

The marine basin existing in the Northern Caucasus in the Bajocian age, raised to the rank of subprovince, is characterized by rich generic and species composition. In the complex of ammonites, as already mentioned, representatives of *Phyloceras* and *Lytoceras* take a significant place. Below, a list of genera that allowed to specify certain issues of palaeobiogeography is given. They include the genera inhabiting the Middle European region - *Toxolioceras, Sonninia, Witchellia, Stephanoceras, Strenoceras, Pseudogarantiana, Garantiana, Parkinsonia, Praebigotites, Prorsisphinctes*, as well as the genera predominantly occurring in the Mediterranean region – *Spiroceras, Reynesella, Hyperlioceras, Dorsetensia, Lissoceras, Emileia, Otoites, Normannites, Stemmatoceras, Cadomites, Sphaeroceras* and *Leptosphinctes* [16, 17, 18]. Migration routes and the dispersal of ammonites in the North Caucasian subprovince were the same as in the Aalenian age, i.e. across the Danish-Polish and pre-Dobrogea troughs and the Mediterranean fauna migrated across the Balkans, Carpathians and the Crimea.

From the foregoing we can draw the following conclusions: Quantitatively the richest Sinemurian complex of ammonites occurs in the regions of the Lesser Caucasus and the Southern slope of the Greater Caucasus. In the South Caucasus, in comparison with the North Caucasus, in the composition of Sinemurian-Pliensbachian ammonite fauna the Mediterranean genera prevail. Toarcian-Aalenian ammonite complexes of the Southern Caucasus are poorer to some extent than those of the North Caucasus. The composition of ammonites is mixed. It is represented by the Mediterranean and Middle European forms, as well as by the forms common to both biogeographic units. In the formation of ammonite complexes of the Caucasus a major role is played by their exchange with other basins, particularly with the seas of South and North-Western Europe.

Thus, on the basis of palaeobiogeographical analysis of the Early Jurassic-Bajocian ammonite fauna of the Caucasus, the routes of migration and areas of their dispersal can be specified and therefore, the boundaries of paleobiogeographical basins of the considered region be determined.

გეოლოგია

კავკასიის ადრეიურულ-ბაიოსური დროის ზღვიური აუზების პალეობიოგეოგრაფიული დარაიონება ამონიტების მიხედვით

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*** საქართველოს ტექნიკური უნივერსიტეტი, გეოლოგიის დეპარტამენტი, თბილისი*

კავკასიის ადრეიურულ-ბაიოსური ამონიტების ეკოლოგიისა და ევოლუციური განვითარების შესწავლის საფუძველზე დადგინდა ამ ცხოველთა მიგრაციის გზები და განსახლების ადგილები. დაზუსტდა პალეობიოგეოგრაფიული საზღვრები და დასაბუთებულია ადრეიურულ-აალენურ დროს კავკასიის ტერიტორიაზე 4 პალეობიოგეოგრაფიული რაიონის არსებობა: 1. მცირე კავკასიონის; 2. სამხრეთ კავკასიის მთათაშუა არე; 3. კავკასიონის სამხრეთი ფერდობის და 4. ჩრდილოეთ კავკასიის. ბაიოსური საუკუნის ბოლოს მოხდა ამონიტური ფაუნის საგრძნობი დიფერენციაცია და ახალი ოჯახებისა და გვარების წარმოშობა. დასაბუთებულია არსებული პალეობიოგეოგრაფიული რაიონების უფრო მაღალ, ქვეპროვინციის რანგში აყვანა და ნახიჩევანის ქვეპროვინციის არსებობა.

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The Regularities of the Electrolytic Dissociation of 1,1-Cyclopentane and 1,1-Cyclohexanedicarboxylic Acids

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ABSTRACT. The parameters of electrolytic dissociation of 1,1-Cyclopentane and 1,1-Cyclohexanedicarboxylic acids in their dilute (0.0001-0.01M) solutions were determined with the aid of original accurate and empirical equations suggested by the authors. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *dissociation constant, 1,1-cyclopentanedicarboxylic acid, 1,1-cyclohexanedicarboxylic acid, degrees of dissociation, dissociation step, hydrogen ion concentration.*

Cyclopentane and cyclohexanedicarboxylic acids are used as pharmaceutical intermediates and also in such demanding markets as automotive, transportation, maintenance and aerospace. They are also the members of positive resist radiation-sensitive and coating resinous compositions. It should be noted that their useful properties are directly connected with the peculiarities of electrolytic dissociation of these acids.

In this communication the regularities of dissociation of 1,1-cyclopentanedicarboxylic (CPCA) and 1,1-cyclohexanedicarboxylic (CHCA) acids are determined with the aid of an original method suggested by the authors for analysis of the complex equilibria of dissociation of weak multibasic organic acids with the "overlapping" equilibria effect [1-3].

It was shown by us that the mass action law equations for the m dissociation step of weak multibasic organic acid H_nA may be written as follows:

$$K_m = \frac{c(\alpha_m - \alpha_{m+1})\sum_{m=1}^{n}\alpha_m}{\alpha_{m-1} - \alpha_m} F_m = \frac{c\alpha'_m(1-\alpha'_{m+1})\sum_{m=1}^{n}\alpha'_m\alpha_{m-1}}{1-\alpha'_{m-1}} F_m, \qquad (1)$$

where K_m is the thermodynamic dissociation constant of m step, c is a total (analytical) concentration of acid, α_m, α_{m+1} and α_{m-1} are the usual degrees of dissociation of corresponding steps, α'_m, α'_{m+1} and α'_{m-1} are the "partial" degrees of dissociation (this term was first suggested by the authors [3]), F_m is the quotient of the activity coefficients for m step:

$$F_m = \frac{f_{H^+} f_{H_{n-m}A^{m-}}}{f_{H_{n-(m-1)}A^{(m-1)-}}}. \qquad (2)$$

In the case of dibasic CPCA and CHCA the equations for the law of dilution for both dissociation steps may be presented as follows (according to equation (1)):

$$K_1=\frac{c(\alpha_1^2-\alpha_2^2)}{1-\alpha_1}F_1=\frac{c\alpha_1^2[1-(\alpha_2')^2]}{1-\alpha_1}F_1, \quad (3)$$

$$K_2=\frac{c\alpha_2(\alpha_1+\alpha_2)}{\alpha_1-\alpha_2}F_2=\frac{c\alpha_1\alpha_2'(1+\alpha_2')}{1-\alpha_2'}F_2. \quad (4)$$

According to equations (3) and (4) the degrees of dissociation, α_1, α_2 and α_2' can be evaluated successively by iterative solution of following quadratic equations:

$$\alpha_1=\frac{1}{2}\left[-\frac{K_1}{cF_1}+\sqrt{\left(\frac{K_1}{cF_1}\right)^2+4\left(\alpha_2^2+\frac{K_1}{cF_1}\right)}\right], \quad (5)$$

$$\alpha_2=\frac{1}{2}\left[-\left(\frac{K_2}{cF_2}+\alpha_1\right)+\sqrt{\left(\frac{K_2}{cF_2}+\alpha_1\right)^2+\frac{4K_2\alpha_1}{cF_2}}\right], \quad (6)$$

$$\alpha_2'=\frac{1}{2}\left[-\left(1+\frac{K_2}{\alpha_1cF_2}\right)+\sqrt{\left(1+\frac{K_2}{\alpha_1cF_2}\right)^2+\frac{4K_2}{\alpha_1cF_2}}\right]. \quad (7)$$

The values of the "partial" degree of dissociation α_2' may be also determined with the aid of α_1 and α_2 values:

$$\alpha_2'=\frac{\alpha_2}{\alpha_1}. \quad (8)$$

The values of the activity coefficient of hydrogen ions and mono- and dianions may be approximated with the aid of the Debye-Huckel equation:

$$\lg f_i=-\frac{z_i^2A\sqrt{I}}{1+a_iB\sqrt{I}}, \quad (9)$$

where a_i is the cation-anion distance of closest approach, A and B are constants depending on the properties of water at given temperature, z_i is the charge of ion. The ionic strength $I=c(\alpha_1+2\alpha_2)=c\alpha_1(1+2\alpha_2')$. The activity coefficient of undissociated acid is assumed to be unity.

With the aid of the dissociation degree values we may also calculate the values of concentration of all dissociated and undissociated forms of both acids:

$$[H^+]=c(\alpha_1+\alpha_2)=c\alpha_1(1+\alpha_2'), \quad (10)$$

$$[HA^-]=c(\alpha_1-\alpha_2)=c\alpha_1(1-\alpha_2'), \quad (11)$$

$$[A^{2-}]=c\alpha_2=c\alpha_1\alpha_2', \quad (12)$$

$$[H_2A]=c(1-\alpha_1). \quad (13)$$

The equations (5)-(8) were used for the calculation of the values of usual and "partial" degrees of dissociation for CPCA and CHCA in their dilute (0.0001-0.01M) solutions. The K_1 and K_2 values were taken from [4]: K_1=5.9·10^{-4}; K_2=8.3·10^{-5} (CPCA); K_1=3.5·10^{-4}; K_2=7.8·10^{-5} (CHCA). The values of the constants of equation (9) were taken from [5]. The calculated values of α_1, α_2, α_2' and pH at 25°C are presented in Tables 1 (CPCA) and 2 (CHCA).

The equations (10)-(13) allow the determination of the intervals of the acid concentration in which various charged and uncharged forms of acid prevail. The conditions of equality of these concentrations are:

$$[HA^-]=[H_2A]:\alpha_1=\frac{1+\alpha_2}{2}=\frac{1}{2-\alpha_2'}, \quad (14)$$

$$[A^{2-}]=[H_2A]:\alpha_1=1-\alpha_2=\frac{1}{\alpha_2'+1}, \quad (15)$$

$$[H^+]=[H_2A]:\alpha_1=\frac{1-\alpha_2}{2}=\frac{1}{\alpha_2'+2}, \quad (16)$$

$$[HA^-]=[A^{2-}]:\alpha_1=2\alpha_2, \quad (17)$$

$$\alpha_2'=0.5. \quad (18)$$

The data of Table 1 show that in the $c\le 0.001$M interval the [HA$^-$] value in the CPCA solutions exceeds the [H$_2$A] value (the inequality $\alpha_1>\frac{1+\alpha_2}{2}$ is fulfilled). Up to c = 0.0002M the [A^{2-}] value also

Table 1. The parameters of electrolytic dissociation of 1,1-cyclopentanedicarboxylic acid in its dilute solutions at 25°C

Acid concentration, M	α_1	α_2	α'_2	pH
0.0001	0.8903	0.3665	0.4117	3.907
0.0002	0.8053	0.2407	0.2989	3.687
0.0004	0.6971	0.1475	0.2116	3.481
0.0006	0.6280	0.1079	0.1718	3.366
0.0008	0.5782	0.08566	0.1481	3.286
0.001	0.5399	0.07129	0.1320	3.226
0.002	0.4269	0.03949	0.09250	3.045
0.004	0.3283	0.02161	0.06582	2.871
0.006	0.2788	0.01486	0.05330	2.773
0.008	0.2474	0.01145	0.04628	2.704
0.01	0.2251	$9.34\cdot10^{-3}$	0.04151	2.651

exceeds $[H_2A]$ value (the inequality $\alpha_1 > 1-\alpha_2$ is fulfilled). Up to $c = 0.0015$M the $[H^+]$ value exceeds $[H_2A]$ value (the inequality $\alpha_1 > \frac{1-\alpha_2}{2}$ is fulfilled). In all studied regions of the CPCA concentration monoanion remains the predominant anion (the inequalities $\alpha_1 > 2\alpha_2$ and $\alpha'_2 < 0.5$ are fulfilled).

According to the data of Table 2 in the solutions of CHCA up to c = 0.0005M the $[HA^-]$ value exceeds the $[H_2A]$ value (inequality $\alpha_1 > \frac{1+\alpha_2}{2}$ are fulfilled). Up to $c \approx 0.0002$M the inequality $\alpha_1 > 1-\alpha_2$ is fulfilled (the $[A^{2-}]$ value exceeds the $[H_2A]$ value). The $[H^+]$ value exceeds the $[H_2A]$ value up to c = 0.0009M (inequality $\alpha_1 > \frac{1-\alpha_2}{2}$ is fulfilled). In the CHCA solutions monoanion also remains the predominant anion (the inequalities $\alpha_1 > 2\alpha_2$ and $\alpha'_2 < 0.5$ are fulfilled in all studied regions of the c values).

Taking into account the comparative complexity of calculations with the aid of equations (5)-(7), we suggest also simple empirical equations for fast approximate calculation of the $\alpha_1, \alpha_2, \alpha'_2$ and pH values in the dilute solutions of CPCA and CHCA.

1,1-cyclopentanedicarboxylic acid

$$\alpha_1 = 0.10888c^{-0.235}, \quad (19)$$

$$\alpha_2 = 4.742\cdot10^{-4}c^{-0.729}, \quad (20)$$

Table 2. The parameters of electrolytic dissociation of 1,1-cyclohexanedicarboxylic acid in its dilute solutions at 25°C

Acid concentration, M	α_1	α_2	α'_2	pH
0.0001	0.8366	0.3441	0.4113	3.934
0.0002	0.7305	0.2226	0.3047	3.727
0.0004	0.6099	0.1351	0.2215	3.535
0.0006	0.5387	0.09852	0.1829	3.428
0.0008	0.4897	0.07809	0.1595	3.353
0.001	0.4531	0.06494	0.1433	3.297
0.002	0.3496	0.03595	0.1028	3.126
0.004	0.2639	0.01949	0.07385	2.961
0.006	0.2222	0.01353	0.06089	2.867
0.008	0.1961	0.01043	0.05319	2.800
0.01	0.1777	$8.51\cdot10^{-3}$	0.04787	2.750

$$\alpha_2' = 4.3551 \cdot 10^{-3} c^{-0.494}, \qquad (21)$$

$$pH = 1.188 - 0.677 \lg c. \qquad (22)$$

1,1-cyclohexanedicarboxylic acid

$$\alpha_1 = 0.07261 c^{-0.269}, \qquad (23)$$

$$\alpha_2 = 4.4259 \cdot 10^{-4} c^{-0.727}, \qquad (24)$$

$$\alpha_2' = 6.0954 \cdot 10^{-3} c^{-0.458}, \qquad (25)$$

$$\mathrm{pH} = 1.367 - 0.64 \lg c. \qquad (26)$$

The equations for α_1 and α_2 may be used in the interval c=0.0001-0.001M; the equations for α_2' and pH may be used in the interval c=0.0001-0.01M.

ფიზიკური ქიმია

1,1-ციკლოპენტან- და 1,1-ციკლოჰექსანდიკარბონმჟავების ელექტროლიტური დისოციაციის კანონზომიერებები

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ავტორების მიერ შემოთავაზებული ორიგინალური განტოლებების დახმარებით გათვლილია 1,1-ციკლოპენტან- და 1,1-ციკლოჰექსანდიკარბონმჟავების ელექტროლიტური დისოციაციის ორივე საფეხურის ჩვეულებრივი და "პარციალური" ხარისხების, pH-ის, მონო- და დიანიონებისა და არადისოცირებული მჟავას მოლეკულების კონცენტრაციების სიდიდეები ორივე მჟავას განზავებული ხსნარების კონცენტრაციის ინტერვალში 0.0001-0.01M. დადგენილია სხვადასხვა დამუხტული და დაუმუხტავი ნაწილაკების დომინირების კონცენტრაციული უბნები. შემოთავაზებულია აგრეთვე მარტივი ემპირიული განტოლებები დისოციაციის ხარისხებისა და ხსნართა ჰH-ის მნიშვნელობების სწრაფი მიახლოებითი გათვლისათვის.

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Polyurethanes on the Basis of Card-Type Polycyclic Bisphenols and Different Diisocyanates

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(Presented by Academy Member Shota Samsoniya)

ABSTRACT. Card-Type secondary diols are synthesized by means of oxyalkylation of bisphenols. Linear homogeneous polyurethanes are obtained through interaction of diols and diisocyanates. Their physical and chemical properties, thermal and heat-resistance are studied, as well as resistance in respect of radiation emanation of polymeric compositions obtained on their basis. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *diols, diisocyanates, polyuretanes, bisphenols, card-type, oxyalkylation.*

Among the homogeneous linear polyurethanes specific interest attaches to polyurethanes, which are simultaneously characterized by good solubility in trivial organic solvents and by high thermal stability. With this in view for the synthesis of polyurethanes the authors used the card-type diols of diverse structure [1-3].

The advantage of the polyurethanes synthesized on the basis of such type diols is that they are characterized by good solubility in organic solvents irrespective of the diisocyanate structures and high thermal stability. It is conditioned by the presence of non-coplanar structured polycyclic substitutes in the polymer chain near the central carbon atom, which hinders the free movement of two phenol nuclei, which are linked to the carbon atom of volumetric cyclic groups. Due to the same reason, shifting of cyclic fragments of polymer molecules with respect to each other is complicated. It results in the increase of high heat resistance stability of the polymer. At thesame time large size of the above stated groups hinders the packed placing of polymer chains. Therefore, irrespective of the great concentracion of aromatic hydrocarbons in the macromolecule, polymers are well dissolved in a number of organic solvents.

The present paper deals with the synthesis and of norbornane type group containing secondary diol-5,5-bis(4-β-oxypropoxyphenyl)-haxahydro-4,7-methylenindanylidene its phenyl substituted derivatives and polyurethanes obtained on the basis of various diisicyanates.

Synthesis of polyurethanes is based on migration polymerization reaction. Addition via migration of hydrogen atom of hydroxyl in the chain of growing

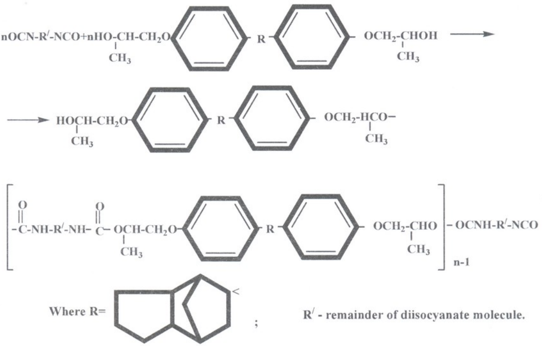


Fig. Polyurethanes on the basis of diol-5,5-bis(4-β-oxypropoxyphenyl)-haxahydro-4,7- methylenindanyliden and different diisocyanates.

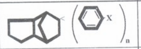
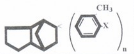
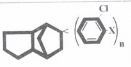
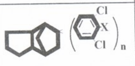
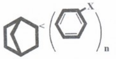
molecule into the nitrogen atom of isocyanate group is realized. This process proceeds in stages with gradual increase of molecular mass. The reaction proceeds according to the scheme (Fig.).

Polyurethane synthesis is carried out in chlorobenzol medium. Initial concentration of the yielded components are: 1 mol/l, mole ratio 1 : 1, reaction duration- 8 h, temperature 130 °C. Due to the fact that during the process of reaction the ends of the growing polymer chain contain isocyanate and hydroxyl groups, at the end of the reaction, during purification of the obtained products the amine group is formed as a result of interaction of isocyanate group with water. At the interaction of amine group with isocyanate group the spatial structure is formed, as a result of which polymer loses the solubility. Therefore at the end of the reaction, isocyanate groups are blocked by adding single atom alcohol. After removal of chlorobenzol the polymer is dissolved in chloroform and precipitated in ethyl ether. Isolated polymer is dried in vacuum thermostat at 70-80 °C.

The authors obtained polyurethanes on the basis of the diol synthesized by us-5,5-bis(4-β-oxypropoxyphenyl)hexahydro-4-7-methylindanylidene [1] and its phenylsubstituted derivatives: 5,5-bis(3-methyl-4-β-oxypropoxyphenyl) hexahydro-4,7-methylindanyliden [2], 5,5-bis(3-chloro-4-β–oxypropoxyphenyl) hexahydro-4,7-methylindayliden [3] and5,5-bis(3,5-dichloro-4-β-oxypropoxyphenyl)haxahydro-4,7-methylenindanyliden [3, 4] and various diisocyanates, via migration polymerization.

The properties of polyurethanes are presented (Table). As seen from the thermal mechanical studies of the synthesized polyurethanes, they are characterized by high thermal stability, which, as stated above, is conditioned by non-coplanar card type substitute existing at the central carbon atom in diol component of macromolecule [4]. The studies showed that growth of volume of norbornane type card cycles in diol molecule results in the increase of the softening point of polyurethanes [5]. For example softening-points of polyurethanes obtained on the basis of diol [1] and 2,4-toluylene and 4,4/-diphenyl methane diisocyanate are 240-265 °C and 250-270 °C, respectively. While for the polyurethanes obtained on the basis of 2,2-bis-(4-β-oxypropoxyphenyl) norbornyliden and same diisocyanates (see polymer 5), where the norbornane cycle substituted at the central

Table. Properties of polyurethanes synthesized on the basis of 5,5[1]-bis-(4-β-oxypropoxyphenyl) hexahydro-4,7-methylenindanyliden, its phenyl-substituted derivatives and various diisocyanates[x)]

№	Chemical structure of diols	1,6-hexamethylene-diisocyanate		2,4-toluylenediisocyanate		4,4[1]-diphenylmethane-diisocyanate	
		η reduced to dl/g	Softening point,^{0}C	η reduced to dl/g	Softening point,^{0}C	η reduced to dl/ g	Softening point,^{0}C
1		0.30	140-150	0.65	240-265	0.65	250-270
2		0.40	120-130	0.80	210-225	0.65	225-240
3		0.30	115-130	0.40	215-230	0.40	210-230
4		0.45	105-110	0.55	220-235	0.30	220-250
5		0.30	85-105	0.65	155-200	0.75	165-218

Where X= $OCH_2\text{-}\underset{OH}{CH}\text{-}CH_3$

carbon atom of diol component of polymer chain is of smaller size. Polymer softening temperature correspondingly was 155-200 °C and 165-218 °C (1).

Substitution of hydrogen atom in ortho-position by methyl group [2] and by Cl-atoms [3] in phenol nucleus of the diol component of macromolecules affects the thermal stability of polyurethanes, In all cases, the softening point of polyurethanes somewhat decreases [5]. Simultaneously fire resistance of polymers increases as a result of substitution of chlorine atoms.

Thermal stability of polyurethanes is also affected by diisocyanate structure. Substitution of aliphatic diisocyanate by aromatic one, results in the increase of thermal stability of polyurethanes. For example, for the polyurethanes obtained on the basis of [1] diols at the substitution of aliphatic -1,6-hexamethylendiisocyanate by aromatic - 2,4-toluylene and 4,4′-diphenylmethanes, the softening point is increased from 140-150°C to 250-270°C respectively.

According to the roentgen-structural analysis of polyurethanes the polymers synthesized by us are characterized by amorphous structure, which is conditioned by large size of card cycle of indan possessing non-coplanar structure substituted at the central carbon atom of bisphenol in diol fragment of the macromolecule. Due to the same reason the synthesized polyurethanes are characterized by good solubility in organic solvents, which enables to process polymers from the concentrated solutions.

The method of dynamic thermal-gravimetric analysis proved that decrease of polyurethane mass of synthesized polymers by 10% takes place at 250-300°C, while active decomposition starts at 350-400°C. Above this temperature, polymers are decomposed completely without formation of coke.

On the basis of the synthesized diols radiation resistant polymers are obtained, which are used as matrices in boron containing compounds possessing absorption properties.

ორგანული ქიმია

პოლიურეთანები კარდული ტიპის პოლიციკლური ბისფენოლებისა და სხვადასხვა დიიზოციანატების ბაზაზე

ე. გავაშელიძე*, ნ. მაისურაძე*, ნ. დოხტურიშვილი*, გ. პაპავა*, ნ. გელაშვილი*, ზ. მოლოდინაშვილი*, მ. გურგენიშვილი*, ი. ჩიტრეკაშვილი*

ი. ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტის პ.მელიქიშვილის ფიზიკური და ორგანული ქიმიის ინსტიტუტი

(წარმოდგენილია აკადემიის წევრის შ. სამსონიას მიერ)

სინთეზირებულია კარდული ტიპის მეორეული დიოლები ბისფენოლების ოქსიალკილირებით პროპილენოქსიდის საშუალებით.

დიოლების და დიიზოციანატების ურთიერთმოქმედებით მიღებულია ხაზოვანი ერთგვაროვანი პოლიურეთანები და შესწავლილია მათი ფიზიკურ-მექანიკური თვისებები, თერმო- და თბომედეგობა, მათ საფუძველზე მიღებული პოლიმერული კომპოზიციების რადიაციული გამოსხივებისადმი მედეგობა.

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N-Lactosylation of Amino Benzoic Acids

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(Presented by Academy Member Devi Ugrekhelidze)

ABSTRACT. The N-lactosylation of isomeric amino benzoic acids by D-lactose is studied. N-m-Carboxyphenyl-β-D-lactosyl amine and N-p-Carboxyphenyl-β-D-lactosyl amine are synthesized and characterized. *© 2012 Bull. Georg. Natl. Acad. Sci.*

***Key words**: N-lactosylation, amino benzoic acids, D-lactose.*

O-, m-, and p-amino benzoic acids carry out important physiological functions. P-amino benzoic acid is vitamin for many microorganisms, and the necessary precursor for biosynthesis of folic acid. P-amino benzoic acid is applied to treatment of such diseases, as scleroderma [1], Personae's disease [2]. As a component, it is widely used in protective anti-sun creams [3]. All three isomers of amino benzoic acids are antimut agens [4]. Such derivatives of amino benzoic acids which, after penetration into the action area, purposefully decompose with liberation of an active ingredient are successfully applied. [5]. It is necessary to note that such derivatives of amino benzoic acids frequently cause specific physiological effects which considerably differ from the physiological effects of initial amino benzoic acids [6], and for this reason, researchers carry out intensive investigations with the purpose of revealing such derivatives. One of the ways of synthesis of such derivatives is N-glycosylation of amino benzoic acids.

One of the prospective ways of preparation of N-glycosides of amino benzoic acids is direct interaction of aldoses and amino benzoic acids in aqueous or alcoholic or aqueous-alcoholic solutions, in cold or at heating, in the presence of the catalyst (the base or acid) or without it. This way of synthesis is intensively investigated, and so has not developed into a preparative method [7]. The basic barrier in the process of this synthesis is the accompanying melanoidin reaction, as a result of which N-glycosides, formed by N-glycosylation of amino benzoic acids, are transformed into a mix of melanoidine products. N-glycosylation of amino benzoic acids by reducing disaccharides is investigated only for maltose [8]; In this work we investigated the reaction of N-lactosylation of amino benzoic acids.

The reaction between o-, m-, p-amino benzoic acids and D-lactose was carried out in 96% ethanol medium, under the reflux, in the presence of small quantities of water and catalyst (glacial acetic acid).

Table 1. Formation of N-lactosides by reaction of D-lactose with o-, m-, p-amino benzoic acids

No.	N-Lactosyl amine	Yield, (%)	M. P., °C
I	N-o-Carboxyphenyl-D-lactosyl amine	0	–
II	N-m-Carboxyphenyl-β-D-lactosyl amine	30.0	138-140
III	N-p-Carboxyphenyl-β-D-lactosyl amine	50.0	105-106

The synthesized N-lactosides were purified by means of recrystallization (ethanol, diethyl ether), and their purity was checked by the method of TLC and paper chromatography. The identification of synthesized N-lactosides was carried out by a method of elementary analysis, by infrared spectra (UR-20, in KBr), ^{13}C-nuclear magnetic resonance spectra (Bruker NM-250 MGH, standard $(CD_3)_2SO$), and melting points. The results are shown in Table 1.

The data of Table 1 show that from m- and ï-isomers of amino benzoic acid the corresponding N-lactosides are formed; however, despite the change of the key parameters of reaction, we did not manage to obtain the desirable N-lactoside from o-amino benzoic acid. It is known that the process of N-glycosylation is significantly influenced by the basic nature of the reacting amine, and those factors which define the stability of sugar conformation, the more the basicity of the amine, the more actively it participates in the reaction of N-glycosylation; however subsequent transformations of the formed N-glycosides also proceed actively (Amadori rearrangement, Maillard reaction, deamination-decarboxylation, etc [9]).

By interaction of lactose with p-amino benzoic acid, the carboxylic group - because of its negative inductive and negative mesomeric effects - reduces the density of the electron cloud of the nitrogen atom, thereby reducing its basic nature. In this case, the N-lactoside of p-amino benzoic acid, because of its high stability, is formed with higher yield than the N-lactoside of m-amino benzoic acid. At reaction of the o-amino benzoic acid with D-lactose it is not possible to isolate the desirable product – appropriate lactoside; it is possible to assume that because of spatial effects, reaction between them does not occur. The most specific reaction typical of all oligosaccharides is their hydrolysis with cleavage of glycoside bonds and formation of monosaccharides. The infra-red and ^{13}C PMR spectra of synthesized N-lactosides (II, III) show that in these compounds the 1→4 β-glycoside bond is preserved between A and B carbohydrate rings.

The following were characteristic regions of infra-red absorption spectra of synthesized N-carboxyphenyl-D-lactosyl amines: 3380-2600 cm^{-1} (valence vibrations of O-H bond of carbohydrate); 2850-2900 cm^{-1} (valence vibrations of C-H bonds of carbohydrate); 1490-1510 cm-1 (absorption of N-glycoside bond of N-lactosylamines); 1450-1200 cm^{-1} (deformation vibrations of C-H and C-O-H); 1130-1150 cm^{-1} (vibrations of carbohydrate ring: C-O, C-C, C-O-C); 1020-1100 cm^{-1} (valence vibrations of C_1-N bond af anomeric centre at C_1); 700-1000 cm^{-1} (vibrations of carbohydrate ring); 750-900 cm-1 (deformation vibrations of C_1-H bonds of anomeric centre at C_1); 900-930 and 740-770 cm^{-1} (absorption of cyclic pyranose forms of N-glycosylamines) [10].

The ^{13}C PMR spectra of compounds II and III are divided into three basic ranges: the carbon atoms of both carbohydrate rings A and B are located in the range of 60-100 ppm respectively. From them the shift in the weakest field at 60.47 ppm and 60.47 ppm assigns to carbon atoms 6 and 6 of A and B ring. Chemical shifts at 103.85 and 103.82 correspond to C1 carbon atoms of ring A, which through β-glycoside bond are linked to C4 carbon atoms of ring B [11, 12]. C1 atoms of a rings B, which by amino-glycoside bond are linked to a benzene ring, resonate at 84.82 and 83.68 ppm; according to our data the

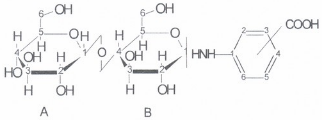

Fig. 1. Carboxyphenyl-D-lactosyl amines

Table 2. ^{13}**C PMR spectra of synthesized N-carboxyphenyl lactosylamines. Chemical shifts of C atoms, ppm.**

Number of C atom	Carbohydrate ring A	Carbohydrate ring B	Aromatic ring
N-m-carboxyphenyl-β-D-lactosylamine			
C1	103.82	84.82	147.29
C2	73.27	68.20	114.10
C3	75.83	75.56	131.34
C4	70.65	80.75	118.06
C5	75.28	72.79	128.93
C6	60.47	60.47	117.04
N-p-carboxyphenyl-β-D-lactosylamine			
C1	103.85	83.68	151.30
C2	73.26	68.20	112.27
C3	75.84	75.55	130.93
C4	70.61	80.73	118.68
C5	75.36	72.70	130.93
C6	60.46	60.46	112.27

appropriate glycoside bond has a β-configuration.

Synthesis of N-carboxyphenyl lactosylamines. A mixture of 3.42 g (0.01 M) of D-lactose, 1.47 g (0.01 M) of o-, m-, or p-amino benzoic acid, 15 ml of 96% ethanol, 0.5 ml of water, and 0.3 ml of a glacial acetic acid was heated up in a boiling water bath to full dissolution of initial products. The mixture cooled up to room temperature 50 ml of diethyl ether was added and after blending left for the night at room temperature. The precipitated crystals were filtered, ground with 96% ethanol, diethyl ether added to the mixture and after careful blending the precipitate was filtered. The purity of the synthesized N-carboxyphenyl lactosylamines was checked by TLC.

ორგანული ქიმია

ამინობენზოის მჟავების N-ლაქტოზილირება

რ. კუბლაშვილი*, მ. ლაბარტყავა*, ქ. გიორგაძე*, ნ. ქარქაშაძე*

* *ი. ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტის პ. მელიქიშვილის სახ. ფიზიკური და ორგანული ქიმიის ინსტიტუტი*

(წარმოდგენილია აკადემიის წევრის დ. უგრეხელიძის მიერ)

შესწავლილია იზომერული ამინობენზოის მჟავების N-ლაქტოზილირება D-ლაქტოზით. სინთეზირებულია და დახასიათებულია N-M-კარბოქსიფენილ-β-D-ლაქტოზილამინი და N-P კარბოქსიფენილ-β-D-ლაქტოზილამინი.

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Hydraulic Engineering

Determination of the Darcy Coefficient at Pressure Flow of Non-Newtonian Fluid in the Pipe

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ABSTRACT. The methodology of hydraulic calculation of the pressure losses along the length at the motion of non-Newtonian fluid with flow core in a round pipe is presented in the paper. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *non-Newtonian fluid, flow core, pressure loss along the length.*

Non-Newtonian fluids, where together with viscosity some additional properties appear, such as limit pressure shift, at the reaching of which the medium begins to flow as Newtonian fluid, attract special interest due to their wide spread.

An attempt is made to determine the value of Darcy coefficient at the motion of non-Newtonian fluid with flow core.

In the Figure the calculation scheme of the distribution of the velocities of the pressure motions of non-Newtonian fluid with core of the flow in pipeline of round form is presented. Such a motion is necessary to consider in the system of (x,r) where "Ox" is directed along the axis of the pipe. Let us denote the radius of the pipe r_c, core radius – r_0.

The rheologic law of the non-Newtonian fluid motion is often described by the Shvedov-Bingham equation [1]:

$$\tau = \tau_0 \pm \mu \frac{du}{dr}, \tag{1}$$

where τ – friction stress at the point of the pipe cross-section, τ_0 – limit stress of the shift after reaching of which the flow of the medium begins, μ – dynamic coefficient of structural viscosity, $\frac{du}{dr}$ – shift speed, r – flowing radius of the point.

Physical explanation of specific properties of such media is based on the presence in them of some inner hard structure at rest, which resists outside impact up to $\tau < \tau_0$ (i.e. fluidity is absent) and the medium behaves as a solid body. The medium begins "to flow" when $\tau > \tau_0$, making it unlike viscous non-Newtonian fluids. At the motion of these flows the medium sticks to the wall, as the result of which the gradient of velocity is observed in the contact surface of the flow with the pipe. Not rarely such a medium is characterized by flow core with undestroyed structure. Flow core on straight-line parts of the pipe behaves as a pivot (quasi-solid body). To such media existing in practice we refer sewage hyperconcen-

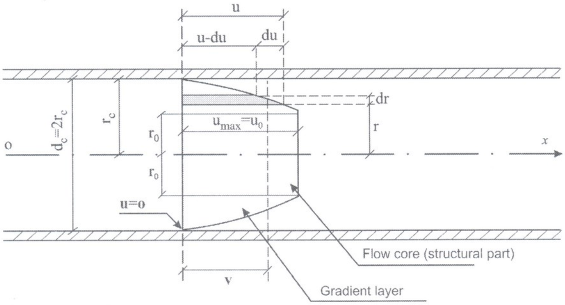

Fig. Scheme of distribution of the pressure flow velocities of non-Newtonian fluid with core of the flow in the pipeline of round cross-section.

trated alluvial debris flows (in draw-off tunnels), clayish and cement slurry (used for washing out of the oil field wells), concrete motion in pressure pipelines of concrete drift, etc.

Naturally, it is not possible to calculate Darcy (λ) coefficient in the mentioned cases according to the known dependences of hydraulics.

Considering that $\tau = \gamma RI = \gamma \frac{r}{2} I$, where R is hydraulic radius of the pipe, γ – specific weight of non-Newtonian fluid, I – hydraulic gradient, instead of (1) we get

$$\gamma \frac{r}{2} I = \tau_0 - \mu \frac{du}{dr}. \qquad (2)$$

Integration of (2) with account of boundary conditions, when $r=r_c$ then $u=0$ gives

$$u = \frac{\gamma I}{4\mu}\left(r_c^2 - r^2\right) - \frac{\tau_0}{\mu}\left(r_c - r\right). \qquad (3)$$

At $\tau_0 = 0$ dependence (3) takes generally known form of characteristics of laminar motion of non-Newtonian fluid [2]:

$$u = \frac{\gamma I}{4\mu}\left(r_c^2 - r^2\right). \qquad (4)$$

At $r = r_0$ from (3) we obtain the velocity of core motion (structural part):

$$u = u_0 = u_{\max} = \frac{\gamma I}{4\mu}\left(r_c^2 - r_0^2\right) - \frac{\tau_0}{\mu}\left(r_c - r_0\right). \qquad (5)$$

Elementary medium discharge in the pipe will be

$$dQ = u d\omega = \frac{\gamma I}{4\mu}\left(r_c^2 - r^2\right) 2\pi r dr - \frac{\tau_0}{\mu}\left(r_c - r\right) 2\pi r dr, \qquad (6)$$

where $d\omega$ is area of elementary live cross-section.

Taking into account that the first member of the right part of (6) characterizes elementary discharge of gradient layer of the flow (i.e. discharge in the limits of a ring between radii r_0 and r_c) the second part presents the flow core discharge from 0 to r_0, then after integration of (6) we shall get the value of total medium discharge in the pipe with radius r_c:

$$Q = \frac{\pi}{\mu}\left[\frac{I\gamma}{8}\left(r_c^4 - 2r_c^2 r_0^2 + r_0^4\right) + 2\tau_0\left(\frac{r_c r_0^2}{2} - \frac{r_0^3}{3}\right)\right]. \qquad (7)$$

At $r_0=0$, i.e. for non-Newtonian fluid, (7) takes the form of [2], i.e. we get the Poiseuille dependence:

$$Q = \frac{\pi I r_c^4 \gamma}{8\mu}. \qquad (8)$$

Table. **Number values of** $N = f\left(\frac{d_0}{d_c}\right)$**:**

$\beta \approx d_0/d_c$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
N	0.98	0.95	0.91	0.893	0.896	0.928	0.992	1.085	1.2025

Taking into consideration that $\tau_0 = \gamma \frac{r_0}{2} I$, expression (7) takes the form:

$$Q = \frac{\gamma \pi I}{2\mu}\left[\frac{r_c^4}{4} - \frac{r_c^2 r_0^2}{2} + r_c r_0^3 - \frac{5}{12} r_0^4\right]. \quad (7')$$

Average velocity of the motion of non-Newtonian fluid with flow core in the round pipe will be:

$$V = \frac{Q}{\pi\ r_c^2} = \frac{\gamma I}{2\mu}\left(\frac{r_c^2}{4} - \frac{r_0^2}{2} + \frac{r_0^3}{r_c} - \frac{5}{12}\frac{r_0^4}{r_c^2}\right). \quad (9)$$

Changing the radius of the pipe via diameter of the pipe d and taking into account $\gamma = \rho g,\ \upsilon = \frac{\mu}{\rho}$, where υ is coefficient of kinematic viscosity, g – acceleration of gravity force, ρ – density, instead of (9) we shall get

$$V = \frac{Igd_c^2}{32\upsilon}\left(1 - 2\frac{d_0^2}{d_c^2} + 4\frac{d_0^3}{d_c^3} - \frac{5}{3}\frac{d_0^4}{d_c^4}\right). \quad (10)$$

Taking into account that hydraulic gradient on straight-line parts of the pipe $I = \frac{h_l}{l}$, where l is the length of the straight-line pipe, h_l – the value of loss by friction along the length, we can write:

$$h_l = \frac{32lV\nu}{gd_c^2\left(1 - 2\beta^2 + 4\beta^3 - \frac{5}{3}\beta^4\right)}, \quad (11)$$

where $\beta = \frac{d_0}{d_c}$ is a relative diameter.

Considering that the general expression for losses of the pressure by length [1,2]:

$$h_l = \lambda \frac{l}{d_c}\frac{V^2}{2g}, \quad (12)$$

Comparing (11) and (12) and determining λ we obtain

$$\lambda = \frac{64\nu}{Vd_c N}, \quad (13)$$

where

$$N = 1 - 2\beta^2 + 4\beta^3 - \frac{5}{3}\beta^4. \quad (14)$$

Or expressing (13) via the Reynolds number $\mathrm{Re} = \frac{Vd_c}{\upsilon}$ instead of (13) we get

$$\lambda = \frac{16}{\mathrm{Re} \cdot N}. \quad (15)$$

Thus, pressure losses at friction in straight-line parts of the pipe at the motion of non-Newtonian fluid with flow core reach the maximum at $d_0 = \frac{d_c}{2}$, which should be taken into consideration while choosing force devices (plants) of such systems.

ჰიდროტექნიკა

მილსადენებში არანიუტონური სითხის დაწნევიანი მოძრაობის დროს დარსის კოეფიციენტის განსაზღვრა

ო. ნათიშვილი*, ვ. თევზაძე**

** აკადემიკოსი, საქართველოს მეცნიერებათა ეროვნული აკადემია, თბილისი*
*** საქართველოს ტექნიკური უნივერსიტეტის წყალთა მეურნეობის ინსტიტუტი, თბილისი*

წრიული კვეთის მილსადენებში არანიუტონური სითხეების მოძრაობის შემთხვევებისათვის დარსის კოეფიციენტის დასადგენად შემოთავაზებულია ჰიდრავლიკური გაანგარიშების მეთოდიკა ნაკადში გულის წარმოქმნისას, რომელიც თავის მხრივ მყარი ტანის მსგავსად გადაადგილდება.

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Human and Animal Physiology

Effect of Age Determination and Athletic Training Factors on Heart Rate

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(Presented by Academy Member Nodar Kipshidze)

ABSTRACT. The decrease dynamics of the heart rate of judokas with an increase of age is discussed in the paper. The degree of influence of age and training factors on heart rate decrease is estimated. It was found that over a 10-year period (8 to 18 years of age) in untrained individual's heart rate decreases on the average by 18 units, which is due to age, while in persons training in judo this index decreases on the average by 23.97 units. In the given age range, for judokas data of heart rate decrease by additional 5.97 units compared with the untrained persons (23.97-18) is subjected to the effect of training factor.

Key words: *heart rate, age, athletic training.*

Method of evaluation of heart rate as a variable is widely used. It is regarded as one of the indicators of a body adaptive capacity – as the body adaptation to different environmental factors significantly depends on cardiovascular reactions and optimal performance of regulatory mechanisms [1].

Observation of heart rate is widely used in diagnostic tests of body functioning, determining the exercise difficulty and estimation of chronotropic impact. It is a relatively easily measured parameter and its use is especially important in sports practice.

Training factor affects the human body functional systems. In particular, training causes changes in cardiovascular system functional status. Trained human heart operates differently at rest compared with that of untrained one. The first noticeable change is increase in the volume of the heart.

According to morphological changes, the trained person's heart, unlike that of an untrained one, experiences functional changes as well and starts to work in a more economical mode. The number of its contractions (pulse rate) in athletes with high physical conditions, trained for endurance, may vary from 28 to 40 beats. This factor can be explained as the result of heart muscle hypertrophy and left ventricular systolic blood volume increase, caused by its strong contraction, which, instead of 50-80 ml (which can be found in untrained individuals) is equal to 100-150 ml [2, 3].

At rest the heart rate is variable. It mostly depends on the age and in most adult persons makes 70 on the average. Heart rate along with age is significantly

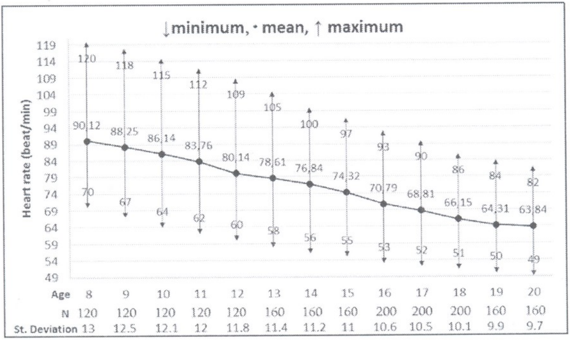

Fig. Descriptive data of heart rate at rest by age in judokas.

affected by the specificity of sports activity. In well trained wrestlers it can make 60-65 beats per minute. As for the teens, on the heart rate number there are still different opinions [4, 5], putting on the agenda the necessity of similar data determination.

Heart rate depends on age, sex, qualification, sports type, intensity of the load [6] and the time of the day [4]. For example, in the period between 2 am and 2 pm at rest it may increase by 13.8% (65 - to 74). Heart rate is also significantly affected by endurance exercise. In untrained persons, after each one-week endurance training the heart rate may reduce by 1 stroke and this process will continue for several weeks. The basic mechanism of this reduction has not yet been fully explored, but the exercise seems to cause parasympathetic system activation and sympathetic system inactivation due to which a relatively low heart rate is observed in trained individuals [4].

Research Aim. To examine the impact of judo training on the heart rate.

Objectives. 1. Studying of heart rate at rest in 8 to 20-year-old judokas.

2. Comparison of the frequency of heart rate of judokas to that of untrained persons at certain age levels.

Research Methods

Subjects. a research was conducted in 2009-2011 in Georgia, in the various specialized training sportsmen groups, from 8 to 20-year-old 2000 male judokas. Their distribution by age is presented in the Figure.

Measurements. heart rate at rest was measured by BP 3AX1 (Microlife), in standing position.

Statistical Data Processing. by the statistical method the data were processed in the computer program SPSS 19. The ANOVA test was used to determine the dependence of heart rate decrease on the age (8-20 years). Quantitative data are presented as mean, standard deviation, maximum and minimum values. The level of significance was set at $p<0.01$.

Research Results and Discussion.

The data obtained by the research are presented in the Figure.

Minimum and maximum values heart rate at rest quantitatively for each next age has the trend to decrease. With respect of data dispersion percentage the picture does not change and at the age of 8 it makes 71.4% (70 and 120), and at the age of 20- 67.3% (49 and 82).

The heart rate at rest, presented as mean values by ages, is the highest at the age of 8 years, and makes 90.12 beats per minute. This index drops by 26.28 units at the age of 20 and makes 63.84 beats per minute; meaning decrease in heart rate by 41.2%.

Our results of the heart rate at rest were compared to the data of [5] obtained in 8-18-years old untrained persons. It was found that at the age of 8, the heart rate frequency was identical to each other. The difference was in rates of decrease in next age groups.

According to the results of S. Fleming et al. [5] the heart rate from 8 to 18 years drops from 90 to 72 in average and reduces by 18 units, or 25%. According to our data, the obtained picture is different and the heart rate from 8 to 18 years reduces from 90.12 to 66.15 on average, implying the data reduction by 23.97 units, i.e. 36.2% on average. The data obtained show that the index of heart rate in judokas from 8 to 18 years reduces by 5.97 beats (23.97-18) more than in untrained persons.

Conclusion

From 8 to 18 year the index of heart rate decrease by 18 units on average is caused by age (untrained individuals). After this limit data reduction by 5.97 units is subjected to the training factor (judokas).

ადამიანის და ცხოველთა ფიზიოლოგია

გულისცემის სიხშირეზე ასაკობრივი დეტერმინაციისა და სპორტული წვრთნის ფაქტორების ზეგავლენა

გ. ზუბიტაშვილი*, დ. ჩიტაშვილი*

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(წარმოდგენილია აკადემიკოს ნ. ყიფშიძის მიერ)

სტატიაში განხილულია ძიუდოში მოვარჯიშე სპორტსმენთა გულისცემის სიხშირის კლების დინამიკა ასაკის მატებასთან დაკავშირებით. შეფასებულია ასაკისა და წვრთნის ფაქტორების ზეგავლენის ხარისხი გულისცემის სიხშირის კლებაზე. დადგენილია, რომ 10-წლიან პერიოდში (8-დან 18 წლამდე) უვარჯიშებელ პირებში გულისცემის სიხშირე საშუალოდ 18 ერთეულით მცირდება, რაც ასაკობრივად დეტერმინირებულია, ძიუდოისტებში კი ეს მონაცემი საშუალოდ 23.97 ერთეულით განიცდის კლებას. მოცემულ ასაკობრივ დიაპაზონში უვარჯიშებლებთან შედარებით ძიუდოისტთათვის გულისცემის სიხშირის დამატებით 5.97 ერთეულით კლების მაჩვენებელი (23.97-18) გაწვრთნილობის ფაქტორის გავლენას ექვემდებარება.

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Entomology

Effect of Entomoparasitic Nematodes *Steinernema feltiae* on Fern Scale (*Pinnaspis aspidistrae* Sign.)

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(Presented by Academy Member Irakli Eliava)

ABSTRACT. Interrelation between fern scale (*Pinnaspis aspidistrae* Sign.) and entomoparasitic nematodes (*Steinernema feltiae*) is studied under laboratory conditions. Susceptibility of fern scale worms to juvenile suspension is evaluated and time and percent of the worm mortality are determined.

Key words: *fern scale, entomoparasitic nematodes,* Steinernema feltiae, *plant protection.*

Fern scale (*Pinnaspis aspidistrae* Sign.: *Coccidia*) damages greatly ornamental and house plants (Figs.1,2). The pest dwells on the underside of the leaf and covers itself with thorax. In the case of thick habitation of pests the leaf turns yellow and falls. Fern scale is polyphage; it damages ornamental plants, such as house ferns (*Nephrolepis exaltata* Schott), dragon-tree (*Draeacena draco* L.), brier (*Smilax exelsa* L.), etc. [1]. Fern scale propagates in West Asia, North and South America; it was detected in Europe and Georgia.

The objective of the work was to study the action of entomoparasitic nematodes (*S. feltiae*) on fern scale (*P. aspidistrae* Sign.) under laboratory conditions.

Entomoparasitic nematodes are harmless to humans, animals, plants; they present effective biological agents to control pests. Among them the nematode *S. feltiae* is used against different pests such as North American white worm (*Hypantria cunea*), Colorado potato beetle (*Leptinotarsa decemlineata*), greenhouse whitefly (*Trialeurodes vaporariorum*) and others.

S. feltiae carries an associated bacterium (*Xenorhabdus* species). Their coaction causes mortality of insects. In the experiment a *S. feltiae* strain was used, produced by e~nema Co., Germany.

A pest's organism as a food source is an important medium to reproduce parasitic nematodes. Life cycle of the nematode *S. feltiae* includes the egg, four juvenile stages (J_1-J_4) and the adult. After the second stage juveniles are covered with a protective film-cuticle, they leave the cadaver in search of a new host. Nematodes enter a pest's body via the mouth, anus or respiratory openings and starts to feed. In the body of the host nematodes release symbiotic bacteria which kill insects within 24-72 hours. In the intestines of a host nematodes produce an amphimictic generation (male and female nematodes).

Fig. 1, 2. Colonies of fern scales (*Pinnaspis aspidistrae*) on laurel leaves

Cultivation of *S. feltiae* at 24-25°C under laboratory conditions was conducted on late age worms of Honeycomb Moth (*Galleria mellonella*) [2].

Individuals of fern scale were collected in spring-summer at the Centre of Ornamental Gardening, Tbilisi, on laurel plants, where the density of pest's habitation was 100-120 individuals per leaf on the average.

Laurel leaves inhabited by fern scales were placed into Petri dishes with wet filter paper on the bottom. Suspension of entomoparasitic nematode was used to infest scales with the following concentrations: 500, 1000 and 1500 IJs/ml. Experiments carried out were replicated 3 times. Laurel leaves in control Petri dishes were treated with distilled water (Fig.3). Records were made on the 3rd, 5th and 7th days. The percent of pests' mortality was determined by Abbott formula [3]. Dead pests infested by nematodes were placed on special water-jacketed cuvette holders in small Petri dishes with filter papers. On the 12th day of pest invasion nematodes leave the host body for cuvettes where nematode biomass was collected (Figs. 4,5).

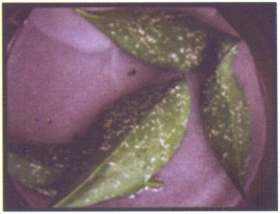

Fig. 3. Infested worms of fern scales

Fig. 4. Action of *S.feltiae* on fern scale worms

Fig. 5. Larvae of *S. feltiae* leaving the body of fern scale

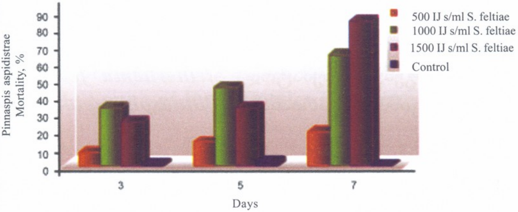

Fig. 6. Mortality of fern scale caused by entomoparasitic nematode *S. feltiae,* in %, at different concentrations of nematode suspension.

Suspension of *S. feltiae* was stored in the refrigerator at 4-6°C. Nematodes were acclimated at room temperature (21-23 °C). The biomass obtained could be used in 6-10 hours.

Results of investigations of the interaction of fern scales and nematodes *S. feltiae* showed that worms of fern scale were resistant to nematode suspension 500 IJs/ml. Mortality of worms in 3 replications was insignificant. At the same time, among infested worms of fern scales live imagoes were detected. In the case of suspension concentration 1000 IJs/ml mortality of worms reached 65% in 3 replications. In the case of nematode suspension concentration 1500 IJs/ml percent of the worm mortality on the 7th day after invasion reached 85% (Fig. 6).

Thus, it was established that against fern scales it is possible to use entomoparasitic nematodes *S. feltiae* in the form of high concentration suspension (1000-1500 IJs/ml) which, in our opinion, significantly decreases the number of fern scales in the ground and greenhouses.

It should be noted that we have not found any literature data on interrelation between fern scale (*Pinnaspis aspidistrae* Sign.) and entomoparasitic nematodes *S. feltiae*.

ენტომოლოგია

ენტომოპათოგენური ნემატოდა *Steinernema feltiae*-ს მოქმედება გვიმრის ფარიანაზე (*Pinnaspis aspidistrae* Sign.)

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ნაშრომში ლაბორატორიულ პირობებში შესწავლილია ენტომოპარაზიტული ნემატოდას (*S. feltiae*) მოქმედება გვიმრის ფარიანაზე (*P.aspidistrae*). შეფასებულია გვიმრის ფარიანას მატლების მიმღებიანობა სხვადასხვა კონცენტრაციის ნემატოდური სუსპენზიის მიმართ. პროცენტულად განსაზღვრულია ფარიანას მატლების სიკვდილიანობა.

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Zoology

Influence of Ecological Factors on the Formation of Nematode Fauna of Bark Beetles (*Coleoptera*: *Scolitidae*)

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ABSTRACT. Bioecology of nematodes invading host beetle is studied in order to identify prospective species of nematodes to be used against pests as biological methods.

The influence of ecological factors on the composition and quantity of nematode fauna of bark beetles is considered. © *2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *nematodes, bark beetle, Taphrorychus bicolor, Ips typographus.*

Today nematodes present biologically evolutionary species of fauna. They have occupied the whole biosphere and dwell in all biotopes. Free-lived and parasitic nematodes are known. Free-lived nematodes dwell in seas, fresh water and soil; parasitic nematodes live in organisms of men, vertebrate and invertebrate animals, and plants.

The relationship between pests and nematodes can be various, but mainly a pest is a host for all nematodes. The tissue of beetle's organs is a life medium for nematodes. Entering a pest through its body openings a parasite entonematode actually infests and kills it, feeding on its tissue or cadaver. When the food resource within the dead pest comes to end, nematodes exit and begin searching for a new host.

Such a lifestyle of nematodes allows us to use them against pests damaging significantly forests and gardens. It should be noted here that bark beetles of the family *Scolitidae* order *Coleoptera* are more harmful among the pests.

Formation of nematode fauna of bark beetles is influenced by both abiotic and biotic factors.

In order to establish the influence of abiotic factors, such as temperature and humidity, on the composition and quantitative changes in nematode fauna an experiment was carried out under natural conditions. During the experiment we used the so-called decoy-trees populated with bark beetles. Observations were made under various temperature and humidity conditions. Temperature as one of the main ecological factors, first of all, affects the activity

Table 1. Temperature influence on species and quantity of nematodes of typographer bark beetles *Ips typographus*

Nematodes	Quantity of nematodes					
	Tbilisi Daily average temperature t=15.7 °C			Gombori Daily average temperature t=7.8 °C		
	Small (unit)	Medium (unit)	Big (>50)	Small (unit)	Medium (unit)	Big (>50)
Bursaphelenchus teratospicularis	-	-	+	+	-	-
Parasitorhabditis bicoloris	-	-	+	+	-	-
Parasitaphelenchus bicoloris	-	+	-	+	-	-
Cryptaphelenchus bicoloris	-	+	-	+	-	-
Panagrolaimus scheucherae	-	-	+	-	+	-

Table 2. Temperature influence on species and quantity of nematodes of typographer bark beetles *Ips typographus*

Nematodes	Quantity of nematodes					
	Decoy tree at the forest edge. Daily average temperature t=30-35 °C			Decoy tree in the forest heart. Daily average temperature t=24-26 °C		
	Small (unit)	Medium (unit)	Big (>50)	Small (unit)	Medium (unit)	Big (>50)
Parasitorhabditis obtusa	+	-	-	-	+	-
Ektaphelenchus typographi	+	-	-	-	+	-
Cryptaphelenchus macrogaster	+	-	-	-	-	+
Bursaphelenchus eidmani	+	-	-	-	-	+
Micoletzkya buetschlii	-	+	-	-	+	-

of a host-beetle, which in turn conditions the biological activity of nematodes of specific and transition groups (commensals, ecto- and endoparasites). Temperature acts indirectly on nematodes related to beetles, and acts directly on nematodes living in wormhole dust. In order to study the temperature effect on quantitative dynamics of nematodes we researched nematode fauna of two-coloured beech bark beetle (*Taphrorychus bicolor*) under various temperature regimes. With this aim in May, 2011, simultaneous observations were made in Tbilisi and Gombori (Eastern Georgia). Daily temperature was 15.7 °C in Tbilisi and 7.8 °C in Gombori. The study of nematode fauna registered in egg and larvae galleries (wormhole dust) showed that in Tbilisi, where the temperature was optimal for the activation of beetle, the quantity of nematodes in wormhole dust was much more than that in Gombori (Table 1).

The result obtained showed that under optimal conditions latent larvae existing in beetle`s organism became active and go through beetle galleries.

Influence of temperature on host-beetle in respect of quantitative dynamics of nematodes was studied on typographer bark beetles (*Ips typographus*). Decoy-trees populated with bark beetles used in experiments were at first placed at the forest edge in sunlight where maximal temperature was 30-35 °C. After a certain period of time the tree was replaced into the forest heart, where the maximal temperature was 24-26 °C. In 2-3 weeks study of egg and larvae galleries (wormhole dust) of host-beetle showed that quantity of nematodes was more on decoy-tree in the forest heart than at the edge of the forest. This fact shows that environmental conditions for vitality and more activity of nematodes are better in the forest heart (Table 2).

Table 3. Influence of humidity on species and quantity of nematodes

Nematodes	Quantity of nematodes					
	Beetles habitation on the decoy tree in sunlight. Humidity 48-50%			Beetles habitation on the decoy tree in the shadow. Humidity 78-80%		
	Small (unit)	Medium (unit)	Big (>50)	Small (unit)	Medium (unit)	Big (>50)
Bursaphelenchus carpus	-	+	-	-	-	-
Sychnolylenchus intricati	-	-	+	+	-	-
Goodeus scolity	-	-	+	-	-	-
Parasitorhabditis malii	-	-	+	+	-	-
Sticktylus pseudobtusus	-	+	-	+	-	-
Panagrolaimus regidus	-	-	+	+	-	-
Panagrobelus coronatus	-	-	-	-	-	+
Panagrelus redivivus	-	-	-	-	-	+
Pabdontolaimus haslacheri	-	-	-	-	+	-
Rhabditis sp.	-	-	-	-	-	+

The result obtained confirms that temperature has a significant effect on the quantity of nematodes of bark beetles and no effect on the species content of nematode fauna.

Another important ecological factor is humidity [3]. We have studied the effect of humidity on apple bark beetles (*Scolytus mali*) in Gori district. During the experiment an apple-tree populated with apple bark beetles was used as decoy for beetles.

The quantity of nematodes in egg and larvae galleries (wormhole dust) was controlled simultaneously under various humidity conditions. We studied wormhole dust on the decoy-tree at the sunny side where the humidity was 48-50% and at the shady side, where the humidity was 78-80%. In the first case ecto- and endoparasites of specific and transition groups (*Bursaphelenchus, Goodeyus, Stictylus, Sychnotylenchus, Parasitorhabditis*) were found in the wormhole dust made by apple bark beetles. Nematodes of nonspecific group were not detected here. In the second case, when humidity was rather high, ecto- and endoparasites of specific group completely disappear, nematodes of transition group were left in single examples, but nematodes of nonspecific group were reproduced (Table 3).

Thus, changes in humidity causes nematode change in species composition and their quantity. Quantity of nonspecific nematodes increases with the increase of humidity due to the fact that they live in saprobe habitat. At places where moisture needed for the formation of microflora is abundant, decay process goes on more intensively, but such conditions are not advantageous for specific nematodes.

It should be noted that the increase of nematode quantity depends on the sum of effective temperatures and on resources nematodes feed on.

Just nonspecific nematodes adapt themselves well to that microflora which is formed in beetle galleries (wormhole dust) under the wood bark and at different places of nematodes' habitat under equivalent optimal temperature and humidity conditions.

ზოოლოგია

ეკოლოგიური ფაქტორების გავლენა ქერქიჭამიების (*Coleoptera: Scolitidae*) ნემატოდოფაუნის ფორმირებაზე

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წარმოდგენილ ნაშრომში შესწავლილია ხოჭო-მასპინძლის ორგანიზმში შეჭრილი ნემატოდების ბიოეკოლოგია მავნე მწერების წინააღმდეგ ბრძოლის ბიოლოგიური მეთოდისათვის გამოსადეგი ნემატოდების პერსპექტიული სახეობების გამოვლენის მიზნით. განხილულია ეკოლოგიური ფაქტორების გავლენა ქერქიჭამიების ნემატოდოფაუნის სტრუქტურასა და რაოდენობაზე.

ექსპერიმენტის შედეგად დადასტურდა, რომ ტემპერატურა მნიშვნელოვნად ცვლის ქერქიჭამიების ნემატოდების რაოდენობას, მაგრამ ნემატოდოფაუნის სახეობრივ შემადგენლობაზე გავლენას არ ახდენს, ხოლო ტენიანობის ცვლილება განაპირობებს როგორც ნემატოდოფაუნის სახეობრივ შედგენილობას, ისე რაოდენობრივ ცვალებადობას.

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Genetics and Selection

Crossability of Endemic Species and Aboriginal Varieties of Georgian Wheat and Traits in F1

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ABSTRACT. The research shows that Georgia is a primary centre of the origin and diversity of cultivated wheat, distinguished from other countries by a high level of endemism. It is proved that Georgian endemic species of wheat have played an important role in the evolution of the genus Triticum and process of wheat selection on a global scale. New species, genera, cultivars and varieties of wheat have been obtained on the basis of wheat species endemic to Georgia. Their genotype bears genes which allow to obtain wheat species of a new type with high immunity and quality features. Issues of crossability of endemic species of Georgian wheat with other species as well as with aboriginal and selection varieties of soft wheat, germination capacity of obtained hybrid grains and viability of plants of the first generation are discussed in the present paper. Peculiarities of inheritance of economically important morphological traits in the first generation of plants are shown. *© 2012 Bull. Georg. Natl. Acad. Sci.*

***Key words**: crossability, genotype, lethal genes.*

It has been established by Georgian and foreign scientists that countries of Western Asia (Georgia, Azerbaijan, Turkey, Iran, Syria, Israel), where a total of 892 species and varieties of wheat have been identified and registered, dominate in the number of species and varieties of wheat among Western Asian and Central Asian (Turkmenistan, Afghanistan, India, Pakistan) centres. Number of species and varieties of wheat registered in countries of Western Asia is 2.5 times and even higher than in countries of Central Asia [1-3].

Among countries of Western Asia those of the South Caucasus are distinguished for the highest number of wheat species and varieties. Turkey holds the second position in this respect [1-3]. Georgia holds the first position by the number of wheat species and especially by the number of cultivated endemic species of wheat, being at the same time distinguished for a high level of endemism [1-12].

The following cultivated natural endemic species have been found in Georgia (Western part of the country): Chelta Zanduri (*T. timopheevii* – 2n=28), Georgian Asli (*T. georgicum* – 2n=28), Makha (*T.macha* – 2n=42). The latter is a combined species, comprising two species: Gvatsa Makha (*T. tubolicum*) and Chelta Makha (*T. imereticum*), hexaploid Zanduri

(*T. zhukoskyi* – 2n=42) and Dika (*T. carthlicum* – 2n=28). Varieties of the latter species have been reported from Armenia and Turkey as well [1-12].

Study of Georgian endemic species of wheat, investigation of the processes of their evolution has allowed to conclude that a wide process of origination of wheat species took place on the territory of Georgia, which had a great impact on the process of evolution of the wheat genus. To the present day new species of wheat are created with the participation of *T.carthlicum* and *T. timopheevii* [1-3,12]. The above discussed facts show that investigation of genetic and selection values of Georgian endemic wheat species is significant not only from theoretical-scientific viewpoint, but is very important from the practical viewpoint too.

Material and Methods

The following wheat species - Georgian endemics (*T. dicoccoides, T.durum, T. turgidum, T. polonicum*); varieties (var. *timopheevii*, var. *viticalosum*, var. *chvamlicum*, var. *stramineum*, var. *rubiginosum*, var. *fuliginosum*, var. *zhukovskyi*, var. *subletschumicum*, var. *macha*, var. *palaeoimereticum*, var. *palaeo-colhicum*) and 32 aboriginal and selection varieties of Georgian wheat, held at the gene depository of the Department of Genetics, Selection and Seed Farming of the Georgian Agrarian Institute (later re-organized to the Georgian Agrarian University) have been used as initial material for the trials.

Methods of intraspecific hybridization, back-crossing and genetic analysis have been applied in our studies. Pollination of castrated spikes was carried out using methods of free (natural), compulsory (artificial) and compulsory (artificial)-free (natural) pollination. In order to obtain each combination 100 flowers were castrated and pollinated.

The obtained hybrid grains were sown outdoors, in the field. Their growth, inventory and monitoring were carried out according to the generally accepted methods.

Results and Discussion

Analysis of experimental material obtained as a result of crossing endemic and other species of Georgian wheat with aboriginal and selection varieties of soft wheat has shown that the process of crossing proceeds without any kind of difficulties.

It became clear that Georgian aboriginal varieties of soft wheat reveal non-uniformity while crossing with tetraploid and hexaploid species, i.e. they are heterogenous. The percentage of the setting of hybrid grains was found to be dependent on the female parent species participating in the process of crossing. When crossing tetraploid and hexaploid species, the setting of hybrid grains is high if tetraploid species is pollinated with the pollen grains of a hexaploid species. The rate of hybrid grains setting greatly depends on ecological-genetic peculiarities i.e. the genotype of the soft wheat variety. Crossing of East Georgian varieties of soft wheat with tetraploid and hexaploid species proceeds with difficulty, while crossing of West Georgian varieties is easier. The highest number of hybrid grains is obtained when the hybrid varieties of soft wheat participate in the process of crossing. At reciprocal crossing of species with different chromosome numbers the difference is noted in the extent of filling of hybrid grains of the first generation. Filling of hybrid grains is higher when the species with higher chromosome number participates in crossing as a female form (Table 1).

Study of the germination capacity of grains of the first hybrid generation has revealed a certain regularity between the rate of grain setting and germinability while crossing varieties of Georgian soft wheat with tetraploid wheat species (Table 2). When hybrid grains setting is high, germination percent of grains decreases and vice versa. Germination capacity of hybrid grains is significantly higher when soft wheat participates in crossing as a female form. Germination capacity of such hybrid grains is to a great extent dependent on the genotype of a soft wheat. Germination capacity of hybrid grains is higher when heterozygous soft wheat participates in the

Table 1. Setting of hybrid grains

NN	Hybrid combination	Number of pollinated flowers	Number of grain setting	Percentage of grain setting, %	±*m*	*t*
1	Georgian varieties of soft wheat x *T. durum* var. *coerulescens* canescens	5600	1461	26.1 (21.8-29.5) P=3.8	0.94	
	Reversed combination	5600	2978	53.1 (45.6-60.2) P=2.0	1.03	18.1
2	Georgian varieties of soft wheat x *T. turgidum* var. *striatum*	8000	2061	24.8 (20.0-31.4) P=4.1		
	Reversed combination	8000	3491	43.6 (41.7-54.5) P=2.7	1.3	11.8
3	Georgian varieties of soft wheat x *T. polonicum* var. *villosum*	8000	315	39.4 (32.0-50.0) P=3.4	1.01	
	Reversed combination	8000	479	59.8 (40.0-70.0) P=2.3	0.96	10.4
4	Georgian varieties of soft wheat x *T. carthlicum* var. *stramineum*	7000	1443	20.6 (11.2-27.8) P=3.4	0.71	
	Reversed combination	7000	2956	42.2 (28.6-52.2) P=2.5	1.03	14.05
5	Georgian varieties of soft wheat x *T. dicoccoides* var. *arabicum*	6300	1156	18.3 (11.8-24.6) P=2.9	0.56	
	Reversed combination	6300	819	28.8 (21.8-37.0) P=2.9	0.92	8.04
6	Georgian varieties of soft wheat x *T. Zhukovskyi*	5600	475	8.61 (3.3-14.7) PP=2.4	0.98	
	Reversed combination	5600	621	11.5 (5.4-17.9) P=2.1	1.02	3.4
7	Georgian varieties of soft wheat x *T. timopheevii*	1600	67	4.1 (3.3-5.0) P=2.0	0.7	
	Reciprocal combination	1600	35	2.5 (1.5-2.75) PP=2.3	0.5	2.3
8	*T. carthlicum* x *T. turgidum*	1400	462	33.0 (30.1-40.2) P=3.5	1.23	
	Reversed combination	1400	578	41.2 (39.5-47.0) P=2.7	1.08	4.5
9	*T.carthlicum* x *T.dicoccoides*	1100	316	28.7 (27.7-29.5) P=3.1	0.93	
	Reversed combination	1100	401	36.5 (35.5-38.0) P=2.3	0.81	5.6
10	*T.carthlicum* x *T.dicoccoides*	1800	610	33.0 (30.5-36.5) P=2.8	0.96	
	Reversed combination	1800	767	42.6 (41.5-45.5) P=1.8	1.18	4.8
11	*T. turgidum* x *T. durum*	700	1400	20.0 (18-23) P=4.3	0.87	
	Reversed combination	700	169	24.1 (22-28) P=4.3	1.0	2.2
12	*T. durum* x *T. dicoccoides*	1500	383	25.5 (25-26) P=4.2	1.1	
	Reversed combination	1500	459	30.6 (29.3-31.7) P=4.1	1.1	3.2

process of crossing. Hybrid grains, obtained as a result of crossing Georgian soft wheat with *T.timopheevii* or *T. Zhukovskyi* are characterized by a comparatively high viability if the female form in crossing process is *T. timopheevii* or *T. Zhukovskyi*. Grains of intraspecific hybrids are characterized by high germination capacity when the cultivated tetraploid or hexaploid species are both parental forms. Germination capacity of hybrid grains obtained as a result of crossing cultivated species of wheat with *T. carthlicum* is high when *T. carthlicum* is the maternal form while crossing (Table 2).

The following general regularity has been revealed in the survival capacity of intraspecific hybrid plants of the first generation (Table 2): there is a certain relationship between the setting of hybrid grains in species crossing and the capacity for survivability of plants of the first generation. When the percentage of hybrid grain setting is high, germination capacity and survival rate of plants of the first generation decrease and vice versa. The number of survived plants of the first generation is greatly affected by the level of expression of lethal genes.

Study of morphological traits in the hybrids of the first generation allowed to establish that dominant character of such traits as the absence of an awn, red or black coloration, red color of grain, downiness of leaves, fragility of spike.

Study of the duration of the vegetation period in hybrids of the first generation allowed to establish that inheritance of this index is of intermediate character.

Resistance to diseases in hybrids of the first generation is of dominant character when the hybrid

Table 2. Some indices of viability of hybrids of the first generation

NN	Hybrid combination	Germination %	Survival capacity %
1	Georgian varieties of soft wheat x *T. durum* var. coerulescens	59.4 ± 1.2 (55.5–66.4) P= 1.98	50.1 ± 1.6 (48–60.4) P= 3.2
	Reversed combination	30.7 ± 0.89 (28.2–36.8) P=2.3	25.1 ± 0.94 (23.2–30.9) P=3.6
2	Georgian varieties of soft wheat x *T. turgidum* var. striatum	54.1 ± 1.8 (48.2–61.6) P= 2.8	40.8 ± 0.78 (38.5–45.2) P= 2.4
	Reversed combination	25.9 ± 1.4 (21.1–32.1) P=4.04	20.9 ± 0.64 (17.6–25.3) P=3.1
3	Georgian varieties of soft wheat x *T. carthlicum* var. stramineum	61.6 ± 1.6 (60.1–65.4) P= 2.5	40.5 ± 1.2 (25.1–58.2) P= 3.1
	Reversed combination	34.5 ± 1.46 (32.7–37.2) P=4.2	28.0 ± 0.9 (13.5–39.7) P=3.5
4	Georgian varieties of soft wheat x *T. dicoccoides* var. arabicum	58.8 ± 1.36 (49.8–60.6) P= 2.44	2.9 ± 0.32 (2.0–5.1) P= 4.7
	Reversed combination	28.4 ± 0.86 (25.5–29.9) P=3.1	2.3 ± 0.11 (1.9–4.9) P=3.1
5	*T. carthlicum* x *T. turgidum*	93.0 ± 1.52 P= 1.6	57.1 ± 1.9 P= 3.3
	Reversed combination	72.4 ± 2.5 P=3.4	36.8 ± 2.0 P=3.3
6	*T. carthlicum* X *T. durum*	91.9 ± 2.1 P= 2.3	53.9 ± 1.7 P= 3.2
	Reversed combination	75.8 ± 2.4 P=3.2	40.5 ± 1.4 P=3.5
7	Georgian varieties of soft wheat x *T. timopheevii*	35.6 ± 1.2 P= 1.7	12.3 ± 1.6 P= 1.9
	Reversed combination	46.3 ± 1.5 P=1.9	24.4 ± 1.9 P=1.7
8	Georgian varieties of soft wheat x *T. zhukovskyi*	40.5 ± 1.9 P= 1.8	20.5 ± 1.9 P= 2.1
	Reversed combination	50.2 ± 2.1 P=2.1	29.5 ± 1.4 P=1.8

is obtained with participation of a resistant variety of soft wheat, or when *T. timopheevii* or *T. Zhukovskyi* are one of the parental forms.

Establishment of regularities according to the plant height, productive tillering, spike length, number of spikelets developed on a spike and number of grains in it, mass of single spike and mass of 1000 grains does not seem possible because of lethal or sublethal combinations obtained as a result of the effect of genes responsible for hybrid necrosis, red hybrid chlorosis and hybrid dwarfness. Heterosis was marked in the inheritance of plant height, productive tillering, spike length and number of spiklets per spike in the plants which survived from the effects of these genetic phenomena, while depression was revealed in the inheritance of such traits as number of grains per spike and mass of 1000 grains.

Heterosis is evident also in the process of inheritance of the content of protein and essential amino acids – lysine and tryptophan – in the first generation of hybrid grains. Level of heterosis is higher when one of the paternal forms while obtaining the hybrid is *T. carthlicum* (Dika). *T. timopheevii* (Chelta Zanduri) and *T. zhukoskyi* (hexaploid Zanduri).

Conclusions

The following has been revealed as a result of study of peculiarities of inheritance of crossability, germination capacity of hybrid grains and survival capacity of plants in combinations obtained as a result of reciprocal crossing of endemic and other species of Georgian wheat with varieties of soft wheat and issues of inheritance of traits of parental forms in the first generation:

1. Endemic and other species of Georgian wheat easily cross with varieties of soft wheat and do not require additional manipulations, but hybrid combinations do differ from each other by the level of this index.

2. Setting of hybrid grains is higher when species with lower number of chromosomes is pollinated with pollen grains derived from species with higher chromosome number.

3. Varieties of Kartli ecotype of soft wheat reveal more isolation in crossing with any of wheat species, while the hybrid varieties of soft wheat belonging to the West Georgian ecotype reveal more closeness.

4. Viability of hybrid plants and productivity is higher when a maternal form with higher chromosome number is pollinated with pollen grains belonging to species with small chromosome number.

5. Survival capacity of hybrid combinations and productivity is higher when soft wheat or endemic species of wheat (*T. carthlicum. T.timopheevii. T. Zhukovskyi. T. georgicum* or *T. macha*) play the role of a maternal plant.

6. Strength and doses of lethal complementary genes contained in the genotype of paternal forms are responsible for the level of viability and productivity of hybrid plants.

7. In the hybrids of the first generation dominant (absence of an awn, downiness, red and black coloration, red color of grain, keratoid consistence of grain, fragility of spike) and recessive traits (presence of an awn, white color, absence of downiness, white color of awn, flour-like consistence) have been established.

8. It has been established in hybrids of the first generation that inheritance of the duration of vegetation period is of intermediate character. Heterosis is expressed in spike length, number of spikelets and productive tillering, protein content of grains, while depression is marked in such indices as number of grains per spike, mass of single spike and mass of 1000 grains.

გენეტიკა და სელექცია

საქართველოს ხორბლის ენდემური სახეობების აბორიგენულ ჯიშებთან შეჯვარებადობა და ნიშან-თვისებები F1-ში

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ნაშრომში ნაჩვენებია, რომ საქართველო კულტურული ხორბლის წარმოშობის, ფორმათა წარმოქმნის და მრავალფეროვნების პირველადი ცენტრია, მსოფლიოს სხვა ქვეყნებს შორის გამოირჩევა ენდემიზმის მაღალი დონით. დასაბუთებულია, რომ საქართველოს ხორბლის უნიკალურმა ენდემურმა სახეობებმა ძალიან დიდი როლი შეასრულეს *Triticum*-ის გვარის ევოლუციაში და ხორბლის მსოფლიო სელექციაში. მათ საფუძველზე მიღებულია ხორბლის ახალი სახეობები, გვარები, კულტურები და ჯიშები. მათ გენოტიპშია ისეთი გენები, რომლებიც განაპირობებენ სრულიად ახალი ტიპის იმუნური, მაღალხარისხოვანი, ადაპტაციის მაღალი უნარის მქონე, ეკოლოგიურად უსაფრთხო პროდუქციის, მაღალ და მყარმოსავლიანი ჯიშების მიღებას. ნაშრომში განხილულია, საქართველოს ხორბლის ენდემური სახეობების სხვა სახეობებთან და რბილი ხორბლის აბორიგენულ და სელექციურ ჯიშებთან შეჯვარებადობის უნარიანობა, ჰიბრიდული მარცვლების და პირველი თაობის მცენარეთა სიცოცხლისუნარიანობა, პირველი თაობის მცენარეებში მშობლიური ფორმების მორფოლოგიური და სამეურნეო ნიშან-თვისებათა მემკვიდრეობის თავისებურებანი.

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Pharmacochemistry

Study of Poly[Oxy-1-Carboxy-2-(3,4-Dihydroxyphenyl) Ethylene] from *Symphytum asperum, S.caucasicum, S.officinale, Anchusa italica* by Circular Dichroism

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(Presented by Academy Member Devi Ugrekhelidze)

ABSTRACT. Comparative study of chirality of two carbon atoms C1 and C2 of poly[oxy-1-carboxy-2-(3,4-dihydroxyphenyl)ethylene] (POCDPE) preparations from different species of Boraginaceae family - *Symphytum asperum, S.caucasicum, S.officinale, Anchusa italica* revealed that they have one and the same absolute configuration in all preparations. It has been established that chiral atoms of POCDPE have either (1*R*,2*R*) or (1*S*,2*S*) configurations but not (1*R*,2*S*) and (1*S*,2R) configurations and consequently the denomination of the polymer is poly[oxy-(1*R*)-1-carboxy-(2*R*)-2-(3,4-dihydroxyphenyl)ethylene] or poly[oxy-(1*S*)-1-carboxy-(2*S*)-2-(3,4-dihydroxyphenyl)ethylene]. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *poly[3-(3,4-dihydroxyphenyl)glyceric acid], poly[oxy-1-carboxy-2-(3,4-dihydroxyphenyl) ethylene], 3-(3,4-dihydroxyphenyl)glyceric acid, Symphytum asperum, S.caucasicum, S.officinale, Anchusa italica, circular dichroism.*

In previous papers we have reported about the isolation and identification of biologically active poly[3-(3,4-dihydroxyphenyl)glyceric acid] (PDPGA) that is poly[oxy-1-carboxy-2-(3,4-dihydroxyphenyl) ethylene] (POCDPE) from *Symphytum asperum* – POCDPE-SA, *S. caucasicum* – POCDPE-SC, *S. officinale* – POCDPE-SO and *Anchusa italica* – POCDPE-AI, respectively [1-8] (Fig. 1). The repeating unit of POCDPE – 3-(3,4-dihydroxyphenyl)glyceric acid residue contains two asymmetric carbon atoms C1 and C2 (Fig. 2), but we had not yet any information concerning the chirality of these centers.

Besides, the enantiomers of 3-(3,4-dihydroxyphenyl)glyceric acid – the monomer of PDPGA – (+)-(2*R*,3*S*)-2,3-dihydroxy-3-(3,4-dihydroxyphenyl)-propionic acid [(2*R*,3*S*)-DDPPA] (Fig. 3) and (–)-(2*S*,3*R*)-2,3-dihydroxy-3-(3,4-dihydroxyphenyl)-propionic acid [(2*S*,3*R*)-DDPPA] (Fig. 4) were synthesized from *trans*-caffeic acid [9].

Results and Discussion. Circular dichroism (CD) spectra of (2*R*,3*S*)-DDPPA and (2*S*,3*R*)-DDPPA are given in Figs. 5 and 6, respectively.

CD spectra show positive Cotton effect at the wavelength 195 nm and negative one at the wave-

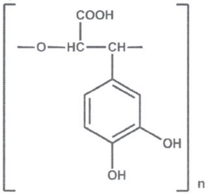

Fig. 1. Poly[oxy-1-carboxy-2-(3,4-dihydroxyphenyl) ethylene] (POCDPE).

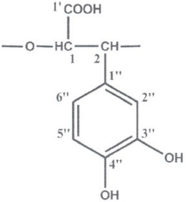

Fig. 2. Repeating unit of POCDPE-SA, POCDPE-SC, POCDPE-SO and POCDPE-AI.

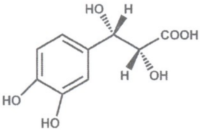

Fig. 3. (+)-(*2R,3S*)-2,3-dihydroxy-3-(3,4-dihydroxyphenyl)-propionic acid [(*2R,3S*)-DDPPA].

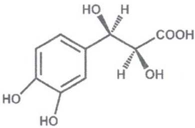

Fig. 4. (−)-(2*S*,3*R*)-2,3-dihydroxy-3-(3,4-dihydroxyphenyl)-propionic acid [(2*S*,3*R*)-DDPPA].

lengths 209, 228 and 275 nm for (2*R*,3*S*)-DDPPA (Fig. 5) and vice versa for (2*S*,3*R*)-DDPPA (Fig. 6). Thus, Cotton effects in CD spectra of (2*R*,3*S*)-DDPPA and (2*S*,3*R*)-DDPPA have opposite profiles (Fig. 7).

The CD spectra of POCDPE-SA, POCDPE-SC, POCDPE-SO and POCDPE-AI have similar profiles and show positive Cotton effects at 194, 214, 280, 286 nm and a negative one at 204, 236 nm (Figs. 8 and 9). These data confirm that two chiral carbon atoms C1 and C2 of the repeating unit of POCDPE-SA, POCDPE-SC, POCDPE-SO and POCDPE-AI (Fig. 2) have one and the same absolute configuration.

Cotton effects in CD spectra of (2*R*,3*S*)-DDPPA and (2*S*,3*R*)-DDPPA are shifted about 5-11 nm

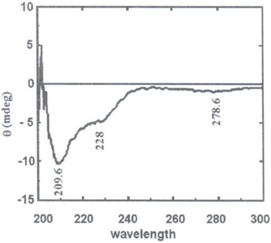

Fig. 5. CD spectrum of (*2R,3S*)-DDPPA.

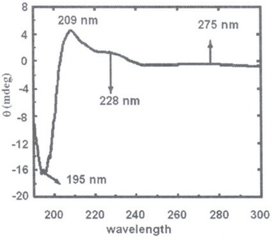

Fig. 6. CD spectrum of (2*S*,3*R*)-DDPPA.

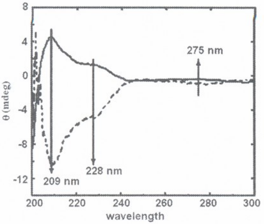

Fig. 7. *Comparison of CD spectra of (2R,3S)-DDPPA* (- - -) and (2*S*,3*R*)-DDPPA (——)

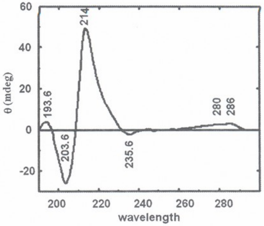

Fig. 8. CD spectrum of POCDPE-SA.

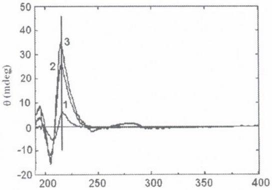

Fig. 9. CD spectra of POCDPE-SC(1), POCDPE-SO(2) and POCDPE-AI(3).

towards lower wavelengths vs Cotton effects in CD spectra of POCDPE-SA, POCDPE-SC, POCDPE-SO and POCDPE-AI.

Cotton effects at 195, 209, 228 and 275 nm in CD spectra of monomers – (2*R*,3*S*)-DDPPA and (2*S*,3*R*)-DDPPA correspond to Cotton effects at the 204, 214, 236 and 286 nm in CD spectra of POCDPE-SA, POCDPE-SC, POCDPE-SO and POCDPE-AI. Besides, as the profiles of CD spectra of polymers (Figs. 8 and 9) do not coincide with CD spectra of (2*R*,3*S*)-DDPPA and (2*S*,3*R*)-DDPPA (Fig. 7) we can exclude (1*R*,2*S*) and (1*S*,2*R*) configurations for chiral atoms C1 and C2 in the repeating units of POCDPE-SA, POCDPE-SC, POCDPE-SO and POCDPE-AI (Fig. 2). Hence, we can suppose that C1 and C2 atoms have either (1*R*,2*R*) or (1*S*,2*S*) configurations and consequently denomination of polymer is poly[oxy-(1*R*)-1-carboxy-(2*R*)-2-(3,4-dihydroxyphenyl)ethylene] or poly[oxy-(1*S*)-1-carboxy-(2*S*)-2-(3,4-dihydroxyphenyl)ethylene].

Establishment of the absolute configuration of these C atoms is a topic for further research.

Materials and Methods. *CD spectra were performed* on a Jasco J-715 instrument (Jasco Co, Tokyo, Japan) equipped with Peltier temperature control system. For all measurements, 1 mm path length quartz cells, 1 nm bandwidth, 0.2 nm resolution, 1 s response and a scan speed of 50 nm/min for each spectrum were used.

CD spectrum of (2*R*,3*S*)-DDPPA (C = 0.56×10^{-3} M, H_2O): $\Delta\varepsilon_{275} = -1$, $\Delta\varepsilon_{228} = -5$, $\Delta\varepsilon_{209} = -10$, $\Delta\varepsilon_{195} = +5$.

CD spectrum of (2*S*,3*R*)-DDPPA (C = 0.5×10^{-3} M, H_2O): $\Delta\varepsilon_{275} = -0.4$, $\Delta\varepsilon_{228} = +1.8$, $\Delta\varepsilon_{209} = +4.8$, $\Delta\varepsilon_{195} = -16$.

CD spectrum of POCDPE-SA (C = 1.2 mg/10 ml, H_2O): $\Delta\varepsilon_{286} = +3$, $\Delta\varepsilon_{236} = -2$, $\Delta\varepsilon_{214} = +50$, $\Delta\varepsilon_{204} = -28$, $\Delta\varepsilon_{194} = +4$.

ფარმაკოქიმია

Symphytum asperum-ის, *S.caucasicum*-ის, *S.officinale*-ს, *Anchusa italica*-ს პოლი[ოქსი-1-კარბოქსი-2-(3,4-დიჰიდროქსიფენილ)ეთილენის] შესწავლა წრიული დიქროიზმის მეთოდით

ვ. ბარბაქაძე*, მ. მერლანი*, ლ. ამირანაშვილი*, ლ. გოგილაშვილი*, კ. მულკიჯანიანი*

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(წარმოდგენილია აკადემიის წევრის დ. უგრეხელიძის მიერ)

Boraginaceae-ს ოჯახის სხვადასხვა სახეობების (*Symphytum asperum, S. caucasicum, S.officinale* და *Anchusa italica)* პოლი[ოქსი-1-კარბოქსი-2-(3,4-დიჰიდროქსიფენილ)ეთილენის] (პოკდფე) პრეპარატების ორი ნახშირბად ატომის C1 და C2 ქირალობის შედარებითმა შესწავლამ აჩვენა, რომ ყველა პრეპარატში მათ აქვთ ერთი და იგივე აბსოლუტური კონფიგურაცია. დადგენილია, რომ პოკდფე-ს ქირალურ ატომებს აქვთ ან (1R,2R) ან (1S,2S) კონფიგურაციები, მაგრამ არა (1R,2S) და (1S,2R) კონფიგურაციები და აქედან გამომდინარე პოლიმერის სახელწოდება შეიძლება იყოს პოლი[ოქსი-(1R)-1-კარბოქსი-(2R)-2-(3,4-დიჰიდროქსიფენილ)ეთილენი] ან პოლი[ოქსი-(1S)-1-კარბოქსი-(2S)-2-(3,4-დიჰიდროქსიფენილ)ეთილენი].

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Ecology

Bioecological Peculiarities of Introduced Ornamental Plants in Batumi Botanical Garden

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ABSTRACT. The view of green infrastructure of the Black Sea coast in Georgia depends much on ornamental plantations. A lot of evergreens and deciduous ornamental plants were brought to Ajara for decorative greenery of resorts and populated areas. Being situated in the subtropical zone of Georgia - on the Black Sea shore - Ajara has beautiful climatic conditions for most subtropical plants. Nevertheless, natural conditions cause some problems to the introduction of ornamental plants and their adaptation to the environment. The selection, reproduction, implantation and bioecology of ornamental plants were studied. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *Forsythia viridissima* Lindl., *Gardenia Jasminoides* J.Ellis, *biocological pecularities.*

The main problem of plant introduction is to develop proper environmental conditions for plants. Acclimation transforms inner biological processes in plants in accordance with new environmental conditions. It is also important for a plant to be adapted to the soil which makes the introducing process easier.

A plant has an inner vital force which provides for its adaptation to different ecological conditions. According to S.Sokolov [1] introduction of plants is a complex of means and methods, which help the plant to acclimate.

During the life period a plant reveals sensibility to the environment. Practical importance of introduction is considerable. It permits to study plants under abiotic and new enviromental conditions. Introduction is considered to be a source of reproduction of natural populations. Their biological, ecological, physiological and biochemical characteristics are studied. The reasons of weakening of plants must be determined.

Introduction of plants on the Black Sea coast of Ajara was started in the 1880s by M.Dalfons, P.Tatarinov, A.Krasnov, I.Gordeziani, G.Mkheidze, N.Sharashidze and others [2]. Favorable climate and rich soil at the Black Sea coast of Ajara create favorable conditions for introduction of plants from different subtropical regions.

Purposeful introduction of plants at the Batumi Botanical Garden began in 1912, when the park was established, and it continues to the present day.

The aim of the study. We aimed to study bio-ecological peculiarities of ornamental shrubs

Fig. 1. Flowering shrub of *Forsythia viridissima* Lindl.

Fig. 2. Flowering *Gardenia Jasminoides* J.Ellis

greenstem Forsythia (*Forsythia viridissima* Lindl) and evergreen Gardenia Jasminoides (*Gardenia Jasminoides* J.Ellis), Practical recomendations to introduce them into the green planting (laying out the parks, etc.) in Ajara were worked out.

Materials and Methods. Phenological observations were made according to the Beidemann method [3]. The following phenophases: swelling and breaking of buds, appearing of flowers, seed ripening, leaf fall, end of vegetation - were included in the programme of phenological observations.

Forsythia (*Forsythia viridissima* Lindl.: *Oleaceae*) is native to Central and East China. It is a deciduous ornamental bush about 4 m high with upright dark green branches (Fig.1).

Leaves are lanceolate-oblong 7-15 cm long, dark green, dentate, acuminated at the tip, wedge-shaped at the butt. Floral buds develop at the end of February-early March (10-14 °C). The flowers are produced in mid-March-mid-April before the leaves. They are large, slightly bell-shaped, sulphureous, 1-3 flowers with erect or bent petiole are clustered, weak fragrance; duration of flowering 25-30 days. Fruit is an egg-shaped small box 1.5 cm in length containing several winged seeds. Forsythia flowers regularly but fruits rarely. The fruit ripens in September-October (Table). The plant was introduced at Batumi Botanical Garden in 1912 [2, 4].

Forsythia viridissima Lindl. grows fast and flowers best in full or partial sun. It grows well in moist soils. Soil should be well drained

The forsythia plant really benefits from pruning. Regular pruning keeps the plants within bounds. Pruning promotes better branching and flowering in future years.

Deciduous Forsythia is distinguished for its ornamentality. Forsythia bushes are attractive singly and in small groups on lawns due to their early and long florescence; they may be used as hedgerow as well. Branchlets cut in January-February and put into water for 8-10 days will make nice bunches with gold flowers which can decorate any interior.

Forsythia is resistant to pests. At bacteriosis eradication is recommended. Due to ornamental features Forsythia is used widely in laying out parks on the Black Sea coast of Ajara.

Gardenia Jasminoides (*Gardenia Jasminoides* J.Ellis: *Rubiaceae*) is native to China, Japan. Gardenia Jasminoides is a bush of 2 m high with branched stem (Fig.2).

Leaves are large, wide lanceolate, leather-like, shiny dark green with entire edges, pointed at tip and narrowed at the basis. Flowers are highly fragrant, large, 10 cm in diameter, single or bunches (3-5 flowers), located at the branch top or at the leaf pocket. Flowers start with white, becoming yellowish later. Gardenia flowers in summer. The fruit is berry [4,5].

Gardenia likes warm, light and sunny climate, grows well at open sunny sites. Sudden change of temperature causes flower fall. The ground must be

Table. Phenophases of *Forsythia viridissima* Lindl. and *Gardenia Jasminoides* J.Ellis under conditions of the Batumi Botanical Garden

Phenophases \ Species		Forsythia	Gardenia Jasminoides
Time of bud break		2nd half of February	Early June
Flowering	Beginning	Mid-March	Early July
	Ending	Mid-April	End of August
Fruit maturity		September-October	Mid-October
Growth of sprouts	Beginning	Early May	2nd half of June
	Ending	1st half of September	Early September
Length of 1-year growth, cm		14-17	7-10

acidic, permanently humid, needs moderate irrigation with lukewarm and soft water. Cold water causes leaf yellowing. During budding from spring till the end of summer it needs regular feeding with chemical fertilizer 20 g / l water. Feeding is not needed in autumn-winter. Budding begins at 16-18 °C; vegetation sprouts begin to grow intensively at 20-24 °C and floral buds develop.

The best time in Ajara to transplant gardenia is spring. Gardenia bush is fluffy, regulation of sprout length is necessary. Usually it is done after the end of flowering or when transplanting. One third of annual growth length must be cut. By pruning a Gardenia bush can be given any form. Rooting of cuttings goes fast in spring, transplantation needs acidic soil. During flowering the soil must be moderately moist.

Forsythia is easily reproduced by green or hardwood cuttings and by layering. Gardenia is reproduced by top sprouts, semi-wood cuttings and seeds. In order to study the conditions under which these plants are well reproduced we took green cuttings of Forsythia in June and semi-green cuttings of Gardenia in September (each 100 samples). The Forsythia cuttings were planted in the moist sand at 15-20 °C and the Gardenia cutttings - in the sand mixed with peat at 20-25 °C. In order to speed up rooting a solution of the growth stimulator "Kornevin" was used. Index of their germination was 100%.

Results. The study and observations allow to conclude that under conditions of Batumi Botanical Garden Forsythia and Gardenia have high adaptation ability, fast growth rhythm, regular flowering and fruit bearing, high productivity, which allows to use them in ornamental greenery of the sea coast of Ajara singly and in small groups.

ეკოლოგია

ინტროდუცირებულ დეკორატიულ მცენარეთა ბიოეკოლოგიური თავისებურებანი ბათუმის ბოტანიკური ბაღის პირობებში

ე. მაჭუტაძე

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(წარმოდგენილია აკადემიის წევრის ა. კორახაშვილის მიერ)

საქართველოს შავიზღვისპირეთის დეკორატიული ინფრასტრუქტურის იერსახე ბევრადაა დამოკიდებული ინტროდუცირებულ დეკორატიულ მცენარეთა ნარგაობაზე. ქალაქების, საკურორტო პარკების და დასახლებული ადგილების გასამწვანებლად წარმატებით გამოიყენება როგორც მარადმწვანე, ისე ფოთოლმცვენი დეკორატიული მცენარეები. ინტროდუცირებულ დეკორატიულ მცენარეთა მრავალმხრივი მნიშვნელობიდან გამომდინარე, ობიექტურად ისახება მისი ბუნებრივი გავრცელების პირობებიდან შეცვლილ გარემო პირობებში კომპლექსურად გამოყენების ამოცანა, რომლის დროსაც ყურადღება ეთმობა დეკორატიულ მცენარეთა ბიოეკოლოგიური საკითხების შესწავლას, შერჩევას, გამრავლებას და დანერგვას.

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Palaeobiology

A New Site of the Neogene Vertebrate Fauna from Kaspi District

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ABSTRACT. Hipparion fauna is of major importance for dating the Neogene fossil-bearing sediments. There are a number of sites with Hipparion fauna in the Southern Caucasus. Out of them only two are dated as Upper Sarmatian: Eldari (Azerbaijan) and Iaghludja (Georgia). Fossil bearing Sarmatian sediments were recently found in Kaspi district, Georgia. Fossil fauna is not diverse, however, key taxa undoubtedly date this site as Late Sarmatian. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *Neogene, vertebrate fauna, Sarmatian, Caucasus, Kavtiskhevi.*

Several years ago the head of the Kaspi archeological expedition Dr. Z. Makharadze informed us that in the environs of Kavtiskhevi village in the sandstones he had seen fossil animal bones. We got interested in this information, a paleontological expedition: D. Lordkipanidze, A. Vekua, G. Megrelishvili, G. Bidzinashvili, Z. Meskhi, J. Agusti and O. Oms immediately went to Kavtiskhevi. The information provided by Makharadze was confirmed and we started explorations and fossil collection.

Geological setting of the region

The research area is located in the central part of Kartli intermontane depression, in the Kura basin, between the villages of Metekhi and Dzegvi. Strata building of the region encompasses almost all stratigraphic units starting with the Upper Cretaceous (Turonian-Santonian) up to the Quaternary sediments inclusively.

Neogene sediments are well represented in the area and have two major lithological facies. The lower facies is represented by a succession of sandstones and claystones and is known as Sakaraulo horizon [1]. Faunistically this horizon was described by G. Kharatishvili [2].

The upper facies is mainly represented by Maikopian clays (Kotsakhuri horizon) which are conformably followed by an almost complete Middle Miocene


a. b.

Fig. 1. General view of the Kavtiskhevi site (a) and fossil-bearing sediments (b).

stratigraphic section.

The scope of our interest lies in Late Neogene fossil-bearing sediments and our discussion will deal with them.

Geologists identify three different horizons in the Sarmatian sediments:

– Early Sarmatian – blue sandstone-mudstone layers, with frequent inclusion of dense greywacke carbonate sandstones with alteration of gravelite lenses. These sediments are exposed on the northern slopes of the Vakistavi mountain and to the North of the Sachite mountain.

– Middle Sarmatian – characterized by variegated facies and mainly found in the central part of Vakistavi-Sachite series. The Middle Sarmatian sediments in general are represented by alterations of sandstones and loam. Lamellar structure of clays is well observable. Sandstones are dense and characterized by carbonate cementation. Rhythmic alterations of shallow sea and continental facies are well observable on these sediments, which are reflected in the lithological diversity and variety of colours. Continental coloured clays and seashore sandstones with midlayers of microconglomerates follow one another in the section. According to geologists, the entire formation can be considered as an analogue of David-Garedji coloured formation which is dated as Sarmatian.

– Late Sarmatian in the region is represented by continental facies: mainly by thick layers of sandstones with middle layers of carbonate gravelites. Loams are brownish, without structure, weakly sandy. Upper Sarmatian sediments are distributed in the environs of Vakistavi-Sachite hills (North- East of the town of Kaspi, along the left bank of the Kura river).

Magnetostratigraphic sampling of the the Kavtiskhevi area took place at two locations. First, at the homonymous macromammals site, which is an isolated outcrop. Second, in a continuous section found at the trench of the road leading to Kaspi, to the north of the Kavtiskhevi village. The site can be easily tracked to the middle of the section, which accounts for some 160 meters of clays and sandstones. In the upper part of the section (135 m thick), a level with microfauna (KJ17 - KJ 18) is also found. Both section and macromammals site are found in strongly dipping strata.

The general paleomagnetic behavior displays a strong post-tilting remagnetization affecting most samples, but in some cases, a high temperature characteristic component can be recognized, being previous to tilting. Thus, in the lower part of the Kavtiskhevi section (including the macromammals site), basically reverse polarities are found. Towards the upper part of the section, some normal polarity samples are found, including the micrommals site KJ17 - KJ 18. Regarding correlation to the standard polarity scale, the occurrence of lower Vallesian (MN-9), indicates that KJ17-18 is located within the normal chron of C5n (9.8 to 10.9 Ma), while the macromammals site would be in the younger part of C5r (lasting from 10.9 to 11.9 Ma).

Paleontological material

As was anticipated, Hipparion fauna was discovered in Kavtiskhevi. This faunal complex was widely distributed during Mio-Pliocene all over the world, except Australia and South America. Neogene faunas with open landscape animals and three digit horses (*hipparions*) are known as Hipparion faunas. Hipparions are the dominant faunal elements in these faunas.

The beginning of the Tertiary is marked by intensification of xerophytisation, which becomes especially remarkable starting with the Middle Miocene. Vegetation characteristic of savannas and steppes becomes dominant, forested areas diminish. This universal process unavoidably induced essential changes in faunal composition. Animals get adapted to and get used to coarse fodder, which is followed by corresponding changes in their bodies, especially tooth masticatory apparatus. Teeth become high crowned, enamel becomes more folded and thicker, accessory elements appear on teeth masticatory surface [3]. These transformations reinforce the stability of teeth against abrasion.

Hipparion faunas in Eastern Georgia are found at the sites: David-Garedji, Dzedzvtakhevi, Arkneti, Bazaleti and Kvabebi. Among them David-Garedji fauna is especially diverse.

It was not a difficult task to find fossil-bearing sediments in Kavtiskhevi. With the aid of archeologists we quickly founda fossil-bearing horizon. Unfortunately, we failed to find any significant accumulation. Very eroded fossils are found in lenses. Isolated teeth and limb bones of *Hipparion* represent the majority of fossils. Bones are found between coarse sandstones, rather cemented and are difficult to prepare. Fossil-bearing layer has latitudinal distribution and creates a positive relief against the background of claystone and sandstone layers.

Kavtiskhevi fauna is evidently Hipparion fauna and is represented by accompanying faunal elements. Despite the poor composition of Kavtiskhevi fauna (this shortcoming can be improved as a result of excavations), the already found material is very important for dating geological structures of Eastern Georgia as well as for reconstruction of Sarmatian geographic situation. The discovered fossils belong to the following animals:

Reptilia: *Testudo* sp.

Mammalia:

- Giraffidae: *Achtiaria borissiaki* Alexeev
- Bovidae: *Gazella* sp., *Tragocerus* aff. *leskovitschi* Borissiak
- Cervidae: *Cervus* sp.
- Suidae: *Microstonyx erimanthius* (Roth et Wagner)
- Equidae: *Hipparion* aff. *eldaricum* Gabunia
- Hyaenidae: *Crocuta* sp.
- Proboscidea: *Choerolophodon pentelici* (Gaudry et Lartet)

It was noted above that in the Kavtiskhevi fossil material mainly *Hipparion* remains are found (mandibles, isolated teeth and limb bones). Fortunately, among these finds are almost comlete hipparion mandible (Kav. 5, Fig. 2, Tab. 2) and left hemimandible (Kav. 85). All the teeth are in place on the complete mandible including canines and incisors.

About 10 isolated moderately and strongly worn upper teeth of *Hipparion* were found. Morphological features are well preserved, which allows us to make taxonomic identification of the Kavtiskhevi *Hipparion*.

Kavtiskhevi *Hipparion* is of a moderate size. Upper teeth are of a moderate size, protocone index is moderate (P^4-19.7; M^1-M^2 – 27.4). Protocone is somewhat elongated and relatively low. Double loop

Fig. 2. *Hipparion* cf. *eldaricum* from Kavtiskhevi, mandible (Kav. 5), occlusal view.

of lower teeth has open wings and has a typical shape characteristic of hipparions. Outer bosom is rather deep, especially in molars.

The length of the horizontal ramus of the mandible from incisors to the ascending ramus is 247 mm; P2-M3 alveolar length is 142.5 mm; width of the mandible at the premolars is 75 mm; height in front of P2 is 45 mm. P1 is not present and has never been present. Diastema between canine and P2 is rather long (61 mm). This indicates that the muzzle of the Kavtiskhevi hipparion was of a moderate size. Incisors are well developed, with closed loops on the occlusal surface. Canine is small, cylindrical and is placed immediately next to the I3. Accessory elements are not observable. Molars are rather large (MP index on Kav. 5 - 93.5, and on Kav. 89 - 93.6). With this character Kavtiskhevi hipparion resembles *Hipparion elegans* found in the Chachuna steppe (Eastern Georgia).

Hipparion limb bones are few among the collected fossils. The only relatively complete metacarpal (Kav. 101, Fig 3) is strongly damaged, yet it is possible to measure it and to observe some morphological characteristics. Metacarpal is relatively long; its total length is 222 mm; transversal diameter of the proximal end is 90.7 mm, and of the distal end – 34.0 mm. Kavtiskhevi *Hipparion* is remarkably smaller than Eldari and David Gareji hipparions according to dimensions of metacarpal as well as of the teeth, but larger than Dzedzvtakhevi *Hipparion* (*H. elegans*) [4].

Dimensions and morphological features of dentition and metacarpal reveals some similarities with Eldari hipparion, however there is not a complete coincidence in morphological characteristics between these two forms. Yet, despite this fact, we still attribute Kavtiskhevi hipparion to the Eldari form and define it as *H.* aff. *eldaricum* Gabunia.

The presence of mastodon in the Kavtiskhevi fauna is of stratigraphic significance and dates the

Fig. 3. *Hipparion* aff. *eldaricum* from Kavtiskhevi, Mc (Kav. 101), dorsal view.

Fig. 4. *Choerolophodon pentelici* from Kavtiskhevi, DP2 (Kav.2, to the left) and dp2 (Kav.3, to the right), occlusal view.

fauna as Sarmatian-Maeotian. It should be noted that only two mastodon teeth are present in the collection – DP2 (Kav. 2) and dp2 (Kav. 3) (Fig.4). Nevertheless these teeth possess taxonomically important features which allows to attribute these fossils to *Choerolophodon pentellici*. *Choerolophodon pentellici* is present in Neogene sediments of the Caucasus, Moldova, Ukraine, Greece and other European localities and is considered to be a form characteristic of Sarmatian-Maeotian age.

The fossil suidae – *Microstonyx erimanthius* (Fig. 5) is another stratigraphically important element in Kavtiskhevi fauna. As a rule, this form is found in Sarmatian faunas, especially in the Caucasus. Presence of *Achtiaria*, *Tragocerus* and *Gazella* convincingly attest to the Upper Sarmatian age of Kavtiskhevi, but due to scarcity of material we will not dwell on their description and distribution.

The presence of *Cricetulodon* sp. in Kavtiskhevi, in a layer above a large fossil mammal site confidently dates the site as Late Sarmatian. In Spain *Cricetulodon* is a typical form from the early Vallesian, MN9, which is correlated with the early Tortonian (sites of Can Ponsic and Can Llobateres). *Cricetulodon* is present in a number of sites from western Turkey, such as Bayraktepe 1 (MN8) and Mahmutkoy (MN9).

Fig. 5. *Microstonyx erimanthius* from Kavtiskhevi, M_3 (Kav.114), occlusal view.

In 1966 a group of geologists (L. Gabunia, K. Matskhonashvili, D. Chkheidze) found a rather rich fossil vertebrate site on the right bank of the river Kura. This site is known as Iaghluja site in the paleontological literature. Based on its faunal composition, they established the age of Iaghluja site as Upper Sarmatian and published a preliminary faunal list [5]. Relatively later this site was excavated and its fauna was studied by G. Meladze [6]. According to this author, the following taxa are present in Iaghluja:

Reptiles: *Testudo* sp.; Aves: *Rustaviornis georgicus*; Mammalia: Melinae sp., *Premophitis* ex gr. *maeotica*, *Adcrocuta eximia*, *Machairodontinae* gen.?, *Choerolophodon pentellici*, *Hipparion* cf. *eldaricum*, Chalicotheriinae gen.?, *Procapreolus* sp., *Oioceros* sp., *Dicerorhinus* sp., *Aceratherium* sp., *Microstonyx* sp., *Oioceross* aff. *atropatenes*, *Gazella* sp., *Paraoioceros improvisus*, *Tragocerus* sp., *Palaeotragus* sp. [7]. It is evident from this faunal list that Iaghluja fauna has many elements in common with the Kavtiskhevi fauna. Geologically, they come from the same stratigraphic layers and represent one faunal complex.

Animals of different ecological niches are present in the Kavtiskhevi fauna, yet, open savanna and steppe animals predominate. This indicates that

Table 1. Measurements and comparison of the upper teeth of *Hipparion* aff. *eldaricum* from Kavtiskhevi.

	Hipparion aff. *eldaricum* Kavtiskhevi	*Hipparion eldaricum* Eldari Gadjiev, 1997	*Hipparion garedzicum* Garedzi Gabunia, 1951	*Hipparion elegans* Pavlodar Gromova, 1952
DMD P^3	22.4	27.6	24	22.6
DVL P^3	23	24.7	22.8	22.8
L protocone	5.4	–	–	–
H protocone	3.4	–	–	–
Protocone index	24.1	36.9	30.2	28.4
DMD M^1	27.9	26.1	21.2	19.4
DVL M^1	20.2	21.3	22.3	20.2
L protocone	5.5	–	–	–
H protocone	1.9	–	–	–
Protocone index	19.7	34.4	31.8	33.6
DMD M^2	28.1	–	–	–
DVL M^2	23.1	–	–	–
L protocone	7.7	–	–	–
H protocone	2.7	–	–	–
Protocone index	27.4	–	–	–
DMD M^3	24.5-26.3	22.5	–	19.8
DVL M^3	20-21.4	17.2	–	17.3
L protocone	6.1; 6.6	–	–	–
H protocone	2.7; 4.5	–	–	–
Protocone index	26.9; 25.2	–	–	35

Table 2. Measurements and comparison of the lower teeth of *Hipparion* aff. *eldaricum* from Kavtiskhevi site

	Hipparion aff. *eldaricum* Kavtiskhevi mandible Kav.5	*Hipparion* aff.*eldaricum* Kavtiskhevi mandible Kav. 85	*Hipparion eldaricum Eldari Gabunia, 1959*	*Hipparion garedzicum Garedzi Gabunia, 1951*	*Hipparion elegans Pavlodar Gromova, 1952*
L P_2-M_3	142.5	135.2	–	–	–
L P_2-P_4	73.8	70	–	–	–
L M_1-M_3	69	65.5	–	–	–
M/P index	93.5	93.6	–	–	–
DMD P_2	27	25	27.2	31.0	25.5
DVL P_2	16.3	15.5	15.1	15.7	13.5
DMD P_3	23.4	21.8	24.2	25.2	21.2
DVL P_3	15.1	15.5	16.2	16.7	15.3
DMD P_4	23.0	22.0	–	–	–
DVL P_4	14.4	14.8	–	–	–
DMD M_1	20.0	18.2	27.8	26.4	19.2
DVL M_1	15.9	14	14.8	15.3	12.5
DMD M_2	19.5	20.4	–		–
DVL M_2	13.7	15.2	–		–
DMD M_3	26.5	26.3	27.8		22.9
DVL M_3	11.7	16.7	13.1		10.7

relatively open landscape developed in this region at the end of the Sarmatian and beginning of Maeotian. Climate most probably was relatively warm and humid.

Paleontological excavations in Kavtiskhevi continue. It is very probable that many interesting fossils will be unearthed.

პალეობიოლოგია

ნეოგენურ ხერხემლიანთა ახალი ადგილსაპოვებელი კასპის რაიონში

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ჰიპარიონულ ფაუნას დიდი მნიშვნელობა აქვს ნეოგენური ფაუნის შემცველი ნალექების დათარიღებისათვის. სამხრეთ კავკასიაში ჰიპარიონული ფაუნის ადგილსაპოვებელი რამდენიმეა, მაგრამ მათგან ზედა სარმატული მხოლოდ ორია – ელდარი (აზერბაიჯანი) და იაღლუჯა (საქართველო). ახლახან კასპის რაიონში აღმოჩნდა ზედა სარმატული ფაუნის შემცველი ნალექები. ფაუნა არაა მრავალფეროვანი, მაგრამ დამახასიათებელი ფორმები ზუსტად ათარიღებენ მას ზედა სარმატულად.

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History

Georgia and Armenia's Relations in 1918-1920 and Georgia's Constituent Assembly

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(Presented by Academy Member David Muskhelishvili)

ABSTRACT. **The paper highlights the position of the Constituent Assembly of Georgia on the relations between Georgia and Armenia in 1918-1920. It is emphasized that the relations of the Democratic Republic of Georgia with the Republic of Armenia during the period of their independent existence were rather complicated and volatile.**

Georgia and Armenia acknowledged each other's sovereignty only in March of 1919, and on November 3 of the same year they also concluded two agreements in Tbilisi. The first one envisioned the solution of debatable questions by obligatory arbitration, as for the other, it envisioned free transit for Armenia's goods on Georgia's railway.

The Constituent Assembly, the supreme legislative body of the Democratic Republic of Georgia ratified both agreements with the majority of votes in spite of the resistance of the oppositional parties. Armenia's Parliament also confirmed the agreements successfully.

Georgia's Constituent Assembly immediately reacted to the issues of the day, connected with the relations with Armenia and expressed its impartial opinion. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *Georgia, Armenia, Constituent Assembly, free transit, arbitration agreement.*

The relations of the Democratic Republic of Georgia with the Republic of Armenia during the period of their independent existence were rather complicated and changeable. These two states that appeared almost simultaneously (in May of 1918) in the same city (Tbilisi) acknowledged one another only *de facto* and abstained from recognizing each other *de jure* for ten months. The main reason of this was that there were no fixed frontiers between the two newly established republics. In spite of the fact that the Georgian state was established on its historic territory it could not escape the unjust claims of its neighbors, among them of Armenia.

The authorities of the Democratic Republic of Georgia began establishing the state frontiers of the country according to the historic, economic, strategic and other parameters. As the publicist of the newspaper "Sakartvelo" said, "Georgia took a definite and quite correct position from the very start; I do not want anything of anybody else, and I will not give you anything belonging to me. Georgia has not declared any claims to the historic territory of any of its neighboring nations, or strategic points of these states, and it has not touched the vital interests of any of its neighboring peoples. Georgia demands the same kind of attitude of the neighboring peoples [1: 1]," but the neighboring states acted differently.

The governmental circles of Armenia requested that the frontiers in the districts, considered to be debatable, should be fixed according to the principle of ethnic and real population in them. They wanted to take the Lore-Borchalo, Akhaltsikhe and Akhalkalaki districts from our country. They had been conditioned by the colonial policy of the Russian Empire and it was not the result of the natural development of these parts of the country.

The authorities of Armenia did not take part in the work of the conference of south Caucasian republics, appointed in Tbilisi, and tried to get hold of the disputed territories by war.

On December 9, 1918 Armenia's armed forces invaded the Borchalo district and occupied its main strategic points in several days. The government of Armenia presented an ultimatum to Georgia that it should free the territory, including Tbilisi [2].

The Georgian regular army and the units of people's guards liberated the districts, occupied by the enemy in a short time and restored the *status quo ante bellum* [3]. After the Georgian armed detachments had made the enemy retreat the commanders of the army of the Allies, deployed in the Southern Caucasus, intervened. Following their advice, the military operations were stopped on December 31 and the disputed territory was declared the neutral zone of Lore. Before the question was finally solved the Georgian and Armenian armies were to stand there in turn [4:13].

The above-mentioned incident was called a military conflict between Georgia and Armenia, the war of two neighboring republics. The bloody clash whose victims exceeded 1000 people on both sides [5: 460-462], let alone other damage, did not settle the contradiction of interests between the neighboring states. Later on the Prime Minister of the Armenian Republic O. Kajaznuni said, "The war made us think of many things. We had fought against the neighbor with whom we should have had the closest contacts. It was Georgia through which we got in touch with the outer world. We felt it and we really wanted to have friendly relations with the Georgians, but we were unable to do it. Alongside the positions of Georgia's government, a certain part was played by the fact that we were weak, politically undeveloped and lacked the ability of governing the country" [6:24].

The leaders of Georgia and Armenia placed their hopes on the Paris Peace Conference in vain too. The leaders of the allied states, busy with large-scale matters, did not show great interest in settling the debatable questions of the small Caucasian republics, It was the economic necessity that stimulated the drawing together and partner cooperation between Georgia and Armenia that were on the way of building independent states.

More attention was paid to diplomatic formalities in Tbilisi and Yerevan. In March 1919 when the Constituent Assembly, Georgia's supreme legislative body, began to work the government of the Armenian Republic, Armenia's Parliament and the National Council of Armenians living in Georgia sent telegrams of congratulation to mark this event. The Presidium of the Constituent Assembly considered these congratulations to be an especially important and noteworthy fact and in contrast to many other telegrams, given to the press for publication, they were read at the plenary meeting straightaway [7: 32, March 14].

The Constituent Assembly of Georgia answered the received congratulation telegrams with telegrams of thanks.

The democratic Republic of Georgia was the first to make a step towards a mutual juridical recognition. On March 8, 1919 Georgia's government recognized the Republic of Armenia as an independent state *de jure*, confirming it with a special note. On March 24 of the same year the minister of foreign affairs of Armenia S. Tigranyan sent an answering note to his Georgian colleague: "I am firmly convinced that the republics, established on the territory of Transcaucasia, have common interests and goals in a number of questions, which are very important for their steady existence and prosperity., the Republic of Armenia is glad and sees solidarity and the confirmation of interrelations in this act. For its part the government of Armenia thinks it their duty to confirm before your government that it considered and considers the Democratic Republic of Georgia an independent state" [8].

"The claims have become moderate. The psychology of confrontation between states has died down. The wishes have been put in order. It has become clear to everybody who had the right to demand what, and who could give up what... The energy of self-sacrifice and readiness for sacrifice has made it clear what rights each of us have and shed light on unjustified violence as well," the Georgian press remarked concerning the fact of Armenian and Georgian relations [1: 1].

Two important agreements were concluded in Tbilisi on November 3, 1919. According to the first one Georgia and Armenia undertook to solve all disputes, existing at present or that might arise between these states in future, by agreement, but if no agreement was reached to solve it by obligatory arbitration. The other document envisaged free transit of Armenian goods on the Georgian railway for three years.

The agreements were signed by the Minister of Internal Affairs and the Defence Minister of Georgia N. Ramishvili, the Deputy Chairman of Georgia's Constituent Assembly S. Mdivani, the representative of the government of the Republic of Armenia S. Mamikonyan and member of Armenia's Parliament Khachaturyan. Both agreements were subjected to ratification. The exchange of the instruments of ratification was to take place in the capital of Georgia within two weeks.

Upon proper consideration of the agreements, the committee of foreign affairs and the juridical committee of Georgia's Constituent Assembly passed the agreements to the supreme legislative body for ratification. The Constituent Assembly convened a special session on November 14, 1919 to ratify the agreements.

The Social-Democrat K. Japaridze, member of the committee of foreign affairs of the Assembly presented the question for discussion at the plenary meeting.

The deputies discussed the good and bad sides of the agreements separately. Japaridze said, "Last year's sad conflict has been forgotten since an agreement was reached with Armenia, the conflict entered our history as a dark spot. Some people thought that this event meant relations between Georgia and Armenia were spoilt for ever. But such a thing cannot happen to nations that are joined and interlocked by historic destiny" [7: 4, Nov. 14].

The Minister of Foreign Affairs Evgeni Gegechkori spoke on behalf of the government of Georgia. According to him, owing to the concluded agreements, Georgia would be acknowledged by the whole of democratic Europe as a republic of progressive, cultural principles. "This agreement is a guarantee that the disaster that happened between our country and Armenia will not occur again. It is of great real value, and it depends upon both of our peoples that we should use this beautiful sharp weapon as people's interests demand."

"I cannot imagine such an arrangement of Transcaucasia where one nation will be happy and free and others will be wretched and enslaved... Solidarity of nations is the means that will bring prosperity to our republics and will establish the necessary conditions for our existence and development." Gegechkori stated [7: 7-8, Nov. 14].

The socialist-federalist Samson Pirtskhalava approved of the agreement, concluded with Armenia, welcoming it. "Considering the past life of Georgia, the wars that devastated half of Georgia and destroyed the cultural creations of the nation, considering the important questions facing Georgia, we have no right to choose any other way, but the way of agreement with the neighboring peoples," said the deputy [7: 9, 19 XI].

The socialist-revolutionary Leo Shengelaia called the agreement a new victory of democracy and the first step made towards the welfare of the Georgian and Armenian peoples. "The agreement is a splendid proof of the fact that the expectations of nationalists did not come true... Neither the war nor the chauvinistic poison killed the aspiration to democracy of both nations, their aspiration to solidarity, establishing good relations between themselves... No arms will judge us henuforth. War and imperialistic policy are rejected. We believe that the working democracy of Georgia and Armenia will not deviate from this path," he said [7: 12, Nov. 14].

Ter-Stepanyants spoke on behalf of the Dashnaktsyutun. He greeted the agreement and emphasized the following, "This great issue will facilitate the solidarity of Georgia and Armenia's democracy" [7: 17, Nov. 14].

While discussing the agreement, socialists of every colour accentuated only the significance of the peoples' solidarity and cooperation. As for the right-wing opposition – the National-Democratic and National Parties of Georgia, they considered the question from the viewpoint of national and state interests.

The chairman of the National-Democratic faction Giorgi Gvazava focused attention on the international resonance of the Georgian-Armenian agreement and he found it positive from that point of view. "Our situation today is such that this agreement will be considered very important in Europe. Many false and spurious rumors are spread in Europe about us and the peoples of Transcaucasia in general. When they hear about it in Paris where the questions on nations are decided, of course, such a direction of our policy will make a good impression. They will change their opinion of us. From this point of view this act is a good step," Gvazava remarked [7: 14-15, Nov. 14].

The leader of Georgia National Party – Grigol Veshapeli compared the agreement, presented for ratification, with the military-defensive agreement, concluded with Azerbaijan in June 1919, emphasizing that: "An agreement should not begin with economic questions. It should begin with the political alliance which will make it the duty of the participating countries to defend their neighbor from the aggression of a foreign force." Herewith, Veshapeli expressed his suspicion that "such an agreement will remain a declaration and will not have any real political significance in the life of the Caucasian nations before our neighbor – the Armenian nation – acknowledges that it is absolutely necessary to establish a political alliance of defense of the Caucasian republics whose aim will be the defense and consolidation of the freedom of independent national states" [7: 12, Nov. 14].

The chairman of the National-Democratic Party – Spiridon Kedia found the agreement on free transit defective and considered it damaging for Georgia. "Transit is one of the important sources of Georgia's Treasury and... our government has given away such a factor to Armenia gratis," he remarked. This oppositionist deputy also noted that the government had not paid any attention to the fact of the transit of military material to Armenia via Georgian territory, which he thought quite inadmissible.

The speaker Konstantine Japaridze defended the position of the majority, saying: "If you do not make any concessions, you will not be able to reach an agreement. This concession, was prompted by our wish to have good-neighborly relations with Armenia and we are sure that all the disagreement will soon come to an end, the whole Caucasian democracy will stand on the grounds of common interests and will guarantee our free existence and future" [7: 19-20, Nov. 14].

The ratification of the agreements was put to vote separately. The Parliament factions of the National-Democratic and National Parties abstained from voting. The presented documents were ratified by the majority of votes. The Armenian Parliament too carried out the ratification successfully.

The High Commisioner of Britain in the Caucasus – Oliver Wardrop sent the following telegram to the

Minister of Foreign Affairs of the Democratic Republic of Georgia in connection with this important step towards establishing good-neighborly relations between Georgia and Armenia: "Your Excellency, I hasten to congratulate You and Your colleagues on those splendid agreements whose copies you sent me today, showing Your good will. Accept my deepest regards" [9: 4, Nov. 7].

Both agreements, concluded on November 3, 1919 between Georgia and Armenia came into force officially after the ratification. The free transit, which was mainly an expression of good will on the part of Georgia, was put into operation immediately. Georgia implemented the term of the agreement fully. As for the obligatory arbitration, the government of the Democratic Republic of Georgia did not violate the terms of the agreement in this case either, and was true to them to the end. But the leaders of the Armenian Republic again dithered in this respect, expressed reluctance in solving the problem by international norms, and on every possible occasion presented territorial claims to Georgia, as, for instance at the San Remo conference in April, 1929, etc.

The Democratic Republic of Georgia carried out the course of close cooperation with Armenia and Azerbaijan and the Republic of Mountain Peoples, of "the Caucasian unity," and the unity of economic and military defense of the Region [10: 23].

The Armenian Dashnak government also spoke much of the solidarity of neighboring nations, but when it came to the practical realization of this solidarity, they retreated and refused to join the Georgian-Azerbaijan agreement on defense, aimed at averting the threat coming from the volunteers' army in the first place. The dual nature of the political authorities of the Armenian state was exposed by the Georgian periodical press and quite a few deputies of the supreme legislative body. The "Ertoba" ["Unity"] newspaper wrote: "Georgia, Azerbaijan and the mountain peoples see and feel the coming danger very well. Only the Ararat Republic, blinded by the policy of Dashnaktsyutun, does not see it" [11].

In spite of such disposition, the Georgian political spectrum welcomed the *de facto* recognotion of the Republic of Armenia by the Entente. On receiving this information, the minister of foreign affairs, Evgeni Gegechkori, addressing the meeting of the Constituent Assembly on January 23, 1919, said: "I am quite sure that the international recognition of the Armenian republic will cause great satisfaction of our people and it will be received as very pleasant news. Today it is confirmed once more that the fate and interests of the Caucasian nations are intertwined. The happiness and joy of one nation must be the happiness and joy of the other. It is the circumstance that makes it our duty to fight and act together."

The Constituent Assembly sent a telegram of congratulations to the Parliament of Armenia in connection with the recognition of their independence [9: 17, Jan. 23].

Thus, the Constituent Assembly of Georgia immediately reacted to all the issues of the day of the neighboring Armenia, the issues, connected with their relations as well, and openly expressed its opinion: it condemned what was to be condemned and was unacceptable, and welcomed everything that was reasonable and expedient. The supreme legislative body of the country approved and supported the good-neighborly policy of Georgia's government, among them with the Republic of Armenia as well.

ისტორია

საქართველო-სომხეთის ურთიერთობა 1918-1920 წლებში და საქართველოს დამფუძნებელი კრება

გ. მამაცაშვილი

ი. გოგებაშვილის სახელმწიფო უნივერსიტეტი, თელავი

(წარმოდგენილია აკადემიკოს დ. მუსხელიშვილის მიერ)

სტატიაში შესწავლილია 1918-1920 წლებში საქართველოს დემოკრატიული რესპუბლიკის ურთიერთობა სომხეთის რესპუბლიკასთან და მისი ასახვა საქართველოს უმაღლეს საკანონმდებლო ორგანოში — დამფუძნებელ კრებაში.

საზღვრების გაუმიჯნაობის გამო ორი მეზობელი სახელმწიფოს ურთიერთობა მათი სახელმწიფოებრივი დამოუკიდებლობის პერიოდში საკმაოდ რთული და ცვალებადი იყო.

სომხეთის მხრიდან საქართველოსათვის წაყენებულ ტერიტორიულ პრეტენზიებს 1918 წლის დეკემბერში ამ ქვეყნებს შორის სისხლიანი შეტაკება მოჰყვა. ომმა მეზობელი სახელმწიფოების ინტერესთა წინააღმდეგობა ვერ მოაგვარა, სადავო ტერიტორია დროებით ლორეს ნეიტრალურ ზონად გამოცხადდა.

საქართველომ და სომხეთმა ერთმანეთის სუვერენიტეტი მხოლოდ 1919 წლის მარტში აღიარეს, იმავე წლის 3 ნოემბერს კი თბილისში ორი ხელშეკრულებაც დადეს. პირველი მათგანი დავების სავალდებულო არბიტრაჟით გადაწყვეტას, ხოლო სომხეთის ტვირთებისათვის საქართველოს რკინიგზაზე თავისუფალ ტრანზიტს ითვალისწინებდა.

დამფუძნებელმა კრებამ, ოპოზიციური პარტიების წინააღმდეგობის მიუხედავად, ორივე ხელშეკრულება ხმების უმრავლესობით დაამტკიცა. რატიფიკაცია წარმატებით განახორციელა სომხეთის პარლამენტმაც.

საქართველოს დამფუძნებელი კრება სომხეთთან ურთიერთობის ყველა საჭირბოროტო საკითხზე მყისიერად რეაგირებდა და გამოთქვამდა თავის პირუთვნელ აზრს.

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Linguistics

On Megrelian Verse

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ABSTRACT. The versification system of Megrelian verse is identified and its specificity is shown. The place of Megrelian verse in Common Georgian versification is defined. Comparative analysis with other nonliterary language Georgian verse is offered. The relation of Megrelian verse with the stage of development of Common Georgian versification, reconstructed as the initial (pre-first) stage, is discussed.

Key words: *Megrelian verse, Literary Georgian verse, binary unit.*

1. The basis of Megrelian verse system is a binary structural unit, which consists of two segments, as differential elements. This unit can be considered in two ways. Either as a constituent of a line, as in binary unit, let us refer to it as a binomial (in this case, a line will be a two-binomial structural unit); e.g. (4+4) + (4+4) (Megrelian texts have been published repeatedly [1-5]):

p'at'onepi gamigonit ate čkimi guriš č'ua
"*Gentlemen, please understand my heartache*".

Or it can be considered as a line (in this case it materially coincides with a binomial; on this coincidence and the corresponding diachronic law, see [6:22-23]), e.g. 4+4:

p'at'onepi gamigonit
ate čkimi guriš č'ua

In the present article it will be considered as a line. Other, higher level units, in particular, stanza, will not be discussed here.

The structure of the main, basic binary unit is: n+k, where n ≥ k. In Megrelian this n+k is 4+4; it may be 4+3, which occurs more frequently in heterometric stanzas alongside 4+4; rarely there may be 4+2, and extremely rarely 4+1 and 4+0 (all three cases in practice belong to exceptions and atypical structures). In these records figures denote syllabic length of each constituent of a binary unit; + indicates rhythmic boundary between two constituents; rhythmic boundary is the main generating factor of the Georgian verse system.

In Megrelian verse 5+5 also occurs, but, firstly, it is very rare, and, secondly, it is not an organic element of the Megrelian system, being a result of the influence of Literary Georgian.

Thus, the first member in a binary unit is always 4, and the second – 4, 3, 2, 1 (n+k, where n=4, k=4, 3, 2, 1, 0).

At the same time, a conclusion can be made from the very beginning: the principal structure of Megrelian verse is identical to that of Literary Georgian verse, in which units are of binary structure, including the smallest rhythmic unit (binomial), which has the following structure:

n+k<=> (n-2)+(k+2),
where n ≥ k.

In this symbolic record, which reflects the Megrelian situation as well, it is noted that a binomial, along with the basic variety, also has an alternant, in which the rhythmic boundary is moved back by two syllables [6:31} as compared to the main variety (Silagadze 1987: 31). This phenomenon is recorded in Megrelian verse, in particular, the structure 4+4 is replaced by 2+6; e.g.:

vai iši mumašeni 4+4 *"Woe to his father*
mapaš mozoğuašeni 2+6 *due to the arrival of the king."*

All the above-mentioned is a conclusion of preliminary character, which needs correcting. Namely, the above-mentioned interpretation reflects the initial system of Megrelian verse as a member of the Common Georgian system; the system presented in this way in principle coincides with the common Georgian system.

The "initial system" in this case means that it was not realized in this form finally, in particular, it was corrected according to the peculiarities of Megrelian speech (in detail see below). As regards its definition in this form above, it is possible in two ways: a) taking into account some facts that represent statistical exceptions and ultimately do not reflect typical Megrelian verse (this refers to the constructions 4+2, 4+1, 4+0); b) with certain reconstruction.

The point is that the system of Megrelian verse by its essence is two-leveled: the initial system + local specificity.

2. As regards local specificity, it is entirely based on the fact that in Megrelian – ordinary speech – there are two optional variants/styles of pronunciation/speaking. This phenomenon is known and described, particularly – in detail see [5: 356-366].

The phenomenon lies in the fact that in Megrelian speech there are doublets for the absolute majority of words/forms; what is most important, they are of different syllabic length.

Specifically, these are the following cases:

a) in the word auslaut after a consonant there may appear a narrow vowel *i, u* or the so-called neutral/irrational vowel. As a result, there are always two variants of a word, with two different number of syllables: *tak/taki* – *"here", č'aruns/č'arunsi* – *"he writes",* and so forth. b) the opposite phenomenon: if a word ends with a narrow vowel, it may always be used by the loss of this vowel as well, accordingly, it will be shorter by one syllable: *koči/koč* – *"man," doč'aru/ doč'ar* – *"he wrote",* etc. c) the so-called long, more exactly, double vowel optionally is always substituted by one vowel: *kovotxii/kovotxi* – *"I lent smth to smb", vaoko* → *vaako/vako* – *"He does not want",* etc.

In a phrase the above-discussed phenomenon yields a large number of different variants of syllabic length. For example, the phrase (see [5, 358]) *"žiri rčinu koči mursi"* (8 syllables) can be represented as follows: *"žir rčin koč murs"* (4 syllables), *"žiri rčin koč murs"* (5), *"žir rčin koči mursi"* (6), *"žiri rčin koči mursi"* (7).

In all the above-mentioned cases, as noted, there are two parallel forms for a word. This phenomenon in a verse creates an opportunity of different lengths and structures for a given line, one of which (maybe more than one) gives a marked form from the viewpoint of versification.

3. As mentioned above, the phenomenon of two styles is known and described. The objective is to determine how this phenomenon is activated in verse, in particular, what specific function it has.

In Megrelian verse, in contrast with ordinary speech, according to its function it is not a phenomenon of style, but that of metrics.

Thus, two styles of pronunciation create a certain regulatory mechanism in verse, the function of which is different depending on whether it functions in the first or the second segment.

Specifically, the following two rules can be identified.

a) In the first segment of a binary unit (line) (in *n* part of n+k structure) this phenomenon, if necessary, regulates the number of syllables of the given meter. A positional variant of syllabic length of a segment is created, which is unique for the given meter and all other possible variants are excluded. Thus, in the first segment the functioning of this phenomenon is mandatory, otherwise, the metric scheme will be violated. For example:

5+4 → 4+4:

xoği gilurcu didi kami → xoğ gilurcu didi kami

("a bull is going with big horns").

b) This rule is actual for the second constituent of a binary unit as well: here too, if necessary, it regulates the number of syllables. At the same time, we can formulate a rule which is specific to the second segment of a line (for *k* part of $n + k$ structure). In the second segment its functioning has the same character as in ordinary speech, where the use of parallel forms of words is optional. In particular, this time in verse more than one variants of syllabic length appear for the second segment, each of which is of equal force, none of them is rejected. This time the most important thing is not to violate the basic rule of a binary unit, according to which, the second segment should not be longer than the first one ($n \geq k$). In fact, in this way one of the main features of Megrelian verse is formed – that there are different optional variants for a meter given in it (see below).

Finally, in the first segment of the potential opportunities only one is activated – which is necessary for the structure; in the second segment several variants of equal force emerge.

4. The peculiar character of Megrelian verse is obvious at the time of its comparison with Literary Georgian verse.

It was noted above that in Georgian literary verse, as well as in Megrelian verse, the basic structural unit (binomial, line) is n+k structure. The system of this unit in Georgian verse is represented by three subsystems:

I	II	III
5+5	4+4	3+3
5+4	4+3	3+2
5+3	4+2	3+1
5+2	4+1	3+0
5+1	4+0	5+0

In Megrelian in place of the structure represented above there is only the following:

4+4
4+3
4+2
4+1
4+0

In other words: what in Literary Georgian versification is one subsystem within a system consisting of three subsystems, in Megrelian is a single system.

This situation in itself constitutes the specificity of Megrelian verse. But neither this formulation is accurate and reflecting fully the main characteristic of Megrelian verse. Obviously, the following provision may be formulated: the specificity of Megrelian verse is created by the fact that, unlike Georgian literary verse (as well as Georgian dialectal verse), in the Megrelian system practically one meter is represented 4+4 (n=k) with its allometres (as far as I know, this term is not used in scholarly circulation): 4+4, 4+3 (also 4+2, 4+1).

Such a conclusion is based on these arguments: a) in the Megrelian system n+k is 4+k (i.e., the structure in which the rhythmic boundary is after 4 syllables); unlike the Georgian literary system, it is not familiar with structures 5+k and 3+k (i.e. those in which the rhythmic boundary is elsewhere – after 5 and 3 syllables). b) What is most important, the great part of Megrelian verse material shows specimens which can be optionally interpreted in two ways – as 4+4 and 4+3. When such specimens are discovered where one, autonomous structure of line is represented (4+4, more rarely 4+3), as a rule, inside them appear certain fragments or separate lines which include parallel forms. (As regards 4+2 and 4+1 structures, they are rarely found, mostly next to 4+4; along with this, in some cases the analysis can reveal the initial structure 4+4).

Thus, the provision may be formulated that the Megrelian versification is based on one meter, which is represented by allometres (/alternants/particular manifestations). Allometres in this case means not variants conditioned by a certain mechanism, which have their space/position of use, but that these variants are applicable optionally/voluntarily.

If this conclusion is controversial because in the Megrelian verse text corpus there may be discovered some specimens (of insignificant quantity), which do not correspond to 4+4 structure, in particular, they correspond to 5+5 structure, then the provision can be formulated as follows: the initial system of Megrelian verse is based on one meter (4+4), i.e. the system which is typical and reflects the specificity of Megrelian proper, not innovations (on innovations, see below.).

This situation creates a fundamental difference from Georgian literary verse: where in Georgian literary verse (as well as in dialectal) there are autonomous meters 4+4, 4+3, etc., in Megrelian there is one meter with its alternants. This specificity of Megrelian verse is completely due to the peculiarity of Megrelian speech – two styles.

The picture of realization of Megrelian versification meter is as follows: on the one hand, we have verse specimens, based on some one binary structure (one allometre): 4+4 or 4+3 (as noted above, in such cases in some fragments of a given verse text parallel structures will be revealed as well). On the other hand, we have specimens that are interpreted optionally (/ are read/comprehended) in the form of one structure as well as the other, – they may be interpreted either in one way or in the other.

The specificity of Megrelian verse that a verse text gives an opportunity to be read in several ways – two, three or more, is reflected in practice in the following way.

I

šio, čit'i, kopurini,	4+4	*"shoo, bird, fly,*
kemeuği tena tisi,	4+4	*bring it to her,*
dio xesi kemeči do,	4+4	*first hand it to her,*
uk'ul kaağudi p'isi.	4+4 (2+6)	*then kiss her on her mouth".*

II

šio, čit'i, kopurin,	4+3
kemeuği tena tis,	4+3
dio xes kemeči do,	4+3 (2+5)
uk'ul kaağudi p'is.	4+3 (2+5)

III

šio, čit'i, kopurini,	4+4
kemeuği tena tis,	4+3
dio xesi kemeči do,	4+4
uk'ul kaağudi p'is.	4+3 (2+5)

Other variants are also possible when 4+3 is only in the third line.

At the same time, this variant is impossible:

šio, čit', kopurin,	3+3
kemeuğ tena tis.	3+3

This option is not possible, because the meter structure 4+k is violated in it.

Thus, the realization in verse of variants determined by different styles is limited to a certain degree (as compared with normal speech). The following limiting rule can be formulated (which is a concrete definition of the above-formulated rules). In general, functioning of the mechanism of two styles of speech in Megrelian verse is permissible only within the limits if the meter structure is observed/is not violated. In particular, a) in the first segment of a line (binomial) the four-syllable length (n=4) must be observed (not be violated); b) in the second segment of a line (binomial) free variation within the four-syllable length must be observed (not be violated) (k=4, 3). Finally:

$n+k$

where $n=4$, $k \in \{4,3\}$,

accordingly, $(4+4) \sim (4+3)$.

This symbolic record adequately reflects the specificity of Megrelian verse; cf. for Literary Georgian:

n+k

where n=4, k=4,3,2,1,0.

When defining the system of Megrelian verse, above all, the fact should be taken into consideration that there exists Literary Georgian versification (/ Common Georgian versification) one of the realizations of which is Megrelian verse. From these positions, as noted above, in the system of Megrelian verse we should see two levels: the first – the initial system, which in principle is the same as the Common Georgian system; the second – the specificity which underlies the initial system.

The above-mentioned qualification and symbolic record reflect, on the one hand, the initial system, and, on the other, its correction, determined by the specificity of Megrelian speech.

The initial system of Megrelian verse is the system of the basic binary unit of Common Georgian, in particular, its subsystem which begins with 4+4: 4+4, 4+3, 4+2, 4+1, 4+0. Correcting, determined by Megrelian speech, is the action of two styles at the time of realization of this subsystem, which implies emergence of two parallel forms for every word, the difference between the syllabic lengths of which is one syllable; in verse, namely in the structure 4+4, this gives free alternation of the constructions 4+4 and 4+3 (difference in one syllable of syllabic length); as a result, the situation is created which neutralizes the metric opposition (4+4): (4+3), due to which the Megrelian system is not a system of independent meters 4+4, 4+3, but a single system with allometers: 4+4 ~ 4+3.

The reality of the described picture is indicated by the answer to the following question: Why were 4+2 and 4+1 (as well as 4+0) not included in the system? Again due to the factor of two styles of speech: on the one hand, for 4+4, as for the basic structure, a variant differing in one syllable will be 4+3, on the other hand, the constructions 4+2 and 4+1 create 2- and 3-syllable differences.

Specification of the symbolic record leads to the following principal conclusion: no matter what kind of allometres a line includes (i.e., whatever the length of the second segment may be in conditions of the four-syllable length), this is the same meter 4+4. It makes no difference for bearers of verse speech whether at a given passage in a verse 4+4 or 4+3 will be read. In other words, as in Megrelian verse koči/koč, č'aruns/č'arunsi, etc. are the same, so in verse 4+4, 4+3 (as well as 4+2, 4+1) are the same.

The conclusion can also be formulated in the following way: when speaking about allometres we actually go beyond the reality of the Georgian literary (/Common Georgian) system. As regards the reality of Megrelian verse proper, here we have one meter at the time of implementation of which it does not matter by means of which "allometre" a given verse text will be interpreted. At the metric level the Megrelian system practically is familiar only with the rule n=k.

Finally, the meter functioning in Megrelian verse is 4+4 of the Common Georgian system, specifically realized in Megrelian.

The conclusion on the principled nature of Megrelian is interesting from the viewpoint of one question – the typological qualification of Georgian versification. Here the main point is that for Megrelian verse it does not matter how many syllables there are in a line (seven or eight), the most important is the position of the rhythmic boundary (after 4 syllables). This situation is one more argument for the fact that the typology of Georgian versification is determined by the rhythmic boundary (and not by the number of syllables, the more so – by stress; see [7]).

5. It may be noted that the existence of two styles in Megrelian is due to the fact that Megrelian is an oral language (not a literary language, written language), and as such (as a living language), it is not subject to the experience of standardization/normalization at all, and in particular by the literary way (written recording). The literary language for Megrelian is the Georgian literary language (which does not have the so-called literary dialect – personally for me this is a essential provision; it, as a single/common language, was created in a far earlier period).

In the same context, the above-discussed peculiarity of Megrelian verse is due to the fact that it represents oral speech/folklore. Obviously the following general provision will be correct: folklore (in our case – folk verse), is a dialectal (i.e. non-literary language) form. Thus, Megrelian verse is folk/dialectal verse. It should be stressed that "dialectal" in this case does not refer to a dialect proper, but to a "non-literary" language, i.e. everything that is different from the literary language, whether it is a dialect proper, or, in the case of our reality, Megrelian, Laz and Svan, which can be called branches of common Georgian. In this case we have one opposition – the Georgian literary language: Megrelian/Laz/Svan/dialects.

Megrelian verse typologically is Georgian folk/dialectal verse.

Folklore, as the so-called collective creation (see [8, 373]), along with everything else, differs from literary (i.e. individual) creation in that in it fuwer forms are presented (quantitatively and qualitatively): folklore is a lesser form than literature, in this regard it is more stable (cf.[9, 91-92]); namely, folk verse, this is verse with a less form/less meter than literary.

In this context the following provision can be formulated: Georgian literary versification is conventionally a complete system; Georgian folk/dialectal verse (including Megrelian) is incomplete.

This means that a single, Georgian (Common Georgian) system is considered.

The following provision is correct: there is one, Georgian (Common Georgian) versification – a single system that is implemented by the following representatives: Georgian literary verse, folk/dialectal verse (Megrelian, Laz, Svan verse, verse of the dialects of the Georgian literary language).

Georgian (/Common Georgian) Versification

Georgian literary verse | Georgian dialectal/folk verse (verse of the Megrelian, Laz, Svan branches; verse of dialects)

Within the diagram, the left side – Georgian literary verse – is opposed by the entire right side; at the same time, "Georgian literary verse" is Common Georgian literary verse (which is literary verse for branches and dialects):

Common Georgian literary verse: { Verse of Megrelian, Laz, Svan branches; verse of dialects }

In this case, the following reasoning is of principled character: for dialectal/folk verse there exists a single/Common Georgian literary verse. (Similar as there is, on the one hand, functionally one/common Georgian literary language, represented by full realization – in the form of the graphic/written language and in the form of the common oral language, and on the other hand, everything else – the Megrelian, Laz, Svan branches, dialects. Along with this, this language is also common because it is created by everyone, on the common basis; as far as I know, I. Javakhishvili was the first to express this view, see [10: 154]).

If we return to Megrelian verse, the following is noteworthy: Megrelian verse is a member of Georgian (Common Georgian) versification. Here the following reasoning is of principle: the language – in this case not as prosody, but text material – is not important: by the prosodic peculiarities that are characteristic of Megrelian, a member of the Common Georgian versification is created; or: this prosody serves to create a specific member of Georgian versification, because, as mentioned, we have a single Georgian/Common Georgian versification, evidenced by a certain number of representatives.

6. The fact that Megrelian verse is one of the representatives of Georgian versification is formally clearly demonstrated by the following experiment. Let us take again the basic system of Literary Georgian versification (that of a binomial, line) with three subsystems, and move it to Megrelian.

In Megrelian practice the first and third columns will make gaps. We can fill these gaps at the expense of a certain transformation of some actually existing Megrelian verse text. For example, let us transform the stanza

iro mulas ič'aruki,	*"You always write that you are coming,*
iro mulas ičina.	*You always send word that you are coming.*
komic'ii, golvapiro,	*Tell me please,*
mužamiša giğina	*how long shall I wait for you"*

so that the first constituent of the line is 5-syllable, and the second – 3-syllable:

si iro mulas ič'aruk,	5+3
si iro mulas ičina.	5+3
komic'i, goluapiro,	5+3
ma mužamiša giğina.	5+3

Before us is a normal Georgian verse – normal Megrelian verse (for bearers of Megrelian verse speech, who at the same time are bearers of common Georgian verse speech, such a verse will be a marked form). This is 5 +3, the so-called *low shairi* of Georgian versification. But at the same time it is a metric structure, which is not realized in Megrelian verse.

In the context of this experiment the following question can be posed: Which is the reality within which Megrelian verse should be considered? In other words, this is a problem of material: Which is the material that can be considered as a typical Megrelian verse and can be described and characterized according to it. The answer, apparently, is that it is the material/reality according to which Megrelian verse is described as a system, based on meter 4+4 (plus allometres) and on rules of its realization.

The point is that some specimens which exist in reality and are based on other metric structure (in particular, 5+5), firstly, show an insignificant number, secondly, they obviously reflect the situation of the new period and represent a result of the influence of Georgian literary verse. Most importantly, these are specimens in which the action of two styles characteristic of Megrelian speech is not reflected, in particular – free variation within the number of syllables characteristic of the second segment of a line.

It is possible to draw a conclusion: the material that reflects typical Megrelian verse ends where the action of two styles ends, which creates in verse various free variants for one and the same metric structure.

As to the indicated rare specimens, they should be regarded as an attempt of filling the gaps which were discussed above. From the positions of principled reasoning, it is a conscious act in order to introduce innovations (innovations – on the basis of Georgian literary verse), – in fact, individual creation. In this way, it is possible to increase the number of Megrelian verse meters, but such specimens will not create typical Megrelian verse. More exactly: they do not belong to the initial system of Megrelian verse (i.e. the system which undergoes innovations). It may also be in this way: this is no longer a typologically folk/dialectal verse, i.e. Megrelian verse as a peculiar, original variant of Georgian versification.

In the above-mentioned context general reasoning is also possible. At present (in the conditions of the present-day general realities), the opposition folklore: literature (in particular, folk verse: literary verse) is transformed and is qualitatively different. From the viewpoint of forecast an opinion may be offered: the tendency to abolishing this opposition is developing.

7. For a full description it is interesting to discuss one more question – relation of Megrelian verse to other non-literary Georgian verse, in particular, to verse of the dialects.

Here the main point is that, unlike the dialects (as well as the Literary Georgian language), *low shairi,* based on 5+3 structure, is not realized in Megrelian.

The entire picture is as follows: in west-Georgian verse 4+4 (*high shairi*) is dominant, 5+3 is less used, which predominates in east-Georgian verse, along with this, in east-Georgian mountain verse actually only 5+3 is represented only. In Megrelian there is 4+4 (with its alternants), 5+3 does not occur at all. Thus, Megrelian opposes the overall picture: a) opposes west-Georgian, b) strongly opposes east-Georgian, and c) completely opposes east-Georgian mountain verse (Khevsur, Pshavian).

By which feature can this situation be characterized? In this case, attention mostly attaches to one feature – 5+3 is a binary structure, based on the inequality of the constituents: the first constituent is longer than the second (n>k). As we can see, Megrelian verse rejects such a structure. In this context the important thing is that, on the one hand, in innovations Megrelian generally allows meters in which the rhythmic boundary is after 5 syllables, but introduces only those in which the first constituent is equal with the second 5+5 (n=k); on the other hand, it rejects such a structure having the same boundary which is based on the inequality of the constituents, in particular – in which the first constituent is longer than the second one (n>k) – 5+3.

8. This situation has a direct connection with one more question. This is the relation of Megrelian verse to the stage of diachrony of Common Georgian versification which is reconstructed as the pre-first (pre-fixed) stage in the development (here not the common-Kartvelian pre-system is implied, but exactly the stage which must have preceded all the known, confirmed stages in the development of Georgian literary versification).

Reconstruction shows here the following picture [6: 93-107; 11: 163-164]. The initial (pre-fixed) stage of Georgian versification is a state in which in a basic, low-level, binary unit of the system (as well as in binary units of all other levels) only the rule of two equal constituents n=k is at work, i.e. the second rule n>k does not apply (finally, the universal rule n≥k, formed later, does not apply), on the basis of which, e.g. *low shairi* having the 5+3 structure is constructed.

As regards Megrelian verse from this viewpoint, such a binary unit which is made up of two equal constituents functions (in a specific form) exactly in Megrelian.

In this respect, the fact is especially noteworthy that 5+3 with unequal constituents (*low shairi*) does not occur in Megrelian at all, a) which is dominant in east-Georgian verse, and is the only one in mountain verse; b) what is the most important, which is realized and is very popular at the first stage of the development of Georgian literary versification. Of these two meters, realized at the very first stage, united under the heading of "*shairi*", Megrelian chooses the meter 4+4, based on the principle of equality n=k, and categorically rejects the meter 5+3, based on the principle of inequality.

The following conclusion can be made: 1) the situation confirmed in Megrelian verse (resp. in its initial system) directly reflects the state (stage) when in Georgian versification (being common) the principle of division into unequal constituents in a binary unit was not realized, only the principle of equal constituents functioned; 2) the rhythmic/metric form of Megrelian verse developed earlier than the east-Georgian state, based on the dominance of 5+3.

Thus, the reconstructed (initial, starting) stage of Georgian verse, based on the principle of two equal constituents, at the time of verification is confirmed by the versification picture of Megrelian verse, which reflects the oldest stage of the development of Georgian versification (versification proper).

ენათმეცნიერება

მეგრული ლექსის შესახებ

ა.სილაგაძე

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მეგზური ავტორთათვის

1. წარმოდგენილი სტატია დაბეჭდილი უნდა იყოს A4 ფორმატის თეთრ ქაღალდზე ორი ინტერვალით, ზედა და ქვედა არეები – 20-20 მმ, გვერდითი არეები - 30-30 მმ. სტატიის სიგრძე არ უნდა აღემატებოდეს 12 ნაბეჭდ გვერდს, ფორმულების, ცხრილებისა და სურათების ჩათვლით.
2. სტატია წარმოდგენილი უნდა იყოს ინგლისურ ენაზე მთავარი რედაქტორის სახელზე. სტატიას უნდა ახლდეს აგრეთვე:
   - ცნობები ავტორის (ავტორების) შესახებ, მოცემულ შრომაში მისი წვლილის აღნიშვნით;
   - აკადემიის წევრის წარდგინება ან ორი კვალიფიციური რეცენზია სხვადასხვა ინსტიტუტიდან (თუ ავტორი არ არის აკადემიის წევრი); მიმართვა რედაქციის სახელზე გამოქვეყნების თხოვნით იმ დაწესებულებიდან, სადაც შესრულებულია შრომა;

   სტატიას განიხილავს სარედაქციო საბჭო და გადაწყვეტს მისი გამოქვეყნება-არგამოქვეყნების საკითხს.
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   - სამუშაო ადგილი, საფოსტო მისამართი, ელექტრ. ფოსტა;
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   - რეზიუმე;
   - საკვანძო სიტყვები;
   - ტექსტი. ტექსტის ბოლოს დაფინანსების წყაროს და მადლობების ჩამონათვალი;
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4. ელექტრონული ვერსია: CD-ROM, შრიფტი შრიფტი Times New Roman. ტექსტი: Supply – Word, Word Perfect, RTF. ფორმულები შესრულებული უნდა იყოს Equation 3.0 ან MathType 5.0 equation პროგრამის გამოყენებით. ცხრილები და ნახაზები წარმოდგენილი უნდა იყოს ცალ-ცალკე ფაილად, ნახაზები უნდა შესრულდეს TIFF, EPS, მაღალი ხარისხის PDF, JPEG (არანაკლებ 200 dpi), ცხრილები: Word, Word Perfect, RTF.
5. სტატიის ტიპოგრაფიული თუ ელექტრონული სახით გამოქვეყნებაზე ავტორმა უნდა განაცხადოს თანხმობა;
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   სარედაქციო საბჭო არ განიხილავს ნაშრომს, რომელიც მოთხოვნების მიხედვით არ იქნება გაფორმებული.
   ავტორს შეუძლია ნიმუშად გამოიყენოს ჟურნალის რომელიმე ნომერი.
6. ნაშრომის გამოქვეყნება უფასოა.

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